Harmonic Forms on Manifolds with Non-Negative Bakry-Émery-Ricci Curvature

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Abstract

In this paper we prove that on a complete smooth metric measure space with non-negative Bakry-Émery-Ricci curvature if the space of weighted $L^2$ harmonic one-forms is non-trivial then the weighted volume of the manifold is finite and universal cover of the manifold splits isometrically as the product of the real line with an hypersurface.

Keywords: Harmonic forms, Non-negative Bakry-Émery-Ricci curvature, Smooth metric measure spaces.

1 Introduction

The theory of $L^2$ harmonic forms together with harmonic functions has been used to study the geometry and topology of complete manifolds (for example in [8, 9, 10, 11]). The theory of smooth metric measure spaces has also been attracting some interest due to its connection with the Ricci flow (see [2]) and as an independent research topic: there are works about volume estimates [16, 14], essential spectrum of the drifted Laplacian [15], etc. In [11] Lott discussed (among other things) the topology of compact smooth metric measure spaces with non-negative Bakry-Émery-Ricci curvature using harmonic forms, and in [12] Munteanu and Wang obtained geometrical and topological results when these manifolds are non-compact using harmonic functions (see also [13, 14]). In this paper we investigate harmonic forms on non-compact smooth metric measure spaces with non-negative Bakry-Émery-Ricci curvature.

Recall that a smooth metric measure space $(M, g, e^{-f} dv)$ is a Riemannian manifold $(M, g)$ together with a smooth function $f$ and a measure $e^{-f} dv$. The Bakry-Émery-Ricci curvature is defined by the formula

$$\text{Ric}_f = \text{Ric} + \text{Hess} f.$$

A differential form $\omega$ is called an $L^2_f$ differential form if

$$\int_M |\omega|^2 e^{-f} dv < \infty.$$
It is well known that the formal adjoint of the exterior derivative $d$ with respect to the $L^2$ inner product is given by the formula

$$\delta_f = \delta + \iota\nabla f,$$

where $\iota\nabla f$ denotes the interior product with the vector field $\nabla f$ (see [1] or Lemma [2,1]). The $f$-Hodge Laplacian is defined by the formula

$$\Delta_f = -(d\delta_f + \delta_f d).$$

The space of $L^2_f$ harmonic one-forms is the set of all $L^2_f$ one-forms $\omega$ satisfying the equation

$$\Delta_f \omega = 0.$$

In this article we prove the following result (Theorem 4.1).

**Theorem 1.1.** Let $(M^n, g, e^{-f}dv)$ be a complete non-compact smooth metric measure space with non-negative Bakry-Émery-Ricci curvature. If the space of $L^2_f$ harmonic one-forms is non-trivial then the weighted volume of $M^n$ is finite, that is

$$\text{vol}_f(M^n) = \int_{M^n} e^{-f}dv < \infty,$$

and the universal covering splits isometrically as $\tilde{M}^n = \mathbb{R} \times N^{n-1}$.

A corresponding result for compact manifolds is discussed in [11] (Theorem 1, item 3). In particular we obtain the following result (Corollary 4.2) concerning the vanishing of $L^2_f$ harmonic forms when the Bakry-Émery-Ricci curvature is non-negative and the first eigenvalue of the $f$-Laplacian is positive, which applies to non-trivial gradient Ricci steady solitons (see Corollary 4.3).

**Corollary 1.2.** Let $(M, g, e^{-f}dv)$ be a complete non-compact smooth metric measure space with non-negative Bakry-Émery-Ricci curvature. If the first eigenvalue of the $f$-Laplacian is positive then the space of $L^2_f$ harmonic one-forms is trivial.

We also obtain the following vanishing result (Corollary 4.4) for the space of $L^2_f$ harmonic forms when the function $f$ is bounded.

**Corollary 1.3.** Let $(M, g, e^{-f}dv)$ be a complete non-compact smooth metric measure space with non-negative Bakry-Émery-Ricci curvature. If the function $f$ is bounded then the space of $L^2_f$ harmonic one-forms is trivial.

