Wound D-branes in Gepner Models

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Abstract

We propose a new prescription of how to represent D-branes in Gepner models in more general homology classes than those in the previous constructions. The central role is played by a certain projection acting on the Recknagel-Schomerus boundary states. Consequently, the boundary states are in most cases no longer a sum of products of \(N = 2\) Ishibashi states, but nevertheless preserve spacetime supersymmetry and satisfy the Cardy condition. We demonstrate these in the \((k = 1)^3\) Gepner model in detail, and construct boundary states for D-branes wound around arbitrary rigid 1-cycles on the corresponding 2-torus. We also emphasize the necessity of some angle-dependent transformations in identifying a proper free-field realization for each brane tilted at an angle. In particular, this is essential for the Witten index to give the correct intersection numbers between the different D-branes.

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1 Introduction

One of the important issues in the study of D-branes is their stability in Calabi-Yau compactifications. (See [1, 2] for a quick overview on the subject.) In general, a spectrum of BPS states in a weakly coupled theory is not necessarily the same as the one at strong couplings, as we know from many field-theory examples. Relations between D-branes among different points in Calabi-Yau moduli space were first studied in the quintic case [3] by utilizing the knowledge of monodromies of the periods [4] and the boundary states in Gepner models [5]. Related developments may be found in [6]-[13].

Gepner models [14] are a useful way to describe a compactification on a specific Calabi-Yau manifold which (typically) has a small volume. Therefore, the boundary-state representation of D-branes in Gepner models, pioneered by Recknagel and Schomerus [5], provides us a tractable framework for analyzing D-branes in the stringy (strong sigma-model coupling) regime. (Earlier seminal and other related works are [15]-[29].)

Although very useful, their proposed boundary states for D-branes at the Gepner point, called rational boundary states, do not exhaust all possible supersymmetric D-branes [30] in the Calabi-Yau space. For example, in the \((k=1)^3\) Gepner model, which describes a compactification upon a certain flat 2-torus, it is known [20] that the Recknagel-Schomerus A-boundary states can represent only D1-branes \(^1\) wrapped around either of the shortest nontrivial 1-cycles. Since it is straightforward (as we see in section 3) to represent D1-branes wound around arbitrary rigid 1-cycles in terms of the toroidal CFT, our present technology of constructing D-brane boundary states in Gepner models is clearly unsatisfactory. \(^2\)

In this paper, we give an answer to the basic question of how these infinitely many D1-branes wound around various 1-cycles are described as boundary states in terms of the Gepner-model language. Our strategy is to first calculate the open-string partition functions between parallel and intersecting D1-branes at arbitrary angles on a general 2-torus by means of the toroidal CFT approach, and then construct boundary states in the \((k=1)^3\) Gepner model so that the partition functions between them coincide with the ‘geometric’ partition functions for the corresponding 2-torus.

Although we discuss in detail only this simplest example of Gepner models, it al-

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\(^1\)For definiteness, we assume here that there are no spacelike noncompact Neumann directions.

\(^2\)Also, in the quintic case, pure D0-branes have no corresponding rational boundary states at the Gepner point, but this does not necessarily mean their nonexistence \([3]\). (See \([31]\) for more recent discussions.) Another example is the D-branes wrapping the cycles induced by the resolution of the singularity \([22]\).
ready presents the essential features and immediately indicates how we may generalize the Recknagel-Schomerus states in more complicated Gepner models which correspond to more general Calabi-Yau compactifications. Some of the notable features of our construction include:

- A projection operation acting on a Recknagel-Schomerus state, which leaves only a subsector satisfying a certain $U(1)$-charge condition; this condition is linear and orthogonal to the total $U(1)$-charge condition relevant to supersymmetry. Consequently the resulting new boundary states are also supersymmetric.

- A treatment of partition functions in terms of $\beta$-orbits; this allows us to easily (or rather automatically) incorporate the Cardy condition (integrality of the coefficients of the open-channel partition function), as well as exhibits manifest supersymmetry.

- A suitable identification of a proper free-field realization for each brane tilted at an angle; to distinguish one tilted D1-brane from another, it turns out that one must identify a set of free fields realizing $N = 2$ minimal models for each brane through some angle-dependent transformations.

Owing to the projection, the boundary states are in general no longer a sum of products of $N = 2$ Ishibashi states.

At this point, we would like to clarify the difference between our prescription and the work [24], where the construction in [5] was extended by twisting the ‘simple currents’ (the spectral-flow and the ‘fermion-aligning’ operators) as well as taking into account their ‘fixed point’, and the expressions of [3, 22] were recovered as special cases. As mentioned there, the former amounts to shifting the modulus of the total $U(1)$ current; such a shift also appears in this paper as a part of the angle-dependent transformation. The most significant difference is that our prescription includes a projection with respect to another $U(1)$ which is orthogonal to the total $U(1)$ current; here the order of the cyclic group for the projection may be arbitrarily large depending on the length of the cycle. Another new aspect of our prescription is the angle-dependent rotation (which is also orthogonal to the total $U(1)$) in identifying the proper free-field realization for each tilted D1-brane. We show that this is indispensable for the Witten index to give the correct intersection numbers between different D1-branes.

This paper is organized as follows. In section 2, we give a brief summary of the known construction of boundary states in general Gepner models. In section 3, we calculate the open-string partition functions in the toroidal CFT formulation. In section 4, we
construct boundary states in the \((k=1)^3\) Gepner model which reproduce the geometric partition functions and the intersection numbers. In section 5, we outline the construction of boundary states in the \((k=2)^2\) model. The last section summarizes our results and also includes a discussion on the application to general Gepner models.

2 Boundary states in Gepner models

We begin by briefly reviewing how D-branes are described in Gepner models [5]. Consider a Gepner model defined by \(r\) tensor product of \(N = 2\) minimal models with level \(k_j\) \((j = 1, \ldots, r)\), describing a compactification of type II string theory to \((d + 2)\)-dimensional spacetime. Although we will mainly consider the case for \(d = 6\), the case for \(d = 2\) can be treated in parallel as well. We assume \(d = 2\) or 6 in the following.

Recknagel and Schomerus assumed that the A-boundary states in this model are given by

\[
|\alpha > _A \equiv |S_0; (L_j, M_j, S_j) > _A
= \frac{1}{k^A_0} \sum_{\lambda, \mu} B^\lambda_\mu |\lambda, \mu >>_A, \tag{2.1}
\]

\[
B^\lambda_\mu = (-1)^{s_0^2} e^{-\pi i s_0 S_0} \prod_{j=1}^r \left( \frac{\sin \frac{\pi (l_j+1)(L_j+1)}{k_j+2}}{\sin \frac{1}{2} \pi \frac{l_j+1}{k_j+2}} e^{\pi i \frac{m_j M_j}{k_j+2} \frac{s_j S_j}{2}} \right). \tag{2.2}
\]

\(|\lambda, \mu >>_A\) is a product of \(r \ N = 2\) A-type Ishibashi states

\[
|\lambda, \mu >>_A = |s_0 >> \prod_{j=1}^r |l_j, m_j, s_j >>_A \tag{2.3}
\]

with the collective labels

\[
\lambda = \{l_1, \ldots, l_r\}, \quad \mu = \{s_0; m_1, \ldots, m_r; s_1, \ldots, s_r\}, \tag{2.4}
\]

where

\[
l_j = 0, \ldots, k, \quad m_j \in \mathbb{Z}_{2(k_j+2)}, \quad s_j \in \mathbb{Z}_4 \quad (j = 1, \ldots, r) \tag{2.5}
\]

are the labels of the irreducible representations of the \(N = 2\) superconformal algebra. \(s_0 \in \mathbb{Z}_4\) labels the irreducible representations of the \(SO(d)\) current algebra. The normalizations of the Ishibashi states are

\[
|\bar{l}, \bar{m}, \bar{s} > \equiv |\bar{q}^{\frac{N-2}{2}} |l, m, s >>_A
= \delta_l^{(2(k+2))} \delta_{s,s}^{(4)} \chi_m^l (\bar{\tau}, 0), \tag{2.6}
\]

\[
|\bar{s}_0 > \equiv \delta_{s_0,s_0}^{SO(d)} (\bar{\tau}), \tag{2.7}
\]
where \( \chi^L_s(\tilde{\tau}, z) \) and \( \chi^\text{SO}(d)_s(\tilde{\tau}) \) (\( \tilde{q} = e^{2\pi i \tilde{\tau}} \)) are the \( N = 2 \) minimal and the \( \text{SO}(d) \) characters, respectively. \( \chi^L_s(\tilde{\tau}, z) = 0 \) if \( l + m + s \neq 0 \mod 2 \). \( \delta^{(N)}_{m,m'} \) is the delta function on \( \mathbb{Z}_N \).

The ‘CPT-conjugate’ boundary state is defined by

\[
\Lambda < \Theta \alpha | \equiv \frac{1}{\kappa_A} \sum_{\lambda, \mu}^\beta B^{\lambda \mu}_\alpha \Lambda \ll \lambda, \mu | = | \Lambda.
\tag{2.8}
\]

This operation is almost equivalent to taking the hermitian conjugate, with the exception that the factor \((-1)^{\frac{\Theta}{2}}\) remains unchanged; it is so arranged that the partition function with itself becomes an alternating summation (and hence supersymmetric). The partition function between the two boundary states \( \alpha \) and \( \tilde{\alpha} \) is calculated as follows:\footnote{\textsuperscript{3} There is a typo in the modular transformation in \cite{14}, and that has also been transmitted to many places in the literature; the correct prefactor of \( S_{(l,m,s),(l',m',s')}^{(k)} \) is \( \frac{1}{2(k+2)} \) \footnote{\textsuperscript{\textsuperscript{3}}} and not \( \frac{1}{\sqrt{2(k+2)}} \).}

\[
Z^{A}_{\tilde{\alpha} \alpha}(\tau) \equiv \Lambda < \Theta \tilde{\alpha} | q^{L_0 - \frac{\Theta}{2}} | \Theta \alpha >_A
= \frac{2^r}{\kappa_{\tilde{\alpha}} \kappa_A} \sum_{\lambda, \mu}^\beta B^{\lambda \mu}_\alpha \lambda, \mu (\tilde{\tau})
= \frac{2^r}{\kappa_{\tilde{\alpha}} \kappa_A} \sum_{\lambda, \mu}^\beta \sum_{\lambda', \mu'} \lambda, \mu (\tilde{\tau})
= \sum_{\lambda', \mu'} \sum_{\nu_0 = 0, 1} \sum_{\nu_1, \ldots, \nu_r = 0, 1} (-1)^{\nu_0} \delta^{(4)}_{\nu_0, 2 + \tilde{S}_0 - \tilde{S}_0 - 2} \sum_{j=1}^r \nu_j
\cdot \prod_{j=1}^r \left( N_j^L \delta_{m_j', m_j - \nu_0} \delta^{(4)}_{s_j', s_j - \nu_0 + 2} \chi_{\lambda', \mu'}^{(l, m, s)} (\tilde{\tau}), \tag{2.9}\right)
\]

where \( \tilde{\tau} = -1/\tau \), \( K \equiv \text{lcm}(4, 2(k_j + 2)) \) and

\[
\Lambda \ll \lambda, \mu | q^{L_0 - \frac{\Theta}{2}} | \lambda, \mu >_A = \chi_{\lambda, \mu}^{(\tilde{\tau})}
\equiv \chi_{s_0, s'_0}^{\text{SO}(d)} (\tilde{\tau}) \prod_{j=1}^r \chi_{l_j, m_j}^{l_j, s_j} (\tilde{\tau}, 0). \tag{2.10}
\]

