Recursive formulas for \( _2F_1 \) and \( _3F_2 \) hypergeometric series

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Abstract

Recursive formulas extending some known \( _2F_1 \) and \( _3F_2 \) summation formulas by using contiguous relations have been obtained. On the one hand, these recursive equations are quite suitable for symbolic and numerical evaluation by means of computer algebra. On the other hand, sometimes closed-forms of such extensions can be derived by induction. It is expected that the method used to obtain the different recursive equations can be applied to extend other hypergeometric summation formulas given in the literature.

Keywords: generalized hypergeometric functions, hypergeometric summation formulas, contiguous hypergeometric identities, recursive hypergeometric formulas

Mathematics Subject Classification: 33C05, 33C20

1 Introduction

Hypergeometric and generalized hypergeometric functions have many applications as solutions of problems concerning mathematics [1][2] and physics [3][4]. It is worth noting that whenever these functions can be expressed in terms of gamma functions, the results are very important in many applications and also from a theoretical point of view. However, few summation theorems are available in the literature [5]. Thereby, in the last decades, many publications have devoted to broaden these classical results. For instance, it is well-known that by repeated application of the contiguous relations of the hypergeometric function \( _2F_1 (a, b; c; z) \) [6 Sect. 15.5(ii)], any function \( _2F_1 (a + k, b + \ell; c + m; z) \), in which \( k, \ell, \) and \( m \) are integers, can be expressed as a linear combination of \( _2F_1 (a, b; c; z) \) and any of its contiguous functions, with coefficients that are rational functions of \( a, b, c, \) and \( z \). Also, by systematic exploitation of the relations between contiguous functions given by generalized hypergeometric functions \( _pF_q \) [7 Sect. 48], we can find in [8] a generalization of Watson’s...
theorem, and in [9] a generalization of Dixon’s theorem. However, these generalizations extend these theorems for a finite number of integer numbers summed to certain parameters of the corresponding hypergeometric functions.

In this paper, we provide recursive formulas in order to extend some known summation formulas of \(2F_1\) hypergeometric function at arguments \(z \neq 1\), and \(3F_2\) hypergeometric function at argument \(z = 1\). These recursive formulas are very suitable for symbolic computation and numerical evaluation by using computer algebra, and they are not restricted to a finite number of cases, as in the papers stated above. Despite the fact that these recursive formulas might be automatically computed using creative telescoping (also called Wilf-Zeilberger’s theory [5 Sects. 3.10&11]) following the method describe in [10] for non-terminating series, we propose here a much more simple approach using contiguous relations. Also, it is sometimes possible to derive general expressions in closed-form by induction from the corresponding recursive equation. In fact, the method described in this paper is quite general and can be used for many other summation formulas given in the literature, so that here we present some selected examples.

This article is organized as follows. In Section 2 we derive some recursive formulas for \(2F_1\) hypergeometric function at arguments \(z = \frac{1}{2}, 2, -1\). Section 3 extends some known \(3F_2\) hypergeometric summation formulas at argument unity, such as Pfaff-Saalschutz sum, Watson’s sum and Dixon’s sum, among others. Finally, the conclusions are collected in Section 4.

2 \(2F_1\) recursive formulas

In this Section we consider Gauss’s hypergeometric series, defined as

\[
2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| z \right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{m! (c)_m} z^m, \tag{1}
\]

where \((\alpha)_m = \Gamma (\alpha + m)/\Gamma (\alpha)\) denotes the Pochhammer symbol. When \(a\) or \(b\) are negative integers, the series (1) terminates, so that it converges. If it is not so, (1) is absolutely convergent when \(|z| < 1\), except when the parameter \(c = 0, -1, -2, \ldots\) In the case in which \(|z| = 1\), the series is absolutely convergent when \(\text{Re}(c - a - b) > 0\), conditionally convergent when \(-1 < \text{Re}(c - a - b) \leq 0\) and \(z \neq 1\); and divergent when \(\text{Re}(c - a - b) \leq -1\).

In order to obtain the recursive formulas stated below, we will use known contiguous relations connecting Gauss’s hypergeometric series \(2F_1 (a, b; c; z)\) with other two hypergeometric series which vary their parameters \(a, b, c, \pm 1\) unit. The idea is to particularize the parameters in the contiguous relation in such a way that we can define a recursive relation in which the initial iteration \((k = 0)\) is given by a known summation formula. Iterating the recursive relation and with the aid of computer algebra is quite easy to obtain summation formulas for other integers \(k\).

