Implementing fanout, parity, and Mod gates via spin exchange interactions

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Abstract

We show that, for any \( n > 0 \), the Heisenberg interaction among \( 2n \) qubits (as spin-1/2 particles) can be used to exactly implement an \( n \)-qubit parity gate, which is equivalent in constant depth to an \( n \)-qubit fanout gate. Either isotropic or nonisotropic versions of the interaction can be used. We generalize our basic results by showing that any Hamiltonian (acting on suitably encoded logical qubits), whose eigenvalues depend quadratically on the Hamming weight of the logical qubit values, can be used to implement generalized \( \text{Mod}_q \) gates for any \( q \geq 2 \).

This paper is a sequel to quant-ph/0309163 and resolves a question left open in that paper.

1 Introduction

Let \( \mathcal{H} \) be the Hilbert space of \( n \) qubits, where \( n \geq 1 \). The fanout operator \( F_n : \mathcal{H} \rightarrow \mathcal{H} \), depicted in Figure 1, copies the (classical) value of a single qubit to \( n \) other qubits. Unbounded fanout is usually taken for granted in models of classical Boolean circuits, even when the fanin of gates is bounded. They cannot be taken for granted in quantum circuits, however, since copying the value of a quantum bit to \( n - 1 \) other bits requires significant nonlocal interactions. Unbounded fanout gates have been shown to be a surprisingly powerful primitive for quantum computation, allowing one to reduce the depth of a circuit computing, say, the Quantum Fourier Transform (QFT) to essentially constant depth [10]. The quantum part of Shor’s factoring algorithm can thus be implemented in constant depth if unbounded fanout gates are available. This result about the power of fanout is especially important because in most of the significant proposals for implementing quantum circuits to date, long computations will surely be difficult to maintain due to decoherence, current quantum error correction techniques notwithstanding. Shallow quantum circuits may prove to be, at least in the short term, the only realistic model of feasible quantum computation, and fanout gates would increase their power significantly.

Without some quantum gate with unbounded width (arity), it is not clear that any nontrivial decision problem can be computed by \( o(\log n) \)-depth quantum circuits with bounded error. This is
certainly true if we only allow a one-qubit output measurement (then the output can only depend on \(2^{\log n}\) input qubits—see [5] for a discussion), but it also seems to be true even if we allow a computational-basis measurement of an arbitrary number of qubits at the end [6]. Even if we allow unbounded AND gates (generalized Toffoli gates), it is not clear what we can do in sublogarithmic depth. We do know that we cannot approximate fanout gates this way [5].

To summarize, fanout gates are an extremely useful, and perhaps necessary, primitive for allowing small-depth quantum circuits to solve useful problems. Furthermore, implementing fanout with a conventional quantum circuit requires logarithmic depth, even if unbounded AND gates are allowed. Therefore, implementing a fast fanout gate will require an unconventional approach.

We provide such an approach here by showing that the fanout operator arises easily by evolving qubits via a simple and well-studied Hamiltonian, the spin-exchange or Heisenberg interaction, together with a modest amount of encoding and decoding of qubits (which only requires constant-width gates and constant depth). Our results answer positively a question by I. L. Chuang, who asked how certain forms of the Heisenberg interaction, which are implementable in the laboratory, may be useful for quantum computation [3, 4]. In particular, we show that the fanout gate on \(n\) logical qubits can be achieved exactly by encoding them into \(2n\) physical qubits (each a spin-1/2 particle), then applying the Heisenberg interaction to the encoded qubits. The interaction need not be isotropic; both isotropic and nonisotropic versions of the interaction work equally well.

In [7], we showed that a variant of the Heisenberg interaction, where the Hamiltonian is proportional to the square of the \(z\)-component of the total spin, can implement parity easily (without encoding). When applied to three qubits, this interaction yields an “inversion on equality gate” \(I_=-\), defined by

\[
I_- |xyz\rangle = \begin{cases} 
- |xyz\rangle & \text{if } x = y = z, \\
|xyz\rangle & \text{otherwise}.
\end{cases}
\]

\(I_-\) and single-qubit gates together form a universal set of gates. Recently, implementation of \(I_-\) as well as the three-qubit parity and fanout gates in NMR using the above Hamiltonian has been reported [8]. Our current paper affirmatively answers a question left open in [7] as to whether parity/fanout can be implemented using more common forms of the Heisenberg interaction, involving \(x\)-, \(y\)-, and \(z\)-components of the total spin.
In Section 2 we define the general Heisenberg interaction between \( n \) identical spins, as well as the special case of interest to us. In Section 3.1 we give an implementation of the \((r+1)\)-bit parity gate, depicted in Figure 2 where \( r = n/2 \). The fanout gate arises by placing Hadamard gates on each qubit on both sides of the parity gate \((2(r+1)\) Hadamard gates in all), and thus implementing parity is equivalent to implementing fanout. We also show in Section 3.2 how different qubit encoding schemes can reduce the ratio \( n/r \) to be arbitrarily close to one. In Section 4, we generalize our results in two ways: (1) any Hamiltonian whose eigenvalues depend quadratically on the Hamming weight of the logical qubits can be used to implement parity, and hence fanout, and (2) any such Hamiltonian can implement generalized Mod\(_q\)-gates directly for any \( q \geq 2 \).

