PARITY FACTORS I: GENERAL KOTZIG-LOVÁSZ DECOMPOSITION FOR GRAFTS

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ABSTRACT. This paper is the first from a series of papers that establish a generalization of the basilica decomposition for cardinality minimum joins in grafts. Joins in grafts are also known as $T$-joins in graphs, where $T$ is a given set of vertices, and minimum joins in grafts can be considered as a generalization of perfect matchings in graphs provided in terms of parity. The basilica decomposition is a canonical decomposition applicable to general graphs with perfect matchings, and the general Kotzig-Lovász decomposition is one of the three central concepts that compose this theory. The classical Kotzig-Lovász decomposition is a canonical decomposition for a special class of graphs known as factor-connected graphs and is famous for its contribution to the study of the matching polytope and lattice. The general Kotzig-Lovász decomposition is a nontrivial generalization of its classical counterpart and is applicable to general graphs with perfect matchings. As a component of the basilica decomposition theory, the general Kotzig-Lovász decomposition has contributed to the derivation of further results in matching theory, such as a characterization of barriers or an alternative proof of the tight cut lemma. In this paper, we present an analogue of the general Kotzig-Lovász decomposition for minimum joins in grafts.

1. INTRODUCTION

This paper is the first in a series of papers [9, 10] that establish the basilica decomposition [7, 12, 13] for grafts. In this paper, we present the general Kotzig-Lovász decomposition [7, 12, 13, 17–20] for grafts.

The general Kotzig-Lovász decomposition is a canonical decomposition of graphs that describes the structure of perfect matchings. In the theory of matchings (1-matchings), canonical decompositions, such as the Gallai-Edmonds [4, 6] or Dulmage-Mendelsohn [1–3] decompositions, are fundamental tools that constitute the basis of the theory [21]. In matching theory, something is said to be canonical if it is determined uniquely for a given graph. A canonical decomposition is a decomposition determined uniquely for a graph and thus provides us with a comprehensive view of the structure of all maximum matchings. Canonical decompositions are therefore strong tools for analyzing matchings.

The classical Kotzig-Lovász decomposition [17, 20] was proposed for a particular class of graphs with perfect matchings known as factor-connected graphs. This class of graphs is essential to polyhedral studies of perfect matchings [21, 22]. The classical Kotzig-Lovász decomposition was used for deriving important combinatorial results for factor-connected graphs, such as the two ear theorem and tight cut lemma [21], both of which, along with the classical Kotzig-Lovász decomposition itself, have

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been used for deriving central results regarding the perfect matching polytope and lattice \[21\,22\].

Recently, the classical Kotzig-Lovász decomposition was generalized for general graphs with perfect matchings \[7\,12\,13\]. We call this the general Kotzig-Lovász decomposition or just the Kotzig-Lovász decomposition. The general Kotzig-Lovász decomposition is proposed as one of the three main components of the theory of basilica decomposition \[7\,12\,13\]. The basilica decomposition is a canonical decomposition applicable to general graphs with perfect matchings and consists of the following three main concepts:

(i) the general Kotzig-Lovász decomposition,

(ii) the cathedral order, a canonical partial order between factor-components, and

(iii) the canonically determined relationship between (i) and (ii).

The basilica decomposition has been used for deriving alternative proofs \[15\,16\] of classical theorems such as Lovász’s cathedral theorem for saturated graphs \[20\] and Edmonds et al.’s tight cut lemma \[5\]. It is also used for providing a characterization of barriers \[11\,14\,21\] and a generalization of the Dulmage-Mendelsohn decomposition \[1\,8\] for arbitrary graphs \[8\], which was originally for bipartite graphs.

Perfect matchings can be generalized into T-joins \[21\,22\]. Given a graph and a set of vertices \(T\), a T-join is a set of edges \(F\) such that each vertex is adjacent to an odd number of edges from \(F\) if and only if it is a vertex from \(T\). A graph has a T-join if and only if each connected component has an even number of vertices from \(T\).

The concept of T-joins in graphs are also known as joins in grafts; a graft is defined as a tuple of graph and set \(T\) that can possess a T-join. In this paper, we often use this terminology of joins and grafts. Even in a graph with T-joins, the minimum T-join problem, namely, which set of edges can be a T-join with the minimum number of edges, is not trivial. If a given graph has a perfect matching and \(T\) is equal to its vertex set, then minimum T-joins of this graph coincide with perfect matchings. That is, minimum T-joins are a generalization of perfect matchings that is provided in terms of parity. The minimum T-join problem includes well-known classical problems such as the Chinese postman problem \[21\,22\]. Additionally, T-joins are known to be closely related to famous open problems in graph theory such as Tutte’s 4-flow conjecture \[22\].

