The relativistic phase space and Newman-Penrose basis

Yaakov Friedman
Jerusalem College of Technology
P.O.B. 16031 Jerusalem 91160
Israel

Abstract

We define a complex relativistic phase space which is the space $\mathbb{C}^4$ equipped with the Minkowski metric and with a geometric tri-product on it. The geometric tri-product is similar to the triple product of the bounded symmetric domain of type IV in Cartan’s classification, called the spin domain. We show that there are two types of tripotents — the basic elements of the tri-product in the relativistic phase space. We construct a spectral decomposition for elements of this space. A description of compatibility of element of the relativistic phase space is given. We show that the relativistic phase space has two natural bases consisting of compatible tripotents. The first one is the natural basis for four-vectors and the second one is the Newman-Penrose basis. The second one determines Dirac bi-spinors on the phase space. Thus, the relativistic phase space has similar features to the quantum mechanical state space.

1 Introduction

In [6] we introduced a complex relativistic phase space as follows. We equip the space $\mathbb{C}^4$ with an inner product based on the Lorentz metric and define a new geometric tri-product on it. This space is used to represent both space-time coordinates and the four-momentum of an object. The real part of the inner product extends the notion of an interval, while the imaginary part extends the symplectic structure of the classical phase space. We construct both spin $1$ and spin $1/2$ representations of the Poincaré group by natural operators of the tri-product on the phase space.

We study now the algebraic properties of the complex relativistic phase space. We define and describe the tripotents — the basic elements of the tri-product and show that there are only two types of such tripotents. The first type will be called maximal tripotents and the second one minimal ones. We define orthogonality of tripotents and show that each element can be decomposed as a linear combination of orthogonal tripotents. This show that the elements of
the relativistic phase space have a decomposition similar to the decomposition of states in quantum mechanics.

We define the notion of compatibility of tripotents, which is slightly weaker than the one used in quantum mechanics. We give a characterization when two tripotents are compatible. We show that the relativistic phase space has two natural bases consisting of compatible tripotents. The first one is the natural basis for four-vectors and the second one is the Newman-Penrose basis.

2 A complex relativistic phase space

For description of classical motion of a point-like body, the classical phase space, composed from space position and the 3-momentum, is used. A symplectic structure on the phase space provides information on the scalar product and the antisymmetric symplectic form of this space. For the classical phase space this structure can be expressed efficiently by introducing the complex structure and a scalar product under which the classical phase space becomes equal to \( \mathbb{C}^3 \), see [8] and [1].

For description of the motion of a relativistic point-like body we may use the relativistic phase space. This is obtained by adding the time and the energy variables to the classical phase space. So, the relativistic phase space may be identified with \( \mathbb{C}^4 \), in which the real part expresses the 4-momentum and the imaginary part represents the space-time position of a point-like body. To extend the symplectic structure to this space, we will use the complex-valued scalar product introduced by E. Cartan, see [3]. Since the scalar product needs to provide information on the interval of the 4-vectors, we replace the Euclidian metric, used usually, with the Lorentzian one.

We define a scalar product \( \langle \cdot | \cdot \rangle \) on \( \mathbb{C}^4 \) with natural basis \( \{ u_\mu \} \) as follows: For two arbitrary vectors \( a, b \in \mathbb{C}^4 \) it is given by

\[
\langle a | b \rangle = \eta_{\mu\nu} a^\mu b^\nu,
\]

(1)

where \( a = a^\mu u_\mu \) and \( b = b^\nu u_\nu \). Evidently, this scalar product is bilinear and symmetric. For an arbitrary element \( a \in \mathbb{C}^4 \), the scalar square is given by

\[
a^2 = \langle a | a \rangle = \eta_{\mu\nu} a^\mu a^\nu,
\]

(2)

which is a complex number, not necessary positive or even real.

