Two-sided bounds on some output-related quantities in linear stochastically excited vibration systems with application of the differential calculus of norms

L. Kohaupt

Abstract: A linear stochastic vibration model in state-space form, $\dot{x}(t) = Ax(t) + b(t)$, $x(0) = x_0$, with output equation $x_S(t) = Sx(t)$ is investigated, where $A$ is the system matrix and $b(t)$ is the white noise excitation. The output equation $x_S(t) = Sx(t)$ can be viewed as a transformation of the state vector $x(t)$ that is mapped by the rectangular matrix $S$ into the output vector $x_S(t)$. It is known that, under certain conditions, the solution $x(t)$ is a random vector that can be completely described by its mean vector, $m_x(t)$, and its covariance matrix, $P_x(t)$. If matrix $A$ is asymptotically stable, then $m_x(t) \rightarrow 0$ ($t \rightarrow \infty$) and $P_x(t) \rightarrow P_S$ ($t \rightarrow \infty$), where $P_S$ is a positive (semi-)definite matrix. Similar results will be derived for some output-related quantities. The obtained results are of special interest to applied mathematicians and engineers.

Keywords: linear stochastic vibration system excited by white noise with output equation; output-related mean vector; output-related covariance matrix; two-sided bounds; differential calculus of norms

AMS subject classifications: 34D05; 34F05; 65L05

ABOUT THE AUTHOR
Ludwig Kohaupt received the equivalent to the master’s degree (Diplom-Mathematiker) in Mathematics in 1971 and the equivalent to the PhD (Dr.phil.nat.) in 1973 from the University of Frankfurt/Main. From 1974 until 1979, Kohaupt was a teacher in Mathematics and Physics at a Secondary School. During that time (from 1977 until 1979), he was also an auditor at the Technical University of Darmstadt in Engineering Subjects, such as Mechanics, and especially Dynamics. From 1979 until 1990, he joined the Mercedes-Benz car company in Stuttgart as a Computational Engineer, where he worked in areas such as Dynamics (vibration of car models), Cam Design, Gearing, and Engine Design. Then, in 1990, Dr. Kohaupt combined his preceding experiences by taking over a professorship at the Beuth University of Technology Berlin (formerly known as TFH Berlin). He retired on April 01, 2014.

PUBLIC INTEREST STATEMENT
When a dynamical system with solution vector $x$ of length $n$ describes an engineering problem, only a few components of $x$ are needed, as a rule. But, nevertheless, the whole pertinent initial value problem must be solved. In order to obtain only the components of interest, one defines an output matrix, say $S$, that selects them from $x$ by defining the new output vector $x_S := Sx$ showing its importance. This equation is called output equation. For example, if the engineer wants to apply only the first, second, and $n$th component, then one defines $S$ as $S = [e_1, e_2, e_n]^T$ where $e_i$ means the $i$th unit vector for $i = 1, 2, n$. In the present paper, new two-sided estimates on the mean vector and covariance matrix pertinent to the output vector $x_S$ in linear stochastically excited vibration systems are derived that parallel those associated with $x$ obtained recently.
1. Introduction

In order to make the paper more easily readable for a large readership, we first introduce the notions of output vector and output equation common to engineers. When a dynamical system with solution vector \( x \) of length \( n \) describes an engineering problem, only a few components of \( x \) are needed, as a rule. But, nevertheless, the whole pertinent initial value problem must be solved. In order to obtain only the components of interest, one defines an output or transformation matrix, say \( S \), that selects them from \( x \) by defining the output \( x_S := S x \). This equation is called output equation. For example, if the engineer wants to use only the first, second, and \( n \)th component, then one defines \( S \) as

\[
S = [e_1, e_2, e_n]^T
\]

where \( e_i \) means the \( i \)th unit vector for \( i = 1, 2, n \). In other words, by employing the output equation \( x_S = Sx \), a subset of components can be selected from the whole set of degrees of freedom which is usually necessary in practice. Of course, one can also define \( S \) such that it forms linear combinations of components of \( x \). Whereas, in the preceding paper Kohaupt (2015b), the whole vector \( x \) was analyzed, in the present paper, it is replaced by the output \( x_S \). The given comments on \( x_S \) show why this is important.

In this paper, a linear stochastic vibration model of the form \( x(t) = Ax(t) + b(t) \), \( x(0) = x_0 \), with output equation \( x_S(t) = Sx(t) \) is investigated, where \( A \) is a real system matrix, \( b(t) \) white noise excitation, and \( x_0 \) an initial vector that can be completely characterized by its mean vector \( m_0 \) and its covariance matrix \( P_0 \). Likewise, the solution \( x(t) \), also called response, is a random vector that can be described by its mean vector \( m_x(t) := m_{xtrl} \) and its covariance matrix, \( P_x(t) := P_{xtrl} \). For asymptotically stable matrices \( A \), it is known that \( m_x(t) \to 0 \ (t \to \infty) \) and \( P_x(t) \to P \ (t \to \infty) \), where \( P \) is a positive (semi-)definite matrix. Similarly, for the output or transformed quantity \( x_S(t) \), one has \( m_{x_S}(t) \to 0 \ (t \to \infty) \) and \( P_{x_S}(t) \to P_S \ (t \to \infty) \) with a positive (semi-)definite matrix \( P_S \). The asymptotic behavior of \( m_x(t) \) and \( P_x(t) \) was studied in Kohaupt (2015b).

In this paper, we investigate the asymptotic behavior of \( m_{x_S}(t) \) and \( P_{x_S}(t) \). As appropriate norms for the investigation of this problem, again the Euclidean norm for \( m_{x_S}(t) \) and the spectral norm for \( P_{x_S}(t) \) is the respective natural choice; both norms are denoted by \( \| \cdot \|_2 \).

The main new points of the paper are

- the determination of two-sided bounds on \( m_{x_S}(t) \) and \( P_{x_S}(t) \);
- the derivation of formulas for the right norm derivatives \( D^+_{x_S}(t) = P_{x_S} \); \( k = 0, 1, 2 \);
- the application of these results to the computation of the best constants in the two-sided bounds.

Special attention is paid to conditions ensuring the positiveness of the constants in the lower bounds when \( S \) is only rectangular and not square regular.

The paper is structured as follows.

In Section 2, the linear stochastically excited vibration model with output equation is presented. Then, in Section 3, the transformed quantities \( m_{x_S}(t) \) and \( P_{x_S}(t) \) are determined from \( m_x(t) \) and \( P_x(t) \), respectively, by appropriate use of the output matrix \( S \) as transformation matrix. Section 4 derives two-sided bounds on \( x_S(t) = Sx(t) \) with \( x(t) = Ax(t), x(0) = x_0 \), as a preparation to derive two-sided bounds on \( m_{x_S}(t) \) in Section 6. Section 5 determines two-sided bounds on \( \Phi_x(t) := S \Phi(t) \) with \( \Phi(t) = A \Phi(t), \Phi(0) = E \), as a preparation to derive two-sided bounds on \( P_{x_S}(t) \) in Section 7. Section 8 studies the local regularity of \( P_{x_S}(t) \). Then in Section 9, as the main result, formulas for the right norm derivatives \( D^+_{x_S}(t) = P_{x_S} \); \( k = 0, 1, 2 \) are obtained. Section 10, for the specified data in the stochastically exited model, presents applications, where the differential calculus of norms is employed by computing the best constants in the new two-sided bounds on \( m_{x_S}(t) \) and \( P_{x_S}(t) \). In Section 11, conclusions are drawn. The Appendix A contains sufficient algebraic conditions that ensure the positiveness of the constants in the lower bounds when \( S \) is only rectangular and not square regular. Finally, we comment on the References. The author’s papers on the differential
calculus of norms are contained in Kohaupt (1999, 2001, 2002, 2003, 2004a, 2004b, 2005, 2006, 2007a, 2007b, 2007c, 2008a, 2008b, 2008c, 2008d, 2009a, 2009b, 2009c, 2010a, 2010b, 2011, 2012, 2013, 2015a, 2015b). The articles Bhatia and Elsner (2003), Benner, Denißen, and Kohaupt (2013, 2016), and Whidborne and Amer (2011) refer to some of the author’s works. The publications Coppel (1965), Dahlquist (1959), Desoer and Haneda (1972), Hairer, Nørset, and Wanner (1993), Higueras and Garcia-Celayeta (1999, 2000), Hu and Hu (2000), Lozinskiǐ (1958), Pao (1973a, 1973b), Söderlind and Mattheij (1985), Ström (1972, 1975) contain subjects on the logarithmic norm which was the starting point of the author’s development of the differential calculus of norms. The References Bickley and McNamee (1960), Kučera (1974), and Ma (1966) were important for the author’s article on the equation VA + A∗V = μV in Kohaupt (2008a). The publications Achieser and Glasman (1968), Heuser (1975), Kantorovich and Akilov (1982), Kato (1966), and Taylor (1958) are textbooks on functional analysis useful, for instance, in the proofs of the theorems in Section 5. The books Golub and van Loan (1989), Niemeyer and Wermuth (1987), and Stummel and Hainer (1980) contain chapters on Matrix Theory and Numerical Mathematics valuable in connection with the subject of the present paper. The books Müller and Schiehlen (1985), Thomson and Dahleh (1998), and Waller (1975) are on engineering dynamical systems. In paper Guyan (1965), a reduction method for stiffness and mass matrices is discussed, a method that is still in use nowadays. Last, but not least, Kloeden and Platen (1992) is a standard book on the numerical solution of stochastic differential equations.

2. The linear stochastically excited vibration system with output equation

In order to make the paper as far as possible self-contained, we summarize the known facts on linear stochastically excited systems. In the presentation, we closely follow the line of Müller and Schiehlen (1985, Sections 9.1 and 9.2).

So, let us depart from the deterministic model in state-space form

\[
\dot{x}(t) = Ax(t) + b(t), \quad t > 0, \quad x(0) = x_0,
\]

\[
x_S(t) = Sx(t)
\]

with system matrix \( A \in \mathbb{R}^{n \times n} \), the state vector \( x(t) \in \mathbb{R}^n \) and the excitation vector \( b(t) \in \mathbb{R}^n \), \( t \geq 0 \), the output matrix \( S \in \mathbb{R}^{l \times n} \), and the output vector \( x_S(t) \). We call (2) output equation. It can be understood as a transformation making of \( x(t) \) the transformed quantity \( x_S(t) \) by applying the transformation matrix \( S \) to \( x(t) \).

Now, we replace the deterministic excitation \( b(t) \) by a stochastic excitation in the form of white noise. Thus, \( b(t) \) can be completely described by the mean vector \( m_b(t) \) and the central correlation matrix \( N_b(t, \tau) \) with

\[
m_b(t) = 0,
\]

\[
N_b(t, \tau) = Q \delta(t - \tau),
\]

where \( Q = Q_b \) is the \( n \times n \) intensity matrix of the excitation and \( \delta(t - \tau) \) the \( \delta \)-function (more precisely, the \( \delta \)-functional).

From the central correlation matrix, for \( \tau = t \) one obtains the positive semi-definite covariance matrix

\[
P_b(t) = N_b(t, t).
\]

At this point, we mention that the definition of a real positive semi-definite matrix includes its symmetry.
When the excitation is white noise, the deterministic initial value problem (1) can be formally maintained as the theory of linear stochastic differential equations shows. However, the initial state $x_0$ must be introduced as Gaussian random vector,

$$x_0 \sim (m_0, P_0),$$

which is to be independent of the excitation; here, the sign $\sim$ means that the initial state $x_0$ is completely described by its mean vector $m_0$ and its covariance matrix $P_0$. More precisely: $x_0$ is a Gaussian random vector whose density function is completely determined by $m_0$ and $P_0$ alone.

