Comments on Quiver Gauge Theories and Matrix Models

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Abstract

Dijkgraaf and Vafa have conjectured the correspondences between topological string theories, $N = 1$ gauge theories and matrix models. By the use of this conjecture, we calculate the quantum deformations of Calabi-Yau threefolds with ADE singularities from ADE multimatric models. We obtain the effective superpotentials of the dual quiver gauge theories in terms of the geometric engineering for the deformed geometries. We find the Veneziano-Yankielowicz terms in the effective superpotentials.

December 2002
1. Introduction

String theories give us a lot of useful methods in order for us to understand various gauge theories. For example, the AdS/CFT correspondence [1] and the gauge/gravity correspondence [2] are known well. A-model topological string theories correspond to Chern-Simons gauge theories by the gauge/gravity correspondence.

There are mirrors symmetry between the A-model topological string theories and the B-model topological string theories. Recently Dijkgraaf and Vafa have proposed the correspondences between the B-model topological string theories, $N = 1$ supersymmetric gauge theories and large $N$ matrix models [3-5]. In other words, these correspondences are the mirror duals of the gauge/gravity correspondence. The $N = 1$ gauge theory is constructed by adding certain superpotential to the $N = 2$ gauge theory. In the description of D-brane configurations, the $N = 1$ gauge theory is realized by D5-branes wrapped on two-cycles in Calabi-Yau manifolds. When the two-cycles are blown down, new three-cycles emerge. Three-form $RR$ and $NSNS$ forms then appear instead of the D5-branes. This is called geometric transition [6-10]. On the Calabi-Yau manifold after the geometric transition, there are two kinds of three cycles, which are compact $A_i$-cycles and non-compact $B_i$-cycles. We define periods,

$$ i = \frac{1}{2 i} = \frac{\theta F_0}{\theta i} = \frac{Z}{B_i}; \quad i = \frac{1}{2 i} = \frac{1}{g_s} ; \quad (1.1)$$

where is a holomorphic three-form on the Calabi-Yau threefold and $F_0$ is a prepotential. In terms of these periods we can write down the effective superpotential of the dual gauge theory,

$$ W_e = \sum_{i} \left( N_i 2 ~ 1 \right) ; \quad (1.2)$$

where $N_i$ is a number of D-branes and $i$ is a gauge coupling.

It is proposed that the effective superpotential can be reproduced by the matrix model with certain tree level superpotential $W(\theta) [5]$. The partition function of the matrix model is

$$ Z = \mathcal{P} \exp \left( \frac{1}{g_s} W(\theta) \right) . \text{Fixing } S_1 = N_1 g_s, \text{ we take the limit } N_1 \to 1; g_s \to 1, \text{ and the partition function then leads to } Z = \exp \left( g_s^2 g_2 F_g(S_1) \right). \text{ The free energies } F_g \text{ are the contributions of genus } g \text{ diagram s. In particular } F_0 \text{ is derived from the planar diagram s. If we can calculate the partition function, we obtain the free energy, } \log Z . \text{ We can identify } F_0 \text{ with the prepotential of the dual gauge theory and we obtain the exact effective superpotential in terms of (1.1) and (1.2). Since the perturbative analyses in the}$$
Matrix models lead to the exact results in the dual gauge theories, this new derivation of the superpotentials is powerful. A lot of works on this subject have been done [1] [54].

In this paper we consider $N = 1$ quiver gauge theories and matrix models. The matrix models are ADE multi-matrix models, which have been studied in [51][52]. The quiver gauge theories are realized by the string theories on the Calabi-Yau manifolds with ADE singularities. The $N = 2$ quiver gauge theories lead to the $N = 1$ theories by the additional superpotentials, while the dual Calabi-Yau geometries are deformed. These deformations are reproduced by the matrix model [4]. Since we systematically introduce a lot of gauge symmetries to the quiver gauge theories, they are interesting also for realistic particle theories [23].

In Section 2, we will analyse the quantum deformations of the ADE singularities in the matrix model side. In terms of the deformed geometries, we will calculate the superpotentials of the dual quiver gauge theories. Section 3 is devoted to the conclusions and the some comments on left problems.

2. Effective superpotentials of quiver gauge theories

Before discussing on the quiver gauge theories and the multi-matrix models, we will give a brief review on a one-matrix model [3]. Let us consider an $N \times N$ Hermitian matrix $M$. The partition function of the one-matrix model with the tree level superpotential $W(\theta)$ is

$$Z = \prod_{I} \exp \frac{1}{g_s} W(\theta) ; \quad (2.1)$$

where we set that $W(\theta)$ is a degree $n$ polynomial of $\theta$. In terms of the $N$ eigen values $\theta_I$ ($I = 1; \ldots; N$) of $M$, we can rewrite the partition function (2.1) as

$$Z = \prod_{I} \exp \left( \theta_I \prod_{I < J} \log \left( \frac{\theta_I}{\theta_J} \right) \right) \prod_{I} \exp \frac{1}{g_s} W(\theta_I) ; \quad (2.2)$$

