One-sided sharp thresholds for homology of random flag complexes

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Abstract
We prove that the random flag complex has a probability regime where the probability of nonvanishing homology is asymptotically bounded away from zero and away from one. Related to this main result, we also establish new bounds on a sharp threshold for the fundamental group of a random flag complex to be a free group. In doing so, we show that there is an intermediate probability regime in which the random flag complex has fundamental group that is neither free nor has Kazhdan’s property (T).

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1 | INTRODUCTION

One of the most well-known and important phase transitions in the Erdős–Rényi random graph model $G(n, p)$ is the phase transition for the giant component and emergence of cycles at $p = 1/n$ first proved in [11]. While sharper versions examining the behavior of random graphs in the critical window exist, to motivate the results here, we state the Erdős–Rényi phase transition theorem as follows.

**Theorem 1.1** (Erdős–Rényi [11]). For $G \sim G(n, c/n)$ with $c \in (0, \infty)$ constant, one has the following.

(i) If $c < 1$, then with high probability, $G \sim G(n, c/n)$ has all connected components of order $O(\log n)$ and the probability that $G$ contains cycles is

$$1 - \sqrt{1 - c \exp(c/2 + c^2/4)}.$$
(ii) If \( c > 1 \), then with high probability, \( G \sim G(n, c/n) \) has a unique giant component of order \( \Theta(n) \) and the rest of the components are of order \( O(\log n) \) and with high probability \( G \) has cycles.

Because there are almost always cycles above the threshold but below threshold, there is a positive probability that cycles are present and a positive probability they are not, the cycle threshold in \( G(n, p) \) may be described as a one-sided sharp threshold. Here, we study a higher dimensional analog of this one-sided sharp threshold for the emergence of cycles in the random clique complex model first introduced by Kahle [14]. For any graph \( G \), the clique complex of \( G \), also call the flag complex of \( G \), is the simplicial complex whose faces are the cliques of \( G \). In other words, the clique complex of \( G \) is the maximal simplicial complex \( X \) so that the 1-skeleton of \( X \), denoted as \( X^{(1)} \), is \( G \). The random clique complex model \( X(n, p) \) is sampled as the clique complex of the Erdős–Rényi random graph \( G(n, p) \).

Cycles in \( G(n, p) \) generalize to higher dimensions as nonvanishing homology groups in \( X(n, p) \). A short argument using classic results about inclusion of cliques in \( G(n, p) \) can be used to show that \( X \sim X(n, c/\sqrt{n}) \) for \( c > \sqrt{d+1} \) with high probability has nonvanishing \( d \)th homology group (for any choice of coefficients). The details of such an argument can be found in [15]. The \( \sqrt{d+1} \) lower bound was recently improved by Kanazawa [17].

On the other hand, a result of [14] establishes that if \( p = n^{-1/(d+\varepsilon)} \) for any fixed \( \varepsilon > 0 \), then with high probability, \( X \sim X(n, p) \) has vanishing \( d \)th homology group. As the main result here, we improve on this lower bound by showing a probability regime where the probability of that the \( d \)th homology group vanishes is bounded away from zero and away from one.

**Theorem 1.2.** For \( d \geq 2 \), there exists an explicit constant \( \varepsilon_d \) so if \( c < \varepsilon_d \), then \( \beta_d(X, \mathbb{K}) \), for any coefficient field \( \mathbb{K} \), for \( X \sim X(n, c/\sqrt{n}) \) is asymptotically Poisson distributed with mean

\[
\frac{c^{2(d+1)d}}{2^{d+1}(d+1)!}.
\]

The Poisson distribution comes from counting particular subcomplexes in \( X \sim X(n, c/\sqrt{n}) \). The boundary of the \((d+1)\)-dimensional cross-polytope is the clique complex of the complete \((d+1)\)-partite graph with two vertices in each part, and standard results about random graphs tell us that \( X \sim X(n, c/\sqrt{n}) \) for \( c \) constant will contain Poisson-distributed copies of the boundary of the \((d+1)\)-dimensional cross-polytope. For \( c \) a small enough constant, we show that these cross-polytope boundaries are the only subcomplexes of \( X(n, c/\sqrt{n}) \) that can carry homology in degree \( d \).

In fact, we actually prove something stronger. The proof of Theorem 1.2 builds on a result of Malen [19] about \( d \)-collapsibility of random clique complexes. We show that in the studied regime, \( X \sim X(n, c/\sqrt{n}) \) asymptotically almost surely can be collapsed to a complex whose pure \( d \)-dimensional part is a face-disjoint union of \((d+1)\)-cross-polytope boundaries. We explain this in more detail in Section 2.

Because we show a collapsibility result, the proof of Theorem 1.2 implies that for \( c \) sufficiently small, \( X \sim X(n, c/\sqrt{n}) \) asymptotically almost surely has free fundamental group. This improves on a result of Costa, Farber, and Horak [9] that \( X(n, n^{-\alpha}) \) has free fundamental group if \( \alpha > 1/2 \), but nonfree fundamental group if \( 1/3 < \alpha < 1/2 \). Here, the assumption of \( \alpha > 1/3 \) comes from the threshold for simply connectivity of \( X(n, p) \), due to [9] and [14]. In Section 6, we adapt some of the methods for the proof of Theorem 1.2 combined with results of [9] to also improve on the
upper bound for the threshold for the fundamental group to be nonfree. Our result regarding the fundamental group is the following theorem.

**Theorem 1.3.** For $c < \frac{1}{64}$ and $p < c/\sqrt{n}$, asymptotically almost surely $X \sim X(n, p)$ has that $\pi_1(X)$ is free. For $C > \sqrt{3}$, $\alpha > 1/3$, and $C/\sqrt{n} < p < n^{-\alpha}$, asymptotically almost surely $X \sim X(n, p)$ has that $\pi_1(X)$ is not free.

To establish the upper bound in Theorem 1.3, we study “almost asphericity” of random flag complexes. This is a natural analog of a notion previously studied in the Linial–Meshulam model by Costa and Farber [7]. Theorem 1.3 together with a result of Kahle [15] implies the following theorem that settles a conjecture of Costa, Farber, and Horak that there is an intermediate regime where $\pi_1(X)$ for $X \sim X(n, p)$ is not free and does not have Kazhdan’s property (T).

**Theorem 1.4.** For any fixed $\varepsilon > 0$ if

$$\left(\frac{3 + \varepsilon}{n}\right)^{1/2} < p < \left(\frac{(3/2 - \varepsilon)\log n}{n}\right)^{1/2},$$

then with high probability, $X \sim X(n, p)$ has that $\pi_1(X)$ is not free and does not have property (T).

We then close with some discussion about what the true sharp thresholds for the discussed properties might be.

## 2 BACKGROUND

A series of four papers [1–3, 18] study the generalization of the Erdős–Rényi phase transition to the Linial–Meshulam model. Recall that for $d \geq 1$ fixed, one samples $Y \sim Y_d(n, p)$ in the Linial–Meshulam model by starting with the complete $(d - 1)$-complex on $n$ vertices and including each $d$-dimensional face independently with probability $p$; the $d = 1$ case is then exactly the Erdős–Rényi random graph model. For $d \geq 2$, there are (at least) two ways to generalize the existence of cycles in a graph to an analogous property in a $d$-complex: non-$d$-collapsibility and the nonvanishing of the $d$th homology group.

A face in a simplicial complex is said to be free if it is properly contained in only one other face. The removal of a free face and its unique coface is called an elementary collapse, and a complex is $d$-collapsible, provided that there exists a sequence of elementary collapses that eliminates all faces of dimension at least $d$. For a graph, a free vertex is simply a leaf, and so, a graph is acyclic if and only if it is 1-collapsible. On the other hand, an elementary collapse is a homotopy equivalence, so a $d$-collapsible complex will have no homology in degree $d$ or larger; however, it is well known that the reverse implication holds only for $d = 1$.

For $Y_d(n, p)$, $d$-collapsibility and nonvanishing of the $d$th homology group have different thresholds, and the former is established by [3] and [1] and the latter by [2] and [18]. For each value of $d$, both thresholds are in the regime $p = c/n$. The critical constants $\gamma_d$ and $c_d$ for each threshold are given explicitly by solutions to certain transcendental equations with $\gamma_d$ being of order $\log d$ for $d \to \infty$ and $c_d$ being asymptotically very slightly smaller than $d + 1$. 
Both of these thresholds in $Y_d(n, p)$ are one-sided sharp thresholds. For $p = c/n$, one can show that the expected number of copies of the boundary of the $(d + 1)$-simplex, $\partial \Delta_{d+1}$, in $Y_d(n, c/n)$ is Poisson distributed with mean $c^{d+2}/(d + 2)!$. Since $Y_d(n, p)$ is always $d$-dimensional, a $(d + 1)$-simplex boundary is always a nontrivial homology class and an obstruction to $d$-collapsibility. Thus, the results of [1–3, 18] can be summarized as follows.

- For $0 < c < \gamma_d$, $Y \sim Y_d(n, c/n)$ is $d$-collapsible with probability asymptotic to $\exp(-c^{d+2}/(d + 2)!)$ [3].
- For $\gamma_d < c < c_d$, $Y \sim Y_d(n, c/n)$ is not $d$-collapsible with high probability [1], but the $d$th homology group with real coefficients is generated by embedded copies of $\partial \Delta_{d+1}$, in particular $\beta_d(Y; \mathbb{R})$ is asymptotically Poisson distributed with mean $c^{d+2}/(d + 2)!$ [18].
- For $c_d < c$, $Y \sim Y_d(n, c/n)$ has nonvanishing $d$th homology group for any choice of coefficients with high probability [2].

The key to the arguments that establish both thresholds is the local behavior of $Y \sim Y_d(n, p)$ at a $(d - 1)$-dimensional face and the possibility of large cores, complexes that are obstructions to $d$-collapsibility. Regarding the local behavior at $(d - 1)$-faces, which is complementary to what we cover here, Kanazawa [17] proves results about the local behavior in the clique complex setting to establish asymptotics for the Betti numbers.

The separate challenge that we focus on here though is to adapt an argument counting $d$-dimensional cores in $X \sim X(n, p)$. The Linial–Meshulam model $Y_d(n, p)$ produces complexes that are $d$-dimensional and $d$-complexes are easier to $d$-collapse than complexes of larger dimension creating some obstacles to overcome.

A $d$-dimensional complex is said to be a core, provided that each $(d - 1)$-dimensional face is contained in at least two $d$-dimensional faces, so a $d$-complex is $d$-collapsible if and only if it does not contain a $d$-dimensional core. This means that a greedy approach to elementary collapses will tell us whether or not a given $d$-complex is $d$-collapsible. For $d$-collapsibility of a given $k$-complex with $k > d$, the situation becomes more complicated. For example, any simplex is collapsible in the usual sense to a vertex, so, in particular, it is $d$-collapsible for any $d$; however, if the collapses are chosen in the wrong way, it is possible to get stuck. An explicit example of this phenomenon is shown [5] where Benedetti and Lutz show a way to collapse the 7-simplex to a triangulation of the dunce hat on eight vertices, which is not 2-collapsible. In Section 3, we describe how we work around this potential possibility of getting stuck to prove Theorem 1.2.

A paper of Malen [19] studies $d$-collapsibility in $X(n, p)$. His main result shows that for $\alpha > 1/d$, with high probability, $X \sim X(n, n^{-\alpha})$ is $d$-collapsible. This result was applied recently by Dochtermann and the present author [10] to establish results on random coedge ideals. Some of the background for that article motivates the questions considered here, as the $p = n^{1/d}$ regime for $d$ an integer is related to boundary cases for theorems proved in [10]. In [10], we consider a conjecture of Erman and Yang [12] about normal distribution of Betti numbers of random coedge ideals. Random coedge ideals correspond in a natural way to random flag complexes, and in [12], Erman and Yang prove a normal distribution of the first row of the Betti table of the random coedge ideal for $G \sim G(n, c/n)$ and $c < 1$, that is, for $p$ close to, but below, the Erdős–Rényi phase transition. A challenge to generalizing their conjecture to other rows was that there was not an established one-sided sharp phase transition for homology in dimensions larger than one for $X \sim X(n, p)$.

