ON FUJITA’S SEMI-AMPLENESS IN THE RANK ONE CASE

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Abstract. We prove that a conjecture of Fujita on the semi-ampleness is true in the case of rank one direct summand, though it is wrong in higher rank case by Catanese and Dettweiler.

1. Introduction

Let $f : X \to Y$ be a surjective morphism with connected fibers between smooth projective complex varieties. Then the direct image sheaf $f_*\omega_{X/Y}$ of the relative canonical sheaf has a semi-positivity property that it is a direct sum of a generically ample sheaf and a unitary flat locally free sheaf ([4]). Fujita asked whether the unitary flat part is semi-ample ([7]). But Catanese and Dettweiler found a series of counter-examples to this question ([2], [3]). They showed that there are rank 2 unitary flat direct summands for which the monodromy representations have infinite images, hence not semi-ample.

We prove in this paper that the rank 1 case is different:

Theorem 1.1. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected geometric fibers over $\mathbb{C}$. Let $V = \mathcal{O}_Y(L)$ be an invertible sheaf on $Y$ which is a direct summand of the direct image sheaf $f_*\mathcal{O}_X(K_{X/Y})$. Assume that $L$ is numerically trivial. Then there exists a positive integer $m$ such that $\mathcal{O}_Y(mL) \cong \mathcal{O}_Y$.

For this purpose, we will use the theory of rank one local systems by Simpson [13], [14] as well as the Hodge theory of Deligne [6]. The examples by Catanese and Dettweiler shows that the higher rank local systems are quite different from the rank one case.

As an application, we prove that the moduli part of the canonical bundle formula ([10]) is torsion if it is numerically trivial:

Theorem 1.2. Let $L$ be the moduli part as in Theorem 1.1. Assume that $L$ is numerically trivial. Then $L$ is torsion in the sense that there exists a positive integer $m$ such that $mL \sim 0$.

We note that the local system $V = \mathcal{O}_Y(L)$ does not underly a variation of Hodge structures, and the “moduli part” does not come from the moduli
of Hodge structures. If the generic fiber has trivial canonical bundle, then the fiber space $f$ becomes isotrivial by the local Torelli theorem, and the theorem is clear. But the theorem works more generally, for example in the case where the geometric genus of the generic fiber is equal to one and the moduli moves without changing the periods.

We denote by $\sim$ the linear equivalence of divisors while $\sim_{\mathbb{Q}}$ the $\mathbb{Q}$-linear equivalence. We work over $\mathbb{C}$.

2. Preliminaries

2.1. semipositivity. We recall the semi-positivity theorem:

**Theorem 2.1** (Fujita decomposition [4]). Let $f : X \to Y$ be a proper surjective morphism with connected geometric fibers from a compact Kähler manifold to a smooth projective variety. Then there is an orthogonal direct sum decomposition of the direct image sheaf $f_*\omega_{X/Y} = U \oplus W$ with respect to the natural $L^2$ hermitian metric, where $U$ is a unitary flat locally free sheaf and $W$ is a generically ample torsion free sheaf, a sheaf whose restriction to the generic curve section is ample.

The flat connection on the summand $U$ is the Gauss-Manin connection when restricted to an open subset of $Y$ over which $f$ is smooth. There is a monodromy representation $\rho : \pi_1(Y, y_0) \to \text{Aut}(V_{y_0})$ for any indecomposable direct summand $V$ of $U$, where $y_0 \in Y$ is an arbitrarily fixed base point. We note that the flat connection extends over whole $Y$. Catanese and Dettweiler ([2], [3]) proved that the images of $\rho$ are not necessarily finite if the rank of $V$ is larger than 1 in a series of examples (cf. Subsection 2.3). On the other hand, we will prove in this paper that the image of $\rho$ is always finite if rank $V = 1$.

2.2. moduli part. In [10], we defined the moduli part of the canonical bundle formula, which is a generalization of Kodaira’s canonical bundle formula ([12]) for elliptic surfaces (cf. [9]):

**Theorem 2.2** ([10] Theorem 2). Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected geometric fibers. Let $P = \sum P_j$ and $Q = \sum Q_l$ be normal crossing divisors on $X$ and $Y$ respectively such that $f^{-1}(Q) \subset P$. Let $D = \sum d_j P_j$ be a $\mathbb{Q}$-divisor such that $d_j < 1$ for all $j$. Let $\Delta = \sum \delta_l Q_l$ be the smallest $\mathbb{Q}$-divisor on $Y$ such that

$$P - D \geq f^*(Q - \Delta).$$

Assume the following conditions:

1. $f : X \to Y$ and its restrictions $f : P^{[m]} \to Y$ for all $m > 0$ are smooth over $Y \setminus Q$, where $P^{[m]}$ is the disjoint union of all the intersections of distinct $m$ irreducible components $P_j$ which are mapped by $f$ onto $Y$. 

