GENERATING NEW PERFECT-FLUID SOLUTIONS
FROM KNOWN ONES

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Abstract

Stationary perfect-fluid configurations of Einstein’s theory of gravity are studied. It is assumed that the 4-velocity of the fluid is parallel to the stationary Killing field, and also that the norm and the twist potential of the stationary Killing field are functionally independent. It has been pointed out earlier by one of us (I.R.) that for these perfect-fluid geometries some of the basic field equations are invariant under an $SL(2,R)$ transformation. Here it is shown that this transformation can be applied to generate possibly new perfect-fluid solutions from existing known ones only for the case of barotropic equation of state $\rho + 3p = 0$. In order to study the effect of this transformation, its application to known perfect-fluid solutions is presented. In this way, different previously known solutions could be written in a single compact form. A new derivation of all Petrov type D stationary axisymmetric rigidly rotating perfect-fluid solutions with an equation of state $\rho + 3p = \text{constant}$ is given in an appendix.

PACS number: 04.20.Jb, 04.40.+c

1. Introduction

In spite of great efforts, no stationary axisymmetric perfect-fluid solution of Einstein’s equations has been found which would be appropriate for describing, in the framework of general relativity, rapidly-rotating compact massive stars. In fact, only a small number of stationary axisymmetric perfect-fluid solutions without higher symmetries are known [1-8] (see also references therein). On the other hand, owing to the nonlinear character of Einstein’s equations, only the knowledge of as much solutions as possible could provide us with a deeper insight into the methods that might help in obtaining astrophysically relevant solutions of Einstein’s equations. Therefore, every new solution generating method is welcome in general relativity.

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Here we would like to present and apply a technique by which one can generate new solutions from previously known ones. This technique is, in fact, a generalization of a method given by Geroch [9] by which out of any source-free solution of Einstein's equations possessing a non-null Killing vector field one could get a new one-parameter family of vacuum solutions. It has been recently shown by one of us [10], that an analogous transformation can be successfully used for spacetimes possessing a non-null Killing field and certain kinds of matter fields. In particular, some of the basic field equations for perfect-fluid matter sources having 4-velocity parallel to a timelike Killing field possess exactly the same type of invariance as the vacuum field equations [10,11]. A perfect fluid with 4-velocity parallel to a timelike Killing field is “rigid”, i.e., it is expansion- and shear-free. Consequently, such a fluid seems to be far from being appropriate for astrophysical applications. However, as it was shown by Geroch and Lindblom [15], perfect fluids of this kind represent equilibrium configurations of dissipative relativistic fluids. Thereby, the study of these models is of obvious astrophysical interest. Note that this “rigidity” assumption introduces considerable simplifications of the basic field equations.

In this paper we consider stationary perfect-fluid configurations with 4-velocity parallel to the stationary Killing field. The norm and the twist potential of the Killing field are assumed to be functionally independent. In section 2 we recall some of the basic notions and results of the projection (or 3-dimensional) formalism of general relativity developed for spacetimes with a non-null Killing field. Then, we present the field equations in geometrically preferred local coordinates. In section 3 a transformation by which one can generate new solutions from known ones will be recalled. In particular, we determine the conditions under which this transformation can be applied to the selected perfect-fluid spacetimes. It turns out that the original perfect fluid has to possess the barotropic equation of state \( \rho + 3p = 0 \) which is invariant under the action of the transformation. Finally, in section 4, we apply the developed technique to all existing suitable stationary perfect-fluid solutions. Accordingly, in section 4, first we recall the most general family of known stationary axisymmetric perfect-fluid solutions for which the equation of state is \( \rho + 3P = 0 \), the flow is rigid in the above sense and the twist potential and the norm of the Killing field are functionally independent. These solutions belong to the large family consisting of all Petrov type D stationary axisymmetric rigidly-rotating perfect-fluid metrics with the equation of state \( \rho + 3P = \text{constant} \), obtained by Senovilla [1]. Note that this

\[ \text{1 The invariance properties of the basic field equations for electrically charged rigid perfect fluids were studied earlier by Kramer et al. [12, 13]. Note, however, that their considerations were restricted to the static case exclusively. An application of their approach was the derivation of a regular static charged interior Reissner-Nordström solution from the interior Schwarzschild metric [14]. As far as we know, the results used in the present paper to analyze the invariance properties of the field equations are new and they were published first in Refs. [10,11].] \]
family contains as a subfamily each previously known stationary axisymmetric perfect-fluid geometry with the equation of state $\rho + 3p = constant$. Several other solutions were found later by Mars and Senovilla which are also of Petrov type D stationary perfect-fluid solutions – one of them possesses two commuting timelike Killing fields while the other is a stationary axisymmetric solution – and the equation of state is $\rho + 3p = 0$ for both of them. In appendix 1 these solutions are shown to be subclasses of the ones given by Senovilla. Finally, a new form of all Petrov type D stationary axisymmetric rigidly rotating metrics with an equation of state $\rho + 3p = constant$ is presented in appendix 2. The solutions yielded by the application of the generalized Geroch transformation to the original Senovilla metrics can be related easily to these general but relatively compact and simple metrics. Moreover, this family of solutions contains on equal footing both of the original classes of the Senovilla solutions.

