POWERS OF SYMMETRIC DIFFERENTIAL OPERATORS I.

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Abstract. Let $L$ be a linear symmetric differential operators on $L^2(\mathbb{R})$ whose domain is the Schwartz test function space, $\mathcal{S}$. For the majority of this paper, it is assumed that the coefficient of $L$ are polynomial functions on $\mathbb{R}$. We will give criteria on the polynomial coefficients of $L$ which guarantees that $L$ is essentially self-adjoint, $\bar{L} \geq -CI$ for some $C < \infty$, and that $\mathcal{S}$ is a core for $(L + C)^r$ for all $r \geq 0$. Given another polynomial coefficient differential operator, $\tilde{L}$, we will further give criteria on the coefficients $L$ and $\tilde{L}$ which implies operator comparison inequalities of the form $(\tilde{L} + C)^r \leq C r (L + C)^r$ for all $0 \leq r < \infty$. The last inequality generalized to allow for an added parameter, $\hbar > 0$, in the coefficients is used to provide a large class of operators satisfying the hypotheses in [3] where a strong form of the classical limit of quantum mechanics is shown to hold.

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1. Introduction

Let $L^2(\mathbb{R}) := L^2(\mathbb{R}, dx)$ be the Hilbert space of square integrable complex valued functions on $\mathbb{R}$ relative to Lebesgue measure, $dx$. The inner product on $L^2(\mathbb{R})$ is taken to be

$$\langle u, v \rangle := \int_{\mathbb{R}} u(x) \overline{v}(x) \, dx \quad \text{for } u, v \in L^2(\mathbb{R}) \tag{1.1}$$

and the corresponding norm is $\|u\| = \sqrt{\langle u, u \rangle}$. [Note that we are using the mathematics convention that $\langle u, v \rangle$ is linear in the first variable and conjugate linear in the second.] .

**Notation 1.1.** Let $C^\infty(\mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{C})$ denote the smooth functions from $\mathbb{R}$ to $\mathbb{C}$, $C^\infty_c(\mathbb{R})$ denote those $f \in C^\infty(\mathbb{R})$ which have compact support, and $\mathcal{S} := \mathcal{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$ be the subspace of Schwartz test functions, i.e. those $f \in C^\infty(\mathbb{R})$ such that $f$ and its derivatives vanish at infinity faster than $|x|^{-n}$ for all $n \in \mathbb{N}$.

**Notation 1.2.** Let $C^\infty(\mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{C})$. Also, let $\partial : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ denote the differentiation operator, i.e. $\partial f(x) = f'(x) = \frac{d}{dx} f(x)$.

**Notation 1.3.** Given a function $f : \mathbb{R} \to \mathbb{C}$, we let $M_f g := fg$ for all functions $g : \mathbb{R} \to \mathbb{C}$, i.e. $M_f$ denotes the linear operator given by multiplication by $f$. Notice that if $f \in C^\infty(\mathbb{R})$ then we may view $M_f$ as a linear operator from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$.

For the purposes of this paper, a $d^{th}$-order linear differential operator on $C^\infty(\mathbb{R})$ with $d \in \mathbb{N}$ is an operator $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ which may be expressed as

$$L = \sum_{k=0}^{d} M_{a_k} \partial^k = \sum_{k=0}^{d} a_k \partial^k \tag{1.2}$$

for some $\{a_k\}_{k=0}^{d} \subset C^\infty(\mathbb{R}, \mathbb{C})$. The symbol of $L$, $\sigma = \sigma_L$, is the function on $\mathbb{R} \times \mathbb{R}$ defined by

$$\sigma_L(x, \xi) := \sum_{k=0}^{d} a_k(x) (i\xi)^k. \tag{1.3}$$

**Remark 1.4.** The action of $L$ on $C^\infty_c(\mathbb{R})$ completely determines the coefficients, $\{a_k\}_{k=0}^{d}$. Indeed, suppose that $x_0 \in \mathbb{R}$ and $0 \leq k \leq d$ and let $\varphi(x) := (x - x_0)^k \chi(x)$ where $\chi \in C^\infty_c(\mathbb{R})$ such that $\chi = 1$ in a neighborhood of $x_0$. Then an elementary computation shows $k! \cdot a_k(x_0) = (L \varphi)(x_0)$. In particular if $L \varphi \equiv 0$ for all $\varphi \in C^\infty_c(\mathbb{R})$ then $a_k \equiv 0$ for $0 \leq k \leq d$ and hence $L \varphi \equiv 0$ for all $\varphi \in C^\infty(\mathbb{R})$.

**Definition 1.5** (Formal adjoint and symmetry). Suppose $L$ is a linear differential operator on $C^\infty(\mathbb{R})$ as in Eq. (1.2). Then $L^\dagger : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ denote the formal adjoint of $L$ given by the differential operator,

$$L^\dagger = \sum_{k=0}^{d} (-1)^k \partial^k M_{a_k} \text{ on } C^\infty(\mathbb{R}). \tag{1.4}$$

Moreover $L$ is said to be symmetric if $L^\dagger = L \text{ on } C^\infty(\mathbb{R})$. 


Remark 1.6. Using Remark [1.4] one easily shows \( L \) may alternatively be characterized as that unique \( \delta^h \)-order differential operator on \( C^\infty (\mathbb{R}) \) such that

\[
(Lf,g) = \langle f, L^1 g \rangle \quad \text{for all } f, g \in C^\infty_c (\mathbb{R}).
\] (1.5)

From this characterization it is then easily verified that:

1. The dagger operation is an involution, in particular \( L^{\dagger \dagger} = L \) and if \( S \) is another linear differential operator on \( C^\infty (\mathbb{R}) \), then \( (LS)^{\dagger} = S^{\dagger}L^{\dagger} \).
2. \( L \) is symmetric iff \( \langle Lf, g \rangle = \langle f, Lg \rangle \) for all \( f, g \in C^\infty_c (\mathbb{R}) \).

Proposition [2.2] below shows if \( \{a_k\}_{k=0}^d \subset C^\infty (\mathbb{R}, \mathbb{R}) \), then \( L = L^1 \) iff \( d = 2m \) is even and there exists \( \{b_l\}_{l=0}^m \subset C^\infty (\mathbb{R}, \mathbb{R}) \) such that

\[
L = L (\{b_l\}_{l=0}^m) := \sum_{l=0}^{m} (-1)^l \partial^l b_i (x) \partial^l.
\] (1.6)

The factor of \((-1)^l\) is added for later convenience. The coefficients \( \{b_l\}_{l=0}^m \) are uniquely determined by \( \{a_k\}_{k=0}^2 \) (the even coefficients in Eq. (1.2)) and in turn the coefficients \( \{a_k\}_{k=0}^{2m} \) are determined by the \( \{b_l\}_{l=0}^m \), see Theorem 2.7 and Lemma 2.4 respectively. We say that \( L \) is written in divergence form when \( L \) is expressed as in Eq. (1.6).

From now on let us assume that \( \{a_k\}_{k=0}^d \subset C^\infty (\mathbb{R}, \mathbb{R}) \) and \( L \) is given as in Eq. (1.2). For each \( n \in \mathbb{N} \), \( L^n \) is a \(dn\) order differential operator on \( C^\infty (\mathbb{R}) \) and hence there exists \( \{A_k\}_{k=0}^{2mn} \subset C^\infty (\mathbb{R}, \mathbb{C}) \) such that

\[
L^n = \sum_{k=0}^{dn} A_k \partial^k.
\] (1.7)

If we further assume that \( L \) is symmetric (so \( d = 2m \) for some \( m \in \mathbb{N}_0 \)), then by Remark 1.6 \( L^n \) is a symmetric \( 2mn \)-order differential operator. Therefore by Proposition 2.2 there exists \( \{B_l\}_{l=0}^{mn} \subset C^\infty (\mathbb{R}, \mathbb{R}) \) so that \( L^n \) may be written in divergence form as

\[
L^n = \sum_{l=0}^{mn} (-1)^l \partial^l B_l \partial^l.
\] (1.8)

Information about the coefficients \( \{A_k\}_{k=0}^{2mn} \) and \( \{B_l\}_{l=0}^{mn} \) in terms of the divergence form coefficients \( \{b_l\}_{l=0}^m \) of \( L \) may be found in Propositions 3.7 and Proposition 3.8 respectively.

Let \( \mathbb{R}[x] \) be the space of polynomial functions in one variable, \( x \), with real coefficients.

Remark 1.7. If the coefficients, \( \{a_k\}_{k=0}^{d=2m} \), of \( L \) in the Eq. (1.2) are in \( \mathbb{R}[x] \), then \( L \) and \( L^1 \) are both linear differential operator on \( C^\infty (\mathbb{R}) \) which leave \( S \) invariant. Moreover by simple integration by parts Eq. (1.5) holds with \( C^\infty _c (\mathbb{R}) \) replaced by \( S \), i.e. \( \langle Lf, g \rangle = \langle f, L^1 g \rangle \) for all \( f, g \in S \).

Notation 1.8. For the remainder of this introduction we are going to assume \( L \) is symmetric (\( L = L^1 \)), \( L \) is given in divergence form as in Eq. (1.6) with \( \{b_l\}_{l=0}^m \subset \mathbb{R}[x] \), and we now view \( L \) as an operator on \( L^2 (\mathbb{R}, m) \) with \( D(L) = S \subset L^2 (\mathbb{R}, m) \).

In other words, we are going to replace \( L \) by \( L|_S \).

The main results of this paper will now be summarized in the next two subsections.
1.1. Essential self-adjointness results.

**Theorem 1.9.** Let \( m \in \mathbb{N} \), \( \{b_l\}_{l=0}^{m} \subset \mathbb{R}[x] \) with \( b_m(x) \neq 0 \) and assume:

1. either \( \inf_{x} b_l(x) > 0 \) or \( b_l \equiv 0 \) and
2. \( \deg(b_l) \leq \max\{\deg(b_0), 0\} \) whenever \( 1 \leq l \leq m \). [The zero polynomial is defined to be of degree \(-\infty\).]

If \( L \) is the unbounded operator on \( L^2(\mathbb{R}) \) as in Notation 1.8, then \( L^n \) (for which \( D(L^n) \) is still \( \mathcal{S} \)) is essentially self-adjoint for all \( n \in \mathbb{N} \).

**Remark 1.10.** Notice that assumption 1 of Theorem 1.9 implies \( \deg(b_l) \) is even and the leading order coefficient of \( b_l \) is positive unless \( b_l \equiv 0 \).

**Definition 1.11** (Subspace Symmetry). Let \( S \) be a dense subspace of a Hilbert space \( \mathcal{K} \) and \( A \) be a linear operators on \( \mathcal{K} \). Then \( A \) is said to be symmetric on \( S \) if \( S \subseteq D(A) \) and

\[
\langle A\psi, \psi \rangle_{\mathcal{K}} = \langle \psi, A\psi \rangle_{\mathcal{K}} \quad \forall \psi \in S.
\]

The equality is equivalent to say \( A|_S \subseteq (A|_S)^* \) or \( A \subseteq A^* \) if \( D(A) = S \).

**Remark 1.12.** Using Remark 1.6 it is easy to see that \( L \) with polynomial coefficients is symmetric on \( \mathcal{C}^\infty(\mathbb{R}) \) as in Definition 1.8 if and only if \( L \) is symmetric on \( \mathcal{S} \) as in Definition 1.11.

We now introduce three different partial ordering on symmetric operators on a Hilbert space.

**Notation 1.13.** Let \( S \) be a dense subspace of a Hilbert space, \( \mathcal{K} \), and \( A \) and \( B \) be two densely defined operators on \( \mathcal{K} \).

1. We write \( A \preceq_S B \) if both \( A \) and \( B \) are symmetric on \( S \) (Definition 1.11) and
   \[
   \langle A\psi, \psi \rangle_{\mathcal{K}} \leq \langle B\psi, \psi \rangle_{\mathcal{K}} \quad \forall \psi \in S.
   \]
2. We write \( A \preceq B \) if \( A \preceq D(B) \), i.e. \( D(B) \subset D(A) \), \( A \) and \( B \) are both symmetric on \( D(B) \), and
   \[
   \langle A\psi, \psi \rangle_{\mathcal{K}} \leq \langle B\psi, \psi \rangle_{\mathcal{K}} \quad \forall \psi \in D(B).
   \]
3. If \( A \) and \( B \) are non-negative (i.e. \( 0 \preceq A \) and \( 0 \preceq B \)) self adjoint operators on a Hilbert space \( \mathcal{K} \), then we say \( A \preceq B \) if and only if \( D(\sqrt{B}) \subset D\left(\sqrt{A}\right) \) and
   \[
   \left\| \sqrt{A}\psi \right\| \leq \left\| \sqrt{B}\psi \right\| \quad \forall \psi \in D\left(\sqrt{B}\right).
   \]

There is a sizable literature dealing with similar essential self-adjointness in Theorem 1.9, see for example [2, 9, 15]. Suppose that \( b_2, b_1, \) and \( b_0 \) are smooth real-valued functions of \( x \in \mathbb{R} \) and \( T \) is an differential operator on \( \mathcal{C}_c^\infty(\mathbb{R}) \subseteq L^2(\mathbb{R}) \) defined by,

\[
T = -\partial b_2(x) \partial + b_0(x) + i(b_1(x) \partial + \partial b_1(x)).
\]

Kato [3] shows \( T^n \) is essentially self-adjoint for all \( n \in \mathbb{N} \) when \( b_2 = 1, b_1 = 0 \) and \( -a - b|x|^2 \preceq \mathcal{C}_c^\infty(\mathbb{R}) \) \( T \) for some constants \( a \) and \( b \). Chernoff [2] gives the same conclusion under certain assumptions on \( b_2 \) and \( T \). For example, Chernoff’s assumptions would hold if \( b_2, b_1 \) and \( b_0 \) are real valued polynomial functions such that \( \deg(b_2) \leq 2 \) and \( b_2 \) is positive and \( T \) is semi-bounded on \( \mathcal{C}_c^\infty(\mathbb{R}) \). In contrast, Theorem 1.9 allows for higher order differential operators but does not allow for
non-polynomial coefficients. [However, the methods in this paper can be pushed further in order to allow for certain non-polynomial coefficients.]

There are also a number of results regarding essential self-adjointness in the pseudo-differential operator literature, the reader may be referred to, for example, [4, 11, 12, 18, 21, 20]. In fact, our proof of Theorem 1.9 will be an adaptation of an approach found in Theorem 3.1 in [11].

1.2. Operator Comparison Theorems. Motivated by considerations involved in taking the classical limit of quantum mechanics in [3] and the important paper by [7], we will define a scaled version of $L$ (see Notation 1.14) where for any $\h > 0$ we make the following replacements in Eq. (1.6),

$$x \rightarrow \sqrt{\h} M x$$

and

$$\partial \rightarrow \sqrt{\h} \partial.$$  (1.9)

For reasons explained in Theorem A.2 of the appendix, we are lead to consider a more general class of operators parametrized by $\h > 0$.

Notation 1.14. Let

$$\{b_l, (\cdot) : 0 \leq l \leq m \text{ and } \h > 0\} \subset \mathbb{R}[x],$$  (1.10)

and then define

$$L_h = L \left( \left\{ b^l_h, \sqrt{\h} (\cdot) \right\} \right)^m_{l=0} = \sum_{l=0}^m (-\h)^l \partial^l b_l, \sqrt{\h} (\cdot) \partial^l \text{ on } S.$$  (1.11)

We now record an assumption which is needed in a number of the results stated below.

Assumption 1. Let $m \in \mathbb{N}_0$. We say $\{b_l, (\cdot)\}_{l=0}^m \subset \mathbb{R}[x]$ and $\eta > 0$ satisfies Assumption 1 if the following conditions hold.

1. For $0 \leq l \leq m$, $b_l, (\cdot) = \sum_{j=0}^{2m_l} \alpha_{l,j} (\h) x^j$ is a real polynomial of $x$ where $\alpha_{l,j}$ is a real continuous function on $[0, \eta]$.

2. For all $0 < \h < \eta$,

$$2m_l = \deg(b_l, (\cdot)) \leq \deg(b_{l-1}, (\cdot)) = 2m_{l-1} \text{ for } 1 \leq l \leq m.$$  (1.12)

3. We have,

$$c_{bm} := \inf_{x \in \mathbb{R}, 0 < \h < \eta} b_m, (\h) > 0$$

and

$$c_{\alpha} := \min_{0 \leq l \leq m} \inf_{0 < \h < \eta} \alpha_{l,2m_l} (\h) > 0,$$  (1.13)

i.e. $b_m, (\h)$ is uniformly in $x \in \mathbb{R}$ and $0 < \h < \eta$ positive and leading orders, $\alpha_{l,2m_l} (\h)$, of all $b_l, \in \mathbb{R}[x]$ are uniformly strictly positive.

Remark 1.15. Conditions (1) and (3) of Assumption 1 implies there exists $A \in (0, \infty)$ so that

$$\min_{0 \leq l \leq m} \inf_{0 < \h < \eta} \inf_{|x| \geq A} b_l, (\h) > 0.$$%

Furthermore, if $k \geq 1$, $0 \leq l_1, \ldots, l_k \leq m$, and $q_h (x) = b_{l_1, h} (x) \ldots b_{l_k, h} (x) \in \mathbb{R}[x]$, then

$$q_h (x) = \sum_{i=0}^{2M} Q_i (\h) \cdot x^i$$
where \( M = m_1 + \ldots + m_k \), each of the coefficients, \( Q_i(h) \) is uniformly bounded for \( 0 < h < \eta \), and

\[
\inf_{0 < h < \eta} Q_{2M}(h) = \inf_{0 < h < \eta} \alpha_{n_{1}, m_1, \ldots, \alpha_{n_{k}, m_k}}(h) \geq c_{\alpha} > 0.
\]

From these remarks one easily shows \( \inf_{0 < h < \eta} \inf_{x \in \mathbb{R}} q_n(x) > -\infty \).

The second main goal of this paper is to find criteria on two symmetric differential operators \( L_h \) and \( \tilde{L}_h \) so that for each \( n \in \mathbb{N} \), there exists \( K_n < \infty \) such that

\[
L_h^n \preceq_K K_n \left( \tilde{L}_h^n + I \right).
\]

(As usual \( I \) denotes the identity operator here and \( \preceq_K \) is as in Notation \( [13] \). For some perspective let us recall the Löwner-Heinz inequality.

**Theorem 1.16** (Löwner-Heinz inequality). If \( A \) and \( B \) are two non-negative self-adjoint operators on a Hilbert space, \( K \), such that \( A \leq B \), then \( A^r \leq B^r \) for \( 0 \leq r \leq 1 \).

Löwner proved this result for finite dimensional matrices in \([10]\) and Heinz extended it to bounded operators in a Hilbert space in \([6]\). Later, both Heinz in \([6]\) and Kato in \([Theorem 2 of 8]\) extended the result for unbounded operators, also see \([Proposition 10.14 of 17]\). There is a large literature on so called “operator monotone functions,” e.g. \([14, 5, 19, 1]\) and \([13, \text{Theorem 18}]\). It is well known (see \([17, \text{Section 10.3}]\) for more background) that \( f(x) = x^r \) is not an operator monotone for \( r > 1 \), see \([17, \text{Example 10.3}]\) for example. This indicates that proving operator inequalities of the form in Eq. \((1.15)\) is somewhat delicate. Our main result in this direction is the subject of the next theorem.

**Theorem 1.17** (Operator Comparison Theorem). Suppose that \( L_h \) and \( L_h \) are two linear differential operators on \( S \) given by

\[
\tilde{L}_h = \sum_{l=0}^{m_\ell} (-h)^l \partial^l b_{l,h}(\sqrt{h}x) \partial^l \quad \text{and} \quad L_h = \sum_{l=0}^{m_L} (-h)^l \partial^l b_{l,h}(\sqrt{h}x) \partial^l,
\]

with polynomial coefficients, \( \{ \bar{b}_{l,h}(x) \}_{l=0}^{m_\ell} \) and \( \{ b_{l,h}(x) \}_{l=0}^{m_L} \) satisfying Assumption \([7]\) with constants \( \eta_{\bar{L}} \) and \( \eta_L \) respectively. Let \( \eta = \min\{ \eta_{\bar{L}}, \eta_L \} \). If we further assume that \( m_\ell \leq m_L \) and there exists \( c_1 \) and \( c_2 \) such that

\[
|b_{l,h}(x)| \leq c_1 (b_{l,h}(x) + c_2) \quad \forall \ 0 \leq l \leq m_\ell \ \text{and} \ 0 < h < \eta,
\]

then for any \( n \in \mathbb{N} \) there exists \( C_1 \) and \( C_2 \) such that

\[
L_h^n \preceq_S C_1 \left( \tilde{L}_h^n + C_2 \right) \text{ for all } 0 < h < \eta,
\]

**Corollary 1.18.** If \( \{ b_{l,h}(x) \}_{l=0}^{m_L} \) and \( \eta > 0 \) satisfy Assumption \([7]\) then there exists \( C \in \mathbb{R} \) such that \( CI \preceq_S K_h \) for all \( 0 < h < \eta \).

