Abstract

In a recent paper, A. Valentini tried to obtain Born’s principle as a result of a subquantum heat death, using classical \( H \)-theorem and the definition of a proper quantum \( H \)-theorem within the framework of Bohm’s theory. In this paper, we shall show the possibility of solving the problem of action-reaction asymmetry present in Bohm’s theory by modifying Valentini’s procedure. However, we get his main result too.

I. INTRODUCTION

Bohm’s theory is a casual interpretation of quantum mechanics that was initially introduced by de Broglie\cite{1} and then developed by Bohm\cite{2}. This theory is claimed to be equivalent to the standard quantum theory without having the conceptual problems of the

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latter[3]. Yet, there are some difficulties with this theory. One of these difficulties is the action-reaction (AR) problem[4,5]. In this theory the velocity of a particle is given by

\[ \dot{x} = \frac{\hbar}{m} \text{Im}(\nabla \psi) = \frac{\nabla S}{m} \]  

(1)

where \( S \) is the phase of the wave function \( \psi = \text{Re}^i \pi \). The wave function \( \psi \) itself is a solution of Schrödinger equation:

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi \]  

(2)

By substituting \( \text{Re}^i \pi \) for \( \psi \) in (2), we shall have the following equations

\[ \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = 0 \]  

(3)

\[ \frac{\partial R^2}{\partial t} + \nabla \cdot \left( \frac{\nabla S}{m} R^2 \right) = 0 \]  

(4)

where (3) is the classical Hamilton-Jacobi equation with an additional term, \( Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \), which is called quantum potential. By differentiating (1) with respect to \( t \) and making use of (3), we obtain

\[ m\ddot{x} = -\nabla (Q + V) \]

which shows that the \( \psi \)-wave affects particle’s motion through the \( -\nabla Q \) term. To secure the action-reaction symmetry, we expect the presence of a term corresponding to particle’s reaction on the wave, in the wave equation of motion (2). This term is not present there[5].

Another difficulty with Bohm’s theory is Born’s statistical principle. In this theory the field \( \psi \) enters as a guiding field for the motion of particles but at same time it is required by the experimental facts to represent a probability density through \( |\psi|^2 \). In this paper we try to obtain Born’s statistical principle and try to solve the AR problem as well.
Our paper is organized as follows: After reviewing Valentini’s quantum $\mathcal{H}$-theorem in section 2, we introduce our new quantum $\mathcal{H}$-theorem in section 3 and, finally, we show how the AR problem is solved by our procedure.

II. QUANTUM $\mathcal{H}$-THEOREM

Recently, A. Valentini has derived the relation $\rho = |\psi|^2$ as the result of a statistical subquantum $\mathcal{H}$-theorem[6]. He looked for a proper quantum function $f_N$ that, like the classical $N$-particle distribution function, satisfies Liouville’s equation. He considered an ensemble of $N$-body systems which could be described by the wave function $\Psi$ and the distribution function $P$. Because $|\Psi|^2$ and $P$ must equalize during the assumed heat death, in general $P \neq |\Psi|^2$ and one can write

$$P(X, t) = f_q(X, t) |\Psi(X, t)|^2$$

where $f_q$ measures the ratio of $P$ to $|\Psi|^2$ at the point $X (\vec{x}_1 \cdots \vec{x}_N)$ at time $t$. Since both $|\Psi|^2$ and $P$ satisfy the continuity equation ($|\Psi|^2$ due to its being a solution of Schrödinger equation and $P$ by its very definition) one can easily show that $f_q$ satisfies the following equation

$$\frac{\partial f_q}{\partial t} + \dot{X} \cdot \vec{\nabla} f_q = 0$$

where $X$ means $\vec{x}_1 \cdots \vec{x}_N$ as before. Thus, Valentini defined his quantum $\mathcal{H}$-function in the following way

$$\mathcal{H}_q = \int d^{3N}X |\Psi(X, t)|^2 f_q(X, t) \ln f_q(X, t)$$

The only difference with the classical one is that $f_q$ is defined in the configuration space while the classical $f_N$ is defined in the phase space and $d^{3N}X d^{3N}P \rightarrow |\Psi(X, t)|^2 d^{3N}X$. Valentini
used Ehrenfest’s coarse-graining method[7]. One can easily shows that $\frac{d\hat{H}_q}{dt} \leq 0$, where $\hat{H}_q$ is the coarse-graind $\mathcal{H}$-function and the equality holds in the equilibrium state, where $\bar{f}_q = 1$ or $\bar{P} = |\Psi|^2$. Here $\bar{P}$ and $|\Psi|^2$ are coarse grained forms of $P$ and $|\Psi|^2$ respectively. Valentini termed this process a subquantumic heat death. Then, he showed that if a single particle is extracted from the large system and prepared in a state with wavefunction $\psi$, its probability density $\rho$ will be equal to $|\psi|^2$, provided that $P = |\Psi|^2$ holds for the large system. Notice that a one-body system not in quantum equilibrium can never relax to quantum equilibrium (when it is left to itself). But any one-body system is extracted from a large system.

