Closed timelike curves re-examined

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Abstract
Examples are given of the creation of closed timelike curves by choices of coordinate identifications. Following Gödel’s prescription, it is seen that flat spacetime can produce closed timelike curves with structure similar to that of Gödel. In this context, coordinate identifications rather than exotic gravitational effects of general relativity are shown to be the source of closed timelike curves. Removing the periodic time coordinate restriction, the modified Gödel family of curves is expressed in a form that retains the timelike and spacelike character of the coordinates. With these coordinates, the nature of the timelike curves is clarified. A helicoidal surface unifies the families of timelike, spacelike and null curves. In all of these, it is seen that as in ordinary flat spacetime, periodicity in the spatial position does not naturally carry over into closure in time. Thus, the original source of serious scientific speculation regarding time machines is seen to be misconceived.

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1 Introduction

While the notion of time travel has excited the imagination of the general public for decades if not centuries, the possibility that there was a potential for its realization as
a part of serious scientific investigation is usually attributed to the discovery of closed
timelike curves (CTC’s) in the Gödel universe of general relativity[1]. A CTC is defined
as a timelike future-directed curve (i.e. always evolving in time within the future light
cone), reuniting with a spacetime point of its earlier history and hence recycling end-
lessly. The general view has been and continues to the present that the scope of exotic
gravitational phenomena via general relativity is the source of Gödel’s closed timelike
curves. In this paper, we show that the essential source is actually a rather unnatu-
ral choice of identifying spacetime points and that the natural choice does not lead to
CTC’s.

The Gödel spacetime, describing a type of rotating universe with no expansion, is a
particular example of the generic class given by[2]

\[ ds^2 = -f^{-1} [e^{\nu}(dz^2 + dr^2) + r^2d\phi^2] + f(d\bar{t} - wd\phi)^2 \] (1)

where \( f, \nu \) and \( w \) are functions of \( r \) and \( z \) with the coordinates having the ranges

\[ -\infty < z < \infty, \quad 0 \leq r, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < \bar{t} < \infty \] (2)

and \( \phi = 0 \) and \( \phi = 2\pi \) being identified. The standard argument is the following: the
metric component

\[ g_{\phi\phi} = -f^{-1}(r^2 - f^2w^2) \] (3)

changes sign at the point where \( f^2w^2 = r^2 \) and hence \( \phi \) becomes a timelike coordinate
for

\[ f^2w^2 > r^2. \] (4)

In this case, the spacetime curve

\[ \bar{t} = \bar{t}_0, \quad r = r_0, \quad \phi = \phi, \quad z = z_0 \] (5)
with $z_0$, $r_0$, $\bar{t}_0$ being constants has been created as a CTC as a result of the now-timelike coordinate $\phi$ having $\phi = 0$ and $\phi = 2\pi$ still being identified as was the case when $\phi$ was spacelike.

There are essential problems with such an approach: while the interpretation of the nature of the curve for

$$f^2 w^2 < r^2$$

as being one of a closed spacelike curve on a constant time slice is unassailable, in case (4), the metric has two timelike coordinates $\bar{t}$ and $\phi$. One coordinate $\bar{t}$ is held fixed while the other coordinate $\phi$ is allowed to run. In spite of the fact that there is nothing inherently wrong in coordinatizing a spacetime with more than one timelike coordinate (Synge provides an example of doing so with four timelike coordinates), it is not conducive to clarity to have a timelike coordinate held fixed in the description of a timelike curve. In what follows, we explore the ramifications of imposing a periodicity upon the timelike coordinate $\phi$.

It has often been remarked that one can artificially create a CTC in $1+1$ Minkowski spacetime with the usual coordinate ranges and metric

$$ds^2 = dt^2 - dx^2$$

provided one imposes the condition that the points $(t, x)$ and $(t + t_0, x)$ ($t_0$ is a constant) are identified. With this periodic identification in the time coordinate, the essential nature of the spacetime is altered from its standard form. This kind of identification plays a key role in what follows.

