Cyclic Codes and Sequences from a Class of Dembowski-Ostrom Functions

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Abstract

Let $q = p^n$ with $p$ be an odd prime. Let $0 \leq k \leq n - 1$ and $k \neq n/2$. In this paper we determine the value distribution of following exponential(character) sums
\[
\sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^n(\alpha x^{p^{3k}+1}+\beta x^{p^k+1})} \quad (\alpha \in \mathbb{F}_{p^m}, \beta \in \mathbb{F}_q)
\]
and
\[
\sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^n(\alpha x^{p^{k}+1}+\beta x^{p^{k}+1}+\gamma x)} \quad (\alpha \in \mathbb{F}_{p^m}, \beta, \gamma \in \mathbb{F}_q)
\]
where $\text{Tr}_1^n : \mathbb{F}_q \to \mathbb{F}_p$ and $\text{Tr}_1^m : \mathbb{F}_{p^m} \to \mathbb{F}_p$ are the canonical trace mappings and $\zeta_p = e^{\frac{2\pi i}{p}}$ is a primitive $p$-th root of unity. As applications:

(1). We determine the weight distribution of the cyclic codes $C_1$ and $C_2$ over $\mathbb{F}_{p^t}$ with parity-check polynomials $h_2(x)h_3(x)$ and $h_1(x)h_2(x)h_3(x)$ respectively where $t$ is a divisor of $d = \gcd(n, k)$, and $h_1(x)$, $h_2(x)$ and $h_3(x)$ are the minimal polynomials of $\pi^{-1}$, $\pi^{-(p^k+1)}$ and $\pi^{-(p^{3k}+1)}$ over $\mathbb{F}_{p^t}$ respectively for a primitive element $\pi$ of $\mathbb{F}_q$.

(2). We determine the correlation distribution among a family of $m$-sequences.

Index terms: Exponential sum, Cyclic code, Sequence, Weight distribution, Correlation distribution
1 Introduction

Basic results on finite fields could be found in [19]. These notations are fixed throughout this paper except for specific statements.

- Let $p$ an odd prime, $p^r = (-1)^{\frac{p-1}{2}} p$, $q = p^n$ and $\mathbb{F}_p$, $\mathbb{F}_q$ be the finite fields of order $p$, $q$ respectively. Let $\pi$ be a primitive element of $\mathbb{F}_q$.

- Let $\text{Tr}_i : \mathbb{F}_{p^i} \rightarrow \mathbb{F}_{p^j}$ be the trace mapping, $\zeta_p = \exp(2\pi \sqrt{-1}/p)$ be a $p$-th root of unity and $\chi(x) = \zeta_p^{\text{Tr}_1(x)}$ be the canonical additive character on $\mathbb{F}_q$.

- Let $k$ be a positive integer, $1 \leq k \leq n - 1$ and $k \notin \{\frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\}$. Let $d = \gcd(n, k)$, $q_0 = p^d$, $q_0^* = (-1)^{\frac{q_0-1}{2}} q_0$ and $s = n/d$. Let $t$ be a divisor of $d$ and $n_0 = n/t$.

- Let $m = n/2$ (if $n$ is even) and $\mu = (-1)^{m/d}$.

Let $C$ be an $[l, k, d]$ linear code and $A_i$ be the number of codewords in $C$ with Hamming weight $i$. The weight distribution $\{A_i\}_{i=0}^l$ is an important research object for theoretical and application interests (see Fitzgerald and Yucas [7], McEliece [20], McEliece and Rumsey [21], van der Vlugt [26], Wolfmann [28] and the references therein).

For a cyclic code, the Hamming weight of each codeword can be expressed by certain combination of general exponential(character) sums (see Feng and Luo [5], [6], Luo and Feng [13], [14], Luo, Tang and Wang [15], Luo [16], van der Vlugt [27], Yuan, Carlet and Ding [30], Zeng, Hu, Jia, Yue and Cao [31], Zeng and Li [32]). More exactly speaking, let $t \mid n$, $C$ be the cyclic code over $\mathbb{F}_{p^t}$ with length $l = q - 1$ and parity-check polynomial,

$$h(x) = h_1(x) \cdots h_u(x) \quad (u \geq 2)$$

where $h_i(x) \ (1 \leq i \leq e)$ are distinct irreducible polynomials in $\mathbb{F}_{p^t}[x]$ with degree $e_i \ (1 \leq i \leq u)$, then $\dim_{\mathbb{F}_p} C = \sum_{i=1}^u e_i$. Let $\pi^{-s_i}$ be a zero of $h_i(x)$, $1 \leq s_i \leq q - 2 \ (1 \leq i \leq u)$. Then the codewords in $C$ can be expressed by

$$c(\alpha_1, \cdots, \alpha_u) = (c_0, c_1, \cdots, c_{l-1}) \quad (\alpha_1, \cdots, \alpha_u \in \mathbb{F}_q)$$

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where \( c_i = \sum_{\lambda=1}^{n} \text{Tr}_p^n(\alpha_\lambda \pi^{i\lambda}) \) (0 \leq i \leq n - 1). Therefore the Hamming weight of the codeword \( c = c(\alpha_1, \cdots, \alpha_u) \) is

\[
w_H(c) = \# \{ i | 0 \leq i \leq l - 1, c_i \neq 0 \}
= l - \# \{ i | 0 \leq i \leq l - 1, c_i = 0 \}
= l - \frac{1}{p^t} \sum_{i=0}^{l-1} \sum_{a \in \mathbb{F}_{p^t}} \zeta_p^{\text{Tr}_p^i(a \cdot \text{Tr}_p^n(\sum_{\lambda=1}^{n} \alpha_\lambda \pi^{i\lambda})))}
= l - \frac{l}{p^t} - \frac{1}{p^t} \sum_{a \in \mathbb{F}_{p^t}^*} \sum_{x \in \mathbb{F}_q^*} \zeta_p^{\text{Tr}_p^i(af(x))}
= l - \frac{l}{p^t} + \frac{p^t-1}{p^t} - \frac{1}{p^t} \sum_{a \in \mathbb{F}_{p^t}^*} S(a\alpha_1, \cdots, a\alpha_u)
= p^{n-t}(p^t-1) - \frac{1}{p^t} \sum_{a \in \mathbb{F}_{p^t}^*} S(a\alpha_1, \cdots, a\alpha_u)
\]\[ (1) \]

where \( f(x) = \alpha_1 x^{s_1} + \alpha_2 x^{s_2} + \cdots + \alpha_u x^{s_u} \in \mathbb{F}_q[x], \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}, \mathbb{F}_{p^t}^* = \mathbb{F}_{p^t} \setminus \{0\} \), and

\[
S(\alpha_1, \cdots, \alpha_u) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_p^i(\alpha_1 x^{s_1} + \cdots + \alpha_u x^{s_u})}.
\]

In this way, the weight distribution of cyclic code \( \mathcal{C} \) can be derived from the explicit evaluating of the exponential sums

\[
S(\alpha_1, \cdots, \alpha_u) \quad (\alpha_1, \cdots, \alpha_u \in \mathbb{F}_q).
\]

Let \( h_1(x), h_2(x) \) and \( h_3(x) \) be the minimal polynomials of \( \pi^{-1}, \pi^{-(p^t+1)} \) and \( \pi^{-(p^t+1)} \) over \( \mathbb{F}_{p^t} \) respectively. Then \( \deg h_i(x) = n_0 \) for 1 \( \leq i \leq 3 \).

Let \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) be the cyclic codes over \( \mathbb{F}_{p^t} \) with length \( l = q - 1 \) and parity-check polynomials \( h_2(x)h_3(x) \) and \( h_1(x)h_2(x)h_3(x) \) respectively. Then we know that the dimensions of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) over \( \mathbb{F}_{p^t} \) are \( 2n_0 \) and \( 3n_0 \) respectively.

A Dembowski-Ostrom function on \( \mathbb{F}_q \) is a \( \mathbb{F}_q \)-linear combination of \( x^{p^i+p^j} \) with \( 0 \leq i \leq n-1 \). Let \( f_{\alpha,\beta}(x) = \alpha x^{p^i+1} + \beta x^{p^j+1} \) for \( \alpha, \beta \in \mathbb{F}_q \). Define the exponential sums

\[
T(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_p^i(f_{\alpha,\beta}(x))}
\]\[ (2) \]
and for $\gamma \in \mathbb{F}_q$,

$$S(\alpha, \beta, \gamma) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^n(f_{\alpha, \beta}(x) + \gamma x)}.$$  (3)

Then the weight distribution of $C_1$ and $C_2$ can be completely determined if $T(\alpha, \beta)$ and $S(\alpha, \beta, \gamma)$ are explicitly evaluated.

Another application of $S(\alpha, \beta, \gamma)$ is to calculate the correlation distribution of corresponding sequences. Let $F$ be a collection of $p$-ary m-sequences of period $q - 1$ defined by

$$F = \{ (a_i(t))_{t=0}^{q-2} \mid 0 \leq i \leq L - 1 \}$$

The correlation function of $a_i$ and $a_j$ for a shift $\tau$ is defined by

$$M_{i,j}(\tau) = \sum_{\lambda=0}^{q-2} \zeta_p^{a_i(\lambda)-a_j(\lambda+\tau)} \quad (0 \leq \tau \leq q - 2).$$

In this paper, we will study the collection of sequences

$$F = \left\{ a_{\alpha, \beta} = \{ a_{\alpha, \beta}(\pi^\lambda) \}_{\lambda=0}^{q-2} \mid \alpha, \beta \in \mathbb{F}_q \right\}$$  (4)

where $a_{\alpha, \beta}(\pi^\lambda) = \text{Tr}_1^n(\alpha \pi^\lambda(p^{3k}+1) + \beta \pi^\lambda(p^{k+1} + \pi^\lambda)).$

Then the correlation function between $a_{\alpha_1, \beta_1}$ and $a_{\alpha_2, \beta_2}$ by a shift $\tau$ ($0 \leq \tau \leq q - 2$) is

$$M_{(\alpha_1, \beta_1),(\alpha_2, \beta_2)}(\tau) = \sum_{\lambda=0}^{q-2} \zeta_p^{a_{\alpha_1, \beta_1}(\lambda)-a_{\alpha_2, \beta_2}(\lambda+\tau)}$$

$$= \sum_{\lambda=0}^{q-2} \zeta_p^{\text{Tr}_1^n(\alpha_1 \pi^\lambda(p^{3k}+1) + \beta_1 \pi^\lambda(p^{k+1} + \pi^\lambda)) - \text{Tr}_1^n(\alpha_2 \pi((\lambda+\tau)(p^{3k}+1) + \beta_2 \pi((\lambda+\tau)(p^{k+1} + \pi^\lambda + \tau))}

= S(\alpha', \beta', \gamma') - 1$$  (5)

where

$$\alpha' = \alpha_1 - \alpha_2 \pi^{\tau(p^{3k}+1)}, \quad \beta' = \beta_1 - \beta_2 \pi^{\tau(p^{k+1})}, \quad \gamma' = 1 - \pi^\tau.$$  (6)

Pairs of non-binary m-sequences with few-valued cross correlations have been extensively studied for several decades, see Charpin [1], Dobbertin, Helleseth, Kumar and Martinsen [4], Gold [8], Helleseth [9, 10], Helleseth and Kumar [11], Helleseth, Lahtonen and Rosendahl [12], Kasami [17, 18], Rosendahl [23, 24] and Trachtenberg [25], Xia and Zeng [29] and references therein.
In [31], the exponential sums $T(\alpha, \beta)$ and $S(\alpha, \beta, \gamma)$ for $n/d$ odd have been evaluated. As an application, the weight distribution to the associated $p$-ary cyclic code $C_2$ is determined. Our paper focuses on the case $n/d$ is even. Moreover, we will determine the weight distribution of $C_1$ and $C_2$ for $t \mid d$. Meanwhile, the correlation distribution of sequences in $\mathcal{F}$ can also be calculated explicitly.

