Anharmonic Bloch Oscillations in the Optical Waveguide Array

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Abstract

The anharmonic Bloch oscillations of a light beam in the array of optical waveguides are considered. The coupling modes model (CMM) with the second order interaction is used to describe the effect analytically. The formula obtained predicts explicitly the path of the optical beam, in particular, the positions of the turning points are found. A total agreement of this formula with the numerical simulation is confirmed.

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I. INTRODUCTION

Today the arrays of coupling optical waveguides attract the increasing interest since they allow to control the behavior of the optical signal effectively. Besides, such systems exhibit many interesting phenomena similar to those that may occur in solids, such as Bloch oscillations, Zener tunneling, Anderson localization, dynamic localization, etc.

Usually, for the numerical simulations of the optical phenomena in the arrays of interacting waveguides the coupling modes model (CMM) is used. This model describes the behavior of light quite exactly if the coupling is enough weak.

But the theoretical research can be considerably simplified if the numerical simulation can be replaced with an analytical formula. In particular, for the periodical arrays of the identical waveguides the equations of CMM can be solved analytically. The analytical solutions were obtained for one-dimensional and two-dimensional arrays, and the long-range interaction was taken into account.

Another problem which can be solved analytically is the array of waveguides with the optical properties (such as radii or refractive indices) gradually changing as one passes from one waveguide to another. This problem was considered in. One of the main results of this work is the analytical expression for the path of the optical beam. The obtained formula predicts that the path of the optical beam takes the periodical oscillating form. This analytical result was confirmed with the experiments. This phenomenon is known as optical Bloch oscillations. It is the optical analog of the Bloch oscillations of an electron in an ordinary crystal placed in an external electric field.

Generally, the optical Bloch oscillations, as well as the other optical effects, are investigated for the plane arrays of the waveguides, since such arrays are quite simple for fabrication. Besides, such systems are the most convenient for the numerical simulation, since the coupling of the adjacent waveguides only should be taken into account.

However, today the increasing attention is drawn to the more complex structures, namely zigzag arrays of waveguides (see Fig. 1). Such arrays are interesting due to the significant role of the second order coupling. In particular, in works the influence of the second order coupling on the effects of diffraction and the formation of solitons was investigated. Besides, recently the investigations of the optical Bloch oscillations in zigzag arrays were published. It was shown that in such systems the signal propagates along a complex periodic trajectory. This phenomenon is known as a non-trivial, or anharmonic Bloch oscillations.
The anharmonic Bloch oscillations are the subject of our work. In this paper we generalize the results of [2] to the case of the zigzag arrays. We consider the system of equations of the CMM taking into account the second order coupling. We find the analytical solution of this system of equations and obtain the formula for the path of the optical beam. The obtained formula allows to predict the geometric parameters of the anharmonic Bloch oscillations, such as the period of the optical beam path and the positions of the turning points. We demonstrate that the analytical formula is consistent with the numerical calculations.

II. ANALYTICAL DERIVATION OF THE ANHARMONIC BLOCH OSCILLATIONS BY MEANS OF THE COUPLING MODES MODEL

Here we represent the analytical derivation of the optical beam path in a zigzag array. Our derivation is based on the coupling mode equations with second order coupling. Note that the calculation below is a generalization of that represented in [2], where the ordinary Bloch oscillations are considered.

We start from the equation of the coupling modes

\[
\left(i \frac{d}{dz} + \beta_0^{(0)} + \alpha j\right) a_j(z) + \gamma_1 \left(a_{j-1}(z) + a_{j+1}(z)\right) + \gamma_2 \left(a_{j-2}(z) + a_{j+2}(z)\right) = 0. \tag{1}
\]

Here

\[
\beta_0^{(0)} = \beta_0^{(0)} + \alpha j, \tag{2}
\]

For the simplicity, we assume that \(\beta_0^{(0)} = 0\). Note that \(\beta_0^{(0)}\) can be easily excluded from the equations by the replacement \(a_j(z) = a'_j(z) e^{i\varphi_0(z)}\).