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1Having 4 timelike coordinates refers to having the diagonal terms of $[g_{ij}]$ all being positive. If it is a physical spacetime, the signature, derived from its eigenvalues, will still have the signs $(+−−−)$. 
As a second example, we consider flat 3+1 spacetime in cylindrical polar coordinates

\[ ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2 \]  

with the standard coordinate ranges and where \( \phi = 0 \) and \( \phi = 2\pi \) are identified as usual, i.e.

\[ (t, r, 0, z) = (t, r, 2\pi, z) \]  

We retain the identification in \( \phi \) for 0 and 2\( \pi \) as we effect the transformation

\[ \bar{t} = t + a\phi, \quad \bar{\phi} = \phi, \quad \bar{r} = r, \quad \bar{z} = z \]  

where \( a \) is a constant. \(^2\) The metric becomes

\[ ds^2 = d\bar{t}^2 - dr^2 - 2ad\bar{t}d\phi - (r^2 - a^2)d\phi^2 - dz^2 \]  

which is precisely of the type (1) but with constant values globally for \( f, w \) and \( \nu \).

Here we shall follow the standard argument to conclude that our spacetime, with metric (11), contains closed timelike \( \phi \) curves \((\bar{t}_0, r_0, \phi, z_0)\) for \( r_0^2 < a^2 \). First, the indication of the timelike character is the positive sign of the \( g_{\phi\phi} \) component of (11). Second, the indication of closure follows from the imposed closure characteristic of the \( \phi \) coordinate,

\[ (\bar{t}, r, 0, z) = (\bar{t}, r, 2\pi, z). \]  

\(^2\) If we wish to preserve the structure of the spacetime, we must apply this transformation to \( \bar{\phi} \) and re-express it in the new coordinates. To do so, first define \( P_1 = (t_1, r_1, 0, z_1) \) and \( P_2 = (t_1, r_1, 2\pi, z_1) \). Then transform \( P_2 \) into \( \bar{P}_2 = (\bar{t}_2, r_2, \phi_2, z_2) \) and using \( t_1 = \bar{t}_1 - a\phi_1 \), write the transformed \( \bar{P}_2 \) in terms of \( \bar{t}_1 \). Finally, apply \( \bar{P}_1 = \bar{P}_2 \) leading to the result of \((\bar{t}, r, 0, z) = (\bar{t} + 2\pi a, r, 2\pi, z)\). It should be noted that, while this identification appears somewhat unusual in the new coordinates, it will not induce any singularity in the curvature similar to the vertex of a cone. Thus this particular system differs from a circumnavigated cosmic string.
Figure 1: The lightcone becomes narrow for $r \gg a$.

Note that this identification is not equivalent to (9). By the analysis of the lightcones, we develop the standard figure depicting the transition from closed spacelike to null to timelike curves.

In the examination of the lightcone structure, we will see in what follows that these $\phi$-curves are indeed spacelike for certain values of $r$ and timelike for others. In the original coordinates, the null vectors in the $\phi$ direction have “velocities”

$$\left(\frac{d\phi}{dt}\right)_{\text{null}} = \pm \frac{1}{r}$$  \hspace{1cm} (13)

whereas in the new coordinates, they are

$$\left(\frac{d\phi}{d\bar{t}}\right)_{\text{null}} = \frac{1}{a \pm r}.$$  \hspace{1cm} (14)

\textsuperscript{3}Here, the velocities are measured in radians per unit time.
Figure 2: The critical value of $r \to a$ results in the lightcone touching the $\phi$ axis.

