Random Attractor For Stochastic Lattice FitzHugh-Nagumo System Driven By $\alpha$-stable Lévy Noises

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Abstract: The present paper is devoted to the existence of a random attractor for stochastic lattice FitzHugh-Nagumo system driven by $\alpha$-stable Lévy noises under some dissipative conditions.

Keywords: Synchronization; Lévy noise; Skorohod metric; random attractor; càdlàg random dynamical system.

1 Introduction

We consider the following stochastic lattice FitzHugh-Nagumo system (SLFNS)

\[
\begin{align*}
\frac{du_i}{dt} &= u_{i-1} - 2u_i + u_{i+1} - \lambda u_i + f_i(u_i) - v_i \\
&\quad + h_i + \sum_{j=1}^{N} \varepsilon_j u_i \diamond \frac{dL_j}{dt}, \\
\frac{dv_i}{dt} &= \varpi v_i + g_i + \sum_{j=1}^{N} \varepsilon_j v_i \diamond \frac{dL_j}{dt}, \\
u(0) &= u_0 = (u_{i0})_{i \in \mathbb{Z}}, v(0) = v_0 = (v_{i0})_{i \in \mathbb{Z}}
\end{align*}
\]

(1.1)

where $\mathbb{Z}$ denotes the integer set, $u_i \in \mathbb{R}$, $\lambda, \varpi$ are positive constants, $h_i, g_i \in \mathbb{R}$, $f_i$ are smooth functions satisfying some dissipative conditions, $\varepsilon_j \in \mathbb{R}$ for $j = 1, \ldots, N$, $L^j$ are mutually independent $\alpha$-stable Lévy motions ($1 < \alpha < 2$), and $\diamond$ denotes the Marcus sense in the stochastic term, $\frac{d}{dt}$ is right-hand derivative of $\cdot(t)$ at $t$, $\ell^2 = (\ell^2, \langle \cdot, \cdot \rangle, \| \cdot \|)$ denotes the regular space of infinite sequences.

As we all known, noises involved in realistic systems will play an important role as intrinsic phenomena rather than just compensation of defects in deterministic models. Stochastic lattice dynamical systems (SLDS) arise naturally in a wide variety of applications where the spatial structure has a discrete character and random influences or uncertainties are taken into account. For the recent research of SLDS, we can see e.g. [Bates et al.(2006), Huang(2007), Caraballo & Lu (2008), Zhao & Zhou(2009), Han et al.(2011)] for the first- or second-order lattice dynamical systems with white noises in regular (or weight) space of infinite sequences, see

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e.g. [Gu (2013), Gu & Li (2013)] for the first-order lattice dynamical systems driven by fractional Brownian motions, see [Gu & Ai (2014)] for the first-order lattice dynamical systems with non-Gaussian noises.

When there are no noises terms, form similar to (1.1) is the discrete of the FitzHugh-Nagumo system which arose as modeling the signal transmission across axons in neurobiology (see [Jones (1984)]). Lattice FitzHugh-Nagumo system was used to stimulate the propagation of action potentials in myelinated nerve axons (see [Elmer & Van Vleck (2005)]). Gaussian processes like Brownian motion have been widely used to model fluctuations in engineering and science. When lattice FitzHugh-Nagumo system perturbed by additive or multiplicative white noises, the existence of random attractors has been proved in [Huang (2007), Gu et al. (2012)].

To the best of our knowledge, there are no results on the system when it is perturbed by a non-Gaussian noise (in terms of Lévy noise).

In fact, some complex phenomena involve non-Gaussian fluctuations with peculiar properties such as anomalous diffusion (mean square displacement is a nonlinear power law of time) [Bouchaud & Georges (1990)] and heavy tail distribution (non-exponential relaxation) [Yoneyzawa (1996)]. For this topic, we can refer to [Shlesinger et al. (1995), Scher et al. (1991), Herrchen (2001), Ditllevsen (1999)] for more details. A Lévy motion $L_t$ is a non-Gaussian process with independent and stationary increments, i.e., increments $\Delta L_t = L_{t+\Delta t} - L_t$ are stationary and independent for any non overlapping time lags $\Delta t$. Moreover, its sample paths are only continuous in probability, namely, $\mathbb{P}(|L_t - L_{t_0}| \geq \epsilon) \to 0$ as $t \to t_0$ for any positive $\epsilon$. With a suitable modification, these path may be taken as càdlàg, i.e., paths are continuous on the right and have limits on the left. This continuity is weaker than the usual continuity in time. Indeed, a càdlàg function has at most countably many discontinuities on any time interval, which generalizes the Brownian motion to some extent (see e.g. [Applebaum (2004)]). As a special case of Lévy processes, the symmetric $\alpha$-stable Lévy motion plays an important role among stable processes just like Brownian motion among Gaussian processes. A stochastic process $\{L_t, t \geq 0\}$ is called the $\alpha$-stable Lévy motions if (i) $L_0 = 0$ a.e., (ii) $L$ has independent increments, and (iii) $L_t - L_s \sim \mathbb{S}_\alpha((t - s)^{\frac{1}{\alpha}}, \beta, 0)$ for $0 \leq s < t < \infty$ and for some $0 < \alpha \leq 2, -1 \leq \beta \leq 1$, where $\mathbb{S}_\alpha(\sigma, \beta, \nu)$ denotes the $\alpha$-stable distribution with index of stability $\alpha$, scale parameter $\sigma$, skewness parameter $\beta$ and shift parameter $\nu$; in particular, $\mathbb{S}_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2)$ denotes the Gaussian distribution. For more details on $\alpha$-stable distributions, we can refer to [Sato (1999)]. It is worth mentioning that when $\alpha = 2$, we have the standard Brownian motion, which the Marcus sense stochastic terms (see e.g. [Marcus (1981)]) reduce to the Stratonovich stochastic terms and the existence of a random attractor for system (1.1) has been considered in [Gu et al. (2012)]. For the further development on Lévy motions, we can refer to the recent
monographs Applebaum (2004), Peszat & Zabczyk (2007).

