I. INTRODUCTION

The fact that in dimension $D > 5$ the totally symmetric tensor fields are not enough to cover all the irreducible representations of the Poincaré group has motivated the study of fields with mixed symmetry [1, 2] belonging to “exotic” representations of the Poincaré group. Additional interest in such fields arises because it is quite natural to expect that in the low energy limit the superstring theory should reduce to a consistent interacting supersymmetric theory of massless and massive higher spin fields ($s \geq 2$) arising from higher dimensions. This proliferation of “exotic” mixed symmetry fields poses the question of identifying different representations that can describe the same spin, possible in different phases with respect to a weak/strong coupling limit. This is precisely the subject of duality, which has been profusely studied along the years in many different contexts [3, 4]. In the massless case, dual formulations for higher spin ($s \geq 2$) fields in arbitrary dimensions have been derived from a first order parent action [5] based upon the Vasiliev action [6]. In this case, when the original description of the gauge fields in dimension $D$ is in terms of totally symmetric tensors, dual theories in terms of mixed symmetry tensors corresponding to Young tableaux having one column with $(D-3)$ boxes plus $(s-1)$ columns with one box have been obtained [5]. A discussion of duality for massless spin two in arbitrary dimensions consistent with the Vasiliev formulation [6] has also been presented in Ref. [5]. Furthermore, the method of the global symmetry extension [5] has been applied to the dualization of massless spin two fields in arbitrary dimensions [9]. An extension of these results to an AdS background was given in Ref. [10].

Dual formulations for massive higher spin fields are not as well explored. Because massive spin two fields naturally appear in brane-world models, there is an increasing interest in the understanding of alternative descriptions of massive gravitons in arbitrary dimensions. Of the many approaches available to produce dual theories we work with the parent Lagrangian method. Basically, in the case of a spin two field, this method is based on a first order action including both the standard linear graviton field $e_{ab}$ together with the corresponding dual field. The individual actions are recovered after eliminating the unwanted field using its equations of motion. In this way, on the one hand we recover the Fierz-Pauli (FP) theory and on the other the proposed dual formulation. It is known that a dimensional reduction of a massless spin two theory in $D$ dimensions leads to a massive spin two theory in $(D-1)$ dimensions [11]. Since the parent action for massless spin two field is known in dimension $D$, we investigate the resulting parent action in $(D-1)$ dimensions arising from a process of dimensional reduction by compactifying one dimension in a circle. Such a reduced parent action will describe a massive spin two field and we will derive the corresponding dual theory from it. Even though a mass is present, the reduced parent action inherits all the gauge symmetries of the original massless theory in $D$ dimensions, so that we end up with a Stueckelberg-like formulation. In this way, the resulting dual actions written in terms of the propagating fields are only obtained after following a mixture of two steps. (1) On the one hand we need to specify the required gauge fixings that still leave the resulting Lagrangians in the same gauge orbit, thus making them equivalent via gauge transformations and/or field redefinitions. This means that a unique Lagrangian is obtained after choosing a specific point in the gauge orbit. (2) On the other hand, and following the basic idea of the parent Lagrangian approach, we perform a series of field eliminations via their equations of motions.
motion. It is precisely this last process that produces inequivalent final Lagrangians that nevertheless describe the same number of degrees of freedom. This aspect of the construction is most clearly seen when the parent Lagrangian has no gauge freedom and each of the fields is eliminated to produce the corresponding non-equivalent dual actions. That is to say, we can expect that alternative field elimination among the remaining auxiliary fields after different gauge fixing will produce non-equivalent final dual Lagrangians.

In other words, at the level of the gauge invariant theory we only know for sure that we have $D(D-3)/2$ independent degrees of freedom, which will reorganize themselves according to the way the gauge and field eliminations are selected. Hence, the method is not free from ambiguities, which basically originate from these choices. An alternative Stueckelberg-like approach has been developed by Zinoviev and suffers from the same type of ambiguities. There are additional ways of compactifying the extra dimension, which are not discussed in this work.

The paper is organized as follows: in Section II we set our conventions and the strategy to carry out the dimensional reduction from $D$ to $(D-1)$ dimensions. Also we show that such reduction produces the Fierz-Pauli theory in $(D-1)$ dimensions when starting from the corresponding massless spin two action in $D$ dimensions. In Section III we start from the massless parent action of Refs. 1, 2 in $D$ dimensions and dimensionally reduce it to a massive parent action in $(D-1)$ dimensions. From this massive parent action we show in Section IV that it is possible to obtain, via different gauge fixings and field eliminations, alternative parent actions containing either a symmetric $(e^{(ab)})$ or a non-symmetric $(e_{ab})$ standard spin two field. In Section V we construct the corresponding dual theories for the massive standard spin two field in arbitrary dimensions. In $D = 4$ and for $e^{(ab)}$ we recover one of the families described in Ref. 16, while for the non-symmetric case we recover the action proposed in Ref. 11. Section VI contains some comments, which summarize the paper. Finally in the Appendix we set the mass parameter equal to zero in the $D = 4$ massive parent action, obtained from the massless five-dimensional one, and exhibit two different gauge fixing, which reshuffles the original five degrees of freedom into the sum of spin two, one and zero non interacting theories. One of such gauge choices leads to a rather unexpected Stueckelberg-like formulation of the massless spin one field.

II. DIMENSIONAL REDUCTION OF A MASSLESS $s = 2$ FIELD FROM $D$ TO $(D - 1)$ DIMENSIONS IN FLAT SPACE-TIME

The action for a massless spin two field in a $D$ dimensional flat space time is

$$S_D^0 = \int d^D x \left[ -\partial C e^{(BA)} \partial C e^{(BA)} + \partial C e^{(BA)} - 2\partial M e^{N} e^{(NM)} + 2\partial M e^{(MA)} \partial N e^{(NA)} \right],$$

while for a massive spin two field the action is the same plus the Fierz-Pauli mass term

$$S^D_\mu = S_D^0 - \mu^2 \int d^D x \left( e^{(AB)} e^{(AB)} - e^2 \right).$$

In both cases $e^{(AB)}$ is a symmetric tensor, $e^{(AB)} = e^{(BA)}$ and we are using the metric $diag(-, +, +, +, +)$. In the massless case there is a local symmetry, related to an arbitrary change of coordinates $x^A \rightarrow x^A + \xi^A(x)$, given by $e^{(AB)} \rightarrow e^{(AB)} + (\partial A \xi^B + \partial B \xi^A)$. A complete gauge fixing implies 2D constraints. For example, as it is usually done in $D = 4$, we can fix this symmetry such that $\partial A e^{(AB)} = 0$ (D constraints), but still remains a symmetry corresponding to the transformations that maintain these relations unaltered, i.e. the ones that satisfy $\partial B e^{B} = 0$. The fixing of this last symmetry leads to $D$ additional constraints. Thus the number of degrees of freedom for the spin two massless field is

$$f^D_0 = \frac{D}{2} (D+1) - 2D = \frac{D}{2} (D-3).$$

In the case of a massive field there is no gauge symmetry due to the mass term, but its Euler-Lagrange equations yield $(D+1)$ constraints, $\partial A e^{(AB)} = 0$ and $e^A = e = 0$, so that the number of degrees of freedom is

$$f^D_\mu = \frac{D}{2} (D+1) - (D+1) = \frac{1}{2} (D+1) (D-2).$$

Notice that the massless spin two field in $D$ dimensions has the same number of degrees of freedom that the massive field in $D - 1$ dimensions, $f^D_\mu = f^{D-1}_\mu$. This suggest a relation between both fields via dimensional reduction. This point is explored in the following.

