Generic Regular Decompositions for Parametric Polynomial Systems

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Abstract
This paper presents a generalization of our earlier work in [19]. In this paper, the two concepts, generic regular decomposition (GRD) and regular-decomposition-unstable (RDU) variety introduced in [19] for generic zero-dimensional systems, are extended to the case where the parametric systems are not necessarily zero-dimensional. An algorithm is provided to compute GRDs and the associated RDU varieties of parametric systems simultaneously on the basis of the algorithm for generic zero-dimensional systems proposed in [19]. Then the solutions of any parametric system can be represented by the solutions of finitely many regular systems and the decomposition is stable at any parameter value in the complement of the associated RDU variety of the parameter space. The related definitions and the results presented in [19] are also generalized and a further discussion on RDU varieties is given from an experimental point of view. The new algorithm has been implemented on the basis of DISCOVERER [28] with Maple 16 and experimented with a number of benchmarks from the literature.

Keywords: parametric polynomial system, regular-decomposition-unstable variety, generic regular decomposition

1 Introduction

As is well known, solving parametric polynomial system plays a key role in many application fields such as automated geometry theorem deduction, stability analysis of dynamical systems, robotics and so on. To solve a parametric system symbolically, a basic idea is to transform the system into new systems with special structures or properties so that the solutions of the original system can be handled via studying the solutions of the new systems, which is relatively easy. Remarkable examples of such methods are the algorithms for computing comprehensive Gröbner systems (CGS) and comprehensive Gröbner bases (CGB) [26, 12, 15, 16, 17, 18]. The methods based on triangular decompositions are another kind of such examples [1, 4, 7, 11, 13, 20, 22, 27, 29, 31, 42].

Since Wu’s work [27], lots of well-known methods based on triangular decompositions have been proposed. An essential concept, “regular chain” (or “normal chain”), and algorithms for computing regular chain decomposition have been introduced by Kalkbrener [11] and Yang and Zhang, [32] independently. For parametric systems, Gao and Chou proposed a method in [7] for identifying all parametric values for which a given system has solutions and giving the solutions by $p$–chains without a partition of the parameter space. Wang gave an efficient algorithm for computing regular system decomposition [22, 23, 25], which is a generalization of regular chain decomposition. The concept of comprehensive triangular decomposition (CTD) introduced by Chen et al. in [4] is an analogue of the CGS for solving parametric polynomial systems.

Two new concepts, generic regular decomposition and regular-decomposition-unstable (RDU) variety for generic zero-dimensional systems, are introduced in [19] and an algorithm is proposed for computing a generic regular decomposition and the associated RDU variety of a given generic...
zero-dimensional system simultaneously. The solutions of the given system can be expressed by finitely many zero-dimensional regular chains if the parameter value is not on the RDU variety. The so-called weakly relatively simplicial decomposition (WRSD) plays a crucial role in the algorithm, which is based on the theories of subresultants.

In this paper, the concepts, generic regular decomposition (GRD) and regular-decomposition-unstable (RDU) variety, introduced in [19] for generic zero-dimensional systems are extended to the case where the parametric systems are not necessarily zero-dimensional. An algorithm is provided to compute GRDs and the associated RDU varieties of parametric systems simultaneously on the base of the algorithm for generic zero-dimensional systems proposed in [19]. Then the solutions of any parametric system can be represented by the solutions of finitely many regular systems and the decomposition is stable at any parameter value in the complement of the associated RDU variety of the parameter space. The new algorithm has been implemented on the base of DISCOVERER [28] with Maple 16 and experimented with a number of benchmarks from the literature [4, 9, 12, 15, 16]. Empirical results are also presented to show the good performance of the algorithm. In other words, this paper presents a generalization of our earlier work in [19].

First of all, we need to introduce the idea proposed in [19] briefly. For a given generic zero-dimensional system $P$ with $n$ variables and $d$ parameters, we considered the parameters as “constants” and proposed Algorithm RDUForZD for computing a so-called generic regular decomposition $T$ of $P$ in $K[U]$ such that $V_{K[U]}(P) = \bigcup_{T \in T} V_{K[U]}(T)\setminus H$, where $T$ is a set of regular chains. At the same time, the algorithm would obtain a parametric polynomial such that the regular decomposition was stable at any parametric point outside the variety (called RDU variety) generated by the parametric polynomial. Roughly speaking, “stable at a parametric point” means that the regular decomposition remains after we substitute the point for the parameters in $P$ and $T$ (see Definition 5). As a result, the original generic zero-dimensional system is “solved” except for the case where parameters are on the RDU variety. That is why the decomposition is called generic regular decomposition.

Now we would like to show some new ideas of this paper. If the given system is not generic zero-dimensional, to obtain a decomposition with similar properties as in the zero-dimensional case, we choose to express the solutions of the system by finitely many regular systems [23] instead of regular chains. So we need to generalize the concept “generic regular (chain) decomposition” introduced in [19] into “generic regular (system) decomposition” (see Definitions 2, 3 and 5). For solving a positive dimensional system, a natural idea is to view some variables as parameters and call recursively the algorithm for generic zero-dimensional systems proposed in [19]. However, to prove the correctness of this procedure, we need to study the properties of characteristic sets under specifications carefully (see Lemma 2 and Corollary 2). Besides, it is worth to notice that we have two different interpretations for the results computed by Algorithm 2DToRC and both of them play a key role in the proof of the correctness (see Lemma 5). Finally, we give an algorithm which, for any parametric system $P$, computes a finite set $TH$ of regular systems in $K[U][X]$ and a polynomial $B \in K[U]$, such that

1. $V_{K[U]}(P) = \bigcup_{[T, H] \in TH} V_{K[U]}(T)\setminus H$; and

2. for any $a \in K \setminus V(B)$, $V(P(a)) = \bigcup_{[T, H] \in TH} V(T(a))\setminus H(a)$ and $[T, H]$ specializes well at $a$ for any $[T, H] \in TH$.

Please see Algorithm 4 in this paper for more details. What’s more, for different orderings of variables, the efficiency of the algorithm can be different and the RDU varieties can be totally different.

At the end of this section, it is worth to point out that the algorithm provided in this paper has a different feature compared to some existing algorithms. The algorithm for computing regular system decomposition proposed in [22, 23] uses the so-called variable elimination, which computes a main branch at first and then gets the other branches one by one. An incremental algorithm, introduced in [4, 5] for computing regular chain decomposition, computes a regular chain decomposition for some polynomials in the given system at first and then intersects the other polynomials with the regular chains one by one. The algorithm proposed in this paper makes use of a hierarchical strategy. From an experimental point of view, different strategies are suitable for different benchmarks (see Section 4).
The paper is organized as follows. Section 2 provides basic definitions and concepts that are needed to understand the main algorithm. Section 3 contains the main algorithm, namely Algorithm 4, and some relative subalgorithms. Also we review the description of algorithms in our former article. Besides, proofs for these algorithms are presented in this section. Some illustrative examples, the empirical data and comparison with previous work along with several implementation details are presented in Section 4. Section 5 concludes the paper with a discussion on our future work along this direction.