In this paper, we present a generalization of the general Kotzig-Lovász decomposition for T-joins. This result is, in fact, a component of the entire theory of the basilica decomposition for T-joins; in this paper and its sequels \[9\,10\], the three central concepts of the basilica decomposition theory are generalized for T-joins. Our result in this paper includes an old announcement by Sebő \[23\], that is, a generalization of the classical Kotzig-Lovász decomposition for T-joins. Considering the analogical relationship between perfect matchings and T-joins, we believe that various consequences will be obtained from our result, such as T-join analogues of

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\[1\] This is not a trivial generalization; the general Kotzig-Lovász decomposition is not the mere disjoint union of the classical Kotzig-Lovász decomposition of each factor-component but is a refinement of it.

\[2\] As with the relationship between the classical and general Kotzig-Lovász decompositions for perfect matchings, the general Kotzig-Lovász decomposition for T-joins is not a trivial extension but a refinement of the T-join analogue of the classical one.
those results derived from the classical or general Kotzig-Lovász decompositions or
the basilica decomposition theory.

The reminder of this paper is organized as follows: In Section 2, we introduce
the basic notation and definitions used in this paper. In Section 3 we explain the exact
statement of the original general Kotzig-Lovász decomposition for graphs with
perfect matchings. In Section 4 we introduce preliminary concepts and facts regarding
distances in grafts. In Section 5 we provide observation for factor-connected grafts
regarding distances to be used in Section 6. In Section 6, we prove the main result of
this paper, the analogue of the general Kotzig-Lovász decomposition for minimum
joins in grafts.

2. Definitions

2.1. Notation. For basic notation, we mostly follow Schrijver [22]. We list in the
following exceptional or non-standard definitions. The symmetric difference of two
sets A and B, that is, \((A \setminus B) \cup (B \setminus A)\) is denoted by \(A \triangle B\). As usual, a singleton
\(\{x\}\) is often denoted simply by \(x\). We treat paths and circuits as graphs. That is,
a circuit is a connected graph in which every vertex is of degree two. A path is a
connected graph in which every vertex is of degree no more than two, and at least
one vertex is of degree less than two. A graph with a single vertex and no edges is
regarded as a path. Let \(G\) be a graph. The vertex set and the edge set of \(G\) are
denoted by \(V(G)\) and \(E(G)\), respectively. The set of connected components of \(G\) is
denoted by \(\mathcal{C}(G)\). Let \(X \subseteq V(G)\). The subgraph of \(G\) induced by \(X\) is denoted by
\(G[X]\). The graph \(G[V(G) \setminus X]\) is denoted by \(G - X\). The contraction of \(G\) by \(X\) is
denoted by \(G/X\). The set of edges joining \(X\) and \(V(G) \setminus X\) is denoted by \(\delta_G(X)\).
The set of edges that span \(X\), that is, those with both ends in \(X\), are denoted by
\(E_G[X]\). Given \(F \subseteq E(G)\), \(G - F\) denotes the subgraph of \(G\) obtained by deleting
\(F\) without removing any vertices. Given a set of disjoint subgraphs \(\mathcal{K}\) of \(G\), \(G/\mathcal{K}\)
denotes the graph obtained by contracting each \(K \in \mathcal{K}\) into a vertex. That is, if
we let \(\mathcal{K} = \{K_1, \ldots, K_l\}\), where \(l \geq 1\), then \(G/\mathcal{K} = G/V(K_1)/\cdots/V(K_l)\).
For simplicity, we identify the vertices or edges of \(G/\mathcal{K}\) with the corresponding items
of \(G\). For example, if \(e \in E(G)\) is an edge joining \(K_1\) and \(K_2\), we also denote by \(e\)
the corresponding edge of \(G/\mathcal{K}\) joining \([K_1]\) and \([K_2]\), where \([K_1]\) and \([K_2]\) are the
contracted vertices corresponding to \(K_1\) and \(K_2\), respectively.

