Characters in Conformal Field Theories
from Thermodynamic Bethe Ansatz

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Abstract. We propose a new $q$-series formula for a character of parafermion conformal field theories associated to arbitrary non-twisted affine Lie algebra $\widehat{g}$. We show its natural origin from a thermodynamic Bethe ansatz analysis including chemical potentials.

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1. Introduction

Recently new aspects in conformal field theories (CFTs) are being recognized through studies of thermodynamic limit of integrable models such as 1d quantum spin chains and (1 + 1)d factorized scattering systems. In these analysis, the Rogers dilogarithm function

\[ L(x) = -\frac{1}{2} \int_0^x \left( \frac{\log(1 - y)}{y} + \frac{\log y}{1 - y} \right) dy \]  

plays a key role that connects thermodynamic quantities in those models to the CFT data, most notably, central charges and scaling dimensions. For example, the following conjecture emerged \[1\],\[2\],\[3\],\[4\] from the restricted solid-on-solid (RSOS) type \[5\],\[6\],\[7\] spin chains:

\[ \frac{\ell \dim g}{g\vee + \ell} - r = \frac{6}{\pi^2} \sum_{(a,m) \in G} L(f_m^{(a)}), \]

where the lhs is the central charge \( c_{PF} \) of the parafermion (PF) CFT \[8\],\[9\] associated to an affine Lie algebra \( \hat{g} \) with rank \( r \), level \( \ell \) and dual Coxeter number \( g\vee \). (See \[11\] for a generalization of (2) including the scaling dimensions.) The set \( G \) is given by (5) and \( f_m^{(a)} \) is the unique solution to the simultaneous algebraic equation in the range \( 0 < f_m^{(a)} < 1 \),

\[ f_m^{(a)} = \prod_{(b,k) \in G} (1 - f_k^{(b)}) K_{ab}^{mk} \text{ for } (a,m) \in G, \]

\[ K_{ab}^{mk} = \left( \min(t_b m, t_a k) - \frac{mk}{\ell} \right) (\alpha_a | \alpha_b), \]

with the notations specified later. Needless to say, the equation of such form as well as the appearance of the dilogarithm are reflecting rich structures encoded in the integrable models. Eq.(2) is thereby connecting the two fundamental ingredients; the CFT data \[11\] on the lhs which is of affine Lie algebraic origin and the intricate formula on the rhs occurring from thermodynamics of the integrable models.

The purpose of this Letter is to put forward such a connection even further based on the thermodynamic Bethe ansatz (TBA) \[12\],\[2\],\[3\],\[4\],\[13\],\[14\]. We shall propose a new \( q \)-series formula for a PF character, which is essentially equivalent to a string function \[15\] of any non-twisted affine Lie algebra \( \hat{g} \) at any level \( \ell \in \mathbb{Z}_{\geq 1} \). It has a surprisingly simple form and seems to reveal an interesting structure of the PF modules. When \( q \to 1^- \), the \( q \)-series formula leads to (2) by comparing the asymptotics on both sides with the method of \[16\]. Thus our new proposal (9) may be viewed as a “lift” of (2) to a PF character formula.
in the sense of [17], [18]. More importantly, we point out that the $q$–series formula arises naturally from the spectra of the TBA-originated effective central charge [10] involving dilogarithms. The key is to observe a one to one correspondence between the independent states in the Hilbert space of the PF CFT and the ways of analytic continuations of the dilogarithm. The idea provides a new insight toward a structural correspondence between CFTs and TBA hence its presentation also consists of our main aim in this Letter. We remark that for the special case $\hat{g} = A_1^{(1)}$, our $q$–series formula coincides with that in [19].

