Arrangements associated to chordal graphs and limits of colored braid groups

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Abstract

Let $G$ be a chordal graph, $X(G)$ the complement of the associated complex arrangement and $\Gamma(G)$ the fundamental group of $X(G)$. We show that $\Gamma(G)$ is a limit of colored braid groups over the poset of simplices of $G$. When $G = G_T$ is the comparability graph associated with a rooted tree $T$, a case recently investigated by the first author, the result takes the following very simple form: $\Gamma(G_T)$ is a limit over $T$ of colored braid groups.

Keywords: Braid groups, chordal graphs, complex hyperplane arrangements, limits over a poset, rooted trees.

Introduction

In [5], the first author investigated certain properties of the arrangements of hyperplanes associated to rooted trees. As pointed out by Stanley, these arrangements are special cases of arrangements associated to chordal graphs. Let $G$ be such a graph, let $X(G)$ denote the complement of the associated complex arrangement and let $\Gamma(G)$ be the fundamental group of $X(G)$.

In this paper, we show that $\Gamma(G)$ can be described, in a compact manner, as a limit of colored braid groups over the poset $S(G)$ of simplices of $G$ (that is, the subsets of the set of vertices for which the induced subgraph is complete). In the special case where $G = G_T$ is the comparability graph associated with a rooted tree $T$, the maximal simplices of $G_T$ are, of course, parametrized by the leaves of $T$ and the result can be reformulated as follows: $\Gamma(T) := \Gamma(G_T)$ is a limit over $T$ of colored braid groups.

The plan of the article is as follows. In Section 1, we recall several facts about chordal graphs, graphical arrangements, fibrations, pull-backs and limits. After these preliminaries, we state and prove, in Section 2, the main theorem. Finally, in Section 3, we consider the case of rooted trees, where the results take a simpler and slightly more precise form.
1 Recollections

1.1 Chordal graphs

Let $G = (V, E)$ be a finite graph, where $V$ is the set of vertices and $E$ the set of edges. A cycle $C$ in $G$, of length $k \geq 3$, is a sequence $v_1, \ldots, v_k$ of pairwise distinct vertices such that $\{v_k, v_1\}$ and $\{v_i, v_{i+1}\}$, for $i = 1, \ldots, k - 1$, are edges. A chord of $C$ is an edge $\{v_i, v_j\}$ distinct from the previous ones, i.e., such that $v_i$ and $v_j$ are not consecutive vertices in the cycle.

One says that $G$ is chordal (or a rigid circuit graph) if every cycle of length $\geq 4$ admits at least one chord.

Any subset $S$ of $V$ determines an induced subgraph, whose edges are the edges of $G$ connecting two elements of $S$. By abuse of language, we will most of the time make no distinction between a subset of $V$ and the corresponding subgraph. From the definition above it is clear that if $G$ is chordal then so is the subgraph induced by any subset $V'$ of $V$.

On the other hand, one says that a subset $S$ of $V$ is a simplex of $G$ if the induced subgraph is complete, that is, if $\{s, s'\}$ is an edge for all $s \neq s'$ in $S$.

1.2 Simplicial vertices and PEO’s

For any vertex $v$, let $N(v)$ denote the set of its neighbours, that is, the vertices connected to $v$. One says that $v$ is a simplicial vertex if $N(v)$ is a simplex. In this case, $N(v) \cup \{v\}$ is a simplex, too.

Following [3], we say that an ordering $v_1, v_2, \ldots, v_n$, where $n = |V|$, of the vertices of $G$ is a perfect elimination ordering (PEO in short) if $v_i$ is a simplicial vertex of $G \setminus \{v_1, \ldots, v_{i-1}\}$, for all $i = 1, \ldots, n$. (This is called a vertex elimination order in [6]; we prefer the above terminology, which we find more suggestive.)

Of course, in a complete graph any vertex is simplicial and any ordering of the vertices is a PEO.

For future reference, let us record the following lemma and corollary.

**Lemma** 1) A vertex is simplicial if and only if it belongs to a unique maximal simplex of $G$.

2) Suppose that $G$ is chordal but not complete. Then $G$ admits at least two nonadjacent simplicial vertices. In particular, if $S$ is a simplex of $G$, then $G \setminus S$ contains a simplicial vertex of $G$.

