Color Transparency and Saturation in QCD

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1. Introduction

The concepts of “color transparency” and “saturation”.

Deep inelastic scattering (DIS), HERA 1992 to 2007:

DIS at low values of

\[ x \equiv x_{bj} \simeq \frac{Q^2}{W^2}, \text{ where} \]
\[ 5 \cdot 10^{-4} \leq x \leq 10^{-1} \]
\[ 0 \leq Q^2 \leq 100 \text{GeV}^2 \]
Low-x Scaling

Empirically: \[ \eta(W^2, Q^2) \equiv \frac{Q^2 + m_0^2}{\Lambda_{sat}^2(W^2)}. \]

\[ \Lambda_{sat}^2(W^2) \sim (W^2)^C. \]

\[ \sigma_{\gamma p}(W^2, Q^2) = \sigma_{\gamma p}(\eta(W^2, Q^2)) \]
\[ \sim \sigma^{(\infty)} \left\{ \begin{array}{ll}
\ln\frac{1}{\eta(W^2, Q^2)}, & \text{for } \eta(W^2, Q^2) \ll 1 \\
\frac{1}{\eta(W^2, Q^2)}, & \text{for } \eta(W^2, Q^2) \gg 1
\end{array} \right\} \]

Schildknecht, Surrow, Tentyukov (2000)
The limit of $\eta(W^2, Q^2) \to 0$, or $W^2 \to \infty$ at $Q^2$ fixed

$$\lim_{W^2 \to \infty, Q^2 \text{ fixed}} \frac{\sigma_{\gamma^*p}(\eta(W^2, Q^2))}{\sigma_{\gamma^*p}(\eta(W^2, Q^2 = 0))} = \lim_{W^2 \to \infty, Q^2 \text{ fixed}} \frac{\ln \left( \frac{\Lambda_{\text{sat}}^2(W^2)}{m_0^2} \right) \ln \left( \frac{m_0^2}{Q^2 + m_0^2} \right)}{\ln \left( \frac{\Lambda_{\text{sat}}^2(W^2)}{m_0^2} \right)} = 1 + \lim_{W^2 \to \infty, Q^2 \text{ fixed}} \frac{\ln \left( \frac{m_0^2}{Q^2 + m_0^2} \right)}{\ln \left( \frac{\Lambda_{\text{sat}}^2(W^2)}{m_0^2} \right)} = 1.$$ 

$\sigma_{\gamma^*p}(\eta(W^2, Q^2 = 0)) = \sigma_{\gamma p}(W^2)$

D. Schildknecht, DIS 2001 (Bologna)

### Table

| $Q^2[GeV^2]$ | $W^2[GeV^2]$ | $\frac{\sigma_{\gamma^*p}(\eta(W^2, Q^2))}{\sigma_{\gamma p}(W^2)}$ |
|--------------|--------------|--------------------------------------------------|
| 1.5          | $2.5 \times 10^7$ | 0.5                                              |
|              | $1.26 \times 10^{11}$ | 0.63                                             |
Aim of the present talk

- Low-x scaling of $\sigma_{\gamma^* p}(\eta(W^2, Q^2))$,

- $\Lambda_{sat}^2 \sim (W^2)^{C_2}$, $C_2 = 0.29$ (DGLAP evolution),

- $F_L(x, Q^2) = 0.27 F_2(x, Q^2), \ (Q^2 > \Lambda_{sat}^2(W^2))$

consequence of the interaction of

$q\bar{q}$ color-dipole fluctuations of the photon with the proton.

- Connection with gluon-distribution function and perturbative (DGLAP) evolution.
- No evidence for parton recombination and non-linear saturation in DIS.
2. Photon-hadron interactions: Late 1960’s, early 1970’s.

1960’s Vector Meson Dominance

\[ p p \rightarrow \rho^0, \omega, \phi \]

J.J. Sakurai (1960, ...)

Shadowing in \( \gamma A \) interactions

\[ \gamma \rightarrow \rho^0, \ldots \]

Leo Stodolsky (1967)
1969 DIS SLAC-MIT Collaboration

Bjorken scaling,

Feynman, parton model
GENERALIZED VECTOR DOMINANCE AND INELASTIC ELECTRON–PROTON SCATTERING *

J. J. SAKURAI and D. SCHILDKNECHT **
Department of Physics, University of California, Los Angeles, USA

Received 30 March 1972

* We propose a model of inelastic electron–proton scattering which takes into account the coupling of the photon to higher–mass vector states. Both the virtual photon–proton cross section $\sigma_T$ (predicted with essentially no adjustable parameters) and the $q^2$ dependence of $R$ are in exceedingly good agreement with the SLAC–MIT data in the diffraction region.
1989 Shadowing EMC Collaboration

