Uniform and Bernoulli measures on the boundary of trace monoids

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Abstract

Trace monoids and heaps of pieces appear in various contexts in combinatorics. They are also natural models to describe the executions in asynchronous systems. The presence of commuting pieces and the absence of a global clock make it challenging to add a probabilistic layer to the model. We introduce and study the class of Bernoulli probability measures, among which the uniform ones, that we claim to be the simplest adequate measures on infinite traces. To explicit them, we strongly rely on trace combinatorics with the Möbius polynomial in the key role.

An elementary challenge with no elementary solution

Let the axis $y = 0$ of an Euclidean plane form the ground. Consider the family of heaps obtained as follows: arbitrary many occurrences of three types of $2 \times 1$ dominoes, or pieces, named $a, b, c$ and centered respectively on $x = 1, x = 3, x = 2$, fall vertically until they reach the ground or get blocked by previously piled up pieces. Allow for heaps constructed with infinitely many pieces. How would you choose an infinite heap uniformly at random? More precisely, is there a probability measure on the set of infinite heaps such that for any two finite heaps $u$ and $v$ with the same number of pieces, the probabilities of seeing $u$ or $v$ at the bottom are the same? We claim that obvious attempts fail to achieve this goal.

A first attempt consists in choosing a piece in $\{a, b, c\}$ with probabilities $p_a, p_b, p_c$, to let it fall, and to redo this independently and repeatedly. We invite the reader to check by hand that the probability measure thus induced on infinite heaps does not have the uniform property, whatever the choice of $p_a, p_b, p_c$.

A second attempt consists in choosing, for each integer $n \geq 0$, one heap at random and uniformly among all heaps of $n$ pieces. But then, if $\mu_n$ denotes the uniform distribution on heaps of $n$ pieces, the sequence $(\mu_n)_{n \geq 0}$ is not consistent. Indeed, consider the heaps of size 1 below on the left, of which there are exactly three, and the heaps of size 2 on the right, of which there are exactly eight:

\begin{itemize}
  \item $a$
  \item $b$
  \item $c$
\end{itemize}

\begin{itemize}
  \item $a$
  \item $b$
  \item $c$
\end{itemize}

\begin{itemize}
  \item $a$
  \item $b$
  \item $c$
\end{itemize}

\begin{itemize}
  \item $a$
  \item $b$
  \item $c$
\end{itemize}
With obvious notations for heaps, the consistency condition already fails for \((\mu_1, \mu_2)\):
\[
\frac{1}{3} = \mu_1(a) \neq \mu_2(aa) + \mu_2(ab) + \mu_2(ac) = 3 \times \frac{1}{8}.
\]

There is therefore no probability measure on infinite heaps that induces the family \((\mu_n)_{n\geq 0}\). The second attempt has failed.

We seemingly face the Cornelian dilemma of choosing between consistent but non-uniform probabilities (first attempt), or uniform but non-consistent probabilities (second attempt).

By specializing the theory developed in the paper, we get the existence of a natural solution to this problem. It relies on trace combinatorics with the Möbius polynomial in the leading role.

The dilemma is solved by playing on a variable which was thought to be fixed: the total mass of finite marginals. Indeed, the uniform probability that we construct on infinite heaps induces a uniform measure on heaps of a given length whose mass exceeds 1. As for the consistency conditions, they do not hold and they are replaced by compatibility conditions based on the inclusion-exclusion principle.

**Introduction**

Trace monoids are finitely presented monoids with commutation relations between some generators, that is to say, relations of the form \(a \cdot b = b \cdot a\).

Trace monoids have first been studied in combinatorics under the name of partially commutative monoids [10], before attracting a lasting interest from the computer science community since they constitute a simple, yet non-trivial, model of concurrent systems [21]. In this context, the commutation relations between generators are interpreted as the co-occurrence of parallel actions, typically parallel access to distributed databases or parallel events in networked systems. Textbooks on the subject are available [12,13], and the domain is still very active both in combinatorics and computer science [4,9].

The elements of a trace monoid are called traces. They can be seen as an extended notion of words, where some letters are allowed to commute with each other. Traces carry several notions which are transposed from words. In particular, traces have a natural notion of length, the number of letters, and they are partially ordered by the prefix relation. A trace monoid can be embedded into a compact metric space where the boundary elements are infinite traces, which play the same role with respect to traces than infinite words play with respect to words.

This paper introduces and studies a class of probability measures on traces. Let \(\mathcal{M}\) be a trace monoid and let \(\partial\mathcal{M}\) be the corresponding set of infinite traces. The **elementary cylinder** \(\uparrow u\) associated with a trace \(u \in \mathcal{M}\) is defined as the set of infinite traces for which \(u\) is a possible finite prefix. A randomization of traces is defined by a probability measure \(\mathbb{P}\) on the space \((\partial\mathcal{M}, \mathcal{F})\), where \(\mathcal{F}\) is the \(\sigma\)-algebra generated by the elementary cylinders. A **Bernoulli** measure is defined as a probability measure satisfying the following *memoryless* property:

\[
\mathbb{P}(\uparrow (u \cdot v)) = \mathbb{P}(\uparrow u) \cdot \mathbb{P}(\uparrow v).
\]

(1)
A Bernoulli measure is entirely determined by the parameters \((p_\alpha)_{\alpha \in \Sigma}\) where \(\Sigma\) is the set of generators and \(p_\alpha = P(\uparrow \alpha)\) is the elementary probability of \(\alpha\). Note that the occurrence of \(\alpha\) does not prevent a parallel letter \(\beta\) to occur concurrently, and the existence of such asynchronous events is characteristic of the model. If the trace monoid is actually a free monoid, we recover the classical Bernoulli measures on infinite words, where the letters occur successively without parallelism.

The case of uniform measures provides a specially interesting situation, where all \(p_\alpha\) are equal, hence, where the cylinders \(\uparrow u\) are given the same probability when \(u\) ranges over traces of the same length.

It is not obvious that probability measures satisfying (1) should exist. The difficulty to overcome is that the usual notion of time has vanished: for two parallel actions, associated with commuting letters, it makes no sense to say that one has occurred before the other. This is reflected by the fact that, for two different traces \(u\) and \(v\) of the same length, the cylinders \(\uparrow u\) and \(\uparrow v\) are not necessarily disjoint. For instance, in the example of the introductory challenge, we have: \((\uparrow a) \cap (\uparrow b) = (\uparrow ab)\).

Natural questions are: do Bernoulli measures exist? And among them, do uniform measures exist? Put differently: what are the conditions on the parameters \((p_\alpha)_{\alpha \in \Sigma}\) to determine a Bernoulli measure? Expliciting the question for the case of uniform measures: what is the condition on \(p\) to determine a uniform probability measure \(P\) of the form \(P(\uparrow u) = p^{|u|}\) for any trace \(u\), where \(|u|\) is the length of \(u\)? Furthermore, provided that Bernoulli and uniform measures do exist, can effective realizations of them be given?

**Results.** We answer the above questions for irreducible trace monoids, those whose dependence graph is connected. First, Bernoulli measures always exist. The parameters \((p_\alpha)_{\alpha \in \Sigma}\) need to obey normalization conditions associated with the Möbius polynomials of the structure. In particular, we prove the existence and uniqueness of the uniform measure which is given by \(P_0(\uparrow u) = p_0^{|u|}\) where \(p_0\) is the unique root of smallest modulus of the Möbius polynomial. We also obtain the following metric characterization of the uniform measure: if a measure \(P\) on the boundary of a trace monoid is such that \(P(\uparrow u)\) only depends on the length \(|u|\), then \(P\) is the above measure, that is \(P = P_0\).

Furthermore, we establish a realization theorem by proving that Bernoulli measures correspond to some particular Markov chains on the Cartier-Foata decomposition of traces. The transition matrix of the Markov chain is explicitly given in function of the parameters \((p_\alpha)_{\alpha \in \Sigma}\) defining the Bernoulli measure. The relationship between the intrinsic parameters \((p_\alpha)_{\alpha \in \Sigma}\) and the transition matrix is based on a general Möbius transform in the sense of Rota [23].

The Möbius polynomial appears in all the results. This is coherent with the long standing view that the Möbius polynomial captures global information on the trace monoid. A striking result in this vein is that the number of traces of length \(n\) is of order \((1/p_0)^n\). Therefore it is natural for \(p_0^{|u|}\) to play a role as a normalization factor for the uniform measure. The really unexpected result is that \(p_0^{|u|}\) corresponds exactly to a probability measure. But the theory that we develop also highlights a local and more precise combinatorial role of the Möbius polynomial. As we have to replace the usual consistency relation between finite marginals by the more general inclusion-exclusion principle, the
non-trivial terms that arise when developing the associated expression are shown to be weighted by the coefficients of the Möbius polynomial.

The uniform measure that we construct is closely related to two classical objects: the Parry measure and the Patterson-Sullivan measure. The *Parry measure* is the measure of maximal entropy on a sofic subshift, that is, roughly, the “uniform” measure on the “infinite paths” in a finite automaton \[19\]. Traces can be represented by their Cartier-Foata decompositions which are recognized by a finite automaton having an associated Parry measure. The limitation in this approach is that the link with the combinatorics of the trace monoid remains hidden in the construction. In a sense, our results reveal the inherent combinatorial structure of the Parry measure. The *Patterson-Sullivan measure* is also a uniform measure, which is classically constructed on the “boundary at infinity” of some geometric groups \[17\]. The proof of its existence is non-constructive and is based on the Poincaré series of the group, which, in the context of the trace monoid, is simply \( \sum_{u \in \mathcal{M}} z^{|u|} \). Using that the Poincaré series of \( \mathcal{M} \) is the inverse of the Möbius polynomial, we get an explicit and combinatorial identification of the Patterson-Sullivan measure for trace monoids.

**Motivations.** First, traces and heaps of pieces are ubiquitous in combinatorics, see \[27\]. Performing random uniform sampling of traces is therefore a natural issue. For instance, the goal may be to evaluate the *speedup*, that is the asymptotic average density of heaps \[6, 18\]. Our results enable to tackle these questions. Using the realization by Markov chains, we get efficient uniform sampling algorithms. As for the computation of the speedup, our construction greatly simplifies the approaches in \[6, 18\].

Second, in concurrency theory, it is relevant to add a probabilistic layer to asynchronous models. Indeed, asynchronous systems are known to suffer from the “state-space explosion” problem, a central issue in the highly investigated field of verification of critical systems. The randomization can be used for the design of testing procedures by providing quantitative guarantees for the fair exploration of trajectories. This is known as statistical model checking. Classical approaches often and roughly consist in translating the asynchronous model into a sequential one and randomizing it using Markov processes. Among the existing formalisms, let us quote Rabin’s probabilistic automata \[22\], probabilistic process algebra \[19\], or stochastic Petri nets \[15\]. We follow a completely orthogonal approach: our Bernoulli measures are directly defined on the asynchronous model and do not rely on a sequentialization of events.

Last, from a mathematical perspective, our results provide the first discrete framework, outside trees \[11\], where the Patterson-Sullivan measure is explicitly identified. This measure is different from the ones induced by random walks which cannot be explicitly described \[26\].

**Organization of the paper.** The paper is organized in three parts. Part I exposes the framework and the results. It contains the statements of the main results but no proofs. Within Part I, Section 6 illustrates the results through a study of two concrete examples; and Section 7 shows an application of our results to the computation of the “speedup” of trace monoids. Part II introduce auxiliary tools. Part III is devoted to the proofs of the results stated in Part I. Last, the concluding Section 15 provides perspectives for future work.
Part I
Framework and results

1. Trace monoids and their boundary

An independence pair is an ordered pair $(\Sigma, I)$, where $\Sigma$ is a finite set, referred to as the alphabet and whose elements are called letters, and $I \subset \Sigma \times \Sigma$ is an irreflexive and symmetric binary relation on $\Sigma$. To each independence pair is attached another ordered pair $(\Sigma, D)$, called the associated dependence pair, where $D$ is defined by $D = (\Sigma \times \Sigma) \setminus I$, which is a symmetric and reflexive relation on $\Sigma$. Two letters $\alpha, \beta \in \Sigma$ such that $(\alpha, \beta) \in I$ are said to be parallel, denoted by $\alpha \parallel \beta$.

To each independence pair $(\Sigma, I)$ is associated the finitely presented monoid $M(\Sigma, I) = \langle \Sigma \mid \alpha \cdot \beta = \beta \cdot \alpha \text{ for } (\alpha, \beta) \in I \rangle$.

Denoting by $\Sigma^*$ the free monoid generated by $\Sigma$, the monoid $M = M(\Sigma, I)$ is thus the quotient monoid $\Sigma^*/\mathcal{R}$, where $\mathcal{R}$ is the congruence relation on $\Sigma^*$ generated by $(\alpha \beta, \beta \alpha)$, for $(\alpha, \beta)$ ranging over $I$. Such a monoid $M$ is called a trace monoid, and its elements are called traces. The concatenation in $M$ is denoted by the dot “·”, the unit element in $M$, the empty trace, is denoted 0. The trace monoid $M$ is said to be non-trivial if $\Sigma \neq \emptyset$. By convention, we only consider non-trivial trace monoids throughout the paper, even if not specified.

Viennot’s heap of pieces interpretation of trace monoids [27] connects them with the introductory challenge. In this interpretation, a trace is identified with the heap obtained from any representative word as follows: each letter corresponds to a piece that falls vertically until it is blocked; a letter is blocked by all other letters but the ones which are parallel to it. We illustrate this in Figure 1 for the example monoid $M_1 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$. The heap model of $M_1$ is the very one considered in the introductory challenge.

![Figure 1: Two congruent words and the resulting heap (trace) for $M_1$](image)
The trace monoid $\mathcal{M}$ is said to be *irreducible* whenever the associated dependence pair $(\Sigma, D)$, as an undirected graph, is connected.

Note that a given trace monoid $\mathcal{M}(\Sigma, I)$ determines the independence pair $(\Sigma, I)$ up to isomorphism, and hence the dependence pair $(\Sigma, D)$ as well, which makes the definition of an irreducible trace monoid meaningful.