\[
\left(i \frac{d}{dz} + \alpha j\right) a_j(z) + \gamma_1 \left(a_{j-1}(z) + a_{j+1}(z)\right) + \gamma_2 \left(a_{j-2}(z) + a_{j+2}(z)\right) = 0. \tag{3}
\]
We perform a Fourier transform of $a_j(z)$:

$$
\tilde{a}(k, z) = \frac{1}{\sqrt{2\pi}} \sum_j a_j(z) \, e^{-ikj},
$$

(4)

$$
a_j(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dk \, \tilde{a}(k, z) \, e^{ikj}.
$$

(5)

From Eq. (4) one can obtain an equation for $\tilde{a}(k, z)$:

$$
\left( i \frac{d}{dz} + \frac{1}{\alpha} \frac{d}{dk} \right) \tilde{a}(k, z) + 2 \left( \gamma_1 \cos k + \gamma_2 \cos 2k \right) \tilde{a}(k, z) = 0.
$$

(6)

Let us consider the eigenmode with a certain $\beta$:

$$
\tilde{a}_\beta(k, z) = \tilde{a}_\beta(k) \, e^{i\beta z}.
$$

(7)

Substituting (7) to (6), one obtains:

$$
\left( \frac{i}{\alpha} \frac{d}{dk} - \beta + 2\gamma_1 \cos k + 2\gamma_2 \cos 2k \right) \tilde{a}_\beta(k) = 0.
$$

(8)

Equation (8) possesses the following solution:

$$
\tilde{a}_\beta(k) = \exp \left\{ -\frac{i}{\alpha} \left( \beta k - 2\gamma_1 \sin k - \gamma_2 \sin 2k \right) \right\}.
$$

(9)

The function $\tilde{a}_\beta(k)$ has to be periodic in $k$. Thus, $\beta = \alpha n$, where $n$ is integer. The function $\tilde{a}_\beta(k)$ for a certain $n$ is

$$
\tilde{a}_n(k) = \exp \left\{ -i n k + \frac{2i\gamma_1}{\alpha} \sin k + \frac{i\gamma_2}{\alpha} \sin 2k \right\}.
$$

(10)

Now let us turn to the general solution of Eq. (6). Any solution $\tilde{a}(k, z)$ can be represented as a superposition of eigenmodes (7):

$$
\tilde{a}(k, z) = \sum_n C_n \tilde{a}_n(k) \, e^{i\alpha n z}.
$$

(11)

The coefficients $C_n$ can be obtained from the boundary conditions $\tilde{a}(k, z = 0) = \tilde{a}^0(k)$. From (11) it follows that

$$
\tilde{a}^0(k) = \sum_n C_n \tilde{a}_n(k) = \sum_n C_n \exp \left\{ -i n k + \frac{2i\gamma_1}{\alpha} \sin k + \frac{i\gamma_2}{\alpha} \sin 2k \right\}.
$$

(12)
Thus,

\[ C_n = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \tilde{a}^0(k) \exp \left\{ ink - \frac{2i\gamma_1}{\alpha} \sin k - \frac{i\gamma_2}{\alpha} \sin 2k \right\}. \]  
\hspace{1cm} (13)

Substituting (13) to (12), one obtains:

\[ \tilde{a}(k, z) = \sum_n \int_{-\pi}^{\pi} \frac{dk'}{2\pi} \tilde{a}^0(k_{\text{in}}(k' - k + \alpha z) \times \]
\[ \times \exp \left\{ \frac{2i\gamma_1}{\alpha} \left( \sin k - \sin k' \right) + \frac{i\gamma_2}{\alpha} \left( \sin 2k - \sin 2k' \right) \right\}. \]  
\hspace{1cm} (14)

Since

\[ \sum_n e^{in(k' - k + \alpha z)} = 2\pi \delta(k' - k + \alpha z), \]  
\hspace{1cm} (15)

Eq. (14) takes the form:

\[ \tilde{a}(k, z) = \tilde{a}^0(k - \alpha z) \times \]
\[ \times \exp \left\{ \frac{2i\gamma_1}{\alpha} \left( \sin k - \sin (k - \alpha z) \right) + \frac{i\gamma_2}{\alpha} \left( \sin 2k - \sin 2(k - \alpha z) \right) \right\}. \]  
\hspace{1cm} (16)

