Numerical models in applied plasmadynamic and plasmastatic problems

K V Brushlinskii\textsuperscript{1,3} and E V Stepin\textsuperscript{1,2,4}

1 Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya sq. 4, 125047 Moscow, Russia
2 Author to whom any correspondence should be addressed.
3 brush@keldysh.ru
4 eugene.v.stepin@gmail.com

Abstract. We present a review of mathematical models and computations in the two plasma physic projects: creation of powerful plasma accelerators and researches of equilibrium plasma configurations in magnetic traps. Steady-state transonic plasma flows in accelerators are investigated in the plasmadynamic models. The recent time works deal with studies of the longitudinal magnetic field influence on the plasma acceleration process in the channels of various geometry. Equilibrium magnetoplasma configurations in the traps are researched in plasmastatic models based on the numerical solution of problems with the Grad-Shafranov equation. Configurations are investigated in the galateya-traps with current carrying conductors immersed into the plasma. In the recent time works, configurations in toroidal magnetic traps are studied in comparison with their analogues straightened into the cylinder.

1. Introduction

Various plasma physical problems essentially use mathematical models of processes to be investigated and computations by means of highly efficient computers. They can lighten and quicken theoretical problem resolution and allow to save on expensive and sometimes impossible experiments. In this paper, we consider the models of actual problems relating to the two projects in the modern plasma techniques. They both were pioneered by prof. A.I. Morozov. A lot of works in theoretical justification, experimental realization and relevant computation of physical processes in plasma plants were carried out successfully under his supervision and with his personal participation.

Two types of models are considered. Plasmadynamic ones deal with investigations of plasma flows in coaxial channels of plasma accelerators. They are described by means of two-dimensional time-dependent problems with the magnetohydrodynamic (MHD) equations. Their numerical solution promoted to create the quasi-stationary high-current plasma accelerator (QSPA) and became a part of the de Laval nozzle magnetic analogue theory. Ongoing researches are concentrated on the characteristics of flows in the presence of a longitudinal magnetic field that is induced additionally by means of external current carrying conductors. The interest is focused on the influence of the channel geometry, i.e., of the curvature of electrodes forming the channel, on the acceleration process and on the flow character with different plasma velocity ratio to the different magnetosonic speeds.
Plasmastatic models take part in studies of equilibrium plasma configurations in magnetic traps that are one of the basic investigation objects in the nuclear fusion problem. If the trap can assume any symmetry (plane, axial or helical), the equilibrium models in the same MHD-approach are based on boundary value problems with the scalar elliptic type differential Grad-Shafranov equation for the magnetic flux function. One of the peculiarities of these problems lies in the fact that they are in some sense underdetermined, because they include two arbitrary functions describing the plasma pressure and electric current distributions among magnetic surfaces. Secondly, some nontrivial questions on the solution existence, uniqueness and stability are related to these problems. Such questions are common in the large class of reaction and diffusion interaction mathematical models. Their analysis is closely connected with spectral properties of linearized equation differential operators. Plasmastatic models of equilibrium configurations in traps are discussed below with examples of galateya-traps, in which current carrying conductors, creating the magnetic field of various geometry, are immersed into the plasma volume. In the recent works, computation results for configurations in toroidal traps have been compared with ones for their analogues straightened into the cylinder.

2. Numerical models of flows in plasma accelerator channels

2.1. Coaxial plasma accelerator scheme

The subject matter of plasma acceleration by a magnetic field is permanently present in the scientific literature, beginning with the theoretical paper by A.I. Morozov [1] and with the first experiments by L.A. Artsimovich et al. [2] published in 1958. It is concentrated largely on the elaboration of coaxial plasma accelerators (figure 1).

![Figure 1. Scheme of plasma accelerator.](image)

Plasma enters the channel, having a ring-shaped form, between two electrodes and is accelerated in the axial direction by means of the Ampere force owing to the interaction of the radial electric current and the azimuthal magnetic field. The channel is nozzle-shaped: the cross-section area decreases in its input part and increases in the output one. Therefore, plasma can be accelerated in the channel up to the velocity values exceeding the fast magnetosonic speed.

