Minimal complex surfaces with Levi-Civita Ricci-flat metrics

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In memory of Professor Lu Qi-Keng

Abstract. This is a continuation of our previous paper [14]. In [14], we introduced the first Aeppli-Chern class on compact complex manifolds, and proved that the $(1,1)$ curvature form of the Levi-Civita connection represents the first Aeppli-Chern class which is a natural link between Riemannian geometry and complex geometry. In this paper, we study the geometry of compact complex manifolds with Levi-Civita Ricci-flat metrics and classify minimal complex surfaces with Levi-Civita Ricci-flat metrics. More precisely, we show that minimal complex surfaces admitting Levi-Civita Ricci-flat metrics are Kähler Calabi-Yau surfaces and Hopf surfaces.

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1. Introduction

In this paper, we study the relationship between Riemannian manifolds and complex manifolds by using various metric connections and their curvature tensors.

Let $(X, h)$ be a Hermitian manifold and $g$ be the background Riemannian metric. It is well-known that, when $(X, h)$ is not Kähler, the relation between the Riemannian geometry $(X, g)$ and the complex geometry $(X, h)$ is extremely complicated. Indeed, on the Hermitian holomorphic tangent bundle $(T^{1,0}X, h)$, there are two typical metric compatible connections:

(1) the Chern connection $\nabla$, i.e. the unique connection $\nabla$ compatible with the Hermitian metric and also the complex structure $\bar{\partial}$;
(2) the Levi-Civita connection $\nabla^{LC}$, i.e. the restriction of the complexified Levi-Civita connection on $T_{C}X$ to the holomorphic tangent bundle $T^{1,0}X$.

From the definition, it is quite obvious that the Levi-Civita connection $\nabla^{LC}$ is a representative of the Riemannian geometry of $(X, g)$. It is also well-known that when $(X, h)$ is not Kähler,
$\nabla$ and $\nabla^{LC}$ are not the same. The complex geometry of the Chern connection is extensively investigated in the literatures by using various methods (e.g. [3, 7, 8, 9, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25]). However, the complex geometry of the Levi-Civita connection is not well understood although it has rich Riemannian geometry structures.

In [14], we introduced the first Aeppli-Chern classes for holomorphic line bundles. Let $L \rightarrow X$ be a holomorphic line bundle over $X$. The first Aeppli-Chern class is defined as

$$c_1^{AC}(L) = [-\sqrt{-1}\partial\bar{\partial} \log h]_A \in H^{1,1}_A(X)$$

where $h$ is an arbitrary smooth Hermitian metric on $L$ and the Aeppli cohomology is

$$H^{p,q}_A(X) := \frac{\text{Ker}\partial\bar{\partial} \cap \Omega^{p,q}(X)}{\text{Im}\partial \cap \Omega^{p,q}(X) + \text{Im}\bar{\partial} \cap \Omega^{p,q}(X)}.$$ 

For a complex manifold $X$, $c_1^{AC}(X)$ is defined to be $c_1^{AC}(K_X^{-1})$ where $K_X^{-1}$ is the anti-canonical line bundle of $X$. Note that, for a Hermitian line bundle $(L, h)$, the classes $c_1(L)$ and $c_1^{AC}(L)$ have the same $(1, 1)$-form representative $\Theta^h = -\sqrt{-1}\partial\bar{\partial} \log h$ (in different classes). It is well-known that on a Hermitian manifold $(X, \omega)$, the first Chern-Ricci curvature $\text{Ric}(\omega) = -\sqrt{-1}\partial\partial^* \log \det(\omega)$ represents the first Chern class $c_1(X)$. As an analog, we proved in [14, Theorem 1.1] that the first Levi-Civita Ricci curvature $\mathfrak{R}\text{ic}(\omega)$ represents the first Aeppli-Chern class $c_1^{AC}(X)$. It is obvious that $c_1(X) = 0$ implies $c_1^{AC}(X) = 0$. Hence, it is very natural to study non-Kähler Calabi-Yau manifolds by using the first Aeppli-Chern class $c_1^{AC}(X)$ and the first Levi-Civita Ricci curvature $\mathfrak{R}\text{ic}(\omega)$. By the celebrated Calabi-Yau theorem ([26]), a compact Kähler manifold has $c_1(X) = 0$ if and only if it has a Kähler metric with Ricci-flat metric, i.e. $\text{Ric}(\omega) = 0$. It is easy to see that if $\mathfrak{R}\text{ic}(\omega) = 0$, then $c_1^{AC}(X) = 0$. There is a natural question analogous to the Calabi conjecture:

**Question 1.1.** On a compact complex manifold $X$, if $c_1^{AC}(X) = 0$, does there exist a smooth Levi-Civita Ricci-flat Hermitian metric $\omega$, i.e. $\mathfrak{R}\text{ic}(\omega) = 0$?

