Transient dynamics of double quantum dots coupled to two reservoirs

Takahisa Fukadai and Tomohiro Sasamoto

Department of Physics, Tokyo Institute of Technology, 2-12-1
Okayama, Meguro-ku, Tokyo, 152-8550, Japan

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Abstract

We study time dependent property of double quantum dots coupled to two reservoirs using the non-equilibrium Green function method. For arbitrary time dependent bias, we derive an expression of the time-dependent electron density of a dot and the current between these dots in the wide band limit approximation. For the special case of a constant bias, we calculate the electron density and the current numerically. As a result, we find that these quantities oscillate and the number of crests in a single period of the current changes depending on the bias voltage. We also obtain an analytical expression of the relaxation time, which expresses how fast the system converges to its steady state. From the expression, we find out that the relaxation time becomes constant when the coupling strength between the dots are large enough in comparison with the difference of coupling strength between the dots and the reservoirs.

1 Introduction

Electron transport is a typical example of non-equilibrium phenomena and has been useful to develop the understanding of the non-equilibrium statistical physics. The great work about the electron transport was accomplished
by Landauer and Büttiker. They derive a simple formula in which the conductance between leads is expressed in terms of simple physical quantities such as transmission coefficient. Although the formula is derived for non-interacting electrons with several assumptions, it has been used widely due to its applicability and simplicity. An extension of Landauer-Büttiker formula to interacting case was done in with the non-equilibrium Green function method in these studies, the time-dependent current of a system coupled to two leads is expressed in terms of interacting Green functions, even in the case where arbitrary time dependent biases are applied to leads. For the special case where the system and reservoirs are in non-equilibrium steady state, the expression of the current reproduces the LB formula.

The approach used in these studies is based on the partitioned approach, which means a system and reservoirs are in independent equilibrium states characterized by a chemical potential $\mu_\alpha$ and an inverse temperature $\beta_\alpha$ with $\alpha$ representing a system and a reservoir at an initial time $t_0$. After the time $t_0$, a bias voltage is applied to each reservoir and couplings between the system and the reservoirs are added. When we only consider the steady state, this approach is justified because all effects of the initial condition disappear in the steady state. However, this approach has a problem when we focus on the transient dynamics where effects of the initial condition cannot be ignored and in a real experiment we just switch on the bias, not the bias and the coupling. Therefore, the approach is not suitable to investigate dynamics or time-dependent non-equilibrium property of nanosystems. The appropriate treatment of this problem may be setting the system and the reservoirs in the same equilibrium state at the initial time where the couplings between the system and the reservoirs are already added. This approach is called the partition-free approach. This improvement was introduced in and applied in . In the study, the authors calculate the Green functions including the effects of the Coulomb interaction with an approximation of the self-energy to conserve physical quantities properly such as energy with the partition-free approach. Without the Coulomb interaction, transport properties are calculated exactly in the wide band limit approximation (WBLA). The way of calculation in is employed to study the case where an arbitrary time dependent bias is applied to reservoirs and to investigate graphene nano-ribbons.

All of the studies above are about the transport through a single system such as a single quantum dot. For the case of the quantum transport through a system consisting of several subsystems such as double quantum
dots (DQD), one can expect more variety of phenomena arise due to the couplings between the system and the reservoirs or their geometrical configuration. From this point of view, many studies about quantum transport through DQD have been conducted experimentally \[14, 17\] and theoretically \[20, 24\]. There are many analytical studies until now, however, the analysis have been mainly carried out in the stationary and special cases. For example, the slave boson approach \[18, 19\], which have been used widely in the studies of DQD, assumes that the Coulomb interaction between electrons are so large and only one electron can exist in each dot. With this approach and mean-field approximation, many studies about the steady state of the double dots have been made which are displayed in the parallel, serial or T-shaped geometry \[20, 23\]. For the special case where the parameters satisfy the Yang-Baxter relations, the Hamiltonian of two dots including Coulomb interaction are exactly solvable with the Bethe ansatz \[24\]. To understand the transient dynamics of DQD are also important for several reasons. One is that DQD can be applied to quantum computation \[25\] and studying how the decoherence happens is necessary in this context. Moreover, it is interesting as a non-equilibrium phenomenon because the competition of the initial correlation effects and the coupling between the dots exists. Until now, theoretical investigation of the transient dynamics of DQD are mainly conducted with master equation \[20\] or quantum master equation \[27\], which cannot treat quantum effects like coherence properly. Some analysis including quantum effect has been conducted by numerics \[28, 32\].

In this paper, to understand the dynamics of DQD analytically, we study the transient dynamics of the double quantum dots in serial geometry with the non-equilibrium Green function method and the partition-free approach in the wide band limit approximation. We obtain an exact expression of the electron density of a dot and the current between DQD with arbitrary parameters in the case where the Coulomb interaction is irrelevant. Experimentally, the case in which high bias voltages are applied to the reservoirs or the total system is set at a high temperature may be an example of our analysis. For the quenched case, where a constant bias voltage is applied to a reservoir suddenly, we calculate these quantities numerically and understand the dependence of these quantities to parameters. From these expressions, we also obtain the relaxation time from any initial condition to the steady state, which is useful to understand how the decoherence is affected by certain parameters.

The paper is organized as follows: we first introduce the non-equilibrium
Green function method in section 2, then we briefly review the previous study of the single dot case \cite{12} in section 3 to highlight the difference between the analysis of the previous case and our case. In section 4, we show the analysis of the double dots, where the second-order partial differential equation for Green functions are derived. We solve the equations and get the expression of the Green functions. We numerically calculate the electron density and the current in section 5 for the quenched case where a bias voltage is applied to a reservoir suddenly.

2 Non-equilibrium Green function

Non-equilibrium Green function approach \cite{6, 7} is a method to calculate physical quantities in non-equilibrium many-body quantum systems. It is especially useful to calculate non-equilibrium physical quantities perturbatively. First, we consider a general case and explain the formalism of the non-equilibrium Green function.

We consider the case where there is a central region surrounded by reservoirs. In the central region, there are some subsystems. The total Hamiltonian is represented as $H_0$ at initial time $t_0$. About the initial state, we assume the subsystems in the central region and the reservoirs are already coupled and they are in the same thermal-equilibrium state characterized by an inverse temperature $\beta$ and a chemical potential $\mu$. After the initial time, the perturbative term $V(t)$ is added to the initial Hamiltonian: $H(t) = H_0 + V(t)$. A bias voltage applied to a reservoir is an example of the perturbation. We assume that the total Hamiltonian can be expressed in the matrix form: $H(t) = \sum_{ij} h_{ij}(t) d_i^\dagger d_j$. The operators $d_i^\dagger, d_i$ express the creation and annihilation operators of the system or the reservoirs. Here, we define the Konstantinov-Perel’ Contour $\gamma$, which is an extension of Keldysh contour to include effects of the initial state, as Fig.1. The contour consists of three paths: $C_- : \{t_0 \rightarrow \infty\}, C_+ : \{\infty \rightarrow t_0\}$ and $C_M : \{t_0 \rightarrow t_0 - i\beta\}$. The non-equilibrium Green function on the contour is defined as

$$G_{ij}(z_1, z_2) := -i \frac{1}{Z} \text{Tr}[e^{-\beta H_0} \mathcal{T}_c[d_i(z_1)d_j^\dagger(z_2)]], \quad (1)$$

where $Z = \text{Tr}e^{-\beta H_0}$ is the partition function at the initial time. Here $z = t_-, t_+, t_0 - i\tau$ represents a position on the contour and time. $t_-$ and $t_+$ express a time $t$ on the path $C_-$ and $C_+$ respectively. $\tau(0 < \tau < \beta)$ is an imaginary
time on the path $C_M$. The operators appearing in (1) are defined as

$$d_i(z) := \begin{cases} d_{H,i}(t), & z \in C_- \oplus C_+, \\ d_M^i(\tau), & z \in C_M, \end{cases}$$

where $d_{H,i}(t) := e^{i \int_{t_0}^t ds H(s)} e^{-i \int_{t_0}^t ds H(s)}$ is the operator in the Heisenberg picture and $d_M^i(\tau)$ is the Matsubara operator $d_M^i(\tau) := e^{H_0 \tau} e^{-H_0 \tau}$. $T_c$ is the contour ordering operator according to the direction of the arrow on the contour. The non-equilibrium Green function (1) takes several forms according to the positions of two arguments $z_1$ and $z_2$. We represent these Green functions as Tab.2. As an example, let us consider the case where

| Symbol      | Position on the contour $\gamma$ | Name       |
|-------------|----------------------------------|------------|
| $G_{ij}^\gamma(t_1, t_2)$ | $z_1, z_2 \in C_-$            | Greater    |
| $G_{ij}^{\gamma+}(t_1, t_2)$ | $z_1, z_2 \in C_+$               | Greater    |
| $G_{ij}^{\gamma<}(t_1, t_2)$ | $(z_1 > z_2) \land (z_1, z_2 \in C_- \oplus C_+)$ | Greater    |
| $G_{ij}^{\gamma>}(t_1, t_2)$ | $(z_1 < z_2) \land (z_1, z_2 \in C_- \oplus C_+)$ | Lesser     |
| $G_{ij}^M(t_1, \tau_2)$  | $(z_1 \in C_M) \land (z_2 \in C_- \oplus C_+)$ | Right      |
| $G_{ij}^M(\tau_1, t_2)$  | $(z_1 \in C_M) \land (z_2 \in C_- \oplus C_+)$ | Left       |

Tab. 1: Definition of the Green functions

these two arguments appearing in the definition (1) are in the part of contour
\[ z_1, z_2 \in C_- \bigoplus C_+ \text{ and } z_1 \text{ is in front of } z_2 \text{ on the contour. In this case, since the argument } z_1 \text{ is always in front of } z_2, T_c \text{ exchanges these two operators and then } (-1) \text{ is multiplied. Therefore, the non-equilibrium Green function takes the form } G_{ij}^{<}(t_1, t_2) = \frac{i}{Z} \text{Tr}[e^{-\beta H_0} d_{H,i}(t_2) d_{H,j}(t_1)] \text{ for } (z_1 < z_2) \land (z_1, z_2 \in C_- \bigoplus C_+). \] 

This function is called the lesser Green function. We also define retarded/advanced Green functions as

\[
G_{ij}^r(t_1, t_2) := \theta(t_1 - t_2) [G_{ij}^{>}(t_1, t_2) - G_{ij}^{<}(t_1, t_2)], \quad (2) \\
G_{ij}^a(t_1, t_2) := -\theta(t_2 - t_1) [G_{ij}^{>}(t_1, t_2) - G_{ij}^{<}(t_1, t_2)], \quad (3)
\]

where \( \theta(t) \) is the Heaviside function: \( \theta(t) = 1(t > 0), 0(t < 0). \)

We define the Hamiltonian on the contour \( h_{ij}(z) \) as

\[
h_{ij}(z) := \begin{cases} h_{ij}(t), & z \in C_- \bigoplus C_+ \\ h_{ij}(t_0), & z \in C_M. \end{cases}
\]

With the contour \( \gamma \) and the non-equilibrium Green function on the contour, we can use the diagram technique even in the non-equilibrium case. For details, see [33].

## 3 Single dot case

Before we discuss the double dot case, which is of our main interest, we briefly review the case of a single dot which has the only one energy level and is coupled to two reservoirs. This is a special case of [12]. In this case, we analyze the equation of motion for the non-equilibrium Green function. The equation is self-consistent, but it becomes solvable after employing the WBLA. By solving the equation of motion, we can obtain a closed-form expression of the non-equilibrium Green function. This section will be useful to understand the differences between the analysis of the single dot case and that of the double dots case, where we cannot obtain a closed-form expression of the non-equilibrium Green function by solving the equation of motion with the WBLA.

### 3.1 Definition of system

We consider the case where the total system consists of a quantum dot in a central region and two reservoirs. At an initial time \( t_0 \), the dot and the
reservoirs are in the thermal-equilibrium state characterized by an inverse
temperature $\beta$ and a chemical potential $\mu$. After the time $t \geq t_0$, a bias
voltage $V_\alpha(t)$ is applied to the reservoir $\alpha$ and then the total system tends to
a non-equilibrium state. The Hamiltonian of the total system is represented
as
$$H(t) = \sum_{k\alpha} \epsilon_{k\alpha}(t) d_{k\alpha}^\dagger d_{k\alpha} + \epsilon_1(t) d_1^\dagger d_1 + V, \quad (4)$$
where $\epsilon_{k\alpha}(t)$ and $\epsilon_1(t)$ are defined as $\epsilon_{k\alpha}(t) = \epsilon_{k\alpha} - \mu(t < t_0), \epsilon_{k\alpha} + V_\alpha(t)(t \geq t_0)$ and $\epsilon_1(t) = \epsilon_1 - \mu(t < t_0), \epsilon_1(t \geq t_0)$. $\epsilon_{k\alpha}$ is the $k$-th eigenvalue of the
Hamiltonian of the reservoir $\alpha$ and $\epsilon_1$ is the eigenvalue of the dot. $d_{k\alpha}, d_{k\alpha}^\dagger$
are the creation and the annihilation operators of the reservoir $\alpha$ and $d_1, d_1^\dagger$
are those of the dot. In this case, the particles are Fermion and thus these
operators satisfy the anti-commutation relations: $\{d_i, d_j^\dagger\} = \delta_{ij}, \{d_i, d_j\} = \{d_i^\dagger, d_j^\dagger\} = 0$, where the index $i$ denotes the index of a reservoir $\alpha$ or the dot: $i = k\alpha, 1$. Each term of the total Hamiltonian has the following meaning:
the first term is the (diagonalized) Hamiltonian of the reservoirs, the second
is the Hamiltonian of the dot, the third is the coupling between the dot and
the reservoirs. We assume that the coupling term takes the form,
$$V = \sum_{k\alpha} T_{k\alpha,1} d_{k\alpha}^\dagger d_1 + T_{1,k\alpha} d_1^\dagger d_{k\alpha}.$$ We use a matrix representation of the Hamiltonian $h_{ij}(t)$ of the total system
to match (4). For example, when the indices take $i = k\alpha$ and $j = k'\alpha$, the
value of $h_{ij}(t)$ is
$$h_{k\alpha,k'\alpha}(t) = \begin{cases} (\epsilon_{k\alpha} - \mu) \delta_{k,k'}, & t < t_0, \\ (\epsilon_{k\alpha} + V_\alpha(t)) \delta_{k,k'}, & t \geq t_0. \end{cases}$$
We define the matrix $h_{\alpha\alpha}(t)$ by $[h_{\alpha\alpha}(t)]_{k,k'} = h_{k\alpha,k'\alpha}(t)$. Similarly, we define
other matrices $h_{\alpha1}(t), h_{1\alpha}$ and $h_{11}(t)$.

3.2 Equation of motion and previous result
By differentiating the definition of each Green function and using the com-
mutation relations, we derive the equations of motion for the non-equilibrium
Green functions as
\[
\begin{align*}
[i \frac{d}{dz_1} - h_{11}(z_1)]G_{11}(z_1, z_2) &= \delta(z_1, z_2) + \sum_\alpha h_{1\alpha} G_{\alpha 1}(z_1, z_2), \\
G_{11}(z_1, z_2) \left[-i \frac{d}{dz_2} - h_{11}(z_2)\right] &= \delta(z_1, z_2) + \sum_\alpha G_{1\alpha}(z_1, z_2) h_{\alpha 1},
\end{align*}
\]
where we use the matrix representation of the non-equilibrium Green function \([G(z_1, z_2)]_{ij} = G_{ij}(z_1, z_2)\). The operator \(\frac{d}{dz}\) is the differential operator which acts from the right side.

We need to know an expression of the lesser Green function to get an expression of one-particle quantities such as the electron density of the dot. To obtain an expression of the lesser Green function, we need to solve the equations of motion with boundary conditions, the KMS(Kubo-Martin-Schwinger) conditions \([34, 35]\). All Green functions must satisfy the KMS conditions, which arise from the condition that the initial state is in thermal equilibrium. The KMS conditions are expressed as,
\[
\begin{align*}
G(z_1, t_0 - i\beta) &= -G(z_1, t_0), \\
G(t_0 - i\beta, z_2) &= -G(t_0, z_2),
\end{align*}
\]
which are directly derived from the definition (1). We note that we can choose another initial condition as in the partitioned approach where the system and the reservoirs are in different equilibrium states at the initial time. However it sometime leads to unphysical behavior of physical quantities \([9]\). Since there are unknown functions in the equations of motion for \(G_{11}(z_1, z_2), \) \(5\) and \(6\), we cannot obtain a closed-form expression of the \(G_{11}(z_1, z_2)\). Now we express solutions of \(7\) and \(8\) as
\[
\begin{align*}
G_{\alpha 1}(z_1, z_2) &= \int_\gamma d\bar{z} g_{\alpha\alpha}(z_1, \bar{z}) h_{\alpha 1}(\bar{z}) G_{11}(\bar{z}, z_2), \\
G_{1\alpha}(z_1, z_2) &= \int_\gamma d\bar{z} G_{11}(z_1, \bar{z}) h_{1\alpha}(\bar{z}) g_{\alpha\alpha}(\bar{z}, z_2),
\end{align*}
\]
where $g_{\alpha\alpha}$ is the non-perturbative Green function of the reservoir $\alpha$ which obeys the equation of motion

$$\left[ i \frac{d}{dz_1} - h_{\alpha\alpha}(z_1) \right] g_{\alpha\alpha}(z_1, z_2) = \delta(z_1, z_2),$$

and $\int_{\gamma} d\bar{z}$ is the integration on the contour $\gamma$. This equation of motion is equal to the one for the isolated reservoir $\alpha$, since there is only the delta function in the RHS of the equation. Actually, we can check that the expressions (10) and (11) are solutions of the equations of motion (7) and (8) satisfying the KMS conditions. Then, by substituting (10) and (11) into (5) and (6), finally we obtain the equation of motion for $G_{11}(z_1, z_2)$,

$$\left[ i \frac{d}{dz_1} - h_{11}(z_1) \right] G_{11}(z_1, z_2) = \delta(z_1, z_2) + \int_{\gamma} d\bar{z} \Sigma_{11}(z_1, \bar{z}) G_{11}(\bar{z}, z_2),$$

$$G_{11}(z_1, z_2) \left[ -i \frac{d}{dz_2} - h_{11}(z_2) \right] = \delta(z_1, z_2) + \int_{\gamma} d\bar{z} G_{11}(z_1, \bar{z}) \Sigma_{11}(\bar{z}, z_2), \tag{12}$$

where $\Sigma_{11}$ is the embedded self-energy defined as

$$\Sigma_{11}(z_1, z_2) = h_{1\alpha}(z_1) g_{\alpha\alpha}(z_1, z_2) h_{\alpha 1}(z_2). \tag{13}$$

At this point, we obtain a closed-form expression of the embedded self-energy because it is expressed in terms of the known parameters and the non-perturbative Green function, which is calculated easily. Now we use the WBLA at this point. This approximation means that the energy bands of reservoirs are so wide and only the electrons on the Fermi energy are transported. Mathematically, the WBLA is equivalent to the substitution: $\sum_k \rightarrow \int \frac{dk}{2\pi} \rightarrow \rho(\epsilon_F) \int d\epsilon_k$ where $\rho(\epsilon_F)$ is the density of state in the vicinity of Fermi energy. Therefore, the level width $\Gamma_{\alpha}(\omega) = 2\pi \sum_k T_{1,\alpha} T_{k\alpha} \delta(\omega - \epsilon_{k\alpha})$ becomes a constant $\Gamma_{\alpha}$ and the embedded self-energy takes a simple form in this approximation. For example, the retarded self-energy $\Sigma_{11}^{r}(t_1, t_2) := \theta(t_1 - t_2)(\Sigma_{11}^{>}(t_1, t_2) - \Sigma_{11}^{<}(t_1, t_2))$ in the WBLA is written as

$$\Sigma_{11}^{r}(t_1, t_2) = -\frac{i}{2} \Gamma \delta(t_1 - t_2),$$

where $\Gamma = \sum_{\alpha} \Gamma_{\alpha}$ is the total level width. The lesser $\Sigma_{11}^{<}(t_1, t_2)$ and the greater $\Sigma_{11}^{>}(t_1, t_2)$ self-energies are defined such as the lesser and greater
Green function in section 2. For the derivation, see Appendix A. The calculation of the self-energy in Appendix A is for the case of the double quantum dots, but the self-energy for the single dot case is obtained in the same way.

