Hierarchical Wave Functions of Fractional Quantum Hall Effect on the Torus

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ABSTRACT

One kind of the hierarchical wave functions of Fractional Quantum Hall Effect on the torus is constructed. We find that the wave functions closely relate to the wave functions of generalized Abelian Chern-Simons theory.

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The past ten years have seen a great deal of interest in Fractional Quantum Hall Effect not only in the group of the condense matter physicists, but also in the group of the mathematical physicists. In especial the relation between FQHE and Chern-Simons-Conformal Field Theory (Chern-Simons-Conformal Field Theory plays an important role in the recent developments of the theoretical physics), was uncovered in the past few years [1,2,3,4,5,6,7, etc.]. FQHE is a place which can realize some theoretical ideas in the theoretical physics on nature. For example, the fractionally charged quasiparticle excitations obey fractional statistics. Such kinds of the excitations are called anyons, which were discovered theoretically by J.M. Leinaas and J. Myrheim and made popular by F. Wilczek ([8] and references in it; see also [9]). In this paper, we shall investigate the relation between FQHE and Chern-Simons theory.

The Laughlin wave functions [10] of FQHE on compact surfaces, for example, sphere and torus at the filling \( \frac{1}{m} \) with \( m \) being an odd integer has been obtained in [11] by F.D.M. Haldane and in [12] by F.D.M. Haldane and E.H. Rezayi. The hierarchical wave functions of FQHE have been discussed in [11,13]. The hierarchical wave functions of FQHE have been investigated most recently in [4,5] (on the sphere, those wave function can be obtained by following the method developed in [14,15]). Then we may ask whether it is possible to extend the result in [12] to get the hierarchical wave functions on the torus.

The results of this paper are as follows. (i) the result in [12] is reviewed. The ground wave functions are reformulated in a new bases. Then we proceed to get the hierarchical wave functions on the torus (this kind of wave function on the plane has been studied in [5]). The degeneracy of the wave functions is obtained and the result agrees with the prediction in [6,7]. (ii)The wave functions are found to be closely related with the wave functions of the generalized Abelian Chern-Simons theory [16].

Following [12], we consider a magnetic field with potential \( \mathbf{A} = -By \hat{x} \), the
wave function describing a electron in the lowest Landau level has the form

$$\psi(x, y) = e^{-\frac{By}{2} f(z)}, \quad (1)$$

where $f(z)$ is the holomorphic function, and the units $e = 1, \hbar = 1$ are used. The lagrangian of the electron in the magnetic field is

$$L = \sum_{i=1, 2} \frac{1}{2} m (v^i)^2 + A^i v^i, \quad (2)$$

where $L$ is invariant up to a total time derivative under the translations. The corresponding Noether currents due to the translations are

$$t_x = m \dot{x} - By, t_y = m \dot{y} + Bx. \quad (3)$$

The conjugate momenta are

$$p_x = m \dot{x} - By, p_y = m \dot{y}. \quad (4)$$

So

$$t_x = p_x, t_y = p_y + Bx. \quad (5)$$

They commute with Hamiltonian

$$H = \frac{1}{2m} [(p_x + By)^2 + (p_y)^2], \quad (6)$$

with the commutations $[x, p_x] = i, [y, p_y] = i$ when the theory is quantized. By identifying $z \sim z + m + n\tau$ with $\tau = \tau_1 + i\tau_2$ and $\tau_2 \geq 0$, we obtain a torus specified by $\tau$. The consistent boundary conditions imposed on the wave function of the electron on this torus are

$$e^{i\tau_2} \psi = e^{i\phi_1} \psi, e^{i\tau_1 t_x + i\tau_2 t_y} \psi = e^{i\phi_2} \psi, \quad (7)$$

with the condition $\tau_2 B = 2\pi \Phi$, and $\Phi$ is an integer, which will insure that $e^{i\tau_2}, e^{i\tau_1 t_x + i\tau_2 t_y}$ commute with each other for the consistence of the equation (7).
By using the relation
\[ e^{i\tau_1 x + i\tau_2 y} = e^{-i\tau_2 x^2 / 2\tau_1} e^{i\tau_1 p_x + i\tau_2 p_y} e^{i\tau_2 y^2 / 2\tau_1}, \]
(7) can be written as
\[ f(z + 1) = e^{i\phi_1} f(z), f(z + \tau) = e^{i\phi_2} e^{-i\pi\Phi(2z + \tau)} f(z). \]  
(8)

For the many-particle wave functions, the condition of the equation (8) is imposed on every particle. Our task is to seek Laughlin-Jastrow wave functions for FQHE.