In section 2 we prove that $L^2_f$ harmonic forms of any degree are closed and co-closed, in section 3 we prove Bochner’s formula and Kato’s inequality for $L^2_f$ harmonic one-forms, in section 4 we prove Theorem 4.1 and some corollaries.

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2 \(L^2_f\) Harmonic Forms are Closed and Co-Closed

In this section we extend to smooth metric measure spaces the known fact that \(L^2\) harmonic forms of any degree are closed and co-closed.

Recall that the dot product of differential forms (of the same degree) on a Riemannian manifold with a volume element \(dv\) is defined by

\[(ω · η)dv = ω ∧ *η.\]

The \(L^2_f\) inner product of \(L^2_f\) differential forms is defined by

\[(ω, η)_{L^2_f(M)} = \int_M ω · η e^{-f} dv.\]

Notice that the \(L^2_f\) inner product is also defined between any two differential forms if one of them is compactly supported.

First we prove for completeness the known fact that \(δ_f\) is the formal adjoint of \(d\).

**Lemma 2.1.** Let \(ω\) be a \((p-1)\)-form and \(η\) be a \(p\)-form on a smooth metric measure space \((M, g, e^{-f} dv)\). If one of these differential forms is compactly supported then the following identity holds

\[(dω, η)_{L^2_f(M)} = (ω, δ_f η)_{L^2_f(M)}.\]  \((2.1)\)

**Proof.** Assume one of these differential forms compactly supported, say \(ω\). We have

\[
(dω, η)_{L^2_f(M)} = \int_M dω · η e^{-f} dv \\
= \int_M dω ∧ *η e^{-f} \\
= \int_M (d(e^{-f} ω) ∧ *η + e^{-f} df ∧ ω ∧ *η) \\
= \int_M d(e^{-f} ω ∧ *η) + \int_M ω ∧ ((-1)^p d * η + (-1)^{p-1} df ∧ *η) e^{-f}.
\]

Using Stokes theorem and the identities (see [7] for a list of formulas)

\[(-1)^p d * η = *δ η\]

and

\[df ∧ *η = (-1)^{p-1} *i_∇ f η\]

we get identity \((2.1)\). \(\square\)

Now we prove that \(L^2_f\) harmonic forms of any degree are closed and co-closed. Here we adapt from Carron [4].
Lemma 2.2. Every $L^2_f$ harmonic form (of any degree) on a complete smooth metric measure space $(M, g, e^{-f}dv)$ is closed and co-closed. In other words,

\[ d\omega = 0 \quad (2.2) \]

and

\[ \delta f \omega = 0. \quad (2.3) \]

Proof. We can choose a smooth function $\phi$ on $M$ such that $\phi = 1$ in $B_R$, $\phi = 0$ in $M \setminus B_{2R}$ and $|\nabla \phi| \leq \frac{2}{R}$ in $B_{2R} \setminus B_R$. Here $B_R$ denotes a ball with center in a fixed point and radius $R$. We have

\[ |d(\phi \omega)|^2_{L^2_f(M)} = |d\phi \wedge \omega|^2_{L^2_f(M)} + (d\phi^2 \wedge \omega, d\omega)_{L^2_f(M)} + |\phi d\omega|^2_{L^2_f(M)} \]

\[ = |d\phi \wedge \omega|^2_{L^2_f(M)} + (d(\phi^2 \omega), d\omega)_{L^2_f(M)} \]

\[ = |d\phi \wedge \omega|^2_{L^2_f(M)} + (\phi^2 \omega, \delta f d\omega)_{L^2_f(M)}. \quad (2.4) \]

