Three Dimensional Lattice-Boltzmann Model for Electrodynamics

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In this paper we develop a 3D Lattice-Boltzmann model that recovers in the continuous limit the Maxwell equations for macroscopic mediums. The model can successfully reproduce the propagation of the electromagnetic waves in dielectric mediums and waveguide, the skin effect, the electrical dipole radiation and the electromagnetic response of a resonant cavity.

Keywords: Electrodynamics; Numerical methods; Electrodynamics simulation

I. INTRODUCTION

To simulate the electromagnetic fields inside materials is a fundamental tool in optics and electrodynamics, specially when boundary conditions are so complex that an analytical solution is impossible. Some examples of this phenomena are the design of antennas and the study of electrical discharges across media, among many others. For this reason, numerous numerical methods for the Maxwell equations have been implemented. One of these methods is the FDTD (Finite-difference time-domain) [1], which is based on finite differences and works pretty well, but consumes huge amounts of computational resource. Another one is the ray tracing method [2], that is only useful when the electromagnetic signals are of a single frequency. In addition, one of the actual problems in computational electrodynamics is the simulation of electromagnetic pulses in heterogenous media, that is one of the main advantages of lattice-Boltzmann methods.

One of the numerical methods for simulating fluids is Lattice Boltzmann (LB) [3], which was developed from lattice-gas automata. Lattice-Boltzmann simulations are performed on regular grids of many cells and a small number of velocity vectors per cell, each one associated to a set of density distribution functions, which evolve and spread together to the neighbor cells according to the collisional Boltzmann equation. As far as we know, there is still not a lattice-Boltzmann model for electrodynamics, but the effects of Maxwell equations on plasmas has been included (in the form of the diffusion equation for the magnetic field) as part of LB models for magnetohydrodynamics. The first LB model for studying plasmas reproduces the resistive magnetohydrodynamic equations and was developed by Chen [4] as an extension of the Lattice-Gas model developed by Chen and Matthaeus [6] and Chen, Matthaeus and Klein [7]. This LB model uses 37 velocity vectors per cell on a square lattice and is developed for two dimensions. Thereafter, Martinez, Chen and Matthaeuss [8] decreased the number of velocity vectors from 37 to 13, which made easier a future 3D extension. Some of the first LB models for magnetohydrodynamics in 3D were developed by Bryan R. Osborn [9] and Breiyanini and Valoungeorgi [10]. They used 19 vectors on a cubic lattice for the fluid, plus 7 vectors for the magnetic field, which makes a total number of 26 vectors per cell. By following a different path, Fogaccia, Benzi and Romanelli [11] introduced a 3D LB model for simulating turbulent plasmas in the electrostatic limit. All these models reproduce the resistive magnetohydrodynamic equations for a single fluid. Nevertheless, in a recent work [12], we have introduced a 3D LB model that reproduces the two fluids theory for plasmas. This model uses a cubic lattice and a set of 39 velocity vectors and includes explicitly the Maxwell equations in the vacuum.

In this paper, we introduce a 3D lattice-Boltzmann model that recovers the Maxwell equations in material media. The model is based on our previous develop for magnetohydrodynamics [12], but it is able to reproduce the behavior of electromagnetic fields inside dielectrics, magnets and conductors. The model uses 33 vectors per cell and 50 probability density functions, and performs well in a wide range of tests. Section II describes the model, with the evolution rules and the equilibrium expressions for the 50 density functions, plus the procedure to compute both the charge and current densities and the electric and magnetic fields. The Chapman-Enskog expansion showing how these rules recover the electrodynamics is developed in Appendix A. In order to validate the model, we simulate in section III the reflection of electromagnetic pulses on dielectric media, the propagation of electromagnetic waves along waveguide, the skin effect on conductors, the radiation pattern of an oscillating electrical dipole and the natural frequencies of a resonant cavity. The main results and conclusions are summarized in section IV.