The stochastic response of the system (1) is formally given by

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - \tau)b(\tau)d\tau,$$

where $\Phi(t)$ besides the fundamental matrix $\Phi(t) = e^{At}$ and the initial vector $x_0$ a stochastic integral occurs.

It can be shown that the stochastic response $x(t)$ is a non-stationary Gauss–Markov process that can be described by the mean vector $m_x(t) = m_{x(t)}$ and the correlation matrix $N_x(t, \tau) = N_{x(t), x(\tau)}$. For $\tau = t$, we get the covariance matrix $P_x(t) = P_{x(t)}$.

If the system is asymptotically stable, the properties of first and second order for the stochastic response $x(t)$ we need are given by

$$m_x(t) = \Phi(t)m_0,$$

$$P_x(t) = \Phi(t)(P_0 - P)\Phi^T(t) + P,$$

where the positive semi-definite $n \times n$ matrix $P$ satisfies the Lyapunov matrix equation

$$AP + PA^T + Q = 0.$$  

This is a special case of the matrix equation $AX + XB = C$, whose solution can be obtained by a method of Ma (1966). For the special case of diagonalizable matrices $A$ and $B$, this is shortly described in Kohaupt (2015b, Appendix A.1).

For asymptotically stable matrices $A$, one has $\lim_{t \to \infty} \Phi(t) = 0$ and thus by (7) and (8),

$$\lim_{t \to \infty} m_x(t) = 0$$

and

$$\lim_{t \to \infty} P_x(t) = P.$$  

In Kohaupt (2015b), we have investigated the asymptotic behavior of $m_x(t)$ and $P_x(t) - P$.

In this paper, we want to derive formulas for $m_x(t)$ and $P_x(t)$ corresponding to those of (7) and (8) and study their asymptotic behavior. This will be done in the next five sections, that is, in Sections 3–7.

3. The output-related quantities $m_{xS}(t)$ and $P_{xS}(t)$

In this section, we determine the output-related quantities $m_{xS}(t)$ and $P_{xS}(t)$ from the corresponding quantities $m_x(t)$ and $P_x(t)$ by appropriate use of the output matrix $S$ as the transformation matrix.
The results of this section are known to mechanical engineers, but are added for the sake of completeness, especially for mathematicians.

One obtains the following lemma.

**Lemma 1** *(Formulas for $m_x(t)$ and $P_{x_S}(t)$)*

Let $S \in \mathcal{R}^{l \times n}$ and $x(t)$ the solution vector of (1). Further, let $x_s(t)$ be given by (2), i.e.

$$x_s(t) = Sx(t).$$

Then, one has

$$m_{x_S}(t) = \Phi_S(t)m_0,$$

$$P_{x_S}(t) = SP_S(t)S^T = \Phi_S(t)(P_0 - P)\Phi_S^T(t) + P_S,$$

with

$$\Phi_S(t) := S\Phi(t)$$

and

$$P_S := SPS^T.$$  

**Proof**

(i) One has

$$m_{x_S}(t) = E(x_s(t)) = E(Sx(t)) = S(E(x(t))) = Sm_S(t),$$

where here $E$ denotes the expectation of a random vector. Using (7), this leads to (11).

(ii) Next, we show that, for the central correlation matrices $N_{x_S}(t, r)$ and $N_{x_S}(t, r)$, the identity

$$N_{x_S}(t, r) = SN_{x_S}(t, r)S^T$$

holds. This is because

$$N_{x_S}(t, r) = E\{[x_S(t) - m_{x_S}(t)][x_S(r) - m_{x_S}(r)]^T\}$$

$$= E\{S[x(t) - m_x(t)][S(x(r) - m_x(r))]^T\}$$

$$= E\{S[x(t) - m_x(t)][x(r) - m_x(r)]^T S^T\}$$

$$= SE\{[x(t) - m_x(t)][x(r) - m_x(r)]\}S^T$$

$$= SN_{x_S}(t, r)S^T.$$

Thus, (15) is proven. Setting $r = t$, this implies

$$P_{x_S}(t) = SP_S(t)S^T.$$  

**Taking into account (8), this leads to (12).**

**Remark** Let the system matrix $A$ be asymptotically stable. Then, $\Phi(t) \to 0$, $(t \to \infty)$ and thus from (12) and (13),
as well as

\[4. \text{Two-sided bounds on } x_S(t) = Sx(t) \text{ with } \dot{x}(t) = Ax(t), \ x(t_0) = x_0, \]

In this section, we discuss the deterministic case \( x_S(t) = Sx(t) \) with \( \dot{x}(t) = Ax(t), \ x(t_0) = x_0 \), as a preparation for Section 6. There, two-sided bounds on \( m x_S(t) \) will be given based on those for \( x_S(t) \) here.

For the positiveness of the constants in the lower bounds, we discuss two cases: the special case when matrix \( A \) is diagonalizable and the case of a general square matrix \( A \).

Let \( S \in \mathbb{C}^{n \times n} \) and

\[ x_S(t) = Sx(t). \tag{17} \]

We obtain

**THEOREM 1** (Two-sided bound on \( x_S(t) = Sx(t) \) by \( e^{\lambda_S[A](t-t_0)} \))

Let \( A \in \mathbb{C}^{n \times n}, 0 \neq x_0 \in \mathbb{C}^n \), and \( x(t) \) be the solution of the initial value problem \( \dot{x}(t) = Ax(t), \ x(t_0) = x_0 \). Let \( \| \cdot \| \) be any vector norm.

Then, there exists a constant \( X_{S,0} \geq 0 \) and for every \( \epsilon > 0 \) a constant \( X_{S,\epsilon}(\epsilon) > 0 \) such that

\[ X_{S,0} e^{\lambda_S[A](t-t_0)} \leq \| x_S(t) \| \leq X_{S,\epsilon}(\epsilon) e^{\lambda_S[A]+\varepsilon}[t-t_0], \ t \geq t_0, \tag{18} \]

where \( \lambda_S[A] \) is the spectral abscissa of \( A \) with respect to \( x_0 \).

If \( A \) is diagonalizable, then \( \epsilon = 0 \) may be chosen, and we write \( X_{S,1} \) instead of \( X_{S,\epsilon}(\epsilon = 0) \).

If \( S \) is square and regular, then \( X_{S,0} > 0 \).

**Proof**

One has

\[ 0 \| x(t) \| \leq \| x_S(t) \| \leq \| S \| \| x(t) \|. \tag{19} \]

Further, according to Kohaupt (2006, Theorem 7), there exists a constant \( X_0 > 0 \) and for every \( \epsilon > 0 \) a constant \( X_\epsilon(\epsilon) > 0 \) such that

\[ X_0 e^{\lambda_S[A]+\varepsilon}[t-t_0] \leq \| x_S(t) \| \leq X_\epsilon(\epsilon) e^{\lambda_S[A]+\varepsilon}[t-t_0], \ t \geq t_0. \tag{20} \]

Combining (19) and (20) leads to (18) with \( X_{S,0} = 0 \).

Further, if \( S \) is square and regular, then instead of (19) we get

\[ 1/\| S^{-1} \| \| x(t) \| \leq \| x_S(t) \| \leq \| S \| \| x(t) \|. \tag{21} \]

Thus, apparently, \( X_{S,0} = X_0/\| S^{-1} \| > 0 \) can be chosen in (18).
An interesting and important question is under what conditions the constant $X_{5,0}$ is positive when $S$ is only rectangular, but not necessarily square and regular. To assert that $X_{5,0}$ is positive, additional conditions have to be imposed. We consider two cases.

**Case 1: Diagonizable matrix $A$** In this case, we need the following hypotheses on $A$ from Kohaupt (2011, Section 3.1).

1. **(H1)** $m = 2n$ and $A \in \mathbb{R}^{m \times m}$,
2. **(H2)** $T^{-1}AT = J = \text{diag}(\lambda_k)_{k=1,\ldots,m}$, where $\lambda_k = \lambda_k(A)$, $k = 1, \ldots, m$ are the eigenvalues of $A$,
3. **(H3)** $\lambda_i = \lambda_i(A) \neq 0$, $i = 1, \ldots, m$,
4. **(H4)** $\lambda_i \neq \lambda_j$, $i \neq j$, $i,j = 1, \ldots, m$,
5. **(HS)** the eigenvectors $p_1, \ldots, p_n, \bar{p}_1, \ldots, \bar{p}_n$ form a basis of $\mathbb{C}^m$.

As a preparation to the subsequent derivations, we collect some definitions resp. notations and representations for the solution vector $x(t)$ from Kohaupt (2006, 2011).

**Representation of the basis $x_k^{(i)}(t), x_k^{(j)}(t), k = 1, \ldots, n$**

Under the hypotheses (H1), (H2), and (HS), from Kohaupt (2011), we obtain the following real basis functions for the ODE $\dot{x} = Ax$:

$$
\begin{align*}
x_k^{(i)}(t) &= e^{\lambda_k(t-t_0)} \left[ \cos \lambda_k(t-t_0)p_k^{(i)} - \sin \lambda_k(t-t_0)\bar{p}_k^{(j)} \right], \\
x_k^{(j)}(t) &= e^{\lambda_k(t-t_0)} \left[ \sin \lambda_k(t-t_0)p_k^{(i)} - \cos \lambda_k(t-t_0)\bar{p}_k^{(j)} \right],
\end{align*}
$$

\[ \text{for } k = 1, \ldots, n, \]  \( (22) \)

where

$$
\lambda_k = \lambda_k^{(r)} + i \lambda_k^{(i)} = \text{Re} \lambda_k + i \text{Im} \lambda_k,
$$

$$
p_k = p_k^{(r)} + i p_k^{(i)} = \text{Re} p_k + i \text{Im} p_k,
$$

$k = 1, \ldots, m = 2n$ are the decompositions of $\lambda_k$ and $p_k$ into their real and imaginary parts. As in Kohaupt (2011), the indices are chosen such that $\lambda_{nk} = \lambda_{nk}^{(r)}$, $p_{nk} = \bar{p}_{nk}$, $k = 1, \ldots, n$.

