Exploration in Structured Reinforcement Learning

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Abstract

We address reinforcement learning problems with finite state and action spaces where the underlying MDP has some known structure that could be potentially exploited to minimize the exploration rates of suboptimal (state, action) pairs. For any arbitrary structure, we derive problem-specific regret lower bounds satisfied by any learning algorithm. These lower bounds are made explicit for unstructured MDPs and for those whose transition probabilities and average reward functions are Lipschitz continuous w.r.t. the state and action. For Lipschitz MDPs, the bounds are shown not to scale with the sizes \(S\) and \(A\) of the state and action spaces, i.e., they are smaller than \(c \log T\) where \(T\) is the time horizon and the constant \(c\) only depends on the Lipschitz structure, the span of the bias function, and the minimal action sub-optimality gap. This contrasts with unstructured MDPs where the regret lower bound typically scales as \(SA \log T\). We devise DEL (Directed Exploration Learning), an algorithm that matches our regret lower bounds. We further simplify the algorithm for Lipschitz MDPs, and show that the simplified version is still able to efficiently exploit the structure.

1 Introduction

Real-world Reinforcement Learning (RL) problems often concern dynamical systems with large state and action spaces, which make the design of efficient algorithms extremely challenging. This difficulty is well illustrated by the known regret fundamental limits. The regret compares the accumulated reward of an optimal policy (aware of the system dynamics and reward function) to that of the algorithm considered, and it quantifies the loss incurred by the need of exploring sub-optimal (state, action) pairs to learn the system dynamics and rewards. In online RL problems with undiscounted reward, regret lower bounds typically scale as \(SA \log T\) or \(\sqrt{SAT}\)\(^1\), where \(S\), \(A\), and \(T\) denote the sizes of the state and action spaces and the time horizon, respectively. Hence, with large state and action spaces, it is essential to identify and exploit any possible structure existing in the system dynamics and reward function so as to minimize exploration phases and in turn reduce regret to reasonable values. Modern RL algorithms actually implicitly impose some structural properties either in the model parameters (transition probabilities and reward function, see e.g. [Ortner and Ryabko, 2012]) or directly in the \(Q\)-function (for discounted RL problems, see e.g. [Mnih et al., 2015]). Despite the successes of these recent algorithms, our understanding of structured RL problems remains limited.

In this paper, we explore structured RL problems with finite state and action spaces. We first derive problem-specific regret lower bounds satisfied by any algorithm for RL problems with any arbitrary structure. These lower bounds are instrumental to devise algorithms optimally balancing exploration and exploitation, i.e., achieving the regret fundamental limits. A similar approach has

\(^1\)The first lower bound is asymptotic in \(T\) and problem-specific, the second is minimax. We ignore here for simplicity the dependence of these bounds in the diameter, bias span, and action sub-optimality gap.

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been recently applied with success to stochastic bandit problems, where the average reward of arms exhibits structural properties, e.g. unimodality [Combes and Proutiere, 2014], Lipschitz continuity [Magureanu et al., 2014], or more general properties [Combes et al., 2017]. Extending these results to RL problems is highly non trivial, and to our knowledge, this paper is the first to provide problem-specific regret lower bounds for structured RL problems. Although the results presented here concern ergodic RL problems with undiscounted reward, they could be easily generalized to discounted problems (under an appropriate definition of regret).

Our contributions are as follows:

1. For ergodic structured RL problems, we derive problem-specific regret lower bounds. The latter are valid for any structure (but are structure-specific), and for unknown system dynamics and reward function.
2. We analyze the lower bounds for unstructured MDPs, and show that they scale at most as \( \frac{(H+1)^2}{T} S A \log T \), where \( H \) and \( \delta_{\min} \) represent the span of the bias function and the minimal state-action sub-optimality gap, respectively. These results extend previously known regret lower bounds derived in the seminal paper [Burnetas and Katehakis, 1997] to the case where the reward function is unknown.
3. We further study the regret lower bounds in the case of Lipschitz MDPs. Interestingly, these bounds are shown to scale at most as \( \frac{(H+1)^2}{T} S_{lip} A_{lip} \log T \) where \( S_{lip} \) and \( A_{lip} \) only depend on the Lipschitz properties of the transition probabilities and reward function. This indicates that when \( H \) and \( \delta_{\min} \) do not scale with the sizes of the state and action spaces, we can hope for a regret growing logarithmically with the time horizon, and independent of \( S \) and \( A \).
4. We propose DEL, an algorithm that achieves our regret fundamental limits for any structured MDP. DEL is rather complex to implement since it requires in each round to solve an optimization problem similar to that providing the regret lower bounds. Fortunately, we were able to devise simplified versions of DEL, with regret scaling at most as \( \frac{(H+1)^2}{T} S_{lip} A_{lip} \log T \) and \( \frac{(H+1)^2}{T} S_{lip} A_{lip} \log T \) for unstructured and Lipschitz MDPs, respectively. In absence of structure, DEL, in its simplified version, does not require to compute action indexes as done in OLP [Tewari and Bartlett, 2008], and yet achieves similar regret guarantees without the knowledge of the reward function. DEL, simplified for Lipschitz MDPs, only needs, in each step, to compute the optimal policy of the estimated MDP, as well as to solve a simple linear program.
5. Preliminary numerical experiments (presented in the appendix) illustrate our theoretical findings. In particular, we provide examples of Lipschitz MDPs, for which the regret under DEL does not seem to scale with \( S \) and \( A \), and significantly outperforms algorithms that do not exploit the structure.

## 2 Related Work

Regret lower bounds have been extensively investigated for unstructured ergodic RL problems. [Burnetas and Katehakis, 1997] provided a problem-specific lower bound similar to ours, but only valid when the reward function is known. Minimax regret lower bounds have been studied e.g. in [Auer et al., 2009] and [Bartlett and Tewari, 2009]: in the worst case, the regret has to scale as \( \sqrt{DSAT} \) where \( D \) is the diameter of the MDP. In spite of these results, regret lower bounds for unstructured RL problems are still attracting some attention, see e.g. [Osband and Van Roy, 2016] for insightful discussions. To our knowledge, this paper constitutes the first attempt to derive regret lower bounds in the case of structured RL problems. Our bounds are asymptotic in the time horizon \( T \), but we hope to extend them to finite time horizons using similar techniques as those recently used to provide such bounds for bandit problems [Garivier et al., Jun. 2018]. These techniques address problem-specific and minimax lower bounds in a unified manner, and can be leveraged to derive minimax lower bounds for structured RL problems. However we do not expect minimax lower bounds to be very informative about the regret gains that one may achieve by exploiting a structure (indeed, the MDPs leading to worst-case regret in unstructured RL comply to many structures).

There have been a plethora of algorithms developed for ergodic unstructured RL problems. We may classify these algorithms depending on their regret guarantees, either scaling as \( \log T \) or \( \sqrt{T} \). In absence of structure, [Burnetas and Katehakis, 1997] developed an asymptotically optimal, but involved, algorithm. This algorithm has been simplified in [Tewari and Bartlett, 2008], but remains more complex than our proposed algorithm. Some algorithms have finite-time regret guarantees...
scaling as $\log T$ [Auer and Ortner, 2007], [Auer et al., 2009], [Filippi et al., 2010]. For example, the authors of [Filippi et al., 2010] propose KL-UCRL an extension of UCRL [Auer and Ortner, 2007] with regret bounded by $O_{	ext{max}} S^2 A \log T$. Having finite-time regret guarantees is arguably desirable, but so far this comes at the expense of a much larger constant in front of $\log T$. Algorithms with regret scaling as $\sqrt{T}$ include UCRL2 [Auer et al., 2009], KL-UCRL with regret guarantees $O(DS\sqrt{AT})$, REGAL.C [Bartlett and Tewari, 2009] with guarantees $O(HS\sqrt{AT})$. Recently, the authors of [Agrawal and Jia, 2017] managed to achieve a regret guarantee of $O(D\sqrt{SAT})$, but only valid when $T \geq S^3 A$.

Algorithms devised to exploit some known structure are most often applicable to RL problems with continuous state or action spaces. Typically, the transition probabilities and reward function are assumed to be smooth in the state and action, typically Lipschitz continuous [Ortner and Ryabko, 2012], [Lakshmanan et al., 2015]. The regret then needs to scale as a power of $T$, e.g. $T^{2/3}$ in [Lakshmanan et al., 2015] for 1-dimensional state spaces. An original approach to RL problems for which the transition probabilities belong to some known class of functions was proposed in [Osband et al., 2017]. Algorithms with regret guarantees of $O_{	ext{max}} S \sqrt{T}$ can be the Lebesgue measure; alternatively, if rewards take values in $[0, 1]$, $\lambda$ can be the sum of Dirac measures at 0 and 1.

### 3 Models and Objectives

We consider an MDP $\phi = (p_\phi, q_\phi)$ with finite state and action spaces $S$ and $A$ of respective cardinalities $S$ and $A$. $p_\phi$ and $q_\phi$ are the transition and reward kernels of $\phi$. Specifically, when in state $x$, taking action $a$, the system moves to state $y$ with probability $p_\phi(y|x, a)$, and a reward drawn from distribution $q_\phi(\cdot|x, a)$ of average $r_\phi(x, a)$ is collected. The rewards are bounded, w.l.o.g., in $[0, 1]$. We assume that for any $(x, a)$, $q_\phi(\cdot|x, a)$ is absolutely continuous w.r.t. some measure $\lambda$ on $[0, 1]^2$.

The random vector $Z_t := (X_t, A_t, R_t)$ represents the state, the action, and the collected reward at step $t$. A policy $\pi$ selects an action, denoted by $\pi_t(x)$, in step $t$ when the system is in state $x$ based on the history captured through $H^T_t$, the $\sigma$-algebra generated by $(Z_1, \ldots, Z_{t-1}, X_t)$ observed under $\pi$: $\pi_t(x)$ is $H^T_t$-measurable. We denote by $\Pi$ the set of all such policies.

#### Structured MDPs.

The MDP $\phi$ is initially unknown. However we assume that $\phi$ belongs to some well specified set $\Phi$ which may encode a known structure of the MDP. The knowledge of $\Phi$ can be exploited to devise (more) efficient policies. The results derived in this paper are valid under any structure, but we give a particular attention to the cases of (i) Unstructured MDPs: $\phi \in \Phi$ if for all $(x, a)$, $p_\phi(\cdot \mid x, a) \in \mathcal{P}(S)$ and $q_\phi(\cdot \mid x, a) \in \mathcal{P}([0, 1])$; (ii) Lipschitz MDPs: $\phi \in \Phi$ if $p_\phi(\cdot \mid x, a)$ and $r_\phi(x, a)$ are Lipschitz-continuous w.r.t. $x$ and $a$ in some metric space (we provide a precise definition in the next section).

#### The learning problem.

The expected cumulative reward up to step $T$ of a policy $\pi \in \Pi$ when the system starts in state $x$ is $V^\pi_T(x) := \mathbb{E}_x^{\pi}[\sum_{t=1}^T R_t]$, where $\mathbb{E}_x^{\pi}[\cdot]$ denotes the expectation under policy $\pi$ given that $X_1 = x$. Now assume that the system starts in state $x$ and evolves according to the initially unknown MDP $\phi \in \Phi$ for given structure $\Phi$, the objective is to devise a policy $\pi \in \Pi$ maximizing $V^\pi_T(x)$ or equivalently, minimizing the regret $R^\pi_T(x)$ up to step $T$ defined as the difference between the cumulative reward of an optimal policy and that obtained under $\pi$:

$$R^\pi_T(x) := V^\pi_T(x) - V^\pi_T(x)$$

where $V^\pi_T(x) := \sup_{\pi \in \Pi} V^\pi_T(x)$.

#### Preliminaries and notations.

Let $\Pi_D$ be the set of stationary (deterministic) policies, i.e. when in state $X_t = x$, $f \in \Pi_D$ selects an action $f(x)$ independent of $t$. $\phi$ is communicating if each pair of states are connected by some policy. Further, $\phi$ is ergodic if under any stationary policy, the resulting Markov chain $(X_t)_{t \geq 1}$ is irreducible. For any communicating $\phi$ and any policy $\pi \in \Pi_D$.

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\footnote{$\lambda$ can be the Lebesgue measure; alternatively, if rewards take values in $[0, 1]$, $\lambda$ can be the sum of Dirac measures at 0 and 1.}

\footnote{$\mathcal{P}(S)$ is the set of distributions on $S$ and $\mathcal{P}([0, 1])$ is the set of distributions on $[0, 1]$, absolutely continuous w.r.t. $\lambda$.}
We further define the set of $\Pi_D$, we denote by $g_\phi^*(x)$ the **gain** of $\pi$ (or long-term average reward) started from initial state $x$: $g_\phi^*(x) := \lim_{T \to \infty} \frac{1}{T} V^\pi_T(x)$. We denote by $\Pi^*(\phi)$ the set of stationary policies with maximal gain: $\Pi^*(\phi) := \{ f \in \Pi_D : g_\phi^f(x) = g_\phi^*(x) \ \forall x \in S \}$, where $g_\phi^f(x) := \max_{\pi \in \Pi} g_\phi^\pi(x)$. If $\phi$ is communicating, the maximal gain is constant and denoted by $g_\phi^\pi(x)$. The **bias function** $h^f_\phi$ of $f \in \Pi_D$ is defined by $\| h^f_\phi \| := C \lim_{T \to \infty} \mathbb{E}[\sum_{t=1}^T (R_t - g_\phi^f(X_t))]$, and quantifies the advantage of starting in state $x$. We denote by $B^\phi$ and $B^\phi_*$, respectively, the Bellman operator under action $a$ and the optimal Bellman operator under $\phi$. They are defined by: for any $h : S \to \mathbb{R}$ and $x \in S$,

\[
(B^\phi h)(x) := r(x, a) + \sum_{y \in S} p(y|x, a) h(y) \quad \text{and} \quad (B^\phi_* h)(x) := \max_{a \in A}(B^\phi h)(x).
\]

Then for any $f \in \Pi_D$, $g^f_\phi$ and $h^f_\phi$ satisfy the **evaluation equation**: for all state $x \in S$, $g^f_\phi(x) + h^f_\phi(x) = (B^\phi f)(x)$. Furthermore, $f \in \Pi^*(\phi)$ if and only if $g^f_\phi$ and $h^f_\phi$ verify the **optimality equation**:

\[
g^f_\phi(x) + h^f_\phi(x) = (B^\phi_* f)(x).
\]

We denote by $h^*_\phi$ the bias function of an optimal stationary policy\(^4\), and by $H$ its span $H := \max_{x, y} h^*_\phi(x) - h^*_\phi(y)$. For $x \in S, h : S \to \mathbb{R}$, and $\phi \in \Phi$, let $O(x; h, \phi) = \{ a \in A : (B^\phi h)(x) = (B^\phi h)(x) \}$. For ergodic $\phi$, $h^*_\phi$ is unique up to an additive constant. Hence, for ergodic $\phi$, the set of optimal actions in state $x$ under $\phi$ is $O(x; \phi) := O(x; h^*_\phi, \phi)$, and $\Pi^*(\phi) = \{ f \in \Pi_D : f(x) \in O(x; \phi) \ \forall x \in S \}$. Finally, we define for any state $x$ and action $a$,

\[
\delta^\phi(x, a) := (B^\phi_* h^*_\phi)(x) - (B^\phi h^*_\phi)(x).
\]

This can be interpreted as the long-term regret obtained by initially selecting action $a$ in state $x$ (and then applying an optimal stationary policy) rather than following an optimal policy. The minimum gap is defined as $\delta_{\min} := \min_{x \in S, a \geq 0} \delta^\phi(x, a)$.

We denote by $\mathbb{R}_+ = \mathbb{R}_+ \cup \{ \infty \}$. The set of MDPs is equipped with the following $\ell_\infty$-norm:

\[
\| \phi - \psi \| := \max_{x, a \in S \times A} \| \phi(x, a) - \psi(x, a) \|, \quad \text{where} \quad \| \phi(x, a) - \psi(x, a) \| := |r(x, a) - r(x, a)| + \max_{y \in S} |p(y|x, a) - p(y|x, a)|.
\]

The proofs of all results are presented in the appendix.

### 4 Regret Lower Bounds

In this section, we present an (asymptotic) regret lower bound satisfied by any **uniformly good learning algorithm**. An algorithm $\pi \in \Pi$ is uniformly good if for all ergodic $\phi \in \Phi$, any initial state $x$ and any constant $\alpha > 0$, the regret of $\pi$ satisfies $R_T^\pi(x) = o(T^\alpha)$.

To state our lower bound, we introduce the following notations. For $\phi$ and $\psi$, we denote $\phi \ll \psi$ if the kernel of $\phi$ is absolutely continuous w.r.t. that of $\psi$, i.e., $\forall \mathcal{E}, P_\psi[\mathcal{E}] = 0 = P_\phi[\mathcal{E}]$. For $\phi$ and $\psi$ such that $\phi \ll \psi$ and $(x, a)$, we define the KL-divergence between $\phi$ and $\psi$ in state-action pair $(x, a)$ $KL_{\phi||\psi}(x, a)$ as the KL-divergence between the distributions of the next state and collected reward if the state is $x$ and $a$ is selected under these two MDPs:

\[
KL_{\phi||\psi}(x, a) = \sum_{y \in S} p_\phi(y|x, a) \log \frac{p_\phi(y|x, a)}{p_\psi(y|x, a)} + \int_0^1 q_\phi(r|x, a) \log \frac{q_\phi(r|x, a)}{q_\psi(r|x, a)} \lambda(dr).
\]

We further define the set of **confusing MDPs** as:

\[
\Delta_\phi = \{ \psi \in \Phi : \phi \ll \psi, \ (i) \ \text{KL}_{\phi||\psi}(x, a) = 0 \ \forall x, \forall a \in O(x; \phi) ; \ (ii) \ \Pi^*(\phi) \cap \Pi^*(\psi) = \emptyset \}.
\]

This set consists of MDP $\psi$’s that (i) coincide with $\phi$ for state-action pairs where the actions are optimal (the kernels of $\phi$ and $\psi$ cannot be statistically distinguished under an optimal policy); and such that (ii) the optimal policies under $\psi$ are not optimal under $\phi$.

\(^4\)In case of $h^*_\phi$ is not unique, we arbitrarily select an optimal stationary policy and define $h^*_\phi$.
Theorem 1. Let \( \phi \in \Phi \) be ergodic. For any uniformly good algorithm \( \pi \in \Pi \) and for any \( x \in S \),
\[
\liminf_{T \to \infty} \frac{R_T(x)}{\log T} \geq K_\Phi(\phi),
\]
where \( K_\Phi(\phi) \) is the value of the following optimization problem:
\[
\inf_{\eta \in \mathcal{F}_\Phi(\phi)} \sum_{(x,a) \in S \times A} \eta(x,a)\delta^*(x,a;\phi),
\]
where \( \mathcal{F}_\Phi(\phi) := \{ \eta \in \mathbb{R}_+^{S \times A} : \sum_{(x,a) \in S \times A} \eta(x,a)\text{KL}_{\phi|\psi}(x,a) \geq 1, \forall \psi \in \Delta_\Phi(\phi) \} \).

The above theorem can be interpreted as follows. When selecting a sub-optimal action \( a \) in state \( x \), one has to pay a regret of \( \delta^*(x,a;\phi) \). Then the minimal number of times any sub-optimal action \( a \) in state \( x \) has to be explored scales as \( \eta^*(x,a) \log T \) where \( \eta^*(x,a) \) solves the optimization problem (2). It is worth mentioning that our lower bound is tight, as we present in Section 5 an algorithm achieving this fundamental limit of regret.

The regret lower bound stated in Theorem 1 extends the problem-specific regret lower bound derived in [Burnetas and Katehakis, 1997] for unstructured ergodic MDPs with known reward function. Our lower bound is valid for unknown reward function, but also applies to any structure \( \Phi \). Note however that at this point, it is only implicitly defined through the solution of (2), which seems difficult to solve. The optimization problem can actually be simplified, as shown later in this section, by providing useful structural properties of the feasibility set \( \mathcal{F}_\Phi(\phi) \) depending on the structure considered. The simplification will be instrumental to quantify the gain that can be achieved when optimally exploiting the structure, as well as to design efficient algorithms.

