The distribution of 3D superconductivity near the second critical field

Ayman Kachmar\textsuperscript{1} and Marwa Nasrallah\textsuperscript{2,3}

\textsuperscript{1} Department of Mathematics, Lebanese University, Hadath, Lebanon
\textsuperscript{2} Faculty of Sciences, Lebanese University, Section IV, Bekaa, Lebanon
\textsuperscript{3} Lebanese International University, Beirut, Lebanon

E-mail: ayman.kashmar@gmail.com and marwa.nasrallah@liu.edu.lb

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Abstract

We study the minimizers of the Ginzburg–Landau energy functional with a uniform magnetic field in a three dimensional bounded domain. The functional depends on two positive parameters, the Ginzburg–Landau parameter and the intensity of the applied magnetic field, and acts on complex-valued functions and vector fields. We establish a formula for the distribution of the $L^2$-norm of the minimizing complex-valued function (order parameter). The formula is valid in the regime where the Ginzburg–Landau parameter is large and the applied magnetic field is close to and strictly below the second critical field— the threshold value corresponding to the transition from the superconducting to the normal phase in the bulk of the sample. Earlier results are valid in 2D domains and for the $L^4$-norm in 3D domains.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper, we derive a formula displaying the distribution of the density of the superconducting electron pairs (Cooper pairs) in a superconducting sample. Such a formula has been obtained in [17] when the sample occupies a cylindrical domain with an infinite height. The novelty here is that the sample is allowed to occupy any bounded three dimensional domain with a smooth boundary.
Our results are valid for type-II superconductors within the Ginzburg–Landau theory. In this theory, a superconducting sample is distinguished by a material parameter \( \kappa > 0 \). \( \kappa \) is called the Ginzburg–Landau parameter. When the sample is placed in a magnetic field, we will denote the intensity of the magnetic field by the positive parameter \( H > 0 \). As \( H \) varies, the state of superconductivity in the sample will undergo several phase transitions that we outline below:

- There is a first critical value \( H_{C_1} > 0 \) such that, if \( H < H_{C_1} \), the sample remains in a perfect superconducting state and repels the applied magnetic field.
- There is a second critical value \( H_{C_2} > H_{C_1} \) such that, if \( H_{C_1} < H < H_{C_2} \), then the applied magnetic field penetrates the sample at point defects and these point defects are in the normal (non-superconducting) state. The rest of the sample is in the superconducting state.
- There is a third critical value \( H_{C_3} > H_{C_2} \) such that, if \( H_{C_2} < H < H_{C_3} \), then the bulk of the sample is in the normal state and the surface of the sample is in the superconducting state.
- If \( H > H_{C_3} \), all the sample is in the normal state.

We refer to the book of de Gennes [9] for the physical background. The distribution of bulk superconductivity under applied magnetic fields with intensity near \( H_{C_2} \) has been investigated by Abrikosov in [1]. Besides the presence of superconductivity in the bulk of the sample, it is uniformly distributed and penetrated by vortices that are arranged on a uniform lattice (see [1]). It has been conjectured that the lattice should be triangular, yet no mathematically rigorous justification of this is available. However, the results available in [21, 24] contain strong indications supporting the correctness of this conjecture.

Using the Ginzburg–Landau model and rigorous mathematical methods, the critical values (fields) \( H_{C_1}, H_{C_2} \) and \( H_{C_3} \) are identified in the large \( \kappa \) regime. For samples occupying infinite cylindrical domains, we refer to the papers [2, 3, 6–8, 13, 20, 23] and the two monographs [10, 22]. The sharp distribution of surface superconductivity is obtained recently in [6–8]. In particular, it is obtained that a superconducting current flows along the surface while the bulk of the sample is the normal state.

For general three dimensional domains, we refer to the papers [5, 11, 12, 14, 15, 18, 19]. The value \( H_{C_2} \) is called the second critical field. Existing results suggest that \( H_{C_2} \sim \kappa \) as \( \kappa \to \infty \), for samples with Ginzburg–Landau parameter \( \kappa \) (see [2, 13, 20]). The situation we discuss in this paper corresponds to an applied magnetic field of intensity close to and strictly below the critical field \( H_{C_2} \) (see assumption 1.1 below).

Suppose that the superconducting sample occupies a domain \( \Omega \subset \mathbb{R}^3 \). The state of the superconductivity is described using a complex-valued function \( \psi : \Omega \to \mathbb{C} \) and a vector field \( A : \Omega \to \mathbb{R}^3 \). The function \( \psi \) is called the Ginzburg–Landau parameter and the vector field \( A \) is called the magnetic potential. The quantity \( |\psi|^2 \) measures the density of the superconducting electron pairs (Cooper pairs) hence when \( \psi(x) \approx 0 \) the sample is in the normal state at \( x \). At equilibrium, the configuration \((\psi, A)\) minimizes the Ginzburg–Landau energy.

If the region \( \Omega \) is an infinite cylinder with cross section \( U \subset \mathbb{R}^2 \) and the applied magnetic field is parallel to the cylinder’s axis, then \( \psi \) and \( A \) can be reduced to functions defined on \( U \). In this case, under the assumptions

\[
\kappa \to \infty \quad \text{and} \quad \kappa^{-1/2} \ll 1 - \frac{H}{\kappa} \ll 1,
\]

the density \( |\psi|^2 \) satisfies (see [17])
Here, as explained earlier, \( \kappa \) and \( H \) are two positive parameters, and \( b_0 = (0,0,1) \) is the profile and direction of the (uniform) applied magnetic field.

Let us introduce the space \( H^1_{\text{div,F}}(\mathbb{R}^3) \) of vector fields defined as follows

\[
H^1_{\text{div,F}}(\mathbb{R}^3) = \{ A : \mathbb{R}^3 \to \mathbb{R}^3 : \text{div} A = 0, \quad \text{and} \quad A - F \in H^1(\mathbb{R}^3) \},
\]

where \( F \) is the following magnetic potential

\[
F(x) = (-x_2/2, x_1/2, 0), \quad \forall \ x = (x_1, x_2, x_3) \in \mathbb{R}^3,
\]

and the space \( H^1(\mathbb{R}^3) \) is the homogeneous Sobolev space, i.e. the closure of \( C^\infty_c(\mathbb{R}^3) \) under the norm \( u \mapsto \|u\|_{H^1(\mathbb{R}^3)} := \|\nabla u\|_{L^2(\mathbb{R}^3)} \).

The energy in (1.3) will be minimized over the space \( H^1(\Omega; \mathbb{C}) \times H^1_{\text{div,F}}(\mathbb{R}^3) \). Actually, this is the natural ‘energy’ space for the functional in (1.3), see [10]. We thereby introduce the following ground state energy

\[
E_{\text{gs,ul}}(\kappa, H) = \inf \{ \mathcal{E}^{3D}(\psi, A) : (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div,F}}(\mathbb{R}^3) \},
\]

For a given \( \kappa \) and \( H \), we will call a minimizer of the functional (1.3) a configuration \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div,F}}(\mathbb{R}^3)\) satisfying \( \mathcal{E}^{3D}(\psi, A) = E_{\text{gs,ul}}(\kappa, H) \). Obviously, such a configuration will depend on \( \kappa \) and \( H \). To emphasize this dependence, we will denote such minimizers by \((\psi, A)_{\kappa,H}\).

Note that a minimizer \((\psi, A)_{\kappa,H}\) is a critical point of the functional in (1.3), i.e.

\[
\forall (\phi, a) \in H^1(\Omega; \mathbb{C}) \times C^\infty_c(\mathbb{R}^3, \mathbb{R}^3), \quad \frac{d}{dr} \mathcal{E}^{3D}(\psi + t\phi, A) \bigg|_{t=0} = 0 \text{ and } \frac{d}{dr} \mathcal{E}^{3D}(\psi, A + ta) \bigg|_{t=0} = 0.
\]

More precisely, a critical point \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div,F}}(\mathbb{R}^3)\) is a weak solution of the Ginzburg–Landau equations,
\[-(\nabla - i\kappa A)^2 \psi = \kappa^2 (1 - |\psi|^2) \psi \quad \text{in } \Omega \]
\[
\text{curl}^2 A = -\frac{1}{\kappa^2} \text{Im}(\overline{\psi}(\nabla - i\kappa A)\psi) \mathbf{1}_{\Omega} \quad \text{in } \mathbb{R}^3
\]
\[\nu \cdot (\nabla - i\kappa A)\psi = 0 \quad \text{on } \partial \Omega.\]

where \(\mathbf{1}_{\Omega}\) is the characteristic function of the domain \(\Omega\), and \(\nu\) is the unit interior normal vector of \(\partial \Omega\). Recall that \(\text{curl} A = \nabla \times A\) and \(\text{curl}^2 A = \nabla \times \nabla \times A\).

Minimizers of the functional in (1.3) are studied in [12, 14]. Under the assumption in (1.1), if \((\psi, A)_{\kappa, \kappa}\) is a minimizer of the functional in (1.3), then

\[
\int_{\Omega} |\psi|^4 \, dx = -2E_{\text{Ab}}|\partial \Omega| \left( 1 - \frac{H}{\kappa} \right)^2 + a \left( 1 - \frac{H}{\kappa} \right)^2. \tag{1.8}
\]

We will improve this formula in theorem 1.2 below. We will work under the following assumption:

**Assumption 1.1.**

\(\bullet \ \alpha : \mathbb{R}_+ \to \mathbb{R}_+\) and \(\beta : \mathbb{R}_+ \to \mathbb{R}_+\) are two functions satisfying

\[\lim_{\kappa \to \infty} \alpha(\kappa) = \infty, \quad \lim_{\kappa \to \infty} \beta(\kappa) = 0 \quad \text{and} \quad \alpha(\kappa) \leq \beta(\kappa) \kappa^{1/2}\] in a neighborhood of \(\infty\).

\(\bullet \ \kappa > 0\) and \(H > 0\) satisfy \(\alpha(\kappa) \kappa^{-1/2} \leq 1 - \frac{H}{\kappa} \leq \beta(\kappa)\).

The heuristics behind assumption 1.1 can be explained as follows. Saying that \(0 < 1 - \frac{H}{\kappa} \leq \beta(\kappa)\) yields that \(1 - \frac{H}{\kappa} \to 0\) as \(\kappa \to \infty\). This means that \(H\) is close to the second critical field \(H_C\) (see [13, 23]). On the other hand, saying that \(1 - \frac{H}{\kappa} \leq \alpha(\kappa) \kappa^{-1/2}\) yields that the formula in (1.8) is valid, hence bulk superconductivity is present. Consequently, \(H\) is strictly below the critical field \(H_C\). Let us mention that by violating the condition \(1 - \frac{H}{\kappa} \gg \kappa^{-1/2}\), the formula in (1.8) has to be replaced by (see [13])

\[
\int_{\Omega} |\psi|^4 \, dx = -2E_{\text{surf}}|\partial \Omega| \kappa^{-1} - 2E_{\text{Ab}}|\Omega| \left( 1 - \frac{H}{\kappa} \right)^2 + a \left( \max \left( 1 - \frac{H}{\kappa} \right)^2, \kappa^{-1} \right),
\]

where \(E_{\text{surf}} < 0\) is a universal constant. Hence, the bulk contribution is dominant for \(1 - \frac{H}{\kappa} \gg \kappa^{-1/2}\).

In this paper, we will prove the following theorem (compare with (1.8)):

**Theorem 1.2 (Sharp bound in \(L^4\)-norm).** There exist \(\kappa_0 > 0\) and a function \(\text{err} : [0, \infty) \to (0, \infty)\) such that:

\(\bullet \ \lim_{\kappa \to \infty} \text{err}(\kappa) = 0;\)

\(\bullet \ \text{the following inequality holds}\)

\[
\frac{1}{|Q_\kappa|} \int_{Q_\kappa} |\psi|^4 \, dx \leq -2E_{\text{Ab}} \left( 1 - \frac{H}{\kappa} \right)^2 + \left( 1 - \frac{H}{\kappa} \right)^2 \text{err}(\kappa).\] \tag{1.9}

where

\(\bullet \ E_{\text{Ab}}\) is the Abrikosov constant introduced below in theorem 2.3;

\(\bullet \ \kappa \gg \kappa_0\) and \((\kappa, H)\) satisfy assumption 1.1;
– $(\psi, A)$ is a solution of (1.7);
– $Q_\kappa$ is any cube of side length $\kappa^{-1/2}$ and satisfying $Q_\kappa \subset \{ \text{dist}(x, \partial \Omega) > 2\kappa^{-1/2} \}$.

Note that the conclusion in theorem 1.2 has been known in the following cases:

• when $Q_\kappa$ is replaced by the whole domain $\Omega$ but without specifying the (sharp) constant $E_{\text{Ab}}$ (see [4, theorem 3.3]);
• when $Q_\kappa$ is replaced by any open subset $D \subset \Omega$ such that $D$ is independent of $\kappa$ and with a smooth boundary (see [12]).

In light of (1.8), we observe that the constant $E_{\text{Ab}}$ in (1.9) is optimal. In fact, if the inequality in (1.8) holds for all critical points $(\psi, A)$ but with the constant $-2E_{\text{Ab}}$ replaced by a constant $C, < -2E_{\text{Ab}}$, then (1.8) yields a contradiction for large values of $\kappa$.

