GLOBAL ASYMPTOTICAL STABILITY OF THE COEXISTENCE FIXED POINT OF A RICKER-TYPE COMPETITIVE MODEL

CHUNQING WU
School of Mathematics and Physics
Changzhou University
Changzhou, 213164, China

PATRICIA J. Y. WONG
School of Electrical and Electronic Engineering
Nanyang Technological University
639798, Singapore

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Abstract. We shall obtain the parameter region that ensures the global asymptotical stability of the coexistence fixed point of a Ricker-type competitive model. The parameter region can be illustrated graphically and examples of such regions are presented. Our result partially answers an open problem proposed by Elaydi and Luis [3] and complements the very recent work by Balreira, Elaydi and Luis [1].

1. Introduction. Discrete time population models are favored by many researchers due to the fact that data is collected in discrete time intervals and many species have non-overlapping generations. For example, Luis, Elaydi and Oliveira [10] have studied systematically the following Ricker-type competitive model

\[
\begin{align*}
x_{n+1} &= x_n \exp(K - x_n - ay_n), \\
y_{n+1} &= y_n \exp(L - bx_n - y_n),
\end{align*}
\]  

(1)

where \( K, L, a, b \) are all positive constants. The results or phenomena, such as the local asymptotical stability of the equilibria, the competitive exclusion principle, the various bifurcation scenarios and also the center manifolds, have been obtained or analyzed.

Ricker-type discrete models are one of the most important models in population biology. They are very simple in the form, but very complicated in the dynamics [11]. While system (1) is a competitive model of Ricker-type, there are also predator-prey models and host-parasite models of this type [5, 6]. A general form of model (1) can be presented as

\[
\begin{align*}
x_{n+1} &= x_n \exp(K - a_{11}x_n - a_{12}y_n), \\
y_{n+1} &= y_n \exp(L - a_{21}x_n - a_{22}y_n),
\end{align*}
\]  

(2)

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* Corresponding author: P.J.Y. Wong.
where $K$, $L$, $a_{ij}$, $i, j = 1, 2$ are all positive constants. The permanence and/or stability of model (2) have been studied in [4, 8, 12, 13, 14]. Note that model (2) is autonomous and is for two species. A non-autonomous version of (2) has been discussed in [2] for two species, and in [14, 16] for $m$ ($m \geq 2$) species.

Global asymptotical stability of the positive equilibrium of a theoretical population model is one of the important research aspects possibly due to the characteristic of maintaining the topological patterns under fluctuations of the environment to some extent. The dynamics of the system will tend to the coexistence fixed point when there are small environmental changes provided this fixed point is globally asymptotically stable. It reflects the self restoration ability of the ecosystem to small environmental changes. Hence, the global asymptotical stability of the equilibria of a theoretical population model has received numerous attention in the literature. In [3], an open problem on the global asymptotical stability of (1) is proposed as the following conjecture.

**Conjecture 1.** (Conjecture 1.2 in [3]) If $a, b > 0$, $ab < 1$, $bK < L < K/a$ and

$$4(ab - 1) + 2(1 - a)L + 2(1 - b)K \leq (aL - K)(bK - L) < (1 - a)L + (1 - b)K, \quad (3)$$

then the coexistence fixed point $(x^*, y^*)$ of (1),

$$(x^*, y^*) = \left( \frac{aL - K}{ab - 1}, \frac{bK - L}{ab - 1} \right), \quad (4)$$

is globally asymptotically stable for all $(K, L) \in \text{Int}(S_1)$, where $S_1$ is the region in the $K$-$L$ parameter space satisfying (3) and $bK < L < K/a$.

In this paper, we shall investigate the global asymptotical stability of the positive equilibrium $(x^*, y^*)$ of the Ricker-type competitive model (1). Specifically we shall obtain a $K$-$L$ parameter region in which the coexistence fixed point $(x^*, y^*)$ of (1) is globally asymptotically stable. Our result partially answers the open problem proposed in Conjecture 1, moreover it complements the very recent result of Balreira, Elaydi and Lúis [1] – the details will be presented and illustrated with figures in the main section, a comparison with the related results of [1] will also be included (see Remarks 2 and 3).

