Geometric Langlands Correspondence Near Opers

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Dedicated to C.S. Seshadri on his 80th birthday

Abstract

Let $G$ be a complex, connected semi-simple Lie group, $^LG$ its Langlands dual group, $\text{Bun}_G$ the moduli stack of $G$-bundles on a smooth projective curve $\Sigma$ over $\mathbb{C}$, $\text{Loc}_{^LG}$ the moduli stack of flat $^LG$-bundles on $\Sigma$. Beilinson and Drinfeld have constructed an equivalence between the category of coherent sheaves on $\text{Loc}_{^LG}$ supported scheme-theoretically at the locus of opers and the category of $\mathcal{D}$-modules on $\text{Bun}_G$ admitting a certain global presentation. We generalize it to an equivalence between the derived category of coherent sheaves on $\text{Loc}_{^LG}$ supported at the formal neighborhood of the locus of opers and the localization at $\mathcal{D}$ of the derived category of $\mathcal{D}$-modules on $\text{Bun}_G$ (and an appropriate equivalence of abelian categories).

1. Introduction

The most complete and satisfying form of the geometric Langlands correspondence for a complex reductive group $G$ and a smooth projective complex curve $\Sigma$ has the form of an equivalence of two categories. One of them is the derived category of quasicoherent sheaves on $\text{Loc}_{^LG}$ (possibly, modified) and the other is the derived category of $\mathcal{D}$-modules on $\text{Bun}_G$ (possibly, modified). In the case that $G$ is the multiplicative group, such an equivalence has been proved independently by G. Laumon [L] and M. Rothstein [R]. For general reductive groups, this has been proposed as a meta-conjecture by A. Beilinson and V. Drinfeld (see, e.g., Sects. 4.4, 4.5, 6.2 of [F] for a survey). Recently, important progress has been made: a precise conjectural statement of such an equivalence was proposed by D. Arinkin and D. Gaitsgory [AG], and its proof was outlined [G2] in the simplest non-abelian case of $G = GL_2$. However, there is still no proof known in the general case.
In a different direction, using a link between the category of modules over the affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) at the critical level and the category of suitably twisted \( \mathcal{D} \)-modules on \( \text{Bun}_G \), Beilinson and Drinfeld obtained an equivalence between special pieces in the two categories \([BD]\) (see Sect. 9.5 of [F] for a survey). On one side, this is the category of those coherent sheaves on \( \text{Loc}_L G \) that are supported scheme-theoretically at the locus of \textit{opers} in \( \text{Loc}_{\ell} G \) (the definition is recalled in §3 below) and on the other side, we have the category of \( \mathcal{D} \)-modules \( \mathcal{M} \) on \( \text{Bun}_G \) admitting finite global presentations

\[
(\mathcal{D} \otimes \mathcal{K}^{-1/2})^p \to (\mathcal{D} \otimes \mathcal{K}^{-1/2})^q \to \mathcal{M}.
\]

Here, \( \mathcal{K} \) is the canonical line bundle of \( \text{Bun}_G \). The appearance of a square root \( \mathcal{K}^{1/2} \) (related to the fact that the \( \hat{\mathfrak{g}} \)-modules are at the critical level) makes it more convenient to switch to the category of projective \( \mathcal{D} \)-modules with the same curvature as \( \mathcal{K}^{1/2} \). We call it the category of Spin \( \mathcal{D} \)-modules. Note that on a smooth variety, these are the sheaves of modules over the twisted version

\[
\mathcal{D}^s := \mathcal{K}^{1/2} \otimes_{\mathcal{O}} \mathcal{D} \otimes_{\mathcal{O}} \mathcal{K}^{-1/2}
\]

of \( \mathcal{D} \), which is the algebra of differential operators on \( \mathcal{K}^{1/2} \). (On a stack such as \( \text{Bun}_G \), this is a bit more subtle.) But, at any rate, tensoring with the chosen \( \mathcal{K}^{1/2} \) gives an equivalence between ordinary and Spin \( \mathcal{D} \)-modules.

Note that if \( G \) is not simply-connected, there are several choices of square roots of \( \mathcal{K} \) over \( \text{Bun}_G \): on each component of \( \text{Bun}_G \), the choices form a torsor over the \( \mathbb{Z}/2 \)-valued characters of \( \pi_1(\text{Bun}_G) = H^1(\Sigma; \pi_1 G) \). However, a preferred choice of \( \mathcal{K}^{1/2} \), the \textit{Pfaffian} of the cohomology of the universal adjoint bundle over \( \text{Bun}_G \), arises from a choice of a theta-characteristic (Spin structure) on \( \Sigma \): see [LS] for the construction. The same choice of theta-characteristic appears on the Langlands dual side, in the definition of opers. The relevant categories for different theta-characteristics are equivalent.

The construction of \([BD]\) relies on the double quotient realization

\[
\text{Bun}_G \simeq G[\Sigma^0]\{G((z))/G[[z]],
\]

where \( \Sigma^0 = \Sigma \setminus \{\sigma\} \), and on the description of the center of a completion of \( U(\hat{\mathfrak{g}}) \) at the critical level due to B. Feigin and the first author \([FF]\). We will recall some needed definitions below, though our discussion in Sections 2 and 3 of the background to the problem will be brief; a more detailed discussion may be found in Part III of [F].

It is natural to ask whether the Beilinson–Drinfeld construction \([BD]\) could be extended to an equivalence of larger subcategories on both sides of the Langlands correspondence. The main difficulty in doing this is that, unlike the flag varieties of Lie groups (and conjecturally of loop groups), the double quotient stack \( \text{Bun}_G \) is not \( \mathcal{D} \)-affine; therefore, the functor of \textit{global sections} is not faithful. Faithfulness fails even at the level of derived categories, which are sometimes more forgiving in this respect. The problem become even more pronounced on the Langlands dual side, where the \( \mathcal{D}^s \)-module \( \mathcal{D}^s \) is expected to correspond to the (possibly degree-shifted, depending on one’s conventions) structure sheaf \( \mathcal{O}_{\text{Op}_{\ell} G} \) of the subvariety of opers in \( \text{Loc}_{\ell} G \). The functor \( R\Gamma(\text{Bun}_G;\underline{\_}) \), on \( \mathcal{D}^s \)-modules corresponds to \( R\text{Hom}(\mathcal{O}_{\text{Op}_{\ell} G};\underline{\_}) \) on \( \text{Coh}(\text{Loc}_{\ell} G) \), and the latter loses all information away from \( \text{Op}_{\ell} G \).
Nevertheless, in this note we extend the Beilinson–Drinfeld construction to the formal neighborhood of the locus $\text{Op}_L^G$ of $L^G$-opers inside $\text{Loc}_L^G$. Our first step is the computation of the derived endomorphism algebra of the structure sheaf $\mathcal{D}^*$, and its identification with the Ext-algebra of the structure sheaf $\mathcal{O}_{\text{Op}^G_L}$ of opers in $\text{Loc}_L^G$. Both algebras are strictly commutative ($A_\infty$-equivalent to strictly commutative algebras), and this suffices to establish an (abstract) equivalence of formal deformation theories of $\mathcal{D}^*$ within the category of $\mathcal{D}^*$-modules on $\text{Bun}_G$, with that of $\text{Op}_L^G$ within $\text{Loc}_L^G$. On general grounds, this implies an (abstract) extension of the Beilinson–Drinfeld construction to the formal neighborhoods of $\mathcal{D}^*$ and $\mathcal{O}_{\text{Op}^G_L}$ in the corresponding categories and derived categories. (The Ext-algebras are the Koszul duals, relative to $\mathcal{D}^*$, respectively $\text{Op}_L^G$, of the formal neighborhoods of the two objects.)

An interesting question is to find a canonical isomorphism between the respective categories. Indeed, the original construction in [BD] is essentially canonical, and it intertwines the Hecke functors on the $\text{Bun}_G$ side and certain functors on the $\text{Loc}_L^G$ side (see also [AG]). Recall that Hecke functors $H_{x,V}$ are parametrized by points $x \in \Sigma$ and representations $V$ of $L^G$. A Hecke eigensheaf is a $\mathcal{D}^*$-module $\mathcal{M}$ which, under the action of $H_{x,V}$, gets tensored with the fiber at $x$ of the flat $V$-bundle over $\Sigma$ associated to a particular $L^G$-local system. That local system is then the “Hecke eigenvalue” of $\mathcal{M}$, a point in $\text{Loc}_L^G$. This pins down the Langlands correspondence at the level of generic points of $\text{Loc}_L^G$. Beilinson and Drinfeld identify the algebra of global sections $\Gamma(\text{Bun}_G; \mathcal{D}^*)$ with the functions on $\text{Op}_L^G$, and the resulting “decomposition” of the sheaf $\mathcal{D}^*$ over $\text{Op}_L^G \subset \text{Loc}_L^G$ turns out to be nothing but the spectral decomposition under the Hecke action.

In the present paper, our generalization of their construction is only canonical to first order: first-order deformations of $\mathcal{D}^*$ in the category $\mathcal{D}^*_{\text{Bun}_G}$-mod are canonically identified with those of $\text{Op}_L^G$ in $\text{Loc}_L^G$. Since the eigensheaf quotients of $\mathcal{D}^*$ correspond to points of $\text{Op}_L^G$, we believe that they should deform to Hecke eigensheaves to all orders\(^1\) with their Hecke eigenvalues being the formal paths in $\text{Loc}_L^G$ starting at points in $\text{Op}_L^G$. Confirming this, and showing that we thus extend the Beilinson–Drinfeld spectral decomposition formally to all orders, requires a finer analysis which will be made in a follow-up paper.