First, we generalize Gauss’s second theorem. In [11] we can find an extension of Gauss’s second theorem for \(j = 0\) and some particular values of \(k\).
Theorem 1 If $k \in \mathbb{N}$ and $j = 0, 1$, then the following recursive equation holds true:

$$G_k(a, b, j) = G_{k-1}(a, b, j) + \frac{a}{a + b + 1 + j}G_{k-1}(a + 1, b + 1, j), \quad (2)$$

where

$$G_k(a, b, j) = \binom{a}{a+b+k} \binom{1}{\frac{1}{2}},$$

and, according to Gauss’s second theorem [6, Eqn. 15.4.28]

$$G_0(a, b, 0) = \sqrt{\pi} \frac{\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{b+1}{2}\right)}, \quad a + b \neq -1, -2, \ldots,$$

and according to [6, Eqn. 15.4.29]

$$G_0(a, b, 1) = 2\sqrt{\pi} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a+2}{2}\right) \Gamma\left(\frac{b+1}{2}\right)} + \frac{a}{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{b+1}{2}\right)} \binom{1}{\frac{1}{2}}.$$

Proof. Consider the contiguous relation [12, Eqn. 9.2.13]

$$\binom{\alpha, \beta + 1}{\gamma} z = \binom{\alpha, \beta}{\gamma} z + \frac{\alpha z}{\gamma} \binom{\alpha + 1, \beta + 1}{\gamma + 1} z,$$

and set $\alpha = a$, $\beta = b + k$, $\gamma = \frac{a+b+j+1}{2}$, and $z = \frac{1}{2}$. □

From the above recursive equation (2), we prove the following identity.

Theorem 2 If $k \in \mathbb{N}$, then

$$\binom{a, a+k}{a+1} \binom{1}{\frac{1}{2}} = 2^a \binom{2^{k-1} - k - 1}{a+1} \binom{2 - k, a + 1}{a + 2} \binom{-1}{1}, \quad (4)$$

where notice that the hypergeometric sum on the RHS of (4) is a finite sum.

Proof. Taking $b = a$ and $j = 1$, (2) is reduced to

$$G_k(a) = G_{k-1}(a) + \frac{a}{2(a+1)}G_{k-1}(a+1), \quad (5)$$

where

$$G_k(a) = \binom{a, a+k}{a+1} \binom{1}{\frac{1}{2}}, \quad (6)$$

and according to [13, Eqn. 7.3.7(16)]

$$G_0(a) = 2^{a-1} a \left[ \psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right) \right], \quad (7)$$
where $\psi(z)$ denotes the digamma function. Now, from (5) and (7), and with the aid of computer algebra, perform the first iterations as:

$$
G_1(a) = 2^a, \\
G_2(a) = 2^a \left( 2 - \frac{1}{a+1} \right), \\
G_3(a) = 2^a \left( 4 - \frac{2}{a+1} - \frac{2}{a+2} \right), \\
G_4(a) = 2^a \left( 8 - \frac{3}{a+1} - \frac{6}{a+2} - \frac{3}{a+3} \right),
$$

thus, we can establish the conjecture

$$
G_k(a) = 2^a \left( 2^{k-1} - \sum_{i=1}^{k-1} \frac{(k-1)!}{(i-1)! (k-i-1)! (a+i)} \right), \tag{8}
$$

which can be proved by induction. Finally, rewrite (8) expressing the sum therein as a hypergeometric function and match the result to (6), to obtain (4).

Next, we generalized the result given in [14], which is given in table form for $k = -1, -2, -3, -4, -5$.

**Theorem 3** If $k \in \mathbb{Z}^-$ and $n \in \mathbb{Z}^+$, then the following recursive equation holds true:

$$
G_k(n,a) = 2 \left( a + k - 1 \right) (2a + k - n - 1) G_{k+1}(n,a) - \frac{2a + k - 2}{k} G_{k+1}(n,a-1), \tag{9}
$$

where

$$
G_k(n,a) = _2F_1 \left( \frac{-n,a}{2a-1+k} \right),
$$

and, according to [15]

$$
G_0(n,a) = \Gamma \left( a - \frac{1}{2} \right) \left[ \frac{1+(-1)^n}{2} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( a + \frac{n+1}{2} \right)} - \frac{1-(-1)^n}{2} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( a + \frac{n+1}{2} \right)} \right].
$$

