Approximating Shepp’s constants for the Slepian process

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Abstract

Slepian process $S(t)$ is a stationary Gaussian process with zero mean and covariance $ES(t)S(t') = \max\{0, 1 - |t - t'|\}$. For any $T > 0$ and $h > 0$, define $F_T(h) = \Pr\{\max_{t \in [0,T]} S(t) < h\}$ and the constants $\Lambda(h) = -\lim_{T \to \infty} \frac{1}{T} \log F_T(h)$ and $\lambda(h) = \exp\{-\Lambda(h)\}$: we will call them ‘Shepp’s constants’. The aim of the paper is construction of accurate approximations for $F_T(h)$ and hence for the Shepp’s constants. We demonstrate that at least some of the approximations are extremely accurate.

Keywords: Slepian process, extreme value theory, boundary crossing probability

1. Introduction

Let $S(t), t \in [0, T]$, be a Gaussian process with mean 0 and covariance

$ES(t)S(t') = \max\{0, 1 - |t - t'|\}$.

This process is often called Slepian process. For any $h > 0$ and $x < h$, define

$F_T(h|x) := \Pr\{\max_{t \in [0,T]} S(t) < h \mid S(0) = x\}$ ;

if $x \geq h$ we set $F_T(h|x) = 0$. Assuming that $x$ has Gaussian distribution $N(0,1)$, and hence the stationarity of the process $S(t)$, we average $F_T(h|x)$ and thus define

$F_T(h) := \int_{-\infty}^{h} F_T(h|x)\varphi(x)dx,$

where $\varphi(x) = (2\pi)^{-1/2}\exp\{-x^2/2\}$.

Key results on the boundary crossing probabilities for the Slepian process have been established by L.Shepp in [1]. In particular, L.Shepp has derived an explicit formula for $F_n(h)$ with $n$ integer, see (2.1) below. As this explicit formula is quite complicated, in (3.7) in the same paper, L.Shepp has conjectured the existence of the following constant (depending on $h$)

$\Lambda(h) = -\lim_{n \to \infty} \frac{1}{n} \log F_n(h)$ (1.2)

and raised the question of constructing accurate approximations and bounds for this constant.

The importance of this constant is related to the asymptotic relation

$F_T(h) \simeq \text{const}[\lambda(h)]^T$ as $T \to \infty,$ (1.3)

where $\lambda(h) = \exp\{-\Lambda(h)\}$. We will call $\Lambda(h)$ and $\lambda(h)$ ‘Shepp’s constants’. In this paper, we are interested in deriving approximations for $F_T(h)$ in the form (1.3) and hence for the Shepp’s

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constants. In formulation of approximations, we offer approximations for $F_T(h)$ for all $T > 2$ and hence approximations for $\Lambda(h)$ and $\lambda(h)$.

One can apply general results shown in [2, 3] and formula (2.1.3) in [4] to approximate $\lambda(h)$ but these results only show that $\lambda(h) \to 1$ as $h \to \infty$ and therefore are of no use here. On the other hand, the following approximation, also derived from general principles, is simple but more useful.

**Approximation 0. Poisson Clumping Heuristic, see [5]:**

$$F_T(h) \simeq \exp(-h\varphi(h)T), \quad \Lambda^{(0)}(h) = h\varphi(h), \quad \lambda^{(0)}(h) = e^{-h\varphi(h)}.$$ 

For large $h$, this approximation is quite good, see Tables 1 and 2 below.

In Section 2 we derive several new approximations for $F_T(h)$ and $\lambda(h)$. In Section 3 we provide numerical results showing that at least some of the approximation derived in Section 2 are extremely accurate. Section 4 contains some minor technical details and Section 5 delivers conclusions.

