Jónsson Cardinals, Erdős Cardinals, and the Core Model

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Abstract

We show that if there is no inner model with a Woodin cardinal and the Steel core model $K$ exists, then every Jónsson cardinal is Ramsey in $K$, and every $\delta$-Jónsson cardinal is $\delta$-Erdős in $K$.

In the absence of the Steel core model $K$ we prove the same conclusion for any model $L[E]$ such that either $V = L[E]$ is the minimal model for a Woodin cardinal, or there is no inner model with a Woodin cardinal and $V$ is a generic extension of $L[E]$.

1 Introduction

It is well known that every Ramsey cardinal is a Jónsson cardinal but surprisingly it appears to be unknown whether every Jónsson cardinal is Ramsey. It seems unlikely that this converse implication holds in general, but Kunen [Kun70] showed that it is true in $L[\mu]$, and this was extended in [Mit79] (see also [Jen81]) to show that if $0^+$ does not exist then every cardinal which is Jónsson, even in $V$, is Ramsey in $K$. In this note we catch up with recent advances in core model theory by extending this result to Steel’s core model [Ste96], provided that this core model exists and there is no model with a Woodin cardinal.

Our main motivation in thinking about this problem came from an interest in the possibility of a covering lemma for a model with a Woodin

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cardinal. This interest led to two extensions of the basic result: First, we consider cardinals $\kappa$ which are $\delta$-Jónsson for a regular cardinal $\delta \leq \kappa$. A $\delta$-Jónsson cardinal, which is the same as a Jónsson cardinal except that the elementary substructure is only required to have order type $\delta$, comes up in stationary tower forcing \cite{MSW} at a Woodin cardinal: $\alpha$ is $\delta$-Jónsson if and only if there is a condition in the stationary tower forcing which forces that $i(\delta) = \alpha$, where $i$ is the generic embedding. We show, under the same conditions as above, that every $\delta$-Jónsson cardinal is $\delta$-Erdős in $K$. To some extent we are following Jensen in this: in \cite{Jen81}, Jensen extended \cite{Mit79} by showing that if there are no models with a measurable cardinal then every $\delta$-Erdős cardinal is $\delta$-Erdős in the Dodd-Jensen core model for one measurable cardinal.

The relevance of these results to Woodin cardinals would seem to be limited by the fact that the relevant Steel core models do not exist in the presence of a Woodin cardinal, and (so far as we know) may not exist even if there is no model with a Woodin cardinal. However, if $V = L[\mathcal{E}]$ or $V$ is a generic extension of a model $L[\mathcal{E}]$ then the model $L[\mathcal{E}]$ looks like a core model. We use this resemblance to show that if $L[\mathcal{E}]$ is a minimal model for a Woodin cardinal then $L[\mathcal{E}]$ satisfies that every $\delta$-Jónsson cardinal is $\delta$-Erdős, and that if $L[\mathcal{E}]$ does not contain a class model for a Woodin cardinal then the same is true of cardinals which are $\delta$-Jónsson in a generic extension of $L[\mathcal{E}]$.

Before stating the main theorem, we give precise definitions of $\delta$-Jónsson and $\delta$-Erdős cardinals.

**Definition 1.1.** If $\delta$ and $\kappa$ are cardinals with $\delta < \kappa$, then $\kappa$ is said to be $\delta$-Jónsson if for each first order structure $\mathcal{A}$ in a countable language with universe $\kappa$ there is an elementary substructure $\mathcal{A}' \prec \mathcal{A}$ with universe $\kappa$ such that order type($\mathcal{A}'$) = $\delta$.

We say that $\kappa$ is $\kappa$-Jónsson if it is Jónsson, that is, if for every structure $\mathcal{A}$ as above there is an elementary substructure $\mathcal{A}' \prec \mathcal{A}$ with universe $\kappa$ such that $|\mathcal{A}'| = \kappa$ but $\mathcal{A}' \neq \mathcal{A}$.

The following definition is due to Baumgartner \cite{Bau77}:

**Definition 1.2.** If $\delta$ and $\kappa$ are cardinals, with $\delta \leq \kappa$, then $\kappa$ is $\delta$-Erdős if for every structure $\mathcal{A}$ in a countable language with universe $\kappa$ and for every closed and unbounded subset $C$ of $\kappa$ there is a set $D \subset C$ of order type $\delta$ which is a normal set of indiscernibles for $\mathcal{A}$.
By a normal set of indiscernibles we mean a set $D$ such that for every $n$-ary function $f$ which is definable in $\mathcal{A}$ without parameters, either $f(d_0, \ldots, d_{n-1}) \geq d_0$ for every $\vec{d} = (d_0, \ldots, d_{n-1}) \in [D]^n$ or else the value of $f(\vec{d})$ is constant for $\vec{d} \in [D]^n$. It should be noted this is equivalent to the seemingly stronger definition which requires the same property to hold for all functions which are definable in $\mathcal{A}$ from parameters smaller than $d_0$.

It is shown in [Bau77] that every Ramsey cardinal $\kappa$ is $\kappa$-Erdős.

**Theorem 1.3 (Main Theorem).** Suppose that $L[\mathcal{E}]$ is a fully iterable model constructed from an good sequence of extenders as in [MS94], and assume that any of the following three conditions are true:

1. There is no class model with a Woodin cardinal, and $L[\mathcal{E}]$ is equal to the Steel core model $K$.

2. There is no class model with a Woodin cardinal, and $V$ is a generic extension of $L[\mathcal{E}]$.

3. $V = L[\mathcal{E}]$ is a minimal model for a Woodin cardinal: that is, if $\eta = \text{len}(\mathcal{E})$ then $\eta$ is Woodin in $L[\mathcal{E}]$, but all of the models $L[\mathcal{E}|\alpha]$ for $\alpha < \eta$ are fully iterable.

Then every cardinal $\kappa$ which is a $\delta$-Jónsson cardinal in $V$, where $\delta$ is any uncountable regular cardinal, is $\delta$-Erdős in $L[\mathcal{E}]$.

The statement in clause (3) that a model $L[\mathcal{E}]$ is minimal requires not only that there is no $\alpha < \text{len}(\mathcal{E})$ such that $L[\mathcal{E}|\alpha]$ satisfies that $\alpha$ is Woodin, but also that there is no iteration tree $T$ on $L_\alpha[\mathcal{E}|\alpha]$ such that $L[\mathcal{E}^T]$ satisfies that there is a Woodin cardinal, where $\mathcal{E}^T$ is the extender sequence stabilized by $T$. This means that $\mathcal{E}^T$ is the union of the set of sequences $\mathcal{E}_\nu|\rho_\nu$, for $\nu$ on the main branch of $T$, where $\mathcal{E}_\nu$ is the extender sequence of the $\nu$th tree of $T$ and $\mathcal{E}_\nu|\rho_\nu = \mathcal{E}_{\nu'}|\rho_\nu$ for all $\nu' > \nu$ in the main branch of $T$. The assumption in hypotheses (1) and (2) that there is no class model with a Woodin cardinal is required to ensure iterability: we need to know that every iteration tree in $V$ on $L[\mathcal{E}]$ has a branch in $V$.

We begin by reducing hypothesis (3) of the theorem to hypothesis (2), thereby eliminating the one case in which there is a Woodin cardinal.

**Lemma 1.4.** Suppose that $L[\mathcal{E}]$ is a minimal model for a Woodin cardinal and that $\kappa$ is $\delta$-Jónsson but not $\delta$-Erdős in $L[\mathcal{E}]$ for some $\delta \leq \kappa$ which is
regular and uncountable in $L[\mathcal{E}]$. Then there is $\alpha < \text{len}(\mathcal{E})$ so that $L[\mathcal{E}|\alpha]$ also satisfies that $\kappa$ is $\delta$-Jónsson but not $\delta$-Erdős.

Proof. Let $L[\mathcal{E}]$, $\kappa$ and $\delta$ be as in the hypothesis of the lemma, and let $\eta = \text{len}(\mathcal{E})$. Then $\kappa \leq \eta$, since there are no $\delta$-Jónsson cardinals in $L[\mathcal{E}]$ above $\eta$.

If $\kappa < \eta$ then we can take $\alpha = \kappa^+ < \eta$. Then $\kappa$ is $\delta$-Jónsson, but not $\delta$-Erdős, in $L[\mathcal{E}|\alpha]$ since $\mathcal{P}(\alpha) \cap L[\mathcal{E}] = \mathcal{P}(\alpha) \cap L[\mathcal{E}|\alpha]$.

If $\delta < \kappa = \eta$ then the hypothesis of the lemma is false since the set of measurable cardinals is stationary in $\eta$, which implies that $\eta$ is $\delta$-Erdős in $L[\mathcal{E}]$.

We will finish the proof of the lemma by showing that if $\delta = \kappa = \eta$ then the structure $L_{\eta}[\mathcal{E}]$ is a Jónsson algebra in $L[\mathcal{E}]$, so that the hypothesis of the lemma is false since $\eta$ is not Jónsson.

**Lemma 1.5.** Suppose $V = L[A]$ where $A \subset \kappa$ and $\kappa$ is a regular cardinal. Then $L_\kappa[A]$ is a Jónsson algebra, and hence $\kappa$ is not a Jónsson cardinal.

Proof. Suppose to the contrary that there is an $X \prec L_\kappa[A]$ such that $|X| = \kappa$ but $\kappa \not\subseteq X$. Choose such an $X$ with $\alpha = \text{inf}(\kappa \setminus X)$ as small as possible, and let $\pi: L_\kappa[A'] \cong X \prec L_\kappa[A]$.

We claim that $\pi$ can be extended to an elementary embedding $\tilde{\pi}: L[A'] \to \text{ult}(L[A'], \pi, \kappa) = L[A]$. For this it is sufficient to show that $\text{ult}(L[A'], \pi, \kappa)$ is defined and well founded.

To show that $\text{ult}(L[A'], \pi, \kappa)$ is defined we need to show that every bounded subset of $\kappa$ in $L[A']$ is a member of $L_\kappa[A']$. Suppose $x \subset \delta < \kappa$ and $x \in L_\xi[A']$. Then there is $Z \prec L_\xi[A']$ with $x \in Z$ such that $|Z| < \kappa$ and $\delta \leq \eta = Z \cap \kappa = \text{sup}(Z \cap \kappa)$. Let $k: L_\xi[A' \cap \eta] \cong Z$ be the inverse of the transitive collapse. Then $x \in L_\xi[A' \cap \eta] \in L_\kappa[A']$.

A similar argument shows that $\text{ult}(L[A'], \pi, \kappa)$ is well founded: Otherwise pick functions $f_n$ and ordinals $\alpha_n < \kappa$ so that $\langle \tilde{\pi}(f_n)(\alpha_n) : n < \omega \rangle$ is a descending sequence of ordinals. Pick $Z$ as above with $\{ \alpha_n : n < \omega \} \cup \{ f_n : n < \omega \} \subset Z$, and define $k: L_\xi[A' \cap \eta] \cong Z$ as before. Then $L_\xi[A' \cap \eta] \in L_\kappa[A']$, and $\pi(L_\xi[A' \cap \eta])$ is ill founded, which is absurd since it is a member of $L_\kappa[A]$.