**Proof.** Consider the following contiguous relations [12 Eqns. 9.2.7&14]

$$
\gamma (\gamma + 1) _2F_1 \left( \frac{\alpha, \beta}{\gamma} \left| z \right. \right) = \gamma (\gamma - \alpha + 1) _2F_1 \left( \frac{\alpha, \beta + 1}{\gamma + 2} \left| z \right. \right) + \alpha [\gamma - (\gamma - \beta) z] _2F_1 \left( \frac{\alpha + 1, \beta + 1}{\gamma + 2} \left| z \right. \right), \tag{10}
$$
and
\[
2F_1 \left( \begin{array}{c} \alpha, \beta + 1 \\ \gamma + 1 \end{array} \bigg| z \right) - 2F_1 \left( \begin{array}{c} \alpha, \beta \\ \gamma \end{array} \bigg| z \right) (11)
\]
\[
= \frac{\alpha (\gamma - \beta) z}{\gamma (\gamma + 1)} 2F_1 \left( \begin{array}{c} \alpha + 1, \beta + 1 \\ \gamma + 2 \end{array} \bigg| z \right).
\]

Therefore, eliminating the hypergeometric function on the RHS of (10) and (11), we arrive at
\[
\gamma 2F_1 \left( \begin{array}{c} \alpha, \beta \\ \gamma \end{array} \bigg| z \right) - (\gamma - \alpha + 1) (\gamma - \beta) z 2F_1 \left( \begin{array}{c} \alpha, \beta + 1 \\ \gamma + 1 \end{array} \bigg| z \right)
\]
\[
= [\gamma - (\gamma - \beta) z] 2F_1 \left( \begin{array}{c} \alpha, \beta + 1 \\ \gamma + 1 \end{array} \bigg| z \right).
\]

Substitute now \( \alpha = -n, \beta = a - 1, \gamma = 2(a - 1) + k \), and \( z = 2 \) to obtain (9).

Finally, we generalize Kummer’s theorem. This case has been discussed more extensively in [16].

**Theorem 4** If \( k \in \mathbb{N} \), then
\[
G_{k+1} (a, b) = \frac{b + k}{k} G_k (a, b) - \frac{2b (a + k + 1)}{k (a - b + 1)} G_k (a + 2, b + 1),
\]
where
\[
G_k (a, b) = 2F_1 \left( \begin{array}{c} a + k, b \\ a - b + 1 \end{array} \bigg| -1 \right),
\]
and, according to [16]
\[
G_1 (a, b) = \frac{\sqrt{\pi}}{2^{a+1}} \frac{1}{\Gamma \left( \frac{a + 1}{2} \right) \Gamma \left( \frac{a + 1}{2} - b \right) + \Gamma \left( \frac{a + 1}{2} \right) \Gamma \left( \frac{a + 1}{2} - b \right)}.
\]

**Proof.** From the identity [6] Eqn. 15.5.20] and the differentiation formula [6] Eqn. 15.5.1], setting \( \alpha \to \alpha + 1 \), we arrive at
\[
z (1 - z) \left( \frac{\alpha + 1}{\gamma} \right) 2F_1 \left( \begin{array}{c} \alpha + 2, \beta + 1 \\ \gamma + 1 \end{array} \bigg| z \right)
\]
\[
= (\gamma - \alpha + 1) 2F_1 \left( \begin{array}{c} \alpha, \beta \\ \gamma \end{array} \bigg| z \right) + (\alpha + 1 - \gamma + \beta z) 2F_1 \left( \begin{array}{c} \alpha + 1, \beta \\ \gamma \end{array} \bigg| z \right).
\]

Thereby, taking \( \alpha = a + k, \beta = b, \gamma = a - b + 1, \) and \( z = -1 \), we arrive at [16].
Note that $\forall k = 0, 13$ reduces to Kummer’s summation formula [6, Eqn. 15.4.26], but in this case, the recursive equation (12) collapses. However, it is worth noting that in [16] we find a closed-form of (13), which reads as

$$G_k(a, b) = \frac{\Gamma(1 + a - b)}{2\Gamma(a + k)} \sum_{m=0}^{k} \binom{k}{m} \frac{\Gamma\left(\frac{a+k+m}{2}\right)}{\Gamma\left(\frac{a-k+m}{2} - b + 1\right)}.$$ (14)

