HARMONIC FUNCTIONS OF GENERAL GRAPH LAPLACIANS

BOBO HUA AND MATTHIAS KELLER

Abstract. We study harmonic functions on general weighted graphs which allow for a compatible intrinsic metric. We prove an $L^p$ Liouville type theorem which is a quantitative integral $L^p$ estimate of harmonic functions analogous to Karp’s theorem for Riemannian manifolds. As corollaries we obtain Yau’s $L^p$-Liouville type theorem on graphs, identify the domain of the generator of the semigroup on $L^p$ and get a criterion for recurrence. As a side product, we show an analogue of Yau’s $L^p$ Caccioppoli inequality. Furthermore, we derive various Liouville type results for harmonic functions on graphs and harmonic maps from graphs into Hadamard spaces.

1. Introduction

The study of harmonic functions is a fundamental topic in various areas of mathematics. An important question is which subspaces of harmonic functions are trivial, that is, they contain only constant functions. Such results are referred to as Liouville type theorems. In Riemannian geometry $L^p$-Liouville type theorems for harmonic functions were studied for example by Yau [Yau76], Karp [Kar82], Li-Schoen [LS84] and many others. Karp’s criterion was later generalized by Sturm [Stu94] to the setting of strongly local regular Dirichlet forms. Over the years there were several attempts to realize an analogous theorem for graphs, see Holopainen-Soardi [HS97], Rigoli-Salvatori-Vignati [RSV97], Masamune [Mas09] and most recently Hua-Jost [HJ13]. In all these works normalized Laplacians were studied (often with further restrictions on the vertex degree) and certain criteria, all weaker than Karp’s integral estimate, were obtained. The main challenge when considering graphs is the non-existence of a chain rule and, moreover, the fact that for unbounded graph Laplacians the natural graph distance is very often not the proper analogue to the Riemannian distance in manifolds. In this paper, we use the newly developed concept of intrinsic metrics on graphs to prove an analogue to Karp’s theorem for general Laplacians on weighted graphs. Thus, we generalize all earlier results on graphs not only with respect to the generality of the setting but also by recovering the precise analogue of Karp’s criterion.

Harmonic maps are very important nonlinear objects in geometric analysis studied thoroughly by many authors (e.g. Eells-Sampson [ES64], Schoen-Yau [SY76], Hildebrandt-Jost-Widman [HJW81]). In this paper, we adopt a definition of harmonic maps between metric measure spaces introduced by Jost [Jos94, Jos97a, Jos97b, Jos98]. In particular, we study harmonic maps from graphs into Hadamard spaces (i.e. globally non-positively curved spaces, also called CAT(0)-spaces), studied also by [KS01, IN05, JT07], and prove Liouville type theorems in this context. For various Liouville theorems on manifolds, we refer to [Che80, Ken90, Tam95, CTW96] and references therein. We prove the finite-energy Liouville theorem for
harmonic maps from graphs into Hadamard spaces analogous to the one in Cheng-Tam-Wang [CTW96] on manifolds.

In what follows we first state and discuss our results and refer for details and precise definitions to Section 2. Our framework are weighted graphs over a discrete measure space \((X, m)\) introduced in [KL12] which includes non locally finite graphs, (see also [Soa94]). In this setting a pseudo metric is called intrinsic if the energy measures of distance functions can be estimated by the measure of the graph (see Definition 2.2). We further call such a pseudo metric compatible if it has finite jump size and the weighted vertex degree is bounded on each distance ball (see Definition 2.3). As the boundedness of the weighted vertex degree is implied by finiteness of distance balls which is equivalent to metric completeness in the case of a path metric on a locally finite graph, see [HKMW13 Theorem A.1], this assumption can be seen as an analogue of completeness in the Riemannian manifold case. Similarly, Sturm [Stu94] asks for precompactness of balls.

Our first main result is the following analogue to Karp’s \(L^p\) Liouville theorem [Kar82 Theorem 2.2], whose proof is given in Section 3.2. A function is called (sub)harmonic if it is in the domain of the formal Laplacian and the formal Laplacian applied to this function is pointwise (less than or) equal to zero, (see Definition 2.1). We denote by \(1_{B_r}\) the characteristic function of the balls \(B_r\), \(r \geq 0\), which are taken with respect to an intrinsic metric about a fixed vertex \(o \in X\).

**Theorem 1.1** (Karp’s \(L^p\) Liouville theorem). Assume a connected weighted graph allows for a compatible intrinsic metric. Then every non-negative subharmonic function \(f\) satisfying

\[
\inf_{r_0 > 0} \int_{r_0}^{\infty} \frac{r}{\|1_{B_r}\|^p_p} dr = \infty,
\]

for some \(p \in (1, \infty)\), is constant.

Clearly, the integral in the theorem above diverges, whenever \(0 \neq f \in L^p(X, m)\). Thus, as an immediate corollary, we get Yau’s \(L^p\) Liouville theorem [Yau76].

**Corollary 1.2** (Yau’s \(L^p\) Liouville theorem). Assume a connected weighted graph allows for a compatible intrinsic metric. Then every non-negative subharmonic function in \(L^p(X, m)\), \(p \in (1, \infty)\), is constant.

**Remark 1.3.** (a) The results above imply the corresponding statements for harmonic functions by the simple observation that \(f_+, f_-\) and \(|f|\) of a harmonic function \(f\) are non-negative and subharmonic.

(b) Harmonicity of a function is independent of the choice of the measure \(m\). Hence, for any non-constant harmonic function \(f\) on \(X\), we may find a sufficiently small measure \(m\) such that \(f \in L^p(X, m)\) for any \(p \in (0, \infty)\), see [Mas09]. Our theorem states that if we impose the restriction of compatibility on the measure and the metric, then the \(L^p\) Liouville theorem holds for \(1 < p < \infty\).

(c) Theorem 1.1 generalizes all earlier results on graphs [HS97 RSV97 Mas09] for the case \(p \in (1, \infty)\). Not only that our setting is more general – as the natural graph distance is always a compatible intrinsic metric to the normalized Laplacian – but also our criterion is more general. In particular, if \(f\) satisfies

\[
\limsup_{r \to \infty} \frac{1}{r^2 \log r} \|f1_{B_r}\|^p_p < \infty,
\]
then the integral in Theorem 1.1 diverges. Thus, Theorem 1.1 is stronger than [HJ13] Theorem 1.1 (which had only $r^2$ rather than $r^2 \log r$ in the denominator). The authors of [HJ13] observed that for the normalized Laplacian the case $p \in (1, 2]$ can already be obtained by their techniques, (see [HJ13] Remark 3.3). Here, the missing cases $p \in (2, \infty)$ are treated by adopting a subtle lemma in [HS97]. Moreover, our techniques would also allow a statement such as [HJ13] Theorem 1.1 for the cases $p < 1$.

(d) In [KL12] discrete measure spaces $(X, m)$ with the assumption that every infinite path has infinite measure are discussed (this assumption is denoted by (A) in [KL12]). It is not hard to see that for connected graphs over $(X, m)$ every non-negative subharmonic function $L^p(X, m), p \in [1, \infty)$, is trivial. In fact, from every non-constant positive subharmonic function we can extract a sequence of vertices such that the function values increase along this sequence (compare [KL12, Lemma 3.2 and Theorem 8]). Since this path has infinite measure, the function is not contained in $L^p(X, m), p \in [1, \infty)$. Thus, the only interesting measure spaces are those that contain an infinite path of finite measure.