To be specific we will consider the reduction from $D$ to $(D - 1)$ dimensions by compactifying one of the spatial coordinates, $y$, on a circle $S^1$ of radius $L$ so that the remaining space continues to be flat. We denote the indices of the
$D$ dimensional tensors with capital letters $(A, B, ...M = 0, 1, ..., D - 1)$ and reserve the lower case ones $(a, b, ...m, ... = 0, 1, 2, 3, ...D - 2)$ to the $(D - 1)$ dimensional tensors. The spatial dimension to be reduced by compactification is denoted by the index $(D - 1)$ so that $A = (a, D - 1)$ and $X^M = (x^m, x^{(D - 1)} = y)$. The basic idea in the reduction is to rewrite any $D$ dimensional action in terms of this splitting $A = (a, D - 1)$. We expand all the fields in $D$ dimensions as a Fourier series of the form

$$Ψ_{AB...}^{RS...}(X^M) = \sum_n Ψ^{(n)}_{AB...}^{RS...}(x^m)e^{in\mu y/L},$$

and we consider a mode with $n/L = \mu$. In this case the coordinate dependence of a $D$ dimensional real tensor $Φ_{AB...}^{RS...}$ is written as

$$Φ_{AB...}^{RS...}(X^M) = \sqrt{\frac{\mu}{4\pi}} Φ_{AB...}^{RS...}(x^m)e^{i\mu y} + \sqrt{\frac{\mu}{4\pi}} Φ_{AB...}^{RS...}(x^m)e^{-i\mu y},$$

where $\mu$ has dimension of mass and will become the mass coefficient for the four dimensional massive fields.

The tensorial transformation under $(D - 1)$-parity $(y \rightarrow -y)$ is defined by

$$Φ_{AB...}^{RS...}(x^m, y) \rightarrow Φ_{AB...}^{RS...}(x^m, -y).$$

Each $(D - 1)$ index will induce an overall minus sign in the fields $Φ_{AB...}^{RS...}(x^m, y)$ under this transformation, thus making the corresponding $x$-dependent component to become real when the number of indices with value $(D - 1)$ is even, and purely imaginary in the case it has an odd number of indices with this value. Thus, for example, when the field has no indices with this value we get

$$Φ_{ab...}^{rs...}(x^m, y) = \sqrt{\frac{\mu}{\pi}} Φ_{ab...}^{rs...}(x^m) \cos \mu y.$$

When the field has one index with this value, we denote such components by

$$Φ_{(D-1)b...}^{rs...}(x^m) = -iΦ_{b...}^{rs...}(x^m)$$

where the tensor with a tilde has only $(D - 2)$-dimensional indices and it is real. In this way we write

$$Φ_{(D-1)b...}^{rs...}(x^m, y) = \sqrt{\frac{\mu}{\pi}} Φ_{b...}^{rs...}(x^m) \sin \mu y.$$

It is clear that the expressions (8) and (10) can be generalized to any tensor having an even or odd number of subindices $(D - 1)$. In general we will use different names for these reduced tensors, dropping the indices with value $(D - 1)$.

After the $(a, D - 1)$ separation has been made in the coordinates and fields, the resulting four dimensional action is obtained by performing the integration of $y$ over a circle. The only surviving contributions come from

$$\oint dy \cos^2 \mu y = \oint dy \sin^2 \mu y = \frac{\pi}{\mu}.$$

In the sequel we denote any function $Σ(x^m, y)$ by $Σ(x, y)$.

We will show now that this dimensional reduction applied to the $D$-dimensional massless spin two field, actually yields the massive $(D - 1)$-dimensional FP theory. We start from the action for the massless spin two field in $D$ dimensions

$$S_D = \frac{1}{2} \int d^Dx \int dy \left( -∂_A e^{(MN)}∂^A e_{(MN)} + 2∂_M e^{(MN)}∂^A e_{(AN)} - 2∂_Me^{(MN)}∂_Ne + ∂_A e∂^A e \right),$$

with $e^{(MN)} = e_{(NM)}$, which is invariant under the gauge transformations

$$δe^{(MN)} = ∂_M ξ_N + ∂_N ξ_M.$$  

The dimensional reduction is implemented in term of the fields $e_{mn}(x)$, $a_m(x)$, $φ(x)$, defined by

$$e_{(mn)}(x, y) = \sqrt{\frac{\mu}{\pi}} e_{(mn)}(x) \cos \mu y,$$

$$e_{((D-1)n)}(x, y) = \sqrt{\frac{\mu}{\pi}} a_m(x) \sin \mu y,$$

$$e_{((D-1)(D-1))}(x, y) = \sqrt{\frac{\mu}{\pi}} φ(x) \cos \mu y,$$
while the gauge transformations \([13]\) are translated into
\[
\delta e_{(mn)} = \partial_m \xi_n + \partial_n \xi_m, \quad \delta a_m = \partial_m \xi - \mu \xi_m, \quad \delta \varphi = 2 \mu \xi(x).
\]
with
\[
\xi_m(x, y) = \sqrt{\frac{\mu}{\pi}} \xi_m(x) \cos \mu y, \quad \xi_{(D-1)}(x, y) = \sqrt{\frac{\mu}{\pi}} \xi(x) \sin \mu y.
\]

Redefining
\[
\tilde{a}_m = a_m - \frac{1}{2 \mu} \partial_m \varphi,
\]
the reduced action is
\[
S_{(D-1)} = \frac{1}{2} \int d^Dx \left\{ -\partial_a e^{(mn)} \partial^a e_{(mn)} + 2 \partial_m e^{(mn)} \partial^a e_{(an)} - 2 \partial_m e^{(mn)} \partial_n e + (\partial_a e)^2 
\right.
- \mu^2 \left[ (e^{(mn)} + \frac{1}{\mu} (\partial_m \tilde{a}_n + \partial_n \tilde{a}_m))^2 - \left( e + \frac{2}{\mu} \partial^a \tilde{a}_n \right)^2 \right],
\]
where \(e = e_m^m\). The action \([20]\) remains invariant under the induced gauge transformations
\[
\delta e_{(mn)} = \partial_m \xi_n + \partial_n \xi_m, \quad \delta \tilde{a}_n = -\mu \xi_n.
\]

To get the FP action we can now fix the gauge, choosing \(\xi_n(x)\) such that \(\tilde{a}_n = 0\). This leaves us with \(e_{(mn)}\) as the remaining degrees of freedom, with the standard \((D-1)\)-dimensional action
\[
S_{(D-1)} = \frac{1}{2} \int d^Dx \left\{ -\partial_a e^{(mn)} \partial^a e_{(mn)} + 2 \partial_m e^{(mn)} \partial^a e_{(an)} - 2 \partial_m e^{(mn)} \partial_n e + (\partial_a e)^2 - \mu^2 \left( e^{(mn)} e^{(mn)} - e^2 \right) \right\}.
\]

