On tilting complexes over blocks covering cyclic blocks

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ABSTRACT
Let $p$ be a prime number, $k$ an algebraically closed field of characteristic $p$, $\tilde{G}$ a finite group, and $G$ a normal subgroup of $\tilde{G}$ having a $p$-power index in $\tilde{G}$. Moreover let $B$ be a block of $kG$ with a cyclic defect group and $\tilde{B}$ be the unique block of $k\tilde{G}$ covering $B$. We study tilting complexes over the block $\tilde{B}$ and show that the block $\tilde{B}$ is a tilting-discrete algebra. Moreover we show that the set of all tilting complexes over $\tilde{B}$ is isomorphic to that over $B$ as partially ordered sets.

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1. Introduction

In representation theory of finite groups, there is a well-known and important conjecture called Broué's abelian defect group conjecture.

Conjecture 1.1. Let $k$ be an algebraically closed field of characteristic $p > 0$, $G$ a finite group, $B$ a block of the group algebra $kG$ with defect group $D$, and $b$ the Brauer correspondent of $B$ in $kN_G(D)$. If $D$ is abelian, then the block $B$ is derived equivalent to $b$.

There are many cases that the Broué's abelian defect group conjecture holds, for example $D$ is a cyclic group [22, 26], $D$ is a Klein four group [11, 23], $D$ is a $p$-solvable group [14] and more.

It is known that Broué's abelian defect group conjecture does not hold generally without the assumption that the defect group $D$ is abelian. However, even if the defect group $D$ is not abelian, it is thought that the similar statement holds in some situations and that how we may state the nonabelian version conjecture. On the other hand, in general, a derived equivalence between blocks induces a perfect isometry (see [9, Proposition 4.10] or [24, Theorem 9.2.9]). Hence, it is expected that two blocks should be derived equivalent if there is a perfect isometry between the blocks, here we recall that an isometry $I : \mathbb{Z}\text{Irr}(B) \to \mathbb{Z}\text{Irr}(b)$ is called a perfect isometry if it satisfies separability condition and integrality condition (further details can be seen in [8, Définition(2) 1.1]). In regards to this, we know the following result.

Theorem 1.2 ([15, Theorem 1.1 (iv)]). Let $k$ be an algebraically closed field of characteristic $p > 2$. Let $\tilde{G}$ be a finite group with a Sylow $p$-subgroup $\tilde{P}$ isomorphic to $C_p^n \times C_p$ for an integer $n \geq 2$. Let $\tilde{B}_0$ be the principal block of $k\tilde{G}$ and $\tilde{b}_0$ the principal block of $kN_{\tilde{G}}(P)$, where $P$ is a subgroup of $\tilde{P}$ isomorphic to $C_p^n$. Then there exists an isometry $I : \mathbb{Z}\text{Irr}(\tilde{B}_0) \to \mathbb{Z}\text{Irr}(\tilde{b}_0)$ satisfying the separability condition.
In addition to the notation in Theorem 1.2, let G be a normal subgroup such that its index in \( \tilde{G} \) is \( p \) and that \( G \) has a Sylow \( p \)-subgroup isomorphic to \( C_{p^n} \) (the existence of such a finite group is ensured by [15, Theorem 1.1 (i)]). Moreover, let \( B_0 \) be the principal block of \( kG \) and \( b_0 \) the principal block of \( kN_G(P) \). The isometry \( I \) in Theorem 1.2 is constructed from the perfect isometry \( I \) between \( B_0 \) and \( b_0 \) naturally, so we can strongly expect that the isometry \( \tilde{I} \) is in fact a perfect isometry. In particular, our interests are perfect isometries between \( \tilde{B}_0 \) and \( \tilde{b}_0 \) obtained by the ones between \( B_0 \) and \( b_0 \) coming from the derived equivalence between \( B_0 \) and \( b_0 \). Indeed we know various derived equivalences between \( B_0 \) and \( b_0 \) by [17, 18, 22, 23, 25] because \( B_0 \) and \( b_0 \) have cyclic groups isomorphic to \( C_{p^n} \) as defect groups, which means they are Brauer tree algebras with the same number of simple modules and the same multiplicity (see [6]). Moreover it may be said that there are derived equivalences between \( B_0 \) and \( b_0 \) induced from the ones between \( B_0 \) and \( b_0 \) and that the perfect isometries are in fact induced by the derived equivalences between \( \tilde{B}_0 \) and \( \tilde{b}_0 \). From these, one of our ambitions is showing that the block \( B_0 \) is derived equivalent to \( \tilde{b}_0 \) by using the one between \( B_0 \) and \( b_0 \) we know a lot of examples of. For the proof, it is essential to find a tilting complex over \( \tilde{B}_0 \) such that its endomorphism algebra in the homotopy category is Morita equivalent to \( \tilde{b}_0 \) by [21, Theorem 6.4]. Hence, it is important to classify the tilting complexes over \( \tilde{B}_0 \) for our ambitions.

After that, in [3] Aihara-Iyama introduced a partial order on the set of all silting complexes over \( \Lambda \), where \( \Lambda \) means a finite dimensional algebra (see Definition 2.10). Moreover, if \( \Lambda \) is a symmetric algebra, then any silting complex over \( \Lambda \) is a tilting complex over \( \Lambda \) (see Proposition 2.9). Hence, the set of all tilting complexes over the block \( B \) has the partially ordered structure because block algebras of finite groups are symmetric algebras. Therefore, our first aim is to give a classification of tilting complexes over the block \( B_0 \). On the other hand, in [3] they introduced silting mutations which are operations producing other silting complexes from the ones, and defined silting mutation quivers (Definition 2.13). In that paper, they show that the silting mutation quiver is equal to the Hasse quiver of all silting complexes on the partial order above [3, Theorem 2.35]. Another aim of this paper is to give a further development of the mutation theory of silting objects. Starting with [3], the mutation theory of silting objects in the derived categories over finite-dimensional algebras has been studied by many researchers, but there are few studies of the theory for the modular representation theory. Hence we aim at investigating the silting objects of the derived categories over certain block algebras. In particular, we consider the following two questions:

- When is a block algebra silting(tilting)-discrete algebra (Definition 2.15)?
- When do two block algebras have the same silting (tilting) mutation quiver?