Since $k(\alpha) > \alpha$ and $\tilde{\pi}: L[A'] \to L[A]$ is an elementary embedding, there is a set $X' \in L[A']$ such that $|X'| = \kappa$ and $X' \prec L_\kappa[A']$ but $\alpha' = \text{inf}(\kappa \setminus X') < \alpha$.

\footnote{NOTE: This replaces an earlier incorrect proof. It is so easy that it must have been known before, and I will probably replace this by a reference, once I find one.}

4
Now set $X'' = \pi'' X'$. Then $|X''| = |X'| = \kappa$ and $X'' \prec L_\alpha[A]$, but $\alpha' \notin X''$. This contradicts the minimality of $\alpha$, and hence completes the proof of the lemma. \hfill \Box

This finishes the proof of lemma 1.4.

It follows that it is sufficient to prove that the conclusion of the main theorem follows from hypothesis (1) or (2) of that theorem. Before doing so we complete the introduction by discussing some of the notation and background theory used in the proof.

**Iteration trees, $L[\mathcal{E}]$, and the Core model**

The basic sources are \cite{MS94b} for $L[\mathcal{E}]$ and \cite{Ste96} for Steel's core model $K$. The primary aim of this section is to clarify the notation which we will use and to state some basic results concerning iteration trees, $\varphi$-minimal structures $L[\mathcal{E}]$ and the core $K$.

A **phalanx** of length $\theta + 1$ is a pair $(\bar{\rho}, \vec{M})$ where $\rho = (\rho_\nu : \nu < \theta)$ is a continuous increasing sequence of ordinals and $\vec{M} = (M_\nu : \nu \leq \theta)$ is a sequence of premice such that if $\alpha < \beta \leq \theta$ then $M_\alpha$ agrees with $M_\beta$ up to $\rho_\alpha$.

We will write $U \oplus U'$ for a phalanx which has $U$ as an initial segment. For the special case where $U'$ is a phalanx of length 1, that is, a single premouse $\mathcal{R}$, we will write $U \oplus_\lambda \mathcal{R}$ for the phalanx obtained by truncating $U$ as necessary, and then adding $\mathcal{R}$ onto the end. If $U = (\bar{\rho}, \vec{M})$ then this means that

$$U \oplus_\lambda \mathcal{R} = \left(\bar{\rho} \upharpoonright \nu \setminus \lambda, \vec{M} \upharpoonright (\nu + 1) \setminus \lambda \mathcal{R}\right)$$

where $\nu$ is the least ordinal such that $\rho_\nu > \lambda$.

Any iteration tree has an underlying phalanx, which is simply the sequence of models $M_\nu$ and ordinals $\rho_\nu$ of the tree. We will normally use the same symbol for a tree and for its associated phalanx.

We will use the term **iteration tree** both for a normal iteration tree, with a single root, and for an **iteration tree on a phalanx**, which Steel calls a **pseudo-iteration tree**. An iteration tree $\mathcal{T}$ on a phalanx $U = (\bar{\rho}, \vec{M})$ is an iteration tree in the normal sense, except that the underlying phalanx of $\mathcal{T}$ has the form $U \oplus \mathcal{T}'$ and the roots of $\mathcal{T}$ are exactly the members of $U$. 

5
Thus, suppose that $F_\nu$ is the extender to be used at stage $\nu < \text{len}(\mathcal{T})$ of the construction of $\mathcal{T}$, and that $\eta = \text{crit}(F_\nu) < \rho_\alpha$ for some $\alpha < \text{len}(\mathcal{U})$. Let $\alpha$ be the largest ordinal such that $\rho_\alpha \geq \eta$. Then $\nu+1$ is at the second level of the tree, being an immediate successor to the root $\alpha$, and $N_{\nu+1} = \text{ult}(\mathcal{M}_\alpha, F_\nu)$.

We will use the symbol $\mathcal{\mathcal{T}_\nu}$ to denote the tree ordering on an iteration tree $\mathcal{T}$; regarding it as an ordering on either the models of the tree or on their indices depending on which is more convenient for the exposition. Thus if $\mathcal{M}_\nu$ is the $\nu$th model of the tree $\mathcal{T}$ then the two formulas $\nu \mathcal{\mathcal{T}_\nu}$ and $\mathcal{M}_\nu \mathcal{\mathcal{T}_\nu}$ mean the same thing.

**Iteration trees and comparisons.** If $\mathcal{T}$ is an iteration tree without drops on its main branch then we write $i^\mathcal{T}$ for the embedding along the main branch. If $\mathcal{T}$ is a normal iteration tree and $P$ is its last model then $i^\mathcal{T}: N_0 \rightarrow P$. If $\mathcal{T}$ is an iteration tree on a phalanx $\mathcal{U}$ then $i^\mathcal{T}: N_\nu \rightarrow P$, where $N_\nu$ is the unique member of $\mathcal{U}$ such that $N_\nu \mathcal{\mathcal{T}_P}$.

If $\mathcal{U}$ and $\mathcal{U'}$ are two phalanxes then the comparison of $\mathcal{U}$ and $\mathcal{U'}$ yields iteration trees $\mathcal{T}$ on $\mathcal{U}$ and $\mathcal{T'}$ on $\mathcal{U'}$ such that if $P$ and $P'$ are the last models of $\mathcal{T}$ and $\mathcal{T'}$ then one of the models $P$ and $P'$ is an initial segment of the other.

The following lemma gives some basic standard facts about this comparison. See, for example, [MS94b].

**Lemma 1.6.** Suppose that $\mathcal{U}$ and $\mathcal{U'}$ are compared using trees $\mathcal{T}$ and $\mathcal{T'}$, with last models $P$ and $P'$.

1. At most one of the trees $\mathcal{T}$ and $\mathcal{T'}$ has a drop on its main branch, and if $\mathcal{T}$ has such a drop then $P'$ is an initial segment of $P$.

2. Suppose that the trees $\mathcal{T}$ and $\mathcal{T'}$ are not both trivial and that there is no drop in the main branch of either tree, and let $\alpha = \min(\text{crit}(i^\mathcal{T}), \text{crit}(i^\mathcal{T'}))$. Then $i^\mathcal{T} | \mathcal{P}(\alpha) \neq i^{\mathcal{T}'} | \mathcal{P}(\alpha')$.

The proof of the conclusion of the main theorem is essentially the same under the assumption of either hypothesis (1) or (2). The next two lemmas express the facts we need about the relevant models: The notion of $\varphi$-minimality and lemma [1.8] are used for hypothesis (1), while lemma [1.9] is used for hypothesis (2).

**Definition 1.7.** If $\varphi$ is a sentence of set theory then a model $L[\mathcal{E}]$ is said to be $\varphi$-minimal if $L[\mathcal{E}] \models \varphi$, but no model $L[\mathcal{E} \upharpoonright \alpha]$ or $L_{\vartheta}[\mathcal{E} \upharpoonright \alpha]$ with $\alpha < \text{len}(\mathcal{E})$ satisfies $\varphi$. 6
Notice that only the first two clauses of the following lemma assume \( \varphi \)-minimality.

**Lemma 1.8.** Suppose that \( \mathcal{E} \) is a good sequence of extenders as in [MS94b] and \( L[\mathcal{E}] \) is fully iterable.

1. If \( i : L[\mathcal{E}] \to P \) is elementary and \( L[\mathcal{E}] \) is \( \varphi \)-minimal for some sentence \( \varphi \) then, \( P \) is also \( \varphi \)-minimal.

2. Suppose that \( L[\mathcal{E}] \) is \( \varphi \)-minimal and \( T \) is an iteration tree with last model \( P \). If there are no drops on the main branch of \( T \) then \( P \) is \( \varphi \)-minimal; while if there are any drops on the main branch of \( T \) then neither \( P \) nor any initial segment of \( P \) satisfies \( \varphi \).

3. Suppose that \( i : L[\mathcal{E}] \to P \) and \( j : L[\mathcal{E}] \to Q \) are elementary embeddings which are definable in a generic extension of \( L[\mathcal{E}] \), and that \( P \) is an initial segment of \( Q \). Then \( P = Q \) and \( i = j \).

4. More generally, suppose that the elementary embeddings

\[
L[\mathcal{E}] \xrightarrow{k} L[\mathcal{E}'] \xrightarrow{i} P \xrightarrow{j} L[\mathcal{E}]
\]

are definable in a generic extension of \( L[\mathcal{E}] \) and that \( k \) is generated by \( \rho = \min(\text{crit}(i), \text{crit}(j)) \), that is,

\[
L[\mathcal{E}'] = \{ k(f)(\nu) : f \in L[\mathcal{E}] \text{ and } \nu < \rho \}.
\]

Then \( i = j \).

5. Suppose that \( \mathcal{M} \) is a mouse with projectum \( \alpha \) which agrees with \( L[\mathcal{E}] \) up to \( \alpha \), and that the phalanx \( (L[\mathcal{E}], (\alpha, \mathcal{M})) \) is fully iterable. If \( \mathcal{M} \) is a member of a generic extension of \( L[\mathcal{E}] \) then \( \mathcal{M} \) is a member of \( L[\mathcal{E}] \).

**Proof.** The proof of clause (1) is immediate, and clause (2) can be proved by a straightforward induction on the length of the iteration tree \( T \).

Suppose for the sake of contradiction that clause (3) is false. The assertion that clause (3) is false is a first order statement \( \varphi \) over \( L[\mathcal{E}] \), so we can suppose that \( L[\mathcal{E}] \) is \( \varphi \)-minimal. Now pick a partial order \( \mathbb{P} \in L[\mathcal{E}] \), a \( L[\mathcal{E}]-\text{generic} \) set \( G \subseteq \mathbb{P} \) and embeddings \( i : L[\mathcal{E}] \to P \) and \( j : L[\mathcal{E}] \to Q \) in \( L[\mathcal{E}][G] \) witnessing the failure of clause (3). Let \( x \) be the least set, in the
order of construction of $L[\mathcal{E}]$, such that $i(x) \neq j(x)$. We may suppose that $P, G$ and $i$ and $j$ were chosen so that $x$ is as small as possible; thus $x$ is definable in $L[\mathcal{E}]$.

Then $P$ and $Q$ are both $\varphi$-minimal by clause (1), and hence $P = Q$. But then $i(x) = j(x)$ since $x$ is definable in $L[\mathcal{E}]$. This contradicts the choice of $x$ and hence completes the proof of clause (3).

Clause (4) follows from clause (3): every member $x$ of $L[\mathcal{E}']$ can be written in the form $x = k(f)(a)$ where $f \in L[\mathcal{E}]$ and $a \in [\alpha]^{<\omega}$. Then

$$i(x) = i(k(f)(a)) = (i(k))(i(a)) = (j(k))(j(a)) = j(k(f)(a)) = j(x),$$

since $i k = j k$ by clause (3) and $i(a) = j(a) = a$.