Let us point out that the derivation in [12, 14] of the upper bound in (1.8) relies on the estimate in [4] to control the error terms. However, the proof we give to theorem 1.2 does not use ingredients from [4] but instead uses theorem 2.7 in this paper, which displays a new formulation of the Abrikosov constant in terms of a non-linear eigenvalue problem. This approach has been used for 2D samples in [16].

Our next result is an asymptotic formula of the $L^2$-distribution of the minimizing order parameters.

**Theorem 1.3 (Distribution of the density).** Let $D \subset \Omega$ be an open set such that $|\partial D| = 0$ and $\overline{D} \subset \Omega$. Suppose that $H$ is a function of $\kappa$ satisfying

\[ H \ll \kappa \quad \text{and} \quad \kappa^{-3/8} \ll \frac{1}{\kappa} \ll 1. \]  

(1.10)

If $(\psi, A)_{\kappa, H}$ is a minimizer of the functional in (1.3), then as $\kappa \to \infty$,

\[ \frac{1}{|D|} \int_D |\psi|^2 \, dx = -2E_{\text{Ab}} \Bigl( 1 - \frac{H}{\kappa} \Bigr) + o \left( 1 - \frac{H}{\kappa} \right). \]  

(1.11)

Here $E_{\text{Ab}} \in [-1, 0)$ is the universal constant defined in theorem 2.3 below.

Note that in (1.11), the domain $D$ is fixed and independent of $\kappa$. Moreover, the assumption that $\overline{D} \subset \Omega$ yields that the boundary of $D$ does not touch the boundary of $\Omega$, in accordance with the assumptions appearing in theorem 1.2.

The conclusion in theorem 1.3 is consistent with the formula in (1.2) but is valid under the more restrictive assumption in (1.10). One reason that prevented us of proving (1.11) under the assumption in (1.1) is the lack of the upper bound

\[ \|\psi\|_{L^2(\Omega_\kappa)} \leq C \left( 1 - \frac{H}{\kappa} \right)^{1/2} \quad (\Omega_\kappa = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \gg \kappa^{-1} \}). \]  

(1.12)

This upper bound is known to hold in 2D domains (see [13]). Since we were not able to prove (1.12) in 3D domains, we used the estimate in theorem 1.2 as a substitute. The price we paid is the restrictive assumption in (1.10). The technical reasons that led us to the assumption in (1.10) are explained in remark 4.5.

The formula in (1.11) supports the claim that the density of superconductivity $|\psi|^2$ is uniformly distributed in the bulk of the sample. This is in accordance with the physical behavior below the second critical field. But many important things are still missing, even in two dimensional samples, for example the vortex structure of the order parameter or estimates of the superconducting currents. Any such additional information must be welcome.
While the formulas in (1.8) and (1.11) are complementary, the philosophy behind their derivation is quite different. The formula in (1.8) is related to the Ginzburg–Landau energy as follows: Multiplying the first G–L equation in (1.7) by $\psi$ then integrating by parts yields
\[-\frac{\kappa^2}{2} \int_{\Omega} |\psi|^4 \, dx = \int_{\Omega} \left( |\nabla - i\kappa H A| \psi |^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{4} |\psi|^4 \right) \, dx.
\]
Thus one can estimate the integral of $|\psi|^4$ by estimating the G–L energy in (1.3). The later is estimated by constructing trial functions to obtain upper bounds, and then by comparing with simplified reference functionals to get matching lower bounds.

The estimate of the integral of $|\psi|^2$ is based on the following principle. Since the integral of $|\psi|^4$ is small (see (1.8)), the non-linearity in the G–L energy (1.3) is weak. Thus, the kinetic energy term should be approximated using the first eigenvalue of the magnetic Laplacian as follows
\[
\int_{\Omega} (|\nabla - i\kappa H A| \psi |^2 \approx \lambda(\kappa H A) \int_{\Omega} |\psi|^2 \, dx.
\]
That way, the energy of $\psi$ in (1.3) becomes
\[
\int_{\Omega} \left( \lambda(\kappa H A) - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) \, dx.
\]
Minimizing this energy yields the Abrikosov constant $E_{Ab}$. Consequently, we get a relation between the Abrikosov constant and the integrals of $|\psi|^2$ and $|\psi|^4$. Using the existing estimate of the integral of $|\psi|^4$, we deduce the formula for the integral of $|\psi|^2$. However, to take care of the error terms arising along the various approximations described above, a lot of technical work is needed.

The rest of the paper is devoted to the proof of theorems 1.2 and 1.3. It is organized as follows:

- Section 2 reviews various limiting energies studied in [12] and concludes with the proof of theorem 2.7. Theorem 2.7 is new and not among the results in [12].
- Section 3 is devoted to the proof of theorem 1.2. It uses theorem 2.7 as a key ingredient.
- Section 4 establishes the asymptotics of the Ginzburg–Landau energy in cubes with small lengths. The main conclusion here is summarized in corollary 4.8. The assumption in (1.10) is needed in this section.
- Section 5 finishes the proof of theorem 1.3. We prove an energy asymptotics for the density in cubes with small lengths as well, see theorem 5.3.

Remark on the notation

The parameters $\kappa$ and $H$ are allowed to vary in such a manner that $H/\kappa \in [c_1, c_2]$, where $0 < c_1 < c_2$ are fixed constants. Whenever the letter $C$ appears, it denotes a positive constant that is independent of $\kappa$ and $H$. Such a constant may depend on the domain $\Omega$, the constants $c_1, c_2$, etc. The value of $C$ might change from one formula to another.

In the proofs, the notation $o(1)$ stands for an expression that depends on $\kappa$ and $H$ such that $o(1) \to 0$ as $\kappa \to \infty$. However, this expression is independent of the choice of a minimizing/critical configuration $(\psi, A)_{\kappa, H}$ of the functional in (1.3), but it depends on the constants $c_1, c_2$, the domain $\Omega$, etc. Sometimes we do local arguments in, say, a ball or a square of center $x_0$ and radius $\ell$. In such arguments, the quantity $o(1)$ is independent of the center $x_0$. 

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Finally, by writing $a(\kappa) \approx b(\kappa)$, we mean that the positive functions $a(\kappa)/b(\kappa)$ and $b(\kappa)/a(\kappa)$ are bounded in a neighborhood of $\kappa = \infty$. In particular, our assumption on $\kappa$ and $H$ can be expressed as $H \approx \kappa$.

2. Limiting energies

2.1. Two-dimensional limiting energy

Let $b > 0$ and $D$ be an open subset in $\mathbb{R}^2$. We define the following reduced Ginzburg–Landau functional,

$$H^1(D) \ni u \mapsto G_{b,D}(u) = \int_D \left( |\nabla - iA_0|u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx,$$

(2.1)

where

$$A_0(x_1, x_2) = \frac{1}{2}(-x_2, x_1), \quad (x = (x_1, x_2) \in \mathbb{R}^2).$$

(2.2)

Given $R > 0$, we denote by $K_R = (-R/2, R/2)^2$ the square of side length $R$ and center $0$. Let us introduce the following ground state energy

$$m_0(b, R) = \inf_{u \in H^1(K_R; \mathbb{C})} G_{b,K_R}(u).$$

(2.3)

It is proved in [2, 12, 23] that, for all $b \geq 0$, there exists $g(b) \in [\frac{1}{2}, 0]$ such that

$$g(b) = \lim_{R \to \infty} \frac{m_0(b, R)}{R^2},$$

(2.4)

and that the function $[0, \infty) \ni b \mapsto g(b) \in [-1/2, 0]$ is continuous, non-decreasing, $g(0) = -\frac{1}{2}$ and $g(b) = 0$ for all $b \geq 1$. Moreover, there exists a universal constant $\alpha \in (0, 1/2)$ such that, for all $b \in [0, 1]$

$$\alpha(b - 1)^2 \leq |g(b)| \leq \frac{1}{2}(b - 1)^2.$$  

(2.5)

Also, for all $R \geq 1$ and $b \in [0, 1]$, it holds the estimate

$$g(b) \leq \frac{m_0(b, R)}{R^2} \leq g(b) + \frac{C}{R},$$

(2.6)

2.2. The 2D periodic Schrödinger operator with uniform magnetic field

Let $R > 0$ and $K_R = (-R/2, R/2) \times (-R/2, R/2)$. In this section we assume that

$$R^2 \in 2\pi \mathbb{N}.$$  

We introduce the following space

$$E_R = \{ u \in H^1_0(\mathbb{R}^2; \mathbb{C}) : u(\cdot + R, \cdot) = e^{i\theta_0/2}u(x_1, x_2), \quad u(x_1, x_2 + R) = e^{-i\pi/2}u(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2 \}. $$

(2.7)

Recall the magnetic potential $A_0$ in (2.2). Consider the operator
\[ P_R^{2D} = - (\nabla - i A_0)^2 \]  
(2.8)

with form domain \( \mathcal{E}_R \) introduced in (2.7). More precisely, \( P_R^{2D} \) is the self-adjoint realization associated with the closed quadratic form

\[ \mathcal{E}_R \ni f \mapsto Q_R^{2D}(f) = \| (\nabla - i A_0)f \|_{L^2(\mathbb{K}_R)}. \]

The operator \( P_R^{2D} \) has a compact resolvent. We denote by \( \{ \mu_i(P_R^{2D}) \}_{i \geq 1} \) the increasing sequence of its eigenvalues. The following proposition may be classical in the spectral theory of Schrödinger operators, but we refer to [2] or [3] for a simple proof.

**Proposition 2.1.** The operator \( P_R^{2D} \) has the following properties:

1. \( \mu_1(P_R^{2D}) = 1 \), and \( \mu_2(P_R^{2D}) = 3 \).
2. The space \( L_R = \text{Ker}(P_R^{2D} - 1) \) is finite dimensional and \( \dim L_R = \frac{p^2}{2\pi} \).

Consequently, denoting by \( \Pi_1 \) the orthogonal projection on the space \( L_R \) in \( L^2(\mathbb{K}_R) \) and by \( \Pi_2 = \text{Id} - \Pi_1 \), we have for all \( f \in D(P_R^{2D}) \),

\[ (P_R^{2D} \Pi_1 f, \Pi_1 f)_{L^2(\mathbb{K}_R)} \geq 3 \| \Pi_2 f \|_{L^2(\mathbb{K}_R)}^2. \]  
(2.9)

The next lemma is a consequence of the existence of the spectral gap between the first two eigenvalues of \( P_R^{2D} \). It is proved in [13, lemma 2.8].

**Lemma 2.2.** Let \( p \geq 2 \). There exists a constant \( C_p > 0 \) such that for any \( \gamma \in (0, 1/2) \), and \( u \in D(P_R^{2D}) \) satisfying

\[ Q_R^{2D}(f) - (1 + \gamma) \| f \|_{L^2(\mathbb{K}_R)}^2 \leq 0 \]  
(2.10)

the following estimate holds:

\[ \| u - \Pi_1 u \|_{L^2(\mathbb{K}_R)} \leq C_p \gamma^{\gamma} \| u \|_{L^2(\mathbb{K}_R)}. \]  
(2.11)

Here \( \Pi_1 \) is the projection on the space \( L_R \).

### 2.3. The Abrikosov energy

We introduce the following energy functional (the Abrikosov energy):

\[ F_R(v) = \int_{\mathbb{K}_R} \left( \frac{1}{2} |v|^4 - |v|^2 \right) \mathrm{d}x. \]

The energy \( F_R \) will be minimized on the space \( L_R \), the (finite dimensional) eigenspace of the first eigenvalue of the periodic operator \( P_R^{2D} \),

\[ L_R = \{ u \in \mathcal{E}_R : P_R^{2D} u = u \}. \]

For all \( R > 0 \), let

\[ c(R) = \min \{ F_R(u) : u \in L_R \}. \]  
(2.12)

The following theorem is proved in [2, 12]:

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Theorem 2.3. There exists a constant $E_{ab} \in [-1/2, 0]$ such that

$$E_{ab} = \lim_{R \to \infty} \frac{c(R)}{R^2} = \lim_{b \to 1} \frac{g(b)}{(b - 1)^2}.$$ 

We collect one more estimate from [17, proposition 3.1 and theorem 3.5]. There exist two constants $C > 0$ and $\epsilon_0 \in (0, 1)$ such that, for all $b \in (1 - \epsilon_0, 1)$ and $R \geq 2$,

$$m_0(b, R) \leq (1 - b)^2 c(R) + C(1 - b) R.$$  

(2.13)

2.4. Three-dimensional limiting energy

Let $b > 0$, $\mathcal{D}$ be an open subset in $\mathbb{R}^3$ and

$$\forall \ u \in H^1(\mathcal{D}), \quad F_{b, \mathcal{D}}(u) = \int_{\mathcal{D}} \left( |b|(|\nabla - iF|u|^2 - |u|^2) + \frac{1}{2} |u|^4 \right)dx,$$

where $F$ is the magnetic potential introduced in (1.5). For all $R > 0$, we denote by $Q_R = K_R \times (-R/2, R/2)$ and

$$M_0(b, R) = \inf_{u \in \mathcal{M}_{Q(R)}} F_{b, Q_R}(u).$$

(2.15)

The next lemma displays the connection between the two and three dimensional ground state energies, $m_0(b, R)$ and $M_0(b, R)$. It is taken from [12, theorem 2.14].