The discharge time of the power source is far above the flow time, and hence the researches are focused on the steady-state flows being obtained in time after the relaxation process.

Parameters of the accelerator allow to consider plasma as a continuous medium and to use the magnetohydrodynamic (MHD) equations in the mathematical models. It is convenient to deal with them in the dimensionless form: all values are measured by means of the units composed from dimension constants in the problem statement (see, e.g., [3]).
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) &= 0 \\
\rho \frac{d\mathbf{v}}{dt} + \nabla p &= \mathbf{j} \times \mathbf{H} \\
\rho \frac{d\varepsilon}{dt} + p \text{ div } \mathbf{v} &= \mathbf{v} j^2 \\
\frac{\partial \mathbf{H}}{\partial t} &= \text{rot} (\mathbf{v} \times \mathbf{H}) - \text{rot} (\mathbf{v} j),
\end{align*}
\]

where \( \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \), \( p = (\gamma - 1) \rho \varepsilon = \frac{\beta}{2} \rho T \), \( \mathbf{j} = \text{rot} \mathbf{H} \).

The equations include two dimensionless parameters
\[
\beta = \frac{8\pi p_0}{H_0^2}, \quad \nu = \frac{c^2}{4\pi \sigma r_0 \nu_0} = \frac{1}{\text{Re}_m},
\]
where \( p_0, H_0, r_0, \nu_0 \) are typical values of corresponding variables, \( \sigma \) is the plasma electric conductivity. Their physical meaning is the ratio of the gas and magnetic pressures at the channel input and the magnetic viscosity, inverse to the magnetic Reynolds number.

The accelerator geometry and plasma flows admit to be restricted by the axial symmetry assumption: \( \partial / \partial \varphi \equiv 0 \) in the cylindrical coordinates \((r, \varphi, z)\).

2.2. Plasma acceleration in the transverse magnetic field

In most of plants that were constructed and experimentally investigated, the plasma acceleration took place only in the own magnetic field, transversal with respect to the flow. In this case
\[
H_z \equiv H_r \equiv 0, \quad \nu_y \equiv 0.
\]

Two-dimensional initial-boundary problems with the equations (1) are formulated in the area \((z, r)\) at the figure 1.

The boundary conditions at the input correspond to the parameters of plasma flowing into the channel and to the value of the radial electric current:
\[
z = 0: \quad \rho = 1, \quad T = 1, \quad H_\varphi = r_0 / r,
\]
where \( r_0 \) is the average radius value at the input cross-section. The direction \( \nu_r / \nu_z \) of flow trajectories are also given. At the output \((z = 1)\), a condition for the parabolic equation with a magnetic field \( \mathbf{H} \) can be given only in the case of a finite plasma conductivity \((\nu > 0)\). Conditions for the velocity are not required, if we assume that the flow is supersonic at the channel output. Conditions on the electrodes \( r = r_t(z), \quad r = r_r(z) \), forming the channel, assume them to be impermeable and equipotential:
\[
\nu_\tau = 0, \quad j_\varphi = 0,
\]
where \( \tau \) and \( n \) are tangent and normal directions at the boundary.

To solve the problem more convenient, we recommend to straighten the curvilinear boundaries by means of the new variables \((z, y)\):
\[
z = z, \quad r = (1 - y)r_t(z) + yr_r(z).
\]

In the problem solving, a steady-state flow regime of most interest relaxes in time. Therefore, initial conditions at \( t = 0 \) may be rather arbitrary as long as they provide the initial plasma acceleration along the \( z \)-axis. The problem statement and its numerical solving method are discussed in [3].
Basic simple properties of steady-state plasma flows in channels are studied in the infinite conductivity approach \((v = 0)\). The computation results are related to the acceleration efficiency and to its dependence on the channel geometry and on the problem parameters. The radial range of the velocity is evaluated. A zone with compressed and heated plasma behind the conical shock wave is researched at the axis of a channel with the shortened central electrode. These results comply with the experimental data in a range of moderate values of the discharge current. After appreciable current increasing, the computation results still show the expected flows with increasing acceleration parameters, but they differ from the experimental ones. In reality, the flow is no longer regular near the electrodes: the current in plasma deviates from the radial direction and forms some vortexes, and plasma density decreases near anode that prevents the acceleration to be efficient. The flow character strongly depends on the electrode polarity that is not taken into account in the MHD-approach.