This paper is presented as follows. In Section 2 we introduce some preliminaries. In Section 3 we will study the value distribution of $T(\alpha, \beta)$ (that is, which value $T(\alpha, \beta)$ takes on and which frequency of each value) and the weight distribution of $C_1$. In Section 3 we will determine the value distribution of $S(\alpha, \beta, \gamma)$, the correlation distribution among the sequences in $\mathcal{F}$, and then the weight distribution of $C_2$. Most lengthy details are presented in several appendixes. The main tools are quadratic form theory over odd characteristic finite fields, some moment identities on $T(\alpha, \beta)$ and a class of Artin-Schreier curves on finite fields which we have employed in [13] and [15]. We will focus our study on the odd prime characteristic case and the binary case will be investigated in a following paper.

2 Preliminaries

We follow the notations in Section 1. The first machinery to determine the values of exponential sums $T(\alpha, \beta)$ and $S(\alpha, \beta, \gamma)$ defined in (2) and (3) is quadratic form theory over $F_{q_0}$.

Let $H$ be an $s \times s$ symmetric matrix over $F_{q_0}$ and $r = \text{rank } H$. Then there exists $M \in \text{GL}_s(F_{q_0})$ such that $H' = MHM^T$ is diagonal and $H' = \text{diag}(a_1, \cdots, a_r, 0, \cdots, 0)$ where $a_i \in F_{q_0}^*$ ($1 \leq i \leq r$). Let $\Delta = a_1 \cdots a_r$ (we assume $\Delta = 1$ when $r = 0$) and $\eta_0$ be the quadratic (multiplicative) character of $F_{q_0}$. Then $\eta_0(\Delta)$ is an invariant of $H$ under the conjugate action of $M \in \text{GL}_s(F_{q_0})$.

For the quadratic form

$$F : F_{q_0}^s \rightarrow F_{q_0}, \quad F(x) = XHX^T \quad (X = (x_1, \cdots, x_s) \in F_{q_0}^s), \quad (7)$$

we have the following result (see [13], Lemma 1).

**Lemma 1.** (i). For the quadratic form $F = XHX^T$ defined in (7), we have

$$\sum_{X \in F_{q_0}^s} \zeta_p(Tr(F(X))) = \begin{cases} \eta_0(\Delta)q_0^{s-r/2} & \text{if } q_0 \equiv 1 \pmod{4}, \\ i^{r} \eta_0(\Delta)q_0^{s-r/2} & \text{if } q_0 \equiv 3 \pmod{4}. \end{cases}$$

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(ii). For $A = (a_1, \cdots, a_s) \in \mathbb{F}_{q_0}^s$, if $2YH + A = 0$ has solution $Y = B \in \mathbb{F}_{q_0}^s$,
then $\sum_{X \in \mathbb{F}_{q_0}^n} \zeta_p^{Tr_d^s(F(X) + AX^T)} = \zeta_p^c \sum_{X \in \mathbb{F}_{q_0}^n} \zeta_p^{Tr_d^s(F(X))}$ where $c = -Tr_d^s(BHB^T) = \frac{1}{2} Tr_d^s(AB^T) \in \mathbb{F}_p$.
Otherwise $\sum_{X \in \mathbb{F}_p^n} \zeta_p^{Tr_d^s(F(X) + AX^T)} = 0$.

In this correspondence we always assume $d = \gcd(n, k)$. Then the field $\mathbb{F}_q$ is a vector space over $\mathbb{F}_{q_0}$ with dimension $s$. We fix a basis $v_1, \cdots, v_s$ of $\mathbb{F}_q$ over $\mathbb{F}_{q_0}$. Then each $x \in \mathbb{F}_q$ can be uniquely expressed as

$$x = x_1v_1 + \cdots + x_sv_s \quad (x_i \in \mathbb{F}_{q_0}).$$

Thus we have the following $\mathbb{F}_{q_0}$-linear isomorphism:

$$\mathbb{F}_q \cong \mathbb{F}_{q_0}^s, \quad x = x_1v_1 + \cdots + x_sv_s \mapsto X = (x_1, \cdots, x_s).$$

With this isomorphism, a function $f : \mathbb{F}_q \to \mathbb{F}_{q_0}$ induces a function $F : \mathbb{F}_{q_0}^s \to \mathbb{F}_{q_0}$ where for $X = (x_1, \cdots, x_s) \in \mathbb{F}_{q_0}^s$, $F(X) = f(x)$ with $x = x_1v_1 + \cdots + x_sv_s$. In this way, function $f(x) = Tr_d^n(\gamma x)$ for $\gamma \in \mathbb{F}_q$ induces a linear form

$$F(X) = Tr_d^n(\gamma x) = \sum_{i=1}^s Tr_d^n(\gamma v_i)x_i = A_\gamma X^T \quad (8)$$

where $A_\gamma = (Tr_d^n(\gamma v_1), \cdots, Tr_d^n(\gamma v_s))$, and $f_{\alpha,\beta}(x) = Tr_d^n(\alpha x^p + \beta x^k)$ for $\alpha, \beta \in \mathbb{F}_q$ induces a quadratic form

$$F_{\alpha,\beta}(X) = Tr_d^n(\alpha x^p + \beta x^k)$$

$$= Tr_d^n \left( \alpha \left( \sum_{i=1}^s x_i v_i^p \right) \left( \sum_{i=1}^s x_i v_i \right) + \beta \sum_{i=1}^s x_i v_i^k \right) \left( \sum_{i=1}^s x_i v_i \right)$$

$$= \sum_{i,j=1}^s Tr_d^n \left( \alpha v_i^p v_j + \beta v_i^k v_j \right) x_ix_j = XH_{\alpha,\beta}X^T \quad (9)$$

where

$$H_{\alpha,\beta} = (h_{ij})_{s \times s} \text{ and } h_{ij} = \frac{1}{2} Tr_d^n \left( \alpha \left( v_i^p v_j + v_i v_j^p \right) + \beta \left( v_i^k v_j + v_i v_j^k \right) \right) \text{ for } 1 \leq i, j \leq s.$$
From Lemma 1, in order to determine the values of
\[ T(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_p(\alpha x^{3k+1} + \beta x^{k+1})} = \sum_{X \in \mathbb{F}_{q_0}^m} \zeta_p^{\text{Tr}_p(X H_{\alpha, \beta} X^T)} \]
and
\[ S(\alpha, \beta, \gamma) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_p(\alpha x^{3k+1} + \beta x^{k+1} + \gamma x)} = \sum_{X \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}_p(X H_{\alpha, \beta} X^T + A \gamma X^T)} \quad (\alpha, \beta, \gamma \in \mathbb{F}_q), \]
we need to determine the rank of \( H_{\alpha, \beta} \) over \( \mathbb{F}_{q_0} \) and the solvability of \( \mathbb{F}_{q_0} \)-linear equation \( 2X H_{\alpha, \beta} + A \gamma = 0 \).

Define \( d' = \gcd(n, 2k) \). Then an easy observation shows
\[ d' = \begin{cases} 
2d, & \text{if } n/d \text{ is even;} \\
d, & \text{otherwise.} \end{cases} \tag{10} \]

Now we could determine the possible ranks of \( H_{\alpha, \beta} \).

**Lemma 2.** For \( \alpha, \beta \in \mathbb{F}_q \) and \( (\alpha, \beta) \neq (0, 0) \), let \( r_{\alpha, \beta} \) be the rank of \( H_{\alpha, \beta} \). Then we have

(i). if \( d' = d \), then the possible values of \( r_{\alpha, \beta} \) are \( s, s - 1, s - 2 \).

(ii). if \( d' = 2d \), then the possible values of \( r_{\alpha, \beta} \) are \( s, s - 2, s - 4, s - 6 \).

**Proof.** For (i), see [31]. For (ii), see Appendix A. \( \square \)

In order to determine the value distribution of \( T(\alpha, \beta) \) for \( \alpha, \beta \in \mathbb{F}_q \), we need the following result on moments of \( T(\alpha, \beta) \).

**Lemma 3.** For the exponential sum \( T(\alpha, \beta) \),

(i). \[ \sum_{\alpha, \beta \in \mathbb{F}_q} T(\alpha, \beta) = p^{2n}; \]

(ii). \[ \sum_{\alpha, \beta \in \mathbb{F}_q} T(\alpha, \beta)^2 = \begin{cases} 
p^{2n} & \text{if } d' = d \text{ and } p^d \equiv 3 \pmod{4}, 
(2p^n - 1) \cdot p^{2n} & \text{if } d' = d \text{ and } p^d \equiv 1 \pmod{4}, 
(p^{n+d} + p^n - p^d) \cdot p^{2n} & \text{if } d' = 2d; \end{cases} \]

(iii). if \( d' = 2d \), then \[ \sum_{\alpha, \beta \in \mathbb{F}_q} T(\alpha, \beta)^3 = (p^{n+3d} + p^n - p^{3d}) \cdot p^{2n}. \]
Proof. see Appendix A. □

In the case \( d' = 2d \), we could determine the explicit values of \( T(\alpha, \beta) \). To this end we will study a class of Artin-Schreier curves. A similar technique has been applied in Coulter [2], Theorem 6.1.

**Lemma 4.** Suppose \((\alpha, \beta) \in (\mathbb{F}_q \times \mathbb{F}_q) \backslash \{0,0\}\) and \( d' = 2d \). Let \( N \) be the number of \( \mathbb{F}_q \)-rational (affine) points on the curve

\[
\alpha x^{p^k+1} + \beta x^{p^{k}+1} = y^{p^d} - y.
\]

(11)

Then

\[ N = q + (p^d - 1) \cdot T(\alpha, \beta). \]

Proof. see Appendix A. □

Now we give an explicit evaluation of \( T(\alpha, \beta) \) in the case \( d' = 2d \).

**Lemma 5.** Assumptions as in Lemma 4 and let \( n = 2m \), then

\[
T(\alpha, \beta) = \begin{cases} 
\mu p^m, & \text{if } r_{\alpha,\beta} = s \\
-\mu p^{m+d}, & \text{if } r_{\alpha,\beta} = s - 2 \\
\mu p^{m+2d}, & \text{if } r_{\alpha,\beta} = s - 4 \\
-\mu p^{m+3d}, & \text{if } r_{\alpha,\beta} = s - 6.
\end{cases}
\]

where \( \mu = (-1)^{m/d} \).

