ABSTRACT

We present a way for calculating the Lagrangian path integral measure directly from the Hamiltonian Schwinger–Dyson equations. The method agrees with the usual way of deriving the measure, however it may be applied to all theories, even when the corresponding momentum integration is not Gaussian. Of particular interest is the connection that is made between the path integral measure and the measure in the corresponding 0-dimensional model. This allows us to uniquely define the path integral even for the case of Euclidean theories whose action is not bounded from below.

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1. Introduction

The Schwinger–Dyson equations lie at the hart of the functional formalism of quantum field theory. Given the complete set of basic amplitudes (i.e. Feynman rules) for the propagator $\Delta_{ij}$, and the vertices $\gamma_{ijk}$, $\gamma_{ijkl}$, . . . as well as the sources $j_i$, we can calculate the Green’s functions of the theory: $G_i$, $G_{ij}$, $G_{ijk}$, . . . The connection between the basic amplitudes and the Green’s functions is given by an infinite set of coupled equations called the Schwinger–Dyson equations. They are simply the consequence of the basic linearity for the addition of amplitudes. In their simplest form the SD equations are given in terms of the generating functionals $Z[j] = \sum_{m=0}^{\infty} i^m m! G_{i_1i_2...i_m} j_{i_1} j_{i_2} ... j_{i_m}$, (1.1)

which generates the Green’s functions, and

$$\hat{I}[\phi] = \frac{1}{2} \phi \Delta^{-1}_{ij} \phi_j + \frac{1}{3!} \gamma_{ijk} \phi_i \phi_j \phi_k + \frac{1}{4!} \gamma_{ijkl} \phi_i \phi_j \phi_k \phi_l + \ldots$$ (1.2)

which generates the Feynman rules. The Schwinger–Dyson equations now take the simple form reminiscent of the classical equations of motion

$$\left( \frac{\partial \hat{I}}{\partial \phi_i} \bigg|_{\phi = \frac{1}{\hbar} \frac{\partial}{\partial \phi_i}} + j_i \right) Z[j] = 0 .$$ (1.3)

This is a linear (functional) differential equation for $Z[j]$. Note that the Fourier transform of (3) is just the Feynman path integral representation of $Z[j]$. The semi–classical limit of $Z[j]$ is dominated by configurations near $\frac{\partial I}{\partial \phi_i} = 0$. From this we see that we may write

$$\hat{I}[\phi] = I[\phi] + \frac{\hbar}{i} M[\phi] ,$$ (1.4)

where $I[\phi]$ is the classical action of the theory, and $M[\phi]$ is the measure term. Though this connection is beautiful this is as far as the usual functional formalism...
takes us – namely there is no way to determine the measure term. The only way to do this is to make connection with the operator formalism. From it we find an expression for the generating functional in terms of a Hamiltonian path integral.

\[
Z[\phi] = \int [dpdq] \exp \left( \frac{i}{\hbar} \int dt \left( \dot{q} p - H(q, p) + jq \right) \right),
\]  

(1.5)

Here the measure is trivial. The Lagrangian expressions, including the corresponding measure, are obtained by doing the momentum path integral.

It is the aim of this paper to provide an alternate way for calculating the measure inside the functional formalism. To do this we shall use the Hamiltonian form of the Schwinger–Dyson equations. In this way we shall determine a differential equation that is satisfied by the measure, and solve it for various instructive models. In the process we shall learn about which boundary conditions one must impose on (3) to uniquely pick out a solution. This will enable us to also tackle unstable field theories: Euclidean theories whose action is not bounded from below, or conversely Minkowski theories whose energy is not bounded from below. A proto–typical theory is Einstein gravity in Euclidean space. Several authors have looked at unstable field theories\[^{1,2,3,4}\], and found that the answer is an analytic extension of the path integral, where one deforms the contour of integration of the path integral. The choice of contour was dictated by the specific model. For example, in [2] David determined the contour for his matrix model approach to strings from the requirement that his non–perturbative results match the well–known perturbative string results. The nice thing about the SD approach to the measure is that it uniquely picks out which contour one should use.
2. The Basic Formalism