D. Schildknecht (1973)
C. Bilchak and D. Schildknecht (1989)

1994 HERA
DIS for $x_{bj} \ll 0.1$
High-mass diffractive production
(“rap-gap” events).
Life time of hadronic fluctuations $\gamma^* \rightarrow \rho^0, \gamma^* \rightarrow q\bar{q}$

i) Four-momentum-conserving transition to virtual state, e.g. $\rho^0$, $q\bar{q}$ state

\[ p^\mu = q^\mu, \]
\[ p^2 = q^2 < 0, \quad p^2 \neq M_{q\bar{q}}^2, \]

Propagator:
\[ \frac{1}{-q^2 + M_{q\bar{q}}^2} = \frac{1}{Q^2 + M_{q\bar{q}}^2}. \]
ii) Equivalently: Three-momentum-conserving transition to on-shell $q\bar{q}$ state

\[ \vec{p} = \vec{q}; \]

\[ p^2 = M_{qq}^2; \quad q^2 = (q^0)^2 - (\vec{q})^2 < 0; \quad Q^2 = -q^2; \]

\[ \Delta E = p^0 - q^0 = \frac{M_{qq}^2 + Q^2}{p^0 + q^0} \]

\[ \cong \frac{M_{qq}^2 + Q^2}{2q^0}. \]

\[ \tau = \frac{1}{\Delta E} = \frac{2M_p\nu}{Q^2 + M_{qq}^2} \frac{1}{M_p} \gg \frac{1}{M_p}. \]

$(q\bar{q})p$ interaction cross section dependent on $W$ ($Q^2$ and $x$ dependence excluded).
Modern picture of low-x DIS:

i) $q\bar{q}$ internal structure

\[ \gamma^* \xrightarrow{\vec{k}_\perp, z(1-z)} q \quad \bar{q} \]

Nikolaev, Zakharov (1991)

ii) $q\bar{q}$-dipole interaction

\[ \gamma^* \xrightarrow{\gamma^* + \ldots} \]

Low (1975)
Nussinov (1975)
Invariant mass of $q\bar{q}$ state

\[ k^2 = k'^2 = m_q^2 = 0 \]
\[ M_{q\bar{q}}^2 = (k + k')^2 = (2k^0_{C.M.})^2 \]
\[ = 4 \frac{k^2}{\sin^2 \vartheta_{C.M.}} \]

In terms of $z$:

\[ k^3 = zq^3; \]
\[ k'^3 = (1 - z)q^3; \]
\[ M_{q\bar{q}}^2 = \frac{k^2}{z(1-z)}; \]
\[ \sin^2 \vartheta_{C.M.} = 4z(1 - z) \]
3. The Color Dipole Picture (CDP).

The longitudinal and the transverse photoabsorption cross section

\[ \gamma^* p \rightarrow \gamma^* p \quad \rightarrow \quad (q\bar{q}) p \rightarrow (q\bar{q}) p \]

channel 1:

channel 2:
A) \( \sigma_{\gamma^*_L,T}(W^2, Q^2) = \int dz \int d^2\vec{r}_\perp |\psi_{L,T}(\vec{r}_\perp, z(1-z), Q^2)|^2 \ \sigma_{(q\bar{q})p}(\vec{r}_\perp, z(1-z), W^2) \)

Remarks: i) \( |\psi_{L,T}(\vec{r}_\perp, z(1-z), Q^2)| \): Probability for \( \gamma^*_L,T \rightarrow q\bar{q} \) fluctuation

ii) \( \sigma_{(q\bar{q})p}(\vec{r}_\perp, z(1-z), W^2) \): \((q\bar{q})p\) cross section dependent on \( W^2 \) (not on \( x \equiv \frac{Q^2}{W^2} \))

B) Gauge-invariant two-gluon coupling:

\[ \sigma_{(q\bar{q})p}(\vec{r}_\perp, z(1-z), W^2) = \int d^2\vec{l}_\perp \tilde{\sigma}(\vec{l}_\perp^2, z(1-z), W^2) \left( 1 - e^{-i \vec{l}_\perp \cdot \vec{r}_\perp} \right) \]

Nikolaev, Zakharov (1991)

Cvetic, Schildknecht, Shoshi (2000)
Equivalently, in terms of the variables:

\[ \vec{r}'_\perp = \sqrt{z(1-z)} \vec{r}_\perp, \]

\[ \vec{l}'_\perp = \frac{\vec{l}_\perp}{\sqrt{z(1-z)}}, \]

