RELATING SIGNED KAZHDAN-LUSZTIG POLYNOMIALS AND CLASSICAL KAZHDAN-LUSZTIG POLYNOMIALS

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Abstract. Motivated by studying the Unitary Dual Problem, a variation of Kazhdan-Lusztig polynomials was defined in [Yee08] which encodes signature information at each level of the Jantzen filtration. These so called signed Kazhdan-Lusztig polynomials may be used to compute the signatures of invariant Hermitian forms on irreducible highest weight modules. The key result of this paper is a simple relationship between signed Kazhdan-Lusztig polynomials and classical Kazhdan-Lusztig polynomials: signed Kazhdan-Lusztig polynomials are shown to equal classical Kazhdan-Lusztig polynomials evaluated at \(-q\) rather than \(q\) and multiplied by a sign. This result has applications to finding the unitary dual for real reductive Lie groups since Harish-Chandra modules may be constructed by applying Zuckerman functors to highest weight modules.

1. Introduction

Classifying irreducible unitary representations of a group, known as the Unitary Dual Problem, is an open problem that is important for its wide ranging applications. In particular, it is a necessary component of a programme in abstract harmonic analysis articulated by I.M. Gelfand in the 1930s for solving difficult problems in disparate areas of mathematics. Gelfand’s general philosophy is to formulate the solution as the solution to a corresponding algebraic problem which, in turn, may be solved by decomposition into simpler (though possibly infinitely many) problems.

The most general approach towards solving the Unitary Dual Problem for real reductive Lie groups has been to first identify a broader set of representations: the Hermitian representations, which accept invariant Hermitian forms. By computing the signatures of these invariant Hermitian forms and then identifying when the forms are definite, one obtains a classification of the unitary representations. The cases for which the Unitary Dual Problem is solved are limited.

In [Yee08], signed Kazhdan-Lusztig polynomials for semisimple Lie algebras were introduced and used to give formulas for signature characters of invariant Hermitian forms on irreducible highest weight modules. While unitary highest weight modules have been identified by the work of Enright-Howe-Wallach, understanding signatures of all Hermitian representations is important since Harish-Chandra modules may be constructed by applying Zuckerman or Bernstein functors to highest weight modules. Identifying the irreducible unitary representations of a real reductive Lie group is equivalent to classifying irreducible Harish-Chandra modules. While the Zuckerman functor is known to preserve unitarity in

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certain circumstances ([Vog84], [Wal84]), it does not preserve unitarity in general, hence the need to understand signatures of all Hermitian highest weight modules. This paper and [Yee] provide dramatic simplifications to the formulas in [Yee08] for signed Kazhdan-Lusztig polynomials and signatures of invariant Hermitian forms on irreducible highest weight modules in the equal rank case. Amazingly, signed Kazhdan-Lusztig polynomials are equal to classical Kazhdan-Lusztig polynomials evaluated at $-q$ up to a sign. Specifically:

**Main Theorem:** Let $g_0$ be a real equal rank semisimple Lie algebra with complexification $g$, $\theta$ a Cartan involution of $g_0$, and let $h_0$ be a $\theta$-stable Cartan subalgebra with complexification $h$. Let $\lambda \in h^*$ be antidominant and let $x$ be in the integral Weyl group of $\lambda$ such that the Verma module $M(x\lambda)$ admits an invariant Hermitian form (more details within the paper). Then:

$$P_{x, y}(q) = (-1)^{\epsilon(x\lambda - y\lambda)} P_{x, y}(-q)$$

where $\epsilon$ is the $\mathbb{Z}_2$-grading on the imaginary root lattice. (That is, $\epsilon(\mu)$ is the parity of the number of non-compact roots in an expression for $\mu$ as an integral linear combination of roots.) The polynomial on the left hand side is a signed Kazhdan-Lusztig polynomial while the polynomial on the right hand side is a classical Kazhdan-Lusztig polynomial.

The format of the paper is as follows.

Sections 2 and 3 contain a synopsis of signature character theory for Verma modules and for irreducible highest weight modules.

In section 4, we simplify the formulas for the signs $\epsilon$ which appear in the formulas in Sections 2 and 3.

Section 5 contains the proof of the main theorem.

In Section 6, we discuss upcoming work.

2. **Signature Characters for Invariant Hermitian Forms on Irreducible Verma Modules**

In this section, we will restrict our attention to the equal rank case although more general formulas appeared in [Yee05]. This streamlines the exposition as it eliminates the additional complications which arise in the non-equal rank case.