**Proof.** Define \( \tilde{L}_h = I \), i.e. we are taking \( m_\ell = 0 \) and \( \bar{b}_{0,h}(x) = 1 \). It then follows from Theorem \((1.17)\) with \( n = 1 \) that there exists \( C_1, C_2 \in (0, \infty) \) such that

\[
I = \tilde{L}_h \preceq_S C_1 L_h + C_2 \quad \text{and hence} \quad L_h + C_2 \preceq_S C_1^{-1} I.
\]

A similar result to Theorem \((1.17)\) may be found in the \([Theorem 1.1 of 19]\). The paper \([19]\) compares the standard Laplacian \(-\Delta\) with an operator of \( H_0 \) in the form of \(-\sum_{i,j}^d \partial_i c_{ij}(x) \partial_j \) with coefficients \( \{ c_{ij} \}_{i,j=1}^d \) lying in a Sobolev spaces.
$W^{m,1,\infty}(\mathbb{R}^d)$ for some $m \in \mathbb{N}$ and $\mathcal{D}(H_0) = W^{\infty,2}(\mathbb{R}^d)$. The theorem shows that if $H_0$ is a symmetric, positive and subelliptic of order $\gamma \in (0,1]$ then $\mathcal{H}_0$ is positive self-adjoint and for all $\alpha \in \left[0, \frac{m+1+\gamma-1}{2}\right]$, there exists $C_\alpha$ such that

$$(-\Delta)^{2\alpha} \leq C_\alpha (I+\mathcal{H}_0)^{2\alpha}$$

As a corollary of Theorem 1.9 and aspects of the proof of Theorem 1.17 given in Section 6.3 below, we have the following corollaries which are proved in subsection 6.3.

**Corollary 1.19.** Suppose $\{b_{l,l}(x)\}_{l=0}^m \subset \mathbb{R}[x]$ and $\eta > 0$ satisfies Assumption 4. $L_h$ is the operator in the Eq. (1.11), and suppose that $C \geq 0$ has been chosen so that $0 \leq \mathcal{L}_h + CI$ for all $0 < h < \eta$. (The existence of $C$ is guaranteed by Corollary 1.18.) Then for any $0 < h < \eta$, $L_h + CI$ is a non-negative self-adjoint operator on $L^2(m)$ and $\mathcal{S}$ is a core for $(\mathcal{L}_h + C)^r$ for all $r \geq 0$.

**Corollary 1.20.** Suppose that $L_h$ and $L_h$ are two linear differential operators and $\eta > 0$ as in Theorem 1.17. If $C \geq 0$ and $\mathcal{C} \geq 0$ are chosen so that $L_h + C \geq I$ and $\mathcal{L}_h + \bar{C} \geq 0$ (as is possible by Corollary 1.18), then $\bar{L}_h + \bar{C}$ and $\bar{L}_h + C$ are non-negative self adjoint operators and for each $r \geq 0$ there exists $C_r$ such that

$$(\bar{L}_h + \bar{C})^r \leq C_r (\bar{L}_h + C)^r \forall 0 < h < \eta.$$  \hspace{1cm} (1.18)

**Definition 1.21** (Number operator). The number operator, $\mathcal{N}_h$ on $L^2(\mathbb{R})$ is defined as the closure of

$$-\frac{1}{2} \partial^2 + \frac{1}{2} x^2 - \frac{1}{2} t$$

on $\mathcal{S}$. \hspace{1cm} (1.19)

For any $h > 0$ we let $\mathcal{N}_h := h\mathcal{N}$ which commonly known as the harmonic oscillator Hamiltonian.

The properties of the number operator are very well known. The next theorem summarizes two such basic properties which we need for this paper.

**Theorem 1.22.** The number operator, $\mathcal{N}_h$, is non-negative (i.e. $0 \leq \mathcal{N}$) self-adjoint operator on $L^2(\mathbb{R})$.

**Proof.** The self-adjointness of $\mathcal{N}$ is an easy consequence of Theorem X.28 in [15] or may be seen as a consequence of Corollary 1.19. The positivity is a simple consequence of the fact that $\mathcal{N}|_{\mathcal{S}} = a_1^\dagger a_1$ where $a_1$ and $a_1^\dagger$ are annihilation and creation operators (for $h = 1$) as given in Eq. (A.1) of the appendix. \hspace{1cm} ■

The next corollary is a direct consequence from Corollaries 1.19 and 1.20 where $\bar{L}_h = \mathcal{N}_h$ in Eq. (1.18).

**Corollary 1.23.** Suppose $m \geq 1$, $\{b_{l,l}(\cdot)\}_{l=0}^m \subset \mathbb{R}[x]$ and $\eta > 0$ satisfy Assumption 4 and $L_h$ is the operator on $\mathcal{S}$ defined in Eq. (1.11).If $C \geq 0$ is chosen so that $I \leq \mathcal{S}_h + C$ (see Corollary 1.18), then:

1. $\bar{L}_h + C$ is a non-negative self-adjoint operator on $L^2(m)$ for all $0 < h < \eta$.
2. $\mathcal{S}$ is a core for $(\bar{L}_h + C)^r$ for all $r \geq 0$ and $0 < h < \eta$.
3. If we further supposed $\text{deg}(b_{0,h}) \geq 2$, then there exists $C_r > 0$ such that

$$\mathcal{N}_h^r \leq C_r (\bar{L}_h + C)^r$$  \hspace{1cm} (1.20)

for all $0 < h < \eta$ and $r \geq 0$.

1By the spectral theorem one shows $D((\bar{L}_h)^r) = D((\bar{L}_h + C)^r)$.  

7
2. A Structure Theorem for Symmetric Differential Operators

Remark 2.1. It is useful to observe if $A$ and $B$ are two linear transformation from a vector space, $V$, to itself, then

$$AB^2 + B^2A = 2BAB + [B, [B, A]],$$

where $[A, B] := AB - BA$ denotes the commutator of $A$ and $B$.

Proposition 2.2. Suppose $\{a_k\}_{k=0}^d \subset C^\infty(\mathbb{R}, \mathbb{R})$ and $L$ is the $d^{th}$-order differential operator on $C^\infty(\mathbb{R})$ as defined in Eq. (1.2). If $L$ is symmetric according to Definition 1.5 (i.e. $L = L^\dagger$ where $L^\dagger$ is as in Eq. (1.4)), then $d$ is even (let $m = d/2$) and there exists $\{b_l\}_{l=0}^m \subset C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$L = \sum_{l=0}^m (-1)^l \partial^l M_{b_l} \partial^l$$

where $M_{b_l}$ is as in Notation 1.3. Moreover, $b_m = (-1)^m a_{2m} = (-1)^m a_d$.

Proof. Since $L = L^\dagger$, we have

$$L = \frac{1}{2} (L + L^\dagger) = \frac{1}{2} \sum_{k=0}^d [a_k \partial^k + (-1)^k \partial^k M_{a_k}]$$

$$= \frac{1}{2} [a_d \partial^d + (-1)^d \partial^d M_{a_d}] + \text{[diff. operator of order $d - 1$]}. \quad (2.2)$$

If $d$ were odd, then $(-1)^d = -1$ and hence (using the product rule),

$$\frac{1}{2} [a_d \partial^d + (-1)^d \partial^d M_{a_d}] = \frac{1}{2} [M_{ad}, \partial^d]$$

$$= \text{[diff. operator of order $d - 1$]}$$

which combined with Eq. (2.2) would imply that $L$ was in fact a differential operator of order no greater than $d - 1$. This shows that $L$ must be an even order operator.

Now knowing that $d$ is even, let $m := d/2 \in \mathbb{N}$. From Eq. (2.2), we learn that

$$L = \frac{1}{2} \sum_{k=0}^{2m} [a_k \partial^k + (-1)^k \partial^k M_{a_k}]$$

$$= \frac{1}{2} [a_{2m} \partial^{2m} + \partial^{2m} M_{a_{2m}}] + R$$

where $R$ is given by

$$R = \frac{1}{2} \sum_{k=0}^{2m-1} [M_{a_k} \partial^k + (-1)^k \partial^k M_{a_k}] .$$

Moreover by Remark 1.6, $R$ is still symmetric. As in the previous paragraph $R$ is in fact an even order differential operator and its order is at most $2m - 2$. Using Remark 2.1 with $A = M_{a_{2m}}$, $B = \partial^{2m}$, and $V = C^\infty(\mathbb{R})$, we learn that

$$\frac{1}{2} [a_{2m} \partial^{2m} + \partial^{2m} M_{a_{2m}}] = \partial^{2m} M_{a_{2m}} \partial^m + \frac{1}{2} [\partial^m, [\partial^m, M_{a_{2m}}]]$$

$$= \partial^m M_{a_{2m}} \partial^m + \text{[diff. operator of order at most $2m - 2$]}. $$

Combining the last three displayed equations together shows

$$L = \partial^m a_{2m} \partial^m + S$$
where \( S = L - \partial^m a_{2m} \partial^m \) is a symmetric (by Remark 1.6) even order differential operator or at most \( 2m - 2 \). It now follows by the induction hypothesis that

\[
S = \sum_{l=0}^{m-1} (-1)^l \partial^l M_{b_l} \partial^l \implies L = \sum_{l=0}^{m} (-1)^l \partial^l M_{b_l} \partial^l
\]

where \( b_m := (-1)^m a_{2m} \).

Our next goal is to record the explicit relationship between \( \{a_k\}_{k=0}^{2m} \) in Eq. (1.2) and \( \{b_k\}_{k=0}^{m} \) in Eq. (2.1).

**Convention 2.3.** To simplify notation in what follows, for \( k, l \in \mathbb{Z} \), let

\[
\binom{l}{k} := \begin{cases} \frac{l!}{k!(l-k)!} & \text{if } 0 \leq k \leq l \\ 0 & \text{otherwise.} \end{cases}
\]

**Lemma 2.4.** If \( \{a_k\}_{k=0}^{2m} \cup \{b_l\}_{l=0}^{m} \subset C^\infty (\mathbb{R}, \mathbb{R}) \) and

\[
\sum_{l=0}^{m} (-1)^l \partial^l b_l (x) \partial^l = \sum_{k=0}^{2m} a_k (x) \partial^k, \tag{2.4}
\]

then

\[
a_k := \sum_{l=0}^{m} \binom{l}{k-l} (-1)^l \partial^{2l-k} b_l. \tag{2.5}
\]

**Proof.** By the product rule,

\[
\partial^l M_{b_l} = \sum_{r=0}^{l} \binom{l}{r} (\partial^{l-r} b_l) \partial^r,
\]

and therefore,

\[
\sum_{l=0}^{m} (-1)^l \partial^l M_{b_l} \partial^l = \sum_{l=0}^{m} \sum_{r=0}^{l} \binom{l}{r} (-1)^l (\partial^{l-r} b_l) \partial^{l+r}
\]

\[
= \sum_{k=0}^{2m} \left[ \sum_{l=0}^{m} \sum_{r=0}^{m} \binom{l}{r} (-1)^l (\partial^{l-r} b_l) 1_{k=l+r} \right] \partial^k
\]

\[
= \sum_{k=0}^{2m} \left[ \sum_{l=0}^{m} \binom{l}{k-l} (-1)^l (\partial^{2l-k} b_l) \right] \partial^k.
\]

Combining this result with Eq. (2.4) gives the identities in Eq. (2.5).

Let us observe that the binomial coefficient of \( a_l \) is zero unless \( 0 \leq k - l \leq l \), i.e. \( l \leq k \leq 2l \). To emphasize this restriction, we may write Eq. (2.5) as

\[
a_k = \sum_{l=0}^{m} 1_{l \leq k \leq 2l} \binom{l}{k-l} (-1)^l \partial^{2l-k} b_l. \tag{2.6}
\]

Taking \( k = 2p \) in Eq. (2.6) and multiplying the result by \( (-1)^p = (-1)^{-p} \) leads to the following corollary.

**Corollary 2.5.** For \( 0 \leq p \leq m \),

\[
(-1)^p a_{2p} = \sum_{l=0}^{m} 1_{p \leq l \leq 2p} \binom{l}{2p-l} (-1)^{(l-p)} \partial^{2(l-p)} b_l. \tag{2.7}
\]
We will see in Theorem 2.7 below that the relations in Eq. (2.4) may be used to uniquely write the \( \{ b_l \}_{l=0}^m \) in terms of linear combinations for the \( \{ a_{2k} \}_{k=0}^n \). In particular, this shows if the operator \( L \) described in Eq. (1.2) is symmetric then \( \{ b_l \}_{l=0}^m \) is completely determined by the \( a_k \) with \( k \) even.

2.1. The divergence form of \( L \).

Notation 2.6. For \( r, s, n \in \mathbb{N}_0 \) and \( 0 \leq r, s \leq m \), let

\[
C_n(r, s) = \sum_{r<s} \left( \frac{k_1}{2r-k_1} \right) \left( \frac{k_2}{2k_1-k_2} \right) \cdots \left( \frac{k_{n-1}}{2k_{n-2}-k_{n-1}} \right) \left( \frac{s}{2k_{n-1} - s} \right)
\]

where the sum is over \( r < k_1 < k_2 < \cdots < k_{n-1} < s \). We also let

\[
K_m(r, s) = \sum_{n=1}^{m-1} (-1)^n C_n(r, s).
\] (2.8)

In particular, \( C_n(0, s) = C_n(m, s) = K_m(0, s) = K_m(m, s) = 0 \) for all \( 0 \leq s \leq m \).

Theorem 2.7. If Eq. (2.4) holds then

\[
(-1)^r b_r = a_{2r} + \sum_{r<s\leq m} K_m(r, s) \partial^{2(s-r)} a_{2s} \quad \forall \ 0 \leq r \leq m.
\] (2.9)

Proof. For \( x \in \mathbb{R} \) let \( b(x) \) and \( a(x) \) denote the column vectors in \( \mathbb{R}^{m+1} \) defined by

\[
b(x) = (\langle -1 \rangle^0 b_0(x), \langle -1 \rangle^1 b_1(x) \ldots, \langle -1 \rangle^m b_m(x))^{tr} \text{ and } a(x) = (a_0(x), a_2(x), a_4(x), \ldots, a_{2m}(x))^{tr}.
\]

Further let \( U \) be the \( (m+1) \times (m+1) \) matrix with entries \( \{ U_{r,k} \}_{r,k=0}^m \) which are linear constant coefficient differential operators given by

\[
U_{r,k} := 1_{r<k\leq 2r} \left( \frac{k}{2r-k} \right) \partial^{2(k-r)}.
\]

Notice that by definition, \( U_{r,k} = 0 \) unless \( k > r \) and \( U_{0,k} = 0 \) for \( 0 \leq k \leq m \). Hence \( U \) is nilpotent and \( U^m = 0 \). Further observe that Eq. (2.7) may be written as

\[
a_{2r} = (-1)^r b_r + \sum_{r<k\leq m} \left( \frac{k}{2r-k} \right) (-1)^k \partial^{2(k-r)} b_k
\]

\[
= (-1)^r b_r + \sum_{r<k\leq m} U_{r,k} (-1)^k b_k
\]

or equivalently stated \( a = (I + U)b \). As \( U \) is nilpotent with \( U^m = 0 \), this last equation may be solved for \( b \) using

\[
b = (I + U)^{-1} a = a + \sum_{n=1}^{m-1} (-U)^n a.
\] (2.10)

In components this equation reads,

\[
(-1)^r b_r = a_r + \sum_{n=1}^{m-1} (-1)^n \sum_{s=0}^{m} U_{r,s}^n a_{2s}
\] (2.11)
However, with the aid of Lemma 2.8 below and the definition of $K_m(r, s)$ in Eq. (2.8) it follows that

$$
\sum_{n=1}^{m-1} (-1)^n U_{r,s}^n = \sum_{n=1}^{m-1} (-1)^n C_n(r, s) \partial^2(s-r) = K_m(r, s) \partial^2(s-r)
$$

which combined with Eq. (2.11) and the fact that $K_m(r, s) = 0$ unless $0 < r < s \leq m$ proves Eq. (2.9). □

**Lemma 2.8.** Let $1 \leq n \leq m$ and $0 \leq r, s \leq m$, then $U^n = 0$ and

$$
U^n_{r,s} = C_n(r, s) \partial^2(s-r).
$$

**Proof.** By definition of matrix multiplication,

$$
U^n_{r,s} = \sum_{k_1, \ldots, k_{n-1}=1}^{m} 1 \leq r < k_1 \leq 2 \left(\frac{k_1}{2r - k_1}\right) \partial^2(k_1-r) \frac{k_1}{2k_1 - k_2} \partial^2(k_2-k_1) \ldots
$$

$$
\ldots 1 \leq k_{n-1} < k_n \leq 2k_{n-1} \frac{k_n}{2k_{n-1} - k_n} \partial^2(k_n-k_{n-1}) \frac{s}{2k_{n-2} - k_{n-1}} \partial^2(s-r)
$$

$$
= \sum_{r < k_1 < k_2 < \ldots < k_{n-1} < s} \left(\frac{k_1}{2r - k_1}\right) \left(\frac{k_2}{2k_1 - k_2}\right) \ldots \left(\frac{k_{n-1}}{2k_{n-2} - k_{n-1}}\right) \left(\frac{s}{2k_{n-2} - k_{n-1}}\right)
$$

$$
= C_n(r, s) \partial^2(s-r).
$$

□

3. The structure of $L^n$

In this section let us fix a $2m$ – order symmetric differential operator, $L$, acting on $C^\infty(\mathbb{R})$ which can be written as in both of the equations (1.2) and (2.1) where the coefficients, $\{a_k\}^2_{k=0}$ and $\{b_l\}^m_{l=0}$ are all real valued smooth functions on $\mathbb{R}$. If $n \in \mathbb{N}$, $L^n$ is a $2mn$ – order symmetric linear differential operator on $C^\infty(\mathbb{R})$ and hence there exits $\{A_k\}^{2mn}_{k=0} \subset C^\infty(\mathbb{R}, \mathbb{R})$ and (using Proposition 2.2) $\{B_l\}^{mn}_{l=0} \subset C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$
L^n = \sum_{k=0}^{2mn} A_k \partial^k = \sum_{l=0}^{mn} (-1)^{\ell} \partial^\ell B_\ell \partial^\ell.
$$

(3.1)

Our goal in this section is to compute the coefficients $\{A_k\}^{2mn}_{k=0}$ in terms of the coefficients $\{b_l\}^m_{l=0}$ defining $L$ as in Eq. (2.1). It turns out that it is useful to compare $L^n$ to the operators which is constructed by writing out $L^n$ while pretending that the coefficients $\{a_k\}^{2m}_{k=0}$ or $\{b_l\}^{m}_{l=0}$ are constant. This is explained in the following notations.

**Notation 3.1.** For $n \in \mathbb{N}$ and $m \in \mathbb{N}$, let $\Lambda_m := \{0, 1, \ldots, m\} \subset \mathbb{N}_0$ and for $j = (j_1, \ldots, j_n) \in \Lambda^m_n$, let $|j| = j_1 + j_2 + \cdots + j_n$. If $k = (k_1, \ldots, k_n) \in \Lambda^m_n$ is another multi-index, we write $k \leq j$ to mean $k_i \leq j_i$ for $1 \leq i \leq n$. [We will use this notation when $m = \infty$ as well in which case $\Lambda_\infty = \mathbb{N}_0$.]