Our main result in this paper, the cohomology calculation of $\S$ goes back some 15 years. Although we lectured on it on numerous occasions, it took us a long time to write it up. Our reluctance was due to the tantalizing proximity of a much stronger result which would follow, if only some general facts about Hecke eigensheaves were known. Indeed, our cohomology calculation and unobstructedness results provide the Jacobian test for étaleness of the “Hecke eigenvalue” morphism, from the moduli stack of simple Hecke eigensheaves to $\text{Loc}_L^G$. The categorical Langlands correspondence predicts an isomorphism\(^2\) between the moduli stack of the former objects and $\text{Loc}_L^G$; furthermore, at the level of points, it has been largely proved for $\text{GL}(n)$ [FGV, G1]. If we had known a priori that the aforementioned moduli stack were locally of finite type, and that the Hecke eigenvalue morphism were algebraic, its étale property near $\mathcal{D}^*$ would follow.

\(^1\)This would be automatic if we knew that these eigensheaves are irreducible, but this is not known at present.
\(^2\)This prediction must be adjusted at the singular points of $\text{Loc}_L^G$, see [AG].
Unfortunately, these hoped-for properties of the moduli of Hecke eigen-$\mathcal{D}^*$-modules are still unsettled. What’s even worse is that it is not known whether the Hecke eigensheaves constructed in [BD] are in fact irreducible. This is why the extent of our results is not quite what we had originally hoped for. We do have a non-trivial statement nonetheless, the one about the formal neighborhood of opers, which we present here.

It is a special pleasure to contribute this paper to a collection honoring C. S. Seshadri, whose work on vector and principal bundles on the Riemann surfaces has been a cornerstone for so much rich and beautiful mathematics developed over the past few decades.

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1.1 Remark. To avoid any confusion, we note that there are actually two different versions of the categorical Langlands correspondence. Indeed, recent work by Kapustin and Witten [KW] in 4D gauge theory also predicts an equivalence of categories, which is purely topological in nature and is not sensitive to the algebraic structure of $\Sigma$. It is different from the one we discuss here (and the one in [AG]). The reason why there are are two equivalences is that there are two algebraic structures on $\text{Loc}_{\mathcal{D}G}$, the Betti structure, which identifies $\text{Loc}_{\mathcal{D}G}$ with the representation variety of $\pi_1(\Sigma)$ in $LG$ and the de Rham structure of flat bundles (they are linked pointwise via the monodromy of a connection). The former is not sensitive to the algebraic structure of $\Sigma$, but the latter is – and in fact, $\Sigma$ can be recovered from it. Correspondingly, there are two algebraic structures on the moduli of (regular holonomic) $\mathcal{D}$-modules with singular support in Laumon’s global nilpotent cone, which are related by the Riemann-Hilbert correspondence: the Betti structure, seeing the monodromy representation and extension data between the various strata, and the de Rham structure arising from the underlying $\mathcal{D}$-module. The second is definitely sensitive to the algebraic structure on $\Sigma$, whereas the former is not believed to be so. These algebraic structures are meant to match pairwise under the Langlands correspondence. Indeed, any natural construction of the correspondence would make such a match, but we don’t have a natural construction of the correspondence at present.

2. Warm-up: the case $G = LG = \text{GL}(1)$

We describe the GL(1) case of the geometric Langlands equivalence, due to Laumon [L] and Rothstein [R], and in the process identify the portion which corresponds to the Beilinson–Drinfeld construction. Here, we opt for a simplification which ignores the global automorphism group GL(1) of line bundles over $\Sigma$, as well as the group of components of Pic($\Sigma$). (These two groups are interchanged under the Langlands duality.)

(2.1) Fourier-Mukai transform. Let $A$ be an Abelian variety and $A^\vee$ its dual, $\mathcal{D} \rightarrow A \times A^\vee$ the Poincaré bundle and $\pi, \pi^\vee$ the projections of the product to the two factors. The Fourier-
Mukai transform

\[ \Phi : \mathcal{D}coh(A) \to \mathcal{D}coh(A^\vee), \quad \Phi(\mathcal{I}) := R\pi_*^{\vee}(\mathcal{P} \otimes \pi^*\mathcal{I}) \]

is an equivalence of the derived categories (with arbitrary decorations: +, -, b).

Following Laumon and Rothstein, let us enhance this as follows. The cotangent bundle \( T^\vee A \) is trivial, projecting by a map \( p \) to the vector space \( H := T^\vee_0 A \). There is a similar projection \( A^\vee \times H \to H \). We can construct a Fourier-Mukai equivalence \( \mathcal{D}coh(T^\vee A) \cong \mathcal{D}coh(A^\vee \times H) \) relative to \( p \) by the same formula, from the correspondence diagram

\[ \begin{array}{ccc}
T^\vee A \times A^\vee & \xrightarrow{T^\vee \pi} & T^\vee A \\
\downarrow{p \times \text{Id}} & & \downarrow{H \times A^\vee} \\
H \times A^\vee & & 
\end{array} \]

(2.2) Deformation. There is a distinguished deformation of \( \mathcal{D}coh(T^\vee A) \), the non-commutative deformation of \( T^\vee A \) defined by the natural symplectic form (‘quantization’). One natural implementation as a genuine deformation, as opposed to a formal one, is the category of coherent \( \mathcal{D} \)-modules over \( A \). There would be a 1-parameter deformation, but the scaling of the cotangent bundle identifies all non-zero deformed categories in the family.

Recall now that \( T^\vee_0 A \) is naturally identified with the complex conjugate of \( H = T^\vee_0 A \). There results a 1-parameter deformation \( A^p \) of the space \( A^\vee \times H \) — this time, in the commutative world — to an affine bundle over \( A^\vee \), classified by \( \text{Id} \in H^1(A^\vee; \mathcal{O} \otimes H) \). The key observation of Laumon and Rothstein is that the Fourier-Mukai equivalence relative to \( p \) deforms to an equivalence between the deformed categories \( \mathcal{D}_{\text{coh}}\mathcal{D}_A\text{-mod} \) and \( \mathcal{D}coh(A^p) \).

(Here \( \mathcal{D}_{\text{coh}}\mathcal{D}_A\text{-mod} \) is the derived category of coherent \( \mathcal{D}_A \)-modules.)

It is important to note that \( A^p \) no longer projects onto \( H \); instead, the fiber of \( A^p \to A^\vee \) over \( 0 \in A^\vee \) is identified with \( H \), as a “section” of the no longer existent projection. Intrinsically, \( A^p \) is the moduli of flat holomorphic line bundles on \( A \), and the Poincaré bundle \( \mathcal{P} \) deforms to the universal bundle \( \mathcal{P}' \to A \times A^p \), which has a flat connection along the \( A \)-factor. The deformed Fourier-Mukai transform uses the push-forward of the de Rham complex of \( \mathcal{P}' \otimes \mathcal{M} \):

\[ \Phi^p : \mathcal{D}\mathcal{D}_A\text{-mod} \to \mathcal{D}coh(A^p), \quad \Phi^p(\mathcal{M}) := R\pi_*^{\vee}(\Omega^\bullet(\mathcal{P}' \otimes \pi^*\mathcal{M}), d). \]

A homological algebra exercise shows that this construction becomes the old \( \Phi \) relative to \( p \), when the deformation degenerates back to the “classical” product case.

(2.3) The abelian Beilinson–Drinfeld construction. Let now \( A \) be the Jacobian \( \text{Pic}^0 \) of our curve \( \Sigma \). In this case, \( A^p \) can also be identified with the moduli space of GL(1)-local systems on \( \Sigma \). Note that, as \( A \) is principally polarized, \( A \) is isomorphic to \( A^\vee \), but this does not yet play a role. Let \( \text{Op}_G \subset A^p \) be the fiber over \( 0 \), which is the space \( H^0(\Sigma; \Omega^1) \) of differentials; let us call it the space of GL(1)-opers.

\footnote{Replacing GL(1) by a torus would make \( A^\vee \) into the moduli space of principal bundles for the dual torus.}
One easily determines that $\Phi^\flat(D) = \mathcal{O}_{\text{Op}_L G}[-\dim A]$, and that

$$\text{End}_{DA}(D) = \Gamma(A; D) = \text{Sym} T_0 A = \mathbb{C}[\text{Op}_L G] = \text{End}_{\mathcal{F}_{\text{coh}(\text{Loc}_G)}}(\mathcal{O}_{\text{Op}_L G}).$$  (2.4)

Exploiting this and the flatness of $\mathcal{D}$ over its global sections, we obtain the following

**2.5 Theorem.** (i) Any $\mathcal{D}$-module $\mathcal{M}$ which admits a finite free global presentation $\mathcal{D}^{\oplus p} \to \mathcal{D}^{\oplus q} \to \mathcal{M}$ has a finite free global $\mathcal{D}$-resolution.

(ii) The Fourier transform $\Phi^\flat(\mathcal{M})$ of such a module is a coherent sheaf supported on $\text{Op}_L G$, shifted into degree $\dim A$.

(iii) $\Phi^\flat$ restricts to an equivalence between the abelian category of $\mathcal{D}$-modules with free global presentations to $\text{Coh}(\text{Op}_L G)$. Thereunder, the functor $\Gamma(A; \_)$ corresponds to $\Gamma(\text{Op}_L G; \_)$.

(iv) The structure sheaf $C_\omega$ of a point $\omega \in \text{Op}_L G$ corresponds to the $\mathcal{D}$-module $D \otimes \Gamma(D) C_\omega$.

Note, in passing, that the $\mathcal{D}$-modules in the theorem all have trivial underlying vector bundles.

**2.6 Remark.** The $\mathcal{D}$-modules $\mathcal{D} \otimes \Gamma(\mathcal{D}) C_\omega$ have the following Hecke eigensheaf property. A degree zero divisor $D \subset \Sigma$ defines a line $\omega^D$, the tensor product of the fibers over $D \subset \Sigma$ of the flat line bundle classified by $\omega$. Let $T_D : A \to A$ denote the translation by $\mathcal{O}(D)$. (these translations are the abelian Hecke correspondences.) There exist, then, functorial isomorphisms

$$T_D^*(\mathcal{D} \otimes \Gamma(\mathcal{D}) C_\omega) \equiv (\mathcal{D} \otimes \Gamma(\mathcal{D}) C_\omega) \otimes \omega^D,$$

coherently additive in $D$.