### III. DIMENSIONAL REDUCTION OF THE D DIMENSIONAL MASSLESS \(s = 2\) PARENT ACTION.

It is well known that the first order parent action
\[
S = -\frac{1}{2} \int d^{(D-1)}y \left[ Y_{C[AB]} (\partial_A e_{BC} - \partial_B e_{AC}) - Y_{C[AB]} Y^{B[AC]} + \frac{1}{D-2} Y_A Y^A \right], \quad Y_{[AB]} = Y_A,
\]
with \(Y_{C[AB]} = -Y_{C[B]A}\), generates massless dual theories for the spin two field in \(D\) dimensions \([5]\). The field \(Y_{C[AB]}\) has \(D^2(D-1)/2\) independent components while \(e_{BC} \neq e_{CB}\) accounts for \(D^2\), which give a total of \(D^2(D + 1)/2\) independent components. The above action is invariant under the gauge transformations (local Lorentz transformations)
\[
\delta Y_{C[AB]} = - \left[ \partial_C \omega_{[AB]} + \partial_D \omega^{[BD]} \eta^{AC} + \partial_D \omega^{[DA]} \eta^{BC} \right],
\]
\[
\delta Y_A = -(D-2) \partial^D \omega_{[DA]}, \quad Y_A = Y_{C[AB]} \eta_{BC},
\]
\[
\delta e_{BC} = \omega_{[BC]},
\]

\(\delta Y_{D[AB]} = D^2 \left( \partial^A \xi_B - \partial^B \xi_A \right) \left( \eta^{AD} \partial^B - \eta^{BD} \partial^A \right) \partial_C \xi^C + \eta^{BD} \partial^2 \xi^A - \eta^{AD} \partial^2 \xi^B\),
\[
\delta Y_A = (D-2) \left( \partial^2 \xi^A - \partial^A \partial_C \xi^C \right),
\]
\[
\delta e_{AB} = \partial_A \xi_B + \partial_B \xi_A.
\]

According to Ref. \([5]\), these gauge symmetries are independent of the number of dimensions.
The dimensional reduction is performed via the following redefinitions for the fields

\[ Y^{c[ab]}(x, y) = \sqrt{\mu} Y^{c[ab]}(x) \cos \mu y, \quad Y^{(D-1)[ab]}(D-1)](x, y) = \sqrt{\mu} Z^b \cos \mu y, \]

\[ Y^{(D-1)[ab]}(x, y) = \sqrt{\mu} V^{[ab]} \sin \mu y, \quad Y^{c[b(D-1)]}(x, y) = \sqrt{\mu} W^{bc} \sin \mu y, \]

\[ e_{ab}(x, y) = \sqrt{\mu} e_{ab}(x) \cos \mu y, \quad e_{(D-1)(D-1)}(x, y) = \sqrt{\mu} S(x) \cos \mu y, \]

\[ e_{a(D-1)}(x, y) = \sqrt{\mu} B_a(x) \sin \mu y, \quad e_{(D-1)a}(x, y) = \sqrt{\mu} A_a(x) \sin \mu y, \]

which reshuffles the original independent components in the following way

\[ Y^{C[AB]} \rightarrow D^2(D-1)/2 \]

\[ \begin{cases} 
Y^{c[ab]} \rightarrow [(D-1)^2(D-2)/2], \\
V^{[ab]} \rightarrow [(D-1)(D-2)/2], \\
W^{bc} \rightarrow [(D-1)^2], \\
Z^b \rightarrow [D-1] 
\end{cases}, \]

\[ e_{BC} \rightarrow D^2 \quad \begin{cases} 
e_{ab} \rightarrow (D-1)^2, \\
A_a \rightarrow (D-1), \\
B_a \rightarrow (D-1), \\
S \rightarrow 1 \end{cases}. \]

Also the \( D(D+1)/2 \) gauge parameters are reorganized according to

\[ \omega_{[ab]}(x, y) = \sqrt{\mu} \omega_{[ab]}(x) \cos \mu y, \quad \omega_{(D-1)a}(x, y) = \sqrt{\mu} \omega_a(x) \sin \mu y, \]

\[ \xi_a(x, y) = \sqrt{\mu} \xi_a(x) \cos \mu y, \quad \xi_{(D-1)}(x, y) = \sqrt{\mu} \xi(x) \sin \mu y. \]

The corresponding gauge transformations in the \((D-1)\) dimensional fields associated to the \((D-1)(D-2)/2\) parameters \( \omega_{[ab]} \) and the \((D-1)\) parameters \( \omega_a \) can be rewritten as:

\[ \delta e_{[ab]}(x) = 0, \quad \delta e_{[ab]}(x) = \omega_{[ab]}(x), \]

\[ \delta B_a(x) = -\omega_a(x), \quad \delta A_a(x) = \omega_a(x), \quad \delta S = 0, \]

\[ \delta Y^{c[ab]}(x) = -\left( \partial_c \omega^{[ab]} + \partial_m \omega^{[bm]} \eta^{ac} + \partial_m \omega^{[ma]} \eta^{bc} \right) - \mu \left( \omega^a \eta^{bc} - \omega^b \eta^{ac} \right), \]

\[ \delta Y^a(x) = -\left( \partial_a \omega^{[ab]} + 3 \partial_m \omega^{[ma]} \right) - 3 \mu \omega^a, \]

\[ \delta V^{[ab]} = \mu \omega^{[ab]}, \quad \delta Z^a = -\partial_m \omega^{[ma]}, \]

\[ \delta W^{ac} = \epsilon^{abc} \omega^a - \partial_m \omega^{[ma]} \eta^{bc}. \]

Let us notice that \( \partial_a \delta W^{ac} = \partial^c \delta_a \omega^a - \partial^c \partial_m \omega^a = 0 \). The remaining gauge transformations, given by the \((D-1)\) parameters \( \xi^a \) and the parameter \( \xi \) are:

\[ \delta e_{[ab]}(x) = \partial_a \xi_b + \partial_b \xi_a, \quad \delta e_{[ab]}(x) = 0, \]

\[ \delta B_a(x) = \delta A_a(x) = \partial_a \xi - \mu \epsilon_a, \quad \delta S = 2 \mu \epsilon, \]

\[ \delta Y^{c[ab]}(x) = \partial^c \left( \partial^a \xi^b - \partial^b \xi^a \right) + (\eta^{ac} \partial^b - \eta^{bc} \partial^a) \partial_m \epsilon^m + \partial^2 \left( \eta^{bc} \xi^a - \eta^{ac} \xi^b \right), \]

\[ \delta Y^a(x) = 2 \partial^a \xi^a - 2 \partial^a \partial_m \epsilon^m - 3 \mu \partial^a \xi - 3 \mu^2 \xi^a, \]
\[ \delta V^{[ab]} = -\mu \left( \partial^a \xi^b - \partial^b \xi^a \right), \quad \delta Z^a = \partial^2 \xi^a - \partial^a \partial_m \xi^m, \] (48)

\[ \delta W^{ad} = \partial^d \partial^a \xi - \eta^{ad} \partial^2 \xi + \mu \left( \partial^d \xi^a - \eta^{ad} \partial_m \xi^m \right). \] (49)

Again we have here \( \partial_a \delta W^{ac} = 0 \). In the above \( \partial^2 = -\partial_0^2 + \nabla^2 \) denotes the \((D-1)\)-dimensional D’Alambertian.

