Dependence of the BEC transition temperature on interaction strength: A perturbative analysis

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Abstract

We compute the critical temperature $T_c$ of a weakly interacting uniform Bose gas in the canonical ensemble, extending the criterion of condensation provided by the counting statistics for the uniform ideal gas. Using ordinary perturbation theory, we find in first order $(T_c - T_c^0)/T_c^0 = -0.93a\rho^{1/3}$, where $T_c^0$ is the transition temperature of the corresponding ideal Bose gas, $a$ is the scattering length, and $\rho$ is the particle number density.

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I. INTRODUCTION

It may well look like a long-solved text book exercise, but the variation of the dilute Bose gas’ critical temperature with the interaction strength has not yet found a conclusive answer. To date, all authors assume a continuous behavior in the limit of weak interaction, \( \lim_{a \to 0} T_c = T_c^0 \), where \( T_c^0 \) is the transition temperature of the non–interacting system, and \( a \) is the s–wave scattering length. However, the sign, the proportionality constant \( c \), and the exponent \( \eta \) in the expression for the shift in the critical temperature at fixed density \( \rho \)

\[
\frac{T_c - T_c^0}{T_c^0} = \pm c \left[ a^3 \rho \right]^{\eta},
\]

are still subject to considerable debate. Early calculations by Fetter and Walecka [1] and Toyoda [2] predict a decrease in temperature, \( T_c < T_c^0 \) (Yet, it should be noted that the expression derived by Fetter and Walecka yields zero for a point potential.). However, more recent calculations indicate the opposite [3–10]. Concerning the exponent, one finds in the literature a set of predicted rational values which range from \( \eta = 1/6 \) [2], [3], [11] to \( \eta = 1/2 \) [12]. The most recent analytical investigations converge towards the value \( \eta = 1/3 \) [4], [5], [6], [7], i. e. they predict a linear dependence of the temperature shift on the scattering length. This result is also backed by Monte Carlo simulations [8], [9], and by an ingenious extrapolation of experimental data for the strongly interacting condensed He4 in vycor glass [14]. Still, the result of Toyoda \( \eta = 1/6 \) continues to find support [3]. The proportionality constant, finally, has been predicted to assume a variety of values, which for \( \eta = 1/3 \) range from \( c = 0.3 \) [8] to \( c = 5.1 \) [14]. The Paris group most recent numerical analysis, for example, points at \( c = 2.3 \) [4], a value which is close to the theoretical prediction of Baym and collaborators [7], while the extrapolation of the experimental data on He4 in vycor glass favors \( c = 5.1 \), which is closer to an early prediction \( c = 4.66 \) of Stoof [1].

It is frequently maintained that ordinary perturbation theory can not be applied as it is plagued by seemingly unsurmountable infrared divergencies (See [7]). We point out that this conclusion is based on the implicit assumption that the grand–canonical statistics, which is
governed by a chemical potential, is a sensible approximation to the real system, i.e. a system where – as a matter of principle – not the chemical potential, but rather the total number of particles is fixed, possibly at very large a value. While this assumption of thermodynamic equivalence does indeed hold in a system with sufficiently strong interactions, it must be rejected in the limit $a \to 0$. In this limit the grand-canonical statistics implies fluctuations of the ground state occupation, which for temperatures at and below $T_c$ turn out to be extravagantly large, $\Delta n_0 \sim \mathcal{O}(N)$ [15–17]. It is these unphysical fluctuations which doom to failure any attempt to reliably compute the shift of the Bose gas critical temperature, in the non-interacting limit $a \to 0$, when resorting to ordinary perturbation theory in the grand-canonical ensemble.

As the ground state giant fluctuations are easily traced back to the fluctuations in the total number of particles, which in the grand-canonical statistics turn into an unacceptable $\Delta N \sim N$ for $T \leq T_c^0$, a safe way out is to resort to statistical ensembles where the total number of particles is not allowed to fluctuate. In the canonical and microcanonical ensembles, for example, the ground state fluctuations of the non-interacting system exhibit a scaling $\Delta n_0 \sim \mathcal{O}(N^{2/3})$ [15] which – although still anomalous – turns out to be sufficiently suppressed for ordinary perturbation theory to be applicable.