The goal of this article is to establish the existence of a random attractor for SLFNS with the nonlinearity $f_i$ under some dissipative conditions and driven by $\alpha$-stable Lévy noises with $\alpha \in (1, 2)$. By virtue of an Ornstein-Uhlenbeck process with a stationary solution, we transform system (1.1) into a conjugated random integral equation (with a solution in the sense of Carathéodory). Here, we assume that $1 < \alpha < 2$ since this is the only case where the solutions of the Ornstein-Uhlenbeck equations for $\alpha$-stable Lévy noises are stationary, which is vital to our purpose. For the case of $0 < \alpha < 1$, there will be a new challenge for us for future research.

The paper is organized as follows. In Sec. 2, we recall some basic concepts in random dynamical systems. In Sec. 3, we give a unique solution to system (1.1) and make sure that the solution generates a random dynamical system. We establish the main result, that is, the existence of a random attractor generated by system (1.1) in Sec. 4.

2 Random dynamical systems and random attractors

For the reader’s convenience, we introduce some basic concepts related to random dynamical systems and random attractors, which are taken from Arnold(1998), Chueshov(2002), Han et al.(2011). Let $(\mathbb{E}, \| \cdot \|_\mathbb{E})$ be a separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

**Definition 2.1.** A stochastic process $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ is a continuous random dynamical system (RDS) over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ if $\varphi$ is $(\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(\mathbb{E}), \mathcal{B}(\mathbb{E}))$-measurable, and for all $\omega \in \Omega$,

(i) the mapping $\varphi(t, \omega) : \mathbb{E} \mapsto \mathbb{E}$, $x \mapsto \varphi(t, \omega)x$ is continuous for every $t \geq 0$,

(ii) $\varphi(0, \omega)$ is the identity on $\mathbb{E}$,

(iii) (cocycle property) $\varphi(s + t, \omega) = \varphi(t, \theta_s \omega)\varphi(s, \omega)$ for all $s, t \geq 0$.

**Definition 2.2.** (i) A set-valued mapping $\omega \mapsto B(\omega) \subset \mathbb{E}$ (we may write it as $B(\omega)$ for short) is said to be a random set if the mapping $\omega \mapsto \text{dist}_\mathbb{E}(x, B(\omega))$ is measurable for any $x \in \mathbb{E}$, where $\text{dist}_\mathbb{E}(x, D)$ is the distance in $\mathbb{E}$ between the element $x$ and the set $D \subset \mathbb{E}$.

(ii) A random set $B(\omega)$ is said to be bounded if there exist $x_0 \in \mathbb{E}$ and a random variable $r(\omega) > 0$ such that $B(\omega) \subset \{x \in \mathbb{E} : \|x - x_0\|_\mathbb{E} \leq r(\omega), x_0 \in \mathbb{E}\}$ for all $\omega \in \Omega$.

(iii) A random set $B(\omega)$ is called a compact random set if $B(\omega)$ is compact for all $\omega \in \Omega$.

(iv) A random bounded set $B(\omega) \subset \mathbb{E}$ is called tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$, $\lim_{t \to +\infty} e^{-\gamma t} d(B(\theta_{-t} \omega)) = 0$ for all $\gamma > 0$, where $d(B) = \sup_{x \in B} \|x\|_\mathbb{E}$. A random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$, $\lim_{t \to +\infty} \sup_{\theta \in \mathbb{R}} e^{-\gamma t} r(\theta_{-t} \omega) = 0$ for all $\gamma > 0$. 

3
We consider an RDS \( \{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega} \) over \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \) and \( \mathcal{D}(\mathbb{E}) \) the set of all tempered random sets of \( \mathbb{E} \).

**Definition 2.3.** A random set \( K \) is called an absorbing set in \( \mathcal{D}(\mathbb{E}) \) if for all \( B \in \mathcal{D}(\mathbb{E}) \) and a.e. \( \omega \in \Omega \) there exists \( t_B(\omega) > 0 \) such that

\[
\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega) \quad \text{for all } t \geq t_B(\omega).
\]

**Definition 2.4.** A random set \( A \) is called a global random \( \mathcal{D}(\mathbb{E}) \) attractor (pullback \( \mathcal{D}(\mathbb{E}) \) attractor) for \( \{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega} \) if the following hold:

(i) \( A \) is a random compact set, i.e. \( \omega \mapsto d(x, A(\omega)) \) is measurable for every \( x \in \mathbb{E} \) and \( A(\omega) \) is compact for a.e. \( \omega \in \Omega \);

(ii) \( A \) is strictly invariant, i.e. for \( \omega \in \Omega \) and all \( t \geq 0 \), \( \varphi(t, \omega)A(\omega) = A(\theta_t\omega) \);

(iii) \( A \) attracts all sets in \( \mathcal{D}(\mathbb{E}) \), i.e. for all \( B \in \mathcal{D}(\mathbb{E}) \) and a.e. \( \omega \in \Omega \), we have

\[
\lim_{t \to +\infty} d(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) = 0,
\]

where \( d(X, Y) = \sup_{x \in X} \inf_{y \in Y} \| x - y \|_2 \) is the Hausdorff semi-metric \((X \subseteq \mathbb{E}, Y \subseteq \mathbb{E})\).