2.2. Matchings. Let \(G\) be a graph. A set \(M \subseteq E(G)\) is a matching if each
\(v \in V(G)\) satisfies \(|\delta_G(v) \cap M| \leq 1\). A matching is maximum if it has the maximum
number of edges. A matching \(M\) is perfect if \(|\delta_G(v) \cap M| = 1\) for each \(v \in V(G)\).
A perfect matching is also referred to as a 1-factor. A 1-factor is a maximum
matching, however the converse does not hold. A graph is said to be factorizable if
it has a 1-factor.

2.3. Grafts and Joins. Let \((G, T)\) be a pair of a graph \(G\) and a set \(T \subseteq V(G)\).
A join of \((G, T)\) is a set \(F \subseteq E(G)\) such that \(|\delta_G(v) \cap F|\) is odd for each \(v \in T\), and
is even for each \(v \in V(G) \setminus T\). The pair \((G, T)\) is called a graft if each connected
component of \(G\) has an even number of vertices from \(T\). When discussing a graft
\((G, T)\), we often treat the items or properties of \(G\) as if they are from \((G, T)\).
For example, we call an edge of \(G\) an edge of \((G, T)\). The following statement is
classical and can be confirmed rather easily by parity arguments; see Lovász and
Plummer [21] or Schrijver [22].
**Fact 2.1.** The pair \((G, T)\) has a join if and only if it is a graft.

A minimum join is a join with the minimum number of edges. We denote by \(\nu(G, T)\) the number of edges in a minimum join of a graft \((G, T)\).

**Remark 2.2.** A join of a graft \((G, T)\) is often referred to as a T-join of \(G\).

### 2.4. Relationship between 1-Factors and Joins.

**Observation 2.3.** If \(G\) is a factorizable graph and \(T = V(G)\), then any \(F \subseteq E(G)\) is a minimum join of the graft \((G, T)\) if and only if \(F\) is a 1-factor of \(G\).

That is, minimum joins of grafts are a generalization of 1-factors of factorizable graphs.

### 3. General Kotzig-Lovász Decomposition for 1-Factors

In this section, we explain the original general Kotzig-Lovász decomposition for 1-factors. An edge from a factorizable graph is allowed if it can be contained in a 1-factor. Two vertices \(u\) and \(v\) are said to be factor-connected if there is a path between \(u\) and \(v\) in which every edge is allowed. A factorizable graph is said to be factor-connected if every two vertices are factor-connected. A factor-connected component or factor-component is a maximal factor-connected subgraph. Hence, a factorizable graph consists of its factor-components, which are disjoint, and edges joining distinct factor-components, which are non-allowed. The concept of factor-components can be defined in an alternative manner as follows. Let \(G\) be a factorizable graph. Let \(\hat{M}\) be the set of allowed edges of \(G\). A factor-component is a subgraph of \(G\) induced by \(V(C)\), where \(C\) is a connected component of a subgraph of \(G\) determined by \(\hat{M}\).

**Definition 3.1.** Let \(G\) be a factorizable graph. A binary relation \(\sim_G\) over \(V(G)\) is defined as follows: for \(u, v \in V(G)\), \(u \sim_G v\) if \(u\) and \(v\) are factor-connected and \(G - u - v\) is not factorizable.

**Theorem 3.2** (Kita [7,12,13]; Kotzig [17,19], Lovász [20]). For any factorizable graph \(G\), the binary relation \(\sim_G\) is an equivalence relation over \(V(G)\).

The family of equivalence classes by \(\sim_G\) is called the general Kotzig-Lovász decomposition or simply the Kotzig-Lovász decomposition of a factorizable graph \(G\). The restricted statement of Theorem 3.2 in which \(G\) is a factor-connected graph was found by Kotzig [17,19] and Lovász [20]. In this case, we often call the structure the classical Kotzig-Lovász decomposition. Although a factorizable graph is made up of factor-components, the general Kotzig-Lovász decomposition of a factorizable graph \(G\) is not the mere disjoint union of the classical Kotzig-Lovász decomposition of each factor-component of \(G\). As will be observed in Section 6, the first one is, in general, a refinement of the second one.