The first part is an almost immediate consequence of the definitions, see e.g. [3] Lemma 3.1]. For the second part, we refer to [3] Lemma 2.2] or [10] Lemma 4.2].

Since any induced subgraph of a chordal graph is still chordal, this implies immediately the
Corollary Suppose that $G$ is chordal and let $S$ be any simplex in $G$. Then there exists a PEO $v_1, \ldots, v_n$ of $G$ such that the vertices not in $S$ appear first, i.e., such that $\{v_1, \ldots, v_k\} = V \setminus S$, where $k = |V \setminus S|$.

It turns out that, conversely, the existence of a PEO implies that $G$ is chordal, see [10, Th. 4.1] or [3, Th. 2.3]. We will not need this fact.

1.3 Fibrations

We gather in this subsection several concepts and results from homotopy theory that will be used in the sequel.

By a space we mean a pointed, Hausdorff topological space and maps are base-point preserving continuous maps. Let $B$ be a pathwise-connected space, with base point $b_0$. A continuous map $f : E \to B$ is called a Hurewicz fibration, or simply fibration, if it has the homotopy lifting property with respect to any space.

Firstly, the composite of two fibrations is again a fibration. Secondly, if $f$ is a locally trivial fibration, it is a Hurewicz fibration, at least when the base space $B$ is paracompact, e.g. metrisable. Thirdly, if $f$ is a fibration, all fibers $f^{-1}(b)$ are homotopy equivalent to $F := f^{-1}(b_0)$. Fourthly, assuming for simplicity that $F$ is pathwise connected, there is a long exact sequence of homotopy groups

$$
\cdots \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1.
$$

Moreover, if $f$ is split, i.e., admits a section, then (†) breaks into split short exact sequences. For all this see, for example, [15, Chap.7] or [18, §§I.7 & IV.8].

We will also need the following functorial property of the exact homotopy sequence of a fibration, see [11, Chap.6]. Consider a (strict) pull-back diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\tilde{g}} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{g} & B.
\end{array}
$$

By this, we mean that $W$ identifies under $(\tilde{f}, \tilde{g})$ with the closed subset of $Y \times X$ consisting of those pairs $(y, x)$ such that $g(y) = f(x)$. Suppose further that $f$ is a fibration, and let $\tilde{F} := \tilde{f}^{-1}(b_0)$ and $\tilde{F} := \tilde{f}^{-1}(y_0)$.

Then $\tilde{f}$ is also a fibration and, under the identification above, the map $\tilde{F} \to F$ induced by $\tilde{g}$ is the identity. For simplicity, assume further that $F$ is pathwise connected. Then, one has a commutative ladder

$$
\begin{array}{ccc}
\cdots & \to & \pi_2(Y) \\
g & \downarrow & \downarrow 1 \\
& \downarrow \tilde{g} & \downarrow g \\
\cdots & \to & \pi_2(B) \\
\end{array}
\quad
\begin{array}{ccc}
\pi_1(F) & \to & \pi_1(W) \\
& \xrightarrow{f} & \pi_1(Y) \to 1 \\
\pi_1(F) & \to & \pi_1(X) \\
& \xrightarrow{f} & \pi_1(B) \to 1.
\end{array}
$$
Corollary If \( \pi_1(F) \to \pi_1(X) \) is injective (e.g., if \( \pi_2(B) = \{1\} \), or if \( f \) admits a section), then so is \( \pi_1(F) \to \pi_1(W) \). In this case, \( \pi_1(W) \) is the pull-back of the diagram

\[
\pi_1(X) \xrightarrow{f} \pi_1(B) \xleftarrow{g} \pi_1(Y).
\]

1.4 Graphical arrangements and locally trivial fibrations

Let \( G = (V, E) \) be a finite graph. It determines an arrangement \( \mathcal{A}(G) \) in \( \mathbb{C}^G \), whose hyperplanes are given by the linear forms \( x_i - x_j \), when \( \{i, j\} \in E \). Let us denote by \( X(G) \) the complement of \( \mathcal{A}(G) \) and by \( \Gamma(G) \) the fundamental group of \( X(G) \).

