A Generalized Jaynes-Cummings Hamiltonian and Supersymmetric Shape-Invariance

A. N. F. Aleixo*
Department of Physics, University of Wisconsin; Madison, Wisconsin 53706 USA

A. B. Balantekin†
Max-Planck-Institut für Kernphysik, Postfach 103980, D-69029 Heidelberg, Germany

M. A. Cândido Ribeiro‡,
Departamento de Física - Instituto de Biociências, Letras e Ciências Exatas
UNESP, São José do Rio Preto, SP - Brazil

Abstract

A class of shape-invariant bound-state problems which represent two-level systems are introduced. It is shown that the coupled-channel Hamiltonians obtained correspond to the generalization of the Jaynes-Cummings Hamiltonian.

03.65.-w,03.65.Fd

*Permanent address: Instituto de Física, Universidade Federal do Rio de Janeiro, RJ - Brazil. Electronic address: aleixo@nucth.physics.wisc.edu

†Permanent address: Department of Physics, University of Wisconsin, Madison, Wisconsin 53706 USA. Electronic address: baha@nucth.physics.wisc.edu

‡Electronic address: macr@df.ibilce.unesp.br
I. INTRODUCTION

Supersymmetric quantum mechanics [1,2] deals with pairs of Hamiltonians which have the same energy spectra, but different eigenstates. A number of such pairs of Hamiltonians share an integrability condition called shape invariance [3]. Although not all exactly-solvable problems are shape-invariant [4], shape invariance, especially in its algebraic formulation [5–7], is a powerful technique to study exactly-solvable systems.

Supersymmetric quantum mechanics is generally studied in the context of one-dimensional systems. The partner Hamiltonians

\[ \hat{H}_1 = \hat{A}^\dagger \hat{A}, \]
\[ \hat{H}_2 = \hat{A} \hat{A}^\dagger, \]

are most readily written in terms of one-dimensional operators

\[ \hat{A} \equiv W(x) + \frac{i}{\sqrt{2m}} \hat{p}, \] (1.2a)
\[ \hat{A}^\dagger \equiv W(x) - \frac{i}{\sqrt{2m}} \hat{p}, \] (1.2b)

where \( W(x) \) is the superpotential. Attempts were made to generalize supersymmetric quantum mechanics and the concept of shape-invariance beyond one-dimensional and spherically-symmetric three-dimensional problems. These include non-central [8], non-local [9], and periodic [10] potentials; a three-body problem in one-dimension [11] with a three-body force [12]; N-body problem [13]; and coupled-channel problems [14,15]. It is not easy to find exact solutions to these problems. For example, in the coupled-channel case a general shape-invariance is only possible in the limit where the superpotential is separable [15] which corresponds to the well-known sudden approximation in the coupled-channel problem [16].

Our goal in this article is to introduce a class of shape-invariant coupled-channel problems which correspond to the generalization of the Jaynes-Cummings Hamiltonian [17].

II. SHAPE INVARIANCE

The Hamiltonian \( \hat{H}_1 \) of Eq. (1.1) is called shape-invariant if the condition

\[ \hat{A}(a_1) \hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2) \hat{A}(a_2) + R(a_1), \] (2.1)

is satisfied [3]. In this equation \( a_1 \) and \( a_2 \) represent parameters of the Hamiltonian. The parameter \( a_2 \) is a function of \( a_1 \) and the remainder \( R(a_1) \) is independent of the dynamical variables such as position and momentum. As it is written the condition of Eq. (2.1) does not require the Hamiltonian to be one-dimensional, and one does not need to choose the ansatz of Eq. (1.2). In the cases studied so far the parameters \( a_1 \) and \( a_2 \) are either related by a translation [14,15] or a scaling [19]. Introducing the similarity transformation that replaces \( a_1 \) with \( a_2 \) in a given operator

