ABSTRACT

We show that two-mode squeezed vacuum-like states may be engineered in the Bohm-Madelung formalism by adequately choosing the phase of the wavefunction. The difference between our wavefunction and the one of the squeezed vacuum states is given precisely by the phase we choose.

Keywords Time dependent coupled harmonic oscillator · Bohm potential · Entangled states · two-mode squeezed states

1 Introduction

Engineering non-classical states of a quantum mechanical system has been a main goal since the discovery of quantum mechanics [1]. Non-classical states are of interest as they show less fluctuations in a given canonical variable (at the expense of the other) than that of coherent states that have fluctuations referred to as the standard quantum limit. In coupled harmonic oscillators, one of the most studied non-classical states that most labs try to generate are the so-called two-mode squeezed vacuum states [2] because of their several applications; among them, the enhancement of two-photon spectroscopy [3]. Generation of two-mode squeezed vacuum states and in particular of the superposition of two-mode squeezed states [4–6] may be used for quantum information processing and quantum sensing [7]. Coupled time dependent harmonic oscillators may be studied by using Ermakov-Lewis [8–11] (invariant) techniques as done by several authors [12–14].

In this manuscript we look at the problem of generation of two-moded squeezed states by shaping the phase of the wavefunction. In fact, methods to tailor the wavefunction of a quantum state based on quantum teleportation have been studied [15]. In particular, we will use an operator approach to the Madelung-Bohm theory that, to the best of our knowledge, have not been approached using either operator techniques or group theory. We solve the Madelung-Bohm continuity or probability conservation equation via such approach and, by using a particular form of the wavefunction’s
phase, i.e., a tailored wavefunction, we are able to show that we may generate non-classical states of two-coupled time dependent harmonic oscillators, namely, two-mode squeezed vacuum states.

2 Madelung-Bohm equations

Let us consider the Schrödinger equation

\[ i \frac{\partial \psi(x,y,t)}{\partial t} = -\frac{1}{2m} \frac{\partial^2 \psi(x,y,t)}{\partial x^2} - \frac{1}{2m} \frac{\partial^2 \psi(x,y,t)}{\partial y^2} + V(x,y,t) \psi(x,y,t) \]  

(1)

with \( \psi(x,y,t) \) the wavefunction of the quantum mechanical system, \( V(x,y,t) \) an external arbitrary potential which can be time dependent and where we set \( \hbar = 1 \). We adopt the Madelung-Bohm approach to quantum mechanics [16–18] and write the wavefunction in terms of a polar decomposition

\[ \psi(x,y,t) = A(x,y,t)e^{iS(x,y,t)} \]  

(2)

which gives rise to two equations, for the real and imaginary components of the Schrödinger equation, namely, a modified Hamilton–Jacobi equation

\[ \frac{1}{2m} \dot{S}_x^2 + \frac{1}{2m} \dot{S}_y^2 + V_B + V + S_t = 0, \]  

(3)

and the continuity (probability conservation) equation

\[ \frac{\partial A}{\partial t} + \frac{1}{m} \left( S_x \frac{\partial A}{\partial x} + S_y \frac{\partial A}{\partial y} \right) + \frac{1}{2m} \left( S_{xx}A + S_{yy}A \right) = 0, \]  

(4)

where the sub-indices represent partial derivatives, where we have omitted the dependence in \( x, y, t \) of all functions, and we have introduced the Bohm potential, which is defined as

\[ V_B(x,y,t) = -\frac{1}{2ma(x,y,t)} \left[ \frac{\partial^2 A(x,y,t)}{\partial x^2} + \frac{\partial^2 A(x,y,t)}{\partial y^2} \right]. \]  

(5)

3 Operator solution to Bohm potential in 3D

The probability equation (4) may be rewritten as a Schrödinger-like equation

\[ \frac{\partial A}{\partial t} = -\frac{1}{2m} \left( 2S_x \frac{\partial}{\partial x} + 2S_y \frac{\partial}{\partial y} + S_{xx} + S_{yy} \right) A. \]  

(6)

To obtain the two-mode squeezed vacuum-like states, we set the phase

\[ S(x,y,t) = m\dot{v}(t)[r\frac{x^2+y^2}{2} + xy] + \mu(t), \]  

(7)

where \( r \) is a parameter that will be determined later. Thus, we rewrite Eq. (6) as

\[ \frac{\partial}{\partial t} A = -iv \left[ (rx+y)\hat{p}_x + (ry+x)\hat{p}_y \right] -iv |A|, \]  

