Fermionic Zero Modes on Domain Walls

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We study fermionic zero modes in the domain wall background. The fermions have Dirac and left- and right-handed Majorana mass terms. The source of the Dirac mass term is the coupling to the scalar field \( \Phi \). The source of the Majorana mass terms could also be the coupling to the scalar field \( \Phi \) or the vacuum expectation value of some other field acquired in a phase transition well above the phase transition of the field \( \Phi \). We derive the fermionic equations of motion and find the necessary and sufficient conditions for a zero mode to exist. We also find the solutions numerically.

In the absence of the Majorana mass terms, the equations are solvable analytically. In the case of massless fermions a zero energy solution exists and we show that although this mode is not discretely normalizable it is Dirac delta function normalizable and should be viewed as part of a continuum spectrum rather than as an isolated zero mode.

I. INTRODUCTION

The vacuum structure in theories with spontaneous symmetry breaking is very rich. There exists topologically stable configurations of gauge and Higgs fields known as monopoles, strings and domain walls. Also, classical configurations which are not topologically stable — non-topological solitons — could exist and be stable because of dynamical reasons. Such objects may play an important role in the evolution of our universe.

Extensive research has been done on the interaction of fermions with solitons and many new and interesting phenomena have been discovered. For example fermionic zero modes on strings are responsible for string superconductivity [4]. Quark and leptonic zero modes have an important effect on the stability of the non-topological electroweak strings [2–4]. Some work has been done also on fermionic zero modes on domain walls, but only Dirac fermions have been considered [8–10].

Recent experimental results [14] strongly suggest that neutrinos, although very light, have a finite mass. Being neutral, neutrinos can have Majorana masses in addition to the usual Dirac mass terms. This paper study zero modes of a particle with both Dirac and Majorana mass terms in the background of a domain wall.

In section 2 we present the Lagrangian of a theory where the coupling to the real scalar field \( \Phi \) is the source of all mass terms. We derive the equations of motion. In section 3 we solve the equations analytically in two asymptotic regimes, near the origin and far from the origin, and find the necessary and sufficient condition for the zero mode to exist. We also solve the equations numerically. If the Majorana mass terms are absent it is possible to solve the equations analytically and our result agrees with the results in literature [2, 4]. We discuss the zero energy solutions in the case of massless fermions. In section 4 we repeat the procedure of section 3 for the case when one or both Majorana masses are spatially homogeneous. The phenomenological significance of obtained results is discussed in section 5.

II. LAGRANGIAN, ANSATZ AND EQUATIONS OF MOTION

We consider a Lagrangian in 3+1 dimensions involving the real scalar field \( \Phi \) and two chiral spinor fields \( \psi_L \) and \( \psi_R \):

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{\lambda}{4} (\Phi^2 - \eta^2)^2 + \nonumber \\
+ i \psi_L \lambda_{\mu} \partial_{\mu} \psi_L + i \psi_R \gamma_{\mu} \partial_{\mu} \psi_R - \nonumber \\
- (k_1 \Phi \psi_L \psi_R + k_2 \Phi \psi_R \psi_L^c + k_3 \psi_L \psi_L + h.c.)
\]

(1)

where \( \lambda, \eta, k_1, k_2 \) and \( k_3 \) are real and positive (if \( \psi_L \) and \( \psi_R \) are four component Dirac spinors we can absorb all phases of \( k_1, k_2 \) and \( k_3 \) into phases of spinor components).

\( \psi^c \equiv C \psi^T \), where \( C \) is the charge conjugation matrix, is introduced in (2) in order to have the most general mass terms for fermions [8].

The scalar potential in (1) has two minima, \( \Phi_{min} = \pm \eta \), and in the absence of fermions supports the domain wall solution:

\[
\Phi_0 = \eta \tanh(\sqrt{\eta} y)
\]

(2)

for a domain wall located in the \( xz \) plane. The resulting fermionic equations of motion in a domain wall background are:

\[
i \gamma^\mu \partial_{\mu} \psi_L = k_1 \Phi_0 \psi_R + k_3 \Phi_0 \psi_L^c \\
i \gamma^\mu \partial_{\mu} \psi_R = k_1 \Phi_0 \psi_L + k_2 \Phi_0 \psi_R^c
\]

(3)