In the following, the optimization problem: \( \inf_{\eta \in \mathcal{F}} \sum_{(x,a) \in S \times A} \eta(x,a)\delta^*(x,a;\phi) \) is referred to as \( P(\phi,F) \); so that \( P(\phi, \mathcal{F}_\Phi(\phi)) \) corresponds to (2).

The proof of Theorem 1 combines a characterization of the regret as a function of the number of times \( N_T(x,a) \) up to \( T \) (state, action) pair \( (x,a) \) is visited, and of the \( \delta^*(x,a;\phi) \)'s, and change-of-measure arguments as those recently used to prove in a very direct manner regret lower bounds in bandit optimization problems [Kaufmann et al., 2016]. More precisely, for any uniformly good algorithm \( \pi \), and for any confusing MDP \( \psi \in \Delta_\Phi(\phi) \), we show that the exploration rates required to statistically distinguish \( \psi \) from \( \phi \) satisfy \( \liminf_{T \to \infty} \frac{1}{\log T} \sum_{(x,a) \in S \times A} \mathbb{E}_{x,a}[N_T(x,a)] \text{KL}_{\phi|\psi}(x,a) \geq 1 \) where the expectation is taken w.r.t. \( \phi \) given any initial state \( x_1 \). The theorem is then obtained by considering (hence optimizing the lower bound) all possible confusing MDPs.

4.1 Decoupled exploration in unstructured MDPs

In the absence of structure, \( \Phi = \{ \psi : p_\psi(\cdot|x,a) \in \mathcal{P}(S), q_\psi(\cdot|x,a) \in \mathcal{P}([0,1]), \forall (x,a) \} \), and we have:

Theorem 2. Consider the unstructured model \( \Phi \), and let \( \phi \in \Phi \) be ergodic. We have:
\[
\mathcal{F}_\Phi(\phi) = \{ \eta \in \mathbb{R}_+^{S \times A} : \forall (x,a) s.t. a \notin \mathcal{O}(x;\phi), \eta(x,a)\text{KL}_{\phi|\psi}(x,a) \geq 1, \forall \psi \in \Delta_\Phi(x,a;\phi) \}
\]
where \( \Delta_\Phi(x,a;\phi) := \{ \psi \in \Phi : \text{KL}_{\phi|\psi}(y,b) = 0 \ \forall (y,b) \neq (x,a) \) and \( (B^\phi_k h^\phi_k(x)) > g^\phi_k + h^\phi_k(x)) \} \).

The theorem states that in the constraints of the optimization problem (2), we can restrict our attention to confusing MDPs \( \psi \) that are different than the original MDP \( \phi \) only for a single state-action pair \( (x,a) \). Further note that the condition \( (B^\phi_k h^\phi_k(x)) > g^\phi_k + h^\phi_k(x)) \) is equivalent to saying that action \( a \) becomes optimal in state \( x \) under \( \psi \) (see Lemma 1(i) in [Burnetas and Katehakis, 1997]). Hence to obtain the lower bound in unstructured MDPs, we may just consider confusing MDPs \( \psi \) which make an initially sub-optimal action \( a \) in state \( x \) optimal by locally changing the kernels and rewards of \( \phi \) at \( (x,a) \) only. Importantly, this observation implies that an optimal algorithm \( \pi \) must satisfy \( \mathbb{E}_{x,a}[N_T(x,a)] \sim T / \inf_{\psi \in \Delta_\Phi(x,a;\phi)} \text{KL}_{\phi|\psi}(x,a) \). In other words, the required level of exploration of the various sub-optimal state-action pairs are decoupled, which significantly simplifies the design of optimal algorithms.

To get an idea on how the regret lower bound scales as the sizes of both state and action spaces, we can further provide an upper bound of the regret lower bound. One may easily observe that
\[ F_{un}(\phi) \subset F_{\Phi}(\phi) \]

where

\[ F_{un}(\phi) = \left\{ \eta \in \mathbb{R}^S \times A : \eta(x,a) \left( \frac{\delta^*(x,a;\phi)}{H+1} \right)^2 \geq 2, \forall (x,a) \text{ s.t. } a \notin \mathcal{O}(x;\phi) \right\}. \]

From this result, an upper bound of the regret lower bound is

\[ K_{un}(\phi) := \frac{2(H+1)^2}{\delta_{\min}} S A \log T, \]

and we can devise algorithms achieving this regret scaling (see Section 5).

**Theorem 2** relies on the following decoupling lemma, actually valid under any structure \( \Phi \).

**Lemma 1.** Let \( \mathcal{U}_1, \mathcal{U}_2 \) be two non-overlapping subsets of the (state, action) pairs such that for all \( (x,a) \in \mathcal{U}_0 := \mathcal{U}_1 \cup \mathcal{U}_2, a \notin \mathcal{O}(x;\phi) \). Define the following three MDPs in \( \Phi \) obtained starting from \( \phi \) and changing the kernels for (state, action) pairs in \( \mathcal{U}_1 \cup \mathcal{U}_2 \). Specifically, let \( (p, q) \) be some transition and reward kernels. For all \( (x,a) \), define \( \psi_j, j \in \{0, 1, 2\} \) as

\[ (p_{\psi_j}(.|x,a), q_{\psi_j}(.|x,a)) = \begin{cases} (p(.|x,a), q(.|x,a)) & \text{if } (x,a) \in \mathcal{U}_j, \\ (p_{\phi}(.|x,a), q_{\phi}(.|x,a)) & \text{otherwise.} \end{cases} \]

Then, if \( \Pi^*(\phi) \cap \Pi^*(\psi_0) = \emptyset \), then \( \Pi^*(\phi) \cap \Pi^*(\psi_1) = \emptyset \) or \( \Pi^*(\phi) \cap \Pi^*(\psi_2) = \emptyset \).

### 4.2 Lipschitz structure

Lipschitz structures have been widely studied in the bandit and reinforcement learning literature. We find it convenient to use the following structure, although one could imagine other variants in more general metric spaces. We assume that the state (resp. action) space can be embedded in the \( d \) (resp. \( d' \)) dimensional Euclidean space: \( S \subset [0,D]^d \) and \( A \subset [0,D']^d \). We consider MDPs whose transition kernels and average rewards are Lipschitz w.r.t. the states and actions. Specifically, let \( L, L' > 0, \alpha, \alpha' > 0 \), and

\[ \Phi = \left\{ \psi : p_{\psi}(.|x,a) \in \mathcal{P}(S), q_{\psi}(.|x,a) \in \mathcal{P}([0,1]) : (L1)-(L2) \text{ hold}, \forall (x,a) \right\}, \]

where

\[ (L1) \quad \| p_{\psi}(.|x,a) - p_{\psi}(.|x',a') \|_1 \leq L d(x,x')^\alpha + L' d(a,a')^{\alpha'}, \]

\[ (L2) \quad | r_{\psi}(x,a) - r_{\psi}(x',a') | \leq L d(x,x')^\alpha + L' d(a,a')^{\alpha'}. \]

Here \( d(.,.) \) is the Euclidean distance, and for two distributions \( p_1 \) and \( p_2 \) on \( S \) we denote by \( \| p_1 - p_2 \|_1 = \sum_{y \in S} | p_1(y) - p_2(y) | \).

**Theorem 3.** For the model \( \Phi \) with Lipschitz structure (L1)-(L2), we have \( F_{\text{lip}}(\phi) \subset F_{\Phi}(\phi) \) where \( F_{\text{lip}}(\phi) \) is the set of \( \eta \in \mathbb{R}^S \times A \) satisfying for all \( (x', a') \) such that \( a' \notin \mathcal{O}(x', \phi) \),

\[ \sum_{x \in S} \sum_{a \notin \mathcal{O}(x, \phi)} \eta(x,a) \left( \frac{\delta^*(x', a'; \phi)}{H+1} - 2 \left( L d(x,x')^\alpha + L' d(a,a')^{\alpha'} \right) \right)^2 \geq 2 \quad (3) \]

where we use the notation \( |u|_+ := \max\{0, u\} \) for \( u \in \mathbb{R} \). Furthermore, the optimal values \( K_{\Phi}(\phi) \) and \( K_{\text{lip}}(\phi) \) of \( P(\phi, F_{\Phi}(\phi)) \) and \( P(\phi, F_{\text{lip}}(\phi)) \) are upper bounded by \( 8\frac{(H+1)^3}{\delta_{\min}} S_{\text{lip}} A_{\text{lip}} \) where

\[ S_{\text{lip}} := \min \left\{ S, \left( \frac{D' \sqrt{d}}{\delta_{\min} (S L (H+1))} \right)^d \right\}, \quad \text{and} \quad A_{\text{lip}} := \min \left\{ A, \left( \frac{D' \sqrt{d}}{\delta_{\min} (S L (H+1))} \right)^{1/\alpha'} + 1 \right\}^{d'}. \]

The above theorem has important consequences. First, it states that exploiting the Lipschitz structure optimally, one may achieve a regret at most scaling as \( \frac{(H+1)^3}{\delta_{\min}} S_{\text{lip}} A_{\text{lip}} \log T \). This scaling is independent of the sizes of the state and action spaces provided that the minimal gap \( \delta_{\min} \) is fixed, and provided that the span \( H \) does not scale with \( S \). The latter condition typically holds for fast mixing models or for MDPs with diameter not scaling with \( S \) (refer to [Bartlett and Tewari, 2009] for a precise connection between \( H \) and the diameter). Hence, exploiting the structure can really yield significant regret improvements. As shown in the next section, leveraging the simplified structure in \( F_{\text{lip}}(\phi) \), we may devise a simple algorithm achieving these improvements, i.e., having a regret scaling at most as \( K_{\text{lip}}(\phi) \log T \).
Algorithm 1 DEL(γ)

**input** Model structure Φ

- Initialize $N_t(x) ← \mathbb{I}[x = X_1]$, $N_t(x,a) ← 0$, $s_t(x) ← 0$, $p_t(y \mid x,a) ← 1/|S|$, $r_t(x,a) ← 0$ for each $x,y \in S$, $a \in \mathcal{A}$, and $φ_t$ accordingly.

**for** $t = 1, \ldots, T$ **do**

- For each $x \in S$, let $G_t(x) := \{a \in A : N_t(x,a) \geq \log^2 N_t(x)\}$, $φ_t^* := φ_t(C_t)$, $h_t^*(x) := h_{φ_t^*}(x)$, $ζ_t := \frac{1}{1 + \log \log t}$ and $γ_t := (1 + γ)(1 + \log t)$.
- **if** $∀a ∈ \mathcal{O}(x; φ_t^*)$, $N_t(X_t,a) < \log^2 N_t(X_t) + 1$ **then**
  - **Monotize:** $A_t ← A_t^{\text{mt}} := \arg \min_{a \in \mathcal{O}(x; φ_t^*)} N_t(X_t,a)$.
  - **else** if $∃a ∈ \mathcal{A}$ s.t. $N_t(X_t,a) < \frac{\log N_t(X_t)}{1 + \log \log N_t(X_t)}$ **then**
    - **Estimate:** $A_t ← A_t^{\text{pt}} := \arg \min_{a \in \mathcal{A}} N_t(X_t,a)$.
  - **else** if $(N_t(x,a)/N_t)(x,a) ∈ \mathcal{S} ∧ \mathcal{A}$ **then**
    - **F:** $F_t := \mathcal{F}_φ(φ_t; C_t, ζ_t)$.
    - **end if**
    - **Exploit:** $A_t ← A_t^{\text{sp}} := \arg \min_{a \in A, N_t(x,a) ≤ η_t(x,a)} N_t(X_t,a)$.
    - **end if**
- **else**
  - For each $(x,a) ∈ \mathcal{S} × \mathcal{A}$, let $δ_t(x,a) := δ^*(x,a; φ_t, C_t, ζ_t)$.
  - **if** $F_t := \mathcal{F}_φ(φ_t; C_t, ζ_t) \cap \{η : η(x,a) = \infty \text{ if } δ_t(x,a) = 0\} = \emptyset$ **then**
    - Let $η_t(x,a) = \infty$ if $δ_t(x,a) = 0$ and $η_t(x,a) = 0$ otherwise.
  - **else**
    - Obtain a solution $η_t$ of $\mathcal{P}(δ_t,F_t)$: $\inf_{η ∈ \mathcal{F} \sum_{(x,a) ∈ \mathcal{S} × \mathcal{A}} η(x,a) δ_t(x,a)}$.
    - **end if**
  - **Exploire:** $A_t ← A_t^{\text{sp}} := \arg \min_{a \in A, N_t(x,a) ≤ η_t(x,a)} N_t(X_t,a)$.
  - **end if**
- **end for**

5 Algorithms

In this section, we present DEL (Directed Exploration Learning), an algorithm that achieves the regret limits identified in the previous section. Asymptotically optimal algorithms for generic controlled Markov chains have already been proposed in [Graves and Lai, 1997], and could be adapted to our setting. By presenting DEL, we aim at providing simplified, yet optimal algorithms. Moreover, DEL can be adapted so that the exploration rates of sub-optimal actions are directed towards the solution of an optimization problem $P(φ, \mathcal{F}(φ))$ provided that $\mathcal{F}(φ) ⊂ \mathcal{F}_φ(φ)$ (it suffices to use $\mathcal{F}(φ_t)$ instead of $\mathcal{F}_φ(φ_t)$ in DEL). For example, in the case of Lipschitz structure Φ, running DEL on $\mathcal{F}_\text{lip}(φ)$ yields a regret scaling at most as $(H+1)^s_{\text{min}} \text{Sup} \mathcal{A}_\text{lip} \log T$.

The pseudo-code of DEL with input parameter $γ > 0$ is given in Algorithm 2. There, for notational convenience, we abuse the notations and redefine $\log t$ as $1[t ≥ 1] \log t$, and let $∞ · 0 = 0$. $φ_t$ refers to the estimated MDP at time $t$ (using empirical transition rates and rewards). For any non-empty correspondence $C : \mathcal{S} → \mathcal{A}$ (i.e., for any $x$, $C(x)$ is a non-empty subset of $\mathcal{A}$), let $φ(C)$ denote the restricted MDP where the set of actions available at state $x$ is $C(x)$. Then, $g_{φ(C)}$ and $h^*_{φ(C)}$ are the (optimal) gain and bias functions corresponding to the restricted MDP $φ(C)$. Given a restriction defined by $C$, for each $(x,a) ∈ \mathcal{S} × \mathcal{A}$, let $δ^*(x,a; φ, C) := (B^*_{φ(C)} h^*_{φ(C)})(x) − (B^1 h^*_{φ(C)})(y)$ and $H^*(φ(C)) := \max_{x,y \in S} h^*_{φ(C)}(x) − h^*_{φ(C)}(y)$. For $ζ ≥ 0$, let $δ^*(x,a; φ, C, ζ) := 0$ if $δ^*(x,a; φ, C) ≤ ζ$, and let $δ^*(x,a; φ, C, ζ) := δ^*(x,a; φ, C)$ otherwise. For $ζ ≥ 0$, we further define the set of confusing MDPs $Δ_φ(φ; C, ζ)$, and the set of feasible solutions $\mathcal{F}_φ(φ; C, ζ)$ as:

$$\Delta_φ(φ; C, ζ) := \left\{ ψ ∈ \Phi \cup \{φ\} : φ ≪ ψ \land (i) \text{ KL}_{φ;ψ}(x,a) = 0 \forall x, ∀a ∈ \mathcal{O}(x; φ(C)); (ii) \text{ KL}_{φ;ψ}(x,a) ≥ 1 \land (iii) \exists (x,a) ∈ \mathcal{S} × \mathcal{A} \text{ s.t. } a / ∈ \mathcal{O}(x; φ(C)) \right\}$$

$$\mathcal{F}_φ(φ; C, ζ) := \{ η ∈ \mathbb{R}_+^{|S|} : \sum_{x ∈ S} \sum_{a ∈ \mathcal{A}} η(x,a) \text{ KL}_{φ;ψ}(x,a) ≥ 1, \forall ψ ∈ \Delta_φ(φ; C, ζ) \}.$$
Similar sets $\mathcal{F}_{un}(\phi; C, \zeta)$ and $\mathcal{F}_{lip}(\phi; C, \zeta)$ can be defined for the cases of unstructured and Lipschitz MDPs (refer to the appendix), and DEL can be simplified in these cases by replacing $\mathcal{F}_\phi(\phi; C, \zeta)$ by $\mathcal{F}_{un}(\phi; C, \zeta)$ or $\mathcal{F}_{lip}(\phi; C, \zeta)$ in the pseudo-code. Finally, $\mathcal{P}(\delta, \mathcal{F})$ refers to the optimization problem $\inf_{\eta \in \mathcal{F}} \sum_{(x,a) \in S \times A} \eta(x, a) \delta(x, a)$.

DEL combines the ideas behind OSSB [Combes et al., 2017], an asymptotically optimal algorithm for structured bandits, and the asymptotically optimal algorithm presented in [Burnetas and Katehakis, 1997] for RL problems without structure. DEL design aims at exploring sub-optimal actions no more than what the regret lower bound prescribes. To this aim, it essentially solves in each iteration $t$ an optimization problem close to $\mathcal{P}(\phi_t, \mathcal{F}_\phi(\phi_t))$ where $\phi_t$ is an estimate of the true MDP $\phi$. Depending on the solution and the number of times apparently sub-optimal actions have been played, DEL decides to explore or exploit. The estimation phase ensures that certainty equivalence holds. The “monotonization” phase together with the restriction to relatively well selected actions were already proposed in [Burnetas and Katehakis, 1997] to make sure that accurately estimated actions only are selected in the exploitation phase. The various details and complications introduced in DEL ensure that its regret analysis can be conducted. In practice (see the appendix), our initial experiments suggest that many details can be removed without large regret penalties.

**Theorem 4.** For a structure $\Phi$ with Bernoulli rewards and for any ergodic MDP $\phi \in \Phi$, assume that: (i) $\phi$ is in the interior of $\Phi$ (i.e., there exists a constant $\zeta_0 > 0$ such that for any $\zeta \in (0, \zeta_0)$, $\psi \in \Phi$ if $||\phi - \psi|| \leq \zeta$ and $\psi \ll \phi$), (ii) the solution $\eta^*(\phi)$ is uniquely defined for each $(x, a)$ such that $a \notin O(x; \phi)$, (iii) continuous at $\phi$ (i.e., for any given $\varepsilon > 0$, there exists $\zeta(\varepsilon) > 0$ such that for all $\zeta \in (0, \zeta(\varepsilon))$, if $||\psi - \phi|| \leq \zeta \max_{(x,a):a\notin O(x;\phi)} \|\eta^*(x,a;\psi,\zeta) - \eta^*(x,a;\phi)\| \leq \varepsilon$ where $\eta^*(\psi, \zeta)$ is solution of $\mathcal{P}(\delta^*(\psi, A, \zeta), \mathcal{F}_\phi(\psi, A, \zeta))$, and $\eta^*(x, a; \phi)$ that of $\mathcal{P}(\phi, \mathcal{F}_\phi(\phi)))$. Then, for $\pi = \text{DEL}(\gamma)$ with any $\gamma > 0$, we have:

$$\limsup_{T \to \infty} \frac{R_T^\pi(\phi)}{\log T} \leq (1 + \gamma)K_{\Phi}(\phi). \quad (4)$$

For Lipschitz $\Phi$ with (L1)-(L2) (resp. unstructured $\Phi$), if $\pi = \text{DEL}$ uses in each step $t$, $\mathcal{F}_{lip}(\phi_t; C_t, \zeta_t)$ (resp. $\mathcal{F}_{un}(\phi_t; C_t, \zeta_t)$) instead of $\mathcal{F}_\phi(\phi_t; C_t, \zeta_t)$, its regret is asymptotically smaller than $(1 + \gamma)K_{\text{lip}}(\phi) \log T$ (resp. $(1 + \gamma)K_{\text{un}}(\phi) \log T$).

In the above theorem, the assumptions about the uniqueness and continuity of the solution $\eta^*(\phi)$ could be verified for particular structures. In particular, we believe that they generally hold in the case of unstructured and Lipschitz MDPs. Also note that similar assumptions have been made in [Graves and Lai, 1997].

