A $\hbar$-deformed Virasoro Algebra as Hidden Symmetry of the Restricted sine-Gordon Model

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Abstract

As the Yangian double with center, which is deformed from affine algebra by the additive loop parameter $\hbar$, we get the commuting relation and the bosonization of quantum $\hbar$-deformed Virasoro algebra. The corresponding Miura transformation, associated screening operators and the BRST charge have been studied. Moreover, we also construct the bosonization for type I and type II intertwiner vertex operators. Finally, we show that the commuting relation of these vertex operators in the case of $p' = r - p = r - 1$ and $\hbar = \pi$ actually give the exact scattering matrix of the Restricted sine-Gordon model.

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1 Introduction

Recently, more and more attention has been paid on the studies of q-deformation of infinite dimensional algebra—q-deformed affine algebra\[1\], q-deformed virasoro algebra and W-algebra\[3,4\]. These q-deformed algebra would play the same role in the studies of completely integrable models which are integrably perturbed from the critical ones (conformal invariant models) as the non-deformed algebra in the studies of conformal field theories (CFT). From the bosonization for q-deformed affine algebra $U_q(sl_2)$ and its vertex operators, Jimbo et al succeeded in giving the correlation functions of XXZ model both in the bulk case\[7\] and the boundary case\[8\]. Feigin and Frenkel obtained the q-deformed virasoro algebra, W-algebra, corresponding “screening operators” and quantum q-deformed Miura transformations from the quantization for the q-deformation of the classical virasoro and W-algebra\[4,5\]. On the other hand, Awata et al also constructed the quantum q-deformed Virasoro and W-algebra, associated “screening operators” and related Miura transformation from the studies of Macdonald symmetric functions\[8\]. The studies of bosonization

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for q-deformed virasoro and W-algebra has been begun in Ref. [2] and Ref. [9]. Actually, Asai, Jimbo et al. has constructed the bosonization for vertex operators of q-deformed W-algebra \( \phi_{\mu}(z) \) in Ref. [9].

However, there exists another important deformation of infinite dimensional algebra which also plays an important role in the completely integrable field theories (In order to comparison with q-deformation, we call it as \( \bar{h} \)-deformation). This deformation was originated by Drinfeld in studies of Yangian [1]. Simirnov suggested that the Yangian double would be the dynamical non-abelian symmetry algebra for SU(2)-invariant Thirring model [16]. Recently progress has been achieved by Khoroshkin et al. [11] and Iohara et al. [12] in the studies of Yangian double and its central extension. The bosonization for Yangian double with center and its vertex operators make it possible to describe the structure of model (local operators, the asymptotic states etc.) in terms of the representation theory of Yangian double with center [11]. It also ensure that the correlation functions of SU(2)-invariant Thirring model can be given explicitly in the integral forms. In this paper, we will given a new deformation of Virasoro algebra—\( \bar{h} \)-deformed Virasoro algebra (HDVA) which has the similar relation as the Yangian algebra and the q-deformed affine algebra. When the deformed parameter \( \bar{h} \rightarrow 0 \), this HDVA will degenerate to the usual Virasoro algebra. The HDVA is quantum version of some classical Poisson algebra which is constructed by \( \bar{h} \)-deformed Suggware construction of Yangian double with center at the critical point [4,5] (the classical version of this algebra has been studied by our colleague and the paper is in preparation). We also show that when the deformed parameter \( \bar{h} = \pi \), this HDVA is the hidden non-abelian dynamical symmetry algebra of the Restricted sin-Gordon model (This can be seen from the state space [14] and scattering matrix [15,17]). Moreover, we construct the bosonization for \( \bar{h} \)-deformed virasoro algebra and its type I and type II vertex operators. The commuting relation for these vertex operators (in the case of \( p' = r \), \( p = r - 1 \) and \( \bar{h} = \pi \)) actually give the exact scattering matrix of the Restricted sine-Gordon [13,14,15,17]. It is well-known that the similar phenomena has occured in the studies of braid relations for the vertex operators (primary fields) of Virasoro algebra in CFT [20].