Indeed, as we shall demonstrate in this letter, first order perturbation theory in the canonical ensemble yields the following shift in the critical temperature (where $\lambda_0$ is the De Broglie thermal wave length at $T = T_c^0$):

$$\frac{T_c - T_c^0}{T_c^0} = \frac{2}{5} \left[ \frac{8\pi}{3\zeta(3/2)} \right] \frac{a}{\lambda_0}$$

$$\approx -0.93 \left[ a^3 \rho \right]^{1/4},$$

which – contrary to some early expectations – is neither zero nor infinite.
II. THE HAMILTONIAN

We consider a uniform system of \( N \) weakly interacting bosons in a volume \( V = L^3 \), imposing periodic boundary conditions. The Hamiltonian reads

\[
\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}},
\]

(4)

where \( \hat{H}_0 \) is the Bose gas kinetic energy,

\[
\hat{H}_0 = \sum_k \varepsilon_k \hat{n}_k, \quad \varepsilon_k = \frac{\hbar^2 k^2}{2m},
\]

(5)

and \( \hat{H}_{\text{int}} \) describes the particle pair interaction,

\[
\hat{H}_{\text{int}} = \frac{u}{2N} \sum_{pq} \hat{b}_p^\dagger \hat{b}_q^\dagger \hat{b}_{q-k} \hat{b}_{p+k}, \quad u = \frac{4\pi \hbar^2 aN}{mV}.
\]

(6)

Here \( \mathbf{k} = (2\pi/L)\mathbf{n} \) is a wave vector, with \( \mathbf{n} \) a vector of integers, \( \hat{b}_k, \hat{b}_k^\dagger \) are bosonic particle annihilation and creation operators, \( \hat{n}_k = \hat{b}_k^\dagger \hat{b}_k \) is the associated number operator, \( m \) denotes the particle mass, and \( a \) denotes the s–wave scattering length.

III. THE COUNTING STATISTICS

We shall be working at fixed density \( \rho = N/V \) (equivalently: fixed specific volume \( v = \rho^{-1} \)), but variable total number of particles \( N \) (and, concomitantly, variable system volume \( V \)). The first issue to be faced is to provide a definition of the transition temperature, which – as we recall – only acquires the meaning of a critical temperature in the thermodynamic limit \( N \to \infty, V \to \infty, \rho = N/V \) constant.

We base our definition on the counting statistics of the zero–momentum state,

\[
P_n(\beta; N) = \frac{1}{Z(\beta; N)} \text{Tr} \left\{ \delta_{\hat{n}_0,n} e^{-\beta \hat{H}} \delta_{\hat{N},N} \right\},
\]

(7)

which is the probability to find \( n \) particles (out of \( N \) total particles) in the zero–momentum state \( \mathbf{p} = \hbar \mathbf{k} = 0 \). Here, \( \hat{N} = \sum_k \hat{n}_k \) is the operator for the total number of particles, \( \delta_{a,b} \) is the Kronecker delta, and \( Z(\beta; N) \) is the canonical partition function,
\[ Z(\beta; N) = \text{Tr} \left\{ e^{-\beta \hat{H}} \delta_{\hat{N},N} \right\}. \] (8)

In the non-interacting limit, the counting statistics for high temperatures is a strictly decreasing function of \( n \), i.e. \( P_n > P_{n+1} \). For sufficiently low temperatures it displays a single peak at \( n \sim \langle n \rangle \sim O(N) \). Assuming that a system of weakly interacting bosons behaves correspondingly, we introduce the auxiliary function

\[ \tilde{D}(\beta; N) \equiv \text{Tr} \left\{ [\delta_{\hat{n}_0,0} - \delta_{\hat{n}_0,1}] e^{-\beta \hat{H}} \delta_{\hat{N},N} \right\}. \] (9)

The cross-over from the high-temperature regime, where \( \tilde{D} > 0 \), to the low-temperature regime, where \( \tilde{D} < 0 \), is assumed to occur for a certain value \( \beta = \beta_s \), which is defined by the relation

\[ \tilde{D}(\beta_s; N) = 0. \] (10)

For fixed density \( \rho \), and fixed scattering length \( a \), the solution of this equation depends on the total number of particles \( N \): \( \beta_s = \beta_s(N) \). We stipulate that, in the thermodynamic limit, the cross-over temperature \( T_s = (k_B \beta_s)^{-1} \) coincides with the critical temperature of Bose–Einstein condensation:

\[ \lim_{N \to \infty} T_s(N) = T_c. \] (11)

This identification, being non-trivial for an interacting system, will be verified below for the non-interacting case.