Let \( v \in V \). Let us denote by \( p_v \) the restriction to \( X(G) \) of the natural projection \( \mathbb{C}^G \to \mathbb{C}^{G \setminus v} \). Clearly, it maps \( X(G) \) onto \( X(G \setminus v) \) and, for every \( z = (z_u)_{u \in G \setminus v} \in X(G \setminus v) \), one has

\[
p_v^{-1}(z) = \{ u \in C \mid x \neq z_u, \ \forall u \in N(v) \}.
\]

Set \( \nu(z) = \{|z_u | u \in N(v)\} \). Of course, if \( p_v \) is a locally trivial fibration then \( \nu(z) = |N(v)| \) for all \( z \in X(G \setminus v) \). It is well-known that the converse holds, see for example [11 Lemma 1] or [4 Prop.2]. This fact is also used implicitly in the proof of [16 Th. 2.9], where a considerably more general result is proved (see also [13 Th. 5.11]). Since the details are not easily found in the literature (see, however, the proof of [7 Th.1]), we recall the argument for the reader’s (and for our own) convenience.

Lemma Let \( v \) be a simplicial vertex with \( N \) neighbours. Then the projection \( p_v : X(G) \to X(G \setminus v) \) is a split, locally trivial smooth fibration, with fiber \( \mathbb{C} \) minus \( N \) points.

Proof. Let us choose a numbering \( u_1, \ldots, u_n \) of the vertices of \( G \setminus v \) such that \( N(v) = \{ u_1, \ldots, u_N \} \), and denote the corresponding coordinates \( z_u \) on \( X(G \setminus v) \) simply by \( z_i \).

Let \( z^0 \in X(G \setminus v) \). Since \( v \) is a simplicial vertex, one has \( z_i^0 \neq z_j^0 \) for \( i \neq j \) in \( [1, N] \). Pick \( \varepsilon > 0 \) such that the open disks \( D(z_i^0, 2\varepsilon) \) are disjoint, and let \( U \) denote the open subset of \( X(G \setminus v) \) defined by \( |z_i - z_i^0| < \varepsilon \), for \( i = 1, \ldots, N \).

Let \( \theta : \mathbb{C} \to [0, 1] \) be a smooth function such that \( \theta(z) = 1 \) if \( |z| \leq \varepsilon \) and \( \theta(z) = 0 \) if \( |z| \geq 3\varepsilon/2 \). The map \( V : U \times \mathbb{C} \to \mathbb{C} \) given by

\[
V(z, x) = \sum_{i=1}^N \theta(x - z_i^0)(z_i - z_i^0)
\]
is clearly $C^\infty$. For each $z$, it defines a vector field $V_z$ on $\mathbb{C}$, which depends smoothly on the parameter $z$, vanishes outside the union of the disks $D(z_i^0, 3\varepsilon/2)$, and coincides on each $D(z_i^0, \varepsilon)$ with the constant vector field $z_i^0 - z_i^0$. Since $V_z$ has compact support, the flow $\psi(z, t, x)$ corresponding to the initial condition $\psi(z, 0, x) = x$, is defined for all values of $t$.

Set $\phi(z, x) = \psi(z, 1, x)$ and $\phi_z(x) = \phi(z, x)$. Then $\phi$ is $C^\infty$ and it is not difficult to see that, for every $z$, $\phi_z$ is a diffeomorphism of $\mathbb{C}$ which equals the identity outside the union of the disks $D(z_i^0, 3\varepsilon/2)$ and maps each $z_i^0$ to $z_i$. Moreover, $\phi_z^{-1}(x)$ is nothing but $\psi(z, -1, x)$ and hence the map $\mu : (z, x) \mapsto \phi_z^{-1}(x)$ is $C^\infty$, too. Therefore, the maps

$$f : U \times (\mathbb{C} \setminus \{z_1^0, \ldots, z_N^0\}) \to p_v^{-1}(U) : g$$

given by $f(z, x) = (z, \phi(z, x))$ and $g(z, x) = (z, \mu(z, x))$ are inverse diffeomorphisms. This proves that $p_v$ is a smooth locally trivial fibration.

Moreover, by [1, Lemma 1] or [8, Prop. 2.4], $p_v$ admits a section $\sigma_v$. Indeed, following [8], one may take $\sigma_v(z) = \left(z, (1 + \sum_{i=1}^N |z_i|^2)^{1/2}\right)$. This completes the proof of the lemma.