The generating functional written as a Hamiltonian path integral is given by

\[ Z[j, k] = \int [dpdq] \exp \left( \frac{i}{\hbar} \int dt (p\dot{q} - H(q, p) + jq + kp) \right) , \tag{2.1} \]

where we have for later convenience added a source term for the momenta. The Schwinger–Dyson equations are easily derived from the identities

\[ 0 = \int [dpdq] \frac{\delta}{\delta q} \exp \left( \frac{i}{\hbar} \int dt (p\dot{q} - H(q, p) + jq + kp) \right) \]
\[ 0 = \int [dpdq] \frac{\delta}{\delta p} \exp \left( \frac{i}{\hbar} \int dt (p\dot{q} - H(q, p) + jq + kp) \right) . \]

This gives us

\[ \left( \dot{P} + \frac{\partial H(Q, P)}{\partial Q} - j \right) Z[j, k] = 0 \]
\[ \left( \dot{Q} - \frac{\partial H(Q, P)}{\partial P} + k \right) Z[j, k] = 0 , \tag{2.2} \]

where we have introduced \( P = \frac{\hbar}{i} \frac{\delta}{\delta k} \), and \( Q = \frac{\hbar}{i} \frac{\delta}{\delta j} \). The above Schwinger–Dyson equations look just like the classical Hamiltonian equations of motion. The only difference is that we have the following non-zero commutators

\[ [P, k] = \frac{\hbar}{i} \]
\[ [Q, j] = \frac{\hbar}{i} . \tag{2.3} \]

Note that in this formalism \( P \) and \( Q \) commute.

We will now use the above equations to derive the Lagrangian path integral measure. As an example let us look at a model whose Hamiltonian is simply

\[ H(q, p) = \frac{1}{2} p^2 + V(q) . \tag{2.4} \]
In this case the SD equations read
\[
\begin{align*}
\left( \dot{P} + V'(Q) - \dot{\phi} \right) Z[j, k] &= 0 \\
\left( \dot{Q} - P + k \right) Z[j, k] &= 0 ,
\end{align*}
\]
(2.5)

Differentiating the second of these equations with respect to time, and then adding this to (5a) we get an equation for \( Q \) alone
\[
\left( \ddot{Q} - V'(Q) - (j - \dot{k}) \right) Z[j, k] = 0 .
\]
(2.6)

The action for this model is \( I[q] = \int dt \left( \frac{1}{2} \dot{q}^2 - V(q) \right) \). By introducing \( J = j - \dot{k} \) we may write (6) as
\[
\left( \frac{\delta I}{\delta Q} + J \right) Z[j, k] = 0 ,
\]
(2.7)

which is just the Lagrange formalism Schwinger–Dyson equation. Fourier transforming this we get
\[
Z[j, k] = \int [dq] \exp \left( \frac{i}{\hbar} \int dt \left( \frac{1}{2} \dot{q}^2 - V(q) + (j - \dot{k})q \right) \right) .
\]

We can now turn off the source for momenta. The generating functional \( Z[j] = Z[j, k = 0] \) equals
\[
Z[j] = \int [dq] \exp \left( \frac{i}{\hbar} \int dt \left( \frac{1}{2} \dot{q}^2 - V(q) + jq \right) \right) .
\]
(2.8)

We have just derived the well known result that the path integral measure is trivial for models whose Hamiltonian is of the simple form given in (4).