**Proposition 2.5.** (See [Han et al. (2011)] ) Suppose that

(a) there exists a random bounded absorbing set \( K(\omega) \in \mathcal{D}(\ell^2), \omega \in \Omega \), such that for any \( B(\omega) \in \mathcal{D}(\ell^2) \) and all \( \omega \in \Omega \), there exists \( T(\omega, B) > 0 \) yielding \( \varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega) \) for all \( t \geq T(\omega, B) \);

(b) the RDS \( \{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega} \) is random asymptotically null on \( K(\omega) \), i.e., for any \( \epsilon > 0 \), there exist \( T(\epsilon, \omega, K) > 0 \) and \( I_0(\epsilon, \omega, K) \in \mathbb{N} \) such that

\[
\sup_{u \in K(\omega), |t| > I_0(\epsilon, \omega, K(\omega))} |\varphi(t, \theta_{-t}u, \omega)|^2 \leq \epsilon^2, \quad \forall t \geq T(\epsilon, \omega, K(\omega)). \tag{2.1}
\]

Then the RDS \( \{\varphi(t, \omega, \cdot)\}_{t \geq 0, \omega \in \Omega} \) possesses a unique global random \( \mathcal{D}(\ell^2) \) attractor given by

\[
\hat{A}(\omega) = \bigcap_{t \geq T(\omega, K)} \bigcup_{t \geq T(\omega, K)} \varphi\left(t, \theta_{-t}\omega, K(\theta_{-t}\omega)\right). \tag{2.2}
\]

3 SLFNS driven by \( \alpha \)-stable Lévy noises

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, where \( \Omega = \mathcal{S}(\mathbb{R}, \ell^2) \) with Skorokhod metric as the canonical sample space of càdlàg functions defined on \( \mathbb{R} \) and taking values in \( \ell^2 \), \( \mathcal{F} := \mathcal{B}(\mathcal{S}(\mathbb{R}, \ell^2)) \) the associated Borel \( \sigma \)-field and \( \mathbb{P} \) is the corresponding (Lévy) probability measure on \( \mathcal{F} \) which is given by the distribution of a two-sided Lévy process with paths in \( \mathcal{S}(\mathbb{R}, \ell^2) \), i.e. \( \omega(t) = L_t(\omega) \).
Let $\theta_t \omega (\cdot) = \omega (\cdot + t) - \omega (t), t \in \mathbb{R}$, then the mapping $(t, \omega) \to \theta_t \omega$ is continuous and measurable (see [Arnold (1998)]), and the (Lévy) probability measure is $\theta$-invariant, i.e. $\mathbb{P}(\theta_t^{-1}(\hat{A})) = \mathbb{P}(\hat{A})$ for all $\hat{A} \in \mathcal{F}$ (see [Applebaum (2004)]).

For convenience, we now formulate system (1.1) as a stochastic differential equation in $\ell^2 \times \ell^2$. For $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, define $A, B, B^*$ to be linear operators from $\ell^2$ to $\ell^2$ as follows:

$$(Au)_i = -u_{i-1} + 2u_i - u_{i+1},$$
$$(Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i, \quad i \in \mathbb{Z}.$$ 

It is easy to show that $A = BB^* = B^*B$, $(B^*u, u') = (u, Bu')$ for all $u, u' \in \ell^2$, which implies that $(Au, u) \geq 0$.

Let $f_i \in C(\mathbb{R})$ satisfy the conditions that $\sup_{i \in \mathbb{Z}} |f'(u)|$ is bounded for $u$ in bounded sets and $f_i(x)x \geq 0$ for all $x \in \mathbb{R}$. Let $\hat{f}$ be the Nemytski operator associated with $f_i$, for $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, then $\hat{f}(u) \in \ell^2$ and $\hat{f}$ is locally Lipschitz from $\ell^2$ to $\ell^2$ (see [Bates et al. (2006), Caraballo & Lu (2008)]). In the sequel, when no confusion arises, we identify $\hat{f}$ with $f$.

Let $E = \ell^2 \times \ell^2$, for $\Psi = (u, v) \in E$, denote the norm $\|\Psi\|^2 := \|\Psi\|_E^2 = \|u\|^2 + \|v\|^2$. Then system (1.1) can be interpreted as a system of integral equations in $E$ for $t \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\begin{cases}
  u(t) = u(0) + \int_0^t (-Au(s) - \lambda u(s) + f(u(s)) - v(s) + h)ds \\
  + \sum_{j=1}^N \int_0^t \varepsilon_j u(s) \circ dL^j_t, \\
  v(t) = v(0) + \int_0^t (qu(s) - \omega v(s) + g)ds \\
  + \sum_{j=1}^N \int_0^t \varepsilon_j v(s) \circ dL^j_t,
\end{cases}
$$

(3.1)

where the stochastic integral is understood to be in the Marcus sense.

