A Solvable Tensor Field Theory

R. Pascalie*

Université de Bordeaux, LaBRI, CNRS UMR 5800, Talence, France, EU
Mathematisches Institut der Westfälischen Wilhelms-Universität, Münster, Germany, EU

March 8, 2019

Abstract

We solve the closed Schwinger-Dyson equation for the 2-point function of a tensor field theory with a quartic melonic interaction, in terms of Lambert’s W-function, using a perturbative expansion and Lagrange-Bürmann resummation. Higher-point functions are then obtained recursively.

1 Introduction

Tensor models have regain a considerable interest since the discovery of their large $N$ limit (see [1], [2, 3] or the book [4]). Recently, tensor models have been related to the Sachdev-Ye-Kitaev model [5] [6, 7, 8] in [9] and [10] which is a promising toy-model for understanding black holes through holography (see also [11, 12], the lectures [13] and the review [14]).

In this paper we study a specific type of tensor field theory (TFT) [1]. More precisely, we consider a $U(N)$-invariant tensor models whose kinetic part is modified to include a Laplacian-like operator (this operator is a discrete Laplacian in the Fourier transformed space of the tensor index space). This type of tensor model has originally been used to implement renormalization techniques for tensor models (see [16], the review [17] or the thesis [18] and references within) and has also been studied as an SYK-like TFT [19]. Recently, the functional Renormalization Group (FRG) as been used in [20] to investigate the existence of a universal continuum limit in tensor models, see also the review [21]. This is also closely related to the Polchinski’s equation for TFT [22]. Our study provides a complementary non-perturbative tool to these two approaches.

The Ward-Takahashi identity (WTI) for TFT, first appeared in [23] and has been fully established in [24]. It was used in [25] to derive the tower of exact Schwinger-Dyson equations (SDE) with connected boundary graph. Their large $N$ limit was established in [25]. Then the tower of SDE with a disconnected boundary graph was derived in [26]. Numerical methods were used in [27] for a $\phi_3^4$ just renormalizable tensor model to study the solutions of closed SDE for the 2- and 4-point functions.

Let us also mention here that the WTI has been already successfully used to study the SDE in the context of matrix models of non-commutative quantum field theory - see [28] and [29]. In particular, the closed SDE 2-point function for the non-commutative and 2 dimensional $\lambda \phi^4$ has

---

*romain.pascalie@u-bordeaux.fr

1 Not to be confused with tensor fields living on a space-time such as in [15]
been solved in \cite{30} using and resuming a perturbative expansion. The building block of this solution is the Lambert-W function.

Our paper is organised as follows. In the following section we describe the setup of our work, namely the action of the model, the boundary graph expansion of the free energy and the 2-point function SDE in the large $N$ limit. The third section is dedicated to the analysis of the perturbative expansion of the 2-point function which leads us to consider the model with one quartic melonic interaction. In the fourth section we perform the resummation of the perturbative expansion, in order to obtain the non-perturbative solution of the SDE. We then discuss shortly the higher-point functions before giving some concluding remarks. In the appendix we obtain recurrence relations on the number appearing in the perturbative expansion which translate into formulas involving Stirling numbers.

\section{The model}

Let us consider a complex rank-3 bosonic tensor field theory with an action of the form

\begin{equation}
S[\varphi, \bar{\varphi}] = \sum_{x} \varphi^{x}(1 + |x|^{2})\varphi^{x} + \frac{\lambda}{N^{2}} \sum_{c=1}^{3} \sum_{a,b} \varphi^{a} \varphi^{b_{c}} \varphi^{a_{c}} \varphi^{a},
\end{equation}

with $x = (x_{1}, x_{2}, x_{3}) \in \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}^{3}$, $|x|^{2} = x_{1}^{2} + x_{2}^{2} + x_{3}^{2}$ and $a_{c}b_{c} = (a_{1}, \ldots, a_{c-1}, b_{c}, a_{c+1}, \ldots, a_{D})$ for a $D$-tuple. Here $\varphi$ is a rank-3 bosonic tensor. Note that the quartic melonic interaction terms $G_{B}(x) = \delta_{i,j} \delta_{j,k} \delta_{p,i} \delta_{p,k}$. Here $p = (p_{1}, p_{2}, \ldots, 1)^{3}$ is a momentum triplet determined by the boundary graph $B$. For instance, the boundary graph $V_{1}$ denoting the pillow graph for the colour 1 (see figure 1), $J(V_{1})(x, y) = J_{x}J_{y}J_{x_{1}}J_{y_{1}}J_{x_{2}}J_{y_{2}}J_{x_{3}}$.

A 2$k$-point function for a connected boundary graph $B$ is

\begin{equation}
G_{B}^{(2k)}(X) = \frac{N^{-\alpha(B)}}{Z_{0}} \prod_{i=1}^{k} \left( \frac{\delta}{\delta J_{p_{i}}} \frac{\delta}{\delta J_{x_{i}}} \right) Z[J, \bar{J}] \bigg|_{J = \bar{J} = 0},
\end{equation}

where $\partial_{\text{int}}$ is the set of boundary graphs associated to the interaction terms, $V(B)$ is the number of vertices of $B$, $X = (x^{1}, \ldots, x^{k}) \in \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}^{3k}$, $\text{Aut}(B)$ is the symmetry group of the graph $B$, $J(B)(X) = J_{x_{1}} \ldots J_{x_{k}}J_{p_{1}} \ldots J_{p_{k}}$. Here $p_{i} = p^{i}(X) \in \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}^{3}$ is a momentum triplet determined by the boundary graph $B$. For instance, the boundary graph $V_{1}$ denoting the pillow graph for the colour 1 (see figure 1), $J(V_{1})(x, y) = J_{x}J_{y}J_{x_{1}}J_{y_{1}}J_{x_{2}}J_{y_{2}}J_{x_{3}}$.