**Notation 2.1.** We use the following notation in this section:

- $g_0$ is a real equal rank semisimple Lie algebra
- $\theta$ is a Cartan involution on $g_0$ inducing the decomposition $g_0 = k_0 \oplus p_0$
- $h_0 = t_0 \oplus a_0$ is the Cartan decomposition of a $\theta$-stable Cartan subalgebra
- omitting the subscript 0 indicates complexification
- $b = h \oplus n$ is a Borel subalgebra giving positive roots $\Delta^+(g, h)$ and $g = n \oplus h \oplus n^-$ is the corresponding triangular decomposition
- $\Lambda_r$ is the root lattice
- $\rho$ is one half the sum of the positive roots
- $\alpha_1, \ldots, \alpha_n$ are the simple roots and $s_1, \ldots, s_n$ the corresponding simple reflections
- $W$ is the Weyl group and $\mathcal{C}_0$, the fundamental chamber, is chosen to be antidominant
- $\lambda \in h^*$
- $M(\lambda) = U(g) \otimes_{U(h)} \mathbb{C}_{\lambda - \rho}$ is the Verma module of highest weight $\lambda - \rho$ with canonical generator $v_{\lambda - \rho}$
- applies to elements of $g$ and $h^*$ denotes complex conjugation relative to the real form $g_0$
- $H_{\alpha,N}$ denotes the affine hyperplane $H_{\alpha,N} = \{ \lambda \in h^* : (\lambda, \alpha^\vee) = n \}$ where $\alpha \in \Delta(g,h)$ and $n \in \mathbb{Z}$.
- $W_a$ is the affine Weyl group with fundamental (antidominant) alcove $A_0$

**Definition 2.2.** An invariant Hermitian form on a representation $V$ is a sesquilinear pairing $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that

$$\langle X \cdot v, w \rangle + \langle v, \overline{X} \cdot w \rangle = 0$$

for every $v, w \in V$ and $X \in g$.

**Proposition 2.3.** ([Yee05], p. 641) The Verma module $M(\lambda)$ admits a non-trivial invariant Hermitian form if and only if $\theta(b) = b$ and $\lambda - \rho$ is imaginary.

Such a form is unique up to a real scalar. If $\langle v_{\lambda-\rho}, v_{\lambda-\rho} \rangle = 1$, then the form is called the Shapovalov form and is denoted by $\langle \cdot, \cdot \rangle_{\lambda}$. Henceforth, for the remainder of this paper, we fix $b$ and $\lambda$ to satisfy these conditions.

Because of invariance, the weight space decomposition of the Verma module is an orthogonal decomposition, making the notion of the signature of the Shapovalov form reasonable although the Verma module is infinite-dimensional. Thus:

**Definition 2.4.** If the Shapovalov form $\langle \cdot, \cdot \rangle_{\lambda}$ is non-degenerate and has signature $(p(\mu), q(\mu))$ on the $\lambda - \mu - \rho$ weight space, the signature character of the Shapovalov form on $M(\lambda)$ is:

$$ch_s M(\lambda) = \sum_{\mu \in \Lambda^+} (p(\mu) - q(\mu)) e^{\lambda - \mu - \rho}.$$

The radical of the Shapovalov form is precisely the unique maximal proper submodule of $M(\lambda)$. The Shapovalov determinant formula states the following:

**Proposition 2.5.** The determinant of a matrix representing the Shapovalov form on the $\lambda - \rho - \mu$ weight space, up to a constant, is

$$\prod_{\alpha \in \Delta^+(g,h)} \prod_{n=1}^{\infty} ((\lambda, \alpha^\vee) - n)^{P(\mu-n\alpha)}$$

where $P$ denotes the Kostant partition function.

Thus the Shapovalov form is degenerate precisely on the reducibility hyperplanes $H_{\alpha,n}$ where $\alpha$ and $n$ are positive. In any region bounded by these hyperplanes, the Shapovalov form stays non-degenerate, whence the signature remains constant.

In [Wal84], Nolan Wallach used an asymptotic argument to determine the signature on the largest of these regions:

**Theorem 2.6.** ([Wal84], Lemma 2.3) Let $\lambda$ be imaginary and $b$ be $\theta$-stable. If $(\lambda, \alpha^\vee) < 1$ for every positive root $\alpha$, then

$$ch_s M(\lambda) = \frac{e^{\lambda - \rho}}{\prod_{\alpha \in \Delta^+(t,h)} (1 + e^{-a}) \prod_{\alpha \in \Delta^+(p,h)} (1 - e^{-a})}.$$
As a historical note, Wallach showed that the Zuckerman functor applied to these modules produced unitary representations.

Formulas for signature characters for all irreducible Verma modules admitting invariant Hermitian forms may be found in [Yee05]. The proof uses the following philosophy found in [Vog84]. Within any region bounded by reducibility hyperplanes, the signature remains constant. The goal is to understand how signatures change as you cross a reducibility hyperplane into another region. If you cross only one reducibility hyperplane at a time, the structure of the corresponding Jantzen filtration (see 4.2 for the definition) is simple and the signature changes by the signature of the radical, which is also a Verma module:

Lemma 2.7. ([Yee05], Proposition 3.2) Suppose \( \lambda_t : (-\delta, \delta) \rightarrow \mathfrak{h}^* \) is a path for which every \( M(\lambda_t) \) admits a non-degenerate invariant Hermitian form. Suppose \( M(\lambda_t) \) is irreducible for \( t \neq 0 \) and \( \lambda_0 \) belongs to the reducibility hyperplane \( H_{\alpha,n} \) but not to any other reducibility hyperplane. Suppose for \( t \in (0, \delta) \) that is in the positive half space \( \lambda_t \in H_{\alpha,n}^+ \) while for \( t \in (-\delta, 0) \), \( \lambda_t \in H_{\alpha,n}^- \). Let \( t_1 \in (0, \delta) \) and let \( t_2 \in (-\delta, 0) \). Then:

\[
ch_s M(\lambda_{t_1}) = e^{\lambda_{t_1} - \lambda_{t_2}} ch_s M(\lambda_{t_2}) + 2\varepsilon(H_{\alpha,n}, \lambda_0) ch_s M(\lambda_{t_1} - n\alpha).
\]

for some sign \( \varepsilon(H_{\alpha,n}, \lambda_0) = \pm 1 \). We can extend the definition of \( \varepsilon \) to other affine hyperplanes by setting \( \varepsilon(H_{\alpha,n}, \lambda_0) = 0 \) when the only affine hyperplane \( \lambda_0 \) belongs to is \( H_{\alpha,n} \) and \( H_{\alpha,n} \) is not a reducibility hyperplane.