To obtain results on global asymptotical stability of the equilibrium, appropriate Lyapunov functions are usually constructed [9]. However, it is not easy to find a feasible Lyapunov function since the methods to construct these functions vary from different models [7]. Another method to establish the global asymptotical stability of the equilibrium is to verify that this equilibrium is both locally stable as well as globally attractive [15]. In the next section we shall employ this method to obtain the global asymptotical stability of the positive equilibrium of (1).

2. **Global asymptotical stability.** In this section, we shall establish the global asymptotical stability of the equilibrium of model (1) by showing that this equilibrium is both locally stable as well as globally attractive [15]. To begin, we consider (1) with initial values

$$x_0 > 0, \quad y_0 > 0. \quad (5)$$

The first lemma is to ensure the *permanence* of (1). To be specific, model (1) with initial values (5) is said to be permanent if there exist positive constants $\alpha$, $\alpha^*$, $\beta$ and $\beta^*$ such that

$$\alpha \leq \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n \leq \beta$$
and

\[ \alpha^* \leq \liminf_{n \to \infty} y_n \leq \limsup_{n \to \infty} y_n \leq \beta^* \]

hold for each solution of model (1) with initial values (5).

Lemma 2.1. If

\[ \exp(K-1) < \frac{L}{b} \quad \text{and} \quad \exp(L-1) < \frac{K}{a}, \]

then model (1) with (5) is permanent.

Proof. Noting that \( x_{n+1} \leq x_n \exp(K-x_n) \) and the maximum of the function \( x \exp(K-x) \) is \( \exp(K-1) \), we find

\[ \limsup_{n \to \infty} x_n \leq \exp(K-1). \]

Similarly, since \( y_{n+1} \leq y_n \exp(L-y_n) \), we also have

\[ \limsup_{n \to \infty} y_n \leq \exp(L-1). \]

Next, we shall prove that there exists positive constant \( l \), which will be determined later, such that

\[ \liminf_{n \to \infty} x_n \geq l. \]

From (6), there exists \( \epsilon > 0 \) such that \( \exp(L-1) + \epsilon < \frac{K}{a} \), or equivalently

\[ K - a[\exp(L-1) + \epsilon] > 0. \]

Noting (8), we can choose \( N_0 \) sufficiently large such that

\[ y_n \leq \exp(L-1) + \epsilon, \quad n \geq N_0. \]

It then follows from the first equation of (1) that

\[ x_{n+1} = x_n \exp(K-x_n - ay_n) \geq x_n \exp(K-x_n - a(\exp(L-1) + \epsilon)), \quad n \geq N_0. \]

There are two cases to consider.

Case 1. Suppose there exists an \( N_1 (\geq N_0) \) such that \( x_{N_1+1} \leq x_{N_1} \). From (11) \( \mid n = N_1 \), it follows that \( K - x_{N_1} - a(\exp(L-1) + \epsilon) \leq 0 \), i.e., \( x_{N_1} \geq K - a(\exp(L-1) + \epsilon) \). Using this and also (7) in (11) \( \mid n = N_1 \) gives

\[ x_{N_1+1} \geq (K - a(\exp(L-1) + \epsilon)) \exp(K - x_{N_1} - a(\exp(L-1) + \epsilon)) \]

\[ \geq (K - a(\exp(L-1) + \epsilon)) \times \exp(K - \exp(K-1) - a(\exp(L-1) + \epsilon)) \equiv l. \]

Next, we shall show that

\[ x_n \geq l, \quad n \geq N_1. \]

Suppose that (13) is not true. Then, there exists an \( N_2 (\geq N_1) \) such that \( x_{N_2} < l \). In view of (12), it is clear that \( N_2 \geq N_1 + 2 \). Let \( N_2 \) be the smallest integer satisfying \( x_{N_2} < l \). Therefore, we have \( x_{N_2-1} \geq l \). However, similar to the derivation of (12), we get \( x_{N_2} \geq l \), which is a contradiction. Hence, (13) holds.