2 Preliminaries

The Heisenberg interaction describes the way particles in the same general location affect each other by the magnetic moments arising from their spin angular momenta. Given a system of \( m \) identical labeled spins described by vector operators \( \vec{S}_1, \ldots, \vec{S}_m \), the Hamiltonian \( E \) of the system is a weighted sum of the energies of all the pairwise interactions, plus a term for any external magnetic field (assumed to be in the \( z \)-direction):

\[
E = - \sum_{i<j} J_{i,j} \vec{S}_i \cdot \vec{S}_j + \alpha \sum_i (\vec{S}_i)_z,
\]

where the \( J_{i,j} \) and \( \alpha \) are constants.\(^1\) In this paper, we will show how this interaction can implement fanout in the special case where all the \( J_{i,j} \) are equal. In this case, \( E \) is related to the squared magnitude of the total spin of the system.

We will assume here that physical qubits are implemented as spin-1/2 particles, with \(|0\rangle\) being the spin-up state (in the positive \( z \)-direction) and \(|1\rangle\) being the spin-down state (in the negative \( z \)-direction) and \( \alpha \) being the product of the magnetic field strength and the gyromagnetic ratio for the individual spins (see [12], §21.3 for example).

\(^1\)The \( J_{i,j} \) are usually assumed to be positive, appropriate for ferromagnetic interactions which give the lowest energy when spins are aligned in parallel. The value of \( \alpha \) is the product of the magnetic field strength and the gyromagnetic ratio for the individual spins (see [12], §21.3 for example).
z-direction). Given a system of \( m \) qubits labeled \( 1, \ldots, m \), we define

\[
\begin{align*}
J_x &= \frac{1}{2} \sum_{i=1}^{m} X_i \\
J_y &= \frac{1}{2} \sum_{i=1}^{m} Y_i \\
J_z &= \frac{1}{2} \sum_{i=1}^{m} Z_i
\end{align*}
\]

where \( X_i, Y_i, \) and \( Z_i \) are the three Pauli operators acting on the \( i \)th qubit. \( J_x, J_y, \) and \( J_z \) give the total spin in the \( x \), \( y \), and \( z \)-directions, respectively. The squared magnitude of the total spin angular momentum of the system is given by the observable

\[
J^2 = J_x^2 + J_y^2 + J_z^2.
\]

Note that

\[
J^2 = \frac{3m}{4} I + \frac{1}{2} \sum_{1 \leq i < j \leq m} (X_i X_j + Y_i Y_j + Z_i Z_j) = \frac{3m}{4} I + \sum_{i < j} \vec{S}_i \cdot \vec{S}_j,
\]

where \( \vec{S}_i = \frac{1}{2} (X_i, Y_i, Z_i) \) is the vector observable giving the spin of the \( i \)th qubit. It is then clear that, in the absence of an external magnetic field, \( J^2 \) is linearly related to the energy \( E \) above. This is an isotropic Heisenberg interaction. To account for an external field in the \( z \)-direction, we define, for any real \( \alpha \),

\[
H_\alpha = -J^2 + \alpha J_z.
\]

This is the case of the Heisenberg interaction where all the \( J_{i,j} \) are unity.\(^2\) Our methods can also accommodate an extra term in the Hamiltonian proportional to \( J_z^2 \) with no additional effort (see Section 3.1), so we consider the more general Hamiltonian

\[
H_{\alpha,\beta} = -J^2 + \alpha J_z + \beta J_z^2
\]

for any real \( \alpha \) and \( \beta \) such that \( \beta \neq 1 \). We will evolve the system of \( m \) qubits using \( H_{\alpha,\beta} \) as the Hamiltonian. A related Hamiltonian \( J_z^2 \) is used in [7] to implement fanout; this is an easy case, since each computational basis state is already an eigenstate of \( J_z^2 \) and thus the implementation requires no encoding of qubits. Using the current Hamiltonian \( H_{\alpha,\beta} \) is more complicated and requires encoding logical qubits into groups of physical qubits, so that the tensor product of all the physical qubits encoding a logical basis state will be an eigenstate of \( H_{\alpha,\beta} \).