The notion of independence clique is central in the combinatorics of trace monoids. An element $c \in \mathcal{M}$ is said to be a clique if it is of the form $c = \alpha_1 \cdots \alpha_n$ for some integer $n \geq 0$ and for some letters $\alpha_1, \ldots, \alpha_n \in \Sigma$ such that $i \neq j \Rightarrow (\alpha_i, \alpha_j) \in I$. The set of cliques is denoted $\mathcal{C}_\mathcal{M}$, or simply $\mathcal{C}$. The set $\mathcal{C} \setminus \{0\}$ of non-empty cliques is denoted $\mathcal{C}_\mathcal{M}$, or $\mathcal{C}$. Noting that a letter may occur at most once in any representative word of a clique, we identify a clique with the set of letters occurring in any of its representative words. In the heap representation, each layer of a heap is a clique.

Two cliques $c, c' \in \mathcal{C}$ are said to be parallel whenever $c \times c' \subseteq I$, which is denoted $c \parallel c'$. This relation extends to cliques the parallelism relation defined on letters. For each clique $c \in \mathcal{C}$, we consider the sub-monoid $\mathcal{M}_c \subseteq \mathcal{M}$ defined as follows:

$$\Sigma_c = \{ \alpha \in \Sigma : \alpha \parallel c \}, \quad I_c = I \cap (\Sigma_c \times \Sigma_c), \quad \mathcal{M}_c = \mathcal{M}(\Sigma_c, I_c). \quad (2)$$

The length of a trace $u \in \mathcal{M}$ is defined as the length of any of its representative words in the free monoid $\Sigma^*$, and is denoted by $|u|$. Obviously, length is additive on traces, and 0 is the unique trace of length 0. The length of a trace corresponds to the number of pieces in the associated heap.

We consider the left divisibility relation of $\mathcal{M}$, denoted $\leq$, and defined by:

$$\forall u, v \in \mathcal{M}, \quad u \leq v \iff \exists w \in \mathcal{M} \quad v = u \cdot w.$$  

Trace monoids are cancellative [10]. This justifies the notation $v - u$ to denote the unique trace $w \in \mathcal{M}$ such that $v = u \cdot w$ whenever $u \leq v$ holds. The two properties mentioned above for the length of traces imply that $(\mathcal{M}, \leq)$ is a partial order.

Informally, infinite traces correspond to heaps with a countably infinite number of pieces. Following [11], a formal way to define infinite traces associated to $\mathcal{M}$ is to consider the completion of $\mathcal{M}$ with respect to least upper bound (l.u.b.) of non-decreasing sequences in $(\mathcal{M}, \leq)$. Say that a sequence $(u_k)_{k \geq 0}$ is non-decreasing in $\mathcal{M}$ if $u_k \leq u_{k+1}$ holds for all integers $k \geq 0$. Let $(\mathcal{H}, \preceq)$ be the pre-ordered set of all non-decreasing sequences in $\mathcal{M}$ equipped with the Egli-Milner pre-ordering relation, defined as follows:

$$(u_k)_{k \geq 0} \preceq (u'_k)_{k \geq 0} \iff \forall k \geq 0 \quad \exists k' \geq 0 \quad u_k \leq u'_{k'}.$$  

Finally, let $(\mathcal{W}, \preceq)$ be the collapse partial order associated with $(\mathcal{H}, \preceq)$. The elements of $\mathcal{W} = \mathcal{W}(\Sigma, I)$ are called generalized traces. Intuitively, any non-decreasing sequence in $\mathcal{M}$ defines a generalized trace, and two such sequences are identified whenever they share the same l.u.b. in a universal l.u.b.-completion of $\mathcal{M}$. In particular, there is a natural embedding of partial orders $\iota : \mathcal{M} \rightarrow \mathcal{W}$ which associates to each trace $u \in \mathcal{M}$ the generalized trace represented by the constant sequence, equal to $u$. In the heap model, generalized traces correspond to heaps with countably many pieces, either finitely or infinitely many.
By construction, any generalized trace is the l.u.b. in $W$ of a non-decreasing sequence of traces in $M$. Furthermore, $(W, \leq)$ is shown to be closed with respect to l.u.b. of non-decreasing sequences, and also to enjoy the following compactness property: for every trace $u \in M$ and for every non-decreasing sequence $(u_k)_{k \geq 0}$ in $M$ such that $\bigvee \{u_k : k \geq 0\} \geq u$ holds in $W$, there exists an integer $k \geq 0$ such that $u_k \geq u$ holds in $M$. This property is used to reduce problems concerning generalized traces to problems concerning traces.

The boundary of $M$ is defined as the measurable space $(\partial M, \mathcal{F})$. The set $\partial M$ is defined by $\partial M = W \setminus M$, the set of infinite traces. For any trace $u \in M$, the elementary cylinder of base $u$ is the non-empty subset of $\partial M$ defined by $\uparrow u = \{\xi \in \partial M : u \leq \xi\}$; and $\mathcal{F}$ is the $\sigma$-algebra on $\partial M$ generated by the countable collection of all elementary cylinders.

2. Finite measures on the boundary

In this section, we point out two basic facts which are valid for any finite measure on the boundary of a trace monoid $M$.

First, it is known, see [5, p.150], that any two traces $u, v \in M$ have a l.u.b. $u \lor v$ in $M$ if and only if there exists a trace $w \in M$ such that $u \leq w$ and $v \leq w$, in which case $u$ and $v$ are said to be compatible. Using the compactness property mentioned above, we deduce:

$$\forall u, v \in M \quad \uparrow u \cap \uparrow v = \begin{cases} \uparrow (u \lor v), & \text{if } u \text{ and } v \text{ are compatible}, \\ \emptyset, & \text{otherwise}. \end{cases}$$

(3)

In particular, two different elementary cylinders $\uparrow u$ and $\uparrow v$ may have a non-empty intersection, even if $u$ and $v$ have the same length. For instance, for the monoid $M_1 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$ of the introductory challenge, we have the identity $\uparrow a \cap \uparrow b = \uparrow (a \cdot b)$.

It follows from (3) that the collection of elementary cylinders, to which is added the empty set, forms a $\pi$-system which generates $\mathcal{F}$. This implies that a measure of finite mass on $(\partial M, \mathcal{F})$ is entirely determined by its values on elementary cylinders.

Second, we highlight a relation satisfied by any finite measure on the boundary (proof postponed to §12).

• Proposition 2.1—Let $\lambda$ be a measure of finite mass defined on the boundary $(\partial M, \mathcal{F})$ of a trace monoid $M$. Then:

$$\forall u \in M \quad \sum_{c \in \Sigma_M} (-1)^{|c|} \lambda(\uparrow (u \cdot c)) = 0.$$  

(4)

If $M = \Sigma^*$ is a free monoid, corresponding to the trivial independence relation $I = \emptyset$, then the only non-empty cliques are the letters of the alphabet $\Sigma$. In this case, if $\lambda_k$ is the marginal distribution of $\lambda$ on the words of length $k \geq 0$, the relation (4) is equivalent to $\lambda_k(u) = \sum_{\alpha \in \Sigma} \lambda_{k+1}(u \cdot \alpha)$, the usual consistency relation between marginals. For general trace monoids however, the sum in (4) contains terms for cliques of length $\geq 2$. This relates with [6].

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3. Valuations and Bernoulli measures

Our central object of study is introduced in the following definition. Throughout the paper, $\mathbb{R}_+^*$ denotes the set of positive reals.

**Definition 3.1**—Let $\mathcal{M}$ be a trace monoid. We say that a probability measure $\mathbb{P}$ on $(\partial \mathcal{M}, \mathcal{F})$ is a Bernoulli measure if it satisfies:

$$\forall u \in \mathcal{M} \quad \mathbb{P}(\uparrow u) > 0,$$

$$\forall u, v \in \mathcal{M} \quad \mathbb{P}(\uparrow (u \cdot v)) = \mathbb{P}(\uparrow u)\mathbb{P}(\uparrow v).$$

(5) (6)

The characteristic numbers of $\mathbb{P}$ are defined by $p_\alpha = \mathbb{P}(\uparrow \alpha)$ for $\alpha \in \Sigma$.

A Bernoulli measure $\mathbb{P}$ is entirely determined by its characteristic numbers, since, by (6), the value of $\mathbb{P}$ on all elementary cylinders is determined by the characteristic numbers. The characteristic numbers appear thus as the natural family of parameters of a Bernoulli measure.

The main property of Bernoulli measures, condition (6), corresponds to a memoryless property on traces. Note that, if $\mathcal{M} = \Sigma^*$ is a free monoid, then $(\partial \mathcal{M}, \mathcal{F})$ is the standard sample space of infinite sequences with values in $\Sigma$, and measures satisfying (6) are indeed the standard Bernoulli measures corresponding to i.i.d. processes.

Condition (5) is there for convenience and does not involve any loss of generality. Indeed, it will be satisfied when restricting ourselves to the submonoid generated by those $\alpha \in \Sigma$ such that $p_\alpha > 0$.

Say that a function $f : \mathcal{M} \to \mathbb{R}_+^*$ which satisfies:

$$\forall u, v \in \mathcal{M} \quad f(u \cdot v) = f(u)f(v),$$

(7)

is a valuation, and we insist that $f$ only takes positive values. The numbers defined by $p_\alpha = f(\alpha)$ are called the characteristic numbers of the valuation, and it is readily seen that for any family of positive numbers $(q_\alpha)_{\alpha \in \Sigma}$, there exists a unique valuation with $(q_\alpha)_{\alpha \in \Sigma}$ as characteristic numbers. By definition, if $\mathbb{P}$ is a Bernoulli measure on $(\partial \mathcal{M}, \mathcal{F})$, then the function $u \in \mathcal{M} \mapsto \mathbb{P}(\uparrow u)$ is a valuation, that is said to be induced by $\mathbb{P}$.

We recall next the notion of Möbius polynomial and the notion of Möbius transform of functions. The general notion of Möbius transform for partial orders has been introduced by Rota [23], and we particularize it to trace monoids.

Considering a trace monoid $\mathcal{M}$ and any real-valued function $f : \mathcal{C} \to \mathbb{R}$, the Möbius transform of $f$ is the function $h : \mathcal{C} \to \mathbb{R}$ defined by:

$$h(c) = \sum_{c' \in \mathcal{C} : c \leq c'} (-1)^{|c'|-|c|} f(c').$$

(8)

By convention, if $f : \mathcal{M} \to \mathbb{R}_+^*$ is a valuation, the Möbius transform of $f$ is defined as the Möbius transform of its restriction $f|_\mathcal{C}$.

For each letter $\alpha \in \Sigma$, let $X_\alpha$ be a formal indeterminate, and let $\mathbb{Z}[\Sigma]$ be the ring of polynomials over $(X_\alpha)_{\alpha \in \Sigma}$. The multi-variate Möbius polynomial associated to $(\Sigma, I)$ is $\mu_{\mathcal{M}} \in \mathbb{Z}[\Sigma]$ defined by:

$$\mu_{\mathcal{M}} = \sum_{c \in \mathcal{C}_{\mathcal{M}}} (-1)^{|c|} \prod_{\alpha \in c} X_\alpha.$$

(9)
The evaluation of the polynomial $\mu_M$ over a family of real numbers $(p_\alpha)_{\alpha \in \Sigma}$ is obtained by substituting the real numbers $p_\alpha$ to the indeterminates $X_\alpha$ in the above expression. The result is denoted $\mu_M((p_\alpha)_{\alpha \in \Sigma})$.

When considering a valuation $f : M \to R_+^*$, with characteristic numbers $(p_\alpha)_{\alpha \in \Sigma}$, the Möbius transform $h$ of $f$ has the following simple expression involving the evaluation of Möbius polynomials:

$$\forall c \in \mathcal{C} \quad h(c) = f(c) \mu_M_c((p_\alpha)_{\alpha \in \Sigma}),$$

where $M_c$ is the sub-monoid defined in (2). The expression (10) derives immediately from the change of variable $c' = c \cdot \delta$, for $\delta$ ranging over $\mathcal{C}_M$, in the defining sum (8) for $h(c)$, and using the multiplicative property (7). Two particular instances of (10) shall be noted: for $c = 0$, we obtain $h(0) = \mu_M((p_\alpha)_{\alpha \in \Sigma})$ since $f(0) = 1$, and if $c \in \mathcal{C}_M$ is maximal then $h(c) = f(c)$ since $\mathcal{C}_M = \{0\}$.

**Definition 3.2**—Let $M = M(\Sigma, I)$ be a trace monoid. A valuation $f : M \to R_+^*$ is a Möbius valuation if its Möbius transform $h : \mathcal{C} \to M$ satisfies the following two conditions:

(a) $h(0) = 0$,  

(b) $\forall c \in \mathcal{C}_M \quad h(c) > 0$.

Equivalently, if $(p_\alpha)_{\alpha \in \Sigma}$ are the characteristic numbers of $f$, then $f$ is a Möbius valuation if and only if:

$$\begin{align*}
(a) & \quad \mu_M((p_\alpha)_{\alpha \in \Sigma}) = 0, \\
(b) & \quad \forall c \in \mathcal{C}_M \quad \mu_M((p_\alpha)_{\alpha \in \Sigma}) > 0.
\end{align*}$$

Our first main result transfers the initial problem of determining Bernoulli measures to the new problem of determining Möbius valuations.

**Theorem 3.3**—Let $M(\Sigma, I)$ be an irreducible trace monoid. Then:

1. The valuation induced by any Bernoulli measure on the boundary of $M$ is a Möbius valuation.

2. If $f : M \to R_+^*$ is a Möbius valuation, there exists a unique Bernoulli measure on $(\partial M, \mathcal{F})$ such that $P(\uparrow u) = f(u)$ for all $u \in M$.

Although Theorem 3.3 provides valuable information, it does not state the existence of Möbius valuations—and thus of Bernoulli measures. We will give a positive result on this crucial point in §5.