Now let us give a brief review about the definition of \( \theta \) function. The standard \( \theta \) function is defined as
\[ \theta(z|\tau) = \sum_n \exp(\pi in^2\tau + 2\pi inz), n \subset \text{integer}. \]  
(9)

More generally, \( \theta \) function on the lattice [17] is given by
\[ \theta(z|e, \tau) = \sum_v \exp(\pi iv^2\tau + 2\pi iv \cdot z), \]  
(10)

where \( v \) is a vector on a \( l \)-dimension lattice, \( v = \sum_{i=1}^l n_i e_i \), with \( n_i \) being integers, \( e_i \cdot e_j = \Lambda_{ij} \) and \( z = z_i e_i \). The \( \theta \) function in the equation (9) is the \( \theta \) function defined by (10) with \( l = 1, e_1 \cdot e_1 = 1 \). Furthermore we define
\[ \theta \left[ \begin{array}{c} a \\ b \end{array} \right] \left( z|e, \tau \right) = \sum_v \exp(\pi i(v + a)^2\tau + 2\pi i(v + a) \cdot (z + b)), \]  
(11)

where \( a, b \) are arbitrary vectors on the lattice. Only the positive lattice will be considered later, which is defined as \( x_i \Lambda_{ij} x_J > 0 \) for any nonzero real \( x_i \). The positive lattice will insure that \( \theta \) function in the equation (10) is well defined. The dual lattice \( e_i^* \) is defined as
\[ e_i^* \cdot e_j = \delta_{ij}, \]  
(12)

and they have inner products as \( e_i^* \cdot e_j^* = \Lambda_{i,j}^{-1} \).
It can be verified that

\[
\theta_{\left[\begin{array}{c} a \\ b \end{array}\right]}(z + e_i | e, \tau) = e^{2\pi i a \cdot e_i} \theta_{\left[\begin{array}{c} a \\ b \end{array}\right]}(z | e, \tau), \tag{13}
\]

\[
\theta_{\left[\begin{array}{c} a \\ b \end{array}\right]}(z + \tau e_i | e, \tau) = \exp [-\pi i \tau e_i^2 - 2\pi i e_i \cdot (z + b)] \theta_{\left[\begin{array}{c} a \\ b \end{array}\right]}(z | e, \tau),
\]

\[
\theta_{\left[\begin{array}{c} a \\ b \end{array}\right]}(z + e_i^* | e, \tau) = e^{2\pi i a \cdot e_i^*} \theta_{\left[\begin{array}{c} a \\ b \end{array}\right]}(z | e, \tau), \tag{13}
\]

\[
\theta_{\left[\begin{array}{c} a \\ b \end{array}\right]}(z + \tau e_i^* | e, \tau) = \exp [-\pi i \tau (e_i^*)^2 - 2\pi i e_i^* \cdot (z + b)] \theta_{\left[\begin{array}{c} a + e_i^* \\ b \end{array}\right]}(z | e, \tau),
\]

and

\[
\theta_{\left[\begin{array}{c} a + e_i \\ b + e_j^* \end{array}\right]}(z | e, \tau) = \exp(2\pi i a \cdot e_j^*) \theta_{\left[\begin{array}{c} a \\ b \end{array}\right]}(z | e, \tau). \tag{14}
\]