The representation of the Dirac matrices we will use is:

\[
\gamma^0 = \begin{pmatrix} \tau^3 & 0 \\ 0 & -\tau^3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i \tau^2 & 0 \\ 0 & -i \tau^2 \end{pmatrix}, \nonumber \\
\gamma^2 = \begin{pmatrix} -i \tau^1 & 0 \\ 0 & i \tau^1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \nonumber \\
C \equiv i \gamma^2 \gamma^0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(4)
where $\tau^i, i = 1,2,3$ are the Pauli matrices. In this representation, a four-component Dirac fermion has left and right-handed components of the form: $\psi^L_i = (\alpha, \beta, -\alpha, -\beta)$ and $\psi^R_i = (\gamma, \delta, \gamma, \delta)$.

For a domain wall located in the $xz$ plane, $\Phi_0$ is a function of $y$ only and we look for a solution to (9) of the form:

$$
\begin{align*}
\alpha &= a(y)e^{i\omega t-ik_xx-ik_zz} \\
\beta &= b(y)e^{i\omega t-ik_xx-ik_zz} \\
\gamma &= c(y)e^{i\omega t-ik_xx-ik_zz} \\
\delta &= d(y)e^{i\omega t-ik_xx-ik_zz}
\end{align*}
$$

$a(y), b(y), c(y)$ and $d(y)$ can be taken to be real functions of $y$. We are interested in zero energy solutions to equation (3) such that all spinor components fall off exponentially outside the string core (large $y$) and are well-behaved (nonsingular) at the origin (small $y$).

After setting $\omega = k_x = k_z = 0$, we get two identical sets of equations:

$$
\begin{align*}
d' &= k_1 \Phi_0 d + k_3 \Phi_0 a \\
d'' &= k_1 \Phi_0 a + k_2 \Phi_0 d
\end{align*}
$$

$$
\begin{align*}
b' &= k_1 \Phi_0 c + k_3 \Phi_0 b \\
c' &= k_1 \Phi_0 c + k_2 \Phi_0 c
\end{align*}
$$

III. SOLUTIONS

Let us first analyze the system (3) for $y \geq 0$ ($y \leq 0$ is analogous because of the symmetry under $y \rightarrow -y$).

No analytic solution to the system could be found, but we can learn something about the structure of solutions by looking at their asymptotic behavior. Neglecting a domain wall back reaction to fermions, $\Phi_0 \rightarrow \eta$ at large $y$. Assuming the asymptotic behavior $a \sim a_0 e^{s y}$ and $d \sim d_0 e^{t y}$ ($a_0, d_0, s$ and $t$ are arbitrary real numbers) we get following conditions:

$$
s = t, \quad s^2 - s\eta(k_2 + k_3) + k_2 k_3 \eta^2 - k_1^2 \eta^2 = 0 \quad (8)
$$

The solutions to this second order polynomial equation for $s$ are

$$
s_{\pm} = \frac{1}{2} \eta \left[ (k_2 + k_3) \pm \sqrt{(k_2 - k_3)^2 + 4 k_1^2} \right] \quad (9)
$$

Both solutions for $s$ are real. $s_+$ is positive, and is giving rise to exponentially growing mode. $s_-$ is negative if and only if $k_2k_3 < k_1^2$, giving rise to exponentially decaying mode. Only exponentially decaying modes are physically acceptable. So, we conclude that the necessary condition for the zero mode to exist on a domain wall is $k_2k_3 < k_1^2$.

We should note that this result is very similar to the conjecture in (3) about the existence of neutrino zero modes on electroweak strings.

The analysis for $y \leq 0$ is analogous. For $y < 0$ we have $a \sim a_0 e^{-s|y|}$, so only positive $s$ leads to a physically acceptable result. But we also have $\eta \rightarrow -\eta$ in (3) and the necessary condition for the existence of a zero mode stays the same.