It turns out that \( \varepsilon(H_{\alpha,n}, \lambda_0) \) stays constant over \( \lambda_0 \) in a given Weyl chamber, so we let \( \varepsilon(H_{\alpha,n}, s) \) be that value in the Weyl chamber \( s\mathfrak{C}_0 \).

\( \varepsilon(H_{\alpha,n}, s) \) is computed in [Yee05]:

Theorem 2.8. ([Yee05] Theorems 6.1.12, 5.2.18) Let \( \gamma \) be a positive root and let \( \gamma = s_{i_1} \cdots s_{i_{k-1}} \alpha_k \) be such that \( ht(s_{i_1} \cdots s_{i_{k-1}} \alpha_k) \) decreases as \( j \) increases. Let \( w_{\gamma} = s_{i_1} \cdots s_{i_k} \). If \( \gamma \) hyperplanes are reducibility hyperplanes on \( s\mathfrak{C}_0 \) and if \( \gamma \) does not form a type \( G_2 \) root system with other roots, then:

\[
\varepsilon(H_{\gamma,N}, s) = \frac{(-1)^{N\#\{\text{noncompact } \alpha_j : |\alpha_j| \geq |\gamma|\}}}{\times \left( -1 \right)^{\#\{\beta \in \Delta(w_{\gamma}^{-1}) : |\beta| = |\gamma|, \beta \neq \gamma, \text{ and } \beta, s_{\beta} \gamma \in \Delta(s^{-1})\}} \times \left( -1 \right)^{\#\{\beta \in \Delta(w_{\gamma}^{-1}) : |\beta| \neq |\gamma| \text{ and } \beta, -s_{\beta} s_{\gamma} \beta \in \Delta(s^{-1})\}}.
\]
Let \( \alpha_1 \) and \( \alpha_2 \) be the long and short simple roots for a type \( G_2 \) root system, respectively. Let \( \delta_\alpha = 1 \) if \( \alpha \) is compact, and let it be \(-1\) if \( \alpha \) is non-compact. We have:

| Hyperplane | Weyl Chamber \( sC_0 \) |
|------------|--------------------------|
| \( H_{\alpha_1,N} \) | \( s_1 \) | \( s_1s_2 \) | \( s_1s_2s_1 \) | \( s_1s_2s_1s_2 \) | \( s_1s_2s_1s_2s_1 \) | \( s_1s_2s_1s_2s_1s_2 \) |
| \( s_1s_2 \) | \( s_1s_2s_1 \) | \( s_1s_2s_1s_2 \) | \( s_1s_2s_1s_2s_1 \) | \( s_1s_2s_1s_2s_1s_2 \) |
| \( s_1s_2s_1 \) | \( s_1s_2s_1s_2 \) | \( s_1s_2s_1s_2s_1 \) | \( s_1s_2s_1s_2s_1s_2 \) |
| \( s_1s_2s_1s_2 \) | \( s_1s_2s_1s_2s_1 \) | \( s_1s_2s_1s_2s_1s_2 \) | \( s_1s_2s_1s_2s_1s_2s_1 \) |
| \( s_1s_2s_1s_2s_1 \) | \( s_1s_2s_1s_2s_1s_2 \) | \( s_1s_2s_1s_2s_1s_2s_1 \) | \( s_1s_2s_1s_2s_1s_2s_1s_2 \) |
| \( s_1s_2s_1s_2s_1s_2 \) | \( s_1s_2s_1s_2s_1s_2s_1s_2 \) | \( s_1s_2s_1s_2s_1s_2s_1s_2s_1 \) |

Since alcoves defined by the action of the affine Weyl group are bounded by affine hyperplanes of the form \( H_{\alpha,n} \), where \( \alpha \) is a root and \( n \in \mathbb{Z} \), the signature within the interior of an alcove is constant and it makes sense to define \( \varepsilon(A,A') \) for adjacent alcoves \( A, A' \) separated by the affine hyperplane \( H_{\alpha,n} \) where for \( \lambda \in A \) and \( \lambda' \in A' \):

\[
ch_s M(\lambda) = e^{\lambda - \lambda'} ch_s M(\lambda') + 2\varepsilon(A,A') ch_s M(\lambda - n\alpha).
\]

Observe that \( \varepsilon(A,A') = -\varepsilon(A',A) \).

