Electromagnetic anomaly in the presence of electric and chiral magnetic conductivities in relativistic heavy-ion collisions

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We study the spacetime evolution of electric (E) and magnetic (B) fields along with the electromagnetic anomaly (E · B) in the presence of electric (σ) and chiral magnetic (σ₅) conductivities in Au+Au collisions at √sNN = 200 GeV. By comparing to the Lienard-Wiechert solutions with zero conductivities, we observe a symmetry breaking of the electromagnetic field in a conducting medium with respect to the reaction plane. The decay of the field is also significantly decelerated after the conductivities are introduced. Similar effects are also found for the dipole structure of E · B as well as the quadrupole structure of (E · B)B, which may finally affect the charge separation of the elliptic flow coefficient of hadrons observed in high-energy nuclear collisions.

I. INTRODUCTION

The high-speed movement of charged nuclei in noncentral relativistic heavy-ion collisions can produce strong electromagnetic field. Its magnitude can be estimated via eB ∼ γvZe²/R², whose peak value can reach the order of 10¹⁸ Gauss in Au+Au collisions at the BNL Relativistic Heavy-Ion Collider (RHIC), and 10¹⁹ Gauss in Pb+Pb collisions at the CERN Large Hadron Collider (LHC) [1–4]. This provides a unique environment to investigate properties of nuclear matter under extreme electromagnetic field, such as the anomalous transport effects in the quark-gluon plasma (QGP) produced by the energetic nuclear collisions [5–10]. The electromagnetic field can also cause separation of particles with opposite charges, as reflected by the charge-odd directed flow coefficient found in both theoretical calculations [11–17] and experimental measurements [18, 19], although a precise agreement between theory and experiment is still an ongoing effort.

With the presence of an external electric field E, one would expect a vector current induced in a conducting matter according to the Ohm’s law,

\[ j_\mu = \sigma E_\mu, \]

with σ being the electric conductivity and \( j_\mu \) being the electric current. Meanwhile, inside a plasma composed of chiral fermions, vector current \( j_\mu \) and axial current \( j_5^\mu \) can also be induced by magnetic field B. If chiral anomaly, or a nonzero axial chiral charge potential \( \mu_A \), exists, a vector current \( j_\mu = \langle \bar{\psi} \gamma_\mu \gamma^5 \psi \rangle \) will be induced by the imbalance between left and right handed quarks as \[ j_\mu^{\text{CME}} = \sigma_5 \mu_A B, \]

where \( \sigma_5 \) is known as the chiral magnetic conductivity given by \( \sigma_5 \approx eN_c/2\pi² \) with \( N_c \) being the number of colors. This is known as the chiral magnetic effect (CME). With the mass term included, the axial anomaly equation reads \[ \partial_\nu j_5^\nu = 2i(\bar{\psi} \gamma_5 \psi) - \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \]

\[ - \frac{g^2}{16\pi^2} G_{\mu\nu} G_{\rho\sigma}, \]

from which one can observe three terms of contributions to the axial charge. The first term on the right hand side comes from the finite quark mass originated from the chiral symmetry breaking, while the second and third terms correspond to the QED and QCD anomalies respectively. The QED anomaly, usually represented by E · B, will be the main focus of this work.

Similarly, a nonzero vector chemical potential \( \mu_A \) would induce an axial current \( j_A = \langle \bar{\psi} \gamma^\mu \gamma^5 \psi \rangle \) in the presence of the external B field [25, 26] :

\[ j_A^{\text{CSE}} = \sigma_5 \mu_A B, \]

which causes the axial charge separation along the B and is known as the chiral separation effect (CSE).

Chiral magnetic and chiral separation effects have been investigated within various approaches, such as hydrodynamics, kinetic theory, holographic QCD and lattice QCD [27–33]. It has been proposed that the nonzero vector and axial charges can mutually induce each other, leading to a collective excitation in the QGP known as the chiral magnetic wave (CMW) [21, 34, 35]. The charge quadrupole structure associated with this CMW can further result in different elliptic flow coefficients \( v_2 \) between positive and negative charged particles [22, 36], as observed by the STAR experiment [37]. Nevertheless, it has been suggested in Ref. [38] that even without the formation of CMW, the dipole shape of the E · B distribution in the transverse plane can already generate...
an electric quadrupole moment when being coupled to the magnetic field $\mathbf{B}$. This provides an alternative direction for understanding the $v_2$ separation between opposite charges, considering the negative result on the recent search for the CME at RHIC [39]. Moreover, this novel mechanism does not need a finite baryon density ($\rho_V$) to drive the charge separation of $v_2$, therefore may lead to different beam energy dependence of this charge separation, which can be further tested by the beam energy scan program at RHIC. The space-averaged electromagnetic field and electromagnetic anomaly weighted by energy density have been further investigated in Ref. [40].

