Plane problem of contact interaction for foundation
with multilayer nonuniform coating

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Abstract.
Present article is devoted to solving the problem of contact interaction between a rigid punch and a viscoelastic aging foundation with a thin coating, which consists of several elastic layers. The nonuniformities of layer depend on longitudinal coordinate. They can be described by discontinuous or rapidly changing functions. The basic mixed integral equation and integral additional conditions for this problem are presented. The analytic solutions for all versions of the problem are presented.

1. Introduction
Solutions for contact problems for foundations with nonuniform coatings were constructed in [1, 2]. It were the problems for viscoelastic foundation with one surface nonuniform coating (onelayer coating). This paper devoted to the plane contact problem for foundation with coating, which consist of several different longitudinally nonuniform layers.

2. Statement of the plane contact problem
Viscoelastic aging layer of a thickness $H$ with a thin elastic multilayer coating lies on a rigid basis (number of layers equal to $N$). All layers have different thickness $h_k$ ($k = 1, 2, \ldots, N$). These layers have longitudinal nonuniformity, i.e. elastic characteristics of each layer depends on longitudinal coordinate, but constant in depth. These nonuniformities can be described even by discontinuous functions. We also assume that the rigidities of coating layers are less than the rigidity of the lower layer or they are of the same order of magnitude. There is smooth contact or perfect contact between layers and between the lower layer and the rigid base. We consider the case of plane strain.

At time $\tau_0 \geq \tau_{\text{lower}}$ ($\tau_{\text{lower}}$ is a time of lower layer production), the force $P(t)$ starts to indent a rigid smooth punch into the surface of such a foundation. This punch width equal to $2a$ and this width much more than the total thickness of the coating layers, i.e. $2a \gg h_1 + h_2 + \ldots + h_N$. The function $g(x)$ describes distance between contact surfaces in nondeformable state and called backlash function. This function describes punch base form in this problem. We assume that we apply such force that the contact area coincides with the punch width, i.e. equal to $2a$. It is a problem with known contact area.
\[ q(x, t) = \frac{2(1 - \nu_{\text{lower}}^2)}{\pi} \left[ \int_{-a}^{a} k_{pl}(x - \xi) \frac{q(\xi, t)}{E_{\text{lower}}(t - \tau_{\text{lower}})} d\xi + \int_{\tau_0}^{t} K(t - \tau_{\text{lower}}, \tau - \tau_{\text{lower}}) \int_{-a}^{a} k_{pl}(x - \xi) \frac{q(\xi, \tau)}{E_{\text{lower}}(\tau - \tau_{\text{lower}})} d\xi d\tau \right] \]

where \( q(x, t) \) is contact pressure under the punch, \( R_k(x) \) are contact rigidities of the coating layers depend on Young moduli and Poisson’s ratio of this layers \([3, 6]\): in the case of a smooth contact between layers \( k \) and \( k - 1 \) (or between first layer and upper viscoelastic layer if \( k = 1 \)), we have \( R_k(x) = E_k(x)/[1 - \nu_k(x)] \) and in the case of an perfect contact, \( R_k(x) = E_k(x)[1 - \nu_k(x)]/[1 - \nu_k(x) - 2\nu_k^2(x)] \). \( E_k(x) \) and \( \nu_k(x) \) are the Young modulus and Poisson’s ratio of the \( k \)-th coating layer; \( E_{\text{lower}}(t) \) and \( \nu_{\text{lower}} \) are the Young modulus and Poisson’s ratio of lower layer; \( k_{pl}(x - \xi)/H \) is known kernel of the plane contact problem, which has the form \( k_{pl}(s) = \int_{0}^{\infty} [L(u)/u \cos(2u)] du \), where function \( L(u) \) depend on conditions between upper layer and rigid base: in the case of smooth contact \( L(u) = [\cosh(2u) - 1]/[\sinh(2u) + 2u] \) and in the case of perfect contact \( L(u) = [2k \sinh(2u)/[2k \cosh(2u) + 4u^2 + 1 + k^2]] \) \((k = 3 - 4\nu_{\text{lower}})\); \( K(t, \tau) \) is creep kernel which has a form (see \([4-6]\)) \( K(t, \tau) = E_{\text{lower}}(\tau) \frac{\tau}{2} \left[ 1/E_{\text{lower}}(\tau) + C_{\text{lower}}(t, \tau) \right] \). \( C_{\text{lower}}(t, \tau) \) is the tensile creep functions; \( e(t) \) is force eccentricity.

