AN OBSERVATION ON HIGHEST WEIGHT CRYSTALS

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1. INTRODUCTION

As shown in the paper of Stembridge \cite{Ste03}, crystal graphs can be characterized by their local behavior. In this paper, we observe that a certain local property on crystals forces a more global property. In type $A$, this statement says that if a node has a single parent and single grandparent, then there is a unique walk from the highest weight node to it. In other classical types, there is a similar (but necessarily more technical) statement. This walk is obtained from the associated level 1 perfect crystal, $B^{1,1}$. (It is unique unless the Dynkin diagram contains that of $D_4$ as a subdiagram.)

This crystal observation was motivated by representation-theoretic behavior of the affine Hecke algebra of type $A$, which is known to be captured by highest weight crystals of type $A^{(1)}$ by the results in \cite{Gro}. As discussed below, the proofs in either setting are straightforward, and so Grojnowski’s theorem linking the two phenomena is not needed. However, the result is presented here for crystals as one can say something in all types (Grojnowski’s theorem is only in type $A$), and because the statement seems more surprising in the language of crystals than it does for affine Hecke algebra modules.

2. CRYSTALS

We begin by reviewing some of the definitions and notation for crystal graphs, but assume the reader is familiar with crystals and with root systems. For a more comprehensive and complete discussion, see \cite{Kas95}.

In the following, we fix a root system of finite or affine type. $I$ indexes the simple roots (and the nodes of the corresponding Dynkin diagram); $P$ is the weight lattice; $P^*$ is the coroot lattice with canonical pairing $\langle \ , \ \rangle$. The simple roots are $\alpha_i \in P$, and simple coroots are $h_i \in P^*$. The fundamental weights are denoted $\Lambda_i$ and satisfy $\langle h_i, \Lambda_j \rangle = \delta_{ij}$. The matrix $[a_{ij}]$ where $a_{ij} = \langle h_i, \alpha_j \rangle$ is the corresponding Cartan matrix.
A *crystal* is a set of nodes $B$, endowed with the following maps

\[
\begin{align*}
\text{wt} : B & \to P \\
\varepsilon_i : B & \to \mathbb{Z} \cup \{-\infty\} \\
\varphi_i : B & \to \mathbb{Z} \cup \{-\infty\} \\
\tilde{e}_i : B & \to B \cup \{0\} \\
\tilde{f}_i : B & \to B \cup \{0\}.
\end{align*}
\]

The maps satisfy the following axioms:

- $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle \quad \forall i \in I, b \in B.$
- If $\tilde{e}_i b \neq 0$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$, $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$.
- If $\tilde{f}_i b \neq 0$, then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$, $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$.
- For $a, b \in B$, $a = \tilde{f}_i b$ if and only if $b = \tilde{e}_i a$.
- If $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Given the crystal data, we can draw the associated *crystal graph*. It is a directed graph with nodes $B$, and $I$-colored arrows given by

\[
b \xrightarrow{i} a
\]

when $a = \tilde{f}_i b$, or equivalently when $b = \tilde{e}_i a$.

In all of the following, we will make the extra assumption that our crystal $B$ is a highest weight crystal. Consequently, we can read the data of

\[
\begin{align*}
\varepsilon_i(b) &= \max \{n \geq 0 \mid \tilde{e}_i^n b \neq 0\} \\
\varphi_i(b) &= \max \{n \geq 0 \mid \tilde{f}_i^n b \neq 0\},
\end{align*}
\]

off of the crystal graph, encoded in the following picture

\[
\begin{array}{c}
\bullet \xrightarrow{i} \bullet \xrightarrow{i} \cdots \bullet \xrightarrow{i} b \xrightarrow{i} \bullet \xrightarrow{i} \cdots \\
\varepsilon_i(b) \quad \varphi_i(b)
\end{array}
\]

We will also use the notation $\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i$. Thus $\varepsilon(b)$ describes the “in”-arrows leading to the node $b$.

Below, we will be interested in describing certain cases where $\varepsilon(b) = \Lambda_i$ and $\varepsilon(\tilde{e}_i b) = \Lambda_j$. (However we will not put any restrictions on “out”-arrows.)
2.1. Extra terminology. We introduce some terminology below.

Let’s say that a node $a$ is singular if

$$\sum_{i \in I} m_i \leq 1, \text{ where } \varepsilon(a) = \sum_{i \in I} m_i \Lambda_i. \quad (2.1)$$

Notice equation (2.1) implies there is at most one $i \in I$ such that $\tilde{e}_i a \neq 0$. In particular, highest weight nodes satisfy (2.1). In the crystal graph, we picture singular nodes as having a single “in”-arrow leading to it (and any arrow preceding that one carries a different color), but there is no restriction on its “out”-arrows.

If $\tilde{e}_i(a) = b$ for some $i$, we shall say $b$ is a parent of $a$. We will define ancestor inductively by saying parents are ancestors and parents of ancestors are also ancestors.

3. Kashiwara’s Theorem for Highest Weight Crystals

In all the following theorems, we fix a root system and assume $B$ is a fixed highest weight crystal of that type.

The crystal graph $B$ comes from an integrable highest weight module $V$ of the associated Lie algebra or quantum enveloping algebra. We appeal to theorems of Kashiwara that ensure the existence of a global basis $\{G(b) \mid b \in B\}$ of $V$. In the following $e_i$ will denote a Chevalley generator, and $e_i^{(m)}$ its divided power.

We first give a remark (in Section 5) of [Kas93] as the following useful lemma. One should compare it to the statement $\tilde{w}(\tilde{e}_i b) = \tilde{w}(b) + \alpha_i$.

**Lemma 3.1.** When $\tilde{e}_i b \neq 0$,

$$\varepsilon(\tilde{e}_i b) = \varepsilon(b) + \sum_{j \in I} m_j \Lambda_j, \text{ where } m_i = -1, \quad 0 \leq m_j \leq -a_{ij}. \quad (3.1)$$

In general, we have no control over the value $m_j$ takes in the range $0 \leq m_j \leq -a_{ij}$ for $j \neq i$. Below, we will be interested in describing certain cases where we can force a single $m_j = 1$ and the rest zero. In other words, we want that $\varepsilon(b) = \Lambda_i$ and $\varepsilon(\tilde{e}_i b) = \Lambda_j$.

We list some immediate corollaries to this lemma.

**Corollary 3.2.** Let $b \in B$ and suppose $a_{ij} = 0$. Then $\tilde{e}_j b = 0 \implies \tilde{e}_j \tilde{e}_i b = 0$.