Here, we want to modify Valentini’s procedure so that one can directly obtain Born’s principle for a one-body system. In this case we will be forced to use Boltzmann’s procedure, and that naturally leads to a solution of the AR problem.

III. AN ALTERNATIVE QUANTUM $\mathcal{H}$-THEOREM

Consider an ensemble of one-body systems. Suppose that all these systems are in the $\psi$ state and that their distribution function is $\rho$. Furthermore, suppose that at $t = 0$ we have $\rho \neq |\psi|^2$, and define

$$\rho(\vec{x}, t) = f_q(\vec{x}, t) |\psi(\vec{x}, t)|^2$$  \hspace{1cm} (5)

In Valentini’s procedure the complexity of systems leads to heat death, but our systems are simple (one-body) ones. Thus, if we want to have heat death, we must assume that $f_q$ satisfies a quantum Boltzmann equation

$$\frac{\partial f_q}{\partial t} + \hat{x} \cdot \vec{\nabla} f_q = J(f_q)$$  \hspace{1cm} (6)

where $J(f_q)$ is related to the particle reaction on its associated wave. Since $\rho$ satisfies a
continuity equation, as the result of its definition, thus (5) and (6) imply that $|\psi|^2$ does not satisfy the continuity equation any more. In fact, the continuity equation for $|\psi|^2$ is changed to

$$\frac{\partial |\psi|^2}{\partial t} + \vec{\nabla} \cdot \left( \frac{\vec{\nabla} S}{m} |\psi|^2 \right) = -\frac{J(f_q)}{f_q} |\psi|^2$$

(7)

This means that $\psi$ is a solution of a nonlinear Schrödinger equation. If we want to have the quantum potential in its regular form, i.e. $(-\frac{\hbar^2}{2m} \nabla^2 R)$, we must choose the nonlinear term in a particular form. The proper selection is

$$i\hbar (\frac{\partial}{\partial t} + g(f_q)) \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi$$

(8)

where $g(f_q)$ is a real function of $f_q$. Now with the substitution $\psi = Re^{i\phi}$ we shall have

$$\frac{\partial S}{\partial t} + \frac{(\vec{\nabla} S)^2}{2m} - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = 0$$

(9)

$$\frac{\partial |\psi|^2}{\partial t} + \vec{\nabla} \cdot \left( \frac{\vec{\nabla} S}{m} |\psi|^2 \right) = -2g(f_q) |\psi|^2$$

(10)

By comparing (10) with (7) we have $g(f_q) = \frac{1}{2} \frac{J(f_q)}{f_q}$. Now, we have to select $g(f_q)$ in such a way that it leads to the equality of $\rho$ and $|\psi|^2$ (i.e. $f_q = 1$). Some requirements for such a function is:

1- It must be invariant under $t \rightarrow -t$.

2- It must change its sign for $f_q = 1$.

If we fine systems for which subquantum heat death has not occurred,[8], we shall obtain the actual form of the $g(f_q)$ function. A proper selection is $g(f_q) = \alpha(1 - f_q)$ where $\alpha$ is a constant. Then, (10) gives (with $\alpha = \frac{1}{2}$)

$$\frac{\partial |\psi|^2}{\partial t} + \vec{\nabla} \cdot \left( \frac{\vec{\nabla} S}{m} |\psi|^2 \right) = (f_q - 1) |\psi|^2$$

(11)
Now, consider the right hand side of (11) as the source of $|\psi|^2$ field. At any point of space where $|\psi|^2 < \rho$ (i.e. $f_q > 1$), the source of $|\psi|^2$ is positive and therefore $|\psi|^2$ increases at that point. On the other hand, at any point of space where $|\psi|^2 > \rho$ (i.e. $f_q < 1$), the source of $|\psi|^2$ is negative and therefore $|\psi|^2$ decreases at that point. The variation of $|\psi|^2$ continues until $|\psi|^2$ becomes equal to $\rho$. After that, since both $|\psi|^2$ and $\rho$ evolve under the same velocity field ($\frac{\vec{S}}{m}$), they remain equal. Here our reasoning is not exact. To prove $\rho \to |\psi|^2$ exactly, we introduce the following $H_q$ function:

$$H_q = \int d^3 x (\rho - |\psi|^2) \ln \left( \frac{\rho}{|\psi|^2} \right)$$ (12)

