On Norm of Elementary Operator: An Application of Stampfli’s Maximal Numerical Range

Denis Njue Kingangi

Department of Mathematics & Computer Science, University of Eldoret, Eldoret, Kenya

Email address: dankingangi2003@yahoo.com

To cite this article:
Denis Njue Kingangi. On Norm of Elementary Operator: An Application of Stampfli’s Maximal Numerical Range. Pure and Applied Mathematics Journal. Vol. 7, No. 1, 2018, pp. 6-10. doi: 10.11648/j.pamj.20180701.12

Received: February 4, 2018; Accepted: March 8, 2018; Published: March 27, 2018

Abstract: Many researchers in operator theory have attempted to determine the relationship between the norm of an elementary operator of finite length and the norms of its coefficient operators. Various results have been obtained using varied approaches. In this paper, we attempt this problem by the use of the Stampfli’s maximal numerical range.

Keywords: Elementary Operator, Maximal Numerical Range, Rank-One Operator

1. Introduction

Properties of elementary operators have been investigated in the recent past under varied aspects. Their norms have been a subject of interest for research in operator theory. Deriving a formula to express the norm of an arbitrary elementary operator in terms of its coefficient operators remains a topic of research in operator theory. In the current paper, the concept of the maximal numerical range is applied in determining the lower bound of the norm of an elementary operator consisting of two terms, and also to determine the conditions under which the norm of this operator is expressible in terms of its coefficient operators in $B(H)$. Specifically, the Stampfli’s maximal numerical range is employed in arriving at our results.

Let $H$ be a complex Hilbert space and $B(H)$ be the set of bounded linear operators on $H$. We define an elementary operator,

$$E_n: B(H) \rightarrow B(H), W \mapsto \sum_{i=1}^{n} T_i W S_i,$$

for all $W \in B(H)$ where $T_i, S_i$ are fixed elements of $B(H)$. When $n = 1$, then we obtain a basic elementary operator,

$$E_1: B(H) \rightarrow B(H), W \mapsto TWS,$$

for all $W \in B(H)$ and $T, S$ fixed in $B(H)$. We denote the basic elementary operator by $M_{T,S}$. When $n = 2$, we obtain the elementary operator of length two, whereby,

$$E_2(W) = T_1 W S_1 + T_2 W S_2,$$

for all $W \in B(H)$ and $T_i, S_i$ fixed in $B(H)$ for $i = 1, 2$.

The Jordan elementary operator, $U_{T,S}$, is defined as;

$$U_{T,S}: B(H) \rightarrow B(H), W \rightarrow TWS + SWT,$$

for all $W \in B(H)$ and $T, S$ fixed in $B(H)$, is an example of elementary operators.

Let $A$ be an algebra. A derivation is a function $\Delta: A \rightarrow A$ for which $\Delta(x y) = x(\Delta y) + (\Delta x)y$ for all $x, y \in A$. If there is an $a \in A$ such that $\Delta x = xa - ax$ for all $x \in A$, then $a$ is called an inner derivation. A derivation is another example of elementary operators.

The Stampfli’s maximal numerical range of $T \in B(H)$ is the set,

$$W(T) = \{ \lambda \in \mathbb{C}: (Tx_n, x_n) \rightarrow \lambda, ||x_n|| = 1, ||T x_n|| \rightarrow ||T|| \},$$

while the maximal numerical range $W_a(T^*S)$ of $T^*S$ relative to $S$ is defined as,

$$W_a(T^*S) = \{ \lambda \in \mathbb{C}: \exists x_n \in H, ||x_n|| = 1, \lim_{n \rightarrow \infty} (T^*S x_n, x_n) = \lambda, ||S x_n|| = ||S|| \},$$

where $T^*$ is the Hilbert adjoint of $T$.

For any $x, y \in H$, the rank one operator, $x \otimes y \in B(H)$, is defined by $(x \otimes y)(z) = (z, y)x$, for all $z \in H$.

This paper has determined the norm of the elementary operator of length two.