II. 3D LATTICE-BOLTZMANN MODEL FOR ELECTRODYNAMICS

In a simple Lattice-Boltzmann model [3], the D-dimensional space is divided into a regular grid of cells. Each cell has Q vectors \( \vec{v}_i \) that links the cell with its
neighbors, and each vector is associated to a distribution function \( f_i \). The distribution function evolves according to the Boltzmann equation,

\[
f_i(\vec{x} + \vec{v}_i, t + 1) - f_i(\vec{x}, t) = \Omega_i(\vec{x}, t) \quad ,
\]

where \( \Omega_i(\vec{x}, t) \) is a collision term, which is usually taken as a time relaxation to some equilibrium density, \( f_i^{eq} \). This is known as the Bhatnagar-Gross-Krook (BGK) operator \([13]\),

\[
\Omega_i(\vec{x}, t) = \frac{1}{\tau} \left( f_i(\vec{x}, t) - f_i^{eq}(\vec{x}, t) \right) \quad ,
\]

where \( \tau \) is the relaxation time and \( f_i^{eq}(\vec{x}, t) \) is the equilibrium function. The equilibrium function is chosen in such a way, that (in the continuum limit) the model simulates the actual physics of the system.

For our 3D model, we use a cubic regular grid of lattice constant \( \delta x = \sqrt{2}c\delta t \), with \( c \) the light speed in vacuum (\( c \approx 3 \times 10^8 \text{m/s} \)); in other words, \( c = 1/\sqrt{2} \) in normalized lattice units (time unit = \( \delta t \), spatial unit = \( \delta x \)). There are 13 velocity vectors (figure 1), 13 different vectors for the electric field vectors \( \vec{e}_{i}^{p} \), and 7 different vectors for the magnetic field vectors \( \vec{b}_{i}^{p} \), where \( p \) indicates the plane of location.

The equilibrium function is chosen in such a way, that (in the continuum limit) the model simulates the actual physics of the system.

\[
f_i \rightarrow \text{equilibrium function.}
\]

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The macroscopic variables are computed as follows:

\[ \vec{D} = \sum_{i=1}^{4} \sum_{p=0}^{2} \sum_{j=0}^{1} G^{p(0)}_{ij} \vec{e}^{p}_{ij} , \]  \hfill (6a)

\[ \vec{B} = \sum_{i=1}^{4} \sum_{p=0}^{2} \sum_{j=0}^{1} G^{p(1)}_{ij} \vec{b}^{p}_{ij} , \]  \hfill (6b)

\[ \rho_c = G^{(0)}_0 + \sum_{i=1}^{4} \sum_{p=0}^{2} \sum_{j=0}^{1} G^{p(0)}_{ij} \rho^{p}_{ij} , \]  \hfill (6c)

\[ \vec{E} = \frac{\vec{D}}{\epsilon_r} , \]  \hfill (6d)

\[ \vec{H} = \frac{\vec{B}}{\mu_r} , \]  \hfill (6e)

\[ \vec{J} = \sigma \vec{E} , \]  \hfill (6f)

where \( \vec{D}, \vec{E} \) and \( \vec{J} \) are subsidiary fields that represent the displacement field, the electric field and the total current density before external forcements, respectively (the actual mean fields, including external forcements, are described below). In addition, \( \vec{B} \) is the induction field, \( \vec{H} \) is the magnetic field, \( \rho_c \) is the total charge density and \( \epsilon_r, \mu_r \) and \( \sigma \) are the dielectric constant, the permeability constant and the conductivity of the medium, respectively.

For the evolution of these distribution functions we follow the proposal of Zhaoli Guo, Chuguang Zheng and Baochang Shi that includes external forcements in the lattice Boltzmann equations, as follows:

\[ G^{p(r)}_{ij}(\vec{x} + \vec{e}^p_i, t + 1) - G^{p(r)}_{ij}(\vec{x}, t) = \Omega^{p(r)}_{ij}(\vec{x}, t) + T^{p(r)}_{ij} , \]  \hfill (7)

\[ G^{(r)}_0(\vec{x}, t + 1) - G^{(r)}_0(\vec{x}, t) = \Omega^{(r)}_0(\vec{x}, t) + T^{(r)}_0 , \]  \hfill (8)

where \( T^{(r)}_{ij} \) and \( T^{(r)}_0 \) are forcement coefficients \((r=0,1)\), defined by

\[ T^{p(r)}_{ij} = 0 , \]  \hfill (9a)

\[ T^{(r)}_0 = 0 . \]  \hfill (9b)

The mean electric field, \( \vec{E}' \), is defined by

\[ \vec{E}' = \vec{E} - \frac{\mu_0}{4\epsilon_r} \vec{J} , \]  \hfill (10)

and the mean density current vector \( \vec{J}' \) is defined by

\[ \vec{J}' = \sigma \vec{E}' , \]  \hfill (11)