From the equation of motion (12) and the embedded self-energy in the WBLA, we can get the following equations of motion of the Matsubara, right/left and lesser Green functions in the WBLA:

\[
\begin{align*}
-\frac{d}{d\tau_1} - h_{11}^{\text{eff}}(\tau_1, \tau_2) &= i(\tau_1 - \tau_2) + \left(\Sigma_{11}^M * G_{11}^M\right)(\tau_1, \tau_2), \\
\left(i \frac{d}{dt_1} - h_{11}^{\text{eff}}\right)G_{11}^I(t_1, \tau_2) &= \left(\Sigma_{11}^I * G_{11}^M\right)(t_1, \tau_2), \\
G_{11}^I(\tau_1, t_2) \left(-i \frac{d}{dt_2} - h_{11}^{\text{eff}}\right) &= \left(G_{11}^M * \Sigma_{11}^I\right)(t_1, \tau_2), \\
\left(i \frac{d}{dt_1} - h_{11}^{\text{eff}}\right)G_{11}^< (t_1, t_2) &= \left(\Sigma_{11}^< * G_{11}^a + \Sigma_{11}^I * G_{11}^I\right)(t_1, \tau_2),
\end{align*}
\]

where \(h_{11}^{\text{eff}}\) is the effective energy of the system: \(h_{11}^{\text{eff}} := h_{11} - i\Gamma/2\). \((f \cdot g)(t, t')\) is the convolution of \(f\) and \(g\) and \((f * g)(\tau, \tau')\) is the imaginary time convolution of \(f\) and \(g\): \((f \cdot g)(t, t') = -i \int_0^\infty ds f(t, s)g(s, t')\), \((f * g)(\tau, \tau') = \int_0^\beta ds f(\tau, s)g(s, \tau')\). In this derivation, we use the Langreth rule [36] to convert the integration in the most RHS of (12) to an integral over the real axis. Since we have an expression of the Matsubara self-energy, we can obtain an expression of the Matsubara Green function from (14) by applying the Fourier transformation. Then, the resulting equations are the inhomogeneous first-order differential equations and thus we can solve these equations. Details of these calculations are given in [12]. Finally, we obtain the following expression of the lesser Green function

\[
G_{11}^<(t_1, t_2) = e^{-ih_{11}^{\text{eff}}(t_1 - t_0)} \int_{-\infty}^{\infty} d\omega f(\omega - \mu) \sum_\alpha \left[K_\alpha(t_1, t_0; \omega) \Gamma_\alpha G_\alpha(\omega) - G_\alpha^\dagger(\omega) \Gamma_\alpha K_\alpha^\dagger(t_2, t_0; \omega) + iK_\alpha(t_1, t_0; \omega) \Gamma_\alpha K_\alpha^\dagger(t_2, t_0; \omega) + iA_\alpha(\omega)\right] e^{i(h_{11}^{\text{eff}})(t_2 - t_0)},
\]

where \(G^\dagger(\omega) = (\omega - h_{11}^{\text{eff}})^{-1}\) and \(G_\alpha(\omega) = (\omega - h_{11}^{\text{eff}})^{-1}\) are Fourier transformed expressions of the retarded/advanced Green functions.
\[ \{e^{\beta \omega} + 1\}^{-1} \] is the Fermi distribution function. Here, we introduce the spectral function \( A_\alpha(\omega) = G^r(\omega) \Gamma_\alpha G^a(\omega) \) and the function
\[ K_\alpha(t, t_0; \omega) = \int_{t_0}^{t} ds e^{-i(\omega - h_{11}^{\text{eff}})(s-t_0)} e^{-i \psi_\alpha(s; t_0)}, \]
which includes all the effects from the biased voltage \( \psi_\alpha(t, t_0) := \int_{t_0}^{t} ds V_\alpha(s) \).

Since the electron density is expressed as \( \rho(t) = -iG_1^{\text{<}}(t, t) \), we calculate the electron density almost directly. With the concrete expression (17), we calculate the electron density numerically for the quenched case where a constant bias voltage is applied suddenly: \( V_L = 4, V_R = 0 \). We take other parameters as \( \epsilon = 1, \Gamma_L = \Gamma_R = 1/2 \) and consider two cases with \( \beta = 1 \) as a high temperature case and \( \beta = 100 \) as a low temperature case. For the case \( \beta = 1 \), we can see that the electron density approaches the limiting value exponentially (Fig. 2). By contrast, the behavior is different for \( \beta = 100 \) (Fig. 3). It behaves as almost exponential, but slightly fluctuates. By looking at the concrete expression of the lesser Green function in (17), we can figure out that the difference arises from the existence of the functions that are multiplied to the exponential function \( e^{-i h_{11}^{\text{eff}}(t_1-t_0)} \). Therefore, we conclude that the effect of these functions multiplied to the exponential function decrease when the temperature becomes larger. This property also holds for the double dots case as in the discussion of the relaxation time in chapter 5. We can also see that the electron density does not oscillate for both cases. This is because we consider the dot having only one energy level. When we consider the case of the double dots, however, an oscillation appears even though each dot has only one energy level. See Fig. 5 in the next section for this argument.
4 Double dots case

4.1 Definition of system

Next, we consider the case of the double quantum dots, which is the main target of our study. Fig. 4 is a schematic diagram of this case. In the case, there are two quantum dots, not one, in the central region. Since there are the coupling between the dots, we can expect that richer phenomena will arise than those of the single dot. We assume the initial condition is the same as in the case of the single dot in section 4. At an initial time $t_0$, the dots and the reservoirs are in the thermal-equilibrium state characterized by an inverse temperature $\beta$ and a chemical potential $\mu$. After the time $t \geq t_0$, a bias voltage $V_{\alpha}(t)$ is applied to each reservoir $\alpha$ and the system gets into a non-equilibrium state. The Hamiltonian of the total system is represented as

$$H(t) = \begin{cases} 
\sum_{\alpha=L,R,k} (\epsilon_{k\alpha} - \mu) d_{k\alpha}^{\dagger} d_{k\alpha} + \sum_{n=\{1,2\}} (\epsilon_n - \mu) d_{n}^{\dagger} d_{n} + V_{1L} + V_{2R} + T_d, & t < t_0 \\
\sum_{\alpha=L,R,k} (\epsilon_{k\alpha} + V_{\alpha}(t)) d_{k\alpha}^{\dagger} d_{k\alpha} + \sum_{n=\{1,2\}} \epsilon_n d_{n}^{\dagger} d_{n} + V_{1L} + V_{2R} + T_d, & t \geq t_0 
\end{cases}$$

(18)

where $\epsilon_{k\alpha}$ is the $k$-th eigenvalue of the Hamiltonian of the reservoir $\alpha$ and $\epsilon_n$ is the eigenvalue of the dot $n (n = 1, 2)$. We assume the Hamiltonian of the reservoirs can be diagonalized and each dot has just one energy level. $d_{k\alpha}, d_{k\alpha}^{\dagger}$ are the creation and the annihilation operators of a reservoir $\alpha$ and $d_{n}, d_{n}^{\dagger}$ are those of the dot $n$. The transported particles are Fermion and
thus the operators satisfy the anti-commutation relations: $\{d_i, d_j^\dagger\} = \delta_{ij}$, $\{d_i, d_j\} = \{d_i^\dagger, d_j^\dagger\} = 0$. Each term of the Hamiltonian has the meaning as follows: the first term is the (diagonalized) Hamiltonian of the reservoirs, the second is the Hamiltonian of the dots, the third is the coupling between the dot1 and the left reservoir, the fourth is the coupling between the dot2 and the right reservoir. The fifth, which does not appear in the case of the single dot, is the coupling between the two dots. Here, we assume the interactions are written in the forms:

\[ V_{i\alpha} = \sum_{k\alpha,n} T_{k\alpha,n} d_{k\alpha}^\dagger d_n + T_{n,k\alpha} d_n^\dagger d_{k\alpha}, \]

\[ T_d = t_{12} d_1^\dagger d_2 + t_{21} d_2^\dagger d_1. \]

We note that we are interested in how the fact that the central system consists of two subsystems affects physical quantities. By this reason, we consider the region where Coulomb interaction can be ignored. If there is the effect of Coulomb interaction, there would be more interesting phenomena, but it would be difficult to tell whether the cause of a phenomenon is from the fact that the system consists of some subsystems or the Coulomb interaction. Experimentally, this approximation may be justified when the energy scale of electrons is larger than that of the Coulomb interaction, i.e., when a high bias voltage is applied or when a system is at a high temperature. We represent the Hamiltonian of the total system \((18)\) in the same way as the single dot case

\[ H(t) = \sum_{ij} h_{ij}(t) d_i^\dagger d_j, \]

where the index \(i\) can take \(k\alpha\) or \(n\) and the elements of \(h_{ij}(t)\) are defined to match \((18)\). We also define the contour \(\gamma\) and the non-equilibrium Green function in the same way as in section 3.

### 4.2 Equation of motion for non-equilibrium Green function

By differentiating the definition of each Green function in Tab.1 and using the anticommutation relations for Fermion, we can derive these equations of
motion for the non-equilibrium Green functions:

\[
\begin{align*}
\left[i \frac{d}{dz_1} - h_{11}(z_1) \right] G_{11}(z_1, z_2) &= \delta(z_1, z_2) + h_{1L} \cdot G_{L1}(z_1, z_2) + t_{12} G_{21}(z_1, z_2), \\
G_{11}(z_1, z_2) \left[-i \frac{d}{dz_2} - h_{11}(z_2) \right] &= \delta(z_1, z_2) + h_{1L} \cdot G_{L1}(z_1, z_2) + G_{12}(z_1, z_2) t_{21}, \\
\left[i \frac{d}{dz_2} - h_{22}(z_1) \right] G_{22}(z_1, z_2) &= \delta(z_1, z_2) + h_{2R} \cdot G_{R2}(z_1, z_2) + t_{21} G_{12}(z_1, z_2), \\
G_{22}(z_1, z_2) \left[-i \frac{d}{dz_2} - h_{22}(z_2) \right] &= \delta(z_1, z_2) + h_{2R} \cdot G_{R2}(z_1, z_2) + G_{21}(z_1, z_2) t_{12}.
\end{align*}
\] (19)

In the case of the double dots, we must consider the equations of motion for not only \(G_{11}\) and \(G_{22}\), but also \(G_{12}\) and \(G_{21}\) for analysis:

\[
\begin{align*}
\left[i \frac{d}{dz_1} - h_{11}(z_1) \right] G_{12}(z_1, z_2) &= h_{1L} \cdot G_{L2}(z_1, z_2) + t_{12} G_{22}(z_1, z_2), \\
G_{12}(z_1, z_2) \left[-i \frac{d}{dz_2} - h_{22}(z_2) \right] &= G_{1R}(z_1, z_2) \cdot h_{R2} + G_{11}(z_1, z_2) t_{12}, \\
\left[i \frac{d}{dz_1} - h_{22}(z_1) \right] G_{21}(z_1, z_2) &= h_{2R} \cdot G_{R1}(z_1, z_2) + t_{21} G_{11}(z_1, z_2), \\
G_{21}(z_1, z_2) \left[-i \frac{d}{dz_2} - h_{11}(z_2) \right] &= G_{2R}(z_1, z_2) \cdot h_{R1} + G_{22}(z_1, z_2) t_{21},
\end{align*}
\] (20)

where we use the matrix representation of the non-equilibrium Green function \(\{G(z_1, z_2)\}_{ij} = G_{ij}(z_1, z_2)\). As in the case of the single dot in section 3, all
Green functions must satisfy the KMS (Kubo-Martin-Schwinger) conditions
\[
\begin{align*}
G(z_1, t_0 - i\beta) &= -G(z_1, t_0), \\
G(t_0 - i\beta, z_2) &= -G(t_0, z_2).
\end{align*}
\] (29)

We can construct solutions of the equations, (27) and (28) satisfying the KMS conditions in the form
\[
\begin{align*}
G_{L1} &= \int_\gamma d\bar{z} g_{LL}(z_1, \bar{z}) h_{L1} G_{11}(\bar{z}, z_2), \\
G_{R1} &= \int_\gamma d\bar{z} g_{RR}(z_1, \bar{z}) h_{R2} G_{21}(\bar{z}, z_2),
\end{align*}
\] (30)

where \( g_{\alpha\alpha} \) is the non-perturbative Green function of the reservoir \( \alpha \) which obeys the equation of motion,
\[
\left[ i \frac{d}{dz_1} - h_{11}(z_1) \right] g_{\alpha\alpha}(z_1, z_2) = \delta(z_1, z_2),
\]
and \( \int_\gamma d\bar{z} \) is the integration on the contour \( \gamma \). Finally, by substituting (30) and (31) into (19) and (21), we get
\[
\begin{align*}
\left[ i \frac{d}{dz_1} - h_{11}(z_1) \right] G_{11}(z_1, z_2) &= 1\delta(z_1, z_2) \\
&+ \int_\gamma d\bar{z} \Sigma_{11}(z_1, \bar{z}) G_{11}(\bar{z}, z_2) + t_{12} G_{21}(z_1, z_2),
\end{align*}
\] (32)

\[
\begin{align*}
\left[ i \frac{d}{dz_1} - h_{22}(z_1) \right] G_{21}(z_1, z_2) &= \int_\gamma d\bar{z} \Sigma_{22}(z_1, \bar{z}) G_{21}(\bar{z}, z_2) + t_{21} G_{11}(z_1, z_2) \\
\Sigma_{11}(z_1, \bar{z}) &= h_{1L} \cdot g_{LL}(z_1, \bar{z}) h_{L1}, \\
\Sigma_{22}(z_1, \bar{z}) &= h_{2R} \cdot g_{RR}(z_1, \bar{z}) h_{R2}.
\end{align*}
\] (33)

In comparison to the single dot case, we find that new terms appear in the rightmost hand sides of the equations of motion for the system (32) (33).

Hence, we cannot solve the equations in the same way as for the single dot case where we obtained the solution by integrating the equation of motion. However, there are only two unknown functions, \( G_{11} \) and \( G_{21} \), in the two differential equations (32) and (33). Therefore, in principle, by substituting one equation into the other, we can derive a second-order differential equation...
for the non-equilibrium Green function in closed form. We can expect that we obtain a concrete expression of the non-equilibrium Green function by integrating the second-order differential equation. This is the basic idea of our analysis. Although this idea is very simple, a phenomenon which cannot be seen in the analysis of the single dot appears. In the second-order partial differential equation, a new quantity called the pseudo self-energy appears. The problem is that a divergence appears in a second-order partial differential equation when one computes the new quantity straightforwardly. We solve this problem by changing the way of calculation of the quantity. We explain this problem and how we solve it in Appendix E.

4.3 Second-order differential equations of the non-equilibrium Green functions and the pseudo self-energy

Actually, for example, we can get second-order differential equations for the lesser Green function of the system from (32) and (33). See Appendix D for details of the derivation,

\[
\left[ -\frac{d^2}{dt^2} - i(\epsilon_1 + \epsilon_2) \frac{d}{dt} + (\epsilon_1 \epsilon_2 - t_{12} t_{21}) \right] G_{11}^<(t_1, t_2) = \int_\gamma d\bar{z} \Sigma_{11}^+(t_1, \bar{z}) G_{11}^+(\bar{z}, t_2) - i \left( \Sigma_{22}^r(t_1, t_0) G_{11}^<(t_0, t_2) + \Sigma_{22}^<(t_1, t_0) G_{11}^a(t_0, t_2) \right) - \Sigma_{22}^<(t_1, t_2),
\]

(35)

\[
G_{11}^<(t_1, t_2) \left[ -\frac{d^2}{dt^2} + i(\epsilon_1 + \epsilon_2) \frac{d}{dt} + (\epsilon_1 \epsilon_2 - t_{12} t_{21}) \right] = \int_\gamma d\bar{z} \Sigma_{11}^-(t_1, \bar{z}) \Sigma_{11}^+(\bar{z}, t_2) - i \left( G_{11}^r(t_1, t_0) \Sigma_{22}^<(t_0, t_2) + G_{11}^<(t_1, t_0) \Sigma_{22}^a(t_0, t_2) \right) + \Sigma_{22}^<(t_1, t_2),
\]

(36)

where we define a new quantity \( \tilde{\Sigma}_{11}(z_1, z_2) \), \( \Sigma_{11}(z_1, z_2) \) as

\[
\tilde{\Sigma}_{11}(z_1, z_2) := \left( i \frac{d}{dz_1} - h_2(z_1) \right) \Sigma_{11}(z_1, z_2) + \Sigma_{22}(z_1, z_2) \left( -i \frac{d}{dz_2} - h_1(z_2) \right) - \int_\gamma d\bar{z} \Sigma_{22}(z_1, \bar{z}) \Sigma_{11}(\bar{z}, z_2),
\]

(37)
\[
\bar{\Sigma}_{11}(z_1, z_2) := \left( i \frac{d}{dz_1} - h_1(z_2) \right) \Sigma_{22}(z_1, z_2) + \Sigma_{11}(z_1, z_2) \left( -i \frac{d}{dz_2} - h_2(z_2) \right)
- \int d\bar{z} \Sigma_{11}(z_1, \bar{z}) \Sigma_{22}(\bar{z}, z_2). \tag{38}
\]

By comparing (35) and (36) with (12), we see that \(\tilde{\Sigma}_{11}(z_1, z_2)\), \(\bar{\Sigma}_{11}(z_1, z_2)\) appear in place of the self-energy in RHS of (12). Although they are similar in this sense, \(\tilde{\Sigma}_{11}(z_1, z_2), \bar{\Sigma}_{11}(z_1, z_2)\) include terms whose dimensions are \(\epsilon^2\) such as \(h_2(z_1)\Sigma_{11}(z_1, z_2)\), not \(\epsilon\). Thus, we simply call them the pseudo self-energies here. We have not understood their physical meaning yet. We note that the pseudo self-energies do not satisfy the Langreth rule, which the usual self-energy satisfies and we used for the integration of the equation of motion in the single dot case. For the pseudo self-energies, we have to apply a modified Langreth rule. We explain the rule in Appendix B.

For analysis of the equation (35) or second-order partial differential equations for other Green functions that will appear later, we must know the expressions of the embedded self-energy and the pseudo self-energy. We calculate these quantities with the WBLA. Within the WBLA, the self-energy takes the simple form that is the same as the single dot case \[12\]. For example, the retarded/advanced self-energies are represented as

\[
\Sigma_{11}^r(t_1, t_2) = -\frac{i}{2} \Gamma_L \delta(t_1 - t_2), \quad \Sigma_{22}^r(t_1, t_2) = -\frac{i}{2} \Gamma_R \delta(t_1 - t_2),
\Sigma_{11}^a(t_1, t_2) = \frac{i}{2} \Gamma_L \delta(t_1 - t_2), \quad \Sigma_{22}^a(t_1, t_2) = \frac{i}{2} \Gamma_R \delta(t_1 - t_2). \tag{39}
\]

The expressions of other self-energies are given in Appendix A. The new quantities, the pseudo self-energies, are calculated from the expressions of the self-energy and the definition \[37\]. But, we must pay attention when we calculate the retarded/advanced parts of the pseudo self-energy. This is because a diverging term appears in the second-order differential equations of the retarded/advanced part if we calculate the pseudo self-energies directly. This technical problem is explained in Appendix E. Finally, the retarded pseudo self-energy is expressed as

\[
\hat{\Sigma}_{11}^r(t_1, t_2) := \theta(t_1 - t_2)(\hat{\Sigma}_{11}^>_{11}(t_1, t_2) - \hat{\Sigma}_{11}^<_{11}(t_1, t_2))
= \frac{1}{2} \left( \Gamma_{L,11} + \Gamma_{R,22} \right) \frac{d}{dt(t_1 - t_2)} \delta(t_1 - t_2) + \frac{i}{2} \left( \epsilon_2 \Gamma_{L,11} + \epsilon_1 \Gamma_{R,22} \right) \delta(t_1 - t_2)
- (h_{1L} h_{1L} + h_{2R} h_{2R}) \delta(t_1 - t_2). \tag{40}
\]
In the WBLA, $\bar{\Sigma}$ takes the same form as $\tilde{\Sigma}$. Details of the derivation of the retarded pseudo-self energy and other pseudo-self energies are given in Appendix D.