2. Derivation of approximations for $F_T(h)$ and $\lambda(h)$

2.1. Shepp’s explicit formula for $F_n(h|x)$

2.1.1. Shepp’s formula

The following formula is the result (2.15) in [1]:

$$F_n(h|x) = \frac{1}{\varphi(x)} \int_{D_x} \det|\varphi(y_{j+1} + h)|_{j=0}^{n} dy_2 \ldots dy_{n+1}, \quad (2.1)$$

where $T = n$ is a positive integer, $D_x = \{y_2, \ldots, y_{T+1} \mid h - x < y_2 < y_3 < \ldots < y_{n+1}\}$, $y_0 = 0, y_1 = h - x$. L. Shepp in [1] has also derived explicit formulas for $F_T(h|x)$ with non-integral $T > 0$ but these formulas are more complicated and are realistically applicable only for small enough $T$ (say, $T \leq 3$).

From (2.1) we straightforwardly obtain

$$F_1(h|x) = \Phi(h) - \frac{\varphi(h)}{\varphi(x)} \Phi(x),$$

$$F_1(h) = \int_{-\infty}^{h} F_1(h|x) \varphi(x) dx = \Phi^2(h) - \varphi(h) \left[h\Phi(h) + \varphi(h)\right], \quad (2.2)$$

where $\Phi(x) = \int_{-\infty}^{x} \varphi(t) dt$. Derivation of explicit formulas for $F_T(h|x)$ and $F_T(h)$ with $T \leq 1$ is relatively easy as the process $S(t)$ is conditionally Markovian in the interval $[0, 1]$, see [5]. Formula (2.2) has been first derived in [7].

In what follows, $F_2(h)$ and $F_2(h|0)$ (in addition to $F_1(h)$ and $F_1(h|0)$) play important roles. The expressions for both, $F_2(h)$ and $F_2(h|0)$, are more complicated than expressions for $F_1(h)$ and $F_1(h|0)$. Nevertheless, these expressions can be reduced to a one-dimensional integrals and further approximated as shown in Appendix; see Section 4.

2.1.2. An alternative representation of the Shepp’s formula (2.1)

Let $T = n$ be a positive integer, $y_0 = 0, y_1 = h - x$. For $i = 0, 1, \ldots, n$ we set $s_i = h + y_i - y_{i+1}$ with $s_0 = x$. It follows from Shepp’s proof of (2.1) that $s_0, s_1, \ldots, s_n$ have the meaning of the values of the process $S(t)$ at the times $t = 0, 1, \ldots, n$: $S(i) = s_i (i = 0, 1, \ldots, n)$. The range of the variables $s_i$ is $(-\infty, h)$. The variables $y_1, \ldots, y_{n+1}$ are expressed via $s_0, \ldots, s_n$ by $y_k = kh - s_0 - s_1 - \cdots - s_{k-1}$ ($k = 1, \ldots, n+1$) with $y_0 = 0$. Changing the variables, we obtain the following equivalent expression for the probability (2.1):

$$F_n(h|x) = \frac{1}{\varphi(x)} \int_{-\infty}^{h} \ldots \int_{-\infty}^{h} \det|\varphi(s_i + a_{1,j})|_{i,j=0}^{n} ds_1 \ldots ds_n, \quad (2.3)$$

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For this transition density, Expression (2.3) for the probability $F_n(h|x)$ implies that the function

$$p(s_0, s_1, \ldots, s_n) = \frac{1}{\varphi(s_0) F_n(h|s_0)} \det |\varphi(s_i + a_{i,j})|^n_{i,j=0}.$$  (2.4)

is the joint probability density function for the values $S(0), S(1), \ldots, S(n)$ under the condition $S(t) < h$ for all $t \in [0, n]$.

Since $s_n$ is the value of $S(n)$, the formula (2.4) also shows the transition density from $s_0 = x$ to $s_n$ conditionally $S(t) < h$ for all $t \in [0, n]$:

$$p^{(n)}_h(x \rightarrow s_n) = \frac{1}{\varphi(x)} \int_{-\infty}^{h} \cdots \int_{-\infty}^{h} \det |\varphi(s_i + a_{i,j})|^n_{i,j=0} ds_1 \ldots ds_{n-1}.$$  (2.5)

For this transition density, $\int_{-\infty}^{h} p^{(n)}_h(x \rightarrow z) dz = F_n(h|x)$.

