Spatial-decay of solutions to the quasi-geostrophic equation with the critical and supercritical dissipation

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Abstract

The initial value problem for the two-dimensional dissipative quasi-geostrophic equation derived from geophysical fluid dynamics is studied. The dissipation of this equation is given by the fractional Laplacian. It is known that the half Laplacian is a critical dissipation for the quasi-geostrophic equation. The global existence of solutions upon the suitable condition is also well known, and that solutions of a fractional dissipative equation decay with a polynomial order as the spatial variable tends to infinity. In this paper, far field asymptotics of solutions to the quasi-geostrophic equation are given in the critical and the supercritical cases. Those estimates are derived from the energy methods for the difference between the solution and its asymptotic profile.

Keywords: quasi-geostrophic equation, anomalous diffusion, asymptotic profiles, decay estimates, spatial decay

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1. Introduction

The quasi-geostrophic equation is derived from the model of geophysical fluid dynamics (see [11]). Here we consider the following initial value problem:

\[
\begin{align*}
\partial_t \theta + (-\Delta)^{\alpha/2} \theta + \nabla \cdot (\theta \nabla^\perp \psi) &= 0, & t > 0, & x \in \mathbb{R}^2, \\
(-\Delta)^{1/2} \psi &= \theta, & t > 0, & x \in \mathbb{R}^2, \\
\theta(0, x) &= \theta_0(x), & x \in \mathbb{R}^2,
\end{align*}
\]

where \(0 < \alpha < 2\), \(\nabla^\perp = (-\partial_2, \partial_1)\) and the initial temperature \(\theta_0\) is the given function. The real valued function \(\theta\) denotes the potential temperature and \(\nabla^\perp \psi\) is the fluid velocity. The quasi-geostrophic equation is important also in meteorology. When \(\alpha = 1\), this case is called critical, since its structure is quite similar to 3D Navier–Stokes equations. The case \(\alpha > 1\) and \(\alpha < 1\) are called subcritical and supercritical respectively. In the subcritical case, it is well known that solutions exist uniquely and globally in time without any smallness conditions for initial data. In the critical and the supercritical cases, uniqueness and global existence are discussed with small initial data in scale invariant spaces by many authors (see [9, 10, 13, 21, 22, 32]). The global regularity in the critical case is proved in [6, 12, 26]. Finite time blow-up of solutions with non small initial data is also discussed for the supercritical case in [28].

The next issue is to consider the asymptotic behavior of solutions as \(t\) or \(|x|\) goes to infinity. In this paper we treat the global regular solution of (1) which satisfies

\[
\theta \in C([0, \infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))
\]

and

\[
\int_{\mathbb{R}^2} \theta(t, x) dx = \int_{\mathbb{R}^2} \theta_0(x) dx \quad \text{and} \quad \|\theta(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(1 + t)^{-2/\alpha}
\]

for \(t > 0\). Those properties are confirmed initial data that is small and smooth. Moreover, if the initial data is in \(H^\sigma(\mathbb{R}^2)\) for some \(\sigma > 2\) and sufficiently small, then

\[
\|(-\Delta)^{\sigma/2} \theta(t)\|_{L^2(\mathbb{R}^2)} \leq C(1 + t)^{-\frac{\sigma}{2} - \frac{2}{d}}
\]

holds for \(t > 0\) (see proposition 2.5 in section 2).

Asymptotic behavior of the solutions of an equation of this type as \(t \to +\infty\) is discussed in several preceding works (see for example [1, 7, 14, 15, 17, 19, 20, 24, 27, 34]). In this paper we study spatial decay of the solution of (1) by employing the following method. For an unknown function \(\varphi\), and a given and bounded function \(\Phi\), now we assume that \(\|x|^{\mu} (\varphi - \Phi)\|_{L^2(\mathbb{R}^2)} < +\infty\) and \(\|x|^{\mu} \Phi\|_{L^2(\mathbb{R}^2)} = +\infty\), where \(\mu\) is some positive constant. Then we can say that \(\Phi\) has the same spatial profile as \(\varphi\). This idea firstly is applied to the Navier–Stokes flow and an asymptotic profile of the velocity as \(|x| \to +\infty\) is derived (see [3, 4]). The solution of the quasi-geostrophic equation of subcritical case is estimated by the general theory via [5] which is developed for parabolic-type equations. Namely, if \(1 < \alpha < 2\), then the solution fulfills that \(\|x|^{\mu} (\varphi - MG_\alpha(t))\|_{L^2(\mathbb{R}^2)} \leq C\) and \(\|x|^{\mu} G_\alpha(t)\|_{L^2(\mathbb{R}^2)} = +\infty\) for \(1 + \alpha \leq \mu < 2 + \alpha\) and \(t > 0\), where \(M = \int_{\mathbb{R}^2} \theta_0(x) dx\) and \(G_\alpha(t) = (2\pi)^{-1} e^{-t|\cdot|^2} \mathcal{F}^{-1} |\cdot|^{\alpha-1}\).
\[ \theta(t) = G_\alpha(t) \ast \theta_0 - \int_0^t \nabla G_\alpha(t-s) \ast (\theta \nabla (-\Delta)^{-1/2} \theta)(s) \, ds. \]  

(5)

In the subcritical case, \( \nabla G_\alpha(t-s) \) in the nonlinear term is integrable in \( s \in (0, t) \). Therefore the \( L^p - L^q \) estimate for \( \nabla G_\alpha \) leads to the assertion. But, in the case \( 0 < \alpha \leq 1 \), this term has a singularity. Indeed, since the scaling property gives that 
\[ \| \nabla G_\alpha(t-s) \|_{L^p(R^2)} = (t-s)^{-2(1-1/p)/\alpha-1/\alpha} \| \nabla G_\alpha(1) \|_{L^p(R^2)}, \]
\( \nabla G_\alpha(t-s) \) is not integrable at \( s = t \). To avoid this singularity, we use a new weighted estimate (see proposition 2.6) which firstly is developed for the drift-diffusion equation (see [37]). The goal of this paper is to derive spatial decay of the solution of (1) for \( 0 < \alpha \leq 1 \). The main assertion is as follows.