In this work, we extend these previous studies [38, 40] on the spatial distribution of $\mathbf{E} \cdot \mathbf{B}$ to a more realistic nuclear medium that includes both electric and chiral magnetic conductivities. It has been found that the decay of the electromagnetic field can be significantly decelerated when conductivities are introduced [11, 41–46]. Symmetry breaking has also been suggested for the field with respect to the reaction plane after including the conductivities [43]. We will follow Ref. [43] to further investigate the time evolution of the spatial distribution of the electromagnetic field in the presence of electric and chiral magnetic conductivities. In particular, effects on the dipole structure of the electromagnetic anomaly and the electric quadrupole pattern will be discussed in detail.

This work will be organized as follows. We will first provide a brief review on the solution of the electromagnetic field in both conducting and non-conducting media in Sec. II. Numerical results of the spacetime evolution of the electromagnetic field will be presented in Sec. III, and compared between with and without including electric and chiral magnetic conductivities. In Sec. IV, we will discuss effects of conductivities on the electromagnetic anomaly and the electric quadrupole moment. A summary and outlook will be presented in Sec. V.

II. CALCULATION OF ELECTROMAGNETIC FIELD

A: Non-conducting System ($\sigma = \sigma_\chi = 0$)

For a non-conducting system, or vacuum, where both electric and chiral magnetic conductivities are zero ($\sigma = \sigma_\chi = 0$), we evaluate the electromagnetic field according to the Lienard-Wiechert potential [1, 47] as

$$E(t, \mathbf{x}) = \frac{e}{4\pi} \sum_n \frac{(1 - v_n^2) \mathbf{R}_n}{(\mathbf{R}_n^2 - (\mathbf{R}_n \times \mathbf{v}_n)^2)^{3/2}}, \quad (5)$$

$$B(t, \mathbf{x}) = \frac{e}{4\pi} \sum_n \frac{(1 - v_n^2) (\mathbf{v}_n \times \mathbf{R}_n)}{(\mathbf{R}_n^2 - (\mathbf{R}_n \times \mathbf{v}_n)^2)^{3/2}}, \quad (6)$$

where $\mathbf{R}_n = \mathbf{x} - \mathbf{x}_n$ is the relative position vector between the field point $\mathbf{x}$ under discussion and the source point $\mathbf{x}_n$, and $\mathbf{x}_n$ and $\mathbf{v}_n$ are respectively the position and velocity of the $n$-th proton in the colliding nuclei at the current time $t$. Note that the above equations are valid when each source charge is traveling with a constant velocity. Otherwise, the original form of the Lienard-Wiechert fields [2] using the retarded time should be applied.

B: Conducting System ($\sigma \neq 0, \sigma_\chi \neq 0$)

The QGP matter produced in heavy-ion collisions is a conducting medium. The in-medium electromagnetic field can be solved using the Maxwell equations with both electric ($\sigma$) and chiral magnetic ($\sigma_\chi$) conductivities included:

$$\nabla \cdot \mathbf{F} = \left\{ \begin{array}{l} \rho_{\text{ext}} / \epsilon \rightarrow \mathbf{E} \\
0 \rightarrow \mathbf{B}, \end{array} \right. \quad (7)$$

$$\nabla \times \mathbf{F} = \left\{ \begin{array}{l} -\partial_t \mathbf{B} \\
\partial_t \mathbf{E} + \mathbf{J}_{\text{ext}} + \sigma \mathbf{E} + \sigma_\chi \mathbf{B} \rightarrow \mathbf{E}, \end{array} \right. \quad (8)$$