We can use standard arguments to prove clause (5) from clauses (1–4). We will give a fairly complete proof here in order to remind the reader of the techniques which will be applied later in the paper in slightly different contexts. As the following diagram indicates, we begin by comparing $L[\mathcal{E}]$ with the phalanx $(\alpha, (L[\mathcal{E}], \mathcal{M}))$, using trees $T$ on $L[\mathcal{E}]$ and $U$ on $(\alpha, (L[\mathcal{E}], \mathcal{M}))$. We use wavy arrows in the diagram since we do not know whether the indicated embeddings exist:

$$L[\mathcal{E}] \sim T \sim P$$

$$(\alpha, (L[\mathcal{E}], \mathcal{M})) \sim U \sim Q$$

Thus the first two models of the tree $U$ are $N_0 = L[\mathcal{E}]$ and $N_1 = \mathcal{M}$. The tree $U$ has two roots $0$, and $1$, which means that $0$ and $1$ are incomparable in the tree ordering $\leq_U$ of $U$, while for every $\nu < \text{len}(U)$ either $0 \leq_U \nu$ or $1 \leq_U \nu$. We will say for short that every model $\mathcal{M}_\nu$ of $U$ is either above $L[\mathcal{E}]$ or above $\mathcal{M}$.

The ordinal $\alpha$ is used as if it were the length of an extender used to obtain $N_1$. This means that if $\nu < \text{len}(U)$ then $0$ is an immediate predecessor of $\nu + 1$ in $U$, so that $N_{\nu+1} = \text{ult}(L[\mathcal{E}], F_\nu)$, if and only if crit($F_\nu$) $< \alpha$, where $F_\nu \in N_\nu$ is the extender which is to be used to define $N_{\nu+1}$.

First, notice that if the last model $Q$ of $U$ lies above $\mathcal{M}$ and is an initial segment of $P$ then $\mathcal{M} \in L[\mathcal{E}]$, as required. To see this, first note that since $Q$ is an initial segment of $P$, lemma [1.1](1) implies that there is no drop on the main branch of $U$, so that the embedding $i'^d$ is defined. Furthermore $i'^d|\alpha$
is the identity since $Q$ is above $(\alpha, \mathcal{M})$ in $\mathcal{U}$. Hence the master code $A$ of $\mathcal{M}$ is still definable in $Q$, and hence is a member of $P$. But since $L[\mathcal{E}]$ and $\mathcal{M}$ agree up to $\alpha$, it follows that $A \in L[\mathcal{E}]$, and thus $\mathcal{M} \in L[\mathcal{E}]$ since it is definable from $A$.

Thus it is sufficient to show that $Q$ lies above $\mathcal{M}$ in $\mathcal{U}$, and that $Q$ is an initial segment of $P$. We will first show that $Q$ lies above $\mathcal{M}$. Suppose to the contrary that $Q$ lies above $L[\mathcal{E}]$. Then there is no drop in the main branch of either tree: suppose for example that there is a drop in the main branch of $\mathcal{T}$. Then no initial segment of $P$ satisfies $\varphi$. But $Q$ is an initial segment of $P$ and hence also fails to satisfy $\varphi$. This is a contradiction since the embedding $i^\mathcal{U}: L[\mathcal{E}] \to Q$ is defined and so $Q \models \varphi$ by elementarity. Thus the main branch of $\mathcal{T}$ does not drop, and the same argument shows that the main branch of $\mathcal{U}$ does not drop either. It follows by clause (3) that $i = j$, but this contradicts clause 2 of lemma 1.6. Thus $Q$ lies above $\mathcal{M}$ in $\mathcal{U}$.

Now we can finish the proof by showing that $Q$ is an initial segment of $P$. Suppose to the contrary that $P$ is a proper initial segment of $Q$. This implies that there are no drops on the main branch of $\mathcal{T}$, so $P$ is a proper class. Then $Q$ is a proper class, and since it lies above $\mathcal{M}$, which is a set, it follows that $\text{len}(\mathcal{U}) = \text{Ord}$ and that there is a closed unbounded class $C$ of ordinals which are critical points of the embeddings along the main branch of $\mathcal{U}$, and hence are inaccessible in $Q$. Furthermore there is a closed unbounded subclass $C' \subset C$ such that $i^\mathcal{T} \upharpoonright \alpha \subset \alpha$ for $\alpha \in C'$, and since the universe is a set generic extension of $L[\mathcal{E}]$ there is a closed unbounded subclass $C'' \subset C'$ which is definable in $L[\mathcal{E}]$. If $\alpha$ is the $\omega$th member of $C''$ then $\text{cf}(\alpha) = \omega$ in $L[\mathcal{E}]$, and hence in $P$. This is a contradiction because $\text{cf}^Q(\alpha) = \alpha$, but $Q$ contains $P$.

This completes the proof of lemma 1.8.

The next lemma is used instead of lemma 1.8 to prove the conclusion of the main theorem from hypothesis (2). In Steel’s terminology, it asserts that the initial segments of the core model are very sound.

Lemma 1.9 (Steel [Ste96]). Let $K = L[\mathcal{F}]$ be the Steel core model and let $\lambda$ be any ordinal. Then there is a model $W = L[\mathcal{E}]$ such that

1. $\mathcal{E} \upharpoonright \lambda = \mathcal{F} \upharpoonright \lambda$.

2. (W is universal) If $W$ is compared with any iterable phalanx $\mathcal{U}$, then the last model of the tree on $\mathcal{U}$ is an initial segment of the last model
of the tree on \( W \). Furthermore, if \( i: W \to W' \) is elementary and \( W' \) is iterable then \( W' \) is also universal.

3. In particular, if \( \mathcal{M} \) is a mouse with projectum \( \alpha \leq \lambda \) which agrees with \( W \) up to \( \alpha \), and if the phalanx \( (\alpha, (W, \mathcal{M})) \) is fully iterable, then \( \mathcal{M} \) is an initial segment of \( W = L[\mathcal{E}] \), that is, \( \mathcal{M} = J_\nu[\mathcal{E}] \) for some ordinal \( \nu \).

4. If \( i, j: W \to P \) for some iterable model \( P \) then \( i|K_\lambda = j|K_\lambda \).

2 Proof of the main theorem

Notation and Summary

As pointed out earlier, it is sufficient to prove the conclusion of theorem 1.3 from hypotheses (1) and (2). For the rest of this paper we assume that \( W = L[\mathcal{E}] \) is an iterable model, that there is no iterable class model with a Woodin cardinal, and that \( \delta \leq \kappa \) are cardinals in \( W \) such that \( \delta \) is regular and uncountable and \( \kappa \) is \( \delta \)-Jónsson. Furthermore we assume for hypothesis (1) that \( V \) is a generic extension of \( W \), so that we can use lemma 1.8, and that \( W \) is \( \varphi \)-minimal for the assertion that the conclusion of the main theorem fails in some generic extension of \( W \). We assume for hypothesis (2) that the core model \( K \) exists, and that \( W \) is a model agreeing with \( K \) up to \( \kappa + \) which satisfies the conditions of lemma 1.9. In either case we will show that \( \kappa \) is \( \delta \)-Erdős in \( W \). This gives a direct proof of the conclusion of the theorem from hypothesis (2), and a proof by contradiction from hypothesis (1).

Since we are trying to show that \( \kappa \) is \( \delta \)-Erdős in \( L[\mathcal{E}] \), let us fix an arbitrary structure \( A \in L[\mathcal{E}] \) in a countable language with universe \( \kappa \), and let \( C \in L[\mathcal{E}] \) be a closed unbounded subset of \( \kappa \). We will find a set \( D \in L[\mathcal{E}] \) of normal indiscernibles for \( A \) such that \( \delta \not\subseteq X \) but \( X \cap \kappa \) has order-type \( \delta \).

**Proposition 2.1.** There is a set \( X \) satisfying \( \{ \delta, \kappa, C, A \} \subset X \) and \( (X, \mathcal{E}) \prec (H_\lambda, \mathcal{E}) \) such that \( \delta \not\subseteq X \) but \( X \cap \kappa \) has order-type \( \delta \).

**Proof.** First take an elementary substructure \( (X^*, \mathcal{E}) \) of \( (H_\lambda, \mathcal{E}) \) such that \( \{ \delta, \kappa, C, A \} \cup \kappa \subset X^* \) and \( |X^*| = \kappa \). Now let \( f: \kappa \cong X^* \) and use \( f \) to code \( (X^*, \mathcal{E}) \) into a structure \( \mathcal{B} \) with universe \( \kappa \) such that if \( Z \subset \kappa \) is the universe of an elementary substructure of \( \mathcal{B} \) then \( f"Z \) is the universe of an elementary substructure of \( (X^*, \mathcal{E}) \) with the property that \( \kappa \cap f"Z = Z \) and \( \{ \delta, \kappa, C, A \} \subset f"Z \).
If \( \delta = \kappa \) then \( \kappa \) is Jónsson and hence \( B^* \) has a elementary substructure with universe \( Z \) such that \( |Z| = \kappa \) but \( Z \not\in \kappa \). If \( \delta < \kappa \) then \( \kappa \) is \( \delta \)-Jónsson and there is an elementary substructure \( B \) with universe \( Z \not\in \delta \) such that order type \( Z = \delta \), so that order type \( Z \cap \delta < \delta \) and hence \( \delta \not\in B^* \). In either case \( X = f"Z \) is the universe of an elementary substructure of \((X^*, \mathcal{E})\) with the required properties.

Now let \( X \) be as given by the proposition and let \( N \) be a transitive set with \( \pi: N \cong X \), so that \( \text{crit}(\pi) < \delta \). This situation is similar to the situation at the start of the proof of the weak covering lemma [MSS94], and it will be useful to compare the two proofs. Our substructure \( X \) differs from that used in the proof of the covering lemma in two significant ways. The first is that we cannot assume that \( "X \subseteq X \), as in the proof of the covering lemma (nor have we been able to use Fodor’s lemma to avoid countable closure, as Dodd and Jensen do) The other difference partially counterbalances the first: \( \delta = \pi^{-1}(\kappa) \) is a regular cardinal. The closure condition \( "X \subseteq X \) is used several times in the proof of the covering lemma, and we will deviate from the proof of the covering lemma only when it is necessary to work around this lack of closure.

As in the proof of the covering lemma, we set \( \overline{W} = \pi^{-1}(W) \), and compare the two models \( W \) and \( \overline{W} \) using iteration trees \( T \) on \( W \) and \( U \) on \( \overline{W} \), continuing this comparison until the final models of the two trees agree up to \( \pi^{-1}(\kappa) \). At this point the proof of the weak covering lemma uses a rather complicated induction to reach two important conclusions: (i) the sequence \( \overline{W} \) is never moved in the comparison, so that the tree \( U \) is trivial, and (ii) if \( \mathcal{M}_\phi \) is the final model of \( T \) then the model \( \mathcal{R}_\phi = \text{ult}(\mathcal{M}_\phi, \pi, \kappa) \) is iterable.

