Application of Unloading Fractured Rock Mass under Uniaxial Tension Based on Endochronic Theory

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Abstract. In this paper, the constitutive relation of fractured rock mass under uniaxial tension is derived, and the corresponding numerical solution is obtained. It is found that the kernel function with the Proney expansion can fit well with the experimental data. When the parameters of the kernel function are taken from the experimental data, the constitutive equation can be simplified in the initial loading stage, which is in good agreement with Jin Fen's experimental data under unloading condition. The simplified equation of the endochronic theory is well verified by the experimental data.

1. Introduction
Endochronic theory by Valanis [1] has been widely used since its inception in 1971 to model the inelastic behavior of materials. An important component of endochronic theory is an internal time, not real time, but dependent on the properties of the material and its deformation history, which is used to describe the mechanical response using the historical dependence of the whole deformation and temperature. Valanis first applies the endochronic constitutive equation to the complex strain history of aluminum and copper [2]. In 1975, Valanis and Wu [3] applied the endochronic theory to the cyclic creep and relaxation of metals. In 1974, Valanis discussed the effect of prestrain on the cyclic strength weakening of metals, and the results are consistent with the observed phenomena [4]. The same time, Bazant applied the endochronic theory to the inelastic behavior of concrete and geological materials [5]. Kosinski and Wu are the first to apply the endochronic theory to the plastic dynamic problem. In 1977 [6] they considered the effect of strain rate history in the constitutive equation, and analyzed the steady viscoplastic wave problem. In this paper, they calculated the dynamic wave pattern of aluminum plate by using the tree branch method and compared it with the experimental results. In 1978, they also analyzed and calculated the one-dimensional plastic wave propagation problem [7]. Another characteristic of endochronic theory [3] is that it depends on the internal time of the mechanical response which defines the plastic deformation in the complete deformation history, so that the theory can describe the complex material behavior more accurately Especially when loading is complex (such as non-proportional loading, unloading, or cyclic loading) . Therefore, the endochronic theory can be well applied to the unloading mechanical analysis of fractured rock mass.

Unloading crack is an important factor affecting the sability of rock slope. At present, great achievements have been made at home and abroad on the Mechanical State of rock mass under loading. The mature theories and new sciences such as plastic mechanics, fracture mechanics, damage mechanics, rheology, new mathematical theory and neural network have been introduced into the
study of rock mechanics. It is well known that there are a large number of dispersed micro-cracks in rock, and the nucleation, propagation and convergence of micro-cracks have a significant impact on the mechanical properties of rock materials. Up to now, no study of applying endochronic theory to build energy equation of unloading rock mass and the application and expansion of endochronic theory in this field. In this paper, the constitutive relation of fractured rock mass under uniaxial tension is deduced, and the corresponding numerical solution is obtained and verified.

2. Endochronic Theory
Valanis began to refine the original theory in 1979. Endochronic is also defined as a quantity related to the length of the integral path in the plastic strain space. On this basis, endochronic theory has been successfully applied to the study of the properties of different materials, such as Valanis, Bazant, Bakhshiani, etc. Therefore, according to the form invariance law of the constitutive equation \[ \sigma = \int_0^z B(z-z') \frac{d\tilde{\varepsilon}}{dz'} dz' \]

\[ dz = d\zeta / f(\zeta) \]

\[ d\zeta^2 = P_{ijkl} d\tilde{\varepsilon}_{ij}^p d\tilde{\varepsilon}_{j}^p \] or \[ d\zeta^2 = d\tilde{\varepsilon}^p \] (3)

Where \( f(\zeta) \) as an enhanced function; \( \tilde{\varepsilon} \), \( \zeta \) are an endochronic variable, a function of generalized strain; \( \zeta \) is an endochronic scale, \( \zeta \) is an endochronic measure.

