A minimalistic pure spinor sigma-model in AdS

Andrei Mikhailov†

Instituto de Física Teórica, Universidade Estadual Paulista
R. Dr. Bento Teobaldo Ferraz 271, Bloco II – Barra Funda
CEP:01140-070 – São Paulo, Brasil

Abstract

The $b$-ghost of the pure spinor formalism in a general curved background is not holomorphic. For such theories, the construction of the string measure requires the knowledge of the action of diffeomorphisms on the BV phase space. We construct such an action for the pure spinor sigma-model in $AdS_5 \times S^5$. From the point of view of the BV formalism, this sigma-model belongs to the class of theories where the expansion of the Master Action in antifields terminates at the quadratic order. We show that it can be reduced to a simpler degenerate sigma-model, preserving the AdS symmetries. We construct the action of the algebra of worldsheet vector fields on the BV phase space of this minimalistic sigma-model, and explain how to lift it to the original model.

† on leave from Institute for Theoretical and Experimental Physics, ul. Bol. Cheremushkinskaya, 25, Moscow 117259, Russia
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1 Introduction

The $b$-ghost of the pure spinor formalism in a general curved background is only holomorphic up to a $Q$-exact expression [1]. The construction of the string measure for such theories was suggested in [2, 3]. It requires the knowledge of the action of the group of worldsheet diffeomorphisms on the BV phase space. For a vector field $\xi$ on the worldsheet (= infinitesimal diffeomorphism) let $\Phi_\xi$ be the BV Hamiltonian generating the action of $\xi$ on the BV phase space. Then, the string measure is, schematically:

$$\exp (S_{BV} + \sigma + \Phi_F)$$

where:

- $S_{BV}$ is the worldsheet Master Action
- $\sigma$ is the generating function of the variations of the Lagrangian submanifold (for the standard choice of the family, this is just the usual $\int \mu_zb_{zz} + \mu_{\bar{z}}b_{\bar{z}z}$)
- $F$ is the curvature of the connection on the equivalence class of worldsheet theories, considered as a principal bundle over the space of theories modulo diffeomorphisms

It is not completely trivial to construct $\Phi_\xi$ for the pure spinor superstring in AdS. One of the complications is the somewhat unusual form of the pure spinor part of the action. Schematically:

$$S_{\lambda w} = \int w_L (\partial_- + A_-)\lambda_L + w_R (\partial_+ + A_+)\lambda_R + Sw_L \lambda_L w_R - \lambda_R$$

where $S$ is a linear combination of Ramond-Ramond field strengths. Notice that the conjugate momenta $w_L$ and $w_R$ only enter through their $(1,0)$ and $(0,1)$ component, respectively. We can try to integrate out $w$, ending up with a “standard” kinetic term for ghosts:

$$\frac{\partial_- \lambda_L \partial_+ \lambda_R}{S\lambda_L \lambda_R}$$

Notice that $S$ landed in the denominator. It would seem that the theory depends quite irregularly on the Ramond-Ramond field, but this is not true. All physics sits at $\lambda = 0$, and the $w\lambda w\lambda$ term is in some sense subleading.
In this paper we will show, closely following \[4, 5\], that the pure spinor terms (2) can actually be removed by reduction to a smaller BV phase space, keeping intact all the symmetries of $AdS_5 \times S^5$. The resulting action is degenerate, and therefore can not be immediately used for quantization. On the other hand, it is simpler than the original action. In particular, the action of worldsheet diffeomorphisms in this reduced BV phase space is rather transparent, although the explicit expression Eq. (107) is somewhat involved. We then explain how to lift this action to an action on some quantizable theory which is basically the same as the original pure spinor sigma-model of \[10\].

For the case of flat spacetime, the formal expressions are somewhat more complicated. The construction of the action of diffeomorphisms is a work in progress with Renann Lipinski \[6\].

**Plan of the paper** We begin in Section 2 with the general discussion of the reduction procedure when a BV Master Action is a quadratic-linear functional of antifields. In Section 3 we apply this to the case of pure spinor superstring in $AdS_5 \times S^5$. In Sections 4 we construct the action of diffeomorphisms in the minimalistic sigma-model. Then in Section 5 we construct the action of diffeomorphisms on the BV phase space of the non-degenerate theory, which is essentially equivalent (quasiisomorphic) to the original sigma-model. Sections 6 and 7 contain summary and generalizations, and Section 8 open problems.

### 2 Master Actions quadratic-linear in antifields

Suppose that the BV phase space is an odd cotangent bundle, \textit{i.e.} is of the form $\Pi T^*N$ for some manifold $N$ (the “field space”), and the Master Action is of the form:

$$S_{BV} = S_{cl}(\phi) + Q^a(\phi)\phi_a^* + \frac{1}{2} \tilde{\pi}^{ab}(\phi)\phi_a^*\phi_b^* \quad (4)$$

The term linear in antifields, $Q^a(\phi)\phi_a^*$, defines an odd vector field $Q$ on $N$:

$$Q = Q^a(\phi) \frac{\partial}{\partial \phi^a} \quad (5)$$
Similarly, $\pi^{ab}$ defines an even bivector $\pi$ on $N$:

$$\pi = \pi^{ab}(\phi) \frac{\partial}{\partial \phi^a} \wedge \frac{\partial}{\partial \phi^b}$$  \hspace{1cm} (6)$$

The Master Equation $\{ S_{BV}, S_{BV} \} = 0$ implies:

$$Q^2 + [\pi, S_{cl}] = 0$$  \hspace{1cm} (7)

$$[Q, \pi] = 0$$  \hspace{1cm} (8)

$$[\pi, \pi] = 0$$  \hspace{1cm} (9)

where $[\cdot, \cdot]$ is the Schouten bracket of polyvector fields:

$$Q^2 = \frac{1}{2} [Q, Q]^a \frac{\partial}{\partial \phi^a}$$  \hspace{1cm} (10)

where $[Q, Q]^a = 2Q^b \frac{\partial}{\partial \phi^b} Q^a$  \hspace{1cm} (11)

$$[Q, \pi] = [Q, \pi]^{ab} \frac{\partial}{\partial \phi^a} \wedge \frac{\partial}{\partial \phi^b}$$  \hspace{1cm} (12)

where $[Q, \pi]^{ab} = Q^c \frac{\partial}{\partial \phi^c} \pi^{ab} - 2\pi^{[a|c]} \frac{\partial}{\partial \phi^c} Q^{b]}$  \hspace{1cm} (13)

$$[\pi, \pi] = [\pi, \pi]^{abc} \frac{\partial}{\partial \phi^a} \wedge \frac{\partial}{\partial \phi^b} \wedge \frac{\partial}{\partial \phi^c}$$  \hspace{1cm} (14)

where $[\pi, \pi]^{abc} = 4\pi^{[a|d]} \frac{\partial}{\partial \phi^d} \pi^{[bc]}$  \hspace{1cm} (15)

Eq. (7) says that the BRST operator $Q$ is only nilpotent on-shell [7].

We will show that under some conditions, this theory can be reduced to a simpler theory which has BRST operator nilpotent off-shell (and therefore its Master Action has no quadratic terms $\phi^* \phi^*$).

The case when $\pi$ is non-degenerate Let us first consider the case when the Poisson bivector $\pi^{ab}$ is nondegenerate. Eq. (8) implies that an odd function $\psi \in \text{Fun}(N)$ locally exists, such that $Q = [\pi, \psi]$. Suppose that $\psi$ is also defined globally. Let us consider the canonical transformation of
Darboux coordinates generated by $\psi$:

\[(\phi, \phi^*) \to (\tilde{\phi}, \tilde{\phi}^*)\]

\[
\phi^a = \tilde{\phi}^a
\]

\[
\phi_a^* = \tilde{\phi}_a^* + \frac{\partial}{\partial \tilde{\phi}^a} \psi(\tilde{\phi})
\]

More geometrically: $\tilde{\phi}$ and $\tilde{\phi}^*$ (functions on $\Pi^*T^*N$) are pullbacks of $\phi$ and $\phi^*$ by the flux of the Hamiltonian vector field $\{\psi, -\}$ by the time $1$. (The flux integrates to Eqs. (16) because $\psi$ only depends on $\phi$, and therefore the velocity of $\phi^*$ is $\phi^*$-independent.)

In the new coordinates:

\[
S = \tilde{S}_{\text{cl}} + \frac{1}{2} \pi^{ab}(\phi) \phi^a \phi^b
\]

where \(\tilde{S}_{\text{cl}} = S_{\text{cl}} + \frac{1}{2} \pi^{ab} \partial_a \psi \partial_b \psi\)

The $\phi^*$-linear term is gone! The Master Equation implies that $[\tilde{S}_{\text{cl}}, \pi] = 0$. Since we assumed that $\pi$ is nondegenerate, this implies:

\[
\tilde{S}_{\text{cl}} = \text{const}
\]

The case of degenerate $\pi$ We are actually interested in the case when $\pi$ is degenerate. Let $\mathcal{P} \subset TN$ be the distribution tangent to symplectic leaves of $\pi$:

\[
\mathcal{P} = \im \pi \subset TN
\]

This distribution is integrable because $\pi$ satisfies the Jacobi identity. We also assume that $Q$ is transverse to $\mathcal{P}$:

\[
Q \notin \mathcal{P}
\]

Let us also consider the distribution $\mathcal{P} + Q$ which is generated by elements of $\mathcal{P}$ and by $Q$. Eqs. (16) and (17) imply that $\mathcal{P} + Q$ is also integrable. Let us assume the existence of a 2-form $\omega$ on each integrable surface of $\mathcal{P} + Q$.

\[^1\text{this} \omega \text{ is even}; \text{it should not be confused with the odd symplectic form of } \Pi^*T^*N\]

\[^2\text{it is enough to define} \omega \text{ on each integrable surface of } \mathcal{P} + Q; \text{it does not have to be defined on the whole } N\]

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and a function $\psi \in \text{Fun}(N)$ which satisfy:

\begin{align*}
\pi_\omega \pi &= \pi \quad (22) \\
\omega_\pi \omega &= \omega \quad (23) \\
d\omega|_{P+Q} &= 0 \quad (24) \\
(\iota_Q \omega - d\psi)|_P &= 0 \quad (25)
\end{align*}

