The Beilinson-Bernstein correspondence for quantized enveloping algebras

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Abstract

Theory of the quantized flag manifold as a quasi-scheme (non-commutative scheme) has been developed by Lunts-Rosenberg [12]. They have formulated an analogue of the Beilinson-Bernstein correspondence using the \( q \)-differential operators introduced in their earlier paper [11]. In this paper we shall establish its modified version using a class of \( q \)-differential operators, which is (possibly) smaller than the one in [11].

0 Introduction

Let \( G \) be a connected, simply-connected semisimple algebraic group over the complex number field \( \mathbb{C} \), and let \( B \) and \( B^- \) be Borel subgroups of \( G \) such that \( H = B \cap B^- \) is a maximal torus of \( G \). Denote the Weyl group by \( W \) and the character group of \( H \) by \( \Lambda \). We choose a system of positive roots \( \Delta^+ \subset \Lambda \) as the weights of \( \text{Lie}(B,B^-) \). Let \( \Lambda^+ \) be the set of dominant integral weights. We have \( \Lambda = \sum_{i=1}^{\ell} \mathbb{Z}\omega_i, \ \Lambda^+ = \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\omega_i \), where \( \{\omega_1, \ldots, \omega_\ell\} \) denotes the set of fundamental weights. For \( \lambda \in \Lambda^+ \) we denote the irreducible (left) \( G \)-module with highest weight \( \lambda \) by \( V^1(\lambda) \).

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The algebraic variety $B = B^{-\setminus}G$ is called the flag manifold for $G$. It has an affine open covering

$$B = \bigcup_{w \in W} U_w,$$

$$U_w = B^{-\setminus}B^w = \text{Spec}(R^1_w).$$

We have a closed embedding of $B$ into $\mathbb{P}(V^1(\omega_1)^*) \times \cdots \times \mathbb{P}(V^1(\omega_\ell)^*)$ given by $B^w g \mapsto ([v_1 g], \ldots, [v_\ell g])$, where $v_i$ is a non-zero element of the (right) $G$-module $V^1(\omega_i)^* = \text{Hom}_C(V^1(\omega_i), C)$ satisfying $v_i h = \omega_i(h)v_i$ for $h \in H$.

Hence we have $B = \text{Proj}_\mathbb{Z}(A^1)$ for a $\mathbb{Z}$-graded ring $A^1$. The graded ring $A^1$ is described as follows. Let $C[G]$ denote the coordinate algebra of $G$. One has a natural $G$-bimodule structure on $C[G]$. Then we have the identification

$$A^1 = \{ \varphi \in C[G] \mid \varphi g = \varphi (g \in [B^-, B^-]) \},$$

and the grading $A^1 = \bigoplus_{\lambda \in \Lambda^+} A^1(\lambda)$ by $\Lambda(\simeq \mathbb{Z}_\ell)$ is given by

$$A^1(\lambda) = \{ \varphi \in A^1 \mid \varphi h = \lambda(h)\varphi (h \in H) \} \quad (\lambda \in \Lambda^+).$$

In this paper we shall be concerned with the $q$-analogue $B_q$ of the flag manifold $B$ introduced by Lunts-Rosenberg [12]. Let $U = U_q(g)$ denote the corresponding simply-connected quantized enveloping algebra. It is a Hopf algebra over the field $F = \mathbb{Q}(q^{1/\ell_0})$, where $q$ is transcendental and $\ell_0$ is an appropriate positive integer. See [1] below for the precise definition. We note that the Cartan part of $U$ is identified with the group algebra of the weight lattice $\Lambda$. It is well-known that a $q$-analogue $C_q[G]$ of $C[G]$ is defined as a Hopf algebra dual to $U$. Using $C_q[G]$ one can naturally define $q$-analogues $A$ and $R_w$ of $A^1$ and $R^1_w$ respectively. $A$ is a $\Lambda$-graded $F$-algebra and $R_w$ is an $F$-algebra; however, they are non-commutative. Hence in order to give meanings to

$$B_q = \text{Proj}_\Lambda(A) = \bigcup_{w \in W} U_{w,q}, \quad U_{w,q} = \text{Spec}(R_w)$$

we need the notion of non-commutative algebraic varieties.

A starting point of the non-commutative algebraic geometry is the general fact that a (commutative) scheme $X$ is uniquely determined from the category $\text{Mod}(\mathcal{O}_X)$ of quasi-coherent sheaves (Rosenberg [18]). It was Manin who proposed to consider generalization of the category $\text{Mod}(\mathcal{O}_X)$ in the non-commutative setting. There already exist several works along the line of this Manin’s idea. Theory of non-commutative projective schemes for non-commutative graded rings are developed by Artin-Zhang [1], Manin [15], Verevkin [20], Rosenberg [17]. Theory of general non-commutative schemes
equipped with non-commutative affine open covering is also given by Rosenberg [R]. The quantized flag manifold $B_q$ is a non-commutative projective scheme as well as a general non-commutative scheme (quasi-scheme) in the sense of Rosenberg.

Let us give a description of $B_q$ as a non-commutative projective scheme. For a ring $R$ we denote the category of left $R$-modules by $\text{Mod}(R)$. For a $\Lambda$-graded ring $R = \bigoplus_{\lambda \in \Lambda} R(\lambda)$ a left $R$-module $M$ equipped with the decomposition $M = \bigoplus_{\lambda \in \Lambda} M(\lambda)$ of $R(0)$-submodules is called a $\Lambda$-graded left $R$-module if $R(\xi) M(\lambda) \subset M(\lambda + \xi)$ for any $\lambda, \xi \in \Lambda$. We denote the category of $\Lambda$-graded left $R$-modules by $\text{Mod}_\Lambda(R)$. Let $\text{Tor}_\Lambda(A)$ be the full subcategory of $\text{Mod}_\Lambda(A)$ consisting of $M \in \text{Mod}_\Lambda(A)$ such that for any $m \in M$ there exists some $\mu \in \Lambda^+$ satisfying $A(\xi) m = \{0\}$ for any $\xi \in \mu + \Lambda^+$. Then we can define an abelian category $\text{Mod}(\mathcal{O}_{B_q})$ of “quasi-coherent sheaves on $B_q$” as the quotient

$$\text{Mod}(\mathcal{O}_{B_q}) = \text{Mod}_\Lambda(A)/\text{Tor}_\Lambda(A).$$

Moreover, “the global section functor”

$$\Gamma : \text{Mod}(\mathcal{O}_{B_q}) \to \text{Mod}(\mathbb{F})$$

is defined as follows. The natural functor

$$\omega^* : \text{Mod}_\Lambda(A) \to \text{Mod}(\mathcal{O}_{B_q}) = \text{Mod}_\Lambda(A)/\text{Tor}_\Lambda(A)$$

admits a right adjoint functor

$$\omega_* : \text{Mod}(\mathcal{O}_{B_q}) \to \text{Mod}_\Lambda(A),$$

and $\Gamma$ is defined by $\Gamma(M) = (\omega_* M)(0)$. We can also define the higher cohomology groups $H^i(M)$ for an object $M$ of $\text{Mod}(\mathcal{O}_{B_q})$ by $H^i(M) = (R^i \Gamma)(M)$ using the right derived functors.

Now let us consider “$D$-modules on $B_q$”. The ring $\hat{D}$ of $q$-differential operators acting on the graded algebra $A$ is defined by Lunts-Rosenberg [11, 12] as follows. For $\varphi \in A$ let $\ell_\varphi, r_\varphi \in \text{End}_\mathbb{F}(A)$ denotes the left multiplication and the right multiplication respectively; i.e. $\ell_\varphi(\psi) = \varphi \psi, r_\varphi(\psi) = \psi \varphi$ for $\psi \in A$. Define an increasing sequence

$$\{0\} = F^{-1} \hat{D} \subset F^0 \hat{D} \subset F^1 \hat{D} \subset \cdots \subset \text{End}_\mathbb{F}(A)$$

of $\mathbb{F}$-subspaces of $\text{End}_\mathbb{F}(A)$ inductively by

$$F^p \hat{D} = \sum_{\varphi_1, \varphi_2 \in A, \lambda, \mu \in \Lambda} \ell_{\varphi_1} (F^p \hat{D})_{\lambda, \mu} \ell_{\varphi_2},$$

where $(F^p \hat{D})_{\lambda, \mu}$ consists of $d \in \text{End}_\mathbb{F}(A)$ satisfying
(a) \(d(A(\xi)) \subset A(\xi + \lambda)\) for any \(\xi \in \Lambda\),
(b) \(d\ell_{\varphi} - q^{(\mu, \xi)} \ell_{\varphi} d \in F^{p-1} \tilde{D}\) for any \(\xi \in \Lambda\) and \(\varphi \in A(\xi)\).
Here, \((, ) : \Lambda \times \Lambda \to \mathbb{Q}\) is the restriction of a standard symmetric bilinear form on \(\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda\). Set
\[
\tilde{D} = \bigcup_p F^p \tilde{D} \subset \text{End}_\mathbb{F}(A).
\]
Then \(\tilde{D}\) is a subalgebra of \(\text{End}_\mathbb{F}(A)\), and its \(\Lambda\)-grading \(\tilde{D} = \bigoplus_{\lambda \in \Lambda} \tilde{D}(\lambda)\) is given by
\[
\tilde{D}(\lambda) = \{d \in \tilde{D} \mid d(A(\xi)) \subset A(\xi + \lambda)\}.
\]
It seems to be a hard task to determine how large \(\tilde{D}\) is. Anyway \(\tilde{D}\) contains \(\ell_{\varphi}, r_{\varphi}\) for \(\varphi \in A\), operators \(\partial_u (u \in U)\) given by the natural action of \(U\) on \(A\), and the degree operators \(\sigma_\lambda (\lambda \in \Lambda)\) given by \(\sigma_\lambda A(\xi) = q^{(\lambda, \xi)} \text{id}\). We denote by \(D\) the subalgebra of \(\tilde{D}\) generated by the operators \(\ell_{\varphi}, r_{\varphi}\) (\(\varphi \in A\)), \(\partial_u (u \in U)\), \(\sigma_\lambda (\lambda \in \Lambda)\). It is shown using the universal \(R\)-matrix that \(D\) is generated by \(\ell_{\varphi}\) (\(\varphi \in A\)), \(\partial_u (u \in U)\), \(\sigma_\lambda (\lambda \in \Lambda)\).

Let \(\lambda \in \Lambda\). We define the category \(\text{Mod}(\tilde{D}_{\mathcal{B}_q}, \lambda)\) of “modules over the sheaf of rings of twisted differential operators \(\tilde{D}_{\mathcal{B}_q, \lambda}\)” by
\[
\text{Mod}(\tilde{D}_{\mathcal{B}_q, \lambda}) = \text{Mod}_{\Lambda, \lambda}(\tilde{D})/\text{Tor}_{\Lambda, \lambda}(\tilde{D}),
\]
where \(\text{Mod}_{\Lambda, \lambda}(\tilde{D})\) denotes the category of \(\Lambda\)-graded left \(\tilde{D}\)-modules \(M\) satisfying \(\sigma_\mu [M(\xi) = q^{(\mu, \xi)} \text{id}\) for any \(\mu, \xi \in \Lambda\), and \(\text{Tor}_{\Lambda, \lambda}(\tilde{D})\) is its full subcategory consisting of objects of \(\text{Mod}_{\Lambda, \lambda}(\tilde{D})\) which belong to \(\text{Tor}_{\Lambda}(A)\) as \(\Lambda\)-graded \(A\)-modules. Here, we identify \(A\) with a graded subring of \(\tilde{D}\) by \(A \ni \varphi \mapsto \ell_{\varphi} \in \tilde{D}\).
Define \(\tilde{D}_\lambda \in \text{Mod}_{\Lambda, \lambda}(\tilde{D})\) by
\[
\tilde{D}_\lambda = \tilde{D} / \sum_{\mu \in \Lambda} \tilde{D}(\sigma_\mu - q^{(\mu, \lambda)}).
\]
Since \(\sigma_\mu\) belongs to the center of \(\tilde{D}(0)\) we have an \(\mathbb{F}\)-algebra structure on \(\tilde{D}_\lambda(0)\). Then the global section functor \(\Gamma : \text{Mod}(\mathcal{O}_{\mathcal{B}_q}) \to \text{Mod}(\mathbb{F})\) induces a left exact functor
\[
\tilde{\Gamma}_\lambda : \text{Mod}(\tilde{D}_{\mathcal{B}_q, \lambda}) \to \text{Mod}(\tilde{D}_\lambda(0)).
\]
Denote the Verma module for \(U\) with highest weight \(\lambda\) by \(T(\lambda)\) and its annihilator in \(U\) by \(J_\lambda\). By Joseph [8] the ideal \(J_\lambda\) is generated by its intersection with the center of \(U\). We have canonical \(\mathbb{F}\)-algebra homomorphisms
\[
U/J_\lambda \to \tilde{D}_\lambda(0) \to \tilde{\Gamma}_\lambda(\omega^* \tilde{D}_\lambda).
\]
Let \(\rho \in \Lambda\) denote the half sum of the positive roots.
Theorem 0.1 (Lunts-Rosenberg). If \( \lambda + \rho \in \Lambda^+ \), then the functor \( \tilde{\Gamma}_\lambda \) is exact.

Conjecture 0.2 (Lunts-Rosenberg). If \( \lambda \in \Lambda^+ \), then \( \tilde{\Gamma}_\lambda(M) = 0 \) implies \( M = 0 \).

Conjecture 0.3 (Lunts-Rosenberg). For any \( \lambda \in \Lambda \) we have \( U/J_\lambda \simeq \tilde{D}_\lambda(0) \simeq \tilde{\Gamma}_\lambda(\omega^* \tilde{D}_\lambda) \).

By a standard argument Theorem 0.1, Conjecture 0.2 and Conjecture 0.3 for \( \lambda \in \Lambda^+ \) imply the following analogue of the Beilinson-Bernstein correspondence (Beilinson-Bernstein [2]).

Conjecture 0.4 (Lunts-Rosenberg). For \( \lambda \in \Lambda^+ \) \( \tilde{\Gamma}_\lambda \) induces the equivalence of categories:

\[
\text{Mod}(\tilde{D}_{\mathcal{B}_q,\lambda}) \simeq \text{Mod}(U/J_\lambda).
\]

We can define \( \text{Mod}(\mathcal{D}_{\mathcal{B}_q,\lambda}), D_\lambda, \Gamma_\lambda : \text{Mod}(\mathcal{D}_{\mathcal{B}_q,\lambda}) \to \text{Mod}(D_\lambda(0)) \) and \( U/J_\lambda \to D_\lambda(0) \to \Gamma_\lambda(\omega^* D_\lambda) \) using \( D \) instead of \( \tilde{D} \). Our main result is the following.

Theorem 0.5. Conjecture 0.2 is true.

Theorem 0.6. Theorem 0.1, Conjecture 0.2 and Conjecture 0.3 for \( \lambda + \rho \in \Lambda^+ \) are all true for \( D \); that is,

(i) If \( \lambda + \rho \in \Lambda^+ \), then the functor \( \Gamma_\lambda \) is exact.

(ii) If \( \lambda \in \Lambda^+ \), then \( \Gamma_\lambda(M) = 0 \) implies \( M = 0 \).

(iii) For any \( \lambda \in \Lambda \) we have \( U/J_\lambda \simeq D_\lambda(0) \).

(iv) If \( \lambda + \rho \in \Lambda^+ \), then we have \( D_\lambda(0) \simeq \Gamma_\lambda(\omega^* D_\lambda) \).

We can deduce the following from Theorem 0.6.

Corollary 0.7. For \( \lambda \in \Lambda^+ \) \( \Gamma_\lambda \) induces the equivalence of categories:

\[
\text{Mod}(\mathcal{D}_{\mathcal{B}_q,\lambda}) \simeq \text{Mod}(U/J_\lambda).
\]

In Lunts-Rosenberg [12] it is noted that a \( q \)-analogue of the formula

\[
(0.3) \quad \Gamma(\mathcal{B}, (\mathcal{O}_B \otimes \mathcal{C} V^1(\mu)) \otimes \mathcal{O}_B M) \simeq V^1(\mu) \otimes \mathcal{C} \Gamma(\mathcal{B}, M),
\]

implies Theorem 0.5. We can show it using basic properties of universal \( R \)-matrices (Proposition 3.13 below), from which we obtain Theorem 0.5. The
proofs of Theorem 0.6 (i) and (ii) are similar to those for Theorem 0.1 and Theorem 0.3 respectively. Our proof of Theorem 0.6 (iii) and (iv) is similar to that of the corresponding fact for Lie algebras given by Borho-Brylinski [3]. The proof by Borho-Brylinski uses the structure of the annihilators of Verma modules and a result of N. Conze-Berline and M. Duflo. In the quantum setting both the structure theorem of the annihilators of Verma modules and an analogue of the theorem of N. Conze-Berline and M. Duflo are already obtained by Joseph [8], [10]. Theorem 0.6 (iii) follows from the result about the annihilators of Verma modules easily; however, unlike the Lie algebra case Joseph’s theorem giving an analogue of a result by N. Conze-Berline and M. Duflo does not immediately imply Theorem 0.6 (iv) since $U_q(\mathfrak{g})$ is not locally finite with respect to the adjoint action. We overcome the difficulty by the arguments used in the proof of Theorem 0.6 (i), where the assumption $\lambda + \rho \in \Lambda^+$ is necessary.

Let us give a comment in order to justify the usage of $D$ instead of $\tilde{D}$. Let $\tilde{D}^1$ and $D^1$ denote the subalgebras of $\text{End}_C(A^1)$ corresponding to $\tilde{D}$ and $D$ in the ordinary enveloping algebra situation. The algebra $D^1$ is in fact a proper subalgebra of $\tilde{D}^1$ by Bernstein-Gelfand-Gelfand [4, Example 2]; however, the corresponding categories $\text{Mod}(\tilde{D}^1, \lambda)$ and $\text{Mod}(D^1, \lambda)$ defined similarly to (0.2) are equivalent since the corresponding rings of differential operators are isomorphic locally on $B$.

We note that Theorem 0.6 and Corollary 0.7 for $\mathfrak{g} = \mathfrak{sl}(2)$ is due to Hodges [7]. We also note that a different approach to the Beilinson-Bernstein correspondence for the quantized enveloping algebras is given in Joseph [9].

In this paper we shall use the following notation for a Hopf algebra $H$ over a field $K$. The multiplication, the unit, the comultiplication, the counit, and the antipode of $H$ are denoted by

\begin{align}
(0.4) & \quad m_H : H \otimes_K H \to H, \\
(0.5) & \quad \eta_H : K \to H, \\
(0.6) & \quad \Delta_H : H \to H \otimes_K H, \\
(0.7) & \quad \epsilon_H : H \to K, \\
(0.8) & \quad S_H : H \to H
\end{align}

respectively. The subscript $H$ will often be omitted. For $n \in \mathbb{Z}_{>0}$ we denote by

$$\Delta_n : H \to H^\otimes n+1$$

the algebra homomorphism given by

$$\Delta_1 = \Delta, \quad \Delta_n = (\Delta \otimes \text{id}_{H^\otimes n-1}) \circ \Delta_{n-1},$$
and write
\[ \Delta_n(h) = \sum_{(h)_n} h(0) \otimes \cdots \otimes h(n). \]

1 Quantum groups

1.1 Quantized enveloping algebras

Let \( \mathfrak{g} \) be a finite-dimensional semisimple Lie algebra over \( \mathbb{C} \) and let \( \mathfrak{h} \) be its Cartan subalgebra. We denote by \( \Delta \subset \mathfrak{h}^*, \Lambda \subset \mathfrak{h}^*, W \subset GL(\mathfrak{h}^*) \) the set of roots, the weight lattice and the Weyl group respectively. We fix a set of simple roots \( \{ \alpha_i \}_{i \in I} \). Let \( \Delta^+ \subset \mathfrak{h}^*, \Lambda^+ \subset \mathfrak{h}^*, \{ \varpi_i \}_{i \in I} \subset \mathfrak{h}^*, \{ s_i \}_{i \in I} \subset W \) denote the corresponding set of positive roots, dominant weights, fundamental weights, and simple reflections respectively. Set
\[ Q^+ = \sum_{\alpha \in \Delta^+} \mathbb{Z}_{\geq 0} \alpha = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathfrak{h}^*. \]

Let \( \rho \in \mathfrak{h}^* \) be the half sum of positive roots. We denote the longest element of \( W \) by \( w_0 \). We fix a \( W \)-invariant symmetric bilinear form
\begin{equation}
(\ , \ ) : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathbb{C}
\end{equation}
satisfying \( (\alpha_i, \alpha_i) \in \mathbb{Q}_{>0} \) for any \( i \in I \). For \( i \in I \) we set
\begin{equation}
\alpha_i^\vee = 2\alpha_i / (\alpha_i, \alpha_i) \in \mathfrak{h}^*.
\end{equation}

Take a positive integer \( \ell_0 \) satisfying
\begin{equation}
\ell_0(\alpha_i, \alpha_i) \subset 2\mathbb{Z} \quad (i \in I), \quad \ell_0(\Lambda, \Lambda) \subset \mathbb{Z},
\end{equation}
and let \( \mathbb{F} = \mathbb{Q}(q^{1/\ell_0}) \) be the rational function field over \( \mathbb{Q} \) with variable \( q^{1/\ell_0} \).

In this paper \( \otimes \) stands for \( \otimes_{\mathbb{F}} \).

For \( n \in \mathbb{Z}_{\geq 0} \) we set
\[ [n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}], \quad [n]_t! = [n]_t[n - 1]_t \cdots [2]_t[1]_t \in \mathbb{Z}[t, t^{-1}]. \]

The simply-connected quantized enveloping algebra \( U = U_q(\mathfrak{g}) \) is an associative algebra over \( \mathbb{F} \) with the identity element 1 generated by the elements
\[ k_\lambda (\lambda \in \Lambda), \ e_i, f_i \ (i \in I) \] satisfying the following defining relations:

\begin{align*}
(1.4) \quad & k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda+\mu} \quad (\lambda, \mu \in \Lambda), \\
(1.5) \quad & k_\lambda e_i k_\lambda^{-1} = q^{(\lambda,\alpha_i)} e_i, \quad (\lambda \in \Lambda, i \in I), \\
(1.6) \quad & k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda,\alpha_i)} f_i \quad (\lambda \in \Lambda, i \in I), \\
(1.7) \quad & e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I), \\
(1.8) \quad & \sum_{n=0}^{1-a_{ij}} (-1)^n e_i^{(1-a_{ij}-n)} e_j e_i^{(n)} = 0 \quad (i, j \in I, i \neq j), \\
(1.9) \quad & \sum_{n=0}^{1-a_{ij}} (-1)^n f_i^{(1-a_{ij}-n)} f_j f_i^{(n)} = 0 \quad (i, j \in I, i \neq j),
\end{align*}

where \( q_i = q^{(\alpha_i,\alpha_i)/2}, k_i = k_{\alpha_i}, a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \) for \( i, j \in I \), and

\[ e_i^{(n)} = e_i^n/[n]_q!, \quad f_i^{(n)} = f_i^n/[n]_q! \]

for \( i \in I \) and \( n \in \mathbb{Z}_{\geq 0} \). Algebra homomorphisms \( \Delta : U \to U \otimes U, \epsilon : U \to \mathbb{F} \) and an algebra anti-automorphism \( S : U \to U \) are defined by:

\begin{align*}
(1.10) \quad & \Delta(k_\lambda) = k_\lambda \otimes k_\lambda, \\
(1.11) \quad & \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \\
(1.12) \quad & \epsilon(k_\lambda) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0, \\
(1.13) \quad & S(k_\lambda) = k_\lambda^{-1}, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i,
\end{align*}

and \( U \) is endowed with a Hopf algebra structure with the comultiplication \( \Delta \), the counit \( \epsilon \) and the antipode \( S \).

We define subalgebras \( U^0, U^\geq 0, U^\leq 0, U^+, U^- \) of \( U \) by

\begin{align*}
(1.13) \quad & U^0 = \langle k_\lambda \mid \lambda \in \Lambda \rangle, \\
(1.14) \quad & U^\geq 0 = \langle k_\lambda, e_i \mid \lambda \in \Lambda, i \in I \rangle, \\
(1.15) \quad & U^\leq 0 = \langle k_\lambda, f_i \mid \lambda \in \Lambda, i \in I \rangle, \\
(1.16) \quad & U^+ = \langle e_i \mid i \in I \rangle, \\
(1.17) \quad & U^- = \langle f_i \mid i \in I \rangle.
\end{align*}

Note that \( U^0, U^\geq 0, U^\leq 0 \) are Hopf subalgebras of \( U \), while \( U^+ \) and \( U^- \) are not Hopf subalgebras.

The following result is standard.

**Proposition 1.1.** (i) \( \{k_\lambda \mid \lambda \in \Lambda \} \) is an \( \mathbb{F} \)-basis of \( U^0 \).
(ii) $U^+$ (resp. $U^-$) is isomorphic to the $\mathbb{F}$-algebra generated by $\{e_i \mid i \in I\}$ (resp. $\{f_i \mid i \in I\}$) with defining relation (1.3) (resp. (1.6)).

(iii) $U^{\geq 0}$ (resp. $U^{\leq 0}$) is isomorphic to the $\mathbb{F}$-algebra generated by $\{e_i, k_\lambda \mid i \in I, \lambda \in \Lambda\}$ (resp. $\{f_i, k_\lambda \mid i \in I, \lambda \in \Lambda\}$) with defining relations (1.4), (1.5), (1.8) (resp. (1.4), (1.5), (1.9)).

(iv) The linear maps

$$U^- \otimes U^0 \otimes U^+ \to U \leftarrow U^+ \otimes U^0 \otimes U^-,$$
$$U^+ \otimes U^0 \to U^{\geq 0} \leftarrow U^0 \otimes U^+, \quad U^- \otimes U^0 \to U^{\leq 0} \leftarrow U^0 \otimes U^-$$

induced by the multiplication are all isomorphisms.

We define Hopf algebra homomorphisms

(1.18) \[ \pi^+: U^{\geq 0} \to U^0, \quad \pi^- : U^{\leq 0} \to U^0 \]

by $\pi^+(k_\lambda) = k_\lambda$ ($\lambda \in \Lambda$), $\pi^+(e_i) = \pi^-(f_i) = 0$ ($i \in I$).

For $\gamma \in Q^+$ we set

$$U_{\pm}^{\pm\gamma} = \{x \in U^{\pm} \mid k_\lambda x k_\lambda^{-1} = q^{\pm(\lambda, \gamma)} x \ (\lambda \in \Lambda)\}.$$ 

We have

$$U^{\pm} = \bigoplus_{\gamma \in Q^+} U_{\pm}^{\pm\gamma}.$$ 

1.2 Representations

Let $H$ be a Hopf algebra over a field $\mathbb{K}$. For left $H$-modules $V_1, V_2$ we endow $V_1 \otimes_{\mathbb{K}} V_2$ with a left $H$-module structure by

(1.19) \[ h(v_1 \otimes v_2) = \Delta(h)(v_1 \otimes v_2) \quad (h \in H, v_1 \in V_1, v_2 \in V_2). \]

For a left $H$-module $V$ its dual space $V^* = \text{Hom}_\mathbb{K}(V, \mathbb{K})$ is endowed with a structure of a right $H$-module (i.e., a left $H^{\text{op}}$-module, where $H^{\text{op}}$ denotes the algebra opposite to $H$) by

(1.20) \[ \langle v^* h, v \rangle = \langle v^*, hv \rangle \quad (v^* \in V^*, h \in H, v \in V), \]

where $\langle , \rangle : H^* \times H \to \mathbb{K}$ denotes the canonical pairing.

For $\lambda \in \Lambda$ we define an algebra homomorphism

(1.21) \[ \chi_\lambda : U^0 \to \mathbb{F} \]
by \( \chi_\lambda(k_\mu) = q^{(\lambda, \mu)}(\mu \in \Lambda) \). We can extend it to algebra homomorphisms

\[
(1.22) \quad \chi^+: U^\leq \to \mathbb{F}, \quad \chi^-: U^\geq \to \mathbb{F}
\]

by \( \chi^\pm = \chi_\lambda \circ \pi^\pm \).

For a left (resp. right) \( U \)-module \( V \) and \( \lambda \in \Lambda \) the subspace

\[
(1.23) \quad V_\lambda = \{ v \in V \mid tv = \chi_\lambda(t)v \ (t \in U^0) \}
\]

of \( V \) is called the weight space with weight \( \lambda \). Those \( \lambda \in \Lambda \) such that \( V_\lambda \neq \{0\} \) are called the weights of \( V \). For a left (or right) \( U \)-module \( V \) which is a direct sum of finite-dimensional weight spaces we define its character by the formal sum

\[
\text{ch}(V) = \sum_{\lambda \in \Lambda} \dim V_\lambda e^\lambda.
\]

We denote by \( \text{Mod}^f(U) \) (resp. \( \text{Mod}^f(U^{op}) \)) the abelian category whose objects are finite-dimensional left (resp. right) \( U \)-modules which are direct sums of weight spaces.

