The normalized Laplacian and related indexes of graphs with edges blew up by cliques

Qi Ma, Zemin Jin*
Department of Mathematics, Zhejiang Normal University
Jinhua 321004, P.R. China

Abstract
In this paper, we introduce the clique-blew up graph $CL(G)$ of a given graph $G$, which is obtained from $G$ by replacing each edge of $G$ with a complete graph $K_n$. We characterize all the normalized Laplacian spectrum of the graph $CL(G)$ in terms of the given graph $G$. Based on the spectrum obtained, the formulae to calculate the multiplicative degree-Kirchhoff index, the Kemeny’s constant and the number of spanning trees of $CL(G)$ are derived well. Finally, the spectrum and indexes of the clique-blew up iterative graphs are present.

Key Words: adjacent matrix; normalized Laplacian; multiplicative degree-Kirchhoff index; Kemeny’s constant; spanning tree.

AMS subject classification (2010): 05C50, 05C76.

1 Introduction

1.1 Notions and definitions
We consider a simple and connected graph $G = (V(G), E(G))$ with $n$ vertices and denote the vertex set of $G$ by $V(G) = \{1, 2, \cdots, n\}$. For any two adjacent vertices $i$ and $j$, we denote it by $i \sim j$. Denote the degree of a vertex $i$ by $d_i$ in $G$. Let $A_G$ be the adjacency matrix of $G$, where the $(i, j)$-entry equals to 1 if $i \sim j$ and 0 otherwise. Clearly $A_G$ is an $n \times n$ matrix. Let $D_G = \text{diag}(d_1, d_2, \cdots, d_n)$ is the diagonal matrix of vertex degrees of $G$, where $d_i$ is the degree of $i$ in $G$. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of $G$. Given a matrix $M$, let $M(i, j)$ denote the $(i, j)$-entry of $M$. For the eigenvalue $\lambda$ of the matrix $M$, denote by $m_M(\lambda)$ the multiplicity of $\lambda$ in $M$.

Given a graph $G$, one can always define the random walk on $G$ as a Markov chain $X_n, n \geq 0$. The probability of jumping from the current vertex $i$ to another vertex $j$ is $p_{ij}$, where $p_{ij} = \frac{1}{d_i}$ if $i \sim j$ are adjacent and $p_{ij} = 0$ otherwise, i.e.,

$$p_{ij} = \begin{cases} \frac{1}{d_i}, & \text{if } i \sim j, \\ 0, & \text{otherwise}. \end{cases}$$

The matrix $P_G = (p_{ij})_{n \times n}$ is the transition probability matrix for the random walk defined on $G$. It is clear that $P_G = D_G^{-1}A_G$. The normalized Laplacian matrix of the graph $G$ is defined to be

$$L_G = I - D_G^{\frac{1}{2}}P_GD_G^{-\frac{1}{2}},$$

*Corresponding author. Email: silvester.ma@outlook.com (Ma), zeminjin@zjnu.cn (Jin)
where $I$ is an $n \times n$ identity matrix. Let $\delta_{ij}$ be the Kronecker delta, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. According to the definition of $L_G$, we have that:

$$L_G(i, j) = \delta_{ij} - \frac{A_G(i, j)}{\sqrt{d_i d_j}}.$$ 

The eigenvalues of $L_G$ are non-negative because $L_G$ is Hermitian to $I - P_G = D_G^{-1}L_G$. For the $n$ eigenvalues of $L_G$, we label them by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Define the normalized Laplacian spectrum on $L_G$ of the graph $G$ as $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

Often the normalized Laplacian spectrum of graphs can be used to characterize parameters of graphs, see [6]. Recently, one of very interesting applications of the spectrum of graphs is to study the the electric network. Klein and Randić [19] proposed a new distance function called resistance distance between two vertices in graphs. Assume that there is a unit resistor on every edge of the graph $G$. When we attach a battery at two vertices $i$ and $j$, the resistance distance between $i$ and $j$, denoted by $r_{ij}$, is the electrical resistance between $i$ and $j$ in $G$. For more recent results about resistance distances, one can refer to [20, 27]. Chen and Zhang [5] proposed a new index called the multiplicative degree-Kirchhoff index (see [9]) which is defined as $K^{f^*}(G) = \sum_{i<j} d_i d_j r_{ij}$. There is a close relationship between the multiplicative degree-Kirchhoff index and the normalized spectrum. In recent years, more and more results relating to the normalized Laplacian spectrum and the multiplicative degree-Kirchhoff index of some graphs have been obtained, see [3, 9, 10, 13, 16, 17, 24, 29, 30].

### 1.2 Backgrounds

Many graph invariants, including the multiplicative degree-Kirchhoff index, the Kemeny’s constant, the number of spanning trees, can be calculated in term of the spectrum of the graph. In recent years, some researchers focused on blowing up all the edges of a given graph by replacing each edge with some another graph. The spectrum of the resulting graph always can be characterized in term of the given graph.