Given our formula for \( \varepsilon(H_{\alpha,n}, s) \), we know how signatures change as we cross reducibility hyperplanes. We cross one reducibility hyperplane at a time until we reach the region where signatures are known by Wallach’s work. We find by induction:

**Theorem 2.9.** ([Yee05] Theorem 4.6) Let:
- \( \tau : W_a \to W \) denote the group homomorphism defined by sending \( w = ts \) to \( t \) where \( t \) is translation by an element of the root lattice and \( s \) is an element of \( W \)
- \( \tilde{\tau} : W_a \to W \) be defined by sending \( w \) to \( \tilde{w} \) where \( \tilde{w}C_0 \) is the Weyl chamber containing \( wA_0 \)
- \( \lambda \in wA_0 \) such that \( M(\lambda) \) admits a nondegenerate invariant Hermitian form
- \( R(\mu) = \prod_{\alpha \in \Phi^+} (1 + e^{-\alpha}) \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \)
- \( wA_0 = C_0 \overset{r_1}{\to} C_1 \overset{r_2}{\to} \cdots \overset{r_{\ell}}{\to} C_\ell = \tilde{w}A_0 \) a path from \( wA_0 \) to \( \tilde{w}A_0 \) where the \( C_i \)'s are the alcoves on the path and the \( r_i \)'s are the reflections through the affine hyperplanes separating the alcoves

Then

\[
ch_s M(\lambda) = \sum_{I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, \ell\}} \varepsilon(I) 2^{\mid I \mid} R(r_{i_1}r_{i_2} \cdots r_{i_k}r_{i_{k-1}} \cdots r_{i_1}\lambda)
\]

where \( \varepsilon(\{\}) = 1 \) and \( \varepsilon(I) = \varepsilon(C_{i_1-1}, C_{i_1}) \varepsilon(C_{i_2-1}, C_{i_2}) \cdots \varepsilon(C_{i_k-1}, C_{i_k}) \) if \( I \neq \{\} \).
3. Simplifying the Sign Formulas ε

Notation 3.1. We fix our notation for this section.

- γ ∈ Δ⁺(g, h)
- N ∈ ℤ⁺
- w ∈ W
- α is a simple root with corresponding simple reflection s
- x ∈ W satisfies xs > x.

In this section, we compute ε in the cases which appear in the recursion formulas for computing signed Kazhdan-Lusztig polynomials from [Yee08].

We begin by showing that the second and third terms in the expression in Theorem 2.8 for ε(H_{xα,N}, xs) are 1. This requires a careful study of wγ and w_{xα} which was defined in 2.8.

Recall that Δ(w) = {α ∈ Δ⁺ | wα < 0}. If w = s_{i_1}s_{i_2}⋯s_{i_k} is a reduced expression, then Δ(w⁻¹) = {α_{i_1}, s_{i_1}α_{i_2}, ⋯, s_{i_1}s_{i_2}⋯s_{i_{k-1}}α_{i_k}}. Note that w depends on the choice of expression γ = s_{i_1}s_{i_2}⋯s_{i_{k-1}}α_{i_k}, but for all choices, γ ∈ Δ(w⁻¹). We need the following technical lemmas:

Lemma 3.2. There exists a choice of w_{xα} such that Δ(w_{xα}⁻¹) ⊂ Δ((xs)⁻¹).

Proof. The proof is by double induction on the length of x and the height of xα.

First induction: on the length of x.

First base case: ℓ(x) = 0. If x = 1, then w_{xα} = s_α and Δ(w_{xα}⁻¹) = {α} = Δ(xs). (Note that the only possible height for xα is 1.)

First inductive hypothesis: Assume for all x of length less than or equal to k and x < xs that there exists w_{xα} such that Δ((xs)⁻¹) ⊃ Δ(w_{xα}⁻¹).

Induction step: We prove this by induction on ht(xα).

Second base case: If the height of xα is 1, then w_{xα} = s_{xα} has length 1 and Δ(s_{xα}) = {xα}. Since sx⁻¹xα = sα = −α, we have xα ∈ Δ((xs)⁻¹).

Consider x' of length k + 1, x'α > 0. If ht(x'α) > 1, then there exists a simple reflection t = s_µ such that x' = tx for some x of length k and ht(x'α) = ht(txα) > ht(xα) > 0. By our inductive hypothesis, there exists w_{txα} with Δ(w_{txα}⁻¹) ⊂ Δ((txs)⁻¹). Since txα > 0, we see that txs > tx. Since tx > x, we have txs > xs. Therefore

Δ((txs)⁻¹) = Δ(sx⁻¹tsx) = tΔ((xs)⁻¹) ∪ {µ}.

Choose w_{txα} = tw_{xα}. From Lemma 5.3.3 of [Yee05], tw_{xα} > w_{xα} and therefore

Δ(w_{txα}⁻¹) = Δ((tw_{xα})⁻¹) = tΔ(w_{xα}⁻¹) ∪ {µ}.

It follows that Δ(w_{txα}⁻¹) ⊂ Δ((txs)⁻¹). □

Lemma 3.3. If β ∈ Δ(w_γ⁻¹), then (β, γ) > 0. In particular, β and γ are not orthogonal.