(e) Sturm [Stu94] proves an analogue for Karp’s theorem for weakly subharmonic functions. This might seem stronger, however, in our setting on graphs weak solutions of equations are automatically solutions, [HKLW12, Theorem 2.2 and Corollary 2.3].

Corollary 1.2 allows us to explicitly determine the domain of the generator $L_p$ of the semigroup on $L^p(X, m)$. We denote by $\Delta$ the formal Laplacian with formal domain $F$. (For definitions see Section 2.2). The proof of the corollary below is given in Section 3.4.

**Corollary 1.4 (Domain of the $L^p$ generators).** Assume a connected weighted graph allows for a compatible intrinsic metric. Then, for $p \in (1, \infty)$, the generator $L_p$ is a restriction of $\Delta$ and

$$D(L_p) = \{ u \in L^p(X, m) \cap F \mid \Delta u \in L^p(X, m) \}.$$

**Remark 1.5.** (a) The corollary above generalizes [HKMW13, Theorem 1] to the case $p \in (1, \infty)$ and settles the question in [HKMW13, Remark 3.6]. Moreover, it complements [KL12, Theorem 5].

(b) It would be interesting to know whether there is a Liouville type theorem for functions in $D(L_p)$ without the assumption of compatibility on the metric.

We get furthermore a sufficient criterion for recurrence analogous to [Kar82, Theorem 3.5] and [Stu94, Theorem 3] which generalizes for example [DK81, Theorem 2.2], [RSV97, Corollary B], [Woe00, Lemma 3.12], [Gri09, Corollary 1.4], [MUW12, Theorem 1.2] on graphs. For a characterization of recurrence see Proposition 3.3 in Section 3.4 where also the proof of the corollary below is given.

**Corollary 1.6 (Recurrence).** Assume a connected weighted graph allows for a compatible intrinsic metric. If

$$\int_1^\infty \frac{r}{m(B_r)}dr = \infty,$$

then the graph is recurrent.

Contrary to the normalized Laplacian, [HJ13] Theorem 1.2], there is no $L^1$ Liouville type theorem in the general case. However, for stochastic complete graphs...
(see Section 3) we have the following analogue to [Gri88, Theorem 3], [Stu94, Theorem 2]. The proof following [Gri99] is given in Section 4. We also give counter-examples to $L^1$ Liouville theorem which complement the counter-examples from manifolds, [Chu83, LS84].

**Theorem 1.7** (Grigor’yan’s $L^1$ theorem). Assume a connected graph is stochastically complete. Then, every non-negative superharmonic function in $L^1(X, m)$ is constant.

For vertices $x, y \in X$ that are connected by an edge, we denote a directed edge by $xy$ and the positive symmetric edge weight by $\mu_{xy}$. We define $\nabla_{xy}f = f(x) - f(y)$.

The following $L^p$ Caccioppoli-type inequality is a side product of our analysis. Such an inequality was proven in [HS97, HJ13, RSV97] for bounded operators. The classical Caccioppoli inequality is the case $p = 2$, which can be found for graphs in [CG98, LX10, HKMW13].

**Theorem 1.8** (Caccioppoli-type inequality). Assume a connected weighted graph allows for a compatible intrinsic metric and $p \in (1, \infty)$. Then, there is $C > 0$ such that for every non-negative subharmonic function $f$ and all $0 < r < R - 3s$

$$\sum_{x,y \in B_r} \mu_{xy}(f(x) \vee f(y))^{p-2} |\nabla_{xy}f|^2 \leq \frac{C}{(R-r)^2} \|f1_{B_R \setminus B_r}\|_p^p,$$

where $s$ is the jump size of the intrinsic metric (see Section 2.3).

**Remark 1.9.** (a) The theorem above allows for a direct proof of Corollary 1.2 confer [HJ13, Corollary 3.1].

(b) For $p \geq 2$, we can strengthen the inequality by replacing $(f(x) \vee f(y))^{p-2}$ on the left hand side by $f^{p-2}(x) + f^{p-2}(y)$, see Remark 3.2 in Section 3.3 where the theorem is proven.

The following quantitative consequence of Theorem 1.1 which is a generalization of Corollary 1.2 has various corollaries that are stated and proven in Section 5. For an intrinsic metric $\rho$ and a fixed vertex $o \in X$ let

$$\rho_1 = 1 \vee \rho(\cdot, o).$$

**Theorem 1.10.** Assume a connected weighted graph allows for a compatible intrinsic metric $\rho$. If a non-negative subharmonic function $f$ satisfies

$$f \in L^p(X, m\rho_1^{-2}),$$

for some $p \in (1, \infty)$, then $f$ is constant.

Next, we turn to harmonic maps from graphs into Hadamard spaces, (see Section 6 in particular Definition 6.1). We prove the following consequence of Karp’s theorem in Section 6.

**Theorem 1.11** (Karp’s theorem for harmonic maps). Assume a connected weighted graph allows for a compatible intrinsic metric $\rho$. Let $u$ be a harmonic map into a Hadamard space $(Y, d)$. If there are $p \in (1, \infty)$ and $y \in Y$ such that

$$d(u(\cdot), y) \in L^p(X, m\rho_1^{-2}),$$

then $u$ is bounded. Moreover, if $m\rho_1^{-2}(X) = \infty$ or $y$ is in the image of $u$, then $u$ is constant.
Finally, we turn to harmonic functions and maps of finite energy, (for definitions see Section 2.2 and Section 6.2). The two theorems below stand in close relationship to the celebrated theorem of Kendall [Ken88, Theorem 6], (confer [HK91, KS08]). Our first result in this line is a direct consequence of Theorem 6.3 and it is an analogue to Cheng-Tam-Wang [CTW96, Theorem 3.1].

**Theorem 1.12.** Assume that on a graph every harmonic function of finite energy is bounded. Then, every harmonic map from the graph into an Hadamard space is bounded.

The second result in this line is an analogue to [CTW96, Theorem 3.2]. An Hadamard space is called locally compact if for any point there exists a precompact neighborhood.

**Theorem 1.13.** Assume that on a graph every bounded harmonic function is constant. Then, every finite-energy harmonic map from the graph into a locally compact Hadamard space is constant.

The paper is organized as follows. In the next section, we introduce the involved concepts and recall some basic inequalities. Section 3 is devoted to the proofs of Theorem 1.1, Theorem 1.8 and the corollaries above. The proof of Theorem 1.7 and counter-examples to an \(L^1\)-Liouville type statement are given in Section 4. In Section 5 we prove Theorem 1.10 and derive various corollaries. Harmonic maps from graphs into Hadamard spaces are discussed in Section 6. Theorem 1.11 is proven in Section 6.1 and Theorems 1.12 and 1.13 are proven in Section 6.2. Several applications are discussed in Section 6.3.

Throughout this paper \(C\) always denotes a constant that might change from line to line. Moreover, we use the convention that \(\infty \cdot 0 = 0\), (which only appears in expressions such as \(f^{-q}(x)\nabla_{xy}f\) with \(f(x) = f(y) = 0\) and \(q > 0\)).

