Some aspects of the quark-antiquark Wilson loop formalism in the NRQCD framework

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Starting from the NRQCD Lagrangian the heavy quark-antiquark potential is written in terms of field strength insertions on a static Wilson loop. The relevant matching coefficients are given at the present status of knowledge. The short-range, perturbatively dominated, behaviour of the spin-dependent terms is discussed.

1. NRQCD AND THE WILSON LOOP FORMALISM

Heavy quark bound states provide an extremely difficult but at the same time appealing system to test QCD. The difficulties are obvious. One is the mixing of different energy scales. This is a typical feature of any bound state problem in quantum field theory and makes tricky even a purely perturbative solution of it. An other conceptual difficulty is connected with the nonperturbative nature of low-energy QCD. This suggests that nonpertubative contributions have to be taken into account in almost all QCD bound states. The reason why heavy mesons are appealing is that the existence of an expansion parameter (the inverse of the mass \( m \) in the Lagrangian and the velocity \( v \) of the quark as a dynamical defined power counting parameter) makes possible to handle the first difficulty and to keep under control the second one. The tool is provided by NRQCD \[ \] . This is an effective theory equivalent to QCD and obtained from QCD by integrating out the hard energy scale \( m \). The Lagrangian comes from the original QCD Lagrangian via a Foldy–Wouthuysen transformation. The ultraviolet regime of QCD (at energy scale \( m \)) is perturbatively encoded order by order in the coupling constant \( \alpha_s \) in the matching coefficients which appear in front of the new operators of the effective theory. This ensures the equivalence between the effective theory and the original one at a given order in \( 1/m \) and \( \alpha_s \). At order \( 1/m^2 \) the NRQCD Lagrangian describing a bound state between a quark of mass \( m_1 \) and an antiquark of mass \( m_2 \) is

\[
L = Q_1^\dagger \left( iD_0 + c_2^{(1)} \frac{D^2}{2m_1} + c_4^{(1)} \frac{D^4}{8m_1^2} + c_6^{(1)} \frac{\sigma \cdot B}{2m_1} \right) + \text{antiquark terms (1 ↔ 2)} \\
+ \frac{d_1}{m_1 m_2} Q_1^\dagger Q_2 Q_1 Q_1 + \frac{d_2}{m_1 m_2} Q_1^\dagger \sigma Q_2 Q_2^\dagger \sigma Q_1 \\
+ \frac{d_3}{m_1 m_2} Q_1^\dagger T^a Q_2 Q_2^\dagger T^a Q_1 \\
+ \frac{d_4}{m_1 m_2} Q_1^\dagger T^a \sigma Q_2 Q_2^\dagger T^a \sigma Q_1.
\]

This is the relevant Lagrangian in order to calculate the bound state observables up to order \( O(v^4) \). A discussion of the operators appearing in (1) in terms of powers of the quark velocity can be found in [2]. The coefficients \( c_2^{(j)}, c_4^{(j)}, \ldots \) are evaluated at a matching scale \( \mu \) for a particle of mass \( m_j \).

Nonperturbative contributions to the heavy meson observables can be evaluated directly from the Lagrangian (1) via lattice simulations [1]. Typically, since the hard degrees of freedom have been integrated out explicitly, the needed lattice cut-off \( \mu \) is expected to be larger (smaller in terms of energy) than the usual one with a clear reduction in the computation time. Despite the ad-
vantages, there are also some drawbacks in this method. In particular in this way we do not learn very much on our "analytic" knowledge on the QCD vacuum structure. Moreover computations on coarse lattices are not always under control. Therefore it is worthwhile to use the Lagrangian of NRQCD as a starting point and to work out the quark-antiquark interaction in the so-called Wilson loop formalism \[\ldots\]. The advantage in doing so is that all the nonperturbative dynamics will be contained in gauge field averages of field strength insertions on a static Wilson loop. These can be very easily evaluated by means of some QCD vacuum model \[\ldots\], or via traditional lattice simulations \[\ldots\] providing in this way a powerful method in order to discriminate between different models. The derivation of the quark-antiquark potential in the Wilson loop formalism from the NRQCD Lagrangian was first suggested in this context in \[\ldots\] and is discussed with details in \[\ldots\]. Here we present only some results. The heavy quark-antiquark potential (assumed that it exists) is given by