where $\rho_{\text{ext}}$ and $\mathbf{J}_{\text{ext}}$ are external charge and current densities, and $\mathbf{F}$ denotes either electric ($\mathbf{E}$) or magnetic ($\mathbf{B}$) field. Considering that all source charges propagate along the $z$-axis, one can obtain the following algebraic solutions of the electromagnetic field using the Green’s function method in the cylindrical coordinates [43]:

$$B_\phi(t, \mathbf{x}) = \frac{Q}{4\pi} \frac{v_x \gamma x_T}{\Delta^{3/2}} \left(1 + \frac{\sigma \gamma}{2} \sqrt{\Delta}\right) e^A,$$

$$B_r(t, \mathbf{x}) = -\sigma \frac{Q}{4\pi} \frac{v_y \gamma y_T}{\Delta^{3/2}} e^A \left[\gamma (vt - z) + A \sqrt{\Delta}\right],$$

$$B_z(t, \mathbf{x}) = \sigma \frac{Q}{4\pi} \frac{v_y \gamma}{\Delta^{3/2}} e^A \left[\Delta \left(1 - \frac{\sigma \gamma}{2} \sqrt{\Delta}\right) + \gamma^2 (vt - z)^2 \left(1 + \frac{\sigma \gamma}{2} \sqrt{\Delta}\right)\right], \quad (9)$$

in which $\Delta$ and $A$ are defined as $\Delta \equiv \gamma^2 (vt - z)^2 + x_T^2$, and $A \equiv (\sigma \gamma / 2) \left[\gamma (vt - z) - \sqrt{\Delta}\right]$, with $x_T$ being the magnitude of the transverse coordinate $x_T = \sqrt{x^2 + y^2}$, and

$$E_\phi(t, \mathbf{x}) = \sigma \frac{Q}{8\pi} \frac{v^2 \gamma^2 x_T}{\Delta^{3/2}} e^A \left[\gamma (vt - z) + A \sqrt{\Delta}\right],$$

$$E_r(t, \mathbf{x}) = \frac{Q}{4\pi} e^A \left\{ \gamma x_T \left(1 + \frac{\sigma \gamma}{2} \sqrt{\Delta}\right) \right. \left. - \sigma \frac{v_x}{x_T} e^{-\sigma (t-z/v)} \left[1 + \frac{\gamma (vt - z)}{\sqrt{\Delta}}\right] \right\},$$

$$E_z(t, \mathbf{x}) = \frac{Q}{4\pi} \left\{ e^A \left[\gamma (vt - z) + A \sqrt{\Delta} + \sigma \gamma \Delta\right] + \frac{\sigma^2}{\gamma v^2} e^{-\sigma (t-z/v)} \Gamma (0, -A) \right\}, \quad (10)$$

with $\Gamma (0, -A)$ being the incomplete gamma function defined as $\Gamma (a, z) = \int_z^\infty t^{a-1} e^{-t} dt$. One may verify
that Eqs. (9) and (10) above return to the previous Lienard-Wiechert solution with vanishing \( \sigma \) and \( \sigma_\chi \).

In this work, we use the Monte-Carlo (MC) Glauber model developed by the PHOBOS Collaboration \[48\] to calculate the spatial distribution of the source charges. A two-step calculation is performed in this model. First, for a given impact parameter \( b \), the centers of projectile and target nuclei are located at \( x = \pm b/2 \) with the impact parameter defined in the \( x \)-direction and the beams in the \( z \)-direction. The positions of nucleons in the two nuclei are determined stochastically. The Woods-Saxon distribution is taken for the density profile of nucleons in each nucleus

\[
\rho(r, \theta) = \frac{\rho_0}{1 + \exp \left[ \frac{R(\theta) - \rho_0}{R} \right]} \left[ 1 + w \left( \frac{R(\theta)}{R} \right)^2 \right],
\]

where \( \rho_0 \) denotes the nuclear density at the nucleus center, \( R \) is the surface thickness parameter, and \( R(\theta) = R_0 [1 + \beta_2 Y_{20}(\theta) + \beta_3 Y_{30}(\theta)] \) is the nuclear radius in which \( Y_{n0}(\theta) \) are spherical harmonic functions. Here, the parameters \( \beta_2 \), \( \beta_3 \) and \( w \) determine the deviation from a spherical nucleus. These nucleons are then assumed to propagate along straight trajectories (in \(+/-z\) directions). For each pair of nucleons, one from projectile and one from target, a collision between them takes place if their distance \( d \) (in the transverse plane) satisfies

\[
d \leq \sqrt{\frac{\sigma_{\text{inel}}^{NN}}{\pi}},
\]

where \( \sigma_{\text{inel}}^{NN} \) is the inelastic cross section of nucleon-nucleon collisions. Those nucleons that participate in collisions are labelled as “participants” while those that do not participate in collisions are labelled as “spectators”.