Figure 3: The lightcone structure of $r = a/2$ where the $\phi$ direction is timelike.
As shown in figure 1, in the limit as \( r \to \infty \), the lightcone (14) becomes very narrow (similar to the case in the original coordinates (13)). At the critical radius \( r \to a^+ \), the lightcone dips and touches the \( \phi \) axis as seen in figure 2. Figure 3 illustrates the structure for \( r_0 < a \); the \( \phi \) curve is enclosed within the lightcone. Together, these can be combined to construct a diagram similar to that which displays the curves of the Gödel universe as shown in the standard texts (see for example [5]). Figure 4 would indicate that for \( r_0 < a \), there are closed timelike curves.

Consider the facts which imply that the \( \phi \)-curve is a closed timelike curve for a fixed \( r_0 < a \). The curve is always timelike, and hence the proper time flows monotonically and never becomes imaginary, i.e. the curve does not reverse and proceed into the past lightcone. If we transform the “cylindrical coordinates” \((\bar{t}, r, \phi, z)\) into a more familiar

\[ T_p(\mathcal{M}_4) \] 

The \( \mathcal{M}_4 \) chart does not have any “\( \phi \) axis.” The \( \phi \) axis refers to the \( \mathbf{e}_\phi \) direction in \( T_p(\mathcal{M}_4) \) for a particular point \( p \). At each point \( p \) in the 4-dimensional (flat) spacetime manifold \( \mathcal{M}_4 \), there is a tangent vector space \( T_p(\mathcal{M}_4) \), where all vectors \( \mathbf{u} = u^i \mathbf{e}_i \) reside. However, only the \( \mathbf{e}_\phi \) and \( \mathbf{e}_t \) directions are of interest because our curve will not have any \( \mathbf{e}_r \) or \( \mathbf{e}_z \) components. The lightcone is defined as a set of all vectors \( \mathbf{u} = u^i \mathbf{e}_i \in T_p(\mathcal{M}_4) \) such that \( g(\mathbf{u}, \mathbf{u}) = g_{ij} u^i u^j = 0 \). In the metric (8) and (11), it is obvious that a single-component vector \( \mathbf{u} = u^\phi \mathbf{e}_\phi \) cannot be a null vector. Thus, when we refer to null vectors in the “\( \phi \) direction” (as opposed to the \( r \) or \( z \) directions), we are referring to all vectors in \( T_p(\mathcal{M}_4) \) having both \( \mathbf{e}_\phi \) and \( \mathbf{e}_t \) components. It is useful to compare our curve with the lightcone at each point. It should be emphasized that in such a comparison, we first pick a particular point \( p \) and then work within the tangent vector space \( T_p(\mathcal{M}_4) \). In this vector space it is sensible to speak of the “\( \phi \) axis” whereas the \( \mathcal{M}_4 \) manifold does not have any “\( \phi \) axis.”
“cartesian coordinates” $(\tilde{t}, \bar{x}, \bar{y}, \bar{z})$, we find that the $\phi$ curve follows the trajectory

\[
\begin{align*}
\tilde{t} &= \tilde{t}_0 \\
\bar{x} &= r_0 \cos \phi \\
\bar{y} &= r_0 \sin \phi \\
\bar{z} &= z_0 \\

ds^2 > 0 \quad \text{(time-like)} \quad \forall \phi \in [0, 2\pi]
\end{align*}
\]

and this timelike curve returns to the original location in spacetime as a CTC.

However, we recall that the original spacetime, with metric (8) and standard coordinate ranges and identifications, is simply ordinary flat spacetime. The metric (11) was derived simply from a coordinate transformation. The essential element that led to the CTC in this flat space was the continued demand that $\phi$ exhibit closure even when it became a timelike coordinate.

To illustrate a more natural choice of identification for these curves, we transform the light cones of figure 4 back into the original fiducial $(t, r, \phi, z)$ coordinates. This is illustrated in figure 5. One can see why these lightcones in figure 4 appear to tilt in terms of the $(\tilde{t}, \phi)$ coordinates as $r$ varies. The curves $t + a\phi = \tilde{t}_0, \ r = r_0, \ z = z_0$ being helices, are inside and outside the lightcone for $r_0 < a$ and $r_0 > a$, respectively. If we do not continue to impose closure in $\phi$ when it becomes a timelike coordinate, the CTC characteristic is absent.