Proof. If s_{i_1}⋯s_{i_k} is a reduced expression for w_γ given by Lemma 5.3.3 of [Yee05], then β = s_{i_1}s_{i_2}⋯s_{i_{j-1}}α_{i_j} for some 1 ≤ j < k and γ = s_{i_1}s_{i_2}⋯s_{i_{k-1}}α_{i_k}. Thus (β, γ) = (α_{i_j}, s_{i_j}s_{i_{j+1}}⋯s_{i_{k-1}}α_{i_k}) > 0 since ht(s_{i_j}s_{i_{j+1}}⋯s_{i_{k-1}}α_{i_k}) > ht(s_{i_{j+1}}s_{i_{j+2}}⋯s_{i_{k-1}}α_{i_k}). (recall that w_γ was defined to have this height property.) □

Definition 3.4. Let S_{w_{xα}} = {β ∈ Δ(w_{xα}⁻¹) : |β| = |xα|, β ≠ xα, and β, s_βxα ∈ Δ((xs)⁻¹)}. (See the second term in the formula for ε.)
Lemma 3.5. Suppose $w_{xα}$ is constructed as in Lemma 3.2. Then $S^2_{w_{xα}} = \{\}$.

Proof. Suppose $β ∈ S^2_{w_{xα}}$. Since $β$ and $xα$ are the same length, they generate a type $A_2$ root system. Since $(β, xα) > 0$ by Lemma 3.3, therefore $s_βxα = xα − β$. To belong to $S^2_{w_{xα}}$, $β ∈ Δ((xs)^{-1}) ⇒ sx^{-1}β < 0$. However, $sx^{-1}s_βxα = sx^{-1}(xα − β) = −α − sx^{-1}β$ which must be positive since $α$ is simple and $sx^{-1}β < 0$. This contradicts the condition $s_βxα ∈ Δ((xs)^{-1})$, so $β ∉ S^2_{w_{xα}}$.

Definition 3.6. Let $S^3_{w_{xα}} = \{β ∈ Δ((xs)^{-1}) : |β| ≠ |xα| and β, −s_βs_{xα}β ∈ Δ((xs)^{-1})\}$. (See the third term in the formula for $ε$.)

Lemma 3.7. Suppose $w_{xα}$ is constructed as in Lemma 3.2 and that $xα$ does not form a type $G_2$ root system with any other roots. Then $S^3_{w_{xα}} = \{\}$.

Proof. Suppose $β ∈ S^3_{w_{xα}}$. Since $xα$ and $β$ generate a type $B_2$ root system with $(xα, β) > 0$, therefore either $−s_βs_{xα}β = xα − β$ if $xα$ is long, or $−s_βs_{xα}β = 2xα − β$ if $xα$ is short. In the former case, arguing as in the proof of the previous lemma, we may see that it is impossible for both $−s_βs_{xα}β ∈ Δ((xs)^{-1})$ to be simultaneously satisfied, contradicting $β ∈ S^3_{w_{xα}}$. In the latter case, observe that $β, −s_βs_{xα}β ∈ Δ((xs)^{-1})$ yield the formulas $sx^{-1}β < 0$ and $sx^{-1}(2xα − β) = −2α − sx^{-1}β < 0$. Noting that $α$ is short and simple and $−sx^{-1}β$ is long and positive, using standard constructions of root systems such as those on p. 64 of [Hum72] it is apparent that a positive long root minus twice a short simple root cannot give a negative root-contradiction. Therefore $S^3_{w_{xα}} = \{\}$.

We arrive at the following:

Theorem 3.8. Let $x ∈ W$ and let $α$ be a simple root with simple reflection $s$ such that $xs > x$. Then:

1. If $H_{α,N}$ intersects $wC_0$, then $ε(H_{α,N}, w) = 1$ if $α$ is compact, and $(-1)^N$ if $α$ is noncompact.
2. $ε(H_{α,N}, xs) = (-1)^{ε(Nxα)}$.

Proof. (1): This is just Lemma 5.2.17 of [Yee05].

(2): First, we see from Theorem 2.8 that the result holds for $α$ which forms a type $G_2$ root system with other roots, so it suffices to prove the theorem for other settings. From our previous two lemmas, it suffices to prove that if $w_{xα} = s_{i_1} ⋯ s_{i_k}$, then $\#\{\text{noncompact } α_{i_j} : |α_{i_j}| ≥ |xα|\} ≡ ε(xα)$ (mod 2). We prove this result by induction on $k$. Observe first that the set we count makes no reference to the Weyl chamber. Clearly if $k = 1$, then $xα$ is simple and the result follows immediately. Otherwise, suppose $k > 1$ and let $β = s_{i_2} ⋯ s_{i_{k-1}}α_{i_k}$. We see that we may select $w_{β} = s_{i_2}s_{i_3}⋯s_{i_k}$. We may assume by induction that $\#\{\text{noncompact } α_{i_j} : |α_{i_j}| ≥ |β|, 2 ≤ j ≤ k\} ≡ ε(β)$. Observe that $|β| = |xα| = |α_{i_k}|$. If $|α_{i_1}| < |xα| = |β|$, then $2(β, α_{i_1})/(|α_{i_1}|^2)$ is even (recall we already settled the case of type $G_2$ roots), so $ε(xα) = ε(s_{i_1}β) = ε(β)$. The sets we count for $xα$ and for $β$ are the same, so the result holds for $xα$. If $|α_{i_1}| ≥ |xα| = |β|$, then $xα = s_{i_1}β = β + α_{i_1}$, from which the result follows immediately for $xα$.