In this work, we use \( \sigma_{\text{inel}}^{NN} = 42 \text{ mb} \), \( \rho_0 = 0.17 \text{ fm}^{-3} \), \( R = 6.38 \text{ fm} \), \( d = 0.535 \text{ fm} \) and \( w = 0 \) for Au-Au collisions at \( \sqrt{s_{NN}} = 200 \text{ GeV} \) at RHIC. We define the initial time \( (t = 0) \) as the moment when the two oppositely moving nuclei collide. In Fig. 1, we illustrate our initial charge distribution based on this IC Glauber approach, left for the view on the transverse plane, and right for the view on the reaction plane. Purple and green dots represent participant nucleons from the two colliding nuclei, while blue and red represent spectators that do not participate in inelastic scatterings. For each nucleon, we use the probability \( Z/A \) (79/197 for Au nucleus) to determine whether it is a proton that contributes to the electromagnetic field we discuss. Protons in both participants and spectators are taken into account for evaluating the electromagnetic field. Minor difference has been found between considering only spectators and all nucleons. For instance, for \( b = 10 \text{ fm} \), protons in spectators alone yield about 6% smaller \( B_y \) compared to protons in the whole nucleus. For calculating the electromagnetic field in the rest of this study, we average over 50,000 MC Glauber events for each impact parameter setup to obtain a smooth geometric distribution of the source charges.

### III. SPATIAL DISTRIBUTIONS OF ELECTROMAGNETIC FIELDS

In this section we present our numerical results on the spatial distribution of the electric and magnetic fields, compared between zero and finite electric \( (\sigma) \) and chiral magnetic \( (\sigma_\chi) \) conductivity cases. Based on the previous discussions in Sec. II, the MC Glauber model is used to obtain the spacetime evolution profile of electric charges for Au+Au collisions at \( \sqrt{s_{NN}} = 200 \text{ GeV} \) with different impact parameters. The Maxwell equations are solved for the electromagnetic fields with finite \( \sigma \) and \( \sigma_\chi \), while the Lienard-Wiechert solution is taken for the case of \( \sigma = \sigma_\chi = 0 \).

Shown in Figs. 2 - 5 are the spatial distributions of \( eE_x, eE_y, eB_x, eB_y \) respectively in Au+Au collisions at \( \sqrt{s_{NN}} = 200 \text{ GeV} \). In each figure, we present the Lienard-Wiechert solution for vacuum in the first row and results for finite conductivities in the second row. Here the conductivities are taken as \( \sigma = 5.8 \text{ MeV} \) and \( \sigma_\chi = 1.5 \text{ MeV} \) as used in Ref. \[43\]. Note that the value of \( \sigma = 5.8 \text{ MeV} \) is consistent with the lattice QCD result around the top temperature of the QGP at RHIC \[49, 50\], which is expected to decrease together with the medium temperature as the QGP expands. In addition, there is no direct guidance of how to choose the value of \( \sigma_\chi \) yet. Since the analytical solution of the electromagnetic field – Eqs. (9) and (10) – is obtained in the limit of \( \sigma_\chi \ll \sigma \) in Ref. \[43\], we take the value assigned in this original work. In the present study, we will only focus on investigating the effects of the electric and chiral magnetic conductivities with the given values above. Constraints on these values will be explored in a follow-up study where we connect electromagnetic effects to experimental observables.

In each row of the figures, two snapshots of time evolution are presented. For the vacuum cases, due to the rapid decay of the electromagnetic field, we present results for \( t = 0.1 \text{ fm}/c \) and \( 0.3 \text{ fm}/c \). On the other hand, \( t = 0.1 \text{ fm}/c \) and \( 1.1 \text{ fm}/c \) are presented for the finite conductivity cases in which the decay speed is much slower. For each snapshot, results for two impact parameters, \( b = 4 \text{ fm} \) and \( 8 \text{ fm} \) are shown.