2. Basic notions and the field equations

We consider perfect-fluid spacetimes, $(M, g_{ab})$, admitting a timelike Killing field, $\xi^a$. The mass density, the pressure and the 4-velocity of the fluid are denoted by $\rho$, $p$ and $u^a$, respectively. We assume, furthermore, that the 4-velocity is parallel to the timelike Killing field $\xi^a$, which implies

$$u^a = (-v)^{-\frac{1}{2}} \xi^a.$$  \hspace{1cm} (2.1)

Then, the flow is rigid in the sense that it is expansion- and shear-free. As mentioned above, this assumption on rigidity is compatible with the general strategy that one would like to find a faithful description of equilibrium states of dissipative relativistic fluids.

It is known that the entire geometrical content of a spacetime possessing a timelike Killing field can be represented on the 3-dimensional space of Killing orbits by a triple $(v, \omega_a, h_{ab})$ where $v$ is the norm of the Killing field,

$$v = \xi^a \xi_a,$$  \hspace{1cm} (2.2)

$\omega_a$ is the twist of the Killing field,

$$\omega_a = \epsilon_{abcd} \xi^b \nabla_c \xi^d,$$  \hspace{1cm} (2.3)

(here $\epsilon_{abcd}$ is the 4-volume element), while $h_{ab}$ is a Riemannian metric defined as

$$h_{ab} = -vg_{ab} + \xi_a \xi_b.$$  \hspace{1cm} (2.4)

For the selected stationary perfect-fluid configurations $\omega_a$ can be expressed (at least locally) as the gradient of a function, $\omega$, called the twist potential

$$\omega_a = D_a \omega.$$  \hspace{1cm} (2.5)
Throughout this paper we shall assume that the above defined functions \( v \) and \( \omega \) are functionally independent, i.e.,

\[
D[\omega D_{\mu}]v \neq 0. \tag{2.6}
\]

(The complementary case, when \( v \) and \( \omega \) are functionally dependent, will be considered in a subsequent paper.) Whenever eq. (2.6) is satisfied, the basic field equations can be simplified by the introduction of geometrically preferred local coordinates, \((x^1, x^2, x^3)\) \([10,11]\). Then we have that the functions \( v \) and \( \omega \) and also – by virtue of the field equations and the equation of state which is of the form \( \rho = \rho(p) \) for the considered rigid perfect fluids – the functions \( \rho \) and \( p \) depend only on the coordinates \( x^1 \) and \( x^2 \). In these local coordinates the set of equations describing a stationary perfect fluid with 4-velocity parallel to the stationary Killing field reduces to \([10,11]\)

\[
2(R_{AB} - \frac{R_{33} h_{AB}}{h_{33}}) = v^{-2}(\partial_A v \partial_B v + \partial_A \omega \partial_B \omega), \tag{2.7}
\]

\[
R_{33} = 16\pi v^{-1} p h_{33}, \tag{2.8}
\]

\[
\partial_A p + \frac{1}{2}(\rho + p) \frac{\partial_A v}{v} = 0, \tag{2.9}
\]

where \( R_{ab} \) is the Ricci tensor associated with the three dimensional metric \( h_{ab} \) and the capital Latin indices take the values 1, 2 while the Greek ones take 1, 2, 3. As it follows from the results of Refs. \([10,11]\), for the stationary perfect-fluid geometries under consideration it is sufficient to solve the above (reduced) set of field equations, because the functional independence of \( v \) and \( \omega \) ensures that any solution of this set satisfies the complete system of Einstein’s equations.

3. The transformation

In this section first the symmetry properties of eq. (2.7) will be studied. In particular, it will be recalled that a one-parameter family of new solutions of eq. (2.7) can be associated with any given solution of this equation. Then, the restrictions on the applicability of the related transformation – imposed by the remaining field equations, (2.8) and (2.9) – will be determined.