4.4 Integration of second-order partial differential equations

4.4.1 Matsubara Green function

With (32), (33) on the vertical part of the contour, we can derive a second-order partial differential equation for the Matsubara Green function $G_{11}^M(\tau_1, \tau_2)$ in the same way as (35),

$$\left[\frac{d^2}{d\tau_1^2} + (\epsilon_1 - \mu + \epsilon_2 + \mu) \frac{d}{d\tau_1} + ((\epsilon_1 - \mu)(\epsilon_2 - \mu) - t_{12}t_{21})\right] G_{11}^M(\tau_1, \tau_2)
= \left(\tilde{\Sigma}_{11}^M * G_{11}^M\right)_{(\tau_1, \tau_2)} - \Sigma_{22}^M(\tau_1, \tau_2) - i(\epsilon_2 - \mu)\delta(\tau_1 - \tau_2) - \frac{d}{d(\tau_1 - \tau_2)}\delta(\tau_1 - \tau_2).$$  (41)

By expanding the Matsubara Green function $G_{11}^M(\tau_1, \tau_2)$, the Matsubara self-energy and $\delta(\tau_1 - \tau_2)$ into the Matsubara sum, we get

$$G_{11}^M(\omega_q) = \begin{cases} 
\frac{\omega_q - \epsilon_{1,\text{eff}}}{\omega_q - (\epsilon_1 M_{,\text{eff}} + \epsilon_2 M_{,\text{eff}}) - t_{12}t_{21}}, & \text{Im}\omega_q > 0, \\
\frac{\omega_q - \epsilon_{1,\text{eff}}}{\omega_q - (\epsilon_1 M_{,\text{eff}} + \epsilon_2 M_{,\text{eff}}) - t_{12}t_{21}}, & \text{Im}\omega_q < 0,
\end{cases}  \quad (42)$$

$$G_{11}^M(\tau_1, \tau_2) = \frac{i}{\beta} \sum_q e^{-\omega_q(\tau_1 - \tau_2)} G_{11}^M(\omega_q),  \quad (43)$$

where $\epsilon_{1,\text{eff}} = \epsilon_1 - \mu - i/2\Gamma_{L,11}$ and $\epsilon_{2,\text{eff}} = \epsilon_2 - \mu - i/2\Gamma_{R,22}$. When we consider the case where the dot2 does not exist and therefore $\epsilon_{2,\text{eff}} = t_{12} = t_{21} = 0$, the Matsubara Green function becomes $G_{11}^M(\omega_q) = \{\omega_q + \epsilon_{1,\text{eff}}\}$, which is the same result as in the single dot case [12]. Other Matsubara Green functions are calculated similarly and the results are

$$G_{22}^M(\omega_q) = \begin{cases} 
\frac{\omega_q - \epsilon_{1,\text{eff}}}{\omega_q - (\epsilon_1 M_{,\text{eff}} + \epsilon_2 M_{,\text{eff}}) - t_{12}t_{21}}, & \text{Im}\omega_q > 0, \\
\frac{\omega_q - \epsilon_{1,\text{eff}}}{\omega_q - (\epsilon_1 M_{,\text{eff}} + \epsilon_2 M_{,\text{eff}}) - t_{12}t_{21}}, & \text{Im}\omega_q < 0,
\end{cases}  \quad (44)$$
\[ G_{12}^M(\omega_q) = \begin{cases} \frac{t_{12}}{\omega_q^2 - (\epsilon_1^{M,eff} + \epsilon_2^{M,eff})\omega_q + (\epsilon_1^{M,eff} \epsilon_2^{M,eff} - t_{12} t_{21})}, & \text{Im} \omega_q > 0, \\ \frac{t_{12}}{\omega_q^2 - ((\epsilon_1^{M,eff})^\dagger + (\epsilon_2^{M,eff})^\dagger)\omega_q + ((\epsilon_1^{M,eff})^\dagger (\epsilon_2^{M,eff})^\dagger - t_{12} t_{21})}, & \text{Im} \omega_q < 0, \end{cases} \]

\[ G_{21}^M(\omega_q) = \begin{cases} \frac{t_{21}}{\omega_q^2 - (\epsilon_1^{M,eff} + \epsilon_2^{M,eff})\omega_q + (\epsilon_1^{M,eff} \epsilon_2^{M,eff} - t_{12} t_{21})}, & \text{Im} \omega_q > 0, \\ \frac{t_{21}}{\omega_q^2 - ((\epsilon_1^{M,eff})^\dagger + (\epsilon_2^{M,eff})^\dagger)\omega_q + ((\epsilon_1^{M,eff})^\dagger (\epsilon_2^{M,eff})^\dagger - t_{12} t_{21})}, & \text{Im} \omega_q < 0. \end{cases} \]

The lead-dot Matsubara Green functions, \( G_{L1}^M \) and \( G_{1L}^M \), are determined from the solutions (30), (31) as

\[ G_{L1}(\tau_1, \tau_2) = (g_{LL}^M * h_{L1}(\tau) G_{11}^M(\tau_1, \tau_2)), \quad (47) \]

\[ G_{1L}(\tau_1, \tau_2) = (G_{11}^M h_{1L}(\tau) * g_{LL}^M)(\tau_1, \tau_2), \quad (48) \]

\[ G_{L2}(\tau_1, \tau_2) = (g_{LL}^M * h_{L1}(\tau) G_{12}^M(\tau_1, \tau_2)), \quad (49) \]

\[ G_{2L}(\tau_1, \tau_2) = (G_{21}^M h_{1L}(\tau) * g_{LL}^M)(\tau_1, \tau_2), \quad (50) \]

where we use the Langreth rule.

### 4.4.2 Retarded/advanced Green function

By differentiating the definition of the retarded/advanced Green functions and using the equations (32), (33) with the modified Langreth rule, we can derive the second-order differential equations for the retarded/advanced Green functions

\[ - \frac{d^2}{dt_1^2} - i(\epsilon_1^{eff} + \epsilon_2^{eff}) \frac{d}{dt_1} + (\epsilon_1^{eff} \epsilon_2^{eff} - t_{12} t_{21}) \] 
\[ \quad \times G_{11}^\alpha(t_1, t_2) 
\quad = i \frac{d}{d(t_1 - t_2)} \delta(t_1 - t_2) - \epsilon_2^{eff} \delta(t_1 - t_2), \quad (51) \]

\[ - \frac{d^2}{dt_1^2} - i(\epsilon_1^{eff})^\dagger + (\epsilon_2^{eff})^\dagger \frac{d}{dt_1} + ((\epsilon_1^{eff})^\dagger(\epsilon_2^{eff})^\dagger - t_{12} t_{21}) \] 
\[ \quad \times G_{11}^\alpha(t_1, t_2) 
\quad = i \frac{d}{d(t_1 - t_2)} \delta(t_1 - t_2) - (\epsilon_2^{eff})^\dagger \delta(t_1 - t_2), \quad (52) \]
where $\epsilon_{1}^{\text{eff}} := \epsilon_{1} - i/2\Gamma_{L,11}$ and $\epsilon_{2}^{\text{eff}} := \epsilon_{2} - i/2\Gamma_{R,22}$ are the effective energies of the dots. In this case, the additional terms which are the second and the third terms in (35) and (36) disappear. The derivations of (51) (52) are in Appendix E.

From now on, we just focus on the equation (51), as we can solve the equation (52) in the same way. General solution of the homogeneous equation (51) is a linear combination of $e^{-ik_{1}(t_{1}-t_{2})}$ and $e^{-ik_{2}(t_{1}-t_{2})}$, where $k_{1}$ and $k_{2}$ are the solutions of the characteristic equation and are defined as

$$k_{1} = p - \sqrt{p^{2} - q}, \quad k_{2} = p + \sqrt{p^{2} - q},$$

$$p = \frac{\epsilon_{1}^{\text{eff}} + \epsilon_{2}^{\text{eff}}}{2}, \quad q = \epsilon_{1}^{\text{eff}}\epsilon_{2}^{\text{eff}} - t_{12}t_{21}.$$ 

We assume $G_{11}^{r}(t_{1}, t_{2})$ takes the following form:

$$G_{11}^{r}(t_{1}, t_{2}) = -i\theta(t_{1} - t_{2}) \left( C_{1} e^{-ik_{1}(t_{1}-t_{2})} + C_{2} e^{-ik_{2}(t_{1}-t_{2})} \right). \quad (53)$$

By substituting (53) into the equation (51), we can confirm that the function is actually a particular solution when $C_{1}$ and $C_{2}$ satisfy the conditions

$$C_{1} = \frac{k_{2} - \epsilon_{1}^{\text{eff}}}{k_{2} - k_{1}}, \quad C_{2} = -\frac{k_{1} - \epsilon_{1}^{\text{eff}}}{k_{2} - k_{1}}.$$ 

When we consider the case where there are only the dot1 and the reservoir L and therefore $\epsilon_{2} = t_{12} = t_{21} = 0$, the equation (51) takes the following form:

$$\left[ -\frac{d^{2}}{dt_{1}^{2}} - i\epsilon_{1}^{\text{eff}} \frac{d}{dt_{1}} \right] G_{11}^{r}(t_{1}, t_{2}) = i\frac{d}{d(t_{1} - t_{2})}\delta(t_{1} - t_{2}).$$

In this case, the solution is $G_{11}^{r}(t_{1}, t_{2}) = -i\theta(t_{1} - t_{2}) e^{-i\epsilon_{1}^{\text{eff}}(t_{1}-t_{2})}$, which is the same expression for the single dot case [12]. Similarly, we get the solution of the equation (52) expressed as

$$G_{11}^{a}(t_{1}, t_{2}) = i\theta(t_{2} - t_{1}) \left( C_{1}^{\dagger} e^{-ik_{1}^{\dagger}(t_{1}-t_{2})} + C_{2}^{\dagger} e^{-ik_{2}^{\dagger}(t_{1}-t_{2})} \right). \quad (54)$$

### 4.4.3 Right/left Green function

By differentiating the definition of the right/left Green functions of the dot1 and using the equations (32) (33) and the modified Langreth rule, we can de-
rive the second-order differential equations of the right/left Green functions,

\[
\left[-\frac{d^2}{dt^2} - i(\epsilon_1^{eff} + \epsilon_2^{eff}) \frac{d}{dt} + (\epsilon_1^{eff} \epsilon_2^{eff} - t_{12} t_{21})\right] G_{11}^\dagger(t, \tau) = \left(\tilde{\Sigma}_{11}^\dagger * G_{11}^{M}\right)_{(\tau, t)} + \Sigma_{22}^\dagger(t, \tau),
\]

(55)

\[
G_{11}^\dagger(\tau, t) \left[-\frac{d^2}{dt^2} + i((\epsilon_1^{eff})^\dagger + (\epsilon_2^{eff})^\dagger) \frac{d}{dt} + ((\epsilon_1^{eff})^\dagger(\epsilon_2^{eff})^\dagger - t_{12} t_{21})\right] = \left(G_{11}^M * \Sigma_{11}^\dagger\right)_{(\tau, t)} + \Sigma_{22}^\dagger(\tau, t).
\]

(56)

With the knowledge of the usual second-order differential equation, the equation (55) is solved with boundary conditions: \(G_{11}^\dagger(t_0, \tau) = G_{11}^{M}(0^+, \tau)\) and \(i \frac{d}{dt} G_{11}^\dagger(t, \tau)|_{t=t_0} = \epsilon_1 G_{11}^{M}(0^+, \tau) + t_{12} G_{21}^{M}(0^+, \tau)\) as

\[
G_{11}^\dagger(t, \tau) = b_1(\tau)e^{-ik_1(t-t_0)} + b_2(\tau)e^{-ik_2(t-t_0)}
\]

\[\left.+ \frac{i}{k_2 - k_1} \int_{t_0}^{t} ds \left\{ \left(\tilde{\Sigma}_{11}^\dagger * G_{11}^{M}\right)_{(s, \tau)} - \Sigma_{22}^\dagger(s, \tau) \right\} (e^{-i k_1(t-s)} - e^{-i k_2(t-s)}) \right\},
\]

(57)

where

\[
b_1(\tau) = -\frac{1}{k_2 - k_1} \{(\epsilon_1 - k_2)G_{11}^{M}(0^+, \tau) + \Sigma_{11}^M * G_{11}^{M}(0^+, \tau) + t_{12} G_{12}^{M}(0^+, \tau)\},
\]

\[
b_2(\tau) = \frac{1}{k_2 - k_1} \{(\epsilon_1 - k_1)G_{11}^{M}(0^+, \tau) + \Sigma_{11}^M * G_{11}^{M}(0^+, \tau) + t_{12} G_{12}^{M}(0^+, \tau)\}.
\]

We can calculate other right and left Green functions similarly and the results are

\[
G_{11}^\dagger(\tau, t) = c_1(\tau)e^{i(k_1)^\dagger(t-t_0)} + c_2(\tau)e^{i(k_2)^\dagger(t-t_0)}
\]

\[\left.- \frac{i}{(k_2 - k_1)^\dagger} \int_{t_0}^{t} ds \left\{ \left(G_{11}^{M} * \tilde{\Sigma}_{11}^\dagger\right)_{(s, \tau)} - \Sigma_{22}^\dagger(s, \tau) \right\} (e^{i(k_1)^\dagger(t-s)} - e^{i(k_2)^\dagger(t-s)}) \right\},
\]

(58)

\[
c_1(\tau) = -\frac{1}{(k_2 - k_1)^\dagger} \{(\epsilon_1 - k_2)^\dagger G_{11}^{M}(\tau, 0^+) + (G_{11}^{M} * \Sigma_{11}^M)_{(\tau, 0^+)} + G_{12}^{M}(\tau, 0^+) t_{21}\},
\]

\[
c_2(\tau) = \frac{1}{(k_2 - k_1)^\dagger} \{(\epsilon_1 - k_1)^\dagger G_{11}^{M}(\tau, 0^+) + (G_{11}^{M} * \Sigma_{11}^M)_{(\tau, 0^+)} + G_{12}^{M}(\tau, 0^+) t_{21}\},
\]
\[ G_{12}^1(t, \tau) = d_1(\tau)e^{-ik_1(t-t_0)} + d_2(\tau)e^{-ik_2(t-t_0)} \]
\[ + \frac{i}{k_2 - k_1} \int_{t_0}^{t} ds \left\{ (\Sigma_{11}^M * G_{12}^M(s, \tau)) (e^{-ik_1(t-s)} - e^{-ik_2(t-s)}) \right\}, \quad (59) \]

\[ d_1(\tau) = -\frac{1}{k_2 - k_1} \{ (\epsilon_1 - k_2)G_{12}^M(0^+, \tau) + (\Sigma_{11}^M * G_{12}^M)(0^+, \tau) + t_{12}G_{22}^M(0^+, \tau) \}, \]
\[ d_2(\tau) = \frac{1}{k_2 - k_1} \{ (\epsilon_1 - k_1)G_{12}^M(0^+, \tau) + (\Sigma_{11}^M * G_{12}^M)(0^+, \tau) + t_{12}G_{22}^M(0^+, \tau) \}. \]

The right Green functions between the lead and the system are determined from the solutions (30), (31) as
\[ G_{1L}^r(t, \tau) = (G_{11}^r \cdot h_{1L}g_{LL}^1 + G_{11}^r * h_{1L}g_{LL}^M)(t, \tau), \quad (60) \]
\[ G_{L1}^r(t, \tau) = (g_{LL}^1h_{1L} \cdot G_{11}^r + g_{LL}^1h_{1L} * G_{11}^r)(t, \tau). \quad (61) \]

### 4.4.4 Lesser Green function

The second-order differential equation for the lesser Green function is written in the following form:
\[ G_{11}^{\leq}(t_1, t_2) \left[ -\frac{d^2}{dt_2^2} + i((\epsilon_1^{\text{eff}})^\dagger + (\epsilon_2^{\text{eff}})^\dagger) \frac{d}{dt_2} + ((\epsilon_1^{\text{eff}})^\dagger(\epsilon_2^{\text{eff}})^\dagger - t_{12}t_{21}) \right] \]
\[ = \left( G_{11}^r \cdot \Sigma_{11}^c + G_{11}^r * \Sigma_{11}^l \right)_{(t_1, t_2)} - iG_{11}^r(t_1, t_0)\Sigma_{22}^{\leq}(t_0, t_2) + \Sigma_{22}^{\leq}(t_1, t_2), \quad (62) \]
where we apply the modified Langreth rule to (35). We can solve this equation in the same way as for the right/left Green functions and the solution is expressed as
\[ G_{11}^{\leq}(t_1, t_2) = l_1(t_1)e^{ik_1^1(t_2-t_0)} + l_2(t_1)e^{ik_2^1(t_2-t_0)} - \frac{i}{(k_2 - k_1)^\dagger} \int_{t_0}^{t_2} ds \]
\[ \times \{ (G_{11}^r \cdot \Sigma_{11}^c + G_{11}^r * \Sigma_{11}^l)_{(t_1, s)} - iG_{11}^r(t_1, t_0)\Sigma_{22}^{\leq}(t_0, s) + \Sigma_{22}^{\leq}(t_1, s) \} (e^{ik_1^1(t_2-s)} - e^{ik_2^1(t_2-s)}). \quad (63) \]

These coefficients \( l_1(t), l_2(t) \) are determined from two boundary conditions
\[ G_{11}^{\leq}(t_1, t_0) = G_{11}^1(t_1, 0^+) \] and \( G_{11}^{\leq}(t_1, t_2) \big|_{t_2=t_0} = \epsilon_1 G_{11}^1(t_1, 0^+) + G_{1L}^1(t_1, 0^+) \).
\[ h_{L1} + G_{12}^1(t_1, 0^+)t_{21}, \]

\[ l_1(t) = -\frac{1}{(k_2 - k_1)^{\dagger}} \{ (\epsilon_1 - k_2^1)G_{11}^1(t, 0^+) + \left( G_{11}^r \cdot \Sigma_{11}^1 + G_{11}^1 \ast \Sigma_{11}^M \right)_{(t, 0^+)} + G_{12}^1(t, 0^+)t_{21} \}, \]

\[ l_2(t) = \frac{1}{(k_2 - k_1)^{\dagger}} \{ (\epsilon_1 - k_1^1)G_{11}^1(t, 0^+) + \left( G_{11}^r \cdot \Sigma_{11}^1 + G_{11}^1 \ast \Sigma_{11}^M \right)_{(t, 0^+)} + G_{12}^1(t, 0^+)t_{21} \}, \]

where we use the expression \( G_{1L}^1(t, \tau) \cdot h_{L1} = \left( G_{11}^r \cdot \Sigma_{11}^1 + G_{11}^1 \ast \Sigma_{11}^M \right)_{(t, \tau)} \), which is derived from (60) and the definition of the self-energy.