For the electric field presented in Figs. 2 and 3, one observes its magnitude decreases as the impact parameter increases. It is maximized at the most central collisions, as has been shown in Ref. \[38\]. As time evolves, the electric field spreads out in space with a decreasing magnitude. Comparing between the upper and lower panels, one can observe the electric field decays much slower when the conductivities \( \sigma \) and \( \sigma_\chi \) are present. It is interesting to note that when \( \sigma = \sigma_\chi = 0 \), the spatial distributions of both \( |E_x| \) and \( |E_y| \) appear symmetric with respect to both \( x = 0 \) and \( y = 0 \) axes. However, for finite conductivities, these distributions are only symmetric about the \( x = 0 \) axis but asymmetric about \( y = 0 \). This could be understood with the non-zero azimuthal component \( E_\phi \) with the presence of \( \sigma_\chi \). Similar to the illustration provided in Ref. \[43\], if one assumes one pro-
ton travels in $-\hat{z}$ at $(-a, 0, 0)$ while another travels in $+\hat{z}$ at $(a, 0, 0)$, the $E_r$ components they generate according to Eq. (10) will contribute to the same sign of $E_x$ at two symmetric locations with respect to the $y = 0$ axis, e.g. $(b, c, 0)$ and $(b, -c, 0)$, while opposite sign of $E_y$ at these two locations. To the contrary, the $E_\phi$ components from the two moving charges will generate opposite sign of $E_x$ while same sign of $E_y$ at the two locations above. As a result, the finite $E_\phi$ breaks the original symmetry of $E_x(y) = E_x(-y)$ and $E_y(y) = -E_y(-y)$ at zero conductivities.

Some similar features can be observed in Figs. 4 and 5 for the spatial distributions of the magnetic field, such as the slower decay of $|B_x|$ and $|B_y|$ and their spread into space after finite $\sigma$ and $\sigma_\chi$ are included, as well as their broken symmetry with respect to the reaction plane when finite conductivities are present. However, different from the electric field, it is the radial component $B_r$ in Eq. (9), determined by $\sigma_\chi$, that breaks the original $B_x(y) = -B_x(-y)$ and $B_y(y) = B_y(-y)$ symmetry at zero conductivities. In addition, different patterns of the spatial distribution can also be found between electric and magnetic fields. And opposite to the electric field, the magnetic field increases as the impact parameter increases.

In Fig. 6, we present the spatial distributions of the electric (upper panels) and magnetic (lower panels) field in the longitudinal direction.Unlike the transverse components, even in the presence of the electric and chiral magnetic conductivities, $E_z$ and $B_z$ distributions still appear symmetric about both $x = 0$ and $y = 0$ axes. This could be understood with the $B_z$ and $E_z$ components directly given by Eqs. (9) and (10). Two protons moving along $\pm \hat{z}$ at $(\pm a, 0, 0)$ yield $F_z(y) = F_z(-y)$ and
Figure 3. (Color online) The spatial distributions of $eE_y$ (in the unit of $m_e^2$) in 200 AGeV Au+Au collisions, compared between zero vs. finite conductivities, different impact parameters and evolution times.

Figure 4. (Color online) The spatial distributions of $eB_x$ (in the unit of $m_e^2$) in 200 AGeV Au+Au collisions, compared between zero vs. finite conductivities, different impact parameters and evolution times.

$F_z(x) = -F_z(-x)$, with $F_z$ for both $B_z$ and $E_z$. And compared to their corresponding transverse components, the magnitudes of electric and magnetic fields are much smaller in the longitudinal direction, as has been suggested in Refs. [1, 2, 51].

