Algebraic structures related to reflection equations

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Abstract

Quadratic algebras related to the reflection equations are introduced. They are quantum group comodule algebras. The quantum group $F_q(GL(2))$ is taken as the example. The properties of the algebras (center, representations, realizations, real forms, fusion procedure etc) as well as the generalizations are discussed.

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1. Introduction

The progress in understanding the algebraic roots of the quantum integrability achieved in last decades has already resulted in introducing several new algebraic objects, such as the Yang-Baxter equation (YBE), quantum groups and quantum algebras, exchange and quadratic algebras. It seems that the list should be enriched with another item: the reflection equation (RE), which has arisen recently in several independent contexts.

The RE reads

\[ R(u - v)K(u)R(u + v)K(v) = K(v)R(u + v)K(u)R(u - v) \]  

where \( K(u) \) is a square \( N \times N \) matrix, \( K^1 = K \otimes \text{id}_N \), \( K^2 = \text{id}_N \otimes K \) (the notation is usual in the quantum inverse scattering method), and \( R(u) \) is a solution to the YBE. One can also consider (1) as defining relations for the associative algebra \( \mathcal{A} \) generated by the elements of the matrix \( K(u) \) [1].

Being introduced in [2] as the equation describing factorized scattering on a half-line, the RE and the related algebra \( \mathcal{A} \) had found soon quite different applications to the quantum current algebras [3] and to the integrable models with non-periodic boundary conditions [1, 4].

Since the constant (i.e. not including the spectral parameter \( u \)) solutions to the YBE are extensively used in the quantum group theory [5], it is quite natural to study a version of the RE without spectral parameter. Though such an equation and the related algebras (for different \( R \) matrices) appeared already in several papers [5–12], they were not distinguished as separate objects of study until recently.

In the present paper we collect the basic facts which are necessary for any systematic study of the RE. We restrict ourselves to the case of the simplest quantum group \( F_q(GL(2)) \) and choose the following form of RE

\[ RKK^1Rt_1K^2 = K^2Rt_1K^1R \]  

The paper is organized as follows. After introducing the basic definitions and notation the general properties of the quadratic algebra \( \mathcal{A} \) defined by (2) are described in section 2. They are discussed and partially proved in section 3. In section 4 few comments are given on the representations of \( \mathcal{A} \). In section 5 some equivalent variants of RE are pointed out as well as the relations between \( \mathcal{A} \) and other algebras. In conclusion, we discuss some generalizations of RE and its applications.
2. Definitions

The quantum group $F \equiv F_q(GL(2))$ can be defined as the associative algebra generated by four elements $a, b, c, d$ and the relations

\begin{align*}
ab &= qba, & bd &= qdb, & [a, d] &= \omega bc, \\
ac &= qca, & cd &= qdc, & [b, c] &= 0,
\end{align*}

$q$ being a complex parameter and $\omega = q - q^{-1}$. Introducing matrix $T$

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

and using the notation $T^1 \equiv T \otimes \text{id}$, $T^2 \equiv \text{id} \otimes T$ one can rewrite the relations (3) in the compact form [5]

\[
R_{12}^T T = T_{21} R^T
\]

where the R matrix is given by

\[
R = \begin{pmatrix} q & 1 \\ \omega & 1 \\ q \end{pmatrix}, \quad R^{t_1} = \begin{pmatrix} q & \omega \\ 1 & 1 \\ q \end{pmatrix}
\]

and $t_1$ means transposition with respect to the first space in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Remark. If we replace the R matrix in (4) by $\mathcal{P} R^{-1} \mathcal{P}$, where $\mathcal{P}$ is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$, the commutation relations (3) remain the same. The consequences of this fact are discussed in section 6.2.

The Hopf algebra structure is defined on $F$ after introducing comultiplication map $\Delta : F \rightarrow F \otimes F$, counit map $\varepsilon : F \rightarrow \mathbb{C}$ and coinverse map $s : F \rightarrow F$ by the formulae

\[
\Delta(T) = T_1 T_2, \quad \varepsilon(T) = \text{id}, \quad s(T) = T^{-1}
\]

where $(T_1 T_2)_{ij} \equiv \sum_{k=1}^2 T_{ik} \otimes T_{kj}$.