2.2. Approximating $\lambda(h)$ through eigenvalues of integral operators

2.2.1. One-step transition

In the case $n = 1$ we obtain from (2.5):

$$p^{(1)}_h(x \rightarrow z) = \frac{1}{\varphi(x)} \det \begin{pmatrix} \varphi(x) & \varphi(x-h+z) \\ \varphi(h) & \varphi(z) \end{pmatrix} = \varphi(z) \left[ 1 - e^{-(h-z)(h-x)} \right]$$  (2.6)

with $z = s_1 < h$.

Let $\lambda_1(h)$ be the largest eigenvalue of the the integral operator with kernel (2.6):

$$\lambda_1(h)p(z) = \int_{-\infty}^{h} p(x) p^{(1)}_h(x \rightarrow z) dx, \; z < h,$$

where eigenfunction $p(x)$ is some probability density on $(-\infty, h]$. The Ruelle-Krasnosel'skii-Perron-Frobenius theory of bounded linear positive operators (see e.g. Theorem XIII.43 in [8]) implies that the maximum eigenvalue $\lambda$ of the operator with kernel $K(x, z) = p^{(1)}_h(x \rightarrow z)$ is simple, real and positive and the eigenfunction $p(x)$ can be chosen as a probability density.

Similarly to what we have done below in Section 2.2.2, we can suggest computing good numerical approximations to $\lambda_1(h)$ using Gauss-Legendre quadrature formulas. However, we suggest to use (4.15) from [9] instead; this helps us to obtain the following simple but rather accurate approximation to $\lambda_1(h)$:

$$\lambda_1(h) = \Phi(h) + \varphi(h)/h - \varphi(h)[\varphi(h) + h\Phi(h)] / \left[ \Phi(h) - e^{-h^2/2} / 2 \right].$$

**Approximation 1:** $F_T(h) \simeq F_1(h) \left[ \lambda_1(h) \right]^{T-1} \quad (T \geq 1)$; $\Lambda^{(1)}(h) = -\log \lambda_1(h)$, $\lambda^{(1)}(h) = \hat{\lambda}_1(h)$.

2.2.2. Transition in a twice longer interval

Consider now the interval $[0, 2]$. We could have extended the method of Section 2.2.1 and used the eigenvalue (square root of it) for the transition $s_0 \rightarrow s_2$ with transition density expressed in (2.5) with $n = 2$. This would improve Approximation 1 but this improvement is only marginal. Instead, we will use another approach: we consider the transition $s_1 \rightarrow s_2$ but use the interval $[0, 1]$ just for setting up the initial condition for observing $S(t)$ at $t \in [1, 2]$. 

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For $n = 2$, the expression \[2.4\] for the joint probability density function for the values $S(0), S(1), S(2)$ under the condition $S(t) < h$ for all $t \in [0, 2]$ has the form

$$p(s_0, s_1, s_2) = \frac{1}{\varphi(s_0) F_2(h|s_0)} \det \begin{pmatrix} \varphi(s_0) & \varphi(s_0-h+s_1) & \varphi(s_0-2h+s_1+s_2) \\ \varphi(h) & \varphi(s_1) & \varphi(s_1+2h-s_2) \\ \varphi(2h-s_1) & \varphi(h) & \varphi(s_2) \end{pmatrix}.$$  

Denote by $p_1(z)$, $z < h$, the ‘non-normalized’ density of $S(1)$ under the condition $S(t) < h$ for all $t \in [0, 1]$ that satisfies $\int_{-\infty}^h p_1(z)\,dz = F_1(h)$. Using \[2.6\], we obtain

$$p_1(z) = \int_{-\infty}^h p_h^{(1)}(x \to z)\varphi(x)\,dx = \Phi(h)\varphi(z) - \Phi(z)\varphi(h).$$