where $(\theta)$ is the Vandermonde determinant, $\prod_{I < J} (\theta_I / \theta_J)$. When we describe the partition function as $Z = \prod_{I} \exp \left( \frac{1}{g_s} \sum_{I} W(\theta_I) \right)$, the effective action $\hat{S}$ is denoted by

$$\hat{S} = \frac{1}{g_s} \sum_{I} W(\theta_I) + 2 \sum_{I < J} \log (\theta_I / \theta_J) ; \quad (2.3)$$
From the action (2.3), the equations of motion for $i$ are written down as

$$\frac{1}{g_s} W^0 (i) - 2 g_s \sum_{j \neq i} \frac{1}{X_{ij}} = 0; \quad (2.4)$$

We introduce a resolvent,

$$\chi (x) \frac{1}{N} \frac{X^N}{I} \frac{1}{x} \frac{1}{I} = W^0 (x) \quad (2.5)$$

which is useful in the matrix model technology [53, 54]. The physical meaning of the resolvent is a loop operator and we can easily derive a loop equation from (2.4) in terms of the resolvent. From (2.4), we define the function,

$$\chi (x) = W^0 (x) \frac{1}{N} \frac{X^N}{I} \frac{1}{x} \frac{1}{I} = W^0 (x) \quad (2.6)$$

where $S$ is the 't Hooft coupling $N g_s$. $\chi (x)$ is not a polynomial, but, in large $N$ limit, $y (x)^2$ is given by $y (x)^2 = W^0 (x)^2 + f_n (x)$, where $f_n (x)$ is a degree $n$ polynomial.

In the context of large $N$ duality and geometric transitions [33], the dual Calabi-Yau geometry after the deformation is denoted by

$$u^2 + v^2 + y^2 + W^0 (x)^2 + f_n (x) = 0; \quad (2.7)$$

We then consider the one-form

$$y (x) dx = \frac{P}{W^0 (x)^2 + f_n (x)} dx; \quad (2.8)$$

The periods $\{1.2\}$ are described as

$$\frac{1}{2} \int_{A_i} \chi (x) dx = i; \quad \frac{1}{2} \int_{B_i} \chi (x) dx = \Omega W_0 = i; \quad (2.9)$$

in terms of the one-form (2.3). Without the deformation, $y (x)$ in (2.3) is equal to $W^0 (x)$. Since the function (2.4) derived from the matrix model can be identified with $y (x)$ in (2.3), $2 \chi (x)$ in (2.5) leads to $f_n (x)$ in (2.7), in other words, $f_n (x)$ is regarded as the contribution of loop operators in the matrix model. Adding $f_n (x)$ is called a quantum deformation.
Let us now consider the ADE singularities and the quiver gauge theories. The Calabi-Yau threefolds with the ADE singularities are realized by the resolution of two dimensional ADE singularities. The boxes over x-plane are denoted by

\[ A_n \quad u^2 + v^2 + \sum_{i=1}^{n-1} \left( y_i (x) \right) = 0; \quad t_i = 0; \quad (2.10)\]

\[ D_n \quad u^2 + v^2 y + \frac{1}{y} \sum_{i=1}^{n-1} \left( y_i (x)^2 \right) \quad t_i (x)^2 + 2 \quad v t_i (x) = 0; \quad (2.11)\]

\[ E_6 \quad u^2 + v^3 + y^4 + 2 (x) v y^2 + 5 (x) v y + 6 (x) y^2 + 8 (x) v + 9 (x) y + 12 (x) = 0; \quad (2.12)\]

\[ E_7 \quad u^2 + v^3 + v y^3 + 2 (x) v^2 y + 6 (x) y^2 + 8 (x) v y + 10 (x) y^2 + 12 (x) v + 14 (x) y + 18 (x) = 0; \quad (2.13)\]

\[ E_8 \quad u^2 + v^3 + y^5 + 2 (x) v y^3 + 8 (x) v y^2 + 12 (x) y^3 + 14 (x) v y + 18 (x) y^2 + 20 (x) v + 24 (x) y + 30 (x) = 0; \quad (2.14)\]

where \( i \) are functions of \( t_i (x) \) and are explicitly written down in [55]. For these brations we can describe the one-forms \( y_i (x) dx \) as

\[ A_n \quad y_i = t_{i+1}; \quad i = 1; \quad \ldots; \quad n; \quad \ldots; \quad (2.15)\]

\[ D_n \quad y_i = t_{i+1}; \quad i = 1; \quad \ldots; \quad n; \quad 1; \quad n = y_{1+1} \quad t_{1+1}; \quad (2.16)\]

\[ E_n \quad y_i = t_{i+1}; \quad i = 1; \quad \ldots; \quad n; \quad 1; \quad n = y_{1+1} + t_2 + t_3; \quad (2.17)\]

If we calculate the periods [2.3] for the one-forms \( y_i (x) dx \) in [2.15], [2.16] and [2.17], we obtain the effective superpotentials of the quiver gauge theories.