As by-product, we can obtain an interesting identity, inserting (14) in the recursive formula (12). Direct substitution yields

$$k+1 \sum_{m=0}^{k} \binom{k+1}{m} \frac{\Gamma\left(\frac{a+k+m+1}{2}\right)}{\Gamma\left(\frac{a-k+m+1}{2} - b\right)} = \sum_{m=0}^{k} \binom{k}{m} \left(a + k - \frac{bm}{k}\right) \frac{\Gamma\left(\frac{a+k+m}{2}\right)}{\Gamma\left(\frac{a-k+m}{2} - b + 1\right)}.$$ (15)

Now, recast (15) as

$$\frac{\Gamma\left(\frac{a+k+1}{2}\right)}{\Gamma\left(\frac{a-k+1}{2} - b\right)} + \sum_{m=0}^{k} \binom{k}{m} \frac{(k+1)(a+k+m)}{2(m+1)} \frac{\Gamma\left(\frac{a+k+m}{2}\right)}{\Gamma\left(\frac{a-k+m}{2} - b + 1\right)} = \sum_{m=0}^{k} \binom{k}{m} \left(a + k - \frac{bm}{k}\right) \frac{\Gamma\left(\frac{a+k+m}{2}\right)}{\Gamma\left(\frac{a-k+m}{2} - b + 1\right)}.$$ (16)

It is worth noting that (16) can be proven using computer algebra.

### 3 $\mathbf{3F_2}$ recursive formulas

The generalized hypergeometric series $\mathbf{pF_q}$ is a natural generalization of the Gauss’s series $\mathbf{2F_1}$, and is defined as

$$\mathbf{pF_q} \left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| z \right) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m (b_1)_m \cdots (b_q)_m}{m!} z^m.$$ (17)

If any $a_j, j = 1 \ldots p$ is a negative integer, then the series (17) terminates, thus it converges. If (17) is not a terminating series, then it converges $\forall |z| < \infty$, if $p \leq q$; and $\forall |z| < 1$, if $p = q + 1$. Also, (17) diverges $\forall z \neq 0$, if $p > q + 1$. If $p = q + 1$ and $|z| = 1$, then the series (17) is absolutely convergent when $\text{Re} \left(\sum_{j=1}^{p} b_j - \sum_{j=1}^{q} a_j\right) > 0$; conditionally convergent when $z \neq 1$ and $-1 < \text{Re} \left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) < 0$; and divergent when $\text{Re} \left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) \leq -1$. 

\[6\]
Theorem 5 If \(k \in \mathbb{N}\), then

\[
G_k(a, b, c, d) = G_{k-1}(a, b, c, d) + \frac{ab}{c(d+1)}G_{k-1}(a+1, b+1, c+1, d+1),
\]

where

\[
G_k(a, b, c, d) = \binom{a, b, c + k + 1}{d + 1, c} 1,
\]

and where, renaming parameters in \([17]\), we have

\[
G_0(a, b, c, d) = \frac{\Gamma(d + 1)\Gamma(d - a - b)}{c\Gamma(d - a + 1)\Gamma(d - b + 1)}[a(b - c) + c(d - b)].
\]

Proof. Consider the contiguous relation \([5, \text{Eqn. 3.7.9}]\), exchanging the parameters \(\alpha \leftrightarrow \gamma\),

\[
\binom{\alpha, \beta, \gamma + 1}{\delta, \varepsilon} 1
\]

\[
= \binom{\alpha, \beta, \gamma}{\delta, \varepsilon} 1 + \frac{\alpha\beta}{\delta\varepsilon}\binom{\alpha + 1, \beta + 1, \gamma + 1}{\delta + 1, \varepsilon + 1} 1.
\]

and perform the substitutions \(\alpha = a, \beta = b, \gamma = c + k + 1, \delta = d + 1\) and \(\varepsilon = c\), to obtain the recursive formula \([13]\). \(\blacksquare\)

Remark 6 Notice that \(G_0(a, b, c, c)\) reduces to Gauss’s summation formula \([6, \text{Eqn. 15.4.20}]\).

From the above recursive equation \([13]\), we obtain the following identity.