Then the transition density from $x = s_1$ to $z = s_2$ under the condition $S(t) < h$ for all $t \in [0, 2]$ is achieved by integrating $s_0$ out and renormalising the joint density:

$$q_h(x \to z) = \frac{1}{p_1(x)} \int_{-\infty}^h \det \begin{pmatrix} \varphi(s_0) & \varphi(s_0-h+x) & \varphi(s_0-2h+x+z) \\ \varphi(h) & \varphi(x) & \varphi(x+z-h) \\ \varphi(2h-x) & \varphi(h) & \varphi(z) \end{pmatrix} ds_0.$$

$$= \frac{1}{\Phi(h)\varphi(x) - \Phi(x)\varphi(h)} \det \begin{pmatrix} \Phi(h) & \Phi(x) & \Phi(x+z-h) \\ \varphi(h) & \varphi(x) & \varphi(x+z-h) \\ \varphi(2h-x) & \varphi(h) & \varphi(z) \end{pmatrix}.$$  

Let $\lambda_2(h)$ be the largest eigenvalue of the integral operator with kernel $q_h$:

$$\lambda_2(h) q(z) = \int_{-\infty}^h q(x) q_h(x \to z)\,dx, \quad z < h,$$

where eigenfunction $q(x)$ is some probability density on $(-\infty, h]$. Similarly to the case $n = 1$, $\lambda_2(h)$ is simple, real and positive eigenvalue of the operator with kernel $K(x, z) = q_h(x \to z)$ and the eigenfunction $q(x)$ can be chosen as a probability density.

In numerical examples below we approximate $\lambda_2(h)$ using the methodology described in p.154. It is based on the Gauss-Legendre discretization of the interval $[-C, h]$, with some large $C > 0$, into an $N$-point set $x_1, \ldots, x_N$ (the $x_i$’s are the roots of the $N$-th Legendre polynomial on $[-C, h]$), and the use of the Gauss-Legendre weights $w_i$ associated with points $x_i$; $\lambda_2(h)$ and $q(x)$ are then approximated by the largest eigenvalue and associated eigenvector of the matrix $D^{1/2} A D^{1/2}$, where $D = \text{diag}(w_i)$, and $A_{i,j} = q_h(x_i \to x_j)$. If $N$ is large enough then the resulting approximation $\hat{\lambda}_2(h)$ to $\lambda_2(h)$ is arbitrarily accurate.

**Approximation 2:** $F_T(h) \simeq F_2(h) \left[\hat{\lambda}_2(h)\right]^{T-2}$ ($T \geq 2$); $\Lambda^{(2)}(h) = -\log \hat{\lambda}_2(h)$, $\lambda^{(2)}(h) = \hat{\lambda}_2(h)$.

2.2.3. **Quality of Approximations 1 and 2**

Approximation 1 is more accurate than Approximation 0 but it is still not accurate enough. This is related to the fact that the process $S(t)$ is not Markovian and the behaviour of $S(t)$ on the interval $[i, i + 1]$ depends on all values of $S(t)$ in the interval $[i - 1, i]$ and not only on the value $s_i = S(i)$, which is a simplification we used for derivation of Approximation 1. Approximation 2 corrects the bias of Approximation 1 by considering twice longer intervals $[i - 1, i + 1]$ and using the behaviour of $S(t)$ in the first half of the interval $[i-1, i+1]$ just for setting up the initial condition at $[i, i+1]$. As shown in Section 3 Approximation 2 is much more accurate than Approximations 0 and 1. The approximations developed in the following section also carefully consider the dependence of $S(t)$ on its past; they could be made arbitrarily accurate (on expense of increased computational complexity).
2.3. Further approximations taking into account the non-Markovianity of \( S(t) \)