If we want to match the exponentially decaying large $y$ solutions in a given pair of equations to the solutions at the origin, we must know how many solutions for small $y$ are well-behaved. Neglecting a domain wall back reaction to fermions, $\Phi_0 \sim f_0 y + f_2 y^2 + \ldots$ for small $y$. Assuming that $a \sim a_0 y^s, d \sim d_0 y^t$ ($a_0$ and $b_0$ are arbitrary real numbers while $s$ and $t$ are nonnegative real numbers) for small $y$, we find that the leading orders are $s = 0, t = 0$. System (3) is invariant under parity ($y \rightarrow -y$), so all corrections to the solution which are parity violating are zero. We can write the general well-behaved solution in the form:

$$
\begin{pmatrix}
a \\
b
\end{pmatrix} = \begin{pmatrix}
a_0 y^0 + a_2 y^2 + a_4 y^4 + \ldots \\
d_0 y^0 + d_2 y^2 + d_4 y^4 + \ldots
\end{pmatrix}
$$

Substituting (10) into (3) and equating coefficients of the same order in $y$ we see that only two coefficients are independent (say $a_0$ and $d_0$), while all others are functions of the first two). This means that there are two linearly independent well-behaved solutions. For the system of two linear first order differential equations we should have two solutions in total, so we conclude that both solutions are well-behaved.

Obviously, all the arguments given for $y \geq 0$ hold for $y \leq 0$ too, so we will confine our following discussion to positive $y$. Each of the two well-behaved solutions at the origin matches to a unique linear combination of both solutions at large $y$. If both solutions at large $y$ are bad (exponentially growing), then there is no solution which is well-behaved everywhere. If one of them is good (exponentially decaying) there is always one well-behaved solution everywhere — good solution at large $y$ which matches to a linear combination of two good solutions at the origin. In this case, the necessary condition for the zero mode to exist, $k_2k_3 < k_1^2$, is also sufficient. Indeed, numerically solving the system (3), we find that there is one well-behaved solution (Figure (3)). We took $k_1 \eta = 2, k_2 \eta = 1$ and $k_3 \eta = 1$ which satisfies the necessary and sufficient condition for a zero mode to exist. We also set $\sqrt{2\eta} = 1$. 


we look at the massless fermion zero mode (ω → 0 limit of a plane wave solution) as an isolated zero mode, it is not normalizable. But, this state is actually part of a continuum spectrum of the theory and is Dirac delta function normalizable.

IV. CONSTANT MAJORANA MASS TERMS

Probably the most interesting case is when the source of the Majorana masses $M_L$ and $M_R$ is not the coupling with the field Φ. The Majorana mass terms can be taken to be spatially homogeneous and presumed to arise from the vacuum expectation value of some field acquired in a phase transition above the phase transition of the field Φ. In this case we set $k_2 Φ \equiv M_R$ and $k_3 Φ \equiv M_L$ in Lagrangian (6).

We can carry out a similar analysis to the one done in section 3. The equations of motion now are:

$$a' = k_1 Φ_0 d + M_L a$$
$$d' = k_1 Φ_0 a + M_R d$$
$$b' = k_1 Φ_0 c + M_L b$$
$$c' = k_1 Φ_0 b + M_R c$$

An analysis of the large $y$ behavior for $y \geq 0$ is analogous to the previous case. Substituting $k_2 η \equiv M_R$, $k_3 η \equiv M_L$ in (6) and demanding negative solutions for $s$ we find the necessary condition for a zero mode to exist: $M_L M_R < M_D^2$, where we just formally write $k_1 η = M_D$. For $y \leq 0$, we demand positive solutions for $s$ and see that if $M_L M_R < M_D^2$ there is one decaying mode.

An analysis of the behavior near the origin is slightly different than before since the constant $M_L$ and $M_R$ terms break the $y \rightarrow -y$ symmetry of the equations (13) and parity breaking corrections to the leading order in a solution are also present (odd powers in $y$). Thus, the general well-behaved solution to the system (13) can be written as:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_0 y^0 + a_1 y + a_2 y^2 + a_3 y^3 + \ldots \\ d_0 y^0 + d_1 y + d_2 y^2 + d_3 y^3 + \ldots \end{pmatrix}$$

Substituting (17) into (13) we see that two coefficients are independent (say $a_0$ and $d_0$) which means that there are two linearly independent well-behaved solutions. This conclusion is valid for both positive and negative $y$. The matching of well-behaved solutions at large and small $y$ is similar to system (3). However, the matching of a given pair of solutions for $a(y)$ and $d(y)$ for $y \geq 0$ and $y \leq 0$ at $y = 0$ is nontrivial. There is no symmetry $y \rightarrow -y$ and $\frac{a(y=0^+)}{d(y=0^+)} \neq \frac{a(y=0^-)}{d(y=0^-)}$ (Figure 4). Numerical analysis shows that $\frac{a(y=0^+)}{d(y=0^+)} = \frac{a(y=0^-)}{d(y=0^-)}$ only if $M_L = M_R$. We conclude that if $M_L M_R < M_D^2$, where $M_L = M_R$, there is
one solution which is well-behaved everywhere. Numerically solving the system \([13]\) setting \(k_1\eta = 2\), \(M_R = 1\), \(M_L = 1\) and \(\sqrt{\frac{\eta}{2}} = 1\), we find one well-behaved solution (Figure (3)).