In \cite{25}, we used the Ward-Takahashi identity established in \cite{24} to determine the Schwinger-Dyson equation for 2$k$-point function with a connected boundary graph $B$. Then in \cite{31} we conjectured a general formula for the scalings

$$\alpha(B) = 3 - B - 2g - 2k,$$
Figure 1: Two connected Feynman graphs and the associated boundary graphs. Dotted lines correspond to the free propagator and solid lines to the contraction of the indices of the tensors. To each external leg of a Feynman diagram is associated an external vertex so that the open graph is bipartite. These vertices are exactly the vertices of the boundary graph. An edge of colour \(c\) in the boundary graph, corresponds to a path between two external leg in the Feynman graph, which alternates between dotted lines and lines of colour \(c\). White and black vertex in a boundary graph \(B\), correspond in \(J(B)\) to the sources \(J\) and \(\bar{J}\) respectively. In the figure a) the boundary graph \(V_1\) is connected. In fig. b) the boundary graph \(m|m\) is disconnected.

where \(2k\) is the number of vertices of \(B\), \(B\) is its number of connected components and \(g\) its genus.

Let us mention here that the coefficient \(\alpha(B)\) does not depend on the choice of colouring that can be made for the respective bubble. For the three pillow graphs one has: \(\alpha(V_1) = \alpha(V_2) = \alpha(V_3)\).

Following [31], for \(N = \frac{N}{\Lambda}\) and using

\[
\lim_{N \to \infty} \frac{\Lambda}{N} \sum_{k=1}^{N} f\left(\frac{k\Lambda}{N}\right) = \int_{0}^{\Lambda} dx f(x),
\]

the SDE for the 2-point function writes

\[
G^{(2)}(x) = \left(1 + |x|^2 + 2\lambda \sum_{c=1}^{3} \int_{0}^{\Lambda} dq_c G^{(2)}(q_c x_c)\right)^{-1}.
\]

The aim of this paper is to solve the 2-point function of this type of models, in the limit \(\Lambda = \infty\). We will mainly study the case with \(c = 1\) only, which is essential for solving the SDE, as we will see in the next section.

3 Perturbative expansion

In this section we will to compute the first orders of the perturbative expansion of the 2-point function. We use a Taylor subtraction scheme to renormalize the UV divergences. For simplicity, let us plug in equation (7), the following expansion of the 2-point function

\[
G^{(2)}(x) = \sum_{n \geq 0} \lambda^n G^{(2)}_n(x),
\]

in order to obtain a recursive equation for \(n \geq 1\), which writes:

\[
G^{(2)}_n(x) = -\frac{2}{|x|^2 + 1} \sum_{c=1}^{3} \int dq_c \sum_{k=0}^{n-1} G^{(2)}_k(q_c x_c) G^{(2)}_{n-k-1}(x).
\]
3.1 Model with the 3 quartic melonic interactions

Using the recursive equation \( [9] \), we get

\[
G_0^{(2)}(x) = \frac{1}{1 + |x|^2},
\]

\[
G_1^{(2)}(x) = -\frac{2}{(1 + |x|^2)^2} \sum_{c=1}^{3} \int dq_c \left( \frac{1}{1 + |q_c x_c|^2} - \frac{1}{1 + |q_c|^2} \right)
= \frac{\pi}{2(1 + |x|^2)^2} \sum_{c=1}^{3} \log (x_c^2 + 1),
\]

\[
G_2^{(2)}(x) = -\frac{2}{1 + |x|^2} \sum_{c=1}^{3} \int dq_c \left\{ \left( \frac{1}{1 + |q_c x_c|^2} - \frac{1}{1 + |q_c|^2} \right) \frac{\pi}{2(1 + |x|^2)^2} \sum_{d=1}^{3} \log (x_d^2 + 1) \right\}
+ \frac{\pi}{2(1 + |x|^2)^2} \sum_{d=1}^{3} \log ((q_d x_d)_d + 1),
\]

\[
= -\frac{1}{1 + |x|^2} \left( \sum_{c=1}^{3} \sum_{d=1}^{3} \frac{\pi^2 \log (x_c^2 + 1) \log (x_d^2 + 1)}{4(1 + |x|^2)} - \sum_{c=1}^{3} \frac{\pi \log (x_c^2 + 1)}{2(x_c^2 + 1)} \right)
- \frac{\pi^2}{2} \sum_{c=1}^{3} \log \left( \frac{1}{4} (x_c^2 + 1) \right)
+ 2 \tan^{-1}(x_c) \right) \right).}
\]

(13)

We can remark that the last term in \( G_2^{(2)}(x) \) is the only term not containing powers of logarithms. It comes from the last term of (12) for \( d \neq c \), which graphically corresponds to figure 2. This suggests that if we look at a model with only 1 pillow interaction, such graphs cannot exist, and the perturbative expansion should only be made of powers of logarithms.