As a well-known consequence of the lemma (see [8]), one obtains that the long exact homotopy sequence of the locally trivial fibration $p_v$ gives isomorphisms $\pi_i(X(G)) \cong \pi_i(X(G \setminus v))$ for $i \geq 2$, along with a split exact sequence

$$1 \to F_N \to \Gamma(G) \to \Gamma(G \setminus v) \to 1,$$

where $F_N$ is the free group on $N$ generators, $N = |N(v)|$.

### 1.5 Chordal graphs and fibrations

Return to $X(G)$, the complement of the arrangement associated to our graph $G$. By the previous paragraph, if $v$ is a simplicial vertex of $G$, then $X(G) \to X(G \setminus v)$ is a locally trivial fibration with a section.

Assume now that $G$ is chordal. Of course then, by iterated use of the previous lemma one obtains that $\mathcal{A}(G)$ is of fiber-type in the sense of Falk and Randell [8] and hence that $X(G)$ is a $K(\pi, 1)$ space.

Moreover, since a composite of fibrations (resp. sections thereof) is a fibration (resp. a section thereof), one obtains, using Corollary [1.2] the following corollary, which will prove useful later.

**Corollary** Let $S$ be a subset of $V$ such that there exists a PEO of $G$ beginning with the elements of $V \setminus S$. Then $X(G) \to X(S)$ is a split fibration, whose fiber and base are $K(\pi, 1)$ spaces. This is the case, in particular, if $S$ is a simplex of $G$. 

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Remark. It follows from [14, Th. 2.2 & Cor. 2.4] that the projection \( X(G) \to X(S) \) is actually a locally trivial smooth fibration. This is a general result about the projection corresponding to any modular element in the lattice of an arbitrary hyperplane arrangement, whose proof relies on Thom’s first isotopy lemma. We will not need this more general result.

1.6 Colored braid groups

Let \( \mathcal{I} \) denote the category whose objects are finite sets and whose morphisms are the injective maps. For any finite set \( I \), let \( C_I \) denote the complete graph on \( I \) and let \( P(I) := \Gamma(C_I) = \pi_1(X(C_I)) \).

Thus, \( P(I) \) is isomorphic to the braid group on \( n \) colored strands, where \( n = |I| \), see [9] or [2, §1.4].

**Proposition** The assignment \( I \mapsto P(I) \) is a contravariant functor from \( \mathcal{I} \) to the category of groups.

**Proof.** For every subset \( J \) of \( I \), one has the natural projection \( \phi_{IJ} : X(C_I) \to X(C_J) \) and the associated group homomorphism \( \rho_{IJ} := \pi_1(\phi_{IJ}) \), from \( P(I) \) to \( P(J) \). Since, clearly, \( \phi_{KJ} \circ \phi_{IJ} = \phi_{KI} \), when \( K \subset J \subset I \), the result follows from the functoriality of \( \pi_1 \).

1.7 Limits

Let \( \mathcal{S} \) be a finite poset and let \( F \) be a contravariant functor from \( \mathcal{S} \) to the category of groups. That is, one is given a morphism \( \rho_{ts} : P(s) \to P(t) \) whenever \( t \leq s \), such that \( \rho_{ut}\rho_{ts} = \rho_{us} \) if \( u \leq t \leq s \).

Then, the limit of \( F \) over \( \mathcal{S} \) is the following subgroup of \( \prod_{s \in \mathcal{S}} F(s) \) :

\[
\lim_{\mathcal{S}} F := \left\{ (g_s)_{s \in \mathcal{S}} \in \prod_{s \in \mathcal{S}} F(s) \mid \rho_{ts}(g_s) = g_t, \ \forall t \leq s \right\}.
\]

**Remarks.** 1) This is, of course, a special case of the general notion of limit, see for instance [12, Chap.V] or [17, Appendix A.5]. The above definition is sufficient for our purposes.

2) Denoting by \( M(\mathcal{S}) \) the set of maximal elements of \( \mathcal{S} \), \( \lim_{\mathcal{S}} F \) may also be regarded as a subgroup of the (smaller) product \( \prod_{s \in M(\mathcal{S})} F(s) \).

