A LOWER BOUND FOR THE GONALITY CONJECTURE

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Abstract. For every integer \( k \geq 3 \) we construct a \( k \)-gonal curve \( C \) along with a very ample divisor of degree \( 2g + k - 1 \) (where \( g \) is the genus of \( C \)) to which the vanishing statement from the Green-Lazarsfeld gonality conjecture does not apply.

The gonality conjecture due to Green and Lazarsfeld \([GL86]\) states that for \( C \) a smooth complex projective curve of genus \( g \geq 1 \) and gonality \( k \geq 2 \), and \( L \) a globally generated divisor on \( C \) of sufficiently large degree, one has the following vanishing criterion in Koszul cohomology:

\[
K_{i,1}(C, L) \neq 0 \iff 1 \leq i \leq h^0(C, L) - k - 1.
\]

This conjecture was proved two years ago by Ein and Lazarsfeld \([EL15]\), who moreover provided the sufficient lower bound \( \deg L \geq g^3 \). In the meantime this was improved to \( \deg L \geq 4g - 3 \) by Rathmann \([Ra16]\).

It is likely that this bound can be improved further, although Green and Lazarsfeld already noted \([GL86\text{ p. 86}]\) that one at least needs

\[
\deg L \geq 2g + k - 1
\]

because of the non-vanishing of \( K_{g-1,1}(K_C + D) \), for any divisor \( D \) of rank 1 and degree \( k \) on \( C \). Very recently Farkas and Kemeny \([FK16]\) showed that if \( C \) is sufficiently generic inside the moduli space of \( k \)-gonal curves of genus \( g \) then the bound \((2)\) is sufficient for the vanishing criterion \((1)\) to hold. Moreover they conjectured that this should be true for \textit{all} curves whose genus is large enough when compared to the gonality, a statement which they called the effective gonality conjecture. Results due to Green \([Gr84\text{ Thm. 3.c.1}]\) resp. Teixidor i Bigas \([Te07\text{ Prop. 3.8}]\) imply that this is indeed the case for trigonal resp. tetragonal curves of genus \( g > 3 \) resp. \( g > 6 \).

In this note we aim for an improved delimitation of the foregoing considerations, through the following result.

**Theorem.** For each \( k \geq 3 \) there exists a curve \( C \) of genus \( g = k(k - 1)/2 \) along with a very ample divisor \( L \) of degree \( 2g + k - 1 \) such that

\[
K_{h^0(C, L) - k, 1}(C, L) \neq 0.
\]

In particular, the bound \((2)\) is not sufficient for the gonality conjecture to apply.

The main conclusions to be drawn are that on the one hand, at most, one can hope to improve Rathmann’s bound to \( \deg L \geq 2g + k \), and that on the other hand \( g > k(k - 1)/2 \) is a necessary lower bound in the statement of Farkas and
Kemeny’s effective gonality conjecture. In particular one observes that the special cases implied by the works of Green and Teixidor i Bigas are sharp.

The construction is very short and explicit. Namely we let \( C \) be a smooth projective plane curve of degree \( k + 1 \) and let \( L \) be the effective divisor cut out by a curve of degree \( k - 1 \). Note that the gonality of \( C \) equals \( k \) by [Se87, Prop. 3.13] and that the degree of \( L \) equals \( (k - 1)(k + 1) = 2g + k - 1 \), as announced. This automatically entails that \( L \) is very ample. Now the standard exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-C) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0
\]

can be easily converted into

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(k - 1) \to \mathcal{O}_C(L) \to 0.
\]

Taking cohomology and using that \( H^0 \) and \( H^1 \) vanish in the case of \( \mathcal{O}_{\mathbb{P}^2}(-2) \) we end up with an isomorphism

\[
H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k - 1)) \to H^0(C, L),
\]

which shows that \( h^0(C, L) = \binom{k+1}{2} \) and that the embedding

\[
C \hookrightarrow \mathbb{P}^{\binom{k+1}{2}}
\]

lands inside the \((k - 1)\)-th Veronese surface. This implies that we have an injection

\[
K^{(k+1)}_{k-1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k - 1)) \hookrightarrow K^{(k+1)}_{k-1}(C, L) = K_{h^0(C, L)-k,1}(C, L).
\]

But the former space is non-trivial, from which our theorem follows. One reference for this last fact is [Gr84, App.], which can be applied to the decomposition \( \mathcal{O}_{\mathbb{P}^2}(k - 1) = \mathcal{O}_{\mathbb{P}^2}(k - 2) \otimes \mathcal{O}_{\mathbb{P}^2}(1) \). Alternatively we refer to [CCDL16, Thm. 1.8] for a more direct statement.

**Remark.** We work over \( \mathbb{C} \) because for instance [EL15] does so as well, but the theorem presented above is valid over any algebraically closed field, with the same proof.

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