BOUNDEDNESS OF OPERATORS IN BILATERAL GRAND LEBESGUE SPACES

with exact and weakly exact constant calculation.

E.Ostrovsky\textsuperscript{a}, L.Sirota\textsuperscript{b}

\textsuperscript{a} Corresponding Author. Department of Mathematics and computer science, Bar-Ilan University, 84105, Ramat Gan, Israel.

E - mail: eugostrovsky@list.ru

\textsuperscript{b} Department of Mathematics and computer science. Bar-Ilan University, 84105, Ramat Gan, Israel.

E - mail: sirota@zahav.net.il

Abstract. In this article we investigate an action of some operators (not necessary to be linear or sublinear) in the so-called (Bilateral) Grand Lebesgue Spaces (GLS), in particular, double weight Fourier operators, maximal operators, imbedding operators etc.

We intend to calculate an exact or at least weak exact values for correspondent imbedding constant.

We obtain also interpolation theorems for GLS spaces.

We construct several examples to show the exactness of offered estimations.

In two last sections we introduce anisotropic Grand Lebesgue Spaces, obtain some estimates for Fourier two-weight inequalities and calculate Boyd’s multidimensional indices for this spaces.

\textbf{Key words and phrases}: Grand and ordinary Lebesgue Spaces (GLS), anisotropic spaces, bilateral estimates, exact and asymptotical examples, rearrangement invariant (r.i.) spaces, Fourier transform, weight, Hardy-Littlewood and Sobolev’s imbedding theorems, integrals and series, integral, differential and pseudodifferential operators, fractional derivatives, moment inequalities, equivalent norms, upper and lower estimations.

\textit{Mathematics Subject Classification 2000}. Primary 42Bxx, 4202; Secondary 28A78, 42B08.

1 Introduction. Notations. Problem Statement.

Let \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) be a two measurable spaces with sigma-finite non-trivial measures \(\mu, \nu\). For the measurable real valued functions \(f(x), x \in X, f : X \to\)
$g(y), \ y \in Y, g : Y \to R$ the symbols $|f|_p = |f|_p(X, \mu)$ and correspondingly $|g|_q = |g|_q(Y, \nu)$ will denote the usually $L_p$ and $L_q$ norms:

$$|f|_p = |f|_{L_p}(X, \mu) = \left[ \int_X |f(x)|^p \mu(dx) \right]^{1/p}, \ p \geq 1. \quad (1.1a)$$

$$|g|_q = |g|_{L_q}(Y, \nu) = \left[ \int_Y |g(y)|^q \nu(dy) \right]^{1/q}, \ q \geq 1. \quad (1.1b)$$

Let $U$ be an operator, not necessary to be linear or sublinear, defined on arbitrary measurable simple bounded function $f : X \to R$ with values in the set of measurable functions $g : Y \to R$. We impose the following important condition on the operator $U$.

**Condition A, ”moment inequality”.** There exist an interval $(a, b), 1 \leq a < b \leq \infty$, $(c, d), 1 \leq c < d \leq \infty$, and strictly monotonically continuous function $q = q(p), q : (a, b) \to (c, d)$, for which the operator $U$ is bounded as the operator from the space $L_p(X) = L_p(X, \mu)$ onto the space $L_q(Y) = L_q(Y, \nu)$:

$$|Uf|_{q(p)} \leq K(p) |f|_p, \ \forall p \in (a, b) \Rightarrow K(p) < \infty. \quad (A)$$

We assume hereafter that the interval $(a, b)$ is the maximal interval for which the inequality $(A)$ there holds.

We denote the inverse function to the function $p \to q(p)$ as $r = r(q); r : (c, d) \to (a, b), q(r(p)) = p$.

Further, we understand as the constant $K(p)$ its minimal value, namely: $K(p) = |U| \left( L_p \to L_q(p) \right)$ or equally

$$K(p) = \sup_{f \in L_p(X), f \neq 0} \left[ \frac{|Uf|_{q(p)}}{|f|_p} \right] = |U|_{p \to q(p)}. \quad (A)$$

There are many examples of operators satisfying the condition (inequality) $(A)$ with calculated the exact (or exactly evaluated) value of constant $K(p)$: classical Hardy-Littlewood inequalities [23], with its Bradley’s generalizations [7]; integral [47], [11], chapter 5, section 11, p. 567-580; in particular, convolution operators (regular and singular) [2], oscillating integrals [80], potentials [30], p. 270-271; Sobolev’s imbedding operators with modern generalizations [42], [43]; Poincare-Sobolev’s inequalities [1], [18], [86], maximal operators with Muckenhoupt’s [45] generalization, Hilbert’s transforms [4], chapter 3, Fourier’s transform operators (discrete and continuous) [6], [9], [10], [85], [87]; pseudodifferential (Fourier integral) operators [83] etc.

For instance, let us consider the following integral operator:

$$U_a[f](x) = \int_R |t|^\mu h(x \cdot t)f(t)dt,$$

$$1 \leq p \leq q, \ 1/q = \mu - 1/p, 1/\sigma := 1 + \mu - 2/p.$$
\[ K(p) = \left[ \int_R |t|^{-\sigma(p-1)/p} |h(t)|^{\sigma} \, dt \right]^{1/\sigma}, \]

see [47], p. 222.

Let us concern the so-called potential operators, of a Riesz’s type (non-homogeneous):

\[ P[f](x) = \int_{R^n} a(x) \, b(y) \, |x - y|^{\beta-n} \, f(y) \, dy. \]

Let

\[ \frac{p-1}{p} + \frac{1}{p_0} < 1, \quad \frac{1}{q} + \frac{1}{q_0} < 1, \quad \beta \in (0, n), \]

\[ \frac{1}{p} = \frac{1}{q} + \frac{1}{p_0} - \frac{\beta}{n}. \]

It is known that

\[ |P|(L_p \rightarrow L_q) \leq C \cdot |a|_{p_0} \cdot |b|_{q_0}, \]

see [30].

For the following modification of Fourier transform:

\[ \hat{f}(y) = \int_{R^n} e^{-2\pi ixy} f(x) dx \]

W.Beckner in [2] proved the following estimates with exact constants computations:

\[ |\hat{f}|_{p_1} \leq A(p) \cdot |f|_p, \quad p \in (1, 2], \quad p_1 = p/(p-1), \]

where

\[ A(p) \overset{\text{def}}{=} \left[ \frac{p^{1/p}}{(p_1)^{1/p_1}} \right]^{n/2}; \]

\[ |f \ast g|_r \leq (A(p)A(q)A(r_1))^{n} \cdot |f|_p \cdot |g|_q, \quad 1/r = 1/p + 1/q - 1. \]

S.K.Pichorides in [75] proved the following inequality for the Hilbert’s transform \( H[f] \) with sharp value of constant:

\[ |H[f]|_p \leq \Lambda(p) \cdot |f|_p, \quad 1 < p < \infty, \]

\[ \Lambda(p) = \tan(\pi/(2p)), \quad 1 < p \leq 2; \quad \Lambda(p) = \cot(\pi/(2p)), \quad 2 < p < \infty. \]

Analogous estimates are known for Sobolev’s (Hardy-Littlewood-Sobolev) imbedding theorems, see [82]. We refer here only the so-called fractional Sobolev’s inequality [26], [27]:

\[ |f|_q \leq K_S(p) \cdot \left| \left[ \sqrt{-\Delta} \right]^{s} [f] \right|_p, \]
\[
\left[\sqrt{-\Delta}\right]^s f(y) \overset{def}{=} F^{-1}\left((|x|^s F[f])(x)\right)(y),
\]

\[K_S(p) = \pi^{s/2} \cdot \frac{\Gamma((n-s)/2)}{\Gamma((n+s)/2)} \cdot \left\{ \frac{\Gamma(s)}{\Gamma(n/2)} \right\}^{s/n},\]

\[0 < s < n, \quad 1 < p < n/s, \quad q = \frac{pn}{n - sp}, \quad \Delta[f](x_1, x_2, \ldots, x_n) = \sum_{m=1}^{n} \frac{\partial^2}{\partial x_m^2} f.\]

Next example: pseudodifferential operators, on the other terms: Fourier integral operators. We refer to the book of Taylor M.E. [84], article [46].

Consider for example the pseudodifferential operator \(P(D)\) of a view

\[P(D)[f](x) = \int_{R^n} e^{ix \xi} p(\xi) F[f](\xi) \, d\xi.\]

If

\[\forall k : |k| \leq \text{Ent}(n/2) + 1 \Rightarrow \sup_{R>0} \int_{R<\xi<2R} |\xi|^{|k|D^k P(\xi) |^2 d\xi < \infty,}\]

\(\text{Ent}(z)\) denotes the integer part of a variable \(z\), then \(P(D) : L_p \to L_p, \ p \in (1, \infty)\) and

\[|P(D)|(L_p \to L_p) \leq C \cdot \frac{p^2}{p - 1}.\]

Too modern results are obtained in [14].

Other example: weight inequalities for Fourier transform; we follow B.Muckenhoupt [45].

Denote as usually

\[F[f](y) = (2\pi)^{-n/2} \int_{R^n} e^{ixy} f(x) \, dx.\]

Let \(u = u(y), \ v = v(x)\) be two positive integrable functions (weights). We cite here the inequality of a view:

\[\left[\int_{R^n} |F[f](y)|^q u(y) \, dy\right]^{1/q} \leq K_M(p, q) \left[\int_{R^n} |f(x)|^p v(x) \, dx\right]^{1/p}, \ p, q \in (1, \infty).\]

Suppose here that \(p \leq q; \ the \ alternative \ case \ will \ be \ studied \ further.\)

For instance, let \(p \in (1, 2]\); we denote for some positive constants \(A, B\)

\[I(A, B) = \sup_{r>0} \left\{ \left[\int_{u(y)>B} u(y) \, dy\right] \cdot \left[\int_{v(x)<A} v(x)^{-1/(p-1)} \, dx\right] \right\}^{1/p}.\]

Muckenhoupt in [45] proved in particular that for \(q = p/(p - 1)\)
\[ \left[ \int_{\mathbb{R}^n} |F[f](y)|^q u(y) \, dy \right]^{1/q} \leq \frac{CI(A,B)}{p-1} \cdot \left[ \int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx \right]^{1/p}, \ p, q \in (1, \infty). \]

This inequality may be rewritten as follows.

\[ \left[ \int_{\mathbb{R}^n} |F[f](y)|^q u(y) \, dy \right]^{1/q} \leq \inf_{A,B>0} \frac{CI(A,B)}{p-1} \cdot \left[ \int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx \right]^{1/p}, \ p, q \in (1, \infty). \]

For the classical Calderon-Zygmund singular integral operators \( U \) (and its commutators with vector fields) are known the asymptotically exact up to multiplicative constants of a view

\[ |U|(L_p \to L_p) \leq C \frac{p^2}{p-1}, \ p \in (1, \infty). \]

see [74], [41].

Analogous estimates for different modifications of Fourier integral operators for example for the maximal Fourier operator

\[ |\sup_N S_N[f]|_p \leq C \frac{p^4}{(p-1)^2} |f|_p, \ p \in (1, \infty), \]

where \( S_N[f](x) \) denotes the \( N \) \( \text{-th} \) partial Fourier sum for the function \( f : [0,2\pi] \to \mathbb{R} \) see, e.g. in the book of Reyna [76].

