Exclusive decays $\chi_{cJ} \to K^*(892)K$ within the effective field theory framework

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Abstract

We study hadronic decays $\chi_{cJ} \to K^*(892)\bar{K}$ within the effective field theory framework. We consider colour-singlet and colour-octet contributions using (p)NRQCD effective theory and study their properties. We show that infrared singularities in collinear integrals of the colour-singlet amplitudes can be absorbed into the colour-octet matrix element. The heavy quark spin symmetry allows us to establish a relation between the colour-octet matrix elements and to compute hard spin symmetry breaking corrections which are free from any infrared divergencies. We apply obtained corrections for a phenomenological description of the measured branching fractions.
1 Introduction

A study of heavy quark systems like charmonium and bottomonium remains one of the most interesting topics in particle physics already for many years. Many new interesting experimental results have been obtained by BABAR, BELL, BESII and BESIII collaborations during last years. In particular, many new data about various exclusive decays have been collected and many new results are expected in the future. On the other side, a theoretical description of various exclusive decay channels remains puzzling, see e.g. discussions in reviews [1, 2] and references there in. Often, underlying hadronic dynamics is very complicated, involves non-perturbative effects which are even difficult to include into a systematic theoretical description. One of such problematic contributions is the colour-octet mechanism [1, 3]. In inclusive processes such contributions are described as unknown long-distance matrix elements [4] but for exclusive decays a systematic description of such mechanism is still not well understood [1, 2]. Some attempts to build a framework for the description of colour-octet matrix elements in exclusive decays can be found in Refs. [5–7]. In Ref. [8] it is suggested that the colour-octet configuration may play an important role for understanding of the well known \( \rho\pi \)-puzzle.

In present work we consider hadronic decays \( \chi_{cJ} \rightarrow K^* \bar{K} \) which are interesting because they are suppressed in QCD but have sufficiently large decay widths. The branching fractions of these decays have been measured by BES collaboration [9, 10]. In Table 1 we collect experimental results for the branching ratios from [11]. The amplitudes for these decays do not vanish due to the SU(3) breaking in QCD and the experimental results indicate that such symmetry breaking effects are sufficiently large.

Another interesting point is that the dominant amplitudes in both decays are suppressed according to helicity selection rule [12–14]. Therefore the decay amplitudes are sensitive to the higher Fock components of the meson wave functions. This effect is natural for \( \chi_{c2} \) and can also be easily established for \( \chi_{c1} \). Hence sufficiently large values of the measured decay rates also imply the strong violation of the helicity selection rule. In this respect these processes could be similar to decay \( J/\psi \rightarrow \rho\pi \) and probably have resembling underlying decay mechanism.

In Ref. [15] it is suggested that amplitude for \( \chi_{c2} \rightarrow K^* \bar{K} \) decay is dominated by long distance decay mechanism which can be accounted through the model with intermediate mesonic loops. The obtained numerical estimates are about factor two smaller then the experimental results. The second decay \( \chi_{c1} \rightarrow K^* \bar{K} \) has not yet been discussed in the literature, we could not find any theoretical predictions for the corresponding decay width.

In present work we consider both decays within the effective field theory framework. We apply NRQCD [4, 16] and potential NRQCD (pNRQCD) [17–22] effective theories and soft collinear effective theory (SCET) [23–28] in order to describe decays of \( P \)-wave quarkonia into \( K^* \bar{K} \) mesons. An advantage of this framework is in opportunity to apply the heavy quark spin symmetry (HQSS) which allows one to relate various matrix elements and to constrain a contribution associated with the colour-octet mechanism. The latter can play an important role in understanding of underlying mechanism of \( P \)-wave quarkonia decays [1, 5, 6].

We shall compute the colour-singlet contributions which involve different twist-2 and twist-3 \( K \)-meson light-cone distribution amplitudes (DAs). It will be shown that such contributions can not provide a consistent theoretical description because of infrared (IR) divergencies appearing in the collinear convolution integrals. Hence a naive collinear factorisation is violated in this case. Such situation often arises in description of amplitudes involving the higher Fock components of a hadronic wave functions. A systematic description of such endpoint divergencies still remains challenging to theory.

The structure of the IR divergencies allows one to conclude that they can be related with the colour-octet matrix element. A similar situation has been observed in analysis of \( B \rightarrow \chi_{cJ} \bar{K} \) decay in Ref. [31]. In this work it was shown that arising IR endpoint singularities in the colour-singlet contribution can be
absorbed into a colour-octet operator matrix element computed in the Coulomb limit. Following the same idea we shall define and compute the relevant colour-octet matrix elements in the Coulomb limit. We shall show that the endpoint singularities arising in the colour-singlet amplitudes can be absorbed into colour-octet matrix elements. Next, we shall obtain the HQSS relation between the colour-octet matrix elements of $P$-wave charmonia. Such constrain are dictated by the structure of the effective Lagrangian where the interactions with the heavy quark spin are suppressed. The relation between the colour-octet matrix elements allows one to obtain a consistent factorisation scheme for the spin symmetry breaking terms in the colour-singlet matrix elements. This description is quite similar to the physical subtraction scheme used for the description of various amplitudes in $B$-decays in Refs. [29,30].

Our paper is organised as follows: in Sec. 2 we set up the notation and define kinematics and amplitudes. In Sec. 3 we compute various colour-singlet contributions and study their properties. Sec. 4 is devoted to analysis of the colour-octet contributions in the Coulomb limit. In Sec. 5 we discuss effects provided by the symmetry-breaking corrections and estimate a contribution of the colour-octet matrix elements. Then we conclude in Sec. 6.

2 Kinematics, notation and decay amplitudes

The decay amplitudes $\chi_{cJ} \rightarrow \bar{K} + K^*$ are defined as

$$\langle \bar{K}(k)K^*(p); \text{out} | \text{in}; \chi_{cJ}(P) \rangle = i(2\pi)^4 \delta(P - p - k) \mathcal{M}_{\chi_{cJ} \rightarrow \bar{K}K^*}. \quad (1)$$

In what follow we use the frame where heavy meson is at rest and the $z$-axis is chosen along the momenta of outgoing particles

$$P = M(1, 0) = M\omega, \quad (2)$$

where $M$ is charmonium mass and $\omega$ denotes charmonium four-velocity. For any vector with subscript $V$ denote the indices transverse to velocity direction $\omega$: $V^{\mu} \cdot \omega = 0$. The momenta of outgoing mesons read

$$k = (k_0, 0, 0, k_z), \quad p = (p_0, 0, 0, p_z), \quad (3)$$

with (for simplicity, in the following we use $m_{\bar{K}} \equiv m_P$ and $m_{K*} \equiv m_V$)

$$k_0 = \frac{M^2 + m_P^2 - m_V^2}{2M}, \quad p_0 = \frac{M^2 - m_P^2 + m_V^2}{2M}, \quad (4)$$

$$k_z = -p_z = \frac{1}{2M} \left[ \left( M^2 - (m_V - m_P)^2 \right) \left( M^2 - (m_V + m_P)^2 \right) \right]^{1/2}. \quad (5)$$

Assuming that the heavy quark mass is sufficiently large $m \gg \Lambda_{QCD}$ one obtains

$$k \simeq m(1, 0, 0, 1) = 2m^2 \frac{n}{2}, \quad p \simeq m(1, 0, 0, -1) = 2m^2 \frac{n}{2}, \quad (6)$$

where we introduced auxiliary light-cone vectors $n$ and $\bar{n}$ with $(n\bar{n}) = 2$. Any four-vector $V^\mu$ can be decomposed as

$$V^\mu = (V \cdot n) \frac{\bar{n}^\mu}{2} + (V \cdot \bar{n}) \frac{\bar{n}^\mu}{2} + V_\perp^\mu, \quad (7)$$

where $V_\perp$ denotes the components which are transverse to the light-like vectors: $(V_\perp \cdot n) = (V_\perp \cdot \bar{n}) = 0$. In particular, in the rest frame

$$\omega = \frac{1}{2} (n + \bar{n}), \quad \omega^2 = 1. \quad (8)$$

In the following we also use the following short notations

$$g_\perp^{\mu\nu} = g^{\mu\nu} - \frac{1}{2} (n^\mu \bar{n}^\nu + n^\nu \bar{n}^\mu), \quad i\epsilon_\perp^{\mu\nu\alpha\beta} = \frac{1}{2} i\epsilon^{\mu\nu\alpha\beta} n^\alpha \bar{n}^\beta, \quad (9)$$

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with $\varepsilon_{0123} = 1$. The decay amplitudes can be parametrised as

$$M_{\chi_{c1} \to \bar{K}K^*} = (\epsilon_\chi \cdot k) (\epsilon_\bar{V}^\perp \cdot k) \frac{m_V}{M^2} A_1^\parallel + (\epsilon_\chi^\perp \cdot \epsilon_\bar{V}) \frac{(kP)}{M} A_1^\perp, \quad (10)$$

$$M_{\chi_{c2} \to \bar{K}K^*} = \frac{\epsilon_\chi^{\mu\nu} k_\nu \varepsilon_{\mu\alpha\beta\rho}}{(kp)} \frac{k^\beta p^\rho}{p^2} A_2^\parallel. \quad (11)$$

where $\epsilon_\chi$ and $\epsilon_\bar{V}$ denote polarisation vectors of the charmonium and vector meson, respectively. The polarisation vectors satisfy

$$\sum_\lambda (\epsilon_\chi^{(\lambda)})_{\mu} (\epsilon_\chi^{(\lambda)})_{\nu}^* = -g_{\mu\nu} + P_{\mu} P_{\nu}/M^2, \quad (12)$$

$$\sum_\lambda (\epsilon_\chi^{(\lambda)})_{\mu} (\epsilon_\chi^{(\lambda)})_{\rho}^* = \frac{1}{2} G_{\mu\rho} - \frac{1}{3} G_{\mu\sigma} G_{\nu^\sigma}, \quad (13)$$

where $G_{\mu\nu} = g_{\mu\nu} - P_{\mu} P_{\nu}/M^2$, the normalization is such that $(\epsilon_\chi^{(\lambda)})_{\mu\nu} (\epsilon_\chi^{(\lambda)})_{\mu\nu}^* \equiv (\epsilon_\chi^{(\lambda)})_{\mu} (\epsilon_\chi^{(\lambda)})_{\mu}^* \equiv \delta_{\lambda\lambda'}$ and similar for the $K^*$ vector meson.

The kinematical factors in Eqs. (10) and (11) are chosen in order to have dimensionless amplitudes $A_i$. One can easily see that amplitude $A_1^\parallel$ describes decay of the longitudinally polarised $\chi_{c1}$ while the amplitudes $A_{1,2}^\perp$ correspond to transversely polarised $\chi_{c1,2}$. The expressions for decay widths read

$$\Gamma[\chi_{c1} \to \bar{K}K^*] = \frac{\sqrt{3}}{8 \pi} \frac{k_0^2}{M^2} \left( |A_1^\parallel|^2 + \frac{1}{2} \frac{(pk)^2}{M^4} |A_1^\perp|^2 \right), \quad (14)$$

$$\Gamma[\chi_{c2} \to \bar{K}K^*] = \frac{\sqrt{5}}{8 \pi} \frac{k_0^2}{M^2} |A_2^\perp|^2 \left( 1 - \frac{m_P^2}{k_0^2} \right) \left( 1 - \frac{m_P^2 m_V^2}{(kp)^2} \right). \quad (15)$$

### 3 Colour-singlet contributions

#### 3.1 Colour-singlet contribution to amplitude $A_1^\parallel$

The computation of the colour-singlet contribution is quite standard, corresponding contribution is described by the diagrams in Fig. 1. The heavy quark and antiquark annihilate at short distance of order $1/m$ into the two highly virtual gluons which further create light quark-antiquark pairs forming the final mesons. An average size of the charmonium is of order $1/mv$ where $v$ is the heavy quark velocity in the rest frame. Since $mv \ll m$ the colour-singlet decay amplitude is proportional to the heavy meson wave function at the origin. Corresponding contribution can be described by a matrix element of the appropriate colour-singlet operator in NRQCD framework. The details about the required matrix elements are given in Appendix A.

