Quantum Field Theory in Light-Front coordinates

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Abstract

Canonical formulation of quantum field theory on the Light Front (LF) is reviewed. The problem of constructing the LF Hamiltonian which gives the theory equivalent to original Lorentz and gauge invariant one is considered. We describe possible ways of solving this problem: (a) the limiting transition from the equal-time Hamiltonian in a fastly moving Lorentz frame to LF Hamiltonian, (b) the direct comparison of LF perturbation theory in coupling constant and usual Lorentz-covariant Feynman perturbation theory. Gauge invariant regularization of LF Hamiltonian via introducing a lattice in transverse coordinates and imposing periodic boundary conditions in LF coordinate $x^-$ for gauge fields on the interval $|x^-| < L$ is considered. We find that LF canonical formalism for this regularization avoid usual most complicated constraints connecting zero and nonzero modes of gauge fields.

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1 Introduction

V. A. Fock has elaborated a beautiful method to describe state vectors in quantum field theory. The corresponding vector space is now called the Fock space [1]. This method plays very important role in quantum field theory in Light-Front (LF) coordinates [2]:

\[ x^\pm = (1/\sqrt{2})(x^0 \pm x^3), x^1, x^2, \]

where \( x^0, x^1, x^2, x^3 \) are Lorentz coordinates. The \( x^+ \) plays the role of time, and canonical quantization is carried out on a hypersurface \( x^+ = \text{const} \). The advantage of this scheme is connected with the positivity of the momentum \( P^- \) (translation operator along \( x^- \) axis), which becomes quadratic in fields on the LF. As a consequence the lowest eigenstate of the operator \( P^- \) is both physical vacuum and the “mathematical” vacuum of perturbation theory [3]. Using Fock space over this vacuum one can solve stationary Schroedinger equation with Hamiltonian \( P^+ \) (translation operator along \( x^+ \) axis) to find the spectrum of bound states. The problem of describing the physical vacuum, very complicated in usual formulation with Lorentz coordinates, does not appear here. Such approach, based on solving Schroedinger equation on the LF, is called LF Hamiltonian approach. It attracts attention for a long time as a possible mean for solving Quantum Field Theory problems.

While giving essential advantages, the application of LF coordinates in Quantum Field Theory leads to some difficulties. The hyperplane \( x^+ = \text{const} \) is a characteristic surface for relativistic differential field equations. It is not evident without additional investigation that quantization on such hypersurface generates a theory equivalent to one quantized in the usual way in Lorentz coordinates [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. This is in particular essential because of the special divergences at \( p^- = 0 \) appearing in LF quantization scheme. Beside of usual ultraviolet regularization one has to apply special regularization of such divergences. We will consider the following simplest prescription of such regularization:

(a) cutoff of momenta \( p^- \)

\[ |p^-| \geq \varepsilon, \quad \varepsilon > 0; \quad (1.1) \]

(b) cutoff of the \( x^- \)

\[ -L \leq x^- \leq L. \quad (1.2) \]

with periodic boundary conditions in \( x^- \) for all fields.

The regularization (b) discretizes the spectrum of the operator \( P^- \) (\( p^- = \pi n/L \), where \( n \) is an integer). This formulation is called sometime ”Discretized LF Quantization” [14]. Fourier components of fields, corresponding to \( p^- = 0 \) (and usually called ”zero modes”) turn out to be dependent variables and must be expressed in terms of nonzero modes via solving constraint equations (constraints). These constraints are usually very complicated, and solving of them is a difficult problem.

The prescriptions of regularization of divergences at \( p^- = 0 \) described above are convenient for Hamiltonian approach, but both of them break Lorentz invariance and the prescription (a) breaks also the gauge invariance. Therefore the equivalence of LF and original Lorentz (and gauge) invariant formulation can be broken even in the limit of
removed cutoff. To avoid this inequivalence some modification of usual renormalization procedure may be necessary, see for example [15] and [12].

The problem of constructing the LF Hamiltonian which gives a theory equivalent to original Lorentz and gauge invariant one turned out to be rather difficult. Now it is solved only for nongauge field theories [12]. We will describe possible approaches to this problem.

In Sect. 2 we give basic relations of quantum field theory in LF coordinates. In Sect. 3 we consider the limiting transition from fast moving Lorentz frame to the LF. This transition relates the formulations in Lorentz and LF coordinates [12]. In Sect. 4 we investigate the relation between LF perturbation theory in coupling constant and usual Lorentz-covariant Feynman perturbation theory. For Yukawa model this investigation shows how to construct LF Hamiltonian giving the theory perturbatively equivalent to original one [12]. For gauge theories the methods developed in Sect. 3 and Sect. 4 do not give the required LF Hamiltonian due to specific difficulties. In Sect. 5 we consider gauge invariant ultraviolet regularization of LF Hamiltonian via introducing a lattice in transverse coordinates $x^1, x^2$ and taking complex matrix variables for gauge fields on links of the lattice [16]. We find that LF canonical formalism for gauge theories with this regularization avoid usual most complicated constraints connecting zero and nonzero modes.

2 Formal canonical quantization of Field Theory on the Light Front and the problem of bound states.

In order to find the bound state spectrum in some field theory quantized on the LF the following system of equations is usually solved:

$$ P_+ |\Psi\rangle = P'_+ |\Psi\rangle, \quad (2.1) $$

$$ P_- |\Psi\rangle = P'_- |\Psi\rangle, \quad (2.2) $$

$$ P_\perp |\Psi\rangle = 0, \quad (2.3) $$

where $P_\perp = \{P_1, P_2\}$. The mass of bound state is equal to

$$ m = \sqrt{2P'_+ P'_-}. \quad (2.4) $$

It was taken into account that nonzero components of metric tensor in LF coordinates are

$$ g_{++} = g_{--} = 1, \quad g_{11} = g_{22} = -1. \quad (2.5) $$

The operators $P_-, P_\perp$ are quadratic in fields, and the solution of equations (2.2), (2.3) is not difficult. The problem is in solving the Schroedinger equation (2.1). Physical vacuum $|\Omega\rangle$ is lowest eigenstate of the operator $P_-$, and

$$ P_+ |\Omega\rangle = 0, \quad (2.6) $$
\[ P_- |\Omega\rangle = 0, \quad (2.7) \]
\[ P_\perp |\Omega\rangle = 0. \quad (2.8) \]

To fulfil equations (2.6), (2.7) one should subtract, if it is necessary, corresponding renormalizing constants from the operators \( P_+, P_- \). The \( |\Omega\rangle \) plays simultaneously the role of mathematical vacuum of Fock space. A solution \(|\Psi\rangle\) of Schrodinger eqn. (2.1) belong to this space.

Expressions for the operators \( P_+, P_- \) can be obtained by canonical quantization on the LF. Let us describe the procedure of such quantization via some examples, without analyzing so far the question about the equivalence of appearing theory and the original Lorentz-covariant one. It is assumed that in addition to explicit regularization of divergences at \( p_- = 0 \) necessary ultraviolet regularization is implied.

### 2.1 Scalar self-interacting field in (1+1)-dimensional space-time.

Peculiarities of LF quantization are well seen even in this simple example. We have only LF coordinates \( x^+, x^- \). The Lagrangian is equal to

\[ L = \int dx^- \left( \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \lambda \varphi^4 \right), \quad (2.9) \]

or

\[ L = \int dx^- \left( \frac{1}{2} \partial_+ \varphi \partial_- \varphi - \frac{1}{2} m^2 \varphi^2 - \lambda \varphi^4 \right), \quad (2.10) \]

The "time" derivative \( \partial_- \varphi \) enters into this Lagrangian only linearly. For the transition to canonical theory it is sufficient to rewrite the expression \( \int dx^- \frac{1}{2} \partial_+ \varphi \partial_- \varphi \) in standard form. To achieve this let us take the Fourier decomposition

\[ \varphi(x^-) = (2\pi)^{-\frac{1}{2}} \int_0^\infty dk |2k|^{-\frac{1}{2}} \left( a(k) \exp(-ikx^-) + a^+(k) \exp(ikx^-) \right), \quad (2.11) \]

where \( k \equiv k_- \), \( \varphi(x^-) \equiv \varphi(x^+, x^-) \), \( a(k) \equiv a(x^+, k) \). The Lagrangian (2.10) takes the form

\[ L = \int_0^\infty dk \left( \frac{a^+(k)\dot{a}(k) - a(k)\dot{a}^+(k)}{2i} \right) - H, \quad (2.12) \]

where \( \dot{a} \equiv \partial a / \partial x^+ \) and

\[ H = \int dx^- \left( \frac{1}{2} m^2 \varphi^2 + \lambda \varphi^4 \right). \quad (2.13) \]
Here we have used the equality
\[ \int_0^\infty dk \int_0^\infty dk' \delta(k + k')k' \left( a(k)\dot{a}(k') - a^+(k)\dot{a}^+(k') \right) = 0. \] (2.14)

It is implied that the function \( \varphi(x) \) in (2.13) is expressed in terms of \( a^+(k) \) and \( a(k) \) with the help of formulae (2.10). "Time" derivatives \( \dot{a}(k), \dot{a}^+(k) \) enter into Lagrangian \( L \) in a form standard for canonical theory. Therefore one can interpret after quantization the \( a^+(k) \) and \( a(k) \) as creation and annihilation operators satisfying the following commutation relations at fixed \( x^+ \) and \( k > 0, k' \geq 0 \):
\[ [a(k), a^+(k')] = \delta(k - k'), \quad [a(k), a(k')] = 0, \] (2.15)

It is also seen that the \( H \) is LF Hamiltonian, i.e. \( H = P_+ \).

We have also the formulae
\[ P_\mu = \int \! dx^- T_{-\mu}, \] (2.16)
where the energy-momentum tensor \( T_{\nu\mu} \) is equal to
\[ T_{\nu\mu} = \partial_\nu \varphi \partial_\mu \varphi - g_{\nu\mu} \mathcal{L}. \] (2.17)

Via this relation one can reproduce the expression (2.13) for \( P_+ \equiv H \), and obtain the equality
\[ P_- = \int \! dx^- (\partial_- \varphi)^2 = \frac{1}{2} \int \! dk k \left( a^+(k)a(k) + a(k)a^+(k) \right). \] (2.18)

The lowest eigen state of the operator \( P_- \) is the physical vacuum \( |\Omega\rangle \) for which
\[ a(k)|\Omega\rangle = 0 \] (2.19)
at any \( k \). It is seen that vacuum expectation values \( \langle \Omega | P_- |\Omega\rangle, \langle \Omega | P_+ |\Omega\rangle \) are infinite. The renormalization can be got by taking normal ordered forms : \( P_+ : : , : P_- : \) with respect to the operators \( a^+, a \) (the symbol :: means as usual that operators \( a^+ \) stand everywhere before of operators \( a \)). Normal ordering of the \( \lambda \varphi^4 \) term in the Hamiltonian \( P_+ \) leads also to renormalization of the mass. In the following, writing \( P_+, P_- \), we mean the expressions : \( P_+ : , : P_- : \), satisfying conditions (2.6), (2.7). Normal ordering of the operators \( P_+, P_- \) allows to avoid all ultraviolet divergences in this simple model.

2.2 Theory of interacting scalar and fermion fields in \((3+1)-\) dimensional space-time (Yukawa model).

The Lagrangian of the model is
\[ L = \int \! d^2x^+ dx^- \left( \overline{\psi} (i\gamma^\mu \partial_\mu - M) \psi + \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{2} g \overline{\psi} \gamma^5 \varphi - \lambda \varphi^3 - \lambda \varphi^4 \right), \] (2.20)
where $M$ is the fermion mass, $m$ is the boson mass, $\bar{\psi} = \psi^+\gamma^0$, $\varphi = \varphi^+$; $g$, $\lambda$, $\lambda'$ are coupling constants. Here and so on $\mu, \nu, \ldots = +, -, 1, 2$; $i, k, \ldots = 1, 2$; $x^\perp \equiv (x^1, x^2)$. For Dirac’s $\gamma$-matrices we use:

$$
\gamma^0 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & I \\ iI & 0 \end{pmatrix}, \quad \gamma^\perp = \begin{pmatrix} -i\sigma^\perp & 0 \\ 0 & i\sigma^\perp \end{pmatrix},
$$

(2.21)

where $I$ is a unit $2 \times 2$ matrix, $\sigma^\perp \equiv \{\sigma^1, \sigma^2\}$, $\sigma^i$ are Pauli matrices.

We introduce 2-component spinors $\chi$, $\xi$, writing

$$
\psi = \begin{pmatrix} \chi \\ \xi \end{pmatrix}.
$$

The Lagrangian $L$ can be written in the form

$$
L = \int d^2x^\perp dx^- \left( i\sqrt{2}\chi^+\partial_+\chi + i\sqrt{2}\xi^+\partial_-\xi + (i\xi^+ (\sigma^i \partial_i - M) \chi + \text{H.c.}) + \partial_+\varphi\partial_-\varphi - \frac{1}{2}\partial_i\varphi\partial_i\varphi - \frac{1}{2}m^2\varphi^2 - ig\varphi (\xi^+\chi - \chi^+\xi) - \lambda\varphi^3 - \lambda\varphi^4 \right),
$$

(2.22)

where H.c. means Hermitian conjugation. The variation of this Lagrangian with respect to $\chi^+$ leads to the equation

$$
\sqrt{2}\partial_+\xi = - (\sigma^i \partial_i - M) \chi + g\varphi \chi.
$$

(2.23)

This equation does not contain derivatives in $x^\perp$ and therefore is a constraint. One should solve it with respect to $\xi$ and substitute the result into the Lagrangian. In doing this we must invert the operator $\partial_-$ which becomes an operator of multiplication $ik_-$ after Fourier transformation. Inverse operator $(ik_-)^{-1}$ has singularity at $k_- = 0$. To avoid this singularity we introduce the regularization (1.1). For any function $f(x^-) \equiv f(x^+, x^-, x^\perp)$ we define Fourier transform

$$
f(x^-) = \frac{1}{\sqrt{2\pi}} \int dk_- e^{ik_-x^-} \tilde{f}(k_-),
$$

(2.24)

where $\tilde{f}(k_-) \equiv \tilde{f}(x^+, k_-, x^\perp)$, and put

$$
[f(x^-)] \equiv \frac{1}{\sqrt{2\pi}} \int dk_- e^{ik_-x^-} \tilde{f}(k_-), \quad |k_-| \geq \varepsilon > 0.
$$

(2.25)

We insert into the Lagrangian (2.22) the variables $[\chi]$, $[\chi^+]$, $[\xi]$, $[\xi^+]$, $[\varphi]$ instead of $\chi$, $\chi^+$, $\xi$, $\xi^+$, $\varphi$ and obtain the constraint equation

$$
\sqrt{2}\partial_- [\xi] = - (\sigma^i \partial_i - M) [\chi] + g[[\varphi][\chi]]
$$

(2.26)

instead of (2.23). Its solution is

$$
[\xi] = \frac{1}{\sqrt{2}} \partial_-^{-1} \left( - (\sigma^i \partial_i - M) [\chi] + g[[\varphi][\chi]] \right),
$$