Xie et al. [29] initially replaced each edge of a graph $G$ with a triangle. They added a parallel path of lengths two between each two adjacent vertices. The spectrum of the normalized Laplacian of the new graph are characterized in term of $G$. Later, Wang et al. [28] generalized the result of [29] by replacing each edge with $k$ triangles, i.e., they added $k$ edge-disjoint paths of length two between each two adjacent two vertices. Li and Hou [21] blew up each edge of $G$ to a 4-cycle by adding a new path of length three between each two adjacent vertices. The resulting graph is called the quadrilateral graph $Q(G)$. Huang and Li [14] further added $k$ paths of length three between each two adjacent vertices to get the so-called $k$-quadrilateral graph $Q^k(G)$ of $G$. Luckily, the normalized Laplacian spectra of these resulting graphs can be characterized completely in term of the initial graph $G$. As applications, one can calculate the multiplicative degree-Kirchhoff index, the Kemeny’s constant and the number of spanning trees of of these graphs again in term of the initial graph.

Pan et al. [26] introduced an analogue method to replace the edges of a graph. They added a triangle or a 4-cycle between each two adjacent vertices and connected these
vertices in a suitable way. The subdivision graph was considered in [30]. More ideas to blow up the edges of a given graph were studied in [8, 13]. The authors [11, 15, 22, 25] considered the graph chains, which are obtained by replacing only one edge of the given graph iteratively with some special structures. In addition to the spectra of the obtained graphs above, the authors [4, 12, 28, 31] studied the hitting times of the random walks on these graphs.

2 Preliminaries

Throughout all the paper, let \( n \geq 3 \) and \( G \) be a simple and connected graph with \( N_0 \) vertices and \( E_0 \) edges. For any edge \( e \), we add \( n - 2 \) vertices, \( k^e_i, i = 1, 2, \cdots, n - 2 \), so that all these vertices together with the end-vertices of \( e \) form a \( K_n \). The resulting graph is called the clique-blew up graph and written by \( CL(G) \). The Figure 1 gives an example of the clique-blew up graph for \( G = K_3 \) and \( n = 5 \).

![Figure 1: The graph \( G = K_3 \) and its clique-blew up graph \( CL(G) \) for \( n = 5 \).](image)

Let \( V_N \) be the set of all the newly added vertices in \( CL(G) \) and \( V_O \) be the set of the vertices inherited from \( G \). That is, the vertex set \( V(CL(G)) \) of \( CL(G) \) is the union of \( V_N \) and \( V_O \). We denote by \( N_1 \) the total number of vertices and \( E_1 \) the total number of edges of \( CL(G) \). It is clear that \( E_1 = \frac{n(n-1)E_0}{2} \) and \( N_1 = N_0 + (n-2)E_0 \).

**Lemma 2.1** [6] Let \( G \) be a connected graph with \( n \) vertices, and \( \mathcal{L}_G \) be the normalized Laplacian matrix of \( G \). The normalized Laplacian spectrum of \( G \) is \( \sigma = \{ 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \} \). We have

(i) \( \frac{2m}{n} \leq \lambda_n \leq 2 \) with \( \lambda_n = 2 \) if and only if \( G \) is bipartite;

(ii) If \( G \) is bipartite, then for any eigenvalue \( \lambda_i \) of \( \mathcal{L}_G \), \( 2 - \lambda_i \) is also an eigenvalue of \( \mathcal{L}_G \) and \( m_{\mathcal{L}_G}(\lambda_i) = m_{\mathcal{L}_G}(2 - \lambda_i) \).

By determining the spectrum on the normalized Laplacian of \( G \), the specific calculation formulae of the multiplicative degree-Kirchhoff index, the Kmeneny’s constant and the number of spanning trees of graph \( G \) can be listed as follows.

**Lemma 2.2** Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges, \( \sigma = \{ 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \} \) is the spectrum on the normalized Laplacian \( \mathcal{L}_G \) of \( G \). Then

(i) [5] The multiplicative degree-Kirchhoff index of \( G \) is \( Kf^{*}(G) = 2m \sum_{i=2}^{n} \frac{1}{\lambda_i} \).

(ii) [2] The Kemeny’s constant of \( G \) is \( K_e(G) = \sum_{i=2}^{n} \frac{1}{\lambda_i} \).

(iii) [6] The number \( \tau(G) \) of spanning trees of \( G \) is \( \tau(G) = \frac{1}{2m} \prod_{i=1}^{n} d_i \cdot \prod_{k=2}^{n} \lambda_k \).

(iv) From (i) and (ii), we have \( Kf^{*}(G) = 2mK_e(G) \).
3 The normalized Laplacian spectrum of $CL(G)$

For the clique-blew up graph $CL(G)$ of $G$, the normalized Laplacian of $CL(G)$ is denoted by $L_C$. Denote the degree of the vertex $i \in V(CL(G))$ by $d_i'$. Let $A_C$ be the adjacency matrix of $CL(G)$ and $D_C$ be the degree matrix of $CL(G)$. Let $N_G = D_G^{-\frac{1}{2}} A_G D_G^{-\frac{1}{2}}$ and $N_C = D_C^{-\frac{1}{2}} A_C D_C^{-\frac{1}{2}}$. For the incidence matrix of a connected graph, we have the following result.