Now let us look at a bit more complicated example. We consider a model with
Hamiltonian given by
\[ H(q, p) = \frac{1}{2} G^{-1}(q)p^2 + V(q). \]  
\[ (2.9) \]

The Hamiltonian SD equations are now
\[
\begin{align*}
\left( \frac{\dot{P}}{2} - \frac{1}{2} G^{-2}(Q)G'(Q)P^2 + V'(Q) - j \right) Z[j, k] &= 0 \\
\left( \frac{\dot{Q}}{2} - G^{-1}(Q)P + k \right) Z[j, k] &= 0.
\end{align*}
\]  
\[ (2.10) \]

We may write the second equation as \( PZ = G(\dot{Q} + k)Z \) and use this to get rid of the \( P \) terms in the first equation. Therefore
\[
P^2Z = P(G\dot{Q} + Gk)Z = (G\dot{Q} + Gk)PZ + [P, k]GZ = ((G\dot{Q} + Gk)^2 + \frac{\hbar}{i} G)Z,
\]
as well as
\[
\dot{P}Z = (G'\dot{Q}(\dot{Q} + k) + G(\ddot{Q} + \dot{k}))Z.
\]

Substituting this into (10a), and setting \( k = 0 \) we get
\[
(G\ddot{Q} + \frac{1}{2} G'\dot{Q}^2 + V' - \frac{1}{2} \frac{\hbar}{i} (\ln G)' - j) Z[j] = 0.
\]  
\[ (2.11) \]

This equation can be written as
\[
\left( \frac{\delta \tilde{I}}{\delta \dot{Q}} + j \right) Z[j] = 0,
\]  
\[ (2.12) \]
where \( \tilde{I} = I + \frac{\hbar}{i} M \). The first term is the classical action \( I[q] = \int dt \left( \frac{1}{2} G(q)\dot{q}^2 - V(q) \right) \), while the second term gives a contribution to the measure and is given by \( M = \int dt \ln \sqrt{G} \). Fourier transforming (12) gives us
\[
Z[j] = \int \prod_t (dq(t)\sqrt{G(q)}) \exp \left( \frac{i}{\hbar} (I + \int dt \dot{q}j) \right),
\]  
\[ (2.13) \]
which again agrees with the standard derivation of the Lagrangian path integral in which one performs the Gaussian momentum integration in the Hamiltonian path integral.
The generalization of the previous example to more variables gives us the \( \sigma \)-model

\[
L = \frac{1}{2} G_{\alpha \beta}(q) \dot{q}^\alpha \dot{q}^\beta .
\] (2.14)

The Hamiltonian is given in terms of the inverse metric \( G_{\alpha \beta} \), and equals \( H = \frac{1}{2} G_{\alpha \beta} p_\alpha p_\beta \). The SD equations become

\[
\left( \dot{P}_\alpha + \frac{1}{2} G^{\gamma \delta}_{\alpha} P_\gamma P_\delta - j_\alpha \right) Z[j, k] = 0 \\
\left( \dot{Q}^\alpha - G^{\alpha \beta} P_\beta + k^\alpha \right) Z[j, k] = 0 .
\] (2.15)

Just as in the previous example it is a simple exercise to get rid of the \( P \) terms and derive the Lagrangian SD equation. It may be compactly written as

\[
\left( \frac{\delta I}{\delta Q^\alpha} - i \hbar \frac{1}{\sqrt{G}} \partial_\alpha \sqrt{G} + j_\alpha \right) Z[j] = 0 ,
\] (2.16)

where \( G = \text{det} \, G_{\alpha \beta} \). The corresponding path integral has the familiar form

\[
Z[j] = \int \prod_t (dq(t) \sqrt{G(q)}) \exp \left( \frac{i}{\hbar} \left( I + \int dt j_\alpha q^\alpha \right) \right) .
\] (2.17)

From these examples it is obvious that the generalization from 1-dimensional field theory, \textit{i.e.} quantum mechanics, to \( d \)-dimensional field theory is trivial. The \( d \)-dimensional expressions just contain more dummy labels.
3. Non-Trivial Examples

In this section we will look at models whose Hamiltonians are not quadratic in $p$. To begin with we look at the Hamiltonian

$$H = \frac{1}{3} p^3. \tag{3.1}$$

Obviously the energy will not be bounded from below, however, let us not worry about this for the moment. Later we will see that the SD equations will be able to make a sensible theory out of (1). The Hamiltonian SD equations are

$$\begin{align*}
(\dot{P} - j)Z &= 0 \\
(\dot{Q} - P^2 + k)Z &= 0. \tag{3.2}
\end{align*}$$