After substituting the fields (50) in the \( D \) dimensional action (23) and performing the integration with respect to the fifth coordinate \( y \) we obtain the following dimensionally reduced parent action in \((D-1)\) dimensions

\[
S = -\frac{1}{2} \int d^4 x \left\{ Y^{[ab]} \left( \partial_a e_{bc} - \partial_b e_{ac} \right) - Y^{[ab]} Y^{[bc]} + \frac{1}{D-2} Y a Y^a 
+ V^{[ab]} (\partial_a B_b - \partial_b B_a) + 2 W^{ac} a_{ac} + 2 \mu W^{ac} e_{ac} - 2V^{[ab]} W^{ab} - W_{bc} W^{cb} 
+ 2Z^a \partial_a S - 2 \mu Z^a B_a + \frac{2}{D-2} Z a Y^a - \frac{D-3}{D-2} Z a Z^a + \frac{1}{D-2} W^2 \right\},
\] (50)

where \( W = W^a_a \). This parent action contains \( D^2 (D+1)/2 \) fields and \( D(D+1)/2 \) arbitrary functions to be gauge fixed. After the gauge fixing \( D (D^2-1)/2 \) variables remain. Going from the \( D (D^2-1)/2 \) remaining variables to the final \( D(D-3)/2 \) degrees of freedom requires the elimination via equations of motion of some of the remaining variables, which act as auxiliary fields. The gauge fixed Lagrangians are equivalent in the sense that all of them are in a gauge orbit, but the subsequent elimination of auxiliary fields depends on the gauge fixing and breaks this equivalence.

According to the gauge transformations (49) and (45), the fields \( A_a, B_a \) and \( S \) are pure gauge fields and can be completely fixed by an adequate choice of \( \omega_a, \xi_a \) and \( \xi \). With this partial gauge fixing in the action (50), \( W_{bc}, V_a \) and \( Z_a \) are purely algebraic fields, and thus all the dynamics is contained in the fields \( Y^{[ab]} \) and \( e_{ac} \). The remaining gauge symmetry, related to \( \omega^{[ab]} \), can be used either to set zero the antisymmetric part of \( e_{ac} \), in which case \( V^{[ab]} \) becomes a Lagrange multiplier for \( W_{bc} \), or to fix \( V^{[ab]} \), in which case \( e_{ac} \) has no definite symmetry. These two possibilities are considered in the following section.

IV. GAUGE FIXING AND AUXILIARY VARIABLE ELIMINATION IN THE PARENT ACTION

Gauge invariance is preserved by the above dimensional reduction. In fact, we have explicitly verified that the action (50) is invariant under the full set of gauge transformations (48)-(49). In this sense, the action (50) is of the Stueckelberg type, being of similar character than those obtained in Refs. [12, 13]. In the following we explore the two gauge fixings mentioned at the end of the preceding section, followed by the subsequent elimination of auxiliary variables.

A. GAUGE FIXING LEADING TO A PARENT ACTION WITH SYMMETRICAL \( e^{(bc)} \)

In this case we fix the gauge by choosing the infinitesimal parameters \( \omega^{[ab]}, \omega_a, \xi_a, \xi \) as follows. We have the transformations

\[ \tilde{e}^{[ab]} = e^{[ab]} + \omega^{[ab]}, \] (51)

\[ \tilde{S} = S + 2\mu \xi, \] (52)

\[ \tilde{A}_a = A_a + \omega_a + \partial_a \xi - \mu \xi_a, \] (53)

\[ \tilde{B}_a = B_a - \omega_a + \partial_a \xi - \mu \xi_a, \] (54)

where we are temporarily denoting the gauge transformed fields by a bar. We take \( \omega^{[ab]} \) such that \( \tilde{e}^{[ab]} = 0 \), i.e. only the symmetric part \( e^{(ab)} \) survives. Besides, we choose the remaining parameters in such a way that

\[ \tilde{S} = \tilde{A}_a = \tilde{B}_a = 0. \] (55)

This can be done by taking

\[ \xi = -\frac{1}{2\mu} S, \quad \omega_a = \frac{1}{2} (B_a - A_a), \quad \xi_a = \frac{1}{2\mu} (A_a + B_a) - \frac{1}{4\mu^2} \partial_a S. \] (56)
Thus the gauge fixed parent action becomes
\[
S = -\frac{1}{2} \int d^4x \left\{ Y^{c[ab]} (\partial_a e_{bc} - \partial_b e_{ac}) + 2\mu W^{ac} e_{(ac)} \\
- Y^{c[a]b} Y^{b[ac]} + \frac{1}{D-2} Y_a Y^a - W_{bc} W^{cb} + \frac{1}{D-2} W^2 - \frac{D-3}{D-2} Z_a Z^a \\
- 2W^{ab} V_{[ab]} + \frac{2}{D-2} Z_a Y^a \right\} ,
\]
(57)
where the bars of the gauge transformed fields have been dropped. Here \( V_{[ac]} \) acts as a Lagrange multiplier that produces the constraint
\[
W^{[ac]} = 0 ,
\]
(58)
which is immediately implemented by just leaving the symmetric part of \( W^{ab} , W^{(ab)} \), in the action. We still have some auxiliary fields that can be eliminated from the action. They are \( W^{(ab)} \) itself and \( Z_a \), which are algebraically determined by their equations of motion
\[
Z^a = \frac{1}{(D-3)} Y^a ,
\]
(59)
\[
W_{(bc)} = \mu (e_{(bc)} - \eta_{bc} e) , \quad W = -(D-2) \mu e.
\]
(60)
Substituting in (57), our final expression for the \( (D-1) \)-dimensionally reduced massive parent action is
\[
S = -\frac{1}{2} \int d^4x \left\{ Y^{c[ab]} (\partial_a e_{bc} - \partial_b e_{ac}) - Y_{[ab]} Y^{b[ac]} + \frac{1}{D-3} Y_a Y^a + \mu^2 (e_{(bc)} e_{(bc)} - e^2) \right\} ,
\]
(61)
with \( e = e_{(ab)} \eta^{ab} \). By eliminating \( Y^{(ab)c} \) we recover the FP action, and the elimination of \( e_{(bc)} \) leads to a dual action of the form discussed in Ref. [13] for \( D = 4 \) dimensions.

B. GAUGE FIXING LEADING TO A PARENT ACTION WITH A NON-SYMMETRICAL \( e_{bc} \)

Next we apply a gauge fixing partially similar to the one of the preceding section. We still fix the gauge in such a way that
\[
\tilde{S} = 0 = \tilde{A}_a = 0 = \tilde{B}_a ,
\]
(62)
but the gauge freedom in \( \omega^{[ab]} \) is used to set
\[
V^{[ab]} = 0 ,
\]
instead of \( e_{[bc]} = 0 \), and thus \( e_{bc} \) have no definite symmetry. The parent action results in
\[
S = -\frac{1}{2} \int d^4x \left\{ Y^{c[ab]} (\partial_a e_{bc} - \partial_b e_{ac}) - Y_{[ab]} Y^{b[ac]} + \frac{1}{D-2} Y_a Y^a \\
+ 2\mu W^{ac} e_{ac} - W_{bc} W^{cb} + \frac{1}{D-2} W^2 \\
+ \frac{2}{D-2} Z_a Y^a - \frac{D-3}{D-2} Z_a Z^a \right\} .
\]
(63)
Note that \( W^{bc} \) is not constrained to have a definite symmetry. As before, we eliminate \( Z_a \) and \( W^{bc} \) using the corresponding equations of motion. The case of \( Z_a \) is the same as in the previous section so that we obtain
\[
S = -\frac{1}{2} \int d^4x \left\{ Y^{c[ab]} (\partial_a e_{bc} - \partial_b e_{ac}) - Y_{[ab]} Y^{b[ac]} + \frac{1}{D-3} Y_a Y^a \\
+ 2\mu W^{ac} e_{ac} - W_{bc} W^{cb} + \frac{1}{D-2} W^2 \right\} ,
\]
(64)
Finally we get \[ W_{cb} = \mu (\epsilon_{bc} - \eta_{bc} \epsilon), \quad W = -(D - 2) \mu \epsilon. \]

Finally we get
\[
S = \frac{1}{2} \int d^4x \left[ Y^{[ab]} (\partial_a \epsilon_{bc} - \partial_b \epsilon_{ac}) - Y_{e[ab]} Y^{bc[ac]} + \frac{1}{D-3} Y_a Y_a - \mu^2 (\epsilon_{bc} \epsilon_{cb} - \epsilon^2) \right].
\]

as the final parent action in this sequence of gauge fixings and field eliminations, which is analogous to the one obtained in the preceding section, but with \( \epsilon_{bc} \) without a definite symmetry. This is precisely the action obtained in Ref. [7].