We do not reach either of these two conclusions. The inability to prove that \( U \) is trivial is merely a nuisance; it will be dealt with in the proof but for clarity we ignore it in this summary. Our inability to prove that \( \mathcal{R}_\phi \) is iterable, on the other hand, requires a fundamental change in the proof. To see what changes are necessary, let us look at the two basic cases which come up in the proof of the covering lemma:

**Case 1 (\( \mathcal{M}_\phi \) is a set).** In this case there must be a drop somewhere along the main branch \( b \) of \( T \), so that there is a \( \nu < \phi \) in \( b \) such that the \( \nu \)th model \( \mathcal{M}_\nu \) of \( T \) has cardinality less than \( \delta \). In this case we can use the fact that \( \pi^{-1}(\kappa) = \delta \), a regular cardinal, to show that \( T \) has length \( \delta \) and hence generates the required set of indiscernibles of order type \( \delta \). There is no need for \( \mathcal{R}_\phi \) to be iterable in this case.
Case 2 ($\mathcal{M}_\phi$ is a weasel, that is, a proper class). In this case we will use an argument taken from the proof of the weak covering lemma in [MSS 94] to show that $i_{0,\phi}^\tau(\rho) > \delta$, where $i_{0,\phi}^\tau$ the embedding along the main branch of $\mathcal{T}$ and $\rho = \text{crit}(i_{0,\phi}^\tau)$. Since $\mathcal{T}$ only uses extenders of length less than $\delta$ we can again use the fact that $\delta$ is a regular cardinal to conclude that $\mathcal{T}$ has length $\delta$ and hence generates the desired set of indiscernibles. The problem is that the argument taken from the covering lemma depends heavily on the assumption that $\mathcal{R}_\phi = \text{ult}(\mathcal{M}_\phi, \pi, \kappa)$ is iterable. We will work around this difficulty by noticing that if $\mathcal{R}_\phi$ is not iterable then this failure must have been evidenced in some earlier model $\mathcal{M}_\nu$ on the main branch of the tree $\mathcal{T}$, and in fact in a structure $Q < \mathcal{M}_\nu$ with $|Q| < \delta$. In this case we will modify the construction of the tree $\mathcal{T}$ by dropping at stage $\nu$ to the premouse $Q$.

If there is any drop on the main branch of the modified tree $\mathcal{T}$ then we are in case (1) and there is no need for $\mathcal{R}_\phi$ to be iterable. On the other hand, if there is no such drop then $\mathcal{R}_\phi$ is iterable and we can use the argument from the covering lemma.

For lack of a better term will will call the modified tree $\mathcal{T}$ a quasi-iteration tree.

We are now almost ready to begin the first half of the actual proof, which is the construction of the trees $\mathcal{T}$ on $W$ and $\mathcal{U}$ on $\overline{W}$. We also use the embedding $\pi$ to copy $\mathcal{U}$ to a tree $\tilde{\mathcal{U}}$ on $W$, using the shift lemma of Martin and Steel [MS94a]. The trees $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are ordinary iteration trees in the sense of [MS94b] and hence present no difficulties concerning iterability; however the tree $\mathcal{T}$ is not a standard iteration tree and hence requires special treatment. The verification of the iterability of $\mathcal{T}$ is in lemma 2.2, which is one of two lemmas which are proved after the description of the construction of the trees.

The following diagram gives the maps between the trees $\mathcal{T}$, $\mathcal{U}$ and $\tilde{\mathcal{U}}$:

\[
\begin{array}{ccc}
\mathcal{T} : & M_\nu & \xrightarrow{i_{\nu,\nu'}} M_{\nu'} \\
\mathcal{U} : & N_\nu & \xrightarrow{j_{\nu,\nu'}} N_{\nu'} \\
\pi_\nu & \xrightarrow{\pi_\nu} & \pi_{\nu'} \\
\tilde{\mathcal{U}} : & \tilde{N}_\nu & \xrightarrow{\tilde{j}_{\nu,\nu'}} \tilde{N}_{\nu'}
\end{array}
\]

The horizontal maps are only defined if $\nu'$ is above $\nu$ in the relevant tree and there is no drop in the branch between $\nu$ and $\nu'$. 

12
The construction of $\mathcal{T}$, $\mathcal{U}$ and $\tilde{\mathcal{U}}$.

For the rest of this proof $\pi : N \cong X \prec (H_\lambda, \mathcal{E})$ will be as in lemma 2.1. We will write $\overline{W} = \pi^{-1}[X \cap W]$. The trees $\mathcal{T}$, $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are defined by recursion on their lengths. We are not using padded iteration trees, and hence the trees need not have the same length. We will write $W = \pi^{-1}[\mathcal{X}]$. The trees $\mathcal{M}_\nu$ for the $\nu$th model of $\mathcal{T}$, and we will write $\mathcal{N}_\nu$ and $\tilde{\mathcal{N}}_\nu$ for the $\nu$th model of $\mathcal{U}$ and $\tilde{\mathcal{U}}$ respectively. The construction starts with $\mathcal{M}_0 = W$, $\mathcal{N}_0 = \overline{W}$, $\tilde{\mathcal{N}}_0 = W$, and $\pi_0 = \pi : \mathcal{N}_0 \to \tilde{\mathcal{N}}_0$.

Suppose that during the course of the recursion we have already defined an initial segment $\mathcal{T} \upharpoonright \phi$ of $\mathcal{T}$ and initial segments $\mathcal{U} \upharpoonright \theta$ and $\mathcal{\tilde{U}} \upharpoonright \theta$ of $\mathcal{U}$ and $\tilde{\mathcal{U}}$ respectively. At the next stage of the recursion we will extend one or both of the trees $\mathcal{T}$ and $\mathcal{U}$. There are three cases:

Case 1 (At least one of $\phi$ or $\theta$ is a limit ordinal). If $\theta$ is a limit ordinal then we extend the tree $\mathcal{U} \upharpoonright \theta$ by taking the unique well founded branch $b$ of $\mathcal{U} \upharpoonright \theta$. This unique well founded branch exists because $\mathcal{U}$ is a standard iteration tree on the iterable model $\overline{W}$. The tree $\tilde{\mathcal{U}}$ also has a unique well founded branch $\tilde{b}$, which must be the branch corresponding to $b$ since otherwise the preimage of $\tilde{b}$ would be a second well founded branch through $\mathcal{U}$. Thus we can define $\pi_\theta : \mathcal{N}_\theta \to \tilde{\mathcal{N}}_\theta$ to be the direct limit of the maps $\pi_\nu : \mathcal{N}_\nu \to \tilde{\mathcal{N}}_\nu$ for $\nu \in b$. This direct limit is defined since $\mathcal{U}$ and $\tilde{\mathcal{U}}$ have the same underlying tree and the maps $\pi_\nu$ commute with the respective tree embeddings.

If $\phi$ is a limit ordinal, then we similarly have to pick a well founded branch $b$ of $\mathcal{T} \upharpoonright \phi$. Since $\mathcal{T}$ is not a standard iteration tree, we cannot use the general theory as in the last paragraph. Instead we use the following lemma, the proof of which is deferred until after the construction of the trees $\mathcal{T}$ and $\mathcal{U}$.

Lemma 2.2. There is a function $b$ such that if $\mathcal{T}$ is defined at limit ordinals $\phi$ by setting $[0, \phi] = b(\mathcal{T} \upharpoonright \phi)$ then every model $\mathcal{M}_\nu$ of $\mathcal{T}$ is well founded.

This concludes case 1. In the remaining cases both $\phi$ and $\theta$ are successor ordinals, say $\phi = \gamma + 1$ and $\theta = \gamma' + 1$. Let $\alpha$ be the largest ordinal such that $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$ agree up to $\alpha$, and let $\tilde{\alpha} = \sup \pi_{\gamma'}^{\alpha}$ and $\mathcal{R} = \hbox{ult}(\mathcal{M}_\gamma, \pi_{\gamma'}, \tilde{\alpha})$. Case 2 ($\mathcal{M}_\phi$ is a weasel and the phalanx $\tilde{\mathcal{U}} \oplus \mathcal{R}$ is not iterable). This is the case in which the definition of $\mathcal{T}$ differs from a normal comparison. We will define $\mathcal{M}_\phi$ to be a premouse of cardinality less than $\delta$, thus ensuring by brute force that if $\phi$ is on the main branch of $\mathcal{T}$ then the iteration along this
main branch will yield a set of indiscernibles of cardinality \( \delta \). The following lemma will be proved later along with lemma 2.4.

**Lemma 2.3.** If there is an ill-behaved tree on \( \tilde{U} \oplus \tilde{\alpha} \), then there is an elementary substructure \( Q \) of \( M_\phi \), with \( \alpha \subset Q \) and \( |Q| = |\alpha| \), such that there is an ill-behaved tree on \( \tilde{U} \oplus \tilde{\alpha} \) ult(\( Q, \pi_\theta, \tilde{\alpha} \)).

The case hypothesis asserts that the hypothesis of the lemma is true, and we define \( M_\phi \) to be the transitive collapse of the elementary substructure \( Q \) of \( M_\gamma \) given by the lemma. We extend the tree ordering by letting \( \nu \prec^T \phi \) if and only if \( \nu < \phi \) and \( \nu \prec^T \gamma \). We will regard this as a drop and hence leave the embedding \( i_{\nu,\phi} \) undefined for \( \nu \prec^T \phi \). The ordinal \( \rho_\phi \) associated with this stage of the tree (which would be \( \text{len}(E_\phi) \) in the normal successor case) is defined to be the larger of \( \sup_{\nu<\phi} \rho_\nu \) and the least ordinal \( \beta \) such that there is an ill-behaved tree on the phalanx \( \tilde{U} \oplus \tilde{\beta} \) ult(\( M_\phi, \pi_\theta, \tilde{\beta} \)), where \( \tilde{\beta} = \sup \pi_\theta “ \beta \).

We will call the node \( \phi \) of \( T \) a special successor node, and we will call \( \{\gamma, \phi\} = \{\gamma, \gamma + 1\} \) a special pair. Notice that \( \gamma \) and \( \gamma + 1 \) have the same set of predecessors in \( T \), even if \( \gamma \) is a limit ordinal. This is impossible in a standard iteration tree.

**Case 3 (Neither case 1 nor case 2 holds).** This case is completely standard. Let \( \mathcal{E}_\gamma \) and \( \mathcal{F}_\gamma \) be the extender sequences in \( M_\gamma \) and \( N_\gamma \) respectively, and let \( \beta \) be the least ordinal such that \( \beta \in \text{domain}(\mathcal{E}_\phi) \cup \text{domain}(\mathcal{F}_\theta) \), and \( (\mathcal{E}_\phi)_\beta \neq (\mathcal{F}_\theta)_\beta \) if \( \beta \in \text{domain}(\mathcal{E}_\phi) \cap \text{domain}(\mathcal{F}_\theta) \). If \( \beta \) is in the domain of \( \mathcal{E}_\gamma \) then extend \( T \) as follows: let \( \nu \) be the least ordinal such that \( \rho_\nu > \text{crit}(E) \), let \( M_\gamma^* \) be the largest initial segment of \( M_\nu \) such that \( E \) is an extender on \( M_\nu \), and set \( M_\theta = \text{ult}(M_\gamma^*, E) \). Similarly if \( B \) is in the domain of \( \mathcal{F}_\gamma \) then extend \( U|\theta \) by setting \( N_\theta = \text{ult}(N_\nu^*, F) \) and use the shift lemma to define \( \tilde{N}_\theta = \text{ult}(\tilde{N}_\nu^*, \pi_\theta(F)) \) in \( \tilde{U} \), where \( \tilde{N}_\nu^* = \tilde{N}_\nu \) if \( N_\nu^* = N_\nu \), and \( \tilde{N}_\theta^* = \pi_\nu(N_\theta^*) \) if \( N_\theta^* \) is a proper initial segment of \( N_\nu \).

