Constructions of Binary Optimal Locally Repairable Codes via Intersection Subspaces

Wenqin Zhang, Deng Tang, Yuan Luo

Abstract

Locally repairable codes (LRCs), which can recover any symbol of a codeword by reading only a small number of other symbols, have been widely used in real-world distributed storage systems, such as Microsoft Azure Storage and Ceph Storage Cluster. Since the binary linear LRCs can significantly reduce the coding and decoding complexity, the construction of binary LRCs is of particular interest. To date, all the known optimal binary linear LRCs with the locality $2^b$ ($b \geq 3$) are based on the so-called partial spread which is a collection of the same dimensional subspaces with pairwise trivial, i.e., zero-dimensional intersection.

In this paper, we concentrate on binary linear LRCs with disjoint local repair groups. We construct dimensional optimal binary linear LRCs with locality $2^b$ ($b \geq 3$) and minimum distance $d \geq 6$ by employing intersection subspaces deduced from the direct sum vs. the traditional partial spread construction. This method will increase the number of possible repair groups of LRCs as many as possible, and thus efficiently enlarge the range of the construction parameters while keeping the largest code rates compared with all known binary linear LRCs with minimum distance $d \geq 6$ and locality $2^b$ ($b \geq 3$).

Index Terms

Locally repairable codes, distributed storage systems, intersection space.

I. INTRODUCTION

Efficient distributed storage systems (DSSs) provides access to data by storing them in a distributed manner across several storage nodes. However, in the individual unreliability of storage nodes, it could lead to generating failures. To repair the data from failed nodes, the simplest solution is the straightforward replication of data packets across different disks. The replication has good parallel reading ability which is very important for hot data that needed to be read frequently. Unfortunately, replication has very high storage overhead, so it is unsuitable where there is accelerated and relentless data growth.

Erasure coding is used in cases such as they are employed in Windows Azure Storage [1] and the Facebook Analytics Hadoop cluster [2] in consideration of the higher fault-tolerance values and lower storage overheads. Erasure coding calculates the redundancy out of the original data, which can be used to repair data in the failed case. Maximum distance separable (MDS) erasure codes such as Reed-Solomon (RS) codes are optimal in terms of storage overhead. In the case of single node failure, erasure coding requires connecting a large subset of surviving nodes which

W. Zhang, D. Tang and Y. Luo are with School of Electronic Information and Electrical Engineering, Shanghai Jiao Tong University, Shanghai, 200240, China.
Email: wenqin_zhang@sjtu.edu.cn, dengtang@sjtu.edu.cn, luoyuan@cs.sjtu.edu.cn (Corresponding author).
will lead to increase the complexity of network traffic and many input/output (I/O) operations. Consequently, the regenerating codes [3] and codes with locality (known more commonly as locally repairable codes) [4] were introduced in such a scenario. Regenerating codes attempt to minimize the number of transmitted symbols, meanwhile the aim of locally repairable codes (LRCs) is to optimize the number of disk reads required to repair a single failed node. In this paper, we focus on the construction of locally repairable codes with good parameters.

A. Code with locality and known results

Let \( q \) be a power of an arbitrary prime and \( \mathbb{F}_q \) be the finite field with \( q \) elements. A code \( C \) of \( \mathbb{F}_n^q \) is called an \([n, k, d]_q\) linear code, if it has length \( n \), dimension \( k \), and minimum distance \( d \).

The \( i \)-th code symbol of a linear code \( C \) is said to have locality \( r \) if \( c_i \) in each codeword \( c \in C \) can be represented as a function of \( r \) other symbols. This means that any single erased symbol can be recovered by downloading at most \( r \) other symbols. The set of such \( r \) symbols that can repair the \( i \)-th symbol is called a “recovery set”. This linear code \( C \) is denoted as an \([n, k, d]_q\) locally repairable code with locality \( r \). However, if \( q = 2 \), we omit \( q \) from the notation \([n, k, d]_q\) for simplicity.

LRCs are well studied and many works have been done (see, for example, in [4]–[10]) to explore the relationship between the parameters \( n, k, d, r \). Consider an \([n, k, d]_q\) LRC with locality \( r \) for information symbols, Gopalan et al. [4] proved the well-known Singleton-like bound as:

\[
d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2, \tag{1}
\]

where \( \lceil \cdot \rceil \) stands for the ceiling function. An LRC is said to be \( d \)-optimal if it satisfies (1) with equality for given \( n, k, r \). Similarly, we say an LRC is \( r \)-optimal if \( r \) is the smallest integer satisfying (1) for given \( n, k, d \). Note that when \( r = k \), the bound in (1) specializes to the classical Singleton bound \( d \leq n - k + 1 \). In the last few years, many constructions of optimal LRCs achieving bound in (1) have been presented. In [11], Tamo et al. proposed optimum LRCs over a finite field of size \( q \geq n + 1 \). In [12], Hao et al. proposed optimum LRCs with \( d = 3, 4 \) over a finite field of size \( q \geq r + 2 \) which achieves the bound in (1). In addition, linear code constructions that achieve the above bound based on Reed-Solomon codes, among other techniques, have been recently discovered in [11], [13]–[18]. Although the bound in (1) certainly holds for all LRCs, it is not tight in many cases. Recently, it was improved in [19] and [8].

However, the size of the code alphabet, which is an important parameter, is neglected in these works. In practice, codes over small alphabets particularly attract more attention in the application of storage because of their ease implementation. So it is desirable to derive new bounds considering the alphabet size. In 2013, Cadamb and Mazumdar derived a new bound for \([n, k, d]_q\) LRCs which took the size of the alphabet into account [20]. This bound is known as C-M bound. They showed that the dimension \( k \) of an \([n, k, d]_q\) LRC \( C \) over \( \mathbb{F}_q \) with locality \( r \) is upper bounded by

\[
k \leq \min_{t \in \mathbb{Z}^+} \left\{ tr + k_{opt}^{(q)}(n - t(r + 1), d) \right\}, \tag{2}
\]

where \( k_{opt}^{(q)}(n, d) \) is the largest possible dimension of a code with length \( n \) for a given alphabet size \( q \) and a given minimum distance \( d \). This bound is applied to both linear and nonlinear
codes. The C-M bound can be attained by binary simplex codes [6]. Later in [20], [21], the explicit constructions of the family of binary LRCs are proposed which achieve the bound in (2). However, because the exact value of $k_{\text{opt}}^{(0)}(n, d)$ can only be obtained in a limited case with relatively short code length, it is difficult to apply the C-M bound to evaluate the optimality of general LRCs.

In 2018, Agarwal et al. [22] derived a linear programming (LP) bound for LRCs under the setting of disjoint local repair groups. Although the LP bound strictly improves the C-M bound in many parameters, neither of them is explicitly computable, which causes much trouble in getting exact values from the bounds. Later, Wang et al. [8] presented a sphere-packing bound for binary LRCs based on disjoint local repair groups, which serves as a generalization of the bounds in [4], [23]. This bound is advantageous in two ways compared to the previous bounds. It is tighter than the C-M bound for binary linear LRCs with long code lengths. In addition, the inequality of the bound is expressed in an explicit form.

