SU(4) Coherent Effects to the Canted Antiferromagnetic Phase in Bilayer Quantum Hall Systems at \( \nu = 2 \)

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In bilayer quantum Hall (BLQH) systems at \( \nu = 2 \), three different kinds of ground states are expected to be realized, i.e. a spin polarized phase (spin phase), a pseudospin polarized phase (ppin phase) and a canted antiferromagnetic phase (C-phase). An SU(4) scheme gives a powerful tool to investigate BLQH systems which have not only the spin SU(2) but also the layer (pseudospin) SU(2) degrees of freedom. In this paper, we discuss an origin of the C-phase in the SU(4) context and investigate SU(4) coherent effects to it. We show peculiar operators in the SU(4) group which do not exist in \( SU_{\text{spin}}(2) \otimes SU_{\text{ppin}}(2) \) group play a key role to its realization. It is also pointed out that not only spins but also pseudospins are “canted” in the C-phase.

I. INTRODUCTION

Recently bilayer quantum Hall (BLQH) systems have much attention due to their peculiar phenomena stemmed from many isospin degrees of freedom. BLQH systems have not only the spin SU(2) but also the layer (pseudospin or ppin) SU(2) degrees of freedom. The interlayer interactions bring many interesting phenomena in BLQH systems [1]. For instance, skyrmion excitations arise not only in a spin space but also in a pseudospin space [2, 3]. A Josephson-like tunneling current across the layers has been observed [4], which is predicted a decade ago [5, 6]. An SU(4) formalism naturally incorporates the spin SU(2) and pseudospin (ppin) SU(2) degrees of freedom. To use such a larger group, one can easily treat various ground states and excitations from a unified point of view [7, 8].

In BLQH systems at \( \nu = 2 \), three different kinds of phases are expected to be realized. One is a fully spin polarized ferromagnetic phase, which we call the spin phase. Second one is a spin singlet phase, which we call the ppin phase because it is also a fully pseudospin polarized one. They have been observed experimentally [9, 10]. The last one is a canted antiferromagnetic phase (C-phase), which is predicted [11, 12, 13] between the spin and ppin phase in a phase diagram. The C-phase is an exotic phase, where spins on the front layer and the back layer exhibit ferromagnetic properties in the perpendicular direction and simultaneously antiferromagnetic properties to the two dimensional layer directions.

The spin and ppin phases can be understood as different ground states from an SU(4) unified point of view. Excitations on two phases are determined by the breaking patterns of the SU(4) symmetry [8]. Thus the SU(4) formalism has played essential roles to understand physical properties of these phases. However, SU(4) analysis on the C-phase has not been accomplished at all. The aim of this paper is to investigate the C-phase in the SU(4) context.

Demler and Das Sarma analyzed the C-phase with the use of an effective boson theory [14]. Their Hamiltonian has not included the interlayer exchange interaction. MacDonald et al. also investigated the system based on the Hartree Fock method [15]. However isospin exchange interactions have been dropped in the approximation. Both interlayer and isospin interactions are important in BLQH systems because they bring rich physics as stated above.

We improve their arguments with the use of more realistic SU(4) Hamiltonian, which includes the interlayer and SU(4) isospin exchange interactions fully [7, 8]. It is shown that the SU(4) coherence disturbs the C-phase realization. Peculiar SU(4) operators which do not exist in \( SU_{\text{spin}}(2) \otimes SU_{\text{ppin}}(2) \) act as an order parameter of the C-phase. We refer them as \( U \)-operators, which are essential for the C-phase realization.

In Section II we review the derivation of the microscopic Landau-site Hamiltonian [7, 8] for the BLQH system, which is an anisotropic SU(4) nonlinear sigma model in a continuum limit. In Section III we make a group theoretical study of isospin states in BLQH systems. At \( \nu = 2 \) they belong to a 6-dimensional representation of SU(4). They are referred as Schwinger bosons [16]. With the use of the boson picture, we construct an effective boson theory just like [17]. Classifying them with respect to the subgroup \( SU_{\text{spin}}(2) \otimes SU_{\text{ppin}}(2) \), we introduce three different kinds of ground states, i.e. the spin phase, the ppin phase and the C-phase. It is shown that not only spins but also ppins are “canted” in the C-phase. We discuss a crucial role of \( U \)-operators for the realization of the C-phase. In Section IV we investigate SU(4) coherent effects to the C-phase. The SU(4) coherence reduces the C-phase region and disturbs its realization. We give physical interpretations of this effect. Section V is devoted to summary and discussion.