\( \chi_{\lambda', \mu'}^{(l, m, s)} (\tau) \) is also given by an obvious formula. \( S_{s_0, s'_0}^{\text{SO}(d)}, S_{(l,m,s),(l',m',s')}^{(k)} \) defined by

\[
S_{s_0, s'_0}^{\text{SO}(d)} = \frac{1}{2} e^{-\frac{\pi i}{4} \delta_{s_0, s'_0}},
\]

\[
S_{(l,m,s),(l',m',s')}^{(k)} = \frac{1}{2(k+2)} \sin \left( \frac{\pi (l + 1)(l' + 1)}{k + 2} \right) e^{\pi i \frac{m_m' - m_m'}{k+2}}. \tag{2.11}
\]
are the modular transformation matrices

$$
\chi_{s_0}^{SO(d)} \left( \frac{-1}{\tau} \right) = \sum_{s'_0 \in \mathbb{Z}_4} S_{s_0, s'_0}^{SO(d)} \chi_{s'_0}^{SO(d)}(\tau),
$$

$$
\chi_{l,s}^{SO(d)} \left( \frac{-1}{\tau}, \frac{z}{\tau} \right) = \sum_{l'=0}^{k} \sum_{s' \in \mathbb{Z}_4} \sum_{m' \in \mathbb{Z}_2(k+2)} \sum_{s'' \in \mathbb{Z}_4} S_{l,m,s}(l', m', s'') e^{\frac{\pi i}{\tau} \sum_{s''} \chi_{s''}(\tau, z)}.
$$

(2.12)

$N_{L_j L_j}^{L_j}$ is the $SU(2)$ fusion coefficients. Finally, we have chosen the normalization constant $\kappa^A_\alpha$ as

$$
\kappa^A_\alpha = \left( \frac{2^{r+1} \prod_{j=1}^r (k_j + 2)}{K} \right)^{1/2}
$$

(2.13)
in the last expression.

An important observation is that the alternating summation over $\nu_0$ of the product of characters (2.9) is precisely the spectral-flow ($\beta$-) orbit, which was originally introduced in [14] in the construction of modular invariant partition functions for closed superstring compactifications. The partition function $Z_{\tilde{\alpha}\alpha}^A(\tau)$ vanishes if the ‘initial condition’ of the flow

$$
(\tilde{S}_0 - S_0 + 2; \tilde{M}_j - M_j, \tilde{S}_j - S_j)
$$

satisfies the $\beta$-condition

$$
-\frac{d}{8}(\tilde{S}_0 - S_0) + \frac{1}{2} \sum_{j=1}^r \left( \frac{\tilde{M}_j - M_j}{k_j + 2} - \frac{\tilde{S}_j - S_j}{2} \right) \in \mathbb{Z}
$$

(2.15)

for $d = 2$ or 6. If (2.14) satisfies (2.15), then so do all the other orbits in $Z_{\tilde{\alpha}\alpha}^A(\tau)$ automatically, and hence it is manifestly supersymmetric. In particular, if $\alpha = \tilde{\alpha}$, then

$$
Z_{\alpha\alpha}^A(\tau) = \sum_{\lambda', \mu'} \sum_{\nu_0=0}^{K-1} \sum_{\nu_1, \ldots, \nu_r=0,1} (-1)^{\nu_0} \delta^{(4)}_{s_0', 2 - \nu_0 - 2 \sum_{j=1}^r \nu_j} \prod_{j=1}^r \left( N_{L_j L_j}^j \delta_{m_j' \nu_0, m_j' \nu_0 + 2 \nu_j} \delta^{(4)}_{s_j', -\nu_0 + 2 \nu_j} \right) \chi_{\lambda', \mu'}(\tau).
$$

(2.16)

The open-channel partition function between identical boundary states is thus given by the sum of $\beta$-orbits with labels

$$
(2 - 2 \sum_{j} \nu_j; 0, 2 \nu_j).
$$

(2.17)

Since (2.9) depends on $(S_0; M_j, S_j)$, $(\tilde{S}_0; \tilde{M}_j, \tilde{S}_j)$ only through their differences, the partition function for a boundary state with itself always reduces to (2.16), irrespective of the specific values of $(S_0; M_j, S_j)$. This means that the boundary states of the type (2.1) can represent only finitely many D-branes in different geometric configurations.


3 Partition functions of D-branes on general tori

In this section, we derive the partition functions between arbitrary pairs of D-branes on a general 2-torus. There are two types of partition functions depending on the relative positions of D-branes: parallel or tilted at an angle. We construct boundary states in the \((k = 1)^3\) Gepner model in section 4 so that they reproduce the geometric partition functions given in this section for the corresponding torus \((SU(3)\) torus).

3.1 Parallel branes

In this subsection, we derive the partition functions for parallel D\(_p\)-branes on a 2-torus \((\times\) eight-dimensional Minkowski space) and find the corresponding geometric boundary states. We assume that one of the \(p\)-dimensional worldvolume winds around a 1-cycle labeled by the relatively prime winding numbers \((p, q)\) (Figure 1).

**The open-string channel**

The partition function is defined in the open-string channel by a one-loop amplitude

\[
Z^{tot} = (\log(\det H^{(o)}))^ {-1/2} = \int_0^\infty \frac{dt}{2t} \exp(-2\pi t H^{(o)}),
\]

where \(H^{(o)}\) is the open-string hamiltonian, \(\exp(-2\pi t H^{(o)})\) is the trace over the degrees of freedom of the open string ending on the two parallel branes. In the following, we assume that the two branes are on top of each other.

The calculation of the trace is straightforward except for the summation of the momenta and winding modes. We first give the result:

\[
Z^{tot}_{\theta = \theta_{p,q}} = 2V_p \int_0^\infty \frac{dt}{2t} (8\pi^2 t \alpha')^{-\frac{p}{2}} \frac{1}{\eta^6(\tau)} \left[ \frac{1}{\eta^2(\tau)} Z^{open}_{\theta = \theta_{p,q},0}(\tau) \frac{1}{2} \left( \frac{\theta_3^4(\tau,0) - \theta_2^4(\tau,0) - \theta_1^4(\tau,0)}{\eta^4(\tau)} \right) \right],
\]

with

\[
Z^{open}_{\theta = \theta_{p,q},0}(\tau) = \sum_{m, n \in \mathbb{Z}} q^{G_{11}p^2 + 2G_{12}pq + G_{22}q^2} (m^2 - 2B_{12} mn + (G + B_{12}) n^2)(\tau = it).
\]

\((\tau = it)\). \(2V_p\) and \((8\pi^2 t \alpha')^{-p/2}\) come from the integral of the coordinates and the momenta of the noncompact directions, while the modular forms are the oscillator sum for all the transverse directions with the GSO projection.

Let us sketch the derivation of the formula (3.3). Let \(G_{\mu\nu}\) and \(B_{\mu\nu}\) be the metric and the antisymmetric-tensor-field background of the torus. We normalize the metric so that...
the radius of the $X^1$ direction is $\sqrt{G_{11}}\sqrt{\alpha'}$ and the volume of the torus is $\sqrt{G}(2\pi\sqrt{\alpha'})^2$ ($G$ is the determinant of the metric). For example, the backgrounds corresponding to the $SU(3)$ and $SU(2)^2$ tori in this unit are

$$SU(3)\text{ torus : } G_{11} = G_{22} = 1, \quad G_{12} = B_{12} = 1/2,$$
$$SU(2)^2\text{ torus : } G_{11} = G_{22} = 1, \quad G_{12} = B_{12} = 0.$$ 

At the boundaries ($\sigma = 0, \pi$), the open string satisfies

$$\delta X^\mu(G_{\mu\nu}\partial_\sigma X^\nu + B_{\mu\nu}\partial_\tau X^\nu) = 0, \quad (3.4)$$

or equivalently

$$\partial_\sigma \tilde{X}_\theta^a = 0, \quad (3.5)$$

Here we introduced the reference local Lorentz coordinates

$$\tilde{X}_\theta^a = e^a\mu X^\mu \quad (a = 1, 2), \quad (3.6)$$

where the zweibein $e^a\mu$ (and its inverse $e^\mu a$) is chosen so that the directions of $\tilde{X}_\theta^1 = 0$ and $X^1$ coincide, i.e., one of the sides of the torus lies on the $\tilde{X}_\theta^1 = 0$ direction:

$$e^a\mu = \frac{1}{\sqrt{G}} \begin{pmatrix} G_{11} & G_{12} \\ 0 & \sqrt{G} \end{pmatrix}, \quad e^\mu a = \frac{1}{\sqrt{G}G_{11}} \begin{pmatrix} \sqrt{G} & -G_{12} \\ 0 & G_{11} \end{pmatrix}. \quad (3.7)$$

The boundary condition (3.5) is decomposed for the $(p, q)$ brane in terms of the rotated coordinates as

$$\partial_\sigma \tilde{X}_{\theta=\theta_{p,q}}^1 = 0, \quad \partial_\tau \tilde{X}_{\theta=\theta_{p,q}}^2 = 0, \quad (3.8)$$

where

$$\begin{pmatrix} \tilde{X}_{\theta=\theta_{p,q}}^1 \\ \tilde{X}_{\theta=\theta_{p,q}}^2 \end{pmatrix} = R(\theta_{p,q}) \begin{pmatrix} \tilde{X}_{\theta=0}^1 \\ \tilde{X}_{\theta=0}^2 \end{pmatrix}, \quad R(\theta_{p,q}) \equiv \begin{pmatrix} \cos \theta_{p,q} & \sin \theta_{p,q} \\ -\sin \theta_{p,q} & \cos \theta_{p,q} \end{pmatrix}, \quad (3.9)$$

$$\sin \theta_{p,q} = \frac{q\sqrt{G}/\sqrt{G_{11}}}{\sqrt{G_{11}p^2 + 2G_{12}pq + G_{22}q^2}}, \quad \cos \theta_{p,q} = \frac{p\sqrt{G_{11}} + qG_{12}/\sqrt{G_{11}}}{\sqrt{G_{11}p^2 + 2G_{12}pq + G_{22}q^2}}. \quad (3.10)$$

Then, the momentum is quantized in units of the inverse of the $(p, q)$ cycle’s length for the Neumann direction, while the length of the open string is quantized in units of the distance between the nearest trajectories for the Dirichlet direction:

$$\hat{p}_{\theta=\theta_{p,q}} = \frac{m}{\sqrt{(G_{11}p^2 + 2G_{12}pq + G_{22}q^2)\alpha'}},$$
$$\hat{X}_{\theta=\theta_{p,q}}^2 (\tau, \pi) = \tilde{X}_{\theta=\theta_{p,q}}^2 (\tau, 0) + \frac{n2\pi\sqrt{\alpha'}\sqrt{G}}{\sqrt{G_{11}p^2 + 2G_{12}pq + G_{22}q^2}}. \quad (3.11)$$
\( (m, n \in \mathbb{Z}) \) where
\[
\hat{p}_{\theta=\theta_{p,q}}^a = \frac{1}{2\pi \alpha'} \int_0^\pi d\sigma [\partial_\tau \hat{X}_{\theta=\theta_{p,q}}^a + (R(\theta_{p,q})\hat{B}R(\theta_{p,q})^{-1})_{ab} \partial_\sigma \hat{X}_{\theta=\theta_{p,q}}^b], \tag{3.12}
\]
with \( \hat{B}_{ab} = e_a^\mu e_b^\nu B_{\mu\nu} \). Solving these quantization conditions under the boundary conditions (3.8), we get the following expansions
\[
\hat{X}_{\theta=\theta_{p,q}}^1 = \frac{(m - B_{12}n)}{\sqrt{G_{11}p^2 + 2G_{12}pq + G_{22}q^2}} \tau + (\text{the oscillator part}),
\]
\[
\hat{X}_{\theta=\theta_{p,q}}^2 = \frac{n}{\sqrt{G_{11}p^2 + 2G_{12}pq + G_{22}q^2}} \sigma + (\text{the oscillator part}). \tag{3.13}
\]
Substituting them into the momentum-winding part of the open-string hamiltonian and taking the summation, we have (3.3).