Case 2. Suppose \( x_{n+1} > x_n \) for all \( n \geq N_0 \). In this case, it is clear that

\[ \lim_{n \to \infty} x_n = l_0 (< \infty). \]

Now, let \( n \to \infty \) in the first equation of (1), we get

\[ \lim_{n \to \infty} (K - x_n - ay_n) = 0. \]
Using (14) and (10), we find \( \liminf_{n \to \infty} (K - x_n - ay_n) \geq K - l_0 - a(\exp(L - 1) + \epsilon) \), and in view of (15) it follows that
\[
l_0 \geq K - a(\exp(L - 1) + \epsilon).
\]

Further, since \( \exp(K - 1) \geq K \), we have
\[
\exp(K - \exp(K - 1) - a(\exp(L - 1) + \epsilon)) < 1.
\]

Consequently, noting the definition of \( l_\epsilon \) (see (12)) we find
\[
\lim_{n \to \infty} x_n = l_0 \geq K - a(\exp(L - 1) + \epsilon) > l_\epsilon.
\]

In both cases above (see (13) and (16)), if we denote \( \lim_{\epsilon \to 0} l_\epsilon = l \), then it is clear that (9) holds and \( l > 0 \).

By a similar argument, we can show that
\[
\liminf_{n \to \infty} y_n \geq m > 0,
\]
where \( m = (L - b \exp(K - 1)) \exp(L - \exp(L - 1) - b \exp(K - 1)) \).

It now follows from (7)–(9) and (17) that model (1) with (5) is permanent. To confirm further, it is possible to verify that (9) holds and \( l \leq \exp(K - 1) \) and \( m \leq \exp(L - 1) \).

Indeed, since \( \exp(K - 1) \geq K \), we have \( \exp(K - \exp(K - 1) - a \exp(L - 1)) < 1 \) and so
\[
l = \lim_{\epsilon \to 0} l_\epsilon = (K - a \exp(L - 1)) \exp(K - \exp(K - 1) - a \exp(L - 1)) < (K - a \exp(L - 1)) < K \leq \exp(K - 1).
\]

Likewise, it can be shown that \( m \leq \exp(L - 1) \). This completes the proof.

From [10], it is known that if
\[
bK < L < K/a, \ ab < 1,
\]
or
\[
K/a < L < bK, \ ab > 1,
\]
is satisfied, then model (1) has a unique positive equilibrium (coexistence fixed point) \( (x^*, y^*) \) given by (4). Further, it is shown in [10] that this positive equilibrium is a saddle if (19) holds, and the following result.

**Lemma 2.2.** [10] If (18) and (3) hold, then the positive equilibrium \( (x^*, y^*) \) of (1) is locally asymptotically stable.

In the following, we denote
\[
G_0 = \{(x, y) \mid x \geq 0, \ y \geq 0, \ K - x - ay \leq 0, \ L - bx - y \leq 0\}.
\]

**Lemma 2.3.** Suppose that (18) holds, then the maximum \( M_1 \) of the function \( f(x, y) = x \exp(K - x - ay) \) in the region \( G_0 \) is
\[
(1) \ M_1 = \max\{x^*, \ \frac{L}{b} \exp(K - \frac{L}{b})\} \ \text{if} \ 1 < \frac{1}{1-ab} \leq x^*;
(2) \ M_1 = \max\{x^*, \ \frac{L}{b} \exp(K - \frac{L}{b})\} \ \text{if} \ 1 \leq \frac{L}{b} \leq \frac{1}{1-ab};
(3) \ M_1 = \max\{\frac{1}{1-ab} \exp(K - aL - 1), \ \frac{L}{b} \exp(K - \frac{L}{b})\} \ \text{if} \ x^* < \frac{1}{1-ab} < \frac{L}{b};
(4) \ M_1 = \exp(K - 1) \ \text{if} \ \frac{L}{b} < 1 < \frac{1}{1-ab}.
\]

**Proof.** It is obvious that \( f(x, y) \) is bounded in \( G_0 \). The maximum \( M_1 \) is obtained by direct computation, the details are given in the Appendix.
Lemma 2.4. Suppose that (18) holds, then the maximum $M_2$ of the function $g(x, y) = y \exp(L - bx - y)$ in the region $G_0$ is

1. $M_2 = \max\{y^*, \frac{K}{a} \exp(L - \frac{K}{a})\}$ if $1 < \frac{1}{1-ab} \leq y^*$;
2. $M_2 = \max\{y^*, \frac{K}{a} \exp(L - \frac{K}{a})\}$ if $1 \leq \frac{K}{a} \leq \frac{1}{1-ab}$;
3. $M_2 = \max\{\frac{1}{1-ab} \exp(L - bK - 1), \frac{K}{a} \exp(L - \frac{K}{a})\}$ if $y^* < \frac{1}{1-ab} < \frac{K}{a}$;
4. $M_2 = \exp(L - 1)$ if $\frac{K}{a} < 1 < \frac{1}{1-ab}$.