For any \( s \in \mathcal{S} \), set \( D(s) := \{ t \mid t \leq s \} \). Further, if \( s \) is a maximal element of \( \mathcal{S} \), define its cone

\[ C(s) = \{ t \leq s \mid \forall u \in \mathcal{S}, t \leq u \Rightarrow u \leq s \}. \]

Then, the following lemma is straightforward and its proof left to the reader.
Lemma Let \( s \) be a maximal element of \( S \) and let \( T \) be a subset of \( C(s) \) containing \( s \). Then \( \lim F \) is isomorphic to the pull-back of the diagram

\[
F(s) \to \lim_{D(s) \setminus T} F \leftarrow \lim_{S \setminus T} F.
\]

2 Limits of colored braid groups

2.1 The poset of simplices and the theorem

Again, we return to our chordal graph \( G = (V, E) \) and recall that \( \Gamma(G) := \pi_1(X(G)) \). Let us denote by \( S(G) \) the poset of non-empty simplices of \( G \), ordered by inclusion. Observe that \( S(G) \) is a meet-semilattice.

We will regard \( S(G) \) as a subposet of the lattice of subsets of \( V \). That is, we identify each simplex of \( G \) with the underlying subset of \( V \). Then, recall from 1.6 the functor \( P : \mathcal{I} \to \{ \text{Groups} \} \). We can now state our main result.

Theorem Let \( G \) be a chordal graph. Then \( \Gamma(G) \) is the limit over \( S(G) \) of the functor \( P \), that is,

\[
\Gamma(G) = \lim_{S(G)} P = \left\{ (\gamma S)_{S \in S(G)} \in \prod_{S \in S(G)} P(S) \mid \rho_{TS}(\gamma S) = \gamma T, \quad \forall T \leq S \right\}.
\]

The theorem is of course true if \( G \) is complete or, more generally, a disjoint union of complete graphs. The proof will be by induction on the number of vertices of \( G \).

2.2 A pull-back diagram of spaces

Let \( s_0 \) be a simplicial vertex of \( G \) and let \( S = N(s_0) \cup \{ s_0 \} \); it is the unique maximal simplex containing \( s_0 \) (see Lemma 1.2). Let us introduce

\[
S^0 := \{ s \in S \mid \forall t \in V, \quad \{ s, t \} \in E \Rightarrow t \in S \}.
\]

Note that \( S^0 \) is not empty since \( s_0 \in S^0 \).

With obvious notation, there is a pull-back diagram

\[
\begin{array}{ccc}
C^G & \to & C^G \setminus S^0 \\
\downarrow & & \downarrow \\
C^S & \to & C^S \setminus S^0.
\end{array}
\]

Lemma The previous diagram induces a pull-back diagram

\[
\begin{array}{ccc}
X(G) & \to & X(G \setminus S^0) \\
\downarrow & \phi & \downarrow f \\
X(S) & \to & X(S \setminus S^0).
\end{array}
\]
Proof. One observes first that the lemma is implied by the following claim: if \( \{i, j\} \) is an edge of \( G \) then \( i, j \in S \) or \( i, j \in G \setminus S^0 \).

To prove this last assertion, suppose that \( \{i, j\} \) is an edge of \( G \). It is of the desired type if \( i, j \in S \) or \( i, j \not\in S^0 \). So, the only case to worry about is the case where \( i \in S^0 \) and \( j \not\in S \). But then \( \{i, j\} \) is not an edge, by the definition of \( S^0 \). This proves the assertion and, hence, the lemma.

2.3 Completion of the proof

Thus, (\( \ast \)) is a pullback diagram. Moreover, since \( S \setminus S^0 \) is a simplex of \( G \setminus S^0 \), it follows from Corollary 1.5 that \( f \) is a split fibration. (We will see below that, for a slightly different, though similar, reason, \( \phi \) is also a split fibration, but we do not need this for the moment).

Therefore, one deduces from Corollary 1.3 that \( \Gamma(G) \) is the pull-back of the diagram

\[
\begin{array}{ccc}
\Gamma(S) & \longrightarrow & \Gamma(S \setminus S^0) \\
\downarrow & & \downarrow \\
P(S) & \longrightarrow & \lim_{S(G \setminus S^0)} P.
\end{array}
\]

Lemma 1) Every simplex of \( G \) which meets \( S^0 \) is contained in \( S \).
2) Every element of \( S^0 \) is a simplicial vertex of \( G \).

Proof. Let \( s \in S^0 \). By the very definition of \( S^0 \), no neighbour of \( s \) belongs to \( G \setminus S \). This implies the first assertion, and the second follows immediately.

