Theory and Solutions of Heat Pulse Method for Determining Soil Thermal Properties

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Abstract. Information on thermal properties of soil is of paramount importance for environmental and earth science, and engineering. The heat pulse (HP) method has become the key technology for accurate determination of soil thermal properties and a variety of other physical properties (e.g., water content, bulk density, and water flux) in both laboratory and field environments. The HP method is a transient method that is commonly based on the analytical solutions to the radial heat flow equation when a line-heat source is applied. Over the past few decades, great endeavors have been devoted to advance the HP method. For example, the evolution and development in probe design, data logging equipment, data interpretation and computing capability has remarkably improved the accuracy and ease of use for determining soil thermal properties. However, there is a lack of study collating and synthesizing the development of the theory/solutions to obtain thermal properties of soil using the HP method. In this paper, we review the fundamental theories and solutions of the HP method, including differences and similarities of theories and applications between instantaneous line heat source (ILHS) and short-duration line heat-source (SLHS), between dual-probe heat-pulse (DPHP) and single-probe heat-pulse (SPHP) methods, and between the non-linear model fit (NMF) method and single point (SPM) method for data interpretation. In addition, the numerical solutions and semi-analytical solutions are also presented to provide heat pulse users information for selecting the best-fit method to meet their goals.

1. Introduction

The importance of soil thermal properties (i.e., soil thermal diffusivity $\kappa$, heat capacity $C_v$, and soil thermal conductivity $\lambda$) have long been recognized by scientists [1-3]. Knowledge of soil thermal properties is required by disciplines such as earth and environmental, and engineering. Because soil thermal properties largely determine surface energy balance and soil thermal regime [4, 5]; affect soil
physical, chemical, and biological processes [6-10], freezing and thawing processes and snowmelt infiltration [11-16]. Therefore, soil thermal properties are required parameters in many climate models or land surface model at global or community scales [17-19]. Soil thermal properties are also used to predict soil bulk density or soil water contents [20-25]. In addition, soil thermal properties are critical parameters to design and construct geotechnical and geo-environmental applications such as resource and transportation development, geothermal energy resources and energy piles [26-31], recovery of natural methane gas hydrates and geological CO$_2$ sequestration [32], and radioactive waste disposal [33-35]. Furthermore, thermal properties play a critical role in determining heat flow of an extraterrestrial body [36-41]. As a result, thermal property measurements were implemented for many extraterrestrial mission, including Apollo 15 and 17 mission and the recent Phoenix lander to the Moon [42, 43], Rosetta spacecraft to the Comet 67P [36], and the InSight mission to the Mars [36].

In consideration of the importance, a variety of methods and apparatus have been developed and applied to determine thermal properties of soil. These technologies generally can be categorized as steady-state (or interchangeably stationary) heat flow method and the transient (or interchangeably non-stationary) heat flow method [25, 44]. Some of the examples for steady-state methods are radial and axial methods, guarded hot plate (GHP), and absolute and comparative methods [45-48]. The steady-state method maintains a constant temperature gradient across the tested soil sample to reach unidirectional heat flux conditions, which is required by the steady heat flow theory [3, 49]. Soil thermal properties are obtained by comparison of experimental data (e.g., soil temperature profiles or heat flux) with solution to the Fourier or Laplace equation [50]. Transient method simulates radial heat flow released by a heating source of different shapes (e.g., line, cylindrical, plane, point, or spherical). It records change of soil temperature due to the release of short duration of heat release (hereafter, heat pulse) liberating from a heating source (hereafter, heater). By fitting the solutions of Fourier’s equation (i.e., heat conduction equation) to the measured temperature change with time (T-t data) [50, 51], soil thermal properties can be obtained. The line heat source based transient method in literature is termed heat pulse (HP) method, which is alternatively named cylindrical thermal probe [3], non-steady-state method, thermal conductivity probe [52, 53], transient hot-wire method, transient line source [54], differentiated line heat source [47], hot needle method, or probe/needle method. In contrast to the steady-state method, the HP method has proved to be more reliable for measurement of soil thermal properties with respect to accuracy, energy consumption, measurement time, cost of equipment and maintenance, and ease of use for lab and field applications [25, 55, 56].

Advancements of the HP method in improved determination of soil thermal property have been associated with remarkable development and improvement in data logging equipment, probe design and construction, data interpretation and computing ability from the 1950s [25]. Dual-probe heat-pulse (DPHP) method and single-probe heat-pulse (SPHP) method are the main two categories of HP method [25, 57]. Probes of SPHP and DPHP method consist of a heating element (heater) and minimum one temperature sensing element (temperature sensor). The heater is used to provide heating source and the temperature sensor is used to record temperature change. For the SPHP, the temperature sensor and heater are mounted together within the same needle, while the temperature sensor(s) and heater are mounted in separate needles for DPHP. The cylindrical probe developed to measure soil thermal conductivity by de Vries [55] has been recognized as the pioneering prototype of SPHP probes, while the most widely utilized DPHP probes are derived from the probe developed by Campbell et al. [58]. For SPHP method, a relatively long period of heat pulse is applied [49, 55, 59], while the DPHP method uses a shorter duration of heat pulse (e.g., 8~15s for DPHP vs ≥ 1min for SPHP) [24, 25, 58, 60, 61].

For SPHP method, $\lambda$ is obtained according to the slope of a straight line resulted from the plotting of the temperature change versus logarithmic function of time, log(t) [55]. The DPHP method can simultaneously determine $C_p$, $\kappa$, and $\lambda$. Campbell et al. [58] estimated $C_p$ based on theory of instantaneous line heat source (ILHS). However, study of Kluitenberg et al. [62] showed that the ILHS theory results in overestimated $C_p$, and this overestimation can be solved by the use of short duration line heat pulse (SLHS) theory presented by Carslaw and Jaeger [51]. After that, Bristow et al. [63]
proposed the single point (SPM) method to simultaneously estimate $C_v, \kappa$ and $\lambda$ by extracting the maximum rise of temperature and its corresponding time from the recorded $T-t$ data. It was found that the SPM method cannot cope well with sparse and noise $T-t$ data with flat peak. Therefore, Bristow et al. [64] presented the nonlinear model fit (NMF) method that determines thermal properties of soil by fitting the recorded $T-t$ data. These studies served for subsequent applications and developments of this method in soil science and other disciplines. Error analysis of probe designs and data interpretation methods were well documented for reference [62, 65-68].