There are exist 4 different versions of the problem: 1) the settlement and tilt angle of the punch are given (i.e., the right-hand side of the equation is given); 2) the punch settlement and the moment of the load application are given; 3) the tilt angle of the punch and the force of the load application are given; 4) the force and the moment of the load application are given.
3. Dimensionless form and connected problems

Make a change of variables in (1) according to the formulas

\[ x^* = \frac{x}{a}, \quad \xi^* = \frac{\xi}{a}, \quad t^* = \frac{t}{\tau_0}, \quad \tau^* = \frac{\tau}{\tau_0}, \quad \tau^*_{\text{lower}} = \frac{\tau_{\text{lower}}}{\tau_0}, \quad \lambda = \frac{H}{a}. \]

\[ \delta^*(t^*) = \frac{\delta(t)}{a}, \quad \alpha^*(t^*) = \alpha(t), \quad g^*(x^*) = \frac{g(x)}{a}, \quad c^*(t^*) = \frac{E_{\text{lower}}(t - \tau^*_{\text{lower}})}{E_0}. \]

\[ m^*(x^*) = \frac{E_0}{2a(1 - \nu_{\text{lower}}^2)} \sum_{k=1}^{N} h_k R_k(x^*), \quad q^*(x^*, t^*) = \frac{2(1 - \nu_{\text{lower}}^2)q(x, t)}{E_{\text{lower}}(t - \tau^*_{\text{lower}})}, \]

\[ P^*(t^*) = \frac{2P(t)(1 - \nu_{\text{lower}}^2)}{E_{\text{lower}}(t - \tau^*_{\text{lower}})^2}, \quad M^*(t^*) = \frac{2P(t)\epsilon(t)(1 - \nu_{\text{lower}}^2)}{E_{\text{lower}}(t - \tau^*_{\text{lower}})a^2}, \]

\[ V^* f(t^*) = \int_{1}^{t^*} K^*(t^*, \tau^*) f(\tau^*) d\tau^*, \quad K^*(t^*, \tau^*) = K(t - \tau^*_{\text{lower}}, \tau - \tau^*_{\text{lower}})\tau_0, \]

\[ F^* f(t^*) = \int_{-1}^{1} k_{\text{pl}}^*(x^*, \xi^*) f(\xi^*) d\xi^*, \quad k_{\text{pl}}^*(x^*, \xi^*) = \frac{1}{\pi} k_{\text{pl}} \left( \frac{x^* - \xi^*}{H} \right) = \frac{1}{\pi} k_{\text{pl}} \left( \frac{x^* - \xi^*}{\lambda} \right). \]

If we omit the asterisks in formulas we obtain the following mixed integral equation with the additional condition in dimensionless form:

\[ c(t)m(x)q(x, t) + (I - V)Fq(x, t) = \delta(t) + \alpha(t)x - g(x), \]

\[ \int_{-1}^{1} q(\xi, t) d\xi = P(t), \quad \int_{-1}^{1} q(\xi, t)\xi d\xi = M(t), \quad x \in [-1, 1], \quad t \geq 1. \quad (2) \]

These equations are similar to dimensionless equations of the problem for foundation with one nonuniform coating [1, 2, 8]. The construction of their analytical solutions is based on special orthogonal projection method [7]. So, the solution for our problem will have similar form. We will present only final formulas for all versions of the problems.