The first question is one of the themes of silting theory, and many researchers gave various examples of silting-discrete algebras, which are characterized by the property that all silting complexes are compact up to equivalence, for example Brauer graph algebras whose Brauer graphs have at most one odd cycle and no cycle of even length [1], representation-finite piecewise hereditary algebras [4], representation-finite symmetric algebras [4], preprojective algebras of Dynkin type [5], derived-discrete algebras with finite global dimension [10] and algebras of dihedral, semidihedral and quaternion type [12] and more. If we get a positive answer of the first question and if we get a silting-discrete block, hence tilting-discrete block, then we understand that all tilting complexes are connected to each other by iterated mutation and this enables us to understand of the derived category of the modules over the block (see Theorem 2.16).

Also the answer of the second question lets us enable to understand the tilting complexes over a block algebra from those over the other one. Hence, we may reduce the consideration of the easier block to find a suitable tilting complex when we want to find a suitable tilting complex. Thus, it makes sense to consider the above two questions.

On these questions, Koshio and the author gave a partial answer in [16]. In the paper, it is shown that the number of two-term tilting complexes over \( B_0 \) is finite and that the induction functor gives the isomorphism as partially ordered sets between two-term tilting complexes over \( B_0 \) and the ones over \( \tilde{B}_0 \) by using \( \tau \)-tilting theory introduced by Adachi-Iyama-Reiten [2], here two-term tilting complexes means tilting complexes with vanishing cohomologies in degrees other than 0 and \(-1\).
Theorem 1.3 ([16, Theorem 1.2]). Let $\tilde{G}$ be a finite group, $G$ a normal subgroup such that the index $|\tilde{G} : G|$ is a $p$-power, $k$ an algebraically closed field of characteristic $p > 0$, $B$ a block of $kG$, and $\tilde{B}$ the unique block of $k\tilde{G}$ covering $B$. Assume $\tilde{B}$ satisfies the following conditions:

1. Any indecomposable $B$-module is $I_{\tilde{G}}(B)$-invariant.
2. $B$ is a $\tau$-tilting finite algebra, that is $B$ has a finite number of two-term tilting complexes.

Then the induction functor $(-)^{\uparrow \tilde{G}} : \text{mod } B \rightarrow \text{mod } \tilde{B}$ gives an isomorphism between the set of two-term tilting complexes over $B$ and that of $\tilde{B}$ as the partially ordered sets.

Here, we remark that in the setting of Theorem 1.2, the blocks $B_0$ and $\tilde{B}_0$ satisfy the condition in Theorem 1.3. Moreover, we remark that if an algebra is a tilting-discrete algebra, then the number of two-term tilting complexes over the algebra is finite. Thus, the above theorem is a partial solution of the questions and would be helpful for the solutions for the above questions. In this paper we generalize Theorem 1.3 and give the following result under the more general assumption than the case in Theorem 1.2 in conjunction with giving positive answers of above questions.

Theorem 1.4. (see Theorems 3.4 and 3.5) Let $\tilde{G}$ be a finite group, $G$ a normal subgroup such that the index $|\tilde{G} : G|$ is a $p$-power, $k$ an algebraically closed field of characteristic $p > 0$, $B$ a block of $kG$, and $\tilde{B}$ the unique block of $k\tilde{G}$ covering $B$. Assume $\tilde{B}$ satisfies the following conditions:

1. Any indecomposable $B$-module is $I_{\tilde{G}}(B)$-invariant.
2. $B$ is a $\tau$-tilting-discrete algebra.
3. Any algebra derived equivalent to $B$ has a finite number of two-term tilting complexes.

Then $\tilde{B}$ is a tilting-discrete algebra. Moreover the induction functor $(-)^{\uparrow \tilde{G}} : K^b(\text{proj } B) \rightarrow K^b(\text{proj } \tilde{B})$ induces an isomorphism between $\text{tilt } B$ and $\text{tilt } \tilde{B}$ as partially ordered sets, here $\text{tilt } B$ and $\text{tilt } \tilde{B}$ mean the set of all tilting complexes over $B$ and $\tilde{B}$, respectively.

Now we return to the case in which we are interested, that is, the covered block $B$ of $kG$ has a cyclic defect group. Then the conditions (i), (ii), and (iii) of Theorem 1.4 are satisfied automatically (see Proposition 3.7), so we can apply Theorem 1.4 to the block $B$. Moreover, in the setting of Theorem 1.4, since the principal block $B_0$ of $kG$ is the unique block covered by the principal block $\tilde{B}_0$ of $k\tilde{G}$, the following theorem would be helpful for the consideration for the situation of Theorem 1.2.

Theorem 1.5. (see Theorem 3.8) Let $\tilde{G}$ be a finite group having $G$ as a normal subgroup with its index in $\tilde{G}$ a $p$-power. Let $B$ be a block of the finite group $G$ with cyclic defect group and $\tilde{B}$ the unique block of $k\tilde{G}$ covering $B$. Then the following hold.

1. $\tilde{B}$ is a tilting-discrete algebra.
2. The induction functor $(-)^{\uparrow \tilde{G}} : \text{tilt } B \rightarrow \text{tilt } \tilde{B}$ induces an isomorphism of partially ordered sets.

In this paper, we use the following notation and terminology. Here, modules are finitely generated right modules unless otherwise stated. For a finite group $G$, a subgroup $H$ of $G$ and a $kH$-module $U$, we denote by $U^{\uparrow G}$ the induced module $U \otimes_{kH} kG$.

For a finite dimensional algebra $\Lambda$, we denote by $|\Lambda|$ the number of the isomorphism classes of simple $\Lambda$-modules. We denote by $\text{mod } \Lambda$ the category of finitely generated right $\Lambda$-modules, and by $\text{proj } \Lambda$ the category of finitely generated projective $\Lambda$-modules. We denote by $K^b(\text{proj } \Lambda)$ the bounded homotopy category of $\text{proj } \Lambda$. For $C \subseteq K^b(\text{proj } \Lambda)$, we denote by $\text{add } C$ the smallest full subcategory of $K^b(\text{proj } \Lambda)$ which contains $C$ and is closed under finite co-products, summands and isomorphisms.