For a better illustration of the field configuration, we present the two-dimensional vector fields of $E_T$ (upper panels) and $B_T$ (lower panels) in the transverse plane at $z = 0$ in Fig. 7, in which the contour plots are for the magnitude of $|E_T|$ and $|B_T|$. One can observe a clear broken symmetry of both $|E_T|$ and $|B_T|$ about the $y = 0$ axis after finite conductivities are introduced. The breaking appears stronger for the magnetic field than the electric field. Moreover, as shown by the vector field, one can clearly see the zero electric field near the origin $(0, 0, 0)$, while a finite magnetic field along $-\hat{y}$. Note that without conductivity, the magnetic field follows the $-\hat{y}$ direction along the $x = 0$ axis. However, its direction changes after conductivities are introduced, especially when the position is away from the origin. This would affect the inner product between electric and magnetic fields, as will be discussed in the following section.
IV. SPATIAL DISTRIBUTIONS OF $\mathbf{E} \cdot \mathbf{B}$

With the separate results of $\mathbf{E}$ and $\mathbf{B}$ fields above, we further investigate the spatial distribution of their inner product $(\mathbf{E} \cdot \mathbf{B} = E_x B_x + E_y B_y + E_z B_z)$, which is directly related to the generation of the electric quadrupole moment. We keep contributions from both transverse and longitudinal components for the inner product for completeness, though one may also neglect the longitudinal part [38] due to its relatively small contribution.

In Fig. 8, we first present the spatial distribution of the angle between electric and magnetic fields in the transverse plane. For the most central collisions ($b = 0$), one may consider orthogonality between $E_T$ and $B_T$ due to the vanishing magnitude of the magnetic field. At finite impact parameter, $E_T$ and $B_T$ are orthogonal to each other around the $y = 0$ axis, but appear parallel or antiparallel around the $x = 0$ axis. In vacuum, one expects to see anti-parallel alignment in the $y > 0$ half plane while parallel alignment in the $y < 0$ half plane. However, after conductivities are introduced, parallel configuration can also be observed in the $y > 0$ half plane. This is mainly
due to the direction flip of the magnetic field, as has been discussed in Fig. 7. As time evolves, the pattern of these angular distributions expand outwards in the transverse plane. A faster expansion is seen in vacuum than in a conducting medium.

Shown in Fig. 9 is the distribution of $\mathbf{E} \cdot \mathbf{B}$, compared between zero vs. finite conductivities, and different impact parameters and evolution times. One naturally expect to see zero values for the most central collisions, while finite value for peripheral collisions. Symmetric distributions are observed with respect to the reaction plane (or the $y = 0$ axis) for the Lienard-Wiechert solution, as presented in Ref. [38]. However, these distributions become asymmetric after finite $\sigma$ and $\sigma_x$ are included. Despite the asymmetric distribution, one can still observe a dipole structure of $\mathbf{E} \cdot \mathbf{B}$, i.e., opposite signs in the $y > 0$ and $y < 0$ half planes. The generation of this dipole structure can be understood with the angular distributions in Fig. 8, where the electric and magnetic fields are generally parallel to each other in the $y < 0$ half plane.
The in-plane direction, giving rise to the charge separation of heavy-ion collisions, while negative charges into the out-of-plane direction, resulting in an electric quadrupole moment in the end. This  

\[ \frac{E}{B} \]

is also seen in the

\[ y > 0 \]

region, the small magnitude of \( J_y \) makes \( J_x \) still consistent with the findings proposed in Ref. [38].

To further investigate how the conductivities quantitatively affect the electric quadrupole moment, in Fig. 11 we compare the participant number (\( N_{\text{part}}\)) dependence of \( E \cdot B \) between the Lienard-Wiechert solution and the solution of the Maxwell equations with finite conductivities. Results are shown for different locations at the initial time. In the upper panel, we observe that at \((0, -4 \text{ fm}, 0)\), \( E \cdot B \) from with and without conductivities share similar shape of the \( N_{\text{part}} \) dependence. It first increases and then decreases as \( N_{\text{part}} \) increases, since the electric field is small at large impact parameter (small \( N_{\text{part}} \)) while the magnetic field is small at small impact parameter (large \( N_{\text{part}} \)). On the other hand, the magnitude of \( E \cdot B \) with conductivities is about 9 times smaller than that without conductivities at the initial time. In addition, while the absolute value of \( E \cdot B \) with zero conductivity are symmetric at \((0, -4 \text{ fm}, 0)\) and \((0, 4 \text{ fm}, 0)\) (the black and red curves overlap each other), such symmetry is broken (between the blue and green curves) after finite conductivities are introduced. Similar findings have also been confirmed in the lower panel for the locations of \((0, \pm 6 \text{ fm}, 0)\).