2 Preliminaries

The following paragraphs give a brief outline of the vocabulary and tools we will be using throughout the paper. All concepts without precise definitions can be found in [3, 27, 31]. \( \mathbb{R} \) and \( \mathbb{C} \) stand for the field of real numbers and the field of complex numbers, respectively.

Suppose \( \{u_1, \ldots, u_d, x_1, \ldots, x_n\} \) is a set of indeterminates with a given order \( u_1 < \ldots < u_d < x_1 < \ldots < x_n \) where \( u_1, \ldots, u_d \) and \( x_1, \ldots, x_n \) are the sets of parameters and variables, respectively. Let \( U = \{u_1, \ldots, u_d\} \) and \( X = \{x_1, \ldots, x_n\} \). Suppose \( K \) is a field and \( \overline{K} \) denotes its algebraic closure. Let \( K[U] \) be the ring of polynomials in \( U \) with coefficients in \( K \) and \( K(U) \) be the rational function field. A non-empty finite subset \( P \) of \( K[U][X] \) is said to be a polynomial system or system. If \( P \subseteq K[U][X] \), it is a parametric polynomial system or parametric system. If \( P \subset K[X] \), it is a constant polynomial system or constant system.

For a non-empty finite subset \( P \subset K[U][X] \), \( (P)_{K[U][X]} \) denotes the ideal generated by \( P \) in \( K[U][X] \) and \( \sqrt{(P)_{K[U][X]}} \) denotes the radical of \( (P)_{K[U][X]} \). For any \( F \in K[U][X] \setminus \{0\} \), \( \text{deg}(F) \) denotes the degree of \( F \). The class of \( P \) in \( K[U][X] \), denoted by \( \text{cls}(F) \), is either a non-contradictory ascending chain or a contradictory chain. A non-empty finite set \( \{u_1, \ldots, u_d, x_1, \ldots, x_n\} \) is a triangular set in \( K(U) \). A non-empty finite set \( \{u_1, \ldots, u_d, x_1, \ldots, x_n\} \) is a triangular set in \( K(U) \).

The paper is organized as follows. Section 2 provides basic definitions and concepts that are needed to understand the main algorithm. Section 3 contains the main algorithm, namely Algorithm 4, and some relative subalgorithms. Also we review the description of algorithms in our former article. Besides, proofs for these algorithms are presented in this section. Some illustrative examples, the empirical data and comparison with previous work along with several implementation details are presented in Section 4. Section 5 concludes the paper with a discussion on our future work along this direction.
Successive Pseudo Remainder. For two polynomials $F$ and $P$ in $K[U][X]$ ($\overline{K}(X)$) and a variable $x \in X$, the pseudo remainder of $F$ pseudo-divided by $P$ w.r.t. $x$ is denoted by $\text{prem}(F, P, x)$. Particularly, $\text{prem}(F, P, \text{mvar}(P))$ is denoted by $\text{prem}(F, P)$. For a polynomial $F \in K[U][X]$ ($\overline{K}(X)$) and a triangular set $T = \{T_1, \ldots, T_r\}$ in $K[U][X]$ ($\overline{K}(X)$), the successive pseudo remainder $\overline{22}$ of $F$ w.r.t. $T$ is denoted by $\text{prem}(F, T)$, namely

$$\text{prem}(F, T) = \text{prem}(\ldots \text{prem}(F, T_r, T_{r-1}), \ldots, T_1).$$

For a finite set $P \subset K[U][X]$ ($\overline{K}(X)$), $\text{prem}(P, T)$ denotes the set $\{\text{prem}(F, T) \mid F \in P\}$.

Successive Resultant. For two polynomials $F$ and $P$ in $K[U][X]$ ($\overline{K}(X)$) and a variable $x \in X$, the resultant $\overline{22}$ of $F$ and $P$ w.r.t. $x$ is denoted by $\text{res}(F, P, x)$. Particularly, $\text{res}(F, P, \text{mvar}(P))$ is denoted by $\text{res}(F, P)$. For a polynomial $F \in K[U][X]$ ($\overline{K}(X)$) and a triangular set $T = \{T_1, \ldots, T_r\}$ in $K[U][X]$ ($\overline{K}(X)$), the successive resultant $\overline{22}$ of $F$ w.r.t. $T$ is denoted by $\text{res}(F, T)$, namely

$$\text{res}(F, T) = \text{res}(\ldots \text{res}(F, T_r, T_{r-1}), \ldots, T_1).$$

Regular Chain. A triangular set $T = \{T_1, \ldots, T_r\}$ in $K[U][X]$ ($\overline{K}(X)$) is said to be a regular chain in $K[U][X]$ ($\overline{K}(X)$), if $I_T \neq \emptyset$ and for each $i (1 < i \leq r)$, $\text{res}(T_i, \{T_{i-1}, \ldots, T_1\}) \neq \emptyset$. If $T$ is a regular chain in $K[U][X]$ ($\overline{K}(X)$) and $\text{mvar}(T) = X$, $T$ is a zero-dimensional regular chain.

Regular System. Let $T \subset K[U][X]$ ($\overline{K}(X)$) be a regular chain and $H \in K[U][X]$ ($\overline{K}(X)$). If $\text{res}(H, T) \neq 0$, then $[T, H]$ is said to be a regular system in $K[U][X]$ ($\overline{K}(X)$).

Proposition 1. $\overline{B}$ If $[T, H]$ is a regular system in $K[U][X]$, then $V_{K[U]}(T \setminus H) \neq \emptyset$.

Assignment Homomorphism. For each $a = (a_1, \ldots, a_d) \in \overline{K}^d$, $\phi_a : K[U][X] \rightarrow \overline{K}(X)$ is a homomorphism such that $\phi_a(F) = F(a, X)$ for all $F \in K[U][X]$ and we denote $\phi_a(F)$ by $\overline{F}(a)$. For a non-empty finite set $P \subset K[U][X]$, $P(a)$ denotes the set $\{\overline{F}(a) \mid F \in P\}$ and remark that $P(a) = \emptyset$ if $P = \emptyset$.

Characteristic Set And Wu’s Method. An ascending chain $C$ in $K[U][X]$ is a characteristic set of $P$ in $K[U][X]$ if $C \subset (P)_{K[U][X]}$ and $\text{prem}(P, C) = \{0\}$. Theorem $\overline{1}$ below is the so-called well-ordering principle.