(8)

by using the momentum operators \( \hat{p}_x = -i\frac{\partial}{\partial x} \) and \( \hat{p}_y = -i\frac{\partial}{\partial y} \). Its formal solution is

\[ A(x,y,t) = e^{-iv[(rx+y)\hat{p}_x + (ry+x)\hat{p}_y -iv]}A_0(x,y) = e^{-iv[(x\hat{p}_x + y\hat{p}_y) + (x\hat{p}_y + y\hat{p}_x -iv)]}A_0(x,y) \]  

(9)

where \( A_0(x,y) = A(x,y,t = 0) \) is the initial condition which determines that we must have \( v(t = 0) = 0 \). It may be seen from the above equation that the relevant operators are \( y\hat{p}_x + x\hat{p}_y \) and \( x\hat{p}_x + y\hat{p}_y \) (a product of squeeze operators [19]) and as they commute, we may write (9) as

\[ A(x,y,t) = e^{-ivx}e^{-ivy}e^{-iv(y\hat{p}_x + x\hat{p}_y)}A_0(x,y). \]  

(10)

We introduce now the annihilation and creation operators for both x and y dimensions as usual

\[ \hat{a} = \frac{x + i\hat{p}_x}{\sqrt{2}}, \quad \hat{a}^\dagger = \frac{x - i\hat{p}_x}{\sqrt{2}}. \]  

(11)
and
\[ \hat{b} = \frac{y + i\hat{p}_y}{\sqrt{2}}, \quad \hat{b}^\dagger = \frac{y - i\hat{p}_y}{\sqrt{2}}. \]

and we can cast the formal solution Eq. (10) as
\[ A(x, y, t) = \exp(-2rv) \exp \left[ \frac{rV}{2} (\hat{a}^2 - \hat{a}^{12} + \hat{b}^2 - \hat{b}^{12}) \right] \exp \left[ v (\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b}) \right] A_0(x, y); \]

it is worth to note two aspects: first, that the operator \( \exp \left[ rV (\hat{a}^2 - \hat{a}^{12} + \hat{b}^2 - \hat{b}^{12}) \right] \) is the well known squeeze operator in two dimensions, and second, that the first acting operator, the one most to the right, does not depend on \( r \). Thus, if we understand the action of this last operator, we have a solution to our problem.

In Appendix A, we show how to factorize the evolution operator \( e^{i\hat{a}\hat{b}^\dagger - i\hat{a}\hat{b}} \), and using that factorization we get
\[ A(x, y, t) = \frac{\exp(-2rv)}{\cosh v} \exp \left[ \frac{rV}{2} (\hat{a}^2 - \hat{a}^{12} + \hat{b}^2 - \hat{b}^{12}) \right] \exp \left[ \tanh(v) (\hat{a}^\dagger \hat{b}^\dagger) \right] \exp \left[ -\ln(\cosh v) (\hat{a}\hat{b} + \hat{b}^\dagger\hat{a}^\dagger) \right] \exp \left[ -\tanh(v) (\hat{a}\hat{b}) \right] A_0(x, y). \]

We would like to underline that since we have set the phase of the wave function, Eq. (7), and we have also found the amplitude of the same wave function, Eq. (14), we can get the external potential \( V(x, y, t) \) clearing it from Eq. (3); below we will get an explicit expression for a given initial condition. We want also to remark that the wave function (2), with Eq. (7) and Eq. (3), is a solution of the Schrödinger equation (1) with the corresponding external potential explained in the previous paragraph.

### 4 Two-mode squeezed vacuum states

Let us consider as initial condition the separable initial state \( A_0(x, y) = \phi_0(x)\phi_0(y) \) with
\[ \phi_n(\eta) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{n^2}{2}} H_n(\eta), \]

being \( H_n(\eta) \) the Hermite polynomial of order \( n \) and which are neither more nor less than the proper functions of the harmonic oscillator Hamiltonian.

#### 4.1 Case \( r = 0 \)

As we explain above, if we establish \( A(x, y, t) \) for \( r = 0 \), we have the complete solution as soon as the other operator is just the squeeze operator. Thus, if we make \( r = 0 \) in (14) and we introduce there the initial condition \( A_0(x, y) = \phi_0(x)\phi_0(y) \), the first and second acting operators will give simply \( A_0(x, y) \) as \( \hat{a}\phi_0(x) = 0 \) and \( \hat{b}\phi_0(y) = 0 \). Therefore, if we use the fact that the operators \( \hat{a} \) and \( \hat{b} \) are the ladder operators for the functions \( \phi_n(x) \) and \( \phi_n(y) \), respectively, we may write
\[ A(x, y, t) = \frac{1}{\cosh v} e^{\tanh(v)\hat{a}\hat{b}^\dagger} \phi_0(x)\phi_0(y) \]
\[ = \frac{1}{\cosh v} \sum_{n=0}^{\infty} \tanh^n(v) \phi_n(x)\phi_n(y), \]

that is a highly entangled state for \( v > 0 \).