3.2 Model with 1 quartic melonic interaction

Indeed, for only the pillow for the colour 1 as an interaction, we get:

\[
G^{(2)}(x) = \frac{1}{1 + |x|^2} + \frac{\pi \lambda}{2(1 + |x|^2)^2} \log (x_1^2 + 1) + \frac{(\pi \lambda)^2}{4(1 + |x|^2)^2} \left( \frac{\log^2 (x_1^2 + 1)}{(1 + |x|^2)} - \frac{\log (x_1^2 + 1)}{(x_1^2 + 1)} \right) + O(\lambda^3).
\]

(14)
We computed the expansion up to order 9 in the coupling using Mathematica.

In this case, we can notice that only two types of integrals appear:

\[
\int \mathrm{d}q_1 \left( \frac{1}{1 + |q_1 x_1|^2} - \frac{1}{1 + |q_1|^2} \right) = -\frac{\pi}{4} \log (x_1^2 + 1),
\]

\[
\int \mathrm{d}q_1 \left( \frac{1}{1 + |q_1|^2} \right)^n = \frac{\pi (1 + x_1^2)^{1-n}}{4(n-1)} \quad \text{for } n > 1.
\]

Hence, we can compute easily higher orders in the loop expansion, which suggest the following form for all order \( n \) in the coupling

\[
G_n^{(2)}(x) = \left( \frac{\pi}{2} \right)^n \left( \frac{\log n (1 + x_1^2)}{1 + |x|^2} \right)^{n+1} + \left( \frac{-1}{1 + x_1^2} \right)^n \sum_{k=1}^{n-1} (-1)^k \log^k (1 + x_1^2) \sum_{m=1}^{k} a_{n,k,m} \frac{(1 + x_1^2)^m}{(1 + |x|^2)^{m+1}} \right),
\]

where we conjecture that the numbers \( a_{n,k,m} \) are

\[
a_{n,k,m} = \left( \frac{n-1}{m-1} \right) \frac{m!}{k!} |s_{n-m,n-k}| = (-1)^{k-m}(n-1)!m \frac{s_{n-m,n-k}}{(n-m)!k!},
\]

where \( s_{n,k} \) are the Stirling numbers of the 1st kind. Using the change of variable \( j = n - m \), we have

\[
b_{n,k,j} = (-1)^{k+j-n}(n-1)!(n-j) \frac{s_{j,n-k}}{j!k!}.
\]

Noting that \( s_{j,n-k} = 0 \) if \( j < n - k \) and if \( k = 0 \) or \( k = n \), we can write the sum on \( j \) from 1 to \( n-1 \) and the sum on \( k \) from 0 to \( n \). This leads to the following expression:

\[
G_n^{(2)}(x) = \left( \frac{\pi}{2} \right)^n \left( \frac{\log n (1 + x_1^2)}{1 + |x|^2} \right)^{n+1} + \sum_{k=1}^{n-1} \sum_{m=1}^{k} \frac{s_{j,n-k}}{j!k!} \frac{(-1)^{(n-j)}(n-j)}{1 + |x|^2} \log^k (1 + x_1^2)
\]

The structure of the perturbative expansion is similar to the one studied in [30]. In the next section, we will sum the expansion following the same method.

### 4 Resummation

In this section we perform the resummation of the perturbative expansion to obtain an explicit expression for the 2-point function. Let us use the formulas

\[
(-1)^j s_{j,n-k} = \left. \frac{1}{(n-k)!} \frac{d^{n-k}}{du^{n-k}} \frac{\Gamma(j-u)}{\Gamma(-u)} \right|_{u=0},
\]

\[
\log^k (1 + x_1^2) = \left. \frac{d^k}{du^k} (1 + x_1^2)^u \right|_{u=0}.
\]

\(^2\text{We computed the expansion up to order 9 in the coupling using Mathematica.}\)
to rewrite the second term of the RHS of (20) as

\[
\left(\frac{\pi}{2}\right)^n \frac{n!}{j!n} \frac{1}{(1 + |x|^2)^{n+j}(1 + x_1^2)^j} \frac{n}{k} \left(\frac{d^{n-k}}{du^{n-k}} \frac{\Gamma(j - u)}{\Gamma(-u)} \right) \left(\frac{d^k}{d u^k} (1 + x_1^2)^u\right)_{u=0} \]

\[
= \left(\frac{\pi}{2}\right)^n \frac{n!}{j!n} \frac{1}{(1 + |x|^2)^{n+j}(1 + x_1^2)^j} \frac{d^n}{du^n} \frac{\Gamma(j - u)}{\Gamma(-u)} (1 + x_1^2)^u \bigg|_{u=0} .
\]  

(23)

Then using

\[
\frac{d^n}{du^n} \left(\frac{\Gamma(j - u)}{\Gamma(-u)} (1 + x_1^2)^u\right) = \frac{d^n}{du^n} \left(-1\right)^j (1 + x_1^2)^j \frac{d^j}{d(x_1^2)^j} \log^n (1 + x_1^2),
\]