Shown in Fig. 12 is the zone averaged \(|E \cdot B|\) as a function of the participant number at the initial time. The average is conducted over the geometric overlapping region between the two colliding nuclei, i.e., region that simultaneously satisfies \((x - b/2)^2 + y^2 < r_A^2 \) and \((x + b/2)^2 + y^2 < r_A^2 \), with \( b \) being the impact parameter and...
$r_A$ being the nucleus radius parameter taken as 6.5 fm here. Uncertainties from taking different $r_A$ values and applying different average schemes have been discussed in Ref. [38] and found small. In the upper panel of Fig. 12, we first follow Ref. [38] to present the zone averaged value of $\mathbf{E} \cdot \mathbf{B}$ in the $y < 0$ half plane. Similar to previous results at a specific location, the zone averaged $\mathbf{E} \cdot \mathbf{B}$ share a similar shape with respect to $N_{\text{part}}$ between zero and finite conductivities, although the magnitude at the initial time becomes much smaller after conductivities are included.

Since the electromagnetic field in a conducting medium is asymmetric about the reaction plane, averaging in the $y < 0$ half plane is no longer a good representation of the dipole structure of $\mathbf{E} \cdot \mathbf{B}$ over the whole overlapping region. Therefore, in the lower panel of Fig. 12, we compare different average schemes for the finite conductivity scenario. Visible difference can be observed between averaging $\mathbf{E} \cdot \mathbf{B}$ over the $y < 0$ half plane (black curve) and averaging $\mathbf{E} \cdot \mathbf{B}$ over the whole overlapping region (red curve). In this lower panel, the shape of the average $|\mathbf{E} \cdot \mathbf{B}|$ is also compared to the slope parameter of the charge separation of the hadron $v_2$ measured by the STAR Collaboration [37] (purple), as proposed in Ref. [38]. The slope parameter $r$, defined via $v_2(\pi^\pm) = v_{\text{base}}(\pi^\pm) \mp r_{A_{\text{ch}}}/2$ with the charge asymmetry of the collision system given by $A_{\text{ch}} = (N_+ - N_-)/(N_+ + N_-)$, quantifies the different $v_2$ between $\pi^+$ and $\pi^-$. As shown in the figure, the average $|\mathbf{E} \cdot \mathbf{B}|$ shares a similar $N_{\text{part}}$ dependence to the measured $r$ parameter, implying the QED anomaly ($\mathbf{E} \cdot \mathbf{B}$) could be a possible source for the separation of $v_2$ between positive and negative charges. Since the electric quadrupole moment is a more direct cause of the charge separation of $v_2$, we also present the zone average of $|\langle \mathbf{E} \cdot \mathbf{B}\rangle B_y|$ in the figure (blue curve). Indeed, a better qualitative agreement is obtained with the shape of the measured $r$ parameter. Nevertheless, a quantitative description of the experimental data would require coupling the electromagnetic field with the QGP expansion (e.g. the hydrodynamic model). This is beyond the scope of the present work and will be left for a future exploration.

In the end, we study the time evolution of $\mathbf{E} \cdot \mathbf{B}$ in Fig. 13. Events with participant number between 130 and 140 are selected here (corresponding to an impact parameter around 8 fm) for Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV. Results at different locations, $(0, 3 \text{ fm}, 0)$

Figure 11. (Color online) The value of $e^2 \mathbf{E} \cdot \mathbf{B}$ (in the unit of $m_+^2$) as a function of $N_{\text{part}}$ at the initial time, and positions of $r = (0, \pm 4 \text{ fm}, 0)$ (upper panel) and $r = (0, \pm 6 \text{ fm}, 0)$ (lower panel) in 200 AGeV Au+Au collisions, compared between zero and finite conductivities.

