Unconventional superfluidity in Bose-Fermi Mixtures

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Pairing between fermions that attract each other, reveal itself to the macroscopic world in the form of superfluidity. Since the discovery of fermionic superfluidity \cite{1}, intense search has been going on to find various unconventional forms of fermion pairing \cite{2} as well as to increase the transition temperature \cite{3}. Here, we show that a two dimensional mixture of single-component fermions and dipolar bosons allows to reach experimentally feasible superfluid transition temperatures for non-standard pairing symmetries. Excitations in these superfluids are anyonic and their statistics depends on the order of their permutations, i.e is non-Abelian. Our results provide for the first time an example of a highly tunable system which exhibits various kind of pairing symmetry and high transition temperature. Additionally, they provide a playground to observe anyonic excitations and their braiding properties.

Attraction between fermionic particles favours pairing of fermions resulting in superfluidity of the system. The paired fermions, known as Cooper pairs, can have different kind of internal symmetries. The common ones found in nature have $s$-wave and $d$-wave like internal structure and conserves parity and time reversal symmetry. Also, Cooper pairs with chiral $p_\pm + ip_\pm$-wave internal structure have been proposed for the observed superfluidity of electrons in Strontium Ruthenate \cite{4}. This kind of pairing breaks the time-reversal symmetry. Spin-less chiral $p$-wave superfluid state has formal resemblance with the “Pfaffian” state proposed in relation to the fractional quantum Hall state with filling factor $5/2$ \cite{5,6,7}. When confined in a two dimensional geometry, excitations in the chiral $p$-wave superfluid become non-abelian anyons.

Anyons are particles living in a two dimensional plane that under exchange do not behave as bosons or fermions. In some cases, exchange of two such particles depends on the order of the exchange \cite{8,9,10}. Particles obeying such laws are called non-Abelian anyons. Apart from fundamental interest on the occurrence of such particles, non-Abelian anyons find remarkable applications in the field of quantum information for quantum memories and fault-tolerant quantum computation \cite{11}. Recently it has been shown that quasi-particles living in stable vortex excitations of chiral two-dimensional $p$-wave spinless superfluids obey non-Abelian statistics \cite{12,13}. Using $p$-wave Feshbach resonances in fermionic ultracold atoms, such superfluids can be realized in principle, but this procedure is very difficult because of non-elastic loss processes \cite{14}.

Bose-Fermi mixtures are another candidate for creating superfluidity in fermions via boson mediated interactions and have formal resemblance with phonon mediated superconductivity in metals \cite{1}. It was found, however, when the bosons and single-component fermions are completely mixed, the maximum possible transition temperature is of the order of $10^{-5} T_F$ for $p$-wave pairing, where $T_F$ is the Fermi Temperature \cite{14}. Any attempt to increase the transition temperature by increasing the boson-fermion interaction strength or fermionic density results in phase separation between the mixture. Hence it is experimentally hardly possible to study this phenomenon for unconventional superfluidity in ultracold atomic systems.

Here, we show a possible way of overcoming these difficulties. We study the property of superfluidity in Bose-Fermi mixtures, where bosons are interacting via long-ranged dipolar interactions. We show that the transition temperature for $p$-wave superfluidity can become comparable to the Fermi energy. We find that other more exotic Cooper pairs with $f$- and $h$-wave internal symmetries are possible in certain range of Fermi energies without bosons and fermions separating. To the best of our knowledge, for the first time a system is proposed where conventional pairing mechanism gives rise to different exotic internal structures of the Cooper pairs with strong interaction in respective angular momentum channels. In addition, we study the excitations in chiral states of the odd-wave superfluids and point out their non-Abelian anyonic nature.

Experimentally, an available bosonic species, where prominent dipolar interaction can be achieved using Feshbach resonance is Cr$^{52}$ \cite{17,18}. Another route towards achieving dipolar condensate is to experimentally realize quantum degenerate heteronuclear molecules \cite{19} which have permanent electric moment. Thus in the near future a quantum degenerate mixture of dipolar bosons and fermions will be achievable experimentally.