**Lemma 3.1** [7] Let $B$ be the incidence matrix of a connected graph $G$ with $n$ vertices. Then

$$\text{rank}(B) = \begin{cases} n - 1, & \text{if } G \text{ is bipartite,} \\ n, & \text{if } G \text{ is non-bipartite.} \end{cases}$$

At first, we consider the eigenvalue and its eigenvector in the graph $CL(G)$. Let $v = (v_1, v_2, \ldots, v_N)^T$ be an eigenvector with respect to the eigenvalue $\lambda$ of $L_C$, i.e.,

$$L_C v = (I - N_C) v = \lambda v. \tag{1}$$

For any vertex $u \in V(CL(G))$, the Eq. (1) indicates that

$$(1 - \lambda) v_u = \sum_{p=1}^{N_1} N_C(u, p) v_p = \sum_{p=1}^{N_1} \frac{A_C(u, p)}{\sqrt{d_u' d_p'}} v_p. \tag{2}$$

For any vertex $i \in V_G$, denote by $N_O(i)$ the set of neighbors of $i$ in $G$. Let $e$ be an edge with end vertices $i$ and $j$ in $G$. By the construction of $CL(G)$ and Eq. (2), we have

$$(1 - \lambda) v_i = \sum_{j \in N_O(i)} \frac{v_j}{\sqrt{d_i' d_j'}} + \sum_{e \in E(G)} \frac{v_e}{\sqrt{d_i' d_j'}}$$

$$= \sum_{j \in N_O(i)} \frac{v_j}{(n - 1) \sqrt{d_id_j}} + \sum_{e \in E(G)} \frac{v_e}{n - 1} \sum_{i=1}^{n-2} \frac{v_{k_i e}}{\sqrt{d_i' d_k_i}} \tag{3}$$

Similarly, for the new vertices $v_{k_1 e}$ and $v_{k_2 e}$ corresponding to the edge $e \in E(G)$ with end vertices $i$ and $j$, we have

$$(1 - \lambda) v_{k_1 e} = \frac{v_i}{\sqrt{d_i' d_{k_1 e}}} + \frac{v_j}{\sqrt{d_j' d_{k_1 e}}} + \frac{v_{k_2 e}}{\sqrt{d_{k_2 e} d_{k_1 e}}} + \frac{v_{k_3 e}}{\sqrt{d_{k_3 e} d_{k_2 e}}} + \cdots + \frac{v_{k_{n-2} e}}{\sqrt{d_{k_{n-2} e} d_{k_{n-3} e}}} \tag{4}$$

$$(1 - \lambda) v_{k_2 e} = \frac{v_i}{\sqrt{d_i' d_{k_2 e}}} + \frac{v_j}{\sqrt{d_j' d_{k_2 e}}} + \frac{v_{k_1 e}}{\sqrt{d_{k_1 e} d_{k_2 e}}} + \frac{v_{k_3 e}}{\sqrt{d_{k_3 e} d_{k_1 e}}} + \cdots + \frac{v_{k_{n-2} e}}{\sqrt{d_{k_{n-2} e} d_{k_{n-3} e}}} \tag{5}$$

The following lemma shows the relationship between the normalized Laplacian eigenvalues of $CL(G)$ and $G$. 

4
Lemma 3.2 Let $\lambda$ be an eigenvalue of $L_C$ such that $\lambda \neq \frac{n}{n-1}$ and $\frac{2}{n-1}$. Then $(n-1)\lambda$ is an eigenvalue of $L_G$ with $m_{L_G}(\lambda) = m_{L_C}((n-1)\lambda)$.

Proof: Let $v = (v_1, v_2, \ldots, v_N)^T$ be an eigenvector with respect to the eigenvalue $\lambda$ of $L_C$. Let $e \in E(G)$ with end vertices $i$ and $j$. Since $\lambda \neq \frac{n}{n-1}$, from Eqs. (4) and (5), we have $v_k^i = v_k^j$. For the same reason, we can easily get

$$v_k^i = v_k^j = \cdots = v_k^{n-2}. \quad (6)$$

For convenience, let $v_k^i = x_e$. Substituting Eq. (6) into Eqs. (3) and (4), we have

$$n - 2 \sum_{i = 0}^{n-2} v_i^j = (n - 1)(2 - n\lambda + \lambda) v_i^j + \sum_{i = 0}^{n-2} v_i^j \sqrt{d_i d_j} \quad (7)$$

Combining Eqs. (7) and (8), for $\lambda \neq \frac{n}{n-1}$ and $\frac{2}{n-1}$, it follows that

$$(1 - \lambda)v_i = \left(\frac{n - 2}{(n - 1)(2 - n\lambda + \lambda)}\right) v_i + \sum_{j \in N_G(i)} w_j \frac{n - n\lambda + \lambda}{(n - 1)(2 - n\lambda + \lambda)} \sqrt{d_i d_j} v_j, \quad (8)$$

i.e.,

$$(1 - (n-1)\lambda)v_i = \sum_{j \in N_G(i)} \frac{w_j}{\sqrt{d_i d_j}}. \quad (9)$$

holds for $\lambda \neq \frac{n}{n-1}$ and $\frac{2}{n-1}$.

From Eq. (9), it is obvious that $1-(n-1)\lambda$ is an eigenvalue of the matrix $N_G$ for $\lambda \neq \frac{n}{n-1}$ and $\frac{2}{n-1}$. So for any eigenvalue $\lambda$ ($\lambda \neq \frac{n}{n-1}$ and $\frac{2}{n-1}$) and a corresponding eigenvector $v$ of $L_C$, $(n-1)\lambda$ and $(v_i^j)^T_{i \in V_G}$ are an eigenvalue and a corresponding eigenvector of $L_G$, respectively. This implies that $m_{L_G}((n-1)\lambda) \geq m_{L_C}(\lambda)$.

