Higher order topological invariants from the
Chern-Simons action

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Abstract. It is well known that for a field theory with Chern-Simons action, expectation values of Wilson line operators are topological invariants. The standard result is expressed in terms of the Gaussian linkings of closed curves defining the operators. We show how a judicious choice of Wilson lines leads to higher order topological linkings.

1. Introduction

Our understanding of knots and links in topological field theory began with the work of Polyakov (Polyakov, 1988) who showed that the expectation value of a Wilson line in Chern-Simons theory gives the Gaussian linking of the components of a link. This work was soon followed by Witten’s (Witten, 1989) extensive exploration of expectation values of Wilson lines in non-abelian Chern-Simons theory which he showed is a natural framework for understanding Jones polynomials from knot theory. Perturbative expansion of the expectation values leads to higher order linking and the corresponding link polynomials (Guadagnini et al., 1989). Our approach is somewhat different. Wilson loops in the abelian Chern-Simons theory also implicitly contains the requisite apparatus for the study of higher order linking, and we show here that this can be facilitated by a set of gauges (Buniy & Kephart, 2008a, b, 2006) tailored to the higher order linking problem. We begin with a discussion of intersection number and its relation to Gaussian linking. Next we review the minimal required background from Poincaré duality and de Rham theorem needed to carry out our calculations. We then introduce a special set of gauge potentials that allow us to arrive at our main result—higher order linking invariants at all orders. We conclude with detailed computations for several examples where linkings of various orders appear.

2. Intersection and linking numbers

Suppose C and C’ are disjoint oriented closed curves in $\mathbb{R}^3$, and S and S’ are surfaces such that $\partial S = C$ and $\partial S’ = C’$. By smooth deformations of C which leave it in $\mathbb{R}^3 \setminus C’$, we can change the number of points of intersection of C and S’. To find a quantity which is invariant under such deformations, we note that additional points that appear due to the deformations come in pairs, and the intersections of C and S’ have opposite orientations for points in each pair (and likewise for disappearing pairs). If C and S’ intersect transversely at a point P, we define the intersection index $I(C, S’, P)$ to be equal 1 or −1 depending on the relative orientation of C and
We define the intersection number $I(C, S')$ as the sum of the intersection indices over all points of intersection,

$$I(C, S') = \sum_{P \in C \cap S'} I(C, S', P). \tag{1}$$

It is clear that $I(C, S')$ is invariant under the above deformations as contributions due to additional pairs of points cancel.

One can also deform $S'$ into $\tilde{S}'$ and notice that $I(C, S') = I(C, \tilde{S}')$, where $\partial S' = \partial \tilde{S}' = C'$. This means that the intersection number depends only on $C$ and $C'$; for this reason we call it the linking number, $L(C, C') = I(C, S')$. Examination of relevant orientations gives

$$L(C, C') = (4\pi)^{-1} \int_{C \times C'} \sum_{a, b, c} \varepsilon_{abc} \frac{(x - x')^a}{\|x - x'\|^3} dx^b \wedge dx^c. \tag{2}$$

### 3. Duality

From Poincaré duality and de Rham theorem (Griffiths & Harris, 1978; Hatcher, 2002), for a closed curve $C$ in three dimensions, there exists a closed 2-form $F$ such that for any 1-form $B$ we have

$$\int_C B = \int_{\mathbb{R}^3} B \wedge F. \tag{3}$$

We call $(C, F)$ a dual set. Since $B$ is arbitrary, it is clear that $F$ has support only on $C$, i.e. $\text{supp} F = C$. Since $\mathbb{R}^3$ is simply connected, there exists a 1-form $A$ such that $F = dA$. The Stokes theorem then gives

$$\int_S dB = \int_{\mathbb{R}^3} dB \wedge A. \tag{4}$$

This means that there is a particular solution $A$ such that

$$A(x) = \int_{y \in S} \delta(x - y) \sum_a dx^a + dy^a. \tag{5}$$

This is a singular gauge with $\text{supp} A = S$. We can similarly introduce a dual set $(C', F')$ such that $F' = dA'$ and for any 1-form $B'$ we have

$$\int_{C'} B' = \int_{\mathbb{R}^3} B' \wedge F'. \tag{6}$$

Taking $B = A'$, $B' = A$ and using the Stokes theorem again, we find

$$\int_C A' = \int_{C'} A = \int_{\mathbb{R}^3} A \wedge dA' = L(C, C'). \tag{7}$$