The second of these equations may be written as $P^2 Z = (\dot{Q} + k)Z$. Now we are faced with a problem. In order to get rid of the $P$ dependence of the first SD equation we need to know how $P$ acts on the generating functional. Instead of this we are given how $P^2$ acts on $Z$. If $P$ and $k$ commuted then the answer would be simply $PZ = \sqrt{\dot{Q} + k}Z$. In fact, as we shall see, this indeed holds when we take $\hbar \to 0$. From its definition we have $P = \hbar \frac{\delta}{\delta k}$, so that what we have is actually

$$\frac{\delta^2}{\delta k^2}Z[j, k] = -\frac{1}{\hbar^2} (k + \dot{Q})Z[j, k].$$

Let us note that this is actually just an ordinary differential equation: $j$ is just a label, and at the same time $\dot{Q}$ is just a constant as far as $k$ differentiation is concerned. Writing $C$ instead of $\dot{Q}$ we have

$$\frac{d^2}{dk^2}Z = -\frac{1}{\hbar^2} (k + C)Z. \tag{3.3}$$

We don’t really need to solve this – all we need is to find $PZ$. Because of this we
impose

\[
\frac{dZ}{dk} = \frac{i}{\hbar} F(k) Z .
\]  

(3.4)

Differentiating this and using (3) we find that \( F \) satisfies the Riccati equation

\[
-i\hbar \frac{dF}{dk} + F^2 = k + C .
\]  

(3.5)

There are two general ways for dealing with Riccati equations. The first is to write \( F \propto W' W \) and choose the constant of proportionality in such a way that \( W \) obeys a linear differential equation of second order. This is however just our starting equation (3), so this doesn’t help us. The second way to solve Riccati equations leads to the general solution when any particular solution is known. Again this is of no use since we know no obvious particular solution of (5). Equation (5), however, does have a natural small parameter in it, and we can find perturbative solutions, \( i.e. \) solutions written in terms of a power series in \( \hbar \). We write \( F = F_0 + \hbar F_1 + \hbar^2 F_2 + \ldots \) Equation (5) now gives

\[
F_0^2 = k + C
\]

\[-iF_0' + 2F_0 F_1 = 0
\]

\[-iF_1' + F_1^2 + 2F_0 F_2 = 0
\]

\[
\ldots
\]

Choosing the + sign for \( F_0 \) we get

\[
F_0 = \sqrt{k + C}
\]

\[
F_1 = \frac{i}{4} (k + C)^{-1}
\]

\[
F_2 = \frac{5}{32} (k + C)^{-5/2}
\]

\[
\ldots
\]

This gives us

\[
PZ = \left( (k + \dot{Q})^{1/2} + \frac{i\hbar}{4} (k + \dot{Q})^{-1} + \frac{5\hbar^2}{32} (k + \dot{Q})^{-5/2} + \ldots \right) Z .
\]

Differentiating this, substituting into the first SD equation and setting \( k = 0 \) we
get
\[
\left( \frac{1}{2} \dot{Q}^{-1/2} \ddot{Q} - \frac{i}{4} \hbar \dot{Q}^{-2} \ddot{Q} - \frac{25}{64} \hbar^2 \dot{Q}^{-7/2} \ddot{Q} + \ldots - j \right) Z[j] = 0 . \tag{3.6}
\]
This is simply
\[
\left( \frac{\delta \hat{I}}{\delta Q} + j \right) Z[j] = 0 ,
\]
where
\[
\hat{I} = I + \frac{i}{4} \hbar \int dt \ln \dot{q} - \frac{5}{48} \hbar^2 \int dt \dot{q}^{-3/2} + \ldots \tag{3.7}
\]
To one loop the path integral may be written as
\[
Z[j] = \int \prod_t \left( dq(t) \dot{q}^{-1/4} \right) \exp \left( \frac{i}{\hbar} (I + \int dt jq) \right) . \tag{3.8}
\]