Existence of $\psi$ satisfying Eq. (25) locally follows from Eqs. (22) and (24), because they imply $d(\iota_Q \omega)|_P = 0$. But we also require this $\psi$ to be a globally well-defined function on $N$. Contracting $\iota_Q \omega - d\psi$ with $\pi_\omega$ we find that:

$$Q - [\pi, \psi] \in \ker (\omega|_{P+Q}) \quad (26)$$

Let us define the new odd vector field:

$$\tilde{Q} = Q - [\pi, \psi] \quad (27)$$

Eq. (24) implies that $\ker (\omega|_{P+Q})$ is an integrable distribution inside an integral surface of $P + Q$. Therefore Eq. (26) implies that $\tilde{Q}^2$ is proportional to $\tilde{Q}$, i.e. exists a function $\zeta$ such that: $\tilde{Q}^2 = \zeta \tilde{Q}$. In fact $\zeta = 0$, since $\tilde{Q}^2 \in P$ and $\tilde{Q} \not\in P$. We conclude:

$$\tilde{Q}^2 = 0 \quad (28)$$

Let us consider the canonical transformation (16) of Darboux coordinates generated by $\psi$. With these new Darboux coordinates:

$$S_{BV} = S_{cl} - \frac{1}{2} \omega(Q, Q) + (Q - [\pi, \psi])^a \tilde{\phi}_a^* + \frac{1}{2} \pi^{ab} \tilde{\phi}_a^* \tilde{\phi}_b^* \quad (29)$$

Notice that the new “classical action”:

$$\tilde{S}_{cl} = S_{cl} - \frac{1}{2} \omega(Q, Q) \quad (30)$$

is automatically constant on symplectic leaves of $\pi$. Also, it follows that $\tilde{Q}$ consistently defines an odd nilpotent vector field on the moduli space of symplectic leaves of $\pi$. These facts follow from $\{S_{BV}, S_{BV}\} = 0$. To summarize:

$$S_{BV} = \tilde{S}_{BV} + \frac{1}{2} \pi^{ab}(\phi) \tilde{\phi}_a^* \tilde{\phi}_b^* \quad (31)$$

where

$$\tilde{S}_{BV} = \tilde{S}_{cl}(\chi) + \tilde{Q}(\chi)^m \chi_m^* \quad (32)$$

where $\chi$ is coordinates on the space of symplectic leaves of $\pi$. We therefore constructed a new, simpler theory, on the space of symplectic leaves of $\pi$.

This theory can be interpreted as the result of integrating out $w^*$. 
Oversimplified example  We will now illustrate the relation by a toy sigma-model (we will actually run the procedure “in reverse”). Let $\Sigma$ be a two-dimensional worldsheet. Let us start with:

$$S_{BV} = S_{cl} + \int_{\Sigma} \lambda \theta^*$$  \hspace{1cm} (33)

where $S_{cl}$ does not not depend neither on the fermionic field $\theta^a$ nor on the bosonic field $\lambda^a$. (It depends on some other fields $\phi^\mu$.) This action is highly degenerate; the path integral $\int [d\lambda][d\theta][d\phi] e^{S_{cl}(\phi)}$ is undefined (infinity from integrating over $\lambda$ times zero from integrating over $\theta$). To regularize $\infty \times 0$, let us introduce a new field-antifield pair $w, w^*$, where $w$ is a bosonic 1-form on the worldsheet and $w^*$ is a fermionic 1-form on the worldsheet:

$$w = w_+ dz + w_- d\bar{z}$$ \hspace{1cm} (34)

$$w^* = w^*_+ dz + w^*_- d\bar{z}$$ \hspace{1cm} (35)

$$\omega_{BV} = \int_{\Sigma} dw^* \wedge dw$$ \hspace{1cm} (36)

(where $d$ is the field space differential, not the worldsheet differential). Let us add $(w^*)^2$ to the BV action:

$$S_{BV} = S_{cl} + \int_{\Sigma} \lambda \theta^* + \int \omega^* \wedge w^*$$ \hspace{1cm} (37)

(Notice that this $\int w^* \wedge w^*$ does not involve the worldsheet metric.) This corresponds to:

$$\omega = \pi^{-1} = \int_{\Sigma} dw \wedge dw$$ \hspace{1cm} (38)

(again, $d$ is the field space differential, not the worldsheet differential). In this case $P$ is the subspace of the tangent space generated by $\frac{\partial}{\partial w}$, and $Q$ is generated by $\lambda \frac{\partial}{\partial \theta}$. Then, shift the Lagrangian submanifold by a gauge fermion:

$$\Psi = \int_{\Sigma} w \wedge d\theta$$ \hspace{1cm} (39)

This results in the new classical action:

$$S_{cl}^{new} = S_{cl} + \int_{\Sigma} w \wedge d\lambda + \int_{\Sigma} d\theta \wedge d\theta$$ \hspace{1cm} (40)
This is, still, not a quantizable action (the kinetic term for $\theta$ is a total derivative). One particular way of choosing a Lagrangian submanifold leading to quantizable action is to treat $w_+$ and $w_-$ asymmetrically (pick a worldsheet complex structure), see section on A-model in AKSZ\cite{8} and Section 5.3 of this paper. This requires more than one flavour of $w$.

3 Pure spinor superstring in $AdS_5 \times S^5$

3.1 Notations

We follow the notations in \cite{9}. The superconformal algebra $g = \mathfrak{psu}(2,2|4)$ has $\mathbb{Z}_4$-grading:

$$g = g_0 + g_1 + g_2 + g_3$$

Bars over subindices are to remind that they are mod 4. Geometrically, $g_2$ can be identified with the tangent space to the bosonic $AdS_5 \times S^5$, which is the direct sum of the tangent space to $AdS_5$ and the tangent space to $S^5$:

$$T(AdS_5 \times S^5) = T(AdS_5) \oplus T(S^5)$$

Therefore elements of $g_2$ are vectors from this tangent space. We can also consider the tangent space to the full superspace $M$:

$$M = \text{super}(AdS_5 \times S^5) = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

$$T\left(\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}\right) = g_1 \oplus g_2 \oplus g_3$$

— this is a direct sum of three vector bundles. We parametrize a point in $M$ by $g \in PSU(2,2|4)$ modulo the equivalence relation:

$$g \simeq hg \quad \text{for all} \quad h \in SO(1,4) \times SO(5)$$

We are identifying representations of $g_0 = \text{Lie}(SO(1,4) \times SO(5))$, such as $g_1, g_2, g_3$, with the corresponding vector bundles over the coset space (43).

In fact, the worldsheet field $\lambda_L$ takes values in the fibers of $g_3$ and $\lambda_R$ takes values in the fibers of $g_1$. The pure spinor conditions define the cones $C_L$ and $C_R$: $C_L, C_R$
Here \( \{\ldots\} \) denotes the anticommutator (the Lie superalgebra operation) of elements of \( \mathfrak{g} \). It should be be confused with neither the odd Poisson bracket, nor the even Poisson bracket corresponding to \( \pi^{ab} \) of Section 2. Again, we identify \( C_L \) and \( C_R \) as bundles over super-AdS. (They are not vector bundles, because their fibers are cones and not linear spaces.) We will denote:

\[
PS \text{ AdS} \quad \quad PS \text{ AdS}_5 \times S^5 = \frac{C_L \times C_R \times PSU(2,2|4)}{SO(1,4) \times SO(5)}
\]

where the prefix PS on the LHS stands for “Pure spinors” (and on the RHS for “Projective” and “Special”).

In Appendix [A] we construct \( PSU(2,2|4) \)-invariant surjective maps of bundles (“projectors”):

\[
P_{31} : (\mathfrak{g}_3 \times C_L) \to TC_L \\
P_{13} : (\mathfrak{g}_1 \times C_R) \to TC_R
\]

They are rational functions of \( \lambda_L \) and \( \lambda_R \).

### 3.2 Standard action

The action of the AdS sigma-model has the following form [10]:

\[
S_0 = \int dz \, d\bar{z} \, \text{Str} \left( \frac{1}{2} J_{2+}J_{2-} + \frac{3}{4} J_{1+}J_{3-} + \frac{1}{4} J_{3+}J_{1-} + w_{1+}D_{0-}\lambda_3 + w_{3-}D_{0+}\lambda_1 - N_{0+}N_{0-} \right)
\]

where \( J_n \) are the \( \mathfrak{g}_n \)-components of \( J = -dgg^{-1} = J_+ dz + J_- d\bar{z} \). We write \( \lambda_3 \) instead of \( \lambda_L \) and \( \lambda_1 \) instead of \( \lambda_R \), just to highlight the \( \mathbb{Z}_4 \)-grading. (And also because neither \( \lambda_L \) is strictly speaking left-moving, nor is \( \lambda_R \) right-moving.) The covariant derivative \( D_{0\pm} \) is defined as follows:

\[
D_{0\pm} = \partial_\pm + [J_{0\pm}, \cdot]
\]
Since $\lambda_3$ and $\lambda_1$ both satisfy the pure spinor constraints, the corresponding conjugate momenta are defined up to “gauge transformations”:

\begin{align}
\delta_2 w_1^+ &= [\nu_2^+, \lambda_3] \\
\delta_3 w_3^- &= [\nu_2^-, \lambda_1]
\end{align}

(53) (54)

where $\nu_2$ and $u_2$ are arbitrary sections of the pullback to the worldsheet of $g_2$. The BRST transformations are defined up to gauge transformations corresponding to the equivalence relation (45). It is possible to fix this ambiguity so that:

\begin{align}
Q\lambda_L &= Q\lambda_R = 0 \quad & (55) \\
Qg &= (\lambda_L + \lambda_R)g \\
Qw_1^+ &= -J_1^+, \quad Qw_3^- = -J_3^-
\end{align}

(56) (57)

The first line in Eq. (51) is by itself not BRST invariant. Modulo total derivatives, its BRST variation is:

\[
Q \int d\tau d\sigma \text{Str}\left( \frac{1}{2} J_{2^+} J_{2^-} + \frac{3}{4} J_{1^+} J_{3^-} + \frac{1}{4} J_{3^+} J_{1^-} \right) = \\
= \int d\tau d\sigma \text{Str}\left( -D_0^+ \lambda_1 J_{3^-} - D_0^- \lambda_3 J_{1^+} \right)
\]

(58)

This cancels with the BRST variation of the second line in Eq. (51).