**Proof of lemmas 2.2 and 2.3**

This completes the construction of the trees \( T, U \) and \( \tilde{U} \) except for the proof of lemmas 2.2 and 2.3.

**Proof of lemma 2.2.** We define an auxiliary iteration tree \( \tilde{T} \), together with an embedding from \( T \) to \( \tilde{T} \). Since \( \tilde{T} \) is a standard iteration tree on \( W \), all of its models are well founded. We show that at each limit ordinal \( \phi < \text{len}(T) \)
there is a unique branch of $T$ which maps to the well founded branch of $\overline{T}$, and we will take $b(\overline{T}|\phi)$ to be this unique branch.

The embedding consists of a map $\sigma: \text{len}(T) \to \text{len}(\overline{T})$, together with embeddings $t_\nu: M_\nu \to \overline{M}_{\sigma(\nu)}$ for each ordinal $\nu < \text{len}(T)$. The map $\sigma$ is one to one and order preserving, with the exception that both members of a special pair of $T$ map to the same node of $\overline{T}$. The construction is by recursion on the nodes $\phi$ of $T$. Suppose that the construction has been carried out up to $\phi$, so we have already constructed $T|\phi$ together with the map $\sigma|\phi: \phi \to \theta$ and the maps $t_\alpha$ for $\alpha < \phi$, and assume that the following conditions are satisfied for all nodes $\alpha, \beta < \phi$:

1. $\sigma(\alpha) \leq_{\overline{T}} \sigma(\beta)$ if and only if either $\alpha \leq T \beta$ or else there is $\alpha' \leq T \beta$ such that $\{\alpha, \alpha'\}$ is a special pair.
2. If $\alpha < \beta$ then $t_\alpha|\rho_\alpha = t_\beta|\rho_\alpha$.
3. If there is a normal drop in $[0, \alpha)_T$, so that $M_\alpha$ is an iterate of a mouse, then $t_\alpha$ is a weak $\text{deg}(\alpha)$-embedding in the sense of [MS94b], and otherwise (if $M_\alpha$ is a weasel or else the only drop has been at a special successor) $t_\alpha$ is an elementary embedding.

**Case 1** ($\phi = \gamma+1$ is a standard successor node of $T$). This case is standard. Set $\sigma(\phi) = \sigma(\gamma) + 1$, set $\overline{E}_{\sigma(\gamma)} = t_\gamma(E_\gamma)$ and set $\overline{M}_{\sigma(\phi)} = \text{ult}\left(\overline{M}_{\sigma(\alpha)}, \overline{E}_{\sigma(\alpha)}\right)$ where $\overline{M}_{\sigma(\alpha)}$ is defined as usual. Standard arguments show that the induction hypotheses are true at $\phi$.

**Case 2** ($\phi = \gamma + 1$ is a special successor node of $T$). In this case we set $\sigma(\phi) = \sigma(\gamma)$. By the construction of $T$ there is an elementary embedding $s_\gamma: M_\phi \to M_\gamma$, so we can set $t_\phi = t_\gamma s_\gamma$. Again, it is straightforward to verify that the induction hypotheses are true at $\phi$.

**Case 3** ($\phi$ is a limit ordinal). Since clause (1) holds for $\alpha, \beta < \phi$ we have $\phi = \sup \sigma^{\nu} \phi$, so we can set $\sigma(\phi) = \phi$. Since $\overline{T}|\phi$ is a standard iteration tree there is a unique well founded branch $\overline{b}$ through $\overline{T}|\phi$. Extend $\overline{T}$ as usual by letting $\overline{M}_\nu = \overline{M}_{\overline{b}}$, the direct limit of $\overline{T}$ along the branch $\overline{b}$. Now set

$$b = \bigcup \{ [0, \nu)_T : \sigma(\nu) \in \overline{b} \} = \{ \nu : \exists \nu' \in \sigma^{-1}(\overline{b}) (\nu <_T \nu') \}.$$ 

First notice that $b \subset \sigma^{-1}(\overline{b})$, since if $\nu \in b$ then there is $\nu' \in \sigma^{-1}(\overline{b})$ such that $\nu <_T \nu'$, so $\sigma(\nu) <_{\overline{T}} \sigma(\nu') \in \overline{b}$. To see that $b$ is linearly ordered, suppose
\[ \nu_0, \nu_1 \in b \text{ with } \nu_0 < \nu_1. \] If \( \nu_0 \not< \tau \nu_1 \) then clause (1) implies that there is \( \nu'_0 \) such that \( \{ \nu_0, \nu'_0 \} \) is a special pair and \( \nu'_0 \leq \tau \nu_1 \). Since \( \nu_0 \in b \) there is \( \nu'_1 \) such that \( \sigma(\nu'_1) \in \bar{b} \) and \( \nu_0 < \tau \nu'_1 \), but this is impossible because in this case \( \sigma(\nu_1) \) and \( \sigma(\nu'_1) \) would be incomparable in \( \bar{T} \).

Hence \( b \) is a cofinal branch of \( T|\phi \), so we can define \( M_\phi = M_b \), the limit of \( T \) along the branch \( b \). Now define \( t_\phi \) to be the limit of the embeddings \( t_\nu \) for \( \nu \in b \). Again, it is straightforward to verify the induction hypotheses.

This completes all three cases, and hence the proof of lemma 2.2. \( \square \)

We are now ready to finish part one of the proof by proving lemma 2.3. The proof is essentially the same as that of the following lemma, which is needed for part two of the proof, so we will combine the two proofs. This technique of applying Martin-Steel iterability theorem \[MS94a\] is due to Woodin.

**Lemma 2.4.** Suppose \( M_\phi \) and \( N_\theta \) are the last models of \( T \) and \( U \), so that \( M_\phi \) and \( N_\theta \) are matched up to \( \delta \), and suppose that \( M_\phi \) is a weasel. Then there are no ill behaved trees on the phalanx \( \tilde{U} \oplus \tilde{\delta} \uparrow \text{ult}(M_\phi, \pi_\theta, \tilde{\delta}) \), where \( \tilde{\delta} = \sup \pi_\theta \alpha \).

**Proof of lemmas 2.3 and 2.4.** For lemma 2.3 we are given models \( M_\phi \) and \( N_\theta \) from the trees \( T \) and \( U \) and an ordinal \( \alpha < \delta \) such that \( M_\phi \) and \( N_\theta \) agree up to \( \alpha \). In the case of lemma 2.4 we take \( M_\phi \) and \( N_\theta \) to be the final models of the trees \( T \) and \( U \), and we take \( \alpha = \delta \).

The map \( \pi \) was used to copy the tree \( U|\theta \) on \( L[\mathcal{E}] \) to a tree \( \tilde{U} \) on \( L[\mathcal{E}] \), with copy map \( \pi_\nu : N_\nu \rightarrow \tilde{N}_\nu \) for each model \( N_\nu \) of \( U \). Set \( \tilde{\alpha} = \sup \pi_\alpha \alpha \) and \( \tilde{\mathcal{R}} = \uparrow \text{ult}(M_\phi, \pi_\theta, \tilde{\delta}) \), and assume that there is a ill behaved tree on \( \tilde{\alpha} \oplus \tilde{\mathcal{R}} \). In case of lemma 2.3 the tree is part of the hypothesis, while for lemma 2.4 we assume the existence of such an ill behaved tree towards a proof by contradiction.

Let \( Z_1 \prec \mathcal{P}(H_\lambda) \) be an elementary substructure such that \( \{ A, C, W, X \} \subset Z_1 \), and such that

- For lemma 2.3: \( \alpha + 1 \subset Z_1 \) and \( |Z_1| = |\alpha| \). We set \( \eta = \alpha \).
- For lemma 2.4: \( Z_1 \cap \delta = \sup(Z_1 \cap \delta) = |Z_1| \). We set \( \eta = Z_1 \cap \delta \).

Notice that everything which has been mentioned is definable from members of \( Z_1 \), and hence is in \( Z_1 \). We set \( \bar{\eta} = \sup \pi^\alpha \eta \).

Now let \( \psi_1 : M_1 \cong Z_1 \) be the transitive collapse, and let \( Q = \psi_1^{-1}(M_\phi) \cong M_\phi \cap Z_1 \). The major part of the proof of these lemmas is the proof of the following claim:
Claim 2.5. There is an ill founded tree $S_1$ on $\tilde{U} \oplus \tilde{\eta}$ ult($Q, \pi_\theta, \tilde{\eta}$).

In the case of lemma 2.3, this claim is exactly what is required. Before proving claim 2.5 we show that lemma 2.4 also follows from this claim.

Proof of 2.4 from 2.5. Note that $\eta \in [0, \phi)_T$ since $[0, \phi)_T$ is closed and unbounded in $\delta$. We claim that $Z_1 \cap M_\phi \subset \text{range}(i_{\eta, \phi})$, so that the embedding $\psi_1 : Q \cong Z_1 \cap M_\phi \prec M$ can be factored

$$\psi_1 : Q \xrightarrow{k} M_\eta \xrightarrow{i_{\eta, \phi}} M_\phi$$

where $k = (i_{\eta, \phi})^{-1} \psi_1$.

To see that $Z_1 \cap M_\phi \subset \text{range}(i_{\eta, \phi})$, first notice that every member of $M_\phi$ is in range($i_{\nu, \phi}$) for some $\nu \in [0, \phi)_T$. Hence any member $x$ of $M_\phi \cap Z_1$ is in range($i_{\nu, \phi}$) for some $\nu \in [0, \phi)_T \cap Z_1 = [0, \phi)_T \cap \eta = [0, \eta)_T$, since $\eta \in [0, \phi)_T$, and hence $x \in \text{range}(i_{\eta, \phi})$.

The embedding $k : Q \rightarrow M_\eta$ induces a map

$$\tilde{k} : \text{ult}(Q, \pi_\theta, \tilde{\eta}) \rightarrow \text{ult}(M_\eta, \pi_\theta, \tilde{\eta})$$

which in turn can be used to copy $S_1$ to a tree $S_2$ on $\tilde{U} \oplus \tilde{\eta}$ ult($M_\eta, \pi_\theta, \tilde{\eta}$) which is necessarily also ill behaved. Now $\pi_\theta | \eta = \pi_\theta | \eta$, where $\mathcal{N}_\theta$ is the stage which the tree $U$ had reached at the time $M_\eta$ was being considered in tree $\mathcal{T}$. Thus $\tilde{U} \oplus \tilde{\eta}$ ult($M_\eta, \pi_\theta, \tilde{\eta}$) is the tree $\tilde{U} \oplus \tilde{\alpha}$ $\mathcal{R}$ of case (2) of the construction of the trees, so the existence of the ill behaved tree $S_2$ would have caused the tree $\mathcal{T}$ to drop at stage $\eta$. Furthermore $\rho_\eta = \eta = \bigcup_{\nu < \eta} \rho_\nu$, so the final model $M_\phi$ of the tree must be above the second member $\eta + 1$ of the special pair, rather than above $\eta$. This contradicts the assumption that $M_\phi$ is a weasel and hence completes the proof of lemma 2.4 from claim 2.5.