6 Extensions and Future Work

It is worth extending the approach developed in this paper to the case of structured discounted RL problems (although for such problems, there is no ideal way of defining the regret of an algorithm). There are other extensions worth investigating. For example, since our framework allows any kind of structure, we may specify our regret lower bounds for structures stronger than that corresponding to Lipschitz continuity, e.g., the reward may exhibit some kind of unimodality or convexity. Under such structures, the regret improvements might become even more significant. Another interesting direction consists in generalizing the results to the case of communicating MDPs. This would allow us for example to consider deterministic system dynamics and unknown probabilistic rewards.

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A The DEL Algorithm

In this section, we present DEL, and state its asymptotic performance guarantees. DEL pseudo-code is given in Algorithm 2. There, for notational convenience, we abuse the notations and redefine log \( t \) as \( \lfloor t \geq 1 \lceil \log t \). \( \phi_t \) refers to the estimated MDP at time \( t \) (e.g. using empirical transition rates). For non-empty correspondence \( C : S \rightarrow A \) (i.e., for any \( x \), \( C(x) \) is a non-empty subset of \( A \)), let \( \phi(C) \) denote the restricted MDP where the set of actions available at state \( x \) is limited to \( C(x) \). Then, \( \phi_\delta(C) \) and \( \phi_\epsilon(C) \) are the (optimal) gain and bias functions corresponding to the restricted MDP \( \phi(C) \), respectively. Given a restriction defined by \( C \), for each \((x, a) \in S \times A \), let \( \delta^*(x, a; \phi, C) := (B^\phi(C)h^\phi(C))(x) - (B^\phi(C))h^\phi(C)(y) \) and \( H_\phi(C) := \max_{x,y \in S} h^\phi(C)(x) - h^\phi(C)(y) \). For \( \zeta \geq 0 \), let \( \delta^*(x, a; \phi, C, \zeta) := 0 \) if \( \delta^*(x, a; \phi, C) \leq \zeta \), and let \( \delta^*(x, a; \phi, C, \zeta) := \delta^*(x, a; \phi, C) \) otherwise. For \( \zeta \geq 0 \), we further define the set of confusing MDPs \( \Delta_\phi(C, \zeta) \), and the set of feasible solutions \( \mathcal{F}_\phi(C, \zeta) \):

\[
\Delta_\phi(C, \zeta) := \left\{ \psi \in \Phi \cup \{ \phi \} : \phi \ll \psi, \ (i) \ \frac{\text{KL}_{\phi|\psi}(x, a)}{H_\phi(C) + 1} = 0 \ \forall x, \forall a \in \mathcal{O}(x; \phi(C)) \right. \\
\left. \ (ii) \ \exists x, a \in \mathcal{O}(x; \phi(C)) \text{ s.t. } a \not\in \mathcal{O}(x; \phi(C)) \text{ and } \delta^*(x, a; \phi, C, \zeta) = 0 \right\}
\]

\[
\mathcal{F}_\phi(C, \zeta) := \left\{ \eta \in \mathbb{R}_+^{S \times A} : \sum_{x \in S} \sum_{a \in A} \eta(x, a) \text{KL}_{\phi|\psi}(x, a) \geq 1 \ \forall \psi \in \Delta_\phi(C, \zeta) \right\}
\]

For the unstructured and Lipschitz MDPs, we simplify the feasible solution set as \( \mathcal{F}_\text{un}(\phi; C, \zeta) \) and \( \mathcal{F}_\text{lip}(\phi; C, \zeta) \), respectively, defined as:

\[
\mathcal{F}_\text{un}(\phi; C, \zeta) := \left\{ \eta \in \mathbb{R}_+^{S \times A} : \eta(x, a) \left( \frac{\delta^*(x, a; \phi, C, \zeta)}{H_\phi(C) + 1} \right)^2 \geq 2 \ \forall (x, a) \text{ s.t. } a \not\in \mathcal{O}(x; \phi(C)) \right\}
\]

\[
\mathcal{F}_\text{lip}(\phi; C, \zeta) := \left\{ \eta \in \mathbb{R}_+^{S \times A} : L_{\text{lip}}(x', a'; \phi, C, \zeta) \geq 2 \psi(x', a') \text{ s.t. } a' \not\in \mathcal{O}(x'; \phi(C)) \right\}
\]

where

\[
L_{\text{lip}}(x', a'; \phi, C, \zeta) := \sum_{x \in S} \sum_{a \in A} \eta(x, a) \left( \left[ \frac{\delta^*(x', a'; \phi, C, \zeta)}{H_\phi(C) + 1} - 2 \left( L_\delta(x, x')^a + L_\delta(a, a')^{a'} \right) \right]_+ \right)^2.
\]

B Numerical Experiments

In this section, we briefly illustrate the performance of a simplified version of the DEL algorithm on a simple example constructed so as to comply to a Lipschitz structure. Our objective is to investigate the regret gains obtained by exploiting a Lipschitz structure, and we compare the performance of our two simplified versions of DEL with \( \gamma = 1 \) and \( \zeta = 0 \), one solving \( P(\phi_t, \mathcal{F}_\text{un}(\phi_t; C_t, \zeta)) \) in step \( t \), and the other solving \( P(\phi_t, \mathcal{F}_\text{lip}(\phi_t; C_t, \zeta)) \).

The RL problem. We consider a toy MDP whose states are partitioned into two clusters \( S_1, S_2 \) of equal sizes \( S/2 \). Both states and actions are embedded into \( \mathbb{R} \):

- The states in cluster \( S_1 \) (resp. \( S_2 \)) are randomly generated in the interval \([\zeta, 0]\) (resp. \([1, 1 + \zeta]\)) for some \( \zeta \in (0, 1) \);
- In each state there are two possible actions: \( s = 0 \) (stands for stay) and \( m = 1 \) (stands for move).

The transition probabilities depend on the states only through their corresponding clusters, and are given by: for \( \epsilon \in (0, 0.5) \),

\[
p(y|x, a) = \begin{cases} \frac{2(1-\epsilon)}{S} & \text{if } (x, y, a) \in \Gamma_p, \\ \frac{2\epsilon}{S} & \text{otherwise} \end{cases}
\]

\[\text{(5)}\]
Algorithm 2 DEL(\( \gamma \))

**input** Model structure \( \Phi \)

Initialize \( N_t(x) \leftarrow 1[x = X_t], N_t(x,a) \leftarrow 0, s_t(x) \leftarrow 0, p_t(y \mid x, a) \leftarrow 1/|S|, r_t(x, a) \leftarrow 0 \) for each \( x, y \in S, a \in A \), and \( \phi_t \) accordingly.

for \( t = 1, \ldots, T \) do

For each \( x \in S \), let \( C_t(x) := \{ a \in A : N_t(x,a) \geq \log^2 N_t(x) \} \), \( \phi_t' := \phi_t(C_t), h_t'(x) := h_t^*(x), \zeta_t := \frac{1}{1 + \log \log t} \) and \( \gamma_t := (1 + \gamma)(1 + \log t) \)

if \( \forall a \in O(x; \phi_t'), N_t(x_t, a) < \log^2 N_t(x_t) + 1 \) then

Monotonize: \( A_t \leftarrow A_t^{\text{mt}} := \arg \min_{a \in O(x; \phi_t')} N_t(x_t, a) \).

else if \( \exists a \in A \) s.t. \( N_t(x_t, a) < \frac{\log N_t(x_t)}{1 + \log \log N_t(x_t)} \) then

Estimate: \( A_t \leftarrow A_t^{\text{st}} := \arg \min_{a \in A} N_t(x_t, a) \).

else if \( \left( \frac{N_t(x,a)}{N_t} : (x,a) \in S \times A \right) \in F_{\phi_t; C_t}(\zeta_t) \) then

Exploit: \( A_t \leftarrow A_t^{\text{exp}} := \arg \min_{a \in O(x; \phi_t')} N_t(x_t, a) \).

else

For each \( (x,a) \in S \times A \), let \( \delta_t(x,a) := \delta^*(x,a; \phi_t, C_t, \zeta_t) \).

if \( F_t := F_{\phi_t; C_t}(\zeta_t) \cap \{ \eta : \eta(x,a) = \infty \text{ if } \delta_t(x,a) = 0 \} = \emptyset \) then

Let \( \eta_t(x,a) = \infty \) if \( \delta_t(x,a) = 0 \) and \( \eta_t(x,a) = 0 \) otherwise.

else

Obtain a solution \( \eta_t \) of \( P(\delta_t, F_t) : \inf_{\eta \in F_t} \sum_{(x,a) \in S \times A} \eta(x,a) \delta_t(x,a) \)

end if

Explo: \( A_t \leftarrow A_t^{\text{exp}} := \arg \min_{a \in A; \eta_t(x,a) \leq \eta_t(x_t,a) \gamma_t} N_t(x_t, a) \).

end if

Select action \( A_t \), and observe the next state \( X_{t+1} \) and the instantaneous reward \( R_t \).

Update \( \phi_{t+1} \):

\[
N_{t+1}(x) \leftarrow N_t(x) + \mathbb{1}[x = X_{t+1}], \quad N_{t+1}(x,a) \leftarrow N_t(x,a) + \mathbb{1}[(x,a) = (X_t, A_t)],
\]

\[
p_{t+1}(y \mid x,a) \leftarrow \begin{cases} N_{t+1}(x,a)/p_t(y \mid x,a), y = X_{t+1} & \text{if } (x,a) = (X_t, A_t), \forall x,y \in S, a \in A \\ p_t(y \mid x,a) & \text{otherwise} \end{cases}
\]

\[
r_{t+1}(x,a) \leftarrow \begin{cases} N_{t+1}(x,a)/r_t(x,a) & \text{if } (x,a) = (X_t, A_t) \\ r_t(x,a) & \text{otherwise} \end{cases}
\]

end for

where

\[
\Gamma_p := \{ (x,y,a) : a = s, \exists i \in \{1,2\}, x,y \in S_i \} \cup \{ (x,y,a) : a = m, \exists i \in \{1,2\}, x \in S_i, y \notin S_i \}.
\]

In words, when the agent decides to move, she will end up in a state uniformly sampled from the other cluster with probability \( 1 - \epsilon \); when she decides to stay, she changes state within her cluster uniformly at random. We take \( \epsilon > 0 \) to ensure irreducibility. For numerical experiments we take \( \epsilon = 0.1 \) and \( \zeta = 0.1 \). The reward is obtained according to the following deterministic rule:

\[
r(x,a) = \begin{cases} 1 & \text{if } (x,a) : a = m \text{ and } x \in S_1, \\ 0 & \text{otherwise}. \end{cases} \quad (6)
\]

A reward is collected when the agent is in cluster \( S_1 \) and decides to move. The optimal stationary strategy consists in moving in each state.

Figure 1 presents the regret of the two versions of our DEL algorithm. Clearly, exploiting the structure brings a very significant performance improvement and the gain grows as the number of states increases, as predicted by our theoretical results. Observe that the regret after \( T = 50k \) steps under the version of DEL exploiting the Lipschitz structure barely grows with the number of states, see Figure 1(b), which was also expected.
C Proof of Theorem 1

Notations and preliminaries. Let $N_T(x) = \sum_{t=1}^{T} \mathbb{I}[X_t = x]$ and $N_T(x,a) = \sum_{t=1}^{T} \mathbb{I}[X_t = x, A_t = a]$ denote the number of times $x$ and $(x,a)$ have been visited up to step $T$. For any $\psi \in \Phi$ and any initial state $x_1$, we denote by $P_{\psi|x_1}$ and $E_{\psi|x_1}$ the probability measure and expectation under $\psi$ conditioned on $X_1 = x_1$. The regret up to step $T$ starting in state $x_1$ under $\pi$ and $\psi$ is denoted by $R_{T,\psi}^\pi(x_1)$. To emphasize the dependence on the MDP $\psi$, we may leverage the same arguments as those used in the proof of Proposition 1 of [Burnetas and Katehakis, 1997] to establish a connection between the regret of an algorithm $\pi \in \Pi$ under $\psi$ and $N_T(x,a)$. Specifically, for any $x_1$,

$$
R_{T,\psi}^\pi(x_1) = \sum_{x \in S} \sum_{a \not\in \mathcal{O}(x,\psi)} E_{\psi|x_1}^\pi[N_T(x,a)]\delta^\pi(x,a;\psi) + O(1), \text{ as } T \to \infty.
$$

In addition, due to the ergodicity of $\psi$, we can also prove as in Proposition 2 in [Burnetas and Katehakis, 1997] that there exists constants $C, \rho > 0$ such that for any $x \in S$, $\pi \in \Pi$,

$$
\mathbb{P}^\pi_{\psi|x_1}[N_T(x) \leq \rho T] \leq C \cdot \exp(-\rho T/2).
$$

Change-of-measure argument. Let $\pi$ be a uniformly good algorithm, and $x_1$ an initial state. For any bad MDP $\psi \in \Delta_S(\phi)$, the argument consists in (i) relating the log-likelihood of the observations under $\phi$ and $\psi$ to the expected number of times sub-optimal actions are selected under $\pi$, and (ii) using the fact that $\pi$ is uniformly good to derive a lower bound on the log-likelihood.

(i) Define by $L$ the log-likelihood of the observations up to step $T$ under $\phi$ and $\psi$. We can use the same techniques as in [Kaufmann et al., 2016, Garivier et al., Jun. 2018] (essentially an extension of Wald’s lemma):

$$
E_{\phi|x_1}^\pi[L] = \sum_{x,a} E_{\phi|x_1}^\pi[N_T(x,a)]KL(\phi|x_1,\pi|\psi|x_1,a).
$$

The so-called data processing inequality [Garivier et al., Jun. 2018] yields for all event $\mathcal{E}$ in $\mathcal{H}_T^\pi$:

$$
E_{\phi|x_1}^\pi[L] \geq kl(P_{\phi|x_1}[\mathcal{E}], P_{\psi|x_1}[\mathcal{E}]),
$$

where for $u, v \in [0, 1]$, $kl(u,v) := u \log \frac{u}{v} + (1-u) \log \frac{1-u}{1-v}$.

Combine with (9), this leads to:

$$
\sum_{x,a \not\in \mathcal{O}(x,\phi)} E_{\phi|x_1}^\pi[N_T(x,a)]KL(\phi|x_1,\pi|\psi|x_1,a) \geq KL(P_{\phi|x_1}[\mathcal{E}], P_{\psi|x_1}[\mathcal{E}]).
$$

Note that in the above sum, we removed $a \in \mathcal{O}(x,\phi)$ since $KL(\phi|x_1,\pi|\psi|x_1,a) = 0$ if $a \in \mathcal{O}(x,\phi)$.
We first prove the decoupling lemma.\footnote{We prove the lemma by contradiction. Assume that $\Pi^*(\phi) \cap \Pi^*(\psi_1) \neq \emptyset$ and $\Pi^*(\phi) \cap \Pi^*(\psi_2) \neq \emptyset$. Let $\Pi^*(\phi, \psi_1, \psi_2) := \Pi^*(\phi) \cap \Pi^*(\psi_1) \cap \Pi^*(\psi_2)$. It is sufficient to show that $\Pi^*(\phi, \psi_1, \psi_2) \neq \emptyset$.

Indeed, this implies $\Pi^*(\phi) \cap \Pi^*(\psi_1) \neq \emptyset$. Note that any policy $f \in \Pi^*(\phi)$ has the same gain and bias function under $\phi, \psi_0, \psi_1, \psi_2$ since the modifications of $\phi$ are made on suboptimal (state, action) pairs. Specifically,

$$g^f_{\psi_0} = g^f_{\psi_1} = g^f_{\psi_2} = g^f_\phi \quad \text{and} \quad h^f_{\psi_0} = h^f_{\psi_1} = h^f_{\psi_2} = h^f_\phi.$$}

where for the first and second terms in the last inequality, we used (8) and Markov inequality, which converges to

$$\rho$$

Indeed, otherwise $\pi$ would not be uniformly good. Now define the event $\mathcal{E}$ as:

$$\mathcal{E} := \left[ N_T(x) \geq \rho T, \sum_{a \notin \mathcal{O}(x, \phi)} N_T(x, a) \leq \sqrt{T} \right],$$

where the constant $\rho$ is chosen so that (8) holds under $\phi$ and $\psi$. Using a union bound, we have

$$1 - \mathbb{P}^\pi_{\phi|x_1}[\mathcal{E}] \leq \mathbb{P}^\pi_{\phi|x_1}[N_T(x) \leq \rho T] + \mathbb{P}^\pi_{\phi|x_1}\left[ \sum_{a \notin \mathcal{O}(x, \phi)} N_T(x, a) \geq \sqrt{T} \right] \leq C \cdot \exp(-\rho T/2) + \frac{\mathbb{E}^\pi_{\phi|x_1}\left[ \sum_{a \notin \mathcal{O}(x, \phi)} N_T(x, a) \right]}{\sqrt{T}} \ (11)$$

where for the first and second terms in the last inequality, we used (8) and Markov inequality, respectively. Since $\pi$ is uniformly good, $\mathbb{E}^\pi_{\phi|x_1}[\sum_{a \notin \mathcal{O}(x, \phi)} N_T(x, a)] = o(T^\alpha)$ for all $\alpha > 0$, the last term of (11) converges to 0, i.e., $\mathbb{P}^\pi_{\phi|x_1}[\mathcal{E}] \to 1$ as $T \to \infty$. Using Markov inequality, it follows that

$$\mathbb{E}^\pi_{\phi|x_1}[\mathcal{E}] \leq \mathbb{E}^\pi_{\psi|x_1}\left[ N_T(x) - \sum_{a \notin \mathcal{O}(x, \phi)} N_T(x, a) \geq \rho T - \sqrt{T} \right] \leq \frac{\mathbb{E}^\pi_{\phi|x_1}[\sum_{a \notin \mathcal{O}(x, \phi)} N_T(x, a)]}{\rho T - \sqrt{T}}$$

which converges to 0 because of our choice of $x$. Combining $\mathbb{P}^\pi_{\phi|x_1}[\mathcal{E}] \to 1$ and $\mathbb{P}^\pi_{\psi|x_1}[\mathcal{E}] \to 0$,

$$\frac{\text{kl}(\mathbb{P}^\pi_{\phi|x_1}[\mathcal{E}], \mathbb{P}^\pi_{\psi|x_1}[\mathcal{E}])}{\log T} \sim \frac{1}{\log T} \log \left( \frac{\mathbb{P}^\pi_{\phi|x_1}[\mathcal{E}]}{\mathbb{P}^\pi_{\psi|x_1}[\mathcal{E}]} \right) \geq \frac{1}{\log T} \log \left( \frac{\rho T - \sqrt{T}}{\mathbb{E}^\pi_{\phi|x_1}[\sum_{a \notin \mathcal{O}(x, \phi)} N_T(x, a)]} \right)$$

which converges to 1 as $T$ grows large due to our choice of $x$. Plugging this result in (10), we get:

$$\lim \inf_{T \to \infty} \frac{1}{\log T} \sum_{x, \phi \notin \mathcal{O}(x, \phi)} \mathbb{E}^\pi_{\phi|x_1}[N_T(x, a)] \text{KL}_{\phi|\psi}(x, a) \geq 1. \ (12)$$

Combining the above constraints valid for any $\psi \in \Delta_\phi(\psi)$ and (7) concludes the proof of the theorem.

D Proof of Theorem 2

We first prove the decoupling lemma.