Now define the associative algebra $\mathcal{A}$ by four generators $\alpha, \beta, \gamma, \delta$, which can be thought of as the elements of the matrix $K$

\[
K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]
and by the quadratic relations (2) or, more explicitly,

\[
\begin{align*}
[\alpha, \beta] &= \omega \alpha \gamma, \\
[\alpha \gamma] &= q^2 \gamma \alpha, \\
[\alpha, \delta] &= \omega (q \beta + \gamma) \gamma, \\
[\beta, \gamma] &= 0, \\
[\beta, \delta] &= \omega \gamma \delta, \\
\gamma \delta &= q^2 \delta \gamma.
\end{align*}
\]  

The algebra \( \mathcal{A} \) has the following properties.

1. \( \mathcal{A} \) is a Poincaré-Birkhoff-Witt (PBW) algebra which means that the linear space spanned by the monomials of order \( p \) in the generators \( \alpha \beta \gamma \delta \) has the same dimension as in the commutative case that is \( (p + 3)!/p!3! \).

2. \( \mathcal{A} \) is an \( F \)-comodule-algebra that is there exists a map \( \varphi \) (coaction of \( F \)) \( \varphi : \mathcal{A} \to F \otimes \mathcal{A} \) which is consistent with the comultiplication \( \Delta \)

\[
(\Delta \otimes \text{id}) \circ \varphi = (\text{id} \otimes \varphi) \circ \varphi
\]  

(7)

and the counit \( \varepsilon \)

\[
(\varepsilon \otimes \text{id}) \circ \varphi = \text{id}
\]  

(8)

and, besides, is an algebra homomorphism.

By virtue of the duality between \( F = F_q(\text{GL}(2)) \) and the quantum algebra \( \mathcal{U}_q(sl(2)) \) the dual map \( \varphi^* : \mathcal{U}_q(sl(2)) \otimes \mathcal{A} \to \mathcal{A} \) defines the structure of \( \mathcal{U}_q(sl(2)) \)-module algebra on \( \mathcal{A} \).

3. The center of \( \mathcal{A} \) is generated by two elements (for generic \( q \), not root of unity)

\[
c_1 = \beta - q \gamma, \\
c_2 = \alpha \delta - q^2 \beta \gamma
\]  

(9)

4. There are three real forms of \( \mathcal{A} \) consistent with three known real forms of \( F \): \( F_q(U(2)), F_q(U(1, 1)) \) and \( F_q(GL(2, \mathbb{R})) \).

5. The two-sided ideal generated in \( \mathcal{A} \) by the relation \( c_1 = 0 \) is invariant under the coaction \( \varphi \) and the corresponding quotient algebra is isomorphic to the quantum homogeneous space.
3. Discussion and Proofs

1. In order to prove the PBW-property of $\mathcal{A}$ it is necessary to verify that any monomial in $\alpha \beta \gamma \delta$ can be expanded uniquely into the sum of alphabetically ordered monomials. The possibility of alphabetical ordering is easy to establish using the commutation relations (6). As for the linear independence of alphabetically ordered monomials, it is sufficient to verify it only for cubic monomials [13, 14].

2. The (left) coaction of $\mathcal{F}$ on $\mathcal{A}$ is defined on the generators by the formula

$$\varphi(K) = TKT^t$$

where $t$ stands for the matrix transposition, and is extended to the whole algebra $\mathcal{A}$ as an algebra homomorphism. For example,

$$\varphi(\beta) = a\alpha + bc\gamma + ad\beta + bd\delta$$

Verification of the $\mathcal{F}$-comodule-algebra axioms (7), (8) is a matter of direct calculation based on the commutation relation (4) and the equivalent relations

$$RT_1^t T_2^t = T_2^t T_1^t R, \quad T_1^t R_1^t T_2^t = T_2^t R_1^t T_1^t$$

obtained from (4) using transpositions and the symmetries

$$\mathcal{P} R^t \mathcal{P} = R, \quad [\mathcal{P}, R^t_1] = 0$$

of the R matrix (5). Here superscript $t$ means total transposition and $\mathcal{P}$ is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The duality between the quantum group $\mathcal{F} = F_q(GL(2))$ and the quantum algebra $\mathcal{U}_q(sl(2))$ [5, 15] implies that $\mathcal{A}$ is also a $\mathcal{U}_q(sl(2))$-module algebra.