This paper is organized as follows. In Section 2 we state several notation and results on block theory and silting theory. In Section 3 we give the proof of the main theorems.
2. Preliminaries

2.1. Block theory

In this section, let $k$ be an algebraically closed field of characteristic $p > 0$. We denote by $G$ a finite group, and by $kG$ the trivial module of $kG$, that is, a one-dimensional vector space on which each element in $G$ acts as the identity. We recall the definition of blocks of group algebras. The group algebra $kG$ has a unique decomposition

$$kG = B_1 \times \cdots \times B_n$$

into a direct product of subalgebras $B_i$ each of which is indecomposable as an algebra. Then each direct product component $B_i$ is called a block of $kG$. For any indecomposable $kG$-module $M$, there exists a unique block $B_i$ of $kG$ such that $M = MB_i$ and $MB_j = 0$ for all $j \in \{1, \ldots, n\} - \{i\}$. Then we say that $M$ lies in the block $B_i$ or that $M$ is a $B_i$-module. Also we denote by $B_0(G)$ the principal block of $kG$, that is, the unique block of $kG$ which does not annihilate the trivial $kG$-module $kG$.

First, we recall the definition of defect groups of blocks of finite group algebras and their properties.

**Definition 2.1.** Let $B$ be a block of $kG$. A minimal subgroup $D$ of $G$ which satisfies the following condition is uniquely determined up to conjugacy in $G$: the $B$-bimodule epimorphism

$$B \otimes_{kD} B \twoheadrightarrow B \quad (b_1 \otimes_{kD} b_2 \mapsto b_1 b_2)$$

is a split epimorphism. We call the subgroup a defect group of the block $B$.

The following results are well known (for example, see [6]).

**Proposition 2.2.** For the principal block $B_0(G)$ of $kG$, its defect group is a Sylow $p$-subgroup of $G$.

**Proposition 2.3.** For a block $B$ of $kG$ and a defect group $D$ of $B$, the following are equivalent:

1. $D$ is a nontrivial cyclic group;
2. $B$ is of finite representation type and is not semisimple;
3. $B$ is a Brauer tree algebra.

**Definition 2.4.** Let $H$ be a subgroup of $G$. For a $kH$-module $U$, we denote by $U^G := U \otimes_{kH} kG$ the induced module of $U$ from $H$ to $G$. Also, for a complex $X = (X^i, d^i)$, we denote by $X^G$ the complex $(X^i \otimes_{kH} kG, d^i \otimes_{kH} kG)$. This induces a functor from $K^b(\text{proj } kH)$ to $K^b(\text{proj } kG)$.

We will consider the case where $G$ is a normal subgroup of a finite group $	ilde{G}$ and has a $p$-power index. The following is the complex version of Green’s indecomposability theorem.

**Proposition 2.5** (see [13]). Let $G$ be a normal subgroup of a finite group $	ilde{G}$ of a $p$-power index. If $X$ is a bounded indecomposable complex of $kG$-modules, then the induced complex $X^G$ of $k\tilde{G}$-modules is also a bounded indecomposable complex of $k\tilde{G}$-modules.

**Proof.** The proof of [6, Theorem 8.8] works for complexes, hence, we get the result. \qed

**Proposition 2.6** ([20, Corollary 5.5.6]). Let $G$ be a normal subgroup of $\tilde{G}$, and $B$ a block of $G$. If the index of $G$ in $\tilde{G}$ is a $p$-power, then there exists a unique block of $k\tilde{G}$ covering $B$.

**Remark 2.7.** Let $G$ be a normal subgroup of a finite group $	ilde{G}$ of a $p$-power index, $B$ a block of $kG$, and $\tilde{B}$ the unique block of $k\tilde{G}$ covering $B$. Then by Propositions 2.5 and 2.6, for any indecomposable complex
of $K^b(\text{proj } B)$, we can easily show that the induced complex $X^\uparrow \bar{G}$ is an indecomposable complex of $K^b(\text{proj } \bar{B})$.

### 2.2. Silting mutations

In this section, $\Lambda$ means a finite-dimensional algebra unless otherwise stated. We say that a complex $P$ of $K^b(\text{proj } \Lambda)$ is basic if $P$ is isomorphic to a direct sum of indecomposable complexes which are mutually non-isomorphic.

**Definition 2.8.** Let $P$ be a complex of $K^b(\text{proj } \Lambda)$. We say that the complex is a silting complex (or tilting complex, respectively) if the following conditions are satisfied:

1. $\text{Hom}_{K^b(\text{proj } \Lambda)}(P, P[n]) = 0$ for any $n > 0$ (for any $n \neq 0$, respectively).
2. It holds that thick $P = K^b(\text{proj } \Lambda)$, where thick $P$ is the smallest thick subcategory containing add $P$.

By the above definition, it holds that tilting complexes are silting complexes, but the converse does not hold generally. By the following proposition, the silting complexes over symmetric algebras are tilting complexes. In particular, silting complexes over block algebras are tilting complexes.

**Proposition 2.9** ([3, Example 2.8]). If $\Lambda$ is a finite-dimensional symmetric algebra, then any silting complex over $\Lambda$ is a tilting complex.

In [3, Definition 2.10, Theorem 2.11], it is shown that there is a partial order on the set of silting complexes.

**Definition 2.10.** Let $P$ and $Q$ be silting complexes of $K^b(\text{proj } \Lambda)$. We define a relation $\geq$ between $P$ and $Q$ as follows:

$$P \geq Q : \iff \text{Hom}_{K^b(\text{proj } \Lambda)}(P, Q[i]) = 0 \text{ (for any } i > 0).$$

Then the relation $\geq$ gives a partial order on silt $\Lambda$, where silt $\Lambda$ means the set of isomorphism classes of basic silting complexes over $\Lambda$.

We recall the definition of mutations for silting complexes of $K^b(\text{proj } \Lambda)$ [3, Definition 2.30, Theorem 3.1].

**Definition 2.11.** Let $P$ be a basic silting complex of $K^b(\text{proj } \Lambda)$ and decompose it as $P = X \oplus M$. We take a triangle

$$X \xrightarrow{f} M' \rightarrow Y \rightarrow$$

with a minimal left (add $M$)-approximation $f$ of $X$. Then the complex $\mu_X^{-}(P) := Y \oplus M$ is a silting complex in $K^b(\text{proj } \Lambda)$ again. We call the complex $\mu_X^{-}(P)$ a left mutation of $P$ with respect to $X$. If $X$ is indecomposable, then we say that the left mutation is irreducible. We define the (irreducible) right mutation $\mu_X^{+}(P)$ dually. Mutation will mean either left or right mutation.

