The Rank-One Separable Interaction Kernel for Nucleons with Scalar Propagator

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Abstract—In the paper the covariant kernel of the nucleon-nucleon interaction of particles with scalar propagators is analyzed. The Bethe–Salpeter equation for the $T$ matrix is considered in the rank-one separable kernel. The parameters of the kernel for the specific partial-wave channels explicitly connect with the observables low energy scattering parameters and phase shifts, deuteron binding energy. Covariant separable kernels for the partial-wave channels with total angular momentum $J = 0\left(J^p_0, J^P_0\right)$ and $J = 1\left(J^p_1, J^P_1, J^p_3, J^P_3\right)$ are constructed.

1. INTRODUCTION

Investigation of few-nucleon systems is very important for understanding the strong interactions. For a consistent covariant description of the nucleon-nucleon ($NN$) interactions the relativistic Bethe–Salpeter (BS) equation [1] is commonly used. The BS approach with the separable kernel of interaction gives a good description of elastic and inelastic electromagnetic processes with the deuteron [2]. For instance, the BS formalism facilitates an analysis of the role of P-waves (negative energy partial-wave components of the BS amplitude) in the electromagnetic properties of the deuteron and its comparison with the nonrelativistic treatment [3]. Furthermore, the covariant BS approach makes it possible to analyze off-mass-shell effects and contributions of the relativistic two-body currents [4].

Another important topic of the physics of strong interaction is studying the three-nucleon systems and hadron–deuteron reactions. The relativistic three-particle systems are described by the Faddeev equations within the BS approach—the so called Bethe–Salpeter–Faddeev equations. A general consideration of the three-nucleon systems is a rather difficult task. So some simplification should be done at the first stage of such investigations.

First of all the only pair-interactions are taken into account while the three-body forces are considered. Second is using the covariant separable kernel of $NN$ interaction. In this case the system of the 6-fold integral equations transfer to system of the 2-fold integral equations with generalized potentials. The third assumption is to treat all nucleons having equal masses and the scalar propagators instead of spinor ones. The spin-isospin structure of the nucleons is taken into account by using the so-called recoupling-coefficient matrix.

In the paper the covariant kernel of the $NN$ interaction of the particles with scalar propagators for the partial-wave channels with the total angular momentum $J = 0,1$ is analyzed. Obtained results will be used in the future calculations of the three-nucleon systems.

The paper is organized as following: in Section 2 the formalism is given, the details of the calculations and results are presented in Section 3 and conclusion is given in Section 4.

2. FORMALISM

The Bethe–Salpeter equation for the nucleon-nucleon $T$ matrix is written as

$$T\left(\hat{p}', \hat{p}; \hat{P}\right) = V\left(\hat{p}', \hat{p}; \hat{P}\right)$$

$$+ \int \frac{d^4 \hat{k}V\left(\hat{p}', \hat{k}; P\right)S\left(\hat{k}; \hat{P}\right)T\left(\hat{k}, \hat{p}; \hat{P}\right)}{4\pi^4}.$$

(1)

Here the total four-momentum $\hat{P} = (\hat{p}_1 + \hat{p}_2)$ and the relative four-momentum $\hat{p} = (\hat{p}_1 - \hat{p}_2)/2$ are introduced, and $V$ is the kernel of $NN$ interaction (for details, see reference [2]).
The nucleons in the equation are treated as spin-one-half particles with scalar propagators

\[ S(\hat{k}; \hat{p}) = [(\hat{p}/2 + \hat{k}) - m_N^2 + i0]^{-1} \]

\[ \times [(\hat{p}/2 - \hat{k}) - m_N^2 + i0]^{-1}. \]  

(2)