The algebra $\mathcal{U}_q(sl(2))$ is generated by three elements $H, X_+, X_-$ and the relations

$$[H, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}$$

or in the matrix form

$$R^\pm L^\pm \hat{L}^\varepsilon = \hat{L}^\varepsilon L^\pm R^\pm \quad \forall \varepsilon \in \{+, -\}$$
where
\[ L^+ = \begin{pmatrix} q^H & \omega X_- \\ 0 & q^{-H} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{-H} & 0 \\ -\omega X_+ & q^H \end{pmatrix} \leq (12) \]
and
\[ R^+ = q^{-1/2}PRP, \quad R^- = q^{1/2}R^{-1} \]

The pairing \( \langle \cdot, \cdot \rangle \) between \( F \) and \( \mathcal{U}_q(sl(2)) \) is described by the relations [5]
\[ \langle L^\pm, T^\mp \rangle = R^\pm, \quad \langle L^\pm, T^\mp T^\mp \rangle = R^\pm R^\mp, \ldots \]
where, as usual, the subscripts mark the spaces where the corresponding R matrices act nontrivially.

The above formulas allow to calculate the corresponding action \( \varphi^* \) of \( \mathcal{U}_q(sl(2)) \) on \( \mathcal{A} \):
\[ \varphi^*(L^\pm)K \rightarrow R^\pm K (R^\pm)^{t_2} \]
or, more explicitly,
\[ \varphi^*(X_\pm) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ q^{-1}\alpha & \beta + q^{-1}\gamma \end{pmatrix} \]
\[ \varphi^*(X_\pm) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} q\beta + \gamma & q\delta \\ \delta & 0 \end{pmatrix} \]
\[ \varphi^*(q^H) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} q\alpha & \beta \\ \gamma & q^{-1}\delta \end{pmatrix} \]

3. Before we proceed to the discussion of the center of \( \mathcal{A} \) let us recall few facts about the quantum group \( F \). The quantum determinant \( \det_q T \) generating the center of \( F \) can be constructed via the fusion procedure technique [5, 16]
\[ P_-\hat{T}\hat{T} = P_-\hat{T}\hat{T}P_- = \det_q TP_- \leq (13) \]

The rank one projector \( P_- \) and the complementary rank three projector \( P_+ \) in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) are defined by the spectral decomposition of the modified R matrix [17]
\[ \hat{R} \equiv \mathcal{P}R = qP_+ - q^{-1}P_- \leq (14) \]

The commutativity of \( \det_q T \) with \( T \) and its group-like property \( \Delta(\det_q T) = \det_q T \otimes \det_q T \) follow from the relation (4), e.g.
\[ \Delta(\det_q T) = P_-\frac{1}{T_1\hat{T}_2\hat{T}_1\hat{T}_2}P_- \hat{T}_1\hat{T}_2 = P_-\hat{T}_1\hat{T}_2P_-\hat{T}_1\hat{T}_2 = \det_q T \otimes \det_q T \]
The quantum group $F_q(\text{GL}(2))$ is well known to possess the invariant quadratic form $\varepsilon_q$

$$\varepsilon_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \quad T\varepsilon_q T^t = T^t \varepsilon_q T = \varepsilon_q \text{det}_q T \quad (15)$$

The last relation, rewritten as \(\sum_{mn}(\varepsilon_q^t)_{mn}T_{ni}T_{jm} = (\varepsilon_q^t)_{ji}\), allows to introduce the trace operation for $K$ matrices which is invariant under the quantum group coaction (10) up to $\text{det}_q T$ factor

$$\text{tr}_q K \equiv \text{tr}_q\varepsilon_t K = \text{tr}_q TKT^t/\text{det}_q T, \quad T \in F_q(\text{GL}(2)) \quad (16)$$