As we have shown in [14, Theorem 1.2], $\mathfrak{R}\text{ic}(\omega) = 0$ is equivalent to

$$\text{Ric}(\omega) = \frac{1}{2}(\partial\partial^* \omega + \bar{\partial}\bar{\partial}^* \omega).$$

The equation (1.2) is not the Monge-Ampère type equation since there are also non-elliptic second order derivatives on the right hand side. As it is well-known, it is particularly challenging to solve such equations. Instead of solving the equation (1.2) directly, we use several observations in [24] to study the geometry of the equation (1.2) and obtain necessary conditions to solve (1.2). See Corollary 5.1, Theorem 5.2 and Theorem 5.4 in Section 5 for more details. By using these necessary conditions and Kodaira-Enriques’ classification (e.g. [2, p. 244]) of minimal complex surfaces, we obtain:

**Theorem 1.2.** Let $X$ be a minimal complex surface. Suppose $X$ admits a Levi-Civita Ricci-flat Hermitian metric $\omega$. Then $X$ lies in one of the following

1. Enriques surfaces;
2. bi-elliptic surfaces;
3. $K3$ surfaces;
4. 2-tori;
5. Hopf surfaces.
It worths to point out that we do not show every Hopf surface admitting a Levi-Civita Ricci-flat metric. We only construct such metrics on diagonal Hopf surfaces (see Theorem 7.3). We conjecture that all Hopf surfaces can support Levi-Civita Ricci-flat metrics. On the other hand, every Hopf surface \( X \) has \( c_1(X) = 0 \in H^2(X, \mathbb{R}) \). However, it is easy to show that \( X \) can not support Chern-Ricci flat Hermitian metrics, i.e. Hermitian metrics \( \omega \) with \( \text{Ric}(\omega) = 0 \).

As an application of Theorem 1.2, we obtain the following example which indicates that we need extra constraints to solve Question 1.1 in general:

**Corollary 1.3.** Let \( X \) be a Kodaira surface or an Inoue surface. Then

\[
c_1(X) = c_1^{\text{BC}}(X) = c_1^{\text{AC}}(X) = 0.
\]

However, \( X \) does not admit a Levi-Civita Ricci-flat Hermitian metric.

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2. Preliminaries

2.1. Chern connection on complex manifolds. Let \((X, \omega_g)\) be a compact Hermitian manifold. There exists a unique connection \( \nabla \) on the holomorphic tangent bundle \( T^{1,0}X \) which is compatible with the Hermitian metric and also the complex structure. This connection \( \nabla \) is called the Chern connection. The Chern connection \( \nabla \) on \((T^{1,0}X, \omega_g)\) has curvature components

\[
R_{i\bar{k}j\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}.
\]

The (first) Chern-Ricci form \( \text{Ric}(\omega_g) \) of \((X, \omega_g)\) has components

\[
R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{k}j\bar{l}} = -\frac{\partial^2 \log \det(g)}{\partial z^i \partial \bar{z}^j}
\]

which also represents the first Chern class \( c_1(X) \) of the complex manifold \( X \). The Chern scalar curvature \( s_g \) of \((X, \omega_g)\) is given by

\[
s_g = \text{tr}_{\omega_g} \text{Ric}(\omega_g) = g^{i\bar{j}} R_{i\bar{j}}.
\]

The total scalar curvature of \( \omega_g \) is

\[
\int_X s_g \omega_g^n = n \int \text{Ric}(\omega_g) \wedge \omega_g^{n-1},
\]

where \( n \) is the complex dimension of \( X \).
2.2. **Bott-Chern classes and Aeppli classes.** The Bott-Chern cohomology and the Aeppli cohomology on a compact complex manifold $X$ are given by

$$
H^{p,q}_{BC}(X) := \frac{\text{Ker} \cap \Omega^{p,q}(X)}{\text{Im} \overline{\partial} \cap \Omega^{p,q}(X)} \quad \text{and} \quad H^{p,q}_{A}(X) := \frac{\text{Ker} \overline{\partial} \cap \Omega^{p,q}(X)}{\text{Im} \overline{\partial} \cap \Omega^{p,q}(X) + \text{Im} \overline{\partial} \cap \Omega^{p,q}(X)}.
$$

Let $\text{Pic}(X)$ be the set of holomorphic line bundles over $X$. As similar as the first Chern class map $c_1 : \text{Pic}(X) \to H^{1,1}_A(X)$, there is a first Aeppli-Chern class map

$$(2.4) \quad c_1^{AC} : \text{Pic}(X) \to H^{1,1}_A(X).$$

Given any holomorphic line bundle $L \to X$ and any Hermitian metric $h$ on $L$, its curvature form $\Theta_h$ is locally given by $-\sqrt{-1} \partial \overline{\partial} \log h$. We define $c_1^{AC}(L)$ to be the class of $\Theta_h$ in $H^{1,1}_A(X)$. For a complex manifold $X$, $c_1^{AC}(X)$ is defined to be $c_1^{AC}(K_X^{-1})$ where $K_X^{-1}$ is the anti-canonical line bundle. The first Bott-Chern class $c_1^{BC}(X)$ can be defined similarly.

2.3. **Special manifolds.** Let $X$ be a compact complex manifold.

1. A Hermitian metric $\omega_g$ is called a Gauduchon metric if $\partial \overline{\partial} \omega_g^{n-1} = 0$. It is proved by Gauduchon ([10]) that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to scaling).

2. A Hermitian metric $\omega_g$ is called a balanced metric if $d\omega_g^{n-1} = 0$ or equivalently $d^*\omega_g = 0$. On a compact complex surface, a balanced metric is also Kähler, i.e. $\omega_g = 0$. It is well-known many Hermitian manifolds can not support balanced metrics, e.g. Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$. It is also obvious that balanced metrics are Gauduchon.

3. $X$ is called a Calabi-Yau manifold if $c_1(X) = 0 \in H^2(X, \mathbb{R})$.

It is obvious that,

$$c_1^{BC}(X) = 0 \implies c_1(X) = 0 \implies c_1^{AC}(X) = 0,$$

and on compact Kähler manifolds or manifolds supporting the $\partial \overline{\partial}$-lemma ([14, Corollary 1.4]), they are equivalent.