II. LANDAU-SITE HAMILTONIAN

We analyze electrons in the lowest Landau level in BLQH systems. One Landau site contains four electron states distinguished by the SU(4) isospin index
The Coulomb interaction is given by

\begin{equation}
H_C = \sum_{\alpha, \beta = \uparrow, \downarrow} \frac{1}{2} \int d^2 x d^2 y V_{\alpha \beta}(x - y) \rho_\alpha(x) \rho_\beta(y),
\end{equation}

where \( V_{\uparrow \downarrow} = V_{\downarrow \uparrow} = e^2/4\pi \varepsilon r \) is the intralayer Coulomb interaction, while \( V_{\uparrow \uparrow} = V_{\downarrow \downarrow} = e^2/4\pi \varepsilon \sqrt{r_x^2 + d^2} \) is the interlayer Coulomb interaction with the interlayer separation \( d \). The Coulomb interaction is decomposed into two terms, \( H_C = H_C^+ + H_C^\pm \), with

\begin{align}
H_C^+ &= \frac{1}{2} \int d^2 x d^2 y V_+(x - y) \rho(x) \rho(y), \\
H_C^\pm &= \frac{1}{2} \int d^2 x d^2 y V_-(x - y) \Delta \rho(x) \Delta \rho(y),
\end{align}

where \( V_\pm = \frac{1}{2}(V_{\uparrow \downarrow} \pm V_{\downarrow \uparrow}) \); \( \rho(x) \) is the total density, \( \Delta \rho(x) \) is the density difference between the front and back layers. \( H_C^+ \) respects the SU(4) symmetry, while \( H_C^\pm \) does not.

By using the von-Neumann formalism \[ \text{[2E]} \], the exchange term in \[ \text{[2D]} \] is rewritten as a direct product of \( \text{SU}_{\text{spin}}(2) \) and \( \text{SU}_{\text{ppin}}(2) \) operators,

\begin{equation}
H_X = \sum_{(i,j)} J_{ij} \left( S(i) \cdot S(j) + \frac{1}{4} n(i) n(j) \right) \odot \left( \sum_a \frac{P_a(i) P_a(j) + \frac{1}{4} n(i) n(j)}{J_{ij}} \right)
= -2 \sum_{(i,j)} \left( J_{ij}^+ S(i) \cdot S(j) + J_{ij}^- P_a(i) P_a(j) + J_{ij}^0 n(i) n(j) \right) U_{ab}(i) U_{ab}(j) + \frac{1}{4} J_{ij} n(i) n(j),
\end{equation}

where \( J_{ij}^+ = J_{ij}^y = J_{ij}^t = J_{ij}^\pm - J_{ij}^- \), and \( J_{ij}^- = J_{ij}^0 \equiv J_{ij}^x + J_{ij}^y \). The exchange integral \( J_{ij}^\pm \) is given by

\begin{equation}
J_{ij}^\pm = \frac{1}{2} \int d^2 x d^2 y \varphi_i^*(x) \varphi_j^*(y) V_\pm(x - y) \varphi_i(y) \varphi_j(x).
\end{equation}

In the Hamiltonian \[ \text{[2D]} \], \( \sum_{(i,j)} \) stands for the sum over all pairs of sites indexed by \( i \) and \( j \); \( n(i) \) is the electron number operator, \( n \equiv \sum_\sigma c_\sigma^\dagger(i) c_\sigma \), at each site \( i \). The group SU(4) is generated by the Hermitian, traceless, \( 4 \times 4 \) matrices, which consist of \((2^4 - 1)\) independent generators; \( S_a, P_a, U_{ab} \equiv 2 S_a \otimes P_b \). The subgroup \( \text{SU}_{\text{spin}}(2) \otimes \text{SU}_{\text{ppin}}(2) \) is generated by \( \text{S, P} \). The \( U_{ab} \)-generators are peculiar ones, which do not exist in the \( \text{SU}_{\text{spin}}(2) \otimes \text{SU}_{\text{ppin}}(2) \) group. They mix the \( \text{SU}_{\text{spin}}(2) \) multiplets with the \( \text{SU}_{\text{ppin}}(2) \) ones.

The explicit forms of the SU(4) generators are given as follows,

\begin{equation}
S = S^\uparrow + S^\downarrow, \quad P = P^\uparrow + P^\downarrow,
\end{equation}

where

\begin{align}
S^\uparrow_a &= (c_i^\uparrow, c_i^\dagger) \frac{\tau_a}{2} (c_{i\dagger}^\uparrow, c_i^\dagger), \quad S^\downarrow_a = (c_i^\downarrow, c_i^\dagger) \frac{\tau_a}{2} (c_{i\dagger}^\downarrow, c_i^\dagger), \\
P^\uparrow_a &= (c_i^\uparrow, c_i^\dagger) \frac{\tau_a}{2} (c_{i\dagger}^\uparrow, c_i^\dagger), \quad P^\downarrow_a = (c_i^\downarrow, c_i^\dagger) \frac{\tau_a}{2} (c_{i\dagger}^\downarrow, c_i^\dagger),
\end{align}

and

\begin{equation}
U_{ab} = 2 S_a \otimes P_b = \frac{1}{2} (c_i^\uparrow, c_i^\dagger) (c_{i\dagger}^\uparrow, c_{i\dagger}^\dagger) \tau_a \otimes \tau_b \frac{1}{2} (c_i^\uparrow, c_i^\dagger) (c_{i\dagger}^\uparrow, c_{i\dagger}^\dagger),
\end{equation}

\( c_\sigma(i) \) is the annihilation operator of the electron with isospin \( \sigma \) at Landau site \( i \).