### 4. Preliminaries on Distances

**4.1. Fundamentals of Distances.** In this section, we introduce the concept of the distance between two vertices in a graft where the edge weight is determined by a given minimum join and explain its basic properties. Let \((G, T)\) be a graft, and let \(F\) be a minimum join of \((G, T)\).
**Definition 4.1.** For each edge $e \in E(G)$, let $w_P(e) := -1$ if $e \in F$, and let $w_P(e) := 1$ otherwise. Given a subgraph $C$ of $G$, which is typically a path or circuit, $w_P(C)$ denotes $\sum_{e \in E(C)} w_P(e)$. For $u, v \in V(G)$, the distance between $u$ and $v$ in $(G, T)$ regarding $F$ is the minimum value of $w_P(P)$, where $P$ is taken over all paths between $u$ and $v$, and is denoted by $\lambda(u, v; F; G, T)$. Note that if $u = v$, then $\lambda(u, v; F; G, T) = 0$.

The distance between two vertices might be defined on the basis of trails instead of paths. However, the next statement shows that if $F$ is a minimum join, the concepts of distances defined by paths and trails coincide.

**Proposition 4.2.** Let $(G, T)$ be a graft, and let $F \subseteq E(G)$. Then, $F$ is a minimum join of $(G, T)$ if and only if $w_P(C) \geq 0$ holds for every circuit $C$.

The next lemma states that the distances between two vertices does not depend on which minimum join is given.

**Lemma 4.3 (Sebő [23]).** Let $(G, T)$ be a graft and $F$ be a minimum join of $(G, T)$. Then, for any $x, y \in V(G)$ with $x \neq y$, $\lambda(x, y; F; G, T) = \nu(G, T \triangle \{x, y\}) - \nu(G, T)$.

Under Lemma 4.3 we abbreviate $\lambda(x, y; F; G, T)$ to $\lambda(x, y; G, T)$ for any $x, y \in V(G)$.

### 4.2. Comb-Bipartite Grafts

In this section, we introduce the concept of comb-bipartite grafts and their basic properties.

**Definition 4.4.** We say that a graft $(G, T)$ is bipartite if $G$ is a bipartite graph. We call $A$ and $B$ color classes of a bipartite graft $(G, T)$ if $A$ and $B$ are color classes of $G$. We say that a graft $(G, T)$ is comb-bipartite if $G$ is a bipartite graph with color classes $A$ and $B$, the color class $B$ is a subset of $T$, and $\nu(G, T) = |B|$. Here, we call $A$ and $B$ the spine and tooth sets of the comb-bipartite graft $(G, T)$, respectively.

The notion of comb-bipartite grafts is closely related to comb-critical towers introduced by Sebő [23]. A tower is a pair of a connected graph and a set of an odd number of vertices. We introduce comb-bipartite grafts so that we can discuss the property of grafts.

By definition, the next statement about comb-bipartite grafts is easily confirmed.

**Lemma 4.5.** The following three properties are equivalent for a bipartite graft $(G, T)$ with color classes $A$ and $B$.

(i) The graft $(G, T)$ is comb-bipartite with spine set $A$ and tooth set $B$.
(ii) For any minimum join $F$ of $(G, T)$, each $v \in B$ satisfies $|\delta_G(v) \cap F| = 1$.
(iii) For some minimum join $F$ of $(G, T)$, each $v \in B$ satisfies $|\delta_G(v) \cap F| = 1$.

Accordingly, the next lemma follows.

**Lemma 4.6.** Let $(G, T)$ be a comb-bipartite graft with spine set $A$ and tooth set $B$, and let $F$ be a minimum join of $(G, T)$. If $P$ is a path with ends $s \in A$ and $t \in B$ with $w_P(P) = -1$, then $|\delta_P(v) \cap F| = 1$ holds for any $v \in V(P) \cap B$.

### 4.3. Sebő’s Distance Decomposition

In this section, we present a known statement about distances that is taken from a profound theorem by Sebő [23]. Given a specific vertex $r$ in a graft, we can determine a partition of the vertex set according to the distance from $r$. Sebő provided the property of distances by revealing the structure of this partition.
Definition 4.7. Let \((G, T)\) be a graft and \(F\) be a minimum join of \((G, T)\). Let \(r \in V(G)\). We define \(U_0(r) := \{x \in V(G) : \lambda(r, x; G, T) = 0\}\). We also define \(U_-(r) := \{x \in V(G) : \lambda(r, x; G, T) < 0\}\) and \(U_{\leq 0}(r) := U_0(r) \cup U_-(r)\). We denote by \(Q_r\) the connected component of \(G[U_{\leq 0}(r)] - E[U_0(r)]\) that contains \(r\). Further, we denote by \(Q'_r\) the graph obtained from \(Q_r\) by deleting all edges from \(E[U_0(r)]\) and contracting each connected component of \(Q_r - U_0(r)\) into one vertex, namely, \(Q'_r := Q_r / C(Q_r - U_0(r)) - E[U_0(r)]\). For each \(K \in C(Q_r - U_0(r))\), let \([K]\) be the contracted vertices of \(Q'_r\) that correspond to \(K\), and let \(T'_r = (T \cap V(Q_r) \cap U_0) \cup \{[K] : K \in C(Q_r - U_0(r))\}\). Note that \((Q'_r, T'_r)\) is a bipartite graft.