The last considered in this section example belongs to E.M.Stein and G.Weiss [79]. Consider the following singular integral operator \( S \) with homogeneous kernel \( K = K(x, y), \ x, y \in \mathbb{R}^n : \)

\[ S[f](x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \]

where \( K(x, y) \geq 0, \)

\[ \forall \lambda > 0 \Rightarrow K(\lambda x, \lambda y) = \lambda^{-n} K(x, y), \]

\[ \forall Y \in SO(n) \Rightarrow K(Y x, Y y) = K(x, y). \]

Proposition:

\[ |S|(L_p \to L_p) = \int_{\mathbb{R}^n} K(x, e_1) |x|^{-n/p} \, dx, \ e_1 = (1, 0, 0, \ldots, 0). \]

This result generalized classical inequalities of Hardy-Littlewood.

Our aim is extrapolation of the ”moment” inequality (A) on the so-called Grand Lebesgue Spaces (GLS) with exact or at least weak exact constants calculations.
The paper is organized as follows. In the next section we recall used facts about Grand Lebesgue Spaces, formulate and prove the main result of this paper.

In the third section we obtain the GLS boundedness of the so-called maximal transform. The fourth section is devoted to the weight Fourier operators boundedness in GLS spaces.

In the next section we consider the interpolation inequalities for operators acting in GLS spaces. We based here on the classical interpolation results belonging to Riesz-Thorin and Marcinkiewicz.

In the sixth section we generalize obtained results on the “multidimensional” case and prove the exactness of the relation between parameters in the Lebesgue spaces Fourier and convolution weight operators inequalities, by means of a so-called ”dilation method”. The next section is devoted to the generalization of obtained results on the anisotropic spaces; we prove also the exactness of our estimates.

The object of the eighth section is generalization of previous results on the case when the weight functions are arbitrary continuous regular varying.

The last section contains some concluding remarks.

2 Grand Lebesgue Spaces. Main result.

Let \((X, A, \mu)\) be again measurable space with sigma-finite non-trivial measure \(\mu\). We recall in this section for readers conventions some definitions and facts from the theory of GLS spaces.

Recently, see [15], [16], [17], [28], [29], [34],[39], [48], [49] etc. appear so-called Grand Lebesgue Spaces \(GLS = G(\psi) = G\psi = G(\psi; A, B), \ A, B = \text{const}, \ A \geq 1, \ A < B \leq \infty,\) spaces consisting on all the measurable functions \(f : X \to R\) with finite norms

\[
||f||G(\psi) \overset{def}{=} \sup_{p \in (A,B)} \left[ ||f||_p / \psi(p) \right]. \tag{2.1}
\]

Here \(\psi(\cdot)\) is some continuous positive on the open interval \((A, B)\) function such that

\[
\inf_{p \in (A,B)} \psi(p) > 0, \ \psi(p) = \infty, \ p \notin (A, B). \tag{2.2}
\]

We will denote

\[
\text{supp}(\psi) \overset{def}{=} (A, B) = \{p : \psi(p) < \infty, \} \tag{2.3}
\]

The set of all \(\psi\) functions with support \(\text{supp}(\psi) = (A, B)\) will be denoted by \(\Psi(A, B)\).

This spaces are rearrangement invariant, see [4], and are used, for example, in the theory of probability [34], [48], [49]; theory of Partial Differential Equations [16], [29]; functional analysis [17], [28], [39], [49], [60], [81]; theory of Fourier series [48], theory of martingales [49], mathematical statistics [52], [53], [54], [55], [56], [57], [58], [59]; theory of approximation [65] etc.
Notice that in the case when $\psi(\cdot) \in \Psi(A, B)$, a function $p \to p \log \psi(p)$ is convex, and $B = \infty$, then the space $G\psi$ coincides with some exponential Orlicz space. Conversely, if $B < \infty$, then the space $G\psi(A, B)$ does not coincide with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marzinkieievicz etc.

**Remark 2.0.** If we define the degenerate $\psi_r(p), r = \text{const} \geq 1$ function as follows:

$$\psi_r(p) = \infty, \quad p \neq r; \quad \psi_r(r) = 1$$

and agree $C/\infty = 0, C = \text{const} > 0$, then the $G\psi_r(\cdot)$ space coincides with the classical Lebesgue space $L_r$.

Let $\xi : X \to R$ be measurable function such that for some constants $(A, B), 1 \leq A < B \leq \infty$, $|\xi|_p < \infty$. The natural function for the function $\xi = \xi(x) \psi_\xi(p)$ may be defined by formula

$$\psi_\xi(p) := |\xi|_p, \quad p \in (A, B).$$

Analogously, let $\xi = \xi(t, x), t \in T, T$ is arbitrary set, be a family of a measurable functions such that

$$\exists(A, B) : 1 \leq A < B \leq \infty, \forall p \in (A, B) \Rightarrow \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.$$ 

The natural function $\xi : \xi(x) \psi_\xi(p)$ for the family $\xi(\cdot)$ denotes by definition

$$\psi_\xi(p) := \sup_{t \in T} |\xi(t, \cdot)|_p, \quad p \in (A, B).$$

**Theorem 2.1.** Let $f \in G\psi$; then

$$||U[f]||_{G\psi_1} \leq 1 \times ||f||_{G\psi},$$

and the constant "1" in the inequality (2.5) is the best possible.

**Proof of the upper estimate** is very simple. Let $f \in G\psi$; we can suppose $||f||_{G\psi} = 1$. It follows from the direct definition of the norm in the GLS that

$$\forall p \in (a, b) \Rightarrow |f|_{p, X} \leq \psi(p).$$

We obtain from the condition (A)
\[ |U[f]|q(p) \leq K(p)\psi(p), \]
or equally
\[ |U[f]|q \leq K(r(q))\psi(r(q)) = \psi_1(q) = \psi_1(q)||f||G\psi, \]
\[ ||U[f]||G\psi_1 = \sup_{q \in (c,d)} ||U[f]|q/\psi_1(q)|| \leq ||f||G\psi. \]

**Proof of the lower estimate.** We denote
\[ V(\psi, f) = \left[ \frac{||U[f]||G\psi_1}{||f||G\psi} \right], \mathcal{V} = \sup_{\psi \in G\psi(a,b)} \sup_{f \in G\psi} V(\psi, f), \]
and define as usually
\[ ||U||_{(G\psi \to G\psi_1)} = ||U|| = \sup_{f \in G\psi, f \neq 0} \left[ \frac{||U[f]||G\psi_1}{||f||G\psi} \right], \]
then
\[ ||U|| = \sup_{f \in G\psi, f \neq 0} \left[ \frac{\sup_p |U[f]|q(p)/\psi_1(q(p))}{\sup_p |f|p/\psi(p)} \right]. \]

The proposition or theorem (2.1) may be rewritten as follows: \( \mathcal{V} = 1; \) we know that \( \mathcal{V} \leq 1; \) it remains to prove an opposite inequality.

Let us choose
\[ q_0 = \arg \sup_{q \in (c,d)} \sup_{f \in G\psi, f \neq 0} \left[ \frac{|U[f]|q}{K(r(q))|f|r(q)} \right], \]
\[ f_0 = \arg \sup_{f \in G\psi} \left[ \frac{|U[f]|q_0}{K(r(q_0))|f|r(q_0)} \right], \psi_0(p) = |f_0|p = \psi_{f_0}(p), \]
\[ \psi_{0,1}(q) = K(r(q)) \times \psi_0(r(q)). \]
It follows from the definition of the function \( K = K(p) \) that
\[ |U[f_0]|q_0 = K(r(q_0))|f_0|r(q_0), \]
and we can suppose without loss of generality that the function \( f_0 \) here exists.

Further,
\[ \mathcal{V} = \sup_{\psi \in G\psi(a,b)} \sup_{f \in G\psi} V(\psi, f) = \sup_{\psi \in G\psi(a,b)} \sup_{f \in G\psi} \left[ \frac{||U[f]||G\psi_1}{||f||G\psi} \right] = \sup_{\psi \in G\psi(a,b)} \sup_{f \in G\psi} \left[ \frac{\sup_q |U[f]|q/\psi_1(q)}{\sup_p |f|p/\psi(p)} \right] \geq \frac{\sup_q |U[f_0]|q/\psi_{0,1}(q)}{\sup_p |f_0|p/\psi_0(p)} \geq \]
as long as \( f_0 \neq 0 \).

In the case when the function \( f_0 \) does’nt exist, we conclude that for arbitrary \( \epsilon \in (0, 1/2) \) there exists a family of a measurable functions \( f_\epsilon = f_\epsilon(x) \) for which

\[
|U[f_\epsilon]|_q > (1 - \epsilon) K(r(q_\epsilon)) |f_\epsilon|_{r(q_\epsilon)},
\]

and therefore

\[
V \geq 1 - \epsilon.
\]

**Remark 2.1.** In all known examples the value \( p_0 \) tends to the boundaries of the set \((a, b)\); may be \( p_0 \to \infty \) if \( b = \infty \).

**Remark 2.2.** There are many cases when \( q = p \) and following \((c, d) = (a, b)\), for example Hilbert transform or maximal Fourier operators etc. Obviously, here

\[
\psi_1(p) = K(p) \psi(p), \ p \in (a, b).
\]

**Remark 2.3.** If we know instead the exact value of constant \( K(p) \) only the weak inequality of a view

\[
0 \leq c_1 \leq \sup_{p \in (a, b)} \frac{|U|_{p \to q(p)}}{K(p)} \leq C_2 < \infty,
\]

then evidently

\[
c_1 \leq ||U||(G\psi \to G\psi_1) \leq C_2,
\]

(the ”weak value” of constant.)

**Remark 2.4.** It may be investigated analogously the case of inequality

\[
|U_1[f]|_{q(p)} \leq K(p) |U_2[f]|_p, \ \forall p \in (a, b) \Rightarrow K(p) < \infty,
\]

where \( U_1, U_2 \) are some operators and the second operator \( U_2 \) is not invertible. For example, the inequality (B) is true for Sobolev’s imbedding theorems.

**Remark 2.5.** We consider here the case when the value \( q = q(p) \) is’nt unique. More exactly, let for some interval \((a, b)\), \( 1 \leq a < b \leq \infty \) of the values \( p \) there exists a non-trivial interval \( Q(p) = (Q_1(p), Q_2(p)) \), \( 1 \leq Q_1(p) < Q_2(p) \leq \infty \) of the values \( q \) and positive finite function \( K_Q(p, q) \) such that

\[
|U[f]|_q \leq K_Q(p, q) |f|_p.
\]

Let for some non-zero function \( \psi \in \Psi(a, b) f \in G\psi \), then

\[
|U[f]|_q \leq K_Q(p, q) \psi(p) \ ||f||G\psi,
\]

and let \( \nu = \nu(q) \) be another function from the set \( \Psi(c, d) \). We obtain from (2.10) after dividing on \( \nu(q) \cdot ||f||G\psi \)
\[
\frac{|U[f]|_q}{\nu(q) \|f\|G\psi} \leq \frac{K(p, q) \psi(p)}{\nu(q)}.
\]

We get taking supremum over \( q \):

\[
\frac{||U[f]||G\nu}{\|f\|G\psi} \leq \sup_q \left[ \frac{K(p, q) \psi(p)}{\nu(q)} \right].
\] (2.11)

As long as the inequality is true for each values \( p \), we conclude eventually:

\[
||U[f]||G\nu \leq \inf_{p \in (a,b)} \sup_{q \in Q(p)} \left[ \frac{K(p, q) \psi(p)}{\nu(q)} \right] \cdot ||f||G\psi =: \overline{K} \|f\|G\psi,
\] (2.12)

if obviously \( \overline{K} < \infty \).