As discussed in the introduction, the proof of Theorem 1.2 proceeds by proving something stronger related to $d$-collapsibility. Toward formally stating our $d$-collapsibility result, we recall
that the pure $d$-dimensional part of a simplicial complex $X$ is the subcomplex of $X$ whose facets are the $d$-dimensional faces of $X$, and we say that a simplicial complex $X$ is almost $d$-collapsible if there is a sequence of elementary collapses from $X$ to a complex $Y$ so that $Y$ is at most $d$-dimensional and the pure $d$-dimensional part of $Y$ is a face disjoint union of copies of the $(d + 1)$-dimensional cross-polytope boundary. For convenience, we let $\vartriangleleft_d$ denote the $(d + 1)$-dimensional cross-polytope boundary. Because we are dealing with flag complexes throughout, we typically would not distinguish between $\vartriangleleft_d$ as a $d$-complex and its 1-skeleton as a graph. The stronger version of Theorem 1.2 that we prove is the following.

**Theorem 2.1.** For $c < \frac{1}{2^{2d+1}d}$ with high probability $X \sim X(n, c/\sqrt{n})$ is almost $d$-collapsible.

If a complex is almost $d$-collapsible, then its $d$th homology group is the free abelian group with one generator for each remaining copy of $\vartriangleleft_d$ after the collapsing sequence erases everything else of dimension $d$ and higher. So then, the Poisson distribution part of Theorem 1.2 comes from enumeration of copies of $\vartriangleleft_d$ that are not collapsed away by collapsing $X$. This part comes from classic random graph theory.

Recall that the essential density of a graph $H$ is

$$
\rho(H) = \max\{e(H')/\nu(H') \mid H' \subseteq H\},
$$

and that a graph is strictly balanced, provided that the essential density is attained by the graph itself and not by any proper subgraphs. A classic result of Bollobás [6] shows that for any finite strictly balanced graph $H$, the number of copies of $H$ embedded in $G(n, c/n^{1/\rho(H)})$ for $c$ fixed is asymptotically Poisson distributed with mean:

$$
\frac{c^e(H)}{|\text{Aut}(H)|}.
$$

For more background on embedding subgraphs into random graphs, see, for example, [13, Chapter 5].

It is straightforward to check that $\vartriangleleft_d$ is strictly balanced with essential density $d$. The number of automorphisms of $\vartriangleleft_d$ is $(d + 1)!2^{d+1}$; this can be easily seen by counting the number of automorphisms of the complement of $\vartriangleleft_d$ that is a matching on $2(d + 1)$ vertices. Therefore, the number of copies of $\vartriangleleft_d$ embedded in $X(n, c/\sqrt{n})$ is asymptotically distributed as a Poisson random variable with mean

$$
\frac{c^{2d(d+1)}}{2^{d+1}(d + 1)!}.
$$

We have to be a bit careful though as not every copy of $\vartriangleleft_d$ necessarily survives the collapsing process. For example, a noninduced copy of $\vartriangleleft_d$ could potentially be collapsed away; however, in proving Theorem 1.2 from Theorem 2.1, we show that copies of $\vartriangleleft_d$ that do not contribute to $d$th homology are negligible in the limit.

**Remark 2.2.** Theorem 2.1 and a simple first moment argument enumerating copies of $\vartriangleleft_d$ in an Erdős–Rényi random graph imply that if $p = o(n^{-1/\alpha})$, then $X \sim X(n, p)$ is asymptotically almost surely $d$-collapsible. This strengthens the result of Malen [19] that $\alpha > 1/d$ implies that $X \sim$
\(X(n, n^{-\alpha})\) is asymptotically almost surely \(d\)-collapsible and an earlier result of Kahle [14] that \(\alpha > 1/d\) implies \(X \sim (n, n^{-\alpha})\) asymptotically almost surely has no homology above dimension \((d-1)\).

3 | OVERVIEW OF THE PROOF

The approach to proving Theorem 2.1 is to count obstructions to \(d\)-collapsibility. Typically, a \(d\)-dimensional core is defined to be a \(d\)-dimensional complex so that every \((d-1)\)-dimensional face belongs to at least two \(d\)-dimensional faces. Here though we drop the “\(d\)-dimensional complex” assumption. We say that a simplicial complex \(Z\) is a \(d\)-core, provided that every \((d-1)\)-dimensional face of \(Z\) belongs to at least two \(d\)-dimensional faces. Note that usage of “\(d\)-core” here differs from the definition readers may be familiar with from hypergraph literature.

Naively, we might try to show that the only \(d\)-cores are the embedded copies of \(\varnothing_d\) and that this implies that the complex collapses so that the only \(d\)-dimensional faces left are those in copies of \(\varnothing_d\). This will not work, however, because, unlike \(Y_d(n, c/n)\), \(X(n, c/\sqrt[n]{n})\) in general has dimension larger than \(d\). In particular, for \(d \geq 2\), the expected number of \((d+1)\)-dimensional faces in \(X(n, c/\sqrt[n]{n})\) is

\[
\binom{n}{d+2}\left(\frac{c}{\sqrt[n]{n}}\right)^{d+2} \approx \frac{c}{(d+2)!} n^{(d+1)(d+1)/2} \to \infty.
\]

And the boundary of a \((d+1)\)-simplex is a \(d\)-core.

To describe precisely how we will collapse \(X(n, c/\sqrt[n]{n})\), we first introduce some terminology and notation that is fairly standard. For a \(d\)-complex \(Y\), the dual graph of \(Y\), denoted as \(\mathcal{G}(Y)\), is defined to be the graph whose vertices are the \(d\)-dimensional faces of \(Y\) with edges between two vertices if and only if the corresponding faces meet at a \((d-1)\)-dimensional face. A \(d\)-dimensional strongly connected complex is a pure \(d\)-complex \(Y\) so that \(\mathcal{G}(Y)\) is connected. We will collapse \(X \sim X(n, c/\sqrt[n]{n})\) by collapsing each of its \(d\)-dimensional strongly connected components separately, and showing that these collapsing moves may be applied in a coherent way to collapse all the faces of dimension \(d\) from \(X\). This is similar to the approach taken in [19]. The first step is a slight strengthening of Lemma 3.3 of [19], given here with proof as Lemma 3.2.

In order to state this lemma, we introduce a definition and some notation.

**Definition 3.1.** For a simplicial complex \(Y\), the \(k\)-flag closure of \(Y\) denoted as \(\text{flag}_k(Y)\) is the maximal simplicial complex whose \(k\)-skeleton is the same as the \(k\)-skeleton of \(Y\). In other words, \(\text{flag}_k(Y)\) is the simplicial complex obtained from \(Y\) by adding the requirement that the simplex \(\sigma\) is included if its \(k\)-skeleton belongs to \(Y\). We say that a complex \(Y\) is a \(k\)-flag complex if \(Y = \text{flag}_k(Y)\).

Outside of this section and Section 4, we will most often deal with the usual flag closure \(\text{flag}_1(Y)\), so we will use the notation \(\overline{Y}\) for \(\text{flag}_1(Y)\).

Even though we are dealing with flag complexes throughout, the higher dimensional flag closures are necessary for the next lemma. Lemma 3.2 is false in dimensions larger than 2 if \(\text{flag}_{d-1}\) is replaced by \(\text{flag}_1\). To see why we could take \(d = 3\) and have two strongly connected \(3\)-complexes \(A\) and \(B\) with \(A\) as just a single tetrahedron and \(B\) as some \(3\)-complex that contains all the edges of \(A\) but none of the triangles of \(A\). Then, the flag closure of \(A\) and \(B\) both contain \(A\), but the \(2\)-flag closure of \(B\) does not contain \(A\).
Lemma 3.2. Let $X$ be a clique complex and $S_1$ and $S_2$ be strongly connected components of $X^{(d)}$. Then, $\dim(\text{flag}_{d-1}(S_1) \cap \text{flag}_{d-1}(S_2)) \leq (d-1)$ and if $\sigma \in \text{flag}_{d-1}(S_1) \cap \text{flag}_{d-1}(S_2)$ is $(d-1)$-dimensional, then $\sigma$ is maximal in at least one of $\text{flag}_{d-1}(S_1)$ and $\text{flag}_{d-1}(S_2)$.

Proof. Suppose that $\tau \in \text{flag}_{d-1}(S_1) \cap \text{flag}_{d-1}(S_2)$ is a $d$-simplex. By the definition of the $(d-1)$-flag closure, $\tau^{(d-1)}$, which is $\partial \tau$, belongs to $S_1$ and to $S_2$. But the intersection of $S_1$ and $S_2$ is at most $(d-2)$-dimensional as the strongly connected components of $X^{(d)}$ partition both $d$-faces and nonmaximal $(d-1)$-dimensional faces.

Next, suppose that $\sigma$ is a $(d-1)$-dimensional face in $\text{flag}_{d-1}(S_1)$ and $\text{flag}_{d-1}(S_2)$ with $\sigma \subseteq \tau_1 \in \text{flag}_{d-1}(S_1)$ and $\sigma \subseteq \tau_2 \in \text{flag}_{d-1}(S_2)$, $\tau_1, \tau_2$ both $d$-dimensional. Then, the $(d-1)$-skeleton of $\tau_1$ belongs to $S_1$ and the $(d-1)$-skeleton of $\tau_2$ belongs to $S_2$. Moreover, as $X$ is a flag complex, $\tau_1$ belongs to $S_1$ and $\tau_2$ belongs to $S_2$. But then we can move from $\tau_1$ to $\tau_2$ in $X^{(d)}$ by passing through $\sigma$. This contradicts maximality of $S_1$ and $S_2$. □

When $d$-collapsing $X$, we only have to consider the subcomplex of $X$ obtained by deleting maximal faces of dimension at most $d-1$. So, we look at the appropriate flag closure of $X^{(d)}$, and $X^{(d)}$ itself has a canonical decomposition into strongly connected components. We prove the following using Lemma 3.2 to show that it suffices to consider each strongly connected component of $X^{(d)}$ separately.

Lemma 3.3. Let $X$ be a flag complex with $S_1, \ldots, S_m$ the strongly connected components of $X^{(d)}$. If $\text{flag}_{d-1}(S_i)$ is almost $d$-collapsible for each $i$, then $X$ is almost $d$-collapsible.

Proof. By Lemma 3.2, $\text{flag}_{d-1}(S_1), \ldots, \text{flag}_{d-1}(S_m)$ partitions the faces of $X$ of dimension at least $d$. Therefore, any collapsing move that does not involve a face of dimension at most $(d-1)$ only affects one of the $\text{flag}_{d-1}(S_i)$'s. A collapsing move $(\sigma, \tau)$ in $\text{flag}_{d-1}(S_i)$ collapsing a $(d-1)$-face $\sigma$ into a $d$-face $\tau$ means that $\sigma$ is not maximal in $\text{flag}_{d-1}(S_i)$, so by Lemma 3.2, $\sigma$ is maximal in all the $\text{flag}_{d-1}(S_j)$ other than $\text{flag}_{d-1}(S_i)$ that contain it. Since we are only interested in almost $d$-collapsibility, maximal $(d-1)$-dimensional faces in $\text{flag}_{d-1}(S_j)$ do not affect the ability to $d$-collapse, so we do not need to use $\sigma$ to almost $d$-collapse any of the other $\text{flag}_{d-1}(S_j)$'s. Thus, we may run a collapsing sequence on $\text{flag}_{d-1}(S_1)$ that arrives a face disjoint union of copies of $\varnothing_d$. Such a collapsing sequence only changes the other $\text{flag}_{d-1}(S_i)$'s by possibly removing maximal $(d-1)$-dimensional faces, which leaves all the necessary collapsing moves available to almost $d$-collapse each $\text{flag}_{d-1}(S_i)$ separately. After separately collapsing each $\text{flag}_{d-1}(S_1), \ldots, \text{flag}_{d-1}(S_m)$, all the $d$-faces remaining belong to copies of $\varnothing_d$. Moreover, as $\text{flag}_{d-1}(S_1), \ldots, \text{flag}_{d-1}(S_m)$ partition $d$-faces of $X$ and copies of $\varnothing_d$ remaining in each individual $\text{flag}_{d-1}(S_i)$ are face disjoint from one another, the remaining copies $\varnothing_d$ after collapsing $X$ are also face disjoint from one another. □

Having reduced the problem to collapsing each $d$-dimensional strongly connected component separately, we have two lemmas that will constitute the bulk of the work to prove Theorem 2.1.

Lemma 3.4. For $c \leq \frac{1}{2 + 2d}$ with high probability, $X \sim X(n, c / \sqrt[4]{n})$ has no strongly connected $d$-dimensional subcomplex on more than $C \log n$ edges for $C$ large enough.
Lemma 3.5. For $d \geq 2$, if $H$ is a $d$-complex where the link of each $(d-2)$-dimensional face is an induced subgraph $H^{(1)}$, and $H^{(1)}$ has essential density

$$\rho(H^{(1)}) < d + \frac{1}{4 + 4d},$$

then $\text{flag}_{d-1}(H)$ is almost $d$-collapsible.