Proof. Consider the \( \mathbb{F}_q \)-rational (affine) points on the Artin-Schreier curve in Lemma 4. It is easy to verify that \((0, y)\) with \( y \in \mathbb{F}_p \) are exactly the points on the curve with \( x = 0 \). If \((x, y)\) with \( x \neq 0 \) is a point on this curve, then so are \((tx, t^{p^d+1}y)\) with \( t^{p^{2d}-1} = 1 \) (note that \( p^{3k+1} + 1 \equiv p^k + 1 \equiv p^d + 1 \pmod{p^{2d} - 1} \) since \( 3k/d \) and \( k/d \) are both odd by (10)). In total, we have

\[ q + (p^d - 1)T(\alpha, \beta) = N \equiv p^d \pmod{p^{2d} - 1} \]

which yields

\[ T(\alpha, \beta) \equiv 1 \pmod{p^d + 1}. \]

We only consider the case \( r_{\alpha,\beta} = s \) and \( m/d \) is odd. The other cases are similar. In this case \( T(\alpha, \beta) = \pm p^m \). Assume \( T(\alpha, \beta) = p^m \). Then \( p^d + 1 \mid p^m - 1 \) which contradicts to \( m/d \) is odd. Therefore \( T(\alpha, \beta) = -p^m \). □
Remark. (i). Our treatment improve the technique in [2], [3], in which the case \((p, d) = (3, 1)\) is dealt with in a different manner.

(ii). Applying Lemma 5 to Lemma 4, we could determine the number of rational points on the curve \((11)\).

3 Exponential Sums \(T(\alpha, \beta)\) and Cyclic Code \(C_1\)

Define  
\[
N_i = \{ (\alpha, \beta) \in \mathbb{F}_q \times \mathbb{F}_q \mid (0, 0) \} \mid r_{\alpha, \beta} = s - i \} .
\]
Then \(n_i = |N_i|\) for \(i = 0, 2, 4, 6\).

According to the possible values of \(T(\alpha, \beta)\) given by Lemma 1 (setting \(F(X) = X H_{\alpha, \beta} X^T = \text{Tr}^d(\alpha x^{p^{3k+1}} + \beta x^{p^{k+1}})\)), we define that for \(\varepsilon = \pm 1\),  
\[
N_{\varepsilon,i} = \begin{cases}  
\{(\alpha, \beta) \in \mathbb{F}_q^2 \mid (0, 0) \} \mid T(\alpha, \beta) = \varepsilon p^\frac{n+id}{2} \} & \text{if } n + id \text{ is even}, \\
\{(\alpha, \beta) \in \mathbb{F}_q^2 \mid (0, 0) \} \mid T(\alpha, \beta) = \varepsilon \sqrt{p^n p^\frac{n+id-1}{2}} \} & \text{if } n + id \text{ is odd}
\end{cases}
\]
where \(p^* = (-1)^{\frac{p-1}{2}} p\) and \(n_{\varepsilon,i} = |N_{\varepsilon,i}|\).

Recall \(q_0^* = (-1)^{\frac{q_0-1}{2}} q_0\). In this section we prove the following results.

Theorem 1. The value distribution of the multi-set \(\{T(\alpha, \beta) \mid \alpha, \beta \in \mathbb{F}_q\}\) is shown as following.

(i). For the case \(d' = d\),

| values          | multiplicity                                           |
|-----------------|--------------------------------------------------------|
| \(\sqrt{q_0} q_0^\frac{n+1}{2}, -\sqrt{q_0} q_0^\frac{n-1}{2}\) | \(\frac{1}{2} p^{2d}(p^n - p^{n-d} - p^{n-2d} + 1)(p^n - 1)/(p^{2d} - 1)\) |
| \(p^\frac{n+1}{2}\)                          | \(\frac{1}{2} p^\frac{n-1}{2} (p^{n-d} + 1)(p^n - 1)\)               |
| \(-p^\frac{n+1}{2}\)                          | \(\frac{1}{2} p^\frac{n-1}{2} (p^{n-d} - 1)(p^n - 1)\)               |
| \(\sqrt{q_0} q_0^\frac{n+1}{2}, -\sqrt{q_0} q_0^\frac{n-1}{2}\) | \(\frac{1}{2} (p^n - 1)(p^{n-d} - 1)/(p^{2d} - 1)\)       |
| \(p^n\)                                     | 1                                                      |

(ii). For the case \(d' = 2d\),
Proof. see [31] for (i) and Appendix B for (ii).

\[ \text{Remark.} \quad \text{(i) In the case } k \in \left\{ \frac{n}{4}, \frac{n}{2}, \frac{3n}{4} \right\}, \text{the exponential sum } T(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} \chi((\alpha x^k + \beta)x^{k+1}) \text{ has been extensively studied, for example, see [2], [19].} \]

\[ \text{(ii) In the case } k \in \left\{ \frac{n}{3}, \frac{2n}{3} \right\}, \text{the exponential sum } T(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} \chi(\alpha x^k + \beta x^{k+1}) \text{ is a special case in [15]. In the case } k \in \left\{ \frac{n}{3}, \frac{2n}{3} \right\}, \text{the exponential sum } T(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} \chi(\alpha x^2 + \beta x^{k+1}) \text{ is a special case in [13].} \]

Recall that $t$ is a divisor of $d$ and $C_1$ is the cyclic code over $\mathbb{F}_{p^t}$ with parity-check polynomial $h_2(x)h_3(x)$ where $h_2(x)$ and $h_3(x)$ are the minimal polynomials of $\pi^{-(p^k+1)}$ and $\pi^{-(p^{k+1})}$, respectively.

**Theorem 2.** Suppose that $1 \leq k \leq n-1$ and $k \notin \left\{ \frac{n}{4}, \frac{n}{2}, \frac{3n}{4} \right\}$. Then the weight distribution \( \{A_0, A_1, \cdots, A_{q-1} \} \) of the cyclic code $C_1$ over $\mathbb{F}_{p^t}$ (\( p \geq 3 \)) with length \( q-1 \) is shown as following.

\[ (i) \text{ For the case } d = d' \text{ and } d/t \text{ are both odd, then } \dim_{\mathbb{F}_p} C_1 = 2n_0 \text{ and } A_i = 0 \text{ except for} \]

| \( \mu p^m \) | \( \text{multiplicity} \) |
|----------------|--------------------------|
| \( -\mu p^{m+d} \) | \( \frac{(p^n-1)(p^{n+6d}-p^{n+4d}-\mu p^{m+5d}+\mu p^{m+4d}+p^{6d})}{(p^d+1)(p^{2d}+1)} \) |
| \( \mu p^{m+2d} \) | \( \frac{(p^{m-2d}-\mu)(p^{m-3d}+\mu p^{m-2d}+\mu p^d)}{(p^d+1)(p^{2d}+1)} \) |
| \( -\mu p^{m+3d} \) | \( \frac{(p^{m-2d}-\mu)(p^{m-3d}+\mu p^{m-2d}+\mu p^d)}{(p^d+1)(p^{2d}+1)} \) |
| \( p^n \) | 1 |

where $\mu = (-1)^{m/d}$.
(ii). For the case $d = d'$ and $d/t$ is even, then $\dim_{\mathbb{F}_p} C_1 = 2n_0$ and $A_i = 0$ except for

| $i$                                                                 | $A_i$                                                                 |
|----------------------------------------------------------------------|----------------------------------------------------------------------|
| $(p^t - 1)(p^{n-t} - p^{n-d} \frac{d}{2})$                           | $\frac{1}{2}(p^n - 1)(p^{n-d} - 1)/(p^{2d} - 1)$                      |
| $(p^t - 1)(p^{n-t} - p^{n-d} \frac{d}{2} - t)$                       | $\frac{1}{2}p^{n-1}(p^{n-d} + 1)(p^n - 1)$                           |
| $(p^t - 1)(p^{n-t} - p^{n-d} \frac{d}{2} - 2)$                       | $\frac{1}{2}p^{2d}(p^n - p^{n-d} - p^{n-2d} + 1)(p^n - 1)/(p^{2d} - 1)$|
| $(p^t - 1)(p^{n-t} + p^{n-d} \frac{d}{2} - t)$                       | $\frac{1}{2}p^{2d}(p^n - p^{n-d} - p^{n-2d} + 1)(p^n - 1)/(p^{2d} - 1)$|
| $(p^t - 1)(p^{n-t} + p^{n-d} \frac{d}{2} - 2)$                       | $\frac{1}{2}(p^n - 1)(p^{n-d} - 1)/(p^{2d} - 1)$                      |
| 0                                                                     | 1                                                                    |

(iii). For the case $d' = 2d$ and $k \notin \{ \frac{n}{6}, \frac{5n}{6} \}$, then $\dim_{\mathbb{F}_p} C_1 = 2n_0$ and $A_i = 0$ except for

| $i$                                                                 | $A_i$                                                                 |
|----------------------------------------------------------------------|----------------------------------------------------------------------|
| $(p^t - 1)(p^{n-t} + \mu p^{n+3d-t})$                               | $\frac{(p^{n-2d} - \mu)(p^{n-d} - \mu)(p^n - 1)}{(p^d + 1)(p^{2d} - 1)(p^{3d} + 1)}$ |
| $(p^t - 1)(p^{n-t} - \mu p^{n+2d-t})$                               | $\frac{(p^{n-d} + \mu)(p^{n-d} + \mu)(p^n - 1)}{(p^d + 1)(p^{3d} - 1)}$ |
| $(p^t - 1)(p^{n-t} + \mu p^{n-d-t})$                                | $\frac{(p^n - 1)(p^{n-d} + \mu)(p^{n-d} - \mu)(p^n - 1)}{(p^d + 1)(p^{2d} - 1)}$ |
| $(p^t - 1)(p^{n-t} - \mu p^{n-t})$                                  | $\frac{(p^n - 1)(p^{n-d} + \mu)(p^{n-d} - \mu)(p^n - 1)}{(p^d + 1)(p^{2d} - 1)}$ |
| 0                                                                     | 1                                                                    |

(iii). For the case $d' = 2d$ and $k \in \{ \frac{n}{6}, \frac{5n}{6} \}$, then $\dim_{\mathbb{F}_p} C_1 = 3n_0/2$ and $A_i = 0$ except for

| $i$                                                                 | $A_i$                                                                 |
|----------------------------------------------------------------------|----------------------------------------------------------------------|
| $(p^t - 1)(p^{n-t} + \frac{n}{3} \frac{d}{2}-t)$                    | $(p^n - 1)/(p^d + 1)$                                               |
| $(p^t - 1)(p^{n-t} - \frac{n}{3} \frac{d}{2}-t)$                    | $p^\frac{d}{2}(p^{n-d} + 1)(p^n - 1)/(p^d + 1)$                     |
| $(p^t - 1)(p^{n-t} - \frac{n}{3} \frac{d}{2} - 2)$                  | $p^\frac{d}{2}(p^{n-d} - 1)(p^n - 1)/(p^d + 1)$                     |
| 0                                                                     | 1                                                                    |

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Proof. see Appendix B.

Remark. (1). In the case \( d = d' \). Since \( \gcd(p^n - 1, p^k + 1) = 2 \), the first \( l' = \frac{n-1}{2} \) coordinates of each codeword of \( C_1 \) form a cyclic code \( C'_1 \) over \( \mathbb{F}_{p^t} \) with length \( l' \) and dimension \( 2n_0 \). Let \( (A'_0, \cdots, A'_{l'}) \) be the weight distribution of \( C'_1 \), then \( A'_i = A_{2i} \ (0 \leq i \leq l') \).

(2). In the case \( d' = 2d \). Since \( \gcd(p^n - 1, p^k + 1) = p^{d+1} \), the first \( l' = \frac{n-1}{p^{d+1}} \) coordinates of each codeword of \( C_1 \) form a cyclic code \( C'_1 \) over \( \mathbb{F}_{p^t} \) with length \( l' \) and dimension \( 2n_0 \). Let \( (A'_0, \cdots, A'_{l'}) \) be the weight distribution of \( C'_1 \), then \( A'_i = A_{(p^{d+1})i} \ (0 \leq i \leq l') \).

