A Morse-theoretical analysis of gravitational lensing by a Kerr-Newman black hole

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Consider, in the domain of outer communication $M_+$ of a Kerr-Newman black hole, a point $p$ (observation event) and a timelike curve $\gamma$ (worldline of light source). Assume that $\gamma$ (i) has no past end-point, (ii) does not intersect the caustic of the past light-cone of $p$, and (iii) goes neither to the horizon nor to infinity in the past. We prove that then for infinitely many positive integers $k$ there is a past-pointing lightlike geodesic $\lambda_k$ of (Morse) index $k$ from $p$ to $\gamma$, hence an observer at $p$ sees infinitely many images of $\gamma$. Moreover, we demonstrate that all lightlike geodesics from an event to a timelike curve in $M_+$ are confined to a certain spherical shell. Our characterization of this spherical shell shows that in the Kerr-Newman spacetime the occurrence of infinitely many images is intimately related to the occurrence of centrifugal-plus-Coriolis force reversal.

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I. INTRODUCTION

The question of how many images an observer at an event $p$ sees of a light source with worldline $\gamma$ is equivalent to the question of how many past-pointing lightlike geodesics from $p$ to $\gamma$ exist. In spacetimes with many symmetries this question can be addressed, in principle, by directly integrating the geodesic equation. In the spacetime around a non-rotating and uncharged black hole of mass $m$, e.g., which is described by the Schwarzschild metric, all lightlike geodesics can be explicitly written in terms of elliptic integrals; with the help of these explicit expressions, it is easy to verify that in the region outside the horizon, i.e. in the region where $r > 2m$, there are infinitely many past-pointing lightlike geodesics from any event $p$ to any integral curve of the Killing vector field $\partial_t$. This was demonstrated already in 1959 bei Darwin [1]. We may thus say that a Schwarzschild black hole acts as a gravitational lens that produces infinitely many images of any static light source. However, already in the Schwarzschild spacetime the problem becomes more difficult if we want to consider light sources which are not static, i.e., worldlines $\gamma$ which are not integral curves of $\partial_t$.

In this paper we want to investigate this problem for the more general case of a charged and rotating black hole, which is described by the Kerr-Newman metric. More precisely, we want to demonstrate that in the domain of outer communication around a Kerr-Newman black hole, i.e., in the domain outside of the outer horizon, there are infinitely many past-pointing lightlike geodesics from an unspecified event $p$ to an unspecified worldline $\gamma$, with as little restrictions on $\gamma$ as possible. Although the geodesic equation in the Kerr-Newman spacetime is completely integrable, the mathematical expressions are so involved that it is very difficult to achieve this goal by explicitly integrating the geodesic equation. Therefore it is recommendable to use more indirect methods.

Such a method is provided by Morse theory. Quite generally, Morse theory relates the number of solutions to a variational principle to the topology of the space of trial maps. Here we refer to a special variant of Morse theory, developed by Uhlenbeck [2], which is based on a version of Fermat’s principle for a globally hyperbolic Lorentzian manifold $(M,g)$. The trial maps are the lightlike curves joining a point $p$ and a timelike curve $\gamma$ in $M$, and the solution curves of Fermat’s principle are the lightlike geodesics. If $(M,g)$ and $\gamma$ satisfy additional conditions, the topology of the space of trial maps is determined by the topology of $M$. Uhlenbeck’s work gives criteria that guarantee the existence of infinitely many past- or future-pointing lightlike geodesics from $p$ to $\gamma$. In this paper we will apply her results to the domain of outer communication around a Kerr-Newman black hole which is, indeed, a globally hyperbolic Lorentzian manifold. We will show that the criteria for having infinitely many past-pointing timelike geodesics from $p$ to $\gamma$ are satisfied for every event $p$ and every timelike curve $\gamma$ in this region, provided that the following three conditions are satisfied. First, $\gamma$ must not have a past end-point; it is obvious that we need a condition of this kind because otherwise it would be possible to choose for $\gamma$ an arbitrarily short section of a worldline such that trivially the number of past-pointing lightlike geodesics from $p$ to $\gamma$ is zero. Second, $\gamma$ must not intersect the caustic of the past light-cone of $p$; this excludes all cases where $p$ sees an extended image, such as an Einstein ring, of $\gamma$. Third, in the
past the worldline $\gamma$ must not go to the horizon or to infinity. Under these (very mild) restrictions on the motion of the light source we will see that the Kerr-Newman black hole acts as a gravitational lens that produces infinitely many images. Moreover, we will also show that all (past-directed) lightlike geodesics from $p$ to $\gamma$ are confined to a certain spherical shell. For the characterization of this shell we will have to discuss a light-convexity property which turns out to be intimately related to the phenomenon of centrifugal(-plus-Coriolis) force reversal. This phenomenon has been discussed, first in spherically symmetric static and then in more general spacetimes, in several papers by Marek Abramowicz with various coauthors: material which is of interest to us can be found, in particular, in Abramowicz, Carter and Lasota [3]; Abramowicz [4] and Abramowicz, Nurowski and Wex [5].

The paper is organized as follows. In Section II we summarize the Morse-theoretical results we want to use. Section III is devoted to the notions of centrifugal and Coriolis force in the Kerr-Newman spacetime; in particular, we introduce a potential $\Psi_+$ (respectively $\Psi_-$) that characterizes the sum of centrifugal and Coriolis force with respect to co-rotating (respectively counter-rotating) observers whose velocity approaches the velocity of light. In Section IV we discuss multiple imaging in the Kerr-Newman spacetime with the help of the Morse theoretical result quoted in Section II and with the help of the potential $\Psi_\pm$ introduced in Section III. Our results are summarized and discussed in Section V.

II. A RESULT FROM MORSE THEORY

In this section we briefly review a Morse-theoretical result that relates the number of lightlike geodesics between a point $p$ and a timelike curve $\gamma$ in a globally hyperbolic Lorentzian manifold to the topology of this manifold. This result was found by Uhlenbeck [2] and its relevance in view of gravitational lensing was discussed by McKenzie [6]. Uhlenbeck’s work is based on a variational principle for lightlike geodesics (“Fermat principle”) in a globally hyperbolic Lorentzian manifold, and her main method of proof is to approximate trial paths by broken geodesics. With the help of infinite-dimensional Hilbert manifold techniques Giannoni, Masiello, and Piccione were able to rederive Uhlenbeck’s result [7] and to generalize it to certain subsets-with-boundary of spacetimes that need not be globally hyperbolic [8]. In contrast to Uhlenbeck, they start out from a variational principle for lightlike geodesics that is not restricted to globally hyperbolic spacetimes. (Such a Fermat principle for arbitrary general-relativistic spacetimes was first formulated by Kovner [9]; the proof that the solution curves of Kovner’s variational principle are, indeed, precisely the lightlike geodesics was given by Perlick [10].) Although for our purpose the original Uhlenbeck result is sufficient, readers who are interested in technical details are encouraged to also consult the papers by Giannoni, Masiello, and Piccione, in particular because in the Uhlenbeck paper some of the proofs are not worked out in full detail.