**Remark 2.12.** If $\Lambda$ is a finite-dimensional symmetric algebra, then, for any tilting complex $P = X \oplus M$ over $\Lambda$, the complex $\mu_X^{\epsilon}(P)$ is a tilting complex again by Proposition 2.9, where $\epsilon$ means $+$ or $-$.
The silting mutation quiver was introduced by [3, Definition 2.41].

**Definition 2.13 ([3, Definition 2.41])**. The silting mutation quiver of \( K^b(\text{proj } \Lambda) \) is defined as follows:
- The set of vertices is silt \( \Lambda \).
- We draw an arrow from \( P \) to \( Q \) if \( Q \) is an irreducible left mutation of \( P \).

The following proposition shows that the Hasse quiver of the partially ordered set silt \( \Lambda \) is exactly the silting mutation quiver of \( K^b(\text{proj } \Lambda) \).

**Theorem 2.14 ([3, Theorem 2.35])**. For any silting complexes \( P \) and \( Q \) over \( \Lambda \), the following conditions are equivalent:
1. \( Q \) is an irreducible left mutation of \( P \);
2. \( P \) is an irreducible right mutation of \( Q \);
3. \( P > Q \) and there is no silting complex \( L \) satisfying \( P > L > Q \).

From now on, we assume that \( \Lambda \) is a finite-dimensional symmetric algebra unless otherwise stated. In particular, any silting complex over \( \Lambda \) is a tilting complex over \( \Lambda \) by Proposition 2.9.

We recall the definition of tilting-discrete algebras.

**Definition 2.15.** We say that an algebra (which is not necessarily a symmetric algebra) \( \Lambda \) is a tilting-discrete algebra if for all \( \ell > 0 \) and any tilting complex \( P \) over \( \Lambda \), the set

\[ \{ T \in \text{tilt } \Lambda \mid P \geq T \geq P[\ell]\} \]

is a finite set, where tilt \( \Lambda \) means the set of isomorphism classes of basic tilting complexes over \( \Lambda \).

**Theorem 2.16 ([4, Theorem 3.5]).** If \( \Lambda \) is a tilting-discrete algebra, then \( \Lambda \) is a strongly tilting connected algebra, that is, for any tilting complexes \( T \) and \( U \), the complex \( T \) can be obtained from \( U \) by either iterated irreducible left mutation or iterated irreducible right mutation.

There is an equivalent condition on the tilting-discreteness, which plays an important role later.

**Theorem 2.17 ([5, Theorem 1.2]).** For a finite-dimensional self-injective algebra \( \Lambda \), the following are equivalent.

1. \( \Lambda \) is a tilting-discrete algebra.
2. \( 2\text{-tilt}_p \Lambda := \{ T \in \text{tilt } \Lambda \mid P \geq T \geq P[1]\} \) is a finite set for any tilting complex \( P \) which is given by iterated irreducible left tilting mutation from \( \Lambda \).

The following lemma is essential for the proof of Theorem 3.4.

**Lemma 2.18.** Let \( \Lambda \) be a finite-dimensional symmetric algebra and \( P \) a tilting complex over \( \Lambda \). Then the underlying graph of the Hasse quiver of the partially ordered set \( 2\text{-tilt}_p \Lambda \) is a \( |\Lambda| \)-regular graph. Moreover, for any algebra \( \Gamma \) derived equivalent to \( \Lambda \), if the number of two-term tilting complexes over \( \Gamma \) is finite, then \( 2\text{-tilt}_p \Lambda \) is a finite \( |\Lambda| \)-regular graph.

**Proof.** First, we remark that, for any symmetric finite-dimensional algebra \( \Lambda \) and for any tilting complex \( T \) over \( \Lambda \), if it holds that \( \Lambda \geq T \geq \Lambda[1] \), then \( T \) is a two-term tilting complex by the definition of the partial order on tilt \( \Lambda \) (we recall that a complex \( X = (X^i) \) is said to be a two-term complex if \( X^i = 0 \) if \( i \neq 0, -1 \)). Let \( P \) be a tilting complex over \( \Lambda \). Since finite-dimensional symmetric algebras are closed under the derived equivalence, the endomorphism algebra \( \text{End}_{K^b(\text{proj } \Lambda)}(P) \) is also a finite-dimensional symmetric algebra. Now, we consider the derived equivalence \( F : K^-(\text{proj } \Lambda) \to \)
$K^-(\text{proj}\text{End}_{K^b(\text{proj}\Lambda)}(P))$ induced by the tilting complex $P$ over $\Lambda$. The functor induces an isomorphism between the following two partially ordered sets:

- $2\text{-}\text{tilt}_P\Lambda = \{T \in \text{tilt}\Lambda \mid P \geq T \geq P[1]\}$
- $\{T' \in \text{tilt}\text{End}_{K^b(\text{proj}\Lambda)}(P) \mid \text{End}_{K^b(\text{proj}\Lambda)}(P) \supseteq T' \supseteq \text{End}_{K^b(\text{proj}\Lambda)}(P)[1]\}$

The latter is the set of all two-term tilting complexes over $\text{End}_{K^b(\text{proj}\Lambda)}(P)$. Hence it is isomorphic to the support $\tau$-tilting quiver of $\text{End}_{K^b(\text{proj}\Lambda)}(P)$ by [2, Corollary 3.9], so its underlying graph is a $|\Lambda|$-regular graph since the derived equivalence between $\Lambda$ and $\text{End}_{K^b(\text{proj}\Lambda)}(P)$ means $|\text{End}_{K^b(\text{proj}\Lambda)}(P)| = |\Lambda|$ (we recall that two support $\tau$-tilting modules are connected by an arrow in the support $\tau$-tilting quiver if and only if they are mutations of each other and that for each support $\tau$-tilting module $M$ over $\Lambda$ there are $|\Lambda|$ sorts of mutations of $M$). Therefore, the underlying graph of the Hasse quiver of the set 2-tilt$_P\Lambda$ is a $|\Lambda|$-regular graph too.