\section*{I. SYSTEM}

\subsection*{A. Dipolar Bose Condensate}

Our system, as sketched in Fig.1, consists of dipolar bosons mixed with single component fermions confined in a quasi-two dimensional geometry by a harmonic po-
tential with frequency $\omega_z$ and oscillator length $\ell_0$. Here we present a brief overview of dipolar condensates with a focus on our present problem (for details see methods). First, we assume that the bosons are polarized along the $z$ direction. The dipolar interaction reads $V_{dd} = \frac{16\pi g_{dd}}{\ell_0^3}(3k_z^2/k^2 - 1)$ in momentum space, where $g_{dd}$ is the dipole-dipole interaction strength. For atoms $g_{dd} = \mu_0\mu^2/m/4\pi$, and for dipolar molecules $g_{dd} = \mu^2/4\pi\epsilon_0$ where $\mu_m$ and $\mu_e$ are the magnetic moment of the atoms and the electric dipole moment of the molecules, respectively. We assume that the $z$ dependance of the bosonic excitations, the condensate-polarized states). First, we assume that the bosons are polarized along the $z$ direction. The effective dipolar interaction takes the form $V_{eff} = \frac{2g_{dd}}{\ell_0^2}V(k_{\perp})$, where $k_{\perp}^2 = k_x^2 + k_y^2$ and $V(k_{\perp})$ denotes the shape of the interaction. Next, we define a dimensionless dipolar interaction strength $g_{dd} = \frac{8\pi m_b g_{dd} m_0 \ell_0}{5\hbar^2}$ which will be used later, where $\ell_0$ is the ground state oscillator length. $V(k_{\perp})$ is repulsive for small momentum and attractive in the high momentum limit. At low temperature, the ground state of the bosons is a condensate with fluctuations around this state. Generally, the spectrum of these excitations, denoted by $\Omega(k_{\perp})$, can be divided into two parts: i) $\sim k_{\perp}$, phonon spectrum for small momenta and ii) $\sim k_{\perp}^2$, free-particle like spectrum for higher momenta. For the dipolar condensates and $g_{dd}$ greater than a critical value, $\Omega(k_{\perp})$ has a minimum at momentum $k_0$, where $k_0$ is in intermediate momentum regime as shown in Fig. 2b. Following Landau, the excitations around the minimum are called “rotons”. With increasing $g_{dd}$ the excitation energy at $k_0$ decreases and eventually vanishes for a critical particle density as shown in Fig. 2b. When the particle density exceeds that critical value, the excitation energy becomes imaginary at finite momentum and the condensate becomes unstable. The “roton” part of the spectrum is absent in condensates with contact interaction.

B. Effective interaction

In this section we discuss the effect of bosons on fermions. The fermions are interacting with the bosons via short ranged contact interaction of strength $g_{bt}$. The fermionic density dependance on $z$ is given by a normalized Gaussian with width $\ell_f$. We also define the dimensionality parameter for the fermions as $\eta = \mu/\hbar\omega_z$, where $\mu$ is the chemical potential for the fermions. For $\eta < 1$ the fermionic system is quasi-two dimensional, and we can assume $\ell_f \sim \sqrt{\hbar/m/\omega_z}$, i.e the width is given by the oscillator length of the trapping potential. $\eta > 1$ corresponds to three dimensional fermionic system and $\ell_f$ is given by minimizing the Thomas-Fermi energy functional, $\ell_f \sim \eta^{1/4}(\hbar/m_f/\omega_z)^{1/2}$. The fluctuations in fermionic density couple to the density fluctuations present in the Bose condensate. Due to the momentum dependance of the bosonic excitations, the condensate-fermion interaction becomes function of momentum. Integrating out the bosonic degree of freedom results in effective interaction between the fermions,

$$V_{ph}(q_{\perp}, \omega) = \frac{9\eta_0^2\alpha^2}{16\pi R^2} \frac{n_0 q^2}{m_b} \frac{1}{\omega^2(q_{\perp})^2},$$

where $q^2 = 2k_f^2(1 - \cos \phi)$, is the momentum exchange between the interacting particles along the Fermi surface, and $\alpha$ is a number which depends on $R/\ell_f$ (see Methods). This form of interaction has formal resemblance to superconductivity in metals. Due to the momentum dependance, if attractive, $V_{ph}(q_{\perp}, \omega)$ makes the Fermi surface unstable against formation of Cooper pairs with higher internal symmetries. Assuming momentum transfer around Fermi momentum $k_f$, in two-dimensions we can expand Eq. (1) as

$$V_{ph}(k_{\perp}, 0) = \frac{3g_{dd}^2 N_0}{8\pi g_{dd} \omega(0) \ell_0} \sum_{m = -1, 0, 1, \ldots} \lambda_m e^{im\phi},$$