**2.7 Derived version.** The equalities (2.4) enhance to

$$\mathbf{R}\text{End}_{DA}(D) \equiv \mathbf{R}\Gamma(A; D) \equiv \text{Sym} T_0 A \otimes \bigwedge^* T_0^* A \cong \Omega^*[\text{Op}_L G] \cong \mathbf{R}\text{End}_{\mathcal{F}_{\text{coh}(A')}}(\mathcal{O}_{\text{Op}_L G}).$$  (2.8)

We have written $\equiv$ for canonical isomorphisms; whereas, in the isomorphisms with the algebra $\text{Sym} T_0 A \otimes \bigwedge^* T_0 A$ of differential forms on $\text{Op}_L G$, we have invoked the polarization of $A$. We did this twice, left and right: all other terms are canonically isomorphic, because $T_0^* A$ is the normal bundle to $\text{Op}_L G$ in $A^5$.

There is more to the isomorphism (2.8) than first meets the eye, because the two $\mathbf{R}\text{End}$ complexes are more than algebra objects in the derived categories of vector spaces: namely, they have natural $A_\infty$-structures, from the (higher) Yoneda products. It is with these higher structure included that they are isomorphic, as $A_\infty$-algebras, to the skew-commutative algebra $\Omega^*[\text{Op}_L G]^5$.

More generally, for any $\mathcal{M} \in D(DA\text{-mod})$, we find, by invoking the results of Laumon and Rothstein, that

$$\mathbf{R}\text{Hom}_{DA}(D; \mathcal{M}) = \mathbf{R}\Gamma(A; \mathcal{M}) = \mathbf{R}\text{Hom}_{\text{Loc}_G} (\mathcal{O}_{\text{Op}_L G}; \Phi^\flat(\mathcal{M})) [\dim A].$$  (2.9)

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5The linear structures on $A, A^5$ allow one to verify that; but it also follows from the fact that the Fourier transform $\Phi^\flat$ enhances to a functor between the differential graded categories underlying the derived ones.
Local Serre duality identifies the latter with the global sections of the derived restriction of \( \Phi^\flat(M) \) to \( \text{Op}_{LG} \). In this sense, the functor \( R\Gamma \) on \( \text{D}_{\text{coh}}D_A\text{-mod} \) corresponds to the restriction to \( \text{Op}_{LG} \) in \( \text{D} \text{Coh}(A^\flat) \) (followed by global sections, which however do no “damage” as \( \text{Op}_{LG} \) is affine).

The derived version of the abelian Beilinson–Drinfeld correspondence restricts the Fourier transform \( \Phi^\flat \) to the part which can be gleaned from (2.8), (2.9). There are variations of the statement: we can use the abelian categories or the derived categories, or we can use subcategories — thick closures of \( D \) or \( \text{Coh}_{\text{Op}_{LG}} \), respectively — or localize the categories at the respective objects.

2.10 Theorem.  
(i) \( \Phi^\flat \) restricts to an equivalence of triangulated categories, from the thick closure of \( D \) in \( \text{D}(D_A\text{-mod}) \) to that of \( \mathcal{O}_{\text{Op}_{LG}} \) in \( \text{D}(\text{Coh}(\text{Loc}_{LG})) \).

(ii) Restricting to the corresponding abelian categories, we get an equivalence between the categories of nilpotent deformations of objects in Theorem 2.5.(iii).

(iii) The same holds for the localizations at \( D \) and \( \mathcal{O}_{\text{Op}_{LG}} \) of the abelian or derived categories of \( D_A \) and \( \text{Coh}(\text{Loc}_{LG}) \)-modules, respectively.

2.11 Remark. Recall that the localization \( \mathcal{C} \) of a \((\mathbb{C}\text{-linear}) \) category \( \mathcal{C} \) at an object \( x \) is obtained by inverting morphisms \( \varphi : y \to z \) which induce isomorphisms \( \varphi_* : \text{Hom}(x,y) \to \text{Hom}(x,z) \). The functor \( \mathcal{C} \to \text{Mod} - \text{End}(x), y \in \text{Ob}(\mathcal{C}) \mapsto \text{Hom}(x,y) \), is faithful; often, as in our examples, it can be made full and essentially surjective, with the right finiteness conditions on \( \text{End}(x) \)-modules. Thus, \( \text{Coh}(\text{Loc}_{LG}) \) localized at \( \mathcal{O}_{\text{Op}_{LG}} \) is equivalent to the category of coherent sheaves on the formal neighborhood of \( \text{Op}_{LG} \) in \( \text{Loc}_{LG} \). On the other hand, the thick closure of \( \mathcal{O}_{\text{Op}_{LG}} \) in \( \text{Coh}(\text{Loc}_{LG}) \) comprises all successive extensions of \( \mathcal{O}_{\text{Op}_{LG}} \)-modules in \( \text{Coh}(\text{Loc}_{LG}) \), the coherent sheaves supported in some finite-order neighborhood of \( \text{Op}_{LG} \).

3. The constructions of Hitchin and of Beilinson–Drinfeld

We now recall the (semi-)classical construction of N. Hitchin [H] and its quantization due to Beilinson–Drinfeld. One substantial difference with the abelian case is the stack nature of the moduli \( \text{Bun}_G \) of \( G \)-bundles, which cannot be avoided because different \( G \)-bundles have different automorphism groups. (We cannot restrict ourselves to the moduli spaces of semi-stable bundles because they are not preserved by the Hecke correspondences, which are essential for the formulation of the geometric Langlands correspondence.)

(3.1) The cotangent stack. The “classical” version of \( D \) (or of any of its twisted versions) on a manifold \( X \) is the sheaf of \( \mathcal{O}_X \)-algebras \( \text{Sym} T \). For a smooth stack \( \mathfrak{X} \), which is locally a quotient of a smooth manifold by an algebraic group, \( T \) is a complex defined up to a quasi-isomorphism, and there is the corresponding differential graded version of \( \text{Sym} T \). Locally, if \( \mathfrak{X} = X/L \), a model for the tangent complex \( T\mathfrak{X} \) is \([t \overset{\alpha}{\to} TX]\) (in degrees \(-1\) and \(0\)) and the infinitesimal action map \( \alpha \). Accordingly,

\[
\text{Sym}^r T\mathfrak{X} \cong \bigoplus_{s+t=r} \wedge^s t \otimes \text{Sym}^t TX, \partial_\alpha \biggr) .
\]
The differential induced from $\alpha$ makes the total $\text{Sym}$ into a differential graded algebra. The cotangent stack $T^*X := \text{Spec}_X \text{Sym} T X$ is a derived stack (in vector spaces) over $X$. Different presentations of $X$ give equivalent models for $T^*X$.

This applies for instance to the stack $X = \text{Bun}_G$, with some bad and some good news.

- **Bad news:** $\text{Bun}_G$ may only be presented as a quotient locally (on substacks of finite type).
- **Good news:** when $G$ is semi-simple and $\Sigma$ has genus 2 or more, $\text{Sym} T \text{Bun}_G$ is quasi-isomorphic to its degree zero (top) cohomology; so it is not truly a derived stack.

The good news overrides the bad, as it gives a strict model for $T^* \text{Bun}_G$ over $\text{Bun}_G$: we need not delve into coherent systems of quasi-isomorphisms in patching the local descriptions together.

**Henceforth, unless specified otherwise, we will assume that $G$ is semi-simple and the genus of $\Sigma$ is 2 or more.**

(3.2) The Hitchin map. Hitchin [H] constructed a morphism $\chi$ from $T^* \text{Bun}_G$ to a $\mathbb{Z}$-graded vector space $H$, generalizing the projection $p : T^* A \to H$ we used in §2 for $G = GL(1)$.

(Hitchin originally introduced $\chi$ for the moduli space $[T^* \text{Bun}_G]$ of stable objects associated to $T^* \text{Bun}_G$, the stable Higgs bundles; but extending it to the stack is straightforward.) The space $H$, nowadays called the Hitchin base, is isomorphic to

$$H = \bigoplus_{m \in \exp(g)} H^{(m+1)} \simeq \bigoplus_{m \in \exp(g)} H^0 \left( \Sigma; K_{\Sigma}^{\otimes (m+1)} \right),$$

with the direct sum ranging over the exponents of $g$. This isomorphism is not canonical and involves the choice of a set of graded generators of the space of invariant functions on the Lie algebra $g$. The fiberwise $\mathbb{G}_m$-action on $T^* \text{Bun}_G$ is compatible with the $\mathbb{G}_m$-action on $H$ defined by the $\mathbb{Z}$-grading. Hitchin proved that $\chi$ was flat, that its generic fiber within the neutral component of $[T^* \text{Bun}_G]$ was an abelian variety, and that it was a torsor over it within every other component; finally, that these fibers were Lagrangian for the natural symplectic structure on $[T^* \text{Bun}_G]$. These results were extended in [BD] from the moduli space of stable bundles to the stack $T^* \text{Bun}_G$. One important distinction here is the appearance of a gerbe with band $Z(G)$ over the regular fibers of $\chi$ in the stack $T^* \text{Bun}_G$, coming from global automorphisms of $G$-bundles.

The map $\chi$ has a section, which depends on the choice of a Spin structure on $\Sigma$. This section is the classical limit of the Lagrangian variety of opers, recalled below, and is relevant to the Langlands dual Hitchin fibration $^L \chi : T^* \text{Bun}_G(\Sigma; {}^L G) \to {}^L H$. (Remarkably enough, the bases $H$ and $^L H$ can be identified, once we choose an invariant quadratic form $k$ on $g$.)