Theorem 1. $\overline{22}$ There exists an algorithm which, for an input non-empty finite subset $P \subset K[U][X]$, outputs either a contradictory ascending chain meaning that $V_{K[U]}(P) = \emptyset$, or a (non-contradictory) characteristic set $C = \{C_1, \ldots, C_t\}$ such that

$$V_{K[U]}(P) = V_{K[U]}(C \setminus \{C_t\}) \cup \cup_{i=1}^{t} V_{K[U]}(P \cup C \cup \{I_{C_i}\}).$$

On the base of Theorem $\overline{1}$ there exists an algorithm, namely Wu’s method, for computing a finite sequence of ascending chains $C_1, C_2, \ldots, C_m$ ($m \geq 1$) in $K[U][X]$ such that

1. $C_1, C_2, \ldots, C_m$ is a finite sequence of characteristic sets in $K[U][X]$;
2. If $m = 1$, $V_{K[U]}(P) = \emptyset$. Otherwise, suppose $S = \{C_i \mid 1 \leq i \leq m \text{ and } C_i \text{ is a non-contradictory ascending chain}\}$, then $V_{K[U]}(P) = \cup_{C \in S} V_{K[U]}(C \setminus \{C_t\}).$

The set of ascending chains $\{C_1, C_2, \ldots, C_m\}$ above is said to be a Wu’s decomposition or characteristic set decomposition of $P$ in $K[U][X]$.

For any triangular set $T$ in $K[U][X]$, we denote $\#(X) - \#(T)$ by $d(T, X)$.

Definition 1. Let $P$ be a parametric system in $K[U][X]$ and the set $\{C_1, C_2, \ldots, C_m\}$ of ascending chains be a Wu’s decomposition of $P$ in $K[U][X]$. If $d(C_i, X) = 0$ for every non-contradictory ascending chain $C_i$, $P$ is said to be a generic zero-dimensional system. Otherwise, $P$ is said to be a generic positive-dimensional system.

Definition 2. Let $P$ be a parametric system in $K[U][X]$ and $\mathbb{T} = \{(T_1, H_1), \ldots, (T_s, H_s)\}$ be a set of regular systems in $K[U][X]$. $\mathbb{T}$ is said to be a parametric regular system decomposition of $P$ in $K[U][X]$, if $V_{K[U]}(P) = \cup_{i=1}^{s} V_{K[U]}(T_i \setminus H_i)$.

1 The definition of regular system is the same as that introduced in $\overline{I}$ and different from that proposed in $\overline{22}$, see more details in $\overline{4}$. 
Definition 3. Let \([T, H]\) be a regular system in \(K[U][X]\) and \(a \in \overline{K}^d\). If \(T(a)\) is a regular chain in \(\overline{K}[X]\), \(\text{rank}(T(a)) = \text{rank}(T)\) and \(\text{res}(H(a), T(a)) \neq 0\), then we say that the regular system \([T, H]\) specializes well at \(a\).

Definition 4. Let \(P\) be a parametric system in \(K[U][X]\) and \(\mathcal{T}_P = \{[T_1, H_1], \ldots, [T_s, H_s]\}\) be a parametric regular system decomposition of \(P\) in \(K[U][X]\). For any \(a \in \overline{K}^d\), if \(V(P(a)) = \bigcup_{i=1}^s V(T_i(a) \setminus H_i(a))\) and \([T_i, H_i] (1 \leq i \leq s)\) specializes well at \(a\), then \(\mathcal{T}_P\) is said to be stable at \(a\).

Remark that the concept of stable generic regular (chain) decomposition is first introduced in [19].

Definition 5. Let \(\mathcal{T}_P\) be a parametric regular system decomposition of a given parametric system \(P\) in \(K[U][X]\). If there is an affine variety \(V\) in \(\overline{K}^d\) with \(\dim(V) < d\) such that \(\mathcal{T}_P\) is stable at any \(a \in \overline{K}^d \setminus V\), then \(\mathcal{T}_P\) is said to be a generic regular system decomposition of \(P\) and \(V\) is said to be a regular-decomposition-unstable variety (RDU) of \(P\) w.r.t. \(\mathcal{T}_P\).

Definition 6. Let \(T\) be a zero-dimensional regular chain in \(K[U][X]\) and \(P \in K[U][X]\). Suppose \(\mathbb{H}\) and \(\mathbb{G}\) are two finite sets of zero-dimensional regular chains in \(K[U][X]\). If

1. \(V_{K(U)}(T \cup \{P\}) = \bigcup_{H \in \mathbb{H}} V_{K(U)}(H)\),
2. \(V_{K(U)}(T \setminus P) = \bigcup_{G \in \mathbb{G}} V_{K(U)}(G)\),

then \((\mathbb{H}, \mathbb{G})\) is said to be a weakly relatively simplicial decomposition of \(T\) w.r.t. \(P\) in \(K[U][X]\).

In [19], we gave an algorithm for computing weakly relatively simplicial decompositions. Here, we only present its specification but omit the details.

Algorithm 1. \texttt{WRSD}

\textbf{Input:} A zero-dimensional regular chain \(T = \{T_1, \ldots, T_n\}\) in \(K[U][X]\), a polynomial \(P \in K[U][X]\), variables \(X = \{x_1, \ldots, x_n\}\)

\textbf{Output:} \((\mathbb{H}, \mathbb{G}, B)\), where

1. \((\mathbb{H}, \mathbb{G})\) is a weakly relatively simplicial decomposition of \(T\) w.r.t. \(P\) in \(K[U][X]\);
2. \(B\) is a polynomial in \(K[U]\) such that for any \(a \in K^d \setminus V^U(B)\), the weakly relatively simplicial decomposition \((\mathbb{H}, \mathbb{G})\) of \(T\) w.r.t. \(P\) is stable\(^2\) at \(a\).

3 Theory and Algorithm

We first give some notations. Assume that \texttt{Alg} is a name of an algorithm and \(p_1, \ldots, p_t\) is a sequence of inputs of this algorithm. If the output of \texttt{Alg}(\(p_1, \ldots, p_t\)) is a finite sequence \(q_1, \ldots, q_s\), \(q_i\) is denoted by \texttt{Alg}(\(p_1, \ldots, p_t\), \(i\)). Given a finite set \(S = \{s_1, \ldots, s_t\}\) and a map \(\phi\) on \(S\), \(\text{map}(s \to \phi(s), S)\) denotes the set \(\{\phi(s) \mid s \in S\}\).

3.1 Wu’s Decomposition Under Specification

The general idea of Wu’s method is presented in Section 2. Some results on Wu’s decomposition under specification are given in this section.