In the sequel, we choose units so that \( \hbar = 1 \). If \( A \) and \( B \) are both vectors or both operators, we say, \( "A \propto B" \) to mean that \( A = e^{i\theta} B \) for some real \( \theta \), that is, \( A = B \) up to an overall phase factor. We use the same notation with individual components of \( A \) and \( B \), meaning that the phase factor is independent of which component we choose. If \( x \in \{0,1\}^n \) is a bit vector, we let \( \text{wt}(x) \) denote the Hamming weight of \( x \), that is, the number of 1s in \( x \).

\(^2\)This does not lose generality, since constant factors in the energy can be absorbed by adjusting the time of the interaction.
2.1 Spin States

The properties of the $J$-operators are well-known. See, for example, Böhm [2]. We will review the essential ones here. The commutation relations are $[J_x, J_y] = i J_z$, and likewise for the two other cyclic shifts of the indices. $J^2$ commutes with $J_z$, so one may choose an orthonormal basis of the $n$-qubit Hilbert space $\mathcal{H}$ that diagonalizes both simultaneously. Eigenstates of $J^2$ and $J_z$ are traditionally labeled as $|j, m, \ell\rangle$, where $0 \leq j \leq n/2$ and $-j \leq m \leq j$, with $n/2 - j$ and $j - m$ both integers. We have $J_z |j, m, \ell\rangle = m |j, m, \ell\rangle$, and $J^2 |j, m, \ell\rangle = j(j+1) |j, m, \ell\rangle$. The extra parameter $\ell$ is used to give distinct labels to different basis vectors in degenerate eigenspaces of $J^2$ and $J_z$. These basis vectors can be chosen so that, for any value $\ell$ that appears as the third label of some basis vector, the basis vectors labeled by $\ell$ span an irreducible spin representation, that is, a minimal subspace of $\mathcal{H}$ invariant under the action of $J_x$, $J_y$, and $J_z$. This space will be spanned by the basis vectors $|j, -j, \ell\rangle, |j, -j+1, \ell\rangle, \ldots, |j, j-1, \ell\rangle, |j, j, \ell\rangle$, for some $j = j(\ell)$ depending only on the label $\ell$, and is called a spin-$j$ representation. Letting $J_+ = J_x + i J_y$ and $J_- = J_x - i J_y$ be the usual raising and lowering operators, respectively, we may adjust the phases of the basis vectors so that

$$J_+ |j, m, \ell\rangle = \sqrt{j(j+1) - m(m+1)} |j, m+1, \ell\rangle,$$

$$J_- |j, m, \ell\rangle = \sqrt{j(j+1) - m(m-1)} |j, m-1, \ell\rangle.$$  

(This sets the relative phases of states within the representation, but still allows the overall phase of the representation to be adjusted relative to other representations.)

2.2 Number of Spin Representations

An important fact that we will use later is that for each $j$, there are exactly $k_{n,j} := \binom{n}{n/2-j} - \binom{n}{n/2-j-1}$ many spin-$j$ representations in the decomposition of $\mathcal{H}$.\footnote{By convention, if $k < 0$ then $\binom{n}{k} = 0$.} One way to see this is as follows. For any $j \geq 0$ such that $n/2 - j$ is an integer, let $\mathcal{H}_j$ be the eigenspace of $J_z$ with eigenvalue $j$ (if $j > n/2$, then $\mathcal{H}_j$ has dimension zero). Clearly, $\dim(\mathcal{H}_j) = \binom{n}{n/2-j}$, since $\mathcal{H}_j$ is spanned by all the computational basis vectors with Hamming weight $n/2 - j$. The $J_+$ operator maps $\mathcal{H}_j$ into $\mathcal{H}_{j+1}$, and so its kernel on $\mathcal{H}_j$ has dimension at least $k_{n,j} > 0$, given above. Now the space $\ker(J_+) \cap \mathcal{H}_j$ is spanned by the set of all states of the form $|j, j, \ell\rangle$, so there are no less than $k_{n,j}$ distinct values for $\ell$ occurring in the set, and each one labels a distinct spin-$j$ representation. Finally, since each spin-$j$ representation has dimension $2j + 1$ for all $j$, and since $\mathcal{H}$ has $2^n$ dimensions, a simple counting argument shows that there can be no more than $k_{n,j}$ many spin-$j$ representations, either.

It follows that there are a total of $\binom{n}{\lfloor n/2 \rfloor}$ many spin representations in the decomposition of $\mathcal{H}$. Moreover, if $n$ is even, then it is easy to show that the representations are evenly split between those where $n/2 - j$ is even and those where $n/2 - j$ is odd: $\frac{1}{2} \binom{n}{\lfloor n/2 \rfloor}$ representations for each. This fact will be used in Section 3.2.