2. New $q$–series formula

Let $g$ denote one of the classical simple Lie algebras $A_r (r \geq 1), B_r (r \geq 2), C_r (r \geq 1), D_r (r \geq 4), E_6, E_7, E_8, F_4$ and $G_2$. We write $r = \text{rank } g$ and $\hat{g}$ to mean the non-twisted affinization of $g$ [20]. Let $\Delta, \Delta_+, \Pi, h, (\cdot|\cdot)$ denote the root system, the set of positive roots, the set of the simple roots, the Cartan subalgebra, the invariant form on $g$, respectively. The spaces $h$ and $h^*$ are identified via the form $(\cdot|\cdot)$. We employ the normalization $|\text{long root}|^2 = 2$ and set $t_a = 2/(\alpha_a |\alpha_a)$, $\alpha_a^\vee = t_a \alpha_a$ for each simple root $\alpha_a$, where the nodes $1 \leq a \leq r$ on the Dynkin diagram are enumerated according to [20]. The root lattice $Q = \bigoplus \mathbb{Z} \alpha_a$, the coroot lattice $Q^\vee = \bigoplus \mathbb{Z} \alpha_a^\vee$ and the weight lattice $P = (Q^\vee)^*$ are as usual. We find it convenient to label the weights of $\hat{g}$ (mod null root) by its projection onto the classical part $P$. Throughout the Letter we fix an integer $\ell \in \mathbb{Z}_{\geq 1}$ and put $\ell_a = t_a \ell$ and

$$G = \{(a, m) | 1 \leq a \leq r, 1 \leq m \leq \ell_a - 1, a, m \in \mathbb{Z}\} \tag{5}(5)\text{page equation 55}$$

following [4], [10].

Let $L^\Lambda$ denote the integrable $\hat{g}$-module having a level $\ell$ dominant integral weight $\Lambda$ as the highest weight [20]. One can fit the action of the (homogeneous) Heisenberg algebra $\hat{a}$ of rank $r$ on $L^\Lambda$ [19], [4]. The algebra $\hat{a}$ has a basis $\{a_n^x | x \in \Pi, n \in \mathbb{Z}\} \cup \{id\}$. The irreducible module $\Omega^\Lambda$ of PF algebra is isomorphic to the subspace of $L^\Lambda$ consisting of the vectors $v$ such that

$$a_n^x v = 0 \quad \text{for } x \in \Pi, \quad n > 0. \tag{5}(5)\text{page equation 66}$$

The space admits the weight space decomposition

$$\Omega^\Lambda = \bigoplus_{\Lambda \equiv \Lambda \mod Q} \Omega^\Lambda_\Lambda. \tag{5}(5)\text{page equation 77}$$

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The PF currents $\psi^\alpha_n (\alpha \in \Delta)$, which commute with the operators $a^\tau_{\pm n} (n \in \mathbb{Z}_{\geq 1})$, map the elements in $\Omega^A_\lambda$ into another sector $\Omega^A_{\lambda+\alpha}$. The character of $\lambda$-weight sector $\Omega^A_\lambda$ (with variable $q$) is given by [13], [9]

$$\text{ch}(\Omega^A_\lambda) = \eta(q)^r c^A_\lambda(q),$$

where $c^A_\lambda(q)$ is a string function of $\hat{g}$ at level $\ell$ and $\eta(q)$ is the Dedekind eta function. The string function is by definition the character of the (graded) $\lambda$-weight subspace of $L^A$, which is of fundamental importance. So far its explicit formula is not known for general $\hat{g}$ and $\ell$ although several expressions are available in some cases [15], [21], [19]. Let $\bar{\Omega}^A$ be the quotient of the space $\Omega^A$ by the identification $\Omega^A_\lambda \sim \Omega^A_{\lambda + \ell Q^\vee}$, and the Hilbert space of the chiral half of the PF CFT corresponds to the direct sum of $\bar{\Omega}^A$'s.

From now on we shall exclusively consider the vacuum module $\Omega^0$ case and propose the following character formula for each $\lambda$-sector ($\lambda \in Q$):

$$\text{ch}(\Omega^0_\lambda) = q^{-c_{PF}/24} \sum_{\lambda(n) \equiv \lambda \mod \ell Q^\vee} \frac{q^{\mathcal{K}(n)}}{(q)_n},$$

where $\mathcal{K}(n)$ is a function defined by equation 1010.