Definition 7. Let \(P_1\) be a parametric system in \(K[U][X]\) and \(S = \{C_1, \ldots, C_m\}\) be a Wu’s decomposition of \(P_1\) in \(K[U][X]\). Suppose \(L = \{C_{l,1}, C_{l,2}, \ldots, C_{l,k}\}\) is a subset of \(S\) satisfying that

1. \(C_{l,1}\) is a characteristic set of \(P_1\).
2. If \(k \geq 2\), \(C_{l,i} (2 \leq i \leq k)\) is a characteristic set of \(P_i = P_{i-1} \cup C_{l,i-1} \cup \{1_{C_{l,i-1}}\}\) where \(C_{l,i-1} \subset C_{l,i-1}\).

\(^2\)Please see the definition of stable in [19].
(3) If $k = 1$, $C_{t,1}$ is a contradictory ascending chain. Otherwise, $C_{t,i}$ ($1 \leq i \leq k - 1$) is a non-contradictory ascending chain and $C_{t,k}$ is a contradictory ascending chain.

Then $L$ is said to be a line of $S$ and $P^L = \{P_1, \ldots, P_k\}$ the corresponding systems.

**Lemma 1.** Let $P_1$ be a parametric system in $K[U][X]$ and $S = \{C_1, \ldots, C_m\}$ be a Wu’s decomposition of $P_1$ in $K[U][X]$. Let $L = \{C_{t,1}, \ldots, C_{t,k}\}$ be a line of $S$ with corresponding systems $\{P_1, \ldots, P_k\}$. Then for any $a \in \overline{K} \setminus V^L(C_{t,k})$, there exists a polynomial $p_i \in P_i$ such that $p_i(a) \neq 0$ for any $i$ ($1 \leq i \leq k$).

**Proof.** We prove it by induction on the number k of elements of $L$. If $k=1$, it means $L$ contains only one element. Since $C_{t,1}$ is a contradictory ascending chain, we can assume that $C_{t,1} = \{C_1\}$ where $C_{t,1} \in K[U]$ and we know that $C_{t,1} \in (P_1)$. Suppose $P_1 = \{f_{1,1}, \ldots, f_{1,t_1}\}$. Then $C_{t,1}$ can be written as $C_{t,1} = \textstyle \sum_{j=1}^{t_1} h_j f_{1,j}$ where $h_j \in K[U][X]$ for any $j$ ($1 \leq j \leq t_1$). Thus $C_{t,1}(a) = \textstyle \sum_{j=1}^{t_1} h_j(a) f_{1,j}(a)$ holds. Since $C_{t,1}(a) \neq 0$, there must exist some $f_{1,e_1} \in P_1$ such that $f_{1,e_1}(a) \neq 0$. Let $p_1 = f_{1,e_1}$ and we are done.

Now we assume that the conclusion holds when $k < N$ ($N > 1$). Suppose $k = N$. Let $L_2 = \{C_{t,2}, \ldots, C_{t,k}\} \subseteq L$ be a line of Wu’s decomposition of $P_2$. According to the induction hypothesis, for any $a \in \overline{K} \setminus V^L(C_{t,k})$, there exists a polynomial $p_i \in P_i$ such that $p_i(a) \neq 0$ for any $i$ ($2 \leq i \leq k$). As is known to us, $P_2 = P_1 \cup C_{t,1} \cup \{I_{C_1}\}$, if $P_2 \in P_1$, let $p_1 = p_2$ and the conclusion holds obviously. Otherwise, $P_2 \in \{I_{C_1}\} \cup C_{t,1}$. If $P_2 \in C_{t,1} \subset (P_1)$, then $P_2$ can be written as $p_2 = \textstyle \sum_{j=1}^{t_1} h_j f_{1,j}$ where $h_j \in K[U][X]$ for any $j$ ($1 \leq j \leq t_1$). Thus $p_2(a) = \textstyle \sum_{j=1}^{t_1} h_j(a) f_{1,j}(a)$ holds. Since $p_2(a) \neq 0$, there must exist some $f_{1,e_1} \in P_1$ such that $f_{1,e_1}(a) \neq 0$. Let $p_1 = f_{1,e_1}$ and we are done. If $P_2 = \{I_{C_1}\}$, it implies $C_{t,1}(a) \neq 0$. Since $C_{t,1}$ can be written as $C_{t,1} = \textstyle \sum_{j=1}^{t_1} h_j f_{1,j}$ where $h_j \in K[U][X]$ for any $j$ ($1 \leq j \leq t_1$). Thus there must exist some $f_{1,e_1} \in P_1$ such that $f_{1,e_1}(a) \neq 0$. Let $p_1 = f_{1,e_1}$ and the conclusion holds.

With Lemma 1 Corollary 1 holds obviously.

**Corollary 1.** Let $P$ be a parametric system in $K[U][X]$ and $S = \{C_1, \ldots, C_m\}$ be a Wu’s decomposition of $P$ in $K[U][X]$. Let $L = \{C_{t,1}, C_{t,2}, \ldots, C_{t,k}\}$ be a line of $S$. If $P = \{p\}$, containing only one polynomial in $K[U][X]$, $p(a) \neq 0$ for any $a \in \overline{K} \setminus V^L(C_{t,k})$.

**Lemma 2.** Suppose $C = \{C_1, \ldots, C_t\}$ is a non-contradictory ascending chain and a characteristic set of parametric system $P$ in $K[U][X]$. For any $a \in \overline{K}$, $V(P(a)) = V(C(a) \setminus I_C(a)) \cup \cup_{i=1}^{t} V((P(a) \setminus C(a)) \cup I_{C_i}(a))$.

**Proof.** For any $a \in \overline{K}$, if $I_C(a) \equiv 0$, the conclusion holds obviously since $I_C(a) = \textstyle \prod_{i=1}^{t} I_{C_i}(a)$. Now we prove the conclusion when $I_C(a) \neq 0$. According to the definition of characteristic set, we know that $C \subset \langle P \rangle$ and for any $p \in P$, $I_{C_i}^{q_1} \cdots I_{C_i}^{q_t} p = q_1 C_1 \cdots + q_t C_t$ where $q_i \in K[U][X]$ for any $i$ ($1 \leq i \leq t$). Then $C(a) \subset \langle P(a) \rangle$ and $I_{C_i}^{q_1} \cdots \cdot I_{C_i}^{q_t} p(a) = q_1 C_1(a) \cdots + q_t C_t(a)$. Therefore $V(C(a) \setminus I_C(a)) \subset V(P(a)) \subset V(C(a))$ and $V(P(a)) = V(C(a) \setminus I_C(a)) \cup V((P(a) \cup I_C(a)))$. Since $C(a) \subset \langle P(a) \rangle$, $V(P(a) \cup I_C(a)) = V(P(a) \setminus C(a)) = \cup_{i=1}^{t} V((P(a) \setminus C(a)) \cup I_{C_i}(a))$. We are done.