**The closed-string channel**

In the closed-string channel, the partition function is defined as the tree amplitude by using the boundary state:
\[
<\text{tot} B|\Delta|B>^\text{tot} = \frac{\alpha' \pi}{2} \int_0^\infty \frac{dt}{t^2} <\text{tot} B|q^H(c)_B>^\text{tot}, \tag{3.14}
\]
where \( \Delta \) and \( H^{(c)} \) are the propagator and the hamiltonian of the closed string. The boundary state \( |B>^\text{tot} \) is a tensor product of the oscillator part and the zero-mode part; the contributions from the compact directions as well as the overall normalization depend on \( (p, q) \). They are determined by solving the geometric boundary conditions and demanding that the above amplitude is equal to \( Z_{\theta=\theta_{p,q}}^\text{tot} \) (open-closed correspondence).

This is a generalization of, e.g., [20, 33] to arbitrary \( (p, q) \) branes on general tori.

The oscillator part of the boundary state is defined ([34], for example) so that it satisfies the corresponding geometric boundary conditions and provides the supersymmetric amplitude. We refer to this state for the \( (p, q) \) brane as \( |\theta_{p,q}>^\text{osc} \) and normalize such that
\[
<\text{osc} \theta_{p,q}|q^{H^{(c)}}|\theta_{p,q}>^\text{osc} = \frac{1}{\eta^6(\tau)} \cdot \frac{1}{\eta^2(\tau)} \cdot \frac{1}{2} \frac{\partial_4^1(\tau, 0) - \partial_4^1(\tau, 0) - \partial_2^1(\tau, 0)}{\eta^4(\tau)}, \tag{3.15}
\]
where the each factor comes from the noncompact bosons, the compact bosons and the fermions, respectively. The \( \theta_{p,q} \) dependences (3.9) of the compact-boson oscillators in the bra and the ket states cancel out.
The zero-mode parts of the boundary state are as follows. For the noncompact directions, the boundary state \( |\vec{k},\vec{0}\rangle \) has nonzero momenta \( \vec{k} \) (zero momenta \( \vec{0} \)) for the Dirichlet (Neumann) directions, and is normalized as \[33\]

\[ < \vec{k},\vec{0}|\vec{k}',\vec{0} > = V_p(2\pi)^8\nu \delta^{(8-p)}(\vec{k} - \vec{k}'). \]

(3.16)

For the torus directions, in terms of the reference local Lorentz coordinates which are equivalent to the boundary conditions are written as

\[
\begin{align*}
\hat{X}^{p}_{\sigma=0} &= \hat{x}^a + \frac{\alpha'}{2} \hat{p}^a_R(\tau - \sigma) + \frac{\alpha'}{2} \hat{p}^a_R(\tau + \sigma) + \text{(the oscillator part)}, \\
\hat{p}^a_L &= (n_\mu + m_\nu (B_{\nu\mu} + G_{\nu\mu})) \frac{e^a}{\sqrt{\alpha'}}; \quad \hat{p}^a_R = (n_\mu + m_\nu (B_{\nu\mu} - G_{\nu\mu})) \frac{e^a}{\sqrt{\alpha'}}.
\end{align*}
\]

(3.17)

the boundary conditions are written as

\[
\begin{align*}
&\left( \cos \theta_{p,q} (\hat{p}^1_L + \hat{p}^1_R) + \sin \theta_{p,q} (\hat{p}^2_L + \hat{p}^2_R) \right) | B >= 0, \\
&\left( -\sin \theta_{p,q} (\hat{p}^1_L - \hat{p}^1_R) + \cos \theta_{p,q} (\hat{p}^2_L - \hat{p}^2_R) \right) | B >= 0,
\end{align*}
\]

(3.18)

which are equivalent to

\[
(q m^1 - p m^2) | B >= 0, \quad (p n_1 + q n_2) | B >= 0.
\]

(3.19)

Plugging the solution \((m^1, m^2) = (mp, mq), (n_1, n_2) = (-nq, np) \) \((m, n \in \mathbb{Z})\) back into (3.17), we obtain the two-dimensional lattice \( \Lambda_{p,q} : \)

\[
\begin{align*}
\hat{p}^1_L &= \frac{1}{\sqrt{\alpha' G_{11}}} [-n q + m \{ p G_{11} + q (G_{12} - B_{12}) \}], \\
\hat{p}^2_L &= \frac{1}{\sqrt{\alpha' G_{11} G}} [n \{ q G_{12} + p G_{11} \} + m \{ p G_{11} B_{12} + q (G + G_{12} B_{12}) \}], \\
\hat{p}^1_R &= \frac{1}{\sqrt{\alpha' G_{11} G}} [-n q - m \{ p G_{11} + q (G_{12} + B_{12}) \}], \\
\hat{p}^2_R &= \frac{1}{\sqrt{\alpha' G_{11} G}} [n \{ q G_{12} + p G_{11} \} + m \{ p G_{11} B_{12} - q (G - G_{12} B_{12}) \}].
\end{align*}
\]

(3.20)

Thus, the momentum-winding part of the boundary state can be written as

\[
| \theta_{p,q} > = \sum_{\Lambda_{p,q}} | \hat{p}^1_L, \hat{p}^2_L, \hat{p}^1_R, \hat{p}^2_R >.
\]

(3.21)

By evaluating the corresponding part of the hamiltonian \( H_0^{(c)} \) in the amplitude, we obtain

\[
Z_{\theta=\theta_{p,q},0}^{\text{closed}}(\vec{q}) \equiv 0 < \theta_{p,q} | \vec{q}^{(c)} | \theta_{p,q} > = \sum_{n,m \in \mathbb{Z}} \frac{q^{G_{11} p^2 + 2G_{12} p q + G_{22} q^2}}{4 \alpha} (n^2 + 2B_{12} nm + (G + B_{12}^2)m^2),
\]

(3.22)
which, in fact, is related to the open-string one (3.3) via the Poisson resummation:

\[
\frac{1}{\eta^2(\tau)} Z_{\theta=\theta_{p,q},0}^{\text{open}}(\tau) = \frac{1}{\eta^2(\tilde{\tau})} Z_{\theta=\theta_{p,q},0}(\tilde{\tau}), \quad C_{p,q} = \frac{G_{11}p^2 + 2G_{12pq} + G_{22}q^2}{2\sqrt{G}}. \quad (3.23)
\]

The open-closed correspondence

The total boundary state which satisfies the open-closed correspondence is then obtained by

\[
|\theta = \theta_{p,q} >^\text{tot} = \frac{N_p}{2} \sqrt{C_{p,q}} \int \frac{dk}{2\pi} 8^{-p} |\theta_{p,q} >_\text{osc} \times |\theta_{p,q} >_0 \times |\vec{k}, \vec{0} >, \quad (3.24)
\]

where

\[
N_p = \sqrt{2} T_p, \quad T_p \equiv 2^{3-p/2} \pi^{\frac{7}{2}-p} \alpha'^{(3-p)/2}. \quad (3.25)
\]

In fact, by using (3.15), (3.16), (3.22) and (3.23), it is easy to show that

\[
<\theta = \theta_{p,q} | \Delta |\theta = \theta_{p,q} >^\text{tot} = Z_{\theta=\theta_{p,q}}^\text{tot}. \quad (3.26)
\]

Let us discuss the meaning of the prefactor. It is known that \( T_p \) in (3.23) is the normalization factor of the Dp-brane boundary state in ten-dimensional Minkowski space, and is related to its tension as [36]

\[
\tau_p = \frac{1}{\kappa_{10}} T_p \quad (3.27)
\]

(\( \kappa_{10} \) is the ten-dimensional gravitational constant). After compactification on a torus, the gravitational constant reads

\[
\frac{1}{\kappa_8} = \frac{1}{\kappa_{10}} \sqrt{V_{\text{torus}}}, \quad V_{\text{torus}} = \sqrt{G(2\pi \alpha')^2}. \quad (3.28)
\]

Thus, the tension \( \tau_{p,q} \) of the \( (p, q) \) brane in eight dimensions becomes

\[
\tau_{p,q} = \frac{1}{\kappa_8} N_p \sqrt{C_{p,q}} = \tau_p 2\pi \sqrt{\alpha'(G_{11}p^2 + 2G_{12pq} + G_{22}q^2)}, \quad (3.29)
\]

which is consistent with the geometric picture.

3.2 Branes at an angle

In this subsection, we consider the intersection of the \( (\tilde{p}, \tilde{q}) \) brane with the \( (p, q) \) brane at an angle. The one-loop partition function has been obtained [37, 38] for the open strings ending on such branes in a noncompact target space. In this case, the
open-string modes receive angle-dependent twists, and hence no zeromode exists, in particular. Therefore, even when we consider branes intersecting in a compact space, there is no modification of the partition function by either the momentum quantization or the appearance of the winding modes. However, the volume integral in the trace in (3.1) receives the change because the intersection number of the two branes is 1 for the non-compact case, whereas \((\tilde{p}, \tilde{q}) \cdot (p, q) \equiv \tilde{q}p - \tilde{p}q\) for the compact case. We thus multiply (the absolute value of) this extra factor to the total partition function and obtain

\[
Z_{\theta_{p,q}, (p,q)}^{\text{tot}} = 2(\tilde{p}, \tilde{q}) \cdot (p, q)V_p \int_0^\infty \frac{dt}{2t} \left(8\pi^2 t \alpha'\right)^{-\frac{3}{2}} \frac{1}{\eta^6(\tau)},
\]

where \(\Delta \theta = \theta_{p,q} - \theta_{\tilde{p}, \tilde{q}}\). The sign of \(\sin(\Delta \theta \tau)\) in the denominator and that of the intersection number in the numerator cancel out.