Note that the real subspace \( M \) defined by the real vectors of \( \mathbb{C}^4 \) with the bilinear form (2) may be identified with the Minkowski space. The same is true for the pure imaginary subspace \( iM \) defined by pure imaginary vectors. We use the subspace \( M \) to represent the four-vector momentum \( p = p^\mu u_\mu \) and the subspace \( iM \) to represent the space-time coordinates \( ix = x^\mu iu_\mu \) of a point-like body in an inertial system. Thus, the space \( \mathbb{C}^4 \) represents both space-time and the four-momentum as

\[
a = a^\mu u_\mu \quad \text{with} \quad a^\mu = p^\mu + ix^\mu.
\]
For any two vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{C}^4 \) we may consider also a scalar product \( \langle \mathbf{a} | \mathbf{b} \rangle = \eta_{\mu\nu} \bar{a}^\mu \bar{b}^\nu \). The Lorentzian scalar product is the real part

\[
Re \langle \mathbf{a} | \mathbf{b} \rangle = \frac{1}{2} \eta_{\mu\nu} (\bar{a}^\mu \bar{b}^\nu + a^\mu b^\nu), \tag{3}
\]

which is symmetric and extends the Lorentzian product (and the notion of an interval) from the subspaces \( M \) and \( iM \) to \( \mathbb{C}^4 = M \oplus iM \).

The imaginary part defines a skew scalar product

\[
[a, b] = Im \langle \mathbf{a} | \mathbf{b} \rangle = \frac{1}{2i} \eta_{\mu\nu} (\bar{a}^\mu \bar{b}^\nu - a^\mu b^\nu), \tag{4}
\]

which extends the symplectic skew scalar product. This bracket can be used to define the Poisson bracket of two functions and two vector fields. Thus, the space \( \mathbb{C}^4 \) with the scalar product (1) can be used as a basis for a relativistic phase space.

**Definition** Let \( \mathbb{C}^4 \) denote a 4-dimensional complex space with the scalar product (1). A geometric tri-product \( \{ , , \} : \mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}^4 \) is defined for any triple of elements \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) as

\[
d = \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \} = \langle \mathbf{a} | \mathbf{b} \rangle \mathbf{c} - \langle \mathbf{c} | \mathbf{a} \rangle \mathbf{b} + \langle \mathbf{b} | \mathbf{c} \rangle \mathbf{a} \tag{5}.
\]

In the basis \( \{ \mathbf{u}_\mu \} \) definition (5) takes the form

\[
d^\mu = \eta_{\alpha\beta} a^\alpha b^\beta c^\mu - \eta_{\alpha\beta} c^\alpha a^\beta b^\mu + \eta_{\alpha\beta} b^\alpha c^\beta a^\mu. \tag{6}
\]

For any pair of elements \( \mathbf{a}, \mathbf{b} \in \mathbb{C}^4 \) we define a linear map \( D(\mathbf{a}, \mathbf{b}) : \mathbb{C}^4 \to \mathbb{C}^4 \) as

\[
D(\mathbf{a}, \mathbf{b}) \mathbf{c} = \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \}. \tag{7}
\]

It is easy to verify that the geometric tri-product satisfies the following properties:

**Proposition** The tri-product, defined by (5), satisfies:

1. \( \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \} \) is complex linear in all variables \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \).

2. The triple product is symmetric in the pair of outer variables

\[
\{ \mathbf{a}, \mathbf{b}, \mathbf{c} \} = \{ \mathbf{c}, \mathbf{b}, \mathbf{a} \}. \tag{8}
\]

3. For arbitrary \( \mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbb{C}^4 \), the following identity holds

\[
[D(\mathbf{x}, \mathbf{y}), D(\mathbf{a}, \mathbf{b})] = D(D(\mathbf{x}, \mathbf{y}) \mathbf{a}, \mathbf{b}) - D(\mathbf{a}, D(\mathbf{y}, \mathbf{x}) \mathbf{b}). \tag{9}
\]

Properties of the previous proposition are the defining properties for the Jordan triple products associated with a homogeneous spaces, see [9] and [10]. If the Euclidean inner product of \( \mathbb{C}^4 \) is used in the definition (5), this triple product is the triple product of the bounded symmetric domain of type IV.
in Cartan’s classification, called the spin factor. A similar triple product was obtained \cite{5} for the ball of relativistically admissible velocities under the action of the conformal group. As we will see later, the geometric tri-product \cite{5} is useful in defining the action of the Lorentz group on $C^4$.

The space $C^4$ with a form \cite{11} for the given metric tensor and a geometric tri-product \cite{5} will be denoted by $S^4$. As we have seen, the space $S^4$ can be used to represent the space-time coordinates and the relativistic momentum variables. The form \cite{11} on it defines both the interval and the symplectic form and the tri-product may be used to define the action of the Lorentz group. Thus, we propose to call $S^4$ the complex relativistic phase space.