According to Theorem 4 and Lemma 2, we can get Corollary 2 easily.

**Corollary 2.** Let $P$ be a parametric system in $K[U][X]$ and $\{C_1, \ldots, C_m\}$ be a Wu’s decomposition of $P$ in $K[U][X]$. Suppose $S = \{C_i|1 \leq i \leq m\}$ and $C_i$ is a non-contradictory ascending chain and $CS = \{C_i|1 \leq i \leq m\}$ and $C_i$ is a contradictory ascending chain. Then for any $a \in \overline{K} \setminus V^L(\cup_{CS} V^U(S))$, $V(P(a)) = \cup_{CS} V(C(a) \setminus I_C(a))$.

### 3.2 Converting To Regular Systems

Let $T$ be a triangular set in $K[U][X]$. We can compute a set of finite sequence of regular systems $\emptyset \mathcal{S} = \{[T_1, H_2], \ldots, [T_s, H_s]\}$ in $K[U][X]$ and a polynomial $B \in K[U]$ on the basis
of Algorithm $[1]$ such that $V_{K(U)}(T \backslash I_T) = \bigcup_{i=1}^n V_{K(U)}(T_i \backslash H_i)$ and for any $a \in \overline{K}^d \setminus V^U(B)$, TH specializes well at $a$ and $V(T(a) \backslash I_T(a)) = \bigcup_{i=1}^n V(T_i(a) \backslash H_i(a))$. The algorithm is presented as Algorithm $[3]$ which plays a key role in Algorithm $[1]$ proposed in the next section.

Algorithm $[2]$ below was proposed in $[13]$ for zero-dimensional case. We just give its specification here. Our focus in this paper is how to deal with the case where the triangular set is positive-dimensional. So, we need to convert a triangular set with $\text{mvar}(T) \subseteq X$ to a set of regular systems.

**Algorithm 2. ZDToRC**

**Input:** A triangular set $T$ in $K[U][X]$ with $\text{mvar}(T) = X$, variables $X = \{x_1, \ldots, x_n\}$.

**Output:** $[G, B]$, where

1. $G$ is a finite set of zero-dimensional regular chains in $K[U][X]$ such that $V_{K(U)}(T \backslash I_T) = \bigcup_{G \in G} V_{K(U)}(G)$;
2. $B$ is a polynomial in $K[U]$ such that for any $a \in \overline{K}^d \setminus V^U(B)$, $V(T(a) \backslash I_T(a)) = \bigcup_{G \in G} V(G(a))$ and $G$ specializes well at $a$ for any $G \in G$.

According to Algorithm $[2]$ the following Proposition $[2]$ is clear.

**Proposition 2.** Let $T$ be a triangular set in $K[U][X]$ and $\text{ZDToRC}(T, \text{mvar}(T)) = [G, B]$. If $G \neq \emptyset$, $I_T(a) \neq 0$ for $a \not\in V^U(B)$.

Now suppose $T = \{T_1, \ldots, T_l\}$ is a triangular set in $K[U][X]$ and $\text{mvar}(T) \subseteq X$ where variables $X = \{x_1, \ldots, x_n\}$ and parameters $U = \{u_1, \ldots, u_d\}$. It is interesting to show that we have two versions to interpret the relationship between $T$ and the results computed by $\text{ZDToRC}(T, \text{mvar}(T))$. Assume that $\text{ZDToRC}(T, \text{mvar}(T)) = [G, B]$. Let $B(T) = \text{mvar}(T)$ and $F(T) = X \setminus \text{mvar}(T)$. On one hand, $T$ can be regarded as a triangular set in $K[U, F(T)][B(T)]$. At this point, according to Algorithm $[2]$ we know that

1. $V_{K(U,F(T))}(T \backslash I_T) = \bigcup_{G \in G} V_{K(U,F(T))}(G)$;
2. for any $a \in \overline{K}^{d+n-l} \setminus V^{U,F(T)}(B)$, $V(T(a) \backslash I_T(a)) = \bigcup_{G \in G} V(G(a))$ and $G$ specializes well at $a$ for any $G \in G$ if $G \neq \emptyset$.

On the other hand, $T$ can also be regarded as a triangular set in $K[U,F(T)][B(T)]$. Let $K = K(U), \mathcal{U} = F(T)$ and $X = B(T)$. Then according to Algorithm $[2]$

1. $V_{K(U)}(T \backslash I_T) = \bigcup_{G \in G} V_{K(U)}(G)$;
2. for any $a \in \overline{K}^{d+n-l} \setminus V^U(B)$, $V_{K}(T(a) \backslash I_T(a)) = \bigcup_{G \in G} V_{K}(G(a))$ and $G$ specializes well at $a$ for any $G \in G$ if $G \neq \emptyset$.

Remark that the above statements (1) and (3) are exactly the same since $K(U, \mathcal{U}) = K(U)$. As discussed above, we have the following lemma by statements (2) and (4).

**Lemma 3.** Suppose $T$ is a triangular set in $K[U][X]$ and $\text{ZDToRC}(T, \text{mvar}(T)) = [G, B]$. Then $V_{K(U)}(T \backslash I_T \cdot B) = \bigcup_{G \in G} V_{K(U)}(G \cdot B)$ and $V(T(a) \backslash I_T(a) \cdot B(a)) = \bigcup_{G \in G} V(G(a) \cdot B(a))$ for any $a \in \overline{K}^{d} \setminus V^U(B)$.

**Theorem 2.** Algorithm $[3]$ terminates correctly.

**Proof.** For a given triangular set $T = \{T_1, \ldots, T_l\}$ in $K[U][X]$. Suppose $\text{ZDToRC}(T, \text{mvar}(T)) = [G_0, B_0]$.

Firstly, we prove Algorithm $[3]$ terminates. If $B_0 \in K[U]$, it terminates obviously. Otherwise, $B_0 \in K[U][F(T)] \subset K[U][X]$. Assume that $\{C_1, C_2, \ldots, C_m\}$ is the Wu’s decomposition of $\{B_0\}$ in $K[U][X]$ computed by Wu’s method and $\mathcal{S} = \{C_i|1 \leq i \leq m\}$ and $C_i$ is a non-contradictory ascending chain. Obviously $m > 1$ since $B_0 \in K[U][F(T)] \subset K[U]$. For any $C_i \in \mathcal{S}$, let $T_i = T \cup C_i$. Then we know that $\text{mvar}(T) \subseteq \text{mvar}(T_i)$, which means $d(T, X) < d(T_i, X)$. Therefore, it is clearly that Algorithm $[3]$ terminates within finite steps of recursion.