We can alternatively derive this partition function as the amplitude between the boundary states (3.24) with different angles. The amplitude for the oscillator part is

\[
\langle \varphi_{\theta_{\tilde{p}, \tilde{q}}|\tilde{q}}^{\mu, (c)} | \varphi_{\theta_{p,q}} >_{\text{osc}} = \frac{1}{\eta^6(\tau)} \cdot \frac{2 \sin \Delta \theta \eta(\tilde{\tau})}{\eta^4(\tilde{\tau})} \left[ \frac{1}{2} \vartheta_3^2(\tilde{\tau}, 0) \vartheta_3(\tilde{\tau}, \frac{\Delta \theta}{\pi}) - \vartheta_2^2(\tilde{\tau}, 0) \vartheta_2(\tilde{\tau}, \frac{\Delta \theta}{\pi}) - \vartheta_1^2(\tilde{\tau}, 0) \vartheta_1(\tilde{\tau}, \frac{\Delta \theta}{\pi}) \right],
\]

where the states are normalized as in (3.13). The \(\Delta \theta\) dependence arises because the oscillators of the compact bosons (3.3) and their superpartners (including the RR zero modes \(l\)) in the bra and the ket states are relatively rotated. For the momentum-winding part, (3.24) is replaced by

\[
\langle 0 | \varphi_{\theta_{\tilde{p}, \tilde{q}}|\tilde{q}}^{\mu, (c)} | \theta_{p,q} >_0 = 1.
\]

This holds because the momenta (3.20) between the two boundary states coincide only at one lattice point, \(\tilde{p}_L = \tilde{p}_R = 0\). Another identity we use is

\[
\sqrt{C_{\tilde{p}, \tilde{q}}} \sqrt{C_{p,q}} 2 \sin \Delta \theta = (\tilde{p}, \tilde{q}) \cdot (p, q).
\]

Then, we find

\[
\langle \theta_{p,q} | \varphi_{\theta_{\tilde{p}, \tilde{q}}} >_{\text{tot}} = Z_{\theta_{p,q}, (p,q)}^{\text{tot}}\]

as expected. This also implies that the boundary states (3.24) are mutually consistent.

\[4 \cos \Delta \theta \text{ in } \vartheta_2(\tilde{\tau}, \frac{\Delta \theta}{\pi}) \text{ comes from the RR vacuum amplitude.}\]
3.3 Partition functions normalized in terms of CFT

In later sections, we will construct boundary states in the \((k = 1)^3\) and \((k = 2)^2\) Gepner models so that they reproduce the geometric partition functions for the \(SU(3)\) and \(SU(2)^2\) torus, respectively. For this purpose, it is convenient to extract the ‘CFT piece’ from (3.2) and (3.30) by discarding the common kinematical factors (and \(1/\eta^6(\tau)\)) as

\[
Z_{\theta = \theta_{p, q}}^{\text{open}}(\tau) \equiv \frac{1}{\eta^2(\tau)} Z_{\theta = \theta_{p, q}, 0}^{\text{open}}(\tau) \frac{1}{2} \frac{\vartheta_4^3(\tau, 0) - \vartheta_2^3(\tau, 0) - \vartheta_4^3(\tau, 0)}{\eta^4(\tau)},
\]

(3.35)

\[
Z_{\theta = \theta_{p, q}, 0}^{\text{open}}(\tau) \equiv (\tilde{p}, \tilde{q}) \cdot (p, q)
\]

\[
\frac{1}{2} \frac{\vartheta_3^3(\tau, 0) \vartheta_3(\tau, \frac{\Delta_\nu}{\pi} \tau) - \vartheta_2^3(\tau, 0) \vartheta_2(\tau, \frac{\Delta_\nu}{\pi} \tau) - \vartheta_4^3(\tau, 0) \vartheta_4(\tau, \frac{\Delta_\nu}{\pi} \tau)}{\eta^3(\tau) (-i \vartheta_1(\tau, \frac{\Delta_\nu}{\pi} \tau))}.
\]

(3.36)

4 D1-branes in the \((k = 1)^3\) model

4.1 Partition functions

Let us now concentrate on the D1-branes in the \((k = 1)^3\) model. As mentioned before, it describes the compactification on the special torus, with backgrounds

\[
G_{\mu\nu} = \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

(4.1)
in our convention. The Recknagel-Schomerus states are

\[
|\alpha >_A = \frac{1}{2^3 3^4} \sum_{s_0, l_j, m_j, s_j} B^\lambda_{\alpha} \chi_{s_0} |s_0 >= \prod_{j=1}^3 \frac{1}{|l_j, m_j, s_j, \alpha >_A},
\]

(4.2)

\[
B^\lambda_{\alpha} = e^{\pi i \left[\frac{l^2}{2} + \sum_{j=1}^3 l_j l_j + \sum_{j=1}^3 s_j s_j \right]}.
\]

(4.3)

As we have seen in section 2, the partition function for a boundary state \(|\alpha >_A\) with itself can be written as a sum of \(\beta\)-orbits. This motivates us to introduce the notation

\[
\text{Orbit}(s_0; (l_j, m_j, s_j); z_j)
\]

\[
= \sum_{l_0 \in \mathbb{Z}_{12}} (-1)^l \chi_{s_0}^{SO(6)}(\tau) \chi_{l_1, s_1}^{l_0} \chi_{l_2, s_2}^{l_0} \chi_{l_3, s_3}^{l_0} (\tau, z_1, z_2, z_3).
\]

(4.4)

\[\text{Again, no spacelike noncompact Neumann directions are assumed below.}\]

\[\text{Since } \chi_{m+i}^{l-i} = \chi_{m,i}^{l}, \text{ the same state appears } 2^3 = 8 \text{ times in the summation; they all have the same phase factor if } \text{the integer labels } (L_j, M_j, S_j) \text{ satisfy } L_j + M_j + S_j = \text{even. } |\alpha >_A \text{ itself vanishes otherwise.}\]

\[\text{To be precise, this } B^\lambda_{\alpha} \text{ is the phase-factor piece of } (2.2) \text{ and the remaining real constant is absorbed into the prefactor of } (4.3). \text{ But still we use the same symbol for notational simplicity.}\]
The minimal characters are given by
\[ \chi_{l,m}^{s}(\tau, z) = \frac{\delta_{l+m+s,0}}{\eta(\tau)} \Theta_{2m-3s,0}(\tau, \frac{z}{3}). \] (4.5)

The \( SO(6) \) characters are
\[ \chi_{s_0}^{SO(6)}(\tau) = \frac{1}{\eta^2(\tau)} \left[ (\Theta_{-s_0,2}(\tau, 0))^3 + 3\Theta_{-s_0,2}(\tau, 0)(\Theta_{-s_0+2,2}(\tau, 0))^2 \right]. \] (4.6)

\( \text{Orbit}(s_0; (l_j, m_j, s_j); z_j) \) identically vanishes if the \( \beta \)-condition is satisfied
\[ \frac{s_0}{4} + \frac{1}{2} \sum_{j=1}^{3} \left( \frac{m_j}{3} - \frac{s_j}{2} \right) = \frac{1}{2} \quad (\text{mod } \mathbb{Z}) \] (4.7)

and if
\[ z_1 + z_2 + z_3 = 0. \] (4.8)

Indeed, one can then write
\[ \text{Orbit}(s_0; (l_j, m_j, s_j); z_j) = \sum_{(\hat{m}_j)} \sum_{\nu \in \mathbb{Z}_4} \left( -1 \right)^\nu \chi_{s_0-\nu}^{SO(6)}(\tau) \Theta_{\hat{m}_j+4j+\nu,6}(\tau, \frac{z_j}{3}) \] (4.9)

where
\[ (\text{Jacobi}) \equiv \sum_{\nu \in \mathbb{Z}_4} \left( -1 \right)^\nu \chi_{-\nu}^{SO(6)}(\tau) \Theta_{\nu+2,2}(\tau, 0) \] (4.10)

\[ \hat{m}_j \equiv 2m_j - 3s_j \quad (j = 1, 2, 3). \] (4.11)

In the last line (4.12), we have used the multiplication formula for the theta functions.

The \( l_j \) dependences are only through \( \delta_{l_j+m_j+s_j,0} \), and they are omitted in (4.13).

With this notation, the partition function for the boundary state \( |\alpha >_A \) with itself can be written as
\[ Z^A_{\alpha\alpha}(\tau) = \sum_{\nu_1,\nu_2,\nu_3 \in \mathbb{Z}_2} \text{Orbit} \left[ 2 - 2 \sum_{j=1}^{3} \nu_j; (0,0,2\nu_3); 0 \right] \] (4.12)
which \( \theta = 0 \), precisely coincides with \( 2Z_{\theta=0}^{\text{open}}(\tau) \), the open-string partition function (3.3) (3.33) for \( \theta = 0 \). Thus we have

\[
|\theta = 0 > = \frac{1}{\sqrt{2}} |\alpha >_{A},
\]

(4.13)

where \( \alpha = (S_0, (L_j, M_j, S_j)) \) are a set of arbitrary integer labels satisfying \( L_j + M_j + S_j = \text{even} \).

Let us next express

\[
Z_{\theta=\theta_{p,q}}^{\text{open}}(\tau) = (\text{Jacobi}) \cdot \frac{1}{\eta^2(\tau)} \sum_{m,n \in \mathbb{Z}} q^{\frac{m^2+mn+n^2}{p^2+pq+q^2}}
\]

(4.14)

for a general pair of relatively prime integers \( p, q \). The angle \( \theta_{p,q} \) (3.10) of the brane is given by

\[
\sin \theta_{p,q} = \frac{\sqrt{3} q}{\sqrt{p^2 + pq + q^2}}, \quad \cos \theta_{p,q} = \frac{p + q}{\sqrt{p^2 + pq + q^2}}
\]

(4.15)

in the present case. We make use of the equation

\[
\sum_{m,n \in \mathbb{Z}} q^{\frac{m^2+mn+n^2}{p^2+pq+q^2}} = \sum_{\Delta m \in \mathbb{Z}^{p^2+pq+q^2}} q^{\frac{(\Delta m)^2}{p^2+pq+q^2}} \sum_{s,i_1,i_2 \in \mathbb{Z}_2} \Theta_{12s+6i_1,12} \left( \tau, \frac{(p + q)\tau \Delta m}{2(p^2 + pq + q^2)} \right) \cdot \Theta_{4s+2i_1+4i_2,4} \left( \tau, \frac{(p - q)\tau \Delta m}{2(p^2 + pq + q^2)} \right).
\]

(4.16)

Plugging this equation into (4.14) and comparing with (4.9), we find

\[
Z_{\theta=\theta_{p,q}}^{\text{open}}(\tau) = \sum_{\Delta m \in \mathbb{Z}^{p^2+pq+q^2}} q^{\frac{(\Delta m)^2}{p^2+pq+q^2}} \cdot \frac{1}{2} \sum_{\nu_1,\nu_2,\nu_3 \in \mathbb{Z}_2} \text{Orbit} \left( 2 - 2 \sum_{j=1}^{3} \nu_j; (0,0,2\nu_j); \tau z_j(\Delta m) \right)
\]

(4.17)

with

\[
\begin{align*}
z_1(\Delta m) &= \frac{-p - 2q}{p^2 + pq + q^2} \Delta m, \\
z_2(\Delta m) &= \frac{2p + q}{p^2 + pq + q^2} \Delta m, \\
z_3(\Delta m) &= \frac{-p + q}{p^2 + pq + q^2} \Delta m.
\end{align*}
\]

(4.18)

