CONVOLUTION SUMS OF SOME FUNCTIONS ON DIVISORS

HEEKYOUNG HAHN

Abstract. One of the main goals in this paper is to establish convolution sums of functions for the divisor sums 
\[ \tilde{\sigma}_s(n) = \sum_{d \mid n} (-1)^{d-1} d^s \]
and 
\[ \hat{\sigma}_s(n) = \sum_{d \mid n} (-1)^{d-1} n d^s, \]
for certain \( s \), which were first defined by Glaisher. We first introduce three functions 
\( P(q) \), \( E(q) \), and \( Q(q) \) related to \( \tilde{\sigma}_s(n) \), \( \hat{\sigma}_s(n) \), and \( \tilde{\sigma}_3(n) \), respectively, and then we evaluate them in terms of two parameters \( x \) and \( z \) in Ramanujan’s theory of elliptic functions. Using these formulas, we derive some identities from which we can deduce convolution sum identities. We discuss some formulae for determining \( r_s(n) \) and \( \delta_s(n) \), \( s = 4, 8 \), in terms of \( \tilde{\sigma}(n) \), \( \hat{\sigma}(n) \), and \( \tilde{\sigma}_3(n) \), where \( r_s(n) \) denotes the number of representations of \( n \) as a sum of \( s \) squares and \( \delta_s(n) \) denotes the number of representations of \( n \) as a sum of \( s \) triangular numbers. Finally, we find some partition congruences by using the notion of colored partitions.

1. Introduction

In his famous paper [21, pp. 136–162], Ramanujan introduced the three Eisenstein series \( P(q) \), \( Q(q) \) and \( R(q) \) defined for \( |q| < 1 \) by

\[ P(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \]
\[ Q(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \]
\[ R(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \]

where for \( s, n \in \mathbb{N} \),

\[ \sigma_s(n) = \sum_{d \mid n} d^s. \]

As usual, we set \( \sigma_1(n) = \sigma(n) \) and \( \sigma_s(n) = 0 \) if \( n \notin \mathbb{N} \). Ramanujan also proved that (1.1)–(1.3) satisfy the differential equations [21 (30)], [22, p. 142]

\[ q \frac{dP(q)}{dq} = \frac{P^2(q) - Q(q)}{12}, \]
\[ q \frac{dQ(q)}{dq} = \frac{P(q)Q(q) - R(q)}{3}, \]
\[ q \frac{dR(q)}{dq} = \frac{P(q)R(q) - Q^2(q)}{2}. \]

2000 Mathematics Subject Classification. Primary: 11A67; Secondary: 33E05.
After rewriting (1.4) as

\[ P^2(q) = Q(q) + 12q \frac{dP(q)}{dq}, \]

and equating the coefficients of \( q^n \) on both sides, we obtain the arithmetic identity

\[ 12 \sum_{m<n} \sigma(m)\sigma(n-m) = 5\sigma_3(n) - (6n-1)\sigma(n). \tag{1.8} \]

Likewise, from (1.5), we obtain

\[ \sum_{m<n} \sigma(m)\sigma_3(n-m) = 21\sigma_5(n) - (30n-10)\sigma_3(n) - \sigma(n). \tag{1.9} \]

Ramanujan recorded nine identities of the type (1.8) and (1.9) in his notebooks. The history of the convolution sums involving the divisor function \( \sigma_s(n) \) goes back to Glaisher [8], [9], [10]. A most comprehensive treatment of these identities is given in the paper [12]. In their paper [12], Huard, Ou, Spearman and Williams prove many such formulae in an elementary manner by using their generalization of Liouville’s classical formula given in [17]. Recently, Cheng and Williams [5] found further convolution sums of the type

\[ \sum_{m<n} \sigma(4m-3)\sigma(4n-(4m-3)) = 4\sigma_3(n) - 4\sigma_3(n/2). \]

Now define two functions on which we focus in this paper by, for \( s, n \in \mathbb{N} \),

\[ \bar{\sigma}_s(n) = \sum_{d|n} (-1)^{d-1} d^s, \]

\[ \hat{\sigma}_s(n) = \sum_{d|n} (-1)^{\frac{n}{d}-1} d^s, \]

where we set \( \bar{\sigma}_1(n) = \bar{\sigma}(n) \), \( \hat{\sigma}_1(n) = \hat{\sigma}(n) \) and \( \bar{\sigma}_s(n) = \hat{\sigma}_s(n) = 0 \) if \( n \notin \mathbb{N} \). The origin of these functions goes back to Glaisher. In his paper [9], Glaisher defined seven quantities which depend on the divisors of \( n \), including (1.10) and (1.11), and studied the relations among them. He also found expressions for all seven functions in terms of \( \sigma_s(n) \). For instance, the functions \( \bar{\sigma}_s(n) \) and \( \hat{\sigma}_s(n) \) have the formulae [9]

\[ \bar{\sigma}_s(n) = \sigma_s(n) - 2^{s+1} \sigma_s(n/2), \]

\[ \hat{\sigma}_s(n) = \sigma_s(n) - 2\sigma_s(n/2). \]

From the relations (1.12) and (1.13), it is clear that, for all \( n \geq 0 \),

\[ \bar{\sigma}_s(2n+1) = \sigma_s(2n+1) = \hat{\sigma}_s(2n+1). \tag{1.14} \]

One of our goals in the present paper is to establish convolution sums involving \( \bar{\sigma}_s \) and \( \hat{\sigma}_s \) for certain \( s \). So we need to define three functions related to (1.10) and (1.11)
by, for $|q| < 1$,

$$P(q) := 1 + 8 \sum_{n=1}^{\infty} \tilde{\sigma}(n) q^n,$$

(1.15)

$$E(q) := 1 + 24 \sum_{n=1}^{\infty} \hat{\sigma}(n) q^n,$$

(1.16)

$$Q(q) := 1 - 16 \sum_{n=1}^{\infty} \tilde{\sigma}_3(n) q^n.$$

(1.17)

Analogous to (1.4)–(1.6), our three functions (1.15)–(1.17) satisfy the differential equations [11], [19], [20]

$$q \frac{dP(q)}{dq} = \frac{P^2(q) - Q(q)}{4},$$

(1.18)

$$q \frac{dE(q)}{dq} = \frac{E(q)P(q) - Q(q)}{2},$$

(1.19)

$$q \frac{dQ(q)}{dq} = P(q)Q(q) - E(q)Q(q).$$

(1.20)

If we define the related series analogues to [5]

$$P_{r,2}(q) = \sum_{n=0}^{\infty} \tilde{\sigma}(2n + r) q^{2n+r}, \quad r = 0, 1,$$

(1.21)

$$E_{r,2}(q) = \sum_{n=0}^{\infty} \hat{\sigma}(2n + r) q^{2n+r}, \quad r = 0, 1,$$

(1.22)

$$Q_{r,2}(q) = \sum_{n=0}^{\infty} \tilde{\sigma}_3(2n + r) q^{2n+r}, \quad r = 0, 1,$$

(1.23)

then we find many identities involving the series $P_{r,2}(q)$, $E_{r,2}(q)$, $Q_{r,2}(q)$, and the functions $P(q)$, $E(q)$, and $Q(q)$.