**Notation 3.2.** Given $n \in \mathbb{N}$, and $L$ as in Eq. (2.1), let $\{B_\ell\}^{mn}_{\ell=0}$ be $C^\infty(\mathbb{R}, \mathbb{R})$ – functions defined by

$$
B_\ell := \sum_{j \in \Lambda^m_n} 1_{|j| = \ell} b_{j_1} \cdots b_{j_n},
$$

(3.2)
and $L^{(n)}_B$ be the differential operator given by

$$L^{(n)}_B := \sum_{\ell=0}^{nm} (-1)^\ell \partial^\ell B_\ell \partial^\ell. \quad (3.3)$$

It will also be convenient later to set $B_{k/2} \equiv 0$ when $k$ is an odd integer.

**Example 3.3.** If $m = 1$ and $n = 2$, then

$$L = -\partial b_1 \partial + b_0 \text{ and } \quad L^2 = -\partial b_1 \partial^2 b_1 \partial - \partial b_1 \partial b_0 - b_0 \partial b_1 \partial + b_0^2.$$ 

To put $L^2$ into divergence form we repeatedly use the product rule, $\partial V = V \partial + V'$. Thus

$$\partial b_1 \partial b_0 + b_0 \partial b_1 \partial = \partial b_1 b_0 \partial + \partial b_1 b_0' + \partial b_0 b_1 \partial - b_0' b_1 \partial$$

and

$$\partial b_1 \partial^2 b_1 \partial = \partial^2 b_1 \partial b_1 \partial - \partial b_1' \partial b_1 \partial$$

Combining the last three displayed equations together shows

$$L^2 = \partial^2 b_1^2 \partial^2 + \partial \left[ -2b_1 b_0 + (b_1 b_0)' - (b_1')^2 \right] \partial + b_0^2 - (b_0')^2$$

Dropping all terms in Eq. (3.4) which contain a derivative of $b_1$ or $b_0$ shows

$$L^{(2)}_B = \partial^2 b_1^2 \partial^2 - \partial \left[ 2b_1 b_0 \right] \partial + b_0^2. \quad (3.5)$$

**Notation 3.4.** For $j \in \mathbb{N}^n_0$ and $k \in \mathbb{N}^n_0$, let

$$\binom{k}{j} := \prod_{i=1}^{n} \binom{k_i}{j_i}$$

where the binomial coefficients are as in Convention [2.3]

**Lemma 3.5.** If $L$ is as in Eq. (3.4),

$$M_{e^{-\xi (,)}} L M_{e^{\xi (,)}} = \sum_{l=0}^{m} (-1)^l (\partial + i\xi)^l M_{b_l (,)} (\partial + i\xi)^l. \quad (3.6)$$

**Proof.** If $f \in C^\infty (\mathbb{R})$, the product rule gives,

$$\partial_x \left[ e^{i\xi x} f (x) \right] = e^{i\xi x} (\partial_x + i\xi) f (x),$$

which is to say,

$$M_{e^{-\xi (,)}} \partial M_{e^{\xi (,)}} = (\partial + i\xi). \quad (3.7)$$

Combining Eq. (3.7) with the fact that

$$M_{e^{-\xi (,)}} M_{b_i} M_{e^{\xi (,)}} = M_{b_i}$$
allows us to conclude,

\[
M_{e^{-i\xi(x)}} L M_{e^{i\xi(x)}} = \sum_{l=0}^{m} (-1)^l M_{e^{-i\xi(x)}} \partial^l M_{b_l} \partial^l M_{e^{i\xi(x)}}
\]

\[
= \sum_{l=0}^{m} (-1)^l M_{e^{-i\xi(x)}} \partial^l M_{e^{i\xi(x)}} M_{b_l} M_{e^{-i\xi(x)}} \partial^l M_{e^{i\xi(x)}}
\]

\[
= \sum_{l=0}^{m} (-1)^l (\partial + i\xi)^l M_{b_l} (\partial + i\xi)^l.
\]

**Proof.**

First observe that if \( T \) is a function constantly equal to 1. We will often abuse notation and write this last equation as,

\[
T_k(x) := \sum_{q, l, j \in \Lambda^m_n} C_k(q, l, j) (\partial_{x}^l b_{q_l} (x) \partial_{x}^j) \ldots (\partial_{x}^j b_{q_l} \partial_{x}^j) \partial_{x}^j b_{q_l} (x)
\]

where 1 is a function constantly equal to 1. We will often abuse notation and write this last equation as,

\[
T_k(x) := \sum_{q, l, j \in \Lambda^m_n} C_k(q, l, j) (\partial_{x}^l b_{q_l} (x) \partial_{x}^j) \ldots (\partial_{x}^j b_{q_l} \partial_{x}^j) \partial_{x}^j b_{q_l} (x)
\]

**Proposition 3.7** \((A_k = A_k(\{b_l\}_{l=0}^m))\). If \( L \) is given as in Eq. (2.1), then coefficients \( \{A_k\}_{k=0}^{2mn} \) of \( L^n \) in Eq. (3.1) are given by

\[
A_k = 1_{k \in 2\mathbb{N}_0} \cdot (-1)^{k/2} B_k/2 + T_k,
\]

where \( B \) and \( T \) are as in Notations 3.2 and 3.6 respectively. Moreover, if we further assume \( \{b_l\}_{l=0}^m \) are polynomial functions such that

\[
\deg(b_l) \leq \max \{ \deg(b_0) , 0 \} \quad \text{for} \quad 1 \leq l \leq m,
\]

then \( \{B_k\}_{k=0}^{mn} \) and \( \{T_k\}_{k=0}^{2mn} \) are polynomials such that

\[
\deg(T_k) < \max \{ n \deg(b_0) , 0 \} = \max \{ \deg(B_0) , 0 \} \quad \text{for} \quad 0 \leq k \leq 2mn.
\]

**Proof.** First observe that if \( L^n \) is described as in Eq. (3.1), then

\[
\sum_{k=0}^{2mn} A_k(x) (i\xi)^k = \sigma_n(x, \xi) = e^{-i\xi x} L^n_x (e^{i\xi x})
\]

where \( \sigma_n \) := \( \sigma_{L^n} \) is a symbol of \( L^n \) defined in Eq. (1.13) and \( L^n_x \) is a differential operator with respect to \( x \). To compute the right side of this equation, take the \( n^{th} \) power of Eq. (3.9) to learn

\[
M_{e^{-i\xi(x)}} L^n M_{e^{i\xi(x)}} = (M_{e^{-i\xi(x)}} L M_{e^{i\xi(x)}})^n
\]

\[
= \sum_{q_1, \ldots, q_n = 0}^m (\partial + i\xi)^{q_1} M_{b_{q_1}} (\partial + i\xi)^{q_1} \ldots (\partial + i\xi)^{q_n} M_{b_{q_n}} (\partial + i\xi)^{q_n}
\]

\[
= \sum_{q \in \Lambda^m_n} (-1)^{\|q\|} (\partial + i\xi)^{\|q\|} M_{b_{q_1}} (\partial + i\xi)^{q_1} \ldots (\partial + i\xi)^{q_n} M_{b_{q_n}} (\partial + i\xi)^{q_n}.
\]
Applying this result to the constant function $1$ then shows

$$\sigma_n(x, \xi) = e^{-i\xi L^n_x} = M_{e^{-i\xi} L^n_x M_{e^{i\xi}}} 1$$

$$= \sum_{q \in \Lambda_m} (-1)^{|q|} (\partial + i\xi)^{q_1} M_{b_{q_1}} (\partial + i\xi)^{q_2} \ldots (\partial + i\xi)^{q_n} M_{b_{q_n}} (\partial + i\xi)^{q_n} 1.$$ 

Making repeatedly used of the binomial formula to expand out all the terms $(\partial + i\xi)^q$ appearing above then gives,

$$\sum_{k=0}^{2mn} A_k(x)(i\xi)^k = \sigma_n(x, \xi)$$

Looking the coefficient of $(i\xi)^k$ on the right side of this expression shows,

$$A_k(x) := \sum_{q,l,j \in \Lambda_m} \binom{q}{l} \binom{q}{j} (-1)^{|q|} 1_{2[q]|-|l|-|j|=k} \partial_x^{q_1} b_{q_1} (x) \partial_x^{q_1} b_{q_1} (x) \partial_x^{q_n} b_{q_n} (x) \partial_x^{q_n} b_{q_n} (x)$$

$$= \sum_{q,l,j \in \Lambda_m} \binom{q}{l} \binom{q}{j} (-1)^{|q|} 1_{j_1=0} 1_{2[|q|-|l|-|j|=k]} \partial_x^{q_n} b_{q_n} (x) \partial_x^{q_n} b_{q_n} (x)$$

$$= \sum_{q \in \Lambda_m} 1_{|q|=k} (-1)^{|q|} b_{q_n} (x) \ldots b_{q_n} (x)$$

$$+ \sum_{q,l,j \in \Lambda_m} \binom{q}{l} \binom{q}{j} (-1)^{|q|} 1_{j_1=0} 1_{2[|q|-|k|-|j|]=0} \partial_x^{q_n} b_{q_n} (x) \partial_x^{q_n} b_{q_n} (x) \partial_x^{q_n} b_{q_n} (x) \partial_x^{q_n} b_{q_n} (x)$$

which completes the proof of Eq. (3.10). The remaining assertions now easily follow from the formulas for $\{B_{\ell}\}_{\ell=0}^{mn}$ and $\{T_{\ell}\}_{\ell=0}^{2mn}$ in Notations 3.2 and 3.6 and the assumption in Eq. (3.11). 

As we can see from Example 3.3 computing the coefficients $\{B_{\ell}\}_{\ell=0}^{mn}$ in Eq. (3.1) can be tedious in terms of in terms of the coefficients $\{b_{i}\}_{i=0}^{m}$ defining $L$ as in Eq. (2.1). Although we do not need the explicit formula for the $\{B_{\ell}\}_{\ell=0}^{mn}$, we will need some general properties of these coefficients which we develop below.

**Proposition 3.8.** Given $B_{\ell} = B_{\ell}(\{b_{i}\}_{i=0}^{m})$ of $L^n$ in Eq. (3.7). There are constants $\hat{C}(n, \ell, k, p)$ for $n \in \mathbb{N}_0$, $0 \leq \ell \leq mn$, $k \in \Lambda_m^n$ and $p \in \Lambda_m^n$ such that:

1. $\hat{C}(n, \ell, k, p) = 0$ unless $0 < |p| = 2|k| - 2\ell$ and
2. if $L = \sum_{\ell=0}^{mn} \partial^{\ell} b_{\ell}$ then $L^n = \sum_{\ell=0}^{mn} (-1)^{\ell} \partial^{\ell} B_{\ell} \partial^{\ell}$, where

$$B_{\ell} = B_{\ell} + R_{\ell}$$

with $B_{\ell}$ as in Eq. (3.2) and $R_{\ell}$ is defined by

$$R_{\ell} = \sum_{k \in \Lambda_m^n, p \in \Lambda_m^n} \hat{C}(n, \ell, k, p) (\partial^{p_1} b_{k_1}) (\partial^{p_2} b_{k_2}) \ldots (\partial^{p_n} b_{k_n}).$$

Notice that $2|k| - 2\ell \leq 2mn - 2\ell$ and so if $\ell = mn$, we must have $|p| = 2|k| - 2\ell = 0$ and so $\hat{C}(n, \ell, k, p) = 0$. This shows that $R_{mn} = 0$ which can easily be verified independently if the reader so desires.
Proof. By Theorem 2.7, we know that $L^n = \sum_{\ell=0}^{mn} (-1)^\ell \partial^\ell B_\ell \partial^\ell$ where
\[
(-1)^\ell B_\ell = A_{2\ell} + \sum_{\ell < s \leq mn} K_{mn}(\ell,s) \partial^{2(s-\ell)} A_{2s}, \quad 0 \leq \ell \leq mn
\] (3.14)
and $\{A_k\}_{k=0}^{2mn}$ are the coefficients in Eq. (3.1). Using the formula for the $\{A_k\}$ from Proposition 3.7 in Eq. (3.14) implies,
\[
(-1)^\ell B_\ell = (-1)^\ell T_{2\ell} + \sum_{\ell < s \leq mn} K_{mn}(\ell,s) \partial^{2(s-\ell)} \left[ (-1)^s B_s + T_{2s} \right],
\]
i.e. $B_\ell = B_\ell + R_\ell$ where
\[
R_\ell = (-1)^\ell T_{2\ell} + \sum_{\ell < s \leq mn} (-1)^\ell K_{mn}(\ell,s) \partial^{2(s-\ell)} \left[ (-1)^s B_s + T_{2s} \right].
\]

It now only remains to see that this remainder term may be written as in Eq. (3.18).

Recall from Eq. (3.2) that $B_s := \sum_{j \in \Lambda^s_n} 1_{|j|=s} b_{j_1} \ldots b_{j_n}$ and so
\[
\partial^{2(s-\ell)} B_s = \sum_{j \in \Lambda^s_n} 1_{|j|=s} \partial^{2(s-\ell)} [b_{j_1} \ldots b_{j_n}].
\]

For $s > \ell$, $\partial^{2(s-\ell)} [b_{j_1} \ldots b_{j_n}]$ is a linear combination of terms of the form,
\[
(\partial^{p_1} b_{j_1}) (\partial^{p_2} b_{j_2}) \ldots (\partial^{p_n} b_{j_n})
\]
where $0 < |p| = 2(s - \ell) = 2 |j| - 2\ell$ as desired. Similarly, from Eq. (3.9), $T_{2s}$ is a linear combination of monomials of the form,
\[
\partial^{l_1} M_{b_{j_1}} \partial^{l_2} M_{b_{j_1}} \ldots \partial^{l_s} M_{b_{j_n}} \partial^{|q|} b_{q_1}
\]
with $2 |q| - 2s = |l| + |j| > 0$ and $j_1 = 0$.

It then follows that
\[
\partial^{2(s-\ell)} \partial^{l_1} M_{b_{j_1}} \partial^{l_2} M_{b_{j_1}} \ldots \partial^{l_s} M_{b_{j_n}} \partial^{|q|} b_{q_1}
\]
is a linear combination of monomials of the form,
\[
(\partial^{p_1} b_{q_1}) (\partial^{p_2} b_{q_2}) \ldots (\partial^{p_n} b_{q_n})
\]
where
\[
0 < |p| = 2(s - \ell) + |l| + |j| = 2(s - \ell) + 2 |q| - 2s = 2 |q| - 2\ell.
\]

Putting all of these comments together completes the proof. \qed

4. The Essential Self Adjointness Proof

This section is devoted to the proof of Theorem 1.9. Lemma 4.1 records a simple sufficient condition for showing a symmetric operator on a Hilbert space is in fact essentially self-adjoint. For the remainder of this paper, we assume that the coefficients, $\{a_k\}_{k=0}^d$ of $L$ in Eq. (1.2) are all in $\mathbb{R}[x]$ and we now restrict $L$ to $S$ as described in Notation 1.8. The operators, $L^n$, are then defined for all $n \in \mathbb{N}$ and we still have $D(L^n) = S$, see Remark 1.7.

Lemma 4.1 (Self-Adjointness Criteria). Let $\mathbb{L} : \mathcal{K} \rightarrow \mathcal{K}$ be a densely defined symmetric operator on a Hilbert space $\mathcal{K}$ and let $S = D(\mathbb{L})$ be the domain of $\mathbb{L}$. Assume there exists linear operators $T_\mu : S \rightarrow S$ and bounded operators $R_\mu : \mathcal{K} \rightarrow \mathcal{K}$ for $\mu \in \mathbb{R}$ such that;
\begin{enumerate}
  \item $(\mathbb{L} + i\mu)T_\mu u = (I + R_\mu)u$ for all $u \in S$, and
\end{enumerate}
(2) there exists $M > 0$ such that $\|R_\mu\|_{op} < 1$ for $|\mu| > M$.

Under these assumptions, $\mathbb{L}|_S$ is essentially self-adjoint.

**Proof.** $\|R_\mu\|_{op} < 1$ for $|\mu| > M$ is assumed in condition 2 which implies $I + R_\mu$ is invertible. Therefore, if $f \in K$, then $g := (I + R_\mu)^{-1} f \in K$ satisfies $(I + R_\mu) g = f$.

We may then choose $\{g_n\}_{n=1}^\infty \subset S$ such that $g_n \to g$ in $K$. Let $s_n = T_\mu g_n \in S$. We have, by condition 1, that

$$
\|(L + i\mu) s_n - f\| = \|(I + R_\mu) g_n - f\|
\leq \|(I + R_\mu) (g_n - g)\|
\leq \|I + R_\mu\|_{op} \|g_n - g\| \to 0 \text{ as } n \to \infty.
$$

We have thus verified that $\text{Ran}(L + i\mu)|_S$ is dense in $K$ for all $|\mu| > M$ from which it follows that $\mathbb{L}|_S$ is essentially self-adjoint, see for example the corollary on p.257 in Reed and Simon [16].

**Notation 4.2.** Let $\{B_\ell(x)\}_{\ell=0}^{mn}$ be the coefficients defined in Eq. (3.2) and define

$$
\Sigma(x,\xi) := \sum_{\ell=0}^{mn} B_\ell(x) \xi^{2\ell}.
$$

(4.1)

From Eqs. (3.10) and (1.3) the symbol, $\sigma_n(x,\xi) := \sigma_{L^n}(x,\xi)$, of $L^n$ presented as in Eq. (3.11) may be written as

$$
\sigma_n(x,\xi) = \sum_{\ell=0}^{mn} \left[ B_\ell(x) + (-1)^\ell T_{2\ell}(x) \right] \xi^{2\ell} - i \sum_{\ell=1}^{mn} (-1)^\ell T_{2\ell-1}(x) \xi^{2\ell-1} \quad (4.2)
$$

$$
= \Sigma(x,\xi) + \sum_{\ell=0}^{mn} \left[ (-1)^\ell T_{2\ell}(x) \right] \xi^{2\ell} - i \sum_{\ell=1}^{mn} (-1)^\ell T_{2\ell-1}(x) \xi^{2\ell-1}, \quad (4.3)
$$

where the coefficients $\{T_k\}_{k=0}^{2mn}$ are as in Eq. (5.9). More importantly, for our purposes,

$$
\text{Re} \sigma_n(x,\xi) = \Sigma(x,\xi) + \sum_{\ell=0}^{mn} \left[ (-1)^\ell T_{2\ell}(x) \right] \xi^{2\ell}. \quad (4.4)
$$

The following lemma will be useful in estimating all of these functions of $(x,\xi)$.

**Lemma 4.3.** Let $0 \leq k_1 < k_2 < \infty$ and $p(x)$, $q(x)$ and $r(x)$ be real polynomials such that $\deg p \leq \deg q$, $q > 0$, and $r$ is bounded from below.

1. If $\deg(p) < \deg(r)$ or $p$ is a constant function, then, for any $k_1 < k_2$ and $\lambda > 0$, there exists $c_\lambda$ such that

$$
|p(x)\xi^{k_1}| \leq \lambda \left( q(x) |\xi|^{k_2} + r(x) \right) + c_\lambda. \quad (4.5)
$$

2. If $\deg(p) \leq \deg(r)$, then for any $k_1 < k_2$ and $\lambda > 0$, there exists constants $c_\lambda$ and $d_\lambda$ such that

$$
|p(x)\xi^{k_1}| \leq \lambda q(x) |\xi|^{k_2} + c_\lambda r(x) + d_\lambda. \quad (4.6)
$$

**Proof.** Since $\deg p \leq \deg q$ and $q > 0$, $\deg q \in 2\mathbb{N}_0$ and $K := \sup_{x \in \mathbb{R}} |p(x)|/q(x) < \infty$, i.e. $p(x) \leq Kq(x)$. One also has for every $\tau > 0$, there exists $0 < a_\tau < \infty$ such
\[ |\xi^{k_1}| \leq \tau |\xi|^{k_2} + a_r. \]
Combining these estimates shows,
\[
|p(x) \xi^{k_1}| \leq \tau |p(x)| |\xi|^{k_2} + a_r |p(x)|
\]
\[
\leq \tau K q(x) |\xi|^{k_2} + a_r |p(x)|.
\] (4.7)

If \( \deg p < \deg r \), for every \( \delta > 0 \) there exists \( 0 < b_\delta < \infty \) such that \( |p(x)| \leq \delta r(x) + b_\delta \) which combined with Eq. (4.8) implies
\[
|p(x) \xi^{k_1}| \leq \tau K q(x) |\xi|^{k_2} + a_r \left( \delta r(x) + b_\delta \right)
\]
and Eq. (4.9) follows by choosing \( \tau = \lambda/K \) and then \( \delta = \lambda/a_r \) so that \( c_\lambda = a_r b_\delta \).