4. Solutions

4.1. Solution for known punch settlement and tilt angle

Final formulas for the case of known right side have a form

\[ q(x, t) = \frac{1}{m(x)} \sum_{k=0}^{\infty} z_k(t)\Phi_k(x), \]

\[ P(t) = \sum_{k=0}^{\infty} z_k(t)\delta_k, \quad M(t) = \sum_{k=0}^{\infty} z_k(t)\alpha_k, \]

where

\[ \Phi_k(x) = \sum_{m=0}^{\infty} \psi_{km} p_m^*(x), \quad z_k(t) = (I + W_k) \frac{\delta_k \delta(t) + \alpha_k \alpha(t) - g_k}{c(t) + \gamma_k}, \]

\[ \delta_k = \sqrt{J_0} \psi_{k0}, \quad \alpha_k = \frac{J_1}{\sqrt{J_0}} \psi_{k0} + \sqrt{\frac{J_0 J_2 - J_1^2}{J_0}}, \quad g_k = \int_{-1}^{1} \frac{g(\xi)\Phi_k(\xi)}{m(\xi)} d\xi, \]

\[ W_k f(t) = \int_{1}^{t} R_k(t, \tau) f(\tau) d\tau, \quad k = 0, 1, 2, \ldots, \]
polynomials $p^*_m(x)$ and constants $J_0$, $J_1$, $J_2$ can be determined from

$$J_m = \int_{-1}^{1} \frac{\xi^m d\xi}{m(\xi)}, \quad d_{-1} = 1, \quad d_m = \begin{vmatrix} J_0 & \cdots & J_m \\ \vdots & \ddots & \vdots \\ J_m & \cdots & J_{2m} \end{vmatrix},$$

$$p^*_m(x) = \frac{1}{d_{m-1}d_m} \begin{vmatrix} J_0 & J_1 & \cdots & J_m \\ \vdots & \vdots & \ddots & \vdots \\ J_{m-1} & J_m & \cdots & J_{2m-1} \\ x & \cdots & x^m \end{vmatrix}, \quad m = 0, 1, 2, \ldots,$$

coefficients $\psi_{km}$ and $\gamma_k$ can be derived from spectral problem $\sum_{k=0}^{\infty} K_{ml}\psi_{kl} = \gamma_k\psi_{km}$ $(k, m = 0, 1, 2, \ldots)$, coefficients $K_{ml}$ determined by ratios

$$K_{ml} = \int_{-1}^{1} \int_{-1}^{1} \frac{p^*_m(x)K_{pl}(x, \xi)p^*_l(\xi)}{m(x)m(\xi)} dx \, d\xi, \quad m, l = 0, 1, 2, \ldots,$$

and kernels $R_k(t, \tau)$ are resolvents for the kernels $\gamma_kK(t, \tau)/[c(t) + \gamma_k] (k = 0, 1, 2, \ldots)$.

### 4.2. Solution for version with known indenting force and moment

Solution of equations (2) for the case with known indenting force and moment has a form

$$q(x, t) = \frac{1}{m(x)} \left[ z_0(t)p^*_0(x) + z_1(t)p^*_1(x) + \sum_{k=2}^{\infty} z_k(t)\Phi_k(x) \right],$$

$$\alpha(t) = \sqrt{\frac{J_0}{J_0J_2 - J_1^2}} \left\{ g_1 + c(t)z_1(t) + (I - V) \left[ K_{10}z_0(t) + K_{11}z_1(t) + \sum_{k=2}^{\infty} K^{(1)}_{k}z_k(t) \right] \right\},$$

$$\delta(t) = \frac{1}{\sqrt{J_0}} \left\{ g_0 + c(t)z_0(t) + (I - V) \left[ K_{00}z_0(t) + K_{01}z_1(t) + \sum_{k=2}^{\infty} K^{(0)}_{k}z_k(t) \right] \right\} - \alpha(t) \frac{J_1}{J_0},$$

where

$$z_0(t) = \frac{P(t)}{\sqrt{J_0}}, \quad z_1(t) = \frac{J_0M(t) - J_1P(t)}{\sqrt{J_0(J_0J_2 - J_1^2)}}, \quad \Phi_k(x) = \sum_{m=2}^{\infty} \psi_{km}p^*_m(x),$$