2 \( \bar{h} \)-deformed virasoro algebra and its bosonization

Firstly, we give some review of q-deformed virasoro algebra [4]. This deformed algebra depends on two parameters \( p \) and \( q \) and is generated by current \( T_q(z) \) with the following relations

\[
\begin{align*}
 f_q\left(\frac{w}{z}\right)T_q(w) - f_q\left(\frac{z}{w}\right)T_q(z) & = \frac{(1 - q)(1 - p/q)}{1 - p} \left( \delta\left(\frac{w}{z}\right) - \delta\left(\frac{w}{z}\right) \right) \\
 f_q(x) & = \frac{1}{1 - x} \left( \frac{xq; p^2}{xp^2 - 1; p^2} \right) \left( \frac{x; p^2}{xp^2 - q^2; p^2} \right), \quad (z; p) = \prod_{n=0}^{\infty} (1 - zp^n)
\end{align*}
\]
It can be shown that the $\hbar$-deformation of affine algebra (or Yangian double) can be considered as the operators scaling limit of q-deformation of affine algebra\[2,19\] (but not the scaling limit of bosonic field realization). Thus, we give the HDVA as the operator scaling limit of q-deformation of Virasoro algebra in Eq.(1). The scaling limit is taken as following way

$$z = p^{-\frac{\hbar}{\pi}} \quad q = p^{-\xi}, \quad f(\beta) = \lim_{p \to 1} f_q(p^{\frac{\hbar}{\pi}}) \quad T(\beta) = \lim_{p \to 1} T_q(p^{\frac{\hbar}{\pi}})$$

From direct calculation, we obtain the HDVA which is generated by the current $T(\beta)$ as the following relation

$$f(\beta - \beta_1)T(\beta_1)T(\beta_2) - f(\beta_1 - \beta_2)T(\beta_2)T(\beta_1) = \xi(\xi + 1)(\delta(\beta_1 - \beta_2 + i\hbar) - \delta(\beta_1 - \beta_2 - i\hbar))$$

In order to study the bosonization for HDVA, we will start from the quantum Miura transformation associated with HDVA in the same way as that of the studies for q-deformed W-algebra given by Feigin and Frenkel\[4\]. Let us consider free bosons $\lambda(t)$ with continuous parameter $t \in \mathbb{R} - 0$ which satisfy

$$[\lambda(t), \lambda(t')] = 4 \text{sh} \frac{h}{2} \text{sh} \frac{h(t + t')}{2} \delta(t + t')$$

We also need to define the “Zero mode” operators $P$ and $Q$, which commute with $\lambda(t)$ and satisfy the relation

$$[P, Q] = -i$$

We consider the Fock space $F_p$ of Heisenberg algebra defined by Eq.(3) and Eq.(4) which is generated by the highest weight vector $v_p$. The highest weight vector $v_p$ satisfies

$$\lambda(t)v_p = 0 \quad t > 0 \quad \text{and} \quad Pv_p = pv_p$$

Now we introduce the field $\Lambda(\beta)$

$$\Lambda(\beta) =: \exp\{-\int_{-\infty}^{\infty} \lambda(t)e^{it\beta} dt\}$$

and give the associated quantum Miura transformation (i.e the transformation from $\Lambda(\beta)$ to $\hbar$-deformation Virasoro algebra $T(\beta)$)

$$T(\beta) = \Lambda(\beta + \frac{i\hbar}{2}) + \Lambda^{-1}(\beta - \frac{i\hbar}{2})$$
In order to get the commuting relation for bosonic field, we must give a comment: When one compute the exchange relation of operators, one often encounter an integral as follows
\[ \int_0^\infty F(t)dt \]
which is divergent at \( t = 0 \). Here we adopt the regularization given by Jimbo et al.\[10\]. Namely, the above integral should be understood as the contour integral
\[ \int_C F(t) \frac{\log(-t)}{2i\pi} dt \]
where the contour \( C \) is chosen as in the Ref.\[10\]. After straightforward calculation, we show that the bosonic representation for \( \hbar \)-deformed virasoro current \( T(\beta) \) in Eq. (7) satisfies the definition relation for HDVA as in Eq. (2). Thus, we construct the bosonization for HDVA.

In what follows we will be interested in a particular representations of HDVA in Fock space which corresponds to minimal models of CFT\[13\]. This representations are parameterized by two positive coprime integer numbers \( p', p \) (\( p' > p \)) and two type boson \( a(t), a'(t) \) which are associated with \( \alpha_+ \) series and \( \alpha_- \) series respectively
\[ \xi = \frac{p}{p' - p} , \quad \alpha_0 = \sqrt{\xi(1 + \xi)} \] (8)
\[ \alpha_+ = -\sqrt{\frac{p'}{p}} = \sqrt{\frac{(\xi + 1)}{\xi}} , \quad a(t) = \frac{\lambda(t)}{2sh\frac{1}{2} \xi} \] (9)
\[ \alpha_- = \sqrt{\frac{p}{p'}} = \sqrt{\frac{\xi}{(1 + \xi)}} , \quad a(t) = \frac{\lambda(t)}{2sh\frac{1}{2} (\xi + 1)} \] (10)