Proof of claim 2.3. To find $S_1$, let $Z_0 \prec Z_1$ with $\{A, C, W, X\} \subset Z_0$ and $|Z_0| = \omega$. Then $Z_0$ satisfies that there is an ill behaved tree $S$ on $\tilde{U} \oplus \tilde{\alpha}$ $\mathcal{R}$. Let $M_0 \cong Z_0$ be transitive, with maps

$$\psi : M_0 \xrightarrow{\psi_0} M_1 \xrightarrow{\psi_1} \mathcal{P}(H_\lambda).$$

Set $S_0 = \psi^{-1}(S)$. Then $M_0$ satisfies that $S_0$ is ill behaved, but we need to show that $S_0$ has no branches even in $V$:

Claim 2.6. The tree $S_0$ is ill behaved in $V$.  

17
Proof. Suppose to the contrary that there is a well founded branch through $S_0$ in $V$.

Let $F$ be the limit of the extender sequences of the models in $S$; that is, if $\xi < \text{len}(F)$ then $F | \xi$ is an initial segment of the $\nu$th model $M_\nu^S$ for every sufficiently large $\nu < \text{len}(S) = \eta$; but $F$ itself is not an initial segment of the extender sequence of any of these models. By our assumption there is no class model with a Woodin cardinal, and hence there is $\gamma$ such that $L_\gamma[F]$ satisfies that $\text{len}(F)$ is not a Woodin cardinal. By elementarity $\gamma \in M_0$.

Set $\gamma' = \psi^{-1}(\gamma)$, and let $G$ be $M_0$-generic for the Levy collapse $\text{col}(\omega, \gamma')$ of $\gamma'$ onto $\omega$. In $M_0[G]$ form the tree $A$ of attempts to find a branch through $S_0$ such that the limit along the branch either is well founded or has a well founded part of length at least $\gamma$. Because $S_0$ has a well founded branch in $V$, the tree $A$ has an infinite branch in $V$, and by the absoluteness of well order it follows that $A$ has an infinite branch in $M_0[G]$ as well. Thus $S_0$ has a well founded branch $b$ in $M_0$. By the Martin-Steel iterability theorem [MS94a] there can be at most one branch $b \in M_0[G]$ through $S_0$ which is well founded up to $\gamma$. The uniqueness of $b$, together with the homogeneity of the Levy collapse $G$, implies that $b \in M_0$. This contradicts the fact that $M_0$ satisfies that $S_0$ has no well founded branch, and hence completes the proof of claim 2.6.

In order to complete the proof of claim 2.5, and hence of lemmas 2.3 and 2.4, we copy $S_0$ to a tree on $\tilde{U} \oplus \tilde{\eta} \text{ult}(Q, \pi_\theta, \tilde{\eta})$ as follows:

The tree $S_0$ is on the phalanx

$$\psi^{-1} \left( \tilde{U} \oplus \tilde{\alpha} \text{ ult}(M_\phi, \pi_\theta, \tilde{\alpha}_1) \right).$$

If $P_\nu$ is the $\nu$th member of the phalanx (1) then $\psi_0 | P_\nu : P_\nu \to \psi_0(P_\nu)$, which is the $\psi_0(\nu)$th member of the phalanx

$$\psi_1^{-1} \left( \tilde{U} \oplus \tilde{\alpha} \text{ ult}(M_\phi, \pi_\theta, \tilde{\alpha}) \right) = \psi_1^{-1}(\tilde{U}) \oplus \tilde{\alpha}_1 \text{ ult}(Q, \psi_1^{-1}(\pi_\theta), \alpha_1)$$

where $\tilde{\alpha}_1 = \psi^{-1}(\tilde{\alpha})$. The phalanx (2) can embedded into $\tilde{U} \oplus \tilde{\eta} \text{ult}(Q, \pi_\theta, \tilde{\eta})$ by the map $\psi_1$, and by copying $S_0$ along this embedding we obtain the required ill behaved tree on $\tilde{U} \oplus \tilde{\eta} \text{ult}(Q, \pi_\theta, \tilde{\eta})$.

This completes the proof of claim 2.5 and hence of lemmas 2.3 and 2.4. \qed
The Indiscernibles Generated by \( \mathcal{T} \)

This concludes the construction of the trees \( \mathcal{T} \) and \( \mathcal{U} \), which is the first half of the proof of theorem \linebreak \ref{thm:main}, and we are now ready to use \( \mathcal{T} \) and \( \mathcal{U} \) to show that \( L[E] \) has a set of indiscernibles for the structure \( \mathcal{A} \). First, we will show that the tree \( \mathcal{T} \) has length \( \delta \) and hence generates a cofinal set of indiscernibles for its last model. Using this, we will then show that the tree \( \mathcal{U} \) does not drop along its main branch. This implies that the set \( I \) of indiscernibles from \( \mathcal{T} \) is a set of indiscernibles for every set in \( W \), and in particular for \( \pi^{-1}(\mathcal{A}) \). Finally we will show that the filter on \( L[E] \) generated by \( \pi \) is a member of \( L[E] \), and use this filter inside \( L[E] \) to construct the required set \( D \in L[E] \) of indiscernibles for \( \mathcal{A} \). The next lemma, which shows that \( \mathcal{T} \) generates a set of indiscernibles of size \( \delta \), is the main lemma of this half of the proof.

**Lemma 2.7.** The tree \( \mathcal{T} \) has length \( \delta + 1 \), and there are ordinals \( \nu < \delta \) and \( \rho < \delta \) such that \( i_{\nu,\delta} \) is defined and \( i_{\nu,\delta}(\rho) \geq \delta \).

**Proof.** Let \( \phi + 1 \) be the length of the quasi-iteration tree \( \mathcal{T} \) on \( L[E] \), and let \( \theta + 1 \) be the length of the iteration tree \( \mathcal{U} \) on \( W = \pi^{-1}(L[E]) \). Since \( \delta \) is a cardinal, the proof of the comparison lemma implies that \( \phi, \theta \leq \delta \).

**Claim 2.8.** The final models \( M_\phi \) and \( N_\theta \) of the trees \( \mathcal{T} \) and \( \mathcal{U} \) have size at least \( \delta \), and hence agree up to \( \delta \).

**Proof.** Suppose to the contrary that one of these models has size less than \( \delta \). By the construction of the trees \( \mathcal{T} \) and \( \mathcal{U} \) it follows that that model is an initial segment of the final model of the other tree. In addition, that tree must drop along its main branch, since the each of the roots \( W \) and \( W \) of the two trees have cardinality at least \( \delta \). Lemma \ref{lem:comparison}(1) asserts that this is impossible in any comparison using ordinary iteration trees. We will verify that this is still impossible for the modified comparison using the quasi-iteration tree \( \mathcal{T} \).

The reason why lemma \ref{lem:comparison}(1) is true is that if there is an (ordinary) drop at some node \( \nu + 1 \) in the main branch of one of the trees, say \( \mathcal{T} \), then there is a subset \( x \) of the projectum \( \rho \) of \( M_{\nu+1} \) which is definable in \( M_{\nu+1} \) and is not a member of the corresponding model of the other tree. If we take \( \nu + 1 \) to be last place along the main branch where such a drop occurs, then \( i_{\nu+1,\theta} \) exists, and \( i_{\nu+1,\theta}|_{\rho} \) is the identity. Then \( x \) is definable in \( M_\theta \) and not a member of \( N_\phi \), so \( M_\theta \) cannot be a proper initial segment of \( N_\phi \).

This argument is unaffected by the use of the quasi-iteration tree \( \mathcal{T} \) instead of a normal iteration tree. It follows that \( \mathcal{T} \) must have a special drop
\(\nu + 1\), but no normal drops, on its main branch, and \(\mathcal{M}_\theta\) must be an initial segment of \(\mathcal{N}_\phi\). From the definition of a special drop it follows that the phalanx \(\mathcal{U} \oplus \tilde{\rho} \text{ult}(\mathcal{M}_{\nu+1}, \pi_{\nu'}, \tilde{\rho})\) is not iterable, where \(\nu'\) is the stage in \(\mathcal{T}\) corresponding to \(\nu + 1\) in \(\mathcal{T}\), and \(\tilde{\rho} = \rho_{\nu+1}^\pi\). We will use this lack of iterability just like the set \(x\) in the case of a normal drop: it implies that the phalanx \(\mathcal{U} \oplus \tilde{\delta} \text{ult}(\mathcal{M}_\theta, \pi_\phi, \tilde{\delta})\) is not iterable, where \(\nu'\) is the stage in \(\mathcal{U}\) corresponding to \(\nu + 1\) in \(\mathcal{T}\), and \(\tilde{\rho} = \rho_\phi^\pi\). We will use this lack of iterability just like the set \(x\) in the case of a normal drop: it implies that the phalanx \(\mathcal{U} \oplus \tilde{\delta} \text{ult}(\mathcal{N}_\phi, \pi_\phi, \tilde{\delta})\) is not iterable, but if \(\mathcal{M}_\theta\) is an initial segment of \(\mathcal{N}_\phi\) then \(\text{ult}(\mathcal{M}_\theta, \pi_\phi, \tilde{\delta})\) can be embedded into \(\text{ult}(\mathcal{N}_\phi, \phi_\phi, \tilde{\delta})\), so that the phalanx \(\mathcal{U} \oplus \tilde{\delta} \text{ult}(\mathcal{N}_\phi, \pi_\phi, \tilde{\delta})\) is not iterable. This is absurd, since the later phalanx is actually the standard iteration tree \(\tilde{\mathcal{U}}\) on the iterable model \(\mathcal{W}\).

This contradiction completes the proof of the claim. \(\square\)

Thus the models \(\mathcal{M}_\phi\) and \(\mathcal{N}_\theta\) agree up to \(\delta\). If there is a drop of any kind in the main branch of \(\mathcal{T}\) then \(|\mathcal{M}_\nu| < \delta\) for every sufficiently large \(\nu\) in the main branch of \(\mathcal{T}\). Since \(|\mathcal{M}_\phi| = \delta\) while all of the extenders in \(\mathcal{T}\) have length less than \(\delta\), it follows that \(\phi = \delta\) as required.

Thus we can assume for the remainder of the proof of this lemma that there are no drops in the main branch of \(\mathcal{T}\), and hence \(\mathcal{M}_\phi\) is a weasel. If we set \(\tilde{\delta} = \bigcup \pi_\theta \mu^\mu\) and \(\mathcal{R} = \text{ult}(\mathcal{M}_\delta, \pi_\theta, \tilde{\delta})\) then lemma 2.4 asserts that there are no ill behaved trees on the phalanx \(\tilde{\mathcal{U}} \oplus \tilde{\delta} \mathcal{R}\), so we can compare the models \(\mathcal{R}\) and \(\mathcal{N}_\theta\), using an iteration tree \(\tilde{\mathcal{U}} \oplus \tilde{\delta} \mathcal{R} \oplus \mathcal{W}\) on the phalanx \(\tilde{\mathcal{U}} \oplus \tilde{\delta} \mathcal{R}\) and an iteration tree \(\tilde{\mathcal{U}} \oplus \mathcal{V}\) on the phalanx \(\tilde{\mathcal{U}}\). The comparison takes place as if \(\tilde{\mathcal{U}}\) were simply a phalanx; however we will later make use of the iteration tree structure on \(\tilde{\mathcal{U}}\), regarding \(\tilde{\mathcal{U}} \oplus \tilde{\delta} \mathcal{R} \oplus \mathcal{W}\) as a (nonstandard) iteration tree with two roots \(L[\mathcal{E}]\) and \(\mathcal{R}\).