The above is an anisotropic endochronic theory, coupled with an anisotropic damage constitutive equation. In engineering problems, many materials from the initial isotropic through the damage evolution of anisotropic materials. The effect of anisotropic damage is to include the effective stress in the intrinsic time definition of effective strain and effective inelastic strain, but the material anisotropy is introduced by fourth order tensor if specified \( B \) is in the following isotropic form:

\[ B_{ijkl} = 2\mu(z) \left( \delta_{ik} \delta_{jl} - 1/3 \delta_{ij} \delta_{kl} \right) + K \delta_{ij} \delta_{kl} \]

Where is a constant(\( \kappa = E/3(1-2\nu) \)), the change of volume is elastic, If you replace 4 with 1, you get

\[ \tilde{\sigma}_{ij} = \int_0^z 2\mu(z-z') \frac{d\tilde{\varepsilon}_{ij}}{dz'} dz' + K\tilde{\varepsilon}_{mn} \delta_{ij} \]

\[ \tilde{\varepsilon}_{ij} = \tilde{\varepsilon}_{ij} - 1/3 \tilde{\varepsilon}_{mn} \delta_{ij} \]

From the elastic relation of volume deformation

\[ K\tilde{\varepsilon}_{mn} = \frac{\sigma_{mn}}{3} \]

Solution

\[ S_{ij} = \tilde{\sigma}_{ij} - \frac{\sigma_{mn}}{3} \delta_{ij} = \int_0^z 2\mu(z-z') \frac{d\tilde{\varepsilon}_{ij}}{dz'} dz' \]

The deviatoric stress in Formula 8 is related to the total strain deviation, so it is necessary to establish the relationship between its endochronic theory and the inelastic strain deviation. For isotropic materials, the normal tensor

\[ a_{ijkl} = \frac{1+\nu}{E} \delta_{ik} \delta_{jl} - \frac{\nu}{E} \delta_{ij} \delta_{kl} \]

The Laplace transform of (8) becomes
The above formula is based on an endochronic theory that depends on isotropy in history.

3. Study on the Constitutive Relation of Fracture Rocks under Uniaxial Tension

According to Valanis and Murakami and Read [10], the unloading fractured rock mass under uniaxial tension can be expressed as follows:

\[
\rho(z) = Cz^{-\alpha}
\]  

or \[
\rho(z) = \sum_{r=1}^{n} R_r \exp(-\beta_r z) \quad 0 < \alpha < 1
\]

Where \( C, \alpha \) and \( R_r, \beta_r \) are positive constant, is positive integer. Because the force is on the tensile direction, the following indicators will no longer be used.

In practical engineering calculation it is difficult to deal with the integral form in Formula 8. Substitute equation (13) above into equation (10) to obtain

\[
S = \sum_{r=1}^{n} K_r
\]

Here

\[
K_r = R_r \int_{0}^{z} e^{-\alpha_r(z-z')} \frac{d\varepsilon^p_{z'}}{dz'} dz'
\]

If we take the derivative of this with respect to \( z \) (wechat business), we get

\[
\frac{dK_r}{dz} + \alpha_r K_r = R_r \frac{d\varepsilon^p}{dz}
\]

Formula (16) is the form of linear first-order differential equation instead of the form of integration, so it is unnecessary to remember the entire strain history and the specific value of the kernel function.

Make \( \frac{d\varepsilon^p}{dz} = \frac{\delta\varepsilon^p}{\delta\varepsilon} \), can be obtained from equation (14)

\[
K_r (z_0 + \delta\varepsilon) = K_r (z_0) e^{-\alpha_r \delta\varepsilon} + \frac{R_r}{\alpha_r} (1 - e^{-\alpha_r \delta\varepsilon}) \frac{\delta\varepsilon^p}{\delta\varepsilon}
\]

Substitute equations (12) and (13) into equation (16) to obtain

\[
G\delta\varepsilon + \sum_{r=1}^{n} \left[ \frac{R_r}{\alpha_r} - K_r (z_0) \frac{\delta\varepsilon^p}{\delta\varepsilon} \left(1 - e^{-\alpha_r \delta\varepsilon}\right) \right] = G\delta\varepsilon \frac{\delta\varepsilon}{\delta\varepsilon^p}
\]

From equation (17), it can be seen that the equation and \( \frac{\delta\varepsilon}{\delta\varepsilon^p} \) are related, and there is no need to solve \( \delta\varepsilon^p \) for them.
4. Numerical Solution of Endochronic Constitutive Relation of Fractured Rock Mass under Uniaxial Tension Condition

With the change of \( z \), the stress-strain relationship can be conveniently calculated through (18) and (12). The general expressions of the endochronic constitutive numerical solutions calculated by mathematical software are as follows

\[
d\varepsilon = \frac{1}{2} \left\{ \sum_{i=1}^{n} \alpha_i \cdots \alpha_i \alpha_{i+1} \cdots \alpha_i R_i - \sum_{i=1}^{n} \alpha_i \cdots \alpha_i \alpha_{i+1} \cdots \alpha_i R_i e^{-(\alpha_i \delta_i)} \right\}^{\frac{1}{2}}
\]

\[
/ \alpha_i \cdots \alpha_i \sum_{i=1}^{n} K_i (1 - e^{-(\alpha_i \delta_i)})
\]