In the following, we will derive the quantum deformations of the ADE singularities from the ADE matrix models and consider the effective superpotentials of the dual quiver gauge theories.

2.1. A_n singularity

Firstly we consider the A_n singularities and the A_n quiver gauge theories.

![A_2 quiver diagram](g.1 A_2 quiver diagram)
For the simplest example, let us study an $A_2$ singularity. The $A_2$ quiver diagram is similar to the $A_2$ Dynkin diagram and is represented in fig. 1. We assign a $U(N_i)$ gauge group to the $i$-th node of the $A_2$ diagram. The dual quiver gauge theory consists of the adjoint scalars $1; 2$ and the bifundamentals $Q_{12}; Q_{21}$ transforming in the representation of $(N_1; N_2); (N_2; N_1)$ respectively. The tree level superpotential is

$$W(\mathbf{i}; \mathbf{Q}) = \text{Tr} Q_{12} \mathbf{Q}_{21} + \text{Tr} Q_{21} \mathbf{Q}_{12} + W_1(\mathbf{i}) + W_2(\mathbf{i}) :$$

The supersymmetry of the quiver gauge theory is broken from $N = 2$ to $N = 1$ by inserting the superpotentials $W_i(\mathbf{i})$. The partition function of the matrix model which we should consider is $Z = \int \text{d} \mathbf{Q} \exp \left( \frac{1}{g_s} W(\mathbf{i}; \mathbf{Q}) \right)$. $\mathbf{i}$ is an $N_i \times N_i$ matrix and $Q_{ij}$ is an $N_i \times N_j$ matrix. Integrating $Q_{ij}$ out in the partition function, we obtain the effective action of $\mathbf{i}$. Since $\mathbf{i}$ is the $N_i \times N_i$ matrix, using the eigen values $\mathbf{i} \mathbf{I}$ ($i = 1, 2; I = 1, \ldots, N_i$) we exchange the matrix integrals $\int \text{d} \mathbf{i}$ for the eigen value integrals $\int \text{d} \mathbf{I}$. We then obtain the equations of motion [3],

$$\frac{1}{g_s} W_1(\mathbf{i}; \mathbf{I}) = 2 \mathbf{X}^1 \mathbf{I} \mathbf{I} + \mathbf{X}^2 \mathbf{I} \mathbf{I} + \mathbf{X}^3 \mathbf{I} \mathbf{I} = 0; \quad (2.18)$$

$$\frac{1}{g_s} W_2(\mathbf{i}; \mathbf{I}) = 2 \mathbf{X}^2 \mathbf{I} \mathbf{I} + \mathbf{X}^1 \mathbf{I} \mathbf{I} + \mathbf{X}^3 \mathbf{I} \mathbf{I} = 0; \quad (2.19)$$

The second terms in (2.18) and (2.19) are the contributions of loop effects of $\mathbf{i} \mathbf{I}$, and the third terms come from the contributions of $Q_{ij}$. On the analogy of (2.5), we define resolvents,

$$\mathbf{i}_1(\mathbf{x}) = \frac{1}{N_1} \sum_{\mathbf{I}} \mathbf{X}^1 \mathbf{I} \mathbf{I} \mathbf{x}^{\mathbf{I}}; \quad \mathbf{i}_2(\mathbf{x}) = \frac{1}{N_2} \sum_{\mathbf{I}} \mathbf{X}^2 \mathbf{I} \mathbf{I} \mathbf{x}^{\mathbf{I}}; \quad \mathbf{i}_3(\mathbf{x}) = \frac{1}{N_2} \sum_{\mathbf{I}} \mathbf{X}^3 \mathbf{I} \mathbf{I} \mathbf{x}^{\mathbf{I}}: \quad (2.20)$$

By the way, the classical deformation of the $A_2$ singularity is denoted by

$$u^2 + v^2 + (y \mathbf{c}_1^{(i)}(\mathbf{x}))(y \mathbf{c}_2^{(i)}(\mathbf{x}))(y \mathbf{c}_3^{(i)}(\mathbf{x})) = 0; \quad \mathbf{c}_3^{(i)}(\mathbf{x}) = 0; \quad (2.21)$$

where the deformation parameters $\mathbf{c}_i^{(i)}(\mathbf{x})$ are given by

$$\mathbf{c}_1^{(i)}(\mathbf{x}) = \frac{2W_1(\mathbf{i}; \mathbf{x}) + W_2(\mathbf{i}; \mathbf{x})}{3}; \quad \mathbf{c}_2^{(i)}(\mathbf{x}) = \frac{W_1(\mathbf{i}; \mathbf{x}) + W_2(\mathbf{i}; \mathbf{x})}{3}; \quad \mathbf{c}_3^{(i)}(\mathbf{x}) = \frac{W_1(\mathbf{i}; \mathbf{x}) + 2W_2(\mathbf{i}; \mathbf{x})}{3}; \quad (2.22)$$
From (2.10) and (2.15), on the other hand, from (2.18), (2.19) and (2.20), we obtain the one-form $y_1(x)dx$ which are described as

$$y_1(x) = W_1^0(x) + 2S_1!(x) + S_2!(x)$$
$$y_2(x) = W_2^0(x) + S_1!(x) + 2S_2!(x)$$