Following Uhlenbeck [2], we consider a 4-dimensional Lorentzian manifold $(M, g)$ that admits a foliation into smooth Cauchy surfaces, i.e., a globally hyperbolic spacetime. (For background material on globally hyperbolic spacetimes the reader may consult, e.g., Hawking and Ellis [11].) The fact that the original definition of global hyperbolicity is equivalent to the existence of a foliation into smooth Cauchy surfaces was completely proven only recently by Bernal and Sánchez [12].) Then $M$ can be written as a product of a 3-dimensional manifold $S$, which serves as the prototype for each Cauchy surface, and a time-axis,

$$M = S \times \mathbb{R}. \tag{1}$$

Moreover, this product can be chosen such that the metric $g$ orthogonally splits into a spatial and a temporal part,

$$g = g_{ij}(x, t) \, dx^i \, dx^j - f(x, t) \, dt^2, \tag{2}$$

where $t$ is the time coordinate given by projecting from $M = S \times \mathbb{R}$ onto the second factor, $x = (x^1, x^2, x^3)$ are coordinates on $S$, and the summation convention is used for latin indices running from 1 to 3. (We write (2) in terms of coordinates for notational convenience only. We do not want to presuppose that $S$ can be covered by a single coordinate system.) We interpret the direction of increasing $t$ as the future-direction on $M$. Again following Uhlenbeck [2], we say that the splitting (2) satisfies the metric growth condition if for every compact subset of $S$ there is a function $F$ with

$$\int_{-\infty}^{0} \frac{dt}{F(t)} = \infty \tag{3}$$

such that for $t \leq 0$ the inequality

$$g_{ij}(x, t) \, v^i \, v^j \leq f(x, t) \, F(t)^2 \, G_{ij}(x) \, v^i \, v^j \tag{4}$$

holds for all $x$ in the compact subset and for all $(v^1, v^2, v^3) \in \mathbb{R}^3$, with a time-independent Riemannian metric $G_{ij}$ on $S$. It is easy to check that the metric growth condition assures that for every (smooth) curve $\alpha : [a, b] \rightarrow S$ there is
a function \( T : [a, b] \rightarrow \mathbb{R} \) with \( T(a) = 0 \) such that the curve \( \lambda : [a, b] \rightarrow M = S \times \mathbb{R}, s \mapsto (\alpha(s), T(s)) \) is past-pointing and lightlike. In particular, the metric growth condition assures that from each point \( p \) in \( M \) we can find a past-pointing lightlike curve to every timelike curve that is vertical with respect to the orthogonal splitting chosen. In this sense, the metric growth condition prohibits the existence of particle horizons, cf. Uhlenbeck \[2\] and McKenzie \[6\]. Please note that our formulation of the metric growth condition is the same as McKenzie’s which differs from Uhlenbeck’s by interchanging future and past (i.e., \( t \mapsto -t \)). The reason is that Uhlenbeck in her paper characterizes future-pointing lightlike geodesics from a point to a timelike curve whereas we, in view of gravitational lensing, are interested in past-pointing ones.

For formulating Uhlenbeck’s result we have to assume that the reader is familiar with the notion of conjugate points and with the following facts (see, e.g., Perlick \[13\]). The totality of all conjugate points, along any lightlike geodesic issuing from a point \( p \) into the past, makes up the caustic of the past light-cone of \( p \). A lightlike geodesic is said to have (Morse) index \( k \) if it has \( k \) conjugate points in its interior; here and in the following every conjugate point has to be counted with its multiplicity. For a lightlike geodesic with two end-points, the index is always finite. It is our goal to estimate the number of past-pointing lightlike geodesics of index \( k \) from a point \( p \) to a timelike curve \( \gamma \) that does not meet the caustic of the past light-cone of \( p \). The latter condition is generically satisfied in the sense that, for any \( \gamma \), the set of all points \( p \) for which it is true is dense in \( M \). This condition makes sure that the past-pointing lightlike geodesics from \( p \) to \( \gamma \) are countable, i.e., it excludes gravitational lensing situations where the observer sees a continuum of images such as an Einstein ring.

As another preparation, we recall how the Betti numbers \( B_k \) of the loop space \( L(M) \) of a connected topological space \( M \) are defined. As a realization of \( L(M) \) one may take the space of all continuous curves between any two fixed points in \( M \). The \( k \)th Betti number \( B_k \) is formally defined as the dimension of the \( k \)-th homology space of \( L(M) \) with coefficients in a field \( \mathbb{F} \). (For our purpose we may choose \( \mathbb{F} = \mathbb{R} \).) Roughly speaking, \( B_0 \) counts the connected components of \( L(M) \) and \( B_k \), for \( k > 0 \), counts those “holes” in \( L(M) \) that prevent a \( k \)-sphere from being a boundary. If the reader is not familiar with Betti numbers he or she may consult e.g. \[13\].

After these preparations Uhlenbeck’s result that we want to use later in this paper can now be phrased in the following way.

**Theorem 1.** (Uhlenbeck \[2\]) Consider a globally hyperbolic spacetime \((M, g)\) that admits an orthogonal splitting \[1\], \[4\] satisfying the metric growth condition. Fix a point \( p \in M \) and a smooth timelike curve \( \gamma : \mathbb{R} \rightarrow M \) which, in terms of the above-mentioned orthogonal splitting, takes the form \( \gamma(\tau) = (\beta(\tau), \tau) \), with a curve \( \beta : \mathbb{R} \rightarrow S \). Moreover, assume that \( \gamma \) does not meet the caustic of the past light-cone of \( p \) and that for some sequence \((\tau_i)_{i \in \mathbb{N}}\) with \( \tau_i \rightarrow -\infty \) the sequence \((\beta(\tau_i))_{i \in \mathbb{N}}\) converges in \( S \). Then the Morse inequalities

\[
N_k \geq B_k \quad \text{for all} \quad k \in \mathbb{N}_0
\]

and the Morse relation

\[
\sum_{k=0}^{\infty} (-1)^k N_k = \sum_{k=0}^{\infty} (-1)^k B_k
\]

hold true, where \( N_k \) denotes the number of past-pointing lightlike geodesics with index \( k \) from \( p \) to \( \gamma \), and \( B_k \) denotes the \( k \)-th Betti number of the loop space of \( M \).

**Proof.** See Uhlenbeck \[2\], §4 and Proposition 5.2. \(\square\)

Please note that the convergence condition on \((\beta(\tau_i))_{i \in \mathbb{N}}\) is certainly satisfied if \( \beta \) is confined to a compact subset of \( S \), i.e., if \( \gamma \) stays in a spatially compact set.

The sum on the right-hand side of \(6\) is, by definition, the Euler characteristic \( \chi \) of the loop space of \( M \). Hence, \(6\) can also be written in the form

\[
N_+ - N_- = \chi,
\]

where \( N_+ \) (respectively \( N_- \)) denotes the number of past-pointing lightlike geodesics with even (respectively odd) index from \( p \) to \( \gamma \).

The Betti numbers of the loop space of \( M = S \times \mathbb{R} \) are, of course, determined by the topology of \( S \). Three cases are to be distinguished.

**Case A:** \( M \) is not simply connected. Then the loop space of \( M \) has infinitely many connected components, so \( B_0 = \infty \). In this situation \[6\] says that \( N_0 = \infty \), i.e., that there are infinitely many past-pointing lightlike geodesics from \( p \) to \( \gamma \) that are free of conjugate points.