Notice that we used Lemma 2.1 in equation (2.4). We also have the identity

\[ \delta f (\phi \omega) = \delta (\phi \omega) + \iota \nabla f (\phi \omega) \]

\[ = \phi \delta \omega - \iota \nabla \phi \omega + \phi \iota \nabla f \omega \]

\[ = \phi \delta f \omega - \iota \nabla \phi \omega, \quad (2.5) \]

so

\[ |\delta f (\phi \omega)|^2_{L^2_f(M)} = |\iota \nabla \phi \omega|^2_{L^2_f(M)} - (\iota \nabla \phi \omega, \delta f \omega)_{L^2_f(M)} + |\phi \delta f \omega|^2_{L^2_f(M)} \]

\[ = |\iota \nabla \phi \omega|^2_{L^2_f(M)} + (\delta f (\phi^2 \omega), \delta f \omega)_{L^2_f(M)} \]

\[ = |\iota \nabla \phi \omega|^2_{L^2_f(M)} + (\phi^2 \omega, \delta f d\omega)_{L^2_f(M)}. \quad (2.6) \]

Notice that we used Lemma 2.1 again in equation (2.6). Using the identity

\[ |d\phi \wedge \omega|^2 + |\iota \nabla \phi \omega|^2 = |\nabla \phi|^2 |\omega|^2 \]

and the assumption $\omega$ is an $L^2_f$ harmonic form, adding equations (2.4) and (2.6) we get

\[ |d(\phi \omega)|^2_{L^2_f(M)} + |\delta f (\phi \omega)|^2_{L^2_f(M)} = \int_M |\nabla \phi|^2 |\omega|^2 e^{-f}dv \quad (2.7) \]

Since $\omega$ is an $L^2_f$ differential form sending $R \to \infty$ in equation (2.7) we conclude that

\[ |d\omega|^2_{L^2_f(M)} + |\delta f \omega|^2_{L^2_f(M)} = 0, \]

which implies identities (2.2) and (2.3).
Let us recall classical Bochner’s formula for one-forms
\[ \frac{1}{2} \Delta |\omega|^2 = |\nabla \omega|^2 + \Delta \omega \cdot \omega + \text{Ric}(\omega, \omega) \]
(here we use the same notation to represent the dual of a one-form) and Kato’s inequality for \( L^2 \) harmonic one-forms
\[ |\nabla \omega|^2 \geq \frac{n}{n-1} |\nabla |\omega||^2. \]

In this section we extend these formulas to smooth metric measure spaces.
First we extend Bochner’s formula, which is also found in Lott’s paper [11] (equation 2.10). We thank the referee for pointing out this reference.

**Lemma 3.1.** Let \( \omega \) be a one-form on a smooth metric measure space \((M, g, e^{-f} dv)\). Then the following identity holds
\[ \frac{1}{2} \Delta f |\omega|^2 = |\nabla \omega|^2 + \Delta f \omega \cdot \omega + \text{Ric}_f(\omega, \omega). \]  

**Proof.** By a simple computation we have
\[ \Delta f = \Delta - d\iota_f \nabla f - \iota_f d, \]
so using the classical Bochner’s formula for one-forms we get
\[ \frac{1}{2} \Delta f |\omega|^2 = \frac{1}{2} \Delta |\omega|^2 - \frac{1}{2} \nabla f \cdot \nabla |\omega|^2 \]
\[ = |\nabla \omega|^2 + \Delta \omega \cdot \omega + \text{Ric}(\omega, \omega) - \frac{1}{2} \nabla f \cdot \nabla |\omega|^2 \]
\[ = |\nabla \omega|^2 + \Delta f \omega \cdot \omega + \text{Ric}_f(\omega, \omega) \]
\[ - \frac{1}{2} \nabla f \cdot \nabla |\omega|^2 - \text{Hess} f(\omega, \omega) + d\iota_f \omega \cdot \omega + \iota_f d\omega \cdot \omega. \]

It remains to prove that
\[ - \frac{1}{2} \nabla f \cdot \nabla |\omega|^2 - \text{Hess} f(\omega, \omega) + d\iota_f \omega \cdot \omega + \iota_f d\omega \cdot \omega = 0. \]  