Moreover in 1-dimension lattice with \( e_1 \cdot e_1 = 1 \), we define

\[
\theta_3(z | \tau) = \theta_{\left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right]}(z | \tau), \tag{15}
\]

which is an odd function of \( z \). It can be verified that

\[
\theta_3(z + 1 | \tau) = e^{\pi i} \theta_3(z | \tau),
\]

\[
\theta_3(z + \tau | \tau) = \exp [-\pi i \tau - 2\pi i \cdot (z + \frac{1}{2})] \theta_3(z | \tau). \tag{16}
\]

The Laughlin-Jastrow wave functions on the torus at the filling \( \frac{1}{m} \) with \( m \) being an odd positive integer can be written as

\[
\Psi(z_i) = \exp\left(-\frac{\pi \Phi}{\tau_2} \sum_i y_i^2\right) F(z_i),
\]

\[
F(z_i) = \theta_{\left[\begin{array}{c} a \\ b \end{array}\right]} \left( \sum_i z_i e | e, \tau \right) \prod_{i<j} [\theta_3(z_i - z_j | \tau)]^m, \tag{17}
\]

where \( \theta \) function is on 1-dimension lattice, \( e^2 = m, \ i = 1, 2 \ldots, N \) with \( N \) being
the number of the electrons and $a = a_1e, b = b_1e$. Thus

\[
F(z_i + 1) = (-1)^{N-1}e^{2\pi a_1m}F(z_i),
\]
\[
F(z_i + \tau) = \exp(-\pi(N - 1) - 2\pi imb_1)\exp[-i\pi mN(2z_i + \tau)]F(z_i).
\] (18)

Comparing to the equation (8), we get

\[
\Phi = mN, \phi_1 = \pi(\Phi + 1) + 2\pi n_1 + 2\pi a_1m, \phi_2 = \pi(\Phi + 1) + 2\pi n_2 - 2\pi b_1m.
\] (19)

Thus the solutions of (19) will give $m$ ground wave functions, which can be shown by using the equation (14). Explicitly the solutions are

\[
a_1 = \frac{\phi_1}{2\pi m} + \frac{\Phi + 1}{2m} + \frac{i}{m}, b_1 = -\frac{\phi_2}{2\pi m} + \frac{\Phi + 1}{2m}, i = 0, 1, \ldots, m - 1,
\] (20)

which will give $m$ orthogonal Laughlin-Jastrow wave functions. So there is $m$-fold center-mass degeneracy [12] (its possible physical relevance has been studied in [18]).

We give a remark here. Different ground states are connected by gauge transformation of the magnetic field [9,19]. The degeneracy will disappear and one unique ground state appears by fixing the gauge of the magnetic field. Let us use the above Laughlin wave functions as an example. Now we take the magnetic potential as $A_1 = (-By + \frac{2\pi c_1}{\tau_2})\hat{x}$, $A_2 = \frac{2\pi c_2}{\tau_2}\hat{y}$ with $c_1, c_2$ being constant and $c = ic_1 - c_2$. The magnetic field is independent on constants $c_i$. That $c_i$ can be any constant numbers and the choice of $c_i$ is the choice of the gauge of the magnetic potential. $c$ and $c + m + n$ are gauge equivalent as they are connected by the large gauge transformations on the torus which are generated by $U_1 = \exp(-\frac{2\pi iy}{\tau_2})$ and $U_2 = \exp[\frac{\pi(\tau_2 - \tau_2)}{\tau_2}]$. If we take $A_1 = (-By + \frac{2\pi c_1}{\tau_2})\hat{x}$, $A_2 = \frac{2\pi c_2}{\tau_2}\hat{y}$, the wave
functions will be

\[ \Psi(z_i) = \exp\left(\frac{\pi}{2\tau_2}[ce^* + \sum_i (\bar{z}_i - z_i)e]^2\right) \]
\[ \cdot \exp\left(-\frac{\pi \Phi}{\tau_2} \sum_i y_i^2 - \frac{\pi}{2\tau_2} \left[\sum_i (\bar{z}_i - z_i)e\right]^2\right) F(z_i), \]
\[ F(z_i) = \theta\left[a\right] \left[\sum_i z_i e - ce^*[e, \tau]\right] \prod_{i<j} \left[\theta_3(z_i - z_j|\tau)\right]^m, \]

where \( e^* = \frac{1}{e} \) and \( a, b \) are remain unchanged and still given by (20). Then we find that the above wave functions will be invariant up to a phase under the related large gauge transformation of \( U_1 \) and will change from one to another under the related large gauge transformation of \( U_2 \). So we can say that the degeneracy will disappear by fixing the gauge of the magnetic field. The definition of the large gauge transformation on the wave functions can be found in [19] or formula (4.72) in [9], see also [20] (but pay attention on the different notation).