(2.27)
where the operator $\partial_{-}^{-1}$ is completely defined by the condition
\[ \partial_{-}^{-1}[f] = [\partial_{-}^{-1}[f]]. \tag{2.28} \]
After Fourier transformation the operator $\partial_{-}^{-1}$ is replaced by $(ik_{-})^{-1}$.
Substituting the expression (2.27) into the Lagrangian (where all fields $\chi, \chi^{+}, \ldots$ are replaced with $[\chi], [\chi^{+}], \ldots$) we come to the result:
\[ L = \int d^{2}x^{+}d^{-} \left( i\sqrt{2} [\chi^{+}] \partial_{+} [\chi] + \partial_{-}[\varphi][\partial_{+}[\varphi] + \right. \]
\[ \left. + \frac{1}{\sqrt{2}} \left( (\sigma^{i}\partial_{i} - M) [\chi] - g([\varphi][\chi]) \right)^{+}(-i\partial_{-})^{-1}\left( (\sigma^{k}\partial_{k} - M) [\chi] - g([\varphi][\chi]) \right) - \right. \]
\[ \left. - \frac{1}{2} \partial_{i}[\varphi][\partial_{i}[\varphi] - \frac{1}{2} m^{2}[\varphi]^{2} - \lambda[\varphi]^{3} - \lambda[\varphi]^{4} \right), \tag{2.29} \]
As in Sect. 2a time derivatives $\partial_{+}[\chi], \partial_{+}[\varphi]$ enter into the Lagrangian (2.29) linearly. Therefore to go to canonical formalism it is sufficient to find a standard form for the expression
\[ i\sqrt{2}[\chi^{+}]\partial_{+}[\chi] + \partial_{-}[\varphi^{+}][\partial_{+}[\varphi] \]
(before quantization the quantities $\chi^{+}, \chi$ are elements of Grassman algebra). We write
\[ [\varphi(x^{-})] = (2\pi)^{-1/2} \int_{\xi}^{\infty} dk_{-} (2k_{-})^{-1/2}(a(k_{-}) \exp(-ik_{-}x^{-}) + a^{+}(k_{-}) \exp(ik_{-}x^{-})) \tag{2.30} \]
\[ [\chi_{r}(x^{-})] = (2\pi)^{-1/2} 2^{-1/4} \int_{\xi}^{\infty} dk_{-} (b_{r}(k_{-}) \exp(-ik_{-}x^{-}) + c_{r}^{+}(k_{-}) \exp(ik_{-}x^{-})) \tag{2.31} \]
\[ [\chi_{r}^{+}(x^{-})] = (2\pi)^{-1/2} 2^{-1/4} \int_{\xi}^{\infty} dk_{-} (c_{r}(k_{-}) \exp(-ik_{-}x^{-}) + b_{r}^{+}(k_{-}) \exp(ik_{-}x^{-}) \tag{2.32} \]
where
\[ [\varphi(x^{-})] \equiv [\varphi(x^{+}, x^{-}, x^{+})], \quad a(k_{-}) \equiv a(x^{+}, k_{-}, x^{+}) \]
et cetera, $r = 1, 2$. The Lagrangian (2.23) takes the form
\[ L = \int d^{2}x^{+} \int_{\xi}^{\infty} dk_{-} (a(k_{-})\dot{a}^{+}(k_{-}) - a^{+}(k_{-})\dot{a}(k_{-}) - \]
\[ -b_{r}(k_{-})\dot{b}_{r}^{+}(k_{-}) - b_{r}^{+}(k_{-})\dot{b}_{r}(k_{-}) - c_{r}(k_{-})\dot{c}_{r}^{+}(k_{-}) - c_{r}^{+}(k_{-})\dot{c}_{r}(k_{-})) - H \tag{2.33} \]
where
\[ H = \left( -\frac{1}{\sqrt{2}} (\sigma^{i}\partial_{i} - M) [\chi] - g([\varphi][\chi]) \right)^{+}(-i\partial_{-})^{-1}\left( (\sigma^{k}\partial_{k} - M) [\chi] - g([\varphi][\chi]) \right) + \right. \]
\[ \left. + \frac{1}{2} \partial_{i}[\varphi][\partial_{i}[\varphi] + \frac{1}{2} m^{2}[\varphi]^{2} + \lambda[\varphi]^{3} + \lambda[\varphi]^{4} \right). \tag{2.34} \]
It is assumed that the quantities \( \chi, [\chi^+] \) in the formulae (2.34) are expressed in terms of \( b, b^+, c, c^+, a, a^+ \) with the help of (2.33), (2.31), (2.32).

It follows from (2.33) that \( a^+, a, b^+, b, c^+, c \) play a role of creation and annihilation operators. After quantization they satisfy the commutation relations (at \( x^+=\text{const} \)):

\[
[a(k_-, x^\perp), a^+(k'_-, x'^\perp)]_\pm = \delta(k_- - k'_-) \delta^2(x^\perp - x'^\perp),
\]

\[
[b(k_-, x^\perp), b^+(k'_-, x'^\perp)]_\pm = \delta(k_- - k'_-) \delta^2(x^\perp - x'^\perp),
\]

\[
[c(k_-, x^\perp), c^+(k'_-, x'^\perp)]_\pm = \delta(k_- - k'_-) \delta^2(x^\perp - x'^\perp),
\]

where \([x, y]_\pm = xy \pm yx\). The remaining (anti)commutators are equal to zero. It is seen that the quantity \( H \) is LF Hamiltonian \( (H = P_+) \). The operator of the momentum \( P_- \) is equal to

\[
P_- = \int d^2x T_- = \frac{1}{2} \int d^2x^\perp \int_\infty dk_- k_-(a^+(k_-)a(k_-) + a(k_-)a^+(k_-) + b^+(k_-)b(k_-) - b(k_-)b^+(k_-)) + c^+(k_-)c_r(k_-) - c_r(k_-)c^+(k_-)).
\]

The quantities \( P_+ \equiv H \) and \( P_- \) should be normally ordered with respect to creation and annihilation operators. The lowest eigenstate of the momentum \( P_- \) is the physical vacuum. It is defined by conditions

\[
a(k_-, x^\perp)|\Omega\rangle = 0, \quad b(k_-, x^\perp)|\Omega\rangle = 0, \quad c(k_-, x^\perp)|\Omega\rangle = 0, \quad \forall x^\perp, k_-. \tag{2.39}
\]

The equalities (2.6), (2.7) become true after normal ordering of the operators \( P_+ \) and \( P_- \). For the \( P_- \) it is seen from the formulae (2.38), and for the \( P_+ \) it follows from the fact that every term of \( P_+ \) contains a \( \delta \)-function of difference between the sum of momenta \( k_- \) of creation operators and the sum of momenta \( k_- \) of annihilation operators. Due to the positivity of all momenta \( k_- \), in our regularization scheme every term of the \( P_+ \) contains at least one annihilation operator. Therefore for normally ordered \( P_+ \) we have \( P_+|\Omega\rangle = 0 \).

The model under consideration requires ultraviolet regularization. It can be done in different ways. We consider this question together with the renormalization problem in Sect. 3 and 4.

### 2.3 The U(N)-theory of pure gauge fields.

We consider the \( U(N) \) rather than the \( SU(N) \) theory because it is technically more simple. The transition to the \( SU(N) \) can be done easily. Gauge field is described by Hermitian matrices

\[
A_{\mu}(x) = A^+_{\mu}(x) \equiv \{A^{ij}_{\mu}(x)\}, \tag{2.40}
\]
where $\mu = +,-,1,2$; $i,j = 1,2,\ldots, N$. Let us assume, that for the indexes $i,j$ and analogous the usual rule of summation on repeated indexes is not used, and where it is necessary the sign of a sum is indicated. Field strengths tensor is

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ig [A_{\mu}, A_{\nu}], \quad (2.41)$$

and gauge transformation has the form

$$A_{\mu} \rightarrow A'_{\mu} = U^+ A_{\mu} U + \frac{i}{g} U^+ \partial_{\mu} U, \quad U^+ U = I. \quad (2.42)$$

To escape a breakdown of gauge invariance we apply the regularization of the type (1.2) with periodic boundary conditions

$$A_{\mu}(x^+, -L, x^+) = A_{\mu}(x^+, L, x^+), \quad (2.43)$$

on the interval $-L \leq x^- \leq L$. All Fourier modes of $A_{\mu}(x)$ in $x^-$ must be kept, including zero modes (at $k^- = 0$).

The Lagrangian has the form

$$L = -\frac{1}{2} \int d^2 x^\perp \int_{-L}^{L} dx^- \text{Tr} (F_{\mu\nu} F^{\mu\nu}), \quad (2.44)$$

or

$$L = \int d^2 x^\perp \int_{-L}^{L} dx^- \text{Tr} \left( F_{++}^2 + 2 F_{-k} F_{+k} - \frac{1}{2} F_{kk'} F_{kk'} \right), \quad (2.45)$$

where $k, k' = 1,2$. Time derivatives $\partial_+ A_k$ are present only in the term

$$\text{Tr} (2(\partial_- A_k - \partial_k A_- - ig[A_- , A_+]) \partial_+ A_k).$$

This expression can be put in standard canonical form only after fixing the gauge in a special form of the type $A_- = 0$ because then the term above becomes similar to that of scalar field theory $\text{Tr} (2(\partial_- A_k) \partial_+ A_k)$. However not every field, periodic in $x^-$, can be transformed to the $A_- = 0$ gauge. Indeed, the loop integral

$$\Gamma(x^+, x^+) = \text{Tr} \left\{ \text{P exp} \left( i \int_{-L}^{L} dx^- A_-(x^+, x^-, x^+) \right) \right\}, \quad (2.46)$$

where the symbol 'P' means the ordering of operators along the $x^-$, is a gauge invariant quantity. If this integral is not equal to $N$ for some field then the gauge $A_- = 0$ is not possible for this field. Therefore we choose more weak gauge condition

$$A_{ij}^i = 0, \quad i \neq j, \quad \partial_- A_{ii}^i = 0, \quad (2.47)$$
and put

\[ A_{ii}^\parallel(x) = v^i(x^+, x^\perp). \] (2.48)

The gauge (2.47) breaks not only the local gauge invariance but unlike the \( A_\parallel = 0 \) gauge, also global gauge invariance (it remains only the abelian subgroup of gauge transformations not depending on \( x^- \)). This has some technical inconvenience but now any periodic field can be described in the gauge (2.47). Furthermore, if we restrict the class of possible periodic fields by the condition \( A_\parallel = 0 \), disregarding the describe consideration, we come to canonical theory with even more complicate constraints if zero modes are taken into account [17, 18, 19].

From the point of view of LF canonical formalism the variables \( A_{ij}^\parallel \) are "coordinates". Therefore one can restrict their values by the condition (2.47) directly in the Lagrangian without loosing any equations of motion. Let us introduce the denotations

\[ D^- A_{ij}^\parallel = (\partial_\parallel - ig(v^i - v^j)) A_{ij}^\parallel, \] (2.49)

\[ D^- A_{ii}^\parallel = (\partial_\parallel - ig(v^i - v^i)) A_{ii}^\parallel, \] (2.50)

where obviously

\[ D^- A_{ii}^\parallel = \partial_\parallel A_{ii}^\parallel. \] (2.51)

Also for any function \( f(x^-) \equiv f(x^+, x^-, x^\perp) \) periodic in \( x^- \) we denote

\[ f(0) = \frac{1}{2L} \int_{-L}^{L} dx^- f(x^-), \] (2.52)

\[ [f(x^-)] = f(x^-) - f(0). \] (2.53)

Obviously,

\[ \int_{-L}^{L} dx^- [f(x^-)] = 0. \] (2.54)

After gauge fixing (2.47) the Lagrangian (2.45) takes the form

\[
L = \int d^2x \int_{-L}^{L} dx^- \left\{ 2 \sum_i (\partial_\parallel [A_{ii}^\parallel]) \partial_\parallel [A_{ii}^\parallel] + 2 \sum_{i,j,i \neq j} (D^- A_{ij}^\parallel) \partial_\parallel A_{ii}^\parallel + \sum_i (\partial_\parallel [A_{ii}^\parallel])^2 + \sum_{i,j,i \neq j} (D^- A_{ii}^\parallel) D^- A_{ii}^\parallel + 2 \sum_i [A_{ii}^\parallel] \left[ \partial_\parallel [A_{ii}^\parallel] - ig \sum_{j', j' \neq i} (A_{ii}^{j'} D^- A_{k j'}^\parallel - (D^- A_{k j'}^\parallel) A_{k j'}^{i}) \right] \right\} +
\]
Here we have ignored some unessential surface terms.

Variation of the Lagrangian in \([A^ii_+], A^ij_+ at i \neq j\) leads to constraints, the solution of which can be written in the form

\[
[A^ii_+] = \partial^{-2}_- \left[ \partial_+ [A^ii_k] - ig \sum_{j', j' \neq i} \left( A^{ij'}_k D_- A^{j'i}_k - (D_- A^{ij'}_k) A^{j'i}_k \right) \right], \tag{2.56}
\]

\[
A^{ij'}_k \bigg|_{i \neq j} = D^{-2}_- \left( \partial_+ D_- A^{ij'}_k - ig \sum_{j'} \left( A^{ij'}_k D_- A^{j'i}_k - (D_- A^{ij'}_k) A^{j'i}_k \right) \right). \tag{2.57}
\]

The operator \(\partial^{-1}_-\) is completely defined, as before, by the condition (2.28) being well defined on functions \([f(x)]\). The operator \(D^{-1}_-\) after Fourier transformation in \(x^-\) is reduced to the multiplication by \((i(k_+ - g(v^i - v^j)))^{-1}\). Therefore it has, in general, no singularities for any \(k_- = n(\pi/L)\) with integer \(n\).

Substituting the expressions (2.56), (2.57) into the Lagrangian (2.55), we exclude from it the quantities \([A^ii_+]\) and \(A^ij_+ at i \neq j\). The variation of the Lagrangian in \(A^ii_{+0}\) leads to the constraint

\[
Q^{ii}(x^+, x^-) \equiv -2 \left( 2L \partial_+ \partial_- v^i + ig \int_{-L}^L dx^- \sum_{j', j' \neq i} \left( A^{ij'}_k D_- A^{j'i}_k - (D_- A^{ij'}_k) A^{j'i}_k \right) \right) = 0. \tag{2.58}
\]

It is a first class constraint which can be posed on physical state vectors after quantization. Therefore we can keep the term with this constraint in the Lagrangian.