### 4.5 Calculation of physical quantities

In the last subsection 4.4, we obtained the concrete expressions of the non-equilibrium Green functions. In this section, we explain how the physical quantities are expressed with the non-equilibrium Green functions. A definition of the electron density at the dot \( i \) is \( \rho_i(t) := \langle d^\dagger_{H,i}(t)d_{H,i}(t) \rangle \). With the non-equilibrium Green function, the density is expressed as

\[ \rho_i(t) = -iG_{ii}^<(t, t), \]

where we use the definition of the lesser Green function \( G_{ii}^<(t, t') = i \langle d^\dagger_{H,i}(t)d_{H,i}(t) \rangle \). This relation says that the lesser Green function is directly relevant to the electron density. We define the current from the dot \( i \)

\[ J_i(t) := -\frac{d}{dt}\rho_i(t), \quad (64) \]

and from the reservoir \( \alpha \) as

\[ J_\alpha(t) := -\frac{d}{dt}\rho_\alpha(t), \]

where we define the electron density of the reservoir in the same way as that of the dot: \( \rho_\alpha(t) := \sum_{k\alpha} \langle d^\dagger_{H,\alpha}(t)d_{H,\alpha}(t) \rangle \). We can find a representation of the current from the left reservoir in terms of the non-equilibrium Green function,

\[ J_L(t) = -\frac{d}{dt}\rho_L(t) \]

\[ = -2\text{Re}[G_{1L}^< h_{L1}], \]
where we use the Heisenberg equation of the Hamiltonian \([18]\) for the derivative of the operators. By substituting the expression of the lesser Green function \([61]\) and using the Langreth rule, we can express the current in terms of the Green function:

\[
J_L(t) = -2\Re \left[ (\Sigma^\dagger_{11} \cdot G_{11}^\dagger)(t,t) + (\Sigma^r_{11} \cdot G_{11}^r)(t,t) + (\Sigma^\omega_{11} \cdot G_{11}^\omega)(t,t) \right] \tag{65}
\]

For discussions below, we represent the three terms \(-2\Re (\Sigma^\dagger_{11} \cdot G_{11}^\dagger)(t,t)\], \(-2\Re (\Sigma^r_{11} \cdot G_{11}^r)(t,t)\], and \(-2\Re (\Sigma^\omega_{11} \cdot G_{11}^\omega)(t,t)\] as \(J_{L1}(t)\), \(J_{L2}(t)\), and \(J_{L3}(t)\) respectively.

We can obtain the current between the dots as follows. Because the particles in the left reservoir only move to the dot1, the current between the left reservoir and the dot1 equal to the current from the left reservoir: \(J_L(t) = J_{L\rightarrow 1}(t)\). The current from the dot1 equals the sum of the current from the dot1 to the reservoir \(L\) and the dot2: \(J_1(t) = J_{1\rightarrow L} + J_{1\rightarrow 2}\). Therefore, by summing these two relations, the current from the dot1 to the dot2 is expressed as

\[
J_{1\rightarrow 2}(t) = J_1(t) + J_L(t),
\]

where we use \(J_{L\rightarrow 1}(t) = -J_{1\rightarrow L}(t)\). With this relation, we can get an expression of the current between the dots.

5 Numerical results

With the expressions obtained in subsection 4.5, we calculate the electron density of the dot1 and the current between the dots numerically. Here, we consider the quenched case, where a constant bias voltage is applied to the left reservoir \(V_L(t) = V_L\) suddenly at an initial time \(t_0\) and \(V_R = 0\). We consider the two cases: \(V_L = 4\) as a high bias voltage and \(V_L = 2\) as a low bias voltage. In both cases, the inverse temperature is taken to be \(\beta = 1\). The energy levels of the dots are \(\epsilon_1 = \epsilon_2 = 1\). The energies of the two subsystems must take the same value because of energy conservation \([26]\). The couplings between the subsystems and the reservoirs are symmetric \(\Gamma_L = \Gamma_R = 1/2\) and the couplings between the subsystems are \(t_{12} = t_{21} = 1\). Throughout our numerical computations, we use a representation of a physical quantity that is expressed in terms of the integration with respect to frequency \(\omega\) such as \([17]\) to reduce computational cost. Details of this rearrangement for the
electron density is given in Appendix F. The current between the dots is rewritten in the same way. Results are shown in figures from Fig.5 to Fig.7. Fig.5 shows the electron density of the dot1 and Fig.6, 7 show the currents between the dots. First we see the graph of the electron density of the dot1 in Fig.5. The electron density oscillates in our DQD case. This cannot be

\[ V_L = 4 \]

steady for

\[ V_L = 2 \]

steady for

\[
\begin{align*}
\rho_1(t) & \quad V_L = 4 \\
\rho_1(t) & \quad steady \ for \ V_L = 4 \\
\rho_1(t) & \quad V_L = 2 \\
\rho_1(t) & \quad steady \ for \ V_L = 2 \\
\end{align*}
\]

Fig. 5: the electron density of the dot1 and time for \( V_L = 2 \) and \( V_L = 4 \). The straight line is the steady state value of the electron density that is calculated from the analytical expression of the density.

seen in the case of the single dot (Fig.2). Since the electron density is directly relevant to the lesser Green function \( \rho_{11}(t) = -i G_{11}^<(t,t) \), the cause of the difference between the cases is the different behavior of the lesser Green function. In the single dot case, the expression of the lesser Green function is (17). If we take these two times to be the same \( t_1 = t_2 = t \), then the imaginary part of the exponential \( e^{-i k_1(t_1-t_0)} \) vanishes because of another factor \( e^{i(k_1^+t_1(t_2-t_0))} \). Therefore, the oscillation does not appear in the single dot case. In the double dots case, the expression of the lesser Green function is (63). Unlike the single dot case, there are two kinds of exponentials which have the different exponents \( k_1 \) and \( k_2 \). Therefore, the imaginary parts of these exponential functions do not vanish and the oscillation appear.

Next we consider the current between the dots. Depending on the bias voltage, the current behaves differently. For the low bias case of \( V_L = 2 \), we can see that there is a single crest in a period (Fig.6). In contrast, for the high bias case of \( V_L = 4 \), there are two crests in a period (Fig.7). To understand
the origin of this difference, we investigate the current from the left reservoir \( J_L(t) \) and the dot1 \( J_1(t) \), because the current between the dots are the sum of these two quantities. Their behavior is shown in Fig.8 and Fig.9. From these figures, we can understand that the dependence of the current between the dots to the bias voltage arises due to the dependence of the current from the left reservoir. As we will mention later in the discussion of the relaxation time, the two crests in a period of the current from the left reservoir appear due to the existence of \( k_1 \) and \( k_2 \), which are the solutions of the characteristic equation of the second-order differential equation for the retarded Green function \[ \text{(51)}. \] Since the necessity to investigate the second-order partial differential equations arises due to the existence of the coupling between the dots, we conclude that the two wave crests in the current between the dots is a manifestation of the fact that the system consists of two subsystems.

We also investigate the relaxation time. When a physical quantity \( A(t) \) behaves as \( A(t) \approx e^{-\frac{t}{\tau}} \), \( \tau \) is called the relaxation time. The notation \( \approx \) means that we ignore algebraic functions multiplied to the exponential. From the expression \[ \text{(63)}, \]
we can see that the relaxation time is mainly determined by the imaginary parts of \( k_1 \) or \( k_2 \). For the case \( \epsilon_1 = \epsilon_2 = \epsilon \), these exponents take the following forms:

\[
\begin{align*}
k_1 &= \epsilon - \frac{\Gamma_L + \Gamma_R}{4} i - \sqrt{|t_{12}|^2 - \frac{1}{16} (\Gamma_L - \Gamma_R)^2}, \\
k_2 &= \epsilon - \frac{\Gamma_L + \Gamma_R}{4} i + \sqrt{|t_{12}|^2 - \frac{1}{16} (\Gamma_L - \Gamma_R)^2}.
\end{align*}
\]

The expressions of these imaginary parts can change, because the inside of
the square root can be both negative or positive. When we consider the case $|t_{12}| < 1/4|\Gamma_L - \Gamma_R|$, these imaginary parts are written as $\text{Im}k_1 = -(\Gamma_L + \Gamma_R + \sqrt{\Gamma_L - \Gamma_R})^2 - 16|t_{12}|^2)/4$ and $\text{Im}k_2 = -(\Gamma_L - \Gamma_R - \sqrt{\Gamma_L - \Gamma_R})^2 - 16|t_{12}|^2)/4$. In this case, the relaxation time $\tau$ is about $-1/\text{Im}k_2 = 4/(\Gamma_L + \Gamma_R - \sqrt{\Gamma_L - \Gamma_R})^2 - 16|t_{12}|^2)$. The relaxation time depends on the coupling strength between the dots $|t_{12}|$ and it becomes smaller as $|t_{12}|$ gets larger. This is intuitively a natural result. For the case $|t_{12}| > 1/4|\Gamma_L - \Gamma_R|$, however, we can see an interesting fact. In this case, these imaginary parts take the same form as $\text{Im}k_1 = \text{Im}k_2 = -(\Gamma_L + \Gamma_R)/4$ and $\tau$ is about $4/(\Gamma_L + \Gamma_R)$. This expression tells us that the relaxation time $\tau$ does not depend on $|t_{12}|$. This is against our intuition. Since the coupling strength between the dots determines the probability that an electron in a dot is transported to the other, we can expect that the relaxation time becomes smaller as the coupling strength between the dots gets larger. To test this fact, we calculate the relaxation time from numerical results and compare it with the theoretically expected value. We consider the case where the parameters take the values: $\Gamma_L = \Gamma_R = 1/2$, $\epsilon = 1$, $V_L = 4$ and $\beta = 1$. In this situation, the relaxation time does not depend on the coupling strength $|t_{12}| > 0$. In our analysis, we obtain the relaxation time numerically by fitting the data of the first term of RHS in (65), $J_{L1}$, to exponential function. This is because the other terms converge to the value of steady state faster than the first term and therefore the first term determines the relaxation time. We can see this fact in Fig.10. We calculate the relaxation time for various coupling strength of the dots $|t_{12}|$, 0.5 $\leq |t_{12}| \leq 2$. The result is shown in Fig.11. In this case,
the theoretical value is $\tau = 0.25$. Therefore, we conclude that the relaxation time is actually constant and match the theoretical value. This phenomenon can be understood intuitively as follows. Let us fix the parameters, $\Gamma_L$ and $\Gamma_R$, to satisfy $\Gamma_L > \Gamma_R$. We assume the electrons are transported from the left reservoir to the right. In this situation, we change the coupling constants between the dots. Here we name the average time during which a particle needs to move from a place to its right the waiting time. We note that the coupling constants represent the easiness of moving and therefore determine the waiting time of each place. For example, the coupling constant between the dots determines the waiting time of the left dot. Let us consider the two limiting cases: $|t_{12}| \ll 1/4|\Gamma_L - \Gamma_R|$ and $|t_{12}| \gg 1/4|\Gamma_L - \Gamma_R|$. In the case $|t_{12}| \ll 1/4|\Gamma_L - \Gamma_R|$, the waiting time of the left dot will be much longer than that of the left reservoir or the right dot. Therefore, the relaxation time becomes shorter if we make the coupling constants between dots bigger and the waiting time of the left dot shorter. In this way, the relaxation time depends on the coupling strength between the dots in the case $|t_{12}| \ll 1/4|\Gamma_L - \Gamma_R|$. For the case $|t_{12}| \gg 1/4|\Gamma_L - \Gamma_R|$, the situation is different. Of course, if we make the coupling strength between the dots bigger, the waiting time in the left dot becomes shorter. However, this does not affect the relaxation time. This is because the waiting time of the left reservoir or the right dot is much longer than that of the left dot and therefore moving of a particle from the left dot is usually blocked by a particle in the right dot. In this manner, we
can understand that the relaxation time does not depend on the coupling constant between the dots in the regime $|t_{12}| \gg 1/4|\Gamma_L - \Gamma_R|$. The value $|t_{12}| = 1/4|\Gamma_L - \Gamma_R|$ tells us the transition point.

We also calculate the relaxation time for $\beta = 100$ as a low temperature case and the result is shown in Fig. 12. The relaxation time takes about the theoretical value only in the vicinity of $|t_{12}| = 2$, and it is larger than the theoretical value in the most of the regime $0.5 \leq |t_{12}| < 2$. To understand this fact deeply, we rewrite the expression of the current $J_{L1}(t)$ into the form which is convenient to see the dependence of parameters in the same way as the rearrangement of the electron density in Appendix F. Consequently, the current $J_{L1}(t)$ is decomposed into two terms: $J_{L1}(t) = (C_{L1,a1}(t) + C_{L1,b}(t))e^{i(k_1)t} + (C_{L1,a2}(t) + C_{L1,b}(t))e^{i(k_2)t}$. From this expression, we can conclude that the difference between numerical and theoretical values of the relaxation time is due to the coefficients $C_{L1,a1}$, $C_{L1,a2}$, and $C_{L1,b}$. We numerically calculate these coefficients and the results are shown in figures from Fig. 13 to Fig. 16. Because the coefficients $C_{L1,a1}$ and $C_{L1,a2}$ take the almost same expression, we only consider $C_{L1,a1}$.

From the graphs in Fig. 14 and Fig. 16 we figure out that the periods of the coefficients $\text{Im}(C_{L1,a1})$ and $\text{Im}(C_{L1,b})$ are not influenced by the coupling constant. From the graph in Fig. 15, we see the amplitude of $\text{Re}(C_{L1,b})$ takes a larger value as the coupling constant gets larger, but its frequency does not change. Hence, we can conclude that these coefficients do not affect the relaxation time. By contrast, the coefficient $\text{Re}(C_{L1,a1})$ behaves differently.
From Fig. 13, we see that its period becomes shorter as the coupling constant gets larger. In other words, \( \text{Re} (C_{L1,a1}) \) converges to the steady state value faster as the constant gets larger. Therefore, the influence of these coefficients to the relaxation time \( \tau \) becomes smaller as the coupling constant \( |t_{12}| \) gets smaller. This explains why the relaxation time gradually moves toward the theoretical value \( \tau = 2.5 \) depending on the coupling constant \( |t_{12}| \).

6 Conclusion

We have investigated the transient dynamics of the double quantum dots with the non-equilibrium Green function method in the wide band limit approximation. As a result, we have obtained the analytic expressions of the electron density of a dot and the current between the dots. Based on these results, we have calculated the physical quantities numerically for the quenched case, where a constant bias voltage is applied to a reservoir suddenly. From the numerical computation, we have found that the quantum fluctuation appears in both the current and the electron density. Especially, the qualitative behavior of current between the dots has changed depending on the bias voltage. For the case of a low bias voltage, there is one crest in a period of the oscillation in the current. However, for the case of a high bias voltage, the number of crests in a period changes to two. This difference arises due to the existence of the coupling strength between the dots. In addition, we have calculated the relaxation time and found that the relaxation time becomes
constant when the coupling strength between the dots are large enough. This fact tells us that if a system is an open system, we have to take the effects from the edges into consideration even when we consider an quantity in the bulk.

Until now, we only consider the regime where the Coulomb interaction is irrelevant. Therefore, our next goal is to investigate the transient dynamics of the double quantum dots including the Coulomb interaction. Since many interesting studies have been reported even for the steady case, we can expect that much richer phenomena arise due to the competence of the initial correlation effects and the Coulomb interaction in the transient dynamics of the double quantum dots.

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Appendix

A. Non-perturbative Green function and the self-energy in the WBLA

Non-perturbative Green function

The non-perturbative Green function obeys the equation of motion:

\[
\left[i \frac{d}{dz_1} - h_{\alpha\alpha}(z_1)\right] g_{\alpha\alpha}(z_1, z_2) = \delta(z_1, z_2).
\]

This means that the non-perturbative Green function is the Green function for the case where the reservoir \( \alpha \) evolves independently, or with no interaction. Then we consider the expression of the operator \( d_{k\alpha}(z) \) under the Hamiltonian \( h_{\alpha\alpha} \). Because the time-dependent part of the Hamiltonian \( V_\alpha(t)d_{k\alpha}^\dagger d_{k\alpha} \) and the rest part \( \epsilon_{k\alpha}d_{k\alpha}^\dagger d_{k\alpha} \) are interchangeable, the operator \( d_{k\alpha}(z) \) for \( z \in C_\pm \) is represented as

\[
d_{k\alpha}(t) = d_{k\alpha}e^{-i\phi_{k\alpha}(t,t_0)} = (d_{k\alpha}^\dagger(t))^\dagger,
\]

where we define the function \( \phi_{k\alpha}(t,t_0) = \epsilon_{k\alpha}(t - t_0) + \int_{t_0}^t ds V_\alpha(s) \) for convenience. In the same way, the operator \( d_{k\alpha}(z) \) for \( z = t_0 - i\tau \in C_M \) is expressed as

\[
d_{k\alpha}(t_0 - i\tau) = d_{k\alpha}e^{-(\epsilon_{k\alpha} - \mu)\tau}, \quad d_{k\alpha}^\dagger(t_0 - i\tau) = d_{k\alpha}e^{(\epsilon_{k\alpha} - \mu)\tau}.
\]

We note the fact \( (d_{k\alpha}(t_0 - i\tau))^\dagger \neq d_{k\alpha}^\dagger(t_0 - i\tau) \). From these operators, the non-perturbative Green functions are calculated as

\[
\begin{align*}
[g_{\alpha\alpha}^{\uparrow}(t_1, t_2)]_{k,k'} & = -i\theta(t_1 - t_2)\delta_{kk'}e^{-i\phi_{k\alpha}(t_1, t_2)}, \quad (A.1) \\
[g_{\alpha\alpha}^{\downarrow}(t_1, t_2)]_{k,k'} & = i\theta(t_2 - t_1)\delta_{kk'}e^{-i\phi_{k\alpha}(t_1, t_2)}, \quad (A.2) \\
[g_{\alpha\alpha}^{>}(t_1, t_2)]_{k,k'} & = -i\delta_{kk'}[1 - f(\epsilon_{k\alpha} - \mu)]e^{-i\phi_{k\alpha}(t_1, t_2)}, \quad (A.3) \\
[g_{\alpha\alpha}^{<}(t_1, t_2)]_{k,k'} & = i\delta_{kk'}f(\epsilon_{k\alpha} - \mu)e^{-i\phi_{k\alpha}(t_1, t_2)}. \quad (A.4)
\end{align*}
\]
For the Matsubara component, it is useful to use the representation of the Matsubara sum, which arises from the boundary conditions $g_{aa}(z_1, t_0) = -g_{aa}(z_1, \beta)$, $g_{aa}(t_0, z_2) = -g_{aa}(\beta, z_2)$,

$$[g^M_{aa}(\tau_1, \tau_2)]_{k,k'} = \delta_{kk'} \frac{i}{\beta} \sum_q \frac{e^{-\omega_q (\tau_1 - \tau_2)}}{\omega_q - \epsilon_{ka} + \mu}, \quad (A.5)$$

where $\omega_q = i\pi(2q + 1)/\beta$ is the Matsubara frequency and the summation about $q$ runs over all integers. A derivation is as follows. From the definition,

$$[g^M_{aa}(\tau_1, \tau_2)]_{kk'} = -i \langle T_\tau (d^M_{ka}(\tau_1)(d^\dagger_{k'\alpha})^M) \rangle$$

$$= -i [\theta(\tau_1 - \tau_2)(1 - f(\epsilon_{ka} - \mu)) - \theta(\tau_2 - \tau_1)f(\epsilon_{ka} - \mu)] e^{-\epsilon_{ka} - \mu(\tau_1 - \tau_2)} \delta_{kk'}$$

(\because \text{the expectation is taken in the equilibrium state})

$$= -i [\theta(\tau_1 - \tau_2)(1 - f(\epsilon_{ka} - \mu)) - \theta(-\tau_1)f(\epsilon_{ka} - \mu)] e^{-\epsilon_{ka} - \mu(\tau_1 - \tau_2)} \delta_{kk'}.$$