To start off, note that, just like for the vacuum case, the left hand side of eq. (2.7) depends merely on the 3-dimensional metric, \( h_{ab} \), while the right-hand side is given in terms of the functions \( v \) and \( \omega \). By exactly the same argument as used for the vacuum case in Ref. \([9]\), it can be shown that by starting with a particular solution, \((v_0, \omega_0)\), of eq. (2.7) associated with a fixed 3-dimensional metric, \( h_{ab} \), (and thereby, a fixed set of functions \( R_{AB} - \frac{R_{33} h_{AB}}{h_{33}} \)) one can generate a one-parameter family of solutions, \((v_\tau, \omega_\tau)\), of eq. (2.7). More precisely, let \((v_0, \omega_0)\) be a particular solution; then, the full set of solutions of eq. (2.7) associated with a
fixed 3-metric, \( h_{ab} \), can be given as
\[
v_{\tau} = \frac{v_0}{(\cos \tau - \omega_0 \sin \tau)^2 + v_0^2 \sin^2 \tau},
\]
(3.1)

\[
\omega_{\tau} = \frac{(\omega_0 \cos \tau + \sin \tau)(-\omega_0 \sin \tau + \cos \tau) - v_0^2 \sin \tau \cos \tau}{(\cos \tau - \omega_0 \sin \tau)^2 + v_0^2 \sin^2 \tau}.
\]
(3.2)

These formulas, with \( \tau \) as the only independent parameter, can be derived from the general form of an \( SL(2,\mathbb{R}) \) transformation by factoring out with respect to the trivial two-parameter gauge transformations [9]. Although this transformation may yield gauge inequivalent geometries, which can be shown to form a circle, its repeated applications do not generate new solutions since the transformation then merely yields a rotation of this circle \(^2\).

The most significant difference between the set of vacuum field equations and the field equations for the case under consideration is that for the vacuum problem the appropriate form of eq. (2.7) is the only equation to be solved, while for the perfect-fluid case the basic field variables, beside eq. (2.7), have to satisfy eqs. (2.8) and (2.9), as well. These equations, however, impose non-trivial restrictions on the applicability of the transformation.

To determine these restrictions, let us consider first eq. (2.8). It can be easily seen that if one fixes the 3-geometry, \( h_{ab} \), the quantity \( pv^{-1} \) has to be left intact by the above transformation. While \( v \) transforms according to eq. (3.1), \( p \) has to transform so that the variation of \( p \) compensates the variation of \( v \). This implies that
\[
\frac{p_\tau}{v_\tau} = \frac{p_0}{v_0},
\]
(3.3)

has to hold, where \( p_0 \) and \( v_0 \) denote the pressure and norm of the Killing field for the original perfect-fluid configuration, while \( p_\tau \) and \( v_\tau \) are the corresponding functions for the transformed geometries. This equation represents one of the subsidiary conditions to be satisfied whenever one applies the above transformation.

A further restriction is risen by the Euler-Lagrange equation (2.9). In fact, eqs. (3.1) and (3.3) can be used to determine the transformed mass density, \( \rho_{\tau} \), and, thereby, the transformation law for the equation of state of the fluid. To see this, take the derivative of eq. (3.3)
\[
v_\tau^{-1}(\partial_A p_\tau - p_\tau \frac{\partial_A v_\tau}{v_\tau}) = v_0^{-1}(\partial_A p_0 - p_0 \frac{\partial_A v_0}{v_0}).
\]
(3.4)

Then, using the Euler-Lagrange equation (2.9), one gets from (3.4)
\[
(\rho_\tau + 3p_\tau)\frac{\partial_A v_\tau}{v_\tau^2} = (\rho_0 + 3p_0)\frac{\partial_A v_0}{v_0^2}.
\]
(3.5)

\(^2\) The properties of the transformation are analyzed in detail in Ref. [9] for the vacuum case. Exactly the same analysis applies for the present case.
Finally, substituting the right hand side of (3.1) for \( v_\tau \) into (3.5) one obtains,

\[
(\rho_0 + 3p_0)\partial_A v_0 = \\
= (\rho_\tau + 3p_\tau)\left\{ (\cos \tau - \omega_0 \sin \tau)^2 \sin \tau v_0^2 \partial_A v_0 + 2 \sin \tau v_0 (\cos \tau - \omega_0 \sin \tau) \partial_A \omega_0 \right\}.
\]  

(3.6)

Since \( v_0 \) and \( \omega_0 \) are supposed to be functionally independent, the coefficients of \( \partial_A v_0 \) and \( \partial_A \omega_0 \) in eq. (3.6) must vanish separately. This, however, implies that both of the quantities \( \rho_0 + 3p_0 \) and \( \rho_\tau + 3p_\tau \) must vanish identically. Correspondingly, the above transformation can be applied only to those perfect-fluid configurations that have the barotropic equation of state \( \rho + 3p = 0 \). Furthermore, each of the new solutions yielded by the transformation has to possess this equation of state.