To consider this phenomena in computation, the MHD-model was modified: it includes the finite plasma conductivity \((v > 0)\) and the Hall effect, i.e., the difference between electron and ion velocities:

\[
\rho \left( \mathbf{v}^i - \mathbf{v}^e \right) = \xi \mathbf{j}, \quad \xi = \frac{m_i}{e r_0 \sqrt{4 \pi r_0}},
\]

where \(m_i\) and \(e\) are an ion mass and the electron charge.

Hence, Ohm’s law includes the interaction between the electric current and the magnetic field. The main element of the model modification is the additional term in the right-hand side of the magnetic induction equation in the system (1):

\[
-\xi \text{rot} \left( \mathbf{j} \times \mathbf{H} - \nabla p/\rho \right).
\]

The quantitative measure of the Hall effect is the dimensionless parameter \(\xi\) (see (3)). Now, a steady-state flow regime is established in computation with a current deviation from the radius and electric potential jump near the electrodes, but only under the restriction \(\xi < \xi_{cr}(v)\) with monotone function \(\xi_{cr}(v)\). If \(\xi > \xi_{cr}(v)\), the flow loses stability. In the case of infinite conductivity \((v = 0)\), the initial value problem under discussion may become ill-posed.

The electrode anomalies mentioned above were overcome in the ion (but not electron in the simplest acceleration scheme) current transfer: the transparent electrodes are used instead of the solid ones, and an additional plasma flux enters the channel through the anode compensating the depression caused by the Hall effect. This complication of the device and the regular acceleration regime therein are justified by computations made in terms of the MHD-model taking into account the Hall effect.

Computations of fully ionized flow have been discussed up to now. In addition, some investigations about modelling of the neutral gas ionization process in the input section of accelerator channels should be mentioned (see [3] and the papers by A.N. Kozlov et al. [4, 5]).

The above-mentioned results make up a significant part of the basic theory describing plasma flows in the de Laval nozzle MHD analogue. They promoted also the successful elaboration of some accelerator devices, including the quasi-stationary high-current plasma accelerator (QSPA), that presented experimentally record parameters of velocity and energy of an output stream [6, 7]. Mathematical models and computations of plasma flows in channels are presented in [3, 8, 9] with essential bibliography.

2.3. Longitudinal magnetic field influence on the plasma flow

In the recent time works, the attention is focused on the plasma flows in coaxial channels in the presence of a longitudinal magnetic field, which can be additionally induced by means of external current carrying conductors enclosing the device. In this case, the two-dimensional MHD-model with the equations (1) includes all the three components of the magnetic field \(\mathbf{H}\) and the velocity \(\mathbf{v}\), since the longitudinal field rotates plasma around the channel axis.
The main characteristics of the acceleration and other flow features are studied in the quasi-one-dimensional approach. Steady-state flows in horizontal narrow tubes of the variable cross-section area \( S(z) \) are described by the ordinary differential equations:

\[
\begin{align*}
\frac{d\rho}{dz} &= -\frac{u^2-C_A^2}{(u^2-C_s^2)(u^2-C_r^2)} \rho u_z^2 \frac{dS}{dz} \\
\frac{du}{dz} &= \frac{C_r^2\left(u^2-C_s^2\right)+u^2 H_s^2/\rho}{(u^2-C_s^2)(u^2-C_r^2)} u \frac{dS}{dz} \\
\frac{dH_s}{dz} &= -\frac{H_s u^4}{(u^2-C_s^2)(u^2-C_r^2)} \frac{1}{S} \frac{dS}{dz} \\
\frac{dw}{dz} &= \frac{1}{\rho u} \frac{dH_s}{dz} - \frac{1}{\rho} \frac{dp}{dz} = 0; \quad \frac{d}{dz} H_z S = 0,
\end{align*}
\]