Now we can prove that $c_1$ and $c_2$ defined by (9) lie in the center of $A$. Note that, by definition, $c_1 = \text{tr}_q K$. Take $\text{tr}_q$ of the relation (2) with respect to the first space in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Then, using the fact that the $R$ matrix, as a solution to the YBE, is a representation of the algebra $F$ (4) having the quantum determinant $\text{det}_q R = q$, and the identities (15), (16) one obtains from the left-hand-side of (2) the equality

$$\text{tr}_q^{(1)}(R KR^{t_1})\hat{K}^2 = q c_1(K) \hat{K}^2$$

and from the r.h.s. of (2) respectively

$$\hat{K}^2 \text{tr}_q^{(1)}(R^{t_1} RK) = \hat{K}^2 c_1(K) q$$

which establishes the commutativity of $c_1(K)$ with the generators $K$ of $A$.

In order to obtain the expression for $c_2$ analogous to (13) multiply the RE (2) from the left by the permutation operator $\mathcal{P}$

$$\hat{R}KR^{t_1} \hat{K}^2 = R^{t_1} K \hat{K}^2 \hat{R}$$

and then by the rank one projector $P_-$. Taking into account (14) one gets

$$P_- KR^{t_1} \hat{K}^2 = P_- R^{t_1} \hat{K}^2 P_- = c_2(K) P_-$$

The proof of the commutativity of $c_2(K)$ with the generators $K$ of $A$ uses the same fusion technique as the above proof for $\text{det}_q T$. The fact that the quantum determinants of $R$ and $R^{t_1}$ are numbers

$$P_-^{(12)} R_{32} R_{31} = q P_-^{(12)}, \quad P_-^{(12)} R_{31}^{t_3} R_{32}^{t_3} = q P_-^{(12)}$$
again plays the crucial role.

Rewriting \( q^2 c_2(K) K \) as \[ q^2 P^{(12)}_{-1} K R^{t_1} K^2 = P^{(12)}_{-1} K R^{t_1} R^{t_3}_{32} R^{t_3}_{31} K R_{32} R_{31} \]

using then the YBE for \( R \) and \( R^{t_1} \) and RE for several times one arrives finally to

\[ q^2 K P^{(12)}_{-1} K R^{t_1} K = q^2 K c_2(K) \]

which proves the assertion.

Note that the central elements are transformed homogeneously under the quantum group coaction (10)

\[ \varphi : c_1(K) \rightarrow \det_q T c_1(K), \quad c_2(K) \rightarrow (\det_q T)^2 c_2(K). \] (17)

The transformation law for \( c_1(K) \) follows from (16). The corresponding formula for \( c_2(K) \) follows from (15), (16) and the identity

\[ \text{tr}_q K \mathcal{E}_q K = (1 + q^2) c_2(K) - q (c_1(K))^2 \]

Furthermore

\[ K \mathcal{E}_q K = c_2(K) \mathcal{E}_q - q c_1(K) K. \]

The formula (17) and the following inversion formula for K matrix

\[ K^{-1} = \frac{1}{c_2} \begin{pmatrix} \delta & -\beta + \omega \gamma \\ -q^2 \gamma & \alpha \end{pmatrix} \]

allow to think of \( c_2 \) as the quantum determinant of \( K \).

We have no proof that for generic \( q \) the center of \( A \) is generated by \( c_1, c_2 \) though it is highly plausible conjecture taking into account the analogous results for the quantum group \( F \).

4. Let us discuss now the real forms of \( A \). It is well known that for \( F_q(GL(2)) \) there are three real forms [5]. For \( F_q(U(2)) \) the parameter \( q \) is real \( \bar{q} = q \) and the *-antiinvolution is given by

\[ \{a^*, b^*, c^*, d^*\} = \{d, -qc, -q^{-1}b, a\} \]
For $F_q(U(1, 1))$ again $\bar{q} = q$ and

$$\{a^*, b^*, c^*, d^*\} = \{d, qc, q^{-1}b, a\}$$

For $F_q(GL(2, \mathbb{R}))$ the parameter $q$ is unitary $\bar{q} = 1/q$ and

$$\{a^*, b^*, c^*, d^*\} = \{a, b, c, d\}$$

The corresponding real forms of $\mathcal{A}$ are:

$$F_q(U(2)) : \quad \{\alpha^*, \beta^*, \gamma^*, \delta^*\} = \{q\delta, -\beta, -\gamma, q^{-1}\alpha\}$$

$$F_q(U(1, 1)) : \quad \{\alpha^*, \beta^*, \gamma^*, \delta^*\} = \{q\delta, \beta, \gamma, q^{-1}\alpha\}$$

$$F_q(GL(2, \mathbb{R}) : \quad \{\alpha^*, \beta^*, \gamma^*, \delta^*\} = \{\alpha, \gamma + qc_1, q\gamma, \delta\}$$