3. The Levi-Civita connection on the holomorphic tangent bundle

Let’s recall some elementary settings (e.g. [14, Section 2]). Let $(M, g, \nabla)$ be a $2n$-dimensional Riemannian manifold with the Levi-Civita connection $\nabla$. The tangent bundle of $M$ is also denoted by $T_M$. The Riemannian curvature tensor of $(M, g, \nabla)$ is

$$R(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, W)$$

for tangent vectors $X, Y, Z, W \in T_M$. Let $T_M = T_M \otimes \mathbb{C}$ be the complexification. We can extend the metric $g$ and the Levi-Civita connection $\nabla$ to $T_M$ in the $\mathbb{C}$-linear way. Hence for any $a, b, c, d \in \mathbb{C}$ and $X, Y, Z, W \in T_M$, we have

$$R(aX, bY, cZ, dW) = abcd \cdot R(X, Y, Z, W).$$

Let $(M, g, J)$ be an almost Hermitian manifold, i.e., $J : T_M \to T_M$ with $J^2 = -1$, and for any $X, Y \in T_M$, $g(JX, JY) = g(X, Y)$. The Nijenhuis tensor $N_J : \Gamma(M, T_M) \times
The almost complex structure $J$ is called integrable if $N_J \equiv 0$ and then we call $(M, g, J)$ a Hermitian manifold. We can also extend $T_c M$ in the $\mathbb{C}$-linear way. Hence for any $X, Y \in T_c M$, we still have $g(JX, JY) = g(X, Y)$. By Newlander-Nirenberg’s theorem, there exists a real coordinate system $\{x^i, x^f\}$ such that $z^i = x^i + \sqrt{-1}x^f$ are local holomorphic coordinates on $M$. Moreover, we have $T_c M = T^{1,0} M \oplus T^{0,1} M$ where

$$T^{1,0} M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \right\} \quad \text{and} \quad T^{0,1} M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^m} \right\}.$$ 

Since $T^{1,0} M$ is a subbundle of $T_c M$, there is an induced connection $\nabla^{LC}$ on the holomorphic tangent bundle $T^{1,0} M$ given by

$$\nabla^{LC} = \pi \circ \nabla : \Gamma(M, T^{1,0} M) \xrightarrow{\nabla} \Gamma(M, T_c M \otimes T_c M) \xrightarrow{\pi} \Gamma(M, T_c M \otimes T^{1,0} M).$$

Let $h = (h_{\bar{z}z})$ be the corresponding Hermitian metric on $T^{1,0} M$ induced by $(M, g, J)$. It is obvious that $\nabla^{LC}$ is a metric compatible connection on the Hermitian holomorphic vector bundle $(T^{1,0} M, h)$, and we call $\nabla^{LC}$ the Levi-Civita connection on the complex manifold $M$. It is obvious that, $\nabla^{LC}$ is determined by the following relations

$$\nabla^{LC}_{\frac{\partial}{\partial z^k}} \frac{\partial}{\partial z^p} := \Gamma_{kp}^q \frac{\partial}{\partial z^q} \quad \text{and} \quad \nabla^{LC}_{\frac{\partial}{\partial \bar{z}^k}} \frac{\partial}{\partial \bar{z}^p} := \Gamma_{kp}^q \frac{\partial}{\partial \bar{z}^q},$$

where

$$\Gamma_{ij}^k = \frac{1}{2} h^{k\ell} \left( \frac{\partial h_{\bar{z}z}}{\partial z^i} + \frac{\partial h_{z\bar{z}}}{\partial z^j} - \frac{\partial h_{\bar{z}z}}{\partial \bar{z}^j} \right), \quad \text{and} \quad \Gamma_{ij}^{\bar{k}} = \frac{1}{2} h^{k\ell} \left( \frac{\partial h_{\bar{z}z}}{\partial \bar{z}^i} - \frac{\partial h_{z\bar{z}}}{\partial \bar{z}^j} \right).$$

The curvature tensor $\mathfrak{R} \in \Gamma(M, \Lambda^2 T_c M \otimes T^{1,0} M \otimes T^{1,0} M)$ of $\nabla^{LC}$ is given by

$$\mathfrak{R}(X, Y)s = \nabla^{LC}_X \nabla^{LC}_Y s - \nabla^{LC}_Y \nabla^{LC}_X s - \nabla^{LC}_{\left[ X, Y \right]} s$$

for any $X, Y \in T_c M$ and $s \in T^{1,0} M$. A straightforward computation shows that the curvature tensor $\mathfrak{R}$ has $(1, 1)$ components

$$\mathfrak{R}^{\ell}_{\bar{k}s} = - \left( \frac{\partial r_{ij}^\ell}{\partial \bar{z}^k} - \frac{\partial r_{ik}^\ell}{\partial z^j} + \Gamma_{ik}^s r_{sj}^\ell - \Gamma_{jk}^s r_{is}^\ell \right).$$

**Definition 3.1.** The (first) Levi-Civita Ricci curvature $\mathfrak{Ric}(\omega_h)$ of the Hermitian vector bundle $(T^{1,0} M, \omega_h, \nabla^{LC})$ is

$$\mathfrak{Ric}(\omega_h) = \sqrt{-1} \mathfrak{R}^{(1)} \frac{dz^i}{\bar{z}^j} \wedge \frac{dz^j}{\bar{z}^i}\quad \text{with}\quad \mathfrak{R}^{(1)}_{ij} = \mathfrak{R}^{\ell}_{\bar{\ell}i j}.$$ 

The Levi-Civita scalar curvature $s_{LC}$ of $\nabla^{LC}$ on $T^{1,0} M$ is

$$s_{LC} = h^{\bar{\ell} j} h^{\ell k} \mathfrak{R}^{\ell}_{\bar{k} j}.$$
4. Geometry of the first Aeppli-Chern class

Let’s give a straightforward proof of [14, Theorem 1.2].