As mentioned above, the behaviour of \( S(t) \) on the interval \([i, i+1]\) depends on all values of \( S(t) \) in the interval \([i-1, i]\) and not only on the value \( s_i = S(i) \). The exact value of the Shepp’s constant \( \lambda(h) \) can be defined as the limit (as \( i \to \infty \)) of the probability that \( S(t) < h \) for all \( t \in [i, i+1] \) under the condition \( S(t) < h \) for all \( t \leq i \). Using the formula for conditional probability, we obtain

\[
\lambda(h) = \lim_{i \to \infty} \frac{F_i(h)}{F_{i-1}(h)}.
\]

Waiting a long time without reaching \( h \) is not numerically possible and is not what is really required for computation of \( \lambda(h) \). What we need is for the process \( S(t) \) to (approximately) reach the stationary behaviour in the interval \([i-1, i]\) under the condition \( S(t) < h \) for all \( t < i \). Since the memory of \( S(t) \) is short (it follows from the representation \( S(t) = W(t) - W(t+1) \), where \( W(t) \) is the standard Wiener process), this stationary behaviour of \( S(t) \) is practically achieved for very small \( i \), as is also seen from numerical results of Section 3. Moreover, since ratios \( F_i(h)/F_{i-1}(h) \) are very close to \( F_i(h)/F_{i-1}(h) \) for \( i \geq 2 \), we can use ratios \( F_i(h)/F_{i-1}(h) \) in (2.7) instead. For computing the approximations, it makes integration easier.

The above considerations give rise to several approximations formulated below. We start with simpler approximations which are easy to compute and end up with approximations which are extremely accurate but are harder to compute. We claim that for all \( h \geq 1 \), Approximation 7 has at least seven correct decimal places as the true value of \( \lambda(h) \). However, we would not recommend extremely accurate Approximations 6 and 7 since Approximations 4 and 5 are already very accurate, see Tables 1 and 2, but are much easier to compute. Approximation 3, the simplest in the bunch, is also quite accurate. Note that all approximations for \( F_T(h) \) can be applied for any \( T \geq 2 \).

**Approximation 3:** \[ F_T(h) \approx F_2(h) \left[ \lambda^3(h) \right]^{T-2}, \text{ where } \lambda^3(h) = F_2(h)/F_1(h). \]

**Approximation 4:** \[ F_T(h) \approx F_2(h) \left[ \lambda^4(h) \right]^{T-2}, \text{ where } \lambda^4(h) = F_3(h)/F_2(h). \]

**Approximation 5:** \[ F_T(h) \approx F_2(h) \left[ \lambda^5(h) \right]^{T-2}, \text{ where } \lambda^5(h) = F_3(h)/F_2(h). \]

**Approximation 6:** \[ F_T(h) \approx F_3(h) \left[ \lambda^6(h) \right]^{T-3}, \text{ where } \lambda^6(h) = F_3(h)/F_2(h). \]

**Approximation 7:** \[ F_T(h) \approx F_4(h) \left[ \lambda^7(h) \right]^{T-4}, \text{ where } \lambda^7(h) = F_4(h)/F_3(h). \]

Numerical complexity of these approximations is related to the necessity of computing either \( F_n(h)/F_0 \) or \( F_n(h) \) for suitable \( n \). It follows from (2.3) that \( F_n(h)/F_0 \) is an \( n \)-dimensional integral. Consequently, \( F_n(h) \) is an \((n+1)\)-dimensional integral. In both cases, the dimensionality of the integral can be reduced by one, respectively to \( n-1 \) and \( n \), with no further analytical reduction possible. In view of results of Section 2, computation of Approximations 3 and 4 is easy, computation of Approximation 5 requires numerical evaluation of a one-dimensional integral (which is not hard) but to compute Approximation 7 we need to approximate a three-dimensional integral, which has to be done with high precision as otherwise Approximation 7 is not worth using: indeed, Approximations 4–6 are almost as good but are much easier to compute. As Approximation 7 provides us with the values which are practically indistinguishable from the true values of \( \lambda(h) \), we use Approximation 7 only for the assessment of the accuracy of other approximations and do not recommend using it in practice.