Although great progresses on the development of theories and solutions to estimate thermal properties of soil with the HP method, few studies have systematically reviewed and collated these achievements. In this paper, we review the HP method based theories: instantaneous heat pulse theory and short duration heat pulse theory, SPHP and DPHP methods, SPM and NMF methods, and their differences were compared, the numerical solutions and semi-analytical solutions are also presented in order to heat pulse users with practical guidance to choose method best fit their needs.

2. Theory and solutions for the HP Method

2.1. Theory

Heat conduction in heterogamous and anisotropic soils in Cartesian coordinate system can be expressed as [25, 51]

$$
C_v \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(\lambda_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\lambda_y \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(\lambda_z \frac{\partial T}{\partial z})
$$

(1)

where $C_v$ is soil volumetric heat capacity, J m$^{-3}$ºC$^{-1}$; $C_c = \rho_b C$, $\rho_b$ is soil bulk density, kg m$^{-3}$, and $c$ is soil specific heat capacity, J kg$^{-1}$ºC$^{-1}$; $T$ is temperature in ºC; $t$ is time in s; $x$, $y$, and $z$ in m; $\lambda$ is thermal conductivity of soil, W m$^{-1}$ºC$^{-1}$ or equally J s$^{-1}$m$^{-1}$ºC$^{-1}$.

In a homogenous, isotropic and semi-infinite media, $\lambda$ is assumed to be constant with space and time, rewriting Eq. (1) in the cylindrical coordinate system gives [51]:

$$
\frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right)
$$

(2)

where $k = \lambda \cdot C_v$ is soil thermal diffusivity, m$^2$ s$^{-1}$; $A$ is the azimuth, º; and $r$ is the radial distance from the heater, m. Equation (2) can be simplified for radial heat flow as [25]

$$
\frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right)
$$

(3)

Equation (3) can be solved for analytical solutions with respect to different heat sources, the most commonly used analytical solutions will be discussed in section 2.2.

2.2. Analytical solutions

2.2.1. Solution for instantaneous line heat source (ILHS). For an isotropic and homogeneous medium with an initially uniform temperature, Carslaw and Jaeger [51] presented the solution to Eq. (3) with an instantaneous heat pulse liberating from an infinite line heat-source

$$
\Delta T(r, t) = \frac{Q}{\pi \kappa t} \exp \left( \frac{-r^2}{4\kappa t} \right)
$$

(4)

where $\Delta T(r, t)$ is the temperature change with time at a radial distance of $r$ (m) away from the heater; $Q$ is the amount of heat released by the heater, m$^2$ ºC.
\[ Q = q / \rho_c \]  

(5)

where \( q \) is the amount of heat released per unit length of the heater, J m\(^{-1}\). \( q \) can be determined with Joule’s law.

\( E \) in Eq. (4) and \( E_1(x) \) in Eqs. (7-8) below are symbolic exponential integral functions. They can be described as [55]:

\[ E_1(x) = -Ei(-x) = \int_{x}^{\infty} \frac{\exp(-n)}{n} \, dn = -\gamma - \ln(x) - \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!} \]  

(6)

where \( n \) is an integration variable, \( \gamma = 0.57721 \) is the Euler-Mascheroni number, and \( x = r^2 / (4\kappa t) \).

2.2.2. Solution for short-duration line heat source (SLHS). For initial and boundary conditions: \( T = 0 \) for \( t = 0 \) and \( r \geq 0 \), \( T = 0 \) at \( t > 0 \) and \( r \rightarrow \infty \), and \( -2\pi r \lambda \frac{dt}{dr} = q' \) at \( t > 0 \) and \( r \rightarrow 0 \). \( q' \) is the amount of heat released per unit length and unit time of the heater, W m\(^{-1}\) or J m\(^{-1}\) s\(^{-1}\). The solutions to Eq. (3) at before and after the heat release periods are [55]

\[
\Delta T(r, t) = \begin{cases} 
  \frac{Q'}{4\pi \kappa} E_i \left( \frac{r^2}{4\kappa t} \right) & 0 < t \leq t_0 \quad (a) \\
  \frac{Q'}{4\pi \kappa} \left[ E_1 \left( \frac{r^2}{4\kappa t} \right) - E_1 \left( \frac{r^2}{4\kappa(t-t_0)} \right) \right] & t > t_0 \quad (b)
\end{cases}
\]  

(7)

where \( t_0 \) is the duration of heat pulse, s. Equation (7a) indicates the change of temperature at distance \( r \), \( \Delta T(r, t) \), for a heat pulse application starting at \( t = 0 \) and ending at \( t = t_0 \), while Eq. (7b) describes \( \Delta T(r, t) \) from the ending of the heat pulse, i.e., \( t > t_0 \).

2.2.3. Solution for continuous line heat source (CLHS). The CLHS uses the analytical solution to Eq. (3) during the heating period [51, 55]

\[
\Delta T(r, t) = -\frac{Q'}{4\pi \kappa} E_i \left( -\frac{r^2}{4\kappa t} \right) = \frac{Q'}{4\pi \kappa} E_1 \left( \frac{r^2}{4\kappa t} \right)
\]  

(8)

where, \( Q' = q' / \rho c \) is strength of the heat source per unit time, m\(^2\) °C s\(^{-1}\).

2.3. Derivation of soil thermal properties

The different approaches to derive thermal properties of soil with Eq. (7) demonstrate the fundamental differences between the DPHP and the SPHP methods.