where the dimensionless effective interaction between the fermions in angular momentum channel $m$ is given by

$$\lambda_m = \alpha^2 \frac{m_f^2}{\pi^2 \hbar^2} \frac{\exp(im\phi)\phi/2\pi}{\int_0^{2\pi} \frac{d\phi}{\omega(\phi)}},$$

and $N_0 = m_f/\pi^2\hbar^2$ is the two-dimensional density of states for the fermions. In the present paper, $\lambda_m > 0$ denotes attractive interaction. As we are considering single component fermions, pairing will occur in the odd angular momentum channels. The effective dipole-dipole interaction between the bosons is given by the density dependent term $g_{dd}$. The expression for effective interaction obviously makes sense as long as the excitation frequencies of the bosons are real. To look into the properties of effective interaction, we consider first the interaction strength in $p$-wave channel $\lambda_1$. As shown in Fig. 2b, increasing $g_{dd}$ results in smaller roton gap, and when $g_{dd} \sim 3.61$, the roton minimum touches the zero energy axis. Subsequently the effective fermionic interaction $\lambda_1$ increases and diverges as $g_{dd} \rightarrow 3.61$, as shown in Fig. 2a. Depending on the dimensionality $\eta$, the rate of divergence changes and the interaction can become repulsive. This Feshbach resonances-like characteristics are the novel feature of this system.

Next we look into the variation of $\lambda_m$ as a function of $\eta$ as depicted in Fig. 3. Here for concreteness we consider the particular case corresponding to a Chromium-Potassium mixture with $m_b = 52$ a.m.u. and $m_f = 40$ a.m.u.. Using these masses, the interaction strengths in the channels $m = 1$ ($p$-wave), $m = 3$ ($f$-wave), $m = 5$ ($h$-wave) have been plotted in Fig. 3. As $\eta \rightarrow 1$, interaction in the $p$-wave channel becomes predominant. This trend is also followed when we are inside the three dimensional limit with $\eta > 1$. But, with decreasing dimensionality we find that the predominant interaction channel changes.
FIG. 1: (a) Schematic image of a two-dimensional mixture of dipolar bosons (semi-transparent spheres with arrows) and fermions (blue spheres). (b) The interaction between the bosons and fermions induces interaction between the fermions. This results in superfluidity of the fermions with various angular momenta as indicated by the arrows around the fermions.

FIG. 2: (a) Effective interaction strength in the $p$-wave channel $\lambda_1$ as a function of boson-boson effective interaction strength $g_{3d}$. The green, blue and black lines correspond to $\eta = 0.50, 0.60, 0.90$ respectively. (b) The excitation spectrum $\Omega(k_{\perp})$ of the two dimensional Bose gas plotted as a function of $k_{\perp}$. The blue, black and red lines correspond to $g_{3d} = 2.0, 3.0, 3.61$, respectively.

surprisingly from $m = 1$ to $m = 5$, the $h$-wave channel. Then $\lambda_5$ goes through a maximum attraction around $\eta \sim 0.6$ and the interaction strength in the $p$-wave channel becomes repulsive. In this region, the repulsive $p$-wave interaction will renormalize the mass of the fermions and the superfluid instability will be due to $m = 5$ channel. By decreasing $\eta$ further, both $\lambda_{1,5}$ become repulsive whereas the $f$-wave channel with $m = 3$ becomes attractive. Decreasing $\eta$ further results in a situation, where all the channel have negligible interaction. The interaction strength at individual channel can be increase further by increasing the bosonic density $n_b$ closer to the critical value.

Another obvious contribution to the strength of the interaction is the boson-fermion contact interaction strength $g_{bf}$. The condition for mechanical stability of the Bose-Fermi mixtures reads 

$$\frac{3g_{bf}^2N_0}{8\pi g_{dd}V(0)\ell_0} < 1,$$

which constraints the magnitude of $g_{bf}$. From Fig. 2a and Fig. 3 it is clear that the maximum attractive interactions in each channel can be $\lambda_1, \lambda_2, \lambda_3 > 1$ which are within the strong-coupling regime. Consequently maximum transition temperatures possible in all three channels are close to the fermi temperature. At this point let us compare this to the case of usual boson-fermion mixture, where highest interaction strength possible is $\lambda_1 \sim 1$ [15, 16]. The maximum transition temperature possible is for $p$-wave symmetry and $T_c$ is of the order $\sim \exp(-1/1)\mu \sim 10^{-5}\mu$. Note, then we have an improvement in $T_c$ of the order of $10^5$.