The transitions of the light quarks into final mesons also involve non-perturbative QCD interactions associated with the typical hadronic scale $\Lambda \ll m$. In charmonium rest frame the energies of the outgoing mesons are large of order $m$ and corresponding non-perturbative contributions are described by the light-cone matrix elements which are related to the light-cone wave functions at zero transverse separation, the so-called light-cone distribution amplitudes (DAs). Detailed description of these quantities is also given in Appendix A.

All the matrix elements arising in description of the amplitudes can be estimated according to the power counting with respect to small parameters: velocity $v$ and ratio $\Lambda/m$. At the leading-order we only have contribution to the amplitude $A_1^\parallel$. In this case the soft overlaps with the $in$ and $out$ mesonic states are described by the leading-order NRQCD matrix element and by the leading twist DAs $\phi^{\mu\nu}_2$ and $\phi_{2P}$ where the subscripts $V$ and $P$ denote the vector and pseudoscalar mesons. In the following we always assume $V \equiv K^*$ and $P \equiv K$. The transverse amplitudes $A_{1,2}^\perp$ are suppressed by the power of $\Lambda/m$ due to the helicity conservation in the hard subprocess. As a result they depend on the twist-3 DAs and this provides suppression by the extra power of the small ratio $\Lambda/m$. 
Figure 1: The QCD diagrams describing the colour-singlet mechanism of $\chi_{cJ} \to VP$ decays. The blobs denote various non-perturbative matrix elements.

The computation of the diagrams in Fig.1 with the appropriate operator projections gives the following result

$$A_\parallel^J = -\frac{f_V f_P}{m^2} i \langle O^3P_1 \rangle \left(\frac{\pi \alpha_s(\mu^2)}{N_c}\right)^2 C_F J_\parallel^J(\mu).$$  \hfill (16)

with the collinear convolution integral $(1 - x \equiv \bar{x})$

$$J_\parallel^J(\mu) = \int_0^1 dx \frac{\phi_{2V}(x, \mu)}{x\bar{x}} \int_0^1 dy \frac{\phi_{2P}(y, \mu)}{y\bar{y}} \frac{y - x}{xy + \bar{y}x}. \hfill (17)$$

We also imply standard notation $C_F = (N_c^2 - 1)/(2N_c)$ with $N_c = 3$. The factorisation scale $\mu$ is of order of the hard scale $m$. The definitions of the non-perturbative constants $f_V \parallel$, $f_P$ and $\langle O^3P_1 \rangle$ are given in Appendix A. According to NRQCD counting rules $\langle O^3P_1 \rangle \sim v^4$ and ratio $f_V f_P/m^2 \sim (\Lambda/m)^2$. Hence from Eq. (16) one obtains

$$A_\parallel^J \sim v^4 \left(\frac{\Lambda}{m}\right)^2. \hfill (18)$$

From Eq. (17) one can see that the hard kernel is antisymmetric with respect to interchange $\{x, y\} \rightarrow \{\bar{x}, \bar{y}\}$ and therefore the collinear integral is proportional to antisymmetric combinations $\phi_{2V}(x) - \phi_{2V}(\bar{x})$ or $\phi_{2P}(y) - \phi_{2P}(\bar{y})$ in Eq. (17). Such combination does not vanish for $K$-meson DAs due to $SU(3)$ breaking effects. Using the models for the distribution amplitudes as in Eqs. (106) and (114) we obtain

$$J_\parallel^J = \frac{27}{2} \left(\pi^2 - 4\right) \left(b_1(\mu) - a_{1V}(\mu)\right) + \frac{27}{2} (6\pi^2 - 44) \left(b_1(\mu) a_{2V}(\mu) - a_{1V}(\mu) b_2(\mu)\right), \hfill (19)$$

where $b_{1P}$ and $a_{1V}$ are the DA parameters defined in Appendix A. The moments $a_{1V}$ and $b_1$ vanishes in the exact $SU(3)$ limit and this illustrate the sensitivity of the integral $J_\parallel^J$ to the flavour symmetry violation.

Let us estimate the branching fraction of $\chi_{c1}$ state assuming that the transverse amplitude $A_\perp^I$ is small. In order to obtain numerical estimate we set $m_c = 1.5$ GeV, $\Lambda_{QCD}^{(4)} = 310$ MeV that gives $\alpha_s(2m_c) = 0.29$, for the total width we take the experimental value $\Gamma[\chi_{c1}] = 0.84$ MeV, the numerical values of other parameters are given in Appendix A. Varying the factorisation scale $\mu^2$ between $m_c^2$ and $4m_c^2$ we obtain

$$Br[\chi_{c1} \to K^0K^*(892)^0 + c.c.] = (0.02 - 0.06) \times 10^{-3}. \hfill (20)$$

We see that this value is about two orders of magnitude smaller then the experimental branching fraction, see Table 1. This result allows one to conclude that the dominant numerical contribution is most probably provided by the amplitude $A_\perp^I$. This conclusion is also agree with sufficiently large value of the branching ratio for the $\chi_{c2}$ decay.

### 3.2 Colour-singlet contributions to amplitudes $A_{1,2}^\perp$

The calculation of colour-singlet contributions to the amplitudes $A_{1,2}^\perp$ is more complicated because one has to take into account two different configurations: twist-2 projection onto $K$ state and twist-3 one.
onto $K^*$ meson ($P_2V_3$ configuration) and vice versa. In general, the twist-3 contributions include two-particle and three-particle operators. The matrix elements of three-particle quark-gluon operators are also known as genuine twist-3 contributions. Using QCD equation of motions the matrix elements of two-particle twist-3 operators can be described in terms of twist-2 and twist-3 quark-gluon DAs. In this work we neglect contributions of the three-particle quark-gluon operators for simplicity. Hence we need to consider only the matrix elements of two-particle twist-3 operators neglecting the quark-gluon DAs. Such approximation is also known as Wandzura-Wilczek (WW) approximation. In this case one has to compute the same diagrams as in Fig.1 but with the appropriate twist-3 projections for the collinear matrix elements. In order to make our notations simpler in what follow we do not introduce any specific notation for twist-3 DAs in the WW approximation.

The calculation is quite standard and we skip here the technical details. The following results has been obtained (remind $V \equiv K^*$, $P \equiv \bar{K}$)

$$A_{\pm}^{(0)} = \frac{i \langle 0 | \mathcal{O}^{(3)P_3} | \rangle}{m^3} \left( \frac{\pi \alpha_s}{N_c} \right)^2 C_F 2^{J/2} \left\{ \int \frac{d^4p}{2\tau} \mathcal{J}^{(j)}[P_2V_3] + \int \frac{d^4p}{2\tau} \mathcal{J}^{(j)}[P_2V_2] \right\},$$

(21)

where we introduced the subscript “c” in order to stress the collinear operator structure for the final mesonic state. The collinear convolution integrals $J^{(j)}[P_2V_3]$ read

$$J^{(j)}[P_2V_3] = \frac{1}{8} \int_0^1 dx \frac{\tilde{\sigma}_2^\perp(x)}{xy} \int_0^1 \frac{dx}{x}, \quad C_{\pm}^{(1)}(x,y) = \frac{2\tilde{x} - 1}{xy + \tilde{x}y^2}, \quad C_{\pm}^{(2)}(x,y) = \frac{x(\tilde{x} + y)}{(xy + \tilde{x}y)^2}.$$  

(22)

The hard kernels in Eq. (22) read

$$\tilde{\sigma}_2^\perp(x) = \frac{2\tilde{x}}{(xy + \tilde{x}y^2)} - \frac{\tilde{x}(x + \tilde{y})}{(xy + \tilde{x}y)^2}, \quad \tilde{\sigma}_2^\perp(x) = \frac{2\tilde{x} - 1}{xy + \tilde{x}y^2} + \frac{\tilde{x}(y - \tilde{x})}{(xy + \tilde{x}y)^2},$$

(23)

The function $\Omega(u)$ is defined in Appendix A, see Eq. (117).

For the second convolution integral we obtained

$$J^{(j)}[P_2V_2] = \frac{(-1)^{J+1}}{48} \int_0^1 dx \frac{\tilde{\sigma}_2^\perp(x)}{xy} \int_0^1 dy \frac{\tilde{\sigma}_2^\perp(y)}{xy} \frac{C^{(j)}_p(x,y)\Delta \phi_{2P}^\sigma(y) + C^{(j)}_{\sigma}(x,y)\Delta \phi_{3P}^\sigma(y) + C^{(j)}_p(x,y)\Delta \phi_{3P}^\sigma(y) + C^{(j)}_{\sigma}(x,y)\Delta \phi_{3P}^\sigma(y)}{yy(xy + \tilde{y}y^2)}$$

(26)

where

$$C^{(j=1)}_p = 6(\tilde{x} - y + 2(y - \tilde{y})(x + \tilde{y})), \quad C^{(j=1)}_{\sigma} = 4(\tilde{x} - y),$$

(27)

$$C^{(j=2)}_p = 6(y - \tilde{x}), \quad C^{(j=2)}_{\sigma} = 4(y - \tilde{x}).$$

(28)

The explicit expressions for DAs $\phi_{2P}^\sigma$ and $\phi_{3P}^\sigma$ are given in Eqs. (107), (108) and (114), the prime denotes derivative with respect to collinear fraction: $\phi'(x) \equiv d/dx\phi(x)$. From Eq. (21) one can easily conclude that the transverse amplitudes behave as

$$A_{\pm}^{(0)} \sim \nu^4 \left( \frac{\Lambda}{m} \right)^3,$$

(30)
and these contributions are suppressed compared to $A \parallel$. On the other side, amplitudes $A \perp [P \nu V]$ include the so-called chiral enhanced coefficient $\mu_p$, see Eq. (105), which is numerically large. Taking into account the real value of the $c$-quark mass one finds that $\mu_p/m \sim 1$ and such corrections can provide a large effect.

The convolution integrals $J_I^{(J)}$ have logarithmic IR-divergencies associated with regions $y \to 0$, $x \to 1$ and $y \to 1$, $x \to 0$. These are the so-called endpoint divergencies which indicate about logarithmic overlap with the ultrasoft domain. In order to single out these divergencies one needs to perform an expansion of the integrands in the corresponding regions.