To prove that this stochastic equation (3.1) generates a random dynamical system, we will transform it into a random differential equation in $E$. Now, we introduce the Ornstein-Uhlenbeck processes in $\ell^2$ on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ given by the random variable

$$
\begin{align*}
  z(\theta_t \omega) & = - \int_{-\infty}^0 e^{s \theta_t} \omega(s)ds, \quad t \in \mathbb{R}, \omega \in \Omega.
\end{align*}
$$

(3.2)

The above integrals exist in the sense of any path with a subexponential growth, and $z$ solves the following Ornstein-Uhlenbeck equation

$$
\begin{align*}
dz + zdt = dL_t, \quad t \in \mathbb{R}.
\end{align*}
$$

(3.3)

In fact, we have the following properties (see Lemma 3.1 in [Gu & Ai (2014)]): (i) There exists a $\{\theta_t\}_{t \in \mathbb{R}}$-invariant subset $\bar{\Omega} \in \mathcal{F}$ of full measure for a.e. $\omega \in \bar{\Omega}$, the random variable

$$
\begin{align*}
z(\omega) & = - \int_{-\infty}^0 e^s \omega(s)ds,
\end{align*}
$$

(3.4)

For convenience, we now formulate system (1.1) as a stochastic differential equation in $\ell^2 \times \ell^2$. For $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, define $A, B, B^*$ to be linear operators from $\ell^2$ to $\ell^2$ as follows:
is well defined and the unique stationary solutions of (3.3) is given by (3.2). Moreover, the mapping \( t \to z(\theta, \omega) \) is càdlàg; (ii) For \( \omega \in \bar{\Omega} \), the sample paths \( \omega(t) \) of \( L_t \) satisfy

\[
\lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0, \quad t \in \mathbb{R}
\]

and

\[
\lim_{t \to \pm \infty} \frac{|z(\theta, \omega)|}{|t|} = \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t z(\theta, \omega(s)) ds = 0.
\]

Now, let \( z_j \) be the associated Ornstein-Uhlenbeck process corresponding to (3.3) with \( L_t^j \) instead of \( L_t \) and denote \( \Lambda(\omega) = e^{\sum_{j=1}^N \varepsilon_j z_j(\omega)} \text{Id}_E \), then \( \Lambda(\omega) \) is clearly a homeomorphism in \( E \) and the inverse operator is well defined by \( \Lambda^{-1}(\omega) = e^{-\sum_{j=1}^N \varepsilon_j z_j(\omega)} \text{Id}_E \). It is easy to verify that \( \|\Lambda^{-1}(\theta, \omega)\| \) has sub-exponential growth as \( t \to \pm \infty \) for \( \omega \in \Omega \). Hence \( \|\Lambda^{-1}\| \) is tempered. Since the mapping of \( \theta \) on \( \bar{\Omega} \) has the same properties as the original one if we choose the trace \( \sigma \)-algebra with respect to \( \bar{\Omega} \) to be denoted also by \( \mathcal{F} \), we can change our metric dynamical system with respect to \( \bar{\Omega} \), and still denoted the symbols by \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \).

Denote \( \xi(\theta_t, \omega) = \sum_{j=1}^N \varepsilon_j z_j(\theta_t, \omega) \), and consider the change in variables

\[
(U(t), V(t)) = \Lambda^{-1}(\theta_t, \omega)(u(t), v(t))
\]

\[
= e^{-\xi(\theta_t, \omega)}(u(t), v(t)),
\]

where \((u, v)\) is the solution of (3.1), then we get the evolution equations with random coefficients but without white noise

\[
\begin{cases}
\frac{dU}{dt} = -\Lambda U - (\lambda - \xi(\theta_t, \omega))U + e^{-\xi(\theta_t, \omega)} f(e^{\xi(\theta_t, \omega)} U) - V + e^{-\xi(\theta_t, \omega)} h, \\
\frac{dV}{dt} = \rho U - (\xi(\theta_t, \omega))V + e^{-\xi(\theta_t, \omega)} g,
\end{cases}
\]

(3.4)

and initial condition \((U(0), V(0)) = (U_0, V_0) \in \mathbb{E}\).

Now, we have the following result:

**Theorem 3.1.** Let \( T > 0 \) and \( \Psi_0 = (U_0, V_0) \in \mathbb{E} \) be fixed, then the following statements hold:

(i) For every \( \omega \in \Omega \), system (3.2) has a unique solution \( \Psi(\cdot, \omega, \Psi_0) = (U(\cdot, \omega, U_0), V(\cdot, \omega, V_0)) \in C([0, T], \mathbb{E}) \) in the sense of Carathéodory.

(ii) For each \( \omega \in \Omega \), the mapping \( \Psi_0 \in \mathbb{E} \mapsto \Psi(\cdot, \omega, \Psi_0) \in C([0, T], \mathbb{E}) \) is continuous, which implies the solution \( \Psi \) of (3.2) continuously depends on the initial data \( \Psi_0 \).