**Remark 2.6.** Let us consider a particular case of inequality (2.9); namely, suppose for any constants \( A, B > 0 \)

\[
|U[f]|_q \leq C \cdot A^{1/q} \cdot B^{-1/p} \|f\|_p, \ p \in (a, b), \ q \in (c, d).
\] (2.13)

We get repeating at the same considerations:

\[
\frac{|U[f]|_q \cdot B^{1/p}}{\nu(q) \psi(p)} \leq C \cdot A^{1/q} \|f\|_p.
\] (2.14)

Recall that the fundamental function of the space \( G\psi \) has a view:

\[
\phi(G\psi, \delta) = \sup_{p \in (a,b)} \left[ \delta^{1/p} \right].
\]

We have taking supremum of the inequality (2.14) over \( q \) and \( p \):

\[
\frac{||U[f]||G\nu}{\phi(G\nu, A)} \leq C \cdot \frac{||f||G\psi}{\phi(G\psi, B)}.
\] (2.15)

The inequality (2.15) was used in the approximation theory, see [65].

**Remark 2.7.** The lower bound in the theorem 2.1 may be obtained very simple if we allow to consider the degenerate \( \psi \) - functions.

**Example of non-uniqueness.**

We return to the article of Muckenhoupt [45], but we represent more general case: \( q \neq p/(p-1) \). Namely, let here \( 1 < p < 2 \) and \( p \leq q < p_1 = p/(p-1) \). Denote by \( u^*(y) \) the non-increasing rearrangement modification of the function \( u(y) \), and define the following integral:

\[
J(A, B) = \sup_{r>0} [J_1(B, r) J_2(A, r)] ,
\]

where

\[
J_1(B, r) = \int_{\{|y u^*(y)|^{p_1/q} > Brx\}} u^*(y) \ dy,
\]
\[ J_2(A, r) = \left[ \int_{\{v(x) < Ar^{-p-1}\}} v(x)^{-1/(p-1)} \, dx \right]^{q/p}. \]

Suppose \( J(A, B) < \infty \) for some constants \( A, B > 0 \) (Muckenhoupt’s condition). We conclude after some computations from the article [45]:

\[
\left[ \int_{\mathbb{R}^n} u(y) |F[f](y)|^q \, dy \right]^{1/q} \leq K_{MM}(p, q) \cdot \left[ \int_{\mathbb{R}^n} v(x) |f(x)|^p \, dx \right]^{1/p}, \tag{2.16}
\]

where

\[
K_{MM}(p, q) = \frac{C(A, B; J(A, B))}{(p - 1)(p_1 - q)}.
\tag{2.17}
\]

### 3 Boundedness of maximal operator in GLS

Let \( U = U_\theta = U_\theta[f], \ \theta \in \Theta \), where \( \Theta = \{\theta\} \) is arbitrary set, be a family of linear (or at least sublinear) operators. The sublinear (in general case) operator of a view

\[
U[f](x) = \sup_{\theta \in \Theta} |U_\theta[f](x)|, \quad x \in X \tag{3.1}
\]

will be called maximal operator for the family \( U_\theta \), if is correctly defined on some Banach functional space of a (measurable) functions \( f : X \to \mathbb{R} \).

We consider in this section the boundedness of operator \( \mathcal{T}[f] \) in Grand Lebesgue Spaces.

Note that the case of maximal Fourier transform and some other singular operators is observed, e.g., in [17], [70].

We consider only the case of classical maximal centered ball Hardy - Littlewood operator in the ordinary Euclidean space \( X = \mathbb{R}^d \):

\[
\mathcal{M}[f](x) = \sup_{B:|B| \in (0, \infty)} \left[ |B|^{-1} \int_B |f(x)| \, dx \right] \tag{3.2}
\]

where ”supremum” is calculated over all Euclidean balls \( B \) with center at the point \( x; \ |B| := \text{meas}(B) \in (0, \infty). \)

Let \( \psi \in G\psi(1, \infty) \); we define

\[
\psi^{(1)}(p) \overset{\text{def}}{=} \psi(p) \cdot \frac{p}{p - 1}, \quad p \in (1, \infty). \tag{3.3}
\]

Note that \( \psi^{(1)}(\cdot) \in G\psi(1, \infty). \)

**Theorem 3.1.** If \( f \in G\psi \), then

\[
||\mathcal{M}[f]||_{G\psi^{(1)}} \leq C_2 ||f||_{G\psi},
\]
for some absolute, i.e. not depending on the dimension $d$ weakly exact non-trivial constant $C_3$.

**Proof.** We will use theorem 2.1, more exactly remark 2.3. The classical result belonging to E.M.Stein E.M and J.O.Str"omberg [78] states that the constant $C_2$ in (2.7) for considered maximal operator $\mathcal{M}$ is bounded over all the dimensions $d$:

$$\sup_d C_2(d) =: C_3 < \infty.$$  

In detail, E.M.Stein E.M and J.O.Str"omberg [78] proved:

$$|\mathcal{M}|_{p \to p} \approx \frac{p}{p - 1}, p \in (1, \infty].$$

The lower estimate for the constant $c_1$ in (2.7) is also uniform over $d$ bounded from below, which may be proved by consideration of an example

$$f_0 = I(||x|| \leq 1), ||x|| = \sqrt{\sum_{i=1}^{d} x_i^2},$$

$$I(x \in A) = I_A(x) = 1, x \in A; I(x \in A) = I_A(x) = 0, x \notin A.$$

We have for the function $f_0$:

$$|\mathcal{M}[f_0]|_p \geq c_1 \frac{p}{p - 1}|f_0|_p.$$  

This completes the proof of theorem 3.1.

### 4 Weight Pitt - Beckner - Okikiolu inequalities for GLS spaces

Let $x = \vec{x} \in \mathbb{R}^n$ be an $n-$dimensional vector, $n = 1, 2, \ldots$ which consists on the $l, l \geq 1$ subvectors $\{x_j, \} j = 1, 2, \ldots, l$:

$$x = (x_1, x_2, \ldots, x_l), x_j = \vec{x}_j \in \mathbb{R}^{m_j}, \dim(x_j) = m_j \geq 1. \quad (4.1)$$

Let also $\alpha = \vec{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ and $\beta = \vec{\beta} = \{\beta_1, \beta_2, \ldots, \beta_l\}$ be two fixed $l-$dimensional vectors such that

$$\alpha_j \in [0, 1), \beta_j \in [0, 1), \alpha_j + \beta_j \leq 1, j = 1, 2, \ldots, l.$$  

We denote as ordinary by $|x_j|$ the Euclidean norm of the vector $x_j$:

$$|x|^{-\alpha} = \prod_{j=1}^{l} |x_j|^{-\alpha_j}$$

and analogously for the vector $y \in \mathbb{R}^n$

$$|y|^\beta = \prod_{j=1}^{l} |y_j|^\beta_j. \quad (4.2)$$
Obviously,
\[ \sum_{j=1}^{l} m_j = l, \quad |x| = (\sum_{j=1}^{l} |x_j|^2)^{1/2}. \]
We define alike to the book Okikiolu [47], p. 313-314, see also [3] the so-called double weight Fourier transform \( F_{\alpha,\beta}[f](x) \) by the following way:
\[
F_{\alpha,\beta}[f](x) = (2\pi)^{-n/2} |x|^\alpha \int_{\mathbb{R}^n} |y|^\beta f(y) e^{ixy} \, dy,
\]
with \( xy \) being the inner product of the vectors \( x, y : xy = \sum_{k=1}^{n} x_k y_k \).

The inequality of a view (relative Lebesgue measures in whole space \( \mathbb{R}^n \))
\[ |F_{\alpha,\beta}[f]|_q \leq K_{PBO}(p) |f|_p \]  \hspace{1cm} (4.4a),
or equally
\[ ||y|^{-\alpha} F[f](y)||_q \leq K_{PBO}(p) ||x|^{\beta} f(x)||_p, \]  \hspace{1cm} (4.4)
is said to be (generalized) weight Pitt - Beckner - Okikiolu (PBO) inequality.

We will understood as customary as the value \( K_{PBO}(p) \) its minimal value:
\[
K_{PBO}(p) = \sup_{f \neq 0, f \in L^p(\mathbb{R}^n)} \left[ \frac{|F_{\alpha,\beta}[f]|_q}{|f|_p} \right].
\]  \hspace{1cm} (4.5)

As before, \( p \in (p_0, p_1), \ p_0, p_1 = \text{const}, \ 1 \leq p_1 < p_2 \leq \infty, \ q \in (q_0, q_1), \ 1 \leq q_0 < q_2 \leq \infty \) and \( q = q(p) \) is some uniquely defined continuous strictly monotonic function (if there exists).

There are many publications about this inequality, see also, for instance, [3], [6], [9], [10], [19], [20], [31], [37], [39], [40], [51], [87] etc.

A very interest application of the PBO inequality in the quantum mechanic are described in the articles [2], [24].

We consider in this section only ”one-dimensional” case, i.e. when \( l = 1 \).
The general case \( l \geq 2 \) will be considered further.

Let us introduce the following important conditions in the domain
\[ p_0 = n/(n - \beta), \ p_1 = \infty, \ q_0 = 1, q_1 = n/\alpha : \]
\[
\beta - \alpha = n - n \left( \frac{1}{p} + \frac{1}{q} \right),
\]  \hspace{1cm} (4.6)
\[ p > p_0, \ 1 \leq q < q_0, \]
which defined uniquely the continuous function \( q = q(p) \).
The condition (4.6) was imposed by many authors, see [2], [19], [40] etc.

Theorem 4.1.
A. The conditions 4.6 is necessary for PBO inequality (4.4).

B. If the condition 4.6 is satisfied, then for $p > p_0$

$$C_1(\alpha, \beta, n) \left[ \frac{p}{p - p_0} \right]^{(\alpha + \beta)/n} \leq K_{PBO}(p) \leq C_2(\alpha, \beta, n) \left[ \frac{p}{p - p_0} \right]^\max(1,(\alpha + \beta)/n).$$  \tag{4.7}

C. Both the boundaries in the inequality (4.7), lower and upper, are attainable.

**Proof. Part B.**

0. The upper bound for the coefficient $K_{PBO}(p)$ follows from the direct computation in the article of W. Beckner [3]. For instance, in the case when $\alpha = \beta$ or correspondingly $q = p/(p-1)$ W. Beckner computed the exact value of this constant.