(24)

and realising that the first term of the rhs of (20) corresponds to \( j = 0 \), we have

\[
G_n^{(2)}(x) = \left(\frac{\pi}{2}\right)^n \frac{\log^n (1 + x_1^2)}{(1 + |x|^2)^{n+1}} + \sum_{j=1}^{n-1} \frac{n - j}{j!n} \frac{(-1)^j (1 + x_1^2)^j}{(1 + |x|^2)^{n+1-j} d(x_1^2)^j} \log^n (1 + x_1^2)
\]

\[
= \left(\frac{\pi}{2}\right)^n \sum_{j=0}^{n-1} \frac{n - j}{j!n} \frac{(-1)^j}{(1 + |x|^2)^{n+1-j} d(x_1^2)^j} \log^n (1 + x_1^2).
\]  

(25)

We then write

\[
\frac{1}{(1 + |x|^2)^{n+1-j}} = \frac{(-1)^n - j}{(n - j)! d(x_1^2)^{n-j} (1 + |x|^2)},
\]

(26)

to get

\[
G^{(2)}(x) = \frac{1}{1 + |x|^2} + \sum_{n=1}^{\infty} \left(\frac{\pi}{2}\right)^n \frac{(-1)^n \lambda^n}{n!} \sum_{j=0}^{n-1} \frac{(-1)^j}{d(x_1^2)^{n-j} (1 + |x|^2)} \frac{d^j}{d(x_1^2)^j} \log^n (1 + x_1^2)
\]

\[
= \frac{1}{1 + |x|^2} - \sum_{n=1}^{\infty} \left(\frac{\pi}{2}\right)^n \frac{\lambda^n}{n!} \frac{d^{n-1}}{d(x_1^2)^{n-1}} \frac{(-\log(1 + x_1^2))^n}{(1 + |x|^2)^2} .
\]  

(27)

To sum this series, we use the Lagrange-Bürmann inversion formula [32] [33]. This formula states that for \( \phi(\omega) \) analytic at \( \omega = 0 \), such that \( \phi(0) \neq 0 \) and \( f(\omega) = x \phi(\omega) \), the inverse function \( g(z) \) of \( f(\omega) \), such that \( z = f(g(z)) \), is analytic at \( z = 0 \) and given by

\[
g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{d \omega^{n-1}} \phi(\omega) \bigg|_{\omega=0} .
\]

(28)

Moreover, for any analytic function \( H(z) \) such that \( H(0) = 0 \),

\[
H(g(z)) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{d \omega^{n-1}} \left( H'(\omega) \phi(\omega)^n \right) \bigg|_{\omega=0} .
\]

(29)

Hence, for \( z = \frac{\pi}{2} \lambda \), \( \phi(\omega) = -\log(1 + \omega + x_1^2) \) and \( H(\omega) = \log \left(1 + \frac{\omega}{(1 + |x|^2)^2}\right) \), equation (28) gives

\[
g(x_1, z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{d(x_1^2)^{n-1}} (-\log(1 + x_1^2))^n ,
\]

(30)

6
such that
\[ z = -\frac{g(x_1, z)}{\log(1 + g(x_1, z) + x_1^2)}. \] (31)
which is solved by
\[ g(x_1, z) = z W \left( \frac{1}{z} e^{\frac{1+g^2}{z}} \right) - 1 - x_1^2. \] (32)
Then, using equation (29), we can write
\[ G(2)\left( x_1 \right) = \frac{1}{1 + |x|^2} - \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{d(x_1^2)^{n-1}} \frac{(-\log(1 + x_1^2))^n}{(1 + |x|^2)^2} = \frac{1}{1 + |x|^2 + g(x_1, z)}. \] (33)
This result can be integrated:
\[ \int dq_1 (G(q_1|x_1) - \frac{1}{1 + |q_1|^2}) = -\frac{\pi}{4} \log \left( 1 + x_1^2 + g(x_1, z) \right). \] (34)
Using (7) for \( c = 1 \), we recover (31).
We have thus proved that (33) is a solution of the Schwinger-Dyson equation
\[ G(2)(x) = \left( 1 + |x|^2 + 2\lambda \int dq_1 (G(q_1|x_1) - \frac{1}{1 + |q_1|^2}) \right)^{-1}. \] (35)
In the limit \( \lambda \to 0 \), using \( W(x) = \log x - \log \log x + o(1) \) we get

\[ \lim_{\lambda \to 0} \frac{\pi \lambda}{2} W \left( \frac{2}{\pi \lambda} e^{\frac{2(1+x_1^2)}{\pi \lambda}} \right) = 1 + x_1^2, \] (36)
so that
\[ \lim_{\lambda \to 0} G(2)(x) = \frac{1}{1 + |x|^2}, \] (37)
and we recover the free propagator, as expected.

5 Higher-point functions
The boundary graphs of the model with 1 quartic melonic interaction have connected components of the form of figure 3. The 2k-point function SDE with connected boundary graph was derived in the section 6 of [25], taking the large \( N \) limit established in [31], we get
\[ G(2k)\left( X \right) = 2\lambda G(2)\left( x_1^1, x_2^2, x_3^3 \right) \sum_{\rho=2}^{k} G(2k-2\rho+2)(x^\rho, \ldots, x^k) \frac{G(2\rho-2)(x_1^1, \ldots, x^{\rho-1}) - G(2\rho-2)(x_1^\rho, x_2^1, x_3^1, \ldots, x^{\rho-1})}{(x_1^1)^2 - (x_1^\rho)^2}. \] (38)
From the solution (33) of the 2-point function SDE, we can recursively obtain any higher-point function with a connected boundary graph.