Lemma 2.4. There exists a universal constant $C > 0$ such that, for all $b > 0$ and $R > 0$, we have

$$Rm_0(b, R) \leq M_0(b, R) \leq (R - 2)m_0(b, R) + C.$$  

(2.16)

Combining (2.6) and (2.16), we deduce the following lemma.

Lemma 2.5. There exists a universal constant $C > 0$ such that for all $R \geq 1$ and $b > 0$,

$$g(b) \leq \frac{M_0(b, R)}{R^3} \leq \frac{R - 2}{R} g(b) + \frac{C}{R}.$$  

As a consequence of lemma 2.5, we may prove:

Lemma 2.6. There exists a constant $C > 0$, such that, if $b \in (0, 1]$, $R > 1$ and $v_{b, R}$ is a minimizer of $F_{b, Q_R}$ (i.e. $F_{b, Q_R}(v_{b, R}) = M_0(b, R)$), then,

$$-2R^2(R - 2)g(b) - CR^2 \leq \int_{Q_R} |v_{b, R}|^4 dx \leq -2R^3 g(b).$$  

(2.17)

Proof. The minimizer satisfies the following equation

$$-b(\nabla - iF)^2 v_{b, R} = (1 - |v_{b, R}|^2) v_{b, R},$$

with Dirichlet boundary conditions on the boundary of $Q_R$.

Multiplying the above equation by $\overline{v_{b, R}}$, integrating over $Q_R$ and performing an integration by parts, it follows that

$$M_0(b, R) = -\frac{1}{2} \int_{Q_R} |v_{b, R}|^4 dx.$$  

Now applying lemma 2.5 finishes the proof of lemma 2.6. □
Now we establish a link between the ground state energy in (2.15) and a non-linear eigenvalue problem. Such a relationship has been discovered in [16] in the two dimensional setting. We define the linear functional

$$F_{b,0}^\text{lin}(u) = \int_D (b|\nabla - i\mathbf{F}|u|^2 - |u|^2) \, dx.$$  

(2.18)

We will minimize this functional in the space of functions satisfying

$$\int_{Q_R} |u|^4 \, dx = 1.$$  

That way, we are led to introduce the following ground state energy

$$\mathcal{M}_0(b,R) = \inf \left\{ \frac{F_{b,0}^\text{lin}(u)}{\left(\int_{Q_R} |u|^4 \, dx\right)^{1/2}} : u \in H_0^1(Q_R) \setminus \{0\} \right\}.$$  

(2.19)

We aim at proving that

$$\lim_{R \to \infty} \frac{\mathcal{M}_0(b,R)}{R^{3/2}} = g_{\text{new}}(b),$$  

(2.20)

where

$$g_{\text{new}}(b) = -\sqrt{-2g(b)}.$$  

Actually, it holds:

**Theorem 2.7.** Let $b \in (0,1)$. There exist two constants $C > 0$ and $R_0 > 1$ such that, for all $R \geq R_0$

$$-(-2g(b))^{1/2} \leq \frac{\mathcal{M}_0(b,R)}{R^{3/2}} \leq 1 - C\left(\frac{-2g(b)}{R}\right)^{1/2} + C\left(-2g(b)\right)^{-1/2}.$$  

(2.21)

In light of theorem 2.7, we infer that

$$g(b) = -\frac{1}{2} \left(\lim_{R \to \infty} \frac{\mathcal{M}_0(b,R)}{R^{3/2}}\right)^2.$$  

**Proof of theorem 2.7.** Upper bound:

We will prove the following inequality

$$\mathcal{M}_0(b,R) \leq -R(R-2)^{1/2} (-2g(b))^{1/2} + CR^{1/2}(-2g(b))^{-1/2}$$  

(2.22)

valid for some universal constant $C$, for all $b \in (0,1)$ and $R$ sufficiently large.

Let $v_{b,R}$ be a minimizer of $\mathcal{M}_0(b,R)$ for the Dirichlet boundary condition. Using the definition of $\mathcal{M}_0(b,R)$, we may write

$$F_{b,0}(v_{b,R}) = \mathcal{M}_0(b,R)$$

$$\geq \mathcal{M}_0(b,R) \left(\int_{Q_R} |v_{b,R}|^4 \, dx\right)^{1/2} + \frac{1}{2} \int_{Q_R} |v_{b,R}|^4 \, dx.$$  

(2.23)
By lemma 2.6, we get for $R$ sufficiently large
\[ M_0(b, R) \geq \mathcal{M}_0(b, R)(-2R^2(R - 2)g(b) - CR^2)_{+}^{1/2} + \frac{1}{2}(-2R^2(R - 2)g(b) - CR^2). \]

We use lemma 2.5 to estimate $M_0(b, R)$ from above. Consequently, we obtain, for some new constant $C' > 0$,
\[ \mathcal{M}_0(b, R)(-2R^2(R - 2)g(b) - CR^2)_{+}^{1/2} \leq -2R^2(R - 2)g(b) + C'R^2. \]

This finishes the proof of the upper bound in (2.22).

**Lower bound:**

We will prove that for all $b \in (0, 1)$ and $R > 1$,
\[ \mathcal{M}_0(b, R) \geq -R^{3/2}(-2g(b))^{1/2}. \tag{2.24} \]

Let $w_{b, R}$ be a minimizer of $\mathcal{M}_0(b, R)$. Let us normalize $w_{b, R}$ as follows
\[ w_{b, R}^* = \frac{\mathcal{F}_{b, Qh}(w_{b, R})}{\|w_{b, R}\|_{L^2(Q)}}, \]

The $L^2$ norm of $w_{b, R}$ satisfies
\[ \|w_{b, R}\|_{L^2(Q)} = R^{3/4}(-2g(b))^{1/4}. \]

By definition of $\mathcal{M}_0(b, R)$, we see that
\[ \mathcal{M}_0(b, R) = \frac{\mathcal{F}_{b, Qh}(w_{b, R})}{\|w_{b, R}\|_{L^2(Q)}} = R^{-3/2}(-2g(b))^{-1/2} \mathcal{F}_{b, Qh}(w_{b, R}). \tag{2.25} \]

We write
\[ \mathcal{F}_{b, Qh}(w_{b, R}^*) = \mathcal{F}_{b, Qh}(w_{b, R}) - \frac{1}{2} \int_{Qh} |w_{b, R}^*|^2 \, dx \\
\geq \mathcal{M}_0(b, R) + R^3 g(b) \\
\geq 2R^2 g(b). \]

Note that in the last inequality, we used lemma 2.5 to write a lower bound for $M_0(b, R)$.

Now, inserting the inequality $\mathcal{F}_{b, Qh}(w_{b, R}^*) \geq 2R^2 g(b)$ in (2.25), we obtain (2.24). \qed

### 3. Proof of theorem 1.2

In the sequel, we will work with the following local energies
\[ \mathcal{E}_d(\psi; A; D) = \int_D \left( (\nabla - i\kappa H A)^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^2 \right) \, dx, \]
\[ \mathcal{E}(\psi; A; D) = \mathcal{E}_d(\psi; A; D) + \kappa^2 H^2 \int_\Omega |\text{curl}(A - F)|^2 \, dx \quad (D \subset \Omega). \tag{3.1} \]
We collect various *a priori* estimates that are useful in the proof of theorem 1.2 (see [10, chapter 10]).

**Lemma 3.1.** If \((\psi, A)\) is a solution of (1.7), then
\[
\|\psi\|_\infty \leq 1,  \tag{3.2}
\]
\[
\|(|\nabla - i\kappa H A|)\psi\|_{C(\bar{\Gamma})} \leq C_1 \sqrt{\kappa H},  \tag{3.3}
\]
and
\[
\|\text{curl}(A - F)\|_{C(\bar{\Gamma})} \leq \frac{C_1}{H}. \tag{3.4}
\]

**Lemma 3.2.** Let \(0 < \Lambda_{\min} < \Lambda_{\max}\) be fixed constants (independent of \((\kappa, H))\). There exist positive constants \(C\) and \(\kappa_0\) such that if
\[
\kappa \geq \kappa_0, \quad \Lambda_{\min} \leq \frac{H}{\kappa} \leq \Lambda_{\max},
\]
and \((\psi, A)\) is a solution of (1.7), then the following is true.

Let \(\ell \in (0, 1)\) and \(Q_0 \subset \Omega\) be a cube of side length \(\ell\), then there exists a function \(\phi \in C^\infty(\overline{Q})\) such that, for all \(x \in Q_0\), we have
\[
|A(x) - F(x) - \nabla \phi(x)| \leq C \frac{\lambda^{1/6}}{\kappa \ell}, \tag{3.5}
\]
where
\[
\lambda = \max\left(\frac{1}{\kappa}, \left(1 - \frac{H}{\kappa}\right)^2\right).
\]

**Proof.** In [14, corollary 4.4], it is proved that \(\|A - F\|_{C^1(\partial Q)} \leq C\kappa^{-1}\lambda^{1/6}\). The conclusion in lemma 3.2 follows by taking \(\phi(x) = (A(x_0) - F(x_0)) \cdot (x - x_0)\) where \(x_0\) is the center of the square \(Q_0\).

**Proof of theorem 1.2.** Let \(\sigma \in (0, 1)\) and \(Q_{\kappa, \sigma}\) be the cube having the same center as \(Q_{\kappa}\) but with side length \((1 + \sigma)\kappa^{1/2}\). Let \(\chi \in C^\infty(\overline{Q_{\kappa, \sigma}})\) be a cut-off function satisfying, for all \(\kappa \geq 1\),
\[
\chi = 1 \quad \text{in} \quad Q_\kappa, \quad 0 \leq \chi \leq 1 \quad \text{and} \quad |\nabla \chi| \leq C\sigma^{-1/2}\kappa^{1/2} \quad \text{in} \quad Q_{\kappa, \sigma}.
\]
An integration by parts and the first equation in (1.7) yield the following localization formula
\[
\mathcal{E}_0(\chi \psi; A; Q_{\kappa, \sigma}) = \kappa^2 \int_{Q_{\kappa, \sigma}} \chi^2 \left(-1 + \frac{1}{2}\chi^2\right)|\psi|^4\,dx + \int_{Q_{\kappa, \sigma}} |\nabla \chi|^2 |\psi|^2\,dx \leq C\sigma^{-1}\kappa^{-1/2}. \tag{3.6}
\]
Note that we have used that the term \((-1 + 1/2\chi^2)\) is negative, the bound on \(|\nabla \chi|\) and that \(|Q_{\kappa, \sigma}| \leq C\kappa^{-3/2}\). Let us introduce the following linear energy
\[
\mathcal{L}_{0,\kappa}(\chi \psi; A) = \int_{Q_{\kappa, \sigma}} \left(|(\nabla - i\kappa H A)\chi \psi|^2 - \kappa^2\chi |\psi|^2\right)\,dx.
\]
Let $\phi$ be the function satisfying (3.5) in $Q_{\kappa, \sigma}$ (i.e. with $\ell = (1 + \sigma)\kappa^{-1/2}$). Using the Cauchy–Schwarz inequality, we write,
\begin{align*}
L_{0, \kappa}(\chi \psi, A) &= L_{0, \kappa}(e^{-i\kappa H_{\rho}^{\phi}} \chi \psi, A - \nabla \phi) \\
&\geq \int_{Q_{\kappa, \sigma}} \left| (1 - \kappa^{-1/2})(\nabla - i\kappa H) e^{-i\kappa H_{\rho}^{\phi}} \chi \psi \right|^2 - \kappa^2 |\chi \psi|^2 - C_{\kappa^{3/2}} \lambda^{1/2} |\chi \psi|^2 \mathrm{d}x.
\end{align*}
(3.7)

Using the expression of $\lambda$ in lemma 3.2 and the assumption on $H$ in theorem 1.2, we get
\begin{align*}
L_{0, \kappa}(\chi \psi, A) &= L_{0, \kappa}(e^{-i\kappa H_{\rho}^{\phi}} \chi \psi, A - \nabla \phi) \\
&\geq \int_{Q_{\kappa, \sigma}} \left| (1 - \kappa^{-1/2})(\nabla - i\kappa H) e^{-i\kappa H_{\rho}^{\phi}} \chi \psi \right|^2 - \kappa^2 |\chi \psi|^2 - C_{\kappa^{3/2}} |\chi \psi|^2 \mathrm{d}x.
\end{align*}
(3.8)

Let $b = (1 - \kappa^{-1/2})H_{\rho}^{\phi}$, and $R = \ell \sqrt{\kappa H}$ and $x_0$ the center of the square $Q_{\kappa, \sigma}$. Apply the change of variables $y = \sqrt{\kappa H}(x - x_0)$ to get
\begin{align*}
L_{0, \kappa}(\chi \psi, A) &\geq \kappa^{5/4} H^{-3/4} M_0(b, R) \|\chi \psi\|_4^2 - C_{\kappa^{3/2}} \|\chi \psi\|_2^2,
\end{align*}
where $M_0(b, R)$ is the energy introduced in (2.19). We use theorem 2.7 to write a lower bound of $M_0(b, R)$ and Hölder inequality to estimate $\|\chi \psi\|_2$. That way we get,
\begin{align*}
L_{0, \kappa}(\chi \psi, A) &\geq -\kappa^{5/4} H^{-3/4} R^{3/2} (2g(b))^{1/2} \|\chi \psi\|_4^2 - C_{\kappa^{3/4}} \|\chi \psi\|_2^2.
\end{align*}
Recall that $\ell = (1 + \sigma)\kappa^{-1/2}$ is the side length of the cube $Q_{\kappa, \sigma}$ and that $R = \ell \sqrt{\kappa H} = (1 + \sigma)\sqrt{\kappa H}$. Note that
\begin{align*}
E_0(\chi \psi; Q_{\kappa, \sigma}) &= L_{0, \kappa}(\chi \psi, A) + \frac{\kappa^2}{2} \|\chi \psi\|_4^2 \\
&\geq -\kappa^{5/4} (1 + \sigma)^{3/2} (2g(b))^{1/2} \|\chi \psi\|_4^2 - C_{\kappa^{3/4}} \|\chi \psi\|_4^2 + \frac{\kappa^2}{2} \|\chi \psi\|_2^4.
\end{align*}