### 3.3 New action

On the other hand, we observe that:

\[
Q \text{Str} (J_1^+ P_{31} J_{3^-}) = \text{Str} (-D_0^+ \lambda_1 J_{3^-} - D_0^- \lambda_3 J_{1^+})
\]

(59)

Notice that the projector drops out on the RHS because $\partial \lambda$ is automatically tangent to the cone. Comparing this to (58) we see that the following expression:

\[
S_0' = \int d\tau d\sigma \text{Str}\left( \frac{1}{2} J_{2^+} J_{2^-} + \frac{3}{4} J_{1^+} J_{3^-} + \frac{1}{4} J_{3^+} J_{1^-} - J_{1^+} P_{31} J_{3^-} \right)
\]

(60)

is BRST invariant. It does not contain neither derivatives of pure spinors, nor their conjugate momenta.
3.4 The $b$-ghost

We define:

$$b_{++} = \frac{\text{STr}((\{J_{3+}, \lambda_3\} - \{J_{1+}, \lambda_1\})J_{2+})}{\text{STr}(\lambda_3\lambda_1)} = \frac{\text{Tr}((\{J_{3+}, \lambda_3\} - \{J_{1+}, \lambda_1\})J_{2+})}{\text{STr}(\lambda_3\lambda_1)}$$

(61)

$$b_{--} = \text{same but with } + \text{ replaced with } -$$

(62)

(See Appendix A for notations. We use the fact that $\text{Str}(A_2B_2) = \text{Str}(A_2B_2\Sigma) = \text{Tr}(A_2B_2)$. ) These expressions satisfy (Appendix B):

$$Qb_{++} = T_{++} \text{ and } Qb_{--} = T_{--}$$

(63)

where

$$T_{++} = \text{Str} \left( \frac{1}{2} J_{2+}J_{2+} + J_{1+}(1 - P_{31})J_{3+} \right)$$

$$T_{--} = \text{Str} \left( \frac{1}{2} J_{2-}J_{2-} + J_{1-}(1 - P_{31})J_{3-} \right)$$

Notice that:

$$S_0' = S_0'' + QB$$

(64)

where

$$B = \int d\tau d\sigma \frac{\text{Tr}((\{J_{3+}, \lambda_3\} - \{J_{1+}, \lambda_1\})J_{2-} + (+ \leftrightarrow -))}{\text{STr}(\lambda_3\lambda_1)}$$

(65)

$$S_0'' = \int \text{STr}(J_1 \wedge (1 - P_{31})J_3 - J_1 \wedge P_{31}J_3)$$

(66)

and $S_0''$ is diffeomorphism-invariant (and therefore degenerate!). The BRST invariance of $S_0''$ can be verified explicitly as follows:

$$QS_0'' = \int \text{STr} \left( [\lambda_3, J_2] \wedge J_3 - [\lambda_1, J_2] \wedge J_1 - D_0\lambda_1 \wedge J_3 + D_0\lambda_3 \wedge J_1 \right) =$$

$$= \int d \text{STr}(\lambda_3J_1 - \lambda_1J_3) = 0$$

(67)

3.5 Gauge fixing $SO(1, 4) \times SO(5)$

Consider the action of the BRST operator given by Eq (84) on $g$. It is nilpotent only up to the $g_{0}$-gauge transformation by $\{\lambda_3, \lambda_1\}$. We have so
far worked on the factorspace by gauge transformations. This means that we think of the group element $g$ and pure spinors $\lambda$ as defined only modulo the gauge transformation:

$$(g, \lambda) \simeq (hg, h\lambda h^{-1}) \quad (68)$$

It turns out that the action of these gauge transformations on the BV phase space is somewhat nontrivial, see Section 5.4. We will now just fix the gauge, postponing the discussion of gauge transformations to Section 5.4. Let us parametrize the group element $g \in PSU(2,2|4)$ by $u, x, \theta$:

$$g = e^u e^{x+\theta} \quad (69)$$

where $u \in g_0$, $x \in g_2$, and $\theta \in g_3 + g_1$, and impose the following gauge fixing condition:

$$u = 0 \quad (70)$$

Since Eq. (70) does not contain derivatives, this gauge is “ghostless”, the Faddeev-Popov procedure is not needed. In this gauge fixed formalism, the BRST operator includes the gauge fixing term (cp. Eqs. (55), (56), (57)):

$$A_0 Qg = (\lambda_3 + \lambda_1 + A_0)g$$

$$Q\lambda_3 = [A_0, \lambda_3], \quad Q\lambda_1 = [A_0, \lambda_1]$$

$$Qw_{1+} = -J_{1+} + [A_0, w_{1+}], \quad Qw_{3-} = -J_{3-} + [A_0, w_{3-}]$$

where $A_0 \in g_0$ is some function of $\theta, \lambda$ and $x$, defined by Eqs. (71) and (70); schematically $A_0 = \{\theta_L, \lambda_1\} + \{\theta_R, \lambda_3\} + \ldots$. This $A_0$ is usually called “the compensating gauge transformation”. It automatically satisfies:

$$QA_0 = -\{\lambda_3, \lambda_1\} + \frac{1}{2}[A_0, A_0] \quad (74)$$

Gauge fixing is only possible locally in $AdS_5 \times S^5$. In order for our constructions to work globally, we will cover $AdS_5 \times S^5$ with patches and gauge-fix over each patch. Then we have to glue overlapping patches. We will explain how to do this in Section 5.4.

### 3.6 In BV language

We will now show that the difference between the original action and the action (60) can be interpreted in the BV formalism as a particular case of the construction outlined in Section 2.
The BRST symmetry of the pure spinor superstring in $AdS_5 \times S^5$ is nilpotent only on-shell. More precisely, the only deviation from the nilpotence arises when we act on the conjugate momenta of the pure spinors:

$$Q^2 w_{1+} = \frac{\delta S_0}{\delta w_{3-}}$$  \hfill (75)
$$Q^2 w_{3-} = \frac{\delta S_0}{\delta w_{1+}}$$  \hfill (76)

(while the action of $Q^2$ on the matter fields is zero even off-shell). This means that the BV Master Action contains a term quadratic in the antifields:

$$S_{BV} = S_0 + \int (QZ^i)Z_i^* + \int (QL)\lambda^* + \int (Qw)w^* + \int \text{Str} \left( w_{1+}^* w_{3-}^* \right)$$  \hfill (77)

In this formula $Z$ and $Z^*$ stand for matter fields $(x$ and $\theta)$ and their antifields, and $S_0$ is given by Eq. (51). The matter fields $Z$ are essentially $x$ and $\theta$ where:

$$J = -dgg^{-1} \quad \text{with} \quad g = e^{x+\theta} , \quad x \in \mathfrak{g}_2, \quad \theta \in \mathfrak{g}_3 \oplus \mathfrak{g}_1 :$$

$$Z = x \quad \text{and} \quad \theta$$  \hfill (78)

Their BRST transformation $QZ^i$ is read from Eq. (71). We observe that the action is of the same type as described in Section 2. The Poisson bivector is:

$$\pi = \int \text{Str} \left( \frac{\partial}{\partial w_{1+}} \wedge \frac{\partial}{\partial w_{3-}} \right)$$  \hfill (79)

The 2-form $\omega$ discussed in Section 2 can be chosen as follows:

$$\omega = \int \text{Str} (dw_{1+} \wedge P_{31} dw_{3-})$$  \hfill (80)

The projector $P_{31}$ is needed to make $\omega$ invariant with respect to the gauge transformations (53) and (54). We take the following generating function $\psi$ satisfying Eq. (25):

$$\psi = \int \text{Str} (w_{1+}P_{31}J_{3-} + w_{3-}P_{13}J_{1+} + w_{1+}[A_0, w_{3-}])$$  \hfill (81)

The new “classical action” $\tilde{S}_{cl}$ is given by Eq (60). (We will provide more details for a slightly more general calculation in Section 3.) It is, indeed,
constant along the symplectic leaves of \( \pi \), as the fields \( w_\pm \) are not present in this new Lagrangian at all. The new BV action is:

\[
\tilde{S}_{BV} = \int d\tau \, d\sigma \, \text{Str} \left( \frac{1}{2} J_{2+} J_{2-} + \frac{3}{4} J_{1+} J_{3-} + \frac{1}{4} J_{3+} J_{1-} - J_{1+} P_{31} J_{3-} + \sum_{Z \in \{x, \theta, \lambda\}} (QZ) Z^* \right)
\]  

(82)

where \( Z^i \) runs over \( \theta, x, \lambda \) and the action of \( Q \) on \( Z^i \) is the same as it was in the original \( \sigma \)-model. The new BV phase space is smaller, it only contains \( \theta, x, \lambda, \theta^*, x^*, \lambda^* \). The BRST operator is now nilpotent off-shell; the dependence of the BV action on the antifields is linear. The fields \( \lambda_{L|R} \) enter only through their combination invariant under local rescalings (they enter through \( P_{31} \)). This in particular implies that the BRST symmetry \( Q \) is now a local symmetry.

Of course, the new action (62) is degenerate.