In order to have the mean density vector \( \vec{J}' \) in terms of the subsidiary fields, we replace Eq. (10) into Eq. (11) to obtain

\[ \vec{J}' = \frac{\sigma}{1 + \frac{\mu_0}{4\epsilon_r}} \vec{E} , \]  \hfill (12)

Next, we adopt BGK collision terms \( \Omega^{p(r)}_{ij} \) and \( \Omega^{(r)}_0 \) of the form

\[ \Omega^{p(r)}_{ij} = -\frac{1}{\tau}(G^{p(r)}_{ij}(\vec{x}, t) - G^{p(r)}_{ij}(\vec{x}, t)) , \]  \hfill (13a)

\[ \Omega^{(r)}_0 = -\frac{1}{\tau}(G^{(r)}_0(\vec{x}, t) - G^{(r)}_0(\vec{x}, t)) \]  \hfill (13b)

where the relaxation time is set to \( \tau = \frac{1}{2} \).

The equilibrium functions for the electromagnetic fields are

\[ G^{p(r)}_{ij}(\vec{x}, t) = \frac{1}{16} \vec{e}^p_i \cdot \vec{J}' + \frac{\epsilon^1 - \epsilon}{4} \vec{E}' \cdot \vec{e}^p_i + \frac{\mu^1 - 1}{8} \vec{B} \cdot \vec{b}^p_i , \]  \hfill (14a)

\[ G^{(r)}_0(\vec{x}, t) = \rho_c . \]  \hfill (14b)

This completes the definition of the lattice Boltzmann model.

The proof that this lattice-Boltzmann model, via a Chapman-Enskog expansion, recovers the Maxwell equations is shown in Appendix A. This model reproduces the following equations:

\[ \frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{J}' = 0 , \]  \hfill (15a)

\[ \nabla \times \vec{E}' = -\frac{\partial \vec{B}}{\partial t} , \]  \hfill (15b)

\[ \nabla \times \vec{H} = \mu_0 \vec{J}' + \frac{1}{c^2} \frac{\partial \vec{D}'}{\partial t} . \]  \hfill (15c)

It is known that we can derive from these equations the following expresions for the divergence of the electromagnetic fields:

\[ \nabla \cdot \vec{D}' = \frac{\rho_c}{\epsilon_0} , \]  \hfill (16a)

\[ \nabla \cdot \vec{B} = 0 . \]  \hfill (16b)

that is, the other two Maxwell equations hold if they are satisfied as initial conditions.
III. APPLICATIONS

A. Dielectric Interface

First, we simulate the propagation of an electromagnetic gaussian pulse crossing an dielectric interface. For this purpose, we took an unidimensional array of \( L \) cells with periodic boundary conditions in the \( z \) coordinate and with each cell being its own neighbor in both directions of the \( x \) and \( y \) coordinates. One half of the simulation space, \( z<L/2 \), is vacuum (\( \epsilon=1.0 \)) and the other half, \( z>L/2 \), represents a dielectric medium with dielectric constant \( \epsilon=2.5 \). For avoiding abrupts changes of the dielectric constant between two neighboring cells we choose the following distribution of the permittivity:

\[
\epsilon = 1.75 + 0.75 \tanh(x - L/2) \quad .
\] (17)

The functional form of the incident gaussian electromagnetic pulse is given by the equations

\[
\vec{E} = (E_0 \exp(-\alpha(z - z_0)^2), 0, 0) \quad ,
\] (18a)

\[
\vec{B} = (0, B_0 \exp(-\alpha(z - z_0)^2), 0) \quad ,
\] (18b)
centered at \( z_0 \). The constant \( \alpha \) fixes the pulse width, \( E_0 \) is the pulse amplitude and the constant \( B_0 \) is related with \( E_0 \) according to the relation \( E_0 = cB_0 \), with \( c \) the vacuum light speed.

For the simulation we choose \( L=200 \), \( c=1/\sqrt{2} \), \( E_0=0.001 \), \( \alpha=0.01 \) y \( z_0=40 \) (in normalized units). The initial condition is shown in figure[4] and the electric field after 140 time steps is shown in figure[5].