1. In order to obtain the lower bound in (4.7), we consider the following example. Let us introduce the following subsets (generalized truncated "segments") of whole space $R^n = \{ x = \vec{x} : \}$

$$D(c_0, c_1, c_2) = \{ x : |x_j| \geq c_0, |x_j|/|x| \in [c_1, c_2], \text{ } j = 1, 2, \ldots, n; \} \tag{4.8}$$

$$G(c_3, c_4, c_5) = \{ y : 0 < |y_j| \leq c_3, |y_j|/|y| \in [c_4, c_5], \text{ } j = 1, 2, \ldots, n; \} \tag{4.9}$$

$$0 < c_0 < \infty; 0 < c_1 < 1 < c_2 < \infty; 0 < c_3 < \infty; 0 < c_4 < 1 < c_5 < \infty;$$

$$D = D(c_1, c_2) = D(1, c_1, c_2); \quad G = G(c_4, c_5) = G(1, c_4, c_5).$$

2. We consider the following example. Let $f_0 = f_0(x), \ x \in R^n$ be even function such that

$$f_0(x) = f_{0,n}(x) = \frac{I_D(x)}{\prod_{j=1}^n |x_j|}. \tag{4.10}$$

We have using multidimensional polar (spherical) coordinates:

$$|x|^\beta f_0(x)|_p^p \leq \int_{D(c_1, c_2)} |x|^{p(\beta-n)} dx \leq C_3(n) \int_{C_1(n)}^\infty r^{n-1+p(\beta-n)} dr, \tag{4.11}$$

as long as inside the domain $D(c_1, c_2)$ is true the following inequality:

$$\prod_{j=1}^n |x_j| \geq C_5(n, c_1, c_2) |x|^n.$$

We conclude taking the integral (4.11) that if $p > n/(n - \beta)$ then
\[ |x|^\beta f_0(x)|_p \leq C_6(n, \beta)(p - n/(n - \beta))^{-1/p} \leq C_7(n, \beta)(p/(p - p_0))^{-(n - \beta)/n}. \quad (4.12) \]

3. We investigate in this pilcrow the behavior of the Fourier transform of the function \( f_0(y) \) as \(|y| \to 0, \ |y| \leq 1\):

\[
F[f_0](y) = \int_D e^{i\eta y} \frac{dx}{\prod_{j=1}^n |x_j|} = C_8(n) \int_D \cos(xy) \left[ \prod_{j=1}^n I(x_j > 0) \right] \frac{dx}{\prod_{j=1}^n |x_j|}.
\]

We restrict ourself only the case when \( y \in G(c_3, c_4, c_5) \).

We obtain after the substitution \( x_j = v_j/y_j \):

\[
F[f_0](y) = C_9 \int_{G(y)} \cos(\sum_j v_j) \frac{dv}{\prod_{j=1}^n |v_j|} \sim C_9 \int_{G(y)} \frac{dv}{\prod_{j=1}^n |v_j|},
\]

where \( G(y) \) is an image of the set \( D \) under our substitution.

Since for all the values \( y, \ y \in G(c_3, c_4, c_5) \) the domain \( G(y) \) contains the set \( D(c_6|y|, c_7, c_8) \), we have for the values \( y \in G(c_3, c_4, c_5) \) using again the multidimensional polar (spherical) coordinates as before:

\[
F[f_0](y) \geq C_{10} \int_{D(c_6|y|, c_7, c_8)} \frac{dv}{\prod_{j=1}^n |v_j|} \geq C_{11} \int_{C_4 \rho^n \rho^{n-1-n}} \rho^n \ | \rho^n d\rho \geq C_{14} \log |y|.
\]

4. We estimate here the left-hand side of inequality (2.7) for our operator. Namely,

\[
|y|^{-\alpha} F[f](y)|_q \geq \int_{0<|y|\leq c_7} C_{15} |y|^{-\alpha q} |\log |y||^q dy \geq C_{16} \int_0^{C_{17}} z^{n-1-\alpha q} |\log z|^q dz \geq C_{18} (n, \alpha) (n - \alpha q)^{-q-1},
\]

or equally

\[
|y|^{-\alpha} F[f_0](y)|_q \geq C_{19} (n, \alpha) (q_0 - q)^{-1-1/q} \geq
\]

\[
C_{20} (n, \alpha, \beta) (p - p_0)^{1-1/q} \geq C_{21} \left[ \frac{p}{p - p_0} \right]^{1-\alpha/n}, \quad (4.14)
\]

here \( q \in [1, n/\alpha) \) and correspondingly \( p \in (n/(n - \beta), \infty) \).

5. We conclude after dividing \(|y|^{-\alpha} F[f_0](y)|_q \) over \(|x|^\beta f(x)|_p \):

\[
K_{PBO}(p) \geq \frac{|y|^{-\alpha} F[f_0](y)|_q}{|x|^\beta f_0(x)|_p} \geq C_{21} (n, \alpha, \beta) \left[ \frac{p}{p - p_0} \right]^{-\beta/n - \alpha/n}. \quad (4.15)
\]
This completes the proof of the proposition B of theorem 4.1.

**Proof of the assertion C.**

**Lower bound.** The lower bound in the inequality (4.7) is attained as \( \alpha \to 0^+, \beta \to 0^+ \), for instance, in the Hausdorff-Young inequality

\[
|F[f]|_{p/(p-1)} \leq C^n |f|_p, \quad p \in (1, 2],
\]

where \( C \) is an absolute constant (calculated by W. Beckner [2], [3]); here \( \alpha = \beta = 0 \).

The upper bound is attained, e.g., in the case when \( \alpha = \beta > 0, q = p/(p-1) \), see [3].

At the same upper estimate may be obtained from the classical Wiener-Paley theorem:

\[
\int_R |y|^{p-2} |F[f](y)|^p \, dy \leq K_{WP}^p(p) \int_R |f(x)|^p \, dx,
\]

see [36], chapter 5, section 5; in this case

\[
K_{WP}(p) \asymp \frac{p}{p-1}, \quad p \in (1, 2].
\]

**Remark 4.1.** Notice that our counter-example isn’t a radial function!

As a corollary: let \( \psi(\cdot) \in G\psi(p_0, \infty) \); we define a new function

\[
\psi_{PBO}(p) = \frac{p\psi(p)}{p - p_0}.
\]

We assert:

\[
||F_{\alpha, \beta}[f]||_{G\psi_{PBO}} \leq C_2(\alpha, \beta, n)||f||_{G\psi}, \quad (4.16)
\]

and the last inequality is weak logarithmically exact.

**Proof of the part A.**

We will use (here and further) the so-called ”dilation method”, introduced, e.g. by G.Talenti [82] for a finding of an optimal constant in the Sobolev’s imbedding theorems. Indeed, let \( f, f : R^n \to R \) be some non-zero function from the Schwartz space \( S(R^n) \), for which the inequality PBO (4.4) there holds. We define the family \( T_\lambda[\cdot] \), \( \lambda \in (0, \infty) \) of linear dilations operators of a view

\[
T_\lambda[f](x) = f(\lambda x), \quad (4.17)
\]

Note that if \( f \in S(R^n) \), then \( \forall \lambda \in (0, \infty) \Rightarrow T_\lambda[f] \in S(R^n) \).

We have:

\[
F[T_\lambda f] = \lambda^{-n} F[f], \quad (4.18)
\]

\[
| |x|^\beta T_\lambda f|_p = \lambda^{-n/p-\beta} | |x|^\beta f|_p, \quad (4.19)
\]
\[ |y|^{-\alpha} F[T_{\lambda} f]|_q = \lambda^{-n+n/q-\alpha} |y|^{-\alpha} F|_q. \quad (4.20) \]

We get after substituting onto (4.4) for the function \( T_{\lambda} f \):

\[ \lambda^{-n+n/q-\alpha} |y|^{-\alpha} F[f]|_q \leq K_{PBO}(p)\lambda^{-n/p-\beta} |x|^{\beta} f|_p, \quad (4.21) \]

As long as the last inequality may be satisfied for all the values \( \lambda \in (0, \infty) \) only in the case when

\[ -n + n/q - \alpha = -n/p - \beta, \]

or equally

\[ \beta - \alpha = n - n(1/p + 1/q), \]

Q.E.D.

5 Interpolation of operators in Grand Lebesgue spaces

Let \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) be again two measurable spaces with sigma-finite non-trivial measures \( \mu, \nu \); we assume in addition that both the measures \( \mu, \nu \) are resonant in the sense described, e.g. in monograph Bennet and Sharpley [4], chapter 1.2. Recall that the notion “resonant” of the measure \( \mu \) means either atomlessness this measure or discreteness \( \mu \) such that all the atoms have equal measures.

A. Classical version.

We consider first of all the interpolation of linear operators in the spirit of the classical theorem belonging to Riesz and Thorin. Let \( p_0, p_1, q_0, q_1 \) be fixed numbers such that \( 1 \leq p_0, p_1, q_0, q_1 < \infty, \ p_0 \leq q_0, p_1 \leq q_1 \).

We can and will suppose without loss of generality that \( q_0 < q_1, \ p_0 < p_1 \).

We consider in this section a linear operator from the set of all measurable functions \( f : X \to \mathbb{R} \) into the set of measurable functions \( f : Y \to \mathbb{R} \) such that

\[ |T[f]|_{p_0} \leq M_0|f|_{q_0}, \ |T[f]|_{p_1} \leq M_1|f|_{q_1}, M_0, M_1 = \text{const}. \quad (5.1) \]

On the other words, the operator \( T \) is simultaneously of strong-type \((p_0, q_0)\) and \((p_1, q_1)\).

Define a functions for \( q \in (q_0, q_1) \):

\[ r_{RT}(q) = \frac{p_0 p_1 (q_1 - q_0) q}{p_0 q_1 (q - q_0) + p_1 q_0 (q_1 - q)}, \]

\[ \Theta(q) = \frac{q_1 (q - q_0)}{q (q_1 - q_0)}, \]

so that
1 − Θ(\(q\)) = \(\frac{q_0(q_1 - q)}{q(q_1 - q_0)}\);

\[ M_\theta(q) = 2M_0^{1-\Theta(q)} M_1^{\Theta(q)}. \]

**Theorem 5.1.** Let \(\psi \in G\psi(p_0, p_1)\); we define a new function

\[ \psi_{RT} = \psi_{RT}(q) = M_\theta(q) \cdot \psi(r_{RT}(q)). \]  

(5.2)

Proposition: for each function from the space \(G\psi(p_0, p_1)\) there holds:

\[ ||T[f]||_{G\psi_{RT}} \leq ||f||_{G\psi}. \]  

(5.3)

**Proof** follows from theorem 2.1 and from the classical theorem of Riesz and Thorin [4], p. 185:

\[ ||T[f]||_q \leq M_\theta(q) \cdot |f|_p, \ p \in (p_0, p_1), \ q \in (q_0, q_1). \]

where

\[ \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}. \]  

(5.4)

**Remark 5.1.** This result is weakly exact, e.g., when \(T\) is Fourier transform on the whole real line (or whole space \(R^n\)). Theorem of Hardy - Young asserts that

\[ ||T[f]||_q \leq C |f|_p, \ p \in (1, 2), \ q = p/(p - 1). \]

Lorentz version.

We recall first of all the definition of the Lorentz spaces over the triple (say) \((X, A, \mu)\). Let the numbers \(p, q\) be a given such that \(1 \leq p \leq \infty, 0 < q \leq \infty\).