\[ \hat{T}(a_1) \hat{O}(a_1) \hat{T}^\dagger(a_1) = \hat{O}(a_2) \] (2.2)
and the operators
\[ \hat{B}_+ = \hat{A}^\dagger(a_1) \hat{T}(a_1) \] (2.3)
\[ \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1) \hat{A}(a_1), \] (2.4)
the Hamiltonians of Eq. (1.1) take the forms
\[ \hat{H}_1 = \hat{B}_+ \hat{B}_-. \] (2.5)
and
\[ \hat{H}_2 = \hat{T} \hat{B}_- \hat{B}_+ \hat{T}^\dagger. \] (2.6)
Using Eq. (2.1) one can also easily prove the commutation relation
\[ [\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1) R(a_1) \hat{T}(a_1) \equiv R(a_0), \] (2.7)
where we used the identity
\[ R(a_n) = \hat{T}(a_1) R(a_{n-1}) \hat{T}^\dagger(a_1), \] (2.8)
valid for any \( n \). The ground state of the Hamiltonian \( \hat{H}_1 \) satisfies the condition
\[ \hat{A} \left| \psi_0 \right\rangle = 0 = \hat{B}_- \left| \psi_0 \right\rangle. \] (2.9)
The \( n \)-th excited state of \( \hat{H}_1 \) is given by
\[ \left| \psi_n \right\rangle \sim (\hat{B}_+)^n \left| \psi_0 \right\rangle \] (2.10)
with the eigenvalue
\[ \varepsilon_n = \sum_{k=1}^{n} R(a_k). \] (2.11)
Note that the eigenstate of Eq. (2.10) needs to be suitably normalized. We discuss the normalization of this state in the next section.

III. GENERALIZATION OF THE JAYNES-CUMMINGS HAMILTONIAN

To generalize the Jaynes-Cummings Hamiltonian to general shape-invariant systems we introduce the operator
\[ \hat{S} = \sigma_+ \hat{A} + \sigma_- \hat{A}^\dagger, \] (3.1)
where
\[ \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2), \] (3.2)
with $\sigma_i$, with $i = 1, 2, 3$, being the Pauli matrices and the operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the shape invariance condition of Eq. (2.1). We search for the eigenstates of $\hat{S}$. It is more convenient to work with the square of this operator, which can be written as

$$\hat{S}^2 = \begin{bmatrix} \hat{T} & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}.$$  (3.3)

Note the freedom of sign choice in this equation, which results in two possible decompositions of $\hat{S}^2$.

We next introduce the states

$$|\Psi\rangle_\pm = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} |m\rangle \\ |n\rangle \end{bmatrix}$$  (3.4)

where $|m\rangle$ and $|n\rangle$ are the abbreviated notation for the states $|\psi_n\rangle$ and $|\psi_m\rangle$ of Eq. (2.10). Using Eqs. (2.7), (3.3) and (3.4) and the fact that the operator $\hat{T}$ is unitary one gets

$$\hat{S}^2 |\Psi\rangle_\pm = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_+ \hat{B}_- + R(a_0) & 0 \\ 0 & \hat{B}_+ \hat{B}_- \end{bmatrix} \begin{bmatrix} |m\rangle \\ |n\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \varepsilon_m + R(a_0) & 0 \\ 0 & \varepsilon_n \end{bmatrix} \begin{bmatrix} |m\rangle \\ |n\rangle \end{bmatrix}.  \quad (3.5)$$

Using Eqs. (2.8) and (2.11) one can write

$$\hat{T} [\varepsilon_m + R(a_0)] \hat{T}^\dagger = \hat{T} [R(a_1) + R(a_2) + \cdots + R(a_m) + R(a_0)] \hat{T}^\dagger$$

$$= R(a_2) + R(a_3) + \cdots + R(a_{m+1}) + R(a_1) = \varepsilon_{m+1}.  \quad (3.6)$$

Hence the states

$$|\Psi_m\rangle_\pm = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} |m\rangle \\ |m+1\rangle \end{bmatrix}, \quad m = 0, 1, 2, \cdots  \quad (3.7)$$

are the normalized eigenstates of the operator $\hat{S}^2$

$$\hat{S}^2 |\Psi_m\rangle_\pm = \varepsilon_{m+1} |\Psi_m\rangle_\pm.  \quad (3.8)$$

One can also calculate the action of the operator $\hat{S}$ on this state

$$\hat{S} |\Psi_m\rangle_\pm = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm \hat{T} \hat{B}_- & |m+1\rangle \\ \hat{B}_+ & |m\rangle \end{bmatrix}.  \quad (3.9)$$