In [9,19], they also show that the phases \( \phi_1, \phi_2 \) need to be fixed if requiring that the wave function under the modular transformations will be transformed covariantly. We will come back to this point at the end of the paper.

The hierarchical FQHE, similar to the one considered by N. Read [5] on the plane, can be characterized by the symmetric matrix \( \Lambda \) with \( \Lambda_{1,1} \) being an odd positive integer and other diagonal elements being even integers. In particular we consider an integer lattice defined by matrix

\[
\Lambda = \begin{pmatrix}
  p_1 & -1 & 0 & \ldots & 0 & 0 \\
  -1 & p_2 & -1 & 0 & \ldots & 0 \\
  0 & -1 & p_3 & -1 & 0 & \ldots \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
  0 & \ldots & 0 & -1 & p_{l-1} & -1 \\
  0 & 0 & \ldots & 0 & -1 & p_l
\end{pmatrix}, \tag{21}
\]

where \( p_1 \) is a positive integer, \( p_i, i = 2, 3, \ldots, l \) are even integers. \( \Lambda \) describes
a $l$-level hierarchical state. The coordinates of the particles are expressed by $z_i^s$, where $z_i^s$ is the coordinate of the $i^{th}$ particle in level $s$, for example, $z_i^1 = z_i$ is the coordinate of the $i^{th}$ electron. The wave functions on the torus are supposed to be

$$F(z_i) = \int \prod_{s=2}^l dz_i^s dz_i^s \theta \left( \sum_{s=1}^l \sum_i z_i^s e_s | e, \tau \right) \prod_{i,j,s(1) \leq s(2)} \left[ \theta_3(z_i^{s(1)} - z_j^{s(2)} | \tau) \right] e_{s(1)} e_{s(2)}.$$  

(22)

The variables $z_i^s, s = 2, 3, \ldots, l$ are integrated in the region of the torus and $i, j, s(1) \leq s(2)$ means that if $s(1) = s(2)$, then we take $i < j$. The explicit form of $e_i$ is not important because the final results only depend on the inner products among $e_i$. In order that the function integrated in (22) is mathematical well defined on the torus, it is required that the integrated function is periodic with coordinates $z_i^s, s = 2, 3, \ldots, l$ around the non-contractible loops of the torus.

In order that the $\theta$ function in (22) be well defined, the matrix $\Lambda$ must be positive definite. So $p_i, i = 2, 3, \ldots, l$ must be positive even integers.

By applying the condition (8), we get one of those requirements,

$$\sum_j \Lambda_{ij} N_j = \begin{cases} \Phi, & \text{if } i = 1; \\ 0, & \text{otherwise,} \end{cases}$$  

(23)

where $N_j$ is the number of the particles in level $j$. The filling factor equals to $\nu = \frac{N_1}{\Phi} = e_1^* \cdot e_1^* = \Lambda_{1,1}^{-1}$, where $N_1$ is the number of the electrons, and we find from (23)

$$\nu = \frac{1}{p_1 - \frac{1}{p_2 - \frac{1}{\cdots - \frac{1}{p_l}}}}.$$  

(24)

It is also possible to construct other hierarchy states [4], and we will discuss it on the torus elsewhere [20]. Here we just remark that the filling factor for this case
where $p_1$ is an odd positive integer and $p_i$ with $i \neq 1$ is a positive even integer. Actually, the filling factors in the equations (24), the filling of charge conjugate state of the state with filling given by (24) (about conjugate state, see [21] and [20]), and (25) cover almost all kinds of fillings realized in the experiments with the filling having an odd denominator [22]. The filling observed in experiment quoted in [22], which does not belong to the above category, is $3_{11}$. The mathematical requirement for the well defined wave functions restricts the possible types of the filling factors of those kinds of hierarchical wave functions and this fact maybe can explain why (25), the filling factors in the equations (24) and the filling factors of its conjugate cover almost all fillings observed in the experiments.