Consider binary linear LRCs with minimum distance $d \geq 5$; the dimension $k$ is actually upper bounded by the largest integer no greater than the following explicit bound [8] given in equation (3). For any $[n, k, d]$ binary linear LRCs with locality $r$ such that $d \geq 5$ and $2 \leq r \leq \frac{n}{2} - 2$, it follows that

$$k \leq \frac{rn}{r + 1} - \min \left\{ \log_2 \left( 1 + \frac{rn}{2} \right), \frac{rn}{(r + 1)(r + 2)} \right\}.$$  \hspace{1cm} (3)

We say a binary linear LRC is $k$-optimal if it satisfies the bound in (3) with equality for given $n$, $d$, and $r$. This paper will focus on a general assumption $n \geq 5(r + 1)(r + 2)$ that will be satisfied in all the main results, and thus the bound in (3) can be further simplified to be

$$k \leq \frac{rn}{r + 1} - \log_2 \left( 1 + \frac{rn}{2} \right).$$

It is known that there are many works related to the theoretic bounds for the LRCs. Apart from the works, there is much literature regarding achieving a large code length $n$ for a given alphabet size (field size) $q$ [11], [24]–[27]. However, compared to $q$-ary LRCs, binary LRCs are known to be advantageous in terms of implementation in practical systems. For the optimal binary linear LRCs, various construction methods have been proposed. In 2017, by the partial spread, Nam et al. constructed a class of binary linear LRCs with minimum distance at least 6 and showed that some examples are optimal with respect to the bound in (2) [28]. Subsequently, Wang et al. constructed an $[n = 2^s - 1, k = \frac{rn}{r + 1} - s, d \geq 6]$ binary LRC with locality $r = 2^h$, which achieved the bound in (3) from generalized Hamming codes in [8]. In [29], Ma et al. proposed a class of optimal binary linear LRCs for $d = 6$, which included the codes given in [8], and they also presented a new $k$-optimal construction for locality 3 and minimum distance 6 from a partial $t$-spread. For any fixed locality $r$ and minimum distance $d$, the coding rate of optimal LRCs becomes larger as the code length becomes larger [3]. Note that most constructions of the binary linear LRCs are based on the partial $t$-spread, namely a set of mutually disjoint $t$-dimensional subspace in $V_m(q)$, which denotes the $m$-dimensional projective space over the finite field $F_q$. Owing to the mutually disjoint subspaces, it is easy to calculate the minimum distance of binary linear LRCs. But, the disadvantage of this approach is the limit to code length. Actually, for given locality, based on the intersection subspace, the optimal binary linear LRCs can be constructed
with code length larger than previously known, but few constructions exist. A more ingenious approach is necessary to cope with the intersection subspace in order to guarantee the minimum distance. For these above reason, it is a challenging and interesting problem to construct the optimal binary linear LRCs by applying the intersection subspace.

B. Our results

By utilizing parity check matrices, we present an explicit construction of binary linear LRCs based on the intersection subspace with minimum distance \( d \geq 6 \) and locality \( r = 2^b \), when \( r+1 \) divides \( n \). These intersection subspaces are designed by the direct sum of subspaces. Those LRCs turn out to be optimal in terms of the bound in (3). Precisely speaking, the following results are obtained.

In fact, we propose an explicit construction of \( k \)-optimal binary linear LRCs with new parameters \( [n = (r+1)\ell, k \geq n-s-\ell-m, d \geq 6] \) (see Construction 1 and Theorem 1 below), where \( \ell = \frac{2^m-1}{2b-s+1}, b \geq 3, 1 \leq s < b \) and \( (2b-s)m \). Furthermore, those binary linear LRCs all can attain the bound in (3) when \( \ell \) belongs to a determined range, so they are \( k \)-optimal (Theorem 2). In the case of \( (2b-s) \nmid m \), we provide construction of optimal binary linear LRCs with parameters \( [n = (r+1)\ell, k = n-\ell-s-m, d \geq 6] \) where \( \ell = \frac{2^m-s-2(2b-s)(2^{z-1})-1}{2(2b-s)-1} \), \( 1 \leq s < b \), and \( z \equiv (m-s) \mod (2b-s) < b \) (Theorem 3). Similar to Theorem 2, a class of \( k \)-optimal LRCs with a wider code length can be obtained from Theorem 3, see Theorem 4. All results of optimal binary linear LRCs in this paper are summarized in Table I.

| Table I: Optimal binary linear LRCs with \( d \geq 6 \) and \( r = 2^b \) |
|---|---|---|
| Theorem 1 | \( n = (r+1)\ell, k = n-s-\ell-m \) | \( \ell = \frac{2^m-1}{2b-s+1}, 1 \leq s < b, (2b-s) \mid m \) |
| Theorem 2 | \( n = (r+1)\ell, k = n-s-\ell-m \) | \( \ell = \frac{2^m-s-1}{2b-s+1} \leq \frac{2^m-1}{2b-s+1} \), \( 1 \leq s < b, (2b-s) \mid m \) |
| Theorem 3 | \( n = (r+1)\ell, k = n-\ell-m-s \) | \( \ell = \frac{2^m-s-2(2b-s)(2^{z-1})-1}{2(2b-s)-1} \leq \frac{2^m-s-2(b-s)(2^{z-1})-1}{2(2b-s)-1} \), \( 1 \leq s < b, (2b-s) \mid m, z \equiv m \mod (2b-s) < b \) |
| Theorem 4 | \( n = (r+1)\ell, k = n-\ell-m-s \) | \( \ell = \frac{2^m-s-2(2b-s)(2^{z-1})-1}{2(2b-s)-1} \leq \frac{2^m-s-2(b-s)(2^{z-1})-1}{2(2b-s)-1} \), \( 1 \leq s < b, (2b-s) \mid m, z \equiv m \mod (2b-s) < b \) |

- those parameters are previously unknown.

Moreover, we compare our results with the state-of-the-art approaches for a fixed locality \( r \) (Table II, Table III). The results show that the optimal LRCs in this work have more flexible parameters \([n, k]\) than those in [8], [29]. In other words, our construction can generate more repair groups such that the code length with the same locality of optimal LRCs is larger. Additionally, by calculating code rate, it can be obtained that the code rate \( R \triangleq k/n \) in our construction is higher than that in [8]. At the end of this paper, a shortening technique will yield the derivation of new binary linear LRCs. By deleting codewords in optimal binary linear LRCs with nonzero values in the last coordinates and then removing the last coordinates from the remaining codewords, we can suggest new parameters from the original binary linear LRCs (Theorem 5).
C. Organization

In Section II, some basic definitions and results on LRCs, parity $t$-spread and intersecting subspace are introduced. In Section III, we present an explicit construction of LRCs and apply this construction to obtain the main results in Theorem 1, Theorem 2, Theorem 3, and Theorem 4. In Section IV, three examples are given to explain the corresponding construction and some tables are listed to show the comparison. In Section V, Theorem 5 proposes the result to shorten binary linear LRCs. Finally, Section VI concludes the paper and Section VII is for the acknowledgement. In addition, some desired matrices of the constructions are displayed in Appendix.

II. Preliminaries

In this section, we introduce some notations and basic results required later in this paper.

- Let $\mathbb{F}_q$ be the finite field with $q$ elements.
- Let $\mathbb{F}_q^n$ be the vector space with dimension $n$ over $\mathbb{F}_q$.
- Suppose that $n$ is a positive integer, we write $[n] = \{1, \cdots, n\}$.
- For any $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{F}_q^n$, the Euclidean inner product of $x$ and $y$ is defined as $x \cdot y = \sum_{i=1}^{n} x_i y_i$.
- For any $x \in \mathbb{F}_q^n$, we use $\text{wt}(x)$ to denote the Hamming weight of $x$, i.e., the number of nonzero coordinates in $x = (x_1, x_2, \cdots, x_n)$.
- The support set of $x$ is denoted by $\text{supp}(x) = \{i \mid x_i \neq 0\}$.

A. Locally repairable codes

Let $C$ be a $[n, k, d]_q$ linear code with length $n$, dimension $k$, and minimum distance $d$ over the finite field $\mathbb{F}_q$. Then, $C$ has a $k \times n$ generator matrix $G$ and an $(n - k) \times n$ parity check matrix $H$. The dual of $C$ is defined by

$$C^\perp = \{ w \in \mathbb{F}_q^n : w \cdot c = 0 \text{ for all } c \in C \}.$$  

The rows of $H$ are the codewords of $C^\perp$. Hence, the $k \times n$ generator matrix $G$ and $(n - k) \times n$ parity check matrix $H$ satisfy $GH^T = 0$, where $T$ denotes the transpose of matrix $H$. There is a well-known distance property of linear codes as follows.

**Lemma 1** ([30],Theorem 4.5.6). Let $C$ be a linear code and let $H$ be a parity check matrix for $C$. Then $C$ has minimum distance $\geq d$ if and only if any $d - 1$ columns of $H$ are linearly independent.

Now, we give the formal definition of linear LRCs.

**Definition 1.** The linear code $C$ is a binary locally repairable code (LRC) with locality $r$ if for any $i \in [n] \triangleq \{1, 2, \cdots, n\}$, there exists a subset $R_i \subset [n] \setminus \{i\}$ with $|R_i| \leq r$ such that the $i$-th symbol $c_i$ can be recovered by $\{c_j\}_{j \in R_i}$, i.e.,

$$c_i = \sum_{j \in R_i} c_j.$$  

The set $R_i$ is called a recovery set for $c_i$.  