\[
V(r) = \lim_{T \to \infty} \frac{1}{T} \log \frac{W}{r} + \left( \frac{S^{(1)} \cdot L^{(1)}}{m_1^2} + \frac{S^{(2)} \cdot L^{(2)}}{m_2^2} \right) \frac{2c_F^+ V'_A(r) + c_S^+ V'_V(r)}{2r} + \frac{S^{(1)} \cdot L^{(2)} + S^{(2)} \cdot L^{(1)}}{m_1 m_2} \frac{c_F^- V'_A(r) + c_S^- V'_V(r)}{2r} + \frac{S^{(1)} \cdot L^{(1)} - S^{(2)} \cdot L^{(2)}}{m_1 m_2} \frac{c_F^- V'_V(r)}{r} + \frac{1}{8} \left( \frac{c_F^{(1)} + c_D^{(2)}}{m_1^2} \right) \left( \Delta V_0(r) + \Delta V_a^E(r) \right) + \frac{1}{8} \left( \frac{c_F^{(1)} + c_D^{(2)}}{m_2^2} \right) \Delta V_a^B(r) + \frac{c_F^{(1)} c_F^{(2)}}{m_1 m_2} \left( \frac{S^{(1)} \cdot r S^{(2)} \cdot r}{r^2} - \frac{S^{(1)} \cdot S^{(2)}}{3} \right) V_5(r)
\]

The "potentials" \(V_1, V_2, \ldots\) are scale dependent gauge field averages of electric and magnetic field strength insertions on the static Wilson loop and are explicitly given in \[\ldots\]. \(W\) is the gauge average of the non-static Wilson loop. The expansion of it around the static Wilson loop gives the static potential \(V_0\) plus velocity (non-spin) dependent terms. \(S^{(j)}\) and \(L^{(j)}\) are the spin and orbital angular momentum operators of the particle \(j\). The matching coefficients are defined as \(2c_{F,S}^{(1)} \equiv c_{F,S}^{(1)} \pm c_{F,S}^{(2)}\) and \(d\) is the relevant contribution to the mixing coming from the four quark operators in Eq. \[\ldots\] and will be given in the next section. Apart from the matching Eq. \(\ldots\) is equivalent to the potential derived in \[\ldots\]. In the next section we will give explicitly the matching coefficients and discuss briefly the relevance of the matching in order to have a short range consistent potential.

2. MATCHING COEFFICIENTS

Since for reparameterization invariance \(c_S = 2c_F - 1\), all the spin dependent potentials given in Eq. \(\ldots\) turn out to depend only on \(c_F\) (if the mass of the particle is irrelevant we will omit to indicate it). This coefficient is known up to two loop in the anomalous dimension \[\ldots\]:

\[
c_F = \left( \frac{\alpha_s(m)}{\alpha_s(\mu)} \right)^{\gamma_0/2\beta_0} \left[ 1 + \frac{\alpha_s(m)}{4\pi} c_1 + \frac{\alpha_s(m) - \alpha_s(\mu)}{4\pi} \left( \frac{\gamma_1^0 \beta_0 - \gamma_0^0 \beta_1}{2\beta_0^2} \right) \right]
\]

where \(\beta_j\) are the usual \(\beta\)-function coefficients, \(\gamma_0 = 2C_A, \gamma_1 = 68C_A^2/9 - 26C_A N_f/9, c_1 = 2(C_A + C_F)\), \(N_f\) is the number of flavors, \(C_F\) is the Casimir of the fundamental representation and \(C_A\) is the Casimir of the adjoint representation. At the lattice scale used in \[\ldots\] the numerical values of this coefficient at the bottom and charm mass are \(c_F(m_b) \approx 1.06 \times (1 + 0.15) = 1.22\) and \(c_F(m_c) \approx 1.27 \times (1 + 0.25) = 1.59\) respectively. The corrections due to the one loop matching are relevant (15% in the bottom case and 25% in the charm case) and therefore of the same order of the next power in the velocity in the Lagrangian \[\ldots\] (usually accepted values are \(\langle v_b^2 \rangle \sim 0.07\) and \(\langle v_c^2 \rangle \sim 0.24\)).

An evaluation of the coefficient \(c_D\) associated with the Darwin term in the NRQCD Lagrangian
is given in [10]:
\[ c_D = \left( \frac{7}{4} - 8 \frac{C_F}{C_A} \right) \left( \frac{\alpha_s(m)}{\alpha_s(\mu)} \right)^{2C_A/3\beta_0} \]
\[-\frac{5}{4} \left( \frac{\alpha_s(m)}{\alpha_s(\mu)} \right)^{2C_A/\beta_0} + \frac{1}{2} + 8 \frac{C_F}{C_A}.\]

This corrects a previous wrong evaluation given in [8]. At the lattice scale used in [7] the numerical values of this coefficient at the bottom and charm mass are \( c_D(m_b) \simeq 0.76 \) and \( c_D(m_c) \simeq -0.08 \) respectively. As pointed out in [7], since the potential \( \Delta V^E_s \) manifests a \( 1/r \) behaviour, this term gives a flavor-dependent contribution to the central potential. However this contribution is suppressed in the bottom case by the bottom mass (see Eq. (2)) and in the charm case by the smallness of the corresponding matching coefficient.