2.3.1. The SPHP method. For the SPHP measurement, contact between heater and temperature sensing element is assumed to be perfect contact [69]. The temperature sensor located at the middle length of the probe and records the temperature change over time. In Eq. (6), if \( t \) is large enough or \( r \) is very small (i.e., \( r^2 / 4\kappa t \ll 1 \)), recorded temperature change the can be approximated as [70, 71])

\[
T(t) \approx \frac{q}{4\pi \lambda} \ln(t) + B
\]  

(9)

where \( B \) is a constant. The corresponding temperature of the heater \( T_1 \) and \( T_2 \) at time \( t_1 \) and \( t_2 \) can be described as [69, 72]:
where $t_c$ is a time correction term to account error sources associated with the SPHP method, including reduction of the Eq. (7), finite probe diameter, contact resistance, and axial heat flow error [47]. Equation (10) generally leads to a linear relationship between logarithmic temperature change over time with a slope of $q'/4\pi \lambda$. $\lambda$ can be calculated for a known $q'$. To apply this method, heat pulse is required to be released for a rather long period of time (e.g., > 60s) to counteract the error sources [73].

\[ T_2 - T_1 = \frac{q'}{4\pi \lambda} \ln \frac{t_2 + t_c}{t_1 + t_c} \]  

\[ (10) \]

Figure 1. Measured temperature change with time in sand for the single-probe heat-pulse (SPHP) method. The strength of heat is 5.77w m$^{-1}$, heating duration is 10 min, soil particle size is < 1mm, soil bulk density is 1.5g cm$^{-3}$, and five soil water contents (0, 0.05, 0.12, 0.17, and 0.27cm$^{-3}$ cm$^{-3}$) were used [25].

Many subsequent work pertaining to investigate effects of thermal contact resistance, axial (non-radial) heat flow, the finite probe dimension, and probe heat capacity have been accounted for to improve this method [51, 71, 74-77]. These theoretical interpretations have remarkably improved the accuracy and precision of estimated soil thermal properties with SPHP.

A modified formula accounting for ‘lumped’ effects of error sources can be described as [55, 56];

\[ \Delta T = \begin{cases} 
\frac{q'}{4\pi \lambda_h} \ln(t + t_c) + m & t \leq t_0 \ [a] \\
\frac{q'}{4\pi \lambda_c} [\ln(t + t'_c) - \ln(t + t'_c - t_0) + m'] & t > t_0 \ [b] 
\end{cases} \]  

\[ (11) \]

where $\lambda_h$ and $\lambda_c$ are soil $\lambda$ at the phases of heating and cooling, respectively; $t_c$ and $t'_c$ indicate time correction terms; and $m$ and $m'$ are constants. The values of $m$ and $t_c$ may be different from $m'$ and $t'_c$. Shiozawa and Campbell [56] recommended the use of the average of $\lambda_h$ and $\lambda_c$ for the best estimates of soil thermal conductivity. A measured example of temperature change in sands with the SPHP method is shown in Figure 1. Reader may consult Wechsler [78] for more details on early time developments of the SPHP method.

2.3.2. The DPHP method. Unlike the SPHP method, DPHP method analyzes the T-t data recorded by the temperature sensor that is located at a small distance (e.g., 6mm) away from the heater. Campbell et al. [58] developed a porotype DPHP probe by use separated stainless steel tubes to house the heating wires and temperature sensor. This DPHP probe and a new data interpretation method were used to estimate $C_v$. The study of Campbell et al. [58] serves the basis for a series of subsequent studies and advance the development of the HP method. Among the various progresses in this method, the nonlinear model fit (NMF) method and single point (SPM) method for data interpretation significantly widen the applications of HP method and will be presented below.
(1) SPM method for analyzing soil thermal properties

Thermal properties of soil with the SPM method can be determined on the basis of either the SLHS solution or the ILHS solution as presented in section 2.2.

1) ILHS solution based SPM method

By differentiating Eq. (4) with respect to time, the maximum temperature change ($\Delta T_m$) at a distance ($r_m$) from the heater can be obtained when the derivative is zero [58, 79]. This gives

$$ t_m = \frac{r_m^2}{4\kappa} $$

(12)

Rearrangement of Eq. (12) gives thermal diffusivity $\kappa$

$$ \kappa = \frac{r_m^2}{4t_m} $$

(13)

Substituting Eq. (12) into Eq. (4) gives the maximum temperature change $\Delta T_m$ [58]

$$ \Delta T_m = Q\left(\frac{e\pi r_m^2}{4t_m}\right) $$

(14)

Because $Q = q/C_v$, rearranging Eq. (14) gives [58]

$$ C_v = \frac{q}{e\pi r_m^2 \Delta T_m} $$

(15)

Since $\lambda = \kappa \cdot C_v$, combining Eq. (13) for $\kappa$ and Eq. (15) for $C_v$ gives

$$ \lambda = \frac{q}{4\pi \Delta T_m t_m} $$

(16)

2) SLHS solution based SPM method

In practice, it is not possible to meet the assumptions of instantaneous heat release to an isotropic, homogeneous, infinite, and isothermal porous medium as required by the ILHS theory. Therefore, SLHS solution for a heating source of short duration is used instead.

![Figure 2. Simulated temperature change with time for instantaneous line source heat pulse (ILHS, Eq.(4) and short-duration line-source heat pulse (SLHS, Eq.(7) solution. t_m and t_m8 are times corresponding to the maximum temperature change ($\Delta T_m$) for instantaneous line heat pulse and line heat pulse of 8s, respectively. Parameters are: r=0.006m, t_o=8s, q=62.5W m^-1, $\kappa=5\times10^{-7}$m^2 s^-1, $C_v=1\times10^6$J m^3 °C^-1 were used for the simulation [25, 63].]
The difference between the SLHS and ILHS solutions is shown in Figure 2. The solution of SLHS results in a delayed $t_w$, but it places small effect on $\Delta T_m$ [62-64]. Previous researches have showed that the feasibility to accurately determine $C_v$ with Eq. (15) [58, 79, 80]. However, it should be noted that than the calculated $C_v$ with the SLHS solution is slightly smaller the calculated $C_v$ using the solution of ILHS [62].

Differentiating Eq. (7) b with time gives $\kappa$ at the time of maximum temperature rise [58, 80]:

$$
\kappa = \frac{r^2}{4t_m} \left( \frac{t_0}{t_m - t_o} \right) \left[ \ln \left( \frac{t_m}{t_m - t_o} \right) \right]^{-1}
$$

(17)

For time $t > t_o$, rearranging Eq. [7] b yields

$$
C_v = \frac{q'}{4\pi \Delta T_m} \left[ E_1 \left( \frac{r^2}{4k(t_m - t_0)} \right) - E_1 \left( \frac{r^2}{4k(t_m - t_0)} \right) \right]
$$

(18)

Equation (18) requires known values of $q'$, $\Delta T_m$, $r$, $t_m$, and $t_0$.