## 2. Set-up and preliminaries

### 2.1. Weighted graphs.

Let \(X\) be a countable discrete set and \(m : X \to (0, \infty)\). Extending \(m\) additively to sets, \((X, m)\) becomes a measure space with a measure of full support. A graph over \((X, m)\) is induced by an edge weight function \(\mu : X \times X \to [0, \infty), (x, y) \mapsto \mu_{xy}\) that is symmetric, has zero diagonal and satisfies

\[
\sum_{y \in X} \mu_{xy} < \infty, \quad x \in X.
\]

If \(\mu_{xy} > 0\) we write \(x \sim y\) and let \(xy\) and \(yx\) be the oriented edges of the graph. We write \(xy \subseteq A\) for a set \(A \subseteq X\) if both of the vertices of the edge \(xy\) are contained in \(A\), i.e., \(x, y \in A\). When we fix an orientation for the edges we denote the directed edges often by \(e\).

We refer to the triple \((X, \mu, m)\) as a weighted graph. We assume the graph is connected, that is for every two vertices \(x, y \in X\) there is a path \(x = x_0 \sim x_1 \sim \ldots \sim x_n = y\).

The spaces \(L^p(X, m), p \in [1, \infty)\), and \(L^\infty(X)\) are defined in the natural way. For \(p \in [1, \infty)\), let \(p^*\) be its Hölder dual, i.e., \(\frac{1}{p} + \frac{1}{p^*} = 1\).
2.2. Laplacians and (sub)harmonic functions. We define the formal Laplacian $\Delta$ on the formal domain

$$F(X) = \{ f : X \to \mathbb{R} \mid \sum_{y \in X} \mu_{xy}|f(y)| < \infty \text{ for all } x \in X \},$$

by

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \in X} \mu_{xy}(f(x) - f(y)).$$

**Definition 2.1** (Harmonic function). A function $f : X \to \mathbb{R}$ is called harmonic (subharmonic, superharmonic) if $f \in F(X)$ and $\Delta f = 0$, $(\Delta f \leq 0$, $\Delta f \geq 0)$.

Obviously, the measure does not play a role in the definition of harmonicity. We denote by $L$ the positive selfadjoint restriction of $\Delta$ on $L^2(X, m)$ which arises from the closure $Q$ of the restriction of the quadratic form $E : \{ X \to \mathbb{R} \} \to [0, \infty]$

$$E(f) = \frac{1}{2} \sum_{x,y \in X} \mu_{xy}|\nabla xy f|^2$$

to $C_c(X)$, the space of finitely supported functions, (for details see [KL12]). Since $Q$ is a Dirichlet form, the semigroup $e^{-tL}$, $t \geq 0$, extends to a $C_0$-semigroup on $L^p(X, m)$, $p \in [1, \infty)$ (resp. a weak $C_0$-semigroup for $p = \infty$). We denote the generators of these semigroups by $L_p$, $p \in [1, \infty)$.

2.3. Intrinsic metrics. Next, we introduce the concept of intrinsic metrics. A pseudo metric is a symmetric map $X \times X \to [0, \infty)$ with zero diagonal which satisfies the triangle inequality.

**Definition 2.2** (Intrinsic metric). A pseudo metric $\rho$ on $X$ is called an intrinsic metric if

$$\sum_{y \in X} \mu_{xy} \rho^2(x, y) \leq m(x), \quad x \in X.$$

If for a function $f : X \to \mathbb{R}$ the map $\Gamma(f) : x \mapsto \sum_{y \in X} \mu_{xy}|\nabla xy f|^2$ takes finite values, then $\Gamma(f)$ defines the energy measure of $f$. Thus, a pseudo metric $\rho$ is intrinsic if the energy measures $\Gamma(\rho(x, \cdot))$, $x \in X$, are absolutely continuous with respect to $m$ with Radon-Nikodym derivative $\frac{d\Gamma(\rho(x, \cdot))}{dm} = \Gamma(\rho(x, \cdot))/m$ satisfying $\Gamma(\rho(x, \cdot))/m \leq 1$.

In various situations the natural graph distance proves to be insufficient for the investigations of unbounded Laplacians, see [Woj09, Woj11, KLW13]. For this reason the concept of intrinsic metrics developed in [FLW] for regular Dirichlet forms received quite some attention as a candidate to overcome these problems. Indeed, intrinsic metrics already have been applied successfully to various problems on graphs [BHK13, BKW, Fol11, Fol12, HKMW13] and related settings [GHM12].

The jumps size $s$ of a pseudo metric is given by

$$s := \sup\{ \rho(x, y) \mid x, y \in X, x \sim y \} \in [0, \infty].$$

From now on, $\rho$ always denotes an intrinsic metric and $s$ denotes its jump size.
We fix a base point \( o \in X \) which we suppress in notation and denote the distance balls by

\[
B_r = \{ x \in X \mid \rho(x, o) \leq r \}, \quad r \geq 0.
\]

Since \( \rho \) takes values in \([0, \infty)\) in our setting, the results are indeed independent of the choice of \( o \). For \( U \subseteq X \), we write \( B_r(U) = \{ x \in X \mid \rho(x, y) \leq r \text{ for some } y \in U \}, \quad r \geq 0 \).

Define the \textit{weighted vertex degree} \( \text{Deg} : X \to [0, \infty) \) by

\[
\text{Deg}(x) = \frac{1}{m(x)} \sum_{y \in X} \mu_{xy}, \quad x \in X.
\]

**Definition 2.3** (Compatible metric). A pseudo metric on \( X \) is called \textit{compatible} if it has finite jump size and the restriction of \( \text{Deg} \) to every distance ball is bounded, i.e., \( \text{Deg}|_{B_r} \leq C(r) < \infty \) for all \( r \geq 0 \).

**Example 2.4.** (a) For any given weighted graph there is an intrinsic path metric defined by

\[
\delta(x, y) = \inf_{x = x_0 \sim \ldots \sim x_n = y} \sum_{i=0}^{n-1} (\text{Deg}(x_i) \vee \text{Deg}(x_{i+1}))^{-\frac{1}{2}}.
\]

This intrinsic metric can be turned into an intrinsic metric \( \delta_r \) with finite jump size \( s = r \) by taking the path metric with edge weights \( \delta(x, y) \wedge r \), \( x \sim y \). In many cases, neither \( \delta_r \) nor \( \delta \) is compatible.

(b) If the measure \( m \) is larger than the measure \( n(x) = \sum_{y \in X} \mu_{xy}, \quad x \in X \), then the natural graph distance (i.e., the path metric with edge weights 1) is an intrinsic metric which is compatible since \( s = 1 \) and \( \text{Deg} \leq 1 \) in this case.

**Remark 2.5.** (a) In view of Example 2.4 (b) it is apparent that [HJ13, Theorem 1.1] is included in Theorem 1.1.

(b) In [HKMW13, Theorem A.1] a Hopf-Rinow type theorem is shown which states that for a locally finite graph a path metric is complete if and only if all balls are finite. Thus, compatibility can be seen as a completeness assumption of the graph.

(c) It is not hard to see that there are graphs that do not allow for a compatible intrinsic metric. However, to a given edge weight function \( \mu \) and a pseudo metric \( \rho \), we can always assign a minimal measure \( m \) such that \( \rho \) is intrinsic, i.e., let \( m(x) = \sum_{y \in X} \mu_{xy} \rho^2(x, y), \quad x \in X \). If \( \rho \) already has finite jump size and all balls are finite, then \( \rho \) is automatically compatible.

