I discuss some aspects of the universality of soft gluon dynamics in semileptonic and radiative decays at the threshold region.

1 Introduction

Let us consider the threshold logarithmic resumming for semi-inclusive B decays $B \to X_q + \langle \text{leptons or } \gamma \rangle$, where $q \equiv u, d, s$. The perturbative calculation of such processes, at the threshold region where $E_X \gg m_X$, is plagued by large logarithms, which originate from the incomplete cancellation of infrared real and virtual gluon emissions. They can be factorized into the form

$$f(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} f_{n,k} \alpha_S^n(Q) \log^k y, \quad y = \frac{Q^2}{m_X^2}$$

where $Q$ is the hard scale of the process: at the threshold we can set $Q = 2E_X$. Both the logarithms and the argument of the running coupling depend on the kinematics of the problem, by means of the hard scale $Q$. The series contains at most two logarithms for each power of $\alpha$, one of soft origin and another one of collinear origin. The terms of the series become large even if $\alpha_S \ll 1$: a truncated perturbative expansion is meaningless and logarithmic resummation is required in order to maintain a valid perturbative expansion.

2 Radiative and Semileptonic Decays

Let us consider first the radiative decay with a real photon in the final state

$$B \to X_s + \gamma.$$
The final hadronic energy is always large and it is of the order of the heavy-flavor mass:

\[ 2E_X = m_b \left( 1 + \frac{m_X^2}{m_b^2} \right) \simeq m_b. \tag{3} \]

Therefore, \( Q = m_b \) can be set and the following resummation formula holds for the invariant mass distribution:

\[
\frac{1}{\Gamma_r} \frac{d \Gamma_r}{dt} = C_r[\alpha_S(m_b)] \sigma(t; \alpha_S(m_b)) + d_r[t; \alpha_S(m_b)],
\]

where \( t \equiv m_X^2/m_b^2 \). The coefficient \( C_r[\alpha_S(m_b)] \) and the remainder function \( d_r[t; \alpha_S(m_b)] \) are short-distance, process dependent functions. They have a reliable perturbative expansion in the QCD fixed coupling \( \alpha_S(m_b) \simeq 0.22 \). By definition, the remainder function does not contain large logarithms in \( t \). It behaves regularly at the threshold; that means having, at the most, integrable singularities:

\[
\lim_{t \to 0} \int_0^t d_r(t'; \alpha_S) dt' = 0 \tag{5}
\]

The last term, \( \sigma(t; \alpha_S(m_b)) \), is the long-distance dominated, QCD form factor. It factorizes the threshold logarithms appearing in the perturbative expansion and takes into account universal long-distance effects in radiative and semileptonic decays.

Let us now consider the decay

\[ B \to X_u + l + \nu. \tag{6} \]

It is possible to obtain a factorized form for the triple differential distribution, the most general distribution in process \( \Gamma_r \) (its integration leads to all other spectra):

\[
\frac{1}{\Gamma_r} \frac{d \Gamma_r}{dx du dw} = C_r[x, w; \alpha(w m_b)] \sigma[u, x, w; \alpha(w m_b)] + d_r[x, u, w; \alpha(w m_b)],
\]

where:

\[
w \equiv \frac{2E_X}{m_b} \quad (0 \leq w \leq 2), \quad x \equiv \frac{2E_l}{m_b} \quad (0 \leq x \leq 1) \tag{8}
\]

and

\[
u \equiv \frac{E_X - \sqrt{E_X^2 - m_X^2}}{E_X + \sqrt{E_X^2 - m_X^2}} = \frac{1 - \sqrt{1 - (2m_X/Q)^2}}{1 + \sqrt{1 - (2m_X/Q)^2}} \simeq \left( \frac{m_X}{Q} \right)^2 \quad (0 \leq u \leq 1). \tag{9}
\]

By passing from the two body radiative decay to the three body semileptonic decay, the distribution acquires a dependence on the charged lepton energy \( E_l \). Moreover, and most important, while in the radiative decay the hard scale \( Q \) is always large (order of \( m_b \)), this is no longer true for the semileptonic decay. We have for instance the kinematical configuration with a large invariant mass for the lepton pair, where \( E_X \) is substantially reduced. Therefore, we cannot set \( Q = m_b \), as in the radiative decay [123].