**IV. PERTURBATIVE ANALYSIS OF \( T_s \)**

We determine \( \beta_s \) using a series expansion in \( \hat{H}_{\text{int}} \). The Dyson series of \( \tilde{D} = \tilde{D}(\beta; N) \) reads

\[ \tilde{D} = \tilde{D}_0 + \tilde{D}_1 + \tilde{D}_2 + \ldots, \] (12)

where \( \tilde{D}_n \equiv \tilde{D}_n(\beta; N) \) is of \( n \)-th order in \( \hat{H}_{\text{int}} \). The first two terms are given by
\[ \tilde{D}_0(\beta; N) = \text{Tr} \left\{ [\delta_{\hat{n}_0,0} - \delta_{\hat{n}_0,1}] e^{-\beta \hat{H}_0} \delta_{\hat{N},N} \right\}, \]  

(13)

\[ \tilde{D}_1(\beta; N) = -\beta \text{Tr} \left\{ [\delta_{\hat{n}_0,0} - \delta_{\hat{n}_0,1}] \hat{H}_{\text{int}} e^{-\beta \hat{H}_0} \delta_{\hat{N},N} \right\}. \]  

(14)

To solve Eq. (10) we set

\[ \beta^{\ast} = \beta^{(0)} + \Delta \beta^{\ast}, \]  

(15)

where \( \beta^{(0)} \) denotes the cross-over inverse temperature of the non-interacting Bose gas, and \( \Delta \beta^{\ast} \) is a correction which is assumed to be small. The defining equation for \( \beta^{(0)} \) reads

\[ \tilde{D}_0(\beta^{(0)}; N) = 0, \]  

(16)

and the shift to leading order

\[ \frac{\Delta \beta^{\ast}}{\beta^{(0)}} = \frac{\tilde{D}_1(\beta; N)}{\beta \tilde{E}_0(\beta; N) \mid_{\beta = \beta^{(0)}}}, \]  

(17)

where

\[ \tilde{E}_0(\beta; N) = -\frac{\partial}{\partial \beta} \tilde{D}_0(\beta; N). \]  

(18)

A. Exact Relations

Observing \( \varepsilon_0 = 0 \), which implies that \( \hat{H}_0 \) does not depend on \( \hat{n}_0 \), we may recast Eq. (13) into the form

\[ \tilde{D}_0(\beta; N) = \text{Tr}_{\text{ex}} \left\{ [\delta_{\hat{n}_{\text{ex},0}} - \delta_{\hat{n}_{\text{ex},1}}] e^{-\beta \hat{H}_0} \delta_{\hat{N}_{\text{ex}},N} \right\}, \]  

(19)

where \( \text{Tr}_{\text{ex}} \) denotes the trace over the occupation of excited states \( k \neq 0 \), and

\[ \hat{N}_{\text{ex}} \equiv \sum_{k \neq 0} \hat{n}_k \]  

(20)

denotes the operator of the number of particles in the excited states. Furthermore, using

\[ \hat{H}_{\text{int}} = \frac{u}{N} \hat{N} (\hat{N} - 1) - \frac{u}{N} \sum_k \hat{n}_k (\hat{n}_k - 1) \frac{2}{2} + \hat{R} \]  

(21)
where $\hat{R}$ has no diagonal elements in the Fock basis, and observing that $[\delta_{\hat{n}_0,0} - \delta_{\hat{n}_0,1}]\hat{n}_0(\hat{n}_0 - 1) = 0$, one finds

$$
\tilde{D}_1(\beta; N) = \frac{u\beta}{N} \text{Tr}_{\text{ex}} \left\{ \left[ \delta_{\hat{N}_{\text{ex}},N} - \delta_{\hat{N}_{\text{ex}},N-1} \right] \sum_{k \neq 0} \frac{\hat{n}_k(\hat{n}_k - 1)}{2} e^{-\beta \hat{H}_0} \right\} - (N - 1) u\beta \tilde{D}_0(\beta; N).
$$

(22)

Note that due to the definition of $\beta_*(0)$ the second terms does not contribute to the shift $\Delta \beta_*$. To proceed, we use the Laplace representation of the Kronecker delta

$$
\delta_{\hat{N}_{\text{ex}},N} = \frac{1}{2\pi i} \int_{-i\pi}^{+i\pi} d\alpha e^{(N-\hat{N}_{\text{ex}})\alpha}
$$