(d) The assumption that \( \text{Deg} \) is bounded on distance balls is equivalent to either of the following assumptions

(i) The restriction of \( \Delta \) to any distance ball (with Dirichlet boundary conditions) is a bounded operator.

(ii) The Radon-Nikodym derivative of the measure \( n \) given by \( n(x) = \sum_{y \in X} \mu_{xy}, \quad x \in X \), with respect to the measure \( m \) is bounded on the distance balls.

The equivalence of (i) follows from Theorem [HKMW12, Theorem 9.3] and the one of (ii) is obvious.
In the subsequent, we will make use of the cut-off function \( \eta = \eta_{r,R}, \) \( 0 \leq r < R, \) on \( X \) given by
\[
\eta = 1 \wedge \left( \frac{R - \rho(x,y)}{R - r} \right)_+.
\]

**Lemma 2.6.** Let \( \eta = \eta_{r,R}, \) \( 0 < r < R, \) be given as above. Then,
(a) \( \eta|_{B_r} \equiv 1 \) and \( \eta|_{X\setminus B_R} \equiv 0. \)
(b) For \( x \in X, \)
\[
\sum_{y \in X} \mu_{xy} |\nabla xy\eta|^2 \leq \frac{1}{(R - r)^2} 1_{B_{R+s}\setminus B_{R-s}}(x)m(x).
\]

**Proof.** (a) is obvious from the definition of \( \eta \) and (b) follows directly from \( |\nabla xy\eta| \leq \frac{1}{R - r} \rho(x,y) 1_{B_{R+s}\setminus B_{R-s}}(x) \) for \( x \sim y \) and the intrinsic metric property of \( \rho. \)

\[\Box\]

2.4. **Green’s formula, Leibniz rules and mean value theorem.** We first prove a Green’s formula which is an \( L^p \) version of the one in [HKMW13].

**Lemma 2.7** (Green’s formula). Let \( p \in [1, \infty), \) \( U \subseteq X \) and assume \( \text{Deg} \) is bounded on \( U. \) Then for all \( f \) with \( f 1_U \in L^p(X, m) \cap F(X) \) and \( g \in L^{p'}(X, m) \) with \( B_s(\text{supp } g) \subseteq U \)
\[
\sum_{x \in X} (\Delta f)(x)g(x)m(x) = \frac{1}{2} \sum_{x,y \in U} \mu_{xy} |\nabla xyf|^2 |\nabla xyg|.
\]

**Proof.** The formal calculation in the proof of Green’s formula is a straightforward algebraic manipulation. To ensure that all involved terms converge absolutely, one invokes Hölder’s inequality and the boundedness assumption on \( \text{Deg} \) (confer the proof of Lemma 3.1 and 3.3 in [HKMW13]).

\[\Box\]

The following Leibniz rules follow by direct computations.

**Lemma 2.8** (Leibniz rules). For all \( x, y \in X, x \sim y \) and \( f, g : X \to \mathbb{R} \)
\[
\nabla_{xy}(fg) = f(y)\nabla_{xy}g + g(x)\nabla_{xy}f
= f(y)\nabla_{xy}g + g(y)\nabla_{xy}f + \nabla_{xy}f \nabla_{xy}g.
\]

A fundamental difference of Laplacians on graphs and on manifolds is the absence of a chain rule in the graph case. In particular, existence of a chain rule can be used as a characterization for a regular Dirichlet form to be strongly local. We circumvent this problem by using the mean value theorem from calculus instead. In particular, for a continuously differentiable function \( \phi : \mathbb{R} \to \mathbb{R} \) and \( f : X \to \mathbb{R}, \) we have
\[
\nabla_{xy}(\phi \circ f) = \phi'(\zeta)\nabla_{xy}f, \quad \text{for some } \zeta \in [f(x) \wedge f(y), f(x) \vee f(y)].
\]

In this paper we will apply this formula to get estimates for the function \( \phi : t \mapsto t^{p-1}, \) \( p \in (1, \infty). \) However, we need a refined inequality as it was already used in the proof of [HS97] Theorem 2.1]. For the convenience of the reader, we include a short proof here.

**Lemma 2.9** (Mean value inequalities). For all \( f : X \to \mathbb{R} \) and \( x \sim y \) with \( \nabla_{xy}f \geq 0, \)
(a) \( \nabla_{xy}f^{p-1} \geq \frac{1}{2}(f^{p-2}(x) + f^{p-2}(y))\nabla_{xy}f, \) for \( p \in [2, \infty), \)
(b) \( \nabla_{xy}f^{p-1} \geq C(f(x) \vee f(y))^{p-2}\nabla_{xy}f, \) for \( p \in (1, \infty), \) where \( C = (p - 1) \wedge 1. \)
Proof. (a) Denote $a = f(y)$, $b = f(x)$. As it is the only non-trivial case, we assume $0 < a < b$. Note that for $p \neq 1$

$$b^{p-1} - a^{p-1} = (b - a)(b^{p-2} + a^{p-2}) + ab(b^{p-3} - a^{p-3}).$$

Thus, the statement is immediate for $p \geq 3$ since the second term on the right side is non-negative in this case. Let $2 \leq p < 3$ and note $a^{p-3} > b^{p-3}$. The function $t \mapsto t^{2-p}$ is convex on $(0, \infty)$ and, thus, its image lies below the line segment connecting $(b^{-1}, b^{p-2})$ and $(a^{-1}, a^{p-2})$. Therefore,

$$a^{p-3} - b^{p-3} \leq \frac{a^{p-3} - b^{p-3}}{(3-p)} = \int_{b^{-1}}^{a^{-1}} t^{2-p} dt \leq (a^{-1} - b^{-1})\left(\frac{(b^{p-2} - a^{p-2})}{2} + a^{p-2}\right)$$

$$= \frac{1}{2ab}(b-a)(a^{p-2} + b^{p-2}).$$

From the equality in the beginning of the proof we now deduce the assertion in the case $2 \leq p < 3$.

(b) The case $p \geq 2$ follows from (a). The case $1 < p \leq 2$ in (b) follows directly from the mean value theorem. \hfill \Box

3. Proofs for harmonic functions

In this section we prove the main theorems and the corresponding corollaries for harmonic functions. It will be convenient to introduce the following orientation on the edges. For a given non-negative subharmonic function $f$, we let $E_f$ be the set of oriented edges $e = e_+e_-$ such that

$$\nabla_{e} f \geq 0, \quad \text{i.e.,} \quad f(e_+) \geq f(e_-).$$

3.1. The key estimate. The lemma below is vital for the proof of Theorem 1.1 and Theorem 1.8.

Lemma 3.1. Let $p \in (1, \infty)$, $0 \leq \varphi \in L^{\infty}(X)$ and $U = B_\epsilon(\text{supp}\, \varphi)$. Assume $\text{Deg}$ is bounded on $U$. Then, for every non-negative subharmonic function $f$ with $f1_U \in L^p(X, m)$,

$$\sum_{e \in E_f} \mu_e f^{-2}(e_+) \varphi^2(e_-)|\nabla_{e} f|^2 \leq C \sum_{e \in E_f, e \subset U} \mu_e f^{p-1}(e_+) \varphi(e_-) |\nabla_{e} f| |\nabla_{e} \varphi|,$$

where $C = 2/((p-1) \wedge 1)$.