Since $L(C, C')$ is a topological invariant, smooth deformations of $C$ which leave it in $\mathbb{R}^3 \setminus C'$ should not change its value. This is possible if and only if $dA'|_{\mathbb{R}^3 \setminus C'} = 0$. If $A'$ is exact, then $L(C, C') = 0$, and so a nontrivial case is when $A'$ is a closed 1-form which is not exact.
As an example with the first order linking, consider the Hopf link (Rolfsen, 1990; Kauffman, 2001) in Fig. 1. We use the singular gauge (5) to compute

\[ L(C_1, C_2) = \int_{\mathbb{R}^3} dA_1 \wedge A_2 = 1. \]  

(8)

This follows since we can choose \( C_1 \cap S_2 = P \), where \( P \) is a point. The first order non-self linking is \( \tilde{L}_1(C, C) = 2 \). For \( C = \bigcup_{1 \leq i \leq N} C_i \), we find

\[ L_1(C, C) = \sum_{i,j} L(C_i, C_j), \]  

(9)

which is in agreement with results in Refs. (Polyakov, 1988; Witten, 1989).

4. Second order fields

Suppose \( \{ C_i \}_{1 \leq i \leq N} \) are disjoint closed curves in \( \mathbb{R}^3 \) and let \( F_i = dA_i \) be dual to \( C_i \). We now construct a dual set \( (C_{ij}, F_{ij}) \) of the second order which satisfies

\[ \int_{C_{ij}} B = \int_{\mathbb{R}^3} B \wedge F_{ij} \]  

(10)

for any 1-form \( B \) and \( i \neq j \). Since \( B \) is arbitrary, it follows that \( \text{supp} F_{ij} = C_{ij} \). The most general 2-form which can be expressed in terms of \( A_i \) and \( A_j \) is

\[ F_{ij} = f_i dA_j - f_j dA_i + g A_i \wedge A_j, \]  

(11)

where \( f_i, f_j \) and \( g \) are arbitrary functions. In order for \( F_{ij} \) to be closed, the functions have to satisfy certain conditions. A requirement \( dF_{ij}|_{\mathbb{R}^3 \setminus (C_i \cup C_j)} = 0 \) gives \( dg|_{\mathbb{R}^3 \setminus (C_i \cup C_j)} = 0 \) and without loss of generality we set \( g = 1 \). Requirements \( dF_{ij}|_{C_i} = 0 \) and \( dF_{ij}|_{C_j} = 0 \) then give

\[ (df_j - A_j)|_{C_i} = 0, \]  

(12)

\[ (df_i - A_i)|_{C_j} = 0. \]  

(13)

Integrating these conditions, we find a constraint \( L(C_i, C_j) = 0 \). This means that the second order field associated with a pair of closed curves can be defined only if the curves are unlinked.

Since \( dF_{ij} = 0 \), there exists a 1-form \( A_{ij} \) such that \( F_{ij} = dA_{ij} \). We seek a solution in the form

\[ A_{ij} = \frac{1}{2}(\gamma_i A_j - \gamma_j A_i). \]  

(14)

\[ ^1 \text{We assume that values of appropriate quantities vanish at infinity, so there is no contribution from boundary terms. Boundary terms also vanish if we use } S^3 \text{ instead of } \mathbb{R}^3. \]
From a requirement $F_{ij} = dA_{ij}$, the unknown functions $\gamma_i$ and $\gamma_j$ are found to satisfy

$$\begin{align*}
(d\gamma_i - A_i)|_{\mathbb{R}^3 \setminus (C_i \cup C_j)} &= 0, \\
& \text{(15)} \\
d\gamma_i|_{C_i} &= 0, \\
& \text{(16)} \\
d\gamma_i - 2A_i|_{C_j} &= 0, \\
& \text{(17)} \\
(\gamma_i - 2f_i)|_{C_j} &= 0, \\
& \text{(18)}
\end{align*}$$