The spectral abscissa of $A$ with respect to the initial vector $x_0 \in \mathbb{R}^n$. Let, $u_k^{(r)}, k = 1, \ldots, m = 2n$ be the eigenvectors of $A^*$ corresponding to the eigenvalues $\bar{\lambda}_k$, $k = 1, \ldots, m = 2n$. Under (H1), (H2), and (HS), the solution $x(t)$ of (1) has the form

$$
x(t) = \sum_{k=1}^{m=2n} c_{1k} p_k e^{\lambda_k^{(r)}(t-t_0)} + \sum_{k=1}^{m=2n} \bar{c}_{1k} \bar{p}_k e^{\lambda_k^{(i)}(t-t_0)},
$$

\[ \text{with uniquely determined coefficients } c_{1k}, k = 1, \ldots, m = 2n. \]  \( (23) \)

Using the relations

$$
c_{2k} = c_{1,n+k} = \bar{c}_{1k}, \quad k = 1, \ldots, n,
$$

\[ \text{(see Kohaupt, 2011, Section 3.1 for the last relation), then according to Kohaupt (2011), the spectral abscissa of } A \text{ with respect to the initial vector } x_0 \in \mathbb{R}^n \text{ is given by} \]  \( (24) \)

$$
\nu_0 := \nu_0[A] = \max_{k=1,\ldots,m=2n} \left\{ \lambda_k^{(r)}(A) \mid x_0 \perp u_k^{(r)} \right\}
$$

$$
= \max_{k=1,\ldots,m=2n} \left\{ \lambda_k^{(r)}(A) \mid c_{1k} \neq 0 \right\}
$$

$$
= \max_{k=1,\ldots,n} \left\{ \lambda_k^{(r)}(A) \mid c_{1k} \neq 0 \right\}
$$

$$
= \max_{k=1,\ldots,n} \left\{ \lambda_k^{(r)}(A) \mid x_0 \perp u_k^{(r)} \right\}
$$

\[ \text{(25)} \]
Index sets In the sequel, we need the following index sets:

\[ J^+_v := \{ k_0 \in \mathbb{N} | 1 \leq k_0 \leq n \text{ and } \lambda^{(i)}_{k_0}(A) = \nu_0 \} \] (26)

and

\[ J^-_v := \{ 1, \ldots, n \} \setminus J_v \]
\[ = \{ k_0 \in \mathbb{N} | 1 \leq k_0 \leq n \text{ and } \lambda^{(i)}_{k_0}(A) < \nu_0 \} \] (27)

Appropriate representation of \( x(t) \) We have

\[ x(t) = \sum_{k=1}^{n} [c^{(r)}_k x^{(r)}_k(t) + c^{(i)}_k x^{(i)}_k(t)] \] (28)

with

\[ c^{(r)}_k = 2 \Re c_{1k}, \quad c^{(i)}_k = -2 \Im c_{1k}, \quad k = 1, \ldots, n \] (cf. Kohaupt, 2011). Thus, due to (28) and (22),

\[ x(t) = \sum_{k=1}^{n} e^{\lambda^{(i)}_{k}(t-t_0)} f^{i}_k(t) \] (29)

with

\[ f^{i}_k(t) := c^{(r)}_k [\cos \lambda^{(i)}_k(t-t_0) p^{(r)}_k - \sin \lambda^{(i)}_k(t-t_0) p^{(i)}_k]
+ c^{(i)}_k [\sin \lambda^{(i)}_k(t-t_0) p^{(r)}_k + \cos \lambda^{(i)}_k(t-t_0) p^{(i)}_k], \] (30)

\[ k = 1, \ldots, n. \]

Appropriate representation of \( y(t) \) and \( \dot{y}(t) \) (needed in the Appendix) Let

\[ p_k := \begin{bmatrix} q_k \cr r_k \end{bmatrix}, \quad p^{(r)}_k := \begin{bmatrix} q^{(r)}_k \cr r^{(r)}_k \end{bmatrix}, \quad p^{(i)}_k := \begin{bmatrix} q^{(i)}_k \cr r^{(i)}_k \end{bmatrix}, \] (31)

with \( q_k, r_k \in \mathbb{C}^n, q^{(r)}_k, r^{(r)}_k, q^{(i)}_k, r^{(i)}_k \in \mathbb{R}^n, k = 1, \ldots, m = 2n \). Then, from (29), (30),

\[ y(t) = \sum_{k=1}^{n} e^{\lambda^{(i)}_{k}(t-t_0)} g^{i}_k(t) \] (32)

with

\[ g^{i}_k(t) := c^{(r)}_k [\cos \lambda^{(i)}_k(t-t_0) q^{(r)}_k - \sin \lambda^{(i)}_k(t-t_0) q^{(i)}_k]
+ c^{(i)}_k [\sin \lambda^{(i)}_k(t-t_0) q^{(r)}_k + \cos \lambda^{(i)}_k(t-t_0) q^{(i)}_k], \] (33)

\[ k = 1, \ldots, n \text{ as well as } \]

\[ z(t) = \dot{y}(t) = \sum_{k=1}^{n} e^{\lambda^{(i)}_{k}(t-t_0)} h^{i}_k(t) \] (34)

with
\[ h_k(t) = c_k^{(r)} [\cos \lambda_k^{(r)}(t-t_0)\lambda_k^{(t)} - \sin \lambda_k^{(r)}(t-t_0)\lambda_k^{(t)}] \\
+ c_k^{(r)} [\sin \lambda_k^{(r)}(t-t_0)\lambda_k^{(t)} + \cos \lambda_k^{(r)}(t-t_0)\lambda_k^{(t)}], \quad k = 1, \ldots, n. \] (35)

Herewith, one obtains for a corresponding estimate on \( x(t) \), compare Kohaupt (2011, (10)).

Now, let
\[ S_{\nu}(t) \neq 0, \quad t \geq t_0, \] (37)

Then, similarly as in Kohaupt (2011, (12)),
\[ \| \sum_{k \in J_{\nu}} S_{\nu}(t) \| \geq \inf_{t \geq t_0} \| \sum_{k \in J_{\nu}} S_{\nu}(t) \| = X_{S_{\nu}}, \quad t \geq t_0. \] (38)

Together with (36), this entails
\[ \| x(t) \| \geq X_{S_{\nu}} e^{|r_{\nu}(t-t_0)|}, \quad t \geq t_1 \geq t_0 \] (39)

with
\[ X_{S_{\nu}} = \frac{R_{S_{\nu}}}{2} > 0 \]

for sufficiently large \( t_1 \). Thus, we obtain

**Theorem 2** (Positiveness of the constant \( X_{S_{\nu}} \) in lower bound if \( A \) diagonalizable)

Let the hypotheses (H1), (H2), and (HS) for \( A \) be fulfilled, \( 0 \neq x_0 \in IR^n, S \in IR^{k \times m} \) be diagonalizable as well as condition (37) be satisfied.

Then, there exists a positive constant \( X_{S_{\nu}} \) such that
\[ X_{S_{\nu}} e^{|r_{\nu}(t-t_0)|} \leq \| x(t) \|, \quad t \geq t_1 \geq t_0. \] (40)

for sufficiently large \( t_1 \geq t_0 \).

If \( X_{S_{\nu}}(t) \neq 0, \quad t \geq t_0 \), then \( t_1 = t_0 \) can be chosen.

**Proof** The last statement is proven similarly as in the proof of Kohaupt (2011, Theorem 1). \( \square \)
Remarks

- As opposed to (37), the relation \(\sum_{k=1}^{\infty} f_k(t) \neq 0, \quad t \geq t_0\), in Kohaupt (2011, (11)) could be proven there and thus needed not be assumed.
- We mention that the quantities \(f_k(t)\) depend on the initial vector \(x_0\) through their coefficients \(c_{p}'^i)^c_k\) (Kohaupt, 2011, (8)). To stress this fact, one can write \(f_k(t) = f_k(t, x_0)\) or \(f_k(t) = f_k_{x_0}(t)\).

Case 2 General square matrix \(A\) In this case, we need the following hypotheses on \(A\) from Kohaupt (2011, Section 3.2).

\[
\begin{align*}
(H1') & \quad m = 2n \text{ and } A \in \mathbb{R}^{m \times m}, \\
(H2') & \quad T^{-1}AT = J = \text{diag}(J_1(\lambda_i))_{i=1,...,n}, \text{ where } J_1(\lambda_i) \in \mathbb{C}^{m \times m} \text{ are the canonical Jordan forms,} \\
(H3') & \quad \lambda_i = \lambda_i(A) \neq 0, \quad i = 1, \ldots, r, \\
(H4') & \quad \lambda_i \neq \lambda_j, \quad i \neq j, \quad i, j = 1, \ldots, r, \\
(H5') & \quad r = 2p, \text{ and the principal vectors} \\
p_1^{(1)}, \ldots, p_m^{(1)}, \ldots, p_1^{(\rho)}, \ldots, p_m^{(\rho)}, p_1^{(1)}, \ldots, p_m^{(1)}, \ldots, p_1^{(\rho)}, \ldots, p_m^{(\rho)} \text{ form a basis of } \mathbb{C}^m.
\end{align*}
\]

In the case of a general square matrix \(A\), we also have to collect some definitions resp. notations and representations of \(x(t)\) from Kohaupt (2006, 2011).

Representation of the basis \(x_1^{(r)}(t), x_2^{(i)}(t), k = 1, \ldots, m, l = 1, \ldots, \rho\)

Under the hypotheses \((H1')\), \((H2')\), and \((H5')\), from Kohaupt (2011) we obtain the following real basis functions for the ODE \(\dot{x} = Ax\):

\[
\begin{align*}
x_1^{(r)}(t) &= e^{\lambda_1(t-t_0)} \left\{ \cos \lambda_1(t-t_0) \left[ \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + \frac{p_1^{(r)}(t-t_0)}{k-1} + \ldots + \frac{p_1^{(r)}(t-t_0)}{k-1} \right] \\
&\quad - \sin \lambda_1(t-t_0) \left[ \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + \frac{p_1^{(r)}(t-t_0)}{k-1} + \ldots + \frac{p_1^{(r)}(t-t_0)}{k-1} \right] \right\}, \\
x_2^{(i)}(t) &= e^{\lambda_1(t-t_0)} \left\{ \sin \lambda_1(t-t_0) \left[ \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + \frac{p_1^{(r)}(t-t_0)}{k-1} + \ldots + \frac{p_1^{(r)}(t-t_0)}{k-1} \right] \\
&\quad + \cos \lambda_1(t-t_0) \left[ \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + \frac{p_1^{(r)}(t-t_0)}{k-1} + \ldots + \frac{p_1^{(r)}(t-t_0)}{k-1} \right] \right\},
\end{align*}
\]

where

\[
p_k^{(i)} = p_k^{(r)} + i p_k^{(i)}
\]

is the decomposition of \(p_k^{(i)}\) into its real and imaginary part.
The spectral abscissa of $A$ with respect to the initial vector $x_0 \in \mathbb{R}^n$ let $u_{k}^{l,r}, \ k = 1, \ldots, m$, be the principal vectors of stage $k$ of $A^*$ corresponding to the eigenvalue $\lambda_l, \ l = 1, \ldots, r = 2\rho$. Under $(H1')$, $(H2')$, and $(HS')$, the solution $x(t)$ of (1) has the form

$$x(t) = \sum_{l=1}^{\rho} \sum_{k=1}^{m} c_{1k}^{(l)} x_k^{(l)}(t) = \sum_{l=1}^{\rho} \sum_{k=1}^{m} [c_{1k}^{(l)} x_k^{(l)}(t) + c_{2k}^{(l)} x_k^{(l)}(t)].$$

(42)

with uniquely determined coefficients $c_{1k}^{(l)}, \ k = 1, \ldots, m, \ l = 1, \ldots, r = 2\rho$. Using the relations

$$c_{1k}^{(l)} = (x_0, u_{k}^{l,r}), \ k = 1, \ldots, m, \ l = 1, \ldots, \rho$$

$$c_{1k}^{(l)} = c_{1k}^{(l+1)} = c_{1k}^{(l)}, \ l = 1, \ldots, \rho$$

(43)

(see Kohaupt, 2011, Section 3.2 for the last relation), then the spectral abscissa of $A$ with respect to the initial vector $x_0 \in \mathbb{R}^n$ is

$$v_0: = v_{x_0}[A] := \max_{l=1, \ldots, 2\rho} \{\lambda_{l}^{(l)}(A) | x_0 \perp M_{\lambda_l^{(l)}(A)}^{(l)} = [u_{1}^{l,r}, \ldots, u_{m}^{l,r}]\}$$

$$= \max_{l=1, \ldots, 2\rho} \{\lambda_{l}^{(l)}(A) | c_{1k}^{(l)} \neq 0 \text{ for at least one } k \in \{1, \ldots, m\}\}$$

$$= \max_{l=1, \ldots, 2\rho} \{\lambda_{l}^{(l)}(A) | c_{1k}^{(l)} \neq 0 \text{ for at least one } k \in \{1, \ldots, m\}\}$$