In the limit \( d \to 0 \), which we call the SU(4) limit, the exchange Hamiltonian \[ \text{[2D]} \] is reduced to an SU(4) invariant form,

\begin{equation}
H_X \to H_X^H = -2 \sum_{(i,j)} J_{ij}^{\pm} \left( S(i) \cdot S(j) + P(i) \cdot P(j) \right) + U_{ab}(i) U_{ab}(j) + \frac{1}{4} n(i) n(j).
\end{equation}

In the SU(4) limit, the interlayer Coulomb potential \( V_{\uparrow \downarrow} \) goes to the intralayer Coulomb potential \( V_{\uparrow \downarrow} \) \( (V_{\downarrow \uparrow}) \) vanishes. The SU(4)-invariant exchange Hamiltonian \[ \text{[2D]} \] is obtained from \( H_C^\pm \).

Meanwhile in the limit \( d \to \infty \) where two layers are sufficiently separated, the pseudospin stiffness \( J_d \) goes to zero. The exchange Hamiltonian \[ \text{[2D]} \] is reduced to the Hamiltonian used in Demler and Das Sarma \[ \text{[15]} \].

\begin{equation}
H_X \to -8 \sum_{(i,j)} J_{ij} \left( S(i) \cdot S(j) + \frac{1}{4} n(i) n(j) \right) \odot \left( P_z(i) \cdot P_z(j) + \frac{1}{4} n(i) n(j) \right)
= -4 \sum_{(i,j)} \left( S^x(i) \cdot S^x(j) + S^y(i) \cdot S^y(j) + \frac{1}{4} n_f(i) n_f(j) + \frac{1}{4} n_b(i) n_b(j) \right),
\end{equation}

where the symmetry of the exchange interaction is \( \text{SU}_{\text{front}}(2) \otimes \text{SU}_{\text{back}}(2) \). Each \( \text{SU}(2) \) represents the spin rotation symmetry on each layer.

The direct interaction is given as

\begin{equation}
H_D = \sum_{i=1}^{N_b} \left( -\Delta Z S_z(i) + \varepsilon_{\text{cap}} P_z(i) P_z(i) - \Delta_{\text{SAS}} P_x(i) \right),
\end{equation}

where \( \Delta Z \) and \( \Delta_{\text{SAS}} \) are the Zeeman and tunneling gaps, while \( \varepsilon_{\text{cap}} \) is the capacitance energy,

\begin{equation}
\varepsilon_{\text{cap}} = \frac{e^2}{4\pi \varepsilon \ell_B} \sqrt{\frac{\pi}{2}} \left( 1 - e^{2\ell_B^2/2\varepsilon} \{ 1 - \text{erf}(d/\sqrt{2}\ell_B) \} \right).
\end{equation}
The total Landau-site Hamiltonian is the sum of the direct term and the exchange term,\
\[ H_{tot} = H_D + H_X. \] (2.12)

### III. GROUND STATES

At \( \nu = 2 \), one Landau site contains two electrons, each of which belongs to the 4-dimensional irreducible representation of SU(4). They form a Schwinger boson, which belongs to the 6-dimensional irreducible representation due to their antisymmetry,
\[ (4 \otimes 4)_{\text{antisymmetric}} = 6. \] (3.1)

In the language of the subgroup SU\(_{\text{spin}}(2) \otimes SU_{\text{ppin}}(2)\), the 6-dimensional irreducible representation is divided into two different irreducible representations
\[ 6 = (3, 1) + (1, 3), \] (3.2)
where 3 is the symmetric representation of SU(2), and 1 is the antisymmetric representation of SU(2). The (3, 1) sector is called a spin-sector and the (1, 3) sector a ppin-sector. The SU\(_{\text{spin}}(2)\) and SU\(_{\text{ppin}}(2)\) operators rotate the spin- and ppin-sectors individually, however they do not mix both sectors. Only the \( U \)-operators can mix them.

#### A. Spin phase and Ppin phase

The ground state and its energy in each sector are obtained by minimizing the Hamiltonian (2.12).