\textbf{Proof of Lemma 1.} We prove the lemma by contradiction. Assume that $\Pi^*(\phi) \cap \Pi^*(\psi_1) \neq \emptyset$ and $\Pi^*(\phi) \cap \Pi^*(\psi_2) \neq \emptyset$. Let $\Pi^*(\phi, \psi_1, \psi_2) := \Pi^*(\phi) \cap \Pi^*(\psi_1) \cap \Pi^*(\psi_2)$. It is sufficient to show

$$\begin{array}{c}
(i) \quad \Pi^*(\phi, \psi_1, \psi_2) \neq \emptyset, \quad \text{and} \quad (ii) \quad \Pi^*(\phi, \psi_1, \psi_2) \subseteq \Pi^*(\psi_0) \ .
\end{array} \ (13)$$

Indeed, this implies $\Pi^*(\phi) \cap \Pi^*(\psi_0) \neq \emptyset$. Note that any policy $f \in \Pi^*(\phi)$ has the same gain and bias function under $\phi, \psi_0, \psi_1, \psi_2$ since the modifications of $\phi$ are made on suboptimal (state, action) pairs. Specifically,

$$g^f_{\psi_0} = g^f_{\psi_1} = g^f_{\psi_2} = g^f_\phi \quad \text{and} \quad h^f_{\psi_0} = h^f_{\psi_1} = h^f_{\psi_2} = h^f_\phi \ . \ (14)$$
To prove (i), the first part of (13), consider a policy \( f' \in \Pi^*(\phi) \cap \Pi^*(\psi_1) \) and a policy \( f'' \in \Pi^*(\phi) \cap \Pi^*(\psi_2) \). Then, from the optimality of \( f' \) under \( \psi_1 \), it follows that for each \( x \in S \),

\[
g_{\psi_1}^{f'} = g_{\psi_1}^{f'} \geq g_{\psi_2}^{f''} = g_{\psi_2}^* \tag{15}
\]

where for the second equality, we use (14). Similarly, we have for each \( x \in S \),

\[
g_{\psi_2}^{f''} \geq g_{\psi_2}^{f'} = g_{\psi_1}^* \tag{16}
\]

Hence \( g_{\psi_1}^* = g_{\psi_2}^* \) and \( \Pi^*(\phi, \psi_1, \psi_2) = \Pi^*(\phi) \cap \Pi^*(\psi_1) = \Pi^*(\phi) \cap \Pi^*(\psi_2) \neq \emptyset \).

We now prove (ii), the second part of (13). Let \( f \in \Pi^*(\phi, \psi_1, \psi_2) \). It is sufficient to show \( g_{\psi_0}^f \) and \( h_{\psi_0}^f(x) \) verify the Bellman optimality equation for model \( \psi_0 \). Using (14) and the optimality of \( f \) under \( \psi_1 \), for all \( a \in A \), if \( (x, a) \notin U_2 \),

\[
r_{\psi_0}(x, f(x)) + \sum_{y \in S} p_{\psi_0}(y|x, f(x)) h_{\psi_0}^f(y) \leq r_{\psi_1}(x, f(x)) + \sum_{y \in S} p_{\psi_1}(y|x, f(x)) h_{\psi_1}^f(y)
\]

for (a) and (c), we used (14) and the fact that the kernels of \( \psi_0 \) and \( \psi_1 \) are the same at every \( (x, a) \notin U_2 \), and for (b), we used the fact that \( g_{\psi_1}^f \) and \( h_{\psi_1}^f(x) \) verify the Bellman optimality equation for \( \psi_1 \). Similarly, using the optimality of \( f \) under \( \psi_2 \), it follows that for \( (x, a) \in U_2 \),

\[
r_{\psi_0}(x, f(x)) + \sum_{y \in S} p_{\psi_0}(y|x, f(x)) h_{\psi_0}^f(y) = r_{\psi_2}(x, f(x)) + \sum_{y \in S} p_{\psi_2}(y|x, f(x)) h_{\psi_2}^f(y)
\]

\[
= r_{\psi_0}(x, a) + \sum_{y \in S} p_{\psi_0}(y|x, a) h_{\psi_0}^f(y) \tag{17}
\]

Combining (16) and (17), for all \( (x, a) \in S \times A \),

\[
r_{\psi_0}(x, f(x)) + \sum_{y \in S} p_{\psi_0}(y|x, f(x)) h_{\psi_0}^f(y) \geq r_{\psi_0}(x, a) + \sum_{y \in S} p_{\psi_0}(y|x, a) h_{\psi_0}^f(y),
\]

which implies that \( g_{\psi_0}^f \) and \( h_{\psi_0}^f(x) \) verify the Bellman optimality equation under model \( \psi_0 \), i.e., \( f \in \Pi^*(\psi_0) \).

**Proof of Theorem 2.** Recall that any policy \( f \in \Pi^*(\phi) \) has the same gain and bias function in \( \psi \) and \( \phi \) since the kernels of \( \phi \) and \( \psi \) are identical at every \( (x, a) \) such that \( a \in O(x; \phi) \). More formally, for any \( f \in \Pi^*(\phi) \),

\[
B_{\phi}^f = B_{\psi}^f, \quad g^*_\phi = g^*_\psi = g^f_{\phi} \quad \text{and} \quad h^\phi_\phi(\cdot) = h^\phi_\psi(\cdot) = h^f_\phi(\cdot).
\]

Next we show that for all \( \psi \in \Delta_\phi(\phi) \),

\[
\Pi^*(\phi) \cap \Pi^*(\psi) = \emptyset \iff \exists (x, a) \text{ such that } (B_{\phi}^a h^a_\phi)(x) > g^f_\phi + h^f_\phi(x). \tag{18}
\]

We prove (18) by contradiction. Consider a policy \( f \in \Pi^*(\phi) \). Suppose that for all \( (x, a) \), \( (B_{\phi}^a h^a_\phi)(x) \leq g^f_\phi + h^f_\phi(x) \). Then, for all \( (x, a) \),

\[
(B_{\phi}^a h^a_\phi)(x) = (B_{\phi}^a h^a_\phi)(x) = g^f_\phi + h^f_\phi(x) \leq \max_{a \in A} (B_{\phi}^a h^a_\phi)(x)
\]

which implies that \( g^f_\phi \) and \( h^f_\phi(x) \) verify the Bellman optimality equation under \( \psi \). Hence, \( f \in \Pi^*(\psi) \) which contradicts to \( \Pi^*(\phi) \cap \Pi^*(\psi) = \emptyset \).
Finally Theorem 2 is obtained by combining the decoupling lemma and (18). Indeed, due to the decoupling lemma, we may restrict $\Delta_\phi(\phi)$ to MDPs obtained from $\phi$ by only changing the kernels in a single state-action pair. \hfill \Box

**Simplification for null structure.** We conclude this section by proving that $\mathcal{F}_{\text{un}}(\phi) \subset \mathcal{F}_{\phi}(\phi)$. Let $\eta \in \mathcal{F}_{\text{un}}(\phi)$, recalling that

$$\mathcal{F}_{\text{un}}(\phi) = \left\{ \eta \in \mathcal{F}_0(\phi) : \eta(x, a) \left( \frac{\delta^*(x, a; \phi)}{H + 1} \right)^2 \geq 2, \forall (x, a) \text{ s.t. } a \notin \mathcal{O}(x; \phi) \right\}.$$ 

We show that $\eta \in \mathcal{F}_{\phi}(\phi)$. To this aim, we need to show that $\forall (x, a) \text{ s.t. } a \notin \mathcal{O}(x; \phi)$,

$$\eta(x, a) \text{KL}_{\phi|\psi}(x, a) \geq 1, \forall \psi \in \Delta_\phi(x, a; \phi).$$

Let $(x, a)$ be such that $a \notin \mathcal{O}(x; \phi)$, which means $a \notin \mathcal{O}(x, h^*_\phi; \phi)$, and $\psi \in \Delta_\phi(x, a; \phi)$. We have, by definition, $(B^*_\phi h^*_\phi)(x) \geq g^*_\phi + h^*_\phi(x)$. Then,

$$\delta^*(x, a; \phi) = (B^*_\phi h^*_\phi)(x) - (B^*_\phi h^*_\phi)(x)$$

$$= r_\psi(x, a) + \sum_{y \in S} (p_\psi(y | x, a) - p_\phi(y | x, a)) h^*_\phi(y)$$

$$\leq \|q_\psi(. | x, a) - q_\phi(. | x, a)\|_1 + H\|p_\psi(. | x, a) - p_\phi(. | x, a)\|_1$$

$$\leq (H + 1)\|\psi(x, a) - \phi(x, a)\|_1$$

where we define

$$\|\psi(x, a) - \phi(x, a)\|_1 := \|q_\psi(. | x, a) - q_\phi(. | x, a)\|_1 + \|p_\psi(. | x, a) - p_\phi(. | x, a)\|_1.$$ 

Finally, Pinsker’s inequality yields:

$$2\text{KL}_{\phi|\psi}(x, a) \geq \|\psi(x, a) - \phi(x, a)\|_1^2 \geq \left( \frac{\delta^*(x, a; \phi)}{H + 1} \right)^2.$$ 

This implies that:

$$\eta(x, a) \text{KL}_{\phi|\psi}(x, a) \geq \frac{\eta(x, a)}{2} \left( \frac{\delta^*(x, a; \phi)}{H + 1} \right)^2 \geq 1$$

where the last inequality is due to the fact that $\eta(x, a) \in \mathcal{F}_{\text{un}}(\phi)$.

**E Proof of Theorem 3**

We prove that $\mathcal{F}_{\text{lp}}(\phi) \subset \mathcal{F}_{\phi}(\phi)$. Let $\eta \in \mathcal{F}_{\text{lp}}(\phi)$. We show that $\eta \in \mathcal{F}_{\phi}(\phi)$. Let $\psi \in \Delta_\phi(\phi)$, then, from (18), there exist $(x', a')$ such that $(B^*_\phi h^*_\phi)(x') \geq g^*_\phi + h^*_\phi(x')$. Then, using the same arguments as at the end of the previous section, we obtain:

$$\|\psi(x', a') - \phi(x', a')\|_1 \geq \frac{\delta^*(x', a'; \phi)}{H + 1}.$$ 

Now for all $(x, a) \in S \times A$,

$$\|\phi(x', a') - \psi(x', a')\|_1 \leq \|\phi(x', a') - \phi(x, a)\|_1 + \|\phi(x, a) - \psi(x, a)\|_1 + \|\psi(x, a) - \psi(x', a')\|_1$$

$$\leq \|\phi(x, a) - \psi(x, a)\|_1 + 2Ld(x, x') + 2L'd(a, a') \alpha.$$ 

(20)
where the first inequality follows from the triangular inequality and the second follows from Lipschitz continuity. This further implies that

\[
\|\phi(x, a) - \psi(x, a)\|_1 \geq \left[ \frac{\delta^*(x', a'; \phi)}{H + 1} - 2 \left( Ld(x, x')^\alpha + L'd(a, a')^{a'} \right) \right]_+.
\]

Hence, using Pinsker’s inequality,

\[
2\text{KL}_{\phi|\psi}(x, a) \geq \left[ \frac{\delta^*(x', a'; \phi)}{H + 1} - 2 \left( Ld(x, x')^\alpha + L'd(a, a')^{a'} \right) \right]_+^2,
\]

which implies that:

\[
\eta(x, a)\text{KL}_{\phi|\psi}(x, a) \geq \frac{\eta(x, a)}{2} \left[ \frac{\delta^*(x', a'; \phi)}{H + 1} - 2 \left( Ld(x, x')^\alpha + L'd(a, a')^{a'} \right) \right]_+^2 \geq 1. \tag{22}
\]

The last inequality follows from \( \eta \in \mathcal{F}_{\text{lip}} \). Thus \( \mathcal{F}_{\text{lip}}(\phi) \subset \mathcal{F}_{\phi}(\phi) \).

Next we derive an upper bound for \( K_\phi(\phi) \). To this aim, we construct a vector \( \eta \geq 0 \) verifying (2b) for our given structure \( \Phi \). Then, we get an upper bound of \( K_\phi(\phi) \) by evaluating the objective function of \( P(\phi, \mathcal{F}_{\phi}(\phi)) \) at \( \eta \).

To construct \( \eta \), we build a sequence \( (X_i)_{i=1,2,...} \) of sets of (state, action) pairs, as well as a sequence \( (x_i)_{i=1,2,...} \) (state, action) pairs, such that for any \( i \geq 1 \), \( X_{i+1} \subset X_i \), and \( (x_i, a_i) \in \arg\max_{(x,a)\in X_i} \delta^*(x, a; \phi) \) (ties are broken arbitrarily).

We start with \( X_1 = \{(x, a): x \in S, a \notin O(x; \phi), i.e., \delta^*(x, a; \phi) > 0\} \). Recursively, for each \( i = 1, 2, ..., \) let

\[
B_i = \left\{ (x, a) \in X_i : Ld(x, x_i)^\alpha + L'd(a, a_i)^{a'} \leq \frac{\delta_{\text{min}}}{4(H + 1)} \right\}, \quad \text{and}
\]

\[
X_{i+1} = X_i \setminus B_i. \tag{23}
\]

Let \( I \) be the first index such that \( X_{I+1} = \emptyset \). Construct \( \eta \) as

\[
\eta(x, a) = \begin{cases} 8 \left( \frac{\delta_{\text{min}}}{H + 1} \right)^{-2} & \text{if } \exists i \in [1, I] \text{ such that } (x, a) = (x_i, a_i), \\ 0 & \text{otherwise}. \end{cases} \tag{24}
\]

Observe that \( \eta \) is strictly positive at only \( I \) pairs, and hence

\[
\sum_{(x,a)\in\delta\times A} \delta^*(x, a; \phi)\eta(x,a) \leq 8(H + 1) \left( \frac{H + 1}{\delta_{\text{min}}} \right)^2 I
\]

since \( \delta^*(x, a; \phi) \leq H + 1 \) for all \((x,a)\). Next, we bound \( I \) using the covering and packing numbers of the hypercubes \([0, D]^d\) and \([0, D')^d\).

**Lemma 2.** The generation of \( X_i \)'s in (23) must stop after \( (S_{\text{lip}}A_{\text{lip}} + 1) \) iterations, i.e., \( I \leq S_{\text{lip}}A_{\text{lip}} \).

The proof of this lemma is postponed at the end of this section. To complete the proof of the theorem, it remains to show that \( \eta \) verifies all the constraints (2b) for the Lipschitz structure \( \Phi \).

Remember that \( \mathcal{F}_{\text{lip}}(\phi) \subset \mathcal{F}_{\phi}(\phi) \). Fix \( \psi \in \mathcal{A}_\phi(\phi) \). There exists \((x', a')\) such that \( a' \notin O(x'; h^*_\phi, \phi) \) and \( a' \in O(x'; h^*_\phi, \psi) \), and such that (19) holds. Let \( i \in \{1, \ldots, I\} \) denote an index such that
where for \((x', a') \in B_i\). Note that such an index \(i\) exists since \((x', a') \in X_1\) and \(X_{I+1} = \emptyset\). Thus, we have:

\[
\sum_{(x,a) \in S \times A} \eta(x,a) \text{KL}_{\phi|\psi}(x,a) \geq \sum_{(x,a) \in S \times A} \frac{\eta(x,a)}{2} \left[ \frac{\delta^*(x', a'; \phi)}{H+1} - 2 \left( L_d(x,x')^\alpha + L'd(a,a')^{\alpha'} \right) \right]^2 + \]

\[
\geq \frac{\eta(x_i,a_i)}{2} \left[ \frac{\delta^*(x', a'; \phi)}{H+1} - 2 \left( L_d(x_i,x')^\alpha + L'd(a_i,a')^{\alpha'} \right) \right]^2 + \]

\[
\geq \frac{\eta(x_i,a_i)}{2} \left[ \frac{\delta^*(x', a'; \phi)}{H+1} - \frac{1}{2} \frac{\delta_{\min}}{H+1} \right]^2 \]

\[
\geq 4 \left( \frac{\delta_{\min}}{H+1} \right)^2 \left( \frac{1}{2} \frac{\delta_{\min}}{H+1} \right)^2 = 1
\]

where the third inequality follows from the fact that \((x', a') \in B_i\). Hence we have verified that \(\eta\) satisfies the feasibility constraint for \(\psi\). Since this observation holds for all \(\psi \in \Delta_\Phi(\phi)\), this completes the proof of Theorem 3.

**Proof of Lemma 2.** A \(\delta\)-packing of a set \(D\) with respect to a metric \(\rho\) is a set \(\{x_1, \ldots, x_n\} \subset D\) such that \(\rho(x_i - x_j) > \delta\) for all different \(i, j \in \{1, \ldots, n\}\). The \(\delta\)-packing number \(I_p(\delta, D, \rho)\) is the cardinality of the largest \(\delta\)-packing. The construction of \(\mathcal{X}_i\) ensures that for different \(i, j \in \{1, \ldots, I\},\)

\[
\ell_{\text{lip}}((x_i, a_i), (x_j, a_j)) > \delta := \frac{\delta_{\min}}{4(H+1)} ,
\]

where for \((x,a), (x', a') \in \mathbb{R}^d \times \mathbb{R}^d,\)

\[
\ell_{\text{lip}}((x,a), (x', a')) := L_d(x,x')^\alpha + L'd(a,a')^{\alpha'} .
\]

Then, we have:

\[
I \leq I_p(\delta, S \times A, \ell_{\text{lip}}) .
\]

To obtain an upper bound of the packing number, we further define the covering number. A \(\delta\)-cover of a set \(D\) with respect to a metric \(\rho\) is a set \(\{x_1, \ldots, x_j\} \subset D\) such that for each \(x \in D\), there exists some \(i \in \{1, \ldots, I\}\) such that \(\rho(x, x_i) \leq \delta\). The \(\delta\)-covering number \(I_c(\delta, D, \rho)\) is the smallest cardinality of \(\delta\)-cover. Then, we have the following relationship between the packing and covering numbers.

**Lemma 3.** For all \(\delta > 0\), \(D, D'\) such that \(D \subset D'\),

\[
I_p(2\delta, D, \rho) \leq I_c(\delta, D, \rho) \leq I_c(\delta, D', \rho) .
\]

The proof of this lemma is provided at the end of the section for completeness. Define the metrics \(\ell_{\max}^{(1)}, \ell_{\max}^{(2)}, \ell_{\max}\) for \(\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d\), respectively, as follows:

\[
\ell_{\max}^{(1)}(x, x') := \left( \frac{1}{\sqrt{d}} \left( \frac{\delta}{2L} \right)^{1/\alpha'} \right)^{-1} \| x - x' \|_{\infty} ,
\]

\[
\ell_{\max}^{(2)}(a, a') := \left( \frac{1}{\sqrt{d}} \left( \frac{\delta}{2L} \right)^{1/\alpha'} \right)^{-1} \| a - a' \|_{\infty} ,
\]

\[
\ell_{\max}((x,a), (x', a')) := \max \left\{ \ell_{\max}^{(1)}(x, x'), \ell_{\max}^{(2)}(a, a') \right\} ,
\]

where \(\| \cdot \|_{\infty}\) is infinite norm. Then, it follows that for any \((x, a), (x', a') \in \mathbb{R}^d \times \mathbb{R}^d,\)

\[
\ell_{\max}((x,a), (x', a')) \leq 1 \implies \ell_{\text{lip}}((x,a), (x', a')) \leq \delta .
\]

Hence, we have

\[
I \leq I_c(\delta, S \times A, \ell_{\text{lip}}) \leq I_c(1, S \times A, \ell_{\max}) \leq I_c(1, S, \ell_{\max}^{(1)}) I_c(1, A, \ell_{\max}^{(2)})
\]

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since for any 1-cover $S'$ of $S$ with metric $\ell_{\max}^{(1)}$ and any 1-cover $A'$ of $A$ with metric $\ell_{\max}^{(2)}$, their Cartesian product $S' \times A' = \{(x, a) : x \in S', a \in A'\}$ is 1-cover of $S \times A$ with metric $\ell_{\max}$. We now study $I_c(1, S, \ell_{\max}^{(1)})$ and $I_c(1, A, \ell_{\max}^{(2)})$. Recalling $S \subset [0, D]^d$ and using Lemma 3, it follows directly that

$$I_c(1, S, \ell_{\max}^{(1)}) \leq I_c(1, [0, D]^d, \ell_{\max}^{(1)}) = I_c\left(\frac{1}{\sqrt{d}}\left(\frac{\delta}{2L}\right)^{1/\alpha}, [0, D]^d, \|\cdot\|_{\infty}\right) \leq \left(\frac{D}{\frac{1}{\sqrt{d}}\left(\frac{\delta}{2L}\right)^{1/\alpha} + 1}\right)^d,$$

which implies

$$I_c(1, S, \ell_{\max}^{(1)}) \leq \min\left\{|S|, \left(\frac{D}{\frac{1}{\sqrt{d}}\left(\frac{\delta}{2L}\right)^{1/\alpha} + 1}\right)^d\right\} = S_{\text{lip}},$$

where we used the fact that $I_c(1, S, \ell_{\max}^{(1)}) \leq |S|$. Similarly, we have

$$I_c(1, A, \ell_{\max}^{(2)}) \leq \min\left\{|A|, \left(\frac{D'}{\frac{1}{\sqrt{d'}}\left(\frac{\delta}{2L'}\right)^{1/\alpha'} + 1}\right)^{d'}\right\} = A_{\text{lip}}.$$

This completes the proof of Lemma 2. \hfill \Box

**Proof of Lemma 3.** Consider a $\delta$-cover $\mathcal{X}$ and a $2\delta$-packing $\mathcal{Y}$ of set $\mathcal{D}$ with respect to metric $\rho$. Then, there is no $x \in \mathcal{X}$ such that $y, y' \in B(\delta, x) = \{x' \in D : \rho(x, x') \leq \delta\}$ for two different $y, y' \in \mathcal{Y}$. Otherwise, we would have $\rho(x, y) \leq \delta$ and $\rho(x, y') \leq \delta$ which implies $\rho(y, y') \leq 2\delta$ from the triangle inequality, and contradicts the fact that $y, y'$ are two different elements of $2\delta$-cover, i.e., $\rho(y, y') > 2\delta$. Thus, the cardinality of $\mathcal{Y}$ cannot be larger than that of $\mathcal{X}$. Due to the arbitrary choice of $\delta$-cover $\mathcal{X}$ and a $2\delta$-packing $\mathcal{Y}$, we conclude that $I_p(2\delta, D, \rho) \leq I_c(\delta, D, \rho)$.