(3). If \( k \in \left\{ \frac{n}{4}, \frac{n}{2}, \frac{3n}{4} \right\} \), then \( \pi = \pi^{-1} \) and the dual code of \( C_1 \) has only one zero. This special case is trivial.

4 Results on Correlation Distribution of Sequences and Cyclic Code \( C_2 \)

Recall \( \phi_{\alpha,\beta}(x) \) in the proof of Lemma 2 and \( N_{i,\varepsilon} \) in the proof of Theorem 1. Finally we will determine the value distribution of \( S(\alpha, \beta, \gamma) \), the correlation distribution among sequences in \( \mathcal{F} \) defined in (4) and the weight distribution of \( C_2 \) defined in Section 1. The following result will play an important role.

Lemma 6. Assume \( t \) be a divisor of \( d \). For any \( a \in \mathbb{F}_{p^t} \) and any \( (\alpha, \beta) \in N_{i,\varepsilon} \) with \( \varepsilon = \pm 1 \), then the number of elements \( \gamma \in \mathbb{F}_q \) satisfying

(i). \( \phi_{\alpha,\beta}(x) + \gamma = 0 \) is solvable\( (\text{choose one solution, say } x_0) \),

(ii). \( \text{Tr}_t^n(\alpha x_0^{p^k} + \beta x_0^{p^h}) = a \)

is

\[
\begin{cases}
  p^{n-id-t} & \text{if } s - i \text{ and } d/t \text{ are both odd, and } a = 0, \\
  p^{n-id-t} + \varepsilon\eta'(a)p^{\frac{n-id-t}{2}} & \text{if } s - i \text{ and } d/t \text{ are both odd, and } a \neq 0, \\
  p^{n-id-t} + \varepsilon(p^t - 1)p^{\frac{n-id-t}{2}} & \text{if } s - i \text{ or } d/t \text{ is even, and } a = 0, \\
  p^{n-id-t} - \varepsilon p^{\frac{n-id-t}{2}} & \text{if } s - i \text{ or } d/t \text{ is even, and } a \neq 0.
\end{cases}
\]

where \( \eta' \) is the quadratic (multiplicative) character on \( \mathbb{F}_{p^t} \).
Proof. see Appendix C.

Let $p^* = (-1)^{\frac{p-1}{2}} p$ and $\left(\frac{\alpha}{p}\right)$ be the Legendre symbol. We are now ready to give the value distribution of $S(\alpha, \beta, \gamma)$.

Theorem 3. The value distribution of the multi-set $\{S(\alpha, \beta, \gamma) \mid \alpha \in \mathbb{F}_p^n, (\beta, \gamma) \in \mathbb{F}_q^2\}$ is shown as following.

(i). If $d' = d$ is odd (that is $n$ is odd), then

| values | multiplicity |
|--------|--------------|
| $\varepsilon \sqrt{p^*p^{\frac{n-1}{2}}}$ | $\frac{1}{2}p^n+2d-1(p^n-p^{n-d}-p^{n-2d+1}+1)(p^n-1)$ |
| $\varepsilon \zeta_p \sqrt{p^*p^{\frac{n-1}{2}}}$ | $\frac{1}{2}p^{2d}(p^{n-1}+\varepsilon \left(\frac{-1}{p}\right)p^{n-\frac{d}{2}})(p^n-p^{n-d}-p^{n-2d+1}+1)(p^n-1)$ |
| $\varepsilon p^{\frac{n-1}{2}}$ | $\frac{1}{2}p^{n-d-1}(p^{n-2d+1} + \varepsilon(p-1))(p^{n-2d+1} + \varepsilon)(p^n-1)$ |
| $\varepsilon \zeta_p \sqrt{p^*p^{\frac{n-1}{2}}}$ | $\frac{1}{2}p^{n-2d-1}(p^{n-1}(p^n-1)(p^{n-1})(p^{n-d-1}+1)$ |
| $\varepsilon \zeta_p \sqrt{p^*p^{\frac{n+1}{2}}}$ | $\frac{1}{2}p^{n-2d-1}+\varepsilon \left(\frac{-1}{p}\right)p^{n-\frac{d-1}{2}}(p^n-1)(p^{n-d-1}+1)$ |
| $0$ | $(p^n-1)(p^{2n-d}+p^{2n-2d}+p^{2n-3d}-p^{n-2d}+1)$ |

where $\varepsilon = \pm 1, 1 \leq j \leq p-1$.

(ii). If $d' = d$ is even, then

| values | multiplicity |
|--------|--------------|
| $\varepsilon p_0^\frac{n+1}{2}$ | $\frac{1}{2}p^{2d}(p^{n-1}+\varepsilon(p-1)p^{n-1})(p^n-p^{n-d}-p^{n-2n+1}+1)(p^n-1)$ |
| $\varepsilon \zeta_p \sqrt{p^*p^{\frac{n+1}{2}}}$ | $\frac{1}{2}p^{2d}(p^{n-1}-\varepsilon p_0^{n-1})(p^n-p^{n-d}-p^{n-2n+1}+1)(p^n-1)$ |
| $\varepsilon p^{\frac{n+1}{2}}$ | $\frac{1}{2}p^{n-d-1}(p^{n-2d+1} + \varepsilon(p-1))(p^{n-2d+1} + \varepsilon)(p^n-1)$ |
| $\varepsilon \zeta_p \sqrt{p^*p^{\frac{n+1}{2}}}$ | $\frac{1}{2}p^{n-2d-1}(p^{n-1}(p^n-1)(p^{n-1})(p^{n-1})+1)$ |
| $\varepsilon \zeta_p \sqrt{p^*p^{\frac{n-1}{2}}}$ | $\frac{1}{2}(p^{n-2d-1} + \varepsilon(p-1)p^{n-2d-1}+1)$ |
| $0$ | $(p^n-1)(p^{2n-d}+p^{2n-2d}+p^{2n-3d}-p^{n-2d}+1)$ |

$p^n$ | 1 |
where \( \varepsilon = \pm 1, 1 \leq j \leq p - 1 \).

(iii). For the case \( d' = 2d \),

$$
\begin{array}{|c|c|}
\hline
\text{values} & \text{multiplicity} \\
\hline
\mu p^m & \frac{(p^n-1)(p^{n-d}+\mu(p-1)p^{n-1}+p^n-d)+p^{m+3d}}{(p^d+1)(p^{d-1})} \\
\hline
\mu q_j p^m & \frac{(p^n-1)(p^{n-d}+\mu p^{m+3d})}{(p^d+1)(p^{d-1})} \\
\hline
-\mu p^{m+d} & \frac{(p^n-1)(p^{n-d}+\mu(p-1)p^{n-d})}{(p^d+1)(p^{d-1})} \\
\hline
-\mu q_j p^{m+d} & \frac{(p^n-1)(p^{n-d}+\mu p^{m+3d})}{(p^d+1)(p^{d-1})} \\
\hline
\mu p^{m+2d} & \frac{(p^n-1)(p^{n-d}+\mu(p-1)p^{n-d})}{(p^d+1)(p^{d-1})} \\
\hline
\mu q_j p^{m+2d} & \frac{(p^n-1)(p^{n-d}+\mu p^{m+3d})}{(p^d+1)(p^{d-1})} \\
\hline
-\mu p^{m+3d} & \frac{(p^n-1)(p^{n-d}+\mu(p-1)p^{n-d})}{(p^d+1)(p^{d-1})} \\
\hline
-\mu q_j p^{m+3d} & \frac{(p^n-1)(p^{n-d}+\mu p^{m+3d})}{(p^d+1)(p^{d-1})} \\
\hline
0 & \frac{(p^n-1)(1-\mu p^{m-d}+p^{m-3d})}{(p^d+1)(p^{d-1})} \\
\hline
p^d & 1 \\
\hline
\end{array}
$$

for \( 1 \leq j \leq p - 1 \).

Proof. For (i) and (ii), see [31]. For (iii), see Appendix C. \( \square \)

In order to give the correlation distribution among the sequences in \( F \), we need an easy observation.

**Lemma 7.** For any given \( \gamma \in \mathbb{F}^* \), when \((\alpha, \beta)\) runs through \( \mathbb{F}_q \times \mathbb{F}_q \), the distribution of \( S(\alpha, \beta, \gamma) \) is the same as \( S(\alpha, \beta, 1) \).

As a consequence of Theorem 1, Theorem 3, and Lemma 7, we could give the correlation distribution amidst the sequences in \( F \).

**Theorem 4.** Let \( 1 \leq k \leq n - 1 \) and \( k \notin \{ \frac{n}{6}, \frac{n}{2}, \frac{5n}{6} \} \). The collection \( F \) defined in [3] is a family of \( p^m \) \( p \)-ary sequences with period \( q - 1 \). Its correlation distribution is given as follows.

(i). For the case \( d' = d \) is odd (that is \( n \) is odd), then
where \( \varepsilon = \pm 1, 1 \leq j \leq p - 1. \)

(ii). For the case \( d' = d \) is even, then

| values | multiplicity |
|--------|--------------|
| \( \varepsilon \sqrt{p^* p} \frac{n-1}{2} - 1 \) | \( \frac{1}{2} p^{n+2d} (p^{2n-1} - 2p^{n-1} + 1) \frac{p^n - p^{n-d} - p^{n-2d} + 1}{p^{2d} - 1} \) |
| \( \varepsilon \zeta_p \sqrt{p^* p} \frac{n-1}{2} - 1 \) | \( \frac{1}{2} p^{n+2d} (p^{n-1} + \varepsilon (\frac{n}{p}) \frac{n-1}{2}) (p^n - 2) \frac{p^n - p^{n-d} - p^{n-2d} + 1}{p^{2d} - 1} \) |
| \( \varepsilon p^{n+2d} - 1 \) | \( \frac{1}{2} p^{n+2d} (\varepsilon (\frac{n}{p}) \frac{n-1}{2} + \varepsilon (p-1))(p^n - 2) \frac{p^n - d - p^{n-d} + 1}{p^{2d} - 1} \) |

where \( \varepsilon = \pm 1, 1 \leq j \leq p - 1. \)

(iii). For the case \( d' = 2d \),

| values | multiplicity |
|--------|--------------|
| \( \varepsilon \sqrt{p^* p} \frac{n-1}{2} - 1 \) | \( \frac{1}{2} p^{n+2d} ((p^{n-1} + \varepsilon (p-1) \frac{n-1}{2})(p^n - 2) + 1) \frac{p^n - p^{n-d} - p^{n-2d} + 1}{p^{2d} - 1} \) |
| \( \varepsilon \zeta_p \sqrt{p^* p} \frac{n-1}{2} - 1 \) | \( \frac{1}{2} p^{n+2d} (\varepsilon (\frac{n}{p}) \frac{n-1}{2} - \varepsilon (\frac{n}{p}) \frac{n-1}{2})(p^n - 2) \frac{p^n - d - p^{n-d} + 1}{p^{2d} - 1} \) |
| \( \varepsilon p^{n+2d} - 1 \) | \( \frac{1}{2} p^{n+2d} (\varepsilon (\frac{n}{p}) \frac{n-1}{2} + \varepsilon (p-1))(p^n - 2) \frac{p^n - d - p^{n-d} + 1}{p^{2d} - 1} \) |
| \( \varepsilon \zeta_p \sqrt{p^* p} \frac{n-1}{2} - 1 \) | \( \frac{1}{2} p^{n+2d} ((p^{n-2d-1} + \varepsilon (p-1) \frac{n-2d-1}{2})(p^n - 2) + 1) \frac{p^n - d - p^{n-d} + 1}{p^{2d} - 1} \) |

where \( \varepsilon = \pm 1, 1 \leq j \leq p - 1. \)