Here the summation in 9 runs over the vectors

$$n = (n^a_m)_{(a,m) \in G} \in (\mathbb{Z}_{\geq 0})^{|G|}$$

under the indicated restriction $\lambda(n) \equiv \lambda \mod \ell Q^\vee$ with

$$\lambda(n) = \sum_{(a,m) \in G} mn^a_m \alpha_a,$$

which is compatible with the invariance property $c^A_\lambda = c^A_{\lambda+\ell Q^\vee}$. Under the above restriction, it can be easily shown that the rhs of 9 contains only non-negative integer powers of $q$ up to an overall factor $q^p$ with $p \equiv -\frac{c_{PF}}{24} - \frac{1}{2} |\lambda|^2_{2r} \mod \mathbb{Z}$. The character of the space $\bar{\Omega}^0$ is now given by the same formula 9 but without any restriction on the $n$-sum other than 12. At present, a proof is not known for 9 for general $\hat{g}$ and $\ell$. However one can verify several
cases directly and observe a wealth of consistency as we shall see below. For \( \widehat{g} = A_1^{(1)} \), some generalizations into different directions have also been conjectured in [18, 22].

Firstly, (9) is indeed valid for \((\widehat{g}, \ell) = (A_1^{(1)}, \text{general})\) as it coincides with the formula in [14]. So is the case \((\widehat{g}, \ell) = (B_r^{(1)}, 1)\) with \(r\) general, where one can actually compute the \(n\)-sum by means of eq.(2.2.6) in [23] and compare the result with that in [14].

The case \((\widehat{g}, \ell) = (G_2^{(1)}, 1)\) can also be proved since the \(q\)-series (9) then reduces to that for \((A_1^{(1)}, 3)\) (cf.[15]). Not to mention, (9) is trivially true for \(\widehat{g} = A_r^{(1)}, D_r^{(1)}\) and \(E_{6,7,8}^{(1)}\) with \(\ell = 1\) when the PF module becomes 1 dimensional. Secondly, we have generated the low power terms in (9) by computer and checked agreements with the known results on the string functions for \((\widehat{g}, \ell) = (A_{2,3}^{(1)}, 2, 3)\) [24], \((C_{3,4}^{(1)}, 1), (F_4^{(1)}, 1)\) and \((E_8^{(1)}, 2)\) [13, 21]. For instance in the last example, the rhs of (9) with \(\lambda = \Lambda_1\) yields \(q^{3/16}(1 + 29q + 288q^2 + 1878q^3 + \cdots)\). This agrees with the \(E_8^{(1)}\) level 2 result \(t_{A_1}^{2\Lambda_1}\) given by eq.(4.4.3a) and Proposition 4.4.1(e) in [21] as an order 9 polynomial of the Virasoro characters. (\(\Lambda_i\) denotes the \(i\)-th fundamental weight.) One may substitute (8) and (9) into the character formula [15] under the principal specialization of \(z\). We have then checked that the resulting \(q\)-series for \(\text{ch}(L^0)\) indeed fulfills the known factorization property [15] up to some power for many examples including \((\widehat{g}, \ell) = (A_{2,3}^{(1)}, 3), (B_3^{(1)}, 2, 3), (D_4^{(1)}, 2, 3), (F_4^{(1)}, 2)\) and \((G_2^{(1)}, 2)\). Thirdly, if (9) is true, then

\[
\min\{\mathcal{K}(\mathbf{n}) \mid \text{(12) and } \lambda(\mathbf{n}) \equiv \lambda \mod \ell Q^\vee \} = n_\lambda^0 - \frac{|\lambda|^2}{2\ell} \tag{15}
\]

must hold by comparing the leading powers on both sides. Here \(n_\lambda^0\) is the minimum eigenvalue of the Virasoro operator \(L_0\) in the \(\lambda\)-weight subspace of \(L^0\) and is equal to the minimum number of roots to express \(\lambda\) as their sum if it is possible within \(\ell\) roots. Our quadratic form \(\mathcal{K}(\mathbf{n})\) (14) has the consistent property to it since