It is easy to verify that these real forms of $\mathcal{A}$ are consistent with those of $\mathcal{F}$ that is $\varphi(K_{ij}^*) = \varphi(K_{ij})^*$.

5. The invariance of the ideal $c_1 = 0$ under the coaction $\varphi$ follows immediately from (17). The resulting quotient algebra is generated by three generators $\alpha, \gamma, \delta$ ($\beta = q\gamma$) and the relations

$$\alpha\gamma = q^2\gamma\alpha, \quad \gamma\delta = q^2\delta\gamma, \quad [\alpha, \delta] = q(q^2 - q^{-2})\!\gamma^2$$

The algebra is isomorphic to the quantum homogeneous space for $F_q(SL(2))$. It is also isomorphic to the subalgebra of $\mathcal{F}$ generated by the elements of the matrix $TT^t$ [5] (see [18] for the $F_q(GL(n))$ case).
4. Representations

The representation theory for the algebra $\mathcal{A}$ is a topic deserving special investigation. We restrict ourselves with few remarks.

There are two one-dimensional representations of $\mathcal{A}$:

$$K^{(0)} = \varepsilon_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \quad K^{(1)} = \begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix}$$ (18)

As shown in the next section, under the additional condition of $\gamma$ being invertible the quotient algebra $\mathcal{A}/(c_2 = 1)$ is isomorphic to the quantum algebra $\mathcal{U}_q(sl(2))$. Hence, the irreducible representations of $\mathcal{U}_q(sl(2))$ are translated into those of $\mathcal{A}$.

Another way to construct representations of $\mathcal{A}$ is to use the coaction $\varphi$ (10) of $F$. Any pair of representations $\pi$ of $\mathcal{A}$ and $\rho$ of $F$

$$\pi : \mathcal{A} \to \text{End}(V), \quad \rho : F \to \text{End}(W)$$

gives rise to another representation of $\mathcal{A}$

$$(\rho \otimes \pi) \circ \varphi : \mathcal{A} \to \text{End}(W \otimes V)$$

It is an open question whether this construction generates new irreducible representation of $\mathcal{A}$ provided $\pi$ and $\rho$ are irreducible. It is also unknown if any irreducible representation of $\mathcal{A}$ can be decomposed into $\pi$ and $\rho$.

Representations of comodule-algebras related to the reflection equations with $R$ matrices corresponding to other quantum (super)groups such as $F_q(GL(m|n))$ were studied in [19].

5. Variants of RE and related algebras

1. Consider the algebra $\mathcal{X}$ defined by the generators $x, y, u, v$ and by the relations

$$xy = qyx, \quad xu = qux, \quad xv = vx + \omega uy,$$

$$uv = qvu, \quad yu = uy, \quad yv = qvy,$$

Introducing the column $X$ and row $Y$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad Y = \begin{pmatrix} u & v \end{pmatrix}$$
one can represent the above commutation relations as the exchange algebras [7, 8, 11]

\[
RX_X^1 X^2 = qX^2 X^1, \quad \tilde{Y} \tilde{Y} R = q\tilde{Y} \tilde{Y}
\]

\[
\tilde{Y} R \tilde{t}_1 X^2 = X^2 Y^1, \quad \tilde{Y} R \tilde{t}_1 \tilde{X} = \tilde{X} Y^2
\]

(19)

(20)

The algebra \( \mathcal{X} \) has a central element \( \zeta = Y \varepsilon_q X = uy - qvx \). The subalgebras of \( \mathcal{X} \) spanned by \( X \) or \( Y \) are isomorphic to the “quantum plane” [13].