3 Conjugation on $S^4$

In order to be able to use effectively the tri-product we will need to introduce an observer-dependent conjugation on the Relativistic Phase Space, defined as follows:

**Definition** Let $C^4$ be the complex 4-dimensional space with a metric tensor $\eta_{\mu\nu}$ on it. For arbitrary element $b = b^\mu u_\mu$ define a conjugate $\hat{b}$ as

$$\hat{b} = \bar{b}^0 u_0 - \bar{b}^j u_j = \hat{b}^\mu u_\mu, \quad j = 1, 2, 3,$$

(10)

with $\bar{b}^\mu$ denoting the complex conjugate of $b^\mu$.

This conjugation is complex conjugate linear and combines complex conjugation with space reversal. Since this conjugation involve space reversal, it is observer dependent and is not Lorentz invariant. But this is the only deviation from Lorentz invariance that we will need. This correspond to generally excepted idea that Quantum Mechanical effects are observer-dependent.

With this conjugation

$$< \hat{b} | b > = \sum |b^\mu|^2 = |b|^2$$

(11)

coincides with the Euclidian norm in $C^4$ and

$$< \hat{a} | \hat{b} > = < a | \bar{b} > .$$

(12)

It can be considered as a map to the dual space under the Euclidian norm. Note that $|b|^2 \geq 0$ and equal to zero only if $b = 0$.

The conjugation is a triple automorphism, meaning

$$\{a, \hat{b}, \hat{c}\} = \{a, b, c\}.$$

(13)

By use of this conjugation, for any element $a \in S^4$ we may introduce a linear operator

$$D(a)c = D(a, \hat{a})c, \quad c \in S^4.$$

(14)
4 Tripotents of the spin space

The tripotents of the tri-product replace the building blocks for binary operations - the idempotents, defined as non-zero elements $p$ that satisfy $p^2 = p$. For a ternary operation, the building blocks are the tripotents defined as:

**Definition** A non-zero elements $u$ of $S^4$ is called a tripotent if $u^{(3)} = \{u, \hat{u}, u\} = D(u)u = u$.

From the definition of a tripotent $u \in S^4$ and the definition of the triple product \[ \{u, \hat{u}, u\} = 2 < \hat{u}|u > u - u^2\hat{u} = 2|u|^2u - u^2\hat{u} = u, \tag{15} \]
or, equivalently,

\[ (2|u|^2 - 1)u - u^2\hat{u} = 0, \iff \alpha u - \beta \hat{u} = 0 \tag{16} \]

for some constants $\alpha, \beta$.

If $\alpha \neq 0$, denote $\nu = \beta/\alpha$. Then $u = \nu\hat{u}$. By taking the conjugate of this equation we get $\hat{u} = \overline{\nu}u = \overline{\nu}u$ implying $|\nu| = 1$. Now $|u|^2 = < \hat{u}|u > = \overline{\nu}u^2$ and by substituting this in (16) we get $u^2\overline{\nu}u = u$, implying $u^2 = \nu$ and $|u|^2 = 1$. Thus, in this case,

\[ u = \nu\hat{u}, \quad |\nu| = 1, \quad u^2 = \nu, \quad |u|^2 = 1. \tag{17} \]

Tripotents satisfying (17) will be called maximal tripotents. If $u$ is a maximal tripotent, then $D(u) = I$ and $Q(u)b = 2 < u|b > u - \nu b$. Note that our basis vectors $u_q$ are maximal tripotents.

We can obtain an explicit form for maximal tripotents in $S^4$. Denoting $\nu = e^{2i\varphi}$ and defining $a = e^{-i\varphi}u = e^{i\varphi}\hat{u}$ we obtain $\hat{a} = e^{i\varphi}\hat{u} = a$. Decompose $a$ into the time and spacial components as $a = a_0u_0 + a_ju_j$ for $j = 1, 2, 3$. Denote the spacial part $\hat{a} = a_ju_j$, which can be be written as $\hat{a} = |\hat{a}|\hat{n}$ with $\hat{n}$ a unit spacial vector belonging to the sphere

\[ S_2 = \{\hat{n} = n^j u_j : n^j \in \mathbb{R}, \sum (n^j)^2 = 1\}. \tag{18} \]

Since $a = \hat{a}$, the value $a_0$ is real and $\hat{a}$ is a purely imaginary vector. Now, $|a|^2 = |u|^2 = 1$ becomes $a_0^2 + |\hat{a}|^2 = 1$. This implies that there is a real number $\psi$ and $\hat{n} \in S_2$ such that $a = \cos \psi u_0 + i \sin \psi \hat{n}$. Thus, $u$ is a maximal tripotent if and only if there are real numbers $\varphi, \psi$ and $\hat{n} \in S_2$ such that

\[ u = e^{i\varphi}a = e^{i\varphi}(\cos \psi u_0 + i \sin \psi \hat{n}), \quad \varphi, \psi \in \mathbb{R}, \quad \hat{n} \in S_2. \tag{19} \]

This implies that the collection of all maximal tripotents form a torus.

