ON THE MENGER COVERING PROPERTY AND D-SPACES

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Abstract. The main results of this paper are:

• It is consistent that every subparacompact space \( X \) of size \( \omega_1 \) is a D-space.
• If there exists a Michael space, then all productively Lindelöf spaces have the Menger property and, therefore, are D-spaces.
• Every locally D-space which admits a \( \sigma \)-locally finite cover by Lindelöf spaces is a D-space.

1. Introduction

A neighbourhood assignment for a topological space \( X \) is a function \( N \) from \( X \) to the topology of \( X \) such that \( x \in N(x) \) for all \( x \). A topological space \( X \) is said to be a D-space [6] if for every neighbourhood assignment \( N \) for \( X \) there exists a closed and discrete subset \( A \subset X \) such that \( N(A) = \bigcup_{x \in A} N(x) = X \).

It is unknown whether paracompact (even Lindelöf) spaces are D-spaces. Our first result in this note answers [7, Problem 3.8] in the affirmative and may be thought of as a very partial solution to this problem.

Our second result shows that the affirmative answer to [19, Problem 2.6], which asks whether all productively Lindelöf spaces are D-spaces, is consistent. It is worth mentioning that our premises (i.e., the existence of a Michael space) are not known to be inconsistent.

Our third result is a common generalization of two theorems from [10].

Most of our proofs use either the recent important result of Aurichi [2] asserting that every topological space with the Menger property is a D-space or the ideas from its proof. We consider only regular topological spaces. For the definitions of small cardinals \( d \) and \( \text{cov}(\mathcal{M}) \) used in this paper we refer the reader to [22].

1 While completing this manuscript, we learned that this result was independently obtained by Hang Zhang and Wei-Xue Shi; see [15].
2. Subparacompact spaces of size \( \omega_1 \)

Following [4] we say that a topological space \( X \) has the property \( E^*_\omega \) if for every sequence \( \langle u_n : n \in \omega \rangle \) of countable open covers of \( X \) there exists a sequence \( \langle u_n : n \in \omega \rangle \) such that \( u_n \in \{u_n\}^{<\omega} \) and \( \bigcup_{n \in \omega} u_n = X \). In the realm of Lindelöf spaces the property \( E^*_\omega \) is usually called the Menger property or \( \bigcup_{n \in \omega} (\mathcal{O}, \mathcal{O}) \); see [21] and references therein.

We say that a topological space \( X \) has property \( D_\omega \) if for every neighbourhood assignment \( N \) there exists a countable collection \( \{A_n : n \in \omega\} \) of closed discrete subsets of \( X \) such that \( X = \bigcup_{n \in \omega} N(A_n) \). Observe that the property \( D_\omega \) is inherited by all closed subsets.

The following theorem is the main result of this section.

**Theorem 2.1.** Suppose that a topological space \( X \) has properties \( D_\omega \) and \( E^*_\omega \). Then \( X \) is a \( D \)-space.

The proof of Theorem 2.1 is analogous to the proof of [3, Proposition 2.6]. In particular, it uses the following game of length \( \omega \) on a topological space \( X \): On the \( n \)th move player \( I \) chooses a countable open cover \( u_n = \{U_{n,k} : k \in \omega\} \) such that \( U_{n,k} \subset U_{n,k+1} \) for all \( k \in \omega \), and player \( II \) responds by choosing a natural number \( n_k \). Player \( II \) wins the game if \( \bigcup_{n \in \omega} U_{n,k} = X \). Otherwise, player \( I \) wins. We shall call this game the \( E^*_\omega \)-game. In the realm of Lindelöf spaces this game is known under the name Menger game. It is well known that a Lindelöf space \( X \) has the property \( E^*_\omega \) if and only if the first player has no winning strategy in the \( E^*_\omega \)-game on \( X \); see [8, 14]. The proof of [14, Theorem 13] also works without any change for non-Lindelöf spaces.

**Proposition 2.2.** A topological space \( X \) has the property \( E^*_\omega \) if and only if the first player has no winning strategy in the \( E^*_\omega \)-game.