Finally, the contributions coming from the four-fermion operators are usually suppressed either in \( \alpha_s \) or in powers of the quark velocity \( v \). Nevertheless under RG transformation the contribution to the spin-spin potential coming from the chromomagnetic operator in the NRQCD Lagrangian mixes with some of the local four quark operators. In order to take into account this mixing the delta contribution to the spin-spin potential has been added in Eq. (2) though it would be suppressed in \( \alpha_s \). The coefficient \( d \) has been evaluated in [8]:
\[ d = \frac{1}{8} \left( \frac{\alpha_s(m_1)}{\alpha_s(m_2)} \right)^{C_A/\beta_0} \left[ 1 - \left( \frac{\alpha_s(m_2)}{\alpha_s(\mu)} \right)^{2C_A/\beta_0} \right].\]

As noticed in [8], the presence of the matching coefficients in the expression for the potential makes possible the agreement in the short range region between the potential derived here with the traditional QCD one loop perturbative calculation, e.g. in [13]. Let us focus on the spin-orbit terms \( V_1 \) and \( V_2 \). Comparing properly with [11] we get for \( V_2 \) the perturbative contribution
\[ V_{2,\text{pert}}(r) = \frac{C_F \alpha_s(\mu)}{r^2} \left\{ 1 + \frac{\alpha_s}{\pi} \left[ -\frac{\beta_0}{12} - \frac{2C_A}{3} \right] \right. \]
\[ + \left. \frac{\beta_0 - C_A}{2} (\log(\mu r) + \gamma_E - 1) \right\},\]
where \( \gamma_E \) is the Euler constant. This expression agrees very well with the lattice measurement of the same quantity shown in Fig. 4. In the same way we get for \( V_1 \) the perturbative contribution
\[ V_{1,\text{pert}}(r) = \frac{C_F \alpha_s(\mu)}{r^2} \left[ 1 + \frac{\alpha_s}{\pi} \left( -\frac{\beta_0}{12} - \frac{2C_A}{3} \right) \right] \]
\[ + \frac{\beta_0 - C_A}{2} (\log(\mu r) + \gamma_E - 1) \].

It is extremely interesting to compare the above expression with the short-range behaviour of the \( V_1 \) potential as given by the lattice measurement shown in Fig. 4. Apart an overall shift proportional to the string tension and therefore of non-perturbative origin the agreement is very good. This is quite significant since the perturbative part of \( V_1 \) is entirely due to loop corrections. As a consequence \( V_1 \) is more sensitive than \( V_2 \) to the matching scale \( \mu \). Notice that at very short distances the function \( -V_{1}', \) just because the \( \log(\mu r) \) term, is expected to become negative, but up to now no lattice data are available in this region.

As a last comment we notice that due to the so-called Gromes relation \( V_2 - V_1' = V_5 \) [12] a \( V_5 \) potential (in the notation of [11]) emerges also in Eq. (3) by collecting the contributions coming from the fourth and fifth line. The perturbative expression we get agrees with that one given in [11]:
\[ V_{5,\text{pert}}(r) = \frac{C_F V_5'(r)}{r} = \frac{\alpha_s}{\pi} \frac{1}{r^3} \left[ C_A C_F \log \frac{m_2}{m_1} \right].\]
Figure 2. The spin dependent potential $-V_1'$ as given by the lattice measurement of [7].

3. CONCLUSIONS

In the framework of NRQCD and at the present status of the matching we have given the expression for the heavy quark potential in terms of field strength insertions on a static Wilson loop. This has the advantage that traditional lattice calculations can be used in order to evaluate nonperturbative contributions. Moreover in this way a comparison between different QCD vacuum models can be performed directly in terms of Wilson loop expectation values. This approach has been developed with some extent in [4]. Here we have emphasized the role played by the matching coefficients in order to make consistent the short range behaviour of the potential that we obtain with the usual scattering matrix derived potential. We noticed that present lattice data are sensitive to one loop corrections and to the matching scale.

As a conclusion, let us mention two open problems. In order to have a 10 % accuracy in the quarkonium spin splitting it is necessary to add to the Lagrangian (3) higher order operators [6]. The inclusion (if possible) of such operators in an expression like Eq. (2) is still to do. Moreover in order to obtain Eq. (2) we have implicitly assumed the existence of a potential. Non-potential terms surely exist in perturbative QCD. How to treat it in a system affected by nonperturbative physics is still unclear. Interesting developments could come from a promising approach recently proposed in [13].

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