(44)

$$= \max_{l=1, \ldots, 2\rho} \{\lambda_{l}^{(l)}(A) | x_0 \perp M_{\lambda_l^{(l)}(A)}^{(l)} = [u_{1}^{l,r}, \ldots, u_{m}^{l,r}]\}$$

Index sets For the sequel, we need the following index sets:

$$J_{\rho} := \{l_0 \in \mathbb{N} | 1 \leq l_0 \leq \rho \text{ and } \lambda_{l_0}^{(l)}(A) = v_0\}$$

(45)

and

$$J_{\rho} := \{1, \ldots, \rho\} \setminus J_{\rho}$$

$$= \{l_0 \in \mathbb{N} | 1 \leq l_0 \leq \rho \text{ and } \lambda_{l_0}^{(l)}(A) < v_0\}.\)$$

(46)

Appropriate representation of $x(t)$ We have

$$x(t) = \sum_{l=1}^{\rho} \sum_{k=1}^{m} [c_{1k}^{(l)} x_k^{(l)}(t) + c_{2k}^{(l)} x_k^{(l)}(t)] \tag{47}$$

with

$$c_{1k}^{(l)} = 2 \Re c_{1k}^{(l)}, \ c_{2k}^{(l)} = -2 \Im c_{1k}^{(l)}, \ k = 1, \ldots, m, \ l = 1, \ldots, \rho$$

(cf. Kohaupt, 2011). Thus, due (47),

$$x(t) = \sum_{l=1}^{\rho} e^{\lambda_{l_0}^{(l)}(A-t)} \sum_{k=1}^{m} f_{1k}^{(l)}(t) \tag{48}$$

with
\[
f_k^{(i)}(t) = c_k^{(i)} \left\{ \cos \lambda_1^{(i)}(t - t_0) \left[ p_1^{(i)} \left( \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + p_{k-1}^{(i)} (t - t_0) + p_k^{(i)} \right) \right] \\
- \sin \lambda_1^{(i)}(t - t_0) \left[ p_1^{(i)} \left( \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + p_{k-1}^{(i)} (t - t_0) + p_k^{(i)} \right) \right] \right\} \\
+ c_k^{(i)} \left\{ \sin \lambda_1^{(i)}(t - t_0) \left[ p_1^{(i)} \left( \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + p_{k-1}^{(i)} (t - t_0) + p_k^{(i)} \right) \right] \\
+ \cos \lambda_1^{(i)}(t - t_0) \left[ p_1^{(i)} \left( \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + p_{k-1}^{(i)} (t - t_0) + p_k^{(i)} \right) \right] \right\} \\
\]

(49)

\[k = 1, \ldots, m, \ l = 1, \ldots, \rho.\]

**Appropriate representation of \(y(t)\) and \(\dot{y}(t)\) (needed in the Appendix)**

Set

\[
p_k^{(i)} = \begin{bmatrix} q_k^{(i)} \\ r_k^{(i)} \end{bmatrix}, \quad p_k^{(i)} = \begin{bmatrix} q_k^{(i)} \\ r_k^{(i)} \end{bmatrix}, \quad p_k^{(i)}(t) = \begin{bmatrix} q_k^{(i)} \\ r_k^{(i)} \end{bmatrix},
\]

(50)

with \(q_k^{(i)}, r_k^{(i)} \in \mathbb{C}^n, q_k^{(i)}, r_k^{(i)} \in \mathbb{R}^n, k = 1, \ldots, m, \ l = 1, \ldots, \rho.\)

Then, from (48), (49)

\[
y(t) = \sum_{l=1}^{\rho} e^{q_l^{(i)}(t-t_0)} \sum_{k=1}^{m} g_k^{(i)}(t)
\]

(51)

with

\[
g_k^{(i)}(t) = c_k^{(i)} \left\{ \cos \lambda_1^{(i)}(t - t_0) \left[ q_1^{(i)} \left( \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + q_{k-1}^{(i)} (t - t_0) + q_k^{(i)} \right) \right] \\
- \sin \lambda_1^{(i)}(t - t_0) \left[ q_1^{(i)} \left( \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + q_{k-1}^{(i)} (t - t_0) + q_k^{(i)} \right) \right] \right\} \\
+ c_k^{(i)} \left\{ \sin \lambda_1^{(i)}(t - t_0) \left[ q_1^{(i)} \left( \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + q_{k-1}^{(i)} (t - t_0) + q_k^{(i)} \right) \right] \\
+ \cos \lambda_1^{(i)}(t - t_0) \left[ q_1^{(i)} \left( \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + q_{k-1}^{(i)} (t - t_0) + q_k^{(i)} \right) \right] \right\} \\
\]

(52)

\[k = 1, \ldots, m, \ l = 1, \ldots, \rho\]

as well as

\[
z(t) = \dot{y}(t) = \sum_{l=1}^{\rho} e^{q_l^{(i)}(t-t_0)} \sum_{k=1}^{m} h_k^{(i)}(t)
\]

(53)

with
\[ h_k^{(i)}(t) = c_k^{(i)} \left\{ \cos \lambda_i^{(i)}(t - t_0) \left[ \frac{\nu_{i1}^{(i)} (t - t_0)^{k-1}}{(k-1)!} + \cdots + \nu_{ik}^{(i)} (t - t_0) + \nu_{ik}^{(i)} \right] ight\} \]

\[ - \sin \lambda_i^{(i)}(t - t_0) \left[ \frac{\nu_{i1}^{(i)} (t - t_0)^{k-1}}{(k-1)!} + \cdots + \nu_{ik}^{(i)} (t - t_0) + \nu_{ik}^{(i)} \right] \]

\[ + c_k^{(i)} \left\{ \sin \lambda_i^{(i)}(t - t_0) \left[ \frac{\nu_{i1}^{(i)} (t - t_0)^{k-1}}{(k-1)!} + \cdots + \nu_{ik}^{(i)} (t - t_0) + \nu_{ik}^{(i)} \right] \right\} \]

\[ + \cos \lambda_i^{(i)}(t - t_0) \left[ \frac{\nu_{i1}^{(i)} (t - t_0)^{k-1}}{(k-1)!} + \cdots + \nu_{ik}^{(i)} (t - t_0) + \nu_{ik}^{(i)} \right] \]

\[ k = 1, \ldots, m, \lambda = 1, \ldots, \rho \]

Now, let

\[ \sum_{i \in S} \sum_{k=1}^{m} S_{k}^{(i)}(t) \neq 0, \quad t \geq t_0. \]

(55)

Then, similarly as in Kohaupt (2011, Section 3.2), there exists a constant \( X_{S,0} > 0 \) such that

\[ \|X_0(t)\| \geq X_{S,0} e^{\nu_0(t-t_0)}, \quad t \geq t_1 \geq t_0 \]

(56)

for sufficiently large \( t_1 \geq t_0 \).

Thus, we obtain

**THEOREM 3 (Positiveness of the constant \( X_{S,0} \) in lower bound if \( A \) a general square)**

Let the hypotheses (H1'), (H2'), and (HS') for \( A \) be fulfilled, \( 0 \neq x_0 \in \mathbb{R}^m, S \in \mathbb{R}^{km}, A \) be a general square matrix as well as condition (55) be satisfied.

Then, there exists a positive constant \( X_{S,0} \) such that

\[ X_{S,0} e^{\nu_0(t-t_0)} \leq \|X_0(t)\|, \quad t \geq t_1 \geq t_0. \]

(57)

for sufficiently large \( t_1 \geq t_0 \).

If \( x_0(t) \neq 0, \quad t \geq t_0 \) then \( t_1 = t_0 \) can be chosen.

**Remark** Sufficient algebraic conditions for (37) resp. (55) will be given in the Appendix; they are independent of the initial vector \( x_0 \) and the time \( t \).

**5. Two-sided bounds on \( \Phi(t) = S\Phi(t) \) with \( \Phi(t) = A\Phi(t), \Phi(0) = E \)**

In this section, we discuss the deterministic case \( \Phi(t) = S\Phi(t) \) with \( \Phi(t) = A\Phi(t), \Phi(0) = E \), as a preparation for Section 7. There, two-sided bounds on \( P(t) = \Phi(t)(P_0 - P)\Phi^T(t) \) will be given based on those for \( \Phi(t) \) here.

Moreover, for the positiveness of the constant in the lower bound, we discuss two cases: the special case when matrix \( A \) is diagonalizable and the case when \( A \) is general square.

We obtain
Theorem 4 (Two-sided bound on $P_{x_1}(t) - P_x$ based on $\|\Phi_2(t)\|_2^2$)

Let $A \in \mathbb{R}^{nxn}$, let $\Phi(t) = e^{At}$ be the associated fundamental matrix with $\Phi(0) = E$ where $E$ is the identity matrix. Further, let $P_0, P \in \mathbb{R}^{nxn}$ be the covariance matrices from Section 2.

Then,

$$q_0 \|\Phi_2(t)\|_2^2 \leq \|P_{x_1}(t) - P_x\|_2 \leq q_1 \|\Phi_2(t)\|_2^2, \quad t \geq 0,$$

where

$$q_0 = \inf_{\|v\|_1 = 1} |((P_0 - P)v, v)|$$

and

$$q_1 = \sup_{\|v\|_1 = 1} |((P_0 - P)v, v)| = \|P_0 - P\|_2.$$

If $P_0 \neq P$, then $q_1 > 0$. If $P_0 - P$ is regular, then

$$q_0 = \|(P_0 - P)^{-1}\|_2^2 > 0.$$

Proof. The proof follows from Kohaupt (2015b, Lemmas 1 and 2) with $C = P_0 - P$ and $\Psi = \Phi_2(t) = \Phi_2(t)$ as well as $\|\Psi(t)\|_2 = \|\Phi_2(t)\|_2$.

Next, we have to derive two-sided bounds on $\|\Phi_2(t)\|_2$. For this, we write

$$S \Phi(t) = S[\varphi_1(t), \ldots, \varphi_n(t)] = [S\varphi_1(t), \ldots, S\varphi_n(t)],$$

where $\varphi_1(t), \ldots, \varphi_n(t)$ are the columns of the fundamental matrix $\Phi(t)$, i.e.

$$\Phi(t) = [\varphi_1(t), \ldots, \varphi_n(t)].$$

Now, $\Phi(t) = A\Phi(t), \Phi(0) = E$, is equivalent to

$$\dot{\varphi}_j(t) = A\varphi_j(t), \varphi(0) = e_j, \quad j = 1, \ldots, n,$$

where $e_j$ is the $j$th unit column vector.

The two-sided bounds on $\Phi_2(t) = S \Phi(t)$ can be done in any norm. Let the matrix norm $| \cdot |_\infty$ be given by

$$|B|_\infty = \max_{i,j=1,\ldots,n} |B_{ij}|, \quad B = (B_{ij}) \in \mathbb{C}^{nxn}.$$ 

Then,

$$|\Phi_2(t)|_\infty = |S \Phi(t)|_\infty = \max_{j=1,\ldots,n} \|S\varphi_j(t)\|_\infty.$$

Thus, the two-sided bound on $\Phi_2(t)$ has been reduced to two-sided bounds on $\|S\varphi_j(t)\|_\infty, j = 1, \ldots, n$.

Similarly to Theorem 1, we obtain

Theorem 5 (Two-sided bound on $\Phi_2(t) = S \Phi(t)$ by $e^{(Al)}$) Let $A \in \mathbb{C}^{nxn}$ and $\Phi(t)$ be the fundamental matrix of $A$ with $\Phi(0) = E$, i.e., let $\Phi(t)$ be the solution of the initial value problem $\dot{\Phi}(t) = A\Phi(t)$, $\Phi(0) = E$ and $\Phi_2(t) = S \Phi(t)$.
Then, there exists a constant \( \varphi_{S,0} \geq 0 \) and for every \( \varepsilon > 0 \) a constant \( \varphi_{S,1}(\varepsilon) > 0 \) such that
\[
\varphi_{S,0} e^{\nu[A]t} \leq \|\Phi_S(t)\| \leq \varphi_{S,1}(\varepsilon) e^{\nu[A]+\varepsilon t}, \quad t \geq 0,
\]
where \( \nu[A] \) is the spectral abscissa of \( A \).