Consider, for instance, the first integral in Eq. (26) which has the following integrand

$$F(x, y) = \frac{\bar{\phi}_{2V}(x) C_\sigma(x, y) \Delta \phi_{3P}^\sigma(y) + \ldots}{x y (x + y)^2},$$

where dots denote the other terms in the numerator of Eq. (26). Using the DA models from Appendix A one easily finds

$$\lim_{x \to 1} \bar{\phi}_{2V}(x) = -\bar{x} \bar{\phi}_{2V}(1), \quad \lim_{x \to 0} \bar{\phi}_{2V}(x) = x \bar{\phi}_{2V}(0),$$

$$\lim_{y \to 0} \Delta \phi_{3P}^\sigma(y) = \rho c^{\Delta} \frac{3}{2} (\alpha + \beta \ln y) \equiv \Delta \phi_{3P}^\sigma(y \sim 0), \quad \alpha = 1 + 2b_2, \quad \beta = 1 + 6b_2,$$

$$\lim_{y \to 1} \Delta \phi_{3P}^\sigma(y) = 6 \Delta \phi_{3P}^\sigma(y \sim 0), \quad \lim_{y \to 1} \Delta \phi_{3P}^\sigma(y) = -6 \Delta \phi_{3P}^\sigma(y \sim 1),$$

This gives

$$F(x, y) \big|_{y \to x \sim 0} = (-1)^{J+1} 4 \bar{\phi}_{2V} \frac{6 \Delta \phi_{3P}^\sigma(y \sim 0)}{(y + x)^2} \equiv F(x \sim 1, y \sim 0),$$

$$F(x, y) \big|_{x \to y \sim 0} = (-1)^{J+1} 4 \bar{\phi}_{2V} \frac{6 \Delta \phi_{3P}^\sigma(y \sim 1)}{(x + y)^2} \equiv F(x \sim 0, y \sim 1).$$

The corresponding convolution integral in Eq. (26) can be written as a sum of the regular and singular terms

$$I_{\text{reg}} = \int_0^1 dx \int_0^1 dy \{ F(x, y) - F(x \sim 1, y \sim 0) - F(x \sim 0, y \sim 1) \},$$

$$I_{\text{sing}} = \int_0^1 dx \int_0^1 dy \{ F(x \sim 1, y \sim 0) + F(x \sim 0, y \sim 1) \}$$

One can easily see that the singular integrals are IR-divergent. In order to regularise them we apply analytic regularisation modifying the heavy quark propagator. We imply that the regularisation is introduced after differentiation with respect to momentum $\Delta \tau$ (which is required by projection on quarkonium $P$-wave state)

$$\frac{1}{[m^2(xy + xy)]^{\nu \epsilon}} \to \frac{\nu^{2\epsilon}}{[m^2(xy + xy)]^{\nu \epsilon}},$$

where $\nu$ is the renormalisation scale. With such regulator one obtains

$$I_{\text{sing}} = (-1)^{J+1} 4 \bar{\phi}_{2V} \frac{ \nu^{2\epsilon} }{[m^2(xy + xy)]^{\nu \epsilon}} \int_0^1 dx \int_0^1 dy \left\{ \frac{6 \Delta \phi_{3P}^\sigma(y \sim 0)}{(y + x)^{2+\epsilon}} - \frac{6 \Delta \phi_{3P}^\sigma(y \sim 1)}{(y + x)^{2+\epsilon}} \right\},$$

$$I_{\text{sing}} = (-1)^{J+1} 4 \bar{\phi}_{2V} \frac{ \nu^{2\epsilon} }{[m^2(xy + xy)]^{\nu \epsilon}} \int_0^1 dx \int_0^1 dy \left\{ \frac{6 \Delta \phi_{3P}^\sigma(y \sim 0)}{(y + x)^{2+\epsilon}} - \frac{6 \Delta \phi_{3P}^\sigma(y \sim 1)}{(y + x)^{2+\epsilon}} \right\},$$
where $t \equiv \nu^2/m^2$ and we used that $\bar{\phi}^{J}_{2V}(1) = -\bar{\phi}^{J}_{2V}(0)$. Simple but lengthy calculation yields

$$I_{\text{sing}} = (-1)^{J+1} \bar{\phi}^{J}_{2V}(1) 72 \rho^K \times \left( \frac{\beta}{\varepsilon^2} - \beta \frac{\ln t}{\varepsilon} - (\alpha - \beta) \frac{1}{\varepsilon} - \beta \frac{1}{2} \ln^2 t - (\alpha - \beta) \ln t + \beta \frac{\pi^2}{12} - (\alpha - \beta) \alpha \ln 2 \right).$$

(42)

The double IR-pole is generated due to presence of the logarithms $\ln y$ and $\ln \bar{y}$ in Eqs. (33) and (35). Hence for the first integral from Eq. (26) we obtain

$$J_{c1}^{(J)}[P_{2V}] = -\frac{1}{48} \int_0^1 dx \int_0^1 dy F(x, y)$$

$$= (-1)^{J} \bar{\phi}^{J}_{2V}(1) \frac{3}{2} \rho^K \left( \frac{\beta}{\varepsilon^2} - \beta \frac{\ln t}{\varepsilon} - (\alpha - \beta) \frac{1}{\varepsilon} - \beta \frac{1}{2} \ln^2 t - (\alpha - \beta) \ln t \right) + \ldots ,$$

(43)

where dots denote the remain finite terms. The same technique is also applied for other convolution integrals in Eqs. (22) and (26). These integrals have IR-divergencies which produce double and single poles in $\varepsilon$.

A study of the structure of the divergent integrals helps to identify an operator which has to be added in order to absorb appearing IR-divergencies. The momenta in the diagrams in Fig. 1 are chosen in such way that the intermediate gluons have momenta $y k + \bar{x} p$ and $\bar{y} k + x p$. Hence in the regions $y \sim x \rightarrow 0$ or $x \sim \bar{y} \rightarrow 0$ one of the gluons has very small virtuality while the second one carries hard momentum. It is natural to assume that the gluon with small momentum is ultrasoft, i.e. in the endpoint region we have $yk + \bar{x}p \sim mv^2$ or $\bar{y}k + xp \sim mv^2$ which is equivalent to $y \sim x \sim v^2$ or $x \sim \bar{y} \sim v^2$. The emission of ultrasoft gluons does not change the virtuality of a soft heavy quark with momentum $p_Q \sim mv$. Therefore in the endpoint domain the momentum of the virtual heavy quark in diagrams in Fig. 1 is not hard anymore. Since this intermediate quark remains soft with $p_Q \sim (mv)$ the corresponding soft propagator generates in the hard kernels of Eqs. (36) and (37) the combinations $(y + \bar{x})^{-2}$ or $(x + \bar{y})^{-2}$, which lead to IR-divergencies in the convolution integrals. In the case the hard subprocess is described by the hard annihilation of the heavy quark-antiquark pair into the light quark-antiquark pair $cc \rightarrow g^* \rightarrow q + \bar{q}$ or $cc \rightarrow g^* \rightarrow s + \bar{s}$ with light-like momenta. Because the annihilation proceeds through the one hard gluon the corresponding heavy quark-antiquark pair is in the colour-octet state. Hence we can conclude that a colour-octet matrix element must be added into the consideration in order to explain IR-divergencies of the colour-singlet contribution. It is clear that such octet contribution must have the same scaling behaviour as the singlet one in Eq. (30).

The mixing of singlet and octet mechanisms in exclusive decays within the effective theory framework has already been studied in Ref. [31]. In present case the situation is similar but a bit more complicated technically because of double IR-poles, see Eq. (42). In the realistic world the colour-octet contribution is non-perturbative because of relatively small charm mass. However in the next section we consider corresponding matrix element in the Coulomb limit which allows one to perform calculations within the pNRQCD framework. Such a consideration helps to verify the matching of the IR- and UV-divergencies between the colour-singlet and colour-octet terms.

So far let us consider some qualitative arguments based on symmetry properties of the effective field theory in the limit $m \rightarrow \infty$ of heavy quark mass. It is well known that the HQSS provides approximate relations between matrix elements for the various states of a given radial and orbital excitation of heavy quarkonium. The violation of the heavy-quark spin symmetry come from higher order terms in effective Lagrangian which are suppressed by powers of $v$. The example of such relations for the wave functions are well known [4] and provide relations as in Eq. (100). Factorising the hard modes for the colour-octet mechanism one obtains matrix elements with more complicated operators but the effective heavy quark Lagrangian is the same. Hence interactions of ultrasoft gluons with the spin of heavy quarks are suppressed and this can provide an approximate relations between the octet matrix elements.

In case of the hard subprocess $cc \rightarrow g^* \rightarrow q + \bar{q}$ which is relevant for our case, the hard factorisation generate the four-quark operators like

$$C_h \bar{q} \gamma_\sigma t^a q \chi_\omega^\dagger \gamma_\tau t^a \psi_\omega,$$

(44)
where $C_h$ is the hard coefficient function, $t^a$ denotes the SU(3) color matrices, $q$ denotes the light quark field, $\psi_\omega$ and $\chi_\omega^\dagger$ denote quark and antiquark four-component spinors in the NRQCD, see more details in Appendix A. Then the colour-octet amplitude is schematically given by the matrix element

$$A_j^{(8)} = C_h \langle K^* \bar{K} | \bar{q} \gamma_\tau t^a q \chi_\omega^\dagger t^a \psi_\omega | J(\omega) \rangle. \quad (45)$$

According to NRQCD counting rules, the bilinear heavy quark operator is of order $v^3$. In order to get a contribution of order $v^4$ which can mix with the colour-singlet contribution in (30) one needs an interaction of order $v$. In pNRQCD Lagrangian such interaction is only described by chromoelectric dipole vertex $\sim \bar{\psi}_\omega(x) \vec{E}(t) \psi_\omega(x)$ which is not sensitive to the heavy quark spin. Therefore we conclude that HQSS can also relate the matrix elements (45) with the different $J = 1, 2$. Below we will obtain that in the weak coupling limit $|p_{uds}| \gg \Lambda$ this implies

$$A_{j=1}^{(8)} = -\frac{1}{\sqrt{2}} A_{j=2}^{(8)}. \quad (46)$$

up to higher order corrections in small velocity $v$. Next important step is the assumption that at given order the total result for the physical amplitude is only given by the sum of the singlet and octet amplitudes and therefore must be IR-finite

$$A_j^+ = A_j^{(0)} + A_j^{(8)}. \quad (47)$$

The compensation of the singularities in Eq. (47) imply that divergent integrals in the colour-singlet amplitudes must satisfy to the same relation (46) as the colour-octet amplitudes. Then the hard contributions which violate spin-symmetry must be well defined and therefore can be computed unambiguously. A similar situation takes plays in B-decays \[29, 30\]. Using Eqs.(46) and (47) in order to exclude colour-octet amplitude $A_j^{(8)}$ one obtains

$$A_1^+ + \frac{1}{\sqrt{2}} A_2^+ = A_1^{(0)} + \frac{1}{\sqrt{2}} A_2^{(0)}. \quad (48)$$

The combination of the colour-octet amplitudes in this equation must be well defined since the lhs is free form any divergencies. This point can be easily verified using results in Eqs.(21). Performing the required analytical calculations we indeed obtain that the combination $A_1^{(0)} + \frac{1}{\sqrt{2}} A_2^{(0)}$ is free from the endpoint divergencies. This observation supports the factorisation formula suggested in Eq.(47). Notice that this compensation works independently for the two different collinear operators $P_2 V_3$ and $P_3 V_2$. The relation (48) is one of the results of this work and will be used in our phenomenological analysis in Sec.5.