In Section 2, we evaluate (1.15), (1.16), (1.17), (1.21), (1.22), and (1.23) in terms of two parameters $x$ and $z$ in Ramanujan’s theory of elliptic functions. Using these formulas, we derive some identities involving Ramanujan’s theta functions. In Section 3, we find representations for certain infinite series related to $P(q)$, $E(q)$, and $Q(q)$. In Section 4, using the evaluations we obtained in Section 2, we derive convolution sums of (1.10) and (1.11). In Section 5, we discuss some formulae for determining $r_s(n)$ and $\delta_s(n)$ in terms of $\tilde{\sigma}_s(n)$ and $\hat{\sigma}_s(n)$, where $r_s(n)$ denotes the number of representations of $n$ as a sum of $s$ squares and $\delta_s(n)$ denotes the number of representations of $n$ as a sum of $s$ triangular numbers. Finally, we find some partition congruences connected with $\tilde{\sigma}_s(n)$ and $\hat{\sigma}_s(n)$ by using the notion of colored partitions.
2. Evaluations and identities involving Ramanujan’s theta functions

To derive the desired identities, we need to use evaluations of theta functions [3, pp. 122–138] to determine the quantities \( P(q^r), E(q^r), Q(q^r), P(-q), E(-q), \) and \( Q(-q), r = 1, 2. \)

If

\[
y = \frac{\pi \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1-x \right)}{\, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right)}, \quad |x| < 1,
\]

where \( _2F_1 \) denotes the Gaussian hypergeometric function, the evaluations are given in terms of, in Ramanujan’s notation,

\[
z := \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right)
\]

and \( x \). The derivative \( y’ \) is given by

\[
\frac{dy}{dx} = -\frac{1}{x(1-x)z^2};
\]

see, for example, Berndt’s book [2, p. 87]. The function \( z := \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) \) satisfies the differential equation [3, p. 120]

\[
\frac{d^2z}{dz^2} = \frac{z}{4x(1-x)} - \frac{(1-2x)dz}{x(1-x)dx}.
\]

From now on, we will denote \( q := e^{-y} \).

Ramanujan’s theta functions \( \varphi(q), \psi(q), \) and \( f(-q) \) [3, Entry 22, p. 36] are defined, for \( |q| < 1 \), by

\[
\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q^2)_{\infty}(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}(-q^2; q^2)_{\infty}},
\]

\[
\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},
\]

\[
f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q; q)_{\infty},
\]

where, as usual, for any complex number \( a \), we write

\[
(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).
\]

Here, the product representations arise from the Jacobi triple product identity [3, Entry 19, p. 35]. In the following lemma, we list the evaluations of the theta functions in terms of \( x \) and \( z \) [3, Entries 10–12, pp. 122–124], which we will employ in a majority of our proofs.
Lemma 2.1. If $y$ and $z$ are defined by (2.1), (2.2), respectively, and $\psi(q)$, $\varphi(q)$, and $f(-q)$ are defined by (2.5), (2.6), and (2.7), respectively, then

\begin{align}
\varphi(q) &= \sqrt{z}, \\
\varphi(-q) &= (1 - x)^{1/4} \sqrt{z}, \\
q^{1/8} \psi(q) &= 2^{-1/2} x^{1/8} \sqrt{z}, \\
q^{1/4} \psi(q^2) &= 2^{-1} x^{1/4} \sqrt{z}, \\
q^{1/24} f(-q) &= 2^{-1/6} (1 - x)^{1/6} x^{1/24} \sqrt{z}.
\end{align}

Using these evaluations, we obtain formulas for $P(q)$, $E(q)$, and $Q(q)$.

Theorem 2.2. If $y$ and $z$ are defined as in (2.1) and (2.2), respectively, and $q := e^{-y}$, then

\begin{align}
P(q) &= z^2 (1 - x) + 4x(1 - x) \frac{dz}{dx}, \\
E(q) &= z^2 (1 + x), \\
Q(q) &= z^4 (1 - x)^2.
\end{align}

Proof of (2.13). In the derivation below, we find that, by using (2.10),

\begin{align*}
P(q) &= 1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{e^{ny} - 1} \\
&= 1 - 8 \frac{d}{dy} \sum_{n=1}^{\infty} (-1)^n \text{Log}(1 - e^{-ny}) \\
&= 1 - 8 \frac{d}{dy} \text{Log} \prod_{n=1}^{\infty} \frac{1 - e^{-2ny}}{1 - e^{-2(n-1)y}} \\
&= -8 \frac{d}{dy} \text{Log} \{ e^{-y/8} \psi(e^{-y}) \},
\end{align*}

where we use the infinite product representation of $\psi(e^{-y})$ in (2.6). If we employ (2.10) and (2.3), then we find that

$$P(q) = 8x(1 - x) z^2 \frac{d}{dx} \text{Log} \{ 2^{-1/2} \sqrt{z} x^{1/8} \}$$

$$= z^2 (1 - x) + 4x(1 - x) \frac{dz}{dx}.$$  

Proof of (2.14). In the derivation below, we employ (2.12) and (2.9) to find that

$$E(q) = 1 + 24 \sum_{n=1}^{\infty} \frac{n}{e^{ny} + 1}$$

$$= 1 - 24 \frac{d}{dy} \sum_{n=1}^{\infty} \text{Log}(1 + e^{-ny})$$
Again using the evaluations (2.9) and (2.12) and applying (2.3), we find that
\[ \mathcal{E}(q) = 24x(1 - x)z^2 \frac{d}{dx} \log \left\{ 2^{-1/6} (1 - x)^{-1/12} x^{1/24} \right\} \]
\[ = z^2(1 + x), \]
which completes our proof.

**Proof of (2.15).** From (1.18), we have
\[ \frac{q}{d} \frac{dP(q)}{dq} = \frac{P(q)^2 - Q(q)}{4}. \]
Thus, by the chain rule, we deduce that
\[ \frac{dP(e^{-y})}{dy} = \frac{Q(e^{-y}) - P(e^{-y})}{4}. \]
Moreover, by (2.3), we derive that
\[ \frac{dP(e^{-y})}{dx} = \frac{1}{x(1-x)z^2} \frac{dP(e^{-y})}{dy}. \]
Hence,
\[ (2.16) \quad - x(1-x)z^2 \frac{dP(e^{-y})}{dx} = \frac{Q(e^{-y}) - P^2(e^{-y})}{4}. \]
Thus we see that we can determine \( Q(e^{-y}) \) from (2.13) and (2.16). Using (2.13) and the hypergeometric differential equation (2.4), we find, upon direct calculation, that
\[ (2.17) \quad \frac{dP(e^{-y})}{dx} = 2(1-x)z \frac{dz}{dx} + 4x(1-x) \left( \frac{dz}{dx} \right)^2. \]
Thus from (2.13), (2.16), and (2.17), we see that
\[ Q(q) = Q(e^{-y}) = \left\{ (1-x)z^2 + 4x(1-x)z \frac{dz}{dx} \right\}^2 \]
\[ - 4x(1-x)z^2 \left\{ 2(1-x)z \frac{dz}{dx} + 4x(1-x) \left( \frac{dz}{dx} \right)^2 \right\}. \]
Upon simplifying, we reach the desired conclusion.

Before proceeding further, we briefly mention the procedure [3, p. 125], called **duplication**, in the theory of elliptic functions. If
\[ (2.18) \quad \Omega(x, e^{-y}, z) = 0, \]
and \( x', y' \), and \( z' \) is another set of parameters such that
\[ \Omega(x', e^{-y'}, z') = 0 \]
and
\[ x = \frac{4\sqrt{x'}}{(1 + \sqrt{x'})^2}, \]
then we can deduce the “new” formula
\[ \Omega \left( \left( \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right)^2, e^{-2y}, \frac{1}{2} z(1 + \sqrt{1 - x}) \right) = 0, \]
from the “old” formula (2.18). This process is called obtaining a formula by duplication. We will use this procedure in many proofs.