The spin-sector (3, 1) consists of spin-triplet pseudospin-singlet states,
\[ |t_1\rangle = |f_+^\uparrow, b_+^\downarrow\rangle, \quad |t_0\rangle = \frac{1}{\sqrt{2}}(|f_+^\uparrow, b_+^\downarrow\rangle + |f_+^\downarrow, b_+^\uparrow\rangle), \]
\[ |t_\downarrow\rangle = |f_+^\downarrow, b_+^\downarrow\rangle. \] (3.3)

\( t_\downarrow \)'s satisfy hardcore bosonic commutation relations. We call them \( t \)-bosons.

The ppin-sector (1, 3) consists of spin-singlet pseudospin-triplet states,
\[ |\tau_+\rangle = |f_+^\uparrow, f_+^\downarrow\rangle, \quad |\tau_0\rangle = \frac{1}{\sqrt{2}}(|f_+^\uparrow, b_+^\downarrow\rangle - |f_+^\downarrow, b_+^\uparrow\rangle), \]
\[ |\tau_\downarrow\rangle = |b_+^\downarrow, b_+^\downarrow\rangle. \] (3.4)

\( \tau_\delta \)'s also satisfy hardcore bosonic commutation relations. We call them \( \tau \)-bosons.

In the spin-sector, the minimum energy is given by
\[ E_{t_\downarrow} = -\Delta_Z - 2J, \] (3.5)
where the spin stiffness \( J \) is given as
\[ J = \frac{1}{16\sqrt{2\pi}} \frac{\varepsilon^2}{4\pi\varepsilon\ell_B}. \] (3.6)

The ground state of the spin-sector is \( \prod_{i=1}^{N_\Phi} |t_\downarrow\rangle_i \), with the ground-state energy \( N_\Phi E_{t_\downarrow} \).

In the ppin-sector, the minimum energy state is
\[ |v_+\rangle = \frac{\cos \theta - \sin \theta}{2} (|\tau_+\rangle + |\tau_-\rangle) + \frac{\cos \theta + \sin \theta}{\sqrt{2}} |\tau_0\rangle, \] (3.7)
where
\[ \tan \theta = \frac{\varepsilon_{\text{cap}}}{2\Delta_{\text{SAS}} + \sqrt{4\Delta_{\text{SAS}}^2 + \varepsilon_{\text{cap}}^2}}, \] (3.8)
with the energy
\[ E_{v_+} = \frac{1}{2} (\varepsilon_{\text{cap}} - \sqrt{4\Delta_{\text{SAS}}^2 + \varepsilon_{\text{cap}}^2}) - (J + J^d \cos^2 (2\theta)). \] (3.9)

The pseudospin stiffness \( J^d \) is given as
\[ \frac{J^d}{J} = -\sqrt{2} \frac{d}{\pi \ell_B} + \left(1 + \frac{d^2}{\ell_B^2}\right) \frac{e^{d^2/2\ell_B^2} - \text{erf}(d/\sqrt{2}\ell_B)}{\sqrt{\pi}}. \] (3.10)

The ground state of the ppin-sector is \( \prod_{i=1}^{N_\Phi} |v_+\rangle_i \), with the ground-state energy \( N_\Phi E_{v_+} \). With the decrease of the interlayer separation \( d \), the \( v_+ \)-boson exchange energy also decreases, while the \( t_\downarrow \)-boson exchange energy does not change keeping the minimum value \(-2J\). The \( v_+ \)-boson exchange energy yields the minimum value \(-2J\) only when two layers coincide.

Consequently, there are two possible ground states, \( |\Phi_S\rangle = \prod_{i=1}^{N_\Phi} |t_\downarrow\rangle_i \) or \( |\Phi_P\rangle = \prod_{i=1}^{N_\Phi} |v_+\rangle_i \). When \( E_{t_\downarrow} < E_{v_+} \), the ground state is \( |\Phi_S\rangle \), which we call the spin phase since all spins are polarized. On the other hand, when \( E_{v_+} < E_{t_\downarrow} \), the ground state is \( |\Phi_P\rangle \), which we call the ppin phase.

#### B. C-phase

Instead of the SU(4) operators given in (2.5) and (2.7), it is convenient to use such raising, lowering operators,
\[ S_+ = \frac{1}{\sqrt{2}} (S_x + iS_y), \quad S_- = \frac{1}{\sqrt{2}} (S_x - iS_y), \]
\[ P_+ = \frac{1}{\sqrt{2}} (P_x + iP_y), \quad P_- = \frac{1}{\sqrt{2}} (P_x - iP_y), \]
\[ U_{++} = 2S_+ P_+, \quad U_{+-} = 2S_- P_-, \]
\[ U_{-+} = 2S_- P_+, \quad U_{--} = 2S_+ P_-, \]
\[ U_{zz} = 2S_z P_z, \quad U_{zz'} = 2S_z \otimes P_z. \] (3.11)