Given an elementary operator $E_2$ on $B(H)$ with fixed operators $T_i, S_i$ on $B(H)$ for $i = 1, 2$, does the relationship...
Let $H$ be a complex Hilbert space, $B(H)$ be the algebra of bounded linear operators on $H$, and $T, S \in B(H)$ be fixed. For a Jordan elementary operator $U_{T,S}$, Mathieu [5], in 1990 proved that in the case of prime $C^*$-algebras, the lower bound of the norm of $U_{T,S}$ can be estimated by

$$
\left\| U_{T,S} \right\| \geq \frac{2}{3} \|T\| \|S\|.
$$

In 1994, Cabrera and Rodriguez [3], proved that

$$
\left\| U_{T,S} \right\| \geq \frac{1}{20412} \|T\| \|S\|,
$$

for prime JB*-algebras.

On their part, Stacho and Zalar [6], in 1996, worked on the standard operator algebra (which is a sub-algebra of $B(H)$) that contains all finite rank operators. They first showed that the operator $U_{T,S}$ actually represents a Jordan triple structure of a $C^*$-algebra. They also showed that if $\mathcal{A}$ is a standard operator algebra acting on a Hilbert space $H$, and $T, S \in \mathcal{A}$, then

$$
\left\| U_{T,S} \right\| \geq 2(\sqrt{2} - 1) \|T\| \|S\|.
$$

They later (1998), proved that

$$
\left\| U_{T,S} \right\| \geq \|T\| \|S\|
$$

for the algebra of symmetric operators acting on a Hilbert space. They attached a family of Hilbert spaces to standard operator algebra and used the inner products in them to obtain their results.

Barraa and Boumazguor [2], in the year 2001 used the concept of the numerical range of $T$ relative to $S$, denoted by $W_S(T^*S)$, to obtain their results. They employed the idea of finite rank operators to prove the following theorem;

Theorem 2.1 Let $H$ be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on $H$. If $T, S \in B(H)$ with $S \neq 0$, then;

$$
\left\| U_{T,S} \right\| \geq \sup_{\lambda \in \mathcal{W}_S(T^*S)} \left\{ \left\| S + \frac{\lambda}{\|S\|} \right\| \right\}.
$$

As a consequent of this, they proved the following corollary;

Corollary 2.2 Let $H$ be a complex Hilbert space and $T, S$ be bounded linear operators on $H$. If $0 \in W_S(T^*S) \cup W_T(S^*T)$, then;

$$
\left\| U_{T,S} \right\| \geq \left\| T\right\| \|S\|.
$$

They also proved the following proposition;

Proposition 2.3 Let $H$ be a complex Hilbert space and $T, S$ be bounded linear operators on $H$. If $\|T\| \|S\| \in W_T(S^*T) \cap W_T(S^*T')$, then;

$$
\left\| U_{T,S} \right\| = 2 \|T\| \|S\|.
$$

Proves to theorems 2.1, 2.2 and 2.3, can be obtained from [2].

3. Norm of Elementary Operator of Length Two

Kingangi et al [4] in 2014 used finite rank operators to determine the norm of the elementary operator $E_2$. Below is the theorem they proved (see theorem 2.5);

Theorem 3.1 Let $H$ be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on $H$. Let $E_2$ be the elementary operator on $B(H)$ defined above. If for an operator $W \in B(H)$ with $\|W\| = 1$ we have $W(x) = x$ for all unit vectors $x \in H$, then;

$$
\|E_2\| = \sum_{i=1}^{2} \|T_i\| \|S_i\|,
$$

for $i = 1, 2$.

For the proof of theorem 3.1, see King’ang’i [4], theorem 2.5.

Odero et al [7], determined the norm of tensor product elementary operator. They showed that for an inner derivation $\Delta_T$, we have $\|\Delta_T\| = 2 \|T\|$ if and only if $0 \in W(T)$. They also proved that if $S, T \in B(H)$, then we have;

$$
\|E_2\| = \sup \{\|TX - XS\| : X \in B(H), \|X\| = 1\}.
$$

Jocic et al [8], proved that if $(T_{\alpha})_{\alpha \in \Lambda}$ and $(S_{\alpha})_{\alpha \in \Lambda}$ are weakly* measurable families of bounded Hilbert space operators such that transfers $X \mapsto \int_{\Lambda} T_{\alpha} X T_{\alpha} d\mu(\alpha)$ and $X \mapsto \int_{\Lambda} S_{\alpha} X S_{\alpha} d\mu(\alpha)$ on $B(H)$ have their spectra contained in the unit disc, then for all bounded operators $X$, we have;

$$
\|\Delta_T X \Delta_S\| \leq \|X - \int_{\Lambda} T_{\alpha} X S_{\alpha} d\mu(\alpha)\|,
$$

where

$$
\Delta_T = S - \lim_{r \to 1} \left( I + \sum_{n=1}^{\infty} r^{2n} \int_{\Lambda} \cdots \int_{\Lambda} |T_{\alpha_1} \cdots T_{\alpha_n}|^2 d\mu(\alpha_1, \ldots, \alpha_n) \right)^{-\frac{1}{2}}
$$
and Δ₂ by analogy.