Applying the process of duplication to (2.13), (2.14), and (2.15), we obtain
\[ P(q^2) = z^2(1 - x) + 2x(1 - x) z \frac{dz}{dx} , \]
\[ E(q^2) = z^2(1 - \frac{1}{2} x) , \]
\[ Q(q^2) = z^4(1 - x). \]

Berndt [3, p. 126] has also described the process of obtaining a new formula from (2.18) by changing the sign of \( q \). If (2.18) holds then the formula
\[ \Omega \left( \frac{x}{x - 1}, -q, z \sqrt{1 - x} \right) = 0 \]
also holds. This result is attributed to Jacobi by Berndt [3, p. 126].

Applying Jacobi’s change of sign procedure to (2.13), (2.14) and (2.15), we deduce that
\[ P(-q) = z^2(1 - 2x) + 4x(1 - x) z \frac{dz}{dx} , \]
\[ E(-q) = z^2(1 - 2x) , \]
\[ Q(-q) = z^4. \]

Simple calculations analogous to [5] show that
\[ P_{0,2}(q) = \frac{1}{16} \left( -2 + P(q) + P(-q) \right) , \]
\[ P_{1,2}(q) = \frac{1}{16} \left( P(q) - P(-q) \right) , \]
\[ E_{0,2}(q) = \frac{1}{48} \left( -2 + E(q) + E(-q) \right) , \]
\[ E_{1,2}(q) = \frac{1}{48} \left( E(q) - E(-q) \right) , \]
\[ Q_{0,2}(q) = \frac{1}{32} \left( -2 - Q(q) - Q(-q) \right) , \]
\[ Q_{1,2}(q) = \frac{1}{32} \left( -Q(q) + Q(-q) \right) . \]
Using (2.13)–(2.15) and (2.24)–(2.26), we obtain the evaluations of the series $P_{r,2}(q)$, $E_{r,2}(q)$ and $Q_{r,2}(q)$ as follows:

**Theorem 2.3.**

(2.33) \[ P_{0,2}(q) = \frac{1}{16}(-2 + (2 - 3x)z^2 + 8x(1 - x)z \frac{dz}{dx}), \]

(2.34) \[ P_{1,2}(q) = \frac{1}{16}xz^2, \]

(2.35) \[ E_{0,2}(q) = \frac{1}{48}(-2 + (2 - x)z^2), \]

(2.36) \[ E_{1,2}(q) = \frac{1}{16}xz^2, \]

(2.37) \[ Q_{0,2}(q) = \frac{1}{32}(2 - (2 - 2x + x^2)z^4), \]

(2.38) \[ Q_{1,2}(q) = \frac{1}{32}x(2 - x)z^4. \]

We note a few results which are used in the next section. Using (2.3) and $q := e^{-y}$, we have

\[ \frac{1}{q} \frac{dq}{dx} = -\frac{dy}{dx} = \frac{1}{x(1 - x)z^2} \]

so that

(2.39) \[ \frac{dq}{dx} = \frac{q}{x(1 - x)z^2}. \]

From (2.3), (2.13), and (2.39), we obtain

\[ \frac{dP(q)}{dq} = \frac{dP(q)}{dx} \frac{dx}{dq} = \frac{q}{x(1 - x)z^2} \left( z^2(1 - x) + 4x(1 - x)z \frac{dz}{dx} \right), \]

\[ = -z^2 + (6 - 10x)z \frac{dz}{dx} + 4x(1 - x)(\frac{dz}{dx})^2 + 4x(1 - x)z \frac{d^2z}{dx^2}, \]

so that

(2.40) \[ q \frac{dP(q)}{dq} = 2x(1 - x)^2z^3 \frac{dz}{dx} + 4x^2(1 - x)^2z^2 \left( \frac{dz}{dx} \right)^2. \]

Similarly, from (2.4), (2.20), and (2.39), we obtain

(2.41) \[ q \frac{dP(q^2)}{dq} = -\frac{x(1 - x)z^4}{2} + 2x(1 - x)^2z^3 \frac{dz}{dx} + 2x^2(1 - x)^2z^2 \left( \frac{dz}{dx} \right)^2. \]
In a similar manner, we find that

\[ q \frac{dE(q)}{dq} = x(1-x)z^4 + 2x(1-x)(1+x)z^3 \frac{dz}{dx}, \]

\[ q \frac{dE(q^2)}{dq} = -\frac{x(1-x)z^4}{2} + x(1-x)(2-x)z^3 \frac{dz}{dx}. \]

Next, using Lemma 2.1 and using (2.13), (2.14), (2.15), (2.20), (2.21), (2.22), (2.24), (2.25), and (2.26), we obtain the following identities.

**Theorem 2.4.** Recall that \( P, E, \) and \( Q \) are defined by (1.15), (1.16), and (1.17), respectively, and that \( \varphi(q) \) and \( \psi(q) \) are defined in (2.5) and (2.6), respectively. Then

\[ Q(q) = \varphi^8(-q), \]

(2.44)

\[ 16\psi^4(q^2) + \varphi^4(q) = E(q), \]

(2.45)

\[ 2E(q^2) + E(q) = 3\varphi^4(q), \]

(2.46)

\[ \varphi^4(q)E(q) + Q(q^2) = 2\varphi^8(q), \]

(2.47)

\[ E(q) - E(q^2) = 24q\varphi^4(q^2), \]

(2.48)

\[ P(q) - P(-q) = 16q\varphi^4(q^2), \]

(2.49)

\[ Q(q) + Q(-q) = 32q(8\psi^8(q^2) - \psi^8(q)), \]

(2.50)

\[ E^2(q) - Q(q) = 64q\psi^8(q). \]

**Proof of (2.44).** The result is clear from (2.9) and (2.13).

**Proof of (2.45).** The equality

\[ 16\psi^4(q^2) + \varphi^4(q) = xz^2 + z^2 = (1+x)z^2 = E(q) \]

follows from (2.8), (2.11) and (2.14).

**Proof of (2.46).** Employing (2.14) and (2.21), we have

\[ 2E(q^2) + E(q) = 3z^2. \]

So the proof is completed by using (2.3).

**Proof of (2.47).** By (2.8), (2.14), and (2.15), we find that

\[ \varphi^4(q)E(q) + Q(q^2) = z^4(1+x) + z^4(1-x) = 2z^4 = 2\varphi^8(q). \]

**Proof of (2.48).** By using (2.14), (2.21), and (2.11) we obtain

\[ E(q) - E(q^2) = \frac{3}{2}xz^2 = 24q\varphi^4(q^2). \]

**Proof of (2.49).** From (2.13) and (2.24), we find that

\[ P(q) - P(-q) = (1-x)z^2 - (1-2x)z^2 = xz^2 = 16q\varphi^4(q^2). \]

**Proof of (2.50).** By the definition of \( Q \), we obtain

\[ Q(q) + Q(-q) = -16 \sum_{n=1}^{\infty} (2n-1)^3 q^{2n-1} \left( \frac{1}{1-q^{2n-1}} + \frac{1}{1+q^{2n-1}} \right). \]
$$= -32 \sum_{n=1}^{\infty} \frac{(2n-1)^3 q^{2n-1}}{1 - q^{4n-2}} = 32q(8\psi^8(q^2) - \psi^8(q)),$$

where we use Example(ii) in [3, p. 139].

**Proof of (2.51).** From (2.14) and (2.15), we see that

$$\mathcal{E}^2(q) - \mathcal{Q}(q) = 4xz^4 = \left(\frac{1}{16}xz^2\right) (64z^2)$$

$$= (q\psi^4(q^2)) \cdot (64\varphi^4(q)),$$

where the last equality follows from (2.8) and (2.10). After employing the fact [3, Entry 25, p. 40]

$$\varphi(q)\psi(q^2) = \psi^2(q),$$

we achieve the desired result.