For any \( \lambda \in \Lambda^+ \) there exists a unique irreducible object \( V(\lambda) \) of \( \text{Mod}^f(U) \) such that \( \lambda \) is a weight of \( V \) and any weight of \( V \) belongs to \( \lambda - Q^+ \). Any object of \( \text{Mod}^f(U) \) is a direct sum of this type of irreducible objects. As in the Lie algebra case the character of \( V(\lambda) \) is given by Weyl’s character formula:

\[
(1.25) \quad \text{ch}(V(\lambda)) = \sum_{w \in W} (-1)^{\text{det}(w)} e^{w(\lambda+\rho)-\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1} \quad (\lambda \in \Lambda^+)
\]

(Lusztig \cite{13}). By \( (1.25) \) we obtain the following.

**Lemma 1.2.** Let \( \gamma \in Q^+ \).

(i) For sufficiently large \( \lambda \in \Lambda^+ \) the linear maps

\[
U^-_{-\gamma} \ni u \mapsto uv_\lambda \in V(\lambda)_{\lambda-\gamma}, \quad U^+_\gamma \ni u \mapsto v^*_\lambda u \in V^*(\lambda)_{\lambda-\gamma}
\]

are bijective. Here, \( v_\lambda \) and \( v^*_\lambda \) are non-zero elements of \( V(\lambda)_\lambda \) and \( V^*(\lambda)_\lambda \) respectively.

(ii) For sufficiently large \( \lambda \in \Lambda^+ \) the linear maps

\[
U^-_\gamma \ni u \mapsto uv_{-\lambda} \in V(-w_0\lambda)_{-\lambda+\gamma}, \quad U^-_{-\gamma} \ni u \mapsto v^*_{-\lambda} u \in V^*(-w_0\lambda)_{-\lambda+\gamma}
\]

are bijective. Here, \( v_{-\lambda} \) and \( v^*_{-\lambda} \) are non-zero elements of \( V(-w_0\lambda)_{-\lambda} \) and \( V^*(-w_0\lambda)_{-\lambda} \) respectively.
Remark 1.3. In this paper the expression “for sufficiently large $\lambda \in \Lambda^+$ ...” means that “there exists some $\xi \in \Lambda^+$ such that for any $\lambda \in \xi + \Lambda^+$ ...”.

For any $V \in \text{Mod}^f(U)$ we have $(V^*)_1 \simeq (V_1)^*$ and hence $\text{ch}(V) = \text{ch}(V^*)$. For $\lambda \in \Lambda^+$ we set $V^*(\lambda) = (V(\lambda))^*$. Any object of $\text{Mod}^f(U^{\text{op}})$ is a direct sum of irreducible submodules isomorphic to $V^*(\lambda)$ for $\lambda \in \Lambda^+$. We note

\begin{align}
V(\lambda)_\lambda &= \{v \in V(\lambda) \mid e_i v = 0 \ (i \in I)\}, \\
V^*(\lambda)_\lambda &= \{v \in V^*(\lambda) \mid v f_i = 0 \ (i \in I)\}.
\end{align}

For a left (resp. right) $U$-module $V$ which is a direct sum of finite-dimensional weight spaces we define its restricted dual $V^\star$ by

\begin{equation}
V^\star = \sum_{\lambda \in \Lambda} (V^0_\lambda)^* \subset V^*.
\end{equation}

It is easily seen that $V^\star$ is a right (resp. left) $U$-submodule of $V^*$. We have

$$(V^\star)^\star \simeq V, \quad \text{ch}(V^\star) = \text{ch}(V).$$

For $\lambda \in \Lambda$ we define left $U$-modules $T(\lambda)$, $T^*(\lambda)$ and right $U$-modules $T_\ell(\lambda)$, $T_r^*(\lambda)$ by

\begin{align}
T(\lambda) &= U/ \sum_{u \in U_{\geq 0}} U(u - \chi^+_{\lambda}(u)), \\
T_\ell(\lambda) &= U/ \sum_{u \in U_{\leq 0}} (u - \chi^-_{\lambda}(u)) U, \\
T^*(\lambda) &= (T_\ell(\lambda))^*, \\
T_r^*(\lambda) &= (T_r(\lambda))^*.
\end{align}

We have

\begin{equation}
\text{ch}(T(\lambda)) = \text{ch}(T^*(\lambda)) = \text{ch}(T_\ell(\lambda)) = \text{ch}(T_r^*(\lambda)) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}.
\end{equation}

1.3 Universal $R$-matrices

There exists a unique bilinear form

\begin{equation}
(\ , \ ) : U_{\geq 0} \times U_{\leq 0} \to \mathbb{F}
\end{equation}

satisfying

\begin{align}
(x, y_1 y_2) &= (\Delta(x), y_1 \otimes y_2) \quad (x \in U_{\geq 0}, y_1, y_2 \in U_{\leq 0}), \\
(x_1 x_2, y) &= (x_2 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in U_{\geq 0}, y \in U_{\leq 0}), \\
(k_\lambda, k_\mu) &= q^{-\langle \lambda, \mu \rangle} \quad (\lambda, \mu \in \Lambda), \\
(k_\lambda, f_i) &= (e_i, k_\lambda) = 0 \quad (\lambda \in \Lambda, i \in I), \\
(e_i, f_j) &= \delta_{ij} / (q_i^{-1} - q_i) \quad (i, j \in I).
\end{align}
(see Tanisaki [19]). For any \( \beta \in Q^+ \) the restriction of (1.31) to \( U_\beta^+ \times U_{-\beta}^- \) is non-degenerate, and we denote the corresponding canonical element of \( U_\beta^+ \otimes U_{-\beta}^- \) by \( \Xi_\beta \). Set

\[
(1.37) \quad \Xi = \sum_{\beta \in Q^+} q^{(\beta,\beta)}(k_\beta^{-1} \otimes k_\beta)\Xi_\beta.
\]

It belongs to a completion of \( U \otimes U \).

Let \( V, V' \in \text{Mod}^f(U) \). We define \( \tau_{V,V'} \in \text{Hom}_F(V \otimes V', V' \otimes V) \) and \( \kappa_{V,V'} \in GL(V \otimes V') \) by \( \tau_{V,V'}(v \otimes v') = v' \otimes v \) and \( \kappa_{V,V'}(v \otimes v') = q^{(\lambda,\mu)}v \otimes v' \) for \( v \in V_\lambda, v' \in V'_\mu \). Set

\[
(1.38) \quad R_{V,V'} = \kappa_{V,V'}^{-1} \circ \Xi \in \text{End}_F(V \otimes V'),
\]
\[
(1.39) \quad R_{V,V'}^\vee = \tau_{V,V'} \circ R_{V,V'} \in \text{Hom}_F(V \otimes V', V' \otimes V).
\]

For morphisms \( f : V_1 \to V_2, f' : V'_1 \to V'_2 \) in \( \text{Mod}^f(U) \) we have

\[
(1.40) \quad (f \otimes f') \circ R_{V_1,V'_1} = R_{V_2,V'_2} \circ (f \otimes f'), \quad (f' \otimes f) \circ R_{V'_1,V'_1} = R_{V'_2,V'_2} \circ (f \otimes f')
\]

by definition.

We shall also use the following properties of \( R_{V,V'} \) (see Drinfeld [5], Lusztig [14], Tanisaki [19]).

**Proposition 1.4.** Let \( V, V', V'' \in \text{Mod}^f(U) \).

(i) \( R_{V,V'} \) is invertible, and its inverse is given by

\[
R_{V,V'}^{-1} = \left( \sum_{\beta \in Q^+} q^{(\beta,\beta)}(1 \otimes k_\beta)(S \otimes \text{id})(\Xi_{\beta}) \right) \circ \kappa_{V,V'}.
\]

(ii) \( R_{V,V'}^\vee \) is an isomorphism of \( U \)-modules.

(iii) The composition of

\[
V \otimes V' \otimes V'' \xrightarrow{\text{id}_V \otimes R_{V',V''}^\vee} V \otimes V'' \otimes V' \xrightarrow{R_{V,V''}^\vee \otimes \text{id}_V} V'' \otimes V \otimes V'
\]

coincides with \( R_{V \otimes V', V''}^\vee \).
1.4 Center

We denote by $\mathfrak{z}$ the center of $U$. Let $\mathbb{F}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{F} e(\lambda)$ be the group algebra of $\Lambda$, and define a linear map $\zeta : \mathfrak{z} \to \mathbb{F}[\Lambda]$ as the composition of

$$
\mathfrak{z} \hookrightarrow U \cong U^- \otimes U^0 \otimes U^+ \xrightarrow{\epsilon^+ \otimes \kappa \otimes \epsilon^+} \mathbb{F} \otimes \mathbb{F}[\Lambda] \otimes \mathbb{F} \cong \mathbb{F}[\Lambda],
$$

where $U \cong U^- \otimes U^0 \otimes U^+$ is given by the multiplication of $U$, $\epsilon^+ : U^+ \to \mathbb{F}$ are the restrictions of the comultiplication $\epsilon : U \to \mathbb{F}$, and $\kappa : U^0 \to \mathbb{F}[\Lambda]$ is the isomorphism given by $\kappa(k_\lambda) = e(\lambda)$ for any $\lambda \in \Lambda$. Define shifted actions of the Weyl group $W$ on $\Lambda$ and $\mathbb{F}[\Lambda]$ by

$$
(1.41) \quad w \circ \lambda = w(\lambda + \rho) - \rho \quad (w \in W, \ \lambda \in \Lambda),
$$

$$
(1.42) \quad w \circ e(\lambda) = q^{(\nu_{\lambda, \rho})} e(w(\lambda)) \quad (w \in W, \ \lambda \in \Lambda)
$$

respectively. Note that the action of $W$ on $\Lambda$ is an affine action and the one on $\mathbb{F}[\Lambda]$ is a linear action.

**Proposition 1.5** (see Tanisaki [19]). $\zeta : \mathfrak{z} \to \mathbb{F}[\Lambda]$ is an injective algebra homomorphism, and its image coincides with

$$
\mathbb{F}[\Lambda]^W = \{ x \in \mathbb{F}[\Lambda] \mid w \circ x = x \ (w \in W) \}.
$$

The isomorphism $\mathfrak{z} \cong \mathbb{F}[\Lambda]^W$ induced by $\zeta$ is called the Harish-Chandra isomorphism.

For $\lambda \in \Lambda$ we define an algebra homomorphism

$$
(1.43) \quad \zeta_\lambda : \mathfrak{z} \to \mathbb{F}
$$

as the composition of $\mathfrak{z} \hookrightarrow \mathbb{F}[\Lambda] \to \mathbb{F}$ where $\mathbb{F}[\Lambda] \to \mathbb{F}$ is given by $e(\mu) \mapsto q^{(\lambda, \mu)}$. The following is result is standard.

**Proposition 1.6.**

(i) For $\lambda_1, \lambda_2 \in \Lambda$ we have $\zeta_{\lambda_1} = \zeta_{\lambda_2}$ if and only if $\lambda_2 \in W \circ \lambda_1$.

(ii) For $\lambda \in \Lambda^+$ and $z \in \mathfrak{z}$ we have $z|V(\lambda) = \zeta_\lambda(z)$ id.

(iii) For $\lambda \in \Lambda$ and $z \in \mathfrak{z}$ the action of $z$ on $T(\lambda)$ and $T^*(\lambda)$ are given by the multiplication by $\zeta_\lambda(z)$.
1.5 Braid group actions

We set

\[ \exp_t(x) = \sum_{n=0}^{\infty} \frac{t^{n(n-1)/2}}{n!} x^n \in \mathbb{Q}(t)[[x]]. \]

Note that

\[ \exp_t(x)^{-1} = \exp_{t^{-1}}(-x). \]

For \( i \in I \) define an operator \( T_i \) on \( V \in \text{Mod}^f(U) \) by

\[ T_i = \exp_{q_{i}^{-1}}(q_i k_i f_i) \exp_{q_{i}^{-1}}(-e_i) \exp_{q_{i}^{-1}}(q_i^{-1} k_i^{-1} f_i) H_i \]

\[ = \exp_{q_{i}^{-1}}(-q_i k_i^{-1} e_i) \exp_{q_{i}^{-1}}(f_i) \exp_{q_{i}^{-1}}(-q_i^{-1} k_i e_i) H_i, \]

where \( H_i \) is the operator on \( V \) which acts by \( q_i^{(\lambda, \alpha_i)}((\lambda, \alpha_i) + 1)/2 \) id on \( V_{\lambda} \) for each \( \lambda \in \Lambda \). This operator coincides with Lusztig’s operator \( T'_{i,1} \) in [14, 5.2].

We shall use the following result later (see Lusztig [14, 5.3]).

**Lemma 1.7.** Let \( V_1, V_2 \in \text{Mod}^f(U) \). As an operator on \( V_1 \otimes V_2 \in \text{Mod}^f(U) \) we have

\[ T_i = \exp_{q_i}(q_i^{-2}(q_i - q_i^{-1}) e_i k_i^{-1} \otimes f_i k_i)(T_i \otimes T_i) \]

\[ = (T_i \otimes T_i) \exp_{q_i}((q_i - q_i^{-1}) f_i \otimes e_i). \]

For \( w \in W \) we choose a minimal expression \( w = s_{i_1} \cdots s_{i_n} \) and set

\[ T_w = T_{i_1} \cdots T_{i_n}. \]

It is known that \( T_w \) does not depend on the choice of a minimal expression and that

\[ T_w(V_\lambda) = V_{w \lambda} \quad (V \in \text{Mod}^f(U), \ \lambda \in \Lambda). \]

For \( i \in I \) we can define an algebra automorphism \( T_i \) of \( U \) by

\[ T_i(k_\mu) = k_{s_i \mu}, \quad (\mu \in \Lambda), \]

\[ T_i(e_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} e_i^{(a_{ij} - k)} e_j e_i^{(k)} & (j \in I, \ j \neq i), \\ -f_i k_i & (j = i), \end{cases} \]

\[ T_i(f_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{k} f_i^{(k)} f_j f_i^{(a_{ij} - k)} & (j \in I, \ j \neq i), \\ -k_i^{-1} e_i & (j = i). \end{cases} \]

For \( w \in W \) we define an algebra automorphism \( T_w \) of \( U \) by \( T_w = T_{i_1} \cdots T_{i_n} \) where \( w = s_{i_1} \cdots s_{i_n} \) is a minimal expression. The automorphism \( T_w \) does not depend on the choice of a minimal expression. It is known that

(1.44) \[ T_w(uv) = T_w(u)T_w(v) \quad (w \in W, \ u \in U, \ v \in V \in \text{Mod}^f(U)). \]
Let $V \in \text{Mod}^f(U^{\text{op}})$. By $V^* \in \text{Mod}^f(U)$ we can define an operator $'T_w$ on $V$ by

\[ ('T_w(v), v^*) = \langle v, T_w(v^*) \rangle \quad (v \in V, v^* \in V^*).\]

**Lemma 1.8.** Let $w \in W$ and let $\lambda \in \Lambda^\pm$.

(i) Let $V \in \text{Mod}^f(U)$. For any $v \in V$ and any $\ell \in V(\lambda)_{\lambda}$ we have

\[ T_w^{-1}(\ell \otimes v) = T_w^{-1}(\ell) \otimes T_w^{-1}(v), \quad T_w(v \otimes \ell) = T_w(v) \otimes T_w(\ell). \]

(ii) Let $V \in \text{Mod}^f(U^{\text{op}})$. For any $v \in V$ and any $\ell \in V^*(\lambda)_{\lambda}$ we have

\[ 'T_w^{-1}(\ell \otimes v) = 'T_w^{-1}(\ell) \otimes 'T_w^{-1}(v), \quad 'T_w(v \otimes \ell) = 'T_w(v) \otimes 'T_w(\ell). \]

**Proof.** We can easily reduce the proof to the rank one case. In the rank one case they follow from Lemma 1.7. 

\[ \square \]

### 1.6 Dual Hopf algebras

Let $H$ be a Hopf algebra over a field $\mathbb{K}$. The dual space $H^* = \text{Hom}_{\mathbb{K}}(H, \mathbb{K})$ is endowed with a structure of an $H$-bimodule by

\[ \langle h_1fh_2, h \rangle = \langle f, h_2hh_1 \rangle \quad (h, h_1, h_2 \in H, f \in H^*). \]

The linear maps $m_{H^*}, \eta_{H^*}, \Delta_{H^*}, \epsilon_{H^*}, S_{H^*}$ induce linear maps

\begin{align*}
(1.45) \quad & m_{H^*} = '\Delta_{H^*} : (H \otimes_{\mathbb{K}} H)^* \to H^*, \\
(1.46) \quad & \eta_{H^*} = '\epsilon_{H} : \mathbb{K} \to H^*, \\
(1.47) \quad & \Delta_{H^*} = 'm_{H} : H^* \to (H \otimes_{\mathbb{K}} H)^*, \\
(1.48) \quad & \epsilon_{H^*} = '\eta_{H} : H^* \to \mathbb{K}, \\
(1.49) \quad & S_{H^*} = 'S_{H} : H^* \to H^*. 
\end{align*}

Note that we have $H^* \otimes_{\mathbb{K}} H^* \subset (H \otimes_{\mathbb{K}} H)^*$.

Let $T$ be a Hopf subalgebra of $H$. We assume that $T$ is commutative and cocommutative. The set $\text{Hom}_{\text{alg}}(T, \mathbb{K})$ of algebra homomorphisms from $T$ to $\mathbb{K}$ is endowed with a structure of an abelian group by

\[ (\varphi \psi)(t) = \sum_{(t)} \varphi(t_{(1)})\psi(t_{(2)}) \quad (\varphi, \psi \in \text{Hom}_{\text{alg}}(T, \mathbb{K}), t \in T). \]

Assume that we are given a subgroup $\Upsilon$ of $\text{Hom}_{\text{alg}}(T, \mathbb{K})$.

Denote by $\text{Mod}_\Upsilon(T)$ (resp. $\text{Mod}_{\Upsilon \times \Upsilon}(T \otimes_{\mathbb{K}} T)$) the subcategory of $\text{Mod}(T)$ (resp. $\text{Mod}(T \otimes_{\mathbb{K}} T)$) consisting of finite-dimensional semisimple $T$-modules (resp. $T \otimes_{\mathbb{K}} T$-modules) whose irreducible factors are contained in $\Upsilon$ (resp. $\Upsilon \times \Upsilon$). Here elements of $\Upsilon$ (resp. $\Upsilon \times \Upsilon$) are identified with isomorphism classes of objects of the category $\text{Mod}(T)$ (resp. $\text{Mod}(T \otimes_{\mathbb{K}} T)$).
Proposition 1.9. The following conditions on \( f \in H^* \) are equivalent.

(a) \( Hf \in \text{Mod}_T(T) \).
(b) \( fH \in \text{Mod}_T(T) \).
(c) \( HfH \in \text{Mod}_{\mathcal{Y}}(T \otimes_K T) \).
(d) There exists a two-sided ideal \( I \) of \( H \) such that \( \langle f, I \rangle = \{0\} \) and \( H/I \in \text{Mod}_{\mathcal{Y} \times \mathcal{Y}}(T \otimes_K T) \).

Proof. We have obviously (c)⇒(a). We obtain (a)⇒(d) by setting \( I = \text{Ker}(H \to \text{End}_K(Hf)) \). From (d) we obtain (c) as a consequence of \( HfH \subset (H/I)^* \). The implications (c)⇒(b)⇒(d) is proved similarly.

We denote by \( H^*_T,\mathcal{Y} \) the set of \( f \in H^* \) satisfying the equivalent conditions in Proposition 1.9.

Proposition 1.10. A Hopf algebra structure on \( H^*_T,\mathcal{Y} \) is induced by the linear maps (1.45), ..., (1.49).

Proof. We need to show \( m_{H^*(H^*_T \otimes_K H^*_T)} \subset H^*_T,\mathcal{Y}, \eta_{H^*}(1) \subset H^*_T,\mathcal{Y}, \Delta_{H^*}(H^*_T,\mathcal{Y}) \subset H^*_T \otimes_K H^*_T, \eta_{H^*}(H^*_T,\mathcal{Y}) \subset H^*_T,\mathcal{Y} \). They are consequences of our assumptions on \( T \) and \( \mathcal{Y} \). Details are omitted.

We denote by \( \text{Mod}_{T,\mathcal{Y}}(H) \) the full subcategory of \( \text{Mod}(H) \) consisting of \( H \)-modules which belong to \( \text{Mod}_{\mathcal{Y}}(T) \) as a \( T \)-module. For \( M \in \text{Mod}_{T,\mathcal{Y}}(H) \) we define a homomorphism of \( H \)-bimodules

\[
\Phi_M : M \otimes_K M^* \to H^*_T,\mathcal{Y}
\]

by \( \langle \Phi_M(v \otimes v^*), h \rangle = \langle v^*, hv \rangle \) for \( h \in H, v \in M, v^* \in M^* \). Elements of \( \text{Im}(\Phi_M) \) are called matrix coefficients of the \( H \)-module \( M \).

We denote by \( \text{Mod}^{\text{irr}}_{T,\mathcal{Y}}(H) \) the set of isomorphism classes of irreducible \( H \)-modules contained in \( \text{Mod}_{T,\mathcal{Y}}(H) \).

Proposition 1.11. (i) We have \( H^*_T,\mathcal{Y} = \sum_{M \in \text{Mod}_{T,\mathcal{Y}}(H)} \text{Im}(\Phi_M) \).

(ii) Assume that \( \text{Mod}_{T,\mathcal{Y}}(H) \) is a semisimple category and that \( \text{End}_K(M) = \mathbb{K}\text{id} \) for any \( M \in \text{Mod}^{\text{irr}}_{T,\mathcal{Y}}(H) \). Then the homomorphism

\[
\bigoplus_{M \in \text{Mod}^{\text{irr}}_{T,\mathcal{Y}}(H)} \Phi_M : \bigoplus_{M \in \text{Mod}^{\text{irr}}_{T,\mathcal{Y}}(H)} M \otimes_K M^* \to H^*_T,\mathcal{Y}
\]

of \( H \)-bimodules is an isomorphism.
Proof. (i) We have obviously $\text{Im} (\Phi_M) \subset H^*_{T,\Upsilon}$ for $M \in \text{Mod}_{T,\Upsilon}(H)$. Let $f \in H^*_{T,\Upsilon}$. Set $M = Hf$. Let $\{v_j\}$ be a basis of $M$ and let $\{v^*_j\}$ be the dual basis of $M^*$. For $h \in H$ we have

$$\langle f, h \rangle = \langle hf, 1 \rangle = \sum_j \langle v^*_j, hf \rangle \langle v_j, 1 \rangle = \sum_j \langle \Phi_M (f \otimes v^*_j), h \rangle \langle v_j, 1 \rangle$$

and hence $f = \sum_j \langle v_j, 1 \rangle \Phi_M (f \otimes v^*_j) \in \text{Im} (\Phi_M)$.

(ii) By (i) we have $H^*_{T,\Upsilon} = \sum_{M \in \text{Mod}_{T,\Upsilon}^{irr}(H)} \text{Im} (\Phi_M)$. For $M \in \text{Mod}_{T,\Upsilon}^{irr}(H)$ $\text{Im} (\Phi_M)$ is a sum of left $H$-modules isomorphic to $M$, and hence we have $H^*_{T,\Upsilon} = \bigoplus_{M \in \text{Mod}_{T,\Upsilon}^{irr}(H)} \text{Im} (\Phi_M)$. Let $M \in \text{Mod}_{T,\Upsilon}^{irr}(H)$. We see that $M \otimes_k M^*$ is an irreducible $H$-bimodule from $\text{End}_H(M) = k \text{id}$, and hence $\Phi_M$ is injective. \qed

1.7 Quantized coordinate algebras

Set

$$F = U^*_{U^0, \Lambda}, \quad F^{\geq 0} = \left( U^{\geq 0} \right)^*_{U^0, \Lambda}, \quad F^0 = \left( U^0 \right)^*_{U^0, \Lambda},$$

where $\Lambda$ is regarded as a subgroup of $\text{Hom}_{\text{alg}}(U^0, k)$ by $\lambda \mapsto \chi_\lambda$. The Hopf algebras $F$, $F^{\geq 0}$, $F^0$ are $q$-analogues of the coordinate algebras of $G$, $B$, $H$ (in the notation of Section 0) respectively.

By Proposition 1.11 (ii) we have

$$F^0 = \bigoplus_{\lambda \in \Lambda} k \chi_\lambda,$$

$$F \cong \bigoplus_{\lambda \in \Lambda^+} V(\lambda) \otimes V^*(\lambda).$$

We see easily that in $F^0$ we have

$$\chi_\lambda \chi_\mu = \chi_{\lambda + \mu} \quad (\lambda, \mu \in \Lambda), \quad \chi_0 = 1.$$ 

In particular, $F^0$ is isomorphic to the group algebra of $\Lambda$.

The Hopf algebra homomorphisms

$$U^0 \hookrightarrow U^{\geq 0} \hookrightarrow U, \quad U^{\geq 0} \xrightarrow{\pi^+} U^0$$

induce Hopf algebra homomorphisms

$$F \xrightarrow{r^+} F^{\geq 0} \xrightarrow{r^0} F^0, \quad F^0 \xrightarrow{t^0} F^{\geq 0}.$$
Since the composition of $U^0 \to U^\geq 0$ and $\pi^+$ is the identity of $U^0$, we have
\begin{equation}
(1.55) \quad r_0^+ \circ i_0^+ = \text{id}.
\end{equation}
In particular, $r_0^+$ is surjective and $i_0^+$ is injective. By $\chi^+_{\lambda} = i_0^+(\chi_\lambda)$ we have
\begin{equation}
(1.56) \quad \chi^+_{\lambda} \chi^+_{\mu} = \chi^+_{\lambda+\mu} \quad (\lambda, \mu \in \Lambda), \quad \chi^+_{0} = 1
\end{equation}
in $F^\geq 0$.

For $\lambda \in \Lambda$ we set
\begin{equation}
(1.57) \quad F^\geq 0(\lambda) = \{ f \in F^\geq 0 \mid ft = \chi_\lambda(t)f \ (t \in U^0) \}.
\end{equation}
We see easily that
\begin{equation}
F^\geq 0(\lambda) \subset F^\geq 0(\lambda + \mu) \quad (\lambda, \mu \in \Lambda), \quad \chi_\lambda \in F^\geq 0(\lambda).
\end{equation}
In particular, $F^\geq 0(0)$ is a subalgebra of $F^\geq 0$.

Set
\begin{equation}
(U^+)^\bullet = \bigoplus_{\gamma \in Q^+} (U^\gamma)^* \subset (U^+)^*.
\end{equation}

**Proposition 1.12.** (i) The linear map $F^\geq 0(0) \otimes F^0 \to F^\geq 0$ given by $f \otimes f' \mapsto f i_0^+(f')$ is an isomorphism.

(ii) For $f \in F^\geq 0(0)$ we have $f|U^+ \in (U^+)^\bullet$. Moreover, the linear map $F^\geq 0(0) \to (U^+)^\bullet \ (f \mapsto f|U^+)$ is an isomorphism.

**Proof.** For $f \in F^\geq 0(0), \ f' \in F^0$ we have
\begin{equation}
(1.58) \quad \langle f i_0^+(f'), tu \rangle = \langle f', t \rangle \langle f, u \rangle \quad (u \in U^+, t \in U^0).
\end{equation}
Indeed, for $u \in U^+, \lambda \in \Lambda$ we have
\begin{align*}
\langle f i_0^+(f'), k_\lambda u \rangle &= \langle f \otimes i_0^+(f'), \Delta(k_\lambda u) \rangle = \langle f \otimes f', (\text{id} \otimes \pi^+) \Delta(k_\lambda u) \rangle \\
&= \langle f \otimes f', k_\lambda u \otimes k_\lambda \rangle = \langle f', k_\lambda \rangle \langle f, u \rangle.
\end{align*}
by
\begin{equation}
(\text{id} \otimes \pi^+) \Delta(u) = u \otimes 1 \ (u \in U^+), \quad (\text{id} \otimes \pi^+) \Delta(k_\lambda) = k_\lambda \otimes k_\lambda \ (\lambda \in \Lambda).
\end{equation}

Recall that $F^\geq 0$ was defined as a subspace of $(U^\geq 0)^*$. By (1.58) it is sufficient to show that $F^\geq 0$ coincides with $F^0 \otimes (U^+)^\bullet$ under the identification $(U^\geq 0)^* = (U^0 \otimes U^+)^*$. It is easy to show $F^\geq 0 \subset F^0 \otimes (U^+)^\bullet$ and $F^\geq 0 \supset F^0 \otimes 1$. 
Hence it is sufficient to show $F_{\geq 0} \supset \chi_\gamma^+ \otimes (U_-^+)^*$ for any $\gamma \in Q^+$. Define $M \in \text{Mod}_{U^0_*}(U_{\geq 0})$ by

$$N = U_{\geq 0}/ \sum_{\lambda \in \Lambda} U_{\geq 0}(k_\lambda - 1), \quad M = N/ \sum_{\gamma - \gamma' \not\in Q^+} N_{\gamma'}.$$

Then the elements of $\chi_\gamma^+ \otimes (U_-^+)^*$ are obtained as matrix coefficients of the $U_{\geq 0}$-module $M$.