The case of disconnected boundary graph is more involved [26] and no general expression of the SDE in the large \( N \) limit have yet been obtained. The simplest equation is the SDE for the 4-point function with disconnected boundary graph, which for only 1 quartic interaction and in the large \( N \) limit reduces to
\[ G^{(4)}_{\text{min}}(x, y) = -2\lambda (G^{(2)}(x))^2 \int dq_2 dq_3 G^{(4)}_{\text{min}}(x_1, q_2, q_3, y). \] (39)
Analysing the perturbative expansion of $G^{(4)}_{m|m}$, we see that there is no contribution at order $\lambda^0$, since the Feynman graph which can contribute (made with two free propagator) is disconnected. Moreover in the appendix of [31], we determined that the first contribution to the perturbative expansion is at order $\lambda^2$ and corresponds to graphs built with 2 different pillow interactions, of the form of the Feynman graph of figure [1]b). In the present case of the model with only 1 quartic melonic interaction, no such graph exists. Then, by plugging an expansion of the form of (5) for the 2- and 4-point functions in (39), we can recursively establish that all order of the perturbative expansion of $G^{(4)}_{m|m}$ are null. Hence, at leading order in the large $N$ limit, $G^{(4)}_{m|m}$ is completely suppressed.

6 Concluding remarks

In this paper we have solved the 2-point function of a tensor field theory with 1 quartic melonic interaction, with building block the Lambert-W function, using a perturbative expansion and a Lagrange-Bürmann resummation. From this result, all higher-point functions with connected boundary graph can be obtained recursively. Moreover, we have shown by a perturbative argument that the 4-point function with a disconnected boundary graph is null at leading order in the large $N$ limit.

A first perspective for this work is the study of higher-point functions with disconnected boundary graph. The general tower of SDE has been determined in [26], now one has to prove the conjecture made in [31], and as in the connected boundary case, take the large $N$ limit of the SDE and solve them. The fact that the 4-point function with a disconnected boundary graph is null may indicate that at least some of the higher-point functions will also be suppressed at leading order in $N$.

Another perspective which appears interesting to us is the study the model with 3 quartic melonic interactions. The perturbative expansion is more involved but other techniques such as the blobbed topological expansion for such tensor model [34] may prove useful.

Acknowledgement

The author would like to thank Raimar Wulkenhaar for his guidance throughout this project, Adrian Tanasa for his advice and comments on the manuscript, and Alexander Hock for helpful discussions. Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044–390685587, Mathematics Münster: Dynamics–
A Recurrence relations

In this section, we will use the recursive equation (9) to determine recurrence relations on the numbers $a_{n,k,m}$. We first perform the integration

$$
\int dq_1 g_p^{(2)}(q_1 x_1) = -\pi/4 \log(1 + x_1^2) \quad \text{if } p = 0,
$$

$$
= \left(\frac{\pi}{2}\right)^{p+1} \frac{\log^p(1 + x_1^2)}{2p(1 + x_1^2)^p} + \frac{(-1)^p}{2(1 + x_1^2)^p} \sum_{m=1}^r \frac{a_{p,m}}{m} \quad \text{if } p > 0,
$$

(41)

where for $p = 1$ the sum on $r$ does not appear. Plugging back the ansatz (17) in the recurrence relation (9) with $c = 1$ gives

$$
G_n^{(2)}(x) = -\frac{2}{|x|^2 + 1} \left\{ -\pi/4 \log(1 + x_1^2) G_{n-1}^{(2)}(x) + \sum_{p=1}^{n-1} \left(\frac{\pi}{2}\right)^{p+1} \frac{\log^p(1 + x_1^2)}{2p(1 + x_1^2)^p} G_{n-p-1}^{(2)}(x) \right\}.
$$

(42)

The first term of (42) gives

$$
\pi \log(1 + x_1^2) \left(\frac{\pi}{2}\right)^{n-1} \frac{\log^{n-1}(1 + x_1^2)}{(1 + |x|^2)^n} + \frac{(-1)^{n-1}}{2(1 + x_1^2)^n} \sum_{k=1}^{n-2} (-1)^k \log^k(1 + x_1^2) \sum_{m=1}^k a_{n-1,k,m} \frac{(1 + x_1^2)^m}{(1 + |x|^2)^{m+1}}
$$

$$
= \left(\frac{\pi}{2}\right)^n \frac{\log^n(1 + x_1^2)}{(1 + |x|^2)^n} + \frac{(-1)^n}{2(1 + x_1^2)^n} \sum_{k=2}^{n-1} (-1)^k \log^k(1 + x_1^2) \sum_{m=2}^k a_{n-1,k-1,m-1} \frac{(1 + x_1^2)^m}{(1 + |x|^2)^{m+1}},
$$