We denote as customary for each measurable function \(f : X \to R\) by \(f^*(t), \ t \in (0, \infty)\), the left inverse function to the distribution function for its absolute value:

\[ f^*(t) = h^{-1}(t), \ h(t) = \mu\{x \in X, \ |f(x)| > t\}. \]

The space \(L_{p,q} = L_{p,q}(X)\) consists, by definition, see [4], p. 213-214, on all the functions \(f, f : X \to R\) with finite norm

\[ |f|_{p,q} = \left[ \int_0^\infty [t^{1/p} f^*(t)]^q dt/t \right]^{1/q}, \]  

(5.5)

if \(q < \infty\), and

\[ |f|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t). \]

Recall that the linear operator \(T\) is said to be of weak type \((p, q)\), if

\[ |T[f]|_{q,\infty} \leq M|f|_{p,1}, \ M = \text{const} < \infty. \]
Theorem 5.2. Let $p_0, p_1, q_0, q_1$ be fixed numbers such that $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0 < q_1 \leq \infty$. Let also $T$ be simultaneously weak type $(p_0, q_0)$ and $(p_1, q_1)$:

$$|T[f]|_{q_i, \infty} \leq M_i|f|_{p_i, 1}, \ i = 0, 1. \quad (5.6)$$

We introduce for each function $\psi \in G\psi(p_0, p_1)$ a new function

$$\psi_M(q) = \psi(r_{HT}(q)) \left[ (q - q_0)(q_1 - q) \right]^{-1}, \ q \in (q_0, q_1).$$

Proposition:

$$||T[f]||_{G\psi_M} \leq C(p_0, p_1, q_0, q_1) \max(M_0, M_1) ||f||_{G\psi}. \quad (5.7)$$

Proof is at the same as the proof of theorem 5.1. We will use only instead the classical theorem of Riesz - Thorin the interpolation theorem belonging to Marcinkiewicz (see [4], p. 226). Namely, if the conditions (5.6) are satisfied, then

$$|T[f]|_q \leq \frac{c(p_0, p_1, q_0, q_1)}{\theta(1 - \theta)} \max(M_0, M_1) |f|_p,$$

where as before $p \in (p_0, p_1)$, $q \in (q_0, q_1)$.

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

Remark 5.2. The assertion of theorem (5.2) is in general case non-improvable, see, e.g. [70], [73].

6 Multidimensional case. ”Dilation method”.

We consider in this section the multidimensional (vector): $l \geq 2$ generalization of PBO inequality (4.4) in the sense of section 4. Namely, we investigate the inequality of a view

$$|y|^{-\bar{\alpha}} F[f](y)|_q \leq K_{n, \bar{\alpha}, \bar{\beta}}(p) \ |x|^{\bar{\beta}} f(x)|_p,$$

or for simplicity

$$|y|^{-\alpha} F[f](y)|_q \leq K_{n, \alpha, \beta}(p) \ |x|^{\beta} f(x)|_p. \quad (6.1)$$

Recall that $x = \bar{x} \in R^n$ be $n$-dimensional vector, $n = 1, 2, \ldots$; which consists on the $l, l \geq 1$ subvectors \{\(x_j\), \(j = 1, 2, \ldots, l\) :}

$$x = (x_1, x_2, \ldots, x_l), \ x_j = \bar{x}_j \in R^{m_j}, \ \dim(x_j) = m_j \geq 1;$$

$$\alpha = \bar{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$$

and $\beta = \bar{\beta} = \{\beta_1, \beta_2, \ldots, \beta_l\}$ be two fixed $l$-dimensional vectors such that
\( \alpha_j \in [0, 1), \beta_j \in [0, 1), j = 1, 2, \ldots, l, \alpha_j + \beta_j \leq 1. \)

We denote as ordinary by \(|x_j|\) the Euclidean norm of the vector \(x_j\);

\[ |x|^{-\alpha} := \prod_{j=1}^{l} |x_j|^{-\alpha_j} \]

and analogously for the vector \(y \in \mathbb{R}^n\)

\[ |y|^\beta := \prod_{j=1}^{l} |y_j|^\beta_j. \]

**Theorem 6.1.**

1. The inequality (6.1) may be satisfied for each function \(f \in S(\mathbb{R}^n)\) only when

\[ \frac{\beta_j - \alpha_j}{m_j} = \text{const}, \quad j \in [1, 2, \ldots, l], \quad (6.2) \]

2. If the condition (6.2) is satisfied, then

\[ \beta_j - \alpha_j = m_j \left( 1 - \frac{1}{p} - \frac{1}{q} \right), \quad (6.3) \]

\[ C_1(\alpha, \beta, n) \prod_{j=1}^{l} \left[ \frac{p}{(p - m_j / (m_j - \beta_j))} \right]^{(\alpha_j + \beta_j)/m_j} \leq K_{n,\alpha,\beta}(p) \leq \]

\[ C_2(\alpha, \beta, n) \prod_{j=1}^{l} \left[ \frac{p}{(p - m_j / (m_j - \beta_j))} \right]^{\max(1, (\alpha_j + \beta_j)/m_j)}, \quad (6.4) \]

\[ p > \max_{j=1,2,\ldots,l} \frac{m_j}{m_j - \beta_j}. \]

**Proof** of the second proposition is at the same as in the one-dimensional case (theorem 4.1); the upper bound for the coefficient \(K_{n,\alpha,\beta}(p)\) may be obtained analogously considerations of W. Beckner [2].

The example \(\hat{f}(x)\) for the lower bound may be constructed as a product of the \(m_j\) dimensional examples:

\[ \hat{f}(x) = \prod_{j=1}^{l} f_{0,m_j}(x_j). \]

It remains to prove the necessity of the condition (6.3).

We will use a multidimensional generalization of dilation method.

It is enough to consider only the two-dimensional case: \(l = 2\). The inequality (6.1) then has a view:

\[ |y_1|^{-\alpha_1} |y_2|^{-\alpha_2} F[f](y) \leq K_{n,\alpha,\beta}(p) \quad |x_1|^\beta_1 |x_2|^\beta_2 f(x), \quad (6.5) \]
Let $\lambda_1, \lambda_2$ be two arbitrary independent positive numbers. We define the multi-dimenional dilation operator $T_{\lambda_1, \lambda_2}[f]$ as follows:

$$T_{\lambda_1, \lambda_2}[f](x_1, x_2) = f(\lambda_1 x_1, \lambda_2 x_2).$$  \hfill (6.6a)

We have:

$$F[T_{\lambda_1, \lambda_2}[f]](y_1, y_2) = \lambda_1^{-m_1} \lambda_2^{-m_2} F[f](y_1/\lambda_1, y_2/\lambda_2);$$ \hfill (6.6b)

$$| |y_1|^{-\alpha_1} |y_2|^{-\alpha_2} F[T_{\lambda_1, \lambda_2}[f]](y_1, y_2)|_q = \lambda_1^{m_1/q - m_1 - \alpha_1} \lambda_2^{m_2/q - m_2 - \alpha_2} | |y_1|^{-\alpha_1} |y_2|^{-\alpha_2} F[f](y_1, y_2)|_q. \hfill (6.6c)$$

$$| |x_1|^{\beta_1} |x_2|^{\beta_2} T_{\lambda_1, \lambda_2}[f](x_1, x_2)|_p = \lambda_1^{-\beta_1 - m_1/p} \lambda_2^{-\beta_2 - m_2/p} | |x_1|^{\beta_1} |x_2|^{\beta_2} f(x_1, x_2)|_p; \hfill (6.6d)$$

We obtain substituting into (6.5) the function $T_{\lambda_1, \lambda_2}[f]$ instead the function $f$:

$$\lambda_1^{m_1/q - m_1 - \alpha_1} \lambda_2^{m_2/q - m_2 - \alpha_2} | |y_1|^{-\alpha_1} |y_2|^{-\alpha_2} F[f](y_1, y_2)|_q \leq \lambda_1^{-\beta_1 - m_1/p} \lambda_2^{-\beta_2 - m_2/p} | |x_1|^{\beta_1} |x_2|^{\beta_2} f(x_1, x_2)|_p. \hfill (6.7)$$

Since the values $\lambda_1, \lambda_2$ are arbitrary positive, the inequality may be satisfied only when

$$m_1/q - m_1 - \alpha_1 = -\beta_1 - m_1/p, \quad m_2/q - m_2 - \alpha_2 = -\beta_2 - m_2/p,$$

or equally

$$(\beta_1 - \alpha_1)/m_1 = 1 - 1/p - 1/q = (\beta_2 - \alpha_2)/m_2. \hfill (6.8)$$

This completes the proof of theorem 6.1.

**Remark 6.1.** At the same result as theorem 6.1, up to simple chaining of variables and functions, may be obtained from the considerations Okikiolu [47], p. 313-318. The condition (6.2) is there presumed (in another notations).

## 7 PBO inequalities for anisotropic Lebesgue spaces.

We consider in this section the case of some generalization of PBO inequality when the condition (6.2) is'nt satisfied.
In order to formulate and prove the generalization of the inequality (6.1), we will use the so-called anisotropic Lebesgue spaces. More detail information about this spaces see in the books [5], chapter 16,17; [38], chapter 11; using for us theory of operators interpolation in this spaces see in [5], chapter 17,18.

Let \( \vec{p} = (p_1, p_2, \ldots, p_l) \) be \( l \) - dimensional vector such that

\[
p_j > m_j / (m_j - \beta_j) =: p_{0j}, \quad q_j \in [1, m_j / \alpha_j) =: q_{0j}, \quad j = 1, 2, \ldots, l.
\]

We denote the set all the values \( \vec{p} \) as \( Q = Q (p_{01}, p_{02}, \ldots, p_{0l}) \).

Let also \( u = u (x) \), \( x \in \mathbb{R}^n \) be measurable function: \( u : \mathbb{R}^n \to \mathbb{R} \).

Recall that the anisotropic Lebesgue space \( L_{\vec{p}} \) consists on all the functions \( f \) with finite norm

\[
|f|_{\vec{p}} \overset{def}{=} \left( \int_{R_1} dx_1 \left( \int_{R_2} dx_2 \cdots \left( \int_{R_l} f(x)^{p_1} dx_1 \right)^{p_2/p_1} \right)^{p_3/p_2} \cdots \right)^{1/p_l}.
\]

Note that in general case

\[
|f|_{p_1, p_2} \neq |f|_{p_2, p_1},
\]

but

\[
|f|_{p,p} = |f|_p.
\]

Observe also that if \( f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2) \) (condition of factorization), then

\[
|f|_{p_1, p_2} = |g_1|_{p_1} \cdot |g_2|_{p_2},
\]

(formula of factorization).

Let \( \psi = \psi (\vec{p}) \) be some continuous positive on the set \( Q = Q (p_{01}, p_{02}, \ldots, p_{0l}) \) function such that

\[
\inf_{\vec{p} \in Q} \psi (\vec{p}) > 0, \quad \psi (p) = \infty, \quad p \notin Q.
\]

We denote the set all of such a functions as \( \Psi_Q \).

The (multidimensional, anisotropic) Grand Lebesgue Spaces \( GLS = G_Q (\psi) = G_Q \psi \) space consists on all the measurable functions \( f : \mathbb{R}^n \to \mathbb{R} \) with finite norms

\[
||f||_{G_Q (\psi)} \overset{def}{=} \sup_{\vec{p} \in Q} \left[ |f| / \psi (\vec{p}) \right].
\]

The object of our investigation in this section is an inequality of a view

\[
|y|^{-\alpha} F[f](y) |q \leq K_{n, \alpha, \beta} (\vec{p}) \quad |x|^{\beta} f(x) |p\]

briefly:

\[
|y|^{-\alpha} F[f](y) |q \leq K_{n, \alpha, \beta} (\vec{p}) \quad |x|^{\beta} f(x) |p,
\]

(the classical version), or correspondingly

\[
||y|^{-\alpha} F[f](y) ||G \psi Q \leq || |x|^{\beta} f(x) ||G \psi Q,
\]
We introduce before the formulation the following notations. The connection between \( p_j \) and \( q_j \), \( j = 1, 2, \ldots, l \) may be described by the formulae
\[
\beta_j - \alpha_j = m_j \left( 1 - \frac{1}{p_j} - \frac{1}{q_j} \right). \tag{7.6}
\]

**Theorem 7.1.**

A. The condition (7.6) is necessary and sufficient for the existing and finiteness of the constant \( K_{n,\alpha,\beta}(\vec{p}) \) for the inequality (7.5).