Photon wave function (e.g. L):

\[ K_0(r'_\perp Q) = \frac{1}{2\pi} \int d^2\vec{k}'_\perp \frac{1}{Q^2 + k'^2} e^{-i\vec{r}'_\perp \cdot \vec{k}'_\perp} \]

\[ \gamma^* q\bar{q} \text{ coupling : } \sum_{\lambda=-\lambda=\pm 1} |j^\lambda_{L,\lambda'}|^2 = 4M_{qq}^2 (d_{10}^1(z))^2, \]

\[ \sum_{\lambda=-\lambda'=\pm 1} |j^\lambda_{T,\lambda'}(+)|^2 = \sum_{\lambda=-\lambda=\pm 1} |j^\lambda_{T,\lambda'}(-)|^2 = 4M_{qq}^2 \frac{1}{2} ((d_{1-1}^1(z))^2 + (d_{11}^1(z))^2). \]
Upon introducing the cross section $\sigma_{(q\bar{q})_{L,T}^j P}(r'_\perp, W^2)$, for $(q\bar{q})_{L,T}^j P$ scattering

A) $\sigma_{L,T}^*_P(W^2, Q^2) = \frac{a}{\pi} \sum_q Q_q^2 Q^2 \int dr'^2 K_{0,1}^2 (r' Q) \sigma_{(q\bar{q})_{L,T}^j P}(r'_\perp, W^2)$.  

Kuroda, Schildknecht (2011)

and

B) $\sigma_{(q\bar{q})_{L,T}^j P}(r'_\perp, W^2) = \int d^2 l'^2 \tilde{\sigma}_{(q\bar{q})_{L,T}^j P}(l'^2, W^2) (1 - e^{-i l'^2 \cdot r'_\perp})$

$= \pi \int d^2 l'^2 \tilde{\sigma}_{(q\bar{q})_{L,T}^j P}(l'^2, W^2) \cdot \left(1 - \frac{\int d^2 l'^2 \tilde{\sigma}_{(q\bar{q})_{L,T}^j P}(l'^2, W^2) J_0(l'_r r'_\perp)}{\int d^2 l'^2 \tilde{\sigma}_{(q\bar{q})_{L,T}^j P}(l'^2, W^2)}\right)$

For fixed dipole size, $r'_\perp$, dominant contribution to dipole cross section

$l'^2 \perp \leq l'^2 \perp \text{Max}(W^2)$. 

18
The Color Dipole Cross Section.

I) Color transparency

\[ 0 < l'_\perp r'_\perp < l'_\perp \text{Max}(W^2) r'_\perp \ll 1, \quad J_0(l'_\perp r'_\perp) \approx 1 - \frac{1}{4}(l'_\perp r'_\perp)^2 \]

\[
\sigma_{(q\bar{q})_{L,T}^1}^{L_p}(r'^2, W^2) = \\
\frac{1}{4} \pi r'^2 \int \tilde{l}'^2 \tilde{l}'^2 \tilde{\sigma}_{(q\bar{q})_{L}^1}^{L_p}(\tilde{l}'^2, W^2) \left\{ \begin{array}{ll} 1, & (r'^2 \ll \frac{1}{l'_\perp \text{Max}(W^2)}) \\
\rho_W, & \end{array} \right. \\
\text{where} \quad \int \tilde{l}'^2 \tilde{l}'^2 \tilde{\sigma}_{(q\bar{q})_{L}^1}^{L_p}(\tilde{l}'^2, W^2) = \rho_w \int \tilde{l}'^2 \tilde{l}'^2 \tilde{\sigma}_{(q\bar{q})_{L}^1}^{L_p}(\tilde{l}'^2, W^2). 
\]

\[
\sigma_{(q\bar{q})_{L,L}^1}^{L_p}(r'^2, W^2) = \frac{1}{4} r'^2 \sigma^{(\infty)}_L(W^2) \Lambda^2_{\text{sat}}(W^2) \left\{ \begin{array}{ll} 1, & (r'^2 \ll \frac{1}{l'_\perp \text{Max}(W^2)}) \\
\rho_W, & \end{array} \right. \\
\text{where} \quad \Lambda^2_{\text{sat}}(W^2) \equiv \frac{\int \tilde{l}'^2 \tilde{l}'^2 \tilde{\sigma}_{(q\bar{q})_{L}^1}^{L_p}(\tilde{l}'^2, W^2)}{\int \tilde{l}'^2 \tilde{\sigma}_{(q\bar{q})_{L}^1}^{L_p}(\tilde{l}'^2, W^2)}. 
\]

Strong cancellation between channel 1 and channel 2.
II) Saturation

$l'_\perp Max(W^2)r'_\perp \gg 1$,

huge integrations range in integral over $dr^2_{\perp}$, many oscillations of $J_0(l'_\perp r'_\perp)$, contribution from channel 2 vanishing

$$\sigma_{(q\bar{q})L,T=1, p}(r^2_{\perp}, W^2) \approx \pi \int d\vec{l}^2_{\perp} \bar{\sigma}_{(q\bar{q})L,T=1, p}(\vec{l}^2_{\perp}, W^2) \equiv \sigma_{L,T}^{(\infty)}(W^2),$$

$$\left( r'^2_{\perp} \gg \frac{1}{l'^2_{\perp Max}(W^2)} \right).$$

Unitarity: $\sigma_{L,T}^{(\infty)}(W^2)$ at most logarithmically dependent on $W^2$.