Finishing this section we provide the analytical result for the collinear integrals which define the symmetry breaking contribution

$$A_{1c}^{(0)} + \frac{1}{\sqrt{2}} A_{2c}^{(0)} \equiv \Delta A_{c}^{(0)}$$

$$= \frac{i \langle O^3 P_j \rangle f_P f_{\perp} m_V}{m^2} \left( \frac{\pi \alpha_s}{N_c} \right)^2 C_F \sqrt{2} \left( J_{c}^{(1)} [P_2 V_3] + J_{c}^{(2)} [P_2 V_3] \right)$$

$$+ \frac{i \langle O^3 P_j \rangle f_P f_{\perp} f^{\perp}}{m^2} \left( \frac{\pi \alpha_s}{N_c} \right)^2 C_F \sqrt{2} \left( J_{c}^{(1)} [P_3 V_2] + J_{c}^{(2)} [P_3 V_2] \right). \quad (49)$$
Using the models of DAs from Appendix A we obtain

\[ J_c^{(1)}[P_2V_3] + J_c^{(2)}[P_2V_3] = \frac{3}{2} b_1 \left\{ \frac{9}{4} (8 - \pi^2) + a_{2V} \frac{9}{16} (11\pi^2 - 108) \right\} \\
+ \frac{3}{2} b_1 a_1^+ \left[ \frac{9}{2} (8 - 3\zeta(3) + \pi^2(1 - 2 \ln 2)) + 9a_{2V} \left( 9(1 - \zeta(3)) + \pi^2 \left( \frac{17}{4} - 6 \ln 2 \right) \right) \right] \\
+ \frac{3}{2} b_1 a_1^+ \left[ 2(8 - \pi^2) + b_2 \frac{27}{16} (6\zeta(3) - 24 + \pi^2(4 \ln 2 - 1)) \right] \\
+ \frac{3}{2} b_1 \lambda_+ \left\{ \frac{3}{4} \left( 8 - \pi^2 \right) + b_2 \left( 9(11\pi^2 - 108) \right) \right\} \\
+ \frac{3}{2} b_1 \lambda_+ a_1^+ \left\{ \frac{3}{4} \left( 6\zeta(3) + 4\pi^2(\ln 2 - 1) \right) - \frac{9}{4} b_2 \left( 12\zeta(3) - 40 + \pi^2 (8\ln 2 - 3) \right) \right\} \\
- a_{2V}^{(1)} \left[ 18\zeta(3) - 40 + \pi^2 (12\ln 2 - 7) + \frac{3}{2} b_2 \left( 72\zeta(3) + 300 + \pi^2 (48\ln 2 - 73) \right) \right] \\
+ \frac{3}{2} \left( -\frac{9}{8} \right) \lambda^+ a_1^+ \left\{ 2 \left( 6 - \zeta(3) - 8 + \pi^2 (4\ln 2 - 3) \right) \right\} + b_2 \left[ 84 + \pi^2 (48\ln 2 - 51) + 72\zeta(3) \right] \right\}, \quad (50)\]

\[ J_c^{(3)}[P_3V_2] + J_c^{(2)}[P_3V_2] = -\frac{9}{8} \mu^K \left\{ 2\pi^2 + a_{2V} \zeta(3) (20 - \pi^2) + b_2 (20 - \pi^2 + a_{2V} (39\pi^2 - 360)) \right\}, \quad (51)\]

where we assume that all DA parameters depend on the factorisation scale \( \mu \). From these results one can explicitly see that in the exact \( SU(3) \) limit the amplitudes vanishes as it must be.

### 3.3 Soft-overlap colour-singlet contribution to amplitudes \( A_{1,2} \)

There is one more contribution which must be included into the description. This contribution appears due to long distance interactions between the outgoing partons and associated with the typical hadronic scale \( \Lambda \). In this case heavy quark and antiquark annihilate at short distances into the light quark-antiquark due to long distance interactions between the outgoing partons and associated with the typical hadronic scale \( \Lambda \). There is one more contribution which must be included into the description. This contribution appears close to the total momenta of the outgoing mesons. The hard-collinear particles interact with the soft and collinear particles in order to produce the final hadronic states. Corresponding subprocess depends on the hard-collinear and soft virtualities of order \( m\Lambda \) and \( \Lambda^2 \), respectively. Such contribution can be included into consideration as a matrix element within the soft collinear effective theory (SCET) framework.

Corresponding diagrams are schematically shown in Fig.2(a), the dashed lines denote the hard-collinear particles which are attached to the blob which denotes the SCET matrix element. We assume that the hard-collinear scale \( m\Lambda \) is not large and consider the SCET matrix elements as non-perturbative objects. We expect that in the limit \( m \to \infty \) these matrix elements are of order \( (\Lambda/m)^3 \), and therefore the corresponding soft-overlap amplitude is of the same order as the hard one, see Eq.(30). This can be directly verified in SCET-II [28] by construction of relevant \( T \)-products or by direct computation of the higher-order diagrams as shown in Fig.2(b). Computing such diagrams one obtains specific collinear endpoint singularities which are related to the SCET matrix elements. Similar calculations are known for the space-like transition form factor in the process \( \gamma^* \rho \to \pi \), see e.g. Refs. [14,32]. The detailed analysis of this point is complicated and for simplicity, we accept the suggested power behaviour of the soft-overlap contribution as reliable assumption.

The soft-overlap matrix elements describe a configuration when one of the partons carries almost total hadronic momentum while the others are soft. Such situation can be interpreted as a soft-overlap of the final hadronic states. For space-like form-factors such scattering picture is also known as a Feynman mechanism [33] and corresponding effect was studied with the help of the light-front wave functions [34].

The hard coefficient functions \( C_{gg} \) are given by the sum of the one-loop diagrams like one shown in Fig.2(a). The resulting expression for the colour-singlet soft-overlap amplitudes can be written as

\[ \mathcal{M}_{x_e \to \bar{K}K^*} \mid _{\text{Fig.2}} \simeq \frac{i \langle O^{(3)P_3} \rangle}{m^3} C_{gg}^{(J)} \left\{ \delta J_1 \epsilon_{\rho} \rho \epsilon_{\chi}^{\perp} + \delta J_2 (\epsilon_{\chi})_{\rho \sigma} \delta \right\} \langle \bar{K}K^* \mid O_{\text{SCET}} \mid 0 \rangle, \quad (52)\]
where $\delta_{ji}$ is Kronecker symbol, the matrix element of the two-particle SCET operator $O_{SCET}$ is defined as

$$\langle K(k)K^*(p, e^*)| \bar{s}_n(0)W_n\gamma^\alpha W_\alpha^\dagger \bar{s}_n(0) - \bar{q}_n(0)W_n\gamma^\alpha W_\alpha^\dagger \bar{q}_n(0) |0\rangle = i\varepsilon^{\alpha}_{\alpha\beta}(e^*_\alpha)^2 m (f^g_{PV} - f^g_{PV}).$$

Here the quark fields $\psi_n, \bar{\psi}_n$ ($\psi = s, q$) and $W_{n, \alpha}$ denote the hard-collinear SCET fields and corresponding hard-collinear Wilson lines

$$\frac{d\psi_n(x) = 0, W_n = P \exp \left\{ ig \int_{-\infty}^{0} ds \, \bar{n} \cdot A(s\bar{n}) \right\}.}$$

and similar for the light-cone sector associated with $\bar{n}$.

The form factors $f^g_{PV}$ describe transition of the hard-collinear quark-antiquark pair to the final hadronic state within the SCET framework. The relative sign minus in Eq. (53) can be understood as an consequence of $C$-parity of the initial state. We see that in the exact $SU(3)$ limit such matrix element vanishes as it must be. The definition (53) is process independent, the similar matrix element may also appear in other hard reactions, for instance, in the wide angle scattering $\gamma\gamma \rightarrow KK^*$ at large energy and momentum transfer. The different combination of these form factors can also appear in the process $e^+e^- \rightarrow \gamma^* \rightarrow KK^*$. Using Eqs. (52) and (53) one can easily find corresponding contributions to the colour-singlet amplitudes $A^{(0)}_f$.

The computation of the hard coefficients $C^{(J)}_{gg}$ is straightforward: one has to compute the box diagrams as in Fig. 2(a) in the appropriate kinematics. The required one-loop integrals are similar to the integrals studied for $\chi_{f} \rightarrow e^+e^-$ decays. We borrow the results from Refs. (35) adding the colour factor and QCD couplings. These integrals have IR-divergencies which are regularised by dimensional regularisation $D = 4 - 2\varepsilon$. Using $\overline{MS}$-scheme one obtains

$$C^{(J=1)}_{gg} = \alpha_s^2 C_F \frac{1}{N_c} \left( \frac{1}{\varepsilon} - \ln \frac{\mu^2}{m^2} - 2 \ln 2 \right), \quad C^{(J=2)}_{gg} = \alpha_s^2 C_F \left\{ \frac{1}{\varepsilon} + \ln \frac{\mu^2}{m^2} + \frac{2}{3} \left( \ln 2 - 1 + i\pi \right) \right\},$$

where $\mu$ is the factorisation scale. The IR-singularities in the hard loop corresponds to the integration domain where one of the gluons becomes ultrasoft. Corresponding IR-poles can be again absorbed into the colour-octet matrix element which will be discussed this in the next section. Therefore the total result for the soft-overlap amplitude is also given by the sum of the colour-singlet and colour-octet matrix elements.

The soft-overlap colour-octet contribution also satisfy Eq. (46) as a part of the total colour-octet amplitude. Therefore one can also apply Eq. (48) in order to define the corresponding HQSS breaking terms. Using Eq. (53) we obtain

$$A^{(0)}_f + \frac{1}{\sqrt{2}} A^{(0)}_2 \equiv \Delta A^{(0)}_2 = \frac{i}{\alpha_s^2 C_F \frac{1}{N_c}} \left( f^g_{PV} - f^g_{PV} \right) \frac{\sqrt{2}}{3} \left( 1 - 2 \ln 2 + i\pi \right).$$

Figure 2: a) An example of the one-loop diagram and schematic factorisation for the soft-overlap colour-singlet contribution. The black square denotes the operator vertex and dashed fermion lines with the blob denote the SCET matrix element defined in Eq. (53). The crossed vertex with the blob describe the NRQCD matrix element. b) The higher order perturbative diagram which have IR-singularities associated with the soft-overlap configuration.
Figure 3: The hard factorisation associated with the annihilation $c\bar{c} \rightarrow g^* \rightarrow q(s) + \bar{q}(\bar{s})$. The crossed circle denotes the vertex of the four-fermion operator and the attached double line describes the heavy quark-antiquark pair.

We again obtain that IR-poles and $\mu$ dependence cancel as it is expected. Notice that this contribution has imaginary part which is related to the two-gluon intermediate cut in the loop diagram. This contribution is also of order $\alpha_s^2$ and have the same power counting behaviour as $A_\perp^{(0)}$ in Eq.(49). If the value of the SCET matrix elements $\sim (f_{PV} - f_{q\bar{q}})$ is sufficiently large then this contribution can not be neglected.

4 Colour-octet contributions in the Coulomb limit

For realistic charmonium the colour-octet matrix elements can be computed only within a non-perturbative framework. In the Coulomb limit the ultrasoft scale is a larger than the typical hadronic scale $mv^2 \gg \Lambda$. In this case quarkonium state can be considered as a weakly bound state with the binding energy $E \sim mv^2$. Therefore QCD perturbation theory can be applied for calculations associated with the ultrasoft scale. The standard framework includes: factorisation of hard modes and transition to NRQCD, to integration over soft and potential gluons and transition to pNRQCD which only contains potential heavy quarks and ultrasoft gluons as degrees of freedom. Such picture can not provide reliable estimates for charmonia but it still allows one to study different theoretical questions. Our aim is to show that the colour-octet matrix elements satisfy to Eq.(46) in the Coulomb limit and to study IR- and UV-singularities in the sum of singlet and octet contributions.