At this level, there is universality among radiative and semi-leptonic decays, meaning that the same QCD form factor \( \sigma \) appears in both distributions [4] and [7], evaluated at the argument \( u \) in the semileptonic case and at \( t \) in the radiative decay. The coupling constant argument is set at the hard scale \( Q = 2E_X \) in both processes; in the radiative decay, that implies it is fixed to \( m_b \).

A systematic logarithmic resummation is consistently done in \( N \)-moment space or Mellin space. For example, the Mellin transform of the radiative spectrum is of the form

\[
\int_0^1 (1 - t)^{N-1} \frac{1}{\Gamma_r} \frac{d \Gamma_r}{dt} dt = C_r(\alpha_S) \sigma_N(\alpha_S) + d_r,N(\alpha_S), \tag{10}
\]
\[ \sigma_N(\alpha_S) \equiv \int_0^1 dt (1 - t)^{N-1} \sigma(t; \alpha_S) \] and similarly for \( d_{r,N}(\alpha_S) \). The threshold region is studied in moment space by taking the limit \( N \to \infty \). It can be shown\(^5\)\(^6\) that the form factor in \( N \)-space has the following exponential structure:

\[ \sigma_N(\alpha_S) = e^{G_N(\alpha_S)}. \] (11)

where \( G_N(\alpha_S) = l g_1(\lambda) + g_2(\lambda) + \alpha_S g_3(\lambda) + \alpha_S^2 g_4(\lambda) + \cdots \) and \( \lambda \equiv \beta_0 \alpha_S \log N \). The \( g_i(\lambda) \) are homogeneous functions of \( \lambda \) and have a series expansion around \( \lambda = 0 \). At LO (leading order) approximation only the function \( g_1 \) is required, at NLO (next-to-leading order) \( g_1 \) and \( g_2 \), at NNLO (next-to-next-to-leading order) \( g_3 \) as well, and so on. The form factor in \( u \) space is recovered by the inverse Mellin transform. It is convenient to define the partially integrated or cumulative form factor \( \Sigma(u; \alpha_S) \):

\[ \Sigma(u; \alpha_S) = \int_0^u du' \sigma(u'; \alpha_S). \] (12)

To NNLO accuracy, one can write\(^7\)

\[ \Sigma(u; \alpha_S) = e^{g_1(\tau) \log 1/u + g_2(\tau) \frac{S}{S|\tau \to 0}}, \] (13)

where \( \tau = \beta_0 \alpha_S \log 1/u \); by definition \( h_1(\tau) \equiv \frac{d}{d\tau} [\tau g_1(\tau)] = g_1(\tau) + \tau g'_1(\tau) \) and

\[ S = e^{\alpha_S g_3(\tau)} \left\{ 1 + \beta_0 \alpha_S g'_2(\tau) \psi [1 - h_1(\tau)] + \frac{1}{2} \beta_0 \alpha_S h'_1(\tau) \left\{ \psi^2 [1 - h_1(\tau)] - \psi' [1 - h_1(\tau)] \right\} \right\}. \]

The QCD cumulative form factor can be written in an exponential form\(^8\)

\[ \Sigma = e^G, \] (14)

where the expansion for the function \( G \) is:

\[ G(u; \alpha_S) = \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} G_{nk} \alpha_S^n \log^k \frac{1}{u}, \] (15)

and \( G_{nk} \) are numerical coefficients.

### 3 Single distributions for semileptonic decays

All double and single distributions in semileptonic decays are obtained by integrating the triple differential distribution\(^7\). Some of these distributions may require also an integration over the hard scale \( Q \) up to \( m_b \); this is different from what happens in the radiative decays, where the hard scale is fixed to \( m_b \) . The decay spectra for\(^9\)\(^10\) can therefore be divided into two classes:\(^2\)^\(^3\)^\(^4\)

1. distributions in which the hadronic energy \( E_X \) is not integrated over. These distributions have the same infrared structure of the hadron invariant mass distribution of the radiative decay and can be related via short-distance coefficients to the photon spectrum\(^4\). They share with the radiative decay the same structure of threshold logarithms. In this sense, we recover universality of long distance effects.