and the same expressions with $i$ and $j$ interchanged. This implies $\gamma_i|_{\mathbb{R}^3 \setminus (\cup_k C_k)} = \int_{\Gamma_i} A_i$, where $\Gamma_i$ is a curve in $\mathbb{R}^3 \setminus (\cup_k C_k)$; this means $\gamma_i$ is a nonlocal quantity. If $S_i \cap C_j = \emptyset$, then $d\gamma_i|_{C_j} = 0$, and so $\gamma_i|_{C_j}$ is a constant. If $S_i \cap C_j \neq \emptyset$, then $S_i \cap S_j = \bigcup_m (S_i \cap S_j)(m)$, where $m$ labels disjoint segments of the intersection; for an example, see Fig. 2. Let $C_{j(m)}$ be the segment of $C_j$ which closes the curve $(S_i \cap S_j)(m)$ and agrees with its orientation; this closed curve is $C_{ij(m)} = C_{j(m)} \cup (S_i \cap S_j)(m)$. We define $C_j' = \bigcup_m C_{j(m)}$ and its complement in $C_j$ is $C_j''$. It follows from the above relations that $\gamma_i|_{C_j'}$ and $\gamma_i|_{C_j''}$ are constants such that $\gamma_i|_{C_j'} - \gamma_i|_{C_j''} = 2$. Without loss of generality, we set $\gamma_i|_{C_j'} = 2$ and $\gamma_i|_{C_j''} = 0$. Using the definition of $F_{ij}$, the duality condition now becomes

$$\int_{C_{ij}} B = -\int_{C_{j}'} B + \int_{C_{j}''} B + \int_{S_i \cap S_j} B. \quad (19)$$

This implies $C_{ij} = C_{i}^{-1} \cup C_{j}' \cup (S_i \cap S_j)$, which agrees with supp $F_{ij} = C_{ij}$. See Fig. 2 for an example of the above construction. The Stokes theorem gives

$$\int_{S_{ij}} dA = \int_M dA \land A_{ij}, \quad (20)$$

where a surface $S_{ij}$ is such that $\partial S_{ij} = C_{ij}$. This means supp $A_{ij} = S_{ij}$.

The above construction requires $L(C_i, C_j) = 0$. Since the curves $C_i$ and $C_j$ are unlinked, the surfaces $S_i$ and $S_j$ can be chosen to be disjoint, in which case $dA_{ij}|_{\mathbb{R}^3 \setminus (C_i \cup C_j)} = 0$. One can similarly proceed to construct higher order fields. For example, if $L(C_{ij}, C_k) = 0$ for all $(i, j, k)$, then we can define the third order 1-forms $A_{ijk}$. In particular, they satisfy

$$dA_{ijk}|_{\mathbb{R}^3 \setminus (C_i \cup C_j \cup C_k)} = (A_i \wedge A_{jk} + A_{ij} \wedge A_k)|_{\mathbb{R}^3 \setminus (C_i \cup C_j \cup C_k)} \quad (21)$$
for \(i \neq j \neq k\).

If all linkings of order \(q\), where \(1 \leq q \leq p - 1\), vanish, then we can similarly construct dual sets \((C_{I_p}, dA_{I_p})\), where \(I_p = (i_1, i_2, \ldots, i_p)\). The quantities \(dA_{I_p}\) are related to what is known in algebraic topology as the Massey products of cohomology groups (Massey, 1959, 1968; Kraines, 1966; O’Neill, 1979; Fenn, 1983); see also Refs. (Monastyrsky & Retakh, 1986; Berger, 1990).

As an example with second order linking, consider the Borromean rings (Rolfsen, 1990; Kauffman, 2001) in Fig. 3. We use the singular gauge to compute

\[
L(C_{ij}, C_k) = \int_{\mathbb{R}^3} dA_i \wedge A_j = 0
\]

for \(i \neq j\). This follows since we can choose \(C_i \cap S_j = \emptyset\) for \(i \neq j\). We take \(C = C_1 \cup C_2 \cup C_3\) and find that the first order non-self linking vanishes,

\[
\tilde{L}_1(C, C) = 2L(C_1, C_2) + 2L(C_2, C_3) + 2L(C_3, C_1) = 0.
\]

We define the second order curves \(\{C_{ij}\}\) for \(i \neq j\). Again we use the singular gauge to compute

\[
L(C_{ij}, C_k) = \int_{\mathbb{R}^3} A_i \wedge A_j \wedge A_k - \frac{1}{2} \int_{C_i} \gamma_j A_k + \frac{1}{2} \int_{C_j} \gamma_i A_k = \varepsilon_{ijk},
\]

where \((i, j, k)\) is a permutation of \((1, 2, 3)\). The integral over \(\mathbb{R}^3\) equals \(\varepsilon_{ijk}\) since we can choose \(S_i \cap S_j \cap S_k = P\), where \(P\) is a point. The integrals over \(C_i\) and \(C_j\) vanish since we can choose \(S_k\) such that \(C_i \cap S_k = \emptyset\) and \(C_j \cap S_k = \emptyset\). Similarly,