The basic relations (11), valid for any finite measure, when applied to a Bernoulli measure $P$, reduce to the following:

$$\forall u \in M \quad \sum_{c \in \mathcal{C}} (-1)^{|c|} P(\uparrow (u \cdot c)) = 0,$$

hence

$$\sum_{c \in \mathcal{C}} (-1)^{|c|} P(\uparrow c) = 0.$$

Developing each $P(\uparrow c)$ as a product of characteristic numbers $p_\alpha$ for $\alpha$ ranging over $c$, yields the relation $\mu_M((p_\alpha)_{\alpha \in M}) = 0$, proving that point (a) in (12) is a necessary condition for $P$ to be a Bernoulli measure. This is the only elementary part in the proof of Theorem 3.3, the rest of the proof is postponed to §5 for point 1 and to §4 for point 2.
4. Cartier-Foata subshift

We introduce now a subshift of finite type based on the Cartier-Foata decomposition of traces. This is the starting point for a realization result in which Bernoulli measures are described as the law of the trajectories of a Markov chain.

Let $\mathcal{M} = \mathcal{M} (\Sigma, I)$ be a trace monoid, and let $D \subseteq \Sigma \times \Sigma$ be the associated dependence relation. A pair $(c, c') \in C_M \times C_M$ of cliques is said to be Cartier-Foata admissible, denoted $c \rightarrow c'$, if:

$$\forall \beta \in c' \exists \alpha \in c \ (\alpha, \beta) \in D.$$ 

For every non-empty trace $u \in M$, there exists a unique integer $n \geq 1$ and a unique sequence of non-empty cliques $(c_1, \ldots, c_n)$ such that $u = c_1 \cdot \ldots \cdot c_n$ and $c_i \rightarrow c_{i+1}$ holds for all $i$ in $\{1, \ldots, n-1\}$. This sequence of cliques, denoted $c_1 \rightarrow \ldots \rightarrow c_n$, is called the Cartier-Foata decomposition or Cartier-Foata normal form of $u$ [10, 27]. In the heap interpretation, this sequence of cliques corresponds to the successive layers of pieces in the heap.

The Cartier-Foata subshift of $\mathcal{M}$ is the subshift of finite type associated with the graph $(C_C, \rightarrow)$. See for instance Figure 2 in §6. The sample space associated with the subshift is the measurable space $(\Omega, G)$ corresponding to the infinite paths in the graph $(C_C, \rightarrow)$. Hence elements $\omega \in \Omega$ are given by infinite sequences $(c_k)_{k \geq 1}$ of non-empty cliques such that $c_k \rightarrow c_{k+1}$ holds for all $k \geq 1$, and $G$ is the $\sigma$-algebra of $\Omega$ induced by the product $\sigma$-algebra, where $C$ is equipped with the discrete $\sigma$-algebra.

The Cartier-Foata decomposition result can be rephrased as the fact that the finite paths in the graph $(C_C, \rightarrow)$ are in bijection with the traces of the monoid. In the same way, infinite paths of $(C_C, \rightarrow)$ correspond naturally to the points of the boundary of the monoid. We postpone the proof of this result to §8 and admit for the time being that there exists a bi-measurable bijection $\Psi : \partial M \rightarrow \Omega$, which associates to each point of the boundary $\xi \in \partial M$ an infinite sequence $(c_k)_{k \geq 1}$ with values in $C_C$, and entirely characterized by the following two properties:

$$\forall k \geq 1 \quad c_k \rightarrow c_{k+1}, \quad \bigvee_{k \geq 1} (c_1 \cdot \ldots \cdot c_k) = \xi. \quad (13)$$

The bijection $\Psi : \partial M \rightarrow \Omega$ induces a bijection $\Psi_* : \mathcal{M}_1 (\partial M, \mathcal{F}) \rightarrow \mathcal{M}_1 (\Omega, \mathcal{G})$ between the associated sets of probability measures. We shall always use this identification.

Both spaces $\partial M$ and $\Omega$ come equipped with their own elementary cylinders: $\uparrow u$ with $u \in M$ for $\partial M$, and $\{\omega \in \Omega : C_1 (\omega) = c_1, \ldots, C_n (\omega) = c_n\}$ with $c_1 \rightarrow \ldots \rightarrow c_n$ for $\Omega$. If $P$ and $Q$ are probability measures on $\partial M$ and $\Omega$ related by $Q = \Psi_* P$, the effective correspondence between the values of $P$ and $Q$ on their respective elementary cylinders is non-trivial. For instance $P(\uparrow u)$ differs in general from $Q(C_1 = c_1, \ldots, C_n = c_n)$ where $c_1 \rightarrow \ldots \rightarrow c_n$ is the Cartier-Foata decomposition of $u$. The correspondence between cylinders is established in details in §8.
Theorem 4.1—Let $\mathcal{M}$ be an irreducible trace monoid.

1. Assume that $P$ is a Bernoulli measure on $(\partial \mathcal{M}, \mathfrak{F})$, with $f_P(\cdot) = \mathbb{P}(\uparrow \cdot)$ the induced valuation. Then, under probability $\mathbb{P}$, the sequence $(C_k)_{k \geq 1}$ of Cartier-Foata cliques is an irreducible and aperiodic Markov chain with values in $\mathfrak{C}$. The law of $C_1$ is the restriction to $\mathfrak{C}$ of the Möbius transform $h : \mathfrak{C} \to \mathbb{R}$ of $f_P$, and $h > 0$ on $\mathfrak{C}$. The transition matrix of the chain is $P = (P_{c,c'})_{(c,c') \in \mathfrak{C} \times \mathfrak{C}}$ given by:

$$P_{c,c'} = \begin{cases} h(c')/g(c), & \text{if } c \rightarrow c' \\ 0, & \text{if } \neg(c \rightarrow c') \end{cases}, \quad \text{with } g(c) = \sum_{c' \in \mathfrak{C} : c \rightarrow c'} h(c'). \quad (14)$$

Furthermore, for any integer $n \geq 1$, if $c_1, \ldots, c_n$ are $n$ non-empty cliques such that $c_1 \rightarrow \ldots \rightarrow c_n$ holds, then:

$$\mathbb{P}(C_1 = c_1, \ldots, C_n = c_n) = f_P(c_1) \cdots f_P(c_n-1)h(c_n). \quad (15)$$

2. Conversely, let $f : \mathcal{M} \to \mathbb{R}^*_+$ be a Möbius valuation, and let $h : \mathfrak{C} \to \mathbb{R}$ be the Möbius transform of $f$. Then the restriction $h|_{\mathfrak{C}}$ is a probability distribution on $\mathfrak{C}$. The Markov chain on $\mathfrak{C}$ with $h|_{\mathfrak{C}}$ as initial law, and with transition matrix $P$ given as in (14) above, induces a Bernoulli measure $\mathbb{P}$ on $\partial \mathcal{M}$ which satisfies $\mathbb{P}(\uparrow u) = f(u)$ for all traces $u$ in $\mathcal{M}$.

It is worth observing that $h|_{\mathfrak{C}}$ is not the stationary distribution of $P$, implying that $(C_n)_n$ is not stationary with respect to $n$ under $\mathbb{P}$. Markovian measures with the property (15) also appear in the context of random walks on some infinite groups, see [20], where they are called “markovian multiplicative”.

5. Uniform measures

So far we have obtained polynomial normalization conditions for the characteristic numbers of Bernoulli measures (§4) and we have identified Bernoulli measures with certain Markov measures on a combinatorial subshift (§4). The reader might have noticed that the actual existence of Bernoulli measures has not yet been proved.

In this section we state the existence of uniform Bernoulli measures, those having all their characteristic numbers identical. We also introduce the weaker notion of uniform measure. An equivalence between uniform measures and uniform Bernoulli measures is stated—a non-trivial result. Then we show how small deformations of the characteristic numbers around the particular value for the uniform measure lead to a continuum of distinct Bernoulli measures.

Let $\mathcal{M}$ be a trace monoid, and assume there exists a Bernoulli measure for which all characteristic numbers are equal, say to some real $p > 0$. Then, according to Theorem 4.1 point 1 and using the formulation stated in (14)–(a), the number $p$ must be a root of the monovariate Möbius polynomial $\mu_{\mathcal{M}}(X) \in \mathbb{Z}[X]$ defined by:

$$\mu_{\mathcal{M}}(X) = \sum_{c \in \mathfrak{C}} (-1)^{|c|}X^{|c|}. \quad (16)$$

We therefore face two questions. First, among the roots of $\mu_M(X)$, which ones correspond indeed to a Bernoulli measure? Such measures, and we shall prove their existence, we call uniform Bernoulli measures.

Obviously, any uniform Bernoulli measure satisfies the following property:

$$\forall u, v \in M \quad |u| = |v| \Rightarrow P(\uparrow u) = P(\uparrow v). \quad (17)$$

We underline that the above property is purely metric. Say that a probability measure on $\partial M$ satisfying (17) is uniform. The second question is: does any uniform measure belong to the class of Bernoulli measures? In other words, does (17) imply the memoryless property $P(\uparrow (u \cdot v)) = P(\uparrow u)P(\uparrow v)$? Note that the answer is clearly affirmative in the case of a free monoid, but much less trivial for a trace monoid.

Next theorem brings answers to the two above questions. The statement requires the knowledge of the following fact, which will be given in an even more precise form below in Theorem 9.1: the Möbius polynomial of an independence pair $(\Sigma, I)$ has a unique root of smallest modulus. This root is real and lies in the open interval $(0, 1)$.

• **Theorem 5.1**—Let $M$ be an irreducible trace monoid, and let $p_0$ be the unique root of smallest modulus of the Möbius polynomial $\mu_M(X)$. Then:

1. There exists a unique uniform Bernoulli measure $P_0$ on $(\partial M, \mathcal{F})$. It is entirely characterized by $P_0(\uparrow u) = p_0^{|u|}$.

2. Any uniform measure is Bernoulli uniform. Hence $P_0$ is also the unique uniform measure on $(\partial M, \mathcal{F})$.

Point 2 in Theorem 5.1 appears as a confirmation of the central role of Bernoulli measures.

Having identified at least one Bernoulli measure for each irreducible trace monoid, we are able to construct many others by considering small variations around the value $(p_0, \ldots, p_0)$ for the family of characteristic numbers.

• **Proposition 5.2**—Let $M = M(\Sigma, I)$ be irreducible with $|\Sigma| > 1$. There exists a continuous family of different Bernoulli measures on $\partial M$.

The proofs of Theorem 5.1 and of Proposition 5.2 are postponed to § 14.

**6. Two illustrative examples**

We illustrate the above results on two specific examples. We first concentrate on $M_1 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$, the example of the introductory challenge—providing a complete description of the Bernoulli measures.

Non-empty cliques of $M_1$ are given by $\{a, b, c, a \cdot b\}$, and the Cartier-Foata subshift is represented in Figure 2. The Möbius polynomials of $M_1$ is $\mu_{M_1}(X) = 1 - 3X + X^2$, with roots $(3 \pm \sqrt{5})/2$. Keeping the same symbols $a, b, c$ to denote the characteristic numbers of a generic valuation $f : M_1 \to \mathbb{R}_+^*$, the Möbius transform $h$ of $f$ is given as follows:

| clique $\gamma$ | 0  | a  | b  | c  | a $\cdot$ b |
|-----------------|----|----|----|----|-------------|
| valuation $f(\gamma)$ | 1  | a  | b  | c  | ab          |
| Möbius transform $h(\gamma)$ | 1  | a $\cdot$ b $- c + ab$ | a $- ab$ | b $- ab$ | c $\cdot ab$ |
Möbius valuations are thus determined, according to Definition 3.2, by the following conditions on parameters:

\[ 1 - a - b - c + ab = 0, \quad 1 - a > 0, \quad 1 - b > 0, \quad c > 0. \] (18)

It follows from Theorem 3.3 that Bernoulli measures on \( \mathcal{M}_1 \) are in bijective correspondence with the set of triples \( (a, b, c) \in (\mathbb{R}_+^*)^3 \) solutions of (18). The set of admissible triples forms a surface, a plot of which is given in Figure 3.

We can easily compute the transition matrix \( P \) associated with a generic Bernoulli measure with parameters \( (a, b, c) \) solution of (18). The normalization factor \( g \) from Theorem 4.1 is equal to 1 on all maximal cliques, which are \( a \cdot b \) and \( c \). We observe that \( g(a) = a - ab + c = 1 - b \), taking into account that \( 1 - a - b - c + ab = 0 \). Similarly, \( g(b) = 1 - a \). According to formula (14) stated in Theorem 4.1 and indexing the rows and columns of the transition matrix...
according to the cliques \((a, b, c, a \cdot b)\) in this order, we get:

\[
P = \begin{pmatrix}
a & 0 & 1-a & 0 \\
0 & b & 1-b & 0 \\
a-ab & b-ab & c & ab \\
a-ab & b-ab & c & ab
\end{pmatrix}
\]

It is readily checked by hand on this example, and this is true in general, that conditions \([15]\) insure that the above matrix has all its entries non negative and that all lines sum up to 1.

According to Theorem 5.1, the only uniform measure \(\mathbb{P}_0\) on \(\mathcal{M}_1\) is determined by the root \(p_0 = (3 - \sqrt{5})/2 = 0.382\ldots\) of \(\mu_{\mathcal{M}_1}\), and satisfies \(\mathbb{P}_0(\uparrow u) = p_0^{|u|}\) for all \(u\) in \(\mathcal{M}\). Note that, for this example, the other root of \(\mu_{\mathcal{M}_1}\) is outside \((0,1)\), so it is immediate that only \(p_0\) can correspond to a probability. But the Möbius polynomial might have several roots within \((0,1)\), as the next example reveals.