4. Comparing Signed Kazhdan-Lusztig Polynomials and Classical Kazhdan-Lusztig Polynomials

We will use the following notation in this section:
which descends naturally to a non-degenerate invariant Hermitian form 

**Lemma 4.3.** (\[Vog84\], Proposition 3.3) Using the notation of the previous definition, let \((p_j, q_j)\) be the signature of \(\langle \cdot, \cdot \rangle_j\). Then:

For small \(t > 0\), the signature is \[
\left(\sum_j p_j, \sum_j q_j\right)
\]

For small \(t < 0\), the signature is \[
\left(\sum_{j \text{ even}} p_j + \sum_{j \text{ odd}} q_j, \sum_{j \text{ odd}} p_j + \sum_{j \text{ even}} q_j\right)
\]

Since Verma modules \(M(\lambda)\) may be viewed as all being realized on the same vector space \(U(n^-)\) and the weight space decomposition is an orthogonal decomposition under the Shapovalov form, analytic paths in the real subspace of \(\mathfrak{h}^*\) of imaginary weights and the Shapovalov forms on the corresponding Verma modules give rise to the Jantzen filtration on a given Verma module. Let \(\lambda\) be antidominant and let \(x \in W_\lambda\), the integral Weyl group. We consider Verma modules of the form \(M(x\lambda)\). It is well-known that the \(j^{th}\) level of the Jantzen filtration of \(M(x\lambda)\) does not depend on the choice of analytic path (proved by Barbasch in \[Bar83\]). Furthermore, the \(j^{th}\) level of the Jantzen filtration, \(M(x\lambda)_{(j)} := M(x\lambda)^{(j)}/M(x\lambda)^{(j+1)}\) is semisimple and a direct sum of modules of the form \(L(y\lambda)\) where \(y \in W_\lambda\) and \(y \leq x\). The Jantzen Conjecture states that the multiplicity of any particular irreducible highest weight module in the \(j^{th}\) level of the Jantzen filtration may be determined by classical Kazhdan-Lusztig polynomials:

**Theorem 4.4.** (\[BB93\]) Jantzen’s Conjecture: Let \(\lambda\) be antidominant, \(x, y \in W_\lambda\). Then:

\[
[M(x\lambda)_{(j)} : L(y\lambda)] = \text{coefficient of } q^{(\ell(x) - \ell(y) - j)/2} \text{ in } P_{\Delta^0_x, \Delta^0_y}(q).
\]
While the vectors in the \( j \)th level of the Jantzen filtration of \( M(x\lambda) \) are independent of the choice of analytic path, the signature of \( \langle \cdot, \cdot \rangle \) on \( M(x\lambda)_{(j)} \) is not. For example, combining Lemma 2.7 and Lemma 4.3, one observes that the signature of \( \langle \cdot, \cdot \rangle \) depends on the direction of the analytic path.

For the purpose of studying signatures, rather than recording multiplicities in the \( j \)th level of the Jantzen filtration, contributions by all \( L(y\lambda) \)'s to the signature of \( \langle \cdot, \cdot \rangle \) are recorded for a particular filtration direction in signed Kazhdan-Lusztig polynomials:

**Definition 4.5.** ([Yee08]) Let \( \lambda \) be antidominant and let \( x, y \in W_\lambda \). Consider the invariant Hermitian forms \( \langle \cdot, \cdot \rangle \) on the various levels of the Jantzen filtration of \( M(x\lambda) \) arising from an analytic path whose direction as \( t \to 0^+ \) is \( \delta \). If

\[
ch_s \langle \cdot, \cdot \rangle = \sum_{y \leq x} a_{\lambda,\delta} w_0^\lambda x, w_0^\lambda y \cdot ch_s L(y\lambda)
\]

where by \( ch_s L(y\lambda) \) we mean the signature of the Shapovalov form, then the value of \( a_{\lambda,\delta} w_0^\lambda x, w_0^\lambda y \) is the same for all \( \delta \) in the interior of the same Weyl chamber. We use the notation \( a_{\lambda,\delta} w_0^\lambda x, w_0^\lambda y \) and \( a_{\lambda,\delta} w_0^\lambda x, w_0^\lambda y \) where \( w \in W \) interchangeably without further comment. Signed Kazhdan-Lusztig polynomials are defined by:

\[
P_{w_0^\lambda x, w_0^\lambda y}^\lambda(q) := \sum_{j \geq 0} a_{w_0^\lambda x, w_0^\lambda y}^\lambda ch_s L(y\lambda)
\]

Note that for small \( t > 0 \), recalling that \(-\rho \in \mathfrak{c}_0\), we have by Lemma 4.3,

\[
e^{\omega_\rho t} ch_s M(x\lambda + w(-\rho)t) = \sum_{y \leq x} P_{w_0^\lambda x, w_0^\lambda y}^\lambda(1)ch_s L(y\lambda).
\]

The left side is known by work in [Yee05]. We would like a formula for \( ch_s L(x\lambda) \), which requires inversion. In [Yee], we will show a simple inversion formula which expresses \( ch_s L(x\lambda) \) as a linear combination of \( ch_s M(y\lambda + w_\lambda^\rho(-\rho)t) \). It vastly improves the inversion formula found in [Yee08].