Proof. From the assumptions $f1_U \in L^p(X, m)$ and $\varphi \in L^{\infty}(X)$, we infer $\varphi^2 f^{p-1} \in L^{p'}(X, m)$ (as $p^* = p/(p-1)$). Thus, compatibility of the pseudo metric implies applicability of Green’s formula with $f$ and $g = \varphi^2 f^{p-1}$. We start by using non-negativity and subharmonicity of $f$ before applying Green’s formula (Lemma 2.7).
and the first and second Leibniz rule (Lemma 2.8)

\[
0 \geq \sum_{x \in X} (\Delta f)(x)(\phi^2 f^{p-1})(x)m(x) = \sum_{e \in E, r \in U} \mu_e \nabla_e f \nabla_e (\phi^2 f^{p-1})
\]

\[
= \sum_{e \in U} \mu_e \nabla_e f \left[ \phi^2 (e^-) \nabla_e f^{p-1} + f^{p-1}(e^+) \nabla_e \phi^2 \right]
\]

\[
= \sum_{e \in U} \mu_e \nabla_e f \left[ \phi^2 (e^-) \nabla_e f^{p-1} + 2f^{p-1}(e^+) \phi (e^-) \nabla_e \phi + f^{p-1}(e^+) |\nabla_e \phi|^2 \right]
\]

\[
\geq C \sum_{e \in U} \mu_e f^{p-2}(e^+) \phi^2 (e^-) |\nabla_e f|^2 + 2 \sum_{e \in U} \mu_e f^{p-1}(e^+) \phi (e^-) \nabla_e f \nabla_e \phi,
\]

where we dropped the third term in the third line since it is non-negative because of $\nabla_e f \geq 0$ and we estimated the first term on the right hand side using the mean value theorem, Lemma 2.9 (b). Absolute convergence of the two terms in the last line can be checked using Hölder’s inequality and the assumptions $f1_U \in L^p(X, m)$, $\phi \in L^\infty(X)$ and boundedness of $\text{Deg}$ on $U$. Hence, we obtain the statement of the lemma.

3.2. Proof of Karp’s theorem.

Proof of Theorem 1.1. Let $p \in (1, \infty)$ and let $f$ be a non-negative subharmonic function. Assume $f1_{B_r} \in L^p(X, m)$ for all $r \geq 0$ since otherwise $\inf_{r \to 0} \int_{B_r} r/\|f1_{B_r}\|_p^p dr = 0$. Let $\eta = \eta_{r+s, R-s}$ with $0 < r < R - 3s$ (see Section 2.3). Then by Lemma 3.1 (applied with $\phi = \eta$) we obtain (noting additionally that $\nabla_x \eta = 0$, $x, y \in B_r$)

\[
\sum_{e \in B_R} \mu_e f^{p-2}(e^+) \eta^2 (e^-) |\nabla_e f|^2 \leq C \sum_{e \in B_R \setminus B_r} \mu_e f^{p-1}(e^+) \eta(e^-) \nabla_e f |\nabla_e \eta|.
\]

Now, the Cauchy-Schwarz inequality, $\sum_{e \in B_R} \mu_e f^p(e^+) |\nabla_x \eta|^2 \leq \sum_{x, y} \mu_{xy} f^p(x) |\nabla_x \eta|^2$ and the cut-off function lemma, Lemma 2.4, yield

\[
\left( \sum_{e \in B_R} \mu_e f^{p-2}(e^+) \eta^2 (e^-) |\nabla_e f|^2 \right)^2 \leq C \left( \sum_{e \in B_R \setminus B_r} \mu_e f^p(e^+) |\nabla_x \eta|^2 \right) \left( \sum_{e \in B_R \setminus B_r} f^{p-2}(e^+) \eta^2 (e^-) |\nabla_e f|^2 \right)
\]

\[
\leq \frac{C}{(R - r)^2} \|f1_{B_R \setminus B_r}\|_p^p \left( \sum_{e \in B_R} \mu_e f^{p-2}(e^+) \eta^2 (e^-) |\nabla_e f|^2 \right).
\]

Let $R_0 \geq 3s$ be such that $f1_{B_{R_0}} \neq 0$ and denote

\[
v(r) = \|f1_{B_r}\|_p, \quad r \geq 0.
\]

Moreover, for $j \geq 0$, let $r_j = 2^j R_0$, $\varphi_j = \eta_{R_j + s, R_{j+1} - s}$ and

\[
Q_{j+1} = \sum_{e \in B_{R_{j+1}}} \mu_e f^{p-2}(e^+) \varphi_j^2 (e^-) |\nabla_e f|^2.
\]

As $\varphi_{j+1} \leq \varphi_j$, we get $Q_j \leq Q_{j+1}$ and together with the estimate above this implies

\[
Q_j Q_{j+1} \leq Q_{j+1}^2 \leq C \frac{v(R_{j+1})}{(R_{j+1} - R_j)^2} (Q_{j+1} - Q_j), \quad j \geq 0.
\]
Since $R_{j+1} = 2R_j$, dividing the above inequality by $\frac{v(R_{j+1})}{R_{j+1}} Q_j Q_{j+1}$ and adding $C/Q_{j+1}$ yield
\[
\frac{R_{j+1}^2}{v(R_{j+1})} + \frac{C}{Q_{j+1}} \leq \frac{C}{Q_j}
\]
and, thus,
\[
\frac{1}{C} \sum_{j=1}^{\infty} \frac{R_{j+1}^2}{v(R_{j+1})} \leq \frac{1}{Q_1}.
\]
Now, the assumption $\int_{R_0}^{\infty} r/v(r)dr = \infty$ implies $\sum_{j=0}^{\infty} \frac{R_j^2}{v(R_j)} = \infty$. Therefore, $Q_1 = 0$. As this is true for all $R_0$ large enough, we have
\[
f^{p-2}(e_+)|\nabla f|^2 = 0,
\]
for all edges $e$. For $p \geq 2$, connectedness clearly implies that $f$ is constant. On the other hand, for $p \in (1, 2]$, we always have $f^{p-2}(e_+) > 0$ and, thus, $f$ is constant. \[\square\]

### 3.3. Proof of the Caccioppoli inequality.

**Proof of Theorem 1.8.** Using Lemma 3.1 and the inequality $ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, $\varepsilon > 0$, we estimate
\[
\sum_{e \in E_f} \mu_e f^{p-2}(e_+) \varphi^2(e_-)|\nabla e f|^2 \leq C \sum_{e \in E_f} \mu_e f^{p-1}(e_+) \varphi(e_-) |\nabla e f| |\nabla \varphi|
\]
\[
\leq \frac{1}{2} \sum_{e \in E_f} \mu_e f^{p-2}(e_+) \varphi^2(e_-)|\nabla e f|^2 + C \sum_{e \in E_f} \mu_e f^p(e_+)|\nabla \varphi|^2.
\]
Letting $\varphi = \eta = \eta_{r+s, R-s}$ with $0 < r < R - 3s$ (from Section 2.3) and using the cut-off function lemma, Lemma 2.6, we arrive at
\[
\sum_{e \in E_f} \mu_e f^{p-2}(e_+) |\nabla e f|^2 \leq C \sum_{e \in E_f} \mu_e f^p(e_+) |\nabla \eta|^2 \leq \frac{C}{(R-r)^2} \|f 1_{B_r\backslash B_s}\|_p^p.
\]
\[\square\]

**Remark 3.2.** In order to obtain the stronger statement for $p \in [2, \infty)$ mentioned in Remark 1.9 (b), we invoke Lemma 2.9 (a) in the proof of Lemma 3.1 instead of Lemma 2.9 (b) and proceed as in the proof above.