The remaining argument is clear because $\text{End}_{K^b(\text{proj}\Lambda)}(P)$ is derived equivalent to $\Lambda$ for any tilting complex $P$ over $\Lambda$.

\begin{corollary}
For a Brauer tree algebra $\Lambda$ and any tilting complex $P$ over $\Lambda$, the set 2-tilt$_P\Lambda$ is a finite $|\Lambda|$-regular graph.
\end{corollary}

\begin{proof}
If $\Lambda$ is a Brauer tree algebra, then, for any tilting complex $P$ over $\Lambda$, the endomorphism algebra $\text{End}_{K^b(\text{proj}\Lambda)}(P)$ is a Brauer tree algebra again. Hence, the number of elements in 2-tilt$_P\Lambda$ which is isomorphic to the set of all two-term tilting complex over $\text{End}_{K^b(\text{proj}\Lambda)}(P)$ is finite by [7, Theorem 1.1] or [4, Theorem 5.2]. Hence, we can apply Lemma 2.18 and complete the proof.
\end{proof}

\section{Main theorem and its proof}

\subsection{Notation and assumption}

In this section, we use the following notation. Let $k$ be an algebraically closed field of characteristic $p > 0$. We denote by $G$ a finite group, and by $\tilde{G}$ a finite group having $G$ as a normal subgroup and assume that the index of $G$ in $\tilde{G}$ is $p^n$ for some positive integer $n$. We denote by $B$ a block of $kG$. Then by Proposition 2.6, there exists a unique block of $k\tilde{G}$ covering $B$. We denote the unique block of $k\tilde{G}$ by $\tilde{B}$. Also we denote by $I_{\tilde{G}}(B) := \{\tilde{g} \in \tilde{G} \mid \tilde{g}B\tilde{g}^{-1} = B\}$ the inertial group of the block $B$ in $\tilde{G}$. Moreover we assume the following condition is satisfied:

\begin{itemize}
\item[(Inv)] Any $B$-module is $I_{\tilde{G}}(B)$-invariant, that is, for any $B$-module $U$ and any $\tilde{g} \in I_{\tilde{G}}(B)$, it holds that $U\tilde{g} \cong U$ as $B$-modules.
\end{itemize}

\begin{lemma}
The condition (Inv) on the invariance of modules implies that of complexes of $K^b(\text{proj} B)$.
\end{lemma}

\begin{proof}
Assume that the condition (Inv) is satisfied and that any complex of $K^b(\text{proj} B)$ of length less than or equal to $n$ is $I_{\tilde{G}}(B)$-invariant. Take an arbitrary complex $(X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} X_n)$ of projective $B$-modules, and let $Y$ be the truncated complex $(X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} X_n)$. Then, by the assumption we have that

$$Y\tilde{g} = (X_1\tilde{g} \xrightarrow{d_1} X_2\tilde{g} \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} X_n\tilde{g}) \cong Y$$

for any $\tilde{g} \in I_{\tilde{G}}(B)$, where $d_i' : X_i\tilde{g} \to X_{i+1}\tilde{g}$ is the homomorphism which maps $x_i\tilde{g}$ to $d_i(x_i)\tilde{g}$ for each $i$. We need to show that

$$X\tilde{g} = (X_0\tilde{g} \xrightarrow{d_1'} X_1\tilde{g} \xrightarrow{d_2'} X_2\tilde{g} \xrightarrow{d_3'} \cdots \xrightarrow{d_{n-1}'} X_n\tilde{g}) \cong X$$

We easily see that $\text{Im} d_i' = (\text{Im} d_i)\tilde{g}$ for each $i$. Hence, we have that $\text{Im} d_0 \cong \text{Im} d_0'$ by the condition (Inv). By the injectivity of $I(\text{Im} d_0)$, we have the extension $\alpha'_0 : I(\text{Im} d_0) \to I(\text{Im} d_0')$ of the isomorphism
Im \( d_0 \) \( \cong \) Im \( d'_0 \), where \( I(\text{Im } d_0) \) and \( I(\text{Im } d'_0) \) mean injective envelopes of \( I(\text{Im } d_0) \) and \( I(\text{Im } d'_0) \), respectively, that is we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Im } d_0 & \xrightarrow{i_0} & I(\text{Im } d_0) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\text{Im } d'_0 & \xrightarrow{\alpha'_1} & I(\text{Im } d'_0).
\end{array}
\]

Since the homomorphism \( \alpha'_1 \circ i_0 \) is a monomorphism and \( i_0 : \text{Im } d_0 \to I(\text{Im } d_0) \) is an essential monomorphism, we have the homomorphism \( \alpha'_1 \) is a monomorphism. Hence, we have that \( \alpha'_1 \) is an isomorphism because \( \dim I(\text{Im } d_0) = \dim I(\text{Im } d'_0) \). Since \( I(\text{Im } d_0) \) is a direct summand of \( X_1 \) and \( I(\text{Im } d'_0) \) is that of \( X_1\hat{g} \), we have an isomorphism \( \alpha'_1 : X_1/I(\text{Im } d_0) \to X_1\hat{g}/I(\text{Im } d'_0) \) by Krull-Schmidt theorem. Using two direct decompositions \( X_1 = I(\text{Im } d_0) \oplus X_1/I(\text{Im } d_0) \) and \( X_1\hat{g} = I(\text{Im } d'_0) \oplus X_1\hat{g}/I(\text{Im } d'_0) \), we define the homomorphism

\[
\alpha_1 : X_1 = I(\text{Im } d_0) \oplus X_1/I(\text{Im } d_0) \xrightarrow{\begin{bmatrix} \alpha'_1 & 0 \\ 0 & \alpha''_1 \end{bmatrix}} I(\text{Im } d'_0) \oplus X_1\hat{g}/I(\text{Im } d'_0) = X_1\hat{g}.
\]

Obviously the homomorphism \( \alpha_1 \) is an isomorphism.