The simple inversion formula is related to the main theorem of this paper, which we now proceed to state and prove:

**Theorem 4.6.** Let \( \lambda \) be antidominant, and let \( x, y \in W_\lambda \). Then signed Kazhdan-Lusztig polynomials are related to classical Kazhdan-Lusztig polynomials by:

\[
P_{x,y}^\lambda(q) = (-1)^{\epsilon(x\lambda-y\lambda)} P_{x,y}(-q)
\]

where \( \epsilon \) is the \( \mathbb{Z}_2 \)-grading on the (imaginary) root lattice. Specifically, \( \epsilon(\mu) \) is the parity of the number of non-compact roots in an expression for \( \mu \) as an integral linear combination of roots.

**Proof.** We prove this theorem by induction and by comparing recursive formulas for computing classical and signed Kazhdan-Lusztig polynomials after substituting appropriate simplifications determined in this paper.

Classical Kazhdan-Lusztig polynomials may be computed using \( P_{x,x}(q) = 1 \), \( P_{x,y}(q) = 0 \) if \( x \not\leq y \), and the following recursive formulas where \( s = s_\alpha \) where \( \alpha \in \Pi_\lambda \):
a) If $ys > y$ and $xs > x \geq y$ then:

$$P_{w_\lambda^0, w_\lambda^0 y} = P_{w_\lambda xs, w_\lambda^0 y}$$

a') If $sy > y$ and $sx > x \geq y$ then:

$$P_{w_\lambda^0, w_\lambda^0 y} = P_{w_\lambda^0, w_\lambda^0 y}$$

b) If $y > ys$ and $x < xs$ then:

$$P_{w_\lambda^0, w_\lambda^0 y} + qP_{w_\lambda^0, w_\lambda^0 y} = \sum_{z \in W_\lambda | z < zs} a_{w_\lambda^0, w_\lambda^0 y} q^\ell(x) \ell(y) + 1 P_{w_\lambda z, w_\lambda^0 y}$$

Now we study recursive formulas for signed Kazhdan-Lusztig polynomials from [Yee08].

First, we must note some errata:
- The formula before Proposition 4.6.6 uses incorrect notation. It should say:

$$ch_s U_\lambda M(z\lambda + \delta t)_0 = \text{sgn}(\bar{c}t^r z, c)_0 \text{ch}_s L(z\lambda)$$

- Substituting the above into the $j^{th}$ level formula from Proposition 4.6.5 of [Yee08], the sign of the first term on the right hand side gives rise to the coefficient for the final term on the right hand side of Proposition 4.6.6. Proposition 4.6.6 should actually state:

If $x, y \in W_\lambda$ are such that $x < xs$ and $y > ys$ and $x > y$, then:

$$\text{sgn}(\bar{c}t^r z, c)_0 P_{w_\lambda^0, w_\lambda^0 y}(q) + \text{sgn}(\bar{c}t^r z, c)_0 P_{w_\lambda^0, w_\lambda^0 y}(q)$$

$$= \sum_{z \in W_\lambda | z < zs} \text{sgn}(\bar{c}t^r z, c)_0 a_{w_\lambda^0, w_\lambda^0 y} q^\ell(x) \ell(y) + 1 P_{w_\lambda^0, w_\lambda^0 y}(q)$$

Note that only the sign in the last term on the right side of the formula changed.

- Due to the above correction, formula b) in Theorem 4.6.10 of [Yee08] becomes:

$$-(-1)^{c((\lambda,\alpha^\vee) x\alpha)} P_{w_\lambda^0, w_\lambda^0 y}(q) + \text{sgn}(\delta, x\alpha^\vee) q P_{w_\lambda^0, w_\lambda^0 y}(q)$$

$$= \sum_{z \in W_\lambda | z < zs} \text{sgn}(\delta, x\alpha^\vee) a_{w_\lambda^0, w_\lambda^0 y} q^\ell(x) \ell(y) + 1 P_{w_\lambda^0, w_\lambda^0 y}(q) - (-1)^{c((\lambda,\alpha^\vee) y\alpha)} P_{w_\lambda^0, w_\lambda^0 y}(q).$$

The final term in the right side of the formula changed to $-(-1)^{c((\lambda,\alpha^\vee) y\alpha)} P_{w_\lambda^0, w_\lambda^0 y}(q)$.

Just as $\text{sgn}(\bar{c}t^r z, c)_0 = (-1)^{c((\lambda,\alpha^\vee) x\alpha)}$ for $x < xs$, so must $\text{sgn}(\bar{c}t^r z, c)_0 = (-1)^{c((\lambda,\alpha^\vee) y\alpha)}$ for $ys < y$.

Substituting Theorem 3.8 into Theorem 4.6.10 of [Yee08] with the erratum in the case $w = w_\lambda^0$ and using invariance of the inner product on $b^*$ under the Weyl group, we see that signed Kazhdan-Lusztig polynomials may be computed using $P_{x, x}(q) = 1$, $P_{x, y}(q) = 0$ if $x \not< y$, and the following recursive formulas where $s = s_\alpha$ where $\alpha \in \Pi_\lambda$:
a) If \( y < s \) and \( x > x \geq y \) then:

\[
P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)}P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(q)
\]

a') If \( y > s \) and \( x > x \geq y \) then:

\[
P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(q)
\]

b) If \( y > y > s \) and \( x < x \) then:

\[
\left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(q) + qP_{w^0_{X},w^0_{Y}}(q)\]
\[
= q \sum_{z \in W_{\lambda}} a_{w^0_{X},w^0_{Y}}(q) \frac{\left(\epsilon(z) - \epsilon(y) - 1\right)}{2} P_{w^0_{X},w^0_{Y}}(q)
\]
\[
- \left(-1\right)^{\epsilon((y,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(q).
\]

Observe that the theorem holds for \( x = 1 \) and \( x = y \). We now prove our theorem by induction on \( x \): that is, if the theorem holds for \( x \), then it holds for \( x \) and therefore it holds in general.