The second inequality in the lemma is straightforward. \hfill \Box

**F Proof of Theorem 4**

We analyze the regret under $\pi = \text{DEL}$ algorithm when implemented with the original feasible set $\mathcal{F}_c(\phi; c, \zeta)$. Extending the analysis to the case where DEL runs on the simplified feasible sets $\mathcal{F}_{\text{un}}(\phi; c, \zeta)$ and $\mathcal{F}_{\text{lip}}(\phi; c, \zeta)$ can be easily done.

For $T \geq 1, \varepsilon > 0, x \in S$ and $a \in A$, define the following random variables:

$$W^{(1)}_T(x, a; \varepsilon) := \sum_{t=1}^T \mathbb{1}[\langle X_t, A_t \rangle = (x, a), E_t(\varepsilon), (\mathcal{B}_\phi^a h_t^\varepsilon)(x) \leq (\mathcal{B}_\phi^a h_t^\varepsilon)(x) - 2\varepsilon]$$

$$W^{(2)}_T(x, a; \varepsilon) := \sum_{t=1}^T \mathbb{1}[\langle X_t, A_t \rangle = (x, a), E_t(\varepsilon), (\mathcal{B}_\phi^a h_t^\varepsilon)(x) > (\mathcal{B}_\phi^a h_t^\varepsilon)(x) - 2\varepsilon]$$

$$W^{(3)}_T(\varepsilon) := \sum_{t=1}^T \mathbb{1}[-E_t(\varepsilon)]$$

where we use the standard notation $\neg \mathcal{U}$ to represent the event that $\mathcal{U}$ does not occur, where we recall that $h_t^\varepsilon := h_{\phi_t}$ is the bias function of the restricted estimated model $\phi_t^\varepsilon = \phi_t(c_t)$ at time $t$, and where the event $E_t(\varepsilon)$ is defined as:

$$E_t(\varepsilon) := \{\Pi^*(\phi_t^\varepsilon) \subseteq \Pi^*(\phi) \text{ and } |r_t(x, a) - r_\phi(x, a)| + |h_t^\varepsilon(x) - h_\phi^\varepsilon(x)| \leq \varepsilon \forall x \in S, \forall a \in \mathcal{O}(x; \phi_t^\varepsilon)\}.$$
From the above definitions, we have:

$$R_T^n(x_1) \leq \sum_{(x,a): \not \in \mathcal{O}(x; \phi)} \delta^*(x, a; \phi) \mathbb{E}_{\phi|x_1}^{n} \left[ W_T^{(1)}(x, a; \varepsilon) \right] \quad (25a)$$

$$+ \sum_{(x,a): \not \in \mathcal{O}(x; \phi)} S \mathbb{E}_{\phi|x_1}^{n} \left[ W_T^{(2)}(x, a; \varepsilon) \right] \quad (25b)$$

$$+ S \mathbb{E}_{\phi|x_1}^{n} \left[ W_T^{(3)}(\varepsilon) \right]. \quad (25c)$$

The multiplicative factor $S$ in the last two terms arises from the fact that $\max_{(x,a)} \delta^*(x, a; \phi) \leq S$ when the magnitude of the instantaneous reward is bounded by 1. Next we provide upper bounds of each of the three terms in (25).

### A. Upper bounds for (25a) and (25b).

To study the first two terms in (25), we first make the following observation. When $Z$ each $(t)$ continuous at $\phi$, then the solution $\eta^*(\phi)$ is unique for each $(x,a)$ such that $a \not \in \mathcal{O}(x; \phi)$; and (iii) continuous at $\phi$. Then, for any $(x,a) \in \mathcal{S} \times \mathcal{A}$ such that $a \not \in \mathcal{O}(x; \phi)$,

$$\sum_{t=1}^{T} \mathbb{E}_{\phi|x_1}^{n} \left[ W_T^{(1)}(x, a; \varepsilon) \right] \leq o(\log T) + \sum_{t=1}^{T} \mathbb{E}_{\phi|x_1}^{n} \left[ Z_t^{(1)}(x, a; \varepsilon) \right]$$

where the events $Z_t^{(1)}(x, a; \varepsilon)$ and $Z_t^{(2)}(x, a; \varepsilon)$ are defined as:

$$Z_t^{(1)}(x, a; \varepsilon) := \{ (X_t, A_t) = (x, a), \mathcal{E}_t(\varepsilon), \mathcal{E}_t^{\text{est}}, (B_{\phi t}^n h_t^n)(x) \leq (B_{\phi t}^n h_t^n)(x) - 2\varepsilon \}$$

$$Z_t^{(2)}(x, a; \varepsilon) := \{ (X_t, A_t) = (x, a), \mathcal{E}_t(\varepsilon), \mathcal{E}_t^{\text{est}}, (B_{\phi t}^n h_t^n)(x) > (B_{\phi t}^n h_t^n)(x) - 2\varepsilon \}.$$

The following lemma is proved in Section F.1, and deals events $Z_t^{(1)}(x, a; \varepsilon)$.

**Lemma 4.** For structure $\Phi$ with Bernoulli rewards and an ergodic MDP $\phi \in \Phi$, consider $\pi = \text{DEL}(\gamma)$ for $\gamma > 0$. Suppose that (i) $\phi$ is in the interior of $\Phi$; (ii) the solution $\eta^*(\phi)$ is unique for each $(x,a)$ such that $a \not \in \mathcal{O}(x; \phi)$; and (iii) continuous at $\phi$. Then, for any $(x,a) \in \mathcal{S} \times \mathcal{A}$ such that $a \not \in \mathcal{O}(x; \phi)$,

$$\lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\sum_{t=1}^{T} \mathbb{P}_{\phi|x_1} \left[ Z_t^{(1)}(x, a; \varepsilon) \right]}{\log T} \leq (1 + \gamma)\eta^*(x, a; \phi).$$

The following lemma is proved in Section F.2, and deals events $Z_t^{(2)}(x, a; \varepsilon)$. Its proof relies on the following observation. When $Z_t^{(2)}(x, a; \varepsilon)$ occurs for sufficiently small $\varepsilon$, the facts that $\mathcal{E}_t(\varepsilon)$ holds and that $(B_{\phi t}^n h_t^n)(x) < (B_{\phi t}^n h_t^n)(x) - 2\varepsilon$ imply that $\phi_t(x,a)$ does not estimate $\phi(x,a)$ accurately. The lemma then follows from concentration arguments.
Lemma 5. For structure $\Phi$ with Bernoulli rewards and an ergodic MDP $\phi \in \Phi$, consider $\pi = \text{DEL}(\gamma)$ for $\gamma > 0$. Then, there exists $\varepsilon_2 > 0$ such that for any $(x, a) \in S \times A$ such that $a \notin O(x; \phi)$ and $\varepsilon \in (0, \varepsilon_2)$,

$$\sum_{t=1}^{T} \mathbb{P}_{\phi}^{\pi}[Z_{t}^{(2)}(x, a; \varepsilon)] = o(\log T) \quad \text{as } T \to \infty.$$  

B. Upper bound for (25c). The last term in (25) is concerned with the regret generated when $E_{t}(\varepsilon)$ does not occur. It is upper bounded in the following lemma proved in Section F.3. To establish this result, we use a similar argument as that in Proposition 5 of [Burnetas and Katehakis, 1997]. Intuitively, we show that by the design of the algorithm, the restricted bias function $h_{t}'$ is monotonically improved so that it eventually converges to the optimal bias function $h_{t}$ with high probability. In this analysis, we provide a more sophisticated concentration inequality than the one in [Burnetas and Katehakis, 1997]. This concentration inequality is particularly important to bound the regret generated in the exploitation phase.

Lemma 6. For structure $\Phi$ with Bernoulli rewards and an ergodic MDP $\phi \in \Phi$, consider $\pi = \text{DEL}(\gamma)$ for $\gamma > 0$. Suppose $\phi$ is in the interior of $\Phi$, i.e., there exists a constant $\zeta > 0$ such that for any $x \in (0, \zeta)$, $\psi \in \Phi$ if $\| \phi - \psi \| \leq \zeta$. Then, there exists $\varepsilon_3 > 0$ such that for any $\varepsilon \in (0, \varepsilon_3)$,

$$\mathbb{P}_{\phi}^{\pi}[-E_{T}(\varepsilon)] = o(1/T) \quad \text{as } T \to \infty.$$  

We provide the proof of Lemma 6 in Section F.3. Now, we are ready to complete the proof of Theorem 4. Combining Lemma 4, (27) and (26), we get

$$\sum_{x \in S} \sum_{a \notin O(x; \phi)} \delta^{*}(x, a; \phi) \left( \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\mathbb{E}_{\phi}^{\pi}[W_{T}^{(1)}(x, a; \varepsilon)]}{\log T} \right) \leq (1 + \gamma) \sum_{x \in S} \sum_{a \in A} \delta^{*}(x, a; \phi)\eta^{*}(x, a; \phi)

= (1 + \gamma)K_{x}(\phi).$$

Similarly, combining Lemma 5 with (27) and (26), it follows that for sufficiently small $\varepsilon \in (0, \min\{\varepsilon_2, \varepsilon_3\})$,

$$\limsup_{T \to \infty} \frac{\mathbb{E}_{\phi}^{\pi}[\sum_{x \in S} \sum_{a \notin O(x; \phi)} SW_{T}^{(2)}(x, a; \varepsilon)]}{\log T} = 0.$$

From Lemma 6, we have that for sufficiently small $\varepsilon \in (0, \min\{\varepsilon_2, \varepsilon_3\})$,

$$\limsup_{T \to \infty} \frac{\mathbb{E}_{\phi}^{\pi}[W_{T}^{(3)}(\varepsilon)]}{\log T} = 0.$$

Therefore, recalling the decomposition of regret bound in (25), we conclude the proof of Theorem 4.

\[\Box\]

F.1 Proof of Lemma 4

To establish the lemma, we investigate the event $Z_{t}^{(1)}(x, a; \varepsilon)$ depending on whether $F_{t}$ is empty or not, and on whether $\phi_{t}$ is a good approximation of $\phi$. To this aim, for any given $t > 0$ and $\zeta > 0$, define the event $B_{t}(\zeta) := \{x, a) \in S \times A : B_{t}(x, a; \zeta) \}$ where for each $(x, a) \in S \times A, B_{t}(x, a; \zeta) := \{\|\phi_{t}(x, a) - \phi(x, a)\| \leq \zeta\}$. Fix $(x, a) \in S \times A$ such that $a \notin O(x; \phi)$. By the continuity assumption made in Theorem 4, we have:

$$\sum_{t=1}^{T} \mathbb{E}_{\phi}^{\pi}[Z_{t}^{(1)}(x, a; \varepsilon), F_{t} \neq \emptyset, \zeta_{t} < \zeta(\varepsilon), B_{t}(\zeta_{t})]

\leq \mathbb{E}_{\phi}^{\pi}\left[\sum_{t=1}^{T} \mathbb{I}[(X_{t}, A_{t}) = (x, a), N_{t}(x, a) \leq \eta(x, a)\eta_{t}, F_{t} \neq \emptyset, \zeta_{t} < \zeta(\varepsilon), B_{t}(\zeta_{t})]\right]

\leq \mathbb{E}_{\phi}^{\pi}\left[\sum_{t=1}^{T} \mathbb{I}[(X_{t}, A_{t}) = (x, a), N_{t}(x, a) \leq (\eta^{*}(x, a; \phi) + \varepsilon)\gamma_{t}]\right]

\leq (\eta^{*}(x, a; \phi) + \varepsilon)\gamma_{t} + 2.$$
where the second inequality is from the continuity of $\eta^*(\phi)$, and the last inequality is from a simple counting argument made precise in the following lemma [Burnetas and Katehakis, 1997] (Lemma 3 therein):

**Lemma 7.** Consider any (random) sequence of $Z_t \in \{0, 1\}$ for $t > 0$. Let $N_T := \sum_{t=1}^T 1[Z_t = 1]$. Then, for all $N > 0$, $\sum_{t=1}^T 1[Z_t = 1, N_t \leq N] \leq N + 1$ (point-wise if the sequence is random).

**Proof of Lemma 7.** The proof is straightforward from rewriting the summation as follows:

$$
\sum_{t=1}^T 1[Z_t = 1, N_t \leq N] = \sum_{t=1}^T \sum_{n=1}^N 1[Z_t = 1, N_t = n] 
= \sum_{n=1}^N \sum_{t=1}^T 1[Z_t = 1, N_t = n] \leq N + 1
$$

where the last inequality is from the fact that $\sum_{t=1}^T 1[Z_t = 1, N_t = n] \leq 1$. \hfill \Box

Since $\lim_{T \to \infty} \frac{\log T}{T} = (1 + \gamma)$ for all $x \in S$, we obtain:

$$
\lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{\mathbb{P}_x[\mathcal{Z}_t^{(1)}(x, a; \varepsilon) \neq \emptyset, \mathcal{F}_t \neq \emptyset, \mathcal{C}_t < \zeta(\varepsilon), \mathcal{B}_t(\zeta_t)]}{\log T} = (1 + \gamma) \eta^*(x, a; \phi).
$$

Hence, to complete the proof of Lemma 4, it suffices to show that

$$
\sum_{t=1}^T \mathbb{P}_{x|1}[\mathcal{Z}_t^{(1)}(x, a; \varepsilon), \mathcal{F}_t = \emptyset] = O(1) \tag{29}
$$

$$
\sum_{t=1}^T \mathbb{P}_{x|1}[\mathcal{Z}_t^{(1)}(x, a; \varepsilon), \mathcal{F}_t \neq \emptyset, -\mathcal{B}_t(\zeta_t)] = o(\log T) \tag{30}
$$

since $\sum_{t=1}^T \mathbb{P}_{\phi'|1}[\zeta_t > \zeta(\varepsilon)] = O(1)$.

To prove (29), observe that on the event $\mathcal{Z}_t^{(1)}(x, a; \varepsilon)$ for sufficiently large $t \geq e^{\varepsilon}$, i.e., $\zeta_t < \varepsilon$, for $b \in \mathcal{O}(x, \phi_t)'$, we have

$$
\delta^*(x, a; \phi_t, \zeta_t) = (B_{\phi_t}^0, h_t^0)(x) - (B_{\phi_t}^0, h_t^0)(x) 
\geq (B_{\phi_t}^0, h_t^0)(x) - (B_{\phi_t}^0, h_t^0)(x) + 2\varepsilon 
\geq -|B_{\phi_t}^0, h_t^0(x) - (B_{\phi_t}^0, h_t^0)(x)| + 2\varepsilon 
\geq -|(r_t(x, b) - r_0(x, b)) + |h_t'(x) - h_t^0(x))| + 2\varepsilon 
\geq \varepsilon > \zeta_t
$$

where the first, second, and fourth inequalities are from that on the event $\mathcal{Z}_t^{(1)}(x, a; \varepsilon)$, $(B_{\phi_t}^0, h_t^0)(x) \leq h_t^0(x) - 2\varepsilon$, $\mathcal{O}(x, \phi_t)' \subseteq \mathcal{O}(x, \phi)$, and $|r_t(x, b) - r_0(x, b)| + |h_t'(x) - h_t^0(x)| \leq \varepsilon$, respectively, and the last one is from the choice of $t$ such that $\zeta_t = 1/(1 + \log \log t) < \varepsilon$. Therefore, when $\mathcal{Z}_t^{(1)}(x, a; \varepsilon)$ occurs for sufficiently large $t \geq e^{\varepsilon}$,

$$
\delta_t(x, a) > \zeta_t > 0. \tag{31}
$$

If $\mathcal{F}_t$ is empty, from the design of DEL algorithm, $\delta_t(x, a) > 0$ implies that $\eta_t(x, a) = 0$ and thus $(x, a)$ is not selected in the exploration phase. This concludes the proof of (29) as $\sum_{t=1}^T \mathbb{P}_{\phi'|1}[\zeta_t > \varepsilon] \leq e^{\varepsilon} = O(1)$.

To show (30), observe that when $\mathcal{Z}_t^{(1)}(x, a; \varepsilon)$ and $\mathcal{F}_t \neq \emptyset$ occur, for $t \geq e^{\varepsilon}$ combining (31) and Lemma 8 given below, we get:

$$
\eta_t(x, a) \leq 2SA \left(\frac{S + 1}{\zeta_t}\right)^2. \tag{32}
$$
Lemma 8. Consider a structure $\Phi$, an MDP $\phi \in \Phi$, a non-empty correspondence $C : S \rightarrow A$, and $\zeta > 0$. If $F_{\Phi}(\phi; C, \zeta)$ is non-empty and there exists $(x, a) \in S \times A$ such that $\delta^*(x, a; \phi, C, \zeta) > 0$, then $\eta^*(x, a; \phi, C, \zeta) \leq 2SA \left( \frac{S + 1}{\zeta} \right)^2$ where $\eta^*(x, a; \phi, C, \zeta)$ is a solution of $P(\delta^*(\phi, C, \zeta), F_{\Phi}(\phi; C, \zeta))$.

Proof of Lemma 8. Using the same arguments as those used in Theorem 2 to show that $F_{\Phi}(\phi) \subset F_{\Phi}(\phi)$, one can easily check that $\frac{1}{S\epsilon} \sum_{t=0}^{T} \sum_{a \in A} |Z^t_i(x, a; \epsilon)|^2 \leq 2SA \left( \frac{S + 1}{\zeta} \right)^2$. Now let $\eta$ be defined as $\eta(x, a) = \infty$ if $\delta^*(x, a; \phi, C, \zeta) = 0$ and $\eta(x, a) = 2 \left( \frac{S + 1}{\zeta} \right)^2$ otherwise. Then $\eta \in F_{\Phi}(\phi; C, \zeta) \subset F_{\Phi}(\phi; C, \zeta)$. We deduce that the optimal objective value of $P(\delta^*(\phi, C, \zeta), F_{\Phi}(\phi; C, \zeta))$ is upper-bounded by

$$\sum_{(x, a) \in S \times A} \eta^*(x, a; \phi, C, \zeta) \delta^*(x, a; \phi, C, \zeta) \leq \sum_{(x, a) \in S \times A} \eta(x, a) \delta^*(x, a; \phi, C, \zeta) \leq 2SA \left( \frac{S + 1}{\zeta} \right)^2.$$

Using the optimality of $\eta^*(\phi, C, \zeta)$ and (33), we conclude that for $(x, a) \in S \times A$ such that $\delta^*(x, a; \phi, C, \zeta) > 0, \eta^*(x, a; \phi, C, \zeta) \leq 2SA \left( \frac{S + 1}{\zeta} \right)^2$. 