Proof. Clearly $g(x, y)$ is bounded in $G_0$. The proof is similar to that of Lemma 2.3.

The next result establishes the global attractiveness of the positive equilibrium $(x^*, y^*)$.

Lemma 2.5. Let (6) and (18) hold. If

$$M_1 \leq x^*, \quad M_2 \leq y^*, \quad l_1 \neq 1, \quad l_2 \neq 1,$$

and for $(x_n, y_n)$ that satisfies (1),

$$\liminf_{n \to \infty} x_n = l_1 \neq 1, \quad \liminf_{n \to \infty} y_n = l_2 \neq 1,$$

then the positive equilibrium $(x^*, y^*)$ of (1) is globally attractive in the interior of the first quadrant.

Proof. Under assumptions (6) and (18), model (1) with (5) is permanent (Lemma 2.1) and has a positive equilibrium $(x^*, y^*)$ [10]. Hence, there exists positive constants $l_1$, $l_2$, $L_1$ and $L_2$ such that

$$0 < l_1 = \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n = L_1,$$

$$0 < l_2 = \liminf_{n \to \infty} y_n \leq \limsup_{n \to \infty} y_n = L_2. \quad (23)$$

Noting that

$$l_1 = \liminf_{n \to \infty} x_{n+1} = \liminf_{n \to \infty} [x_n \exp(K - x_n - ay_n)]$$

$$\geq \{\liminf_{n \to \infty} x_n\} \{\liminf_{n \to \infty} \exp(K - x_n)\} \{\liminf_{n \to \infty} \exp(-ay_n)\}$$

$$= l_1 \exp(K - L_1) \exp(-aL_2)$$

and likewise $l_2 \geq l_2 \exp(L - bL_1 - L_2)$, it follows that

$$K - L_1 - aL_2 \leq 0, \quad L - bL_1 - L_2 \leq 0, \quad (24)$$

and hence the point $(L_1, L_2)$ lies in $G_0$.

Now we define a region $G_\delta$ in the first quadrant as follows: $G_\delta \subset G_0$ and

$$G_\delta = \{(x, y) \mid x \geq 0, \ y \geq 0, \ K_\delta - x - ay \leq 0, \ L_\delta - x - by \leq 0\} \quad (25)$$

where the lines $e_1 : K_\delta - x - ay = 0$ and $e_2 : L_\delta - x - by = 0$ are parallel to the lines $s_1 : K - x - ay = 0$ and $s_2 : L - bx - y = 0$, respectively, such that the distances of $e_1$ to $s_1$ and $e_2$ to $s_2$ are both $\delta > 0$. Moreover, $e_1$ and $e_2$ are closer to the origin than the lines $s_1$ and $s_2$. The regions $G_0$ and $G_\delta$ are illustrated in Figure 1.

With $f(x, y)$ and $g(x, y)$ defined in Lemmas 2.3 and 2.4, we note that model (1) is simply

$$\begin{cases}
  x_{n+1} = f(x_n, y_n), \\
  y_{n+1} = g(x_n, y_n).
\end{cases} \quad (26)$$
Let the maximum of \( f(x, y) \) and \( g(x, y) \) in \( G_\delta \) be denoted as \( M_{1\delta_i} \) and \( M_{2\delta_i} \), respectively. For any \( \epsilon_1 > 0 \) sufficiently small, from (21) there exists \( \delta_1 > 0 \) such that \( M_{1\delta_i} < x^* + \epsilon_1 \) and \( M_{2\delta_i} < y^* + \epsilon_1 \). Hence, noting (23) and (26) we get

\[
L_1 \leq M_{1\delta_i} < x^* + \epsilon_1, \quad L_2 \leq M_{2\delta_i} < y^* + \epsilon_1. \tag{27}
\]

Noting that \( \epsilon_1 \) is arbitrarily given and the limit of \( G_{\delta_1} \) is \( G_0 \) as \( \delta_1 \to 0 \), hence, with the continuity of the functions concerned, we have \( M_{1\delta_1} \to M_1 \) and \( M_{2\delta_1} \to M_2 \). It follows from (27) that

\[
L_1 \leq M_1 \leq x^*, \quad L_2 \leq M_2 \leq y^*. \tag{28}
\]