We consider the action of HDVA on the Fock space \( F_{l,k} \)
\[ PF_{l,k} = \alpha_{l,k} F_{l,k} , \quad \alpha_{l,k} = \alpha_+ l + \alpha_- k , \quad 0 \leq l \leq p , \quad 0 \leq k \leq p' \] (11)
as bosonic operator \( T(\beta) \). This action is highly reducible and we have to throw out some states from Fock module to obtain the irreducible component. The procedure depends on BRST charge \( Q_\pm \) which are defined as follows
\[ Q_+ = \oint_{C_+} d\beta S^+(\beta) f(\beta, \alpha_0 P) , \quad Q_- = \oint_{C_-} d\beta S^-(\beta) f'(\beta, \alpha_0 P) \] (12)
screening current \( S^+(\beta) = e^{-2i\alpha_+ Q} : exp\{ -\int_{-\infty}^{\infty} a(t)(e^{\frac{1}{2}\beta} + e^{-\frac{1}{2}\beta}) e^{it\beta} dt \} : \) (13)
screening current \( S^-(\beta) = e^{-2i\alpha_- Q} : exp\{ -\int_{-\infty}^{\infty} a'(t)(e^{\frac{1}{2}\beta} + e^{-\frac{1}{2}\beta}) e^{it\beta} dt \} : \) (14)
\[ f(\beta, \alpha_0 P) = \frac{\sin\pi(\frac{\beta}{p} - \frac{1}{2}) - \alpha_0 P}{\sin\pi(\frac{\beta}{p} + \frac{1}{2})} \] (15)
\[
f'(\beta, \alpha_0 P) = \frac{\sin \pi \left( \frac{i\beta}{\hbar(\xi+1)} + \frac{1}{2(1+\xi)} + \frac{\alpha_0 P}{1+\xi} \right)}{\sin \pi \left( \frac{i\beta}{\hbar(1+\xi)} - \frac{1}{2(1+\xi)} \right)}
\]

where the integration contours are chosen as follows: the contour \( C_+ \) enclose the poles \( \beta = \frac{i\hbar}{2} - i\hbar \xi n (0 \leq n) \), the contour \( C_- \) enclose the poles \( \beta = -\frac{i\hbar}{2} + i\hbar (\xi + 1)n (0 \leq n) \). An important properties of screening charges \( Q_\pm \) is that they commute with the HDVA current \( T(\beta) \) and have BRST properties acting on the Fock space \( F_{l,k} \)

\[
Q_+^P |_{F_{i,k}} = 0 \quad Q_-^P |_{F_{i,k}} = 0
\]

As a result, we could have the following complexes

\[
\cdots \xrightarrow{X_{-2} = Q_0^+} F_{2p-l,k} \xrightarrow{X_{-1} = Q_{p-1}^+} F_{l,k} \xrightarrow{X_0 = Q_0^+} F_{-l,k} \xrightarrow{X_1 = Q_{p-1}^+} \cdots
\]

\[
\cdots \xrightarrow{Y_{-2} = Q_k} F_{l,2p' - k} \xrightarrow{Y_{-1} = Q_{p'-k}} F_{l,k} \xrightarrow{Y_0 = Q_k} F_{l,-k} \xrightarrow{Y_1 = Q_{p'-k}} \cdots
\]

we assume the following cohomological properties

\[
\text{Ker} X_j / \text{Im} X_{j-1} = 0 \quad \text{if} \quad j \neq 0
\]

\[
\text{Ker} Y_j / \text{Im} Y_{j-1} = 0 \quad \text{if} \quad j \neq 0
\]

\[
\text{Ker} X_0 = \text{Ker} Y_0 \quad \text{Im} X_{-1} = \text{Im} Y_{-1}
\]

\[
L_{l,k} = \text{Ker} X_0 / \text{Im} X_{-1} = \text{Ker} Y_0 / \text{Im} Y_{-1}
\]

where \( L_{l,k} \) is an irreducible representation of HDVA.

In order to relate with the Restricted sin-Gordon model, we should take \( \hbar = \pi \). In the following part of this paper we restrict us to the case of \( \hbar = \pi \), but the vertex operators ((21)—(24)) and their commuting relations ( (25)—(32)) are easy to generalize to the generic \( \hbar \). If \( p = r - 1 \) and \( p' = r \), the above space \( L_{l,k} \) will be the space of the Restricted sine-Gordon model [14].