A factor of 2 comes from the fact that only a half of the choices of \( \nu_j \in \mathbb{Z}_2 \) represent independent \( \beta \)-orbits.
\(Z_{\theta=\theta_{p,q}}^{\text{open}}(\tau)\) can thus be obtained by first shifting \(z_j\) of each minimal character in \(Z_{\theta=0}^{\text{open}}(\tau)\) by a fractional unit times \(\tau \Delta m\), and then summing over \(\Delta m \in \mathbb{Z}_{p^2+pq+q^2}\). Such a shift of \(z\) linear in \(\tau\) is known as a ‘spectral flow’. Note that the \(\beta\)-orbit itself is also a spectral-flow orbit of the total \(U(1)\) current. These two kinds of spectral flows are orthogonal to each other since

\[
z_1(\Delta m) + z_2(\Delta m) + z_3(\Delta m) = 0. \tag{4.19}
\]

### 4.2 Ishibashi states in different realizations

To construct boundary states reproducing the partition functions (4.17), we will first generalize the ordinary \(N = 2\) Ishibashi states to a certain one-parameter family of them. We restrict ourselves to the \(k = 1\) case. The A-type boundary states \(|B \gg A\rangle\) are defined to be the states such that

\[
(L_n - \overline{L}_{-n})|B \gg A\rangle = (J_n - \overline{J}_{-n})|B \gg A\rangle = (G^\pm_r + i\eta \overline{G}^\mp_{-r})|B \gg A\rangle = 0 \tag{4.20}
\]

\((\eta = \pm 1)\). If \(k = 1\), the \(N = 2\) superconformal currents are realized by a single free boson \(\phi(z)\) as

\[
T(z) = -\frac{1}{2} (\partial \phi)^2, \quad J(z) = \frac{i}{\sqrt{3}} \partial \phi, \quad G^\pm(z) = \sqrt{\frac{2}{3}} e^{\pm i\sqrt{3}\phi(z)} \tag{4.21}
\]

(and similar expressions obtained by the replacement \(\phi(z) \rightarrow \overline{\phi}(\overline{z})\) for \(T(\overline{z}), J(\overline{z})\) and \(G^\mp(\overline{z})\)). Their mode expansions are

\[
T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}, \quad J(z) = \sum_{n \in \mathbb{Z}} \frac{J_n}{z^{n+1}}, \quad G^\pm(z) = \sum_{r \in \mathbb{Z}+\nu} \frac{G^\pm_r}{z^{r+3/2}} \tag{4.22}
\]

\((\nu = \frac{1}{2} or 0\) depending on NS or R\) and

\[
\partial \phi(z) = \sum_{n \in \mathbb{Z}} \frac{a_n}{z^{n+1}}. \tag{4.23}
\]

The anti-holomorphic fields (with \(\overline{\cdot}\)’s) are also expanded in a similar fashion.

The first two A-boundary conditions (4.20) are equivalent to the Dirichlet boundary condition for the free boson \(\phi + \overline{\phi}\)

\[
(a_n - \overline{a}_{-n})|B \gg A\rangle = 0. \tag{4.24}
\]

In general, an Ishibashi state is obtained by first gluing every state of an irreducible representation with its anti-linear transform, and then summing over all states in the
whole representation space. Thus, an $N=2$ A-type Ishibashi state is not a single
Dirichlet boundary state of the free boson, but a sum of such states whose momenta lie
on the momentum lattice of the realization. The relative phase factor of each state with
different momentum in the summation is determined by the last equation of (4.20).

Let the boundary be at $|z|=1$. Writing the boundary values of $\phi(z)$, $\bar{\phi}(\bar{z})$ as functions
of $\zeta = -i \log z$, the Dirichlet condition (4.24) translates into
\[ \partial_\zeta (\phi + \bar{\phi}) = 0. \tag{4.25} \]
The constant value of $\phi + \bar{\phi} \equiv x$ at the boundary is an undetermined integration constant,
being a modulus of the free-boson boundary state. The $G^\pm$ boundary conditions in (4.20)
fix this modulus to be
\[ x = 0 \tag{4.26} \]
since
\[ e^{\pm \sqrt{3} \phi(\zeta)} + \eta e^{\mp \sqrt{3} \bar{\phi}(\zeta)} = 0. \tag{4.27} \]
(Using the label $s$, $\eta = \pm 1$ is absorbed into the overall constant of the $N=2$ boundary
states with definite ‘G-parity’.) Therefore, defining the free-boson vacua as
\[ |\gamma> \equiv e^{i \gamma (\phi(z)+\bar{\phi}(\bar{z}))}|0>, \]
\[ a_0 |\gamma> = |\bar{\alpha}_0 |\gamma> = \gamma |\gamma> , \tag{4.28} \]
an $N=2$ Ishibashi state can be written in a sum of free-boson Dirichlet states
\[ |l, m, s \gg_A = \sum_{n \in \mathbb{Z}} \exp \left( \sum_{k=1}^{\infty} \frac{a_{-k \pi -k}}{k} \right) |\gamma_{m,s} + 2 \sqrt{3} n >, \tag{4.29} \]
where
\[ \gamma_{m,s} = \sqrt{3} \left( \frac{m}{3} - \frac{s}{2} \right) \tag{4.30} \]
is the momentum of the $N=2$ vacuum state.

We next turn on the modulus $(\phi + \bar{\phi})|_{\text{boundary}} = x \neq 0$. The $G^\pm$ boundary conditions
then change into
\[ e^{\pm \sqrt{3} \phi(\zeta)} + \eta e^{\pm \sqrt{3} \bar{\phi}(\zeta)} e^{\mp \sqrt{3} \phi(\zeta)} = 0 \iff (G^+_r + i \eta e^{\pm \sqrt{3} \bar{\phi}(\zeta)} G^-_r)|B \gg_A, x = 0. \tag{4.31} \]
Thus, $|B \gg_{A,x}$ (whose defining relations are (4.20) with the last conditions replaced by (1.31)) is not an A-boundary state in the realization (1.21). However, if we use, for instance

$$\phi'(z) = \phi(z), \quad \overline{\phi'}(\tau) = \overline{\phi}(\tau) - x$$

(4.32)

for the new realization

$$G'^\pm(z) = \sqrt{2/3} e^{\pm i\sqrt{3} \phi'(z)}, \quad G'^\pm(\tau) = \sqrt{2/3} e^{\pm i\sqrt{3} \phi(\tau)},$$

(4.33)

it is clear that the modes of $G'^\pm(z), G'^\pm(\tau)$ satisfy the usual A-boundary conditions.

Let us now define a new one-parameter family of Ishibashi states $|l, m, s; \nu \gg_A$ by the equation

$$|l, m, s; \nu \gg_A = \sum_{n \in \mathbb{Z}} e^{\sum_{k=1}^\infty \frac{1}{k} e^{-i x (\gamma_{m,s} + 2\sqrt{3} n)} \gamma_{m,s} + 2\sqrt{3} n},$$

(4.34)

where $\nu \in \mathbb{R}$ is the modulus

$$x = -\frac{2\pi \nu}{\sqrt{3}}.$$  

(4.35)

One of the important effects of the modulus is that it causes a shift in the ‘$z$’ argument of characters in the transition amplitude

$$A \ll \tilde{l}, \tilde{m}, \tilde{s}; \tilde{\nu}, \nu \gg \Theta_{l, m, s; \nu \gg_A} = \sum_{n \in \mathbb{Z}} e^{1/3 \sum_{k=1}^\infty \frac{1}{k} e^{-i x (\gamma_{m,s} + 2\sqrt{3} n)} \gamma_{m,s} + 2\sqrt{3} n},$$

(4.36)

which replaces the relation (2.6). Clearly, the original A-boundary state $|l, m, s \gg_A$ is a special case for which $\nu$ is set equal to 0.

To summarize, $|l, m, s; \nu \gg_A$ is obtained by giving a phase factor of $e^{2\pi i \nu Q}$ to every state in the Ishibashi state $|l, m, s \gg_A$, depending on the $U(1)$ charge $Q$ of each state. $|l, m, s; \nu \gg_A$ is an A-boundary state for the $N = 2$ generators in a realization in which the free-boson modulus is shifted by $\nu$.

4.3 Boundary states at general angles

We will now construct the boundary state for D1-branes tilted at an angle of $\theta_{p,q}$ by using the one-parameter family of $N = 2$ Ishibashi states we have constructed in the

\footnote{More precisely, it should read $| \cdots \gg_{A,x}$ since it satisfies the same boundary conditions as $|B \gg_{A,x}$ does.}

However, if we use, for instance

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which replaces the relation (2.6). Clearly, the original A-boundary state $|l, m, s \gg_A$ is a special case for which $\nu$ is set equal to 0.

To summarize, $|l, m, s; \nu \gg_A$ is obtained by giving a phase factor of $e^{2\pi i \nu Q}$ to every state in the Ishibashi state $|l, m, s \gg_A$, depending on the $U(1)$ charge $Q$ of each state. $|l, m, s; \nu \gg_A$ is an A-boundary state for the $N = 2$ generators in a realization in which the free-boson modulus is shifted by $\nu$.

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\footnote{More precisely, it should read $| \cdots \gg_{A,x}$ since it satisfies the same boundary conditions as $|B \gg_{A,x}$ does.}
previous subsection. Suppose that we replace each Ishibashi state \(|l_j, m_j, s_j \gg A\) in \(|\alpha \gg A\) \((2.11)\) by one with a shifted modulus \(|l_j, m_j, s_j ; z_j \gg A\). Define

\[
|\alpha ; z \gg A \equiv \frac{1}{K_A} \sum_{\lambda, \mu} B^{\lambda, \mu}_\alpha |\lambda, \mu ; z \gg A, \]

\[
|\lambda, \mu ; z \gg A \equiv |s_0 \gg \prod_{j=1}^{r} |l_j, m_j, s_j ; z_j \gg A, \]

(4.37)

where the coefficients \(B^{\lambda, \mu}_\alpha\) are the same as before (given by \((2.2)\)). Using the formula for the modular transformations \((2.12)\), a calculation similar to \((2.9)\) leads us to the expression

\[
Z^A_{(\tilde{\alpha} ; \tilde{z}) (\alpha ; z)} (\tau) \equiv A < \Theta \tilde{\alpha} ; \tilde{z} | q^{L_0 - \frac{c}{24}} | \alpha ; z \gg A
\]

\[
= \sum_{K=1}^{K-1} \sum_{\lambda', \mu'} \sum_{\nu_0 = 0}^{\lambda'} \sum_{\nu_1 = 0}^{\mu'} (-1)^{\nu_0} \delta^{(4)}_{s_0, 2 + \tilde{s_0} - s_0 - 2 \nu_0 - 2} \sum_{j=1}^{r} \nu_j \prod_{j=1}^{r} \left( \delta_{l_j, \tilde{l}_j} \delta_{m_j, \tilde{m}_j} \delta_{s_j, \tilde{s}_j} \right) \sum_{j=1}^{r} \frac{\nu_j}{2(k_j + 2) (k_j + 2) (\tilde{k}_j - z_j)^2} \chi^{\lambda', \mu'} (\tau, \tau (z_j - \tilde{z}_j)), \]

(4.38)

where

\[
A < \lambda, \mu ; z | q^{L_0 - \frac{c}{24}} | \lambda, \mu ; z \gg A \equiv \chi^{\lambda, \mu} (\tilde{\tau}, z - \tilde{z})
\]

\[
\equiv \chi^{SO(d)}_{s_0} (\tilde{\tau}) \prod_{j=1}^{r} \chi^{l_j, s_j}_{m_j} (\tau, z_j - \tilde{z}_j), \]

(4.39)

and

\[
\chi^{\lambda', \mu'} (\tau, \tau (z - \tilde{z})) = \chi^{SO(d)}_{s_0'} (\tau) \prod_{j=1}^{r} \chi^{l_j, s_j'}_{m_j} (\tau, \tau (z_j - \tilde{z}_j)).
\]

(4.40)

Thus we see that each minimal character gets a spectral flow in the open channel because of the modulus shift.