**Theorem 4.1.** Let \((X, \omega)\) be a compact Hermitian manifold. Then the first Levi-Civita Ricci form \(\mathfrak{Ric}(\omega)\) represents the first Aeppli-Chern class \(c_1^{AC}(X)\) in \(H_{\Lambda}^{1,1}(X)\). Moreover, we have the Ricci curvature relation

\[
\mathfrak{Ric}(\omega) = \text{Ric}(\omega) - \frac{1}{2} (\partial \bar{\partial} \omega + \bar{\partial} \partial \omega),
\]

and the scalar curvature relation

\[
s_{LC} = s_C - \langle \partial \bar{\partial} \omega, \omega \rangle.
\]

**Proof.** It is easy to show that

\[
\partial \bar{\partial} \omega = 2 \sqrt{-1} \Gamma^i_k \, dz^i
\]

and so

\[
- \frac{\partial \bar{\partial} \omega + \bar{\partial} \partial \omega}{2} = \sqrt{-1} \left( \frac{\partial \Gamma^i_k}{\partial z^i} + \frac{\partial \Gamma^i_k}{\partial \bar{z}^i} \right) \, dz^i \wedge d\bar{z}^i.
\]

On the other hand, by formula (3.4), we have

\[
\mathfrak{R}_{ij} = \mathfrak{R}_{ij}^k = - \frac{\partial \Gamma^k_i}{\partial z^j} + \frac{\partial \Gamma^k_j}{\partial z^i}.
\]

Moreover, we have

\[
\left( - \frac{\partial \Gamma^k_i}{\partial \bar{z}^j} + \frac{\partial \Gamma^k_j}{\partial \bar{z}^i} \right) - \left( - \frac{\partial \Gamma^k_i}{\partial z^j} + \frac{\partial \Gamma^k_j}{\partial z^i} \right) = - \frac{\partial \Gamma^k_i}{\partial z^j} - \frac{\partial \Gamma^k_j}{\partial z^i} = - \frac{\partial^2 \log \det(g)}{\partial z^i \partial \bar{z}^j},
\]

which establishes formula (4.1). It is easy to show \(\langle \partial \bar{\partial} \omega, \omega \rangle = \langle \bar{\partial} \partial \omega, \omega \rangle\). \(\square\)

**Lemma 4.2.** Let \((X, \omega)\) be a compact Hermitian manifold with complex dimension \(n\). Suppose \(f \in C^\infty(X, \mathbb{R})\) and \(\omega_f = e^f \omega\). Then we have

\[
\partial^f \omega_f = \partial \omega + \sqrt{-1} (n-1) \partial f \quad \text{and} \quad \bar{\partial}^f \omega_f = \bar{\partial} \omega - \sqrt{-1} (n-1) \bar{\partial} f,
\]

where \(\partial^f, \bar{\partial}^f\) are the adjoint operators with respect to the metric \(\omega\) and \(\omega_f\) respectively.

**Definition 4.3.** The Kodaira dimension \(\kappa(L)\) of a line bundle \(L\) is defined to be

\[
\kappa(L) := \limsup_{m \to +\infty} \frac{\log \dim \! C^0(X, L^m)}{\log m}
\]

and the **Kodaira dimension** \(\kappa(X)\) of \(X\) is defined as \(\kappa(X) := \kappa(K_X)\) where the logarithm of zero is defined to be \(-\infty\).

**Theorem 4.4.** Let \((X, \omega)\) be a compact Hermitian manifold. Suppose the Levi-Civita scalar curvature \(s_{LC}\) of \(\omega\) is positive, then \(K_X\) is not pseudo-effective and \(\kappa(X) = -\infty\).
Proof. Let \( \omega_f = e^f \omega \) be the Gauduchon metric in the conformal class of \( \omega \). Then by Lemma 4.2 and Theorem 4.1, we have

\[
\text{Ric}(\omega_f) - \frac{\partial \bar{\partial} f \omega_f + \overline{\partial \partial^* f \omega_f}}{2} = \text{Ric}(\omega) - \frac{\partial \bar{\partial}^* \omega + \overline{\partial \partial^* \omega}}{2} - \sqrt{-1} \partial \bar{\partial} f.
\]

Moreover, we have

\[
\int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = \int_X \left( \text{Ric}(\omega) - \sqrt{-1} \partial \bar{\partial} f + \frac{\partial \bar{\partial}^* \omega_f + \overline{\partial \partial^* \omega_f}}{2} \right) \wedge \omega_f^{n-1}
= \int_X \text{Ric}(\omega) \wedge \omega_f^{n-1} + \frac{1}{2} \left( \| \overline{\partial_f} \omega_f \|_{\omega_f}^2 + \| \partial_f \omega_f \|_{\omega_f}^2 \right)
= \frac{1}{n} \int_X e^{(n-1)f} \cdot s_{\text{LC}} \cdot \omega^n + \frac{1}{2} \left( \| \overline{\partial_f} \omega_f \|_{\omega_f}^2 + \| \partial_f \omega_f \|_{\omega_f}^2 \right).
\]

(4.8)

Suppose the Levi-Civita scalar curvature \( s_{\text{LC}} > 0 \), then the total Chern scalar curvature of the Gauduchon metric \( \omega_f \) is strictly positive, i.e.

\[
\int_X s_f \cdot \omega_f^n = \frac{1}{n} \int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} > 0.
\]

By [24, Theorem 1.1] and [24, Corollary 3.3], we know \( K_X \) is not pseudo-effective and \( \kappa(X) = -\infty \).

\( \square \)

**Corollary 4.5.** Let \( X \) be a compact complex manifold with \( c_{BC}^1(X) = 0 \), then there is no Hermitian metric with positive Levi-Civita scalar curvature.