where \( u = \nu_z, \ w = \nu_r, \ C_s^2 = \gamma p/\rho, \ C_r, \ C_A \) are the slow, fast and Alfven magnetic sound speeds in the \( z \) – direction [3]. The equations have two singularities at \( u = C_s \) and \( u = C_r \), unlike gas and even MHD in the transverse field flows. Therefore, continuous solutions may be, in general, only subsonic or supersonic with respect to \( C_s \) and \( C_r \). The most interesting for plasma accelerating transonic flows may exist in nozzles, where plasma velocity crosses the sound speed in the minimum cross-section at \( S'(z) = 0 \). Moreover, in horizontal tubes, the transition over \( C_A \) is impossible: if \( u = C_A \) in any point, the velocity is identically equals to the Alfven one with a constant density. The steady-state flows are classified in 9 types [3, 10]. Four types are super-Alfvenic \( (u > C_A > C_s) \): supersonic \( (u > C_r) \), subsonic \( (u < C_r) \) and transonic relative to \( C_r \), accelerating or decelerating. The fifth type is the above-mentioned Alfvenic, whose peculiarities and relaxation process are considered in [11]. The other four types are sub-Alfvenic \( (u < C_A < C_r) \) and differ from one another by the ratio between the velocity \( u \) and the slow sound speed \( C_s \).

A two-dimensional MHD-model with the longitudinal magnetic field is realized in computations of flows in the nozzle-type channels of plasma accelerators [3, 12, 13]. The researches are aimed to the steady-state super-Alfvenic transonic accelerating flows and to their dependence on the ratio between the longitudinal and the transverse field typical values and on the preshaped electrode curvature. It follows from the computation results that the admissible in super-Alfvenic flows longitudinal field can to press plasma to the external electrode and to deflect the current from the radial direction in the opposite to the Hall effect way, i.e., it is capable to neutralize its negative influence on the plasma acceleration. The comparison of flows in the channels with several electrode geometry allows to assert that the acceleration with convex-inward central electrode is more efficient then in the opposite case with convex external one.

3. **Equilibrium plasma configurations in the magnetic traps**

Plasmastatic models of plasma equilibrium in the magnetic field in traps use the same continuous medium mechanics approach. Equilibrium plasma configurations are entirely defined by the distribution of the pressure \( p \), magnetic field \( \mathbf{H} \) and current density \( \mathbf{j} \) in the trap space. They satisfy the equilibrium MHD-equation and Maxwell ones:

\[
\nabla p = \mathbf{j} \times \mathbf{H}; \quad \mathbf{j} = \text{rot} \mathbf{H}; \quad \text{div} \mathbf{H} = 0
\]
Configuration models based on these equations are simple enough, well-known and being used widely in two-dimensional problems allowing any symmetry of the trap and its elements. Toroidal traps with axial symmetry $\partial / \partial \phi \equiv 0$ in the cylinder coordinates are of special interest. In this case, the system of equations (5) may be reduced to the single scalar Grad-Shafranov equation for the magnetic flux function $\psi$ [14, 15]:

$$
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} + r^2 \frac{\partial p}{\partial \psi} + I \frac{\partial I}{\partial \psi} = 0,
\end{align*}
$$

where $r_{H_r} = -\frac{\partial \psi}{\partial z}$, $r_{H_z} = \frac{\partial \psi}{\partial r}$, $I = r_{H_r}$.