We have already expressed the partition function \(Z^{\text{open}}_{\theta = \theta_{p,q}} (\tau) \) \((4.14)\) in a sum over spectral-flowed \(\beta\)-orbits \((4.17)\), so we use \(|\alpha ; z \gg A\) as a building block for the \(\theta = \theta_{p,q}\) boundary state. Writing it out explicitly,

\[
|\alpha ; z \gg A = \frac{1}{2^{3+3+i}} \sum_{s_0, l_j, m_j, s_j} B^{\lambda, \mu}_\alpha |s_0 \gg \prod_{j=1}^{3} |l_j, m_j, s_j ; z_j \gg A, \]

(4.41)
where $B_{\lambda,\mu}^{\alpha}$ is given by (4.3). In this case the partition function simply reduces to

$$Z_A^{(\alpha,\tilde{z})(\alpha,z)}(\tau) = q^{\frac{1}{6} \sum_{j=1}^{3} (z_j - \tilde{z}_j)^2} \sum_{\nu_1, \nu_2, \nu_3 \in \mathbb{Z}_2} \text{Orbit} \left( 2 - 2 \sum_{j=1}^{3} \nu_j; (0,0,2\nu_j); \tau(z_j - \tilde{z}_j) \right),$$

(4.42)

which becomes identical (up to an overall factor) to each spectral-flowed orbit in the summation (4.17), provided that $z_j$ are set equal to $z_j(\Delta m)$ (4.18). Fixing the normalization correctly, we find that

$$|\theta = \theta_{p,q} > \equiv \frac{1}{\sqrt{2(p^2 + pq + q^2)}} \sum_{\Delta m \in \mathbb{Z}_{p^2 + pq + q^2}} |\alpha ; z(\Delta m) >_A$$

(4.43)

precisely yields

$$Z_{\theta=\theta_{p,q}}^{\text{open}}(\tau) = < \theta = \theta_{p,q}| \theta > \equiv q^{\frac{1}{60} - \frac{2\pi}{24}} |\theta = \theta_{p,q} >.$$  

(4.44)

and therefore represents a D-brane tilted at an angle of $\theta_{p,q}$. Thus we have obtained, starting from the Recknagel-Schomerus states, the new ones reproducing the geometric D1-brane partition functions at arbitrary angles.

### 4.4 Projected RS boundary states

Let us examine the relation of our new boundary states to the Recknagel-Schomerus states more closely. As we saw in subsect.4.2, each state contained in $|l,m,s;z>_A$ acquires a phase factor of $e^{2\pi izQ}$, depending on its $N = 2U(1)$ charge $Q$. If we define

$$P_{p,q}|\lambda,\mu >_A \equiv \frac{1}{p^2 + pq + q^2} \sum_{\Delta m \in \mathbb{Z}_{p^2 + pq + q^2}} |\lambda,\mu ; z(\Delta m) >_A$$

$$= \frac{1}{p^2 + pq + q^2} \sum_{\Delta m \in \mathbb{Z}_{p^2 + pq + q^2}} |s_0 \gg \prod_{j=1}^{3} |l_j, m_j, s_j ; z_j(\Delta m) >_A$$

(4.45)

for $|\lambda,\mu >_A$ satisfying the $\beta$-condition, then $P_{p,q}$ acts as a projection operator selecting only states such that

$$\sum_{j=1}^{3} Q_j \cdot z_j(\Delta m = 1) \in \mathbb{Z},$$

(4.46)

where

$$Q_j = \frac{m_j}{3} - \frac{s_j}{2} + 2n_j \quad (n_j \in \mathbb{Z})$$

(4.47)
is the $U(1)$ charge of the $j$th $N = 2$ minimal model. Using this $P_{p,q}$, the boundary state $|\theta = \theta_{p,q}\rangle$ can be compactly written in the form

$$|\theta = \theta_{p,q}\rangle = \sqrt{p^2 + pq + q^2} P_{p,q}|\theta = 0\rangle.$$  \hfill (4.48)

The $U(1)$-charge lattice points of $|\lambda, \mu \gg_\Lambda$ get ‘thinned out’ after the projection (Figure 2). The projected boundary state $P_{p,q}|\theta = 0\rangle$ is no longer a sum of products of Ishibashi states in general, unless

$$\frac{2 \left[ (-p - 2q)n_1 + (2p + q)n_2 + (-p + q)n_3 \right]}{p^2 + pq + q^2} \in \mathbb{Z} \quad (4.49)$$

holds for any integers $n_j$. \textbf{(4.49)} has a solution if and only if $p^2 + pq + q^2 = 1, 3$, which respectively correspond to $\theta = 0, \frac{\pi}{6}$ (mod $\frac{\pi}{3}$). In these special cases the boundary state $|\theta = \theta_{p,q}\rangle$ reduces to a single ($\theta = 0$) or a sum ($\theta = \frac{\pi}{6}$) of Recknagel-Schomerus state(s), but otherwise it cannot be written in such forms.

The density of the $U(1)$-charge lattice gets thinner as the length of the brane becomes longer. On the contrary, the open-channel spectrum gets richer due to the modular transformation, which is in agreement with the physical interpretation.

### 4.5 Proper free-field realizations

We have shown how the geometric D-brane partition functions on a torus are reproduced from the boundary states in the $(k = 1)^3$ Gepner model. But this is not the end of the story. Suppose that we consider the partition function between the boundary states with \textit{different} angles $|\theta = \theta_{p,q}\rangle$ and $|\theta = \theta_{p,q}'\rangle$. Although one would expect

$$\langle \theta = \theta_{p,q}' | q^{L_0 - \frac{c}{12}} | \theta = \theta_{p,q}\rangle$$

\hfill (4.50)

to coincide with the geometric answer, it does not yield the correct result $Z_{\theta_{p,q}, \theta_{p,q}'}^{\text{open}}(\tau)$ \textbf{(3.36)} if the calculation is done using the formula \textbf{(4.39)}.

To gain insight into the problem, let us examine how the Gepner model reconstructs the momentum-winding lattices of the geometric boundary states. We consider the $\theta = 0$ case first. Its momentum lattice is the most dense, and the corresponding boundary state on the Gepner-model side is the one given by the Recknagel-Schomerus construction, a collection of all possible combinations of $N = 2$ boundary states that satisfy the $\beta$-condition. Each of these $N = 2$ A-boundary states further decomposes into a sum of free-boson Dirichlet states, whose momenta $\gamma_j$ ($j = 1, 2, 3$) lie on the cubic lattice

$$\begin{align*}
(\gamma_1, \gamma_2, \gamma_3) &= \frac{1}{\sqrt{12}}(\hat{m}_1 + 12n_1, \hat{m}_2 + 12n_2, \hat{m}_3 + 12n_3), \\
n_j &\in \mathbb{Z} \quad (j = 1, 2, 3).
\end{align*} \quad \hfill (4.51)$$

20
The whole momentum lattice of the $\theta = 0$ boundary state is the direct sum of (4.51) over all possible integers $\hat{m}_j$ ($j = 1, 2, 3$) that fulfill the $\beta$-condition

$$\sum_{j=1}^{3} \hat{m}_j = -3s_0 + 6 \quad (\text{mod } 12).$$

(4.52) $\hat{m}_j$ must be all even or all odd. If a particular set of $\hat{m}_j$ satisfy these conditions, then

$$(\hat{m}_1, \hat{m}_2, \hat{m}_3) + k_1(2, 0, -2) + k_2(0, 2, -2)$$

also do for any $k_1, k_2 \in \mathbb{Z}_6$. Thus, the two-dimensional sub-lattice consisting of the points on the plane

$$\sum_{j=1}^{3} \gamma_j = 0$$

is given by

$$(\gamma_1, \gamma_2, \gamma_3) = \frac{1}{\sqrt{12}} \left( (\hat{m}_1^{(0)}, \hat{m}_2^{(0)}, \hat{m}_3^{(0)}) + k_1(2, 0, -2) + k_2(0, 2, -2) \right),$$

(4.55) where $k_1, k_2$ may now take any integer values. $(\hat{m}_1^{(0)}, \hat{m}_2^{(0)}, \hat{m}_3^{(0)})$ is some reference lattice point, $(0, 0, 0)$ for instance. It is then easy to check that the shift

$$(\hat{m}_1^{(0)}, \hat{m}_2^{(0)}, \hat{m}_3^{(0)}) \rightarrow (\hat{m}_1^{(0)}, \hat{m}_2^{(0)}, \hat{m}_3^{(0)}) + \nu (1, 1, 1) \quad (\nu \in \mathbb{Z})$$

(4.56) exhausts all the points of the $\theta = 0$ momentum lattice. The $\theta = 0$ lattice is, thus, a direct product of the one-dimensional lattice

$$(\gamma_1, \gamma_2, \gamma_3) = \frac{\nu}{\sqrt{12}} (1, 1, 1), \quad (\nu \in \mathbb{Z})$$

(4.57) and the two-dimensional one (4.55), which are orthogonal to each other.