Proof. If \( c_{BC}^1(X) = 0 \), then by [17, Theorem 1.3], there exists a smooth Gauduchon metric \( \omega_g \) with \( \text{Ric}(\omega_g) = 0 \). Hence for any other Gauduchon metric \( \omega_G \), we have

\[
\int_X \text{Ric}(\omega_G) \wedge \omega_G^{n-1} = 0.
\]

Suppose \( \omega \) is a Hermitian metric with positive Levi-Civita scalar curvature \( s_{\text{LC}} \) and \( \omega_f \) is the Gauduchon metric in the conformal class of \( \omega \), then by (4.8), we have

\[
\int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = \frac{1}{n} \int_X e^{(n-1)f} \cdot s_{\text{LC}} \cdot \omega^n + \frac{1}{2} \left( \| \overline{\partial_f} \omega_f \|_{\omega_f}^2 + \| \partial_f \omega_f \|_{\omega_f}^2 \right) > 0,
\]

which is a contradiction. \( \square \)

5. Compact complex manifolds with Levi-Civita Ricci-flat metrics

Let’s recall that, a Levi-Civita Ricci-flat metric is a Hermitian metric satisfying \( \mathfrak{Ric}(\omega) = 0 \), or equivalently, by formula (4.1)

\[
\text{Ric}(\omega) = \frac{\partial \bar{\partial}^* \omega + \overline{\partial \partial^* \omega}}{2}.
\]

The first obstruction for the existence of Levi-Civita Ricci-flat Hermitian metric is the top first Chern number:
Corollary 5.1. Suppose $c_1^{\text{AC}}(X) = 0$, then the top intersection number $c_1^n(X) = 0$. In particular, if $X$ has a Levi-Civita Ricci-flat Hermitian metric $\omega$, then $c_1^n(X) = 0$.

Proof. By definition, if $c_1^{\text{AC}}(X) = 0$, then
\[ \text{Ric}(\omega) = \overline{\partial} A + \partial B \]
where $A$ is a $(1,0)$-form and $B$ is a $(0,1)$-form. Hence
\[ c_1^n(X) = \int_X (\text{Ric}(\omega))^n = \int_X (\text{Ric}(\omega))^{n-1} \wedge (\overline{\partial} A + \partial B) = 0 \]
since $\text{Ric}(\omega)$ is both $\partial$ and $\overline{\partial}$-closed. $\square$

Theorem 5.2. Let $X$ be a compact complex manifold. Suppose $\omega$ is a Levi-Civita Ricci-flat Hermitian metric. Then either
1. $\kappa(X) = -\infty$; or
2. $\kappa(X) = 0$ and $(X,\omega)$ is conformally balanced with $K_X$ a holomorphic torsion, i.e. $K_X^{\otimes m} = O_X$ for some $m \in \mathbb{Z}^+$. 

Proof. Let $\omega_f = e^f \omega$ be the Gauduchon metric in the conformal class of $\omega$. Then by formula (4.8), the total Chern scalar curvature of $\omega_f$ is
\[ \int_X s_f \cdot \omega_f^n = n \int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = n \| \overline{\partial} f \omega_f \|_{\omega_f}^2, \]
since the Levi-Civita scalar curvature $s_{LC} = 0$. Suppose $\overline{\partial} f \omega_f \neq 0$, then
\[ \int_X s_f \cdot \omega_f^n > 0. \]
By [24, Corollary 3.3], we have $\kappa(X) = -\infty$. On the other hand, if $\overline{\partial} f \omega_f = 0$, i.e. $(X,\omega)$ is conformally balanced. Then the total Chern scalar curvature of the Gauduchon metric
\[ \int_X s_f \cdot \omega_f^n = 0. \]
Then by [24, Theorem 1.4], we have $\kappa(X) = -\infty$ or $\kappa(X) = 0$, and when $\kappa(X) = 0$, $K_X$ is a holomorphic torsion. $\square$

Definition 5.3 ([4]). Let $X$ be a compact complex manifold. $X$ is said to satisfy the $\partial \overline{\partial}$-lemma if the following statement holds: if $\eta$ is $d$-exact, $\partial$-closed and $\overline{\partial}$-closed, it must be $\partial \overline{\partial}$-exact. In particular, on such manifolds, for any pure-type form $\varphi \in \Omega^{p,q}(X)$, if $\varphi$ is $\partial \overline{\partial}$-closed and $\partial$-exact, then it is $\partial \overline{\partial}$-exact.

Let $\mu : \tilde{X} \rightarrow X$ be a modification between compact complex manifolds. If the $\partial \overline{\partial}$-lemma holds for $\tilde{X}$, then the $\partial \overline{\partial}$-lemma also holds for $X$. In particular, Moishezon manifolds and manifolds in Fujiki class $\mathcal{C}$ support the $\partial \overline{\partial}$-lemma. For more details, we refer to [4, 1] and also the references therein.