Theorem 7 If \(k = 0, 1, 2\ldots\), then

\[
\binom{a, b, c + k + 1}{d + 1, c} 1
\]

\[
= \frac{(-1)^k\Gamma(d + 1)\Gamma(d - a - b - k)\Gamma(a + b - d + 1)}{c\Gamma(d - a + 1)\Gamma(d - b + 1)}
\]

\[
\times \left\{[a(b - c) + c(d - b)]\binom{-k, a, b}{c + 1, a + b - d + 1} 1\right\}
\]

\[
+ \frac{ab(c - d)k}{(c + 1)(a + b - d + 1)}\binom{1 - k, a + 1, b + 1}{c + 2, a + b - d + 2} 1\right\},
\]

where notice that on the RHS of \([22]\) the hypergeometric sums are finite sums.
Proof. With the aid of computer algebra, we can iterate the first terms of the recursive equation (18), starting from (20), obtaining:

\[
G_1(a, b, c, d) = -\frac{\Gamma(d+1)\Gamma(d-a-b-1)}{c\Gamma(d-a+1)\Gamma(d-b+1)} \left\{ (a+b-d-1)[a(b-c)+c(d-b)] - \frac{ab[a(b-c)-(b+1)c+(c+1)d]}{c+1} \right\},
\]

\[
G_2(a, b, c, d) = \frac{\Gamma(d+1)\Gamma(d-a-b-2)}{c\Gamma(d-a+1)\Gamma(d-b+1)} \left\{ (a+b-d-1)(a+b-d-2)[a(b-c)+c(d-b)] - \frac{2ab(a+b-d-2)[a(b-c)-(b+1)c+(c+1)d]}{c+1} + \frac{a(a+1)b(b+1)[a(b-c)-(b+2)c+(c+2)d]}{(c+1)(c+2)} \right\}.
\]

Therefore, we can conjecture the following general form,

\[
G_k(a, b, c, d) = \frac{(-1)^k\Gamma(d+1)\Gamma(d-a-b-k)}{c\Gamma(d-a+1)\Gamma(d-b+1)} \sum_{j=0}^{k} (-1)^j \binom{a+b-d+j+1}{c+1}_j (a)_j (b)_j^k \times [a(b-c)-(b+j)c+(c+j)d],
\]

which can be proved by induction. Finally, split the sum given in (23) in two sums and recast them as hypergeometric sums, to obtain,

\[
G_k(a, b, c, d) = \frac{(-1)^k\Gamma(d+1)\Gamma(d-a-b-k)\Gamma(a+b-d+k+1)}{c\Gamma(d-a+1)\Gamma(d-b+1)\Gamma(a+b-d+1)} \left\{ \frac{[a(b-c)+c(d-c)]}{(c+1)} _3F_2 \left( \begin{array}{c} -k, a, b \\ c+1, a+b-d+1 \end{array} \right| 1 \right\} + \frac{ab(c-d)k}{(c+1)(a+b-d+1)} _3F_2 \left( \begin{array}{c} 1-k, a+1, b+1 \\ c+2, a+b-d+2 \end{array} \right| 1 \right\}.
\]

Finally, match (24) to (19), to obtain (22). ■

Next, we consider an extension of Pfaff-Saalschütz summation formula.
Theorem 8  If $k, n \in \mathbb{N}$, then

$$G_k(n, a, b, c) = G_{k-1}(n, a+1, b+1) \quad (25)$$

$$+ \frac{nb}{(c+k)(a+b+1-n-c)} G_{k-1}(n-1, a+1, b+1, c+2),$$

where

$$G_k(n, a, b, c) = \binom{c-k}{-n, a, b}_{c+k, a+b+1-n-c},$$

and, according to Pfaff-Saalschutz balanced sum [6, Eqn. 16.4.3],

$$G_0(n, a, b, c) = \frac{(c-a)_n}{(c)(c-a-b)_n} (26)$$

Proof. Consider again the contiguous relation given in [5, Eqn. 3.7.9], exchanging the parameters $\alpha \leftrightarrow \beta$, thus

$$\binom{\alpha, \beta+1, \gamma}{\delta, \varepsilon}_1 = \binom{\alpha, \beta, \gamma}{\delta, \varepsilon}_1 + \frac{\alpha \gamma}{\delta \varepsilon} \binom{\alpha+1, \beta+1, \gamma+1}{\delta+1, \varepsilon+1}_1. (27)$$

Now, performing the substitutions $\alpha = -n$, $\beta = a$, $\gamma = b$, $\delta = c+k$, and $\varepsilon = a+b+1-n-c$, we arrive at the recursive equation (25). $\blacksquare$