**Corollary 3.3.** Let $a, b \in B$ both be singular nodes, and suppose $b$ is a parent of $a$, with $b = \tilde{e}_i a$. Then $\tilde{e}_j b \neq 0 \implies a_{ij} < 0$.

**Theorem 3.4 ([Kas93]).** Let $b \in B$ and suppose $\tilde{e}_i^{(m)} b \neq 0$ but $\tilde{e}_i^{(m+1)} b = 0$. Then

$$e_i^{(m)} G(b) = G(\tilde{e}_i^{(m)} b) \quad \text{and} \quad e_i^{(m+1)} G(b) = 0.$$
As a corollary to this theorem, employing the Serre relations, we can deduce several properties of singular nodes. We review the Serre relations below.

Fix \(i, j \in I, i \neq j\). Let \(\ell = 1 - \langle h_i, \alpha_j \rangle = 1 - a_{ij}\). Then

\[
\sum_{k=0}^{\ell} e_i^{(k)} e_j e_i^{(\ell-k)} = 0.
\]

**Corollary 3.5.**  
(1) Suppose \(a_{ij} = 0\). Then \(\tilde{e}_j b = 0, \tilde{e}_i^2 b = 0 \implies \tilde{e}_j (\tilde{e}_i b) = 0\).

(2) Suppose \(a_{ij} = -1\). Then \(\tilde{e}_j b = 0, \tilde{e}_i^2 b = 0 \implies \tilde{e}_i (\tilde{e}_j \tilde{e}_i b) = 0\).

(3) Suppose \(a_{ij} = -2\). Then \(\tilde{e}_j b = 0, \tilde{e}_i^2 b = 0 \implies \tilde{e}_i (\tilde{e}_j \tilde{e}_i \tilde{e}_j \tilde{e}_i b) = 0\). Also \(\tilde{e}_i \tilde{e}_j b = 0, \tilde{e}_i^2 \tilde{e}_j b = 0 \implies \tilde{e}_i^2 \tilde{e}_j b = 0\).

If in addition \(a_{ji} = -1\), then \(\tilde{e}_j b = 0, \tilde{e}_i^2 b = 0 \implies \tilde{e}_i^2 \tilde{e}_j b = 0\) and \(\tilde{e}_j (\tilde{e}_i \tilde{e}_j \tilde{e}_i b) = 0\).

**Proof.**  
(1) This follows directly from Corollary 3.2, which is a stronger statement. (We note one may also prove this using Theorem 3.4 in a manner similar to the subsequent cases.)

(2) From the Serre relations for \(a_{ij} = -1\), we know that \((e_i^{(2)} e_j - e_i e_j e_i + e_j e_i^{(2)})(G(b)) = 0\). Applying Theorem 3.4, \(\tilde{e}_j b = 0 \implies e_j G(b) = 0 \) and \(e_i G(b) = G(\tilde{e}_i b)\). Hence we get \(0 = e_i e_j e_i G(b) = e_i e_j G(\tilde{e}_i b)\).

Kashiwara’s equation 5.3.8 in [Kas93] gives \(e_i G(b)\) as a linear combination of \(G(\tilde{e}_i b)\) and \(G(b')\) where \(\varphi_k(b') \leq \varphi_k(b)\) for all \(k \in I\). Iterating this, we get that \(0 = e_i e_j G(\tilde{e}_i b)\) is a linear combination of \(G(\tilde{e}_i \tilde{e}_j \tilde{e}_i b)\) and terms \(G(b')\). It is straightforward (using equation \(8.1\)) to show the restrictions on \(b'\) can only be satisfied if \(\varepsilon_i (\tilde{e}_i \tilde{e}_j \tilde{e}_i b) \leq -a_{ij} - 1 = 0\). But this forces \(\tilde{e}_i \tilde{e}_j \tilde{e}_i b = 0\). In the case there are no such \(b'\), we then get \(G(\tilde{e}_i \tilde{e}_j \tilde{e}_i b) = 0\), so again \(\tilde{e}_i \tilde{e}_j \tilde{e}_i b = 0\).

(3) The conditions on \(b\) give us \(e_j G(b) = 0, e_i G(b) = G(\tilde{e}_i b)\), \(e_j G(\tilde{e}_i b) = 0\), and \(e_j e_i G(b) = G(\tilde{e}_j \tilde{e}_i b)\). The Serre relations imply \(0 = e_i^{(2)} e_j e_i G(b) = e_i^{(2)} G(\tilde{e}_j \tilde{e}_i b)\) which implies \(\tilde{e}_j \tilde{e}_i b = 0\). (In particular, this also implies \(e_i e_j e_i G(b) = G(\tilde{e}_i \tilde{e}_j \tilde{e}_i b)\).) For the second case, we get \(0 = e_i^{(3)} e_j G(b) = e_i^{(3)} G(\tilde{e}_j b)\) so that \(\tilde{e}_j \tilde{e}_i b = 0\).

For the final statement, the proof of the first implication follows immediately from equation \(8.1\). For the second, as \(a_{ji} = -1\), \(e_j e_i e_j e_i G(b) = e_j e_i G(\tilde{e}_i b) = e_j^{(2)} e_i G(\tilde{e}_i b) + e_i e_j^{(2)} G(\tilde{e}_i b) = 0\). So \(0 = e_j e_i e_j G(\tilde{e}_i b) = e_j G(\tilde{e}_j \tilde{e}_j \tilde{e}_i b)\), yielding \(\tilde{e}_j \tilde{e}_i \tilde{e}_j \tilde{e}_i b = 0\). 

\(\Box\)
We remark that there are similar statements for \( a_{ij} = -3, -4 \), but they do not translate into interesting statements about singular nodes as the other cases do in Theorem 3.7 below.

Case (1) of Corollary 3.5 says that if \( a_{ij} = 0 \) and we see

\[
\bullet \xrightarrow{i} \bullet \rightarrow \bullet \xrightarrow{j} b,
\]

we do not see \( \bullet \xrightarrow{j} \bullet \rightarrow \bullet \xrightarrow{i} b \).

Compare this with the fact that \( a_{ij} = 0 \) means that in the Dynkin diagram we see

\[
\circ j \circ i \text{ and not } \circ j \circ i
\]

in the crystal. Similarly, when \( a_{ij} = -1 \) and we see

\[
\bullet \xrightarrow{j} \bullet \xrightarrow{i} \bullet \rightarrow \bullet \xrightarrow{j} b,
\]

we do not see \( \bullet \xrightarrow{i} \bullet \xrightarrow{j} \bullet \rightarrow \bullet \xrightarrow{i} b \).