Of course, this decomposition of the momentum lattice just rephrases the multiplication formula for the theta functions in section 4, though we are now working in the closed channel. The extracted one-dimensional lattice (4.57) is nothing but the $N = 2$ total $U(1)$-charge lattice, yielding the free-fermion theta in the partition function. Also, the two-dimensional lattice (4.55) coincides with the momentum-winding lattice of the compact bosons on the torus. The latter can be verified as follows: Let $\phi^{j}_{\theta = 0}(z)$ ($j = 1, 2, 3$) be the free bosons that realize the $k = 1$ minimal models and describe the boundary
state for the $\theta = 0$ D-brane in the Gepner model. Change the orthogonal basis of the free bosons as
\[
\begin{pmatrix}
\varphi^1_{\theta=0} \\
\varphi^2_{\theta=0} \\
\varphi^3_{\theta=0}
\end{pmatrix} \equiv \begin{pmatrix}
\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
\phi^1_{\theta=0} \\
\phi^2_{\theta=0} \\
\phi^3_{\theta=0}
\end{pmatrix}.
\] (4.58)

$\varphi^3_{\theta=0}$ is the total $U(1)$ current. We also define $\bar{\varphi}^j_{\theta=0}(z)$ and $\bar{\varphi}^j_{\theta=0}(z)$ ($j = 1, 2, 3$) as their anti-holomorphic counterpart, being related by the same transformation as (4.58). The vertex operator of the bosons $\phi^j_{\theta=0}(z)$ carrying the momenta (4.55) (with $(\hat{m}^{(0)}_1, \hat{m}^{(0)}_2, \hat{m}^{(0)}_3) = (\nu, \nu, \nu)$) becomes
\[
\exp \left( i \sum_{j=1}^3 \gamma_j \bar{\varphi}^j_{\theta=0} \right) = \exp \left( i \sqrt{3} \left( \frac{1}{2} k_2 \varphi^1_{\theta=0} + \left( k_1 + \frac{k_2}{2} \right) \varphi^2_{\theta=0} + i \frac{\nu}{2} \varphi^3_{\theta=0} \right) \right) (4.59)
\]

$(k_1, k_2 \in \mathbb{Z})$, and similarly for the anti-holomorphic bosons. Thus, $\varphi^3_{\theta=0}$ and $\varphi^3_{\theta=0}$ indeed are the bosonized complex fermions. On the other hand, the momentum-winding lattice of the compact bosons
\[
(\hat{X}^1_{\theta=0,L}, \hat{X}^2_{\theta=0,L}, \hat{X}^1_{\theta=0,R}, \hat{X}^2_{\theta=0,R})
\] (4.60)
is
\[
\sqrt{2} \left( \frac{\sqrt{3}}{2} m, n + \frac{m}{2}; -\frac{\sqrt{3}}{2} m, n + \frac{m}{2} \right) (m, n \in \mathbb{Z}) (4.61)
\]
$(\Lambda_{1,0} (3.20))$, so this allows us to identify
\[
\begin{align*}
\hat{X}^1_{\theta=0,L} &= -\varphi^1_{\theta=0}, \\
\hat{X}^2_{\theta=0,L} &= \varphi^2_{\theta=0}, \\
\hat{X}^1_{\theta=0,R} &= \varphi^1_{\theta=0}, \\
\hat{X}^2_{\theta=0,R} &= \varphi^2_{\theta=0}.
\end{align*}
\] (4.62)
Thus we see that the three free bosons of the $(k = 1)^3$ Gepner model combine into a single complex free fermion and a pair of T-dual coordinates of the torus. The minus sign in (4.62) arises because the Neumann-Dirichlet boundary states are described here by only the A-boundary states.

We next consider the lattice for a general angle $\theta_{p,q}$. As we saw in the previous subsection, it is obtained from the $\theta = 0$ lattice by the projection (4.47). Leaving only the points that satisfy (4.46) yields the lattice
\[
(\gamma_1, \gamma_2, \gamma_3) = \frac{1}{\sqrt{12}} \left( (\hat{m}^{(0)}_1, \hat{m}^{(0)}_2, \hat{m}^{(0)}_3) + 2l_1(p, q, -p - q) + 2l_2(-q, p + q, -p) \right) (4.63)
\]
\( (l_1, l_2 \in \mathbb{Z}) \). The vertex operator reads

\[
\exp \left( i \sum_{j=1}^{3} \gamma_j \phi^j_{\theta = \theta_{p,q}} \right) = \exp \left( i \frac{2(p^2 + pq + q^2)}{3} \left( -\frac{\sqrt{3}}{2} l_2 \phi^1_{\theta = \theta_{p,q}} + \left( l_1 + \frac{l_2}{2} \right) \phi^2_{\theta = \theta_{p,q}} + \frac{i}{2} \phi^3_{\theta = \theta_{p,q}} \right) \right), \tag{4.64}
\]

where \( \phi^j_{\theta = \theta_{p,q}} (j = 1, 2, 3) \) are again the free bosons in which the \( N = 2 \) minimal models are realized, but here the distinction from those for the \( \theta = 0 \) D-brane is already anticipated. \( \phi^j_{\theta = \theta_{p,q}} \) are given by the relation

\[
\begin{pmatrix}
\phi^1_{\theta = \theta_{p,q}} \\
\phi^2_{\theta = \theta_{p,q}} \\
\phi^3_{\theta = \theta_{p,q}}
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_{p,q} & -\sin \theta_{p,q} \\
\sin \theta_{p,q} & \cos \theta_{p,q}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
\phi^1_{\theta = \theta_{p,q}} \\
\phi^2_{\theta = \theta_{p,q}} \\
\phi^3_{\theta = \theta_{p,q}}
\end{pmatrix}, \tag{4.65}
\]

The anti-holomorphic bosons \( \bar{\phi}^j_{\theta = \theta_{p,q}}, \bar{\phi}^j_{\theta = \theta_{p,q}} \) satisfy the same relation. To compare with the geometric momentum-winding lattice at \( \theta = \theta_{p,q} \), we go to the local Lorentz frame \( \mathbf{B.39} \). Writing the lattice \( \Lambda_{p,q} \mathbf{B.20} \) as the momentum-winding lattice of the bosons

\[
(\hat{X}^1_{\theta = \theta_{p,q}, L}, \hat{X}^2_{\theta = \theta_{p,q}, L}, \hat{X}^1_{\theta = \theta_{p,q}, R}, \hat{X}^2_{\theta = \theta_{p,q}, R}) \tag{4.66}
\]

in this frame, we obtain

\[
\sqrt{\frac{2(p^2 + pq + q^2)}{3}} \left( \frac{\sqrt{3}}{2} m, n + \frac{m}{2}; -\frac{\sqrt{3}}{2} m, n + \frac{m}{2} \right) \tag{4.67}
\]

\((m, n \in \mathbb{Z})\). Therefore, comparing with \( \mathbf{I.64} \), we may again identify the geometric and the Gepner-model bosons through the relations

\[
\left\{
\begin{array}{l}
\hat{X}^1_{\theta = \theta_{p,q}, L} = -\phi^1_{\theta = \theta_{p,q}}; \\
\hat{X}^2_{\theta = \theta_{p,q}, L} = -\phi^2_{\theta = \theta_{p,q}};
\end{array}
\right.
\quad
\left\{
\begin{array}{l}
\hat{X}^1_{\theta = \theta_{p,q}, R} = \phi^1_{\theta = \theta_{p,q}}; \\
\hat{X}^2_{\theta = \theta_{p,q}, R} = \phi^2_{\theta = \theta_{p,q}}.
\end{array}
\right. \tag{4.68}
\]

We are at last in a position to understand why the naive evaluation of \( \mathbf{I.50} \) did not give \( Z_{\theta_{p,q}}^{\text{open}} (\mathbf{B.30}) \). Owing to the sign-flip which occurs in the left-moving part \( \mathbf{I.68} \), the free bosons \( \phi^a_{\theta = \theta_{p,q}} \) and \( \bar{\phi}^a_{\theta = \theta_{p,q}} (a = 1, 2) \) rotate in the opposite directions

\[
\phi^a_{\theta = \theta_{p,q}} = R(-\Delta \theta)^a b \phi^b_{\theta = \theta_{p,q}}; \quad \bar{\phi}^a_{\theta = \theta_{p,q}} = R(\Delta \theta)^a b \bar{\phi}^b_{\theta = \theta_{p,q}} \tag{4.69}
\]

with

\[
R(\Delta \theta)^a b = \begin{pmatrix}
\cos \Delta \theta & \sin \Delta \theta \\
-\sin \Delta \theta & \cos \Delta \theta
\end{pmatrix}, \quad \Delta \theta = \theta_{p,q} - \theta_{p,q}. \tag{4.70}
\]
Therefore, if \( \theta_{p,q} \neq \theta_{p,q} \), an A-boundary state written in \( \varphi^a_{\theta=\theta_{p,q}} \), \( \overline{\varphi}^a_{\theta=\theta_{p,q}} \) does not have equal left-right momenta in \( \varphi^a_{\theta=\theta_{p,q}} \), \( \overline{\varphi}^a_{\theta=\theta_{p,q}} \), except for the zero-momentum state. Another striking consequence of the relations (4.69) is that the free bosons \( \phi^j_{\theta=\theta_{p,q}} \), \( \overline{\phi}^j_{\theta=\theta_{p,q}} \) realizing individual \( N=2 \) minimal models are not the same if the angle of the D-brane is different.

To distinguish one tilted D-brane from another, one must specify which free fields are used to realize the \( N=2 \) minimal models!

Taking into account the rotation (4.69), we now write
\[
|\theta = \theta_{p,q} > = \sqrt{p^2 + pq + q^2} P_{p,q}|\theta = 0 >_{p,q},
\]
where by \( |\theta = 0 >_{p,q} \) we denote the previous \( \theta = 0 \) boundary state but made up of the particular realization \( \phi^j_{\theta=\theta_{p,q}} \), \( \overline{\phi}^j_{\theta=\theta_{p,q}} \) \((j = 1, 2, 3)\). We also write the product of Ishibashi states \( |\lambda, \mu ; z >_{p,q} \) in the same meaning. Then the amplitude
\[
\hat{\theta}_{p,q} \ll \lambda, \mu ; \tilde{z} | q^{L_0 - \frac{\Delta}{2}} |\lambda, \mu ; z >_{p,q} \quad (4.72)
\]
is not just a product of characters, but instead given, for \((\lambda, \mu) = (s_0; (l_j, m_j, s_j))\), by
\[
(4.72) = \chi_{s_0}^{SO(6)}(\tilde{\tau}) \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^{2i\Delta\theta})(1 - q^n e^{-2i\Delta\theta})}{1 - q^n} \Theta_{-3s_0 + 6 + 24r, 18(\tilde{\tau}, \frac{\Delta\theta}{3\pi})} (4.73)
\]
where
\[
\sum_{j=1}^{3} \tilde{m}_j = -3s_0 + 6 + 12r, \quad \tilde{m}_j = 2m_j - 3s_j, \quad r \in \mathbb{Z}_3. \quad (4.74)
\]
The second factor of (4.73) comes from the oscillator excitations of \( \varphi^a_{\theta=\theta_{p,q}} \), \( \overline{\varphi}^a_{\theta=\theta_{p,q}} \) and \( \varphi^3_{\theta=\theta_{p,q}} \), \( \overline{\varphi}^3_{\theta=\theta_{p,q}} \) \((a = 1, 2)\). By construction, the calculations are the same as before for the compact bosons. Also, we have no theta here because the momentum overlap occurs only at one point. The \( z \) or \( \tilde{z} \) dependence thus disappears. The last factor is the contribution from \( \varphi^3_{\theta=\theta_{p,q}} \), \( \overline{\varphi}^3_{\theta=\theta_{p,q}} \) and \( \varphi^3_{\theta=\theta_{p,q}} \), \( \overline{\varphi}^3_{\theta=\theta_{p,q}} \). The denominator is from the ordinary oscillator contribution, whereas the theta function comes from the momentum overlap of \( \varphi^3_{\theta=\theta_{p,q}} \) and \( \varphi^3_{\theta=\theta_{p,q}} \). Here we have identified these bosons through the modulus shift
\[
\varphi^3_{\theta=\theta_{p,q}} + \overline{\varphi}^3_{\theta=\theta_{p,q}} = \varphi^3_{\theta=\theta_{p,q}} + \overline{\varphi}^3_{\theta=\theta_{p,q}} + 2\Delta\theta \quad (4.75)
\]
in order to take into account the rotation of the compact fermions, so that the level-18 theta acquires \( \Delta\theta/3\pi \). Thus, in all, we obtain the equality (4.73). Using this formula,
we calculate the partition function as