2.3 Spin States versus Computational Basis States

Finally, we mention how some of the spin states relate to computational basis states. There is one spin-$n/2$ representation in the decomposition, namely, the completely symmetric representation,
which is spanned by the states

\[ |n/2, n/2 - k \rangle = \binom{n}{k}^{-1/2} \sum_{\text{wt}(x) = k} |x \rangle \]

for integer \( k \) with \( 0 \leq k \leq n \). (This equation sets the overall phase of the spin-\( n/2 \) representation.) A key point in this paper is to note that \( |\frac{n}{2}, \frac{n}{2} \rangle = |0^n \rangle \) is a tensor product of single qubits. This means that some spin states involve little or no entanglement among qubits and thus can be prepared using only reasonably local interactions. More generally, suppose we group some of the \( n \) qubits into disjoint pairs \((i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)\) for some \( 0 \leq p \leq n/2 \) (that is, \( i_1, j_1, \ldots, i_p, j_p \in \{1, \ldots, n\} \) and are all pairwise distinct), then we form a state \( |\psi \rangle \) by putting each pair \((i_k, j_k)\) of qubits into the singlet state \( |0, 0 \rangle = (|10 \rangle - |01 \rangle)/\sqrt{2} \) and each of the rest of the (unpaired) qubits into the \( |0 \rangle \) state, i.e.,

\[ |\psi \rangle = |0, 0\rangle_{i_1, j_1} \cdots |0, 0\rangle_{i_p, j_p} |0^{n-2p}\rangle_S = |0, 0\rangle_{i_1, j_1} \cdots |0, 0\rangle_{i_p, j_p} |j, j\rangle_S, \]

(4)

where \( j = n/2 - p \), and \( S \) is the set of all unpaired qubits. Then it is easy to check that \( |\psi \rangle \) is an eigenstate of \( J_z \) (with eigenvalue \( n/2 - p \)) and is in \( \ker(J_+) \). It follows that \( |\psi \rangle \) is also an eigenstate of \( J^2 \), because \( 0 = J_- J_+ |\psi \rangle = (J_z^2 - J_z) |\psi \rangle \). Many of the states \(|j, j, \ell\rangle\) where \( j = n/2 - p \) can be defined this way, but not all—different choices of the \( p \) pairs do not always produce states that are orthogonal to each other, or even linearly independent.

### 3 Main Results

#### 3.1 Parity Gate by Heisenberg Interactions

Recall the Hamiltonian \( H_{\alpha, \beta} \) of (1) on the space \( \mathcal{H} \) of \( n \) qubits labeled 1, \ldots, \( n \). \( H_{\alpha, \beta} \) commutes with \( J^2 \), so it has eigenvectors \(|j, m, \ell\rangle\) with respective eigenvalues \(-j(j + 1) + \alpha m + \beta m^2\).

Let \( x = x_1 \cdots x_r \) be a vector of \( r \) bits, where \( r \) is some number no greater than \( n \). We wish to encode the \( r \)-qubit computational basis state \(|x\rangle\) into an \( n \)-qubit eigenstate \(|j, m, \ell\rangle\) of \( H_{\alpha, \beta} \) so that \( j \) depends linearly on \( wt(x) \), and we want to do this by using gates that act on as few qubits as possible. The easiest way to accomplish this is to have \( n = 2r \) and create encoded states of the form given by (4). One may encode each input qubit (with an ancilla) into two qubits, sending \(|00\rangle\) to \(|0_L\rangle := |00\rangle\) and sending \(|10\rangle\) to \(|1_L\rangle := (|10\rangle - |01\rangle)/\sqrt{2} \), the singlet state. A simple circuit—one of many—for this is shown in Figure 3 which defines the encoding operator \( E \). (Since \( E^\dagger = E \), we will also use \( E \) to decode.) Clearly, there are other operators that will work just as well, since \( E \) is underdetermined. We encode the \( i \)th input with ancilla into the qubits \( 2i - 1 \) and \( 2i \) of \( \mathcal{H} \). Thus \(|x_1 \cdots x_r\rangle\) maps to \(|x_L\rangle := |x_1L\rangle \otimes \cdots \otimes |x_rL\rangle\), and this state is in the form of (4), where the set \( S \) consists of all physical qubits encoding the \(|0_L\rangle\) states, i.e., \( S = \bigcup_{x_i=0} \{2i - 1, 2i\} \). If \( x \neq y \), then

![Figure 3: A two-qubit encoder.](image-url)
clearly \( \langle x_L | y_L \rangle = 0 \), so we can assume without loss of generality that \( |x_L\rangle = |j(x), j(x), x\rangle \), where \( j(x) := n/2 - \text{wt}(x) \), and \( x \) itself is used for the label.