It is important to emphasize that there is no further restriction on the applicability of the above transformation (see eqs. (3.1) and (3.2)), risen by eqs. (2.8) and (2.9). In fact, for these perfect-fluid geometries with equation of state \( \rho + 3p = 0 \) the Euler-Lagrange equation can be integrated. It is easy to check that the general solution of eq. (2.9) for this equation of state is \( p v^{-1} = constant \) which, by virtue of eqs. (2.8) and (3.3), implies (in accordance with the fact that \( h_{ab} \) is kept fixed) that \( p_\tau v_\tau^{-1} = p_0 v_0^{-1} = constant \). Hence, for these perfect-fluid configurations the functional forms of \( \rho_\tau \) and \( p_\tau \) will not be explicitly presented. They can be obtained by simply multiplying \( v_\tau \) by constant factors.

4. Getting solutions from existing ones

In this section the above transformation is applied to known stationary perfect-fluid solutions of Einstein’s equations with equation of state \( \rho + 3p = 0 \). The main steps of the procedure of getting new solutions are the following: Start with a stationary perfect-fluid solution with equation of state \( \rho_0 + 3p_0 = 0 \). Determine, first, the functions \( v_0 \) and \( \omega_0 \) as well as the fixed 3-metric, \( h_{ab} \), which are the input data for our procedure. Then, determine the functional form of \( v_\tau \) and \( \omega_\tau \) using eqs. (3.1) and (3.2). Finally, the 4-dimensional line element can be given, in virtue of eq. (16.22) of Ref. [13], by the integration of the relevant form of eq. (16.23) of Ref. [13].

The stationary perfect-fluid solutions which can be used as input for the above procedure are, in fact, rare. There is a large class containing Petrov type D stationary axisymmetric perfect-fluid solutions with the equation of state \( \rho + 3p = constant \), given by Senovilla [1], which includes as a special subcase the Wahlquist solution [13] and certain families of solutions given earlier by Kramer [5,6] and by Mars and Senovilla [3].

\[3\] It was not mentioned originally by the authors in Ref. [3] that two of the solutions given there represent, actually, a subclass of the previously published Senovilla metrics.
The general form of all Petrov type D stationary axisymmetric rigidly rotating perfect-fluid metrics with an equation of state $\rho + 3p = \text{constant}$ can be written [1] as

$$ds^2 = v_0(dt + A_0d\phi)^2 + (V - W)[G^{-1}dx^2 + H^{-1}dy^2 + c^2GH(G - H)^{-1}d\phi^2]$$

(4.1)

where the norm of the stationary Killing field, $(\partial/\partial t)^a$, is

$$v_0 = \frac{H - G}{V - W},$$

(4.2)

furthermore,

$$A_0 = \frac{c(HV - GW)}{V - W},$$

(4.3)

and $c$ is an arbitrary constant. Here the functions $V = V(x)$, $W = W(y)$, $G = G(x)$ and $H = H(y)$ satisfy eqs. (16) and (17) of Ref. [1] (see also eqs. (A2.1) and (A2.2) of the present paper). The Senovilla metrics are divided into two classes. The first one, class I with $\frac{dV}{dx} \neq 0$, coincides for a special setting of the parameters [1] with the Wahlquist solution [13], while in class II with $\frac{dV}{dx} = 0$, one can also find metrics given previously by Kramer [5,6] and by Mars and Senovilla [3]. For the Senovilla metrics the functions $V$, $W$, $G$ and $H$ satisfy eqs. (18), (19a) and (19b) of Ref. [1] for class I and eqs. (18), (20a) and (20b) of Ref. [1] for class II. The general solutions of these equations for the case of $\rho + 3p = 0$ (which corresponds to the vanishing of the parameter $a$ in these equations) can be given as follows:

For class I:

$$V(x) = m_0 + M \sin(4bx + x_0)$$

(4.4)

$$W(y) = m_0 + M \cosh(4by + y_0)$$

(4.5)

$$G(x) = s_0 + S \sin(4bx + x_0) + N_1 \cos(4bx + x_0)$$

(4.6)

$$H(y) = s_0 + N_2 \sinh(4by + y_0) + S \cosh(4by + y_0)$$

(4.7)

with

$$M = \frac{1}{2d} \sqrt{m^2 - 4b^2n}, \quad m_0 = \frac{n}{2d}, \quad s_0 = \frac{3}{4b}, \quad S = \frac{3}{8b} \sqrt{\frac{m + 2b^2h}{m^2 - 4b^2n}},$$

where $m$, $n$, $s$ and $h$ are the constants used in the field equations of Ref. [1] and, finally, $x_0$, $y_0$, $N_1$ and $N_2$ are constants of integration.\(^4\)