By Proposition 1.12 (i) we obtain

$$(1.59)\quad F_{\geq 0} = \bigoplus_{\lambda \in \Lambda} F_{\geq 0}(\lambda), \quad F_{\geq 0}(\lambda) = F_{\geq 0}(0) \chi_\lambda^+ (\lambda \in \Lambda).$$

**Proposition 1.13.** The Hopf algebra homomorphism $r_+ : F \to F_{\geq 0}$ is surjective.

**Proof.** Identify $F_{\geq 0}$ with $F^0 \otimes (U^+)^*$ by Proposition 1.12. We see easily that $\chi_\lambda \otimes 1 \in \text{Im}(r_+)$ for any $\lambda \in \Lambda$. Hence it is sufficient to show that for any $\gamma \in Q^+$ we have $\text{Im}(r_+) \supset \chi_{\lambda - \lambda + \gamma} \otimes (U^+_-)^*$ for sufficiently large $\lambda \in \Lambda^+$. Let $v_\lambda$ be a non-zero element of $V(w_0 \lambda)_{-\lambda}$. By Lemma 1.2 (ii) the linear map $U^+_- \ni u \mapsto uv_\lambda \in V(-w_0 \lambda)_{-\lambda + \gamma}$ is bijective when $\lambda$ is sufficiently large. Hence elements of $\chi_{\lambda - \lambda + \gamma} \otimes (U^+_-)^*$ are obtained as the matrix coefficients of the $U_{\geq 0}$-module $V(-w_0 \lambda)$.

## 2 Quantized flag manifolds

### 2.1 Homogeneous coordinate algebras

We set

$$(2.1)\quad A = \{ \varphi \in F \mid \varphi u = e(u)\varphi (u \in U^-) \}.$$  

**Lemma 2.1.** (i) $A$ is a subalgebra of $F$.

(ii) $A$ is a left $U$-submodule of $F$.

(iii) We have $\Delta_F(A) \subset A \otimes F$.

(iv) The multiplication $A \otimes A \to A$ is a homomorphism of $U$-modules; i.e.

$$(2.2)\quad u(\varphi_0 \varphi_1) = \sum_{(u)} (u(0)\varphi_0)(u(1)\varphi_1) \quad (\varphi_0, \varphi_1 \in A, u \in U).$$
Proof. (i) For $\varphi \in F$ we have $\varphi \in A$ if and only if $\varphi f_i = 0$ for any $i \in I$. We have

$$\langle 1 f_i, u \rangle = \langle 1, f_i u \rangle = \epsilon(f_i u) = 0$$

for any $u \in U$, and hence $1 \in A$. For $\varphi_1, \varphi_2 \in A$ we have

$$\langle (\varphi_1 \varphi_2) f_i, u \rangle = \langle \varphi_1 \varphi_2, f_i u \rangle = \langle \varphi_1 \otimes \varphi_2, \Delta(f_i u) \rangle$$

$$= \langle \varphi_1 \otimes \varphi_2, (f_i \otimes k_i^{-1} + 1 \otimes f_i) \Delta(u) \rangle$$

$$= \langle \varphi_1 f_i \otimes \varphi_2 k_i^{-1}, \Delta(u) \rangle + \langle \varphi_1 \otimes \varphi_2 f_i, \Delta(u) \rangle = 0$$

for any $i \in I$ and $u \in U$, and hence $\varphi_1 \varphi_2 \in A$.

The statement (ii) is obvious from the definition.

For $\varphi \in A, u \in U^+, u_1, u_2 \in U$ we have

$$\langle ((\Delta(\varphi))(u \otimes 1), u_1 \otimes u_2 \rangle = \langle \Delta(\varphi), uu_1 \otimes u_2 \rangle = \langle \varphi, uu_1 u_2 \rangle = \langle \varphi, u_1 u_2 \rangle$$

$$= \epsilon(u) \langle \varphi, u_1 u_2 \rangle = \epsilon(u) \langle \Delta(\varphi), u_1 \otimes u_2 \rangle$$

and hence $((\Delta(\varphi))(u \otimes 1) = \epsilon(u)\Delta(\varphi)$. It follows that $\Delta(\varphi) \in A \otimes F$. The statement (iii) is proved.

(iv) For $u' \in U$ we have

$$\langle u(\varphi_0 \varphi_1), u' \rangle = \langle \varphi_0 \varphi_1, u'u \rangle = \langle \varphi_0 \otimes \varphi_1, \Delta(u')\Delta(u) \rangle$$

$$= \sum_{(u)} \langle u(0)\varphi_0 \otimes u(1)\varphi_1, \Delta(u') \rangle \sum_{(u)} \langle (u(0)\varphi_0)(u(1)\varphi_1), u' \rangle.$$

By Lemma 2.1 (iii) we obtain an algebra homomorphism

$$\Delta : A \to A \otimes F$$

by restricting $\Delta_F$ to $A$.

For $\lambda \in \Lambda$ we set

$$A(\lambda) = \{ \varphi \in F \mid \varphi u = \chi^\lambda(u)\varphi \ (u \in U^{\geq 0}) \}.$$

By (1.27) we have

$$A \simeq \bigoplus_{\lambda \in \Lambda^+} V(\lambda) \otimes V^*(\lambda)_\lambda$$

under the isomorphism (1.53). Hence we have

$$A(\lambda) \simeq \begin{cases} V(\lambda) & (\lambda \in \Lambda^+) \\ 0 & \text{(otherwise)} \end{cases}$$
as a $U$-module, and
\begin{equation}
A = \bigoplus_{\lambda \in \Lambda^+} A(\lambda).
\end{equation}

For each $\lambda \in \Lambda^+$ we fix
\begin{equation}
v_{\lambda}^* \in V^*(\lambda) \setminus \{0\}.
\end{equation}
Then an isomorphism
\begin{equation}
f_{\lambda} : V(\lambda) \to A(\lambda)
\end{equation}
of $U$-modules is defined by
\begin{equation}
\langle f_{\lambda}(v), u \rangle = \langle v_{\lambda}^*, uv \rangle \quad (v \in V(\lambda), \, u \in U).
\end{equation}

Let $\lambda, \mu \in \Lambda^+$. Then the $U$-module $V(\lambda) \otimes V(\mu)$ (resp. the right $U$-module $V^*(\lambda) \otimes V^*(\mu)$) contains $V(\lambda + \mu)$ (resp. $V^*(\lambda + \mu)$) with multiplicity one. Let
\[i_{\lambda,\mu} : V^*(\lambda + \mu) \to V^*(\lambda) \otimes V^*(\mu)\]
be the embedding of $U$-modules such that $i_{\lambda,\mu}(v_{\lambda+\mu}^*) = v_{\lambda}^* \otimes v_{\mu}^*$, and denote the corresponding projection by
\begin{equation}
p_{\lambda,\mu} : V(\lambda) \otimes V(\mu) \to V(\lambda + \mu).
\end{equation}

**Lemma 2.2.** For $\lambda, \mu \in \Lambda^+$ we have
\[f_{\lambda}(v_0) f_{\mu}(v_1) = f_{\lambda+\mu}(p_{\lambda,\mu}(v_0 \otimes v_1)) \quad (v_0 \in V(\lambda), \, v_1 \in V(\mu)).\]
In particular, the multiplication of the algebra $A$ induces a surjective homomorphism $A(\lambda) \otimes A(\mu) \to A(\lambda + \mu)$ of $U$-modules.

**Proof.** For $u \in U$ we have
\begin{align*}
\langle f_{\lambda}(v_0) f_{\mu}(v_1), u \rangle &= \langle f_{\lambda}(v_0) \otimes f_{\mu}(v_1), \Delta(u) \rangle = \sum_{(u)} \langle f_{\lambda}(v_0), u(0) \rangle \langle f_{\mu}(v_1), u(1) \rangle \\
&= \sum_{(u)} \langle v_{\lambda}^*, u(0) v_0 \rangle \langle v_{\mu}^*, u(1) v_1 \rangle = \langle v_{\lambda}^* \otimes v_{\mu}^*, u(v_0 \otimes v_1) \rangle \\
&= \langle (i_{\lambda,\mu}(v_{\lambda+\mu}^*), u(v_0 \otimes v_1)) = \langle v_{\lambda+\mu}^*, p_{\lambda,\mu}(u(v_0 \otimes v_1)) = \langle v_{\lambda+\mu}^*, u p_{\lambda,\mu}(v_0 \otimes v_1) \rangle \\
&= \langle f_{\lambda+\mu}(p_{\lambda,\mu}(v_0 \otimes v_1)), u \rangle.
\end{align*}
Hence we have $f_{\lambda}(v_0) f_{\mu}(v_1) = f_{\lambda+\mu}(p_{\lambda,\mu}(v_0 \otimes v_1))$. □
Hence $A$ is a Λ-grade $F$-algebra with $A(0) = F1$.

By Joseph [9] we have the following.

**Proposition 2.3 (Joseph).** (i) $A$ is a domain, i.e., if $\varphi \psi = 0$ for $\varphi, \psi \in A$, then we have $\varphi = 0$ or $\psi = 0$.

(ii) $A$ is left and right noetherian.

For a ring (resp. Λ-graded ring) $R$ we denote by $\text{Mod}(R)$ (resp. $\text{Mod}_\Lambda(R)$) the category of left $R$-modules (resp. Λ-graded left $R$-modules). For $M \in \text{Mod}_\Lambda(R)$ and $\nu \in \Lambda$ we define $M[\nu] \in \text{Mod}_\Lambda(R)$ by

$$(M[\nu])(\lambda) = M(\lambda + \nu).$$

### 2.2 Category of quasi-coherent sheaves

For $M \in \text{Mod}_\Lambda(A)$ we denote by $\text{Tor}(M)$ the graded $A$-submodule of $M$ consisting of $m \in M$ such that $A(\lambda)m = \{0\}$ for sufficiently large $\lambda \in \Lambda^+$. Let $\text{Tor}_\Lambda(A)$ be the full subcategory of $\text{Mod}_\Lambda(A)$ consisting of $M \in \text{Mod}_\Lambda(A)$ satisfying $M = \text{Tor}(M)$. Note that $\text{Tor}_\Lambda(A)$ is closed under taking subquotients and extensions in $\text{Mod}_\Lambda(A)$. Let $\Sigma$ denote the collection of morphisms $f$ in $\text{Mod}_\Lambda(A)$ satisfying $\text{Ker}(f), \text{Coker}(f) \in \text{Tor}_\Lambda(A)$. Then we define the abelian category $\mathcal{M}(A)$ of “quasi-coherent sheaves” on the “quantized flag manifold $B_q$” by

$$\mathcal{M}(A) = \frac{\text{Mod}_\Lambda(A)}{\text{Tor}_\Lambda(A)} = \Sigma^{-1}\text{Mod}_\Lambda(A)$$

(see Gabriel-Zisman [6] and Popescu [16] for the notion of localization of categories). For $\nu \in \Lambda$ we denote by

$$\mathcal{M}(A) \ni M \mapsto M[\nu] \in \mathcal{M}(A)$$

the exact functor induced by $\text{Mod}_\Lambda(A) \ni M \mapsto M[\nu] \in \text{Mod}_\Lambda(A)$.

Let

$$\omega^* : \text{Mod}_\Lambda(A) \to \mathcal{M}(A)$$

be the canonical localization functor. We have the following by the definition.

**Lemma 2.4.** (i) $\omega^*$ is an exact functor.

(ii) Let $f$ be a morphism in $\text{Mod}_\Lambda(A)$. Then $\omega^* f$ is an isomorphism if and only if the kernel and the cokernel of $f$ belong to $\text{Tor}_\Lambda(A)$.  

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By Popescu [16, Ch4, Corollary 6.2] we have the following.

**Proposition 2.5.** The abelian category $\mathcal{M}(A)$ has enough injectives.

It is shown using Proposition 2.5 that there exists an additive functor

$$\omega_* : \mathcal{M}(A) \to \text{Mod}_\Lambda(A)$$

which is right adjoint to $\omega^*$ (see Popescu [16, Ch4, Proposition 5.2]). Note that $\omega_*$ is left exact. By Popescu [16, Ch4, Proposition 4.3] we have the following.

**Proposition 2.6.** The canonical morphism $\omega^* \circ \omega_* \to \text{Id}$ is an isomorphism.

**Corollary 2.7.** Let $M \in \text{Mod}_\Lambda(A)$. Set $N = \omega_* \omega^* M$ and let $f : M \to N$ be the canonical morphism. Then $N$ and $f$ are uniquely characterized by the following properties.

(a) Ker$(f)$ and Coker$(f)$ belong to Tor$_\Lambda(A)$.

(b) Tor$(N) = \{0\}$.

(c) Any monomorphism $N \to L$ with $L/N \in \text{Tor}_\Lambda(A)$ is a split morphism.

**Proof.** Let $f : M \to N = \omega_* \omega^* M$ be the canonical morphism. We have also a canonical morphism $g : \omega^* \omega_* (\omega^* M) \to \omega^* M$, and the composition $g \circ \omega^* f : \omega^* M \to \omega^* M$ is equal to id$_{\omega^* M}$ by the definition of the adjoint functors. Since $g$ is an isomorphism by Proposition 2.6, $\omega^* f$ is also an isomorphism. This implies (a). If $T$ is a subobject of $N = \omega_* \omega^* M$ belonging to Tor$_\Lambda(A)$, then we have

$$\text{Hom}(T, N) = \text{Hom}(T, \omega_* \omega^* M) \simeq \text{Hom}(\omega^* T, \omega^* M) = \text{Hom}(0, \omega^* M) = 0,$$

and hence $T = 0$. The statement (b) is proved. To show (c) it is sufficient to show that the homomorphism $\text{Hom}(L, N) \to \text{Hom}(N, N)$ induced by $N \to L$ is surjective. By $\omega^* M \simeq \omega^* N \simeq \omega^* L$ we have

$$\text{Hom}(L, N) \simeq \text{Hom}(\omega^* L, \omega^* M) \simeq \text{Hom}(\omega^* M, \omega^* M)$$

and similarly $\text{Hom}(N, N) \simeq \text{Hom}(\omega^* M, \omega^* M)$. Hence we have $\text{Hom}(L, N) \simeq \text{Hom}(N, N)$.

Assume that the conditions (a), (b), (c) are satisfied for some $f : M \to N$. By (a) $\omega^* f$ is an isomorphism. Let $h : N \to \omega_* \omega^* N$ be the canonical morphism, and define $g : N \to \omega_* \omega^* M$ by $g = (\omega_* \omega^* f)^{-1} \circ h$. Then the kernel and the cokernel of $g$ belong to Tor$_\Lambda(A)$. Hence by (b) we have Ker$(g) = 0$. By applying (c) to $g$ we see that Coker$(g)$ is isomorphic to a subobject of $\omega_* \omega^* M$, and hence Coker$(g) = 0$. \qed

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We define the “global section functor”

\[(2.13)\]
\[\Gamma : \mathcal{M}(A) \rightarrow \text{Mod}(\mathcal{F}).\]

by \(\Gamma(M) = (\omega_* M)(0)\). Note that \(\Gamma\) is left exact. Its right derived functors are denoted by

\[(2.14)\]
\[H^k = R^k\Gamma : \mathcal{M}(A) \rightarrow \text{Mod}(\mathcal{F}).\]

### 2.3 Affine open covering

For each \(w \in W\) and \(\lambda \in \Lambda^+\) we fix a non-zero element \(c^w_\lambda\) of \(A(\lambda)_{w^{-1}\lambda}\). Note that we have \(\dim A(\lambda)_{w^{-1}\lambda} = 1\). By Lemma 2.2 we have \(c^w_\lambda c^w_\mu \in \mathcal{F} \times c^w_{\lambda+\mu}\) for any \(w \in W\) and \(\lambda, \mu \in \Lambda^+\), and hence

\[(2.15)\]
\[S_w = \bigcup_{\lambda \in \Lambda^+} \mathcal{F} \times c^w_\lambda\]

is a multiplicative subset of \(A\) for any \(w \in W\). Moreover, we have the following.

**Proposition 2.8 (Joseph [9]).** Let \(w \in W\).

(i) \(S_w\) satisfies the left and right Ore conditions in \(A\).

(ii) The canonical homomorphism \(A \rightarrow S_w^{-1}A\) is injective.

We shall give a proof of Proposition 2.8 different from the one in [9]. We need the following.

**Lemma 2.9.** Let \(w \in W\), \(\mu \in \Lambda^+\), and fix \(v_{w^{-1}\mu} \in V(\mu)_{w^{-1}\mu} \setminus \{0\}\).

(i) For any \(\lambda \in \Lambda^+\)

\[p_{\lambda,\mu}|V(\lambda) \otimes v_{w^{-1}\mu} : V(\lambda) \otimes v_{w^{-1}\mu} \rightarrow V(\lambda + \mu)\]

is injective.

(ii) Let \(\gamma \in Q^+\). For sufficiently large \(\lambda \in \Lambda^+\) we have

\[p_{\lambda,\mu}(V(\lambda)_{w^{-1}(\lambda-\gamma)} \otimes v_{w^{-1}\mu}) = V(\lambda + \mu)_{w^{-1}(\lambda+\mu-\gamma)}.\]
Let us first show (a). We may assume $V_p$ for some $\lambda, \eta \in \Lambda$.

Lemma 2.9 (ii).

Proof of Proposition 2.8. It is sufficient to show $\dim V(\lambda_{\lambda - \gamma} \otimes v_\mu) = 0$. Assume $e_i v \otimes v_\mu = e_i (v \otimes v_\mu) \in N$ for any $i \in I$. Thus we have $e_i v = 0$ for any $i \in I$ by the hypothesis of induction. Hence we obtain $v = 0$.

(ii) By (i) the linear map

$$p_{\lambda, \mu} | V(\lambda_{\lambda - \gamma}) \otimes v_\mu : V(\lambda_{\lambda - \gamma}) \otimes v_\mu \to V(\lambda + \mu)_{\lambda + \mu - \gamma}.$$  

is injective. Hence it is sufficient to show $\dim V(\lambda_{\lambda - \gamma}) = \dim V(\lambda + \mu)_{\lambda + \mu - \gamma}$ when $\lambda$ is sufficiently large. This follows from Lemma 1.2. □

Proof of Proposition 2.8. It is sufficient to show the following.

(a) For any $\varphi \in A$ and $s \in S_w$ there exists some $t \in S_w$ and $\psi \in A$ satisfying $t \varphi = \psi s$.

(b) For any $\varphi \in A$ and $s \in S_w$ there exists some $t \in S_w$ and $\psi \in A$ satisfying $\varphi t = s \psi$.

(c) If $\varphi s = 0$ for $\varphi \in A$ and $s \in S_w$, then we have $\varphi = 0$.

(d) If $s \varphi = 0$ for $\varphi \in A$ and $s \in S_w$, then we have $\varphi = 0$.

Let us first show (a). We may assume $s = c_{\lambda}^w$ and $\varphi = f_\eta(m)$ for some $\lambda, \eta \in \Lambda^+$ and $m \in V(\eta)$. We may further assume that $m \in V(\eta_{w^{-1}(\eta - \gamma)})$ for some $\eta \in Q^+$. Then we need to find $\mu, \xi \in \Lambda^+$ and $n \in V(\xi)$ such that $c_{\lambda}^w f_\eta(m) = f_\xi(n)c_{\lambda}^w$. By Lemma 2.2 it is sufficient to show the existence of $\mu, \xi \in \Lambda^+$ and $n \in V(\xi)$ such that $\mu + \eta = \xi + \lambda$ and $p_{\mu, \eta} (v_{w^{-1}\mu} \otimes m) = p_{\xi, \lambda} (n \otimes v_{w^{-1}\lambda})$, where $v_{w^{-1}\lambda}$ and $v_{w^{-1}\mu}$ are non-zero elements of $V(\lambda)_{w^{-1}\lambda}$ and $V(\mu)_{w^{-1}\mu}$ respectively. Take sufficiently large $\xi \in \Lambda^+$ and set $\mu = \xi + \lambda - \eta \in \Lambda^+$. By $p_{\mu, \eta} (v_{w^{-1}\mu} \otimes m) \in V(\xi + \lambda)_{w^{-1}(\xi + \lambda - \gamma)}$ the assertion follows from Lemma 2.9 (ii).

Let us show (c). We may assume that there exists $\lambda, \mu \in \Lambda^+, \gamma \in Q^+$ and $m \in V(\mu)_{w^{-1}(\mu - \gamma)}$ such that $s = c_{\lambda}^w$ and $\varphi = f_\mu(m)$. By Lemma 2.2 we have $p_{\mu, \lambda} (m \otimes v_{w^{-1}\lambda}) = 0$, where $v_{w^{-1}\lambda}$ is a non-zero element of $V(\lambda)_{w^{-1}\lambda}$. Then we obtain $m = 0$ by Lemma 2.9 (ii). Hence $\varphi = 0$.

The statements (b) and (d) are proved similarly. □

Since $S_w$ consists of homogeneous elements, $S_w^{-1}A$ is a $\Lambda$-graded $F$-algebra. We define an $F$-algebra $R_w$ by

$$R_w = (S_w^{-1}A)(0).$$
Note that for any \( \lambda \in \Lambda S^{-1}_w A(\lambda) \) is a free left (right) \( R_w \)-module generated by \( c^w_{\lambda_1}(c^w_{\lambda_2})^{-1} \) where \( \lambda = \lambda_1 - \lambda_2 \) with \( \lambda_1, \lambda_2 \in \Lambda^+ \). In particular, we have an equivalence of categories

\[
\text{Mod}_\Lambda(S^{-1}_w A) \simeq \text{Mod}(R_w)
\]

given by

\[
\text{Mod}_\Lambda(S^{-1}_w A) \ni M \mapsto M(0) \in \text{Mod}(R_w),
\]

\[
\text{Mod}(R_w) \ni N \mapsto S^{-1}_w A \otimes_{R_w} N \in \text{Mod}_\Lambda(S^{-1}_w A).
\]

For any \( M \in \text{Tor}_\Lambda(A) \) we have \( S^{-1}_w M = 0 \) by the definition of \( \text{Tor}_\Lambda(A) \), and hence the localization functor \( \text{Mod}_\Lambda(A) \to \text{Mod}_\Lambda(S^{-1}_w A) \) induces an exact functor

\[
(2.17) \quad j^*_w : \mathcal{M}(A) \to \text{Mod}(R_w)
\]

for any \( w \in W \). \( \text{Mod}(R_w) \) is regarded as the category of “quasi-coherent sheaves on the affine open subset \( U_{w,q} = \text{Spec}(R_w) \) of \( B_q \)".

**Proposition 2.10 (Lunts-Rosenberg [12]).** \( \mathcal{M}(A) \) is a quasi-scheme with Zariski cover \( \{j^*_w | w \in W\} \) in the sense of Rosenberg [18]. In particular, a morphism \( f \) in \( \mathcal{M}(A) \) is an isomorphism if and only if \( j^*_w f \) is an isomorphism for any \( w \in W \).

**Remark 2.11.** An essential part in the proof of Proposition 2.10 is to show the following fact: For any \( \mu \in \Lambda^+ \) one has

\[
\sum_{w \in W} A(\lambda)c^w_{\mu} = A(\lambda + \mu)
\]

for any sufficiently large \( \lambda \in \Lambda^+ \). The proofs of this fact given in Joseph [9] and Lunts-Rosenberg [12] both use the reduction to the case \( q = 1 \). One needs another proof in order to define the quantized flag manifold at roots of unity as a quasi-scheme.

**Corollary 2.12.** For \( M \in \text{Mod}_\Lambda(A) \) the canonical homomorphism

\[
\omega_\star \omega^* M \to \bigoplus_{w \in W} S^{-1}_w M
\]

is injective.
By Proposition 2.10 one can use the general results in Rosenberg [18] for quasi-schemes. Especially we have a description of the cohomology groups in terms of certain Čech cohomology groups.

Using the Čech cohomology groups and the arguments involving the reduction to the case $q = 1$ Lunts-Rosenberg [12, III, §4] obtained the following analogues of Serre’s theorem and the Borel-Weil theorem.

**Proposition 2.13 (Lunts-Rosenberg [12]).** Let $f : M \to N$ be an epimorphism in $\text{Mod}_A(A)$. Assume that $M, N$ are finitely-generated as $A$-modules. Then the homomorphism

$$\Gamma(\omega^*M[\lambda]) \to \Gamma(\omega^*N[\lambda])$$

is surjective for sufficiently large $\lambda \in \Lambda^+$.

**Proposition 2.14 (Lunts-Rosenberg [12]).** For $\lambda \in \Lambda$ we have

$$\Gamma(A[\lambda]) = \begin{cases} A(\lambda) & (\lambda \in \Lambda^+) \\ 0 & (\lambda \notin \Lambda^+) \end{cases}.$$  

In other words we have $A \simeq \omega_*\omega^*A$.

### 2.4 Schubert varieties

The contents of this subsection will not be used in the sequel.

Let $w \in W$. Since $F$ is a sum of finite-dimensional right $U$-submodules contained in $\text{Mod}^f(U^{\text{op}})$, we can define the operator $^tT_w : F \to F$. Let $\tilde{\epsilon}_w : F \to F$ be the linear map defined by $\tilde{\epsilon}_w(\varphi) = (^tT_w(\varphi), 1)$ for $\varphi \in F$, and define

$$\epsilon_w : A \to \mathbb{F} \quad \text{(2.18)}$$

to be the composition of the inclusion $A \hookrightarrow F$ with $\tilde{\epsilon}_w$.

**Lemma 2.15.** $\epsilon_w$ is an algebra homomorphism, and we have $\epsilon_w(S_w) \subseteq \mathbb{F}^\times$.

**Proof.** By $^tT_w(1) = 1$ we have $\epsilon_w(1) = 1$. Let us show $\epsilon_w(\varphi \psi) = \epsilon_w(\varphi)\epsilon_w(\psi)$ for $\varphi, \psi \in A$. We may assume that $\varphi \in A(\lambda), \psi \in A(\mu)$ for $\lambda, \mu \in \Lambda^+$. Then we have $\varphi = f_\lambda(v_1), \psi = f_\mu(v_2)$ for some $v_1 \in V(\lambda), v_2 \in V(\mu)$. For $u \in U$ we have

$$\langle \varphi, u \rangle = \langle v_\lambda^*, u v_1 \rangle, \quad \langle \psi, u \rangle = \langle v_\mu^*, u v_2 \rangle, \quad \langle \varphi \psi, u \rangle = \langle v_\lambda^* \otimes v_\mu^*, u(v_1 \otimes v_2) \rangle,$$
and hence

\[ \epsilon_w(\varphi) = \langle t^T_w(v^*_\lambda), v_1 \rangle, \quad \epsilon_w(\psi) = \langle t^T_w(v^*_\mu), v_2 \rangle, \]

\[ \epsilon_w(\varphi \psi) = \langle t^T_w(v^*_\lambda \otimes v^*_\mu), v_1 \otimes v_2 \rangle. \]

Hence we obtain \( \epsilon_w(\varphi \psi) = \epsilon_w(\varphi) \epsilon_w(\psi) \) by Lemma 1.8. For \( \lambda \in \Lambda^+ \) take \( v_{w^{-1}_\lambda} \in V(\lambda)_{w^{-1}_\lambda} \setminus \{0\} \) such that \( c^w_\lambda = f_\lambda(v_{w^{-1}_\lambda}) \). Then we have \( \langle c^w_\lambda, u \rangle = \langle v^*_\lambda, uv_{w^{-1}_\lambda} \rangle \) for \( u \in U \), and hence

\[ \epsilon_w(c^w_\lambda) = \langle t^T_w(v^*_\lambda), v_{w^{-1}_\lambda} \rangle = \langle v^*_\lambda, T_w(v_{w^{-1}_\lambda}) \rangle \neq 0. \]

By \( \epsilon_w(S_w) \subset F^x \) the algebra homomorphism \( \epsilon_w \) is uniquely extended to

(2.19) \[ \epsilon_w : S_w^{-1} A \to F. \]

We define an algebra homomorphism

(2.20) \[ \Phi_w : A \to F^{\geq 0} \]

as the composition of

\[ A \xrightarrow{r_+} A \otimes F \xrightarrow{\epsilon_w \otimes r_+} F \otimes F^{\geq 0} = F^{\geq 0}. \]

**Lemma 2.16.** (i) Elements of \( \Phi_w(S_w) \) are invertible in \( F^{\geq 0} \).

(ii) We have \( \Phi_w(A(\lambda)) \subset F^{\geq 0}(w^{-1}_\lambda) \) for any \( \lambda \in \Lambda \).