Equations (17-18) can be combined to calculate $\lambda$ given that $\lambda = \kappa C_v$ [81]

$$
\lambda = \frac{q}{4\pi \Delta T_m} \left( E_1 \left( \frac{\ln [t_m/(t_m - t_0)]}{t_0/(t_m - t_0)} \right) \right) - E_1 \left( \frac{\ln [t_m/(t_m - t_0)]}{t_0/t_m} \right)
$$

(19)

In addition, Knight and Kluitenberg [82] presented an alternative approach for Eq. (18) by only keeping the first five terms of the Taylor series expansion:

$$
C_v = \frac{q'}{e\pi r^2 \Delta T_m} \left( 1 - \frac{\varepsilon^3}{8} \left( \frac{1}{3} + \varepsilon \left[ \frac{1}{3} + \varepsilon \left( \frac{7}{8} + \frac{7}{3} \varepsilon \right) \right] \right) \right)
$$

(20)

where $\varepsilon = t_0/t_m$. The advantage of Eq. [20] over Eq. [18] is that it does not contain an exponential integral function. It is not usually incorporated in data logger function libraries or spreadsheet software packages. Knight and Kluitenberg [82] also showed that $C_v$ determined from Eq. (20) is more accurate than that from Eq. [18].

The SPM method is easy to apply. However, the SPM method requires accurate and precise value of $\Delta T_m$ and $t_m$, which make the method suitable for flat and broad peaks, noisy or sparse data. In addition, estimates of $C_v$ and $\lambda$ are prone to errors in the single point values of $\Delta T_m$, because the thermal diffusivity and conductivity are sensitive to $t_m$ error resulting from uncertainties in probe needle spacing [65, 66].

(2) NMF method for analyzing soil thermal properties

Bristow et al. [64] proposed the nonlinear regression method to determine $C_v$ and $\kappa$ so to overcome the limitations of the SP method. By best fitting the experimental data of $\Delta T(t)$ with Eq. (7) with the nonlinear fitting procedure, the sum of squared errors (SSE) can be minimized[64]. Figure 3 shows the analysis of $\Delta T(t)$ data with the SPM and the NMF methods. NMF method may underestimate the peak $\Delta T(t)$ but overestimate the measured $\Delta T(t)$ at late part of the cooling period if the whole dataset are used for fitting. To overcome this problem, Bristow et al. [64] recommended the use of a subset (~3/4) of the measured $\Delta T(t)$ data for the NMF method. This subset of data could also better represent the ILHS solution and the influences of finite probe on thermal properties of soil [83, 84]. Later, a nonlinear optimization method was presented to minimize the SSE [60, 83]. The NMF or weighted NMF can be achieved with programs such as Excel, MathCAD or Matlab. Researchers also presented some ready for use programs such as the “HPC” [85] and “INV-WATFLX” [86]).
Figure 3. Determination of the soil thermal properties using single point (SPM) and non-linear fitting (NMF) method based on the instantaneous line heat source (ILHS) solution and short-duration line heat source (SLHS) solution. The associated parameters used are: \( r=0.0068 \text{m} \), \( t_0=8 \text{s} \), and \( q'=11.5 \text{W m}^{-1} \) \[25\].

2.4. Numerical simulation
The analytical solutions of HP method have gained wide acceptance due to its ease of use and reasonably accurate estimates of soil thermal properties \[25\]. While the numerical modeling offers opportunities to estimate soil water contents and temperature profile in addition to thermal properties when the initial and boundary conditions of the study become too complicated when the use of analytical solutions is inadequate \[67, 87, 88\].

Finite difference and finite element are the most commonly used numerical simulation methods. Care should be taken to set the initial and boundary conditions and the discretization selection of the time and space steps. Studies \[83, 87-92\] are among the best examples of numerically solving for thermal properties and other physical properties of soil.

2.5. Semi-analytical solutions
As stated above, although with wide applications, the analytical solutions are not capable of accounting for the influences of probe properties such as probe dimension, probe thermal properties, and thermal contact resistance. These probe effects can be taken into account by the Numerical computing, but they are complex to be applied by beginners. Alternatively, semi-analytical solutions may be for accurate estimate with moderate complexity. Some of the good examples of semi-analytical solutions are listed blow for SPHP and DPHP methods, respectively.

2.5.1. Solutions for SPHP method. For an infinite cylindrical heat source inserted in a homogeneous infinite soil. The cylindrical heat pulse probe has a thermal contact conductance \( H \) (\( \text{W} \cdot \text{m}^{-2} \cdot \text{°C}^{-1} \)) between the heater and its surrounding soil particles. If heat is continuously liberated at a constant heating rate of \( q \) (\( \text{W} \cdot \text{m}^{-1} \)), the radial heat flow equations are given as \[70\]

\[
\frac{\partial^2 T_2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial T_2}{\partial \rho} = \frac{1}{\kappa} \frac{\partial T_2}{\partial t} \quad r < \rho < \infty \quad t > 0
\]

\[
T_1 = T_2 = 0 \quad t = 0
\]

\[
-\lambda \frac{\partial T_2}{\partial \rho} = H(T_1 - T_2) \quad \rho = r \quad t > 0
\]

\[
-\lambda \frac{\partial T_2}{\partial \rho} = \frac{q}{2\pi \rho} - \frac{\rho c_p}{2} \frac{\partial T_1}{\partial t} \quad \rho = r \quad t > 0
\]
\( T_2 \) is bounded as \( \rho \to \infty \) \hspace{1cm} (25)

where \( T_1 \) and \( T_2 \) are the temperature (\(^\circ\text{C}\)) of the probe and soil, respectively; \( C_p \) is the probe heat capacity (\( J \cdot m^{-3} \cdot ^\circ\text{C}^{-1} \), subscript \( p \) indicates probe), and \( t \) is time (s). The temperature increase at the surface of the cylindrical probe is \([70, 71, 75, 77]\):

\[
T_1(t) = \frac{8q}{\pi^3 r_0^3 C_p^2 k^2} \int_0^\infty \frac{1 - e^{-ktu^2}}{u^3(r^2 + Q^2)} \, du \hspace{1cm} (26)
\]

where \( P \) and \( Q \) are:

\[
P = uJ_0(r_0u) + J_1(r_0u) \left( \frac{u^2}{\lambda_1} - \frac{2}{c_t r_0 \kappa} \right) \hspace{1cm} (27)
\]

\[
Q = uY_0(r_0u) + Y_1(r_0u) \left( \frac{u^2}{\lambda_1} - \frac{2}{c_t r_0 \kappa} \right) \hspace{1cm} (28)
\]

where \( u \) is a variable of integration; \( Y_0 \) and \( Y_1 \) are the zero- and first-order Bessel functions of the second kind, respectively. \( J_0 \) and \( J_1 \) are the zero- and first-order Bessel functions of the first kind; and, respectively.