The first step is the factorisation of the hard modes, it is described by the tree level diagrams associated with the subprocess $c\bar{c} \rightarrow g^* \rightarrow q + \bar{q}$ or $c\bar{c} \rightarrow g^* \rightarrow s + \bar{s}$ which is shown graphically in Fig. 3. Two diagrams correspond to the two different regions $y \sim \bar{x} \sim v^2$ or $x \sim \bar{y} \sim v^2$ in the collinear integrals. Hence for the octet amplitudes we get

$$i\mathcal{M}_{\chi_{ij} \rightarrow K^* \bar{K}}^{(8)} = C_g \left\{ \bar{s}_n(0)W_n \gamma^\alpha t^a W_n^\dagger \bar{q}_n(0) + \bar{q}_n(0)W_n \gamma^\alpha t^a W_n^\dagger \bar{s}_n(0) \right\} \chi^\dagger \gamma^\alpha t^a \psi |\chi_{cJ}\rangle,$$

with the hard coefficient function

$$C_g = \frac{i\alpha_s(\mu)\pi}{m^2}. \quad (59)$$

In Eq.(58) we also use notation for the collinear fields and Wilson lines as in Eq.(54).

The next step is transition to pNRQCD. To our accuracy the matching of the vector NRQCD operator in (58) onto pNRQCD operator is trivial

$$\chi^\dagger \gamma^\alpha t^a \psi |NRQCD = \chi^\dagger \gamma^\alpha t^a \psi |pNRQCD \quad (60)$$

Therefore we can easily pass to the calculation of the matrix element (58) in pNRQCD.

To leading-order accuracy in $\alpha_s$ one can consider two different sets of Feynman diagrams: tree level and one-loop graphs which are shown in Fig. 4 and Fig. 5.

The tree diagrams has the following structure. The initial $P$-wave bound state decays through the chromoelectric dipole interaction into ultrasoft gluon and bounded colour-octet quark-antiquark pair. The insertion of this interaction vertex costs one power of the small velocity $v$ therefore the total octet contribution is of order $v^4$ (remind, the $S$-wave vector operator in [55] is of order $v^3$). The colour-octet quark-antiquark pair propagates a distance $\sim 1/(mv^2)$ and annihilates into light quark-antiquark pair with momenta of order $k$ and $p$. The colour-octet propagator is described by the non-relativistic Coulomb Green function $G_C$. The virtual ultrasoft gluon creates ultrasoft quark-antiquark pair which have to be combined with the collinear quark and antiquark in order to create a collinear operator describing the long
distance overlap with outgoing mesonic states. Since $p_{us}^2 \gg \Lambda^2$ the ultrasoft particles still can be matched onto collinear degrees of freedom. However corresponding collinear fractions are small, of order $v^2$ as it follows from the momentum conservation. Therefore corresponding collinear matrix elements describe the asymmetric collinear configurations where one parton carries the small collinear fraction $x \sim v^2$. This is exactly the endpoint configuration which provides the IR-singularities in the colour-singlet matrix element. The resulting expression for such diagram must be expanded with respect to small collinear fractions keeping only those terms which provide contribution of order $v^4$. The long distance dynamics associated with the hadronic scale $\Lambda$ is still described by the DAs given by the collinear matrix elements. Such situation is a consequence of existence of two well separated scales $m v^2 \gg \Lambda$ in the Coulomb limit.

The one-loop diagrams in Fig. 5 describe the colour-octet contribution associated with the soft-overlap amplitude. In this case the ultrasoft gluon interacts with the collinear quark or antiquark creating the colourless quark-antiquark operator. Further interactions of the hard-collinear particles is only associated with the typical hadronic scale $\Lambda$ and can be described as matrix elements of the SCET operator which is shown by blob in Fig.5. As in Sec. 3 we consider these matrix elements as non-perturbative quantities. Obviously, such contribution is also of order $v^4$.

Technically the calculation of the pNRQCD diagrams in Fig.4 is similar to calculation in Ref. [31] and useful technical details can be found in this work. The analytical expression for the colour-octet amplitude can be written as

$$M_{\chi_{cJ} \to K^* \bar{K}}^{(8)} = \left. \left. \frac{2\pi^2}{m^2} \alpha_s(\mu_{us}) \alpha_s(\mu) \frac{C_F}{N_c} \right|_{\text{Fig.4}} \int_0^1 dx \int_0^1 dy \left( \theta(x < \eta) \theta(y < \eta) D_s^{\alpha\beta}(x, y) + \theta(x < \eta) \theta(y < \eta) D_q^{\alpha\beta}(x, y) \right) \right\vert_{\text{Fig.4}}$$

The second line of Eq. (61) describes the subdiagram with light quarks and ultrasoft gluon propagators (written explicitly in the denominators). The $\theta$ functions restrict the integrations regions over quark fractions and the cut-off $\eta$ must be understood as UV-regulator. The functions $D_s$ and $D_q$ consist of contributions of the light-quark vertices and DAs. A calculation of these contributions is the same as for the colour-singlet case but in addition one has to expand of the collinear integrands with respect to small fractions in the regions $y \sim \bar{x} \sim v^2$ or $x \sim \bar{y} \sim v^2$ as it is indicated by the appropriate $\theta$-functions.
The third line of Eq. (61) describes the heavy quark subdiagram. The relative heavy quark momentum $\Delta_T$ is of order $mv$, in what follow we assume that $\Delta \equiv |\Delta|$. The momentum space radial wave function of $P$-wave state reads

$$
\tilde{R}_{21}(\Delta) = iR'_{21}(0) = \frac{16\pi\gamma_B\Delta}{(\Delta^2 + \gamma_B^2/4)^2}, \quad \gamma_B = \frac{1}{2}m_c\alpha_sC_F.
$$

(62)

where $R'_{21}(0)$ is the derivative of the position radial wave function at the origin. The trace over Dirac indices includes the projectors on $P$-wave state

$$
\Delta_1 = \frac{1}{2\sqrt{2}} \frac{\Delta^\mu}{\Delta} [\gamma_\rho, \gamma_\lambda] \gamma_5, \quad \Delta_2 = -\epsilon_\mu^\nu \Delta_\tau^\mu \gamma_\tau^\nu.
$$

(63)

The factor $\gamma_{\perp o}$ is the Dirac part of the vertex of the octet $S$-wave operator in Eq. (68). Simple calculation yields

$$
\frac{1}{4} \text{tr} [\Lambda J(1 - \bar{\psi}) \gamma_{\perp o}(1 + \psi)] = \frac{\Delta_T \cdot (n - \bar{n})}{2} \frac{1}{\Delta} \left( \delta_{J1}\sqrt{2}i\epsilon^\mu_{\rho\alpha}e^\rho_{\chi} + \delta_{J2}2(\epsilon_{\chi\alpha\beta}\bar{n}^\beta) \right).
$$

(64)

The expression in the brackets in Eq. (64) gives the dependence on the total momentum $P$ of $\Delta$. Substituting (64) and (68) into Eq. (61), and using rotation invariance in order to reduce $\Delta$ and by octet Coulomb Green function $G$ and 2nd term in Eq. (67), while the remnant higher order contributions are suppressed by the factor $1/(2N_c)$ for each colour-octet exchange. For our purpose it is also enough to consider a simplified approximation which is given by no-gluon exchange leading term in Eq. (67). This gives

$$
D_{\alpha}^\beta(E,\Delta_T) = (\omega^\beta k_\alpha - g_\alpha^\beta(\omega k_\alpha)) \frac{\partial}{\partial \Delta_T^\alpha} \int \frac{d^3\Delta'}{(2\pi)^3} G_C(\Delta_T, \Delta_T'; E - (\omega k_\alpha))
$$

(65)

and by octet Coulomb Green function $G_C$

$$
G_{(8)}(\Delta_T, \Delta_T'; E) = \frac{-2m^2(\Delta - \Delta')}{E + \Delta_T^2/m} + \frac{g^2}{2N_C} \frac{1}{E + \Delta_T^2/m} \frac{1}{(\Delta - \Delta')^2} \frac{1}{E + \Delta_T^2/m} + O(g^4).
$$

(66)

The full expression can be found in Refs. [45, 46]. In the various calculations, see e.g. Refs. [31, 47–49], it has been observed that the dominant numerical impact is provided by the relatively simple first term in Eq. (67), while the remnant higher order contributions are suppressed by the factor $1/(2N_c)$ for each colour-octet exchange. For our purpose it is also enough to consider a simplified approximation which is given by no-gluon exchange leading term in Eq. (67). This gives

$$
D_{\alpha}^\beta(E,\Delta_T) = (\omega^\beta k_\alpha - g_\alpha^\beta(\omega k_\alpha)) \frac{2\Delta_T^\alpha}{m} \frac{1}{[E - (\omega k_\alpha) + \Delta_T^2/m + i\varepsilon]^2},
$$

Substituting (64) and (68) into Eq. (61), and using rotation invariance in order to reduce $\Delta_T^\alpha \Delta_T^\beta \rightarrow -\Delta_T^2 g_\alpha^\beta/3$ one obtains

$$
\mathcal{M}_{\chi_E \rightarrow K^+K^-}^{(8)} = \delta_{J1}\sqrt{2}i\epsilon^\rho_{\mu\alpha}e^\rho_{\chi} + \delta_{J2}2(\epsilon_{\chi\alpha\beta}\bar{n}^\beta) \frac{-2m^2}{m^2} \alpha_s(\mu_{as})\alpha_s(\mu)\frac{C_F}{N_c^2} \int \frac{dx}{4\pi m} \int \frac{dy}{4\pi m} \left( \frac{\theta(\bar{x} < \eta)\theta(y < \eta)}{(yk + xp)^2} D_{\alpha}^{\beta}(x, y) + \frac{\theta(x < \eta)\theta(\bar{y} < \eta)}{(yk + xp)^2} D_{\alpha}^{\beta}(x, y) \right)
$$

$$
\times \sqrt{N_c} \sqrt{M_X} \int \frac{d^3\Delta}{(2\pi)^3} \tilde{R}_{21}(\Delta) \frac{\Delta_T^\alpha}{3} \frac{\bar{n}^\beta(nk_\alpha - n^\beta(\bar{n}k_\alpha))}{[E - (\omega k_\alpha) + \Delta_T^2/m + i\varepsilon]^2}.
$$

(69)
The dependence on the total momentum $J$ is only given by expression in brackets in the first line. The trace structures in $D^{\alpha \beta}_{\kappa \rho}$ in Eq. (69) allows one to conclude that the resulting expression can be rewritten in the following form

$$
\mathcal{M}^{(8)}_{\chi, J \to K^* K} = \left\{ \delta J_1 \sqrt{2} i e m_\chi e_\rho + \delta J_2 (e_\chi)_{\alpha \beta} \bar{n}^\beta \right\} (-i) \epsilon_\rho^{\alpha \beta} (e_V)_\rho m J_{us}
$$

(70)

$$
= \left\{ \delta J_1 (e_\chi e_V^\rho) - \delta J_2 (e_\chi)_{\alpha \beta} \bar{n}^\beta \epsilon_\rho^{\alpha \beta} (e_V)_\rho \right\} m (-1)^J 2^{J/2} J_{us},
$$

(71)

where $J_{us}$ is the universal convolution integral. Comparing the last equation with the definitions of the scalar amplitudes in Eqs. (10) and (11) one obtains

$$
\mathcal{A}_{Jc}^{(8)} = (-1)^J 2^{J/2} J_{us}.
$$

(72)

This result leads to Eq. (46) which was proposed in the previous section. In order to obtain Eq. (72) we used approximate expression for the Coulomb Green function but the given derivation can be also extended to the case of exact colour-octet propagator.