Compare this to the fact that when \( a_{ij} = -1 \) we see \( \circ j \circ i \) in the Dynkin diagram but

\[
\text{not } \circ i \circ j \text{ nor } \circ j \circ i \text{ nor } \circ i \rightarrow j
\]

which we should associate to \( \circ i \rightarrow j \) in the former case, and to the folding of \( \circ i \rightarrow j \) in the latter cases. In Theorem 3.7 below, we shall see that requiring certain singularity conditions on nodes forces the colors on their in-arrows to behave as a directed "path" or walk would on the Dynkin diagram, as suggested above. Choosing \( a, b \) with \( \varepsilon(a) = \Lambda_i, \varepsilon(b) = \Lambda_j \) and \( b \) the parent of \( a \) puts an "orientation" on the Dynkin diagram. As the Dynkin diagram’s vertices correspond to arrows in the crystal, we really are making a statement about a graph dual to the Dynkin diagram. It turns out the correct notion of duality in this setting is exactly captured in an associated level 1 perfect crystal.

Below, we recap, case by case, the consequences of Corollary 3.5 on all of the ancestors of a singular node \( a \) whose parent is also singular. We describe all walks on the crystal, from the highest weight node \( v \) to \( a \). These walks are described exactly by walks on the level 1 perfect crystal \( B^{1,1} \). These crystals are displayed in the body of the proof as well as in the appendix. (In type \( A_n^{(1)} \) we also need the perfect crystal \( B_n^{1,1} \) obtained by reversing all arrows in \( B^{1,1} \). In type \( A_1^{(1)} \) we require the grandparent to be singular as well.)
A necessary, but not sufficient, condition for both a node and its parent to be singular is that it has the form $\bar{f}_{i_1}\bar{f}_{i_2}\cdots\bar{f}_{i_k}\mathbf{v}$, where $i_1\rightarrow i_2\rightarrow\cdots\rightarrow i_k$ is a consecutive sequence of arrows in $B^{1,1}$. (The theorem also describes which nodes of this form are not singular.) This means that we can give a case by case description of the node’s ancestors, but a global statement about the walks from $\mathbf{v}$ to $\mathbf{a}$. The local nature of singularity means that the result in affine type follows from that in finite type (in small rank), and so we structure the statements and proofs of the following theorem accordingly.

**Theorem 3.6.** Let $B$ be a highest weight crystal with highest weight node $\mathbf{v}$ of type $A_n, n \geq 1$, $A_n^{(1)}(1), n \geq 2$, $A_n^{(2)}(2), n \geq 2$, $A_n^{(2)}(1), n \geq 2$, $A_{2n-1}(2), n \geq 3$, $B_n, n \geq 2$, $B_n^{(1)}, n \geq 3$, $C_n, n \geq 2$, $C_n^{(1)}, n \geq 2$, $D_n, n \geq 4$, $D_n^{(1)}, n \geq 4$, $D_{n+1}(2), n \geq 2$. Suppose $\mathbf{a} \in B$ is a singular node with singular parent. Then

$$\mathbf{a} = \bar{f}_{i_1}\bar{f}_{i_2}\cdots\bar{f}_{i_k}\mathbf{v},$$

only when $i_1\rightarrow i_2\rightarrow\cdots\rightarrow i_k$ is a consecutive sequence of arrows in the level 1 perfect crystal $B^{1,1}$ (or $B^{n,1}$ in type $A$) of appropriate type, omitting 0-arrows in finite type. If the Dynkin diagram does not contain that of $D_4$ as a subdiagram, then this sequence is unique. (In type $A_1^{(1)}$, we get the same conclusion if we also require $\mathbf{a}$ also have singular grandparent.)

**Theorem 3.7.**

1. Let $B$ be a highest weight crystal of type $A_n, n \geq 1$ or of type $A_n^{(1)}, n \geq 2$.

   Suppose $\mathbf{a}, \mathbf{b} \in B$ are both singular nodes with $\mathbf{b} = \bar{e}_i\mathbf{a}$. Then all ancestors of $\mathbf{a}$ are singular. There is a unique walk (on the directed graph) from the highest weight node $\mathbf{v} \in B$ to $\mathbf{a}$, given by $\mathbf{a} = \bar{f}_i\bar{f}_{i+1}\bar{f}_{i+2}\cdots\bar{f}_{i+k}\mathbf{v}$, where subscripts are taken mod $n$.

2. Let $B$ be a highest weight crystal of type $A_1^{(1)}$.

   Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c} \in B$ are all singular nodes with $\mathbf{b} = \bar{e}_i\mathbf{a}$, $\mathbf{c} = \bar{e}_j\mathbf{b}$. (Necessarily, $i \neq j$.) Then all ancestors of $\mathbf{a}$ are singular. There is a unique walk from the highest weight node $\mathbf{v}$ to $\mathbf{a}$, given by $\mathbf{a} = \bar{f}_i\bar{f}_j\bar{f}_i\bar{f}_j\cdots\mathbf{v}$.

3. Let $B$ be a highest weight crystal of type $C_n, n \geq 2$.

   Suppose $\mathbf{a}, \mathbf{b} \in B$ are both singular nodes with $\mathbf{b} = \bar{e}_i\mathbf{a}$. Then all ancestors of $\mathbf{a}$ are singular. There is a unique walk from the highest weight node $\mathbf{v}$ to $\mathbf{a}$, given by the following possibilities:

   a) $\mathbf{a} = \bar{f}_{i_1}\bar{f}_{i_2}\cdots\mathbf{v}$.

   b) $\mathbf{a} = \bar{f}_i\bar{f}_{i+1}\bar{f}_{i+2}\cdots\bar{f}_{n-1}\bar{f}_n\bar{f}_{n-1}\bar{f}_{n-2}\cdots\mathbf{v}$.
(4) Let \( B \) be a highest weight crystal of type \( B_n, n \geq 2 \).

Suppose \( a, b \in B \) are both singular nodes with \( b = \bar{e}_ia \).
Then all but one of the ancestors of \( a \) are singular. If there is a non-singular ancestor \( c \), it satisfies \( \varepsilon(c) = 2\Lambda_n \).

There is a unique walk from the highest weight node \( v \) to \( a \), given by the following possibilities:

(a) \( a = \tilde{f}_i\tilde{f}_i\pm\tilde{f}_i\pm\cdots v \).
(b) \( a = \tilde{f}_i\tilde{f}_{i+1}\cdots\tilde{f}_{n-1}\tilde{f}_n\tilde{f}_{n-2}\cdots v \).

(5) Let \( B \) be a highest weight crystal of type \( D_n, n \geq 4 \).

Suppose \( a, b \in B \) are both singular nodes with \( b = \bar{e}_ia \).
Then all but one of the ancestors of \( a \) are singular. If there is a non-singular ancestor \( c \), it satisfies \( \varepsilon(c) = \Lambda_{n-1} + \Lambda_n \).