The theoretical predictions of the amplitudes of the transmitted and reflected pulses can be calculated from

\[
\frac{E_0'}{E_0} = \frac{2}{\sqrt{\frac{\epsilon_2}{\epsilon_1} + 1}} \ y
\] (19a)

\[
\frac{E_0''}{E_0} = \frac{\sqrt{\frac{\epsilon_2}{\epsilon_1} - 1}}{\sqrt{\frac{\epsilon_2}{\epsilon_1} + 1}} \quad ,
\] (19b)

For our case these predictions give a ratio between the transmitted amplitude and the incident amplitude of 0.7751 and a ratio between the reflected and incident amplitudes of 0.2249. The values measured in the simulation are 0.7750 and 0.2249, respectively. This results correspond to errors smaller than 1% with the theoretical results. The results of this simulation show us that the model is able to reproduce the propagation of electromagnetic fields when dielectric media are present. This simulation takes less than 1 minute in a single Pentium IV de 3.0 GHz.

B. Skin Effect

For reproducing the skin effect, we simulate an electromagnetic plane wave that arrives perpendicular to a conductor material. For this purpose we construct a unidimensional space of \( L \) cells as before, with zero conductivity for \( z<L/4 \) and \( \sigma_0 \) conductivity for \( z>L/4 \) is \( \sigma_0 \). In order to avoid abrupt changes between neighboring cells we choose the following distribution for the conductivity:

\[
\sigma = \sigma_0 (1 + \tanh(x - L/4)) \quad .
\] (20)

In order to generate an incoming plane wave we set at \( z=0 \) an electric oscillator by imposing there an electromagnetic field given by

\[
\vec{E}' = (E_0 \sin(\omega t), 0, 0) \quad ,
\] (21a)

\[
\vec{B}' = (0, B_0 \sin(\omega t), 0) \quad ,
\] (21b)

where \( E_0 \) is the wave amplitude, \( cB_0=E_0 \) and \( \omega \) is the angular frequency of the wave. Some simulation snapshots
FIG. 6: Electric field $E$ as a function of the $z$ component with a conductor position $z>250$ at $t=635$ time steps. The solid line is the electric field and the dashed line is the theoretical exponential decrease.

FIG. 7: Electric field $E$ as a function of the $z$ component with a conductor position $z>250$ at $t=647$ time steps. The solid line is the electric field and the dashed line is the theoretical exponential decrease.

FIG. 8: Electric field $E$ as a function of the $z$ component with a conductor position $z>250$ at $t=670$ time steps. The solid line is the electric field and the dashed line is the theoretical exponential decrease.

Theoretical expression for the amplitude of the oscillating electric field inside the conductor is

$$E = E_0 \exp\left(-x/\Delta\right), \quad (22)$$

where $E_0$ is the amplitude of the electric field at the outside and $\Delta$ is known as the skin thickness. For good conductors this thickness is given by

$$\Delta = \sqrt{\frac{2}{\sigma\mu_0\mu_r\omega}}. \quad (23)$$

Here, $\sigma$ is the conductivity, $\mu_0$ is the permeability of free space, $\mu_r$ is the relative permeability, and $\omega$ is the angular frequency.

The simulation results are shown in Figures 6, 7, and 8. The simulation values are (in normalized units): $L=1000$, $E_0=0.001$, $\omega=\pi/100$, $c=1/\sqrt{2}$ and $\sigma_0=10^6$.

The theoretical expression for the amplitude of the oscillating current density is

$$E = E_0 \exp\left(-x/\Delta\right), \quad (22)$$

where $E_0$ is the amplitude of the electric field at the outside and $\Delta$ is known as the skin thickness. For good conductors this thickness is given by

$$\Delta = \sqrt{\frac{2}{\sigma\mu_0\mu_r\omega}}. \quad (23)$$

One can observe that the amplitude of the electric field oscillation follows in excellent agreement the theoretical prediction. This simulation takes less than 1 minute in a single Pentium IV de 3.0 GHz.

C. Electric Dipole

To simulate an electric dipole we construct an array of $100 \times 100 \times 100$ cells with free boundary conditions (each limit cell takes itself as his own missing neighbor). In the center of this array we insert a small oscillating current density in the $z$ direction,

$$J_z = J'_0 \sin\left(\frac{2\pi}{T} t\right), \quad (24)$$

where $J'_0$ is the amplitude of the current density and $T$ is the oscillation. For avoiding any abrupt change of the physical quantities between two neighboring cells we choose actually a gaussian functional form for the amplitude of the current density $J'_0$,

$$J'_0 = J_0 \exp(-0.75[(x-50)^2 + (y-50)^2 + (z-50)^2]) \quad (25)$$

The period was set to $T=25.0$ time steps and the amplitude to $J_0=0.0001$ (in automaton units).