Again, iterating the recursive equation (25), we can generalize Pfaff-Saalschutz summation formula as follows:

Theorem 9  If $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots$, then

$$\binom{-n, a, b}{c+k, a+b+1-n-c}_1 = \binom{c-a}_n (c-b)_n (c-a-b)_n \sum_{j=0}^k (-1)^j \binom{k}{j} 
\frac{(b)_j (n-j+1) \Gamma(c-b+n+k-j)}{(c-a)_j \Gamma(c-a+j)} (28)$$

where $k = 0$ matches Pfaff-Saalschutz summation formula (26).

Proof. Computing the first iterations of (25), starting from (26), we can conjecture that

$$G_k(n, a, b, c) = \frac{\Gamma(n-a-c)}{(c-b+k)(c+k)_n (c-a-b)_n} \sum_{j=0}^k (-1)^j \binom{k}{j} (b)_j (n-j+1) \Gamma(c-b+n+k-j) \frac{\Gamma(c-a+j)}{(c-a)_j} (29)$$

which can be proven by induction. Finally, the sum given in (28) can be rewritten as a terminating hypergeometric sum, obtaining (27). $\blacksquare$
Theorem 10 If $k \in \mathbb{N}$, we have the recursive formula
\[
G_{k+1}(a, b, c) = \frac{a(a - 1 - 2(b + c + k))}{k(b + c + k)} G_k(a + 1, b + 1, c + 1)
\]
\[
- \frac{(a - b)(a - c)}{k(b + c + k)} G_k(a - 1, b, c),
\]
where
\[
G_k(a, b, c) = \binom{3}{F_2}{a, b + k, c + k \left| a - b + 1, a - c + 1 \right)}
\]
and where, taking $i = 1$ and $j = 0$ in [9], we have
\[
G_1(a, b, c) = \frac{\Gamma(1 + a - b) \Gamma(1 + a - c)}{2^{a+1}bc \Gamma(a - 2c) \Gamma(a - b - c)} \begin{bmatrix}
\Gamma\left(\frac{a+1}{2} - c\right) \Gamma\left(\frac{b}{2} - c\right) \\
\Gamma\left(\frac{a+1}{2}ight) \Gamma\left(\frac{b}{2} - b\right)
\end{bmatrix}
\]
\[
\Gamma\left(\frac{a+1}{2} - b - c\right) \Gamma\left(\frac{a}{2} - c\right).
\]

Proof. Consider the contiguous relation [5 Eqn. 3.7.12], performing the substitution $\alpha \rightarrow \alpha + 1$,
\[
\delta \varepsilon \binom{3}{F_2}{\alpha + 1, \beta, \gamma \left| \delta, \varepsilon \right)}
\]
\[
= (\alpha + 1)(\delta + \varepsilon - \alpha - \beta - \gamma - 2) \binom{3}{F_2}{\alpha + 2, \beta + 1, \gamma + 1 \left| \delta + 1, \varepsilon + 1 \right)}
\]
\[
+ (\delta - \alpha - 1)(\varepsilon - \alpha - 1) \binom{3}{F_2}{\alpha + 1, \beta + 1, \gamma + 1 \left| \delta + 1, \varepsilon + 1 \right)},
\]
and also the contiguous relation [5 Eqn. 3.7.9]
\[
\binom{3}{F_2}{\alpha + 1, \beta, \gamma \left| \delta, \varepsilon \right)} = \binom{3}{F_2}{\alpha, \beta, \gamma \left| \delta, \varepsilon \right)} + \frac{\beta \gamma}{\delta \varepsilon} \binom{3}{F_2}{\alpha + 1, \beta + 1, \gamma + 1 \left| \delta + 1, \varepsilon + 1 \right)}.
\]

Thereby, eliminating the hypergeometric sum of the RHS of (30) and (31), we arrive at
\[
\delta \varepsilon \binom{3}{F_2}{\alpha, \beta, \gamma \left| \delta, \varepsilon \right)}
\]
\[- (\alpha + 1)(\delta + \varepsilon - \alpha - \beta - \gamma - 2) \binom{3}{F_2}{\alpha + 2, \beta + 1, \gamma + 1 \left| \delta + 1, \varepsilon + 1 \right)}
\]
\[
= [(\delta - \alpha - 1)(\varepsilon - \alpha - 1) - \beta \gamma] \binom{3}{F_2}{\alpha + 1, \beta + 1, \gamma + 1 \left| \delta + 1, \varepsilon + 1 \right)}.
\]