Since $(X - Y) \ln \left( \frac{X}{Y} \right) \geq 0$ for all $X, Y \geq 0$, we have $H_q \geq 0$ – the equality being relevant to the case $\rho = |\psi|^2$. If we show that for the forgoing $J(f_q)$ one has $\frac{dH_q}{dt} \leq 0$, where again the equality is to relevant to $\rho = |\psi|^2$ (i.e. $f_q = 1$) state, we have shown that $|\psi|^2$ becomes equal to $\rho$ finally. We write (12) in the form

$$H_q = \int d^3 x |\psi|^2 (f_q - 1) \ln f_q = \int d^3 x |\psi|^2 G(f_q)$$

where $G(f_q) = (f_q - 1) \ln f_q$. Now we have for $\frac{dH_q}{dt}$

$$\frac{dH_q}{dt} = \int d^3 x \left\{ -\nabla \cdot (\frac{\dot{\vec{x}}}{|\psi|^2} G(f_q)) - J(f_q) \left[ \frac{G(f_q)}{f_q} - \frac{\partial G(f_q)}{\partial f_q} \right] |\psi|^2 \right\}$$ (13)

where we have done an integration by parts. Passing to the limit of large volumes and dropping the surface term in (13) leads to

$$\frac{dH_q}{dt} = \int d^3 x \frac{J(f_q)}{f_q} \{(f_q - 1 + \ln f_q) |\psi|^2 \}$$

The quantity in $\{\}$, is negative for $f_q < 1$ and positive for $f_q > 1$. Now, since the $J(f_q)$ (i.e. $f_q(1 - f_q)$) is positive for $f_q < 1$ and negative for $f_q > 1$, the integrand is negative or zero for all values of $f_q$ and we have

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\[
\frac{dH_q}{dt} \leq 0
\]
where the equality holds for \( f_q = 1 \). The existence of a quantity that decreases continuously to its minimum value in \( \rho = |\psi|^2 \) (i.e. \( f_q = 1 \)) guarantees Born’s principle.

**IV. THE ACTION-REACTION PROBLEM**

In the classical gravity, matter fixes space-time geometry and correspondingly matter’s motion is determined by the space-time geometry. This means that matter and space-time affect each other. Thus, the action-reaction symmetry is preserved. In fact, the nonlinearity of Einstein equations is the result of this mutual action-reaction. In the same way, mutual action-reaction between wave and particle in Bohm’s theory, leads to a nonlinear Schrödinger equation. But nonlinearity does not necessarily mean particle reaction on the wave. Indeed nonlinear terms must contain some information about particle’s position too.

In the last section we showed that Born’s principle can be a result of the presence of a nonlinear term, in the Schrödinger equation, with special characteristics. In this section we want to show that the existence of such a term solves the AR problem in a satisfactory way.

The nonlinear Schrödinger equation with our choice of \( g(f_q) \) function (i.e. \( \alpha(1 - f_q) \)) becomes

\[
\hbar \left( \frac{\partial}{\partial t} + \alpha \left( 1 - \frac{\rho}{|\psi|^2} \right) \right) \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi \quad (14)
\]

In this equation, \( i\hbar \alpha \left( 1 - \frac{\rho}{|\psi|^2} \right) \), indicates particle’s reaction on the wave, and as we expect this nonlinear term contains information about particle’s position through \( \rho \). To clarify this point, we write the distribution function \( \rho \) as

\[
\rho(\vec{x}) = \sum_{i=1}^{N} \delta(\vec{x} - \vec{x}_i)
\]
where $\vec{x}_i$ is the position of i-th particle of the ensemble. Then, the $\rho$-distribution is determined from the position of the chosen particle and the other members of the ensemble. If we have only one particle instead of an ensemble of particles, then (14) changes to

$$i\hbar\left(\frac{\partial}{\partial t} + \alpha \frac{\delta(\vec{x} - \vec{x}_i)}{|\psi|^2}\right)\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$$

Now it is clear that particle’s position affects wave evolution, directly. Thus, there is no AR problem. In an actual experiment we always confront an ensemble of particles and for such ensembles the heat death has already occurred. Thus the nonlinear term, that is an indicator of particle’s reaction on the wave, is eliminated. This means that the AR problem is a result of the establishment of Born’s principle. In fact this is similar to the argument that Valentini presents[9] to show that the signal-locality (i.e. the absence of practical instantaneous signalling) and the uncertainty principle are valid if and only if $\rho = |\psi|^2$.

V. CONCLUSION

We showed that the Born’s principle can be the result of the presence of a nonlinear term in schrödinger equation, with special characteristics. Then, we showed that this nonlinear term is indicator of particle’s reaction on the wave. Thus, in this way, we have not only obtained Born’s principle (Bohm’s postulate) but also we have solved the AR problem.

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