OPTIMAL RETIREMENT TIME AND CONSUMPTION WITH THE VARIATION IN HABITUAL PERSISTENCE

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ABSTRACT. In this paper, we study the individual’s optimal retirement time and optimal consumption under habitual persistence. Because the individual feels equally satisfied with a lower habitual level and is more reluctant to change the habitual level after retirement, we assume that both the level and the sensitivity of the habitual consumption decline at the time of retirement. We establish the concise form of the habitual evolutions, and obtain the optimal retirement time and consumption policy based on martingale and duality methods. The optimal consumption experiences a sharp decline at retirement, but the excess consumption raises because of the reduced sensitivity of the habitual level. This result contributes to explain the “retirement consumption puzzle”. Particularly, the optimal retirement and consumption policies are balanced between the wealth effect and the habitual effect. Larger wealth increases consumption, and larger growth inertia (sensitivity) of the habitual level decreases consumption and brings forward the retirement time.

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1. Introduction

In empirical evidence, we observe a sharp decline in the consumption at retirement, which cannot be explained by the classical Merton (1969, 1983) model of lifetime optimal consumption. Literatures explain this “retirement consumption puzzle” as the results of more household productions, less work-related expenses, restricted retirement incomes and more medical expenses, etc (cf. Banks, Blundell and Tanner (1998), Bernheim, Skinner and Weinberg...
Hurd and Rohwedder (2003), Hurst (2003), Schwerdt (2005), Blau (2008)). However, the explanations are still a bit controversial and need more theoretical evidence. In this paper, we try to establish the individual’s optimal retirement decision and consumption problem under habitual persistence to clear the controversy in some extent, and establish some practical settings on the habitual parameters. The results are explanatory to the empirical evidence.

The habit persistence was originally studied by Pollak (1970) and Ryder and Heal (1973). The habitual level is the weighted average of the past consumption and only the excess consumption produces utility. Basically, the optimization problems are divided into two categories according to whether the consumption can be lower than the habitual level. Detemple and Zapatero (1991), Chapman (1998), Yu (2015) and Guan, Liang and Yuan (2021) study the optimal consumption problem under the additive habit framework. And Shrikhande (1997) and Detemple and Karatzas (2003) study the problem under the non-addictive framework. Under this framework, it is usually assumed that there exist the minimal consumption constraints (cf. Lim, Lee and Shin (2018) and Angoshtari, Bayraktar and Young (2020)). We assume that the actual consumption can be lower than the habitual level and the downward deviation is bounded for the practical considerations. In this circumstance, the CRRA (Constant Relative Risk Averse) utility is not well defined. Inspired by Curatola (2017) and Bilsen, Laeven and Nijman (2020), which study the optimal consumption problem under loss aversion and habitual reference, we explore the S-shaped utility in this paper.

The individual’s life cycle is divided into two periods. He/She earns wages and makes contributions to the social endowment insurance before retirement, and receives benefits after retirement. Hence the individual’s wealth process is naturally described by two-stage stochastic differential equations, as in Chen, Hentschel and Xu (2018). For the retirement period, the individual’s income usually declines due to the low income replacement rate. However, the individual has more leisure time in this period. As such, several ways are implemented to depict the utility of leisure. In Banks, Blundell and Tanner (1998) and Karlstrom, Palme and Svensson (2004), a weight parameter larger than one is set to the retirement consumption. This leads to a jump in the
consumption at retirement and contributes to explain the “retirement consumption puzzle”. Then, Chen, Hentschel and Xu (2018) assume a leisure weight related to the retirement time and obtain interesting findings.

It is still very challenging to describe the utility of leisure in the framework of habitual persistence. According to the statistical results of the survey, we observe that the individual feels equally satisfied with a lower actual consumption level after retirement. This may be explained as that the individual has more leisure time and makes more household productions. We model this effect as the shrinking of the habitual level at the time of retirement. Under this assumption, the same actual consumption leads to more excess consumption and greater utility after retirement. Besides, we find that the individuals are reluctant to change their habitual levels after retirement. We observe that one time large (low) consumption has limited impacts on the habitual consumption level, which is caused by the income constraint and hard to change habit. As such, we assume that the sensitivity of the habitual consumption declines after retirement. Under the relatively abundant wealth scenario in practice, we expect a spiral increase of the habitual consumption level. Because the inertia of the habitual growth rate declines after retirement, the same actual consumption leads to greater utility.