4 Action of diffeomorphisms

4.1 Formulation of the problem

Let \( L_*^2 \) be the BV Hamiltonian generating the left shift by elements of \( g_2 \); if \( f \) is any function of \( g \), then:

\[
\{ \text{Str} (A_2 L_*^2) , f \}_{BV} (g) = \left. \frac{d}{dt} \right|_{t=0} f \left( e^{tA_2} g \right)
\]  

(83)

The \( L_*^0, L_*^1 \) and \( L_*^3 \) are defined similarly. In particular:

\[
\{ \tilde{S}_{BV}, \_ \} = \int \text{Str} (\lambda_3 L_*^1 + \lambda_1 L_*^3)
\]  

(84)

With these notations, when \( X \) and \( Y \) are two even elements of \( g \),

\[
\left\{ \int \text{Str}(X L^*), \int \text{Str}(Y L^*) \right\}_{BV} = - \int \text{Str}([X,Y] L^*)
\]  

(85)

(Even elements are generators of \( g_2 \) and \( g_0 \), and also the generators of \( g_3 \) and \( g_1 \) multiplied by a Grassmann odd parameter.)
The infinitesimal action of diffeomorphisms is generated by the following BV Hamiltonian $V_\xi$:

\[
V_\xi = \int \text{Str} \left( (\iota_\xi D_0 \lambda_3)^* \lambda_1^* + (\iota_\xi D_0 \lambda_1) \lambda_3^* - (\iota_\xi J_3) L_1^* - (\iota_\xi J_1) L_3^* - (\iota_\xi J_2) L_2^* \right)
\]

where $D_0 \lambda = d\lambda + [J_0, \lambda]$  \hspace{1cm} (86)

In this section we will construct $\Phi_\xi$ such that:

\[
V_\xi = \{ \tilde{S}_{\text{BV}}, \Phi_\xi \}_{\text{BV}}
\]

It is very easy to construct such $\Phi_\xi$ if we don’t care about the global symmetries of $\text{AdS}_5 \times S^5$. (Something like $\Phi_\xi = \frac{\delta}{\delta \lambda} V_\xi$.) But we will construct a $\Phi_\xi$ invariant under the supersymmetries of $\text{AdS}_5 \times S^5$, i.e. invariant under the right shifts of $g$. We believe that such an invariant construction has better chance of satisfying the equivariance conditions of [2, 3] at the quantum level, because supersymmetries restrict quantum corrections. In particular, the equivariance condition must require that the $\Phi_\xi$ correspond, in some sense, to a primary operator.

Comment on gauge transformations  \hspace{1cm} In this Section we discuss vector fields on the factorspace $PS \text{ AdS}$ defined Eq. (48). They are the same as $SO(1, 4) \times SO(5)$-invariant vector fields on $\mathcal{C}_L \times \mathcal{C}_R \times PSU(2, 2|4)$ modulo $SO(1, 4) \times SO(5)$-invariant vertical vector fields. All the formulas here are modulo vertical $SO(1, 4) \times SO(5)$-invariant vector fields.

4.2 Subspaces associated to a pair of pure spinors  \hspace{1cm} We use the notations of Section A.5. For $X_3 \in [g_{2L}, \lambda_1]$ and $X_1 \in [g_{2R}, \lambda_3]$, let $T_2(X_1 + X_2)$ denote the map:

\[
T_2 : [\lambda_1, g_{2L}] \oplus [\lambda_3, g_{2R}] \rightarrow g_{2L} \oplus g_{2R}
\]

\[
T_2([\lambda_1, v_{2L}] + [\lambda_3, v_{2R}]) = v_{2L} + v_{2R}
\]

(This is a direct sum of two completely independent linear maps.) For a pair notations Sec A
$I_3 \oplus I_1 \in T^\perp C_R \oplus T^\perp C_L$ we decompose:

$$I_3 \oplus I_1 = (I_3^{\text{split}} \oplus I_1^{\text{split}}) + (I_3^{\ker} \oplus I_1^{\ker}) + (I_3^{\text{coker}} \oplus I_1^{\text{coker}})$$

(91)

where $I_3^{\text{split}} \oplus I_1^{\text{split}} \in [\lambda_1, g_{2L}] \oplus [\lambda_3, g_{2R}]$  

(92)

$I_3^{\ker} \oplus I_1^{\ker} \in \ker \left[ T^\perp C_R \oplus T^\perp C_L \xrightarrow{(+, (\lambda_3, -) \oplus (\lambda_1, -))} g_2 \right]$  

(93)

$I_3^{\text{coker}} \oplus I_1^{\text{coker}} \in \text{coker} \left[ g_2 \{\lambda_3 \mapsto (\lambda_1, -) \: T^\perp C_R \oplus T^\perp C_L \} \right]$  

(94)

where we must use a special representative of the cokernel:

$$I_3^{\ker} = \frac{1}{2} \text{Tr}(\lambda_3 I_3 - \lambda_1 I_1) \left[ \lambda_3, [\lambda_1, \lambda_1]_{\text{STL}} \right]$$

(95)

$$I_1^{\ker} = \frac{1}{2} \text{Tr}(\lambda_3 I_3 - \lambda_1 I_1) \left[ \lambda_3, [\lambda_1, \lambda_1]_{\text{STL}} \right]$$

(96)

$$I_3^{\text{coker}} = \frac{1}{2} \text{Tr}(\lambda_3 I_3 + \lambda_1 I_1) \left[ \lambda_1, [\lambda_3, \lambda_1]_{\text{STL}} \right]$$

(97)

$$I_1^{\text{coker}} = -\frac{1}{2} \text{Tr}(\lambda_3 I_3 + \lambda_1 I_1) \left[ \lambda_3, [\lambda_1, \lambda_1]_{\text{STL}} \right]$$

(98)

Similarly, any $I_2 \in g_2$ (assumed to be both TL and STL) can be decomposed:

$$I_2 = I_2^L + I_2^R + I_2^{\ker} + I_2^{\text{coker}}$$

(99)

where $I_2^L \in g_{2L}$  

(100)

$I_2^R \in g_{2R}$  

(101)

$I_2^{\ker} \in \ker \left[ g_2 \{\lambda_3 \mapsto (\lambda_1, -) \: T^\perp C_R \oplus T^\perp C_L \} \right]$  

(102)

$I_2^{\text{coker}} \in \text{coker} \left[ T^\perp C_R \oplus T^\perp C_L \xrightarrow{(+, (\lambda_3, -) \oplus (\lambda_1, -))} g_2 \right]$  

(103)

Explicitly:

$$I_2^{\ker} = \frac{\text{Tr}(I_2[\lambda_3, \lambda_1])}{\text{Tr}([\lambda_3, \lambda_1]_{\text{STL}}^2)} \left[ \lambda_3, \lambda_1 \right]_{\text{TL}}$$

(104)

$$I_2^{\text{coker}} = \frac{\text{Tr}(I_2[\lambda_3, \lambda_1])}{\text{Tr}([\lambda_3, \lambda_1]_{\text{STL}}^2)} \left[ \lambda_3, \lambda_1 \right]_{\text{STL}}$$

(105)

---

3For example, $I_3^{\text{split}}$ denotes the component of $I_3$ which belongs to $[\lambda_1, g_{2L}]$; the label “split” is because we could not invent any better notation.
4.3 Construction of $\Phi_\xi$

The generating function $V_\xi$ of the infinitesimal worldsheet diffeomorphisms (= vector fields) $\xi = \xi^r \partial_r + \xi^\sigma \partial_\sigma$, given by Eq. (86), is BV-exact:

$$V_\xi = \{ \tilde{S}_{\text{BV}}, \Phi_\xi \}$$ (106)

$$\Phi_\xi = - \int \text{Str} \left( (P_{31} \iota_\xi J_3) \lambda_3^* + (P_{13} \iota_\xi J_1) \lambda_3^* + \right.$$ (107)

$$\left. + \left( T_2 \left( \iota_\xi j_3^{\text{split}} + \iota_\xi j_1^{\text{split}} \right) + A[\lambda_3, \lambda_1]\right) L_2^* + \right.$$ (116)

$$+ B \left( [\lambda_1, [\lambda_3, \lambda_1]]L_3^* - [\lambda_3, [\lambda_3, \lambda_1]]L_1^* \right) \right)$$

where

$$\iota_\xi J = - \xi^\alpha \partial_\alpha g^{-1}$$

$$A = \frac{1}{2} \frac{\text{Tr} \left( \lambda_3 (1 - P_{31}) \iota_\xi J_3 - \lambda_1 (1 - P_{13}) \iota_\xi J_1 \right)}{\text{Tr}([\lambda_3, \lambda_1]_{\text{STL}})^2}$$

$$B = \frac{\text{STr}(\lambda_3 \lambda_1)}{\text{STr}(\lambda_3) \text{STr}(\lambda_3, \lambda_1)_{\text{STL}}^2}$$ (108)

The coefficients $A$ and $B$ satisfy:

$$(QA)[\lambda_3, \lambda_1]_{\text{STL}} = \iota_\xi J_2^{\text{coker}}$$ (109)

$$(A) [\lambda_1, [\lambda_3, \lambda_1]]_{\text{STL}} + [\lambda_3, [\lambda_3, \lambda_1]]_{\text{STL}} = \iota_\xi J_3^{\text{ker}} + \iota_\xi J_1^{\text{ker}}$$ (110)

$$(QB) ([\lambda_1, [\lambda_3, \lambda_1]]_{\text{STL}} - [\lambda_3, [\lambda_3, \lambda_1]]_{\text{STL}}) = \iota_\xi J_1^{\text{coker}} + \iota_\xi J_3^{\text{coker}}$$ (111)

$$B (\{ \lambda_3, [\lambda_1, [\lambda_3, \lambda_1]]_{\text{STL}} \} - \{ \lambda_1, [\lambda_3, [\lambda_3, \lambda_1]]_{\text{STL}} \}) = \iota_\xi J_2^{\text{ker}}$$ (112)

Eq. (106) follows from:

$$\iota_\xi J = P_{31} \iota_\xi J_3 + P_{13} \iota_\xi J_1 +$$ (113)

$$+ \iota_\xi j_3^{\text{split}} + \iota_\xi j_1^{\text{split}} + \iota_\xi j_3^{\text{ker}} + \iota_\xi j_1^{\text{ker}} +$$ (114)

$$+ (QB) ([\lambda_1, [\lambda_3, \lambda_1]]_{\text{STL}} - [\lambda_3, [\lambda_3, \lambda_1]]_{\text{STL}}) +$$ (115)

$$+ \iota_\xi j_2^{\text{split}} + \iota_\xi j_3^{\text{ker}} + (QA)[\lambda_3, \lambda_1]_{\text{STL}}$$ (116)