\[
< \theta = \theta_{p,q} | \bar{q}^{L_0 - \frac{c}{24}} | \theta = \theta_{p,q} > \\
= \sqrt{p^2 + \tilde{p} \tilde{q} + \bar{q}^2} \sqrt{p^2 + pq + q^2} \theta_{p,q} < \theta = 0 | \bar{q}^{L_0 - \frac{c}{24}} | \theta = 0 > _{\theta_{p,q}} \\
= \sqrt{p^2 + \tilde{p} \tilde{q} + \bar{q}^2} \sqrt{p^2 + pq + q^2} \left( \frac{1}{\sqrt{2}} \right)^2 \left( \frac{2^3}{2^{\frac{10}{3}} \pi^3} \right)^2 \\
\cdot \sum_{s_0 \in \mathbb{Z}_4} \sum_{r \in \mathbb{Z}_3} (1)^{\delta_s} \cdot \frac{\theta_{\tilde{q}}^{s_0} \chi_{\tilde{s}}^{SO(6)}(\tilde{\tau}) \Theta_{-s_0 + 6 + 12r, 18}(\tilde{\tau}, \frac{\Delta \theta}{3 \pi})}{\prod_{n=1}^{\infty} \left( (1 - \tilde{q}^n e^{2i \Delta \theta}) (1 - \tilde{q}^n e^{-2i \Delta \theta}) (1 - \tilde{q}^n) \right)} \\
= \sqrt{\left( p^2 + \tilde{p} \tilde{q} + \bar{q}^2 \right) \left( p^2 + pq + q^2 \right)} \frac{3}{4} \\
\cdot \sum_{s_0 \in \mathbb{Z}_4} (1)^{\delta_s} \cdot 2 \sin \Delta \theta \cdot \frac{\chi_{\tilde{s}}^{SO(6)}(\tilde{\tau}) \Theta_{s_0 + 2, 2}(\tilde{\tau}, \frac{\Delta \theta}{\pi})}{\psi_1(\tilde{\tau}, \frac{\Delta \theta}{\pi})}.
\]

(4.76)

The projection operator \( P_{p,q} \) drops out since it is inert for the states having zero \( \varphi_{\theta = \theta_{p,q}} \) momenta \( (\alpha = 1, 2) \). By a modular transformation, the final expression (4.76) precisely reproduces \( Z_{\theta_{p,q}, \theta_{p,q}}^{open}(\tau) \) (3.36) in section 3.

### 4.6 The Witten index

The angle-dependent identification (4.69) is also essential for the Witten index to give the correct intersection numbers of the cycles. The open-string Witten index for the D-branes tilted at angles of \( \theta = \theta_{p,q} \) and \( \theta_{p,q} \) \( (p, q) \neq (p, q) \) is defined by the transition amplitude between the corresponding boundary states with the \((-1)^{F_L}\) insertion in the closed channel

\[
I_{\theta_{p,q}, \theta_{p,q}} \equiv \text{R, int} < \theta = \theta_{p,q} | (-1)^{F_L} \bar{q}^{L_0 - \frac{c}{24}} | \theta = \theta_{p,q} > \text{R, int},
\]

(4.77)

where \text{R, int} indicates that we only consider the internal part of the boundary state in the Ramond sector. The total fermion number is

\[
F_L = \frac{1}{2} + \frac{1}{6} \sum_{j=1}^{3} \tilde{m}_j \in \mathbb{Z}.
\]

(4.78)

Using (4.73) with the \( SO(6) \) character being removed, \( I_{\theta_{p,q}, \theta_{p,q}} \) is similarly calculated as

\[
I_{\theta_{p,q}, \theta_{p,q}} = \sqrt{p^2 + \tilde{p} \tilde{q} + \bar{q}^2} \sqrt{p^2 + pq + q^2} \theta_{p,q} \text{R, int} < \theta = 0 | (-1)^{F_L} \bar{q}^{L_0 - \frac{c}{24}} | \theta = 0 > _{\theta_{p,q}} \\
= \sqrt{p^2 + \tilde{p} \tilde{q} + \bar{q}^2} \sqrt{p^2 + pq + q^2} \left( \frac{1}{\sqrt{2}} \right)^2 \left( \frac{2^3}{2^{\frac{10}{3}} \pi^3} \right)^2
\]

Using (4.78) with the \( SO(6) \) character being removed, \( I_{\theta_{p,q}, \theta_{p,q}} \) is similarly calculated as
\[ \sum_{s_0=\pm 1} \sum_{r \in \mathbb{Z}} (-1)^{s_0+\frac{1}{2}+\frac{r}{6} \sum_{j=1}^{3} \hat{m}_j} \cdot \frac{\tilde{q}^{-\frac{1}{2}} \Theta_{-3s_0+6+12r, 18}(\tilde{r}, \frac{\Delta m}{3\tilde{r}})}{\prod_{n=1}^{\infty} ((1 - \tilde{q}^n e^{2i\Delta m})(1 - \tilde{q}^n e^{-2i\Delta m})(1 - \tilde{q}^n))} \]

\[ = \sqrt{\left( p^2 + \tilde{p} q + \tilde{q}^2 \right) (p^2 + pq + q^2)} \cdot 2i \sin \Delta \theta \]

which is precisely \((i \text{ times})\) the intersection number of two wound branes on the torus. If we take the total (internal) \(U(1)\) charge \(J_{\theta}^{N=2} = \frac{1}{6} \sum_{j=1}^{3} \hat{m}_j\) in place of \(F_L\) in the definition, we get an integer index without \(i\).

5 Comments on the \((k = 2)^2\) model

In this section, we briefly outline how to construct in the \((k = 2)^2\) Gepner model the boundary states for arbitrary \((p, q)\) branes on the corresponding \(SU(2)^2\) torus. In this case, the geometric partition function for parallel branes (3.35) is given by

\[ Z_{\text{open}}^{\theta = \theta_{p,q}}(\tau) = \text{(Jacobi)} \cdot \frac{1}{\eta^2(\tau)} \sum_{m,n \in \mathbb{Z}} q^{\frac{m^2+n^2}{p^2+q^2}} \sum_{\Delta m \in \mathbb{Z}^2} q^{\frac{\Delta m^2}{p^2+q^2}} \sum_{\nu \in \mathbb{Z}_4} (-1)^\nu \chi_{-\nu}^{SO(6)}(\tau) \]

\[ \cdot \frac{1}{\eta^3(\tau)} \Theta_{\nu+2,2}(\tau, 0) \sum_{s \in \mathbb{Z}_2} \Theta_{2s,2}(\tau, \frac{(p-q)\tau \Delta m}{p^2+q^2}) \Theta_{2s,2}(\tau, \frac{(p+q)\tau \Delta m}{p^2+q^2}). \]

As in the case of the \(SU(3)\) torus, this partition function can be thought of as obtained from a suitable projection of the \(\theta = 0\) partition function in the closed-string channel.

Let us discuss the corresponding projection in the Gepner model. It was found [20] that the partition function for \(\theta = 0\) \((p = 1, q = 0)\) is reproduced from the amplitude (2.16) for the Recknagel-Schomerus boundary states. This is shown by rearranging the product of two \(k = 2\) minimal model characters

\[ \chi_m^{l,s}(\tau, z) = c_{m-s}(\tau) \Theta_m(\tau, \frac{z}{2}) \]

in the amplitude as a triple product of theta functions. One of the last two theta’s in (5.1) comes from the product of the string functions \(c_{m-s}(\tau)\), which are essentially the partition functions for the parafermions in the free-field realizations. The important point is, for a general \(\theta_{p,q}\), that the \(\Delta m\) dependence in those theta’s results in a particular projection in the closed-string channel after summing over \(\Delta m\). Therefore, in the \((k = 2)^2\) case, we see that not only the \(U(1)\) charges but also the parafermions are subject to the projection.
The easiest way to handle such a projection is to bosonize the parafermions, which are real free fermions in this model. Once we rewrite the Recknagel-Schomerus boundary state for \( \theta = 0 \) in terms of the free bosons, the boundary state for \( \theta_{p,q} \) is obtained by the projection similarly to the \((k = 1)^3\) case. The bosonization also enables us to specify the free-field realizations, which is needed to reproduce the partition functions at angles. The details of the construction will be reported elsewhere.

6 Conclusions

We found that, to represent infinitely many wound D1-branes in the \((k=1)^3\) Gepner model, we can no longer keep, in general, the ‘structure’ of \( N = 2 \) Ishibashi states. We need to project out some states to have a rich spectrum in the open channel, but then this forces us to give up imposing any fixed A-boundary conditions on the individual \( N = 2 \) minimal models. The projected boundary state is obtained by writing the Recknagel-Schomerus states in terms of free-boson boundary states with moduli and summing over shifts. This shift is orthogonal to the total \( U(1) \) charge, and therefore does not break spacetime supersymmetry. Each boundary state in the summation can be regarded as an \( N = 2 \) A-boundary state in some realization, but the projected state as a whole is not.

Another unexpected aspect of the new boundary states is the necessity of the proper realization for each angle of the D-brane. We found it necessary, in particular, for the Witten index to correctly yield the intersection numbers, at least for the Gepner-model descriptions of the toroidal compactifications. After all, to describe infinitely many supersymmetric D-branes, we need more information than just an algebraic \( N = 2 \) Ishibashi state has, that is, in which free fields it is realized.

One might then ask, “Why have the consistent intersection numbers been obtained so far without taking into account such ‘angle-wise’ realizations in the literature?” Our explanation for this is that the parafermions in those models carry some information on the angles of the branes, and probably the previous discussions were the special cases in which no distinction among the realizations was necessary.

It would be extremely interesting to study whether a similar projection in general Gepner models yields new boundary states representing D-branes wound around more general supersymmetric cycles of the Calabi-Yau. We again start from a Recknagel-Schomerus boundary state. The guiding principle is that its constituent states get projected and thinned out depending on the \( N = 2 \) \( U(1) \) charges they carry. The resulting boundary state is supersymmetric if and only if the shift of the \( U(1) \)-boson modulus is orthogonal.
to the total $U(1)$ current. To proceed further, we must consider the following:

First, we need to have some convenient realization for the parafermions to systematically project out some of the states from the parafermion state space. The parafermion piece of the $N = 2$ free fields becomes trivial if $k = 1$, which allowed us in the $(k = 1)^3$ model to only consider the projection for the $U(1)$ bosons. As is already clear in the $(k = 2)^2$ case, not only the free-boson piece but also the parafermion piece of the states are subject to the projection, in a correlated manner. In the $(k = 2)^2$ example, the two real fermions ($= Z_2$ parafermions) are conveniently bosonized, and get projected similarly to the $(k = 1)^3$ model. In general cases, the Wakimoto or the coset realization might be of use for this purpose. Another thing we have to worry about is how to determine the relations among the free-field realizations for different projected boundary states. This will be hard because we have no analogue of the toroidal CFT description in general. One possible criterion is to require that the boundary states have integral Witten indices among themselves.

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Figure 1 The \((p, q)\) cycle.
Figure 2 The momentum lattice of the boundary state for $s_0 = 0$ representing (a) the $\theta = 0$ D-brane. (b) The $\theta = \frac{\pi}{6}$ D-brane. The lattice points form lines parallel to the total $U(1)$ direction ($\beta$-orbit). (The axes are scaled by 6.) They are thinned out due to the projection.