Theorem 5.4. Let $X$ be a compact complex manifold and $\omega$ be a Levi-Civita Ricci-flat Hermitian metric. If $X$ supports the $\partial \overline{\partial}$-Lemma, then $(X,\omega)$ is conformally balanced and $K_X$ is unitary flat.
Proof. By formula (4.1), we have
\[
(5.3) \quad \text{Ric}(\omega) = \frac{\partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega}{2},
\]
since \( \mathfrak{Ric}(\omega) = 0 \). Note that \( \text{Ric}(\omega) \) is \( \partial \)-closed and \( \bar{\partial} \)-closed, and so we have
\[
\bar{\partial} \partial^* \omega = 0.
\]
Moreover, if \( X \) supports the \( \partial \)-Lemma, then the \( \bar{\partial} \)-closed and \( \partial \)-exact \( (1, 1) \)-form \( \partial \partial^* \omega \) is \( \bar{\partial} \)-exact, i.e. there exists a smooth function \( \varphi \) such that
\[
\partial \partial^* \omega = \partial \partial \varphi.
\]
Therefore, \( \text{Ric}(\omega) = \sqrt{-1} \partial \bar{\partial} F \) where \( F = -\frac{\varphi}{2\sqrt{-1}} \in C^\infty(X, \mathbb{R}) \). It is obvious that the Hermitian metric \( e^{\frac{F}{2}} \omega \) is Chern Ricci-flat, i.e. \( K \) is unitary flat. Moreover, for any Gauduchon metric \( \omega_G \) on \( X \), we have
\[
(5.4) \quad \int_X \text{Ric}(\omega_G) \wedge \omega_G^{n-1} = \int_X \text{Ric}(\omega) \wedge \omega^{n-1} = \int_X \sqrt{-1} \partial \bar{\partial} F \wedge \omega_G^{n-1} = 0.
\]
Let \( \omega_f = e^f \omega \) be the Gauduchon metric in the conformal class of \( \omega \), then by (5.2), we have
\[
\int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = \| \overline{\partial}_f \omega_f \|_{\omega_f}^2 = 0,
\]
that is \( \overline{\partial}_f \omega_f = 0 \). \( \square \)

6. Classification of minimal complex surfaces with Levi-Civita Ricci-flat metrics

In this section, we investigate the Levi-Civita Ricci-flat metrics on minimal complex surfaces and prove Theorem 1.2.

**Theorem 6.1.** Let \( X \) be a minimal complex surface with \( \kappa(X) \geq 0 \). Suppose \( X \) admits a Levi-Civita Ricci-flat Hermitian metric \( \omega \). Then \( X \) is Kähler surface of Calabi-Yau type, i.e. \( X \) is exactly one of the following

1. an Enriques surface;
2. a bi-elliptic surface;
3. a K3 surface;
4. a torus.

**Proof.** By Theorem 5.2, we have \( \kappa(X) \leq 0 \). Hence, we only need to consider minimal surfaces with \( \kappa(X) = 0 \). By the Kodaira-Enriques’ classification of minimal surfaces (e.g. [2, p. 244]), a minimal surface with \( \kappa(X) = 0 \) has torsion canonical line bundle \( K \), i.e. \( K^{m} = \mathcal{O} \) for some \( m \in \mathbb{Z}^+ \). Hence, there exists a Hermitian metric \( \omega_0 \) with \( \text{Ric}(\omega_0) = 0 \). Let \( \omega \) be the Hermitian metric with \( \mathfrak{Ric}(\omega) = 0 \) and \( \omega_f = e^f \omega \) be the Gauduchon metric in the conformal class of \( \omega \). Then
\[
(6.1) \quad \int_X \text{Ric}(\omega_f) \wedge \omega_f = \int_X \text{Ric}(\omega_0) \wedge \omega_f = 0.
\]
Hence, by (5.2), we obtain \( \overline{\partial}_f \omega_f = 0 \). Since \( \dim X = 2 \), we have \( d \omega_f = 0 \), i.e. \( X \) is a Kähler surface. According to the Kodaira-Enriques’ classification, \( X \) is either an Enriques surface, a
bi-elliptic surface, a K3 surface or a torus. All these surfaces are Kähler surfaces of Calabi-Yau type, and all Kähler Calabi-Yau metrics are Levi-Civita Ricci-flat.

**Remark 6.2.** The Hermitian metric $\omega$ with $\mathcal{R}(\omega) = 0$ in Theorem 6.1 is not necessarily Kähler. Indeed, let $\omega_{\text{CY}}$ be a Calabi-Yau Kähler metric on $X$. Then for any non constant smooth function $f \in C^\infty(X, \mathbb{R})$, we can construct a non-Kähler Levi-Civita Ricci-flat metric. By Yau’s theorem, there exists a Kähler metric $\omega_0$ such that

$$\omega_0^2 = e^{-f} \omega_{\text{CY}}^2.$$

Let $\omega = e^f \omega_0$. Then $\omega$ is a non-Kähler metric with Levi-Civita Ricci-flat curvature. Indeed,

$$\mathcal{R}(\omega) = \mathcal{R}(\omega_0) - 2\sqrt{-1} \partial \bar{\partial} f - \frac{\partial \bar{\partial}_0 \omega_0^2 + \partial \bar{\partial}_1 \omega_0^2 + \sqrt{-1} \partial \bar{\partial} f}{2} + \sqrt{-1} \partial \bar{\partial} f$$

$$= \mathcal{R}(\omega_{\text{CY}}) + \sqrt{-1} \partial \bar{\partial} f - 2\sqrt{-1} \partial \bar{\partial} f - \frac{\partial \bar{\partial}_0 \omega_0^2 + \partial \bar{\partial}_1 \omega_0^2 + \sqrt{-1} \partial \bar{\partial} f}{2} + \sqrt{-1} \partial \bar{\partial} f$$

$$= 0,$$

where we use Lemma 4.2 in the second identity.

**Theorem 6.3.** Let $X$ be a minimal complex surface with $\kappa(X) = -\infty$. Suppose $X$ admits a Levi-Civita Ricci-flat Hermitian metric $\omega$. Then $X$ is a Hopf surface.

**Proof.** According to the Kodaira-Enriques’ classification [2, p. 244], $X$ is one of the following

1. a minimal rational surface;
2. a minimal surface of class VII;
3. a ruled surface of genus $g \geq 1$.