For $G = GL(1)$, this slice is the fiber $0 \times H$ from the previous section. For $G = SL(2)$, the slice sends $h \in H^0(\Sigma; K_{\Sigma}^{\otimes 2})$ to the pair consisting of the bundle $E := K_{\Sigma}^{1/2} \oplus K_{\Sigma}^{-1/2}$ and

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6It is easy to describe $H$ canonically in terms of the graded vector space $\text{Spec}(\mathbb{C}[g]^G)$, but the above description will suffice for us.
the cotangent vector ("Higgs field")

\[
\begin{bmatrix}
0 & h \\
1 & 0
\end{bmatrix} : E \to E \otimes K_\Sigma.
\]

For general \(g\), the slice can be constructed using the principal embedding \(\mathfrak{sl}(2) \to g\) and the Kostant slice for the adjoint action.

Choose now a non-degenerate invariant quadratic form \(k\) on \(g\), and a corresponding ample line bundle \(\Theta_k\) on \(\text{Bun}_G\). (If \(g\) is a simple Lie algebra, then the form \(k\) is unique up to scalar; there can be some torsion ambiguity in defining \(\Theta_k\), but this will not be relevant.) Hitchin proved the following for the moduli space of stable Higgs bundles. With our assumptions on \(G\) and \(\Sigma\), the codimension of the unstable part of the stack is too high to have an effect, so we state the result directly for the stack.

For each \(\gamma \in \pi_1 G\), let \(\text{Bun}_G^\gamma\) be the corresponding component of \(\text{Bun}_G\).

### 3.3 Theorem (H). Assume that \(G\) is simply-connected. Then,

(i) All global functions on \(T^\vee \text{Bun}_G^\gamma\) are lifted from the base: \(H^0(T^\vee \text{Bun}_G^\gamma; \mathcal{O}) \cong \mathbb{C}[H]\).

(ii) \(H^1(T^\vee \text{Bun}_G^\gamma; \mathcal{O}) \cong \mathbb{C}[H] \otimes H^\vee \cong \Omega^1[H]\).

(iii) Specifically, the generators of \(H^1\) over \(H^0\) arise from \(H^0\) by contracting the vector fields defined by the linear Hamiltonians with \(c_1(\Theta_k) \in H^1(T^\vee \text{Bun}_G^\gamma; \Omega^1)\).

### 3.4 Remark. The tangent bundles to the regular fibers of \(\chi\) may be identified with \(H^\vee\) using the symplectic structure on \(T^\vee \text{Bun}_G\). The fibers of \(R^1 \chi_*(\mathcal{O})\) are therefore isomorphic to \(\overline{H}\); we identify them with \(H^\vee\) by using \(c_1(\Theta_k)\).

The result on \(H^0\) immediately implies an equivalence between the abelian category of \(\mathcal{O}\)-modules on \(T^\vee \text{Bun}_G\) with a finite global presentation by \(\mathcal{O}\) and the abelian category of coherent sheaves with support on the Hitchin slice in \(T^\vee \text{Bun}_{L_L}G\). The components of \(T^\vee \text{Bun}_G\) labeled by \(\pi_1 G\) get interchanged with the characters of the \(Z(L_L)\)-gerbe over the slice. In Theorem 4.2 below, we will generalize Hitchin’s result to all cohomologies; this will affirm the derived versions of this equivalence.

### (3.5) Deformations. Hitchin’s construction extends to deformations of the stacks \(T^\vee \text{Bun}_G\) and \(T^\vee \text{Bun}_{L_L}G\) in the manner analogous to \(\S 2.2\). It is easier to describe the analogue of \(\mathcal{A}'\), in which the connected components of the stack \(T^\vee \text{Bun}_{L_L}G\), fibered in vector spaces over the components of \(\text{Bun}_{L_L}G\), deform into fibrations in affine spaces, according to the class \([k^{-1}] \in H^1(\text{Bun}_{L_L}G; \Omega^1)\). (Recall that this group is also \(H^2(\text{Bun}_{G}(\Sigma; L_L); \mathbb{C})\), and the quadratic form \(k^{-1}\) on \(L_L g\) dual to \(k\) on \(g\) defines a class in the latter.) The total stack of this relative affine deformation is isomorphic to the stack \(\text{Loc}_{L_L}G\) of \(L_L\)-local systems.

The relevant non-commutative deformation of \(T^\vee \text{Bun}_G\) converts \(\text{Sym} T\) into the sheaf \(\mathcal{D}^*\) of Spin differential operators (differential operators on \(\mathcal{H}^{1/2}\)). This has an increasing filtration whose associated graded is \(\text{Sym} T\text{Bun}_G\).

\(^7\)One can deform to any twisted form of \(\mathcal{D}\), but only the Spin version gives an interesting answer to the questions we are interested in.
(3.6) Technical aside on \( \mathcal{D} \) of a stack. A (derived) category of \( \mathcal{D} \)-modules on an Artin stack \( \mathfrak{X} \) can be defined in a number of ways. We can use crystals, or (complexes of) \( \mathcal{O} \)-modules over the de Rham site \( dR(\mathfrak{X}) \) [ST]. The sheaf \( \mathcal{D}_X \) of differential operators is more peculiar. It is a distinguished object in \( D(\mathcal{D}_X\text{-mod}) \); in a local presentation \( \mathfrak{X} = X/L \), a model for its underlying \( \mathcal{O} \)-module is the Chevalley complex of \( \mathcal{D}_X \) and the Lie algebra \( \mathfrak{l} \). This has a natural increasing filtration, whose associated graded is \( \text{Sym}[\mathfrak{l}] \rightarrow TX \) of \( \mathfrak{X} \).

This \( \mathcal{D}_X \) is not a sheaf of algebras over the smooth site of \( \mathfrak{X} \). Even for a manifold \( X \), the extension of \( \mathcal{D}_X \) to the smooth site by pull-backs is not a sheaf of algebras. However, over any open substack \( \mathfrak{U} \subset \mathfrak{X} \), \( H^*(\mathfrak{U}; \mathcal{D}) \) is an algebra, isomorphic to \( \text{Ext}^*_{\mathcal{D}(\mathfrak{U})}(\mathcal{D}, \mathcal{D}) \). One characterization of \( \mathcal{D} \), which makes the isomorphism automatic, is \( p_!\mathcal{O} \), where \( p_! \) is the (derived) left adjoint of the pull-back along the projection \( p : \mathfrak{X} \rightarrow dR(\mathfrak{X}) \) [BD]. The Ext-description enhances to give an \( A_{\infty} \)-algebra structure with the Yoneda products.

Twisted analytic versions of \( \mathcal{D}_X \) are classified by \( H^1(\mathfrak{X}; \mathcal{O}_{\mathfrak{X}}/\mathbb{C}) \). On a manifold, \( \mathcal{O}_{\mathfrak{X}}/\mathbb{C} \cong (\mathcal{O}_{\mathfrak{X}}^\times)/\mathbb{C} \) is the sheaf of algebra automorphisms of \( \mathcal{D} \) fixing \( \mathcal{O} \), and 1-cocycles allow the patching together of local copies of \( \mathcal{D} \) by means of isomorphisms fixing \( \mathcal{O} \). The same group is the analytic hyper-cohomology \( \mathcal{H}^2(\mathfrak{X}; (\Omega^1_{\mathfrak{X}}, d)) \) of the truncated de Rham complex. When Hodge decomposition holds — or else, in the algebraic case, with log poles at infinity — the latter becomes the Hodge filtered part \( F^{}H^2(\mathfrak{X}; \mathbb{C}) \).

The algebraic version of the same hyper-cohomology defines twisted algebraic versions of \( \mathcal{D} \). The latter are not locally isomorphic to \( \mathcal{D} \) in the Zariski topology, but to twisted versions \( \mathcal{D}^\varphi \) satisfying \([X, Y] = \varphi(X, Y)\) for vector fields \( X, Y \) and (locally defined) closed 2-forms \( \varphi \). A class in \( \mathcal{H}^2(\mathfrak{X}; (\Omega^1_{\mathfrak{X}}, d)) \) is represented, in a Čech covering \( U_i \), by 1-forms \( \eta_{ij} \) on \( U_{ij} \) and closed 2-forms \( \varphi_i \) on \( U_i \), satisfying the Čech cocycle condition for the \( \eta \) and the relations \( d\eta_{ij} = \varphi_i - \varphi_j \). The transformation \( X \mapsto X + \eta_{ij}(X) \) extends to algebra isomorphisms patching together the \( \mathcal{D}^\varphi \). Our situation is simplified by the isomorphism \( H^2(\text{Bun}_G; \mathbb{C}) \cong H^1(\text{Bun}_G; \Omega^1) \), for semi-simple \( G \); the twisted versions of \( \mathcal{D} \) over \( \text{Bun}_G \) are classified by invariant quadratic forms on \( \mathfrak{g} \), and are interpreted as differential operators on (complex) powers of the theta-line bundle.

(3.7) Opers. The final ingredient in the quantization of Hitchin’s construction is the space of opers in \( \text{Loc}_{L^G} \). There are several isomorphic components, and just for the Hitchin section, pinning down one of them requires a choice of \( K_{\Sigma}^{1/2} \). For \( L^G = \text{PSL}(2) \), the opers are the local systems whose underlying \( \text{SL}(2) \)-bundle is the unique (up to isomorphism) non-trivial extension \( K_{\Sigma}^{1/2} \rightarrow E \rightarrow K_{\Sigma}^{-1/2} \).

For general \( L^G \), they are the local systems whose underlying principal bundle \( P \) is defined from \( E \) by the inclusion of the principal \( \text{SL}(2) \) subgroup in \( L^G \). Recall now that \( \text{Loc}_{L^G} \) has an algebraic symplectic structure, defined by the pairing on forms in any invariant quadratic form on \( L^G \).