Suppose that \( \beta \neq 1 \), and let
\[
t := \frac{\pi}{2|\beta - 1|}.
\]
We let
\[
U := e^{-itH_{\alpha, \beta}},
\]
the unitary operator resulting from evolving the qubits with \( H_{\alpha, \beta} \) for time \( t \). For fixed input vector \( x = x_1 \cdots x_r \in \{0, 1\}^r \), let \( k = \text{wt}(x) \), and for \( b \in \{0, 1\} \) let \( x^b := x_1 \cdots x_{r-1} b \). To compute the parity of \( k \) with a quantum circuit on input \( |x\rangle \), we first run qubit \( r \) through a Hadamard gate to produce the state
\[
|\varphi_x\rangle := \frac{|x^0\rangle + (-1)^{x^r}|x^1\rangle}{\sqrt{2}}.
\]
Next, we encode each qubit as described above to obtain
\[
|\varphi_{x,L}\rangle := \left| x^{j_0}_L \right\rangle + \frac{(-1)^{x^r}}{\sqrt{2}} \left| x^{j_1}_L \right\rangle = \frac{|j_0, j_0, x^0\rangle + (-1)^{x^r}|j_1, j_1, x^1\rangle}{\sqrt{2}},
\]
where we have set \( j_0 := j(x^b) = n/2 - \text{wt}(x^b) \). Observing that \( j_1 = j_0 - 1 \), we see that \( |\varphi_{x,L}\rangle \) is a balanced superposition of two spin states, one with even \( j \) and the other with odd \( j \). We now apply \( U \) to \( |\varphi_{x,L}\rangle \). Noting that \( H_{\alpha, \beta}|j, j, \ell\rangle = ( (\beta - 1)j^2 + (\alpha - 1)j) |j, j, \ell\rangle \) for any \( j \), we set
\[
\gamma := \frac{\alpha - 1}{\beta - 1},
\]
and define
\[
|\eta_{j,\ell}\rangle := U|j, j, \ell\rangle = e^{-i\pi(j^2 + \gamma j)/2}|j, j, \ell\rangle
\]
for any \( j \) and \( \ell \), where for convenience, we are letting
\[
s := \frac{\beta - 1}{|\beta - 1|} = \begin{cases} 1 & \text{if } \beta > 1, \\ -1 & \text{if } \beta < 1. \end{cases}
\]
We have
\[
U|\varphi_{x,L}\rangle = (|\eta_{j_0, x^0}\rangle + (-1)^{x^r}|\eta_{j_1, x^1}\rangle) / \sqrt{2}
\]
\[
= \left[ \exp(-i\pi(j_0^2 + \gamma j_0)/2) |j_0, j_0, x^0\rangle + (-1)^{x^r} \exp(-i\pi(j_1^2 + \gamma j_1)/2) |j_1, j_1, x^1\rangle \right] / \sqrt{2}
\]
\[
= |x_1, L\rangle \otimes \cdots \otimes |x_{r-1}, L\rangle \otimes |\Psi_x\rangle,
\]
where
\[
|\Psi_x\rangle = \frac{e^{-i\pi(j_0^2 + \gamma j_0)/2}|0_L\rangle + (-1)^{x^r} e^{-i\pi(j_1^2 + \gamma j_1)/2}|1_L\rangle}{\sqrt{2}}
\]
\[
= \frac{e^{-i\pi(j_0^2 + \gamma j_0)/2}}{\sqrt{2}} \left[ |0_L\rangle + (-1)^{x^r} e^{-i\pi(2j_0 + 1 - \gamma)/2}|1_L\rangle \right]
\]
\[
= \frac{1}{\sqrt{2}} \left[ |0_L\rangle + (-1)^{x^r + \text{wt}(x)} e^{i\pi(\gamma - 1)/2}|1_L\rangle \right]
\]
\[
= \frac{1}{\sqrt{2}} \left[ |0_L\rangle + (-1)^{r + \text{wt}(x)} e^{i\pi(\gamma - 1)/2}|1_L\rangle \right]
\]
is the state of the $(2r-1)$st and $(2r)$th qubits. (Recall that $j_1 = j_0 - 1$, and that $j_0 = \frac{n}{2} - \text{wt}(x^0) = r - \text{wt}(x) + x_r$.)

For any $y = y_1 \cdots y_r \in \{0,1\}^r$, it is easy to see that if $\text{wt}(x)$ and $\text{wt}(y)$ have opposite parity, then $|\Psi_y\rangle$ and $|\Psi_x\rangle$ are orthogonal: it follows immediately from (9–12) that

$$2\langle \Psi_y | \Psi_x \rangle \propto 1 + (-1)^{\text{wt}(x) + \text{wt}(y)},$$

which is zero if $\text{wt}(x)$ and $\text{wt}(y)$ have opposite parity. This analysis shows that we have isolated the parity information in the $r$th logical qubit. We decode just the two physical qubits corresponding to this qubit to obtain the state

$$E |\Psi_x\rangle = \frac{e^{-i\pi(j_0^2 + j_0) / 2}}{\sqrt{2}} \left[ |00\rangle + (-1)^{r + \text{wt}(x)} e^{i\pi(\gamma - 1) / 2} |10\rangle \right]$$

$$\propto \frac{1}{\sqrt{2}} \left[ |0\rangle + (-1)^{r + \text{wt}(x)} e^{i\pi(\gamma - 1) / 2} |1\rangle \right] \otimes |0\rangle.$$

The second of these qubits is the restored ancilla. We then apply two gates, $V$ followed by $H$, to the first qubit, where

$$V := \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi(2r+\gamma-1)/2} \end{bmatrix},$$

and $H$ is the Hadamard transform. This yields $|\text{wt}(x) \mod 2\rangle$ as the state of the first qubit, up to some unconditional phase factor.