\(^4\) Note that the functional form of $W(y)$ given by eq. (4.5) differs from the original expression of $W(y)$ in Ref. [1]. After checking the functional form of $W(y)$ given by eq. (21) of Ref. [1] we have found that it does not satisfy eq. (18) of Ref. [1] unless the second term of the right-hand side of eq. (21) of Ref. [1] is divided by the constant $C_1$ used there. Since the function $H(y)$ depends on the explicit form of $W(y)$ (see eq. (19b) of Ref. [1]), to have the correct solution given by eq. (4.7) for class I, one has to alter the function $H(y)$ correspondingly. Furthermore, in the last term of the expression for $G(x)$ for class I in Ref. [1] instead of the factor $(4bx + C_4)$ there should be $(-4bx + C_4)$ in order to have the correct solution of the basic field equations.
For class II:

\[ V(x) = \frac{m}{2b^2} = \text{constant} \]  \hspace{1cm} (4.8)

\[ W(y) = L_1 e^{-4by} + \frac{m}{2b^2} \]  \hspace{1cm} (4.9)

\[ G(x) = \frac{h}{16b^2} - L_3 \cos(4bx) - L_4 \sin(4bx) \]  \hspace{1cm} (4.10)

\[ H(y) = \frac{h}{16b^2} + L_2 e^{-4by} \]  \hspace{1cm} (4.11)

where \( L_1, L_2, L_3 \) and \( L_4 \) are arbitrary constants.

Using eqs. (4.2)-(4.7) the norm of the Killing field, \( v_0 \), and the twist potential, \( \omega_0 \), for class I can be shown to be

\[ v_0 = -\frac{1}{M} \left[ S + \frac{N_1 \cos(4bx + x_0) - N_2 \sinh(4by + y_0)}{\sin(4bx + x_0) - \cosh(4by + y_0)} \right], \]  \hspace{1cm} (4.12)

\[ \omega_0 = \frac{1}{M} \frac{N_2 \cos(4bx + x_0) + N_1 \sinh(4by + y_0)}{\sin(4bx + x_0) - \cosh(4by + y_0)}. \]  \hspace{1cm} (4.13)

The same quantities for class II are

\[ v_0 = -\frac{1}{L_1} \left( L_3 \cos(4bx) + L_4 \sin(4bx) \right) e^{4by} - \frac{L_2}{L_1}, \]  \hspace{1cm} (4.14)

\[ \omega_0 = \frac{1}{L_1} e^{4by} (-L_3 \sin(4bx) + L_4 \cos(4bx)). \]  \hspace{1cm} (4.15)

Here the twist potentials have been determined from the relevant form of eq. (16.23) of Ref. [13].

The families of perfect-fluid configurations obtained from the Senovilla-metrics are given as

\[ ds^2 = v_\tau (dt + A_\tau d\phi)^2 - \frac{1}{v_\tau} \left[ \frac{G - H}{G} dx^2 + \frac{G - H}{H} dy^2 + c^2 GH d\phi^2 \right], \]  \hspace{1cm} (4.16)

where \( G \) and \( H \) are the original functions given by eqs. (4.6)-(4.7) and eqs. (4.10)-(4.11) for class I and class II, respectively. Moreover, the actual forms of \( v_\tau \) and \( \omega_\tau \) can be calculated by using eqs. (3.1) and (3.2) with \( v_0 \) and \( \omega_0 \) determined by eqs. (4.12) and (4.13) for class I and by eqs. (4.14) and (4.15) for class II. Finally, the function \( A_\tau = A_\tau(x,y) \) is the solution of the system

\[ \frac{\partial A_\tau}{\partial x} = cH v_\tau^{-2} \frac{\partial \omega_\tau}{\partial y}, \]  \hspace{1cm} (4.17)

\[ \frac{\partial A_\tau}{\partial y} = -cG v_\tau^{-2} \frac{\partial \omega_\tau}{\partial x}. \]  \hspace{1cm} (4.18)

Using expression (3.1) for \( v_\tau \), it turns out that the general solution of eqs. (4.17) and (4.18) can be written as

\[ A_\tau = \frac{HW_\tau - GV_\tau}{H - G}. \]  \hspace{1cm} (4.19)
where $V_\tau$ and $W_\tau$ are the transformed versions of the functions $V(x)$ and $W(y)$. It is also true, that $v_\tau$ can be expressed as $v_\tau = (H - G)/(V_\tau - W_\tau)$. The functions $V_\tau$ and $W_\tau$ are

\begin{align}
V_\tau(x) &= f(\tau) + C_{1\tau} \sin(4bx + x_0) + C_{2\tau} \cos(4bx + x_0) \\
W_\tau(y) &= f(\tau) + C_{3\tau} \sinh(4by + y_0) + C_{4\tau} \cosh(4by + y_0)
\end{align}

(4.20)

where $f(\tau)$ is an arbitrary function of $\tau$ with the property $f(0) = m_0$ while the factors $C_{1\tau}, C_{2\tau}, C_{3\tau}$ and $C_{4\tau}$ are given by the following expressions.

