Do quantum strategies always win? A case study in the entangled quantum penny flip game

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In a seminal paper, Meyer[1] described the advantages of quantum game theory by looking at the classical penny flip game. A player using quantum strategy can win against a classical player almost 100% of the time. Here we make a slight modification of the quantum game, with the two players sharing an entangled state to begin with. We then analyse two different scenarios, one in which the quantum player makes unitary transformations to her qubit while classical player uses a pure strategy of either flipping or not flipping the state of his qubit. In the second scenario we have the quantum player making similar unitary transformations while the classical player makes use of a mixed strategy wherein he either flips or not with some probability “p”. We show that in the second scenario, 100% win record of a quantum player is drastically reduced and for a particular probability “p” the classical player can even win against the quantum player.

I. INTRODUCTION

Game theory is an extremely interesting and sophisticated field which holds within its ambit the power to resolve conflicts, propose new strategies in making war and peace, understanding various games from zero sum games like tick-tack-toe to non zero sum games like Prisoners dilemma. However, in the past 15 years a new addition to game theory has come into being. This is the story of Quantum games. It deals with how to quantize the previously played classical games or propose new games which explicitly use quantum mechanical phenomena like entanglement, nonlocality and quantum interference. A player may use quantum mechanics as a viable strategy to defeat his/her opponent in any classical game. In this context Meyer was one of the first to detect and prove that Quantum game theory has the following rules:

Classical Penny flip game as introduced by Meyer in his paper[1] has the following rules:

- Players P and Q each have a penny.
- Initial state of the penny is heads (say).
- Each player can choose to either flip or not flip the penny and if in the end the state is heads, Q wins else P wins.
- Players cannot see the current state of the penny.
- Sequence of actions: Q → P → Q
- If final state is heads, Q wins else P wins

However there have been works which have commented that quantum games dont offer anything new and whatever they promise can be replicated via classical correlated equilibrium[4]. In this work we apply this hypothesis to the entanglement based quantum penny flip game. We indeed see that a classical player can outwit her quantum opponent by using a particular mixed strategy which is not possible in the non-entanglement based quantum penny flip game. The entanglement quantum penny flip game could be mistaken for the non-entanglement based quantum penny flip game. We indeed see that quantum strategies if used by a player in the classical Penny flip game can help her outwit her opponent 100% to nil. This led to a flourishing field of quantum games. Previous works include quantum prisoners dilemma[2], including an experimental implementation of the Prisoner’s dilemma in a NMR quantum computer[3]. One of the key motivations for playing games, in the quantum world, comes from the possibility of re-formulating quantum communication protocols, and algorithms, in terms of games between quantum and classical players[5]. In fact the aforesaid reference claims that there exists a basic relationship between quantum algorithms and quantum games and research on quantum games can lead to new quantum algorithms[6].

II. CLASSICAL GAME

The matrix form of the game is:

$$\begin{pmatrix}
NN & NF & FN & FF \\
(-1,1) & (1,-1) & (1,-1) & (-1,1) \\
(1,-1) & (-1,1) & (-1,1) & (1,-1)
\end{pmatrix}$$ (1)
where N is not flipping and F is flipping. Both players have \( \frac{1}{2} \) as their winning probability.

Matching pennies is an example of a strictly competitive or zero sum game which has no pure strategy Nash equilibrium. The numbers in the matrix above are the payoffs for either player, first index is for P and second is for Q. For example \((-1, 1)\) means P loses a penny while Q gains a penny as end state is Heads.

### III. QUANTUM PENNY FLIP GAME

The Quantum penny flip game as described by Meyer in Ref[1] is as follows. The starship Enterprise is facing a calamity. This is when Q appears on the bridge and offers to rescue the ship if Captain P can beat her at a simple game: Q produces a penny and asks the captain to place it in a small box, head up. Then Q, followed by P, followed by Q, dip their fingers into the box, without looking at the penny, and either flip it over or leave it as it is. After Q’s second turn they open the box and Q wins if the penny is head up. Q wins every time they play, using the following “quantum” strategy:

\[
\begin{align*}
|0\rangle &\quad \text{Q does } H \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
|1\rangle &\quad \text{P does } X \quad \text{or } I \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
|0\rangle &\quad \text{Q does } H \quad |0\rangle 
\end{align*}
\]

Here 0 denotes ‘head’ and 1 denotes ‘tail’, \( H = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \) is the Hadamard transformation, \( I \) means leaving the penny alone and the action with \( X = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) flips the penny over.