Thus: Property of dipole interaction:

$$\lim_{r'^2_{\perp fixed}} \lim_{W^2 \to \infty} \sigma_{(q\bar{q})L,T=1, p}(r'_\perp, W^2) = \lim_{r'^2_{\perp \to \infty}} \lim_{W^2 fixed} \sigma_{(q\bar{q})L,T=1, p}(r'^2_{\perp}, W^2)$$
Photoabsorption Cross Section

Due to $K_{0,1}(r'_\perp Q) \sim \frac{\pi}{2r'_\perp Q} e^{-2r'_\perp Q}$, $(r'_\perp Q \gg 1)$, cross section determined by

$$r''_\perp < \frac{1}{Q^2}.$$  

At fixed $Q^2$,

either $r''_\perp < \frac{1}{Q^2} < \frac{1}{\Lambda_{sat}(W^2)}$, color transparency: $Q^2 \gg \Lambda_{sat}^2(W^2)$

or $\frac{1}{\Lambda_{sat}(W^2)} < r''_\perp < \frac{1}{Q^2}$, saturation: $\Lambda_{sat}^2(W^2) \ll Q^2$.

$$\sigma_{\gamma^*p}(W^2, Q^2) = \sigma_{\gamma^*p}(\eta(W^2, Q^2)) = \frac{\alpha}{\pi} \sum_q Q_q^2 \left\{ \begin{array}{ll} \sigma_T^{(\infty)}(W^2) \ln \frac{1}{\eta(W^2, Q^2)}, & (\eta(W^2, Q^2) \ll 1) \quad \text{(sat.)}, \\ \frac{1}{6}(1 + 2\rho)\sigma_L^{(\infty)}(W^2) \frac{1}{\eta(W^2, Q^2)}, & (\eta(W^2, Q^2) \gg 1), \quad \text{(col.tr.)} \end{array} \right.$$  

$$\eta(W^2, Q^2) = \frac{Q^2 + m_0^2}{\Lambda_{sat}^2(W^2)}$$

Color-gauge-invariant $q\bar{q}$ (dipole) interaction with gluon field in the nucleon implies low-x scaling.
Low-x Scaling

Empirically: \[ \eta(W^2, Q^2) \equiv \frac{Q^2 + m_0^2}{\Lambda_{sat}^2(W^2)}, \]

\[ \Lambda_{sat}^2(W^2) \sim (W^2)^{C_2} \]

\[ \sigma_{\gamma^*p}(W^2, Q^2) = \sigma_{\gamma^*p}(\eta(W^2, Q^2)) \]

\[ \sim \sigma^{(\infty)} \begin{cases} \frac{ln}{\eta(W^2, Q^2)} & , \text{ for } \eta(W^2, Q^2) \ll 1 \\ \frac{1}{\eta(W^2, Q^2)} & , \text{ for } \eta(W^2, Q^2) \gg 1 \end{cases} \]

Schildknecht, Surrow, Tentyukov (2000)
The $W$-dependence

\[ F_2(x, Q^2) \approx \frac{Q^2}{4\pi^2\alpha} (\sigma_{\gamma^*p}(W^2, Q^2) + \sigma_{\gamma^*p}(W^2, Q^2)) \]

\[ = \frac{\sum_q Q_q^2}{4\pi^2} \int dz \int d\vec{l}_\perp^2 \tilde{\sigma}(\vec{l}_\perp^2, z(1-z), W^2)(1 + 2\rho) \]

\[ = F_2(W^2) \quad \text{for} \quad x < 0.1. \]
The limit of $\eta(W^2, Q^2) \to 0$, or $W^2 \to \infty$ at $Q^2$ fixed

$$\lim_{W^2 \to \infty, Q^2 \text{ fixed}} \frac{\sigma_{\gamma^* p}(\eta(W^2, Q^2))}{\sigma_{\gamma^* p}(\eta(W^2, Q^2 = 0))} = \lim_{Q^2 \to \infty, W^2 \text{ fixed}} \frac{\ln \left( \frac{\Lambda_{\text{sat}}^2(W^2)}{m_0^2} \right)}{\ln \left( \frac{\Lambda_{\text{sat}}^2(W^2)}{m_0^2} \right)} = 1 + \lim_{Q^2 \to \infty, W^2 \text{ fixed}} \frac{\ln \frac{m_0^2}{Q^2 + m_0^2}}{\ln \frac{\Lambda_{\text{sat}}^2(W^2)}{m_0^2}} = 1.$$  