On the other hand, for any eigenvalue $(n-1)\lambda$ ($(n-1)\lambda \neq 0, 2$) and a corresponding eigenvector $(v_i^j)^T_{i \in V_G}$ of $L_G$, the value $\lambda$ is an eigenvalue of $L_C$. Also, the vector determined by $(v_i^j)^T_{i \in V_G}$ and Eq. (8) is a corresponding eigenvector for the eigenvalue $\lambda$ of $L_C$. Hence $m_{L_G}((n-1)\lambda) \leq m_{L_C}(\lambda)$. So we have that $m_{L_G}((n-1)\lambda) = m_{L_C}(\lambda)$. The proof is completed. \[\square\]

Now we give a complete representation about the normalized Laplacian eigenvalues and corresponding eigenvectors of $CL(G)$ as follows.

Theorem 3.3 Let $G$ be a simple connected graph with $N_0$ vertices and $E_0$ edges and $CL(G)$ be the clique-blew up graph of $G$. The normalized Laplacian spectrum of $CL(G)$ can be obtained as following

(i) The value 0 is an eigenvalue of $L_C$ with the multiplicity 1;

(ii) If $\lambda$ ($\lambda \neq 0, 2$) is an eigenvalue of $L_G$, then the value $\frac{\lambda}{n-1}$ is an eigenvalue of $L_C$ and $m_{L_C}(\frac{\lambda}{n-1}) = m_{L_G}(\lambda)$;

(iii) If $G$ is non-bipartite, then the value $\frac{2}{n-1}$ is an eigenvalue of $L_C$ with the multiplicity $E_0 - N_0$;

(iv) If $G$ is bipartite, then the value $\frac{2}{n-1}$ is an eigenvalue of $L_C$ with the multiplicity $E_0 - N_0 + 1$;

(v) The value $\frac{n}{n-1}$ is the eigenvalue of $L_C$ with the multiplicity $(n-3)E_0 + N_0$. 

5
Proof: (i) It is obvious from Lemma 2.1.
(ii) It follows from Lemma 3.2 that the statement holds obviously.

Since each eigenvalue $\lambda (\lambda \neq \frac{2}{n-1}, \frac{n}{n-1})$ of $\mathcal{L}_C$ and its multiplicity have been determined in the statements above, here we only need to consider the eigenvalue $\lambda \in \{\frac{2}{n-1}, \frac{n}{n-1}\}$.

Let $v = (v_1, v_2, \cdots, v_{N_1})^T$ be an eigenvector with respect to the eigenvalue $\lambda = \frac{2}{n-1}$ of $\mathcal{L}_C$. Let $i \in V_O$ and $e \in E(G)$ with end vertices $i$ and $j$. For $n \geq 3$, from Eqs. (10) and (15), we have $\frac{n-2}{n-1}(v_{k_1} - v_{k_2}) = 0$, that is to say, $v_{k_1} = v_{k_2}$. For the same reason, we can easily get

\[ v_{k_1} = v_{k_2} = \cdots = v_{k_{n-1}}. \tag{10} \]

For convenience, let $v_{k_i} = x_e$. Substituting Eq. (10) and $\lambda = \frac{2}{n-1}$ into Eqs. (3) and (4), we have that

\[ \frac{n - 3}{n - 1} v_i = \sum_{j \in N_O(i)} \frac{v_j}{(n - 1)\sqrt{d_id_j}} + \sum_{e \in E(G) \text{ is incident with } i} \frac{n - 2}{n - 1} \frac{x_e}{\sqrt{d_i}} \tag{11} \]

and

\[ \frac{v_i}{\sqrt{d_i}} = -\frac{v_j}{\sqrt{d_j}}. \tag{12} \]

(iii) Let $G$ be non-bipartite. Suppose that $C$ is an odd cycle in $G$ of length $l$ with its vertices $i_1, i_2, \ldots, i_l$ in turn. By Eq. (12), we have

\[ \frac{v_{i_1}}{\sqrt{d_{i_1}}} = -\frac{v_{i_2}}{\sqrt{d_{i_2}}} = \frac{v_{i_3}}{\sqrt{d_{i_3}}} = \cdots = \frac{v_{i_l}}{\sqrt{d_{i_l}}} = -\frac{v_{i_1}}{\sqrt{d_{i_1}}}, \]

which implies that $v_{i_k} = 0, k = 1, 2, \cdots, l$. Since $G$ is connected, it holds that

\[ v_i = 0 \text{ for all } i \in V_O. \tag{13} \]

Together with Eq. (11), we have that for $i \in V_O$

\[ \sum_{e \in E(G) \text{ is incident with } i} x_e = 0. \tag{14} \]

Therefore, the eigenvectors $v = (v_1, v_2, \ldots, v_{N_1})^T$ associated with $\lambda = \frac{2}{n-1}$ can be determined by Eqs. (10), (14) and (15). Notice that $v_{k_1} = v_{k_2} = \cdots = v_{k_{n-1}} = x_e$. Let $x = (x_e)^T$ which is an $E_0$ dimensional vector by the construction of $\mathcal{C}(G)$. It is easy to see that Eq. (14) is equivalent to the equation system $Bx = 0$, where $B$ is the incident matrix of $G$. By Lemma 3.3, for $Bx = 0$, the number of solutions in its basic solution system is $E_0 - N_0$ when $G$ is non-bipartite, i.e., $m_{\mathcal{L}_C}(\frac{2}{n-1}) = E_0 - N_0$.