It includes two functions $p(\psi)$ and $I(\psi)$, which are to be given at the problem statement by means of any additional information about the desired solution. The arbitrary choice of these two functions makes boundary problems with the equation (6) underdetermined. The reason of this fact is that the strict derivation of this equation from (5) doesn’t use the magnetic displacement equation (1) in equilibrium $\text{rot} \, \mathbf{j} = 0$, i.e., doesn’t take into account the finite plasma conductivity. Hence, the Grad-Shafranov equation describes the idealized model of infinitely conductive plasma. This ideal situation must be considered as quasi-equilibrium, because it spreads out due to magnetic field diffusion (1) with small value $\nu \ll 1$. The functions $p(\psi)$ and $I(\psi)$ can be defined only in the configuration formation process, and they depend on the formation conditions [16, 17]. Nevertheless, boundary problems with the Grad-Shafranov equations are employed successfully in investigations of ideal equilibrium configuration models, where the functions $p(\psi)$ and $I(\psi)$ are being chosen in accordance with a requirement to the trap under consideration.

As an example, we refer the models of so-called galateya-traps with current carrying conductors immersed into the plasma [18, 19]. In the Galateya-belt, the magnetic field is induced by two parallel ring-shaped conductors inside the plasma torus [20]. Main features of equilibrium configurations and their dependence on the problem parameters are considered in the trap analogue straightened into the cylinder with two parallel conductors. It admits the plane symmetry, and the corresponding variant of the Grad-Shafranov equation is

$$
\Delta \psi + \frac{dp}{d\psi} + I \frac{dI}{d\psi} = 0,
$$

where $H_x = \frac{\partial \psi}{\partial y}$, $H_y = -\frac{\partial \psi}{\partial x}$, $I = H_z$.

Boundary problems with it are considered in the cylinder cross-section by the plane $z = \text{const}$, having the square or the circular form as usual. The boundary is supposed to be impermeable for the magnetic field, and that means the condition $\psi = \psi_\Gamma = \text{const}$ there. It is convenient not to separate the conductor territories that would make the problem solving area multiply connected. We consider it as a simply connected one, and the conductors are included in the model by means of the additional term in the left-hand side of the equation (7):

$$
\int_{j_{\text{ex}}(x, y)} = j_0 \sum_{k=1}^2 \exp \left( -\frac{(x - x_k)^2 + (y - y_k)^2}{r_k^2} \right),
$$

where $(x_k, y_k)$ are the conductor centers, $r_k$ is their notional radius, and the coefficient $j_0$ is chosen from the condition that the integral $\int j_{\text{ex}} \, dx \, dy$ over the every conductor area corresponds to the value of the current in it.

In the cylinder problem, $I(\psi) = H_z = 0$. The function $p(\psi)$ is chosen from the condition that the hot plasma in the trap would not contact with the outer boundary and with conductors, i.e., would be
confined by the magnetic field only. We arrange it in the center and along the magnetic separatrix passing through it (figure 2a), for example, by means of the function

\[ p(\psi) = p_0 \exp \left( -\left( \frac{\psi - \psi_0}{q} \right)^2 \right), \]

where \( \psi_0 = \psi(0,0) \) is the value of the solution required in the center.

![Figure 2](image1.jpg)

**Figure 2.** Magnetic field (a) and pressure (b) in the cylindrical Galateya-belt trap.

The boundary problem is solved using the relaxation method [3]. In computation series, equilibrium magnetoplasma configurations in the cylindrical Belt have been obtained [21, 22]. They are disposed in the center and have a form of a curvilinear quadrangle with convex-inward boundaries and narrow strips surrounding the conductors attached to it (figure 2b). The equilibrium is obtained in relaxation only under the restriction

\[ p_0 < p_0^{cr} \]  \( (8) \)

for maximum pressure measured in the magnetic unit that is proportional to the square value of the current in the conductors. Its physical meaning is that the magnetic field of a fixed magnitude can’t confine plasma with an arbitrary high pressure. The same restriction is connected with the mathematical nature of the problem. The relaxation method solving the boundary problem of the type

\[ \Delta \psi + g(\psi) = 0, \psi|_t = \text{const} \]  \( (9) \)

converges, if its error, being the solution of the linearized problem

\[ \frac{\partial u}{\partial t} + L[u] = 0; \ L[u] = -\Delta u - g'(\psi)u, \ u|_t = 0, \]

tends to zero at \( t \to \infty \). This can take place only if the self-adjoint operator \( L[u] \) is positively definite. Therefore, the restriction (8) is referred not to the numerical method choice but to the spectral character of the linearized problem. Restrictions of the same nature are common for mathematical models of a large class of reaction and diffusion interaction processes (e.g., the theory of burning) using semi-linear elliptic equation of the type (9) [3].