**Case B:** \( M \) is simply connected but not contractible to a point. Then for all but finitely many \( k \in \mathbb{N}_0 \) we have \( B_k > 0 \). This was proven in a classical paper by Serre \[15\], cf. McKenzie \[6\]. In this situation \[6\] implies \( N_k > 0 \) for all but

In this case, the Kerr-Newman metric describes the spacetime around a rotating black hole with mass \( m \), charge \( q \), and specific angular momentum \( a \). The Kerr-Newman metric contains the Kerr metric (\( a = 0 \)), the Reissner-Nordström metric (\( q = 0 \)) and the Schwarzschild metric (\( q = 0 \) and \( a = 0 \)) as special cases which are all discussed, in great detail, in Chandrasekhar [17]; for the Kerr metric we also refer to O’Neill [18].

By (10), the equation \( \Delta = 0 \) has two real roots,

\[
r_{\pm} = m \pm \sqrt{m^2 - a^2 - q^2} ,
\]

which determine the two horizons. We shall restrict to the region

\[
M_{+} : \quad r_{+} < r < \infty ,
\]

which is usually called the domain of outer communication of the Kerr-Newman black hole. On \( M_{+} \), the coordinates \( \varphi \) and \( \theta \) range over \( S^2 \), the coordinate \( t \) ranges over \( \mathbb{R} \), and the coordinate \( r \) ranges over an open interval which is diffeomorphic to \( \mathbb{R} \); hence \( M_{+} \simeq S^2 \times \mathbb{R}^2 \).

From now on we will consider the spacetime \( (M_{+}, g) \), where \( g \) denotes the restriction of the Kerr-Newman metric with (10) to the domain \( M_{+} \) given by (12). For the sake of brevity, we will refer to \( (M_{+}, g) \) as to the exterior Kerr-Newman spacetime. As a matter of fact, \( (M_{+}, g) \) is a globally hyperbolic spacetime; the Boyer-Lindquist time coordinate \( t \) gives a foliation of \( M_{+} \) into Cauchy surfaces \( t = \text{constant} \). Together with the lines perpendicular to these surfaces, we get an orthogonal splitting of the form (12). Observers with worldlines perpendicular to the surfaces \( t = \text{constant} \) are called zero-angular-momentum observers or locally non-rotating observers. In contrast to the worldlines perpendicular to the surfaces \( t = \text{constant} \), the integral curves of the Killing vector field \( \partial_t \) are not timelike on all of \( M_{+} \); they become spacelike inside the so-called ergosphere which is characterized by the inequality \( \Delta < a^2 \sin^2 \theta \). For \( a \neq 0 \) it is impossible to find a Killing vector field which is timelike on all of \( M_{+} \); in this sense, the exterior Kerr-Newman spacetime is not a stationary spacetime.

In the rest of this section we discuss the notions of centrifugal force and Coriolis force for observers on circular orbits around the axis of rotational symmetry in the exterior Kerr-Newman spacetime \( (M_{+}, g) \). For background information on these notions we refer to the work of Marek Abramowicz and his collaborators [3, 4, 7] which was mentioned.
already in the introduction. For our discussion it will be convenient to introduce on $M_+$ the orthonormal basis

$$E_0 = \frac{1}{\rho \sqrt{\Delta}} \left( (r^2 + a^2) \partial_t + a \partial_\phi \right),$$

$$E_1 = \frac{1}{\rho \sin \theta} \left( \partial_\phi + a \sin^2 \theta \partial_t \right),$$

$$E_2 = \frac{1}{\rho} \partial_\theta, \quad E_3 = \frac{\sqrt{\Delta}}{\rho} \partial_r,$$  \hspace{1cm} (13)

whose dual basis is given by the covector fields

$$-g(E_0, \cdot) = \frac{\sqrt{\Delta}}{\rho} \left( dt - a \sin^2 \theta \, d\phi \right),$$

$$g(E_1, \cdot) = \frac{\sin \theta}{\rho} \left( (r^2 + a^2) \, d\phi \right. - a \, dt \Big),$$

$$g(E_2, \cdot) = \rho \, d\theta, \quad g(E_3, \cdot) = \frac{\rho}{\sqrt{\Delta}} \, dr.$$  \hspace{1cm} (14)

Henceforth we refer to the integral curves of the timelike basis field $E_0$ as to the worldlines of the standard observers in $(M_+, g)$. For later purpose we list all non-vanishing Lie brackets of the $E_i$.

$$[E_0, E_2] = -\frac{a^2}{\rho^3} \cos \theta \sin \theta \, E_0,$$

$$[E_0, E_3] = \left( \frac{r - m}{\rho \sqrt{\Delta}} - \frac{r \sqrt{\Delta}}{\rho^3} \right) E_0 + \frac{2 a \sin \theta}{\rho^3} \, E_1,$$

$$[E_1, E_2] = \left( \frac{\rho^2 + a^2 \sin^2 \theta}{\rho^3 \sin \theta} \right) \cos \theta \, E_1 - \frac{2 a \sqrt{\Delta} \cos \theta}{\rho^3} \, E_0,$$

$$[E_1, E_3] = \frac{r \sqrt{\Delta}}{\rho^3} \, E_1,$$

$$[E_2, E_3] = \frac{r \sqrt{\Delta}}{\rho^3} \, E_2 + \frac{a^2 \cos \theta \sin \theta}{\rho^3} \, E_3.$$  \hspace{1cm} (15)

For every $v \in [0, 1]$, the integral curves of the vector field

$$U = \frac{E_0 \pm v \, E_1}{\sqrt{1 - v^2}}$$  \hspace{1cm} (16)

can be interpreted as the worldlines of observers who circle along the $\phi$-lines around the axis of rotational symmetry of the Kerr-Newman spacetime. The number $v$ gives the velocity (in units of the velocity of light) of these observers with respect to the standard observers. For the upper sign in (16), the motion relative to the standard observers is in the positive $\phi$-direction and thus co-rotating with the black hole (because of our assumption $a \geq 0$), for the negative sign it is in the negative $\phi$-direction and thus counter-rotating. Please note that $g(U, U) = -1$, which demonstrates that the integral curves of $U$ are parametrized by proper time.

In general, $U$ is non-geodesic, $\nabla_U U \neq 0$, i.e., one needs a thrust to stay on an integral curve of $U$. Correspondingly, relative to a $U$-observer a freely falling particle undergoes an “inertial acceleration” measured by $-\nabla_U U$. To calculate this quantity, we write

$$-g(\nabla_U U, E_i) = -U g(U, E_i) + g(U, \nabla_U E_i) = -U g(U, E_i) + g(U, [U, E_i]),$$  \hspace{1cm} (17)

The first term on the right-hand side vanishes, and the second term can be easily calculated with the help of (16) and (15), for $i=0,1,2,3$. We find

$$-g(\nabla_U U, \cdot) = A_{grav} + A_{Cor} + A_{cent}$$  \hspace{1cm} (18)
where the covector fields

\[
A_{\text{grav}} = \frac{\Delta r - \rho^2(r - m)}{\rho^2\Delta} dr + \frac{a^2}{\rho^2} \sin \vartheta \cos \vartheta \, d\vartheta ,
\]

\[
A_{\text{Cor}} = \pm \frac{v}{1 - v^2} \frac{2 a \sqrt{\Delta}}{\rho^2} \left( \frac{r}{\Delta} \sin \vartheta \, dr + \cos \vartheta \, d\vartheta \right) ,
\]

\[
A_{\text{cent}} = \frac{v^2}{(1 - v^2)} \left( \frac{2 r \Delta - \rho^2(r - m)}{\rho^2\Delta} dr + \frac{(\rho^2 + 2 a^2\sin^2\vartheta) \cos \vartheta}{\rho^2 \sin \vartheta} \, d\vartheta \right)
\]

give, respectively, the gravitational, the Coriolis, and the centrifugal acceleration of a freely falling particle relative to the \(U\)-observers. (Multiplication with the particle’s mass gives the corresponding “inertial force.”) Here the decomposition of the total inertial acceleration into its three contributions is made according to the same rule as in Newtonian mechanics: The gravitational acceleration is independent of \(v\), the Coriolis acceleration is odd with respect to \(v\), and the centrifugal acceleration is even with respect to \(v\). In [19], it was shown that, according to this rule, gravitational, Coriolis and centrifugal acceleration are unambiguous whenever a timelike 2-surface with a timelike vector field has been specified; here we apply this procedure to each 2-surface \((r, \vartheta) = \text{constant}\) with the timelike vector field \(E_0\).