Figure 12. (Color online) The zone-averaged $|\mathbf{E} \cdot \mathbf{B}|$ (in the unit of $m_+^2$) and $|\langle \mathbf{E} \cdot \mathbf{B}\rangle B_y|$ (in the unit of $m_+^2$) in the geometric overlapping region between colliding nuclei at the initial time of 200 AGeV Au+Au collisions, as a function of $N_{\text{part}}$, compared between zero and finite conductivities (upper panel), and between different average schemes and the slope parameter $r$ measured by the STAR Collaboration [37] (lower panel).
and (3 fm, 3 fm, 0), are presented and compared between zero and finite conductivity scenarios. One can observe although the zero conductivity scenario starts with a larger $E \cdot B$ than the finite conductivity scenario, as has also been observed previously in Figs. 11 and 12, the former decays much faster than the latter. Therefore, including finite $\sigma$ and $\sigma_\chi$ helps extend the influence of the electromagnetic field to a much later evolution stage of the QGP. Since stronger elliptic flow of the medium will be developed towards later time, introducing the electric and chiral magnetic conductivities may also quantitatively enhance the charge separation of $v_2$, or the slope parameter $r$.

## V. SUMMARY AND OUTLOOK

In this work, we have conducted a systematic study on the effects of the electric ($\sigma$) and chiral magnetic ($\sigma_\chi$) conductivities on the spacetime evolution of the electromagnetic fields generated in high-energy nuclear collisions. By coupling the charge distribution from a MC Glauber model with the solution of the Maxwell equations that include both $\sigma$ and $\sigma_\chi$, or its zero conductivity limit (Lienard-Wiechert), we have calculated the time evolution of the spatial distributions of electric ($E$) and magnetic ($B$) fields, together with the electromagnetic anomaly ($E \cdot B$) and the electric quadrupole moment $(E \cdot B)B$ at both zero and finite conductivities.

Our results show that although the electromagnetic field in vacuum is about an order of magnitude stronger than that in a conducting medium at the initial time, the former decays much faster than the latter. Additionally, in the transverse plane, while $|E|$ and $|B|$ appear symmetric about both $x = 0$ and $y = 0$ axes at zero conductivities, a broken symmetry about the $y = 0$ axis, or the reaction plane, is observed after finite conductivities are introduced. This symmetry breaking is mainly from the non-vanishing azimuthal component with the presence of $\sigma_\chi$ for the electric field $E_\tau$, while from the non-vanishing radial component with the presence of $\sigma_\chi$ for the magnetic field $B_\tau$. The magnitudes of the longitudinal components of both $E$ and $B$ appear much smaller than their transverse component, while no symmetry breaking is observed for $E_\zeta$ and $B_\zeta$ after finite conductivities are introduced. A clear dipole structure for $E \cdot B$ and a quadrupole pattern for $(E \cdot B)B$ are still observed in our results although they are both distorted compared to the vacuum scenario due to the symmetry breaking of $E$ and $B$ fields in a conducting medium. Since the magnitude of $E$ decreases, while the magnitude of $B$ increases as the impact parameter increases, one can observe a non-monotonic dependence (first increase and then decrease) of $E \cdot B$ and $(E \cdot B)B$ with respect to the nucleon participant number in heavy-ion collisions. These dependences are found qualitatively consistent with the STAR data on the slope parameter $r$ as a function of the participant number, indicating the QED anomaly could be an underlying mechanism that drives the $v_2$ separation between positive and negative charges. Since $(E \cdot B)B$ is more directly related to the electric quadrupole moment that gives rise to the charge separation, it appears to agree with the experimental data better than $E \cdot B$.

While this work provides a more quantitative understanding of the spacetime evolution of electromagnetic field and electromagnetic anomaly in relativistic heavy-ion collisions, it should be further improved in several directions. For instance, it is necessary to couple these profiles of electromagnetic field to hydrodynamic models or transport models for a more direct comparison to the charged particle observables, from which one may draw more solid conclusion about whether the QED anomaly is the key mechanism of the charge separation of the hadron $v_2$. In addition, we assumed constant values of $\sigma$ and $\sigma_\chi$ in the present study, which should vary as the QGP expands. Last but not least, apart from the slope parameter $r$ of the $v_2$ separation, there exist other observables that may help place more stringent constraints on the electromagnetic field inside a conducting medium, such as the directed flow coefficient $(v_1)$ of heavy quarks, whose precise theoretical description still remains a challenge with simplified modelings of the electromagnetic field in literature. We will extend our study to these aspects in our upcoming efforts.

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