We can choose a local orthonormal basis \( e_1, \ldots, e_n \) with dual basis \( \theta_1, \ldots, \theta_n \) and assume the connection forms \( \theta_{ij} \) vanish on a fixed point. Writing \( \omega = \omega_i \theta_i \), we have
\[ \frac{1}{2} \nabla f \cdot \nabla |\omega|^2 = f_i \omega_j \omega_{ji} \]  
and
\[ \text{Hess} f(\omega, \omega) = \omega_i \omega_j f_{ij}. \]
Computing
\[\delta f \omega + \iota \nabla f d\omega = d(f_i \omega_i) + i \nabla f (\omega_i j \wedge \theta_i)\]
\[= \omega_i df_i + f_i d\omega_i + \omega_{ij} f_j \theta_i - \omega_{ij} f_i \theta_j\]
\[= \omega_i f_{ij} \theta_j + f_i \omega_{ij} \theta_j + \omega_{ij} f_j \theta_i - \omega_{ij} f_i \theta_j\]
\[= (\omega_i f_{ij} + f_i \omega_{ji}) \theta_j,\]
we obtain
\[\delta f \omega \cdot \omega + \iota \nabla f d\omega \cdot \omega = \omega_i \omega_j f_{ij} + f_i \omega_j \omega_{ji}.\] (3.5)

Equations (3.3), (3.4) and (3.5) imply equation (3.2). \qed

Now we extend Kato’s inequality.

**Lemma 3.2.** Let \(\omega\) be an \(L^2\) harmonic one-form on a smooth metric measure space \((M^n, g, e^{-f} dv)\). Then the following inequality holds
\[|\nabla \omega|^2 \geq \frac{1}{n-1} \left( |\nabla f| - |\nabla f \cdot \omega|^2 + |\nabla \omega|^2 \right).\] (3.6)
Moreover, if equality holds in (3.6) then
\[\nabla \omega = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_2 & 0 \\ 0 & 0 & \ldots & 0 & \lambda_2 \end{pmatrix}\] (3.7)
where \(\lambda_1 = \nabla f \cdot \omega - (n-1)\lambda_2\).

**Proof.** We can choose a local orthonormal basis \(e_1, \ldots, e_n\) with dual basis \(\theta_1, \ldots, \theta_n\). Writing
\[\omega = \sum_{i=1}^n \omega_i \theta_i\]
we have
\[d\omega = \sum_{i,j=1}^n \omega_{ij} \theta_j \wedge \theta_i\]
and
\[\delta f \omega = -\sum_{i=1}^n \omega_{ii} + \sum_{i=1}^n \omega_i f_i.\]
By Lemma 2.2 we know that \(L^2_f\) harmonic forms are closed and co-closed therefore
\[\omega_{ij} = \omega_{ji}\] (3.8)
for \(i, j = 1, \ldots, n\) and
\[\sum_{i=1}^n \omega_{ii} = \nabla f \cdot \omega.\] (3.9)
In the rest of the proof we adapt from Li and Wang [9]. We can set $e_1 = \frac{\omega}{|\omega|}$. Computing

$$|\nabla |\omega| |^2 = 4 \sum_{i=1}^{n} (\omega_1 \omega_1)^2$$

$$= 4|\omega|^2 \sum_{i=1}^{n} (\omega_1)^2,$$

we obtain

$$|\nabla |\omega||^2 = \sum_{j=1}^{n} \omega_{1j}^2. \quad (3.10)$$

Using identities (3.8), (3.9) and (3.10) we get

$$|\nabla \omega|^2 = \omega_{11}^2 + \sum_{j=2}^{n} \omega_{1j}^2 + \sum_{j=2}^{n} \omega_{2j}^2 + \sum_{i=2}^{n} \omega_{ii}^2 + \sum_{i,j=2, i \neq j}^{n} \omega_{ij}^2$$

$$\geq \omega_{11}^2 + 2 \sum_{j=2}^{n} \omega_{1j}^2 + \frac{1}{n-1} \left( \sum_{i=2}^{n} \omega_{ii} \right)^2$$