In a break with the traditional terminology and notation, in this paper we will denote by \( \text{Op}_{L^G} \subset \text{Loc}_{L^G} \) one of the components of the space of opers (specifying this component may require a choice of \( K_{\Sigma}^{1/2} \)).

Beilinson and Drinfeld have proved [BD] the following:
3.8 Proposition. \( \text{Op}_{L^G} \subset \text{Loc}_{L^G} \) is a smooth Lagrangian subvariety contained in the smooth, Deligne–Mumford part of \( \text{Loc}_{L^G} \). It carries a trivial gerbe with band \( Z(L^G) \), and the associated space is an affine space over \( H \).

The last part follows from the computation

\[ \Gamma(\Sigma, \text{ad}_P \otimes K_\Sigma) \cong H, \]

which uses the Kostant slice for the adjoint action. When combined with the grading on \( H \), the proposition supplies an increasing filtration on the algebra of functions \( \mathbb{C}[\text{Op}_{L^G}] \).

Recall that the components \( \text{Bun}_G^\gamma \) of \( \text{Bun}_G \) are labeled by \( \gamma \in \pi_1 G \).

3.9 Theorem ([BD]). There is a canonical isomorphism of algebras \( \Gamma(\text{Bun}_G^\gamma; \mathcal{D}^s) \cong \mathbb{C}[\text{Op}_{L^G}] \) of global Spin differential operators on \( \text{Bun}_G^\gamma \) with the polynomial functions on \( \text{Op}_{L^G} \).

3.10 Remark. Beilinson and Drinfeld prove more: for a point \( \omega \in \text{Op}_{L^G} \), with residue field \( \mathbb{C}_\omega \), the \( \mathcal{D}^s \)-module \( \mathcal{D}_\omega^s := \mathcal{D}^s \otimes_{\Gamma(\mathcal{D}^s)} \mathbb{C}_\omega \) is regular holonomic, with singular support in Laumon’s nilpotent cone, and is a Hecke eigensheaf whose eigenvalue is the local system \( \omega \). Thus, \( \mathcal{D}_\omega^s \) is assigned to \( \omega \) under the geometric Langlands correspondence.

As in our discussion for Theorem 2.5, this gives

3.11 Corollary. (i) Any \( \mathcal{D}^s \)-module admitting a finite free global presentation by copies of \( \mathcal{D}^s \) has a finite free global \( \mathcal{D}^s \)-resolution.

(ii) The corresponding presentation in \( \text{Coh}(\text{Op}_{L^G}) \) defines a coherent sheaf, and this assignment gives an equivalence from the abelian category of \( \mathcal{D}^s \)-modules on \( \text{Bun}_G \) with free global presentations to that of coherent \( \mathcal{O}_{\text{Op}_{L^G}} \)-modules.

(iii) Thereunder, the functor \( \Gamma(\text{Bun}_G; \_\_ \_\_) \) corresponds to \( \Gamma(\text{Op}_{L^G}; \_\_ \_\_) \).

Under this equivalence, different components of \( \text{Bun}_G \) correspond to the characters of the \( Z(L^G) \)-gerbe over \( \text{Op}_{L^G} \) under Corollary 3.11.

4. Computation of higher cohomologies

We now generalize the Beilinson–Drinfeld theorem 3.9 by describing the higher cohomology.

(4.1) The classical case. We first extend Hitchin’s calculation. Loosely phrased, our result asserts that singularities in the fibers of the Hitchin map \( \chi \) are invisible to the cohomology of \( \mathcal{O} \). For simplicity, we assume here that \( G \) is simply-connected (so that \( L^G \) is of adjoint type). The result extends by matching components of \( T^\vee \text{Bun}_G \) and characters of \( Z(L^G) \).

4.2 Theorem. We have an isomorphism \( H^\bullet(T^\vee \text{Bun}_G; \mathcal{O}) \cong \Omega^\bullet[H] \) with the algebraic differentials on the Hitchin base \( H \). The grading induced by the \( \mathbb{C}^\times \)-action on \( T^\vee \) comes from the natural grading on \( H \) on the even copy, shifted down by 1 for the odd copy. This isomorphism depends on \( k \) in the following way: rescalings of the restriction of \( k \) to a simple factor of \( g \) corresponds to rescalings of the appropriate summand in \( \Omega^1[H] \).
Proof. The calculation parallels the arguments in [FHT], especially §8 and §9. We refer the reader to the latter for more background and the details of the methods (as they do involve infinite-dimensional vector bundles over a stack of infinite type).

Choosing a point $\sigma \in \Sigma$ with a local coordinate $z$ and letting $\Sigma^o := \Sigma \setminus \{\sigma\}$, we can present the stack $\text{Bun}_G$ as the double quotient $G[[z]]\{G((z))\}/G[\Sigma^o]$. We rewrite this as $G[[z]]\{X$, with the thick flag variety $X := G((z))/G[\Sigma^o]$. However, we choose to present the tangent complex by a different, 2-step resolution

$$g[\Sigma^o] \xrightarrow{\partial = \text{Ad}_\phi} g((z))/g[[z]]$$

with the differential equal to the adjoint twist of the obvious inclusion at the point $\phi \in G((z))$. The second term is a trivial bundle over $X$, whereas the first is associated to the adjoint action of $G[\Sigma^o]$.

We seek the $G[[z]]$-equivariant hyper-cohomology of the differential graded algebra

$$H^q (\text{Bun}_G; \Sym^r T) = \bigoplus_{s+t=r} H^q_{G[[z]]} (X; \Lambda^s g[\Sigma^o] \otimes \Sym^t(g((z))/g[[z]]))$$

(4.3)

with the generating bundle $g[\Sigma^o]$ placed in cohomological degree $-1$ and differential induced from $\partial$. The Sym-factor carries the adjoint action of $G[[z]]$.

Filtering by $s$-degree gives a spectral sequence $E_1^{-s,q} \Rightarrow H^{q-s}$ which we now compute. Let us first recall why, as in §9.4 of [FHT], we have a key factorization of the $E_1$ term,

$$E_1^{-s,q} = \bigoplus_u H^{q-u}_{G[[z]]} (\Sym^{r-s}(g((z))/g[[z]])) \otimes H^u (X; \Lambda^s g[\Sigma^o]).$$

(4.4)

The product above would a priori be the $E_1$ term in the Leray spectral sequence for the fibration $\text{Bun}_G \to BG[[z]]$. However, we claim that the second factor $H^*(X; \Lambda^s g[\Sigma^o])$ is a free skew-commutative algebra, trivial as a $G((z))$-representation. If so, then the Leray $E_1$ term is freely generated over the base by classes which extend to the total space, and so (4.4) is the true $E_1$ term for (4.3) and not just the first page of its Leray sequence.

Our claim about $H^*(X; \Lambda^s g[\Sigma^o])$ is a variant of [FHT], Theorem D, where we have replaced the differentials $\Omega^1(\Sigma^o; g)$ in the exterior algebra generators by the functions $g[\Sigma^o]$. Our functorial construction of the generating cocycles in [FHT] §9 allows in fact for a twist by any line bundle $L \to \Sigma^o$ in those coefficients, with a corresponding twist appearing in the cohomology generators. More precisely, the general statement identifies $H^u (X; \Lambda^s \Omega^1(\Sigma^o; g \otimes L))$ with the free bi-graded skew-commutative algebra generated by copies of

$$\Gamma (\Sigma^o; \mathcal{L}^{s,m}) \quad \text{and} \quad \Gamma (\Sigma^o; \mathcal{L}^{s,m+1} \otimes K_\Sigma)$$

(4.5)

in degrees $(s, u) = (m, m)$ and $(m + 1, m)$, as $m \in \exp(g)$. The parity is given by $u - s$.

Theorem D of [FHT] had $\mathcal{L} = \mathcal{O}$.

In the present case, $\mathcal{L} = T\Sigma^o$, the spaces in each pair (4.5) are both isomorphic to $\Gamma (\Sigma^o; (T\Sigma^o)^{s,m})$, and with respect to (4.4), $s = r, u = q$, giving $(s, q, r) = (m, m, m), (m+1, m, m+1)$.
Similarly, Theorem A of [FHT] (with the same twist in the generators) describes the cohomology $H^q_{G[[z]]} \left( \text{Sym}^t(g((z))/g[[z]]) \right)$ as the skew-commutative algebra freely generated by copies of

$$\{\mathbb{C}((z))/\mathbb{C}[[z]]\} \otimes (\partial/\partial z)^{\otimes m} \quad (4.6)$$

placed in bi-degrees $(t, q) = (m, 1)$ and $(m + 1, 0)$, as $m$ ranges over $\exp(g)$. Parity is given by $q$. With $s = u = 0$ in (4.4), these correspond to $(s, q, r) = (0, 1, m), (0, 0, m + 1)$.

The generators in (4.5) and (4.6) come in pairs of matching $r$-degrees $m, m + 1$, and with spectral sequence bi-degrees $(-s, q)$ equal to $(-m, m)$ and $(0, 1)$ for the former, and $(-m - 1, m)$ and $(0, 0)$ for the latter. There is the possibility of some obvious leading differentials in the spectral sequence for (4.3) given by the restriction maps

$$\Gamma (\Sigma^o; (T\Sigma)^{\otimes m}) \to \{\mathbb{C}((z))/\mathbb{C}[[z]]\} \otimes (\partial/\partial z)^{\otimes m} \quad (4.7)$$

from (4.5) to (4.6), on pages $m$ and $m + 1$ respectively. That these are indeed the leading differentials can be seen in the description of the generating cocycles via contraction with the Atiyah class, as in §8 and §9 of [FHT]; the same calculation applies here verbatim.