2. distributions in which the hadronic energy is integrated over and therefore all the hadronic energies contribute. The distributions in this class have an infrared structure different among themselves and from the hadron invariant mass distribution of the radiative decay. The structure of the threshold logarithms is not the same of\(^4\); there is no universality of long distance effects.
The single distribution in the hadron energy $E_X$ belongs to the first class, while the single distributions in $m_X$, in $E_l$ and in the light cone variable $p_+ = E_X - |\vec{p}_X|$ ($\vec{p}_X$ defined as the three-momentum of the final hadron state $X_0$) belong to the second group.

Let us consider, for instance, the single distribution in the hadron energy, obtained by integrating the triple differential distribution in $u$ and $x$:}

$$\frac{1}{2\Gamma}\frac{d\Gamma}{dw} = C_{W1}(\alpha_S) \left\{ 1 - C_{W2}(\alpha_S) \Sigma[w - 1; \alpha_S(m_b)] + H(w; \alpha_S) \right\} \quad (w > 1). \quad (16)$$

$C_{W1}(\alpha_S)$ and $C_{W2}(\alpha_S)$ are short distance coefficient functions and $H(w; \alpha_S)$ is a remainder function, vanishing in $w = 1$: they all have standard $\alpha_S$ expansions. We may consider the parts of the spectrum for $w < 1$ and $w > 1$ as two different spectra, merging in the point $w = 1$. The interesting case occurs at $w > 1$, when resummation is necessary; in fact, before the Sudakov shoulder, at $w < 1$, there are no large logarithms, and the spectrum can be written as an ordinary $\alpha_S$ expansion. The hadron energy distribution contains $\Sigma$ defined in (12), i.e. just the integral of the form factor $\sigma$ entering the radiative decay spectrum. The hadron energy spectrum is therefore directly connected to the radiative decay.

This is not necessarily true in other kinds of semileptonic spectra, as f.i. the distribution in the invariant hadron mass squared, that is in the variable $t = m_X^2/m_q^2 \simeq u w^2$. Such distribution is obtained by the triple differential one (7) by integrating in $x$ and afterwards in $u$ and $w$, with the appropriate kinematical constraints. The integration in $w$ changes the logarithmic structure of the distribution. It is still possible to find a resummed expression for the spectrum, with a form factor factorizing large logarithms at all orders in perturbation theory, but the form factor is now effective, i.e. process dependent. After the choice of a factorization scheme, we can write:

$$\frac{1}{\Gamma}\frac{d\Gamma}{dt} = C_T(\alpha_S) \sigma_T(t; \alpha_S) + d_T(t; \alpha_S), \quad (17)$$

where:

$$\sigma_T(t; \alpha_S) = \int_0^1 du/(2\sqrt{t}u) C_H \left( \sqrt{t/u}; \alpha_S(w m_b) \right) \sigma \left[ u; \alpha_S(w m_b \sqrt{t/u}) \right]. \quad (18)$$

$C_T(\alpha_S)$, $C_H(w; \alpha_S)$ are short distance coefficient functions; $d_T(t; \alpha_S)$ is a remainder function.

There is no simple connection between such semileptonic spectrum and the hadron mass distribution in radiative decays. They have different logarithmic structures. This can be explicitly checked by building a cumulative form factor $\Sigma_T(t; \alpha_S)$ analogous to (12). In a minimal factorization scheme it is also possible to build the analogous of exponentiated expression (13). In any case, by expanding $\Sigma_T(t; \alpha_S)$ in a logarithmic series, and comparing the coefficients of the logarithms with the coefficients $G_{nk}$ in (15), one finds that they differ already at $O(\alpha_S)$ (in the single logarithm term).

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