For class I:

\begin{align}
C_{1\tau} &= \frac{1}{M} \left[ \sin^2 \tau (S^2 - N_1^2 - N_2^2) + M^2 \cos^2 \tau \right], \\
C_{2\tau} &= \frac{2 \sin \tau}{M} \left[ \sin \tau N_1 S - \cos \tau N_2 M \right], \\
C_{3\tau} &= \frac{2 \sin \tau}{M} \left[ \sin \tau N_2 S + \cos \tau N_1 M \right], \\
C_{4\tau} &= \frac{1}{M} \left[ \sin^2 \tau (S^2 + N_1^2 + N_2^2) + M^2 \cos^2 \tau \right].
\end{align}

(4.22)

(4.23)

(4.24)

(4.25)

For class II:

\begin{align}
C_{1\tau} &= \frac{-2 \sin \tau}{L_1} \left( L_2 L_4 \sin \tau + L_1 L_3 \cos \tau \right), \\
C_{2\tau} &= \frac{-2 \sin \tau}{L_1} \left( L_2 L_3 \sin \tau - L_1 L_4 \cos \tau \right), \\
C_{3\tau} &= \frac{2(L_2^2 + L_3^2)}{L_1} \sin^2 \tau - L_1, \\
C_{4\tau} &= \frac{-2(L_2^2 + L_3^2)}{L_1} \sin^2 \tau + L_1.
\end{align}

(4.26)

(4.27)

(4.28)

(4.29)

It is straightforward to check that for both classes I and II the identity $C_{4\tau}^2 = C_{1\tau}^2 + C_{2\tau}^2 + C_{3\tau}^2$ holds. Using this relationship, it can be seen that the transformed solutions $V_\tau$, $W_\tau$, $G$ and $H$ also satisfy the original field equations (16) and (17) of Ref. [1] \textsuperscript{5}. Therefore, we can state that the transformed metrics represent Petrov type D stationary axisymmetric rigidly-rotating perfect-fluid configurations, i.e. the solutions obtained by the application of the transformation are, actually, not new. In particular, it can be shown that the metrics obtained from class I include (for special settings of the parameters) both of the original classes ($\frac{dV}{dx} \neq 0$ and $\frac{dV}{dx} = 0$) of the Senovilla metrics, while the metrics obtained from class II comprise a subclass of this family. Note that although the transformation does not yield new solutions from the Senovilla metrics, it is not a gauge transformation. This follows from the fact that it maps both class I and class II onto a strictly larger subclass of the entire family of solutions.

\textsuperscript{5} See also the general form of the functions $V, W, G$ and $H$ given in appendix 2.

9
5. Conclusions

Stationary perfect-fluid configurations with 4-velocity parallel to the stationary Killing field were considered for which the norm and the twist potential of the Killing field are functionally independent. It was shown that by the use of ‘effective’ $SL(2,\mathbb{R})$ transformations new perfect-fluid solutions can be obtained from known ones whenever the equation of state is $\rho + 3p = 0$. By applying the relevant method to known geometries with this equation of state we obtained stationary perfect-fluid solutions of Einstein’s equations which contain in a compact form all the configurations we used as input for our procedure. Although these solutions are not gauge related to the ones we started with, there are no new perfect-fluid configurations among them, since all special cases were given previously.

Note, finally, that the perfect-fluid solutions to which our procedure has been applied here possess at least two commuting Killing fields. However, we would like to emphasize that our method postulates merely the existence of a single timelike Killing field.

Appendix 1

Here it is shown that the two families of metrics given by formulas (27) and (42) of Ref. [3] are, in fact, subclasses of class II of Senovilla metrics.

The case of a Mars-Senovilla metric with two timelike Killing fields

A stationary rigid perfect-fluid solution with equation of state $\rho + 3p = 0$ was found by Mars and Senovilla [3]. This spacetime is also of Petrov type D and its line element is

$$ds^2 = \alpha \beta a^2 y^2 dt^2 - \frac{x^2}{\alpha \beta} (aydT + \alpha ay dt)^2 + \frac{a^2 y^2 dx^2}{\delta^2 - \frac{a^2 x^2}{\beta^2} (x^2 - 2\beta^2)} + dy^2,$$  \hspace{1cm} (A1.1)

where $\alpha > 0$, $\beta > 0$, $\delta$ and $a$ are arbitrary constants. This solution possesses two commuting Killing vector fields, $\left(\frac{\partial}{\partial t}\right)^a$ and $\left(\frac{\partial}{\partial T}\right)^a$, associated with the coordinates $t$ and $T$, respectively. It is an interesting feature of this solution that both of the commuting Killing fields are timelike. The 4-velocity of the fluid is parallel to the Killing field $\left(\frac{\partial}{\partial t}\right)^a$.