Q’s quantum strategy of putting the penny into the equal superposition of ‘head’ and ‘tail’, on his first turn, means that whether Picard flips the penny over or not, it remains in an equal superposition which Q can rotate back to ‘head’ by applying \( H \) again since \( H = H^{-1} \). So Q always wins when they open the box. Thus playing the penny flip game with a quantum strategy enables the player to win against one who is playing classical, 100% of the time. Of course one may ask what if both play quantum? In that case as shown by Meyer, the quantum advantage vanishes, see Theorem 2 of Ref[1].

A two-person zero-sum game need not have a quantum/quantum equilibrium.

### IV. QUANTUM PENNY FLIP WITH ENTANGLEMENT

The game that we introduce here has the following interpretation, we begin with a maximally entangled state and allow P and Q to make moves. If the final state of the game is a maximally entangled state then Q wins, and if it is a non maximally entangled state then its a draw, if its a separable state then P wins. Allow P to make classical moves, i.e. either \( I \) or \( X \) and Q to make Hadamard transforms (or any unitary operations) on their own qubit. This game should not be confused with the quantum two penny game[7] wherein there are two heads and two tails, here the heads of the penny is the “entangled state” and tails is the “separable state”. It is the entanglement analog of the classical penny flip game with the addition of a “draw”.

Consider the initial state of the system as the entangled state:

\[
|B\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \quad (2)
\]

Sequence of actions is : Q \( \rightarrow \) P \( \rightarrow \) Q.

#### A. Step 1

So, if Q does a Hadamard transformation on his qubit we get:

\[
H \otimes I \left( \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \right) = \frac{1}{2} (|00\rangle + |01\rangle - |10\rangle + |11\rangle) \quad (3)
\]

So the current state of the game is:

\[
|\psi\rangle = \frac{1}{2} (|00\rangle + |01\rangle - |10\rangle + |11\rangle) \quad (4)
\]

#### B. Step 2

Now its P’s turn to make a move, which means either leaving the state unchanged or applying the flip \( X \) operation, when applying the flip the state becomes:

\[
I \otimes X \left( \frac{1}{2} (|00\rangle + |01\rangle - |10\rangle + |11\rangle) \right) = \frac{1}{2} (|01\rangle + |00\rangle - |11\rangle + |10\rangle) \quad (5)
\]

So the current state of the game is either of:

\[
|\psi1\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle) \quad (6)
\]

or

\[
|\psi\rangle = \frac{1}{2} (|00\rangle + |01\rangle - |10\rangle + |11\rangle) \quad (7)
\]

#### C. Step 3

Q’s final move on the above states leaves us with:

\[
H \otimes I \left( \frac{1}{\sqrt{2}} (|00\rangle + |01\rangle + |10\rangle - |11\rangle) \right) = |B1\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \quad (8)
\]

or

\[
H \otimes I \left( \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle) \right) = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle) \quad (9)
\]

So finally we have the either the following states, depending on whether \( P \) had flipped or not flipped his qubit:

- \( |B\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle) \)
- \( |B1\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \)
Matching pennies with entangled states and one player having quantum strategies while the other is classical, like the original version by Meyer gives a definite win to Q as both the states above are maximally entangled Bell states. The game here is about whether player Q having all quantum strategies at his hand can keep the state maximally entangled, whereas P with classical moves can or cannot reduce the entanglement. If in the end all the states obtained are maximally entangled then Q wins, if they are separable then P wins. What the game essentially shows is that it is not possible for player P with pure classical strategy, Q still uses the pure quantum strategy. Since contrary to the previous case, P now uses a mixed classical strategy which entails flipping the state of his qubit with probability $p$ or not flipping. The state after P’s move then is: $\rho_2 = p(I \otimes X)\rho_1(I \otimes X)^\dagger + (1-p)(I \otimes I)\rho_1(I \otimes I)^\dagger$.

### V. QUANTUM PENNY FLIP WITH ENTANGLEMENT: WHEN P PLAYS WITH A MIXED STRATEGY

Now we allow for P using albeit classical but mixed strategy. This entails P with probability $p$ flipping the state of his qubit and with probability $1-p$ leaving it as it is. In this case as before P and Q start with a maximally entangled Bell pair. Q makes the first move, again a Hadamard. Next is P’s turn and as defined earlier flips with probability $p$. Finally Q does a Hadamard. At the end the final state is observed for the amount of entanglement. As before P and Q share a Bell pair. But contrary to the previous case, P now uses a mixed classical strategy, Q still uses the pure quantum strategy. Since P uses a mixed strategy we have to take recourse to density matrices to explain the results.