Recall that the link of a face $\sigma$ in a simplicial complex $H$ is the subcomplex of $H$ given by

$$\text{lk}_H(\sigma) := \{ \tau \setminus \sigma \mid \sigma \subseteq \tau \in H \}.$$

Of these two lemmas, Lemma 3.5 is the easier to prove, so we prove it in Section 4 and then prove Lemma 3.4 in Section 5. First, we show how these two lemmas imply Theorem 2.1.

Proof of Theorem 2.1. By Lemma 3.3, it suffices to show that each $d$-dimensional strongly connected component of $X \sim X(n, c/\sqrt{n})$ is almost $d$-collapsible after taking the $(d-1)$-flag closure. By Lemma 3.4, with high probability, $X \sim X(n, c/\sqrt{n})$ has no $d$-dimensional strongly connected component $S$ so that the graph of $S$ has more than $C \log n$ edges for some $C = C(d, c)$ sufficiently large. So, there is no $d$-dimensional strongly connected component on more than $2C \log n$ vertices.

Now if we can prove that with high probability, every $d$-dimensional strongly connected component on at most $3C \log n$ vertices has essential density (of its 1-skeleton, not its dual graph) less than $d + \frac{1}{4 + 4d}$, then we can finish the proof using Lemma 3.5. The probability that $X \sim X(n, c/\sqrt{n})$ has a $d$-dimensional strongly connected component on at most $3C \log n$ vertices of essential density at least $d + \frac{1}{4 + 4d}$ is at most the probability that $G \sim G(n, c/\sqrt{n})$ has a subgraph $H$ on at most $3C \log n$ vertices with $|E(H)| \geq (d + \frac{1}{4 + 4d})|V(H)|$.

By a standard first moment argument, the expected number of subgraphs of $G(n, c/\sqrt{n})$ with $v$ vertices at least $(d + \epsilon)v$ edges for any $\epsilon > 0$ is at most

$$\binom{n}{v} (v^2)^{(d+\epsilon)v} \left(\frac{c}{\sqrt{n}}\right)^{(d+\epsilon)v}.$$

By taking a sum over $1 \leq v \leq 3C \log n$, we have the probability that $X \sim X(n, c/\sqrt{n})$ contains a strongly connected component on at most $3C \log n$ vertices with density larger than $(d + \epsilon)$ is at most

$$\sum_{v=1}^{3C \log n} \binom{n}{v} (v^2)^{(d+\epsilon)v} \left(\frac{c}{\sqrt{n}}\right)^{(d+\epsilon)v} \leq \sum_{v=1}^{3C \log n} \left(\frac{c^2d^{2d+2\epsilon-1}e^{cd+\epsilon}}{n^{\epsilon}}\right)^v \leq \sum_{v=1}^{\infty} \left(\frac{c^2d^{2d+2\epsilon-1}e^{cd+\epsilon}}{n^{\epsilon}}\right)^v = o(1).$$

Thus, with high probability, each $d$-dimensional strongly connected component of $X \sim X(n, c/\sqrt{n})$ is on at most $2C \log n$ vertices and has essential density less than $d + \frac{1}{4 + 4d}$. Lastly,
in order to apply Lemma 3.5, we check that necessary link condition holds for each strongly connected connected component of $X^{(d)}$; we will see that this holds simply because $X$ is a flag complex rather than for any reason having to do with the randomness.

As $X$ is a flag complex, if $S$ is a strongly connected component of $X^{(d)}$, then for any $(d - 2)$-face $\sigma$ of $S$, we just have to verify that its link within $S$, which is necessarily 1-dimensional as we consider $X^{(d)}$, is an induced subgraph of $X$. Suppose that $v_1, v_2$ belong to $\text{lk}_S(\sigma)$ and that $\{v_1, v_2\}$ is an edge of $X$. Then, $\{v_1\} \cup \sigma$ and $\{v_2\} \cup \sigma$ both belong to $S$ and the 1-skeleton of $\{v_1, v_2\} \cup \sigma$ belongs to $X$. Therefore, $\{v_1, v_2\} \cup \sigma$ belongs to $X^{(d)}$ and in particular belongs to some strongly connected component of $X^{(d)}$. As the strongly connected components also partition the nonmaximal $(d - 1)$-faces, and at least two $(d - 1)$-faces of $\{v_1, v_2\} \cup \sigma$ belong to $S$, $\{v_1, v_2\} \cup \sigma$ also belongs to $S$ and so $\{v_1, v_2\}$ is an edge of $\text{lk}_S(\sigma)$.

Thus, by Lemma 3.5, each $d$-dimensional strongly connected component of $X$ is almost $d$-collapsible, so by Lemma 3.3, $X$ is almost $d$-collapsible.

Now we show that Theorem 2.1 implies Theorem 1.2.

Proof of Theorem 1.2. For $c < \varepsilon_d = \frac{1}{2^{d+1}d}$, $X \sim X(n, c/{\sqrt[2d+1]{n}})$ is asymptotically almost surely almost $d$-collapsible. If $X$ is almost $d$-collapsible, then after collapsing, the pure $d$-dimensional part of the remaining complex is a face disjoint union of copies of $\diamondsuit_d$. Thus, $\beta_d(X; \mathbb{K})$ is at most the number of copies of $\diamondsuit_d$ in $X$.

Next, we give a lower bound that $\beta_d(X; \mathbb{K})$ related to the number of copies of $\diamondsuit_d$. Each $\diamondsuit_d$ is a triangulation of $S^d$, so each is a $d$-cycle, so it suffices to bound the probability that each is a $d$-boundary. If $K$ is a set of $2d + 2$ vertices on which there is a copy of $\diamondsuit_d$ in $X$, then a sufficient condition for that copy of $\diamondsuit_d$ to not be a $d$-boundary is that one of the $d$-faces of $\diamondsuit_d$ is maximal in $X$. That is, one of the $d$-faces of the flag complex induced on $K$ is not contained in a $(d + 1)$-dimensional face.

If a copy of $\diamondsuit_d$ contains a $d$-face that is not maximal, then either that copy of $\diamondsuit_d$ is not induced or there exists a vertex $v$ outside the $2d + 2$ vertices of $\diamondsuit_d$ that is adjacent exactly to the $(d + 1)$-vertices of a $d$-face of $\diamondsuit_d$. A noninduced copy of $\diamondsuit_d$ is a graph $H$ with $2d + 2$ vertices and at least $2d(d + 1) + 1$ edges, so the expected number of copies of $H$ in $G \sim G(n, c/{\sqrt[2d+1]{n}})$ is $O(n^{-1/d}) = o(1)$. On the other hand, the expected number of copies of $\diamondsuit_d$ that meet a $(d + 1)$-simplex at a $d$-simplex is $O(n^{-1/d})$ too. Thus, with high probability

$$\# \text{ copies of } \diamondsuit_d - o(1) \leq \beta_d(X; \mathbb{K}) \leq \# \text{ copies of } \diamondsuit_d.$$  

So, asymptotically, $\beta_d(X; \mathbb{K})$ is equal to the number of copies of $\diamondsuit_d$. By the Bollobás result discussed earlier, the number of copies of $\diamondsuit_d$ in $X$ is Poisson distributed with mean $c^{2d(d+1)/(2^{d+1}(d + 1)!)}$.

4 | SIMPLICITY OF SMALL COMPONENTS

In this section, we prove Lemma 3.5. For this proof and elsewhere, we need a few more definitions. We have already recalled the definition of a link. We also recall that a complex is pure if all of its maximal faces are of the same dimension. For a vertex $v$ in a complex $X$, if $\text{lk}_X(v)$ is $(d - 1)$-collapsible with collapsing sequence $(\sigma_1, \tau_1), \ldots, (\sigma_t, \tau_t)$ with each $\sigma_i$ a free face of $\tau_i$ and all $\sigma_i, \tau_i \in \text{lk}_X(v)$, to $d$-collapse around $v$ means to apply the collapsing sequence $(\sigma_1 \cup \{v\}, \tau_1 \cup \{v\}), \ldots, (\sigma_t \cup \{v\}, \tau_t \cup \{v\})$.  

□
\[\{v\}, \tau_t \cup \{v\}\) to \(X\). By this collapsing strategy, we see that if \(\text{lk}_X(v)\) is \((d-1)\)-collapsible and \(X \setminus \{v\}\) is \(d\)-collapsible, then \(X\) is \(d\)-collapsible. Indeed, we can \(d\)-collapse \(X\) around \(v\) and then \(v\) is not contained in any faces of dimension larger than \((d-1)\), but nothing in \(X \setminus \{v\}\) has been changed, so we can freely attempt to \(d\)-collapse \(X \setminus \{v\}\) after \(d\)-collapsing around \(v\).

Given the assumptions of Lemma 3.5, for brevity, we say that a \(d\)-complex \(H\) satisfies the \(\star\)-condition if the link of each of its \((d-2)\)-faces is an induced subgraph of \(H\). We begin with a generalization of the well-known fact that \(\triangledown_d\) is the only non-\(d\)-collapsible flag complex on fewer than \(2d + 3\) vertices.

**Lemma 4.1.** For \(d \geq 2\), if \(H\) is a \(d\)-complex on at most \(2d + 2\) vertices which satisfies the \(\star\)-condition, then either

\[(i) \text{ flag}_{d-1}(H)\) is \(d\)-collapsible, or
\[(ii) H = \triangledown_d.\]

The proof of this lemma will make use of the following for the induction.

**Lemma 4.2.** For \(d \geq 3\), if \(H\) is a pure \(d\)-complex that satisfies the \(\star\)-condition, then for any vertex \(v\) of \(H\), \(\text{lk}(v)\) is a pure \((d-1)\)-complex that also satisfies the \(\star\)-condition. Moreover, \(\text{lk}_{\text{flag}_{d-1}(H)}(v) = \text{flag}_{d-2}(\text{lk}_H(v))\).

**Proof.** Obviously, since \(H\) is pure \(d\)-dimensional, \(\text{lk}_H(v)\) is pure \((d-1)\)-dimensional. We show that it satisfies the \(\star\)-condition (for faces of codimension 2, that is, dimension \(d-3\)). Let \(\sigma\) be a \((d-3)\)-face of \(\text{lk}_H(v)\), and suppose that \(\{u_1, u_2\}\) is an edge of \(\text{lk}_H(v)\) with \(\sigma \cup \{u_1\}\) and \(\sigma \cup \{u_2\}\) in \(\text{lk}_H(v)\). Then, \(u_1\) and \(u_2\) are both in the link of the \((d-2)\)-face \(\sigma \cup \{v\}\), and \(\{u_1, u_2\}\) is an edge of \(H\), so by the \(\star\)-condition on \(H\), \(\sigma \cup \{v, u_1, u_2\}\) is a face of \(H\), so \(\sigma \cup \{u_1, u_2\}\) belongs to \(\text{lk}_H(v)\) and so \(\text{lk}_{\text{flag}_{d-1}(H)}(v)\) is an induced subgraph of the graph of \(\text{lk}_H(v)\).

Now we verify that \(\text{lk}_{\text{flag}_{d-1}(H)}(v) = \text{flag}_{d-2}(\text{lk}_H(v))\). Suppose that \(\tau \in \text{lk}_{\text{flag}_{d-1}(H)}(v)\), then \(\tau \cup \{v\}\) belongs to \(\text{flag}_{d-1}(H)\), so the \((d-1)\)-skeleton of \(\tau \cup \{v\}\) belongs to \(H\). Thus, the \((d-2)\)-skeleton of \(\tau\) belongs to \(\text{lk}_H(v)\), so \(\tau \in \text{flag}_{d-2}(\text{lk}_H(v))\). On the other hand, suppose that \(\tau \in \text{flag}_{d-2}(\text{lk}_H(v))\). Then, the \((d-2)\)-skeleton of \(\tau\) belongs to \(\text{lk}_H(v)\). Therefore, every \((d-1)\)-face of \(\tau \cup \{v\}\) that contains \(v\) belongs to \(H\). Now for any \(\sigma \subset \tau\) of dimension \((d-1)\), we have that the \((d-2)\)-skeleton of \(\sigma\) belongs to \(\text{lk}_H(v)\). Thus, it will suffice to show that \(\text{lk}_H(v)\) cannot contain an empty \((d-1)\)-simplex (which is the boundary of a simplex on \(d\) vertices without the simplex itself being included). This will imply that \(\sigma\) belongs to \(\text{lk}_H(v)\) and hence to \(H\) that will allow us to conclude that the \((d-1)\)-skeleton of \(\tau \cup \{v\}\) belongs to \(H\) so \(\tau \in \text{lk}_{\text{flag}_{d-1}(H)}(v)\).