\[
\min\{\mathcal{K}(\mathbf{n}) \mid \text{(12) and } \lambda(\mathbf{n}) \equiv \lambda \mod \ell Q^\vee \} = 1 - \frac{|\lambda|^2}{2\ell} \tag{16}
\]

is valid for any positive root \(\lambda\), which is a special case of (15). Finally, we remark that (9) is also consistent in that it leads to the dilogarithm conjecture (2) by comparing the asymptotics on both sides as \(q \to 1^-\). To see this, we firstly note that the leading divergence of the lhs in (9) is \((\bar{q})^{-c_{PF}/24} (\bar{q} = e^{-2\pi i/\tau})\) when \(q = e^{2\pi i \tau} \to 1^-\) [15]. As for the rhs,
one can apply the argument in [16, 17] to get a crude estimate \((q) - \sum L(f^{(a)})/4\pi^2\), from which (2) follows. In particular, (3) arises essentially from the “saddle-point condition” \(q^n_m = 1 - f^{(a)}_m\) with respect to \(n^{(a)}_m\).

Before closing this section, let us discuss how our \(q\)-series (4) will indicate a basis structure in the PF module in the light of the earlier works [19, 25]. The space \(\Omega^0\) is certainly spanned by the vectors

\[ T^\gamma \psi_{-k_1}^{\beta_1} \cdots \psi_{-k_j}^{\beta_j} v_0 \quad (\gamma \in \ell \mathbb{Q}, \beta_i \in \Delta_+), \]

where \(v_0\) is the highest weight vector and \(T^\gamma\) is the translation isomorphism \(T^\gamma : \Omega^0_\lambda \rightarrow \Omega^0_{\lambda + \gamma}\). Furthermore by introducing a lexicographic ordering in this set, we can choose Poincaré-Birkhoff-Witt type vectors among them as a spanning set of \(\Omega^0\). To illustrate the idea let us take the example \((\hat{g}, \ell) = (A^{(1)}_2, 2)\) with \(\lambda = \alpha_1 + \alpha_2 \in \Delta\) and consider the \(n^{(1)}_1 = n^{(2)}_1 = 1\) term \(q^{1/2} / (q_1(q_1))\) in (4) (apart from \(q^{-c_{PF}/24}\)). In view of (13) and the restriction \(\lambda(n) \equiv \lambda \mod \ell \mathbb{Q}\), it corresponds to the character of the subspace of \(\Omega^0_{\alpha_1 + \alpha_2}\) spanned by the vectors

\[ \psi_{-k_1}^{\alpha_1 + \alpha_2} v_0 \quad (k \geq 1), \]
\[ \psi_{-k_1}^{\alpha_1} \psi_{-k_2}^{\alpha_2} v_0 \quad (k_1 > k_2 \geq 1), \]
\[ \psi_{-k_1}^{\alpha_2} \psi_{-k_2}^{\alpha_1} v_0 \quad (k_1 \geq k_2 \geq 1), \]

since their contributions amount to it as

\[ q^{-1/2} \left( \frac{q}{(q_1)} + \frac{q^3}{(q_2)} + \frac{q^2}{(q_1(q_2))} \right) = \frac{q^{1/2}}{(q_1(q_1))}. \]

Here the prefactor \(q^{-1/2}\) comes from \(-|\alpha_1 + \alpha_2|^2/2\ell = -1/2\). In general non-trivial relations exist among the operators \(\prod_i \psi_{-k_i}^{\beta_i}\) if \((\sum \beta_i |\Lambda_j|) \geq \ell\) for some fundamental weight \(\Lambda_j\), hence one must eliminate some spanning vectors to get a real basis. We leave it as an interesting future problem.