**Proof.** (i) Let \( \lambda \in \Lambda^+ \). Take \( v_{w^{-1}_\lambda} \in V(\lambda)_{w^{-1}_\lambda} \setminus \{0\} \) such that \( c^w_\lambda = f_\lambda(v_{w^{-1}_\lambda}) \). Then for \( u \in U \) and \( x \in U^{\geq 0} \) we have

\[ \langle (\text{id} \otimes r_+) \circ \overline{\Delta}(c^w_\lambda), u \otimes x \rangle = \langle c^w_\lambda, ux \rangle = \langle v^*_\lambda, uxv_{w^{-1}_\lambda} \rangle. \]

Therefore, we have

\[ \langle \Phi_w(c^w_\lambda), x \rangle = \langle t^T_w(v^*_\lambda), xv_{w^{-1}_\lambda} \rangle = \chi^+_{w^{-1}_\lambda}(x) \langle t^T_w(v^*_\lambda), v_{w^{-1}_\lambda} \rangle. \]

By \( t^T_w(v^*_\lambda) \in V^*(\lambda)_{w^{-1}_\lambda} \setminus \{0\} \) we obtain \( \Phi_w(c^w_\lambda) = \chi^+_{w^{-1}_\lambda} \) up to a non-zero constant multiple. The statement (i) is proved.

The proof of (ii) is similar and omitted. \( \square \)

By Lemma 2.16 (ii) \( \ker(\Phi_w) \) is a graded ideal of \( A \). The \( \Lambda \)-graded \( F \)-algebra

(2.21) \[ A_w = A/ \ker(\Phi_w) \]

is a \( q \)-analogue of the homogeneous coordinate algebra of the Schubert variety corresponding to \( w \in W \).
3 Quasi-coherent sheaves with $U$-actions

3.1 The algebra $\tilde{U}$

Recall that $A$ is a left $U$-module by Lemma 2.1 (ii). We sometimes write this action of $U$ on $A$ by

$$U \otimes A \rightarrow A \quad (u \otimes \varphi \mapsto \partial_u(\varphi)).$$

We have also a left $A$-module structure on $A$ given by the left multiplication. By Lemma 2.1 (iv) we have

$$\partial_u(\varphi \psi) = \sum_{(u)} \partial_{u(0)}(\varphi) \partial_{u(1)}(\psi) \quad (u \in U, \varphi, \psi \in A),$$

and hence $A$ is a module over the $\mathbb{F}$-algebra $\tilde{U}$ generated by the elements

$$\{ \pi | a \in A \} \cup \{ \pi | u \in U \}$$

satisfying the fundamental relations:

$$\varphi_1 \varphi_2 = \varphi_1 \varphi_2 \quad (\varphi_1, \varphi_2 \in A),$$

$$u_1 u_2 = u_1 u_2 \quad (u_1, u_2 \in U),$$

$$u \varphi = \sum_{(u)} \partial_{u(0)}(\varphi) u_{(1)} \quad (u \in U, \varphi \in A).$$

For $u \in U$ and $\varphi \in A$ we have

$$\sum_{(u)} u_{(1)} \partial_{S^{-1}u(0)}(\varphi) = \sum_{(u)} \partial_{u(1)}(\varphi) u_{(2)} = \sum_{(u)} e(u(0)) \varphi u_{(1)} = \varphi u_{(1)}$$

by (3.3), and hence

$$\varphi u_{(1)} = \sum_{(u)} u_{(1)} \partial_{S^{-1}u(0)}(\varphi) \quad (u \in U, \varphi \in A).$$

By a similar calculation we see that (3.4) implies (3.3). Hence we can replace (3.4) with (3.3) in defining $\tilde{U}$. Rewriting (3.3) and (3.4) in terms of generators of $U$ we obtain

$$\bar{k}_\lambda \varphi = \partial_{k_\lambda}(\varphi) k_\lambda \quad (\lambda \in \Lambda, \varphi \in A),$$

$$\bar{e}_i \varphi = \partial_{k_i}(\varphi) e_i + \partial_{e_i}(\varphi) \quad (i \in I, \varphi \in A),$$

$$\bar{f}_i \varphi = \bar{\varphi} \bar{f}_i + \partial_{f_i}(\varphi) k_i^{-1} \quad (i \in I, \varphi \in A),$$

$$\bar{\varphi} \bar{k}_\lambda = \bar{k}_\lambda \partial_{k^{-1}_\lambda}(\varphi) \quad (\lambda \in \Lambda, \varphi \in A),$$

$$\bar{\varphi} \bar{e}_i = \bar{e}_i \partial_{k^{-1}_i}(\varphi) - \bar{e}_i k_i^{-1}(\varphi) \quad (i \in I, \varphi \in A),$$

$$\bar{\varphi} \bar{f}_i = \bar{f}_i \bar{\varphi} - k_i^{-1} \partial_{k_i f_i}(\varphi) \quad (i \in I, \varphi \in A).$$
Proposition 3.1. The linear maps

\[ i_1 : A \otimes U \to \tilde{U} \quad (\varphi \otimes u \mapsto \overline{\varphi u}), \quad i_2 : U \otimes A \to \tilde{U} \quad (u \otimes \varphi \mapsto \overline{u \varphi}) \]

are bijective.

Proof. We can define an \( F \)-algebra structure on \( A \otimes U \) by

\[ (\varphi \otimes u)(\varphi' \otimes u') = \sum_{(u_1)} \varphi \partial_{u(0)}(\varphi') \otimes u_{(1)}u'. \]

Then an algebra homomorphism \( j : \tilde{U} \to A \otimes U \) is defined by

\[ j(\overline{\varphi}) = \varphi \otimes 1 \quad (\varphi \in A), \quad j(\overline{u}) = 1 \otimes u \quad (u \in U). \]

Moreover, \( i_1 \) is an algebra homomorphism, and we have \( j \circ i_1 = \text{id}, i_1 \circ j = \text{id} \). Thus \( i_1 \) is an isomorphism. Similarly, \( i_2 \) is an isomorphism. \( \square \)

We shall regard \( A \) and \( U \) as subalgebras of \( \tilde{U} \) by the embeddings

\[ A \hookrightarrow \tilde{U} \quad (\varphi \mapsto \overline{\varphi}), \quad U \hookrightarrow \tilde{U} \quad (u \mapsto \overline{u}), \]

and we sometimes write \( \varphi \) and \( u \) for \( \overline{\varphi} \) and \( \overline{u} \).

Note that \( \tilde{U} \) is naturally a \( \Lambda \)-graded \( F \)-algebra by

\[ \tilde{U}(\lambda) = A(\lambda)U = UA(\lambda) \quad (\lambda \in \Lambda), \]

and \( A \) is an object of \( \text{Mod}_\Lambda(\tilde{U}) \).

Proposition 3.2. Let \( w \in W \).

(i) \( S_w \) satisfies the left and right Ore conditions in \( \tilde{U} \).

(ii) The canonical homomorphism \( \tilde{U} \to S_w^{-1}\tilde{U} \) is injective.

Proof. It is sufficient to show the following.

(a) For any \( d \in \tilde{U} \) and \( s \in S_w \) there exists some \( t \in S_w \) and \( d' \in \tilde{U} \) satisfying \( td = d's \).

(b) For any \( d \in \tilde{U} \) and \( s \in S_w \) there exists some \( t \in S_w \) and \( d' \in \tilde{U} \) satisfying \( dt = sd' \).

(c) If \( sd = 0 \) for \( d \in \tilde{U} \) and \( s \in S_w \), then we have \( d = 0 \).

(d) If \( ds = 0 \) for \( d \in \tilde{U} \) and \( s \in S_w \), then we have \( d = 0 \).
In proving (a) and (b) we only need to deal with the cases \( d = \varphi \in A \) and \( d = u \in U \). The case \( d = \varphi \in A \) is already known since \( S_w \) satisfies the left and right Ore conditions in \( A \). The case \( d = u \in U \) is a consequence of (3.5), ..., (3.10) and the case \( d = \varphi \in A \). The statements (c) and (d) follow from Proposition 3.1 since \( S_w \) satisfies the left and right Ore conditions in \( A \) and since \( A \to S_w^{-1} \) is injective.

By Proposition 3.2 (i) we have

\[
S_w^{-1} A \otimes_A \tilde{U} \simeq S_w^{-1} \tilde{U} \simeq \tilde{U} \otimes_A S_w^{-1} A.
\]

Moreover, for any \( M \in \text{Mod}_A(\tilde{U}) \) we have

\[
S_w^{-1} M = S_w^{-1} A \otimes_A M = S_w^{-1} \tilde{U} \otimes \tilde{U} M \in \text{Mod}_A(S_w^{-1} \tilde{U}).
\]

In particular, \( S_w^{-1} A \) is a \( U \)-module. We write the action of \( U \) on \( S_w^{-1} A \) by

\[
U \otimes S_w^{-1} A \to S_w^{-1} A \quad (u \otimes \varphi \mapsto \partial_u(\varphi)).
\]

**Lemma 3.3.** For any \( w \in W \) we have

\[
\varphi = \sum_{(u)} \partial_{u(0)}(\varphi) u(1) \quad (u \in U, \ \varphi \in S_w^{-1} A)
\]

in \( S_w^{-1} \tilde{U} \).

**Proof.** Set \( \mathcal{X} = \{ k_\lambda, e_i, f_i \mid \lambda \in \Lambda, \ i \in I \} \). For \( u \in U \) and \( s \in S_w \) we have

\[
\varepsilon(u) 1 = \partial_u(1) = \partial_u(ss^{-1}) = (us)(s^{-1}) = \sum_{(u)} \partial_{u(0)}(s) \partial_{u(1)}(s^{-1}).
\]

Considering the case \( u \in \mathcal{X} \) we obtain

\[
\begin{align*}
\partial_{k_\lambda}(s) \partial_{k_\lambda}(s^{-1}) &= 1 \quad (\lambda \in \Lambda), \\
\partial_{e_i}(s) s^{-1} + \partial_{k_i}(s) \partial_{e_i}(s^{-1}) &= 0 \quad (i \in I), \\
\partial_{f_i} s^{-1} + \partial_{k_i^{-1}}(s) + s \partial_{f_i}(s^{-1}) &= 0 \quad (i \in I).
\end{align*}
\]

We can rewrite them as

\[
\begin{align*}
\partial_{k_\lambda}(s^{-1}) \partial_{k_\lambda}(s) &= 1 \quad (\lambda \in \Lambda), \\
\partial_{e_i}(s^{-1}) s + \partial_{k_i}(s^{-1}) \partial_{e_i}(s) &= 0 \quad (i \in I), \\
\partial_{f_i}(s^{-1}) \partial_{k_i^{-1}}(s) + s^{-1} \partial_{f_i}(s) &= 0 \quad (i \in I),
\end{align*}
\]

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which gives

\[(3.15) \quad \sum_{(u)} \partial_{u(0)}(s^{-1})\partial_{u(1)}(s) = \epsilon(u)1 \quad (u \in \mathcal{X}).\]

Hence for \(u \in \mathcal{X}\) and \(s \in S_w\) we have

\[
\left(\sum_{(u)} \partial_{u(0)}(s^{-1})u(1)\right) s = \sum_{(u)} \partial_{u(0)}(s^{-1})\partial_{u(1)}(s)u(2) = \sum_{(u)} \epsilon(u(0))u(1) = u
\]

in \(S_w^{-1}U\). Here, we have used \((3.15)\) and the fact that for any \(u \in \mathcal{X}\) \(\Delta(u)\) is a sum of the elements of \(\mathcal{X} \otimes \mathcal{X}\). Hence we have

\[
us^{-1} = \sum_{(u)} \partial_{u(0)}(s^{-1})u(1) \quad (u \in \mathcal{X}).
\]

Let \(\varphi \in S_w^{-1}A\). We can write it as \(\varphi = s^{-1}\psi\) for \(s \in S_w, \psi \in A\). Then we obtain

\[(3.16) \quad u\varphi = us^{-1}\psi = \sum_{(u)} \partial_{u(0)}(s^{-1})u(1)\psi = \sum_{(u)} \partial_{u(0)}(s^{-1})\partial_{u(1)}(\psi)u(2)\]

for any \(u \in \mathcal{X}\). In particular, we have

\[(3.17) \quad \partial_u(\varphi) = (u\varphi)(1) = \sum_{(u)} \partial_{u(0)}(s^{-1})\partial_{u(1)}(\psi)\partial_u(1) = \sum_{(u)} \partial_{u(0)}(s^{-1})\partial_u(1)\psi\]

for any \(u \in \mathcal{X}\). By \((3.16), (3.17)\) we obtain \((3.14)\) for \(u \in \mathcal{X}\). Since \(\mathcal{X}\) generates \(U\), \((3.14)\) holds for any \(u \in U\).

**Remark 3.4.** We can give a more conceptual proof of Lemma 3.3 using the right \(A\)-module structures on \(A\) and \(S_w^{-1}A\) given by the right multiplications.

**Proposition 3.5.** Let \(w \in W\). The \(U\)-module structure on \(S_w^{-1}A\) is characterized by the following conditions:

(a) The inclusion \(A \hookrightarrow S_w^{-1}A\) is a homomorphism of \(U\)-modules.

(b) For \(u \in U, \varphi_0, \varphi_1 \in S_w^{-1}A\) we have \(\partial_u(\varphi_0\varphi_1) = \sum_{(u)} \partial_{u(0)}(\varphi_0)\partial_{u(1)}(\varphi_1)\).

**Proof.** The \(U\)-module structure \((3.13)\) obviously satisfies the condition (a), and it satisfies (b) by Lemma 3.3.

Assume that we are given a \(U\)-module structure

\[U \otimes S_w^{-1}A \rightarrow S_w^{-1}A \quad (u \otimes \varphi \mapsto \partial_u(\varphi))\]
satisfying (a) and (b). For \( s \in S_w \) and \( u \in U \) we have

\[
e(\epsilon(u)1) = \partial_u(1) = \partial_u(s^{-1}s) = \sum_{(u_i)} \partial_{a(u)}(s^{-1})\partial_{a(1)}(s)
\]

by (a) and (b). In particular, we obtain

\[
\begin{align*}
\partial_{k_{\lambda}}(s^{-1})\partial_{k_{\lambda}}(s) &= 1 \\
\partial_{e_i}(s^{-1})s + \partial_{k_{\lambda}}(s^{-1})\partial_{e_i}(s) &= 0 \\
\partial_{f_i}(s^{-1})\partial_{k_{i}^{-1}}(s) + s^{-1}\partial_{f_i}(s) &= 0
\end{align*}
\]

\((\lambda \in \Lambda), (i \in I)\).

Hence we have

\[
\begin{align*}
\partial_{k_{\lambda}}(s^{-1}) &= (\partial_{k_{\lambda}}(s))^{-1} \\
\partial_{e_i}(s^{-1}) &= -(\partial_{k_{i}}(s))^{-1}\partial_{e_i}(s)s^{-1} \\
\partial_{f_i}(s^{-1}) &= -s^{-1}\partial_{f_i}(s)(\partial_{k_{i}^{-1}}(s))^{-1}
\end{align*}
\]

\((\lambda \in \Lambda), (i \in I)\).

By (b) we have

\[
\begin{align*}
\partial_{k_{\lambda}}(s^{-1}\psi) &= (\partial_{k_{\lambda}}(s))^{-1}\partial_{k_{\lambda}}(\psi), \\
\partial_{e_i}(s^{-1}\psi) &= -(\partial_{k_{i}}(s))^{-1}\partial_{e_i}(s)s^{-1}\psi + (\partial_{k_{i}}(s))^{-1}\partial_{e_i}(\psi), \\
\partial_{f_i}(s^{-1}\psi) &= -s^{-1}\partial_{f_i}(s)(\partial_{k_{i}^{-1}}(s))^{-1}\partial_{k_{i}^{-1}}(\psi) + s^{-1}\partial_{f_i}(\psi).
\end{align*}
\]

for any \( \psi \in A \). It implies that the action of the generators of \( U \) on \( S_w^{-1}A \) is uniquely determined from the \( U \)-module structure of \( A \). The uniqueness of the \( U \)-module structure on \( S_w^{-1}A \) satisfying (a) and (b) is verified.

\[\square\]

### 3.2 Global section functors

In this subsection \( C \) denotes a \( \Lambda \)-graded \( \mathbb{F} \)-algebra satisfying

\begin{align*}
(3.18) & \quad C \text{ contains } A \text{ as a } \Lambda \text{-graded subalgebra}, \\
(3.19) & \quad S_w \text{ satisfies the left and right Ore conditions in } C \text{ for any } w \in W, \\
(3.20) & \quad A(\lambda)C(\mu) \subset C(\mu)A(\lambda) \text{ for any } \lambda, \mu \in \Lambda.
\end{align*}

Note that \( A \) and \( \tilde{U} \) satisfy the conditions \((3.18), (3.19), (3.20)\). Another example will be the \( \Lambda \)-graded \( \mathbb{F} \)-algebra \( D \), which will be introduced in Section 4.1.

We have an obvious exact functor

\[
(3.21) \quad F_* : \text{Mod}_\Lambda(C) \to \text{Mod}_\Lambda(A)
\]
given by restricting the action of $C$ to $A$. Its left adjoint functor is given by

$$F^* : \text{Mod}_\Lambda(A) \to \text{Mod}_\Lambda(C) \quad (M \mapsto C \otimes_A M).$$

The condition (3.20) implies the following.

**Lemma 3.6.** Let $M \in \text{Mod}_\Lambda(C)$. Then $\text{Tor}(M)$, which is a priori an object of $\text{Mod}_\Lambda(A)$, is in fact an object of $\text{Mod}_\Lambda(C)$.

Set

$$\mathcal{M}(C) := \frac{\text{Mod}_\Lambda(C)}{\text{Tor}_\Lambda(C)} = \Sigma_C^{-1} \text{Mod}_\Lambda(C),$$

where $\text{Tor}_\Lambda(C)$ denotes the full subcategory of $\text{Mod}_\Lambda(C)$ consisting of $M \in \text{Mod}_\Lambda(C)$ with $F_* M \in \text{Tor}_\Lambda(A)$, and $\Sigma_C$ is the collection of morphisms $f$ in $\text{Mod}_\Lambda(C)$ satisfying $F_* f \in \Sigma$. Let

$$\omega_C^* : \text{Mod}_\Lambda(C) \to \mathcal{M}(C)$$

be the localization functor. Similarly to the case $C = A$, the abelian category $\mathcal{M}(C)$ has enough injectives and we have a left exact functor

$$\omega_{C*} : \mathcal{M}(C) \to \text{Mod}_\Lambda(C)$$

which is right adjoint to $\omega_C^*$. The canonical morphism $\omega_C^* \circ \omega_{C*} \to \text{Id}$ is an isomorphism and we have the following.

**Lemma 3.7.** Let $M \in \text{Mod}_\Lambda(C)$. Set $N = \omega_{C*} \omega_C^* M$ and let $f : M \to N$ be the canonical morphism. Then $N$ and $f$ are uniquely characterized by the following properties.

(a) $\text{Ker}(f)$ and $\text{Coker}(f)$ belong to $\text{Tor}_\Lambda(C)$.

(b) $\text{Tor}(N) = \{0\}$.

(c) Any monomorphism $N \to L$ with $L/N \in \text{Tor}_\Lambda(C)$ is a split morphism.

We define a left exact functor

$$\Gamma_C : \mathcal{M}(C) \to \text{Mod}(C(0))$$

by $\Gamma_C(M) = (\omega_{C*} M)(0)$.

By the universal property of the localization functor there exists uniquely an exact functor

$$\tilde{F}_* : \mathcal{M}(C) \to \mathcal{M}(A)$$
such that the following diagram is commutative:

\[
\begin{array}{cccc}
\text{Mod}_\Lambda(C) & \xrightarrow{F_*} & \text{Mod}_\Lambda(A) \\
\omega_C & \downarrow & \omega_* \\
\mathcal{M}(C) & \xrightarrow{F_*} & \mathcal{M}(A).
\end{array}
\]

**Lemma 3.8.** There exists uniquely an additive functor

\[(3.28)\quad \tilde{F}^*: \mathcal{M}(A) \rightarrow \mathcal{M}(C)\]

such that the following diagram is commutative:

\[
\begin{array}{cccc}
\text{Mod}_\Lambda(A) & \xrightarrow{F_*} & \text{Mod}_\Lambda(C) \\
\omega_* & \downarrow & \omega_C \\
\mathcal{M}(A) & \xrightarrow{F_*} & \mathcal{M}(C).
\end{array}
\]

**Proof.** By the universal property of the localization functor it is sufficient to show \(F^*(\Sigma) \subset \Sigma_C\). Let \(f: M \rightarrow N\) be a morphism in \(\Sigma\). The corresponding homomorphism \(S^{-1}_w A \otimes_A M \rightarrow S^{-1}_w A \otimes_A N\) is an isomorphism for any \(w \in W\). By Proposition 2.10 we have only to show that the corresponding morphism \(S^{-1}_w A \otimes_A C \otimes_A M \rightarrow S^{-1}_w A \otimes_A C \otimes_A N\) is an isomorphism for any \(w \in W\). This follows from \(S^{-1}_w C \simeq S^{-1}_w A \otimes_A C \simeq C \otimes_A S^{-1}_w A\).

**Lemma 3.9.** The following diagram is commutative:

\[
\begin{array}{cccc}
\mathcal{M}(C) & \xrightarrow{\tilde{F}_*} & \mathcal{M}(A) \\
\omega_{C*} & \downarrow & \omega_* \\
\text{Mod}_\Lambda(C) & \xrightarrow{F_*} & \text{Mod}_\Lambda(A).
\end{array}
\]

**Proof.** We have a sequence of morphisms

\[F_* \circ \omega_{C*} \rightarrow \omega_* \circ \omega^* \circ F_* \circ \omega_{C*} = \omega_* \circ \tilde{F}_* \circ \omega^* \circ \omega_{C*} = \omega_* \circ \tilde{F}_* .\]

Hence it is sufficient to show that \(g: F_* \omega_{C*} M \rightarrow \omega_* \omega^* F_* \omega_{C*} M\) is an isomorphism for any \(M\). By

\[\omega^* \circ F_* \circ \omega_{C*} = \tilde{F}_* \circ \omega^* \circ \omega_{C*} = \tilde{F}_* = \omega^* \circ \omega_* \circ \tilde{F}_* ,\]

\(\omega^* g\) is an isomorphism, and hence \(\text{Ker}(g), \text{Coker}(g) \in \text{Tor}_A(A)\).

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By $\text{Tor}(F_*\omega_C, M) = 0$ and $\text{Ker}(g) \in \text{Tor}_A(A)$ $g$ is a monomorphism. Hence it is sufficient to show that the injective homomorphism

$$g_L : \text{Hom}(L, F_*\omega_C, M) \rightarrow \text{Hom}(L, \omega_*\omega^*F_*\omega_C, M)$$

is surjective for any $L$. By

$$\text{Hom}(L, \omega_*\omega^*F_*\omega_C, M) \simeq \text{Hom}(\omega^*L, \omega_*F_*\omega_C, M) \simeq \text{Hom}(\omega_C^*F^*L, M) \simeq \text{Hom}(F^*\omega_*L, M)$$

$\text{Hom}(L, F_*\omega_C, M)$ and $\text{Hom}(L, \omega_*\omega^*F_*\omega_C, M)$ depend only on $\omega^*L$ Let $a \in \text{Hom}(L, \omega_*\omega^*F_*\omega_C, M)$. Set $L_1 = \text{Ker}(L \rightarrow \text{Coker}(g))$. Since $\text{Coker}(g)$ belongs to $\text{Tor}_A(A)$ we have $L/L_1 \in \text{Tor}_A(A)$, and hence we have $\omega^*L_1 \simeq \omega^*L$. Therefore, we can replace $L$ with $L_1$. Then $a|L_1$ is clearly contained in the image of $g_{L_1}$.

In view of Lemma 3.9 we shall often drop the subscript $C$ in $\omega_C$ and $\Gamma_C$ and write them simply as $\omega_*$ and $\Gamma$.

Let $K$ be a full subcategory of $\text{Mod}_A(C)$ closed under taking subquotients in $\text{Mod}_A(C)$, and set

$$\mathcal{K} = K/\text{Tor}_A(C) \cap K = \Sigma^{-1}K,$$

where $\Sigma$ is the collection of morphisms in $K$ whose kernel and cokernel belong to $\text{Tor}_A(C) \cap K$, and denote by $\overrightarrow{\omega} : K \rightarrow \mathcal{K}$ the localization functor. Let $j : K \rightarrow \text{Mod}_A(C)$ be the embedding. By the universal property of the localization functor we have a functor $\overrightarrow{j} : \mathcal{K} \rightarrow \mathcal{M}(C)$ such that the following diagram commutes:

$$\begin{array}{ccc}
K & \xrightarrow{j} & \text{Mod}_A(C) \\
\overrightarrow{\omega} \downarrow & & \downarrow \omega_C \\
\mathcal{K} & \xrightarrow{\overrightarrow{j}} & \mathcal{M}(C).
\end{array}$$

**Lemma 3.10.** $\overrightarrow{j}$ is fully faithful.

**Proof.** We need to show that the canonical homomorphism

$$(3.29) \quad \text{Hom}_K(\overrightarrow{\omega}^*M, \overrightarrow{\omega}^*N) \rightarrow \text{Hom}_{\mathcal{M}(C)}(\omega_C^*M, \omega_C^*N)$$

is an isomorphism for $M, N \in K$. We may assume $\text{Tor}(M) = \text{Tor}(N) = \{0\}$.

Assume that $\overrightarrow{\omega}^*f \circ (\overrightarrow{\omega}^*s)^{-1}$ belongs to the kernel of $(3.29)$, where $f : L \rightarrow N$ is a morphism in $K$ and $s : L \rightarrow M$ belongs to $\Sigma$. By $\text{Tor}(M) = \text{Tor}(N) = \{0\}$
we have $\text{Tor}(L) = \text{Ker}(s) \subset \text{Ker}(f)$, and hence we may assume that $\text{Tor}(L) = \{0\}$ by replacing $L$ with $L/\text{Tor}(L)$. Since $\omega_C^* f \circ (\omega_C^* s)^{-1}$ belongs to the kernel of (3.29), we have $\omega_C^* f \circ (\omega_C^* s)^{-1} = 0$, and hence $\omega_C^* f = 0$. It means that there exists $t : R \to L$, which belongs to $\Sigma_C$, such that $f \circ t = 0$. By $\text{Tor}(L) = \{0\}$ we have $\text{Tor}(R) = \text{Ker}(t)$. Hence we can assume $\text{Ker}(t) = \{0\}$ by replacing $R$ with $R/\text{Tor}(R)$. Then $R$ is a subobject of $L$, and hence $t$ belongs to $\Sigma$. It follows that $\omega_C^* f = 0$, and hence $\omega_C^* f \circ (\omega_C^* s)^{-1} = 0$.

3.3 The vector bundle $E^\mu$ associated to $V(\mu)$

The contents of Sections 3.3, 3.4 except for Proposition 3.13 below are due to Lunts-Rosenberg [12]. Some of the proofs are also included for the convenience of the readers.

Let $\mu \in \Lambda^+$. Following Lunts-Rosenberg [12] we shall define an $A$-bimodule $E^\mu$, which is a $q$-analogue of the vector bundle $\mathcal{O}_B \otimes_C V^1(\mu)$.

Set

\[(3.30) \quad E^\mu = V(\mu) \otimes A.\]

We endow $E^\mu$ with a right $A$-module structure by $(v \otimes \varphi) \psi = v \otimes \varphi \psi$ for $v \in V(\mu)$ and $\varphi, \psi \in A$. We identify $E^\mu$ with $A \otimes V(\mu)$ via the isomorphism

\[\eta = R_{A,V(\mu)}^\vee : A \otimes V(\mu) \to V(\mu) \otimes A = E^\mu,\]

and define a left $A$-module structure on $E^\mu = A \otimes V(\mu)$ by $\psi(\varphi \otimes v) = \psi \varphi \otimes v$ for $v \in V(\mu)$ and $\varphi, \psi \in A$. Note that $\eta$ is well-defined since $A$ is a sum of $U$-submodules belonging to $\text{Mod}^f(U)$.