Up to the positive factor \(v/(1 - v^2)\), the sum of Coriolis and centrifugal acceleration is equal to

\[
Z_\pm(v) = \pm \frac{2 a \sqrt{\Delta}}{\rho^2} \left( \frac{r}{\Delta} \sin \vartheta \, dr + \cos \vartheta \, d\vartheta \right) + v \left( \frac{2 r \Delta - \rho^2(r - m)}{\rho^2\Delta} dr + \frac{(\rho^2 + 2 a^2\sin^2\vartheta) \cos \vartheta}{\rho^2 \sin \vartheta} \, d\vartheta \right).
\]

If we exclude the Reissner-Nordström case \(a = 0\), the Coriolis force dominates the centrifugal force for small \(v\). To investigate the behavior for \(v\) close to the velocity of light, we consider the limit \(v \to 1\). By a straight-forward calculation we find that

\[
Z_\pm(v) \xrightarrow{v \to 1} \frac{\sin \vartheta}{\rho^2\Delta} \left( \rho^2 + a^2 \pm a \sqrt{\Delta} \sin \vartheta \right)^2 \, d\Psi_\pm ,
\]

where

\[
d\Psi_\pm = \frac{2 r \Delta - (r - m) \rho^2 \pm 2 a r \sqrt{\Delta} \sin \vartheta}{\sqrt{\Delta} \sin \vartheta \left( \rho^2 + a^2 \pm a \sqrt{\Delta} \sin \vartheta \right)^2} dr
\]

\[
+ \frac{(\rho^2 + 2 a^2\sin^2\vartheta \pm 2 a \sqrt{\Delta} \sin \vartheta)}{\sin^2 \vartheta \left( \rho^2 + a^2 \pm a \sqrt{\Delta} \sin \vartheta \right)^2} \, d\vartheta
\]

is the differential of the function

\[
\Psi_\pm = \frac{-1 + \sqrt{\Delta}}{\sqrt{\rho^2 + a^2 \pm a \sin \vartheta}} \, \sin \vartheta .
\]

Because of \(\sin \vartheta\) in the denominator, both \(\Psi_-\) and \(\Psi_+\) are singular along the axis. \(\Psi_+\) is negative on all of \(M_+\) whereas \(\Psi_-\) is negative outside and positive inside the ergosphere.

From [23] we read that, in the limit \(v \to 1\), the sum of Coriolis and centrifugal force is perpendicular to the surfaces \(\Psi_\pm = \text{constant}\) and points in the direction of increasing \(\Psi_\pm\). In this limit, we may thus view the function \(\Psi_+\) (or \(\Psi_-\), resp.) as a Coriolis-plus-centrifugal potential for co-rotating (or counter-rotating, resp.) observers. The surfaces \(\Psi_\pm = \text{constant}\) are shown in Figure 11.

It is not difficult to see that \(\Psi_\pm\) is independent of the family of observers with respect to which the inertial accelerations have been defined, as long as their 4-velocity is a linear combination of \(\partial_t\) and \(\partial_\varphi\). We have chosen the standard observers; a different choice would lead to different formulas for the inertial accelerations [19, 20] and [21], but to the same \(\Psi_\pm\). For the sake of comparison, the reader may consult Nayak and Vishveshwara [21] where the inertial accelerations are calculated with respect to the zero angular momentum observers. Also, it should be mentioned that the potentials \(\Psi_+\) and \(\Psi_-\), or closely related functions, have been used already by other authors. The quantities \(\Omega_{\pm}\), e.g., introduced by de Felice and Usseglio-Tomasset [21] in their analysis of physical effects related to centrifugal force reversal in the equatorial plane of the Kerr metric, are related to our potentials by \(\Omega_{\pm} = \mp \Psi_\pm|_{\vartheta = \pi/2}\).
In the Reissner-Nordström case \( a = 0 \), the Coriolis acceleration vanishes identically and

\[
\Psi = \Psi_+ = \Psi_- = -\sqrt{r^2 - 2mr + q^2 + 2mr} \sin \vartheta
\]

(26)

is a potential for the centrifugal acceleration in the sense that \( A_{\text{cent}} \) is a multiple of \( d\Psi \). In this case, the surfaces \( \Psi = \text{constant} \) coincide with what Abramowicz calls the von Zeipel cylinders. Abramowicz’s Figure 1 in [4], which shows the von Zeipel cylinders in the Schwarzschild spacetime, coincides with the \( a \to 0 \) limit of our Figure [4] which shows the surfaces \( \Psi_+ = \text{constant} \) and \( \Psi_- = \text{constant} \) in the Kerr spacetime. (The notion of von Zeipel cylinders has also been defined in the Kerr metric, see [22], for observers of a specified angular velocity. However, this angular-velocity-dependent von Zeipel cylinders are not related to the potentials \( \Psi_+ \) and \( \Psi_- \) in the Kerr spacetime.)

By construction, the function \( \Psi_{\pm} \) has the following property. If we send a lightlike geodesic tangential to a \( \varphi \)-line in the positive (respectively negative) \( \varphi \)-direction, it will move away from this \( \varphi \)-line in the direction of the negative gradient of \( \Psi_+ \) (respectively \( \Psi_- \)). Thus, each zero of the differential \( d\Psi_+ \) (respectively \( d\Psi_- \)) indicates a co-rotating (respectively counter-rotating) circular lightlike geodesic, i.e., a "photon circle". By \( \Psi_{\pm} \), \( d\Psi_{\pm} \) vanishes if

\[
\cos \vartheta = 0 \quad \text{and} \quad 2r \Delta - (r - m) \rho^2 = 2a r \sqrt{\Delta} \sin \vartheta = 0.
\]

(27)

By writing \( \Delta \) and \( \rho^2 \) explicitly, we see that (27) is true at \( \vartheta = \pi / 2 \) and \( r = r_{\pm}^{ph} \), where \( r_{\pm}^{ph} \) is defined by the equation

\[
(r_{\pm}^{ph})^2 = \frac{3m}{2} r_{\pm}^{ph} + 2a^2 + 2q^2 = 2a \left( r_{\pm}^{ph} \right)^2 - 2m r_{\pm}^{ph} + a^2 + q^2.
\]

(28)

For \( 0 < \sqrt{a^2 + q^2} < m \), \( r_{\pm}^{ph} \) has exactly one solution for each which satisfies

\[
r_+ < r_{\pm}^{ph} < \frac{3m}{2} + \sqrt{\frac{9m^2}{4} - 2q^2} < r_- < 2m + 2 \sqrt{m^2 - q^2}.
\]

(29)

So there is exactly one co-rotating photon circle in \( M_+ \), corresponding to the critical point of \( \Psi_+ \) at \( r_+^{ph} \), and exactly one counter-rotating photon circle in \( M_- \), corresponding to the critical point of \( \Psi_- \) at \( r_-^{ph} \), see Figure 1. (The relation of photon circles to centrifugal-plus-Coriolis force in the limit \( v \to 1 \) is also discussed by Stuchlík, Hledík and Jurán [23]; note, however, that their work is restricted to the equatorial plane of the Kerr-Newman spacetime throughout.)