The gauge fixed actions [57] and [63] are equivalent in the usual sense of gauge theories, but in each case the additional elimination of auxiliary variables follows a different pattern. For this reason, although both parent actions lead to the same action for \( \epsilon_{\{bc\}} \), after eliminating \( Y^{[ab]} \), they yield different dual theories after eliminating either \( \epsilon_{\{bc\}} \) or \( \epsilon_{bc} \). The case discussed in this subsection reproduces the Curtright-Freund [1] dual theory when restricted to \( D = 4 \).

V. DUAL THEORIES

In this section we show that the two sequences of gauge fixings and field eliminations proposed above lead to the standard FP theory on one hand, but to completely different dual actions on the other. Once we have obtained the massive parent action from dimensional reduction we set \((D - 1)\) to \( D \) and relabel the tensor indices with capital letters.

A. THE CASE OF A SYMMETRICAL \( \epsilon_{\{BC\}} \)

In a flat \( D \)-dimensional space-time we take
\[
S = \frac{1}{2} \int d^Dx \left[ Y^{C[AB]} (\partial_A \epsilon_{BC} - \partial_B \epsilon_{AC}) + Y_{C[AB]} Y^{B[AC]} - \frac{1}{D - 2} Y_A Y_A - \mu^2 (\epsilon_{AB} \epsilon^{(AB)} - \epsilon^2) \right],
\]
as our parent action. Here the fields are \( \epsilon_{\{BC\}} = \epsilon_{(CB)} \) and \( Y^{C[AB]} = -Y^{C[B]} \), which have \( D(D + 1)/2 \) and \( D^2(D - 1)/2 \) components respectively.

Eliminating \( Y^{C[AB]} \) using its Euler-Lagrange equations
\[
Y_{C[AB]} = -(\partial_A \epsilon_{BC} - \partial_B \epsilon_{AC}) + (\partial_A e - \partial_M e_{AM}) \eta_{BC} - (\partial_B e - \partial_M e_{BM}) \eta_{AC}, \quad Y_A = (D - 2) (\partial_A e - \partial_B e_{AB}),
\]
yields finally to
\[
S = \frac{1}{2} \int d^D x \left[ -\partial_C e_{BA} \partial_C e^{BA} + \partial_C e_C \partial_C e - 2 \partial_M e_N e_{NM} + 2 \partial_M e_{MA} \partial_N e^{NA} - \mu^2 (e_{AB} e^{AB} - \epsilon^2) \right],
\]
which is precisely the FP action in \( D \) dimensions.

To obtain the dual action we eliminate \( \epsilon^{(BA)} \) from its equations of motion obtained from [66]. It is convenient to introduce the decomposition
\[
Y_{R[PQ]} = C_{R[PQ]} + A_{[PQR]},
\]
where the field \( A_{[PQR]} \), which has \( D(D - 1)(D - 2)/6 \) independent components, is completely antisymmetric in all indices, while \( C_{R[PQ]} \) satisfies the cyclic identity
\[
C_{R[PQ]} + C_{P[QR]} + C_{Q[RP]} = 0, \quad \leftrightarrow \quad C_{C[AB]} \epsilon^{ABCN_1...N_D} = 0.
\]

This splitting works because the number of constraints arising from the cyclic identity is precisely \( D(D - 1)(D - 2)/6 \).
In terms of this new field, the action (60) results

\[ S = \frac{1}{2} \int d^Dx \left[ (\partial^C e^{BA} - \partial^B e^{CA}) C_{A[C}B] + \frac{1}{2} C_{C[AB]} C^{C[AB]} - A_{[ABC]} A^{[ABC]} - \frac{1}{(D - 2)} C_A C^A - \mu^2 (e_{AB} e^{AB} - \epsilon^2) \right], \]

where \( C^A = C^B_{[AB]} \). In the above we have used the cyclic identity to rewrite the quadratic terms in \( C_{C[AB]} C^{C[AB]} \) in the form \( C_{C[AB]} C^{C[AB]} \). The field \( A_{[ABC]} \) decouples, leading to \( A_{[ABC]} = 0 \) in virtue of its equations of motion. In order to make future contact with Refs. [15] [16] we introduce the Hodge-dual of \( C^{C[AB]} \)

\[ T_P^{[Q_1 Q_2 ... Q_{D-2}]} = \frac{1}{2} C_P^{[AB]} \epsilon_{ABQ_1 Q_2 ... Q_{D-2}}, \]

which is a tensor of rank \((D - 1)\) completely antisymmetric in its last \((D - 2)\) indices. The resulting action corresponding to the field \( T_P^{[Q_1 Q_2 ... Q_{D-2}]} \) will be taken as the dual version of the original FP formulation. We can invert (73) obtaining

\[ C_P^{[AB]} = - \frac{1}{(D - 2)!} T_P^{[Q_1 Q_2 ... Q_{D-2}]} \epsilon_{Q_1 Q_2 ... Q_{D-2} AB}, \]

\[ C^A = - \frac{1}{(D - 2)!} \epsilon^{Q_1 Q_2 ... Q_{D-2} A S T} S_{[Q_1 Q_2 ... Q_{D-2}]} \]

Notice that the cyclic identity of \( C_P^{[AB]} \) leads to the traceless condition

\[ T_P^{[P Q_1 Q_2 ... Q_{D-3}]} = 0. \]

Let us remark that the kinetic part of the action for the field \( T_P^{[Q_1 Q_2 ... Q_{D-2}]} \) will arise from the terms containing \( \epsilon_{(AB)} \) in (72), while the corresponding mass terms are contained in the remaining pieces with the field \( C_{A[C}B] \). In other words \( S = S_{KIN} + S_{MASS} \), with

\[ S_{KIN}(\epsilon, T) = \frac{1}{2} \int d^Dx \left[ 2 \partial^C e^{(BA)} C_{[CB]A} - \mu^2 \left[ \epsilon_{(AB)} e^{(AB)} - \epsilon^2 \right] \right], \]

\[ S_{MASS}(T) = \frac{1}{2} \int d^Dx \left[ \frac{1}{2} C_{C[AB]} C^{C[AB]} - \frac{1}{D - 2} C_A C^A \right]. \]

The mass contribution produces

\[ S_{MASS}(T) = - \frac{1}{2(D - 2)!} \int d^Dx \left[ \frac{D - 3}{D - 2} T_A^{[Q_1 Q_2 ... Q_{D-2}]} T^{A[Q_1 Q_2 ... Q_{D-2}]} + T_A^{[BM_1 M_2 ... M_{D-2}]} T_B^{[A M_1 M_2 ... M_{D-2}]} \right]. \]