The partial-wave decomposition of the equation (in the rest frame of the two-nucleon system) leads to the following form:

\[ i_{LL}(p_0', p', p_0, p; s) = v_{LL}(p_0, p', p_0, p; s) + \frac{i}{4\pi} \sum_L \int dk_0 \int k^2 dk \nu_{LL}(p_0, p', k_0, k; s) \times S(k_0, k; s)i_{LL}(k_0, k, p_0, p; s). \]  

(3)

Here \( s = p^2 \), \( t \) and \( v \) are the partial-wave decomposed \( T \) matrix and kernel \( V \) and \( e_\delta = \sqrt{k^2 + m^2} \). For the singlet (uncoupled triplet) case \( (L = J) \) there is only one term in the sum and there are two terms for the coupled triplet case \( (L = J \pm 1) \).

To solve the Eq. (3) the separable form (rank-one) for the partial-wave decomposed kernels of interactions is assumed

\[ v_{LL}(p_0', p', p_0, p; s) = \lambda g^{[L]}(p_0', p')g^{[L]}(p_0, p), \]  

where \( \lambda \) and \( g \) are the parameters and form factors of the model. Then the \( T \) matrix can be written as

\[ i_{LL}(p_0', p', p_0, p; s) = \tau(s)g^{[L]}(p_0', p')g^{[L]}(p_0, p), \]  

with the function \( \tau(s) \)

\[ \tau(s) = 1/(\lambda^{-1} + h(s)), \]  

(6)

while the function \( h(s) \) has the following form:

\[ h(s) = \sum_L h_L(s) = \frac{-i}{4\pi} \int dk_0 \int k^2 dk \sum_L [g_L^{[L]}(k_0, k)]^2S(k_0, k; s). \]  

(7)

To construct the rank-one \( NN \) kernel the covariant generalization of the Yamaguchi [5] functions for \( g^{[L]}(k_0, k) \) are used in the following form

\[ g^{[L]}(k_0, k) = \frac{C_L(k_0^2 - k^2)}{(k_0^2 - k^2 - \beta_L^2 + i0)}, \]  

\[ g^{[L]}(p_0', p') = \frac{\sqrt{-k_0^2 + k_0'^2}}{(k_0^2 - k_0'^2 - \beta_L^2 + i0)}^{-2}, \]  

\[ g^{[L]}(p_0, p) = \frac{\beta_L^2(k_0^2 - k^2)}{(k_0^2 - k^2 - \beta_L^2 + i0)^2}, \]  

(10)

where \( \beta_L \) and \( C_L \) are the model parameters.

\[ \lambda^{-1} = -h(s = M_b^2). \]  

(13)

2.1. Deuteron and \( NN \)-Scattering Observables

The on-mass-shell \( t(s) \) matrix can be expressed through the following observables:

(1) in the singlet (uncoupled triplet) channel

\[ t(s) = t(0, \overline{p}, 0, \overline{p}, s) = -\frac{8\pi\sqrt{s}e^{i\delta}}{\overline{p}} \]  

(11)

(2) in the coupled triplet channel

\[ t(s) = \frac{4\pi\sqrt{s}}{\overline{p}} \left( \cos 2\delta - 1 \right) \frac{i}{\sin 2e^{i\delta(s)} \cos 2e^{i\delta(-s)}}, \]  

(12)

with \( \overline{p} = \sqrt{s/4 - m^2} = \sqrt{mT_{lab}/2} \). We introduced scattering phase shifts \( \delta = \delta_{L=J}, \delta_c = \delta_{L=J-1}, \delta_x = \delta_{L=J+1} \) and mixing parameter \( \epsilon \).

If the bound state exists in the partial-wave channel there is a simple pole on the total momentum squared in the \( T \) matrix. Using Eq. (6) one can write \( (M_b = 2m_N - E_b, \text{ where } E_b \text{ is the energy of the bound state}) \):

\[ \lambda^{-1} = -h(s = M_b^2). \]  

(13)

The normalization condition for the deuteron vertex functions can be written as

\[ \frac{d}{dP}(h_L(s) + h_P(s)) = 2P_L(p_L + p_P), \]  

(14)

where \( s = P^2 = M_b^2 \) and \( p_L \) is a partial-wave state pseudoprobability.