**Lemma 3.11.**  (i) For $\varphi, \psi \in A$ and $e \in E^\mu$ we have $(\varphi e) \psi = \varphi (e \psi)$.

(ii) We have a commutative diagram

\[
\begin{array}{ccc}
V(\mu) & \longrightarrow & V(\mu) \\
\downarrow & & \downarrow \\
A \otimes V(\mu) & \longrightarrow & V(\mu) \otimes A,
\end{array}
\]

where the left (right) vertical arrow is given by $v \mapsto 1 \otimes v$ (resp. $v \mapsto v \otimes 1$).
Proof. (i) Let $m : A \otimes A \to A$ be the multiplication of the algebra $A$. It is sufficient to show that the composition of

$$A \otimes V(\mu) \otimes A \xrightarrow{\eta \otimes \text{id}_A} V(\mu) \otimes A \otimes A \xrightarrow{\text{id}_V(\mu) \otimes m} V(\mu) \otimes A$$

gives the left action of $A$ on $E^\mu = V(\mu) \otimes A$. This is equivalent to showing that

$$(\text{id}_V(\mu) \otimes m) \circ (\eta \otimes \text{id}_A) \circ (\text{id}_A \otimes \eta) : A \otimes A \otimes V(\mu) \to V(\mu) \otimes A$$

coincides with

$$\eta \circ (m \otimes \text{id}_V(\mu)) : A \otimes A \otimes V(\mu) \to V(\mu) \otimes A.$$ 

Indeed we have

$$
\begin{align*}
(\text{id}_V(\mu) \otimes m) \circ (\eta \otimes \text{id}_A) \circ (\text{id}_A \otimes \eta) &= (\text{id}_V(\mu) \otimes m) \circ (\mathcal{R}_{A,V(\mu)} \otimes \text{id}_A) \circ (\text{id}_A \otimes \mathcal{R}_{A,V(\mu)}) \\
&= (\text{id}_V(\mu) \otimes m) \circ \mathcal{R}_{A \otimes A,V(\mu)} \\
&= \mathcal{R}_{A,V(\mu)} \circ (m \otimes \text{id}_V(\mu)) \\
&= \eta \circ (m \otimes \text{id}_V(\mu)).
\end{align*}
$$

Here the second equality is a consequence of Proposition 1.4 (iii), and the third equality follows from (1.40).

(ii) Note that $A(0) = \mathbb{F}1$ is isomorphic to the trivial $U$-modules $V(0)$. We need to show that $\mathcal{R}_{V(0),V(\mu)} : V(0) \otimes V(\mu) \to V(\mu) \otimes V(0)$ is equal to $\text{id}_V(\mu)$ under the identification $V(0) \otimes V(\mu) \simeq V(\mu) \otimes V(0) \simeq V(\mu)$ of $U$-modules. This follows from the definition of $\mathcal{R}_{V(0),V(\mu)}$. \[\Box\]

By Lemma 3.11 (i) $E^\mu$ is an $A$-bimodule. By Lemma 3.11 (ii) we have a canonical embedding $V(\mu) \hookrightarrow E^\mu$ such that the actions of $A$ on $E^\mu$ from the left and the right induce $A \otimes V(\mu) \simeq E^\mu$ and $V(\mu) \otimes A \simeq E^\mu$ respectively.

For $\lambda \in \Lambda$ we set

$$E^\mu(\lambda) = V(\mu) \otimes A(\lambda) \subset E^\mu.$$ 

By $\eta(A(\lambda) \otimes V(\mu)) = V(\mu) \otimes A(\lambda)$ we have $E^\mu(\lambda) = A(\lambda) \otimes V(\mu)$ under the identification $E^\mu = A \otimes V(\mu)$. Moreover, we have

$$E^\mu = \bigoplus_{\lambda \in \Lambda^+} E^\mu(\lambda), \quad A(\xi) E^\mu(\lambda) \subset E^\mu(\xi + \lambda), \quad E^\mu(\lambda) A(\xi) \subset E^\mu(\xi + \lambda).$$

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Let $M \in \text{Mod}_\Lambda(A)$. We have a natural left $A$-module structure on $E^\mu \otimes_A M \simeq V(\mu) \otimes M$ induced from the left action of $A$ on $E^\mu$. Moreover, we have $E^\mu \otimes_A M \in \text{Mod}_\Lambda(A)$ by $(E^\mu \otimes_A M)(\lambda) = V(\mu) \otimes M(\lambda)$, and we obtain an exact functor

\begin{equation}
E^\mu \otimes_A (\bullet) : \text{Mod}_\Lambda(A) \to \text{Mod}_\Lambda(A)
\end{equation}

sending $M$ to $E^\mu \otimes_A M$.

**Lemma 3.12.** The functor (3.32) induces $E^\mu \otimes_A (\bullet) : \mathcal{M}(A) \to \mathcal{M}(A)$.

**Proof.** It is sufficient to show that for any $M \in \text{Tor}_\Lambda(A)$ we have $E^\mu \otimes A_M \in \text{Tor}_\Lambda(A)$. This follows from $A(\xi)(V(\mu) \otimes m) \subset V(\mu) \otimes A(\xi)m$ for any $m \in M$.

**Proposition 3.13.** For any $M \in \mathcal{M}(A)$ we have

\[ E^\mu \otimes_A \omega_\nu \overline{M} \simeq \omega_\nu (E^\mu \otimes_A \overline{M}) \]

**Proof.** Choose a filtration

\[ V(\mu) = V^n \supset V^{n-1} \supset \cdots \supset V^1 \supset V^0 = \{0\} \]

of $V(\mu)$ consisting of $U^{\leq 0}$-submodules $V^k$ satisfying $\dim V^k/V^{k-1} = 1$, and consider the corresponding filtration

\[ E^\mu = E^n \supset E^{n-1} \supset \cdots \supset E^1 \supset E^0 = \{0\} \]

of the right $A$-module $E^\mu = V(\mu) \otimes A$ given by $E^k = V^k \otimes A$. By the definition of the left $A$-module structure on $E^\mu$, especially by the fact that $\Xi$ belongs to a completion of $U \otimes U^{\leq 0}$, we see that $E^k$ is a left $A$-submodule for any $k$. Let $\nu_k \in \Lambda$ be the weight of $V^k/V^{k-1}$ and take $\nu_k \in V(\mu)_{\nu_k}$ such that $V^k = \mathbb{F}v_k \oplus V^{k-1}$. Let $\tau_k$ be the corresponding element of the $A$-bimodule $E^k/E^{k-1}$. For any $\varphi \in A(\lambda)_{\xi}$ we have $\varphi \tau_k = q^{-(\nu_k, \xi)}\tau_k \varphi$ by the explicit form of $\Xi$.

For $\nu \in \Lambda$ we define an automorphism $h_\nu$ of the graded $\mathbb{F}$-algebra $A$ by $h_\nu(\varphi) = q^{-(\nu, \xi)}\varphi$ for $\varphi \in A(\lambda)_{\xi}$. For $N \in \text{Mod}_\Lambda(A)$ we define $h_\nu^* N \in \text{Mod}_\Lambda(A)$ by

\begin{align*}
    h_\nu^* N & \simeq N \quad (h_\nu^*(n) \leftrightarrow n) \quad \text{as grade } \mathbb{F}-\text{modules} \\
    \varphi(h_\nu^*(n)) & = h_\nu^*(h_\nu(\varphi)n) \quad (\varphi \in A, \ n \in N).
\end{align*}

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Then we have \( E^k/E^{k-1} \otimes_A N \simeq h^*_v \cdot N \) as a graded \( A \)-module. Note that \( h^*_v \) induces exact functors
\[
h^*_v : \text{Mod}_A(A) \to \text{Mod}_A(A), \quad \tilde{h}^*_v : \mathcal{M}(A) \to \mathcal{M}(A)
\]
satisfying \( \omega_* \tilde{h}^*_v = h^*_v \omega_* \).

Let \( \overline{M} \in \mathcal{M}(A) \). We have morphisms
\[
\Phi^k : E^k \otimes_A \omega_* \overline{M} \to \omega_* (E^k \otimes_A \overline{M}), \\
\Psi^k : E^k/E^{k-1} \otimes_A \omega_* \overline{M} \to \omega_* (E^k/E^{k-1} \otimes_A \overline{M})
\]
in \( \text{Mod}_A(A) \) functorial with respect to \( \overline{M} \) such that
\[
\omega^* \Phi^k : \omega^* (E^k \otimes_A \omega_* \overline{M}) \to \omega^* \omega_* (E^k \otimes_A \overline{M}), \\
\omega^* \Psi^k : \omega^* (E^k/E^{k-1} \otimes_A \omega_* \overline{M}) \to \omega^* \omega_* (E^k/E^{k-1} \otimes_A \overline{M})
\]
are isomorphisms. Note that \( E^k \otimes_A (\bullet) \) and \( E^k/E^{k-1} \otimes_A (\bullet) \) on \( \mathcal{M}(A) \) are defined similarly to \( E^n \otimes_A (\bullet) \) on \( \mathcal{M}(A) \).

Note that \( \Psi^k \) is an isomorphism by
\[
E^k/E^{k-1} \otimes_A \omega_* \overline{M} \simeq h^*_v \cdot \omega_* \overline{M} \simeq \omega_* h^*_v \cdot \overline{M} \simeq \omega_* (E^k/E^{k-1} \otimes_A \overline{M}).
\]

Let us show that \( \Phi^k \) is an isomorphism. The surjectivity is proved by induction on \( k \) using the following commutative diagram whose rows are exact.

\[
\begin{array}{cccccc}
0 & \longrightarrow & E^{k-1} \otimes_A \omega_* \overline{M} & \longrightarrow & E^k \otimes_A \omega_* \overline{M} & \longrightarrow & E^k/E^{k-1} \otimes_A \omega_* \overline{M} & \longrightarrow & 0 \\
|| & & \phi^k \downarrow & & \phi^k \downarrow & & \phi^k \downarrow \\
0 & \longrightarrow & \omega_* (E^{k-1} \otimes_A \overline{M}) & \longrightarrow & \omega_* (E^k \otimes_A \overline{M}) & \longrightarrow & \omega_* (E^k/E^{k-1} \otimes_A \overline{M}) & \longrightarrow & 0 \\
\end{array}
\]

Since \( \omega^* \Phi^k \) is an isomorphism, \( \text{Ker}(\Phi^k) \) belongs to \( \text{Tor}_A(A) \). Hence, in order to prove that \( \Phi^k \) is injective, we have only to show \( \text{Tor}(E^k \otimes_A \omega_* \overline{M}) = \{0\} \).

By \( \text{Tor}(\omega_* \overline{M}) = \{0\} \) (see Corollary 2.17) it is sufficient to show \( \text{Tor}(E^k \otimes_A N) = \{0\} \) for any \( N \) with \( \text{Tor}(N) = \{0\} \). The image of \( \text{Tor}(E^k \otimes_A N) \) under \( E^k \otimes_A N \to E^k/E^{k-1} \otimes_A N \) is contained in \( \text{Tor}(E^k/E^{k-1} \otimes_A N) \simeq \text{Tor}(h^*_v \cdot N) = \{0\} \), and hence \( \text{Tor}(E^k \otimes_A N) \subset \text{Tor}(E^{k-1} \otimes_A N) = \{0\} \) by induction on \( k \).

We have obtained the desired result by considering the case \( k = n \).

\[\square\]

**Remark 3.14.** By Proposition 3.13 we obtain
\[
\Gamma(V(\mu) \otimes \overline{M}) = \Gamma(E^n \otimes_A \overline{M}) = E^n \otimes_A \Gamma(\overline{M}) = V(\mu) \otimes \Gamma(\overline{M})
\]
for any object \( M \) of \( \mathcal{M}(A) \). As noted in Lunts-Rosenberg [12, IV, 6.6 Remark] this implies Theorem 0.5.

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3.4 Filtration of $E^\mu$

Define a left $U$-module structure on $E^\mu = V(\mu) \otimes A$ by
\[
u(v \otimes \varphi) = \sum_{(u)_1} u(0)v \otimes u(1)\varphi \quad (u \in U, v \in V(\mu), \varphi \in A).
\]

Since $\eta$ is an isomorphism of $U$-modules, we have
\[
u(\varphi e \psi) = \sum_{(u)_2} (u(0)\varphi)(u(1)e)(u(2)\psi) \quad (u \in U, \varphi, \psi \in A, e \in E^\mu).
\]

For $M \in \text{Mod}_A(\tilde{U})$ a left $U$-module structure on $E^\mu \otimes_A M$ is defined by
\[
u(e \otimes m) = \sum_{(u)_1} u(0)e \otimes u(1)m \quad (u \in U, e \in E^\mu, m \in M),
\]
and it gives a $\Lambda$-graded left $\tilde{U}$-module structure on $E^\mu \otimes_A M$. Moreover $E^\mu \otimes_A (\bullet)$ induces an exact functor
\[
u^\mu \otimes_A (\bullet) : \mathcal{M} (\tilde{U}) \to \mathcal{M} (\tilde{U}).
\]

We fix $\lambda_0 \in \Lambda^+$ such that
\[
u \lambda_0 + \nu \in \Lambda^+ \quad \text{for any weight } \nu \text{ of } V(\mu),
\]
and set
\[
u^r = V(\mu) \otimes \bigoplus_{\lambda \in \lambda_0 + \Lambda^+} A(\lambda) \subset V(\mu) \otimes A = E^\mu.
\]

The following is obvious from the definition.

**Lemma 3.15.** $\nu^r$ is a graded $(A, A)$-submodule and a $U$-submodule of $E^\mu$. Moreover, $E^\mu / \nu^r$ belongs to $\text{Tor}_A(\Lambda)$ as a graded left $A$-module.

We fix a labeling $\{\nu_1, \ldots, \nu_r\}$ of the set of distinct weights of $V(\mu)$ such that $\nu_i - \nu_j \in \Lambda^+$ implies $i \geq j$. In particular, $\nu_1$ is the lowest weight $w_0\mu$ and $\nu_r$ is the highest weight $\mu$. Set $m_j = \dim V(\mu)_{\nu_j}$. For $\lambda \in \lambda_0 + \Lambda^+$ we have
\[
u^\mu (\lambda) \simeq \bigoplus_{j=1}^r V(\lambda + \nu_j)^{\otimes m_j}
\]
as $U$-modules since
\[	ext{ch}(\nu^\mu (\lambda)) = \text{ch}(V(\mu) \otimes A(\lambda)) = \sum_{j=1}^r m_j \text{ch}(V(\lambda + \nu_j))
\]
by Weyl's character formula. Define a filtration

\[ (3.38) \quad \{0\} = E^\mu_0 \subset E^\mu_1 \subset \cdots \subset E^\mu_r = E^\mu \]

of \( E^\mu \) consisting of graded \( U \)-submodules by

\[ E^\mu_k(\lambda) \simeq \bigoplus_{j=1}^{k} V(\lambda + \nu_j)^{\oplus m_j} \quad (\lambda \in \lambda_0 + \Lambda^+) \]

**Lemma 3.16.** \( E^\mu_k \) is an \((A, A)\)-submodule of \( E^\mu \).

**Proof.** Let \( \lambda \in \lambda_0 + \Lambda^+ \) and let \( T \) be a \( U \)-submodule of \( \overline{E}^\mu(\lambda) = V(\mu) \otimes A(\lambda) \) isomorphic to \( V(\lambda + \nu_l) \). Let \( \xi \in \Lambda^+ \). Then \( TA(\xi) \) is the image of \( T \otimes A(\xi) \) under the homomorphism \( \overline{E}^\mu(\lambda) \otimes A(\xi) \ni e \otimes \varphi \mapsto e \varphi \in \overline{E}^\mu(\lambda + \xi) \) of \( U \)-modules. Hence \( TA(\xi) \) is a \( U \)-submodule of \( \overline{E}^\mu(\lambda + \xi) \) whose weights are contained in \( \lambda + \xi + \nu_l - \Lambda^+ \). It follows that \( TA(\xi) \subset \bigoplus_{j=1}^{k} V(\lambda + \xi + \nu_j)^{\oplus m_j} \), and hence \( E^\mu_k \) is a right \( A \)-submodule. The assertion about the left module structure is proved similarly. \( \square \)

For \( k = 1, \ldots, r \) we set

\[ A(k) = \bigoplus_{\lambda \in \lambda_0 + \nu_k + \Lambda^+} A(\lambda). \]

It is a graded \((A, A)\)-submodule and a \( U \)-submodule of \( A \).

**Lemma 3.17.**

(i) There exists an isomorphism

\[ \Phi : \overline{E}^\mu_k/E^\mu_{k-1} \rightarrow A(k)[\nu_k]^{\oplus m_k} \]

of graded right \( A \)-modules and \( U \)-modules.

(ii) Identify \( A(k)[\nu_k]^{\oplus m_k} \) with \( A(k)[\nu_k] \otimes \mathbb{F}^{m_k}. \) Then there exists a group homomorphism \( \tau : \Lambda \rightarrow GL_{m_k}(\mathbb{F}) \) such that \( \Phi(\varphi v) = (id \otimes \tau(\xi)) \varphi \Phi(v) \)

for any \( \varphi \in A(\xi), v \in \overline{E}^\mu_k/E^\mu_{k-1}. \)

**Proof.** For simplicity we set \( M = \overline{E}^\mu_k/E^\mu_{k-1} \) and \( N = A(k)[\nu_k]^{\oplus m_k}. \)

(i) As \( U \)-modules we have

\[ M(\lambda) \simeq N(\lambda) \simeq \begin{cases} V(\lambda + \nu_k)^{\oplus m_k} & (\lambda \in \lambda_0 + \Lambda^+) \\ 0 & (\text{otherwise}) \end{cases}. \]
For \( \lambda \in \lambda_0 + \Lambda^+, \xi \in \Lambda^+, c \in A(\xi) \setminus \{0\} \) we have linear maps
\[
\begin{align*}
(3.39) & \quad M(\lambda)_{\lambda + \nu_k} \ni v \mapsto vc \in M(\lambda + \xi)_{\lambda + \xi + \nu_k}, \\
(3.40) & \quad N(\lambda)_{\lambda + \nu_k} \ni v \mapsto vc \in N(\lambda + \xi)_{\lambda + \xi + \nu_k}.
\end{align*}
\]

Let us show that they are isomorphisms. Considering the dimensions it is sufficient to show that they are injective. Since \( A \) is a domain, (3.40) is injective. Set \( \tilde{M} = \tilde{E}^\mu_k \). Then the projection \( \tilde{E}^\mu_k \to \tilde{E}^\mu_k/\tilde{E}^\mu_k-1 \) induces \( M(\lambda)_{\lambda + \nu_k} \cong M(\lambda)_{\lambda + \nu_k} \). Hence the injectivity of (3.39) follows from the injectivity of \( M(\lambda)_{\lambda + \nu_k} \to M(\lambda + \xi)_{\lambda + \xi + \nu_k} \), which is a consequence of \( \tilde{E}^\mu_k \subseteq V(\mu) \otimes A \).

Hence there exists a family \( \beta_\lambda : M(\lambda)_{\lambda + \nu_k} \to N(\lambda)_{\lambda + \nu_k} \) of linear isomorphisms satisfying \( \beta_\lambda(v)c = \beta_\lambda(v)c \) for any \( \lambda \in \lambda_0 + \Lambda^+, \xi \in \Lambda^+, v \in M(\lambda)_{\lambda + \nu_k}, c \in A(\xi) \). From this we obtain an isomorphism \( \Phi : M \to N \) of graded \( U \)-modules given by
\[
\Phi(uv) = u\beta_\lambda(v) \quad (u \in U, \lambda \in \lambda_0 + \Lambda^+, v \in M(\lambda)_{\lambda + \nu_k}).
\]

It remains to show the commutativity of the diagram:
\[
\begin{array}{ccc}
M(\lambda) \otimes A(\xi) & \longrightarrow & M(\lambda + \xi) \\
\downarrow \Phi \otimes \text{id} & & \downarrow \Phi \\
N(\lambda) \otimes A(\xi) & \longrightarrow & N(\lambda + \xi)
\end{array}
\]
for \( \lambda \in \lambda_0 + \Lambda^+, \xi \in \Lambda^+ \). Here the horizontal arrows are given by the right \( A \)-module structures. In particular, they are homomorphisms of \( U \)-modules. Therefore the assertion follows from the commutativity of
\[
\begin{array}{ccc}
M(\lambda)_{\lambda + \nu_k} \otimes A(\xi) & \longrightarrow & M(\lambda + \xi)_{\lambda + \xi + \nu_k} \\
\downarrow \Phi \otimes \text{id} & & \downarrow \Phi \\
N(\lambda)_{\lambda + \nu_k} \otimes A(\xi) & \longrightarrow & N(\lambda + \xi)_{\lambda + \xi + \nu_k}.
\end{array}
\]

(ii) Similarly to the proof of (i) it is sufficient to show that there exists a group homomorphism \( \tau : \Lambda \to GL_mk(\mathbb{F}) \) such that \( \Phi(cv) = (\text{id} \otimes \tau(\xi))c\Phi(v) \) for any \( \xi \in \Lambda^+, \lambda \in \lambda_0 + \Lambda^+, c \in A(\xi), v \in M(\lambda)_{\lambda + \nu_k} \).

Let \( \xi \in \Lambda^+ \). Considering the linear isomorphisms (3.39), (3.40) for \( \lambda = \lambda_0 \) we obtain \( \tau(\xi) \in GL_mk(\mathbb{F}) \) such that \( \Phi(cm) = (\text{id} \otimes \tau(\xi))c\Phi(m) \) for any \( c \in A(\xi), m \in M(\lambda_0)_{\lambda_0 + \nu_k} \). Let \( c \in A(\xi), c' \in A(\eta), m \in M(\lambda_0)_{\lambda_0 + \nu_k} \). Then we have
\[
\begin{align*}
\Phi(cm)c' &= \Phi((cm)c') = \Phi(cm)c' = ((\text{id} \otimes \tau(\xi))c\Phi(m))c' \\
&= (\text{id} \otimes \tau(\xi))c\Phi(m)c' = (\text{id} \otimes \tau(\xi))c\Phi(mc'),
\end{align*}
\]
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and hence we obtain \( \Phi(cm) = (\text{id} \otimes \tau(\xi))c\Phi(m) \) for any \( \xi \in \Lambda^+ \), \( \lambda \in \lambda_0 + \Lambda^+ \), \( c \in A(\xi) \), \( m \in M(\lambda)_{\lambda + \nu_k} \). We have \( \tau(0) = \text{id} \) and \( \tau(\xi)\tau(\xi') = \tau(\xi + \xi') \) for any \( \xi, \xi' \in \Lambda^+ \), and hence \( \tau \) is extended to a group homomorphism \( \tau : \Lambda \to GL_{m_k}(\mathbb{F}) \).

**Lemma 3.18.** Let \( f : E_1 \to E_2 \) be a morphism of \( \Lambda \)-graded \( A \)-bimodules such that \( \text{Ker}(f), \text{Coker}(f) \in \text{Tor}_A(A) \) as graded left \( A \)-modules. Let \( M \in \text{Mod}_A(A) \), and let \( \tilde{f} : E_1 \otimes_A M \to E_2 \otimes_A M \) be the corresponding morphism in \( \text{Mod}_A(A) \). Then \( \text{Ker}(\tilde{f}), \text{Coker}(\tilde{f}) \in \text{Tor}_A(A) \).

**Proof.** By the assumption we have \( S_{w}^{-1}E_1 \cong S_{w}^{-1}E_2 \) for any \( w \in W \). Thus

\[
S_{w}^{-1}(E_1 \otimes_A M) \cong S_{w}^{-1}E_1 \otimes_A M \cong S_{w}^{-1}E_2 \otimes_A M \cong S_{w}^{-1}(E_2 \otimes_A M).
\]

Hence we have \( \omega^*(E_1 \otimes_A M) \cong \omega^*(E_2 \otimes_A M) \) by Proposition 2.10. This is equivalent to \( \text{Ker}(\tilde{f}), \text{Coker}(\tilde{f}) \in \text{Tor}_A(A) \).

Hence we have the following.

**Lemma 3.19.** \( E^* \otimes_A (\cdot) \) and \( \overline{E}^* \otimes_A (\cdot) \) are isomorphic as functors from \( \mathcal{M}(A) \) to \( \mathcal{M}(A) \) and from \( \mathcal{M}(\overline{U}) \) to \( \mathcal{M}(\overline{U}) \).

Let \( M \in \text{Mod}_A(A) \) (resp. \( \text{Mod}_A(\overline{U}) \)). By Lemma 3.19 we have \( \omega^*(\overline{E}^* \otimes_A M) = \omega^*(E^* \otimes_A M) \). Hence the filtration

\[
\{0\} = \overline{E}_0^* \subset \overline{E}_1^* \subset \ldots \subset \overline{E}_r^* = \overline{E}^* \subset E^*
\]

of \( E^* \) induces the sequence

\[
\{0\} = \omega^*(\overline{E}_0 \otimes_A M) \to \omega^*(\overline{E}_1 \otimes_A M) \to \cdots \to \omega^*(\overline{E}_r \otimes_A M) = \omega^*(E^* \otimes_A M)
\]

of morphisms in \( \mathcal{M}(A) \) (resp. \( \mathcal{M}(\overline{U}) \)).

**Lemma 3.20.** Let \( M \in \text{Mod}_A(A) \) (resp. \( \text{Mod}_A(\overline{U}) \)).

(i) We have the exact sequence:

\[
0 \to \omega^*((\overline{E}_k^*/\overline{E}_{k-1}^*) \otimes_A M) \to \omega^*(\overline{E}_k^* \otimes_A M) \to \omega^*(\overline{E}_k^*/\overline{E}_{k-1}^*) \otimes_A M) \to 0.
\]

in \( \mathcal{M}(A) \) (resp. \( \mathcal{M}(\overline{U}) \)).

(ii) We have an isomorphism

\[
\omega^*((\overline{E}_k^*/\overline{E}_{k-1}^*) \otimes_A M) \cong \omega^*M[\nu_k]^{\oplus m_k}.
\]

in \( \mathcal{M}(A) \) (resp. \( \mathcal{M}(\overline{U}) \)).
Proof. The statements for $\tilde{U}$ easily follow from those for $A$, and hence we shall only consider the case $M \in \text{Mod}_A(A)$.

(i) Since $(\bullet) \otimes_A M$ is right exact, we have an exact sequence

$$0 \to K \to \mathcal{E}_{k-1}^i \otimes_A M \to \mathcal{E}_k^i \otimes_A M \to (\mathcal{E}_k^i/\mathcal{E}_{k-1}^i) \otimes_A M \to 0$$

for some $K \in \text{Mod}_A(A)$. Since $\omega^*$ is exact, it is sufficient to show $\omega^*K = 0$. This is equivalent to $S_{w}^{-1}K = 0$ for any $w \in W$, which is also equivalent to the injectivity of $S_{w}^{-1}(\mathcal{E}_{k-1}^i \otimes_A M) \to S_{w}^{-1}(\mathcal{E}_k^i \otimes_A M)$ for any $w \in W$. Thus we have only to show $S_{w}^{-1}\text{Tor}_1^A(\mathcal{E}_k^i/\mathcal{E}_{k-1}^i, M) = 0$ for any $w \in W$. By Lemma 3.17 there exists an exact sequence

$$0 \to \mathcal{E}_k^i/\mathcal{E}_{k-1}^i \to F \to C \to 0$$

of graded $A$-bimodules, where $F$ is isomorphic to $A[\nu_k]^{\oplus m_k}$ as a graded right $A$-module and $C$ belongs to $\text{Tor}_A(A)$ as a graded left $A$-module. By $C \in \text{Tor}_A(A)$ we have $S_{w}^{-1}C = 0$ and hence

$$S_{w}^{-1}\text{Tor}_1^A(\mathcal{E}_k^i/\mathcal{E}_{k-1}^i, M) = \text{Tor}_A^n(S_{w}^{-1}(\mathcal{E}_k^i/\mathcal{E}_{k-1}^i), M) = \text{Tor}_A^n(S_{w}^{-1}F, M)$$

$$= S_{w}^{-1}\text{Tor}_1^A(F, M) = 0$$

for any $n \neq 0$.