Suppose $X$ is a Kähler surface, i.e. $X$ is a minimal rational surface or a ruled surface of genus $g \geq 1$. Then the $\partial \bar{\partial}$-Lemma holds on $X$. Hence by Theorem 5.4, $K_X$ is unitary flat and $\kappa(X) = 0$ which a is contradiction. Hence $X$ is non-Kähler, i.e. $X$ is a minimal surface of class VII. By Corollary 7.1, Theorem 7.2 and Theorem 7.3 in the next section, we complete the proof. □

**Corollary 6.4.** Let $X$ be a Kodaira surface or an Inoue surface. Then

$$c_1^{BC}(X) = c_1^{AC}(X) = 0.$$

However, $X$ does not admit a Levi-Civita Ricci-flat Hermitian metric.

**Proof.** It is well-known ([2, p. 244]) that a Kodaira surface is a non-Kähler surface with torsion canonical line bundle $K_X$. Hence $c_1^{BC}(X) = c_1^{AC}(X) = 0$ and $\kappa(X) = 0$. By Theorem 6.1, it has no Levi-Civita Ricci-flat Hermitian metric. For an Inoue surface, one has $b_2(X) = 0$. Hence, $c_1(X) = c_1^{AC}(X) = 0$. □
7. The Levi-Civita Ricci-flat metrics on minimal surfaces of class VII

A class VII surface is a minimal compact complex surface with $b_1 = 1$ and $\kappa(X) = -\infty$. There are three classes of them:

1. Hopf surfaces: whose universal cover is $\mathbb{C}^2 - \{0\}$, or equivalently a class VII surface with $b_2 = 0$ and contains a curve;
2. Inoue surfaces: a class VII surface has $b_2 = 0$ and contains no curves;
3. all class VII surfaces with $b_2 > 0$.

**Corollary 7.1.** On VII surfaces with $b_2 > 0$, there is no Levi-Civita Ricci-flat Hermitian metrics.

**Proof.** It is well-known that on minimal VII surfaces we have

$$c_1^2(X) = -b_2.$$

Corollary 7.1 follows from Corollary 5.1. \qed

7.1. Inoue surfaces. It is well-known ([11]) that an Inoue surface is a quotient of $\mathbb{H} \times \mathbb{C}$ by a properly discontinuous group of affine transformations where $\mathbb{H}$ is the upper half-plane. There are three types of Inoue surfaces:

1. Inoue surfaces $S_M$. Let $M$ be a matrix in $\text{SL}_3(\mathbb{Z})$ admitting one real eigenvalue $\alpha > 1$ and two complex conjugate eigenvalues $\beta \neq \overline{\beta}$. Let $(a_1, a_2, a_3)$ be a real eigenvector of $M$ corresponding to $\alpha$ and let $(b_1, b_2, b_3)$ be an eigenvector of $M$ corresponding to $\beta$. Then $X = S_M$ is the quotient of $\mathbb{H} \times \mathbb{C}$ by the group of affine automorphisms generated by

$$g_0(w, z) = (\alpha w, \beta z),$$
$$g_i(w, z) = (w + a_i, z + b_i), \quad i = 1, 2, 3.$$

2. Inoue surfaces $X = S^+_N, p, q, r; t$ are defined as the quotient of $\mathbb{H} \times \mathbb{C}$ by the group of affine automorphisms generated by

$$g_0(w, z) = (\alpha w, z + t),$$
$$g_i(w, z) = (w + a_i, z + b_i w + c_i), \quad i = 1, 2,$$
$$g_3(w, z) = \left( w, z + \frac{b_1 a_2 - b_2 a_1}{r} \right),$$

where $(a_1, a_2)$ and $(b_1, b_2)$ are the eigenvectors of some matrix $N \in \text{SL}_2(\mathbb{Z})$ admitting real eigenvalues $\alpha > 1, \alpha^{-1}$. Moreover $t \in \mathbb{C}$ and $p, q, r (r \neq 0)$ are integers, and $(c_1, c_2)$ depends on $(a_i, b_i), p, q, r$ (see [11]).

3. Inoue surfaces $X = S^-_N, p, q, r; t$ have unramified double cover which are Inoue surfaces of type $S^+_N, p, q, r; t$.

**Theorem 7.2.** On Inoue surfaces, there is no Levi-Civita Ricci-flat Hermitian metrics.

**Proof.** Suppose $\omega$ is a Levi-Civita Ricci-flat Hermitian metric on the Inoue surface $X$. Let $\omega_f = e^f \omega$ be the Gauduchon metric in the conformal class of $\omega$, then by formula (4.8), the total Chern scalar curvature of $\omega_f$ is