Some useful identities

$$\text{STr} ([\lambda_3, [\lambda_3, \lambda_1]]_{\text{STL}}(1 - P_{31}) \iota_\xi J_3) = - \frac{\text{STr}(\lambda_3 \lambda_1)}{2} \text{Tr} (\lambda_3 (1 - P_{31}) \iota_\xi J_3)$$ (117)
\{λ_1, [λ_3, λ_1]_{STL}\} = -\{λ_3, [λ_1, λ_3]_{STL}\} =
\frac{1}{2}[λ_1, λ_3]_{TL} Str(λ_3 λ_1) + \frac{1}{8} ((Str(λ_3 λ_1))^2 - 2Tr[λ_1, λ_3]^2) 1 =
\frac{1}{2}[λ_1, λ_3]_{TL} Str(λ_3 λ_1) - \frac{1}{4} Tr([λ_3, λ_1]_{STL})^2 1 \quad (118)

\text{STr}\left(\left[λ_1, [λ_3, λ_1]_{STL}\right] \left[λ_3, [λ_3, λ_1]_{STL}\right]\right) =
= - \frac{1}{2} Str(λ_3 λ_1) Str\left([λ_1, λ_3]_{STL}[λ_1, \overline{λ_3}]_{TL}\right) =
= - \frac{1}{2} Str(λ_3 λ_1) Tr([λ_3, λ_1]_{STL})^2 \quad (119)

Notice that we have $\text{Tr}([λ_3, λ_1]_{STL})^2$ in denominators. At the same time:

\text{STr}([λ_3, λ_1]_{STL})^2 = \text{STr}([λ_3, λ_1])^2 = 0 \quad (120)

5 Regularization

The “minimalistic action” (82) cannot be regularized in a way that would preserve the symmetries of $AdS_5 \times S^5$; it is impossible to choose a $PSU(2, 2|4)$-invariant Lagrangian submanifold so that the restriction of the Master Action of Eq. (82) to it be non-degenerate. Let us therefore return to the original action of Eqs. (51), (77), but in a way preserving the worldsheet diffeomorphisms. The construction is somewhat similar to the description of the topological A-model in [8].

5.1 Adding more fields

Add a pair of bosonic 1-form fields $ω_3$ and $ω_1$, taking values in $g_3$ and $g_1$, respectively, and their antifields $ω_3^*$ and $ω_1^*$, also 1-forms:

$$ω_{BV} = \int \text{STr} (dω_3^* \wedge dω_1 + dω_1^* \wedge dω_3) \quad (121)$$
(where \(d\) is the differential in the field space, not on the worldsheet!). In other words, for any “test 1-forms” \(f_1\) and \(g_3\):

\[
\begin{align*}
\left\{ \int \text{Str} \ f_1 \wedge \omega_3^* , \int \text{Str} \ \omega_1 \wedge g_3 \right\} &= \int \text{Str} \ f_1 \wedge g_3 \\
\left\{ \int \text{Str} \ g_3 \wedge \omega_1 , \int \text{Str} \ \omega_3^* \wedge f_1 \right\} &= \int \text{Str} \ g_3 \wedge f_1
\end{align*}
\]

(122)  

(123)

We define the BV Master Action as follows:

\[
\tilde{S}_{BV} = \tilde{S}_{BV} + \int \text{Str} (\omega_1^* \wedge \omega_3^*)
\]

(124)

and the BV Hamiltonian for the action of diffeomorphisms as follows:

\[
\hat{\mathcal{V}}_\xi = \{ \tilde{S}_{BV} , \hat{\Phi}_\xi \}_{BV}
\]

(125)

\[
\hat{\Phi}_\xi = \Phi_\xi + \int \text{Str} (\omega_3 \wedge \mathcal{L}_\xi \omega_1)
\]

(126)

where \(\mathcal{L}_\xi\) is the Lie derivative.

The expression \(\int \text{Str} (\delta \omega_3 \wedge \delta \omega_1)\) defines a symplectic structure on the space of 1-forms with values in \(g_{\text{odd}}\). The expression \(\int \text{Str} (\omega_3^* \wedge \omega_1^*)\) is the corresponding Poisson bivector. The Lie derivative preserves this (even) symplectic structure, and \(\int \text{Str} (\omega_3 \wedge \mathcal{L}_\xi \omega_1)\) is the corresponding Hamiltonian.

5.2 A canonical transformation

Let us do the canonical transformation by a flux of the following odd Hamiltonian:

\[
\Psi_{(0)} = \int \text{Str} [A_0, \omega_3] \wedge \omega_1 = - \int \text{Str} [A_0, \omega_1] \wedge \omega_3
\]

(127)

This is the Hamiltonian of \([A_0, \_]\) in the same sense as \(\int \text{Str} (\omega_3 \wedge \mathcal{L}_\xi \omega_1)\) is the Hamiltonian of \(\mathcal{L}_\xi\); we again use the same procedure of passing from Eq. (4) to Eq. (29), actually in reverse.

The effect of the flux of \(\Psi_{(0)}\) on the BV Master Action \(\tilde{S}_{BV}^+\) of Eq. (124)
\[ \tilde{S}_{BV}^+ = \tilde{S}_{BV} + \int \text{STr} \omega_3^* \wedge \omega_1^* \]

becomes

\[
\tilde{S}_{BV}^+ = \tilde{S}_{BV} + \int \text{STr} \left( \left( \{\lambda_3, \lambda_1\} - \frac{1}{2}[A_0, A_0] \right), \omega_3 \right) \wedge \omega_1 + 
\int \text{STr} (\omega_3^* + [A_0, \omega_3]) \wedge (\omega_1^* - [A_0, \omega_1]) = \\
\tilde{S}_{BV}^+ + \int \text{STr} \{\lambda_3, \lambda_1\} \{\omega_3, \wedge \omega_1\} + 
\int \text{STr} \left( [A_0, \omega_3] \wedge \omega_1^* + [A_0, \omega_1] \wedge \omega_3^* \right) 
\]

Notice that the terms of the form $A^2 \omega^2$ cancelled. This is automatic, because such terms would contradict the Master Equation (the bracket $\{\omega^* \omega^*, A^2 \omega^2\}$ would have nothing to cancel against).

The purpose of this canonical transformation was, essentially, to introduce the compensator term $[A, \omega]$ into the action of $Q$ on $\omega$, cp. Eq. (71). We will discuss this in a more general context in Section 5.4.

We are now ready to construct the Lagrangian submanifold.

### 5.3 Constraint surface and its conormal bundle

The configuration space $X$ of this new theory is parametrized by $g, \lambda_3, \lambda_1, \omega_3 \pm$ and $\omega_1 \pm$. Let us consider a subspace $Y \subset X$ defined by the constraints:

\[
(1 - P_{13})\omega_1 = 0 \\
(1 - P_{31})\omega_3 = 0 \\
\omega_1 = 0 \\
\omega_3 = 0
\]

Consider the odd conormal bundle $\Pi T^\perp Y$ of $Y \subset X$ in the BV phase space $\Pi T^* X$. As any conormal bundle, this is a Lagrangian submanifold. The restriction of $\tilde{S}_{BV}^+$ on this Lagrangian submanifold is still degenerate. $\Pi T^\perp Y$

But let us deform it by the following generating function:

\[
\Psi = \int \text{STr} \left( \omega_3 P_{13} J_{1+} + \omega_1 P_{31} J_{3-} \right)
\]
The restriction of $\tilde{\mathcal{L}}_{BV}^\prime$ to this deformed Lagrangian submanifold is equal to:

$$\int \text{STr} \left( \frac{1}{2} J_{2+} J_{2-} + \frac{3}{4} J_{1+} J_{3-} + \frac{1}{4} J_{3+} J_{1-} + w_{1+} D_0 - \lambda_3 + w_{3} - D_0 + \lambda_1 + N_0 + N_0 - + \omega_{3+}^* \omega_{1-}^* + \omega_{3-} \omega_{1+}^* \right)$$  \quad (132)

where $N_0^+ = \{w_{1+}, \lambda_3\}$, $N_0^- = \{w_{3-}, \lambda_1\}$,

$$w_{1+} = P_{13} \omega_{1+} \text{ and } w_{3-} = P_{31} \omega_{3-} \quad (133)$$

Notice that the terms:

$$\int \text{STr} \left( [A_0, \lambda_3] \lambda_3^* + [A_0, \lambda_1] \lambda_1^* + [A_0, \omega_{3+}] \omega_{1-}^* + [A_0, \omega_{3-}] \omega_{1+}^* + [A_0, \omega_{1+}] \omega_{3-}^* + [A_0, \omega_{1-}] \omega_{3+}^* \right)$$  \quad (134)

vanish on $T^4 Y$. Indeed, the vector field:

$$[A_0, \lambda_3] \frac{\partial}{\partial \lambda_3} + [A_0, \lambda_1] \frac{\partial}{\partial \lambda_1} + [A_0, \omega_{3+}] \frac{\partial}{\partial \omega_{3+}} + [A_0, \omega_{3-}] \frac{\partial}{\partial \omega_{3-}} + [A_0, \omega_{1+}] \frac{\partial}{\partial \omega_{1+}} + [A_0, \omega_{1-}] \frac{\partial}{\partial \omega_{1-}}$$  \quad (135)

is tangent to the constraint surface (130); the conormal bundle, by definition, consists of those one-forms which vanish on such vectors. The term $\omega_{3+}^* \omega_{1-}^* + \omega_{3-} \omega_{1+}^*$ computes the contribution to the action from the fiber $\Pi T^4 Y$. The coordinates of the fiber enter without derivatives, and decouple.

We therefore return to the original $AdS_5 \times S^5$ action of Eq. (51).

But now we understand how the worldsheet diffeomorphisms act, at the level of BV phase space.

### 5.4 Gluing charts

In our construction we used a lift of $AdS_5 \times S^5$ to $PSU(2,2|4)$ (Section 3.5). This is only possible locally. Therefore, we have to explain how to glue together overlapping patches. This is a particular case of a general construction, which we will now describe.
The idea is to build a theory which is locally (on every patch of $B$) a direct product of two theories $S_\phi$ and $S_w$:

$$S_{tot} = S_\phi + S_w = S_{cl}(\phi) + Q^\mu(\phi)\phi^*_\mu + \frac{1}{2}w^*(\omega^{-1})^{ab}w^*_b$$  \hspace{0.5cm} (136)

but transition functions between overlapping patches mix $\phi$ and $w$.