**Theorem 1.1.** Let \( 0 < \alpha \leq 1, \sigma > 2, \theta_0 \in H^\sigma(R^2), |x|\theta_0 \in L^1(R^2) \) and \( |x|^2 \theta_0 \in L^2(R^2) \cap L^\infty(R^2) \) and \( \| \theta_0 \|_{H^\sigma(R^2)} \) be sufficiently small, depending on \( \alpha \) and \( \sigma \). Assume that the solution \( \theta \) of (1) fulfills (2) and (3). Then

\[ \left\| |x|^2(\theta(t) - MG_\alpha(t)) \right\|_{L^2(R^2)} \leq \begin{cases} C(\log(2+t))^{3/2}, & \alpha = 1, \\ C(\log(2+t))^{1/2}, & 0 < \alpha < 1 \end{cases} \]

holds for \( t > 0 \), where \( M = \int_{R^2} \theta_0(x) \, dx \).

Here \( M \in R \) since \( \| \theta_0 \|_{L^2(R^2)} \leq C \| (1 + |x|^2) \theta_0 \|_{L^2(R^2)} < +\infty \). The smallness of \( \theta_0 \) is required in order to give (4) and employed in (19). We emphasize that \( \| |x|^2 G_\alpha(t) \|_{L^2(R^2)} = +\infty \) for \( t > 0 \) since \( |x|^{2+\alpha} G_\alpha(t,x) \to t \alpha 2^{\alpha-1} \pi^{-2} \sin(\alpha \pi/2) \Gamma(1+\alpha/2) \) as \( |x| \to +\infty \) (see [2]).

Therefore theorem 1.1 states that the decay-rate of \( \theta \) as \( |x| \to +\infty \) is represented by \( MG_\alpha \). As mentioned above, similar inequalities as in theorem 1.1 has been proved in [5] with \( \alpha > 1 \). We note that the left-hand side of the inequality is bounded by a positive constant (no logarithmic growth) in [5]. The authors expect that the logarithmic growth in theorem 1.1 can be excluded, since the solution of the linear problem yields that \( \| |x|^2(G_\alpha(t) \ast \theta_0 - MG_\alpha(t)) \|_{L^2(R^2)} \leq C \) for \( t > 0 \) and \( 0 < \alpha < 2 \) (see lemma 2.7 in section 2). Namely, it is expected that spatial decay structure of the quasi-geostrophic equation in the critical and supercritical cases is similar as in the subcritical.

In the proof of theorem 1.1, we put \( v = \theta - G_\alpha \ast \theta_0 \) for the solution \( \theta \) of (1). Namely \( v \) is the nonlinear term on (5). Theorem 1.1 is shown mainly by applying the energy methods. Throughout this procedure, some basic properties of \( G_\alpha \) and the Riesz transform via the harmonic analysis are used. Moreover, we employ the weighted estimate and the estimate for the derivations of \( \theta \) which also are derived from the energy methods (see propositions 2.5 and 2.6). In particular, we estimate \( \| |x|^2 \theta \|_{L^2(R^2)} \) with \( q > 2/\alpha \) in proposition 2.6 by using the altered energy method. We note that \( \| |x|^2 \theta \|_{L^2(R^2)} \) is estimated in [37] for solutions to the drift-diffusion equation with fractional dissipation. In the proof of propositions 2.5 and 2.6, the commutator estimates play a crucial role.

Spatial decay of solutions is an important problem not only for the quasi-geostrophic equation but also in studies for several models. We mention the porous media equation, the modified quasi-geostrophic equation and the modified porous media equation as an example (see [8, 30, 38]).

**Notation.** We define the Fourier transform and its inverse by \( \mathcal{F}[\varphi](\xi) = (2\pi)^{-1} \int_{R^2} e^{-ix \cdot \xi} \varphi(x) \, dx \) and \( \mathcal{F}^{-1}[\varphi](x) = (2\pi)^{-1} \int_{R^2} e^{ix \cdot \xi} \varphi(\xi) \, d\xi \), where \( i = \sqrt{-1} \). We denote the derivations by \( \partial_i = \partial/\partial \xi_i, \partial_j = \partial/\partial x_j \) \( (j = 1, 2) \), \( \nabla = (\partial_1, \partial_2) \), \( \nabla^2 = (-\partial_2, \partial_1) \), \( \Delta = \partial_x^2 + \partial_y^2 \) and \( (-\Delta)^{\alpha/2} \varphi = \mathcal{F}^{-1}[(|\xi|^\alpha \mathcal{F}[\varphi])] \). Also we define \( (-\Delta)^{-\sigma/2} \varphi = \mathcal{F}^{-1}[(|\xi|^{-\sigma} \mathcal{F}[\varphi])] \) for
0 < σ < 2. The Hölder conjugate of $1 \leq p \leq \infty$ is denoted by $p'$, i.e. $1/p + 1/p' = 1$. The Riesz transform is defined by $R_j \varphi = \partial_j (-\Delta)^{-1/2} \varphi = \mathcal{F}^{-1}[i \xi_j |\xi|^{-1} \mathcal{F}[\varphi]] \ (j = 1, 2)$. For $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2 = (\mathbb{N} \cup \{0\})^2$, $|\beta| = \beta_1 + \beta_2$. For some operators $A$ and $B$, we denote the commutator by $[A, B] = AB - BA$. Various non-negative constants are denoted by $C$.

2. Preliminaries

In this section, we prepare some inequalities for several functions and the solution. From the scaling property, the fundamental solution fulfills that
\[ G_\alpha(t, x) = t^{-2/\alpha} G_\alpha(1, t^{-1/\alpha} x) \] (6)
for $(t, x) \in (0, +\infty) \times \mathbb{R}^2$. Furthermore
\[ |\nabla \beta G_\alpha(1, x)| \leq C_{\beta} (1 + |x|^2)^{-1 - \frac{\beta}{2}} \] (7)
is satisfied for $\beta \in \mathbb{Z}_+^2$ and $x \in \mathbb{R}^2$. When $\alpha = 1$, this estimate is clear since $G_\alpha$ is the Poisson kernel in this case. For the case $0 < \alpha < 1$, we use the following Hörmander–Mikhlin inequality.