Formula a): if by the induction hypothesis

\[
P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q)
\]

and \( y < s \) and \( x > x \geq y \), then formula a) gives

\[
P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q)
\]

since \( x s \lambda = s_x x \lambda = x \lambda = (x \lambda,\alpha)\).

Formula a'): if by the induction hypothesis

\[
P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q)
\]

and \( y > y > s \) and \( x > x \geq y \), then

\[
P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q)
\]

since \( x s \lambda = x \lambda = (x \lambda,\alpha)\).

Formula b): Suppose \( y > y > s \) and \( x < x \). If by the induction hypothesis:

\[
P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q),
\]

\[
P_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q),
\]

\[
a_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q),
\]

\[
a_{w^0_{X},w^0_{Y}}(q) = \left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q),
\]

\[
\left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q)
\]

(recall that we can in fact apply induction to \( z \) since \( z \leq x \), or else \( P_{w^0_{X},w^0_{Y}}(-q) = 0 \), formula b) may be rewritten:

\[
\left(-1\right)^{\epsilon((x,\lambda)\alpha)\alpha}P_{w^0_{X},w^0_{Y}}(-q)
\]

SQL query result:
Rearranging, we obtain:

\[
P_{w_\lambda^0, x, w_\mu^0}^\lambda(q) = (-1)^{\epsilon(w_\lambda^0 (xs\lambda - y\lambda))}\left[q P_{w_\lambda^0, x, w_\mu^0}^\lambda((-q)) - q \sum_{z \in W_\lambda | z < zs} a_{w_\lambda^0 z, w_\mu^0 y, 1}((-q))^{(z) - (y) - 1} P_{w_\lambda^0 z, w_\mu^0 y}^\lambda((-q)) + P_{w_\lambda^0 x, w_\mu^0 y}^\lambda((-q))\right].
\]

Observe that up to multiplication by \((-1)^{\epsilon(w_\lambda^0 (xs\lambda - y\lambda))}\), the right hand side is simply the formula for \(P_{w_\lambda^0, x, w_\mu^0}^\lambda(q)\) arising from classical formula b) with \(-q\) in place of \(q\), whence

\[
P_{w_\lambda^0, x, w_\mu^0}^\lambda(q) = (-1)^{\epsilon(w_\lambda^0 (xs\lambda - y\lambda))} P_{w_\lambda^0, x, w_\mu^0}^\lambda((-q))
\]

and the theorem holds in general by induction. \(\square\)

5. Conclusion

Although the classification of unitary highest weight modules has been solved by work of Enright-Howe-Wallach, it would be interesting to recover the classification using signed Kazhdan-Lusztig polynomials and the formulas in this paper and in [Yee]. Cohomological induction applied to highest weight modules produces Harish-Chandra modules for which signatures can be recorded using signed Kazhdan-Lusztig-Vogan polynomials. Techniques used to identify unitary representations among highest weight modules may very well have analogues for Harish-Chandra modules.

It would also be interesting to investigate signed Kazhdan-Lusztig polynomials for the non-equal rank case.

References

[Bar83] Dan Barbasch. Filtrations on Verma modules. Ann. Sci. École Norm. Sup. (4), 16(3):489–494, 1983.

[BB93] A. Beilinson and J. Bernstein. A proof of Jantzen Conjectures, volume 16 of Adv. Soviet Math. Amer. Math. Soc., Providence, RI, 1993.

[Hum72] James E. Humphreys. Introduction to Lie Algebra and Representation Theory. Number 9 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1972.

[Vog84] David A. Vogan. Unitarizability of certain series of representations. Annals of Mathematics, 120:141–187, 1984.

[Wal84] Nolan R. Wallach. On the unitarizability of derived functor modules. Invent. Math., 78(1):131–141, 1984.

[Yee] Wai Ling Yee. Classical and signed Kazhdan-Lusztig polynomials: Character multiplicity inversion by induction. Preprint. arxiv ID.

[Yee05] Wai Ling Yee. The signature of the Shapovalov form on irreducible Verma modules. Representation Theory, 9:638–677, 2005.

[Yee08] Wai Ling Yee. Signatures of invariant Hermitian forms on irreducible highest weight modules. Duke Mathematical Journal, 142:165–196, 2008.

[Yee09a] Wai Ling Yee. Signed Kazhdan-Lusztig polynomials for compact real forms. Technical report, University of Windsor, Windsor, Ontario, 2009.

[Yee09b] Wai Ling Yee. Signed Kazhdan-Lusztig polynomials for compact real forms: Erratum. Technical report, University of Windsor, Windsor, Ontario, 2009.
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