Consider the trace monoid \(\mathcal{M}_2 = \mathcal{M}(\Sigma_2, I_2)\) with \(\Sigma_2 = \{a_1, \ldots, a_5\}\), and which associated dependency relation \(D_2\) is depicted in Figure 4. Then the Möbius polynomial is \(\mu_{\mathcal{M}_2}(X) = 1 - 5X + 5X^2\), with roots \(q_0 = 1/2 - \sqrt{5}/10\) and \(q_1 = 1/2 + \sqrt{5}/10\). Hence, \(\mu_{\mathcal{M}_2}\) has its two roots within \((0,1)\), so we need to use the full statement of Theorem 5.1 point 1 in order to rule out \(q_1\) and retain only \(q_0\) as defining a uniform measure.

This can be double-checked by hand on this example. Consider the valuation \(f(u) = q_1^{|u|}\). Let \(h\) be the Möbius transform of \(f\). We have, for \(i \in \{1, \ldots, 5\}\): \(h(a_i) = q_1(1-2q_1) = -q_1/\sqrt{5} < 0\). Therefore, the valuation \(f\) is not Möbius and \(q_1\) does not qualify to define a uniform measure.

7. Computing the speedup

In a trace monoid, what is the “average parallelism”? Or what is the average speedup of the parallel execution time compared to the sequential one? Or what is the average density of a heap? The questions are natural and have been extensively studied, see \([6, 7, 18, 24]\). Obviously the probability assumptions need to be specified for the questions to make sense.

Let \(\tau(u)\) denote the height of a trace \(u \in \mathcal{M}\), which is equivalently defined either as the number of cliques in the Cartier-Foata decomposition of \(u\), or as
the height of the heap of pieces associated with $u$, and can also be interpreted as the “parallel execution time” of $u$.

One standard approach is to define the average parallelism as the quantity $\lambda_M = \lim_{n \to \infty} n / \tau(x_1, x_2, \ldots, x_n)$ where $(x_n)_n$ is an independent and uniformly distributed sequence of $\Sigma$-valued random variables. The limit exists indeed and is constant with probability one [24]. The quantity $\lambda_M$ is non-algebraic except for small trace monoids [18], and is NP-hard even to approximate [7].

Another approach is as follows. For all integer $n \geq 0$, set $\mathcal{M}_n = \{ u \in \mathcal{M} : |u| = n \}$ and let $\mu_n$ be the uniform probability distribution on the finite set $\mathcal{M}_n$. Let the speedup of $\mathcal{M}$ be the limit, assuming existence:

$$
\gamma_M = \lim_{n \to \infty} \mathbb{E}_{\mu_n} \left[ \frac{|u|}{\tau(u)} \right] = \lim_{n \to \infty} \frac{n}{\# \mathcal{M}_n} \sum_{u \in \mathcal{M}_n} \frac{1}{\tau(u)} \tag{19}
$$

In [18], the limit in (19) is shown to exist and to be an algebraic number. Here is a sketch of the proof when $\mathcal{M}$ is irreducible.

Define the bivariate generating series $L(x, y) = \sum_{u \in \mathcal{M}} x^{|u|} y^\tau(u)$. Observe that $L(x, 1) = G_M(x)$ is the generating series of $\mathcal{M}$, and is thus the inverse of the M"obius polynomial, according to Theorem 9.1 point 1 (§9). Let $p_0$ be the unique root of smallest modulus of the M"obius polynomial $\mu_M(x)$, see point 2 of Theorem [11]. Then $p_0$ is the dominant singularity of $L(x, 1)$. Using standard manipulations on generating series, we obtain:

$$
\gamma_M = \frac{p_0 \cdot [L(x, 1) \cdot (p_0 - x)]_{x=p_0}}{[(\partial L/\partial y)(x, 1) \cdot (p_0 - x)^2]_{x=p_0}}.
$$

The above expression is tractable to a certain extent since the series $L(x, y)$ can be shown to be rational [18, Prop. 4.1].

Using the results of the present paper, we obtain a simpler tractable expression for $\gamma_M$, in a more direct way that we now sketch.

The existence of the uniform probability measure $P$ on $\partial \mathcal{M}$ enables us to reinterpret the limit in (19) as the expected density of an infinite uniform heap. Now recall that, under $P$, the sequence $(C_n)_n$ of cliques is an irreducible and aperiodic Markov chain. According to the Ergodic Theorem for Markov chains (e.g., [2], §3.4), the density of infinite heaps is $P$-almost surely constant and equal to the expected density $\gamma_M$. In other terms, we have, $P$-almost surely,

$$
\gamma_M = \lim_{n \to \infty} \frac{|C_1 \cdots C_n|}{\tau(C_1 \cdots C_n)} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n |C_k| = \sum_{c \in \varepsilon_M} \pi(c) |c|,
$$

where $\pi$ is the stationary distribution of $(C_n)_n$ , that is, the solution of $\pi P = \pi$, $\pi(\cdot) > 0$, $\sum_{c \in \varepsilon_M} \pi(c) = 1$, for the matrix $P$ defined in Theorem [11].

For instance, for the trace monoids $\mathcal{M}_1$ and $\mathcal{M}_2$ analyzed in §8, we obtain:

$$
\gamma_{\mathcal{M}_1} = \frac{35 - 5 \sqrt{5}}{22} = 1.0827 \cdots, \quad \gamma_{\mathcal{M}_2} = \frac{29 - \sqrt{5}}{22} = 1.2165 \cdots.
$$

So parallelism increases the speed of execution by about 8% in the monoid $\mathcal{M}_1$ and by about 22% in the monoid $\mathcal{M}_2$. 

15
8. Elementary cylinders and sequences of cliques

Let $M = M(\Sigma, I)$ be a trace monoid. In this section, we establish the correspondence between points of the boundary $\partial M$, and infinite sequences of non-empty cliques satisfying the Cartier-Foata condition. We also describe how the order on traces transposes to their Cartier-Foata normal form.

To each non-empty trace $u \in M$ of Cartier-Foata normal form $d_1 \to \ldots \to d_n$, we associate an infinite sequence $(c_k)_{k \geq 1}$ of cliques, defined as follows: $c_k = d_k$ if $1 \leq k \leq n$, and $c_k = 0$ for $k > n$. The sequence $(c_k)_{k \geq 1}$ is called the extended Cartier-Foata decomposition of $u$, abbreviated xCF.

Recall from §7 that $\tau (u)$, the height of the trace $u$, is defined as the number of cliques in the Cartier-Foata normal form of $u$. Equivalently, $\tau (u)$ is the number of non-empty cliques in the xCF decomposition of $u$.

The Cartier-Foata decomposition establishes, for each integer $n \geq 1$, a bijection between the set of traces of height $n$ and a subset of the product $C^n$. How the ordering between traces is read on their Cartier-Foata decompositions is the topic of next results. In particular, the obtained order on Cartier-Foata sequences is strictly coarser in general than the order induced by the product order on $C^n$.

In the following result and later on, we make use of the notion of parallel cliques introduced in §4 by writing $c \parallel c_1, \ldots, c_n$ if $c \parallel c_i$ for all integers $i \in \{1, \ldots, n\}$.

- **Lemma 8.1**—Let $M = M(\Sigma, I)$ be a trace monoid and let $u, v \in M$ be two non-empty traces. Let $d_1 \to \ldots \to d_n$ and $d_1 \to \ldots \to d_p$ be the Cartier-Foata decompositions of $u$ and of $v$. Then $u \leq v$ if and only if $n \leq p$ and there are $n$ cliques $\gamma_1, \ldots, \gamma_n$ such that, for all $i \in \{1, \ldots, n\}$:

1. $\gamma_i \parallel c_i, \ldots, c_n$; and
2. $d_i = c_i \cdot \gamma_i$.

**Proof.** If the Cartier-Foata normal forms of $u$ and $v$ satisfy the properties stated in points 1, 2, then an easy induction argument shows that:

$$v = c_1 \cdot \ldots \cdot c_n \cdot (\gamma_1 \cdot \ldots \cdot \gamma_n) \cdot d_{n+1} \cdot \ldots \cdot d_p,$$

which implies that $u \leq v$.

Conversely, the proof is simple using the heap of pieces intuition. Here is the main argument. Assume that $u \leq v$, and let $w$ be such that $v = u \cdot w$. If $w = 0$, the result is trivial. Otherwise, let $\delta_1 \to \ldots \to \delta_r$ be the Cartier-Foata
normal form of \( w \). Apply the following recursive construction: pick a letter \( \alpha \in \delta_1 \), and move \( \alpha \) from \( \delta_1 \) to the clique \( c_i \) where \( i \) is the smallest index such that \( \alpha \parallel c_i, \ldots, c_n \). If there is no such index \( i \), the letter \( \alpha \) stays in \( \delta_1 \). Then repeat the operation, until all letters of \( \delta_1 \) have been dispatched. Once this is done, recursively apply the same procedure to \( \delta_2 \) up to \( \delta_r \). Some cliques among the \( \delta_i \) might entirely vanish during the procedure. The whole procedure yields the Cartier-Foata normal form of \( u \cdot w \), under the requested form. \( \square \)

For each integer \( p \geq 0 \), we define the \( p \)-cut operation as the mapping \( \kappa_p : \mathcal{M} \to \mathcal{M} \), \( u \mapsto \kappa_p(u) = c_1 \cdot \ldots \cdot c_p \), where \((c_k)_{k \geq 1} \) is the \( \text{xCF} \) decomposition of \( u \).

- **Corollary 8.2**—Let \( \mathcal{M} \) be a trace monoid, and let \( u, v \in \mathcal{M} \) be two traces. Then \( u \leq v \) if and only if \( u \leq \kappa_{\tau(u)}(v) \).

  **Proof.** If \( u \leq \kappa_{\tau(v)}(v) \), then \( u \leq v \) since \( \kappa_n(v) \leq v \) holds trivially for any integer \( n \geq 0 \). Conversely, assume that \( u \leq v \). Then the Cartier-Foata normal forms \( c_1 \to \ldots \to c_n \) and \( d_1 \to \ldots \to d_p \) of \( u \) and \( v \) satisfy properties 1 and 2 in Lemma 8.1. We have \( \tau(u) = n \) and

  \[ \kappa_{\tau(u)}(v) = (c_1 \gamma_1) \cdot \ldots \cdot (c_n \gamma_n) = (c_1 \cdot \ldots \cdot c_n) \cdot \gamma_1 \cdot \ldots \cdot \gamma_n \geq u. \]

  The proof is complete. \( \square \)

- **Corollary 8.3**—Let \( \mathcal{M} \) be a trace monoid. Let \((c_k)_{k \geq 1}\) and \((d_k)_{k \geq 1}\) be the \( \text{xCF} \) decompositions of two traces \( u \) and \( v \) of \( \mathcal{M} \). If \( u \leq v \), then \( c_k \leq d_k \) for all integers \( k \geq 1 \).

  **Proof.** The inequality \( c_k \leq d_k \) is trivial if \( c_k = 0 \). And for \( c_k \neq 0 \), then \( c_k \) belongs to the Cartier-Foata normal form of \( u \), and \( c_k \leq d_k \) follows from Lemma 8.1 point 2. \( \square \)

We lift the \( \text{xCF} \) decomposition to generalized traces as follows.

- **Lemma 8.4**—Let \( \mathcal{M} = \mathcal{M}(\Sigma, I) \) be a trace monoid. Then, for every generalized trace \( u \in \mathcal{W}(\Sigma, I) \), there exists a unique infinite sequence of cliques \((c_i)_{i \geq 1}\) such that \( u = \bigvee_{k \geq 1} (c_1 \cdot \ldots \cdot c_k) \) and \( c_k \to c_{k+1} \) holds for all integers \( k \geq 0 \).

  **Proof.** The result is clear if \( u \notin \partial \mathcal{M} \). And if \( u \in \partial \mathcal{M} \), we consider a non-decreasing sequence \((u_n)_{n \geq 0}\) in \( \mathcal{M} \) such that \( u = \bigvee_{n \geq 0} u_n \). Then, if \((c_{n,k})_{k \geq 1}\) is the \( \text{xCF} \) decomposition of \( u_n \), it follows from Corollary 8.3 that \((c_{n,k})_{n \geq 0}\) is non-decreasing in \( \mathcal{C} \) for each integer \( k \geq 1 \), and thus eventually constant, say equal to \( c_k \). Routine verifications using the compactness property stated in §4 show that the sequence \((c_k)_{k \geq 1}\) thus defined is the only adequate sequence. \( \square \)

Say that the sequence \((c_k)_{k \geq 1}\) associated to a generalized trace \( u \in \mathcal{W}(\Sigma, I) \) as in Lemma 8.4 is the \( \text{xCF decomposition} \) of \( u \). For each \( \xi \in \partial \mathcal{M} \), the sequence \((c_k)_{k \geq 1}\) is the unique sequence of non-empty cliques announced in 10.

Recall that we have defined \( \Omega \) as the set of infinite paths in the graph \((\mathcal{C}, \rightarrow)\) of non-empty cliques. Mapping each point \( \xi \in \partial \mathcal{M} \) to its \( \text{xCF} \) decomposition defines a well-defined application \( \Psi : \partial \mathcal{M} \to \Omega \), which is the mapping
announced in \(\S\)\textsuperscript{18} It is bijective; its inverse is given, if \(\omega \in \Omega\) has the form \(\omega = (c_k)_{k \geq 1}\), by:

\[
\Psi^{-1}(\omega) = \bigvee_{k \geq 1} (c_1 \cdots c_k).
\]

The following result explores how elementary cylinders transpose through the xCF decomposition, connecting the two different points of view on the boundary elements: the intrinsic point of view through elementary cylinders, and the effective point of view through sequences of cliques.

**Proposition 8.5**—Let \(\mathcal{M}\) be a trace monoid. For any element \(\xi \in \partial \mathcal{M}\), denote by \((C_k(\xi))_{k \geq 1}\) the xCF decomposition of \(\xi\). Let \(n \geq 1\) be an integer, and let \(c_1, \ldots, c_n \in \mathcal{C}\) be \(n\) cliques such that \(c_1 \to \ldots \to c_n\) holds. Put \(v = c_1 \cdots c_{n-1}\) and \(u = v \cdot c_n\). Then the following equalities of subsets of \(\partial \mathcal{M}\) hold:

\[
\uparrow (c_1 \cdots c_n) = \{ \xi \in \partial \mathcal{M} : C_1(\xi) \cdot \ldots \cdot C_n(\xi) \geq u \}, \quad (20)
\]

\[
\{ \xi \in \partial \mathcal{M} : C_1(\xi) = c_1 \land \ldots \land C_n(\xi) = c_n \} = \uparrow u \setminus \bigcup_{c \leq c_n} \uparrow (v \cdot c), \quad (21)
\]

where \(c_n < c\) means \(c_n \leq c\) and \(c_n \neq c\).