Technical assumption: in this Section, just to simplify formulas, we assume that all $\phi$ and $w$ are bosons.

Consider the following data, consisting of two parts. The first part is a Lie group $H$, and a principal $H$-bundle $E$ with base $B$. Suppose that $B$ comes with a nilpotent vector field $Q \in \text{Vect}(B)$ and a $Q$-invariant action $S_{cl} \in \text{Fun}(B)$. Then $S_B(\phi,\phi^*) = S_{cl}(\phi) + Q^\mu(\phi)\phi^*_\mu$ satisfies the Master Equation on the BV phase space $\Pi T^*B$. The second part of the data is a symplectic vector space $W$ which is a representation of $H$. This means that $W$ is equipped with an even $H$-invariant symplectic form $\omega$.

Let us cover $B$ with charts $\{U_i | i \in I\}$ and trivialize $E$ over each chart:

$$p^{-1}(U_i) \simeq U_i \times H$$  \hspace{0.5cm} (137)

At the intersection $U_i \cap U_j$ we identify $(\phi, h_i) \in U_i \times H$ with $(\phi, h_j) \in U_j \times H$ if

$$h_j = u_{ji}(\phi)h_i$$  \hspace{0.5cm} (138)

All this comes from $E \overset{H}{\to} B$. We will now construct a new odd symplectic manifold, which is locally $\Pi T^*U_j \times \Pi T^*W$, with some transition functions, which we will now describe.

Transition functions Let $\mathfrak{h}$ be the Lie algebra of $H$. For each $\alpha \in \text{Map}(B, \mathfrak{h})$ consider the following BV Hamiltonian:

$$\chi_\alpha = \{S_{tot}, F_\alpha\}$$  \hspace{0.5cm} (139)

where

$$F_\alpha = -\frac{1}{2}w^b \rho_*(\alpha(\phi))^a_b \omega_{ac} w^c$$  \hspace{0.5cm} (140)

Here $\rho_*$ is the representation of the Lie algebra corresponding to the representation $\rho$ of the group, and $\omega$ is the symplectic form of $W$. Eq. (140) defines $F_\alpha$ as the Hamiltonian of the infinitesimal action of $\alpha$ on $w$, i.e. the “usual”
(even) moment map. (Here we use our assumption that \(\omega\) is \(H\)-invariant.) The explicit formula for \(\chi_\alpha\) is:

\[
\chi_\alpha = \rho_*(\alpha(\phi))^a_b w^b w^*_a - \frac{1}{2} w^b \rho_*(Q\alpha(\phi))^a_b \omega_{ac} w^c
\]  

(141)

Notice that:

\[
\{\chi_\alpha_1, F_{\alpha_2}\} = - F_{[\alpha_1, \alpha_2]}
\]  

(142)

The flux of the BV-Hamiltonian vector field \(\{\chi_\alpha, \cdot\}\) is a canonical transformation, and Eq. (139) implies that this canonical transformation is a symmetry of \(S_{\text{tot}}\). This canonical transformation does not touch \(\phi^*\), it only acts on \(\phi^*, w, w^*\). We identify \((\phi, \phi^*_i, w_i, w^*_i)\) on chart \(U_i\) with \((\phi, \phi^*_j, w_j, w^*_j)\) on chart \(U_j\) when \((\phi^*_j, w_j, w^*_j)\) is the flux of \((\phi^*_i, w_i, w^*_i)\) by the time 1 along the vector field \(\{\chi_{\alpha_{ji}}, \cdot\}\) where \(\alpha_{ji}\) is the log of \(u_{ji}\), i.e. \(u_{ji} = e^{\alpha_{ji}}\). Explicitly:

\[
w^a_j = \rho(u_{ji})^a_b w^b_i
\]  

(143)

\[
w^a_j = \rho(u_{ji}^{-1})^a_b w^b_i - \omega_{ab} \rho(u_{ji})^b_c w^c_i
\]  

(144)

\[
\phi^*_{j\mu} = \phi^*_{i\mu} - w^*_j \rho_* \left( \frac{\partial u_{ji}}{\partial \phi^*_{ji}} u_{ji}^{-1} \right)^a_b w^b_j - \frac{1}{2} w^a_j \omega_{ab} \partial_{\partial \phi^*_{ji}} \rho_* (Q u_{ji} u_{ji}^{-1})^b_c \omega^c_j
\]  

(145)

These gluing rules are consistent on triple intersections because of Eq. (142).

**Lagrangian submanifold** Eqs. (144) and (145) look somewhat unusual. In particular, the “standard” Lagrangian submanifold \(\phi^* = w^* = 0\) is not well-defined, because it is incompatible with our transition functions. One simple example of a well-defined Lagrangian submanifold is \(w = \phi^* = 0\). We will now give another example, which repairs the ill-defined \(w^* = \phi^* = 0\).

The construction requires a choice of a *connection* in the principal bundle \(E \overset{H}{\rightarrow} B\). To specify a connection, we choose on every chart \(U_i\) some \(h\)-valued 1-form \(A_{i\mu}\), with the following identifications on the intersection \(U_i \cap U_j\):

\[
\frac{\partial}{\partial \phi^*_{ji}} + A_{j\mu}(\phi) = u_{ji}(\phi) \left( \frac{\partial}{\partial \phi^*_{ji}} + A_{i\mu}(\phi) \right) (u_{ji}(\phi))^{-1}
\]  

(146)

and in particular:

\[
Q \rho(u_{ji})^a_b + \rho_*(Q^\mu A_{j\mu})^a_b \rho(u_{ji})^c_i - \rho(u_{ji})^a_c \rho_*(Q^\mu A_{i\mu})^c_b = 0
\]  

(147)

\(^*\)In BV formalism, there is no such thing as the standard Lagrangian submanifold. We invented this notion to denote the one where all antifields (w.r.to some Darboux coordinates) are zero. This is often a useful starting point to construct Lagrangian submanifolds.
On every chart, let us pass to a new set of Darboux coordinates, by doing the canonical transformation with the following gauge fermion:

\[ \Psi_i = \frac{1}{2} w^a_i \omega_{ab} Q^\mu(\phi) \rho_*(A_{i\mu}(\phi))_b^c w^c_i \]  

(148)

Notice that \( \Psi_i \) does not depend on antifields; therefore this canonical transformation is just a shift:

\[ \tilde{w}^a_i = w^a_i \]  

(149)

\[ \tilde{w}^*_i a = w^*_i a - \omega_{ab} Q^\nu(\phi) \rho_*(A_{i\nu}(\phi))_c^b w^c_i \]  

(150)

\[ \tilde{\phi}_{*i\mu} = \phi_{*i\mu} - \frac{1}{2} \frac{\partial}{\partial \phi^\mu} \left[ w^a_i \omega_{ab} Q^\nu(\phi) \rho_*(A_{i\nu}(\phi))_c^b w^c_i \right] \]  

(151)

This canonical transformation does not preserve \( S_{\text{BV}} \), therefore the expression for the action will be different in different charts, see Eq. (129). In particular, it will contain the term \( \tilde{w}^* Q^\mu \rho_*(A_{i\mu}) \tilde{w} \), which means that the action of the BRST operator on \( \tilde{w} \) involves the connection. On the other hand, the transition functions simplify:

\[ \tilde{w}^a_j = \rho (u_{ji}(\phi))^b_a \tilde{w}^b_i \]  

(152)

\[ \tilde{w}^*_j a = \rho (u_{ji}(\phi))^{-1}_b^a \tilde{w}^*_b \]  

(153)

\[ \tilde{\phi}_{*j\mu} = \phi_{*j\mu} - \tilde{w}^*_c \rho (u_{ji}(\phi))^{-1}_a ^c \left( \frac{\partial}{\partial \phi^\mu} \rho (u_{ji}(\phi))_b^a \right) \tilde{w}^b_i \]  

(154)

These are the usual transition functions of the odd cotangent bundle \( \Pi T^* W \), where \( W \) is the vector bundle with the fiber \( W \), associated to the principal vector bundle \( E^H \to B \).

In particular, the “standard” Lagrangian submanifold \( \tilde{w}^* = \tilde{\phi}^* = 0 \) is compatible with gluing. The corresponding BRST operator is defined by the part of the BV action linear in the antifields:

\[ Q_{\text{BRST}} = Q^\mu \frac{\partial}{\partial \phi^\mu} + Q^\nu \rho_*(A_{\nu})_b^a \tilde{w}^b \frac{\partial}{\partial \tilde{w}^a} \]  

(155)

After this canonical transformation of Eqs. (149), (150) and (151), the new \( S_{\text{cl}} \) is such that this \( Q_{\text{BRST}} \) is nilpotent on-shell.
**Gluing together** $\Phi_\xi$. Let us consider the relation between the functions $\hat{\Phi}_\xi$ defined by Eq. \((126)\) on two overlapping charts. It is enough to consider the case of infinitesimal transition function, *i.e.* $u_{ji} = 1 + \epsilon \alpha_{ji}$, where $\epsilon$ is infinitesimally small. With $F_\alpha$ defined in Eq. \((140)\), the difference between $\hat{\Phi}_\xi$ on two coordinate charts is:

$$\delta_{ji} \hat{\Phi}_\xi = \{S_{\text{tot}}, F_{\alpha_{ji}}, \hat{\Phi}_\xi\} = -\{S_{\text{tot}}, \hat{\Phi}_\xi, F_{\alpha_{ji}}\} + \{S_{\text{tot}}, \{F_{\alpha_{ji}}, \hat{\Phi}_\xi\}\} \quad (156)$$

The first term on the RHS is zero:

$$\{S_{\text{tot}}, \hat{\Phi}_\xi, F_{\alpha_{ji}}\} = 0 \quad (157)$$

since $F_\alpha$ is diffeomorphism-invariant. Let us study the second term. We have:

$$\{F_{\alpha_{ji}}, \hat{\Phi}_\xi\} = \{F_{\alpha_{ji}}, \Phi_\xi\} = \frac{1}{2} w^a \omega_{ab} \Phi_\xi^\mu \frac{\partial}{\partial \phi^\mu} (\alpha_{ji}) c^b w^c =$$

$$= -\frac{1}{2} \left( w^a \omega_{ab} \Phi_\xi^\mu (\delta_{ji} A_\mu - [\alpha_{ji}, A_\mu]) c^b w^c \right) =$$

$$= -F_{\Phi_\xi^\mu \delta_{ji}, A_\mu} + \frac{1}{2} w^a \omega_{ab} \Phi_\xi^\mu [\alpha_{ji}, A_\mu] c^b w^c =$$

$$= -F_{\Phi_\xi^\mu \delta_{ji}, A_\mu} - \{S_{\text{tot}}, F_{\alpha_{ji}}\}, F_{\Phi_\mu A_\mu}\} \quad (158)$$

where $A_\mu$ is any connection, transforming as in Eq. \((146)\). Therefore the following expression:

$$\hat{\Phi}_\xi' = \hat{\Phi}_\xi + \{S_{\text{tot}}, F_{\Phi_\mu A_\mu}\} \quad (159)$$

is consistent on intersections of patches.