Using Mehler’s formula [21-24]
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\rho}{2} \right)^n H_n(x)H_n(y) = \frac{1}{\sqrt{1 - \rho^2}} \exp \left[ -\rho^2 \frac{(x^2 + y^2)}{2} - 2\rho xy \right], \]

and taking \( \rho = \tanh v \), we finally obtain
\[ A(x, y, t) = \frac{1}{\sqrt{\pi}} \exp \left[ -\frac{x^2 + y^2}{2} - \frac{(x^2 + y^2)\tanh^2 v - 2\rho xy\tanh v}{\sech^2 v} \right]. \]
4.2 Case \( r \neq 0 \)

In the case \( r \neq 0 \), we need to apply the squeeze operator given in (10) to the above equation, which readily gives

\[
A(x,y,t) = \frac{1}{\sqrt{\pi}} \exp(-r\nu) \exp\left[ -\frac{x^2 + y^2}{2} e^{-2r\nu} - \frac{(x^2 + y^2)\tanh^2 \nu - 2xy\tanh \nu}{\text{sech}^2 \nu} e^{-2r\nu}\right].
\] (19)

From expression (5), we can now write explicitly the Bohm potential

\[
V_B(x,y,t) = -\frac{1}{2m} \left[ (x^2 + y^2)\cosh(4\nu)e^{-4r\nu} - 2xy\sinh(4\nu)e^{-4r\nu} - 2\cosh(2\nu)e^{-2r\nu}\right],
\] (20)

and clearing \( V(x,y,t) \) from (3) the external potential

\[
V(x,y,t) = \frac{1}{2} \left( x^2 + y^2 \right) \left[ -m (r^2 + 1) \dot{\nu}^2 - m\dot{\nu} - 2\cosh(4\nu)e^{-4r\nu}\right]
\]

\[
+ xy \left[ -2mr\dot{\nu}^2 - m\dot{\nu} + 2\sinh(4\nu)e^{-4r\nu}\right] + 2\cosh(2\nu)e^{-2r\nu} - \dot{\mu},
\] (21)

that correspond to a time dependent potential for two coupled time dependent harmonic oscillators [20].

Thus, the wavefunction (2) with phase (7) and amplitude (19) is solution of the Schrödinger equation (1) with external potential (21).

Both potentials, the Bohm (20) and the external (21), are paraboloids in the space. The levels curves of both potentials are conics sections. Using elementary techniques of analytic geometry, we find that the discriminant of the level curves of the Bohm potential is \( D = -\frac{e^{-10\nu(t)} \cosh(2\nu(t))}{4m^3} \) which is always different from zero, so the level curves are always non-degenerated conics, and the minor of the element 33 of the matrix of the corresponding quadratic form is \( \frac{e^{-8\nu(t)}}{4m^2} \) which is always positive, and thus the level curves will be in all cases ellipses. So independently of \( \nu, \mu \) and of the values of the parameters, the Bohm potential will have always the same qualitative behavior; it is also worth to note that we have the same conduct for \( r \neq 0 \) than for \( r = 0 \).

In the case of the external potential, Eq. (21), the situation is far more complicated and the level curves can be any kind of conic included degenerated cases which can be parallel straight lines; this open the possibility of the appearing of some kind of phase transitions, but we will not explore that possibility in this work.

4.3 A first example

Let us consider \( m = 1, r = 0, \nu(t) = t \) and \( \mu(t) = 0 \). The wave function results to be

\[
\psi(x,y,t) = \frac{1}{\sqrt{\pi}} \exp \left[ -\frac{1}{2} \cosh(2t) \left( x^2 + y^2 \right) + xy\sinh(2t) + ixy \right].
\] (22)
Figure 1: The square of the absolute value of the wave function $|\psi(x,y,t)|^2$ for $m = 1$, $r = 0$, $\nu(t) = t$, $\mu(t) = 0$ at different times.

the Bohm potential is

$$V_B(x,y,t) = -\frac{1}{2}(x^2 + y^2) \cosh(4t) + xy \sinh(4t) + \cosh(2t), \quad (23)$$

and the external potential is

$$V(x,y,t) = \frac{x^2 + y^2}{2} \left[ \cosh(4t) - 1 \right] - xy \sinh(4t) - \cosh(2t). \quad (24)$$

In Fig. 1, we plot the square of the wavefunction for different times. Starting from a (radial) Gaussian function with no preferred angle, it squeezes to give a well defined angle. Squeezing occurs for a variable at 45 degrees where the wavefunction produces a well defined position.