A strategy of the first player in the \( E^*_\omega \)-game may be thought of as a map \( \Upsilon : \omega^{<\omega} \to \mathcal{O}(X) \), where \( \mathcal{O}(X) \) stands for the collection of all countable open covers of \( X \). The strategy \( \Upsilon \) is winning if \( X \neq \bigcup_{n \in \omega} U_{z,n,z(n)} \) for all \( z \in \omega^{\omega} \), for which \( \Upsilon(s) = \{U_{s,k} : k \in \omega\} \in \mathcal{O}(X) \).

We are in a position now to present the proof of Theorem 2.1.

**Proof.** We shall define a strategy \( \Upsilon : X \to \mathcal{O}(X) \) of the player \( I \) in the \( E^*_\omega \)-game on \( X \) as follows. Set \( F_0 = X \). The property \( D_\omega \) yields an increasing sequence \( \langle A_{\emptyset,k} : k \in \omega \rangle \) of closed discrete subsets of \( F_0 \) such that \( X = \bigcup_{k \in \omega} N(A_{\emptyset,k}) \). Set \( \Upsilon(\emptyset) = u_0 = \{N(A_{\emptyset,k}) : k \in \omega\} \).

Suppose that for some \( m \in \omega \) and all \( s \in \omega^{\leq m} \) we have already defined a closed subset \( F_s \) of \( X \), an increasing sequence \( \langle A_{s,k} : k \in \omega \rangle \) of closed discrete subsets of \( F_s \), and a countable open cover \( \Upsilon(s) = u_s \) of \( X \) such that \( u_s = \{(X \setminus F_s) \cup N(A_{s,k}) : k \in \omega\} \).

Fix \( s \in \omega^{m+1} \). Since \( X \) has the property \( D_\omega \), so does its closed subspace \( F_s := X \setminus \bigcup_{i \in m+1} N(A_{s,(i,s(i))}) \), and hence there exists an increasing sequence \( \langle A_{s,k} : k \in \omega \rangle \) of closed discrete subsets of \( F_s \) such that \( F_s \subset \bigcup_{k \in \omega} N(A_{s,k}) \). Set \( \Upsilon(s) = u_s = \{(X \setminus F_s) \cup N(A_{s,k}) : k \in \omega\} \). This completes the definition of \( \Upsilon \).

Since \( X \) has the property \( E^*_\omega \), \( \Upsilon \) is not winning. Thus there exists \( z \in \omega^{\omega} \) such that \( X = \bigcup_{n \in \omega} (X \setminus F_{z|n}) \cup N(A_{z|n,z(n)}) \). By the inductive construction, \( X \setminus F_0 = \emptyset \) and \( X \setminus F_{z|n} = \bigcup_{i<n} N(A_{z|i,z(i)}) \) for all \( n > 0 \). It follows from above that \( X = \bigcup_{n \in \omega} N(A_{z|n,z(n)}) \). In addition, \( A_{z|n,z(n)} \subset F_{z|n} = X \setminus \bigcup_{i<n} N(A_{z|i,z(i)}) \)
for all \( n > 0 \), which implies that \( A := \bigcup_{n \in \omega} A_{\geq n,z(n)} \) is a closed discrete subset of \( X \). It suffices to note that \( N(A) = X \). \( \square \)

We recall from [5] that a topological space \( X \) is called subparacompact if every open cover of \( X \) has a \( \sigma \)-locally finite closed refinement.

**Lemma 2.3.** Suppose that \( X \) is a subparacompact topological space which can be covered by \( \omega_1 \)-many of its Lindelöf subspaces. Then \( X \) has the property \( D_\omega \).

**Proof.** Let \( \mathcal{L} = \{ L_\xi : \xi < \omega_1 \} \) be an increasing cover of \( X \) by Lindelöf subspaces, let \( \tau \) be the topology of \( X \), and let \( N : X \to \tau \) be a neighbourhood assignment. Construct by induction a sequence \( \langle C_\alpha : \alpha < \omega_1 \rangle \) of (possibly empty) countable subsets of \( X \) such that

1. \( L_0 \subset N(C_0) \);
2. \( C_\alpha \cap N(\bigcup_{\xi < \alpha} C_\xi) = \emptyset \) for all \( \alpha < \omega_1 \); and
3. \( L_\alpha \setminus N(\bigcup_{\xi < \alpha} C_\xi) \subset N(C_\alpha) \) for all \( \alpha < \omega_1 \).