Since \((L_1, L_2) \in G_0\), together with \( L_1 \leq x^* \) and \( L_2 \leq y^* \) (from (28)), it is seen from Figure 1 that we must have \( L_1 = x^* \) and \( L_2 = y^* \), i.e.,

\[
\limsup_{n \to \infty} x_n = L_1 = x^*, \quad \limsup_{n \to \infty} y_n = L_2 = y^*. \tag{29}
\]

In the sequel, we proceed to prove

\[
\liminf_{n \to \infty} x_n = l_1 = x^*, \quad \liminf_{n \to \infty} y_n = l_2 = y^*. \tag{30}
\]

Under the assumption (22), there are four cases to consider.

**Case 1.** \( l_1 < 1, l_2 < 1 \). Noting that the function \( x \exp(K - x) \) is increasing in the interval \([0, 1]\), we find

\[
l_1 = \liminf_{n \to \infty} x_{n+1} \geq \left\{ \liminf_{n \to \infty} [x_n \exp(K - x_n)] \right\} \left\{ \liminf_{n \to \infty} \exp(-ay_n) \right\} \geq l_1 \exp(K - l_1) \exp(-aL_2) = l_1 \exp(K - l_1 - aL_2)
\]

and hence

\[
K - l_1 - aL_2 \leq 0. \tag{31}
\]
Similarly, we get
\[ L - bL_1 - l_2 \leq 0. \] \hspace{1cm} (32)
In view of (29), substituting \( L_1 = x^* \) and \( L_2 = y^* \) into (31) and (32), we get \( l_1 \geq K - ay^* = x^* \) and \( l_2 \geq L - bx^* = y^* \). Coupling with (23), we obtain \( l_1 = L_1 = x^* \) and \( l_2 = L_2 = y^* \).

**Case 2.** \( l_1 < 1, l_2 > 1 \). Noting that the function \( x \exp(K - x) \) is decreasing in the interval \([1, \infty)\), we get
\[ l_2 = \liminf_{n \to \infty} y_{n+1} \geq \left\{ \liminf_{n \to \infty} [y_n \exp(L - y_n)] \right\} \left\{ \liminf_{n \to \infty} \exp(-bx_n) \right\} \]
\[ \geq L_2 \exp(L - L_2) \exp(-bL_1) \]
\[ = L_2 \exp(L - y^* - bx^*) = L_2 \]
where we have also made use of (29). In view of (23), it follows that \( l_2 = L_2 = y^* \). The proof of \( l_1 = L_1 = x^* \) is similar to Case 1.

**Case 3.** \( l_1 > 1, l_2 < 1 \). The proof is similar to Case 2.

**Case 4.** \( l_1 > 1, l_2 > 1 \). As in Case 2, we obtain \( l_1 \geq L_1 \) and \( l_2 \geq L_2 \). Hence, noting (23) and (29) we have \( l_1 = L_1 = x^* \) and \( l_2 = L_2 = y^* \).

In all the four cases, we have shown that (30) holds. Coupling with (29), we have \( \lim_{n \to \infty} x_n = x^* \) and \( \lim_{n \to \infty} y_n = y^* \). Hence, \((x^*, y^*)\) is globally attractive in the interior of the first quadrant.

From Lemmas 2.2 and 2.5, we have the following main result.

**Theorem 2.6.** Suppose that the conditions (3), (6), (18), (21) and (22) are satisfied, then the positive equilibrium \((x^*, y^*)\) of model (1) is globally asymptotically stable in the interior of the first quadrant.

**Proof.** Since the conditions of Lemma 2.2 and 2.5 are satisfied, the positive equilibrium \((x^*, y^*)\) of model (1) exists, which is both locally asymptotically stable and globally attractive in the interior of the first quadrant. Therefore, \((x^*, y^*)\) is globally asymptotically stable in the interior of the first quadrant. \(\square\)

**Remark 1.** Noting that
\[ \frac{1}{1 - ab} \exp(K - aL - 1) \geq \frac{K - aL}{1 - ab} = x^* \]
and
\[ \frac{1}{1 - ab} \exp(L - bK - 1) \geq \frac{L - bK}{1 - ab} = y^*, \]
the condition (21) cannot be feasible in case (3) of both Lemmas 2.3 and 2.4. Also, since 
\[ \exp(K - 1) \geq K \geq x^* \] and \[ \exp(L - 1) \geq L \geq y^*, \]
the condition (21) cannot be feasible in case (4) of both Lemmas 2.3 and 2.4. Hence, only the cases (1) and (2) of both Lemmas 2.3 and 2.4 are feasible for Theorem 2.6.