Wafula, et al [9], considered normally represented elementary operators. They proved that the norm of an elementary operator is equal to the largest singular value of the operator itself. They also proved that, if the Jordan elementary operator UTₜₛ, we have:

\[ \|UTₜₛ\|_{op} ≥ 2\sqrt{2 - t}\|T\|\|S\|, \]

where \( S, T ∈ B(H) \).

In 2017, King’ang’i [1] employed the concept of the maximal numerical range of \( T'S \) relative to \( S \) to determine the lower norm of an elementary operator of length two. He proved the following theorem (see theorem 3.1):

Theorem 3.2. Let \( E₂ \) be an elementary operator of length two on \( B(H) \). Then,

\[ \|E₂\| ≥ \sup_{x,y\in W₁(S₂S₁)} \left\{ \|S₁||T₁ + \bar{\lambda}||S₂||T₂\| \right\}, \]

where \( S₁, T₁ \) are fixed elements of \( B(H) \) for \( i = 1,2 \).

He also determined the conditions on which the norm of \( E₂ \) is expressible in terms of the norms of its coefficient operators by proving the following theorems (see corollary 3.2 and theorem 3.3):

Corollary 3.3 Let \( H \) be a complex Hilbert space and \( TᵢSᵢ \) be bounded linear operators on \( H \) for \( i = 1,2 \). Let \( 0 ∈ W₁(S₁S₂)∪W₂(S₁S₂) \). Then, \( \|E₂\| ≥ \|T₁\|\|S₁\| \), where \( E₂ \) is as defined earlier.

Theorem 3.4 Let \( H \) be a complex Hilbert space and \( TᵢSᵢ \) be bounded linear operators on \( H \) for \( i = 1,2 \). Let \( E₂ \) be an elementary operator of length two. If \( \|T₁\|\|T₂\| \in W₁(T₁T₂) \) and \( \|S₁\|\|S₂\| \in W₂(S₁S₂) \), then, \( \|E₂\| = \Sigma^₂₁\|Tᵢ\|\|Sᵢ\| \).

Below, we present more results on the norm of this operator by employing the concept of the Stampfli’s maximal numerical range. In theorem 3.5, we determine the lower bound of the norm of the operator \( E₂ \) while in theorem 3.6 we determine the conditions necessary to express the norm of \( E₂ \) in the form \( \|E₂\| = \Sigma^₂₁\|Tᵢ\|\|Sᵢ\| \).

Theorem 3.5 Let \( E₂ \) be an elementary operator on \( B(H) \) and \( S₁, S₂ ∈ B(H) \). If \( λᵢ ∈ W₀(Sᵢ) \) for each \( λᵢ ∈ C, i = 1,2 \), then we have \( \|E₂\| ≥ \sup_{λᵢ∈W₀(Sᵢ)} (\Sigma^₂₁ λᵢTᵢ) \). Let \( \|E₂\| ≥ \sup_{x,y\in W₁(S₁S₂)} \left\{ \Sigma^₂₁ λᵢTᵢ \right\} \).

Proof. Let \( \{xₙ\}_{n=1}^∞ \) be a sequence of unit vectors in a Hilbert space \( H \) and \( y = \text{span}_H \{xₙ\} \) be any vector in \( H \) for a unit vector \( y ∈ H \), defined by \( y = \sum_{i=1}^∞ \lambdaᵢTᵢy \).

Thus, \( \|E₂\| ≥ \sup_{x,y\in W₁(S₁S₂)} (\Sigma^₂₁ λᵢTᵢ) \), or \( \|E₂\| ≥ \sup_{λᵢ∈W₀(Sᵢ)} (\Sigma^₂₁ λᵢTᵢ) \), and this completes the proof.

In the next theorem, the condition necessary for the norm of the elementary operator \( E₂ \) to be equal to the sum of the product of the norms of the corresponding coefficient operators in its definition is given.