In genus $\geq 2$ and for semi-simple $g$ (when all $m > 0$), the map in (4.7) is injective with cokernel $H^1(\Sigma; (T\Sigma)^{\otimes m})$. After resolving the leading differentials, we thus get for $H^q(Bun_G; \text{Sym} T)$ the free skew-commutative algebra on copies of

$$H^1(\Sigma; (T\Sigma)^{\otimes m})$$

in cohomology degrees $q = 0, 1$ and symmetric degrees $m + 1$ and $m$, respectively. Now, we know from Hitchin’s construction that these generators survive to $E_\infty$ in the sequence; therefore, no further differentials can occur and the theorem is proved.

(4.8) Quantization. We now deform the graded $\mathcal{O}_{Bun_G}$-algebra $\text{Sym} T_{Bun_G}$ to the $\mathcal{O}_{Bun_G}$-module $\mathcal{D}^s$. The latter is filtered by the order of the operator, and the associated graded sheaf is $\text{Sym} T_{Bun_G}$. The canonical isomorphism $H^\bullet(Bun_G; \mathcal{D}^s) = \text{Ext}^\bullet_{\mathcal{O}_{Bun_G}}(\mathcal{D}^s; \mathcal{D}^s)$ makes the former into an algebra deforming $H^\bullet(Bun_G; \text{Sym} T)$.

The algebra of differential forms on opers, $\Omega^\bullet[\text{Op}_{LG}]$, is also filtered, using the $H$-affine structure on $\text{Op}_{LG}$ (see Proposition 3.8 and below). Recall that we have fixed a non-degenerate invariant quadratic form $k$ on $g$. We then have an isomorphism

$$\Omega^\bullet[\text{Op}_{LG}] \cong \text{Ext}^\bullet_{\text{Loc}_{LG}}(\mathcal{O}_{\text{Op}_{LG}}; \mathcal{O}_{\text{Op}_{LG}})$$

which is obtained using the $(k$-dependent) symplectic structure on $\text{Loc}_{LG}$, which identifies the normal and cotangent bundles to $\text{Op}_{LG}$. The following supplies the higher cohomologies in Theorem 3.9.

4.9 Theorem. There is a canonical isomorphism of filtered algebras

$$H^\bullet(Bun_G; \mathcal{D}^s) = \text{Ext}^\bullet_{\text{Loc}_{LG}}(\mathcal{O}_{\text{Op}_{LG}}; \mathcal{O}_{\text{Op}_{LG}}).$$

Using the $(k$-dependent) isomorphism with $\Omega^\bullet[\text{Op}_{LG}]$, the associated graded map becomes the isomorphism of Theorem 4.2 (multiplied by $(-1)^{\deg}$).
4.10 Remark. The use of the form $k^{-1}$ on $L^g$, concealed in the symplectic form on $\text{Loc}_{L^G}$, is the source of the $k$-dependence in Theorem 4.2.

We prove the theorem in two steps. First, we partially refine our calculation in Theorem 4.2:

4.11 Proposition. The odd generators $H^\vee$ of $H^1(\text{Bun}_G; \text{Sym} T)$ over $H^0$ have a preferred lift to $H^1(\text{Bun}_G; \mathcal{D}^s)$. Furthermore, the associated graded algebra of $H^*(\text{Bun}_G; \mathcal{D}^s)$ (with respect to the filtration inherited from the order filtration on $\mathcal{D}^s$) is naturally isomorphic to $H^*(\text{Bun}_G; \text{Sym} T)$.

Then, in the next section, we ascertain the commutativity of $H^*(\text{Bun}_G; \text{Sym} T)$. A quantum version of the construction in Theorem 4.2 will present $H^*(\text{Bun}_G; \mathcal{D}^s)$ as quotient of a commutative algebra; in fact, Theorem 5.1 will refine Theorem 4.9 to an isomorphism of $A_\infty$ algebras. This will identify the formal neighborhood of $\mathcal{D}^s$ within $\mathcal{D}^s$-modules on $\text{Bun}_G$ with that of $\mathcal{O}_{\text{Op}_{L^G}}$ inside $\text{Coh}(\text{Loc}_{L^G})$, canonically to the first order.

Proof of Proposition 4.11. In Lemma 4.13 below, we will show that the cohomologies of $\text{Sym} T$ and $\mathcal{D}^s$ on the (infinite type) stack $\text{Bun}_G$ come from a finite type substack. Allowing this for now, the increasing order filtration on $\mathcal{D}^s$, with $\text{Gr}(\mathcal{D}^s) = \text{Sym} T$, leads to a spectral sequence with the first term $E^{p,q}_{1} = H^{p+q}(\text{Bun}_G; \text{Sym}^{-p} T) \Rightarrow H^{p+q}(\text{Bun}_G; \mathcal{D}^s)$. We will show its collapse by verifying the survival of all $E_1$ generators. Theorem 3.9 addresses $H^0(\text{Bun}_G; \mathcal{D}^s) = \mathbb{C}[\text{Op}_{L^G}]$. We will now construct a space of $H^1$ classes isomorphic to $H^\vee$, lifting the symbols of Hitchin’s $H^1$ generators.

Those generators arise by contracting the linear (on $H$) Hamiltonian vector fields with the Chern class $c_1(\Theta_k) \in H^1(\Omega^1)$ (Theorem 3.3). Now, if $c_1 = d\kappa$, this would be the Poisson bracket with $\kappa$. There is no such $\kappa \in H^1(\text{Bun}_G; \mathcal{O})$, but it does exist in the analytic $H^1(\text{Bun}_G; \mathcal{O}_{an}/\mathbb{C})$ (see §3.6), as seen from the long exact sequence

$$\cdots \rightarrow H^1(\text{Bun}_G; \mathcal{O}_{an}) \rightarrow H^1(\text{Bun}_G; \mathcal{O}_{an}/\mathbb{C}) \rightarrow H^2(\text{Bun}_G; \mathbb{C}) \rightarrow H^2(\text{Bun}_G; \mathcal{O}_{an}) \rightarrow \cdots$$

and the vanishing of $H^{>0}(\text{Bun}_G; \mathcal{O}_{an})$. This suffices for the bracket interpretation, because $\mathbb{C}$ is central: the Poisson bracket lifts to a commutator operation

$$[, ]: H^*(\mathcal{O}_{an}/\mathbb{C}) \otimes H^*(\mathcal{D}^s) \rightarrow H^*(\mathcal{D}^s),$$

and similarly to any twisted version of $\mathcal{D}$.

To avoid a GAGA comparison (which does apply to the stack $\text{Bun}_G$ but requires additional discussion), we describe this operation in the algebraic category, where a similar operation exists for classes in the Hodge filtered part $\mathbb{H}^*(\Omega^1 \xrightarrow{\partial} \Omega^2 \xrightarrow{\partial} \cdots)$ of de Rham cohomology. Morally, the operation is algebraic because the commutator of a differential operator with the anti-derivative in $\mathcal{O}_{an}/\mathbb{C}$ of an algebraic hyper-cohomology class is algebraic. The precise definition, for a class $\mathbb{H}^1$, is the following. Recall from §3.6 that such a
class $(\eta, \varphi)$ defines a twisted form $\mathcal{D}^{(\eta, \varphi)}$ of $\mathcal{D}$. (We actually need to deform $\mathcal{D}^s$ but opt for less notational clutter.) We form the 1-parameter family $\mathcal{D}^t$ over $\mathbb{C}[t]$ defined by $t(\eta, \varphi)$ and consider the long exact sequence

$$\cdots \to H^p(\mathcal{D}^t) \xrightarrow{i} H^p(\mathcal{D}^t) \xrightarrow{\pi} H^p(\mathcal{D}) \xrightarrow{\delta} H^{p+1}(\mathcal{D}^t) \xrightarrow{i} \cdots$$

induced from $\mathcal{D}^t \xrightarrow{i} \mathcal{D}^t \xrightarrow{\pi} \mathcal{D}$. The composition $\pi \circ \delta : H^p(\mathcal{D}) \to H^{p+1}(\mathcal{D})$ is the desired operation. (The reader will recognize it as the first differential in a spectral sequence $H^*(\mathcal{D})[[t]] \Rightarrow H^*(\mathcal{D}^t)$ for the deformation $\mathcal{D} \rightsquigarrow \mathcal{D}^t$.) Applied to our Chern class $c_1(\Theta_k) \in H^1(\text{Bun}_G; \Omega^1 \to \Omega^2)$, this lifts Hitchin’s contraction on symbols, giving us the desired generators in $H^1(\text{Bun}_G; \mathcal{D}^s)$ and concludes the proof. □

4.12 Remark. On the Langlands dual side in Theorem 4.9, moving away from Spin becomes the non-commutative deformation $\text{Loc}_{GL}^\prime$ of $\text{Loc}_{GL}$ generated by the symplectic form. In the deformed algebra, the algebra $\text{Ext}_{\text{Loc}_{GL}}^*(\mathcal{O}_{\text{Op}_{GL}}; \mathcal{O}_{\text{Op}_{GL}})$ collapses to $\mathbb{C}$: the first differential of the spectral sequence of this deformation becomes de Rham’s operator on $\Omega^*(\text{Op}_{GL})$, under the isomorphism in Theorem 4.9, and the spectral sequence collapses to $\mathbb{C}$ on the second page. So, for all twists $\mathcal{D}^t \neq \mathcal{D}^s$ of $\mathcal{D}$, $H^*(\text{Bun}_G; \mathcal{D}^t) \cong \mathbb{C}$.

We conclude with the technical result promised in the proof of the proposition above.

4.13 Lemma. The cohomologies $H^*(\text{Bun}_G; \text{Sym} T)$ and $H^*(\text{Bun}_G; \mathcal{D}^t)$ are computed correctly on some finite union of Atiyah-Bott strata of $\text{Bun}_G$, for any twisted form $\mathcal{D}^t$ of $\mathcal{D}$.