If we do not mind the extra phase factor, which only depends on $\gamma$, $r$, $s$, and $\text{wt}(x_0)$, then we can simply use $E$ to decode the other pairs of physical qubits, and we thus obtain a circuit that computes parity. To cleanly and exactly match the parity gate defined in Figure 2, however, we may first copy the parity information onto a fresh qubit, then undo the previous computation. The circuit for the latter operation is shown in Figure 4. The gate on the left in Figure 4 is an $(r+1)$-qubit parity gate.

Figure 4: Circuit to implement parity with Heisenberg interactions.
Remark. The circuit of Figure 4 uses the $U^\dagger$ gate. Any unitary gate $U'$ that agrees with $U^\dagger$ on the subspace $\mathcal{H}'$ of $\mathcal{H}$ spanned by vectors of the form $|j, j, \ell\rangle$ can substitute for $U^\dagger$ to implement parity exactly. We would like to implement $U'$ by evolving the same Hamiltonian $H_{\alpha, \beta}$ for some positive length of time. We can do this if there is a $u > 0$ such that $e^{-iuH_{\alpha, \beta}}$ fixes all vectors in $\mathcal{H}'$. Then we can implement $U'$ by evolving $H_{\alpha, \beta}$ for time $ku - t$ where $t$ is given by (5) and $k$ is some integer such that $ku \geq t$, i.e., $U' := e^{-i(ku-t)H_{\alpha, \beta}}$. By (3), it can be shown that such a $u$ exists if and only if the $\gamma$ of (7) is rational. For arbitrary real $\gamma$, we can still implement $U'$ by evolving the altered Hamiltonian $H_{\alpha', \beta'}$ for some suitable $\alpha', \beta'$ and time $t'$. Assuming $\beta' \neq 1$ and letting $\gamma' := \frac{\alpha'}{\beta'} - 1$, it follows from (3) by a straightforward argument that for $t' \geq 0$, the operator $U'' := e^{-i\gamma'H_{\alpha', \beta'}}$ is a suitable replacement for $U^\dagger$ (i.e., $U''U$ fixes all vectors in $\mathcal{H}'$) if and only if there is an integer $\ell$ such that (i) $(2\ell + 1)(\gamma' + 1) + s(\gamma + 1)$ is an integer multiple of four, (ii) $2\ell + 1$ and $\beta' - 1$ have the same sign, and (iii) $t' = \frac{\pi(2\ell+1)}{2(\beta'-1)}$.

### 3.2 More Compressed Encodings

In the previous section we encoded each logical qubit into two physical qubits before applying $U$. By encoding groups of logical qubits, we can reduce the physical-to-logical qubit number ratio as close as we want to unity. Fix an integer $c > 0$, and let $d$ be the smallest even integer such that $\binom{d}{c} \geq 2^c$. We can compute parity as before by dividing the logical qubits into groups of $c$ qubits each (assume for convenience that $r$ is a multiple of $c$), and encoding each group into a group of $d$ physical qubits, yielding a ratio of $d/c$. Let $E'$ be such an encoder, depicted in Figure 5. Our only requirement for $E'$ is that it map each state $|x0^{d-c}\rangle$, where $x \in \{0,1\}^c$, into a state of the form $|j_x, j_x, \ell_x\rangle$ where $j_x$ has the same parity as $d/2 - \text{wt}(x)$ and $\ell_x \neq \ell_y$ if $x \neq y$. By our discussion in Section 2.2, there are enough spin representations on $d$ qubits to allow this, so such an $E'$ exists.

Now by the considerations of Section 2.3 we see that each input basis state $|x\rangle$ with $x \in \{0,1\}^r$ is thus encoded into a superposition of spin states $a_1|j_1, j_1, \ell_1\rangle + a_2|j_2, j_2, \ell_2\rangle + \ldots$, where all the $j_i$ are integers with parity equal to that of $n/2 - \text{wt}(x)$. We then see by linearity that we can simulate the parity gate with a circuit identical to that shown in Figure 4 except that $E$ is replaced with $E'$ or $E'^\dagger$ as appropriate, and each encoding group has $d$ physical qubits.
Since \( \frac{d}{d/2} \equiv 2^d/\sqrt{d} \), we see that
\[
\frac{d}{c} = \frac{d}{d - \frac{1}{2} \log_2 d} + O(1),
\]
which shows the trade-off between the size of \( E' \) and the ratio \( d/c \).