**Claim 4.3.** For \(d \geq 2\), if \(H\) is \(d\)-complex that satisfies the \(\star\)-condition, then \(H\) cannot contain an empty \(d\)-simplex.

**Proof of claim.** Suppose that \(H\) is \(d\)-dimensional with the \(\star\)-condition and \(H\) contains all the \((d-1)\)-faces of \([v_0, \ldots, v_d]\). Then, \(H\) contains \([v_0, \ldots, v_{d-2}, v_{d-1}], [v_0, \ldots, v_{d-2}, v_d]\), and \([v_{d-1}, v_d]\), thus the link of the \((d-2)\)-face \([v_0, \ldots, v_{d-2}]\) contains the edge \([v_{d-1}, v_d]\) by the \(\star\)-condition, and so, \([v_0, \ldots, v_{d-2}, v_{d-1}, v_d]\) belongs to \(H\). □

By the claim and the fact that \(\text{lk}_H(v)\) satisfies the \(\star\)-condition, we have that \(\text{lk}_H(v)\) cannot contain an empty \((d-1)\)-simplex, and so, we finish the proof. □
**Proof of Lemma 4.1.** The proof is by induction on $d$. The $d = 2$ case follows from the well-known fact that the octahedron $\Diamond_2$ is the only flag complex on fewer than seven vertices that is not 2-collapsible. Now suppose that $d > 2$ and $H$ is a vertex minimal $d$-complex satisfying the $*$-condition on at most $2d + 2$ vertices so that its $(d - 1)$-flag closure is not $d$-collapsible. We may also assume that $H$ is pure $d$-dimensional. Indeed, if $H$ has a maximal face of dimension less than $(d - 1)$, then that face may be removed without affecting the structure of the $d$-faces in $\text{flag}_{d-1}(H)$ and also not affecting the $*$-condition on $H$. If $H$ has a maximal face of dimension $(d - 1)$, that face may be deleted without breaking the $*$-condition as deleting a maximal face of dimension $(d - 1)$ can only delete vertices from links of $(d - 2)$-faces. Moreover, such a deletion does not affect the structure of the $d$-faces in $\text{flag}_{d-1}(H)$ since any $d$-face of $\text{flag}_{d-1}(H)$ whose facets are all maximal in $H$ is an empty $d$-simplex in $H$ contradicting Claim 4.3.

Now assuming pure $d$-dimensional and vertex minimality if $H$ has a vertex $v$ so that $\deg(v) < 2d$, then $\text{lk}_H(v)$ is a pure $(d - 1)$-complex that also satisfies the $*$-condition by Lemma 4.2. Thus, by induction, $\text{lk}_H(v)$ must have $(d - 1)$-collapsible $(d - 2)$-flag closure. Therefore, by the second part of Lemma 4.2, taking the $(d - 1)$-flag closure of $H$ allows us to $d$-collapse around $v$ and then use vertex minimality to $d$-collapse $\text{flag}_{d-1}(H)$. Thus, every vertex of $H$ has degree at least $2d$, in particular $H$ has at least $2d + 1$ vertices. There are now two possibilities for $H$, either

(i) $H$ has a vertex of degree exactly $2d$ and has $2d + 2$ vertices, or

(ii) the 1-skeleton of $H$ is a complete graph on $2d + 1$ or $2d + 2$ vertices.

In the former case, $H$ contains two vertices $w$ and $w'$ so that $w$ and $w'$ are not adjacent to each other but are both adjacent to the remaining $2d$ vertices. By induction then, $\text{lk}_H(w)$ is a copy of $\Diamond_{d-1}$ and $\text{lk}_H(w')$ is a copy of $\Diamond_{d-1}$ on the same vertex set. From here, its suffices to show that they must both be the same copy of $\Diamond_{d-1}$. This follows in a fairly straightforward way from the $*$-condition. Suppose that $u$ and $u'$ are antipodal vertices in $\text{lk}_H(w) = \Diamond_{d-1}$ but $\{u, u'\}$ is an edge of $H$. Then taking some $(d - 3)$-face $\tau$ in $\text{lk}_H(w) \cap \text{lk}_H(u) \cap \text{lk}_H(u')$ (which must exist since this intersection is a copy of $\Diamond_{d-2}$), we have $\tau \cup \{w, u\}$, $\tau \cup \{w, u'\}$, and $\{u, u'\}$ all present in $H$ with $\tau \cup \{w\}$ of dimension $d - 2$ so $\tau \cup \{w, u, u'\}$ belongs to $H$ which means that $\{u, u'\}$ is an edge of $\text{lk}(w)$, but this is a contradiction. So, $H$ is the suspension of $\Diamond_{d-1}; H = \Diamond_d$.

Now we turn to the case that $H^{(1)}$ is a complete graph. If $H^{(1)}$ is a complete graph, then the $*$-condition implies that every $(d - 2)$-face of $H$ has link whose 1-skeleton is complete. However, this means that when we take the $(d - 1)$-flag closure, the link of every $(d - 2)$-face is a simplex.

To see this, we can apply the second part of Lemma 4.2 inductively to conclude that if $\sigma$ is a $k$-face of $H$, then $\text{lk}_{\text{flag}_{d-1}(H)}(\sigma) = \text{flag}_{d-1-(k+1)}(\text{lk}_H(\sigma))$. Indeed, the $k = 0$ case is already part of Lemma 4.2. For the inductive step, we note that if $\dim(\sigma) = k$, then $\sigma = \tau \cup \{w\}$ for a vertex $w$ and $\tau$ a $(k - 1)$-face with $\text{lk}_H(\tau)$ pure $(d - k)$-dimensional that satisfies the $*$-condition. Therefore,

$$
\text{flag}_{d-1-(k+1)}(\text{lk}_H(\sigma)) = \text{flag}_{d-1-(k+1)}(\text{lk}_{\text{lk}_H(\tau)}(w))
$$

$$
= \text{lk}_{\text{flag}_{d-1-(k+1)}(\text{lk}_H(\tau))}(w) \text{ by part (2) of Lemma 4.2}
$$

$$
= \text{lk}_{\text{lk}_{\text{flag}_{d-1}(H)}(\tau)}(w) \text{ by induction}
$$

$$
= \text{lk}_{\text{flag}_{d-1}(H)}(\sigma).
$$

In particular, $\text{lk}_{\text{flag}_{d-1}(H)}(\tau)$ is an ordinary flag complex for $\tau$ a $(d - 3)$-dimensional face. As the link of a $(d - 2)$-face is the link of a vertex within the link of a $(d - 3)$-face, we have that the link of every $(d - 2)$-face is flag. On the other hand, the link of every $(d - 2)$-face has complete
1-skeleton. Thus, the link of every \((d - 2)\) face is a simplex. The proof is then complete by the following claim that shows that \(\text{flag}_{(d-1)}(H)\) is not just \(d\)-collapsible, it is \((d - 1)\)-collapsible. Note that the possibility of links that are empty simplices is included which is why we do not necessarily have full collapsibility.

**Claim 4.4.** If \(X\) is a complex so that the link of every face of dimension \(k\) is a simplex, then \(X\) is \((k + 1)\)-collapsible.

**Proof.** By induction. If \(k = 0\), then the link of every vertex is a simplex. If there is a nonisolated vertex \(v\), then we can collapse around it so that we are left with a single vertex \(w\) in the link of \(v\) and then we can collapse \(v\) into \(\{v, w\}\). This has the effect of deleting \(v\) from the complex and deleting a vertex preserves the link condition. We proceed until we have only isolated vertices.

Now suppose that the claim holds for \(k - 1\) and we verify it for \(k\). Take \(v\) to be a vertex of \(X\), then \(\text{lk}_X(v)\) is a complex so that the link of every face of dimension \(k - 1\) is a simplex. By induction, then the link of \(v\) is \(k\)-collapsible. So, we collapse around \(v\) and now have that \(v\) does not belong to any \((k + 1)\)-faces. Thus, as far as \((k + 1)\)-collapsibility is concerned, we can simply delete \(v\) from \(X\) and preserve the link condition. 

\[\square\]

This finishes the proof of Lemma 4.1.

**Proof of Lemma 3.5.** Let \(H\) be an inclusion minimal counterexample, so \(H\) is pure \(d\)-dimensional and vertex minimal. If \(H\) has a vertex \(u\) so that \(\text{lk}_H(u) = \text{flag}_{(d-2)}(\text{lk}_H(u))\) is \((d - 1)\)-collapsible, then we may \(d\)-collapse \(\text{flag}_{(d-1)}(H)\) around \(u\). As this removes all \(d\)-faces containing \(u\) and the graph induced on \(H \setminus \{u\}\) also satisfies the density condition and \(H \setminus \{u\}\) satisfies the \(*\)-condition, so its \((d - 1)\)-flag closure is almost \(d\)-collapsible by vertex minimality. Therefore, \(H\) is almost \(d\)-collapsible. By Lemma 4.1, it follows that \(\deg(u) \geq 2d\) must hold for any vertex \(u\) of \(H\). Moreover, the following claim holds for \(H\) too.

**Claim 4.5.** Every vertex of \(H\) of degree exactly \(2d\) has a neighbor of degree larger than \(2d\).

**Proof of claim.** Let \(u\) be a vertex of \(H\) of degree exactly \(2d\). Then, by vertex minimality and Lemma 4.1, \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\) is \(\Diamond_{d-1}\) (strictly speaking from what is proved here so far this only applies to \(d \geq 3\), but the \(d = 2\) case is obvious; the only non 1-collapsible flag complex on four vertices is the 4-cycle \(\Diamond_1\)).

We want to show that some vertex in \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\) must have degree larger than \(2d\) in \(H\). For contradiction, suppose that every vertex in \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\) has degree \(2d\) in \(H\). Within \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\), each vertex has degree \(2d - 2\) since \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\) is the graph of \(\Diamond_{d-1}\). Thus, each vertex in \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\) can have exactly one neighbor other than \(u\) outside \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\) by our assumption that all have total degree \(2d\) in \(H\).

Let \(w_1, w_2\) be two vertices in \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\) so that \(w_1\) is adjacent to \(u_1, w_2\) is adjacent to \(u_2\) with \(u_1\) and \(u_2\) outside of \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\). If \(w_1\) and \(w_2\) are connected by an edge, then the edge \(\{w_1, w_2\}\) must belong to a \((d - 1)\)-dimensional face \(\sigma \in \text{lk}_{\text{flag}_{(d-1)}(H)}(u)\), since \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u) = \Diamond_{d-1}\) is pure \((d - 1)\)-dimensional. Now \(\sigma\) is contained in at least two \(d\)-dimensional faces, one of which must be \(\sigma \cup \{u\}\). Otherwise if \(\sigma\) is only contained in \(\sigma \cup \{u\}\), then \((\sigma, \sigma \cup \{u\}\)) is a permitted collapsing move that leaves \(\text{lk}_{\text{flag}_{(d-1)}(H)}(u)\) \((d - 1)\)-collapsible. So, we can \(d\)-collapse around \(u\) and then use
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minimality to \(d\)-collapse (\(\text{flag}_{d-1}(H) \setminus \{u\}) \setminus \sigma\) (which is still \((d - 1)\)-flag since \(\sigma\) is a maximal \((d - 1)\)-face in \(H \setminus \{u\}\)).

Now the only neighbor of \(w_1\) outside of \(\text{lk}_d(H)(u)\) is \(u_1\), so the only other possible \(d\)-face that contains \(\sigma\) is \(\sigma \cup \{u_1\}\). But the same line of reasoning applied to \(w_2\) implies that \(\sigma \cup \{u_2\}\) is the only possibility for a \(d\)-face different from \(\sigma \cup \{u\}\) to contain \(\sigma\). Thus, \(u_1 = u_2\).