In the Reissner-Nordström case, \( a = 0 \), we have \( r_{\pm}^{ph} = r_{\pm}^{ph} = \frac{3m}{2} + \sqrt{\frac{9m^2}{4} - 2q^2} \) (cf., e.g., Chandrasekhar [17], p.218). If we keep \( m \) and \( q \) fixed and vary \( a \) from 0 to the extreme value \( \sqrt{m^2 - q^2} \), \( r_+^{ph} \) decreases from \( \frac{3m}{2} + \sqrt{\frac{9m^2}{4} - 2q^2} \) to \( m \) whereas \( r_-^{ph} \) increases from \( \frac{3m}{2} + \sqrt{\frac{9m^2}{4} - 2q^2} \) to \( 2m + 2 \sqrt{m^2 - q^2} \). As an aside, we mention that, although \( r_+^{ph} \) and \( r_+^{ph} \) both go to \( m \) in the extreme case, the proper distance between the co-rotating photon circle at \( r_+^{ph} \) and the horizon at \( r_+ \) does not go to zero; for the case \( q = 0 \) this surprising feature is discussed in Chandrasekhar [17], p. 340.

From (24) we can read the sign of \( \partial_r \Psi_{\pm} \) at each point. We immediately find the following result.

**Proposition 1.** Decompose the exterior Kerr spacetime into the sets

\[
M_{\text{in}} : \quad 2r \Delta - (r - m) \rho^2 < -2a r \sqrt{\Delta} \sin \vartheta
\]

(30)

\[
K : \quad -2a r \sqrt{\Delta} \sin \vartheta \leq 2r \Delta - (r - m) \rho^2 \leq 2a r \sqrt{\Delta} \sin \vartheta
\]

(31)

\[
M_{\text{out}} : \quad 2a r \sqrt{\Delta} \sin \vartheta < 2r \Delta - (r - m) \rho^2,
\]

(32)

so \( M_+ = M_{\text{in}} \cup K \cup M_{\text{out}} \), see Figure 1. Then

\[
\partial_r \Psi_+ < 0 \quad \text{and} \quad \partial_r \Psi_- < 0 \quad \text{on} \ M_{\text{in}},
\]

(33)

\[
\partial_r \Psi_+ < 0 \quad \text{and} \quad \partial_r \Psi_- > 0 \quad \text{on the interior of} \ K,
\]

(34)

\[
\partial_r \Psi_+ > 0 \quad \text{and} \quad \partial_r \Psi_- > 0 \quad \text{on} \ M_{\text{out}}.
\]

(35)

The inequality \( \partial_r \Psi_+ > 0 \) is true for both signs if and only if, for \( v \) sufficiently large, the sum of Coriolis and centrifugal force is pointing in the direction of increasing \( r \) for co-rotating and counter-rotating observers. An equivalent condition is that the centrifugal force points in the direction of increasing \( r \) and dominates the Coriolis force for \( v \) sufficiently large. This is the situation we are familiar with from Newtonian physics. According to Proposition 1, however, in the Kerr-Newman spacetime this is true only in the region \( M_{\text{out}} \). In the interior of the intermediate
region \( K \) the direction of centrifugal-plus-Coriolis force for large \( v \) is reversed for counter-rotating observers while still normal for co-rotating observers. In the region \( M_{in} \), finally, it is reversed both for co-rotating and for counter-rotating observers.

The relevance of the sets \( M_{out} \), \( M_{in} \) and \( K \) in view of lightlike geodesics is demonstrated in the following proposition.

**Proposition 2.** (a) In the region \( M_{out} \), the radius coordinate \( r \) cannot have other extrema than strict local minima along a lightlike geodesic.

(b) In the region \( M_{in} \), the radius coordinate \( r \) cannot have other extrema than strict local maxima along a lightlike geodesic.

(c) Through each point of \( K \) there is a spherical lightlike geodesic. (Here “spherical” means that the geodesic is completely contained in a sphere \( r = \text{constant} \).)

**Proof.** Let \( X \) be a lightlike and geodesic vector field on \((M_+, g)\), i.e., \( g(X, X) = 0 \) and \( \nabla_X X = 0 \). To prove (a) and (b), we have to demonstrate that the implication

\[
X_r = 0 \Rightarrow X X r > 0
\]

is true at all points of \( M_{out} \) and that the implication

\[
X_r = 0 \Rightarrow X X r < 0
\]

is true at all points of \( M_{in} \). Here \( X_r \) is to be read as “the derivative operator \( X \) applied to the function \( r \)”. The condition \( \nabla_X X = 0 \) implies

\[
X X r = X dr(X) = X \left( \frac{\sqrt{\Delta}}{\rho} g(E_3, X) \right) = \frac{\sqrt{\Delta}}{\rho} g(\nabla_X E_3, X) + \left( X \frac{\sqrt{\Delta}}{\rho} \right) g(E_3, X),
\]

where we have used the basis vector field \( E_3 \) from (13) and (14). Using these orthonormal basis vector fields, we can write \( X \) in the form

\[
X = E_0 + \cos \alpha E_1 + \sin \alpha E_2
\]

at all points where \( X r = 0 \). (A non-zero factor of \( X \) is irrelevant because \( X \) enters quadratically into the right-hand side of (38).) Then (38) takes the form

\[
\frac{\rho \sqrt{\Delta}}{\sqrt{\Delta}} X X r = g(\nabla_{E_0} E_3, E_0) + \sin \alpha \left( g(\nabla_{E_1} E_3, E_0) + g(\nabla_{E_2} E_3, E_2) \right) + \cos \alpha \left( g(\nabla_{E_1} E_3, E_0) + g(\nabla_{E_2} E_3, E_1) \right) + \sin^2 \alpha g(\nabla_{E_2} E_3, E_2) + \cos^2 \alpha g(\nabla_{E_1} E_3, E_1) = g([E_0, E_3], E_0) + g([E_0, E_3], E_0) + g([E_0, E_3], E_1) + g([E_0, E_3], E_1) + \sin^2 \alpha g([E_2, E_3], E_2) + \cos^2 \alpha g([E_1, E_3], E_1).
\]

If we insert the Lie brackets from (15) we find

\[
\rho^4 X X r = 2 r \Delta - (r - m)^2 + 2 a r \sqrt{\Delta} \sin \vartheta \cos \alpha.
\]

Now we compare this expression with (30), (31) and (32). If \( \cos \alpha \) runs through all possible values from \(-1 \) to \( 1 \), the right-hand side of (31) stays positive on \( M_{out} \) and negative on \( M_{in} \). This proves part (a) and part (b). At each point of \( K \) there is exactly one value of \( \cos \alpha \) such that the right-hand side of (31) vanishes. This assigns to each point of \( K \) a lightlike direction such that the integral curves of the resulting direction field are spherical lightlike geodesics. This proves part (c).