If $\alpha = 0$, then from equation (16) we get $\beta = 0$ and thus $u^2 = 0$ and $|u|^2 = 1/2$. Moreover, any element $u$ satisfying these two properties will be a tripotent. Such a tripotent will be called a minimal tripotent.

\[ u - \text{minimal tripotent} \iff u^2 = 0, \ |u|^2 = 1/2. \tag{20} \]
Because minimal tripotents $u$ fulfill $u^2 = 0$, they are called null-vectors in the literature. Note, though, that not all null-vectors are minimal tripotents.

We separate the real and imaginary parts of $u$ as $u = x + iy$ and use that

$$
u^2 = x^2 - y^2 + 2i < x|y > = 0$$

we get $x^2 = y^2$ and $< x|y > = 0$. Separating the time and space components of the real and imaginary parts of $u$ we get $< x|y > = x_0y_0 - < x|y > = 0$. If we denote $\lambda = x^2 = y^2$ then

$$(x_0y_0)^2 = | < x|y > |^2 = (|x|^2 + \lambda)(|y|^2 + \lambda) \geq |x|^2|y|^2.$$ 

The Cauchy-Schwartz inequality for vectors in $\mathbb{R}^3$ excludes $\lambda > 0$ implying that both $x$ and $y$ are space-like or light-like elements of $M$. If $x$ and $y$ are light-like, then from the equality in the Cauchy-Schwartz inequality follow that the vectors $\vec{x}$ and $\vec{y}$ are collinear. From $< x|y > = 0$ and $x^2 = y^2 = 0$ follow that $y = \alpha x$ for some constant $\alpha$. Since a minimal tripotent $u$ satisfy $|u|^2 = 1/2$ there are a real constant $\psi$ and $\vec{n} \in S_2$ such that

$$u = \frac{e^{i\psi}}{2}(u_0 + \vec{n}), \quad \psi \in \mathbb{R}, \quad \vec{n} \in S_2. \quad (21)$$

An arbitrary minimal tripotent satisfies

$$u = x + iy, \quad x, y \in M, \quad x^2 = y^2 \leq 0, \quad < x|y > = 0. \quad (22)$$

If $u$ is a minimal tripotent, the operator $D(u)$, defined by (14) acts as follows:

$$D(u)a = \frac{1}{2}a - < u|a > \hat{u} + < \hat{u}|a > u. \quad (23)$$

It follows from this that if $a$ is an eigenvector of $D(u)$ corresponding to eigenvalue $\alpha$, then:

$$D(u)a = \alpha a, \quad \alpha = \left\{ \begin{array}{l l}
1, & \text{if } a = \lambda u, \quad \lambda \in \mathbb{C} \\
1/2, & \text{if } < u|a > = 0 \text{ and } < \hat{u}|a > = 0 \\
0, & \text{if } a = \lambda \hat{u}, \quad \lambda \in \mathbb{C}
\end{array} \right. \quad (24)$$

This implies that if $u$ is a minimal tripotent then the spectrum of the operator $D(u)$ is the set $\{1, 1/2, 0\}$.

The following table summarizes the properties of the two types of tripotents in $S^4$.

| Type     | $|u|^2|$ | $|\nu|^2$ | sp$D(u)$ | $Q(u)a$ | Decomposition |
|----------|----------|----------|----------|---------|---------------|
| Maximal  | $1$      | $1$      | $1$      | $(2\hat{P}_u - \nu)a$ | $\{1\}$  |
| Minimal  | $0$      | $\frac{1}{2}$ | $\{1, \frac{1}{2}, 0\}$ | $2\hat{P}_u a$ | $\{22\}$ |

Table 1. Types of tripotents in $S^4$.

The operator $P_u$ in the table is defined by $P_u b = < u|b > u$. 