(43)

where we sent $k \to k + 1$ and $m \to m + 1$ to get to the last line. The second term of (42) gives

$$
-\frac{2}{|x|^2 + 1} \left\{ \sum_{p=1}^{n-1} \left(\frac{\pi}{2}\right)^{p+1} \frac{\log^p(1 + x_1^2)}{2p(1 + x_1^2)^p} \left(\frac{\pi}{2}\right)^{n-p} \log^{n-p-1}(1 + x_1^2) \right\}
$$

$$
+ \sum_{p=1}^{n-3} \left(\frac{\pi}{2}\right)^{p+1} \frac{\log^p(1 + x_1^2)}{2p(1 + x_1^2)^p} \left(\frac{\pi}{2}\right)^{n-3} \frac{(-1)^{n-p-1}}{2(1 + x_1^2)^n} \sum_{k=1}^{n-p-2} (-1)^k \log^k(1 + x_1^2) \sum_{m=1}^k a_{n-p-1,k,m} \frac{(1 + x_1^2)^m}{(1 + |x|^2)^{m+1}}
$$

$$
= -\left(\frac{\pi}{2}\right)^n \frac{\log^{n-1}(1 + x_1^2)}{(1 + |x|^2)^n} \sum_{p=1}^{n-3} \frac{(-1)^{n-p}}{p} \frac{1}{(1 + x_1^2)^n} \sum_{k=1}^{n-p-1} (-1)^k \log^k(1 + x_1^2) \sum_{m=1}^k a_{n-p-1,k,m} \frac{(1 + x_1^2)^m}{(1 + |x|^2)^{m+1}} \sum_{p=1}^{n-3} \frac{(-1)^{n-p}}{p} \frac{1}{(1 + x_1^2)^n} \sum_{k=1}^{n-p-1} (-1)^k \log^k(1 + x_1^2) \sum_{m=1}^k a_{n-p-1,k,m} \frac{(1 + x_1^2)^m}{(1 + |x|^2)^{m+1}}
$$

(44)

Setting $r = p + k$ in the line of the previous equation, let us rewrite the double sum as

$$
\sum_{k=1}^{n-3} \sum_{r=k+1}^{n-2} \frac{(-1)^r}{r-k} \log^r(1 + x_1^2) a_{n-r+k-1,k,m} = \sum_{r=2}^{n-2} \sum_{k=1}^{r-1} \frac{(-1)^r}{r-k} \log^r(1 + x_1^2) a_{n-r+k-1,k,m}.
$$

(45)
Then we send $m \to m + 1$ and rewrite double sum to get
\[
\sum_{k=1}^{r-1} \sum_{m=2}^{k+1} a_{n-r+k-1,k,m-1} \frac{(1 + x_1^2)^m}{r - k} = \sum_{m=2}^{r-1} \sum_{k=1}^{m-1} a_{n-r+k-1,k,m-1} \frac{(1 + x_1^2)^m}{r - k}.
\]
Hence, sending $p \to n - p$ and collecting the results we get
\[
- \left( \frac{\pi}{2} \right)^n \log^{n-1}(1 + x_1^2) \sum_{p=1}^{n-1} \frac{1}{n - p} \frac{(1 + x_1^2)^p}{(1 + |x|^2)^{p+1}}
+ \left( \frac{\pi}{2} \right)^n \sum_{r=2}^{n-2} (-1)^r \log^r(1 + x_1^2) \sum_{m=2}^{r-1} \frac{a_{n-r+k-1,k,m-1}}{r - k} \frac{(1 + x_1^2)^m}{(1 + |x|^2)^{m+1}}.
\]
The third term of (42) gives
\[
- \frac{2}{|x|^2 + 1} \left\{ \left( \frac{\pi}{2} \right)^{p+1} \frac{(1 + x_1^2)^p}{(1 + |x|^2)^{p+1}} \sum_{r=1}^{p-1} (-1)^r \log^r(1 + x_1^2) \sum_{m=1}^{r} \frac{a_{p,r,m}}{m} \left( \frac{\pi}{2} \right)^{n-p-1} \log^{n-p-1}(1 + x_1^2)
+ \sum_{p=2}^{n-3} \left( \frac{\pi}{2} \right)^{p+1} \frac{(-1)^p}{(1 + |x|^2)^{p+1}} \sum_{r=1}^{p-1} (-1)^r \log^r(1 + x_1^2) \sum_{m=1}^{r} \frac{a_{p,r,m}}{m} \left( \frac{\pi}{2} \right)^{n-p-1} \log^{n-p-1}(1 + x_1^2)
\right. \left\}.
\]
The first term of equation (48) gives
\[
- \left( \frac{\pi}{2} \right)^n \sum_{p=1}^{n-1} \sum_{r=1}^{p} (-1)^{p+r} \log^{n-p+r-1}(1 + x_1^2) \sum_{m=1}^{r} \frac{a_{p,r,m}}{m} \frac{(1 + x_1^2)^{n-p}}{(1 + |x|^2)^{n-p+1}}
= - \left( \frac{\pi}{2} \right)^{n} \sum_{p=1}^{n-1} \sum_{r=1}^{n-p-r} \frac{(-1)^p}{(1 + |x|^2)^{n-p-r}} \sum_{m=1}^{r} \frac{a_{p,r,m}}{m} \frac{(1 + x_1^2)^{n-p-r-k}}{(1 + |x|^2)^{n-p-r-k+1}}.
\]
by setting $k = p - r$. Then by setting $l = n - 1 - k$ and rewriting the sums we get
\[
\left( \frac{\pi}{2} \right)^{n} \sum_{r=1}^{n-2} (-1)^{r} \log^{r-k-1}(1 + x_1^2) \sum_{m=1}^{r} \frac{a_{n-1+r-k-1,m}}{m} \frac{(1 + x_1^2)^{r-k+1}}{(1 + |x|^2)^{r-k+2}}
= \left( \frac{\pi}{2} \right)^{n} \sum_{r=1}^{n-2} (-1)^r \log^l(1 + x_1^2) \sum_{m=1}^{r} \frac{a_{l+m-1,m}}{m} \frac{(1 + x_1^2)^{l+r+1}}{(1 + |x|^2)^{l+r+2}}.
\]
Then we set $k = l - r + 1$ and obtain
\[
\left( \frac{\pi}{2} \right)^{n} \sum_{r=1}^{n-2} (-1)^r \log^l(1 + x_1^2) \sum_{m=1}^{r} \frac{a_{n-r-k-1,m}}{m} \frac{(1 + x_1^2)^{k}}{(1 + |x|^2)^{k+1}}.
\]
The second term of (48) gives, by rewriting the sums,
\[
\left( \frac{\pi}{2} \right)^{n} \sum_{r=1}^{n-2} (-1)^r \log^{r+k}(1 + x_1^2) \sum_{m=1}^{r} \frac{a_{n-r-k-1,m}}{m} \frac{(1 + x_1^2)^{k}}{(1 + |x|^2)^{k+2}}
= \left( \frac{\pi}{2} \right)^{n} \sum_{r=1}^{n-2} \sum_{k=1}^{n-2} (-1)^{k+r} \log^{k+r}(1 + x_1^2) \sum_{m=1}^{r} \frac{a_{n-r-k-1,m}}{m} \frac{(1 + x_1^2)^{k}}{(1 + |x|^2)^{k+2}}.
\]
First by setting \( q = k + r \) and by several rewriting of the sums we get