Now we must put in the standard canonical form the terms of the Lagrangian

\[
\int d^2 x \int dx^- \left\{ 2 \sum_i (\partial_- [A^{ii}_k]) \partial_+ [A^{ii}_k] + 2 \sum_{i, j, i \neq j} (D_- A^{ij}_k) \partial_+ A^{ji}_k \right\}. \tag{2.59}
\]

It can be reached by going to Fourier transform

\[
[A^{ii}_k(x^-)] = \frac{1}{2\sqrt{2L}} \sum_{k_- = \pi/L}^{\infty} k_-^{-1/2} \left\{ a^i_k(k_-) \exp(-ik_-x^-) + a^{i+}_k(k_-) \exp(ik_-x^-) \right\}, \tag{2.60}
\]
where we sum over \( k_− = n\pi/L \), \( n = 1, 2, \ldots \), and

\[
A^i_j(x^-)|_{i \neq j} = \frac{1}{2\sqrt{2L}} \left\{ \sum_{k>_g(v^i-v^j)} \left( k_− - g(v^i - v^j) \right)^{-1/2} a^i_j(k_-) \exp(ik_-x^-) + \sum_{k>_g(v^j-v^i)} \left( k_− - g(v^j - v^i) \right)^{-1/2} a^i_j(k_-) \exp(-ik_-x^-) \right\},
\]

where we sum over all \( k_− = n\pi/L \) satisfying corresponding inequalities. The expression (2.59) takes the form

\[
(2i)^{-1} \int d^2x^\perp \left\{ \sum_i \sum_{k_−=\pi/L} \left( a^i_k(k_-)\partial_+(a^i_k)^+(k_-) - a^i_k(k_-)\partial_+a^i_k(k_-) \right) + \sum_{i,j, i \neq j} \sum_{k>_g(v^i-v^j)} \left( a^i_j(k_-)\partial_+a^{ij+}_k(k_-) - a^{ij+}_k(k_-)\partial_+a^{ij}_k(k_-) \right) \right\}.
\]

Further, in the Lagrangian (2.55) there is a part

\[
L_v = \int d^2x^\perp \left\{ 2L \sum_i (\partial_+v^i)^2 - 4L \sum_i (\partial_kA^{ij}_{k(0)})\partial_+v^i \right\}.
\]

The "momentum" conjugated to \( v^i \) is

\[
\mathcal{P}^i = \frac{\delta L_v}{\delta (\partial_+v^i)} = 4L \left( \partial_+v^i - \partial_kA^{ij}_{k(0)} \right),
\]

Hence,

\[
\partial_+v^i = \frac{1}{4L} \mathcal{P}^i + \partial_kA^{ij}_{k(0)}.
\]

The corresponding part of the Hamiltonian equals to

\[
H_v = \int d^2x^\perp \sum_i (\mathcal{P}^i\partial_+v^i) - L_v = \int d^2x^\perp 2L \sum_i \left( \frac{\mathcal{P}^i}{4L} + \partial_kA^{ij}_{k(0)} \right)^2,
\]

and the corresponding part of canonical Lagrangian is

\[
L_v = \int d^2x^\perp \sum_i (\mathcal{P}^i\partial_+v^i) - H_v.
\]

Excluding from the Lagrangian (2.55) the quantities \([A^i_k] \) and \( A^{ij}_+ \) (at \( i \neq j \)) vie the eqns. (2.60), (2.61), replacing the terms (2.59) by the expression (2.62) and the part (2.63)

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by the expression (2.67), we obtain the result

\[ L = (2i)^{-1} \int d^2 x^\perp \left\{ \sum_i \sum_{k_0} \left( a_i^k(k_-) \partial_+ a_i^k(k_-) - a_i^{i+}(k_-) \partial_+ a_i^i(k_-) \right) + \right. \\
+ \left. \sum_{i,j} \sum_{k_0} \left( a_i^{ij}(k_-) \partial_+ a_i^{ij}(k_-) - a_i^{i+j}(k_-) \partial_+ a_i^{ij}(k_-) \right) + \right. \\
+ \left. \sum_i P_i^i \partial_+ v^i + \sum_i A_{i+}^{ii} Q_{i+}^{ii} \right\} - H, \]

where \( Q_{i+}^{ii} \) are defined by (2.58), and the Hamiltonian \( H = P_+ \) is equal to

\[ H = \int d^2 x^\perp \int dx^- \left\{ \sum_i \left( \partial_- [A_i^{ii}] \right)^2 + \sum_{i,j} \left( D_- A_i^{ij} \right) D_- A_i^i + \frac{1}{2} \sum_{i,j} F_{kl}^{ij} F_{kl}^{ij} \right\} + \right. \\
+ \left. 2L \int d^2 x^\perp \sum_i \left( \frac{P_i^i}{4L} + \partial_k A_k^{ii} \right)^2. \]

It is implied that instead of the quantities \( [A_i^{ii}] \) and \( A_i^{ij} \) (at \( i \neq j \)) one uses the expressions (2.56), (2.57) and the \( A_i^{ij} \) are expressed in terms of \( a_i^k, a_i^k, a_i^{ij}, a_i^{i+j}, \) (at \( i \neq j \)) and of \( A_k^{ij} \) with the help of eqns. (2.60), (2.61) and

\[ A_i^{ii} = [A_i^{ii}] + A_i^{ii}. \]

It is seen from the formulae (2.68) that \( a_i^{i+}, a_i^k, a_i^{ij}, a_i^{i+j} \) play the role of creation and annihilation operators. After quantization they satisfy the following commutation relations (at \( x^+ = const \)):

\[ [a_i^k(k_-, x^+), a_i^{i+}(k_-, x^{+\perp})]_\_ = \delta^{ij} \delta_{kl} \delta_{k_- k_-'} \delta^2(x^+ - x^{+\perp}), \]

\[ [a_i^{ij}(k_-, x^+), a_i^{i+j}(k_-, x^{+\perp})]_\_ = \delta^{ij} \delta_{kl} \delta_{k_- k_-'} \delta^2(x^+ - x^{+\perp}), \quad i \neq j, \quad i' \neq j'. \]

Also we have

\[ [P_i^i(x^+), v^j(x^{+\perp})]_\_ = -i \delta^{ij} \delta^2(x^+ - x^{+\perp}). \]

Remaining commutators are equal to zero.

The operator of the momentum \( P_- \), defined by

\[ P_- = \int d^2 x \int dx^- T_{-}, \]

is equal to

\( (2.68) \), and the Hamiltonian \( H = P_+ \) is equal to

\( (2.69) \).

(2.70)
acts on physical states $|\Psi\rangle$, satisfying the condition

$$Q^{ii}(x^\perp)|\Psi\rangle = 0,$$

(2.75)
equivalent to the canonical operator

$$P_{\text{can}} = \int d^2x \left( \sum_i \sum_{k_-=\pi/L}^\infty k_- a_i^k(k_-)a_i^k(k_-) + \sum_{i,j,i\neq j} \sum_{k_->g(v^i-v^j)} k_- a_i^k(k_-)a_i^j(k_-) \right),$$

(2.76)
where the normal ordering was made.

Physical vacuum $|\Omega\rangle$ satisfies the relations

$$a_i^k(k_-,x^\perp)|\Omega\rangle = 0,$$

(2.77)
$$a_i^j(k_-,x^\perp)|\Omega\rangle = 0, \quad i \neq j,$$

(2.78)
and the condition (2.75).

This scheme is connected with the following essential difficulty. The zero modes $A_{k(0)}^{ii}(x^\perp)$ are present in the Lagrangian (2.68) and in the Hamiltonian (2.69) but the derivatives $\partial_+ A_{k(0)}^{ii}$ are absent there. Therefore new constraints arise

$$\frac{\delta H}{\delta A_{k(0)}^{ii}(x^\perp)} = 0.$$

(2.79)
These constraints are of the 2nd class and they must be solved with respect to $A_{k(0)}^{ii}$ and then the $A_{k(0)}^{ii}$ have to be excluded from the Hamiltonian. The constraints (2.79) are very complicated and explicit resolution of them is practically impossible. The application of Dirac brackets does not simplify this.

Due to this difficulty a practical calculation usually ignores all zero modes from the beginning. It makes the approximation worse. It is interesting that in the framework of lattice regularization it is possible to overcome the difficulties caused by the constraints (2.79) [16]. This question will be considered in Sect. 5.

## 3 Limiting transition from the theory in Lorentz coordinates to the theory on the Light Front

To clarify the connection between the theory in Lorentz coordinates in Hamiltonian form and analogous theory on the LF we perform the limiting transition from one to the other. Here this transition is considered in the fixed frame of Lorentz coordinates by introducing states that move at a speed close to the speed of light in the direction of the $x^3$ axis. Constructing the matrix elements of the Hamiltonian between such states and studying the limiting transition to the speed of light (an infinite momentum), we can derive information about the Hamiltonian in the light-like coordinates. This information
also takes into account the contribution from intermediate states with finite momenta. Here, we illustrate the results of such an investigation using \((1 + 1)\)-dimensional theory of scalar field with the \(\lambda \varphi^4\)-interaction. Instead of \(x^3\) we denote analogous space coordinate by \(x^1\). The generalization of the method to \((3 + 1)\)-dimensional Yukawa model is discussed briefly at the end of this section. The limiting transition studied here is accomplished approximately by subjecting the momenta \(p_1\) to an auxiliary cutoff that separates fast modes of the fields (with high \(p_1\) values) from slow modes (with finite \(p_1\) values). This cutoff is parametrized in terms of the quantities \(\Lambda, \Lambda_1,\) and \(\delta\) and the limiting-transition parameter \(\eta (\eta > 0, \eta \to 0)\): we have \(\eta^{-1} \Lambda_1 \geq |p_1| \geq \eta^{-1} \delta\) for the fast modes and \(p_1 \leq \Lambda\) for the slow modes \((\Lambda \gg \delta)\). For \(\eta \to 0\), the inequality \(\eta^{-1} \delta > \Lambda\) holds, so that the above momentum intervals are separated. The field modes with the momenta \(\eta^{-1} \delta > |p_1| > \Lambda\) are discarded. This procedure is justified by the fact that the resulting Hamiltonian in the limit \(\eta \to 0\) reproduces the canonical light-front Hamiltonian (without zero modes) when only the fast modes are taken into account and is consistent with conventional Feynman perturbation theory for \(\delta \to 0\). Therefore, even an approximate inclusion of the other (slow) modes may provide a description of nonperturbative effects, such as vacuum condensates. The effective light-front Hamiltonian obtained here for the model under consideration differs from the canonical Hamiltonian only by the presence of the vacuum expectation value of the scalar field and by an additional renormalization of the mass of this field. The renormalized mass involves the vacuum expectation value of the squared slow part of the field. Masses of bound states can be found by solving Schrodinger equation

\[
P_+|\Psi\rangle = \frac{m^2}{2p_-}|\Psi\rangle,
\]

with obtained Hamiltonian \(P_+\).

We start from the standard expression for the Hamiltonian of scalar field \(\varphi(x)\) in \((1 + 1)\)-dimensional space-time in Lorentz coordinates \(x^\mu = (x^0, x^1)\), at \(x^0 = 0\):

\[
H =: \int d^1x \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_1 \varphi)^2 + \frac{m^2}{2} \varphi^2 + \lambda \varphi^4 \right) :,
\]

where \(\Pi(x^1)\) are the variables that are canonically conjugate to \(\varphi(x^1) \equiv \varphi(x^0 = 0, x^1)\), and the symbol : : of the normal ordering refers to the creation and annihilation operators \(a\) and \(a^+\) that diagonalize the free part of the Hamiltonian in the Fock space over the corresponding vacuum \(|0\rangle\). These operators are given by

\[
\varphi(x^1) = \frac{1}{\sqrt{4\pi}} \int dp_1 (m^2 + p_1^2)^{-1/4} \left[ a(p_1) \exp(-ip_1 x^1) + h.c. \right] ,
\]

\[
\Pi(x^1) = \frac{-i}{\sqrt{4\pi}} \int dp_1 (m^2 + p_1^2)^{-1/4} \left[ a(p_1) \exp(-ip_1 x^1) - h.c. \right] ,
\]

where \(a(p_1)|0\rangle = 0\).
To investigate the limiting transition to the light-front Hamiltonian (defined at \(x^+ = 0\)), it is more convenient to go over from the Hamiltonian \((3.2)\) to the operator \(H + P_1 = \sqrt{2}P_+\), where the momentum \(P_1\) has the form
\[
P_1 = \int dp_1 a^+(p_1) a(p_1) p_1. \tag{3.5}
\]

Applying the above parametrization of high momenta in terms of \(\eta, \eta \to 0\), to \(p_1\) we can then consider the transition to an infinitely high momentum of states as a limit of the corresponding Lorentz transformation with parameter \(\eta\). To be more specific, we have \(p_1 \to (-\eta \sqrt{2})^{-1} q_-\), where \(q_-\) is a finite momentum in the light-like coordinates, and
\[
\lim_{\eta \to 0} \left((\eta \sqrt{2})^{-1} p'_1 |(H + P_1)_{x_0=0} | p_1\right) = \langle q'_- |(P_+)_{x^+ = 0} | q_-\rangle. \tag{3.6}
\]

It follows that the eigenvalues \(E_+\) of the operator \((P_+)_{x^+ = 0}\) that correspond to the momentum \(q_-\) are obtained as the corresponding limit of the eigenvalues \(E(\eta)\) of the operator \((H + P_1)_{x_0=0}\) at momentum \(p_1\):
\[
E_+ = \lim_{\eta \to 0} (\eta \sqrt{2})^{-1} E(\eta). \tag{3.7}
\]

In the following, we consider this limiting transition as a part of the eigenvalue problem for the operator \(H + P_1\), using perturbation theory in the parameter \(\eta\). Separating the Fourier modes of the field into fast and slow ones, as is indicated above, and neglecting the region of intermediate momenta \((\eta^{-1} \delta \leq |p_1| \leq \eta^{-1} \Lambda_1\) is the region of the fast modes, and \(|p_1| \leq \Lambda\) is the region of the slow modes), we can substantially simplify this perturbation theory. The \(\eta\) dependence of the field operators and Hamiltonian can then be determined by making, in the region of fast momenta, the change of the variables as
\[
p_1 = \eta^{-1} k, \quad a(p_1) = \sqrt{\eta} \tilde{a}(k), \quad \delta \leq |k| \leq \Lambda_1, \quad [\tilde{a}(k), \tilde{a}^+(k')] = \delta(k - k'). \tag{3.8}
\]

The fast part \([\varphi(x^1)]_f\) of the operator \(\varphi(x^1)\) is estimated as
\[
[\varphi(x^1)]_f = \tilde{\varphi}(y) + O(\eta^2), \quad y = \eta^{-1} x^1, \tag{3.9}
\]
\[
\tilde{\varphi}(y) = (4\pi)^{-1/2} \int dk |k|^{-1/2} [\tilde{a}(k) \exp(-iky) + h.c]. \tag{3.10}
\]

We denote the slow part of the field \(\varphi\) by \(\tilde{\varphi}\) \((\varphi = [\varphi]_f + \tilde{\varphi})\). Substituting formulas \((3.8)-(3.10)\) into Hamiltonian \((3.2)\), we obtain
\[
H + P_1 = \eta^{-1} \left(h_0 + \eta h_1 + \eta^2 h_2 + \ldots\right), \tag{3.11}
\]
\[
h_0 = 2 \int_{\delta}^{\Lambda_1} dk \tilde{a}^+(k) \tilde{a}(k)k, \tag{3.12}
\]
\[ h_1 = (H + P_1)_{\varphi=\bar{\varphi},\Pi=\bar{\Pi}} \equiv (\bar{H} + \bar{P}_1), \]  