### 3. Representations of certain infinite series

In this section, we derive some representations of the infinite series connected with the functions $\mathcal{P}(q), \mathcal{E}(q)$, and $\mathcal{Q}(q)$.

**Theorem 3.1.** We have

$$1 - 24 \sum_{n=1}^{\infty} \frac{2n - 1}{e^{(2n-1)y} + 1} = (1 - 2x)z^2.$$

**Proof.** From (2.14) and (2.21), we find that

$$2\mathcal{E}(q^2) - \mathcal{E}(q) = 2(1 - \frac{x}{2})z^2 - (1 + x)z^2 = (1 - 2x)z^2.$$

On the other hand, by the definition of $\mathcal{E}(q)$ in (1.10), we know that

$$2\mathcal{E}(q^2) - \mathcal{E}(q) = 2\left(1 + 24 \sum_{n=1}^{\infty} \frac{n}{e^{2ny} + 1}\right) - \left(1 + 24 \sum_{n=1}^{\infty} \frac{n}{e^{ny} + 1}\right)$$

$$= 1 + 24 \sum_{n=1}^{\infty} \frac{2n}{e^{2ny} + 1} - 24 \sum_{n=1}^{\infty} \left(\frac{2n}{e^{2ny} + 1} + \frac{2n - 1}{e^{(2n-1)y} + 1}\right)$$

$$= 1 - 24 \sum_{n=1}^{\infty} \frac{2n - 1}{e^{(2n-1)y} + 1}.$$

□

**Remark.** We can compare this result with some of the results in [3, Entry 13, p. 127].

For example [3, (viii)], we have

$$1 + 24 \sum_{n=1}^{\infty} \frac{n}{e^{ny} + 1} = (1 + x)z^2.$$
By using the representations for $P(q)$, $Q(q)$ and $R(q)$ and their algebraic relations, Berndt [3] also lists further representations, such as

\begin{align}
(3.3) & \quad 1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^5}{e^{ny} - 1} = (1 - x)(1 - x^2)z^6, \\
(3.4) & \quad 17 - 32 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^7}{e^{ny} - 1} = (1 - x)^2(17 - 2x + 17x^2)z^8, \\
(3.5) & \quad 1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n n^5}{e^{2ny} - 1} = (1 - x)(1 - \frac{1}{2}x)z^6, \\
(3.6) & \quad 17 - 32 \sum_{n=1}^{\infty} \frac{(-1)^n n^7}{e^{2ny} - 1} = (1 - x)(17 - 17x + 2x^2)z^8.
\end{align}

**Theorem 3.2.** We have

\begin{align}
(3.7) & \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3}{\sinh(ny)} = \frac{1}{8}x(1 - x)z^4, \\
(3.8) & \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^5}{\sinh(ny)} = \frac{1}{8}x(1 - x)(1 - 2x)z^6, \\
(3.9) & \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^7}{\sinh(ny)} = \frac{1}{16}x(1 - x)(2 - 17x + 17x^2)z^8.
\end{align}

**Proof.** We use the elementary fact

\begin{equation}
\frac{1}{x - 1} - \frac{1}{x^2 - 1} = \frac{x}{x^2 - 1} = \frac{1}{x - x^{-1}}.
\end{equation}

To prove \((3.7)\), we simply use the definition of $Q$ and \((3.10)\) to obtain

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3}{e^{ny} - e^{-ny}} = -\frac{1}{16} \left \{ Q(e^{-y}) - Q(e^{-2y}) \right \} = \frac{1}{16}x(1 - x)z^4,
\]

where we used \((2.15)\) and \((2.22)\) in the last equality. For \((3.8)\), by \((3.10)\), the sum to be evaluated is equal to

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^5}{e^{ny} - e^{-ny}} = \frac{1}{8} \left \{ (1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n n^5}{e^{ny} - 1}) - (1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n n^5}{e^{2ny} - 1}) \right \}
\]

\[
= \frac{1}{8} \left \{ (1 - x)(1 - x^2)z^6 - (1 - x)(1 - \frac{1}{2}x)z^6 \right \}
\]

\[
= \frac{1}{8}x(1 - x)(1 - 2x)z^6,
\]

where we employ \((3.3)\) and \((3.5)\) to derive \((3.8)\). In a similar manner, we can deduce \((3.9)\) by using \((3.10)\) and applying \((3.4), (3.6)\). \qed
Applying the duplication process to (3.7)–(3.9), respectively, gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3}{\sinh(2ny)} = \frac{1}{32} \sqrt{1-x(1-\sqrt{1-x})^2} z^4,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^5}{\sinh(2ny)} = \frac{1}{64} \sqrt{1-x(1-\sqrt{1-x})^2(x-2+6\sqrt{1-x})z^6},$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^7}{\sinh(2ny)} = \frac{1}{512} (1-x)(1-\sqrt{1-x})^2(76\sqrt{1-x} - 30(2-x) + x^2)z^8.$$  

**Remark.** We can compare the above results with some results in [3, Entry 15, p. 132]. For example, Berndt proved that

$$\sum_{n=1}^{\infty} \frac{n^3}{\sinh(ny)} = \frac{1}{8} x z^4.$$  

4. SOME CONVOLUTION SUMS OF $\tilde{\sigma}_s(n)$ AND $\tilde{\sigma}_s(n)$

We begin this section by recalling again the three differential equations satisfied by $P(q)$, $E(q)$, and $Q(q)$:

$$q \frac{dP(q)}{dq} = \frac{P^2(q) - Q(q)}{4},$$

$$q \frac{dE(q)}{dq} = \frac{E(q)P(q) - Q(q)}{2},$$

$$q \frac{dQ(q)}{dq} = P(q)Q(q) - E(q)Q(q).$$

It is then easy to show that the following convolution sums follow from (4.1)–(4.3).

**Theorem 4.1.**

$$4 \sum_{m<n} \tilde{\sigma}(m)\tilde{\sigma}(n-m) = -\tilde{\sigma}_3(n) + (2n-1)\tilde{\sigma}(n),$$

$$24 \sum_{m<n} \tilde{\sigma}(m)\tilde{\sigma}(n-m) = -2\tilde{\sigma}_3(n) + (6n-3)\tilde{\sigma}(n) - \tilde{\sigma}(n),$$

$$16 \sum_{m<n} (\tilde{\sigma}(m) - 3\tilde{\sigma}(m))\tilde{\sigma}_3(n-m) = 2n\tilde{\sigma}_3(n) + \tilde{\sigma}(n) - 3\tilde{\sigma}(n).$$

**Proof.** We can rewrite (4.1) as

$$P^2(q) = Q(q) + 4q \frac{dP(q)}{dq}.$$  

Then we have

$$\left(1 + 8 \sum_{n=1}^{\infty} \tilde{\sigma}(n)q^n\right)^2 = \left(1 - 16 \sum_{n=1}^{\infty} \tilde{\sigma}_3(n)q^n\right) + 32 \sum_{n=1}^{\infty} n\tilde{\sigma}(n)q^n.$$
Equating the coefficients of \( q^n \) on both sides of (1.7), we obtain (1.4). In a similar manner, the remaining two convolution sums (1.5) and (1.6) can be derived from (1.2) and (1.3), respectively.

\[ \sum_{m<n} \hat{\sigma}(m)\hat{\sigma}(n-m), \]

which will be mentioned in the following theorem.