We insert this into (3.6) to get
\begin{align*}
\kappa^{5/4} \left( -(1 + \sigma)^{3/2} (2g(b))^{1/2} - C\kappa^{-1/2} + \frac{\kappa^{3/4}}{2} \|\chi \psi\|_2^2 \right) \|\chi \psi\|_4^2 \leq C\sigma^{-1}\kappa^{-1/2}.
\end{align*}
(3.9)

Two cases may occur:

**Case I:**
\begin{align*}
\left( -(1 + \sigma)^{3/2} (2g(b))^{1/2} - C\kappa^{-1/2} + \frac{\kappa^{3/4}}{2} \|\chi \psi\|_2^2 \right) \leq \kappa^{-1/2}
\end{align*}

**Case II:**
\begin{align*}
\left( -(1 + \sigma)^{3/2} (2g(b))^{1/2} - C\kappa^{-1/2} + \frac{\kappa^{3/4}}{2} \|\chi \psi\|_2^2 \right) \geq \kappa^{-1/2}.
\end{align*}

In both cases, we infer from (3.9),
\begin{align*}
\|\chi \psi\|_4^2 \leq (1 + \sigma)^{3/2} \kappa^{-1/2} (2g(b))^{1/2} + C\sigma^{-1}\kappa^{-5/4}.
\end{align*}
(3.10)
Since $\chi = 1$ in $Q_\kappa \subset Q_{\kappa,0}$ and $|Q_\kappa| = \kappa^{-3/2}$, it follows that
\[
\left( \frac{1}{|Q_\kappa|} \int_{Q_\kappa} |\psi|^4 \, dx \right)^{1/2} \leq (1 + \sigma)^{3/2}(-g(b))^{1/2} + C\sigma^{-1}\kappa^{-1/2}.
\] (3.11)

This yields the conclusion in theorem 1.2 once we choose $\sigma = \left(1 - \frac{H}{\kappa}\right)^{1/2}$. In fact, assumption 1.1 ensures that
- $\sigma \ll 1$ and $\sigma^{-1}\kappa^{-1/2} \ll 1 - \frac{H}{\kappa}$;
- $b = (1 - \kappa^{-1/2})\frac{H}{\kappa} \to 1$ so that by theorem 2.3, $g(b) = E_{Ab}(b - 1)^2 + (b - 1)^2o(1)$.

We will need to work with boxes rather than cubes only. These boxes are defined in:

**Definition 3.3.** Let $0 < \ell, L < 1$. By a $(\ell, L)$ box we mean a cuboid of the form
\[
Q_{\ell,L} = (-\ell/2, \ell/2) \times (-\ell/2, \ell/2) \times (-L/2, L/2) + x_0,
\]
for some point $x_0 \in \mathbb{R}^3$ (the center of the box).

Note that, a $(\ell, L)$ box for which $L = \ell$ is simply a cube of side length $\ell$. Later on, we will need general boxes instead of cubes. We will decompose a 3D domain into horizontal sections uniformly separated by a distance $L$. Then in every horizontal section, we perform a decomposition into squares of length $\ell$. The result will be a decomposition of the 3D domain into $(\ell, L)$ boxes rather than cubes. We apply this type of decomposition in theorem 5.3. We can handle the error terms in the proof of theorem 5.3 by selecting $L$ and $\ell$ with distinct scales.

We record the following simple corollary of theorem 1.2:

**Corollary 3.4.** There exist two constants $C > 0$ and $\kappa_0 > 0$ such that the following estimate
\[
\int_{Q_{\ell,L}} |\psi|^4 \, dx \leq C\ell^2L\left(1 - \frac{H}{\kappa}\right)^2,
\]
is valid as long as assumption 1.1 is satisfied and
- $\kappa \geq \kappa_0$;
- $\kappa^{-1/2} \leq \ell, L < 1$;
- $Q_{\ell,L} \subset \{\text{dist}(x, \partial \Omega) \geq 2\kappa^{-1/2}\}$ is a $(\ell, L)$-box.

Furthermore, it holds,
\[
\limsup_{\kappa \to \infty} \left(1 - \frac{H}{\kappa}\right)^{-2} \frac{1}{|Q_{\ell,L}|} \int_{Q_{\ell,L}} |\psi|^4 \, dx \leq -2E_{Ab}.
\] (3.12)

**Corollary 3.5.** Under the assumptions in theorem 1.2,
\[
\limsup_{\kappa \to \infty} \left(1 - \frac{H}{\kappa}\right)^{-2} \frac{1}{|D|} \int_{D} |\psi|^4 \, dx \leq -2E_{Ab},
\] (3.13)
where $D \subset \Omega$ is any fixed open subset such that $\partial D \subset \partial \Omega$ and $|\partial D| = 0$.

The proof of corollary 3.5 follows from theorem 1.2 by covering the domain $D$ with a collection of cubes with side length $\kappa^{-1/2}$. Since $D$ is independent of $\kappa$ and does not touch the
boundary, the cubes can be selected such that they are away from $\partial \Omega$ by a distance at least $2\kappa^{-1/2}$ so that (1.9) holds. Moreover, since $|\partial D| = 0$, the number of cubes needed to cover $D$ is asymptotically equal to $|D| \kappa^{-3/2}$.

4. Energy asymptotics

In the sequel, we will work with the local energies introduced in (3.1). In particular, we prove

- Proposition 4.1, which is an upper bound on the local energy; and
- Proposition 4.3, stating a lower bound on the local energy.

Propositions 4.1 and 4.3 hold under assumption 1.1. But the remainder terms we obtain are of the correct order under the additional assumption 4.6 that we state below. Assumption 4.6 arises in a natural manner in light of the discussion given in remark 4.5. Working under this assumption yields corollary 4.8 below, which contains sharp asymptotic expansions for the local energy and the local $L^2$-norm of the order parameter.

We will use the notation introduced below.

**Notation 4.1.** For every $\ell \in (0, 1)$, we let $Q_\ell \subset \Omega$ be a cube of side length $\ell$ and $\chi_\ell \in C^\infty_c(Q_\ell)$ be a cut-off function satisfying

$$
\chi_\ell = 1 \text{ in } Q_{\ell - \frac{1}{\sqrt\kappa}}, \quad 0 \leq \chi_\ell \leq 1, \quad |\nabla \chi_\ell| \leq c\sqrt{\kappa H} \quad \text{and} \quad |\Delta \chi_\ell| \leq c^2 \kappa H \text{ in } Q_\ell,
$$

where $c > 0$ is a universal constant.

Note that the function $\chi_\ell$ varies on the length scale $1/\sqrt{\kappa H}$. We get this scale naturally as we will rescale the Ginzburg–Landau equations through a transformation of the form $x \mapsto x_0 + \sqrt{\kappa H}x$ for some fixed point $x_0$. Such a rescaling transforms a ‘small’ cube of center $x_0$ and side length $\ell$ to a ‘large’ cube of center 0 and radius $R = \ell \sqrt{\kappa H} \gg 1$. In particular, the cube $Q_{\ell - \frac{1}{\sqrt\kappa}}$ is transformed to the large square with center 0 and radius $R = 1$. Writing (4.1) in the new scale, we get a function that varies on the length scale 1.

**Proposition 4.1.** There exist two constants $\kappa_0 > 1$ and $C > 0$ such that the following inequalities holds

$$
\frac{(1 - \delta)}{|Q_\ell|} E_0(\chi_\ell e^{icH_0}, F; Q_\ell)
\leq \frac{1}{|Q_\ell|} E_0(\chi_\ell \psi, A; Q_\ell) + C(\delta \kappa + \delta^{-1/2} \ell^2 [\kappa - H]^{2/3})[\kappa - H]
\leq \frac{1}{|Q_\ell|} E_0(\psi, A; Q_\ell) + C(\ell^{-1/2} \kappa^{1/2} + \delta \kappa + \delta^{-1/3} \ell^2 [\kappa - H]^{2/3})[\kappa - H],
$$

where

- $\delta \in (0, 1)$, $\kappa \geq \kappa_0$ and $(\kappa, H)$ satisfy assumption 1.1;
- $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times \overline{H}^1_{div}(\mathbb{R}^3)$ is a solution of (1.7);
- $\kappa^{-1/2} \leq \ell < 1$, $Q_\ell$ and $\chi_\ell$ are as in notation 4.1;
- $F$ is the magnetic potential introduced in (1.5);
- $\phi \in C^\infty(Q_\ell)$ is the smooth function in lemma 3.2.
Proof. Step 1: Lower bound on $E_d(\psi, A; Q_t)$. The aim of this step is to prove the estimate in (4.3) below. Since $\chi_t = 1$ in $Q_t - \frac{1}{\sqrt{\delta t}}$, it holds the simple decomposition

$$E_d(\chi_t \psi, A; Q_t) = E_d(\psi, A; Q_t - \frac{1}{\sqrt{\delta t}}) + E_d(\chi_t \psi, A; Q_t \setminus Q_t - \frac{1}{\sqrt{\delta t}}).$$ (4.2)

Straightforward calculations yield

$$\int_{Q \setminus Q_t - \frac{1}{\sqrt{\delta t}}} |(\nabla - i\kappa H A)\chi_t \psi|^2 \, dx$$

$$= \int_{Q \setminus Q_t - \frac{1}{\sqrt{\delta t}}} |\chi_t(\nabla - i\kappa H A)\psi|^2 \, dx + \int_{Q \setminus Q_t - \frac{1}{\sqrt{\delta t}}} |\nabla \chi_t|^2 |\psi|^2 \, dx$$

$$+ 2 \text{Re} \left\{ \int_{Q \setminus Q_t - \frac{1}{\sqrt{\delta t}}} \chi_t \left( \nabla \chi_t \cdot (\nabla - i\kappa H A) \psi \right) \, dx \right\}$$

$$= \int_{Q \setminus Q_t - \frac{1}{\sqrt{\delta t}}} |\chi_t(\nabla - i\kappa H A)\psi|^2 \, dx - \int_{Q \setminus Q_t - \frac{1}{\sqrt{\delta t}}} |\psi|^2 \chi_t \Delta \chi_t \, dx.$$ (4.3)

Note that the term $\chi_t \left( \nabla \chi_t \cdot (-i\kappa H A) \psi \right)$ is pure imaginary since $\chi_t$ is real-valued. We insert the estimates in corollary 3.4 into the aforementioned formula to obtain

$$\int_{Q \setminus Q_t - \frac{1}{\sqrt{\delta t}}} |(\nabla - i\kappa H A)\chi_t \psi|^2 \, dx \leq \int_{Q \setminus Q_t - \frac{1}{\sqrt{\delta t}}} |(\nabla - i\kappa H A)\psi|^2 \, dx + C \epsilon^{-1/2} \kappa^{1/2} |\kappa - H| \epsilon^3.$$

We insert this into (4.2). After a rearrangement of the terms we get

$$E_d(\psi, A; Q_t) \leq E_d(\psi, A; Q_t) + \kappa^2 \int_{Q_t} (1 - \chi_t^2) |\psi|^2 \, dx + C \epsilon^{-1/2} \kappa^{1/2} |\kappa - H| \epsilon^3.$$

We estimate the term $\int_{Q_t} (1 - \chi_t^2) |\psi|^2 \, dx$ using the assumption on the support of $1 - \chi_t$, the Cauchy–Schwarz inequality and the estimate in corollary 3.4. That way we get

$$E_d(\psi, A; Q_t) \leq E_d(\psi, A; Q_t) + C \epsilon^{-1/2} \kappa^{1/2} |\kappa - H| \epsilon^3.$$ (4.3)

Step 2: Replacing $A$ by $F$.