Equation [26] is complicated for fitting to data for soil thermal properties; so inversely numerical computation was employed instead [93]. An asymptotic expansion can be an alternative simplification for the modified Bessel functions. Blackwell [70] and Waite \textit{et al}. [77] derived the small- and large time approximations for the temperature at a probe surface. The small-time solution is

\[
T(t) \sim \frac{q}{\pi r_0^3 C_p} \left[ t - \lambda_p^2 t + \frac{16 \lambda_p^2 \sqrt{\kappa}}{15 \sqrt{c_T} r_0 \lambda} t^{5/2} + o(t^3) \right], \quad t \ll \frac{r_0^2}{\kappa} \hspace{1cm} (29)
\]

Equation [29] can also be described as [77]

\[
T(t) \sim Z_1 t - Z_2 Z_3 t^2 + Z_4 Z_5 t^{5/2}, \quad t \ll \frac{r_0^2}{\kappa} \hspace{1cm} (30)
\]

where \( Z_1, Z_2, \) and \( Z_3 \) are fitting parameters for the \( T(t) \) relationship in the small-time domain; and \( \lambda_p = (qZ_2)/(\pi r_0 Z_1) \).

The large time approximation is [70]

\[
T(t) \sim A \ln(t) + B + \frac{C \ln(t) + D}{t} \hspace{1cm} (31)
\]

where \( A, B, C, \) and \( D \) are fitted parameters for the measured \( T(t) \) relationship in the large-time domain; \( C \ln(t) + D/t \to 0 \) when \( t \gg \frac{r_0^2}{\kappa} \). By fitting the \( T(t) \) relationship with the small time domain and large time domain to obtain the fitting parameters, \( \lambda \) and \( \kappa \) can be calculated based on the relationships

\[
A = \frac{q}{4 \pi \lambda} \hspace{1cm} (32)
\]

\[
B = \frac{q}{4 \pi \lambda} \left[ \ln(\kappa) - 2 \ln(\rho_0) + \ln(4) - \gamma + \frac{2\lambda}{\rho_0 \lambda_p} \right] \hspace{1cm} (33)
\]

Arrangement of Eqs. (32) and (33) gives
\[ \kappa = \frac{r_0^2}{4} \exp \left( \frac{\beta}{\lambda} + \gamma - \frac{2\lambda}{r_0^2} \right) \] (34)

When the cylindrical probe/heater is hollow, the \( T(t) \) relationship is expressed as [70, 71]

\[ T(t) \sim \frac{q}{\kappa p_0 c_p} \left\{ -\Delta_2 + \left[ 1 - \frac{2(\Delta_1-\Delta_2)\lambda_p}{c_p r_0} \right] t - \frac{\lambda_p r^2}{c_p r_0} + \frac{16\lambda_p^2 \sqrt{\pi}}{15v \pi \kappa c_p r_0} t^{5/2} + o(t^2) \right\} \frac{r_0^2}{\kappa} \ll t \ll \frac{r_0^2}{\kappa} \] (35)

where \( \Delta_1 \) and \( \Delta_2 \) are

\[ \Delta_1 = \frac{1}{\kappa_p} \left[ \frac{r_0^2 - 3r_1^2}{8} + \frac{1}{2} \ln \left( \frac{r_0}{r_1} \right) \left( \frac{r_0^2}{r_0^2 - r_1^2} \right) \right] \] (36)

\[ \Delta_2 = \frac{1}{\kappa_p} \left[ \frac{r_0^2 + r_1^2}{8} - \frac{1}{2} \ln \left( \frac{r_0}{r_1} \right) \left( \frac{r_0^2 - r_1^2}{r_0^2 + r_1^2} \right) \right] \] (37)

where \( r_1 \) represents the inner radius of the hollow needle, \( \lambda_p \) and \( \kappa_p \) are the thermal conductivity and thermal diffusivity of the probe material, respectively. The large-time approximation \( (t \gg r_0^2/\kappa) \) for the hollow probe is the same as Eq. (31).

Blackwell [70] provided the time domain solution for \( \Delta T(t) \) of the heat source as Eq. (26), while Kristiansen [94] provided a time domain solution for \( \Delta T(r, t) \) in the soil surrounding the heating probe [71, 92]:

\[ T(r, t) = \frac{a b^2}{\lambda} \int_0^\infty \left[ 1 - \exp \left( ru^2 \right) \right] \frac{Q' r_0(u) - P' Y_0(u)}{u^2 (u^2 + a^2)} \, du \] (38)

where,

\[ P' = u J_0(u) - \lambda (hu^2 - \alpha) J_1(u) \] (39)

\[ Q' = u Y_0(u) - \lambda (hu^2 - \alpha) Y_1(u) \] (40)

where the dimensionless parameters: contact resistance term \( (j_i) \), radius \( (\eta) \), and time \( (\tau) \), and

\[ h = \lambda / (b H), \eta = r / b, \tau = 4 \kappa t / b^2, \alpha = 2 \pi b^2 \rho c / C_p \] (41)

where \( \alpha \) is twice the ratio of the \( \rho_b c \) of the medium to that of the sensor. \( S / \pi b^2, C_p \) is the probe heat capacity per unit length. For large values of \( H \), soil temperature at large times can be approximated as [95]

\[ T(\tau, \eta) \approx \frac{q}{4 \pi^2 \lambda} \left[ \ln(\tau) - 2 \ln(\eta) - \gamma + \frac{2}{\tau} \ln(\tau) - \gamma + \frac{1}{\tau} \left[ \frac{2}{\tau} - 2 \ln(\eta) + \eta^2 \right] + o(\tau^{-2}) \right] \] (42)

\[ T(\tau, \eta) = \begin{cases} \frac{q}{4 \pi^2 \lambda} \left[ 2 \ln \left( \eta_1 / \eta_2 \right) + \frac{2}{\tau} \left[ 2 \ln \left( \eta_2 / \eta_1 \right) + \eta_1^2 - \eta_2^2 \right] \right] & 0 < \tau \leq \tau_0 \\
\frac{q}{4 \pi^2 \lambda} \left( \frac{\tau_0}{(\tau_0 \tau - \tau)} \right) \left[ 2 \ln \left( \eta_1 / \eta_2 \right) + \eta_1^2 - \eta_2^2 \right] \right] & \tau > \tau_0 \] (43)