In Gödel’s [II] spacetime, the metric

\[
ds^2 = a^2 \left( dt^2 - dr^2 + \frac{1}{2} e^{2\phi} d\phi^2 + 2e^{\phi} d\tilde{t} d\tilde{\phi} - d\bar{z}^2 \right) \quad (15)
\]
Figure 4: In the \((\tilde{t}, \phi)\) coordinates, the tipping light cones produce a CTC for \(r < a\). The boxes at the bottom follow the curves for constant \(\tilde{t}\).

is expressed with timelike coordinates \(\tilde{t}, \tilde{\phi}\) globally. To present this in a physically desirable globally explicit 3+1 form, the transformation

\[
\begin{align*}
\tilde{t} &= t + \frac{r\phi}{2} (1 - \ln r) + \frac{1}{2} \ln r \\
\tilde{r} &= r\phi \\
\tilde{\phi} &= -\frac{1}{2} e^{-r\phi} \ln r \\
\tilde{z} &= z
\end{align*}
\]

is applied. The metric becomes

\[
\frac{ds^2}{a^2} = dt^2 - \left[ \phi^2 + \frac{1}{8r^2} (r\phi \ln r - 1)^2 \right] dr^2 - \left[ \frac{3}{4} r^2 + \frac{1}{8} (r \ln r)^2 \right] d\phi^2 - dz^2 \\
- \frac{1}{4} \left( 8r\phi + r\phi (\ln r)^2 - \ln r \right) dr d\phi + r dt d\phi.
\]
Figure 5: Again, the boxes are used as visual aids to illustrate the evolution of the curves. By contrast with the previous figure, the boxes here are at constant $t$. In the $(t, \phi)$ coordinate system, the spacelike, null and timelike curve are seen as a unified family of curves advancing monotonically in time $t$. Evolving curves never close in terms of $t$ and hence there are no CTC’s with the periodic time restriction removed. Here, the fixed $\bar{t} = \bar{t}_0$ surface is actually helicoidal.
The price to pay for achieving this more desirable form is the introduction of \( \phi \) dependence in the metric, a phenomenon that is familiar from other situations in general relativity. The identification

\[
(\bar{t}, \bar{r}, 0, \bar{z}) = (\bar{t}, \bar{r}, 2\pi, \bar{z})
\]

is transformed, following the procedure as in footnote 2, to

\[
(t, 1, \phi, z) = (t + 2\pi(1 - \phi)e^\phi, e^{-4\pi e^\phi}, \phi e^{4\pi e^\phi}, z)
\]

and in this form, there is no suggestion of any identification of spacetime points. One might object that there was no motivation to identify the \( \bar{\phi} \) end-points in the first Gödel form (15) whereas in the second form [1]

\[
ds^2 = 4a^2 \left(dT^2 - dR^2 + (\sinh^4 R - \sinh^2 R) d\Phi^2 + 2\sqrt{2} \sinh^2 R dT d\Phi - dZ^2 \right)
\]

there is a motivation to identify \( \Phi = 0 \) and \( \Phi = 2\pi \) because of the transformation

\[
e^\phi = \cosh 2R + \cos \Phi \sinh 2R
\]

\[
\tilde{\phi} e^\phi = \sqrt{2} \sin \Phi \sinh 2R
\]

\[
\tan \left(\frac{\Phi}{2} + \frac{\bar{t} - 2T}{2\sqrt{2}}\right) = e^{-2R} \tan \frac{\Phi}{2},
\]

i.e. \( \Phi = 0 \) and \( \Phi = 2\pi \) are mapped to the same point due to the sin, cos and tan of \( \Phi \) terms in the transformation. Because of this, one might argue that the \( \Phi \)-curve is naturally closed (as well as being timelike). To counter this argument, consider a simple Minkowski spacetime \((t, x)\) mapped to \((p, q)\) using