There are at most two walks from the highest weight node \( v \) to \( a \), given by the following possibilities.

(a) \( a = \tilde{f}_i\tilde{f}_i\pm\cdots v \).
(b) \( a = \tilde{f}_i\tilde{f}_{i+1}\cdots\tilde{f}_{n-2}\tilde{f}_{n-1, n}\tilde{f}_{n-2}\cdots v \) (In this case, we get two walks.)

(6) Let \( B \) be a highest weight crystal of type \( C_n^{(1)}, n \geq 2, A_n^{(2)}, n \geq 2, A_n^{(2)\dagger}, n \geq 2, \) or \( D_{n+1}^{(2)}, n \geq 2 \).

Suppose \( a, b \in B \) are both singular nodes with \( b = \bar{e}_ia \). Then ancestors \( c \) of \( a \) are either singular or they satisfy \( \varepsilon(c) = 2\Lambda_n \) in types \( A_n^{(2)\dagger}, D_{n+1}^{(2)} \), \( \varepsilon(c) = 2\Lambda_0 \) in types \( A_n^{(2)}, D_{n+1}^{(2)} \).

In all cases, there is a unique walk from the highest weight node \( v \) to \( a \), given by the following possibilities:

\[ a = \tilde{f}_i\tilde{f}_{i_2}\cdots\tilde{f}_{i_k}v, \]

where \( i \rightarrow i_2 \rightarrow \cdots \rightarrow i_k \) is a consecutive sequence of arrows in the level 1 perfect crystal \( B_n^{1,1} \) of appropriate type.

(7) Let \( B \) be a highest weight crystal of type \( D_n^{(1)}, n \geq 4, A_n^{(2)}, n \geq 3, \) or \( B_n^{(1)} \).

Suppose \( a, b \in B \) are both singular nodes with \( b = \bar{e}_ia \). Then ancestors \( c \) of \( a \) are either singular or they satisfy \( \varepsilon(c) = 2\Lambda_n \) in type \( B_n^{(1)} \), \( \varepsilon(c) = \Lambda_{n-1} + \Lambda_n \) in type \( D_n^{(1)} \), \( \varepsilon(c) = \Lambda_1 + \Lambda_0 \) in types \( D_n^{(1)}, B_n^{(1)} \).
Walks from the highest weight node $v \in B$ to $a$, described by the following (infinite) possibilities.

$$a = \tilde{f}_i \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} v,$$

where $\xrightarrow{i \to i_2 \to \cdots \to i_k}$ is a consecutive sequence of arrows in $B^{1,1}$.

We remark that in cases not included above, such as exceptional types, or type $A_n(2)$, that requiring a certain number of consecutive singular nodes either gives many possible complicated walks from the highest weight node or none at all. At the end of this paper we have a short discussion regarding type $E_6$.

Proof. \([1] [A_n, A_n^{(1)}] \) We have $\varepsilon(a) = \Lambda_i$, and either $b = v$ or $\varepsilon(b) = \Lambda_j$ with $j$ connected to $i$ in the Dynkin diagram by Corollary 3.3. In this case $j = i \pm 1$, taking $j \text{mod} n$ if necessary. Applying this corollary again, $\tilde{e}_k(\tilde{e}_i \tilde{e}_j b) = 0$ unless $k = j \pm 1$. By case \([2] \) of Corollary 3.5 $0 = \tilde{e}_i(\tilde{e}_j \tilde{e}_i a) = \tilde{e}_i \tilde{e}_j b$, so we must have $k = i \pm 1$ and either $\tilde{e}_j b = v$ or $\varepsilon(\tilde{e}_j b) = \Lambda_k$. Hence we can inductively apply this argument to the pair $b$ and $\tilde{e}_j b$. As $B$ is a highest weight crystal, this process must eventually terminate at $\tilde{e}_{i \pm m} \cdots \tilde{e}_{i \pm 1} a = v$ which is equivalent to $a = \tilde{f}_i \tilde{f}_{i+1} \tilde{f}_{i+2} \cdots \tilde{f}_{i+m} v$.

The above sequence of consecutively colored arrows exactly corresponds to a sequence of arrows on the following perfect crystals.

As we only care about the arrow labels, we omit the node labels that are usually also pictured in the crystals.

The reader should compare the above perfect crystals to the Dynkin diagrams

and consider the discussion below Corollary 3.5. Observe that arrows being consecutive in the perfect crystal correspond to vertices being adjacent in the Dynkin diagram.

\([2] [A_1^{(1)}] \) We note that only in this case do we require three consecutive singular nodes. As above, we necessarily have $\varepsilon(a) = \Lambda_i$, $\varepsilon(b) = \Lambda_j$, $\varepsilon(c) = \Lambda_i$ (or $c = v$) by Corollary 3.3. If we can show $\varepsilon(\tilde{e}_i c) = \Lambda_j$ or that $\tilde{e}_j \tilde{e}_i c = 0$ (forcing $\tilde{e}_i c = v$), we will be done by a
By Corollary 3.3, \( \tilde{e} \) is again captured in the perfect crystal of type \( A_1 \) which is represented as:

\[
A_1^{(1)} : \begin{array}{c}
\circ \\
0 \\
\bullet \\
1
\end{array}
\]

This proof is similar to that of case (1). We need only consider the case that \( a \) and \( b = \tilde{e}_{n-1}a \) are singular with \( \varepsilon(a) = \Lambda_{n-2}, \varepsilon(b) = \Lambda_{n-1} \). Let \( c = \tilde{e}_n b \). We claim either \( c = v \) or \( \varepsilon(c) = \Lambda_{n-1} \).

Again, we draw the Dynkin diagram

\[
C_n : \begin{array}{c}
1 \\
2 \\
\cdot \cdot \cdot \\
n-1 \\
n
\end{array}
\]

and show the perfect crystal of type \( C_n^{(1)} \) with the 0-arrow removed, which is suggestive of picturing the double arrow as a folding. (Note that we recover the same graph reversing orientation of all arrows.)

Note that the conclusions (a), (b) can also be expressed as \( a = f_1 f_{i+1} f_{i+2} \cdots f_{i+k} v, \) so long as subscripts are taken mod \( 2n \), and one sets \( f_{n+m} := f_{n-m} \) for \( 0 < m < n \).