Let $\phi \in C_0^\infty(\Omega_t)$ be the function satisfying the estimate in (3.5). Using the gauge invariance and the Cauchy–Schwarz inequality, we get

$$E_d(\chi_t \psi, A; Q_t) = E_d(\chi_t \psi e^{iH\phi}, A - \nabla \phi; Q_t)$$

$$\geq (1 - \delta) E_d(\chi_t \psi e^{iH\phi}, F; Q_t) - (C\delta^{-1} \kappa^2 H^2 \|A - F - \nabla \phi\|_{L^2(Q_t)}^2 + \delta \kappa^2) \int_{Q_t} |\psi|^2 \, dx.$$ (4.3)

Using the estimates in corollary 3.4 and (3.5) we get,

$$E_d(\chi_t \psi, A; Q_t) \geq (1 - \delta) E_d(\chi_t \psi e^{iH\phi}, F; Q_t) - C \left( \delta^{-1} \kappa^2 \left(1 - \frac{H}{\kappa}\right)^{3/2} \epsilon^3 + \delta \kappa |\kappa - H| \epsilon^3 \right).$$

Inserting this into (4.3), we finish the proof of proposition 4.1. □
Remark 4.2. In the setting of proposition 4.1, let \( R = \ell \sqrt{\kappa H} \). The change of variables \( \kappa \mapsto \ell R H \), lemma 2.4 and (2.6) yield

\[
\frac{1}{|Q|} \mathcal{E}_0(\psi, A; Q) \geq \kappa^2 g \left( \frac{H}{\kappa} \right)
\]

Furthermore, under assumption 1.1, we know that \( H/\kappa \rightarrow 1_- \), and by theorem 2.3,

\[
\kappa^2 g \left( \frac{H}{\kappa} \right) = E_{\text{ab}}[\kappa] - H^2 + [\kappa - H]^2 o(1).
\]

Proposition 4.3. There exist positive constants \( C > 0 \) and \( \kappa_0 > 1 \) such that the following inequality holds

\[
\frac{\mathcal{E}_0(\psi, A; Q)}{|Q|} \leq (1 + \delta) \left( 1 - \frac{2}{R} \right) [\kappa - H]^2, c(R) \left( \varepsilon^{-1} + \kappa - H \right) \leq 1 + \delta \kappa + \delta^{-1} \varepsilon^2 \left( [\kappa - H]^2 + \varepsilon^{-1/2} \right) [\kappa - H],
\]

where

- \( \delta \in (0, 1) \), \( \kappa \geq \kappa_0 \) and \( (\kappa, H) \) satisfy assumption 1.1;
- \( (\psi, A) \in H^1(\Omega; \mathbb{C}) \times \mathcal{H} \) is a minimizer of the functional in (1.3);
- \( \kappa^{-1/2} \leq \ell < 1 \); \( Q \subset \{ \text{dist}(x, \partial \Omega) \geq 2\kappa^{-1/2} \} \) is a cube of side length \( \ell \);
- \( R = \ell \sqrt{\kappa H} \) and \( c(R) \) is the energy introduced in (2.12).

Proof. Let \( x_0 \) be the center of \( Q \). Without loss of generality, we may assume that \( x_0 = 0 \) so that we reduce to the case

\[
Q = (-\ell/2, \ell/2) \times (-\ell/2, \ell/2) \times (-\ell/2, \ell/2) \subset \{ \text{dist}(x, \partial \Omega) > \kappa^{-1+\delta} \}.
\]

In light of lemma 3.2, we may assume, after performing a gauge transformation, that the magnetic potential satisfies,

\[
|A(x) - F(x)| \leq C\kappa^{-3} \left( 1 - \frac{H}{\kappa} \right)^{1/3} \ell,
\]

where \( F \) is the magnetic potential introduced in (1.5).

Let \( b = H/\kappa \), \( R = \ell \sqrt{\kappa H} \) and \( v_R \in \mathcal{H}_0(Q_R) \) be a minimizer of the functional in (2.15), i.e. \( F_{b, Q_R}(v_R) = M_0(b, R) \).

Let \( \chi_R \in C^\infty_c(\mathbb{R}^3) \) be a cut-off function such that

\[
0 \leq \chi_R \leq 1, \ |\nabla \chi_R| \leq C \quad \text{in} \quad \text{supp} \ \chi_R \subset Q_{R+1}, \ \chi_R = 1 \quad \text{in} \ Q_R,
\]

for some universal constant \( C \). Let \( \eta_R(x) = 1 - \chi_R(x \sqrt{\kappa H}) \) for all \( x \in \mathbb{R}^3 \). We introduce the function (see [23])

\[
\varphi(x) = 1 - \chi_R(x \sqrt{\kappa H}) + \eta_R(x) \psi(x), \quad (x \in \Omega).
\]


Note that the function $\varphi$ satisfies
\[
\varphi(x) = \begin{cases} 
\eta_{b}(x) & \text{if } x \in Q, \\
\eta_{g}(x)\psi(x) & \text{if } x \in Q_{\ell} \setminus \frac{1}{\sqrt{\kappa H}} \setminus Q, \\
\psi(x) & \text{if } x \in \Omega \setminus Q_{\ell} \setminus \frac{1}{\sqrt{\kappa H}}. 
\end{cases}
\] (4.7)

That is, we defined $\varphi$ to be the actual minimizer $\psi$ away from the cube $Q$. In the cube $Q_{\ell}$, $\varphi$ is the approximate minimizer $\eta_{b}(x)\sqrt{\kappa H}$. We will need the following lemma:

**Lemma 4.4.** There exist constants $C, \kappa_0 > 0$ such that, for all $\delta \in (0, 1)$ and $\kappa \geq \kappa_0$,
\[
\mathcal{E}(\varphi, A; \Omega) \leq \mathcal{E}(\psi, A; \Omega \setminus Q_{\ell}) + \left(1 + \delta \right) \frac{1}{b\sqrt{\kappa H}} M_{0}(b, R) + r(\kappa) 
\] (4.8)
where the functional $\mathcal{E}$ is defined in (3.1), $M_{0}(b, R)$ is defined in (2.15) and $r(\kappa)$ is
\[
r(\kappa) = C(\delta \kappa + \delta^{-1} \kappa^{1/3} \kappa - H)^{2/3} + \kappa^{-1/2} \kappa^{1/2} |\kappa - H| \kappa^{3}. 
\] (4.9)

Now we proceed in the proof of proposition 4.3. By the definition of the minimizer $(\psi, A)$, we have
\[
\mathcal{E}(\varphi, A; \Omega) \leq \mathcal{E}(\psi, A; \Omega). 
\]
Since $\mathcal{E}(\psi, A; \Omega) = \mathcal{E}(\psi, A; \Omega \setminus Q_{\ell}) + \mathcal{E}_{0}(\psi, A; Q_{\ell})$, then (4.8) yields,
\[
\mathcal{E}_{0}(\psi, A; Q_{\ell}) \leq (1 + \delta) \frac{1}{b\sqrt{\kappa H}} M_{0}(b, R) + r(\kappa), 
\]
where $r(\kappa)$ is given in (4.9). Dividing both sides by $|Q_{\ell}|$ and using lemma 2.4 and (2.13), we finish the proof of proposition 4.3. □

**Proof of lemma 4.4.** Recall the Ginzburg–Landau energy $\mathcal{E}_{0}$ defined in (3.1). We may write
\[
\mathcal{E}(\varphi, A; \Omega) = \mathcal{E}_{1} + \mathcal{E}_{2} 
\] (4.10)
where
\[
\mathcal{E}_{1} = \mathcal{E}(\varphi, A; \Omega \setminus Q_{\ell}), \quad \mathcal{E}_{2} = \mathcal{E}_{0}(\varphi, A; Q_{\ell}) 
\] (4.11)
Let us start by estimating $\mathcal{E}_{1}$ from above. We write
\[
\mathcal{E}_{1} = \mathcal{E}(\psi, A; \Omega \setminus Q_{\ell}) + \mathcal{R}(\psi, A), 
\] (4.12)
where
\[
\mathcal{R}(\psi, A) = \mathcal{E}(\eta_{b}(x)\psi, A; Q_{\ell} \setminus \frac{1}{\sqrt{\kappa H}} \setminus Q) - \mathcal{E}_{0}(\psi, A; Q_{\ell} \setminus \frac{1}{\sqrt{\kappa H}} \setminus Q). 
\]
An integration by parts yields
\[
R(\psi, A) = \frac{\kappa^2}{2} \int_{Q_1} \left( \eta_R^2(x) - 2\eta_R^2(x)H \right) |\psi|^2 \, dx + \kappa^2 \int_{Q_1} |\nabla \eta_R|^2 |\psi|^2 \, dx.
\]
Using that \(0 \leq \eta_R \leq 1\) together with the estimate \(|\nabla \eta_R| \leq C \sqrt{\kappa H}\) and corollary 3.4, we get
\[
R(\psi, A) \leq C \ell^{-1/2} \kappa^{1/2}[\kappa - H] \ell^3.
\]
Now, we estimate the energy \(E_2\) in (4.11). Using the Cauchy-Shwarz inequality and (4.4), we write for all \(\delta \in (0, 1)\),
\[
E_2 \leq (1 + \delta) \int_{Q_1} \left( |(\nabla - i \kappa H F) \varphi|^2 - \kappa^2 |\varphi|^2 + \frac{\kappa^2}{2} |\varphi|^4 \right) \, dx
\]
\[
+ C \left( \delta \kappa^2 + \delta^{-1} \kappa^2 \left( 1 - \frac{H}{\kappa} \right)^{2/3} \ell^2 \right) \int_{Q_1} |\varphi|^2 \, dx.
\]
Now we use that \(\varphi = \psi_{\kappa}(x)\) in \(Q_1\), the estimate in lemma 2.6 and (2.5) to write,
\[
E_2 \leq (1 + \delta) \int_{Q_1} \left( |(\nabla - i \kappa H F) \varphi|^2 - \kappa^2 |\varphi|^2 + \frac{\kappa^2}{2} |\varphi|^4 \right) \, dx
\]
\[
+ C \left( \delta \kappa^2 [\kappa - H] \ell^3 + \delta^{-1} \kappa^2 \left( 1 - \frac{H}{\kappa} \right)^{2/3} \ell^5 \right).
\]
Since \(\varphi(x) = \psi_{\kappa}(x)\) in \(Q_1\), \(b = H/\kappa\) and \(R = \ell \sqrt{\kappa H}\), a change of variables yields
\[
\int_{Q_1} \left( |(\nabla - i \kappa H F) \varphi|^2 - \kappa^2 |\varphi|^2 + \frac{\kappa^2}{2} |\varphi|^4 \right) \, dx = \frac{1}{b \sqrt{\kappa H}} M_0(b, R).
\]
Inserting this into (4.14) then collecting (4.13) and (4.10), we finish the proof of (4.8).

**Remark 4.5 (On the remainder terms in propositions 4.1 and 4.3).** Let \(\mu = \kappa^{1/2}(1 - \frac{H}{\kappa})\). Assumption 1.1 can be reformulated as \(1 \ll \mu \ll \kappa^{1/2}\). For the inequalities announced in propositions 4.1 and 4.3 to be useful, the remainder terms have to be negligible compared to the principal terms. We will see that this will force us to assume an additional condition on \(\mu\), i.e. on the relation between the magnetic field \(H\) and the G–L parameter \(\kappa\). However, the extra assumption that we will derive is dictated by our proof of proposition 4.3, hence it is technical and does not have a physical motivation.
We need to choose the parameters $\delta$ and $\ell$ such that the following terms
\[ \delta\kappa, \quad \ell^{-1/2}\kappa^{1/2}, \quad \ell^{-1}, \quad \kappa^{-1}\ell^{-3}[\kappa - H]^{-1}, \quad \delta^{-1}\kappa^{1/3}\ell^2[\kappa - H]^{2/3} \]
are of the order $o([\kappa - H])$. Note that:
(i) propositions 4.1 and 4.3 are valid when $\kappa^{-1} \ll \ell \ll 1$ and $\delta \ll 1$;
(ii) $[\kappa - H] = \kappa^{1/2}\mu$;
(iii) to obtain that $\ell^{-1/2}\kappa^{1/2} = o([\kappa - H])$, we need to assume that $\mu^{-2} \ll \ell \ll 1$;
(iv) since $1 \ll \mu \ll \kappa^{1/2}$, the condition assumed in (iii) yields $\ell^{-1} = o([\kappa - H])$ and $\kappa^{-1}\ell^{-3}[\kappa - H]^{-1} = o([\kappa - H])$;
(v) to obtain that $\delta = o([\kappa - H])$, we need to assume that $\delta \ll \kappa^{-1/2}\mu$. If we write $\delta = B\kappa^{-1/2}$, then we get the condition $B \ll \mu$;
(vi) now, if $\delta^{-1}\kappa^{1/3}\ell^2[\kappa - H]^{2/3} = o([\kappa - H])$, then the condition in (v) yields $\ell \ll B^{1/2}\mu^{1/6}\kappa^{-1/3}$;
(vii) the conditions in (iii) and (vi) are compatible when $\mu^{-2} \ll B^{1/2}\mu^{1/6}\kappa^{-1/3}$, i.e. when $\mu \gg B^{-3/12}\kappa^{2/13}$;
(viii) using the condition on $B$ in (v), we get from (vii) the condition $\mu \gg \mu^{-3/13}\kappa^{2/13}$, which is equivalent to $\mu \gg \kappa^{1/8}$.

Recall that $\mu = \kappa^{1/2}(1 - \frac{H}{\kappa})$. Consequently, (viii) yields that $1 - \frac{H}{\kappa} \gg \kappa^{-3/8}$, thereby motivating the Technical assumption 4.6.

**Assumption 4.6.**

- $a : \mathbb{R}_+ \to \mathbb{R}_+$ and $b : \mathbb{R}_+ \to \mathbb{R}_+$ are two functions satisfying
  \[ \lim_{\kappa \to \infty} a(\kappa) = \infty, \quad \lim_{\kappa \to \infty} b(\kappa) = 0 \quad \text{and} \quad a(\kappa)\kappa^{-3/8} \leq b(\kappa) \text{ in a neighborhood of } \infty. \]
- $\kappa > 0$ and $H > 0$ satisfy $a(\kappa)\kappa^{-3/8} \leq 1 - \frac{H}{\kappa} \leq b(\kappa)$.