(2.23)

Since the classical deformations are given by $y_1^{\text{cl}}(x) = t_1^\text{cl} = W_1^0(x)$ and $y_2^{\text{cl}}(x) = t_2^\text{cl} = W_2^0(x)$, the terms including $!_1$ in (2.23) imply quantum effects. We can define the quantum deformation of the $A_2$ singularity \[ \] as

$$u^2 + v^2 + (y - t_1(x))(y - t_2(x))(y - t_3(x)) = 0; \quad t_i(x) = 0$$

(2.24)

so that $t_i(x)$ satisfy $y_1(x) = t_2(x)$, $y_2(x) = t_3(x)$, and $y_2(x) = t_3(x)$. Actually $t_i(x)$ can be written down as

$$t_i(x) = t_i^\text{cl}(x) + !_1(x); \quad t_2(x) = t_2^\text{cl}(x) + !_1(x) + 2S_2!(x); \quad t_3(x) = t_3^\text{cl}(x) + 2S_2!(x)$$

(2.25)

So far we have given a brief review on the geometry of the $A_2$ quiver \[ \]. We will generalize the above discussions to the $A_n$ quiver and calculate the effective superpotentials of $A_n$ quiver gauge theories. We will consider, in particular, the quadratic tree level superpotentials.

$$A_n$$

\[ g.2 A_n \text{ quiver diagram} \]

The $A_n$ quiver diagram is represented in $g.2$. Since the diagram has $n$ nodes, we assign a $U(N_i)$ gauge group to each $i$-th node. The dual theory is the $A_n$ quiver gauge theory which consists of the adjoint scalars $i$ and the bifundamental matters $Q_{i+1;i+1;i}$. The tree level superpotential is

$$W(\; ; Q) = \sum_{i=1}^{X^1} \text{Tr}(Q_{i,i+1} Q_{i,i+1} \cdots Q_{i+1;i+1;i} Q_{i+1;i+1} Q_{i,i+1}) + \sum_{i=1}^{X^0} \text{Tr}W_i(\; ; Q)$$

where $W_i(\; ; Q)$ is a polynomial of $i$. We integrate out $Q_{i,j}$ in the partition function,

$$Z = \prod_{i=1}^{X^1} \prod_{i>j}^{X^0} dQ_{i,j} \exp \frac{1}{g_s} W(\; ; Q)$$

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and we can rewrite it in terms of the eigen values \( i; I = 1; \ldots , i \), \( \mathcal{M} \). We then obtain

\[
Z = \sum_{i=1}^{Z} d_{i} Q \frac{1}{\det(1_i 1_i 1_i + i+1)} \exp \frac{1}{g_s} W (i; Q) \]

\[
Z = \sum_{i=1}^{Z} d_{i} Q \frac{1}{\det(1_i 1_i 1_i 1_i + i+1)} \exp \frac{1}{g_s} W (i; Q) \]

\[
\text{When we describe the partition function as } R \mathcal{Q} d_{i} e^{s}, \text{ the effective action } \hat{S} \text{ is denoted by}

\[
\hat{S} = \frac{1}{g_s} \sum_{i=1}^{\mathcal{Q}} d_{i} \text{ e}^{s} \frac{1}{2 \log(i_{i} i_{j}) + \log(i_{i} i_{i+1})}:
\]

\[
\text{The equations of motion for the action } (2.27) \text{ become}

\[
\frac{1}{g_s} W_0 (i; j; \mu) = \frac{1}{2} \sum_{j=1; j \neq i}^{\mathcal{Q}} \frac{1}{j_{i} j_{j} + j_{i+1} j_{j+1}} = 0; \quad j = 2; \quad n \quad 1;
\]

\[
\frac{1}{g_s} W_0 (j; n; \mu) = \frac{1}{2} \sum_{i=1; i \neq j}^{\mathcal{Q}} \frac{1}{i_{i} i_{j} + i_{i+1} i_{j+1}} = 0; \quad j = 2; \quad n \quad 1;
\]

We also introduce the resolvents,

\[
(2.29)
\]

and the 't Hooft couplings \( s_{1} \). Note that, in the context of the large N duality, we \( x_{S_{1}} \) and take the limit where \( N_{1} \) go to infinity. We then read the following functions from (2.28) as

\[
\begin{align*}
y_1 (x) &= W_0 (x) \quad 2s_{1} !_{1} (x) + s_{2} !_{2} (x); \\
y_{j} (x) &= W_0 (x) \quad 2s_{j} !_{j} (x) + s_{j+1} !_{j+1} (x); \quad j = 2; \quad n \quad 1; \\
y_{n} (x) &= W_0 (x) \quad 2s_{n} !_{n} (x) + s_{n+1} !_{n+1} (x);
\end{align*}
\]
\( y_i(x) \) include the quantum deformations of \( A_n \) singularity and \( y_i(x)dx \) denote the deformed one-forms. Note that \( y_0^i = W_0^i \) are regarded as the classical deformations.