Then we use these expressions in the Fourier transform about the Matsubara frequency:

$$\frac{1}{i} \int_0^\beta d\tau \theta(\tau)e^{(\omega_q - \epsilon_{ka} + \mu)\tau} = \frac{1}{\omega_q - \epsilon_{ka} + \mu},$$

$$\frac{1}{i} \int_0^\beta d\tau \theta(-\tau)e^{(\omega_q - \epsilon_{ka} + \mu)\tau} = 0.$$

By substituting these expressions in the definition, we get

$$[g^M_{aa}(\tau_1, \tau_2)]_{kk'} = \frac{1}{\beta} \sum_q \frac{e^{-\omega_q (\tau_1 - \tau_2)}(1 - f(\epsilon_{ka} - \mu)) (1 + e^{-\beta(\epsilon_{ka} - \mu)})}{\omega_q - \epsilon_{ka} + \mu} \delta_{kk'}$$

$$= \frac{i}{\beta} \sum_q \frac{e^{-\omega_q (\tau_1 - \tau_2)}}{\omega_q - \epsilon_{ka} + \mu}.$$

The right/left non-perturbative Green functions are calculated with these results,

$$[g_{aa}(t, \tau)]_{k,k'} = [g_{aa}(0, \tau)]_{k,k'} e^{-i\phi_{ka}(t, \tau_0)} = \delta_{kk'} \frac{i}{\beta} \sum_q \frac{e^{\omega_q \tau}}{\omega_q - \epsilon_{ka} + \mu}, \quad (A.6)$$

$$[g^M_{aa}(\tau, t)]_{k,k'} = [g^M_{aa}(\tau, 0)]_{k,k'} e^{i\phi_{ka}(t, \tau_0)} = \delta_{kk'} \frac{i}{\beta} \sum_q \frac{e^{-\omega_q \tau}}{\omega_q - \epsilon_{ka} + \mu}. \quad (A.7)$$

These are the same as in [12].
The self-energy in the WBLA

From the definition of the self-energy \(34\) and the expressions of the non-perturbative Green function obtained above, we can calculate the self-energy. For the retarded part, by the Fourier transform, we get

\[
\Sigma_{11}^r(t_1, t_2) = \theta(t_1 - t_2) \left[ \Sigma_{11}^{\tau}(t_1, t_2) - \Sigma_{11}^{\tau >}(t_1, t_2) \right] \\
= \theta(t_1 - t_2) \sum_{k,k'} T_{1,kL} \left( \left[ g_{LL}^{>}(t_1, t_2) \right]_{k,k'} - \left[ g_{LL}^{<}(t_1, t_2) \right]_{k,k'} \right) T_{k'L,1} \quad (\because \text{By definition} (34)) \\
= \theta(t_1 - t_2) \sum_k T_{1,kL} \left( -i e^{-i\psi_L(t_1,t_2)} \right) T_{kL,1} \quad (\because (B.3) \text{ and } (B.4)) \\
= -i e^{-i\psi_L(t_1,t_2)} \sum_k T_{1,kL} T_{kL,1} \theta(t_1 - t_2) e^{-i\epsilon_{kL}(t_1-t_2)} \\
= e^{-i\psi_L(t_1,t_2)} \int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} \sum_k \frac{T_{1,kL} T_{kL,1}}{\omega + i\theta - \epsilon_{kL}} \quad (\because \text{From the formula below}) \\
= e^{-i\psi_L(t_1,t_2)} \int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} \left[ \Lambda_L(\omega) - \frac{i}{2} \Gamma_L(\omega) \right] \quad (\because \frac{1}{x \pm i\theta} = \mathcal{P} \left( \frac{1}{x} \right) \mp i\pi \delta(x),)
\]

where we define

\[
\Lambda_L(\omega) = \mathcal{P} \int \frac{d\omega'}{2\pi} \frac{\Gamma_L(\omega')}{\omega - \omega'}, \\
\Gamma_L(\omega) = 2\pi \sum_k T_{1,kL} T_{kL,1} \delta(\epsilon_{kL} - \omega),
\]

and \(\mathcal{P}\) represents the Cauchy principal part. In the derivation, we use the Fourier transformation which is defined as \(F(\omega) = \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}\). The inverse Fourier transformation is expressed as \(f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} F(\omega) e^{-i\omega t}\). In this
definition, the Fourier transformation of \( \theta(t)e^{-iat}, a \in \mathbb{R} \) is expressed as
\[
\int_{-\infty}^{\infty} dt \theta(t)e^{-iat} = \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} dt \theta(t)e^{-(\delta - i(\omega - a))t} (\because \omega \to \omega + i\delta) \\
= \lim_{\delta \to 0^+} \int_{0}^{\infty} dt e^{-\delta t + i(\omega - a)t} \\
= \lim_{\delta \to 0^+} \left[ \frac{1}{-\delta + i(\omega - a)} e^{-(\delta t + i(\omega - a)t)} \right]_{0}^{\infty} \\
= \lim_{\delta \to 0^+} \frac{1}{-\delta + i(\omega - a)} \\
= \lim_{\delta \to 0^+} \frac{1}{i \omega + i\delta - a} \\
= -\frac{1}{i \omega + i\delta - a}.
\]

Therefore, we get the formula
\[
\theta(t_1-t_2)e^{-ic_{kL}(t_1-t_2)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} \frac{e^{-i\delta \omega}}{\omega + i\delta - c_{kL}}.
\]

In the WBLA, the line width \( \Gamma_L(\omega) \) does not depend on the energy \( \omega \) and \( \Lambda_L(\omega) \) becomes 0: \( \Gamma_L(\omega) \sim \Gamma_L, \Lambda_L = 0. \) Then, the retarded self-energy in the WBLA takes the form:
\[
\Sigma^{r}_{11}(t_1, t_2) = -\frac{i}{2} \Gamma_{L,11} \delta(t_1 - t_2),
\]
which is \([39]\). Other self-energies are calculated similarly and the results are
\[
\Sigma^{M}_{11}(\tau_1, \tau_2) = -\frac{\Gamma_{L,11}}{2\beta} \sum_{q} \xi_{q} e^{-\omega_{q}(\tau_1 - \tau_2)}, \quad (A.8)
\]
\[
\Sigma^{M}_{22}(\tau_1, \tau_2) = -\frac{\Gamma_{R,22}}{2\beta} \sum_{q} \xi_{q} e^{-\omega_{q}(\tau_1 - \tau_2)}, \quad (A.9)
\]
\[
\Sigma^{<}_{11}(t_1, t_2) = i\Gamma_{L,11} e^{-i\psi_L(t_1,t_2)} \int \frac{d\omega}{2\pi} f(\omega - \mu) e^{-i\omega(t_1-t_2)}, \quad (A.10)
\]
\[
\Sigma^{<}_{22}(t_1, t_2) = i\Gamma_{R,22} e^{-i\psi_R(t_1,t_2)} \int \frac{d\omega}{2\pi} f(\omega - \mu) e^{-i\omega(t_1-t_2)}, \quad (A.11)
\]
\[
\Sigma^{>}_{11}(t_1, t_2) = -i\Gamma_{L,11} e^{-i\psi_L(t_1,t_2)} \int \frac{d\omega}{2\pi} [1 - f(\omega - \mu)] e^{-i\omega(t_1-t_2)}, \quad (A.12)
\]
\[
\Sigma^{>}_{22}(t_1, t_2) = -i\Gamma_{R,22} e^{-i\psi_R(t_1,t_2)} \int \frac{d\omega}{2\pi} [1 - f(\omega - \mu)] e^{-i\omega(t_1-t_2)}, \quad (A.13)
\]
\[ \Sigma_{11}^r(\tau, t) = \frac{i}{\beta} \Gamma_{L,11} e^{i \psi_L(t,t_0)} \sum \frac{e^{-i \omega \tau}}{2\pi} \frac{e^{i \omega(t-t_0)}}{\omega_q - \omega + \mu}, \quad (A.14) \]
\[ \Sigma_{22}^r(\tau, t) = \frac{i}{\beta} \Gamma_{R,22} e^{i \psi_R(t,t_0)} \sum \frac{e^{-i \omega \tau}}{2\pi} \frac{e^{i \omega(t-t_0)}}{\omega_q - \omega + \mu}, \quad (A.15) \]
\[ \Sigma_{11}^a(\tau, t) = \frac{i}{\beta} \Gamma_{L,11} e^{-i \psi_L(t,t_0)} \sum \frac{e^{i \omega \tau}}{2\pi} \frac{e^{-i \omega(t-t_0)}}{\omega_q - \omega + \mu}, \quad (A.16) \]
\[ \Sigma_{22}^a(\tau, t) = \frac{i}{\beta} \Gamma_{R,22} e^{-i \psi_R(t,t_0)} \sum \frac{e^{i \omega \tau}}{2\pi} \frac{e^{-i \omega(t-t_0)}}{\omega_q - \omega + \mu}, \quad (A.17) \]

B. Modified Langreth rule for the pseudo self-energy in the WBLA

The Langreth rules \[36\] are written as
\[ \Sigma^r_{11}(t_1, t_2) = \Sigma^{-\gamma}_{11}(t_1, t_2) - \Sigma^\gamma_{11}(t_1, t_2), \quad (B.1) \]
\[ \Sigma^a_{11}(t_1, t_2) = \Sigma^{-\gamma}_{11}(t_1, t_2) - \Sigma^\gamma_{11}(t_1, t_2). \quad (B.2) \]

The rules express the relations between the retarded or the advanced part of the self-energy and other parts of the self-energy. The rules are derived straightforwardly from the fact that the self-energies are defined on the contour \( \gamma \). Therefore, the rules also hold for the quantities defined on the contour such as the non-equilibrium Green functions. Since the pseudo self-energies \( \tilde{\Sigma}_{11} \) are expressed with the quantities on the contour, one may expect the pseudo self-energy follows the Langreth rule. However, we cannot apply the Langreth rule to the pseudo self-energies. This is because there is a product of quantities depending on the contour \( \gamma \) in its definition \[37\]. Instead, the following modified Langreth rules hold for the pseudo self-energy:
\[ \tilde{\Sigma}^r_{11} = \tilde{\Sigma}^{-\gamma}_{11}(t_1, t_2) - \tilde{\Sigma}^\gamma_{11}(t_1, t_2) + (\Sigma^r_{11} \cdot \Sigma^r_{22})(t_1, t_2) - \delta(t_1 - t_2)(h_{1L} h_{L1} + h_{2R} h_{R2}), \quad (B.3) \]
\[ \tilde{\Sigma}^a_{11} = \tilde{\Sigma}^{-\gamma}_{11}(t_1, t_2) - \tilde{\Sigma}^\gamma_{11}(t_1, t_2) + (\Sigma^r_{22} \cdot \Sigma^r_{11})(t_1, t_2) - \delta(t_1 - t_2)(h_{1L} h_{L1} + h_{2R} h_{R2}). \quad (B.4) \]

When we calculate the quantities including the pseudo self-energies, we use the modified Langreth rules.
The derivation is as follows. To derive the modified Langreth rule (B.3), we compare the expression of the retarded part of the pseudo self-energy \( \tilde{\Sigma}_{11}^{\text{r}} := \theta(t_1 - t_2)(\tilde{\Sigma}_{11}^{>} - \tilde{\Sigma}_{11}^{<}) \) and other parts of the pseudo self-energy \( \tilde{\Sigma}_{11}^{--} - \tilde{\Sigma}_{11}^{<} \). By substituting the equation of motion for the non-perturbative Green function into the derivative of the self-energies in the definition of the pseudo self-energy (37), we get an expression
\[
\tilde{\Sigma}_{11}(z_1, z_2) = h_{1L} h_{2L} g_{LL}(z_1, z_2) h_{L1} + h_{2R} h_{RR} g_{RR}(z_1, z_2) h_{R2} - \epsilon_1 \Sigma_{11}(z_1, z_2) - \epsilon_2 \Sigma_{22}(z_1, z_2) \\
- \int_{\gamma} dz \Sigma_{22}(z_1, \bar{z}) \Sigma_{11}(\bar{z}, z_2) + \delta(z_1, z_2)(h_{1L} h_{L1} + h_{2R} h_{R2}). \tag{B.5}
\]
From (B.5), we can obtain expressions of \( \tilde{\Sigma}_{11}^{\text{r}} \) and \( \tilde{\Sigma}_{11}^{--} - \tilde{\Sigma}_{11}^{<} \). If there were only the first four terms in (B.5), we can apply the ordinary Langreth rule, because the non-perturbative Green function and the self-energy satisfy the standard Langreth rule. But a modification is necessary due to the existence of the fifth \( \int_{\gamma} dz \Sigma_{22}(z_1, \bar{z}) \Sigma_{11}(\bar{z}, z_2) \) and the sixth term \( \delta(z_1, z_2)(h_{1L} h_{L1} + h_{2R} h_{R2}) \). Therefore, we only consider the expressions of the fifth and the sixth terms of \( \tilde{\Sigma}_{11}^{\text{r}} \) and \( \tilde{\Sigma}_{11}^{--} - \tilde{\Sigma}_{11}^{<} \) and compare them. In fact, the fifth term for \( \tilde{\Sigma}_{11}^{\text{r}} \) is 0 in the WBLA
\[
\theta(t_1 - t_2) \int_{\gamma} dz (\Sigma_{22}^{+}(t_1, \bar{z}) \Sigma_{11}^{+}(\bar{z}, t_2) - \Sigma_{22}^{-}(t_1, \bar{z}) \Sigma_{11}^{-}(\bar{z}, t_2)) = 0, \tag{B.6}
\]
where we use the expression of the self-energy in the WBLA, which is calculated in Appendix A. The sixth term vanishes for the pseudo retarded self-energy due to the Heaviside function in the definition. For \( \tilde{\Sigma}_{11}^{--} - \tilde{\Sigma}_{11}^{<} \), the fifth term is written as
\[
\int_{\gamma} dz (\Sigma_{22}^{-}(t_1, \bar{z}) \Sigma_{11}^{-}(\bar{z}, t_2) - \Sigma_{22}^{+}(t_1, \bar{z}) \Sigma_{11}^{+}(\bar{z}, t_2)) = (\Sigma_{22}^{r} \cdot \Sigma_{11}^{r})(t_1, t_2), \tag{B.7}
\]
where we use the Langreth rule. The notation \( \Sigma_{22}^{r}(t_1, \bar{z}) \) means that the argument \( z_1 \) is on the contour \( C_- \) and the argument \( z_2 \) is not identified where it is on the contour \( \gamma \). The sixth term remains unchanged. Therefore, we can conclude that the difference between \( \tilde{\Sigma}_{11}^{--} - \tilde{\Sigma}_{11}^{<} \) and \( \tilde{\Sigma}_{11}^{r} \) is \( (\Sigma_{22}^{r} \cdot \Sigma_{11}^{r})(t_1, t_2) - \delta(t_1 - t_2)(h_{1L} h_{L1} + h_{2R} h_{R2}) \). This means that the modified Langreth rule (B.3) holds. The modified Langreth rule for the advanced pseudo self-energy (B.4) is derived in the same way.
C. The pseudo self-energy in the WBLA

Matsubara pseudo self-energy

From the definition of the pseudo self-energy (37), we get

\[ \tilde{\Sigma}_{11}^{M}(\tau_1, \tau_2) = \frac{1}{i} \left[ \frac{d}{d\tau_1} \right] g_{LL}^{M}(\tau_1, \tau_2) h_{L1} + h_{2R} g_{LL}^{M}(\tau_1, \tau_2) \left( \frac{d}{d\tau_2} \right) h_{R2} \]
\[ - (\epsilon_1 - \mu) \Sigma_{22}^{M}(\tau_1, \tau_2) - (\epsilon_2 - \mu) \Sigma_{11}^{M}(\tau_1, \tau_2) - (\Sigma_{22} * \Sigma_{11})_{(\tau_1, \tau_2)} \]
\[ = (i) + (ii) + (iii) + (iv) + (v). \quad (C.1) \]

The first term is computed as

\[ (i) = \Gamma_{L,11}^{i} \frac{i}{\beta} \sum_{q} \int \frac{d\omega}{2\pi} \frac{\omega_q}{\omega_q - \omega + \mu} e^{-\omega_q(\tau_1 - \tau_2)} \]
\[ = -\frac{\Gamma_{L,11}}{2\beta} \sum_{q} \omega_q \xi_q e^{-\omega_q(\tau_1 - \tau_2)}, \quad (C.2) \]

where we use the expression

\[ \int \frac{d\omega}{2\pi} \frac{1}{\omega_q - \omega + \mu} = \frac{i}{2} \xi_q. \]

The function \( \xi_q \) is defined as \( \xi_q = 1 \) (Im \( \omega_q < 0 \)), \(-1 \) (Im \( \omega_q > 0 \)). In the same way, the second term is written as

\[ (ii) = -\frac{\Gamma_{R,22}}{2\beta} \sum_{q} \omega_q \xi_q e^{-\omega_q(\tau_1 - \tau_2)}. \quad (C.3) \]

With the expressions of the Matsubara self-energies [A.8], [A.9] calculated in Appendix A, we get

\[ (iii) = (\epsilon_1 - \mu) \frac{\Gamma_{R,22}}{2\beta} \sum_{q} \omega_q e^{-\omega_q(\tau_1 - \tau_2)}, \quad (C.4) \]
\[ (iv) = (\epsilon_2 - \mu) \frac{\Gamma_{L,11}}{2\beta} \sum_{q} \omega_q e^{-\omega_q(\tau_1 - \tau_2)}. \quad (C.5) \]
The fifth term is calculated with the relation \( \int_0^\beta d\tau e^{-(\omega_q - \omega'_q)\tau} = \beta \delta_{q,q'} \) and the result is
\[
(v) = i \frac{\Gamma_{L,11} \Gamma_{R,22}}{4\beta} \sum q e^{-\omega_q(\tau_1 - \tau_2)}. \tag{C.6}
\]

With the expressions from (C.2) to (C.6), we get the Matsubara pseudo self-energy (52):
\[
\tilde{\Sigma}^M_{11}(\tau_1, \tau_2) = \sum q e^{-\omega_q(\tau_1 - \tau_2)} \left[ -\frac{\xi_q}{2\beta} \left\{ (\omega_q - \epsilon_1 + \mu) \Gamma_{R,22} + (\omega_q - \epsilon_2 + \mu) \Gamma_{L,11} \right\} + i \frac{\Gamma_{L,11} \Gamma_{R,22}}{4\beta} \right]. \tag{C.7}
\]

Retarded/Advanced pseudo self-energy

The retarded pseudo self-energy is defined as \( \tilde{\Sigma}^r_{11}(t_1, t_2) = \theta(t_1 - t_2)(\tilde{\Sigma}_{11}(t_1, t_2) - \tilde{\Sigma}^<_{11}(t_1, t_2)) \). With the definition of the pseudo self-energy (57), this is written in the form
\[
\tilde{\Sigma}^r_{11}(t_1, t_2) = \theta(t_1 - t_2) \left[ \left( i \frac{d}{dt_1} - \epsilon_2 \right) (\tilde{\Sigma}^>_{11}(t_1, t_2) - \tilde{\Sigma}^<_{11}(t_1, t_2)) \right. \\
+ (\Sigma^>_{22}(t_1, t_2) - \Sigma^<_{22}(t_1, t_2)) \left( -i \frac{d}{dt_2} - \epsilon_1 \right) \\
- \int_\gamma dz (\Sigma^+_{22}(t_1, z) \Sigma^<_{11}(z, t_2) - \Sigma^<_{22}(t_1, z) \Sigma^+_{11}(z, t_2)) \right] \\
= (vi) + (vii) + (viii). \tag{C.8}
\]

With the definition of the retarded self-energy, we rewrite the first term (vi) as
\[
\theta(t_1 - t_2) \left( i \frac{d}{dt_1} - \epsilon_2 \right) (\tilde{\Sigma}^>_{11}(t_1, t_2) - \tilde{\Sigma}^<_{11}(t_1, t_2)) = \left( i \frac{d}{dt_1} - \epsilon_2 \right) \Sigma^r_{11}(t_1, t_2) \\
- i \delta(t_1 - t_2) (\Sigma^>_{11}(t_1, t_2) - \Sigma^<_{11}(t_1, t_2)). \tag{C.9}
\]

With the expression of the self-energies, (A.10) and (A.12), we get
\[
(\Sigma^>_{11}(t_1, t_2) - \Sigma^<_{11}(t_1, t_2)) = -i h_{1L} h_{L1} e^{-i\psi_L(t_1, t_2)}. \]

Therefore, (C.9) is expressed in a form:
\[
\theta(t_1 - t_2) \left( i \frac{d}{dt_1} - \epsilon_2 \right) (\tilde{\Sigma}^>_{11}(t_1, t_2) - \tilde{\Sigma}^<_{11}(t_1, t_2)) = \left( i \frac{d}{dt_1} - \epsilon_2 \right) \Sigma^r_{11}(t_1, t_2) - h_{1L} h_{L1} \delta(t_1 - t_2). \tag{C.10}
\]
In the same way, (vii) is represented as

$$\theta(t_1-t_2) (\Sigma_{22}^>(t_1, t_2) - \Sigma_{22}^<(t_1, t_2)) \left(-i \frac{d}{dt_2} - \epsilon_1\right) = \Sigma_{22}^r(t_1, t_2) \left(-i \frac{d}{dt_2} - \epsilon_1\right) - h_{2R} h_{R2} \delta(t_1-t_2).$$

(C.11)

Then we use the expression of the retarded self-energy in the WBLA (39).