(iii). For the case \( d' = 2d \),
| values          | multiplicity                                                                 |
|-----------------|-----------------------------------------------------------------------------|
| $\mu p^m - 1$   | $p^{2n} \left( (p^n-2)(p^{n-1}+\mu(p-1)p^{m-1})+1 \right) \left( p^{n+6d-3p^n-3d+\mu p^m+5d-\mu p^m+6d+\mu p^m+6d+8d \right)$ |
| $\mu C_p^j p^m - 1$ | $p^{2n} \left( (p^{n-2})(p^{n-1}-\mu p^{m-1})(p^{n+6d-3p^n-3d+\mu p^m+5d-\mu p^m+6d+\mu p^m+6d+8d) \right)$ |
| $-\mu p^{m+d} - 1$ | $p^{2n} \left( (p^{n-2d-1}-\mu(p-1)p^{m-1})(p^{n-2}+1) \left( p^{n+3d}+p^{n+2d-d}+p^{n-d}+p^{n-2d-\mu p^m+3d+\mu p^m+p^d \right) \right)$ |
| $-\mu C_p^j p^{m+d} - 1$ | $p^{2n} \left( (p^{n-2d-1}+\mu p^{m-d-1})(p^{n-2})(p^{n+3d}+p^{n+2d-d}+p^{n-d}+p^{n-2d-\mu p^m+3d+\mu p^m+p^d} \right)$ |
| $\mu p^{m+2d} - 1$ | $p^{2n} \left( (p^{n-d}+\mu)(p^{n-2}+1) \left( p^{n+d}+p^{m-2d-\mu p^d}+p^{n-4d-1+\mu(p-1)p^{m-2d-1}} \right) \right)$ |
| $\mu C_p^j p^{m+2d} - 1$ | $p^{2n} \left( (p^{n-d}+\mu)(p^{n-d}+p^{m-2d-\mu p^d}+p^{n-4d-1+\mu(p-1)p^{m-2d-1}} \right)$ |
| $-\mu p^{m+3d} - 1$ | $p^{2n} \left( (p^{m-2d}-\mu)(p^{n-2}+1)(p^{n-d}+\mu)(p^{n-6d-1}-\mu(p-1)p^{m-3d-1}) \right)$ |
| $-\mu C_p^j p^{m+3d} - 1$ | $p^{2n} \left( (p^{m-2d}-\mu)(p^{m-d}+\mu)(p^{n-6d-1}+\mu p^{m-3d-1})(p^{n-2}) \right)$ |
| $-1$            | $p^{2n} \left( p^n-2 \right) \left( 1 - \mu p^{2d-m} - \mu p^{3n-8d} + p^{n-2d} \right)$ |
|                 | $\frac{p^{2n}+p^{2n-9d+\mu p^{m-3d-1}+\mu p^{m-5d-\mu p^m-4d-p^{n-6d}}}}{p^{n+1}}$ |

for $1 \leq j \leq p - 1$ and $\mu = (-1)^{m/d}$.

Recall that $C_2$ is the cyclic code over $\mathbb{F}_{p'}$, with parity-check polynomial $h_1(x)h_2(x)h_3(x)$ where $h_1(x)$, $h_2(x)$ and $h_3(x)$ are the minimal polynomials of $\pi^{-1}$, $\pi^{-(p^{k+1})}$ and $\pi^{-(p^{3k+1})}$ respectively. Here we are ready to determine the weight distribution of $C_2$.

**Theorem 5.** For $n \geq 3,k \notin \{ 0, \frac{2n}{3}, \frac{n}{2}, \frac{3n}{4}, \frac{5n}{6} \}$, the weight distribution $\{ A_0, A_1, \ldots, A_{q-1} \}$ of the cyclic code $C_2$ over $\mathbb{F}_{p'} (p \geq 3)$ with length $q - 1$ and dim$_{p'} C_1 = 3n_0$ is shown as follows:

(i). the case $d' = d$ and $d/t$ is odd, 

17
\[
\begin{array}{|c|c|}
\hline
i & A_i \\
\hline
(p^t - 1)p^{n-t} - p^{\frac{n+d-t}{2}} & \frac{1}{2} p^{\frac{n-2d-t}{2}} (p^t - 1)(p^{\frac{n-2d-t}{2}} + 1) \frac{(p^{n-d-1})p^{n-1}}{p^{d-1}} \\
(p^t - 1)(p^{n-d} - p^{\frac{n+d-t}{2}}) & \frac{1}{2} p^{n-d-t}(p^{\frac{n-d}{2}} + 1)(p^{\frac{n-d}{2}} + p^t - 1)(p^n - 1) \\
(p^t - 1)p^{n-t} - p^{\frac{n-d-t}{2}} & \frac{1}{2} p^{n-d-t}(p^t - 1)(p^{n-d} - 1)(p^n - 1) \\
(p^t - 1)p^n - t - p^{\frac{n-d-t}{2}} & \frac{1}{2} p^{n-2d-t}(p^t - 1)(p^{\frac{n-d}{2}} + 1)(p^n - p^{n-d} - p^{n-2d} + 1) \frac{p^{n-1}}{p^{d-1}} \\
(p^t - 1)p^{n-t} & (p^n - 1)(p^{2n-t} + p^{2n-d} - p^{2n-d-t} - p^{2n-2d} + p^{2n-3d} \\
& - p^{2n-3d-t} + p^{n-t} - p^{n-2d} + p^{n-2d-t} + 1) \\
(p^t - 1)p^n - t - p^{\frac{n-d-t}{2}} & \frac{1}{2} p^{n-2d-t}(p^t - 1)(p^{\frac{n-d}{2}} - 1)(p^n - p^{n-d} - p^{n-2d} + 1) \frac{p^{n-1}}{p^{d-1}} \\
(p^t - 1)p^{n-t} + p^{\frac{n-d-t}{2}} & \frac{1}{2} p^{n-d-t}(p^t - 1)(p^{n-d} - 1)(p^n - 1) \\
(p^t - 1)(p^{n-d} + p^{\frac{n+d-t}{2}}) & \frac{1}{2} p^{n-d-t}(p^{\frac{n-d}{2}} - 1)(p^{\frac{n-d}{2}} + p^t - 1)(p^n - 1) \\
(p^t - 1)p^n - t + p^{\frac{n-2d-t}{2}} & \frac{1}{2} p^{n-2d-t}(p^t - 1)(p^{\frac{n-2d-t}{2}} - 1) \frac{(p^{n-d-1})p^{n-1}}{p^{d-1}} \\
0 & 1 \\
\hline
\end{array}
\]

(ii). the case \(d' = d\) and \(d/t\) is even,
$$
\begin{array}{|c|c|}
\hline
i & A_i \\
\hline
(p^t - 1)(p^{n-t} - p^{\frac{n}{2} + d-t}) & \frac{1}{2}p^{\frac{n}{2} - d-t}(p^{\frac{n}{2} - d} + p^t - 1)(p^{n-d} - 1)_{p^{2n-1}-1} \\
(p^t - 1)p^{n-t} - p^{\frac{n}{2} + d-t} & \frac{1}{2}p^{\frac{n}{2} - d-t}(p^t - 1)(p^{\frac{n}{2} - d} + 1)(p^{n-d} - 1)_{p^{2n-1}-1} \\
(p^t - 1)(p^{n-t} - p^{\frac{n}{2} + d-t}) & \frac{1}{2}p^{n-d-t}(p^t - 1)(p^{\frac{n}{2} - d} + 1)(p^{n-d} - 1)_{p^{2n-1}-1} \\
(p^t - 1)p^{n-t} - p^{\frac{n}{2} - t} & \frac{1}{2}p^{n-d-t}(p^t - 1)(p^{n-d} - 1)(p^{n-1}) \\
(p^t - 1)(p^{n-t} - p^{\frac{n}{2} - t}) & \frac{1}{2}p^{\frac{n}{2} + 2d-t}(p^{\frac{n}{2}} + p^t - 1)(p^{n-p^{n-d} - p^{n-2d} + 1})_{p^{2n-1}-1} \\
(p^t - 1)p^{n-t} + p^{\frac{n}{2} - t} & \frac{1}{2}p^{\frac{n}{2} + 2d-t}(p^t - 1)(p^{\frac{n}{2}} - 1)(p^{n-p^{n-d} - p^{n-2d} + 1})_{p^{2n-1}-1} \\
(p^t - 1)(p^{n-t} + p^{\frac{n}{2} - t}) & \frac{1}{2}p^{\frac{n}{2} + 2d-t}(p^{\frac{n}{2}} - p^t + 1)(p^{n-p^{n-d} - p^{n-2d} + 1})_{p^{2n-1}-1} \\
(p^t - 1)p^{n-t} + p^{\frac{n}{2} - t} & \frac{1}{2}p^{n-d-t}(p^t - 1)(p^{n-d} - 1)(p^{n-1}) \\
(p^t - 1)(p^{n-t} + p^{\frac{n}{2} - d - t}) & \frac{1}{2}p^{\frac{n}{2} - d-t}(p^t - 1)(p^{\frac{n}{2} - d} - 1)(p^{n-d} - 1)_{p^{2n-1}-1} \\
(p^t - 1)(p^{n-t} + p^{\frac{n}{2} - d - t}) & \frac{1}{2}p^{\frac{n}{2} + d-t}(p^t - 1)(p^{\frac{n}{2} - d} - p^t + 1)(p^{n-d} - 1)_{p^{2n-1}-1} \\
0 & 1 \\
\hline
\end{array}
$$

(iii). the case $d' = 2d$,
\[ \begin{array}{|c|c|}
\hline
i & A_i \\
\hline
(p^t - 1) (p^{n-t} - \mu p^{m-t}) & \left( p^{n-t} + \mu (p^t - 1)p^{m-t} \right) (p^{n-t} - 1) (p^{n+6d-p^{m+4d}+p^{n+d}+\mu p^{m+5d}+\mu p^{m+4d}+p^{6d}} \\
(p^t - 1) p^{n-t} + \mu p^{m-t} & \left( p^{n-t} - 1 \right) (p^{n+6d-p^{m+4d}+p^{n+d}+\mu p^{m+5d}+\mu p^{m+4d}+p^{6d}} \\
(p^t - 1) (p^{n-t} + \mu p^{m+d-t}) & \left( p^{n-t} - 1 \right) (p^{n+4d+p^{m+2d}+p^{n-d}+p^{n-2d}+\mu p^{m+3d}+\mu p^{m}+p^{3d}} \\
(p^t - 1) p^{n-t} - \mu \mu p^{m+d-t} & \left( p^{n-t} - 1 \right) (p^{n+3d+p^{n+2d}+p^{n-d}+p^{n-2d}+\mu p^{m+3d}+\mu p^{m}+p^{3d}} \\
(p^t - 1) (p^{n-t} - \mu p^{m+2d-t}) & \left( p^{n-t} - 1 \right) (p^{n+4d}+p^{n+2d}+\mu (p^{m+4d}+p^{n+2d}+\mu p^{m}+p^{3d}) (p^{n-1}} \\
(p^t - 1) p^{n-t} + \mu p^{m+2d-t} & \left( p^{n-t} - 1 \right) (p^{n+3d+\mu p^{m+2d}+p^{n-d}+\mu p^{m+3d}+\mu p^{m}+p^{3d}} \\
(p^t - 1) (p^{n-t} + \mu p^{m+3d-t}) & \left( p^{n-t} - 1 \right) (p^{n+3d+p^{n-d}+p^{n-2d}+\mu p^{m+3d}+\mu p^{m}+p^{3d}} \\
(p^t - 1) p^{n-t} - \mu p^{m+3d-t} & \left( p^{n-t} - 1 \right) (p^{n+4d+p^{m+3d}+\mu p^{m-d}+\mu p^{m-2d}+\mu p^{m}+p^{3d}} \\
(p^t - 1)p^{n-t} & \left( p^{n+2n-9d} - \mu p^{3m} + \mu p^{m-d} + \mu p^{m-3d} - \mu p^{m-3d} - \mu p^{m-4d} + p^{n-6d} + p^{d+1}\right) \frac{p^{n-1}}{p^t + 1} \\
\hline
p^n & 1 \\
\hline
\end{array} \]

where \( \mu = (-1)^{\frac{m}{a}} \).