Now we prove the correctness by induction on the recursive depth $h$. If $h = 1$, according to Algorithm $[3]$, $B_0 \in K[U]$ and the conclusion follows from Lemma $[3]$. Assume that the conclusion holds for $h < N$ ($N > 1$). When $h = N$, $B_0 \in K[U][F(T)] \subset K[U]$. We can assume
Algorithm 3. TSToRS

**Input:** A triangular set $T = \{T_1, \ldots, T_l\} \subset K[U][X]$, variables $X = \{x_1, \ldots, x_n\}$.

**Output:** $[G, B]$, where

1. $G$ is a finite set of regular systems in $K[U][X]$ such that $V_{K[U]}(T \setminus I_T) = \cup_{G, H \in G} V_{K[U]}(G \setminus H)$;
2. $B$ is a polynomial in $K[U]$, for any $a \in K[U] \setminus V^U(B)$, $V(T(a) \setminus I_T(a)) = \cup_{G, H \in G} V(G(a) \setminus H(a))$ and $[G, H]$ specializes well at $a$ for any $[G, H] \in G$ if $G \neq \emptyset$.

1. $W := \emptyset$
2. $G := \emptyset$
3. if $W \in K[U]$ then
   4. return $[G, W]$
5. Compute a Wu's decomposition $\{C_1, C_2, \ldots, C_m\}$ of $\{W\}$ in $K[U][X]$ by Wu's method
6. $B := 1$
7. for $i = 1 \rightarrow m$
   8. if $C_i$ is a contradictory ascending chain then
      9. $B := B \cdot \text{op}(C_i)$
   else
      10. $T_i := T \cup C_i$
11. $G := G \cup \text{TSToRS}(T_i, X)\bigcup B := B \cdot \text{TSToRS}(T_i, X)\bigcup$\bigcup $[G, B]$\bigcup

that $\{C_1, C_2, \ldots, C_m\}$ is the Wu's decomposition of $\{B_0\}$ in $K[U][X]$ computed by Line 5 in Algorithm 3. Let $N = \{1 \leq i \leq m \text{ and } C_i \text{ is a non-contradictory ascending chain}\}$ and $CN = \{1, 2, \ldots, m\} \setminus N$. For any $i \in N$, let $T_i = T \cup C_i$ and then $T_i$ is a triangular set in $K[U][X]$. Suppose $C_j = \{C_j\}$ ($j \in CN$), $\text{TSToRS}(T_1, \text{mvar}(T_1)) = [G_1, B_1]$ for any $i \in N$. Then according to Algorithm 3 $G = \cup_{G \in G_0} [G, B_0] \cup \cup_{i \in N} G_i$ and $B = \prod_{j \in CN} C_j \cdot \prod_{i \in N} B_i$. By Lemma 3, Wu's method and the induction hypothesis, we get

$$V_{K[U]}(T \setminus I_T) = V_{K[U]}(T \setminus B_0) \cup V_{K[U]}(T \setminus \{B_0\} \setminus I_T)$$

$$= \cup_{G \in G_0} V_{K[U]}(G \setminus B_0) \cup (V_{K[U]}(T \setminus I_T) \cap V_{K[U]}(B_0))$$

$$= \cup_{G \in G_0} V_{K[U]}(G \setminus B_0) \cup (V_{K[U]}(T \setminus I_T) \cap \cup_{i \in N} V_{K[U]}(C_i \setminus I_C))$$

$$= \cup_{G \in G_0} V_{K[U]}(G \setminus B_0) \cup (\cup_{i \in N} V_{K[U]}(T_i \setminus I_T_i))$$

$$= \cup_{G \in G_0} V_{K[U]}(G \setminus B_0) \cup (\cup_{i \in N} \cup_{G, H \in G_i} V_{K[U]}(G \setminus H)).$$

Therefore, the statement (1) in the specification of Algorithm 3 holds.

For any $a \in K[U] \setminus V^U(B)$, $C_j(a) \neq 0$ and $B_i(a) \neq 0$ for any $i \in N$ and $j \in CN$. Thus by Corollary 3 and the induction hypothesis, we get

$$V(T(a) \setminus I_T(a)) = \cup_{i \in N} V(C_i(a) \setminus I_C_i(a)).$$

By Lemma 3, we know that $[G, H]$ specializes well at $a$ for any $i \in N$. Therefore, the statement (2) in the specification of Algorithm 3 holds. □
Remark 1. By Algorithm $\mathcal{A}$ for any regular system $[T, H]$ in the first output of Algorithm $\mathcal{A}$ we know that $\text{res}(I_T, T)$ is a factor of $H$. If $B \in K[U]$ is the second output of Algorithm $\mathcal{A}$ by Corollary 2 we know that $V_{K[U]}(\text{res}(I_T, T)) \subset V_{K[U]}(H)$ and $V^U(\text{res}(I_T, T)) \subset V^U(H) \subset V^U(B)$.

3.3 Computing RDU

We present the main result in this section. Algorithm $\mathcal{A}$ shows how to compute a generic regular system decomposition and the associated RDU of a given system simultaneously.

Algorithm 4. RDU

Input: A parametric system $P$ in $K[U][X]$, variables $X = \{x_1, \ldots, x_n\}$.

Output: $[\mathcal{T}, B]$, where
(1) $\mathcal{T}$ is a finite set of regular systems in $K[U][X]$ such that $V_{K[U]}(P) = \bigcup_{[T, H] \in \mathcal{T}} V_{K[U]}(T[H])$,
(2) $B$ is a polynomial in $K[U]$ for any $a \in \mathcal{K}[U \setminus V^U(B)], V(P(a)) = \bigcup_{[T, H] \in \mathcal{T}} V(T(a) \setminus H(a))$ and $[T, H]$ specializes well at $a$ for any $[T, H] \in \mathcal{T}$.

1. Compute a Wu’s decomposition $\{C_1, \ldots, C_m\}$ of $P$ in $K[U][X]$ by Wu’s method
2. $B := 1$
3. $\mathcal{T} := \emptyset$
4. For $i = 1 \rightarrow m$
   5. If $C_i$ is a contradictory ascending chain then
5.1. $B := B \cdot \text{op}(C_i)$
6. Else
   7. $W := \text{TSToRS}(C_i, X)$
   8. $\mathcal{T} := \mathcal{T} \cup W_1$, $B := B \cdot W_2$
9. return $[\mathcal{T}, B]$

Theorem 3. Algorithm $\mathcal{A}$ terminates correctly.

Proof. The termination follows from the termination of Algorithm $\mathcal{A}$ We only need to show the correctness. In fact, the statement (1) in the specification of Algorithm $\mathcal{A}$ follows from Wu’s method and Algorithm $\mathcal{A}$ and the statement (2) follows from Corollary 2 and Algorithm $\mathcal{A}$.