Set \( C = \bigcup_{\alpha < \omega_1} C_\alpha \). The subparacompactness of \( X \) yields a closed cover \( \mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n \) of \( X \) which refines \( \mathcal{U} = \{ N(x) : x \in C \} \) and such that each \( \mathcal{F}_n \) is locally finite. Since every element of \( \mathcal{U} \) contains at most countably many elements of \( C \), so do elements of \( \mathcal{F} \). Therefore for every \( F \in \mathcal{F}_n \) such that \( C \cap F \neq \emptyset \) we can write this intersection in the form \( \{ x_{n,F,m} : m \in \omega \} \). Now it is easy to see that \( A_{n,m} := \{ x_{n,F,m} : F \in \mathcal{F}_n, C \cap F \neq \emptyset \} \) is a closed discrete subset of \( X \) and \( \bigcup_{n,m \in \omega} A_{n,m} = C \). \( \square \)

**Remark 2.4.** What we have actually used in the proof of Lemma 2.3 is the following weakening of subparacompactness: every open cover \( \mathcal{U} \) which is closed under unions of its countable subsets admits a \( \sigma \)-locally finite closed refinement. We do not know whether this property is strictly weaker than subparacompactness.

**Corollary 2.5.** Let \( X \) be a countably tight paracompact topological space of density \( \omega_1 \). Then \( X \) has the property \( D_\omega \).

**Proof.** Let \( \{ x_\alpha : \alpha < \omega_1 \} \) be a dense subspace of \( X \). Since \( X \) has countable tightness, \( X = \bigcup_{\alpha < \omega_1} \{ x_\xi : \xi < \alpha \} \). It suffices to note that the closure of any countable subspace of a paracompact space is Lindelöf. \( \square \)

It is well known [9] Theorem 4.4] (and it easily follows from corresponding definitions) that any Lindelöf space of size \( \theta \) has the Menger property. The same argument shows that every topological space of size \( \theta \) has the property \( E^*_\omega \). Combining this with Theorem 2.1 and Lemma 2.3, we get the following corollary, which implies the first of the results mentioned in our abstract.

**Corollary 2.6.** Suppose that \( X \) is a subparacompact topological space of size \( |X| < \theta \) which can be covered by \( \omega_1 \)-many of its Lindelöf subspaces. Then \( X \) is a \( D \)-space.

### 3. Concerning the existence of a Michael space

A topological space \( X \) is said to be productively Lindelöf if \( X \times Y \) is Lindelöf for all Lindelöf spaces \( Y \). It was asked in [19] whether productively Lindelöf spaces are \( D \)-spaces. The positive answer to the above question has been proved consistent,
and in a stream of recent papers (see the list of references in [19]) several sufficient set-theoretical conditions were established. The following statement gives a uniform proof for some of these results. In particular, it implies [16] Theorems 5 and 7] and [1 Corollary 4.5] and answers [17, Question 15] in the affirmative.

A Lindelöf space $Y$ is called a Michael space if $\omega_1 \times Y$ is not Lindelöf.

**Proposition 3.1.** If there exists a Michael space, then every productively Lindelöf space has the Menger property.

We refer the reader to [1], where the existence of a Michael space was reformulated in a combinatorial language and a number of set-theoretical conditions guaranteeing the existence of Michael spaces were established.

In the proof of Proposition 3.1 we shall use set-valued maps; see [13]. By a set-valued map $\Phi$ from a set $X$ into a set $Y$ we understand a map from $X$ into $\mathcal{P}(Y)$ and write $\Phi : X \Rightarrow Y$ (here $\mathcal{P}(Y)$ denotes the set of all subsets of $Y$). For a subset $A$ of $X$ we set $\Phi(A) = \bigcup_{x \in A} \Phi(x) \subset Y$. A set-valued map $\Phi$ from a topological spaces $X$ to a topological space $Y$ is said to be

- **compact-valued** if $\Phi(x)$ is compact for every $x \in X$;
- **upper semicontinuous** if for every open subset $V$ of $Y$ the set $\Phi^{-1}_c(V) = \{ x \in X : \Phi(x) \subset V \}$ is open in $X$.