To show that this metric is isometric to class II of Senovilla metrics change the variables as

$$dt \to \sqrt{\frac{\alpha}{\beta}} dt, \quad dx \to \sqrt{G} dx, \quad dy \to \sqrt{H} dy, \quad d\phi \to dT$$  \hspace{1cm} (A1.2)

and set the functions in the Senovilla metrics to

$$G(x) = Q^2(x), \quad V(x) = 0, \quad H(y) = \beta^2 = constant, \quad W(y) = -e^{-2ay},$$  \hspace{1cm} (A1.3)
where $\alpha$ and $\beta$ are constants of the Mars-Senovilla metric. Substituting the above expression for $G(x)$ in eq. (20a) of Ref. [1] and changing the constants of Ref. [1] to

$$h = 4a^2, \quad b = \frac{1}{2} \beta, \quad c = \frac{1}{\sqrt{\alpha \beta}} \quad \text{(A1.4)}$$

we get the corresponding equation of Ref. [3] for $Q(x)$ as given at the bottom of page 3063 in Ref. [3]. From this point, the same way as in Ref. [3], one obtains the metric in the form given by eq. (42) of Ref. [3].

Finally, note that the “axial” Killing vector of the Senovilla metrics, with the settings (A1.2)-(A1.4), becomes, actually, a timelike Killing vector.

The case of a stationary axisymmetric Mars-Senovilla metric

Another stationary axisymmetric Petrov type D perfect-fluid solution with equation of state $\rho + 3p = 0$ was also found by Mars and Senovilla (see case I of Ref. [3]). Originally, this solution was thought of and described as a differentially rotating perfect fluid. However, the authors informed us, that, although the line element is correct (see eq. (27) of Ref. [3]), this stationary perfect-fluid solution is, in fact, rigidly rotating with 4-velocity parallel to the stationary Killing field.

The line element of this spacetime is

$$ds^2 = -\beta^2 \dot{H}^2 dt^2 + X^2 \left( \frac{d}{H} d\phi + \beta \omega H dt \right)^2 + \frac{dX^2}{H^2(2\epsilon a^2 X^2 - \omega^2 X^4 + L_0)} + \frac{dy^2}{H^2}, \quad \text{(A1.5)}$$

where $\beta$, $\omega$, $a$ and $L_0$ are arbitrary constants and $\epsilon$ is a sign [3]. Here $H$ is a function of the variable $y$ satisfying

$$\dot{H} = \epsilon a^2 H, \quad \text{(A1.6)}$$

where the dot denotes derivative with respect to $y$.

To show that this metric can be obtained from class II of Senovilla metrics, first, change the variables as

$$dt \to \frac{dt}{\sqrt{\beta}}, \quad dx \to \sqrt{G} dx, \quad dy \to \sqrt{H_s} dy \quad \text{(A1.7)}$$

in the Senovilla metrics where $H_s(y)$ is the function given by eq. (4.11). Furthermore, set

$$G(x) = \omega^2 L^2(x), \quad V(x) = 0, \quad H_s(y) = \frac{H^2(y)}{H^2(y)}, \quad W(y) = -H^{-2}(y), \quad \text{(A1.8)}$$

where $\beta$ and $\omega$ are constants of the Mars-Senovilla metric. Then, setting the constants of the Senovilla metrics as

$$h = 4a^2, \quad b = \frac{1}{2}, \quad c = \frac{1}{\omega} \quad \text{(A1.9)}$$
one can see that the functions \(G(x), V(x), H_s(y)\) and \(W(y)\) satisfy eqs. (18) and (20a-b) of Ref. [1] whenever the function \(L(x)\) satisfies eq. (25) of Ref. [3] and \(H(y)\) satisfies the equation given below eq. (27) of Ref. [3].

**Appendix 2**

Here we are going to reflect briefly on a method yielding solutions of the field equations (16) and (17) of Ref. [1]. This way one gets the general form of all stationary axisymmetric Petrov type D rigidly rotating perfect-fluid metrics with an equation of state \(\rho + 3p = \text{constant}\) possessing a simpler functional form than the original solutions given in Ref. [1].

The metric is given by eq. (4.1) and the basic field equations for the functions \(V = V(x), W = W(y), G = G(x)\) and \(H = H(y)\) (see eqs. (16) and (17) of Ref. [1]) are

\[
V'^2 + W'^2 = 2(V - W)[8b^2(V - W) + V''], \tag{A2.1}
\]

\[
12a(V - W) = 16b^2(G - H) + V''\frac{G - H}{V - W} - \frac{V'G' + W'H'}{V - W} + G'', \tag{A2.2}
\]

where the prime denotes derivation with respect to the argument. The general solutions of these equations can be obtained as follows.

Taking the derivative of eq. (A2.1) with respect to the variable \(x\) and using the fact that for a non-singular metric \(V - W \neq 0\) we get

\[
(V'''' + 16b^2V') = 0. \tag{A2.3}
\]

The general solution of this equation is

\[
V(x) = V_0 + C_1 \sin(4bx) + C_2 \cos(4bx), \tag{A2.4}
\]

where \(V_0, C_1\) and \(C_2\) are constants.