By substituting the expression into (C.10) (C.11), we get

$$\theta(t_1-t_2) \left( i \frac{d}{dt_1} - \epsilon_2 \right) (\Sigma_{11}^>(t_1, t_2) - \Sigma_{11}^<(t_1, t_2)) + (\Sigma_{22}^>(t_1, t_2) - \Sigma_{22}^<(t_1, t_2)) \left(-i \frac{d}{dt_2} - \epsilon_1\right) = \Sigma_{11}^r(t_1, t_2) \left(-i \frac{d}{dt_2} - \epsilon_1\right) - h_{1L} h_{L1} + h_{2R} h_{R2} \delta(t_1-t_2).$$

(C.12)

By the substitution of (C.12) and (B.7) in (C.8), we get the expression of the retarded pseudo self-energy (40). The advanced pseudo self-energy is calculated in the same way and the result is

$$\tilde{\Sigma}_{11}^a(t_1, t_2) := -\theta(t_2-t_1) (\Sigma_{11}^>(t_1, t_2) - \Sigma_{11}^<(t_1, t_2))$$

$$= \frac{1}{2}(\Gamma_{L,11} + \Gamma_{R,22}) \frac{d}{d(t_1-t_2)} \delta(t_1-t_2) + i \frac{1}{2}(\epsilon_2 \Gamma_{L,11} + \epsilon_1 \Gamma_{R,22}) \delta(t_1-t_2)$$

$$- (h_{1L} h_{L1} + h_{2R} h_{R2}) \delta(t_1-t_2).$$

(C.13)

**Left/right pseudo self-energy**

From the definition (37), the left pseudo self-energy is represented as

$$\tilde{\Sigma}_{11}^l(\tau, t) = \left(-\frac{d}{d\tau} - \epsilon_2 + \mu\right) \Sigma_{11}^l(\tau, t) + \Sigma_{22}^r(\tau, t) \left(-i \frac{d}{dt} - \epsilon_1\right)$$

$$- \int d\bar{z} \Sigma_{22}(\tau, \bar{z}) \Sigma_{11}(\bar{z}, t)$$

$$= (ix) + (x) + (xi).$$

(C.14)
From the expressions (A.14), (A.15) of the left self-energy in the WBLA, the first and second terms are written as

\[ \langle \text{i}\rangle = \frac{i}{\beta} \Gamma_{L,11} e^{i\psi_L(t,t_0)} \sum_q (\omega_q - \epsilon_2 + \mu)e^{-\omega_q \tau} \int \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t_0)}}{\omega_q - \omega + \mu}, \quad (C.15) \]

\[ \langle \text{x}\rangle = \frac{i}{\beta} \Gamma_{R,11} e^{i\psi_R(t,t_0)} \sum_q e^{-\omega_q \tau} \int \frac{d\omega}{2\pi} \frac{\omega + V_R(t) - \epsilon_1 e^{i\omega(t-t_0)}}{\omega_q - \omega + \mu} \quad (C.16) \]

With the Langreth rule and the expression of the retarded and the advanced self-energy in the WBLA (39), we get

\[ \langle \text{xi}\rangle = -\frac{i}{2} \Gamma_{L,11}(\tau,t) \Sigma_{22}^\dagger - \left( \Sigma_{22}^M \ast \Sigma_{11}^\dagger \right)_{(\tau,t)}. \quad (C.17) \]

We can derive the expression of the left pseudo self-energy from (C.14)-(C.16):

\[ \tilde{\Sigma}_{11}(\tau_1, t_2) = \frac{i}{\beta} \Gamma_{L,11} e^{i\psi_L(t_1,t_0)} \sum_q (\omega_q - \epsilon_2 + \mu)e^{-\omega_q \tau} \int \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t_0)}}{\omega_q - \omega + \mu} + \frac{i}{\beta} \Gamma_{R,22} e^{i\psi_R(t_1,t_0)} \]

\[ \times \sum_q e^{-\omega_q \tau} \int \frac{d\omega}{2\pi} \frac{\omega + V_L(t) - \epsilon_1 e^{i\omega(t-t_0)}}{\omega_q - \omega + \mu} - \frac{i}{2} \Gamma_{L,11}(\tau_1,t_2) - \left( \Sigma_{22}^M \ast \Sigma_{11}^\dagger \right)_{(\tau_1,t_2)}. \quad (C.18) \]

The right pseudo self-energy is derived in the same way as the left pseudo self-energy and it is expressed as

\[ \tilde{\Sigma}_{11}(t_1, \tau_2) = \frac{i}{\beta} \Gamma_{L,11} e^{-i\psi_L(t_1,t_0)} \sum_q e^{\omega_q \tau} \int \frac{d\omega}{2\pi} \frac{\omega + V_L(t) - \epsilon_2 e^{-i\omega(t-t_0)}}{\omega_q - \omega + \mu} + \frac{i}{\beta} \Gamma_{R,22} e^{-i\psi_R(t_1,t_0)} \]

\[ \times \sum_q (\omega_q - \epsilon_1 + \mu)e^{\omega_q \tau} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t_0)}}{\omega_q - \omega + \mu} + \frac{i}{2} \Gamma_{R,22}(\Sigma_{11}^\dagger(t_1, \tau_2) - \left( \Sigma_{22}^M \ast \Sigma_{11}^\dagger \right)_{(t_1,\tau_2)}. \quad (C.19) \]
Lesser/Greater pseudo self-energy

From the definition \( \Sigma_{11}^\tau(t_1, t_2) \), the lesser pseudo self-energy is represented as

\[
\tilde{\Sigma}_{11}^\tau(\tau, t) = \left( i \frac{d}{dt} - \epsilon_2 \right) \Sigma_{11}^\tau(t_1, t_2) + \Sigma_{22}^\tau(t_1, t_2) \left( -i \frac{d}{dt} - \epsilon_1 \right)
\]

\[ - \int d\bar{z} \Sigma_{22}(t_1, \bar{z}) \Sigma_{11}^+(\bar{z}, t_2) \]

\[ = (\text{xii}) + (\text{xiii}) + (\text{xiv}). \quad (C.20) \]

By differentiating the representations, \( (A.10) \) and \( (A.11) \), of the left self-energy in the WBLA, we get

\[ (\text{xii}) = i \Gamma_{L,11} e^{-i\psi_{L}(t_1, t_2)} \int \frac{d\omega}{2\pi} (\omega + V_L(t_1) - \epsilon_2) f(\omega - \mu) e^{-i\omega(t_1 - t_2)}, \quad (C.21) \]

\[ (\text{xiii}) = i \Gamma_{R,22} e^{-i\psi_{R}(t_1, t_2)} \int \frac{d\omega}{2\pi} (\omega + V_R(t_2) - \epsilon_1) f(\omega - \mu) e^{-i\omega(t_1 - t_2)}. \quad (C.22) \]

Using the expressions of the retarded/advanced self-energies in WBLA \( (39) \) and the Langreth rule, the third term is written as

\[ (\text{xiv}) = \frac{i}{2} (\Gamma_{R,22} \Sigma_{11}^\tau - \Gamma_{L,11} \Sigma_{22}^\tau) - \left( \Sigma_{22} \Sigma_{11}^{\dagger} \right)_{(t_1, t_2)}. \quad (C.23) \]

Actually, the second term in the representation above equals to 0:

\[
\left( \Sigma_{22} \Sigma_{11}^{\dagger} \right)_{(t_1, t_2)} = -i \int_0^\beta d\tau \frac{i}{\beta} \Gamma_{R,22} e^{-i\psi_{R}(t_1, t_2)} \sum_q e^{\omega_q \tau} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1 - t_2)}}{\omega_q - \omega + \mu}
\]

\[
\times \frac{i}{\beta} \Gamma_{L,11} e^{i\psi_{L}(t_2, t_0)} \sum_{q'} e^{-\omega_{q'} \tau} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1 - t_2)}}{\omega_{q'} - \omega + \mu}
\]

\[ = 0, \]

where we use the expressions of the left/right self-energies in the WBLA \( (A.14), (A.17) \). From these results which we found above, we derive an expression of the lesser pseudo self-energy:

\[
\tilde{\Sigma}_{11}^\tau(t_1, t_2) = i \Gamma_{L,11} e^{-i\psi_{L}(t_1, t_0)} \int \frac{d\omega}{2\pi} (\omega + V_L(t_1) - \epsilon_2) f(\omega - \mu) e^{-i\omega(t_1 - t_0)} + i \Gamma_{R,22} e^{-i\psi_{R}(t_1, t_0)} \int \frac{d\omega}{2\pi} (\omega + V_R(t_2) - \epsilon_1) f(\omega - \mu) e^{-i\omega(t_1 - t_0)} + \frac{i}{2} (\Gamma_{R,22} \Sigma_{11}^\tau(t_1, t_2) - \Gamma_{L,11} \Sigma_{22}^\tau(t_1, t_2)). \quad (C.23) \]
The greater pseudo self-energy is calculated in the same way as
\[ \overline{\Sigma}_{11}^{>}(t_1, t_2) = -i \Gamma_{L,11} e^{-i \psi_L(t_1,t_0)} \int \frac{d\omega}{2\pi} (\omega + V_L(t_1) - \epsilon_2) \left[ 1 - f(\omega - \mu) \right] e^{-i\omega(t-t_0)} \]
\[- i \Gamma_{R,22} e^{-i \psi_R(t_1,t_0)} \int \frac{d\omega}{2\pi} (\omega + V_R(t_2) - \epsilon_1) \left[ 1 - f(\omega - \mu) \right] e^{-i\omega(t-t_0)} \]
\[ + \frac{i}{2} \left( \Gamma_{R,22} \Sigma_{11}^{>}(t_1, t_2) - \Gamma_{L,11} \Sigma_{22}^{>}(t_1, t_2) \right). \]  
(C.24)

D. Derivation of the second-order partial differential equation

Lesser Green function

Here, we show a derivation of the second-order partial differential equation for the lesser Green function (35). The derivation starts from the equations of motion for \( G_{11}^{<} \) and \( G_{21}^{<} \) of the lesser part:
\[ \left[ i \frac{d}{dt_1} - \epsilon_1 \right] G_{11}^{<}(t_1, t_2) = \int d\tilde{\omega} \Sigma_{11}^{-}(t_1, \tilde{\omega}) G_{11}^{+}(\tilde{\omega}, t_2) + t_{12} G_{21}^{<}(t_1, t_2), \]  
(D.1)
\[ \left[ i \frac{d}{dt_1} - \epsilon_2 \right] G_{21}^{<}(t_1, t_2) = \int d\tilde{\omega} \Sigma_{22}^{-}(t_1, \tilde{\omega}) G_{21}^{+}(\tilde{\omega}, t_2) + t_{21} G_{11}^{<}(t_1, t_2). \]  
(D.2)

To substitute the differential equation of \( G_{21}^{<} \) into the equation of \( G_{11}^{<} \), we rewrite the equation (D.1) into the form:
\[ G_{21}^{<}(t_1, t_2) = \frac{1}{t_{12}} \left\{ \left[ i \frac{d}{dt_1} - \epsilon_1 \right] G_{11}^{<}(t_1, t_2) - \int d\tilde{\omega} \Sigma_{11}^{-}(t_1, \tilde{\omega}) G_{11}^{+}(\tilde{\omega}, t_2) \right\}, \]  
(D.3)
\[ i \frac{d}{dt_1} G_{21}^{<}(t_1, t_2) = \frac{1}{t_{12}} \left\{ \left[ -\frac{d^2}{dt_1^2} - i \epsilon_1 \frac{d}{dt_1} \right] G_{11}^{<}(t_1, t_2) - i \int d\tilde{\omega} \Sigma_{11}^{-}(t_1, \tilde{\omega}) G_{11}^{+}(\tilde{\omega}, t_2) \right\}. \]  
(D.4)
By substituting these expressions \((D.3)\) \((D.4)\) into \((D.2)\) and multiplying \(t_{12}\), we get
\[
\left[ \frac{d^2}{dt_1^2} - i(\epsilon_1 + \epsilon_2) \frac{d}{dt_1} + (\epsilon_1 \epsilon_2 - t_{12} \tau_{21}) \right] G_{11}^\gamma (t_1, t_2) \\
= \int_\gamma \bar{d}z \left( i \frac{d}{dt_1} - \epsilon_2 \right) \Sigma_1 \left( t_1, \bar{z} \right) G_{11}^\gamma (\bar{z}, t_2) + t_{12} \int_\gamma \bar{d}z \Sigma_2 (t_1, \bar{z}) G_{21}^\gamma (\bar{z}, t_2).
\]
\[(D.5)\]

With the Langreth rule, the rightmost term is expressed as
\[
t_{12} \int_\gamma \bar{d}z \Sigma_{22}^- (t_1, \bar{z}) G_{21}^+ (\bar{z}, t_2) = t_{12} \left( \Sigma_{22}^- \cdot G_{21}^- - \Sigma_{22}^- \cdot G_{21}^+ + \Sigma_{22}^\gamma \cdot G_{21}^\gamma \right)_{(t_1, t_2)}.
\]
\[(D.6)\]

With the equation of motion for \(G_{11}\) \((32)\), we rewrite \(G_{21}\) in \((D.6)\) in terms of \(G_{11}\). From \((32)\), we get
\[
G_{21}^- (t_1, t_2) = \frac{1}{t_{12}} \left\{ \left[ i \frac{d}{dt_1} - \epsilon_1 \right] G_{11}^- (t_1, t_2) - \int_\gamma \bar{d}z \Sigma_{11}^{-} (t_1, \bar{z}) G_{11}^\gamma (\bar{z}, t_2) \right\},
\]
\[
G_{21}^+ (t_1, t_2) = \frac{1}{t_{12}} \left\{ \left[ i \frac{d}{dt_1} - \epsilon_1 \right] G_{11}^+ (t_1, t_2) - \int_\gamma \bar{d}z \Sigma_{11}^+ (t_1, \bar{z}) G_{11}^\gamma (\bar{z}, t_2) + \delta (t_1 - t_2) \right\},
\]
\[
G_{21}^\gamma (\tau_1, t_2) = \frac{1}{t_{12}} \left\{ \left[ -\frac{d}{dt_1} - \epsilon_1 + \mu \right] G_{11}^\gamma (\tau_1, t_2) - \int_\gamma \bar{d}z \Sigma_{11} (\tau_1, \bar{z}) G_{11}^\gamma (\bar{z}, t_2) \right\}.
\]

With these equations, we get these expressions:
\[
t_{12} \left( \Sigma_{22}^- \cdot G_{21}^- \right)_{(t_1, t_2)} = \int_{t_0}^\infty dt \Sigma_{22}^- (t_1, t) \left\{ \left[ i \frac{d}{dt} - \epsilon_1 \right] G_{11}^- (t, t_2) - \int_\gamma \bar{d}z \Sigma_{11}^- (t, \bar{z}) G_{11}^\gamma (\bar{z}, t_2) \right\},
\]
\[(D.7)\]
\[
- t_{12} \left( \Sigma_{22}^- \cdot G_{21}^+ \right)_{(t_1, t_2)} = - \int_{t_0}^\infty dt \Sigma_{22}^- (t_1, t) \left\{ \left[ i \frac{d}{dt} - \epsilon_1 \right] G_{11}^+ (t, t_2) \\
- \int_\gamma \bar{d}z \Sigma_{11}^+ (t, \bar{z}) G_{11}^\gamma (\bar{z}, t_2) \right\} - \Sigma_{22}^-,
\]
\[(D.8)\]
\[
t_{12} \left( \Sigma_{22}^\gamma \cdot G_{21}^\gamma \right)_{(t_1, t_2)} = -i \int_0^\beta d\tau \Sigma_{22}^\gamma (t_1, \tau) \left\{ \left[ -\frac{d}{d\tau} - \epsilon_1 + \mu \right] G_{11}^\gamma (\tau, t_2) \\
- \int_\gamma \bar{d}z \Sigma_{11} (\tau, \bar{z}) G_{11}^\gamma (\bar{z}, t_2) \right\}.
\]
\[(D.9)\]
Integrating by part, the first terms of these expressions are written as

\[
\int_{t_0}^{\infty} dt \Sigma_{22}^{-}(t_1, t) i \frac{d}{dt} G_{11}^{<}(t, t_2) = i \left[ \Sigma_{22}^{-}(t_1, t) G_{11}^{<}(t, t_2) \right]_{t_0}^{\infty} + \int_{t_0}^{\infty} dt \left( -i \frac{d}{dt} \right) \Sigma_{22}^{-}(t_1, t) G_{11}^{<}(t, t_2),
\]

\[
\int_{t_0}^{\infty} dt \Sigma_{22}^{<}(t_1, t) i \frac{d}{dt} G_{11}^{++}(t, t_2) = i \left[ \Sigma_{22}^{<}(t_1, t) G_{11}^{++}(t, t_2) \right]_{t_0}^{\infty} + \int_{t_0}^{\infty} dt \left( -i \frac{d}{dt} \right) \Sigma_{22}^{<}(t_1, t) G_{11}^{++}(t, t_2),
\]

\[
\int_{0}^{\beta} d\tau \Sigma_{22}^{\dagger}(t_1, \tau) \left( -\frac{d}{d\tau} \right) G_{11}^{\dagger}(\tau, t_2) = \int_{0}^{\beta} d\tau \frac{d}{d\tau} \Sigma_{22}^{\dagger}(t_1, \tau) G_{11}^{\dagger}(\tau, t_2). \quad (D.10)
\]

The boundary term \( \left[ \Sigma_{22}^{<}(t_1, \tau) G_{11}^{\dagger}(\tau, t_2) \right]_{0}^{\beta} \) in (D.10) vanishes due to the KMS condition \( (29) \). With these expressions and \( (D.7)-(D.9), (D.6) \) is written as

\[
t_{12} \int d\bar{z} \Sigma_{22}^{-}(t_1, \bar{z}) G_{11}^{+}(\bar{z}, t_2)
= \int_{\gamma} d\bar{z} \left[ \Sigma_{22}^{-}(t_1, \bar{z}) \left( -i \frac{d}{d\bar{z}} - h_{1}(\bar{z}) \right) - \int d\bar{z} \Sigma_{22}^{-}(t_1, \bar{z}) \Sigma_{11}(\bar{z}, \bar{z}) \right] G_{11}^{+}(\bar{z}, t_2)
+ i \left[ \Sigma_{22}^{-}(t_1, t) G_{11}^{<}(t, t_2) - \Sigma_{22}^{<}(t_1, t) G_{11}^{++}(t, t_2) \right]_{t_0}^{\infty} - \Sigma_{22}^{<}(t_1, t_2). \quad (D.11)
\]

By substituting (D.11) into (D.5) and using the definition of the pseudo self-energy, we get the second-order partial differential equation for the lesser Green function \( (35) \).