(3.13)

\[ h_2 = \int dk \left( \frac{m^2}{2|k|} \right) \tilde{a}^+(k)\tilde{a}(k) + : \lambda \int dy \left[ \tilde{\varphi}^4(y) + 4\tilde{\varphi}(0)\tilde{\varphi}^3(y) + 6\tilde{\varphi}^2(0)\tilde{\varphi}^2(y) \right]:. \quad (3.14) \]

Prior to performing integration with respect to \( y \), we formally expanded the operators \( \tilde{\varphi}(x^1) = \tilde{\varphi}(\eta y) \) in Taylor series in the variable \( \eta y \) and estimated their orders in the parameter \( \eta \) at fixed \( y \). Such estimates can be justified at least in Feynman perturbation theory. The operator \((\bar{H} + \bar{P}_1)\) in (3.13) is defined in such a way that its minimum eigenvalue is zero. Let us consider perturbation theory in the parameter \( \eta \) for the equation

\[ (H + P_1)|f\rangle = E|f\rangle \]  

(3.15)

under the condition that the states \(|f\rangle\) have, as in formula (3.4), a negative value of \( P_1 \) proportional to \( \eta^{-1} \) for \( \eta \to 0 \) and also describe the states with a finite mass. The expansions of the quantity \( E \) and the vector \(|f\rangle\) in power series in \( \eta \) can then be written as

\[ E = \eta^{-1} \sum_{n=2}^{\infty} \eta^n E_n, \quad |f\rangle = \sum_{n=0}^{\infty} \eta^n |f_n\rangle. \]  

(3.16)

We arrive at the system of equations

\[ h_0|f_0\rangle = 0, \]  

(3.17)

\[ h_0|f_1\rangle + h_1|f_0\rangle = 0, \]  

(3.18)

\[ h_0|f_2\rangle + h_1|f_1\rangle + (h_2 - E_2)|f_0\rangle = 0, \quad \ldots. \]  

(3.19)

To describe solutions to these equations, we use the basis generated by the fast-field operators \( \tilde{a}^+(k) \) over the vacuum \(|0\rangle\) and the slow-field operators \( \tilde{\varphi} \) and \( \tilde{\Pi} \) over the vacuum \(|v\rangle\) that corresponds to the Hamiltonian \((\bar{H} + \bar{P}_1)\). The vectors of this basis can be symbolically represented as

\[ \tilde{a}^+ \ldots \tilde{a}^+|0\rangle \tilde{\varphi} \ldots \tilde{\varphi} \tilde{\Pi} \ldots \tilde{\Pi}|v\rangle. \]  

(3.20)

By virtue of (3.12), the manifold of solutions \(|f_0\rangle\) to equation (3.14) is reduced to the set of vectors (3.20), which do not contain the operators \( \tilde{a}^+(k) \) with \( k \geq \delta \). Let \( P_0 \) be the projection operator onto this set. According to equation (3.18), we then have

\[ P_0 h_0|f_1\rangle = (\bar{H} + \bar{P}_1)|f_0\rangle = 0. \]  

(3.21)

Equation (3.21) requires that the vectors \(|f_0\rangle\) be the lowest eigenstates of the operator \((\bar{H} + \bar{P}_1)\), that is, linear combinations of basis vectors (3.20) including neither the operators
\( \tilde{a}^+(k) \) with \( k \geq \delta \) nor the operators \( \tilde{\varphi} \) and \( \tilde{\Pi} \). We denote the projection operator on this set of vectors \( (3.20) \) by \( P'_0 \). To determine the quantity \( E_2 \) which we are interested in, it is sufficient to consider the \( P'_0 \)-projection of equation \( (3.19) \). Taking into account \( (3.17) \), \( (3.20) \), and \( (3.21) \), we find that \( E_2 \) appears as a solution to the eigenvalue problem

\[
\mathcal{P}'_0 h_2 | f_0 \rangle = E_2 | f_0 \rangle.
\]

Thus, in accordance with \( (3.6) \) and \( (3.7) \), the operator \( \mathcal{P}'_0 h_2 \mathcal{P}'_0 \) plays the role of the effective light-front Hamiltonian \( \mathcal{P}'_0 \). Substituting formula \( (3.14) \) for the operator \( \mathcal{P}'_0 \) into the expression for \( \mathcal{P}'_0 \), we take into account that, between the projection operators \( \mathcal{P}'_0 \), the contribution of the field modes with positive momenta \( (k \geq \delta) \) vanishes and that the products of the operators of the slow part of the field can be replaced with their expectation values for the vacuum \( |v \rangle \). In addition, we note that, under the Lorentz transformation corresponding to the limiting transition \( \eta \to 0 \) in formula \( (3.6) \), the variable \( y \) goes over into the light-like coordinate \( y^- = -y/\sqrt{2} \), the momenta \( k \) go over into the light-like momenta \( q_- = -\sqrt{2}k \), and the corresponding coordinate \( y^+ \) vanishes at \( x^0 = 0 \) (for finite values of \( y^- \)). Going over to the operators \( A(q_-) = 2^{-1/2} \tilde{a}(-k) \) for \( k \leq -\delta \) (\( q_- \geq \delta \sqrt{2} \)) and to the corresponding field \( \Phi(y^+ = 0, y^-) = \tilde{\varphi}_-(y) \), where \( \tilde{\varphi}_- \) is the part of the \( \tilde{\varphi} \) containing only the modes with negative momenta \( (k \leq -\delta) \), we obtain the effective Hamiltonian \( \mathcal{P}'_0 \) in the form

\[
\mathcal{P}'_0 = \int dy^- \left\{ \frac{1}{2} \left[ m^2 + 12\lambda \langle \tilde{\varphi}^2 \rangle_v \right] \Phi^2 + 4\lambda \langle \tilde{\varphi} \rangle_v \Phi^3 + \lambda \Phi^4 \right\}.
\]

where \( \langle \ldots \rangle_v \) is the expectation value for the vacuum \( |v \rangle \). Expression \( (3.23) \) coincides with the canonical effective Hamiltonian if, in the latter, we take into account the shift of the field by the constant \( \langle \tilde{\varphi} \rangle_v \) and the change in the mass squared by \( 12\lambda [\langle \tilde{\varphi}^2 \rangle_v - \langle \tilde{\varphi} \rangle_v^2] \).

Analogous results were obtained for Yukawa model in \((3 + 1)\)-dimensional space-time \( [9] \). In regularization of Pauli-Villars type, introducing a number of nonphysical fields with very large masses. The absence of essential difference between the Hamiltonian obtained via limiting transition and canonical LF Hamiltonian is connected with this choice of regularization. Other regularizations can lead to more complicated results.

This method of limiting transition can not be directly expanded to gauge theories, because the approximations used for nongauge theories are not justified.

4 Comparison of Light Front perturbation theory with the theory in Lorentz coordinates

As is already known, canonical quantization in LF, i.e., on the \( x^+ = \text{const} \) hypersurface, can result in a theory not quite equivalent to the Lorentz-invariant theory (i.e., to the standard Feynman formalism). This is due, first of all, to strong singularities at zero values of the "light-like" momentum variables \( Q_- = \frac{1}{\sqrt{2}}(Q_0 - Q_3) \). To restore the equivalence with a Lorentz-covariant theory, one has to add unusual counter-terms to the formal
canonical Hamiltonian for the LF, \( H = P_+ \). These counter-terms can be found by comparing the perturbation theory based on the canonical LF formalism with Lorentz-covariant perturbation theory. This is done in the present section. The light-front Hamiltonian thus obtained can then be used in nonperturbative calculations. It is possible, however, that perturbation theory does not provide all of the necessary additions to the canonical Hamiltonian, as some of these additions can be nonperturbative. In spite of this, it seems necessary to examine this problem within the framework of perturbation theory first.

For practical purposes a stationary noncovariant light-front perturbation theory, which is similar to the one applied in nonrelativistic quantum mechanics, is widely used. It was found [20, 21, 22] that the ”light-front” Dyson formalism allows this theory to be transformed into an equivalent light-front Feynman theory (under an appropriate regularization). Then, by re-summing the integrands of the Feynman integrals, one can recast their form so that they become the same as in the Lorentz-covariant theory. (This is not the case for diagrams without external lines, which we do not consider here.) Then, the difference between the light-front and Lorentz-covariant approaches that persists is only due to the different regularizations and different methods of calculating the Feynman integrals (which is important because of the possible absence of their absolute convergence in pseudo-Euclidean space). In the present section, we concentrate on the analysis of this difference.

A light-front theory needs not only the standard UV regularization, but also a special regularization of the singularities \( Q_\perp = 0 \). In our approach, this regularization (by method (1.1)) eliminates the creation operators \( a^+(Q) \) and annihilation operators \( a(Q) \) with \( |Q_\perp| < \varepsilon \) from the Fourier expansion of the field operators in the field representation. As a result, the integration w.r.t. the corresponding momentum \( Q_- \) over the range \((-\infty, -\varepsilon) \cup (\varepsilon, \infty)\) is associated with each line before removing the \( \delta \)-functions. Different propagators are regularized independently, which allows the described re-arrangement of the perturbation theory series. On the other hand, this regularization is convenient for further nonperturbative numerical calculations with the light-front Hamiltonian, to which the necessary counter-terms are added (the ”effective” Hamiltonian). We require that this Hamiltonian generate a theory equivalent to the Lorentz-covariant theory when the regularization is removed. Note that Lorentz-invariant methods of regularization (e.g., Pauli-Villars regularization) are far less convenient for numerical calculations and we shall only briefly mention them.

The specific properties of the light-front Feynman formalism manifest themselves only in the integration over the variables \( Q_\pm = \frac{1}{\sqrt{2}}(Q_0 \pm Q_3) \), while integration over the transverse momenta \( Q_\perp \equiv \{Q_1, Q_2\} \) is the same in the light-front and the Lorentz coordinates (though it might be nontrivial because it requires regularization and renormalization). Therefore, we concentrate on a comparison of diagrams for fixed transverse momenta (which is equivalent to a two-dimensional problem).

In this section, we propose a method that allows one to find the difference (in the limit \( \varepsilon \to 0 \)) between any light-front Feynman integral and the corresponding Lorentz-covariant integral without having to calculate them completely. Based on this method, a procedure is elaborated for constructing an effective Hamiltonian in LF in any order of perturbation
theory. The procedure can be applied to all nongauge field theories, as well as to Abelian and non-Abelian gauge theories in the gauge $A_- = 0$ with the vector meson propagator chosen according to the Mandelstam-Leibbrandt prescription $[23, 24]$. The question of whether the additional components of the Hamiltonian that arise can be combined into a finite number of counter-terms must be dealt with separately in each particular case.

Application of this formalism to the Yukawa model makes it possible to obtain the effective light-front Hamiltonian in a closed form. The result agrees with the conclusions of the work $[20]$, where a comparison was made of the light-front and Lorentz-covariant methods via calculating self-energy diagrams in all orders of perturbation theory and other diagrams in lowest orders. Conversely, for gauge theories (both Abelian and non-Abelian), it was found that counter-terms of arbitrarily high order in field operators must be added to the effective Hamiltonian. This result may turn out to be wrong if the contributions to the counter-terms are mutually canceled. This calls for further investigation, but such possibility appears to be very unlikely.

What we have said above does not depreciate the light-front formalism as applied to gauge theories. This is because the only requirement concerning the light-front Hamiltonian is that it correctly reproduces all gauge-invariant quantities rather than the off-mass-shell Feynman integrals in a given gauge. However, renormalization of the light-front Hamiltonian turns out to be a difficult problem and it requires new approaches. We do not examine the possibilities of changing the light-front Hamiltonian by introducing new nonphysical fields by a method different from the Pauli-Villars regularization $[25]$ or the possibilities of using gauges more general than $A_- = 0$ with the Mandelstam-Leibbrandt propagator. These points also need to be investigated further.

4.1 Reduction of light-front and Lorentz-covariant Feynman integrals to a form convenient for comparison

Let us examine an arbitrary IPI Feynman diagram. We fix all external momenta and all transverse momenta of integration, and integrate only over $Q_+$ and $Q_-:

$$F = \lim_{\varepsilon \to 0} \int \frac{d^2 Q^i}{\prod_i(2Q_+Q_- - M_i^2 + i\varepsilon)}.$$  (4.1)

We assume that all vertices are polynomial and that the propagator has the form

$$\frac{z(Q)}{Q^2 - m^2 + i\varepsilon}, \quad \text{or} \quad \frac{z(Q) Q_+}{(Q^2 - m^2 + i\varepsilon)(2Q_+Q_- + i\varepsilon)},$$  (4.2)

where $z(Q)$ is a polynomial. A propagator of the second type in (4.2) arises in gauge theories in the gauge $A_- = 0$ if the Mandelstam-Leibbrandt formalism $[23, 24]$ with the vector boson propagator

$$\frac{1}{Q^2 + i\varepsilon} \left( g_{\mu\nu} - \frac{(\delta^+_{\mu} Q_\nu + Q_\mu \delta^+_{\nu})Q_+}{2Q_+Q_- + i\varepsilon} \right),$$

20
is used. In eq. (4.1) either $M^2_i = m^2_i + Q^2_i \neq 0$, where $m_i$ is the particle mass, or $M^2_i = 0$.

The function $f$ involves the numerators of all propagators and all vertices with the necessary $\delta$-functions, that include the external momenta $p^k$ (the same expression without the $\delta$-functions is a polynomial, which we denote by $\tilde{f}$). We assume for the diagram $F$ and for all of its subdiagrams that the conditions

$$\omega_i < 0, \quad \omega_+ < 0,$$

(4.3)

hold, where $\omega_+$ is the index of divergence w.r.t. $Q_+$ at $Q^i_+ \neq 0 \forall i$, and $\omega_i$ is the index of divergence in $Q_+$ and $Q_-$ (simultaneously); $Q_\pm = \frac{1}{\sqrt{2}}(Q_0 \pm Q_3)$. The diagrams that do not meet these conditions should be examined separately for each particular theory (their number is usually finite). We seek the difference between the value of integral (4.1) obtained by the Lorentz-covariant calculation and its value calculated in light-front coordinates (light-front calculation).