**Theorem 4.2.**

\[ 36 \sum_{m<n} \hat{\sigma}(m)\hat{\sigma}(n-m) = \begin{cases} -3\tilde{\sigma}(n) + 3\tilde{\sigma}_3(n), & \text{if } n \text{ is odd,} \\ -3\tilde{\sigma}(n) - 5\tilde{\sigma}_3(n) + 4\tilde{\sigma}_3(n/2), & \text{if } n \text{ is even.} \end{cases} \]

**Proof.** By using (2.14), (2.15), (2.22) and (2.38), we can easily derive the identity

\[ \mathcal{E}^2(q) = z^4(1+x)^2 \\
= z^4(5(1-x)^2 - 4(1-x) + 4x(2-x)) \\
= 5\mathcal{Q}(q) - 4\mathcal{Q}(q^2) + 128\mathcal{Q}_{1,2}(q). \]

Equating coefficients of \( q^n \) gives the desired evaluation. \( \square \)

**Remark.** We point out that certain of the convolution sums considered here can be evaluated from known results in an elementary manner. For example, by using the relation (1.13), we have that

\[ \sum_{m<n} \hat{\sigma}(m)\hat{\sigma}(n-m) = \sum_{m<n} (\sigma(m) - 2\sigma(m/2))(\sigma(n-m) - 2\sigma((n-m)/2)) \]

\[ = \sum_{m<n} \sigma(m)\sigma(n-m) - 2 \sum_{m<n} \sigma(m/2)\sigma(n-m) \]

\[ - 2 \sum_{m<n} \sigma((n-m)/2)\sigma(m) + 4 \sum_{m<n} \sigma(m/2)\sigma((n-m)/2) \]

\[ = A(n) - 4B(n) + 4A(n/2), \]

where

\[ A(n) = \sum_{m<n} \sigma(m)\sigma(n-m) \]

and

\[ B(n) = \sum_{m<n/2} \sigma(m)\sigma(n-2m). \]

The values of \( A(n) \) and \( B(n) \) are given in [12].

**Theorem 4.3.**

\[ 16 \sum_{m<n} \tilde{\sigma}(m)\tilde{\sigma}_3(n-m) = -\tilde{\sigma}_5(n) + 2(n-1)\tilde{\sigma}_3(n) + \tilde{\sigma}(n). \]
Remark. Note that the identities (4.4) and (4.9) are analogues of the identities (1.8) and (1.9), respectively, which we mentioned in Section 1. The identity (4.4) was also proved by Glaisher [9] by theory of the elliptic functions.

Using the formulas given in Section 2, for \( r \neq s \) and \( r, s \in \{1, 2\} \), we determine the products \( \mathcal{P}(q^r)\mathcal{P}(q^s) \) and \( \mathcal{P}(q^r)\mathcal{E}(q^s) \) as linear combinations of \( \mathcal{Q}(q) \), \( \mathcal{Q}(q^2) \) and the derivatives of \( \mathcal{P}(q) \), \( \mathcal{P}(q^2) \), \( \mathcal{E}(q) \), and \( \mathcal{E}(q^2) \).

**Theorem 4.4.** We have

\[
\mathcal{P}(q^r)\mathcal{P}(q^s) = \mathcal{Q}(q^2) + q \frac{d\mathcal{P}(q)}{dq} + 2q \frac{d\mathcal{P}(q^2)}{dq},
\]

(4.11)

\[
\mathcal{P}(q^r)\mathcal{E}(q) = \mathcal{Q}(q^2) + \frac{1}{3} \left( q \frac{d\mathcal{E}(q)}{dq} + 2q \frac{d\mathcal{E}(q^2)}{dq} \right) + \left( q \frac{d\mathcal{P}(q)}{dq} - 2q \frac{d\mathcal{P}(q^2)}{dq} \right),
\]

(4.12)

\[
\mathcal{P}(q)e(q) \mathcal{E}(q^2) = \frac{1}{2} (3\mathcal{Q}(q^2) - \mathcal{Q}(q)) + 2q \frac{d\mathcal{E}(q^2)}{dq}.
\]

(4.13)

We just give the proof of (4.11), since the remaining proofs are similar.

**Proof of (4.11).** By (2.22), (2.40), and (2.41), we have

\[
\mathcal{Q}(q^2) + q \frac{d\mathcal{P}(q)}{dq} + 2q \frac{d\mathcal{P}(q^2)}{dq} = (1 - x)z^4 + 2x(1 - x)^2z^3 \frac{dz}{dx} + 4x^2(1 - x)^2z^2 \left( \frac{dz}{dx} \right)^2
\]

\[
- x(1 - x)z^4 + 4x(1 - x)^2z^3 \frac{dz}{dx} + 4x^2(1 - x)^2z^2 \left( \frac{dz}{dx} \right)^2
\]

\[
= (1 - x)^2z^4 + 6x(1 - x)^2z^3 \frac{dz}{dx} + 8x^2(1 - x)^2z^2 \left( \frac{dz}{dx} \right)^2
\]

\[
= \mathcal{P}(q)\mathcal{P}(q^2),
\]

where we simply calculate the product of (2.13) and (2.20). This completes the proof of (4.11). The remaining formulas can be proved similarly.

Equating the coefficients of \( q^n \) on both sides in the three identities in Theorem 4.4 we obtain the next theorem.

**Theorem 4.5.** We have

\[
8 \sum_{m<n/2} \tilde{\sigma}(m)\tilde{\sigma}(n - 2m) = - \tilde{\sigma}(n/2) + (n - 1)\tilde{\sigma}(n) + (2n - 1)\tilde{\sigma}(n/2),
\]

(4.14)

\[
24 \sum_{m<n/2} \tilde{\sigma}(m)\tilde{\sigma}(n - 2m) = - 2\tilde{\sigma}(n/2) + (2n - 3)\tilde{\sigma}(n) + 4n\tilde{\sigma}(n/2)
\]

(4.15)
We have

\[ E(n) = \frac{1}{12} \sum_{m \leq n/2} m \sigma(m) = \sigma(n) - 3\sigma(n/2) + (6n - 3)\sigma(n/2) - \sigma(n). \]

(4.16)

The next theorem shows that for \( r \in \{0, 1\} \) and \( s \in \{1, 2\} \), the products of the form \( \mathcal{P}_{r,2}(q)(-1 + \mathcal{P}(q^s)) \) and \( \mathcal{E}_{r,2}(q)(-1 + \mathcal{P}(q^s)) \) can be expressed as linear combinations of \( \mathcal{P}(q), \mathcal{P}(q^2), \mathcal{E}(q), \mathcal{E}(q^2), \mathcal{Q}(q), \mathcal{Q}(q^2) \), and the derivatives of \( \mathcal{P}(q), \mathcal{P}(q^2), \mathcal{E}(q), \) and \( \mathcal{E}(q^2) \). A MAPLE program was run to determine the identities.