This means that for every pair of adjacent vertices \(w_1\) and \(w_2\), their unique neighbor outside of \(\text{lk}_d(H)(u)\) other than \(u\) is the same vertex. But since \(d \geq 2\), \(\text{lk}_d(H)(u)\) is a connected graph, so there is some single vertex \(u'\) outside of \(\text{lk}_d(H)(u)\) with \(u \neq u'\) so that \(u'\) is adjacent to all neighbors of \(u\). Thus, we have found a copy of \(\vartriangle_d\) in \(H\). This copy of \(\vartriangle_d\) is attached to the complex induced by deleting \(u\) and all vertices in \(\text{lk}_d(H)(u)\), at \(u'\). So, \(H\) is a wedge sum of a smaller \(d\)-complex \(H''\) and a copy of \(\vartriangle_d\). Collapsing moves in \(\text{flag}_{d-1}(H'')\) do not affect the attached \(\vartriangle_d\), and by minimality, \(\text{flag}_{d-1}(H'')\) is almost \(d\)-collapsible, so \(\text{flag}_{d-1}(H)\) is almost \(d\)-collapsible too. This contradicts our choice of \(H\); thus, each vertex of degree \(2d\) in \(H\) is adjacent to a vertex of degree larger than \(2d\).

\(\square\)

Let \(t \geq 1\) denote the number of vertices of \(H\) of degree larger than \(2d\). By the claim, each vertex of degree \(2d\) has a neighbor of degree larger than \(2d\), so there are at least \(V(H) - t\) edges from the set of vertices of degree \(2d\) to the set of vertices of degree larger than \(2d\). Thus, the degree sum

\[
\sum_{\{v \in X \mid \text{deg}(v) \geq 2d + 1\}} \text{deg}(v) \geq |V(H)| - t.
\]

And

\[
\sum_{\{v \in X \mid \text{deg}(v) = 2d\}} \text{deg}(v) = 2d(|V(H)| - t).
\]

Thus, the number of edges of \(X\) is at least

\[
\frac{(2d + 1)(|V(H)| - t)}{2}.
\]

On the other hand, the number of edges is also at least

\[
\frac{(2d + 1)t + 2d(|V(H)| - t)}{2} = \frac{2d|V(H)| + t}{2}.
\]

So,

\[
|E(H)| \geq \max \left\{ \frac{2d|V(H)| + t}{2}, \frac{(2d + 1)(|V(H)| - t)}{2} \right\}.
\]

These two values are equal when \(t = \frac{|V(H)|}{2 + 2d}\) which means that for every value of \(t \geq 1\), the number of edges of \(H\) is at least

\[
\frac{2d|V(H)| + |V(H)|/(2 + 2d)}{2} = \left(d + \frac{1}{4 + 4d}\right)|V(H)|.
\]

But this contradicts the density assumption on \(H\) as a minimal counterexample. \(\square\)
5 | ENUMERATION OF LARGE COMPONENTS

Here, we prove Lemma 3.4. The argument splits into two parts. We first rule out graphs of large $d$-dimensional strongly connected components that contain $i$-faces of large degree for some $i < (d - 1)$, where degree of an $i$-dimensional face is taken to be the number of $(i + 1)$-dimensional faces that contain it. Then, we enumerate the number of possible $d$-dimensional strongly connected complexes that have all $i$-faces having small degree, and apply a first moment argument.

For the first case, subcomplexes with faces of large degree may be ruled out by examining the global structure of $X \sim X(n, c/\sqrt{n})$. To motivate this, let us first consider the case that $d = 2$. In this case, the degree of a vertex in $X \sim X(n, c/\sqrt{n})$ is distributed as a binomial with $n - 1$ trials and success probability $c\sqrt{n}$, and so, by Chernoff bound, we can conclude that with high probability, every vertex has degree at most $(1 + \epsilon)c\sqrt{n}$ for any $\epsilon > 0$. Thus, no subcomplex of $X$ can have any vertex of degree larger than $(1 + \epsilon)c\sqrt{n}$ with high probability, and so, in particular, $X$ cannot contain a two-dimensional strongly connected subcomplex whose graph has a vertex of degree larger than $(1 + \epsilon)c\sqrt{n}$.

Now we make this precise and prove the following for general $d \geq 2$.

**Lemma 5.1.** For any $X \sim X(n, c/\sqrt{n})$ with high probability for every $1 \leq i < d$ and every $\epsilon > 0$, the maximum degree of an $(i - 1)$-dimensional face is at most $(1 + \epsilon)c^{i-1-i/d}$.

**Proof.** Given an $(i - 1)$-dimensional face $\sigma$, the degree of $\sigma$ in $X$ conditioned on $\sigma$ being included in $X$ in the first place is distributed as a binomial random variable with $n - i$ trials and success probability $c\sqrt{n}$. Therefore, the expected degree of $\sigma$ given that $\sigma$ is included in $X$ is asymptotically $c^{i-1-i/d}$. Thus, by Chernoff bound, the probability that $\deg(\sigma) > (1 + \epsilon)c^{i-1-i/d}$ is at most $\exp(-Cn^{1-i/d})$ where $C$ is a constant that depends on $c$ and $\epsilon$. Thus, taking a union bound over all $(\binom{n}{i})$ possible $(i - 1)$-dimensional faces, we have that with high probability, there is no $i$-dimensional face of degree larger than $(1 + \epsilon)c^{i-1-i/d}$, and then, we take a further union bound over all $1 \leq i < d$. \hfill $\square$

It follows immediately from Lemma 5.1 that for any $\epsilon > 0$, with high probability, $X \sim X(n, c/\sqrt{n})$ has no strongly connected $d$-dimensional subcomplexes for which there exists $1 \leq i < d$ so that some $(i - 1)$-dimensional face of the subcomplex has degree larger than $(1 + \epsilon)c^{i-1-i/d}$. We will say that $Y$ a subcomplex of $X$ is $c$-bounded if the degree of every $(i - 1)$-dimensional face for $1 \leq i < d$ is at most $c^{i-1-i/d}$. We next count the number of $c$-bounded strongly connected $d$-complexes $Y$ with $Y^{(1)}$ having $r$ edges.

**Lemma 5.2.** For any $0 < c < 1$ and $d \geq 2$, the number of graphs $H$ on $[n]$ with $r$ edges so that $H$ is the graph of a $c$-bounded strongly connected $d$-dimensional complex is at most

$$2^{d+1}n^{(d+1)/2}(2^{1+2d}d\sqrt{n})^r.$$

The approach we take to prove this lemma is inspired by the proof of Theorem 4.1 of [3] where the Linial–Meshulam model is considered rather than the random flag complex model.
Proof. We consider a procedure to construct a minimal $d$-dimensional strongly connected complex with $r$ edges and at most $n$ vertices by adding $d$-faces one at a time. As the complex we end up with is strongly connected, we may construct it by starting with a single $d$-face and at each step adding a $d$-face that contains an already existing $(d-1)$-face that itself belongs to an existing $d$-face. We will always add our $d$-faces in this way, but this is not quite enough to get the bound we want because even if we add $d$-faces according to this rule, there will be in general many different orderings of $d$-faces that produce the same complex. Additionally, we know that we want $r$ edges at the end, but that does not tell us precisely how many steps we need since the number of steps is determined by the number of $d$-faces. To limit the amount of overcounting, we add a few more requirements to the procedure. First, we order the $(d-1)$-faces of the complete $(d-1)$-complex on $[n]$. Throughout the process, $(d-1)$-faces of the complex we have constructed so far will be marked as saturated or unsaturated. At each step, the ordering and saturated/unsaturated markings tell us a well-defined $(d-1)$-face $\sigma$ and we grow from $\sigma$ by adding some new $d$-face $\sigma \cup \{v\}$ to our complex.

If a $(d-1)$-face $\sigma$ is unsaturated that means that it is eligible to be selected later in the process when we will add $\sigma \cup \{v\}$ for some vertex $v$ so that some edge of $\sigma \cup \{v\}$ is missing right, we add $\sigma \cup \{v\}$ to $\sigma$. Saturated means that $\sigma$ is ineligible to grow from for the rest of the process, though new $d$-dimensional cofaces of $\sigma$ may still be added in moves that add $\sigma \cup \{v\}$ but grow from some other $(d-1)$ face or moves that grow from $\sigma$, but simply fill in an existing $d$-simplex boundary.

At the first step, we simply take an arbitrary $d$-face and mark each of its $(d-1)$-faces as unsaturated or saturated. At each subsequent step, we pick the minimum unsaturated $(d-1)$-face $\sigma$ according to the ordering on edges and add a new $d$-face $\sigma \cup \{y\}$ where $y$ is a vertex, unless there is an empty $d$-simplex boundary in the complex, which we will discuss later. Of course, we also add any missing edges of $\sigma \cup \{y\}$ as well. Other than the first move that adds one $d$-face and $(d+1)$ edges, this procedure partitions the remaining moves into $d+1$ types: moves that add no edges (i.e., moves that simply fill in an empty $d$-simplex whose $1$-skeleton is already present), moves that add a 1 edge, moves that add 2 edges, and so on, and moves that add $d$ edges. If we denote the number of these moves in a single instance of the procedure, respectively, as $m_0, m_1, m_2, \ldots, m_d$, we observe that the number of edges in the complex we end up with is $(d+1) + m_1 + 2m_2 + \cdots + dm_d$, and the number of $d$-faces is $1 + m_0 + m_1 + m_2 + \cdots + m_d = m$.

For a complex $K$ constructed by this procedure, we let $\Xi_i(K)$ for $i \in \{0, \ldots, d\}$ denote the number of ways to extend $K$ by adding a single $d$-face in a way that adds exactly $i$ edges and by $\Xi(K)$ the sum $\Xi_0(K) + \Xi_1(K) + \Xi_2(K) + \cdots + \Xi_d(X)$.

Except for moves where we add no new edges, after we have grown from the minimal $(d-1)$-face $\sigma$ to $\sigma \cup \{y\}$, we decide for each unsaturated edge of $\sigma \cup \{y\}$ whether or not to flip it to saturated. We additionally need one more rule for the procedure to avoid too large of an overcount and use the fact that we build flag complexes. We order the $\binom{n}{d+1}$ $d$-faces on $[n]$ and whenever we have a complex $K$ that has an empty $d$-simplex, we choose the smallest empty $d$-simplex in $K$ according to the ordering and fill it in growing from its minimal $(d-1)$-face, and we do not flip unsaturated $(d-1)$-faces to saturated. So, we can effectively skip over these moves.

Subject to these rules, we choose a sequence $(a_1, \ldots, a_{i-1})$ where each $a_i$ is 1, 2, \ldots, or $d$ with $m_k$ denoting the number of $i$ so that $a_i = k$ so that $(d+1) + m_1 + 2m_2 + \cdots + dm_d = r$. We will count the number of ways to run this procedure by steps $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_t$, where $K_1$ is an arbitrary $d$-face and $K_{i+1}$ is obtained from $K_i$ by a move that adds $a_i$ edges and afterward filling in all $d$-faces whose $1$-skeleton is present, and $t$ is such that $K_t$ will have $r$ edges.

For a given sequence $(a_1, \ldots, a_{i-1})$, we let $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_t$ be a run of our procedure from a single triangle $K_1$ to a complex with $r$ edges $K_r$ where we add $a_i$ edges to go from $K_i$ to $K_{i+1}$. We
wish to bound
\[
\prod_{i=1}^{t-1} \Xi_{a_1}(K_i).
\]

At each step, the \((d-1)\)-face we grow from is deterministic; it is the minimal unsaturated \((d-1)\)-face in \(K_i\). By the \(c\)-bounded condition if \(a_i = k < d\), then the number of ways to extend the selected \((d-1)\)-face at \(K_i\) is at most \(2^{d+1} \binom{d}{k} c^{d-k} n^{1-(d-k)/d} \leq 2^{d+1} 2^d c^{d-k} n^{k/d}\). Indeed, in order to add only \(k\) new edges, we have to pick \(v\) to be a common neighbor of some \(d-k\) vertices of \(\sigma\) and add \(\sigma \cup \{v\}\). If \(a_i = d\), then there are trivially at most \(n\) ways to extend \(K_i\). After adding the new \(d\)-face, we choose for each of its \((d-1)\)-faces whether to flip from unsaturated to saturated.