### 3.4. Proof of the corollaries.

**Proof of Corollary 1.2 (Yau’s $L^p$ Liouville theorem).** Clearly the integral in Theorem 1.1 diverges if $f \in L^p(X, \mu)$. \[\square\]

**Proof of Corollary 1.4 (Domain of the $L^p$ generators).** Let $f \in L^p(X, \mu) \cap F(X)$ be such that $(\Delta + 1)f = 0$. Since the positive and negative part $f_+, f_-$ of $f$ are non-negative, subharmonic and in $L^p(X, \mu)$, they must be constant by Corollary 1.2. This implies $f_+ \equiv 0$ and, thus, $f \equiv 0$. Now, the proof of the corollary works literally line by line as the proof of [KL12] Theorem 5. \[\square\]

For the proof of Corollary 1.6 we recall the following well known equivalent conditions for recurrence.

**Proposition 3.3 (Characterization of recurrence).** Let a connected graph $X$ be given. Then the following are equivalent.
(i) For the transition matrix $P$ with $P_{x,y} = \mu_{xy}/\sum_{z \in X} \mu_{xz}$, $x, y \in X$, we have $\sum_{n=0}^{\infty} P^{(n)}(x,y) = \infty$ for some (all) $x, y \in X$, where $P^{(n)}$ denotes the $n$-th power of $P$.

(ii) For $m \equiv 1$ and some (all) $x, y \in X$, we have $\int_{0}^{\infty} e^{-tL} \delta_x(y) dt = \infty$, where $\delta_x = 1$ if $x = y$ and zero otherwise.

(iii) For all $m$ and some (all) $x, y \in X$, we have $\int_{0}^{\infty} e^{-tL} \delta_x(y) dt = \infty$.

(iv) Every bounded superharmonic (or subharmonic) function is constant.

(v) Every non-negative superharmonic function is constant.

(vi) Every superharmonic (or subharmonic) function of finite energy is constant.

(vii) $\text{cap}(x) := \inf \{E(f) | f \in C_c(X), f(x) = 1 \} = 0$ for some (all) $x \in X$.

A graph is called recurrent if one of the equivalent statements of Proposition 3.3 is satisfied.

Proof. The equivalence (i)$\Leftrightarrow$(ii) is shown in [Sch12, Theorem 6] (confer [Che04, Theorem 4.34]). The equivalences (ii)$\Leftrightarrow$(vi)$\Leftrightarrow$(iii) are in [Sch12, Theorem 2 and Theorem 9] (confer [Son94, Theorem 3.34]). The equivalences (i)$\Leftrightarrow$(v)$\Leftrightarrow$(vii) are found in [Woe00, Theorem 1.16, Theorem 2.12]. The equivalence (iv)$\Leftrightarrow$(v) follows since every non-negative superharmonic function $f$ can be approximated by the bounded superharmonic functions $f \wedge n$, $n \geq 1$. $\square$

Proof of Corollary 1.7 (Recurrence). Theorem 1.7 implies that any non-negative bounded subharmonic function $f$ is constant provided $\inf_{x} \int_{r_0}^{\infty} r/m(B_r) dr = \infty$ since $\|f1_{B_r}\|_p \leq \|f\|_p m(B_r)$, $r \geq 0$. By Proposition 3.3 the graph is recurrent. $\square$

4. $L^1$-Liouville theorem and counter-examples

In this section we deal with the borderline case of the $L^p$ Liouville theorem $p = 1$. We first prove Theorem 1.7 which deals with the stochastic complete case and then give two examples which show that there is no $L^1$ Liouville theorem for non-negative subharmonic functions in the general case.

A graph is called stochastically complete if $e^{-tL}1 = 1$, where 1 denotes the function that is constantly one on $X$. For the relevance of the concept see [Gri99, KL12, Woj09]. The proof of Theorem 1.7 follows along the lines of the proof of [Gri99, Theorem 13.2].

Proof of Theorem 1.7. If the graph is recurrent, then there are no non-constant non-negative superharmonic functions by Proposition 3.3. So assume the graph is not recurrent which implies $G(x,y) = \int_{0}^{\infty} e^{-tL} \delta_x(y) dt < \infty$, $x, y \in X$, again by Proposition 3.3. Let $K_n$, $n \geq 0$, be a sequence of finite sets exhausting $X$ and $G_n(x,y) = \int_{0}^{\infty} e^{-tL_n} \delta_x(y) dt$, where $L_n$ are the finite dimensional operators arising from the restriction of the form $Q$ to $C_c(K_n)$. By domain monotonicity, [KL12, Proposition 2.6 and 2.7] the semigroups $e^{-tL_n}$ converge monotonously increasing to $e^{-tL}$ and, hence, $G_n(x,y) \leq G(x,y)$ for $x, y \in K_n$, and $G_n \nearrow G$, $n \to \infty$, pointwise. By direct calculation for any $x \in K_n$

$$L_n G_n(x,y) = \int_{0}^{\infty} L_n e^{-tL_n} \delta_x(y) dt = \int_{0}^{\infty} \partial_t e^{-tL_n} \delta_x(y) dt = [e^{-tL_n} \delta_x(y)]_0^\infty = \delta_x(y)$$

and, hence, $G_n(x,\cdot)$ are harmonic on $K_n \setminus \{x\}$, $n \geq 0$.

Let $u$ be a non-trivial non-negative superharmonic function which is strictly positive by the minimum principle [KL12, Theorem 8]. Let $U \subseteq X$ be finite with
Proof of Theorem 1.10. Let $G = (X, \mu, m)$ be an infinite line graph, i.e., $X = \mathbb{Z}$ and $x \sim y$ if $|x - y| = 1$ for $x, y \in \mathbb{Z}$. Define the edge weight by $\mu_{xy} = 2^{-|x|+|y|}$ for $x \sim y$ and the measure $m$ by $m(x) = (|x| + 1)^{-2^{2-|x|}}, x \in \mathbb{Z}$, which implies $m(X) < \infty$. The intrinsic metric $\delta$ (introduced in Example 2.4) is compatible as it satisfies $\delta(x, x+1) = C(|x|+1)^{-1}$ and, thus, $\sum_{x=-\infty}^{\infty} \delta(x, x+1) = \infty$. However, the function $f$ defined as

$$f(x) = \text{sign}(x)(2^{|x|} - 1), \quad x \in \mathbb{Z},$$

is harmonic and, clearly, $f \in L^p(X, m), p \in (0, 1]$.

Example 4.2 (Infinite volume). We can extend the above example to the infinite volume case. Let $G'$ be the graph from above and $G''$ be a locally finite graph of infinite volume which allows for a compatible path metric. We glue $G''$ to the vertex $x = 0$ of the graph $G$ by identifying a vertex in $G''$ with $x = 0$. Next, we extend the path metrics in the natural way and obtain (by renormalizing the edge weights of the metric at the edges around $x = 0$ if necessary) again a compatible intrinsic metric and the graph has infinite volume. Moreover, we extend $f$ on $G'$ from above by zero to $G''$ and obtain a harmonic function which is in $L^p, p \in (0, 1]$.