**Remark 2.** Let \( D \) be the \( K-L \) parameter region obtained in Theorem 2.6, and \( S_1 \) be the \( K-L \) parameter region associated with Lemma 2.2 [10] (note that \( S_1 \) is also the parameter region proposed in Conjecture 1 [3]). To be specific, \( D \) is the intersection of conditions (3), (6), (18), (21) and (22), and the global asymptotical stability of
(x*, y*) is guaranteed for (K, L) ∈ D. On the other hand, S_1 is described by (3) and (18), and by Lemma 2.2 the positive equilibrium (x*, y*) is locally asymptotically stable for (K, L) ∈ S_1. Logically we should have D ⊂ S_1. Indeed, by setting a = 0.2 and b = 0.25, we plot S_1 and D in Figure 2 and observe that D ⊂ S_1.

**Remark 3.** It is shown recently in [1] (Theorem 6.1) that the local stability of the positive equilibrium (x*, y*) implies its global stability under certain conditions. The results are stated below.

**Theorem 6.1.** [1] Let 1 < K, L < 2 in model (1). Suppose that the coexistence fixed point (x*, y*) is locally asymptotically stable. Further, assume the following conditions:

(a) The region R_1 is contained in the region Γ_1;
(b) For all m ≠ n, LC^m_1 ∩ LC^n_1 = ∅.

Then, the positive equilibrium (x*, y*) of (1) is globally asymptotically stable with respect to the interior of the first quadrant.

**Lemma 6.2.** [1] If

\[
K \in \left( \frac{a + 1 - 2a\sqrt{b}}{1-ab}, \frac{a + 1 + 2a\sqrt{b}}{1-ab} \right), \quad L \in \left( \frac{1+b-2b\sqrt{a}}{1-ab}, \frac{1+b+2b\sqrt{a}}{1-ab} \right),
\]  

then condition (a) of Theorem 6.1 holds.

Here, R_1, Γ_1 and LC^1_m are defined as follows:

\[
R_1 = \left\{ (x,y) \mid x > 0, y > 0, y \leq \frac{1-x}{1-(1-ab)x}, \ x < \frac{1}{1-ab} \right\};
\]

\[
Γ_1 = \{(x,y) \mid x > 0, y > 0, K - x - ay > 0, \ L - bx - y > 0\};
\]

the curve LC^1_m is defined as the image of rank (m + 1) of LC^1_{-1} of the map \( F \)

\[
LC^1_m = F^{m+1}(LC^1_{-1}), \ m = -1, 0, 1, 2, \ldots
\]
Figure 3. Region D, enclosed by the three red curves, for different values of \( a \) and \( b \). (3a) \( a = 0.2, b = 1.1 \); (3b) \( a = 0.4, b = 0.4 \); (3c) \( a = 0.5, b = 0.4 \); (3d) \( a = 0.5, b = 0.5 \), here \( D \) is just a point \((2, 2)\).

where

\[
LC_{-1}^1 = \left\{(x, y) \mid x > 0, \ y > 0, \ y = \frac{1 - x}{1 - (1 - ab)x}, \ x < \frac{1}{1 - ab}\right\}
\]

and \( F: \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) is the Ricker competition map given by

\[
F(x, y) = (x \exp(K - x - ay), \ y \exp(L - bx - y)).
\]

Let us consider the case when \( a = 0.2 \) and \( b = 0.25 \). To guarantee the global asymptotical stability of the positive equilibrium \((x^*, y^*)\), one computes from (33) the ranges of \( K \) and \( L \) as

\[
1.0526 < K < 1.4737 \quad \text{and} \quad 1.0804 < L < 1.5512
\]

in order to fulfill condition (a) of Theorem 6.1 of [1]. Hence, when the parameters \( K \) and \( L \) are outside of the above ranges, say \((K, L) = (1.5, 2.1)\), then Theorem 6.1 of [1] may not be applicable to obtain the global asymptotical stability of the positive equilibrium \((x^*, y^*)\). On the other hand, it is easy to verify numerically that \((K, L) = (1.5, 2.1)\) is a point in the parameter region \( D \) obtained by Theorem 2.6 in this paper (refer to Figure 2(b)). Thus, Theorem 2.6 is able to establish the global asymptotical stability of the positive equilibrium \((x^*, y^*)\) when \((K, L) = (1.5, 2.1)\).