$$z_k(t) = (I + W_k)(I - V)[z_0(t)K^{(0)}_k + z_1(t)K^{(1)}_k] - \frac{g_k}{c(t) + \gamma_k}, \quad W_kf(t) = \int_{-1}^{t} R_k(t, \tau)f(\tau) \, d\tau,$$

$$g_0 = \int_{-1}^{1} \frac{g(\xi)p^*_0(\xi)}{m(\xi)} \, d\xi, \quad g_1 = \int_{-1}^{1} \frac{g(\xi)p^*_1(\xi)}{m(\xi)} \, d\xi, \quad g_k = \int_{-1}^{1} \frac{g(\xi)\Phi_k(\xi)}{m(\xi)} \, d\xi,$$

$$K^{(0)}_k = \sum_{m=2}^{\infty} K_{0m}\psi_{km}, \quad K^{(1)}_k = \sum_{m=2}^{\infty} K_{1m}\psi_{km}, \quad k = 2, 3, 4, \ldots,$$

polynomials $p^*_m(x)$ and constants $J_0$, $J_1$, $J_2$ can be determined from (4), coefficients $\psi_{km}$ and $\gamma_k$ can be derived from spectral problem $\sum_{k=2}^{\infty} K_{ml}\psi_{kl} = \gamma_k\psi_{km}$ $(m, l = 2, 3, 4, \ldots)$, coefficients $K_{ml}$ determined by ratios (5), and kernels $R_k(t, \tau)$ are resolvents for the kernels $\gamma_kK(t, \tau)/[c(t) + \gamma_k]$ $(k = 2, 3, 4, \ldots)$.
4.3. Solution for the case with known indenting force and punch tilt angle

Solution for this case has a form

\[ q(x, t) = \frac{1}{m(x)} \left[ z_0(t) p_0^*(x) + \sum_{k=1}^{\infty} z_k(t) \Phi_k(x) \right], \]

\[ \delta(t) = \frac{1}{J_0} \left\{ g_0 + c(t) z_0(t) + (I - V) \left[ K_{00} z_0(t) + \sum_{k=1}^{\infty} K_k^{(0)} z_k(t) \right] \right\} - \alpha(t) \frac{J_1}{J_0}, \]

\[ M(t) = z_0(t) \frac{J_1}{J_0} + \sum_{k=1}^{\infty} z_k(t) \alpha_{1k}, \]

where

\[ z_0(t) = \frac{P(t)}{J_0}, \quad \Phi_k(x) = \sum_{m=1}^{\infty} \omega_{km} p_m^*(x), \quad z_k(t) = (I + W_k) \frac{\alpha_{1k} \alpha(t) - (I - V) z_0(t) K_k^{(0)} - g_k}{c(t) + \gamma_k}, \]

\[ g_0 = \int_{-1}^{1} \frac{g(\xi) p_0^*(\xi)}{m(\xi)} d\xi, \quad g_k = \int_{-1}^{1} \frac{g(\xi) \Phi_k(\xi)}{m(\xi)} d\xi, \quad \alpha_{1k} = \sqrt{\frac{J_0 J_2 - J_1^2}{J_0 J_2}} \psi_{k1}, \]

\[ K_k^{(0)} = \sum_{n=1}^{\infty} K_{0n} \psi_{kn}, \quad W_k f(t) = \int_{1}^{t} R_k(t, \tau) f(\tau) d\tau, \quad k = 1, 2, 3, \ldots, \]

polynomials \( p_m^*(x) \) and constants \( J_0, J_1, J_2 \) can be determined from (4), coefficients \( \psi_{km} \) and \( \gamma_k \) can be derived from spectral problem \( \sum_{l=1}^{\infty} K_{ml} \psi_{kl} = \gamma_k \psi_{km} (m, l = 1, 2, 3, \ldots) \), coefficients \( K_{ml} \) determined by ratios (5), and kernels \( R_k(t, \tau) \) are resolvents for the kernels \( \gamma_k K(t, \tau)/(c(t) + \gamma_k) \) \( (k = 1, 2, 3, \ldots) \).