$$= \omega_{11}^2 + 2 \sum_{j=2}^{n} \omega_{1j}^2 + \frac{1}{n-1} (\omega_{11} + \nabla f \cdot \omega)^2$$

$$= \frac{n-1}{n-1} \omega_{11}^2 + 2 \sum_{j=2}^{n} \omega_{1j}^2 + \frac{1}{n-1} (\nabla f \cdot \omega)^2 - \frac{2}{n-1} \omega_{11} \nabla f \cdot \omega$$

$$\geq \frac{n-1}{n-1} |\nabla |\omega||^2 + \frac{1}{n-1} (\nabla f \cdot \omega)^2 - \frac{2n-1}{n-1} |\nabla |\omega|| \cdot |\nabla f \cdot \omega|$$

$$= \frac{1}{n-1} \left( |\nabla |\omega|| - |\nabla f \cdot \omega| \right)^2 + |\nabla |\omega||^2.$$

Matrix (3.7) follows by analyzing intermediate inequalities. \qed

Remark 3.3. The following inequality holds under the same assumptions of Lemma 3.2:

$$|\nabla \omega|^2 \geq \frac{n-1}{n-1} |\nabla |\omega||^2 + \frac{1}{n-1} (\nabla f \cdot \omega)^2 - \frac{2n-1}{n-1} |\omega|^2 (\nabla f \cdot \omega) \nabla \omega(\omega, \omega). \quad (3.11)$$

4 Non-Negative Bakry-Émery-Ricci Curvature

In this section we prove some results concerning smooth metric measure spaces with non-negative Bakry-Émery-Ricci curvature.

First we prove the following theorem.

Theorem 4.1. Let $(M^n, g, e^{-f} dv)$ be a complete non-compact smooth metric measure space with non-negative Bakry-Émery-Ricci curvature. If the space of
L^2 f-harmonic one-forms is non-trivial then the weighted volume of M^n is finite, that is
\[ \text{vol}_f(M^n) = \int_{M^n} e^{-f} dv < \infty, \]
and the universal covering splits isometrically as \( \tilde{M}^n = \mathbb{R} \times N^{n-1} \).

**Proof.** Fix a non-trivial \( L^2_f \) harmonic one-form \( \omega \) on \( M \). Since the Bakry-Émery-Ricci curvature is non-negative and the one-form \( \omega \) is \( f \)-harmonic by Lemma 3.1 we have
\[ \frac{1}{2} \Delta_f |\omega|^2 \geq |\nabla \omega|^2, \]
so
\[ |\omega| \Delta_f |\omega| \geq |\nabla \omega|^2 - |\nabla| \omega ||^2, \]
hence by lemma 3.2 we have
\[ |\omega| \Delta_f |\omega| \geq 0. \quad (4.1) \]

We can choose a smooth function \( \phi \) on \( M \) such that \( \phi = 1 \) in \( B_R \), \( \phi = 0 \) in \( M \setminus B_{2R} \) and \( |\nabla \phi| \leq \frac{2}{R} \) in \( B_{2R} \setminus B_R \). Here \( B_R \) denotes a ball with center in a fixed point and radius \( R \). Multiplying inequality (4.1) by \( \phi^2 \) and integrating by parts we get
\[ \int_M |\nabla \omega|^2 \phi^2 e^{-f} dv \leq -2 \int_M |\omega| \phi \nabla |\omega| \cdot \nabla \phi e^{-f} dv. \]

Using Cauchy-Schwarz and Young’s inequalities we get
\[ \frac{1}{2} \int_M |\nabla |\omega||^2 \phi^2 e^{-f} dv \leq 2 \int_M |\omega|^2 |\nabla \phi|^2 e^{-f} dv. \]