and Cartan operators; \( S_z, P_+, U_{zz} = 2S_z \otimes P_z \). \( S_+(-) \) raises (lowers) the spin \( z \)-component by one; \( S_+(-)|\tau_0\rangle = |t_{\downarrow(\uparrow)}\rangle \). Similarly, \( P_+(-) \) raises (lowers) the ppin \( z \)-component by one; \( P_+(-)|\tau_0\rangle = |\tau_{+(-)}\rangle \). \( U_{++}, U_{+-}, U_{-+} \) and \( U_{--} \) change not only the spin but also the ppin
z-component by one and mix the spin-sector with the ppin-sector, for instance \( U_{++} |t_t\rangle = |r_+\rangle \). The \( U_{++}, U_{+-} \) and \( U_{+z} \) change the ppin-sector states to the spin-sector states, while the \( U_{--}, U_{-+} \) and \( U_{--} \) do the spin-sector states to the ppin-sector states.

We focus on a Hilbert space spanned by the lowest energy states \( |t_t\rangle, |v_v\rangle \) and search for the lowest energy in this subspace. We abbreviate \( |t_t\rangle, |v_v\rangle \) as \( |t\rangle, |v\rangle \) for simplicity. The effective one body operator which operates on this subspace is represented as

\[
O^{eff} = \sum_i \langle b_i | O(i) | b'_i \rangle b^\dagger_i b'_i, \tag{3.12}
\]

where \( b=t,v \). The matrix representations of the SU(4) operators \( S_a, P_a, U_{ab} \) are

\[
S_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.13a}
\]

\[
P_+ = P_-^\dagger = \frac{\cos(2\theta)}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.13b}
\]

\[
U_{++} = U_{--} = -U_{+-} = -U_{+z} = \frac{\cos \theta - \sin \theta}{2} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \tag{3.13c}
\]

\[
U_{+z} = U_{-z} = \frac{\cos \theta + \sin \theta}{2} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{3.13d}
\]

Matrices of other SU(4) operators are zeros. Due to the mixing of the spin- and the ppin-sectors by \( U_{ab} \)-operators, only the \( U_{ab} \) matrices give off-diagonal elements, which represent the interchange of \( t \) and \( v \)-bosons. The matrix of the direct term \( \Delta H_D \) is given as

\[
H_D = \begin{pmatrix} E^D_t & 0 \\ 0 & E^D_v \end{pmatrix}, \tag{3.14}
\]

where

\[
E^D_t = -\Delta_z, \tag{3.15a}
\]

\[
E^D_v = -\sqrt{\Delta_{SAS}^2 + \left( \frac{\varepsilon_{cap}}{2} \right)^2 + \frac{\varepsilon_{cap}}{2}}. \tag{3.15b}
\]

Next, we consider the matrix elements for the exchange interaction term \( \Delta X \). First we rewrite the exchange interaction term with the use of raising and lowering operators. The spin-spin interaction is rewritten as

\[
S(i) \cdot S(j) = S_{++}(i)S_{--}(j) + S_{+-}(i)S_{+-}(j) + S_{-+}(i)S_{-+}(j). \tag{3.16}
\]

Similarly the ppin-ppin interaction is

\[
P(i) \cdot P(j) = P_{++}(i)P_{--}(j) + P_{+-}(i)P_{+-}(j) + P_{-+}(i)P_{-+}(j). \tag{3.17}
\]

The other exchange interaction is

\[
U_{ab}(i) \cdot U_{ab}(j) = U_{++}(i)U_{--}(j) + U_{+-}(i)U_{+-}(j) + U_{-+}(i)U_{-+}(j) + U_{zz}(i)U_{zz}(j).
\]

(3.18)

To reproduce the exchange interaction \( \Delta X \), the spin and the ppin stiffness are needed in front of these operators. Each of \( U_{++}, U_{--}, U_{+-}, U_{-+} \) exchange interaction parts needs the ppin stiffness \( J^d \) as its coefficient, while each of \( U_{++}, U_{--}, U_{+-}, U_{-+} \) parts does the spin stiffness \( J \). The difference between these two coefficients represents the magnitude of the SU(4)-noninvariant interaction.

Two body operators are effectively described as

\[
O^{eff} = \sum_{ij} \langle b_i O(i) | b'_j \rangle | b'_j \rangle | b'_i \rangle = \sum_{ij} \langle b_i O(i) | b'_j \rangle \otimes | b_j O(j) | b'_j \rangle | b'_i \rangle. \tag{3.19}
\]

One can easily obtain matrix elements of two body operators from the direct product of one body operators. Using the basis of two body Hilbert space: \( |tt\rangle, |tv\rangle, |vt\rangle, |vv\rangle \), we take matrix elements of the exchange interaction term \( \Delta X \). With the use of \( \Delta X \), the matrix of the exchange interaction is written as

\[
H_X \equiv -J
\]

\[
- \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & J^+ + J^- \sin(2\theta) & 0 & 0 \\ 0 & 0 & J^d \cos^2(2\theta) & 0 \\ 0 & 0 & 0 & J^d \cos^2(2\theta) \end{pmatrix}, \tag{3.20}
\]

with \( J^\pm \equiv \frac{1}{2}(J \pm J^d) \). The first term \(-J\) comes from the density-density interaction term. The \((1,1)\) and \((4,4)\) diagonal components in the matrix come from the spin and the ppin exchange interactions individually. Meanwhile the off-diagonal components come from the SU(4) peculiar exchange term \( U_{ab}(i)U_{ab}(j) \). These off-diagonal elements yield an origin of the C-phase.