4.4 A second example

In this second example, we set $m = 1$, $r = 1$, $\nu(t) = t^2$ and $\mu(t) = 0$, and we obtain for the wave function

$$\psi(x,y,t) = \frac{1}{\sqrt{\pi}} \exp \left[ -e^{-2t^2} \cosh(2t^2) \frac{x^2 + y^2}{2} + e^{-2t^2} \sinh(2t^2) xy + it \left( x + y \right)^2 - t^2 \right], \quad (25)$$
Figure 2: The square of the absolute value of the wave function $|\psi(x,y,t)|^2$ for $m = 1$, $r = 1$, $v(t) = t^2$, $\mu(t) = 0$ at different times.

for the Bohm potential

$$V_B(x,y,t) = -\frac{1}{2}e^{-4t^2}(x^2 + y^2)\cosh(4t^2) + e^{-4t^2}xy\sinh(4t^2) + e^{-2t^2}\cosh(2t^2),$$

and for the external potential

$$V(x,y,t) = \left[-4t^2 + \frac{1}{2}e^{-4t^2}\cosh(4t^2) - 1\right](x^2 + y^2) + \left[-8t^2 - e^{-4t^2}\sinh(4t^2) - 2\right]xy - e^{-2t^2}\cosh(2t^2).$$

In Fig. 2, we plot the square of the wavefunction for different times. Starting from a (radial) Gaussian function with no preferred angle, it also squeezes to give a well defined angle. Squeezing occurs for a variable at 45 degrees where the wavefunction becomes well defined in position. However, the effect of $r$ produces not such a good well-defined localization of the wavefunction as the one for $r = 0$. It is clear that the Bohm and external potentials, for both cases $r = 0$ and $r \neq 0$, are given by time dependent coupled harmonic oscillators [12–14].

5 Conclusions

We have presented an operator method to engineering two-mode squeezed vacuum-like states; this method is based in the Madelung-Bohm approach to the Schrödinger equation and uses explicitly the Bohm potential. It is the correct selection of the phase of the wave function what allows us to find these entangled quantum states. The method provides an explicit form for the external time dependent quadratic potential that make possible the generation of the two-mode squeezed vacuum-like states.
A Factorization of the evolution operator

We propose as factorization of the evolution operator \( \exp \left[ \nu \left( \hat{a}^\dagger \hat{b} - \hat{a} \hat{b} \right) \right] \), given in (13), the ansatz

\[
\hat{U}(\nu) = \exp\nu (\hat{a}^\dagger \hat{b} - \hat{a} \hat{b} ) = \exp f_1(\nu) \hat{a}^\dagger \hat{b} \exp f_2(\nu)(\hat{a} \hat{b}^\dagger + \hat{b}^\dagger \hat{a}) \exp f_3(\nu) \hat{a} \hat{b}.
\]

The derivative of the evolution operator with respect to \( \nu \) gives, on the one hand,

\[
\frac{d\hat{U}(\nu)}{d\nu} = (\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}) \hat{U}(\nu),
\]

and, on the other hand,

\[
\frac{d\hat{U}(\nu)}{d\nu} = \frac{d f_1}{d\nu} \hat{a}^\dagger \hat{b}^\dagger \exp f_1(\nu) \hat{a}^\dagger \hat{b} \exp f_2(\nu)(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) \exp f_3(\nu) \hat{a} \hat{b} + \frac{d f_2}{d\nu} \exp f_1(\nu) \hat{a}^\dagger \hat{b} \exp f_2(\nu)(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) \exp f_3(\nu) \hat{a} \hat{b}
\]

\[
+ \frac{d f_3}{d\nu} \exp f_1(\nu) \hat{a}^\dagger \hat{b}^\dagger \ exp f_2(\nu)(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) \exp f_3(\nu) \hat{a} \hat{b},
\]

where, if we use the identity \( e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + \frac{1}{3!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \ldots \), we obtain the set of coupled differential equations

\[
1 = \frac{d f_1}{d\nu} - 2 f_1 \frac{d f_2}{d\nu} + f_1 f_3 \frac{d f_3}{d\nu} e^{-2 f_2}, \quad 0 = \frac{d f_2}{d\nu} - f_1 \frac{d f_3}{d\nu} e^{-2 f_2}, \quad 0 = -\frac{d f_3}{d\nu} e^{-2 f_2},
\]

subject to the initial conditions \( f_1(0) = f_2(0) = f_3(0) = 0 \). It is not difficult to show that the solution of the above system of equations is

\[
f_1(\nu) = \tanh \nu, \quad f_2(\nu) = -\ln \cosh \nu, \quad f_3(\nu) = -\tanh \nu.
\]

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