$$\sigma_{\gamma^* p}(\eta(W^2, Q^2 = 0)) = \sigma_{\gamma p}(W^2)$$

D. Schildknecht, DIS 2001 (Bologna)

\[
\lim_{W^2 \to \infty, Q^2 \text{ fixed}} \frac{F_2(x \equiv Q^2/W^2, Q^2)}{\sigma_{\gamma p}(W^2)} = \frac{Q^2}{4\pi^2\alpha}.
\]

| $Q^2$ [GeV$^2$] | $W^2$ [GeV$^2$] | $\frac{\sigma_{\gamma^* p}(\eta(W^2, Q^2))}{\sigma_{\gamma p}(W^2)}$ |
|----------------|----------------|-------------------------------|
| 1.5            | $2.5 \times 10^7$ | 0.5                           |
|                | $1.26 \times 10^{11}$ | 0.63                         |
Observation by Caldwell

\[ \sigma_{\gamma^* p}(W^2, Q^2) = \sigma_0(Q^2) \left( \frac{W^2}{Q^2} \right)^{\lambda_{\text{eff}}(Q^2)} \equiv \sigma_0(Q^2) l^{\lambda_{\text{eff}}(Q^2)} \]

\(Q^2\)-independent limit at approximately \(W^2 \simeq 10^9 Q^2\).
The \((Q^2, W^2)\) plane
The longitudinal-to-transverse ratio

\((q\bar{q})^{J=1}_{L,T}\) states: \(\gamma_{L,T}^* \rightarrow (q\bar{q})^{J=1}_{L,T}\)

\[
\sigma_{x_{L,T}p}(W^2, Q^2) = \alpha \sum_q Q_q^2 \frac{1}{Q_q^2} \left\{ \int d\vec{l}'_{\perp} \vec{r}'_{\perp} \sigma_{(q\bar{q})}^{J=1}_L(\vec{r}'_{\perp}, W^2), \right. \\
\left. 2 \int d\vec{l}'_{\perp} \vec{r}'_{\perp} \sigma_{(q\bar{q})}^{J=1}_T(\vec{r}'_{\perp}, W^2). \right. 
\]

\[
\rho_{W} = \frac{\int d\vec{l}'_{\perp} \vec{r}'_{\perp} \sigma_{(q\bar{q})}^{J=1}_L(\vec{r}'_{\perp}, W^2)}{\int d\vec{l}'_{\perp} \vec{r}'_{\perp} \sigma_{(q\bar{q})}^{J=1}_T(\vec{r}'_{\perp}, W^2)}. 
\]

Numerical value of \(\rho_{W} = \rho\):

\[
\vec{l}'_{\perp}^2 = z(1-z)\vec{l}'_{\perp}^2 
\]

\[
\langle \vec{l}'_{\perp}^2 \rangle_{L,T}^{\vec{l}'_{\perp}^2 = \text{const}} = \vec{l}'_{\perp}^2 \left\{ 6 \int dz z^2 (1-z)^2 = \frac{4}{20} \vec{l}'_{\perp}^2, \right. \\
\left. \frac{3}{2} \int dz z(1-z)(1-2z(1-z)) = \frac{3}{20} \vec{l}'_{\perp}^2. \right. 
\]

Uncertainty principle:

\[
\rho_{W} = \frac{\langle r_{\perp}^2 \rangle_T}{\langle r_{\perp}^2 \rangle_L} = \frac{\langle \vec{l}'_{\perp}^2 \rangle_L}{\langle \vec{l}'_{\perp}^2 \rangle_T} = \frac{4}{3} \equiv \rho. 
\]

\[
R = \frac{1}{2\rho} = \begin{cases} 0.5 & \text{for } \rho = 1, \\ \frac{3}{8} = 0.375 & \text{for } \rho = \frac{4}{3}. \end{cases} \text{ ad hoc, helicity independence} \quad \text{Kuroda, Schildknecht (2008)}
\]
$F_2 = 0.27 N N L H 1 P D F 2 0 0 0$
4. The CDP, the Gluon Distribution Function and Evolution.

\[
F_L(x, Q^2) = \frac{\alpha_s(Q^2)}{3\pi} \sum_q Q_q^2 \cdot 6I_g(x, Q^2),
\]

where \( I_g(x, Q^2) \equiv \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 \left( 1 - \frac{x}{y} \right) y g(y, Q^2). \)

\[
F_L(\xi_L x, Q^2) = \frac{\alpha_s(Q^2)}{3\pi} \sum_q Q_q^2 G(x, Q^2). 
\]

\[
F_2(x, Q^2) = \frac{5}{18} x \sum (x, Q^2).
\]

\[
\frac{\partial F_2(\xi_2 x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{3\pi} \sum_q Q_q^2 G(x, Q^2). 
\]

Cooper-Sarkar et al. (1988)

rescaling factors:

(\(\xi_L, \xi_2\)) \(\simeq (0.40, 0.50)\)

(\(\xi_L, \xi_2\)) = (0.45, 0.40) for specific Prytz (1993)

Prytz (1993)

gluon distribution.