**Retarded Green function**

By definition, the retarded Green function is expressed as

\[
G_{11}^{r}(t_1, t_2) = \theta(t_1 - t_2)(G_{11}^{>}(t_1, t_2) - G_{11}^{<}(t_1, t_2)).
\]

We can get the following relations by differentiating the above definition,

\[
i \frac{d}{dt_1} G_{11}^{r}(t_1, t_2) = i\delta(t_1 - t_2)(G_{11}^{>}(t_1, t_2) - G_{11}^{<}(t_1, t_2)) + i\theta(t_1 - t_2) \frac{d}{dt_1} (G_{11}^{>}(t_1, t_2) - G_{11}^{<}(t_1, t_2))
= \delta(t_1 - t_2) + i\theta(t_1 - t_2) \frac{d}{dt_1} (G_{11}^{>}(t_1, t_2) - G_{11}^{<}(t_1, t_2)),
\]

\[
(D.12)
\]
\[-\frac{d^2}{dt_1^2}G''_{11}(t_1, t_2) = i\frac{d}{dt_1}\left(\delta(t_1 - t_2) + \theta(t_1 - t_2) \frac{d}{dt_1}(G''_{11}(t_1, t_2) - G''_{11}(t_1, t_2))\right)\]
\[= i\frac{d}{d(t_1 - t_2)}\delta(t_1 - t_2) - \delta(t_1 - t_2) \frac{d}{dt_1}(G''_{11}(t_1, t_2) - G''_{11}(t_1, t_2))\]
\[= \theta(t_1 - t_2) \frac{d^2}{dt_1^2}(G''_{11}(t_1, t_2) - G''_{11}(t_1, t_2))\]
\[= i\frac{d}{d(t_1 - t_2)}\delta(t_1 - t_2) + \epsilon_1\delta(t_1 - t_2) - \theta(t_1 - t_2) \frac{d^2}{dt_1^2}(G''_{11}(t_1, t_2) - G''_{11}(t_1, t_2)),\]
(D.13)

where we use the relations

\[\langle G''_{11}(t_1, t_2) - G''_{11}(t_1, t_2) \rangle|_{t_1 = t_2} = -i,
\]
\[i\frac{d}{dt_1}(G''_{11}(t_1, t_2) - G''_{11}(t_1, t_2))|_{t_1 = t_2} = -i\epsilon_1.\]

The first relation is derived by substituting the definition of the lesser \(G''_{11}(t_1, t_2) = i\langle d'_{H,1}(t_2) d_{H,1}(t_1) \rangle\) and greater Green function \(G''_{11}(t_1, t_2) = -i\langle d_{H,1}(t_1) d'_{H,1}(t_2) \rangle\) and using the commutation relation for Fermion. The second follows from the equation of motion for the lesser and the greater Green function derived from (19) and using the commutation relation again. With these relations, we obtain

\[\left[-\frac{d^2}{dt_1^2} - i(\epsilon_1 + \epsilon_2) \frac{d}{dt_1} + (\epsilon_1\epsilon_2 - t_{12}t_{21}) \right] G''_{11}(t_1, t_2) = i\frac{d}{d(t_1 - t_2)}\delta(t_1 - t_2) = -\epsilon_2\delta(t_1 - t_2)\]
\[+ \theta(t_1 - t_2) \left[-\frac{d^2}{dt_1^2} - i(\epsilon_1 + \epsilon_2) \frac{d}{dt_1} + (\epsilon_1\epsilon_2 - t_{12}t_{21}) \right] (G''_{11}(t_1, t_2) - G''_{11}(t_1, t_2)).\]
(D.14)

Since we want to obtain a differential equation of \(G''_{11}\) in closed form, next we rewrite the last term of RHS in (D.14) into the form which is expressed with \(G''_{11}\). To do it, first we apply the Langreth rule, the modified Langreth rules (B.3), (B.4), and use the second-order partial differential equations of the lesser (35) and the greater Green functions for the last term. Then it is
finally decomposed into the five terms:

\[
\theta(t_1 - t_2) \left[ -\frac{d^2}{dt_1^2} - i(\epsilon_1 + \epsilon_2) \frac{d}{dt_1} + (\epsilon_1 \epsilon_2 - t_{12} t_{21}) \right] (G_{11}^\gamma(t_1, t_2) - G_{11}^\zeta(t_1, t_2))
\]

\[
= \theta(t_1 - t_2) \int \Gamma_{i1}^+ (t_1, z) G_{11}^- (z, t_2) - \tilde{\Sigma}_{i1}^- (t_1, z) G_{11}^+ (z, t_2)) - \Sigma_{22}^r (t_1, t_2)
\]

\[
= \theta(t_1 - t_2) \left[ (\tilde{\Sigma}_{i1}^r \cdot G_{11}^- - \tilde{\Sigma}_{i1}^- \cdot G_{11}^r) - (\tilde{\Sigma}_{i1}^{++} \cdot G_{11}^r - \tilde{\Sigma}_{i1}^{<} \cdot G_{11}^{++}) \right]_{(t_1, t_2)} - \Sigma_{22}^r (t_1, t_2)
\]

\[
= \theta(t_1 - t_2) \left[ (\tilde{\Sigma}_{i1}^r \cdot (G_{11}^a + G_{11}^r) - \Sigma_{22}^r (t_1, t_2) \right.
\]

\[
+ \delta(t_1 - t)(h_{1L} h_{1L} + h_{2R} h_{2R}) \right) \cdot G_{11}^r + \tilde{\Sigma}_{i1}^r \cdot (G_{11}^a + G_{11}^r) \right]_{(t_1, t_2)} - \Sigma_{22}^r (t_1, t_2)
\]

\[
= \theta(t_1 - t_2) \left[ (\tilde{\Sigma}_{i1}^r - \tilde{\Sigma}_{i1}^r) \cdot G_{11}^a \right]_{(t_1, t_2)} + \theta(t_1 - t_2) \left[ \tilde{\Sigma}_{i1}^r \cdot (G_{11}^r - G_{11}^a) \right]_{(t_1, t_2)}
\]

\[
- \theta(t_1 - t_2) \left[ (\tilde{\Sigma}_{i1}^r \cdot \Sigma_{22}^r (t_1, t_2) \cdot (G_{11}^r - G_{11}^a) \right]_{(t_1, t_2)} + (h_{1L} h_{1L} + h_{2R} h_{2R}) G_{11}^r (t_1, t_2) - \Sigma_{22}^r (t_1, t_2)
\]

\[
= R_1 + R_2 + R_3 + R_4 - \Sigma_{22}^r (t_1, t_2). \quad \text{(D.15)}
\]

From the third line to the fourth, we use the Langreth rule and the modified Langreth rule. With the concrete expression of the self-energy and the pseudo self-energy, we can express \( R_1 \) as

\[
R_1 = \theta(t_1 - t_2) \int_{t_0}^{\infty} dt (\tilde{\Sigma}_{i1}^+ - \tilde{\Sigma}_{i1}^-) G_{11}^a (t, t_2)
\]

\[
= \theta(t_1 - t_2) \int_{t_0}^{\infty} dtr_1 (t_1, t) \frac{d}{d(t_1 - t)} \delta(t_1 - t) G_{11}^a (t_1, t_2),
\]

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where we use $\theta(t_1 - t_2)G_{11}^a(t_1, t_2) = 0$ and define $r_1(t, s) := \Gamma_{L,11}e^{-i\psi_L(t,s)} + \Gamma_{R,22}e^{-i\psi_R(t,s)}$. By integrating by part, we get
\[
R_1 = \theta(t_1 - t_2) \left[ r(t_1, t)G_{11}^a(t, t_2)\delta(t_1 - t) \right]_{t_0}^\infty \\
- \theta(t_1 - t_2) \int_{t_0}^\infty dt \frac{d}{d(t_1 - t)} \{ r_1(t_1, t)G_{11}^a(t, t_2) \} \delta(t_1 - t) \\
= -\theta(t_1 - t_2) \int_{t_0}^\infty dt \frac{d}{dt} \{ G_{11}^a(t, t_2) \} r_1(t_1, t) \delta(t_1 - t) dt \\
= \theta(t_1 - t) r(t_1, t_1) \frac{d}{dt_1} G_{11}^a(t_1, t_2).
\]
By differentiating the definition of $G_{11}^a(t_1, t_2)$ and substitute it into the expression above, we finally obtain
\[
R_1 = -i\theta(t_1 - t_2) \delta(t_2 - t_1) r_1(t_1, t_1) \\
= -\frac{i}{2} (\Gamma_{L,11} + \Gamma_{R,22}) \delta(t_1 - t_2),
\]
where we use the fact $\int dt \theta(t) \delta(t) f(t) = \int dt \frac{1}{2} \delta(t) f(t)$ for an arbitrary test function $f$. For $R_2$, with the expression of the retarded pseudo self-energy \[40\], we can see that it consists of three terms:
\[
R_2 = \theta(t_1 - t_2) \left[ \tilde{\Sigma}_{11}^r \cdot (G_{11}^> - G_{11}^<) \right]_{(t_1, t_2)} \\
= \theta(t_1 - t_2) \int_{t_0}^\infty \left[ \frac{1}{2} (\Gamma_{L,11} + \Gamma_{R,22}) \frac{d}{d(t_1 - t)} \delta(t_1 - t) \\
+ \frac{i}{2} (\epsilon_2 \Gamma_{L,11} + \epsilon_1 \Gamma_{R,22}) \delta(t_1 - t) - (h_{1L}h_{L1} + h_{2R}h_{R2}) \delta(t_1 - t) \right] \{ G_{11}^>(t_1, t_2) - G_{11}^<(t_1, t_2) \} \\
= r_{2,1} + r_{2,2} + r_{2,3}.
\]
Each term is calculated as follows:
\[
r_{2,1} = \theta(t_1 - t_2) \int_{t_0}^\infty dt \frac{1}{2} (\Gamma_{L,11} + \Gamma_{R,22}) \frac{d}{d(t_1 - t)} \delta(t_1 - t) \{ G_{11}^>(t_1, t_2) - G_{11}^<(t_1, t_2) \} \\
= \theta(t_1 - t_2) \left[ \frac{1}{2} (\Gamma_{L,11} + \Gamma_{R,22}) \delta(t_1 - t_2) \{ G_{11}^>(t_1, t_2) - G_{11}^<(t_1, t_2) \} \right]_{t_0}^\infty \\
- \theta(t_1 - t_2) \frac{1}{2} (\Gamma_{L,11} + \Gamma_{R,22}) \int_{t_0}^\infty \frac{d}{d(t_1 - t)} \{ G_{11}^>(t_1, t_2) - G_{11}^<(t_1, t_2) \} \delta(t_1 - t) \\
= \frac{1}{2} (\Gamma_{L,11} + \Gamma_{R,22}) \theta(t_1 - t_2) \frac{d}{dt_1} \{ G_{11}^>(t_1, t_2) - G_{11}^<(t_1, t_2) \},
\]
(D.16)
\[ r_{2,2} = \theta(t_1 - t_2) \int_{t_0}^{\infty} dt \frac{i}{2}(\epsilon_2 \Gamma_{L,11} + \epsilon_1 \Gamma_{R,22}) \delta(t_1 - t)(G_{11}^r(t_1, t_2) - G_{11}^s(t_1, t_2)) \]
\[ = \frac{i}{2}(\epsilon_2 \Gamma_{L,11} + \epsilon_1 \Gamma_{R,22})G_{11}^r(t_1, t_2), \]  
(D.18)

\[ r_{2,3} = -\theta(t_1 - t_2) \int_{t_0}^{\infty} dt (h_{1L} h_{L1} + h_{2R} h_{R2}) \delta(t_1 - t)(G_{11}^r(t_1, t_2) - G_{11}^s(t_1, t_2)) \]
\[ = -(h_{1L} h_{L1} + h_{2R} h_{R2})G_{11}^r(t_1, t_2). \]  
(D.19)

By taking the sum of the terms (D.17)-(D.19), we obtain the expression of \( R_2 \):

\[ R_2 = \frac{1}{2}(\Gamma_{L,11} + \Gamma_{R,22})\theta(t_1 - t_2) \frac{d}{dt_1}(G_{11}^r(t_1, t_2) - G_{11}^s(t_1, t_2)) \]
\[ + \frac{i}{2}(\epsilon_2 \Gamma_{L,11} + \epsilon_1 \Gamma_{R,22})G_{11}^r(t_1, t_2) - (h_{1L} h_{L1} + h_{2R} h_{R2})G_{11}^r(t_1, t_2) \]
\[ = \frac{1}{2}(\Gamma_{L,11} + \Gamma_{R,22}) \frac{d}{dt_1}G_{11}^r(t_1, t_2) + \frac{i}{2}(\Gamma_{L,11} + \Gamma_{R,22})\delta(t_1 - t_2) \]
\[ + \frac{i}{2}(\epsilon_2 \Gamma_{L,11} + \epsilon_1 \Gamma_{R,22})G_{11}^r(t_1, t_2) - (h_{1L} h_{L1} + h_{2R} h_{R2})G_{11}^r(t_1, t_2). \]  
(D.20)

In the WBLA, the term in \( R_3 \) is represented as \((\Sigma_{11}^r, \Sigma_{22}^r)(t,s) = -1/4\Gamma_{L,11} \Gamma_{R,22}\delta(t-s)\). Then we get the expression of \( R_3 \):

\[ R_3 = -\theta(t_1 - t_2) \int_{t_0}^{\infty} dt (\Sigma_{11}^r \cdot \Sigma_{22}^r)(t_1, t)(G_{11}^r(t, t_2) - G_{11}^s(t, t_2)) \]
\[ = \frac{1}{4} \Gamma_{L,11} \Gamma_{R,22} G_{11}^r(t_1, t_2). \]  
(D.21)

By substituting the expressions (D.16), (D.20) and (D.21) into (D.15), we obtain

\[ \theta(t_1 - t_2) \left[ -\frac{d^2}{dt_1^2} - i(\epsilon_1 + \epsilon_2) \frac{d}{dt_1} + (\epsilon_1 \epsilon_2 - t_1 t_2) \right] (G_{11}^r(t_1, t_2) - G_{11}^s(t_1, t_2)) \]
\[ = \frac{i}{2}(\epsilon_2 \Gamma_{L,11} + \epsilon_1 \Gamma_{R,22}) - \frac{i}{2}(\Gamma_{L,11} \Gamma_{R,22})G_{11}^r(t_1, t_2) \]
\[ + \frac{1}{2}(\Gamma_{L,11} + \Gamma_{R,22}) \frac{d}{dt_1}G_{11}^r(t_1, t_2) + \frac{i}{2} \Gamma_{R,22} \delta(t_1 - t_2) \]
\[ = (\epsilon_1 \epsilon_2 - \epsilon_1^{eff} \epsilon_2^{eff})G_{11}^r(t_1, t_2) + \frac{1}{2}(\Gamma_{L,11} + \Gamma_{R,22}) \frac{d}{dt_1}G_{11}^r(t_1, t_2) + \frac{i}{2} \Gamma_{R,22} \delta(t_1 - t_2), \]  
(D.22)
where we define $\epsilon_1^{\text{eff}} := \epsilon_1 - i/2\Gamma_{L,11}$ and $\epsilon_2^{\text{eff}} := \epsilon_2 - i/2\Gamma_{R,22}$. With (D.22), we can rewrite (D.14) into the form which is a differential equation for $G_{11}^r(t_1, t_2)$ in closed-form. This is the second-order partial differential equation for $G_{11}^r(t_1, t_2)$, or (51).

E. On a problem for calculating the retarded pseudo self-energy

The definition of the pseudo self-energy is (37)

$$\tilde{\Sigma}_{11}(z_1, z_2) := \left(i \frac{d}{dz_1} - h_2(z_1)\right) \Sigma_{11}(z_1, z_2) + \Sigma_{22}(z_1, z_2) \left(-i \frac{d}{dz_2} - h_1(z_2)\right)$$

$$- \int d\bar{z} \Sigma_{22}(z_1, \bar{z}) \Sigma_{11}(\bar{z}, z_2),$$

where the self-energies are defined as $\Sigma_{11}(z_1, \bar{z}) = h_{1L} \cdot g_{LL}(z_1, \bar{z}) h_{L1}$, $\Sigma_{22}(z_1, \bar{z}) = h_{2R} \cdot g_{RR}(z_1, \bar{z}) h_{R2}$. From this definition, we can calculate the retarded part of the pseudo self-energy straightforwardly in the following way. However, as we will see soon, there is a problem of divergence because the term $\theta(t) \frac{d}{dt} \delta(t)$ appears in this calculation. The difficulty can be avoided by changing the way of calculation as in Appendix D.