The simulation results are shown in Figures 9 and 10. Figure 9 shows the equipotential lines after 56 time steps. Figure 10 shows the amplitude of the radiated magnetic field as a function of the $x$ component and theoretical values given by

$$B = k^2 P x \sqrt{1 - \frac{1}{k^2 x^2}}. \quad (26)$$

Here, $k$ is the magnitude of the wave vector, $k=\omega/c$, with $c$ the light speed in vacuum and $\omega$ the oscillation angular frequency. The magnitude $P$ of the electric dipolar...
TABLE I: Simulational $B_{\text{exp}}$ and theoretical $B_{\text{th,co}}$ magnetic field amplitudes for an oscillating electric dipole (all values in automaton units).

| Simulational amplitude $B_{\text{exp}}$ | Theoretical amplitude $B_{\text{th,co}}$ | Error (%) |
|-----------------------------------------|---------------------------------------|----------|
| 23.33                                   | 23.01                                 | 1.4      |
| 9.53                                    | 9.37                                  | 1.7      |
| 6.12                                    | 6.07                                  | 0.8      |
| 4.54                                    | 4.51                                  | 0.7      |

momentum is computed as

$$P = \frac{J_0 \Sigma}{\omega}, \quad (27)$$

where $\Sigma$ is the effective volume of the dipole. Table I shows the simulation peaks and the theoretical values for the magnetic field amplitude. Again, the simulation matches the theoretical predictions with an accuracy of less than 2%. This illustrates the possibilities of the method to simulate antennas. This simulation takes 40 minutes in a single Pentium IV de 3.0 GHz.

D. Waveguide. Microstrip

In this section we simulate a microstrip waveguide (see figure 11). For this purpose, we chose a 3D array of $100 \times 50 \times 50$ cells with free boundary conditions and we insert two parallel metallic layers of conductivity $\sigma$, one wider than the other. The dimensions, depicted in figure 11 are (in normalized units): $t=5$, $d=20$, $W=50$ and $\epsilon_r=1$. The signal enters into the waveguide by forcing the cells at $y=0$ to have the electric and magnetic fields of a plane wave (Eqs. (21)) with $E_0=0.01$, $c=1/\sqrt{2}$ and $\omega=\pi/100$ (see figure 13). The conductivity is smoothly changed along three cells, as in Sec. (see figure 12).

Figure 14 shows the $z$ component of the electric field at $y=25$ after 467 time steps. The voltage and current along the waveguide at the same time are drawn in figure 15. The electric voltage $V(z,t)$ and current $I(z,t)$ are in phase (plane wave). The waveguide impedance $Z$, computed as $Z=V/I$, gives us $Z=70.73 \ \Omega$. The theoretical value for the impedance of an infinite microstrip waveguide is given by (16).
FIG. 12: Electric conductivity as a function of $x$ and $z$ at $y=25$

FIG. 13: Signal configuration that enter in the waveguide.

$$Z = \frac{Z_0}{2\pi \sqrt{2(1 + \varepsilon_r)}} \ln \left[ 1 + \frac{4d}{\omega_{eff}} \left( \frac{14 + \frac{\omega_{ef}}{\varepsilon_r}}{11 \omega_{eff}} + \sqrt{\left( \frac{14 + \frac{\omega_{ef}}{\varepsilon_r}}{11 \omega_{eff}} \right)^2 + \pi^2 \left( 1 + \frac{1}{\varepsilon_r} \right)} \right) \right], \quad (28)$$

where

$$\omega_{eff} = W + \frac{1 + \frac{1}{\varepsilon_r}}{2\pi} \ln \left[ 4e \left( \left( \frac{t}{d} \right)^2 + \left( \frac{1}{\pi \frac{W}{d} + \frac{1}{\varepsilon_r}} \right)^2 \right)^{-\frac{1}{2}} \right], \quad (29)$$

With this equation we obtain a theoretical value of $Z=72.6\Omega$. This gives us a 3% difference between the simulation and the theoretical prediction. This simulation takes 90 minutes in a single Pentium IV de 3.0 GHz.