If \( A \) is diagonalizable, then \( \varepsilon = 0 \) may be chosen, and we write \( \varphi_{S,1} \) instead of \( \varphi_{S,1}(\varepsilon) = 0 \).

If \( S \) is square and regular, then \( \varphi_{S,0} > 0 \).

Proof From (18) and (63), there exist constants \( \varphi_{S,0,j} \geq 0 \) and for every \( \varepsilon > 0 \) constants \( \varphi_{S,1,j}(\varepsilon) > 0 \) such that
\[
\varphi_{S,0,j} e^{\nu[A]t} \leq \|S f_j(t)\| \leq \varphi_{S,1,j}(\varepsilon) e^{\nu[A]t}, \quad t \geq 0.
\]

Define
\[
\varphi_{S,0} := \min_{j=1,\ldots,n} \varphi_{S,0,j}
\]
and
\[
\varphi_{S,1}(\varepsilon) := \max_{j=1,\ldots,n} \varphi_{S,1,j}(\varepsilon).
\]

Then, taking into account (64) and the relation
\[
\max_{j=1,\ldots,n} \nu[A] = \nu[A]
\]
(cf. Kohaupt, 2006, Proof of Theorem 7) as well as the equivalence of norms in finite-dimensional spaces, the two-sided bound (65) follows. The rest is clear from Theorem 1.

Corresponding to Theorems 2 and 3, we obtain the following two theorems.

**Theorem 6** (Positiveness of the constant \( \varphi_{S,0} \) in lower bound if \( A \) diagonalizable) Let the hypotheses \( (H1), (H2), \) and \( (HS) \) for \( A \) be fulfilled, \( S \in \mathbb{R}^{l \times m}, A \) be diagonalizable as well as condition (37) be satisfied with \( f_j(t) = f_{k,j}(t), j = 1, \ldots, m \).

Then, there exists a positive constant \( \varphi_{S,0} \) such that
\[
\varphi_{S,0} e^{\nu[A]t} \leq \|\Phi_S(t)\|, \quad t \geq t_1 \geq t_0.
\]
for sufficiently large \( t_1 \geq t_0 \).

If \( \Phi_S(t) \neq 0, t \geq t_0 \), then \( t_1 = t_0 \) can be chosen.

**Theorem 7** (Positiveness of the constant \( \varphi_{S,0} \) in lower bound if \( A \) general square) Let the hypotheses \( (H1'), (H2'), \) and \( (HS') \) for \( A \) be fulfilled, \( S \in \mathbb{R}^{l \times m}, A \) be a general square matrix as well as condition (55) be satisfied.

Then, there exists a positive constant \( \varphi_{S,0} \) such that
\[
\varphi_{S,0} e^{\nu[A]t} \leq \|\Phi_S(t)\|, \quad t \geq t_1 \geq t_0.
\]
for sufficiently large \( t_1 \geq t_0 \).

If \( \Phi_S(t) \neq 0, t \geq t_0 \), then \( t_1 = t_0 \) can be chosen.
6. Two-sided bounds on \( m_s(t) \)

According to Equation (11), we have

\[ m_s(t) = \Phi_s(t)m_0, \quad t \geq 0. \]

Now, \( x_s(t) = Sx(t) = S\Phi(t)x_0 = \Phi_s(t)x_0 \) in Theorem 1. Assuming \( m_0 \neq 0 \) and choosing the Euclidean norm \( \| \cdot \| = \| \cdot \|_2 \) as well as \( m_0 \) instead of \( x_0 \), we therefore obtain from Theorem 1 for every \( \epsilon > 0 \) the two-sided bound

\[
\mu_{s,0} e^{\gamma_{s}^\epsilon t} \leq \| m_s(t) \|_2 \leq \mu_{s,1}(\epsilon) e^{\gamma_{s}^{\epsilon} t + K}, \quad t \geq 0,
\]

for constants \( \mu_{s,0} \geq 0 \) and \( \mu_{s,1}(\epsilon) > 0 \). Sufficient conditions for \( \mu_{s,0} > 0 \) are obtained by Theorems 2 and 3 when replacing there \( x_0 \) by \( m_0 \).

7. Two-sided bounds on \( P_s(t) - P_s = \Phi_s(t)(P_0 - P)\Phi_s'(t) \)

Based on Theorems 4 and 5, we obtain

**Corollary 1** (Two-sided bounds on \( P_s(t) - P_s \)) Let \( A \in \mathbb{R}^{n \times n} \), let \( \Phi(t) = e^{At} \) be the associated fundamental matrix with \( \Phi(0) = E \), where \( E \) is the identity matrix, as well as \( S \in \mathbb{R}^{m \times n} \) and \( \Phi_s(t) = S\Phi(t) \). Further, let \( P_0, P \in \mathbb{R}^{m \times n} \) be the covariance matrices from Section 2.

Then, there exists a constant \( p_{s,0} \geq 0 \) and for every \( \epsilon > 0 \) a constant \( p_{s,1}(\epsilon) > 0 \) such that

\[
p_{s,0} e^{2\epsilon t} \leq \| P_s(t) - P_s \|_2 \leq p_{s,1}(\epsilon) e^{\epsilon t}, \quad t \geq 0.
\]

If \( P_0 - P \) and \( S \) are regular, then \( p_{s,0} > 0 \).

**Remark** If \( S \in \mathbb{R}^{m \times n} \) is not square regular, under additional conditions stated in Theorems 6 and 7, it can also be asserted that \( p_{s,0} > 0 \).

8. Local regularity of the function \( \| P_s(t) - P_s \|_2 \)

We have the following lemma which states – loosely speaking – that for every \( t_0 \geq 0 \), the function \( t \mapsto \| \Delta P_s(t) \|_2^2 = \| P_s(t) - P_s \|_2^2 = \| \Phi_s(t)(P_0 - P)\Phi_s'(t) \|_2^2 \) is real analytic in some right neighborhood \([t_0, t_0 + \Delta t_0] \).

**Lemma 1** (Real analyticity of \( t \mapsto \| P_s(t) - P_s \|_2^2 \) on \([t_0, t_0 + \Delta t_0] \)) Let \( t_0 \in \mathbb{R}^+ \). Then, there exists a number \( \Delta t_0 > 0 \) and a function \( t \mapsto \Delta P_s(t) \), which is real analytic on \([t_0, t_0 + \Delta t_0] \) such that

\[
\Delta P_s(t) = \| \Delta P_s(t) \|_2 = \| P_s(t) - P_s \|_2 = \| \Phi_s(t)(P_0 - P)\Phi_s'(t) \|_2, \quad t \in [t_0, t_0 + \Delta t_0].
\]

**Proof** Based on \( \| \Delta P_s(t) \|_2 \) = \( \max(\| \lambda_{\max}(\Delta P_s(t)) \|, \| \lambda_{\min}(\Delta P_s(t)) \|) \), the proof is similar to that of Kohaupt (2002, Lemma 1). The details are left to the reader.

9. Formulas for the norm derivatives \( D_k^k \| P_s(t) - P_s \|_2 \), \( k = 0, 1, 2 \)

Let \( A \in \mathcal{A} \), and \( C \in \mathcal{A} \) with \( C^* = C \). As in Kohaupt (2015b, Section 7), we set

\[ \Psi(t) = \Phi(t)C\Phi^*(t), \quad t \geq 0. \]

Let \( S \in \mathcal{A} \) and define \( \Psi_s(t) = S\Psi(t)S^*, \quad t \geq 0. \)

Then, \( \Psi_s(t) = S\Phi(t)C\Phi^*(t)S^*, \quad t \geq 0. \)
Similarly as in Kohaupt (2015b, Section 7), for \( t_0 \in \mathbb{R}_0^+ \),

\[
\Psi_s(t) = S \Phi(t) C \Phi^*(t) S^* = \sum_{j=0}^{\infty} S \Phi(t_0) B_j \Phi^*(t_0) S^* \frac{(t-t_0)^j}{j!},
\]

with

\[
B_j = \sum_{k=0}^{j} \binom{j}{k} A^{j-k} C A^k,
\]

\( j = 0, 1, 2, \ldots \), and thus

\[
\Psi(t) = T_s^{(0)} + T_s^{(1)}(t-t_0) + T_s^{(2)}(t-t_0)^2 + \ldots
\]

with

\[
T_s^{(k)} = S T_s^{(k)} S^*, \quad k = 0, 1, 2, \ldots
\]

where the quantities \( T_s^{(k)} \), \( k = 0, 1, 2, \ldots \) are defined in Kohaupt (2015b, Section 7).

Consequently, one obtains the formulas for

\[
D_k^k \| P_s x_s (t) - P_s \|_2, \quad k = 0, 1, 2, \ldots
\]

from those for \( D_k^k \| P_s x_s (t) - P_s \|_2, \quad k = 0, 1, 2, \ldots \) when replacing \( T_s^{(k)} \) by \( T_s^{(k)} \), \( k = 0, 1, 2, \ldots \).

10. Applications
In this section, we apply the new two-sided bounds on \( \| P_s x_s (t) - P_s \|_2 \) obtained in Section 7 as well as the differential calculus of norms developed in Sections 8 and 9 to a linear stochastic vibration model with output equation for asymptotically stable system matrix and white noise excitation vector.

In Subsection 10.1, the stochastic vibration model as well as its state-space form is given, in Subsection 10.2 the transformation matrix \( S \) in chosen and in Section 10.3 the data are specified. In Section 10.4, the positiveness of the constants \( X_{n,0} \) and \( \varphi_{s,0} \) in the lower bounds is verified. In Section 10.5, computations with the chosen data are carried out, such as the computation of \( P \) and \( P_0 - P \) as well as the computation of the curves \( y = D_k^k \| P_s x_s (t) - P_s \|_2, \quad k = 0, 1, 2, \ldots \) and of the curve \( y = \| P_s x_s (t) - P_s \|_2 \) along with its best upper and lower bounds for the two ranges \( t \in [0;5] \) and \( t \in [5;26] \). In Section 10.6, computational aspects are shortly discussed.

10.1. The stochastic vibration model and its state-space form
Consider the multi-mass vibration model in Figure 1.

The associated initial-value problem is given by

\[
M \ddot{y} + B \dot{y} + Ky = f(t), \quad y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0
\]

where \( y = [y_1, \ldots, y_n]^T \) and \( f(t) = [f_1(t), \ldots, f_n(t)]^T \) as well as
Here, $y$ is the displacement vector, $f(t)$ the applied force, and $M$, $B$, and $K$ are the mass, damping, and stiffness matrices, as the case may be.

In the state-space description, one obtains

$$\dot{x}(t) = Ax(t) + b(t), \quad x(0) = x_0,$$

with $x = [y^T, z^T]^T$, $z = \dot{y}$, and $x_0 = [y_0^T, z_0^T]^T$, $z_0 = \dot{y}_0$ where the initial vector $x_0 = [y_0^T, z_0^T]^T$ is characterized by the mean vector $m_0$ and the covariance matrix $P_0$.

The system matrix $A$ and the excitation vector $b(t)$ are given by

$$A = \begin{bmatrix} 0 & E \\ -M^{-1}K & -M^{-1}B \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ -M^{-1}f(t) \end{bmatrix},$$

respectively. The vector $x(t)$ is called state vector.