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It is well known that two different approaches are used to construct LRC, the generator matrix approach [4] and the parity check matrix approach [12]. Next, we will introduce the parity check matrix approach to construct the LRCs. In order to find a suitable parity check matrix involving locality, we begin with a simple lemma.

**Lemma 2** ([7]). An LRC has locality \( r \) if and only if for every coded symbol there exists a codeword \( x \) in \( C^\perp \) whose support set \( \text{supp}(x) \) contains \( i \) and the size of \( \text{supp}(x) \) is at most \( r + 1 \).

An LRC is said to have disjoint \( \ell \) local repair groups if there exist \( \ell \triangleq \frac{n}{r+1} \) vectors \( h_1,h_2,\ldots,h_\ell \) of \( C^\perp \), such that \( \text{wt}(h_i) = r + 1 \) and \( \text{supp}(h_i) \cap \text{supp}(h_j) = \emptyset \) for any \( 1 \leq i \neq j \leq \ell \). In this paper, we will only consider LRCs with disjoint repair group. Let \( C \) be an \( [n,k] \) binary linear LRC with the parity check matrix \( H \) from the dual code \( C^\perp \). The parity check matrix can be represented as follows:

\[
H = \begin{pmatrix}
H_1 & H_2 & \ldots & H_\ell \\
H_G & H_G^2 & \ldots & H_G^\ell
\end{pmatrix},
\]

where \( H_i \) is an \( \ell \times (r + 1) \) matrix whose \( i \)-th row is the all-one vector and the other rows are all-zero vectors and \( H_G^i \) denotes the \( i \)-th \( (n - k - \ell) \times (r + 1) \) matrix for \( 1 \leq i \leq \ell \). Clearly, the lower submatrix \( H_G = [H_G^1,H_G^2,\ldots,H_G^\ell] \) will determine minimum distance \( d \) of the code \( C \). Note that, in the following sections, the column vector \( h_j^i \) is to denote a column of \( H_G^i \), which is different from the meaning of \( h_j \).

**B. Intersection subspace**

Let \( V_m(q) \) be an \( m \)-dimensional vector space over the finite field \( \mathbb{F}_q \). Here, for simplicity, we omit the notation \( q \) in \( V_m(q) \), when \( q = 2 \). The set of all \( t \)-dimensional subspaces of \( V_m(q) \) is called a Grassmannian and denoted by \( G_q(m,t) \). In particular, a code is called \( t \)-intersecting if the intersection between any two codewords is exactly \( t \) elements. A subspace code is a collection of subspaces of a vector space over a finite field \( V_m(q) \), under the so-called subspace distance, defined as \( d_s(U,V) = \dim(U + V) - \dim(U \cap V) \), where \( U, V \) are two codewords (subspaces).

Two \( t \)-dimensional subspaces \( U \) and \( V \) which belong to \( G_q(m,t) \) are said to trivially intersect or disjoint if they only have a zero-dimensional intersection. A partial \( t \)-spread of \( V_m(q) \) is a collection \( S = \{W_1,W_2,\ldots,W_\ell\} \) of \( t \)-dimensional subspaces from \( G_q(m,t) \) such that \( W_i \cap W_j = \{0\} \) for \( 1 \leq i < j \leq \ell \). We call \( \ell \) the size of the partial \( t \)-spread \( S \). If \( t \) divides \( m \) and \( \bigcup_{i=0}^{\ell-1} W_i = V_m(q) \), then the partial spread is called a \( t \)-spread. The number of \( t \)-subspaces in the largest partial spread in \( V_m(q) \) will be denoted by \( \mu_q(m,t) \). One challenging question is to find the maximum partial \( t \)-spread. There are few results related to \( \mu_q(m,t) \), see below.

**Lemma 3** ([31]). If \( t \) is a divisor of \( m \) and \( \ell = \frac{q^m - 1}{q^t - 1} \), then there exists a \( t \)-spread of \( V_m(q) \) with \( \ell \) subspaces, all of dimension \( t \).

**Lemma 4** ([32]). Let \( m \equiv z \mod t \). Then, for all \( q \), we have

\[
\mu_q(m,t) \geq \frac{q^m - q^t(q^z - 1) - 1}{q^t - 1}.
\]
III. CONSTRUCTION BY THE INTERSECTION SUBSPACE

In this section, we begin with a lemma that is essential for us to construct the parity check matrix of binary LRCs with minimum distance \( d \geq 6 \). Then we call a matrix as a desired matrix if it satisfies some special conditions. Finally, applying the intersection subspaces and a desired matrix, we give constructions of \( k \)-optimal binary LRCs with disjoint local repair groups. The parameters of the optimal LRCs are derived in Theorem 1 and Theorem 3, respectively. Additionally, we will give two examples to explain the corresponding constructions in Section IV.

**Lemma 5 ( [28] ).** Consider a binary linear LRC defined by the parity check matrix \( H \) in (4). If the columns of \( H \) satisfy the following three conditions, then the LRC has minimum distance \( d \geq 6 \):

1. No two column vectors from matrix \( H^j_i \) sum to zero for \( i \in [\ell] \);
2. No four column vectors from matrix \( H^j_i \) sum to zero for \( i \in [\ell] \);
3. No four column vectors consisting of two columns from matrix \( H^j_i \) and the other two columns from matrix \( H^j_i \) sum to zero for \( i \neq j \in [\ell] \).

Hence, a straightforward way for the binary linear LRC with minimum distance \( d \geq 6 \) is to construct each submatrix \( H^j_i \) that satisfies the above conditions. Below, we will use the intersection subspace to obtain such a collection of submatrices \( H^j_i \) for \( 1 \leq i \leq \ell \).

Suppose \( W_1, W_2, \ldots, W_\ell \) are \( t \)-dimensional subspaces of a vector space \( V_m \) such that \( W_i \cap W_j = \{0\} \) for \( i \neq j \in [\ell] \). Let \( \{e^i_1, e^i_2, \ldots, e^i_s\} \) be a basis of the subspace \( W_i \), where \( e^i_j = (e^i_{1j}, e^i_{2j}, \ldots, e^i_{mj})^T \in \mathbb{F}_2^m \) for \( j \in [t] \). We shall write the coordinates of the vector \( e^i_j \) as the \( j \)-th column of an \( m \times t \) matrix \( G_{W_i} \), i.e., \( G_{W_i} = [e^i_1, e^i_2, \ldots, e^i_t] \).

**Definition 2.** Let \( G_U = [u_1, u_2, \ldots, u_s] \) be an \( s \times s \) matrix over \( \mathbb{F}_2 \) with full column rank, where the column vector \( u_i = (u_{i1}, u_{i2}, \ldots, u_{is})^T \in \mathbb{F}_2^s \). Let \( U \) be the span of the columns of \( \left( \begin{array}{c} G_U \\ 0_{m \times 1} \end{array} \right) \), where \( 0_{m \times 1} \in \mathbb{F}_2^m \) is the all-zero column vector. Define an \((s + m) \times (s + t)\) matrix \( G_{M_i} \) over \( \mathbb{F}_2 \) as follows.

\[
G_{M_i} = \left( \begin{array}{cc} G_U & 0_{s \times t} \\ 0_{m \times s} & G_{W_i} \end{array} \right) = \left( \begin{array}{cccccccc} u_1 & u_2 & \cdots & u_s & 0_{s \times 1} & 0_{s \times 1} & \cdots & 0_{s \times 1} \\ 0_{m \times 1} & 0_{m \times 1} & \cdots & 0_{m \times 1} & e^i_1 & e^i_2 & \cdots & e^i_t \end{array} \right).
\]

In the special case of \( s = 0 \), we have \( G_{M_i} = G_{W_i} \). Furthermore, a vector space spanned by the columns of \( G_{M_i} \) is denoted as \( M_i \).