(23)

and perform the trace $\text{Tr}_{\text{ex}}$. We then face

$$
\tilde{D}_0(\beta; N) = \frac{1}{2\pi i} \int_{-i\pi}^{i\pi} d\alpha \left[ 1 - e^{-\alpha} \right] e^{\tilde{F}(\alpha)},
$$

(24)

$$
\tilde{D}_1(\beta; N) = \frac{u\beta}{N} \frac{1}{2\pi i} \int_{-i\pi}^{i\pi} d\alpha \left[ 1 - e^{-\alpha} \right] \left( \sum_{k \neq 0} n_k^2(\alpha) \right) e^{\tilde{F}(\alpha)} - (N - 1) u\beta \tilde{D}_0(\beta; N),
$$

(25)

where $n_k(\alpha) = n_k(\alpha; \beta, N)$,

$$
n_k(\alpha; \beta, N) = \frac{1}{e^{\alpha + \beta \varepsilon_k} - 1},
$$

(26)

and $\tilde{F}(\alpha) \equiv \tilde{F}(\alpha; \beta, N)$,

$$
\tilde{F}(\alpha; \beta, N) = N\alpha + N\frac{\nu}{\lambda^3} \tilde{g}_{5/2}(\alpha).
$$

(27)

Here $\lambda \equiv \lambda(\beta)$ is the thermal De Broglie wave length,

$$
\lambda(\beta) = \sqrt{2\pi \hbar^2 / (mk_B T)},
$$

(28)

and $\tilde{g}_{5/2}(\alpha) \equiv \tilde{g}_{5/2}(\alpha; \beta, N)$ is a discrete predecessor of a Bose integral function,

$$
\tilde{g}_{5/2}(\alpha; \beta, N) = -\frac{\lambda^3}{N\nu} \sum_{k \neq 0} \ln \left[ 1 - e^{-\alpha - \beta \varepsilon_k} \right].
$$

(29)

Upon identifying $\alpha = -\beta \mu$, where $\mu$ denotes the chemical potential in the grand–canonical ensemble, we note that for fixed $\alpha$ the corresponding value $\tilde{F}(\alpha)$ is nothing but the ideal Bose gas grand–canonical free energy, and $n_k(\alpha)$ is the grand–canonical mean occupation.
B. Continuum Approximation

For sufficiently large $N$, and for the interesting range of thermal De Broglie wavelength such that $\lambda^3 \rho \ll N$, we may invoke the continuum approximation, and replace the discrete sum over momenta by an integral, $\sum_{k \neq 0} \rightarrow \frac{Nv}{(2\pi)^3} \int d^3k$.

We then obtain

$$\tilde{F}(\alpha; \beta, N) \longrightarrow F(\alpha; \beta, N) = N\alpha + N \frac{v}{\lambda^3} g_{5/2}(\alpha),$$

$$\sum_{k \neq 0} n_k^2(\alpha; \beta, N) \longrightarrow N \frac{v}{\lambda^3} \left( g_{1/2}(\alpha) - g_{3/2}(\alpha) \right),$$

where the $g_\sigma(\alpha)$ denote the Bose–Einstein integral functions

$$g_\sigma(\alpha) = \frac{1}{\Gamma(\sigma)} \int_0^\infty dx \frac{x^{\sigma-1}}{e^{x+\alpha} - 1}.$$

Concomitant with the above replacements we also have:

$$\tilde{D}_0 \longrightarrow D_0 = \int \frac{d\alpha}{2\pi i} \left[ 1 - e^{-\alpha} \right] e^{F(\alpha; \beta, N)},$$

$$\beta \tilde{E}_0 \longrightarrow \beta E_0 = N \frac{v}{\lambda^3} \frac{3}{2} I_{5/2}(\beta; N),$$

$$\tilde{D}_1 \longrightarrow D_1 = \frac{v}{\lambda^3} \beta u \left[ I_{1/2}(\beta; N) - I_{3/2}(\beta; N) \right] - (N - 1) \beta u D_0,$$

where

$$I_\sigma(\beta; N) = \int \frac{d\alpha}{2\pi i} \left[ 1 - e^{-\alpha} \right] g_\sigma(\alpha) e^{F(\alpha; \beta, N)}.$$