If \( \deg p \leq \deg r \), then there exists \( C_1, C_2 < \infty \) such that \( |p(x)| \leq C_1 r(x) + C_2 \) which combined with Eq. (4.8) with \( \tau \) and Eq. (4.9) shows,
\[
|p(x) \xi^{k_1}| \leq \lambda q(x) |\xi|^{k_2} + a_\lambda/K \left( C_1 r(x) + C_2 \right)
\]
from which Eq. (4.10) follows. \( \blacksquare \)

With the use of Lemma 4.3 the following Lemma helps us to estimate the growth of \( T_k(x) \) (see Notation 3.6) and its derivatives of \( L^n \) in Eq. (3.4) for \( 0 \leq k \leq 2mn \)

**Lemma 4.4.** Suppose that \( \{b_j\}_{t=0}^m \) are polynomials satisfying the assumptions in Theorem 1.9. Then for each \( 0 \leq k \leq 2mn \), \( \beta \in \mathbb{N}_0 \), and \( \delta > 0 \), there exists \( C = C(k, \beta, \delta) < \infty \) such that
\[
\left( 1 + |\xi|^k \right) \left( 1 + |x|^{\beta} \right) |\partial_x^\beta T_k(x)| \leq \delta \Sigma(x, \xi) + C(k, \beta, \delta). \] (4.9)

**Proof.** If \( \deg b_0 \leq 0 \), then by condition 2 in Theorem 1.9 it follows that \( \{b_j\}_{t=0}^m \) are all constant in which case \( T_k \equiv 0 \) and the Lemma is trivial. So for the rest of the proof we assume \( \deg b_0 > 0 \).

According to Eq. (3.2), \( T_k \) may be expressed as a linear combination of terms of the form, \( (\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n}) \), where \( j \) and \( p \) are multi-indices such that \( 2 |j| - k = |p| > 0 \). If \( j \) and \( p \) are multi-indices such that \( 2 |j| - k = |p| > 0 \) and \( (\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n}) \neq 0 \), then \( b_{j_1} \cdots b_{j_n} \) is strictly positive and
\[
\deg (b_{j_1} \cdots b_{j_n}) \geq \deg [(\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n})] + |p| > 0.
\]
Given the term, \( b_{j_1} \cdots b_{j_n} \), appears in \( B_{|j|} \), we conclude that \( B_{|j|} \) is strictly positive and
\[
\deg ((\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n})) < \deg (b_{j_1} \cdots b_{j_n}) \leq \deg (B_{|j|}).
\]
Moreover from condition 2 in Theorem 1.9 \( \deg b_j \leq \deg b_0 \) for all \( j \) and therefore we also have
\[
\deg ((\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n})) < \deg (b_{j_1} \cdots b_{j_n}) \leq \deg (b_0^n) = \deg B_0.
\]
Moreover, for any \( r, \beta \in \mathbb{N}_0 \) with \( r \leq \beta \) we still have
\[
\deg \{ x^r \partial_x^\beta [(\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n})] \} \leq \deg ((\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n}))
\]
\[
< \min \{ \deg (B_{|j|}), \deg (B_0) \}.
\]
Hence by substituting
\[
p(x) = x^r \partial_x^\beta [(\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n})], \quad q(x) = B_{|j|}(x) \quad \text{and} \quad r(x) = B_0(x)
\]
in Lemma 4.3 for every \( \lambda > 0 \) there exists \( C_\lambda < \infty \) such that
\[
|x^r \partial_x^\beta [(\partial^{p_1} b_{j_1}) \cdots (\partial^{p_n} b_{j_n})] \xi^k| \leq \lambda \left[ B_{|j|} (x) \xi^{|j|} + B_0 (x) \right] + C_\lambda
\]
\[
\leq \lambda \cdot \Sigma (x, \xi) + C_\lambda
\]
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and similarly, 
\[ |x^r \partial^2_x \left[ (\partial^{p_1} b \ldots) (\partial^{p_n} b) \right]| \leq \lambda \Sigma (x, \xi) + C. \]
These last two equations with \( r = 0 \) and \( r = \beta \) combine to show, for all \( \lambda > 0 \), there exists \( C \lambda < \infty \) such that 
\[ \left( 1 + |\xi|^k \right) \left( 1 + |x|^\beta \right) \left| \partial^2_x \left[ (\partial^{p_1} b \ldots) (\partial^{p_n} b) \right] \right| \leq 4 \left( \lambda \Sigma (x, \xi) + C \right). \]
By using this result in Eq. (3.9), one then sees there is a constant \( K < \infty \) such that 
\[ \left( 1 + |\xi|^k \right) \left( 1 + |x|^\beta \right) \left| \partial^2_x \left[ (\partial^{p_1} b \ldots) (\partial^{p_n} b) \right] \right| \leq K \Sigma (x, \xi) + K C. \]
Equation (4.6) now follows by replacing \( \lambda \) by \( \delta / K \) in the above equation. □

The following Lemma is to study the growth of \( B_\ell (x) \) (see Notation 3.2) and its derivatives of \( L^\alpha \) in Eq. (3.1) for \( \beta \in \mathbb{N}_0 \) and \( m \alpha \).

**Lemma 4.5.** Again suppose that \( \{b_\ell\}_{\ell=0}^\infty \) are polynomials satisfying the assumptions in Theorem 1.9. For all \( \ell \in \mathbb{N}_0 \), and \( \beta \in \mathbb{N}_0 \), there exists \( C < \infty \) such that 
\[ \left| \partial^\alpha_x B_\ell (x) \right| \left( |\xi|^{2\ell} + 1 \right) \left( |x|^{\beta} + 1 \right) \leq C \Sigma (x, \xi) + C. \tag{4.10} \]
Moreover, if we assume \( b_0 \) is not the zero polynomial, then we may drop the second \( C \) in Eq. (4.10), i.e. there exists \( C < \infty \) such that 
\[ \left| \partial^\alpha_x B_\ell (x) \right| \left( |\xi|^{2\ell} + 1 \right) \left( |x|^{\beta} + 1 \right) \leq C \Sigma (x, \xi). \tag{4.11} \]

**Proof.** Case 1. If \( b_0 = 0 \) then by the assumption 2 of Theorem 1.9 each \( b_\ell \) is a constant for \( 1 \leq \ell \leq m \) and therefore \( \partial^\alpha_x B_\ell (x) = 0 \) for all \( \beta > 0 \), i.e. \( B_\ell \) are constant for all \( \ell \). Moreover, if \( \beta = 0 \), from the definition of \( \Sigma (x, \xi) \) in Eq. (4.11) it follows that \( B_\ell x^{2\ell} \leq \Sigma (x, \xi) \) and hence 
\[ |B_\ell (x)| \left( |\xi|^{2\ell} + 1 \right) = B_\ell \left( \xi^{2\ell} + 1 \right) \leq \Sigma (x, \xi) + B_\ell \]
and so Eq. (4.10) holds with \( C = \max_{1 \leq \ell \leq m} \max (1, B_\ell) \).

Case 2. If \( b_0 \neq 0 \), let us assume \( B_\ell \) is not the zero polynomial for otherwise there is nothing to prove. Since \( x^\beta \partial^\alpha_x B_\ell (x) \) and \( \partial^\alpha_x B_\ell (x) \) both have degree no more than \( \deg B_\ell \) and \( B_\ell > 0 \), we may conclude there exists \( C < \infty \) such that 
\[ \left( 1 + |x|^{\beta} \right) \left| \partial^\alpha_x B_\ell (x) \right| = \left| \partial^\alpha_x B_\ell (x) \right| + \left| x^\beta \partial^\alpha_x B_\ell (x) \right| \leq C B_\ell (x). \tag{4.12} \]
Multiplying this equation by \( \xi^{2\ell} \) then shows, 
\[ \left( 1 + |x|^{\beta} \right) \left| \partial^\alpha_x B_\ell (x) \right| \xi^{2\ell} \leq C B_\ell (x) \xi^{2\ell} \leq C \Sigma (x, \xi) \tag{4.13} \]
while \( \deg (B_\ell) \leq \deg (B_0) \) and \( B_0 > 0 \), then there exists \( C_1 < \infty \) such that 
\[ B_\ell (x) \leq C_1 B_0 (x) \leq C_1 \Sigma (x, \xi) \]
which combined with Eq. (4.12) shows 
\[ \left( 1 + |x|^{\beta} \right) \left| \partial^\alpha_x B_\ell (x) \right| \leq C_1 \Sigma (x, \xi). \]
This estimate along with Eq. (4.13) then completes the proof of Eq. (4.10) with no second \( C \). □

**Notation 4.6.** For any non-negative real-valued functions \( f \) and \( g \) on some domain \( U \), we write \( f \lesssim g \) to mean there exists \( C > 0 \) such that \( f (y) \leq C g (y) \) for all \( y \in U \).
The following result is an immediate corollary of Proposition 3.7 and Lemmas 4.4 and 4.5.

**Corollary 4.7.** Suppose that \( \{ b_l \}_{l=0}^m \) are polynomials satisfying the assumptions in Theorem 1.9. If \( \{ A_k \}_{k=0}^{2mn} \) are the coefficients of \( L^\alpha \) as in Eq. (3.1), then for all \( \beta \in \mathbb{N}_0 \) and \( 0 \leq k \leq 2mn \),

\[
| \partial_\xi^\beta x^A_k (x) | \left( 1 + | \xi |^k \right) \left( 1 + | x |^\beta \right) \lesssim \Sigma (x, \xi) + 1. \tag{4.14}
\]

The next lemma is a direct consequence of Lemmas 4.4.

**Lemma 4.8.** Let \( L \) be the operator in Eq. (1.6), where we now assume that \( \{ b_l \}_{l=0}^m \) are polynomials satisfying the assumptions of Theorem 1.9. Then there exists \( c > 0 \) such that the following hold:

\[
\sum_{j=0}^{mn} | T_{2j} (x) \xi^{2j} | \leq \sum_{k=0}^{2mn} | T_k (x) \xi^k | \leq \frac{1}{2} \left( \Sigma (x, \xi) + c \right), \quad \text{and} \tag{4.15}
\]

\[
\frac{3}{2} \Sigma (x, \xi) + \frac{1}{2} c \geq \text{Re} \sigma_n (x, \xi) \geq \frac{1}{2} \Sigma (x, \xi) - \frac{1}{2} c. \tag{4.16}
\]

Alternatively, adding \( c \) to both sides of Eq. (4.16) shows

\[
\frac{3}{2} (\Sigma (x, \xi) + c) \geq \text{Re} \sigma_{n+c} (x, \xi) \geq \frac{1}{2} (\Sigma (x, \xi) + c). \tag{4.17}
\]

A key point is that \( \text{Re} \sigma_n (x, \xi) := \text{Re} \sigma_{L^\alpha} (x, \xi) \) (see Notation 4.2) is bounded from below.

**Notation 4.9.** For the rest of this section, we fix a \( c > 0 \) as in Lemma 4.8 and then define \( L := L^\alpha + c \) where \( D (L) = \mathcal{S} \) and \( \{ b_l \}_{l=1}^m \) in Eq. (1.6) satisfies the assumptions in Theorem 1.9. According to the definition of symbol in Eq. (1.3),

\[
\sigma_L (x, \xi) := \sigma_{L+c} (x, \xi) = \sigma_n (x, \xi) + c
\]

Because of our choice of \( c > 0 \) we know that

\[
\kappa := \inf_{(x, \xi)} \text{Re} \sigma_L (x, \xi) > 0. \tag{4.18}
\]

**Corollary 4.10.** For all \( l, k \in \mathbb{N}_0 \),

\[
| \partial_\xi^l \partial_x^k \sigma_L (x, \xi) | \left( 1 + | \xi |^l \right) \left( 1 + | x |^k \right) \lesssim \Sigma (x, \xi) + 1 \tag{4.19}
\]

or equivalently,

\[
\frac{\partial_\xi^l \partial_x^k \sigma_L (x, \xi) }{ \text{Re} \sigma_L (x, \xi) } \lesssim \frac{1}{ (1 + | \xi |^l ) (1 + | x |^k )} \tag{4.20}
\]

if Eq. (4.17) is applied.

**Proof.** We have

\[
\partial_\xi^l \partial_x^k \sigma_L (x, \xi) = \sum_{j=l}^{2mn} \binom{j}{l} \frac{j!}{(j-l)!} \partial_x^k A_j (x) \xi^{j-l}
\]
and therefore,

\[ \left| \partial_x^l \partial_\xi^k \sigma_L (x, \xi) \right| (1 + |\xi|)^j (1 + |x|)^k \]
\[ \leq \sum_{j=l}^{2mn} \frac{j!}{(j-l)!} \left| \partial_x^l A_j (x) \right| \left| \xi^{j-l} \right| (1 + |\xi|)^j (1 + |x|)^k \]
\[ \lesssim \sum_{j=l}^{2mn} \left| \partial_x^l A_j (x) \right| \left( 1 + |\xi|^j \right) \left( 1 + |x|^k \right) \]
\[ \lesssim \Sigma (x, \xi) + 1. \]

The last step is asserted by Corollary 4.7. Equation (4.20) follows directly from Eqs. (4.17) and (4.19). \[ \blacksquare \]

By the Fourier inversion formula, if \( \psi \in \mathcal{S} \), then

\[ \psi (x) = \int_\mathbb{R} \hat{\psi} (\xi) e^{ix\xi} d\xi, \quad (4.21) \]

where \( \hat{\psi} \) is the Fourier transform of \( \psi \) defined by

\[ \hat{\psi} (\xi) = \frac{1}{2\pi} \int_\mathbb{R} e^{-iy\xi} \psi (y) dy. \quad (4.22) \]

Recall that, with these normalizations, that

\[ \| \psi \| = \sqrt{2\pi} \| \hat{\psi} \| \quad \forall \psi \in L^2 (\mathbb{R}). \quad (4.23) \]

Letting \( \mu \in \mathbb{R} \) and then applying \( L + i\mu \) to Eq. (4.21) gives the following pseudo-differential operator representation of \( (L + i\mu) \psi \),

\[ (L + i\mu) \psi (x) = \int_\mathbb{R} \left[ \sigma_L (x, \xi) + i\mu \right] e^{ix\xi} \hat{\psi} (\xi) d\xi. \quad (4.24) \]

Let \( \kappa \) be as in Eq. (4.18), it follows that for any \( \mu \in \mathbb{R} \),

\[ |\sigma_L (x, \xi) + i\mu| \geq |\text{Re} \sigma_L (x, \xi)| \geq \kappa > 0 \]

for all \( (x, \xi) \in \mathbb{R}^2 \). Therefore, the following integrand in Eq. (4.25) is integrable for \( u \in \mathcal{S} \) and we may define

\[ (T_\mu u) (x) = \int_\mathbb{R} \frac{1}{\sigma_L (x, \xi) + i\mu} e^{ix\xi} \hat{u} (\xi) d\xi. \quad (4.25) \]

Furthermore, we will show that \( T_\mu \) actually preserves \( \mathcal{S} \) later in this section (see Proposition 4.15).

**Notation 4.11.** If \( \{ q_k (x) \}_{k=0}^j \) is a collection of smooth functions and

\[ q (x, \theta) = \sum_{k=0}^j q_k (x) \theta^k, \quad (4.26) \]

then \( q (x, \partial) \) is defined to be the \( j \)-th order differential operator given by

\[ q (x, \partial) := \sum_{k=0}^j q_k (x) \partial_x^k. \quad (4.27) \]
Similarly, for $\xi \in \mathbb{R}$, we let
\[
q(x, \frac{1}{i} \partial_x + \xi) := \sum_{k=0}^{j} q_k(x) \left( \frac{1}{i} \partial_x + \xi \right)^k.
\] (4.28)

For the proofs below, recall from Eq. (3.7) that
\[
q(\partial_x) M_{e^{i \xi \cdot x}} = M_{e^{i \xi \cdot x}} q(\partial_x + i \xi).
\] (4.29)
whenever $q(\theta)$ is a polynomial in $\theta$.

**Lemma 4.12.** Let $q(x, \theta)$ be as in Eq. (4.26) where the coefficients $\{q_k(x)\}_{k=0}^{j}$ are now assumed to be polynomials in $x$. Further let
\[
(Su)(x) := \int_{\mathbb{R}} \Gamma(x, \xi) e^{i \xi \cdot x} u(\xi) d\xi \quad \forall u \in \mathcal{S},
\] (4.30)
where $\Gamma(x, \xi)$ is a smooth function such that $\Gamma(x, \xi)$ and all of its derivatives in both $x$ and $\xi$ have at most polynomial growth in $\xi$ for any fixed $x$. Then
\[
(q(x, \partial_x) Su)(x) = \int_{\mathbb{R}} e^{i \xi \cdot x} q(i \partial_\xi, \partial_x + i \xi) [\Gamma(x, \xi) \hat{u}(\xi)] d\xi,
\] (4.31)
where
\[
q(i \partial_\xi, \partial_x + i \xi) := \sum_{k=0}^{j} q_k(i \partial_\xi) (\partial_x + i M_\xi)^k.
\] (4.32)

**Proof.** Using Eq. (4.29) we find,
\[
(q(x, \partial) Su)(x) = \int_{\mathbb{R}} q(x, \partial_x) [\Gamma(x, \xi) \hat{u}(\xi) e^{i \xi \cdot x}] d\xi
\]
\[
= \sum_{k=0}^{j} \int_{\mathbb{R}} q_k(x) \partial_x^k [\Gamma(x, \xi) \hat{u}(\xi) e^{i \xi \cdot x}] d\xi
\]
\[
= \sum_{k=0}^{j} \int_{\mathbb{R}} q_k(x) e^{i \xi \cdot x} (\partial_x + i \xi)^k [\Gamma(x, \xi) \hat{u}(\xi)] d\xi
\]
\[
= \sum_{k=0}^{j} \int_{\mathbb{R}} [q_k(-i \partial_\xi) e^{i \xi \cdot x}] (\partial_x + i \xi)^k [\Gamma(x, \xi) \hat{u}(\xi)] d\xi
\]
\[
= \sum_{k=0}^{j} \int_{\mathbb{R}} e^{i \xi \cdot x} q_k(i \partial_\xi) (\partial_x + i \xi)^k [\Gamma(x, \xi) \hat{u}(\xi)] d\xi
\]
\[
= \int_{\mathbb{R}} e^{i \xi \cdot x} q(i \partial_\xi, \partial_x + i \xi) [\Gamma(x, \xi) \hat{u}(\xi)] d\xi.
\]
We have used the assumptions on $\Gamma$ to show; (1) that $\partial_x$ commutes with the integral giving the first equality above, and (2) that
\[
\xi \rightarrow (\partial_x + i \xi)^k [\Gamma(x, \xi) \hat{u}(\xi)] \in \mathcal{S}
\]
which is used to justify the integration by parts used in the in the second to last equality. ■
Lemma 4.13. Suppose that \( f \in C^\infty (\mathbb{R}^j, (0, \infty)) \), then for every multi-index, \( \alpha = (\alpha_1, \ldots, \alpha_j) \in \mathbb{N}_0^j \) with \( \alpha \neq 0 \) there exists a polynomial, \( P_\alpha \), with no constant term such that

\[
\frac{\partial^\alpha f}{f} = \frac{1}{f} P_\alpha \left( \left\{ \frac{\partial^\beta f}{f} : 0 < \beta \leq \alpha \right\} \right)
\]

where \( \partial^\alpha := \partial_{\alpha_1} \cdots \partial_{\alpha_j} \). Moreover, \( P_\alpha \left( \left\{ \frac{\partial^\beta f}{f} : 0 < \beta \leq \alpha \right\} \right) \) is a linear combination of monomials of the form \( \prod_{i=1}^k \frac{\partial^{\beta(i)} f}{f} \) where \( \beta(i) \in \mathbb{N}_0^j \) for \( 1 \leq i \leq k \) and \( 1 \leq k \leq |\alpha| \) such that \( \sum_{i=1}^k \beta(i) = \alpha \), and \( \beta(i) \neq 0 \) for all \( i \).