Proof. see Appendix C. \( \square \)

**Remark.** (i). The case (i) and (ii) with \( t = 1 \) has been shown in [31], Theorem 2.

(ii). If \( k = n/6 \) or \( 5n/6 \), then \( C_2 \) has dimension \( 5n/2 \). Its weight distribution has been determined in [13].

## 5 Appendix A

**Proof of Lemma 2 (ii):** For \( Y = (y_1, \cdots, y_s) \in \mathbb{F}_{q_0}^s \), \( y = y_1 v_1 + \cdots + y_s v_s \in \mathbb{F}_q \), we know that

\[ F_{\alpha, \beta}(X + Y) - F_{\alpha, \beta}(X) - F_{\alpha, \beta}(Y) = 2X H_{\alpha, \beta} Y^T \]  (12)

is equal to

\[ f_{\alpha, \beta}(x + y) - f_{\alpha, \beta}(x) - f_{\alpha, \beta}(y) = \text{Tr}_d \left( y^{p^{3k}}(\alpha^{p^{3k}} x^{p^{6k} + \beta p^{3k} x^{p^{6k}} + \beta p^{2k} x^{p^{2k}} + \alpha x}) \right) \]  (13)
Let
\[ \phi_{\alpha,\beta}(x) = \alpha x^{3k} + \beta x^{2k} + \alpha x. \] (14)

Therefore,
\[ r_{\alpha,\beta} = r \iff \text{the number of common solutions of } XH_{\alpha,\beta}Y^T = 0 \text{ for all } Y \in \mathbb{F}_q^s \text{ is } q_0^{s-r}, \]
\[ \iff \text{the number of common solutions of } \text{Tr}^n_d \left(y^{3k} \cdot \phi_{\alpha,\beta}(x)\right) = 0 \text{ for all } y \in \mathbb{F}_q \text{ is } q_0^{s-r}, \]
\[ \iff \phi_{\alpha,\beta}(x) = 0 \text{ has } q_0^{s-r} \text{ solutions in } \mathbb{F}_q. \]

Fix an algebraic closure \( \mathbb{F}_{p^\infty} \) of \( \mathbb{F}_p \), since the degree of \( p^{2k} \)-linearized polynomial \( \phi_{\alpha,\beta}(x) = p^{6k} \) and \( \phi_{\alpha,\beta}(x) = 0 \) has no multiple roots in \( \mathbb{F}_{p^\infty} \) (this fact follows from \( \phi'_{\alpha,\beta}(x) = \alpha \in \mathbb{F}_q^* \)), then the zeroes of \( \phi_{\alpha,\beta}(x) \) in \( \mathbb{F}_{p^\infty} \), say \( V \), form an \( \mathbb{F}_{p^{2k}} \)-vector space of dimension 3. Note that \( \gcd(n, 2k) = 2d \). Then \( V \cap \mathbb{F}_{p^n} \) is a vector space on \( \mathbb{F}_{p^{gcd(n,2k)}} = \mathbb{F}_{p^d} \) with dimension at most 3 since any elements in \( \mathbb{F}_q \) which are linear independent over \( \mathbb{F}_{p^d} \) are also linear independent over \( \mathbb{F}_{p^{2k}} \) (see [25], Lemma 4). Since \( \mathbb{F}_{p^d} \) could be regarded as a 2-dimensional vector space over \( \mathbb{F}_{p^d} \), then the possible values of \( r_{\alpha,\beta} \) is \( s, s-2, s-4 \) and \( s-6 \) for \( (\alpha, \beta) \in \mathbb{F}_q^2 \setminus \{(0,0)\} \). □

**Proof of Lemma 3** (i). We observe that
\[
\sum_{\alpha,\beta \in \mathbb{F}_q} T(\alpha, \beta) = \sum_{\alpha,\beta \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^d(\alpha x^{p^{3k+1}} + \beta x^{p^{k+1}})}
= \sum_{x \in \mathbb{F}_q} \sum_{\alpha, \beta \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^d(\alpha x^{p^{3k+1}})} \sum_{\beta \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^d(\beta x^{p^{k+1}})} = q \cdot \sum_{\alpha, \beta \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^d(\alpha x^{p^{3k+1}})} = p^{2n}.
\]

(ii). We can calculate
\[
\sum_{\alpha,\beta \in \mathbb{F}_q} T(\alpha, \beta)^2 = \sum_{x, y \in \mathbb{F}_q} \sum_{\alpha, \beta \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^d(\alpha x^{p^{3k+1}} + y^{p^{3k+1}})} \sum_{\beta \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^d(\beta (x^{p^{k+1}} + y^{p^{k+1}}))}
= M_2 \cdot p^{2n}
\]

where \( M_2 \) is the number of solutions to the equation
\[
\begin{cases}
  x^{p^{3k+1}} + y^{p^{3k+1}} = 0 \\
  x^{p^{k+1}} + y^{p^{k+1}} = 0
\end{cases}
\] (15)

If \( xy = 0 \) satisfying (15), then \( x = 0 \). Otherwise \( (x/y)^{p^{2k}} = (x/y)^{p^{k+1}} = -1 \) which yields that \( (x/y)^{p^{2k} - 1} = 1 \). Denote by \( x = ty \). Since \( \gcd(2k, n) = d' \), then \( t \in \mathbb{F}_p^{d'} \).
• If \(d' = d\), then \(t \in \mathbb{F}_{p^d}^*\) and (13) is equivalent to \(x^2 + y^2 = 0\). Hence \(t^2 = -1\). There are two or none of \(t \in \mathbb{F}_{p^d}^*\) satisfying \(t^2 = -1\) depending on \(p^d \equiv 1\) (mod 4) or \(p^d \equiv 3\) (mod 4). Therefore

\[
M_2 = \begin{cases} 
1 + 2(q - 1) = 2q - 1, & \text{if } p^d \equiv 1 \pmod{4} \\
1, & \text{if } p^d \equiv 3 \pmod{4}.
\end{cases}
\]

• If \(d' = 2d\), then by (10) we get (15) is equivalent to \(x^{p^d+1} + y^{p^d+1} = 0\). Then we have \(t^{p^d+1} = -1\) which has \(p^d+1\) solutions in \(\mathbb{F}_{p^d}^*\). Therefore

\[
M_2 = (p^d + 1)(p^n - 1) + 1 = p^{n+d} + p^n - p^d.
\]

(iii). We have

\[
\sum_{\alpha, \beta \in \mathbb{F}_q} T(\alpha, \beta)^3 = M_3 \cdot q^2 \quad \text{where}
\]

\[
M_3 = \# \left\{ (x, y, z) \in \mathbb{F}_q^3 \mid x^{p^{3k+1}} + y^{p^{3k+1}} + z^{p^{3k+1}} = 0, x^{p^{k+1}} + y^{p^{k+1}} + z^{p^{k+1}} = 0 \right\}
\]

\[
= M_2 + T' \cdot (q - 1)
\]

and \(T'\) is the number of \(\mathbb{F}_q\)-solutions of

\[
\begin{cases} 
x^{p^{3k+1}} + y^{p^{3k+1}} + 1 = 0 \\
x^{p^{k+1}} + y^{p^{k+1}} + 1 = 0.
\end{cases}
\]

Canceling \(y\) we have \((x^{p^{3k+1}} + 1)^{p^{k+1}} = (x^{p^{k+1}} + 1)^{p^{3k+1}}\) which is equivalent to

\[(x^{p^{4k}} - x)(x^{p^{4k}} - x^{p^{3k}}) = 0.
\]

Therefore \(x^{p^{4k}} = x\) or \(x^{p^{4k}} = x^{p^{3k}}\).

• If \(n/d \equiv 2\) (mod 4), then \(x \in \mathbb{F}_{p^{2d}}\) and symmetrically \(y \in \mathbb{F}_{p^{2d}}\). Hence (17) is equivalent to \(x^{p^d+1} + y^{p^d+1} + 1 = 0\) which is the well-known Hermitian curve on \(\mathbb{F}_{p^{2d}}\). It follows that \(T' = p^{3d} - p^d\).
If \( n/d \equiv 0 \pmod{4} \), then \( x \in \mathbb{F}_{p^d} \) and hence \( y \in \mathbb{F}_{p^d} \). In this case
\[
(x^{p^k+1} + y^{p^k+1} + 1)^{p^k} = x^{p^{3k}+1} + y^{p^{3k}+1} + 1
\]
and then (17) is equivalent to
\[
(x^{d+1} + y^{d+1} + 1) = 0
\]
which is a minimal curve on \( \mathbb{F}_{p^d} \) with genus \( \frac{1}{2} p^d (p^d - 1) \).

Hence \( T' = p^d + 1 - p^d (p^d - 1) p^{2d} - (p^d + 1) = p^{3d} - p^d \).

Anyway, \( M_3 = (p^{n+d} + p^n - p^d) + (p^n - 1)(p^3d - p^d) = p^{n+3d} + p^n - p^{3d} \). \( \square \)

**Remark.** For the case \( d' = d \), \( \sum_{\alpha, \beta \in \mathbb{F}_q} T(\alpha, \beta)^3 \) can also be determined, but we do not need this result.

**Proof of Lemma 4.** We get that
\[
qN = \sum_{\omega \in \mathbb{F}_{q^d}} \sum_{x,y \in \mathbb{F}_q} \zeta_p \left( \left( \alpha x^{p^k+1} + \beta x y^{p^k+1} - y^{p^d} \right) \right)
\]
\[
= q^2 + \sum_{\omega \in \mathbb{F}_{q^d}} \sum_{x \in \mathbb{F}_q} \zeta_p \left( \left( \alpha x^{p^k+1} + \beta x y^{p^k+1} \right) \right) \sum_{y \in \mathbb{F}_q} \zeta_p \left( y^{p^d} - \omega \right)
\]
\[
= q^2 + q \sum_{\omega \in \mathbb{F}_{q^d}} \sum_{x \in \mathbb{F}_q} \zeta_p \left( \left( \alpha x^{p^k+1} + \beta x y^{p^k+1} \right) \right)
\]
where the 3-rd equality follows from that the inner sum is zero unless \( \omega^{p^d} - \omega = 0 \), i.e. \( \omega \in \mathbb{F}_{q^d} \).