(iii) Equation (3.3) generates a continuous RDS \((\varphi(t))_{t \geq 0}\) over \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\), where \( \varphi(t, \omega, \Psi_0) = \Psi(t, \omega, \Psi_0) \) for \( \Psi_0 \in \mathbb{E}, t \geq 0 \) and for all \( \omega \in \Omega \). Moreover, \( \psi(t, \omega, \Psi_0) = \Lambda(\theta_t, \omega)\varphi(t, \omega, \Lambda^{-1}(\omega)\Psi_0) \) for \( \Psi_0 \in \mathbb{E}, t \geq 0 \) and for all \( \omega \in \Omega \), then \( \psi \) is another RDS for which the process \((\omega, t) \to (\psi(t, \omega, \Psi_0))\) solves (3.1) for any initial condition \( \Psi_0 \in \mathbb{E} \).
Proof. (i) Let \( F(t, U) = e^{-\xi(\theta t \omega)} f(e^{\xi(\theta t \omega)} U) \), for any fixed \( T > 0 \) and \( \Psi_0 \in E \) and let \( U_1, U_2 \in Y \), where \( Y \) is a bounded set in \( E \), we have

\[
\| F(t, U_1) - F(t, U_2) \|
\leq C_Y \| v_1 - v_2 \|
\]

where \( C_Y \) is a constant only depending on \( Y \). This implies that the mapping \( F(t, U) \) is locally Lipschitz with respect to \( U \) and the Lipschitz constant is uniformly bounded in \([0, T]\). By the standard arguments, we know that (3.4) possesses a local solution \( \Psi(t, \omega, \Psi_0) \in C([0, T_{\text{max}}), E) \), where \([0, T_{\text{max}})\) is the maximal interval of existence of the solution of (3.4). Next, we need to show that the local solution is a global one. By taking the inner products of \( U \) and \( V \) respectively in \( \ell^2 \) with the two equations in system (3.4), we have

\[
\frac{d}{dt} (\| U \|^2 + \frac{1}{\varrho} \| V \|^2) 
\leq -(\delta - 2\xi(\theta t \omega)) (\| U \|^2 + \frac{1}{\varrho} \| V \|^2) 
\]

\[+ \frac{1}{\delta} (\| h \|^2 + \frac{1}{\varrho} \| g \|^2) e^{-2\xi(\theta t \omega)}, \tag{3.5}\]

where \( \delta = \min \{\lambda, \varpi\} \). By virtue of the special Gronwall lemma (see Lemma 2.8 in [Robinson (2001)]), it yields that

\[
\| \Psi(t) \|^2 \leq e^{-\delta t + \frac{1}{\delta} \int_0^t \xi(\theta r \omega) dr} \| \Psi_0 \|^2 + c_1 (\| h \|^2 + \| g \|^2) 
\cdot e^{-\delta t + \frac{1}{\delta} \int_0^t \xi(\theta r \omega) dr} \int_0^t e^{-2\xi(\theta r \omega) + \delta s - 2 \int_0^s \xi(\theta r \omega) dr} ds,
\]

where \( c_1 = \frac{\max(1, \frac{1}{\varrho})}{\delta \min(1, \frac{1}{\varrho})} \). Denote

\[
a(\omega) = 2 \int_0^T \| \xi(\theta s \omega) \| ds
\]

and

\[
b(\omega) = c_1 \max_{t \in [0, T]} \{ (\| h \|^2 + \| g \|^2) 
\cdot e^{-\delta t + \frac{1}{\delta} \int_0^t \xi(\theta r \omega) dr} \int_0^t e^{-2\xi(\theta r \omega) + \delta s - 2 \int_0^s \xi(\theta r \omega) dr} ds \}.
\]

Due to the properties of the Ornstein-Uhlenbeck process, we know that \( a(\omega), b(\omega) \) are well-defined. Then we have

\[
\| \Psi(t) \|^2 \leq \| \Psi_0 \|^2 e^{a(\omega)} + b(\omega),
\]

which implies that the solution \( \Psi \) is defined in any interval \([0, T]\).
(ii) Let \( \Phi_0 = (\bar{U}_0, \bar{V}_0), \Psi_0 = (U_0, V_0) \in E \), and \( \Phi(t) := (\bar{U}(t, \omega, \bar{U}_0), \bar{V}(t, \omega, \bar{V}_0)), \Psi(t) := (U(t, \omega, U_0), V(t, \omega, V_0)) \) be two solutions of (3.4). By denoting \( Z(t) = \Phi(t) - \Psi(t) \), we have

\[
\frac{d}{dt} \|Z(t)\|^2 
\leq 2e^{-\xi(\theta_t \omega)}\|f(e^{\xi(\theta_t \omega)}\Phi(t)) - f(e^{\xi(\theta_t \omega)}\Psi(t))\|\|Z\|
+ 2\xi(\theta_t \omega)\|Z\|^2
\leq 2(L Y' + \xi(\theta_t \omega))\|Z\|^2 \leq \kappa\|Z\|^2,
\]

where \( \kappa = 2(L Y' + \max_{t \in [0, T]} |\xi(\theta_t \omega)|) \) is well-defined, and \( L Y' \) denotes the Lipschitz constant of \( f \) corresponding to a bounded set \( Y' \in E \) where \( \Phi \) and \( \Psi \) belong to. By the Gronwall lemma again, we obtain

\[
\|Z(t)\|^2 \leq e^{\kappa t}\|Z(0)\|^2,
\]

and consequently

\[
\sup_{t \in [0, T]} \|\Phi(t) - \Psi(t)\|^2 \leq e^{\kappa T}\|\Phi_0 - \Psi_0\|^2.
\]

If \( \Phi_0 = \Psi_0 \), then the above inequality indicates that the uniqueness and continuous dependence on the initial data of the solutions of (3.4).