Lemma 2.1 (Hörmander–Mikhlin inequality [18, 31]). Let $N \in \mathbb{Z}_+$, $0 < \mu \leq 1$ and $\lambda = N + \mu - 2$. Assume that $\varphi \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ satisfies the following conditions:
\[ \begin{align*}
&\bullet \ |\nabla^\gamma \varphi| \in L^1(\mathbb{R}^2) \text{ for any } \gamma \in \mathbb{Z}_+^2 \text{ with } |\gamma| \leq N; \\
&\bullet \ |\nabla^\gamma \varphi(\xi)| \leq C_\gamma |\xi|^{|\gamma|} \text{ for } \xi \neq 0 \text{ and } \gamma \in \mathbb{Z}_+^2 \text{ with } |\gamma| \leq N + 2.
\end{align*} \]

Then
\[ \sup_{x \neq 0} (|x|^{2+\lambda} |\mathcal{F}^{-1}[\varphi](x)|) < +\infty \]
holds.

The proof and the details of this lemma is in [35]. We confirm (7) when $0 < \alpha < 1$. If $|\beta| = 2k$ for $k \in \mathbb{Z}_+$, then we put $\varphi(\xi) = \nabla (-\Delta)^k (\xi^\beta e^{-|\xi|^2})$. Then $\varphi$ satisfies the conditions in lemma 2.1 with $N = 1$, $\mu = \alpha$ and $\lambda = \alpha - 1$. Hence, since $\nabla^\beta G_\alpha(1, x)$ is bounded, we see (7). When $|\beta| = 2k + 1$, we put $\varphi(\xi) = (-\Delta)^{k+1} (\xi^\beta e^{-|\xi|^2})$ in the above procedure and derive (7). The following inequality plays an important role in the energy method.

Lemma 2.2 (Stroock–Varopoulos inequality [29]). Let $0 \leq \alpha \leq 2$, $q \geq 2$ and $f \in W^{\alpha, q}(\mathbb{R}^2)$. Then
\[ \int_{\mathbb{R}^2} |f|^q - 2 f (-\Delta)^{\alpha/2} f dx \geq \frac{2}{q} \int_{\mathbb{R}^2} \left| (-\Delta)^{\alpha/4} |f|^q / 2 \right|^2 dx \]
holds.

For the proof of this lemma, see [13, 21]. The fractional integral $(-\Delta)^{-\sigma/2}$ for $0 < \sigma < 2$ is defined by $(-\Delta)^{-\sigma/2} \varphi = \mathcal{F}^{-1}[|\xi|^{-\sigma} \mathcal{F}[\varphi]]$ and represented by
\[ (-\Delta)^{-\sigma/2} \varphi(x) = \gamma_\sigma \int_{\mathbb{R}^2} \frac{\varphi(y)}{|x - y|^{2-\sigma}} dy \] (8)
for some constant $\gamma_\sigma$ (see [36, 39]). For this integral we see the following inequality of the Sobolev type.

**Lemma 2.3 (Hardy–Littlewood–Sobolev’s inequality [36, 39]).** Let $0 < \sigma < 2$, $1 < p < 2/\sigma$ and $1/p_\ast = 1/p - \sigma/2$. Then there exists a positive constant $C$ such that

$$
\|(-\Delta)^{-\sigma/2}\varphi\|_{L^{p_\ast}(\mathbb{R}^d)} \leq C_{\text{HLS}} \|\varphi\|_{L^p(\mathbb{R}^d)}
$$

for any $\varphi \in L^p(\mathbb{R}^d)$.

The following estimate is due to [25].

**Lemma 2.4 (Kato–Ponce’s commutator estimates [22, 25]).** Let $s > 0$ and $1 < p < \infty$. Assume that the solution $\psi$ of (1) satisfies (2) and (3). Then (4) holds.

**Proposition 2.5.** Let $\sigma > 0$, $\theta_0 \in H^s(\mathbb{R}^d)$ and $\|\theta_0\|_{H^s(\mathbb{R}^d)}$ be small, depending on $\alpha$ and $\sigma$. Assume that the solution $\theta$ of (1) satisfies (2) and (3). Then (4) holds.

**Proof.** Similar argument as in the proof of proposition 2.7 in [37] leads that

$$
\frac{1}{2} \int (1 + t)^{2\gamma} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^d)} + \int_0^t (1 + s)^{2\gamma} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^d)} \, ds
$$

$$
= \frac{1}{2} \|(-\Delta)^{\gamma/2}\theta_0\|^2_{L^2(\mathbb{R}^d)} + \gamma \int_0^t (1 + s)^{2\gamma - 1} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^d)} \, ds
$$

$$
- \int_0^t (1 + s)^{2\gamma} \int_{\mathbb{R}^d} (-\Delta)^{\gamma/2}\theta \left[(-\Delta)^{\gamma/2}, \nabla^\perp \psi\right] \cdot \nabla \theta \, dyds
$$

(9)

for $0 < \gamma < \sigma$ and $\gamma > (1 + \sigma)/\alpha$. For the last term, we apply lemma 2.4 with $p = 2\ast$ and $p_2 = p_3 = 2\ast$, where $1/2\ast = 1/2 - \alpha/4$, and $p_1 = p_4 = 2/\alpha$, and lemma 2.3, then

$$
\left| \int_{\mathbb{R}^d} (-\Delta)^{\gamma/2}\theta \left[(-\Delta)^{\gamma/2}, \nabla^\perp \psi\right] \cdot \nabla \theta \, dy \right| \leq 2C_{\text{KP}} \|\nabla\theta\|_{L^{2/\alpha}(\mathbb{R}^d)} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^d)}
$$

$$
\leq 2C_{\text{KP}} C_{\text{HLS}} \|(-\Delta)^{1-\alpha/2}\theta\|_{L^{2}(\mathbb{R}^d)} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^d)},
$$