Encoding with an odd number of physical qubits per group is also possible, and may sometimes lead to a slightly better trade-off. For example, there are enough spin representations on five physical qubits to encode three logical qubits.

**4 Generalized Mod Gates from Any Quadratic Hamiltonian**

In this section, we show how to implement a Mod\(_q\) gate directly, for any \( q \geq 2 \), using any Hamiltonian whose eigenvalues depend quadratically on the Hamming weights of the inputs. More specifically, we assume a Hamiltonian \( G_n \) acting on \( n \) qubits, real constants \( a_n, b_n, c_n \) with \( a_n > 0 \), and an encoding procedure \( E \) such that, for any computational basis state \( |x\rangle \) over an appropriate number of qubits, \( E(|x\rangle|00\cdots0\rangle) \) is an eigenstate of \( G_n \) with eigenvalue \( a_n w^2 + b_n w + c_n \), where \( w = \text{wt}(x) \) and \( |00\cdots0\rangle \) is some ancilla state. Under these assumptions, we construct circuits implementing Mod\(_q\) gates for any constant \( q \geq 2 \), using evolution under \( G_n \). It is already known that the Mod\(_q\) gates for all \( q \geq 2 \) are constant-depth equivalent to each other \[9\], so in effect, we already have a Hamiltonian simulation of any Mod\(_q\) gate, via our implementation of the parity (Mod\(_2\)) gate and the simulation in \[9\]. Our approach here is much more direct, however. Furthermore, it is only marginally more difficult conceptually to generalize our simulation to all \( q \), rather than just \( q = 2 \).

Our present development also subsumes the results in \[7\], where we implemented parity \((q = 2)\) using the Hamiltonian \( J_z^2 \).

Fix \( q \geq 2 \). We consider \( q \) to be constant. The Mod\(_q\) gate is a classical gate that acts on \( r \) control bits and a target bit. The target bit is flipped iff the Hamming weight of the control bits is not a multiple of \( q \). We will actually simulate a more powerful version of this gate, the generalized Mod\(_q\) gate, which has \( r \) control bits and \( q - 1 \) target bits \( t_1, \ldots, t_{q-1} \). If \( w \) is the Hamming weight of the control bits, then the target bits \( t_1, \ldots, t_i \) are all flipped, where \( i = w \mod q \), and the other target bits are left alone. Figure \[8\] shows how to simulate a (standard) Mod\(_q\) gate with a circuit using two generalized Mod\(_q\) gates and a CNOT gate.

We use some \( G_n \) to implement the generalized Mod\(_q\) gate via the circuit shown in Figure \[7\].

Given an initial basis state \( |x\rangle = |x_1 \cdots x_r\rangle \) of the control qubits, we first prepare \( q - 1 \) ancilla qubits into a state
\[
|\varphi\rangle := \sum_{j=0}^{q-1} c_j |1^j 0^{q-1-j}\rangle,
\]
where the \( c_j \) are any fixed scalars such that \( |c_j| = 1/\sqrt{q} \) (we may take \( c_j = 1/\sqrt{q} \) for all \( j \), for example). Note that \( |x\rangle|\varphi\rangle \) is a superposition of basis states with respective Hamming weights \( \text{wt}(x), \text{wt}(x) + 1, \ldots, \text{wt}(x) + q - 1 \). By assumption, we have an encoder \( E \) that maps each computational basis state \( |y\rangle \) with \( y \in \{0,1\}^{r+q-1} \) (possibly with additional ancillae) to a state \( |y_L\rangle \) over some number \( n \) of qubits such that
\[
G_n|y_L\rangle = (a_n \text{wt}(y)^2 + b_n \text{wt}(y) + c_n)|y_L\rangle.
\]
Figure 6: Simulating a standard \(\text{Mod}_q\) gate using generalized \(\text{Mod}_q\) gates. There are \(r\) control qubits, and the ancillae on the right are the qubits labeled \(t_1, t_2, \ldots, t_{q-1}\), i.e., the target qubits of the generalized \(\text{Mod}_q\) gates.