(iii) The continuity of \( \varphi \) is due to (i) and (ii). The measurability of \( \psi \) follows from the properties of \( \Lambda \). Here, we only remain to prove the conjugacy between \( \varphi \) and \( \psi \). The verification by chain rule is routine and thus be omitted. The proof is complete.

\[ \square \]

4 Existence of a global random attractor

In this section, we will prove the existence of a global random attractor for system (1.1). Since the random dynamical systems \( \varphi \) and \( \psi \) are conjugated, we only have to consider the RDS \( \varphi \).

Firstly, we have our main result

**Theorem 4.1.** The SLDS \( \varphi \) generated by system (3.4) has a unique global random attractor.

In order to prove Theorem 4.1, we will use Proposition 2.5. We first need to prove there exists an absorbing set for \( \varphi \) in \( D(E) \). Next, we will show the RDS \( \varphi \) is random asymptotically null in the sense of (2.1).

**Lemma 4.2.** There exists a closed random tempered set \( K(\omega) \in D(E) \) such that for all \( B \in D(E) \) and a.e. \( \omega \in \Omega \) there exists \( t_B(\omega) > 0 \) such that

\[
\varphi(t, \theta_{-t} \omega)B(\theta_{-t} \omega) \subset K(\omega) \quad \text{for all} \quad t \geq t_B(\omega).
\]

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Proof. Let us start with \( \Psi(t) = \varphi(t, \omega, \Psi_0) \). Then by (3.5), we have
\[
\| \varphi(t) \|^2 \leq e^{-\delta t + 2 \int_0^t \xi(\theta_s) ds} \| \Psi_0 \|^2 + c_1(\| h \|^2 + \| g \|^2)
\]
\[
\cdot e^{-\delta t + 2 \int_0^t \xi(\theta_s) ds} \int_0^t \int_0^s e^{-2 \xi(\theta_s) + \delta s - 2 \int_0^s \xi(\theta_r) dr} ds.
\]
Now, by replacing \( \omega \) with \( \theta_{-t} \omega \) and \( \Psi_0 \) with \( e^{-\xi(\theta_{-t})} \Psi_0 \), respectively, in the expression \( \varphi \), we obtain
\[
\| \varphi(t, \theta_{-t} \omega, e^{-\xi(\theta_{-t})} \Psi_0) \|^2 \leq e^{-\delta t + 2 \int_0^t \xi(\theta_s) ds} \| e^{-\xi(\theta_{-t})} \Psi_0 \|^2
\]
\[
+ c_1(\| h \|^2 + \| g \|^2) e^{-\delta t + 2 \int_0^t \xi(\theta_s) ds} \int_0^t \int_0^s e^{-2 \xi(\theta_s) + \delta s - 2 \int_0^s \xi(\theta_r) dr} ds.
\]
By the properties of the Ornstein-Uhlenbeck process, we know that
\[
\int_0^t e^{-2 \xi(\theta_s) + \delta s - 2 \int_0^s \xi(\theta_r) dr} ds < +\infty.
\]
Consider for any \( \Psi_0 \in B(\theta_{-t} \omega) \), we have
\[
\| \varphi(t, \theta_{-t} \omega, e^{-\xi(\theta_{-t})} \Psi_0) \|^2 \leq e^{-\delta t - 2 \int_0^t \xi(\theta_s) ds} d(B(\theta_{-t} \omega))^2
\]
\[
+ c_1(\| h \|^2 + \| g \|^2) \int_{-\infty}^0 e^{-2 \xi(\theta_s) + \delta s - 2 \int_0^s \xi(\theta_r) dr} ds.
\]
Note that
\[
\lim_{t \to +\infty} e^{-\delta t - 2 \int_0^t \xi(\theta_s) ds} d(B(\theta_{-t} \omega))^2 = 0,
\]

and denote
\[ R^2(\omega) = 1 + c_1(\|h\|^2 + \|g\|^2) \cdot \int_{-\infty}^{0} e^{-2\xi(\theta,\omega)} + \int_{0}^{\infty} e^{-2\xi(\theta,\omega)}d\xi \]

we conclude that
\[ \mathcal{K}(\omega) = \mathcal{B}_\mathcal{E}(0, R(\omega)) \] (4.2)
is an absorbing closed random set. It remains to show that \( \mathcal{K}(\omega) \in \mathcal{D}(\mathcal{E}) \). Indeed, from Definition 2.2 (iv), for all \( \gamma > 0 \), we get
\[ e^{-\gamma t} R^2(\theta_t \omega) = e^{-\gamma t} + c_1 e^{-\gamma t} (\|h\|^2 + \|g\|^2) \cdot \int_{-\infty}^{0} e^{-2\xi(\theta,\omega)} + \int_{0}^{\infty} e^{-2\xi(\theta,\omega)}d\xi \to 0 \]
as \( t \to \infty \),
which completes the proof.