**Claim 2.9.**

1. There are no drops on the main branch of either of the trees \(\tilde{\mathcal{U}} \oplus \tilde{\delta} \mathcal{R} \oplus \mathcal{W}\) or \(\tilde{\mathcal{U}} \oplus \mathcal{V}\).

2. The trees \(\tilde{\mathcal{U}} \oplus \tilde{\delta} \mathcal{R} \oplus \mathcal{W}\) and \(\tilde{\mathcal{U}} \oplus \mathcal{V}\) have the same last model.

3. The final model of \(\tilde{\mathcal{U}} \oplus \tilde{\delta} \mathcal{R} \oplus \mathcal{W}\) is above \(\mathcal{R}\), rather than above \(\mathcal{N}_0\).

**Proof.** The proof of this claim uses the techniques of lemma 2.8(5). The proofs of clauses (1) and (2) are the same as the proof of the corresponding facts in lemma 2.8(5), using the fact that there is no drop in \(\mathcal{T}\) and hence there is an elementary embedding from \(L[\mathcal{E}]\) into \(\mathcal{R}\). It follows that \(\mathcal{R}\) is universal if hypothesis (1) of the main theorem is true, and \(\mathcal{R}\) is \(\phi\)-minimal if hypothesis (2) of the main theorem is true. If there is no drop in the main
branch of $\tilde{U} \oplus \delta R \oplus W$ then the same will be true of the final model $P$ of $W$, since there is an elementary embedding from either $L[\mathcal{E}]$ or $R$ into $P$.

The proof of clause (3) is similar to the proof in lemma 1.8(5) of the fact that $Q$ is not above $L[\mathcal{E}]$. Suppose to the contrary that $P$ lies above the root $\tilde{N}_0 = L[\mathcal{E}]$ of $\tilde{U}$ in the tree $\tilde{U} \oplus \delta R \oplus W$. Let $b$ be the main branch of $\tilde{U} \oplus \delta R \oplus W$, let $c$ be the main branch of $\tilde{U} \oplus V$ and let $\nu \in \text{domain } U$ be the largest member of $b \cap c$. Finally, let $E$ be the first extender used along $b$ after $\nu$, and let $E'$ be the first extender used along $c$ after $\nu$. Then we can write the embedding $i_b$ along the main branch $b$ of $\tilde{U} \oplus \delta R \oplus W$ in the form

$$i_b: L[\mathcal{E}] \xrightarrow{\tilde{u}_{b,\nu}} \tilde{N}_\nu \xrightarrow{i_E} \text{ult}(\tilde{N}_\nu, E) \xrightarrow{k} P,$$

and write the embedding $i_c$ along the main branch of $\tilde{U} \oplus V$ in the form

$$i_c: L[\mathcal{E}] \xrightarrow{\tilde{u}_{c,\nu}} \tilde{N}_\nu \xrightarrow{i_{E'}} \text{ult}(\tilde{N}_\nu, E') \xrightarrow{k'} P.$$

By lemma 1.8(4) or lemma 1.9(4) we get $i_b|K_\kappa = i_c|K_\kappa$. Let $\eta$ be $\min(\text{crit}(E), \text{crit}(E'))$. Then since $\eta < \kappa$ it follows that $k_i^E|\mathcal{P}(\eta) = k'_i^E|\mathcal{P}(\eta)$ and hence one of $E$ and $E'$ is an initial segment of the other. By the proof of lemma 1.8(2) this can never happen if either of the extenders $E$ or $E'$ come from $W$ or $V$, so both of $E$ and $E'$ must come from $U$. This implies that $E = E'$, contradicting the choice of $\nu$. This completes the proof of the claim.

The following diagram illustrates the present situation, with the straight arrows indicating embeddings which are known to exist:

$$
\begin{array}{ccc}
P & & P \\
\downarrow_{i_W} & & \uparrow_{i_W} \\
\mathcal{M}_\phi & \xrightarrow{\pi_\theta} & R \\
\downarrow_{s=\tilde{u} @ V} & & \downarrow_{s=\tilde{u} @ V} \\
L[\mathcal{E}] & \xrightarrow{i_T} & L[\mathcal{E}] \\
\end{array}
$$

The left hand map $i_T$ is defined since there are no drops on the main branch of $\mathcal{T}$. Thus there are two maps from $L[\mathcal{E}]$ to $P$, namely $s = \tilde{u} @ V$ and $t = \tilde{u} @ \pi_\theta i_T$. Then $t|L_\lambda[\mathcal{E}] = s|L_\lambda[\mathcal{E}]$ by lemma 1.8 or 1.9 and in particular $s$ and $t$ have the same critical point $\rho$, with $t(\rho) = s(\rho)$ and $t|\mathcal{P}(\rho) = s|\mathcal{P}(\rho)$. \hfill \square
We claim that \( s(\rho) \geq \tilde{\delta} \). Suppose, to the contrary, that \( s(\rho) < \tilde{\delta} \) and let \( E \) be the first extender used in the main branch of \( \tilde{U} \oplus V \). Then \( \rho = \text{crit}(E) \), and \( \tilde{\delta} > i^E(\rho) > \text{len}(E) \) so \( \text{len}(E) < \tilde{\delta} \). It follows that \( E \) comes from \( \tilde{U} \), so that \( E = \pi_\nu(E) \) for some \( \nu < \theta \), where \( E = E_\nu^\mathcal{U} \). Now let \( \xi = \text{crit}(\pi) = \text{crit}(\pi_\nu) = \text{crit}(\pi_\theta) \). Then \( \rho = \text{crit}(t) \leq \xi \), and it follows that \( \rho < \xi \) since \( \rho = \text{crit}(E) = \pi(\text{crit}(E)) \) and \( \xi \notin \text{range}(\pi) \). Since \( \rho = \text{crit}(t) < \text{crit}(i^W_\theta) \), it follows that \( \rho = \text{crit}(i^T) \) and hence the main branch of \( T \) begins with an extender \( F \) such that \( \text{crit}(F) = \rho \). Since \( \langle t, \rho \rangle \rangle P(\rho) = s \rangle P(\rho) \rangle \) it follows that one of \( F \) and \( E \) is an initial segment of the other, contradicting the construction of \( T \) and \( \mathcal{U} \).

Thus \( s(\rho) \geq \tilde{\delta} \), and since \( t(\rho) = s(\rho) \) we also have \( t(\rho) = i^W_\theta i^T(\rho) \geq \tilde{\delta} \). Since \( i^W \mid \delta \) is the identity it follows that \( \text{crit}(i^T(\rho)) \geq \tilde{\delta} \), and hence \( i^T(\rho) \geq \delta \). But each of the extenders in the tree \( T \) has length less than \( \delta \), so the length of the main branch of \( T \), and hence of \( T \) itself, cannot be smaller than \( \delta \). This completes the proof of lemma 2.7.

Now we can look at the indiscernibles generated by \( T \).

Corollary 2.10. There is no drop in the main branch of \( U \). Furthermore \( j_{0,\theta} \langle \delta \rangle \subset \delta \) and there is a closed and unbounded subset \( I \) of the main branch \( [0, \delta]_T \) of \( T \) satisfying the following four conditions:

1. If \( \nu \in I \) then \( \nu = \text{crit}(i_{\nu,\phi}) \).
2. If \( \nu, \nu' \in I \) with \( \nu < \nu' \) then \( \nu' = i_{\nu,\nu'}(\nu) \).
3. If \( \nu \in I \) then \( j_{0,\theta}^{\langle \nu \rangle} \subset \nu \).
4. Every member of \( \nu \) is regular in \( W \), and is a limit member of \( \pi^{-1}(C) \).

Proof. If there were a drop on the main branch of \( U \) then there would be a model \( N_\nu \) on the main branch of \( U \) with \( |N_\nu| < \delta \). If there is also a drop on the main branch of \( T \) then the proof of the comparison lemma shows that both trees have length less than \( \delta \); while if there is no drop on the main branch of \( T \) then lemma 2.7 implies that \( \text{crit}(E_\nu) = i_{0,\nu}(\rho) \) for every extender \( E_\nu \) used on the main branch of \( T \), and again it follows by the proof of the comparison lemma that both trees have length less than \( \delta \).

Since either case contradicts lemma 2.7, it follows that there is no drop on the main branch of \( U \). The rest of the proof of corollary 2.10 is straightforward.
The indiscernibles $I$ are not true indiscernibles, since they come from ultrapowers by different ultrafilters. If $M$ is any model of the form $L[\mathcal{E}]$ for some good sequence $\mathcal{E}$ then define

$$M \models \nu \in_0 x \iff \begin{cases} \nu \in x & \text{if } o^M(\nu) = 0 \\ x \cap \nu \in U_\nu & \text{if } o^M(\nu) > 0 \end{cases}$$

where $U_\nu$ is the unique order 0 measure on $\nu$ in $M$.

Define $\overline{U}$ to be the set of subsets $x$ of $\delta$ such that $\overline{W} \models \nu \in_0 x$ for all $\nu \in I$. We begin by showing that $\overline{U}$ is a normal measure on $\overline{W}$. We will then use this fact to show that the filter generated in the same way by $\pi^* I$ is an ultrafilter on the model $\mathcal{A}$ for which we need to find indiscernibles.

**Definition 2.11.** If $F$ is a filter on $P(\xi)$ for some ordinal $\xi$ then the $F$-closure of a set $N$ is the smallest set $Y \subset \bigcup_{n<\omega} P(\xi^n)$ such that $N \cap \bigcup_{n<\omega} P(\xi^n) \subset Y$ and, $\{ \vec{v} : \{ \alpha : (\nu_0, \ldots, \nu_{n-1}, \alpha) \in x \} \in F \} \in Y$ whenever $x \in Y$.

**Lemma 2.12.** Let $Y$ be the $\overline{U}$-closure of $\overline{W}$. Then $\overline{U}$ is a normal ultrafilter on $Y$.

**Proof.** First we will show that $P(\delta) \cap N_\theta \subset M_\delta$. Suppose $x \in N_\theta$ and $x \subset \delta$. Then $x \cap \nu \in M_\nu$ for every sufficiently large $\nu \in I$, and hence there is a stationary subset $I' \subset I$ such that for all pairs $\nu < \nu'$ of members of $I'$ we have $i_{\nu,\nu'}(x \cap \nu) = x \cap \nu'$. But then $x = i_{\nu,\delta}(x \cap \nu) \in M_\delta$ where $\nu$ is any member of $I'$.