It is not hard to see that \( M_i \) is a vector subspace of \( V_{m+s} \) with dimension \( s + t \). For each subspace \( M_i \), it can be represented as a direct sum of two subspaces \( U \) and \( W_i \) for \( i \in [\ell] \), i.e., \( M_i = W_i \oplus U \). Hence, it is clear that the intersection of \( M_1, M_2, \ldots, M_\ell \) only consists of the vector subspace \( U \) with the \( s \) dimension. Later, we will exploit these subspaces and a matrix with a special structure to construct binary linear LRCs. In this paper, we call this matrix a desired matrix and give the definition of a desired matrix as follows.
Definition 3. Suppose that $2^t > r$ and $t > s \geq 0$. Let $A_1$ be a $s \times r$ binary matrix and $A_2$ be an $t \times r$ binary matrix such that any two distinct column vectors are linearly independent. Define an $(s + t) \times r$ desired matrix $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = (a_{i,j}) \in \mathbb{F}_2^{(s+t) \times r}$ with full column rank. For $t \geq 3$, the desired matrix $A$ also meets the condition that any 4 columns are linearly independent. Furthermore, $A$ can be viewed as the parity check matrix of an $[r, r - (s + t), \geq 5]$ binary linear code for $t \geq 3$ (See more details in Appendix VIII).

Remark 1. The key idea of intersection subspaces comes from the construction of matrix $G$, while the matrix $A$ will help to expand the range of the construction parameters and control the minimum distance of the binary linear LRCs.

Let $M_1$ and $M_2$ be the two $(s + t)$-dimensional vector spaces over $\mathbb{F}_2$, and spanned by the column of $G_{M_1}$ and $G_{M_2}$ respectively. Let $A$ be a $(s + t) \times r$ desired matrix. For $i \in [\ell]$, the product of matrix $G_{M_i}$ and $A$ is

$$G_{M_i} \cdot A = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1s} \\ u_{21} & u_{22} & \cdots & u_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ u_{s1} & u_{s2} & \cdots & u_{ss} \\ e_{i1}^t & e_{i2}^t & \cdots & e_{it}^t \\ e_{i1}^{t+1} & e_{i2}^{t+1} & \cdots & e_{i(t+t)}^{t+1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{im}^t & e_{im+1}^t & \cdots & e_{int}^t \end{pmatrix} \begin{pmatrix} 0_{t \times s} \\ 0_{m \times s} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s+t} & a_{s+t} & \cdots & a_{s+t} \end{pmatrix},$$

where $0$ denotes a zero matrix. Let $H_{G}^i$ denote an $(s + m) \times (r + 1)$ matrix over $\mathbb{F}_2$ which is equal to the matrix $(0_{(s+m) \times 1}, G_{M_i} \cdot A)$. Note that every column $h_j^i$ of $H_{G}^i$ can be indexed by a pair $(i, j)$ for $1 \leq i \leq \ell$ and $1 \leq j \leq r + 1$, where $h_1^i = (0, 0, \ldots, 0)^T$ and $h_j^i = (\sum_{\ell=1}^s u_{1\ell}a_{ij}, \ldots, \sum_{\ell=1}^s u_{s\ell}a_{ij}, \sum_{\ell=1}^t e_{i\ell}a_{(s+i)j}, \ldots, \sum_{\ell=1}^t e_{im\ell}a_{(s+i)j})^T$. Then, we have the following Lemma 6 with regard to $H_{G}^i$, which plays an important role for the main results of Theorem 1.

Lemma 6. Let $H_{G}^i = (0_{(s+m) \times 1}, G_{M_i} \cdot A)$ for $i \in [\ell]$. The sum of any four columns of $H_{G}$, which consist of two columns from matrix $H_{G}^{i_1}$ and the other two columns from matrix $H_{G}^{i_2}$, is not equal to zero for $i_1 \neq i_2 \in [\ell]$.

Proof. Considering the four columns $h_{j_1}^{i_1}, h_{j_2}^{i_1}, h_{j_3}^{i_2}, h_{j_4}^{i_2}$ in $H_{G}^{i_1}, H_{G}^{i_2}$, without loss of generality, we assume that $h_{j_1}^{i_1} + h_{j_2}^{i_1} + h_{j_3}^{i_2} + h_{j_4}^{i_2} = 0$.

Case (i): Assume that one of the columns is the zero column from $H_{G}^{i_1}$, for example $h_{j_1}^{i_1} = 0$. Then we have

$$\begin{align*}
\sum_{\ell=1}^s u_{1\ell}a_{ij_2} + \sum_{\ell=1}^s u_{1\ell}a_{ij_3} + \sum_{\ell=1}^s u_{1\ell}a_{ij_4} &= 0 \\
\sum_{\ell=1}^s u_{s\ell}a_{ij_2} + \sum_{\ell=1}^s u_{s\ell}a_{ij_3} + \sum_{\ell=1}^s u_{s\ell}a_{ij_4} &= 0 \\
\sum_{\ell=1}^t e_{i\ell}a_{(s+i)j_2} + \sum_{\ell=1}^t e_{i\ell}a_{(s+i)j_3} + \sum_{\ell=1}^t e_{i\ell}a_{(s+i)j_4} &= 0 \\
\sum_{\ell=1}^t e_{m\ell}a_{(s+i)j_2} + \sum_{\ell=1}^t e_{m\ell}a_{(s+i)j_3} + \sum_{\ell=1}^t e_{m\ell}a_{(s+i)j_4} &= 0
\end{align*}$$
From the last $t$ equations, it is shown that

$$
\sum_{\ell=1}^{t} a_{(s+\ell)j_2} \begin{pmatrix}
e^{i_1}_{1\ell} \\
e^{i_1}_{2\ell} \\
\vdots \\
e^{i_1}_{m\ell}
\end{pmatrix} + \sum_{\ell=1}^{t} (a_{(s+\ell)j_3} + a_{(s+\ell)j_4}) \begin{pmatrix}
e^{i_2}_{1\ell} \\
e^{i_2}_{2\ell} \\
\vdots \\
e^{i_2}_{m\ell}
\end{pmatrix} = 0.
$$

$W_{i_1}$ and $W_{i_2}$ are disjoint $t$-dimensional subspaces, i.e., $W_{i_1} \cap W_{i_2} = \{0\}$, which implies that the linear combinations of the basis \{\(e^{i_1}_{1\ell}, e^{i_1}_{2\ell}, \ldots, e^{i_1}_{m\ell}\)\} of $W_{i_1}$ and \{\(e^{i_2}_{1\ell}, e^{i_2}_{2\ell}, \ldots, e^{i_2}_{m\ell}\)\} of $W_{i_2}$ are linearly independent. Thus, we have $\sum_{\ell=1}^{t} a_{(s+\ell)j_2} e^{i_1}_{\ell} = 0$ and $\sum_{\ell=1}^{t} (a_{(s+\ell)j_3} + a_{(s+\ell)j_4}) e^{i_2}_{\ell} = 0$, which shows that $a_{(s+\ell)j_2} = 0$ and $a_{(s+\ell)j_3} + a_{(s+\ell)j_4} = 0$ for all $\ell \in [t]$. We obtain a contradiction to the definition of the desired matrix $A$, so $h_{j_1}^{i_1} + h_{j_2}^{i_1} + h_{j_3}^{i_2} + h_{j_4}^{i_2} \neq 0$. Similarly, the same result holds in the case that $h_{j_1}^{i_1} = h_{j_2}^{i_2} = 0$.