The recent researches deal with a cost of the toroidal trap simplifying substitution by their analogues straightened into the cylinder. The comparison of equilibrium plasma configurations in cylinder and tori of different big radius values is carried on in the plasma cylinder with an electric current (Z-pinch) and in its toroidal modifications, that are a simplest basis of tokamaks. The same analysis for galateya-traps was fulfilled in the Belt and its cylindrical analogue. It has been shown that the cylindrical configurations
being bended into the tori become deformed and shifts in the outer boundary direction. Quantitative estimations of displacement values and configuration parameters and their dependence on the torus big radius have been obtained. These results permit to insert toroidal correction into the information obtained earlier for the cylindrical configurations [22-24].

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References
[1] Morozov A I 1957 JETP 5 (2) 215–20
[2] Artsimovich L A, Luk’ianov S Iu, Podgorny I M, Chuvatin S A 1958 JETP 6 (1) 1–5
[3] Brushlinskii K V 2009 Mathematical and Computational Problems in Magnetohydrodynamics (Moscow: Binom) [in Russian]
[4] Brushlinskii K V, Kozlov A N, Konovalov V S 2015 Comp. Math. And Math. Phys. 55 (8) 1370–80
[5] Kozlov A N and Konovalov V S 2017 Communications in Nonlinear Science and Numerical Simulation 51 169–79
[6] Morozov A I 1990 Sov. J. Plasma Phys. 16 (2) 69–78
[7] Morozov A I 2012 Introduction to Plasma Dynamics (Boca Raton: CRC Press)
[8] Brushlinskii K V and Morozov A I 1980 vol 8 ed by M A Leontovich Reviews of Plasma Physics (NY, London: Consultants Bureau) pp 105–97
[9] Brushlinskii K V, Zaborov A M, Kozlov A N, Morozov A I, Savelyev V V 1990 Sov. J. Plasma Phys. 16 (2) 79–84
[10] Brushlinskii K V and Zhdanova N S 2004 Fluid Dynamics 39 (3) 474–84
[11] Styopin E V 2015 J. Plasma Phys. 81 905810309
[12] Brushlinskii K V and Styopin E V 2017 J. Phys.: Conf. Ser. 937 012007
[13] Brushlinskii K V, Zhdanova N S, Stepin E V 2018 Comp. Math. And Math. Phys. 58 (4) 593–603
[14] Shafraev V D 1958 JETP 6 (3) 545–54
[15] Grad H and Rubin H 1959 Proc. 2-nd UN Int. Conf. on the Peaceful Uses of Atomic Energy, Geneva 31 (NY: Columbia Univ. Press) 190
[16] Brushlinskii K V, Goldich A S, Davydoeva N A 2017 Math. Models Comput. Simul. 9 (1) 60–70
[17] Brushlinskii K V and Chmykhova N A 2014 Vestnik NIYaU MEPhI 3 (1) 40–52 [in Russian]
[18] Morozov A I 1992 Sov. J. Plasma Phys. 18 159
[19] Morozov A I and Savel’ev V V 1998 PHYS-USP 41 (11) 1049–89
[20] Morozov A I and Frank A G 1994 Plasma Phys. Reports 20 879–86
[21] Brushlinskii K V, Gol’dich A S, Desyatova A S 2013 Math. Models Comput. Simul. 5 (2) 156–66
[22] Brushlinskii K V and Goldich A S 2016 Differential Equations 52 (7) 845–54
[23] Brushlinskii K V and Kondratyev I A 2017 J. Phys.: Conf. Ser. 937 012006
[24] Brushlinskii K V and Kondratyev I A 2018 Mathem. Modelirovanie 30 (6) 76–94 [in Russian]