For the ultrasoft integral one obtains

$$
J_{us} = \frac{i \langle O^{(3) P_3} \rangle}{m^3} \alpha_s(\mu) \alpha_s(\mu_{us}) \frac{\pi^2}{N_c^2} C_F \left( \frac{f_V f_P^\mu m^3}{m^3} J_{us}[P_3 V_2] + \frac{f_P f_V m^3}{m^3} J_{us}[P_3 V_3] \right),
$$

(73)

with

$$
J_{us}[P_3 V_2] = m^2 N \int \frac{d^3 \Delta}{(2\pi)^3} \tilde{R}_{21}(\Delta) \Delta \int_1^1 dx \int_0^n dy \frac{\bar{\phi}_{2V}(1) \Delta \phi_{3P}^2(y \sim 0) + \Delta \phi_{2V}^2(1) \phi_{3P}^2(y \sim 0)}{[E - m(y + \bar{x}) + \Delta_2^2/m + i\epsilon]^2},
$$

(74)

$$
J_{us}[P_3 V_3] = \frac{m^2}{2} N \int \frac{d^3 \Delta}{(2\pi)^3} \tilde{R}_{21}(\Delta) \Delta \int_1^1 dx \int_0^n dy \frac{\phi_{2P}^2(0) I[\Delta \Omega] - \Delta \Omega(1) \ln \bar{x} + \Delta \phi_{2P}^2(0) I[\Omega]}{[E - m(y + \bar{x}) + \Delta_2^2/m + i\epsilon]^2},
$$

(75)

where the normalisation factor $N$ is defined as

$$
N = \frac{1}{3 i R_{21}(0)}, \quad N \int \frac{d^3 \Delta}{(2\pi)^3} \tilde{R}_{21}(\Delta) \Delta = 1.
$$

(76)

In Eq. (75) we also defined short notation

$$
I[f] = \int_0^1 du \frac{f(u) - f(1)}{1 - u}.
$$

(77)

Remind that functions $\tilde{f}$ and $\Delta f$ denote symmetric and antisymmetric components, see Eq. (23).

These ultrasoft integrals are divergent and regularised by the UV cut-off $\eta \to \infty$. In order to compute these integrals we must use the same regularisation as for the colour-singlet case. Therefore we introduce the analytical regularisation substituting

$$
\frac{1}{[E - m(y + \bar{x}) + \Delta_2^2/m + i\epsilon]^2} \to \frac{\nu^{2\epsilon}}{[E - m(y + \bar{x}) + \Delta_2^2/m + i\epsilon]^{2+2\epsilon}},
$$

(78)

and take the limit $\eta \to \infty$. Remind, that in Eq. (78) we assume $y \sim \bar{x} \sim \nu^2$ and therefore all terms in the denominator are of order $m \nu^2$.

Consider as example the following term from Eq. (74)

$$
J_{us 1}[P_3 V_3] = m^2 N \int \frac{d^3 \Delta}{(2\pi)^3} \tilde{R}_{21}(\Delta) \Delta \int_1^1 dx \int_0^n dy \frac{\nu^{2\epsilon} \bar{\phi}_{2V}(1) \Delta \phi_{3P}^2(y \sim 0)}{[E - m(y + \bar{x}) + \Delta_2^2/m + i\epsilon]^{2+2\epsilon}}.
$$

(79)
Performing expansion of the integrand in the UV-region $y \approx \bar{x} \rightarrow \infty$ one can see the overlapping with the singular collinear integral in Eq.\((30)\). The regularised integral in Eq.\((79)\) can be easily computed separating the divergent part in the following way

$$J_{us1}[P_3V_2] = m^2 N \int \frac{d^3 \vec{\Delta}}{(2\pi)^3} \hat{R}_{21}(\Delta) \Delta \int_0^\infty dx \int_0^\infty dy \left(\phi_{2V}^\perp(1) \Delta \phi_{1P}^\parallel(y \sim 0)\right) \times \frac{1}{|E - m(y + x + \Delta^2 / m + i\varepsilon)|^2 - \frac{1}{|E - m(y + x + i\varepsilon)|^2}}$$

$$+ m^2 \int_0^\infty dx \int_0^\infty dy \frac{\nu^2 \phi_{2V}^\perp(1) \Delta \phi_{1P}^\parallel(y \sim 0)}{|E - m(y + x) + i\varepsilon|^{2+\varepsilon}}, \quad (80)$$

where we used relation \((76)\). The first integral in \((80)\) is finite and therefore the regularisation is not needed. The integrals over the collinear fractions can be easily computed and one finds \((t = \nu^2 / m^2)\)

$$J_{us1}[P_3V_2] = \frac{-1}{\nu^2} \phi_{2V}^\perp(1) \Delta \phi_{1P}^\parallel(0) \begin{cases} \frac{\beta}{\varepsilon} + \frac{\beta}{\bar{\varepsilon}} (\beta \ln t + \alpha - \beta) + \frac{\beta}{2} \ln^2 t + (\beta - \alpha)(1 - \ln t) - \beta \pi^2 / 6 \\ \frac{\beta}{\varepsilon} \ln^2 \left[\Delta^2 / m^2 - E / m - i0\right] + \alpha \ln \left[\Delta^2 / m^2 - E / m - i0\right] \end{cases} \right\}, \quad (81)$$

with $\alpha$ and $\beta$ defined in Eq.\((33)\). Comparing this result with the expression in Eq.\((43)\) one can see that the poles in logarithms $\ln t$ cancel in the sum $J_{c1}^{(J)}[P_2V_3] + (-1)^J J_{us1}[P_2V_3]$. The computation of the other integrals in Eqs.\((74)\) and \((75)\) is the same, we have checked that the poles and $\mu$-dependence \((\ln t)\) also cancel in the sum of singlet and octet amplitudes. This also demonstrates that our matching is consistent and the endpoint IR singularities of the colour-singlet amplitude can be absorbed into colour-octet contribution as expected. Therefore to a given accuracy the sum of the singlet and octet amplitudes describes the physical amplitude consistently.

The octet integrals in Eqs.\((74)\) and \((75)\) are proportional to derivatives of the DAs and includes the integrals over radial wave functions as in Eq.\((51)\). These integrals generate imaginary part which appears from the region where $\Delta^2 / m < E$ and can be associated with the cut as shown in Fig.\(4\). This imaginary part is the direct consequence of the intermediate colour-octet state. The expression \((51)\) also demonstrates that octet contribution is sensitive to the shape of the wave function while the singlet amplitude depends only from the wave function at the origin. This is qualitative difference between the two terms and it is interesting to understand, at least qualitatively, how this point can affect a description of quarkonium decays.

Consider now the diagrams in Fig.\(5\). Their computation can be done within the same technique as described above. The coupling of the ultrasoft gluon to the hard-collinear quarks is described by the leading-order SCET interactions $\bar{\psi}_n(x) g(n \cdot A_{us}(x\bar{n}n/2)) / 2 \bar{\psi}_n(x)$ and similarly for the $\bar{n}$ light-cone sector. This again yields the result \((72)\) for the corresponding amplitudes

$$A_{J_s}^{(8)} = (-1)^J \frac{1}{\sqrt{2}} J_{us}. \quad (82)$$

The ultrasoft integral in this case reads

$$\tilde{J}_{us} = \frac{i}{m^3} \left\langle 0 | \langle P_0 | \right\rangle \alpha_s(\mu_h) \alpha_s(\mu_{us}) \frac{C_F}{2N_c} \left(f_{P_1}^s - f_{P_3}^q\right) N \int d^3 \vec{\Delta} \hat{R}_{21}(\Delta) \Delta \times \frac{1}{i\pi^2} \int d^D l \frac{1}{|l^2|} \frac{1}{|E - m(l\omega) + \Delta^2 / m + i\varepsilon|^2}. \quad (83)$$

The integral over the ultrasoft momentum $l$ is UV-divergent and we use dimensional regularisation as before, with $d^D l = e^{(\gamma_E + \ln\pi)} m^{D-2} \sqrt{2\pi^{D-4}} d^{D-2} l$. Calculation of this integral yields

$$\frac{1}{i\pi^2} \int d^D l \frac{1}{|l^2|} \frac{1}{|E - m(l\omega) + \Delta^2 / m + i\varepsilon|^2} = \frac{1}{\varepsilon} + \ln \frac{m^2}{\mu^2} + 2 \ln \left(\frac{\Delta^2 / m - E - i0}{m}\right). \quad (84)$$

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Therefore we obtain
\[ A_{J=2,s}^{\perp(8)} = 2 \bar{J}_{us} = \frac{i \langle \mathcal{O}^{(3)P_0} \rangle}{m^4} \alpha_s(\mu) \alpha_s(\mu_{us}) \frac{C_F}{N_c} (f_P - f_Q) \times \left( - \frac{1}{\varepsilon} + \ln \frac{m^2}{\mu^2} + 2 \ln 2 + 2 N \int \frac{d^2 \Delta}{(2\pi)^3} \hat{R}_{21}(\Delta) \Delta \ln \left( \frac{\Delta^2/m - E - i0}{m} \right) \right). \tag{85} \]

Comparing this expression with the hard contribution in (56) we observe that poles in \(1/\varepsilon\) and \(\mu\)-dependence cancel in the sum. The soft-overlap amplitude (85) also has imaginary part which is generated by the cut of ultrasoft propagators in Fig.5.

Let us summarise. The colour-octet amplitudes defined in Eq. (58) are given by the sum
\[ A_{J}^{\perp(8)} = A_{Jc}^{\perp(8)} + A_{Js}^{\perp(8)}. \tag{86} \]

The total decay amplitudes are given by the sum of the singlet and octet amplitudes (47), the singular terms cancel in this sum so that decay amplitude is well defined. This cancellation allows us conclude that various IR-singularities which have been observed in the colour-singlet amplitudes can be absorbed into renormalization of the colour-octet matrix element (58). This matrix element is sensitive to a long-distance behaviour of the quarkonium wave function and have imaginary part due to long distance interactions. Can one get any information about the colour-octet contribution from the experimental data? We try to study this question in the next section.

5 Phenomenology

As we obtained in Sec.3 the colour-singlet amplitude \(A^\parallel\) provides a tiny contribution and can not describe the measured branching ratio. Hence we can suppose that the dominant effect is provided by the transverse amplitudes which are given by the sum of the colour-singlet and colour-octet terms. Suppose that the largest numerical effect is provided by the colour-octet amplitudes \(A_{J}^{\perp(8)}\), i.e. \(A_{J}^{\perp(8)} \gg A_{J}^{\perp(0)}\). In the previous section it was established that these amplitude satisfy to Eq. (16) up to relativistic corrections in velocity \(v\). Using this relation and Eqs. (14) and (15) one obtains
\[ R_{th} = \frac{\Gamma_{\chi_{2} \rightarrow \bar{K}^0 K^{*0} + \text{c.c.}}}{\Gamma_{\chi_{c1} \rightarrow K^0 K^{*0} + \text{c.c.}}} \simeq \left( 1 - \frac{m_P^2}{k_0^2} \right) \left( 1 - \frac{m_P^2 m_L^2}{(kp)^2} \right) \frac{1}{2} \left| \frac{A_{c}^{\perp(8)}}{A_{s}^{\perp(8)}} \right|^2 = 0.55, \tag{87} \]

This estimate includes contribution from the model dependent power suppressed coefficient which yields
\[ \left( 1 - \frac{m_P^2}{k_0^2} \right) \left( 1 - \frac{m_P^2 m_L^2}{(kp)^2} \right) = 0.91. \tag{88} \]

Using the data for neutral mesons \(\bar{K}^0\) and \(K^{*0}\) from Table 1 one finds
\[ R_{exp} = \frac{\text{Br}[\chi_{c2} \rightarrow \bar{K} K^* + \text{c.c.}]}{\text{Br}[\chi_{c1} \rightarrow K K^* + \text{c.c.}]} \frac{\Gamma_{tot}[\chi_{c2}]}{\Gamma_{tot}[\chi_{c1}]} = 0.30 \pm 0.13, \tag{89} \]

where we used \(\Gamma_{tot}[\chi_{c1}] = 0.84\) MeV and \(\Gamma_{tot}[\chi_{c2}] = 1.93\) MeV [1]. The difference between the values \(R_{th}\) and \(R_{exp}\) allows one to suppose that effect from the colour-singlet contribution is not negligible and could help to improve the description. For simplicity we consider only branching fractions of the neutral mesons. The decay amplitudes of the neutral and charged mesons must be the same due to \(SU(2)\) flavour symmetry and data support this conclusion. Therefore a consideration of the charged decays provides similar results.