C. Expansion in $N^{-1/3}$

As $N$ is assumed to be large, $N \gg 1$, it is now tempting to evaluate $D_0, D_1$ in the saddle point approximation. Yet, due to the $O(N^{-2/3})$ proximity of the saddle point to the branch

\footnote{In an extended version of this paper we shall include a systematic study of the corrections to the continuum approximation.}
point of $F$, this procedure is doomed to fail, and a completely different treatment must be developed.\footnote{In field theory, the proximity turns into a confluence which renders meaningless the non–interacting limit of the theory at $T = T_0$.}

Since we expect the integrals to be dominated by the small values of $\alpha$, we resort to the Robinson representation \[19\]:

$$g_\sigma(\alpha) = \Gamma(1 - \sigma)\alpha^{\sigma - 1} + \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \zeta(\sigma - n)\alpha^n.$$  

(37)

Save for the branch cut of $\alpha^{\sigma - 1}$, which runs along the negative real axis, the Robinson expansion converges absolutely for $|\alpha| \leq 2\pi$. Note that the radius of convergence covers the domain of integration in the above $\alpha$–integrals.

Exploiting the Robinson representation, the exponent reads

$$F = \ln C - Y\alpha + \frac{2}{3}X\alpha^{3/2} + X\mathcal{O}(\alpha^2),$$

(38)

where $\ln C = [\zeta(5/2)/(\lambda^3 \rho)]N$ is a constant, $\mathcal{O}(\alpha^2)$ denotes some analytic function, which may be extracted from Eq. (37), and

$$X = \frac{2\sqrt{\pi}}{\zeta(3/2)} \frac{\lambda_0^3}{\lambda^3} N, \quad Y = \left[\frac{\lambda_0^3}{\lambda^3} - 1\right] N,$$

(39)

with $\lambda_0^3 = \zeta(3/2)/\rho$, the thermal De Broglie wave length of the non–interacting gas evaluated at the transition temperature.

We note that for given $N$, the solution of Eq. (37) implies a relation between $X$, which is $O(N)$, and $Y$. As we expect $\lambda_* \to \lambda_0$ in the limit $N \to \infty$, the scaling of $Y_*$ with $N$ is not obvious, yet

$$\epsilon = \frac{Y}{X}$$

(40)

will certainly be small. Introducing the transformation of the integration variable,

$$\alpha \to \tau = \epsilon^{-2}\alpha$$

(41)
the free energy reads

\[ F = \ln C + \Lambda \left( -\tau + \frac{2}{3} \tau^{3/2} \right) + \Lambda \epsilon r_{5/2}(\tau; \epsilon), \tag{42} \]

where

\[ \Lambda = \frac{Y^3}{X^2}, \tag{43} \]

and \( r_{5/2} \) is a regular function,

\[ r_{5/2}(\tau; \epsilon) = \frac{\tau^2}{2\sqrt{\pi}} \sum_{\nu=0}^{\infty} \frac{(-\epsilon^2)^\nu}{(\nu + 2)!} \zeta(1/2 - \nu)\tau^\nu. \tag{44} \]

Since we shall find that \( \Lambda_{*}^{(0)} \sim O(1) \) at the cross-over temperature \( T = T_{*}^{(0)} \), and concomitantly \( \epsilon_{*}^{(0)} \sim O(N^{-1/3}) \) for large \( N \), we may invoke a formal expansion in \( \epsilon \),

\[ \frac{D_0}{C} = \epsilon^4 K_1(\Lambda) + \epsilon^5 \frac{\zeta(1/2)}{4\sqrt{\pi}} \Lambda K_3(\Lambda) + O(\epsilon^6), \tag{45} \]

\[ \frac{I_{1/2} - I_{3/2}}{C} = \frac{\zeta(1/2) - \zeta(3/2)}{C} D_0 + \epsilon^3 \sqrt{\pi} K_{1/2}(\Lambda) + \epsilon^4 \frac{\zeta(1/2)}{4} \Lambda K_{5/2}(\Lambda) + O(\epsilon^5), \tag{46} \]

\[ \frac{I_{5/2}}{C} = \frac{\zeta(5/2)}{C} D_0 - \epsilon^6 \zeta(3/2) K_2(\Lambda) - \epsilon^7 \left[ \frac{\zeta(3/2) \zeta(1/2)}{4\sqrt{\pi}} \Lambda K_4 - \frac{4\sqrt{\pi}}{3} K_{5/2} \right] + O(\epsilon^8), \tag{47} \]

where we have introduced the family of functions

\[ K_\nu(\Lambda) = \frac{1}{2\pi i} \int d\tau \tau^\nu \exp \left\{ \Lambda \left( -\tau + \frac{2}{3} \tau^{3/2} \right) \right\}. \tag{48} \]