II. CHIRALITY AND NON-ABELIAN ANYONS

By considering the superconducting gap equation at low temperature, the gap is maximum when the order parameter breaks time reversal symmetry [2]. From now

- **FIG. 1:**
  - (a) Schematic image of a two-dimensional mixture of dipolar bosons (semi-transparent spheres with arrows) and fermions (blue spheres).
  - (b) Interaction between the bosons and fermions induces interaction between the fermions. This results in superfluidity of the fermions with various angular momenta as indicated by the arrows around the fermions.

- **FIG. 2:**
  - (a) Effective interaction strength in the $p$-wave channel $\lambda_1$ as a function of boson-boson effective interaction strength $g_{3d}$. The green, blue and black lines correspond to $\eta = 0.50, 0.60, 0.90$ respectively.
  - (b) Excitation spectrum $\Omega(k_{\perp})$ of the two dimensional Bose gas plotted as a function of $k_{\perp}$. The blue, black and red lines correspond to $g_{3d} = 2.0, 3.0, 3.61$, respectively.

- **FIG. 3:**
  - Maximum attractive interactions in each channel can be $\lambda_1, \lambda_2, \lambda_3 > 1$ which are within the strong-coupling regime. Consequently maximum transition temperatures possible in all three channels are close to the fermi temperature.
achieved in finding that the condition for non-overlapped states can be well separated vortices is necessary. From Eq. (4) we perform quantum computational task, existence of several quasi-particle operators in that situation is written as

\[ \Delta_m = \Delta_0(\vec{r}) \left[ \frac{k}{k_f} \right]^m e^{im\theta} \]

where \( k_x = k \cos \theta, k_y = k \sin \theta \) and \( \Delta_0(\vec{r}) \) is the center of mass amplitude of the Cooper pairs with \( \vec{r} \) being the center of mass coordinate of the pair. For vortex state \( \Delta_0(\vec{r}) \) is defined as: i) \( \Delta_0(\vec{r}) = 0, r < \xi \) and ii) \( \Delta_0(\vec{r}) = \Delta_0 \exp(i\phi), r \geq \xi \), where \( r = \sqrt{x^2 + y^2} \), \( \tan \phi = y/x \). \( \xi \) is the size of the core of the vortex. The vortex state of the \( p \)-wave superfluids always has a zero-energy bound quasi-particle state. Now we discuss the asymptotic solutions for the zero-energy bound state for \( f \)- and \( h \)-wave order parameters. The quasi-particle states in a single vortex can be found in the limit of large distance from the vortex core by solving the Bogoliubov-DeGennes equation,

\[
H_0u_m + (-i)^m \frac{\Delta_0}{k_f} e^{i\phi/2} \left[ e^{-i\phi} \left( \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \phi} \right) \right]^m e^{i\phi/2} v_m = E u_m
\]

\[
-H_0v_m + (i)^m \frac{\Delta_0}{k_f} e^{-i\phi/2} \left[ e^{i\phi} \left( \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right) \right]^m e^{-i\phi/2} u_m = E v_m,
\]

The asymptotic solution of Eq. (3) with different orbital symmetries read,

\[
\begin{bmatrix}
    u_1 \\
    u_3 \\
    u_5
\end{bmatrix} \sim \begin{bmatrix}
    \exp \left( -\frac{\Delta_r}{\sqrt{2} \Delta_0} r \right) e^{2im\phi} \\
    \exp \left( \frac{m^2 \lambda^2}{\sqrt{2} \Delta_0} r \right) e^{-2im\phi} \\
    \exp \left( -\frac{m^2 \lambda^2}{2\Delta_0} \right) e^{4im\phi}
\end{bmatrix}.
\]

The zero-energy solution for each odd-wave parameter corresponds to different angular momentum channel of the quasi-particles inside a vortex core. These results can also be carried out by applying the “index theorem” \[24\]. For temperature smaller than the energy gap \( \Delta_0^2/\mu \), only the zero energy mode is occupied. The zero-energy state \( \gamma_m = \int d^2 r u_m(r)v_m(r) \), which acts as Majorana fermions \[4, 12, 22\]. \( \gamma_m \) obeys non-Abelian statistics and can be used for quantum computing \[12\]. In order to perform quantum computational task, existence of several well separated vortices is necessary. From Eq. (4) we find that the condition for non-overlapped states can be achieved in \( m = 3 \) states with higher distance between the vortices than the case for p-wave for similar values of gap and fermi energy.