**Proof.** Put \(\mathcal{W} = \mathcal{W}(\Sigma, I)\), and extend the cut operations \(\kappa_p : \mathcal{M} \to \mathcal{M}\) defined above for all integers \(p \geq 0\), to mappings \(\kappa_p : \mathcal{W} \to \mathcal{M}\) in the obvious way. Combining the compactness property with Corollary 8.2 yields:

\[
\forall u \in \mathcal{M} \quad \forall v \in \mathcal{W} \quad u \leq v \iff u \leq \kappa_{\tau(u)}(v). \quad (22)
\]

Applied to \(u = c_1 \cdots c_n\) as in the statement and to \(\xi \in \partial \mathcal{M}\) in place of \(v\), this is \((20)\).

We now prove \((21)\). Set:

\[
A = \{ \xi \in \partial \mathcal{M} : C_1(\xi) = c_1 \land \ldots \land C_n(\xi) = c_n \}, \quad B = \bigcup_{c \leq c_n} \uparrow (v \cdot c).
\]

It is obvious that \(A \subseteq \uparrow u\). We prove that \(A \cap B = \emptyset\). For this, by contradiction, assume there exists \(\xi \in A \cap B\), and let \((\delta_k)_{k \geq 1}\) be the xCF of \(\xi\). Then \(\delta_i = c_i\) for all \(i \in \{1, \ldots, n\}\) since \(\xi \in A\). Let \(c \in \mathcal{C}\) be a clique such that \(c_n < c\) and \(\xi \in \uparrow (v \cdot c)\). Clearly, \(\tau(v \cdot c) = n\). Applying \((22)\) to \(v \cdot c \leq \xi\) we get thus \(c_1 \cdot \ldots \cdot c_{n-1} \cdot c \leq c_1 \cdot \ldots \cdot c_{n-1} \cdot c_n\), and then by left cancellativity of the monoid, \(c \leq c_n\), a contradiction. This proves that \(A \cap B = \emptyset\), and thus the \(\subseteq\) inclusion of \((21)\).

For the converse \(\supseteq\) inclusion, let \(\xi \in \uparrow u \setminus B\), keeping the notation \((\delta_k)_{k \geq 1}\) for its xCF decomposition. Since \(\tau(u) = n\), it follows from \((22)\) that \(c_1 \cdots c_n \leq \delta_1 \cdots \delta_n\). Hence \(\delta_i = c_i \cdot \gamma_i\) for some cliques \(\gamma_1, \ldots, \gamma_n\) as in Lemma 8.1. Using the properties of the cliques \(\gamma_i\)'s, we have \(\xi \geq \delta_1 \cdots \delta_n = c_1 \cdots c_n \cdot \gamma_1 \cdots \gamma_n\). Since \(\xi \notin B\) by assumption, this implies that \(\gamma_i = 0\) for all \(i \in \{1, \ldots, n\}\), and thus \(\xi \in A\).

**Corollary 8.6**—The bijection \(\Psi : \partial \mathcal{M} \to \Omega\) which associates to each point \(\xi \in \partial \mathcal{M}\) its xCF decomposition, is bi-measurable with respect to \((\partial \mathcal{M}, \mathcal{F})\) and \((\Omega, \mathcal{G})\).
Proof. The fact that $\Psi$ is measurable follows from (21). The fact that $\Psi^{-1}$ is measurable follows from (20).

9. Generating series and asymptotics

For $\mathcal{M}$ a trace monoid and $k \geq 0$ an integer, let $\lambda_M(k)$ be the number of traces of length $k$:

$$\lambda_M(k) = \# \{ u \in \mathcal{M} : |u| = k \}.$$ Let $G_M(X)$ be the generating series of $\mathcal{M}$, defined by:

$$G_M(X) = \sum_{u \in \mathcal{M}} X^{|u|} = \sum_{k \geq 0} \lambda_M(k) X^k.$$

The following result is standard, and is the basis of the combinatorial study of trace monoids [10, 14, 18, 27].

**Theorem 9.1**—Let $\mathcal{M}$ be a trace monoid, with $\mu_M(X)$ the mono-variate Möbius polynomial. Then:

1. The following formal identity holds in $\mathbb{Z}[[X]]$:

$$G_M(X) = 1/\mu_M(X).$$

In particular, $G_M(X)$ is a rational series.

2. The polynomial $\mu_M(X)$ has a unique root of smallest modulus, say $p_0$, which is real and lies in $(0, 1)$. The radius of convergence of the power series $G_M(z)$ is thus equal to $p_0$, and the series $G_M(z)$ is divergent at $p_0$.

3. If $N$ is the order of multiplicity of $p_0$ in $\mu_M(X)$, then the following estimates hold for some constant $C > 0$:

$$\lambda_M(k) \sim_{k \to \infty} Ck^{N-1}(1/p_0)^k.$$  (23)

Furthermore, the order of multiplicity of $p_0$ is 1 if $\mathcal{M}$ is irreducible.

**Lemma 9.2**—Let $\mathcal{M}$ be an irreducible trace monoid. Then for any non-empty clique $c \in \mathcal{C}_M$, we have:

$$\lim_{k \to \infty} \lambda_M_c(k)/\lambda_M(k) = 0.$$  (24)

Proof. The argument is rather standard. We sketch it and illustrate it for $\mathcal{M}_1$. Start by considering the Cartier-Foata automaton of $\mathcal{M}$ and transform it by expanding each node corresponding to a clique $c$ of cardinality strictly larger than one, into $|c|$ nodes. The first of the expanded nodes is initial and the last of the expanded nodes is final. The non-expanded nodes are both initial and final. See [18, p.148] for the details of the construction and see Figure 5 for an illustration.

Let $A$ be the resulting automaton. Let $A_c$ be the automaton obtained from $A$ by keeping the same nodes, the same initial and final nodes, but by keeping only the arcs entering into the nodes labelled by the letters of $\Sigma_c$. Admissible paths of length $k$ in $A$ are in bijection with traces of length $k$ in $\mathcal{M}$. And
admissible paths of length $k$ in $A_c$ are in bijection with traces of length $k$ in $M_c$. (Recall that a path in an automaton is admissible if it starts with an initial state and ends up with a final state.) Denote by $A$ the incidence matrix of the automaton $A$, and by $A_c$ the one of $A_c$. By construction, we have:

$$A_c \leq A, \quad A_c \neq A.$$  \hspace{1cm} (25)

According to Lemma 11.1 (see below), since $M$ is irreducible, the matrix $A$ is primitive. So we are in the domain of applicability of [25, Theorem 1.2 p.9 and Theorem 1.1 point (e) p.4], the strong version of Perron-Frobenius Theorem for non-negative matrices. According to it, the strict inequality (25) yields that the spectral radius of $A_c$ is strictly smaller than the spectral radius of $A$. The limit (24) follows.

In Figure 5, we illustrate the construction of the proof for the trace monoid $M_1 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$ by showing the automaton $A$, to be compared with the original automaton depicted in Figure 2. In $A$, the initial nodes are $\{a, b, c, (a \cdot b)_1\}$ and the final nodes are $\{a, b, c, (a \cdot b)_2\}$. For the clique $\{a\}$, for instance, the automaton $A_{\{a\}}$ has the same nodes as $A$ but a single arc: the self-loop around the node $b$.

Figure 5: The expanded automaton $A$.

\begin{itemize}
  \item **Proposition 9.3**—Let $M = M(\Sigma, I)$ be an irreducible trace monoid, and let $p_0$ be the root of smallest modulus of the Möbius polynomial $\mu_M(X)$. Then $\mu_{M_c}(p_0) > 0$ for all non-empty cliques $c \in C_M$.

  \begin{proof}
  By point 3 of Theorem 9.1 and since $M$ is assumed to be irreducible, we have the estimate $\lambda_M(k) \sim C(1/p_0)^k$ for $k \to \infty$. Let $c \in C_M$. If $c$ is maximal, then $\mu_{M_c}(X) = 1$ and thus $\mu_{M_c}(p_0) > 0$ holds trivially. Otherwise, the monoid $M(\Sigma_c, I_c)$ is non-trivial, let $p_c$ be the root of smallest modulus of $\mu_{M_c}$. Then $\lambda_{M_c}(k) \sim C' k^N (1/p_c)^k$ for some constant $C' > 0$ and where $N$ is the order of multiplicity of $p_c$ in $\mu_{M_c}(X)$, by point 3 of Theorem 9.1. In view of Lemma 9.2, it follows that $p_c > p_0$.

  In particular, and by point 2 of Theorem 9.1, $p_0$ is in the open disc of convergence of the series $G_{M_c}(z) = \sum_{k \geq 0} \lambda_{M_c}(k) z^k$. Applying point 1 of Theorem 9.1 to the trace monoid $M_c$, and converting the formal equality into an equality between reals, we get $\mu_{M_c}(p_0) = 1/G_{M_c}(p_0) > 0$, completing the proof.
\end{proof}
\end{itemize}
10. Möbius inversion formula and consequences

The notion of Möbius function of a partial order is due to Rota [23]. Due to the formal equality $G_M(X)\mu_M(X) = 1$, recalled in Theorem 9.1 point 1, the Möbius function, element of the incidence algebra in the sense of Rota, is easily found to be $\mu_M : M \times M \rightarrow \mathbb{Z}$ given by $\mu_M(x, y) = (-1)^{|y|-|x|}$ if $x \leq y$ and if $y - x$ is a clique, and 0 otherwise. From this, we derive the following form of the Möbius inversion formula [23, Prop.2] for trace monoids.

• Proposition 10.1—Let $\mathcal{C}$ be the set of cliques associated with an independence pair $(\Sigma, I)$. If $f, h : \mathcal{C} \rightarrow \mathbb{R}$ are two functions, then $h$ is the Möbius transform of $f$, that is, satisfies:

$$\forall c \in \mathcal{C} \quad h(c) = \sum_{c' \in \mathcal{C} : c \leq c'} (-1)^{|c'| - |c|} f(c'),$$

(26)

if and only if the following holds:

$$\forall c \in \mathcal{C} \quad f(c) = \sum_{c' \in \mathcal{C} : c \geq c'} h(c').$$

(27)

The formula (27) is called the Möbius inversion formula, since it allows to recover any function $f : \mathcal{C} \rightarrow \mathbb{R}$ from its Möbius transform. We give an enhanced version in Proposition 10.4 below which applies outside the mere set $\mathcal{C}$.

• Corollary 10.2—Let $\mathcal{C}$ be the set of cliques associated with an independence pair $(\Sigma, I)$. Let $h : \mathcal{C} \rightarrow \mathbb{R}$ be the Möbius transform of a function $f : \mathcal{C} \rightarrow \mathbb{R}$ such that $f(0) = 1$. Then $h(0) = 0$ if and only if $\sum_{c \in \mathcal{C}} h(c) = 1$.

Proof. The Möbius inversion formula (27) applied to $c = 0$ writes as follows:

$$1 = h(0) + \sum_{c \in \mathcal{C}} h(c),$$

whence the result.

We recall that, by convention, the Möbius transform of a valuation $f : M \rightarrow \mathbb{R}^+_+$ is defined as the Möbius transform of its restriction to $\mathcal{C}_M$.

• Proposition 10.3—Let $h : \mathcal{C} \rightarrow \mathbb{R}$ be the Möbius transform of a valuation $f : M \rightarrow \mathbb{R}^+_+$, where $\mathcal{C}$ is associated to an independence pair $(\Sigma, I)$. Assume that $h(0) = 0$, and let $g : \mathcal{C} \rightarrow \mathbb{R}$ be the function defined by:

$$\forall c \in \mathcal{C} \quad g(c) = \sum_{c' \in \mathcal{C} : c \rightarrow c'} h(c').$$

Then the following formula holds:

$$\forall c \in \mathcal{C} \quad g(c)f(c) = h(c).$$

Proof. The identity $g(0)f(0) = h(0)$ is trivial since $0 \rightarrow c'$ if and only if $c' = 0$, and thus $g(0) = h(0)$, while $f(0) = 1$. For $c \in \mathcal{C}$ a non-empty clique, and by definition of $h$ and of $g$, one has:

$$g(c) = \sum_{c' \in \mathcal{C}} (-1)^{|c'|} f(c') \sum_{\delta \in \mathcal{C} : \delta(c) \cap \delta(c') \neq \emptyset} 1_{\{c \rightarrow \delta\}} 1_{\{\delta \leq c'\}} (-1)^{|\delta|}.$$

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For any $c' \in \mathcal{C}$, the range of $\delta$ in the above sum is \( \{ \delta \in \mathcal{C} : \delta \leq c' \cap \{ \alpha \in \Sigma : c \rightarrow \alpha \} \} \), and the binomial formula yields thus:

\[
\sum_{\delta \in \mathcal{C} : \delta \leq c' \wedge c = \delta} (-1)^{|\delta|} = -\tau_c(c'), \quad \text{with } \tau_c(c') = \begin{cases} 0, & \text{if } c' \parallel c, \\ 1, & \text{if } \neg(c' \parallel c). \end{cases}
\]

We obtain thus:

\[
g(c) = -\sum_{c' \in \mathcal{C}} (-1)^{|c'|} f(c') \tau_c(c'). \tag{28}
\]

The assumption $h(0) = 0$ writes as:

\[
1 + \sum_{c' \in \mathcal{C} : c' \parallel c} (-1)^{|c'|} f(c') + \sum_{c' \in \mathcal{C} : \tau_c(c') = 1} (-1)^{|c'|} f(c') = 0. \tag{29}
\]

Combining (28) and (29) yields:

\[
g(c) = 1 + \sum_{c' \in \mathcal{C} : c' \parallel c} (-1)^{|c'|} f(c'). \tag{30}
\]

We multiply both sides of (30) by $f(c)$ and apply the change of variable $c'' = c \cdot c'$. Using that $f$ is multiplicative, this yields:

\[
f(c)g(c) = f(c) + \sum_{c'' \in \mathcal{C} : c'' > c} (-1)^{|c''| - |c|} f(c'') = h(c),
\]

which was to be proved.