In the light-front calculation, one introduces and then removes the light-front cutoff $|Q_\pm| \geq \varepsilon > 0$:

$$F_{lf} = \lim_{\varepsilon \to 0} \lim_{\omega_+ \to 0} \int_{V_\varepsilon} \prod_i dQ_i^- \int \prod_i dQ_i^+ \frac{f(Q_i^i, p^k)}{\prod_i(2Q_i^+Q_i^- - M^2_i + i\varepsilon)},$$

where $V_\varepsilon = \prod_i ((-\infty, -\varepsilon) \cup (\varepsilon, \infty))$. Here (and in the diagram configurations to be defined below) we take the limit w.r.t. $\varepsilon$, but, generally speaking, this limit may not exist. In this case, we assume that we do not take the limit, but take the sum of all nonpositive power terms of the Laurent series in $\varepsilon$ at the zero point. If conditions (4.3) are satisfied, Statement 2 from Appendix I can be used. This results in the equality

$$F_{lf} = \lim_{\varepsilon \to 0} \lim_{\omega_+ \to 0} \int \prod_i dq_i^+ \int_{V_\varepsilon \cap B_L} \prod_k dq_k^+ \frac{\tilde{f}(Q_i^i, p^k)}{\prod_i(2Q_i^+Q_i^- - M^2_i + i\varepsilon)}.$$  

(4.4)

From here on, the momenta of the lines $Q^i$ are assumed to be expressed in terms of the loop momenta $q^k$, $B_L$ is a sphere of a radius $L$ in the $q^k$-space, and $L$ depends on the external momenta. Now, using Statement 2 from Appendix I, we obtain

$$F_{lf} = \lim_{\varepsilon \to 0} \lim_{\omega_+ \to 0} \lim_{\omega_- \to 0} \int \prod_k dq_k^+ \int \prod_k dq_k^- \frac{\tilde{f}(Q_i^i, p^k) e^{-\gamma} \sum_i Q_i^+ - \beta \sum_i Q_i^-} {\prod_i(2Q_i^+Q_i^- - M^2_i + i\varepsilon)}. \quad (4.5)$$

To reduce the covariant Feynman integral to a form similar to (4.4), we introduce a quantity $\tilde{F}$:

$$\tilde{F} = \lim_{\omega_+ \to 0} \lim_{\omega_- \to 0} \lim_{\omega_\alpha \to 0} \int \prod_k d^2 q_k^+ \frac{\tilde{f}(Q_i^i, p^k) e^{-\gamma} \sum_i Q_i^+ - \beta \sum_i Q_i^-} {\prod_i(2Q_i^+Q_i^- - M^2_i + i\varepsilon)}. \quad (4.6)$$

Let us prove that this quantity coincides with the result of the Lorentz-covariant calculation $F_{cov}$. To this end, we introduce the $\alpha$-representation in the Minkowski space of the
Then we substitute (4.7) into (4.6). Due to the exponentials that cut off \( q_k^+ \), \( q_k^- \) and \( \alpha^i \) the integral over these variables is absolutely convergent. Therefore, one can interchange the integrations over \( q_k^+ \), \( q_k^- \) and \( \alpha^i \). As a result, we obtain the equality

\[
\hat{F} = \lim_{\alpha \to 0} \lim_{\beta \to 0} \lim_{\gamma \to 0} \int_0^\infty \prod \alpha_i \phi(\alpha_i, p^s, \gamma, \beta) e^{-\sum_i \alpha_i}, \tag{4.8}
\]

where

\[
\phi(\alpha_i, p^s, \gamma, \beta) = (-i)^n \tilde{f} \left( \frac{-i}{\partial y_i} \right) \times \int \prod d^2 q^k e^{\sum_i \left[ i\alpha_i(2Q_i^+Q_i^- - M_i^2) + i(Q_i^+ y_i^+ + Q_i^- y_i^-) - \gamma Q_i^+ - \beta Q_i^- \right]} \bigg|_{y_i=0} \tag{4.9}
\]

For the Lorentz-covariant calculation in the \( \alpha \)-representation satisfying conditions (4.3), there is a known expression [26]

\[
F_{\text{cov}} = \lim_{\alpha \to 0} \int_0^\infty \prod \alpha_i \varphi_{\text{cov}}(\alpha_i, p^s) e^{-\sum_i \alpha_i}, \tag{4.10}
\]

where

\[
\varphi_{\text{cov}}(\alpha_i, p^s) = (-i)^n \tilde{f} \left( \frac{-i}{\partial y_i} \right) \times \lim_{\gamma, \beta \to 0} \int \prod d^2 q^k e^{\sum_i \left[ i\alpha_i(2Q_i^+Q_i^- - M_i^2) + i(Q_i^+ y_i^+ + Q_i^- y_i^-) - \gamma Q_i^+ - \beta Q_i^- \right]} \bigg|_{y_i=0} \tag{4.11}
\]

In Appendix 2, it is shown that in (4.8) the limits in \( \gamma \) and \( \beta \) can be interchanged, in turn, with the integration over \( \{\alpha_i\} \), and then with \( \tilde{f} \left( \frac{-i}{\partial y_i} \right) \). After that, a comparison of relations (4.8), (4.9) and (4.10), (4.11), clearly shows that \( \hat{F} = F_{\text{cov}} \). Considering (4.6) and using Statement 1 from Appendix 1, we obtain the equality

\[
F_{\text{cov}} = \lim_{\alpha \to 0} \int \prod \frac{dq^k_+}{B_L} \prod \frac{dq^k_-}{B_L} \frac{\tilde{f}(Q^i, p^s)}{\prod_i(2Q_i^+Q_i^- - M_i^2 + i\alpha)} \tag{4.12}
\]

Expression (4.12) differs from (4.4) only by the range of the integration over \( q_k^k \).
4.2 Reduction of the difference between the light-front and Lorentz-covariant Feynman integrals to a sum of configurations

Let us introduce a partition for each line,

\[
\left( \int_{-\infty}^{\varepsilon} dQ_- + \int_{\varepsilon}^{\infty} dQ_- \right) = \left[ \int dQ_- + (-1) \int_{-\varepsilon}^{\varepsilon} dQ_- \right].
\] (4.13)

We call a line with integration w.r.t. the momentum \( Q_i \) in the range \((-\varepsilon, \varepsilon)\) (before removing the \( \delta \)-functions) a type-1 line, a line with integration in the range \((-\infty, -\varepsilon) \cup (\varepsilon, \infty)\) a type-2 line, and a line with integration over the whole range \((-\infty, \infty)\) a full line. In the diagrams, they are denoted as shown in Figs. la, b, and c, respectively.

\[a\quad b\quad c\quad d\quad e\]

Fig. 1: Notation for different types of lines in the diagrams: "a" is a type-1 line, "b" is a type-2 line, "c" is a full line, "d" is an \( \varepsilon \)-line, and "e" is a \( \Pi \)-line.

Let us substitute partition (4.13) into expression (4.4) for \( F_{lf} \) and open the brackets. Among the resulting terms, there is \( F_{cov} \) (expression (4.12)). We call the remaining terms "diagram configurations" and denote them by \( F_j \). Then we arrive at the relation

\[ F_{lf} - F_{cov} = \sum_j F_j, \]

where

\[ F_j = \lim_{\varepsilon \to 0} \lim_{\varepsilon \to 0} \int \prod_k dq_+^k \int_{V_j^\varepsilon \cap B_L} \frac{f(Q^i; p^s)}{\prod_i (2Q_+^i Q_-^i - M_i^2 + i\varepsilon)}, \]

and \( V_j^\varepsilon \) is the region corresponding to the arrangement of full lines and type-1 lines in the given configuration.

Note that before taking the limit in \( \varepsilon \), Eqs. (4.12) and (4.14) can be used successfully: first, they are applied to a subdiagram and, then, are substituted into the formula for the entire diagram. This is admissible because, after the deformation of the contours described in the proof of Statement 1 from Appendix 1, the integral over the loop momenta \( \{q_+^k\} \) of the subdiagram converges (after integration over the variables \( \{q_-^k\} \) of this subdiagram) absolutely and uniformly with respect to the remaining loop momenta \( \{q_-^{k'}\} \). Therefore, one can interchange the integrals over \( \{q_+^k\} \) and \( \{q_-^{k'}\} \).

Thus, the difference between the light-front and Lorentz-covariant calculations of the diagram is given by the sum of all of its configurations. A configuration of a diagram is the same diagram, but where each line is labeled as a full or type-1 line, provided that at least one type-1 line exists.
4.3 Behavior of the configuration as $\varepsilon \to 0$

We assume that all external momenta $p^s$ are fixed for the diagram in question and

$$p^s_\varepsilon \neq 0, \quad \sum_{s'} p^{s'}_\varepsilon \neq 0,$$

where the summation is taken over any subset of external momenta; all of these momenta are assumed to be directed inward.

Let us consider an arbitrary configuration. We apply the term ”$\varepsilon$-line” to all type-1 lines and those full lines for which integration over $Q_-$ actually does not expand outside the domain $(-r\varepsilon, r\varepsilon)$, where $r$ is a finite number (below, we explain when these lines appear). The remaining full lines are called II-lines. In the diagrams, the $\varepsilon$-lines and II-lines are denoted as shown in Figs. 1d and e, respectively. Note that the diagram can be drawn with lines ”a” and ”c” from Fig. 1 (this defines the configuration unambiguously), or with lines ”d” and ”e” (then the configuration is not uniquely defined).

If among the lines arriving at the vertex only one is full and the others are type-1 lines, this full line is an $\varepsilon$-line by virtue of the momentum conservation at the vertex. The remaining full lines form a subdiagram (probably unconnected). By virtue of conditions (4.13), there is a connected part to which all of the external lines are attached. All of the external lines of the remaining connected parts are $\varepsilon$-lines. Consequently, using Statement 1 from Appendix 1, we can see that integration over the internal momenta of these connected parts can be carried out in a domain of order $\varepsilon$ in size, i.e., all of their internal lines are $\varepsilon$-lines. Thus, an arbitrary configuration can be drawn as in Fig. 2 and integral (4.14), with the corresponding integration domain, is associated with it.

Let us investigate the behavior of the configuration as $\varepsilon \to 0$. From here on, it is convenient to represent the propagator as

$$\tilde{z}(Q) = \frac{z(Q)}{Q^2 - m^2 + i\varepsilon}, \quad \text{where} \quad \tilde{z}(Q) = z(Q) \quad \text{or} \quad \tilde{z}(Q) = \frac{z(Q)}{2Q_- + i\varepsilon/Q_+}. \quad (4.16)$$

rather than as (4.2). Then, in (4.1), $M_1^2 = m^2 + Q_-^2 \neq 0$ and the function $\tilde{f}$ is no longer a polynomial. If the numerator of the integrand consists of several terms, we consider each

![Fig. 2: Form of an arbitrary configuration: Π is the connected subdiagram consisting of II-lines, $\mathcal{E}$ is the subdiagram consisting of $\varepsilon$-lines and, probably, containing no vertices.](image-url)
term separately (except when the terms arise from expressing the propagator momentum $Q_i^\perp$ in terms of loop and external momenta).

We denote the loop momenta of subdiagram $\Pi$ in Fig. 2 by $q^l$ and the others by $k^m$. We make following change of integration variables in (4.14):

$$k^m_m \rightarrow \varepsilon k^m_m. \quad (4.17)$$

Then, the integration over $k^m_m$ goes within finite limits independent of $\varepsilon$. We denote the power of $\varepsilon$ in the common factor by $\tau$ (it stems from the volume elements and the numerators when the transformation (4.17) is made). The contribution to $\tau$ from the expression $1/(2Q_+ + i\varepsilon/Q_+)$ (Eq. (4.16)), which is related to the $\varepsilon$-line, is equal to $-1$. We divide the domain of integration over $k^m_m$ and $q^l_l$ into sectors such that the momenta $Q^\perp_i$ of $\Pi$-lines are separated from zero by an $\varepsilon$-independent constant, the corresponding $\Pi$-line-related propagators and factors from the vertices can be expanded in a series in $\varepsilon$. This expansion commutes with integration.

It is also clear that the denominators of the propagators allow the following estimates under an infinite increase in $|Q_+|:$

$$\left| \frac{1}{2Q_+Q_- - M^2 + i\varepsilon} \right| \leq \begin{cases} 
\frac{1}{c |Q_+|} & \text{for } \Pi\text{-lines,} \\
\frac{1}{\tilde{c} \varepsilon |Q_+|} & \text{for } \varepsilon\text{-lines,} 
\end{cases} \quad (4.18)$$

$$\left| \frac{1}{2Q_- + i\varepsilon/Q_+} \right| \rightarrow \left| \frac{1}{2Q_- + i\varepsilon/(Q_+\varepsilon)} \right| \leq \frac{1}{2|Q_-|}, \quad (4.19)$$

Here $c$ and $\tilde{c}$ are $\varepsilon$-independent constants. Note that for fixed finite $Q_+$, the estimated expressions are bounded as $\varepsilon \rightarrow 0$. After transformation (4.17) and release of the factor $1/\varepsilon$ (in accordance with what was said about the contribution to $\tau$), the $\varepsilon$-line-related expression from (4.16) becomes

$$\left| \frac{1}{2Q_- + i\varepsilon/Q_+} \right| \rightarrow \left| \frac{1}{2Q_- + i\varepsilon/(Q_+\varepsilon)} \right| \leq \frac{1}{2|Q_-|},$$

where a $Q_+$-independent quantity was used for the estimate (this quantity is meaningful and does not depend on $\varepsilon$ because the value of $Q_-$ is separated from zero by an $\varepsilon$-independent constant).

We integrate first over $q^l_l$, $k^m_m$ within one sector and then over $q^l_l$, $k^m_m$ (the latter integral converges uniformly in $\varepsilon$). Let us examine the convergence of the integral over $q^l_l$, $k^m_m$ with canceled denominators of the $\varepsilon$-lines (which is equivalent to estimating the expressions (4.19) by a constant). If it converges, then the initial integral is obviously independent of $\varepsilon$ and the contribution from this sector to the configuration is proportional to $\varepsilon^\tau$.\",null
Let us show that if it diverges with a degree of divergence \( \alpha \), the contribution to the initial integral is proportional to \( \varepsilon^{\tau-\alpha} \) up to logarithmic corrections. To this end, we divide the domain of integration over \( q_l^+, k_m^+ \) into two regions: \( U_1 \), which lies inside a sphere of radius \( \Lambda/\varepsilon \) (\( \Lambda \) is fixed), and \( U_2 \), which lies outside this sphere (recall that in our reasoning, we deal with each sector separately). Now we estimate (4.18) (like (4.19)) in terms of \( \varepsilon^{-\alpha} \) (which is admissible) and change the integration variables as follows:

\[
q_l^+ \rightarrow \frac{1}{\varepsilon} q_l^+, \quad k_m^+ \rightarrow \frac{1}{\varepsilon} k_m^+.
\]

After \( \varepsilon \) is factored out of the numerator and the volume element, the integrand becomes independent of \( \varepsilon \). Thus, the integral converges.

One can choose such \( \Lambda \) (independent of \( \varepsilon \)) that the contribution from the domain \( U_2 \) is smaller in absolute value than the contribution from the domain \( U_1 \). Consequently, the whole integral can be estimated via the integral over the finite domain \( U_1 \). Now we make an inverse replacement in (4.20) and estimate (4.19) by a constant (as above). Since the size of the integration domain is \( \Lambda/\varepsilon \) and the degree of divergence is \( \alpha \), the integral behaves as \( \varepsilon^{-\alpha} \) (up to logarithmic corrections), q.e.d. This reasoning is valid for each sector and, thus, for the configuration as a whole. Obviously,

\[
\alpha = \max_r \alpha_r,
\]

where \( \alpha_r \) is the subdiagram divergence index and the maximum is taken over all subdiagrams \( D_r \) (including unconnected subdiagrams for which \( \alpha_r \) is the sum of the divergence indices of their connected parts). In the case under consideration, \( \alpha_r = \omega_r^+ + \nu_r^+ \), where \( \nu_r^+ \) is the number of internal \( \varepsilon \)-lines in the subdiagram \( D_r \). The quantities \( \omega_r^+ \) are the UV divergence indices of the subdiagram \( D_r \) w.r.t. \( Q_\pm \).