3. Numerical results

In this section we discuss the quality of approximations introduced in Sections 1, 2.2 and 2.3. In Table 1 we present the values of \( \lambda^i(h), i = 0, 1, \ldots, 7 \), for a number of different \( h \). As mentioned above, \( \lambda^7(h) \) is practically the true \( \lambda(h) \) and therefore we compare all other approximations against
$\lambda(7)(h)$. In Table 2 we present the relative errors of all other approximations against $\lambda(7)(h)$; that is, the values $\lambda(i)(h)/\lambda(7)(h) - 1$ for $i = 0, 1, \ldots, 6$. From these two tables we see that Approximations 2–7 are very accurate across all $h$.

Table 1: $\lambda(i)(h), i = 0, 1, \ldots, 7$, for different $h$.

| $\lambda(i)(h)$ | $h=0.5$ | $h=1$ | $h=1.5$ | $h=2$ | $h=2.5$ | $h=3$ | $h=3.5$ | $h=4$ | $h=4.5$ |
|-----------------|---------|-------|---------|-------|---------|-------|---------|-------|---------|
| $\lambda(0)(h)$| 0.838591| 0.785079| 0.823430| 0.897644| 0.957126| 0.996950| 0.999465| 0.999928|
| $\lambda(1)(h)$| 0.413754| 0.596156| 0.762590| 0.885025| 0.955674| 0.986738| 0.996958| 0.999466|
| $\lambda(2)(h)$| 0.366973| 0.563246| 0.746457| 0.879719| 0.954522| 0.986534| 0.996933| 0.999464|
| $\lambda(3)(h)$| 0.360246| 0.564075| 0.748323| 0.880358| 0.954548| 0.986534| 0.996930| 0.999463|
| $\lambda(4)(h)$| 0.365730| 0.562888| 0.746559| 0.879831| 0.954556| 0.986570| 0.996939| 0.999464|
| $\lambda(5)(h)$| 0.367994| 0.564526| 0.747173| 0.879946| 0.954565| 0.986571| 0.996939| 0.999464|
| $\lambda(6)(h)$| 0.367922| 0.564312| 0.747109| 0.879945| 0.954566| 0.986571| 0.996939| 0.999464|
| $\lambda(7)(h)$| 0.368114| 0.564385| 0.747119| 0.879945| 0.954566| 0.986571| 0.996939| 0.999464|

Table 2: Relative errors of $\lambda(i)(h), i = 0, 1, \ldots, 6$, against $\lambda(7)(h)$.

A plot of the relative errors can be seen in Figure 1a, where the number next to the line corresponds to the approximation. Approximations 2, 4, 6 and 7 are monotonically increasing across all $h$ and suggest very accurate lower bounds for the true $\lambda(h)$. Approximations 0 and 1 appear to provide upper bounds for $\lambda(h)$ for all $h$.

As mentioned in Section 2.2.3, Approximation 1 is not as accurate as Approximations 2–7 because it does not adequately take into account the non-Markovianity of $S(t)$. In Figure 1b we...
have plotted $\lambda^{(0)}(h)$ (dotted red line), $\lambda^{(1)}(h)$ (dashed red line) and $\lambda^{(6)}(h)$ (solid green line) for a range of interesting $h$. Visually, all $\lambda^{(i)}(h)$ with $i = 2, 4, 5, 6, 7$ would be visually indistinguishable from each other on the plot in Figure 1b and $\lambda^{(3)}(h)$ would be very close to them.