Next result is a generalization of the Möbius inversion formula (27). Whereas the original Möbius inversion formula is valid for any function $f : \mathcal{C}_M \rightarrow \mathbb{R}$, the generalized version applies to valuations only.

Let $f : \mathcal{M} \rightarrow \mathbb{R}^*$ be a valuation. In (27), the Möbius transform of $f$ was defined as a function $h : \mathcal{C}_M \rightarrow \mathbb{R}$. Here, we extend the domain of definition of $h$ to the whole monoid $\mathcal{M}$ as follows. If $u \in \mathcal{M}$ is a non-empty trace, we write $u = v \cdot c$, where $c \in \mathcal{C}_M$ is the last clique in the Cartier-Foata normal form of $u$, and $v$ is the unique trace such that $u = v \cdot c$ holds. The extended Möbius transform $h : \mathcal{M} \rightarrow \mathbb{R}$ is then defined by:

\[
\forall u \in \mathcal{M} \quad h(u) = f(v)h(c). \tag{31}
\]

**Proposition 10.4**—Let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a valuation defined on a trace monoid. Let $u \in \mathcal{M}$ be a non-empty trace, and let $\mathcal{M}(u)$ denote the set:

\[
\mathcal{M}(u) = \{ u' \in \mathcal{M} : \tau(u') = \tau(u), \ u \leq u' \}, \tag{32}
\]

where $\tau(\cdot)$ is the height function defined in §8. Then we have the identity:

\[
\sum_{u' \in \mathcal{M}(u)} h(u') = f(u), \tag{33}
\]

where $h : \mathcal{M} \rightarrow \mathbb{R}$ is the extended Möbius transform of $f$ defined in (31).
Proof. We fix a non-empty trace \( u \in M \), and we set

\[
S_0 = \sum_{u' \in M(u)} h(u').
\]

Let \( c_1 \to \ldots \to c_n \) be the Cartier-Foata normal form of \( u \), and set \( c = c_n \) and \( v = c_1 \cdot \ldots \cdot c_{n-1} \). We apply Lemma 8.3 to derive:

\[
S_0 = \sum_{(\gamma_1, \ldots, \gamma_n) \in J(c_1, \ldots, c_n)} h(c_1 \cdot \gamma_1 \cdot \ldots \cdot c_n \cdot \gamma_n),
\]

where we have set:

\[
J(c_1, \ldots, c_n) = \left\{ (\gamma_1, \ldots, \gamma_n) \in E^n : \gamma_i \parallel c_i, \ldots, c_n \text{ for } 1 \leq i \leq n, \quad c_1 \cdot \gamma_1 \to \ldots \to c_n \cdot \gamma_n \right\}.
\]

By definition of \( h \), this yields:

\[
S_0 = f(v)S_1, \quad \text{with} \quad S_1 = \sum_{(\gamma_1, \ldots, \gamma_n) \in J(c_1, \ldots, c_n)} f(\gamma_1) \cdot \ldots \cdot f(\gamma_n-1) h(c_n \cdot \gamma_n).
\]

We define, for \( x, y \in E' \):

\[
\lambda(x, y) = \sum_{\delta \in E' : (x \to \delta) \land (\delta \geq y)} h(\delta).
\]

Rewriting \( S_1 \) using the above notation, we get:

\[
S_1 = \sum_{(\gamma_1, \ldots, \gamma_n) \in K(c_1, \ldots, c_n)} f(\gamma_1) \cdot \ldots \cdot f(\gamma_n-1) \lambda(c_{n-1} \cdot \gamma_{n-1}, c_n),
\]

where we have set:

\[
K(c_1, \ldots, c_n) = \left\{ (\gamma_1, \ldots, \gamma_n) \in E'^{n-1} : \gamma_i \parallel c_i, \ldots, c_n \text{ for } 1 \leq i \leq n - 1, \quad c_1 \cdot \gamma_1 \to \ldots \to c_{n-1} \cdot \gamma_{n-1} \right\}.
\]

Applying Lemma 10.5 below yields, for any \( \gamma_{n-1} \) in the scope of the sum defining \( S_1 \):

\[
\lambda(c_{n-1} \cdot \gamma_{n-1}, c_n) = f(c_n) \sum_{\delta \in E' : \delta \parallel c_{n-1} \cdot \gamma_{n-1}, c_n} (-1)^{|\delta|} f(\delta).
\]

Therefore \( S_1 = f(c_n)S_2 \) where \( S_2 \) is defined by:

\[
S_2 = \sum_{(\gamma_1, \ldots, \gamma_n) \in K(c_1, \ldots, c_n)} f(\gamma_1) \cdot \ldots \cdot f(\gamma_n-1)(-1)^{|\delta|} f(\delta).
\]

Since \( S_2 = 1 \) according to Lemma 10.6 below, we conclude that \( S_1 = f(c_n) \) and finally \( S_0 = f(v)f(c_n) = f(u) \), which was to be proved.

In the course of the above proof, we have used the following two lemmas.
Lemma 10.5—If \( x, y \in \mathcal{C} \) are two cliques such that \( x \rightarrow y \) holds, then the quantity \( \lambda(x, y) \) defined in (36) satisfies:

\[
\lambda(x, y) = f(y) \sum_{\delta \in \mathcal{C} : \delta \upharpoonright \{x, y\}} (-1)^{|\delta|} f(\delta).
\]

Proof. By definition of the Möbius transform \( h \), one has:

\[
\lambda(x, y) = \sum_{\delta \in \mathcal{C} : (x \rightarrow y) \cap (y \leq \delta)} h(\delta) = \sum_{z \in \mathcal{C} : y \leq z} (-1)^{|z|} f(z) H(x, y, z) \tag{38}
\]

where \( H(x, y, z) = \sum_{\delta \in \mathcal{C} : (x \rightarrow y) \cap (y \leq \delta \leq z)} (-1)^{|\delta|} \).

Consider \( \delta \) as in the sum defining \( H(x, y, z) \). Since \( x \rightarrow y \) holds by assumption, the following equivalence holds: \( x \rightarrow \delta \iff x \rightarrow (\delta - y) \). The binomial formula yields thus:

\[
H(x, y, z) = \begin{cases} (-1)^{|\delta|}, & \text{if } (z - y) \parallel x, \\ 0, & \text{otherwise.} \end{cases}
\]

Reporting the latter value of \( H(x, y, z) \) in (38) and considering the change of variable \( z = y \cdot \delta \) yields the expected expression for \( \lambda(x, y) \).

Lemma 10.6—For any integer \( n \geq 1 \) and for any cliques \( c_1, \ldots, c_n \) such that \( c_1 \rightarrow \ldots \rightarrow c_n \) holds, the quantity \( S_2 \) defined in (37) satisfies \( S_2 = 1 \).

Proof. We substitute the variable \( \delta' = \delta \cdot \gamma_{n-1} \) to \( \delta \) in the defining sum for \( S_2 \). For each \( \gamma_{n-1} \) in the scope of the sum, one has \( \gamma_{n-1} \parallel c_{n-1}, c_n \), as specified by the definition of \( K(c_1, \ldots, c_n) \). Hence the set \( \{ \delta \in \mathcal{C} : \delta \parallel c_{n-1} \cdot \gamma_{n-1}, c_n \} \) corresponds to the set \( \{ \delta' \in \mathcal{C} : (\delta' \geq \gamma_{n-1}) \land (\delta' \parallel c_{n-1}, c_n) \} \), and the change of variable yields:

\[
S_2 = \sum_{\delta' \in \mathcal{C} : \delta' \parallel c_{n-1}, c_n} (-1)^{|\delta'|} f(\delta') \sum_{(\gamma_1, \ldots, \gamma_{n-2}) \in L(c_1, \ldots, c_n)} f(\gamma_1) \cdots f(\gamma_{n-2}) R(\gamma_{n-2}), \tag{39}
\]

with

\[
L(c_1, \ldots, c_n) = \{(\gamma_1, \ldots, \gamma_{n-2}) \in \mathcal{C}^{n-2} : \gamma_i \parallel c_1, \ldots, c_n \text{ for } 1 \leq i \leq n-2, c_1 \cdot \gamma_1 \rightarrow \ldots \rightarrow c_{n-2} \cdot \gamma_{n-2}\}
\]

and

\[
R(\gamma_{n-2}) = \sum_{\gamma_{n-1} \in \mathcal{C} : \gamma_{n-1} \parallel c_{n-1}, c_n} (-1)^{|\gamma_{n-1}|}. \tag{39}
\]

In the scope of the sum defining \( R(\gamma_{n-2}) \), the condition \( "c_{n-2} \cdot \gamma_{n-2} \rightarrow c_{n-1} \cdot \gamma_{n-1}" \) has been replaced by \( "c_{n-2} \cdot \gamma_{n-2} \rightarrow \gamma_{n-1}" \), which is equivalent since \( c_{n-2} \rightarrow c_{n-1} \) already holds by assumption.

Since \( \delta \parallel c_{n-1}, c_n \), and by the binomial formula, the sum defining \( R(\gamma_{n-2}) \) evaluates as follows:

\[
R(\gamma_{n-2}) = 1_{\{\delta \parallel c_{n-2} \cdot \gamma_{n-2}\}}. \tag{40}
\]
Substituting the right side of (40) into (39), we obtain:

\[ S_2 = \sum_{(\gamma_1, \ldots, \gamma_{n-2}) \in L(c_1, \ldots, c_n) \atop \delta \in \mathcal{E} : \delta(c_{n-2}, c_{n-1}, c_n) \not= 0} f(\gamma_1) \cdots f(\gamma_{n-2})(-1)^{|\delta|} f(\delta) . \]

Applying recursively the same transformation eventually yields:

\[ S_2 = \sum_{\gamma, \delta \in \mathcal{C} : \gamma \parallel c_1, \ldots, c_n} f(\gamma)(-1)^{|\delta|} f(\delta) \sum_{\gamma \leq \delta} (-1)^{|\gamma|} = f(0) = 1, \]

completing the proof. \(\square\)

11. Combinatorial lemmas

The following result is known, see for instance [18, Lemma 3.2]. We provide below an alternative proof.

- **Lemma 11.1**—If \( M \) is an irreducible trace monoid, then \((\mathcal{C}_M, \rightarrow)\) is a connected graph.

**Proof.** Consider the following claim \((\ast)\), which we prove under the hypothesis that \((\Sigma, D)\) is connected:

\((\ast)\) Let \( c \) be a non-empty clique of \((\Sigma, I)\), and let \( \alpha_0 \in \Sigma \) be a letter such that \( \alpha_0 \parallel c \) holds. Then there exists an integer \( p \geq 1 \) and \( p \) non-empty cliques \( \gamma_1, \ldots, \gamma_p \) such that \( \gamma_p = c \cdot \alpha_0 \) and \( c \rightarrow \gamma_1 \rightarrow \ldots \rightarrow \gamma_p \) holds.

Indeed, since \( c \not= 0 \), pick \( \alpha_1 \in c \). Since \((\Sigma, D)\) is assumed to be connected, there is a sequence of letters \( \beta_1, \ldots, \beta_p \in \Sigma \) such that, putting \( \beta_0 = \alpha_1 \), one has \( (\beta_i, \beta_{i+1}) \in D \) for all \( i \in \{0, \ldots, p-1\} \), and \( \beta_p = \alpha_0 \). Next, for each letter \( \alpha \in c \), consider the following integer:

\[ i(\alpha) = \min \{ j \in \{1, \ldots, p\} : \alpha \parallel \beta_j, \beta_{j+1}, \ldots, \beta_p \}. \]

Since \( \beta_p = \alpha_0 \), and since \( \alpha_0 \parallel c \) by assumption, one has indeed \( i(\alpha) \leq p \) for all \( \alpha \in c \). Consider the sequence of cliques \( \gamma_1, \ldots, \gamma_p \) defined as follows:

\[ \forall j \in \{1, \ldots, p\}, \quad \gamma_j = \{\beta_j\} \cup \{\alpha \in c : j \geq i(\alpha)\}. \]

We leave it to the reader to check that \( \gamma_1, \ldots, \gamma_p \) thus defined satisfy the claim \((\ast)\). The statement of the lemma follows easily from the claim. \(\square\)

Next lemma will be a key in proving the uniqueness of uniform measures. We recall first that for any trace monoid \( M = M(\Sigma, I) \), the mirror mapping \( \text{rev} : M \rightarrow M \) is defined as the quotient mapping of the mapping \( \Sigma^* \rightarrow \Sigma^* \) defined on words by \( \text{rev}(\alpha_1 \cdots \alpha_n) = \alpha_n \cdots \alpha_1 \). Given \( u \in M \), the heap of \( \text{rev}(u) \) is obtained from the heap of \( u \) by considering it upside-down. If \( s_1 \rightarrow \ldots
... \to s_k \text{ and } r_1 \to \ldots \to r_{\ell} \text{ are the respective Cartier-Foata decompositions of } u \text{ and } \text{rev}(u), \text{ then:}

\begin{align*}
  k = \ell, \\
  r_k \leq s_1 \\
  r_{k-1} \cdot r_k \leq s_2 \cdot s_1 \\
  \ldots \\
  r_1 \cdot r_2 \cdot \ldots \cdot r_k \leq s_k \cdot s_{k-1} \cdot \ldots \cdot s_1
\end{align*}

(41)

The properties in (41) are easy to visualize using the heap interpretation.