B. If the condition (7.6) is satisfied, and \( \alpha_j, \beta_j \geq 0, \quad \alpha_j + \beta_j \leq 1 \), then the sharp (minimal) value of the coefficient \( K_{n,\alpha,\beta}(\vec{p}) \) satisfies the inequalities
\[
C_1(\alpha, \beta, \vec{m}) \prod_{j=1}^l \left[ \frac{p_j}{(p_j - m_j/(m_j - \beta_j))} \right]^{(\alpha_j + \beta_j)/m_j} \leq K_{n,\alpha,\beta}(\vec{p}) \leq C_2(\alpha, \beta, \vec{m}) \prod_{j=1}^l \left[ \frac{p_j}{(p_j - m_j/(m_j - \beta_j))} \right]^{\max(1, (\alpha_j + \beta_j)/m_j)}, \tag{7.7}
\]

\( \vec{p} \in Q = Q(p_0,1,p_0,2,\ldots,p_0,l) \).

**Proof.** We will use again the multidimensional generalization of dilation method. It is enough to consider only the two-dimensional case: \( l = 2 \). The inequality (7.4) has then a view:
\[
| y_1 |^{-\alpha_1} | y_2 |^{-\alpha_2} F[f](y) |_{q_1,q_2} \leq K_{n,\alpha,\beta}(p) \quad | x_1 |^{\beta_1} | x_2 |^{\beta_2} f(x) |_{p_1,p_2}. \tag{7.8}
\]

Let \( \lambda_1, \lambda_2 \) be two arbitrary independent positive numbers. We define the multidimensional dilation operator \( T_{\lambda_1,\lambda_2}[f] \) as follows:
\[
T_{\lambda_1,\lambda_2}[f](x_1, x_2) = f(\lambda_1 x_1, \lambda_2 x_2). \tag{7.9a}
\]

We have as before:
\[
F[T_{\lambda_1,\lambda_2}[f]](y_1, y_2) = \lambda_1^{-m_1} \lambda_2^{-m_2} F[f](y_1/\lambda_1, y_2/\lambda_2); \tag{7.9b}
\]
\[
| y_1 |^{-\alpha_1} | y_2 |^{-\alpha_2} F[T_{\lambda_1,\lambda_2}[f]](y_1, y_2) |_{q_1,q_2} = \lambda_1^{m_1/q_1 - m_1 - \alpha_1} \lambda_2^{m_2/q_2 - m_2 - \alpha_2} \quad | y_1 |^{-\alpha_1} | y_2 |^{-\alpha_2} F[f](y_1, y_2) |_{q_1,q_2}. \tag{7.9c}
\]

Further,
\[
| x_1 |^{\beta_1} | x_2 |^{\beta_2} T_{\lambda_1,\lambda_2}[f](x_1, x_2) |_{p_1,p_2} =
\]
\[ \lambda_1^{-\beta_1-m_1/p_1} \lambda_2^{-\beta_2-m_2/p_2} \left| x_1 \right|^\beta_1 \left| x_2 \right|^\beta_2 f(x_1, x_2) \mid_{p_1, p_2}; \quad (7.9) \]

We obtain substituting into (7.4) the function \( T_{\lambda_1, \lambda_2} [f] \) instead the function \( f \):

\[ \lambda_1^{m_1/q_1-m_1-\alpha_1} \lambda_2^{m_2/q_2-m_2-\alpha_2} \left| y_1 \right|^{-\alpha_1} \left| y_2 \right|^{-\alpha_2} F[f](y_1, y_2) \mid_{q_1, q_2} \leq \]

\[ \lambda_1^{-\beta_1-m_1/p_1} \lambda_2^{-\beta_2-m_2/p_2} \left| x_1 \right|^\beta_1 \left| x_2 \right|^\beta_2 f(x_1, x_2) \mid_{p_1, p_2}. \quad (7.10) \]

Since the values \( \lambda_1, \lambda_2 \) are arbitrary positive, the inequality may be satisfied only when

\[ m_1/q_1 - m_1 - \alpha_1 = -\beta_1 - m_1/p_1, \]
\[ m_2/q_2 - m_2 - \alpha_2 = -\beta_2 - m_2/p_2, \]

or equally

\[ \beta_1 - \alpha_1 = m_1 (1 - 1/p_1 - 1/q_1), \]
\[ \beta_2 - \alpha_2 = m_2 (1 - 1/p_2 - 1/q_2). \quad (7.11) \]

The example \( \hat{f}(x) \) for the lower bound may be constructed as before by means of factorization property of the anisotropic norm as a product of the \( m_j \) dimensional examples:

\[ \hat{f}(x) = \prod_{j=1}^{l} f_{0, m_j}(x_j). \]

This completes the proof of theorem 7.1.

As a consequence:

**Proposition 7.1.** Let \( \psi_Q \in \Psi_Q \); we define a new function \( \nu_Q = \nu_Q(\vec{p}) \in \Psi_Q, \vec{p} \in Q \) as follows:

\[ \nu_Q(\vec{p}) = \psi_Q(\vec{p}) \cdot \prod_{j=1}^{l} \left[ \frac{p_j}{(p_j - m_j/(m_j - \beta_j))} \right]^{\max(1, (\alpha_j + \beta_j)/m_j)}. \quad (7.12) \]

We assert:

\[ ||| y |^{-\alpha} F[f](y) ||| G\nu_Q \leq C(\vec{\alpha}, \vec{\beta}, \vec{m}) \cdot ||| x |^\beta f(x) ||| G\psi_Q, \quad (7.13) \]

## 8 PBO Inequality with Regular Varying Weight.

Let \( L = L(z), M = M(z), z \in (0, \infty) \) be two slowly varying simultaneously as \( z \to 0 \) and as \( z \to \infty \) continuous positive functions:

\[ \lim_{\lambda \to 0} \frac{L(\lambda z)}{L(\lambda)} = \lim_{\lambda \to \infty} \frac{L(\lambda z)}{L(\lambda)} = 1, \quad (8.1a) \]
\[
\lim_{\lambda \to 0} \frac{M(\lambda z)}{M(\lambda)} = \lim_{\lambda \to \infty} \frac{M(\lambda z)}{M(\lambda)} = 1. \quad (8.1b)
\]

We refer reader to the book of Seneta [77] to the using further facts about regular and slowly varying functions.

We investigate in this section the slight generalization of PBO inequality of a view:

\[
|y|^{-\alpha} M(|y|) F[f](y)|_q \leq K_{LM}(p) |x|^\beta L(|x|) f(x)|_p, \quad (8.2)
\]

so that both the functions

\[
z \to |z|^\beta L(|z|), \quad z \to |z|^{-\alpha} M(|z|)
\]

are regular varying simultaneously as \(z \to 0\) and as \(z \to \infty\).

For example,

\[
L(z) = \max\left(|\log z|^{\theta_1}, 1\right), \quad M(z) = \max(|\log z|^{\theta_2}, 1), \quad \theta_{1,2} = \text{const} \geq 0.
\]

**Theorem 8.1.**

**A.** The inequality (8.2) is true if and only if

\[
\alpha_j, \beta_j \geq 0, \quad \alpha < n, \quad \beta < n,
\]

\[
q < q_0 = n/\alpha, \quad p > p_0 = n/(n - \beta), \quad (8.3a)
\]

\[
\beta - \alpha = n(1 - 1/p - 1/q), \quad L(1/\lambda) \asymp M(\lambda), \quad \lambda \in (0, \infty), \quad (8.3b)
\]

in the sense that

\[
0 < \inf_{\lambda > 0} \frac{L(1/\lambda)}{M(\lambda)} \leq \sup_{\lambda > 0} \frac{L(1/\lambda)}{M(\lambda)} < \infty. \quad (8.3c)
\]

**B.** If the conditions (8.3a), (8.3b), (8.3c) are satisfied, then the function \(K_{LM}(p)\) satisfies at the same restrictions as the function \(K_{PBO}(p)\):

\[
C_3(L, M; \alpha, \beta, n) \left[ \frac{p}{p - p_0} \right]^{(\alpha + \beta)/n} \leq K_{LM}(p) \leq C_4(L, M; \alpha, \beta, n) \left[ \frac{p}{p - p_0} \right]^{\max(1, (\alpha + \beta)/n)}. \quad (8.4)
\]

**Proof.** The estimates (8.4) are proved as before, as in [3], with as the same counterexample. The using in [3] weight interpolation inequalities see, e.g. in the book [4], chapter 4. Namely, if
|U[f]|q_i \leq M_i|f \cdot u_i|_{p_i}, i = 0, 1;

\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},

\frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1},

u = u_0^{1 - \theta} \cdot u_1^\theta, v = v_0^{1 - \theta} \cdot v_1^\theta, u_i, v_i \geq 0,

then

|U[f] \cdot v|_q \leq M_0^{1 - \theta} \cdot M_1^\theta |f \cdot u|_p.

Note only that if \( v_0, v_1 \) are slowly varying, then the function \( v(\cdot) \) is also slowly varying.

It remains to prove the assertion A. We will use as before the dilation method.

Suppose for some function \( f \in S(R^n) \), \( f \neq 0 \)

\[ |[y]^{-\alpha} F[f](y) L(|y|/\lambda) \mid_q \leq K_{LM}(p) \mid |x|^{\beta} f(x) M(|x|)|_p. \tag{8.5} \]

We obtain applying (8.5) to the function \( T_\lambda f(x) = f(\lambda x) \):

\[ \lambda^{-n + n/q - \alpha} |[y]^{-\alpha} F[f](y) L(|y|/\lambda) \mid_q \leq K_{LM}(p) \lambda^{-n/p - \beta} \mid |x|^{\beta} f(x) M(\lambda x) \mid_p. \tag{8.6} \]

As long as the functions \( L, M \) are slowly varying, we conclude that as \( \lambda \to \infty \) or \( \lambda \to 0^+ \):

\[ \lambda^{-n + n/q - \alpha} L(1/\lambda) |[y]^{-\alpha} F[f](y) \mid_q \leq K_{LM}(p) \lambda^{-n/p - \beta} M(\lambda) \mid |x|^{\beta} f(x) \mid_p. \tag{8.7} \]

The last inequality (8.7) may be satisfied only in the case when

\[ -n + n/q - \alpha = -n/p - \beta, \quad L(1/\lambda) \simeq M(\lambda), \]

Q.E.D.