The next theorem is a part of the main result obtained by Sebő [23].

Theorem 4.8 (Sebő [23]). Let \((G, T)\) be a graft and \(F\) be a minimum join of \((G, T)\). Let \(r \in V(G)\).

(i) Then, no edges in \(\delta_G(Q_r)\) are allowed in \((G, T)\).
(ii) No edges of \(Q_r[U_0(r)]\) are allowed in \((G, T)\).
(iii) For each \(K \in C(Q_r - U_0(r))\), \(|\delta_G(K) \cap F| = 1\); let \(r_K \in V(K)\) and \(s_K \in U_0(r)\) be the vertices such that \((s_K, r_K) \in \delta_G(K) \cap F\).
(iv) The graft \((Q'_r, T'_r)\) is comb-bipartite, whose tooth set is \(\{[K] : K \in C(Q_r - U_0(r))\}\), and \((s_K, r_K) : K \in C(Q_r - U_0(r))\) forms a minimum join of \((Q'_r, T'_r)\).
(v) For each \(K \in C(Q_r - U_0(r))\), \(F \cap E(K)\) is a minimum join of the graft \((K, (T \cap V(K)) \triangle \{r_K\}); and,
(vi) for any \(x \in V(K)\), \(\lambda(x, r_K; F \cap E(K); K, (T \cap V(K)) \triangle \{r_K\}) \leq 0\).

In this paper, we refer to the above-mentioned structure of \((G, T)\) as Sebő’s distance decomposition with the root \(r\).

5. Factor-Connectivity and Distance in Grafts

In this section, we introduce a graft analogue of factor-connectivity and show some basic properties of factor-connected grafts regarding distances to be used for proving our main theorem.

Let \((G, T)\) be a graft. An edge \(e \in E(G)\) is allowed if there is a minimum join of \((G, T)\) that contains \(e\). We say that vertices \(u, v \in V(G)\) are factor-connected if \((G, T)\) has a path whose edges are allowed. We say that a graft is factor-connected if any two vertices are factor-connected. We call a maximal factor-connected subgraph of \(G\) a factor-connected component or factor-component, in short, of \((G, T)\). We denote the set of factor-components of \((G, T)\) by \(G(G, T)\). It can easily be observed from the definition that \(G\) consists of factor-components, which are mutually disjoint, and edges joining distinct factor-components, which are not allowed.

We now present some observations about the distance between two factor-connected vertices. Theorem 4.3 implies the next lemma rather immediately.

Lemma 5.1. If \(u\) and \(v\) are factor-connected vertices in a graft \((G, T)\), then \(\lambda(u, v; G, T) \leq 0\) holds.

Proof. Consider Sebő’s distance decomposition with the root \(u\). From Theorem 4.8 we have \(u, v \in V(Q_u)\). As \(V(Q_u) \subseteq U_{\leq 0}(r)\) holds, the statement follows. \(\square\)

From Lemmas 4.6 and 5.1 the next lemma is easily confirmed.

Lemma 5.2. If \((G, T)\) is a factor-connected comb-bipartite graft with tooth set \(B\) and spine set \(A\), then, \(\lambda(x, y; G, T) = -1\) for any \(x \in A\) and any \(y \in B\).
6. General Kotzig-Lovász Decomposition for Joins in Grafts

In this section, we prove our main result in Theorem 6.6, a generalization of the general Kotzig-Lovász decomposition for grafts.

**Definition 6.1.** Let \((G, T)\) be a graft. For \(u, v \in V(G)\), we say that \(u \sim_{(G, T)} v\) if \(u\) and \(v\) are identical or if \(u\) and \(v\) are distinct factor-connected vertices such that \(\nu(G, T \setminus \{u, v\}) = \nu(G, T)\).