Consequently, we obtain the effective Hamiltonian,

\[
H^{eff}_{tot} = E^D_t \sum_i v_i^\dagger v_i - J \sum_{<ij>} v_j^\dagger v_i - J^d \cos^2(2\theta) \sum_{<ij>} v_j^\dagger v_j v_i^\dagger v_i - (J^+ + J^- \sin(2\theta)) \sum_{<ij>} (v_j^\dagger t_j^\dagger t_i^\dagger v_i + t_j^\dagger v_i^\dagger t_i^\dagger v_i).
\]

(3.21)

The last term which comes from the off-diagonal elements of the \( U \)-operators flips the \( t \)-boson to the \( v \)-boson and
vice versa. Due to this effect, $t$-bosons are distributed in the $v$-boson condensation state [Fig. 1].

Using the variation method, we find out the ground states of the effective Hamiltonian. The variational function is

$$\Phi = \prod_i [\alpha^\dagger + \beta v_i] |0\rangle. \tag{3.22}$$

Due to the normalization condition, parameters $\alpha, \beta$ must satisfy such constraint,

$$|\alpha|^2 + |\beta|^2 = 1. \tag{3.23}$$

The expectation value of the reduced Hamiltonian reads

$$E = \langle \Phi | H_{\text{eff}} | \Phi \rangle$$
$$= E_D |\alpha|^2 + E_v |\beta|^2 - J - J|\alpha|^4$$
$$- J^d \cos^2(2\theta) |\beta|^4 - 2(J^+ + J^- \sin(2\theta))|\alpha|^2 |\beta|^2, \tag{3.24}$$

which coincides with the energy in the Demler and Das Sarma in the limit $d \to \infty$,

$$E \to E_D |\alpha|^2 + E_v |\beta|^2 - J - J|\alpha|^4 - J(1+\sin(2\theta))|\alpha|^2 |\beta|^2. \tag{3.25}$$

Meanwhile in the SU(4) limit $d \to 0$, it yields

$$E \to E_D |\alpha|^2 + E_v |\beta|^2 - 2J, \tag{3.26}$$

where we have used the constraint. In this SU(4) limit, only two ground states are allowed. One is the case $|\alpha|^2 |\beta|^2 = 1$, where the ground state energy reads

$$E = -\Delta_Z - 2J. \tag{3.27}$$

This is the spin phase. The other case is $|\alpha|^2 |\beta|^2 = 0, 1$, where the ground state energy reads

$$E = -\Delta_{\text{SAS}} - 2J. \tag{3.28}$$

This is the $ppin$ phase. Note the exchange energies are degenerate in two phases, because the exchange term becomes the SU(4) Casimir operator in the SU(4) limit.

Under the constraint, we minimize the energy. There arise three different kinds of ground states,

$$|\alpha|^2 |\beta|^2 = 1 \text{ if } t_{\text{min}} < 0, \tag{3.29a}$$

$$|\alpha|^2 |\beta|^2 = (\sqrt{1-t_{\text{min}}}, \sqrt{1-t_{\text{min}}}) \text{ if } 0 < t_{\text{min}} < 1, \tag{3.29b}$$

$$|\alpha|^2 |\beta|^2 = (0, 1) \text{ if } 1 < t_{\text{min}}, \tag{3.29c}$$

where

$$t_{\text{min}} = \frac{E_D^2 - E_v^2 - 2J^\dagger(1-\sin^2(2\theta))}{4J^\dagger \sin(2\theta) + 2J^2 \sin^2(2\theta)}. \tag{3.30}$$

With the use of $t_{\text{min}}$, the energy is given as

$$E = E_D^2 - 2J^\dagger(1-\sin^2(2\theta)) + J^2 \sin^2(2\theta)t_{\text{min}}^2. \tag{3.31}$$

At $t_{\text{min}} = 0$, the energy coincides with the spin phase energy. The wave function reads

$$|\Phi_S \rangle = \prod_i v_i |0\rangle. \tag{3.32}$$

Therefore, the spin phase can be interpreted as $t$-boson condensation phase.

At $t_{\text{min}} = 1$, the energy coincides with the $ppin$ phase energy. The wave function reads

$$|\Phi_P \rangle = \prod_i v_i |0\rangle. \tag{3.33}$$

The $ppin$ phase is a $v$-boson condensation phase.