5. Applications of Karp’s theorem

In this section we prove Theorem 1.10 and give several applications which mainly circle around the case of finite measure.

Proof of Theorem 1.10. We assume that for some $p \in (1, \infty)$ the non-negative subharmonic function $f$ is in $L^p(X, m \rho_1^{-2})$ and, hence, $f^p \rho_1^{-2} \in L^1(X, m)$. For large $r_0 \geq 1$, we estimate

$$\int_{r_0}^{\infty} \frac{1}{r^2} \|f 1_{B_r}\|_p^r \, dr \geq \int_{r_0}^{\infty} \frac{1}{r^2} \|f^p \rho_1^{-2} 1_{B_r}\|_1 \, dr \geq C \int_{r_0}^{\infty} \frac{1}{r^2} \, dr = \infty.$$

Hence, Theorem 1.10 implies that $f$ is constant. □

Next, we turn to several consequences of Theorem 1.10. A function $f : X \to \mathbb{R}$ is said to grow less than a function $g : [0, \infty) \to (0, \infty)$ if there are $\beta \in (0, 1)$ and $C > 0$ such that

$$f(x) \leq C g^\beta (\rho_1(x)), \quad x \in X.$$
We say $f$ grows polynomially if $f$ grows less than a polynomial.

We say the measure $m$ has a finite $q$-th moment, $q \in \mathbb{R}$, with respect to an intrinsic metric $\rho$ if

$$\rho_1 \in L^q(X,m),$$

where $\rho_1 = 1 \lor \rho(\cdot,o)$. This assumption implies that all balls have finite measure and if $q \geq 0$ it also implies $m(X) < \infty$.

**Corollary 5.1 (Measures with finite moments).** Assume a connected weighted graph allows for a compatible intrinsic metric and the measure has a finite $q$-th moment, $q \in \mathbb{R}$. Then every non-negative subharmonic function $f$ that grows less than $r \mapsto r^{q+2}$ is constant. In particular, if $q > -2$, then boundedness of $f$ implies $f$ is constant.

**Proof.** If $f$ grows less than $r \mapsto r^{q+2}$, then there is $\varepsilon > 0$ such that $f^{1+\varepsilon} \rho_1^{-2} \leq C \rho_1^q$ on $X$. By the assumption $\rho_1 \in L^q(X,m)$ it follows $f \in L^p(X,m\rho_1^{-2})$ for $p = 1 + \varepsilon$. Hence, the assertion follows from Theorem 1.10. $\square$

Letting $q = 0$ in the above theorem gives the following immediate corollary.

**Corollary 5.2 (Finite measure).** Assume a connected weighted graph allows for a compatible intrinsic metric and $m(X) < \infty$. Then every non-negative subharmonic function $f$ that grows less than quadratic is constant. In particular, $f \in L^\infty(X)$ implies that $f$ is constant.

The final corollary of this section is a consequence of Corollary 1.2.

**Corollary 5.3 (Exponentially decaying measure).** Assume a connected weighted graph allows for a compatible intrinsic metric and $m(X) < \infty$, and there is $\beta > 0$ such that

$$\limsup_{r \to \infty} \frac{1}{r^\beta} \log m(B_{r+1} \setminus B_r) < 0.$$

Then every non-negative subharmonic function that grows polynomially is constant.

**Proof.** If a non-negative subharmonic function $f$ grows polynomially, then there is $q > 0$ such that

$$\|f\|_p^p \leq C \sum_{x \in X} \rho_1^q(x,o)m(x) \leq C \sum_{r=1}^\infty r^q m(B_r \setminus B_{r-1}) + C < \infty$$

by the assumption on the measure. Hence, the theorem follows from Corollary 1.2. $\square$

### 6. Applications to harmonic maps

Harmonic maps between metric measure spaces were introduced by Jost [Jos94, Jos97a, Jos97b, Jos98] and harmonic maps from graphs into Riemannian manifolds or metric spaces have been studied by many authors, e.g. [KS01, IN 05, JT07] and for alternative definitions, see [GS92, KS93, KS97, Stu01, Stu 05].

We use our results concerning the function theory on graphs to derive various Liouville type theorems for harmonic maps from graphs. A particular focus lies on bounded harmonic maps and harmonic maps of finite energy.

Let $(X,\mu,m)$ be a weighted graph. We briefly recall the set up of Hadamard spaces and harmonic maps.
A complete geodesic space \((Y, d)\) is called an \(NPC\) space if it locally satisfies Toponogov’s triangle comparison for non-positive sectional curvature. We refer to Burago-Burago-Ivanov [BBI01], Jost [Jos97b] and Bridson-Haefliger [BH99] for definitions. Here \(NPC\) stands for “non-positive curvature” in the sense of Alexandrov. The space \((Y, d)\) is called an \(Hadamard space\), if the Toponogov’s triangle comparison holds globally, i.e., holds for arbitrary large geodesic triangles. A simply connected \(NPC\) space is an \(Hadamard space\), see [BBI01]. For the sake of simplicity, we only consider \(Hadamard spaces\), also called \(CAT(0)\) spaces, as targets of harmonic maps \(X \to Y\). In the following, we always denote by \(\nu\) the probability measure \(\nu\) of \(\nu\) plays no role in the definition of harmonic maps. In the following, we always denote by \(u : X \to Y\) a harmonic map from a weighted graph into an \(Hadamard space\).

### 6.1. Proof of Theorem 1.11

The proof of Theorem 1.11 is a rather immediate consequence of Theorem 1.10 and the following lemma which is a consequence of Jensen’s inequality and convexity of distance functions on \(Hadamard spaces\).

**Lemma 6.2.** For every harmonic map \(u\) the functions \(X \to [0, \infty), x \mapsto d(u(x), y)\), for fixed \(y \in Y\), are subharmonic.

**Proof.** Jensen’s inequality in \(Hadamard spaces\), see [Stu03, Theorem 6.2], states that for every lower semi-continuous convex function \(g : Y \to [0, \infty)\) and \(\nu \in P^1(Y)\)

\[
g(b(\nu)) \leq \int_Y g(y) \nu(dy).
\]

Now any distance function \(y \mapsto d(y, y_0)\) to a point \(y_0 \in Y\) is convex in an \(Hadamard space\), see e.g. [BBI01, Corollary 9.2.14], which yields the statement. \(\square\)

**Proof of Theorem 1.11.** Combining Theorem 1.10 and Lemma 6.2 yields that \(x \mapsto d(u(x), y)\) is constant. Hence, \(u\) is bounded. If \(m \rho^{-2}(X) = \sum_{x \in X} m(x) \rho^{-2}(x) = \infty\), then a constant function is in \(L^p(X, m \rho^{-2})\) if and only if it is zero. Moreover, if \(y\) is in the image of \(u\) then \(d(u(\cdot), y) \equiv 0\) and hence \(u(x) = y\) for all \(x \in X\). \(\square\)
6.2. Harmonic maps of finite energy. In this section we consider harmonic maps of finite energy and prove Theorem 1.12 and Theorem 1.13 which are analogues to theorems of Cheng-Tam-Wang [CTW96] from Riemannian geometry. We say a harmonic map $u : X \rightarrow Y$ has finite energy if

$$\frac{1}{2} \sum_{x,y \in X} \mu_{xy} d^2(u(x), u(y)) < \infty.$$ 

In order to do so, we need the equivalence of boundedness of finite energy harmonic functions on a graph and that of non-negative subharmonic functions. Recall that a function $f : X \rightarrow \mathbb{R}$ is said to have finite energy if $E(f) < \infty$, see Section 2.2.