By virtue of the relations (19), (20) the matrix

\[
K = XY = \begin{pmatrix} xu & xv \\ yu & yv \end{pmatrix}
\]

satisfies RE (2) the above formula giving thus a homomorphism \( \mathcal{A} \rightarrow \mathcal{X} \). Note that for the obtained realization of \( \mathcal{A} \) one has \( c_2(K) = 0 \) and \( K \varepsilon_q K = \zeta K = -qc_1(K)K \).

2. In the papers [6, 7] another version of RE was obtained

\[
RM R^{-1} M = M \bar{R}^{-1} M \bar{R}, \quad \bar{R} \equiv \mathcal{P} \mathcal{R} \mathcal{P}, \quad M = \begin{pmatrix} \xi & \eta \\ \theta & \tau \end{pmatrix}
\]

(21)

The corresponding algebra \( \mathcal{B} \) is an \( F_q \)-comodule algebra with the coaction

\[
\psi : \mathcal{B} \rightarrow F \otimes \mathcal{B}, \quad \psi(M) = TMT^{-1}
\]

In the case of \( F = F_q(\text{GL}(2)) \) the \( F \)-comodule algebra \( \mathcal{B} \) is isomorphic to \( \mathcal{A} \) because of the relation (15). The isomorphism is given by the formula

\[
K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} -q\eta & \xi \\ -q\tau & \theta \end{pmatrix} = M \varepsilon_q
\]

which implies the relations between the central elements

\[
c_1(K) = \beta - q\gamma = \xi + q^2 \tau \equiv z_1(M)
\]

\[
c_2(K) = \alpha \delta - q^2 \beta \gamma = q^3(\xi \tau - q^{-2} \eta \theta) \equiv q^3 z_2(M)
\]

There is a characteristic equation for the matrix \( M \) also.

Under this isomorphism the q-trace (16) maps into \( \text{tr} DM, \; D = \text{diag}(q^{-1}, q) \) introduced in [5], see also [10].
The algebra $\mathcal{B}$ is connected closely to the quantum algebra $\mathcal{U}_q(sl(2))$ [6,7]. Namely, the matrix $M$ can be realized in terms of $L^\pm$ (12)

$$M = s(L^+)L^-$$

(22)

where $s$ is the antipode in $\mathcal{U}_q(sl(2))$: $s(H) = -H$, $s(X_{\pm}) = -q^{\mp 1}X_{\pm}$.

The formula (22) describes the algebra homomorphism $\chi: \mathcal{B} \to \mathcal{U}_q(sl(2))$

$$\chi(M) = \begin{pmatrix} q^{-2H} + q\omega^2X_-X_+ & -\omega qX_-q^H \\ -\omega q^H X_+ & q^2H \end{pmatrix}$$

for which $z_1(M)$ is proportional to the well known Casimir operator for $\mathcal{U}_q(sl(2))$ and $z_2(M) = 1$ [10].

It is easy to see that the inversion of the homomorphism $\chi$ needs invertibility of the element $\tau$ (which is not the case for the one-dimensional representation $M = K^{(1)}\varepsilon_q$, see (18)). If the element $\tau$ is supposed to be invertible one can introduce in $\mathcal{B}$ a coproduct induced from $\mathcal{U}_q(sl(2))$ [7].

3. We conclude this section with describing the classical limit of $\mathcal{A}$. Let $q = e^{\hbar}$. As $\hbar \to 0$ or $q \to 1$ the algebra $\mathcal{A}$ becomes commutative, its commutator giving rise to the Poisson bracket $[\cdot, \cdot] = -\hbar\{\cdot, \cdot\}$. The commutation relations (2) are transformed into the Poisson brackets relations

$$\{\frac{1}{K}, \frac{2}{K}\} = [r, \frac{1}{K}\frac{2}{K}] + \frac{1}{K}r^{t_1}\frac{2}{K} - \frac{2}{K}r^{t_1}\frac{1}{K}$$

(23)

where $r$ is the classical $r$ matrix $R = 1 + hr + O(h^2)$

$$r = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad r^{t_1} = \begin{pmatrix} 1 & 2 \\ 1 \end{pmatrix}$$