**Theorem 4.6.** We have

(4.17) \[ \mathcal{P}_{0,2}(q)(-1 + \mathcal{P}(q)) = \frac{1}{8} + \frac{1}{16}(\mathcal{Q}(q) + \mathcal{Q}(q^2)) - \frac{1}{24}(\mathcal{E}(q) - 7\mathcal{E}(q^2)) \]
\[ - \frac{1}{2}\mathcal{P}(q^2) + \frac{1}{2}q\frac{d\mathcal{P}(q)}{dq} - \frac{1}{12}\left(q\frac{\mathcal{E}(q)}{dq} + q\frac{\mathcal{E}(q^2)}{dq}\right), \]

(4.18) \[ \mathcal{P}_{0,2}(q)(-1 + \mathcal{P}(q^2)) = \frac{1}{8} + \frac{1}{16}(\mathcal{Q}(q^2) - 3\mathcal{P}(q^2) + \mathcal{E}(q^2)) \]
\[ + \frac{1}{16}\left(q\frac{\mathcal{P}(q)}{dq} + 6q\frac{\mathcal{P}(q^2)}{dq}\right) - \frac{1}{48}\left(q\frac{\mathcal{E}(q)}{dq} + 2q\frac{\mathcal{E}(q^2)}{dq}\right), \]

(4.19) \[ \mathcal{P}_{1,2}(q)(-1 + \mathcal{P}(q)) = \frac{1}{24}(\mathcal{E}(q^2) - \mathcal{E}(q)) + \frac{1}{24}\left(q\frac{\mathcal{E}(q)}{dq} - q\frac{\mathcal{E}(q^2)}{dq}\right), \]

(4.20) \[ \mathcal{P}_{1,2}(q)(-1 + \mathcal{P}(q^2)) = \frac{1}{24}(\mathcal{E}(q^2) - \mathcal{E}(q)) + \frac{1}{24}\left(q\frac{\mathcal{E}(q)}{dq} - q\frac{\mathcal{E}(q^2)}{dq}\right), \]

(4.21) \[ \mathcal{E}_{0,2}(q)(-1 + \mathcal{P}(q)) = \frac{1}{24} + \frac{1}{16}(\mathcal{Q}(q^2) - 3\mathcal{Q}(q)) - \frac{1}{24}\mathcal{P}(q) \]
\[ - \frac{1}{24}\mathcal{E}(q^2) - \frac{1}{12}q\frac{\mathcal{E}(q^2)}{dq}, \]

(4.22) \[ \mathcal{E}_{0,2}(q)(-1 + \mathcal{P}(q^2)) = \frac{1}{24} + \frac{1}{24}(\mathcal{Q}(q^2) - \mathcal{P}(q^2) - \mathcal{E}(q^2)) + \frac{1}{24}q\frac{\mathcal{E}(q^2)}{dq}. \]

Again we just give the proof of (4.17), since the remaining proofs are similar.

**Proof of (4.17).** By (2.13) and (2.22), we have
\[ \mathcal{Q}(q) + \mathcal{Q}(q^2) = 2z^4 - 3xz^4 + x^2z^4, \]
and from (2.14) and (2.21),
\[ \mathcal{E}(q) - 7\mathcal{E}(q^2) = -6z^2 + \frac{9}{2}xz^2, \]
and from (2.42) and (2.43)
\[ q\frac{\mathcal{E}(q)}{dq} + q\frac{\mathcal{E}(q^2)}{dq} = \frac{1}{2}xz^4 - \frac{1}{2}x^2z^4 + 4x(1 - x)z^3\frac{dz}{dx} + x^2(1 - x)z^3\frac{dz}{dx}. \]
Therefore by (2.20), (2.40), and the previous three equalities, we finally obtain
\[
\begin{align*}
\frac{1}{8} + \frac{1}{16}(Q(q) + Q(q^2)) & - \frac{1}{24}(E(q) - 7E(q^2)) - \frac{1}{2}\mathcal{P}(q^2) \\
+ \frac{1}{2}q \frac{d\mathcal{P}(q)}{dq} & - \frac{1}{12}\left( q \frac{E(q)}{dq} + q \frac{E(q^2)}{dq} \right) \\
& = \frac{1}{8} + \frac{1}{16}(2z^4 - 3xz^4 + x^2z^4) - \frac{1}{24}(-6z^2 + 9xz^2) \\
& - \frac{1}{2}\left( z^2(1 - x) + 2x(1 - x)z \frac{dz}{dx} \right) \\
& + \frac{1}{2}\left( 2x(1 - x)^2z^3 \frac{dz}{dx} + 4x^2(1 - x)^2z^2 \left( \frac{dz}{dx} \right)^2 \right) \\
& - \frac{1}{12}\left( \frac{1}{2}x^4 - \frac{1}{2}x^2z^4 + 4x(1 - x)z^3 \frac{dz}{dx} + x^2(1 - x)z^3 \frac{dz}{dx} \right) \\
& = \frac{1}{8} - \frac{1}{4}z^2 + \frac{5}{16}x^2z^2 - x(1 - x)z \frac{dz}{dx} + \frac{1}{8}z^4 - \frac{5}{16}xz^4 + x(1 - x)z^3 \frac{dz}{dx} \\
& + \frac{3}{16}x^2z^4 - \frac{5}{4}x^2(1 - x)z^3 \frac{dz}{dx} + 2x(1 - x) \left( \frac{dz}{dx} \right)^2 \\
& = \mathcal{P}_{0,2}(q)(-1 + \mathcal{P}(q)).
\end{align*}
\]

Equating the coefficients of \( q^n \) on both sides of the six formulas in Theorem 4.6, we obtain the following convolution sums.

**Theorem 4.7.** We have
\[
\begin{align*}
(4.23) \quad 8 \sum_{m<n/2} \tilde{\sigma}(2m)\tilde{\sigma}(n - 2m) & = -\tilde{\sigma}_3(n) - \tilde{\sigma}_3(n/2) + 4n\tilde{\sigma}(n) - 4\tilde{\sigma}(n/2) \\
& -(2n + 1)\tilde{\sigma}(n) + (2n + 7)\tilde{\sigma}(n/2), \\
(4.24) \quad 8 \sum_{m<n/2} \tilde{\sigma}(2m)\tilde{\sigma}(n/2 - m) & = -2\tilde{\sigma}_3(n/2) + n/2\tilde{\sigma}(n) + (3n - 3)\tilde{\sigma}(n/2) \\
& - n/2\tilde{\sigma}(n) - (n - 3)\tilde{\sigma}(n/2), \\
(4.25) \quad 8 \sum_{m<(n+1)/2} \tilde{\sigma}(2m - 1)\tilde{\sigma}(n - (2m - 1)) & = -\tilde{\sigma}_3(n) + \tilde{\sigma}_3(n/2) \\
& + (2n - 1)\tilde{\sigma}(n) - (2n - 1)\tilde{\sigma}(n/2), \\
(4.26) \quad 8 \sum_{m<(n+1)/2} \tilde{\sigma}(2m - 1)\tilde{\sigma}((n + 1)/2 - m) & = (n - 1)\tilde{\sigma}(n) - (n - 1)\tilde{\sigma}(n/2), \\
(4.27) \quad 8 \sum_{m<n/2} \tilde{\sigma}(2m)\tilde{\sigma}(n/2 - m) & = \frac{1}{3}\tilde{\sigma}_3(n) - \tilde{\sigma}_3(n/2) + (2n - 1)\tilde{\sigma}(n/2), \\
(4.28) \quad 8 \sum_{m<n/2} \tilde{\sigma}(2m)\tilde{\sigma}(n/2 - m) & = -\frac{2}{3}\tilde{\sigma}_3(n/2) - \frac{1}{3}\tilde{\sigma}(n/2) + (n - 1)\tilde{\sigma}(n/2).
\end{align*}
\]
5. ON THE REPRESENTATIONS OF INTEGERS AS SUMS OF SQUARES AND TRIANGULAR NUMBERS

It is immediate from the definitions of \( \varphi(q) \) and \( \psi(q) \) in (2.5) and (2.6), respectively, that if

\[
\varphi_s(q) := \sum_{n=0}^{\infty} r_s(n)q^n
\]

and

\[
\psi_s(q) := \sum_{n=0}^{\infty} \delta_s(n)q^n,
\]

then \( r_s(n) \) and \( \delta_s(n) \) are the number of representations of \( n \) as a sum of \( s \) squares and \( s \) triangular numbers, respectively. Clearly \( r_s(0) = \delta_s(0) = 1 \). Here, for each non negative integer \( n \), the triangular number \( T_n \) is defined by

\[
T_n := \frac{n(n+1)}{2}.
\]

By using the representations and identities derived in Section 2, we find expressions for \( r_s(n) \) and \( \delta_s(n) \), \( s = 4, 8 \), as sums of our functions \( \tilde{\sigma}(n) \), \( \hat{\sigma}(n) \), and \( \tilde{\sigma}_3(n) \).