Perturbative solutions like (7) are nice – if there is nothing better arround. However, for this model, we know that the standard treatment of the theory does not work since \( H \) is not bounded from bellow. Let us therefore look at equation (3) again. If we introduce
\[
x = -\hbar^{-2/3}(k + C) , \tag{3.9}
\]
then the equation simplifies to
\[
\frac{d^2 Z}{dx^2} = xZ . \tag{3.10}
\]
This is Airy’s differential equation. Equation (10) represents the \textit{Escherichia coli} in the field of asymptotic expansions, \textit{i.e.} semi-classical expansions\[^{5,6,7}\]. The general solution for \( x \in \mathbb{C} \) can be written as the Airy integral
\[
f(x) = \frac{1}{2\pi i} \int_C e^{tx - \frac{4}{3}t^3} dt . \tag{3.11}
\]
For \( f(x) \) to converge, the integrand must vanish at the end-points. The contour can’t be closed because the integral of an analytic function over such a contour
vanishes. We thus have three topologically distinct contours available corresponding to the end points at infinity with phases $-\frac{2\pi}{3}$, $\frac{2\pi}{3}$, and 0. If we label these points as $x_1$, $x_2$, $x_3$ then contour $C_{ij}$ goes from $x_i$ to $x_j$. In addition we also have $C_{12} + C_{23} + C_{31} = 0$, so that this gives us two independent solution to (10). This is just right since the Airy equation is of second order. The standard choices are the two real independent solutions

\begin{align*}
\text{Ai}(x) &= f_{12}(x) \\
\text{Bi}(x) &= i(f_{23}(x) - f_{31}(x)) .
\end{align*}

(3.12)

Note that (10) is the Schwinger–Dyson equation for the 0-dimensional “path integral” (11), i.e. for a theory with action $I = \frac{1}{3}t^3 - tx$. This is an Euclidean path integral, however, in 0-dimensions there is no difference. By writing $s = it$ we get $f(x) = \frac{1}{2\pi} \int_C ds e^{i(\frac{1}{3}s^3 + sx)}$, which is the corresponding Minkowski expression.

The Airy functions can readily be asymptotically expanded by the method of steepest descent. The saddle points given by $I' = 0$ are at $t = \pm \sqrt{x}$. Paths of steepest descent are given by $\text{Im}(I) = \text{const}$. If we write $t = u + iv$, and look at $x$ real and positive, then the paths of steepest descent passing through the saddle points are $v = 0$ and $v^2 = 3u^2 - 3x$. We have $I''(\pm \sqrt{x}) = \pm 2\sqrt{x}$, so that the left saddle point contributes when going through it along $v^2 = 3u^2 - 3x$, while the right saddle point contributes when we pass through it along $v = 0$. We thus get the asymptotic formulas

\begin{align*}
\text{Ai}(x) &\sim \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \\
\text{Bi}(x) &\sim \frac{1}{\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} .
\end{align*}

(3.13)

If we choose $Z = \text{Bi}(x)$ then using (13) we get $Z \propto (k+C)^{-1/4} e^{\frac{2}{3}(-)^{3/2} \frac{1}{4}(k+C)^{3/2}}$. We have $\text{arg } x \in [-\pi, \pi)$, so that $(-)^{3/2} = i$. Therefore we find

\begin{align*}
\frac{i}{Z} \frac{dZ}{dk} &= -\frac{1}{4}(k+C)^{-1} + \frac{i}{\hbar}(k+C)^{1/2} ,
\end{align*}

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hence

\[ PZ = \left( (k + \dot{Q})^{1/2} + \frac{1}{4} i\hbar (k + \dot{Q})^{-1} \right) Z . \]  

(3.14)

This is in agreement with our perturbative result. The choice of the Ai(x) solution gives us a similar result, but with a wrong sign in front of the classical part of (14). On the other hand if we choose the solution \( Z = Ai(x) + \frac{1}{2} Bi(x) \) then we find

\[ PZ = \left( (k + \dot{Q})^{1/2} \tanh \left( \frac{2}{3} \frac{i}{\hbar} (k + \dot{Q})^{3/2} \right) + \frac{1}{4} i\hbar (k + \dot{Q})^{-1} \right) Z , \]

which doesn’t look at all like our perturbative solution. The above solution differs from the perturbative one by pieces that are smaller than any power of \( \hbar \).