\[
\left( \frac{\pi}{2} \right)^n \frac{(-1)^n}{(1 + x_1^2)^n} \sum_{q=k+1}^{n-3} (-1)^q \log^q(1 + x_1^2) \sum_{k=1}^{n-3} \sum_{l=1}^{n-3} \sum_{p=q-k+1}^{n-2-k} \frac{a_{p,q-k,m}}{m} a_{n-p-1,k,l} \frac{(1 + x_1^2)^{l+1}}{(1 + |x|^2)^{l+2}}
\]

\[
= \left( \frac{\pi}{2} \right)^n \frac{(-1)^n}{(1 + x_1^2)^n} \sum_{q=2}^{n-3} (-1)^q \log^q(1 + x_1^2) \sum_{k=1}^{n-3} \sum_{l=1}^{n-3} \sum_{p=q-k+1}^{n-2-k} \frac{a_{p,q-k,m}}{m} a_{n-p-1,k,l} \frac{(1 + x_1^2)^{l+1}}{(1 + |x|^2)^{l+2}}
\]

\[
= \left( \frac{\pi}{2} \right)^n \frac{(-1)^n}{(1 + x_1^2)^n} \sum_{q=2}^{n-3} (-1)^q \log^q(1 + x_1^2) \sum_{k=1}^{n-3} \sum_{l=1}^{n-3} \sum_{p=q-k+1}^{n-2-k} \frac{a_{p,q-k,m}}{m} a_{n-p-1,k,l} \frac{(1 + x_1^2)^{l+1}}{(1 + |x|^2)^{l+2}}
\]

where we send \( l \to l + 1 \) in the last line.

Now collecting all the results we obtain recurrence relations on \( a_{n,k,m} \):

\[
a_{n,1,1} = a_{n-1,1,1},
\]

\[
a_{n,n,1} = \frac{1}{n-1},
\]

\[
a_{n,n-1,m} = \frac{1}{n-m} + a_{n-1,n-2,m-1}, \quad \text{for } m \in \mathbb{Z}, n-1],
\]

\[
a_{n,n-2,m} = a_{n-1,n-3,m-1} + \sum_{r=m}^{n-3} \frac{a_{r+1,r,m-1}}{n-2-r} + \sum_{l=1}^{n-m-1} \frac{a_{m,n-1-m,l+1}}{l},
\]

for \( m \in \mathbb{Z}, n-2],

\[
a_{n,k,1} = \sum_{l=1}^{k} \frac{a_{n-1,k,l}}{l}, \quad \text{for } k \in \mathbb{Z}, n-3],
\]

\[
a_{n,k,m} = a_{n-1,k-1,m-1} + \sum_{r=m-1}^{k-1} \frac{a_{n-1-r-k,r,m-1}}{k-r} + \sum_{l=1}^{k-m+1} \frac{a_{n-m-k,m+1,l}}{l}
\]

\[
+ \sum_{r=m-1}^{k-r} \sum_{l=1}^{n-2-r} \frac{a_{p,k-r,l} a_{n-p-1,r,m-1}}{l}, \quad \text{for } k \in \mathbb{Z}, n-2] \text{ and } m \in \mathbb{Z}, k.
\]