The individual dynamically controls the optimal asset allocation and consumption policies to achieve the objective, i.e., maximizing the overall utility of the excess consumption. One crucial parameter is the retirement time. Thus, the problem is transformed into a two-stage optimization problem. In the former stage, the individual’s utility is maximized under the assumption that the retirement time is given. In the latter stage, the optimal retirement time is regarded as a pre-commitment policy, which is consistent with the settings in Hey and Lotito (2009), Hey and Panaccione (2011) and Chen, Hentschel and Xu (2018). Literatures like Choi and Shim (2006), Choi, Shim and Shin (2008), Dybvig and Liu (2010), Yang and Koo (2018) and Guan, Liang and Yuan (2021) treat the optimal retirement time as a simultaneously controlled policy and study the optimal stopping time problem with free boundary. However, we believe that the pre-commitment assumption is more practical. In this circumstance, the retirement decisions are made relatively ahead of time, and do not depend on the follow-up situations. This can
be verified by the consistency between the expected and the actual retirement time in the survey.

In order to establish the optimums of the individual’s optimization problem, we introduce the dirac function to represent the dynamics of the habitual level concisely in a unified expression. Using martingale and duality methods, as in Karatzas, Lehoczky, Shreve and Xu (1991), He, Liang, Liu and Ma (2020) and He, Liang and Yuan (2020), we establish the semi-analytical solutions of the problem. Because of the long time horizon of the life cycle optimization, the expectation of the optimal consumption may explode after a long time in the numerical simulations. As such, we originally introduce the certainty equivalence of the actual and the habitual consumption processes timely. In addition, we establish an innovative analytical method to study the impact of the exogenous parameters on the optimal retirement time quantitatively. The theoretical and numerical results show that the optimal consumption is affected by both of the wealth effect and the habitual effect. Particularly, the optimal consumption decision is balanced between the pressure of high habitual consumption level and the time preference. The declines of the level and the sensitivity of the habitual level reduce the pressure and weaken the habitual effect after retirement. The excess consumption rises at the time of retirement accordingly. However, the rise of the excess consumption cannot offset the sharp decline in the habitual consumption. Thus, we observe a decline in the actual consumption at the time of retirement, which is consistent with the empirical evidence of the “retirement consumption puzzle”. Furthermore, larger wealth increases consumption, and larger wage (benefit) postpones (brings forward) the retirement time. Interestingly, larger growth inertia (sensitivity) of the habitual level decreases consumption and brings forward the retirement time. In this circumstance, the habitual effect is the dominance. Early retirement is required to prevent the habitual level from raising too high and reducing overall utility.

The main contributions of this paper are threefold: First, we establish the non-addictive optimal retirement decision and consumption problem under the framework of habitual persistence. Particularly, we model the utility of the retirement leisure as the declines in the level and the sensitivity of the habitual consumption after retirement. Second, we first introduce the dirac function
to establish the unified expression of the habitual level under the settings of variational habitual persistence characters. Through this simple expression and using martingale and duality methods, we establish the semi-analytical optimums of the stochastic control problem. In addition, we quantitatively analyze the relationship between the exogenous parameters and the optimal retirement time. The last, the numerical results confirm the evidence that there exists a sharp decline in the consumption at retirement. It is the comprehensive result of the drop in the habitual consumption and the rise in the excess consumption. Besides, we identify the wealth effect and the habitual effect, which have major impacts on the optimal consumption and the optimal retirement time.

The remainder of this paper is organized as follows: Section 2 formulates the non-addictive optimal consumption problem with the variation in the habitual level and the habitual sensitivity. In Section 3, we establish the optimal control policies and the value function semi-analytically based on dirac function, martingale and duality methods. Section 4 shows the optimal consumption process numerically, which contributes to explain the “retirement consumption puzzle”. In addition, we study the impacts of the parameters on the optimal consumption and the optimal retirement time in this section. The last section concludes the paper.

2. Problem formulation

2.1. Wealth process with the participation of social endowment insurance. We consider the wealth process of an individual who participates in the social endowment insurance. Before retirement, the individual receives wage as the labor income and contributes part of the income as the premium of the social insurance. After retirement, the individual receives benefit from the social insurance. Meanwhile, investment is allowed both before and after retirement. The individual dynamically chooses the asset allocation and consumption policies, as well as the retirement time to maximize the overall utility of the consumption.

First, let \( B = \{ B_t, \ t \geq 0 \} \) be a Brownian motion on complete filtered probability \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), the filtration \( \mathbb{F} = \{ \mathcal{F}_t : \ t \geq 0 \} \) satisfies the usual
conditions and is generated by the Brownian motion $B$, i.e., $\mathcal{F}_t = \sigma\{B_u, \ 0 \leq u \leq t\}, \ t \geq 0$. For simplicity, we assume that the financial market consists of one risk-free asset and one risky asset.