Corollary 3. Let $P$ be a parametric polynomial system, $\text{RDU}(P, X) = [\mathcal{T}, B]$. Then $\mathcal{T}$ is stable at any $a \in \mathcal{K}[U \setminus V^U(B)]$.

The proof of Corollary 3 is similar to that of generic zero-dimensional case. For more details, please see $\text{[7]}$.

4 Examples and Implementation

In this section, we show by examples how our algorithms work. In addition, we run some benchmarks and report comparison to other tools with similar function.

Example 1. Consider the system

$$P = \begin{cases} (ux + 1)z^3 + (vy + 1)z^2 + wxz + 1 \\ ux + 1 \end{cases}$$

where $x$, $y$ and $z$ are variables ($x < y < z$) and $u$, $v$ and $w$ are parameters.

Step 1: According to the first step of Algorithm $\mathcal{A}$ we get Wu’s decomposition $S = \{C_1, C_2, C_3\}$ of $P$ in $\mathcal{R}[u, v, w][x, y, z]$ where $C_1 = \{ux + 1, u + uvy^2 + u^2z - wz\}$, $C_2 = \{ux + 1, vy + 1, u - wz\}$ and $C_3 = \{-u^2w^2w\}$.
Step 2: Let $\emptyset = \emptyset$ and $B = 1$.

Step 3: Because $C_1$ and $C_2$ are both non-contradictory ascending chains and $C_3$ is a contradictory ascending chain, we need to execute $\text{TSToRS}(C_1, [x, y, z])$ and $\text{TSToRS}(C_2, [x, y, z])$.

Step 3.1: According to $\text{TSToRS}$, we execute $\text{ZDToRC}(C_1, \text{mvar}(C_1))$ where $\text{mvar}(C_1) = \{x, z\}$. It returns $W = \{\{ux + 1, u + uvyz^2 + z^2 - u - wz\}, u(v(y + 1))\}$. Since $W_2 \notin K[u, v, w]$, we execute $\text{wusolve}(u(v(y + 1)))$ and it returns $C_{11} = \{vy + 1\}$. $C_{12} = \{u\}$.

Step 3.2: Let $T_1 = T \cup \{vy + 1\}$. Now we need to execute $\text{TSToRS}(T_1)$. When we execute $\text{ZDToRC}(T_1, \text{mvar}(T_1))$, it returns $[\emptyset, vu]$. Since $vu \in K[u, v, w]$, the output of $\text{TSToRS}(C_1, [x, y, z])$ is $\{\{(ux + 1, u + uv)\}, u(v(y + 1))\}$ where $\text{mvar}(C_2) = \{x, y, z\}$. It returns $W = \{\{ux + 1, v + uwxz - u - wz\}, uvw\}$. Since $W_2 \in K[u, v, w]$, the output of $\text{TSToRS}(C_2, [x, y, z])$ is $\{\{(ux + 1, v + u - wz), \text{uw}\}, uvw\}$.

Step 3.3: Let $TH := TH \cup \{\{ux + 1, v + u - wz\}, \text{uw}\}$ and $B = B \cdot uvw$.

Step 3.4: According to $\text{TSToRS}$, we execute $\text{ZDToRC}(C_2, \text{mvar}(C_2))$ where $\text{mvar}(C_2) = \{x, y, z\}$. It returns $W = \{\{ux + 1, v + u - wz\}, \text{uw}\}$. Since $W_2 \in K[u, v, w]$, the output of $\text{TSToRS}(C_3, [x, y, z])$ is $\{(ux + 1, v + u - wz), -uvw\}$.

Step 3.5: Let $TH := TH \cup \{\{ux + 1, vy + 1, u - wz\}, -uvw\}$ and $B = B \cdot uvw$.

Step 3.6: Since $C_3$ is a contradictory ascending chain, we execute $E := B \cdot -v^2w^2w$.

Step 4: Finally, we get $B = u^2v^2w^2$ and $T = \{\{(ux + 1, u + uv)\}, u(v(y + 1))\}$.

In the example, we factor the polynomials and let the polynomial be squarefree in some steps.

With the result above, we get a regular system decomposition of $P$ in $\mathbb{Z}[u, v, w]$. A regular system decomposition $V = \{u, v, w\} \in C^3[uw]$. According to the specification of Algorithm 4 for any $a \in C^3 \setminus V$, $V(P(a)) = \cup_{T \in \text{TSTR}} T(\text{H}(a))$ where $\text{H}(a)$ is a regular system. With the definition of RDU variety, we get that $V$ is the RDU variety of $P$ w.r.t $TH$.

Example 2 is provided by Changbo Chen, which is a good example to show how the orderings of variables affect the results and the efficiency of Algorithm 4.

Example 2. Consider the parametric system

$$P = \begin{cases} d_3d_1r^2 - d_4d_3d_2 + d_2d_1^2 - d_4d_1^2 + d_3d_4 + d_4d_3 + Z - r \cdot t^4 + (-2r_2d_4r + 2r_2d_2r + 2r_2d_1r - 4r_2d_1d_1 + 2r_2d_1d_4 + 2r_2d_4d_2 + 2r_2d_4d_3 + 4r_2d_3d_4 + 2r_2d_4d_3 + 2r_2d_4d_4 + 2r_2d_4d_4 + 2r_2d_4d_4) \cdot t + r_2 + d_4d_3 + d_4d_3 + d_3d_3 \end{cases}$$

where $r, Z, t$ are variables and $r_2, d_3, d_4$ are parameters.

The 3 variables can be ordered in 6 different ways. We tried all these orders when calling Algorithm 4 for this example. We are interested in the RDU varieties output by the algorithm, so we only report the second output of the algorithm.

By calling RDU($P, [r, t, Z]$) and RDU($P, [t, r, Z]$), we get

$$\text{RDU}(P, [r, t, Z]) = r_2d_4$$
$$\text{RDU}(P, [t, r, Z]) = r_2d_4$$

By calling RDU($P, [Z, r, t]$), we get

$$\text{RDU}(P, [Z, r, t]) = (-d_3 + d_4r_2^2 + d_4) = r_2d_4d_3d_3 - 1$$

By calling RDU($P, [r, Z, t]$), we get

$$\text{RDU}(P, [r, Z, t]) = r_2d_4(d_3 - d_4r_2^2 - d_4)$$

By calling RDU($P, [t, Z, r]$), we get

$$\text{RDU}(P, [t, Z, r]) = r_2d_4(d_3^2 - 3r_2^2)(d_4d_3 - 1)$$

RDU($P, [Z, r, t]$) is too huge to be listed here. It has 11 factors and contains 10838 terms (after expanding).
Table 1. Timings (in second) of Example 2 under different orders of variables

| Order                | Time  |
|----------------------|-------|
| [r,t,Z]              | 0.016 |
| [t,r,Z]              | 0.015 |
| [Z,r,t]              | 0.016 |
| [r,Z,t]              | 0.046 |
| [t,Z,r]              | 0.016 |
| [Z,t,r]              | 3.931 |

The timings for the above computation are shown in Table 1. How to choose a suitable order in advance is an interesting topic for our future work.