Analogously may be proved the multidimensional generalization of inequality (8.2). Indeed, let us consider the following inequality:

\[ |[y]^{-\alpha} F[f](y) \prod_{j=1}^l M_j(|y_j|)|_q \leq K_{L,M,n,\alpha,\beta}(\vec{p}) \cdot |[x]^{\beta} f(x) \prod_{j=1}^l L_j(|x_j|)|_{\vec{p}}, \tag{8.8} \]
where \( L_j = L_j(z), \ M_j = M_j(z), \ z \in (0, \infty), \ j = 1, 2, \ldots, l \) are slowly varying simultaneously as \( z \to 0 \) and as \( z \to \infty \) continuous positive functions, \( \vec{p} = \{p_j\}, \ \vec{q} = \{q_j\}, \)

\[
p_j \geq p_{j,0} \overset{\text{def}}{=} \frac{m_j}{m_j - \beta_j}, \quad 1 \leq q_j < q_{j,0} \overset{\text{def}}{=} \frac{m_j}{\alpha_j},
\]

\( 0 \leq \alpha_j, \beta_j < 1. \)

**Theorem 8.2.**

**A.** If the inequality (8.8) there holds for some non-zero function \( f, \ f \in S(R^n) \), then

\[
\frac{\beta_j - \alpha_j}{m_j} = 1 - \frac{1}{p_j} - \frac{1}{q_j}, \quad (8.9)
\]

\[
L_j(1/\lambda) \backsimeq M_j(\lambda). \quad (8.10)
\]

**B.** If the equations (8.9) and (8.10) are satisfied, then

\[
C_5(L, M; \vec{\alpha}, \vec{\beta}, \vec{m}) \prod_{j=1}^{l} \left[ \frac{p_j}{(p_j - m_j/(m_j - \beta_j))} \right]^{(\alpha_j + \beta_j)/m_j} \leq K_{L, M; n, \vec{\alpha}, \vec{\beta}}(\vec{p}); \quad (8.11)
\]

(anisotropic version).

Evidently, if

\[
\frac{\beta_j - \alpha_j}{m_j} = \text{const}, \ j = 1, 2, \ldots, l;
\]

then for the values

\[
p > \max_j \frac{m_j}{m_j - \beta_j}
\]

there holds

\[
|y|^{-\vec{\alpha}} F[f](y) \prod_{j=1}^{l} M_j(|y_j|)|_q \leq K_{L, M; n, \vec{\alpha}, \vec{\beta}}(p) \cdot |x|^{\vec{\beta}} f(x) \prod_{j=1}^{l} L_j(|x_j|)|_p, \quad (8.12)
\]

\[
\frac{\beta_j - \alpha_j}{m_j} = 1 - \frac{1}{p} - \frac{1}{q},
\]

\[
C_5(L, M; \vec{\alpha}, \vec{\beta}, \vec{m}) \prod_{j=1}^{l} \left[ \frac{p}{(p - m_j/(m_j - \beta_j))} \right]^{(\alpha_j + \beta_j)/m_j} \leq K_{L, M; n, \vec{\alpha}, \vec{\beta}}(p).
\]
\[ C_6(L, M; \vec{\alpha}, \vec{\beta}, \vec{\gamma}) \prod_{j=1}^{l} \left[ \frac{p}{p - m_j/(m_j - \beta_j)} \right]^{\max(1, (\alpha_j + \beta_j)/m_j)}. \quad (8.13) \]

(isotropic version).

9 Concluding remarks.

1. We introduce the so-called weight \( L_p \) norm as follows. Let \( \mu \) be arbitrary constant. Denote for the measurable function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \):

\[ |g|_{p, \mu} = \left[ \int_{\mathbb{R}^n} |g(x)|^p |x|^\mu \, dx \right]^{1/p}. \]

Let the parameters \( \alpha, \beta, \lambda, \mu; p, q \) be such that \( n - \lambda > \alpha, \quad q_0 := (n - \lambda)/\alpha > 1, \quad \alpha, \beta \in (0, 1), \ \lambda, \mu \geq 0, \ p > p_0 := (n + \mu)/(n - \beta), q \in [1, q_0), \)

\[ \beta - \alpha = n - n \left( \frac{1}{p + \mu/\beta} + \frac{1}{q + \lambda/\alpha} \right). \]

The proposition of theorem 4.1 may be rewritten after change of parameters as follows. Let us denote

\[ K_{\lambda, \mu}(p) = \sup_{f \neq 0, \in L_{p, \mu}} \left[ \frac{|x|^{-\alpha} F[f]|q-\lambda|}{|x|^{\beta} |f|_{p, \mu}} \right]; \]

then

\[ C_1(\alpha, \beta, \lambda, \mu)[p/(p - p_0)]^{1/q_0+1/p_0} \leq K_{\lambda, \mu}(p) \leq \]

\[ C_2(\alpha, \beta, \lambda, \mu)[p/(p - p_0)]^{\max(1, 1/q_0+1/p_0)}, \ \ p > p_0. \]

2. We introduce and calculate in this pilcrow the multidimensional Boyd’s indices for anisotropic Grand Lebesgue Spaces, which play a very important role in the theory of Fourier series [2], chapter 6.7.

In detail, let \( X = R_+^1 \times R_+^1 \) with ordinary Lebesgue measure. Let \( G_D, \psi, \psi : (a, b) \times (c, d) \rightarrow R_+ \) be some function from the set \( \Psi_D, D = (a, b) \times (c, d), 1 \leq a < b < \infty, 1 \leq c < d \leq \infty \). We introduce the multidimensional Boyd’s indices as follows. Denote for the values \( s, t > 0 \) the multidimensional (in our case- two dimensional) dilation operator

\[ \Delta_{s,t}[f](x, y) = f(x/s, y/t), \]

and define
\[ \overline{\alpha}(G\psi_D) \overset{\text{def}}{=} \lim_{s \to \infty} \frac{\log[||\Delta_{s,t}||G\psi_D]}{\log s}, \]
\[ \underline{\alpha}(G\psi_D) \overset{\text{def}}{=} \lim_{s \to 0^+} \frac{\log[||\Delta_{s,t}||G\psi_D]}{\log s}, \]
\[ \overline{\beta}(G\psi_D) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{\log[||\Delta_{s,t}||G\psi_D]}{\log t}, \]
\[ \underline{\beta}(G\psi_D) \overset{\text{def}}{=} \lim_{t \to 0^+} \frac{\log[||\Delta_{s,t}||G\psi_D]}{\log t}. \]

We conclude analogously to the article [61]:
\[ \overline{\alpha}(G\psi_D) = 1/a, \underline{\alpha}(G\psi_D) = 1/b, \]
\[ \overline{\beta}(G\psi_D) = 1/c, \underline{\beta}(G\psi_D) = 1/d. \]

3. The "dual" version of PBO inequality may be formulated as follows:
\[ | |x|^{-\alpha} f(x)|_q \leq K_{PBO}(p) \ | |y|^{\beta} F[f](y)|_p, \]
where as before
\[ 1 \leq q < n/\alpha, \quad p > n/(n - \beta), \quad \beta - \alpha = n(1 - 1/p - 1/q). \]

4. In the terms of anisotropic norms may be estimated the \( L_p \to L_q \) norm of the integral operators of a view
\[ W[f](x) = \int_Y K(x, y) f(y) \nu(dy), \ x \in X. \]
Namely, it is proved in [30], p. 272 that
\[ |W|(L_p \to L_q) \leq |K|_{q_1, p}, \ q_1 = q/(q - 1), \ q > 1, p \in (1, \infty). \]

References

[1] BADIALE M., TARANTELLO G. A Sobolev-Hardy inequalities with applications to a nonlinear elliptic equations arising in astrophysics, Arch. Rat. Mech. Anal., 163, 4, (2002), 259-293.

[2] BECKNER W. Inequalities in Fourier analysis on \( \mathbb{R}^n \). Proceedings of the National Academy of Science, USA, (1975), V. 72, 638-641.
[3] Beckner W. Pitt’s inequality with sharp convolution estimates. Proc. AMS, v. 136, No 5, (2008), 1871-1875.

[4] Bennett G, Sharpley R. Interpolation of operators. Orlando, Academic Press Inc., (1988).

[5] Besov O.V., Il’in V.P., Nikolskii S.M. Integral representation of functions and imbedding theorems. Vol.2; Scripta Series in Math., V.H.Winston and Sons, (1979), New York, Toronto, Ontario, London.

[6] Boas P.R. Integrability Theory for Trigonometrical Transforms. Springer Verlag, (1967).

[7] J.S.Bradley. Hardy inequalities with mixed norms. Canadian Math. Bull., 21(1978), p. 405-408.

[8] Buldygin V.V., Mushtary D.I., Ostrovsky E.I, Pushalsky M.I. New Trends in Probability Theory and Statistics. Moklas, 1992, Amsterdam, New York, Tokyo.

[9] Chen Y.M. On the integrability of function defined by trigonometrical series. Math. Z., 66, (1956), p. 9-12.

[10] Chen Y.M. Some asymptitic properties of Fourier constants and integration theorems. Math. Z., 68, (1957), p. 227-244.

[11] Dunford N., Schwartz J.T. Linear operators. Interscience Publishers, (1958), New York, London.

[12] Edwards R.E. Fourier Series. A modern Introduction. 1982, v.2; Springer Verlag, 1982. Berlin, Heidelberg, Hong Kong, New York.

[13] Evans L.C. Partial Differential Equations. Second Edition, Graduate Studies in Mathematics, Volume 19,AMS, Providence, Rhode Island, (2010).

[14] D. Dos Santos Ferreira, Staubach W. Global and local regularity of Fourier integral operators on weighted and unweighted Spaces. arXiv:11040234V1 [math.AP] 1Apr 2011.

[15] Capone C., Fiorenza A., Krbec M. On the Extrapolation Blowups in the $L_p$ Scale. Collectanea Mathematica, 48, 2, (1998), 71 - 88.

[16] Fiorenza A. Duality and reflexivity in grand Lebesgue spaces. Collectanea Mathematica (electronic version), 51, 2, (2000), 131 - 148.

[17] Fiorenza A., and Karadzhov G.E. Grand and small Lebesgue spaces and their analogs. Consiglio Nazionale Delle Ricerche, Instituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).
[18] Ghoussoub N., Yuang C. Multiple solutions for quasi-linear PDE involving the critical Sobolev and Hardy exponents. Trans. AMS, 352, (2000), No 12, 5703-5743.

[19] Gorbachov D., Liflyand E. and Tikhonov S. Weighted Fourier Inequalities for Radial Functions. Oberwolfach Preprints, OWP 2009 - 26, 1-17.

[20] Sinnamon G. The Fourier Transform in Weighted Lorentz Spaces. Publ. Math., (2003), v. 47, p. 3 - 29.

[21] Grafakos L., Montgomery-Smith S., and Motunich O. A sharp estimate of constants in maximal inequalities. Studia Math., 134, (1999), 57-67.

[22] Grafakos L., Montgomery-Smith S. Best constants for uncentered maximal functions. arXiv:math(94122v2), [math.FA], 6 Dec 1999.

[23] Hardy G.H., Littlewood J.E., Polya G.P. Inequalities. Cambridge, At the University Press, (1952).

[24] Herbst I.W. Spectrel Theory of the operator $(p^2 + m^2)^{1/2} - Ze^2 / r$, Comm. Math. Phys., 53, (1977), 285-294, MR0436854 (55:9790).

[25] Hernandes E., Weiss G. A First Course on Wavelets. (1996), CRC Press, Boca Raton, New York.

[26] Hichem Hajaiej. Some fractional inequalities and applications to some minimization constrained problems involving a local linearity. arXiv:1104.1414v1 [math.FA] 7 Apr 2011.

[27] Zhongwei Shen and Zhao Peihao. Uniform Sobolev Inequalities and Absolute Continuity of Periodic Operators. Transactions of the AMS., v 3, 60, No 4, (2008), 1741-1758.

[28] Iwaniec T., and Sbordone C. On the integrability of the Jacobian under minimal hypotheses. Arch. Rat.Mech. Anal., 119, (1992), 129 - 143.