Fixing values of the central functions $c_1 = \beta - \gamma$ and $c_2 = \det K = \alpha\delta - \beta\gamma$ one obtains the foliation of the 4-dimensional space spanned by $\alpha\beta\gamma\delta$ into 2-dimensional manifolds on which the Poisson bracket (23) is nondegenerate.
6. Generalizations

1. Let us discuss now the fusion procedure for RE. It does not differ much from the case of the YBE [5, 16]. The peculiarities of the RE case are well seen in the simplest spin-1 case. The R matrix in (4) \( R \in \mathbb{C}^2 \otimes \mathbb{C}^2 \) intertwines two spin-1/2 corepresentations of \( F_q(GL(2)) \). The R matrix in \( \mathbb{C}^2 \otimes \mathbb{C}^3 \) is

\[
R_{1(2)} = P_+ R_{12} R_{12} = P_+ R_{12} R_{12} P_+
\]

where \( P_+ \) is the rank 3 projector (14) from \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) onto \( \mathbb{C}^3 \) and the corresponding spaces are labelled respectively by the indices 2, 2′ and (2). We shall use also

\[
R_{1(2)}^t = P_+ R_{12}^t R_{12}^t P_+
\]

The 3 × 3 matrix \( J \)

\[
J = P_+ K R_{22'} K P_+
\]

satisfy the RE in \( \mathbb{C}^2 \otimes \mathbb{C}^3 \)

\[
R_{1(2)}^t K R_{1(2)}^t J = J R_{1(2)}^t K R_{1(2)}^t
\]

and also the RE in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \)

\[
R_{1(2)}^t J R_{1(2)}^t = J R_{1(2)}^t J R_{1(2)}^t
\]

The entries of the matrix \( J \) generate a subalgebra \( \mathcal{A}' \) of \( \mathcal{A} \) which is also an \( F \)-coideal that is \( \varphi : \mathcal{A}' \to F \otimes \mathcal{A}' \). The analogous fusion procedure for the spectral parameter dependent RE (1) was developed in [20].

There is no doubt that, being properly generalized, the above described fusion procedure should be able to produce solutions to RE in matrices of any dimension corresponding to higher finite-dimensional irreducible corepresentations of \( F_q(GL(2)) \). At least for the one-dimensional representations of \( \mathcal{A} \) (18) the expression for the “universal K matrix” can be written explicitely (see [21] for \( K^{(1)} \) and [12, 19] for \( K^{(0)} \)).

2. Throughout this paper we considered the algebra \( \mathcal{A} \) defined by the relations (2) corresponding to the R matrix (5) of the quantum group \( F_q(GL(2)) \). However, the proof of the comodule algebra property does not use anything besides the commutation
relations (4) and the symmetries (11) of the R matrix. Therefore, to any R matrix satisfying (11) and the related quantum group $F_R$ defined by relations (4) there corresponds some $F_R$-comodule-algebra $\mathcal{A}_R$ defined by relations (2).

The conditions (11), however, are not essential and can be disposed of. Let us extend the algebra $F_R$ introducing another matrix of generators $S$ in addition to $T$ and imposing the commutation relations

\begin{align*}
R^{(1)} T^2 R^{(1)} &= T^2 R^{(1)}, & S R^{(2)} T^2 &= T S R^{(2)}, \\
T R^{(3)} S^2 &= S R^{(3)} T^1, & R^{(4)} S^2 &= S R^{(4)},
\end{align*}

parametrized by four R matrices. It is easy to verify that if $K$ obeys the relation

\begin{align*}
R^{(1)} K R^{(2)} &= 2 R^{(3)} K R^{(4)},
\end{align*}

then the matrix $T K S$ obeys the same relation provided matrix elements of $K$ commute with those of $T$ and $S$.

Suppose now that $S = \sigma(T)$ where $\sigma$ is an antiautomorphism of matrix algebra that is $\sigma(xy) = \sigma(y)\sigma(x)$ for any number matrices $x$, $y$. Then all the R matrices in (25) can be expressed in terms of the matrix $R$ defining the quantum group (4). The resulting algebra is an $F_R$-comodule-algebra with the coaction $K \to TK\sigma(T)$.