Based on the definition of the self-energy, the retarded part of the pseudo self-energy is calculated as

$$\tilde{\Sigma}_{11}^r(t_1, t_2) := \theta(t_1 - t_2)(\tilde{\Sigma}_{11}^>(t_1, t_2) - \tilde{\Sigma}_{11}^<(t_1, t_2))$$

$$\quad = \theta(t_1 - t_2) \left(i \frac{d}{dt_1} - h_2(t_1)\right) h_{1L} \cdot (g_{LL}^>(z_1, \bar{z}) - g_{LL}^<(z_1, \bar{z})) h_{L1}$$

$$\quad \quad + \theta(t_1 - t_2) h_{2R} \cdot (g_{RR}^>(z_1, \bar{z}) - g_{RR}^<(z_1, \bar{z})) h_{R2} \left(-i \frac{d}{dt_2} - h_1(t_2)\right)$$

$$\quad \quad - \theta(t_1 - t_2) \int d\bar{z} \left(\Sigma_{22}^+(t_1, \bar{z}) \Sigma_{11}^-(\bar{z}, t_2) - \Sigma_{22}^-(t_1, \bar{z}) \Sigma_{11}^+(\bar{z}, t_2)\right)$$

$$\quad = \theta(t_1 - t_2) \left(i \frac{d}{dt_1} - h_2(t_1)\right) \sum_{k,k'} T_{1,kL} - i \delta_{kk'} e^{-i\phi_{kL}(t_1, t_2)} T_{kL,1}$$

$$\quad \quad + \theta(t_1 - t_2) \sum_{k,k'} T_{kR,2} - i \delta_{kk'} e^{-i\phi_{kR}(t_1, t_2)} T_{kR,2} \left(-i \frac{d}{dt_2} - h_1(z_2)\right),$$

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where we use the concrete expression of the non-perturbative Green function (A.1) (A.3) and the relation (B.3). Next we rewrite the expression so that it is convenient for applying the WBLA:

\[ \tilde{\Sigma}_{r11}(t_1, t_2) = -i\theta(t_1 - t_2) \left( i \frac{d}{dt_1} - h_2(z_1) \right) \sum_k |T_{1,kL}|^2 e^{-i\phi_{kL}(t_1,t_2)} \]

\[ - i\theta(t_1 - t_2) \sum_k |T_{2,kR}|^2 e^{-i\phi_{kR}(t_1,t_2)} \left( -i \frac{d}{dt_2} - h_1(z_2) \right) \]

\[ = -i\theta(t_1 - t_2) \sum_k |T_{1,kL}|^2 (V_L(t_1) - \epsilon_{kL} - \epsilon_2) e^{-i\phi_{kL}(t_1,t_2)} \]

\[ - i\theta(t_1 - t_2) \sum_k |T_{2,kR}|^2 (V_R(t_2) - \epsilon_{kR} - \epsilon_1) e^{-i\phi_{kR}(t_1,t_2)} \]

\[ = -i\theta(t_1 - t_2) e^{-i\psi_{L}(t_1,t_2)} \sum_k |T_{1,kL}|^2 (V_L(t_1) - \epsilon_{kL} - \epsilon_2) e^{-i\epsilon_{kL}(t_1-t_2)} \]

\[ - i\theta(t_1 - t_2) e^{-i\psi_{R}(t_1,t_2)} \sum_k |T_{2,kR}|^2 (V_R(t_2) - \epsilon_{kR} - \epsilon_1) e^{-i\epsilon_{kR}(t_1-t_2)} \]

\[ = -i\theta(t_1 - t_2) e^{-i\psi_{L}(t_1,t_2)} \int \frac{d\omega}{2\pi} \sum_k 2\pi |T_{1,kL}|^2 \delta(\omega - \epsilon_{kL})(V_L(t_1) - \omega - \epsilon_2) e^{-i\omega(t_1-t_2)} \]

\[ - i\theta(t_1 - t_2) e^{-i\psi_{R}(t_1,t_2)} \int \frac{d\omega}{2\pi} \sum_k 2\pi |T_{2,kR}|^2 \delta(\omega - \epsilon_{kR})(V_R(t_2) - \omega - \epsilon_1) e^{-i\omega(t_1-t_2)}. \]

At this point, we use the WBLA. It is equivalent to the substitution of \( \Gamma_{L,11} \) and \( \Gamma_{R,22} \) by \( \sum_k 2\pi |T_{1,kL}|^2 \delta(\omega - \epsilon_{kL}) \) and \( \sum_k 2\pi |T_{2,kR}|^2 \delta(\omega - \epsilon_{kR}) \) respectively.
Then, the retarded self-energy is written as

\[
\tilde{\Sigma}_{11}^{\tau}(t_1, t_2) = -i\theta(t_1 - t_2)e^{-i\psi_L(t_1, t_2)} \int \frac{d\omega}{2\pi} \Gamma_{L,11}(V_L(t_1) - \omega - \epsilon_2)e^{-i\omega(t_1 - t_2)}
\]

\[-i\theta(t_1 - t_2)e^{-i\psi_R(t_1, t_2)} \int \frac{d\omega}{2\pi} \Gamma_{R,22}(V_R(t_2) - \omega - \epsilon_1)e^{-i\omega(t_1 - t_2)}
\]

\[= -i\theta(t_1 - t_2)\Gamma_{L,11}e^{-i\psi_L(t_1, t_2)} \left((V_L(t_1) - \epsilon_2)\delta(t_1 - t_2) + i\frac{d}{d(t_1 - t_2)}\delta(t_1 - t_2)\right)
\]

\[-i\theta(t_1 - t_2)\Gamma_{R,22}e^{-i\psi_R(t_1, t_2)} \left((V_R(t_2) - \epsilon_1)\delta(t_1 - t_2) + i\frac{d}{d(t_1 - t_2)}\delta(t_1 - t_2)\right)
\]

\[= -i\frac{1}{2} \left(\Gamma_{L,11}(V_L(t_1) - \epsilon_2) + \Gamma_{R,22}(V_R(t_2) - \epsilon_1)\right)\delta(t_1 - t_2)
\]

\[+ \theta(t_1 - t_2)(\Gamma_{L,11}e^{-i\psi_L(t_1, t_2)} + \Gamma_{R,22}e^{-i\psi_R(t_1, t_2)})\frac{d}{d(t_1 - t_2)}\delta(t_1 - t_2),
\]

(E.1)

where we use the relations \(\int \frac{d\omega}{2\pi} e^{-i\omega t} = \delta(t)\) and \(\int dt\theta(t)\delta(t)f(t) = \int dt\frac{1}{2}\delta(t)f(t)\) for an arbitrary test function \(f\). In this expression, the term \(\theta(t)\frac{d}{dt}\delta(t)\) appears. In the expression \([40]\), which actually we use in the calculation of the retarded self-energy, the term does not appear. The problem is that the diverging term appears if one uses this expression including \(\theta(t)\frac{d}{dt}\delta(t)\) for the derivation of the second-order differential equation of retarded Green function. We can see this fact as follows. In Appendix D, we derived the second-order partial differential equation for the retarded Green function. In the derivation, we needed an expression of the retarded pseudo self-energy for the calculation of the term \(r_{2,1}\) in \(R_2\), where we partially integrate a function which is multiplied by \(\frac{d}{dt}\delta(t)\). Therefore, if we use the expression \([E.1]\) including the term \(\theta(t)\frac{d}{dt}\delta(t)\) for the calculation in \(r_{2,1}\), the square of the delta function \(\delta^2(t)\) appears in the partial integration as \(\theta(t)\frac{d}{dt}\delta(t) = \frac{d}{dt}(\theta(t)\delta(t)) - \delta^2(t)\). This is why the diverging term emerges when we use the expression \([E.1]\) and we have to use the expression \([40]\) for calculation of the retarded pseudo self-energy.

The essential difference between two ways of calculation, Appendix C and Appendix E, appears when one employs the WBLA. To understand the difference, we first simplify the problem. Let us take a continuous and differentiable function \(f_N(t)\), which becomes \(\delta(t)\) as \(N \to \infty\). We suppose that \(N \to \infty\) limit corresponds to the WBLA. Let us consider a limit of
\( \theta(t) \frac{d}{dt} f_N(t) \) as \( N \to \infty \). Naturally, we would calculate the limit as

\[
\lim_{N \to \infty} \theta(t) \frac{d}{dt} f_N(t) = \theta(t) \lim_{N \to \infty} \frac{d}{dt} f_N(t) = \theta(t) \frac{d}{dt} \delta(t).
\]

This corresponds to the way of calculation carried out in Appendix E. By contrast, we calculate the same quantity in Appendix C as follows:

\[
\lim_{N \to \infty} \theta(t) \frac{d}{dt} f_N(t) = \lim_{N \to \infty} \left[ \frac{d}{dt} (\theta(t) f_N(t)) - \delta(t) \frac{d}{dt} f_N(t) \right] = \frac{d}{dt} (\theta(t) \delta(t)) - \frac{d}{dt} f_N(t) \bigg|_{t=0} \delta(t),
\]

\[
= \frac{d}{dt} (\theta(t) \delta(t)) - \left. \frac{d}{dt} f_N(t) \right|_{t=0} \delta(t),
\]

\[
= \left. \frac{1}{2} \frac{d}{dt} \delta(t) - \frac{d}{dt} f_N(t) \right|_{t=0} \delta(t). \tag{E.2}
\]

(E.2) corresponds to the expressions (C.9) in Appendix C. Therefore, \( f_N(t) \) corresponds to the term \( \Sigma^{(t)}_{11} (t_1, t_2) - \Sigma^{(t)}_{11} (t_1, t_2) \) in Appendix D. The points in this calculation are that we replace \( \theta(t) \frac{d}{dt} f_N(t) \) by \( \frac{d}{dt} (\theta(t) f_N(t)) - \delta(t) \frac{d}{dt} f_N(t) \) and use the relation \( \theta(t) \delta(t) = \frac{1}{2} \delta(t) \). In this way, we avoid the problem of divergence.

We have not found any reasons to justify this way of calculation physically or mathematically yet, but the expressions of the Matsubara, retarded and advanced Green functions obtained from (40) reproduce the same results in the previous study of the single quantum dot when we take the limit in which the coupling constant between the dots is taken to be zero.

**F. Representation of physical quantities as integrals with respect to frequency**

In this section, we explain how we rewrite the electron density of the dot1 in terms of integration with respect to frequency. The expression is useful to compare the expression of the electron density of the double dots with that of
the single dot (17). In addition, it is convenient for numerical computation. We only explain the rewriting of $G_{11}^1(t,0^+), (G_{11}^1 * \Sigma_{11}^M)(t,0^+)$, and $G_{12}^1(t,0^+)$, which appear in the two coefficients $l_1(t)$ and $l_2(t)$ of the lesser Green function [65].

Throughout the calculations below, we use the following relation about the Matsubara frequency:

$$\frac{i}{\beta} \sum_{q=-\infty}^{\infty} Q(\omega_q) e^{\omega_q 0^+} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega)[Q(\omega^-) - Q(\omega^+)], \quad (F.1)$$

where $Q(\omega)$ is a function which satisfies $\lim_{\omega \to \infty} Q(\omega) e^{-\omega} = 0$ and $f(\omega)$ is the Fermi distribution. This relation is proved by the fact that the Matsubara frequency is the residues of the function $1/(e^{\beta z} + 1)$. We define functions which include effects between the reservoirs and the system:

$$K^i_\alpha (t, t' : \omega) := e^{-ik_i(t-t_0)} \left[ \int_{t_0}^{t} ds e^{-i(\omega-k_i)(s-t')} e^{-i\psi_\alpha(s,t')} \right],$$

$$\tilde{K}^i_\alpha (t, t' : \omega) := e^{-ik_i(t-t_0)} \left[ \int_{t_0}^{t} ds e^{-i(\omega-k_i)(s-t')} V_\alpha(s) e^{-i\psi_\alpha(s,t')} \right].$$

In the following, we express $K^i_\alpha (t, t_0 : \omega)$ as $K^i_\alpha (\omega)$ and represent $G_{11}^M(\omega^-)$ as the Matsubara Green function [42] with the condition that the imaginary part of its variable is negative. $G_{11}^M(\omega^+)$ is also defined similarly.

With the expression of the right Green function [57], $G_{11}^1(t,0^+)$ is written as

$$G_{11}^1(t,0^+) = b_1(0^+) e^{-ik_1(t-t_0)} + b_2(0^+) e^{-ik_2(t-t_0)} - \frac{1}{k_2-k_1} \left[ \int_{t_0}^{t} ds g(s,0^+) (e^{-ik_1(t-s)} - e^{-ik_2(t-s)}) \right], \quad (F.2)$$

where these coefficients $b_1(0^+)$, $b_2(0^+)$ and the function $g(s,0^+)$ are defined as

$$b_1(0^+) = -\frac{1}{k_2-k_1} \{ (\epsilon_1 - k_2) G_{11}^M(0^+,0^+) + (\Sigma_{11}^M * G_{11}^M)(0^+,0^+) + t_{12} G_{12}^M(0^+,0^+) \},$$

$$b_2(0^+) = \frac{1}{k_2-k_1} \{ (\epsilon_1 - k_1) G_{11}^M(0^+,0^+) + (\Sigma_{11}^M * G_{11}^M)(0^+,0^+) + t_{12} G_{12}^M(0^+,0^+) \},$$

$$g(s,0^+) = -i \{ (\Sigma_{11}^M * G_{11}^M)(s,0^+) - \Sigma_{22}^1(s,0^+) \}. $$
Now we explain how we get a representation of which is expressed as an integration over frequency with the relation (F.1). The constant $G_{11}(0^+, 0^+)$ is rewritten as

$$G_{11}^M(0^+, 0^+) = \lim_{\tau_1, \tau_2 \to 0} G_{11}^M(\tau_1, \tau_2)$$

$$= \lim_{\tau_2 \to 0} \frac{i}{\beta} \sum_q e^{\omega_q \tau} G_{11}^M(\omega_q)$$

$$= \int \frac{d\omega}{2\pi} f(\omega - \mu) \left[ G_{11}^M(\omega - \mu) - G_{11}^M(\omega^+ - \mu) \right]. \quad (F.1)$$

Similarly, $(\Sigma_{11}^M * G_{11}^M)(0^+, 0^+)$ is rearranged as

$$(\Sigma_{11}^M * G_{11}^M)(0^+, 0^+) = -i \int_0^\beta d\tau \Sigma_{11}^M(0^+, \tau) G_{11}^M(\tau, 0^+)$$

$$= \lim_{\tau' \to 0} -i \int_0^\beta d\tau \Sigma_{11}^M(\tau', \tau) G_{11}^M(\tau, \tau')$$

$$= \lim_{\tau' \to 0} -i \int_0^\beta d\tau \left( -\frac{\Gamma_{L,11}}{2\beta} \sum_q \xi_q e^{-\omega_q(\tau' - \tau)} \left( \frac{i}{\beta} \sum_{q'} e^{-\omega_{q'}(\tau - \tau')} G_{11}^M(\omega_{q'}) \right) \right)$$

$$= i \frac{\Gamma_{L,11}}{\beta} \sum_q \xi_q e^{\omega_q 0^+} G_{11}^M(\omega_q). \quad \therefore \int_0^\beta d\tau e^{-(\omega_q - \omega_{q'})(\tau - \tau')} = \beta \delta_{q q'}$$

Here we use a relation $\xi_q = 1 - 2U(\text{Im}(\omega_q))$, where $U(t)$ is the unit-step function: $U(t) = 1$ for $t \geq 1$, $U(t) = 0$ for $t < 0$. With this relation and (F.1), we get the expression:

$$(\Sigma_{11}^M * G_{11}^M)(0^+, 0^+) = \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu) \left[ \left\{ 1 - 2U(\text{Im}(\omega - \mu)) \right\} G_{11}^M(\omega - \mu) \right.$$

$$\left. - \left\{ 1 - 2U(\text{Im}(\omega^+ - \mu)) \right\} G_{11}^M(\omega^+ - \mu) \right] \quad (E.1)$$

$$= \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu) \left[ G_{11}^M(\omega - \mu) + G_{11}^M(\omega^+ - \mu) \right]. \quad (F.4)$$

These calculations are the essence of our rearrangement of the lesser Green function. The expression of $G_{21}(0^+, 0^+)$ is obtained in the same way as
We represent the term as
\[ G_{21}^M(0^+, 0^+) = \int \frac{d\omega}{2\pi} f(\omega - \mu)[G_{11}^M(\omega^- - \mu) - G_{11}^M(\omega^+ + \mu)]. \] (F.5)

By substituting (F.3)-(F.5) in \( b_1(0^+) \) and \( b_2(0^+) \), we obtain
\[
b_1(0^+) = -\frac{1}{k_2 - k_1} \{(\epsilon_1 - k_2) \int \frac{d\omega}{2\pi} f(\omega - \mu)[G_{11}^M(\omega^- - \mu) - G_{11}^M(\omega^+ - \mu)]
+ \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G_{11}^M(\omega^- - \mu) + G_{11}^M(\omega^+ - \mu)]
+ t_{12} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G_{21}^M(\omega^- - \mu) - G_{21}^M(\omega^+ - \mu)] \}.
\] (F.6)

\[
b_2(0^+) = \frac{1}{k_2 - k_1} \{(\epsilon_1 - k_1) \int \frac{d\omega}{2\pi} f(\omega - \mu)[G_{11}^M(\omega^- - \mu) - G_{11}^M(\omega^+ - \mu)]
+ \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G_{11}^M(\omega^- - \mu) + G_{11}^M(\omega^+ - \mu)]
+ t_{12} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G_{21}^M(\omega^- - \mu) - G_{21}^M(\omega^+ - \mu)] \}.
\] (F.7)

Next we explain a rewriting of \( \int_{t_0}^t ds g(s, 0^+)(e^{-ik_1(t-s)} - e^{-ik_2(t-s)}) \) in (F.2). We represent the term as \( b_3(t) \). In the same way as the rearrangement of \( b_1(0^+) \) and \( b_2(0^+) \), \( b_3(t) \) is expressed as
\[
b_3(t) = \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G_{11}^M(\omega^- - \mu) - G_{11}^M(\omega^+ - \mu)](e^{-ik_1(t-t_0)} - e^{-ik_2(t-t_0)})
+ \int \frac{d\omega}{2\pi} f(\omega - \mu)\omega G_{11}^M(\omega^- - \mu)[\Gamma_{L,11}(K_L^1(\omega) - K_R^2(\omega)) + \Gamma_{R,22}(K_R^1(\omega) - K_R^2(\omega))]
+ \int \frac{d\omega}{2\pi} f(\omega - \mu)G_{11}^M(\omega^- - \mu)[\Gamma_{L,11}(K_L^1(\omega) - K_L^2(\omega))
- \epsilon_2 \Gamma_{L,11}(K_L^1(\omega) - K_L^2(\omega)) - \epsilon_1 \Gamma_{R,22}(K_R^1(\omega) - K_R^2(\omega))]
+ \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu)G_{11}^M(\omega^- - \mu)\omega [K_L^1(\omega) - K_L^2(\omega) - K_L^1(\omega) + K_R^2(\omega)]
- \Gamma_{R,22} \int \frac{d\omega}{2\pi} f(\omega - \mu)(K_R^1(\omega) - K_R^2(\omega)).
\] (F.8)

With these expressions (F.6)-(F.8), the function \( G_{11}^1(t, 0^+) \) is expressed as
\[
G_{11}^1(t, 0^+) = b_1 e^{-ik_1(t-t_0)} + b_2 e^{-ik_2(t-t_0)} - \frac{1}{k_2 - k_1} b_3(t).
\]
The other functions \((G^t_{11} \ast \Sigma^M_{11})(t,0^+)\) and \(G^t_{12}(t,0^+)\) are rewritten in the same way and the results are

\[
(G^t_{11} \cdot \Sigma^t_{11})(t,0^+) = \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu)(C_1 K^1_L(\omega) + C_2 K^2_L(\omega)),
\]

\[
c_1 = -\frac{1}{k_2 - k_1} \{ (\epsilon_1 - k_2) \int \frac{d\omega}{2\pi} f(\omega - \mu)[G^M_{12}(\omega^- - \mu) - G^M_{12}(\omega^+ - \mu)] \\
+ \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G^M_{12}(\omega^- - \mu) + G^M_{12}(\omega^+ - \mu)] \\
+ t_{12} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G^M_{22}(\omega^- - \mu) - G^M_{22}(\omega^+ - \mu)],
\]

\[
c_2 = \frac{1}{k_2 - k_1} \{ (\epsilon_1 - k_1) \int \frac{d\omega}{2\pi} f(\omega - \mu)[G^M_{12}(\omega^- - \mu) - G^M_{12}(\omega^+ - \mu)] \\
+ \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G^M_{12}(\omega^- - \mu) + G^M_{12}(\omega^+ - \mu)] \\
+ t_{12} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G^M_{22}(\omega^- - \mu) - G^M_{22}(\omega^+ - \mu)],
\]

\[
c_3(t) = \frac{i}{2} \Gamma_{L,11} \int \frac{d\omega}{2\pi} f(\omega - \mu)[G^M_{12}(\omega^- - \mu) - G^M_{12}(\omega^+ - \mu)](e^{-i\epsilon_1(t-t_0)} - e^{-i\epsilon_2(t-t_0)}) \\
+ \int \frac{d\omega}{2\pi} f(\omega - \mu)\omega G^M_{12}(\omega^- - \mu)[\Gamma_{L,11}(K^1_L(\omega) - K^2_L(\omega)) + \Gamma_{R,22}(K^1_R(\omega) - K^2_R(\omega))] \\
+ \int \frac{d\omega}{2\pi} f(\omega - \mu)G^M_{12}(\omega^- - \mu)[\Gamma_{L,11}(K^1_L(\omega) - K^2_L(\omega)) \\
- \epsilon_2 \Gamma_{L,11}(K^1_L(\omega) - K^2_L(\omega)) - \epsilon_1 \Gamma_{R,22}(K^1_R(\omega) - K^2_R(\omega)] \\
+ \frac{i}{2} \Gamma_{L,11} \Gamma_{R,22} \int \frac{d\omega}{2\pi} f(\omega - \mu)G^M_{12}(\omega^- - \mu)[K^1_L(\omega) - K^2_L(\omega) - K^1_R(\omega) + K^2_R(\omega)],
\]

\[
G^t_{12}(t,0^+) = c_1 e^{-ik_1(t-t_0)} + c_2 e^{-ik_2(t-t_0)} - \frac{1}{k_2 - k_1} c_3(t).
\]

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