Proof. We will check the vanishing of local cohomologies on most strata. It will help to run the argument for $\text{SL}(2)$ first, since the general case requires additional book-keeping. An unstable Atiyah-Bott stratum $\mathcal{S}$ of $\text{Bun}_G$ classifies bundles $E$ which are extensions

$$0 \to E' \to E \to E'' \to 0$$

with $\deg E' = - \deg E'' = d > 0$. Choose $d \geq g$, and observe that $\text{Ext}_G^1(E''; E') = 0$, so that $E \cong E' \oplus E''$. The stratum is then a copy of the Jacobian $J$, modulo the semi-direct product group $\text{GL}(1) \rtimes \exp(\mathfrak{h})$, with $\mathfrak{h} := \text{Hom}(E''; E')$. The normal bundle is $\nu := \text{Ext}_G^1(E'; E'')$, with the obvious $\text{GL}(1)$-action and trivial action of $\mathfrak{h}$ (although the latter action is not trivial on higher-order neighborhoods). The local cohomology sheaves

$$\mathcal{H}^*_\mathcal{S}(\text{Bun}_G; \text{Sym} T), \quad \mathcal{H}^*_\mathcal{S}(\text{Bun}_G; \mathcal{D}^t)$$

(4.14)

can be pushed down to the Jacobian $J$, where they have natural associated graded sheaves which are resolved by the $\text{GL}(1)$-invariant part of the Chevalley complex for Lie algebra cohomology of $\mathfrak{h}$

$$\bigwedge^* \mathfrak{h}^\vee \otimes (\text{det}(\nu) \otimes \text{Sym} \nu) \otimes \text{Sym} \nu \otimes \text{Sym} TJ \otimes \bigwedge^* (\mathfrak{h} \oplus \mathfrak{gl}(1));$$

(4.15)

the factor $\bigwedge \mathfrak{h}^\vee$ computes derived $\mathfrak{h}$-invariants, $\text{det} \nu \otimes \text{Sym} \nu$ are the residues along $\mathcal{S}$, and the remaining factors come from the local resolution of the tangent bundle of $\text{Bun}_G$. 15
The GL(1) weights on $\nu$ and $\lambda^\vee$ are positive. In genus $g \geq 2$, $\det(\nu)$ shifts the weights on $\wedge \lambda$ to the positive side as well (using Riemann-Roch and cohomology vanishing), so there are no GL(1)-invariants in (4.14) and the cohomologies with supports on $\mathcal{S}$ vanish.

For a general group, we now repeat the argument with an additional important observation: all Atiyah-Bott strata $\mathcal{S}$, save for a finite number of them, parametrize bundles $F \to \Sigma$ having a destabilizing parabolic reduction $F_P$ to some $P \subset G$ (with Levi component $L$, unipotent radical $U$) with the properties that

$$H^0 (\Sigma; \mathcal{A}_F (u^\vee)) = 0, \quad H^1 (\Sigma; \mathcal{A}_F (u)) = 0,$$

so that $F_P$ actually splits to an $L$-bundle. Note that $F_P$ need not be the maximal destabilizing reduction which defines the stratum $\mathcal{S}$: the latter will only work if the maximally destabilizing coweight — call it $\xi$ — is sufficiently far inside the Weyl chamber face which contains it. (Thus, regular coweights must be some genus-dependent distance away from the walls, whereas coweights on an edge must be some distance from 0.) Instead, we include in $\mathfrak{p}$ all root vectors $e_{\pm \alpha}$ for simple roots $\alpha$ with $0 \leq \alpha(\xi) < 2g$. This partitions the $\xi$'s into neighborhoods of the various faces, and we use the parabolic associated to the smallest nearby face.

We can now repeat the SL(2) discussion, with

$$\mathfrak{h} := H^0 (\Sigma; \mathcal{A}_F (u)), \quad \nu = H^1 (\Sigma; \mathcal{A}_F (u^\vee)),$$

using a central $\text{GL}(1) \subset L$ with positive weights on $u$. The weight on $\det \nu$ dominates the one on $\det \lambda^\vee$, because

$$H^0 (\Sigma; \mathcal{A}_F (u)) \subset H^0 (\Sigma; \mathcal{A}_F (u) \otimes K_{\Sigma}) = H^1 (\Sigma; \mathcal{A}_F (u^\vee)) = \nu^\vee$$

whence we conclude the vanishing of cohomologies with supports. \hfill \Box

4.16 Remark. Our results have analogues in genera 0 and 1, with a graded version of the Hitchin space with odd part $\bigoplus_m H^1(\Sigma; K^m(S^m+1))$. Theorem 4.9 applies, without any changes to the proof; however, the proof of Lemma 4.14 does not go through. Although we believe that Theorem 4.9 does hold, the technical deformation step seems to need a finer argument.

5. Commutativity to all orders and absence of obstructions

We know that $H^*(\text{Bun}_G; \mathcal{D}^*)$ is generated by a copy of $H^\vee$ over $H^0 \cong \mathbb{C}[\text{Op}_{L,G}]$, and that its associated graded algebra is skew-commutative. To finish the proof of Theorem 4.9, we need to verify the skew-commutativity of $H^*(\text{Bun}_G; \mathcal{D}^*)$. We will prove a much stronger statement in the same swoop. Namely, the graded algebra

$$\text{Ext}^*(\text{Bun}_G)(\mathcal{D}^*; \mathcal{D}^*)$$

carries the Yoneda $A_\infty$ multiplication. Complete knowledge of this multiplication captures the formal deformations of $\mathcal{D}^*$, and also the localization of the derived category of Spin $\mathcal{D}$-modules at the object $\mathcal{D}^*$. We will show the strict commutativity of this algebra.
5.1 Theorem. The Yoneda algebra $\text{Ext}^{D}(\text{Bun}_G, D)$ is abstractly $A_\infty$-isomorphic to (the strictly skew-commutative one) $\text{Ext}^\bullet_{\text{Loc}_G}(\mathcal{O}_{\text{Op}_G}; \mathcal{O}_{\text{Op}_G})$.

In particular, formal deformations of $D$ as a $D$-module are unobstructed to all orders and correspond to deformations of $\text{Op}_G \subset \text{Loc}_G$. Theorem 5.1 extends Corollary 3.11 as follows.

5.2 Corollary. (i) We have an equivalence between the thick closure of $D$ in $\text{D}_\text{Bun}_G$-mod and the full subcategory of sheaves in $\text{Coh}(\text{Loc}_G)$ with set-theoretic support on $\text{Op}_G$. (ii) The derived categories of the above are equivalent to the thick closures of $D$ and $\mathcal{O}_{\text{Op}_G}$ in the derived categories of $\text{D}_\text{Bun}_G$-mod and $\text{Coh}(\text{Loc}_G)$, respectively.

5.3 Remark. The earlier identification of $H^0, H^1(\text{Bun}_G; D)$ with $\text{Ext}^0, \text{Ext}^1(\mathcal{O}_{\text{Op}_G})$ gives only the first map in an $A_\infty$ isomorphism. Loosely speaking, we have identified the first neighborhood of $D$ in $\text{D}$-mod with that of $\text{Op}_G$ in $\text{Loc}_G$, and are now proving that the identification can be continued to all orders. We intend to address that and pin the isomorphism down canonically in a follow-up paper.

The proof of Theorem 5.1 relies on its local version, in which $\Sigma$ is replaced by a formal disk. The relevant disk $D$ is the formal neighborhood of $\sigma \in \Sigma$. Namely, Theorem 4.4 from our previous paper [FT] gives an isomorphism of skew-commutative algebras

$$H^\bullet(\mathfrak{g}[[z]], \mathfrak{g}; \mathcal{V}_{\text{crit}}) \cong \Omega^\bullet[\text{Op}_G(D)]$$

where on the left we have the Lie algebra cohomology with coefficients in the vacuum module at critical level $\mathcal{V}_{\text{crit}} := U_{\text{crit}} \hat{\mathfrak{g}} \otimes \mathfrak{g}[[z]] \mathbb{C}$ (see [FT] §2 for the definitions), while on the right $\text{Op}_G(D)$ is the variety of opers on the formal disk. The algebra structure on the left side is explained by the isomorphisms

$$H^\bullet(\mathfrak{g}[[z]], \mathfrak{g}; \mathcal{V}_{\text{crit}}) \cong H^\bullet(\mathfrak{g}(z)), \mathfrak{g}; \text{End}(\mathcal{V}_{\text{crit}})) \cong \text{Ext}^\bullet_{HC(\hat{\mathfrak{g}}, \mathcal{G}[[z]])}(\mathcal{V}_{\text{crit}}; \mathcal{V}_{\text{crit}}),$$

which are proved in [FT]. The Ext groups are computed in a suitably defined category of the Harish-Chandra modules for the pair $(\hat{\mathfrak{g}}, \mathcal{G}[[z]])$ of critical level, and the endomorphism algebra as well as the Lie algebra cochains must be suitably completed. We refer to [FT] for the requisite technical details.

For maximum consequence, we need the $A_\infty$ enhancement of (5.4).

5.5 Proposition. The isomorphism (5.4) may be enriched to one of $A_\infty$ algebras, with the Yoneda structure on the left and the skew-commutative structure on the right.

Proof. The degree zero part of the isomorphism (5.4),

$$H^0(\mathfrak{g}[[z]]; \mathcal{V}_{\text{crit}}) = \text{End}_\mathfrak{g}(\mathcal{V}_{\text{crit}}; \mathcal{V}_{\text{crit}}) \cong \mathbb{C}[\text{Op}_G(D)],$$

was established in [FF], where it is shown to represent the “negative Fourier half” of the center of the (critically twisted) universal enveloping algebra $U_{\text{crit}} \hat{\mathfrak{g}}$, which implies that $\text{Ext}^0$ is strictly central in $\text{Ext}^\bullet$: that is, the latter is an $A_\infty$ algebra over $\text{Ext}^0$. 