3. Origin from TBA

Our proposal (5) for the PF character has stemmed from an analysis based on the TBA type integral equation in [10]

\[ RM_{a}chu = \pi D^{(a)}_{m} + e^{(a)}_{m}(u) + \sum_{(b,k) \in G} \int_{-\infty}^{\infty} dv \Psi^{mk}_{ab}(u - v) \log\left(1 + \exp\left(-\epsilon^{(b)}_{k}(v)\right)\right), \]
which represents interacting “pseudo particles” with energy \( \epsilon_{m}^{(a)}(u) \) labeled by \((a,m) \in G\). Here, \( M_{a} > 0, \pi i D_{m}^{(a)} \) and \( R \) are independent of the rapidity \( u \) and stand for the mass, the chemical potential (cf. \[20\]) and the system size corresponding to the inverse temperature in TBA. The integration kernel \( \Psi_{mk}^{ab}(u) \) decays rapidly when \(|u| \to \infty\) and has been specified in eq.(18) of \[10\]. Here we will not need its explicit form but the properties

\[
\int_{-\infty}^{\infty} \Psi_{mk}^{ab}(u) du = \delta_{ab} \delta_{mk} - K_{mk}^{ab}, \quad \Psi_{mk}^{ab}(u) = \Psi_{mk}^{ab}(-u) = \Psi_{km}^{ba}(u).
\]

Eq.(20) has a similar form to many earlier examples of the TBA equations \[27\],\[13\],\[28\],\[14\],\[29\],\[30\] and is a candidate describing a massive deformation of the level \( \ell \) \( \mathcal{g} \) PF CFT by a certain relevant operator \[31\]. Actually, one can apply the standard TBA technique to show that the free energy

\[
F(R) = -\frac{1}{2\pi} \sum_{a=1}^{r} M_{a} \sum_{m=1}^{\ell_{a}-1} \int_{-\infty}^{\infty} du \ ch\ u \log\left(1 + \exp\left(\epsilon_{m}^{(a)}(u)\right)\right)
\]

has the ultraviolet (UV) asymptotics

\[
F(R) \simeq -\frac{\pi c}{6R} \quad \text{as} \quad R \to 0,
\]

\[
\frac{\pi^{2}}{6} c = \sum_{(a,m) \in G} \left(L(f_{m}^{(a)}) - \frac{\pi i}{2} D_{m}^{(a)} \log(1 - f_{m}^{(a)})\right),
\]

\[
\pi i D_{m}^{(a)} = \log f_{m}^{(a)} - \sum_{(b,k) \in G} K_{mk}^{ab} \log(1 - f_{k}^{(b)}).
\]

Eq.(23) is a characteristic behavior of the CFT with the central charge \( c \) \[32\],\[33\]. In the derivation, we have used \[21\] and put \( \epsilon_{m,+}^{(a)}(u) = \epsilon_{m}^{(a)}(u + \log \frac{2}{R}) \) and passed to the limit \( R \to 0 \) firstly to deduce \( \epsilon_{m,+}^{(a)}(+\infty) = +\infty \) from the assumption \( M_{a} > 0 \). We have also set \( f_{m}^{(a)} = (1 + \exp(\epsilon_{m,+}^{(a)}(-\infty)))^{-1} \), which is natural since the simplest branch choice \( \log f_{m}^{(a)} \), \( \log(1 - f_{m}^{(a)}) \in \mathbb{R} \) for all \((a,m) \in G\) in \[23\] then yields \( D_{m}^{(a)} = 0 \) hence the ground state value \( c = c_{PF} \) by means of the dilogarithm conjecture \[2\]. However, one may allow various branches in \( \epsilon_{m}^{(a)} \) and thereby introduces non-trivial chemical potentials and possibly extracts the excitation spectra as argued in \[34\],\[35\],\[36\],\[26\]. To be more precise and systematic, we introduce the universal covering space \( \mathcal{R} \) of \( \mathbb{C} \setminus \{0,1\} \) and the covering map \( \tilde{i} : \mathcal{R} \to \mathbb{C} \setminus \{0,1\} \), which specifies analytic continuations of the dilogarithm.
The effective central charge $c$ \cite{24} is then to be understood as a function on the set of the points $\tilde{f}_m^{(a)}$ such that $\tilde{i}(\tilde{f}_m^{(a)}) = f_m^{(a)}$, i.e.,