Next, let \( P(\text{Im } d_0) \) and \( P(\text{Im } d'_0) \) be projective covers of \( \text{Im } d_0 \) and \( \text{Im } d'_0 \), respectively. By the projectivity of \( P(\text{Im } d_0) \), we have a homomorphism \( \alpha'_0 \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
P(\text{Im } d_0) & \xrightarrow{\alpha'_0} & \text{Im } d_0 \\
\downarrow{\cong} & & \downarrow{\cong} \\
P(\text{Im } d'_0) & \xrightarrow{\cong} & \text{Im } d'_0.
\end{array}
\]

The dual argument to the previous one gives the homomorphism \( \alpha'_0 \) is an isomorphism. Hence, we have an isomorphism

\[
\alpha_0 : X_0 = P(\text{Im } d_0) \oplus X_0/P(\text{Im } d_0) \xrightarrow{\begin{bmatrix} \alpha'_0 & 0 \\ 0 & \alpha''_0 \end{bmatrix}} P(\text{Im } d'_0) \oplus X_0\hat{g}/P(\text{Im } d'_0) = X_0\hat{g},
\]

where \( \alpha''_0 : X_0/P(\text{Im } d_0) \to X_0\hat{g}/P(\text{Im } d'_0) \) is an isomorphism. By the construction of \( \alpha_0 \) and \( \alpha_1 \), we have the following commutative diagram:

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\alpha_0} & X_1 \\
\downarrow{\cong} & & \downarrow{\cong} \\
X_0\hat{g} & \xrightarrow{\alpha_1} & X_1\hat{g}
\end{array}
\]
Moreover we can take an injective resolution \( I(Y) \) starting with \( \alpha_1 : X_1 \to X_1\widehat{g} \) since \( X_1\widehat{g} \) is an injective envelope of \( X_1 \):

\[
Y : \\
\begin{array}{cccc}
X_1 & \xrightarrow{d_1} & X_2 & \cdots & \xrightarrow{d_{n-1}} & X_n \\
\downarrow{\alpha_1} & & \downarrow{\beta_2} & & \downarrow{\beta_n} \\
I(Y) : & \xrightarrow{\overleftarrow{d_n}} & I(Y)_2 & \cdots & \xrightarrow{\overleftarrow{d_n}} & I(Y)_n
\end{array}
\]

Since \( \beta_2 \circ d_1 \circ \alpha_1^{-1} \circ d'_0 = 0 \), we have a complex

\[
X_0\widehat{g} \xrightarrow{d'_0} X_1\widehat{g} \to I(Y)_2 \to \cdots \to I(Y)_n.
\]

Also, by the assumption that \( Y \cong Y\widehat{g} \), we have that \( I(Y) \cong Y\widehat{g} \), that is we have the following commutative diagram with each \( \gamma_i \) invertible:

\[
\begin{array}{cccc}
I(Y) : & \xrightarrow{\overleftarrow{d_n}} & I(Y)_2 & \cdots & \xrightarrow{\overleftarrow{d_n}} & I(Y)_n \\
\downarrow{\gamma_2} & & \downarrow{\gamma_2} & & \downarrow{\gamma_n} \\
Y\widehat{g} : & \xrightarrow{\overleftarrow{d_n}} & X_2\widehat{g} & \cdots & \xrightarrow{\overleftarrow{d_n}} & X_n\widehat{g}.
\end{array}
\]

Therefore, we have the following commutative diagram:

\[
\begin{array}{cccc}
X : & \xrightarrow{d_0} & X_1 & \xrightarrow{d_1} & X_2 & \cdots & \xrightarrow{d_{n-1}} & X_n \\
\downarrow{\alpha_0} & & \downarrow{\alpha_1} & & \downarrow{\beta_2} & & \downarrow{\beta_n} \\
X_0\widehat{g} : & \xrightarrow{d'_0} & X_1\widehat{g} & \to I(Y)_2 & \cdots & \to I(Y)_n \\
\downarrow{\gamma_2} & & \downarrow{\gamma_2} & & \downarrow{\gamma_n} \\
X\widehat{g} : & \xrightarrow{d'_0} & X_1\widehat{g} & \xrightarrow{d'_1} & X_2\widehat{g} & \cdots & \xrightarrow{d'_n} & X_n\widehat{g}.
\end{array}
\]

Since all vertical homomorphisms are invertible, we get that \( X \cong X\widehat{g} \).

\[\square\]

**Lemma 3.2.** Let \( X \) and \( Y \) be complexes of \( K^b(\text{proj } B) \), and let \( \mathcal{M} \) be a full subcategory of \( K^b(\text{proj } B) \). Then the following hold.

1. For a minimal left \( \mathcal{M} \)-approximation \( f : X \to M \) of \( X \) in \( K^b(\text{proj } B) \), the induced map \( f^{\uparrow \widehat{G}} : X^{\uparrow \widehat{G}} \to M^{\uparrow \widehat{G}} \) is a minimal left \( \mathcal{M}^{\uparrow \widehat{G}} \)-approximation of \( X^{\uparrow \widehat{G}} \) in \( K^b(\text{proj } B) \).
2. For a minimal right \( \mathcal{M} \)-approximation \( f : M \to Y \) of \( M \) in \( K^b(\text{proj } B) \), the induced map \( f^{\uparrow \widehat{G}} : M^{\uparrow \widehat{G}} \to Y^{\uparrow \widehat{G}} \) is a minimal right \( \mathcal{M}^{\uparrow \widehat{G}} \)-approximation of \( Y^{\uparrow \widehat{G}} \) in \( K^b(\text{proj } B) \).

**Proof.** We prove (1), and the other can be proved similarly.

First we show that \( f^{\uparrow \widehat{G}} : X^{\uparrow \widehat{G}} \to M^{\uparrow \widehat{G}} \) is left minimal. Take a morphism \( h : M^{\uparrow \widehat{G}} \to M^{\uparrow \widehat{G}} \) satisfying \( h \circ f^{\uparrow \widehat{G}} = f^{\uparrow \widehat{G}} \). By restricting \( h \circ f^{\uparrow \widehat{G}} = f^{\uparrow \widehat{G}} \) to \( X \otimes_{k\widehat{G}} 1 \cong X \), we have that \( h \circ f = f \), which means that \( h : M \to M \) is an isomorphism by the left minimality of \( f \). Hence, \( h : M^{\uparrow \widehat{G}} \to M^{\uparrow \widehat{G}} \) is an isomorphism too.