![Graph showing numerical solution of system (13) with \(k_1\eta = 2\), \(M_R = 1\), \(M_L = 1\), and \(\sqrt{\frac{\eta}{2}} = 1\).](image)

The domain wall is located at \(y = 0\). The solution is not symmetric with respect to \(y \to -y\).

Let us mention that a similar analysis can be done in the case when just one Majorana mass term is homogeneous, while the second one still comes from a coupling to the field \(\Phi\). Setting \(k_3\Phi \equiv M_L\) in \([14]\) and repeating the procedure from section 3 we find the same form of the necessary condition for a zero mode to exist: \(M_L M_R < M_D^2\), where we just formally write \(k_1\eta = M_D\) and \(k_2\eta = M_R\). However, unlike the previous case, we can not set \(M_L = k_2\Phi_0\) everywhere. Thus, \(\frac{a(y=0^+)}{d(y=0^+)} \neq \frac{a(y=0^-)}{d(y=0^-)}\) for all values of \(M_L\) and there is no matching between the \(y \geq 0\) and \(y \leq 0\) solutions for \(a(y)\) and \(d(y)\). In this case there are no well behaved solutions.

V. PHENOMENOLOGY

By now, we have not addressed the question about the phenomenological validity of the general fermionic mass terms. The Lagrangian \([15]\) could be, in principle, part of some larger theory like the standard model (and various extensions of it) or GUT theories. The specific charges assigned to the fermionic field \(\psi\) would then depend on a specific symmetry group of the model. Mass terms, present in a Lagrangian have to be gauge singlets, thus restricting their possible forms. An illustrative example, phenomenologically valid standard model neutrinos, beside the Dirac mass terms, can only have the right-handed Majorana mass terms. In our notation, this corresponds to the case of \(k_3 = 0\). If the coupling to the scalar field \(\Phi\) gives rise to the right-handed Majorana mass we analyze the results of section 3. Setting \(k_3 = 0\) in \([15]\) we see that the necessary condition \(k_2k_3 < k_1^2\), i.e., \(0 < k_1^2\), is always satisfied. Following the rest of the analysis of section 3, we conclude that in this case there is always one zero mode. However, in this case it would be difficult, without a fine tuning, to explain a large difference between the Dirac and right-handed Majorana masses because the same phase transition gives the scale for both masses.

If the right-handed Majorana mass is spatially homogeneous, then we apply the analysis of the section 4 (which is different from the analysis of the section 3 because there is no symmetry \(y \to -y\)). In the absence of the left-handed Majorana mass, although the necessary condition is satisfied, due to the fact that \(0 = M_L \neq M_R\), the solutions for positive and negative \(y\) do not match at \(y = 0\). In this case, there are no well behaved zero modes.

VI. CONCLUSION

We studied zero energy solutions of the Dirac equation in the background of a domain wall located in the \(xz\) plane. We first considered the case when the vacuum expectation value of the scalar field \(\Phi\) was the source of the Dirac and Majorana mass terms. In the general case, when both the Dirac and Majorana mass terms are present we solved the equations of motion analytically in two asymptotic regimes, large and small \(y\) and found the
necessary and sufficient condition for a zero mode to exist. This condition is very similar to the one conjectured in [9] in the case of neutrino zero modes on electroweak strings. We also solved the equations numerically.

If the Majorana mass terms are absent, it is possible to solve the equations analytically. The result agrees with [5,6].

If the fermions are massless there exists a zero-energy solution which is not discretely normalizable. Solving the non-zero energy equations we showed that this state is actually part of the continuum spectrum and is Dirac delta function normalizable.

We also considered the cases when one or both Majorana terms are spatially homogeneous. If both of them are spatially homogeneous and if $M_L M_R < M_D^2$, with $M_L = M_R$, there is one well behaved solution. If just one is spatially homogeneous there are no well behaved solutions. The phenomenological significance of obtained solutions was discussed.

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