Temperature dependence of soil on the thermal contact resistance disappears to order \( \tau^{-2} \) for Eq. (43) [76]. These solutions are reduced to the line source approximation as the effective sensor radius approaches zero. Temperature difference for the heating and cooling curves can be reformulated as:
\[
T(t) = \begin{cases} 
\frac{q b^2}{16 \pi k} \left[ 2 \ln \left( \frac{\eta_2}{\eta_1} \right) + \eta_1^2 - \eta_2^2 \right] + \frac{a}{2 \pi k} \ln \left( \frac{\eta_2}{\eta_1} \right) & 0 < t \leq t_0 \\
\frac{q b^2}{16 \pi k} \left[ 2 \ln \left( \frac{\eta_2}{\eta_1} \right) + \eta_1^2 - \eta_2^2 \right] \left( \frac{t_0}{t(t-t_0)} \right) & t > t_0 
\end{cases}
\] (44)

Equations [38] to [44] may also apply to dual probe heat pulse [71].

2.5.2. Solutions for DPHP method. In an infinite solid with uniform initial temperature, one analytical solution to Eq. (3) with a pulsed infinite cylindrical line-heat source of radius \( r_0 \) (m) can be expressed as [62]

\[
T(r, t) = \begin{cases} 
\frac{q}{4\pi k} \int_0^\infty \left( \frac{r}{4k(t-t_0)} \right) x^{-1} \exp \left( -x \right) \exp \left( -\left( \frac{r_0^2}{r} \right) x \right) \int_0^{2r_0x/r} dx & 0 < t \leq t_0 \\
\frac{q}{4\pi} \int_0^{r^2/\left(4k(t-t_0)\right)} x^{-1} \exp \left( -x \right) \exp \left( -\left( \frac{r_0^2}{r} \right) x \right) \int_0^{2r_0x/r} dx & t > t_0 
\end{cases}
\] (45)

Where \( x = r^2/[4k(t-t_0)] \) and \( J_0(x) \) is the modified zero order Bessel function.

In an infinite solid with uniform initial temperature, the solution of Eq. (3) with a pulsed finite line heat source of length \( l_0 \) (m) and radius \( r_0 \), the temperature change at \( r \) away from the heater is expressed as [62, 96]

\[
T(r, t) = \begin{cases} 
\frac{q}{4\pi k} \int_0^\infty \left( \frac{r}{4k(t-t_0)} \right) x^{-1} \exp \left( -x \right) \text{erf} \left( \frac{l_0\sqrt{x}}{2r} \right) & 0 < t \leq t_0 \\
\frac{q}{4\pi} \int_0^{r^2/[4k(t-t_0)]} x^{-1} \exp \left( -x \right) \text{erf} \left( \frac{l_0\sqrt{x}}{2r} \right) & t > t_0 
\end{cases}
\] (46)

where \( \text{erf}(x) \) indicates the error function.

Liu et al. [97] derived analytical solutions for a pulsed finite cylindrical heat source placed in a parallelepiped sample of size \( a \) by \( b \) by \( c \). Two types of boundary conditions were used: (1) Zero surface temperature boundary \((t \geq 0, T_S = 0, \sum \) denotes the boundaries of the parallelepiped) and (2) adiabatic or Neumann boundary \((t \geq 0, \frac{\partial T}{\partial n} \bigg|_{T_S} = 0 \). The initial conditions are \( t=0, 0<x<a, 0<y<b, 0<z<c, T=T_0 \). For the Zero surface temperature boundary condition, temperature change is:

\[
T(t) = \begin{cases} 
\frac{4\pi c^2}{V \chi_p} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \left[ \frac{\sin \frac{\pi x}{b} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}}{\sin \frac{\pi x}{b} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}} \right] \left( 1 - \cos \frac{\pi x}{b} \right) \left( 1 - \cos \frac{\pi y}{b} \right) \left( 1 - \cos \frac{\pi z}{c} \right) \right] \exp \left[ -\lambda n^2 \Delta(t-t_0) \right] \frac{1}{n^2} \frac{1}{\eta_1^2} \frac{1}{2} \left( \frac{t^2}{t^2-t_0^2} \right) & 0 < t \leq t_0 \\
\frac{4\pi c^2}{V \chi_p} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \left[ \frac{\sin \frac{\pi x}{b} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}}{\sin \frac{\pi x}{b} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}} \right] \left( 1 - \cos \frac{\pi x}{b} \right) \left( 1 - \cos \frac{\pi y}{b} \right) \left( 1 - \cos \frac{\pi z}{c} \right) \right] \exp \left[ -\lambda n^2 \Delta(t-t_0) \right] \frac{1}{n^2} \frac{1}{\eta_1^2} \frac{1}{2} \left( \frac{t^2}{t^2-t_0^2} \right) & t > t_0 
\end{cases}
\] (47)

where \( V \) is the parallelepiped sample volume. For the adiabatic boundary condition, temperature change is expressed as [97]:

\[
T(x, y, z, t) = \begin{cases} 
\frac{1}{V \chi_p} \left( \frac{\tilde{M} + \tilde{N} + \tilde{P} + \tilde{M} \tilde{N} + \tilde{M} \tilde{P} + \tilde{N} \tilde{P} + \tilde{M} \tilde{N} \tilde{P} + \tilde{M} \tilde{N} \tilde{P} \tilde{M}}{1} \right) & 0 < t \leq t_0 \\
\frac{1}{V^2 \chi_p} \left( \frac{\tilde{M} + \tilde{N} + \tilde{P} + \tilde{M} \tilde{N} + \tilde{M} \tilde{P} + \tilde{N} \tilde{P} + \tilde{M} \tilde{N} \tilde{P} + \tilde{M} \tilde{N} \tilde{P} \tilde{M}}{1} \right) & t > t_0 
\end{cases}
\] (48)

where \( V \) is the volume of the parallelepiped (\( V = a \times b \times c \)), the explicit expressions for \( \tilde{1}, \tilde{1}, \tilde{M}, \tilde{M} \ldots \) can be found in the appendix of [97].