### III. ACKNOWLEDGEMENT

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### IV. METHODS

First we discuss the Hamiltonian describing the bosonic system which is homogeneous in the \( x - y \) plane and trapped in the \( z \) direction by a harmonic potential with frequency \( \omega_z \). In the considered regime of parameters the bosonic density \( n_b(x, y, z) \) is given by

\[
n_b(x, y, z) = \frac{3n_v}{4R_z^2} \left( 1 - \frac{z^2}{R_z^2} \right),
\]

where the Thomas-Fermi radius \( R_z \) is determined variationally. By minimizing the mean field energy of the
Bose condensate within Thomas-Fermi regime we find that $R_z/\ell_0 = (5\ell_{dd}/2)^{1/3}$. After integrating over $z$ dependence of the density profile of bosons, the dipolar interaction takes the form $V_{\text{eff}} = \frac{3\gamma R_z}{2R_b} \mathcal{V}(k_{\perp})$ where

$$
\mathcal{V}(k_{\perp}) = \frac{1}{k_{\perp}} \left[ 4k_{\perp}^2 - 6k_{\perp}^2 - 6(1 + k_{\perp}^2) \exp(-2k_{\perp}) + 6 \right]
- \frac{8}{15} \frac{g}{5\pi g_{dd}},
$$

and $k_{\perp} = k_{\parallel} R_z$ and $g$ is the contact interaction between the bosons which is assumed to be negligible in our case. Subsequently we write the Hamiltonian of the dipolar bosons in the condensed phase, $H_b = \sum_{k_{\perp}} \Omega(k_{\perp}) b_{k_{\perp}}^\dagger b_{k_{\perp}}$, where $b_{k_{\perp}}^\dagger$ and $b_{k_{\perp}}$ are Bogoliubov operators. The excitation spectrum is given in the units of trap frequency,

$$
\Omega^2(k_{\perp} \ell_0) = \frac{|k_{\perp} \ell_0|^4}{4} + g_{dd} \frac{\ell_0}{R_z} \mathcal{V} \left( k_{\perp} \ell_0 \frac{R_z}{\ell_0} \right) \left[ k_{\perp} \ell_0 \right]^2.
$$

Next, we consider the Hamiltonian describing the fermions and the boson-fermion interaction. Kinetic energy for the single component non-interaction fermions, is characterized by the Hamiltonian $H_f = \sum_{k_{\perp}} \left[ \epsilon_f(k_{\perp}) - \mu \right] c_{k_{\perp}}^\dagger c_{k_{\perp}}$, where $c_{k_{\perp}}^\dagger$ and $c_{k_{\perp}}$ are fermionic creation and destruction operator. $\epsilon_f(k_{\perp}) = k_{\perp}^2 / 2m_f$ is the dispersion energy of the fermions with mass $m_f$. The density profile of fermions along the $z$ direction is approximated by a gaussian with width $\ell_f$.

Including the fluctuations in the Bose condensate in the $x - y$ plane, the condensate-fermion interaction Hamiltonian can be written as

$$
H_{bf} = \frac{3g_{bf}}{4\sqrt{\pi R_z}} \alpha \sum_{k_{\perp}, q_{\perp}} \gamma(k_{\perp}) \epsilon_{q_{\perp}}^\dagger \epsilon_{q_{\perp} - k_{\perp}} b_{k_{\perp}}^\dagger (b_{q_{\perp}}^\dagger + b_{-q_{\perp}}),
$$

where the momentum dependent coupling constant is given by $\gamma(k_{\perp}) = \sqrt{2\pi \ell_0 \epsilon_b(k_{\perp}) / \Omega(k_{\perp})}$ and

$$
\alpha = \frac{\ell_f}{R_z} \exp \left( -\frac{R_z^2}{\ell_f^2} \right) \sqrt{\frac{\pi}{2}} \left( \frac{\ell_f^2}{R_z^2} - 2 \right) \text{erf} \left( \frac{R_z}{\ell_f} \right),
$$

with $\text{erf}(\_\_)$ being the error function.
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