We need only consider the case that \( a \) and \( b = \tilde{e}_{n-2}a \) are singular with \( \varepsilon(a) = \Lambda_{n-2}, \varepsilon(b) = \Lambda_{n-1} \). Otherwise it reduces to case (1). Let \( c = \tilde{e}_{n-1} b \). We claim either \( c = v \) or \( \varepsilon(c) = \Lambda_{n-1} \) in which case \( \tilde{e}_{n} c = v \) or \( \varepsilon(c) = 2\Lambda_n \), in which case \( c \) is not singular, but both \( \tilde{e}_{n} c \) and \( \tilde{e}_{n}^2 c \) are singular, and \( \tilde{e}_{n-1} \tilde{e}_{n} c \) is either singular or \( 0 \).

By Corollary 3.3, we know \( \tilde{e}_k c = 0 \) unless \( k = n \) or \( n - 2 \). But case (1) of this theorem rules out the latter. Because \( a_{n,n-1} = -2 \), case (3) of Corollary 3.3, we know \( \tilde{e}_n^2 \tilde{e}_{n-1} b = 0 \), showing the first part of the claim. Now suppose \( \varepsilon(c) = \Lambda_n \). That means \( \tilde{e}_n^2 c = 0 \). Further, \( \tilde{e}_{n-1} \tilde{e}_n c = \tilde{e}_{n-1} \tilde{e}_n \tilde{e}_{n-1} b = 0 \) by case (2) of Corollary 3.3 as \( a_{n-1,n} = -1 \).

By Corollary 3.3, \( \tilde{e}_n \tilde{e}_n c = 0 \) for all \( k \neq n - 1 \), showing \( \tilde{e}_n c = v \) as the crystal \( B \) has a unique highest weight node.

Next suppose \( \varepsilon(c) = 2\Lambda_n \). In particular, notice that \( c \) is not singular. For \( k \neq n - 1, n \), we know \( 0 = \varepsilon_k(c) = \varepsilon_k(\tilde{e}_n c) = \varepsilon_k(\tilde{e}_n^2 c) \) by Lemma 3.1. As above, we still have \( \tilde{e}_{n-1} \tilde{e}_n c = 0 \), so that \( \tilde{e}_n c \) is singular.
To show \( e_n^2 c \) is singular, we need only show \( e_{n-1}^2 e_n^2 c = 0 \). Note

\[
e_{n-1} G(e_n^2 c) = e_{n-1} G(e_n e_{n-1} e_n b) = e_{n-1} e_{n-1} e_n G(b) = (e_{n-1} e_n - \frac{1}{2} e_n e_{n-1} e_n) e_{n-1} G(b) + \left( \frac{1}{2} e_n e_{n-1} e_n e_{n-1} - \frac{1}{2} e_n e_{n-1} e_n e_{n-1} e_n \right) e_n G(b) = 0
\]

by the Serre relations and Theorem 3.3. Hence \( e_n^2 c = 0 \).

We again show the Dynkin diagram, along with the perfect crystal with 0-arrows removed.

\[
B_n : \quad \xymatrix{ & & 1 \ar[rr] & & 2 \ar[rr] & & 3 \ar[ld] & & n-1 \ar[ld] & & n } \\
1 & 2 & 3 & n-1 & n
\]

[D\(n\)] For type \( D_n \), we need only consider the case that \( a \) and \( b = \tilde{e}_{n-3} a \) are singular with \( \varepsilon(a) = \Lambda_{n-3} \), \( \varepsilon(b) = \Lambda_{n-2} \). Let \( c = \tilde{e}_{n-2} b \).

By corollaries 3.3 and 3.5, \( \varepsilon(c) = m_0 \Lambda_n + m_1 \Lambda_{n-1} \) with \( 0 \leq m_1 \leq 1 \).

Let \( m = m_0 + m_1 \). If \( m = 0 \), then \( c = v \) so we are done. If \( m = 1 = m_\ell \), then \( \tilde{e}_{n-\ell} c = v \) and again we are done. Otherwise, let \( d = \tilde{e}_{n} \tilde{e}_{n-1} c \).

Observe \( \varepsilon_n(\tilde{e}_{n-1} c) = \varepsilon_n(c) = 1 \) and \( \varepsilon_{n-1} (\tilde{e}_n c) = 1 \) by Lemma 3.1.

Thus \( G(\tilde{e}_{n} \tilde{e}_{n-1} c) = e_n e_{n-1} G(c) = e_{n-1} e_n G(c) = G(\tilde{e}_{n} \tilde{e}_n c) \), so that \( d = \tilde{e}_{n} \tilde{e}_n c \) as well. If \( k \neq n-2 \), then \( \varepsilon_k(d) = \varepsilon_k(c) = 0 \). We can apply Theorem 3.3 to see \( e_{n-2}^2 d = 0 \), and so \( d \) is singular.

Standard arguments show either \( d = v \) or \( e_{n-2} d \) is also singular. And then this reduces to case [I].

We again show the Dynkin diagram and the perfect crystal with 0-arrows removed.

\[
D_n : \quad \xymatrix{ & & 1 \ar[rr] & & 2 \ar[rr] & & n-1 \ar[ld] & & n } \\
1 & 2 & n-1 & n
\]

[C\(n\), A\(2n\), A\(2n\)\(, D\(n\)\)] The local nature of singularity allows us to apply the results from cases [I], [I], [I] to these types (sometimes reindexing \( i \) for \( n - i \) when encountering \( \Lambda_0 \)).

We list the perfect crystals, and for completeness, the possible walks from \( v \) to \( a \).

\[
C_n^{(1)} (n \geq 2) : \quad \xymatrix{ & 0 \ar[rr] & & 1 \ar[rr] & & 2 \ar[ld] & & n-1 \ar[ld] & & n } \\
0 & 1 & 2 & n-1 & n
\]
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\[ A_{2n}^{(2)}, (n \geq 2) \]

\[ A_{2n+1}^{(2)}, (n \geq 2) \]

\[ D_{n+1}^{(2)}, (n \geq 2) \]

As above, we may apply the results from cases (1), (3), (4), (5) to these types (with appropriate reindexing).
We list the perfect crystals, and for completeness, the possible walks from \( v \) to \( a \).

Below we again use the notation \( \tilde{f}_{n-1,n} \) to stand for either \( \tilde{f}_{n-1}\tilde{f}_n \) or \( \tilde{f}_n\tilde{f}_{n-1} \) in types \( D_n, D_n^{(1)} \) and \( \tilde{f}_{0,1} \) to stand for either \( \tilde{f}_0\tilde{f}_1 \) or \( \tilde{f}_1\tilde{f}_0 \) in types \( D_n^{(1)}, B_n^{(1)} \).

\[ D_n^{(1)}, (n \geq 4) \]

\[ A_{2n}^{(2)}, (n \geq 2) \]

\[ B_n^{(1)}, (n \geq 3) \]

4. Existence

The above theorems consisted of several “uniqueness” statements. The corresponding existence statements also hold.
In Theorem 3.6, we described all sequences \(i_1, i_2, \ldots, i_k\) such that \(a = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} v\), where \(v\) is the highest weight node and we required \(a\) to be singular with singular parent. These possible sequences corresponded to walks \(i_1 \to i_2 \to \cdots \to i_k\) on a perfect crystal. Below we will exhibit a highest weight crystal (one of level 1 or level 2 suffices) and such a node \(a\) for every such walk (excluding of course walks where \(a_{i_1, i_2} \geq 0\), as in that case \(a\) would not be singular).