Accuracy \(\lesssim 0.5\%\).
Using $F_L(x, Q^2) = \frac{1}{2\rho+1} F_2(x, Q^2)$:

$$(2\rho + 1) \frac{\partial}{\partial \ln Q^2} F_2\left(\frac{\xi_2}{\xi_L} x, Q^2\right) = F_2(x, Q^2)$$

i) CDP: $F_2(x, Q^2) = F_2(W^2)$:

$$(2\rho W + 1) \frac{\partial}{\partial \ln W^2} F_2\left(\frac{\xi_L}{\xi_2} W^2\right) = F_2(W^2)$$

ii) Power law

$$F_2(W^2) \sim (W^2)^{C_2} = \left(\frac{Q^2}{x}\right)^{C_2}$$

Compare: “hard Pomeron” solution of DGLAP evolution: $(\frac{1}{x})^\lambda = \text{fixed}$.

“hard Pomeron” Regge: $(\frac{1}{x})^{\epsilon_0 \approx 0.43}$

$$(2\rho W + 1)C_2 \left(\frac{\xi_L}{\xi_2}\right)^{C_2} = 1,$$

$$\rho = \frac{4}{3} : \quad C_2 = \frac{1}{2\rho+1} \left(\frac{\xi_2}{\xi_L}\right)^{C_2} = 0.29 \quad \text{Kuroda, Schildknecht (2005, 2011)}$$

“BFKL-Pomeron”: $(\frac{1}{x})^\lambda = \frac{12\alpha_s}{\pi} \ln 2$

Balitskii, Fadin, Kuraev, Lipatov (1978/79)
\[ F_2(W^2) = f_2 \cdot \left( \frac{W^2}{1 \text{GeV}^2} \right)^{C_2 = 0.29} \]
\[ f_2 = 0.063 \text{ (fitted parameter)} \]

Experimental evidence for \( F_2(x, Q^2) = F_2(W^2 \cong Q^2 / x) \)
and for the prediction of \( C_2 = 0.29 \).
The Gluon Distribution Function

\[ \alpha_s(Q^2)G(x, Q^2) = \frac{3\pi}{\sum_q Q_q^2} F_L(\xi_L x, Q^2) \]

\[ = \frac{3\pi}{\sum_q Q_q^2 (2\rho + 1)} \frac{1}{F_2(\xi_L x, Q^2)} \]

\[ = \frac{3\pi}{\sum_q Q_q^2 (2\rho + 1) \xi_L^{C_2=0.29}} f_2 \left( \frac{W^2}{1\text{GeV}^2} \right)^{C_2=0.29} \]

Comments:

CDP: \( F_{L,2} = F_{L,2} \left( W^2 = \frac{Q^2}{x} \right), \)

\[ \rho = \text{const.} = \frac{4}{3}, \]

\( C_2 = 0.29 \) from evolution

\( f_2 = 0.063 \) fit parameter
Comparison with gluon distributions from Durham data file using $\alpha_s(Q^2) = \alpha_s(Q^2)^{NLO}$
Inversion: $F_2(W^2)$ in terms of gluon distribution:

$$F_2(W^2 = \frac{Q^2}{x}) = \frac{(2\rho + 1) \sum Q^2}{3\pi} \xi_L^2 \alpha_s(Q^2) G(x, Q^2)$$

$$= \frac{(2\rho + 1) \sum Q^2}{3\pi} \frac{1}{8\pi^2} \sigma_L^{(\infty)} \Lambda_{sat}^2(W^2).$$

( upon using $F_2 = f_2 \left( \frac{W^2}{1 GeV^2} \right)^{0.29} = \frac{(2\rho + 1) \sum Q^2}{3\pi} \frac{1}{8\pi^2} \sigma_L^{(\infty)} \Lambda_{sat}^2(W^2).$)

Saturation behavior:

$$F_2(W^2, Q^2) \sim Q^2 \sigma_L^{(\infty)} \ln \frac{\Lambda_{sat}^2(W^2)}{Q^2 + m_0^2}$$

$$\sim Q^2 \sigma_L^{(\infty)} \ln \left( \frac{\alpha_s(Q^2) G(x, Q^2)}{\sigma_L^{(\infty)}(Q^2 + m_0^2)} \right),$$

$$\eta(W^2, Q^2) \gg 1.$$
5. Model for the Dipole Cross Section

Model-independently:

$$\sigma_{\gamma^*p} \sim \begin{cases} \ln\frac{1}{\eta(W^2, Q^2)}, & \eta(W^2, Q^2) \ll 1 \\ \frac{1}{\eta(W^2, Q^2)}, & \eta(W^2, Q^2) \gg 1 \end{cases}$$