Since \( \omega \) is an \( L^2_f \) differential one-form by the monotone convergence theorem sending \( R \to \infty \) we get
\[ \frac{1}{2} \int_M |\nabla |\omega||^2 e^{-f} dv \leq 0, \]
so the function \( |\omega| \) is a constant \( c \). Since
\[ \int_M |\omega|^2 e^{-f} dv = c^2 \text{vol}_f(M) < \infty \]
and \( c \neq 0 \) it follows that the weighted volume \( \text{vol}_f(M) \) is finite. Now we claim that \( \omega \) is parallel. Indeed, since \( |\omega| \) is constant by Bochner’s formula (Lemma 3.1) and the assumption in the Bakry-Émery-Ricci curvature we have
\[ 0 \geq |\nabla \omega|^2, \]
so \( \omega \) is parallel. Now the lifting of \( \omega \) to the universal cover \( \tilde{M} \) is a non-trivial parallel one-form, which concludes the proof by the de-Rham decomposition theorem. \( \square \)
In particular we obtain the following result.

**Corollary 4.2.** Let \((M, g, e^{-f}dv)\) be a complete non-compact smooth metric measure space with non-negative Bakry-Émery-Ricci curvature. If the first eigenvalue of the \(f\)-Laplacian is positive then the space of \(L^2_f\) harmonic one-forms is trivial.

**Proof.** By assumption the first eigenvalue of the \(f\)-Laplacian

\[
\lambda_1(\Delta_f) = \inf_{\phi \in \mathcal{C}_c^\infty(M)} \frac{\int_M |\nabla \phi|^2 e^{-f} dv}{\int_M \phi^2 e^{-f} dv}
\]

is positive. We can choose a smooth function \(\phi\) on \(M\) such that \(\phi = 1\) in \(B_R\), \(\phi = 0\) in \(M \setminus B_{2R}\) and \(|\nabla \phi| \leq \frac{2}{R}\) in \(B_{2R} \setminus B_R\). Then

\[
\lambda_1(\Delta_f) \int_M \phi^2 e^{-f} dv \leq \frac{4}{R^2} \text{vol}_f(M).
\]

We claim that \(\text{vol}_f(M)\) is not finite. Otherwise by the monotone convergence sending \(R \to \infty\) we get \(\lambda_1(\Delta_f) = 0\), a contradiction. Now since \(\text{vol}_f(M) = \infty\) it follows from Theorem 4.1 that the space of \(L^2_f\) harmonic forms is trivial. \(\square\)

Recall that a gradient steady Ricci soliton is a manifold \((M, g)\) together with a smooth function \(f\) satisfying

\[
\text{Ric} + \text{Hess} f = 0.
\]

In this case, it is possible to prove that the scalar curvature \(R\) is non-negative and there is a constant \(a\) such that

\[
R + |\nabla f|^2 = a
\]

(see [2]). It follows that \(|\nabla f| \leq \sqrt{a}\), so the only non-trivial gradient steady Ricci solitons are those with \(a > 0\), which must be non-compact (see [2]). It was proved by Munteanu and Wang in [12] that the first eigenvalue of the \(f\)-Laplacian on non-trivial gradient steady Ricci solitons is positive, more precisely \(\lambda_1(\Delta_f) = \frac{a^2}{4}\), so Corollary 4.2 implies the following result.

**Corollary 4.3.** On a complete non-compact non-trivial gradient steady Ricci soliton \((M, g, f)\) the space of \(L^2_f\) harmonic one-forms is trivial.

It was proved by Wei-Wyllie in [16] that on a complete non-compact smooth metric measure space \((M, g, e^{-f}dv)\) if the function \(f\) is bounded and the Bakry-Émery-Ricci curvature is non-negative then the weighted volume \(\text{vol}_f(M)\) is infinite, so Theorem 4.1 has the following consequence.

**Corollary 4.4.** Let \((M, g, e^{-f}dv)\) be a complete non-compact smooth metric measure space with non-negative Bakry-Émery-Ricci curvature. If the function \(f\) is bounded then the space of \(L^2_f\) harmonic one-forms is trivial.
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