The low-energy parameters—scattering length \( a_L \) and effective range \( r_L \)—are defined by the following equation

\[ \overline{p}^{2L+1}\cot \delta_L(s) = -1/4a_L + \frac{r_L}{2} \overline{p}^2 + C(\overline{p}^3). \]  

(15)

To summarize, the Eq. (5) defines the \( t \) matrix on the mass-shell \( (p_0 = p_0' = 0, p = p' = \overline{p}) \) which is related to \( NN \)-scattering observables and the deuteron scattering phase shifts and low-energy parameters, the bound state energy.

3. CALCULATIONS AND RESULTS

The parameters of the model for definite partial-wave channel \( \lambda_L, \beta_L(C_L) \) can be obtained from the analysis of the \( NN \)-scattering observables. These values are calculated from the on-mass-shell \( t \) matrix, Eq. (5). The two-fold integrations in Eq. (7) can be performed by several ways. In the paper the integration over the \( k_0 \) variable is done by using the Cauchy theorem and the remaining one-fold integration over the \( k \) (or \( e_\delta \)) variable is done numerically. As it is shown in [6] the integration on \( k \) can be performed only for bound state \( \sqrt{s} < 2m_N \) and for elastic \( NN \)-scattering...
with $2m_N < \sqrt{s} < 2(m_N + \beta)$. So, the considered kinetic energy is restricted to the \( T_{\text{lab}} < 4\beta \).

Considering the \( e_k \)-integration one find the pole in the point \( e_k = \sqrt{s}/2 \) in function \( 1/(\sqrt{s}/2 - e_k + i0) \) which should be calculated by using the following formal equation:

$$
\frac{1}{\sqrt{s}/2 - e_k + i0} = \frac{p}{\sqrt{s}/2 - e_k} - i\pi \delta(\sqrt{s}/2 - e_k),
$$

where the first right hand term gives the integral principal value which gives the real part of the function \( h(s) \). The second right hand term gives the imaginary part of \( h(s) \) in the following form

$$
\text{Im} h(s) = \frac{k}{16\sqrt{s}} \sum_L g_L(0, k)^2, \tag{17}
$$

with \( k = \sqrt{s}/4 - m_N^2 \).

The initial values of the parameters for three uncoupled \( P \)-states \( ^1P_0, ^1P_1, ^3P_1 \) and coupled \( ^3S_1 - ^3D_1 \) states are taken from our previous paper for the rank-one separable \( NN \) interaction kernel for nucleon with spinor propagators [8] and they are refitted using the scalar nucleon propagators.

The experimental data for the phase shifts are taken from SAID program http://gwdac.phys.gwu.edu/ and the deuteron energy and low-energy parameter values are taken from Ref. [7].

### 3.1. \(^1S_0\) Partial-Wave State

The parameters of the \(^1S_0\) partial-wave are taken from the paper [9] and given in Table 2 without any changes. The calculated low-energy properties \( (a_L, r_L) \) are given in Table 1 and the phase shifts is shown in Fig. 1.

### 3.2. \(^3P_0, ^1P_1, ^3P_1\) Uncoupled Partial-Wave States

The phase shifts for the uncoupled partial-wave states can be obtained using the Eqs. (5) and (7), namely

$$
\tan \delta_L(s) = \frac{\text{Im} t(s)}{\text{Re} t(s)} = \frac{\text{Im} h(s)}{\lambda_L^{-1} + \text{Re} h(s)}. \tag{18}
$$