**Lemma 4.3.** Let \( \Psi_0(\omega) \in \mathcal{K}(\omega) \) be the absorbing set given by (4.2). Then for every \( \epsilon > 0 \), there exist \( \tilde{T}(\epsilon, \omega, \mathcal{K}(\omega)) > 0 \) and \( \tilde{N}(\epsilon, \omega, \mathcal{K}(\omega)) > 0 \), such that the solution \( \varphi \) of problem (3.4) is random asymptotically null, that is, for all \( t \geq \tilde{T}(\epsilon, \omega, \mathcal{K}(\omega)) \),
\[ \sup_{\Psi \in \mathcal{K}(\omega)} \sum_{|i| > \tilde{N}(\epsilon, \omega, \mathcal{K}(\omega))} |\varphi_i(t, \theta_{-t \omega}, \Psi(\theta_{-t \omega}))|^2 \leq \epsilon^2. \]

**Proof.** Choose a smooth cut-off function satisfying \( 0 \leq \rho(s) \leq 1 \) for \( s \in \mathbb{R}^+ \) and \( \rho(s) = 0 \) for \( 0 \leq s \leq 1 \), \( \rho(s) = 1 \) for \( s \geq 2 \). Suppose there exists a positive constant \( c_0 \) such that \( |\rho'(s)| \leq c_0 \) for \( s \in \mathbb{R}^+ \).

Let \( N \) be a fixed integer which will be specified later, set \( x = (\rho(\frac{|i|}{N}) U_i)_{i \in \mathbb{Z}} \) and \( y = (\rho(\frac{|i|}{N}) V_i)_{i \in \mathbb{Z}} \). Then take the inner product of the two equations in system (3.4) with \( x \) and \( y \) in \( \ell^2 \), respectively, and combine the following two inequalities
\[ (AU, x) = (\tilde{B}U, \tilde{B}x) \geq -\frac{2c_0}{N} \|U\|^2 \geq -\frac{2c_0}{N} \|\varphi\|^2, \]
and
\[ -\infty < -2e^{-\xi(\theta_1 \omega)} \sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N}) f(i \xi(\theta_1 \omega)) U_i \leq 0, \]
Due to the properties of the Ornstein-Uhlenbeck process, there exists a

\[
\frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i|^2 + (\delta - 2\xi(\theta t\omega)) \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i|^2 \leq \frac{c_2}{N} \|\varphi(t, \omega, e^{-\xi(\omega)}\Psi_0)\|^2 + c_1 e^{-\xi(\theta t\omega)} \sum_{|i| \geq N} (|h_i|^2 + |g_i|^2),
\]

where \( c_2 = \frac{4c_0}{\min\{1, \rho\}} \). By using the Gronwall lemma, for \( t \geq T_K = T_K(\omega) \), it follows that

\[
\sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i(t, \omega, e^{-\xi(\omega)}\Psi_0(\omega))|^2 \leq e^{-\delta(t-T_K) + 2 \int_{T_K}^t \xi(\theta s\omega)ds} \cdot \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i(t, \omega, e^{-\xi(\omega)}\Psi_0(\omega))|^2 \tag{4.3}
\]

\[
+ c_2 \int_{T_K}^t e^{-\delta(t-s) + 2 \int_s^t \xi(\theta \tau\omega)ds} \cdot \|\varphi(\tau, \omega, e^{-\xi(\omega)}\Psi_0(\omega))\|^2 d\tau \tag{4.4}
\]

\[
+ c_1 \sum_{|i| \geq N} (|h_i|^2 + |g_i|^2) \int_{T_K}^t e^{-\delta(t-s) + 2 \int_s^t \xi(\theta \tau\omega)ds - \xi(\theta t\omega)} d\tau. \tag{4.5}
\]

Now, substitute \( \theta_t \omega \) for \( \omega \) and estimate each term from (4.3) to (4.5). In (4.1), with \( t \) replaced with \( T_K \) and \( \omega \) with \( \theta_t \omega \), respectively, it follows from (4.3) that

\[
e^{-\delta(t-T_K) + 2 \int_{T_K}^t \xi(\theta s\omega)ds} \cdot \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i(T_K, \theta_t \omega, e^{-\xi(\theta t\omega)}\Psi_0(\theta t\omega))|^2 \leq e^{-\delta t - 2\xi(\theta t\omega) + 2 \int_0^t \xi(\theta \tau\omega)ds} \|\Psi_0\|^2 \]

\[
+ c_1 \int_0^{T_K} e^{-2\xi(\theta s\omega) + \delta(s-t) + 2 \int_s^t \xi(\theta \tau\omega)ds} ds \leq e^{-\delta t - 2\xi(\theta t\omega) + 2 \int_0^t \xi(\theta \tau\omega)ds} \|\Psi_0\|^2 \]

\[
+ c_1 \int_{-t}^{T_K-t} e^{-2\xi(\theta s\omega) + \delta s - 2 \int_0^s \xi(\theta \tau\omega)ds} ds.
\]

Due to the properties of the Ornstein-Uhlenbeck process, there exists a \( T_1(\epsilon, \omega, \mathcal{K}(\omega)) > T_K(\omega) \),
such that if \( t > T_1(\epsilon, \omega, \mathcal{K}(\omega)) \), then

\[
e^{-\delta(t-T_\mathcal{K})+2 \int_{T_\mathcal{K}}^t \xi(\theta_{s-t}\omega)ds} \sum_{i \in \mathbb{Z}} \rho\left(\frac{i}{N}\right) \left| \varphi_i(T_\mathcal{K}, \theta_{-t}\omega, e^{-\xi(\theta_{-t}\omega)}\Psi_0(\theta_{-t}\omega)) \right|^2 \leq \frac{\epsilon^2}{3}.
\]

(4.6)