In view of part (c) of Proposition 2 we refer to the closed region \( K \) as to the photon region of the exterior Kerr-Newman spacetime. Along each spherical lightlike geodesic in \( K \) the \( \vartheta \)-coordinate oscillates between extremal values \( \vartheta_0 \) and \( -\vartheta_0 \), corresponding to boundary points of \( K \), see Figure 2. The \( \varphi \)-coordinate either increases or decreases monotonically. In the Reissner-Nordström case \( a = 0 \), where (20) is a potential for the centrifugal force, the photon
region \( K \) shrinks to the photon sphere \( r = \frac{3}{2} m + \sqrt{\frac{9 m^2}{4} - 2 q^2} \) and Proposition reduces to the known fact that centrifugal force reversal takes place at the photon sphere.

We end this section with a word of caution as to terminology. In part (c) of Proposition we refer to the set \( r = \) constant as to a 'sphere'. This is indeed justified in the sense that, for each fixed \( t \), fixing the radius coordinate \( r \) gives a two-dimensional submanifold of \( M_+ \) that is diffeomorphic to the 2-sphere. Moreover, in our Figures and the sets \( r = \) constant are represented as (meridional cross-sections of) spheres. Note, however, that the Kerr-Newman metric does not induce an isotropic metric on these spheres (unless \( a = 0 \)), so they are not 'round spheres' in the metrical sense.

IV. MULTIPLE IMAGING IN THE KERR-NEWMAN SPACETIME

It is now our goal to discuss multiple imaging in the exterior Kerr-Newman spacetime \((M_+, g)\). To that end we fix a point \( p \) and a timelike curve \( \gamma \) in \( M_+ \) and we want to get some information about the past-pointing lightlike geodesics from \( p \) to \( \gamma \). The following proposition is an immediate consequence of Proposition

**Proposition 3.** Let \( p \) be a point and \( \gamma \) a timelike curve in the exterior Kerr-Newman spacetime. Let

\[
\Lambda : \ r_a < r < r_b
\]

denote the smallest spherical shell, with \( r_+ \leq r_a < r_b \leq \infty \), such that \( p, \gamma \) and the region \( K \) defined by are completely contained in \( \overline{\Lambda} = \text{closure of } \Lambda \) in \( M_+ \). Then all lightlike geodesics that join \( p \) and \( \gamma \) are confined within \( \overline{\Lambda} \).

**Proof.** Along a lightlike geodesic that leaves and re-enters \( \overline{\Lambda} \) the radius coordinate \( r \) must have either a maximum in the region \( M_{\text{out}} \) or a minimum in the region \( M_{\text{in}} \). Proposition makes sure that this cannot happen.

By comparison with Proposition we see that, among all spherical shells whose closures in \( M_+ \) contain \( p \) and \( \gamma \), the shell \( \Lambda \) of Proposition is the smallest shell such that at all points of the boundary of \( \Lambda \) in \( M_+ \) the gradient of \( \Psi_+ \) and the gradient of \( \Psi_- \) are pointing in the direction away from \( \Lambda \). Based on Proposition we will later see that there is a close relation between multiple imaging and centrifugal-plus-Coriolis force reversal in the Kerr-Newman spacetime.

Proposition tells us to what region the lightlike geodesics between \( p \) and \( \gamma \) are confined, but it does not tell us anything about the number of these geodesics. To answer the latter question, we now apply Theorem to the exterior Kerr-Newman spacetime \((M_+, g)\).

**Proposition 4.** Consider, in the exterior Kerr-Newman spacetime \((M_+, g)\), a point \( p \) and a smooth future-pointing timelike curve \( \gamma : \ ] \rightarrow M_+ \), with \( -\infty < \tau_0 < \infty \), which is parametrized by the Boyer-Lindquist time coordinate \( t \), i.e., the \( t \)-coordinate of the point \( \gamma(t) \) is equal to \( t \). Assume (i) that \( \gamma \) does not meet the caustic of the past light-cone of \( p \), and (ii) that for \( t \rightarrow -\infty \) the radius coordinate \( r \) of the point \( \gamma(t) \) remains bounded and bounded away from \( r_+ \). (The last condition means that \( \gamma(t) \) goes neither to infinity nor to the horizon for \( t \rightarrow -\infty \).) Then there is an infinite sequence \( (\lambda_n)_{n \in \mathbb{N}} \) of mutually different past-pointing lightlike geodesics from \( p \) to \( \gamma \). For \( n \rightarrow \infty \), the index of \( \lambda_n \) goes to infinity. Moreover, if we denote the point where \( \lambda_n \) meets the curve \( \gamma \) by \( \gamma(\tau_n) \), then \( \tau_n \rightarrow -\infty \) for \( n \rightarrow \infty \).

**Proof.** We want to apply Theorem to the exterior Kerr-Newman spacetime \((M_+, g)\). To that end, the first thing we have to find is an orthogonal splitting of the exterior Kerr-Newman spacetime that satisfies the metric growth condition. As in the original Boyer-Lindquist coordinates the \( t \)-lines are not orthogonal to the surfaces \( t = \) constant, we change to new coordinates

\[
x^1 = r, \quad x^2 = \vartheta, \quad x^3 = \varphi - u(r, \vartheta) t, \quad t = t,
\]

with

\[
u(r, \vartheta) = \frac{2 m a r}{\rho^2 \Delta + 2 m r (r^2 + a^2)}.
\]

Then the Kerr metric takes the orthogonal splitting form with

\[
\begin{aligned}
g_{ij}(x, t) dx^i dx^j &= \rho^2 \left( \frac{dr^2}{\Delta} + d\vartheta^2 \right) + \\
&+ \frac{\sin^2 \vartheta}{\rho^2} \left( (r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta \right) \left( t \left( \frac{\partial u(r, \vartheta)}{\partial r} dr + \frac{\partial u(r, \vartheta)}{\partial \vartheta} d\vartheta \right) + dx^3 \right)^2
\end{aligned}
\]
Proof. The sequence \((\lambda_n)_{n \in \mathbb{N}}\) of Proposition 4 gives us a sequence \((w_n)_{n \in \mathbb{N}}\) of mutually different lightlike vectors \(w_n \in T_p M_+\) with \(dt(w_n) = -1\) and a sequence \((s_n)_{n \in \mathbb{N}}\) of real numbers \(s_n \geq 0\) such that \( \exp_p(s_nw_n) \) is on \(\gamma\) for all 

**Proposition 5.** Let \(U\) be any open subset of \(M_+\) that contains the region \(K\) defined by \((31)\). Then, if the assumptions of Proposition 4 are satisfied, all but finitely many past-pointing lightlike geodesics from \(p\) to \(\gamma\) intersect \(U\).