Case (ii): The four columns \{\(h_{j_1}^{i_1}, h_{j_2}^{i_1}, h_{j_3}^{i_2}, h_{j_4}^{i_2}\)\} do not contain the zero column vector. Similar to the case (i), we have

$$
\sum_{\ell=1}^{s} (a_{\ell j_1} + a_{\ell j_2} + a_{\ell j_3} + a_{\ell j_4}) \begin{pmatrix}
u_{1\ell} \\
u_{2\ell} \\
\vdots \\
u_{s\ell}
\end{pmatrix} = 0,
$$

(6)

and

$$
\sum_{\ell=1}^{t} (a_{(s+\ell)j_1} + a_{(s+\ell)j_2}) \begin{pmatrix}
e^{i_1}_{1\ell} \\
e^{i_1}_{2\ell} \\
\vdots \\
e^{i_1}_{m\ell}
\end{pmatrix} + \sum_{\ell=1}^{t} (a_{(s+\ell)j_3} + a_{(s+\ell)j_4}) \begin{pmatrix}
e^{i_2}_{1\ell} \\
e^{i_2}_{2\ell} \\
\vdots \\
e^{i_2}_{m\ell}
\end{pmatrix} = 0.
$$

Since the matrix $[u_1, u_2, \ldots, u_s]$ is an $n \times n$ matrix with full column rank, this formula (6) implies that all the coefficients $a_{\ell j_1} + a_{\ell j_2} + a_{\ell j_3} + a_{\ell j_4}$ are equal to zero for $\ell \in [s]$. On the other hand, by the same argument as in the proof of the case (i), we have $\sum_{\ell=1}^{t} (a_{(s+\ell)j_1} + a_{(s+\ell)j_2}) e^{i_1}_{\ell} = 0$ and $\sum_{\ell=1}^{t} (a_{(s+\ell)j_3} + a_{(s+\ell)j_4}) e^{i_2}_{\ell} = 0$. This forces $a_{(s+\ell)j_1} + a_{(s+\ell)j_2} = 0$ and $a_{(s+\ell)j_3} + a_{(s+\ell)j_4} = 0$ for all $\ell \in [t]$. Hence, it is clear that $\sum_{\ell=1}^{4} a_{\ell j_\ell} = 0$ for all $\ell \in [s+t]$. By definition, any four columns of the desired matrix $A$ are linearly independent over $\mathbb{F}_2$. This contradiction completes the proof of the lemma.

Obviously, the key to construct parity check matrix $H$ is the matrix $G_{M_t}$, where the matrix $G_{M_t}$ can be constructed by choosing an identity matrix $G_U$ and matrix $G_{W_t}$ obtained by $t$-spread. Later, we give an explicit construction of binary LRCs with minimum distance $d \geq 6$.

**Construction 1.** Let $r = 2^h$, $s+t = 2b$ and $m \geq 4b$. Let $A$ be a desired matrix of size $2b \times 2b$. Suppose there exists a $(2b-s)$-spread \{\(W_1, W_2, \ldots, W_\ell\)\} of $V_m$ with size $\ell = \frac{2^m-1}{2^b-s-1}$ where $b \geq s \geq 1$ and $(2b-s)|m$. Let $H^*_G = (0_{(s+m)\times 1}, G_{M_t}, A)$ be a matrix of size $(s+m) \times (r+1)$. Then we can construct a binary LRC with the parity check matrix $H$ given in (4).

**Theorem 1.** The code $C$ constructed by the parity check matrix $H$ from Construction 1 is an \([n = (r+1)\ell, k = n-s-\ell-m, d \geq 6]\) binary LRC with locality $r = 2^h$, which is $k$-optimal and attains the bound in (3).
Proof. This proof consists of two parts. In Part 1 we will prove that the dimension $k$ of the corresponding codes achieves the bound (3), i.e., the $k = n - s - \ell - m$. As to Part 2, we will show that the minimum distance $d \geq 6$.

Part 1: It is easy to determine the parameter $n = (r + 1)\ell$, $k \geq \frac{rn}{(r+1)} - s - m$, and $r = 2^b$ by the parity check matrix $H$ in Construction 1, where $\ell = \frac{2m}{2^{2b-s} - 1}$. Clearly, \[
\min \left\{ \log_2 \left( 1 + \frac{rn}{2} \right), \frac{rn}{(r+1)(r+2)} \right\} = \log_2 \left( 1 + \frac{rn}{2} \right) \text{ can be obtained when } m \geq 4b \text{ and } \ell = \frac{2m}{2^{2b-s} - 1}.
\]
Due to bound (3), we have $k \leq \frac{rn}{r+1} - \left[ \log_2 \left( 1 + \frac{rn}{2} \right) \right] = \frac{rn}{r+1} - \left[ \log_2 \left( 1 + 2^{b-1}(2^b+1)\ell \right) \right] = \frac{rn}{r+1} - m - s$ as a result of $2^{m+s-1} < 1 + 2^{b-1}(2^b+1)\ell \leq 2^{m+s}$. Hence, we have $k = n - \ell - s - m$.

Part 2: We focus on proving the minimum distance $d \geq 6$. Notice that the sum of the first $\ell$ rows of $H$ is an all-one vector. Thus the minimum distance of $C$ must be even. To verify that the code $C$ has the minimum distance $d \geq 6$, it suffices to show that the submatrix $H^i_G$ satisfies the conditions in Lemma 5.

Case (i): \( \{h^i_{jz}\}^4_{z=1} \) belong to the same block, i.e., $i_1 = i_2 = i_3 = i_4$.

Without loss of generality, we assume in the contradiction method that $h^i_{j1} + h^i_{j2} + h^i_{j3} + h^i_{j4} = 0$. Similar to the proof of Lemma 6, we obtain $\sum_{s=1}^{\ell} (\sum_{z=1}^{4} a_{s+z:j}) u_s = 0$ and $\sum_{s=1}^{\ell} (\sum_{z=1}^{4} a_{s+z:j}) e^s_{i+1} = 0$. Due to the fact that $u_1, u_2, \ldots, u_s, e^i_{s+1}, \ldots, e^i_{s+t}$ are linearly independent, we have $\sum_{s=1}^{4} a_{s+z:j} = 0$ for $\ell \in [s+t]$, which implies that $a_{j1}, a_{j2}, a_{j3}, a_{j4}$ from $A$ are linearly dependent. However, any 4 columns of $A$ are linearly independent by definition. This leads to a contradiction. Thus for any four columns $\{h^i_{jz}\}^4_{z=1}$, it has $h^i_{j1} + h^i_{j2} + h^i_{j3} + h^i_{j4} \neq 0$. In particular, if $0 \in \{h^i_{jz}\}^4_{z=1}$, then it implies $\sum_{s=1}^{4} a_{s+z:j} = 0$ for $\ell \in [s+t]$ with the same argument. For the same reasons, we have the result that their sum is not zero.

Case (ii): If any two $h^i_{j1}$ and $h^i_{j2}$ vectors are in $H^i_G$, it is obvious that $h^i_{j1} + h^i_{j2} \neq 0$. Therefore, that $H^i_G$ satisfies conditions (1) and (2) in Lemma 5.

Case (iii): Two of $\{h^i_{jz}\}^4_{z=1}$ belong to one block and the other two lie in a different block. Then their sum are not equal to zero by Lemma 6. Hence, $H^i_G$ satisfies the condition (3) in Lemma 5.

From the above work, we show that the lower part $H^i_G$ of $H$ satisfies three conditions in Lemma 5. This completes the proof of Theorem 1.

In fact, our construction can generate a class of optimal binary LRCs with locality $r = 2^b$, when $\ell$ belongs to a determined range. This point is presented in detail in Theorem 2.

**Theorem 2.** Let $r = 2^b$ with $b \geq 3$. Suppose $m, s, b$ are integers such that $m \geq 4b$ and $1 \leq s < b$. If

\[
\frac{2^{m+s-1} - 1}{2^{b-1}(2^b + 1)} < \ell \leq \frac{2^m - 1}{2^{2b-s} - 1},
\]

there exists an $[n = (r+1)\ell, k = r\ell - m - s, d \geq 6]$ binary linear LRC with locality $r = 2^b$, which is optimal with respect to the bound in (3).

**Proof.** By Theorem 1, it is already known that the $C$ is an $[n = (r+1)\ell, k \geq \frac{rn}{r+1} - m - s, d \geq 6]$
binary LRC. Now it needs to show that \( k \leq r\ell - m - s \). Since
\[
\frac{2^{m+s-1} - 1}{2^{b-1}(2^b + 1)} < \ell \leq \frac{2^m - 1}{2^{2b-s} - 1},
\]
we have
\[
2^{m+s-1} < 1 + 2^{b-1}(2^b + 1)\ell \leq \frac{2^{b-1}(2^b + 1)(2^m - 1)}{2^{2b-s} - 1} + 1 \leq 2^{m+s}. \tag{7}
\]
Furthermore, by using the bound (3), the formula (7) implies that
\[
k \leq \frac{rn}{r+1} - \lceil \log_2(1 + \frac{rn}{2}) \rceil = \frac{rn}{r+1} - \lceil \log_2(1 + \frac{r(r+1)\ell}{2}) \rceil = \frac{rn}{r+1} - \lceil \log_2(1 + 2^{b-1}(2^b + 1)\ell) \rceil = \frac{rn}{r+1} - m - s.
\]
Because \( k \geq \frac{rn}{r+1} - m - s \) in Theorem 1, it is shown that \( k = \frac{rn}{r+1} - m - s \), which is \( k \)-optimal with respect to the bound in (3).