The last case is the C-phase, where the wave function is given as

$$|\Phi_C \rangle = \prod_i [\alpha^\dagger + \beta v_i] |0\rangle, \tag{3.34}$$

with the energy. Both $t$-bosons and $v$-bosons simultaneously condense in this phase.

To understand magnetic properties in each phase, we study a spin expectation value in each layer. The $t$-bosons are the highest weight states in the spin triplet. Therefore, in the spin phase, the spins in both layers are fully polarized,

$$\langle \Phi_S | S^i | \Phi_S \rangle = \frac{1}{2} (0, 0, 1), \tag{3.35a}$$

$$\langle \Phi_S | S^b | \Phi_S \rangle = \frac{1}{2} (0, 0, 1). \tag{3.35b}$$

The $v$-bosons condense, which are spin singlet states. Hence, in the $ppin$ phase, no spins are polarized at all, where the expectation values of the spin operators in the front layer and in the back layer are individually 0,

$$\langle \Phi_P | S^i | \Phi_P \rangle = (0, 0, 0), \tag{3.36a}$$

$$\langle \Phi_P | S^b | \Phi_P \rangle = (0, 0, 0). \tag{3.36b}$$

FIG. 1: A $t$-boson hops to the nearest neighbor site by the SU(4) peculiar exchange interactions stemmed from $U$-operators.
In the C-phase, the $t$-bosons and $v$-bosons simultaneously condense. With the use of Eq. (3.31), expectation values of SU(4) isospins with the C-phase wave function are given as

$$\langle S^i_C \rangle = |\alpha|^2, \quad \langle P_x \rangle_C = \cos(2\theta)|\beta|^2,$$
$$\langle U_{xy} \rangle_C = -(\cos \theta - \sin \theta) \text{Im}(\alpha^* \beta),$$
$$\langle U_{xz} \rangle_C = -(\cos \theta + \sin \theta) \text{Re}(\alpha^* \beta),$$
$$\langle U_{yy} \rangle_C = (\cos \theta - \sin \theta) \text{Re}(\alpha^* \beta),$$
$$\langle U_{yz} \rangle_C = -(\cos \theta + \sin \theta) \text{Im}(\alpha^* \beta).$$

(3.37)

Other expectation values of SU(4) isospins are zeros. $U$-operators have non-zero values only in the C-phase, which become a Néel order parameter as we shall see below [Table I].

The spin operators in two layers are given as

$$\langle S^i_C \rangle = \frac{1}{2}(0, 0, |\alpha|^2) - \frac{\cos \theta + \sin \theta}{2} (\text{Re}(\alpha^* \beta), \text{Im}(\alpha^* \beta), 0),$$

(3.38a)

$$\langle S^b_C \rangle = \frac{1}{2}(0, 0, |\alpha|^2) + \frac{\cos \theta + \sin \theta}{2} (\text{Re}(\alpha^* \beta), \text{Im}(\alpha^* \beta), 0),$$

(3.38b)

where we have used the relations, $S^i_a = \frac{1}{2}(S_a + U_{ax}), S^b_a = \frac{1}{2}(S_a - U_{ax})$. The C-phase exhibits antiferromagnetic properties in the two dimensional layer directions, while ferromagnetic properties in the perpendicular direction.

The spin Néel parameter $N_{spin} = S^x - S^b$, which is equivalent to $U$-operator $U_{ax}$, is introduced as an order parameter of the C-phase. In the spin and ppin phases, its expectation values are trivial: $\langle \Phi_S | N_{spin} | \Phi_S \rangle = 0$, while in the C-phase, it yields a nonzero value,

$$\langle N_{spin} \rangle_C = -(\cos \theta + \sin \theta) (\text{Re}(\alpha^* \beta), \text{Im}(\alpha^* \beta), 0).$$

(3.39)

Thus, $N_{spin}$ takes a finite value only in the C-phase, which represents an antiferromagnetic property of spins.

The ppin operators on $x$-polarized two spin-states $|\uparrow_x \rangle$ and $|\downarrow_x \rangle$ are given as

$$\langle P^{x+}_C \rangle = \frac{1}{2} \cos(2\theta)|\beta|^2, 0, 0)$$
$$-\frac{1}{2}(0, (\cos \theta - \sin \theta) \text{Im}(\alpha^* \beta), (\cos \theta + \sin \theta) \text{Re}(\alpha^* \beta)), \quad (3.40a)$$

$$\langle P^{x-}_C \rangle = \frac{1}{2} \cos(2\theta)|\beta|^2, 0, 0)$$
$$+ \frac{1}{2}(0, (\cos \theta - \sin \theta) \text{Im}(\alpha^* \beta), (\cos \theta + \sin \theta) \text{Re}(\alpha^* \beta)), \quad (3.40b)$$

where we have used the relations, $P^{x+}_a = \frac{1}{2}(P_a + U_{ax}), P^{x-}_a = \frac{1}{2}(P_a - U_{ax})$. Physically $P^{x+}_C, P^{x-}_C$ represent the ppin operator on the $x$-spin up, $x$-spin down state individually. For instance, the electron in the front-layer

| Phase | Spin phase | C-phase | Ppin phase |
|-------|------------|---------|-----------|
| Order Parameter | $S$ | $U_{ab}$ | $P$ |