E. Resonant Cavity

As a last test, we simulate a cubic resonant cavity and found its resonance frequencies. The cavity we chose is an array of 50×50×50 cells, corresponding to a size of
5×5×5 cm, with periodic boundary conditions, but imposing a null electric field at the boundary (a perfect conductor). Inside the cavity we set an emmitter point as a single cell with an oscillating electric field in the x direction (the antenna) and a receptor point (the measurement instrument) as a single cell where we measure the electric field amplitude. The emitter point was set at (5, 5, 5) (in cell units) and the receptor point at (5, 5, 45). The oscillation frequency was scanned from 0.0182 to 0.027 oscillations per time step by multiplying each previous value by a constant factor of 0.0072 (a total of 22 different frequencies). Each frequency starts a single run, consisting of 5000 time steps before equilibrium and four whole oscillations to estimate the amplitude of the electric field. Figure 14 shows amplitude of the electric field as a function of frequency. The resonant peaks are clearly identified. Table II compares the values these resonant frequencies with the theoretical predictions computed from [15]\

$$\omega_{nmp} = \frac{c}{\varepsilon_r \mu_r} \sqrt{\left( \frac{m \pi}{L_x} \right)^2 + \left( \frac{m \pi}{L_y} \right)^2 + \left( \frac{n \pi}{L_z} \right)^2},$$

(30)

All differences are smaller than 1%. The whole simulation (22 runs of more than 5000 time steps each on 125000 cells) takes 18 days in a single Pentium IV of 3.0 GHz (actually, we ran the simulation on a cluster of five nodes on two days).

IV. CONCLUSION

In this paper we introduce a three dimensional lattice-Boltzmann model to simulate the Maxwell equations in macroscopic media (dielectric, magnetic and conductor materials). The model actually reproduces the Maxwell equations in the continuous limit, as proved in Appendix A. In order to test the model, we simulated some typical electromagnetic sets, namely: the amplitude of the transmitted and reflected pulses at a dielectric interface, the

| Experimental values | Theoretical values | Error (%) |
|---------------------|--------------------|-----------|
| log($f_{exp}$)      | log($f_{the}$)     | (%)       |
| -1.725              | -1.724             | 0.06      |
| -1.690              | -1.694             | 0.24      |
| -1.660              | -1.669             | 0.54      |
| -1.643              | -1.646             | 0.18      |
| -1.618              | -1.625             | 0.43      |
| -1.590              | -1.589             | 0.57      |

TABLE II: Simulational $f_{exp}$ and theoretical $f_{the}$ resonant frequencies for a cavity of size 50×50×50 cells. The frequencies are in oscillations per time step.
exponential decay in the amplitude of a incident plane wave on a conductor (skin effect), the amplitude (as a function of distance) of the magnetic field radiated by an electric dipole, the characteristic impedance of a microstrip waveguide and the resonant frequencies of a cubic resonant cavity. In all cases, the differences between the simulational results and the theoretical predictions lay between 1% and 3%. These five simulations prove that our LB model works very well in the more different situations.

The model lets us to select the electromagnetic constants (magnetic permeability, electric permittivity and conductivity) for each cell at pleasure. Nevertheless, for avoiding numerical instabilities on the interfaces between two different materials we need to smooth the transition across three cells, approximately. For this purpose we have used tanh functions, but other functional forms can be also employed. This is a standard procedure in many numerical models [1, 2].

The main advantage of our LB model is that it is about ten times faster than FDTD based models. Indeed, a the simulation with FDTD of the propagation of an electromagnetic pulse produced by a discharge antenna on a grid of tantas × tantas × tantas cells takes around Xh in a Pentium IV d. In comparison, our simulation of a magnetic pulse produced by a discharge antenna on a structure × structure × structure cells takes around Xh. For this reason, the LB model seems to be promising in a variety of applications, including among others: the propagation of electromagnetic pulses produced by atmospheric rays or discharge antennas, the light diffraction in objects that have a complex geometry or the sputtering of electromagnetic signals across buildings. The velocity and precision of our lattice-Boltzmann model for electrodynamics are excellent for the design and optimization of antennas, which requires to simulate a large number of configurations, and very specially in the design of pulse antennas, where single-frequency numerical methods fail. All these applications can be thema of future works.

The developments in this paper also show how to introduce antisymmetrical tensors in conservation equations in Lattice-Boltzmann models. These equations are useful to simulate many interesting phenomena, like seismic or torsion waves. These are also topics of future development.

Hereby we have introduced a Lattice-Boltzmann model for electrodynamics that actually reproduces the Maxwell equations in media, with a plenty of future applications. Moreover, this model shows how to construct LB models that fulfills conservation laws with antisymmetric tensors. We hope that this valuable theoretical develop will push the evolution of LB models further away in the horizon of even more exciting applications.