\[
p = x \cos t
\]

\[
q = x \sin t
\]
The metric $ds^2 = dt^2 - dx^2$ becomes

$$ds^2 = \left( \frac{q^2}{(p^2 + q^2)^2} - \frac{p^2}{p^2 + q^2} \right) dp^2 + \left( \frac{p^2}{(p^2 + q^2)^2} - \frac{q^2}{p^2 + q^2} \right) dq^2 - 2pq \left( \frac{p^2 + q^2 + 1}{(p^2 + q^2)^2} \right) dpdq.$$

The light cones are shown in figure 6. The Jacobian of the transformation vanishes only for $x=0$. Consider the curve $p = \cos \tau$, $q = \sin \tau$ which is time-like and naturally closed. Clearly, this timelike curve is simply a segment from $(t, x) = (0, 1)$ to $(2\pi, 1)$ and it would be quite unnatural to identify these two end-points.

Returning to the case of the Gödel spacetime, one might object that in this process, Gödel’s CTC has been artificially removed by making different identifications. However, it is more persuasive to argue that it is the identification in the original Gödel spacetime that is the artificial one. This is reminiscent of our first example where one could choose
to have or not have a CTC, simply through the choice of identification. CTC's did not manifest themselves in the metric, and gravitation via general relativity was not the factor that led to their realization.

We now return to (1) and consider the transformation for the curve where $r, z$ (and hence $f, \nu$ and $w$) are held constant as

$$dt = \tilde{d}t - wd\varphi$$
$$d\Phi = \frac{w^2f - r^2f^{-1}}{2fw}d\varphi - d\tilde{t}.$$  \hfill (16)

The line element for this curve is

$$ds^2 = \frac{f}{(w^2f^2 + r^2)^2} \left((w^2f^2 - r^2)^2dt^2 - 8f^2w^2r^2d\Phi dt - 4f^2w^2r^2d\Phi^2 \right).$$  \hfill (17)

In this system, $t$ is a timelike coordinate and $\Phi$ is a spacelike coordinate regardless of whether (4) or (6) holds. With these coordinates, the integrity of the azimuthal coordinate is maintained explicitly unlike the case with the Gödel approach. Ambiguities of interpretation are removed and hence these coordinates are particularly valuable.

With $\tilde{t}$ held constant, say $\tilde{t} = 0$ for simplicity, we find that the curve has the equation in parametric form

$$t = -w\varphi$$
$$\Phi = \frac{(w^2f - r^2f^{-1})\varphi}{2fw}.$$  \hfill (18)

with parameter $\varphi$.

With the parameter eliminated between the two equations, we see that $\Phi$ is simply a linear function of $t$ with proportionality factor dependent upon the particular $(r, z)$ chosen. At this point, it is natural to ascribe standard geometric (in this case cylindrical
polar-like) character by identifying the spatial points for $\Phi$ (rather than for $\varphi$). However, there is no reason to ascribe periodicity to $t$. The time flows monotonically without looping as it does in conventional flat space. Thus, while the spatial points are retraced ad infinitum, they do so at successively later times. They do so here as in the previous examples in figure 5.

While the imposition of periodicity in $\varphi$ follows naturally from our experience with the symmetry of various spatial geometries, there is no logic that leads us to continue to impose closure in $\varphi$ when it becomes a timelike coordinate. While our experience with nature leads us to deem as physical only those timelike curves that evolve into the future lightcone, there is nothing in our experience that would have us place any apriori demands for periodicity in a timelike coordinate. If gravity were to truly cause closure in a timelike curve, it should do so without injecting periodic character in the timelike coordinate from the outset.