(ii) By the proof of (i) we have $S_{w}^{-1}((\mathcal{E}_k^i/\mathcal{E}_{k-1}^i) \otimes_A M) \simeq S_{w}^{-1}(F \otimes_A M)$ for any $w \in W$, and hence $\omega^*((\mathcal{E}_k^i/\mathcal{E}_{k-1}^i) \otimes_A M) \simeq \omega^*(F \otimes_A M)$. Hence we have only to show that $F \otimes_A M$ is isomorphic to $A[\nu_k]^{\oplus m_k} \otimes_A M$ as a grade left $A$-module. Note that under the identification $F = A[\nu_k] \otimes \mathbb{F}^{m_k}$ of right $A$-modules the left $A$-modules structure on $A[\nu_k] \otimes \mathbb{F}^{m_k}$ induced by the one on $F$ is given by $\varphi(\psi \otimes v) = \varphi \psi \otimes \tau(\lambda)v$ for $\varphi \in A(\lambda)$, $\psi \in A[\nu_k]$, $v \in \mathbb{F}^{m_k}$, where $\tau : \Lambda \to GL_{m_k}(\mathbb{F})$ is a group homomorphism. Thus we have an isomorphism

$$\delta : F \otimes_A M = ((A[\nu_k] \otimes \mathbb{F}^{m_k}) \otimes_A M) \to A[\nu_k]^{\oplus m_k} \otimes_A M = \mathbb{F}^{m_k} \otimes M[\nu_k]$$

of grade left $A$-modules given by

$$\delta(1 \otimes v \otimes m) = \tau(\lambda)^{-1}v \otimes m \quad (v \in \mathbb{F}^{m_k}, \ m \in M[\nu_k](\lambda)).$$

\[\square\]

Considering the cases $k = 1$ and $k = n$ we have obtained, for $M \in \text{Mod}_A(A)$ (resp. $\text{Mod}_A(\tilde{U})$), the canonical monomorphism

$$i_{\mu} : \omega^*M \to \omega^*(E^u \otimes_A M[-w_0\mu])$$

and the canonical epimorphism

$$p_{\mu} : \omega^*(E^u \otimes_A M) \to \omega^*M[\mu]$$

in $\mathcal{M}(A)$ (resp. $\mathcal{M}(\tilde{U})$), which are functorial with respect to $M$. 

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3.5 The affine open subset \( U_{1,q} \)

In this subsection we shall investigate \( S^{-1}_1A \). We denote by \( \iota : A \hookrightarrow S^{-1}_1A \) the canonical algebra homomorphism.

For each \( \lambda \in \Lambda^+ \) take \( v_\lambda \in V(\lambda) \) such that \( \langle v_\lambda^*, v_\lambda \rangle = 1 \). Then we have \( \langle e^1_\lambda, u \rangle = \langle v_\lambda^*, uv_\lambda \rangle \) for any \( u \in U \). For simplicity we write \( c_\lambda \) instead of \( c^1_\lambda \).

Define \( r : A \rightarrow F^{\geq 0} \) as the composition of

\[
A \hookrightarrow F \xrightarrow{r} F^{\geq 0}.
\]

By \( r(c_\lambda) = \chi^+_\lambda \) there exists a unique algebra homomorphism

\[
\theta : S^{-1}_1A \rightarrow F^{\geq 0}
\]

such that \( \theta \circ \iota = r \) (see (3.46)).

**Proposition 3.21.**  
(i) \( \theta \) is an isomorphism of \( \mathbb{F} \)-algebras.

(ii) \( \theta \) is an isomorphism of \( U^{\geq 0} \)-modules, where the \( U^{\geq 0} \)-module structure on \( S^{-1}_1A \) is the restriction of the \( U \)-module structure given in Proposition 3.5.

(iii) \( \theta((S^{-1}_1A)(\lambda)) = F^{\geq 0}(\lambda) \) for any \( \lambda \in \Lambda \).

**Proof.** By definition we have

\[
e_i(c^{-1}_\lambda \psi) = q^{-(\lambda \Lambda_i)}c^{-1}_\lambda (e_i \psi), \quad k_\mu(c^{-1}_\lambda \psi) = q^{-(\lambda \mu)}c^{-1}_\lambda (k_\mu \psi),
\]

\[
e_i(\chi^+_\lambda \varphi) = q^{-(\lambda \Lambda_i)}\chi^+_\lambda (e_i \varphi), \quad k_\mu(\chi^+_\lambda \varphi) = q^{-(\lambda \mu)}\chi^+_\lambda (k_\mu \varphi),
\]

for \( i \in I, \mu \in \Lambda, \lambda \in \Lambda^+, \psi \in A, \varphi \in F^{\geq 0} \). Moreover, \( r \) is a homomorphism of \( U^{\geq 0} \)-modules. Hence \( \theta \) is a homomorphism of \( U^{\geq 0} \)-modules.

By definition we have \( r(A(\lambda)) \subset F^{\geq 0}(\lambda) \) for any \( \lambda \in \Lambda^+ \). In particular, we have \( r(\chi^+_\lambda) \subset F^{\geq 0}(\lambda) \). Since \( \theta \) is a ring homomorphism, we obtain

\[
\theta((S^{-1}_1A)(\lambda)) \subset F^{\geq 0}(\lambda) \quad (\lambda \in \Lambda).
\]

By (3.59) and (3.44) it is sufficient to show that

\[
\theta_0 = \theta \mid (S^{-1}_1A)(0) : (S^{-1}_1A)(0) \rightarrow F^{\geq 0}(0)
\]

is an isomorphism. Assume that \( x \in \text{Ker}(\theta_0) \). There exists some \( \lambda \in \Lambda^+ \) and \( \varphi \in A(\lambda) \) such that \( x = \varphi c^{-1}_\lambda \). Then we have \( \varphi \in \text{Ker}(\theta) \cap A(\lambda) \). Take \( v \in V(\lambda) \) such that \( \varphi = f_\lambda(v) \). By \( f \in \text{Ker}(\theta) \) we have

\[
\langle V^*(\lambda), v \rangle = \langle v^*_\lambda U^{\geq 0}, v \rangle = \langle v^*_\lambda, U^{\geq 0}v \rangle = \{0\}.
\]
This implies \( v = 0 \). Hence \( x = 0 \).

It remains to show the surjectivity of \( \theta_0 \). By Proposition 3.22 (ii) it is sufficient to show that for any \( \gamma \in Q^+ \) there exists some \( \lambda \in \Lambda^+ \) such that the linear map

\[
V(\lambda)_{\lambda - \gamma} \to (U^+)_\gamma^* \quad (v \mapsto \theta(f_\lambda(v)e_\chi^{-1})|U^+_{\gamma})
\]

is surjective. By definition we have

\[
\langle \theta(f_\lambda(v)e_\chi^{-1}), u \rangle = \langle (r_+(f_\lambda(v)))\chi^+_{-\lambda}, u \rangle = \langle (r_+(f_\lambda(v))) \otimes \chi^+_{-\lambda}, \Delta(u) \rangle
\]

\[
= \langle (r_+(f_\lambda(v))), u \rangle = \langle v^*_\chi u, v \rangle.
\]

Hence we obtain the desired result by Lemma 1.2. \( \square \)

**Proposition 3.22.** For any \( \lambda \in \Lambda \) \((S_1^{-1}A)(\lambda)\) is isomorphic to \( T^*(\lambda) \) as a \( U \)-module.

**Proof.** By Proposition 1.12 and Proposition 3.21 we have \( \text{ch}((S_1^{-1}A)(\lambda)) = \text{ch}(T(\lambda)) = \text{ch}(T^*(\lambda)) \). Moreover, the restricted dual \((S_1^{-1}A)(\lambda))^*\) is a rank one free right \( U^+ \)-module. Hence \((S_1^{-1}A)(\lambda))^*\) is isomorphic to \( T_i(\lambda) \) as a right \( U \)-module. It follows that we have \((S_1^{-1}A)(\lambda) \simeq (T_i(\lambda))^* = T^*(\lambda)\). \( \square \)

### 3.6 Coherent sheaves with \( U \)-actions

Let \( \text{Mod}^\circ_\Lambda(\bar{U}) \) be the full subcategory of \( \text{Mod}_\Lambda(\bar{U}) \) consisting of objects \( M \) of \( \text{Mod}_\Lambda(\bar{U}) \) such that \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \); and let \( \text{Mod}^f_\Lambda(\bar{U}) \) be the full subcategory of \( \text{Mod}^\circ_\Lambda(\bar{U}) \) consisting of objects \( M \) of \( \text{Mod}^\circ_\Lambda(\bar{U}) \) such that \( M \) is finitely generated as an \( A \)-module.

Let \( M \in \text{Mod}^f_\Lambda(\bar{U}) \). Since \( M \) is a finitely generated \( A \)-module, we have \( \dim M(\xi) < \infty \) for any \( \xi \in \Lambda \). Hence we have \( M(\xi) \in \text{Mod}^f(\bar{U}) \) for any \( \xi \in \Lambda \).

Let

\[
\mathcal{M}^\circ(\bar{U}) = \frac{\text{Mod}^\circ_\Lambda(\bar{U})}{\text{Tor}_\Lambda(\bar{U}) \cap \text{Mod}^\circ_\Lambda(\bar{U})} = (\Sigma_\bar{U}^\circ)^{-1} \text{Mod}^\circ_\Lambda(\bar{U}),
\]

\[
\mathcal{M}^f(\bar{U}) = \frac{\text{Mod}^f_\Lambda(\bar{U})}{\text{Tor}_\Lambda(\bar{U}) \cap \text{Mod}^f_\Lambda(\bar{U})} = (\Sigma_\bar{U}^f)^{-1} \text{Mod}^f_\Lambda(\bar{U}),
\]

where \( \Sigma_\bar{U}^\circ \) and \( \Sigma_\bar{U}^f \) are the collection of morphisms in \( \text{Mod}^\circ_\Lambda(\bar{U}) \) and \( \text{Mod}^f_\Lambda(\bar{U}) \) whose kernel and cokernel belong to \( \text{Tor}_\Lambda(\bar{U}) \cap \text{Mod}^\circ_\Lambda(\bar{U}) \) and \( \text{Tor}_\Lambda(\bar{U}) \cap \text{Mod}^f_\Lambda(\bar{U}) \) respectively. Denote by

\[
\omega^\circ : \text{Mod}^f_\Lambda(\bar{U}) \to \mathcal{M}^\circ(\bar{U}), \quad \omega^f : \text{Mod}^f_\Lambda(\bar{U}) \to \mathcal{M}^f(\bar{U})
\]
the localization functors.

We shall regard \( \mathcal{M}^0(\tilde{U}) \) and \( \mathcal{M}^I(\tilde{U}) \) as full subcategories of \( \mathcal{M}(\tilde{U}) \) by Lemma 3.10. We sometimes write \( \omega^* \) instead of \( \omega^*_\alpha \) and \( \omega^*_I \).

**Proposition 3.23.** \( \omega^* A[\lambda] \) is an irreducible object of \( \mathcal{M}^I(\tilde{U}) \) for any \( \lambda \in \Lambda \), and any irreducible object of \( \mathcal{M}^I(\tilde{U}) \) is isomorphic to \( \omega^* A[\lambda] \) for some \( \lambda \in \Lambda \).

**Proof.** Let us show that \( \omega^* A[\lambda] \) is irreducible. We may assume that \( \lambda = 0 \). Assume that \( \overline{M} \) is a non-zero subobject of \( \omega^* A \). Set \( M = \omega_* \overline{M} \). Then \( M \) is a subobject of \( A = \omega^* A \) (see Proposition 2.14), and we have \( \overline{M} = \omega^* M \) by \( \omega^* \circ \omega_* = \text{Id} \). Since \( M \) is non-zero, there exists some \( \mu \in \Lambda^+ \) such that \( M(\mu) \neq \{0\} \). Since \( A(\mu) \) is an irreducible \( U \)-module, we have \( M(\mu) = A(\mu) \). It follows that \( M \supseteq \bigoplus_{\xi \in \mu + \Lambda^+} A(\xi) \) by Lemma 2.2, and hence \( A/M \in \text{Tor}_A(A) \). Thus we have \( \overline{M} = \omega^* M = \omega^* A \).

Assume that \( \overline{M} \) is an irreducible object of \( \mathcal{M}^I(\tilde{U}) \). Take \( M \in \text{Mod}^I(\tilde{U}) \) such that \( \omega^* M = \overline{M} \). We may assume \( \text{Tor}(M) = \{0\} \). By \( M \neq \{0\} \) there exists some \( \xi \) such that \( M(\xi) \neq \{0\} \). Note that \( M(\xi) \in \text{Mod}^I(\tilde{U}) \) as a \( U \)-module. Take an irreducible \( U \)-submodule \( V \) of \( M(\xi) \) and set \( N = \tilde{U} V = AV = AV \subseteq M \). By \( N/\text{Tor}(N) \simeq N \neq \{0\} \) we have \( \omega^* N \neq \{0\} \). Hence the irreducibility of \( \overline{M} = \omega^* M \) implies \( \omega^* M = \omega^* N \). Thus we may assume that \( M = \tilde{U} M(\xi) \) and that \( M(\xi) \) is isomorphic to \( V(\mu) \) as a \( U \)-module. Hence \( M \) is isomorphic to a quotient of \( E^\mu[-\xi] = A[-\xi] \otimes V(\mu) \). It follows that \( \overline{M} \) is isomorphic to a quotient of \( \omega^* E^\mu[-\xi] \). By Lemma 3.15 and Lemma 3.20 there exists an increasing sequence

\[
0 = N_0 \subset N_1 \subset \cdots \subset N_n = \omega^* E^\mu
\]

of subobjects of \( \omega^* E^\mu \in \mathcal{M}(\tilde{U}) \) such that for each \( k \) we have \( N_k/N_{k-1} \simeq \omega^* A[\lambda_k] \) for some \( \lambda_k \in \Lambda \). This implies that \( \overline{M} \) is isomorphic to \( \omega^* A[\lambda_k - \xi] \) for some \( k \).

**Lemma 3.24.** For \( M \in \text{Mod}_A^0(\tilde{U}) \) (resp. \( M \in \text{Mod}_A^I(\tilde{U}) \)) we have \( \omega_* \omega^* M \in \text{Mod}_A^0(\tilde{U}) \) (resp. \( M \in \text{Mod}_A^I(\tilde{U}) \)).

**Proof.** Assume that \( M \in \text{Mod}_A^0(\tilde{U}) \). By Corollary 2.12 \( \omega_* \omega^* M \) is a subobject of \( \bigoplus_{w \in W} S_w^{-1} M \). Since \( \bigoplus_{w \in W} S_w^{-1} M \in \text{Mod}_A^0(\tilde{U}) \), we have \( \omega_* \omega^* M \in \text{Mod}_A^0(\tilde{U}) \).

It remains to show that \( \omega_* \overline{M} \) is a finitely generated \( A \)-module for any \( \overline{M} \in \mathcal{M}^I(\tilde{U}) \). By Proposition 2.3 and Proposition 3.23 we may assume that \( \overline{M} = \omega_* A[\lambda] \) for some \( \lambda \in \Lambda \). In this case the assertion follows from Proposition 2.14.

We need the following result later.
LEMMA 3.25. Let \( w \in W \). Let \( M \in \text{Mod}_\Lambda(\tilde{U}) \) and set
\[
\Gamma(\omega^*M)^{\text{fin}} = \{ m \in \Gamma(\omega^*M) \mid \dim FUm < \infty \}.
\]
Then the canonical homomorphism \( \Gamma(\omega^*M)^{\text{fin}} \to (S_w^{-1}M)(0) \) is injective.

PROOF. By Lemma 3.24 we may assume that \( M \simeq \omega_w^*\omega^*M \). Take \( m \in \Gamma(\omega^*M)^{\text{fin}} = M(0)^{\text{fin}} \). Let \( N \) be the \( \tilde{U} \)-submodule of \( M \) generated by \( m \).

Then \( N \) belongs to \( \text{Mod}_\Lambda(\tilde{U}) \), and the canonical morphisms \( \Gamma(\omega^*N) \to \Gamma(\omega^*M) \) and \( (S_w^{-1}N)(0) \to (S_w^{-1}M)(0) \) are injective. Hence we may assume that \( M \in \text{Mod}_\Lambda(\tilde{U}) \). In this case we have \( \Gamma(\omega^*M)^{\text{fin}} = \Gamma(\omega^*M) \) by Lemma 3.24. Assume that there exists a short exact sequence
\[
0 \to \omega^*M_1 \to \omega^*M_2 \to \omega^*M_3 \to 0
\]
in \( \mathcal{M}(\tilde{U}) \). Then we have a commutative diagram:
\[
\begin{array}{cccccc}
0 & \longrightarrow & \Gamma(\omega^*M_1) & \longrightarrow & \Gamma(\omega^*M_2) & \longrightarrow & \Gamma(\omega^*M_3) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S_w^{-1}M_1 & \longrightarrow & S_w^{-1}M_2 & \longrightarrow & S_w^{-1}M_3 & \longrightarrow & 0
\end{array}
\]
whose rows are exact. Hence we may assume that \( M = A[\lambda] \) for some \( \lambda \in \Lambda \) by Proposition 3.23. In this case the assertion is a consequence of Proposition 2.13 and the injectivity of \( A \to S_w^{-1}A \).

4 D-modules

4.1 q-differential operators

For \( \varphi \in A, u \in U, \lambda \in \Lambda \) we define \( \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \in \text{End}_F(A) \) by
\[
\ell_\varphi(\psi) = \varphi\psi, \quad r_\varphi(\psi) = \psi\varphi, \quad \partial_u(\psi) = u\psi, \quad \sigma_\lambda(\psi) = q^{(\lambda,\mu)}\psi
\]
for \( \psi \in A(\mu) \). We define a subalgebra \( D \) of \( \text{End}_F(A) \) by
\[
D = \langle \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A, u \in U, \lambda \in \Lambda \rangle.
\]
For \( \lambda \in \Lambda \) we set
\[
D(\lambda) = \{ d \in D \mid d(A(\xi)) \subset A(\lambda + \xi) \ (\xi \in \Lambda) \}.
\]
Since \( \ell_\varphi, r_\varphi \in D(\lambda) \) for \( \varphi \in A(\lambda) \) and \( \partial_u, \sigma_\lambda \in D(0) \), we have
\[
D = \bigoplus_{\lambda \in \Lambda^+} D(\lambda), \quad D(0) = \langle \partial_u, \sigma_\lambda \mid u \in U, \lambda \in \Lambda \rangle.
\]
In particular, \( D \) is a \( \Lambda \)-graded \( F \)-algebra.
Lemma 4.1. Write

\[ \sum_{\beta \in Q^+} q^{(\beta, \beta)} (1 \otimes k_{\beta})(S \otimes \text{id})(\Xi_{\beta}) = \sum_p x_p \otimes y_p \]

(see Section 1.3 for the notation). Then we have

(4.3) \[ r_\psi = \sum_p \ell_{x_p \psi} \partial y_p \sigma_{-\mu} \] \[ (\psi \in A(\mu)_\eta), \]

(4.4) \[ \ell_\varphi = \sum_p r_{y_p \varphi} \partial x_p \sigma_{-\lambda} \] \[ (\varphi \in A(\lambda)_\xi). \]

Proof. Let \( \varphi \in A(\lambda)_\xi, \psi \in A(\mu)_\eta \). Take \( v_0 \in V(\lambda)_\xi, v_1 \in V(\mu)_\eta \) such that \( \varphi = f_\lambda(v_0), \psi = f_\mu(v_1) \). Set \( \mathcal{R}^\psi = \mathcal{R}^{\psi}_{V(\mu), V(\lambda)} \). By Proposition 1.4 (ii) we have

\[ \langle \varphi \psi, u \rangle = \langle \varphi \otimes \psi, \Delta(u) \rangle = \langle v^*_\lambda \otimes v^*_\mu, u(v_0 \otimes v_1) \rangle \]

By \( v^*_\lambda U^-_{-\beta} = \{0\} \) for \( \beta \in Q^+ \setminus \{0\} \) we have \( \ell^i(\mathcal{R}^\psi)(v^*_\lambda \otimes v^*_\mu) = q^{(\lambda, \mu)} v^*_\mu \otimes v^*_\lambda \). By

\[ \mathcal{R}^{-1}_{V(\mu), V(\lambda)} = \sum_p x_p \otimes y_p \kappa_{V(\mu), V(\lambda)} \]

we have \( (\mathcal{R}^\psi)^{-1}(v_0 \otimes v_1) = q^{(\xi, \eta)} \sum_p x_p v_1 \otimes y_p v_0 \). Hence

\[ \langle \varphi \psi, u \rangle = q^{(\xi, \eta) - (\lambda, \mu)} \sum_p \langle v^*_\mu \otimes v^*_\lambda, u(x_p v_1 \otimes y_p v_0) \rangle \]

By \( v^*_\lambda U^-_{-\beta} = \{0\} \) for \( \beta \in Q^+ \setminus \{0\} \) we have \( \ell^i(\mathcal{R}^\psi)(v^*_\lambda \otimes v^*_\mu) = q^{(\lambda, \mu)} v^*_\mu \otimes v^*_\lambda \). By

\[ \mathcal{R}^{-1}_{V(\mu), V(\lambda)} = \sum_p x_p \otimes y_p \kappa_{V(\mu), V(\lambda)} \]

we have \( (\mathcal{R}^\psi)^{-1}(v_0 \otimes v_1) = q^{(\xi, \eta)} \sum_p x_p v_1 \otimes y_p v_0 \). Hence

\[ \langle \varphi \psi, u \rangle = q^{(\xi, \eta) - (\lambda, \mu)} \sum_p \langle v^*_\mu \otimes v^*_\lambda, u(x_p v_1 \otimes y_p v_0) \rangle \]

It follows that we have

\[ \varphi \psi = q^{(\xi, \eta) - (\lambda, \mu)} \sum_p (\partial_{x_p} \psi)(\partial_{y_p} \varphi). \]

Corollary 4.2. We have

\[ D = \langle \ell_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A, u \in U, \lambda \in \Lambda \rangle = \langle r_\psi, \partial_u, \sigma_\lambda \mid \varphi \in A, u \in U, \lambda \in \Lambda \rangle. \]
We can easily check the following.

\begin{align*}
(4.5) \quad & \ell_{\varphi} \ell_{\psi} = \ell_{\varphi \psi} \quad (\varphi, \psi \in A), \\
(4.6) \quad & \sigma_{\lambda} \sigma_{\mu} = \sigma_{\lambda + \mu} \quad (\lambda, \mu \in \Lambda), \\
(4.7) \quad & \partial_u \partial_{u'} = \partial_{uu'} \quad (u, u' \in U), \\
(4.8) \quad & \sigma_{\lambda} \ell_{\varphi} = q^{(\lambda, \mu)} \ell_{\varphi} \sigma_{\lambda} \quad (\lambda, \mu \in \Lambda, \varphi \in A(\mu)), \\
(4.9) \quad & \sigma_{\lambda} \partial_u = \partial_u \sigma_{\lambda} \quad (\lambda \in \Lambda, u \in U), \\
(4.10) \quad & \partial_u \ell_{\varphi} = \sum_{(u)1} \ell_{u(0)\varphi} \partial_{u(1)} \quad (u \in U, \varphi \in A). \\
\end{align*}

In particular we have homomorphisms

\begin{align*}
\partial : U & \to D \quad (u \mapsto \partial_u), \\
\ell : A & \to D \quad (\varphi \mapsto \ell_{\varphi}), \\
\sigma : F[\Lambda] & \to D \quad (\Lambda \ni \lambda \mapsto \sigma_{\lambda})
\end{align*}

of \( F \)-algebras. Moreover, \( \partial \) and \( \ell \) induces a homomorphism

\begin{align*}
(4.11) \quad & \tilde{U} \to D \quad (\varphi u \mapsto \ell_{\varphi} \partial_u)
\end{align*}

of \( \Lambda \)-graded \( F \)-algebras. In particular, we have

\begin{align*}
(4.12) \quad & \ell_{\varphi} \partial_u = \sum_{(u)1} \partial_{u(1)} \ell_{S^{-1}u(0)\varphi} \quad (u \in U, \varphi \in A).
\end{align*}

by (3.4).

By Proposition \[\text{1.6} \ (\text{ii})\] we have

\begin{align*}
(4.13) \quad & \partial_z = \sigma \circ \zeta(z) \quad (z \in \mathfrak{z})
\end{align*}

where \( \zeta : \mathfrak{z} \to F[\Lambda] \) is as in Section \[\text{1.4} \]

We shall identify \( A \) with a subalgebra of \( D \) via the injective \( F \)-algebra homomorphism \( \ell : A \to D \) (the injectivity follows from Proposition \[\text{2.3} \ (\text{i})\]).

**Proposition 4.3.** Let \( w \in W \).

(i) \( S_w \) satisfies the left and right Ore conditions in \( D \).

(ii) The canonical homomorphism \( D \to S_w^{-1}D \) is injective.

**Proof.** It is sufficient to show the following.

(a) For any \( d \in D \) and \( s \in S_w \) there exists some \( t \in S_w \) and \( d' \in D \) satisfying \( td = d's \).
(b) For any \( d \in D \) and \( s \in S_w \) there exists some \( t \in S_w \) and \( d' \in D \) satisfying \( dt = sd' \).

(c) If \( sd = 0 \) for \( d \in D \) and \( s \in S_w \), then we have \( d = 0 \).

(d) If \( ds = 0 \) for \( d \in D \) and \( s \in S_w \), then we have \( d = 0 \).

The statements (a), (b) is proved similarly to Proposition 3.2. The statement (c) follows from Proposition 2.3 (i). The statement (d) is equivalent to the injectivity of \( D \rightarrow D \otimes_A S_w^{-1}A \). By Lunts-Rosenberg [11, Section 1.2] \( \tilde{D} \rightarrow \tilde{D} \otimes_A S_w^{-1}A \) is injective for a ring \( \tilde{D} \) containing \( D \), and the assertion for \( D \) follows from this.

Hence \( S_w^{-1}D \) is a \( \Lambda \)-graded \( \mathbb{F} \)-algebra, and we have

\[
S_w^{-1}D \cong S_w^{-1}A \otimes_A D \cong D \otimes_A S_w^{-1}A \cong S_w^{-1}A \otimes_A D \otimes_A S_w^{-1}A.
\]

Moreover, for \( M \in \text{Mod}_\Lambda(D) \) we have

\[
S_w^{-1}M = S_w^{-1}D \otimes_D M \in \text{Mod}_\Lambda(S_w^{-1}D).
\]

In particular \( S_w^{-1}A \) is a graded \( S_w^{-1}D \)-module.

Since \( A \) is a sum of \( U \)-submodules contained in \( \text{Mod}^f(U) \), we have the operator \( T_w : A \rightarrow A \). For \( d \in \text{End}_F(A) \) we set

\[
Z_w(d) = T_w^{-1} \circ d \circ T_w \in \text{End}_F(A).
\]

**Proposition 4.4.** \( Z_w \) induces an algebra automorphisms of \( D \) and an algebra isomorphism \( S_1^{-1}D \rightarrow S_w^{-1}D \).

**Proof.** We see easily that \( Z(s \varphi) = \partial_{T_1^{-1}(u)} \), \( Z(s \sigma) = \sigma \) for any \( u \in U \) and \( \sigma \in \Lambda \). For \( \varphi, \psi \in A \) we have

\[
Z(s \varphi)(\psi) = T_1^{-1}(\varphi T_1(\psi)) = m(T_1^{-1}(\varphi \otimes T_1(\psi)))
\]
since the multiplication \( m : A \otimes A \rightarrow A \) is a homomorphism of \( U \)-modules.

By Lemma 1.7 we have

\[
T_1^{-1} = \exp_{q_1^{-1}}(- (q_i - q_i^{-1}) f_i \otimes e_i)(T_1^{-1} \otimes T_1^{-1})
\]
as an operator on \( A \otimes A \). Write \( \exp_{q_1^{-1}}(- (q_i - q_i^{-1}) f_i \otimes e_i) = \sum_p b_p \otimes a_p \).

Then we have

\[
Z(s \varphi)(\psi) = \sum_p \partial_{b_p}(T_1^{-1} \varphi) \partial_{a_p}(\psi),
\]

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and hence

$$Z_{s_i}(\ell_\varphi) = \sum_p \ell_{\partial p}(T^{-1}_i \varphi) \partial_a p \in D$$

for any $\varphi \in A$. Similarly, we can show $Z_{s_i}^{-1}(\ell_\varphi) \in D$ for any $\varphi \in A$. Hence $Z_w$ induces an algebra automorphism of $D$.

It remains to show $Z_w(\ell_{S_i}) = \ell_{S_w}$. It is sufficient to show that for $w \in W$, $i \in I$ such that $w(\alpha_i) \in \Delta^+$ we have $Z_{s_i}(\ell_{S_w}) = \ell_{S_{w_1}}$. Let $\varphi \in S_w$. Take $\lambda \in \Lambda^+$ such that $\varphi \in A(\lambda)_{w^{-1}\lambda} \setminus \{0\}$. Then we have $T^{-1}_i \varphi \in A(\lambda)_{s_i w^{-1}\lambda} \setminus \{0\}$.