From (32), we deduce by design of DEL that, if $Z^t_i(x, a; \epsilon)$ and $F_t \neq \emptyset$ occur, for $t \geq e^{\gamma_t}$, then:

$$N_t(x, a) \leq \eta_t(x, a) \gamma_t \leq 2SA \left( \frac{S + 1}{\zeta} \right)^2 \gamma_t \leq \gamma'_t.$$

where

$$\gamma'_t := 8S^3A(1 + \gamma)(1 + \log \log t)^2 (\log t + 1) > 2SA \left( \frac{S + 1}{\zeta} \right)^2 \gamma_t. \tag{34}$$

Hence defining $B'_t(x, a) := \{(X_t, A_t) = (x, a), N_t(x, a) \leq \gamma'_t, \neg B_t(\zeta_t)\}$, we get:

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{x \in X_t} \left[ Z^t_i(x, a; \epsilon), F_t \neq \emptyset, \neg B_t(\zeta_t) \right] \leq \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{x \in X_t} \left[ Z^t_i(x, a; \epsilon), F_t \neq \emptyset, \neg B_t(\zeta_t), t \geq e^{\gamma_t} \right] + e^{\gamma_t}$$

$$\leq \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{x \in X_t} \left[ B'_t(x, a) \right] + O(1).$$

Using $\rho > 0$ in (8), we check that

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{x \in X_t} \left[ B'_t(x, a) \right]$$

$$\leq \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{x \in X_t} \left[ \min_{y \in S} N_t(y) \geq \rho t, B'_t(x, a) \right] + \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{x \in X_t} \left[ \min_{y \in S} N_t(y) \leq \rho t \right]$$

$$\leq \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{x \in X_t} \left[ \min_{y \in S} N_t(y) \geq \rho t, B'_t(x, a) \right] + o(\log T)$$

$$\leq \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{(y, b) \in S \times A} \left[ \min_{(y, b) \in S \times A} N_t(y, b) \geq \rho t \frac{\log t}{(1 + \log \log t)^2}, B'_t(x, a) \right] + o(\log T). \tag{35}$$

Here, the second inequality stems from (8) and a union bound (over states). The last inequality follows from the following lemma:
**Lemma 9.** Under DEL algorithm, we have

\[ \sum_{t=1}^{T} \mathbb{I} \left[ \min_{y \in \mathcal{S}} N_t(y) \geq \rho t, \min_{(y,b) \in \mathcal{S} \times \mathcal{A}} N_t(y,b) < \frac{\log t}{(1 + \log \log t)^2} \right] = o(\log T). \] (36)

**Proof of Lemma 9.** For \((x, a) \in \mathcal{S} \times \mathcal{A}\) and \(t\) sufficiently large, we claim the following:

\[ \mathbb{I} \left[ N_t(x) \geq \rho t, N_t(x,a) < \frac{\log t}{(1 + \log \log t)^2} \right] = 0. \] (37)

Using the above claim, we can complete the proof. Indeed:

\[
\sum_{t=1}^{T} \mathbb{I} \left[ \min_{y \in \mathcal{S}} N_t(y) \geq \rho t, \min_{(y,b) \in \mathcal{S} \times \mathcal{A}} N_t(y,b) < \frac{\log t}{(1 + \log \log t)^2} \right]
\leq \sum_{t=1}^{T} \sum_{(x,a) \in \mathcal{S} \times \mathcal{A}} \mathbb{I} \left[ N_t(x) \geq \rho t, N_t(x,a) < \frac{\log t}{(1 + \log \log t)^2} \right]
\leq \sum_{t=1}^{T} \sum_{(x,a) \in \mathcal{S} \times \mathcal{A}} \mathbb{I} \left[ N_t(x) \geq \rho t, N_t(x,a) < \frac{\log t}{(1 + \log \log t)^2} \right] = O(1) \quad \text{as } T \to \infty.
\]

where the first inequality stems from the union bound.

Next we prove the claim (37). Fix \((x, a) \in \mathcal{S} \times \mathcal{A}\) and consider sufficiently large \(t\). Suppose \(N_t(x) \geq \rho t\) and let \(t_0 = \lfloor \rho t/2 \rfloor\). Then, since \(N_{t_0}(x) \leq t_0\), it follows that

\[ N_t(x) - N_{t_0}(x) \geq \rho t - \lfloor \rho t/2 \rfloor \geq \rho t/2. \]

Let \(t_1 = \min\{u \in \mathbb{N}: u \geq [t_0, t], N_u(x) - N_{t_0}(x) = \lfloor \rho t/4 \rfloor\}\) denote the time when the number of visits to state \(x\) after time \(t_0\) reaches \(\rho t/4\). Since \(N_t(x) - N_{t_0}(x) \geq \rho t/2 \geq \lfloor \rho t/4 \rfloor\), there exists such a \(t_1 \in [t_0, t]\). From the construction of \(t_1\), it follows that for all \(u \in [t_1, t], \lfloor \rho t/4 \rfloor \leq N_u(x)\) and

\[ N_t(x) - N_{t_1}(x) = (N_t(x) - N_{t_0}(x)) - (N_{t_1}(x) - N_{t_0}(x)) \geq \rho t/2 - \lfloor \rho t/4 \rfloor \geq \rho t/4. \] (38)

Let \(N_{t_1, t}(x) := \{u \in [t_1, t]: X_u = x, E_u^{\text{mnt}}\}\) be the set of times between \(t_1\) and \(t\) when the state is \(x\) and the algorithm does not enter the monotonization phase and hence checks the condition to enter the estimation phase. For \(u \in N_{t_1, t}(x)\), the condition for the algorithm to enter the estimation phase and select an action with the minimum occurrence is:

\[ \exists b \in \mathcal{A}: N_u(x, b) < \frac{\log \lfloor \rho t/4 \rfloor}{1 + \log \log \lfloor \rho t/4 \rfloor} \] (39)

since from the construction of \(t_1\), for any \(u \in [t_1, t]\), we have \(\frac{\log \lfloor \rho t/4 \rfloor}{1 + \log \log \lfloor \rho t/4 \rfloor} \leq \frac{\log N_u(x)}{1 + \log \log N_u(x)}\).

Now assume that the number of times the algorithm enters the monotonization phase in state \(x\) between \(t_1\) and \(t\) is bounded by \(O(\log t)\). From (38) and (39), we deduce the desired claim (37). Indeed, with the observation (39), the fact that monotonization happens a sublinear number of times implies that the algorithm estimates all actions more than

\[
\frac{\log \lfloor \rho t/4 \rfloor}{1 + \log \log \lfloor \rho t/4 \rfloor} \quad \text{times.}
\]

Actually, the fact that monotonization happens a sublinear number of times and (38) imply that \(|N_{t_1, t}(x)| > A \frac{\log \lfloor \rho t/4 \rfloor}{1 + \log \log \lfloor \rho t/4 \rfloor}\) for sufficiently large \(t\).

Using the following lemma, we bound the number that the algorithm enters the monotonization phase between \(t_1\) and \(t\):

**Lemma 10.** For any action \(a \in \mathcal{A}\) and three different \(u, u', u''\) such that \(u < u' < u''\), suppose that the event \(E^{\text{mnt}}_t \cap \{(X_t, A_t) = (x, a)\}\) occurs for all \(t \in \{u, u', u''\}\). Then, when \(N_u(x) > c\),

\[ N_{u''}(x) - N_u(x) \geq \frac{N_u(x)}{2 \log N_u(x)}. \]
Proof of Lemma 10. Observe that selecting action \( b \) in the monotonization phase at time \( t \) means that
\[
N_t(x, a) \in [\log^2 N_t(x), \log^2 N_t(x) + 1)
\]  
(40)
From the fact that \( u < u' < u'' \), we have \( N_{u''}(x, a) \geq N_u(x, a) + 2 \) and thus using (40):
\[
\log^2 N_u(x) + 2 \leq N_u(x, a) + 2 \leq N_{u''}(x, a) < \log^2 N_{u''}(x) + 1.
\]
We deduce that \( \log^2 N_{u''}(x) - \log^2 N_u(x) > 1 \), and conclude that for \( N_u(x) > \epsilon \),
\[
N_{u''}(x) - N_u(x) \geq \frac{N_u(x)}{2 \log N_u(x)}
\]
since the function \( \log^2 t \) is concave with derivative \( \frac{2 \log t}{t^2} \), i.e., in order to increase \( \log^2 t \) by 1, \( t \) should be increased by more than \( \left( \frac{2 \log t}{t} \right)^{-1} \).
From Lemma 10, it follows that for sufficiently large \( t \),
\[
\sum_{b \in A} \sum_{u=1}^t 1[N_t(x) \geq \rho t, \mathcal{E}_t^{\text{mut}}, (X_t, A_t) = (x, b)] \leq A \max \left\{ 3, 3(N_t(x) - N_t(x)) \left( \frac{2 \log N_t(x)}{N_t(x)} \right) \right\}
\]
\[\leq A \max \left\{ 3, 3(t - \lfloor \rho t/4 \rfloor) \left( \frac{2 \log \lfloor \rho t/4 \rfloor}{\lfloor \rho t/4 \rfloor} \right) \right\}
\]
\[\leq 24A \log t. \quad (41)
\]
For the first inequality, we apply Lemma 10 with the fact that as \( u \) increases, \( N_u(x) \) increases and \( \frac{2 \log N_u(x)}{N_u(x)} \) decreases. The second inequality is from the definition of \( t_1 \) and (38). The last inequality holds for sufficiently large \( t \). We have completed the proof of Lemma 9.
We return to the proof of Lemma 4. Lemma 9 establishes (35). Next we provide an upper bound of (35). To this aim, we use the following concentration inequality [Combes and Proutiere 2014]:
\[\text{Lemma 11. Consider any } \phi, \pi, \epsilon > 0 \text{ with Bernoulli reward distribution. Define } \mathcal{H}_t \text{ the } \sigma\text{-algebra generated by } (Z_s)_{1 \leq s \leq t}. \text{ Let } \mathcal{B} \subset \mathbb{N} \text{ be a (random) set of rounds. Assume that there exists a sequence of (random) sets } (\mathcal{B}(s))_{s \geq 1} \text{ such that (i) } \mathcal{B} \subset \cup_{s \geq 1} \mathcal{B}(s), \text{ (ii) for all } s \geq 1 \text{ and all } t \in \mathcal{B}(s), \text{ (iii) } |\mathcal{B}(s)| \leq 1, \text{ and (iv) the event } t \in \mathcal{B}(s) \text{ is } \mathcal{H}_t\text{-measurable. Then for all } \zeta > 0, \text{ and } x_1, x, y \in \mathcal{S}, a \in \mathcal{A},
\sum_{t \geq 1} \mathbb{P}_{\phi \mid x_1} [t \in \mathcal{B}, \| r_t(x, a) - r_{\phi}(x, a) \| > \zeta] \leq \frac{1}{\epsilon \zeta^2}
\sum_{t \geq 1} \mathbb{P}_{\phi \mid x_1} [t \in \mathcal{B}, \| p_t(y \mid x, a) - p_{\phi}(y \mid x, a) \| > \zeta] \leq \frac{1}{\epsilon \zeta^2}
\]
\[\text{Lemma 11. Combes and Proutiere [2014] provides a proof of the first part. Now the occurrence of a transition under action } a \text{ from state } x \text{ to state } y \text{ is a Bernoulli random variable, and hence the second part of the lemma directly follows from the first.}
\]
\[\text{Let } B'_{t}(x, a) := \{ \min_{(y, b) \in \mathcal{S} \times \mathcal{A}} N_t(y, b) \geq \frac{\log t}{(1 + \log t)^2}, B'_t(x, a) \}. \text{ If at time } t \leq T, \text{ we have the } s\text{-th occurrence of } B'_{t}(x, a), \text{ then it follows that } s \leq \gamma'_t \text{ (since } a \text{ is selected in state } x \text{ at time } t, \text{ and } N_t(x, a) \leq \gamma'_t), \text{ and thus}
\min_{(y, b) \in \mathcal{S} \times \mathcal{A}} N_t(y, b) \geq \frac{\log t}{(1 + \log t)^2} \geq \frac{1}{16S^3A(1 + \gamma)(1 + \log t)^4} \gamma'_t
\]
\[\geq \frac{1}{16S^3A(1 + \gamma)(1 + \log t)^4} s,
\]
where the last inequality follows from \( t \leq T \) and \( s \leq \gamma'_t \). Thus since \( \neg B'_t(\zeta_t) \) holds when \( B'_{t}(x, a) \) occurs, we deduce that the set of rounds where \( B'_{t}(x, a) \) occurs satisfies
\[\{ t : B'_{t}(x, a) \text{ occurs} \} \subset \bigcup_{s \geq 1} \cup_{(y, b) \in \mathcal{S} \times \mathcal{A}} \{ t : s\text{-th occurrence of } B'_{t}(x, a), N_t(y, b) \geq \epsilon s, \| \phi_t(y, b) - \phi(y, b) \| > \zeta_x \},
\]
where \( \epsilon := \frac{1}{16S^3(1+\log \log T)^4} \). Now we apply Lemma 11 to each pair \((y, b)\) with \( \zeta = \zeta_T \), and conclude that:

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{P}_{\phi|x_t} [B_t^\epsilon (x, a)] \leq (SA) \frac{16S^3A(1+\log \log T)^4}{(\zeta_T)^2} = 16S^4(1+\log \log T)^6 = o(\log T)
\]

where the factor \( SA \) in the inequality is from the union bound over all \((y, b) \in S \times A \). This proves (30) and completes the proof of Lemma 4.

### F.2 Proof of Lemma 5

Let \( \varepsilon_2 := \min_{(x,a)\in S \times A} \mathbb{P}_{\phi \in \mathcal{O}(x; \phi)} (B_\phi h_\phi^a(x) - (B_\phi^a h_\phi^a)(x) > 0 \). Fix \((x, a) \in S \times A\) such that \( a \notin \mathcal{O}(x; \phi) \), and \( \varepsilon \in (0, \varepsilon_2/5) \) so that

\[
(B_\phi h_\phi^a(x) - (B_\phi^a h_\phi^a)(x) \leq -5\varepsilon.
\]

(42)

When \( Z_t^{(2)}(x, a; \varepsilon) \) occurs, we have

\[
(B_\phi^a h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) = (B_\phi^a h_\phi^a(x) - (B_\phi^a h_\phi^a)(x) + (B_\phi^a h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) \geq \sum_{y \in S} p_t(y | x, a)(h_\phi^a(y) - h_\phi^a(y)) - 3\varepsilon
\]

(43)

where the first inequality stems from the fact that \((B_\phi^a h_\phi^a)(x) \geq (B_\phi^a h_\phi^a)(x) - 2\varepsilon \) when \( Z_t^{(2)}(x, a; \varepsilon) \) occurs, and the last inequality follows from the fact that \( \mathcal{E}_t(\varepsilon) \) holds when \( Z_t^{(2)}(x, a; \varepsilon) \) occurs.

Let \( \zeta = \frac{\varepsilon}{5S} \). Then, recalling the definition of the event \( B_t(x, a; \zeta) := \{ \|\phi_t(x, a) - \phi(x, a)\| \leq \zeta \} \), when \( B_t(x, a; \zeta) \) occurs, we have

\[
||(B_\phi h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x)| \leq |r_t(x, a) - r_\phi(x, a)| + H_\phi \sum_{y \in S} |p_t(y | x, a) - p_\phi(y | x, a)| 
\]

\[
\leq |r_t(x, a) - r_\phi(x, a)| + S^2 \max_{y \in S} |p_t(y | x, a) - p_\phi(y | x, a)| 
\]

\[
\leq S^2 \|\phi_t(x, a) - \phi(x, a)\| 
\]

\[
\leq \varepsilon
\]

(44)

where for the second inequality, we used \( 0 \leq H_\phi \leq S \).

Now, we can deduce that the events \( Z_t^{(2)}(x, a; \varepsilon) \) and \( B_t(x, a; \zeta) \) cannot occur at the same time, i.e.,

\[
\mathbb{P}_{\phi|x_t} \left[ Z_t^{(2)}(x, a; \varepsilon), B_t(x, a; \zeta) \right] = 0.
\]

(45)

Indeed, when \( Z_t^{(2)}(x, a; \varepsilon) \cap B_t(x, a; \zeta) \) occurs, (43) and (44) imply

\[
(B_\phi h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) = (B_\phi h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) \geq (B_\phi h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) - (B_\phi^a h_\phi^a)(x) \geq -4\varepsilon > -5\varepsilon
\]

which contradicts (42) for our choice of \( \varepsilon \), i.e., \( \varepsilon \in (0, \varepsilon_2/5) \).

Hence, to complete the proof, it is sufficient to show that

\[
\sum_{t=1}^{T} \mathbb{P}_{\phi|x_t} [(X_t, A_t) = (x, a), \neg B_t(x, a; \zeta)] = O(1)
\]

(46)
as we have the following bound:
\[
\sum_{t=1}^{T} \mathbb{P}_{\phi_{x|1}} \left[ Z_{t}^{(2)}(x,a;\varepsilon) \right] = \sum_{t=1}^{T} \mathbb{P}_{\phi_{x|1}} \left[ Z_{t}^{(2)}(x,a;\varepsilon), -\mathcal{B}_{t}(x,a;\zeta) \right] \\
\leq \sum_{t=1}^{T} \mathbb{P}_{\phi_{x|1}} \left[ (X_{t},A_{t}) = (x,a), -\mathcal{B}_{t}(x,a;\zeta) \right]
\]

where the equality follows from (45). (46) is obtained by applying Lemma 11 with \{(X_{t},A_{t}) = (x,a)\}, 1 and \(\frac{\varepsilon}{2\varepsilon'}\) for \(\mathcal{B}, \varepsilon\) and \(\zeta\), respectively. This complete the proof of Lemma 5.

\[\Box\]

**F.3 Proof of Lemma 6**

Recall that:
\[\mathcal{E}_{t}(\varepsilon) := \left\{ \Pi^{\star}(\phi_{t}) \subseteq \Pi^{\star}(\phi) \right\} \text{ and } |r_{t}(x,a) - r_{\phi}(x,a)| + |h_{t}^{r}(x) - h_{\phi}^{r}(x)| \leq \varepsilon \forall x \in \mathcal{S}, \forall a \in \mathcal{O}(x;\phi_{t}) \}.
\]

Hence when \(\mathcal{E}_{t}(\varepsilon)\) occurs, (i) the estimation of the bias function in the restricted MDP \(\phi(C_{t})\) is accurate and (ii) the restricted MDP includes the optimal policies of \(\phi\). We first focus on the accuracy of the estimated bias function, and then show that the gain of the restricted MDP \(\phi(C_{t})\) is monotone increasing and that it eventually includes an optimal policy for the (unrestricted) MDP.

**Estimation error in bias function.** We begin with some useful notations. Let \(K := A^{S}\) be the number of all the possible fixed policies. Fix \(\beta \in \left(0, \frac{1}{K+1}\right)\). For sufficiently large \(t > \frac{1}{K+1}\), divide the time interval from 1 to \(t\) into \((K+1)\) subintervals \(I_{0}, I_{1}, \ldots, I_{K}\) such that \(I_{k} := \{u \in \mathbb{N} : i_{k} \leq u < i_{k+1}\}\) where \(i_{0} := 1\) and \(i_{k} := t + 1 - (K + 1 - k)\left[\frac{t}{K+1}\right]\) for \(k \in \{1, \ldots, K+1\}\). Then, it is easy to check that for each \(k \in \{0, \ldots, K\}\),
\[|I_{k}| = i_{k+1} - i_{k} > \beta t.
\]

Indeed, for \(k = 0, i_{0} - i_{0} = t - K \left[\frac{t}{K+1}\right] \geq \frac{t}{K+1} > \beta t\), and for \(k \in [1, K]\), \(i_{k+1} - i_{k} = \left[\frac{t}{K+1}\right] \geq \frac{t}{K+1} > \beta t\) as \(t > \frac{1}{K+1}\), i.e., each subinterval length grows linearly with respect to \(t\).