We have seen how transformations with $t$ and $\varphi$ of the form $\bar{t} = t + a\varphi$ generate helicoidal surfaces for a fixed time $\bar{t}$ as viewed in the $t,\varphi$ coordinate system. While mixing a non-cyclic variable, $t$ with a cyclic $\varphi$ usually does not create difficulties in other branches of physics, the interpretation is more critical in general relativity.

For a proper interpretation, the nature of the coordinates should not be ambiguous. When the metric is expressed with more than one timelike coordinate, much confusion can result such as in the identification of the endpoints of a curve when one of these coordinates is held fixed. The solution is to transform to a system of coordinates in

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5In this more general case, the identification in $\Phi$ is at 0 and $2\pi(w^2 f - r^2 f^{-1})/(2fw)$. This is seen as an indicator of a singularity at $r = 0$ analogous to the vertex of a cone. Bonnor has argued that the CTC could be interpreted as a torsion singularity 2.
which the true nature of the curves in the spacetime becomes clarified. In our example, unlike figure 4 which basically forces one to require closure whether it is appropriate or not, figure 5 allows the freedom to choose whether or not one end of the curve is identified with the other. This is achieved by finding coordinates which maintain their timelike and spacelike character throughout.

In all the examples, identification of points can induce CTC’s. However, from the vantage point of figure 5 it is seen that there is no more logical basis to ascribe periodicity in time here than there would be to do so in the case of repetitive traversal of a circle in flat space. In this figure, the spacelike, timelike and null curves are all seen as part of the same family composing the helicoidal surface. As seen from equations (17) and (18), it would be equally uncalled for to identify times in case (4) as it would be in case (6). Unlike the standard approach with the Gödel metric where there is an apparent discontinuity in character of the possible types of curves, the helicoidal surface unifies the families with all three families advancing monotonically in time without repetition.

There are common elements throughout these examples: A. The metric itself is insufficient to determine whether a CTC exists in the system. B. The identification of points can induce CTC’s in a system whether that system is simple or complicated.

The vast body of physicists, not to mention the public at large, regard the notion of time travel as nothing more than a figment of science fiction fantasy. The faith is placed in the entropy clock that is mono-directional and non-repetitive. However the time-machine concept has re-surfaced in recent years in various guises, most recently in connection with wormholes (see, for example Visser [6]) and through other exotic channels such as tachyons and colliding or spinning cosmic strings. It is natural to
wonder if this would have occurred had it been appreciated from the outset that the apparently more conservative route to time-travel in normal topology such as with the Gödel metric was not what had been claimed for it, that the curves were closed in time simply because of an imposed periodicity in a timelike coordinate.

After Gödel’s work appeared, Chandrasekhar and Wright (“CW”) [7] showed that the Gödel CTC’s were not geodesics, thus concluding that his paper was in error. Stein [8] countered that Gödel never claimed that his CTC’s were geodesics and recently, Ozsvath and Schucking [9] calculated the acceleration of the CTC’s. There are interesting considerations that arise from this. In testing whether the CTC’s were geodesics, it could be said that CW were simultaneously determining whether the CTC’s could exist solely within the confines of the gravitational model. Once it is determined that acceleration is required, the need for a mechanism for achieving it, namely a machine, arises. While the role of machines in other contexts in general relativity is frequently ignorable (and generally ignored), this is not the case here. Repetition in time is the issue in the present context and the machine must be shown to partake of this repetition for consistency. In [9], there is a discussion of the vast fuel requirements and relativistic velocities to achieve CTC’s but there are further even more serious issues: the accelerating machine must also follow a closed timelike path, i.e. it must really be a “time machine” with all of the machine’s complex elements and interactions experiencing closure in time. While the relatively simple purely gravitational model of Gödel (via the imposition of a periodic time coordinate rather than an effect of gravity), can produce such closure, albeit artificially, this has not been demonstrated to be possible, nor would we ever expect it to be possible, with the complexities of a machine that would change the
metric, quite apart from considerations of entropy flow.

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