By $(s_i w^{-1}\lambda, \alpha_i^\vee) = -(\lambda, w \alpha_i^\vee) \leq 0$ we obtain

$$Z_{s_i}(\ell_\varphi) = \sum_p \ell_{\partial p}(T^{-1}_i \varphi) \partial_a p = \ell_{T^{-1}_i \varphi} \in \ell_{S_{w_1}}.$$

4.2 Category of $D$-modules

Applying results in Section 3.2 to $C = D$ we have the localization functor

$$\omega^* = \omega^* : \text{Mod}_\Lambda(D) \to \mathcal{M}(D) := \frac{\text{Mod}_\Lambda(D)}{\text{Tor}_\Lambda(D)} = \Sigma^{-1}_D \text{Mod}_\Lambda(D)$$

and its right adjoint

$$\omega_* = \omega_* : \mathcal{M}(D) \to \text{Mod}_\Lambda(D).$$

Taking the degree zero part of $\omega_*$ we have the global section functor

$$(4.14) \quad \Gamma : \mathcal{M}(D) \to \text{Mod}(D(0)).$$

Define a functor

$$(4.15) \quad \mathcal{L} : \text{Mod}(D(0)) \to \mathcal{M}(D)$$

by $\mathcal{L}(N) = \omega^*(D \otimes_{D(0)} N)$, and we call it the localization functor. Since $D \otimes_{D(0)} (\bullet) : \text{Mod}(D(0)) \to \text{Mod}_\Lambda(D)$ is left adjoint to $\text{Mod}_\Lambda(D) \ni M \mapsto M(0) \in \text{Mod}(D(0))$, we have the following.

**Lemma 4.5.** The functor $\mathcal{L} : \text{Mod}(D(0)) \to \mathcal{M}(D)$ is left adjoint to $\Gamma : \mathcal{M}(D) \to \text{Mod}(D(0))$. 53
For $\lambda \in \Lambda$ we denote by $\text{Mod}_{\Lambda, \lambda}(D)$ the full subcategory of $\text{Mod}_\Lambda(D)$ consisting of $M \in \text{Mod}_\Lambda(D)$ satisfying $\sigma_\mu|M(\xi) = q^{(\mu, \lambda + \xi)} \text{id}$ for any $\mu, \xi \in \Lambda$. Let

$$\mathcal{M}_\lambda(D) = \frac{\text{Mod}_{\Lambda, \lambda}(D)}{\text{Tor}_\lambda(D) \cap \text{Mod}_{\Lambda, \lambda}(D)} = \Sigma_{D, \lambda}^{-1} \text{Mod}_{\Lambda, \lambda}(D),$$

where $\Sigma_{D, \lambda}$ is the collection of morphisms in $\text{Mod}_{\Lambda, \lambda}(D)$ whose kernel and cokernel belong to $\text{Tor}_\lambda(D) \cap \text{Mod}_{\Lambda, \lambda}(D)$, and denote by

$$\omega^*_\lambda : \text{Mod}_{\Lambda, \lambda}(D) \to \mathcal{M}_\lambda(D)$$

the localization functor.

We shall regard $\mathcal{M}_\lambda(D)$ as a full subcategory of $\mathcal{M}(D)$ by Lemma 3.10 and we often write $\omega^*$ instead of $\omega^*_\lambda$.

**Lemma 4.6.** For $M \in \text{Mod}_{\Lambda, \lambda}(D)$ we have $(R^r \omega_*)(\omega^*M) \in \text{Mod}_{\Lambda, \lambda}(D)$ for any $r$.

**Proof.** For $\mu \in \Lambda$ and $N \in \text{Mod}_\Lambda(D)$ we define $h^\mu_N : N \to N$ by $h^\mu_N|N(\xi) = q^{-(\mu, \xi)} \sigma_\mu|N(\xi)$ for any $\xi \in \Lambda$. We see easily that $h^\mu_N$ is a morphism in $\text{Mod}_\Lambda(D)$. Moreover, $h^\mu_N$ is functorial with respect to $N$ in the sense that for a morphism $f : N_1 \to N_2$ in $\text{Mod}_\Lambda(D)$ we have $h^\mu_{N_2} \circ f = f \circ h^\mu_{N_1}$.

Let us show $(R^r \omega_*)(\omega^*h^\mu_N) = h^\mu_{(R^r \omega_*)(\omega^*N)}$ for any $r$. By the functoriality of $h^\mu_N$ stated above, it is sufficient to show $\omega_*\omega^*h^\mu_N = h^\mu_{\omega_*\omega^*N}$. Let $j : N \to \omega_*\omega^*N$ be the canonical morphism. By the definition of adjoint functors we have $\omega_*\omega^*h^\mu_N \circ j = j \circ h^\mu_N$, and by the functoriality of $h^\mu_N$ we have $h^\mu_{\omega_*\omega^*N} \circ j = j \circ h^\mu_N$. Hence we obtain $\omega_*\omega^*h^\mu_N = h^\mu_{\omega_*\omega^*N}$ by

$$\text{Hom}(\omega_*\omega^*N, \omega_*\omega^*N) \simeq \text{Hom}(\omega^*\omega_*\omega^*N, \omega^*N) \simeq \text{Hom}(\omega^*N, \omega^*N) \simeq \text{Hom}(N, \omega_*\omega^*N).$$

Here we have used $\omega^* \circ \omega_* \simeq \text{Id}$ and the fact that $\omega^*$ is left adjoint to $\omega_*$. By $M \in \text{Mod}_{\Lambda, \lambda}(D)$ we have $h^\mu_M = q^{(\mu, \lambda)} \text{id}$. Hence $h^\mu_{(R^r \omega_*)(\omega^*N)} = (R^r \omega_*)(\omega^*h^\mu_M) = q^{(\mu, \lambda)}(R^r \omega_*)(\omega^* \text{id}) = q^{(\mu, \lambda)} \text{id}$, from which we obtain the desired result.

By Lemma 4.6 the restriction of $\omega_* : \mathcal{M}(D) \to \text{Mod}_\Lambda(D)$ to $\mathcal{M}_\lambda(D)$ gives a left exact functor

$$\omega_{\lambda*} : \mathcal{M}_\lambda(D) \to \text{Mod}_{\Lambda, \lambda}(D).$$

We have

$$\text{Hom}_{\mathcal{M}_\lambda(D)}(\omega_{\lambda*}M, \omega_{\lambda*}N) \simeq \text{Hom}_{\mathcal{M}(D)}(\omega^*M, \omega^*N) \simeq \text{Hom}_{\text{Mod}_{\Lambda, \lambda}(D)}(M, \omega_*\omega^*N) \simeq \text{Hom}_{\text{Mod}_{\Lambda, \lambda}(D)}(M, \omega_{\lambda*}\omega^*N),$$

and hence the functor $\omega_{\lambda*}$ is right adjoint to $\omega^*_\lambda$.  

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4.3 Beilinson-Bernstein correspondence

Define $D_\lambda \in \text{Mod}_{\Lambda, \lambda}(D)$ by

\[(4.18) \quad D_\lambda = D / \sum_{\mu \in \Lambda} D(\sigma_\mu - q^{(\mu, \lambda)}) \]

Since $\sigma_\mu$ for $\mu \in \Lambda$ belongs to the center of $D(0)$ we have a natural $\mathbb{F}$-algebra structure on

\[D_\lambda(0) = D(0) / \sum_{\mu \in \Lambda} D(0)(\sigma_\mu - q^{(\mu, \lambda)}).\]

Since $D(0)$ is generated by the elements of the form $\partial_u, \sigma_\mu$ for $u \in U, \mu \in \Lambda$, we have a natural surjective algebra homomorphism $U \to D_\lambda(0)$. Set

\[J_\lambda = \sum_{z \in J} U(z - \zeta_\lambda(z)).\]

By (4.13) we have $J_\lambda \subset \text{Ker}(U \to D_\lambda(0))$ and hence we obtain a surjective algebra homomorphism

\[(4.19) \quad U/J_\lambda \to D_\lambda(0).\]

By Lemma 4.6 we obtain a left exact functor

\[(4.20) \quad \Gamma_\lambda : \mathcal{M}_\lambda(D) \to \text{Mod}(D_\lambda(0))\]

as the restriction of $\Gamma : \mathcal{M}(D) \to \text{Mod}(D(0))$. By restricting the functor $\mathcal{L} : \text{Mod}(D(0)) \to \mathcal{M}(D)$ to $\text{Mod}(D_\lambda(0))$ we obtain a right exact functor

\[(4.21) \quad \mathcal{L}_\lambda : \text{Mod}(D_\lambda(0)) \to \mathcal{M}_\lambda(D).\]

We see easily the following.

**Lemma 4.7.** The functor $\mathcal{L}_\lambda : \text{Mod}(D_\lambda(0)) \to \mathcal{M}_\lambda(D)$ is left adjoint to $\Gamma_\lambda : \mathcal{M}_\lambda(D) \to \text{Mod}_\lambda(D(0))$.

The rest of this paper is devoted to proving the following theorems.

**Theorem 4.8.**

(i) If $\lambda + \rho \in \Lambda^+$, then $\Gamma_\lambda : \mathcal{M}_\lambda(D) \to \text{Mod}(D_\lambda(0))$ is exact.

(ii) Assume $\lambda \in \Lambda^+$. If $\Gamma_\lambda(M) = 0$ for $M \in \mathcal{M}_\lambda(D)$, then we have $M = 0$.

**Theorem 4.9.**

(i) The canonical homomorphism $U/J_\lambda \to D_\lambda(0)$ is an isomorphism for any $\lambda \in \Lambda$.  

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(ii) Assume $\lambda + \rho \in \Lambda^+$. Then the canonical homomorphism $D_{\lambda}(0) \to \Gamma_\lambda(\omega^*D_{\lambda})$ is an isomorphism.

By a standard argument Theorem 4.8 and Theorem 4.9 imply the following.

**Theorem 4.10.** Assume $\lambda \in \Lambda^+$. Then $\Gamma_\lambda : \mathcal{M}_\lambda(D) \to \text{Mod}(U/J_{\lambda})$ gives an equivalence of categories, and its inverse is given by $L_{\lambda}$.

For the sake of completeness we give a proof of Theorem 4.10 assuming Theorem 4.8 and Theorem 4.9 in the rest of this subsection.

Let $M \in \text{Mod}(\mathcal{D}_{\lambda}(0))$. We can take an exact sequence of the form

$$D_{\lambda}(0)^{\oplus J_1} \to D_{\lambda}(0)^{\oplus J_2} \to M \to 0.$$ 

By Theorem 4.8 (i) the functor $\Gamma_\lambda \circ L_{\lambda}$ is right exact. By Theorem 4.9 we have $\Gamma_\lambda L_{\lambda}(D_{\lambda}(0)) \simeq D_{\lambda}(0)$. Thus we obtain a commutative diagram

$$
\begin{array}{ccc}
D_{\lambda}(0)^{\oplus J_1} & \longrightarrow & D_{\lambda}(0)^{\oplus J_2} \\
\| & & \downarrow \\
D_{\lambda}(0)^{\oplus J_1} & \longrightarrow & D_{\lambda}(0)^{\oplus J_2} \longrightarrow \Gamma_\lambda L_{\lambda}(M) \longrightarrow 0
\end{array}
$$

whose rows are exact. Hence the canonical morphism $M \to \Gamma_\lambda L_{\lambda}(M)$ is an isomorphism. It follows that $\text{Id} \to \Gamma_\lambda \circ L_{\lambda}$ is an isomorphism.

It remains to show that $L_{\lambda} \circ \Gamma_\lambda \to \text{Id}$ is an isomorphism. By Lemma 4.7 the composition of

$$\Gamma_\lambda \to (\Gamma_\lambda \circ L_{\lambda}) \circ \Gamma_\lambda = \Gamma_\lambda \circ (L_{\lambda} \circ \Gamma_\lambda) \to \Gamma_\lambda$$

coincides with $\text{Id}$. Since $\text{Id} \to \Gamma_\lambda \circ L_{\lambda}$ is an isomorphism, the canonical morphism $\Gamma_\lambda \circ (L_{\lambda} \circ \Gamma_\lambda) \to \Gamma_\lambda$ is an isomorphism. Let $N \in \mathcal{M}_\lambda(D)$. Setting $K_1 = \text{Ker}(L_{\lambda}\Gamma_\lambda(N) \to N)$, $K_2 = \text{Coker}(L_{\lambda}\Gamma_\lambda(N) \to N)$ we have an exact sequence

$$0 \to K_1 \to L_{\lambda}\Gamma_\lambda(N) \to N \to K_2 \to 0$$

By applying the exact functor $\Gamma_\lambda$ we obtain

$$0 \to \Gamma_\lambda(K_1) \to \Gamma_\lambda L_{\lambda}\Gamma_\lambda(N) \to \Gamma_\lambda(N) \to \Gamma_\lambda(K_2) \to 0.$$ 

Since $\Gamma_\lambda L_{\lambda}\Gamma_\lambda(N) \to \Gamma_\lambda(N)$ is an isomorphism, we have $\Gamma_\lambda(K_1) = \Gamma_\lambda(K_2) = 0$, which implies $K_1 = K_2 = 0$ by Theorem 4.8 (ii). Hence $L_{\lambda} \circ \Gamma_\lambda \to \text{Id}$ is an isomorphism.
4.4 The key lemma

Sections 4.4 and 4.5 are devoted to the proof of Theorem 4.8. The arguments are identical with those in Lunts-Rosenberg [12] except for the usage of Proposition 3.13, which was a conjecture in [12]. We shall reproduce the arguments in [12] for the convenience of readers.

Let $\lambda \in \Lambda$ and $M \in \text{Mod}_{\Lambda,\lambda}(\tilde{D})$. Let $\mu \in \Lambda^+$. Regarding $M$ as an object of $\text{Mod}_{\Lambda}(\tilde{U})$ via (4.11) we have a monomorphism

$$i_{\mu} : \omega^* M \to \omega^*(E^\mu \otimes_A M[-w_0\mu])$$

and an epimorphism

$$p_{\mu} : \omega^*(E^\mu \otimes_A M) \to \omega^* M[\mu]$$

in $\mathcal{M}(\tilde{U})$ (see Section 3.4). Taking $\Gamma$ we obtain a monomorphism

$$\Gamma(i_{\mu}) : \Gamma(\omega^* M) \to \Gamma(\omega^*(E^\mu \otimes_A M[-w_0\mu]))$$

and a morphism

$$\Gamma(p_{\mu}) : \Gamma(\omega^*(E^\mu \otimes_A M)) \to \Gamma(\omega^* M[\mu])$$

of $U$-modules (note that $\tilde{U}(0) = U$).

**Lemma 4.11.** (i) Assume $\lambda + \rho \in \Lambda^+$. Then there exists a splitting of $\Gamma(i_{\mu})$ as $U$-modules, which is functorial with respect to $M$.

(ii) Assume $\lambda \in \Lambda^+$. Then $\Gamma(p_{\mu})$ is surjective.

**Proof.** By Lemma 4.6 we may assume that the canonical morphism $M \to \omega_* \omega^* M$ is an isomorphism. By Section 3.4 we have a filtration

$$(4.22) \{0\} = \tilde{N}_0 \subset \tilde{N}_1 \subset \cdots \subset \tilde{N}_n \subset \omega^*(E^\mu \otimes_A M)$$

such that $\tilde{N}_k/\tilde{N}_{k-1} \simeq \omega^* M[\nu_k]^{\oplus m_k}$. Here $\nu_k$ and $m_k$ are as in Section 3.4.

We have

$$\omega_+ \omega^*(E^\mu \otimes_A M) \simeq E^\mu \otimes_A \omega_+ \omega^* M \simeq E^\mu \otimes_A M$$

by Proposition 3.13 and hence by taking $\omega_*$ in (4.22) we obtain a filtration

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_n \subset E^\mu \otimes_A M$$

of $E^\mu \otimes_A M \in \text{Mod}_{\Lambda}(\tilde{U})$ with the exact sequence

$$0 \to N_{k-1} \to N_k \to M[\nu_k]^{\oplus m_k} \to R^1 \omega_* \tilde{N}_{k-1}.$$
In particular, the quotient $N_k/N_{k-1}$ is isomorphic to a subobject of $M[\nu_k]^{\oplus m_k}$.
Hence we have
\[
z|(N_k/N_{k-1})(\xi) = \zeta_{\lambda+\nu_k+\xi}(z) \text{id} \quad (z \in \mathfrak{g}, \xi \in \Lambda)
\]
by (4.23).

In general for a $U$-module $V$ and $\xi \in \Lambda$ we set
\[
V[\mathfrak{g}, \zeta_\xi] = \{v \in V \mid \forall z \in \mathfrak{g} \ \exists m \geq 0 \text{ such that } (z - \zeta_\xi(z))^m v = 0\}.
\]

We have $\langle N_k/N_{k-1}(\xi) = ((N_k/N_{k-1})(\xi))[\mathfrak{g}, \zeta_{\lambda+\nu_k+\xi}]$.

(i) Set $\mu' = w_0 \mu$. Note
\[
\begin{align*}
\Gamma(\omega^*(E^\mu \otimes_A M[-\mu'])) & = E^\mu \otimes_A M(-\mu') = N_0(-\mu'), \\
\Gamma(\omega^* M) & = \omega_\ast(\overline{E}_1^\mu \otimes_A M)(-\mu') = N_1(-\mu').
\end{align*}
\]
We have the canonical filtration
\[
\{0\} = N_0(-\mu') \subset N_1(-\mu') \subset \ldots \subset N_n(-\mu') = \Gamma(\omega^*(E^\mu \otimes_A M[-\mu']))
\]
of $\Gamma(\omega^*(E^\mu \otimes_A M[-\mu']))$ consisting of $U$-submodules satisfying
\[
N_k(-\mu')/N_{k-1}(-\mu') = (N_k(-\mu')/N_{k-1}(-\mu'))[\mathfrak{g}, \zeta_{\lambda+\nu_k-\mu'}].
\]

Let us show
\[
(4.23) \quad \zeta_{\lambda+\nu_k-\mu'} = \zeta_\lambda \iff k = 1.
\]

By $\nu_1 = \mu'$ we have $\zeta_{\lambda+\nu_1-\mu'} = \zeta_\lambda$. By Proposition 1.6 (i) there exists some $w \in W$ satisfying $w(\lambda + \rho) = \lambda + \rho + \nu_k - \mu'$. By $\lambda + \rho \in \Lambda^+$ we have $\lambda + \rho - w(\lambda + \rho) \in \mathbb{Q}^+$. Since $\mu'$ is the lowest weight, we have $\nu_k - \mu' \in \mathbb{Q}^+$. Hence we obtain $\nu_k - \mu' = 0$, which implies $k = 1$. (4.23) is proved. From this we obtain the canonical direct sum decomposition
\[
\Gamma(\omega^*(E^\mu \otimes_A M[-\mu'])) = \Gamma(\omega^* M) \oplus \sum_{\xi \in \Lambda, \zeta_\xi \in W_{\nu_k}} (\Gamma(\omega^*(E^\mu \otimes_A M[-\mu']))[\mathfrak{g}, \zeta_\xi],
\]
which gives the desired splitting.

(ii) Note
\[
\begin{align*}
\Gamma(\omega^*(E^\mu \otimes_A M)) & = E^\mu \otimes_A M = N_0(0), \\
\Gamma(\omega^* M[\mu]) & = (\omega_\ast \omega^* M[\mu])(0) = (M[\mu])(0) = M(\mu).
\end{align*}
\]

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Consider the exact sequence
\[ 0 \to N_{n-1}(0) \to N_n(0) \to M(\mu) \to R^1\omega_*\tilde{N}_{n-1}(0). \]
We have
\[(4.24) \quad M(\mu) = (M(\mu))[[\mathfrak{g}, \zeta_{\lambda+\mu}]].\]
By the exact sequence
\[ R^1\omega_*\tilde{N}_{n-1}(0) \to R^1\omega_*\tilde{N}_k(0) \to R^1\omega_*(\tilde{N}_k/\tilde{N}_k)(0) \]
and \( R^1\omega_*(\tilde{N}_k/\tilde{N}_{k-1})(0) \cong R^1\omega_*(\omega^*M)(\nu_k) \oplus m_k \) we obtain
\[ R^1\omega_*\tilde{N}_k(0) = \sum_{r=1}^{k} (R^1\omega_*\tilde{N}_k(0))[[\mathfrak{g}, \zeta_{\lambda+\nu_r}]]. \]
by Lemma 4.6. In particular, we have
\[(4.25) \quad R^1\omega_*\tilde{N}_{n-1}(0) = \sum_{k=1}^{n-1} (R^1\omega_*\tilde{N}_{n-1}(0))[[\mathfrak{g}, \zeta_{\lambda+\nu_k}]].\]
Hence it is sufficient to show
\[(4.26) \quad \zeta_{\lambda+\nu_k} = \zeta_{\lambda+\mu} \iff k = n. \]
We have \( \nu_n = \mu \) and hence \( \zeta_{\lambda+\nu_k} = \zeta_{\lambda+\mu} \). Assume that \( \zeta_{\lambda+\nu_k} = \zeta_{\lambda+\mu} \). By Proposition 1.6 (i) there exists some \( w \in W \) satisfying \( w(\lambda+\rho+\nu_k) = \lambda+\rho+\mu \). By \( \lambda+\rho \in \Lambda^+ \) we have \( \lambda+\rho-w(\lambda+\rho) \in Q^+ \). Since \( \mu \) is the highest weight, we have \( \mu-wv_k \in Q^+ \). Hence we obtain \( \lambda+\rho-w(\lambda+\rho) = 0 \). By \( \lambda \in \Lambda^+ \) we have \( w = 1 \) and hence \( m\mu = v_k \), which implies \( k = n \). (4.26) is proved. \( \square \)

### 4.5 Proof of Theorem 4.8

We first show the following.

**Lemma 4.12.** Let \( M \in \text{Mod}_\Lambda(A) \) and let \( m \in \Gamma(\omega^*M) \). Then there exists a finitely generated graded \( A \)-submodule \( N \) of \( M \) such that \( m \in \Gamma(\omega^*N) \).

**Proof.** Let \( f : M \to \omega_*\omega^*M \) be the canonical morphism. Note that we have \( m \in \Gamma(\omega^*M) = \omega_*\omega^*M(0) \subset \omega_*\omega^*M \). By \( \text{Coker}(f) \in \text{Tor}_\Lambda(A) \) there exists \( \lambda \in \Lambda^+ \) such that \( A(\lambda)m \subset \text{Im}(f) \). Take a finite-dimensional subspace \( V \) of \( M(\lambda) \) such that \( f(V) = A(\lambda)m \). Set \( N = \sum_{v \in V} Av \subset M \), \( N' = Am \subset \omega_*\omega^*M \), and let \( h : N \to N' \) be the morphism induced by \( f \). Note
that \(\omega^*N\) and \(\omega^*N'\) are regarded as subobjects of \(\omega^*M\) and \(\omega^*(\omega_*\omega^*M)\) respectively. Since \(\omega^*f\) is an isomorphism, \(\omega^*h\) is a monomorphism. Let \(\overline{m}\) be the element of Coker\((h)\) corresponding to \(m\). By definition we have \(A(\lambda)\overline{m} = \{0\}\), and hence \(\overline{m}\) belongs to Tor\((\text{Coker}(h))\) by Lemma 2.2. Since Coker\((h)\) is generated by \(\overline{m}\), we have Coker\((h)\) \(\in\) Tor\(_A(\Lambda)\), and hence \(\omega^*h\) is an isomorphism. It follows that \(m \in \Gamma(\omega^*N') = \Gamma(\omega^*N)\).

Let us give a proof of Theorem 4.8 (i)

Let \(\lambda \in \Lambda\) such that \(\lambda + \rho \in \Lambda^+\), and let \(f: M \rightarrow N\) be an epimorphism in \(\mathcal{M}_\Lambda(D)\). We need to show that \(\Gamma(f): \Gamma(M) \rightarrow \Gamma(N)\) is an epimorphism.

We first show that there exist \(M, N \in \text{Mod}_{\Lambda, \lambda}(D)\) and an epimorphism \(f: M \rightarrow N\) such that

\[
(4.27) \quad \text{Tor}(N) = \{0\}, \quad \overline{M} = \omega^*M, \quad \overline{N} = \omega^*N, \quad \overline{f} = \omega^*f.
\]

Take \(M, N \in \text{Mod}_{\Lambda, \lambda}(D)\) such that \(\overline{M} = \omega^*M, \overline{N} = \omega^*N\). We may assume Tor\((N) = \{0\}\). Then there exist \(L \in \text{Mod}_{\Lambda, \lambda}(D)\) and morphisms \(s: L \rightarrow M, f: L \rightarrow N\) such that \(s \in \Sigma_D, \overline{f} = \omega^*f \circ (\omega^*s)^{-1}\). By replacing \(M\) with \(L\) we may assume that \(\overline{f} = \omega^*f\) for some \(f: M \rightarrow N\). Since \(\overline{f}\) is an epimorphism, we have Tor\((\text{Coker}(f)) = \text{Coker}(f)\). Hence by replacing \(N\) with \(\text{Im}(f)\) we may further assume that \(f\) is an epimorphism. The existence of \(M, N, f\) as in (4.27) is proved.

Let \(n \in \Gamma(N)\). We need to show that \(n\) is contained in the image of \(\Gamma(M) \rightarrow \Gamma(N)\). By Lemma 4.12 there exists a finitely generated graded \(A\)-submodule \(N_1\) of \(N\) such that \(n \in \Gamma(\omega^*N_1)\). Take a finitely generated graded \(A\)-submodule \(M_1\) of \(M\) such that \(N_1 = f(M_1)\). By Proposition 2.13 there exists some \(\nu \in \Lambda^+\) such that 

\[
\Gamma(\omega^*(E^\mu \otimes_A M_1[\nu])) \rightarrow \Gamma(\omega^*(E^\mu \otimes_A N_1[\nu]))
\]

is surjective. Set \(\mu = -w_0\nu \in \Lambda^+\). By Proposition 3.13 the vertical arrows of the commutative diagram

\[
\begin{array}{ccc}
E^\mu \otimes_A \Gamma(\omega^*(M_1[\nu])) & \rightarrow & E^\mu \otimes_A \Gamma(\omega^*(N_1[\nu])) \\
\downarrow & & \downarrow \\
\Gamma(\omega^*(E^\mu \otimes_A M_1[\nu])) & \rightarrow & \Gamma(\omega^*(E^\mu \otimes_A N_1[\nu]))
\end{array}
\]

are isomorphism, and hence

\[
\Gamma(\omega^*(E^\mu \otimes_A M_1[\nu])) \rightarrow \Gamma(\omega^*(E^\mu \otimes_A N_1[\nu]))
\]

is surjective.
is surjective. By (3.41) we can regard
\[ \Gamma(\omega^*M_1) \longrightarrow \Gamma(\omega^*N_1) \]
\[ \downarrow \quad \downarrow \]
\[ \Gamma(\omega^*M) \longrightarrow \Gamma(\omega^*N) \]

as a subdiagram of the commutative diagram
\[ \Gamma(\omega^*(E^\mu \otimes_A M_1[\nu])) \longrightarrow \Gamma(\omega^*(E^\mu \otimes_A N_1[\nu])) \]
\[ \downarrow \quad \downarrow \]
\[ \Gamma(\omega^*(E^\mu \otimes_A M[\nu])) \longrightarrow \Gamma(\omega^*(E^\mu \otimes_A N[\nu])). \]

Consider the commutative diagram
\[ \Gamma(\omega^*M) \quad \longrightarrow \quad \Gamma(\omega^*N) \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ \Gamma(\omega^*(E^\mu \otimes_A M[\nu])) \quad \longrightarrow \quad \Gamma(\omega^*(E^\mu \otimes_A N[\nu])). \]

Since \( n \in \Gamma(\omega^*N_1) \), the surjectivity of \( h \) implies \( i_N(n) \in \text{Im}(\ell) \). Then we obtain \( n \in \text{Im}(k) \) by the existence of the canonical splitting of \( i_M \) and \( i_N \) assured in Lemma 4.11 (i). The proof of Theorem 4.8 (i) is complete.

We next show Theorem 4.8 (ii).