Table 2. Comparing RDU and Triangularize

| Number | System         | U   | X   | wusolve | TSToRS | Total | Triangularize |
|--------|----------------|-----|-----|---------|--------|-------|--------------|
| 1.     | Hereman-2      | 1   | 7   | 0.063   | 0.063  | 0.452 |
| 2.     | Hereman-8-8    | 3   | 5   | 0.281   | 0.015  | 0.296 |
| 3.     | Maclane        | 3   | 7   | 0.093   | 0.078  | 0.171 |
| 4.     | MontesS7       | 1   | 3   | 0.967   | 0.015  | 0.982 |
| 5.     | MontesS11      | 3   | 3   | 0.016   | 0.016  | 0.032 |
| 6.     | MontesS12      | 2   | 6   | 0.109   | 0.078  | 0.187 |
| 7.     | MontesS13      | 3   | 2   | 0.016   | 0.016  | 0.032 |
| 8.     | MontesS14      | 1   | 4   | 0.031   | 0.016  | 0.047 |
| 9.     | MontesS15      | 4   | 8   | 0.016   | 0.    | 0.016 |
| 10.    | MontesS16      | 3   | 12  | 0.016   | 0.016  | 0.156 |
| 11.    | MontesS18      | 2   | 3   | 0.296   | 0.031  | 0.327 |
| 12.    | Akhashishimus  | 3   | 6   | 0    | 0.    | 0.047 |
| 13.    | Bronstein      | 2   | 2   | 0.016   | 0.016  | 0.062 |
| 14.    | Cheaters-homotopy-easy | 4 | 3 | 0.358 | 10.749 | 11.107 | 0.047 |
| 15.    | Cheaters-homotopy-hard | 5 | 2 | 0. | 32.867 | 32.867 | 0.062 |
| 16.    | Gerdt          | 3   | 4   | 0.015   | 0.015  | 0.086 |
| 17.    | Lanconelli     | 7   | 4   | 0.047   | 0.047  | 0.094 |
| 18.    | Lazard-ascm2001| 3   | 4   | 0.811   | 0.047  | 0.858 |
| 19.    | Leykin-1       | 4   | 4   | 0.312   | 0.    | 0.312 |
| 20.    | Neural         | 1   | 3   | 0.047   | 0.016  | 0.063 |
| 21.    | Pavelle        | 4   | 4   | 0.109   | 0.062  | 0.171 |
| 22.    | SY14           | 2   | 2   | 0.016   | 0.016  | 0.032 |
| 23.    | Wang93         | 2   | 3   | 0.031   | 0.031  | 0.062 |
| 24.    | zhou3          | 6   | 11  | 0.062   | 0.047  | 0.109 |
| 25.    | zhou4          | 4   | 7   | 0.016   | 0.016  | 0.032 |
| 26.    | KdV            | 15  | 11  | 0.406   | 0.406  | 0.812 |
| 27.    | P3P            | 5   | 2   | 0.031   | 0.031  | 0.062 |
| 28.    | SBCD23         | 1   | 3   | 0.031   | 0.031  | 0.062 |
| 29.    | SBCD24         | 1   | 4   | 0.655   | 0.016  | 0.671 |

We have implemented our algorithms with Maple 16. More specifically, Wu’s method for computing parametric triangular decompositions introduced in Section 2 is implemented as a function wusolve and Algorithm [1] is implemented as a function WRSD. We ran many examples collected from other papers [15, 12, 4]. At the same time, we compare the running time with Triangularize in RegularChains which can compute regular decompositions of given polynomial systems with parameters. Throughout this section, all the results are obtained in Maple 16 using an Intel(R) Core(TM) i5 processor (3.20GHz CPU and 2.5 GB total memory) and Windows 7 (32 bit). The empirical data about timings is presented in Table 2.

In Table 2, the column marked $U$ and $X$ mean the number of parameters and variables, respectively. The column marked wusolve means the time used by wusolve at the first step in RDU (see Algorithm 4). The column marked TSToRS means the time used by TSToRS. The column marked Triangularize means the time used by Triangularize. Some data which shows 0. means that the data is less than 0.001 and ignored by system.

From Table 2 we find that the cost of wusolve takes up a majority of the total time in most examples when solving practical problems as shown. Comparing with Triangularize, for some examples, our algorithm is faster than Triangularize. As for the example Cheaters-homotopy-easy and Cheaters-homotopy-hard, the time used by WRSD seems too much. This is not reasonable and optimization should be done in the future.

*Please find more details on Triangularize from the help document of Maple 16.*
5 Conclusions

The focus of this paper is how to decompose a parametric system into regular systems at "parameter level" and the property of this decomposition under specification. We provide an algorithm for computing GRDs of parametric systems and the related RDU varieties simultaneously no matter the systems are generic zero-dimensional or positive-dimensional, which is a generalization of our earlier work in [19] for generic zero-dimensional case. Then any parametric system in $K[U][X]$ can be decomposed into finitely many regular systems and the decomposition is stable at any parameter value in the complement of the associated RDU variety of the parameter space. That is to say, once we obtain such decomposition, all the solutions of the original system are expressed by some regular systems, except for some possible solutions over the parameter values on the associate RDU variety.

Note that, using Algorithm 4 of generic regular decomposition, we may get a complete decomposition of a given parametric system $P$ step by step. A rough procedure is as follows. Firstly, for certain variables $X$ and parameters $U$, we call $\text{RDU}(P, X)$ and get a set of regular systems and a parametric polynomial $B \in K[U]$. Then, let $U_1 = U \setminus \{u\}$ and $X_1 = X \cup \{u\}$ where $u \in U$ and we call $\text{RDU}(P \cup \{B\}, X_1)$ to get a set of regular systems and a parametric polynomial $B_i \in K[U_i]$. Continuing this iteration, until $U_i = \emptyset$ and $B_i \in K$ for some $i$, we can get a complete decomposition finally. This is what we called hierarchical strategy in Section 1.

What can we benefit from this strategy? According to this method, we can stop at any step of the iteration especially when the computation is hard to be finished. Then we get a generic regular decomposition and a set of parametric polynomials which can determine a RDU variety (low dimensional variety). For some huge problems, limited by the computational capacity of micro computer, we can get a partial solution, which is useful if one cannot get any information by other complete methods. Of course, the procedure should be described clearly and proved to be correct. That is one of our future work.

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