TABLE I: The order parameters in the three phases. By including the C-phase, all SU(4) operators appear as order parameters.

with $x$ spin-up $|f^x \rangle = \frac{1}{\sqrt{2}}(|f \uparrow \rangle + |f \downarrow \rangle)$ is the eigenstate of $P^{x+}_a$ with eigenvalue $1$. In the C-phase, the ppins exhibit antiferromagnetic properties in the two dimensional spin $y$-$z$ polarized plane, while ppin ferromagnetic properties in the spin $x$ direction.

The ppin Néel parameter $N_{ppin} = P^{x+} - P^{x-}$, which is equivalent to $U$-operator $U_{ax}$, is introduced as another order parameter of the C-phase. In the spin and ppin phase, $\langle \Phi_P | N_{ppin} | \Phi_P \rangle = 0$, while in the C-phase,

$$\langle N_{ppin} \rangle_C = -\frac{1}{2}(0, (\cos \theta - \sin \theta) \text{Im}(\alpha^* \beta), (\cos \theta + \sin \theta) \text{Re}(\alpha^* \beta)).$$

(3.41)

Thus, the ppins are also “canted” in the C-phase.

Let us examine the case $\theta=0$, where $|v\rangle$ becomes an eigenstate of the operator $P^z$ with the eigenvalue 1. There are no quantum fluctuations around the $P_z$ axis, where the semiclassical interpretation becomes much clear. The magnitude of the SU(4) isospin expectation value becomes unity. The spin and the ppin Néel order parameters are given as

$$\langle N_{spin} \rangle_C = -(\text{Re}(\alpha^* \beta), \text{Im}(\alpha^* \beta), 0), \quad (3.42a)$$
$$\langle N_{ppin} \rangle_C = -(0, \text{Im}(\alpha^* \beta), \text{Re}(\alpha^* \beta)). \quad (3.42b)$$

They are related to each other with the interchange of $x, z$ component, because in the SU(4) formalism the spin and the ppin are treated equivalently. The interchange of $x, z$ comes from the fact that $|f\rangle$ is an eigenstate of $z$ spin operator $S_z$, while $|v\rangle$ is an eigenstate of $x$ ppin operator $P_x$ at $\theta=0$.

**IV. SU(4) COHERENT EFFECTS TO THE C-PHASE**

With the use of 3.30, we investigate SU(4) coherent effects to the C-phase by varying the interlayer separation $d$ [Fig. 4].

It is obvious from these diagrams, when two layers are close, the C-phase region becomes narrow. For instance, in Fig 4, which is in the case $d = \frac{1}{4} \ell_B$, the C-phase almost vanishes being eaten up by the ppin phase. The SU(4) coherence disturbs the realization of the C-phase.

As we have discussed in Section III, in the SU(4) limit the C-phase disappears and the spin phase suddenly
As we have seen in Subsection III B, the $U$-operators in \( (3.13c) \) change the $v$-states to the $t$-states and vice versa. $U$-operators have such effects to distribute $t$ and the $v$ bosons simultaneously. However the $U$-operators in \( (3.13d) \) work “contrary” to the $U$-operators in \( (3.13c) \), i.e. these two kinds of $U$-operators suppress their effects in each other. The strength of the $U$-operators in \( (3.13c) \) is the ppin stiffness $J_d$, while the strength of the $U$-operators in \( (3.13d) \) is the spin stiffness $J$. The difference between the spin and ppin stiffness generates a net effect to condense $t$ and $v$-bosons simultaneously. When two layers are very close, the ppin stiffness $J_d$ are nearly equal to the spin stiffness $J$. Hence their effects are almost suppressed. Consequently, the C-phase region is very narrow in the SU(4) coherent region.

V. SUMMARY AND DISCUSSION

We have investigated an origin of the C-phase in the SU(4) context. We have seen both spins and ppins are “canted” in the C-phase. The $U$-operators become an order parameter of the C-phase. Using the Schwinger boson picture, we have shown that SU(4) coherence decreases the C-phase region. Microscopically, in the C-phase $t$ and $v$-bosons condense simultaneously. Each $U$-operator has an effect to induce simultaneous condensation of such bosons. However, in the SU(4) coherent region, $U$-operators compete in each other and suppress their effects. Consequently the C-phase is unlikely realized in such a region.

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