(iv) Let $G$ be bipartite. Substituting Eq. (12) into Eq. (11), we have that for $i \in V_O$, 

\[ \sqrt{d_i}v_i = \sum_{e \in E(G) \text{ is incident with } i} x_e. \tag{15} \]

Let $\frac{v_i}{\sqrt{d_i}} = t$. Denote by $X$ and $Y$ the partite sets of the graph $G$ and without loss of generality, let $1 \in X$. Then from Eq. (12), we have that $\frac{v_i}{\sqrt{d_i}} = t$ if $i \in X$, and $\frac{v_i}{\sqrt{d_i}} = -t$ if...
\( i \in Y \). According to Eq. (15), we have that for each \( i \in V_O \),

\[
\sum_{e \in E(G)} x_e - d_i t = 0 \quad \text{if} \quad i \in X,
\]

\[
\sum_{e \in E(G)} x_e + d_i t = 0 \quad \text{if} \quad i \in Y.
\]

Therefore, the eigenvectors \( v = (v_1, v_2, \ldots, v_{N_t})^T \) associated with \( \lambda = \frac{\sqrt{2}}{n-1} \) can be determined by Eqs. (10) (12) and (16). Notice that \( v_{k_1^e} = v_{k_2^e} = \cdots = v_{k_{n-2}^e} = x_e \). Let \( \mathbf{x} = (x_e)^T \) which is an \( E_0 \) dimensional vector by the construction of \( CL(G) \).

For convenience, we assume that the first \( |X| \) rows of the incident matrix \( B \) of \( G \) correspond to the vertices of \( X \), and hence the matrix \( B \) can be written as \( B = \begin{bmatrix} B_X \\ B_Y \end{bmatrix} \). Let \( D_X \) and \( D_Y \) denote the volume vectors which consist of degree sequences of vertices of \( X \) and \( Y \), respectively. Let

\[
C = \begin{bmatrix} B_X \\ B_Y \\ -D_X \\ D_Y \end{bmatrix}.
\]

Hence Eqs. (10) (12) and (16) are equivalent to the equation system \( C(X) = 0 \).

By Lemma \ref{lem:rank}, the rank of \( B \) is \( N_0 - 1 \) when \( G \) is bipartite. Now we need to determine the rank of \( C \). We denote the volume vectors of \( C \) by \( e_1, e_2, \ldots, e_{E_0}, e_0 \) from left to right. Assume that \( e_0 \) is linearly related to the \( e_1, e_2, \ldots, e_{E_0} \), it means that, there exist constants \( c_1, c_2, \ldots, e_{E_0} \) making the following formula true,

\[
e_0 = c_1 e_1 + c_2 e_2 + \cdots + c_{E_0} e_{E_0}.
\]

For every volume of \( C \), there are two entries 1 in \( B_X \) and \( B_Y \), respectively. From Eq. (17), by summing all the first \( |X| \) entries in \( e_0 \), we have \( c_1 + c_2 + \cdots + c_{E_0} = \sum_{i=1}^{N_0} (-d_i) \). For the same reason, we can get \( c_1 + c_2 + \cdots + c_{E_0} = \sum_{i=1}^{N_0} d_i \). This implies that \( \sum_{i=1}^{|X|+1} (-d_i) = \sum_{i=1}^{N_0} d_i \).

Notice that \( d_i > 0 \) for each \( i = 1, 2, \ldots, N_0 \). Hence it is obvious that \( \sum_{i=1}^{|X|+1} (-d_i) = \sum_{i=1}^{N_0} d_i \) is impossible. Thus we get a contradiction. So \( e_0 \) and \( e_1, e_2, \ldots, e_{E_0} \) are linearly independent, i.e., the rank of matrix \( C \) is \( r(C) = r(B) + 1 = N_0 \).

Therefore, the number of solutions in basic solution system of \( C(X) = 0 \) is \( E_0 - N_0 + 1 \) when \( G \) is bipartite, i.e., \( m_{L_C}(\frac{2}{n-1}) = E_0 - N_0 + 1 \).

(v) Substituting \( \lambda = \frac{n}{n-1} \) into Eq. (4), we have

\[
v_{k_1^e} + v_{k_2^e} + v_{k_3^e} + \cdots + v_{k_{n-2}^e} + \frac{v_i}{\sqrt{d_i}} + \frac{v_j}{\sqrt{d_j}} = 0.
\]

For convenience, for each edge \( e_s \in E(G) \), \( s = 1, 2, \ldots, E_0 \), denote by \( i_s \) and \( j_s \) the end
vertices of $e_s$. So, we have the following linear equation system

\[
\begin{align*}
    v_1^{e_1} + v_2^{e_1} + v_3^{e_1} + \cdots + v_{n-2}^{e_1} + \frac{v_1^{e_1}}{d_1^{e_1}} + \frac{v_j^{e_1}}{d_j^{e_1}} &= 0, \\
v_1^{e_2} + v_2^{e_2} + v_3^{e_2} + \cdots + v_{n-2}^{e_2} + \frac{v_1^{e_2}}{d_1^{e_2}} + \frac{v_j^{e_2}}{d_j^{e_2}} &= 0, \\
\vdots \\
v_1^{e_0} + v_2^{e_0} + v_3^{e_0} + \cdots + v_{n-2}^{e_0} + \frac{v_1^{e_0}}{d_1^{e_0}} + \frac{v_j^{e_0}}{d_j^{e_0}} &= 0.
\end{align*}
\]  