**Remark 4.7 (Choice of the parameters).** Suppose that assumption 4.6 holds. We choose the parameters $\delta$ and $\ell$ as follows
\[
\delta = \kappa^{-1/2}\left(1 - \frac{H}{\kappa}\right)^{-1/3} \quad \text{and} \quad \ell = \frac{2\pi}{\sqrt{\kappa H}} \left[\kappa^{1/12}[\kappa - H]^{-4/3}\right] \approx \kappa^{-3/4}\left(1 - \frac{H}{\kappa}\right)^{-4/3}.
\]
Here $\lfloor \cdot \rfloor$ denotes the floor function. It is easy to check that $\delta \ll 1$, $\kappa^{-1} \ll \ell \ll 1$ and the following terms
\[ \delta\kappa, \quad \ell^{-1/2}\kappa^{1/2}, \quad \ell^{-1}, \quad \kappa^{-1}\ell^{-3}[\kappa - H]^{-1}, \quad \delta^{-1}\kappa^{1/3}\ell^2[\kappa - H]^{2/3} \]
are of the order $o([\kappa - H])$.

Collecting propositions 4.1 and 4.3, we get:

**Corollary 4.8.** There exist $\kappa_0 > 0$ and a function $\text{err} : [\kappa_0, \infty) \to (0, \infty)$ such that:

- $\lim_{\kappa \to \infty} \text{err}(\kappa) = 0$;
- the following two inequalities hold
\[ \left| \frac{1}{|Q|} \mathcal{E}_d(\chi; \omega; F; Q) - [\kappa - H]^2 E_{Ab} \right| \leq [\kappa - H]^2 \text{err}(\kappa), \quad (4.15) \]

\[ \left| \frac{1}{|Q|} \int_{Q_l} |\psi|^4 \, dx + 2E_{Ab} \left( 1 - \frac{H}{\kappa} \right)^2 \right| \leq \left( 1 - \frac{H}{\kappa} \right)^2 \text{err}(\kappa), \quad (4.16) \]

where

- \( E_{Ab} \) is the Abrikosov constant introduced in theorem 2.3;
- \( F \) is the magnetic potential in (1.5);
- \( \kappa \gg \kappa_0 \) and \((\kappa, H)\) satisfy assumption 4.6;
- \((\psi, A)\) is a minimizer of (1.3);
- \( \ell = \sqrt{\frac{2\pi}{\kappa H}} |\kappa^{9/12} [\kappa - H]^{-4/3}| \);\n- \( Q_l \subset \{ \text{dist}(x, \partial Q) > 2\kappa^{-1/2} \} \) and \( \chi \) are as in notation 4.1;
- \( \phi \in C^\infty(\overline{Q}) \) is the function defined by lemma 3.2.

**Proof.** Under assumption 4.6, we know that \( \kappa^{-3/8} \ll 1 - \frac{H}{\kappa} \ll 1 \). We choose \( \delta \) as in remark 4.7. Our choice of \( \ell \) is the same as in remark 4.7. Thus, we get that all the remainder terms in proposition 4.1 and 4.3 are of order \( o([\kappa - H]^2) \).

Now, collecting the estimates in proposition 4.1, 4.3 and remark 4.2, we get

\[
(1 - \delta)\kappa^2 \left( \frac{H}{\kappa} \right) \leq \frac{(1 - \delta)}{|Q|} \mathcal{E}_d(\chi; \omega; H; F; Q_l) \leq \mathcal{E}_d(\chi; \omega; A; Q_l) + o([\kappa - H]^2) \leq \frac{c(R)}{R^2} |\kappa - H|^2 + o([\kappa - H]^2),
\]

where \( R = \ell \sqrt{\kappa H} \). Our choice of \( \ell \) ensures that \( R \gg 1 \) and \((2\pi)^{-1}R^2 \in \mathbb{N} \). By applying (2.6) and theorem 2.3, we get (4.15) and

\[
\mathcal{E}_d(\chi; \omega; A; Q_l) \leq \ell^3 |\kappa - H|^2 E_{Ab} + \ell^3 o([\kappa - H]^2). \quad (4.17)
\]

The proof of (4.16) follows from the following localization formula,

\[
\mathcal{E}_d(\chi; \omega; A; Q_l) = \kappa^2 \int_{Q_l} \chi^2 \left( -1 + \frac{1}{2} \chi^2 \right) |\psi|^4 \, dx + \int_{Q_l} |\nabla \chi|^2 |\psi|^2 \, dx.
\]

By inserting (4.17) into the aforementioned formula and by using that \( \chi \ll 1 \) in \( Q_{l - \frac{1}{\ell \sqrt{\kappa H}}} \), we get

\[
\frac{-\kappa^2}{2} \int_{Q_{l - \frac{1}{\ell \sqrt{\kappa H}}}} |\psi|^4 \, dx \leq [\kappa - H]^2 E_{Ab} \ell^3 + \kappa^2 \int_{Q_{l - \frac{1}{\ell \sqrt{\kappa H}}}} |\psi|^4 \, dx - \int_{Q_l} |\nabla \chi|^2 |\psi|^2 \, dx + \ell^3 o([\kappa - H]^2).\]

The estimate in corollary 3.4 yields that

\[
\kappa^2 \int_{Q_{l - \frac{1}{\ell \sqrt{\kappa H}}}} |\psi|^4 \, dx + \int_{Q_l} |\nabla \chi|^2 |\psi|^2 \, dx \leq C \ell^{-1/2} |\kappa - H| \ell^3 = \ell^3 o([\kappa - H]^2).\]

This and theorem 1.2 (also see corollary 3.4) finish the proof of (4.16).
5. Sharp estimate of the \(L^2\)-norm

This section contains three main results:

- Lemma 5.1 regarding the spectral theory of the Landau Hamiltonian with (magnetic) periodic conditions with respect to a box lattice of \(\mathbb{R}^3\);
- Lemma 5.2 and theorem 5.3 regarding the behavior of the minimizers of the functional in (1.3) in cubes with small lengths.

The proof of theorem 1.3 is a simple consequence of the result summarized in theorem 5.3. The proof of theorem 5.3 relies on lemma 5.2 and on the decomposition of a given cube into good and bad boxes. The proof of lemma 5.2 needs the result in lemma 5.1 as a key ingredient.

5.1. The 3D periodic operator

Let \(R > 0\) such that \(R^2 \in 2\pi\mathbb{N}\), \(L > 0\) and \(F\) be the magnetic potential in (1.5). We denote by \(P_{R,L}^{3D}\) the operator

\[
P_{R,L}^{3D} = -(\nabla - iF)^2 \quad \text{in} \quad L^2(\Omega_{R,L}), \quad \Omega_{R,L} = (-R/2, R/2)^2 \times (-L/2, L/2),
\]

with form domain the space \(E_{R,L}^{3D}\) defined as follows

\[
E_{R,L}^{3D} = \{u \in H^1_{0\text{loc}}(\mathbb{R}^3; \mathbb{C}) : u(x_1 + R, x_2, x_3) = e^{-iR\phi_0} u(x_1, x_2, x_3), \quad u(x_1, x_2 + R, x_3) = e^{iR\phi_0} u(x_1, x_2, x_3), \quad u(x_1, x_2, x_3 + L) = u(x_1, x_2, x_3), \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3\}. \quad (5.1)
\]

When \(L = R\), we will omit the reference to \(L\) in the notation and simply write \(P_{R}^{3D}, E_{R}^{3D}\) and \(Q_{R}\).

The operator \(P_{R,L}^{3D}\) has compact resolvent. Its sequence of increasing distinct eigenvalues is denoted by \(\{\mu_j(P_{R,L}^{3D})\}\).

The Fourier transform with respect to the \(x_3\)-variable allows us to separate variables and express the operator \(P_{R,L}^{3D}\) as the direct sum

\[
\bigoplus_{n \in \mathbb{Z}} (P^{2D}_R + (2\pi n L^{-1})^2) \quad \text{in} \quad \bigoplus_{n \in \mathbb{Z}} L^2((-R/2, R/2)^2), \quad (5.2)
\]

where \(P^{2D}_R\) is the operator introduced in (2.8). Consequently, we get

\[
\mu_1(P_{R,L}^{3D}) = 1 \quad \text{and} \quad \mu_2(P_{R,L}^{3D}) = 1 + 4\pi^2 L^{-2}. \quad (5.3)
\]

Let \(\Pi_l\) be the orthogonal projection on \(L_l \subset L^2((-R/2, R/2)^2), the first eigenspace of the operator \(P_{R}^{3D}\) in (2.8). By proposition 2.1, we know that, under the assumption that \(R^2 \in 2\pi\mathbb{N}\), the space \(L_l\) is finite dimensional and the dimension is equal to \(N := R^2/2\pi\). Thus, we may express the orthogonal projection \(\Pi_l\) as follows,

\[
\forall g \in L^2((-R/2, R/2)^2), \quad \Pi_l u = \sum_{m=1}^{N} \langle g, f_m \rangle_{L^2((-R/2, R/2)^2)} f_m,
\]

where \((f_m)\) is an orthonormal basis of the space \(L_l\). That way, we may view \(\Pi_l\) as a projection in the space \(L^2(\Omega_{R,L})\) via the formula
\[ \forall \ u \in L^2(Q_{R,L}), \]
\[ (\Pi u(x_1, x_2, x_3) = \sum_{m=1}^{N} f_m(x_1, x_2) \int_{K_R} u(x_1, x_2, x_3) f_m(x_1, x_2) \, dx_1 \, dx_2, \tag{5.4} \]

where
\[ K_R = (-R/2, R/2) \times (-R/2, R/2). \tag{5.5} \]

We introduce the quadratic form of the operator \( P^D_R \):
\[ Q^D_{R,L}(u) = \int_{Q_{R,L}} |(\nabla - iF)u|^2 \, dx. \tag{5.6} \]

Note that by definition of \( F \) and \( A_0 \) in (1.5) and (2.2) respectively, we observe the following useful inequality,
\[ Q^D_R(u) = \int_{Q_{R,L}} ((\nabla - iA_0)u)^2 + |\partial_x u|^2 - |\partial_y u|^2 \, dx \geq \int_{Q_{R,L}} |(\nabla - iA_0)u|^2 \, dx, \tag{5.7} \]

where \( \nabla_{(n, e)} = (\partial_x, \partial_y) \).

Now, we can prove the 3D analogue of lemma 2.2:

**Lemma 5.1.** Let \( 2 \leq p \leq 6 \). There exists a constant \( C_p > 0 \) such that for any \( \gamma \in (0, 1/2) \), \( R, L > 1 \) and \( u \in E^D_R \), satisfying
\[ Q^D_{R,L}(u) - (1 + \gamma) \|u\|_{L^2(Q_{R,L})}^2 \leq 0 \tag{5.8} \]

then the following estimate holds:
\[ \|u - \Pi u\|_{L^2(Q_{R,L})} \leq C_p \sqrt{\gamma} \|u\|_{L^2(Q_{R,L})}. \]

**Proof.** Let \( \Pi u = u - \Pi_1 u \). It is easy to check that \( \Pi_1 u \) and \( \Pi_2 u \) are orthogonal in \( L^2(Q_{R,L}) \) and that
\[ Q^D_{R,L}(u) - \|u\|_{L^2(Q_{R,L})}^2 = \sum_{i=1}^{3} (Q^D_{R,L}(\Pi_i u) - \|\Pi_i u\|_{L^2(Q_{R,L})}^2). \]

Using (5.7) and (2.9), we get
\[ Q^D_{R,L}(u) - \|u\|_{L^2(Q_{R,L})}^2 \geq \frac{1}{2} Q^D_{R,L}(\Pi u) + \left( \frac{3}{2} - 1 \right) \|\Pi_2 u\|_{L^2(Q_{R,L})}^2. \]

Using the diamagnetic inequality, we get further
\[ Q^D_{R,L}(u) - \|u\|_{L^2(Q_{R,L})}^2 \geq \frac{1}{2} \|\nabla |\Pi_2 u|\|_{L^2(Q_{R,L})}^2 + \frac{1}{2} \|\Pi_2 u\|_{L^2(Q_{R,L})}^2. \]

We insert this into (5.8) to get,
\[ \|\nabla |\Pi_2 u|\|_{L^2(Q_{R,L})}^2 + \|\Pi_2 u\|_{L^2(Q_{R,L})}^2 \leq 2 \gamma \|u\|_{L^2(Q_{R,L})}^2. \]

This finishes the proof of lemma 5.1 once the following Sobolev inequality is established
\[ \forall \ R \geq 1, \forall \ p \in [2, 6], \forall \ f \in E^D_R, \|f\|_{L^p(Q_{R,L})} \leq C_p \|f\|_{L^2(Q_{R,L})}, \tag{5.9} \]
where $C_p$ is a constant independent from $R, L \geq 1$. To prove (5.9), let $f \in E_{3D}^{\text{RL}} \chi \in C^\infty_c(B_\mathbb{R}(0, 6))$ and $\eta \in C^\infty_c(B_\mathbb{R}(0, 6))$ such that

- $\chi = 1$ in $B_\mathbb{R}(0, 3)$ and $\eta = 1$ in $B_\mathbb{R}(0, 3)$;
- $0 \leq \chi \leq 1$ in $B_\mathbb{R}(0, 6)$ and $0 \leq \eta \leq 1$ in $B_\mathbb{R}(0, 3)$.