6
5 Orthogonality of tripotents and Spectral decomposition

Definition Let \( u \) be a tripotent in \( S^4 \). We will say that a tripotent \( w \) is orthogonal (or algebraically orthogonal) to \( u \) if
\[
D(u)w = 0. \tag{25}
\]

If \( u \) is a maximal tripotent, then \( D(u) = I \) and there are no tripotents orthogonal to \( u \). From the definition of a minimal tripotent it follows that if \( u \) is a minimal tripotent, \( \hat{u} \) is also a minimal tripotent and from (24) it follows that \( D(u)\hat{u} = 0 \). So, if \( u \) is a minimal tripotent, the tripotent \( \hat{u} \) is orthogonal to \( u \).

Let \( w \) be an arbitrary tripotent orthogonal to a minimal tripotent \( u \). From the definition of orthogonality and (24) we get
\[
w = \lambda \hat{u}. \tag{26}
\]

From (20) it follows that \( |\lambda| = 1 \). Conversely, if \( w = \lambda \hat{u} \) then from (24) \( D(u)w = 0 \) implying that \( w \) is orthogonal to \( u \).

Thus \( w \) is orthogonal to \( u \) \( \iff \) \( w = \lambda \hat{u}, \lambda \in \mathbb{C}, |\lambda| = 1 \) \( \tag{27} \)

Moreover if
\[
a = \alpha u + \beta w \tag{28}
\]
for some constants \( \alpha, \beta \) then \( D(a) = |\alpha|^2 D(u) + |\beta|^2 D(w) \) and thus
\[
a^{(3)} = D(a)a = |\alpha|^2 \alpha u + |\beta|^2 \beta w. \tag{29}
\]

This shows that orthogonal tripotents behave as two orthogonal projections and hence that \( u + w \) and \( u - w \) are maximal tripotents. We may now introduce a partial order on the set of tripotents. We say that \( u < v \) for a pair of tripotents \( u, v \) if there is a tripotent \( w \) orthogonal to \( u \) such that \( v = u + w \). Under such an ordering the minimal tripotents are minimal and the maximal ones are maximal.

If \( a \) has decomposition (28) and \( w = \lambda \hat{u} \) then \( a^2 = \lambda \alpha \beta \). This shows that \( a^2 \) has the meaning of a determinant. If \( a^2 = 0 \) then it follows from (20) that \( u = \frac{a}{2|a|} \) is a minimal tripotent. This implies, in turn, that \( a = 2|a|u \) has a decomposition (28) with a positive constant \( \alpha = 2|a| \). The following Proposition shows that for any element in \( S^4 \) there is such a decomposition with positive constants.

**Proposition** Let \( a \) be a non-zero element of \( S^4 \) with \( a^2 \neq 0 \). Then there is an orthogonal pair of minimal tripotents \( u \) and \( w \) such that
\[
a = s_1 u + s_2 w, \tag{30}
\]
where the pair of non-negative numbers $s_1, s_2$ (called singular numbers of $a$) can be defined from the equations

$$s_1 \pm s_2 = \sqrt{2|a|^2 \pm 2|a|^2}.$$  

(31)

Moreover, if $s_1 > s_2$ this decomposition is unique.

Proof.

Use the polar decomposition of complex numbers to decompose $a^2 = \lambda|a|^2$ with $|\lambda| = 1$. Since $\hat{a}^2 = \overline{a^2}$ we have $\hat{a}^2 = \bar{\lambda}|a|^2$. Define $b = a + \lambda \hat{a}$. Then $b = \bar{\lambda}b$ and $|b|^2 = \bar{\lambda}b^2 = 2|a|^2 + 2|a|^2$. Similarly define $c = a - \lambda \hat{a}$. Then $\hat{c} = -\lambda c$ and $|c|^2 = -\bar{\lambda}c^2 = 2|a|^2 - 2|a|^2$. Direct calculation shows that $<b|c> = <b|c> = <b|c> = 0$. Obviously, $a = \frac{1}{2}(b + c)$.