where $C_{\text{KP}}$ and $C_{\text{HLS}}$ are defined as in proposition 2.4 with $1/p = 1/2 + \alpha/4$, $1/p_2 = 1/p_3 = 1/2 - \alpha/4$ and $p_1 = p_4 = 2/\alpha$ and lemma 2.3 with $p = 2$ and $\sigma = \alpha$, respectively, and they depend on $\alpha$ and $\sigma$. Here $\|(-\Delta)^{1-\alpha/2}\theta\|_{L^{2}(\mathbb{R}^d)}$ is bounded by $\|\theta_0\|_{H^s(\mathbb{R}^d)}$ (see theorem 3.1 in [22]). Thus, if $\|\theta_0\|_{H^s(\mathbb{R}^d)}$ is sufficiently small, depending on $\alpha$ and $\sigma$, then

$$
\left| \int_{\mathbb{R}^d} (-\Delta)^{\gamma/2}\theta \left[(-\Delta)^{\gamma/2}, \nabla^\perp \psi\right] \cdot \nabla \theta \, dy \right| \leq \frac{1}{2} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^d)},
$$

(10)
By Gagliardo–Nirenberg’s inequality (see \cite{16, 23, 33}) and the Young inequality, we see for the second term of (9) that
\[
\gamma \int_0^1 (1 + s)^{2\gamma - 1} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^2)} \, ds \\
\leq C \delta \int_0^1 (1 + s)^{2\gamma - 1 - \frac{\gamma}{2}} \|\theta\|^2_{L^2(\mathbb{R}^2)} \, ds + \delta \int_0^1 (1 + s)^{2\gamma} \|(-\Delta)^{\frac{\gamma}{2}}\theta\|^2_{L^2(\mathbb{R}^2)} \, ds
\]
for any small $\delta > 0$. We see from (3) that the first term of this inequality is bounded by $(1 + t)^{2\gamma-2/\alpha-2\varsigma/\alpha}$. The estimate (11) is guaranteed if $\varsigma < 2 - \alpha/2$. Hence, by applying (10) and (11) into (9), we see (4) with $\varsigma < 2 - \alpha/2$ instead of $\sigma$. Choosing $\varsigma < 4 - \alpha$, we have from Gagliardo–Nirenberg’s inequality that
\[
\gamma \int_0^1 (1 + s)^{2\gamma - 1} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^2)} \, ds \\
\leq C \delta \int_0^1 (1 + s)^{2\gamma - 1 - \frac{\gamma}{2}} \|(-\Delta)^{\gamma/2}\theta\|^2_{L^2(\mathbb{R}^2)} \, ds + \delta \int_0^1 (1 + s)^{2\gamma} \|(-\Delta)^{\frac{\gamma}{2}}\theta\|^2_{L^2(\mathbb{R}^2)} \, ds
\]
for some $\varsigma < 2 - \alpha/2$. Here (4) with $\varsigma$ instead of $\sigma$ yields that the first term of this inequality is bounded by $(1 + t)^{2\gamma-2/\alpha-2\varsigma/\alpha}$. Thus (9) together with (10) and (12) gives (4) with $\varsigma < 4 - \alpha$ instead of $\sigma$. By repeating this procedure, we can choose $\varsigma = \sigma$ and conclude the proof.

For this estimate (4), suitable conditions for the initial data are discussed in \cite{22}. The solution of (1) is included in the weighted Lebesgue spaces.

**Proposition 2.6.** Let $q > 2/\alpha$ and $|x|^2 \theta_0 \in L^q(\mathbb{R}^2)$. Assume that the solution $\theta$ of (1) satisfies (2) and (3). Then
\[
\| |x|^2 \theta(t) \|_{L^q(\mathbb{R}^2)} \leq C(1 + t)^{\frac{\delta}{2}}
\]
for $t > 0$.

**Proof.** We put $\Theta(t, x) = |x|^2 \theta(t, x)$, then we see
\[
\partial_t \Theta + (-\Delta)^{\alpha/2} \Theta + \nabla \cdot (\theta \nabla^\perp \psi) = \left[(-\Delta)^{\alpha/2}, |x|^2\right] \theta - |x|^2 \nabla \cdot (\theta \nabla^\perp \psi).
\]
Here
\[
\left[(-\Delta)^{\alpha/2}, |x|^2\right] \theta = F^{-1}[\left[|\xi|^\alpha, -\Delta\right]] \theta \\
= F^{-1}[\alpha(\alpha - 1)|\xi|^{\alpha-2}\partial_x^2 + 2\alpha|\xi|^{\alpha-2}\xi \cdot \nabla \theta] \\
= \alpha(\alpha - 1)(-\Delta)^{(\alpha-2)/2} \theta - 2\alpha(-\Delta)^{(\alpha-2)/2} \nabla \cdot (\theta \partial_x) \quad (13)
\]
and $\left[|x|^2, \nabla \cdot (\theta \nabla^\perp \psi)\right] = -2x \cdot (\theta \nabla^\perp \psi)$. Hence
\[
\frac{1}{q} \frac{d}{dt} \|\Theta(t)\|_{L^q(\mathbb{R}^2)}^q + \frac{2}{q} \|(-\Delta)^{\alpha/4}((\Theta|q/2)^2)\|_{L^q(\mathbb{R}^2)}^q \\
\leq \alpha(\alpha - 1) \int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta(-\Delta)^{(\alpha-2)/2}\theta dx \\
- 2\alpha \int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta(-\Delta)^{(\alpha-2)/2}\nabla \cdot (x\theta) dx + 2 \int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta x \cdot (\theta\nabla^\perp \psi) dx.
\]