The correcting term $\{S_{\text{tot}}, F_{\Phi_\mu A_\mu}\}$ is the infinitesimal gauge transformation (see Eqs. \((139)\) and \((140)\)) with the parameter $\Phi_\mu A_\mu$.

**Back to** $\text{AdS}_5 \times S^5$. In our case $B$ is the pure spinor bundle over super-$\text{AdS}_5 \times S^5$; the coordinates $\phi$ are functions from the worldsheet to $P\text{S AdS}_5 \times S^5$ (defined in Eq. \((13)\)). The total space $E$ is the space of maps from the worldsheet to $C_L \times C_R \times P\text{SU}(2, 2|4)$. Notice that $C_L \times C_R \times P\text{SU}(2, 2|4)$ is a principal $H$-bundle over $P\text{S AdS}_5 \times S^5$. It has a natural $P\text{SU}(2, 2|4)$-invariant connection, which for every tangent vector:

$$(\dot{\lambda}_L, \dot{\lambda}_R, \dot{g}) \in T (C_L \times C_R \times P\text{SU}(2, 2|4)) \quad (160)$$
declares its vertical component to be $(\dot{g}g^{-1})\hat{a}$, i.e. the projection of $\dot{g}$ on the denominator of (43) using the Killing metric. This defines, pointwise, the connection on the space of maps.

It is natural to use this connection as $A_\mu$ in Eq. (159).

Notice that we do not need a connection to write the BV Master Action (Eq. (124)). But the connection is needed to construct $\hat{\Phi}'_\xi$ (and also in our construction of the Lagrangian submanifold).

6 Taking apart the AdS sigma model

The standard action given by Eq. (51) depends on the worldsheet complex structure and is polynomial in the pure spinor variables. In the BV formalism, it corresponds to a specific choice of the Lagrangian submanifold. We can change the action to a physically equivalent one, by adding BRST quartets and/or deforming the Lagrangian submanifold. We can ask ourselves, what is the simplest formulation of the theory, in the BV language, preserving the symmetries of $AdS_5 \times S^5$? (Of course, the notion of “being the simplest” is somewhat subjective.) In this paper we gave an example of such a “minimalistic” formulation:

$$ S_{BV} = S_{(g, \lambda)} + S_\omega =$$

$$ = \int STr (J_1 \wedge (1 - P_{31})J_3 - J_1 \wedge P_{31}J_3) +$$

$$ + \int STr (\lambda_3 L^*_1 + \lambda_1 L^*_3) +$$

$$ + \int \frac{1}{2} STr (\omega^*_3 \wedge \omega^*_1) \quad (161) $$

Here $L^*$ are the BV Hamiltonians of the left shift, Eq. (83). The relation of Eq. (161) to the original BV action (77) is through adding BRST quartet (Section 5) and canonical transformations (Eqs. (64), (127), (131)). Subjectively, Eq. (161) is the simplest way of presenting the worldsheet Master Action for $AdS_5 \times S^5$.

The Master Action (161) does not depend on the worldsheet metric. The dependence on the worldsheet metric (through the complex structure) comes later when we choose the Lagrangian submanifold.
The way Eq. (161) is written, it seems that \( w \) is completely decoupled from \( g \) and \( \lambda \). But the transition functions on overlapping charts, described in Section 5.4, do mix the two sets of fields.

The Master Action (161) is non-polynomial in \( \lambda \), because of \( P_{31} \).

7 Generalization

Consider a sigma-model with a target space some supermanifold \( \mathcal{X} \). Suppose that \( \mathcal{X} \) is equipped with a nilpotent odd vector field \( Q \in \text{Vect}(\mathcal{X}) \), generating a gauge symmetry of the sigma-model.

- in minimalistic sigma-models the BRST operator is just an odd nilpotent vector field on the target space

This means that the field configuration \( X(\sigma, \tau) \) has the same action as \( e^{\epsilon(\sigma, \tau)Q}X(\sigma, \tau) \) for an arbitrary odd gauge parameter function \( \epsilon \) on the worldsheet:

\[
S[X] = S[e^{\epsilon Q}X]
\]

(162)

Locally and away from the fixed points of \( Q \) this implies that one of the target space fermionic coordinates completely decouples from the action (the action does not depend on it). In case of pure spinor sigma-model, this gauge symmetry does not account for all degeneracy of the action. All directions in the \( \theta \) space tangent to the pure spinor cones are degenerate directions of the quadratic part of the action.

Let us add an additional scalar field on the worldsheet \( \Lambda(\sigma, \tau) \) and consider the following solution of the Master Equation:

\[
S_{BV} = S + \int \Lambda Q^A(X)X^*_A
\]

(163)

In the pure spinor case \( \mathcal{X} \) is parametrized by \( g \in PSU(2,2|4) \) and \( \lambda_L, \lambda_R \) modulo rescaling (i.e. projective pure spinors).

In Type II pure spinor theory, there are actually two anticommuting BRST symmetries, \( Q_L \) and \( Q_R \), and the term in \( S_{BV} \) linear in antifields is

\[
\int \Lambda_L Q^A_L(X)X^*_A + \Lambda_R Q^A_R(X)X^*_A
\]

(164)

The action \( S \) is given by Eq. (60). Such a theory requires regularization.
The minimalistic sigma-model action is written in terms of the target space metric $G$ and the B-field $B$. For example, the action of Eq. (60) corresponds to:

\[
G = \text{Str} \left( \frac{1}{2} J_2 J_2 + J_1 (1 - P_{31}) J_3 \right) \tag{165}
\]

\[
B = \text{Str} \left( \frac{1}{2} J_1 \wedge J_3 - J_1 \wedge P_{31} J_3 \right) \tag{166}
\]

The existence of the $b$-ghost is equivalent to the metric being the Lie derivative along $Q$ of some symmetric tensor $b$:

\[
G = L_Q b \tag{167}
\]

where $L_Q$ is the Lie derivative along the vector field $Q$. In our case (Appendix B):

\[
b = \text{Tr} \left( \left( \{ J_3, \lambda_3 \} - \{ J_1, \lambda_1 \} \right) J_2 \right) \frac{\text{STr}(\lambda_3 \lambda_1)}{\text{STr}(\lambda_3 \lambda_1)} \tag{168}
\]

As in Section 3.4, the part of the action involving the target space metric $G$ is BRST exact.

8 Open problems

We did not verify that $\Phi_\xi'$ of Eq. (159) satisfies the conditions formulated in [2, 3]. In particular, we may hope for $\{ \Phi_\xi', \Phi_\xi' \} = 0$, but more complicated scenarios are also possible. We believe that the invariance of our construction under the symmetries of $AdS_5 \times S^5$ is important to satisfy those conditions at the quantum level.

We did not explicitly calculate the restriction of the $\Phi_\xi'$ to the standard family of Lagrangian submanifolds, corresponding to the integration over the space of metrics. It can probably be expressed in terms of $\mathcal{O}$ where $\partial b = Q \mathcal{O}$ as calculated in [1]. In any case, it is most likely nonzero, and therefore the string measure of Eq. (1) is not just the product of Beltrami differentials, but involves also the curvature terms $\Phi_F$. 

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A The projector

A.1 Definition

Let \( \pi_A \) and \( \pi_S \) denote the projectors:

\[
\begin{align*}
\pi_A : & \quad T(AdS_5 \times S^5) \to T(AdS_5) \quad \text{projector along} \quad T(S^5) \\
\pi_S : & \quad T(AdS_5 \times S^5) \to T(S^5) \quad \text{projector along} \quad T(AdS_5) \\
\pi_A(v) + \pi_S(v) = v
\end{align*}
\]

For any vector \( v \), we will denote \( \overline{v} \) the difference of its \( AdS_5 \) and \( S^5 \) components:

\[
\overline{v} \overset{\text{def}}{=} \pi_A(v) - \pi_S(v) \quad (172)
\]

The projector \( P_{13} : g_1 \to g_1 \) was defined in \([9]\) as follows:

\[
\begin{align*}
P_{13}A_1 &= A_1 + [S_2, \lambda_3] \\
\{\lambda_1, P_{13}A_1\} &= 0
\end{align*}
\]

where \( S_2 \in g_2 \) is adjusted to satisfy (174). In fact \( P_{13} \) is the projection to the tangent space \( TC_R \) along the space \( T^\perp C_L \) which is orthogonal to \( TC_L \) with respect to the metric defined by Str:

\[
(1 - P_{13})A_1 \in T^\perp C_L \quad (175)
\]

In other words:

\[
0 \to T^\perp C_L \xrightarrow{i} g_1 \xrightarrow{P_{13}} TC_R \to 0 \quad (176)
\]

In Section A.3 we will give an explicit formula for \( P_{13} \) following \([1]\).
A.2 Matrix language

It turns out that computations can often be streamlined by thinking about elements of $g$ literally as $4|4$-matrices. In fact $g$ is a factorspace of $sl(4|4)$ modulo a subspace generated by the unit matrix. Therefore, when talking about a matrix corresponding to an element of $g$, we have to explain every time how we choose a representative. The $\mathbb{Z}_4$ grading of $psl(4|4)$ can be extended to $sl(4|4)$; the unit matrix has grade two. Therefore, the ambiguity of adding a unit matrix only arises for representing elements of $g_2$. To deal with this problem, we introduce some notations. Given a matrix $X$ of grade two, we denote $X_{\text{TL}}$ the corresponding traceless matrix:

$$X_{\text{TL}} = X - \frac{\text{Tr}(X)}{8} \mathbf{1}$$

(Letters “TL” are abbreviation for “traceless”). Also, it is often useful to consider $4|4$-matrices with nonzero supertrace. Such matrices do not correspond to any elements of $g$. For a $4|4$-matrix $Y$ we define:

$$Y_{\text{STL}} = Y - \frac{\text{STr}(Y)}{8} \Sigma$$

where $\Sigma = \text{diag}(1, 1, 1, 1, -1, -1, -1, -1)$

In particular:

$$(Y_{\text{TL}})_{\text{STL}} = (Y_{\text{STL}})_{\text{TL}} = Y - \frac{\text{Tr}(Y)}{8} \mathbf{1} - \frac{\text{STr}(Y)}{8} \Sigma$$

We also define, for any even matrix $Y$:

$$\Sigma Y = Y \Sigma = \Sigma Y$$

This definition agrees with Eq. (172).