The (symmetric positive semi-definite) intensity matrix $Q = Q_b$ is obtained from the (symmetric positive semi-definite) intensity matrix $Q_f$ by

$$Q = Q_b = \begin{bmatrix} 0 & 0 \\ 0 & M^{-1}Q_fM^{-1} \end{bmatrix},$$

See Müller and Schiehlen (1985, (9.65)) and the derivation of this relation in Kohaupt (2015b, Appendix A.5).

**10.2. The transformation matrix $S$ and the output equation $x_s(t) = Sx(t)$**

We depart from the equation of motion in vector form, namely $M \ddot{y} + B \dot{y} + Ky = f(t)$, and rewrite it as

$$\ddot{y}_s(t) = \ddot{y} - M^{-1}f(t) = -M^{-1}K\dot{y}(t) - M^{-1}B\dot{y}(t).$$
Following Müller and Schiehlen (1985, (9.56), (9.57)), for a one-mass model with base excitation, we call $\ddot{y}_a$ the **absolute acceleration** of our vibration system; it can be written as

$$\ddot{y}_a(t) = [-M^{-1}K, -M^{-1}B] \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = S\dot{x}(t) = :x_3(t), \quad t \geq 0,$$

with the **transformation matrix**

$$S = [-M^{-1}K, -M^{-1}B].$$

Our output equation therefore is

$$x_3(t) = S\dot{x}(t),$$

where here $S \in \mathbb{R}^{m \times 2n}$ is a rectangular, but not a square regular matrix.

### 10.3. Data for the model

As of now, we specify the values as

$$m_j = 1, \quad j = 1, \ldots, n$$

$$k_j = 1, \quad j = 1, \ldots, n + 1$$

and

$$b_j = \begin{cases} 1/2, & j \text{ even} \\ 1/4, & j \text{ odd}. \end{cases}$$

Then,

$$M = E$$

$$B = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & \frac{3}{2} \\ \vdots & \vdots \\ -1/2 & \frac{3}{2} \end{bmatrix}$$

(if $n$ is even), and

$$K = \begin{bmatrix} 2 & -1 & \vdots & \vdots & \vdots \\ -1 & 2 & -1 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{bmatrix}$$

We choose $n = 5$ in this paper so that the state-space vector $x(t)$ has the dimension $m = 2n = 10$.

For $m_0^T$, we take

$$m_0 = [m_{y_0}^T, m_{z_0}^T]^T$$

with

$$m_{y_0} = [-1, 1, -1, 1, -1]^T$$
and
\[
m_{z_i} = \begin{cases} 
[0, 0, 0, 0, 0]^T & \text{(Case I)} \\
[-1, -1, -1, -1, -1]^T & \text{(Case II)}
\end{cases}
\]

similarly as in Kohaupt (2002) for \(y_0\) and \(\dot{y}_0\). For the \(10 \times 10\) matrix \(P_0\), we choose
\[
P_0 = 0.01 E.
\]

The white-noise force vector \(f(t)\) is specified as
\[
f(t) = [0, \ldots, 0 f_n(t)]^T
\]
so that its intensity matrix \(Q_f \in IR_n \times n\) with \(q_{f,nn} = q\) has the form
\[
Q_f = \begin{bmatrix}
0 & 0 \\
0 & q_{f,nn}
\end{bmatrix} = q \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} = q E^{(n)}.
\]

We choose
\[
q = 0.01.
\]

With \(M = E\), this leads to (see Kohaupt, 2015b, Appendix A.5)
\[
Q = Q_b = \begin{bmatrix}
0 & 0 \\
0 & q_{f,nn}
\end{bmatrix} = q \begin{bmatrix}
0 & 0 \\
0 & E^{(n)}
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & q
\end{bmatrix} \in IR_{m \times m}.
\]

10.4. Positiveness of the constants \(\chi_{5,0}\) and \(\varphi_{5,0}\) (resp. \(p_{5,0}\))

The eigenvalues of matrix \(A\) are given by

\[
\lambda_i = \lambda_i(A) \quad \text{and} \quad \bar{\lambda}_i = \lambda_i(A^*)
\]

for

\[
\begin{array}{c|c}
\lambda_i & \bar{\lambda}_i \\
\hline
1 & -0.69976063878054 + 1.79598147815975i \\
2 & -0.56266837404074 + 1.61635870164386i \\
3 & -0.37500000000000 + 1.36358901432946i \\
4 & -0.18733162594526 + 0.99452168646559i \\
5 & -0.05023936121946 + 0.51637145071101i \\
\end{array}
\]

and

\[
\lambda_{5+i} = \bar{\lambda}_i, \quad i = 1, \ldots, 5
\]

where the numbering is chosen such that \(\text{Im} \lambda_i > 0, \quad i = 1, \ldots, 5\). So,

\[
\lambda_i \neq \lambda_j, \quad i \neq j, \quad i, j = 1, \ldots, 10.
\]

Thus, matrix \(A\) is diagonalizable. Further, conditions (H1)-(H4), and (HS) are fulfilled. Moreover, we have \(J_{\varphi} = \{5\}\) and

\[
p_5 = \begin{bmatrix}
q_5 \\
r_5
\end{bmatrix} = \begin{bmatrix}
0.172578 + 0.192313i \\
0.316694 + 0.310502i \\
0.365988 + 0.358733i \\
0.316694 + 0.310502i \\
0.193410 + 0.166421i \\
-0.107975 + 0.079453i \\
-0.176245 + 0.147932i \\
-0.203627 + 0.170963i \\
-0.176245 + 0.147932i \\
-0.095652 + 0.091510i
\end{bmatrix}.
\]
Therefore, \(q_5, \bar{q}_5\) are linearly independent. Thus, by Lemma A.1 and Theorem 2 resp. Theorem 6, the constants \(X_{S,0}\) and \(\varphi_{S,0}\) are positive. Therefore, also the constant \(p_{S,0}\) is positive.

10.5. Computations with the specified data

(i) Bounds on \(y = \Phi_S(t) m_0\) in the vector norm \(\| \cdot \|_2\) Upper bounds on \(y = \Phi(t) m_0\) in the vector norm \(\| \cdot \|_2\) for the two cases (I) and (II) of \(m_0\) are given in Kohaupt (2002, Figures 2 and 3). There, we had a deterministic problem with \(f(t) = 0\) and the solution vector \(x(t) = \Phi(t) x_0\), where \(x_0\) there had the same data as \(m_0\) here. We mention that for the specified data, \(\nu_{m_0}[A] = \nu[A] = \alpha\) in both cases (Kohaupt, 2006, p. 154) for a method to prove this. For the sake of brevity, we do not compute or plot the lower or upper bounds and thus the two-sided bounds on \(y = \Phi_S(t) m_0\) but leave this to the reader.

(ii) Computation of \(P\) and \(P_0 - P\) The computation of these matrices was already done in Kohaupt (2015b, Subsection 3). There, we saw that \(P\) is symmetric and \(P_0 - P\) symmetric and regular (but not positive definite). Matrix \(P_0 - P\) is needed to compute the curve \(y = \|P_{x_S}(t) - P_S\|_2 = \|\Phi_S(t)(P_0 - P)\Phi_S(t)^T\|_2\).

(iii) Computation of the curves \(y = D^k_+ \|P_{x_S}(t) - P_S\|_2 = D^k_+ \Phi_S(t)(P_0 - P)\Phi_S(t)^T\|_2\) \(k = 0, 1, 2\) The computation of \(y = D^k_+ \|P_{x_S}(t) - P_S\|_2\) \(k = 0, 1, 2\) for the given data is done according to Section 9 with \(C = P_0 - P\). The pertinent curves are illustrated in Figures 2–4.

We have checked the results numerically by difference quotients. More precisely, setting

\[
\Delta P_{x_S}(t) = P_{x_S}(t) - P_S, \quad t > 0,
\]

and

\[
g(t) = \|\Delta P_{x_S}(t)\|_2 = \|P_{x_S}(t) - P_S\|_2, \quad t > 0,
\]

we have investigated the approximations

![Figure 2. Curve](image-url)
\[ \delta_t g(t) = \frac{g(t+h) - g(t-h)}{2h} \approx D_+ g(t), \ t - h \geq 0, \]

and

\[ \delta_t^2 g(t) = \delta_t (\delta_t g(t)) = \frac{g(t+h) - 2g(t) + g(t-h)}{h^2} \approx D_+^2 g(t), \ t - h \geq 0, \]

as well as

\[ \delta_t Dg(t) = \frac{Dg(t+h) - Dg(t-h)}{2h} \approx D_+^2 g(t), \ t - h \geq 0. \]

For, e.g.

\[ t = 2.5, \]
\[ h = 10^{-5}, \]

we obtain

\[ D_+ g(t) = D_+ \|P_{x_1}(t) - P_{x_2}\|_2 = -0.00789413206599, \]
\[ \delta_t g(t) = \delta_t \|P_{x_1}(t) - P_{x_2}\|_2 = -0.00789413206470, \]

as well as

\[ D_+^2 g(t) = D_+^2 \|P_{x_1}(t) - P_{x_2}\|_2 = 0.00180394234645, \]
\[ \delta_t^2 g(t) = \delta_t^2 \|P_{x_1}(t) - P_{x_2}\|_2 = 0.00180394234645, \]

and

\[ D_+^2 g(t) = D_+^2 \|P_{x_1}(t) - P_{x_2}\|_2 = 0.00180394234645, \]
\[ \delta_t Dg(t) = \delta_t D \|P_{x_1}(t) - P_{x_2}\|_2 = 0.00180394235409, \]

so that the computational results for \( y = D_+^k \|P_{x_1}(t) - P_{x_2}\|, k = 0, 1, 2 \) with \( t = 2.5 \) are well underpinned by the difference quotients. As we see, the approximation of \( D_+^2 g(t) = D_+^2 \|P_{x_1}(t) - P_{x_2}\|_2 \) by \( \delta_t Dg(t) \) is much better than by \( \delta_t^2 g(t) \), which was to be expected, of course.

(iv) Bounds on \( y = P_{x_1}(t) - P_{x_2} = \Phi_{x_1}(t) (P_0 - P) \Phi_{x_2}^T(t) \) in the spectral norm \( \cdot \| \_2 \) Let \( \alpha = \nu(A) \) be the spectral abscissa of the system matrix \( A \). With the given data, we obtain

\[ \alpha = \nu(A) = -0.05023936121946 < 0 \]

so that the system matrix \( A \) is asymptotically stable.

The upper bound on \( y = \|P_{x_1}(t) - P_{x_2}\|_2 = \|\Phi_{x_1}(t) (P_0 - P) \Phi_{x_2}^T(t)\|_2 \) is given by \( y = p_{s,1}(\epsilon) e^{2\alpha_1(\epsilon) t} \), \( t \geq 0 \). Here, \( \epsilon = 0 \) can be chosen since matrix \( A \) is diagonalizable. But, in the programs, we have chosen the machine precision \( \epsilon = \text{eps} = 2^{-52} \approx 2.2204 \times 10^{-16} \) of MATLAB in order not to be bothered by this question. With \( \varphi_{x_1}(t) = p_{s,1}(\epsilon) e^{2\alpha_1(\epsilon) t} \), \( t \geq 0 \), the optimal constant \( p_{s,1}(\epsilon) \) in the upper bound is obtained by the two conditions

\[ \|P_{x_1}(t_c) - P_{x_2}\|_2 = \varphi_{x_1}(t_c) = p_{s,1}(\epsilon) e^{2\alpha_1(\epsilon) t_c}, \]
\[ D_+ \|P_{x_1}(t_c) - P_{x_2}\|_2 = \varphi'_{x_1}(t_c) = 2(\alpha + \epsilon) \varphi_{x_1}(t_c), \]

where \( t_c \) is the place of contact between the curves.