[29] Iwaniec T., P. Koskela P., and Onninen J. Mapping of finite distortion: Monotonicity and Continuity. Invent. Math. 144 (2001), 507 - 531.

[30] Jorgens K. Linear Integral Operators. Pitman Advansed Publishing Program. (1970), Boston-London-Melbourne.

[31] Jurkat W.B., Sampson G. On rearrangement and weight inequalities for the Fourier transform. Indiana Univ. Math.J., 33, (1984), 257-270.

[32] Kaczmarz S., Steinhaus H. Theory der Orthogonalreihen. Chelsea Publishing Company, (1951), New York.

[33] Kantorovich L.V., Akilov G.P. Functional Analysis. (1987) Kluvner Verlag.
[34] Kozachenko Yu. V., Ostrovsky E.I. (1985). The Banach Spaces of random Variables of subgaussian type. *Theory of Probab. and Math. Stat.* (in Russian). Kiev, KSU, **32**, 43 - 57.

[35] Krasnoselsky M.A., Routisky Ya. B. Convex Functions and Orlicz Spaces. P. Noordhoff Ltd, (1961), Groningen.

[36] Krein S.G., Petunin Yu.V., Semenov E.M. Interpolation of linear Operators. New York, (1982).

[37] Leindler L. Generalization of inequality of Hardy and Littlewood. *Acta Sci. Math.*, (Szeged), **31**, (1970), 279-285.

[38] Leoni G. A first Course in Sobolev Spaces. Graduate Studies in Mathematics, v. 105, AMS, Providence, Rhode Island, (2009).

[39] Liflyand E., Ostrovsky E., Sirota L. Structural Properties of Bilateral Grand Lebesgue Spaces. *Turk. J. Math.*, **34** (2010), 207-219.

[40] Liflyand E., Tikhonov S. Extended solution of Boas’ conjecture on Fourier transform. *C.R. Math.*, Vol. 346, Is. 21-22, (2008), 1137-1142.

[41] Lerner A., Ombrosi S. and Perez G. $A_1$ bounds for Calderon-Zygmund operators related to a problem of Muckenhoupt and Wheeden. *Math. Res. Lett.*, **16**, (2009), 149-156.

[42] V.Maz’ya. Sobolev Spaces. Kluvner Academic Verlag, (2002), Berlin-Heidelberg-New York.

[43] D.S.Mitrinovich, J.E. Pecaric and A.M.Fink. Inequalities Involving Functions and Their Integrals and Derivatives. Kluvner Academic Verlag, (1996), Dorderecht, Boston, London.

[44] Melas A.D. The best constant for the centered Hardy-Littlewood maximal inequality. *Annales of Math.*, **157**, (2003, p. 647-688.

[45] Muckenhoupt B. Weighted norm inequalities for the Fourier Transform. *Transactions of Amer. Math.Soc.*, V. 276, No 2, April 1983.

[46] Nagase M. The $L^p$ boundedness of pseudo-differential operators with non-regular symbols. *Comm. PDE*, **2**, (1977), 1045-1061.

[47] Okikiolu G.O. Aspects of the Theory of Bounded Integral Operators in $L^p$ spaces. Academic Press, London, New York, (1971).

[48] Ostrovsky E.I. Exponential Estimations for Random Fields. Moscow - Obninsk, OINPE, 1999 (Russian).

[49] Ostrovsky E. and Sirota L. Moment Banach spaces: theory and applications. *HIAT Journal of Science and Engineering*, **C**, Volume 4, Issues 1 - 2, pp. 233 - 262, (2007).
[50] E. Ostrovsky, L. Sirota. Multidimensional Dilation Operators, Boyd and Shimogaki indices of Bilateral Weight Grand Lebesgue Spaces. arXiv:0809.3011[math.FA] 17 Sep 2008.

[51] Sagher Y. Integrability conditions for the Fourier transform. J. Math. Anal. Appl., 54, (1976), 151-156.

[52] E. Ostrovsky, L. Sirota. Adaptive estimation of multidimensional spectral densities. Proceedings of Institute for Advanced Studies, Arad, Israel, (2005), issue 5, p. 42-48.

[53] E. Ostrovsky and L. Sirota. Adaptive multidimensional-time spectral measurements in technical diagnosis. Communications in dependability and Management (CDQM), Vol. 9, No 1, (2006), pp. 45-50.

[54] E. Ostrovsky and L. Sirota. Moment Banach spaces: theory and applications. HIAT Journal of Science and Engineering, C, Volume 4, Issues 1 - 2, pp. 233 - 262, (2007).

[55] E. Ostrovsky, L. Sirota. Adaptive optimal measurements in the technical diagnostics, reliability theory and information theory. Proceedings 5th international conference on the improvement of the quality, reliability and long usage of technical systems and technological processes, (2006), Sharm el Sheikh, Egypt, p. 65-68.

[56] L. Sirota. Reciprocal Spectrums in Technical Diagnosis. Proceedings of the International Symposium on STOCHASTIC MODELS in RELIABILITY, SAFETY, SECURITY and LOGISTICS, (2005), Sami Shamoon College of Engineering, Beer-Sheva, Israel, p. 328-331.

[57] E. Ostrovsky, E. Rogover, L. Sirota. Adaptive Multidimensional Optimal Signal Energy Measurement against the Background Noise. Program and Book of Abstracts of the International Symposium on STOCHASTIC MODELS in RELIABILITY ENGINEERING, LIFE SCIENCES and OPERATION MANAGEMENT, (SMRLO’10), (2010), Sami Shamoon College of Engineering, Beer-Sheva, Israel, p. 174.

[58] E. Ostrovsky, E. Rogover, L. Sirota. Optimal Adaptive Signal Detection and Measurement. Program and Book of Abstracts of the International Symposium on STOCHASTIC MODELS in RELIABILITY ENGINEERING, LIFE SCIENCES and OPERATION MANAGEMENT, (SMRLO’10), (2010), Sami Shamoon College of Engineering, Beer-Sheva, Israel, p. 175.

[59] E. Ostrovsky, L. Sirota. Adaptive Regression Method in the Technical Diagnostics. Proceedings of the National Conference “Scientific Researches in the Field of the Control and Diagnostics”, Arad, Israel, (2006), Publishing of Institute for Advanced Studies, p. 35-38.
[60] E.Ostrovsky, L.Sirota. SOME SINGULAR OPERATORS IN THE BIDE - SIDE GRAND LEBESGUE SPACES. All-Russia School-Conference for Undergraduate and Postgraduate Students and Young Scientists "Fundamental Mathematics and its Applications in Natural Sciences", Articles, Mathematics, vol. 2, Ufa: BashSU, (2008), pp. 241-249.

[61] E.Ostrovsky, E.Rogover and L.Sirota. Riesz’s and Bessel’s operators in in bilateral Grand Lebesgue Spaces. arXiv:0907.3321 [math.FA] 19 Jul 2009.

[62] E. Ostrovsky and L.Sirota. Weight Hardy-Littlewood inequalities for different powers. arXiv:0901.4609v1[math.FA] 29 Oct 2009.

[63] E. Ostrovsky E. Bide-side exponential and moment inequalities for tail of distribution of Polynomial Martingales. Electronic publication, arXiv: math.PR/0406532 v.1 Jun. 2004.

[64] E.Ostrovsky, E.Rogover and L.Sirota. Integral Operators in Bilateral Grand Lebesgue Spaces. arXiv:09012.7601v1 [math.FA] 16 Dez. 2009.

[65] E.Ostrovsky, L.Sirota. Nikolskii-type inequalities for rearrangement invariant spaces. arXiv:0804.2311v1 [math.FA] 15 Apr 2008.

[66] Ostrovsky E., Sirota L. Universal adaptive estimations and confidence intervals in the non-parametrical statistics. arXiv.mathPR/0406535 v1 25 Jun 2004.

[67] Ostrovsky E., Sirota L. Optimal adaptive nonparametric denoising of Multidimensional-time signal. arXiv:0809.30211v1 [physics.data-an] 17 Sep 2008.

[68] Ostrovsky E. Exponential Orlicz’s spaces: new norms and applications. Electronic Publications, arXiv/FA/0406534, v.1, (25.06.2004.)

[69] Ostrovsky E., Sirota L. Some new rearrangement invariant spaces: theory and applications. Electronic publications: arXiv:math.FA/0605732 v1, 29, (May 2006);

[70] Ostrovsky E., Sirota L. Fourier Transforms in Exponential Rearrange- ment Invariant Spaces. Electronic publications: arXiv:math.FA/040639, v1, (20.6.2004.)

[71] Ostrovsky E., Sirota L. Nikolskii-type inequalities for rearrangement Invariant spaces. Electronic Publications, arXiv:0804.2311 v1 [math.FA] 15 Apr 2008.

[72] Ostrovsky E. Bide-side exponential and moment inequalities for tail of distribution of Polynomial Martingales. Electronic publication, arXiv: math.PR/0406532 v.1 Jun. 2004.
[73] Ostrovsky E., Sirota L. Fourier Transforms in Exponential Rearrangement Invariant Spaces. Electronic Publ., arXiv:Math., FA/040639, v.1, 20.6.2004.

[74] C. Otriz-Caraballo. Quadratic $A_1$ bounds for commutators of singular integrals with BMO functions. arXiv:1104.1069v1, [math.CA] 6 Apr 2011.

[75] Pichorides S.K. On the best values of the constant in the theorem of M.Riesz, Zygmund and Kolmogorov. Studia Math., 44, (1972), 165 - 179.

[76] Juan Arias de Reyna. Pointwise convergence of Fourier Series. Lecture Notes in Mathematics, Springer Verlag, (2002), Berlin-Heidelberg-New-York-Toronto-Tokyo).

[77] Seneta E. Regularly Varying Functions. Springer Verlag, 1985. Russian edition, Moscow, Science, 1985.

[78] Stein E.M., Strömberg J.-O. Behavior of maximal function in $\mathbb{R}^n$ for large $n$. Ark. Math., 21 (1983), 259-269.

[79] Stein E.M.; Weiss G. Fractional integrals on $n$ – dimensional Euclidean space. J. Math. Mach., 7 (1958), 503-514; MR 0098285.

[80] E.M. Stein. Oscillating Integral in Fourier Analysis. In: Beijing Lectures in Harmonic Analysis, Princeton University Press, (1986), p. 307 - 355.

[81] Ledoux M., Talagrand M. (1991) Probability in Banach Spaces. Springer, Berlin, MR 1102015.

[82] Talenti G. Inequalities in Rearrangement Invariant Function Spaces. Nonlinear Analysis, Function Spaces and Applications. Prometheus, Prague, 5, (1995), 177-230.

[83] Taylor M.E. Partial Differential Equations. Applied Math. Sciences, 117, Volume 3, (1996), Springer Verlag.

[84] Taylor M.E. Pseudodifferential Operators. Princeton University Press, Princeton, New Jersey, (1981).

[85] Titchmarsh E.C. Introduction to the Theory of Fourier Integrals. Claredon Press, Second Edition. 1948. Oxford.

[86] Vassilev D. $L^p$ Estimates and Asymptotic Behavior for finite Energy Solutions of Extremals to Hardy-Sobolev Inequalities. Trans. of the AMS, V. 363, No 1, January 2011, p. 37-62.

[87] Yu D.S., Zhou P., Zhou S.P. On $L^p$ integrability and convergence of trigonometric series. Studia Mathematica, 182, 3, (2007), p. 215-226.