Notice that a necessary condition for the existence of the \((2b - s)\)-spread is \((2b - s) \mid m\) in Construction 1. In fact, this condition will restrict the parameters of LRC codes constructed using intersection subspace. For the case of \((2b - s) \nmid m\), we can utilize the partial \(t\)-spread of \(V_m(q)\) to replace the \((2b - s)\)-spread. The size of a maximum partial \(t\)-spread of \(V_m(q)\) is not known when \( t \mid m\). An explicit construction for a partial \(t\)-spread of size \(q^m - q^t(q^{s-1})^{-1}\) is presented in \([32]\), where \( z \equiv m \mod t\). Hence, we have the following theorem.

**Theorem 3.** Let \((2b - s) \nmid m\) and \( m \geq 4b \). Then we obtain an optimal binary linear LRC with parameters \([n = (r + 1)\ell, k = n - \ell - s - m, d \geq 6]\) and locality \( r = 2^b\), where \( \ell = \frac{2^m - 2(2b-s)(2^s-1)}{2(2b-s)-1}, 1 \leq s < b, \text{ and } z \equiv m \mod (2b - s) \leq b\).

**Proof.** By the method analogous to that used in the proof of Theorem 1, we can obtain an LRC code with parameters \([n = (r + 1)\ell, k \geq n - \ell - m - s, d \geq 6]\). Hence, we only need to show that it is an optimal LRC, which achieves the bound in (3):

\[
k \leq \frac{rn}{r+1} - \min \left\{ \log_2(1 + \frac{rn}{2}), \frac{rn}{(r+1)(r+2)} \right\}.
\]

Consider \([n = \frac{2^{m-s} - 2(2b-s)(2^s-1)}{2(2b-s)-1}(2^b + 1), k \geq n - \ell - m - s]\) LRC, it is required to show that \( k \leq n - \ell - s - m, \text{ i.e., } m + s - 1 < \min \left\{ \log_2 \left( 1 + \frac{rn}{2} \right), \frac{rn}{(r+1)(r+2)} \right\} \leq m + s \). Due to \( n > 5(r + 1)(r + 2) \), we have to prove

\[
2^{m+s-1} \leq 1 + \frac{rn}{2} \leq 2^{m+s}. \tag{8}
\]

For the left side of the inequality, it needs to be verified that \((2^{m+s-1} - 1)(2^{2b-s} - 1) < 2^{b-1}(2^b + 1) (2^{m} - 2^{2b-s}(2^s - 1) - 1)\). Since \( m \geq 4b \), \( 0 < z \leq b \) and \( 1 \leq s < b \), we have \( 2^{m+b-1} + 2^{b-s-1} = 2^{m-4b} \cdot 2^{3b-1} + 2^{4b-s-1} > 2^{b+s-1} + 2^{3b-1-s+z} \) Then \((2^{m+s-1} - 1)(2^{2b-s} - 1) < 2^{b-1}(2^b + 1) (2^{m} - 2^{2b-s}(2^s - 1) - 1)\) follows from \( 2^{m+s-1} \geq 2^{b-1} \) and \( 2^{2b-s} > 2^{b-1} \), which supports the left side of formula (8).
The right side of the inequality (8) holds by using similar arguments as in the above paragraph. The proof has been completed. 

Similar to the above analysis of Theorem 2, we can also obtain a class of k-optimal LRCs with locality \( r = 2^b \) from Theorem 3 by using the partial \( t \)-spread in Construction 1. Hence, Theorem 4 follows.

**Theorem 4.** Let \( r = 2^b \) with \( b \geq 3 \) and \( 1 \leq s < b \). Suppose that \( m \geq 4b \) is an integer and \( 1 \leq z \equiv m \mod (2b - s) \leq b \). When

\[
\frac{2m+s-1}{2b-1(2^b+1)} < \ell \leq \frac{2m - 2(2^b-s)(2^z-1) - 1}{2(2b-s) - 1},
\]

the code \( C \) in Theorem 3 is a k-optimal binary linear LRC with parameters \( [n = (r+1)\ell, k = n - \ell - m - s, d \geq 6] \).

**Proof.** It is easy to construct the code \( C \) with parameters \( n = (r+1)\ell, d \geq 6 \) and

\[ k \geq n - \ell - m - s \]

by using the partial \( t \)-spread in Construction 1, where \( \ell = \frac{2m - 2(2^b-s)(2^z-1) - 1}{2(2b-s) - 1} \). In the similar way provided in Theorem 2, we only need to show \( k \leq n - \ell - m - s \) below.

Combining the proof of Theorem 3 with the equation (7) and the condition (9), we derive the following chain of inequalities:

\[
2^{m+s-1} < 1 + 2^{b-1}(2^b+1)\ell \leq \frac{2^{b-1}(2^b+1)(2^m - 2(2^b-s)(2^z-1) - 1)}{2(2b-s) - 1} + 1 \leq 2^{m+s},
\]

which implies \( k \leq n - \ell - m - s \) by using the bound in (3). Therefore, \( k = n - \ell - m - s \), which completes the proof of the theorem.

\[ \Box \]

**IV. Examples and Comparisons**

In this section, we give three examples to illustrate the corresponding construction in detail. Example 1, by Theorem 1, shows how to construct the optimal binary LRC from Construction 1, and Example 2 is a special case when \( s = 0 \). We present Example 3 to illustrate the construction of a k-optimal LRC by partial spread in Theorem 3. In Table II, we list some of the optimal binary LRCs with disjoint local repair groups given by Theorem 1 and Example 3 with \( 3 \leq b \leq 6 \) and \( 12 \leq m \leq 40 \). Furthermore, Table III is listed to compare the previous constructions of optimal binary LRCs, which indicates that our constructions have more flexible parameters \([n, k]\) for given locality \( r \) and the corresponding code rates are higher than those in [8] [29].

**Example 1.** Suppose \( b = 3 \), \( s = 2 \) and \( m = 12 \) in Construction 1. Let \( \{W_1, W_2, \cdots, W_{273}\} \) be a 4-spread of \( V_{12} \). Denote a basis of \( W_i \) by \( \{e_1^i, e_2^i, e_3^i, e_4^i\} \) for \( i \in [273] \). Then we choose a matrix \( G_{M_i} \) and a desired matrix \( A_{6 \times 8} \) as follows.

\[
G_{M_i} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_1^i & e_2^i & e_3^i & e_4^i \end{pmatrix}, \quad A_{6 \times 8} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},
\]

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where \( A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \) and \( A_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \). For example, let \( \alpha \) be a primitive element of \( \mathbb{F}_{12}^2 \). Let \( \ell = \frac{12 - 1}{2^{12} - 1} \) and \( \gamma = \alpha^\ell \). We get a basis \( \{\alpha^0, \alpha^0\gamma, \alpha^0\gamma^2, \alpha^0\gamma^3\} \) and a basis \( \{\alpha^1, \alpha^1\gamma, \alpha^1\gamma^2, \alpha^1\gamma^3\} \) of subspace \( W_1 \), subspace \( W_2 \) respectively. Then we obtain the matrices \( H^1_G \) and \( H^2_G \) as follows by expanding the column vectors of the submatrices \( G_{M_i} \cdot A \ (i = 1, 2) \) with respect to the bases respectively.

\[
H^1_G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad H^2_G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
\]

It can be verified that any 5 columns of \( H \) in (4) are linearly independent, so \( H \) defines a \( k \)-optimal \([2457, 2170, 6]\) binary LRC with locality \( r = 8 \) by Theorem 1.

**Example 2.** Taking \( t = 2 \) and \( s = 0 \) in the above example. Let \( \{W_0, W_1, W_2, W_3, W_4\} \) be a \( 2 \)-spread of \( V_4 \) and \( \{e^i_1, e^i_2\} \) be a basis of subspace \( W_i \). By choosing

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_{M_i} = (e^i_1 \ e^i_2),
\]

we obtain a \( k \)-optimal \([15, 10, 6]\) LRC with locality \( r = 2 \).