(18)

The corresponding coefficient matrix contains the following $E_0 \times (n-2)E_0$ submatrix

\[
\begin{pmatrix}
    1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\
    0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\
    \vdots \\
    0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1
\end{pmatrix}
\]

Clearly, the submatrix above is of rank $E_0$. Hence the number of solutions in a basic solution system of the system (18) is $(n-3)E_0 + N_0$. Therefore, $m_{\mathcal{E}_r}(\frac{n}{n-1}) = (n-3)E_0 + N_0$. This completes the proof of the theorem. \hfill \Box

4 Related indexes and clique-blew up iterative graph

Let $CL_0(G) = G$ and $CL_r(G) = CL(CL_{r-1}(G))$ for $r \geq 1$. The graph $CL_r(G)$ is called the $r$-th clique-blew up iterative graph of $G$. The number of vertices and edges of $CL_r(G)$, $r \geq 0$, are denoted by $N_r$ and $E_r$, respectively. From the iterative method of the clique-blew up graph, we have

\[
E_r = \frac{n(n-1)E_{r-1}}{2} \quad \text{and} \quad N_r = N_{r-1} + (n-2)E_{r-1}.
\]

Hence

\[
E_r = \frac{n^r \cdot (n-1)^r E_0}{2^r} \quad \text{and} \quad N_r = N_0 + \frac{2E_0\left(\frac{n^r \cdot (n-1)^r}{2^r} - 1\right)}{n + 1}.
\]  

(19)

For convenience, denote by $L_r$ the normalized Laplacian of $CL_r(G)$ for $r \geq 0$. Denote by $\sigma_r$ the normalized Laplacian spectrum of $CL_r(G)$ for $r \geq 0$. From Theorem 3.3, we have the following theorem.

**Theorem 4.1** Let $G$ be a simple connected graph. For $r \geq 2$ and $n \geq 3$,

\[
    \sigma_r = \left\{ \frac{x}{n-1} \middle| x \in \sigma_{r-1} \setminus \{0\} \right\} \cup \left\{ 0, \frac{2}{n-1}, \ldots, \frac{n}{n-1} \right\},
\]

where $m_{\mathcal{E}_r}(\frac{x}{n-1}) = m_{\mathcal{E}_{r-1}}(x)$ for $x \in \sigma_{r-1} \setminus \{0\}$, $m_{\mathcal{E}_r}(0) = 1$, $m_{\mathcal{E}_r}(\frac{2}{n-1}) = E_{r-1} - N_{r-1}$ and $m_{\mathcal{E}_r}(\frac{n}{n-1}) = (n-3)E_{r-1} + N_{r-1}$. 

8
Theorem 4.2 Let $G$ be a simple connected graph. For $r \geq 1$ and $n \geq 3$, the multiplicative degree-Kirchhoff index $Kf^*(CL_r(G))$ of the $r$-clique-blew up graph $CL_r(G)$ can be determined by the multiplicative degree-Kirchhoff index $Kf^*(G)$ of the initial graph $G$ as follows

$$Kf^*(CL_r(G)) = \frac{n^r \cdot (n-1)^{2r}}{2r} Kf^*(G) - \frac{n^{r-1} \cdot (n-1)^{2r+1}}{2r}(1 - \frac{1}{(n-1)^r})E_0N_0$$

$$+ \frac{(n-1)^{2r} \cdot n^{r-1}}{2r-1} \left( \frac{3n^r}{2r} - 1 \right) \frac{n^r}{2r-1} + \frac{1}{n+1} \left( \frac{n^r}{2r-1} - n - 1 \right) E_0^2. \quad (20)$$

Proof: Recall the normalized Laplacian eigenvalues of $G$ is $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{N_0}$. Whether $G$ is bipartite or not, we have the following result by Theorem 3.3 and Lemma 2.2 (i)

$$Kf^*(CL_r(G)) = 2E_1 \left( \sum_{i=2}^{N_0} \frac{n-1}{\lambda_i} \frac{n-1}{2} (E_0 - N_0) + \frac{n-1}{n} ((n-3)E_0 + N_0) \right)$$

$$= \frac{1}{2} n(n-1)^2Kf^*(G) + \frac{3}{2} (n-1)^2(n-2)E_0^2 - \frac{1}{2} (n-1)^2(n-2)E_0N_0. \quad (21)$$

From Eqs. (19) and (21) and the definition of the $r$-th clique-blew up iterative graph, we can get

$$Kf^*(CL_r(G)) = \frac{1}{2} n(n-1)^2Kf^*(CL_{r-1}(G)) + \frac{3}{2} (n-1)^2(n-2)E_{r-1}^2$$

$$- \frac{1}{2} (n-1)^2(n-2)E_{r-1}N_{r-1}$$

$$= \frac{n^r \cdot (n-1)^{2r}}{2r} Kf^*(G) - \frac{n^{r-1} \cdot (n-1)^{2r+1}}{2r}(1 - \frac{1}{(n-1)^r})E_0N_0$$

$$+ \frac{(n-1)^{2r} \cdot n^{r-1}}{2r-1} \left( \frac{3n^r}{2r} - 1 \right) \frac{n^r}{2r-1} + \frac{1}{n+1} \left( \frac{n^r}{2r-1} - n - 1 \right) E_0^2. \quad (20)$$