For \(k \in [0, K], x \in \mathcal{S}\) and \(a \in \mathcal{A}\), let \(N_{k}^{r}(x) := N_{i_{k+1}}^{r}(x) - N_{i_{k}}^{r}(x)\) and \(N_{k}^{i}(x,a) := N_{i_{k+1}}^{i}(x,a) - N_{i_{k}}^{i}(x,a)\). Using \(\rho > 0\) in (8), for \(\zeta > 0\), define an event \(\mathcal{D}_{t}(\zeta)\) as
\[\mathcal{D}_{t}(\zeta) := \mathcal{D}_{t}' \cap \mathcal{E}_{t}'(\zeta)
\]
where we let
\[\mathcal{D}_{t}' := \left\{ N_{k}^{r}(x) > \rho \beta t, \forall x \in \mathcal{S}, \forall k \in [0, K] \right\}
\]
\[\mathcal{E}_{t}'(\zeta) := \left\{ \|\phi'_{u} - \phi_{u}(C_{u})\| \leq \zeta \forall u \in [i_{t}, t] \right\}.
\]

When \(\mathcal{D}_{t}(\zeta)\) occurs, then in each subinterval, each state is linearly visited, and after the first subinterval, the estimation on the restricted MDP is accurate, i.e., \(\phi(C_{t}) \simeq \phi_{t}(C_{t})\). Note that \(\mathcal{E}_{t}(\varepsilon)\) bounds the error in the estimated gain and bias functions. Hence, we establish the correspondence between \(\zeta\) in \(\mathcal{D}_{t}(\zeta)\) and \(\varepsilon\) in \(\mathcal{E}_{t}(\varepsilon)\) using the continuity of the gain and bias functions in \(\phi\):

**Lemma 12.** Consider an ergodic MDP \(\phi\) with Bernoulli rewards. Then, for \(\varepsilon > 0\), there exists \(\zeta_{0} = \zeta_{0}(\varepsilon, \phi) > 0\) such that for any \(\zeta \in (0, \zeta_{0})\), policy \(f \in \Pi_{D_{t}}\) and MDP \(\psi\), if \(\|\psi - \phi\| \leq \zeta\) and \(\psi \ll \phi\), then \(\psi\) is ergodic, \(|g_{f}' - g_{\phi}'| \leq \varepsilon\) and \(\|h_{f}' - h_{\phi}'\| \leq \varepsilon\).

The proof of Lemma 12 is in Section F.3.1. Observe that on the event \(\mathcal{D}_{t}(\zeta)\), for \(u \geq i_{t}\), every state is visited more than \(\rho \beta t\), i.e., \(\log_{2} N_{u}(x) \geq \log_{2} \rho \beta t \geq 1\) for all \(x \in \mathcal{S}\) and sufficiently large \(t\), and thus, for all \((x, a) \in \mathcal{S} \times \mathcal{A}\) such that \(a \in C_{u}(x)\), \(\phi_{u}(\varepsilon, \phi)\) is indeed the estimation of \(\phi(x,a)\), i.e., \(\phi'_{u} = \phi_{u}(C_{u}) \ll \phi_{u}(C_{u})\). Then, using Lemma 12, it follows that there exists constant \(t_{0} > 0\) such that for \(t > t_{0}\) and \(\zeta \in (0, \min\{\zeta_{0}(\varepsilon, 2), \phi, \varepsilon/2\})\),
\[\mathcal{D}_{t}(\zeta) \subseteq \{\phi_{u}' \text{ is ergodic } \forall u \in [i_{t}, t]\} \cap \mathcal{E}_{t}'(\varepsilon)
\]
where
\[ E_t''(\varepsilon) := \left\{ |r_{\phi_t}(x, f(x)) - r_{\phi}(x, f(x))| + |h_{\phi_t}(x) - h_{\phi}(x)| \leq \varepsilon \forall u \in [i_1^t, t], \forall f \in \Pi_D(C_u), \forall x \in S \right\}, \]
and where for restriction \( C : S \to A \), we denote by \( \Pi_D(C) \) the set of all the possible deterministic policies on the restricted MDP \( \phi(C) \).

**Monotone improvement.** Based on (48), we can identify instrumental properties of DEL algorithm when \( D_1(\varepsilon) \) occurs:

**Lemma 13.** For structure \( \Phi \) with Bernoulli rewards and an ergodic MDP \( \phi \in \Phi \), consider \( \pi = \text{DEL} \). There exists \( \zeta_1 > 0 \) and \( t_1 > 0 \) such that for any \( \zeta \in (0, \zeta_1) \) and \( t > t_1 \), the occurrence of the event \( D_1(\zeta) \) implies that
\[ \Pi^*(\phi_u) \subseteq \Pi^*(\phi(C_u)), \quad \text{and} \quad \phi^*_{u+1} \geq \phi^*_{u}, \quad \forall u \in [i_1^t, t] \quad (49) \]
where we denote by \( \phi^*_{u} := \phi^*_{\Phi(C_u)} \) and \( h^*_{u} := h^*_{\phi(C_u)} \) the optimal gain and bias functions, respectively, on the restricted MDP \( \phi(C_u) \) with true parameter \( \phi \).

The proof of Lemma 13 is presented in Section F.3.2.

Define the event
\[ \mathcal{M}_t := \left\{ \phi^*_{i+1} > \phi^*_{i} \forall i \in [1, K] \text{ or } \phi^*_{i+1} = \phi^*_{i} = \phi^* \text{ for some } i \in [1, K] \right\}. \]

Then, by selecting \( \zeta \) as in Lemma 13 and (48), we can connect the events \( \mathcal{M}_t \) and \( D_1(\zeta) \) to the event \( E_t(\varepsilon) \) as follows: for \( \zeta \in (0, \min\{\zeta_0(\varepsilon/2, \phi), \varepsilon/2, \zeta_1\}) \) and sufficiently large \( t > t_1 \),
\[ \mathcal{M}_t \cap D_1(\zeta) \subseteq E_t(\varepsilon). \quad (50) \]

On the event \( \mathcal{M}_t \), there must exists \( k \in [1, K + 1] \) such that \( \phi^*_{i} = \phi^* \) since the number \( K \) of subintervals is the number of all the possible policy \( \Pi^* \). In addition, for such a \( k \in [1, K + 1] \), on the event \( D_1(\zeta) \), it follows from Lemma 13 that for all \( u \in [i_k^t, t] \), \( \phi^*_{i} \leq \phi^*_{u} \leq \phi^* \), i.e., \( \phi^*_{i} = \phi^* \) and thus \( \Pi^*(\phi(C_u)) \subseteq \Pi^*(\phi) \). Therefore, when both of the events \( \mathcal{M}_t \) and \( D_1(\zeta) \) occur,
\[ \Pi^*(\phi) \supseteq \Pi^*(\phi(C_1)) \supseteq \Pi^*(\phi_i) \]
(again thanks to Lemma 13). Then, we indeed get \( \mathcal{M}_t \cap D_1(\zeta) \subseteq E_t(\varepsilon) \): the ergodicity of \( \phi_i \) guaranteed from (48) implies that the optimal bias function \( h^*_i \) of \( \phi_i \) is unique, and the event \( E_t''(\varepsilon) \) in (48) always occurs on the event \( D_1(\zeta) \). Thus the estimated bias function \( h^*_i \) is close to \( h^* \).

Using (50) and (47), for small enough \( \zeta \in (0, \min\{\zeta_0(\varepsilon/2, \phi), \varepsilon/2, \zeta_1\}) \) and for large enough \( t > 0 \), we get
\[ \mathbb{P}^\pi_{\phi|x_1} [\neg E_t(\varepsilon)] \leq + \mathbb{P}^\pi_{\phi|x_1} [\neg D_1(\zeta)] \mathbb{P}^\pi_{\phi|x_1} [\mathcal{M}_t, \neg D_1(\zeta)] \leq O(1) + \mathbb{P}^\pi_{\phi|x_1} [\neg D'_t] + \mathbb{P}^\pi_{\phi|x_1} [D'_t, \neg E'_t(\zeta)] + \mathbb{P}^\pi_{\phi|x_1} [\mathcal{M}_t, \neg D_1(\zeta)] \quad (51) \]
where the first and last inequalities are from (50) and (47), respectively. To complete the proof of Lemma 6, we provide upper bounds of each term in the r.h.s. of (51). the first term can be easily bounded. Indeed, using (8) and a union bound, we get for \( t \) sufficiently,
\[ \mathbb{P}^\pi_{\phi|x_1} [\neg D'_t] \leq \sum_{x \in \mathcal{S}} \sum_{k \in [0, K]} \mathbb{P}^\pi_{\phi|x_1} [N^k(x) \leq \rho^2t] = o(1/t) \quad (52) \]
where the last equality is from (8) conditioned on \( X^t_{i_k} \) for each \( k \).

Lemma 14 below deals with the last term.

**Lemma 14.** For structure \( \Phi \) with Bernoulli rewards and an ergodic MDP \( \phi \in \Phi \), consider \( \pi = \text{DEL} \). Suppose \( \phi \) is in the interior of \( \Phi \), i.e., there exists a constant \( \zeta_0 > 0 \) such that for any \( \zeta \in (0, \zeta_0) \), \( \psi \in \Phi \) if \( \|\phi - \psi\| \leq \zeta \). There exists \( \zeta_2 > 0 \) such that for \( \zeta \in (0, \zeta_2) \),
\[ \mathbb{P}^\pi_{\phi|x_1} [D_T(\zeta), \neg \mathcal{M}_T] = o(1/T) \quad \text{as } T \to \infty. \]

We provide the proof of Lemma 14 in Section F.3.3. There, the assumption that \( \phi \) is in the interior of \( \Phi \) plays an important role when studying the behavior of the algorithm in the exploitation phase.

To bound the second term in the r.h.s. of (51), we use the following concentration inequality:
Lemma 15. Consider any \( \pi \) and \( x_1 \in S \). There exist \( C_0, c_0, u_0 > 0 \) such that for any \((x,a) \in S \times A\) and \( u \geq u_0 \),
\[
\mathbb{P}^\pi_{x_1}[|\phi_t(x,a) - \phi(x,a)| > \zeta, N_t(x,a) = u] \leq C_0 e^{-c_0 u}.
\]

Proof of Lemma 15. The proof is immediate from Lemma 4(i) in [Burnetas and Katehakis, 1997], which is an application of Cramer’s theorem for estimating Bernoulli random variables. Let \( \hat{\phi}_t(x,a) \) be the estimator of \( \phi(x,a) \) from \( t \) i.i.d. reward and transition samples when action \( a \) is selected in state \( x \). From Lemma 4(i) in [Burnetas and Katehakis, 1997], there are positive constants \( C(x,a), c(x,a) \), and \( u_0(x,a) \) (which may depend on \((x,a)\)), such that for \( u \geq u_0(x,a) \),
\[
\mathbb{P}^\pi_{x_1}[|\phi_t(x,a) - \phi(x,a)| > \zeta, N_t(x,a) = u] \leq \mathbb{P}[|\hat{\phi}_u(x,a) - \phi(x,a)| > \zeta] \leq C_0(x,a)e^{-c_0(x,a)u}.
\]

We complete the proof by taking \( C_0 := \max_{(x,a) \in S \times A} C_0(x,a) \), \( c_0 := \min_{(x,a) \in S \times A} c_0(x,a) \), and \( u_0 := \max_{(x,a) \in S \times A} u_0(x,a) \).

Now observe that:
\[
\begin{align*}
\mathbb{P}^\pi_{x_1}[D_t', -E'_\zeta] &= \mathbb{P}^\pi_{x_1}[D_t', \|\phi_u(x,a) - \phi(x,a)\| > \zeta, \text{for some } u \in [t'_1, t], x \in S, a \in C_u(x)] \\
&\leq \sum_{u=t'_1}^t \sum_{x \in S} \sum_{a \in C_u(x)} \mathbb{P}^\pi_{x_1}[D_t', \|\phi_u(x,a) - \phi(x,a)\| > \zeta] \\
&\leq \sum_{u=t'_1}^t \sum_{x \in S} \sum_{a \in C_u(x)} \sum_{u' \in \mathbb{S}(x,a)} \mathbb{P}^\pi_{x_1}[\|\phi_u(x,a) - \phi(x,a)\| > \zeta, N_u(x,a) \geq \log^2 N_u(x), \rho \beta t \leq N_u(x) \leq u] \\
&\leq \sum_{u=t'_1}^t \sum_{x \in S} \sum_{a \in C_u(x)} \sum_{u' \in \mathbb{S}(x,a)} \sum_{u'' \in \mathbb{S}(x,a)} \mathbb{P}^\pi_{x_1}[\|\phi_u(x,a) - \phi(x,a)\| > \zeta, N_u(x,a) = u'']
\end{align*}
\]

where the second inequality follows from the definition of \( C_u(x) \) and the fact that on the event \( D_t' \), \( \rho \beta t \leq N_0(x) \leq N_u(x) \) for \( u \in [t'_1, t] \). Then, applying Lemma 15, we have
\[
\begin{align*}
\mathbb{P}^\pi_{x_1}[D_t', -E'_\zeta] &\leq \sum_{u=t'_1}^t \sum_{x \in S} \sum_{a \in C_u(x)} \sum_{u' = \rho \beta t}^{\infty} \sum_{u'' = \log^2 t}^{\infty} C_0 e^{-c_0 u''} \\
&\leq \frac{SAC_0^2 e^{-c_0 \log^2(\rho \beta)}}{1 - e^{-c_0}} \\
&= \frac{SAC_0}{1 - e^{-c_0}} e^{-c_0 \log^2(\rho \beta) - \log(\rho \beta) + \log(t(\log(t + 2 \log(\rho \beta))))} \\
&= \frac{SAC_0}{1 - e^{-c_0}} e^{-c_0 \log^2(\rho \beta) - 2c_0 \log(\rho \beta) - c_0 \log t} = o(1/t).
\end{align*}
\]

Combining Lemma 14, (52) and (53) to (51), we complete the proof of Lemma 6.

\[ \square \]

F.3.1 Proof of Lemma 12

Define two strictly positive constants:
\[
\begin{align*}
\zeta_r(\phi) &:= \min \left\{ \frac{r_\phi(x,a)}{2} : \forall x \in S, \forall a \in A \text{ s.t. } r_\phi(x,a) > 0 \right\} \\
\zeta_p(\phi) &:= \min \left\{ \frac{p_\phi(y \mid x,a)}{2} : \forall x, y \in S, \forall a \in A \text{ s.t. } p_\phi(y \mid x,a) > 0 \right\}
\end{align*}
\]
Then, it is straightforward to show that $\phi \ll \psi$ if $\|\psi - \phi\| < \min\{\zeta_r(\phi), \zeta_p(\phi)\}$ since for any $x, y \in \mathcal{S}$ and $a \in \mathcal{A}$, $p_\phi(y \mid x, a) > 0$ implies that $p_\psi(y \mid x, a) \geq p_\phi(y \mid x, a)/2 > 0$, and $r_\phi(x, a) > 0$ implies that $r_\psi(x, a) \geq r_\phi(x, a)/2 > 0$. Therefore, for sufficiently small $\zeta_0 \leq \min\{\zeta_r(\phi), \zeta_p(\phi)\}$, the above observation and the assumption that $\psi \ll \phi$ ensure the mutual absolute continuity between $\phi$ and $\psi$ and thus the ergodicity of $\psi$.

Now, we focus on the continuity of gain and bias functions for given policy $f$. For notational convenience, let $g^f_\phi$ (resp. $g^f_\psi$) and $h^f_\phi$ (resp. $h^f_\psi$) denote the (column) vector of gain and bias functions, respectively, under $\phi$ (resp. $\psi$). Let $P^f_\phi$ (resp. $P^f_\psi$) and $r^f_\phi$ (resp. $r^f_\psi$) are the transition matrix and reward vector w.r.t. policy $f$ under $\phi$ (resp. $\psi$), respectively. Then, we can write the policy evaluation equations of stationary policy $f$ under $\phi$ and $\psi$ as vector and matrix multiplications, c.f., [Puterman, 1994]:

$$g^f_\phi = P^f_\phi g^f_\phi, \quad h^f_\phi = r^f_\phi - g^f_\phi + P^f_\phi h^f_\phi.$$ 

Similarly $g^f_\psi = P^f_\psi g^f_\psi$ and $h^f_\psi = r^f_\psi - g^f_\psi + P^f_\psi h^f_\psi$. Since both $\phi$ and $\psi$ are ergodic, by forcing $h^f_\phi(x_1) = h^f_\psi(x_1) = 0$ for some $x_1 \in \mathcal{S}$, the bias functions $h^f_\phi$ and $h^f_\psi$ can be uniquely defined. Let $D^f := P^f_\phi - P^f_\psi$ and $d^f := h^f_\phi - h^f_\psi$. Then, $\|D^f\| \leq S\zeta$ where $\|\cdot\|$ is the max norm. Noting that the ergodicity of $\phi$ and $\psi$ further provides the invertibility of $I - P^f_\phi$ and $I - P^f_\psi$. A basic linear algebra, c.f., Lemma 7 in [Burnetas and Katehakis, 1997], leads to that for any $\varepsilon > 0$, $\|d^f\| \leq \varepsilon$ if

$$\|D^f\| \leq \frac{\varepsilon}{\|(I - P^f_\phi)^{-1}\|\|h^f_\phi\| + \varepsilon}$$

where the upper bound is independent of $\psi$. From the above continuity of $h^f_\phi$ (and thus that of $g^f_\phi$) with respect to $\psi$ at $\phi$, we can find $\zeta_0(f, \varepsilon, \phi) > 0$ such that for any $\psi$, $|g^f_\phi - g^f_\psi| \leq \varepsilon$ and $\|h^f_\phi - h^f_\psi\| \leq \varepsilon$ if $\|\psi - \phi\| \leq \zeta_0(f, \varepsilon, \phi) \leq \min\{\zeta_r(\phi), \zeta_p(\phi)\}$. Noting the arbitrary choice of $f \in \Pi_D$, we conclude the proof of Lemma 12 by taking $\zeta_0(\varepsilon, \phi) = \min_{f \in \Pi_D} \zeta_0(f, \varepsilon, \phi)$. \hfill\Box

### F.3.2 Proof of Lemma 13

Let $\varepsilon_1 := \min\{|g^f_\phi - g^f_{f'}| : f, f' \in \Pi_D, g^f_{f'} \neq g^f_\phi\} > 0$. Let $\zeta_1 := \min\{\zeta_0\left(\frac{\varepsilon_1}{2S}, \phi\right), \frac{\varepsilon_1}{2S}\}$ and consider $t$ sufficiently large, i.e., $t > t_0$. For $\zeta \in (0, \zeta_1)$, assume that the event $D_t(\zeta)$ occurs.

**Proof of the first part of (49).** Then, for any $u \in [i^1_t, t]$ and $f \in \Pi_D(\mathcal{C}_u)$, it follows from (48) that for any $x \in \mathcal{S}$,

$$|g^f_{\phi_u} - g^f_{\phi_{u'}}| = |(B^f_{\phi_u}h^f_{\phi_u})(x) - (B^f_{\phi_{u'}}h^f_{\phi_{u'}})(x)| \leq |r^f_{\phi_u}(x, f(x)) - r^f_{\phi_{u'}}(x, f(x))| + \sum_{y \in \mathcal{S}}|h^f_{\phi_u}(y) - h^f_{\phi_{u'}}(y)| \leq S\frac{\varepsilon_1}{2S} = \frac{\varepsilon_1}{2}$$

where the last inequality stems from the definition of $E^f_t\left(\frac{\varepsilon_1}{2S}\right)$ in (48). Then, for any $u \in [i^1_t, t]$, $f \in \Pi^*(\phi_u')$, and $f' \in \Pi_D(\mathcal{C}_u)$, we have:

$$g^f_\phi \geq g^f_{\phi_u'} - \frac{\varepsilon_1}{2} \geq g^f_{\phi_u'} - \frac{\varepsilon_1}{2} \geq g^f_{\phi_{u'}} - \varepsilon_1$$

where the first and last inequalities stem from (54), and the second inequality is deduced from the optimality of $f$ under $\phi_{u'}$. Noting that $f, f' \in \Pi_D(\mathcal{C}_u)$, it follows that $g^f_{\phi(\mathcal{C}_u)} = g^f_\phi \geq g^f_{\phi_{u'}} = g^f_{\phi(\mathcal{C}_u)}$. Hence $f$ is optimal under $\phi(\mathcal{C}_u)$ (the choice of $f' \in \Pi_D(\mathcal{C}_u)$ is arbitrary). This completes the proof of the first part in (49).