Assume \( \lambda \in \Lambda^+ \), and let \( \overline{M} \) be a non-zero object of \( \overline{\mathcal{M}}_A(D) \). Set \( M = \omega \overline{M} \). By \( \omega^*M = \overline{M} \neq 0 \) we have \( M \notin \text{Tor}_A(\lambda) \), and hence \( \Gamma(\omega^*M[\mu]) = \omega \overline{M(\mu)} = M(\mu) \neq 0 \) for sufficiently large \( \mu \in \Lambda^+ \). Hence by Proposition 3.13 and Lemma 4.11 (ii) we have
\[ V(\mu) \otimes \Gamma(\omega^*M) \simeq E^\mu \otimes_A \Gamma(\omega^*M) \simeq \Gamma(\omega^*(E^\mu \otimes_A M)) \neq 0. \]

This implies \( \Gamma(\overline{M}) = \Gamma(\omega^*M) \neq 0 \) as desired.

### 4.6 Verma modules

**Lemma 4.13.** For any \( w \in W \) the canonical homomorphism
\[ S_w^{-1}D \rightarrow \text{End}_F(S_w^{-1}A) \]
is injective.
Proof. Let \( s \in S_w \) and \( d \in D \), and assume that \( s^{-1}d \) belongs to the kernel of (4.28). Then \( d \) belongs to the kernel of (4.28) and hence it is contained in \( \text{Ker}(D \to \text{End}_F(A)) \). Since \( D \) is defined as a subalgebra of \( \text{End}_F(A) \), we have \( d = 0 \) and hence \( s^{-1}d = 0 \).

For \( \lambda \in \Lambda \) we have
\[
S^{-1}_w D_\lambda = S^{-1}_w A \otimes_A (D \otimes_{F[\Lambda]} F) = (S^{-1}_w A \otimes_A D) \otimes_{F[\Lambda]} F = S^{-1}_w D \otimes_{F[\Lambda]} F,
\]
where \( F[\Lambda] \to S^{-1}_w D \) and \( F[\Lambda] \to F \) are given by \( e(\mu) \mapsto \sigma_\mu \) and \( e(\mu) \mapsto q^{(\mu, \lambda)} \) respectively. Hence we have
\[
(S^{-1}_w D_\lambda)(0) = (S^{-1}_w D)(0) \otimes_{F[\Lambda]} F.
\]
Since the image of \( F[\Lambda] \to (S^{-1}_w D)(0) \) is contained in the center of \( (S^{-1}_w D)(0) \), we have an \( F \)-algebra structure on \( (S^{-1}_w D_\lambda)(0) \). Moreover, since the action of \( \sigma_\mu \) on \( (S^{-1}_w A)(\lambda) \) is given by \( q^{(\mu, \lambda)} \text{id} \), we have a natural left \( (S^{-1}_w D_\lambda)(0) \)-module structure on \( (S^{-1}_w A)(\lambda) \). Hence we obtain an algebra homomorphism (4.29)
\[
(S^{-1}_w D_\lambda)(0) \to \text{End}_F((S^{-1}_w A)(\lambda))
\]

In the rest of this subsection we shall consider the case \( w = 1 \).

Lemma 4.14. Let \( s \in S_1 \). The right multiplication \( r_s \in D \) is invertible in \( S^{-1}_1 D \) and the action of \( r_s^{-1} \in S^{-1}_1 D \) on \( S^{-1}_1 A \) is given by the right multiplication of \( s^{-1} \in S^{-1}_1 A \).

Proof. We have \( s \in A(\mu) \setminus \{0\} \) for some \( \mu \in \Lambda^+ \). Then \( r_s = \ell_s \partial_{k_\mu} \sigma_{-\mu} \) by Lemma 4.1. Hence \( r_s^{-1} = \sigma_\mu \partial_{k_{-\mu}} \ell_s^{-1} \in S^{-1}_1 D \). For \( t \in S_1 \) and \( \varphi \in A \) we have
\[
r_s(t^{-1} \varphi) = r_s \ell_t^{-1} \varphi = \ell_t^{-1} r_s(\varphi) = t^{-1} \varphi s
\]
by \( r_s \ell_t = \ell_t r_s \). Thus the action of \( r_s \) on \( S^{-1}_1 A \) is given by the right multiplication of \( s \). Hence its inverse is given by the right multiplication by \( s^{-1} \).

By definition the image \( D \) of the canonical injective algebra homomorphism (4.30)
\[
(S^{-1}_1 D)(0) \to \text{End}_F(S^{-1}_1 A)
\]
is generated by
\[
(a) \quad \partial_u \text{ for } u \in U,
\]
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(b) $\sigma_{\mu}$ for $\mu \in \Lambda$.

(c) $L_\varphi : S_1^{-1}A \to S_1^{-1}A (\psi \mapsto \varphi \psi)$, where $\varphi \in (S_1^{-1}A)(0)$.

By Lemma 4.14 $D$ also contains

(d) $R_\varphi : S_1^{-1}A \to S_1^{-1}A (\psi \mapsto \psi \varphi)$, where $\varphi \in (S_1^{-1}A)(0)$.

Since $D$ preserves $(S_1^{-1}A)(\xi)$ for each $\xi \in \Lambda$, we have

$D \subset \prod_{\xi \in \Lambda} \text{End}_\mathbb{F}(F^{\geq 0}(\xi))$.

Here, we have identified $(S_1^{-1}A)(\xi)$ with $F^{\geq 0}(\xi)$ by Proposition 3.21. In particular, $F^{\geq 0}(\xi)$ is regarded as a $U$-module. Note that $F^{\geq 0}(\xi)$ is isomorphic to $T^*_\mathbb{F}(\xi)$ as a $U$-module by Proposition 3.22.

Set

$\text{End}_\mathbb{F}(F^{\geq 0}(\xi)) = \bigoplus_{\nu \in \Lambda} \{ f \in \text{End}_\mathbb{F}(F^{\geq 0}(\xi)) \mid f(F^{\geq 0}(\xi)_{\mu}) \subset F^{\geq 0}(\xi)_{\mu + \nu} \ (\forall \mu \in \Lambda) \}$,

$\text{End}_\mathbb{F}(F^{\geq 0}(\xi)^\star) = \bigoplus_{\nu \in \Lambda} \{ f \in \text{End}_\mathbb{F}(F^{\geq 0}(\xi)^\star) \mid f(F^{\geq 0}(\xi)^\star_{\mu}) \subset F^{\geq 0}(\xi)^\star_{\mu + \nu} \ (\forall \mu \in \Lambda) \}$.

Then we have an isomorphism of $\mathbb{F}$-algebras

$\text{End}_\mathbb{F}(F^{\geq 0}(\xi)) \simeq (\text{End}_\mathbb{F}(F^{\geq 0}(\xi)^\star))^{\text{op}} \quad (h \leftrightarrow h^*)$

$\langle h(v^*), v \rangle = \langle v^*, h^*(v) \rangle \quad (v^* \in F^{\geq 0}(\xi), v \in F^{\geq 0}(\xi)^\star)$.

Hence we obtain an embedding of $\mathbb{F}$-algebras

$\Theta : D \hookrightarrow \prod_{\xi \in \Lambda} \text{End}_\mathbb{F}(F^{\geq 0}(\xi)^\star)^{\text{op}}, \quad \Theta(d) = (\Theta_\xi(d))_{\xi \in \Lambda}$,

$\langle v^*, (\Theta_\xi(d))(v) \rangle = \langle d(v^*), v \rangle \quad (v^* \in F^{\geq 0}(\xi), v \in F^{\geq 0}(\xi)^\star)$.

Since $F^{\geq 0}(\xi)^\star$ is isomorphic to $T_\mathbb{F}(\xi)$ as a right $U$-module, it is a free right $U^+$-module generated by the element $n_\xi \in F^{\geq 0}(\xi)^\star$ given by

$\langle \varphi, n_\xi \rangle = \langle \varphi, 1 \rangle \quad (\varphi \in F^{\geq 0}(\xi))$.

Here, $\langle \, , \rangle$ in the left hand side (resp. the right hand side) is the canonical paring $F^{\geq 0}(\xi) \times F^{\geq 0}(\xi)^\star \to \mathbb{F}$ (resp. $F^{\geq 0} \times U^{\geq 0} \to \mathbb{F}$). We shall identify
\[ F^{≥0}(ξ)^* \text{ with } U^+ \text{ by } (F^{≥0}(ξ)^* \ni n_ξu \leftrightarrow u \in U^+) \text{. Hence we have an embedding of } \mathbb{F}\text{-algebras} \]
\[ \Theta : \mathcal{D} \hookrightarrow \prod_{ξ \in Λ} \text{End}_\mathbb{F}(U^+)^{\text{op}}, \quad \Theta(d) = (\Theta_ξ(d))_{ξ \in Λ}, \]
\[ \langle \varphi, (\Theta_ξ(d))(u) \rangle = \langle d(\varphi), n_ξu \rangle = \langle d(\varphi), u \rangle \quad (\varphi \in F^{≥0}(ξ), u \in U^+) \text{.} \]

For \( x \in U^+, \mu \in Λ \) we define \( M_x, N_\mu \in \text{End}_\mathbb{F}(U^+) \) by
\[ M_x(u) = ux, \quad N_\mu(u) = k^\mu_\mu k^{-1}_\mu (u \in U^+) \text{.} \]

We define linear maps
\[ F^{≥0}(0) \ni \varphi \mapsto P_\varphi \in \text{End}_\mathbb{F}(U^+), \quad F^{≥0}(0) \ni \varphi \mapsto Q_\varphi \in \text{End}_\mathbb{F}(U^+) \]
by
\[ P_\varphi(u) = \sum_{(u)_1} \langle \varphi, u(0) \rangle u(1) \quad (\varphi \in F^{≥0}(0), u \in U^+), \]
\[ Q_\varphi(u) = \sum_{(u)_1} \langle \varphi, u(1) \rangle k^{-1}_\gamma u(0) \quad (\gamma \in Q^+, \varphi \in F^{≥0}(0)_{−\gamma}, u \in U^+) \text{.} \]

Note that \( P_\varphi(u), Q_\varphi(u) \in U^+ \) by
\[ \Delta(u) = \sum_{\gamma \in Q^+} k_\gamma U^+ \otimes U^+_{\gamma} (u \in U^+), \]
\[ \gamma \neq \delta \Rightarrow \langle F^{≥0}(0)_{−\gamma}, U^+_{\delta} \rangle = \{0\} \text{.} \]

For \( i \in I \) we define \( \varphi_i \in F^{≥0}(0) \) by
\[ \langle \varphi_i, tu \rangle = (u, f_i) \quad (t \in U^0, u \in U^+), \]
where \( (\ ,\ ) \) in the right hand side is the paring \[1.31\]. For \( F \in \text{End}_\mathbb{F}(U^+) \) we define \( \Delta(F) \in \prod_{ξ \in Λ} \text{End}_\mathbb{F}(U^+) \) as the corresponding diagonal element.

**Lemma 4.15.** We have
\[ \Theta_ξ(σ_\mu) = q^{(μ,ξ)} \text{id} \quad (μ \in Λ), \]
\[ \Theta(L_\varphi) = \Delta(P_\varphi) \quad (\varphi \in F^{≥0}(0)), \]
\[ \Theta(R_\varphi) = \Delta(Q_\varphi)\Theta(σ_\gamma) \quad (\gamma \in Q^+, \varphi \in F^{≥0}(0)_{−\gamma}), \]
\[ \Theta(∂x) = \Delta(M_x) \quad (x \in U^+), \]
\[ \Theta(∂k_\mu) = \Delta(N_−\mu)\Theta(σ_\mu) \quad (μ \in Λ), \]
\[ \Theta(∂f_i) = q^{−(α_i,ξ)}\Delta(P_{−α_i}N_α_i)\Theta(σ_{−α_i}) − \Delta(Q_{−α_i})\Theta(σ_α_i) \quad (i \in I). \]
Proof. (4.33) is easy. Let $\psi \in F_{\geq 0}(\xi)$, $u \in U^+$. For $\varphi \in F_{\geq 0}(0)$ we have
\[
\langle \psi, (\Theta_\xi(L_\varphi))(u) \rangle = \langle \varphi \psi, u \rangle = \sum_{(u)} \langle \varphi, u(0) \rangle \langle \psi, u(1) \rangle = \langle \psi, P_\varphi(u) \rangle
\]
and hence we obtain (4.34). Similarly, for $\varphi \in F_{\geq 0}(0) - \gamma$ we have
\[
\langle \psi, (\Theta_\xi(R_\varphi))(u) \rangle = \sum_{(u)} \langle \varphi, u(1) \rangle \langle \psi, u(0) \rangle = \langle \psi, k_\gamma Q_\varphi(u) \rangle = \langle \psi k_\gamma, Q_\varphi(u) \rangle = q^{(\gamma, \xi)} \langle \psi, Q_\varphi(u) \rangle.
\]
(4.35) is proved. For $x \in U$ we have
\[
\langle \psi, (\Theta_\xi(\partial_x))(u) \rangle = \langle x \psi, n_\xi u \rangle = \langle \psi, n_\xi u x \rangle.
\]
In case $x \in U^+$ we have $ux \in U^+$, and we obtain (4.36). In case $x = k_\mu$ for $\mu \in \Lambda$ we have
\[
n_\xi uk_\mu = n_\xi k_\mu (k_\mu^{-1} u k_\mu) = q^{(\xi, \mu)} n_\xi N_{-\mu}(u),
\]
and hence (4.37) is proved. In case $x = f_i$ for $i \in I$ we have
\[
uf_i = \sum_{(u_2)} (u(0), f_i) (u(2), k_i) k_i^{-1} u(1) + \sum_{(u_2)} (u(0), 1) (u(2), k_i) f_i u(1)
\]
\[
- \sum_{(u_2)} (u(0), 1) (u(2), f_i k_i) u(1)
\]
\[
= \sum_{(u_1)} (u(0), f_i) k_i^{-1} u(1) + \sum_{(u_1)} (u(1), k_i) f_i u(0) - \sum_{(u_1)} (u(1), f_i k_i) u(0)
\]
by [19] Lemma 2.1.2, Lemma 2.1.3, and hence if $u \in U^\delta_0$, then
\[
n_\xi uf_i = n_\xi \left( \sum_{(u_1)} (u(0), f_i) k_i^{-1} u(1) \right) - n_\xi \left( \sum_{(u_1)} (u(1), f_i k_i) u(0) \right)
\]
\[
= q^{(\delta - \alpha_i - \xi, \alpha_i)} n_\xi \left( \sum_{(u_1)} (\varphi_i, u(0)) u(1) \right) - n_\xi \left( \sum_{(u_1)} (\varphi_i, u(1)) u(0) \right)
\]
\[
= q^{(\delta - \alpha_i - \xi, \alpha_i)} n_\xi (P_{\varphi_i}(u)) - q^{(\xi, \alpha_i)} n_\xi (Q_{\varphi_i}(u))
\]
and we obtain (4.38).

By (4.29) we obtain an algebra homomorphism
\[
(4.39) \quad (S_1^{-1} D_\lambda)(0) \rightarrow \text{End}_F(T_\lambda(\lambda))^\text{op}.
\]
Proposition 4.16. The algebra homomorphism (4.39) is injective.

Proof. Note that $(S^{-1}_1D_\lambda)(0) = \mathcal{D} \otimes \mathbb{F}[\Lambda] \mathbb{F}$, where $\mathbb{F}[\Lambda] \rightarrow \mathcal{D}$ and $\mathbb{F}[\Lambda] \rightarrow \mathbb{F}$ are given by $e(\mu) \mapsto \sigma_\mu$ and $e(\mu) \mapsto q^{(\mu, \lambda)}$ respectively. The algebra homomorphism (4.39) is induced from $\Theta_\lambda : \mathcal{D} \rightarrow \text{End}_\mathbb{F}(U^+)^{\text{op}}$ under the identification $\text{Tr}(\lambda) = U^+$. Let $\mathcal{D}'$ be the subalgebra of $\prod_{\xi \in \Lambda} \text{End}_\mathbb{F}(U^+)^{\text{op}}$ generated by the elements $\Delta(M_x), \Delta(N_\mu), \Delta(P_\phi), \Delta(Q_\phi)$ for $x \in U^+, \mu \in \Lambda, \phi \in \mathbb{F}^{\geq 0}(0)$. By Lemma 4.15 we have $\mathcal{D}' \subset \mathcal{D}$, and $\mathcal{D}$ is generated as a subalgebra of $\prod_{\xi \in \Lambda} \text{End}_\mathbb{F}(U^+)^{\text{op}}$ by $\mathcal{D}'$ and $\{\sigma_\mu | \mu \in \Lambda\}$. Let us show that the linear map

$$\mathcal{D}' \otimes \mathbb{F}[\Lambda] \ni d \otimes e(\mu) \mapsto d\sigma_\mu \in \mathcal{D}$$

is an isomorphism. The surjectivity is a consequence of the fact that $\sigma_\mu$ belongs to the center of $\mathcal{D}$. Assume that we have $\sum_\mu d_\mu \sigma_\mu = 0$ for $d_\mu \in \mathcal{D}'$. For any $\xi \in \Lambda$ and $u \in U^+$ we have $\sum_\mu q^{(\mu, \xi)}d_\mu(u) = 0$, from which we obtain $d_\mu = 0$ for any $\mu$. Hence $\mathcal{D}' \otimes \mathbb{F}[\Lambda] \rightarrow \mathcal{D}$ is bijective. It follows that

$$(S^{-1}_1D_\lambda)(0) = \mathcal{D} \otimes \mathbb{F}[\Lambda] \mathbb{F} \simeq \mathcal{D},$$

and hence $(S^{-1}_1D_\lambda)(0) \rightarrow \text{End}_\mathbb{F}(T_i(\lambda))^{\text{op}}$ is injective. \hfill \Box

4.7 Proof of Theorem 4.9

For $\lambda \in \Lambda$ consider the sequence

$$(4.40) \quad U/J_\lambda \xrightarrow{\alpha} D_\lambda(0) \xrightarrow{\beta} \Gamma(\omega^* D_\lambda) \xrightarrow{\gamma} (S^{-1}_1D_\lambda)(0) \xrightarrow{\delta} \text{End}_\mathbb{F}(T_i(\lambda))^{\text{op}}$$

of algebra homomorphisms. We shall show that $\alpha$ is an isomorphism for any $\lambda \in \Lambda$ and that $\beta$ is an isomorphism if $\lambda + \rho \in \Lambda^+$. We need the following result due to Joseph [8].

Proposition 4.17 (Joseph). The homomorphism

$$U/J_\lambda \rightarrow \text{End}_\mathbb{F}(T_i(\lambda))^{\text{op}}$$

is injective.

We see easily that $\alpha$ is surjective by the definition of $D_\lambda$. By Proposition 4.17 the composition $\delta \circ \gamma \circ \beta \circ \alpha$ is injective. Hence $\alpha$ is an isomorphism and $\beta$ is injective. It remains to show the surjectivity of $\beta$ in the case $\lambda + \rho \in \Lambda^+$. 

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Assume that we are given an \( \mathbb{F} \)-algebra \( E \) and an algebra homomorphism \( f : U \to E \). We have a left \( U \)-module structure on \( E \)

\[
ue = f(u)e \quad (u \in U, \ e \in E).
\]
given by the left multiplication. We have also another left \( U \)-module structure on \( E \)

\[
ad(u)(e) = \sum_{(u)} f(u(0))ef(Su(1)) \quad (u \in U, \ e \in E)
\]
called the adjoint action. Taking the \( U \)-finite parts

\[
E^\text{fin} = \{ e \in E \mid \dim_\mathbb{F} \text{ad}(U)(e) < \infty \}
\]
of \( E = U, \ D_\lambda(0), \ \Gamma(\omega^*D_\lambda), (S_1^{-1}D_\lambda)(0), \ \text{End}_\mathbb{F}(T_\iota(\lambda))^{\text{op}} \) we obtain

\[
(4.41) \quad U^\text{fin} \xrightarrow{\gamma} D_\lambda(0)^\text{fin} \xrightarrow{\beta} \Gamma(\omega^*D_\lambda)^\text{fin} \xrightarrow{\gamma} (S_1^{-1}D_\lambda)(0)^\text{fin} \xrightarrow{\gamma} (\text{End}_\mathbb{F}(T_\iota(\lambda))^{\text{op}})^\text{fin}
\]

We need the following result of Joseph [10, Theorem 8.3.9 (ii)] which is a \( q \)-analogue of a theorem of N. Conze-Berline and M. Duflo.

**Proposition 4.18 (Joseph).** The homomorphism

\[
U^\text{fin} \to (\text{End}_\mathbb{F}(T_\iota(\lambda))^{\text{op}})^\text{fin}
\]
is surjective.

As shown above \( \beta \) is injective, and hence \( \overline{\beta} \) is injective. Since \( \delta \) is injective by Proposition 4.10 \( \delta \) is also injective. Moreover, we have the injectivity of \( \gamma \) by Lemma 3.25. Hence Proposition 4.18 implies that \( \overline{\beta}, \gamma, \delta \) are isomorphisms.

Now we deduce the surjectivity of \( \beta \) from the surjectivity of \( \overline{\beta} \). The assumption \( \lambda + \rho \in \Lambda^+ \) will be used in the arguments below.

Let \( m \in \Gamma(\omega^*D_\lambda) \). We shall show that \( m \in \text{Im}(\beta) \). By Lemma 4.12 there exists a finitely generated graded \( A \)-submodule \( M \) of \( D_\lambda \) such that \( m \in \Gamma(\omega^*M) \).

Let us show that the canonical morphism

\[
(4.42) \quad M(\nu) \to \Gamma(\omega^*M[\nu])
\]
is surjective for sufficiently large \( \nu \in \Lambda^+ \). Since \( M \) is finitely generated, there exists an epimorphism

\[
\bigoplus_{j=1}^n \mathbb{A}[\xi_j] \to M
\]

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in \text{Mod}_\Lambda(A)$ for some $\xi_1, \ldots, \xi_n \in \Lambda$. Consider the commutative diagram
\[
\begin{array}{ccc}
\bigoplus_{j=1}^n A(\xi_j + \nu) & \longrightarrow & M(\nu) \\
\downarrow & & \downarrow \\
\bigoplus_{j=1}^n \Gamma(\omega^* A(\xi_j + \nu)) & \longrightarrow & \Gamma(\omega^* M[\nu]).
\end{array}
\]
Assume that $\nu \in \Lambda^+$ is sufficiently large. Then the left vertical arrow is an isomorphism by Proposition 2.14, and the lower horizontal arrow is surjective by Proposition 2.13. Hence (4.42) is surjective.

Take $\nu \in \Lambda^+$ such that (4.42) is surjective. Set $\mu = -w_0 \nu$, and consider the following commutative diagram (see (3.41)):
\[
\begin{array}{ccc}
\Gamma(\omega^* M) & \longrightarrow & \Gamma(\omega^*(E^\mu \otimes_A M[\nu])) \\
\downarrow & & \downarrow i \\
\Gamma(\omega^* D_\lambda) & \longrightarrow & \Gamma(\omega^*(E^\mu \otimes_A D_\lambda[\nu])).
\end{array}
\]
Since $\lambda + \rho \in \Lambda^+$, there exists a homomorphism
\[
j : \Gamma(\omega^*(E^\mu \otimes_A D_\lambda[\nu])) \rightarrow \Gamma(\omega^* D_\lambda)
\]
of $U$-modules such that $j \circ i = \text{id}$ by Lemma 4.11. Here, the action of $U$ on $E^\mu \otimes_A D_\lambda$ is given by
\[
(4.43) \quad u \cdot (e \otimes d) = \sum_{(u)} u_{(0)} e \otimes \overline{d}_{(1)} d \quad (u \in U, \ d \in D),
\]
where $\overline{d}$ denotes the element of $D_\lambda$ corresponding to $d$, and the one on $\Gamma(\omega^* D_\lambda)$ is given by the left multiplication. Hence it is sufficient to show $\text{Im}(j \circ k) \subset \text{Im}(\beta)$. By Proposition 3.13 we have
\[
\omega_* \omega^*(E^\mu \otimes_A M[\nu]) \simeq E^\mu \otimes_A \omega_* \omega^*(M[\nu]) \simeq V(\mu) \otimes \omega_* \omega^*(M[\nu]),
\]
and hence $\Gamma(\omega^*(E^\mu \otimes_A M[\nu])) \simeq V(\mu) \otimes \Gamma(\omega^*(M[\nu]))$. Similarly we have $\Gamma(\omega^*(E^\mu \otimes_A D_\lambda[\nu])) \simeq V(\mu) \otimes \Gamma(\omega^*(D_\lambda[\nu]))$. Consider the commutative diagram
\[
\begin{array}{ccc}
V(\mu) \otimes M(\nu) & \longrightarrow & V(\mu) \otimes D_\lambda(\nu) \\
\downarrow & & \downarrow \epsilon \\
V(\mu) \otimes \Gamma(\omega^*(M[\nu])) & \longrightarrow & V(\mu) \otimes \Gamma(\omega^*(D_\lambda[\nu])) \\
\downarrow & & \downarrow \epsilon \\
\Gamma(\omega^*(E^\mu \otimes_A M[\nu])) & \longrightarrow & \Gamma(\omega^*(E^\mu \otimes_A D_\lambda[\nu])).
\end{array}
\]
Since (4.42) is surjective, \( \ell' \) is also surjective. Hence it is sufficient to show that the image of the composition of

\[
V(\mu) \otimes D_\lambda(\nu) \xrightarrow{\ell} \Gamma(\omega^*(E^\mu \otimes_A D_\lambda[\nu])) \xrightarrow{j} \Gamma(\omega^* D_\lambda)
\]

is contained in \( \text{Im}(\beta) \). We regard \( V(\mu) \otimes D_\lambda(\nu), \Gamma(\omega^*(E^\mu \otimes_A D_\lambda[\nu])), \Gamma(\omega^* D_\lambda) \) as right \( D_\lambda(0) \)-modules via the right multiplication of \( D(0) \) on \( D \). Then \( \ell \) and \( \beta \) are homomorphisms of right \( D_\lambda(0) \)-modules by definition. Moreover, \( j \) is also a homomorphism of right \( D_\lambda(0) \)-modules by the following reason. Recall that \( j \) is the projection with respect to the action of the center \( Z \) of \( U \). Here, the action of \( U \) on \( E^\mu \otimes_A D_\lambda(\nu) \) is given by (4.43), and the one on \( \Gamma(\omega^* D_\lambda) \) is given by the left multiplication. Since the action of \( U \) and the right action of \( D_\lambda(0) \) on \( \Gamma(\omega^*(E^\mu \otimes_A D_\lambda[\nu])) \) commute with each other, \( j \) is a homomorphism of right \( D_\lambda(0) \)-modules. Let

\[
r : A(\nu) \to D_\lambda(\nu)
\]

be the composition of \( A(\nu) \hookrightarrow D(\nu) \to D_\lambda(\nu) \). Since \( D(\nu) \) is generated by \( A(\nu) \) as a right \( D(0) \)-module, \( D_\lambda(\nu) \) is generated by \( r(A(\nu)) \) as a right \( D_\lambda(0) \)-module. Hence it is sufficient to show \( (j \circ \ell)(V(\mu) \otimes r(A(\nu))) \subset \text{Im}(\beta) \).

We can regard \( V(\mu) \otimes D_\lambda(\nu) \) as a left \( U \)-module

\[
(4.44) \quad u \cdot (v \otimes d) = \sum_{(u_1)} u(0) v \otimes \overline{\partial u_{(1)} d} \quad (u \in U, \ d \in D(\nu)),
\]

We can also regard \( V(\mu) \otimes D_\lambda(\nu) \) as a left \( U \)-module by the adjoint action defined by

\[
\text{ad}(u)(v \otimes d) = \sum_{(u_2)} u(0) v \otimes \overline{\partial u_{(1)} d \partial s u_{(2)}} \quad (u \in U, \ d \in D(\nu)).
\]

Note that \( j \circ \ell \) is a homomorphism of right \( D_\lambda(0) \)-modules as well as a homomorphism of left \( U \)-modules, where the \( U \)-module structure on \( V(\mu) \otimes D_\lambda(\nu) \) is given by (4.44) and the one on \( \Gamma(\omega^* D_\lambda) \) is given by the left multiplication. Hence \( j \circ \ell \) also preserves the adjoint actions of \( U \). Since \( V(\mu) \otimes r(A(\nu)) \) is stable under the adjoint action of \( U \), we have \( V(\mu) \otimes r(A(\nu)) \subset (V(\mu) \otimes D_\lambda(\nu))^{\text{fin}} \) with respect to the adjoint action. It follows that

\[
(j \circ \ell)(V(\mu) \otimes r(A(\nu))) \subset (j \circ \ell)((V(\mu) \otimes D_\lambda(\nu))^{\text{fin}}) \subset \Gamma(\omega^* D_\lambda)^{\text{fin}} \subset \text{Im}(\beta) \subset \text{Im}(\beta).
\]

The proof of Theorem 4.9 is complete.
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