We recall that the tensor product of crystals \(B_2 \otimes B_1\) is defined by the nodes being the Cartesian product of the nodes of \(B_2\) and \(B_1\), \(\varphi (b_2 \otimes b_1) = \varphi (b_2) + \varphi (b_1)\), and arrows are described by the following rule
\[
\tilde{e}_i (b_2 \otimes b_1) = \begin{cases} 
\tilde{e}_i b_2 \otimes b_1 & \text{if } \varphi_i (b_2) \geq \varepsilon_i (b_1) \\
\varepsilon_i (b_2) & \text{if } \varphi_i (b_2) < \varepsilon_i (b_1)
\end{cases}
\]

Consequently
\[
\varepsilon_i (b_2 \otimes b_1) = \varepsilon_i (b_2) + \max\{0, \varepsilon_i (b_1) - \varphi_i (b_2)\}
\]
\[
\varphi_i (b_2 \otimes b_1) = \varphi_i (b_1) + \max\{0, \varphi_i (b_2) - \varepsilon_i (b_1)\}.
\]

We recall the following theorem.

**Theorem 4.1** ([KK98], [KKM+92]). Let \(\lambda\) be a dominant integral weight of level \(k\), and \(B\) be a perfect crystal of level \(\ell\), and suppose \(k \geq \ell\). Then
\[
B(\lambda) \otimes B \simeq \bigoplus_{b \in B^{\leq \lambda}} B(\lambda + \text{wt}(b))
\]
where \(B^{\leq \lambda} = \{b \in B \mid \varepsilon_i (b) \leq \langle h_i, \lambda \rangle \forall i\}\).

We set
\[
\psi^\lambda_\mu : B(\mu) \to B(\lambda) \otimes (B_1^{1,1})^{\otimes k}
\]
to be the embedding dictated by the above theorem, when it is defined. Observe \((\psi^\lambda_\nu \otimes id^{\otimes k}) \circ \psi^\nu_\mu = \psi^{\lambda+\mu}_{\nu+k}\).

We refer the reader to the appendix for a list of the level 1 perfect crystals \(B_1^{1,1}\) (including \(B_n^{1,1}\) in type \(A\)). There is a standard way of labelling the nodes, but it will be convenient here to ignore that convention, so we have omitted that labelling in the appendix.

Let \(i_1 \to i_2 \cdots \to i_k\) be a walk on \(B_1^{1,1}\) (or \(B_n^{1,1}\) in type \(A\)). Let
\[
m = |\{r \mid a_{r, r+1} \geq 0, 1 \leq r < k\}|
\]
and let
\[
\tilde{i_1} \otimes \tilde{i_2} \otimes \cdots \otimes \tilde{i_k} \in (B_1^{1,1})^{\otimes (k-m)}
\]
be such that $\tilde{i}_r$ is the node

$$
\begin{array}{c}
\tilde{i}_r \\
\rightarrow
\tilde{i}_r \\
\rightarrow
\tilde{i}_{r+1}
\end{array}
$$

with an $i_r$-colored arrow going in and $i_{r+1}$ going out if $a_{i_r,i_{r+1}} < 0$. So long as $k > 1$, these nodes are well-defined and this also determines $\tilde{i}_k$. Observe that in the case $a_{i_1,i_2} \geq 0$, the node we describe is actually $\tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k$. Also note the labelling very much depends on the walk, and that one node can receive many different labels.

**Lemma 4.2.** Let $\tilde{i}_1 \otimes \tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k \in (B^{1,1})^\otimes (k-m)$ be as above. For all $i \in I$,

1. $\varepsilon_i(\tilde{i}_1 \otimes \tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k) = \varepsilon_i(\tilde{i}_1)$
2. $\tilde{e}_i(\tilde{i}_1 \otimes \tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k) = \tilde{e}_i(\tilde{i}_1) \otimes \tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k$
3. $\varphi_i(\tilde{i}_1 \otimes \tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k) = \varphi_i(\tilde{i}_k)$

**Proof.** When $k = 1$ this is immediate. Recall from (4.1), $\varepsilon_i(\tilde{i}_1 \otimes \tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k) = \varepsilon_i(\tilde{i}_1) + \max\{0, \varepsilon_i(\tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k) - \varphi_i(\tilde{i}_1)\}$. By the inductive hypothesis, $\varepsilon_i(\tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k) = \varepsilon_i(\tilde{i}_2)$ (by which we mean the leftmost node in case $a_{i_2,i_3} \geq 0$). If $\varepsilon_i(\tilde{i}_2) = 0$, we are done. If $\varepsilon_i(\tilde{i}_2) \neq 0$, we will show $\varepsilon_i(\tilde{i}_2) - \varphi_i(\tilde{i}_1) \leq 0$.

Consider the following possibilities. First, $i = i_2$ and $\varepsilon_{i_2}(\tilde{i}_2) = 1$. As $\varphi_{i_2}(\tilde{i}_1) \geq 1$, we are done. Second, suppose $i = i_2$ and $\varepsilon_{i_2}(\tilde{i}_2) > 1$. In fact, because we assume $\tilde{i}_1$ contributes to the tensor and $i_2 \rightarrow \tilde{i}_1$ joins $\tilde{i}_1$ to $\tilde{i}_2$, this cannot happen. It would mean $\tilde{i}_2$ does not contribute, and the “leftmost” node we refer to above is in fact $\tilde{i}_3$. We have

$$
\begin{array}{c}
\tilde{i}_1 \\
\rightarrow
\tilde{i}_2 \\
\rightarrow
\tilde{i}_3
\end{array}
$$

as $a_{i_2,i_3} = 2 \geq 0$, so that $\varepsilon_{i_2}(\tilde{i}_2 \otimes \tilde{i}_3 \otimes \cdots \tilde{i}_k) = \varepsilon_{i_2}(\tilde{i}_3) = \varphi_{i_2}(\tilde{i}_1) = 2$.