Introducing the operator $\hat{Q}$

$$\hat{Q}^\dagger = (\hat{B}_+ \hat{B}_-)^{-1/2} \hat{B}_+  \quad (3.10)$$

one can write the normalized eigenstate of $\hat{H}_1$ as

$$|m\rangle = (\hat{Q}^\dagger)^m |0\rangle.  \quad (3.11)$$

Using Eqs. (3.10) and (3.11) one gets
\[ \hat{B}_+ | m \rangle = \sqrt{\varepsilon_{m+1}} | m + 1 \rangle. \] (3.12)

Similarly
\[
\hat{T}{\hat{B}}_- | m + 1 \rangle = \hat{T}{\hat{B}}_- \frac{1}{\sqrt{\hat{B}_- \hat{B}_+}} \hat{B}_+ | m \rangle
= \hat{T} \sqrt{\hat{B}_- \hat{B}_+} | m \rangle
= \hat{T} \sqrt{\varepsilon_m + R(a_0)} | m \rangle
= \sqrt{\varepsilon_{m+1}} \hat{T} | m \rangle.
\] (3.13)

Using Eqs. (3.12) and (3.13), Eq. (3.9) takes the form
\[
\hat{S} | \Psi_m \rangle_\pm = \frac{1}{\sqrt{2}} \sqrt{\varepsilon_{m+1}} \left[ \pm \hat{T} | m \rangle \right]
= \pm \sqrt{\varepsilon_{m+1}} | \Psi_m \rangle_\pm.
\] (3.14)

Eqs. (3.8) and (3.14) indicate that the Hamiltonian
\[ \hat{H} = \hat{S}^2 + \sqrt{\hbar \Omega} \hat{S}, \] (3.15)
where \( \Omega \) is a constant, has the eigenstates \( | \Psi_m \rangle_\pm \)
\[ \hat{H} | \Psi_m \rangle_\pm = \left( \varepsilon_{m+1} \pm \sqrt{\hbar \Omega \sqrt{\varepsilon_{m+1}}} \right) | \Psi_m \rangle_\pm \] (3.16)
with the exception of the ground state. It is easy to show that the ground state is
\[ | \Psi_0 \rangle = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \] (3.17)
with eigenvalue 0. To emphasize the structure of Eq. (3.16) as the generalized Jaynes-Cummings Hamiltonian we rewrite it as
\[ \hat{H} = \hat{A}^\dagger \hat{A} + \frac{1}{2} \left[ \hat{A}, \hat{A}^\dagger \right] (\sigma_3 + 1) + \sqrt{\hbar \Omega} \left( \sigma_+ \hat{A} + \sigma_- \hat{A}^\dagger \right). \] (3.18)

When \( \hat{A} \) describes the annihilation operator for the harmonic oscillator, \( [\hat{A}, \hat{A}^\dagger] = \hbar \omega \),
where \( \omega \) is the oscillator frequency. In this case Eq. (3.18) reduces to the standard Jaynes-Cummings Hamiltonian.

When \( \hat{A}^\dagger \hat{A} \) describes the Morse Hamiltonian, Eq. (3.18) takes the form
\[ \hat{H} = \frac{\hat{p}^2}{2M} + V_0 \left( e^{-2\lambda x} - 2e^{-\lambda x} \right) + \sqrt{V_0} \frac{\hbar \lambda}{\sqrt{2M}} (\sigma_3 + 1) e^{-\lambda x} \]
\[ + \sqrt{\hbar \Omega V_0} \left[ \sigma_1 \left( 1 - \frac{\hbar \lambda}{2\sqrt{2MV_0}} e^{-\lambda x} \right) - \sigma_2 \frac{\hat{p}}{\sqrt{2MV_0}} \right] \] (3.19)
with the energy eigenvalues
\[ E_m = \sqrt{V_0} \frac{\hbar \lambda}{\sqrt{2M}} (m + 1) \left[ 2 - \frac{\hbar \lambda}{\sqrt{2MV_0}} (m + 2) \right] \]
\[ \pm \left\{ \frac{\hbar \Omega}{\sqrt{V_0}} \frac{\hbar \lambda}{\sqrt{2M}} (m + 1) \left[ 2 - \frac{\hbar \lambda}{\sqrt{2MV_0}} (m + 2) \right] \right\}^{1/2}. \] (3.20)

Both harmonic oscillator and Morse potential are shape-invariant potentials where parameters are related by a translation. It is also straightforward to use those shape-invariant potentials where the parameters are related by a scaling \[19\] in writing down Eq. (3.18).