Figure 7: Implementing a generalized \(\text{Mod}_q\) gate using the Hamiltonian \(G_n\). Here, \(U = e^{-itG_n}\) for an appropriate \(t > 0\), and \(R\) is described below. Any extra ancillae used by the encoder \(E\) are not shown.
Thus $E$ maps $|x\rangle|\varphi\rangle$ to the state

$$|\psi_{x,L}\rangle := \sum_{j=0}^{q-1} c_j (x1^j0^{q-1-j})_L.$$

Next we apply $U := e^{-itG_n}$ to $|\psi_{x,L}\rangle$, where $t = \frac{2\pi}{qa_n}$, and $k > 0$ is some fixed integer that is prime to $q$ (we may take $k = 1$, for example). Letting $w := wt(x)$ and $b := n/a_n$ and $c := c_n/a_n$, we have

$$U|\psi_{x,L}\rangle = \sum_{j=0}^{q-1} c_j e^{-i\pi k[(w+j)^2 + b(w+j) + c]/q} (x1^j0^{q-1-j})_L.$$

We decode this state using $E^\dagger$ to obtain the state $|x\rangle|\Psi_w\rangle$, where

$$|\Psi_w\rangle := \sum_{j=0}^{q-1} c_j e^{-i\pi k[(w+j)^2 + b(w+j) + c]/q} 1^j0^{q-1-j}$$

is the state of the $q - 1$ ancilla qubits.

To see that we have isolated the value $v$ mod $q$ in the ancillae, we need only check that, for any integer $v$, $\langle \Psi_v | \Psi_w \rangle = 0$ if $v \not\equiv w \pmod q$. Note that all the states of the form $|\Psi_w\rangle$ lie in a $q$-dimensional subspace $\mathcal{H}''$ of $\mathcal{H}$, spanned by $\{|1^j0^{q-1-j}\rangle | 0 \leq j < q\}$. Let $v$ be the Hamming weight of $z$. Then we have

$$\langle \Psi_v | \Psi_w \rangle = \sum_{j=0}^{q-1} |c_j|^2 \exp (i\pi k[(v+j)^2 + b(v+j) + c - (w+j)^2 - b(w+j) - c]/q)$$

$$= q^{-1} \sum_{j=1}^{q-1} \exp (i\pi k[v^2 - w^2 + 2j(v-w) + b(v-w)]/q)$$

$$\propto q^{-1} \sum_{j=1}^{q-1} \exp (2i\pi jk(v-w)/q)$$

$$= \delta_{(v \mod q), (w \mod q)},$$

where $\delta_{x,y}$ is the Kronecker delta. Thus there is an orthonormal basis $\{|\alpha_j\rangle | 0 \leq j < q\}$ for $\mathcal{H}''$ such that, for all integers $w \geq 0$, there are real values $\theta_w$ such that $|\Psi_w\rangle = e^{i\theta_w}|\alpha_w \mod q\rangle$.

To finish the simulation, we apply to the ancillae some (any) operator $R$ that maps $|\alpha_j\rangle$ to $|1^j0^{q-1-j}\rangle$. We then use a CNOT to copy the $j$th ancilla into the target qubit $t_j$. We then undo all the previous computations to get rid of any conditional phase factors. As was remarked in Section 3.1, the reverse computation uses $U^\dagger$, which can be simulated exactly by evolving via $G_n$ for a positive time only with certain restrictions on the value of $b$ (the value $c$ is unimportant in that it only results in an overall phase factor). If $q \geq 3$, then it is easy to check that for real $u > 0, \exp(-iuG_n) \propto I$ (restricted to the space of encoded vectors) if and only if both $\frac{ua_n(1+b)}{2\pi}$ and $\frac{ua_n(2+b)}{\pi}$ are both integers. The latter conditions hold iff $b$ is rational.

Finally, we note that we may be able to get by with less than the full use of $E^\dagger$ and $E$ on the inside of $U$ and $U^\dagger$ in the circuit of Figure 4. If the encoded state after applying $U$ is not completely entangled, we need only decode the ancillae and those qubits that are entangle with the ancillae, as was done in the circuit of Figure 4.
5 Further Research

We have assumed throughout that the coupling coefficients $J_{i,j}$ of (1) are all equal. Whether this assumption is realistic remains to be seen. It is certainly more likely in the short run that in feasible laboratory setups, the $J_{i,j}$ will not be equal, but can still satisfy certain symmetries. For example, if $n$ identical spin-1/2 particles are arranged in a circular ring, we would expect the Hamiltonian to be cyclically symmetric, i.e., $J_{i,j}$ to depend only on $(i - j) \mod n$. For another example, if the particles are arranged on points in a two- or three-dimensional regular lattice, we would expect translational symmetry of the $J_{i,j}$.

Computing parity in these more realistic situations would be very useful and deserving of further investigation.

Heisenberg interactions also figure prominently in recent proposals for fault-tolerant quantum computation in decoherence-free subspaces (see, for example, [11, 1] and references cited therein). The use of these interactions for this purpose does not appear consistent with our use here, yet it would be helpful to integrate these two approaches, perhaps by encoding logical qubits in a DFS.

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Heisenberg interactions on one-dimensional spin chains are widely studied. It is unlikely, however, that these configurations can be used for parity/fanout, since there are only a linear number of significant terms in the Hamiltonian, and it can be shown that a quadratic term (in $j$) in the Hamiltonian is necessary for our results.
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