The proof is completed. \hfill \square

Theorem 4.3 For $r \geq 1$ and $n \geq 3$, the Kemeny’s constant $K_e(CL_r(G))$ for the random walks on $CL_r(G)$ is as follows

$$K_e(CL_r(G)) = (n-1)^rK_e(G) + \frac{(n-1)^{r+1}}{2n}(\frac{1}{(n-1)^r} - 1)N_0 +$$

$$\left( \frac{3(n-1)^r}{n} \frac{n^r}{2r} - 1 \right) + \frac{(n-1)^{r+1}}{n+1} \left( 1 - \frac{n^{r-1}}{2r-1} + \frac{(n-1)^{r-1}}{n(n+1)} \left( 1 - \frac{1}{(n-1)^{r-1}} \right) \right) E_0.$$

Proof: By Lemma 2.2 (iv) and Eq. (21), it follows that

$$K_e(CL(G)) = \frac{1}{2E_1} Kf^*(CL(G))$$

$$= \frac{n-1}{2E_0} Kf^*(G) + \frac{3(n-1)(n-2)}{2n} E_0 - \frac{(n-1)(n-2)}{2n} N_0 \quad (22)$$

$$= (n-1)K_e(G) + \frac{3(n-1)(n-2)}{2n} E_0 - \frac{(n-1)(n-2)}{2n} N_0.$$

From Eqs. (19) and (22) and the definition of the $r$-th clique-blew up iterative graph, we
can get

$$K_e(CL_r(G)) = (n - 1)K_e(CL_{r-1}(G)) + \frac{3(n - 1)(n - 2)}{2n}E_{r-1} - \frac{(n - 1)(n - 2)}{2n}N_{r-1}$$

$$= (n - 1)^rK_e(G) + \frac{(n - 1)^{r+1}}{2n}\left(\frac{1}{(n - 1)^r} - 1\right)N_0 +$$

$$\left(\frac{3(n - 1)^r}{n}\left(\frac{n^r}{2^r} - 1\right) + \frac{(n - 1)^r}{n + 1}(1 - \frac{n^{r-1}}{2^{r-1}}) + \frac{(n - 1)^{r-1}}{n(n + 1)}(1 - \frac{1}{(n - 1)^{r-1}})\right)E_0.$$

The proof is completed.

**Theorem 4.4** For $r \geq 1$ and $n \geq 3$, the number of spanning trees of $CL_r(G)$ is as follows

$$\tau(CL_r(G)) = 2^{2E_0\alpha - rN_0 - \frac{2E_0}{n^{\alpha}}(2\alpha - r) + r} \cdot n^{2(n - 3)E_0\alpha + rN_0 + \frac{2E_0}{n^{\alpha}}(2\alpha - r) - r} \cdot \tau(G),$$

where $\alpha = \frac{n^{r - (n - 1)^r}}{n^{n - 2} - 1}$.

**Proof:** Let the normalized Laplacian eigenvalues of $CL(G)$ be $0 = \lambda_1' < \lambda_2' \leq \cdots \leq \lambda_{N_1}'$. Whether $G$ is bipartite or not, by Lemma 2.2 (iii) and the definition of $CL(G)$ we have

$$\frac{\tau(CL(G))}{\tau(G)} = \frac{2(n - 1)^{N_0 + (n - 2)E_0 - 1}}{n} \prod_{i=2}^{N_1} \lambda_i.$$  \hspace{1cm} (23)

From Theorem 3.3 we have

$$\prod_{i=2}^{N_1} \lambda_i' = \left(\frac{1}{n - 1}\right)^{N_0 - 1} \cdot \left(\frac{2}{n - 1}\right)^{E_0 - N_0} \cdot \left(\frac{n}{n - 1}\right)^{(n - 3)E_0 + N_0} \prod_{i=2}^{N_0} \lambda_i.$$  \hspace{1cm} (24)

By Eqs. (23) and (24), we have

$$\tau(CL(G)) = 2^{E_0 - N_0 + 1} \cdot n^{(n - 3)E_0 + N_0 - 1} \cdot \tau(G).$$

It follows from the recursive relation that

$$\tau(CL_r(G)) = 2^{E_{r-1} - N_{r-1} + 1} \cdot n^{(n - 3)E_{r-1} + N_{r-1} - 1} \cdot \tau(CL_{r-1}(G))$$

$$= 2^{\sum_{i=0}^{r-1} (E_i - N_i) + r} \cdot \sum_{i=0}^{r-1} (n - 3)E_i - N_i - r \cdot \tau(G)$$

$$= 2^{2E_0\alpha - rN_0 - \frac{2E_0}{n^{\alpha}}(2\alpha - r) + r} \cdot n^{2(n - 3)E_0\alpha + rN_0 + \frac{2E_0}{n^{\alpha}}(2\alpha - r) - r} \cdot \tau(G).$$

The proof is completed. 

**Acknowledgement** This work was supported by National Natural Science Foundation of China (11571320 and 11671366) and Zhejiang Provincial Natural Science Foundation (LY19A010018).
References

[1] A. Banerjee, R. Mehatari, On the normalized spectrum of threshold graphs, Linear Algebra Appl. 530 (2017) 288-304.

[2] S. Butler, Algebraic aspects of the normalized laplacian, in: A.Beveridge, J. Griggs, L. Hogben, G. Musiker, P. Tetali (Eds.), Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications. 159 (2016) 295-315.