\[
\tilde{L}(C_{ij}, C_j) = \int_{\mathbb{R}^3} A_i \wedge A_j \wedge A_j - \frac{1}{2} \int_{C_i} \gamma_j A_j + \frac{1}{2} \int_{C_j} \gamma_i A_j = 0
\]

for any \(i \neq j\). We take \(C = C_1 \cup C_2 \cup C_3 \cup C_{12} \cup C_{23} \cup C_{31}\) and find that the second order non-self linking of \(C\) is

\[
\tilde{L}_2(C, C) = 2L(C_{12}, C_3) + 2L(C_{23}, C_1) + 2L(C_{31}, C_2) = 6.
\]
As an example with the third order linking, consider the Whitehead link (Rolfsen, 1990; Kauffman, 2001; Monastyrsky & Retakh, 1986) in Fig. 4. We use the singular gauge to compute

\[ L(C_1, C_2) = \int_{\mathbb{R}^3} dA_1 \wedge A_2 = 0. \]  

(27)

This follows since we can choose \( C_1 \cap S_2 = P_1 \cup P_2 \), where \( P_1 \) and \( P_2 \) are points at which the contributions of \( dA_1 \wedge A_2 \) to the integral cancel due to the opposite orientations. We take \( C = C_1 \cup C_2 \) and find that the first order non-self linking vanishes, \( \tilde{L}_1(C, C) = 0 \).

We define the second order curve \( C_{12} \) and use the singular gauge to compute

\[ L(C_{12}, C_k) = \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_k - \frac{1}{2} \int_{C_1} \gamma_2 A_k + \frac{1}{2} \int_{C_2} \gamma_1 A_k = 0 \]  

(28)

for \( k = 1 \) or \( k = 2 \). We take \( C = C_1 \cup C_2 \cup C_{12} \) and find that the second order non-self linking vanishes, \( \tilde{L}_2(C, C) = 0 \).

We use the singular gauge to compute the third order self linking,

\[ L(C_{12}, C_{12}) = \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_{12} = 2. \]  

(29)

This follows since \( S_{12} \) is the union of disks \( S_{12(1)} \) and \( S_{12(2)} \) such that \( S_{12(1)} \cap S_{12(2)} = P \), where \( P \) is a point, and we can choose \( S_1 \) and \( S_2 \) such that \( S_1 \cap S_2 \cap S_{12} = P \).

We define the third order curves \( C_{121} \) and \( C_{212} \). We use the singular gauge to compute

\[ L(C_{121}, C_2) = \int_{\mathbb{R}^3} (A_1 \wedge A_{21} + A_{12} \wedge A_1) \wedge A_2 = 2 \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_{12} = 4 \]  

(30)

and similarly \( L(C_{212}, C_1) = 4 \). We take \( C = C_1 \cup C_2 \cup C_{12} \cup C_{121} \cup C_{212} \) and find that the third order non-self linking of \( C \) is

\[ \tilde{L}_3(C, C) = 2L(C_{121}, C_2) + 2L(C_{212}, C_1) = 16. \]  

(31)

5. Path integral

We want to construct a field theory for which expectation values of observables are topological invariants of various orders. For this we need to specify the action and the observables. We choose the Chern-Simons action,

\[ S(B) = \int_{\mathbb{R}^3} B \wedge dB, \]  

(32)
since it has all the necessary topological properties (Witten, 1989). First, it does not depend on
the choice of metric. Second, it can be related to linking numbers in the following way. Suppose
\( \Gamma_\alpha \) is a closed curve on which \( dB \) takes a constant value \( dB_\alpha \). (In order for \( \Gamma_\alpha \) to be closed,
we may need to identify points at infinity by considering \( S^3 \) instead of \( \mathbb{R}^3 \). We also avoid field
configurations with sources, such as monopoles.) Since a union of curves \( \Gamma = \bigcup \alpha \Gamma_\alpha \) densely fills
\( \mathbb{R}^3 \), an arbitrary 1-form \( B \) can be written as \( B = \sum \alpha B_\alpha \). The action becomes
\[
S(B) = \sum_{\alpha,\beta} L(\Gamma_\alpha, \Gamma_\beta),
\]
and so (apart from the choice of measure for \( \Gamma \)) we interpret it as the self-linking \( L(\Gamma, \Gamma) \) of the
set of closed field lines of \( dB \).