Proof. The estimate in Eq. \( \text{(4.34)} \) is elementary from Eq. \( \text{(4.17)} \) in Lemma 4.8 and will be left to the reader. From Lemma 4.13,

\[
\frac{\partial_1 \partial_2 f}{f} = \frac{1}{f} \left[ \frac{\partial_1 f \partial_2 f}{f} - \frac{\partial_1 \partial_2 f}{f} \right].
\]

Corollary 4.14. Let \( \mu \in \mathbb{R} \). If

\[
\Gamma (x, \xi) := \frac{1}{\sigma_L (x, \xi) + i\mu},
\]

then

\[
|\Gamma (x, \xi)| \leq \frac{1}{|\text{Re} \sigma_L (x, \xi)|} \lesssim \frac{1}{b_m (x) \xi^{2m} + b_0^m (x) + c} \lesssim 1
\]

and for any \( \alpha, \beta \in \mathbb{N}_0^j \) with \( \alpha + \beta > 0 \), there exists a constant \( c_{\alpha, \beta} > 0 \) such that

\[
\left| \frac{\partial^\alpha \partial^\beta \Gamma (x, \xi)}{\sigma_L (x, \xi) + i\mu} \right| \leq |\Gamma (x, \xi)| \cdot c_{\alpha, \beta} (1 + |\xi|)^{-\beta} (1 + |x|)^{-\alpha}.
\]

Proof. The estimate in Eq. \( \text{(4.33)} \) is elementary from Eq. \( \text{(4.17)} \) in Lemma 4.8 and will be left to the reader. From Lemma 4.13,

\[
\frac{\partial_x \partial_\xi \Gamma (x, \xi)}{\sigma_L (x, \xi) + i\mu} = \frac{1}{\sigma_L (x, \xi) + i\mu} \cdot \Theta_{(\alpha, \beta)} (x, \xi)
\]

where \( \Theta_{(\alpha, \beta)} (x, \xi) \) is a linear combination of the following functions,

\[
\prod_{j=1}^J \frac{\partial^{k_j} \partial_x^{l_j} \sigma_L (x, \xi)}{\sigma_L (x, \xi) + i\mu}
\]

where \( 0 \leq k_j \leq \alpha, \ 0 \leq l_j \leq \beta, \ k_j + l_j > 0, \ \sum_{j=1}^J k_j = \alpha, \ \sum_{j=1}^J l_j = \beta \) and \( 1 \leq J \leq \alpha + \beta \). The estimate in Eq. \( \text{(4.20)} \) implies,

\[
\prod_{j=1}^J \left| \frac{\partial^{k_j} \partial_x^{l_j} \sigma_L (x, \xi)}{\sigma_L (x, \xi) + i\mu} \right| 
\]

which altogether gives the estimated in Eq. \( \text{(4.35)} \). \( \square \)

Proposition 4.15 \( (T_\mu \text{ preserves } \mathcal{S}) \). If \( T_\mu \) is as defined in Eq. \( \text{(4.27)} \), then \( T_\mu (\mathcal{S}) \subset \mathcal{S} \) for all \( \mu \in \mathbb{R} \).
Lemma 4.16. For all $\mu \in \mathbb{R}$ and $u \in S$,
\[
[\mathbb{L} + i\mu] T_{\mu} u = [I + R_{\mu}] u \tag{4.37}
\]
where
\[
(R_{\mu} u)(x) = \int_{\mathbb{R}} \rho_{\mu}(x, \xi) \hat{u}(\xi) e^{ix\xi} d\xi, \tag{4.38}
\]
\[
\rho_{\mu}(x, \xi) := \left( \sigma_{\bar{L}} \left( x, \frac{1}{i} \partial_{x} + \xi \right) - \sigma_{\bar{L}}(x, \xi) \right) \frac{1}{\sigma_{\bar{L}}(x, \xi) + i\mu}, \tag{4.39}
\]
and $\sigma_{\bar{L}}(x, \frac{1}{i} \partial_{x} + \xi)$ is as in Eq. (4.28).

**Proof.** As $\sigma_{\bar{L}}(x, \xi)$ is a polynomial in the $\xi$–variables with smooth coefficients in the $x$–variables, there is no problem justifying the identity,
\[
([\mathbb{L} + i\mu] T_{\mu} u)(x) = \int_{\mathbb{R}} [\mathbb{L}_{x} + i\mu] \left( \Gamma(x, \xi) e^{ix\xi} \right) \hat{u}(\xi) d\xi, \tag{4.40}
\]
where the subscript $x$ on $\mathbb{L}$ indicates that $\mathbb{L}$ acts on $x$–variables only. Using $\mathbb{L} = \sigma_{\bar{L}} \left( x, \frac{1}{i} \partial_{x} \right)$ along with Eq. (4.29) shows,
\[
[\mathbb{L}_{x} + i\mu] \left( \Gamma(x, \xi) e^{ix\xi} \right) = e^{ix\xi} \left[ \sigma_{\bar{L}} \left( x, \frac{1}{i} \partial_{x} + \xi \right) + i\mu \right] \Gamma(x, \xi) + e^{ix\xi} \left[ \sigma_{\bar{L}} \left( x, \frac{1}{i} \partial_{x} + \xi \right) - \sigma_{\bar{L}}(x, \xi) \right] \Gamma(x, \xi) + e^{ix\xi} \left[ \sigma_{\bar{L}} \left( x, \frac{1}{i} \partial_{x} + \xi \right) - \sigma_{\bar{L}}(x, \xi) \right] \Gamma(x, \xi) + e^{ix\xi}
\]
which combined with Eq. (4.40) gives Eq. (4.37).
Lemma 4.17. \( \rho_{\mu} (x, \xi) \) in Eq. (4.39) can be explicitly written as
\[
\rho_{\mu} (x, \xi) = \sum_{k=0}^{2mn} \sum_{j=1}^{k} \binom{k}{j} A_{k} (x) (i\xi)^{k-j} \partial_{x}^{j} \Gamma (x, \xi) .
\]
where \( A_{k} (x) \) and \( \Gamma (x, \xi) \) as in Eq. (3.1) and Eq. (4.33) respectively. Moreover, there exists \( C < \infty \) independent of \( \mu \) so that
\[
| \rho_{\mu} (x, \xi) | \leq C \frac{1}{1 + |x|} \cdot \frac{1}{1 + |\xi|}.
\]

Proof. Using Eq. (1.7) and the formula of \( \sigma_{L} \) in Notation 4.9, we may write Eq. (4.39) more explicitly as,
\[
\rho_{\mu} (x, \xi) = \sum_{k=0}^{2mn} A_{k} (x) \left[ (\partial_{x} + i\xi)^{k} - (i\xi)^{k} \right] \Gamma (x, \xi)
= \sum_{k=1}^{2mn} \sum_{j=1}^{k} \binom{k}{j} A_{k} (x) (i\xi)^{k-j} \partial_{x}^{j} \Gamma (x, \xi).
\]
Therefore, using the estimate in Eq. (4.35) of Corollary 4.1 with \( \beta = 0 \) and \( \alpha = j \), we learn
\[
| \rho_{\mu} (x, \xi) | \leq \sum_{k=1}^{2mn} \sum_{j=1}^{k} \binom{k}{j} |A_{k} (x)| |\xi|^{k-j} |\Gamma (x, \xi)|
\leq \sum_{k=1}^{2mn} \sum_{j=1}^{k} \binom{k}{j} |A_{k} (x)| |\xi|^{k-j} |\Gamma (x, \xi)| \frac{1}{1 + |x|}.
\]
Moreover, for any \( 1 \leq j \leq k \),
\[
|A_{k} (x)| |\xi|^{k-j} |\Gamma (x, \xi)| \leq \frac{\Sigma (x, \xi) + 1}{1 + |\xi|^{k}} |\xi|^{k-j} |\Gamma (x, \xi)|
\leq \frac{1}{1 + |\xi|^{j}} |\text{Re} \sigma_{L} (x, \xi)|
\leq \frac{1}{1 + |\xi|^{j}} \leq \frac{1}{1 + |\xi|}.
\]
wherein we have used the estimates in Eq. (4.14) with \( \beta = 0 \) in the first step, and the left inequality in Eq. (4.34) in the second step, and Eq. (4.17) in the third step.

We are now prepared to complete the proof of Theorem 1.9. The following notation will be used in the proof.

Notation 4.18. If \( g : \mathbb{R}^2 \to \mathbb{C} \) is a measurable function we let
\[
\| g (x, \xi) \|_{L^{2}(d\xi)} := \left( \int_{\mathbb{R}} |g (x, \xi)|^{2} d\xi \right)^{1/2}
\text{and}
\| g (x, \xi) \|_{L^{2}(dx \otimes d\xi)} := \left( \int_{\mathbb{R}^2} |g (x, \xi)|^{2} dx d\xi \right)^{1/2}.
\]
Proposition 3.8 holds. Therefore, the only thing to prove in the item 1 is Eq. (5.1). From \( j \) and for this 0 \(< \mathcal{L} \)

Eqs. (3.2) and (3.13) respectively.

Proposition 5.1. Suppose that \( \{c_i\} \) of the Fourier transform it follows that

Lemma 4.1 and hence Lemma 4.1 is verified. Thus we have to estimate the operator norm of the error term,

\[
\| \hat{u} \|_{L^2(dx \otimes d\xi)} = \frac{1}{\sqrt{2\pi}} \| \rho \|_{L^2(dx \otimes d\xi)} \cdot \| u \|_{L^2(dx)}
\]

where \( \rho \) is the symbol of \( R \) as defined in Eq. (4.39). Since, by Lemma 4.13 and Eq. (2.11), \( \lim_{\mu \to \pm\infty} \rho (x, \xi) = 0 \) and, from Eq. (4.42), \( \rho \) is dominated by

\[
C (1 + |x|)^{-1} (1 + |\xi|)^{-1} \in L^2(dx \otimes d\xi),
\]

it follows that \( \| R_{\mu} \|_{op} \leq \frac{1}{\sqrt{2\pi}} \| \rho \|_{L^2(dx \otimes d\xi)} \to 0 \) as \( \mu \to \pm\infty \) and in particular, \( \| R_{\mu} \|_{op} < 1 \) when \( |\mu| \) is sufficiently large. Therefore, \( L_{|S} \) is essentially self-adjoint from Lemma 4.1 and hence \( L^n_{|S} = (L - c)_{|S} \) (\( c \) from Notation 1.9) is also essentially self-adjoint.

5. The Divergence Form of \( L^n \) and \( L^n_h \)

Suppose now that \( L \) in Eq. (2.1) with polynomial coefficients \( \{b_i\}_{l=0}^m \) is a symmetric differential operator on \( \mathcal{S} \). In section 3, we have expressed the symmetric differential operators on \( \mathcal{S} \), \( L^n \), in the divergence form with the polynomial coefficients \( \{B_l\} \) as in Eq. (3.1) for \( n \in \mathbb{N} \). The goal of this section is to derive some basic properties of the polynomial coefficients \( \{B_l\} \) and generalize coefficients properties for a scaled version \( L^n_h \) where \( L_h \) in Eq. (1.11).

Proposition 5.1. Suppose that \( \{b_i\}_{l=0}^m \) are real polynomials. Let \( B_l \) and \( R_l \) are in Eqs. (3.2) and (4.13) respectively.

1. If \( \deg b_l \leq \deg b_{l-1} \) for \( 1 \leq l \leq m \), then

\[
\deg (R_l) \leq \deg B_l - 2 \quad \text{and} \quad \deg B_l = \deg B_{l-1} + 2 \quad \text{for} \quad 1 \leq l \leq mn.
\]

2. If we only assume that \( \deg b_l \leq \deg b_{l-1} + 2 \) for \( 1 \leq l \leq m \), then

\[
\deg R_l \leq \deg B_l \quad \text{for} \quad 0 \leq l \leq mn.
\]

Proof. From Eq. (3.12), \( \deg (B_l) = \deg B_l \) follows automatically if \( \deg (R_l) \leq \deg B_l - 2 \) (5.1) holds. Therefore, the only thing to prove in the item 1 is Eq. (5.1). From Proposition 3.8, \( R_l \) is a linear combination of \( (\partial^{p_1} b_{k_1}) (\partial^{p_2} b_{k_2}) \ldots (\partial^{p_n} b_{k_n}) \) with \( 0 < |p| = 2|k| - 2l \). For each index \( k \), there exists \( j \) with \( j \leq k \) such that \( |j| = \ell \).

And for this \( j \) we have

\[
\deg ((\partial^{p_1} b_{k_1}) (\partial^{p_2} b_{k_2}) \ldots (\partial^{p_n} b_{k_n})) \leq \sum_{i=1}^{n} \deg (b_{k_i}) - |p|\]

\[
\leq \sum_{i=1}^{n} \deg (b_{j_i}) - |p|
\]

\[
\leq \deg (B_l) - 2.
\]
wherein we have used $|p| \geq 2$ ($|p|$ is positive even) and

$$\deg(B_\ell) = \max_{|j|=\ell} \deg(b_{j_1} \ldots b_{j_n}) = \max_{|j|=\ell} \sum_{i=1}^n \deg(b_{j_i}).$$

Now suppose that we only assume $\deg(b_{k+1}) \leq \deg(b_k) + 2$ (which then implies $\deg(b_{k+r}) \leq \deg(b_k) + 2r$ for $0 \leq r \leq m - k$). Working as above and remember that $0 < |p| = 2|k| - 2\ell$ and $|j| = \ell$ we find

$$\sum_{i=1}^n \deg(b_{j_i}) - |p| \leq \sum_{i=1}^n [\deg(b_{j_i}) + 2(k_i - j_i)] - |p|$$

$$= \sum_{i=1}^n \deg(b_{j_i}) + 2(|k| - |j|) - |p|$$

$$= \sum_{i=1}^n \deg(b_{j_i}) \leq \deg(B_\ell).$$

5.1. Scaled Version of Divergence Form. We now take $\hbar > 0$ and let $L_{\hbar}$ be defined as in Eq. (1.11) where the $\hbar$ – dependent coefficients, $\{b_{l,h}(x)\}_{l=0}^m$, satisfy Assumption 1. To apply the previous formula already developed (for $\hbar = 1$) we need only make the replacements,

$$b_l(x) \rightarrow \hbar^l b_{l,h} \left(\sqrt{\hbar}x\right) \quad \text{for } 0 \leq l \leq m. \quad (5.2)$$

The result of this transformation on $L^n$ is recorded in the following lemma.

Notation 5.2. Let $x_1, \ldots, x_j$ be variables on $\mathbb{R}$. We denote $\mathbb{R}[x_1, \ldots, x_j]$ be a collection of polynomials in $x_1, \ldots, x_j$ with real-valued coefficients.

Proposition 5.3. Let $n \in \mathbb{N}$, $\hbar > 0$, and $\{b_{l,h}(x)\}_{l=0}^m \subset \mathbb{R}[x]$ and let $L_{\hbar}$ be as in Eq. (1.11). Then $L_{\hbar}^n$ is an operator on $S$ and

$$L_{\hbar}^n = \sum_{\ell=0}^{\min(m,n)} (-\hbar)^\ell \partial^\ell B_{\ell,h} \left(\sqrt{\hbar}x\right) \partial^\ell, \quad (5.3)$$

where

$$B_{\ell,h} := B_{\ell,h} + R_{\ell,h} \in \mathbb{R}[x], \quad (5.4)$$

$$B_{\ell,h} = \sum_{k \in \Lambda_{m}^{n}} 1_{|k| = \ell} b_{k_1,h} b_{k_2,h} \ldots b_{k_n,h}, \quad (5.5)$$

$$R_{\ell,h} = \sum_{k \in \Lambda_{m}^{n}, p \in \Lambda_{m}^{n}} \hat{C}(n, \ell, k, p) \hbar^{|p|} (\partial^{p_1} b_{k_1,h}) \ldots (\partial^{p_n} b_{k_n,h}), \quad (5.6)$$

Proof. Making the replacements $b_l(x) \rightarrow \hbar^l b_{l,h} \left(\sqrt{\hbar}x\right)$ in Eqs. (3.13) and (3.2) shows

$$B_\ell(x) \rightarrow \sum_{j \in \Lambda_{m}^{n}} 1_{|j|=\ell} \left[\hbar^{j_1} b_{j_1,h} \left(\sqrt{\hbar}x\right) \ldots \hbar^{j_n} b_{j_n,h} \left(\sqrt{\hbar}x\right)\right] = \hbar^\ell B_{\ell,h} \left(\sqrt{\hbar}x\right) \quad (5.7)$$

$^2$Below, we use $\hat{C}(n, \ell, k, p) = 0$ unless $0 < |p| = 2|k| - 2\ell$, i.e. $|k| = \ell + |p|/2$. 

and
\[ R_\ell (x) \rightarrow \sum_{k \in \Lambda'_n, p \in \Lambda'_m} \hat{C} (n, \ell, k, p) (-1)^{|k|} \hbar^{|k|+|p|} [(\partial^{p_1} b_{k_1, h}) \ldots (\partial^{p_n} b_{k_n, h})] \left( \sqrt{\hbar} x \right) \]
\[ = \sum_{k \in \Lambda'_n, p \in \Lambda'_m} \hat{C} (n, \ell, k, p) (-1)^{|k|} \hbar^{\ell+|p|} [(\partial^{p_1} b_{k_1, h}) \ldots (\partial^{p_n} b_{k_n, h})] \left( \sqrt{\hbar} x \right) \]
\[ = \hbar^\ell \sum_{k \in \Lambda'_n, p \in \Lambda'_m} \hat{C} (n, \ell, k, p) (-1)^{|k|} \hbar^{|p|} [(\partial^{p_1} b_{k_1, h}) \ldots (\partial^{p_n} b_{k_n, h})] \left( \sqrt{\hbar} x \right) \]
\[ = \hbar^\ell R_{\ell,h} \left( \sqrt{\hbar} x \right). \]
(5.8)

Therefore it follows that
\[ B_\ell (x) = B_\ell (x) + R_\ell (x) \rightarrow \hbar^\ell [B_{\ell,h} + R_{\ell,h}] \left( \sqrt{\hbar} x \right) = \hbar^\ell B_{\ell,h} \left( \sqrt{\hbar} x \right) \]
where \( L^n_h \) is then given as in Eq. (5.3). ■

**Notation 5.4.** Let
\[ L^{(n)}_h = \sum_{\ell=0}^{mn} (-\hbar)^{\ell} \partial^\ell B_{\ell,h} \left( \sqrt{\hbar} (\cdot) \right) \partial^\ell, \text{ and} \]
(5.9)
\[ R^{(n)}_h = \sum_{\ell=0}^{mn-1} (-\hbar)^{\ell} \partial^\ell R_{\ell,h} \left( \sqrt{\hbar} (\cdot) \right) \partial^\ell. \]
(5.10)
as operators on \( S \). Then \( L^n_h \) can also be written as
\[ L^n_h = L^{(n)}_h + R^{(n)}_h \text{ on } S. \]
(5.11)

6. **Operator Comparison**

The main purpose of this section is to prove Theorem 1.17. First off, since the inequality symbol \( \preceq_S \) appears very often in this section which is defined in Notation 1.13, let us recall its definition. If \( A \) and \( B \) are symmetric operators on \( S \) (see Definition 1.11), then we say \( A \preceq_S B \) if
\[ \langle A \psi, \psi \rangle \preceq_S \langle B \psi, \psi \rangle \text{ for all } \psi \in S \]
where \( \langle \cdot, \cdot \rangle \) is the usual \( L^2 (m) \) inner product as in Eq. (1.1).