The colour-singlet HQSS breaking relations have been obtained in Eqs. (49) and (57). Using them we can relate the decay amplitudes as
\[ A_{c}^{\perp} = \Delta A_{c}^{\perp(0)} + \Delta A_{s}^{\perp(0)} - \frac{1}{\sqrt{2}} A_{2}^{\perp}. \tag{90} \]
The absolute value \(|A^2_\perp|\) can be estimated from the width \(\Gamma[\chi\rightarrow K^0\bar{K}^0 + c.c.]\) that gives
\[
|A^2_\perp| = (7.0 \pm 1.5) \times 10^{-3},
\]
(91)
The absolute value \(|A^2_\perp|^2\) also depends on the unknown imaginary phase of amplitude \(A^2_\perp\)
\[
A^2_\perp = |A^2_\perp| e^{i\delta},
\]
(92)and on the unknown difference of the SCET amplitudes \(f^p_\perp - f^q_\perp\) in \(\Delta A^{\perp(0)}_s\), see Eq.(85). We accept these quantities as unknown parameters. Let us rewrite the soft-overlap combination as
\[
f^p_\perp - f^q_\perp = \frac{f_p f^0_P m_V}{m^3} \Delta f,
\]
(93)where factor \(f_p f^0_P m_V/m^3\) introduces a “natural scale” and we assume that the value of parameter \(\Delta f\) is real. We can not provide a solid argumentation about suppression of the imaginary part of \(\Delta f\) and therefore this simplification is just an assumption.

In order to get numerical estimates we use the following non-perturbative input. The models of mesonic DAs, quark masses and numerical estimates for the NRQCD matrix elements are described in Appendix A. Calculating the symmetry breaking corrections \(\Delta A^{\perp(0)}_s\) we use \(n_f = 4\), \(m_c = 1.5\) GeV and set the value of renormalisation scale \(\mu^2 = 2m^2\) in QCD coupling and in the DA parameters, this gives \(\alpha_s(2m^2) = 0.29\). We also apply the leading logarithmic evolution for the DA parameters.

The expression for \(\Delta A^{\perp(0)}_c\) is described in Eqs.(49)-(51) and substituting numerical values of the DA parameters we obtain
\[
\Delta A^{\perp(0)}_c = (-1.56 \pm 0.19) \times 10^{-3},
\]
(94)where the error shows the theoretical uncertainty from the variation of values of the DA parameters. The both contributions in Eq.(49) are negative, the largest numerical impact is provided by the terms proportional to \(SU(3)\)-breaking parameters \(\rho^\perp_F\) and \(\lambda^\perp_s\), see definitions in Eqs.(108) and (118). The chiral enhanced contribution associated with the configuration \(P_3 V_2\) is about factor two larger than the contribution from the operator with \(P_3 V_3\) configuration. Comparing results in (91) and (94) one finds that numerical value of the symmetry breaking correction is few times smaller.

For the symmetry breaking soft-overlap contribution [57] we obtain
\[
\Delta A^{\perp(0)}_s = (-0.10 + 0.13i) \Delta f \times 10^{-3},
\]
(95)where \(\Delta f\) is unknown parameter. In the following we suppose that
\[
|\Delta A^{\perp(0)}_s(\Delta f)| \lesssim |\Delta A^{\perp(0)}_c|,
\]
(96)that implies \(|\Delta f| \lesssim 10\). In this case one obtains, for instance,
\[
\Delta A^{\perp(0)}_s(\Delta f = 4) = (-0.41 + 0.54i) \times 10^{-3}, \quad \Delta A^{\perp(0)}_s(\Delta f = 8) = (-0.81 + 1.1i) \times 10^{-3}
\]
(97)

Numerical estimates of \(R_{th}\) in comparison with the \(R_{exp}\) are shown in Fig.6. Theoretical error band (blue shaded area) corresponds to variation of the DA parameters and \(|A^2_\perp|\) value. We see that for each value of \(\Delta f\) we have sufficiently large interval for the phase \(\delta\) which allows to describe the ratio \(R_{exp}\) within the error bars. The largest numerical effect from the symmetry breaking corrections is provided by the interference with large amplitude \(A^2_\perp\). From Fig.6 we conclude that reliable description of the data for the branching fractions can be done taking into account both colour-octet and colour-singlet amplitudes.

In the previous sections it was shown that IR-singularities which appear in the convolution integrals of the colour-singlet amplitudes can be absorbed into the colour-octet amplitudes. Therefore we can define a regular colour-singlet contribution by subtracting the IR-poles. Using such definition of the colour-singlet amplitudes one can try to estimate the value of colour-octet amplitude from the phenomenological value \(A^2_\parallel\) defined in (91).
Figure 6: Ratio $R_{th}$ (dashed) as a function of angle $\delta$ (in degrees) for different fixed values of parameter $\Delta f$. The experimental value $R_{exp}$ is shown by solid line. The blue and gray shaded areas show theoretical and experimental uncertainties, respectively.

The colour-singlet amplitude is given by sum of the collinear $A_{2c}^{\perp (0)}$ and soft-overlap $A_{2s}^{\perp (0)}$ contributions given in Eqs. (21) and (55), respectively. The analytical expression for the finite part of the collinear integral is given in Appendix B. Both amplitudes depends on the factorisation scales $\mu$ or $\nu$ which are set to $m$. This kills endpoint logarithms $\ln m^2/\mu^2$ which must cancel in the sum of singlet and octet amplitudes. Using the values for parameters $\Delta f$ and $\delta$ as in Fig. 6 we can obtain numerical estimates for real and imaginary parts of the colour-octet amplitude $A_{2}^{\perp (8)}$. The obtained results are shown in Fig. ?? We see that the absolute value of the colour-octet amplitude is always few times larger than the colour-singlet one. But the values of the real and imaginary parts of the octet amplitudes strongly depend on the phase $\delta$.

If the soft-overlap amplitude in Eq. (86) is underestimated then the value of the colour-singlet amplitude can be larger. But even if we take $\Delta f = 20$ the colour-octet corrections remain sufficiently large and important. Therefore at least qualitatively we can definitely conclude that colour-octet mechanism plays very important role in the description of $\chi_{Jc} \rightarrow KK^*$ decays.
Figure 7: The colour-octet amplitude $A_2^{\perp(8)}$ in units $10^{-3}$ as a functions of angle $\delta$ at fixed $\Delta f$. The solid, dot-dashed and dashed lines correspond to the absolute value, real and imaginary parts, respectively. The dotted line shows the absolute value of the colour-singlet amplitude $|A_2^{\perp(0)}|$.

6 Discussion

Motivated by existing experimental data we discussed a description of the decay amplitudes $\chi_c J \to \bar{K}K^*$ within the effective field theory framework. We computed the leading-order amplitude which describes decay $\chi_{cJ} \to \bar{K}K^*$ with longitudinally polarised vector mesons. This amplitude is given by the factorisable matrix element of a colour-singlet operator. However corresponding contribution is tiny providing only few percents from the measured branching ratio. Therefore we expect that the dominant effect is provided by the subleading amplitudes describing decays with transversely polarised mesons $\chi_{cJ\perp} \to \bar{K}K^*_\perp$. A description of the colour-singlet contributions in the corresponding amplitudes involves a combination twist-2 and twist-3 collinear matrix elements because of helicity conservation in the QCD hard subprocess. The total answer is quite complicate and for simplicity the amplitudes were computed in the Wandzura-Wilczek approximation neglecting genuine twist-3 quark-gluon matrix elements. The computed amplitudes depends from the collinear convolution integrals which have infrared endpoint divergences. A study of these singularities clearly indicates about the mixing with the colour-octet operator. The colour-octet matrix element has been studied in the Coulomb limit using pNRQCD framework. We obtained that the UV-singularities of the octet matrix element exactly reproduce the IR-singularities in the colour-singlet operator and therefore these IR-divergencies can be absorbed into colour-octet contribution. Hence a consistent description of decay amplitudes is given by the sum of colour-singlet and colour-octet matrix elements. The effective field theory calculations also allows one to observe mechanism which provides the imaginary part of the colour-octet amplitude. In pNRQCD such imaginary part is generated by the cut of an intermediate state with the bounded heavy quark-antiquark in octet state.

The heavy quark spin symmetry allows one to establish a relation between the colour-octet matrix elements for vector $J = 1$ and tensor $J = 2$ states. This makes possible a computation of the spin symmetry breaking corrections which are free from the IR-divergencies. These corrections were computed using the physical subtraction scheme. We also include in our description a contribution which depends on the unknown long distance matrix element describing the soft-overlap dynamics of the final mesons. Making various assumptions about the value of this matrix element we obtained a reliable description of the branching fraction ratio. In this picture the colour-octet matrix element provides a contribution which is few times larger then the colour-singlet one.

The uncertainties in the given consideration can be improved by reduction of the experimental errors and by obtaining an estimate for the unknown soft-overlap matrix element. Potentially, the model independent information about this quantity can be obtained from the cross section of $\gamma \gamma \to \bar{K}K^*$ process in the kinematical region with $s \sim -t \sim -u \gg \Lambda_{QCD}$.

Aknowlegements

I am grateful to M. Vanderhaeghen for attracting my attention to work [10] and for the discussions.
7 Appendix A. Long distance matrix elements

The NRQCD long distance matrix elements are defined as

\[ \langle 0 \vert \frac{1}{2\sqrt{2}} \chi^\dagger \gamma_\mu \mathcal{D}_\mu^\alpha \left( -i \frac{\gamma_\mu \gamma_\nu}{2} \gamma_5 \psi_\omega \right) \chi_c | \mathcal{O}(3P) \rangle, \]

\[ \langle 0 \vert \chi^\dagger \left( -i \frac{\gamma_\mu \gamma_\nu}{2} \gamma_5 \psi_\omega \right) \chi_c | \mathcal{O}(3P) \rangle, \]

where \( \rho \) is the quark wave function of the charmonium state \( | \chi_c \rangle \). The combination \((\alpha, \beta)\) denotes the symmetrical traceless tensor. These operators are constructed from the quark \( \psi_\omega \) and antiquark \( \chi_c \) four-component spinor fields satisfying \( \psi_\omega = \psi_\omega \), \( \chi_c = -\chi_c \). The constants on the \( rhs \) of Eqs. (98) and (99) are related to the value of the charmonium wave functions at the origin. To leading order in small velocity \( v \) they read

\[ \langle \mathcal{O}(3P) \rangle \approx \sqrt{2N_c} \sqrt{2M_{\chi_c}} \frac{3}{4\pi} R_{21}(0), \]

where \( R_{21}(0) \) is the derivative of the quarkonium radial wave function. The value of this parameter has been estimated in the different potential models, see e.g. Ref. [36]. In this paper we use the value computed for the Buchmüller-Tye potential

\[ m = 1.5 \text{ GeV}, \ |R_{21}(0)|^2 = 0.75 \text{ GeV}^5. \]

The leading twist \( K \) and \( K^* \) meson DAs has been studied in many publications, see e.g. [37,41] and references therein. The new results and updates of twist-3 and twist-4 DAs can be found in Refs. [42,44]. Here, for a convenience of reader, we briefly describe definitions and models which are used in this paper.