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Now, Ext° is freely generated in degree 1 over Ext°. The A∞ structure on Ext° is a deformation of the standard skew-commutative one, deformation being implemented by degree-scaling. It is easier to describe this in a Koszul dual picture, so let us move to the Koszul dual of Ext° over Ext°. This is a deformation of a commutative algebra based on the cotangent bundle of Op_{LG}(D), and it suffices to show that this deformation is trivial. The only possible deformation is to a non-commutative structure, implemented to the leading order by a Poisson bi-vector field along the fibers of T^v(Op_{LG}(D)).

Here is the key observation: any such deformation will obstruct, at some order, displacements of the module supported at the zero-section in the cotangent bundle. Indeed, this is just saying that any deformation of the algebra A of functions on (the neighborhood of 0 in) a vector space will obstruct, at some order, the deformation of the skyscraper sheaf at 0. For, otherwise, we can identify the completion of A with the algebra of functions on that (unobstructed) deformation space, and hence infer the commutativity of A. But we will now show that the zero-section can be displaced, to all orders, in the direction of any exact one-form in Ω^1[Op_{LG}(D)]. (note that we can choose all generators in Ω^1[Op_{LG}(D)] to be exact). This will prove the strict commutativity of the Ext-algebra.

Clearly, it suffices to carry out the deformation in the original algebra Ext. (Koszul duality was merely a convenient depiction of the argument above.) The zero-section structure sheaf is End (V^crit). We will now see that V^crit can be deformed to all orders with any initial direction corresponding to an exact form in Ω^1[Op_{LG}(D)]: then, Hom_{g-hat} (V^crit, V^def) will provide the deformation of the zero-section, proving the proposition.

Choose a ∈ C[Op_{LG}(D)]. Consider the family of universal enveloping algebras U_{c+εk} g-hat parametrized by the varying level c+εk away from critical, and choose a family z(ε) ∈ U_{c+εk} g-hat, with z(0) a central element corresponding to a. We consider the formal 1-parameter family of (critical level) g-hat-module structures ρ_t on V^crit deforming the standard one ρ = ρ_0, defined by

$$ρ_t(ξ) := \lim_{ε→0} \left[ \exp \left( \frac{tz(ε)}{ε} \right) ρ(ξ) \exp \left( -\frac{tz(ε)}{ε} \right) \right].$$

Because z(0) is central, the limit exists. Clearly, t = 0 recovers V^crit. The first order term (in t) of the deformation of ρ is defined by the formula

$$\lim_{ε→0} \frac{1}{ε} [z(ε), ξ].$$

As shown in [FT], (4.3), this class in Ext^1_{HC} (V^crit, V^crit) corresponds to da under the isomorphism (5.4).

5.6 Remark. The first order deformation above is the bracket with z with respect to the Poisson structure on the center of U^crit g [FF].

We conclude with two (closely related) proofs of Theorem 5.1, both of which rely on its local analogue, Proposition 5.5.

First proof of Theorem 5.1. We exhibit the Yoneda Ext-algebra as a quotient of the strictly commutative one H^°(g[[z]], g; V^crit). Let us choose in it a vector subspace of generators
of $H^\bullet(g[[z]], g; V_{\text{crit}})$. Then this implements an $A_\infty$ isomorphism of $H^\bullet(\text{Bun}_G; \mathcal{D}^s)$ with a skew-commutative sub-algebra of $H^\bullet(g[[z]], g; V_{\text{crit}})$.

Returning to the presentation $\text{Bun}_G = G[[z]] \setminus X$, the following “quantum” version of the construction of $\text{Sym} T$, from the proof of Theorem 4.2, presents instead the cohomology of $D_{\text{qs}}$:

$$H^q(\text{Bun}_G; D_{\text{qs}}) = H^q G[[z]](X; (\bigwedge^\bullet g[\Sigma^o] \otimes V_{\text{crit}}; \partial));$$

(5.7)

$\partial$ is the Chevalley differential for the fiber-wise Lie algebra action of $g[\Sigma^o]$ on $V_{\text{crit}}$, twisted at the point $\phi \cdot G[\Sigma^o] \in X$ by the adjoint action of the loop group element $\phi$. If the genus of $\Sigma$ is 2 or greater, this complex is a resolution of its bottom (that is, zeroth) homology, so is in fact equivalent to a $\mathcal{D}^s$-module. Taking symbols recovers the “classical” presentation in (4.3).

There is an obvious action on (5.7) of the center of the completed enveloping algebra $U_{\text{crit}}\hat{g}$. This center surjects onto $H^0(\text{Bun}_G; \mathcal{D}^s)$, and the resulting description of $\Gamma(\text{Bun}_G; D_{\text{qs}})$ is just the Beilinson–Drinfeld Theorem 3.9. More precisely, the action of the center factors through the action on (5.7) of the algebra $\text{End}_{\hat{g}}(V_{\text{crit}})$, the degree zero part of (5.4). When identified with functions on $\text{Op}_{L_G}(D)$, central elements act via their restriction to the subvariety $\text{Op}_{L_G}$ of opers on $\Sigma$.

We want to extend this action to the entire Yoneda Ext-algebra of $V_{\text{crit}}$. More precisely, to any higher Yoneda self-extension of $V_{\text{crit}}$ as a projective Harish-Chandra module for the pair $(\hat{g}, G[[z]])$,

$$V_{\text{crit}} \to E_1 \to \cdots \to E_k \to V_{\text{crit}}$$

(5.8)

we can functorially associate a self-extension of $\mathcal{D}^s$ by using the presentation in (5.7). When $\Sigma$ has genus 2 or greater, this gives a surjection of $H^\bullet(g[[z]], g; V_{\text{crit}})$ onto $H^\bullet(\text{Bun}_G; \mathcal{D}^s)$, because we already get a surjection at the level of symbols.

A potential difficulty is that the complex in (5.7), applied to an arbitrary element of our $HC$ category, need not be concentrated in degree zero. In other words, the functor described by the degree-zero part of the complex of sheaves in (5.7), as a functor from projective $\hat{g}$-modules to $\mathcal{D}^s$-modules, is not exact on the full Harish-Chandra category. It is, however, exact on the full, exact subcategory $EV$ consisting of finite, successive extensions of $V_{\text{crit}}$. (The objects of $EV$ are modules which admit a finite filtration with associated graded a direct sum of copies of $V_{\text{crit}}$.) Induction on the length shows that on $EV$, the localization functor (5.7) is concentrated in degree zero (and exact), so it gives an $A_\infty$ morphism of Ext-algebras. Now, because $\text{Ext}(V_{\text{crit}}; V_{\text{crit}})$ in the Harish-Chandra category is generated by $\text{Ext}^1$, and the commutativity relations are enforced by formal deformations which (to all orders) belong to $EV$, we may assume that the $E_i$ in (5.8) belong to $EV$. In other words, the Ext group computed in $EV$ agrees with the one in the Harish-Chandra category. (For a discussion of Yoneda extension classes in exact categories, see for example [NR].) With this observation, we do get the desired $A_\infty$ surjection.

**Second proof of Theorem 5.1.** This argument follows the idea of the proof of Proposition 5.5; so we will not repeat all details, but only address the parts which need modification. By means of the construction (5.7), the formal deformations of $V_{\text{crit}}$ described in the proof of Thm. 5.1 lead to formal deformations of $\mathcal{D}^s$ as $\mathcal{D}^s$-modules. This shows the existence of
formal deformations of $\mathcal{D}^s$ to all orders, tangent to any direction in $\text{Ext}^1 = \Omega^1[\text{Op}_L G]$ which corresponds to an exact differential.

As before, we Koszul-dualize $\text{Ext}^*_{\mathcal{D}^s}(\mathcal{D}^s; \mathcal{D}^s)$ to obtain a deformation of the formal neighborhood of $\text{Op}_L G$ within its cotangent bundle. Again, to leading order, this deformation is given by a Poisson bi-vector field. The key observation, which concludes the proof, is that the only Poisson structure under which first-order Lagrangian displacements of the zero-section are unobstructed to all orders is a constant multiple of the standard symplectic form. This is shown in the Lemma below. However, deformation in that particular direction would collapse the Ext-algebra to $\mathbb{C}$: see Remark 4.12. This is disallowed by our calculation of $\text{Ext}^*$: so the Poisson structure vanishes, our Koszul dual algebra is strictly commutative, and then so is the $A_\infty$ structure on the original Ext-algebra. \(\square\)

5.9 Lemma. Let $X \subset Y$ be manifolds (affine algebraic or Stein) of dimension $2$ or more, such that the normal bundle of $X$ is identified with $T^\vee X$. Let $\alpha$ be a non-trivial Poisson structure near $X \subset Y$, for which $X$ is involutive. Assume that first-order Lagrangian displacements of $X$ are unobstructed to all orders: that is, for every closed differential $\varphi$ on $X$, there exists a formal 1-parameter family of $\alpha$-involutive submanifolds of $Y$, deforming $X$, and equal to first order to the graph of $\varphi$. Then, in a formal neighborhood of $X$, $Y \cong T^\vee X$ in such a way that $\alpha$ is the standard Poisson structure.

Proof. Degenerate $Y$ to $T^\vee X$ and retain the leading part $\text{gr}(\alpha)$ of $\alpha$. The graphs of all differentials are now involutive. Now, it is easy to check that, on the standard symplectic vector space, a constant Poisson structure for which all standard Lagrangian subspaces are involutive is a multiple of the symplectic Poisson structure. So $\text{gr}(\alpha)$ is a (point-dependent) multiple of the symplectic Poisson structure on $T^\vee X$. But only the constant multiples of the standard Poisson structure on $T^\vee X$ are Poisson, if $\dim X > 1$. So $\text{gr}(\alpha)$ is a constant multiple of the standard Poisson structure. But then, $Y$ is symplectic near $X$, with $X$ Lagrangian and the Lemma becomes the formal version of Weinstein’s tubular neighborhood theorem. \(\square\)

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