**Lemma 6.1.** Let \( \{c_\ell\}_{\ell=0}^{M-1} \subset \mathbb{R} \) be given constants. Then for any \( \delta > 0 \), there exists \( C_\delta < \infty \) such that
\[ \sum_{\ell=0}^{M-1} c_\ell \left( -\hbar \partial^2 \right)^\ell \preceq_S \delta \left( -\hbar \partial^2 \right)^M + C_\delta I \quad \forall \ h > 0. \]
(6.1)

**Proof.** By conjugating Eq. (6.1) by the Fourier transform in Eq. (1.22) (so that \( \frac{1}{i} \partial \rightarrow \xi \)) and Eq. (1.23), we may reduce Eq. (6.1) to the easily verified statement; for all \( \delta > 0 \), there exists \( C_\delta < \infty \) such that
\[ \sum_{\ell=0}^{M-1} c_\ell w^\ell \leq \delta w^M + C_\delta \quad \forall \ w \geq 0. \]
(6.2)

Here, \( w \) is shorthand for \( \hbar \xi^2 \). ■
Lemma 6.2. Let \( I \subset \mathbb{R} \) be a compact interval. Suppose \( \{ p_k(\cdot) \}_{k=0}^{m_p} \) and \( \{ q_k(\cdot) \}_{k=0}^{m_q} \subset C(I, \mathbb{R}) \) where \( m_p \in 2\mathbb{N}_0 \) and \( m_q \in \mathbb{N}_0 \) such that

\[
p(x, y) = \sum_{k=0}^{m_p} p_k(y) x^k \text{ and } q(x, y) = \sum_{k=0}^{m_q} q_k(y) x^k
\]

and \( \delta := \min_{y \in I} p_{m_p}(y) > 0 \). If we further assume \( m_p > m_q \) then for any \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that we have

\[
|q(x, y)| \leq \epsilon p(x, y) + C_\epsilon \text{ for all } y \in I \text{ and } x \in \mathbb{R}.
\]

If \( m_p = m_q \) there exists \( D \) and \( E > 0 \) such that we have

\[
|q(x, y)| \leq D p(x, y) + E \text{ for all } y \in I \text{ and } x \in \mathbb{R}.
\]

Proof. Let \( M \) be an upper bound for \( |p_k(y)| \) and \( |q_l(y)| \) for all \( y \in I, 0 \leq k \leq m_p \) and \( 0 \leq l \leq m_q \). Then for any \( D > 0 \) we have,

\[
|q(x, y)| - D p(x, y) \leq \rho_D(x)
\]

where

\[
\rho_D(x) := M \sum_{k=0}^{m_q} |x|^k - D \delta |x|^{m_p} + DM \sum_{k=0}^{m_p-1} |x|^k.
\]

If \( m_p > m_q \) we see \( \lim_{x \to \pm \infty} \rho_D(x) = -\infty \) for all \( D = \epsilon > 0 \) and hence \( C_\epsilon := \max_{x \in \mathbb{R}} \rho_c(x) < \infty \) which combined with Eq. (6.3) proves Eq. (6.4). If \( m_p = m_q \) and \( D \) is chosen so that \( D \delta > M \), we again will have \( \lim_{x \to \pm \infty} \rho_D(x) = -\infty \) and so \( E := \max_{x \in \mathbb{R}} \rho_D(x) < \infty \) which combined with Eq. (6.5) proves Eq. (6.4).

Lemma 6.3. Suppose that \( \{ \beta_{l,h}(\cdot) \}_{l=0}^{m} \) and \( c_{b_m} > 0 \) satisfy Assumption \( \mathbb{A} \) and \( c_{b_m} > 0 \) is the constant in Eq. (1.13). Let \( n \in \mathbb{N} \) and \( \{ \mathcal{B}_{l,h} \}_{l=0}^{m_n} \) and \( \{ \mathcal{B}_{l,h} \}_{l=0}^{m_m} \) be the polynomials defined in Eqs. (5.4) and (5.5) respectively. Then \( \{ \mathcal{B}_{l,h} \}_{l=0}^{m_n} \) and \( \{ \mathcal{B}_{l,h} \}_{l=0}^{m_m} \) satisfy items 1 and 3 of Assumption \( \mathbb{B} \) and in particular,

\[
\mathcal{B}_{m,n,h} = \mathcal{B}_{m,h} = b_{m,h}^n (c_{b_m})^n.
\]

Moreover, if \( R_{l,h} \) is the polynomial in Eq. (6.6), then for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that

\[
|R_{l,h}(x)| \leq \epsilon \mathcal{B}_{l,h}(x) + C_\epsilon \forall x \in \mathbb{R}, \quad 0 < h < \eta, \quad \& \quad 0 \leq l \leq m_n.
\]

Proof. From Eq. (6.6)

\[
\mathcal{B}_{l,h} = \sum_{k \in \Lambda_n^m} 1_{|k|=l} b_{k_1,1} b_{k_2,2} \ldots b_{k_n,h}
\]

from which it easily follows that \( \mathcal{B}_{l,h} \) is a real polynomial with real valued coefficients depending continuously on \( h \). Thus we have verified that the \( \{ \mathcal{B}_{l,h} \}_{l=0}^{m_n} \) satisfy item 1 of Assumption \( \mathbb{B} \).

The highest order coefficient of the polynomial \( \mathcal{B}_{l,h} \) is a linear combination of \( n \)-fold products among the highest order coefficients of \( \{ \beta_{l,h}(x) \}_{l=0}^{m} \) and hence is still bounded from below by a positive constant independent of \( h \in (0, \eta) \). This observation along with the estimate, \( \mathcal{B}_{m,n,h} = b_{m,h}^n (c_{b_m})^n, \) shows \( \{ \mathcal{B}_{l,h} \}_{l=0}^{m_n} \) also satisfies item 3 of Assumption \( \mathbb{B} \).

Applying Proposition (5.1) with \( b_l(x) \to h^l b_{l,h} \left( \sqrt{h} x \right) \), it follows (making use of Eqs. (5.7) and (5.8)) that

\[
\deg R_{l,h} \leq \deg \mathcal{B}_{l,h} - 2 \text{ for } 0 \leq l \leq m_n \text{ and } 0 < h < \eta.
\]
By Eq. (6.6) in Lemma 6.3, we have
\[ \epsilon \]
From Eq. (6.7) in Lemma 6.3 by taking \( \delta \) and both \( L \) and \( x \) is clear that \( \{ \delta \} \) for all \( 0 < \delta < \eta \) satisfies condition 3 of Assumption 1.

Finally, let \( 0 < \delta < \eta \) be fixed. We learn \( B_{\ell, h} = B_{\ell, h} + R_{\ell, h} \) from Eqs. (5.4). It is clear that \( \{ B_{\ell, h} \}_{\ell=0} \) satisfies the item 1 of Assumption 1. From Eq. (6.8), the highest order coefficient of \( B_{\ell, h} \) and \( B_{\ell, h} \) are the same and Proposition 3.8 shows that \( R_{mn, h} = 0 \) which implies \( B_{mn, h} = B_{mn, h} \). Therefore \( \{ B_{\ell, h} \}_{\ell=0} \) also satisfies the item 3 of Assumption 1.

### 6.1. Estimating the quadratic form associated to \( L^n_h \).

**Theorem 6.4.** Supposed \( \{ b_{l, h} (x) \} \) and \( \eta > 0 \) satisfies Assumption 1 and let \( L_h \) and \( L^n_h \) be the operators in Eqs. (1.11) and (5.9) respectively. Then for any \( n \in \mathbb{N} \), there exists \( C_n < \infty \) so that for all \( 0 < \delta < \eta \) and \( c > C_n \):

\[
3 \left( L_h^{(n)} + c \right) \geq \delta L_h^n + c \geq \frac{1}{2} \left( L_h^{(n)} + c \right)
\]

and both \( L_h^{(n)} + c \) and \( L_h^n + c \) are positive operators.

**Proof.** Let \( \psi \in \mathcal{S} \) and \( 0 < \delta < \eta \). From Eqs. (5.10) and (5.11) we can conclude

\[
| \langle \left( L^n_h - L_h^{(n)} \right) \psi, \psi \rangle | = \left| \langle R_h^{(n)} \psi, \psi \rangle \right| \leq \sum_{\ell=0}^{nm-1} \delta \left| \left( R_{\ell, h} \left( \sqrt{\varphi} (\cdot) \right) \partial^\ell \psi, \partial^\ell \psi \right) \right|
\]

From Eq. (6.7) in Lemma 6.3 by taking \( \epsilon = \frac{1}{2} \) and \( C_c = C_\frac{1}{2} \) we have

\[
| R_{\ell, h} (x) | \leq \frac{1}{2} B_{\ell, h} (x) + C_\frac{1}{2}
\]

for all \( 0 \leq \ell \leq mn - 1 \) and \( h \in (0, \eta) \). With the use of Eq. (6.9), we learn

\[
| \langle \left( L^n_h - L_h^{(n)} \right) \psi, \psi \rangle | \leq \sum_{\ell=0}^{nm-1} \delta \left( \left( B_{\ell, h} \left( \sqrt{\varphi} (\cdot) \right) + C_\frac{1}{2} \right) \partial^\ell \psi, \partial^\ell \psi \right) = \frac{1}{2} \left( \sum_{\ell=0}^{nm-1} \langle -h \rangle^\ell \partial^\ell B_{\ell, h} \left( \sqrt{\varphi} (\cdot) \right) \partial^\ell \psi, \psi \rangle + C_\frac{1}{2} \left( \sum_{\ell=0}^{nm-1} \langle -h \rangle^\ell \partial^{2\ell} \psi, \psi \rangle \right.
\]

By Eq. (6.9) in Lemma 6.3 we have

\[
B_{mn, h} = b_{mn, h} \geq b_{mn} > 0.
\]

So making use of Lemma 6.1 by taking \( \delta = c_{b_{mn}} / 2 \) and \( c_c = C_\frac{1}{2} \), there exists \( C_\delta < \infty \) such that for all \( 0 < \delta < \eta \) and \( \psi \in \mathcal{S} \),

\[
C_\frac{1}{2} \left( \sum_{\ell=0}^{nm-1} \langle -h \rangle^\ell \partial^{2\ell} \psi, \psi \rangle \right) \leq \langle -h \rangle^m \langle \delta \partial^{mn} \psi, \partial^{mn} \psi \rangle + \frac{1}{2} C_\delta \langle \psi, \psi \rangle \leq \frac{1}{2} \langle -h \rangle^m \langle B_{mn, h} \left( \sqrt{\varphi} (\cdot) \right) \partial^{mn} \psi, \partial^{mn} \psi \rangle + \frac{1}{2} C_\delta \langle \psi, \psi \rangle
\]
By combining Eqs. (6.10) and (6.11), we get
\[
|\langle \psi, (L_h^n - L_h^{(n)}) \psi \rangle| \\
\leq \frac{1}{2} \left( \sum_{\ell=0}^{m-1} (-\hbar)^\ell \partial^\ell B_{\ell,h} (\sqrt{\hbar} \partial) \partial^\ell + (-\hbar)^m \partial^m B_{mn,h} (\sqrt{\hbar} \partial) \partial^{mn} \right) \psi, \psi \\
+ \frac{1}{2} C_\delta \langle \psi, \psi \rangle \\
= \frac{1}{2} \left( (L_h^{(n)} + C_\delta) \psi, \psi \right). \tag{6.11}
\]
It is easy to conclude that
\[
\langle (L_h^{(n)} + C_\delta) \psi, \psi \rangle \geq 0.
\]
As a result, for all \( c > C_\delta, 0 < \hbar < \eta \), by the Eq. (6.11), we get
\[
\left| \langle (L_h^n + c) - (L_h^{(n)} + c) \rangle \psi, \psi \right| = \left| \langle (L_h^n - L_h^{(n)}) \psi, \psi \rangle \right| \leq \frac{1}{2} \langle (L_h^{(n)} + c) \psi, \psi \rangle
\]
and the desired result follows. ■

6.2. Proof of the operator comparison Theorem 1.17

The purpose of this subsection is to prove Theorem 1.17. We begin with a preparatory lemma whose proof requires the following notation.

Notation 6.5. For any divergence form differential operator \( L \) on \( S \) described as in Eq. (6.1), we may decompose \( L \) into its top order and lower order pieces, \( L = L^{\text{top}} + L^\prec \) where
\[
L^{\text{top}} := (-1)^m \partial^m M_{hm} \partial^m \quad \text{and} \quad L^\prec := \sum_{l=0}^{m-1} (-1)^l \partial^l M_{h^l} \partial^l. \tag{6.12}
\]

Lemma 6.6. Let \( \{B_{\ell,h}(x)\}_{\ell=0}^M \) be polynomial functions depending continuously on \( \hbar \) which satisfies the conditions 1 and 3 of Assumption [4] and so in particular,
\[
c_{B_{\ell,h}} := \inf \left\{ B_{M,h}(\sqrt{\hbar}x) : x \in \mathbb{R} \text{ and } 0 < \hbar < \eta \right\} > 0. \tag{6.13}
\]

If
\[
K_h = \sum_{\ell=0}^M (-\hbar)^\ell \partial^\ell \quad \text{and} \quad L_h = \sum_{\ell=0}^M (-\hbar)^\ell \partial^\ell M_{B_{\ell,h}(\sqrt{\hbar} \partial)} \partial^\ell
\]
are operators on \( S \) then for all \( \gamma > \frac{1}{c_{B_{\ell,h}}} \), there exists \( C_\gamma < \infty \) such that
\[
K_h \leq_S \gamma L_h + C_\gamma I. \tag{6.14}
\]

Proof. Using the conditions 1 and 3 of Assumption [1] on \( \{B_{\ell,h}(x)\}_{\ell=0}^M \) where \( b_{\ell,h} \) is replaced by \( B_{\ell,h} \), we may choose \( E > 0 \) such that for all \( 0 \leq \ell < M \),
\[
c_{\ell} := \inf \left\{ B_{\ell,h}(\sqrt{\hbar}x) + E : x \in \mathbb{R} \text{ and } 0 < \hbar < \eta \right\} > 0
\]
and therefore,
\[
(-\hbar)^\ell \partial^\ell M_{B_{\ell,h}(\sqrt{\hbar} \partial)} \partial^\ell + E (-\hbar)^\ell \partial^\ell = (-\hbar)^\ell \partial^\ell M_{B_{\ell,h}(\sqrt{\hbar} \partial)} \partial^\ell \geq 0 \forall \ell
\]
and in particular \( L_h^\prec + E K_h^\prec \leq S 0 \) which in turn implies
\[
L_h^{\text{top}} \leq_S L_h^{\text{top}} + L_h^\prec + E K_h^\prec = L_h + E K_h^\prec.
\]
Using this observation and Eq. (6.13) we find,
\[ K_h^{\text{top}} = (-\hbar)^M \delta^{2M} \leq_S \frac{1}{c_{BM}} \sum_{k \in \Lambda_{n_L}^M} \left( \mathcal{L}_h^\text{top} + E K_h^\sim \right). \quad (6.15) \]

By Lemma 6.1 and Eq. (6.15), for any \( \delta > 0 \), there exists \( C_\delta < \infty \) such that for all \( \hbar > 0 \),
\[ K_h^\sim \leq_S \delta K_h^{\text{top}} + C_\delta I \leq_S \frac{1}{c_{BM}} \left( \mathcal{L}_h + E K_h^\sim \right) + C_\delta I. \quad (6.16) \]

Given \( \varepsilon > 0 \) small we may use the previous equation with \( \delta > 0 \) chosen so that \( \varepsilon \geq \frac{\delta}{c_{BM} - \delta E} \) to learn there exists \( C_\varepsilon' < \infty \) such that
\[ K_h^\sim \leq_S \varepsilon \mathcal{L}_h + C_\varepsilon' I. \quad (6.17) \]

Combining this inequality with Eq. (6.15) then shows,
\[ K_h = K_h^{\text{top}} + K_h^\sim \leq_S \frac{1}{c_{BM}} \left( \mathcal{L}_h + E(\varepsilon \mathcal{L}_h + C_\varepsilon' I) \right) + \varepsilon \mathcal{L}_h + C_\varepsilon' I. \]

Thus choosing \( \varepsilon > 0 \) sufficiently small in this inequality allows us to conclude for every \( \gamma > \frac{1}{c_{BM}} \) there exists \( C_\gamma < \infty \) such that Eq. (6.14) holds.

We are now ready to give the proof of Theorem 1.17.

**Proof of Theorem 1.17.** Recall \( \eta := \min \{ \eta_L, \eta_L \} \) defined in Theorem 1.17. By the assumption in Eq. (6.10) of Theorem 1.17,
\[ |\tilde{b}_{l,h}(x)| \leq c_1 (b_{l,h}(x) + c_2) \quad \forall 0 \leq l \leq m_L \text{ and } 0 < h < \eta. \]

Moreover, using items 1 and 3 of Assumption \( \Pi \) by increasing the size of \( c_2 \) if necessary, we may further assume that \( b_{l,h}(x) + c_2 \geq 0 \) for all \( x \in \mathbb{R} \), \( 0 \leq l \leq m_L \), and \( 0 < h < \eta \). Without loss of generality, we may define \( \tilde{b}_{l,h}() = 0 \) for all \( l > m_L \) and hence \( \tilde{B}_{l,h}() = 0 \) for all \( l > m_L \). It then follows that there exists \( E_1, E_2 < \infty \) such that for \( 0 \leq l \leq m_L \),
\[
\tilde{B}_{l,h} \leq |\tilde{B}_{l,h}| \leq \sum_{k \in \Lambda_{n_L}^M} 1_{|k|=l} |\tilde{b}_{k_1,h} \ldots \tilde{b}_{k_n,h}|
\leq \sum_{k \in \Lambda_{n_L}^M} 1_{|k|=l} c_1^l (b_{k_1,h} + c_2) \ldots (b_{k_n,h} + c_2)
\leq \sum_{k \in \Lambda_{n_L}^M} 1_{|k|=l} c_1^l (b_{k_1,h} + c_2) \ldots (b_{k_n,h} + c_2)
= E_1 B_{l,h} + E_2,
\]

wherein we have used Eq. (6.3) in Lemma 6.2 for the last inequality by taking
\[
p(x,h) = B_{l,h} = \sum_{k \in \Lambda_{n_L}^M} 1_{|k|=l} (b_{k_1,h} \ldots b_{k_n,h}) \text{ and } q(x,h) = \sum_{k \in \Lambda_{n_L}^M} 1_{|k|=l} c_1^l (b_{k_1,h} + c_2) \ldots (b_{k_n,h} + c_2),
\]

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where (by Lemma 6.3) $B_{\ell, h}$ is an even degree polynomial with a positive leading order coefficient. Hence if we let $\mathcal{L}^{(n)}_h$ and $\mathcal{L}^{(n)}_\ell$ be as in Eq. (6.19), i.e.

$$\mathcal{L}^{(n)}_h = \sum_{\ell=0}^{m_L} (-h)^\ell \partial^\ell B_{\ell, h} \left( \sqrt{\tilde{h}}(\cdot) \right) \partial^\ell \text{ and } \mathcal{L}^{(n)}_\ell = \sum_{\ell=0}^{m_L} (-h)^\ell \partial^\ell B_{\ell, h} \left( \sqrt{\tilde{h}}(\cdot) \right) \partial^\ell,$$

then it follows directly from Eq. (6.18) that

$$\mathcal{L}^{(n)}_h \preceq S E_1 \mathcal{L}^{(n)}_h + E_2 K_h \text{ where } K_h := \sum_{\ell=0}^{nm_L} (-h)^\ell \partial^\ell.$$

Because of Lemma 6.3, we may apply Lemma 6.6 with $M = nm_L$ and $\mathcal{L}_h = \mathcal{L}^{(n)}_h$ to conclude there exists $\gamma > 0$ and $C < \infty$ such that $K_h \preceq S \gamma \mathcal{L}^{(n)}_h + CI$ and thus,

$$\mathcal{L}^{(n)}_h \preceq S (E_1 + \gamma E_2) \mathcal{L}^{(n)}_h + E_2 CI.$$

By Theorem 6.4, there exists $C_L$ and $C_\ell$ such that

$$\frac{1}{2} \mathcal{L}^{(n)}_h \preceq S L^n_h + C_L \text{ and } \|\mathcal{L}^{(n)}_h\|_C \preceq S \frac{3}{2} \left( (E_1 + \gamma E_2) \mathcal{L}^{(n)}_h + E_2 CI + C_L \right).$$

From these last two inequalities, it follows that $\mathcal{L}^{(n)}_h \preceq S (L^n_h + C_2)$ for appropriately chosen constants $C_1$ and $C_2$.

6.3. Proof of Corollary 1.19 For the reader’s convenience let us restated Corollary 1.19 here.

**Corollary 1.19.** Supposed $\{b_{l, h}(x)\}_{l=0}^m \subset \mathbb{R}[x]$ and $\eta > 0$ satisfies Assumption 1. $L_h$ is the operator in the Eq. (1.17), and suppose that $C \geq 0$ has been chosen so that $0 \preceq S L_h + CI$ for all $0 < h < \eta$. (The existence of $C$ is guaranteed by Corollary 1.18.) Then for any $0 < h < \eta$, $L_h + CI$ is a non-negative self-adjoint operator on $L^2(m)$ and $S$ is a core for $(L_h + C)^r$ for all $r \geq 0$.