Let us set the tree level superpotentials to be the quadratic ones,

\[
W_i(x) = \frac{m_i}{2} x^2; \quad i = 1; \quad n:\]  \hspace{1cm} (2.33)

From \( W_0^i(\mu_i) = 0 \), the classical vacua are \( \mu_i = 0 \). Since the perturbative analyses around these vacua in the matrix model give us the exact effective superpotentials of the \( N = 1 \) dual gauge theories, we approximate \( \mu_i \) to the vacuum expectation values, that is, we set all \( \mu_i \) to be equal to zero. \((2.34)\) then becomes

\[
y_i(x) = m_1 x \quad \frac{2S_1}{x} \quad \frac{S_2}{x} = \frac{m_1}{x} (x \quad \tilde{a}^+_i \quad \tilde{a}_i); \]

where \( a_1 = q^{2S_1} m_1^{-S_2} \). The critical point \( a_1 = 0 \) of \( W_0^1(a_1) = 0 \) is split to the two points \( a_1 \). In the same way, each critical point \( a_j = 0 \) \((j = 2; \quad 3; \quad n) \) is split to the points \( a_j = q^{2S_j} m_j^{-1} S_{j+1}^{-1} \), and \( a_j = 0 \) is split to \( a_n = q^{2S_n} m_n^{-1} \). In other words, every original critical point is resolved to the two points. By the use of these resolved points, the periods \((2.35)\) are described as

\[
i = \frac{1}{2} \log \frac{a_i}{a_i} y_i(x)dx; \quad \frac{1}{2} \log \frac{a_i}{a_i} y_i(x)dx; \quad i = 1; \quad n:\]  \hspace{1cm} (2.34)

Since the \( B_i \)-cycles are non-compact, the cut-off \( \mu_i \) are needed. From \((2.30), (2.31)\) and \((2.32),\) we obtain the periods around the \( B_i \)-cycles,

\[
i = \frac{1}{2} m_1 \quad \frac{3}{m_1} \quad \frac{1}{2} \quad (2S_1 \quad S_2) \quad \frac{1}{2} \quad \log \frac{2S_1}{3} \quad \frac{S_2}{3} \quad \frac{1}{2} \quad (2S_1 \quad S_2) \quad \log \frac{1}{m_1}; \hspace{1cm} (2.35)

j = \frac{1}{2} m_j \quad \frac{3}{m_j} \quad \frac{1}{2} \quad (2S_j \quad S_j \quad S_{j+1}) \quad \frac{1}{2} \quad \log \frac{2S_j}{3} \quad \frac{S_j}{3} \quad \frac{1}{2} \quad (2S_j \quad S_j \quad S_{j+1}) \quad \log \frac{1}{m_j}; \quad j = 2; \quad \mu n 1; \hspace{1cm} (2.36)

n = \frac{1}{2} m_n \quad \frac{3}{m_n} \quad \frac{1}{2} \quad (2S_n \quad S_n \quad S_{n+1}) \quad \frac{1}{2} \quad \log \frac{2S_n}{3} \quad \frac{S_n}{3} \quad \frac{1}{2} \quad (2S_n \quad S_n \quad S_{n+1}) \quad \log \frac{1}{m_n}; \hspace{1cm} (2.37)

We also calculate the periods around the \( A_i \)-cycles as

\[
i = \frac{1}{2} (2S_1 \quad S_2); \hspace{1cm} (2.38)

j = \frac{1}{2} (2S_j \quad S_j \quad S_{j+1}); \quad j = 2; \quad \mu n 1; \hspace{1cm} (2.39)

n = \frac{1}{2} (2S_n \quad S_n \quad S_{n+1}); \hspace{1cm} (2.40)
Using these results and (1.2), we obtain the effective superpotential,

\[
W_e = \frac{1}{2} \sum_{i=1}^{n} N_i m_i \left( \frac{1}{2} N_i (2S_1 S_2) \log \frac{2S_1 S_2}{3} + N_n (2S_n S_{n+1}) \log \frac{2S_n S_{n+1}}{3} \right)
\]

\[
+ i_1 (2S_1 S_2) + n (2S_n S_{n+1}) + \sum_{i=2}^{n} \left( 2S_i S_{i+1} \right) : \quad (2.41)
\]

We can reproduce the Veneziano-Yankielowicz term s [56], which appear in the second term of (2.41). Note that, if we set all \(N_i = N\), \(N_i m_i = 1\), all \(i = \), all \(S_i = S\) and all \(i = \), the effective superpotential is denoted simply by

\[
W_e = \frac{1}{2} nN \left( \log \frac{S}{3} + 2 i S : \right) \quad (2.42)
\]

Note that the constant term \(P \sum_{i=1}^{n} \frac{1}{2} N_i m_i \left( \log \frac{S}{3} \right) \) in (2.41) and \(\frac{1}{2} nN \left( \log \frac{S}{3} \right) \) in (2.42) can be ignored.