By a similar procedure, calculating the derivative of eq. (A2.1) with respect to \(y\), we get

\[
W'(W'''' - 16b^2W') = 0 \tag{A2.5}
\]

which holds either if

\[
W' = 0 \tag{A2.6}
\]

or

\[
W'''' - 16b^2W' = 0. \tag{A2.7}
\]

The general solution of eq. (A2.7) for the case when \(W' \neq 0\) is

\[
W(y) = W_0 + C_3 \sinh(4by) + C_4 \cosh(4by). \tag{A2.8}
\]
The substitution of expressions (A2.4) and (A2.8) into eq. (A2.1) yields the following restrictions on the constants

\[ V_0 = W_0 =: C_0 \quad \text{and} \quad C_4^2 = C_1^2 + C_2^2 + C_3^2. \]  \hspace{1cm} (A2.9)

Substituting \( W' = 0 \) and eq. (A2.4) into eq. (A2.1) we have

\[ W(y) = C_0 \pm \sqrt{C_1^2 + C_2^2} = \text{const}. \]  \hspace{1cm} (A2.10)

To get the solutions to eq. (A2.2), first note that by using

\[ V'' + 16b^2V = 16b^2C_0 \]  \hspace{1cm} (A2.11)

eq. (A2.2) can be recast into the form

\[ 12a(V - W) = 16b^2(G - H)(C_0 - W) - V'G' - W'H' + G''(V - W). \]  \hspace{1cm} (A2.12)

Taking the derivative of eq. (A2.12) with respect to \( x \), using (A2.11) and the fact that \( V - W \neq 0 \) we arrive at

\[ G'' + 16b^2G' = 24aV'. \]  \hspace{1cm} (A2.13)

This equation has the general solution

\[ G(x) = G_0 + K_3 \sinh(4bx) + K_4 \cosh(4bx) - \frac{3a}{4b^2}xV', \]  \hspace{1cm} (A2.14)

where \( G_0, K_3 \) and \( K_4 \) are constants of integration.

In order to determine the function \( H(y) \) we take the derivative of eq. (A2.12) with respect to \( y \) and use

\[ W'' - 16b^2W = 16b^2C_0. \]  \hspace{1cm} (A2.15)

This yields

\[ 24a(V - W)W' = W'(G'' + 16b^2G + H'' - 16b^2H). \]  \hspace{1cm} (A2.16)

First, consider the case when \( W' \neq 0 \). Then from eq. (A2.16) it follows that

\[ H''' - 16b^2H' = -24aW' \]  \hspace{1cm} (A2.17)

the general solution of which is

\[ H(y) = H_0 + K_3 \sinh(4by) + K_4 \cosh(4by) - \frac{3a}{4b^2}yW', \]  \hspace{1cm} (A2.18)

where \( H_0, K_3 \) and \( K_4 \) are constants of integration.
By the substitution of expressions (A2.14) and (A2.18) for $G(x)$ and $H(y)$ into eq. (A2.15) one can see that the constants of integration have to satisfy the following conditions

$$H_0 = G_0 =: K_0 \text{ and } K_4C_4 = K_1C_1 + K_2C_2 + K_3C_3.$$  \hfill (A2.19)

Finally, whenever $W' = 0$ eq. (A2.16) is satisfied identically. Then the substitution of $W = C_0 \pm \sqrt{C_1^2 + C_2^2}$ into eq. (A2.12) yields

$$H(y) = K_0 \pm \frac{C_1K_1 + C_2K_2}{\sqrt{C_1^2 + C_2^2}} = \text{constant}. \hfill (A2.20)$$

Our solutions can be related to the original Senovilla ones by setting the constants $m$, $n$, $s$ and $h$, introduced in the field equations (18)-(20) of Ref. [1], in terms of our constants of integration as follows:

$$m = 2b^2C_0, \quad s = \frac{4}{3}b^2K_0 - 2aC_0, \quad n = b^2(C_0^2 - C_1^2 - C_2^2),$$

$$h = a(C_0^2 - C_1^2 - C_2^2) - \frac{4}{3}b^2(C_0K_0 - C_1K_1 - C_2K_2). \hfill (A2.21)$$

Finally, note that the class II Senovilla metrics can be recovered from our solutions by choosing the constants $C_1 = C_2 = 0$.

Acknowledgments

One of us (I.R.) wishes to thank J. M. M. Senovilla for useful discussions and also for drawing into our attention a number of solutions which could be used in our formalism. Thanks are also due to Á. Sebestyén for helpful discussions. A great number of calculations has been carried out using the algebraic computer programme MAPLE V and also by making use of GRTensor II [18]. This research was supported in parts by the OTKA grants F14196 and T016246.
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