Theorem 3.6 Let \( E₂ \) be an elementary operator on \( B(H) \) (xₙ,y) for all \( x ∈ H \). Recall the Stampfli’ smaximal numerical range of \( T ∈ B(H) \) is the set,

\[ W₀(T) = \{ λ ∈ C; \langle Txₙ, xₙ \rangle → λ, \|xₙ\| = 1, \|Txₙ\| → \|T\| \}. \]

If \( λ₁ ∈ W₀(S₁) \) and \( λ₂ ∈ W₀(S₂) \), then there are sequences \( \{xₙ\}_{n=1}^∞ \) and \( \{yₙ\}_{n=1}^∞ \) of unit vectors in \( H \) such that \( \lim_{n→∞} \langle S₁xₙ, xₙ \rangle = λ₁ \), \( \lim_{n→∞} \|S₁xₙ\| = \|S₁\| \) and \( \lim_{n→∞} \langle S₂xₙ, xₙ \rangle = λ₂ \), \( \lim_{n→∞} \|S₂xₙ\| = \|S₂\| \).

Now, we have;

\[ \|E₂(y⊗xₙ)\| = \left\| \sum_{i=1}^∞ M_{TᵢSᵢ} (y⊗xₙ) \right\| xₙ \]

≤ \sum_{i=1}^∞ \|M_{TᵢSᵢ} \| y⊗xₙ \]

≤ \sum_{i=1}^∞ \|M_{TᵢSᵢ} \| y∧xₙ \]

= \sum_{i=1}^∞ \|M_{TᵢSᵢ} \|

Therefore,

\[ \sum_{i=1}^∞ \|M_{TᵢSᵢ} \| ≥ \sum_{i=1}^∞ \|M_{TᵢSᵢ} (y⊗xₙ) \| xₙ \]

= \left\| (T₁(y⊗xₙ)S₁ + T₂(y⊗xₙ)S₂)xₙ \right\|

= \left\| (T₁(y⊗xₙ)S₁xₙ + T₂(y⊗xₙ)S₂xₙ) \right\|

= \left\| (S₁xₙ, xₙ)T₁y + (S₂xₙ, xₙ)T₂y \right\|

Taking limits as \( n → ∞ \), we obtain;

\[ \|E₂\| ≥ \|M_{TᵢSᵢ} \| \]

and this is true for any \( λ₁ ∈ W₀(S₁) \) and \( λ₂ ∈ W₀(S₂) \), and for any unit vector \( y ∈ H \).

Since \( λᵢ ∈ W₀(Sᵢ) \) for \( i = 1,2 \), \( y ∈ H \) are arbitrarily chosen, then we obtain;

\[ \|E₂\| = \sum_{i=1}^∞ \|M_{TᵢSᵢ} \| \]

and it is obtained that \( E₂ \) is the desired elementary operator.
a sequence \( \{x_n\}_{n=1} \) of unit vectors in \( H \) such that 
\[
\lim_{n \to \infty} (S_i x_n, x_n) = S_i, \quad \lim_{n \to \infty} \|S_i x_n\| = \|i\| \quad \text{for } i = 1, 2,
\]
and there is a sequence \( \{y_n\}_{n=1} \) of unit vectors in \( H \) such that 
\[
\lim_{n \to \infty} (T_i y_n, y_n) = T_i, \quad \lim_{n \to \infty} \|T_i y_n\| = \|T_i\|, \quad \text{for } i = 1, 2.
\]

Now, we have;
\[
\| (E_2(x_n \otimes x_n)) x_n \| = \left\| \left( \sum_{i=1}^{2} M_{T_i, S_i} (x_n \otimes x_n) \right) x_n \right\| 
\leq \left\| \sum_{i=1}^{2} M_{T_i, S_i} (x_n \otimes x_n) \right\| \|x_n\|
\]
\[
\left\| \sum_{i=1}^{2} M_{T_i, S_i} \right\|^2 \geq \left\| \left( \sum_{i=1}^{2} M_{T_i, S_i} (x_n \otimes x_n) \right) x_n \right\|^2
\]
\[
= \| (T_1 (x_n \otimes x_n) S_1 + T_2 (x_n \otimes x_n) S_2) x_n \|^2
\]
\[
= \|T_1 (x_n \otimes x_n) S_1 x_n + T_2 (x_n \otimes x_n) S_2 x_n \|^2
\]
\[
= \| (S_1 x_n, x_n) T_1 x_n + (S_2 x_n, x_n) T_2 x_n \|^2
\]
\[
= \| (S_1 x_n, x_n) T_1 x_n \|^2 + \| (S_2 x_n, x_n) T_2 x_n \|^2 + 2 \Re \langle (S_2 x_n, x_n) T_2 x_n, (S_1 x_n, x_n) T_1 x_n \rangle
\]
\[
= \| (S_1 x_n, x_n) \|^2 \| T_1 x_n \|^2 + \| (S_2 x_n, x_n) \|^2 \| T_2 x_n \|^2 + 2 \Re \langle (S_2 x_n, x_n) T_2 x_n, (S_1 x_n, x_n) T_1 x_n \rangle.
\]