Interpolation between $\eta(W^2, Q^2) < 1$ and $\eta(W^2, Q^2) > 1$ by explicit ansatz for the dipole cross section.

$$\Lambda_{sat}^2(W^2) = const \left( \frac{W^2}{1 GeV^2} \right)^{C_2=0.29}$$

Normalization by $Q^2 = 0$ photoproduction (Regge fit):

$$\sigma^{(\infty)}(W^2) \cong \begin{cases} 30 mb, & \text{(for 3 active flavors,)} \\ 18 mb, & \text{(for 4 active flavors,)} \end{cases}$$
Simple ansatz with $\rho = 1$, \( R = \frac{1}{2\rho} = \frac{1}{2} \):

\[
\sigma_{(q\bar{q})p}(\vec{r}_\perp, z(1 - z), W^2) = \sigma^{(\infty)}(W^2) \left( 1 - J_0 \left( r_\perp \sqrt{z(1 - z)\Lambda_{\text{sat}}(W^2)} \right) \right)
\]

\[
\sigma_{\gamma^*p}(W^2, Q^2) = \sigma_{\gamma^*p}(\eta(W^2, Q^2)) + O \left( \frac{m_0^2}{\Lambda_{\text{sat}}^2(W^2)} \right) = \frac{\alpha R_{e^+e^-}}{3\pi} \sigma^{(\infty)}(W^2) I_0(\eta) + O \left( \frac{m_0^2}{\Lambda_{\text{sat}}^2(W^2)} \right), \quad R_{e^+e^-} = 3 \sum_q Q_q^2.
\]

\[
I_0(\eta(W^2, Q^2)) = \frac{1}{\sqrt{1 + 4\eta(W^2, Q^2)}} \ln \frac{\sqrt{1 + 4\eta(W^2, Q^2)} + 1}{\sqrt{1 + 4\eta(W^2, Q^2)} - 1} \approx \frac{1}{\eta(W^2, Q^2)} + O(\eta \ln \eta), \quad \text{for } \eta(W^2, Q^2) \to \frac{m_0^2}{\Lambda_{\text{sat}}^2(W^2)};
\]

\[
\approx \left\{ \begin{array}{ll}
\frac{1}{2\eta(W^2, Q^2)} + O \left( \frac{1}{\eta^2} \right), & \text{for } \eta(W^2, Q^2) \to \infty,
\end{array} \right.
\]
Generalization to $\rho = \frac{4}{3}$.

Constraint: $m_0^2 \leq M_{qq}^2, M_{qq}'^2 \leq m_1^2(W^2)$;  

$$
\sigma_{\gamma^*p} = \sigma_{\gamma^*p} \left( \eta(W^2, Q^2), \frac{m_0^2}{\Lambda_{sat}(W^2)}, \xi \equiv \frac{m_1^2(W^2)}{\Lambda_{sat}(W^2)} \right),
$$

$$
\eta(W^2, Q^2) = \frac{Q^2 + m_0^2}{\Lambda_{sat}(W^2)},
$$

$$
\Lambda_{sat}(W^2) = C_1 \left( \frac{W^2}{W_0^2} + 1 \right)^{C_2} \cong \text{const} \left( \frac{W^2}{1\text{GeV}^2} \right)^{C_2}
$$

$$
C_1 = 1.95\text{GeV}^2
$$

$$
W_0^2 = 1081\text{GeV}^2
$$

$$
C_2 = 0.27(0.29)
$$

$$
m_0^2 = 0.15\text{GeV}^2
$$

$$
m_1^2(W^2) = \xi \Lambda_{sat}(W^2) = 130 \Lambda_{sat}(W^2)
$$

Normalization by $Q^2 = 0$ photoproduction (Regge fit):

$$
\sigma^{(\infty)}(W^2) \cong \begin{cases} 
30\text{mb}, & \text{for 3 active flavors, } R_{e^+e^-} = 2 \\
18\text{mb}, & \text{for 4 active flavors, } R_{e^+e^-} = \frac{10}{3}
\end{cases}
$$
The approach to saturation.
Comparison with Caldwell 6-parameter 2 P-fit: \( \sigma_{\gamma^*p} = \sigma_0 \frac{M^2}{Q^2 + \Lambda^2} \left( \frac{l}{l_0} \right)^{\epsilon_0 + (\epsilon_1 - \epsilon_0)} \sqrt{\frac{Q^2}{Q^2 + \Lambda^2}} \)

where

\[
l = \frac{1}{2x_{bj} M_p}
\]
Comparison with the experimental data directly

Prabhdeep Kaur (2010)
Saturation limit: \( \lim_{W^2 \to \infty} \frac{F_2(x \approx Q^2/W^2, Q^2)}{\sigma_{\gamma p}(W^2)} = \frac{Q^2}{4\pi^2\alpha} \)