Knight et al. [67] showed that the shape of the HP signals could be remarkably altered by the finite probe characteristics, this change is most significant at early times. Therefore, Knight, et al. (2012) proposed the identical-cylindrical-perfect-conductors (ICPC) solution accounting for the finite probe
dimension and thermal properties of the temperature sensing and heater needles, such that the ICPC solution can be used to simulate the scenario of infinitely long probes [98]

\[ \tilde{T}_c(p) = \frac{q_{ls}(\mu L)}{2\pi \alpha} \left[ \mu \alpha L + \mu \alpha L / 2 \right] \]

(49)

where \( p \) represent the Laplace transform parameter and \( \tilde{T}_c(p) \) indicates the Laplace transform of \( T_c(t) \) for continuous heating scenario; \( L \) indicates the spacing between centerlines of the heater and temperature probes; \( \mu = \sqrt{p/\kappa}; \mu = C_1/C_0; \) and \( I_\ell(x) \) is the \( \ell \)-th order of the modified Bessel function of the second kind. Numerical inversion issued to obtain \( T_c(t) \) with the Laplace domain solution in Eq. (49) when \( 0 < t \leq t_0 \) and \( T_c(t - t_0) \) for \( t > t_0 \) to get \( T(t) \). The Knight et al. [67] approach was showed to work as good as other methods such as the correction method for instantaneous heating scenarios [99] and the theory of spatial weighting function [100]. Recently, Knight et al. [101] improved the Knight et al. [67] approach, they derived a semi-analytical solution accounting for the finite thermal conductivity and finite probe radius as well as the thermal contact resistance.

Yang and Jones [86] presented an improved inverse method based on the work of Endo and Hara [102] and Endo and Hara [103] to simultaneously determine thermal properties and water flux of soil with 5-needle heat pulse probes. The two dimensional flow is expressed as [104]

\[ T(x, y, t) = \begin{cases} \frac{q}{4\pi \lambda} \int_0^t s^{-1} \exp \left[ \frac{(x - V_x s)^2 + (y - V_y s)^2}{4ks} \right] ds & 0 < t \leq t_0 \\ \frac{q}{4\pi \lambda} \int_{t-t_0}^t s^{-1} \exp \left[ \frac{(x - V_x s)^2 + (y - V_y s)^2}{4ks} \right] ds & t > t_0 \end{cases} \]

(50)

Thermal diffusivity (\( \kappa \)), the thermal conductivity (\( \lambda \)), the the heat pulse velocity vector (\( V_x, V_y \)) at \( x \) and \( y \) directions can be determined through fitting the \( T \)-\( t \) data with Eq. (43) using least squares approach such that

\[ E(p) = \sum_{i=1}^N \sum_{j=1}^4 w_i \left| T_{ij}^c(P) - T_{ij}^m \right| = \sum_{i=1}^N \sum_{j=1}^4 w_i R_{ij}^2(P) \]

(51)

where \( p \) is the unknown parameter vector; \( N \) is the number of total measured temperature data of each temperature sensor; \( T_{ij}^c(P) \) indicates the \( i \)-th temperature calculated for the \( j \)-th temperature sensor; \( T_{ij}^m \) is the \( i \)-th temperature measured by the \( j \)-th temperature sensor; \( w_i \) is the weighting parameter for the \( i \)-th temperature observation, and \( R_{ij}^2(P) \) represents the \( i \)-th residual of the \( j \)-th temperature sensor.

Equation [51] can be solved by method of Gauss-Newton-Levenberg-Marquardt [104], where parameters of the \((k+1)\)th iteration are predicted from these of the \( k \)-th iteration

\[ p^{k+1} = p^k + \Delta p = p^k - (J^T W J + \alpha I)^{-1} J^T W R(P) \]

(52)

where \( I \) represents an identity matrix; \( W \) and \( R \) are the weight and residual matrix in Eq. [52], respectively; \( \alpha \) is the Marquardt parameter; and \( J \) is the Jacobian matrix

\[
J = \begin{bmatrix}
\frac{\partial T_1^c}{\partial \kappa} & \frac{\partial T_1^c}{\partial V_x} & \frac{\partial T_1^c}{\partial V_y} & \frac{\partial T_3^c}{\partial \kappa} & \frac{\partial T_3^c}{\partial V_x} & \frac{\partial T_3^c}{\partial V_y} \\
\frac{\partial T_2^c}{\partial \kappa} & \frac{\partial T_2^c}{\partial V_x} & \frac{\partial T_2^c}{\partial V_y} & \frac{\partial T_4^c}{\partial \kappa} & \frac{\partial T_4^c}{\partial V_x} & \frac{\partial T_4^c}{\partial V_y} \\
M & M & M & M & M & M \\
\frac{\partial T_5^m}{\partial \kappa} & \frac{\partial T_5^m}{\partial V_x} & \frac{\partial T_5^m}{\partial V_y} & \frac{\partial T_7^m}{\partial \kappa} & \frac{\partial T_7^m}{\partial V_x} & \frac{\partial T_7^m}{\partial V_y} \\
\end{bmatrix}
\]
Equation (53) requires the use of partial derivatives with respect to the $\kappa$, $\lambda$, $V_x$, and $V_y$ as entries that are:

$$
\frac{\partial T}{\partial \kappa} = \left\{ \begin{array}{ll}
\frac{q}{4\pi\kappa} \int_{-t_0}^{t} s^{-1}G(x,y,s)exp\left(-G(x,y,s)\right)ds & 0 < t \leq t_0 \\
\frac{q}{4\pi\kappa} \int_{-t_0}^{t} s^{-1}G(x,y,s)exp\left(-G(x,y,s)\right)ds & t > t_0
\end{array} \right.
$$