The calculation of the kinetic contribution requires the equations of motion for \( \epsilon_{(AB)} \). Here it is convenient to introduce the field strength \( F^{[Q_1 Q_2 ... Q_{D-2} Q_{D-1}]} \), which is a tensor of rank \( D \), associated with the potential \( T^{A[Q_1 Q_2 ... Q_{D-2}]} \), given by

\[ F^{A[Q_1 Q_2 ... Q_{D-2} Q_{D-1}]} = \frac{1}{(D - 2)!} \delta^{[Q_1 Q_2 ... Q_{D-2} Q_{D-1}]} \partial^A T^{A[Q_1 Q_2 ... Q_{D-2} Q_{D-1}]} , \]

which is completely antisymmetric with respect to the \((D - 1)\) indices inside the square brackets. Here \( \delta^{[A_1 A_2 ... A_{D-2} A_{D-1}]} \) denotes the completely antisymmetric delta symbol. In this way \( F^{A[Q_1 Q_2 ... Q_{D-2} Q_{D-1}]} \) satisfies

\[ \epsilon_{Q_1 Q_2 ... Q_{D-2} Q_{D-1} B} F^{A[Q_{D-1} Q_1 Q_2 ... Q_{D-2}]} = (D - 1)! \epsilon_{Q_1 Q_2 ... Q_{D-2} Q_{D-1} B} \partial^{Q_{D-1}} T^{A[Q_1 Q_2 ... Q_{D-2}]} \]

In terms of the field strength the equations of motion for \( \epsilon_{AB} \) lead to

\[ \epsilon_{(AB)} = \frac{1}{2 \mu^2 (D - 1)!} \left[ \epsilon_{Q_1 Q_2 ... Q_{D-2} Q_{D-1} B} F^{A[Q_{D-1} Q_1 Q_2 ... Q_{D-2}]} + \epsilon_{Q_1 Q_2 ... Q_{D-2} Q_{D-1} A} F^{B[Q_{D-1} Q_1 Q_2 ... Q_{D-2}]} \right. \]

\[ \left. - \frac{2}{(D - 1)} \epsilon_{Q_1 Q_2 ... Q_{D-2} Q_{D-1} E} F^{E[Q_{D-1} Q_1 Q_2 ... Q_{D-2}]} \eta_{AB} \right], \]
Using the field strength we can rewrite the coupling term in (77) as
\[
\int d^D x \left[ \delta^C e^{BA} C_{A|CB} \right] = \frac{1}{(D-1)!} \int d^D x e^B e^{A|Q_1Q_2...Q_{D-2}Q_{D-1}|B} F^{|Q_1Q_2...Q_{D-2}Q_{D-1}|}. \tag{84}
\]

The expressions (82) and (83) imply that
\[
\mu^2 (e_{\{AB\}} - \eta_{AB} e) = \frac{1}{2(D-1)!} (Q_1Q_2...Q_{D-2}Q_{D-1}) F^{|Q_1Q_2...Q_{D-2}Q_{D-1}|} + (A \leftrightarrow B), \tag{85}
\]
which allows us to rewrite the kinetic piece of the action (77) in the convenient form
\[
S_{KIN} = \frac{\mu^2}{2} \int d^D x \left( e^{(AB)} - e^2 \right), \tag{86}
\]
where we finally substitute the expressions of \( e_{\{AB\}} \) as functions of \( F^{|Q_1Q_2...Q_{D-2}Q_{D-1}|} \). The result is
\[
S_{KIN} = - \frac{(D-2)}{2\mu^2 (D-1)!} \int d^D x \left\{ F^{|Q_1Q_2...Q_{D-2}Q_{D-1}|} F^{|Q_1Q_2...Q_{D-2}Q_{D-1}|} - \frac{1}{2} \frac{(D-1)^2}{(D-2)} F^{|AQ_1...Q_{D-3}Q_{D-2}|} F^{|AQ_1...Q_{D-3}Q_{D-2}|} + \frac{(D-1)}{(D-2)} F^{|M[NM_3...M_D]|} F^{|M[NM_3...M_D]|} \right\}. \tag{87}
\]

The final action, dual to FP in arbitrary dimensions, is then
\[
S(T) = - \int d^D x \left\{ (D-2) \frac{(D-1)^2}{(D-2)} F^{|Q_1Q_2...Q_{D-2}Q_{D-1}|} F^{|Q_1Q_2...Q_{D-2}Q_{D-1}|} - \frac{1}{2} \frac{(D-1)^2}{(D-2)} F^{|AQ_1...Q_{D-3}Q_{D-2}|} F^{|AQ_1...Q_{D-3}Q_{D-2}|} + \mu^2 \left\{ \frac{(D-3)}{(D-2)} T^{|Q_1Q_2...Q_{D-2}|} T^{|Q_1Q_2...Q_{D-2}|} + T^{|BM_1M_2...M_{D-3}|} T^{|BM_1M_2...M_{D-3}|} \right\}, \tag{88}
\]
where the original action has been adequately rescaled. Setting \( D = 4 \) in the above action leads to the case \( a = e^2 \) of the general Lagrangian (61) in Ref. [16].

**B. THE CASE OF A NON-SYMMETRICAL \( \epsilon_{AC} \)**

This case is discussed in full detail in Ref. [17] so that we only recall the results here. The starting point here is the parent action
\[
S = - \frac{1}{2} \int d^D x \left\{ Y^{|A|B|C|} \partial_B \epsilon_{AC} - \partial_A \epsilon_{BC} \right\} - Y^{|A|B|C|} Y^{|A|B|C|} + \frac{1}{(D-2)} Y^{|A|Y^{|A|} + \mu^2 (\epsilon_{AB} e^{BA} - e^2). \tag{89}
\]
Here the basic fields are the non-symmetrical \( \epsilon_{BC} \) together with \( Y^{|A|B|C|} = -Y^{[BA]|C} \), with \( D^2 \) and \( D^2 (D-1)/2 \) independent components respectively. As shown in reference [2], the above Lagrangian in the massless case leads to the FP action, in terms of \( \epsilon_{BC} \) only, after \( Y^{|A|B|C|} \) is eliminated via the equations of motion. The massive case is completely analogous because the equations of motion for \( Y^{|A|B|C|} \) do not involve the mass term [17]. Thus, the kinetic energy piece of the action in terms of \( \epsilon_{AB} \) involves the antisymmetric part \( \epsilon_{[AB]} \) only as a total derivative. The mass term contributes with a term proportional to \( \epsilon_{[AB]} e^{[AB]} \), which leads to the equation of motion \( \epsilon_{[AB]} = 0 \). It is rather remarkable that the FP formulation is recovered in spite that \( \epsilon_{AB} \) is non-symmetrical.
To obtain the dual description we eliminate \( e^{BA} \) using the equations of motion obtained from the action \((93)\), leading to the following action for \( Y_{[AB]C} \)

\[
\mu^2 S = \int d^D x \left[ \partial_A Y^{C[AB]} \partial^E Y_{B[EC]} - \frac{1}{D-1} (\partial_A Y^A)^2 + \mu^2 \left( Y_C^{[AB]} Y^{B[AC]} - \frac{1}{D-2} Y_A Y^A \right) \right].
\]

(90)

Next we implement the change of variables

\[
Y^{C[AB]} = \tilde{w}^{C[AB]} + \frac{1}{(D-1)} (\eta^{CB} Y^A - \eta^{CA} Y^B),
\]

(91)

where \( \tilde{w}^{C[AB]} \) has a null trace, \( \tilde{w}^B = \tilde{w}_A^{[AB]} = 0 \), and obtain

\[
S = \frac{1}{2} \int d^D x \left[ \partial_A \tilde{w}^{C[AB]} \partial^E \tilde{w}_{B[CE]} + \mu^2 \left( \tilde{w}^{C[BA]} \tilde{w}_{B[CA]} - \frac{1}{(D-1)(D-2)} Y^A Y_A \right) \right],
\]