2. $f$ is a proper surjective morphism with connected geometric fibers from a compact Kähler manifold to a smooth projective variety.

3. $f$ is smooth outside $Q$.

4. $f$ is a finite morphism over $Y \setminus Q$.

5. $f$ is a branched cover over $Y \setminus Q$.

6. $f$ is a locally trivial fibration over $Y \setminus Q$. 

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40. $f$ is a locally trivial fibration over $Y \setminus Q$.
(2) The natural homomorphism $O_Y \to f_*O_X(R-D)$ is surjective at the generic point of $Y$.

(3) $K_X + D \sim Q f^*(K_Y + \Delta + L)$ for some $Q$-divisor $L$ on $Y$.

Then $L$ is nef.

A surjective morphism $f : X \to Y$ between smooth projective varieties with connected geometric fibers is called an algebraic fiber space. $L$ is called the moduli part of the algebraic fiber space $f$.

**Remark 2.3.** (1) $\delta_l < 1$ for all $l$ because $d_j < 1$ for all $j$.

(2) Let $f : X \to Y$ be a surjective morphism between projective varieties with connected geometric fibers such that $(\bar{X}, B)$ is a KLT pair for some $Q$-divisor $B$ on $\bar{X}$ and such that $K_X + B \sim Q f^* M$ for some $Q$-Cartier divisor $M$ on $Y$ as in [1]. Then there are suitable log resolutions $r_1 : X \to \bar{X}$, $r_2 : Y \to \bar{Y}$ and $P, Q, D$ with a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{r_1} & \bar{X} \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{r_2} & \bar{Y}
\end{array}
$$

such that the conditions of the above theorem are satisfied. Conversely, if the minimal model exists and the abundance conjecture is true, then the situation of the theorem is reduced to this case provided that the Kodaira dimension of the generic fiber is 0.

(3) $D$ may have irreducible components with negative coefficients. The coefficients of the vertical irreducible components are modified by $\Delta$. The moduli part $L$ does not depend on the vertical part of $D$ in the following sense. If we replace $D$ by $D - f^* E$ for an effective $Q$-divisor $E$ on $Y$ supported on $Q$, then $\Delta$ is replaced by $\Delta - E$, and the moduli part $L$ does not change.

On the other hand, the horizontal irreducible components of $D$ with negative coefficients are restricted by the condition (2). This condition means roughly that the geometric genus of the generic fiber is 1. We do not require the minimality of the generic fiber, i.e., the numerical triviality of the log canonical divisor. We note also that the Kodaira dimension of the generic fiber can be larger than 0.

We claim that the moduli part comes from the variations of generic fibers of $f$ in the following theorem, which is a consequence of the fact that the formation of Deligne’s canonical extension is compatible with the base change if the local monodromies are unipotent:

**Theorem 2.4** ([1] Proposition 3.1). Let $f : X \to Y$ and $f' : X' \to Y'$ be algebraic fiber spaces with normal crossing divisors $P, P', Q, Q'$ as in Theorem 2.2. Assume the following conditions:
(1) There is a commutative diagram of morphisms

\[
\begin{array}{ccc}
X' & \xrightarrow{p} & X \\
| f' & \downarrow & | \\
Y' & \xrightarrow{q} & Y.
\end{array}
\]

(2) \(p^{-1}(P) = P'\) and \(q^{-1}(Q) = Q'\) set theoretically.

(3) \(p\) and \(q\) are smooth over \(X \setminus f^{-1}(Q)\) and \(Y \setminus Q\) respectively, and

\[
X' \setminus (f')^{-1}(Q') \cong X \setminus f^{-1}Q \times_{Y \setminus Q} Y' \setminus Q'.
\]

Then the following hold:

(a) Let \(D\) be a \(Q\) divisor on \(X\) satisfying the conditions of Theorem 2.2. Define a \(Q\)-divisor \(D'\) on \(X'\) by \(p^*(K_X + D) = K_{X'} + D'\). Then \(D'\) satisfies the conditions of Theorem 2.2.

(b) Let \(L\) and \(L'\) be the moduli parts of \(f\) and \(f'\) defined by using \(D\) and \(D'\) respectively. Then \(L' \sim_Q q^*L\).

2.3. Catanese-Dettweiler. We recall one of Catanese-Dettweiler’s examples for the comparison to the rank 1 case. They constructed a family of curves \(f : X \to Y\) such that there is a rank 2 unitary flat subbundle of \(f^*\omega_{X/Y}\) whose monodromy representation has infinite image [2].