Consider first the case $c = 0$ or $a = \lambda \hat{a}$. For such $a$ the element $v = a/|a|$ is a maximal tripotent as per (17) and thus, according to (19), has the form $v = e^{i\psi}(\cos \varphi u_0 + i \sin \varphi \vec{n})$ with $\varphi, \psi \in \mathbb{R}$ and a unit vector $\vec{n} \in \mathbb{R}^3$. If we define

$$u = \frac{e^{i(\varphi + \psi)}}{2}(u_0 + \vec{n}), \quad w = \frac{e^{i(\varphi - \psi)}}{2}(u_0 - \vec{n}),$$

then from (21) and (26) it follows that both $u, w$ are minimal orthogonal tripotents and $a = |a|v = |a|(u + w)$. So, in this case $a$ has decomposition (30).

Now consider the case $c \neq 0$. Note that $b \neq 0$ always. Define

$$u = \frac{1}{2} \left( \frac{b}{|b|} + \frac{c}{|c|} \right), \quad \text{and} \quad w = \frac{1}{2} \left( \frac{b}{|b|} - \frac{c}{|c|} \right).$$

From the properties of $b$ and $c$ follows that $|u|^2 = |w|^2 = 1/2$ and $u^2 = w^2 = 0$. Thus from (20) it follows that both $u$ and $w$ are minimal tripotents and since $\hat{u} = \lambda \hat{w}$ they are also orthogonal. The expression

$$a = \frac{1}{2}(b + c) = \frac{|b| + |c|}{2}u + \frac{|b| - |c|}{2}w$$

also defines a decomposition (30) for $a$ in this case with

$$s_1 = \frac{|b| + |c|}{2}, \quad s_2 = \frac{|b| - |c|}{2}$$

being non-negative numbers satisfying (31) and $s_1 > s_2$.

To show the uniqueness of the decomposition in the case $s_1 \neq s_2$, assume that a given $a$ has a decomposition (30) with an orthogonal pair of minimal tripotents $u$ and $w$. From (25) we may assume that $w = \lambda \hat{u}$ with $|\lambda| = 1$ and by use of (20) we get

$$|a|^2 = <s_1 \hat{u} + s_2 \lambda \hat{u}|s_1 \hat{u} + s_2 \lambda \hat{u}> = \frac{1}{2}(s_1^2 + s_2^2)$$

and

$$a^2 = <s_1 \hat{u} + s_2 \lambda \hat{u}|s_1 \hat{u} + s_2 \lambda \hat{u}> = \lambda s_1 s_2.$$
This implies equation (31) for $s_1, s_2$ and that these numbers are defined uniquely. Note that $\lambda$ is the argument of the complex number $a^2$ which is also defined uniquely. Denote $d = a/s_1 = u + aw$ with $1 > \alpha$. Then from (29) we have

$$\lim_{n \to \infty} D(d)^n d = \lim_{n \to \infty} u + \alpha^{2n} w = u$$

implying that $u$ is defined uniquely and therefore the decomposition (30) is unique.

6 Compatible tripotents of the spin space

From Quantum Mechanics we know that it is preferable to work with a basis consisting of compatible observables. Two observables are said to be compatible if the result of measurement of one of them is not affected by the measurement of the second one. Compatibility mean commutativity of spectral projections. For self-adjoint operators this is equivalent to commutativity of the operators representing the observables. To define compatibility for a pair of elements $a, b$ in the spin space $S^4$, we use the fact that there are two operators $D(a, \hat{a}), Q(a)$ on $S^4$ associated with any element $a \in S^4$. Commutativity of spectral projections of two elements in $S^4$ can be replaced with commutativity of the operators associated with them.

**Definition** For any element $a \in S^4$ we denote two operators $G_1(a) = D(a, \hat{a})$ and $G_2(a) = Q(a)$. A pair of non-zero elements $a, b \in S^4$ is said to be compatible if

$$[G_j(a), G_k(b)] = 0 \text{ for any } j, k \in \{1, 2\}.$$  

For any linearly independent pair of tripotents $u, v$ from Table 1 follow that the following are equivalent:

$$[Q(u), Q(v)] = 0 \iff [P_u, P_v] = 0 \iff <\hat{u}|v> = 0,$$

with $P_u b = < u|b> u$. Thus any pair of linearly independent compatible tripotents $u, v$ satisfy $< \hat{u}|v> = 0$. The following Proposition describes when a pair of tripotents are compatible.

**Proposition** Let $u, v$ be a pair of linearly independent tripotents in $S^4$. The pair is compatible if and only if: $< \hat{u}|v> = 0$ and, in addition, if one of the tripotents, say $u$, is a minimal one then the other one $v$ must be an eigenvector of $D(u)$ corresponding to eigenvalue 0 or 1/2.