The binary relation \(\sim_{(G, T)}\) is a generalization of the equivalence relation \(\sim_G\); confirm the following by Observation 2.2.

**Observation 6.2.** Let \((G, T)\) be a graft in which \(G\) is a factorizable graph and the set \(T\) is \(V(G)\). Then, for any \(u, v \in V(G)\), \(u \sim_G v\) holds if and only if \(u \sim_{(G, T)} v\) holds.

In the following, we prove that \(\sim_{(G, T)}\) is an equivalence relation. Note that Lemmas 4.3 and 5.1 imply the following:

**Lemma 6.3.** Let \((G, T)\) be a graft, and let \(u, v \in V(G)\) be factor-connected vertices. Then, \(u \sim_{(G, T)} v\) holds if and only if \(\lambda(u, v; G, T) = 0\); by contrast, \(u \sim_{(G, T)} v\) does not hold if and only if \(\lambda(u, v; G, T) < 0\).

The next two lemmas are provided for Theorem 6.6.

**Lemma 6.4.** Let \((G, T)\) be a graft, let \(r \in V(G)\), and let \(H \in \mathcal{G}(G, T)\) be the factor-component with \(r \in V(H)\). Then, \(V(H) \subseteq Q_r\) holds. In the graft \((Q'_r, T'_r)\), the vertices in \((V(H) \cap U_0(r)) \cup \{[K] : K \in \mathcal{C}(Q'_r - U_0(r))\}\) and \(V(K) \cap V(H)\), that is, the vertices from the subgraph of \(Q'_r\) that corresponds to \(H\), are factor-connected.

**Proof.** From Theorem 4.8 (1), \(V(H) \subseteq Q_r\) follows. From Theorem 4.8 (ii) for any two vertices of \(V(H)\), there is a path whose edges are allowed edges of \((G, T)\) from \(E(H) \setminus E(U_0(r))\). From Theorem 4.8 (iii) the allowed edges of \((G, T)\) that join \(U_0(r)\) and components from \(\mathcal{C}(Q'_r - U_0(r))\) are also allowed in \((Q'_r, T'_r)\). Therefore, the statement follows.

**Lemma 6.5.** Let \((G, T)\) be a graft, let \(F\) be a minimum join of \((G, T)\), and let \(r \in V(G)\). Let \(u \in U_0(r)\) and \(K \in \mathcal{C}(Q'_r - U_0(r))\). If the graft \((Q'_r, T'_r)\) has a path \(P\) between \(u\) and \([K]\) with \(w_F(P) = -1\), then, for any \(v \in V(K)\), the graft \((G, T)\) has a path \(\hat{P}\) between \(u\) and \(v\) with \(w_F(\hat{P}) \leq -1\).

**Proof.** First, note that, according to Theorem 4.8 (iii), \(F\) contains a minimum join of \((Q'_r, T'_r)\). Under Lemma 4.6 for each \(L \in \mathcal{C}(Q'_r - U_0(r))\) with \([L] \in V(F)\), define the following: Let \(\epsilon_L\) be the edge from \(\delta_F([L]) \cap F\); additionally, let \(x_L \in V(G)\) be the end of \(\epsilon_L\) from \(V(L)\) by regarding \(\epsilon_L\) as an edge of \(G\). Furthermore, if \(L \neq K\), then let \(f_L\) be the edge from \(\delta_F([L]) \setminus F\); additionally, let \(y_L \in V(G)\) be the end of \(f_L\) from \(V(L)\) by regarding \(f_L\) as an edge of \(G\). If \(L = K\), then let \(y_L\) be a given vertex \(v \in V(K)\). By Theorem 4.8 (vi) each \(L\) has a path \(R_L\) between \(x_L\) and \(y_L\) with \(w(R_L) \leq 0\). By replacing each \([L]\) with \(Q_L\) over \(P\), we obtain a desired path \(\hat{P}\) with \(w_F(\hat{P}) \leq -1\).

We now prove Theorem 6.6.