Let \(f_1 : X_1 \to Y_1 \cong \mathbb{P}^1\) be a family of curves of genus 6 defined by an equation \(z^7 = y(y-1)(y-x)^4\), where \(x\) is the inhomogeneous coordinates on \(Y_1\). Let \(C = C_x\) be a generic fiber of \(f_1\). Then there is a morphism \(\pi : C \to \mathbb{P}^1\) of degree 7 which ramifies at 0, 1, \(\infty, x\). We have \(K_C \sim \pi^*(-2P + \sum P_i)\) for points \(P, P_0, P_1, P_\infty, P_x \in \mathbb{P}^1\), hence \(g(C) = 6\). We have \(P_0 + P_1 + P_\infty + 4P_x \sim 7P\) on \(\mathbb{P}^1\). We have a direct sum decomposition with respect to the characters of the Galois group \(\mathbb{Z}/(7)\) of \(\pi\):

\[
\pi_* \mathcal{O}_C = \bigoplus_{i=0}^{6} \mathcal{O}_{\mathbb{P}^1}(-i + \left\lfloor 4i/7 \right\rfloor).
\]

By duality,

\[
\pi_* \omega_C = \bigoplus_{i=0}^{6} \omega_{\mathbb{P}^1}(i - \left\lfloor 4i/7 \right\rfloor) = \mathcal{O}_{\mathbb{P}^1}(-2) + \mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(1) + \mathcal{O}_{\mathbb{P}^1}(1).
\]

Then we have \(\dim H^{1,0}(C)_i = 0, 0, 0, 1, 1, 2, 2\) for \(i = 0, 1, 2, 3, 4, 5, 6\), where the subscript \(i\) indicates the corresponding eigensubspace of \(H^{1,0}(C)_i\). We set \(V_i = H^{1,0}(C)_i\).

The direct image sheaf of the constant sheaf also decomposes according to the characters:

\[
\pi_* \mathcal{C} = \bigoplus_{i=0}^{6} \mathcal{L}_i
\]
where $L_i$ is the direct image sheaf of a local system on $\mathbb{P}^1 \setminus \{0, 1, x, \infty\}$ by an open immersion $\mathbb{P}^1 \setminus \{0, 1, x, \infty\} \to \mathbb{P}^1$. From the Hodge decomposition, we obtain

$$H^1(\mathbb{P}^1, L_i) = V_i \oplus \bar{V}_{7-i}.$$ 

Since $V_1 = V_2 = 0$, we have $V_i = H^1(\mathbb{P}^1, L_i)$ for $i = 5, 6$.

If $x \in \mathbb{Y} \setminus \{0, 1, \infty\}$ moves, then $V_5$ and $V_6$ become flat sheaves of rank 2. They correspond to ordinary differential equations of order 2 on $\mathbb{Y} \sim = \mathbb{P}^1$ with only 3 regular singular points, i.e., a hypergeometric differential equation

$$z(1-z)\frac{d^2 w}{dz^2} + [c - (a + b + 1)z] \frac{dw}{dz} - abw = 0$$

for some $a, b, c$. The differential equations for which the monodromies are finite are classified, and we know that our sheaves $V_5$ and $V_6$ have infinite monodromies. If we make a suitable base change $\mathbb{Y} \to \mathbb{Y}$ to make the local monodromies around $\{0, 1, \infty\}$ unipotent, then we obtain flat bundles of rank 2 on $\mathbb{Y}$ with infinite monodromies. We note that the flat bundles $V_i$ are not variations of Hodge structures.

### 2.4. Deligne

We recall a theorem of Deligne:

**Theorem 2.5** ([5] Proposition 2.1). Let $f : X^o \to Y^o$ be a smooth projective morphism of complex varieties. Then $Rf_*Q \cong \sum_i R^i f_*Q[-i] \in$ the derived category of bounded complexes $D^b(Y^o)$.