Denote by \( \mathcal{T} \) the set of simplices of \( G \) which meet \( S^0 \) and recall from Lemma 1.7 the definition of \( \mathcal{C}(S) \), the cone of \( S \). Then, the lemma shows that \( \mathcal{T} \) is contained in \( \mathcal{C}(S) \). Moreover, one has, clearly, \( S(G \setminus S^0) = S(G) \setminus \mathcal{T} \) and \( S(S \setminus S^0) = S(S) \setminus \mathcal{T} \). Therefore, using Lemma 1.7 one deduces from (\( \ast \)) that

\[
\Gamma(G) \cong \lim_{S(G)} P.
\]

This completes the proof of the theorem.

Remark. One deduces from the second assertion of the lemma, coupled with Corollary 1.5, that in fact all maps in the diagram (\( \ast \)) are split fibrations.
3 The case of rooted trees

In the case of rooted trees, the results take an even more precise form, which we describe in this final section.

3.1 The theorem for rooted trees

Let $T$ be a rooted tree, let $S$ be its set of vertices, and let $r$ be the distinguished vertex called the root. For each vertex $s$, let $T_s$ be the linear tree joining $s$ to $r$. Let $\leq$ denote the partial order on $S$ induced by $T$, that is, $s \leq t$ iff $T_s \subseteq T_t$; the maximal elements are exactly the leaves of $T$.

Recall the notation of 1.6 and, for each $s$, set $P(s) := P(T_s)$. If $s' \leq s$, then $T_{s'} \subseteq T_s$ and hence one has a morphism $\rho_{s's} : P(s) \to P(s')$. Therefore, $P$ may be regarded as a functor on the poset $T$.

Let us denote by $\mathcal{A}_T$ the arrangement associated to $T$ (see [5]) and by $\Gamma_T$ the fundamental group of the complement. Thus, if $G_T$ denotes the comparability graph of $T$, one has $\mathcal{A}_T = \mathcal{A}(G_T)$ and $\Gamma_T = \Gamma(G_T)$.

Clearly, the leaves of $T$ are simplicial vertices of $G_T$ and any ordering $v_1, v_2, \ldots$ of the vertices such that each $v_i$ is a leaf of $T \setminus \{v_1, \ldots, v_{i-1}\}$ is a PEO. In particular, $G_T$ is chordal (cf. [5] Lemma 2).

Further, the maximal simplices of $G_T$ are the linear trees $T_f$, for $f$ a leaf of $T$. Moreover, any (finite) intersection of such trees is equal to a tree $T_s$, for some $s \in S$. From this, one easily deduces that Theorem 2.1 can be reformulated as follows.

**Theorem** One has $\Gamma_T \cong \lim_T P$.

3.2 The structure of iterated semi-direct product

Moreover, in the case of rooted trees, the structure of $\Gamma_T$ as an iterated semi-direct product of free groups (which is a general feature of fundamental groups of fiber-type arrangements, see [8]), can be described rather precisely, as follows.

Recall that each fibration $X(T_s) \to X(T_{s'})$ admits a section, say $\tau_{ss'}$. Therefore, each $P(s)$ may be identified to a semi-direct product

$$P(s) \cong K_{s's} \ltimes \tau_{ss'}(P(s')),$$

where $K_{s's} = \ker \rho_{s's}$. Then, one obtains easily the following proposition, whose proof is left to the reader. Define the **height** of any vertex $s$ by $h(s) := |T_s| - 1$ and let $n = n(T)$ be the supremum of the $h(f)$, for $f$ a leaf of $T$.
**Proposition**  There is an isomorphism

\[
\Gamma_T \cong \prod_{h(s_1)=1} Z(s_1) \ltimes \left( \prod_{h(s_2)=2} F_2(s_2) \ltimes \cdots \ltimes \left( \prod_{h(s_n)=n} F_n(s_n) \right) \right)
\]

(iterated semi-direct product, for \( i = 1, \ldots, n \), of direct products, indexed by the vertices of height \( i \), of copies of the free group \( F_i \)), such that, for every \( t \in S \), the projection \( \Gamma_T \to P(t) \) identifies with the natural projection from \( \Gamma_T \) to

\[
Z(s_1(t)) \ltimes \left( F_2(s_2(t)) \ltimes \cdots \ltimes F_h(s_h(t)) \right),
\]

where \( h = h(t) \) and \( (r, s_1(t), \ldots, s_h(t)) \) is the ordered chain of vertices \( \leq t \).

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