**Proof of the second part of (49).** Fix $u \in [i^1_t, t]$. Assume that

$$\Pi_D(\mathcal{C}_{u+1}) \cap \Pi^*(\phi_{u'}) \neq \emptyset.$$ (55)
Then, from the first part of (49), we deduce that:
\[ \Pi_D(C_{u+1}) \cap \Pi^*(\phi_u) \subseteq \Pi_D(C_{u+1}) \cap \Pi^*(\phi(C_u)). \]

Combining this with the assumption (55), we get that \( \Pi_D(C_{u+1}) \cap \Pi^*(\phi(C_u)) \neq \emptyset \), which implies that \( g_{u+1}^* \geq g_u^* \). It remains to prove (55).

Let \( x = X_u \). We first show that:
\[ C_{u+1}(x) \cap O(x; \phi_u') \neq \emptyset. \tag{56} \]

If the algorithm enters the monotonization phase, i.e., the event \( E_{u}^{\text{mnt}} \) occurs, then it selects action \( a = A_u \in C_u(x) \cap O(x; \phi_u') \). We deduce that:
\[ N_u(x, a) = \log^2(N_u(x)), \quad N_{u+1}(x, a) = N_u(x, a) + 1, \quad \text{and} \quad N_{u+1}(x) = N_u(x) + 1 \]
Thus, using the fact that \( \log^2(n) + 1 > \log^2(n + 1) \), we obtain
\[ N_{u+1}(x, a) \geq \log^2(N_u(x)) + 1 \geq \log^2(N_u(x) + 1) = \log^2(N_{u+1}(x)). \tag{57} \]
We have shown that \( a \in C_{u+1}(x) \) and thus \( a \in C_{u+1}(x) \cap O(x; \phi_u') \neq \emptyset \).

In case that the event \( E_{u}^{\text{mnt}} \) does not occur, there must exist an action \( a \in O(x; \phi_u') \) such that \( N_u(x, a) \geq \log^2(N_u(x)) + 1 \). Hence, for (57), we get:
\[ N_{u+1}(x, a) \geq N_u(x, a) \geq \log^2(N_u(x)) + 1 \geq \log^2(N_u(x) + 1) = \log^2(N_{u+1}(x)) \]
which implies \( a \in C_{u+1}(x) \cap O(x; \phi_u') \neq \emptyset \).

Now (56) implies (55) (since for any \( y \in S \) such that \( y \neq x \), \( C_u(y) = C_{u+1}(y) \)). This completes the proof of the second part in (49) and that of Lemma 13. \( \square \)

**F.3.3 Proof of Lemma 14**

We will show that for small enough \( \zeta > 0 \),
\[ \mathbb{P}_{\phi, x_1}^\pi [D_t(\zeta), \neg M_t] = o(1/t) \]
where we recall
\[ M_t := \left\{ g_{t+1}^* > g_k^* \quad \forall k \in [1, K] \right\} \cup \left\{ g_{t+1}^* = g_k^* \quad \text{for some} \quad k \in [1, K] \right\} . \]

For \( x \in S \) and restriction \( C : S \rightarrow A \), define
\[ A^+(x; \phi, C) := \{ a \in A : (B^a h^*(C))(x) > (B^a \phi(C))(x) \} \]
as the set of actions that improve the optimal policy of the restricted MDP \( \phi(C) \) at state \( x \). If \( g_{\phi(C)}^* < g_{\phi}^* \), then there must exist a state \( x \) with non-empty \( A^+(x; \phi, C) \). Let \( \varepsilon_2 := \min\{ (B^a h^f)(x) - (B^a \phi(C))(x) : f \in \Pi_D, x \in S, a \in A^+(x; \phi, \{ f \}) \neq \emptyset \} \). Note that \( \varepsilon_2 > 0 \).

Define an event
\[ M'_t := \{ \mathcal{E}^{\text{exp}}, A^+(X_u; \phi, C_u) \neq \emptyset, \exists u \in [i_1', t] \} . \]
Then, we obtain
\[ \mathbb{P}_{\phi, x_1} D_t(\zeta), \neg M_t] \leq \mathbb{P}_{\phi, x_1} D_t(\zeta), \neg M_t, \neg M'_t] + \mathbb{P}_{\phi, x_1} D_t(\zeta), M'_t] . \]

We first focus on the last term in the above. Let \( \zeta \leq \min\{ \zeta_0, \frac{\varepsilon_2}{3} \} \) where \( \zeta_0 \) is taken from the assumption that \( \phi \) is in the interior of \( \Phi \), and \( t \geq \frac{1}{\beta} e^{\varepsilon_2/3} \) so that \( \zeta \leq \varepsilon_2/3 \) for any \( u \geq i_1' \geq \beta t \).

Suppose that for \( u \in [i_1', t] \) and \( x \in S \), the events \( D_t(\zeta) \) and \( \{ X_u = x, \mathcal{E}^{\text{exp}}, A^+(x; \phi, C_u) \neq \emptyset \} \) occur. From Lemma 13, it directly follows that \( O(x; \phi'_u) \subseteq O(x; \phi, C_u) \). By the definition of the improving action set, \( O(x; \phi, C_u) \cap A^+(x; \phi, C_u) = \emptyset \) and thus \( O(x; \phi'_u) \cap A^+(x; \phi, C_u) = \emptyset \).

Construct \( \psi_u \) such that for each \( (y, b) \in S \times A \),
\[ \psi_u(y, b) = \begin{cases} \phi_u(y, b) & \text{if } b \in O(x; \phi'_u), \\ \phi(y, b) & \text{otherwise.} \end{cases} \]
Note that $\psi_u(C_u) = \phi'_u$ and thus $\|\psi_u - \phi\| \leq \|\phi'_u - \phi(C_u)\| \leq \zeta_u$. This implies $\psi_u \in \Phi$ since $\phi$ is an interior point of $\Phi$. For any $a \in A^+ (x; \phi, C_u) \neq 0$, we get $\delta^*(x, a; \psi_u, C_u, \zeta_u) = 0$ as:

$$
\delta^*(x, a; \psi_u, C_u) = (B^*_{\phi_u} h'_u)(x) - (B^*_{\phi'_u} h'_u)(x) = (B^*_{\phi_u} h'_u)(x) - (B^*_{\phi_u} h'_u)(x) 
\leq \frac{2}{3}\varepsilon_2 + (B^*_{\phi_u} h'_u)(x) - (B^*_{\phi_u} h'_u)(x) 
\leq \frac{2}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_2 \leq \zeta_u
$$

where the second equality is from the construction of $\psi_u$ and the fact that $O(x; \phi'_u) \cap A^+ (x; \phi, C_u) = \emptyset$, i.e., $a \notin O(x; \phi'_u)$; and the first and second inequalities are from (48), the definition of $\varepsilon_2$. We have obtained that $\psi_u \in \Phi$ and $\delta^*(x, a; \psi_u, C_u, \zeta_u) = 0$ for some $a \notin O(x; \phi'_u)$. Therefore, $\psi_u \in \Delta_a (\phi_u; C_u, \zeta_u)$. Recalling the entering condition of the exploitation phase, we establish the following relation:

$$
D_t(\zeta) \cap M'_t \subseteq \left\{ \sum_{x \in S} \sum_{a \in A} N_u(x, a) KL_{\phi_u|x_u}(x, a) \geq \gamma_u \exists u \in [i^*_1, t] \right\}
\subseteq \left\{ \sum_{x \in S} \sum_{a \in A} N_u(x, a) KL_{\phi_u|x_u}(x, a) \geq \gamma_u \exists u \in [i^*_1, t] \right\}
$$

where the last inclusion follows from the construction of $\psi_u$, i.e.,

$$
\sum_{x \in S} \sum_{a \in A} N_u(x, a) KL_{\phi_u|x_u}(x, a) = \sum_{x \in S} \sum_{a \notin O(x; \phi'_u)} N_u(x, a) KL_{\phi_u|x_u}(x, a) 
= \sum_{x \in S} \sum_{a \notin O(x; \phi'_u)} N_u(x, a) KL_{\phi_u|x_u}(x, a) 
\leq \sum_{x \in S} \sum_{a \in A} N_u(x, a) KL_{\phi_u|x_u}(x, a).
$$

As a consequence, applying the following lemma, $P^\pi_{\phi|x_1} [D_t(\zeta), M'_t]$ is bounded by $o(1/t)$.

**Lemma 16.** Consider any $\pi$ and $\phi$ with Bernoulli rewards. Then, for any $\gamma > 0$ and $\rho \in (0, 1)$, as $T \to \infty$,

$$
\sum_{t=\rho T}^{T} P \left[ \sum_{x \in S} \sum_{a \in A} N_t(x, a) KL_{\phi_t|x}(x, a) \geq (1 + \gamma) \log t \right] = o(1/T).
$$

**Proof of Lemma 16.** The proof is an application of Theorem 2 in [Magureanu et al., 2014], which says that for $\gamma' > SA + 1$ and sufficiently large $t > 0$,

$$
P \left[ \sum_{x \in S} \sum_{a \in A} N_t(x, a) KL_{\phi_t|x}(x, a) \geq \gamma' \right] \leq e^{-\gamma'} \left( \frac{(\gamma')^2 \log t \log A}{SA} \right)^{SA} e^{SA+1}.
$$

Hence, putting $(1 + \gamma) \log t$ to $\gamma'$, we obtain

$$
\sum_{t=\rho T}^{T} P \left[ \sum_{x \in S} \sum_{a \in A} N_t(x, a) KL_{\phi_t|x}(x, a) \geq (1 + \gamma) \log t \right] 
\leq \sum_{t=\rho T}^{T} e^{-(1+\gamma) \log t} \left( \frac{(1 + \gamma)^2 (\log t)^3}{SA} \right)^{SA} e^{SA+1} 
\leq \sum_{t=\rho T}^{T} e^{SA+1} \left( \frac{(1 + \gamma)^2 (\log t)^3}{SA} \right)^{SA} \left( \frac{t^{1+\gamma}}{t^{1+\gamma}} \right).
$$

Using that $(\log t)^{3SA}/t^{1+\gamma} = O(1/t^{1+\gamma/2})$ for $t \geq \rho T$ and $\int_{\rho T}^{T} 1/t^{1+\gamma/2} dt \leq 1/(\rho T)^{1+\gamma/2} = o(1/T)$, we conclude the proof of Lemma 16. □
It remains to bound $\mathbb{P}_{\phi|x_1}^\pi \left[ D_{t}(\zeta), \neg \mathcal{M}_t, \neg \mathcal{M}'_t \right]$. From Lemma 13, on the event $D_t(\zeta)$, it is true that $g^*_u$ is non-decreasing in $u \in [i_t^1, i_t^2]$, i.e., $g^*_u \leq g^*_{i_t^2+1}$. Hence,

$$D_t(\zeta) \cap \neg \mathcal{M}_t \subseteq D_t(\zeta) \cap (\cup_{k=1}^K \mathcal{M}_k^t)$$

where

$$\mathcal{M}_k^t := \{ g^*_u = g^*_{i_t^2+1} < g^*_u \}.$$

Then, it suffices to show that for any $k \in [1, K]$, $\mathbb{P}_{\phi|x_1}^\pi \left[ D_t(\zeta), \mathcal{M}_k^t, \neg \mathcal{M}'_t \right] = o(1/t)$. For $k \in [1, K]$, assume that the events $D_t(\zeta), \mathcal{M}_k^t$ and $\neg \mathcal{M}'_t$ occur. Fix $x \in S$ such that $\mathcal{A}^+(x; \phi, \mathcal{C}_k^t) \neq \emptyset$. Since $g^*_u < g^*_u$, such a $x \in S$ must exist. In addition, using the second part of Lemma 13 and recalling (55) with the fact that the ergodic MDPs $\phi(\mathcal{C}_u), \phi(\mathcal{C}_u + 1)$ have unique bias functions, it follows that $\mathcal{A}_x^+ = \mathcal{A}_x^+ + 1$ and $h^*_u = h^*_u + 1 \forall u \in \mathcal{I}_k^t$. Therefore, $\mathcal{A}^+(x; \phi, \mathcal{C}_k^t) \neq \emptyset \forall u \in \mathcal{I}_k^t$. Recalling $N_{k}^t(x) := N_{k+1}^t(x) - N_k^t(x)$ and $\mathcal{N}_k(x, a) := N_{k+1}^t(x, a) - N_k^t(x, a)$, this implies that the algorithm never enters the exploitation phase when $X_u = x$, i.e., on the event $\{ D_t(\zeta), \mathcal{M}_k^t, \neg \mathcal{M}'_t \}$,

$$N_{k+1}^t(x, a) = \sum_{u \in \mathcal{I}_k^t} \mathbb{I} \left[ (X_u, A_u) = (x, a), \neg \mathcal{E}^{x_p} \right]$$

$$= \sum_{u \in \mathcal{I}_k^t} \mathbb{I} \left[ (X_u, A_u) = (x, a), \mathcal{E}^{x_u} \cup \mathcal{E}^{x_u} \right] + \sum_{u \in \mathcal{I}_k^t} \mathbb{I} \left[ (X_u, A_u) = (x, a), \mathcal{E}^{x_u} \right]$$

$$\leq O(\log t) + \sum_{u \in \mathcal{I}_k^t} \mathbb{I} \left[ (X_u, A_u) = (x, a), \mathcal{E}^{x_u} \right]$$

(60)

where the last inequality is obtained since by Lemma 10, the number of times the algorithm enters the monotonization phase is $O(\log t)$ (c.f., (41)), and since by design, the algorithm limits the number of times we enter the estimation phase to $O(\log t / \log \log t)$. Hence, it is enough to show that for $(x, a) \in S \times A$ such that $a \notin \mathcal{A}^+(x; \phi, \mathcal{C}_k^t) \neq \emptyset$,

$$\mathbb{P}_{\phi|x_1}^\pi \left[ D_t(\zeta), \mathcal{M}_k^t, \neg \mathcal{M}'_t, \mathcal{L}_k^t(x, a) \right] = o(1/t)$$

(61)

where we define $\mathcal{L}_k^t(x, a) := \{ N_{k+1}^t(x, a) \geq \frac{\rho \beta t}{2A} \}$ with $N_{k+1}^t(x, a) := \sum_{u \in \mathcal{I}_k^t} \mathbb{I} \left[ (X_u, A_u) = (x, a), \mathcal{E}^{x_u} \right]$. Indeed, for $x \in S$ such that $\mathcal{A}^+(x; \phi, \mathcal{C}_k^t) \neq \emptyset$, if the event $\{ D_t(\zeta), \mathcal{M}_k^t, \neg \mathcal{M}'_t, \mathcal{L}_k^t(x, a) \}$ occurs, then

$$\mathcal{L}_k^t(x, a) = \sum_{a \in \mathcal{A}^+(x; \phi, \mathcal{C}_k^t)} N_{k+1}^t(x, a) - \sum_{a \notin \mathcal{A}^+(x; \phi, \mathcal{C}_k^t)} N_{k+1}^t(x, a)$$

$$\geq \rho \beta t - \sum_{a \notin \mathcal{A}^+(x; \phi, \mathcal{C}_k^t)} N_{k+1}^t(x, a)$$

$$\geq \rho \beta t - \sum_{a \notin \mathcal{A}^+(x; \phi, \mathcal{C}_k^t)} N_{k+1}^t(x, a) - O(\log t)$$

$$\geq \rho \beta t - \frac{\rho \beta t}{2} - O(\log t)$$

where for the second inequality, we use (60). This implies that for sufficiently large $t$, there exists $a \in \mathcal{A}^+(x; \phi, \mathcal{C}_k^t)$ such that $N_{k+1}^t(x, a) \geq \frac{1}{2} \rho \beta t \geq \log^2 t \geq \log^2 N_{k+1}^t(x)$, i.e., $a \in \mathcal{C}_{k+1}$, and thus $g^*_u > g^*_u$, which contradicts to the occurrence of the event $\mathcal{M}_k^t$. Therefore, for sufficiently large $t > 0$ and $x \in S$ such that $\mathcal{A}^+(x; \phi, \mathcal{C}_k^t) \neq \emptyset$,

$$\mathbb{P}_{\phi|x_1}^\pi \left[ D_t(\zeta), \mathcal{M}_k^t, \neg \mathcal{M}'_t \right] \leq \sum_{a \notin \mathcal{A}^+(x; \phi, \mathcal{C}_k^t)} \mathbb{P}_{\phi|x_1}^\pi \left[ D_t(\zeta), \mathcal{M}_k^t, \neg \mathcal{M}'_t, \mathcal{L}_k^t(x, a) \right].$$

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It remains to prove (61). Fix \((x, a) \in S \times A\) such that \(a \notin \mathcal{A}^+(x; \phi, C_i)\). Assume that the events \(D_t(\zeta), M^k_t, \neg M'_t\) occur and \(N^\text{exp}_k(x, a) \geq \frac{\rho \beta t}{4A}\). Let

\[
t_3 := \min \left\{ u \in T^k_i : \sum_{v=i'_{k+1}}^u \mathbb{I}[X_v, A_v] = (x, a), e^\text{exp}_u \geq \frac{\rho \beta t}{4A} \right\}.
\]

From the assumption, \(t_3 \in T^k_i\). Then, using a similar argument as that used to derive (53) and using Lemma 15, we can guarantee \(\| \phi_u(x, a) - \phi(x, a) \| \leq \zeta\) for all \(u > t_3\) with probability \(1 - o(1/t)\) as \(N_u(x, a) \geq \frac{\rho \beta t}{4A} = \Omega(t)\), i.e.,

\[
\mathbb{P}_{\phi|x_1}^\pi [D_t(\zeta), M^k_t, \neg M'_t, L^k_t(x, a), \| \phi_u(x, a) - \phi(x, a) \| > \zeta \exists u \in [t_3, i'_{k+1}]] = o(1/t). \tag{62}
\]

Further, assume that \(\| \phi_u(x, a) - \phi(x, a) \| \leq \zeta, \forall u \in [t_3, i'_{k+1}]\). Then, similarly as in (58), we can deduce that \(\delta^*(x, a; \phi_u, C_u) > \zeta_u\) from the assumption of the correctness of the estimated bias function and (48), and thus

\[
\delta^*(x, a; \phi_u, C_u, \zeta_u) > \zeta_u.
\]

Hence, in the exploration phase, when \(F_u = \emptyset, \eta_u(x, a) = 0\) due to the design of the algorithm, while when \(F_u \neq \emptyset, \eta_u(x, a) \leq 2SA \left( \frac{S + 1}{\zeta_u} \right)^2\) due to Lemma 8. Therefore, recalling the definition of \(\gamma'_u\) in (34), for any \(u \in [t_3, i'_{k+1}]\), on the event \(\{e^\text{exp}_u, X_u = x\}\),

\[
\eta_u(x, a) \gamma_u \leq 2SA \left( \frac{S + 1}{\zeta_u} \right)^2 \gamma_u \leq \gamma'_u = O(\log^2 t)
\]

which implies that for sufficiently large \(t > 0\) such that \(\gamma'_u = O(\log^2 t) < \frac{\rho \beta t}{4A} \leq N_u(x, a) = \Omega(t), \mathbb{I}[e^\text{exp}_u, (X_u, A_u) = (x, a)] = 0\) for all \(u \in [t_3, i'_{k+1}]\) due to the design of the exploration phase. Hence, it follows that

\[
\mathbb{P}_{\phi|x_1}^\pi [D_t(\zeta), M^k_t, \neg M'_t, L^k_t(x, a), \| \phi_u(x, a) - \phi(x, a) \| \leq \zeta \forall u \in [t_3, i'_{k+1}]] = 0.
\]

Combining the above with (62), we have completed the proof of (61) and thus the proof of Lemma F.3.3.