Now, by the CBS inequality, we have that;
\[
\langle (T_2 x_n, T_1 x_n) \rangle \leq \|T_2 x_n\| \|T_1 x_n\| \leq \|T_2\| \|T_1\|.
\]
and hence,
\[
\lim_{n \to \infty} (T_2 x_n, T_1 x_n) = \|T_2\| \|T_1\|.
\]

Therefore, taking limits as \( n \to \infty \), we obtain;
\[
\left\| \sum_{i=1}^{2} M_{T_i, S_i} \right\|^2 \geq \left\| S_1 \right\|^2 \| T_1 \|^2 + \left\| S_2 \right\|^2 \| T_2 \|^2 + 2 \| S_1 \| \| T_2 \| \left\| S_1 \right\| \| T_1 \| \| T_2 \|.
\]

The right hand side is equal to \( \sum_{i=1}^{2} \| S_i \| \| T_i \| \|^2 \). Thus we have;
\[
\sum_{i=1}^{2} M_{T_i, S_i} \geq \sum_{i=1}^{2} \| S_i \| \| T_i \|.
\]

Now, since the inequality \( \sum_{i=1}^{2} M_{T_i, S_i} \leq \sum_{i=1}^{2} \| S_i \| \| T_i \| \)
always hold (by the triangular inequality), then we obtain \( \sum_{i=1}^{2} M_{T_i, S_i} = \sum_{i=1}^{2} \| S_i \| \| T_i \| \).

### 4. Conclusions

In this paper, we have determined the lower bound of the norm of an elementary operator of length two in a C*-algebra \( B(H) \) and using the Stampfli’s maximal numerical range. The conditions in which this norm is equal to the sum of the products of the corresponding coefficient operators has also been considered. One may attempt this problem for an elementary operator consisting of more than two terms.

---

**References**

[1] D. N. King’ang’i, *On the norm of elementary operator of length two*, International Journal of Scientific and Innovative Mathematical Research 5 (2017), 34-38.

[2] M. Baraa and M. Boumazgour, *A lower bound of the norm of the operator \( x \leftrightarrow axb + bxa \)*, Extractamathematicae 16 (2001), 223-227.

[3] M. Cabrera and A. Rodriguez, *Non-degenerate ultraprimejordan-banach algebras: a zelmanorian treatment*, Proc. London. Math. Soc 69 (1994), 576-604.

[4] D. N. Kingangi, J. O. Agure and F. O. Nyamwala, *On the norm of elementary operator*, Advances in Pure Mathematics 4 (2014), 309-316.

[5] M. Mathew, *More properties of the product of two derivations of a c*-algebras*, Bull. Austral. Math. Soc 42 (1990), 115-120.
[6] L. L. Stacho and B. Zalar, On the norm of jordan elementary operators in standard operator algebras, Publ. math. debreen 49 (1996), 127-134.

[7] B. A. Odero, J. A. Ogonji, G. K. Rao, Norms of tensor product elementary operators, International Journal of Multidisciplinary sciences and Engineering 6 (2015), 29-32.

[8] D. R. Jocic et al, Norm inequalities for elementary operators and other inner product type integral transformers with the spectra contained in the unit disc, Filomat 31 (2017), 197-206.

[9] A. M. Wafula, N. B. Okelo, O. Ongati, Norms of normally represented elementary operators, International Journal of Modern Science and Technology 3 (2018), 10-26.

[10] S. Paul and C. Gu, Tensor splitting properties of n-inverse pairs of operators, arXiv, 2015, 1-20.