[3] H. Chen, J. Jost, Minimum vertex covers and the spectrum of the normalized laplacian on trees, Linear Algebra Appl. 437 (4) (2012) 1089-1101.

[4] H.Y. Chen, Hitting times for random walks on subdivision and triangulation graphs, Linear Multilinear Algebra, 66 (2018) 117-130.

[5] H.Y. Chen, F.J. Zhang, Resistance distance and the normalized laplacian spectrum, Discret. Appl. Math. 155 (2007) 654-661.

[6] F.R. Chung, Spectral Graph Theory, American Mathematical Society, RI, 1997.

[7] D. Cvetković, P. Rowlinson, S. Simić, An introduction to the theory of graph spectrta, in: London Mathematical Society Student Texts, Cambridge University, London, 2010.

[8] A. Das, P. Panigrahi, Normalized Laplacian spectrum of some subdivision-joins and R-joins of two regular graphs, AKCE Int. J. Graphs Combin. 15 (2018) 261-270.

[9] L.H. Feng, I. Gutman, G.H. Yu, Degree Kirchhoff index of unicyclic Graphs. MATCH Commun Math Comput Chem. 69 (2013) 629-648.

[10] L.H. Feng, G.H. Yu, W.J. Liu, Futher results regarding the degree Kirchhoff index of graohs, Miskolc Math. Notes 15 (1) (2014) 97-108.

[11] C.L. He, S.C. Li, W.J. Luo, L.Q. Sun, Calculating the normalized Laplacian spectrum and the number of spanning trees of linear pentagonal chains, J. Comput. Appl. Math. 344 (2018) 381-393.

[12] J. Huang, S.C. Li, Expected hitting times for random walks on quadrilateral graphs and their applications, Linear Multilinear Algebra 66 (2018) 2389-2408.

[13] J. Huang, S.C. Li, On the normalized Laplacian spectrum, degree-kirchhoff index and spanning trees of graphs, Bull. Aust. Math. Sco. 91 (2015) 353-367.

[14] J. Huang, S.C. Li, The normalized Laplacians on both k-triangle graph and k-quadrilateral graph with their applications, Appl. Math. Comput. 320 (2018) 213-225.

[15] J. Huang, S.C. Li, X.C. Li, The normalized Laplacian, degree-Kirchhoff index and spanning trees of the linear polyomino chains, Appl. Math. Comput. 289 (2016) 324-334.

[16] J. Huang, S.C. Li, L.Q. Sun, The noamalized Laplacians, degree-kirchhoff index and the spanning trees of linear hexagonal chains, Discret. Appl. Math. 207 (2016) 67-79.

[17] S.B. Huang, J. Zhou, C.J. Bu, Some results on Kirchhoff index and degree-kirchhoff index, MATCH. Commun. Math. Comput. Chem. 75 (2016) 207-222.
[18] J.J. Hunter, The role of Kemeny’s constant in properties of Markov chains. Commun Statist Theor Meth. 43 (2014) 1309-1321.

[19] D.J. Klein, M. Randić, Resistance distance, J. Math. Chem. 12 (1993) 81-95.

[20] J.K. Koolen, G. Markowsky, A collection of results concerning electric resistance and simple random walk on distance-regular graphs. Discrete Math. 339 (2) (2016) 737-744.

[21] D.Q. Li, Y.P. Hou, The normalized Laplacian spectrum of quadrilateral graphs and its applications, Appl. Math. Comput. 297 (2017) 180-188.

[22] S.C. Li, W. Wei, S.Q. Yu, On normalized Laplacians, multiplicative degree-Kirchhoff indices, and spanning trees of the linear [n]phenylenes and their dicyclobutadieno derivatives, Int. J. Quantum Chem. 119 (2019), e25863.

[23] M. Levene, G. Loizou, Kemeny’s constant and the random surfer, The American Mathematical Monthly 109 (2002) 741-745.

[24] Y. Liu, J. Shen, The (normalized) Laplacian eigenvalue of signed graphs, Taiwanese J. Math. 19 (2015) 505-17.

[25] X.L. Ma, H. B, The normalized Laplacians, degree-Kirchhoff index and the spanning trees of hexagonal Möbius graphs, Appl. Math. Comput. 355 (2019) 33-46.

[26] Y.G. Pan, J.P. Li, S.C. Li, et al. On the normalized Laplacians with some classical parameters involving graph transformations, Linear Multilinear Algebra, (2018), DOI:10.1080/03081087.2018.1548556.

[27] M. Somodi, On the Ihara zeta function and resistance distance-based indices. Linear Algebra Appl. 513 (2017) 201-209.

[28] C.Y. Wang, Z.L. Guo, S.C. Li, Expected hitting times for random walks on the k -triangle graph and their applications, Appl. Math. Comput. 338 (2018) 698-710.

[29] P.C. Xie, Z.Z. Zhang, F. Comellas, On the spectrum of the normalized Laplacian of iterated triangulations of graphs, Appl. Math. Comput. 273 (2016) 1123-1129

[30] P.C. Xie, Z.Z. Zhang, F. Comellas, The normalized Laplacian spectrum of subdivisions of a graph, Appl. Math. Comput. 286 (2016) 250-256.

[31] X.M. Zhe, X.D. Zhang, The hitting time of random walk on unicyclic graphs, Linear Multilinear Algebra, (2019), DOI: 10.1080/03081087.2019.1611732.