The proof of the following claim is straightforward.

**Claim 3.2.** (1) Suppose that $X, Y$ are topological spaces, $X$ is Lindelöf, and $\Phi : X \Rightarrow Y$ is a compact-valued upper semicontinuous map such that $Y = \Phi(X)$. Then $Y$ is Lindelöf.

(2) If $\Phi_0 : X_0 \Rightarrow Y_0$ and $\Phi_0 : X_1 \Rightarrow Y_1$ are compact-valued upper semicontinuous, then so is the map $\Phi_0 \times \Phi_1 : X_0 \times X_1 \Rightarrow Y_0 \times Y_1$ assigning to each $(x_0, x_1) \in X_0 \times X_1$ the product $\Phi_0(x_0) \times \Phi_1(x_1)$.

**Proof of Proposition 3.1.** Suppose, contrary to our claim, that $X$ is a productively Lindelöf space which does not have the Menger property and $Y$ is a Michael space. It suffices to show that $X \times Y$ is not Lindelöf.

Indeed, by [23, Theorem 8] there exists a compact-valued upper semicontinuous map $\Phi : X \rightarrow \omega_1$ such that $\Phi(X) = \omega_1$. By Claim 3.2(2) the product $\omega_1 \times Y$ is the image of $X \times Y$ under a compact-valued upper semicontinuous map. By the definition of a Michael space, $\omega_1 \times Y$ is not Lindelöf. By applying Claim 3.2(1) we can conclude that $X \times Y$ is not Lindelöf either. 

By a result of Tall [16] the existence of a Michael space implies that all productively Lindelöf analytic metrizable spaces are $\sigma$-compact. Combining recent results obtained in [11] and [12], we can consistently extend this result to all $\Sigma^1_1$ definable subsets of $2^{\omega_1}$.

**Theorem 3.3.** Suppose that $\text{cov}(\mathcal{M}) > \omega_1$ and that there exists a Michael space. Then every productively Lindelöf $\Sigma^1_1$ definable subset of $2^{\omega_1}$ is $\sigma$-compact.

**Proof.** Let $X$ be a productively Lindelöf $\Sigma^1_1$ definable subset of $2^{\omega_1}$.

If $X$ cannot be written as a union of $\omega_1$-many of its compact subspaces, then it contains a closed copy of $\omega_1$ [12], and hence the existence of the Michael space implies that $X$ is not productively Lindelöf, a contradiction.

Thus $X$ can be written as a union of $\omega_1$-many of its compact subspaces, and therefore it is $\sigma$-compact by [1, Corollary 4.15].
We do not know whether the assumption $\text{cov}(\mathcal{M}) > \omega_1$ can be dropped from Theorem 3.3.

**Question 3.4.** Suppose that there exists a Michael space. Is every coanalytic productively Lindel"of space $\sigma$-compact?

By [13, Proposition 31] the affirmative answer to the question above follows from the Axiom of Projective Determinacy.

### 4. Locally finite unions

**Theorem 4.1.** Suppose that $X$ is a locally $D$-space which admits a $\sigma$-locally finite cover by Lindel"of subspaces. Then $X$ is a $D$-space.

**Proof.** Let $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$ be a cover of $X$ by Lindel"of subspaces such that $\mathcal{F}_n$ is locally finite. Fix $F \in \mathcal{F}_n$. For every $x \in F$ there exists an open neighbourhood $U_x$ of $x$ such that $U_x$ is a $D$-space. Let $C_F$ be a countable subset of $F$ such that $F \subseteq \bigcup_{x \in C_F} U_x$, Then $Z_F = \{F \cap U_x : x \in C_F\}$ is a countable cover of $F$ consisting of closed $D$-subspaces of $X$ such that $F \cap Z$ is dense in $Z$ for all $Z \in Z_F$. It follows from the above that $X$ admits a $\sigma$-locally finite cover consisting of closed $D$-subspaces. Since a union of a locally finite family of closed $D$-subspaces is easily seen to be a closed $D$-subspace, $X$ is a union of an increasing sequence of its closed $D$-subspaces. Therefore it is a $D$-space by results of [3]. □

**Corollary 4.2.** If a topological space $X$ admits a $\sigma$-locally finite locally countable cover by topological spaces with the Menger property, then it is a $D$-space. In particular, a locally Lindel"of space admitting a $\sigma$-locally finite cover by topological spaces with the Menger property is a $D$-space.