#### A. Step 1: The initial state

In the form of density matrices the initial state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ is represented as $\rho_0 = |\psi\rangle \langle \psi| = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

#### B. Step 2: Q makes her move

Q makes an unitary transformation on her part of the shared state. $U_{Q1} = \begin{bmatrix} a & b, \\ b^* & -a^* \end{bmatrix}$. The state after Q’s move then is $\rho_1 = (U_{Q1} \otimes I)\rho_0(U_{Q1} \otimes I)^\dagger$.

#### C. Step 3: P plays mixed

P as we said earlier in contrast to the previous section plays a mixed strategy, which entails flipping the state of his qubit with probability “$p$” or not flipping. The state after P’s move then is: $\rho_2 = p(I \otimes X)\rho_1(I \otimes X)^\dagger + (1-p)(I \otimes I)\rho_1(I \otimes I)^\dagger$.

#### D. Step 4: Q makes her final move

At the end Q makes her final move, which as before has to be an unitary transformation, it further could be same as her first move or different. Thus $U_{Q2} = \begin{bmatrix} \alpha & \beta^* \\ \beta & -\alpha^* \end{bmatrix}$. The state after this final move then is $\rho_3 = (U_{Q2} \otimes I)\rho_2(U_{Q2} \otimes I)^\dagger$. To understand this case of P using mixed, lets analyse this case for Q using the familiar Hadamard transform in both steps 2 and 4. In this special case,

$$\rho_3 = \frac{1}{2} \begin{bmatrix} p & 0 & 0 & -p \\ 0 & 1-p & -1+p & 0 \\ 0 & -1+p & 1-p & 0 \\ -p & 0 & 0 & p \end{bmatrix}.$$  

To check the entanglement content of this final state we take recourse to an entanglement monotone: Negativity, $N$. Negativity is defined as- $N = \sum \lambda_i$, where $\lambda_i$ are the eigenvalues of the partial transpose of $\rho_3$. In Fig. 1 we plot the Negativity. We see at $p = 0.5$, it vanishes implying a completely separable state. Thus proving our contention that there exists a classical mixed strategy which can defeat a quantum strategy. Using another particularly important entanglement measure, concurrence one can see that the reported behavior is same, i.e. entanglement vanishes at $p = 0.5$. Concurrence for a two qubit density matrix $\rho_3$ is defined as follows- we first define a “spin-flipped” density matrix, $\gamma$ as $(\sigma_y \otimes \sigma_y)\rho_3(\sigma_y \otimes \sigma_y)$. Then we calculate the square root of the eigenvalues of the matrix $\rho_3\gamma$ (say $\lambda_1,\lambda_2,\lambda_3,\lambda_4$) in decreasing order. Then, concurrence is:

$$\max (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$$

In Fig. 1 (right) the concurrence is plotted. At $p = 1/2$ concurrence vanishes implying a separable state and a win for P’s mixed classical strategy.

### VI. EXPERIMENTAL REALIZATION AND CONCLUSIONS

To experimentally realize this game we first have a Referee/Arbiter who generates the Bell state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ as shown in Fig. 2. He then hands over this state to P and Q, who now share this maximally entangled state. Q first does a unitary (Hadamard) and then P with probability $p$ flips his state or with probability $1-p$ leaves it unaltered. Finally, Q does another unitary (Hadamard, again). Finally they give their output state to the referee. The referee then takes this entangled state and establishes the entanglement content via the Concurrence. The Concurrence has been shown to be experimentally measurable, as in Ref[8] if the referee finds that the Concurrence vanishes then P is declared winner, on the other hand if the concurrence is one, then Q wins, in the rest of the cases the honors are equally shared. The Quantum penny flip game with entangled particles has nontrivial outcomes as compared to the original quantum penny flip game[1]. In a particular case where classical player
Figure 1. Left: Negativity shows that entanglement vanishes at $p = 1/2$. So by P’s classical moves entanglement is completely destroyed enabling him to win. Right: Concurrence vs $p$ showing that entanglement vanishes at $p = 0.5$.

Figure 2. The quantum circuit for the entangled penny flip game. M denotes measurement of entanglement content via concurrence, see Ref[8].

Our future endeavors include checking for the effect of noise, and phenomena like decoherence in entanglement based games. Also there is a flavor in realizing old things with a new perspective and so we are working towards newer interpretations of quantum games.

VII. ACKNOWLEDGMENTS

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