In Figure 2a we illustrate the rate of convergence of $[\log F_n(h)]/n$ in (1.2) to the Shepp’s constant $\Lambda(h)$ as $n$ increases. The dotted black lines correspond to simulation results obtained by computing the probability $F_n(h)$ with 100,000 simulations for $h = 1.5, 2$ and $3$. The solid red lines correspond to Approximation 4 for $\Lambda(h)$ with the chosen $h$. Figure 2a demonstrates how accurate this computationally cheap approximation is and also demonstrates the importance of the multiplying constant $F_2(h)$ in Approximation 4 to correct for the non-linear behaviour seen for small $n$. In Figure 2b we investigate the rate of convergence and accuracy of convergence to $\Lambda(h)$ using all approximations, where we have fixed $h = 3$. In this figure, Approximations 4, 5, 6 and 7 produce results that are visually indistinguishable to Approximation 2 and hence are not plotted. Therefore, in this figure, Approximation 2 can be considered as giving the true Shepp’s constant $\Lambda(h)$. The number next to the line corresponds to which approximation was used.

(a) Rate and accuracy of convergence to the Shepp’s constant $\Lambda(h)$ using simulations (dotted black line) and using Approximation 4 (solid red) for $h = 1.5, 2$ and $3$.

(b) Rate and accuracy of convergence to $\Lambda(h)$ with $h = 3$ using all approximations (numbers next to line correspond to which approximation has been used).

Figure 2

4. Appendix

4.1. Simplified form of $F_2(h)$ and its approximation

Using (2.1) and changing the order of integration where suitable, $F_2(h)$ can expressed through a one-dimensional integral as follows:

$$F_2(h) = \Phi(h)^3 + \varphi(h)^2\Phi(h) + \frac{\varphi(h)^2}{2} \left[(h^2 - 1)\Phi(h) + h\varphi(h)\right] + \int_{-\infty}^{h} \Phi(y)^2\varphi(2h - y)dy$$

$$- 2\varphi(h)\Phi(h)[h\Phi(h) + \varphi(h)] - \frac{1}{\sqrt{2}} \int_{0}^{\infty} \Phi(h - y)\varphi(\sqrt{2}h)\left[\Phi(\sqrt{2}y) - 1/2\right]dy.$$

Using approximations for $\Phi(t)$, it is possible to approximate $F_2(h)$ very accurately. For example,
using the approximation (see [11])

$$\Phi(t) = \begin{cases} 
0.5 \exp(0.717t - 0.416t^2) & \text{for } t \leq 0 \\
1 - 0.5 \exp(-0.717t - 0.416t^2) & \text{for } t > 0 
\end{cases}$$

(4.1)

we obtain

$$F_2(h) \cong \Phi(h)^3 + \varphi(h)^2\Phi(h) + \frac{\varphi(h)^2}{2} \left[(h^2 - 1)\Phi(h) + h\varphi(h)\right] - 2\varphi(h)\Phi(h) [h\Phi(h) + \varphi(h)]$$

$$+ \Phi(2h) - \Phi(h) - \frac{0.5}{\sqrt{2\pi}} e^{-2h^2} \left[2J(0.916, b, h) - \frac{1}{2} J(1.332, b_1, h) - \frac{1}{\sqrt{2\pi}} V(1.416, b, h)\right]$$

$$+ \frac{2}{\sqrt{2\pi}} V(1, b_2, h) + \frac{1}{\pi} K(1.5, b_2, h) - \frac{1}{2} \left\{ K(1.332, b_3, 0) - \frac{2}{\sqrt{2\pi}} U(1.416, b_4, 0) \right\}, \quad (4.2)$$

where $b = 2h - 0.717, b_1 = b - 0.717, b_2 = 2h, b_3 = b + 2.151, b_4 = b + 1.343$,

$$K(x, y, z) = \frac{\sqrt{\pi} e^{y^2/(4x)}}{\sqrt{x}} \Phi \left(\frac{2xz - y}{\sqrt{2x}}\right), \quad U(x, y, z) = \frac{1}{2x} \left[yK(x, y, z) - e^{z(y-z)}\right], \quad (4.3)$$

$J(x, y, z) = K(x, y, z) - K(x, y, 0)$ and $V(x, y, z) = U(x, y, z) - U(x, y, 0)$.