Finally, substituting in (32) $\alpha = a$, $\beta = b + k$, $\gamma = c + k$, $\delta = a - b + 1$ and $\varepsilon = a - c + 1$, we arrive at the recursive equation (29). \[\blacksquare\]
Proof. Exchanging $\beta \leftrightarrow \gamma$ and $\delta \leftrightarrow \varepsilon$ in the recursive relation given in [5] Sect. 3.7, we have

$$3F_2\left(\frac{\alpha, \beta, \gamma + 1}{\delta, \varepsilon + 1} \bigg| 1\right) = 3F_2\left(\frac{\alpha, \beta, \gamma}{\delta, \varepsilon} \bigg| 1\right) + \frac{\alpha \beta (\varepsilon - \gamma)}{\delta \varepsilon (\varepsilon + 1)} 3F_2\left(\frac{\alpha + 1, \beta + 1, \gamma + 1}{\delta + 1, \varepsilon + 2} \bigg| 1\right).$$

Also, performing the change $\varepsilon \to \varepsilon + 1$ in (23),

$$3F_2\left(\frac{\alpha, \beta, \gamma + 1}{\delta, \varepsilon + 1} \bigg| 1\right) = 3F_2\left(\frac{\alpha, \beta, \gamma}{\delta, \varepsilon + 1} \bigg| 1\right) + \frac{\alpha \beta}{\delta (\varepsilon + 1)} 3F_2\left(\frac{\alpha + 1, \beta + 1, \gamma + 1}{\delta + 1, \varepsilon + 2} \bigg| 1\right).$$

Therefore, equating the above equations, we arrive at

$$3F_2\left(\frac{\alpha, \beta, \gamma + 1}{\delta, \varepsilon + 1} \bigg| 1\right) = 3F_2\left(\frac{\alpha, \beta, \gamma}{\delta, \varepsilon} \bigg| 1\right) - \frac{\alpha \beta \gamma}{\delta (\varepsilon + 1)} 3F_2\left(\frac{\alpha + 1, \beta + 1, \gamma + 1}{\delta + 1, \varepsilon + 2} \bigg| 1\right).$$

Now, substituting $\alpha = a, \beta = b, \gamma = c, \delta = \frac{a + b + 1}{2},$ and $\varepsilon = 2c + k,$ we arrive at the recursive equation (24). ■
Theorem 13 If \( k \in \mathbb{N} \), then we have the following recursive equation:

\[
G_k(a, b, c) = G_{k-1}(a, b, c) + \frac{b}{a + b + 1} \left[ \frac{(a + k)(b + 1)}{(2c + 1)(a + b + 3)} G_{k-1}(a + 2, b + 2, c + 1) + G_{k-1}(a + 1, b + 1, c) \right],
\]

where

\[
G_k(a, b, c) = \,_{3}F_{2} \left( \begin{array}{c}
\frac{a + k, b, c}{a + b + 1, 2c} \\
1
\end{array} \right),
\]

and \( G_0(a, b, c) \) is given by Watson’s sum (34).

Proof. Exchanging \( \alpha \leftrightarrow \beta \) and setting \( \beta \rightarrow \beta + 1, \gamma \rightarrow \gamma + 1, \) and \( \delta \rightarrow \delta + 1 \) in the recursive relation given in [5, Sect. 3.7], we have

\[
3_{F_{2}} \left( \begin{array}{c}
\alpha + 1, \beta + 1, \gamma + 1 \\
\alpha + 1, \gamma + 1
\end{array} \right) = \frac{(\beta + 1)(\delta - \alpha)(\gamma + 1)}{\delta(\delta + 1)(\varepsilon + 1)} \,_{3}F_{2} \left( \begin{array}{c}
\alpha + 1, \beta + 2, \gamma + 2 \\
\alpha + 1, \gamma + 1
\end{array} \right) + 3_{F_{2}} \left( \begin{array}{c}
\alpha, \beta + 1, \gamma + 1 \\
\alpha, \beta + 1, \gamma + 1
\end{array} \right).
\]