This is a system of two nonlinear equations in the two unknowns \( t_c \) and \( p_{s,1}(\epsilon) \). By eliminating \( \varphi_{x_1}(t_c) \), this system is reduced to the determination of the zero of

\[ D_+ \|P_{x_1}(t_c) - P_{x_2}\|_2 - 2(\alpha + \epsilon) \|P_{x_1}(t_c) - P_{x_2}\|_2 = 0, \]
which is a single nonlinear equation in the single unknown $t_c$. For this, MATLAB routine \texttt{fsolve} was used.

After $t_c$ has been computed from the above equation, the best constant $p_{S,1}(\epsilon)$ is obtained from

$$p_{S,1}(\epsilon) = \| P_{S}(t_c) - P_S \|_2 e^{-2(\epsilon + \delta/\gamma)}.$$

**Numerical values for range [0;5]** First, we consider the range [0;5]. From the initial guess $t_{c,0} = 1.4$, the computations deliver the values
To compute the lower bound, we have to notice that the curve $y = \|P_x(t) - P_s\|_2$ has kinks like at $t = 0$. This is not seen in Figures 2–5 in the range $[0;5]$, but in Figure 6 in the range $[5;25]$. Therefore, the point of contact $t_s$ between the lower bound $y = M_{0,0} e^{\alpha t}$ and the curve $y = \|P_x(t) - P_s\|_2$ cannot be determined by the calculus of norms, but must be computed from

$$t_s = \min_{j=1,2,\ldots} \|P_x(t_j) - P_s\|_2,$$

where $t_j, j = 1, 2, \ldots$ are the local minima of $y = \|P_x(t) - P_s\|_2$. In this way, with the initial guess $t_{s,0} = 3.0$, the results are

$t_c = 1.355984,$
$p_{s,1}(\epsilon) = 0.024642.$

To compute the lower bound, we have to notice that the curve $y = \|P_x(t) - P_s\|_2$ has kinks like at $t = 0$. This is not seen in Figures 2–5 in the range $[0;5]$, but in Figure 6 in the range $[5;25]$. Therefore, the point of contact $t_s$ between the lower bound $y = M_{0,0} e^{\alpha t}$ and the curve $y = \|P_x(t) - P_s\|_2$ cannot be determined by the calculus of norms, but must be computed from

$$t_s = \min_{j=1,2,\ldots} \|P_x(t_j) - P_s\|_2,$$

where $t_j, j = 1, 2, \ldots$ are the local minima of $y = \|P_x(t) - P_s\|_2$. In this way, with the initial guess $t_{s,0} = 3.0$, the results are

$t_s = 17.152006,$
$p_{s,0} = 9.403735 \times 10^{-5}.$

In Figure 5, the curve $y = \|P_x(t) - P_s\|_2$ along with the best upper and lower bounds are illustrated with stepsize $\Delta t = 0.01$. The upper bound is valid for $t \geq t_1 = 1.172$.

**Numerical values for range [5;25]** On the range $[5;25]$, the two-sided bounds can be better adapted to the curve $y = \|P_x(t) - P_s\|_2$. From the initial guess $t_{c,0} = 14$, the computations deliver

$t_c = 14.876956,$
$p_{s,1}(\epsilon) = 0.00164024.$

Further, with the initial guess $t_{s,0} = 25$, we obtain

$t_s = 24.860534,$
$p_{s,0} = 3.548384.$
In Figure 6, the curve \( y = \| P_x(t) - P_s \|_2 \) along with the best upper and lower bounds are illustrated with stepsize \( \Delta t = 0.01 \). The upper bound is valid for \( t \geq t_1 \approx 6.031 \).

10.6. Computational aspects
In this subsection, we say something about the computer equipment and the computation time for some operations.

(i) As to the computer equipment, the following hardware was available: an Intel Pentium D (3.20 GHz, 800 MHz Front-Side-Bus, 2x2MB DDR2-SDRAM with 533 MHz high-speed memories). As software package, we used MATLAB, Version 6.5.

(ii) The computation time \( t \) of an operation was determined by the command sequence 
\[
t_0 = \text{clock}; \quad \text{operation}; \quad t = \text{etime(clock)}, 
\]
and it is put out in seconds rounded to two decimal places by Matlab. For the computation of the eigenvalues of matrix \( A \), we used the command 
\[
[XA, DA] = \text{eig(A);}
\]
the pertinent computation time is less than 0.01 s. To determine \( \Phi(t) = e^{At} \), we employed Matlab routine \text{expm}. For the computation of the 501 values \( t, y, y_u, y_l \) in Figure 6, it took \( t(\text{Table for Figure 5}) = 1.17 \) s. Here, \( t \) stands for the time value running from \( t_0 = 0 \) to \( t_e = 25 \) with stepsize \( \Delta t = 0.1 \); \( y \) stands for the value of \( \| P_x(t) - P_s \|_2 \), \( y_u \) for the value of the best upper bound \( p_{S,1}(\varepsilon) e^{2(\alpha + \varepsilon) t} \) and \( y_l \) for the value of the lower bound \( p_{S,0} e^{2\alpha t} \). For the computation of the 2501 values \( t, y, y_u, y_l \) in Figure 6, it took \( t(\text{Table for Figure 6}) = 6.35 \) s.

11. Conclusion
In the present paper, a linear stochastic vibration system of the form \( \dot{x}(t) = Ax(t) + b(t), \ x(0) = x_0, \) with output equation \( x_S(t) = S x(t) \) was investigated, where \( A \) is the system matrix and \( b(t) \) white noise excitation. The output equation \( x_S(t) = S x(t) \) is viewed as a transformation of the state vector \( x(t) \) mapped by the rectangular matrix \( S \) into the output vector \( x_S(t) \). If the system matrix \( A \) is asymptotically stable, then the mean vector \( \bar{m}_x(t) \) and the covariance matrix \( P_x(t) \) both converge with \( m_x(t) \rightarrow 0 \) \( (t \rightarrow \infty) \) and \( P_x(t) \rightarrow P_s \) \( (t \rightarrow \infty) \) for some symmetric positive (semi-)definite matrix \( P_s \). This raises the question of the asymptotic behavior of both quantities. The pertinent investigations are made in the Euclidean norm \( \| \cdot \|_2 \) for \( m_x(t) \) and in the spectral norm, also denoted by \( \| \cdot \|_2 \) for \( P_x(t) - P_s \). The main new points are the derivation of two-sided bounds on both quantities,
the determination of the right norm derivatives $D^k \| P_x(t) - P_S \|_2$, $k = 0, 1, 2$ and, as application, the computation of the best constants in the bounds. In the presentation, the author exhibits the relations between the quantities $m_k(t), P_x(t) - P$, and the formulas for $D^k \| P_x(t) - P \|_2$ on the one hand, and the corresponding output-related quantities $m_k(t), P_x(t) - P_S$, and $D^k \| P_x(t) - P_S \|_2$ on the other hand. As a result, we obtain that there is a close relationship between these quantities. Special attention is paid to the positivity of the constants in the lower bounds if the transformation matrix is only rectangular and not necessarily square and regular. In the Appendix, a sufficient algebraic condition for the positiveness of the constants in the lower bounds is derived that is independent of the initial vector and the time variable. To make sure that the (new) formulas for $D^k \| P_x(t) - P_S \|_2$ are correct, we have checked them by various difference quotients. They underpin the correctness of the numerical values for the specified data.

The computation time to generate the last figure with a $10 \times 10$ matrix $A$ is about 6 seconds. Of course, in engineering practice, much larger models occur. As in earlier papers, we mention that in this case engineers usually employ a method called condensation to reduce the size of the matrices.

We have shifted the details of the positiveness of the constants in the lower bounds to the Appendix in order to make the paper easier to comprehend.

The numerical values were given in order that the reader can check the results.

Altogether, the results of the paper should be of interest to applied mathematicians and particularly to engineers.

Acknowledgements
The author would like to give thanks to the anonymous referees for evaluating the paper and for comments that led to a better presentation of the paper.

Funding
The author received no direct funding for this research.

Author details
L. Kohaupt
E-mail: kohaupt@beuth-hochschule.de
1 Department of Mathematics, Beuth University of Technology Berlin, Luxemburger Str. 10, D-13353 Berlin, Germany.

Citation information
Cite this article as: Two-sided bounds on some output-related quantities in linear stochastically excited vibration systems with application of the differential calculus of norms, L. Kohaupt, Cogent Mathematics (2016), 3: 1147932.

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Appendix 1

Algebraic conditions ensuring the positiveness of $X_{S,0}$ resp. $\varphi_{S,0}$ for $S = [-M^{-1}K, -M^{-1}B]$.

We discuss two cases, namely the case of a diagonalizable matrix $A$ and the case of a general square matrix $A$.

The corresponding Lemmas A.1 and A.2 will deliver sufficient algebraic criteria for the positiveness of $X_{S,0}$ and $\varphi_{S,0}$, as the case may be, that are independent of the initial condition and the time, which is the important point.

The results are of interest on their own.

Case 1: Diagonalizable matrix $A$

Let the hypotheses (H1), (H2), and (HS) be fulfilled. Further, according to (37), we assume

$$\sum_{k \in J_{S,0}} S_{f_k}(t) \neq 0, \quad t \geq t_0, \tag{72}$$

with

$$f_k(t) = \begin{bmatrix} g_k(t) \\ h_k(t) \end{bmatrix}$$

so that

$$S_{f_k}(t) = -M^{-1}g_k(t) - M^{-1}Bh_k(t) = -M^{-1}[g_k(t) + Bh_k(t)], \quad k \in J_{S,0}.$$

Thus, (37) resp. (72) is equivalent to

$$\sum_{k \in J_{S,0}} [g_k(t) + Bh_k(t)] \neq 0, \quad t \geq t_0. \tag{73}$$

We have the following

**Sufficient condition for** $\sum_{k \in J_{S,0}} [g_k(t) + Bh_k(t)] \neq 0, \quad t \geq t_0$:

$$K_{q_k}^{(i)} + Br_k^{(i)}, \quad K_{q_k}^{(ii)} + Br_k^{(ii)}, \quad k \in J_{S,0}, \text{ linearly independent} \tag{74}$$

(see Kohaupt, 2011, Section 5.1 for a similar case).

**Lemma A.1 (Some equivalences of the sufficient algebraic condition)**

Let the conditions (H1), (H2), (H3), and (HS) be fulfilled. Further, let $M, B, K \in IR^{n \times n}$ and $M$ be regular.

Then, the following equivalences are valid:

$$K_{q_k}^{(i)} + Br_k^{(i)}, \quad K_{q_k}^{(ii)} + Br_k^{(ii)}, \quad k \in J_{S,0}, \text{ linearly independent} \tag{75}$$

$\iff$
Proof. The equivalences (75) $\iff$ (76) and (77) $\iff$ (78) are clear. Further, since (76) is equivalent to

$$r_k = \lambda_k q_k,$$

(76) is equivalent to

$$Kq_k + \lambda_k Bq_k, \quad Kq_k + \bar{\lambda}_k \bar{B}q_k, \quad k \in J_{\nu_0}, \text{ linearly independent}.$$

Since

$$\lambda^2_k Mq_k + \lambda_k Bq_k + Kq_k = 0, \quad \bar{\lambda}^2_k \bar{M}q_k + \bar{\lambda}_k \bar{B}q_k + \bar{K}q_k = 0, \quad k \in J_{\nu_0},$$

(80) is equivalent to

$$-\lambda^2_k Mq_k, -\bar{\lambda}^2_k \bar{M}q_k, \quad k \in J_{\nu_0}, \text{ linearly independent}.$$

Since $\lambda_k \neq 0$ due to (H3) and $M$ is regular by assumption, (81) is equivalent to (77).