The price of the risk-free asset $S_0 = \{S_0(t), \ t \geq 0\}$ is given by

$$\frac{dS_0(t)}{S_0(t)} = r dt, \ S_0(0) = s_0,$$

and the price of the risky asset $S_1 = \{S_1(t), \ t \geq 0\}$ follows the stochastic differential equation (abbr. SDE):

$$\frac{dS_1(t)}{S_1(t)} = \mu dt + \sigma dB_t, \ S_1(0) = s_1,$$

where $r$ is the risk-free interest rate, $\mu$ and $\sigma$ are the expected return and the volatility of the risky asset, respectively. $s_0$ and $s_1$ are positive constants.

The wage process $W = \{W(t), \ t \geq 0\}$ satisfies the following SDE:

$$\frac{dW(t)}{W(t)} = \alpha(t) dt + \beta(t) dB_t, \ W(0) = W_0,$$

where $\alpha(t)$ and $\beta(t)$ are the expected growth rate and the volatility of the wage at time $t$.

Next, we establish the settings of the social insurance rules and the retirement time. The individual chooses his/her optimal retirement time $\tau$, which lies within $[\tau_{min}, \tau_{max}]$. $\tau_{min}$ and $\tau_{max}$ are minimal and maximal retirement time required by the government. In addition, we assume that the optimal retirement time is a pre-commitment policy, that is, the individual chooses the optimal time to retire at the initial time rather than choosing it by constantly observing the follow-up situations. The settings are consistent with the ones in Hey and Panaccione (2011) and Chen, Hentschel and Xu (2018).

Before retirement, the individual mandatorily participates in the social endowment insurance, and has the obligation to contribute $k$ proportion of the wage as the insurance premium. After retirement, the individual has the right to receive the benefit from the social insurance. The amount of the benefit depends on the retirement time. Particularly, the individual receives the amount of $g(\tau)De^{\xi t}$ as the benefit at time $t$, where $D$ is the benefit at time 0, $\xi$ is the growth rate of the benefit, which is designed to maintain the purchasing power, and $g(\tau)$ is a penalty variable for early retirement. We assume $g(\tau) = e^{-\zeta(\tau_{st} - \tau)^+}$, where $\tau_{st}$ is the statutory retirement time and $\zeta > 0$ is the...
elastic parameter. As such, \( g(\tau) \) is increasing with respect to \( \tau \) before \( \tau_{st} \) and invariant after \( \tau_{st} \). For early retirement \((\tau < \tau_{st})\), the individual receives discounted benefit and is encouraged to work to the statutory retirement age. For the wage process, we assume \( \alpha(t) = \alpha 1_{\{t \leq \tau_{min}\}} \) and \( \beta(t) = \beta 1_{\{t \leq \tau_{min}\}} \), where \( \alpha \) and \( \beta \) are positive constants. It is a realistic setting that the wage stops rising after a certain time.

The last, we establish the individual’s wealth process. Assume that \( T \) is the maximal survival time. The dynamics of the wealth \( X = \{X_t, 0 \leq t \leq T\} \) is determined by three control variables: the wealth allocated to the risky asset \( \pi = \{\pi_t, 0 \leq t \leq T\} \), the consumption level \( C = \{C_t, 0 \leq t \leq T\} \) and the retirement time \( \tau \). It is natural to assume that \( C : [0, T] \times \Omega \rightarrow [0, \infty) \) and \( \pi : [0, T] \times \Omega \rightarrow [0, \infty) \) are \( \mathcal{F} \)-progressively measurable and satisfy the integrability condition \( \int_0^T (C_t + \pi_t^2) dt < \infty \) almost surely. Besides, \( \tau \) is a pre-commitment control variable, as such, we treat it as a given parameter in the first step of optimization.

The individual’s wealth process satisfies the following SDEs:

When \( 0 \leq t \leq \tau \),
\[
dX_t = rX_t + (\mu - r)\pi_t dt + (1 - k)W_t dt + \sigma \pi_t dB_t - C_t dt,
\]
and when \( \tau \leq t \leq T \),
\[
dX_t = rX_t + (\mu - r)\pi_t dt + g(\tau) De^{\xi_t} dt + \sigma \pi_t dB_t - C_t dt.
\]

In the next subsection, we will establish the individual’s optimization objective and the admissible domain of the control variables.

2.2. Optimization objective with the variation in habitual consumption. In this subsection, we establish the S-shaped utility function of the individual and the variations in the level and the sensitivity of the habitual consumption after retirement, which are better depictions of the reality.