2.2. \(D_n\) singularity

Next let us consider the \(D_n\) singularities. A \(D_4\) singularity appears in the compactifications of F-theory and is discussed also in the context of Dijkgraaf-Vafa conjecture [22].

![g.3 D_n quiver diagram](image-url)
The $D_n$ quiver diagram is represented in Fig. 3. In the same way as the $A_n$ quiver gauge theories, we assign a $U(N)$ gauge group to each $i$-th node. The fields needed in the $D_n$ quiver gauge theories are the adjoint scalars $i$ for $i = 1; \ldots; n$ and the bifundamental matters $Q_{ij}$ for the $i$-th and $j$-th nodes which are linked to each other.

The tree level superpotential is

$$W(\mathcal{Q}) = \sum_{i=1}^{n} \text{Tr}(Q_{i1}^+ i+1 Q_{i+1;i}^+ Q_{i1;i}^+ Q_{i1+1})$$

$$+ \sum_{i=1}^{n} \text{Tr}(Q_{1}^{2n} Q_{n2} 2 Q_{mn} 2 Q_{m2n} 2) + \sum_{i=1}^{n} \text{Tr} W_i(i).$$

Integrating $Q_{ij}$ out and rewriting the matrix integrals of $i$ with the eigen value integrals of $i^j$ which are the eigen values of $i$, we obtain the equations of motion for $i^j$. From these equations of motion, we end the one-form $sy_i(x)dx$ of the deformed $D_n$ singularity,

$$y_1(x) = W_1^0(x) + 2S_{1!1}(x) + S_{2!2}(x);$$

$$y_j(x) = W_j^0(x) + 2S_{j!j}(x) + S_{j+1!j+1}(x) + S_{j+1!j}(x); j = 2; \ldots; n 3;$$

$$y_n(x) = W_n^0(x) + 2S_{n!n}(x) + S_{n+1!n+1}(x);$$

by the use of the resolvents which are defined in the same way as (2.29). In the $D_n$ case different from the $A_n$ case, (2.46) is characteristic, because the $(n-2)$-th node is linked to the three nodes. The one-form $sy_i(x)dx$ include the quantum effects coming from the $D_n$ matrix models.

Let us consider the quadratic superpotential (2.33). We can then calculate the periods around the $B$-cycles as

$$1 = \frac{1}{2} m_1^3 \frac{1}{2} (2S_1 S_2) \log \left( \frac{2S_1}{S_2} \right);$$

$$j = \frac{1}{2} m_j^3 \frac{1}{2} (2S_j S_{j+1}) \log \left( \frac{2S_j}{S_{j+1}} \right); j = 2; \ldots; n 3;$$

$$1 = \frac{1}{2} (2S_j S_{j+1}) \log m_j; j = 2; \ldots; n 3;$$

10
\[ n_2 = \frac{1}{2^m n} \frac{1}{2} (2S_n 2 \ S_n 3 \ S_n 1 \ S_n) \ 1 \ \log \frac{2S_n}{3} \frac{S_n}{n} \frac{S_n}{1} \ S_n \]

\[ \frac{1}{2} (2S_n 2 \ S_n 3 \ S_n 1 \ S_n) \ \log m_\frac{n}{2}; \]

\[ n_1 = \frac{1}{2^m n} \frac{1}{n} \frac{1}{2} (2S_n 1 \ S_n 2) \ 1 \ \log \frac{2S_n}{3} \frac{S_n}{n} \frac{S_n}{1} \ S_n \]

\[ \frac{1}{2} (2S_n 1 \ S_n 2) \ \log m_\frac{n}{1}; \]

\[ n = \frac{1}{2^m n} \frac{1}{3} \frac{1}{2} (2S_n \ S_n 2) \ 1 \ \log \frac{2S_n}{3} \frac{S_n}{n} \frac{S_n}{1} \ S_n \]

\[ \frac{1}{2} (2S_n \ S_n 2) \ \log m_\frac{n}{2}; \]

and the periods around the A-cycles as

\[ n_2 = \frac{1}{2} (2S_1 \ S_2); \]

\[ n_1 = \frac{1}{2} (2S_j \ S_j 1 \ S_j + 1); \quad j = 2; \quad ;n 3; \]

\[ n_2 = \frac{1}{2} (2S_n 2 \ S_n 3 \ S_n 1 \ S_n); \]

\[ n_1 = \frac{1}{2} (2S_n 1 \ S_n 2); \]

\[ n = \frac{1}{2} (2S_n \ S_n 2); \]

From these periods, we can obtain the effective superpotentials in terms of (1.2).