Above, we introduced a quantity \( \tau \), which is equal to the power of \( \varepsilon \) that stems from the numerators and volume elements of the entire configuration. We can write \( \tau = \omega_\pm^- - \mu_\pm^+ + \nu_\pm^+ + \eta_\pm^+ \), where \( \mu_\pm^+ \) is the index of the UV divergence in \( Q_- \) of a smaller subdiagram (probably, a tree subdiagram or a nonconnected one) consisting of \( \Pi \)-lines entering \( D_r \). The term \( \eta_\pm^+ \) is the power of \( \varepsilon \) in the common factor, which, during transformation (4.17), stems from the volume elements and numerators of the lines that did not enter \( D_r \). (It is implied that the integration momenta are chosen in the same way as when calculating the divergence indices of \( D_r \).) Then, up to logarithmic corrections, we have

\[
F_j \sim \varepsilon^\sigma, \quad \sigma = \min_r (\tau, \omega_\pm^- - \omega_\pm^+ - \mu_\pm^+ + \eta_\pm^+).
\]

Consequently, for \( \varepsilon \to 0 \), the configuration is equal to zero if \( \sigma > 0 \). Relation (4.22) allows all essential configurations to be distinguished.

### 4.4 Correction procedure and analysis of counter-terms

We want to build a corrected light-front Hamiltonian \( H^\text{cor}_{lf} \) with the cutoff \( |Q^-_\pm| > \varepsilon \), which would generate Green’s functions that coincide in the limit \( \varepsilon \to 0 \) with covariant Green’s
functions within the perturbation theory. We begin with a usual canonical Hamiltonian in the light-front coordinates $H_{\text{lf}}$ with the cutoff $|Q_i| > \varepsilon$. We imply that the integrands of the Feynman diagrams derived from this light-front Hamiltonian coincide with the covariant integrands after some resummation [20, 21, 22]. However, a difference may arise due to the various methods of doing the integration, e.g., due to different auxiliary regularizations. As shown in Sec. 4.2, this difference (in the limit $\varepsilon \to 0$) is equal to the sum of all properly arranged configurations of the diagram. One should add such correcting counter-terms to $H_{\text{lf}}$, which generates additional “counter-term” diagrams, that reproduce nonzero (after taking limit w.r.t. $\varepsilon$) configurations of all of the diagrams. Were we able to do this, we would obtain the desired $H_{\text{lf}}^{\text{cor}}$. In fact, we can only show how to seek the $H_{\text{lf}}^{\text{cor}}$ that generates the Green’s functions coinciding with the covariant ones everywhere except the null set in the external momentum space (defined by condition (4.15)). However, this restriction is not essential because this possible difference does not affect the physical results.

Our correction procedure is similar to the renormalization procedure. We assume that the perturbation theory parameter is the number of loops. We carry out the correction by steps: first, we find the counterterms to the Hamiltonian that generate all nonzero configurations of the diagrams up to the given order and, then, pass to the next order. We take into account that this step involves the counter-term diagrams that arose from the counter-terms added to the Hamiltonian for lower orders. Thus, at each step, we introduce new correcting counter-terms that generate the difference remaining in this order. Let us show how to successfully look for the correcting counter-terms.

We seek counter-terms by the induction method. It is clear that, in the first order in the number of loops, all nonzero configurations are primary. We add the counter-terms that generate them to the Hamiltonian. Now, we examine an arbitrary order of perturbation theory. We assume that in lower orders, all nonzero configurations that can be derived from the counter-terms, accounting for the above comment, have already been generated by the Hamiltonian.

Let us proceed to the order in question. First, we examine nonzero configurations
with only one loop momentum \( k \) and a number of momenta \( q \) (see the notation above Eq. (4.17)). We break the configuration lines one by one without touching the other lines (so that the ends of the broken lines become external lines). The line break may result in a structure that is not a configuration (if conditions (4.13) are violated); a line break may also result in a zero configuration or in a nonzero configuration. If the first case is realized for each broken line, then the initial configuration is primary and it must be generated by the counter-terms of the Hamiltonian in the order under consideration. If breaking of each line results in either the first or the second case, we call the initial configuration real and it must be also generated in this order.

Assume that breaking a line results in the third case. This means that the resulting configuration stems from counter-terms in the lower orders. Then, after restoration of the broken line (i.e., after the appropriate integration), it turns out that the counter-terms of the lower orders have generated the initial configuration (we take into account the comment on successive application of Eq. (4.14); see the end of Sec. 4.2) with the following distinctions: (i) the broken line (and, probably, some others, if a nonsimply connected diagram arises after breaking the line) is not a \( \Pi \)-line but a type-2 line, due to the conditions \( |p_s^x| > \varepsilon \); (ii) if, after restoration of the broken line, the behavior at small \( \varepsilon \) becomes worse (i.e., \( \sigma \) decreased), then fewer terms than are necessary for the initial configuration were considered in the above-mentioned series in \( \varepsilon \). We expand these arising type-2 lines by formula (4.13) and obtain a term where all of these lines are replaced by \( \Pi \)-lines or other terms where some (or all) of these lines have become type-1 lines. In the latter case, one of the momenta \( q \) becomes the momentum \( k \). We call these terms "repeated parts of the configuration" and analyze them together with the configurations that have two momenta \( k \). In the former case, we obtain the initial configuration up to distinction (ii). We add a counter-term to the Hamiltonian that compensates this distinction (the counter-term diagrams generated by it are called the compensating diagrams).

If there is only one line for which the third case is realized, it turns out that, in the given order, it is not necessary to generate the initial configuration by the counter-terms, except for the compensating addition and the repeated part that is considered at the next step. If there are several lines for which the third case is realized, the initial configuration is generated in lower orders more than once. For compensation, it should be generated (with the corresponding numerical coefficient and the opposite sign) by the Hamiltonian counter-terms in the given order. We call this configuration a secondary one. Next, we proceed to examine configurations with two momenta \( k \) and so on up to configurations with all momenta \( k \), which are primary configurations.

Thus, the configurations to be generated by the Hamiltonian counter-terms can be primary (not only the initial primary configurations but also the repeated parts analogous to them, called primary-like), real, compensating, and secondary. If the theory does not produce either the loop consisting only of lines with \( Q_+ \) in the numerator (accounting for contributions from the vertices) or a line with \( Q_+^{n} \) in the numerator for \( n > 1 \), then real configurations are absent because a line without \( Q_+ \) in the numerator can always be broken without increasing \( \sigma \) (see Eq. (4.22)). It is not difficult to demonstrate that if each
appearing primary, real, and compensating configuration has only two external line, then there are no secondary configurations at all.

The dependence of the primary configuration on external momenta becomes trivial if its degree of divergence $\alpha$ is positive, the maximum in formula (4.21) is reached on the diagram itself, and $\sigma = 0$. Then, only the first term is taken into account in the above-mentioned series. Thus, not all of the $\Pi$-line-related propagators and vertex factors depend on $k_m$ and they can be pulled out of the sign of the integral w.r.t. $\{k^m\}$ in (4.14).

We then obtain

$$F^\text{prim}_j = \lim_{\varepsilon \to 0} \lim_{\varepsilon \to 0} \int \prod_m dk^m_+ \prod_i (2Q^+_i Q^-_i - M^2_i + i\varepsilon) \times \prod_m dk^m_- \prod_k (2Q^+_k Q^-_k - M^2_k + i\varepsilon),$$

where $V_\varepsilon$ is a domain of order $\varepsilon$ in size. Let us carry out transformations (4.17) and (4.20). For the denominator of the $\Pi$-line, we obtain

$$\frac{1}{2( \frac{1}{\varepsilon} \sum k_+ + \sum p_+) (\sum p_-) - M^2 + i\varepsilon} \to \frac{\varepsilon}{2(\sum k_+) (\sum p_-)}.$$ 

Here we neglect terms of order $\varepsilon$ in the denominator because the singularity at $k^m_+ = 0$ is integrable under the given conditions for $\alpha$ and everything can be calculated in zero order in $\varepsilon$ at $\sigma = 0$. Thus, the dependence on external momenta can be completely collected into an easily obtained common factor.

### 4.5 Application to the Yukawa model

The Yukawa model involves diagrams that do not satisfy condition (4.3). These are displayed in Figs. 3a and b. We have $\omega_\parallel = 0$ for diagram "a" and $\omega_+ = 0$ for diagram "b".

![Yukawa model diagrams](image)

Fig. 3: Yukawa model diagrams that do not meet condition (4.3).

Nevertheless, these diagrams can be easily included in the general scheme of reasoning. To this end, one should subtract the divergent part, independent of external momenta, in the integrand of the logarithmically divergent (in two-dimensional space, with fixed internal transverse momenta) diagram "a". We obtain an expression with $\omega_\parallel < 0$ (i.e.,
which converges in two-dimensional space) and \( \omega_+ = 0 \), as in diagram "]b\]. This means that the integral over \( q_+ \) converges only in the sense of the principal value (and it is this value of the integral that should be taken in the light-front coordinates to ensure agreement with the stationary noncovariant perturbation theory). This value can be obtained by distinguishing the \( q_+ \)-even part of the integrand.

Two approaches are possible. One is to introduce an appropriate regularization in transverse momenta and to imply integration over them; then, it is convenient to distinguish the part that is even in four-dimensional momenta \( q \). The other is to keep all transverse momenta fixed; then, the part that is even in longitudinal momenta \( q_\parallel \) can be released. For the Yukawa theory, we use the first approach. For the transverse regularization, we use a "]smearing" of vertices, which is equivalent to dividing each propagator by \( 1 + Q_\perp^2 / \Lambda_\perp^2 \). In four-dimensional space, diagram "]a\] diverges quadratically. Under introduction and subsequent removal of the transverse regularization, the divergent part, which was previously subtracted from this diagram, acquires the form \( C_1 + C_2 p_\perp^2 \).

After separating the even part of the regularized expression, we fix all of the transverse momenta again. Then it turns out that diagrams "]a\] and "]b\] in Fig. 3 meet conditions [1.3] and one can show that after all of the operations mentioned, the exponent \( \sigma \) (see [1.22]) does not decrease for any of their configurations. Hence, they can be included in the general scheme without any additional corrections.

Let us first analyze the primary configurations (see the definition in Sec. 4.4). In the numerators, \( k_- \) appears only in the zero or one power and there are no loops where the numerators of all of the lines contain \( k_- \). Consequently, one always has \( \tau > 0 \), \( \mu^r \leq 0 \), and \( \eta^r \geq 0 \) (see the definitions in Sec. 4.3). Analyzing the properties of the expression \( \omega_- - \omega_+ \) for the Yukawa model diagrams, we conclude from [1.22] that \( \sigma \geq 0 \) always holds. The general form of the nonzero primary configurations with \( \sigma = 0 \) is depicted in Fig. 4. Note that they are all configurations with two external line.

Fig. 4: Nonzero configurations in the Yukawa model: \( p \) is the external momentum, and \( \gamma^+ \) or \( \gamma^- \) symbols on the line indicate that the corresponding term is taken in the numerator of the propagator. In configuration "]b\], the part that is proportional to \( \gamma^+ \) is taken.

Further, it is clear that there are no nonzero real configurations (see the comment at the end of Sec. 4.4), and it can be shown by induction that there are no nonzero compensating or secondary configurations either (the definitions are given in Sec. 4.4 also). Thus, only primary or primary-like configurations can be nonzero and all of them have the form
shown in Fig. 4. It can be shown that their degree of divergence $\alpha$ is positive and the maximum in formula (4.21) is reached for the diagram itself. Thus, the reasoning above and below formula (4.23) applies to them. Then, denoting the configurations displayed in Figs. 4a-d by $D_a - D_d$, we arrive at the equalities $D_a = C_a + \gamma p - C_a$, $D_b = C_b + \gamma p - C_b$, $D_c = C_c$, and $D_d = C_d$, where the expressions $C_a - C_d$ depend only on the masses and transverse momenta, but not on the external longitudinal momenta, and have a finite limit as $\varepsilon \to 0$.

Now we assume that $D_a - D_d$ are not single configurations but are the sums of all configurations of the same form and that integration over the internal transverse momenta has already been carried out, (with the above-described regularization). In four-dimensional space, the diagrams $D_a$ and $D_b$ diverge linearly while $D_c$ and $D_d$ diverge quadratically. Therefore, because of the transverse regularization, the coefficients $C_c$ and $C_d$ in the limit of removing this regularization take the form $C_1 + C_2 p^2_{\perp}$, where $C_1$ and $C_2$ do not depend on the external momenta (neither do $C_a$, $C_b$). Thus, to generate all nonzero configurations by the light-front Hamiltonian, only the expression

$$H_c = \tilde{C}_1 \varphi^2 + \tilde{C}_2 p^2_{\perp} \varphi^2 + \tilde{C}_3 \bar{\psi} \gamma^+ \frac{p_+}{p_-} \psi,$$  (4.24)

should be added, where $\varphi$ and $\psi$ are the boson and fermion fields, respectively, and $\tilde{C}_i$, are the constant coefficients.

Comparing (4.24) with the initial canonical light-front Hamiltonian, one can easily see that the found counter-terms are reduced to a renormalization of various terms of the Hamiltonian (in particular the boson mass squared and the fermion mass squared without changing the fermion mass itself). The explicit Lorentz invariance is absent, which compensates the violation of the Lorentz invariance inherent, in the light-front formalism.

Note that in the framework of the second approach, mentioned at the beginning of this section, one can obtain the same results. The only difference is that in two-dimensional space, the contributions from the configurations displayed in Fig. 3 would additionally depend on external transverse momenta. However, this dependence disappears after integration over internal transverse momenta with the introduction and subsequent removal of an appropriate regularization.

In the Pauli Villars regularization, it is easy to verify that the expression $\omega^- - \omega^+ - \mu^+ + \eta^+$ from (4.22) increases. This is because the number of terms in the numerators of the propagator increases. Then, the contribution from the $\varepsilon$-lines does not change, while the $\Pi$-lines belonging to $D_r$ make zero contribution to $\omega^r - \omega^r$ and $\eta^r$, but $-1$ contribution to $\mu^r$. Since $\tau > 0$, this regularization makes it possible to meet the condition $\sigma > 0$ for the configurations that were nonzero (one additional boson field and one additional fermion field are enough). Then it turns out that the canonical light-front Hamiltonian need not be corrected at all.
4.6 Application to gauge theories

Let us consider a gauge theory (e.g., QED or QCD) in the gauge \( A_\perp = 0 \). The boson propagator in the Mandelstam-Leibbrandt prescription has the form

\[
\frac{1}{Q^2 + i\varepsilon} \left( g_{\mu\nu} - \frac{Q_\mu \delta^+_{\perp} Q_+ + Q_\nu \delta^+_{\perp} Q_+}{2Q_+ Q_- + i\varepsilon} \right).
\]

All of the above reasoning was organized such that it could be applied to a theory like this (with fixed transverse momenta \( Q_\perp \neq 0 \)). It turns out that there are nonzero configurations with arbitrarily large numbers of external lines. An example of such a configuration is given in Fig. 5.