Let us assume that the function \( \tilde{a}^0(k) \) possesses a sharp peak near some \( k_0 \). Then, the function at the right-hand side of Eq. (16) can be expanded into a Taylor series around \( (k - \alpha z) - k_0 \):

\[ \tilde{a}(k, z) = \tilde{a}^0(k - \alpha z) e^{i\phi(z) + i(k - \alpha z) \psi(z)} \]  
\hspace{1cm} (17)

where

\[ \phi(z) = \frac{2\gamma_1}{\alpha} \left( \sin (k_0 + \alpha z) - \sin k_0 \right) + \frac{\gamma_2}{\alpha} \left( \sin 2(k_0 + \alpha z) - \sin 2k_0 \right), \]  
\hspace{1cm} (18)

\[ \psi(z) = \frac{2\gamma_1}{\alpha} \left( \cos (k_0 + \alpha z) - \cos k_0 \right) + \frac{2\gamma_2}{\alpha} \left( \cos 2(k_0 + \alpha z) - \cos 2k_0 \right). \]  
\hspace{1cm} (19)

Substitute (17) to (13):

\[ a_j(z) = e^{i\phi(z) + i(k_0 + \alpha z) j} \int_{-\pi}^{\pi} \frac{dk}{\sqrt{2\pi}} \tilde{a}^0(k - \alpha z) e^{i(k - \alpha z - k_0) (j + \psi(z))}. \]  
\hspace{1cm} (20)

The intensity of the optical beam at the \( j \)-th waveguide is defined as \( I_j(z) = |a_j(z)|^2 \). Thus,

\[ I_j(z) = \left| \int_{-\pi - k_0 - \alpha z}^{\pi - k_0 - \alpha z} \frac{dk_1}{\sqrt{2\pi}} a^0(k_0 + k_1) e^{ik_1(j + \psi(z))} \right|^2, \]  
\hspace{1cm} (21)

where \( k_1 = k - \alpha z - k_0 \).
Since $\tilde{a}^0(k)$ has a sharp peak at $k = k_0$, the dependence of the integral limits on $z$ does not affect the integral value. So, we can write

$$I_j(z) = I(j + \psi(z))$$

(22)

Thus, the trajectory of the optical beam is defined with the equation $j + \psi(z) = \text{const}$. This equation results in

$$j(z) = j_0 - \frac{2\gamma_1}{\alpha} \left( \cos(k_0 + \alpha z) - \cos k_0 \right) - \frac{2\gamma_2}{\alpha} \left( \cos 2(k_0 + \alpha z) - \cos 2k_0 \right).$$

(23)

Eq. (23) represents the trajectory of the optical beam in the zigzag array of waveguides.

III. QUALITATIVE ANALYSIS OF THE ANALYTICAL FORMULA

Let us analyze the properties of the trajectory described with Eq. (23). For the simplicity, we consider the special case $j_0 = 0$ and $k_0 = 0$. For this case Eq. (23) takes the form

$$j(z) = \frac{2\gamma_1}{\alpha} \left( 1 - \cos \alpha z \right) + \frac{2\gamma_2}{\alpha} \left( 1 - \cos 2\alpha z \right).$$

(24)

It is obvious that $j(z)$ is the periodical function with the period $2\pi/\alpha$. Below we consider the form of the trajectory in a single period, $0 \leq z < 2\pi/\alpha$. We assume that the coupling constants $\gamma_1$ and $\gamma_2$ are of the same sign (both positive or both negative). This assumption is consistent with experiments.

Let us find the turning points of the trajectory. For this purpose we have to solve the equation

$$\frac{dj}{dz}(z) = 2\gamma_1 \sin \alpha z + 4\gamma_2 \sin 2\alpha z = 2\gamma_1 \sin \alpha z \left( 1 + \frac{4\gamma_2}{\gamma_1} \cos \alpha z \right) = 0.$$  

(25)

It follows from (25), that the turning points are defined with two equations:

$$\sin \alpha z = 0,$$

(26)

$$\cos \alpha z = -\frac{\gamma_1}{4\gamma_2}.$$  

(27)

We should consider two different cases, namely $\gamma_1/4\gamma_2 > 1$ and $\gamma_1/4\gamma_2 < 1$.