For example, we consider the D\textsubscript{4} quiver gauge theory. The effective superpotential becomes

\[ W_e = \frac{1}{2} N (2S_1 \ S_2) \ 1 \ \log \frac{2S_1}{3} \frac{S_2}{n} \frac{1}{2} N (2S_3 \ S_2) \ 1 \ \log \frac{2S_3}{3} \frac{S_2}{n} \frac{1}{2} N (2S_4 \ S_2) \ 1 \ \log \frac{2S_4}{3} \frac{S_2}{n} \]

\[ \frac{1}{2} N (2S_2 \ S_3 \ S_4) \ 1 \ \log \frac{2S_2}{3} \frac{S_3}{n} \frac{S_4}{n} \]

\[ + i (S_1 + S_3 + S_4 \ S_2); \]

where, for simplicity, we set that all \( m_i = 1, \) all \( N_i = N, \) all \( i = 1 \) and all \( i = 3, \) and the constant term \( \frac{1}{2} N_1^2 m_i^3 \) is ignored. Since the first, third and fourth nodes of the D\textsubscript{4} quiver diagram have a cyclic symmetry, \( S_1; S_3; S_4 \) in the superpotential (2.53) can be replaced with one another.
2.3. $E_n$ singularity

Finally we consider the $E_n$ singularities. In the string theories, the $E_n$ singularities play important roles. For example, $E_8$ $E_8$ gauge symmetry of heterotic string theories are realized by the $E_n$ singularities in the $F$-theory. So it is interesting to analyse the $E_n$ singularities.

\[ E_n \]

\[ g.4 \quad E_n \text{ quiver diagram} \]

The $E_n$ quiver diagram is depicted in $g.4$. Each $i$-th node has a $U(N_i)$ gauge group. In the same way as the $A_n$ and $D_n$ quiver gauge theories, we define the adjoint scalars $i$ and the bifundamental matters $Q_{ij}$. The tree level superpotential in the $E_n$ quiver matrix models is

\[ W (;Q) = \sum_{i=1}^{X^1} \text{Tr}(Q_{i1} Q_{i+1}^1 i+1 Q_{i1}^1 i+1 Q_{i1}^1 i+1 Q_{i1}^1 i+1) \]

\[ + \sum_{i=1}^{X^n} \text{Tr}(Q_{3n}^i n Q_{n3}^i n Q_{n3}^i 3 Q_{3n}) + \sum_{i=1}^{X^n} \text{Tr}W_i(1): (2.60) \]

We integrate $Q_{ij}$ out in the partition function $Z = \int dQ \exp \frac{1}{g_s} W (;Q)$ and obtain the effective action of $i$, which are the eigen values of $i$. We calculate the equations of motion from this effective action and read the following functions,

\[ y_1 (x) = W_1^0 (x) - 2S_1!_1 (x) + S_2!_2 (x); \quad (2.61) \]

\[ y_2 (x) = W_2^0 (x) - 2S_2!_2 (x) + S_1!_1 (x) + S_3!_3 (x); \quad (2.62) \]

\[ y_3 (x) = W_3^0 (x) - 2S_3!_3 (x) + S_2!_2 (x) + S_4!_4 (x) + S_n!_n (x); \quad (2.63) \]

\[ y_j (x) = W_j^0 (x) - 2S_j!_j (x) + S_{j+1}!_{j+1} (x) \]

\[ + S_{j+1}!_{j+1} (x); \quad j = 4; \quad n \quad 2; \quad (2.64) \]

\[ y_{n-1} (x) = W_{n-1}^0 (x) - 2S_{n-1}!_{n-1} (x) + S_n!_n (x); \quad (2.65) \]

\[ y_n (x) = W_n^0 (x) - 2S_n!_n (x) + S_3!_3 (x); \quad (2.66) \]
We also consider the quadratic superpotential (2.33) and assume the eigenvalues in \( \lambda_i(x) \) to be in the vacua, that is, \( \lambda_i = 0 \). Since every critical point of \( \lambda_i(x) \) is then split to the two points by the quantum deformations, we calculate the period (2.34) for the one-form \( y_i(x) dx \). The periods \( i \) around the B-cycles are