Note that there exists a remarkable ambiguity in the choice of the R matrices. Namely, one can make the substitution $R \to P R^{-1} P$ independently in any of the matrices $R^{(i)}$ which does not affect the commutation relations (24), cf. the remark immediately after formula (5). Consequently, there exist at least $2^4 = 16$ a priori nonequivalent algebras sharing the same comodule structure. The interrelations of these algebras will be described in a separate paper.

In the case of $\sigma(T) = T^t$ the R matrices are

\begin{align*}
R^{(1)} &= R & \text{or } \tilde{R}^{-1} \\
R^{(2)} &= R^{t_1} & \text{or } (\tilde{R}^{-1})^{t_1} \\
R^{(3)} &= (\tilde{R})^{t_2} & \text{or } (R^{-1})^{t_2} \\
R^{(4)} &= \tilde{R}^t & \text{or } (R^{-1})^t
\end{align*}

where, as in (21), the notation $\tilde{R} \equiv PRP$ is used. Choosing the first option for every $R^{(i)}$ one obtains from (25) the equation [19]

\begin{align*}
K R^{t_1} R^{t_2} R^{t_3} R^{t_4} = K \tilde{R}^{t_1} \tilde{R}^{t_2} \tilde{R}^{t_3} \tilde{R}^{t_4}
\end{align*}
coinciding with the RE (2) if $R$ has the symmetries (11).

Another example is provided by $\sigma(T) = T^{-1}$. In this case

$$
R^{(1)} = R \quad \text{or} \quad \tilde{R}^{-1}
$$

$$
R^{(2)} = R^{-1} \quad \text{or} \quad \tilde{R}
$$

$$
R^{(3)} = \tilde{R}^{-1} \quad \text{or} \quad R
$$

$$
R^{(4)} = \tilde{R} \quad \text{or} \quad R^{-1}
$$

The choice of the first option for every $R^{(i)}$ leads to the algebra $B$ described in the section 5.2.

3. We conclude our discussion of RE with a remark concerning the very form of the equation. Note that formula (10) for the coaction of $F$ is equivalent to

$$
\varphi : K^{i_1 i_2} \rightarrow T^{i_1}_{j_1} T^{i_2}_{j_2} K^{j_1 j_2}
$$

(27)

where the indices are shown explicitly. Here we use the commutativity of $T$ and $K$ and assume summation over repeated indices.

The formula (27) suggests that $K$ can be thought of as a bivector (contravariant tensor of the second order). In a compact form one can write (27) as

$$
\varphi : K_{12} \rightarrow \frac{1}{T} \cdot T \cdot K_{12}
$$

(28)

(the notation is self-evident).

Analogously, the equation (26) can be rewritten as

$$
R^{i_1 i_2} R^{j_1 j_2} K^{i_1 j_1} K^{i_2 j_2} R^{j_1 j_1} K^{i_2 j_2} = K^{i_1 j_1} R^{j_1 j_1} K^{j_2 j_2} R^{i_2 j_2}
$$

or, using the same notation as in (28)

$$
R_{12} R_{1'2'} K_{11'} K_{22'} = \tilde{R}_{1'2'} \cdot \tilde{R}_{12} \cdot K_{22'} K_{11'}
$$

(29)

Moving two $R$ matrices from the right hand side of (29) to the left hand side one obtains

$$
\tilde{R}_{1'2'} \tilde{R}_{1'2'} R_{12} R_{1'2'} K_{11'} K_{22'} = K_{22'} K_{11'}
$$

(30)

Combining the two indices in $K^{i_1 i_2}$ into one composite index and the four $R$ matrices in (30) into one composite $R$ matrix we observe that the equation (30) describes a sort of exchange algebra (19).
The case $\sigma(T) = T^{-1}$ leads in the same way to a mixed tensor having one upper and one lower index.

This observation opens a road to a far-going generalization of the algebra $\mathcal{A}$. Acting in the same spirit one can introduce algebras corresponding to the quantum tensors with any number of upper and lower indices. The papers [13, 22] where the quantum multilinear algebras are studied might be of relevance in performing this program. We hope to touch this subject in a separate article.

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