Now let $U'$ be the filter defined like $\overline{U}$, but using the order 0 measures $U'_\nu$ from $N_\theta$. That is, $x \in U'$ if and only if $N_\theta \models x \in_0 x$ for every sufficiently large $\nu \in I$. Then $U'$ is an ultrafilter on the $U'$-closure of $M_\delta$ and hence on the $U'$-closure of $N_\theta$. Now we claim that $\overline{U} = \{ x : j_{0,\theta}(x) \in U' \}$. Let $\nu$ be in $I$. If $o^{\overline{W}}(\nu) = \nu$ then $j_{0,\theta}(\nu) = \nu$ by corollary 2.10(3,4), and hence

$$\overline{W} \models \nu \in_0 x \iff \nu \in x \iff \nu \in j_{0,\theta}(x) \iff M_\theta \models \nu \in_0 j_{0,\theta}(x).$$

If $o^{\overline{W}}(\nu) > 0$ then the situation is slightly more complicated. We have $\overline{W} \models \nu \in_0 x$ if and only if $x \cap \nu \in U'_\nu$, and if $o^{N_\theta}(\nu) > 0$ then

$$x \cap \nu \in U'_\nu \iff j_{0,\theta}(x) \cap \nu \in U'_\nu \iff M_\theta \models \nu \in_0 j_{0,\theta}(x).$$

23
On the other hand, if $o^N = 0$ then $j_{\nu, \nu + 1}: M_\nu \to \text{ult}(M_\nu, j_{0, \nu}(\overline{M}_\nu))$ so

$$x \cap \nu \in \overline{U}_\nu \iff \nu \in j_{0, \theta}(x) \iff M_\theta \models \nu \in j_{0, \theta}(x).$$

Since $U'$ is a normal ultrafilter on the $U'$-closure of $N_\theta$ it follows that $U$ is a normal ultrafilter on the $U$-closure $Y$ of $W$. \hfill $\Box$

Now repeat the process in $W$, defining a filter $U$ on $\tilde{\delta} = \sup \pi^I \delta$ by $x \in U$ if and only if $W \models \nu \in_0 x$ for every sufficiently large $\nu \in \pi^I I$.

The filter $U$ need not be an ultrafilter on $W$, but we will find a premouse containing $A$ on which $U$ is an ultrafilter. In order to do so let $m \in X$ be the least mouse such that $A$ and $C$ are members of $m$, let $h$ be the canonical Skolem function of $m$, and let $m^*$ be the transitive collapse of $h^\tilde{\delta}$.

**Lemma 2.13.** The filter $U$ is a normal ultrafilter on the $U$-closure of $m^*$.

*Proof.* First we show that $U$ is an ultrafilter on $m^* \cap \mathcal{P}(\tilde{\delta})$. Let $x$ be an arbitrary subset of $\tilde{\delta}$ in $m^*$. Then there is an ordinal $\alpha < \tilde{\delta}$ such that $x = h(\alpha) \cap \tilde{\delta}$. Now for sufficiently large $\nu, \nu' \in I$ we have

$$W \models \forall \beta < \nu \left( \nu \in_0 \bar{h}(\beta) \iff \nu' \in_0 \bar{h}(\beta) \right).$$

(3)

Pick $\nu_0 \in I$ such that $\alpha < \pi(\nu_0)$ and equation (3) holds for all $\nu' > \nu \geq \nu_0$ in $I$. Then

$$W \models \forall \beta < \pi(\nu) \left( \pi(\nu) \in_0 \bar{h}(\beta) \iff \pi(\nu') \in_0 \bar{h}(\beta) \right),$$

also holds for all $\nu' > \nu \geq \nu_0$. In particular, $\pi(\nu) \in_0 \bar{h}(\alpha) \iff \pi(\nu') \in_0 h(\alpha)$, holds for all such $\nu$ and $\nu'$, so that either $x \in U$ or $\tilde{\delta} \setminus x \in U$. Since $x$ was arbitrary it follows that $U$ is an ultrafilter on $m^*$. A straightforward extension of this argument proves that $U$ is a normal ultrafilter on the $U$-closure of $\mathcal{P}(\tilde{\delta}) \cap m^*$.

**Lemma 2.14.** The $U$-closure $Y$ of $\{x \cap \tilde{\delta} : x \in m^*\}$ is a member of $L[\mathcal{E}]$.

*Proof.* Let $n$ be the least mouse such that there is a subset of $\tilde{\delta}$, definable in $n$, which is not measured by $U$; or if there is no such mouse then let $n = W$. Then $U \cap n$ is a normal measure. We will finish the proof of lemma 2.14 by proving the following claim, which clearly implies lemma 2.14:

**Claim 2.15.** 1. $Y \subset n$, and 2. $U \cap n \in K$. 

24
For clause (1), let \( m^* \) be as above, and let \( i_n^U : m^* \to m_n = \text{ult}_n(m^*, U) = \text{ult}(m^*, U^n) \) be the \( n \)-fold iterated ultrapower, which is defined by lemma 2.13. Notice that every set in \( m_n \) is measured by \( U \), and that
\[
\{ \nu_0 : \{ (\nu_1, \ldots, \nu_{n+1}) : (\nu_0, \ldots, \nu_{n+1}) \in x \cap \delta \} \in U^n \} = \left\{ \nu_0 < \tilde{\delta} : (\nu_0, \delta, i_1^U(\tilde{\delta}), \ldots, i_{n-1}^U(\tilde{\delta})) \in i_n^U(x) \right\} \in m_n
\]
for any set \( x \in X \). Thus \( Y \subset \bigcup_n m_n \), and it will be enough to show that the subsets of \( \tilde{\delta} \) in \( m_n \) are an initial subset of those in \( K \), so that \( m_n \) is an initial segment of \( n \).

For clause (2), consider the model \( n_1 = \text{ult}(n, U) \). Since \( U \cap n \) is definable from \( n_1 \), it will be enough to show that \( n_1 \in W \).

Both clauses follow from the following claim:

**Claim 2.16.** Suppose that \( p \) is a premouse which agrees with \( W \) up to \( \tilde{\delta} \), that every member of \( p \) is definable from parameters in \( \tilde{\delta} \cup p \) for some finite set \( p \), and that the phalanx \( (\tilde{\delta}, (W, p)) \) is iterable. Then \( p \in W \) and \( P^W(\tilde{\delta}) \) is an initial segment of \( P^W(\tilde{\delta}) \).

Claim 2.15 will follow from claim 2.16, provided we can show that the phalanx \( (\tilde{\delta}, (W, n_1)) \) and the phalanxes \( (\tilde{\delta}, (W, m_n)) \) for \( n < \omega \) are all iterable.

**Proof of claim 2.16.** Compare the given phalanx with \( W \), using trees \( S \) on \( (\tilde{\delta}, (W, p)) \) and \( T \) on \( W \). Standard arguments show that the last model of \( S \) must lie above \( p \), that there are no drops on the main branch of \( S \), and that the final model \( P \) of \( S \) is an initial segment of the final model \( Q \) of \( T \). The models \( P \) and \( p \) have the same subsets of \( \tilde{\delta} \), so the subsets of \( \tilde{\delta} \) in \( p \) are an initial segment of those in \( Q \), and hence of those in \( W \). Furthermore, \( p \) is isomorphic to the Skolem hull in \( P \) of \( \tilde{\delta} \cup i^S(p) \), so \( p \in Q \) and hence \( p \in W \). □

Thus it only remains to show \( n_1 \) and \( m_n \) satisfy the iterability conditions. We will give the proof for \( n_1 \); the proof for \( m_n \) is similar.

**Claim 2.17.** The phalanx \( (\tilde{\delta}, (W, n_1)) \) is iterable.

**Proof.** We will suppose that \( (\tilde{\delta}, (W, n_1)) \) is not iterable and find an ill behaved tree on the phalanx \( (\tilde{\delta}, (W, n)) \). This contradicts the fact that \( (\tilde{\delta}, (W, n)) \) is iterable, since \( n \) is a mouse of \( W \), and thus proves claim 2.17.
The construction is similar to that in lemma 2.4. Let $\mathcal{S}$ be an ill behaved tree on $(\tilde{\delta}, (W, n_1))$, and let $Z$ be a countable set containing everything relevant such that $Z \prec H_\tau$ for some sufficiently large $\tau$. Let $\eta = \sup(X \cap \tilde{\delta})$ and let $k: n' \to n_1$ be the inverse of the collapse map of the Skolem hull of $\eta \cup \{\tilde{\delta}\}$ in $n_1$. As in lemma 2.4 we can use the Martin-Steel iterability theorem and a Levy collapse to show that the $\mathcal{S}$ induces a ill behaved tree $\mathcal{S}'$ on $(\eta, (W, n'))$. Since $|Z| = \omega < \delta = \text{cf}(\tilde{\delta})$ there is $\nu \in \pi^*I \setminus \eta$ such that $\nu \in_0 x$ for all $x \in Z \cap U$. If $o(\nu) = 0$ then we can embed $n'$ into $n$ by mapping $\tilde{\delta}$ to $\nu$, and thus we get an ill behaved tree on $(\eta, (W, n'))$, a contradiction.

If $o(\nu) > 0$ then we similarly find an ill behaved tree on $(\eta, (W, \text{ult}(n_0, U_\nu)))$ where $U_\nu$ is the order 0 measure on $\nu$. This finishes the proof of claim 2.17.

This completes the proof of claim 2.15 and hence of lemma 2.14.

We can now complete the proof of the main theorem. It only remains to use the ultrafilter $U$ on $m^*$ in $K$ to define a set of indiscernibles for $\mathcal{A}$ in $K$. Let $h^*$ be the Skolem function of $m^*$ and define $D$ to be the set of ordinals $\alpha \in C \cap \tilde{\delta}$ such that

$$\forall x \in h^*\alpha \cap \mathcal{P}(\tilde{\delta}) \ (\alpha \in x \iff x \in U)$$

and for all $n > 0$ in $\omega$

$$\forall x \in h^*\alpha \cap \mathcal{P}(\tilde{\delta}^{1+n}) \ (\{\tilde{\nu} \in [\tilde{\delta}]^n : (\alpha, \tilde{\nu}) \in x\} \in U^n \iff x \in U^{1+n})$$

Then $D \in K$, and $\pi(\nu) \in_0 D$ for every sufficiently large ordinal $\nu \in I$. Thus every member $\nu$ of $\pi^*I$ either is in $D$ or has $D \cap \nu \in U_\nu$, so $|D| \geq |\pi^*I| = \delta$.

This completes the proof of the main theorem.

3 Some questions

The most basic open question is the problem with which we opened the paper:

**Question 1.** Is is a theorem of ZFC that every Ramsey cardinal is Jónsson?

It would certainly be surprising if this were true, but it is also surprising that no counterexamples are yet known.

It is perhaps more plausible to hope that the restriction of the main theorem to models with no class model of a Woodin cardinal can be eliminated:
**Question 2.** Is it a theorem of ZF that if the core model $K$ exists then every Jónsson cardinal is Ramsey in $K$?

Of course the general notion of “$K$ is the core model” remains to be defined. For the present we could take the problem as referring to core models in the sense of Steel.

Finally we mention one more question:

**Question 3.** Can theorem [1.3] be generalized to singular cardinals $\delta$?

It is easy to see that some such generalization is possible, but it is not clear how much can be said.

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