Before proving this corollary we need to develop a few more tools. From Lemma 6.3 $\{B_{\ell, h}\}_{\ell=0}^{mn} \subset \mathbb{R}[x]$ in Eq. (3.4) satisfies both items 1 and 3 of Assumption 1. Therefore, $B_{\ell, h}$ is bounded below for $0 \leq \ell \leq mn - 1$ and $B_{mn,h} > 0$. We may choose $C > 0$ sufficiently large so that

$$B_{\ell, h} + C > 0 \text{ for } 0 \leq \ell \leq mn - 1 \text{ and } 0 < h < \eta. \quad (6.19)$$

**Notation 6.7.** Let $C > 0$ be chosen so that Eq. (6.19) holds and then define the operator, $\hat{L}_h$, by

$$\hat{L}_h := \sum_{\ell=0}^{mn} (-h)^\ell \partial^\ell \left( B_{\ell, h} \left( \sqrt{\tilde{h}}(\cdot) \right) + C_{1_{l<mn}} \right) \partial^\ell = (-h)^{mn} \partial^mn B_{mn,h} \left( \sqrt{\tilde{h}}(\cdot) \right) \partial^mn + \sum_{\ell=0}^{mn-1} (-h)^\ell \partial^\ell \left( B_{\ell, h} \left( \sqrt{\tilde{h}}(\cdot) \right) + C \right) \partial^\ell$$

with domain, $\mathcal{D}(\hat{L}_h) = S$. 

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Lemma 6.8. There exists \( \tilde{C}_1 \) and \( \tilde{C}_2 > 0 \) such that
\[
\| h \partial^{2M} \psi \| \leq \tilde{C}_1 \| L_h \psi \| + \tilde{C}_2 \| \psi \|
\]
holds for all \( 0 \leq M \leq mn, \ 0 < h < \eta, \ \text{and } \psi \in \mathcal{S} \).

Proof. As in Eq. (4.22), let \( \hat{\psi} \) denote the Fourier transform of \( \psi \in \mathcal{S} \) and recall that \( \| \psi \| = \sqrt{2\pi} \| \hat{\psi} \| \). Hence it follows,
\[
\| h^M \partial^{2M} \psi \| = \sqrt{2\pi} \left\| h^M \xi^{2M} \hat{\psi} (\xi) \right\| \\
\leq \sqrt{2\pi} \left\| \left( \sum_{\ell=0}^{M} h^\ell \xi^{2\ell} \right) \hat{\psi} (\xi) \right\| = \left\| \left( \sum_{\ell=0}^{M} \left( -h \right)^\ell \partial^{2\ell} \right) \psi \right\|. \quad (6.20)
\]
With the same \( C \) in Notation (6.7) and using Eq. (6.19), we can see that
\[
1 \leq (B_{\ell,h} + C_{1,\ell < mn}) + 1 \ \forall \ 0 \leq \ell \leq mn \ \& \ 0 < h < \eta.
\]
Therefore applying the operator comparison Theorem (1.11) with \( \tilde{L}_h = \sum_{\ell=0}^{M} \left( -h \right)^\ell \partial^{2\ell} \), \( L_h = \tilde{L}_h \), and \( n = 2 \), there exists \( C_1 \) and \( C_2 > 0 \) such that for
\[
\left\langle \left( \sum_{\ell=0}^{M} \left( -h \right)^\ell \partial^{2\ell} \right) \psi, \psi \right\rangle \leq C_1 \left\langle \tilde{L}_h^2 \psi, \psi \right\rangle + C_2 \left\langle \psi, \psi \right\rangle \ \forall \ \psi \in \mathcal{S} \ \& \ 0 < h < \eta.
\]
Combining this inequality with Eq. (6.20), shows there exists other constants \( \tilde{C}_1 \) and \( \tilde{C}_2 > 0 \) such that
\[
\| h^M \partial^{2M} \psi \| \leq \left\| \left( \sum_{\ell=0}^{M} \left( -h \right)^\ell \partial^{2\ell} \right) \psi \right\| \leq \tilde{C}_1 \| L_h \psi \| + \tilde{C}_2 \| \psi \|.
\]

Lemma 6.9. Let \( A \) and \( B \) be closed operators on a Hilbert space \( \mathcal{K} \) and suppose there exists a subspace, \( \mathcal{S} \subseteq \mathcal{D}(A) \cap \mathcal{D}(B) \), such that \( \mathcal{S} \) is dense and \( \mathcal{S} \) is a core of \( B \). If there exists a constant \( C > 0 \) such that
\[
\| A \psi \| \leq \| B \psi \| + C \| \psi \| \ \forall \ \psi \in \mathcal{S}, \quad (6.21)
\]
then \( \mathcal{D}(B) \subseteq \mathcal{D}(A) \) and
\[
\| A \psi \| \leq \| B \psi \| + C \| \psi \| \ \forall \ \psi \in \mathcal{D}(B). \quad (6.22)
\]

Proof. If \( \psi \in \mathcal{D}(B) \), there exists \( \psi_k \in \mathcal{S} \) such that \( \psi_k \to \psi \) and \( B \psi_k \to B \psi \) as \( k \to \infty \). Because of Eq. (6.21) \( \{A \psi_k\}_{k=1}^{\infty} \) is Cauchy in \( \mathcal{K} \) and hence convergent. As \( A \) is closed we may conclude that \( \psi \in \mathcal{D}(A) \) and that \( \lim_{k \to \infty} A \psi_k = A \psi \). Therefore Eq. (6.22) holds by replacing \( \psi \) in Eq. (6.21) by \( \psi_k \) and then passing to the limit as \( k \to \infty \). \[ \square \]

Proposition 6.10. Suppose \( \{b_{h,k}(x)\}_{k=0}^{m} \subseteq \mathbb{R}[x] \) and \( \eta > 0 \) satisfies Assumption (7) and \( L_h \) is defined by Eq. (7.11) with \( \mathcal{D}(L_h) = \mathcal{S} \) for \( 0 < h < \eta \). Then \( L_h^n \) is self-adjoint and \( \mathcal{S} \) is a core for \( L_h^n \) for all \( n \in \mathbb{N} \) and \( 0 < h < \eta \). [Note \( L_h^n \) is a well defined self-adjoint operator by the spectral theorem.]
Proof. Recall that $L^n_h$ may be written in divergence form as in Eq. (5.3) where $B_{t,h} = B_{t,h} + R_{t,h}$ and $B_{t,h} \in \mathbb{R}[x]$ and $B_{t,h} \in \mathbb{R}[x]$ are as in Eqs. (5.5) and (5.6) respectively. By Assumption 1, $\deg (b_{l-1}) \leq \deg (b_l)$, which used in combination with the item 1 in Proposition 5.1 and the definition of $B_l$ in the Eq. (5.5) implies,

$$
\deg (B_{t,h}) = \deg (B_{t,h}) \leq \max \{ \deg (b_{0,h}), 0 \} 
\leq \max \{ \deg (B_{0,h}), 0 \} = \max \{ \deg (B_{0,h}), 0 \}.
$$

Each term in $\hat{L}_h$ defined in Notation 6.4 is a positive operator and by Theorem 1.9, $L_h$ is self-adjoint. [Recall that $D(L_h) := \mathcal{S}$.] Moreover by Lemma 6.1 for all $\delta > 0$ there exists $C_\delta < \infty$ such that

$$
\left( \hat{L}_h - L^n_h \right)^2 = \left( \sum_{\ell=0}^{n-1} (-\hbar)^\ell C\partial^{2\ell} \right)^2 \leq S \delta (-\hbar)^n \partial^{4mn} + C_\delta I
$$

(6.23)

which implies,

$$
\left\| \left( \hat{L}_h - L^n_h \right) \psi \right\| \leq \delta \left\| (b)^{mn} \partial^{2mn} \psi \right\| + C_\delta \left\| \psi \right\| \forall \psi \in \mathcal{S}.
$$

This inequality along with Lemma 6.3 then gives

$$
\left\| \left( \hat{L}_h - L^n_h \right) \psi \right\| \leq \delta \left( C_1 \left\| \hat{L}_h \psi \right\| + C_2 \left\| \psi \right\| \right) + C_\delta \left\| \psi \right\|
\leq \delta C_1 \left\| \hat{L}_h \psi \right\| + (\delta C_2 + C_\delta) \left\| \psi \right\| \forall \psi \in \mathcal{S}.
$$

Therefore for any $a > 0$ we may take $\delta > 0$ so that $a := \delta C_1$ and then let $C_a := (\delta C_2 + C_\delta) < \infty$ in the previous estimate in order to show,

$$
\left\| \left( \hat{L}_h - L^n_h \right) \psi \right\| \leq a \left\| \hat{L}_h \psi \right\| + C_a \left\| \psi \right\| \forall \psi \in \mathcal{S}.
$$

(6.24)

As a consequence of this inequality with $a < 1$ and a variant of the Kato-Rellich theorem (see [13, Theorem X.13, p. 163]), we may conclude $L^n_h$ is self-adjoint. As this holds for $n = 1$, we conclude that $\hat{L}_h$ is self-adjoint. By the spectral theorem, $\hat{L}_h$ is also self-adjoint. Since $L^n_h \subset \hat{L}_h$, we know that $\mathcal{D}(\hat{L}_h) \subset \hat{L}_h$ and therefore $\mathcal{D}(L^n_h) = \mathcal{D}(\hat{L}_h) = \mathcal{D}(\hat{L}_h)$ which shows $\mathcal{S}$ is a core for $L^n_h$.

Lemma 6.11. If $A$ is any essentially self-adjoint operator on a Hilbert space $K$ and $q : \mathbb{R} \to \mathbb{C}$ is a measurable function such that, for some constants $C_1$ and $C_2$,

$$
|q(x)| \leq C_1 |x| + C_2 \forall x \in \mathbb{R},
$$

then $D(A)$ is a core for $q(A)$.

Proof. To prove this we may assume by the spectral theorem that $K = L^2(\Omega, \mathcal{B}, \mu)$ and $\hat{A} = M_f$ where $\Omega, \mathcal{B}, \mu$ is a $\sigma$ – finite measure space and $f : \Omega \to \mathbb{R}$ is a measurable function. Of course in this model, $q(\hat{A}) = M_{qf}$. In this case, $D := D(A) \subset D(M_f)$ is a dense subspace of $L^2(\mu)$ such that for all $g \in D(M_f)$ there exists $g_n \in D$ such that $g_n \to g$ and $f g_n \to f g$ in $L^2(\mu)$ as $n \to \infty$. For this same sequence we have

$$
\| q(\hat{A}) g_n - q(\hat{A}) g \|_2 = \| q(f) [g_n - g] \|_2 \leq C_1 \| f [g_n - g] \|_2 + C_2 \| g_n - g \| \to 0
$$

(34)
as \(n \to \infty\). This shows that
\[
q(A)|_{\mathcal{D}(M_f)} \subset q(A)|_{\mathcal{D}} \subset q(A).
\] (6.25)

For \(q \in \mathcal{D}(q(A))\) (i.e. both \(g\) and \(g \cdot q \circ f\) are in \(L^2(\mu)\)), let \(g_n := g1_{|f| \leq n} \in \mathcal{D}(M_f)\). Then \(g_n \to g\) in \(L^2(\mu)\) as \(n \to \infty\) by DCT. Moreover
\[
|g_n q \circ f - q \circ f| = (g1_{|f| \leq n} - g) q \circ f \leq 2|g| |q \circ f| \in L^2(\mu),
\]
and so \(\|g_n q \circ f - q \circ f\|_2 \to 0\) as \(n \to \infty\) by DCT as well. This shows that
\[
q(A) = q(A)|_{\mathcal{D}(M_f)} \quad \text{and hence it now follows from Eq. (6.25) that}
\]
\[
q(A) = q(A)|_{\mathcal{D}(M_f)} \subset q(A)|_{\mathcal{D}} \subset q(A).
\]

Lemma 6.12. Let \(B\) be a non-negative self-adjoint operator on a Hilbert space, \(\mathcal{K}\). If \(S\) is a core for \(B^r\) for some \(n \in \mathbb{N}_0\), then \(S\) is a core for \(B^r\) for any \(0 \leq r \leq n\). [By the spectral theorem, \(B^r\) is again a non-negative self-adjoint operator on \(\mathcal{K}\) for any \(0 \leq r < \infty\).]

Proof. Let \(A = B^n|_S\) so that by assumption \(\bar{A} = B^n\), i.e. \(A\) is essentially self-adjoint. The proof is then finished by applying Lemma 6.11 with \(q(x) = |x|^{r/n}\) upon noticing, \(q(A) = q(B^n) = |B^n|^{r/n} = B^r\).

Proof of Corollary 1.19. Let \(C \geq 0\) be the constant in the statement of Corollary 1.19. It is simple to verify that \(\{b_{l,0} + C1_{l=0}\}_{l=0}^m\) and \(\eta > 0\) satisfies Assumption 1 and therefore applying Proposition 6.10 with \(\{b_{l,0}\}_{l=0}^m\) replaced by \(\{b_{l,0} + C1_{l=0}\}_{l=0}^m\) shows, \(L_0 + C\) is self-adjoint and \(S\) is core for \((L_0 + C)^r\) for all \(n \in \mathbb{N}\) and \(0 < h < \eta\). It then follows from Lemma 6.12 that \(S = S\) is a core for \((L_0 + C)^r\) for all \(0 \leq r \leq n\) and \(0 < h < \eta\). As \(n \in \mathbb{N}\) was arbitrary, the proof is complete.

6.4. Proof of Corollary 1.20. In order to prove Corollary 1.20, we will need a lemma below.

Lemma 6.13. Let \(A\) and \(B\) be non-negative self-adjoint operators on a Hilbert Space \(\mathcal{K}\). Suppose \(S\) is a dense subspace of \(\mathcal{K}\) so that \(S \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)\), \(AS \subseteq S\) and \(BS \subseteq S\). If we further assume that for each \(n \in \mathbb{N}_0\), \(S\) is a core of \(B^n\). Then the following are equivalent:

(1) For any \(n \in \mathbb{N}_0\) there exists \(C_n > 0\) such that \(A^n \preceq S C_n B^n\).

(2) For each \(r \geq 0\), there exists \(C_r\) such that \(A^r \preceq S C_r B^r\).

(3) For each \(v \geq 0\), there exists \(C_v\) such that \(A^v \preceq S C_v B^v\).

Recall the different operator inequality notations, \(\preceq, \succeq\) and \(\preceq\), were defined in Notation 1.13.

Proof. (1 \(\Rightarrow\) 2) \(A^n \preceq S C_n B^n\) implies for all \(\psi \in S\) we have
\[
\left\|\sqrt{A^n} \psi\right\|^2 = (A^n \psi, \psi) \leq C_n (B^n \psi, \psi) = \left\|\sqrt{C_n B^n} \psi\right\|^2.
\]

Note \(S\) is a core of \(C_n B^n\) and hence \(S\) is also a core of \(\sqrt{C_n B^n}\) by taking \(q(x) = |x|\) in Lemma 6.11. By using Lemma 6.9 with \(C = 0\) we have \(\mathcal{D}(\sqrt{B^n}) = \mathcal{D}(\sqrt{C_n B^n}) \subseteq \mathcal{D}(\sqrt{A^n})\) and
\[
\left\|\sqrt{A^n} \psi\right\| \leq \left\|\sqrt{C_n B^n} \psi\right\| \quad \text{for all} \ \psi \in \mathcal{D}(\sqrt{C_n B^n}),
\]
i.e. \( A^n \leq C_n B^n \). It then follows by the Löwner-Heinz inequality (Theorem 1.10) that \( A^{nr} \leq C_n B^{nr} \) for all \( 0 \leq r \leq 1 \). Since \( n \in \mathbb{N} \) was arbitrary, we have verified the truth of item 2.

(2 \( \Rightarrow \) 3) Given item 2., it is easy to verify that \( \mathcal{D}(B^n) = \mathcal{D}(C_vB^n) \subseteq \mathcal{D}(A^n) \) for all \( v \geq 0 \). In particularly, we have \( \mathcal{D}(B^n) \subseteq \mathcal{D}(A^n) \cap \mathcal{D} \left( \sqrt{B^n} \right) \) for any \( v \geq 0 \). Hence, by taking \( r = v \) in item 2,

\[
(A^v \psi, \psi) = \left\| \sqrt{A^v} \psi \right\|^2 \leq \left\| \sqrt{C_v B^n} \psi \right\|^2 = (C_v B^n \psi, \psi) \; \forall \psi \in \mathcal{D}(B^n),
\]

i.e. \( A^v \leq C_v B^n \).

(3 \( \Rightarrow \) 1) The assumption that \( S \subseteq \mathcal{D}(A) \cap \mathcal{D}(B) \), \( AS \subseteq S \) and \( BS \subseteq S \) follows that \( S \subseteq \mathcal{D}(B^n) \cap \mathcal{D}(A^n) \) for all \( n \in \mathbb{N}_0 \). By taking \( v = n \), we learn that \( A^n \leq C_n B^n \) which certainly implies \( A^n \leq S C_n B^n \).

**Proof of the Corollary 1.20.** We first observe that the coefficients, \( \{ b_{l,h}(\cdot) + C_{1l=0} \}_{l=0}^{m} \) and \( \{ \tilde{b}_{l,h}(\cdot) + \tilde{C}_{1l=0} \}_{l=0}^{m} \) still satisfy Assumption 1. Using this observation along with the inequalities, \( L_h + C \geq S I \) and \( \tilde{L}_h + \tilde{C} \geq S 0 \), we may use Corollary 1.19 to conclude both \( \tilde{L}_h + C \) and \( \tilde{L}_h + \tilde{C} \) are non-negative self adjoint operators and \( S \) is a core for \( (\tilde{L}_h + C)^r \) for all \( r \geq 0 \) and all \( 0 < h < \eta \). By the operator comparison Theorem 1.17 with \( b_{l,h} \) replaced by \( b_{l,h} + C_{1l=0} \) and \( \tilde{b}_{l,h} \) replaced by \( \tilde{b}_{l,h} + \tilde{C}_{1l=0} \), for any \( n \in \mathbb{N}_0 \), there exists \( C_1 \) and \( C_2 > 0 \) such that

\[
\left( \tilde{L}_h + \tilde{C} \right)^n \preceq S C_1 \left( (L_h + C)^n + C_2 \right).  \tag{6.26}
\]

Because \( (L_h + C)^n \succeq S I \), we may conclude from Eq. 6.26 that

\[
\left( \tilde{L}_h + \tilde{C} \right)^n \preceq S C_n \left( L_h + C \right)^n \quad \forall \ n \in \mathbb{N}_0,
\]

where \( C_n = C_1 (C_2 + 1) \). By taking \( A = \tilde{L}_h + \tilde{C} \) and \( B = (L_h + C) \) and \( S = S \) in Lemma 6.13, we may conclude that for any \( v \geq 0 \), there exists \( C_v > 0 \) such that Eq. 1.18 holds, i.e. \( \left( \tilde{L}_h + \tilde{C} \right)^v \preceq C_v \left( L_h + C \right)^v \quad \forall \ 0 < h < \eta \).

7. Discussion of the 2nd condition in Assumption 1.

We try to relax conditions 2 in Assumption 1. The degree restriction Eq. 1.12 allows the choice of \( \eta \) independent of a power \( n \) in both Theorem 6.4 and Theorem 1.17. If a weaker condition of the degree restriction is assumed, which is

\[
\deg(b_{l,h}) \leq \deg(b_{l-1,h}) + 2 \text{ for all } 0 < h < \eta \text{ and } 0 \leq l \leq m,
\]

then Theorems 7.2 and 7.3 are resulted where now \( \eta \) does depend on \( n \).