**Proof.** The sequence \((\lambda_n)_{n \in \mathbb{N}}\) of Proposition 4 gives us a sequence \((w_n)_{n \in \mathbb{N}}\) of mutually different lightlike vectors \(w_n \in T_p M_+\) with \(dt(w_n) = -1\) and a sequence \((s_n)_{n \in \mathbb{N}}\) of real numbers \(s_n \geq 0\) such that \( \exp_p(s_nw_n) \) is on \(\gamma\) for all

Clearly, if we restrict the range of the coordinates \(x = (x^1, x^2, x^3)\) to a compact set, we can find positive constants \(A\) and \(B\) such that

\[
g_{ij}(x, t)v^iv^j f(x, t) \leq (A + B |t|)^2 \delta_{ij} v^iv^j.
\]

As \(F(t) = A + B |t|\) satisfies the integral condition \((3)\), this proves that our orthogonal splitting satisfies the metric growth condition. – Our assumptions on \(\gamma\) guarantee that we can find a curve \(\gamma' : \mathbb{R} \to M_+\) which, in terms of our orthogonal splitting, is of the form \(\gamma'(\tau) = (\beta'(\tau), \tau)\) such that \(\gamma'(\tau) = \gamma(\tau)\) for all \(|\tau - \infty, \tau_0|\), with some \(\tau_0 \in \mathbb{R}\). (Introducing \(\gamma'\) is necessary because \(\gamma\) need not be defined on all of \(\mathbb{R}\).) As \(\gamma\) does not meet the caustic of the past light-cone of \(p\), we may assure that \(\gamma'\) does not meet the caustic of the past light-cone of \(p\). As \(\gamma\) does not go to the horizon or to infinity for \(\tau \to -\infty\), the set \(\{\beta'(\tau) | -\infty < \tau < \tau_0\}\) is confined to a compact region. Hence, for every sequence \((\tau_i)_{i \in \mathbb{N}}\) with \(\tau_i \to -\infty\) the sequence \((\beta'(\tau_i))_{i \in \mathbb{N}}'\) must have a convergent subsequence. This shows that all the assumptions of Theorem 4 are satisfied if we replace \(\gamma\) with \(\gamma'\). Hence, the theorem tells us that \(N'_k \geq B_k\), where \(N'_k\) is the number of past-pointing lightlike geodesics with index \(k\) from \(p\) to \(\gamma'\) and \(B_k\) is the \(k\)-th Betti number of the loop space of \(M_+ \simeq S^2 \times \mathbb{R}^2\). As \(M_+ \simeq S^2 \times \mathbb{R}^2\) is simply connected but not contractible to a point, the theorem of Serre guarantees that \(B_k > 0\) and, thus, \(N'_k > 0\) for all but finitely many \(k \in \mathbb{N}\). Hence, for almost all positive integers \(k\) there is a past-pointing lightlike geodesic of index \(k\) from \(p\) to \(\gamma'\). This gives us an infinite sequence \((\lambda_n)_{n \in \mathbb{N}}\) of mutually different past-pointing lightlike geodesics from \(p\) to \(\gamma'\) such that the index of \(\lambda_n\) goes to infinity if \(n \to \infty\). We denote the point where \(\lambda_n\) meets the curve \(\gamma'\) by \(\gamma'(\tau_n)\). What remains to be shown is that \(\tau_n \to -\infty\) for \(n \to \infty\); as \(\gamma\) coincides with \(\gamma'\) on \(|\tau - \infty, \tau_0|\), this would make sure that all but finitely many \(\lambda_n\) arrive indeed at \(\gamma\). So we have to prove that it is impossible to select infinitely many \(\tau_n\) that are bounded below. By contradiction, assume that we can find a common lower bound for infinitely many \(\tau_n\). As the \(\tau_n\) are obviously bounded above by the value of the Boyer-Lindquist time coordinate at \(p\), this implies that the \(\tau_n\) have an accumulation point. Hence, for an infinite subsequence of our lightlike geodesics \(\lambda_n\), the end-points \(\gamma'(\tau_n)\) converge to some point \(q\) on \(\gamma'\). As \(\gamma'\) does not meet the caustic of the past light-cone of \(p\), the past light-cone of \(p\) is an immersed 3-dimensional lightlike submanifold near \(q\). We have thus found an infinite sequence of points \(\gamma'(\tau_n)\) that lie in a 3-dimensional lightlike submanifold and, at the same time, on a timelike curve. Such a sequence can converge to \(q\) only if all but finitely many \(\gamma'(\tau_n)\) are equal to \(q\). So there are infinitely many \(\lambda_n\) that terminate at \(q\). As there is only one lightlike direction tangent to the past light-cone of \(p\) at \(q\), all these infinitely many lightlike geodesics must have the same tangent direction at \(q\). As there are no periodic lightlike geodesics in the globally hyperbolic spacetime \((M_+, g)\), any two lightlike geodesics from \(p\) to \(q\) with a common tangent direction at \(q\) must coincide. This contradicts the fact that the \(\lambda_n\) are mutually different, so our assumption that there is a common lower bound for infinitely many \(\tau_n\) cannot be true.

The proof shows that in Proposition 4 the condition of \(\gamma(\tau)\) going neither to infinity nor to the horizon for \(\tau \to -\infty\) can be a little bit relaxed. It suffices to require that there is a sequence \((\tau_i)_{i \in \mathbb{N}}\) of time parameters with \(\tau_i \to -\infty\) for \(i \to \infty\) such that the spatial coordinates of \(\gamma(\tau_i)\) converge. This condition is mathematically weaker than the one given in the proposition, but there are probably no physically interesting situations where the former is satisfied and the latter is not.

Proposition 4 tells us that a Kerr-Newman black hole produces infinitely many images for an arbitrary observer, provided that the worldline of the light source satisfies some (mild) conditions. At the same time, this proposition demonstrates that the past light-cone of every point \(p\) in the exterior Kerr-Newman spacetime must have a non-empty and, indeed, rather complicated caustic; otherwise it would not be possible to find a sequence of past-pointing lightlike geodesics \(\lambda_n\) from \(p\) that intersect this caustic arbitrarily often for \(n\) sufficiently large. Please note that the last sentence of Proposition 4 makes clear that for the existence of infinitely many images it is essential to assume that the light source exists since arbitrarily early times.

In Proposition 4 we have shown that all lightlike geodesics from \(p\) to \(\gamma\) are confined to a spherical shell that contains the photon region \(K\). We can now show that, under the assumptions of Proposition 4, almost all past-pointing lightlike geodesics from \(p\) to \(\gamma\) come actually arbitrarily close to \(K\).
implication (36) holds on

$$M$$

close to the photon region

$$\lambda$$

must lie in

$$U$$

corresponds to a limit light ray

observer's celestial sphere. This follows immediately from the compactness of the 2-sphere. This accumulation point

This is possible only if

$$\lambda$$

comes arbitrarily close to

$$K$$

or has precisely one turning point. (This result can be deduced, e.g., from Calvani and Turolla [25]). Thus, the case

integrability. Then one can show that along a lightlike geodesic in

centrifugal-plus-Coriolis force. Stronger results are possible if one uses the explicit first-order form of the lightlike

of

has a minimum and a maximum in

minimum. In the first case, both the first and the second derivative of

has to meet

for

sufficiently large. As, by Proposition 2, such minima cannot lie in

the geodesic

λ

has to meet for

sufficiently large and we are done. Therefore, we may assume for the rest of the proof that

does not go to the horizon. So along

the coordinate

must either approach a limit value

or pass through a maximum and a minimum. In the first case, both the first and the second derivative of

must go to zero for

This is possible only if

comes arbitrarily close to

, because, as we know from the proof of Proposition 2, the implication 36 holds on

and the implication 37 holds on

. In the second case, again by Proposition 2, the maximum cannot lie in

and the minimum cannot lie in

, hence, both the maximum and the minimum must lie in

. In both cases we have, thus, found that

and hence all but finitely many

intersect

.