$$\int_X s_f \cdot \omega_f^2 = 2 \int_X \text{Ric}(\omega_f) \wedge \omega_f = 2 \| \partial \overline{\partial} \omega_f \|^2_{\omega_f} \geq 0.$$
We shall show that on each Inoue surface, there exists a smooth Gauduchon metric with non-positive but not identically zero first Chern-Ricci curvature. Indeed, let \((w, z) \in \mathbb{H} \times \mathbb{C}\) be the holomorphic coordinates, then by the precise definition of each Inoue surface (see also [5, 6, 18]), we know the form
\[
\sigma = \frac{dw \wedge dz}{\text{Im}(w)}
\]
descends to a smooth nowhere vanishing \((2, 0)\) form on \(X\), i.e. \(\sigma \in \Gamma(X, K_X)\). Then it induces a smooth Hermitian metric \(h\) on \(K_X\) given by \(h(\sigma, \sigma) = 1\). In the holomorphic frame \(e = dw \wedge dz\) of \(K_X\), we have
\[
h = h(e, e) = \left[\text{Im}(w)\right]^2.
\]
It also induces a Hermitian metric \(h^{-1}\) on \(K_X^{-1}\), and the curvature of \(h^{-1}\) is
\[
-\sqrt{-1}\partial \bar{\partial} \log h^{-1} = \sqrt{-1}\partial \bar{\partial} \log [\text{Im}(w)]^2 = -\frac{1}{2} \frac{dw \wedge d\bar{w}}{[\text{Im}(w)]^2},
\]
which also represents \(c_{BC}(X)\). By Theorem [17, Theorem 1.3], there exists a Gauduchon metric \(\omega_G\) with
\[
\text{Ric}(\omega_G) = -\frac{1}{2} \frac{dw \wedge d\bar{w}}{[\text{Im}(w)]^2} \leq 0.
\]
Hence, for any Gauduchon metric \(\omega\), one has
\[
\int_X \text{Ric}(\omega) \wedge \omega = \int_X \text{Ric}(\omega_G) \wedge \omega < 0
\]
which is a contradiction to (7.1).

### 7.2. Hopf manifolds.

Let’s recall an example in [13, 14, Section 6]. Let \(X = \mathbb{S}^{2n-1} \times \mathbb{S}^1\) be the standard \(n\)-dimensional \((n \geq 2)\) Hopf manifold. It is diffeomorphic to \(\mathbb{C}^n - \{0\}/G\) where \(G\) is cyclic group generated by the transformation \(z \to \frac{1}{z}z\). It has a naturally induced metric \(\omega_0\) given by
\[
\omega_0 = \sqrt{-1} \frac{\delta z^i \delta \bar{z}^j}{|z|^2} dz^i \wedge d\bar{z}^j.
\]

We present a straightforward computation to show (c.f. [14, Theorem 6.2]):

**Theorem 7.3.** The perturbed metric
\[
(\omega_g = \omega_0 - \frac{1}{n} \cdot \sqrt{-1} \partial \bar{\partial} \log |z|^2).
\]
is Levi-Civita Ricci-flat, i.e. \(\text{Ric}(\omega_g) = 0\).

**Proof.** If we write \(\omega_g = \sqrt{-1} g \sigma dz^i \wedge d\bar{z}^i\), then
\[
g_{\bar{\sigma}} = \frac{1}{|z|^2} \left( \frac{n-1}{n} \delta_{ij} + \frac{z_j \bar{z}^j}{n|z|^2} \right), \quad \text{and} \quad g_{\bar{\sigma}} = \frac{1}{|z|^2} \left( \frac{n \delta_{ij} + z_j \bar{z}^j}{n(n-1)} \right).
\]
Let \(\partial^*\) and \(\overline{\partial}^*\) be the adjoint operators with respect to \(\omega_g\) and \(\Lambda\) is the dual operator of \(\omega_g \wedge \bullet\).

A straightforward computation ([14, Lemma 3.3]) shows
\[
\overline{\partial}^* \omega_g = \sqrt{-1} \Lambda \partial \omega_g.
\]
Note that
\[ \partial \omega_g = \partial \omega_0 = -\frac{\sqrt{-1} \delta_{ij} \overline{z}^k}{|z|^4} dz^k \wedge dz^i \wedge d\overline{z}^j. \]

Hence, we obtain
\[
\overline{\partial} \omega_g = \sqrt{-1} \Lambda \partial \omega_g = \sqrt{-1} g^{i\overline{q}} \frac{\partial^i \overline{z}^k}{|z|^4} - \sqrt{-1} g^{q\overline{q}} \delta_{ij} \left| z \right|^{-4} dz^i \wedge dz^j
\]
\[ = \sqrt{-1} \sum_k g^{i\overline{k}} \frac{\partial^i \overline{z}^k}{|z|^4} - \sqrt{-1} \sum_q g^{q\overline{q}} \delta_{ij} \left| z \right|^{-4} dz^k. \]

On the other hand,
\[
(7.4) \sum_k g^{i\overline{k}} \overline{z}^k = |z|^2 \overline{z}^i \quad \text{and} \quad \sum_q g^{q\overline{q}} \overline{z}^q = (n+1)|z|^2 \overline{z}^i.
\]

Hence, we have
\[
\overline{\partial} \omega_g = \sqrt{-1} n \frac{\overline{z}^k dz^k}{|z|^2} = -n \sqrt{-1} \partial \log |z|^2
\]
and
\[
\overline{\partial \partial}^* \omega_g = n \sqrt{-1} \partial \overline{\partial} \log |z|^2.
\]

Therefore
\[
\partial \partial^* \omega_g + \overline{\partial \partial}^* \omega_g = \sqrt{-1} n \partial \overline{\partial} \log |z|^2.
\]

A direct computation shows
\[
\text{det}(g_{ij}) = (1 + \lambda)^{n-1} |z|^{-2n}, \quad \text{and we have}
\]
\[
\text{Ric}(\omega_g) = -\sqrt{-1} \partial \overline{\partial} \log \text{det}(g) = n \cdot \sqrt{-1} \partial \overline{\partial} \log |z|^2.
\]

By Theorem 4.1,
\[
\text{Ric}(\omega_g) = \text{Ric}(\omega_g) - \frac{\partial \partial^* \omega_g + \overline{\partial \partial}^* \omega_g}{2} = 0.
\]

**Remark 7.4.** In this example, we construct a solution to the Levi-Civita Ricci-flat equation on Hopf manifolds. It is natural to ask whether there are more solutions. We expect there are theoretical approaches on the existence of Levi-Civita Ricci-flat metrics on all Hopf manifolds.

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