One further notes that condition (b) of Theorem 6.1 of [1] is difficult to verify due to the infinite number of curves \( LC_m^1 \).

Lastly, we remark that the method used in the present work is different from that of [1].
Appendix. In this section we shall give the detailed proof of Lemma 2.3. Recall that
\[ f(x, y) = x \exp(K - x - ay). \]
First, since \( \lim_{x \to +\infty} f(x, y) = 0 \), \( \lim_{y \to +\infty} f(x, y) = 0 \) and \( \lim_{x,y \to +\infty} f(x, y) = 0 \), together with the fact that \( f(x, y) \) is continuous in the region \( G_0 \), the maximum
of \( f(x, y) \) in \( G_0 \) exists.
We shall try to find the critical points of \( f(x, y) \) in the interior of \( G_0 \). The critical points of \( f(x, y) \) are points that satisfy \( \partial f/\partial x = 0 \) and \( \partial f/\partial y = 0 \). From \( \partial f/\partial x = (1 - x) \exp(K - x - ay) = 0 \), we get \( x = 1 \). However, when \( x = 1 \), the
equation \( \partial f/\partial y = -ax \exp(K - x - ay) = 0 \) has no solution. Hence, no critical
point of \( f(x, y) \) exists in the interior of \( G_0 \), and the maximum of \( f(x, y) \) is attained
at the boundaries of \( G_0 \).
The region \( G_0 \) has four boundary lines (refer to Figure 1): (1) the half line \( x = 0 \)
with \( y \geq K/a \); (2) the half line \( y = 0 \) with \( x \geq L/b \); (3) the line \( s_1 : K - x - ay = 0 \)
with \( 0 \leq x \leq x^* \); (4) the line \( s_2 : L - bx - y = 0 \) with \( x^* \leq x \leq L/b \). We shall
calculate the corresponding maximum of \( f(x, y) \) along these four boundary lines
and then take the maximum of all these.

Case (1). \( x = 0 \) with \( y \geq K/a \). In this case, \( f(x, y) = 0 \) for all \( y \).

Case (2). \( y = 0 \) with \( x \geq L/b \). Here, we have \( f(x, y) = x \exp(K - x) \equiv g(x) \). From \( dg/dx = (1 - x) \exp(K - x) = 0 \), we obtain \( x = 1 \). Note that \( g(x) \) is monotonely increasing (decreasing) when \( x \leq 1 \) \((x \geq 1) \). Hence,
(2)(i) if \( 1 \leq L/b \), \( g_{\text{max}} = g \left( \frac{x}{b} \right) = \frac{1}{b} \exp(K - \frac{1}{b}) \);
(2)(ii) if \( 1 \geq L/b \), \( g_{\text{max}} = g(1) = \exp(K - 1) \).

Case (3). The line \( s_1 : K - x - ay = 0 \) with \( 0 \leq x \leq x^* \). We get \( f(x, y) = x \) on
\( s_1 \). Hence, the maximum of \( f \) is \( x^* \).

Case (4). The line \( s_2 : L - bx - y = 0 \) with \( x^* \leq x \leq L/b \). In this case, substituting
\( y = L - bx \) into \( f(x, y) \) gives \( f(x, y) = x \exp(K - aL - (1 - ab)x) \equiv h(x) \). From
\( dh/dx = (1 - (1 - ab)x) \exp(K - aL - (1 - ab)x) = 0 \), we obtain \( x = \frac{1}{1 - ab} \). Note that
\( h(x) \) is monotonely increasing (decreasing) when \( x \leq \frac{1}{1 - ab} \) \((x \geq \frac{1}{1 - ab}) \). Therefore,
(4)(i) if \( \frac{1}{1 - ab} \leq x^* \), \( h_{\text{max}} = h(x^*) = x^* \);
(4)(ii) if \( x^* \leq \frac{1}{1 - ab} \leq \frac{x}{b} \), \( h_{\text{max}} = h \left( \frac{1}{1 - ab} \right) = \frac{1}{1 - ab} \exp(K - aL - 1) \);
(4)(iii) if \( \frac{x}{b} \leq \frac{1}{1 - ab} \), \( h_{\text{max}} = h \left( \frac{x}{b} \right) = \frac{x}{b} \exp(K - \frac{x}{b}) \).