Third, suppose $i \neq i_2$. Then we must have

$$
\begin{array}{c}
i \\
\rightarrow
\tilde{i}_1 \\
\rightarrow
\tilde{i}_2 \\
\rightarrow
\tilde{i}_3
\end{array}
$$

and $\varepsilon_i(\tilde{i}_3) = \varphi_i(\tilde{i}_1) = 1$. Again, $\tilde{i}_2$ does not contribute.

Computing $\varphi_i$ is similar. The above conclusions come from examining all $B^{1,1}$ and from our definition of the node in $(B^{1,1})^\otimes (k-m)$ that our walk specifies.
The rule for computing $\tilde{e}_i$ of a tensor product gives us the second statement. □

Write $v_\lambda \in B(\lambda)$ for the highest weight node.

**Proposition 4.3.** Let $\tilde{i}_0 \to \tilde{i}_1 \to \cdots \to \tilde{i}_k$ be a walk on $B^{1,1}$ (or $B^{n,1}$ in type $A$). Let $\tilde{i}_0$ be the node such that $\tilde{i}_0 \to \tilde{i}_1$, and let $\lambda = \varepsilon(\tilde{i}_0)$.

Let $\mu = \varphi(\tilde{i}_{k-1})$.

1. $v_\lambda \otimes \tilde{i}_0 \otimes \tilde{i}_1 \otimes \cdots \otimes \tilde{i}_{k-1} = \psi_{k-m}^{\lambda,\mu}(v_\mu)$

2. $v_\lambda \otimes \tilde{i}_1 \otimes \cdots \otimes \tilde{i}_k = \psi_{k-m}^{\lambda,\mu}(a)$, where $a = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} v_\mu$.

3. In particular $a \neq 0$, and if $\tilde{i}_1 \otimes \cdots \otimes \tilde{i}_k$ is singular (with singular parent) so is $a$.

**Proof.** 1. From [14,1], $\varepsilon_i(v_\lambda \otimes \tilde{i}_0 \otimes \tilde{i}_1 \otimes \cdots \otimes \tilde{i}_{k-1}) = \varepsilon_i(v_\lambda) + \max\{0, \varepsilon_i(\tilde{i}_0 \otimes \cdots \otimes \tilde{i}_{k-1}) - \varphi_i(v_\lambda)\} = 0 + \max\{0, \varepsilon_i(\tilde{i}_0) - \varphi_i(v_\lambda)\} = 0$ for all $i$ by our choice of $\lambda$. Hence it is a highest weight node. Lemma 4.2 computes its weight is $\mu$, so it must be the image of $v_\mu$. Notice $\mu$ is of level 1 or 2.

2. We only need show $v_\lambda \otimes \tilde{i}_1 \otimes \cdots \otimes \tilde{i}_k = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} (v_\lambda \otimes \tilde{i}_0 \otimes \tilde{i}_1 \otimes \cdots \otimes \tilde{i}_{k-1})$.

We will induct on $k$.

In the case $k = 1$, $\tilde{e}_i(v_\lambda \otimes \tilde{i}_1) = 0$ if $i \neq i_1$ and $\tilde{e}_{i_1}(v_\lambda \otimes \tilde{i}_1) = v_\lambda \otimes \tilde{i}_1$ is singular with $i_1$ describing the only walks from the appropriate highest weight node to it.

We compute, using the inductive hypothesis,

$\psi_{k-m}^{\lambda,\mu}(a) = \tilde{f}_{i_1}(\psi_{1}^{\lambda,\Lambda_{i_1}} \otimes id^{\otimes(k-1-m)}) \circ \psi_{k-1-m}^{\Lambda_{i_1},\mu}(\tilde{f}_{i_2} \cdots \tilde{f}_{i_k} v_\mu)$

$= \tilde{f}_{i_1}(\psi_{1}^{\lambda,\Lambda_{i_1}} \otimes id^{\otimes(k-1-m)})(v_{\Lambda_{i_1}} \otimes \tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k)$

$= \tilde{f}_{i_1}(v_\lambda \otimes \tilde{i}_0 \otimes \tilde{i}_2 \otimes \cdots \otimes \tilde{i}_k)$

$= v_\lambda \otimes \tilde{i}_1 \otimes \cdots \otimes \tilde{i}_k$

3. This follows from Lemma 4.2 and that $\psi_{k-m}^{\lambda,\mu}$ is an embedding. Note that for $k \geq 3$, so long as $a_{i_1,i_2} < 0, a_{i_2,i_3} < 0$ the node will be singular with singular parent. □

5. Representation-theoretic interpretation in type $A$

In this paper, we studied a singular node whose parent is also singular in highest weight crystals of finite and affine type. The papers [FLO+98a, FLO+98b] characterized all singular nodes in a level one highest weight crystal of type $A^{(1)}$ (and [FLO+99] for higher levels)
by their behavior under tensor product of crystals, and they gave a representation-theoretic interpretation of these singular nodes as answering the Jantzen-Seitz problem. These nodes correspond to irreducible modules of the finite Hecke algebra $H_n^{\text{fin}}$ of type $A$ that remain irreducible on restriction from $H_n^{\text{fin}}$ to $H_{n-1}^{\text{fin}}$ (or for the Ariki-Koike (cyclotomic Hecke) algebras). One may then ask: which irreducible modules of $H_n^{\text{fin}}$ remain irreducible on restriction to $H_{n-2}^{\text{fin}}$?

The above Theorem 3.7 in type $A$ was motivated by the following representation-theoretic fact, which addresses the question just posed. If an irreducible module $M$ of the affine Hecke algebra $H_n$ of type $A$ is irreducible on restriction to $H_{n-2}$, then $M$ is one-dimensional and either a trivial or Steinberg (sign) module. The main theorem of [Gro] says that the above Hecke-theoretic statement and case (1) of Theorem 3.7 are equivalent. However, a purely representation-theoretic proof is as straightforward as the crystal-theoretic proofs above.

Compare the representation-theoretic translation (given below) of the crystal-theoretic proof with the following direct proof communicated by Grojnowski.

Let $H_n$ denote the affine Hecke algebra of type $A$. The algebra depends on a parameter $q$, and when we specialize $q = 1$, we recover the group algebra of the wreath product of $\mathbb{Z}$ with the symmetric group. We denote by $T_i$ the generator of $H_n$ that degenerates to the simple transposition $s_i = (i, i+1)$.