We used the notation $B_\mathbb{R}(0, r)$ for the ball of center 0 and radius $r$ in the Euclidean space $\mathbb{R}^d$.

Note that, since $f \in E_{3D}^{\text{RL}}$, then $f(x)$ can be defined everywhere by (magnetic) periodicity. Let us define

$$g(x) = \chi \left( \frac{x_3}{R} \right) \left( \frac{x_3}{L} \right) |f(x)|, \quad (x = (x_3, x_3) \in \mathbb{R}^3).$$

Clearly, $g$ belongs to the homogeneous Sobolev space and the following Sobolev inequality holds

$$\|g\|_{L^p(\mathbb{R}^3)} \leq C \|\nabla g\|_{L^p(\mathbb{R}^d)}.$$

This yields (5.9) for $p = 6$. Moreover, for $p = 2$, (5.9) is obviously true. It remains to prove the inequality for $2 < p < 6$. But this follows from the following interpolation inequality

$$\|f\|_p \leq \|f\|_{2}^{1-\theta} \|f\|_{6}^{\theta}, \quad \text{where} \quad 0 < \theta < 1, \quad \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{6},$$

which is a simple consequence of Hölder’s inequality.

5.2. Asymptotics of the average of the density

Here we return back to the analysis of the minimizers of the functional in (1.3).

**Lemma 5.2.** There exist $\kappa_0 > 1$, $C > 0$ and a function $\text{err} : [\kappa_0, \infty) \to (0, \infty)$ such that it holds the following

$$\|v - \Pi\|_{L^2((-R/2, R/2)^d)} \leq C \sqrt{1 - \frac{H}{\kappa}},$$

(5.10)

$$\mathcal{E}_d(e^{inH_0} \chi, \psi, Q_1) \geq \frac{1}{\sqrt{\kappa H}} \int_{(-R/2, R/2)^d} \left( 1 - \frac{\kappa}{H} \right) \|\Pi\|_2^2 + \frac{\kappa}{2H} |v|^4 \right) dx,$$

(5.11)

$$\frac{1}{R^2} \int_{(-R/2, R/2)^d} |v|^4 \ dx = -2E_{\text{ab}} \left( 1 - \frac{H}{\kappa} \right)^2 + \left( 1 - \frac{H}{\kappa} \right) \text{err}(\kappa),$$

(5.12)

and

$$\frac{1}{R^2} \int_{(-R/2, R/2)^d} |v|^2 \ dx \geq -2E_{\text{ab}} \left( 1 - \frac{H}{\kappa} \right) \left( 1 - \frac{H}{\kappa} \right) \text{err}(\kappa),$$

(5.13)

where

- $\lim_{\kappa \to \infty} \text{err}(\kappa) = 0$;
- $F$ is the magnetic potential in (1.5);
• $\kappa \geq \kappa_0$ and $(\kappa, H)$ satisfy assumption 4.6;
• $(\psi, \mathbf{A})$ is a minimizer of $(1.3)$;
• $\ell = \sqrt{\frac{2\pi}{\kappa}} \left[ \kappa^{3/12} (\kappa - H)^{-4/3} \right]$;
• $R = \ell \sqrt{\kappa H}$;
• the cube $Q_i \subseteq \{ \text{dist}(x, \partial \Omega) > 2\kappa^{-1/2} \}$ and the function $\chi_\ell$ are as in notation 4.1;
• $\phi \in C^\infty(Q_i)$ is the function defined by lemma 3.2;
• $\Pi_1$ is the projection introduced in (5.4);
• $x_j$ is the center of the cube $Q_i$ and

$$v(x) = (e^{i\kappa \phi} \chi_\ell \psi) \left( x_j + \frac{x}{\sqrt{\kappa H}} \right). \quad (x \in (-R/2, R/2)^d).$$

**Proof.**

**Step 1. Proof of (5.10).**

By a gauge transformation and a translation, we may assume that the center of $Q_i$ is $x_j = 0$. We infer from (4.15) that, for $\kappa$ sufficiently large,

$$\int_{Q_i} (|(\nabla - i\kappa H \mathbf{F}) \chi_\ell \psi e^{i\kappa \phi}|^2 - \kappa^2 |\chi_\ell \psi e^{i\kappa \phi}|^2) \, dx < 0.$$

Performing the change of variables $x \mapsto \sqrt{\kappa H} x$, we get

$$\int_{(-R/2, R/2)^d} (|(\nabla - i\mathbf{F}) v|^2 - (1 + \gamma) |v|^2) \, dx < 0,$$

where $\gamma = \frac{\kappa}{H} - 1 \approx 1 - \frac{H}{\kappa}$. Now the estimate in (5.10) follows simply by applying lemma 5.1.

**Step 2. Proof of (5.11).**

Using a change of variable, the min-max principle and (5.3), we get,

$$\mathcal{E}_0(e^{i\kappa \phi} \chi_\ell \psi; F; Q_i) = \frac{1}{\sqrt{\kappa H}} \int_{(-R/2, R/2)^d} \left( |(\nabla - i\mathbf{F}) v|^2 - \frac{\kappa}{H} |v|^2 + \frac{\kappa}{2H} |v|^4 \right) \, dx \geq \frac{1}{\sqrt{\kappa H}} \int_{(-R/2, R/2)^d} \left( \left( 1 - \frac{\kappa}{H} \right) |\Pi v|^2 + \frac{\kappa}{2H} |v|^4 \right) \, dx.$$  \hspace{1cm} (5.14)

**Step 3. Proof of (5.12).** We perform the change of variable $x \mapsto x/\sqrt{\kappa H}$ to get

$$\frac{1}{R^3} \int_{(-R/2, R/2)^d} |v|^4 \, dx = \frac{1}{\ell^3} \int_{Q_i} |\chi_\ell \psi|^4 \, dx.$$  

We use the estimate in corollary 3.4 coupled with Hölder’s inequality and our choice of $\ell$ to write

$$\int_{Q_i \setminus Q_{i-rac{1}{\sqrt{\kappa H}}}} (\chi_\ell^4 - 1) |\psi|^4 \, dx = \left( 1 - \frac{H}{\kappa} \right)^2 \ell o(1).$$

Now, by corollary 4.8,

$$\int_{Q_i} |\chi_\ell \psi|^4 \, dx = \int_{Q_i} |\psi|^4 \, dx + \int_{Q_i \setminus Q_{i-rac{1}{\sqrt{\kappa H}}}} (\chi_\ell^4 - 1) |\psi|^4 \, dx \tag{5.16}$$

$$= -2 \mathcal{E}_0 \left( 1 - \frac{H}{\kappa} \right)^2 \ell^3 + \left( 1 - \frac{H}{\kappa} \right)^2 \ell o(1).$$
Step 4. Proof of (5.13).

We use (5.12) and the following estimate from corollary 4.8
\[
E_0(e^{i\phi\hat{H}}\chi\psi, F; Q_\ell) = E_{\text{Ab}}[\kappa - H_1^2] + [\kappa - H_1^2] \ell^3 \phi(1)
\]
and infer from (5.14)
\[
- \int_{R(2, R)} |\Pi_1| \leq 2E_{\text{Ab}} \left(1 - \frac{H}{\kappa}\right) R^3 + R^3 \phi(1).
\]
This finishes the proof of (5.13) in light of the estimate in (5.10).

Theorem 5.3. There exist \( \kappa_0 > 1 \) and a function \( \text{err} : [\kappa_0, \infty) \to (0, \infty) \) such that:

- \( \lim_{\kappa \to \infty} \text{err}(\kappa) = 0; \)
- the following inequality hold
\[
\left| \frac{1}{|Q_\ell|} \int_{Q_\ell} |\psi|^2 \, dx + 2E_{\text{Ab}} \left(1 - \frac{H}{\kappa}\right) \right| \leq \left(1 - \frac{H}{\kappa}\right) \text{err}(\kappa), \tag{5.15}
\]

where

- \( \kappa \geq \kappa_0 \) and \((\kappa, H)\) satisfy assumption 4.6;
- \((\psi, A)\) is a minimizer of (1.3);
- \( \ell = \frac{2\pi}{\sqrt{\kappa}} \left[\kappa^{3/4}[\kappa - H]\right]^{1/3}; \)
- the cube \( Q_\ell \subset \text{dist}(x, \partial \Omega) > 2\kappa^{-1/2} \) is as in notation 4.1.

Proof. We will prove (5.15) in two steps by establishing the upper and lower bounds in (5.15) independently.

The lower bound follows easily from lemma 5.2 used with \( R = \ell \sqrt{\kappa} \) and \( \ell \) as defined in theorem 5.3. Namely we use (5.13).

The proof of the upper bound is a bit lengthy. We note for future use that assumption 4.6 yields that \( \ell \approx \kappa^{-3/4}(1 - \frac{H}{\kappa})^{-1/3}. \) We introduce the parameters
\[
\alpha = \left(1 - \frac{H}{\kappa}\right)^{1/16}, \quad \epsilon = \left(1 - \frac{H}{\kappa}\right)^{3/8}, \quad L = \left(1 - \frac{H}{\kappa}\right)^{-5/8}, \quad \ell' = (\kappa - H)^{-1} \epsilon, \quad \text{and} \quad R' = \ell' \sqrt{\kappa} H. \tag{5.16}
\]

Note that these parameters satisfy
\[
\left(1 - \frac{H}{\kappa}\right)^2 R^2 L \ll 1, \quad \kappa^{-1} \ll \ell' \ll \ell \ll 1 \quad \text{and} \quad 1 \ll R' \ll R, \tag{5.17}
\]

and
\[
((\ell')^2 + \kappa^2 L^{-2})\alpha^2 \left(1 - \frac{H}{\kappa}\right) \ll [\kappa - H]^2. \tag{5.18}
\]
$\kappa = \ell R H,$ \hspace{1cm} (5.19)

and $\ell$ is defined in theorem 5.3. We need these new parameters to cover the cube $Q_\ell$ by smaller boxes $Q_{\ell',L,L}$. See Step 1 below and figure 1. Furthermore, the conditions on the parameters $\ell', L, \alpha, \epsilon$ will allow us to handle some error terms, particularly when writing the formulas in (5.22), (5.28), (5.30) and (5.31) below.

**Step 1.**

Let $\widehat{Q}_{\ell',L,L}$ be a family of $(\ell', \frac{1}{\sqrt{H}})$-boxes covering the cube $Q_\ell$ (see definition 3.3). These boxes are constructed as follows. First we cover $Q_\ell$ by $N$ boxes of the form

$$\widehat{Q}_{\ell',L,L} = \left(\frac{-\ell'}{2}, \frac{-\ell'}{2}\right) \times \left(\frac{-L}{2\sqrt{H}}, \frac{L}{2\sqrt{H}}\right) + x_i, \quad x_i \in \mathbb{R}^3.$$ 

We choose these boxes to be disjoint (see figure 1), hence the number $N$ satisfies

$$N \leq C \frac{\ell^3 \sqrt{H}}{\ell'^2 L}. \hspace{1cm} (5.20)$$

Now we choose the boxes $Q_{\ell',L,L}$ by expanding the sides of $\widehat{Q}_{\ell',L,L}$ slightly. Precisely, we take
\[ Q_{\ell,L,i} = \left(-\left(1 + \alpha\right)\frac{\ell^2}{2}, \left(1 + \alpha\right)\frac{\ell^2}{2}\right)^2 \times \left(\left(-\left(1 + \alpha\right)\frac{L}{2\sqrt{\kappa H}}, \left(1 + \alpha\right)\frac{L}{2\sqrt{\kappa H}}\right) + x_i. \]

Consider a partition of unity \((h_i)\) satisfying in \(Q_{\ell}\)
\[ \sum_i h_i = 1, \quad \sum_i |\nabla h_i|^2 \leq C((\ell')^2 + \kappa^2 L^2)\alpha^{-2}, \]
and \(\text{supp } h_i \subset Q_{\ell,L,i}\). This partition of unity exists because the boxes \((Q_{\ell,L,i})\) overlap, see figure 1.

Let the notation be as in lemma 5.2 and denote by \(w = e^{i\omega_{\ell,L} h_i \psi}\).

(5.21)

We have the following decomposition formula, obtained by using the identity \(\sum_i h_i^2 = 1\) and an integration by parts (see [10, p 98, equation (8.10)])
\[ \int_{Q_i} |\nabla - i\kappa H F| w|^2 \, dx = \sum_i \int_{Q_i} |\nabla - i\kappa H F| h_i w|^2 \, dx - \sum_i \int_{Q_i} |\nabla h_i|^2 |\psi|^2 \, dx. \]

Consequently, plugging this identity into the local G–L energy \(E_0(w, F; Q_{\ell})\) and observing that \(\text{supp } h_i \subset Q_{\ell,L,i}\), we get the following inequalities
\[ E_0(w, F; Q_{\ell}) \geq \sum_i E_0(h_i w, F; Q_{\ell,L,i}) - \sum_i \|\nabla h_i|\psi\|^2_{L^2(Q_i)} \geq \sum_i E_0(h_i w, F; Q_{\ell,L,i}) - C((\ell')^2 + \kappa^2 L^2)\alpha^{-2}\|\psi\|^2_{L^2(Q_i)} \geq \sum_i E_0(h_i w, F; Q_{\ell,L,i}) - C((\ell')^2 + \kappa^2 L^2)\alpha^{-2}\ell^3 \left(1 - \frac{H}{\kappa}\right) \quad \text{[by corollary 3.4]} \]
\[ \geq \sum_i E_0(h_i w, F; Q_{\ell,L,i}) - \ell^3 |\kappa - H|^2 o(1) \quad \text{[by (5.18)].} \]

In light of corollary 4.8, we get
\[ \sum_i E_0(h_i w, F; Q_{\ell,L,i}) \leq E_0(\kappa - H)^2 \ell^3 + \ell^3 |\kappa - H|^2 o(1). \]

(5.22)

Step 2.