For any \( \omega \in \mathbb{F}_{q^d} \), by (9) we have \( F_{\omega,\alpha,\omega\beta}(X) = \omega \cdot F_{\alpha,\beta}(X) \), \( H_{\omega,\alpha,\omega\beta} = \omega \cdot H_{\alpha,\beta} \) and \( r_{\omega,\alpha,\omega\beta} = r_{\alpha,\beta} \). From Lemma 1 (i) we know that
\[
T(\omega, \alpha, \beta) = \sum_{X \in \mathbb{F}_{q^d}} \zeta_p \left( \left( X H_{\omega,\alpha,\omega\beta} X^T \right) \right) = \eta_0(\omega)^{r_{\alpha,\beta}} T(\alpha, \beta).
\]

In the case \( d' = 2d \), by Lemma 2 (ii) we get that \( r_{\alpha,\beta} \) is even. Hence \( T(\omega, \alpha, \beta) = T(\alpha, \beta) \) for any \( \omega \in \mathbb{F}_{q^d} \) and \( N = q + (p^d - 1) T(\alpha, \beta) \). \( \square \)

### 6 Appendix B

**Proof of Theorem 1 (ii):**

In the case \( d' = 2d \) (\( n/d \) is even and \( k/d \) is odd) and \( r_{\alpha,\beta} = s, s - 2, s - 4 \) or \( s - 6 \) for \( (\alpha, \beta) \neq (0, 0) \). According to Lemma 1 and Lemma 5, we get that for \( (\alpha, \beta) \in N_i \), \( T(\alpha, \beta) = (-1)^{m/d + 1/2} p^{n + 1/2} \).
Combining Lemma 2 and Lemma 3 we have
\[ n_0 + n_2 + n_4 + n_6 = p^{2n} - 1 \] (19)

\[ n_0 - p^d \cdot n_2 + p^{2d} \cdot n_4 - p^{3d} \cdot n_6 = (-1)^{m/d} p^m (p^n - 1) \] (20)

\[ n_0 + p^{2d} \cdot n_2 + p^{4d} \cdot n_4 + p^{6d} \cdot n_6 = p^n (p^d + 1) (p^n - 1). \] (21)

\[ n_0 - p^{3d} \cdot n_2 + p^{6d} \cdot n_4 - p^{9d} \cdot n_6 = (-1)^{m/d} p^{m+3d} (p^n - 1). \] (22)

Solving the system of equations consisting of (19)–(22) yields the result. □

**Proof of Theorem 2**: From (1) we know that for each non-zero codeword \( c(\alpha, \beta) = (c_0, \cdots, c_{l-1}) \) \( (l = q - 1, c_i = \text{Tr}_1^n(\alpha \pi^{(p^{3k+1})i} + \beta \pi^{(p^k+1)i}) \), 0 \( \leq i \leq l - 1 \), and \((\alpha, \beta) \in \mathbb{F}_q \times \mathbb{F}_q\)), the Hamming weight of \( c(\alpha, \beta) \) is
\[ w_H (c(\alpha, \beta)) = p^{n-t}(p^t - 1) - \frac{1}{p^t} \cdot R(\alpha, \beta) \] (23)

where
\[ R(\alpha, \beta) = \sum_{a \in \mathbb{F}_{p^t}^*} T(a \alpha, a \beta) = T(\alpha, \beta) \sum_{a \in \mathbb{F}_{p^t}^*} \eta_0(a)^{r_{\alpha, \beta}} \]

by Lemma 1 (i).

Let \( \eta' \) be the quadratic (multiplicative) character on \( \mathbb{F}_q \). Then we have

(1). if \( d/t \) or \( r_{\alpha, \beta} \) is even, then \( \sum_{a \in \mathbb{F}_{p^t}^*} \eta_0(a)^{r_{\alpha, \beta}} = \sum_{a \in \mathbb{F}_{p^t}^*} 1 = p^t - 1 \) and \( R(\alpha, \beta) = (p^t - 1) T(\alpha, \beta) \).

(2). if \( d/t \) and \( r_{\alpha, \beta} \) are both odd, then \( \sum_{a \in \mathbb{F}_{p^t}^*} \eta_0(a)^{r_{\alpha, \beta}} = \sum_{a \in \mathbb{F}_{p^t}^*} \eta'(a) = 0 \) and \( R(\alpha, \beta) = 0 \).

Thus the weight distribution of \( C_1 \) can be derived from Theorem 1 and (23) directly. For example, if \( d/t \) is odd and \( d' = d \), then

(1). if \( r_{\alpha, \beta} = s \) and \( T(\alpha, \beta) = p^m \), then \( w_H (c(\alpha, \beta)) = (p^t - 1)(p^{n-t} - p^{m-t}) \).

(2). if \( r_{\alpha, \beta} = s \) and \( T(\alpha, \beta) = -p^m \), then \( w_H (c(\alpha, \beta)) = (p^t - 1)(p^{n-t} + p^{m-t}) \).

(3). if \( r_{\alpha, \beta} = s - 1 \), then \( w_H (c(\alpha, \beta)) = (p^t - 1)p^{n-t} \).

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(4). if \( r_{\alpha, \beta} = s - 2 \) and \( T(\alpha, \beta) = -p^{m+d} \), then \( w_H(c(\alpha, \beta)) = (p^t - 1)(p^{n-t} + p^{m+d-t}) \).

\[ \square \]

7 Appendix C

**Proof of Lemma**

Define \( n(\alpha, \beta, a) \) to be the number of \( \gamma \in \mathbb{F}_q \) satisfying (i) and (ii). From (9) we know that \( XH_{\alpha, \beta}X^T = \text{Tr}_d^n(\alpha x^{p^k+1} + \beta x^{p^k+1}) \). Combining (2), (12) and (13) we can get

\[
2XH_{\alpha, \beta} + A_\gamma = 0 \iff 2XH_{\alpha, \beta}Y^T + A_\gamma Y^T = 0 \text{ for all } Y \in \mathbb{F}_q^s \\
\iff \text{Tr}_d^n(y\phi_{\alpha, \beta}(x)) + \text{Tr}_d^n(\gamma y) = 0 \text{ for all } y \in \mathbb{F}_q \\
\iff \text{Tr}_d^n(y(\phi_{\alpha, \beta}(x) + \gamma)) = 0 \text{ for all } y \in \mathbb{F}_q \\
\iff \phi_{\alpha, \beta}(x) + \gamma = 0. \tag{24}
\]

Let \( x_0, x'_0 \) be two distinct solutions of (i) (if exists). We can get \( x_0 = X_0 \cdot V^T \) and \( x'_0 = X'_0 \cdot V^T \) with \( X_0, X'_0 \in \mathbb{F}_q^s \) and \( V = (v_1, \cdots, v_n) \). Define \( \Delta X_0 = X'_0 - X_0 \) and \( \Delta x_0 = x'_0 - x_0 = X_0 \cdot V^T \). Then

\[
\phi_{\alpha, \beta}(x_0) + \gamma = \phi_{\alpha, \beta}(x'_0) + \gamma = 0
\]
gives us

\[
2X_0H_{\alpha, \beta} + A_\gamma = 2X'_0H_{\alpha, \beta} + A_\gamma = 0
\]
and hence

\[
\Delta X_0 \cdot H_{\alpha, \beta} = 0.
\]

It follows that

\[
X'_0 \cdot H_{\alpha, \beta} \cdot X'_0^T = (X_0 + \Delta X_0) \cdot H_{\alpha, \beta} \cdot (X_0 + \Delta X_0)^T \\
= X_0H_{\alpha, \beta}X_0^T + \Delta X_0 \cdot H_{\alpha, \beta} \cdot (\Delta X_0 + 2X_0) = X_0H_{\alpha, \beta}X_0^T.
\]

Therefore

\[
\text{Tr}_t^m(\alpha x'_0p^{m+1}) + \text{Tr}_t^n(\beta x'_0p^{k+1}) = \text{Tr}_t^d \left( \text{Tr}_t^m(\alpha x_0p^{m+1}) + \text{Tr}_t^n(\beta x_0p^{k+1}) \right) = \text{Tr}_t^d \left( X'_0 \cdot H_{\alpha, \beta} \cdot X'_0^T \right) \\
= \text{Tr}_t^d \left( X_0H_{\alpha, \beta}X_0^T \right) = \text{Tr}_t^d \left( \text{Tr}_t^m(\alpha x_0p^{m+1}) + \text{Tr}_t^n(\beta x_0p^{k+1}) \right) = \text{Tr}_t^m(\alpha x_0p^{m+1}) + \text{Tr}_t^n(\beta x_0p^{k+1}).
\]
Hence \( n(\alpha, \beta, a) \) is well-defined (independent of the choice of \( x_0 \)).

If (i) is satisfied, that is, \( \phi_{\alpha, \beta}(x) + \gamma = 0 \) has solution(s) in \( \mathbb{F}_q \) which yields that \( 2XH_{\alpha, \beta} + A_\gamma = 0 \) has solution(s). Note that rank \( H_{\alpha, \beta} = s - i \). Therefore \( 2XH_{\alpha, \beta} + A_\gamma = 0 \) has \( q_0^i = p^{id} \) solutions with \( X \in \mathbb{F}_{q_0}^s \), which is equivalent to saying \( \phi_{\alpha, \beta}(x) + \gamma = 0 \) has \( p^{id} \) solutions in \( \mathbb{F}_q \). Conversely, for any \( x_0 \in \mathbb{F}_q \), we can determine \( \gamma \) by \( \gamma = -\phi_{\alpha, \beta}(x_0) \). Let \( N(\alpha, \beta, a) \) be the number of \( x_0 \in \mathbb{F}_q \) satisfying (ii). Then we have \( n(\alpha, \beta, a) = N(\alpha, \beta, a)/p^{id} \).

Let \( \chi'(x) = \zeta_p^{\Tr_1(x)} \) with \( x \in \mathbb{F}_{p^t} \) be an additive character on \( \mathbb{F}_{p^t} \) and \( G(\eta', \chi') = \sum_{x \in \mathbb{F}_{p^t}} \eta'(x)\chi'(x) \) be the Gaussian sum on \( \mathbb{F}_{p^t} \). We can calculate

\[
p^i \cdot N(\alpha, \beta, a) = \sum_{x \in \mathbb{F}_q} \sum_{\omega \in \mathbb{F}_{p^t}} \zeta_p^{\Tr_1(\omega(\Tr_p(\alpha x p^k + 1 + \beta x p^{k+1}) - a))} \]

\[
= p^n + \sum_{\omega \in \mathbb{F}_{p^t}} T(\omega \alpha, \omega \beta) \zeta_p^{-\Tr_1(a \omega)} \]

\[
= p^n + T(\alpha, \beta) \cdot \sum_{\omega \in \mathbb{F}_{p^t}} \eta_0(\omega)^{s-i} \chi'(-a \omega)
\]

where the 3-rd equality holds from \([18]\) for any \( \omega \in \mathbb{F}_{p^t}^* \subset \mathbb{F}_{q_0}^* \).

- If \( s - i \) and \( d/t \) are both odd, and \( a = 0 \), then \( \eta_0(\omega)^{s-i} = \eta'(\omega) \) and \( N(\alpha, \beta, 0) = p^{n-t} \).

- If \( s - i \) and \( d/t \) are both odd, and \( a \neq 0 \), then

\[
N(\alpha, \beta, a) = p^{n-t} + \frac{1}{p^t} \cdot T(\alpha, \beta) \cdot \sum_{\omega \in \mathbb{F}_{p^t}^*} \eta_0(\omega) \chi'(-a \omega)
\]

\[
= p^{n-t} + \frac{1}{p^t} \cdot T(\alpha, \beta) \cdot \eta'(-a) \cdot G(\eta', \chi')
\]

\[
= p^{n-t} + \varepsilon \eta'(a) p^\frac{n+id-t}{2}
\]

where the 2-nd equality follows from the explicit evaluation of quadratic Gaussian sums (see \([19]\), Theorem 5.15 and 5.33).