Next, from (4.1) and (4.4), it follows that

\[
\frac{c_2}{N} \int_{T_\mathcal{K}}^t e^{-\delta(t-\tau)+2 \int_{\tau}^t \xi(\theta_{s-t}\omega)ds} \cdot \| \varphi(\tau, \theta_{-t}\omega, e^{-\xi(\theta_{-t}\omega)}\Psi_0(\theta_{-t}\omega)) \|^2 d\tau \leq \frac{c_2}{N} \| \Psi_0 \|^2 (t - T_\mathcal{K}) e^{-\delta t - 2\xi(\theta_{-t}\omega)+2 \int_{0}^t \xi(\theta_{s-t}\omega)ds} + \frac{c_1 c_2}{N} (\| h \|^2 + \| g \|^2) \cdot \int_{T_\mathcal{K}}^t \int_{t}^{\tau} e^{-2\xi(\theta_{s-t}\omega)+\delta(s-t)+2 \int_{\tau}^s \xi(\theta_{r-t}\omega)dr} ds d\tau
\]

\[
\leq \frac{c_2}{N} \| \Psi_0 \|^2 (t - T_\mathcal{K}) e^{-\delta t - 2\xi(\theta_{-t}\omega)+2 \int_{0}^t \xi(\theta_{s-t}\omega)ds} + \frac{c_1 c_2}{N} (\| h \|^2 + \| g \|^2) \cdot \int_{T_\mathcal{K}}^t \int_{t}^{\tau-t} e^{-2\xi(\theta_{s-t}\omega)+\delta s+2 \int_{s}^{\tau} \xi(\theta_{r-t}\omega)dr} ds d\tau.
\]

(4.7)

Thanks to the properties of the Ornstein-Uhlenbeck process, there exist \( T_2(\epsilon, \omega, \mathcal{K}(\omega)) > T_\mathcal{K}(\omega) \) and \( N_1(\epsilon, \omega, \mathcal{K}(\omega)) > 0 \) such that if \( t > T_2(\epsilon, \omega, \mathcal{K}(\omega)) \) and \( N > N_1(\epsilon, \omega, \mathcal{K}(\omega)) \), then

\[
\frac{c_2}{N} \int_{T_\mathcal{K}}^t e^{-\delta(t-\tau)+2 \int_{\tau}^t \xi(\theta_{s-t}\omega)ds} \cdot \| \varphi(\tau, \theta_{-t}\omega, e^{-\xi(\theta_{-t}\omega)}\Psi_0(\theta_{-t}\omega)) \|^2 d\tau \leq \frac{\epsilon^2}{3}.
\]

(4.7)

Since \( h, g \in \ell^2 \), by the properties of the Ornstein-Uhlenbeck process again, we find that there exists \( N_2(\epsilon, \omega, \mathcal{K}(\omega)) > 0 \) such that if \( N > N_2(\epsilon, \omega, \mathcal{K}(\omega)) \), then from (4.5),

\[
c_1 \sum_{|i| \geq N} (|h_i|^2 + |g_i|^2) \cdot \int_{T_\mathcal{K}}^t e^{-\delta(t-\tau)+2 \int_{\tau}^t \xi(\theta_{s-t}\omega)ds-\xi(\theta_{r-t}\omega)} d\tau \leq \frac{\epsilon^2}{3}.
\]

(4.8)

Let

\[
\tilde{T}(\epsilon, \omega, \mathcal{K}(\omega)) = \max\{T_1(\epsilon, \omega, \mathcal{K}(\omega)), T_2(\epsilon, \omega, \mathcal{K}(\omega))\},
\]

\[
\tilde{N}(\epsilon, \omega, \mathcal{K}(\omega)) = \max\{N_1(\epsilon, \omega, \mathcal{K}(\omega)), N_2(\epsilon, \omega, \mathcal{K}(\omega))\}.
\]
Then from (4.6), (4.7) and (4.8), for $t > \tilde{T}(\epsilon, \omega, \mathcal{K}(\omega))$ and $N > \tilde{N}(\epsilon, \omega, \mathcal{K}(\omega))$, we get

$$
\sum_{|i| \geq 2N} |\phi_i(t, \theta_{-t}\omega, e^{-\xi(\theta_{-t}\omega)}\Psi_0(\theta_{-t}\omega))|^2 \\
\leq \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{N}\right)|\phi_i(t, \theta_{-t}\omega, e^{-\xi(\theta_{-t}\omega)}\Psi_0(\theta_{-t}\omega))|^2 \leq \epsilon^2,
$$

which implies the conclusion.

We are now in a position to prove our main result.

**Proof of Theorem 4.1.** The desired result follows directly from Lemmas 4.2 and 4.3 and Proposition 2.5.

**Remark 4.4.** The result may have generalized the existing results (see e.g. [Huang(2007), Gu et al. (2012)]) to some extent. First, càdlàg functions in a more wider sense than continues ones as indicated in Introduction section; Second, here we restrict to $1 < \alpha < 2$, when $\alpha = 2$, the $\alpha$-stable process actually reduces to the standard Brownian motion.

**Remark 4.5.** Recently, some sufficient conditions for the upper-semicontinuity of attractors for random lattice systems perturbed by small white noises have been given in [Zhou (2012)]. Here, it is worth mentioning that all the results on this topic are focus on the SLDS perturbed by the white noises. It will be an interesting question left to future research.

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