Rewriting these equations gives explicit relations on Stirling numbers of the first kind, harmonic numbers and binomial coefficients. Indeed, from equation (58) we recover

\[
\frac{1}{(n-1)!} \frac{n-1}{n-k} = \sum_{l=1}^{k} \frac{1}{(n-l)!} \frac{n-1-l}{n-1-k}, \quad \text{for } k \in \mathbb{Z}, n-3],
\]

which correspond to the equation (6.21) in [35]. Setting \( l = n-2-r, k = n-m-1 \) and sending \( n-3 \to n \), equation (57) gives

\[
H_k = \frac{k+1}{2n+3-k} \sum_{l=1}^{k} \frac{n+1-k+l}{l(k+1-l)}, \quad \text{for } k \in \mathbb{Z}, n].
\]
Sending \( r \to k - l \) and in the last term \( l \to r \) of equation (59), we get

\[
\begin{align*}
((n - 1)m - k(m - 1)) &\times \frac{(n - 2)!}{k!(n - m)!} \left[ \frac{n - m}{n - k} \right] \\
= &\sum_{l=1}^{k-m+1} \frac{1}{(n - m - l)!} \left[ \frac{n - m - l}{n - k - l} \right] \left( \frac{(n - 1 - m)!}{(k - m + 1)!} + \frac{m - 1}{l} \frac{(n - 1 - 2)!}{(k - l)!} \right) \\
+ &\sum_{l=1}^{k-m+1} \frac{m - 1}{l!(k - l)!} \sum_{p=l+1}^{n-2-k+l} \frac{(p - 1)!}{(n - 2 - p)!} \left[ \frac{n - m - p}{n - k - 1 - p + l} \right] \sum_{r=1}^{l} \frac{1}{(p - r)!} \left[ \frac{p - r}{p - l} \right],
\end{align*}
\]

for \( k \in [2, n - 3] \) and \( m \in [2, k] \).

References

[1] R. Gurau. “Colored Group Field Theory”. Commun. Math. Phys. 304 (2011), pp. 69–93. arXiv:0907.2582 [hep-th]

[2] S. Carrozza and A. Tanasa. “\( O(N) \) Random Tensor Models”. Lett. Math. Phys. 106.11 (2016), pp. 1531–1559. arXiv:1512.06718 [math-ph]

[3] D. Benedetti, S. Carrozza, R. Gurau, and M. Kolanowski. “The 1/N expansion of the symmetric traceless and the antisymmetric tensor models in rank three” (2017). arXiv:1712.00249 [hep-th]

[4] R. Gurau. Random Tensors. Ed. by O. U. Press. 2017.

[5] S. Sachdev and J. Ye. “Gapless spin fluid ground state in a random, quantum Heisenberg magnet”. Phys. Rev. Lett. 70 (1993). arXiv:cond-mat/9212030, p. 3339.

[6] A. Kitaev. “A simple model of quantum holography”. talk at KITP.

[7] J. Maldacena and D. Stanford. “Remarks on the Sachdev-Ye-Kitaev model”. Phys. Rev. D94.10 (2016), p. 106002. arXiv:1604.07818 [hep-th].

[8] D. J. Gross and V. Rosenhaus. “All point correlation functions in SYK”. JHEP 12 (2017), p. 148. arXiv:1710.08113 [hep-th].

[9] E. Witten. “An SYK-Like Model Without Disorder” (2016). arXiv:1610.09758 [hep-th].

[10] I. R. Klebanov and G. Tarnopolsky. “Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models”. Phys. Rev. D95.4 (2017), p. 046004. arXiv:1611.08915 [hep-th]

[11] R. Gurau. “The complete 1/N expansion of a SYK–like tensor model”. Nucl. Phys. B916 (2017), pp. 386–401. arXiv:1611.04032 [hep-th]

[12] V. Bonzom, L. Lionni, and A. Tanasa. “Diagrammatics of a colored SYK model and of an SYK-like tensor model, leading and next-to-leading orders”. J. Math. Phys. 58.5 (2017), p. 052301. arXiv:1702.06944 [hep-th]

[13] I. R. Klebanov, F. Popov, and G. Tarnopolsky. “TASI Lectures on Large N Tensor Models”. 2018. arXiv:1808.09434 [hep-th]

[14] N. Delporte and V. Rivasseau. “The Tensor Track V: Holographic Tensors”. 2018. arXiv:1804.11101 [hep-th].
[32] J. L. Lagrange. “Nouvelle méthode pour résoudre des équations littérales par le moyen de séries”. Mém. Acad. Roy. des Sci. et Belles-Lettres de Berlin 24 (1770).

[33] H. Bürmann. “Essai de calcul fonctionnaire aux constantes ad-libitum”. Mem.Inst. Nat. Sci Arts. Sci. Math. Phys. 2 (1799), pp. 316–347.

[34] V. Bonzom and S. Dartois. “Blobbed topological recursion for the quartic melonic tensor model”. J. Phys. A51.32 (2018), p. 325201. arXiv:1612.04624 [hep-th].

[35] R. L. Graham, D. E. Knuth, and O. Patashnik. Concrete Mathematics: A Foundation for Computer Science. 2nd. 1994.