$$\frac{\pi^2}{6}c(S) = \sum_{(a,m) \in G} \left( L(\tilde{f}_m^{(a)}) - \frac{\pi i}{2} \tilde{D}_m^{(a)} \log(1 - \tilde{f}_m^{(a)}) \right), \quad \text{page 6 equation 2626}$$

$$\pi i \tilde{D}_m^{(a)} = \log(\tilde{f}_m^{(a)}) - \sum_{(b,k) \in G} K_{a,b}^{m,k} \log(1 - \tilde{f}_k^{(b)}), \quad \text{page 6 equation 2727}$$

as introduced in eq.(11) of \cite{10} (with $z = 0$ therein). Here, $S$ denotes the collection $\{C_{a,m}\}_{(a,m) \in G}$ of the contours $C_{a,m}$ from an arbitrary base point to $0 < f_m^{(a)} < 1$ in $C \setminus \{0,1\}$ specifying the point $\tilde{f}_m^{(a)}$ on $R$. We warn readers that $C_{a,m}$ here does not mean the integration contour as opposed to the convention in \cite{10}. We fix the branch $-\pi < \text{Im}(\log(\cdot)) \leq \pi$ in $\|1\|$ hence $L(x)$ has the cuts $(-\infty,0]$ and $[1,\infty)$ on the complex $x-$plane. The $\tilde{L}(\cdot)$ and $\log(\cdot)$ in \cite{20,27} stand for the analytic continuations of $L(\cdot)$ and $\log(\cdot)$ to $R$, respectively. Because $c(S)$ actually depends only on the homotopy classes of the contours $C_{a,m}$, we shall parametrize them by the integers $\xi_{m,j}^{(a)}, \eta_{m,j}^{(a)} (j \geq 1)$ as $C_{a,m} = C[f_m^{(a)} | \xi_{m,1}^{(a)}, \xi_{m,2}^{(a)}, \ldots | \eta_{m,1}^{(a)}, \eta_{m,2}^{(a)}, \ldots]$, wherein the notation $C[f|\xi_1,\xi_2,\ldots|\eta_1,\eta_2,\ldots]$ signifies the contour going from the base point to $f$ as follows (Fig.1). It firstly goes across the cut $[1,\infty)$ for $\eta_1$ times then crosses the other cut $(-\infty,0]$ for $\xi_1$ times then $[1,\infty)$ again for $\eta_2$ times, $(-\infty,0]$ for $\xi_2$ times and so on before approaching $f$ finally. Here intersections have been counted as $+1$ when the contour goes across the cut $(-\infty,0]$ (resp. $[1,\infty)$) from the upper (resp. lower) half plane to the lower (resp. upper) and $-1$ if opposite. We call $\xi_j$ and $\eta_j$ the winding numbers and assume that they are all zero for $j$ sufficiently large. From these definitions one deduces the formulas

$$\log(\tilde{f}) = \log f + 2\pi i \sum_{j \geq 1} \xi_j, \quad \log(1 - \tilde{f}) = \log(1 - f) + 2\pi i \sum_{j \geq 1} \eta_j, \quad \text{page 7 equation 2828}$$

$$\tilde{L}(\tilde{f}) = L(f) + \pi i \sum_{j \geq 1} \xi_j \log(1 - f) - \pi i \sum_{j \geq 1} \eta_j \log f - 2\pi^2 \sum_{j \geq 1} \xi_j \left( \sum_{j \geq 1} \eta_j \right) + 4\pi^2 \sum_{j \geq 1} \xi_j (\eta_1 + \cdots + \eta_j), \quad \text{page 7 equation 2929}$$

which make the dependences on the contour $C = C[f|\xi_1,\xi_2,\ldots|\eta_1,\eta_2,\ldots]$ explicit. The collection $S$ of the contours is now equivalently represented by the collection of winding numbers $\{\xi_{m,j}^{(a)}, \eta_{m,j}^{(a)}\}_{(a,m) \in G,j \geq 1}$. By applying \cite{28,29} to \cite{20,27} and using
\[ \log(f_m^{(a)}) = \sum_{(b,k) \in G} K_{ab}^{mk} \log(1 - f_k^{(b)}) \] from (3), one can split the \( c(S) \) into \( S \)-dependent and independent parts. The latter turns out to be \( c_{PF} \) due to (2) and we get \(^1\)