Here we used lemma 2.2 and the fact that \(\int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta \nabla \cdot (\Theta \nabla^\perp \psi) dx = 0\). For \(0 < \gamma < 2/\alpha q\), we multiply this inequality by \((1 + t)^{-\gamma q}\), integrate over \((0, t)\) and give that

\[
(1 + t)^{-\gamma q} \|\Theta(t)\|_{L^q(\mathbb{R}^2)}^q + \gamma q \int_0^t (1 + s)^{-\gamma q - 1} \|\Theta(s)\|_{L^q(\mathbb{R}^2)}^q ds \\
+ 2 \int_0^t (1 + s)^{-\gamma q} \|\Theta\|_{L^q(\mathbb{R}^2)}^q ds
\leq \|x\|_{L^q(\mathbb{R}^2)}^q + \alpha(\alpha - 1) q \int_0^t (1 + s)^{-\gamma q} \int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta(-\Delta)^{(\alpha-2)/2}\theta dx ds \\
- 2\alpha q \int_0^t (1 + s)^{-\gamma q} \int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta(-\Delta)^{(\alpha-2)/2} \nabla \cdot (x\theta) dx ds + 2 \int_0^t (1 + s)^{-\gamma q} \int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta x \cdot (\theta\nabla^\perp \psi) dx ds.
\]

(14)

Hardy–Littlewood–Sobolev’s inequality with \(\sigma = 2 - \alpha\) provides that

\[
\int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta(-\Delta)^{(\alpha-2)/2}\theta dx \leq \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1} \|\Theta(-\Delta)^{(\alpha-2)/2}\theta\|_{L^q(\mathbb{R}^2)}^{q-1}
\leq C \|\theta\|_{L^q(\mathbb{R}^2)} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1} \leq C(1 + t)^{-1 + \frac{\alpha}{q}} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1},
\]

where \(1/r = 1/q + (2 - \alpha)/2\). Thus

\[
\int_0^t (1 + s)^{-\gamma q} \int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta(-\Delta)^{(\alpha-2)/2}\theta dx ds \\
\leq C \int_0^t (1 + s)^{-\gamma q - 1} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1} ds
\leq C \delta \int_0^t (1 + s)^{-\gamma q - 1 + \frac{\alpha}{q}} ds + \delta \int_0^t (1 + s)^{-\gamma q - 1} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1} ds
\]

for \(\delta > 0\). Similarly, for \(1/r = 1/q + 1/2 - \alpha/2\),

\[
\int_{\mathbb{R}^2} |\Theta|^{q-2}\Theta(-\Delta)^{(\alpha-2)/2} \nabla \cdot (x\theta) dx \leq \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1} \\
\leq C \|\theta\|_{L^q(\mathbb{R}^2)} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1} \leq C \|\theta\|_{L^q(\mathbb{R}^2)}^{1/2} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1/2} \leq C(1 + t)^{-1 + \frac{\alpha}{q}} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1/2},
\]

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Therefore
\[
\int_0^t (1 + s)^{-\gamma q} \int_{\mathbb{R}^2} |\Theta|^q \|\Theta\|_{L^q} (-\Delta)^{(\alpha-2)/2} \nabla \cdot (x\theta) \, dx \, ds
\leq C \int_0^t (1 + s)^{-\gamma q - 1 + \frac{\alpha}{q}} \|\Theta\|_{L^q}^{-1/2} \|\theta\|_{L^q}^{1/2} \, ds
\leq C_\delta \int_0^t (1 + s)^{-\gamma q - 1 + \frac{\alpha}{q}} \, ds + \delta \int_0^t (1 + s)^{-\gamma q - 1} \|\Theta\|_{L^q}^{q/2} \, ds.
\]

Since \( \|\nabla^\perp \psi\|_{L^r(\mathbb{R}^2)} = \|(-R_2 \theta, R_1 \theta)\|_{L^r(\mathbb{R}^2)} \leq C \|\theta\|_{L^r(\mathbb{R}^2)} \) for \( 1 < r < \infty \),
\[
\int_{\mathbb{R}^2} |\Theta|^q \|\Theta\|_{L^q} (-\Delta)^{(\alpha-2)/2} \nabla \cdot (\theta \nabla^\perp \psi) \, dx \, ds
\leq C \|\theta\|_{L^r(\mathbb{R}^2)}^{1/2} \|\theta\|_{L^s(\mathbb{R}^2)} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1/2} \leq C(1 + r)^{-\frac{\alpha}{q} + \frac{\alpha}{r}} \|\Theta\|_{L^q(\mathbb{R}^2)}^{q-1/2}.
\]

Hence
\[
\int_0^t (1 + s)^{-\gamma q} \int_{\mathbb{R}^2} |\Theta|^q \|\Theta\|_{L^q} (-\Delta)^{(\alpha-2)/2} \nabla \cdot (\theta \nabla^\perp \psi) \, dx \, ds
\leq C \int_0^t (1 + s)^{-\gamma q - 1 + \frac{\alpha}{q}} \|\Theta\|_{L^q}^{-1/2} \|\theta\|_{L^q}^{1/2} \, ds
\leq C_\delta \int_0^t (1 + s)^{-\gamma q - 1 + \frac{\alpha}{q}} \delta \, ds + \delta \int_0^t (1 + s)^{-\gamma q - 1} \|\Theta\|_{L^q}^{q/2} \, ds.
\]

By applying those inequalities into (14) and choosing \( \delta \) sufficiently small, we conclude the proof. \( \square \)

This proposition is crucial since \( \|x^2 G_\alpha(t)\|_{L^r(\mathbb{R}^2)} = r^{2/\alpha} \|x^2 G_\alpha(1)\|_{L^r(\mathbb{R}^2)} \) and \( \|x^2 G_\alpha(1)\|_{L^r(\mathbb{R}^2)} < +\infty \). We see the spatial decay of the solution of the linear equation in the following lemma.

**Lemma 2.7.** Let \( 0 < \alpha \leq 2 \), then
\[
\|x^2 (G_\alpha(t) + \theta_0 - MG_\alpha(t))\|_{L^r(\mathbb{R}^2)} \leq C_\alpha \left( \|x\theta_0\|_{L^r(\mathbb{R}^2)} + \|x^2 \theta_0\|_{L^s(\mathbb{R}^2)} \right)
\]
holds for \( \theta_0 \in L^1(\mathbb{R}^2) \) with \( |x| \theta_0 \in L^1(\mathbb{R}^2) \) and \( |x|^2 \theta_0 \in L^2(\mathbb{R}^2) \), where \( M = \int_{\mathbb{R}^2} \theta_0(x) \, dx \).