We have seen that the naive expansion in \( \hbar \) automatically picks out one solution of (3). The correct procedure is thus to solve (10). To this we need to add additional physical input that tells us which initial conditions to choose (or in the language of the path integral which contour to choose). This multitude of solutions to the SD equations is always present. For a theory whose action is for example

\[ I = \int dx \left( \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{m^g} \phi^n \right) , \]

the SD equation is a linear (functional) differential equation of \((n-1)\)st order. There are thus \( n-1 \) independent solutions. We naively solve the SD equation by a (functional) Fourier transform. However, in this way we choose a specific contour — the real axis. For \( n \) even this is indeed one of the possible solutions. In fact it is the correct one as we know from the operator formalism. For \( n \) odd the real axis is not one of the allowed contours and we seem to have a problem. As we have seen in the simple example of Airy functions there in fact is no problem – we just have to be careful in choosing the correct contours. What is the problem in such theories is that the standard operator formalism does not work, so we seem to lack a criterion that will tell us which of the allowed contours to choose.

There is a rather natural way around this obstacle. We propose that the correct contour is the unique one that has the correct semi–classical limit. Said another
way – we should choose the contour that has the correct physics up to one loop. Let us see what this means on the example of Airy functions. The naive contour would be the real axis. It is wrong since the path integral doesn’t converge. However, one can still formally calculate its asymptotic expansion. What we find is that it only gets a contribution from the right saddle point \( t = \sqrt{x} \). The left saddle point doesn’t contribute because in going along the real axis it represents a maximum of the action, not a minimum. Now let us look at the true solutions. \( \text{Ai}(x) \) only sees the left saddle point. In the direction of its contour this saddle point is a minimum of the action, so everything is ok, however, this doesn’t agree with the imposed semi–classical results. On the other hand \( \text{Bi}(x) \) only sees the right saddle point. The contributions from the left saddle point cancel for the two contours \( C_{23} \) and \( -C_{32} \). Therefore, \( \text{Bi}(x) \) has precisely the correct semi–classical behaviour. It is easy to see that it is the unique such solution of the Airy differential equation.

We have calculated the measure for our model using the SD equations. The way the measure is usually calculated is by performing the momentum integration in the Hamiltonian path integral. Thus, what we have in fact solved is

\[
\int [dp] e^{\frac{\pi}{\hbar} \int dt (p\dot{q} - \frac{1}{3} p^3)} .
\]

As we have seen the solution was given in terms of the Airy differential equation. This is not surprising. The Airy integral is simply the 0-dimensional version of (15). In fact the relation is stronger since (15) is an integration over \( p \) of an expression that doesn’t contain derivatives. Therefore

\[
\int [dp] e^{\frac{\pi}{\hbar} \int dt (p\dot{q} - \frac{1}{3} p^3)} = \prod_t f\left( -\dot{q}(t) \right) .
\]

Now we come to an important point – the choice of contour of the 0-dimensional integral completely determines the path integral (15). We therefore need to use
the Bi(\(x\)) Airy function in (16). Once we do this we get (to one loop)

\[
Z[j] = \int \prod_t (dq(t) q^{-1/4}) e^{\bar{\pi}^t I},
\]

which is precisely what we had before.