**Proof.**

Consider first the case in which neither of the tripotents $u$ and $v$ are minimal. Thus, each of the tripotents $u$ and $v$ is a maximal tripotent. Since $D(u) = I$ for a maximal tripotent, any commutator with $D(u)$ or with $D(v)$ will vanish. Thus compatibility in such case is equivalent to $[Q(u), Q(v)] = 0$ which is equivalent to the condition $< \hat{u}|v> = 0$.

We may assume that $u$ is a minimal tripotent and $v$ is a tripotent compatible with $u$ satisfying $< \hat{u}|v> = 0$. In this case $D(u)a$ was defined by (23) and...
\[ Q(v)a = (2P_u + \lambda I)a = \langle u|\hat{a} > u. \] Thus, \[ [Q(v), D(u)] = 0 \] if and only if \[ [P_v, D(u)] = 0. \] But

\[ P_vD(u)a = \frac{1}{2} < v|\hat{a} > v + < u|\hat{a} > \hat{v} - < u|\hat{a} > < v|u > v \]

\[ = \frac{1}{2} < v|\hat{a} > v - < \hat{u}|\hat{a} > < v|u > v. \]

A similar calculation will give

\[ D(u)P_va = \frac{1}{2} < v|\hat{a} > v - < v|\hat{a} > < u|v > \hat{u}. \]

Thus, \[ [Q(v), D(u)] = 0 \] in the following 2 cases 1) \( v = \hat{u} \) or 2) \( < u|v > = 0. \)

In case 1), from (24) it follows that \( v \) is an eigenvector of \( D(u) \) corresponding to eigenvalue 0 and is a minimal tripotent orthogonal to \( u. \) In case 2) \( v \) is an eigenvector of \( D(u) \) corresponding to eigenvalue 1/2.

Conversely, if \( u \) is a minimal tripotent and tripotent \( v \) is an eigenvector of \( D(u) \) corresponding to eigenvalue 0, then from (24) it follows that \( v = \lambda \hat{u} \) for some constant \( \lambda. \) From the above observations \( [Q(v), D(u)] = [Q(u), D(v)] = [Q(v), Q(u)] = 0. \) Also in this case we get

\[ D(v)D(u)a = \frac{1}{2}(\frac{1}{4}a - < u|a > v - < v|a > u) = D(u)D(v)a, \quad (32) \]

based on (23) and implying that both \( [D(v)D(u)] = 0 \) and that \( v \) and \( u \) are compatible.

If \( u \) is a minimal tripotent and \( v \) satisfying \( < \hat{u}|v >= 0 \) is an eigenvector of \( D(u) \) corresponding to eigenvalue 1/2, then from (24) it follows that \( < u|v >= 0. \) From the above observations \( [Q(v), D(u)] = 0. \) Since for a maximal tripotent \( D(v) = I, \) in case the tripotent \( v \) is maximal, the tripotents \( v \) and \( u \) are compatible.

It remains to consider the case in which \( v \) is a minimal tripotent. Such a pair of tripotents \((u, v)\) is called a \textit{co-orthogonal} pair of tripotents, which is denoted by \( u \perp v. \) From (23) we get

\[ D(v)D(u)a = \]

\[ \frac{1}{4}a + \frac{1}{2} < \hat{u}|a > u + \frac{1}{2} < \hat{v}|a > v - \frac{1}{2} < u|a > \hat{u} - \frac{1}{2} < v|a > \hat{v} \]

\[ = D(u)D(v)a, \]

implying that \( [D(v)D(u)] = 0 \) and that \( v \) and \( u \) are compatible in this case, as well.
7 Bases of the relativistic phase space

The natural basis $u_\mu$ of the relativistic phase space $S^4$ consists of a family of maximal tripotents satisfying $<\hat{u}_\mu|u_\nu> = 0$ for any indices $\mu \neq \nu$. Thus, from Proposition 6.1 it follows that the natural basis consists a family of compatible maximal tripotents.

With the spin triple product based on the Euclidean metric we had a set of bases, called the spin grid, constructed from a family of compatible minimal tripotents, in addition to the bases from maximal tripotents - see [5] p.121-123 and [4]. If the spin space is of dimension 4, the spin grid is also called an odd quadrangle.