Thus, having selected \((a_1, \ldots, a_{m-1})\),
\[
\prod_{i=1}^{r-1} \Xi_{a_1}(K_i) \leq \left( \prod_{k=1}^{d-1} \left( 2^{d+1} 2^d c^{d-k} n^{k/d} \right) m_k \right) \left( 2^{d+1} n \right)^{m_d}
\]
\[
\leq (2^{1+2d})^{r} c^{d(m_1 + \cdots + m_{d-1}) - m_1 - 2m_2 - \cdots - (d-1)m_{d-1}} \left( \sqrt[d]{n} \right)^{m_1 + 2m_2 + \cdots + dm_d}.
\]

As \(m_1 + 2m_2 + \cdots + dm_d = r - \binom{d+1}{2}\), we have that
\[
c^{d(m_1 + \cdots + m_{d-1}) - m_1 - 2m_2 - \cdots - (d-1)m_{d-1}} \left( \sqrt[d]{n} \right)^{m_1 + 2m_2 + \cdots + dm_d} \leq c^{r(d+1)/2}.
\]

So,
\[
(2^{1+2d})^{r} c^{d(m_1 + \cdots + m_{d-1}) - m_1 - 2m_2 - \cdots - (d-1)m_{d-1}} \left( \sqrt[d]{n} \right)^{m_1 + 2m_2 + \cdots + dm_d} \leq (2^{1+2d})^{r} \left( \sqrt[d]{n} \right)^{r(d+1)/2}.
\]

Recall though that this is after choosing the sequence \((a_1, \ldots, a_{m-1})\) and the initial \(d\)-dimensional face and saturated/unsaturated markings on its \((d-1)\)-face, so the number of \(c\)-bounded strongly connected \(d\)-complexes on \(r\) edges with \([n]\) is at most
\[
(2n)^{d+1} d^{r} (2^{1+2d})^{r} \left( \sqrt[d]{n} \right)^{r(d+1)/2} = 2^{d+1} n^{(d+1)/2} (2^{1+2d} d \sqrt[d]{n})^r.
\]

Having proved Lemmas 5.1 and 5.2, we can prove Lemma 3.4.

\textbf{Proof of Lemma 3.4.} Take \(\varepsilon\) so that \(c + \varepsilon < \frac{1}{2^{1+2d}d}\). By Lemma 5.1, \(X \sim X(n, c / \sqrt[d]{n})\) does not contain a strongly connected \(d\)-dimensional subcomplex (of any size) that is not \((c + \varepsilon)\)-bounded. By Lemma 5.2 for any \(r\), the expected number of \((c + \varepsilon)\)-bounded strongly connected \(d\)-complexes in
Thus, by Markov’s inequality, the probability that there exists \( r \geq C \log n \) so that \( X \sim X(n, c/\sqrt{n}) \) has a \((c + \varepsilon)\)-bounded strongly connected subcomplex on \( r \) edges is at most

\[
2^{d+1} \sum_{r=C \log n}^{\infty} n^{(d+1)/2} (2^{1+2d} dc)^r \leq 2^{d+1} n^{(d+1)/2} (2^{1+2d} dc)^C \log n \left( \frac{1}{1 - 2^{1+2d} dc} \right).
\]

And this is \( o(1) \) for \( C \) large enough since \( 2^{1+2d} dc < 1 \).

With this, we have now completed the proof of Theorem 1.2.

### 6 Freeness of the Fundamental Group

A paper of Costa, Farber, and Horak [9] establishes coarse thresholds for the cohomological dimension of the fundamental group of a random clique complex. Namely, they show that if \( p = n^{-\alpha} \), then the cohomological dimension of \( \pi_1(X) \) for \( X \sim X(n, p) \) asymptotically almost surely is:

- 1 if \( \alpha > 1/2 \)
- 2 if \( 1/2 > \alpha > 11/30 \)
- \( \infty \) if \( 11/30 > \alpha > 1/3 \).

Combined with an earlier result of Kahle [14] that \( 1/3 > \alpha \) implies that \( X \) is simply connected with high probability, this gives the full behavior of the fundamental group of a random flag complex up to the right exponent for the phase transitions. Using Theorem 2.1, we can immediately improve on the lower bound for freeness of \( \pi_1(X) \). Using existing results in the literature and a new result similar to Lemma 3.5, we can also improve on the upper bound, proving Theorem 1.3.

The subcritical case of Theorem 1.3 follows immediately from Theorem 2.1, via the following lemma that is easy to prove.

**Lemma 6.1.** If \( X \) is an almost 2-collapsible flag complex, then \( \pi_1(X) \) is a free group.

**Proof.** Since \( X \) is almost 2-collapsible, \( X \) is homotopy equivalent to a complex of dimension at most 2 whose pure two-dimensional part consists of face disjoint copies of \( \diamondsuit_2 \). If we remove a triangle from each of these copies of \( \diamondsuit_2 \), we do not change the fundamental group, because the boundary of the triangle we remove is nullhomotopic. After removing a triangle from each copy of \( \diamondsuit_2 \), we may collapse the complex to a graph.

The supercritical case of Theorem 1.3 is much more involved, but the necessary lemmas are mostly already in place. To state what we need precisely, we say that a (not necessarily flag) com-
plex $X$ is *almost aspherical* if any complex $\bar{X}$ obtained from $X$ by removing one and only one triangle from each embedded copy of $\diamondsuit_2$ is aspherical, that is, $\pi_j(\bar{X}) = 0$ for all $j \geq 2$. This is similar to the notion of *asphericable* in [7] that establishes a result about the Linial–Meshulam model being aspherical for a particular range of $p$ once a triangle is removed from each tetrahedron boundary. We use the term “almost aspherical” rather than “asphericable” here only for the analogy to “almost $d$-collapsible.” Toward the supercritical part of Theorem 1.3, we prove the following.

**Theorem 6.2.** If $p < n^{-\alpha}$ for $\alpha > 12/25$, then $X \sim X(n, p)$ is almost aspherical with high probability. Moreover, the cohomological dimension of $\pi_1(X)$ is at most 2.

This theorem together with other results in the literature then implies the following theorem that is exactly the supercritical case of Theorem 1.3.

**Theorem 6.3.** For $c > \sqrt{3}$ and $\alpha > 1/3$, if $\frac{c}{\sqrt{n}} < p < n^{-\alpha}$, then $X \sim X(n, p)$ asymptotically almost surely has that $\pi_1(X)$ is not a free group.

For this proof and the proof of Theorem 1.4, we recall the following result of Kahle.

**Theorem 6.4** [15, Theorem 1.2]. Let $\varepsilon > 0$ be fixed and $X \sim X(n, p)$. If
\[
p \geq \left( \frac{(3/2 + \varepsilon) \log n}{n} \right)^{1/2},
\]
then with high probability, $\pi_1(X)$ has property (T), and if
\[
\frac{1 + \varepsilon}{n} \leq p \leq \left( \frac{(3/2 - \varepsilon) \log n}{n} \right)^{1/2},
\]
then with high probability, $\pi_1(X)$ does not have property (T).

**Proof of Theorem 6.3.** Set $\alpha' \in (12/25, 1/2)$. If $c/\sqrt{n} < p < n^{-\alpha'}$, we have that $X \sim X(n, p)$ is almost aspherical and its fundamental group has cohomological dimension at most 2 with high probability by Theorem 6.2. To show the cohomological dimension is at least 2, it suffices to prove that $X$ with a single triangle removed from each $\diamondsuit_2$ has homology in degree 2 with high probability. If this holds, then after removing a face from every copy of $\diamondsuit_2$, the resulting complex $\bar{X}$ has $\pi_1(\bar{X}) = \pi_1(X)$ with $\bar{X}$ a $K(\pi_1(X), 1)$. By uniqueness of the homotopy type of a $K(G, 1)$, if $\pi_1(X)$ is free, then $\bar{X}$ is homotopy equivalent to a bouquet of circles, but this cannot be the case if $\bar{X}$ has homology in dimension 2.

By the first moment method with high probability $X$ contains at most $O(n^{6/25} \log n)$ copies of $\diamondsuit_2$. On the other hand, the expected number of triangles in $X \sim X(n, p)$ is
\[
\mathbb{E}(f_2(X)) = \binom{n}{3} p^3,
\]
the expected number of edges in $X \sim X(n, p)$ is

$$
\mathbb{E}(f_1(X)) = \binom{n}{2} p,
$$

and the expected number of tetrahedra in $X \sim X(n, p)$ is

$$
\mathbb{E}(f_3(X)) = \binom{n}{4} p^6.
$$

Now

$$
\beta_2(X) \geq f_2(X) - f_1(X) - f_3(X).
$$

Moreover, $f_i(X)$ simply counts $(i + 1)$-cliques in $G \sim G(n, p)$, and the number of $(i + 1)$-cliques in $G(n, p)$ is known to be well concentrated around its mean when $p$ is such that the mean tends to infinity. Therefore, with high probability

$$
\beta_2(X) \geq (1 - o(1)) \left( \frac{c^3}{6} - \frac{c}{2} \right) n^{3/2} \geq \delta n^{3/2},
$$

for some constant $\delta > 0$ since $c^2 > 3$. Thus, even after deleting a face from each copy of $\bigtriangleup_d$, the remaining complex still has lots of homology in degree 2. Thus, the cohomological dimension of $\pi_1(X)$ is at least 2. An argument of this type showing that $c > \sqrt{d + 1}$ implies that $X \sim X(n, c/\sqrt{n})$ has homology in degree $d$ also appears in [15]. This handles the case that $p < n^{-\alpha'}$.

On the other hand, if $n^{-\alpha'} < p < n^{-\alpha}$, we have $p > (\frac{2\log n}{n})^{1/2}$ that implies by Theorem 6.4 that $\pi_1(X)$ has property (T). A nontrivial group with property (T) cannot be a free group because property (T) implies finite abelianization (see, e.g., Corollary 1.3.6 of [4]). Moreover, since $p < n^{-\alpha}$, we know by the result of [9] discussed above that $\pi_1(X)$ is nontrivial.

**Remark 6.5.** The lower bound of $\sqrt{3}$ in the statement of Theorem 6.3, and therefore, in the statement of Theorem 1.4, can be improved to around 1.66 using a result of Kanazawa [17]. Kanazawa shows that the second Betti number of $X(n, c/\sqrt{n})$ is of order $n^{3/2}$ for $c$ larger than an explicit constant that is around 1.66. We will say more about this in the concluding remarks, but in the interest of keeping everything as self-contained as possible, we keep the proof here to give the simpler $\Omega(n^{3/2})$ lower bound on $\beta_2$ at a cost of the constant being a bit larger.

**Proof of Theorem 1.4.** Theorem 6.3 and Theorem 6.4 immediately imply Theorem 1.4.

It remains to prove Theorem 6.2. Key to the proof is Theorem 6.1 of [9].

**Lemma 6.6** [9, Theorem 6.1]. Assume that $p = o(n^{-\alpha})$ for $\alpha > 1/3$ fixed. Then, for $X \sim X(n, p)$ and $L = L(\alpha)$ a large enough constant, the 2-skeleton of $X^{(2)}$ has the following property with high probability: Any subcomplex $Y$ of $X^{(2)}$ is aspherical if and only if every subcomplex $Z \subseteq Y$ having at most $L$ edges is aspherical.
The proof of Theorem 6.2 requires showing that \( X \sim X(n, p) \) satisfies several more properties in addition to the necessary and sufficient condition property of Lemma 6.6.

**Lemma 6.7.** For \( X \sim X(n, p) \) with \( p < n^{-\alpha}, \alpha > 12/25 \) with high probability \( X \) satisfies all of the following properties.

(i) \( \text{dim}(X) \leq 4 \).
(ii) Every embedded copy of \( \triangledown_2 \) in \( X \) has all of its triangles maximal. In particular, every embedded copy of \( \triangledown_2 \) is induced.
(iii) \( X \) does not contain a tetrahedron and a 4-simplex that meet at a triangle.
(iv) \( X \) is 3-collapsible.
(v) There exists some constant \( L \) so that any subcomplex \( Y \) of \( X(2) \) is aspherical if and only if every subcomplex \( Z \subseteq Y \) having at most \( L \) vertices is aspherical.
(vi) Every for every subcomplex \( Z \) on at most \( \ell \) vertices for any constant \( \ell \), \( \rho(Z(1)) < 25/12 \).

In establishing that \( 11/30 < \alpha < 1/2 \) \( X \sim X(n, n^{-\alpha}) \) has fundamental group of cohomological dimension 2 in \([9]\), Costa, Farber, and Horak show that there is a subcomplex \( Y \subseteq X(2) \) so that every bounded subcomplex of \( Y \) collapses to a graph and so that \( \pi_1(Y) = \pi_1(X) \). Our approach will be to show that for \( \alpha > 12/25, p < n^{-\alpha} \), the complex \( Y \subseteq X(2) \) obtained from \( X \sim X(n, p) \) by collapsing away everything of dimension larger than 2 from \( X \) has \( \pi_1(Y) = \pi_1(X) \) and every finite subcomplex of \( Y \) is aspherical. While it may be possible to show that every bounded subcomplex of this complex \( Y \) collapses to a graph by adapting the approach in \([9]\), since we only need that every bounded subcomplex is aspherical, we take a different approach based on simple homotopy equivalence.