Let us begin with the case $\gamma_1/4\gamma_2 > 1$. For this case Eq. (27) has no solutions, and all the turning points are defined with Eq. (26) only. In a single period $0 \leq z < 2\pi/\alpha$ Eq. (26) possesses two solutions, $z_O = 0$ and $z_A = \pi/\alpha$. Thus, in one period of the trajectory there are two turning points. The schematic image of the trajectory is represented in Fig. 2(a). In this figure, the
turning points are designated with the letters O and A. Substituting the values \( z_O \) and \( z_A \) to Eq. (24), one can find the number \( j \) of a waveguide where the trajectory changes the direction: \( j_O = 0 \) and \( j_A = 4 \gamma_1 / \alpha \).

Now let us turn to the other case, \( \gamma_1 / 4 \gamma_2 < 1 \). The schematic image of the trajectory for this case is represented in Fig. 2(b). For this case, the both equations (26) and (27) possess the solutions. Thus, in a period \( 0 \leq z < 2 \pi / \alpha \) the trajectory possesses four turning points. Two of them are the solutions of (26), they are designated as O and A, and their coordinates \( \{ z_O, j_O \} \) and \( \{ z_A, j_A \} \) are obtained above. Two other turning points are denoted with B and C. Their coordinates \( z \) are the solutions of Eq. (27): \( z_B = \frac{1}{\alpha} \left( \pi - \arccos \frac{\gamma_1}{4 \gamma_2} \right) \) and \( z_C = \frac{1}{\alpha} \left( \pi + \arccos \frac{\gamma_1}{4 \gamma_2} \right) \). Substituting \( z_B \) and \( z_C \) to (24), one obtains \( j_B = j_C = \frac{4 \gamma_2}{\alpha} \left( 1 + \frac{\gamma_1}{4 \gamma_2} \right)^2 \).

IV. COMPARISON OF THE ANALYTICAL FORMULA WITH THE NUMERICAL CALCULATION

Below we compare the trajectory described with the analytical formula (23) with the numerical calculation based on Eq. (1). For this purpose we take the same parameters as in the paper [21]. The calculations are produced for \( \alpha = 0.2, \gamma_1 = 1 \), and for two values of the parameter \( \gamma_2 = 0 \) and
Figure 3: The trajectory of the optical beam. Left: $\gamma_2 = 0$; right: $\gamma_2 = 0.7$.

For the numerical solution of Eq. (1) we have to specify some boundary conditions $a_j(z)$ for $z = 0$. We take the boundary conditions in the form of the Gaussian beam

$$a_j(0) = e^{-\frac{(j-j_0)^2}{\sigma^2}},$$  \hspace{1cm} (28)

where $j_0 = 10$ and $\sigma = 4$. The Fourier transform $\hat{a}^0(k) = \hat{a}(k, z = 0)$ defined by Eq. (1) takes the form

$$\hat{a}^0(k) = e^{-\frac{k^2}{\sigma^2}}.$$  \hspace{1cm} (29)

This function possesses a sharp peak near $k = 0$. Thus, in the analytical formula (23) we take $k_0 = 0$.

The results of our calculations are represented in Fig. 3. The white trail illustrates the Gaussian beam path calculated numerically by means of Eq. (1). The dotted line is the trajectory calculated with the analytical formula (23). The comparison of the numerical and analytical calculations confirms that the analytical formula describes the trajectory of the Gaussian beam exactly.
V. CONCLUSION

In this paper we investigated the anharmonic Bloch oscillations by means of the coupling modes model. For this problem, the equations of the CMM can be solved analytically. This circumstance allowed us to obtain the analytical expression for the path of the optical beam.

However, to use the obtained formula, one has to define the parameters of the CMM, namely, propagation constants and coupling constants. Typically, the values of these parameters are obtained experimentally.

Recently we proposed a method to calculate the parameters of the CMM by means of the multiple scattering formalism [9]. This allows to simplify significantly the investigation of the optical Bloch oscillations.

Besides, let us notice that the CMM is approximate model, and can be used only for the weak interaction between the waveguides. Otherwise, one should use the exact methods of numerical simulation. In this case, the analytical formula obtained in this work may be inapplicable.

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VI. CONCLUSION

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