The functions \( K_\nu \) obey the recurrence relation

\[ K_{\nu+3/2} = K_{\nu+1} - \frac{\nu + 1}{\Lambda} K_\nu, \tag{49} \]

which is easily proven by expressing \( K \) in terms of \( X \) and \( Y \), using the inverse of the transformation \([41]\).
D. Results

Upon inserting Eq. (45) into Eq. (16), the condition which fixes the cross-over temperature of the non-interacting gas reads

$$K_1(\Lambda^*_0) = 0.$$  \hspace{1cm} (50)

up to corrections $\mathcal{O}(N^{-1/3})$. This equation is easily solved numerically, yielding the result $\Lambda^*_0 = 0.334$. Expressed in terms of temperature we have

$$T_\ast^0 = T_c^0 \left[ 1 + \left( \frac{32\pi \Lambda^*_0}{27\zeta(3/2)^2} \right)^{1/3} \frac{1}{N^{1/3}} + \mathcal{O}(N^{-2/3}) \right]$$  \hspace{1cm} (51)

where $T_c^0$ is the critical temperature of the non-interacting Bose gas. Note that for $N$ finite, the cross-over temperature is slightly higher than the transition temperature of the ideal Bose gas, but in the thermodynamic limit $N \to \infty$ they coincide.

Collecting terms and observing that $D_0(\beta^*_0; N) = 0$, the interaction induced shift reads

$$\frac{\Delta \beta_\ast}{\beta^*_0} = -\frac{8\pi}{3\zeta(3/2) \lambda^*_0} \left[ \frac{K_{1/2}(\Lambda)}{\Lambda K'_2(\Lambda)} \right]_{\Lambda = \Lambda^*_0},$$  \hspace{1cm} (52)

up to corrections of order $\mathcal{O}(N^{-1/3})$. The shift involves the ratio $f = K_{1/2}/(\Lambda K'_2)|_{\Lambda = \Lambda^*_0}$. Exploiting the recurrence relation (49), and observing that $K_1(\Lambda^*_0) = 0$, we find $f = -2/5$. We note that this value is exact as it does not depend on the numerical value of $\Lambda^*_0$.

We are now in the position to consider the thermodynamic limit of Eq. (52). Identifying $\lim_{N \to \infty} T_\ast = T_c$, the result reads

$$\frac{\Delta T_c}{T_c^0} \equiv \frac{T_c - T_c^0}{T_c^0} = -2 \frac{8\pi}{5 \zeta(3/2) \lambda^*_0} \frac{a}{\rho^{1/3}} = -0.93 a \rho^{1/3}.$$  \hspace{1cm} (53)

We thus find a negative shift in the critical temperature, growing linearly with the scattering length. This result can be compared with the other fully analytical prediction existing in the literature, derived by Baym, Blaizot and Zinn-Justin [7]:

$$\frac{\Delta T_c}{T_c^0} = \frac{8\pi}{3\zeta(3/2)^{4/3}} a \rho^{1/3} = 2.33 a \rho^{1/3}.$$  \hspace{1cm} (55)
The two results exhibit the same scaling, but differ for the sign and for a factor $2/5$ in the proportionality constant.

The prediction of Baym et al. has been obtained by evaluating the leading order in the $1/N$ expansion for a $O(N)$ field theory model that coincides with the original Bose Hamiltonian for $N = 2$, and by observing that the final result does not explicitly depend on $N$. However, the result is strictly proven only for large $N$, and whether it is reliable also for $N = 2$ is still an open problem.

On the other hand, our approach based on the counting statistics and on ordinary perturbation theory in the canonical ensemble indicates that in the limit of ultraweak interaction there are contributions, otherwise possibly neglected in other approaches, that tend to suppress quantum effects. Clarification of this issue may be expected from higher order perturbation theory.

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