**Lemma 11.2**—(Hat lemma) Let $M$ be an irreducible trace monoid. Then there exists a trace $w \in M$ with the following property:

$$\forall u, v \in M \quad ([u] = [v]) \land (u \neq v) \implies \uparrow (u \cdot w) \cap \uparrow (v \cdot w) = \emptyset \tag{42}$$

**Proof.** Let $M = M(\Sigma, I)$, and let $D$ be the associated dependence relation. Since $M$ is assumed to be irreducible, we consider a sequence $(\alpha_i)_{1 \leq i \leq q}$ with $\alpha_i \in \Sigma$ such that: 1) every $\alpha \in \Sigma$ occurs at least once in the sequence; and 2) $(\alpha_i, \alpha_{i+1}) \in D$ for all $i \in \{1, \ldots, q-1\}$. We introduce the trace

$$w = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{q-1} \cdot \alpha_q \cdot \alpha_{q-1} \cdot \alpha_{q-2} \cdot \ldots \cdot \alpha_1,$$

and we aim at showing that $w$ satisfies (42).

**Claim (\star).** For all $u \in M$, the first $q$ cliques in the Cartier-Foata normal form of $w \cdot u$ are $\alpha_1 \to \alpha_2 \to \ldots \to \alpha_q$, and the last $q$ cliques in the Cartier-Foata normal form of $u \cdot w$ are $\alpha_q \to \alpha_{q-1} \to \ldots \to \alpha_1$.

We prove the claim (\star). By construction, the Cartier-Foata decomposition of $w$ is

$$\alpha_1 \to \alpha_2 \to \ldots \to \alpha_{q-1} \to \alpha_q \to \alpha_{q-1} \to \ldots \to \alpha_2 \to \alpha_1.$$

Consider the trace $w \cdot u$ for some $u \in M$. Let $d_1 \to \ldots \to d_p$ be the Cartier-Foata decomposition of $w \cdot u$. Applying Lemma 8.1 to traces $w$ and $w \cdot u$, we conclude in particular that $2q - 1 \leq p$, and for all $i \in \{1, \ldots, q\}$, we have $d_i = \alpha_i \cdot \gamma_i$ for some clique $\gamma_i \in C$ such that $\gamma_i \parallel \alpha_i, \ldots, \alpha_1, \alpha_{q-1}, \ldots, \alpha_2, \alpha_1$. Since $\alpha_q, \ldots, \alpha_1$ range over all letters of $\Sigma$, it follows that $\gamma_1 = 0$. So we have proved that $d_1 \to \ldots \to d_q = \alpha_1 \to \ldots \to \alpha_q$.

Now for the second part of the claim (\star), consider the trace $u \cdot w$ for some $u \in M$. We have $\text{rev}(u \cdot w) = \text{rev}(w) \cdot \text{rev}(u) = w \cdot \text{rev}(u)$. According to the above, the Cartier-Foata decomposition of $w \cdot \text{rev}(u)$ starts with $\alpha_1 \to \ldots \to \alpha_q$. According to (41), the $q$ last cliques $d_1 \to \ldots \to d_q$ of $u \cdot w = \text{rev}(w \cdot \text{rev}(u))$ satisfy:

$$d_q \leq \alpha_1, \quad d_{q-1} \cdot d_q \leq \alpha_2 \cdot \alpha_1, \quad \ldots \quad d_1 \cdot \ldots \cdot d_q \leq \alpha_q \cdot \ldots \cdot \alpha_1.$$

Since the $\alpha_i$ are minimal in $C$, it follows that $d_q = \alpha_1$, $d_{q-1} = \alpha_2$, $\ldots$, $d_1 = \alpha_q$, which completes the proof of the claim (\star).

We now come to the proof of (42) for $w$. Let $u, v \in M$ such that $|u| = |v|$ and $\uparrow (u \cdot w) \cap \uparrow (v \cdot w) \neq \emptyset$. According to (41), it follows that $u \cdot w$ and $v \cdot w$ are compatible. Hence there are $u', v' \in M$ such that $u \cdot w \cdot u' = v \cdot w \cdot v'$.
Set $\hat{w} = \alpha_1 \cdot \alpha_2 \cdots \cdot \alpha_{q-1}$, so that $w = \hat{w} \cdot \alpha_q \cdot \text{rev}(\hat{w})$. It follows from the claim $(\ast)$ that the Cartier-Foata decomposition of $u \cdot \hat{w} \cdot \alpha_q$ is of the form $c_1 \rightarrow \ldots \rightarrow c_k \rightarrow \alpha_q$, and the Cartier-Foata decomposition of $\alpha_q \cdot \text{rev}(\hat{w}) \cdot u'$ is of the form $\alpha_q \rightarrow d_1 \rightarrow \ldots \rightarrow d_{\ell}$. Hence the Cartier-Foata decomposition of $u \cdot w \cdot u'$ is obtained by concatenating the ones of $u \cdot w$ and of $u'$. By the same argument, the Cartier-Foata decomposition of $v \cdot w \cdot v'$ is obtained by concatenating the ones of $v \cdot w$ and of $v'$.

Hence, by uniqueness of the Cartier-Foata decomposition of $u \cdot w \cdot u' = v \cdot w \cdot v'$, between the decompositions of $u'$ and of $v'$, one is a suffix of the other. On the other hand, $u \cdot w \cdot u' = v \cdot w \cdot v'$ and $|u| = |v|$ imply $|u'| = |v'|$, and therefore $u' = v'$. Since $\mathcal{M}$ is cancellative, we conclude that $u = v$, which completes the proof. \qed
Part III

Proofs of the main results

12. From Bernoulli measures to Markov chains and Möbius valuations

In this section, we prove Proposition 2.1 and point 1 of Theorem 4.1 and point 1 of Theorem 3.3. The three results correspond to necessary conditions for a probability measure on the boundary of a trace monoid to be Bernoulli. We start with the two latter points.

The setting is the following: we consider an irreducible trace monoid $M = M(\Sigma, I)$, and we assume that $P$ is a Bernoulli measure defined on $(\partial M, \mathfrak{F})$. We consider the valuation $f : M \rightarrow \mathbb{R}$, associated with $P$, defined by $f(u) = P(\uparrow u)$ for $u \in M$, and we let $h : \mathcal{C} \rightarrow \mathbb{R}$ be the Möbius transform of $f$.

We start by proving formula (15) in Theorem 4.1, which implies most of the other affirmations. Hence, let $n \geq 1$ be an integer and let $c_1 \rightarrow \ldots \rightarrow c_n$ be $n$ non-empty cliques. According to formula (21) in Proposition 8.5, one has:

$$P(C_1 = c_1, \ldots, C_n = c_n) = f(u) - \delta, \quad \text{with} \quad \delta = P\left( \bigcup_{c \subset c_n} \uparrow (v \cdot c) \right), \quad (43)$$

where $v = c_1 \cdot \ldots \cdot c_{n-1}$ and $u = c_1 \cdot \ldots \cdot c_n$. For any $\xi \in \partial M$, one has $\xi \in \uparrow (v \cdot c)$ for some clique $c > c_n$ if and only if there is a letter $\alpha || c_n$ such that $\xi \in \uparrow (v \cdot c_n \cdot \alpha)$. Let $\{\alpha_1, \ldots, \alpha_r\}$ be an enumeration of such letters. Applying Poincaré inclusion-exclusion principle, we obtain:

$$\delta = \sum_{k=1}^{r} (-1)^{k+1} \sum_{1 \leq i_1 < \ldots < i_k \leq r} P\left( \uparrow (v \cdot c_n \cdot \alpha_{i_1}) \cap \cdots \cap \uparrow (v \cdot c_n \cdot \alpha_{i_k}) \right).$$

For $i_1, \ldots, i_k$ indices as in the above sum, put $\gamma = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ and $\gamma' = c_n \cdot \gamma$. The related intersection is then equal to $\uparrow (v \cdot c_n \cdot \gamma) = \uparrow (v \cdot \gamma')$. By construction, the cliques $\gamma'$ range over the cliques $c' \in \mathcal{C}$ such that $c' > c_n$, and thus, taking into account that $P(\uparrow \cdot) = f(\cdot)$ is multiplicative, we obtain:

$$\delta = \sum_{c' \in \mathcal{C}} (-1)^{|c'|-|c|+1} f(v) \sum_{c' \in \mathcal{C}} (-1)^{|c'|-|c|+1} f(c').$$

Injecting the above in (43), and writing $f(u) = f(v) f(c_n)$, we get:

$$P(C_1 = c_1, \ldots, C_n = c_n) = f(v) \sum_{c' \in \mathcal{C}} (-1)^{|c'|-|c|} f(c') = f(v) h(c_n),$$

Since $f(v) h(c_n) = f(c_1) \cdots f(c_{n-1}) h(c_n)$, we have the desired result.
As a particular case for \( n = 1 \), it follows at once that \( h|_{\mathcal{C}} \) coincides with the probability distribution of \( C_1 \). Therefore, by the total probability law, 
\[
\sum_{c \in \mathcal{C}} h(c) = 1,
\]
and by Corollary 10.2, \( h(0) = 0 \). It remains only to prove that \( h > 0 \) on \( \mathcal{C} \) to obtain that \( f \) is a Möbius valuation.

For this, let \( c \in \mathcal{C} \), and let \( c' \) be a maximal clique in \( \mathcal{C} \). Since \((\mathcal{C}, \rightarrow)\) is connected according to Lemma 11.1, there exists a sequence \( c_1, \ldots, c_n \) of cliques such that \( c_1 = c \), \( c_n = c' \), and \( c_i \rightarrow c_{i+1} \) holds for all \( i \in \{1, \ldots, n-1\} \). Since \( c' \) is maximal, the definition of the Möbius transform yields \( h(c') = f(c') \), and thus \( \mathcal{P}(C_1 = c_1, \ldots, C_n = c_n) = f(c_1) \cdots f(c_n) > 0 \). This implies that \( h(c) = \mathcal{P}(C_1 = c) > 0 \). We have proved that \( f \) is a Möbius valuation, and completed the proof of point 1 in Theorem 3.3.

Finally, it remains only to show that \((C_n)_{n \geq 1}\) is an aperiodic and irreducible Markov chain with the specified transition matrix, since the law of \( C_1 \) has already been identified as \( h|_{\mathcal{C}} \).

From the general formula proved above, we derive, if \( c_1 \rightarrow \cdots \rightarrow c_n \) holds:
\[
\mathcal{P}(C_1 = c_1, \ldots, C_n = c_n | C_1 = c_1, \ldots, C_{n-1} = c_{n-1}) = \frac{1}{h(c_{n-1})} f(c_{n-1}) h(c_n).
\]
Since \( h(0) = 0 \), it follows from Proposition 10.3 and using the same notation \( g \), that \( h(c_{n-1}) = f(c_{n-1}) g(c_{n-1}) \). Therefore:
\[
\mathcal{P}(C_1 = c_1, \ldots, C_n = c_n | C_1 = c_1, \ldots, C_{n-1} = c_{n-1}) = \frac{h(c_n)}{g(c_{n-1})}.
\]
Since the latter quantity only depends on \((C_{n-1}, C_n)\), it follows that \((C_n)_{n \geq 1}\) is a Markov chain with the transition matrix described in the statement of point 1 of Theorem 4.1.

The chain is irreducible since \((\mathcal{C}, \rightarrow)\) is connected, as already observed. And it is aperiodic since \( c \rightarrow c \) holds for any \( c \in \mathcal{C} \). The proof is complete.

**Proof of Proposition 2.1** Consider any measure of finite mass on \( \partial M \). Let \( u \in M \) be any trace. The elementary cylinder \( \uparrow u \) writes as the union: \( \uparrow u = \bigcup_{\gamma \in \Xi} \uparrow (u \cdot \alpha) \). Applying Poincaré inclusion-exclusion principle as above yields the expected formula 4.

### 13. From Möbius valuations to Bernoulli measures, through Markov chains

In this section, we consider a trace monoid \( M \) equipped with a Möbius valuation \( f : M \rightarrow \mathbb{R}_+^* \), and we establish the existence and uniqueness of a probability measure on \((\partial M, \mathcal{F})\) such that \( f(\cdot) = \mathcal{P}(\uparrow \cdot) \). This corresponds to the proof of point 2 of Theorem 8.3 and of point 2 of Theorem 4.4.

The uniqueness of \( \mathcal{P} \) follows from the remark made in § 2 that elementary cylinders form a \( \pi \)-system generating \( \mathcal{F} \).

For proving the existence of \( \mathcal{P} \), we proceed by considering first the Markov chain on the Cartier-Foata subshift which is necessarily induced by \( \mathcal{P} \), if it exists. Let \( h : \mathcal{C} \rightarrow \mathbb{R} \) be the Möbius transform of the Möbius valuation \( f \). By assumption, \( h(0) = 0 \), and therefore, thanks to Corollary 10.2, \( \sum_{c \in \mathcal{C}} h(c) = 1 \). Since \( h > 0 \) on \( \mathcal{C} \) by assumption, it follows that \( h|_{\mathcal{C}} \) defines a probability distribution on \( \mathcal{C} \).
Furthermore, the normalization factor defined by
\[ g(c) = \sum_{c' \in \mathcal{C} : c \rightarrow c'} h(c') \]
is non-zero on \( \mathcal{C} \). Hence the stochastic matrix \( P = (P_{c,c'})_{(c,c') \in \mathcal{C} \times \mathcal{C}} \) is well defined by
\[
P_{c,c'} = \begin{cases} \frac{h(c')}{g(c)}, & \text{if } c \rightarrow c' \\ 0, & \text{if } \neg(c \rightarrow c') \end{cases}.
\]

Let \( \mathcal{Q} \) be the probability measure on the sample space \((\Omega, \mathcal{G})\), corresponding to the law of the Markov chain on \( \mathcal{C} \) with \( h\big|_{\mathcal{C}} \) as initial measure and with \( P \) as transition matrix. Let finally \( \mathcal{P} \) be the probability measure on \((\partial \mathcal{M}, \mathcal{F})\) associated with \( \mathcal{Q} \). Then we claim that \( \mathcal{P}(\uparrow u) = f(u) \) holds for all traces \( u \in \mathcal{M} \).