**Theorem 5.1.** For each positive integer \( n \), we have

\[
\begin{align*}
(5.3) & \quad r_4(n) = 16\tilde{\sigma}(n/2) + 8\hat{\sigma}(n), \\
(5.4) & \quad \delta_4(n) = \tilde{\sigma}(2n+1), \\
(5.5) & \quad r_8(n) = 16(-1)^{n-1}\tilde{\sigma}_3(n) \\
(5.6) & \quad 8\delta_8(n) = \tilde{\sigma}_3(n+1) - \tilde{\sigma}_3(2(n+1)).
\end{align*}
\]

**Proof of (5.3).** The identity (2.46) is equivalent to the identity

\[
\sum_{n=1}^{\infty} r_4(n)q^n = 48\sum_{n=1}^{\infty} \tilde{\sigma}(n)q^{2n} + 24\sum_{n=1}^{\infty} \hat{\sigma}(n)q^n.
\]

The identity (5.3) follows after equating the coefficients of \( q^n \) on both sides of (5.7).

**Proof of (5.4).** By (2.28) and (2.49), we have

\[
q\psi^4(q^2) = p_{1,2}(q).
\]

Hence we have

\[
q\sum_{n=0}^{\infty} \delta_4(n)q^{2n} = \sum_{n=0}^{\infty} \tilde{\sigma}(2n+1)q^{2n+1},
\]

which is the identity (5.4).

**Proof of (5.5).** It is clear from (2.26) that

\[
\sum_{n=1}^{\infty} r_8(n)q^n = -16\sum_{n=1}^{\infty} \tilde{\sigma}_3(n)(-q)^n.
\]
Proof of (5.6). From (2.11), (2.22), and (2.37), we have

\[ 8q^2 \psi(q^2) = \frac{1}{16} - \frac{1}{16} Q(q^2) - Q_0(q). \]

Hence we derive

\[ \delta_8(n - 1)q^{2n} = \sum_{n=1}^{\infty} \tilde{\sigma}_3(n) q^{2n} - \sum_{n=1}^{\infty} \tilde{\sigma}_3(2n) q^{2n}. \]  

(5.10)

Equating the coefficients of \( q^n \) on both sides of (5.10), we obtain the desired result.

Remarks. Jacobi [13, 14, 15] showed that \( r_4(n) \) is 8 times the sum of the divisors of \( n \) that are not multiples of 4, that is,

\[ r_4(n) = 8(\sigma(n) - 4\sigma(n/4)). \]  

(5.11)

Many proofs of (5.11) have been given; see for example [1], [4, p. 15]. Spearman and Williams [24] gave the simplest arithmetic proof of this formula. If we use \( \tilde{\sigma}(n) = \sigma(n) - 2\sigma(n/2) \) from (1.13), then we note that our expression for \( r_4(n) \) in (5.3) is the same as (5.11). By the fact (1.14), we can express (5.4) as

\[ \delta_4(n) = \sigma(2n + 1). \]  

(5.12)

The formula (5.12) is proved in an elementary way [12, Theorem 10], and in using modular forms [18, Theorem 3]. The evaluation of \( \delta_4(n) \) goes back to Legendre [6], [10]. The formula (5.5) first appeared implicitly in the work of Jacobi [14], and explicitly in the work of Eisenstein [7]. Williams [25] gave an arithmetic proof of this formula by showing that

\[ r_8(n) = 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4). \]

Using the theory of modular forms, Ono, Robins, and Wahl [18, Theorem 5] derive a formula for \( \delta_8(n) \), namely

\[ \delta_8(n) = \sigma_3(n + 1) - \sigma_3((n + 1)/2). \]  

(5.13)

Formula (5.13) is also proved in an elementary way in [12, Theorem 12]. It is not hard to show that (5.13) is the same expression as (5.6). From (1.12), we deduce that

\[ \tilde{\sigma}_3(n) = \sigma_3(n) - 16\sigma_3(n/2). \]

Then

\[ 8\delta_8(n) = \tilde{\sigma}_3(n + 1) - \tilde{\sigma}_3(2(n + 1)) = \sigma_3(n + 1) - 16\sigma_3((n + 1)/2) \]

\[ - (\sigma_3(2(n + 1)) - 16\sigma_3(n + 1)) = 8(\sigma_3(n + 1) - \sigma_3((n + 1)/2)), \]

(5.15)

where in the last equality, we use the identity

\[ \sigma_3(2n) = 9\sigma_3(n) - 8\sigma_3(n/2). \]  

(5.16)

The identity (5.16) can be proved by letting \( n := 2^a N, N \) is odd, and then by considering the cases, \( a = 0 \) and \( a > 0 \). After dividing both sides of (5.13) by 8, we have the desired identity (5.13).
6. Some partition congruences

If \( r \) is a non–zero integer, we define the function \( p_r(n) \) by

\[
\sum_{n=0}^{\infty} p_r(n)q^n := \prod_{n=1}^{\infty} (1 - q^n)^r.
\]

Note that \( p_{-1}(n) = p(n) \), the ordinary partition function. A positive integer \( n \) has \( k \) colors if there are \( k \) copies of \( n \) available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called colored partitions. For example, if 1 is allowed to have 2 colors, say \( r(\text{red}) \), and \( g(\text{green}) \), then all colored partitions of 2 are \( 2, 1_r + 1_r, 1_g + 1_g, 1_r + 1_g \). Letting \( p_{e,r}(n) \) and \( p_{o,r}(n) \) denote the number of \( r \)–colored partitions into an even (respectively, odd) number of distinct parts, it is easy to see that

\[
p_r(n) = p_{e,r}(n) - p_{o,r}(n),
\]

when \( r \) is a positive integer.

We prove a congruence for the function \( \mu(n) \) which is defined by

\[
\sum_{n=0}^{\infty} \mu(n)q^n := \prod_{n=1}^{\infty} (1 - q^n)^8 (1 - q^{2n})^8.
\]

It follows that

\[
\mu(n) = \mu_e(n) - \mu_o(n),
\]

where \( \mu_e(n) \) and \( \mu_o(n) \) are the number of 16–colored partitions into an even (respectively, odd) number of distinct parts, where all the parts of the latter eight colors are even.

**Theorem 6.1.** If \( \mu(n) \) is defined by (6.4),

\[\mu(3n - 1) \equiv 0 \pmod{3} \]

We generally denote by \( J \) an integral power series in \( q \) whose coefficients are integers.