Now, substituting the RHS of (36) in (35), we arrive at

\[
3_{F_{2}} \left( \begin{array}{c}
\alpha, \beta + 1, \gamma \\
\alpha, \beta + 1, \gamma
\end{array} \right) = 3_{F_{2}} \left( \begin{array}{c}
\alpha, \beta, \gamma \\
\alpha, \beta, \gamma
\end{array} \right) + \frac{\delta - \alpha}{\alpha \gamma} \left[ \frac{(\beta + 1)(\delta - \alpha)(\gamma + 1)}{\delta(\delta + 1)(\varepsilon + 1)} \,_{3}F_{2} \left( \begin{array}{c}
\alpha + 1, \beta + 2, \gamma + 2 \\
\alpha + 1, \gamma + 1
\end{array} \right) \right. \\
+ \left. 3_{F_{2}} \left( \begin{array}{c}
\alpha, \beta + 1, \gamma + 1 \\
\alpha, \beta + 1, \gamma + 1
\end{array} \right) \right].
\]

Finally, taking \( \alpha = c, \beta = a + k, \gamma = b, \delta = 2c, \) and \( \varepsilon = \frac{a + b + 1}{2} \), we arrive at the recursive equation (35). □

Finally, we provide and extension of a summation formula given by Bailey. By induction from the recursive formula found, a simple closed-form expression is derived.

Theorem 14 If \( k = 0, 1, 2, \ldots \), then the following recursive equation is satisfied:

\[
G_{k+1}(a, b, c) = \frac{(2c - b + k)(2c - b + k + 1)}{(a - 1)(2c - 2b + k + 1)(c - b + k)} \left[ (c - 1) G_k(a - 1, b - 1, c - 1) - (c - a) G_k(a - 1, b, c) \right],
\]
where

\[ G_k(a, b, c) = \left( \begin{array}{c} a, b, c + 1 \\ 1 + 2c - b + k, c \end{array} \right) \]

\[ = \frac{[(a - 2c)(b - c) + kc] \Gamma(2c - b + k + 1) \Gamma(2c - a - 2b + k)}{c \Gamma(2c - 2b + k + 1) \Gamma(2c - a - b + k + 1)} \right) \cdot (38) \]

**Proof.** Considering the contiguous relation [5, Eqn. 3.7.14], we have

\[ \varepsilon 3F2 \left( \begin{array}{c} \alpha, \beta, \gamma \\ \delta, \varepsilon \end{array} \right) \left[ (\alpha - \delta - \beta) (\delta - \gamma) \right] \right) \frac{\alpha + 1, \beta + 1, \gamma + 1}{\delta + 2, \varepsilon + 1} \right) \right) \]

thus, setting \( \alpha = a - 1, \beta = b - 1, \gamma = c, \delta = 2c - b + k, \) and \( \varepsilon = c - 1, \) we arrive at the recursive equation (37). Also, according to [19, Eqn. 6.4(2)], after renaming the parameters, we have

\[ G_0(a, b, c) = \left( 1 - \frac{a}{2c} \right) \frac{\Gamma(2c - b + 1) \Gamma(2c - a - 2b)}{\Gamma(2c - 2b) \Gamma(2c - a - b + 1)} \right) \right) \]

hence, recursive substitution of (39) in (37), after simplification, yields

\[ G_1(a, b, c) = \left( \frac{[(a - 2c)(b - c) + c] \Gamma(2c - b + 2) \Gamma(2c - a - 2b + 1)}{c \Gamma(2c - 2b + 2) \Gamma(2c - a - b + 2)} \right) \right) \]

and

\[ G_2(a, b, c) = \left( \frac{[(a - 2c)(b - c) + 2c] \Gamma(2c - b + 3) \Gamma(2c - a - 2b + 2)}{c \Gamma(2c - 2b + 3) \Gamma(2c - a - b + 3)} \right) \right) \]

Therefore, we may conjecture the general form given in (38), which can be proved easily by induction using the recursive equation (37).

### 4 Conclusions

We have obtained some recursive formulas to extend some known \( \text{2F}_1 \) and \( \text{3F}_2 \) summation formulas by using contiguous relations. These recursive equations are quite suitable for symbolic and numerical evaluation by means of computer algebra. Moreover, in some cases, namely (4), (22), (27), and (38), we have derived closed-form expressions. Also, as by-product, we have obtained an interesting identity in (10). It is expected that the method used to obtain the different recursive equations can be applied to extend other hypergeometric summation formulas given in the literature.
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