**Proof.** The mean-value theorem yields that
\[
G_\alpha(t) + \theta_0 - MG_\alpha(t) = \int_{|y| < |x|^2/2} \int_0^1 \nabla G_\alpha(t, x - \lambda y) \cdot (-y) \theta_0(y) \, d\lambda \, dy
+ \int_{|y| > |x|^2/2} (G_\alpha(t, x - y) - G_\alpha(t, x)) \theta_0(y) \, dy.
\]

Hence Hausdorff–Young’s inequality with (6) and (7) gives that
\[ \|x\|^2 (G_\alpha(t) * \theta_0 - MG_\alpha(t)) \|_{L^2(\mathbb{R}^2)} \leq C \|x\|^2 \nabla G_\alpha(t) \|_{L^2(\mathbb{R}^2)} \|x\|_{L^1(\mathbb{R}^2)} + C \|G_\alpha(t)\|_{L^1(\mathbb{R}^2)} \|x\|^2 \theta_0 \|_{L^2(\mathbb{R}^2)} + C \|x(G_\alpha(t))\|_{L^2(\mathbb{R}^2)} \|x\|\theta_0 \|_{L^2(\mathbb{R}^2)} \leq C \]

and we conclude the proof. \[ \square \]

3. Proof of main theorem

We prove theorem 1.1. Put \( v = \theta - G_\alpha * \theta_0 \), then

\[ v = - \int_0^t \nabla G_\alpha(t-s) * (\theta \nabla^\perp \psi)(s) ds. \]

For \( 1 \leq p < 2/(1 - \alpha) \), we choose \( r_1 \) and \( r_2 \) such that \( 1 + 1/p = 1/r_1 + 1/r_2 \), \( 2(1 - 1/r_1)/\alpha < 1 \) and \( 1 \leq r_2 < 2 \). Moreover, let \( \varsigma \leq \sigma - 1 \), then we see from Hausdorff–Young’s inequality and proposition 2.5 that

\[
\|(-\Delta)^{\varsigma/2} v\|_{L^p(\mathbb{R}^2)} \leq \int_0^{t/2} \|\nabla (-\Delta)^{\varsigma/2} G_\alpha(t-s)\|_{L^p(\mathbb{R}^2)} \|\theta \nabla^\perp \psi\|_{L^1(\mathbb{R}^2)} ds \\
+ \int_{t/2}^t \|G_\alpha(t-s)\|_{L^1(\mathbb{R}^2)} \|\nabla (-\Delta)^{\varsigma/2} \cdot (\theta \nabla^\perp \psi)\|_{L^2(\mathbb{R}^2)} ds \\
\leq C \int_0^{t/2} (t-s)^{-\frac{\varsigma}{p}(1-\frac{1}{r_1})} \frac{1}{\sigma - 1}(1 + s)^{-\frac{\varsigma}{p}} ds \\
+ C \int_{t/2}^t (t-s)^{-\frac{\varsigma}{p}(1-\frac{1}{r_2})} (1 + s)^{-\frac{\varsigma}{p}} ds.
\]

The right-hand side is bounded by \( Ct^{-2(1-1/p)/\alpha-1/\alpha-\varsigma/\alpha} + Ct^{-2(1-1/r_1)/\alpha} (1 + t)^{-2(1-1/r_1)-3/\alpha-\varsigma/\alpha} \). Hence \( \|(-\Delta)^{\varsigma/2} v\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-1/p)/\alpha-1/\alpha-\varsigma/\alpha} \). Similarly

\[
\|(-\Delta)^{\varsigma/2} v\|_{L^p(\mathbb{R}^2)} \leq C \int_0^t (t-s)^{-\frac{\varsigma}{p}(1-\frac{1}{r_1})} (1 + s)^{-\frac{\varsigma}{p}(1-\frac{1}{r_1})} ds.
\]

Thus \( \|(-\Delta)^{\varsigma/2} v\|_{L^p(\mathbb{R}^2)} \) is bounded as \( t \to +0 \). Therefore

\[
\|(-\Delta)^{\varsigma/2} v\|_{L^p(\mathbb{R}^2)} \leq C(1 + t)^{-\frac{\varsigma}{p}(1-\frac{1}{r_1})} \]

for \( 1 \leq p < 2/(1 - \alpha) \) and \( \varsigma \leq \sigma - 1 \). It also holds that

\[
\begin{cases}
\partial_t v + (-\Delta)^{\alpha/2} v + \nabla \cdot (\theta \nabla^\perp \psi) = 0, & t > 0, \ x \in \mathbb{R}^2, \\
v(0, x) = 0, & x \in \mathbb{R}^2.
\end{cases}
\]

Therefore
\[
\frac{1}{2} \left\| \left| x^2 v(t) \right| \right\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left\| \left( -\Delta \right)^{\alpha/4} \left| x^2 v \right| \right\|_{L^2(\mathbb{R}^2)}^2 \, ds
= 4 \int_0^t \int_{\mathbb{R}^2} |x|^2 \theta \nabla x \cdot \nabla^\perp \psi \, dx \, ds
+ \int_0^t \int_{\mathbb{R}^2} |x|^4 \theta \nabla^\perp x \cdot \nabla^\perp \psi \, dx \, ds - \int_0^t \int_{\mathbb{R}^2} |x|^2 \left| \left( -\Delta \right)^{\alpha/2} \right| v \, dx \, ds.
\]  