Let us continue the discussion of the wave functions (22). Another requirement for the periodicity of the function (22) is

$$
\begin{align*}
e_s \cdot b &= \text{integer},
\phi_1 &= \pi(\Phi + 1) + 2\pi e_1 \cdot a + 2\pi \cdot \text{integer},
\phi_2 &= \pi(\Phi + 1) - 2\pi e_1 \cdot b + 2\pi \cdot \text{integer}.
\end{align*}
$$

By writing $a, b$ as $a = a^s e_s^*$, $b = b^s e_s^*$ in the bases of the lattice $\Lambda^*$ (the inverse lattice of $\Lambda$), (26) become

$$
\begin{align*}
a^1 &= \frac{\phi_1}{2\pi} + \frac{\Phi + 1}{2} + n_1,
b^1 &= -\frac{\phi_2}{2\pi} + \frac{\Phi + 1}{2} + n_2,
\end{align*}
$$

The difference between two solutions of $a, b$ lies on $n^s e_s^*$ with $n^s$ being integers. The two solutions of $a$ will give two orthogonal wave functions, if the difference of
two \(a\) does not lie on \(k^se_s\) with \(k^s\) being integers, the two solutions of \(b\) with the same \(a\) will give the same wave function, as it can be shown by using (14). So all wave functions of Laughlin-Jastrow type are described by the solutions

\[
a = \left[\frac{\phi_1}{2\pi} + \frac{\Phi + 1}{2}\right]e_1^* + \sum_s n^se_s^*,
\]

\[
b = \left[-\frac{\phi_2}{2\pi} + \frac{\Phi + 1}{2}\right]e_1^*,
\]

\[
n^se_s^* \subset \frac{\Lambda^*}{\Lambda}.
\]

\(\Lambda^*/\Lambda\) means the space \(n^se_s^*\) with \(n^s\) being integers and identifying \(a(1) = n^s(1)e_s^*\) with \(a(2) = n^s(2)e_s^*\) if \(a(1) - a(2) = k^se_s\) with \(k^s\) being integers. There are \(\text{det} \Lambda\) solutions to (28). So the degeneracy of the ground states is \(\text{det} \Lambda\), which actually is the denominator of the filling. It has been predicted in [6] by using Ginzburg-Landau-Chern-Simons theory of FQHE and in [7] by using Conformal Field theory. However they [6,7] did not give explicit hierarchical wave functions on the torus.

To consider the excitations of the states given by (22), we take the simplest case as an example, one quasiparticle at \(z\). Thus the wave functions (22) become

\[
F(z_i) = \int \prod_{s=2,i} dz_i^s d\bar{z}_i^s \theta\left[\frac{a}{b}\right]\left(\sum_{s=1}^lz_i^s e_s + zq|e, \tau\right)
\]

\[
\times \prod_{i,j,(1)\leq s(2)} \left[\theta_3(z_i^{(1)} - z_j^{(2)}|\tau)\right]^{e_{s(1)}e_{s(2)}}
\]

\[
\times \prod_{i,s} \left[\theta_3(z - z_i^s|\tau)\right]^{q^se_s},
\]

where \(q = q^se_s^*\) with \(q^s\) being integers. \(q\) will characterize the statistics and charge of the quasiparticle. The equation (23) now changes to

\[
\sum_j \Lambda_{ij}N_j + e_i q = \begin{cases} 
\Phi, & \text{if } i = 1; \\
0, & \text{otherwise}.
\end{cases}
\]
The equation (26), (27), (28) remain unchanged. It can be shown
\begin{align}
F(z + 1 | a, b) &= F(z | a, b) \exp(2\pi ia \cdot q + \pi i \Phi e_1^* \cdot q - \pi i q^2), \\
F(z + \tau q | a, b) &= F(z | a + q, b) \exp(-\pi i q \cdot e_1^* \Phi (2z + \tau) \\
&\quad - 2\pi i q \cdot b + \pi i q^2 - \pi i q \cdot e_1^* \Phi).
\end{align}
(31)