4.4. Solution for the case with known moment and punch settlement

Final formulas are

\[ q(x, t) = \frac{1}{m(x)} \left[ z_0(t) p_0^*(x) + \sum_{k=1}^{\infty} z_k(t) \Phi_k(x) \right], \]

\[ \alpha(t) = \frac{1}{J_2} \left\{ g_0 + c(t) z_0(t) + (I - V) \left[ K_{00} z_0(t) + \sum_{k=1}^{\infty} K_k^{(0)} z_k(t) \right] \right\} - \delta(t) \frac{J_1}{J_2}, \]

\[ M(t) = z_0(t) \frac{J_1}{J_2} + \sum_{k=1}^{\infty} z_k(t) \hat{\delta}_{1k}, \]

where

\[ z_0(t) = \frac{M(t)}{J_2}, \quad \Phi_k(x) = \sum_{m=1}^{\infty} \omega_{km} p_m^*(x), \quad z_k(t) = (I + W_k) \frac{\delta_{1k} \delta(t) - (I - V) z_0(t) K_k^{(0)} - g_k}{c(t) + \gamma_k}, \]

\[ g_0 = \int_{-1}^{1} \frac{g(\xi) p_0^*(\xi)}{m(\xi)} d\xi, \quad g_k = \int_{-1}^{1} \frac{g(\xi) \Phi_k(\xi)}{m(\xi)} d\xi, \quad \delta_{1k} = \sqrt{\frac{J_0 J_2 - J_1^2}{J_2}} \psi_{k1}, \]

\[ K_k^{(0)} = \sum_{m=1}^{\infty} K_{0m} \psi_{km}, \quad W_k f(t) = \int_{1}^{t} R_k(t, \tau) f(\tau) d\tau, \quad k = 1, 2, 3, \ldots, \]
polynomials $\hat{p}_m(x)$ can be determined from

$$
\hat{p}_0(x) = \frac{J_1}{\sqrt{J_0 J_2}} p_0^*(x) + \sqrt{\frac{J_0 J_2 - J_1^2}{J_0 J_2}} p_1^*(x), \quad \hat{p}_1(x) = \sqrt{\frac{J_0 J_2 - J_1^2}{J_0 J_2}} p_0^*(x) - \frac{J_1}{\sqrt{J_0 J_2}} p_1^*(x),
$$

$$
\hat{p}_l(x) = p_l^*(x), \quad l = 2, 3, 4, \ldots,
$$

polynomials $p_m(x)$ and constants $J_0, J_1, J_2$ determines by (4), coefficients $\psi_{km}$ and $\gamma_k$ can be derived from spectral problem $\sum_{l=0}^{\infty} \hat{K}_{ml}\psi_{kl} = \gamma_k \psi_{km}$ ($k, m = 1, 2, 3, \ldots$), coefficients $\hat{K}_{ml}$ determined by ratios

$$
\hat{K}_{ml} = \int_{-1}^{1} \int_{-1}^{1} \frac{\hat{p}_m(x) k_{pl}(x, \xi) \hat{p}_l(\xi)}{m(x)m(\xi)} dx d\xi, \quad m, l = 0, 1, 2, \ldots,
$$

and kernels $R_k(t, \tau)$ are resolvents for the kernels $\gamma_k K(t, \tau)/[c(t) + \gamma_k] \ (k = 1, 2, 3, \ldots)$.

**Main Results and Conclusions**

- We pose and solve plane contact problem for viscoelastic aging base with multilayer nonuniform coatings and rigid punches in the case, where nonuniformities are described by different rapidly changing functions.

- The solutions of all versions of the problem are obtained analytically. In the expressions for the contact stresses, function $m(x)$ and hence contact rigidities of the coating layers is distinguished explicitly. It is allows one to perform effective computations for actual nonuniformities of coating layers. Other known methods don’t allow to do this.

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