**Remark 2.** Note that an LRC with the same parameters of Example 2 was also constructed in Example 2 of [8]. Correspondingly, when \( s = 0 \) in Construction 1 and Theorem 1, we obtain an \([n = \frac{2^m - 1}{2^{t-1}}, k = \frac{r}{r+1} \cdot m, d \geq 6]\) binary LRC with locality \( r = 2^t \), which includes the construction of optimal binary linear LRC in [8] as a special case.

Note that in Example 1, we need to satisfy the condition \((2b-s)|m\). For the case of \((2b-s) \nmid m\), we construct LRC by the partial spread in Construction 1. An example of Theorem 3 is presented as follows.

**Example 3.** Let \( m = 12, b = 3, s = 1 \) and let \( \{W_1, W_2, \ldots, W_{129}\} \) be a partial \( 5 \)-spread of \( \mathbb{F}_2^{12} \) by Lemma 4. Then we obtain an \([n = 1161, k = 1019, d \geq 6]\) LRC with locality \( r = 8 \) by Theorem 3. This code is optimal since it attains the bound (3). Moreover, taking \( 113 \leq \ell \leq 129, \)
the code is a \( k \)-optimal LRC with parameters \([n = (r + 1)\ell, k = n - \ell - m - s, d \geq 6]\) by Theorem 4.

Here we list the parameters of optimal binary LRCs with disjoint local repair groups given by Theorem 1 and Theorem 3 in Table II for \(3 \leq b \leq 6\) and \(12 \leq m \leq 40\), which achieve the maximum value obtained from the bound in (3). The blue values are new parameters of optimal binary LRCs in current paper. The parameters of LRC with the same locality \(r\) in [8] are also listed in Table II.

**TABLE II: Optimal binary linear LRCs with \(d \geq 6\)**

| \(r\)       | \([n, k]\) from Theorem 1             | \([n, k]\) from Theorem 3             | \([n, k]\) in [8]          |
|------------|---------------------------------------|---------------------------------------|-----------------------------|
| 8          | [2457, 2170]                          | [1161, 1019]                          | [585, 508]                  |
|            | [10066329, 8947822]                   | [9801, 8606]                          | [2396745, 2130416]          |
| 16         | [4527185, 4260854]                    | [1122833, 1056760]                    | [1118481, 1052664]          |
|            | [602957989425, 576489872357]          | [1150033, 1082360]                    | [73300775185, 68988964840]  |
| 32         | [562436193, 545392638]                | [338825, 327880]                      | [34636833, 33587202]        |
|            | [34670625, 33619970]                  |                                       |                             |
| 64         | [4276545, 4210724]                    | [68290625, 67239968]                  | [266305, 262184]            |
|            | [1091051585, 1074266140]             |                                       |                             |

**Remark 3.** In [8], the authors chose \(\frac{2^m - 1}{2^b - 1}\) vectors, which span disjoint subspaces respectively, from \(\mathbb{F}_{2^m}\) to construct the parity check matrix. Hence, they used the parity check matrix of a \(2^b\)-ary Hamming code with length \(\frac{2^m - 1}{2^b - 1}\). Comparing with the code rate of the construction for \(r = 2^b\) in [8], we have a larger code rate. For locality \(r = 2^b\), Wang et al. constructed an \([n' = \frac{2^m - 1}{2^b - 1}, k' = \frac{r'}{r+1} - m, d \geq 6]\) LRC with disjoint local repair groups [8]. Hence, their code rate is \(\frac{k'}{n'} = \frac{r}{r+1} - \frac{m}{n}\). In our construction, the length of optimal LRC is \(n = (2^b + 1)\frac{2^m - 1}{2^b - 1}\) for \(0 < s < b\), which is approximately \(2^s\) times than \(n'\), where the dimension \(k = \frac{r}{r+1} - s - m\). Hence, for the same \(b\) and \(r\), it is obvious that the code rate \(\frac{k}{n} = \frac{r}{r+1} - \frac{s}{r+1} - \frac{m}{n} < \frac{s + m}{2^n}\) as a result of \(\frac{r + 1}{2^n}\).

Table III gives the summary of optimal binary linear LRCs with disjoint local repair groups whose minimum distance is 6. We also list the result of Theorem 2 and Theorem 4. The comparison of the number of the disjoint local repair group has shown that optimal binary LRC with wider range of parameters can be obtained from Theorem 2. Here, \(\mu_2(m, 2b)\) denotes the size of a maximum partial \(2b\)-spread in \(\mathbb{F}_{2^m}\).

**TABLE III: Optimal binary \([n = (r + 1)\ell, k, \geq 6]\) LRCs with respect to the bound (3)**

| Ref. | \(r = 2^s \geq 8\) | The number of repair groups | Conditions |
|------|---------------------|----------------------------|------------|
| [8]  | \(2^b\)            | \(\ell = \frac{2^m - 1}{2^b - 1}\) | \(2b \mid m, m \geq 4b\) |
| [29] | \(2^b\)            | \(\frac{2^m - 1}{2^b - 1} + 1 \leq \ell \leq \mu_2(m, 2b)\) | \(m \geq 4b\) |
| Thm 2 | \(2^b\)            | \(\frac{2^m - 1}{2^b - 1} + 1 < \ell \leq \frac{2^m - 1}{2^b - s + 1}\) | \((2b - s) \mid m, m \geq 4b, 1 \leq s < b\) |
| Thm 4 | \(2^b\)            | \(\frac{2^m - 1}{2^b - s + 1} < \ell \leq \frac{2^m - 2(2b - s)(2^s - 1) - 1}{2(2b - s) - 1}\) | \((2b - s) \not\mid m, m \geq 4b, 1 \leq s < b\), \(1 \leq z \equiv m \mod (2b - s) < b\) |
Remark 4. As a comparison, the optimal binary LRCs generated by our constructions have more flexible parameters \( [n, k] \) than those in [8], [29] for a fixed locality \( r = 2^b \). Particularly, if we take \( s = 0 \) in Construction 1, we have \( \frac{2^m - 1}{2^{b + 1} - 1} < \ell \leq \frac{2^m - 1}{2^{b + 1}} \) in Theorem 2. Note that \( \ell = \frac{2^m - 1}{2^{b + 1}} \) in [8] and \( m(2, 2b) \leq \frac{2^m - 1}{2^{b + 1}} \) in [29]. Clearly, the optimal binary linear LRCs constructed in [8] and [29] are included in our construction. For example, let \( b = 3 \) and \( m = 12 \), we will get \( \ell = 65 \) in [8] and \( 56 < \ell \leq 65 \) in [29], while in our construction the range is \( 56 < \ell \leq 65 \) and \( 227 < \ell \leq 273 \). It is shown that we can easily construct more optimal binary LRCs with the same locality which cannot be constructed from [8], [29].

V. Shortening LRC

The shortening technique can be applied to the derivation of binary linear LRCs with new parameters. Let \( S \) be a \( q \)-ary \( [n, k, d] \) linear code. For a fixed \( 1 \leq i \leq n \), form the subset \( S \) of \( C \) consisting of the codewords with the \( i \)-th position equal to 0. Clearly, \( S \) is a subcode of \( C \). Then delete the \( i \)-th position from all the words in \( S \) to form a code \( C' \). Moreover, \( C' \) can be easily verified to be a \( q \)-ary \( [n - 1, k', d'] \) linear code with \( k - 1 \leq k' \leq k \), \( d' \geq d \). Hence, we obtain the following theorem with respect to the shortened LRCs.

Theorem 5. Let \( C \) be an \( [n, k, d] \) binary optimal LRC constructed in Construction 1 such that \( n \geq 2(r + 1) \) and \( k \geq 2r \).

1) An \( [n', k', d'] \) LRC \( C' \) with locality \( r \) can be obtained by shortening \( C \), where the parameters of \( C' \) satisfy \( n' = n - a(r + 1), k' \geq k - ar, \) and \( d' \geq d \).

2) Let \( H^i = \begin{pmatrix} H_i & H^i_G \end{pmatrix} \) for \( i \in [\ell] \). Removing a column of each distinct submatrix \( H^{i_1}, H^{i_2}, \ldots, H^{i_\ell} \) from the parity check matrix \( H \) respectively for \( i_r \in [\ell] \), then we obtain a shorten LRC with parameters \( [n' = n - \tau, k' = k - \tau, d' = d] \).