**Proof.** It is obvious from (1.16) that

\[\mathcal{E}(q) = 1 + 3J.\]

Also \( n^3 - n \equiv 0 \pmod{3} \), and so, from (1.15) and (1.17), we obtain

\[\mathcal{Q}(q) = \mathcal{P}(q) + 3J.\]

Hence

\[
(\mathcal{E}^2(q) - \mathcal{Q}(q))\mathcal{Q}(q) = (\mathcal{E}(q)(1 + 3J) - (\mathcal{P}(q) + 3J))\mathcal{Q}(q)
\]

\[
= \mathcal{E}(q)\mathcal{Q}(q) - \mathcal{P}(q)\mathcal{Q}(q) + 3J.
\]

By (2.44) and (2.51), we find that

\[
(\mathcal{E}^2(q) - \mathcal{Q}(q))\mathcal{Q}(q) = 64q^8(q)\varphi^8(-q) = 64qq\prod_{n=1}^{\infty} (1 - q^n)^8 (1 - q^{2n})^8,
\]
where the last equality comes from the fact [3, p. 39]
\[ \varphi(-q)\psi(q) = f(-q)f(-q^2), \]
where \( f(-q) \) is defined by (2.7). On the other hand, observe that, from (1.17) and (1.20),
\[
16 \sum_{n=1}^{\infty} n\tilde{\sigma}_3(n)q^n = -q \frac{dQ(q)}{dq} = \mathcal{E}(q)Q(q) - \mathcal{P}(q)Q(q).
\]
In summary, by (6.5), (6.6), and (6.7), we conclude that
\[
64 \sum_{n=0}^{\infty} \mu(n)q^{n+1} = 16 \sum_{n=1}^{\infty} n\tilde{\sigma}_3(n)q^n + 3J.
\]
But the coefficient of \( q^{3n} \) on the right side of (6.8) is a multiple of 3. So we obtain
\[ \mu(3n - 1) \equiv 0 \pmod{3}. \]
□

Secondly, we prove a congruence for the function \( \nu(n) \) which is defined by
\[
\sum_{n=0}^{\infty} \nu(n)q^n := \prod_{n=1}^{\infty} (1 - q^{2n})^8(1 + q^n)^8.
\]
Thus \( \nu(n) \) is the number of partitions of \( n \) into 16 colors, 8 appear at most once (say \( S_1 \)), and 8 are even and appear at most once (say \( S_2 \)), weighted by the parity of colors from the set \( S_2 \).

**Theorem 6.2.** If \( \nu(n) \) is defined by (6.9), then
\[ \nu(n - 1) \equiv \tilde{\sigma}_3(n) \pmod{3}. \]

**Proof.** Recall from (2.51) of Theorem 2.4 that
\[
\mathcal{E}^2(q) - Q(q) = 64q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^8}{(1 - q^{2n-1})^8} = 64q \prod_{n=1}^{\infty} (1 - q^{2n})^8(1 + q^n)^8,
\]
where, in the last equality, we used the fact [3, (22.3)]
\[ \prod_{n=1}^{\infty} (1 + q^n) = \prod_{n=1}^{\infty} (1 - q^{2n-1})^{-1}. \]
Then, by (6.9) and (6.10), we deduce that
\[
64 \sum_{n=0}^{\infty} \nu(n)q^{n+1} = 48 \sum_{n=1}^{\infty} \tilde{\sigma}(n)q^n + 576 \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \tilde{\sigma}(m)\tilde{\sigma}(n - m)q^n + 16 \sum_{n=1}^{\infty} \tilde{\sigma}_3(n)q^n.
\]
Comparing the coefficients of \( q^n \) on both sides of the above equation, we obtain the identity
\[ 4\nu(n - 1) = 3\tilde{\sigma}(n) + \tilde{\sigma}_3(n) + 36 \sum_{m=1}^{n-1} \tilde{\sigma}(m)\tilde{\sigma}(n - m). \]
We then deduce that

\[ \nu(n - 1) \equiv \tilde{\sigma}_3(n) \pmod{3}. \]

\[ \square \]

**Acknowledgement.** I am deeply indebted to Professors B. C. Berndt and K. S. Williams for their helpful comments and suggestions.

**REFERENCES**

[1] G. E. Andrews, S. B. Ekhad, and D. Zeilberger, *A short proof of Jacobi’s formula for the number of representations of an integer as a sum of four squares*, Amer. Math. Monthly **100** (1993), 274–276.

[2] B. C. Berndt, *Ramanujan’s Notebooks, Part II*, Springer–Verlag, New York, 1989.

[3] B. C. Berndt, *Ramanujan’s Notebooks, Part III*, Springer–Verlag, New York, 1991.

[4] B. C. Berndt, *Ramanujan’s theory of theta-functions*, CRM proceedings and lecture notes **1** (1993), 1–63.

[5] N. Cheng and K. S. Williams, *Some convolution sums involving divisor function*, Proc. Edinburgh Math. Soc., to appear.

[6] L. Dickson, *Theory of numbers, Vol. III*, Chelsea, New York, 1952.

[7] G. Eisenstein, *Neue Theoreme der höheren Arithmetik*, J. Reine Angew. Math. **35** (1847), 117–136.

[8] J. W. L. Glaisher, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14** (1884), 156–163.

[9] J. W. L. Glaisher, *On certain sums of products of quantities depending upon the divisors of a number*, Mess. Math. **15** (1885), 1–20.

[10] J. W. L. Glaisher, *Expressions for the first five powers of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **15** (1885), 33–36.

[11] H. Hahn, *Eisenstein series, analogues of the Rogers–Ramanujan functions, and partition identities*, Ph. D. Thesis, University of Illinois at Urbana–Champaign, 2004.

[12] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams, *Elementary evaluation of certain convolution sums involving divisor functions*, Number Theory for the Millennium, edited by M. A. Bennett, B. C. Berndt, N. Boston, H. G. Diamond, A. J. Hildebrand, and W. Philipp, A. K. Peters, Natick, Massachusetts, 2002, 229–274.

[13] C. G. J. Jacobi, *Note sur la decomposition d’un nombre donné en quatre carrés*, J. Reine Angew. Math. **3** (1828), 191; Werke, Vol. I, p. 247.

[14] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, 1829; Werke, Vol. I, pp. 49–239.

[15] C. G. J. Jacobi, *De compositione numerorum e quator quadratis*, J. Reine Angew. Math. **12** (1834), 167–172; Werke, Vol. VI, pp. 245–251.

[16] A. M. Legendre. *Traité des fonctions elliptiques et des intégrales Euleriennes, Vol. III*, Huzard-Courcier, Paris, 1828.

[17] J. Liouville, *Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres* (fifth article), J. Math. Pures Appl. **3** (1858), 273–288.

[18] K. Ono, S. Robins, and P. T. Wahl, *On the representation of integers as sums of triangular numbers*, Aequationes Math. **50** (1995), 73–94.

[19] V. Ramamani, *Some Identities Conjectured by Srinivasa Ramanujan in His Lithographed Notes Connected with Partition Theory and Elliptic Modular Functions—Their Proofs—Inter Connection with Various Other Topics in the Theory of Numbers and Some Generalizations*, Doctoral Thesis, University of Mysore, 1970.
[20] V. Ramamani, *On some algebraic identities connected with Ramanujan’s work*, in *Ramanujan International Symposium on Analysis*, N. K. Thakare, ed., Macmillan India, Delhi, 1989, 279–291.

[21] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. 22 (1916), 159–184.

[22] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927, reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.

[23] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[24] B. K. Spearman and K. S. Williams, *The simplest arithmetic proof of Jacobi’s four squares theorem*, Far East J. Math. Sci. (FJMS) 2 (2000), 433–439.

[25] K. S. Williams, *An arithmetic proof of Jacobi’s eight squares theorem*, Far East J. Math. Sci. (FJMS) 3 (2001), 1001–1005.

**Department of Mathematics, University of Rochester, Rochester, NY 14627 USA**

**E-mail address:** hahn@math.rochester.edu