We see from (3) and proposition 2.6 that
\[
\left| \int_{\mathbb{R}^2} |x|^2 \theta \nabla x \cdot \nabla^\perp \psi \, dx \right| \leq C \left\| \left| x^2 \right| \right\|_{L^\infty(\mathbb{R}^2)} \left\| \left| x^2 v \right| \right\|_{L^2(\mathbb{R}^2)} \left\| \nabla^\perp \psi \right\|_{L^2(\mathbb{R}^2)}
\leq C \left\| \theta \right\|_{L^1(\mathbb{R}^2)}^{1/2} \left\| \left| x^2 \theta \right| \right\|_{L^2(\mathbb{R}^2)}^{1/2} \left\| \theta \right\|_{L^1(\mathbb{R}^2)} \left\| \left( -\Delta \right)^{\alpha/4} \left| x^2 v \right| \right\|_{L^2(\mathbb{R}^2)}
\leq C \left( 1 + t \right)^{-\frac{\alpha}{2} + \frac{1}{2}} \left\| \left( -\Delta \right)^{\alpha/4} \left| x^2 v \right| \right\|_{L^2(\mathbb{R}^2)},
\] where $1/2 = 1/2 - \alpha/4$ and $1/r = 1/2 + \alpha/4 - 1/2q$. Hence
\[
\left| \int_0^t \int_{\mathbb{R}^2} |x|^2 \theta \nabla x \cdot \nabla^\perp \psi \, dx \, ds \right| \leq C_\delta + \delta \int_0^t \left\| \left( -\Delta \right)^{\alpha/4} \left| x^2 v \right| \right\|_{L^2(\mathbb{R}^2)}^2 \, ds
\] for $\delta > 0$. For the second term of (17), we have
\[
\int_{\mathbb{R}^2} |x|^4 \theta \nabla^\perp v \cdot \nabla^\perp \psi \, dx \leq C \left\| \left| x^2 \theta \right| \right\|_{L^2(\mathbb{R}^2)} \left\| \left| x^2 \nabla v \cdot \nabla^\perp \psi \right| \right\|_{L^2(\mathbb{R}^2)}.
\] Now $\nabla^\perp \psi = (-R_2 \theta, R_1 \theta)$ and the Hörmander–Mikhlin inequality guarantees that $\left| x^2 \right| R_2 G_\alpha(1) \in L^\infty(\mathbb{R}^2)$. The scaling property of $G_\alpha$ says that $\left\| \left| x^2 \right| R_2 G_\alpha(t) \right\|_{L^\infty(\mathbb{R}^2)} = \left\| \left| x^2 \right| R_2 G_\alpha(1) \right\|_{L^\infty(\mathbb{R}^2)}$. Hence
\[
\left\| \left| x^2 \right| \nabla^\perp \psi \right\|_{L^\infty(\mathbb{R}^2)}^2 \leq \sum_{j=1}^2 \left( \left\| \left| x^2 \right| \left( R_2 G_\alpha \ast \theta_0 \right)(t) \right\|_{L^\infty(\mathbb{R}^2)} \right.
+ \left\| \left| x^2 \right| \int_0^t R_2 G_\alpha(t - s) \ast \left( \nabla \theta \cdot \nabla^\perp \psi \right)(s) \, ds \right\|_{L^\infty(\mathbb{R}^2)}\right)
\leq C \left( 1 + \left\| \theta_0 \right\|_{H^\infty(\mathbb{R}^2)} \right)^2 \left\| \left| x^2 \right| \nabla^\perp \psi \right\|_{L^\infty((0,\infty) \times \mathbb{R}^2)}.
\] Here we used theorem 3.1 in [22] and proposition 2.5 in section 2. The constant $C$ on the second term is estimated by
\[
C_{\text{HLS}} \left\| R_2 G_\alpha(1) \right\|_{L^{1+\mu}(\mathbb{R}^2)} \sup_{t > 0} \int_0^t (t - s)^{-\frac{\alpha}{2} - \frac{\beta}{2} - \frac{\sigma}{2} - \frac{\mu}{2}} \, ds
\] for some $2 - \alpha < \mu < 2$, where $C_{\text{HLS}}$ is the constant in lemma 2.3 with $p = 2$ and $\sigma = \mu - 1$. From the smallness of $\theta_0$, we see that $\left\| \left| x^2 \right| \nabla^\perp \psi \right\|_{L^\infty(\mathbb{R}^2)} \leq C$. By employing (15), we obtain $\left\| \nabla v \right\|_{L^2(\mathbb{R}^2)} \leq C \left( 1 + t \right)^{-\alpha - 2/\alpha}$. Therefore
\[
\left| \int_0^t \int_{\mathbb{R}^2} |x|^4 \theta \nabla v \cdot \nabla^\perp \psi \, dx \, ds \right| \leq C.
\] Since (13) with $v$ instead of $\theta$ holds, the last term of (17) is estimated by
\[
\left| \int_{\mathbb{R}^2} |x|^2 v [ |x|^2, (-\Delta)^{\alpha/2} ] \, dx \right| \leq \left\| |x|^2 v \right\|_{L^2_x(\mathbb{R}^2)} \left\| |x|^2, (-\Delta)^{\alpha/2} v \right\|_{L^2_x(\mathbb{R}^2)} \\
\leq C \left\| (-\Delta)^{\alpha/4} (|x|^2 v) \right\|_{L^2_x(\mathbb{R}^2)} \sum_{|\beta|+|\gamma|=2, |\beta| \geq 1} \left\| (-\Delta)^{\alpha-|\beta|/2} (x^\gamma v) \right\|_{L^2_x(\mathbb{R}^2)},
\]

where \(1/2_s = 1/2 - \alpha/4\) and then \(1/2_s' = 1/2 + \alpha/4\). Moreover constants exist: \(b_{\gamma_1, \gamma_2}\) and \(b_0\) such that

\[
\mathcal{F}[x^\gamma v](t) = \sum_{|\gamma_1|+|\gamma_2|=|\gamma|} b_{\gamma_1, \gamma_2} \int_0^t i \xi (i\nabla)_{\gamma_1} (e^{-(t-s)|\xi|^2}) \cdot (i\nabla)_{\gamma_2} \mathcal{F}[\theta \nabla^\perp \psi](s, \xi) \, ds \\
+ b_0 \int_0^t e^{-(t-s)|\xi|^2} \mathcal{F}[\theta \nabla^\perp \psi](s, \xi) \, ds.
\]