- If \( s - i \) or \( d/t \) is even, and \( a = 0 \), then \( \eta_0(\omega)^{s-i} = 1 \) for any \( \omega \in \mathbb{F}_{p^t}^* \) and

\( N(\alpha, \beta, 0) = p^{n-t} + \varepsilon(p^t - 1)p^\frac{n+id-t}{2} \).

- If \( s - i \) or \( d/t \) is even, and \( a \neq 0 \), then

\[
N(\alpha, \beta, a) = p^{n-t} + \frac{1}{p^t} \cdot T(\alpha, \beta) \cdot \sum_{\omega \in \mathbb{F}_{p^t}^*} \chi'(-a \omega)
\]

\[
= p^{n-t} - \varepsilon p^\frac{n+id}{2} - t.
\]

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Therefore we complete the proof by dividing $p^\delta$. □

**Proof of Theorem 3 (iii):** Define

$$\Xi = \{(\alpha, \beta, \gamma) \in \mathbb{F}_q^3 | S(\alpha, \beta, \gamma) = 0 \}$$

and $\xi = |\Xi|$.

Recall $n_i, H_{\alpha, \beta, \gamma}, A_{\gamma}$ in Section 1 and $N_{i, \varepsilon}, n_{i, \varepsilon}$, in the proof of Lemma 2. Note that $2XH_{0,0} + A_{\gamma} = 0$ is solvable if and only if $\gamma = 0$. If $(\alpha, \beta) \in N_{i, \varepsilon}$, then the number of $\gamma \in \mathbb{F}_q$ such that $2XH_{\alpha, \beta} + A_{\gamma} = 0$ is solvable is $q^2 - 1 = p^n - d$. In the case $d' = 2d$, from Lemma 2 (i) we get that

$$\xi = p^n - 1 + (p^n - p^{n-2d})n_{2,1} + (p^n - p^{n-4d})n_{4,-1}$$

$$= (p^n - 1)\left[1 + (p^{2n} + p^{2n-9d} - \varepsilon p^{3m} + \varepsilon p^{3m-d} + \varepsilon p^{3m-3d} - \varepsilon p^{3m-5d} - \varepsilon p^{3m-7d} + \varepsilon p^{3m-8d} + p^n - p^{n-d} - p^{n-4d} - p^{n-6d})/(p^d + 1)\right]$$

(25)

Assume $(\alpha, \beta) \in N_{i, \varepsilon}$ and $\phi_{\alpha, \beta}(x) + \gamma = 0$ has solution(s) in $\mathbb{F}_q$ (choose one, say $x_0$). Then by Lemma 1 we get

$$S(\alpha, \beta, \gamma) = \zeta_p^{-\text{Tr}_p\left((\alpha x_0^{p^{3k+1}+\varepsilon x_0^{2k+1}})\right)} \cdot T(\alpha, \beta).$$

Applying Lemma 1 for $t = 1$ and Theorem 1 we get the result. □

**Proof of Theorem 4:** Recall $M_{(\alpha_1, \beta_1), (\alpha_2, \beta_2)}(\tau)$ defined in (24) and (26). Fix $(\alpha_2, \beta_2) \in \mathbb{F}_q \times \mathbb{F}_q$, when $(\alpha_1, \beta_1)$ runs through $\mathbb{F}_q \times \mathbb{F}_q$ and $\tau$ takes value from 0 to $q - 2$, $(\alpha', \beta', \gamma')$ runs through $\mathbb{F}_q \times \mathbb{F}_q \times \{\mathbb{F}_q \backslash \{1\}\}$ exactly one time.

For any possible value $\kappa$ of $S(\alpha, \beta, \gamma)$, define

$$s_\kappa = \# \{(\alpha, \beta, \gamma) \in \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q | S(\alpha, \beta, \gamma) = \kappa\}$$

$$s^1_\kappa = \# \{(\alpha, \beta, \gamma) \in \mathbb{F}_q \times \mathbb{F}_q \times \{\mathbb{F}_q \backslash \{1\}\} | S(\alpha, \beta, \gamma) = \kappa\}$$

and

$$t_\kappa = \# \{(\alpha, \beta) \in \mathbb{F}_q \times \mathbb{F}_q | T(\alpha, \beta) = \kappa\}.$$

By Lemma 7 we have

$$s^1_\kappa = \frac{q - 2}{q - 1} \times (s_\kappa - t_\kappa) + t_\kappa = \frac{q - 2}{q - 1} \times s_\kappa + \frac{1}{q - 1} \times t_\kappa.$$

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Define $M_{\kappa}$ to be the number of $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ such that $M_{(\alpha_1, \beta_1), (\alpha_2, \beta_2)} = \kappa$. Hence we get

$$M_{\kappa} = p^{2n} \cdot s_{\kappa} = p^{2n} \cdot \left( \frac{q - 2}{q - 1} \cdot s_{\kappa} + \frac{1}{q - 1} \cdot t_{\kappa} \right).$$

Then the result follows from Theorem 1 and Theorem 3.

**Proof of Theorem 5**: From (1) we know that for each non-zero codeword $c(\alpha, \beta, \gamma) = (c_0, \ldots, c_{q-2})$ $(c_i = \text{Tr}_{t}^n(\alpha \pi^{p^k+1}) + \beta \pi^{(p^k+1)\gamma} + \gamma \pi^i)$, $0 \leq i \leq q - 2$, and $(\alpha, \beta, \gamma) \in F_q \times F_q^2$, the Hamming weight of $c(\alpha, \beta, \gamma)$ is

$$w_H(c(\alpha, \beta, \gamma)) = p^n - t(p^t - 1) - \frac{1}{p^t} \cdot R(\alpha, \beta, \gamma)$$

where

$$R(\alpha, \beta, \gamma) = \sum_{\omega \in F_{p^t}^*} S(\omega \alpha, \omega \beta, \omega \gamma).$$

For any $\omega \in F_{p^t}^* \subset F_{q_0}^*$, we have $\phi_{\alpha, \beta}(x) + \omega \gamma = 0$ is equivalent to $\phi_{\alpha, \beta}(x) + \gamma = 0$. Let $x_0 \in F_q$ be a solution of $\phi_{\alpha, \beta}(x) + \gamma = 0$ (if exist).

(1). If $\phi_{\alpha, \beta}(x) + \gamma = 0$ has solutions in $F_q$, then by Lemma 1 and (18) we have

$$S(\omega \alpha, \omega \beta, \omega \gamma) = \zeta_p^{-\left(\text{Tr}_{t}^n(\omega \alpha x_0^{p^k+1} + \omega \beta x_0^{p^k+1})\right)} T(\omega \alpha, \omega \beta) = \zeta_p^{-\left(\text{Tr}_{t}^n(\omega \alpha x_0^{p^k+1} + \omega \beta x_0^{p^k+1})\right)} \eta_0(\omega)^{r_{a, b}} T(\alpha, \beta).$$

Hence

$$R(\alpha, \beta, \gamma) = T(\alpha, \beta) \sum_{\omega \in F_{p^t}^*} \zeta_p^{-\text{Tr}_{t}(\omega \left(\text{Tr}_{t}^n(\alpha x_0^{p^m+1}) + \beta x_0^{p^k+1}\right))} \eta_0(\omega)^{r_{a, b}}.$$

Fix $(\alpha, \beta) \in N_{i, \varepsilon}$ for $\varepsilon = \pm 1$, and suppose $\phi_{\alpha, \beta}(x) + \gamma = 0$ is solvable in $F_q$. Denote by $\vartheta = \text{Tr}_{t}^n(\alpha x_0^{p^m+1} + \beta x_0^{p^k+1})$. Then

- if $s - i$ and $d/t$ are both odd, and $\vartheta = 0$, then

$$R(\alpha, \beta, \gamma) = T(\alpha, \beta) \sum_{\omega \in F_{p^t}^*} \eta_0(\omega) = 0.$$
– if \( s - i \) and \( d/t \) are both odd, and \( \vartheta \neq 0 \), then by the result of quadratic Gaussian sums

\[
R(\alpha, \beta, \gamma) = T(\alpha, \beta) \eta'(\vartheta) G(\eta', \chi')
\]

\[
= \varepsilon \eta'(\vartheta) p^{\frac{n+i}{2}}
\]

\[
= \begin{cases} 
  p^{\frac{n+i}{2}} & \text{if } \varepsilon = \eta'(\vartheta), \\
  -p^{\frac{n+i}{2}} & \text{if } \varepsilon = -\eta'(\vartheta).
\end{cases}
\]

– if \( s - i \) or \( d/t \) is even, and \( \vartheta = 0 \), then \( \eta_0(\omega)^{\varphi_{\omega, \omega}}(x) + \gamma \) has no solutions in \( \mathbb{F}_q \) which implies that \( \varphi_{\omega, \omega}(x) + \gamma \) also has no solutions in \( \mathbb{F}_q \) for any \( \omega \in \mathbb{F}_q^* \subset \mathbb{F}_q \). Hence \( S(\omega, \omega, \omega) = 0 \) and \( R(\alpha, \beta, \gamma) = 0 \).

Thus the weight distribution of \( \mathcal{C}_2 \) can be derived from Theorem 1, Lemma 6, (25) and (26) directly. □

8 Conclusion and Further Study

In this paper we have studied the exponential sums \( T(\alpha, \beta) \) and \( S(\alpha, \beta, \gamma) \) with \( \alpha, \beta, \gamma \in \mathbb{F}_q \). After giving the value distribution of \( T(\alpha, \beta) \) and \( S(\alpha, \beta, \gamma) \), we determine the correlation distribution among a family of sequences, and the weight distributions of the cyclic codes \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \).

For a monomial Dembowski-Ostrom function, the associated exponential sums have been explicitly determined in [2], [3]. For a general Dembowski-Ostrom function \( f(x) \), Lemma 1 reveals the fact that if the number of the solution of the linearized polynomial related to \( f(x) \) is explicitly calculated, then the exponential sums \( \sum_{x \in \mathbb{F}_q} \chi(f(x)) \) and \( \sum_{x \in \mathbb{F}_q} \chi(f(x) + \gamma x) \) could be evaluated explicitly up to \( \pm 1 \). Thereafter, the correlation distribution of sequences and the weight distributions of the associated cyclic codes are also be determined.

In particular, for the case \( f(x) = x^{p^k+1} + x^{p^k} \) with \( l \geq 5 \) odd, we could get the possible values of \( \sum_{x \in \mathbb{F}_q} \chi(f(x)) \) and \( \sum_{x \in \mathbb{F}_q} \chi(f(x) + \gamma x) \). But the first three
moment identities developed in Lemma 3 is not enough to determine the value distribution. However, we could get the possible weights of the corresponding cyclic codes. New machinery and technique should be invented to attack this problem.

For $p = 2$, the exponential sums $T(\alpha, \beta)$ with $n/d$ odd is well known as Kasami-Welch case. Comparing to the odd characteristic case, the binary case has one advantage since the values of $T(\alpha, \beta)$ and $S(\alpha, \beta, \gamma)$ are all integers and one disadvantage since the binary quadratic form theory is a little harder to handle. We will deal with the binary version of this manuscript in a following paper.

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