Consider \( Q_1^2 = 0.036 \text{ GeV}^2 \)

and \( Q_2^2 = 0.1\text{ GeV}^2 \)

\[
F_2(W^2, Q_2^2 = 0.1\text{ GeV}^2) = \frac{Q_2^2}{Q_1^2} F_2(W^2, Q_1^2 = 0.036\text{ GeV}^2)
= 2.78 F_2(W^2, Q_1^2 = 0.036\text{ GeV}^2).
\]

| \( \frac{1}{W^2} \text{[GeV}^{-2}] \) | \( F_2(W^2, Q_1^2 = 0.036\text{ GeV}^2) \) | \( \frac{Q_2^2}{Q_1^2} F_2(W^2, Q_1^2 = 0.036\text{ GeV}^2) \) |
|---|---|---|
| \( 2 \cdot 10^{-5} \) | \( \approx 0.055 \) | 0.15 |
| \( 10^{-4} \) | \( \approx 0.04 \) | 0.11 |
Inversion: \( F_2(W^2) \) in terms of gluon distribution:

\[
F_2(W^2 = \frac{Q^2}{x}) = \frac{(2\rho + 1)}{3\pi} \sum Q^2_q \xi^2 \alpha_s(Q^2) G(x, Q^2)
\]

\[
\eta(W^2, Q^2) \gg 1.
\]

\[
= \frac{(2\rho + 1)}{3\pi} \sum Q^2_q \frac{1}{8\pi^2} \sigma_L^{(\infty)} \Lambda^2_{sat}(W^2).
\]

Saturation behavior:

\[
F_2(W^2, Q^2) \sim Q^2 \sigma_L^{(\infty)} \ln \frac{\Lambda^2_{sat}(W^2)}{Q^2 + m_0^2}
\]

\[
\sim Q^2 \sigma_L^{(\infty)} \ln \left( \frac{\alpha_s(Q^2) G(x, Q^2)}{\sigma_L^{(\infty)} (Q^2 + m_0^2)} \right), \quad \eta(W^2, Q^2) \ll 1.
\]

Logarithmic dependence on gluon distribution in saturation limit.

(upon using \( F_2 = f_2 \left( \frac{W^2}{1 GeV^2} \right)^{0.29} = \frac{(2\rho+1)}{3\pi} \sum Q^2_q \frac{1}{8\pi^2} \sigma_L^{(\infty)} \Lambda^2_{sat}(W^2). \))
CDP and pQCD-improved parton model
CDP and pQCD-improved parton model
The longitudinal structure function, $F_L(x, Q^2)$
6. Conclusions

Gauge-invariant (two-gluon) interaction of color dipole:

i) Color transparency

\[ \sigma_{(q\bar{q})p}(\vec{r}_\perp^2, W^2) \sim \vec{r}_\perp^2, \]  
strong cancellation between channel 1 and channel 2

relevant for \( \eta(W^2, Q^2) = \frac{Q^2 + m_0^2}{\Lambda_{sat}^2(W^2)} > 1, \)

\[ \Lambda_{sat}^2(W^2) \sim (W^2)^{C_2=0.29} \]

\[ F_L(x, Q^2) = 0.27 F_2(x, Q^2), \]

\[ F_2(x, Q^2) = F_2(W^2 = Q^2/x) \]

\[ \sim \Lambda_{sat}^2(W^2), \quad (10 GeV^2 < Q^2 < 100 GeV^2) \]

\[ \sim \alpha_s(Q^2) G(x, Q^2). \]

Peaceful coexistence between CDP and pQCD-improved parton model
ii) Saturation

\[ \sigma_{(q\bar{q})p}(\vec{r}', W^2) \sim \sigma^{(\infty)}, \]  

contribution from channel 2 has died out,

relevant for \( \eta(W^2, Q^2) < 1, \)

\[ F_2(W^2, Q^2) \sim Q^2 \sigma^{(\infty)} \ln \frac{\Lambda_{\text{sat}}^2(W^2)}{Q^2 + m_0^2} \]

\[ \sim Q^2 \sigma^{(\infty)} \ln \left( \frac{\alpha_s(Q^2)G(x, Q^2)}{\sigma_L^{(\infty)}(Q^2 + m_0^2)} \right), \]

Smooth transition from \( \eta(W^2, Q^2) \gg 1 \) to \( \eta(W^2, Q^2) \ll 1, \) including \( Q^2 = 0. \)

No evidence for non-linear gluon saturation in DIS.

Concrete model, interpolating the regions of \( \eta(W^2, Q^2) > 1 \) and \( \eta(W^2, Q^2) < 1, \) describes experimental data for \( x \lesssim 0.1, \) including \( Q^2 = 0 \) photoproduction.