It remains to show that \( f^{\uparrow \widehat{G}} : X^{\uparrow \widehat{G}} \to M^{\uparrow \widehat{G}} \) is a left \( \mathcal{M}^{\uparrow \widehat{G}} \)-approximation of \( X^{\uparrow \widehat{G}} \) in \( K^b(\text{proj } B) \). We can assume \( I_G(B) = \widehat{G} \) because the unique block of \( k\widehat{G}(B) \) covering \( B \) is Morita equivalent to the block \( \widehat{B} \) of \( k\widehat{G} \) by [20, Theorem 5.5.12]. Let \( Z \) be a complex of \( \mathcal{M} \), and let \( \alpha : X^{\uparrow \widehat{G}} \to Z^{\uparrow \widehat{G}} \) be a morphism in \( K^b(\text{proj } B) \). We show that there exists a morphism \( \beta : M^{\uparrow \widehat{G}} \to Z^{\uparrow \widehat{G}} \) such that \( \alpha = \beta \circ f^{\uparrow \widehat{G}} \),
which means that $f^\dagger G$ is a left $\mathcal{M}^\dagger G$-approximation of $X^\dagger G$ in $K^b(\text{proj} \tilde{B})$. By Frobenius reciprocity and Mackey’s formula, we get isomorphisms

$$
\text{Hom}_{K^b(\text{proj} \tilde{B})}(X^\dagger G, Z^\dagger G) \cong \text{Hom}_{K^b(\text{proj} B)}(X, Z^\dagger G)
\cong \bigoplus_{\tilde{G} \in \tilde{G}/G} \text{Hom}_{K^b(\text{proj} B)}(X, \tilde{Z}^\dagger G)
\cong \text{Hom}_{K^b(\text{proj} B)}(X, Z)^{\oplus p^n},
$$

here the last isomorphism comes from the assumption that any $B$-module is $I_G(B)$-invariant. Hence, we can take the morphisms $(\alpha_1, \ldots, \alpha_{p^n}) \in \text{Hom}_{K^b(\text{proj} B)}(X, Z)^{\oplus p^n}$ corresponding to $\alpha \in \text{Hom}_{K^b(\text{proj} B)}(X^\dagger G, Z^\dagger G)$.

**Theorem 3.4.** Assume that the block $B$ is a tilting-discrete algebra and that for any algebra is a tilting complex. Moreover, by [20, Theorem 5.5.12] the induction functor $F \circ \text{ra n yt i l t i n gc o m p l e x} \overline{T} \overline{\text{over}B}$, the induce d complex $T$.

**Proof.** Let $T$ be a tilting complex over $B$. By the assumption (Inv), Lemma 3.1 means that the complex $T$ is $I_G(B)$-invariant. Hence, applying [19, Proposition 2.1] to $T$, we have that the induced complex $T^\dagger G(B)$ is a tilting complex. Moreover, by [20, Theorem 5.5.12] the induction functor $- \otimes_{kG(B)} k\tilde{G}$ gives Morita equivalence between mod $A$ and mod $\tilde{B}$, where $A$ is the unique block of $kG(B)$ covering $B$. Hence, $(T^\dagger G(B))^\dagger \tilde{G} = T^\dagger \tilde{G}$ is a tilting complex over $\tilde{B}$.

Next we show that, for a tilting complex $\mu_{ik}^\epsilon(T)$, it holds that $(\mu_{ik}^\epsilon(T))^\dagger \tilde{G}$ is isomorphic to $\mu_{ik}^\epsilon(T^\dagger \tilde{G})$, where $\epsilon$ means $+$ or $-$. We prove the case of $\epsilon$ being $-$ because the other case can be shown similarly.

For a decomposition $T = T_{ik} \oplus (\bigoplus_{i \neq ik} T_i)$ of $T$ with $T_{ik}$ indecomposable, we consider the distinguished triangle

$$
T_{ik} \xrightarrow{f} M \xrightarrow{\text{Cone}(f)} \to,
$$

where $f$ is a minimal left approximation of $T_{ik}$ with respect to add$(\bigoplus_{i \neq ik} T_i)$. Then, by Lemma 3.2, the morphism $f^\dagger \tilde{G} : T_{ik}^\dagger \tilde{G} \to M^\dagger \tilde{G}$ is a minimal left approximation of $T_{ik}^\dagger \tilde{G}$ with respect to add$(\bigoplus_{i \neq ik} T_i^\dagger \tilde{G})$.

Hence, we get a distinguished triangle

$$
T_{ik}^\dagger \tilde{G} \xrightarrow{f^\dagger \tilde{G}} M^\dagger \tilde{G} \to \text{Cone}(f^\dagger \tilde{G}) \to
$$

with a minimal left approximation $f^\dagger \tilde{G}$ of $T_{ik}^\dagger \tilde{G}$ with respect to add$(\bigoplus_{i \neq ik} T_i^\dagger \tilde{G})$ by Lemma 3.2.

Moreover we can show that $\text{Cone}(f^\dagger \tilde{G})$ is isomorphic to $\text{Cone}(f)^\dagger \tilde{G}$ easily (for example see [27, Lemma 3.4.10]). Therefore, we get

$$
\mu_{ik}^\circ(T^\dagger \tilde{G}) = \text{Cone}(f^\dagger \tilde{G}) \oplus \bigoplus_{i \neq ik} T_i^\dagger \tilde{G} \cong \text{Cone}(f)^\dagger \tilde{G} \oplus \bigoplus_{i \neq ik} T_i^\dagger \tilde{G} = (\mu_{ik}^\circ(T))^\dagger \tilde{G},
$$

which means that mutations commute with the induction functor in our situation. □

**Theorem 3.4.** Assume that the block $B$ is a tilting-discrete algebra and that for any algebra $\Lambda$ derived equivalent to $B$ the number of two-term tilting complexes over $\Lambda$ is finite. Then the block $\tilde{B}$ of $k\tilde{G}$ is also a tilting-discrete algebra.
Proof. We can assume $I_G(B) = \tilde{G}$ (see the proof of Proposition 3.3). We show that the set 2-tilt\_\_B is a finite set for any tilting complex $\tilde{T}$ given by iterated irreducible left mutation from $\tilde{B}$, which means that $\tilde{B}$ is a tilting-discrete algebra by Theorem 2.17.

By the choice of the tilting complex $\tilde{T}$, there exist a sequence $(i_k, i_{k-1}, \ldots, i_1)$ such that
$$\tilde{T} \cong \langle \mu_{i_k} \mu_{i_{k-1}} \cdots \mu_{i_1} \rangle (\tilde{B}).$$

Moreover, for a tilting complex $T := \langle \mu_{i_k} \mu_{i_{k-1}} \cdots \mu_{i_1} \rangle (B)$ over $B$, we have $T^\uparrow \cong \tilde{T}$ by Proposition 3.3.