In Riemannian geometry such a theorem was first proven in [CTW96, Theorem 1.2]. We give a different proof here in the discrete setting by Royden’s decomposition.

**Theorem 6.3.** For connected weighted graphs every harmonic function with finite energy is bounded if and only if every non-negative subharmonic function with finite energy is bounded.

**Proof.** As positive and negative part of a harmonic function are non-negative and subharmonic functions, boundedness of non-negative subharmonic functions of finite energy implies boundedness of harmonic functions of finite energy.

We now turn to the other direction. By Proposition 3.3 there are no non-constant subharmonic functions of finite energy in the case the graph is recurrent. Therefore, we assume the graph is not recurrent (also called transient in the connected case). Let $f$ be a non-negative subharmonic function with finite energy. Then by the discrete version of Royden’s decomposition theorem, see [Soa94, Theorem 3.69], there are unique functions $g$ and $h$ where $g$ is in the completion of $C_c(X)$ under the norm $\|\varphi\|_o = (E(\varphi) + \varphi(o)^2)^{1/2}$, $\varphi \in C_c(X)$, and $h$ is a harmonic function of finite energy such that

$$f = g + h \quad \text{and} \quad E(f) = E(g) + E(h).$$

By [Soa94] Lemma 3.70, $g \leq 0$ since $g$ is subharmonic. Therefore, $0 \leq f \leq h$. By assumption $h$ is bounded and, therefore, $f$ is bounded.

**Proof of Theorem 1.12.** Let $u : X \rightarrow Y$ be a harmonic map of finite energy. For some fixed $y_0 \in Y$ the function $f = d(u(\cdot), y_0)$ is non-negative and subharmonic by Lemma 6.2. Furthermore, by the triangle inequality and the assumption that $u$ has finite energy we get

$$E(f) = \frac{1}{2} \sum_{x,y} \mu_{xy} (f(x) - f(y))^2 \leq \frac{1}{2} \sum_{x,y \in X} \mu_{xy} d^2(u(x), u(y)) < \infty.$$ 

Now, by Theorem 6.3 we get that $f$ as a non-negative subharmonic function of finite energy must be bounded whenever every harmonic function of finite energy on $X$ is bounded (which is our assumption). Thus, $u$ is bounded.

**Theorem 1.13** is a consequence of Theorem 1.12 and the theorem of Kuwae-Sturm [KS08] below which goes back to Kendall in the manifold case [Ken88, Theorem 6] (confer [HK91, LW98, KS08]). However, although it is not explicitly mentioned in [KS08] one actually needs an additional assumption on the local compactness of the target, i.e., every point has a precompact neighborhood.
Theorem 6.4. (Kendall’s theorem [KS08, Theorem 3.1]) Assume that on a connected weighted graph every bounded harmonic functions is constant. Then, every bounded harmonic map into a locally compact Hadamard space is constant.

Next, we come to the proof of Theorem 1.13.

Proof of Theorem 1.13. Let $f$ be a harmonic function on $X$ of finite energy. By a discrete version of Virtanen’s theorem, see [Soa94, Theorem 3.7], $f$ can be approximated by bounded harmonic functions $f_n$ of finite energy (with respect to the norm $\|\varphi\|_0 = (E(\varphi) + \varphi(o)^2)^{1/2}$). By assumption the functions $f_n$, $n \geq 1$, are constant and, thus, $f$ must be constant. Theorem 1.12 implies now that any harmonic map is bounded and, thus, Theorem 6.4 implies that every harmonic map is constant. □

6.3. Harmonic maps and assumptions on the measure of $X$. In this subsection we collect several quantitative results that follow from what we have proven before.

The first corollary can be seen as an analogue to Yau’s $L^p$-Liouville type theorem.

Corollary 6.5. Assume a connected weighted graph $X$ allows for a compatible intrinsic metric and let $u$ be a harmonic map into an Hadamard space $Y$. If there is $y \in Y$ such that $d(u(\cdot), y) \in L^p(X, m)$ for some $p \in (1, \infty)$, then $u$ is bounded. If additionally $m(X) = \infty$, then $u$ is constant.

Proof. The function $d(u(\cdot), y)$ is subharmonic, by Lemma 6.2, and in $L^p(X, m)$ by assumption. Hence, Corollary 1.2 yields $d(u(\cdot), y)$ is constant. The assumption of infinite measure implies $d(u(x), y) = 0$ for all $x \in X$. □

We say a harmonic map $u$ into an Hadamard space $(Y, d)$ grows less than a function $g : [0, \infty) \to (0, \infty)$ if $d(u(\cdot), y)$ grows less than $g$ for some $y \in Y$ (confer Section 5). The next two corollaries are analogues of Corollary 5.1 and Corollary 5.2.

Corollary 6.6 (Measures with finite moment – harmonic maps). Assume a connected weighted graph allows for a compatible intrinsic metric and the measure has a finite $q$-th moment, $q > -2$. Then every harmonic map into an Hadamard space that grows less that $r \mapsto r^{q+2}$ is constant. In particular, bounded harmonic maps and harmonic maps with finite energy are constant.

Proof. Let $u$ be a harmonic map. Since we assume $q + 2 > 0$, we get by the triangle inequality that for all $y \in Y$ the subharmonic (Lemma 6.2) functions $d(u(\cdot), y)$ grow less than $r \mapsto r^{q+2}$. Hence, by Corollary 6.1 the subharmonic function $d(u(\cdot), y)$ is constant for all $y$ which implies that $u$ is constant. This proves the first assertion. Since $q + 2 > 0$, it is easy to see that every bounded harmonic function on $X$ is constant. The second assertion follows from Theorem 6.4 and Theorem 1.13. □

Corollary 6.7 (Finite measure – harmonic maps). Assume a connected weighted graph allows for a compatible intrinsic metric and $m(X) < \infty$. Then every harmonic map into an Hadamard space that grows less than quadratic is constant. In particular, bounded harmonic maps and harmonic maps with finite energy are constant.

Proof. The statements follow directly by the corollary above putting $q = 0$. □
Finally we say that a harmonic map grows polynomially if it grows less than a polynomial and state a corollary analogous to Corollary 5.3.

**Theorem 6.8 (Exponentially decaying measure – harmonic maps).** Assume a connected weighted graph allows for a compatible intrinsic metric, $m(X) < \infty$ and there is $\beta > 0$ such that
\[
\limsup_{r \to \infty} \frac{1}{r^\beta} \log m(B_{r+1} \setminus B_r) < 0.
\]

Then every harmonic map into an Hadamard space that grows polynomially is constant. In particular, bounded harmonic maps and harmonic maps with finite energy are constant.

**Proof.** The statements follow from Corollary 5.3, Theorem 6.4 and Theorem 1.13. □

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BOBO HUA, MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, 04103 LEIPZIG, GERMANY.

E-mail address: bobohua@mis.mpg.de

MATTHIAS KELLER, EINSTEIN INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, 91904 JERUSALEM, ISRAEL.

E-mail address: mkeller00@ma.huji.ac.il