**Lemma 7.1.** Suppose there exists \( \eta > 0 \) such that \( \deg(b_{l,h}) \leq \deg(b_{l-1,h}) + 2 \) for all \( 0 < h < \eta \) and \( 0 \leq l \leq m \). Let \( B_{\ell,h}(x) \) and \( R_{\ell,h}(x) \) be in Eqs. 5.3 and 5.6 respectively. Then we have

\[
\deg_x(R_{\ell,h}) \leq \deg_x(B_{\ell,h}) \text{ for } h \in (0, \eta) \text{ and } 0 \leq \ell \leq mn. \tag{7.1}
\]

**Proof.** Eq. 7.1 follows immediately if we apply the item 2 in Proposition 5.1 with \( b_l(x) \to h^l b_{l,h} \left( \sqrt{h} x \right) \) where \( h \) is fixed.
Theorem 7.2. Let $L_h$ be an operator in the Eq. (7.17). Suppose $b_{l,h}(x)$ satisfies the conditions 1 and 3 in Assumption I and we assume
\[ \deg(b_{l,h}) \leq \deg(b_{l-1,h}) + 2 \text{ for all } 0 < h < \eta \text{ and } 0 \leq l \leq m, \]  
where $\eta$ is the $\eta$ in Assumption I Then for any $n \in \mathbb{N}_0$, there exists $C_n$ and $\eta_n$ such that for all $0 < h < \eta_n$ and $c > C_n$
\[ \frac{3}{2} \left( L_h^{(n)} + c \right) \geq_{S} L_h^n + c \geq_{S} \frac{1}{2} \left( L_h^{(n)} + c \right). \]

Proof. Let $\psi \in \mathcal{S}$ and $0 < h < \eta$, we have
\[ |\langle L_h^n - L_h^{(n)} \rangle \psi, \psi \rangle| = \left| \langle R_h^{(n)} \psi, \psi \rangle \right| \leq \sum_{\ell=0}^{nm-1} |\langle R_{\ell,h} \partial^{\ell} \psi, \partial^{\ell} \psi \rangle|. \]

where $R_h^{(n)}$ and $R_{\ell,h}$ are still defined in the same way as Eqs. (5.10) and (5.6) respectively. From Lemma (5.11) $\deg_x(B_{\ell,h}) \geq \deg_x(R_{\ell,h})$ where $B_{\ell,h}$ is defined in Eq. (5.5). Note $|p| > 0$ in $R_{\ell,h}$ from Eq. (5.6). Although $\deg(R_{\ell,h})$ can be the same as $\deg(B_{\ell,h})$, the extra $h^{|p|}$ factor in the $R_{\ell,h}$ makes $|R_{\ell,h}|$ decrease more rapidly than $B_{\ell,h}$ as $h$ decrease to 0. As a result, there exist constants $\eta_n > 0$ and $C$ such that
\[ |\langle R_{\ell,h} (x) \rangle| \leq \frac{1}{2} B_{\ell,h} (x) + C \]
for all $0 \leq \ell \leq mn - 1$ and $0 < h < \eta_n$. Therefore
\[ \left| \langle L_h^n - L_h^{(n)} \rangle \psi, \psi \rangle \right| \leq \sum_{\ell=0}^{nm-1} h^\ell \left( \left\langle \frac{1}{2} B_{\ell,h} \left( \sqrt{h} (\cdot) \right) + C \right\rangle \partial^{\ell} \psi, \partial^{\ell} \psi \right) \]
\[ = \frac{1}{2} \sum_{\ell=0}^{nm-1} \left( -h \right)^{\ell} \partial^{\ell} B_{\ell,h} \partial^{\ell} \psi, \psi \right) + C \sum_{\ell=0}^{nm-1} \left( -h \right)^{\ell} \partial^{2\ell} \psi, \psi \right). \]

Then by following the argument in Theorem 6.4 we can conclude that there exists $C_n > 0$ such that for all $0 < h < \eta_n$ and $c > C_n$ we have
\[ \frac{1}{2} \sum_{\ell=0}^{nm-1} \left( \partial^{\ell} B_{\ell,h} \partial^{\ell} \psi, \psi \right) + C \sum_{\ell=0}^{nm-1} \left( -h \right)^{\ell} \partial^{2\ell} \psi, \psi \right) \leq \frac{1}{2} \left( L_h^{(n)} + c \right) \psi, \psi \rangle. \]

The result follows immediately by combing the above inequality and Eq. (7.3). □

As a result, the operator comparison theorem now have choice of $\eta$ depending on a power $n$.

Theorem 7.3. Let
\[ \tilde{L}_h = \sum_{\ell=0}^{m_L} (-h)^k \partial^{k} \tilde{b}_{\ell,h} (\sqrt{h} x) \partial^{k} \text{ and } L_h = \sum_{\ell=0}^{m_L} (-h)^k \partial^{k} b_{\ell,h} (\sqrt{h} x) \partial^{k} \]
be operators on $\mathcal{S}$ satisfies conditions in Theorem 7.2. Denote $\eta_L$ and $\eta_L$ as the $\eta$ of $\tilde{L}_h$ and $L_h$ in Assumption I respectively. If $m_L \leq m_L$ and there exists $c_1$ and $c_2$ such that
\[ |\tilde{b}_{\ell,h} (x)| \leq c_1 (b_{\ell,h} (x) + c_2) \text{ for all } 0 \leq \ell \leq m_L \text{ and } 0 < h < \min \{ \eta_L, \eta_L \}; \]
then for any $n$, there exists $C_1$, $C_2$ and $\eta_n$ such that
\[ \left( \tilde{L}_h \right)^n \leq_{S} \left( L_h^n + C_2 \right) \]
for all $0 < \h < \eta_n$. 

**Proof.** The exact same proof as Theorem 1.17 with the use of Theorem 7.2 instead of Theorem 6.4. □

**Appendix A. Operators Associated to Quantization**

Let $\mathcal{A}$ denote the algebra of linear differential operator on $\mathcal{S}$ which have polynomial coefficients. Remark 1.6 shows that the $\dagger$ operation on $\mathcal{A}$ defined in Eq. (1.4) is an involution of $\mathcal{A}$. For $\h > 0$ (following on p.204 in [15] or Hepp [7]), let $a_h \in \mathcal{A}$ and its formal adjoint, $a_h^\dagger$, be the annihilation and creation operators respectively given by

$$a_h = \sqrt{\h^2 (M_x + \partial_x)} \quad \text{and} \quad a_h^\dagger = \sqrt{\h^2 (M_x - \partial_x)} \quad \text{on } \mathcal{S}.$$  \hspace{1cm} (A.1)

These operators satisfy the commutation relation $[a_h, a_h^\dagger] = \h I$ on $\mathcal{S}$.

Let $\mathbb{R} \langle \theta, \theta^* \rangle$ be the space of non-commutative polynomials over $\mathbb{R}$ in two indeterminants $\{\theta, \theta^*\}$. Thus, given $H(\theta, \theta^*) \in \mathbb{R} \langle \theta, \theta^* \rangle$, there exists $d \in \mathbb{N}$ (the degree of $H(\theta, \theta^*)$ in $\theta$ and $\theta^*$) and coefficients,

$$\bigcup_{k=0}^d \left\{ C_k(\mathbf{b}) \in \mathbb{R} : \mathbf{b} \in \{\theta, \theta^*\}^k \right\}$$

such that $H(\theta, \theta^*) = \sum_{k=0}^d H_k(\theta, \theta^*)$ where

$$H_k(\theta, \theta^*) := \sum_{\mathbf{b}=(b_1,...,b_k)\in\{\theta,\theta^*\}^k} C_k(\mathbf{b}) b_1 \ldots b_k \in \mathbb{R} \langle \theta, \theta^* \rangle.$$  \hspace{1cm} (A.2)

We let $H(\theta, \theta^*)^* \in \mathbb{R} \langle \theta, \theta^* \rangle$ be defined by $H(\theta, \theta^*)^* = \sum_{k=0}^d H_k(\theta, \theta^*)^*$ where

$$H_k(\theta, \theta^*)^* := \sum_{\mathbf{b}=(b_1,...,b_k)\in\{\theta,\theta^*\}^k} C_k(\mathbf{b}) b_1^* \ldots b_k^*$$  \hspace{1cm} (A.3)

and for $\mathbf{b} \in \{\theta, \theta^*\}$,

$$b^* := \begin{cases} \theta^* & \text{if } \mathbf{b} = \theta \\ \theta & \text{if } \mathbf{b} = \theta^*. \end{cases}$$

The operation, $H(\theta, \theta^*) \mapsto H(\theta, \theta^*)^*$ defines an involution on $\mathbb{R} \langle \theta, \theta^* \rangle$ and we say that $H(\theta, \theta^*) \in \mathbb{R} \langle \theta, \theta^* \rangle$ is **symmetric** if $H(\theta, \theta^*) = H(\theta, \theta^*)^*$. If $H(\theta, \theta^*) \in \mathbb{R} \langle \theta, \theta^* \rangle$ is symmetric, then $H \left(a_h, a_h^\dagger \right)$ is a symmetric linear differential operator with polynomial coefficients as in Definition 1.5.

In the following lemmas and theorem let $\mathbb{R} [x]$ and $\mathbb{R} \left[\sqrt{\h}, x\right]$ be as in Notation 1.6.

**Lemma A.1.** If $\h > 0$ and $H \in \mathbb{R} \langle \theta, \theta^* \rangle$ is a noncommutative polynomial with degree $d$, then $H \left(a_h, a_h^\dagger\right)$ can be written as a linear differential operator

$$H \left(a_h, a_h^\dagger\right) = \sum_{l=0}^d \h^l G_l \left(\sqrt{\h}, \sqrt{\h} x\right) \partial_x^l$$

where $G_l \left(\sqrt{\h}, x\right) \in \mathbb{R} \left[\sqrt{\h}, x\right]$ is a polynomial of $\sqrt{\h}$ and $x$ for $0 \leq l \leq d$.  \hspace{1cm} (A.4)
From the previous equation it is easy to see where $g$ such that $x$ because each monomial of $G$ there exists $k$ Summing Eq. (A.4) on polynomial with degree $d$ Using the definition of $a$ and (as in \[3\]) for $b \in \{\theta, \theta^*\}$, 
\[
\hat{b} := \begin{cases} 
    a & \text{if } b = \theta \\
    a^\dagger & \text{if } b = \theta^*.
\end{cases}
\]
Using the definition of $a_h$ and $a_h^\dagger$ in Eq. (A.1), there exists 
\[
\left\{ \hat{C}_k (\varepsilon) \in \mathbb{R} : \varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{\pm 1\}^k \right\}
\]
such that 
\[
H_k \left( a_h, a_h^\dagger \right) = (\hbar)^{k/2} \sum_{\varepsilon \in \{\pm 1\}^k} C(\varepsilon) (x + \varepsilon_1 \partial_x) \ldots (x + \varepsilon_k \partial_x).
\]
From the previous equation it is easy to see 
\[
H_k \left( a_h, a_h^\dagger \right) = (\hbar)^{k/2} \sum_{l=0}^{k} g_{l,k}(x) \partial_x^l
\]
where $g_{l,k} \in \mathbb{R}[x]$ with 
\[
\deg_x (g_{l,k}) \leq k - l.
\]
Summing Eq. (A.4) on $k$ and then switching two sums shows 
\[
H \left( a_h, a_h^\dagger \right) = \sum_{k=0}^{d} \sum_{l=0}^{k} (\hbar)^{k/2} g_{l,k}(x) \partial_x^l = \sum_{l=0}^{d} h^{l/2} \left( \sum_{k=l}^{d} h^{-i} g_{l,k}(x) \right) \partial_x^l.
\]
There exists $G_l \left( \sqrt{\hbar}, x \right) \in \mathbb{R} \left[ \sqrt{\hbar}, x \right]$ such that 
\[
G_l \left( \sqrt{\hbar}, \sqrt{\hbar} x \right) = \sum_{k=l}^{d} h^{l/2} g_{l,k}(x)
\]
because each monomial of $x$ in $g_{l,k}(x)$ can be multiplied with enough $\sqrt{\hbar}$ from $\hbar^{k+l}$ by using Eq. (A.5). \[\blacksquare\]

**Theorem A.2.** If $\hbar > 0$ and $H \in \mathbb{R}[\theta, \theta^*]$ is a symmetric noncommutative polynomial with degree $d$, then there exits $m \in \mathbb{N}_0$ and $\{f_i\}_{i=0}^{m} \subset \mathbb{R} \left[ \sqrt{\hbar}, x \right]$ such that 
\[
H \left( a_h, a_h^\dagger \right) = \sum_{i=0}^{m} (-\hbar)^l f_i \left( \sqrt{\hbar}, \sqrt{\hbar} x \right) \partial_x^l \text{ on } S.
\]

**Proof.** Since $H$ is symmetric, $H \left( a_h, a_h^\dagger \right)$ is symmetric, see Definition 1.5. So by Proposition 2.2 $d = 2m$ for some $m \in \mathbb{N}_0$ and $H \left( a_h, a_h^\dagger \right)$ in Eq. (A.6) may be written in a divergence form 
\[
H \left( a_h, a_h^\dagger \right) = \sum_{i=0}^{m} (-1)^l \partial_x^l M_h \partial_x^l.
\]
By substituting \( a_l(x) = \hbar \frac{g_l}{\sqrt{\hbar x}} \) for \( 0 \leq l \leq d = 2m \) and \( r = l \) in Eq. (2.9) from Theorem 2.7 for all \( 0 \leq l \leq m \), we have

\[
(-1)^l b_l \frac{1}{\hbar^l} = \left[ h^l G_{2l} \left( \sqrt{\hbar}, \sqrt{\hbar x} \right) + \sum_{l<s \leq m} K_m(l, s) \hbar^s \partial^{2(s-l)} G_{2s} \left( \sqrt{\hbar}, \sqrt{\hbar x} \right) \right] \times \frac{1}{\hbar^l}
\]

\[
= G_{2l} \left( \sqrt{\hbar}, \sqrt{\hbar x} \right) + \sum_{l<s \leq m} K_m(l, s) \hbar^{2(s-l)} \left( \partial^{2(s-l)} G_{2s} \right) \left( \sqrt{\hbar}, \sqrt{\hbar x} \right)
\]

By using Lemma A.1 it follows that the R.H.S. in the above equation is a polynomial of \( \sqrt{\hbar} \) and \( \sqrt{\hbar x} \). Therefore, there exists \( f_l \left( \sqrt{\hbar}, x \right) \in \mathbb{R} \left[ \sqrt{\hbar}, x \right] \) such that

\[
(-1)^l b_l = h^l f_l \left( \sqrt{\hbar}, \sqrt{\hbar x} \right)
\]

and hence, using the above equation along with Eq. (A.7), we can conclude that

\[
H \left( a_h, a_h^\dagger \right) = \sum_{l=0}^{m} (-1)^l \partial^l_x M_l \partial^l_x = \sum_{l=0}^{m} h^l \partial^l_x f_l \left( \sqrt{\hbar}, \sqrt{\hbar x} \right) \partial^l_x.
\]

\[\blacksquare\]

**Remark A.3.** The functions, \( f_l \left( \sqrt{\hbar}, x \right) \), in Theorem A.2 are examples of the functions, \( b_i, h \), appearing in Eq. (1.10).

**Example A.4.** Let \( H^c(x, \xi) = x^2 \xi^2 \) be a classical Hamiltonian where \( x \) is position and \( \xi \) is momentum on a state space \( \mathbb{R}^2 \). We would like to lift this to a symmetric polynomial in two symmetric indeterminate \( \hat{q} = \frac{\theta + \theta^*}{\sqrt{2}} \) and \( \hat{p} = \frac{\theta - \theta^*}{i\sqrt{2}} \). The Weyl lift of \( x^2 \xi^2 \) is given by

\[
H (\theta, \theta^*) = \frac{1}{4!} \left( \hat{q}^2 \hat{p}^2 + \text{ all permutations} \right)
\]

\[
= \frac{1}{4!} \cdot 2! \cdot 2! \left[ \begin{array}{c}
\hat{q}^2 \hat{p}^2 + \hat{q} \hat{p} \hat{q} \hat{p} + \hat{p} \hat{q} \hat{p} \hat{q} \\
+ \hat{p} \hat{q} \hat{p} \hat{q} + \hat{q} \hat{p} \hat{q} \hat{p} \\
\end{array} \right]
\]

\[
= \frac{1}{3!} \left[ \begin{array}{c}
\hat{q}^2 \hat{p}^2 + \hat{q} \hat{p} \hat{q} \hat{p} + \hat{p} \hat{q} \hat{p} \hat{q} \\
+ \hat{p} \hat{q} \hat{p} \hat{q} + \hat{q} \hat{p} \hat{q} \hat{p} \\
\end{array} \right] \in \mathbb{R} (\theta, \theta^*)
\]

Making the substitutions

\[
\hat{q} \rightarrow \frac{a_h + a_h^\dagger}{\sqrt{2}} = \sqrt{\hbar} M_x \quad \text{and} \quad \hat{p} \rightarrow \frac{a_h - a_h^\dagger}{i \sqrt{2}} = \frac{\sqrt{\hbar}}{i} \partial.
\]

above gives the Weyl quantization of \( x^2 \xi^2 \) to be

\[
H \left( a_h, a_h^\dagger \right) = -\frac{\hbar^2}{3!} (x^2 \partial^2 + \partial^2 x^2 + x \partial x \partial + \partial x \partial x + x \partial^2 x + \partial x^2 \partial) \quad \text{on} \quad \mathcal{S}
\]

which after a little manipulation using the product rule repeatedly may be written as

\[
H \left( a_h, a_h^\dagger \right) = -\hbar^2 \partial x \partial - \frac{1}{2} \hbar^2 \\
= -\hbar \partial b_{1,h} \left( \sqrt{\hbar x} \right) \partial + b_{0,h} \left( \sqrt{\hbar x} \right)
\]

where \( b_{1,h} (x) = x^2 \) and \( b_{0,h} (x) = -\frac{1}{2} \hbar^2 \).
References

[1] T. Ando and F. Hiai. Inequality between powers of positive semidefinite matrices. *Linear Algebra Appl.*, 208/209:65–71, 1994. 1.2

[2] Paul R. Chernoff. Essential self-adjointness of powers of generators of hyperbolic equations. *J. Functional Analysis*, 12:401–414, 1973. 1.1

[3] Bruce K. Driver and Pun Wai Tong. On the classical limit of quantum mechanics i. 2015. (document) 1.2

[4] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989. 1.1

[5] Masatoshi Fujii and Yuki Seo. Reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities. *Nihonkai Math. J.*, 16(2):145–154, 2005. 1.2

[6] Erhard Heinz. Beiträge zur Störungstheorie der Spektralzerlegung. *Math. Ann.*, 123:415–438, 1951. 1.2

[7] Klaus Hepp. The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.*, 35:265–277, 1974. 1.2

[8] Tosio Kato. Notes on some inequalities for linear operators. *Math. Ann.*, 125:208–212, 1952. 1.2

[9] Tosio Kato. A remark to the preceding paper by Chernoff (“Essential self-adjointness of powers of generators of hyperbolic equations”, *J. Functional Analysis* 12 (1973), 401–414). *J. Functional Analysis*, 12:415–417, 1973. 1.1

[10] Karl Löwner. Über monotone Matrixfunktionen. *Math. Z.*, 38(1):177–216, 1934. 1.2

[11] Michihiro Nagase and Tomio Umeda. On the essential self-adjointness of pseudo differential operators. *Proc. Japan Acad. Ser. A Math. Sci.*, 64:94–97, 1988. 1.1

[12] Michihiro Nagase and Tomio Umeda. Weyl quantized Hamiltonians of relativistic spinless particles in magnetic fields. *J. Funct. Anal.*, 92(1):136–154, 1990. 1.1

[13] Josip Pečarić, Takayuki Furuta, Jadranka Mićić Hot, and Yuki Seo. *Mond–Pečarić method in operator inequalities*, volume 1 of *Monographs in Inequalities*. ELEMENT, Zagreb, 2005. Inequalities for bounded selfadjoint operators on a Hilbert space. 1.2

[14] Gert K. Pedersen. Some operator monotone functions. *Proc. Amer. Math. Soc.*, 36:309–310, 1972. 1.2

[15] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. 1.1 1.2 6.3

[16] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. Functional analysis. 4

[17] Konrad Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, volume 265 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2012. 1.2

[18] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson. 1.1

[19] A. F. M. ter Elst and Derek W. Robinson. Uniform subellipticity. *J. Operator Theory*, 62(1):125–149, 2009. 1.2
[20] Masao Yamazaki. The essential selfadjointness of pseudodifferential operators associated with nonelliptic Weyl symbols with large potentials. Osaka J. Math., 29(2):175–202, 1992.

[21] Maciej Zworski. *Semiclassical analysis*, volume 138 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012.

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