In the following, we shall obtain the maximum of \( f(x, y) \) in \( G_0 \) from Cases (1)–(4)
above. The following statements will be useful in subsequent development.
(a) \( \frac{1}{1 - ab} > 1 \) since \( ab < 1 \) from (18);
(b) \( 0 < x^* < K < \frac{x}{b} \) (refer to Figure 1);
(c) \( \exp(K - 1) \geq K > x^* \);
(d) \( \text{The function } x \exp(K - x) \text{ is monotonely increasing when } 0 \leq x \leq 1 \);
(e) \( \frac{1}{1 - ab} \exp(K - aL - 1) > \frac{1}{1 - ab} (K - aL) = x^* \).

From (b), we have \( 0 < x^* < \frac{x}{b} \). Noting the critical points \( x = 1 \) and \( x = \frac{1}{1 - ab} \)
\((> 1) \) obtained in Cases (2) and (4) respectively, we shall insert these two critical points
among 0, \( x^* \), and \( \frac{x}{b} \). There are 6 possibilities which we list below.
(I) $0 < 1 < \frac{1}{1-ab} < x^* < \frac{L}{b}$. This situation corresponds to Cases (3), (2)(i) and (4)(i), hence

$$f_{\text{max}} = \max \left\{ x^*, \frac{L}{b} \exp \left( K - \frac{L}{b} \right), x^* \right\} = \max \left\{ x^*, \frac{L}{b} \exp \left( K - \frac{L}{b} \right) \right\}.$$

(II) $0 < x^* < \frac{L}{b} < \frac{1}{1-ab}$. This situation corresponds to Cases (3), (2)(i), and (4)(iii), hence

$$f_{\text{max}} = \max \left\{ x^*, \frac{L}{b} \exp \left( K - \frac{L}{b} \right), \frac{L}{b} \exp \left( K - \frac{L}{b} \right) \right\}.$$

(III) $0 < x^* < 1 < \frac{L}{b} < \frac{1}{1-ab}$. This situation corresponds to Cases (3), (2)(i), and (4)(iii), same as (II) above.

(IV) $0 < 1 < x^* < \frac{1}{1-ab} < \frac{L}{b}$. This situation corresponds to Cases (3), (2)(i) and (4)(ii), hence

$$f_{\text{max}} = \max \left\{ x^*, \frac{L}{b} \exp \left( K - \frac{L}{b} \right), \frac{1}{1-ab} \exp(K - aL - 1) \right\}$$

$$= \max \left\{ x^*, \frac{1}{1-ab} \exp(K - aL - 1), \frac{L}{b} \exp \left( K - \frac{L}{b} \right) \right\}$$

by (e).

(V) $0 < x^* < 1 < \frac{1}{1-ab} < \frac{L}{b}$. This situation corresponds to Cases (3), (2)(i), and (4)(ii), same as (IV) above.

(VI) $0 < x^* < \frac{L}{b} < 1 < \frac{1}{1-ab}$. This situation corresponds to Cases (3), (2)(ii) and (4)(iii), hence

$$f_{\text{max}} = \max \left\{ x^*, \exp(K - 1), \frac{L}{b} \exp \left( K - \frac{L}{b} \right) \right\} = \exp(K - 1)$$

by (c) and (d).

From (I)–(VI) above, we see that (I) corresponds to case (1) of Lemma 2.3; (II) and (III) correspond to case (2) of Lemma 2.3; (IV) and (V) correspond to case (3) of Lemma 2.3; and (VI) corresponds to case (4) of Lemma 2.3. The proof of Lemma 2.3 is now complete.

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*E-mail address*: cqwu@cczu.edu.cn
*E-mail address*: ejywong@ntu.edu.sg