\[ c(S) = c_{PF} - 24T(S), \quad (\text{page 7 equation 3030}) \]

\[ T(S) = \mathcal{K}(n) - \sum_{(a,m) \in G} \sum_{j \geq 1} \xi_m^{(a)} (\eta_{m,1}^{(a)} + \cdots + \eta_{m,j}^{(a)}), \quad (\text{page 8 equation 3131}) \]

where \( \mathcal{K}(n) \) is defined in (14) and the vector \( n = (n_m^{(a)})_{(a,m) \in G} \in \mathbb{Z}^{|G|} \) is specified from \( S \) by

\[ n_m^{(a)} = \sum_{j \geq 1} \eta_{m,j}^{(a)}. \quad (\text{page 8 equation 3232}) \]

The formulas (30)-(32) describe the \( S \)-dependence of the effective central charge manifestly.

Let us now investigate the spectra of the \( c(S) \) when \( S \) consists of those contours that intersect the cuts \([1, +\infty)\) and \((-\infty, 0]\) always from the lower half plane to the upper with the former crossed firstly if ever. Respecting (32), such an \( S \) is a collection of \( C_{a,m} \) \((a, m) \in G\) parametrized as

\[ C_{a,m} = \mathcal{C}[f_m^{(a)} | \xi_m^{(a)}, 0, 0, \ldots | 1, \ldots, 1, 0, 0, \ldots] \quad (\text{page 8 equation 3333}) \]

for some \( n_m^{(a)} \geq 0 \) and \( \xi_m^{(a)}, \ldots, \xi_m^{(a)} \leq 0 \). Denote by \( \mathcal{O} \) the totality of such \( S \)'s. Then from (30)-(33), one can compute the spectra of the effective central charge as follows.

\[ \sum_{S \in \mathcal{O}} q^{-c(S)/24} = q^{-c_{PF}/24} \sum_{n_1^{(1)}, \ldots, n_r^{(r)} \geq 0} q^{\mathcal{K}(n)} \prod_{(a,m) \in G} \sum_{\xi_m^{(a)}, \ldots, \xi_m^{(a)} \leq 0} q^{-\sum_{j=1}^{n_m^{(a)}} \xi_m^{(a)}} \]

\[ = q^{-c_{PF}/24} \sum_n q^{\mathcal{K}(n)} \frac{(q)_n}{(q)_n}. \quad (\text{page 8 equation 3434}) \]

According to (3), the last expression is nothing but the character \( \text{ch}(\bar{\Omega}^0) \). In this way, the spectra of the effective central charge (26) leads to the PF CFT character itself. This extends our earlier observation in [10] (with \( \Lambda = 0 \)) further toward a structural correspondence between CFTs and TBA. Namely, the independent states in the Hilbert space \( \bar{\Omega}^0 \)

\[^1\] Using this opportunity we remark that eqs.(9b) and (12d) (with \( z = 0 \)) in [10] are erroneous and should be corrected as eqs.(29) and (31) here, respectively.
are in one to one correspondence to the lifts \((\tilde{f}^{(a)}_m)^{(a,m)} \in G \in \mathcal{R}^{[G]}\) of the point \((f^{(a)}_m)^{(a,m)} \in G\) parametrized by the set \(\mathcal{O}\).