The next example that we look at illustrates another novel aspect of the SD approach to the measure. We will look at a model with Lagrangian

\[
L = \frac{1}{3} \dot{q}^3. \tag{3.17}
\]

The Hamiltonian one gets has momenta to a non-integer power

\[
H = \frac{2}{3} p^{3/2}. \tag{3.18}
\]

The SD equations are now

\[
(\dot{P} - j)Z = 0 \\
(\dot{Q} - P^{1/2} + k)Z = 0. \tag{3.19}
\]

Equation (19b) is in fact an example of a so-called extra-ordinary differential equation \[^{[8,9]}\], \textit{i.e.} one that contains derivatives to a fractional power. There are several ways to make sense of such equations the simplest of which is by using Laplace transforms. The Laplace transform is given by

\[
L(f(t)) = \int_0^{+\infty} dt f(t) e^{-ts}, \tag{3.20}
\]

and its inverse is

\[
L^{-1}(g(s)) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} ds g(s) e^{ts}. \tag{3.21}
\]
From (20) we easily find the Laplace transform of an \( n \)-th derivative to be

\[
L \left( f^{(n)}(t) \right) = sL \left( f^{(n-1)}(t) \right) - f^{(n-1)}(0) .
\]

(3.22)

Iterating this \( m \) times and then setting \( n = m \) we get

\[
L \left( f^{(n)}(t) \right) = s^n L \left( f(t) \right) - f^{(n-1)}(0) - sf^{(n-2)}(0) - \ldots - s^{n-1} f(0) .
\]

(3.23)

It is convenient to define \( f^{(n)}(t) \) for negative \( n \)'s. The inverse of a derivative is an integral, so we may take \( f^{(-1)}(t) = \int_a^t du \ f(u) \). We can uniquely specify this by imposing \( f^{(-1)}(0) = 0 \), which gives \( a = 0 \). Similarly we choose

\[
f^{(n)}(0) = 0 \quad \text{for} \quad n \leq -1
\]

(3.24)

Now we shall take (23) and (24) to be valid for all real values of \( n \). For example for \( n = -1 \) we get

\[
L \left( f^{(-1)}(t) \right) = \frac{1}{s} L \left( f(t) \right) .
\]

We can now use the inverse Laplace transform to see that what we get precisely agrees with the definition of \( f^{(-1)}(t) \) given above. For \( n = \frac{1}{2} \) equation (23) gives

\[
L \left( f^{(1/2)}(t) \right) = \sqrt{s} L \left( f(t) \right) - f^{(-1/2)}(0) .
\]

(3.25)

The inverse Laplace transform of this gives us our definition of \( \frac{d^{1/2}}{dt^{1/2}} \). For example, one can easily show that \( \frac{d^{1/2}}{dt^{1/2}} \frac{1}{\sqrt{t}} = 0 \). From this example it is obvious that in general \( \frac{d^{1/2}}{dt^{1/2}} \frac{d^{1/2}}{dt^{1/2}} f \neq \frac{d}{dt} f \). In fact, as is shown in reference [9] we have \( \frac{d^{1/2}}{dt^{1/2}} \frac{d^{1/2}}{dt^{1/2}} f = \frac{d}{dt} f + Gx^{-3/2} \), where the constant \( G \) is determined via consistency conditions. For example, for our previous example we have \( G = 1/2 \).
We are now finished with this mathematical aside, and are ready to face equation (19b), which may be written as \( \frac{d^{1/2}}{dt^{1/2}} Z = \left( \sqrt{\frac{i}{\hbar}} k + \sqrt{\frac{i}{\hbar}} C \right) Z \). If we set \( k + C = i \hbar^{1/3} t \), then this simplifies to

\[
\left( \frac{d}{dt} \right)^{\frac{1}{2}} Z = t Z .
\] (3.26)

Laplace transforming this and using (25) we get

\[
\sqrt{s} L(Z) - D = L(t Z) = -\frac{d}{ds} L(Z) ,
\] (3.27)

where we have set \( D = Z^{(-1/2)}(0) \). Introducing \( L(Z) = F(s) \) allows us to write the previous equation as

\[
\frac{dF}{ds} = D - \sqrt{s} F(s) .
\] (3.28)