(92)

which clearly shows that the trace of \( Y^{C[BA]} \) is an irrelevant variable that can be eliminated from the Lagrangian using its equation of motion. Thus we finally get

\[
S = \frac{1}{2} \int d^D x \left( \partial_A \tilde{w}^{C[AB]} \partial^E \tilde{w}_{B[CE]} + \mu^2 \tilde{w}^{C[AB]} \tilde{w}_{A[C]B} \right).
\]

(93)

Now we introduce the Hodge-dual of \( \tilde{w}^{C[AB]} \)

\[
T^{P[Q_1Q_2...Q_{D-2}]}_{[AB]} = \frac{1}{2} \tilde{w}^{P[AB]} \epsilon_{ABQ_1Q_2...Q_{D-2}},
\]

(94)

which is a dimension-dependent tensor of rank \((D-1)\) completely antisymmetric in its last \((D-2)\) indices. The resulting action corresponding to the field \( T^{P[Q_1Q_2...Q_{D-2}]}_{[AB]} \) will be taken as the dual version of the original FP formulation. Finally we obtain

\[
S(T) = - \int d^D x \left\{ \frac{1}{(D-1)} F_B^{[AQ_1...Q_{D-2}]} F_C^{[AQ_1...Q_{D-2}] - F_A^{[AQ_1...Q_{D-2}]} F_B^{[BQ_1...Q_{D-2}]} \right. \\
- \mu^2 \left[ T^{P[Q_1...Q_{D-2}]}_{[AB]} T^{P[Q_1...Q_{D-2}]}_{[BC]} - (D-2) T_C^{[CQ_1...Q_{D-3}]} T^{[BQ_2...Q_{D-3}]}_{[AB]} \right] \right\},
\]

(95)

with an adequate rescaling of the original action. The field strength \( F_B^{[AQ_1...Q_{D-2}]} \) has been already introduced in Eq. \((81)\). The field \( T^{[BQ_1...Q_{D-2}]}_{[AB]} \) satisfies the cyclic condition

\[
\epsilon^{ASQ_1...Q_{D-2}} T^{S[Q_1Q_2...Q_{D-2}]} = 0.
\]

(96)

The action \((83)\) reduces to the Curtright-Freund action in four dimensions.

The equations of motion are

\[
\left[ (D-2)! \delta_{[M_1...M_{D-1}]}^{[A_1...A_{D-1}]} \right] \delta_C^B - \delta_{[BQ_2...Q_{D-1}]}^{[M_1...M_{D-1}]} \delta_{[CQ_2...Q_{D-1}]}^{[M_1...M_{D-1}]} \right] \partial^{A_1} \partial_{M_1} T^{C_{[M_2...M_{D-1}]} [M_2...M_{D-1}]} - \mu^2 [(D-2)!]^2 \left[ T^{B_{[A_2...A_{D-1}]} - \frac{1}{(D-3)!} \delta_{[BQ_3...Q_{D-1}]}^{[M_2...M_{D-1}]} T^{C_{[CM_3...M_{D-1}} [CM_3...M_{D-1}]} = 0
\]

(97)

and they imply that the field \( T^{B_{[A_2...A_{D-1}]} \) satisfies the additional constraints \((17)\)

\[
\partial^{D} T^{B}_{[A_3...A_{D-1}]} = 0,
\]

(98)

\[
\partial^{D} T^{B}_{[A_3...A_{D-1}]} = 0,
\]

(99)

\[
\partial^{B} T^{B}_{[A_2...A_{D-1}]} = 0.
\]

(100)

After implementing these constraints the equation of motion reduces to its simplest form

\[
(\partial^2 - \mu^2) T^{B}_{[A_2...A_{D-1}]} = 0.
\]

(101)

The field \( T^{B}_{[A_2...A_{D-1}]} \) in \( D \) dimensions has \( I = D^2 (D-1)/2 \) components, but the constraints \((96,98,100)\) manage to leave just \( \frac{D}{2} (D-1) - 1 \) independent degrees of freedom, which indeed is the same number obtained for \( e_{[AB]} \) in the FP formulation.
VI. FINAL COMMENTS

In this paper we have explored a dimensional reduction from $D$ to $(D-1)$ dimensions in order to produce dual theories for massive spin two fields using the parent action method. We started from the corresponding massless action in the higher dimension and generated the mass parameter via dimensional reduction, thus obtaining a lower dimension massive parent action. The massive parent theory inherits all the gauge symmetries of the parent massless action, so that it becomes a Stueckelberg-like action in dimension $(D-1)$. Although this parent action contains several fields, the degrees of freedom are only contained in two of them, $Y_{C}^{[AB]}$ and $e_{AB}$. Even so, the existence of alternative gauge choices together with alternative auxiliary field eliminations via their equations of motion allowed us to identify two kinds of $(D-1)$ dimensional massive parent actions, corresponding either to a symmetric or a non-symmetric standard spin two field $e_{AB}$. The true degrees of freedom for the resulting Fierz-Pauli theory in terms of the field $e_{AB}$ are contained only in the symmetric piece $e_{(AB)}$, in analogy to the massless case [5, 7]. Nevertheless, important differences arose in the corresponding dual theories. In both cases we constructed the dual theory in terms of a mixed symmetry field $T_{A[B_{1},B_{2},...,B_{n-2}]}$ which final action is written without the use of auxiliary fields. The general results are given in Eqs. (11) and (13), respectively. Let us emphasize that in both cases the dual theory to Fierz-Pauli is constructed in terms of the $(D-1)$-rank tensor $T_{A[B_{1},B_{2},...,B_{n-2}]}$, but subjected either to a traceless condition or to a cyclic identity. Let us recall that in the massless case the dual to the Fierz-Pauli field $e_{(AB)}$ is the $(D-2)$-rank tensor $T_{A_{1}[Q_{2},...,Q_{D-1}]}$ [3]. Notice that this result is analogous to the well known one involving $p$-forms, where the dual fields are a $(D-p-1)$-form for the massive case and a $(D-p-2)$-form for the massless case. In the case of $D=4$ the symmetric case leads to a particular family of dual actions previously found in Ref. [10]. The non-symmetrical case reproduces the dual action proposed by Curtright and Freund in Ref. [1]. This constitutes the first proof that this action is indeed dual to Fierz-Pauli. Finally, as a consistency check of our procedure, we have considered in the Appendix the case $\mu=0$ in the $D=4$ parent action [50]. In this case, via adequate gauge fixings and field eliminations, we recover the sum of the free spin two, one and zero massless actions as expected, making up the original five degrees of freedom we started with. One of the choices provides an unexpected Stueckelberg-like formulation of the massless spin one field.

APPENDIX A: A FOUR DIMENSIONAL EXAMPLE

Let us consider the massless ($\mu=0$) parent action [50] in four dimensions. Since we started from the massless spin two field in five dimensions, which has five independent degrees of freedom we should be able to select the corresponding gauge fixings and field eliminations in such a way to recover the description of uncoupled massless fields of spins 2, 1 and 0, thus providing an alternative way of describing the five original degrees of freedom. In the process we will be lead to a rather unexpected way of presenting an action for the massless spin 1 field, in terms of symmetric tensors.