Simple homotopy equivalence was first described by Whitehead, and is based on elementary collapses and expansions. We have already defined an elementary collapse, and an elementary expansion is simply the reverse of an elementary collapse. That is, if \( X \) is a simplicial complex and \( \sigma \) is a simplex on a subset of the vertices of \( X \) so that exactly one facet \( \tau \) of \( \delta \sigma \) does not belong to \( X \), then the elementary expansion of \( X \) at \( \sigma \) is adding \( \tau \) and \( \sigma \) to \( X \). Observe that an elementary expansion is a homotopy equivalence. Two simplicial complexes are *simple homotopy equivalent*, provided that there is a sequence of collapses and expansions that transform one into the other.

To prove Theorem 6.2, we show that if \( X \) is a flag complex that satisfies the conditions of Lemma 6.7, we can remove a triangle from each copy of \( \triangledown_2 \) in \( X \) and then collapse to a 2-complex \( Y \) with the property that any \( Z \subseteq Y \) on at most \( L \) vertices is simple homotopy equivalent to a graph, and is therefore aspherical. In order to prove this, we first establish the following deterministic result that is similar to Lemma 3.5.

**Lemma 6.8.** If \( Z \) is a simplicial complex with the property that the only triangulated 2-spheres contained in \( Z(2) \) are the boundaries of tetrahedra belonging to \( Z \) and if the essential density of the graph of \( Z \) satisfies

\[
\rho(Z(1)) < 25/12,
\]

then \( Z \) is simple homotopy equivalent to a graph.

For brevity, we say that a simplicial complex whose only embedded 2-spheres are boundaries of included tetrahedra is essentially 2-sphere free. The fact that \( Z \) is essentially 2-sphere free will imply
that, even though $Z$ may not be a flag complex, there is a sequence of expansions that transform the link of any vertex $v$ to a flag complex while maintaining the essentially 2-sphere free property and density property on $Z \setminus \{v\}$. So, if $Z$ has a vertex $v$ with $\operatorname{lk}_Z(v)$ 1-collapsible, then $Z$ cannot be a minimal counterexample. From there, we can basically follow the argument of the proof of Lemma 3.5.

**Proof of Lemma 6.8.** Suppose that $Z$ is essentially 2-sphere free with the density condition. Two immediate observations for the density condition are that $Z$ does not contain the 1-skeleton of a 2-sphere on more than six vertices and that $\dim(Z) \leq 4$. By Euler characteristic, a triangulated 2-sphere on $v$ vertices has density $3 - 6/v$, and $3 - 6v \geq 25/12$ for $v \geq 7$. The dimension of $Z$ is at most 4 because $Z^{(1)}$ cannot contain a 6-clique as a 6-clique has density $15/6 > 25/12$.

Let $v$ be a vertex of $Z$, we claim that there exists a sequence of expansions each adding a pair of faces $(\sigma, \sigma \cup \{v\})$, so that the resulting complex $Z'$ has $\operatorname{lk}_{Z'}(v)$ the flag closure $\operatorname{lk}_Z(v)$ and $Z' \setminus \{v\}$ still satisfies the essentially 2-sphere free and density assumptions of $Z$.

Since $\dim(Z) \leq 4$, $\dim(\operatorname{lk}_Z(v)) \leq 3$, thus if we can show that there is a sequence of expansions first filling in all the triangles of $\operatorname{lk}_Z(v)$ and then from there filling in all empty tetrahedra, that will be sufficient for $Z'$ so that $\operatorname{lk}_{Z'}(v) = \operatorname{lk}_Z(v)$, and all that will remain is to check that $Z' \setminus \{v\}$ still satisfies the assumptions from the statement. Suppose that $\partial \sigma$ is an empty triangle in $\operatorname{lk}_Z(v)$. If $\sigma \notin Z$, then $(\sigma, \sigma \cup \{v\})$ is a valid expansion filling in $\partial \sigma$ in the link. However, if $\partial \sigma$ is an empty triangle in $\operatorname{lk}_Z(v)$ and $\sigma \in Z$, then we have the boundary of $\sigma \cup \{v\}$ in $Z$ without $\sigma \cup \{v\}$ in $Z$. But this is a triangulated 2-sphere that is not the boundary of a tetrahedron in $Z$, so this breaks the essentially 2-sphere free condition. Thus, $\sigma \notin Z$. Therefore, it is possible to fill in the triangles of the link of $v$.

Once the triangles are filled in, we move on to tetrahedra. Empty tetrahedera in the link of $v$ after expanding to fill in all the triangles split into two types. On the one hand, we have empty tetrahedra that contain at least one triangle added by an expansion, and on the other hand, we have empty tetrahedra initially present in $\operatorname{lk}_Z(v)$. For the former case, suppose that $\partial \tau$ is an empty tetrahedron in the link of $v$ with $\sigma$ a triangle of $\tau$ added by an expansion. Since $\sigma$ was added by an expansion, the only tetrahedron that contains $\sigma$ is $\sigma \cup \{v\}$, so, in particular, $\tau$ is not in the complex, so $(\tau, \tau \cup \{v\})$ is an allowed expansion that fills in $\partial \tau$ in the link. If $\partial \tau$ was already in $\operatorname{lk}_Z(v)$ as an empty tetrahedron, then $Z$ being essentially 2-sphere free implies that $\tau$ is not in $Z$. Indeed, if $\tau$ belongs to $Z$ and $\partial \tau \in \operatorname{lk}_Z(v)$, then $\partial(\tau \cup \{v\})$ belongs to $Z$, but the boundary of a 4-simplex contains a 2-sphere on five vertices in its 2-skeleton. It follows that $\tau \notin Z$ and so $(\tau, \tau \cup \{v\})$ is a permitted expansion. Thus, it is possible to expand $Z$ to $Z'$ where $\operatorname{lk}_{Z'}(v) = \operatorname{lk}_Z(v)$.

We claim next that $Z' \setminus \{v\}$ is essentially 2-sphere free. As the expansion moves from $Z$ to $Z'$ never added any edges, the graph of $Z'$ is a subgraph of the graph of $Z$. Thus, $Z'$ cannot contain a sphere on more than six vertices. If $S \subseteq Z' \setminus \{v\}$ is a triangulated 2-sphere on at most six vertices, then at least one triangle $\sigma$ of $S$ belongs to $\operatorname{lk}_{Z'}(v)$. In this case, then $\partial \sigma \in \operatorname{lk}_Z(v)$, but then replacing $\sigma$ in $S$ by the cone over its boundary with cone point, $v$ finds the 1-skeleton of a sphere on seven vertices in $Z$, but we have already ruled that out.

Next, suppose that $S \subseteq Z' \setminus \{v\}$ is a sphere on five vertices. Then some nonempty subset $\{\sigma_1, \ldots, \sigma_t\}$ of the triangles of $S$ belongs to $\operatorname{lk}_{Z'}(v)$. Thus, all the edges of the complex $Y$ generated by $\sigma_1, \ldots, \sigma_t$ belong to $\operatorname{lk}_Z(v)$. If $Y$ contains a vertex of degree at least 4, then $Z$ contains the graph of a 2-sphere $S$ on five vertices with an additional vertex adjacent to all vertices of $S$. Such a graph has 14 edges and 6 vertices and $14/6 > 25/12$ breaking the density condition. Thus, $Y$ must be a subcomplex of $S$ in which all vertices have degree at most 3. But then, $Y$ is either a single
triangle, two triangles meeting at an edge, or a tetrahedron with a face removed. In any of these cases, \( Y^{(1)} \) belongs to \( \text{lk}_Z(u) \) and \( Y \) is a disk. Thus, by replacing \( Y \) with the cone over its boundary with cone point \( u \), we find a copy of a sphere on five or six vertices in \( Z \). But \( Z \) is essentially 2-sphere free.

Lastly, suppose that \( Z' \) contains an empty tetrahedron \( S \). Again, at least one triangle of \( S \) belongs to \( \text{lk}_{Z'}(u) \). Let \( Y \) be the subcomplex of \( S \) generated by the triangles of \( S \) in \( \text{lk}_{Z'}(u) \). If \( Y \) is on at most two triangles, the \( Y \) is a disk with no internal vertices and the boundary of that disk belongs to \( \text{lk}_Z(u) \), so we can find a sphere on five vertices in \( Z \). On the other hand, if \( Y \) is on at least three triangles, then all edges of \( S \) belong to \( \text{lk}_{Z'}(u) \) and therefore to \( \text{lk}_Z(u) \). But after expansion \( \text{lk}_{Z'}(u) \) is a flag complex. Therefore, \( S \) bounds an included tetrahedron of \( Z' \).

Now suppose that \( Z \) is a minimal counterexample to our lemma. If \( Z \) has a vertex \( u \) so that \( \text{lk}_Z(u) \) is 1-collapsible, then we may expand \( Z \) to take the flag closure of the link and obtain \( Z' \). Now \( Z' \) is simple homotopy equivalent to \( Z \) and \( \text{lk}_{Z'}(u) \) is 1-collapsible. Thus, we may 2-collapse \( Z' \) around \( u \) and obtain a new complex \( Z'' \) simple homotopy equivalent to \( Z' \) where \( u \) does not belong to any triangles of \( Z'' \). Thus, \( Z'' \) is \( Z' \setminus \{v\} \) together with \( v \) and edges from \( v \) to \( Z' \setminus \{v\} \).

Moreover, \( Z' \setminus \{v\} \) is essentially 2-sphere free and satisfies the essential density bound. Thus, by minimality of \( Z \), there is a sequence of collapses and expansions in \( Z' \setminus \{v\} \) that transform it to a graph. But then \( Z'' \) is simple homotopy equivalent to a graph, so \( Z \) is too. It follows that \( Z \) does not have a vertex \( u \) so that \( \text{lk}_Z(u) \) is 1-collapsible.

We can now conclude that the minimum degree of \( Z^{(1)} \) is 4. Moreover, if \( Z \) has a vertex of degree 4 whose neighbors all have degree 4 too, then \( Z \) either contains a free edge, so we can perform an elementary collapse and then have a smaller counterexample, or else \( Z \) contains a copy of \( \triangle_2 \) as in the proof of Claim 4.5 violating the essentially 2-sphere free condition. Thus, each vertex of \( Z \) of degree 4 has a neighbor of degree larger than 4, and just as in the proof of Lemma 3.5, we have that \( Z \) has density at least 25/12.

We are now ready to prove Theorem 6.2.

**Proof of Theorem 6.2.** For \( p < n^{-\alpha} \) with \( \alpha > 12/25 \), with high probability, \( X \) satisfies all of the conditions of Lemma 6.7 with \( \ell = L \). Let \( Y \) be obtained from \( X \) by deleting a triangle from each copy of \( \triangle_2 \) and 3-collapsing. Clearly, \( Y \) is two-dimensional and \( \pi_1(Y) = \pi_1(X) \), we prove that \( Y \) is aspherical. We first claim that \( Y^{(2)} \) cannot contain a 2-sphere on at most \( 2L \) vertices.

The 1-skeleton of a triangulated sphere on \( v \geq 4 \) vertices has density \( 3 - 6/v \). By condition (vi), then \( X \) cannot contain a triangulated sphere on more than six vertices and at most \( 2L \) vertices. Thus, \( Y \) cannot contain triangulated sphere of \( v \) vertices for \( 7 \leq v \leq 2L \), so it remains to check spheres on at most six vertices. The only triangulated spheres on at most six vertices are \( \triangle_2 \) and stacked polytopes, that is, spheres that can be obtained from \( \emptyset \Delta_3 \) by successively replacing a triangle by the cone over its boundary (commonly called a bistellar flip). We just have four cases then for triangulated spheres on at most six vertices: the cross-polytope or zero, one, or two successive bistellar flips from the tetrahedron. By construction, \( Y \) does not contain any copies of \( \triangle_2 \).

If \( Y \) contains \( \emptyset \Delta_3 \), then this means that the interior was removed by collapsing it into a four-dimensional simplex. Let \( \sigma \) be such a tetrahedron with \( \sigma \subseteq \tau \) so that \( (\sigma, \tau) \) was an elementary collapse. By condition (iii), however, we have that \( X \) does not contain two 4-simplices that meet at a tetrahedron because such a structure would include a 4-simplex and a tetrahedron that meet at a triangle. Thus, after collapsing away all the 4-simplices every tetrahedron of \( \tau \) other than \( \sigma \) still remains. So, we have a triangulated 3-sphere with a single tetrahedron removed. Now when
we collapse away triangle–tetrahedron pairs, the only way to get started on collapsing $\tau \setminus \sigma$ is to collapse away one of the triangles of $\sigma$, but then $\partial \sigma$ does not belong to $Y$.