Let $M$ be an irreducible module of $H_n$, and suppose $\text{Res}_{H_{n-2}}^{H_n} M$ is an irreducible $H_{n-2}$-module. In particular, the generator $T_{n-1}$ commutes with $H_{n-2}$ and so acts by a scalar on all of $M$, where that scalar is $-1$ or $q$, as $(T_{n-1} + 1)(T_{n-1} - q) = 0$. All of the $T_i$ are conjugate in $H_n$, so all the $T_i$ also act by that same scalar on all of $M$. In the case that scalar is $q$, $M$ must have been a trivial module, and when it is $-1$ we have a Steinberg (sign) module. In particular, $M$ is one-dimensional and $\text{Res}_{H_k}^{H_n} M$ is irreducible for all $k \leq n$. This argument is the correct explanation for the result, but it is unclear what its interpretation is in other types.

In contrast, here is the representation-theoretic version of the crystal-theoretic proof given in case (1) of Theorem 3.7.

We refer the reader to [Gro] and [GV01] for all the definitions (as it is not the main focus of this paper). Let $M$ be an irreducible module of $H_n$. There are operators

\[ e_i : \text{Rep} H_n \rightarrow \text{Rep} H_{n-1} \]
that satisfy $\bigoplus_i e_i M = \text{Res}^{H_n}_{H_{n-1}} M$. Further, if $e_i^2 M = 0$ but $e_i M \neq 0$, then $e_i M$ is an irreducible $H_{n-1}$-module, and conversely. A node being singular corresponds to $\text{Res}^{H_n}_{H_{n-1}} M$ being irreducible. Hence the hypotheses of case (1) of Theorem 3.7 correspond to the assumption that $\text{Res}^{H_n}_{H_{n-1}} M = e_i M$ is irreducible, $e_{i+1} e_i M$ is also irreducible, and $e_j e_i M = 0$ for $j \neq i + 1$. This implies $\text{Res}^{H_n}_{H_{n-2}} M = e_{i+1} e_i M$ is irreducible.

We want to conclude that $e_{i+2} e_{i+1} e_i M$ is also irreducible or zero (so we need only show $e_{i+2} e_{i+1} e_i M = 0$, and that $e_j e_{i+1} e_i M = 0$ for $j \neq i + 2$). These all follow from the fact, shown in [Gro], [GV01], that the $e_i$ satisfy the Serre relations of type A. (For ease of exposition, we omit the case where the parameter $q$ appearing in the definition of $H_n$ is a second root of unity, corresponding to case 2 of Theorem 3.7. We omitted this case in Grojnowski’s direct proof above as well, where one must confront the fact that the $T_i$ may not act semisimply.) The proof here is very close to that of case (1) of Theorem 3.7 and (2) of Corollary 3.5, as they both rely on the Serre relations.

We also point out that this statement is obvious for the representation theory of the symmetric group in characteristic 0. Here, irreducible representations are indexed by partitions, and the branching rule says the restriction of an irreducible can be described by removing certain boxes from the partition. For the restriction of an irreducible module from $S_n$ to $S_{n-1}$ to be irreducible means its partition can have at most one removable box, and hence be a rectangle. But for that rectangle to share the same property, the original shape must have been a single row or column, hence our original representation was the trivial or sign module. We remark that the combinatorics in prime characteristic are appreciably different.

While for symmetric group modules in characteristic 0 this is a classical fact, it seemed a surprising statement for crystals: that two consecutive singular nodes could determine all of their ancestors, and that the perfect crystal $B^{1,1}$ controls all the paths between that node and the highest weight node.

6. Exceptional types

Corollaries 3.2, 3.3, 3.5 say that in simply laced type, if $\varepsilon(a) = \Lambda_i, \varepsilon(\bar{e}_i a) = \Lambda_j$, and $\varepsilon_k(\bar{e}_j \bar{e}_i a) \neq 0$, then we see $\bar{e}_i \bar{e}_j \bar{e}_k$ in the Dynkin diagram. In particular $k \neq i$. In classical types, the possible $v \xrightarrow{i_1} \cdots \xrightarrow{i_k} a$ are in correspondence with walks on $B^{1,1}$. In exceptional types, one can also describe a directed graph the corresponding walks must live on. The directed graph is dictated by equation 3.1 and case
2 of Corollary 3.5. They are very complicated to draw (planarly), so we only give pictures for $E_6$ below. Just as in type $A$, the two graphs below differ by reversing orientation of all arrows.

7. Appendix
The crystal graphs $B^{1,1}$ are listed below.

We also need $B^{n-1,1}$ in type $A$:

- $A_n^{(1)}$
- $B_n^{(1)}$
- $C_n^{(1)}$
- $D_n^{(1)}$
- $A_{2n}^{(2)}$
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REFERENCES

[FLO+98a] O. Foda, B. Leclerc, M. Okado, J.-Y. Thibon, and T. A. Welsh. Combinatorics of solvable lattice models, and modular representations of Hecke algebras. In Geometric analysis and Lie theory in mathematics and physics, volume 11 of Austral. Math. Soc. Lect. Ser., pages 243–290. Cambridge Univ. Press, Cambridge, 1998.

[FLO+98b] Omar Foda, Bernard Leclerc, Masato Okado, Jean-Yves Thibon, and Trevor A. Welsh. RSOS models and Jantzen-Seitz representations of Hecke algebras at roots of unity. Lett. Math. Phys., 43(1):31–42, 1998.

[FLO+99] Omar Foda, Bernard Leclerc, Masato Okado, Jean-Yves Thibon, and Trevor A. Welsh. Branching functions of $A_{n-1}^{(1)}$ and Jantzen-Seitz problem for Ariki-Koike algebras. Adv. Math., 141(2):322–365, 1999.

[Gro] I. Grojnowski. Affine $\mathfrak{sl}_p$ controls the representation theory of the symmetric group and related Hecke algebras.

[GV01] I. Grojnowski and M. Vazirani. Strong multiplicity one theorems for affine Hecke algebras of type $A$. Transform. Groups, 6(2):143–155, 2001.

[Kas93] Masaki Kashiwara. Global crystal bases of quantum groups. Duke Math. J., 69(2):455–485, 1993.

[Kas95] Masaki Kashiwara. On crystal bases. In Representations of groups (Banff, AB, 1994), volume 16 of CMS Conf. Proc., pages 155–197. Amer. Math. Soc., Providence, RI, 1995.

[KK98] Seok-Jin Kang and Masaki Kashiwara. Quantized affine algebras and crystals with core. Comm. Math. Phys., 195(3):725–740, 1998.

[KKM+92] Seok-Jin Kang, Masaki Kashiwara, Kailash C. Misra, Tetsuji Miwa, Toshiki Nakashima, and Atsushi Nakayashiki. Perfect crystals of quantum affine Lie algebras. Duke Math. J., 68(3):499–607, 1992.

[Ste03] John R. Stembridge. A local characterization of simply-laced crystals. Trans. Amer. Math. Soc., 355(12):4807–4823 (electronic), 2003.