To find the parameters \( \lambda_L \) and \( \beta_L \) the procedure to minimize the function

$$
\chi^2(\lambda_L, \beta_L) = \sum_{i=1}^{n} \left( \frac{(\delta_{\exp}^i(s_i) - \delta(s_i))^2}{(\Delta \delta_{\exp}^i(s_i))^2} \right) \tag{19}
$$

is used where \( n \) is the number of the experimental points taken into account. The initial values of the parameters are taken from the paper [8]. In calculation of the function \( \chi^2 \) minimum the kinetic energy maximum is taken: \( T_{\text{lab}}^{\text{max}} = 100 \text{ MeV for } ^3P_0 \)-state and \( T_{\text{lab}}^{\text{max}} = 200 \text{ MeV for } ^1P_1, ^3P_1 \)-states. The calculated parameters are given in Table 2 and phase shifts in Figs. 2–4.

### 3.3. \(^3S_1 - ^3D_1\) Coupled Partial-Wave States

The parameters of the \(^3S_1 - ^3D_1\) coupled partial-wave states are obtained from the parameters in the Ref. [8] using the following procedure:

- the \( C_2 \) parameter is refitted by using the normalization condition (14) to have the \(^3D_1\) partial-wave state pseudoprobability \( p_D = 4.5.6\% \) while the parameters \( \beta_0 \) and \( \beta_2 \) are fixed

| Parameters for the \(^1S_0\) partial-wave channel |
|-----------------------------------------------|
| \( \lambda, \text{ GeV}^4 \) | \( \beta_0, \text{ GeV} \) |
| Exp. from [7] | \(^1S_0\) |
| --- | --- |
| \( \lambda \), \( \text{GeV}^4 \) | \( -1.12087 \) |
| \( \beta_0 \), \( \text{GeV} \) | \( 0.228302 \) |

| Parameters for the \(^3P_0, ^1P_1, ^3P_1\) partial-wave channels |
|---------------------------------------------------------------|
| \( \lambda, \text{ GeV}^6 \) | \( ^3P_0 \) | \( ^1P_1 \) | \( ^3P_1 \) |
| --- | --- | --- | --- |
| \( \lambda \), \( \text{GeV}^6 \) | \( 0.0428572 \) | \( -5.83051 \) | \( -3.68029 \) |
| \( \beta_0 \), \( \text{GeV} \) | \( 0.19904 \) | \( 0.48273 \) | \( 0.44127 \) |

![Fig. 1. The \(^1S_0\) channel phase shifts.](image-url)
the λ parameter is refited by using the eq. (13) to have the deuteron binding energy $E_d = 2.2246$ MeV while all other parameters are fixed.

The calculated parameters and low-energy properties ($a_s$ and $r$) are given in Table 3 and phase shifts are presented in Fig. 5.

### 4. CONCLUSIONS

The rank-one covariant separable kernels of the $NN$ interaction with the spin-one-half particles with the scalar propagators are considered. The parameters for the partial-wave states with the total angular momentum $J = 0, 1$ are obtained from the analysis of the $NN$-scattering observables and deuteron static

| Table 3. Parameters for the $^3S_1 - ^3D_1$ partial-wave channels |
|----------------------|------------------|------------------|------------------|
| $\lambda$, GeV$^4$    | $^3S_1 - ^3D_1$ ($p_d = 4\%$) | $^3S_1 - ^3D_1$ ($p_d = 5\%$) | $^3S_1 - ^3D_1$ ($p_d = 6\%$) |
| $\beta_s$, GeV       | $-1.83756$       | $-1.57495$       | $-1.34207$       |
| $C_2$                | $0.251248$       | $0.246713$       | $0.242291$       |
| $\beta_s$, GeV       | $1.71475$        | $2.52745$        | $3.46353$        |
| $a_s$, fm            | $0.294096$       | $0.324494$       | $0.350217$       |
| $r_s$, fm            | $5.424$          | $5.454$          | $5.454$          |
|                      | $1.756$          | $1.81$           | $1.80$           |
properties. The proposed models for the kernels will be used to investigate the three-nucleon bound states and their dynamic electromagnetic properties (form factors).

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