\[
1 = \frac{1}{2} m_1 \left( \frac{1}{2} (2S_1 \ S_2) \right) 1 \ \log \frac{2S_1}{S_2} \ \frac{1}{2} \ (2S_1 \ S_2) \ \log m_1; \\
2 = \frac{1}{2} m_2 \left( \frac{1}{2} (2S_2 \ S_1 \ S_3) \right) 1 \ \log \frac{2S_2}{S_1 \ S_3} \ \frac{1}{2} \ (2S_1 \ S_2) \ \log m_2; \\
3 = \frac{1}{2} m_3 \left( \frac{1}{2} (2S_3 \ S_2 \ S_4 \ S_5) \right) 1 \ \log \frac{2S_3}{S_2 \ S_4 \ S_5} \ \frac{1}{2} \ (2S_1 \ S_2) \ \log m_3; \\
j = \frac{1}{2} m_j \left( \frac{1}{2} (2S_j \ S_j \ S_{j+1}) \right) 1 \ \log \frac{2S_j}{S_j \ S_{j+1}} \ \frac{1}{2} \ (2S_1 \ S_2) \ \log m_j; \\
n = \frac{1}{2} m_n \left( \frac{1}{2} (2S_n \ S_2) \right) 1 \ \log \frac{2S_n}{S_2} \ \frac{1}{2} \ (2S_1 \ S_2) \ \log m_n; 
\]

and the periods \( i \) around the A-cycles are

\[
1 = \frac{1}{2} (2S_1 \ S_2); \\
2 = \frac{1}{2} (2S_2 \ S_1 \ S_3); \\
3 = \frac{1}{2} (2S_3 \ S_2 \ S_4 \ S_5); \\
j = \frac{1}{2} (2S_j \ S_j \ S_{j+1}); \\
n = \frac{1}{2} (2S_n \ S_2); \\
\]

From these periods and (1.2), we can calculate the effective superpotentials of the \( E_n \) quiver gauge theories. For simplicity, we ignore the terms independent of \( S_i \) and set that
all $m_1 = 1$, all $N_1 = N$, all $i = i_1$ and all $i = i$. For example the effective superpotential of the $E_8$ quiver then becomes

$$W_e = \frac{1}{2} N \left(2S_1 S_2\right) 1 \log \frac{2S_1}{3} S_2 \frac{1}{2} N \left(2S_7 S_8\right) 1 \log \frac{2S_7}{3} S_8$$

$$- \frac{1}{2} N \left(2S_8 S_9\right) 1 \log \frac{2S_8}{3} S_9$$

$$\frac{1}{2} N \left(2S_3 S_2 S_4 S_5 S_6\right) 1 \log \frac{2S_3}{3} S_2 S_4 S_5 S_6$$

$$\frac{1}{2} N \left(\prod_{j=2,4,5,6} \left(2S_j S_1 S_j + 1\right)\right) 1 \log \frac{2S_j}{3} S_1 S_j + 1$$

$$+ \ i \left(S_1 + S_7 + S_8 S_9\right):$$

In this effective superpotential we can also find Veneziano-Yankielowicz terms. Since the third node is linked to the three nodes, $S_3$ is characteristic in the $E_n$ quiver as well as in the $D_n$ quiver.

3. Conclusions and discussions

We have considered the Calabi-Yau manifolds with the ADE singularities. If we calculate the periods of one-forms $y_i(x)dx$ around compact $A$-cycles and non-compact $B$-cycles on the deformed geometry of the Calabi-Yau manifolds, we can obtain the effective superpotentials of the dual gauge theories by the geometric engineering. We have calculated the equations of motion in the ADE multi-matrix models. Since the quantum deformations are derived from the perturbative analyses of the multi-matrix models by the Dijkgraaf-Vafa conjecture, we have found out the quantum deformations of the one-forms $y_i(x)dx$ from those equations of motion in the ADE multi-matrix models.

We have considered the quadratic superpotentials $W_i(y) = \frac{1}{2} m_i Tr y^2$ for the simple examples. Then the classical vacua are $y_\mu = 0$, where $y_\mu$ are the eigenvalues of $y_i$. Since the perturbation theory on these vacua gives rise to the effective superpotential in the dual gauge theory, we have approximated $y_\mu$ in the resolvents $y_i^{-1}$ to the vacua. The original critical point $a_1 = 0$, which is obtained from $(y_1^{cl} = y_1^0(a_1) = 0$, is split to the two points $a_1 , a_2$, which are derived from $y_1(a_1) = 0$ on the deformed geometry. In terms of $a_1$ and the cut-off parameters $i$, we have calculated the periods (2.34). From these periods we have written down the effective superpotentials of the dual quiver gauge theories. We have also found that the effective superpotentials include the Veneziano-Yankielowicz terms.
We have used the approximation, that is, all eigen values of appearing in the resolvents are in the classical vacua. But in order to derive exact effective superpotentials for the $N = 1$ quiver gauge theories, we should achieve the integration of the eigen values in the multi-matrix model partition function.

Though it is difficult to exactly calculate the partition functions of the multi-matrix models, we can analyse the loop expansions of the planar diagrams order by order of the 't Hooft couplings $S_1$ by the use of Feynman diagrams [13]. So we should confirm the expansions in the context of the geometric engineering.

Acknowledgement

I would like to thank B. Taylor for useful comments. This work is supported in part by the Grant-in-Aid for Scientific Research of Professor H. Sonoda, Kobe Univ., from Japan Ministry of Education, Science and Culture (# 14340077).
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