This is readily solved for \( D = 0 \), where we find

\[
F(s) = E e^{-\frac{2}{3} s^{3/2}} ,
\] (3.29)

for constant \( E \). The \( D \neq 0 \) equation has the same solution, only \( E \) becomes a function \( E(s) = D \int^s du \exp \left( \frac{2}{3} u^{3/2} \right) + \text{const} \). Although this is solved in quadratures we can’t go any further because we can’t solve the above integral. Therefore, we will continue working with the \( D = 0 \) solution – note that this choice picks out a specific solution of (26). Taking the inverse Laplace transform of (29) gives

\[
Z = E \frac{1}{2\pi i} \int^{C + i \infty}_{C - i \infty} ds e^{3s - \frac{2}{3} s^{3/2}} .
\] This can be readily asymptotically expanded and one gets \( E \frac{1}{2\pi} e^{\frac{1}{2} t^3} \int_{-\infty}^{+\infty} dx e^{\frac{1}{3} x^{-1} x^2} \). This integral converges for \( \text{Re}(t) < 0 \), which is indeed the case since \( t \) is pure imaginary (with infinitesimal negative real part) when \( k \) is real. Therefore we have \( Z \propto \sqrt{-t} e^{\frac{1}{3} t^3} \). Going back to the original variables we after setting \( k = 0 \)

\[
Z \propto \sqrt{\bar{q}} e^{\frac{1}{3} (\bar{q})^3} .
\] (3.30)

Note that this is precisely what we get by doing the corresponding momentum
path integral – once we choose the correct contour. By steepest descent we find
\[ \int [dp] e^{\frac{i}{\hbar} \int dt (p \dot{q} - \frac{2}{3} p^3/2)} = \prod_t \int dp(t) e^{\frac{i}{\hbar} (p \dot{q} - \frac{2}{3} p^3/2)} \propto \prod_t \sqrt{q} e^{\frac{i}{\hbar} \int dt \frac{1}{2} \dot{q}^2}. \]

This is precisely what we got in (30).

Having derived these results the right way, we will now give a fast, though formal derivation. We start from

\[ P^{1/2}Z = (\dot{Q} + k)Z. \quad (3.31) \]

We next multiply both sides with \( P^{1/2} \). Remember \( P^{1/2}P^{1/2}Z = PZ + Gk^{-3/2} \), so we get

\[ PZ + Gk^{-3/2} = P^{1/2}(\dot{Q} + k)Z = (\dot{Q} + k)^2Z + [P^{1/2}, k]Z. \]

Using the realtion \([V(P), k] = -i\hbar \frac{dV}{dP} \), which is strictly only valid for analytic functions \( V(P) \), we get

\[ PZ + Gk^{-3/2} = (\dot{Q} + k)^2Z - i\hbar P^{-1/2}Z. \quad (3.32) \]

From (31) to order \( \hbar^0 \) we immediately get \( P^{-1/2}Z = (\dot{Q} + k)^{-1}Z \), hence to one loop we have

\[ PZ + Gk^{-3/2} = \left((\dot{Q} + k)^2 - \frac{1}{2} i\hbar (Q + k)^{-1}\right)Z. \quad (3.33) \]

Differentiating this with respect to time, and setting \( k = 0 \) we find

\[ \dot{P}Z = \left(2\ddot{Q}\dot{Q} + \frac{1}{2} i\hbar \dddot{Q} \dddot{Q}^{-2}\right)Z = 0. \quad (3.34) \]

Note that we had to choose \( G = 0 \) in order to get a finite result – in fact that is our consistency condition. The above is precisely what we got previously.

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4. Conclusion

As we have seen, the SD equations offer us a new way to calculate the measure for the Lagrangian path integral. All of our examples concerned quantum mechanical systems, but the generalization to field theory in more than one dimension is trivial. What is not trivial, when one tackles full–fledged field theory, is how to deal with gauge symmetries, and anomalies. Therefore, it will be interesting to extend this work to the treatment of gauge theories, and re–derive the measures obtained by Faddeev–Popov and Batalin–Vilkovisky. Another direction one should go is to try and cast the differential equation for the measure not in terms of the Hamiltonian (as in this paper), but solely in terms of the Lagrangian. Doing this will enable us to complete what Dirac and Feynman have started: To define a complete quantum theory in terms of the Lagrangian, i.e. the action.

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