Proof. 1) Assume that \( H \) is a parity check matrix of \( C \) in Construction 1. The first \( \ell \) rows from \( H \), denoted by \( \{h_1, \ldots, h_\ell\} \), form a set of locality rows of \( C \), where \( h_i \in \mathbb{F}_2^n \) with \( \text{supp}(h_i) = r + 1 \). Consider the first \( \tau \) locality row of \( H \), where \( 1 \leq \tau \leq \ell \). We delete the first \( \tau \) locality rows and the corresponding columns which form the support of \( h_i \) of \( H \). Then we obtain an \( m' \times n' \) submatrix \( H' \) with \( n' = n - \tau(r + 1), m' = n - k - \tau \). Let \( C' \) be the \( [n', k', d'] \) linear code with the parity check matrix \( H' \). Due to \( \text{rank}(H') \leq (n' - m') \), it is easy to infer that \( k' \geq k - \tau r \).

Since \( C' \) is a shortening code of \( C \), \( C' \) is an LRC code with minimum distance \( d' \geq d \). This completes the proof.

2) By the construction of optimal LRCs in Construction 1, it is shown that each submatrix \( H^i \) is generated by a desired matrix \( A \) and a matrix \( G_M \) for \( i \in [\ell] \). In fact, \( A \) can be viewed as a parity check matrix of a linear code with minimum distance \( d = 5 \). Let \( A' \) be an \( [r - 1, r - (s + t), d] \) matrix by deleting a column of \( A \) such that any four columns of \( A' \) are still linearly independent. Then we construct \( H^{i_1}, \ldots, H^{i_\ell} \) by utilizing the matrix \( A' \) and the remaining \((\ell - \tau)\) submatrices by utilizing the matrix \( A \) in Construction 1, where \( \tau \in [\ell] \). Hence we obtain a linear LRC \( C' \) with the parameters \( [n' = n - \tau, k' = k - \tau, d' = d] \). In particular, when \( \tau = \ell \), the locality of LRC is \( r - 1 \); otherwise, the locality of LRC is \( r \).

For the special case of the shortened LRCs, we can modify Constructions 1 as in the following example to construct binary LRC.
Example 4. Let $A'$ be a matrix removing the first column from $A$ in Example 1, i.e.,

$$A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$ 

Then we construct the submatrices $H_G^1$ and $H_G^2$ by $(0, \ G_{M_1} \cdot A')$ and $(0, \ G_{M_2} \cdot A')$ respectively, and construct the $(\ell - 2)$ remaining submatrices $H_G^i$ by $(0, \ G_{M_i} \cdot A)$. Thus we obtain a $[2455, 2168, 6]$ linear LRC by Theorem 5, which is $k$-optimal with respect to the bound in (3).

VI. Conclusion

In this paper, we present an explicit construction of $k$-optimal LRCs with minimum distance $d \geq 6$ by investigating parity check matrices. In general, the optimal binary linear LRCs with minimum distance $d \geq 6$ and locality $r = 2^b$ are constructed by $t$-spread of an $m$-dimensional vector space over $\mathbb{F}_q$, which is a collection of $t$-dimensional subspaces with pairwise trivial. Of interest is the idea of using intersection subspaces to replace the method of $t$-spread. Applying this new idea, we efficiently enlarge the range of new parameters of the optimal LRCs with minimum distance $d \geq 6$ and locality $r = 2^b$. In fact, it yields more repair groups such that the corresponding constructions have more flexible lengths and dimensions. Compared with the previous work with the same locality, the code lengths of our work are larger and the code rates are higher.

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REFERENCES

[1] C. Huang, H. Simitci, Y. Xu, A. Ogus, B. Calder, P. Gopalan, J. Li, and S. Yekhanin, “Erasure coding in windows azure storage,” in Proceedings of the 2012 USENIX conference on Annual Technical Conference, 2012, pp. 2–2.

[2] M. Asteris, D. Papailiopoulos, A. G. Dimakis, R. Vadali, S. Chen, and D. Borthakur, “Xoring elephants: Novel erasure codes for big data,” Proceedings of the VLDB Endowment, vol. 6, no. 5, 2013.

[3] A. G. Dimakis, B. Godfrey, Y. Wu, M. J. Wainwright, and K. Ramchandran, “Network coding for distributed storage systems,” IEEE Transactions on Information Theory, vol. 56, no. 9, pp. 4539–4551, 2010.

[4] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, “On the locality of codeword symbols,” IEEE Transactions on Information Theory, vol. 58, no. 11, pp. 6925–6934, 2012.

[5] O. Olmez, C. Hollanti, M. Grezet, R. Freij-Hollanti, and T. Westerbäck, “Bounds on binary locally repairable codes tolerating multiple erasures,” ETH Zurich, 2018.

[6] V. R. Cadambe and A. Mazumdar, “Bounds on the size of locally recoverable codes,” IEEE Transactions on Information Theory, vol. 61, no. 11, pp. 5787–5794, 2015. [Online]. Available: https://doi.org/10.1109/TIT.2015.2477406

[7] V. Guruswami, C. Xing, and C. Yuan, “How long can optimal locally repairable codes be?” IEEE Transactions on Information Theory, vol. 65, no. 6, pp. 3662–3670, 2019.

[8] A. Wang, Z. Zhang, and D. Lin, “Bounds for binary linear locally repairable codes via a sphere-packing approach,” IEEE Transactions on Information Theory, vol. 65, no. 7, pp. 4167–4179, 2019.
In this section, we give an approach to find the desired matrix $A$. This approach requires the help of the computer program. Note that the desired matrix $A$ can be viewed as the parity check
matrix of a $[2^b, 2^b - 2b, d \geq 5]$ binary linear code. In [33], Chen presented the explicit construction of binary linear code with parameters $[2^b, 2^b - 2b, d \geq 5]$ from shortened nonprimitive cyclic code. But we can not directly use this construction. Because the desired matrix $A$ needs to contain a submatrix, of which any two distinct columns are linearly independent, it is difficult to give an explicit construction of the desired matrix $A$. We have used the computer program MAGMA to search some examples of the desired matrix $A$. we obtained those matrices which imply the existence of these desired matrices. Thus, how to construct more desired matrices $A$ by using theoretical analysis and effective search algorithm, remains an open problem.

We briefly recall the construction of binary linear code with parameters $[2^b, 2^b - 2b, d \geq 5]$. Let $n = 2^b + 1$ and $\alpha$ be a primitive root of $x^n - 1$ with minimal polynomial $M_\alpha(x)$. Clearly, the degree of $M_\alpha(x)$ is $2b$. Define $C$ to be the binary cyclic code of length $n$ generated by $(x - 1)M_\alpha(x)$. It is not hard to show that $\{\alpha^i : i = -2, -1, 0, 1, 2\}$ forms a subset of the roots of the generator polynomial of $C$, so $C$ is an $[n = 2^b + 1, k = 2^b - 2b, d \geq 6]$ binary linear code. The code $C$ can be punctured by deleting one of the check bits to yield a code $C'$ of length $2^b$ with $2b$ check bits and $d \geq 5$. Hence, we can obtain a $[2^b, 2b]$ parity check matrix $A'$ of $C'$. Then we perform the row transformation on the matrix $A'$ such that $A'$ transforms a desired matrix $A$. Below, we give some examples of the desired matrix $A$ obtained by the program. Here, if a $z \times n$ submatrix of the $k \times n$ desired matrix satisfies that any two distinct column vectors from the submatrix are linearly independent, we denote this desired matrix $A$ by an $[n, k]_z$ matrix. See the following examples.

- $[8, 6]_4$ matrix $A$:

\[
\begin{pmatrix}
10000010 \\
01000001 \\
11100000 \\
10010011 \\
01001010 \\
01000111
\end{pmatrix}
\]

- $[16, 8]_5$ matrix $A$:

\[
\begin{pmatrix}
0010010001101011 \\
0010000010001111 \\
0100000100111110 \\
1000000010011110 \\
0101000011111100 \\
0110100001011001 \\
0110011001111110 \\
0000010111011101
\end{pmatrix}
\]

- $[32, 10]_6$ matrix $A$:
• $[64, 12]_8$ matrix $A$:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

• $[128, 14]_{16}$ matrix $A$:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]