Here \(b_0 = 0\) when \(|\gamma| = 0\). Thus

\[
(-\Delta)^{(\alpha-|\beta|)/2} (x^\gamma v) (t) = \sum_{|\gamma_1|+|\gamma_2|=|\gamma|} b_{\gamma_1, \gamma_2} \int_0^t \nabla (-\Delta)^{(\alpha-|\beta|)/2} (x^{\gamma_1} G_{\alpha})(t-s) \ast (x^{\gamma_2} \theta \nabla^\perp \psi)(s) \, ds \\
+ b_0 \int_0^t (-\Delta)^{(\alpha-|\beta|)/2} G_{\alpha}(t-s) \ast (\theta \nabla^\perp \psi)(s) \, ds.
\]

(21)

For some \(r_1\) and \(r_2\) with \(2/(3 + \alpha - |\beta| - |\gamma_1|) < r_1 < 2/(3 - |\beta| - |\gamma_1|)\) and \(1 + 1/2_s' = 1/r_1 + 1/r_2\), we see from (7) and (8) that \(\nabla (-\Delta)^{(\alpha-|\beta|)/2} (x^{\gamma_1} G_{\alpha}) \in L^{r_2} (\mathbb{R}^2) \cap L^{r_1} (\mathbb{R}^2)\), and obtain by the Hausdorff–Young’s inequality and proposition 2.6 that

\[
\left\| \int_0^t \nabla (-\Delta)^{(\alpha-|\beta|)/2} (x^{\gamma_1} G_{\alpha})(t-s) \ast (x^{\gamma_2} \theta \nabla^\perp \psi)(s) \, ds \right\|_{L^{2}_{x}(\mathbb{R}^2)} \\
\leq C \int_0^t \left\| \nabla (-\Delta)^{(\alpha-|\beta|)/2} (x^{\gamma_1} G_{\alpha})(t-s) \right\|_{L^{r_2}_{x}(\mathbb{R}^2)} \left\| x^{\gamma_2} (\theta \nabla^\perp \psi)(s) \right\|_{L^{r_1}_{x}(\mathbb{R}^2)} \, ds \\
+ C \int_0^t \left\| \nabla (-\Delta)^{(\alpha-|\beta|)/2} (x^{\gamma_1} G_{\alpha})(t-s) \right\|_{L^{r_2}_{x}(\mathbb{R}^2)} \left\| x^{\gamma_2} (\theta \nabla^\perp \psi)(s) \right\|_{L^{r_1}_{x}(\mathbb{R}^2)} \, ds \\
\leq C \int_0^t (t-s)^{-\frac{\alpha}{4} - \frac{\beta}{2} + \frac{|\beta|}{4} + \frac{|\gamma_1|}{2}} (1+s)^{-\frac{\alpha}{4} + \frac{|\beta|}{2} + \frac{|\gamma_1|}{2}} \, ds \\
+ C \int_0^t (t-s)^{-\frac{\alpha}{4} - \frac{\beta}{2} + \frac{|\beta|}{4} + \frac{|\gamma_1|}{2}} (1+s)^{-\frac{\alpha}{4} - \frac{\beta}{2} + \frac{|\beta|}{4} + \frac{|\gamma_1|}{2}} \, ds.
\]

Here the decay of \(\nabla (-\Delta)^{(\alpha-|\beta|)/2} (x^{\gamma_1} G_{\alpha})\) in time has been provided from (6). Similarly

\[
\left\| \int_0^t \nabla (-\Delta)^{(\alpha-|\beta|)/2} (x^{\gamma_1} G_{\alpha})(t-s) \ast (x^{\gamma_2} \theta \nabla^\perp \psi)(s) \, ds \right\|_{L^{2}_{x}(\mathbb{R}^2)} \\
\leq C \int_0^t (t-s)^{-\frac{\alpha}{4} - \frac{\beta}{2} + \frac{|\beta|}{4} + \frac{|\gamma_1|}{2}} (1+s)^{-\frac{\alpha}{4} - \frac{\beta}{2} + \frac{|\beta|}{4} + \frac{|\gamma_1|}{2}} \, ds.
\]

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A coupling of them gives that

\[
\left\| \int_0^t \nabla (-\Delta)^{(\alpha-|\beta|)/2}(x^\gamma \psi)(t-s) \ast (x^\gamma \theta \nabla \psi')(s) \, ds \right\|_{L^2(\mathbb{R}^2)} \leq C(1 + t)^{-1/2} L_\alpha(t),
\]

where

\[
L_\alpha(t) = \begin{cases} \log(2 + t), & \alpha = 1, \\ 1, & 0 < \alpha < 1. \end{cases}
\]

We estimate the second term of (21) when \(|\gamma| = 1\), i.e. \(|\beta| = 1\), then the similar argument as above says that

\[
\left\| \int_0^t (-\Delta)^{(\alpha-|\beta|)/2} G_\alpha(t-s) \ast (\theta \nabla \psi)(s) \, ds \right\|_{L^2(\mathbb{R}^2)} \leq C(1 + t)^{-1/2}.
\]

Therefore

\[
\left\| [x^2, (-\Delta)^{\alpha/2}] v \right\|_{L^\infty(\mathbb{R}^2)} \leq C \sum_{|\beta| + |\gamma| = 2, |\beta| \geq 1} \left\| (-\Delta)^{(\alpha-|\beta|)/2}(x^\gamma v)(t) \right\|_{L^2(\mathbb{R}^2)} 
\leq C(1 + t)^{-1/2} L_\alpha(t),
\]

and

\[
\left| \int_0^t \int_{\mathbb{R}^2} |x|^2 v \left[ [x^2, (-\Delta)^{\alpha/2}] \right] v \, dx \, ds \right| 
\leq C_\delta \log(2 + t) L_\alpha(t)^2 + \delta \int_0^t \left\| (-\Delta)^{\alpha/4}(|x|^2 v) \right\|_{L^2(\mathbb{R}^2)}^2 \, ds. \tag{22}
\]

Applying (18), (20) and (22) into (17), and choosing \(\delta\) sufficiently small, we see that

\[
\left\| |x|^2 v \right\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left\| (-\Delta)^{\alpha/4}(|x|^2 v) \right\|_{L^2(\mathbb{R}^2)}^2 \, ds \leq C \log(2 + t) L_\alpha(t)^2.
\]

A coupling of this estimate and lemma 2.7 yields the assertion. \qed

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