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APPENDIX A: CHAPMAN-ENSKOG EXPANSION

The Boltzmann equations for each fluid, Eq. (7) and (9), determine the system evolution. This evolution rule gives in the continuum limit the macroscopic differential equations that the system satisfies. This is known as the Chapman-Enskog expansion.

To develop the Chapman-Enskog expansion, we start by taking the Taylor expansion of these equations until second order in spatial and temporal variables,

\[
\epsilon_i \cdot \nabla G_{ij}^{(r)} + \frac{1}{2} \sum_{\alpha, \beta} \frac{\partial^2 G_{ij}^{(r)}}{\partial x_{\alpha} \partial x_{\beta}} (v_{i\alpha}^{(r)} v_{j\beta}^{(r)})
\]

\[
\frac{\partial G_{ij}^{(r)}}{\partial t} + \frac{\partial}{\partial x} (v_{i}^{(r)} G_{ij}^{(r)}) + \frac{1}{2} \frac{\partial^2 G_{ij}^{(r)}}{\partial t^2} \delta t^2 = -\frac{1}{\tau} (G_{ij}^{(r)} - G_{ij}^{(r)\text{eq}}),
\]

\[
\frac{\partial G_{ij}^{(r)}}{\partial t} + \frac{1}{2} \frac{\partial^2 G_{ij}^{(r)}}{\partial t^2} = -\frac{1}{\tau} (G_{ij}^{(r)} - G_{ij}^{(r)\text{eq}}).
\]

where \(\alpha, \beta = x, y, z\) denote the \(x, y\) and \(z\) component. Next, we expand the distribution functions, the spatial and temporal derivates in a power series of a small parameter, \(\lambda\),

\[
G_{ij}^{(r)} = G_{ij}^{(r)(0)} + \lambda G_{ij}^{(r)(1)} + \lambda^2 G_{ij}^{(r)(2)} + \ldots,
\]

\[
\frac{\partial}{\partial t} = \lambda \frac{\partial}{\partial t_1} + \lambda^2 \frac{\partial}{\partial t_2} + \ldots,
\]

\[
\frac{\partial}{\partial x_{\alpha}} = \lambda \frac{\partial}{\partial x_{\alpha_1}} + \ldots.
\]

It is assumed that only the 0th order terms in \(\epsilon\) of the distribution functions contribute to the macroscopic variables. So, for \(n > 0\) we have

\[
\sum_{i,j,p} G_{ij}^{(r)(n)} \epsilon_{i,p} = 0,
\]

\[
\sum_{i,j,p} G_{ij}^{(r)(n)} \delta_{i,j}^{(p)} = 0.
\]

The main current density \(\vec{J}\) is of the order \(\lambda\), so we can write \(\vec{J} = \lambda \vec{J}_1\). Because \(G_{ij}^{(r)\text{eq}}\) is now function
of $\vec{J}$, we need to develop a Chapman-Enskog expansion of the equilibrium function, too:

$$G^{(r)}_{ij}(\tilde{x}, t) = G^{(r)(0)\text{eq}}_{ij} + \lambda G^{(r)(1)\text{eq}}_{ij} + \lambda^2 G^{(r)(2)\text{eq}}_{ij}. \quad (A7)$$

So, if we replace these results into Eqs. (A1) and (A2), we obtain for the zeroth order in $\lambda$

$$G^{(r)(0)\text{eq}}_{ij} = G^{(r)(0)}_{ij}. \quad (A8a)$$

$$G^{(r)}_{0\text{eq}} = G^{(r)(0)}_{0\text{eq}}. \quad (A8b)$$

For the first order in $\lambda$ of the distribution function we have

$$\vec{v} \cdot \vec{v}_1 G^{(r)(0)}_{ij} + \frac{\partial G^{(r)(0)}_{ij}}{\partial t_1} = -\frac{1}{\tau} (G^{(r)(1)}_{ij} - G^{(r)(1)\text{eq}}_{ij}), \quad (A9a)$$

and for the second order in $\lambda$ we obtain

$$\frac{\partial G^{(r)(0)}_{ij}}{\partial t_1} = -\frac{1}{\tau} (G^{(r)(1)}_{ij} - G^{(r)(1)\text{eq}}_{ij}), \quad (A9b)$$