![Fig. 5: Nonzero configuration with an arbitrarily large number of external lines in a gauge theory. The symbols \( \gamma^\perp \) on the lines and the symbols + or \( \perp \) by the vertices indicate that the corresponding terms \( \gamma^+ \) or \( \gamma^\perp \) are taken in the numerators of propagators and in the vertex factors.](image)

Indeed, using formula (4.22), we can see that for the configuration in Fig. 5, \( \tau = 0 \) and, thus, \( \sigma \leq 0 \), i.e., this is a nonzero configuration. It is also clear that introduction of the Pauli-Villars regularization does not improve the situation because it does not affect \( \tau \).

Thus, within the framework of the above-described method for correcting the canonical light-front Hamiltonian of the gauge theory, an infinite number of counter-terms must be added to the Hamiltonian. Note, however, that the formulated conditions for the vanishing of the configuration are sufficient, but, generally speaking, not necessary. Because of this and because of the possible cancellation of different configurations after integration w.r.t. transverse momenta, the number of necessary counter-terms may be smaller.

5 Transverse lattice regularization of Gauge Theories on the Light Front

Ultraviolet regularization of nonabelian gauge theories via introduction of space-time lattice is widely used in nonperturbative considerations [27]. This regularization is introduced in a gauge invariant way. For canonical formulation on the LF we use transverse lattice regularization of Bardeen-Pearson type [28] which allows the action to be
polynomial in independent variables. As before we formulate the theory on the interval 
\(-L \leq x^- \leq L\) assuming periodic boundary conditions for all fields in \(x^-\). Further we
consider again the \(U(N)\) theory of pure gauge fields because this example is technically
more simple than the \(SU(N)\) theory.

The components of gauge field along continuous coordinates \(x^+, x^-\) can be taken
without modification and related to the sites of the lattice. Transverse components are
described with complex \(N \times N\) matrices \(M_k(x), k = 1, 2\). Each matrix \(M_k(x)\) is related
to the link directed from the site \(x - e_k\) to the site \(x\). The transverse vector \(e_k\) connects
two neighbouring sites on the lattice being directed along the positive axis \(x^k\) (\(|e_k| = a\):

\[
\begin{array}{ccc}
  M_k(x) & \rightarrow & \text{axis } x^k \\
  x - e_k & \rightarrow & x
  \end{array}
\]

The matrix \(M_k^+(x)\) is related to the same link but with opposite direction:

\[
\begin{array}{ccc}
  M_k^+(x) & \rightarrow & \text{axis } x^k \\
  x - e_k & \rightarrow & x
  \end{array}
\]

In this section for the index \(k\) the usual rule of summation on repeated indices is not
used, and where it is necessary the sign of a sum is indicated. The elements of these
matrices are considered as independent variables. For any closed directed loop on the
lattice we can construct the trace of the product of matrices \(M_k(x)\) sitting on the links
and order from the right to the left along this loop. For example the expression

\[
\text{Tr} \left\{ M_2(x)M_1(x - e_2)M_2^+(x - e_1)M_1^+(x) \right\}
\]

is related to the loop shown in fig. 6.

It should be noticed that a trace related to closed loop, consisting of one and the same
link passed in both directions, is not identically unity because the matrices \(M_k\) are not
unitary (see, for example, fig. 7).

The unitary matrices \(U(x)\) of gauge transformations act on the \(M\) and \(M^+\) in the
following way:

\[
M_k(x) \rightarrow M_k'(x) = U(x)M_k(x)U^+(x - e_k),
\]

(5.1)

\[
M_k^+(x) \rightarrow M_k^{'+}(x) = U(x - e_k)M_k^+(x)U^+(x).
\]

(5.2)

To connect these matrices with usual gauge field of continuum theory we can write

\[
M_k(x) = I + gaB_k(x) + igA_k(x), \quad B_k^+ = B_k, \quad A_k^+ = A_k. \quad (5.3)
\]
Here $A_k(x)$ coincide with transverse gauge field components in $a \to 0$ limit; $B_k(x)$ are extra fields which should be switched off in the limit.

An analog of field strength $G_{\mu\nu}$ can be defined as follows:

\[
G_{+\pm} = iF_{\pm\pm}, \quad F_{\pm\pm} = \partial_+ A_- (x) - \partial_- A_+ (x) - ig [A_+(x), A_- (x)],
\]

\[
G_{\pm k} (x) = \frac{1}{ga} [\partial_\pm M_k (x) - ig (A_\pm (x) M_k (x) - M_k (x) A_\pm (x - e_k))],
\]

\[
G_{12}(x) = -\frac{1}{ga^2} [M_1(x) M_2(x - e_1) - M_2(x) M_1(x - e_2)]. \quad (5.4)
\]

Under gauge transformation they transform as follows;

\[
G_{+-}(x) \to G'_{+-}(x) = U(x) G_{+-}(x) U^+(x),
\]

\[
G_{\pm k}(x) \to G'_{\pm k}(x) = U(x) G_{\pm k}(x) U^+(x - e_k),
\]

\[
G_{12}(x) \to G'_{12}(x) = U(x) G_{12}(x) U^+(x - e_1 - e_2). \quad (5.5)
\]

We choose a simplest form of the action having correct naive continuum limit:

\[
S = a^2 \sum_{x^+} \int_{-L}^{L} dx^+ \int dx^- \, \text{Tr} \left[ G_{+-}^+ G_{+-}^- + \sum_k (G_{+k}^+ G_{-k}^- + G_{-k}^+ G_{+k}^-) - G_{12}^+ G_{12}^- \right] + S_m, \quad (5.6)
\]
where the additional term $S_m$ gives an infinite mass to extra fields $B_k$ in $a \to 0$ limit:

\[
S_m = -\frac{m^2(a)}{4g^2} \sum_{x \perp} \int dx^+ \int_{-L}^{L} dx^- \sum_k \text{Tr} \left[ \left( M^+_k(x)M_k(x) - I \right)^2 \right] \xrightarrow{a \to 0} \\
- \frac{m^2(a)}{a} \int d^2x^+ \int dx^+ \int_{-L}^{L} dx^- \sum_k \text{Tr} \left( B^2_k \right), \quad m(a) \xrightarrow{a \to 0} \infty. \tag{5.7}
\]

After fixing the gauge:

\[
\partial_- A_+ = 0, \quad A^i_+ (x) = \delta_{ij} v^j (x^+, x^+) \tag{5.8}
\]

this action can be written in standard canonical form:

\[
S = a^2 \sum_{x^\perp} \int dx^+ \int_{-L}^{L} dx^- \left\{ \sum_i \left[ 2F^i_+ (x) \partial_+ v^i (x) \right] + \frac{1}{(ga)^2} \sum_{i,j} \sum_k \left[ D^- M^{ij+}_k (x) \partial_+ M^{ij}_k (x) + \text{h.c.} \right] + \sum_{i,j} A^i_+ (x) Q^{ij} (x) - \mathcal{H} (x) \right\} \tag{5.9}
\]

where

\[
D^- M^{ij+}_k (x) \equiv \left( \partial_- - i g v^i (x) + i g v^j (x - e_k) \right) M^{ij}_k (x), \\
D^- M^{ij}_k (x) \equiv \left( \partial_- + i g v^i (x) - i g v^j (x - e_k) \right) M^{ij}_k (x), \\
D^- F^{ij}_+ (x) \equiv \left( \partial_- - i g v^i (x) + i g v^j (x) \right) F^{ij}_+ (x), \tag{5.10}
\]

$A^i_+ (x)$ play the role of Lagrange multiplier,

\[
Q^{ij} (x) \equiv 2D^- F^{ij}_+ (x) + \frac{i}{ga^2} \sum_{j'} \sum_k \left[ M^{ij+}_k (x) D^- M^{j'j}_k (x) - M^{jj+}_k (x + e_k) D^- M^{ij}_k (x + e_k) \right] = 0 \tag{5.11}
\]

are gauge constraints and

\[
\mathcal{H} = \sum_{ij} \left( F^{ij}_+ + F^{ij}_- + G^{ij}_+ + G^{ij}_- \right) \tag{5.12}
\]

is the Hamiltonian density.

These constraints can be resolved explicitly by expressing $F^{ij}_+$ in terms of other variables with exception of the zero mode components $F^{ij}_+(0)$ which can not be found from
this constraint equation. The zero mode $Q^i_i(0) (x^+, x^+)$ of the constraint remains unresolved and it is imposed as a condition on physical states:

$$Q^i_i(0) (x^+, x^+) |\Psi_{phys}\rangle = 0.$$  \hfill (5.13)

In order to complete the derivation of the action in canonical form and to extract all independent canonical variables we make the Fourier transformation in $x^+$ of the transverse fields $M^{ij}_k(x)$ as follows:

$$M^{ij}_k(x) = \frac{g}{\sqrt{4L}} \sum_{n=-\infty}^{\infty} \left\{ \Theta \left( p_n + gv^i(x) - gv^j(x_e) \right) M^{ij}_n(x^+, x^+) + \Theta \left( -p_n - gv^i(x) + gv^j(x_e) \right) M^{ij}_n(x^+, x^+) \right\} \times$$

\[ \times \left| p_n + gv^i(x) - gv^j(x_e) \right|^{-1/2} e^{-ip_n x^-}, \] \hfill (5.14)

where

$$\Theta(p) = \begin{cases} 1, & p > 0 \\ 0, & p < 0 \end{cases}, \quad p_n = \frac{\pi n}{L}, \quad n \in \mathbb{Z}. \hfill (5.15)$$

Then the action is

$$S = a^2 \sum_{x^+} \int dx^+ \left\{ \sum_i 4L F^{ii}_+(0) \partial_+ v^i + \frac{i}{a^2} \sum_n \sum_{i,j} \sum_k M^{ij}_n \partial_+ M^{ij}_n + 2L \sum_i A^{ii}_+(0) Q^{ii}_i(0) - \tilde{\mathcal{H}}(x) \right\}. \hfill (5.16)$$

Here the $\tilde{\mathcal{H}}$ is obtained from $\mathcal{H}$ via substitution of the expression

$$F^{ij}_{+-} = \left( F^{ij}_{+-} - \delta^{ij} F^{ii}_{+-}(0) \right) + \delta^{ij} F^{ii}_{+-}(0) \hfill (5.17)$$

where the $F^{ij}_{+-} - \delta^{ij} F^{ii}_{+-}(0)$ are written in terms of $M^{ij}_n, M^{ij+}_n, v^i$ by solving the constraints (5.11) and using eq. (5.14). The $F^{ii}_{+-}(0)$ remain independent. The $G^{ij}_{12}$ are also expressed in terms of $M^{ij}_n, M^{ij+}_n, v^i$ via the eqns. (5.4), (5.14).

We have the following set of canonically conjugated pairs of independent variables:

$$\begin{cases} v^i, \\ \Pi_i = 4L a^2 F^{ii}_{+-}(0) \end{cases}, \quad \begin{cases} M^{ij}_n, \\ iM^{ij+}_n \end{cases}. \hfill (5.18)$$

In the quantum theory these variables become operators which satisfy the usual canonical commutation relations:

$$[v^i(x), \Pi_j(x')]_{x^+} = i\delta_{ij}\delta_{x^+,x'^+},$$

$$[M^{ij}_n(x), M^{ij+}_{n'}(x')]_{x^+} = i \delta^{ij}\delta^{ij'}\delta_{nn'}\delta_{kk'}\delta_{x^+,x'^+}. \hfill (5.19)$$
In this formulation there are no most complicated constraints like (2.79) for the zero modes of the transversal field components. If one goes to the limit $a \to 0$, these constraints reappear in a form which contains quantum operators in a definite order. This order was not clear earlier.

The operators $Q^{i}_{(0)}(x^{\perp}, x^{+})$ have the following form in terms of canonical variables:

$$2LQ^{i}_{(0)}(x^{\perp}, x^{+}) =$$

$$= -\frac{g}{2a^2} \sum_{n} \sum_{i} \sum_{j} \sum_{k} \left[ \varepsilon \left( p_{n} + g v^{j}(x + e_{k}) - g v^{i}(x) \right) M^{ij}_{nk}(x + e_{k}) - \varepsilon \left( p_{n} + g v^{i}(x) - g v^{j}(x - e_{k}) \right) M^{ji}_{nk}(x) \right],$$

(5.20)

where

$$\varepsilon(p) = \begin{cases} 1, & p > 0 \\ -1, & p < 0 \end{cases}.$$  

(5.21)

One can easily construct canonical operator of translations in the $x^{-}$:

$$P_{-}^{\text{can.}} = \sum_{x^{\perp}} \sum_{n} \sum_{i,j} \sum_{k} p_{n} \varepsilon \left( p_{n} + g v^{j}(x) - g v^{i}(x - e_{k}) \right) M^{ij}_{nk}(x) M^{ji}_{nk}(x).$$

(5.22)

This expression differs from the physical gauge invariant momentum operator $P_{-}$ by a term proportional to the constraint. The operator $P_{-}$ is

$$P_{-} = a^{2} \sum_{x^{\perp}} \sum_{k} \int_{-L}^{L} dx^{\perp} \text{Tr} \left( G_{-k}^{+} G_{-k} \right) = P_{-}^{\text{can.}} + 2La^{2} \sum_{x^{\perp}} \sum_{i} v^{i} Q^{i}_{(0)} =$$

$$= \sum_{x^{\perp}} \sum_{n} \sum_{i,j} \sum_{k} \left[ p_{n} + g v^{i}(x) - g v^{j}(x - e_{k}) \right] M^{ij}_{nk}(x) M^{ji}_{nk}(x).$$

(5.23)

Absolute minimum of $P_{-}$ corresponds to states $|v\rangle$ defined by the conditions:

$$M^{ij}_{nk}(x) |v\rangle = 0, \quad v^{i}(x) |v\rangle = \tilde{v}^{i}(x) |v\rangle.$$  

(5.24)

The Fock spaces constructed over these states by application of creation operators $M^{ij}_{nk}^{+}$ form the full Fock space of this LF formulation. Operators $Q^{i}_{(0)}$ are diagonal in this Fock space. In the basis

$$\left\{ \prod_{x^{\perp}} \prod_{n} \prod_{i,j} \prod_{k} \left( M^{ij}_{nk}(x) \right)^{m^{ij}_{nk}(x)} |v\rangle \right\},$$

(5.25)

where $m^{ij}_{nk}(x)$ are nonnegative integer numbers, we can write the constraint equation (5.13) in the form:

$$\sum_{n} \sum_{j} \sum_{k} \left[ \varepsilon \left( p_{n} + g \tilde{v}^{j}(x + e_{k}) - g \tilde{v}^{i}(x) \right) m^{ij}_{nk}(x + e_{k}) - \varepsilon \left( p_{n} + g \tilde{v}^{i}(x) - g \tilde{v}^{j}(x - e_{k}) \right) m^{ij}_{nk}(x) \right] = 0.$$  

(5.26)
One can find the eigenvalue $p_-$ of the momentum $P_-$ for such basis state:

$$p_- = \sum_{x^\perp} \sum_n \sum_{i,j} \sum_k \left| p_n + g\tilde{v}^i(x) - g\tilde{v}^j(x - e_k) \right| m_{nk}^{ij}(x)$$ (5.27)

and require that this value be finite. For the physical states with $p_- = 0$ we can find a state giving minimum to the Hamiltonian. We consider this state as the vacuum state. Here the problem is to solve the reduced Schroedinger equation with variables $v^i$ and $\Pi^i_v$ which are independent of $x_-$.