Table 3 shows that approximation (4.2) is very accurate across all $h$ of interest.

| $h$ | $F_2(h)$ |
|-----|----------|
| 0.5 | 0.085014 | 0.250896 | 0.502268 | 0.744845 | 0.900875 | 0.970790 | 0.993430 | 0.998866 | 0.999849 |
| 1   | 0.084687 | 0.250203 | 0.502097 | 0.744837 | 0.900875 | 0.970790 | 0.993430 | 0.998866 | 0.999849 |

Table 3: Accuracy of approximation (4.2) for $F_2(h)$

Using (2.1), we can express $F_2(h|0)$ as follows:

$$F_2(h|0) = \Phi(h)^2 - \frac{1}{2\varphi(0)} \varphi(h)\Phi(h) - h\varphi(h)\Phi(h) + \frac{1}{\varphi(0)} \int_h^\infty \varphi(h - y)\Phi(2h - y)\varphi(y) dy$$

$$- \frac{1}{\varphi(0)} \int_h^\infty \Phi(h - y)\varphi(2h - y)\varphi(y) dy.$$

Using (4.1), we can obtain the approximation

$$F_2(h|0) \cong \Phi(h)^2 - \frac{1}{2\varphi(0)} \varphi(h)\Phi(h) - h\varphi(h)\Phi(h) + \sqrt{\pi}\varphi \left(\frac{h}{\sqrt{2}}\right) \left[ \Phi \left(2h\sqrt{2} - \frac{h}{\sqrt{2}}\right) - \Phi \left(h\sqrt{2} - \frac{h}{\sqrt{2}}\right) \right]$$

$$- \frac{\varphi(h)}{2} e^{-1.664h^2} \left[e^{-1.434h} [K(1.416, 2.664h + 0.717, 2h) - K(1.416, 2.664h + 0.717, h)]$$

$$- e^{1.434} [K(1.416, 2.664h - 0.717, \infty) - K(1.416, 2.664h - 0.717, 2h)]$$

$$- \frac{e^{0.717h-2.416h^2}}{2\sqrt{2\pi}} \left[ K(1.416, 2.832h - 0.717, \infty) - K(1.416, 2.832h - 0.717, h) \right], \quad (4.4)$$

where $K(x, y, z)$ can be found in (4.3). Table 4 shows that approximation (4.4) is very accurate across all $h$ of interest.

5. Conclusions

In his seminal paper [1], L. Shepp derived explicit formulas for $F_T(h) = \Pr \{ \max_{t \in [0,T]} S(t) < h \}$, the distribution of maximum of the so-called Slepian process $S(t)$. As these explicit formulas are
Table 4: Accuracy of approximation (4.4) for $F_2(h|0)$

| $h$ | $F_2(h|0)$ | $F_2(h|0)$ |
|-----|------------|------------|
| 0.5 | 0.090139   | 0.088648   |
| 1   | 0.303517   | 0.302448   |
| 1.5 | 0.576857   | 0.576602   |
| 2   | 0.800758   | 0.800729   |
| 2.5 | 0.927651   | 0.927648   |
| 3   | 0.979722   | 0.979722   |
| 3.5 | 0.995607   | 0.999264   |
| 4   | 0.999905   | 0.999905   |
| 4.5 | 0.999905   | 0.999905   |

complicated, in the same paper L. Shepp has introduced a constant $\Lambda(h) = -\lim_{T \to \infty} \frac{1}{T} \log F_T(h)$ (which we call Shepp’s constant) measuring the rate of decrease of $F_T(h)$ as $T$ grows; L. Shepp also raised the question of constructing accurate approximations and bounds for this constant. Until now, this question has not been addressed. To answer it, we have constructed different approximations for $F_T(h)$ (and hence for $\Lambda(h)$). We have shown in Section 3 that at least some of these approximations are extremely accurate. We have also provided other approximations that are almost as good but are much simpler to compute.

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