Case 2: General square matrix $A$ Let the hypotheses (H1'), (H2'), (H3'), and (HS') be fulfilled. Further, according to (55), we assume

$$\sum_{l \in J_{\nu_0}} \sum_{k=1}^m f_l^{(i)}(t) \neq 0, \quad t \geq t_0,$$  \hspace{1cm} (82)

with

$$f_l^{(i)}(t) = \begin{bmatrix} g_l^{(i)}(t) \\ h_l^{(i)}(t) \end{bmatrix}$$

so that

$$S_{l}^{(i)}(t) = -M^{-1}K_{l}^{(i)}(t) - M^{-1}B_{l}^{(i)}(t) = -M^{-1}[K_{l}^{(i)}(t) + B_{l}^{(i)}(t)], \quad k = 1, \ldots, m,$$

$l \in J_{\nu_0}$. Thus, (55) resp. (82) is equivalent to

$$\sum_{l \in J_{\nu_0}} \sum_{k=1}^m [K_{l}^{(i)}(t) + B_{l}^{(i)}(t)] \neq 0, \quad t \geq t_0.$$

We have the following

Sufficient condition for $\sum_{l \in J_{\nu_0}} \sum_{k=1}^m [K_{l}^{(i)}(t) + B_{l}^{(i)}(t)] \neq 0, \quad t \geq t_0$:

$$K_{l}^{(i)} + Br_l^{(i)}, \quad K_{l}^{(i)} + Br_l^{(i)}, \quad k = 1, \ldots, m, \quad l \in J_{\nu_0}, \text{ linearly independent}.$$  \hspace{1cm} (84)
Lemma A.2 (Some equivalences of the sufficient algebraic condition) Let the hypotheses \( (H1'), \ (H2'), \ (H3'), \) and \( (HS') \) be fulfilled. Further, let \( M, \ B, \ K \in IR^{m \times n} \), and \( M \) be regular.

Then, the following equivalences are valid:

\[
\begin{align*}
Kq_k^{(l)} + Br_k^{(l)}, \ Kq_k^{(l)} + Br_k^{(l)}, \ k = 1, \ldots, m, \ l \in J_{\nu}, \ \text{linearly independent} & \quad (85) \\
\iff & \quad (86) \\
Kq_k^{(l)}, \ K\tilde{q}_k^{(l)} + Br_k^{(l)}, \ k = 1, \ldots, m, \ l \in J_{\nu}, \ \text{linearly independent} \\
\iff & \quad (87) \\
q_k^{(l)}, \ q_k^{(l)}, \ k = 1, \ldots, m, \ l \in J_{\nu}, \ \text{linearly independent} & \quad (88)
\end{align*}
\]

Proof The equivalences \((85) \iff (86)\) and \((87) \iff (88)\) are clear.

Now we prove the equivalence of \((86) \iff (87)\). From the relations

\[
(A - \lambda E)p_k^{(l)} = p_{k-1}^{(l)}
\]

with

\[
p_k^{(l)} = \begin{bmatrix} q_k^{(l)} \\ r_k^{(l)} \end{bmatrix},
\]

\[
k = 1, \ldots, m, \ l \in J_{\nu}, \ \text{in a first step, we derive associated relations for } q_k^{(l)} \text{ and } r_k^{(l)}, \ k = 1, \ldots, m, \ l \in J_{\nu}.
\]

Now, \((89)\) means

\[
\begin{bmatrix} 0 & E \\ -M^{-1}K & -M^{-1}B \end{bmatrix} \begin{bmatrix} q_k^{(l)} \\ r_k^{(l)} \end{bmatrix} = \lambda I \begin{bmatrix} q_k^{(l)} \\ r_k^{(l)} \end{bmatrix} + \begin{bmatrix} q_{k-1}^{(l)} \\ r_{k-1}^{(l)} \end{bmatrix},
\]

that is,

\[
q_k^{(l)} = \lambda q_k^{(l)} + q_{k-1}^{(l)} \\
- M^{-1}Kq_k^{(l)} - M^{-1}Br_k^{(l)} = \lambda r_k^{(l)} + r_{k-1}^{(l)}
\]

or

\[
Kq_k^{(l)} + Br_k^{(l)} = -M[I\lambda r_k^{(l)} + r_{k-1}^{(l)}]
\]

\[
k = 1, \ldots, m, \ l \in J_{\nu}.
\]

Based on \((93)\) and the assumed regularity of \( M \), we see that \((86)\) is equivalent to

\[
\lambda r_k^{(l)} + r_{k-1}^{(l)} = \lambda r_k^{(l)} + r_{k-1}^{(l)}, \ k = 1, \ldots, m, \ l \in J_{\nu}, \ \text{linearly independent}.
\]

In the second step, we show the equivalence of \((94)\) with the following condition:

\[
r_k^{(l)}, \ r_k^{(l)}, \ k = 1, \ldots, m, \ l \in J_{\nu}, \ \text{linearly independent},
\]
and then in the third step, the equivalence of (95) with (87).

**Equivalence of (94) and (95):**

(94) \(\Rightarrow\) (95): So, let (94) be fulfilled. We write \(r_k^{(l)}\) in the form

\[
r_k^{(l)} = \lambda_k r_k^{(l)} + r_{k-1}^{(l)}
\]

\(k = 1, \ldots, m\), \(l \in J_k\), where the \(r_k^{(l)}\) are also principal vectors of stage \(k\) corresponding to the eigenvalue \(\lambda_k\). This is done as follows:

\[
r_1^{(l)} = \lambda_1 r_1^{(l)} + r_0^{(l)}
\]

where \(r_1^{(l)} = \frac{1}{\lambda_1} r_1^{(l)}\) is a principal vector of 1 (or eigenvector) and \(r_0^{(l)} = 0\). Similarly,

\[
r_2^{(l)} = \lambda_2 r_2^{(l)} + r_1^{(l)}
\]

where \(r_2^{(l)} = \frac{1}{\lambda_2}(r_2^{(l)} - r_1^{(l)})\) is a principal vector of 2 corresponding to the eigenvalue \(\lambda_2\).

Proceeding in this way, using induction, (96) is proven.

Therefore, apart from (94), also the following property must hold:

\[
r_k^{(l)} = \lambda_k r_k^{(l)} + r_{k-1}^{(l)}, \quad r_k^{(l)} = \lambda_k r_k^{(l)} + r_{k-1}^{(l)} \quad k = 1, \ldots, m\), \(l \in J_k\), linearly independent,
\]

which proves (95).

(95) \(\Rightarrow\) (94): Let (95) be fulfilled and

\[
\sum_{i=0}^{m} \sum_{k=1}^{l} \left\{ a_k^{(l)}(\lambda_k r_k^{(l)} + r_{k-1}^{(l)}) + b_k^{(l)}(\lambda_k r_k^{(l)} + r_{k-1}^{(l)}) \right\} = 0.
\]

Fully written, we obtain with \(r_0^{(l)} = 0\),

\[
\sum_{i=0}^{m} \left\{ a_1^{(l)}(\lambda_1 r_1^{(l)} + r_0^{(l)}) + b_1^{(l)}(\lambda_1 r_1^{(l)} + r_0^{(l)})
\right.
\]

\[
+ a_2^{(l)}(\lambda_2 r_2^{(l)} + r_1^{(l)}) + b_2^{(l)}(\lambda_2 r_2^{(l)} + r_1^{(l)})
\]

\[
+ a_3^{(l)}(\lambda_3 r_3^{(l)} + r_2^{(l)}) + b_3^{(l)}(\lambda_3 r_3^{(l)} + r_2^{(l)}) + \ldots
\]

\[
+ a_m^{(l)}(\lambda_m r_m^{(l)} + r_{m-1}^{(l)}) + b_m^{(l)}(\lambda_m r_m^{(l)} + r_{m-1}^{(l)}) \right\} = 0
\]

or regrouping

\[
\sum_{i=0}^{m} \left\{ (a_1^{(l)} + a_2^{(l)}) r_1^{(l)} + (b_1^{(l)} + b_2^{(l)}) r_2^{(l)}
\right.
\]

\[
+ (a_3^{(l)} + a_4^{(l)}) r_3^{(l)} + (b_3^{(l)} + b_4^{(l)}) r_4^{(l)} + \ldots
\]

\[
+ (a_{m-1}^{(l)} + a_m^{(l)}) r_{m-1}^{(l)} + (b_{m-1}^{(l)} + b_m^{(l)}) r_m^{(l)}
\]

\[
+ (a_m^{(l)} + a_1^{(l)}) r_1^{(l)} + (b_m^{(l)} + b_1^{(l)}) r_1^{(l)} \right\} = 0
\]

By assumption, from the last line, we get

\[
a_m^{(l)} + a_1^{(l)} = 0, \quad b_m^{(l)} + b_1^{(l)} = 0
\]
or
\[ a_{m_i}^{(i)} = \beta_{m_i}^{(i)} = 0; \]
further,
\[ a_{m-1}^{(i)} \lambda_j + a_{m_i}^{(i)} = 0, \quad \beta_{m-1}^{(i)} \lambda_j + \beta_{m_i}^{(i)} = 0, \]
leading to
\[ a_{m-1}^{(i)} = \beta_{m-1}^{(i)} = 0. \]
Continuing in this way, we ultimately obtain
\[ a_{k_i}^{(i)} = \beta_{k_i}^{(i)} = 0, \quad k = 1, \ldots, m, \quad l \in J_u, \]
so that (94) is proven.

Further, because of the representation (92), then (95) is equivalent to
\[ \lambda q_{k_i}^{(i)} + \alpha q_{k_i}^{(i)} + \bar{\lambda} q_{k_i}^{(i)} + \bar{\alpha} q_{k_i}^{(i)}, \quad k = 1, \ldots, m, \quad l \in J_u, \text{ linearly independent}. \]  
(98)

Similarly as above, this is equivalent to (87).

On the whole, Lemma A.2 is proven. \(\square\)

**Alternative proof for the positiveness of \(X_{S,0}\) resp. \(\varphi_{S,0}\) when \(S = [-M^{-1}K, -M^{-1}B]\)**

In the special case of \(S = [-M^{-1}K, -M^{-1}B]\), there is an alternative proof for the positiveness of \(X_{S,0}\). This alternative proof is simpler, but the foregoing one is applicable to more general forms of \(S\).

We employ the vector resp. matrix norm \(\| \cdot \|_2\) to obtain
\[ \| X(t) \|_2^2 = \| Sx(t) \|_2^2 = \| -M^{-1}Ky(t) \|_2^2 + \| -M^{-1}B\dot{y}(t) \|_2^2 \]
\[ \geq \| K^{-1}M \|_2^{-1} \| y(t) \|_2^2 + \| B^{-1}M \|_2^{-1} \| \dot{y}(t) \|_2^2 \]
\[ \geq X_{S,0}^2 \| x(t) \|_2^2 \]
with
\[ X_{S,0}^2 = \min \{ \| K^{-1}M \|_2^{-2}, \| B^{-1}M \|_2^{-2} \} \]
so that
\[ \| X(t) \|_2 \geq X_{S,0} \| x(t) \|_2. \]

Due to the equivalence of norms in finite-dimensional spaces, this entails that for every vector norm \(\| \cdot \|\) one has
\[ \| X(t) \| \geq X_{S,0} \| x(t) \| \]
with a positive constant \(X_{S,0}\).

A similar proof for the positiveness of \(\varphi_{S,0}\) is possible. This is left to the reader.