The charge of the quasiparticle is $e_1^* \cdot q$, and the statistics parameter of the quasiparticle $e^{i\theta}$ obtained by interchanging two identical excitations, is given by $\theta = \pi q^2$.

The normalized wave functions of (29) should contain a normalization constant which depends on the coordinate $z$. The normalized wave functions are
\begin{equation}
\Psi_n(z_i, z) = \Psi(z_i, z) \exp \left( - \frac{\Phi e_1^* \cdot q y^2}{\tau_2} \right).
\end{equation}
(32)

The similar discussion about the normalization in the presence of the quasiparticles on the plane and the sphere was discussed in [4,14,15]. This will be important for obtaining the hierarchical wave functions studied in [4], which has also rather clear physical meaning and will be discussed in [20].

It will not be surprising that the wave functions of the FQHE relate to the wave functions of Chern-Simons theory after so many works [1,2,3,4,5,7,9]. The functions, which are integrated by the coordinates of the quasiparticles to get the wave functions of the FQHE, are
\begin{equation}
\Psi_b(z_i^s) = \exp \left( - \frac{\pi \Phi \sum_i y_i^2}{\tau_2} \theta \left[ a \right] \sum_{s=1}^l \sum_i z_i^s e_s | e, \tau \right) \\
\quad \times \prod_{i,j,s(1) \leq s(2)} \left[ \theta_3(z_i^{s(1)} - z_j^{s(2)} | \tau) \right]^{e_{s(1)}^* e_{s(2)}}.
\end{equation}
(33)

They can be written as
\begin{equation}
\Psi_b(z_i^s) = \exp \left( - \frac{\pi (\sum_{i,s} y_i^s e_s)^2}{\tau_2} \right) \theta \left[ a \right] \sum_{s=1}^l \sum_i z_i^s e_s | e, \tau \right) \\
\quad \times \prod_{i,j,s(1) \leq s(2)} \left\{ \frac{1}{2} P(z_i^{s(1)} - z_j^{s(2)} | \tau) e_{s(1)}^* e_{s(2)} \right\} \cdot \text{unitary phase}.
\end{equation}
(34)

If take the magnetic potential as $A_1 = (-By + \frac{2\pi c_1}{\tau_2})\hat{x}$, $A_2 = \frac{2\pi c_2}{\tau_2}\hat{y}$, then (34)be-
comes

\[ \Psi_b(z^s_i) = \exp\left( \frac{\pi}{2\tau_2} [ce_1^* + \sum_i (\bar{z}^s_i - z^s_i)e_s] + \frac{\pi (\sum_i y_i^s e_s)^2}{\tau_2} \right) \]

\[ \cdot \prod_{i,j,s(1) \leq s(2)} \exp \left( \frac{1}{2} P(z^s_i(1) - z^s_j(2)|\tau) e_{s(1)} \cdot e_{s(2)} \right) \]  

\[ \cdot \theta \left[ a \right] \left( \sum_{s=1}^l \sum_i z^s_i e_s - ce_1^*e, \tau \right) \cdot \text{unitary phase}, \tag{35} \]

which also are invariant up to a phase under the large gauge transformation \( U_1 \) and change from one state to another under the large gauge transformation \( U_2 \) (for Laughlin state at simple filling, see the discussion before). The unitary phase in (34)and (35)depends on the coordinates \( z^s_i \). \( P(z) \) is the scalar propagator, satisfying

\[ \frac{1}{4} \nabla^2 P(z) = \bar{\partial} \partial P(z) = \pi(\delta^2(z) - \frac{1}{\tau_2}), \]

and is given by

\[ P(z) = \ln \left| \frac{\theta \left[ z \right]}{\theta \left[ \frac{z}{\lambda} \right]} \frac{2}{\pi} \right|^2 + \frac{\pi}{2\tau_2} (z - \bar{z})^2. \]