If $Y$ contains a triangulated sphere on five vertices, then $X$ has two tetrahedra $\sigma_1, \sigma_2$ meeting at a triangle so that after collapse the boundary of the ball with facets $\sigma_1$ and $\sigma_2$ remains. Thus, one of $\sigma_1$ or $\sigma_2$ must be collapsed away by a tetrahedron–4-simplex collapse. If not then the only way to get started on collapsing $\sigma_1$ and $\sigma_2$ is to collapse at one of the triangles on the boundary that contradicts the boundary remaining in $Y$. By (iii) though, in this case, $\sigma_1 \cup \sigma_2$ must form the vertices of a 4-simplex. Without loss of generality, then $\sigma_1$ is collapsed with this 4-simplex and the sequence of triangle–tetrahedron collapses restricted to the remaining punctured 3-sphere leaves behind a 2-complex that contains a triangulated sphere. But this is a homotopy equivalence between a contractible space and a 2-complex with homology in degree 2, and that is impossible.

Lastly, suppose that $Y$ contains the boundary of three tetrahedra $\sigma_1, \sigma_2, \sigma_3$ meeting at a triangle and $\sigma_2$ and $\sigma_3$ meet at a triangle but $\sigma_1$ and $\sigma_3$ do not meet at a triangle. By (iii), then none of the $\sigma_i$ belong to a 4-simplex, so they are all collapsed away by triangle–tetrahedron elementary collapses, and to get started, we have to use a triangle on the boundary. So, having exhausted all cases, we have proved that $Y$ does not contain a sphere on at most $2L$ vertices.

To show that $Y$ is aspherical, it suffices by (v) to prove that for any $Z \subseteq Y$ on at most $L$ vertices, $Z$ is aspherical. Suppose that $Z$ is such a complex. By (vi), then the graph of $Z$ has essential density less than $25/12$. Additionally, since $Y$ does not contain a triangulated 2-sphere, $Z$ is essentially 2-sphere free, so by Lemma 6.8, $Z$ is simple homotopy equivalent to a graph, and hence, $Z$ is aspherical, and this completes the proof.

Finally we prove Lemma 6.7.

**Proof of Lemma 6.7.** We observe that (iii) implies (i). Let $G_1$ be the graph of a 5-clique together with a vertex $v$ adjacent exactly to a triangle of the 5-clique. Then $G_1$ has 6 vertices and 13 edges, so the expected number of copies of $G_1$ in $G \sim G(n, p)$ is

$$O(n^{6-13(12/25)}) = o(1).$$

So, by Markov’s inequality, (iii) and therefore (i) occur with high probability.

For (ii), let $G_2$ be the graph obtained from the graph of $\diamondsuit_2$ by adding a new vertex $v$ adjacent exactly to a triangle of $\diamondsuit_2$. Then, $G_2$ has 7 vertices and 15 edges, so the expected number of such complexes is $O(n^{7-15\alpha})$ that is $o(1)$ since $\alpha > 12/25 > 7/15$. So, no copy of $\diamondsuit_2$ has a triangle intersecting a tetrahedron with a vertex outside of that copy of $\diamondsuit_2$. The only other way a copy of $\diamondsuit_2$ can contain a nonmaximal triangle is if it is not induced. The expected number of noninduced copies of $\diamondsuit_2$ is $O(n^{6-13\alpha}) = o(1)$ since $\alpha > 12/25 > 6/13$.

Condition (iv) is by the main result of [19] that $\alpha > 1/d$ implies that $X \sim X(n, n^{-\alpha})$ is $d$-collapsible. It is also implied by the $d = 3$ case of Theorem 2.1.

Condition (v) is simply a restatement of Lemma 6.6 of [9].

Condition (vi) is by a standard first moment argument. A graph $H$ with $k$ vertices density at least $25/12$ occurs in $G(n, p)$ with probability $O((n^{1-\alpha(25/12)})^k)$ and as $1 - \alpha(25/12) < 0$, we can sum over all the finitely many graphs on at most $\ell$ vertices with density at least $25/12$ to get a $o(1)$ bound on the expected number of included subgraphs.
7 | CONCLUDING REMARKS

The natural remaining questions are what are the sharp thresholds, should they exist, for:

- \( X \sim X(n, p) \) to go from almost \( d \)-collapsible to not almost \( d \)-collapsible,
- \( X \sim X(n, p) \) to go from having all homology in degree \( d \) generated by copies of \( \diamond_d \) to having homology not generated by copies of \( \diamond_d \), and
- \( X \sim X(n, p) \) to go from \( \pi_1(X) \) free to \( \pi_1(X) \) nonfree.

For the Linial–Meshulam model \( Y_d(n, p) \), the “trivial \( d \)-cycles” are boundaries of a \((d + 1)\)-simplex. These are Poisson distributed and [1, 3] establish the constant \( \gamma_d \) so that \( \frac{Y_d}{n} \) is the sharp threshold for \( Y \sim Y_d(n, p) \) to go from almost \( d \)-collapsible (replacing \( \diamond_d \) in the definition here by \( \partial\Delta_{d+1} \)) to not almost \( d \)-collapsible. The threshold in \( Y \sim Y_d(n, p) \) to go from all homology generated by copies of \( \partial\Delta_{d+1} \) to some homology coming from other subcomplexes is at \( c_d / n \) for an explicit constant \( c_d \) is established in [2, 18]. The proofs rely on approximating the local behavior of \( Y_d(n, p) \) by a Galton–Watson process and counting \( d \)-dimensional cores.

Results in the direction of approximating the local behavior were recently extended to the clique complex setting by Kanazawa [17] who showed that that there is indeed a phase transition in the Betti numbers with real coefficients at \( p = (c / n)^{1/d} \) if \( c > c_d \), then the \( d \)th Betti number is of order \( \Theta(n^{(d+1)/2}) \) and even establishes the right coefficient in terms of \( c \) and if \( c < c_d \), then the \( d \)th Betti number is \( o(n^{(d+1)/2}) \). Though it is still an open problem to show that below this phase transition, any homology in dimension \( d \) is generated by copies of \( \diamond_d \); we leave this as a conjecture here.

**Conjecture 7.1.** For \( X \sim X(n, c / \sqrt{n}) \) if

- \( 0 < c < \sqrt{\gamma_d} \) then with high probability, \( X \) is almost \( d \)-collapsible.
- \( \sqrt{\gamma_d} < c < \sqrt{c_d} \) then with high probability, \( X \) is not almost \( d \)-collapsible, but \( \beta_d(X; \mathbb{R}) \) is generated by embedded copies of \( \diamond_d \).

The constants \( c_d \) and \( \gamma_d \) are described fully in [18], where the following table of the approximation of the first few values also appears:

| \( d \) | 2   | 3   | 4   | 5   |
|-------|-----|-----|-----|-----|
| \( \gamma_d \)  | 2.455 | 3.089 | 3.509 | 3.822 |
| \( c_d \)      | 2.754 | 3.907 | 4.962 | 5.984 |

Asymptotically, \( \gamma_d \) is order \( \log(d) \) and \( c_d \) is very slightly smaller than \( d + 1 \).

Regarding the fundamental group, [20] proves the following for \( Y \sim Y_2(n, p) \). These are still the best known bounds on a sharp threshold for \( \pi_1(Y_2(n, p)) \) to be free.

**Theorem 7.2** [20]. If \( c < \gamma_2 \) and \( Y \sim Y_2(n, c / n) \), then with high probability, \( \pi_1(Y) \) is free, while if \( c > \gamma_2 \) and \( Y \sim Y_2(n, c / n) \), then with high probability, \( \pi_1(Y) \) is not free.
Naturally, Conjecture 7.1 together with Theorem 6.2 would imply that $\pi_1(X(n, p))$ is free if $p < \frac{1.567}{\sqrt{n}}$. And Theorem 6.2 together with Kanazawa’s result implies that $\pi_1(X(n, p))$ is not free if $p > \frac{1.660}{\sqrt{n}}$. Between these two bounds, there does not seem to be good evidence either way for what the right conjecture should be for the true threshold for freeness of the fundamental group. It could even be that there is no sharp threshold for $\pi_1(X)$ to go from free to nonfree.

Finally, the sharp threshold for emergence of $d$th homology in $Y_d(n, p)$ is apparently closely tied to the torsion burst in the random model. The torsion burst is a phenomenon observed experimentally in [16], but as of now no proof exists to explain it. In the experiments, right before the first cycle in $d$th homology, other than a copy of the $(d + 1)$-simplex boundary appears, a huge torsion group appears in the $(d − 1)$st homology group. This torsion group vanishes soon after. A few experiments were conducted in [16] to search for torsion in $X(n, p)$, but no torsion was found. However, based on what we have proved here it seems plausible at least that the reason none was observed in experiments was because the experiments were on too few vertices. Perhaps, when $n$ is large enough, one would observe a torsion burst in $X(n, p)$.

As $Y_d(n, p)$ and $X(n, p)$ are special cases of the multiparameter model first defined in [8], it would be reasonable to examine one-sided sharp (or two-sided sharp) thresholds for $d$-collapsibility and nonvanishing of $d$th homology in the multiparameter model.

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**REFERENCES**
1. L. Aronshtam and N. Linial, *The threshold for $d$-collapsibility in random complexes*, Random Structures Algorithms 48 (2016), no. 2, 260–269.
2. L. Aronshtam and N. Linial, *When does the top homology of a random simplicial complex vanish?*, Random Structures Algorithms 46 (2015), no. 1, 26–35.
3. L. Aronshtam, N. Linial, T. Łuczak, and R. Meshulam, *Collapsibility and vanishing of top homology in random simplicial complexes*, Discrete Comput. Geom. 49 (2013), no. 2, 317–334.
4. B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan’s property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.
5. B. Benedetti and F. H. Lutz, *The dunce hat in a minimal non-extendably collapsible 3-ball*, Electronic Geometry Model No. 2013.10.001, 2013.
6. B. Bollobás, *Threshold functions for small subgraphs*, Math. Proc. Cambridge Philos. Soc. 90 (1981), no. 2, 197–206.
7. A. E. Costa and M. Farber, *The asphericity of random 2-dimensional complexes*, Random Structures Algorithms 46 (2015), no. 2, 261–273.
8. A. Costa and M. Farber, *Random simplicial complexes*, Configuration spaces, Springer INdAM Ser., vol. 14, Springer, Cham, 2016, pp. 129–153.
9. A. Costa, M. Farber, and D. Horak, *Fundamental groups of clique complexes of random graphs*, Trans. London Math. Soc. 2 (2015), no. 1, 1–32.
10. A. Dochtermann and A. Newman, *Random subcomplexes and Betti numbers of random edge ideals*, arXiv: 2104.12882. To appear in IMRN.
11. P. Erdős and A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 5 (1960), 17–61.
12. D. Erman and J. Yang, *Random flag complexes and asymptotic syzygies*, Algebra Number Theory 12 (2018), no. 9, 2151–2166.
13. A. Frieze and M. Karoński, *Introduction to random graphs*, Cambridge University Press, Cambridge, 2016.
14. M. Kahle, *Topology of random clique complexes*, Discrete Math. 309 (2009), no. 6, 1658–1671.
15. M. Kahle, *Sharp vanishing thresholds for cohomology of random flag complexes*, Ann. of Math. (2) 179 (2014), no. 3, 1085–1107.
16. M. Kahle, F. H. Lutz, A. Newman, and K. Parsons, *Cohen-Lenstra heuristics for torsion in homology of random complexes*, Exp. Math. 29 (2020), no. 3, 347–359.
17. S. Kanazawa, *Law of large numbers for Betti numbers of homogeneous and spatially independent random simplicial complexes*, Random Structures Algorithms 60 (2022), no. 1, 68–105.
18. N. Linial and Y. Peled, *On the phase transition in random simplicial complexes*, Ann. of Math. (2) 184 (2016), no. 3, 745–773.
19. G. Malen, *Collapsibility of random clique complexes*, arXiv: 1903.05055.
20. A. Newman, *Freeness of the random fundamental group*, J. Topol. Anal. 12 (2020), no. 1, 29–35.