Let
\[ q(h_i w, F; Q_{\ell,L,i}) = \int_{Q_{\ell,L,i}} (|\nabla - i\kappa H F| h_i w|^2 - \kappa^2 |h_i w|^2) \, dx. \]

(5.23)

We introduce the two sets of indices
\[ J_+ = \{ i : q(h_i w, F; Q_{\ell,L,i}) > 0 \} \quad \text{and} \quad J_- = \{ i : q(h_i w, F; Q_{\ell,L,i}) \leq 0 \}. \]

A box \(Q_{\ell,L,i}\) with \(i \in J_-\) will be called a good box, while for \(i \in J_+\), it is a bad box. For a good box, we can apply lemma 5.1. We will see that most of the boxes are good.
Let \( N_+ = \text{Card}, \mathcal{J}_+ \) and \( N_- = \text{Card}, \mathcal{J}_- \). We will prove that
\[
N_- \leq \frac{\ell^3 \sqrt{\kappa H}}{\ell^2 L} o(1),
\]
and
\[
N_+ = N. o(1).
\]
(5.25)
Since \( N_+ + N_- = N \), (5.25) is a simple consequence of (5.20) and (5.24). The upper bound in (5.24) is a simple consequence of (5.20) since \( \mathcal{N}_- \leq N \).

We turn to the proof of the lower bound in (5.24). We have the trivial lower bound that follows from (2.6) and (2.3), theorem 2.3 and a change of variables
\[
\int \gamma \nabla \cdot F_{i1} \, dx \leq \kappa \frac{H}{\sqrt{\kappa H}} (1 + \alpha) R \leq \kappa \frac{H}{\sqrt{\kappa H}} (1 + \alpha)^2 R^2 L.
\]
Here \( R' = \ell' \sqrt{\kappa H} \). Using theorem 2.3, we get further
\[
\mathcal{E}_d(h_w, F; Q_{p,L,i}) \geq (E_{AB} + o(1))(\kappa - H)^2 \ell^2 L(\kappa H)^{-1/2}.
\]
Inserting this into (5.22) and dropping the positive terms corresponding to \( i \in \mathcal{J}_+ \), we get
\[
N_+ (E_{AB} + o(1))(\kappa - H)^2 \ell^2 L(\kappa H)^{-1/2} \leq \sum_{i \in \mathcal{J}_-} \mathcal{E}_d(h_w, F; Q_{p,L,i}) \leq E_{AB}(\kappa - H)^2 \ell^3 + \ell^3 (\kappa - H)^2 o(1).
\]
Since \( E_{AB} < 0 \), this yields (5.24).

**Step 3.**

We denote by \( x_i \) the center of the box \( Q_{p,L,i} \). If \( i \in \mathcal{J}_- \), the change of the variable \( x \mapsto (x - x_i) \sqrt{\kappa H} \) yields
\[
\int_{Q_{p,L}} \left( (|\nabla - iF|v_i|^2 - (1 + \gamma)|v_i|^2) \right) \, dx \leq 0,
\]
where \( \gamma = 1 - \frac{\alpha}{\ell^2} Q_{p,L} = (- (1 + \alpha) R^2, (1 + \alpha) R^2) \times (- (1 + \alpha) L/2, (1 + \alpha) L/2) \) and
\[
v_i(x) = h_w \left( x_i + \frac{x}{\sqrt{\kappa H}} \right).
\]
(5.26)
We apply lemma 5.1 to obtain
\[
\|v_i - \Pi_i v_i\|_{L^2(Q_{p,L})} \leq C \left( \frac{H}{\kappa} \|v_i\|_{L^2(Q_{p,L})} \right) ^{p/(2-p)}, \quad p \in \{2, 4\},
\]
(5.27)
where \( \Pi_i \) is the projection in (5.4). For \( p = 4 \), we write by Hölder's inequality,
\[
\|v_i - \Pi_i v_i\|_{L^2(Q_{p,L})} \leq C (R^2 L)^{1/4} \left( \frac{H}{\kappa} \|v_i\|_{L^2(Q_{p,L})} \right) ^{1/2} \|v_i\|_{L^2(Q_{p,L})},
\]
(5.28)
by (5.17). Let us introduce the function \( u_i \) as follows,
\[ v_i = \left(1 - \frac{H}{\kappa}\right)^{1/2} u_i. \]  

(5.29)

Since \( R' \gg 1 \), we get (see (2.12) and theorem 2.3)

\[
\int_{Q_{x'}} \left( -|\Pi u_i|^2 + \frac{1}{2} |\Pi u_i|^4 \right) \, dx 
\geq \int_{Q_{x'}} \left( (1 + \alpha) R' \right) \, dx \geq E_{\text{ab}} R'^2 L - R'^2 L_0(1).
\]

Thus, we get,

\[- \sum_{i \in J} \int_{Q_{x'}} |\Pi u_i|^2 \, dx \geq - \frac{1}{2} \sum_{i \in J} \int_{Q_{x'}} |\Pi u_i|^4 \, dx + (E_{\text{ab}} + o(1)) R'^2 L N_0.\]

Using (5.28), (5.24) and \( R' = \ell' \sqrt{\kappa H} \), we get further

\[- \sum_{i \in J} \int_{Q_{x'}} |\Pi u_i|^2 \, dx \geq - \frac{1}{2} (1 + o(1)) \sum_{i \in J} \int_{Q_{x'}} |u_i|^4 \, dx + E_{\text{ab}} \ell'^3 (\kappa H)^{3/2} + \ell'^3 (\kappa H)^{3/2} o(1).\]

(5.30)

In light of (5.29), (5.26) and (5.21), we get by a change of variable transformation

\[
\sum_{i \in J} \int_{Q_{x'}} |u_i|^4 \, dx = (\kappa H)^{3/2} \left(1 - \frac{H}{\kappa}\right)^{-2} \sum_{i \in J} \int_{Q_{x'}} |h \psi|^4 \, dx
\leq (\kappa H)^{3/2} \left(1 - \frac{H}{\kappa}\right)^{-2} \int_{Q_{x'}} |\psi|^4 \, dx \leq - 2 E_{\text{ab}} (\kappa H)^{3/2} \ell'^3 + (\kappa H)^{3/2} \ell'^3 o(1)
\]

by corollary 4.8. Inserting this into (5.30), we get

\[- \sum_{i \in J} \int_{Q_{x'}} |\Pi u_i|^2 \, dx \geq 2 E_{\text{ab}} \ell'^3 (\kappa H)^{3/2} + \ell'^3 (\kappa H)^{3/2} o(1).\]

Now, using (5.27) and (5.24), we may write,

\[
\sum_{i \in J} \int_{Q_{x'}} |u_i|^2 \, dx \leq (1 + o(1)) \sum_{i \in J} \int_{Q_{x'}} |\Pi u_i|^2 \, dx \leq -2 E_{\text{ab}} \ell'^3 (\kappa H)^{3/2} + \ell'^3 (\kappa H)^{3/2} o(1).
\]

(5.31)

Recall the expression of \( u_i \) in (5.29). Performing a change of variable, we get

\[
\int_{Q_{x'}} |u_i|^2 \, dx = \left(1 - \frac{H}{\kappa}\right)^{-1} (\kappa H)^{3/2} \int_{Q_{x', x'' \in J, x'' \in H}} |h \chi \psi|^2 \, dx.
\]

Using (5.24) and (5.25), we get

\[
\sum_{i \in J} \int_{Q_{x'}} |u_i|^2 \, dx = \left(1 - \frac{H}{\kappa}\right)^{-1} (\kappa H)^{3/2} \sum_{j \in J, x'' \in J} \int_{Q_{x', x'' \in J, x'' \in H}} |h \chi \psi|^2 \, dx + o\left(\frac{\ell'^3 \sqrt{\kappa H}}{\ell'^2 L}\right)
= \left(1 - \frac{H}{\kappa}\right)^{-1} (\kappa H)^{3/2} \int_{Q_{x'}} |\chi \psi|^2 \, dx + \ell'^3 (\kappa H)^{3/2} o(1),
\]

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by the definition of $\ell'$ and $L$ in (5.16). We insert this into (5.31) and get,
\[
\int_{Q_i} |\chi_i\psi|^2 \, dx \leq -2E_{\text{Ab}}\ell^3\left(1 - \frac{H}{\kappa}\right) + \ell^3\left(1 - \frac{H}{\kappa}\right) o(1).
\] (5.32)

The estimate in corollary 3.4 and Hölder’s inequality yield
\[
\int_{Q_i} |1 - \chi_i|^2 |\psi|^2 \, dx \leq \int_{Q_i \cap Q_i} |\psi|^2 \, dx \leq \frac{\ell'^2}{(\kappa H)^{1/4}} \left(1 - \frac{H}{\kappa}\right) = o\left(\ell^3\left(1 - \frac{H}{\kappa}\right)\right).
\]
Inserting this into (5.32), we get the upper bound in theorem 5.3.

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References

[1] Abrikosov A A 1957 On the magnetic properties of superconductors of the second group J. Exp. Theor. Phys. 5 1174–82
[2] Aftalion A and Serfaty S 2007 Lowest Landau level approach in superconductivity for the Abrikosov lattice close to HC2 Sel. Math. 13 183–202
[3] Almog Y 2006 Abrikosov lattices in finite domains Commun. Math. Phys. 262 677–702
[4] Almog Y 2004 Non-linear surface superconductivity in three dimensions in the large $\kappa$ limit Commun. Contemp. Math. 6 637–52
[5] Baldo S, Jerrard R L, Orlandi G and Soner M 2012 Convergence of Ginzburg–Landau functionals in 3-d superconductivity Arch. Ration. Mech. Anal. 205 699–752
[6] Correggi M and Rougerie N 2014 On the Ginzburg–Landau functional in the surface superconductivity regime Commun. Math. Phys. 332 1297–343
[7] Correggi M and Rougerie N 2016 Boundary behavior of the Ginzburg–Landau order parameter in the surface superconductivity regime Arch. Ration. Mech. Anal. 219 553–606
[8] Correggi M and Rougerie N 2016 Effects of boundary curvature on surface superconductivity Lett. Math. Phys. 106 445–67
[9] de Gennes P G 1999 Superconductivity of Metals and Alloys (Boulder, CO: Westview Press)
[10] Fournais S and Helffer B 2010 Spectral methods in surface superconductivity Progress in Nonlinear Differential Equations and Their Applications vol 77 (Boston: Birkhäuser)
[11] Fournais S and Helffer B 2009 On the Ginzburg–Landau critical field in three dimensions Commun. Pure Appl. Math. 62 215–41
[12] Fournais S and Kachmar A 2013 The ground state energy of the three dimensional Ginzburg–Landau functional. Part I: bulk regime Commun. PDE 38 339–83
[13] Fournais S and Kachmar A 2011 Nucleation of bulk superconductivity close to critical magnetic field Adv. Math. 226 1213–58
[14] Fournais S, Kachmar A and Persson M 2013 The ground state energy of the three dimensional Ginzburg–Landau functional. Part II. Surface regime J. Math. Pures Appl. 99 343–74
[15] Helffer B and Morame A 2004 Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case) Ann. Sci. École Norm. Sup. 37 105–70
[16] Kachmar A 2016 A new formula for the energy of bulk superconductivity Can. Math. Bull. at press
[17] Kachmar A 2013 The Ginzburg–Landau order parameter near the second critical field SIAM. J. Math. Anal. 46 572–87
[18] Kachmar A 2011 The ground state energy of the three dimensional Ginzburg–Landau functional in the mixed phase \textit{J. Funct. Anal.} \textbf{261} 3328–44

[19] Lu K and Pan X-B 2000 Surface nucleation of superconductivity in 3-dimensions \textit{J. Differ. Equ.} \textbf{168} 386–452

[20] Pan X B 2002 Surface superconductivity in applied magnetic fields above HC2 \textit{Commun. Math. Phys.} \textbf{228} 228–370

[21] Sandier S and Serfaty S 2012 From the Ginzburg–Landau model to vortex lattice problems \textit{Commun. Math. Phys.} \textbf{313} 635–743

[22] Sandier S and Serfaty S 2007 Vortices in the magnetic Ginzburg–Landau model \textit{Progress in Nonlinear Differential Equations and Their Applications} vol 70 (Boston: Birkhäuser)

[23] Sandier E and Serfaty S 2003 The decrease of bulk superconductivity close to the second critical field in the Ginzburg–Landau model \textit{SIAM. J. Math. Anal.} \textbf{34} 939–56

[24] Sigal I M 2015 Magnetic vortices, Abrikosov lattices and automorphic functions \textit{Mathematical and Computational Modeling (Pure and Applied Mathematics)} ed R Melnik (New York: Wiley)