In order to verify the numerical calculation formula in this paper, the experimental data in reference [11] are used, and the specific experimental data are shown in Table 1. The kernel function was expanded in two ways according to the kernel function formula (13), as shown in Figure 1. It was found that the kernel function could be well fitted to \( \rho(z) = Cz^{-\alpha} \) when it was expanded to the fourth order by Proney.

In the model of endochronic theory, the most important thing is to determine the parameters \( R \) and \( \alpha \) in the kernel function. However, it is difficult to obtain the parameters in the form of equation (13a). Valasis has pointed out that if the kernel function takes the form of: \( \rho(z) = Cz^{-\alpha} \), then stress-strain has a relationship during the initial loading process:

\[
\varepsilon = \frac{S}{G} + \left( \frac{1 - \alpha S}{C} \right)^{\frac{1}{\alpha}}
\]

Table 1. Model parameters of different rock samples.

|       | GR19 | SAL9  | SA1   |
|-------|------|-------|-------|
| \( R_i (10^{13} Pa) \) | 0.0818 | 0.0265 | 0.01044 |
| \( R_2 (10^{13} Pa) \) | 0.3372 | 0.1093 | 0.4304  |
| \( R_3 (10^{13} Pa) \) | 0.1286 | 0.0417 | 0.1641  |
| \( R_4 (10^{13} Pa) \) | 1.0334 | 0.3349 | 1.3189  |
| \( \alpha_i (10^7 Pa) \) | 0.0102 | 0.0102 | 0.0102  |
| \( \alpha_j (10^7 Pa) \) | 0.5079 | 0.5079 | 0.5079  |
| \( \alpha_k (10^7 Pa) \) | 2.7481 | 2.7481 | 2.7481  |
| \( \alpha_i (10^7 Pa) \) | 9.2721 | 9.2721 | 9.2721  |
| \( G(10^9 Pa) \) | 5.33 | 1.33  | 3.74   |
| \( C(10^9 Pa) \) | 8.76 | 2.85  | 11.2   |

According to Xu et al., in reference [12], when \( \alpha = 0.5 \), the linear relationship between \( Q^{-\alpha} \) and strain amplitude \( \varepsilon_m \) obtained from a large number of experimental observations can be well satisfied. So let's take \( \alpha = 0.5 \). First, when equation (19) is taken as \( \alpha = 0.5 \), the experimental data of Jin Fengnian [13] is verified, as shown in Figure 2.
It can be seen from figure 2 that the constitutive relation of unloaded fractured rock mass calculated by kernel function formula (20) in endochronic theory is in good agreement with the experimental data of Jin Fengnian fractured rock mass. However, this formula can only be fitted well in the elastic phase, and can only be used in the form of formula (19) in the elastic-plastic phase.

Figure 1. Proney expansion of SAL9 kernel function of sandstone ($\alpha = 0.5$).

Figure 2. Comparison between the theoretical values of granite samples and the experimental data of Jin Feng unloading fractured rock mass.

5. Conclusion
According to the law of form invariance in constitutive equation, derived in the anisotropic endochronic constitutive equation. With anisotropic damage within the isotropic fourth order tensor endochronic theory, resulting in isotropic material the endochronic constitutive equation. Based on the unloading theory, the corresponding endochronic theory is obtained by taking the appropriate kernel function formulas (13a) and (13b). The kernel function generation into the basic constitutive equation, within the internal variable $z$ to fetch wechat business in the form of a linear first-order differential equations is obtained after replaced the original integral form. The simplified solution of the differential equation is obtained by mathematical software.

In the practical application, it is found that the form of the constitutive equation in the kernel function can be simplified to form $\varepsilon = \frac{z}{G} + \left(\frac{1}{G} \frac{d}{d\varepsilon}\right)^{\alpha}$ at the initial loading stage. When $\alpha = 0.5$, it is quite consistent with the experimental data of the fractured rock mass under the unloading condition of Jin fengnian. It shows that the simplified form of the endochronic theory has been verified by laboratory test data.

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