We start from

$$S = \frac{1}{2} \int d^{4}x \left\{ Y^{[ab]} \left( \partial_{a}e_{bc} - \partial_{b}e_{ac} \right) - Y_{c[ab]}Y^{[bc]} + \frac{1}{3} Y_{a}Y^{a} + \frac{2}{3} Z_{a}Z^{a} \right. \right. $$

$$+ \left. V^{[ab]} \left( \partial_{a}B_{b} - \partial_{b}B_{a} \right) - 2V^{[ab]} W_{ab} + 2W^{bc} \partial_{b}A_{c} - \frac{2}{3} Z_{a}Z^{a} + 2Z^{b} \partial_{b}S - W^{bc}W_{cb} + \frac{1}{3} W^{2} \right\} .$$  (A1)

Here $W^{b}_{b} = W$ and the indices $a, b, c, ...$ run from 0 to 3. We fix the gauge parameter $\omega_{ab}$ to eliminate the antisymmetric part of $e_{bc}$, which yields

$$S = \frac{1}{2} \int d^{4}x \left\{ Y^{[ab]} \left( \partial_{a}e_{bc} - \partial_{b}e_{ac} \right) - Y_{c[ab]}Y^{[bc]} + \frac{1}{3} Y_{a}Y^{a} \right. \right. $$

$$+ \left. V^{[ab]} \left( \partial_{a}B_{b} - \partial_{b}B_{a} \right) - 2W_{ac}V^{[ac]} - W^{bc}W_{cb} + \frac{1}{3} W^{b}W^{c} + 2W^{bc} \partial_{b}A_{c} + \frac{2}{3} Z_{a}Y^{a} + 2Z^{b} \partial_{b}S - \frac{2}{3} Z_{a}Z^{a} \right\} .$$  (A2)

Next we eliminate $Z^{a}$ from the corresponding equation of motion

$$Z^{a} = \frac{1}{2} \left( Y^{a} + 3\partial^{a}S \right),$$  (A3)
obtaining

\[
S = \frac{1}{2} \int d^4x \left\{ Y^{[ab]} \left( \partial_a e_{\{bc\}} - \partial_b e_{\{ac\}} \right) - Y_{[ab]} Y^{[ac]} + \frac{1}{2} Y_a Y^a + \frac{3}{2} \partial^a S \partial_a S \\
+ V^{[ab]} \left( \partial_a B_b - \partial_b B_a \right) + 2W^{bc} \partial_b A_c - 2V^{[ac]} W_{ac} - W^{bc} W_{cb} + \frac{1}{3} W^2 \right\}.
\]

(A4)

Our next step is to redefine

\[ e_{\{bc\}} \to e_{\{bc\}} - \frac{1}{2} \eta_{bc} S, \]

in such a way to eliminate the crossed term \( Y^a \partial_a S \) so that (A3) reduces to

\[
S = \frac{1}{2} \int d^4x \left\{ Y^{[ab]} \left( \partial_a e_{\{bc\}} - \partial_b e_{\{ac\}} \right) - Y_{[ab]} Y^{[ac]} + \frac{1}{2} Y_a Y^a + \frac{3}{2} \partial^a S \partial_a S \\
+ V^{[ab]} \left( \partial_a B_b - \partial_b B_a \right) - 2V^{[ac]} W_{ac} - W^{bc} W_{cb} + \frac{1}{3} W^2 - 2W^{bc} \partial_b A_c \right\},
\]

(A5)

which already shows the decoupling of the three sectors of the theory. The above action is invariant under the following gauge transformations, which basically include only the vector sector

\[
\delta A_a = \theta_a, \quad \delta B_a = -\theta_a, \quad \delta V^{[ab]} = 0, \\
\delta W_{ab} = \partial_b \theta_c - \eta_{ac} \partial_a \theta^c, \quad \delta W = -36 \partial_a \theta^a \\
\delta e_{\{bc\}} = 0, \quad \delta Y^{[ab]c} = 0, \quad \delta S = 0.
\]

Next we fix the gauge in two alternative forms that yield the standard massless action for the spin one field.

1. **CASE I**

We choose the parameter \( \theta_a \) in such a way that \( A_a = 0 \), leading to

\[
S = \frac{1}{2} \int d^4x \left\{ Y^{[ab]} \left( \partial_a e_{\{bc\}} - \partial_b e_{\{ac\}} \right) - Y_{[ab]} Y^{[ac]} + \frac{1}{2} Y_a Y^a + \frac{3}{2} \partial^a S \partial_a S \\
+ V^{[ab]} \left( \partial_a B_b - \partial_b B_a - 2W_{ab} \right) - W^{bc} W_{cb} + \frac{1}{3} W^2 \right\}.
\]

(A6)

Here \( V^{[ab]} \) is a Lagrange multiplier, which implies that

\[
W_{ab} = \frac{1}{2} \left( \partial_a B_b - \partial_b B_a \right) \to W = 0.
\]

Substituting in (A6) we recover the standard contribution to the massless spin one field, up to a normalization factor.

\[
S = \frac{1}{2} \int d^4x \left[ Y^{[ab]} \left( \partial_a e_{\{bc\}} - \partial_b e_{\{ac\}} \right) - Y_{[ab]} Y^{[ac]} + \frac{1}{2} Y_a Y^a + \frac{3}{2} \partial^a S \partial_a S + \frac{1}{4} \left( \partial_a B_b - \partial_b B_a \right)^2 \right].
\]

(A7)

2. **CASE II**

Now we fix the parameter \( \theta_a \) such that \( B_a = 0 \) obtaining

\[
S = \frac{1}{2} \int d^4x \left\{ Y^{[ab]} \left( \partial_a e_{\{bc\}} - \partial_b e_{\{ac\}} \right) - Y_{[ab]} Y^{[ac]} + \frac{1}{2} Y_a Y^a + \frac{3}{2} \partial^a S \partial_a S \\
- 2V^{[ac]} W_{ac} - W^{bc} W_{cb} + \frac{1}{3} W^2 - 2W^{bc} \partial_b A_c \right\}.
\]

(A8)
Now the Lagrange multiplier $V^{[ac]}$ implies that $W_{ac} = W_{\{ac\}}$ is symmetrical, yielding

$$S = \frac{1}{2} \int d^4 x \left\{ Y^{c[ab]} \left( \partial_a e_{\{bc\}} - \partial_b e_{\{ac\}} \right) - Y_{c[ab]} Y^{b[ac]} + \frac{1}{2} Y_a Y^a + \frac{3}{2} \partial^a S \partial_a S \right. - W^{(bc)} W_{\{cb\}} + \frac{1}{3} W^2 - W^{(bc)} \left( \partial_b A_c + \partial_c A_b \right) \} \right\}. \quad (A9)$$

Notice that the second line in the above equation must provide an alternative way of presenting the action for a massless spin one field, even though it is written in terms of symmetrical fields. We can verify this statement just by eliminating the field $W_{\{bc\}}$. The corresponding equation of motion produces

$$W_{\{bc\}} = -\frac{1}{2} (\partial_b A_c + \partial_c A_b) + \eta_{bc} \partial_a A^a \quad (A10)$$

and the substitution in $(A9)$ leads indeed to the expected action

$$S = \frac{1}{2} \int d^4 x \left\{ Y^{c[ab]} \left( \partial_a e_{\{bc\}} - \partial_b e_{\{ac\}} \right) - Y_{c[ab]} Y^{b[ac]} + \frac{1}{2} Y_a Y^a + \frac{3}{2} \partial^a S \partial_a S + \frac{1}{4} \left( \partial_b A_c - \partial_c A_b \right)^2 \right\}. \quad (A11)$$

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