If \( C = \bigcup \alpha C_\alpha \) is a union of disjoint closed curves, then an integral \( \int_C B \) is invariant with
respect to deformations of \( C \) if and only if \( dB = 0 \). Since \( dB = 0 \) is the classical equation of
motion for \( S(B) \), the integral is a topological observable (at least in the semiclassical limit).
Proceeding as above, we find
\[
\int_C B = \sum_{\alpha,\beta} L(\Gamma_\alpha, C_\beta),
\]
which we interpret as the linking \( L(\Gamma, C) \) of the sets of closed field lines of \( dB \) with \( C \). Since
the measure in the path integral is \( \exp(iS(B)) \), it is convenient to consider as an observable a
Wilson loop operator
\[
W(C, B) = \exp \left( i \int_C B \right).
\]
We thus need to compute the expectation value
\[
Z(C) = \int DB \exp(iS(B))W(C, B).
\]
Using duality, this becomes
\[
Z(C) = \int DB \exp \left( i \int_{\mathbb{R}^3} B \wedge d(B + A) \right),
\]
where \( dA \) is dual to \( C \). The change of the variable \( B = B' - \frac{1}{\lambda}A \) gives four terms in the exponent.
One term corresponds to the path integral of the measure without a Wilson loop, \( Z(\emptyset) \), the two
mixed terms combine into a boundary term, and the forth is the linking invariant. Ignoring the
boundary term, we find
\[
Z(C) = Z(\emptyset) \exp \left( -\frac{1}{\lambda} i(L(C, C)) \right).
\]
A product of Wilson loop operators can be written as a single Wilson loop operator for the
union of the corresponding loops,
\[
\prod_p \prod_{I_p} W(C_{I_p}, B) = W(C, B),
\]
where \( C = \bigcup_p \cup_{I_p} C_{I_p} \) and \( C_{I_p} \) is a curve of order \( p \). It follows that
\[
L(C, C) = \sum_{p,q} \sum_{I_p,J_q} L(C_{I_p}, C_{J_q}).
\]
Here $L(C_{I_p}, C_{J_q})$ is the first order linking of curves $C_{I_p}$ and $C_{J_q}$. This quantity can also be interpreted as a linking of order $p + q - 1$ of the curves $\{C_i\}_{1 \leq i \leq N}$. In general, $L(C, C)$ is a sum of linkings of various orders,

$$L(C, C) = \sum_r L_r(C, C), \quad (41)$$

where the expressions for several lowest order terms are

$$L_1(C, C) = \sum_{i,j} L(C_i, C_j), \quad (42)$$
$$L_2(C, C) = \sum_{i,j,k} L(C_{ij}, C_k), \quad (43)$$
$$L_{3,1}(C, C) = L_{3,1}(C, C) + L_{3,2}(C, C), \quad (44)$$
$$L_{3,1}(C, C) = \sum_{i,j,k,l} L(C_{ijk}, C_l), \quad (45)$$
$$L_{3,2}(C, C) = \sum_{i,j,k,l} L(C_{ij}, C_{kl}). \quad (46)$$

6. Conclusions

In Chern-Simons theory the first order (Gaussian) linking of two curves $C_1$ and $C_2$ associated with the 1-forms $A_1$ and $A_2$ can be ascribed to the topological properties of the expectation value of the associated Wilson lines. Here we have generalized this idea to the case of higher order linking. First, if $C_1$ and $C_2$ are unlinked, then we can define a new second order 1-form $A_{12}$ via Eq. (14). Unlike $C_1$ and $C_2$, the associated curve $C_{12}$ is not fixed in space, but is however sufficient for our topological needs, which are to investigate its linking with other curves, since this property is invariant. If, for instance, we have a third curve $C_3$ unlinked with both $C_1$ and $C_2$, but linked with $C_{12}$, then we find nonzero second order linking, i.e., a nonvanishing second order topological invariant. The Borromean rings is the simplest topological configuration for which this invariant is nonzero. The argument generalizes to higher order invariants. The Whitehead link is the simplest example with a nonvanishing third order topological invariant. For curves $\{C_i\}_{1 \leq i \leq N}$, with no linking of orders 1, 2, ..., $p - 1$ and nonzero linking of order $p$, the general results are obtained from the path integral of the expectation value of a product of Wilson loop operators which depend on curves of order $p$ in the Chern-Simons gauge theory. One could apply these results to the investigation of higher order link polynomials.

Acknowledgments

We are grateful to the Isaac Newton institute for hospitality. RVB thanks Chapman University for support. The work of TWK was supported by U.S. DoE grant number DE-FG05-85ER40226 and Vanderbilt University College of Arts and Sciences.

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