The first order and the second order terms for the equilibrium function of the electromagnetic fields are obtained by replacing the Eq. (10) into Eq. (14). From this, we can obtain, grouping in $\lambda$ powers,

$$G^{(r)(0)\text{eq}}_{ij}(\tilde{x}, t) = \frac{\epsilon^1}{4} \vec{E} \cdot e_{ij}^p + \frac{\mu^1}{8} \vec{B} \cdot b_{ij}^p, \quad (A11a)$$

$$G^{(r)(1)\text{eq}}_{ij}(\tilde{x}, t) = \frac{\lambda}{16} \vec{v} \cdot \vec{J}_1 - \frac{\lambda \mu_0}{16} \vec{J}_1 \cdot e_{ij}^p, \quad (A11b)$$

$$G^{(r)(2)\text{eq}}_{ij}(\tilde{x}, t) = 0, \quad (A11c)$$

$$G^{(r)(0)\text{eq}}_{ij}(\tilde{x}, t) = \rho_c, \quad (A11d)$$

$$G^{(r)(1)\text{eq}}_{0\text{eq}}(\tilde{x}, t) = G^{(r)(2)\text{eq}}_{0\text{eq}}(\tilde{x}, t) = 0. \quad (A11e)$$

Now, we are ready to determine the equations that the model satisfies in the continuum limit. First, let us consider $\tau = 1/2$. By summing up Eqs. (A9a), (A9b), (A10a) and (A10b) over $i$, $j$ and $p$, we get

$$\frac{\partial \rho_c}{\partial t_1} = 0, \quad (A12)$$

and

$$\frac{\partial \rho_c}{\partial t_2} + \nabla \cdot \vec{J}_1 = 0. \quad (A13)$$

Now, we can add these two equations to obtain

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{J} = 0. \quad (A14)$$

Multiplying the equations (A9a), (A9b), (A10a) and (A10b) by $e_{ij}^p$ and summing up over the index $i$, $j$ and $p$, we obtain for $r=0$

$$\frac{\partial (\epsilon \vec{E})}{\partial t_1} - \frac{1}{2} \vec{v} \times \left( \frac{\vec{B}}{\mu} \right) = -\frac{1}{2} \mu_0 \vec{J}_1, \quad (A15)$$

and

$$\frac{\partial (\epsilon \vec{E})}{\partial t_2} - \frac{\mu_0}{4} \frac{\partial \vec{J}_1}{\partial t_1} = 0. \quad (A16)$$

Summing up these two equations, and taking into account the Eq. (10), we arrive to the first Maxwell equation,

$$\frac{\partial (\epsilon \vec{E})}{\partial t} - \frac{1}{2} \vec{v} \times \left( \frac{\vec{B}}{\mu} \right) = -\mu_0 \frac{1}{2} \vec{J}. \quad (A17)$$

Similarly, multiplying the Eqs. (A9a) and (A10a) by $b_{ij}^p$ and summing up over $i$, $j$ and $p$, we obtain for $r=1$

$$\frac{\partial \vec{B}}{\partial t_1} + \nabla \times \vec{E} = 0, \quad (A18)$$

and

$$\frac{\partial \vec{B}}{\partial t_2} - \frac{1}{2} \vec{v} \times \left( \frac{1}{2} \mu_0 \vec{J}_1 \right) = 0. \quad (A19)$$

Summing up these two equations, we get the second Maxwell equation,

$$\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0. \quad (A20)$$

The others two Maxwell equations can be obtained from the Eqs. (A17) and (A20) how is shown in [17]. If we applying the divergence to the Eqs. (A19) and (A20) we get

$$\frac{\partial (\nabla \cdot \vec{E})}{\partial t} = -\frac{1}{2} \mu_0 \nabla \cdot \vec{J}, \quad (A21)$$

and

$$\frac{\partial (\nabla \cdot \vec{B})}{\partial t} = 0. \quad (A22)$$
and because the continuity equation Eq. (A14), we arrive to
\[ \frac{\partial (\vec{\nabla} \cdot \vec{E}')}{\partial t} = \frac{1}{2\mu_0} \frac{\partial \rho_c}{\partial t} . \] (A23)

Finally, we obtain
\[ \frac{\partial (\vec{\nabla} \cdot \vec{E}' - \frac{1}{2\mu_0}\rho_c)}{\partial t} = 0 . \] (A24)

So, if the initial conditions of the electromagnetic fields satisfies the Maxwell equations
\[ \vec{\nabla} \cdot \vec{B} = 0 , \] (A25)
these equations will be reproduced correctly for all time.

Summarying, the equations (A17), (A20), (A23) and (A26) determine the electromagnetic fields evolution. These are the electrodynamics equations for macroscopic mediums. This complete the proof.

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