**Theorem 6.6.** For any graft \((G, T)\), the binary relation \(\sim_{(G, T)}\) is an equivalence relation on \(V(G)\).
transitivity in the following. Let \( u, v, w \in V(G) \) be such that \( u \sim_{(G,T)} v \) and \( v \sim_{(G,T)} w \). If any two from \( u, v, \) and \( w \) are identical, then the statement obviously holds. Therefore, assume that \( u, v, w \) are pairwise distinct, and suppose, to the contrary, that \( u \sim_{(G,T)} w \) does not hold. Let \( H \) be the factor-component that contains \( u, v, \) and \( w \), and let \( F \) be a minimum join of \((G,T)\). Consider Sebő’s distance decomposition with the root \( u \). Because \( u \sim_{(G,T)} v \) is assumed, Lemma 6.3 implies \( v \in U_0(u) \cap V(Q_u) \). Under the present supposition, Lemma 6.3 implies \( w \in U_-(v) \), and accordingly, there exists \( K \in \mathcal{C}(Q_u - U_0(u)) \) such that \( w \in V(K) \). According to Lemma 6.4 in the comb-bipartite graft \((Q_u', T')\), the three vertices \( u, v, \) and \([K] \) are factor-connected. Therefore, from Lemma 6.4, \( Q_u' \) has a path \( P \) between \( v \) and \([K] \) with \( w_F(P) = -1 \). Thus, from Lemma 6.3, \( G \) has a path \( \hat{P} \) between \( v \) and \( w \) with \( w_F(\hat{P}) \leq -1 \). This contradicts \( v \sim_{(G,T)} w \). Thus, \( u \sim_{(G,T)} w \) is proved.

We call the family of equivalence classes determined by \( \sim_{(G,T)} \) the general Kotzig-
Lovász decomposition of a graft \((G,T)\). We denote this family by \( \mathcal{P}(G,T) \). From
Observation 6.2, the general Kotzig-Lovász decomposition for general grafts is a
generalization of the general Kotzig-Lovász decomposition for factorizable graphs.
Our result also includes, as a special case, the result announced in Sebő [23]; if
we restrict \((G,T)\) to be factor-connected, then \( \mathcal{P}(G,T) \) is a graft analogue of the
classical Kotzig-Lovász decomposition.

By the definition of \( \sim_{(G,T)} \), each equivalence class is contained in the vertex set
of some factor-component of \((G,T)\). Therefore, for each \( H \in \mathcal{G}(G,T) \), the family
\( \{S \in \mathcal{P}(G,T) : S \subseteq V(H)\} \) forms a partition of \( V(H) \). We denote this family by
\( \mathcal{P}(H;G,T) \). However, note that, as is also the case for 1-factors, \( \mathcal{P}(G,T) \) is not a
mere disjoint union of \( \mathcal{P}(H;G,T) \) taken over every \( H \in \mathcal{G}(G,T) \), but has a more
refined structure.

**Observation 6.7.** Let \((G,T)\) be a graft, and let \( H \in \mathcal{G}(G,T) \). Then, the family
\( \mathcal{P}(H;G,T) \) is a refinement of \( \mathcal{P}(H,T \cap V(H)) \). That is, if \( u, v \in V(H) \) satisfy
\( u \sim_{(G,T)} v \), then \( u \sim_{(H,T \cap V(H))} v \) holds; however, the converse does not hold in
general.

**Example 6.8.** For example, consider the graft \((G,T)\) given in Figure 4. For this
graft, \( \mathcal{P}(G,T) \) has 10 members as follows. For each factor-component \( H_1, H_2, \) and
\( H_3 \), \( \mathcal{P}(H_1;G,T) \), \( \mathcal{P}(H_2;G,T) \), and \( \mathcal{P}(H_3;G,T) \) have three, five, and two
members, respectively. If we consider \((H_1,T \cap V(H_1))\) as a single graft, \( \mathcal{P}(H_1,T \cap V(H_1)) \)
has four members as shown in Figure 2. The family \( \mathcal{P}(H_1;G,T) \) given in Figure 4
is a proper refinement of \( \mathcal{P}(H_1,T \cap V(H_1)) \).

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FIGURE 1. Example of a graft \((G, T)\) and its general Kotzig-Lovász decomposition. The black and white vertices represent vertices in \(T\) and not in \(T\), respectively. The thick edges represent allowed edges, and \(\mathcal{G}(G, T) = \{H_1, H_2, H_3\}\). Each equivalence class from \(\mathcal{P}(G, T)\) is indicated by a gray region.

FIGURE 2. The Kotzig-Lovász decomposition of the subgraft \((H_1, T \cap V(H_1))\). The family \(\mathcal{P}(H_1; G, T)\) from Figure 1 is a proper refinement of \(\mathcal{P}(H_1, T \cap V(H_1))\).

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