**IV. CONCLUSIONS**

In this article we introduced a class of shape-invariant bound-state problems which represent two-level systems. The corresponding coupled-channel Hamiltonians generalize the Jaynes-Cummings Hamiltonian. If we take \( \hat{H}_1 \) to be the simplest shape-invariant system, namely the harmonic oscillator, our Hamiltonian, Eq. (3.18), reduces to the standard Jaynes-Cummings Hamiltonian, which has been extensively used to model a single field mode on resonance with atomic transitions.

In this article we only addressed generalization of the Jaynes-Cummings model to other shape-invariant bound state systems. Supersymmetric quantum mechanics has been applied to alpha particle \[20\] and Coulomb \[21\] scattering problems. More recently shape-invariance was utilized to calculate quantum tunneling probabilities \[22\]. It may be possible to generalize our results to such continuum problems. Such an investigation will be deferred to a later publication.

**ACKNOWLEDGMENTS**

This work was supported in part by the U.S. National Science Foundation Grant No. PHY-9605140 at the University of Wisconsin, and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation. A.B.B. acknowledges the support of the Alexander von Humboldt-Stiftung. M.A.C.R. acknowledges the support of Fundação de Amparo à Pesquisa do Estado de São Paulo (Contract No. 98/13722-2). A.N.F.A. acknowledges the support of Fundação Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (Contract No. BEX0610/96-8). A.B.B. thanks to the Max-Planck-Institut für Kernphysik and M.A.C.R. to the Nuclear Theory Group at University of Wisconsin for the very kind hospitality.
REFERENCES

[1] E. Witten, Nucl. Phys. B 185, 513 (1981).
[2] For a recent review see F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).
[3] L. Gendenshtein, Pis'ma Zh. Eksp. Teor. Fiz. 38, 299 (1983) [JETP Lett. 38, 356 (1983)].
[4] F. Cooper, J. N. Ginocchio and A. Khare, Phys. Rev. D 36, 2458 (1987).
[5] A. B. Balantekin, Phys. Rev. A 57, 4188 (1998).
[6] S. Chaturvedi, R. Dutt, A. Gangopadhyay, P. Panigrahi, C. Rasinariu, and U. Sukhatme, Phys. Lett. A 248, 109 (1998).
[7] A. B. Balantekin, M. A. Cândido Ribeiro, and A. N. F. Aleixo, J. Phys. A: Math. Gen. 32, 2785 (1999).
[8] R. Dutt, A. Gangopadhyay, and U. Sukhatme, Am. J. Phys. 65, 400 (1997).
[9] J.-Y. Choi and S.-I. Hong, Phys. Rev. A 60, 796 (1999).
[10] G. Dunne and J. Feinberg, Phys. Rev D 57, 1271 (1998).
[11] D. Z. Freedman and P. F. Mende, Nucl. Phys. B 344, 317 (1990).
[12] A. Khare and R. K. Bhaduri, J. Phys. A: Math. Gen. 27, 2213 (1994).
[13] P. K. Ghosh, A. Khare, and M. Sivakumar, Phys. Rev. A 58, 821 (1998).
[14] R. D. Amado, F. Cannata, and J.-P. Dedonder, Phys. Rev. A 38, 3797 (1988); Int. J. Mod. Phys. A 5, 3401 (1990).
[15] T. K. Das and B. Chakrabarti, J. Phys. A: Math. Gen. 32, 2387 (1999).
[16] A. B. Balantekin and N. Takigawa, Rev. Mod. Phys. 70, 77 (1998).
[17] E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).
[18] C. Chuan, J. Phys. A: Math. Gen. 24, L1165 (1991).
[19] A. Khare and U. Sukhatme, J. Phys. A: Math Gen. 26, L901 (1993); D. Barclay et al., Phys. Rev. A 48, 2786 (1993).
[20] D. Baye, Phys. Rev. Lett. 58, 2738 (1987).
[21] R. Amado, Phys. Rev. A 37, 2277 (1988).
[22] A. N. F. Aleixo, A. B. Balantekin, and M. A. Cândido Ribeiro, J. Phys. A: Math. Gen., in press [quant-ph/9910051].