4. Discussions

We have seen that the whole excitation spectra in the PF vacuum module \(\Omega^0\) is obtainable from the UV free energy (or the effective central charge \(c(S)\)) by a certain analytic continuation procedure. It implies that the ground state energy also possesses informations on the excitations since the latters correspond to just different branches of the former. It will be interesting to seek such a phenomenon in a wider class of models in 2d statistical mechanics and quantum field theories. As for our examples in this Letter, there are at least two routes to possibly explain this phenomenon. The first one is to interpret \(c(S)\) as the expectation value of the symmetry operator under which the corresponding excited state becomes the leading \[23\]. Though this argument has been applied almost exclusively to some primary excitations, one may generalize it by including descendant operators to accommodate the full spectra. The second route is to regard \(c(S)\) as representing finite-size corrections \[32, 33\] to various transfer matrix eigenvalues of critical RSOS-type \[5, 6, 7\] spin chains. In such analyses one treats the integral equations similar to \(20\) originated from the actual functional relations \[2\] among the row to row transfer matrices as done in \[33\] for \(\tilde{g} = A_1^{(1)}\). There, non trivial branch choices have indeed been observed to yield various eigenvalues. Thus our prescription here might be related to such an approach by using the \(U_q(\tilde{g})\) functional relation in \[10\].

The \(q\)-series formula \([3]\) and the computation in the previous section concern the UV limit \(R \to 0\) where the TBA just starts to deform the CFT. So in principle they should allow a “continuous deformation” in some sense which will exhibit rich integrability structures away from criticality.

Finally, though we have considered only the vacuum module case in this Letter, it is natural to expect similar formulas to \([3]\) for general PF modules in the light of the result in \([10]\). We hope to report them in our forthcoming paper \([37]\).

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References

[1] A.N.Kirillov, Zap.Naukh.Seminy.LOMI 164 (1987) 121 and private communications
[2] V.V.Bazhanov and Yu.N.Reshetikhin, Int.J.Mod.Phys. A 4 (1989) 115
[3] V.V.Bazhanov and Yu.N.Reshetikhin, J.Phys. A 23 (1990) 1477
[4] A.Kuniba, Nucl.Phys. B 389 (1993) 209
[5] G.E.Andrews, R.J.Baxter and P.J.Forrester, J.Stat.Phys. 35 (1984) 193
[6] E.Date, M.Jimbo, A.Kuniba, T.Miwa and M.Okado, Nucl.Phys. B 290 [FS20] (1987) 231; Adv.Stud.in Pure Math. 16 (1988) 17
[7] M.Jimbo, A.Kuniba, T.Miwa and M.Okado, Commun. Math. Phys. 119 (1988) 543
[8] V.A.Fateev and A.B.Zamolodchikov, Sov.Phys.JETP 62 (1985) 215
[9] G.E.Andrews, The Theory of Partitions, Addison-Wesley, 1976
[10] E.Date, M.Jimbo, A.Kuniba, T.Miwa and M.Okado, Lett.Math.Phys. 17 (1989) 69; Adv.Stud.in Pure Math. 19 (1989) 149
[11] B.L.Feigin, T.Nakanishi and H.Ooguri, Int.J.Mod.Phys. A 7, Suppl. 1A (1992) 217
[12] P.Fendley, Nucl.Phys. B 285 [FS19] (1987) 619
[13] M.J.Martins, Phys.Rev.Lett. 65 (1990) 2091
[14] F.Ravanini, Phys.Lett. B 282 (1992) 73
[15] A.B.Zamolodchikov, Int.J.Mod.Phys. A 4 (1989) 4235
[16] H.W.J.Blöte, J.L.Cardy and M.P.Nightingale, Phys.Rev.Lett. 56 (1986) 742
I.Affleck, Phys.Rev.Lett. 56 (1986) 746
M.J.Martins, Phys.Rev.Lett. 67 (1991) 419
A.Klümper and P.A.Pearce, J.Stat.Phys. 64 (1991) 13; Physica 183A (1992) 304
T.R.Klassen and E.Melzer, Nucl.Phys. B370 (1992) 511
A.Kuniba, T.Nakanishi and J.Suzuki, in preparation
Fig.1
An example of a contour from a base point $z_0$ to a point $f \in (0, 1)$ in $\mathbb{C} \setminus \{0, 1\}$. Its homotopy type is parametrized as $C[f|−2, 0, \ldots|2, 1, 0, \ldots]$ or also as $C[f|0, −2, 0, \ldots|1, 1, 1, 0, \ldots]$. 