We apply Theorem 2.5 to the case $Y^o = Y \setminus Q$ and $X^o = X \setminus f^{-1}(Q)$ in the situation of Theorem 2.2:

$$Rf_*Q_{X^o} \cong \sum_i R^i f_*Q_{X^o}[-i]$$

By tensoring with $V = \mathcal{O}_Y(L)$, we obtain

$$Rf_*C_{X^o} \otimes V \cong \sum_i R^i f_*C_{X^o} \otimes V[-i].$$

Then

$$H^m(X^o, f^*V) \cong \bigoplus_{p+q=m} H^q(Y^o, R^p f_* f^*V).$$

### 2.5. Simpson

We recall a theory of Simpson on the rank one flat bundles [14]. Let $X$ be a smooth projective variety, and let $M$ be the moduli space of local systems of complex vector spaces of rank 1. The real analytic group $M$ has 3 different structures of algebraic varieties $M_B, M_{DR}$ and $M_{Dol}$:

$$M_B = \text{Hom}(\pi_1(X), \mathbb{C}^*)$$

$$M_{DR} = \{ (L, \nabla) \mid \text{line bundle and algebraic integrable connection} \}$$

$$M_{Dol} = \{ (E, \phi) \mid \text{line bundle with torsion Chern class and } \phi \in H^0(\Omega^1_X) \}.$$ 

The cohomology group $H^p(X, v)$ for $v \in M$ has three different realizations in the following way. $H^p_B(X, v)$ is the singular cohomology with coefficients in $v$, $H_{DR}(X, v)$ is the hypercohomology of the de Rham complex $\Omega^\bullet_X \otimes L$ where
the differentials are given by $\nabla$, and $H_{Dol}(X, v)$ is the hypercohomology of the Dolbeault complex $\Omega^*_X \otimes L$ where the differentials are given by the cup products with $\phi$. It is proved in [6] and [13] that these cohomologies are isomorphic.

A subset $S$ of $M$ is called a triple torus if it is an irreducible algebraic subgroup with respect to the 3 algebraic structures. The dimension of cohomology group $H^p(X, v)$ is an upper semi-continuous function on $v$.

**Theorem 2.6** ([14]). The jumping locus of $\dim H^p(X, v)$ is a finite union of torsion translates of triple tori.

### 3. Proof of theorems

**Proof of Theorem 1.1.** There is a finite covering $\pi: Y' \to Y$ from a smooth projective variety $Y$ such that the induced algebraic fiber space $f': X' \to Y'$ has unipotent local monodromies. We have a direct summand $\pi^* V$ of $f'^* \omega_{X/Y}$, which is a torsion if and only if $V$ is a torsion. Therefore we may assume from the beginning that the local monodromies are unipotent.

The direct summand $V = O_Y(L) \subset f_* \omega_{X/Y}$ is a flat subsheaf with respect to the Gauss-Manin connection. Then the restriction $V|_{Y_0}$ is a flat subbundle of $R^nf_0^*C$. Hence $\dim H^0(Y_0, R^nf_0^*C \otimes v)$ jumps at an isolated point $v = -V$. Then so does $\dim H^n(X_0, f^*v)$ by Theorem 2.5.

Let $E = \bigcup P_j$ be the sum of all irreducible components $P_j$ such that $f(P_j) \neq Y$. Then we have $X_0 = X \setminus E$. Let $E^{[m]}$ be the disjoint union of all the intersections of distinct $m$ irreducible components of $E$. Thus $E^{[1]} = \bigsqcup P_j$ and $E^{[2]} = \bigsqcup (P_{j1} \cap P_{j2})$ and so on. We put $E^{[0]} = X$. Then we have a spectral sequence

$$E_1^{p,q} = H^{2p+q}(X^{[-p]}, f^*v) \Rightarrow H^{p+q}(X_0, f^*v).$$

We have the following version of the upper semi-continuity theorem:

**Theorem 3.1.** Let $f: X \to Y$ be a projective morphism of noetherian schemes, and let $F^\bullet$ be a bounded complex of coherent sheaves on $X$ such that each $F^i$ is flat over $Y$. Then for each $p$, the dimension of the hypercohomology $\dim_{(y)} H^p(X_y, F^\bullet_y)$ is an upper semicontinuous function on $Y$.

**Proof.** A parallel proof to [8] Theorem 12.8 works. The bounded complex $F^\bullet$ is replaced by its Cech resolution $C^\bullet$ which is again bounded and flat as in loc. cit. Proposition 12.2. The rest of the proof is the same. $\square$

In the proof of [11] Theorem 6, the Betti, De Rham and Higgs realizations of the cohomologies $H^p(X_0, f^*v)$ are also described as hyper-cohomologies of complexes which are flat over the moduli space $M$. Therefore their dimensions are upper semi-continuous, and the jumping loci are algebraically defined. Thus the jumping locus of $\dim H^n(X_0, f^*v)$ is a union of torsion translates of triple tori by [14] Theorem 4.2. By summing up, we conclude that $V$ is a torsion. $\square$
Proof of Theorem 1.2. The proof of [10] Theorem 2 says that the pull-back of a positive multiple $m'L$ of the moduli part by a finite surjective morphism corresponds to an invertible sheaf which is a direct summand of the direct image sheaf of the relative canonical sheaf. Therefore $L$ is torsion. □

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