V. DISCUSSION AND CONCLUDING REMARKS

We have proven, with the help of Morse theory, in Proposition 4 that a Kerr-Newman black hole acts as a gravita-
tional lens that produces infinitely many images. We emphasize that we made only very mild assumptions on the
motion of the light source and that we considered the whole domain of outer communication, including the ergosphere.
For the sake of comparison, the reader may consult Section 7.2 of Masiello 24 where it is shown, with the help of
Morse theory, that a Kerr black hole produces infinitely many images. However, Masiello’s work is based on a special
version of Morse theory which applies to stationary spacetimes only; therefore he had to exclude the ergosphere from
the discussion, he had to require that the worldline of the light source is an integral curve of the Killing vector field
\( \partial_t \), and he had to restrict to the case of slowly rotating Kerr black holes, \( 0 < a^2 < a_0^2 \) with some \( a_0 \) that remained unspecified, instead of the whole range \( 0 \leq a^2 \leq m^2 \). On the basis of our Proposition 4 one can show that Masiello’s
\( a_0 \) is equal to \( m/\sqrt{2} \), this is the value of \( a \) where the photon region \( K \) reaches the ergosphere (see Figure 2), i.e. where
\( r_+^{ph} = 2m \). For a Kerr spacetime with \( m \geq a \geq m/\sqrt{2} \) we can find an event
and a t-line in \( M_+ \setminus \{ \text{ergosphere} \} \) that can be connected by only finitely many lightlike geodesics in
\( M_+ \setminus \{ \text{ergosphere} \} \).

If an observer sees infinitely many images of a light source, they must have at least one accumulation point on the
observer's celestial sphere. This follows immediately from the compactness of the 2-sphere. This accumulation point

corresponds to a limit light ray \( \lambda_\infty \). In the proof of Proposition 5 we have demonstrated that \( \lambda_\infty \) comes arbitrarily
close to the photon region \( K \) and that either \( \lambda_\infty \) approaches a sphere \( r = \text{constant} \) or the radius coordinate along \( \lambda_\infty \)
has a minimum and a maximum in \( K \). (In the extreme case \( a^2 + q^2 = m^2 \) the ray \( \lambda_\infty \) may go to the inner boundary
of \( M_+ \).) This is all one can show with the help of Morse theory and the qualitative methods based on the sign of
centrifugal-plus-Coriolis force. Stronger results are possible if one uses the explicit first-order form of the lightlike
geodesic equation in the Kerr-Newman spacetime, making use of the constants of motion which reflect complete
integrability. Then one can show that along a lightlike geodesic in \( M_+ \) the radius coordinate is either monotonous
or has precisely one turning point. (This result can be deduced, e.g., from Calvani and Turolla 24). Thus, the case
that there is a minimum and a maximum in \( K \) is, actually, impossible. As a consequence, the limit light ray \( \lambda_\infty \)
necessarily approaches a sphere \( r = \text{constant} \). By total integrability it must then approach a lightlike geodesic with
the same constants of motion. Of course, this must be one of the spherical geodesics in \( K \). (In the extreme case
\( a^2 + q^2 = m^2 \) the limit ray \( \lambda_\infty \) may approach the circular light ray at \( r^{ph} = m \) which is outside of \( M_+ \).)

Also, it follows from Proposition 4 that the limit curve \( \lambda_\infty \) meets the caustic of the past light cone of \( p \) infinitely
many times. This gives, implicitly, some information on the structure of the caustic. For the Kerr case, \( q = 0 \), it was
shown numerically by Rauch and Blandford 20 that the caustic consists of infinitely many tubes with astroid cross
sections. This result was supported by recent analytical results by Bozza, de Luca, Scarpetta, and Sereno 26.

We have shown, in Proposition 5 that all lightlike geodesics connecting an event \( p \) to a timelike curve \( \gamma \) in the
exterior Kerr-Newman spacetime \( M_+ \) are confined to the smallest spherical shell that contains \( p \), \( \gamma \) and the photon
region \( K \). If \( \gamma \) satisfies the assumptions of Proposition 5, which guarantees infinitely many past-pointing lightlike
geodesics from \( p \) to \( \gamma \), Proposition 5 tells us that all but finitely many of them come arbitrarily close to the photon
Thus, our result that a Kerr-Newman black hole produces infinitely many images is crucially related to the existence of the photon region. If we restrict to some open subset of \( M_{\pm} \) whose closure is completely contained in either \( M_{\text{out}} \) or \( M_{\text{in}} \), then we are left with finitely many images for any choice of \( p \) and \( \gamma \). In Section III we have seen that the decomposition of \( M_{\pm} \) into \( M_{\text{in}} \) and \( M_{\text{out}} \) and the photon region \( K \) plays an important role in view of centrifugal-plus-Coriolis force reversal; if we restrict to an open subset of \( M_{\pm} \) that is contained in either \( M_{\text{out}} \) or \( M_{\text{in}} \), then we are left with a spacetime on which \( \partial_r \Psi_+ \) and \( \partial_r \Psi_- \) have the same sign, i.e., the centrifugal-plus-Coriolis force for large velocities points either always outwards or always inwards. In an earlier paper [28] we have shown that in a spherically symmetric and static spacetime the occurrence of gravitational lensing with infinitely many images is equivalent to the occurrence of centrifugal force reversal. Our new results demonstrate that the same equivalence is true for subsets of the exterior Kerr-Newman spacetime, with the only difference that instead of the centrifugal force alone now we have to consider the sum of centrifugal and Coriolis force in the limit \( v \to 1 \). It is an interesting problem to inquire whether this observation carries over to other spacetimes with two commuting Killing vector fields \( \partial_t \) and \( \partial_\phi \) that span timelike 2-surfaces with cylindrical topology.

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FIG. 1: The surfaces $\Psi_+ = \text{constant}$ (top) and $\Psi_- = \text{constant}$ (bottom) are drawn here for the case $q = 0$ and $a = 0.5 \text{ m}$. The picture shows the (half-)plane $(\varphi, t) = \text{constant}$, with $r \sin \vartheta$ on the horizontal and $r \cos \vartheta$ on the vertical axis. The spheres of radius $r_{ph}^+$ and $r_{ph}^-$ are indicated by dashed lines; they meet the equatorial plane in the photon circles. The boundary of the ergosphere coincides with the surface $\Psi_- = 0$ and is indicated in the bottom figure by a thick line; it meets the equatorial plane at $r = 2m$. 
FIG. 2: The regions $M_{in}$, $K$ and $M_{out}$ defined in Proposition 1 are shown here for the case $q = 0$ and $a = 0.5 m$. Again, as in Figure 1 we plot $r \sin \vartheta$ on the horizontal and $r \cos \vartheta$ on the vertical axis. Some of the spherical lightlike geodesics that fill the photon region $K$ are indicated. $K$ meets the equatorial plane in the photon circles at $r = r_{ph}^+$ and $r = r_{ph}^-$ and the axis at radius $r_c$ given by $r_c^3 - 3 r_c^2 m + r_c \left( a^2 + 2 q^2 \right) + a^2 m = 0$. This picture can also be found as Figure 21 in the online article [29].