First, we observe that, for any integer \( n \geq 1 \) and any sequence of cliques \( \delta_1 \rightarrow \cdots \rightarrow \delta_n \), the following identify holds:
\[
\mathcal{P}(C_1 = \delta_1, \ldots, C_n = \delta_n) = f(\delta_1) \cdots f(\delta_{n-1}) h(\delta_n).
\]

Indeed, \( h(0) = 0 \) by assumption, and this implies \( h = fg \) on \( \mathcal{C} \) according to Proposition 10.3. Using the form of the transition matrix \( P \) and the definition of the initial law of the chain \( (C_n)_{n \geq 1} \), we have thus:
\[
\mathcal{P}(C_1 = \delta_1, \ldots, C_n = \delta_n) = h(\delta_1) \frac{h(\delta_2)}{g(\delta_1)} \cdots \frac{h(\delta_n)}{g(\delta_{n-1})} = f(\delta_1) \cdots f(\delta_{n-1}) h(\delta_n),
\]

which proves (44). We recognize the generalized form of the Möbius transform introduced in (31) for the valuation \( f \), and obtain thus:
\[
\mathcal{P}(C_1 = \delta_1, \ldots, C_n = \delta_n) = h(\delta_1 \cdots \delta_n).
\]

We now prove \( \mathcal{P}(\uparrow u) = f(u) \) for \( u \in \mathcal{M} \). Trivially, \( \mathcal{P}(\uparrow 0) = f(0) = 1 \).

Let \( u \) be a non-empty trace, and let \( n = \tau(u) \) be the height of \( u \). It follows from (20) stated in Proposition 8.5 that we have:
\[
\mathcal{P}(\uparrow u) = \mathcal{P}(\uparrow u) = \mathcal{P}(C_1 \cdots C_n \geq u).
\]

The random trace \( C_1 \cdots C_n \) ranges over traces of height \( n \). Combining (45) and (46) yields thus:
\[
\mathcal{P}(\uparrow u) = \sum_{u' \in \mathcal{M} : \tau(u') = \tau(u), u' \geq u} h(u').
\]

By Proposition 10.4, we deduce that \( \mathcal{P}(\uparrow u) = f(u) \), as claimed. This completes the proofs of point 2 of Theorem 3.3 and of point 2 of Theorem 4.1.

14. Uniform measures: existence and uniqueness

This section is devoted to the proof of Theorem 5.1 and of Proposition 5.2.

We consider a trace monoid \( \mathcal{M} = M(\Sigma, I) \), and we let \( p_0 \) be the unique root of smallest modulus of \( \mu_{\mathcal{M}} \), which is well defined according to Theorem 9.1. Let also \( f_0(u) = p_0^{\lvert u \rvert} \) be the uniform valuation associated to \( p_0 \).
Existence of a uniform Bernoulli measure. We aim at applying Theorem 3.3 to obtain the existence of a probability measure $P$ on $\partial M$ such that $P(\uparrow \cdot) = f_0(\cdot)$ on $\mathcal{M}$.

Accordingly, we only have to check that the uniform valuation $f_0$ is a Möbius valuation. As already noted in §5, if $h : \mathcal{C} \to \mathbb{R}$ is the Möbius transform of $f$, the condition $h(0) = 0$ is equivalent to $p_0$ being a root of $\mu_M$, which is fulfilled. According to the equivalence stated in Definition 3.2, the condition $h > 0$ on $\mathcal{C}$ amounts to check that $\mu_M(c) > 0$ for all $c \in \mathcal{C}$, and this derives from Proposition 9.3. Hence $f_0$ is indeed a Möbius valuation, which implies the existence of the desired probability measure.

Uniqueness of the uniform measure. The uniqueness of uniform probability measures entails the uniqueness of Bernoulli uniform measures, hence we restrict ourselves to proving the following: if $(\gamma_n)_{n \geq 0}$ is a sequence of real numbers such that

$$\forall n \geq 0 \forall u \in \mathcal{M} \quad |u| = n \implies P(\uparrow u) = \gamma_n,$$

then $\gamma_n = p_0^n$ for all $n \geq 0$.

Let $\lambda_n = \lambda_M(n)$ denote the number of traces of length $n$ in $\mathcal{M}$ for all integer $n \geq 0$. Consider the following two generating series:

$$G(X) = \sum_{n \geq 0} \lambda_n X^n, \quad S(X) = \sum_{n \geq 0} \gamma_n X^n.$$

According to Theorem 9.1 point 1 we have $G(X) = 1/\mu_M(X)$ where $\mu_M(X)$ is the Möbius polynomial of $\mathcal{M}$. By developing $G(X)\mu_M(X)$, we obtain in particular:

$$\forall n \geq \max_{c \in \mathcal{C}} |c| \sum_{c \in \mathcal{C}} (-1)^{|c|}\lambda_{n - |c|} = 0.$$

Now let us turn our attention to $S(X)$. According to Proposition 2.1 we have: $\sum_{c \in \mathcal{C}} (-1)^{|c|}P(\uparrow (u \cdot c)) = 0$ for all $u \in \mathcal{M}$. Using (17), it translates as:

$$\forall n \geq 0 \sum_{c \in \mathcal{C}} (-1)^{|c|}\gamma_{n + |c|} = 0.$$

In view of (19) and (50), we are steered to consider $G(X)$ and $S(X)$ as being sort of dual. We are going to build upon this.

Equation (50) can be rewritten as $\gamma_n = \sum_{c \in \mathcal{C}} (-1)^{|c|+1}\gamma_{n + |c|}$. By injecting this identity in $S(X)$, we get

$$S(X) = \sum_{n \geq 0} \left( \sum_{c \in \mathcal{C}} (-1)^{|c|+1}\gamma_{n + |c|} \right) X^n$$

$$= \sum_{c \in \mathcal{C}} (-1)^{|c|+1}X^{-|c|} \left( S(X) - \sum_{i=0}^{n - 1} \gamma_i X^i \right).$$

Collecting the different terms involving $S(X)$, we recognize the coefficients of the Möbius polynomial $\mu_M(X)$ and obtain:

$$S(X)\mu_M(1/X) = \sum_{c \in \mathcal{C}} (-1)^{|c|}X^{-|c|} \left( \sum_{i=0}^{|c| - 1} \gamma_i X^i \right).$$

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Note that this proves already that $S(X)$ is rational.

Set $\ell = \max_{x \in \Sigma} |c|$. Then $\mu_M(X)$ is a polynomial of degree $\ell$. Let $p_0, \ldots, p_{\ell-1}$ be the roots of $\mu_M(X)$, with $p_0 < |p_1| \leq |p_2| \leq \cdots \leq |p_{\ell-1}|$. Denoting by “$\sim$” the proportionality relation, we have $\mu_M(1/X) \propto X^{-\ell} (1-p_0 X) \cdots (1-p_{\ell-1} X)$, which yields:

$$S(X) \propto \frac{P(X)}{(1-p_0 X) \cdots (1-p_{\ell-1} X)} \propto \frac{P(X)}{(X-1/p_0) \cdots (X-1/p_{\ell-1})}. \quad (51)$$

$$P(X) = \sum_{c \in C} (-1)^{|c|} X^{|c|} \sum_{i=0}^{|c|-1} \gamma_i X^i.$$  

We observe that $P$ is a polynomial of degree at most $\ell - 1$.

Let $w$ be a trace as in the hat lemma 11.2 that is, satisfying (42). Set $|w| = q$. Define, for all integers $n \geq 0$, the set $D_n = \{ u \cdot w \mid u \in M_{n-q} \}$ where $M_k = \{ u \in M : |u| = k \}$ for all integers $k \geq 0$. Observe that $D_n \subseteq M_n$ and, by cancellativity of the trace monoid $M$, that $D_n$ is in bijection with $M_{n-q}$. Hence:

$$|D_n| = |M_{n-q}| \sim_{n \to \infty} C_1 (1/p_0)^n, \quad (52)$$

for some constant $C_1 > 0$, according to Theorem 5.1 point 3. The cylinders $\uparrow u$ for $u$ ranging over $D_n$ are disjoint by construction of $w$, we have thus $\sum_{u \in D_n} P(\uparrow u) \leq 1$. But, according to (49), we have $\sum_{u \in D_n} P(\uparrow u) = |D_n| \gamma_n$. So we get $|D_n| \cdot \gamma_n \leq 1$. Using (49), we obtain

$$\forall n \geq 0 \quad \gamma_n \leq C_2 p_0^n, \quad (53)$$

for some constant $C_2 > 0$.

Returning to the expression (51) for $S$, the roots of the denominator are:

$$1/|p_{\ell-1}| \leq 1/|p_{\ell-2}| \leq \cdots \leq 1/|p_1| < 1/p_0.$$  

Hence, would any of the roots $1/p_j$ with $j > 0$ not be a root of the numerator $P$, that would prevent (53) to hold. Since $P$ is of degree at most $\ell - 1$, we deduce that $1/p_{\ell-1}, \ldots, 1/p_1$ are exactly all the roots of $P$, and (51) rewrites as $S(X) = K/(1-p_0 X)$ for some constant $K \neq 0$. Evaluating both members at $X = 0$ yields $K = 1$ since $\gamma_0 = 1$, and thus $S(X) = 1/(1-p_0 X)$. Since $S(X) = \sum_{n \geq 0} \gamma_n X^n$ by definition, we obtain that $\gamma_n = p_0^n$ for all $n \geq 0$, and the proof is complete.

**Proof of Proposition 5.2** Let $p_0$ be the unique root of smallest modulus of the Möbius polynomial. Start with the valuation $f$ defined by $f(\alpha) = p_0$ for all $\alpha \in \Sigma$. By Theorem 5.1 $f$ is a Möbius valuation.

Now consider a collection of reals $\varepsilon = (\varepsilon_\alpha)_{\alpha \in \Sigma}$ such that $p_0 + \varepsilon_\alpha \in (0,1)$ for all $\alpha \in \Sigma$. Set $\varepsilon = 0$ as a shorthand for $\varepsilon = (\varepsilon_\alpha = 0)_{\alpha \in \Sigma}$. Let $f_{\varepsilon}$ be the valuation defined by: $f_{\varepsilon}(\alpha) = p_0 + \varepsilon_\alpha$ for all $\alpha \in \Sigma$. Let $h_{\varepsilon}$ be the associated Möbius transform. The goal is to show that there exist a continuous family of values for $(\varepsilon_\alpha)_{\alpha \in \Sigma}$ such that $f_{\varepsilon}$ is Möbius, that is:

$$(i) : \quad h_{\varepsilon}(0) = 0, \quad (ii) : \quad \forall c \in \mathfrak{C} \quad h_{\varepsilon}(c) > 0.$$  

First, observe that condition (ii) is an open condition and that it is satisfied for $\varepsilon = 0$. So it is still satisfied if $|\varepsilon_\alpha|$ is small enough, for all $\alpha \in \Sigma$.  

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Now let us concentrate on \((i)\). Fix a letter \(a \in \Sigma\). The equation \(h_\varepsilon(0) = 0\) is an affine equation in \(\varepsilon_a\) if the values \(\varepsilon_\alpha\) for \(\alpha \neq a\) are fixed:
\[
(p_0 + \varepsilon_a)A_\varepsilon + B_\varepsilon = 0,
\]
with
\[
A_\varepsilon = \sum_{c \in \mathcal{C}} (-1)^{|c|} \prod_{\alpha \in c, \alpha \neq a} (p_0 + \varepsilon_\alpha), \quad B_\varepsilon = \sum_{c \in \mathcal{C}} (-1)^{|c|} \prod_{\alpha \in c} (p_0 + \varepsilon_\alpha).
\]
Observe that we have:
\[
A_0 = \sum_{c \in \mathcal{C}} (-1)^{|c|} |p_0|^{|c|-1}, \quad B_0 = \sum_{c \in \mathcal{C}} (-1)^{|c|} |p_0|^{|c|}.
\]
We recognize in \(B_0\) the M"obius polynomial of the independence pair \((\Sigma', I')\), with \(\Sigma' = \Sigma \setminus \{a\}\) and \(I' = I \cap (\Sigma' \times \Sigma')\), evaluated at \(p_0\). But \(M\) is irreducible, and as already observed in the proof of Proposition 9.3, the comparison of growth rates entails that \(p_0\) is strictly smaller in modulus than all the roots of the polynomial \(\mu_M(\Sigma', I')\). Hence \(B_0 \neq 0\). Since \(p_0A_0 + B_0 = 0\), we conclude that \(A_0 \neq 0\) and thus \(A_\varepsilon \neq 0\) for \(\varepsilon\) small enough. Consequently, the equation \((p_0 + \varepsilon_a)A_\varepsilon + B_\varepsilon = 0\) has a unique solution in \(\varepsilon_a\) if all \(\varepsilon_\alpha\) are small enough for \(\alpha \neq a\). Since \(|\Sigma| > 1\) by assumption, there is indeed an uncountable number of values for \((\varepsilon_\alpha)_{\alpha \neq a}\) arbitrary close to 0.

We have proved the existence of a continuous family of distinct M"obius valuations, and we conclude by Theorem 3.3.

15. Perspectives for future work

In the present paper, we consider only irreducible trace monoids. The extension to non-irreducible trace monoid works out nicely and the complete description of Bernoulli measures is postponed to a companion paper.

Further developments of this work can be expected. First, it is natural to adapt our construction to trace groups. It would also be interesting to generalize our approach to other monoids or groups. Braid monoids and groups, and more generally Artin monoids and groups of finite Coxeter type, are natural candidates. Indeed, these structures share with trace monoids a similar “combinatorial flavor”.

Another extension consists in studying Markovian measures on infinite traces instead of Bernoulli measures—a problem already considered by one of the authors in restricted contexts [2, 3]. A different direction for extension would be to study Bernoulli measures on the executions of a 1-bounded Petri net. In this case, the executions are described as a regular trace language.
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