### 3 Vertex operators and commuting relations

It is well-known that the vertex operators of Virasoro algebra ( or of q-deformed affine algebra) is of great importance in the CFT[13] (or in the solvable lattice model[7]). Hence, we construct the simplest two type vertex operators: \( Z_a''(\beta) \) for type I and \( Z_a(\beta) \) for type II

\[
V_{(1,0)}(\beta) \equiv Z_+(\beta) = e^{i\alpha Q} : \exp \left\{ \int_{\infty}^{\infty} a(t)e^{it\beta} dt \right\} :
\]

\[
V_{(0,1)}(\beta) \equiv Z_+'(\beta) = e^{i\alpha - Q} : \exp \left\{ - \int_{\infty}^{\infty} a'(t)e^{it\beta} dt \right\} :
\]
where the integration contour are chosen as follows: the contour $C_1$ enclose the poles $\eta = \beta + \frac{i\pi}{2} - i\pi n$ $(0 \leq n)$, the contour $C_2$ enclose the poles $\eta = \beta + \frac{i\pi}{2} + i\pi(1 + \xi)n$ $(0 \leq n)$. Because the other vertex operators $V_{l,k}^{m,n}(\beta)$ (corresponds to $V_{l,k}^{m,n}(z)$ in the Ref.[11]) can be constructed by the symmetric fusion of the simplest two type vertex operators $Z_a(\beta)$ and $Z'_a(\beta)$, here we only consider these two type basal vertex operators.

From the direct calculation, we obtain the commuting relations of vertex operators (or the braid matrix for the primary fields in CFT[13,20])

$$V_{(1,0)}^{(1,0)}(\beta) \equiv Z_-(\beta) = \oint_{C_1} d\eta Z_+(\beta) S^+(\eta) f(\eta - \beta, \alpha_0 P)$$

$$V_{(0,1)}^{(0,1)}(\beta) \equiv Z'_-(\beta) = \oint_{C_2} d\eta Z_+(\beta) S^-(\eta) f'(\eta - \beta, \alpha_0 P)$$

we also find that

$$[Z_a(\beta), Q_-] = 0 \quad , \quad [Z'_a(\beta), Q_+] = 0$$
Due to the cohomological properties Eq.(20), the periodic properties of matrices $U$ and $U'$ with regard to $l$ and $k$, and the Eq.(33), we obtain

$$Z_a(\beta_1)Z_b(\beta_2)_{|L_{l,k}} = \sum_{c,d} Z_c(\beta_2)Z_d(\beta_1)U \left( \begin{array}{cc} l + a + b & l + d \\ l + b & l \end{array} \right) |\beta_1 - \beta_2|_{L_{l,k}}$$

(34)

$$Z'_a(\beta_1)Z'_b(\beta_2)_{|L_{l,k}} = \sum_{c,d} Z'_c(\beta_2)Z'_d(\beta_1)U' \left( \begin{array}{cc} k + a + b & k + d \\ k + b & k \end{array} \right) |\beta_1 - \beta_2|_{L_{l,k}}$$

(35)

$$Z_a(\beta_1)Z'_b(\beta_2)_{|L_{l,k}} = ctg\left( \frac{i(\beta_1 - \beta_2)}{2} + \frac{\pi}{4} \right) Z'_b(\beta_2)Z_a(\beta_1)_{|L_{l,k}}$$

(36)

It is well-known that the Restricted sine-Gordon model is a massive integrable model which is perturbed from the minimal conformal field model with center $C = 1 - \frac{6}{r(r-1)}$ ($4 \leq r$). We find that when $p' = r$ and $p = r - 1$ (i.e. $\xi = r - 1$), besides the space of states are the same, the braid matrix $U \left( \begin{array}{cc} l + a + b & l + d \\ l + b & l \end{array} \right)$ is the exact scattering matrix of the Restricted sine-Gordon model. So, we strongly suggest that the $\hbar$-deformed Virasoro algebra defined in Eq.(2) is the just hidden symmetry in the Restricted sine-Gordon model.

4 Conclusion

In this paper, we obtain the HDVA, its bosonic realization in space $L_{l,k}$ and show that this deformed algebra with the deformed parameter $\hbar = \pi$ is the hidden symmetry of the Restricted sine-Gordon model. The bosonization will make it possible to describe the structure of the Restricted sine-Gordon model in terms of the representation theory of the HDVA and to calculate the correlation functions of the Restricted sine-Gordon model. The correlation functions would be the scaling limit ($q = p^{-\xi}$, $\zeta = p^{\pi}$, $r - 1 = \xi$, $p \rightarrow 1$) of Lukyanov and Pugain’s in the Ref.[2]. We will present this result in the other paper. Of course, the studies of the Restricted sine-Gordon model with integrable boundary condition would be important and the bosonization of HDVA make it accessible. Some results about boundary $q$-deformedVirasoro algebra has been obtained.

The $\hbar$-deformation $W$-algebra, its bosonic representation and its vertex operators are also worthy to be investigated. Some results have been obtained by us which is in preparation. We expect that these deformed $W$-algebra plays the role of symmetry algebra for some integrable field model.

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