**Proof.** The second part is a direct consequence of the first one since every $\sigma$-locally countable family of subspaces of a locally Lindel"of space is locally countable.

To prove the first assertion, note that by local countability every point $x \in X$ has a closed neighbourhood which is a countable union of its subspaces with the Menger property, and hence it has the Menger property itself. Therefore $X$ is a locally $D$-space. It now suffices to apply Theorem 4.1. □

It is known that every Lindel"of $C$-scattered space is $C$-like and that $C$-like spaces have the Menger property; see [20, p. 247] and references therein. Thus Corollary 4.2 implies Theorems 2.2 and 3.1 from [10].

### References

1. Alas, O.; Aurichi, L.F.; Junqueira, L.R.; Tall, F.D., Non-productively Lindel"of spaces and small cardinals, Houston J. Math., to appear.
2. Aurichi, L.F., $D$-spaces, topological games, and selection principles, Topology Proc. 36 (2010), 107–122. MR2591778
3. Borges, C.R.; Wehrly, A.C., A study of $D$-spaces, Topology Proc. 16 (1991), 7–15. MR1206448 (94a:54059)
4. Bukovský, L.; Haleš, J., On Hurewicz properties, Topology Appl. 132 (2003), 71–79. MR1990080 (2004f:03085)
5. Burke, D.K., Covering properties, in: Handbook of Set-Theoretic Topology (K. Kunen, J.E. Vaughan, eds.), North Holland, Amsterdam, 1984, 347–422. MR776628 (86e:54030)
6. Van Douwen, E.K.; Pfeffer, W.F., Some properties of the Sorgenfrey line and related spaces, Pacific J. Math. 81 (1979), 371–377. MR547609 (80h:54027)
7. Gruenhage, G., A survey on $D$ spaces, Contemp. Math., Vol. 533, Amer. Math. Soc., Providence, RI, 2011.
10. Martínez, J.C.; Soukup, L., The $D$-property in unions of scattered spaces, Topology Appl. 156 (2009), 3086–3090. MR2556688 (2010k:54027)

11. Moore, J.T., Some of the combinatorics related to Michael’s problem, Proc. Amer. Math. Soc. 127 (1999), 2459–2467. MR1659914 (2000a:54002)

12. Repický, M., Another proof of Hurewicz theorem, Tatra Mt. Math. Publ., to appear.

13. Repovš, D.; Semenov, P., Continuous selections of multivalued mappings. Mathematics and its Applications, 455. Kluwer Academic Publishers, Dordrecht, 1998. MR1659914 (2000a:54002)

14. Scheepers, M., Combinatorics of open covers. I. Ramsey theory, Topology Appl. 69 (1996), 31–62. MR1378387 (97h:90123)

15. Shi, W.; Zhang, H., A note on $D$-spaces, preprint, 2010.

16. Tall, F.D., Productively Lindelöf spaces may all be $D$, Canad. Math. Bulletin, to appear.

17. Tall, F.D., Lindelöf spaces which are Menger, Hurewicz, Alster, productive, or $D$, Topology Appl., to appear.

18. Tall, F.D., A note on productively Lindelöf spaces, preprint, 2010.

19. Tall, F.D., Set-theoretic problems concerning Lindelöf spaces, preprint, 2010.

20. Telgársky, R., Topological games: On the 50th anniversary of the Banach-Mazur game, Rocky Mountain J. Math. 17 (1987), 227–276. MR892457 (88d:54046)

21. Tsaban, B., Selection principles and special sets of reals, in: Open problems in topology. II (edited By Elliott Pearl), Elsevier Sci. Publ., 2007, pp. 91–108.

22. Vaughan, J., Small uncountable cardinals and topology, in: Open problems in topology (J. van Mill, G.M. Reed, eds.), Elsevier Sci. Publ., 1990, pp. 195–218. MR1078647

23. Zdomskyy, L., A semifilter approach to selection principles, Comment. Math. Univ. Carolin. 46 (2005), 525–539. MR2174530 (2006g:54028)