Definition A family of 4 minimal tripotents $(v, w, \hat{v}, \hat{w})$ in $S^4$ is called a spin grid if

1. the pairs $(v, \hat{v})$ and $(w, \hat{w})$ are orthogonal
2. the pairs $(v, w), (w, \hat{v}), (\hat{v}, \hat{w}), (\hat{w}, v)$ are co-orthogonal
3. $2\{v, w, \hat{v}\} = -\hat{w}$ and $2\{w, \hat{v}, \hat{w}\} = -v$.

The spin grid of the relativistic phase space $S^4$ can be constructed as follows. Start with a minimal light-like tripotent defined by (21) as $v = \frac{1}{\sqrt{2}}(u_0 + u_3)$. The tripotent $\hat{v} = \frac{1}{\sqrt{2}}(u_0 - u_3)$ is orthogonal to it. From Proposition 6.1 it follows that in order for the remaining two basic tripotents to be compatible with $v$ and $\hat{v}$ they must be $-1/2$ eigenvectors of the operators $D(v)$ and $D(\hat{v})$. From (24) it follows that such a tripotent, say $z$, must satisfy $<z|v> = 0$ and $<z|\hat{v}> = 0$. Thus, such $z$ must belong to the subspace $X = \text{span}\{u_1, u_2\}$. In this subspace we do not have light-like minimal tripotents. But, by (22) $w = \frac{1}{\sqrt{2}}(u_1 + iu_2) \in X$ is a minimal tripotent and its orthogonal complement $\hat{w} = \frac{1}{\sqrt{2}}(-u_1 + iu_2)$ is also a minimal tripotent in $X$. Direct calculations show that the family of 4 minimal tripotents $(v, w, \hat{v}, \hat{w})$ form a spin grid.

Such a basis is the known Newman-Penrose basis (see [11]) $(l, m, m, n)$ with

\[
\begin{align*}
1 &= \frac{1}{2}(u_0 + u_3), \quad m = \frac{1}{2}(u_1 + iu_2) \\
\bar{m} &= \frac{1}{2}(u_1 - iu_2), \quad n = \frac{1}{2}(u_0 - u_3).
\end{align*}
\]

The spin grid differ from the NP basis in 1) the normalization constant, 2) the order of the basis elements, which was not critical for the NP basis, and 3) the minus in front of $\bar{m}$, which is needed to be able to extend such a basis to higher dimensions (see [5], p.245). The minus in front of $\bar{m}$ has been added in consideration of the symmetries of spinors (see [10]). Thus, we propose to call spin grid a modified NP basis.

Definition A family of 4 minimal tripotents $(v, w, \hat{v}, \hat{w})$ in $S^4$ is called a Modified Newman-Penrose basis if

\[
\begin{align*}
v &= \frac{1}{2}(u_0 + u_3), \quad w = \frac{1}{2}(u_1 + iu_2)
\end{align*}
\]
\[ \vec{v} = \frac{1}{2}(u_0 - u_3), \quad \vec{w} = \frac{1}{2}(-u_1 + iu_2). \] \tag{34}

Such a basis form a spin grid.

I want to thank Yakov Itin for helpful remarks and suggestions.

References

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[2] E. Cartan, Sur les domaines bornés homogènes de l’espace de \( n \) variables complexes, *Abh. Math. Sem. Univ. Hamburg* 11 (1935), 116–162.

[3] E. Cartan, *The Theory of Spinors*, Dover Publ. New York, 1966.

[4] T. Dang and Y. Friedman, Classification of \( JBW^* \)-triples and applications, *Math. Scand.* 61 (1987), 292–330.

[5] Y. Friedman, *Physical Applications of Homogeneous Balls*, Progress in Mathematical Physics 40 (Birkhäuser, Boston, 2004).

[6] Y. Friedman, *Representations of the Poincaré group on relativistic phase space*, arXiv:0802.0070v1, 2008.

[7] Y. Friedman, B. Russo, A new approach to spinors and some representation of the Lorentz group on them, *Foundations of Physics*, 31(12), (2001), 1733–1766.

[8] G. Kaiser, *Quantum Physics, Relativity, and Complex Spacetime*, North-Holland, Amsterdam, 1990.

[9] O. Loos, *Bounded symmetric domains and Jordan pairs*, University of California, Irvine, 1977.

[10] P. O’Donnell, *Introduction to 2-Spinors in General Relativity*, World Scientific Pub., 2003.

[11] R. Penrose & W. Rindler, *Spinors and space-time*, Cambridge University Press, 1984.