However it can be shown that wave functions (35)are the wave functions of the generalized Abelian Chern-Simons theory (up to a unitary phase which actually relates to singular gauge transformation) on the multidimensional \( R^l/\Lambda \) compact Abelian gauge group, where \( \Lambda \) is the integer lattice with bases \( e_i \) defined by \( e_i \cdot e_j = \Lambda \). The large component of Chern-Simons potential here is equal to \( ce_1^* \) [16]. The wave functions of Abelian Chern-Simons theory on the torus have been studied detailed in [19] and the discussion of the wave functions of the generalized Abelian Chern-Simons theory on the torus can be found in [16]. The lagrangian of the generalized Abelian Chern-Simons theory which we are interested in, is given by

\[ L = \int dv \left[ -\frac{1}{4\pi} \epsilon^{\mu \nu \lambda} A_\mu \cdot \partial_\nu A_\lambda + A_0 \cdot \left( \sum_{s,\lambda} e_s \delta^2(z - z^s_i) - \frac{1}{2\pi} \cdot B e^*_1 \right) \right], \tag{36} \]
where $A_{\mu} = A_{s,\mu}e_s$ and with the condition
\[
\int dv \left( \sum_{s,i} e_s \delta^2(z - z_i^s) - \frac{1}{2\pi} \cdot B e_1^* \right) = 0.
\] (37)

The equation (37) can be written as
\[
\sum_s N_s e_s - \Phi e_1^* = 0,
\] (38)
which is the same as the equation (23). The lagrangian (36) will give the right statistics for the electrons and the quasiparticles. To introduce an excitation, we simply add a term $A_0 \cdot q \delta^2(z - z_q)$, where $q = q^s e_s^*$ with $q^s$ being integers. Then we have
\[
\sum_s N_s e_s + q - \Phi e_1^* = 0,
\] (39)
which is the same as the equation (30).

This kind form of the generalized Abelian Chern-Simons action appears in dual form Ginzburg-Landau-Chern-Simons theory of FQHE [4]. The generalized Abelian Chern-Simons (36) characterizes the essential properties of the hierarchical FQHE, from which we can obtain the degeneracy of the ground state and the excitation spectrums, etc..

To discuss the modular property, let us introduce $\theta_1 = \left[ \frac{\phi_1}{2\pi} + \frac{\Phi + 1}{2} \right] e_1^*$, $\theta_2 = \left[ -\frac{\phi_2}{2\pi} + \frac{\Phi + 1}{2} \right] e_1^*$ (this notation is the same as [16]). If the full modular invariance (covariance) is required, then we need to choose $\theta_1 = \theta_2 = \frac{\epsilon_1^*}{2}$ [16,19]. We may also remark that if $\theta_1, \theta_2$ are zero, the wave functions of the FQHE will be transformed covariantly only under the subgroup of the modular transformation $\tau \rightarrow \frac{\alpha \tau + \beta}{\gamma \tau + \delta}$ which is generated by $\tau \rightarrow \tau + 2$, and $\tau \rightarrow -\frac{1}{\tau}$ (the corresponding coordinate transformation under the modular transformation is $z \rightarrow \frac{z}{\gamma \tau + \delta}$). Above results can be arrived by using the method developed, for example in [16,19].

* This also has been observed (for the case of the filling $\nu = \frac{1}{m}$) in [23].
In summary, we have obtained a kind of the hierarchical Laughlin-Jastrow wave functions on the torus. The degeneracy of the ground states is obtained. The relation of the wave functions with the one in Chern-Simons theory is revealed. We hope that the wave functions of FQHE on the surface with arbitrary genus can be obtained by using the corresponding results in Chern-Simons theory.

The author is grateful to Professor R. Iengo for many stimulating discussions and constant encouragements, and to Dr. Marco A.C. Kneipp for explaining his work [16].

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