On the other hand, for any tilting complex $U$ and for any $i > 0$, from Frobenius reciprocity, Mackey’s formula, the assumption (Inv) and Lemma 3.1, we have the following isomorphisms:
$$\text{Hom}_{K^b(\text{proj } B)}(T^\uparrow U, U^\uparrow[i]) \cong \text{Hom}_{K^b(\text{proj } B)}(T, U \tilde{g}[i])$$
$$\cong \bigoplus_{g \in \tilde{G}/G} \text{Hom}_{K^b(\text{proj } B)}(T, U \tilde{g}[i])$$
$$\cong \text{Hom}_{K^b(\text{proj } B)}(T, U[i]) \oplus p^n.$$

Hence, we have that if $T \geq U$ then $T^\uparrow \geq U^\uparrow$. By a similar argument, we have that if $U \geq T[1]$ then $U^\uparrow \geq T^\uparrow[1]$. Summarizing the above, we have that if $T \geq U \geq T[1]$, then $\tilde{T} \geq U^\uparrow \geq \tilde{T}[1]$. Hence, we get an injection map preserving partial order structures;
$$2\text{-tilt}_T B \mapsto 2\text{-tilt}_T \tilde{B} (U \mapsto U^\uparrow \tilde{G}).$$

Now, both Hasse quivers of 2-tilt\_\_T B and 2-tilt\_\_\_\_\_B are $|B|$-regular graphs by Lemma 2.18 (we remark that $|B| = |\tilde{B}|$ because $\tilde{G}/G$ is a $p$-group), and the two sets have the respective maximal elements $T$ and $\tilde{T}$ and have the respective minimal elements $T[1]$ and $\tilde{T}[1]$. Moreover by Theorem 2.14 and Proposition 3.3, for any tilting complexes $U$ and $V$ in 2-tilt\_\_T B with $U > V$, if there is no tilting complex $L$ over $B$ satisfying that $U > L > V$, then there is no tilting complex $\tilde{L}$ over $\tilde{B}$ satisfying that $U^\uparrow > \tilde{L} > V^\uparrow$. Also, by the assumption and Lemma 2.18, the Hasse quiver of 2-tilt\_\_T B is a finite $|B|$-regular graph. Hence, we get that the above injection map is an isomorphism between the partially ordered sets. Therefore, the set 2-tilt\_\_B is a finite set.

Theorem 3.5. Assume that the block $B$ is a tilting-discrete algebra and that for any algebra $\Lambda$ derived equivalent to $B$ the number of two-term tilting complexes over $\Lambda$ is finite. Then the induction functor $(\cdot)^{\uparrow \tilde{G}} : \text{tilt } B \rightarrow \text{tilt } \tilde{B}$ induces an isomorphism of partially ordered sets.

Proof. By Proposition 3.3, we have the well-defined map $(\cdot)^{\uparrow \tilde{G}} : \text{tilt } B \rightarrow \text{tilt } \tilde{B}$. Let $T_1$ and $T_2$ be tilting complexes over $B$ and assume that $T_1 > T_2$ and that there exists no tilting complex $L$ over $B$ such that $T_1 > L > T_2$. Hence, by Theorem 2.14 and Proposition 3.3 again, we have that $T_1^\uparrow \geq T_2^\uparrow$ and there exists no tilting complex $\tilde{L}$ over $\tilde{B}$ such that $T_1^\uparrow \geq \tilde{L} > T_2^\uparrow$ in $\text{tilt } \tilde{B}$. Also, it is obvious that the map $(\cdot)^{\uparrow \tilde{G}} : \text{tilt } B \rightarrow \text{tilt } \tilde{B}$ is a surjection. Therefore, for a tilting complex $T = (\mu_{i_k} \mu_{i_{k-1}} \cdots \mu_{i_1}) (B)$ over $B$, by Proposition 3.3, we have that $\tilde{T} \cong T^\uparrow \tilde{G}$. □

As an immediate application of Theorem 3.4, we show that the homotopy category $K^b(\text{proj } \tilde{B})$ satisfies a Bongartz-type Lemma.

Theorem 3.6. In the same notation as Theorem 3.4, any pretilting complex $\tilde{P}$ in $K^b(\text{proj } \tilde{B})$ is a partial tilting complex, that is, there exists a pretilting complex $\tilde{P}'$ in $K^b(\text{proj } \tilde{B})$ such that $\tilde{P} \oplus \tilde{P}'$ is a tilting complex in $K^b(\text{proj } \tilde{B})$. □
Proof. By Theorem 3.4, $\tilde{B}$ is a tilting-discrete algebra, that is, the homotopy category $K^b(\text{proj} \tilde{B})$ is a tilting-discrete category. Therefore, we have the result by [5, Theorem 2.15].

In the rest of this section, we assume the block $B$ of $kG$ has a cyclic defect group. In this case, the block $B$ has the following nice properties.

**Proposition 3.7.** Let $B$ be a block of $kG$ with a cyclic defect group. Then we have the following.

1. [6, Section 5] The block $B$ is a Brauer tree algebra.
2. [22, Theorem 4.2] Any algebra derived equivalent to $B$ is a Brauer tree algebra associated to a Brauer tree with the same number of edges and multiplicity as the Brauer tree of $B$.
3. [16, Lemma 3.22] The condition (Inv) holds automatically.
4. [4, Theorem 1.2] $B$ is a tilting-discrete algebra.
5. For any tilting complex $P$ over $B$, the set $2$-tilt$_P B$ is a finite $|B|$-regular graph (see Corollary 2.19).

**Theorem 3.8.** Let $G$ be a finite group and $\tilde{G}$ be a finite group having $G$ as a normal subgroup and assume that the index of $G$ in $\tilde{G}$ is a $p$-power. Let $B$ be a block of the finite group $\tilde{G}$ with cyclic defect group, and $\tilde{B}$ be the unique block of $k\tilde{G}$ covering $B$. Then the following hold.

1. $\tilde{B}$ is a tilting-discrete algebra.
2. The induction functor $(-)^{\uparrow \tilde{G}} : \text{tilt} B \to \text{tilt} \tilde{B}$ induces an isomorphism of partially ordered sets.

Proof. By Proposition 3.7, the blocks $B$ and $\tilde{B}$ satisfy the assumption in Theorems 3.4 and 3.5. Therefore we have the result.
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