A.3 Explicit formula for the projector

In fact $S_2$ is given by the following expression:

$$S_2 = \frac{2}{\text{Str}(\lambda_1 \lambda_3) \{\lambda_1, A_1\}_{\text{STL}}}$$
Notice that \( \{\lambda_1, A_1\}_{\text{STL}} \) is actually both super-traceless and traceless; it is the same as \( \{\lambda_1, A_1\}_{\text{TL}} \) (with the overline extending over “TL”). We have to prove that the \( S_2 \) defined this way satisfies (174). Indeed, we have:

\[
[S_2, \lambda_3] = \frac{2}{\text{Str}(\lambda_1 \lambda_3)} \{\lambda_1, A_1\}_{\text{STL}}, \lambda_3 \tag{183}
\]

and we have to prove (174). We have:

\[
\{\lambda_1, \{\lambda_1, A_1\}_{\text{STL}}, \lambda_3\} = \{\lambda_1, [\Sigma \{\lambda_1, A_1\}_{\text{TL}}, \lambda_3]\} = -\Sigma \{[\lambda_1, \lambda_3], \{\lambda_1, A_1\}_{\text{TL}}\} \tag{184}
\]

Both \( \{\lambda_1, A_1\} \) and \( [\lambda_1, \lambda_3] \) have \( \mathbb{Z}_4 \)-grading two. Let us use:

\[
[\lambda_1, \lambda_3] = \frac{1}{4} \text{Str}(\lambda_1 \lambda_3) \Sigma + [\lambda_1, \lambda_3]_{\text{STL}} \tag{185}
\]

For all grade 2 matrices \( A_2 \) and \( B_2 \) such that \( \text{Tr} A_2 = \text{Tr} B_2 = \text{Str} A_2 = \text{Str} B_2 = 0 \) the following identity holds:

\[
\{A_2, B_2\} = A_2 B_2 + B_2 A_2 = \frac{1}{4} (\text{Str}(A_2 B_2) \Sigma + \text{Tr}(A_2 B_2) 1) \tag{186}
\]

Therefore:

\[
\{\lambda_1, \{\lambda_1, A_1\}, \lambda_3\} \mod 1 = -\frac{1}{2} \text{Str}(\lambda_1 \lambda_3) \{\lambda_1, A_1\}_{\text{TL}} \tag{187}
\]

(where “mod 1” means “modulo the center of \( \text{psl}(4|4) \)”, i.e. up to a multiple of the unit matrix). This proves (174).

The central part of \( \{\lambda_1, \{\lambda_1, A_1\}, \lambda_3\} \) is generally speaking nonzero:

\[
\text{Tr} \{\lambda_1, \{\lambda_1, A_1\}, \lambda_3\} = 2 \text{Tr} \left( \lambda_1 \left[ \{\lambda_1, A_1\}, \lambda_3\right] \right) = -2 \text{Tr} \left( [\lambda_1, \lambda_3]_{\text{STL}} \Sigma \{\lambda_1, A_1\} \right) = -2 \text{Str} \left( [\lambda_1, \lambda_3]_{\text{STL}} \{\lambda_1, A_1\} \right) \tag{188}
\]

In \( \Gamma \)-matrix notations, \( [\lambda_1, \lambda_3]_{\text{STL}} \) is \( (\lambda_1, \Gamma^n \lambda_3) \) and \( \{\lambda_1, A_1\} \) is \( (\lambda_1, \Gamma^m A_1) \).

Let us define (cp. Eq. 182):

\[
S_{2/1} : g_1 \rightarrow g_2
\]

\[
S_{2/1} A_1 = \frac{2}{\text{Str}(\lambda_1 \lambda_3)} \{\lambda_1, A_1\}_{\text{STL}} \tag{190}
\]

\[
S_{2/3} : g_3 \rightarrow g_2
\]

\[
S_{2/3} A_3 = \frac{2}{\text{Str}(\lambda_3 \lambda_1)} \{\lambda_3, A_3\}_{\text{STL}} \tag{191}
\]

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so that:

\[ P_{13} A_1 = A_1 + [S_{2/1} A_1, \lambda_3] \]  
\[ P_{31} A_3 = A_3 + [S_{2/3} A_3, \lambda_1] \]  

A.4 Properties of \( P_{13} \) and \( P_{31} \)

It follows from the definition, that for any \( v_2 \in g \) we have

\[ P_{13}[v_2, \lambda_3] = 0 \]  

Let us verify this explicitly using the definition (173) with the explicit expression for \( S_2 \) given by (182). We have:

\[ P_{13}[v_2, \lambda_3] = [v_2, \lambda_3] + \frac{2}{\text{Str}(\lambda_1 \lambda_3)} \{\{\lambda_1, [v_2, \lambda_3]\}, \lambda_3\} \]  

Consider the expression \[ \{\{\lambda_1, [v_2, \lambda_3]\}, \lambda_3\} \]:

\[ \{\{\lambda_1, [v_2, \lambda_3]\}, \lambda_3\} = [\Sigma\{\lambda_1, [v_2, \lambda_3]\}, \lambda_3] = \]  

\[ = -[\Sigma\{[\lambda_1, \lambda_3], v_2\}, \lambda_3] + [\Sigma\{\lambda_1, v_2\}, \lambda_3], \lambda_3] \]  

Let us consider the first expression on the RHS of (196). Using (185) we rewrite:

\[ -[\Sigma\{[\lambda_1, \lambda_3], v_2\}, \lambda_3] = \]  

\[ = -\frac{1}{4}\text{Str}(\lambda_1 \lambda_3)\Sigma\{\lambda_1, v_2\}, \lambda_3] = -\frac{1}{2}\text{Str}(\lambda_1 \lambda_3) [v_2, \lambda_3] \]  

This cancels with the first term on the RHS of (195). And the second expression on the RHS of (196) is zero:

\[ [\Sigma\{\lambda_1, v_2\}, \lambda_3] = 0 \]  

A.5 Subspaces of \( g \) associated to pure spinors

Consider the decomposition:

\[ g_2 = g_{2L} \oplus g_{2R} \oplus C[\lambda_3, \lambda_1]_{\text{STL}} \oplus C[\lambda_3, \lambda_1]_{\text{TL}} \]
Here $g_{2L}$ is a 4-dimensional subspace Tr-orthogonal to $C[λ_3, λ_1]_{TL}$ and commuting with $λ_3$, and $g_{2R}$ is Tr-orthogonal to $C[λ_3, λ_1]_{TL}$ and commuting with $λ_1$.

Similarly we can refine $T^\perp C_R$ and $T^\perp C_L$:

$$g_3 \supset T^\perp C_R = [g_{2L}, λ_1] \oplus C[[λ_3, λ_1]_{STL}, λ_1]$$  \hfill (201)

$$g_1 \supset T^\perp C_L = [g_{2R}, λ_3] \oplus C[[λ_3, λ_1]_{STL}, λ_3]$$  \hfill (202)

### B  BRST variation of the $b$-tensor

Here we will prove:

$$(Q_L + Q_R)\frac{\text{Tr}\{J_1, λ_1\}J_2}{\text{Str}(λ_3λ_1)} = -\text{Str} \left( \frac{1}{4} J_2 J_2 + \frac{1}{2} J_1 (1 - P_{13}) J_3 \right)$$  \hfill (203)

(Remember that Tr(...)) = Str(...)) In fact, only $Q_L$ contributes; the action of $Q_R$ is zero:

$$Q_R \text{Str} \{J_1, λ_1\}J_2 = -\text{Str} \left( \{J_1, λ_1\}\{J_1, λ_1\}_{TL} \right) = 0$$  \hfill (204)

because $J_1$ is a fermion. Let us compute the action of $Q_L$:

$$Q_L \text{Str} \{J_1, λ_1\}J_2 = -\text{Str} \left( \{J_2, λ_3\}, λ_1 \right) J_2 + \{J_1, λ_1\}\{J_3, λ_3\}_{TL} \Sigma =$$

$$= - \text{Tr} \{J_2, [J_2, λ_3]\} λ_1 + \text{Str} \{J_1, λ_1\}_{STL} \{J_3, λ_3\}$$  \hfill (205)

$$= - \frac{1}{4} \text{Str}(λ_3λ_1) \text{Str}(J_2^2) + \text{Str} \left( J_3 \left[ \{J_1, λ_1\}_{STL}, λ_3 \right] \right)$$  \hfill (206)

$$= - \frac{1}{4} \text{Str}(λ_3λ_1) \text{Str}(J_2^2) - \frac{1}{2} \text{Str}(λ_3λ_1) \text{Str}(J_3(1 - P_{13}) J_1)$$  \hfill (207)

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