In the following we assume that direction \( z \) is light-like \((z^2 = 0)\) and \([z, -z]\) denotes the appropriate Wilson line, see e.g. Ref. [42]. For pseudoscalar state \( \bar{K}(\bar{q}s) \) we need the following matrix elements

\[ \langle \bar{K}(k) | \bar{s}(z) \bar{s}(z) \gamma^\mu \gamma_5 \gamma^\nu \bar{g}(z) \rangle = -i f_{K} k^\mu \int_0^1 dy e^{i(2y-1)(kz)} \phi_{2K}(y), \]

\[ \langle \bar{K}(k) | \bar{s}(z) \bar{s}(z) \gamma_5 \gamma^\nu \bar{g}(z) \rangle = f_{K}^\mu \mu_{K} \int dy e^{i(2y-1)(kz)} \phi_{3K}^\mu(y), \]

\[ \langle \bar{K}(k) | \bar{s}(z) \bar{s}(z) \gamma_5 \gamma^\nu \bar{g}(z) \rangle = \frac{i}{3} f_{K} k^\mu \left( k_{\mu} z_{\nu} - k_{\nu} z_{\mu} \right) \int_0^1 dy e^{i(2y-1)(kz)} \phi_{3K}^\sigma(y). \]

Here

\[ \mu_{K} = m_{K}^2 / (m_s + m_q), \]

and \( m_K, m_s, m_q \) denote the masses of the \( K \)-meson, \( s \) - and \( q = u, d \) quarks, \( f_K \) is the decay constant. The models for the corresponding DAs are given by the sum of the few first Gegenbauer moments \((\bar{y} = 1 - y)\)

\[ \phi_{2K}(y, \mu) = 6\bar{y} \left( 1 + b_1(\mu) C_1^{3/2}(2y - 1) + b_2(\mu) C_2^{3/2}(2y - 1) \right), \]

\[ \phi_{3K}^\mu(x, \mu) = 1 + \rho_{2K}^{3/2} \left( 1 + b_2(\mu) \right) \ln \frac{x}{\bar{x}} + \rho_{3K}^{3/2} b_2(\mu) \left( 6 C_1^{1/2}(2x - 1) + C_3^{1/2}(2x - 1) \right), \]

\[ \phi_{3K}^\sigma(x, \mu) = 6\bar{x} \left( 1 - \rho_{2K} b_2(\mu) C_2^{1/2}(2x - 1) \right) + \rho_{3K} (1 + b_2(\mu)) 9\bar{x} \ln \frac{x}{\bar{x}}, \]

where \( \rho_{2K} = (m_s^2 + m_q^2) / m_K^2 \) in Eqs. (107) and (108) we neglect numerically small terms proportional to \( \rho_{2K}^2 b_1 \). Remind, that in this paper we also neglect the three-particle quark-gluons operators. The evolution of the various parameters are well known and explicit formulas can be found in the given references. In our numerical calculations we use the following values

\[ m_{K^0} = 498 \text{ MeV}, \ m_s(2 \text{ GeV}) = 100 \text{ MeV}, \ m_u \simeq m_d \approx 0. \]
The light-cone matrix elements with $K^*(q\bar{s})$ read

$$\langle K^*(p, e^+) | \bar{q}(z) | z, -z \rangle \sigma_{\mu\nu} s(-z) | 0 \rangle = -if_{K^*}^\parallel m_{K^*} (e^*_\mu p_\nu - e^*_\nu p_\mu) \int_0^1 dx e^{i(2x-1)(pz)} \phi_{2K^*}^\parallel(x),$$

$$\langle K^*(p, e^+) | \bar{q}(z) | z, -z \rangle \gamma_\sigma s(-z) | 0 \rangle = -if_{K^*}^\parallel m_{K^*} \int_0^1 dx e^{i(2x-1)(pz)} \left\{ p_\sigma (e^*_\mu z) \phi_{2K^*}^\parallel(x) + e^*_\sigma \phi_{3K^*}^\parallel(x) \right\},$$

$$\langle K^*(p, e^+) | \bar{q}(z) | z, -z \rangle \gamma_\sigma \gamma_5 s(-z) | 0 \rangle = f_{K^*}^\parallel m_{K^*} \frac{1}{2} i \varepsilon_{\mu\nu\rho\lambda} e^* \rho \lambda' z' \int_0^1 dx e^{i(2x-1)(pz)} \psi_{3K^*}^\parallel(x).$$

The corresponding DA models read

$$\phi_{2K^*}^{\perp, \perp}(u) = 6u\bar{u}(1 + a^{\perp, \perp}_{1K^*} C_{111}^{\perp, \perp}(2u - 1) + a^{\perp, \perp}_{2K^*} C_{111}^{\perp, \perp}(2u - 1),$$

$$\phi_{3K^*}^{\perp, \perp}(u) = \frac{1}{2} \int_0^u \frac{dv}{v} \Omega(v) + \frac{1}{2} \int_u^1 \frac{dv}{v} \Omega(v) + \lambda_+^\perp \phi_{2K^*}^\perp(u),$$

$$\psi_{3K^*}^{\perp, \perp}(u) = 2\bar{u} \int_0^u \frac{dv}{v} \Omega(v) + 2u \int_u^1 \frac{dv}{v} \Omega(v),$$

with

$$\Omega(u) \simeq \phi_{2K^*}^{\perp, \perp}(u) + \lambda_+^\perp \frac{1}{2} (2u - 1) \frac{d}{du} \phi_{2K^*}^\perp(u) + \lambda_-^\perp \frac{1}{2} \frac{d}{du} \phi_{2K^*}^\perp(u),$$

$$\lambda_-^\perp = \frac{f_{K^*}^\perp}{f_{K^*}^{\perp, \perp}} m_s - m_q, \quad \lambda_+^\perp = \frac{f_{K^*}^\perp}{f_{K^*}^{\perp, \perp}} m_s + m_q.$$ 

For the DA parameters we use the numerical update from Ref. [44]

$$f_{K^*}^{\parallel} = 220 \text{ MeV}, \quad f_{K^*}^{\perp, \perp}(1 \text{ GeV}) = 185 \text{ MeV}, \quad m_{K^*0} = 896 \text{ MeV},$$

$$a^{\parallel}_{1K^*}(1 \text{ GeV}) = -0.03 \pm 0.02, \quad a^{\perp, \perp}_{1K^*}(1 \text{ GeV}) = -0.04 \pm 0.03,$$

$$a^{\parallel}_{2K^*}(1 \text{ GeV}) = 0.11 \pm 0.09, \quad a^{\perp, \perp}_{2K^*}(1 \text{ GeV}) = 0.10 \pm 0.08.$$ 

Notice that we define SU(3) breaking coefficients $a^{\parallel, \perp}_{1K^*}$ to be negative because $K^*$ state includes s-antiquark.

8 Appendix B. Analytical results for the collinear convolution integrals

Here we provide results for the convolution integrals which describe colour-singlet amplitudes. These results have been obtained using DAs described in the previous section. All divergent integrals are computed using analytical regularisation prescription as described in the text. The singular terms (poles in $1/\varepsilon$) and corresponding scale dependent logarithms in $m/\mu$ cancel in the sum of the octet and singlet amplitudes and therefore they are not shown for simplicity (this is indicated by the subscript “fin”). For simplicity, we also do not write explicitly $\mu$-dependence of the various DA parameters.
For the colour-singlet integrals in Eqs. (21) we obtain

\[ J_{c}^{(j=2)}[P_{2}V_{3}]_{\text{fin}} = \]

\[ \left\{ \Delta \phi_{2P}(0)\bar{I}[\Omega] + \bar{\phi}_{2P}(0)I[\Delta \Omega] \right\} \frac{1}{2} (1 - \ln 2) + \left\{ \phi'_{2P}(0)\Delta \Omega(1) + \Delta \phi'_{2P}(0)\bar{\Omega}(1) \right\} \frac{1}{2} \left( 1 - \frac{\pi^{2}}{12} \right) \]

\[ + \frac{27}{16} b_{1} \left\{ 8 + 3\pi^{2} - 16\ln 2 + a_{2}^{\|} (93 - 13\pi^{2}/4 - 16\ln 2) \right\} \]

\[ - \frac{27}{16} b_{1} a_{3}^{\perp} \left\{ 6\zeta(3) - 16 + 32\ln 2 + 2\pi^{2}(4\ln 2 - 6) + a_{2}^{\perp} \left[ 36\zeta(3) - 516 + 352\ln 2 + 2\pi^{2}(24\ln 2 - 41) \right] \right\} \]

\[ + \frac{g}{16} a_{1V} \left( 16\ln 2 - 40 - \pi^{2} + \frac{3}{2} b_{2} (260 - 49\pi^{2} + 64\ln 2) \right) \]

\[ + \frac{9}{16} \lambda_{x}^{\perp} \left( -32\ln 2 + 4\pi^{2}\ln 2 + 6\zeta(3) + 3b_{2} \left\{ -64\ln 2 + \pi^{2}(-15 + \frac{32}{3} \ln 2) + 12 (10 + \zeta(3)) \right\} \right) \]

\[ + \frac{9}{16} \lambda_{x}^{\perp} a_{2V} \left\{ 36\zeta(3) + 400 + 2\pi^{2}(5 + 16\ln 2) - 352\ln 2 + 3b_{2} \left[ 72\zeta(3) - 580 - 704\ln 2 + 2\pi^{2}(155 + 48\ln 2) \right] \right\} \]

\[ + \frac{27}{32} a_{1V} \lambda_{x}^{\perp} \left\{ -12\zeta(3) - 80 + 96\ln 2 - 2\pi^{2}(1 + 4\ln 2) + 2b_{2} \left[ -72\zeta(3) + 60 + 576\ln 2 - 3\pi^{2}(19 + 16\ln 2) \right] \right\} . \]

\[ J_{c}^{(j=2)}[P_{3}V_{3}]_{\text{fin}} = \Delta \phi'_{2V}(0)\rho_{K}^{\perp} \left( \frac{3}{2} \right) \left( 15b_{2} + (1 + 6b_{2}) \frac{\pi^{2}}{12} - (1 + 21b_{2}) \ln 2 \right) + \Delta \phi'_{2V}(1) (1 - \ln 2) \]

\[ - 9a_{1V}(2\ln 2 - 4) - \frac{9}{2} \rho_{K}(\pi^{2} + 4\ln 2)(1 + 6a_{2V}) \]

\[ + \frac{27}{4} \rho_{K} b_{2}(1 + 6a_{2V})(20 + 3\pi^{2} - 28\ln 2) . \] (123)

The integrals for the vector state \( J = 1 \) can be obtained using Eqs. [50] and [51].

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