(54)

where,

$$
G(x,y,t) = \frac{(x-V_y t)^2 + (x-V_x t)^2}{4\kappa}
$$

(55)

$$
\frac{\partial T}{\partial \lambda} = \left\{ \begin{array}{ll}
-\frac{q}{4\pi\lambda^2} \int_{0}^{t} s^{-1}exp\left(-G(x,y,s)\right)ds & 0 < t \leq t_0 \\
-\frac{q}{4\pi\lambda^2} \int_{t-t_0}^{t} s^{-1}exp\left(-G(x,y,s)\right)ds & t > t_0
\end{array} \right.
$$

(56)

$$
\frac{\partial T}{\partial V_x} = \left\{ \begin{array}{ll}
\frac{q}{8\pi\kappa} \int_{0}^{t} s^{-1}(x-V_x s)exp\left(-G(x,y,s)\right)ds & 0 < t \leq t_0 \\
\frac{q}{8\pi\kappa} \int_{t-t_0}^{t} s^{-1}(x-V_x s)exp\left(-G(x,y,s)\right)ds & t > t_0
\end{array} \right.
$$

(57)

$$
\frac{\partial T}{\partial V_y} = \left\{ \begin{array}{ll}
\frac{q}{8\pi\kappa} \int_{0}^{t} s^{-1}(x-V_y s)exp\left(-G(x,y,s)\right)ds & 0 < t \leq t_0 \\
\frac{q}{8\pi\kappa} \int_{t-t_0}^{t} s^{-1}(x-V_y s)exp\left(-G(x,y,s)\right)ds & t > t_0
\end{array} \right.
$$

(58)

Then the water flux and volumetric heat capacity can be estimated based on the above calculated $\kappa$, $\lambda$, $V_x$, and $V_y$.

3. Summary

A review of the theories and solutions that can be used to determine thermal properties of soil with the heat pulse (HP) method is presented and discussed. This review covers theories and solutions of heat conduction, the instantaneous line heat source (ILHS), the line heat source of finite duration or the short-duration line heat source (SLHS), and the continuous line heat source (CLHS). Based on these theories, the most commonly used dual-probe heat-pulse (DPHP) method and single-probe heat-pulse (SPHP) method for estimating thermal properties of soil were presented. The numerical and semi-analytical solutions for non-thermal equilibrium conditions are also given.

Although the SPHP and DPHP analyze the recorded temperature change over time, they differ from each other in the solution for heat conduction, heat pulse duration, probe design, and data interpretation [25]. For instance, SPHP is based on the solution of CLHS while the DPHP method is based on the solution of SLHS. Heating duration of the SPHP method is longer than that of DPHP method (e.g., 60 to 600s vs 8s to 15s). Soil thermal property can be determined by plotting the logarithmic temperature change with time or fitting the model for the SPHP methods. For the DPHP, the thermal properties can be obtained with the non-linear model fitting (NMF) method or the single point (SPM) method. Their difference is that NMF gives better estimates than SPM with broad and sparse data with flat peaks or if the data is noisy. Soil thermal properties determined by either the SPM method or the NMF method should be compared and validated with the measured $\Delta T(t)$ data [64]. All three thermal properties, namely thermal diffusivity ($\kappa$), heat capacity ($C_v$), and soil thermal conductivity ($\lambda$) can be determined from the DPHP method, while SPHP method traditionally can only be used to accurately determine $\lambda$. Although the primary use of SPHP method is to estimate thermal conductivity of soil, soil heat capacity and thermal diffusivity were attempted to made by Blackwell [70] and Jaeger [75] under the condition of known or negligibly small contact resistance but with a
limited accuracy [3, 105]. Liu and Si [71] evaluated the use of different data interpretation models to estimate thermal properties of three soils. Their findings demonstrated that a similar dry soil \( \lambda \) could be obtained by both methods with a relative deviation < 6.1%. The estimated \( C_v \) with SPHP method with the perfect conductor model developed by Blackwell [70] agrees well with \( C_v \) taken by the differential scanning calorimetry method, while the DPHP method overestimated the \( C_v \). There is report on the use of one sensor to take both DPHP and SPHP measurements on the same soil sample [73], which make this sensor appealing. But the two methods have distinct energy requirements and care should be taken to design sensors suitable for both methods [106].

The analytical solutions are most commonly used method to estimated thermal properties of soil, but they cannot take into account of the influence of factors such as contact resistance, geometry, heat capacity, and thermal conductivity of probes, and boundary and initial conditions. In addition, HP method assumes a local thermal equilibrium condition during the period of measurement [107], however, local thermal non-equilibrium condition is common under field conditions [80, 108, 109]. These constraints imposed by most analytical solutions can be eliminated by the numerical approach [91, 110]. For instance, Hopmans et al. [83] used HYDRUS 2D to inversely model the influences of DPHP probe geometry on water flux and thermal properties of soil. Mortensen et al. [90] took use of a approach of similar manner in a soil column experiment to analyze the feasibility of multiple-function DPHP to predict hydraulic, thermal, and solute transport. Saito et al. [110] investigated the influences of heat pulse induced vapor and geometry of probe on soil water flux and thermal properties in unsaturated soils with the HYDRUS 2D program. Their results showed that a combination of a larger probe diameter and a stronger heat pulse can improve the estimates of water flux if vapor transport is taken into accounted. These reports and others showed the flexibility and capability of numerical simulations. However, for application cases of moderate complexity, numerical simulations are relatively complex to be applied and therefore semi-analytical solutions are fairy alternatives. The semi-analytical solutions accounting for effects of factors such as contact resistance and probe characteristics for SPHP and DPHP methods were collected from literature and presented in this review.

These theories and solutions serve as the basis for the heat pulse method and considerably advance the HP methods in soil science. However, it should be noted that these solutions are generally for unfrozen soils and their applications in frozen soils is limited by the effects of latent heat or phase change (e.g., ice melting or vaporization). Because the heat pulse method induces ice melting during the measurement and alter soil thermal properties being measured. However, the traditional solutions to the radial heat flow equation assume that conductive heat flux is the only heat transport approach in frozen soils. Such that ice melting was assumed to be negligible for most applications in frozen soils [111], or estimated apparent frozen soil thermal properties by embedding the latent heat flux into the thermal properties of soil [12, 112], or took use of numerical simulations [88, 92, 96, 113]. Semi-analytical solutions accounting for the effect of latent heat are worth more attention in future studies.

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