Analog of this lattice formulation in $(2+1)$-dimensions with the Hamiltonian modified by addition of some effective interaction terms, while ignoring all zero modes in $x_-$, was considered in [29, 30] in the framework of color dielectric model approach [31, 32]. The results of mass spectrum calculation at strong coupling can be fitted to the spectrum obtained with usual Wilson-Polyakov lattice theory [27]. However the continuum limit cannot be reached in this model.

In our lattice formulation the difficulty with going to weak coupling limit ($g \to 0$) becomes more transparent. The resulting Hamiltonian, written in the canonical variables, explicitly contains the terms with zero modes (in $x_-$) having coupling constant in the denominator, so that usual perturbative limit does not exist. For example the vacuum state corresponding to zero vacuum expectation values of $M_k(x)$ does not coincide with continuum theory vacuum where these values must be unity. A possible solution of this problem can be found in a modification of canonical LF formulation excluding zero modes in $x_-$ while modifying the LF Hamiltonian so that the theory remains equivalent to original Lorentz and gauge invariant formulation after removing the regularizations. One can tried to do this by comparison of corresponding perturbation theory series.

6 Conclusion

The LF Hamiltonian approach to Quantum Field Theory briefly reviewed here is an attempt to apply a beautiful idea of Fock space representation for quantum field nonperturbatively in the framework of canonical formulation on the LF. The problem of describing the physical vacuum state becomes formally trivial in this approach because such vacuum state coincides with mathematical vacuum of LF Fock space.

However the breakdown of Lorentz and gauge symmetries due to regularizations generates difficulties in proving the equivalence of LF formalism and usual one in Lorentz coordinates. This problem can be solved for nongauge theories but turns out to be very difficult for gauge theories in special (LF) gauge which is needed here. Nevertheless we hope that these difficulties can be overcome by finding a modified form of canonical LF Hamiltonian which generates the perturbation theory equivalent to usual covariant and gauge invariant one.
Appendix 1

Statement 1. If conditions (4.3) are satisfied, then, for fixed external momenta \( p^s \) and \( p^s_\neq 0 \forall s \), the equality

\[
\lim_{\beta \to 0} \lim_{\gamma \to 0} \int \prod_k dq^k_+ \int \prod_k dq^k_- \frac{\tilde{f}(Q^i, p^s) e^{-\gamma \sum_i Q^2_+ - \beta \sum_i Q^2_-}}{\prod_i(2Q^2_+ Q^2_- - M^2_i + i\varepsilon)} = 
\]

\[
= \int \prod_k dq^k_+ \int \prod_k dq^k_- \frac{\tilde{f}(Q^i, p^s)}{\prod_i(2Q^2_+ Q^2_- - M^2_i + i\varepsilon)},
\]

(A.1.1)

holds while the expressions appearing in (A.1.1) exist and the integral over \( \{q^k_+\} \) on the right-hand side is absolutely convergent. It is assumed that the momenta of lines \( Q^i \) are expressed in terms of loop momenta \( q^k \), \( V_\varepsilon \) is the domain corresponding to the presence of full lines, type-1 lines, and type-2 lines (the definitions are given following formula (4.13)), \( B_L \) is the sphere of radius \( L \), where \( L \geq S \max_s |p^s_\neq| \), and \( S \) is a number depending on the diagram structure.

Let us prove the statement. For each type-1 line in (A.1.1), we perform the following partitioning:

\[
\int_{-\varepsilon}^\varepsilon dQ^-_i = \left[ \int_{-\infty}^{-\varepsilon} dQ^-_i + (-1)^{\sum_i Q^-_i} \left( \int_{-\varepsilon}^\infty dQ^-_i + \int_{\varepsilon}^{-\varepsilon} dQ^-_i \right) \right].
\]

Then both sides of Eq. (A.1.1) become the sum of expressions of the same form in which, however, the domain \( V_\varepsilon \) corresponds to the presence of only full and type-2 lines. It is clear that by proving the statement for this \( V_\varepsilon \) (which is done below), we prove the original statement as well.

Let \( \tilde{B} \) be a domain such that the surfaces on which \( Q^-_i = 0 \) are not tangent to the boundary \( \tilde{B} \). First, we prove that in the expression

\[
\int \prod_k dq^k_+ \int \prod_k dq^k_- \frac{\tilde{f}(Q^i, p^s) e^{-\sum_i Q^2_-}}{\prod_i(2Q^2_+ Q^2_- - M^2_i + i\varepsilon)}
\]

(A.1.2)

the integral over \( \{q^k_-\} \) is absolutely convergent (here the integral over \( \{q^k_+\} \) is finite because \( x \varepsilon > 0, \beta > 0 \)). This becomes obvious (considering conditions (4.3) and the fact that, in type-2 lines, the momentum \( Q^-_i \) is separated from zero) if the contours of the integration over \( \{q^k_-\} \) can be deformed in such a way that the momenta \( Q^-_i \) of the full lines are separated from zero by a finite quantity (within the domain \( V_\varepsilon \cap \tilde{B} \)). In this case, we can repeat the well-known Weinberg reasoning [33]. What can prevent deformation is either a “clamping” of the contour or the point \( Q^-_i = 0 \) falling on the integration boundary.

Let us investigate the first alternative. We divide the domain of integration over \( q^k_+ \) into sectors such that the momenta of all full lines \( Q^i_+ \) have a constant sign within one sector. Let us examine one sector. We take a set of full lines whose \( Q^i_- \) may simultaneously...
vanish. In the vicinity of the point where $Q^i_-$ from this set vanish simultaneously, we bend the contours of the integration over $\{q^k_-\}$ such that these contours pass through the points $Q^i_+ = iB^i$ and the momenta $Q^i_-$ of the type-2 lines do not change. Let $B^i$ be such that $B^i Q^i_+ \geq 0$ for the lines from the set (for $Q^i_+$ from the sector under consideration). It is easy to check that this bending is possible. (Since the contours of integration over $q^k_-$ are bent and $Q^i_-$ are expressed in terms of $q^k_-$, one should only check that such $b^k$ exist, where the necessary $B^i$ are expressed in the same way, i.e., that $B^i$ obey the conservation laws and flow only along the full lines). With this bending, rather small in relation to the deviation and the size of the deviation region, the contours do not pass through the poles because, for the denominator of each line from the set in question, we have

$$\left(2Q^i_+ Q^i_- - M_i^2 + i\omega\right) \rightarrow \left(2Q^i_+ \left(Q^i_- + iB^i\right) - M_i^2 + i\omega\right), \quad Q^i_+ B^i \geq 0,$$

and for the other denominators, the bending takes place in a region separated from the point where the corresponding momenta $Q^i_-$ are equal to 0. Repeating the reasoning for all sets, we can see that there is no contour "clamping".

The other alternative is excluded by the above condition for $\tilde{B}$. To make this clear, one should introduce such coordinates $\xi^\alpha$ in the $q^k_-\text{-space}$ that the boundary of the domain $\tilde{B}$ is determined by the equation $\xi^1 = a = \text{const}$ and then argue as above for the coordinates $\xi^\alpha$ with $\alpha \geq 2$.

After bending the contours, integral (A.1.2) is absolutely convergent in $q^k_+$, $q^k_-$ if the integration in $q^k_+$ is carried out within the sector under consideration. On pointing out that the result, of internal integration in (A.1.2) does not depend on the bending, we add the integrals over all sectors and conclude that (A.1.2) converges in $\{q^k_+\}$ absolutely.

Now let us prove that if $\tilde{B}$ is a quite small, finite vicinity of the point $\{q^k_+\}$ that lies outside the sphere $B_L$, then expression (A.1.2) is equal to zero. We consider the momentum $Q^i_-$ of one line. Flowing along the diagram, it can ramify or it can merge with other momenta. Clearly, two situations are possible: either it flows away completely through external lines, or, probably, after long wandering, part of it, $\tilde{Q}^i_-$, makes a complete loop. The former situation is possible only if $|Q^i_-| \leq \sum \limits_r |p^r_-|$, where all external momenta leaving the diagram (but not entering it) are summed. Obviously, $S$ can be chosen such that for $\{q^k_+\}$ from $\tilde{B}$, a line exists whose momentum violates this condition.

The latter situation results in the existence of a loop, where the inequality $Q^i_- > \tilde{Q}^i_-$ holds for all momenta of its lines and the positive direction of the momenta is along the loop. Then the integral over $q^k_+$ of the loop in question can be interchanged with the integrals over $\{q^k_+\}$ (because it is absolutely and uniformly convergent for all $q^k_+$) and the residue formula can be used to perform this integration. Since, for the loop in question, the momenta $Q^i_-$ of the lines of this loop are separated from zero and are of the same sign, the result is zero. This has a simple physical meaning. If we pass to stationary noncovariant perturbation theory, we find that only quanta with positive $Q_-$ can exist. In this case, external particles with positive $p_-$ are incoming and those with negative $p_-$ are outgoing. Then, the momentum conservation law favors the occurrence of the first situation.
The entire outside space for $\tilde{B}$ can be composed of the above domains $B_L$ (everything converges well at infinity due to the factor $\exp(-\beta \sum_i Q_i^2)$). Thus, on the left-hand side of (A.1.1), one can substitute the integration domain $V_\varepsilon \cap B_L$ for $V_\varepsilon$, set the limit in $\gamma$ under the sign of integration over $\{q_k^\pm\}$ because of its absolute convergence, and also set the limit in $\beta$ under the integration sign because the domain of the integration over $\{q_k^\pm\}$ is bounded. Thus, we obtain the right-hand side. The statement is proved.

**Statement 2.** If $V_\varepsilon$ corresponds to the presence of type-2 lines alone, then, under the same conditions as in Statement 1, the equality

$$ \int_{V_\varepsilon} \prod_k dq_k^+ \int_{V_\varepsilon} \prod_k dq_k^- \frac{\tilde{f}(Q^i, p^s)}{\Pi_i(2Q_i^+Q_i^- - M_i^2 + i\varepsilon)} = $$

$$ = \int_{V_\varepsilon} \prod_k dq_k^+ \prod_k dq_k^- \frac{\tilde{f}(Q^i, p^s)}{\Pi_i(2Q_i^+Q_i^- - M_i^2 + i\varepsilon)} $$

is valid.

The proof of this statement is analogous to the second part of the proof of Statement 1.

**Appendix 2**

**Statement.** If conditions (4.3) are satisfied, the limits in $\gamma$ and $\beta$ in (4.8) can be interchanged (in turn) with the sign of the integral over $\{\alpha_i\}$ and then with $\tilde{f}(-i \frac{\partial}{\partial y_i})$.

To prove this, we define the vectors $\{q_1^+, q_1^-, \ldots, q_l^+, q_l^-\} \equiv S$, $\{Q_1^+, Q_1^-, \ldots, Q_n^+, Q_n^-\} \equiv \mu S + P$, and $\{y_1^+, y_1^-, \ldots, y_n^+, y_n^-\} \equiv Y$, where the vector $P$ is built only from external momenta and $\mu$ is an $l \times n$ matrix of rank $l$, $\mu_{2k-1}^2 = \mu_{2k-1}^2 = 0$, $\mu_{2k}^2 = \mu_{2k-1}^2$. Next, we introduce the following notation:

$$ \tilde{\Lambda}_i = \begin{pmatrix} \gamma & -i\alpha_i \\ -i\alpha_i & \beta \end{pmatrix}, \quad \Lambda = \text{diag}\{\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_n\}, \quad A = \mu^t \Lambda \mu, $$

$$ B = \mu^t \Lambda P - \frac{1}{2} i \mu^t Y, \quad C = -P^t \Lambda P + i Y^t P - i \sum_i \alpha_i M_i^2. $$

Then it follows from (4.9) that

$$ \hat{\phi}(\alpha_i, p^s, \gamma, \beta) = (-i)^n \tilde{f}(-i \frac{\partial}{\partial y_i}) \int d^{2l} S \ e^{-S^t A S - 2B^t S + C} \Big|_{y_i = 0} = $$

$$ = (-i)^n \tilde{f}(-i \frac{\partial}{\partial y_i}) e^{B^t A^{-1} B + C} \frac{\pi^l}{\sqrt{\det A}} \Big|_{y_i = 0}. \quad (A.2.1) $$

The function $\tilde{f}$ is a polynomial and we consider each of its terms separately. Up to a factor, each term has the form $\frac{\partial}{\partial y_{i_1}} \ldots \frac{\partial}{\partial y_{i_n}}$. These derivatives act on $C$ and $B$. The action on $C$ results in the constant factor $iN^t P$, the action on $B$ results in the factor.
−(1/2)iN^t\mu A^{-1}B \text{ or } −(1/4)N_1^t\mu A^{-1}\mu^tN_2 \text{ (the latter is the result of the action of two derivatives; } N, N_1, \text{ and } N_2 \text{ are constant vectors).}

It is necessary to prove the correctness of the following three procedures: (i) setting the limit in \(\gamma\) under the integral sign for fixed \(\beta > 0\); (ii) setting the limit in \(\beta\) for \(\gamma = 0\); (iii) setting the limits in \(\gamma\) and \(\beta\) under the signs of differentiation with respect to \(Y\). In cases (i) and (ii), one must obtain the bounds

\[
|\hat{\varphi}(\alpha_i, p^s, \gamma, \beta)| \leq \varphi'(\alpha_i, p^s, \beta),
\]

(A.2.2)

\[
|\hat{\varphi}(\alpha_i, p^s, 0, \beta)| \leq \varphi''(\alpha_i, p^s),
\]

(A.2.3)

where \(\varphi'\) and \(\varphi''\) are functions integrable (for \(\varphi'\) if \(\beta > 0\)) in any finite domain over \(\alpha_i\), with \(\alpha_i \geq 0\). Then, for case (i), we have

\[
|\hat{\varphi}(\alpha_i, p^s, \gamma, \beta) e^{-\sum \alpha_i}| \leq \varphi'(\alpha_i, p^s, \beta) e^{-\sum \alpha_i},
\]

i.e., a limit on the integrated function arises, and, thus, the limit in \(\gamma\) can be put under the integral sign. The situation is similar for case (ii). It is evident from (A.2.1) that the function \(\hat{\varphi}\) can be singular only if the eigenvalues of matrix \(A\) become zero. On finding the lower bound of these eigenvalues, one can prove through rather long reasoning that bounds (A.2.2), (A.2.3) exist if condition (4.3) is satisfied.

After the limits in \(\gamma\) and \(\beta\) are put under the integral sign, it is not difficult to interchange them with the differentiation with respect to \(Y\). One need do it only for \(\alpha_i > 0\) (for each \(i\)) and, in this case, one can show that the eigenvalues of the matrix \(A\) are nonzero and \(\hat{\varphi}\) is not singular.

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