First, we set up the habitual consumption level of the individual. The habitual behavior was originally studied by Pollak (1970) and Ryder and Heal (1973), which establish the criterion of evaluating consumption, and the habitual level is measured by the weighted average of the past consumption. Composing the S-shaped utility, only the consumption exceeding the habitual level produces positive utility. The habitual level \( h = \{h_t, 0 \leq t \leq T\} \) is
defined by

\[ dh_t = \left[ \psi(t)C_t - \eta(t)h_t \right] dt, \quad t \neq \tau, \]

where \( \psi(t) \) and \( \eta(t) > 0 \) are nonnegative and bounded parameters varying with respect to time \( t \). \( \psi(t) = \eta(t) \) usually holds. Under this assumption, the habitual consumption level is the arithmetic average of the past consumption, as discussed in Bilsen, Laeven and Nijman (2020). Particularly, when the actual consumption equals to the habitual consumption, the habitual level remains unchanged.

Next, we establish the two important variations in the habitual consumption after retirement. Briefly, both the level and the sensitivity of the habitual consumption decline at the time of retirement. The modifications of \( \{\psi(t), 0 \leq t \leq T\} \) and \( \{\eta(t), 0 \leq t \leq T\} \) are based on two observations. The former observation is that the individual has more leisure time and is able to make more household productions. As such, the individual feels equally satisfied when the consumption level declines. Because only the difference between the actual consumption and the habitual consumption produces utility, this observation could be modeled as the shrinking of habitual consumption level after retirement. The latter observation is that the retired individual’s habitual level is sluggish to change over time. The consumption after retirement is influenced by the wealth constraints and the daily routines. It is unusual that one sudden extremely high (low) consumption could change the habitual consumption level a lot. Thus, the sensitivity of the variation in the habitual consumption level also decreases after retirement.

Based on the above discussions, we assume \( h_\tau = lh_{\tau^{-}}, 0 < l < 1 \), which refers to the decline of the habitual consumption level after retirement. Furthermore, we assume \( \psi(t) = \psi, \eta(t) = \eta \) when \( 0 \leq t < \tau \); \( \psi(t) = m\psi, \eta(t) = m\eta \) when \( \tau \leq t \leq T \), where \( \psi \) and \( \eta \) are constants. And \( 0 < m < 1 \) refers to the decline of the sensitivity of the habitual consumption level. As such, the habitual consumption level \( h = \{h_t, 0 \leq t \leq T\} \) is given by

\[
h_t = \begin{cases} 
    h_0 \cdot e^{-\eta t} + \psi \cdot \int_0^t e^{-\eta(t-s)}C_s ds, & 0 \leq t < \tau, \\
    l \cdot h_0 \cdot e^{-m\eta(t-\tau)} - \eta\tau + l \cdot \psi \cdot \int_0^{\tau} e^{-m\eta(t-\tau) - \eta(\tau-s)}C_s ds + m \cdot \psi \cdot \int_\tau^t e^{-m\eta(t-s)}C_s ds, & \tau \leq t \leq T.
\end{cases}
\]
In order to obtain the concise representation of $h_t$, we introduce the dirac function $\delta(x)$, which is a generalized function defined by
\[
\delta(x) = \begin{cases} 
\infty, & x = 0, \\
0, & x \neq 0,
\end{cases}
\]
and
\[
\int_{-\infty}^{+\infty} \delta(x) dx = 1.
\]
By substituting $\eta(s)$ with $\widetilde{\eta}(s) - \ln(l)\delta(s - \tau)$ in (2.1), $h$ is transformed into
\[
h_t = h_0 e^{-\int_0^t \eta(s) ds} + \int_0^t e^{-\int_s^t \eta(u) du} \psi(s) C_s ds.
\]
As such, $h$ has the following concise form:
\[
dh_t = [\psi(t)C_t - \eta(t)h_t] dt, \quad 0 \leq t \leq T.
\]
In addition, $\delta(\cdot)$ in (2.2) can also be an indicator to distinguish the pre-retirement and post-retirement scenarios.

The last, we establish the S-shaped utility function of the individual. As discussed above, only the difference between the actual consumption and the habitual consumption produces utility. Besides, we assume that the actual consumption below the habitual level is allowed. As such, we need a well defined utility function to express positive and negative utilities synthetically, and we naturally choose the S-shaped utility function.

The S-shaped utility function is represented as the compositions of two CRRA utilities:
\[
u(C_s - h_s) = \frac{(C_s - h_s)^{1-\gamma}}{1-\gamma} 1_{\{C_s \geq h_s\}} + (-\kappa) \frac{(h_s - C_s)^{1-\gamma}}{1-\gamma} 1_{\{C_s < h_s\}}, \quad \forall s \in [0, T],
\]
where $\gamma$ refers to the relative risk aversion of the individual, and $\kappa$ is the loss aversion parameter. The second term in (2.4) defines the negative utility when the individual’s actual consumption is below the habitual level. In this circumstance, the individual suffers from the gap of inadequate consumption, as such, it produces negative utility. $\kappa > 1$ represents the psychological phenomenon that the suffering of loss is greater than the happiness of equal gain.
Therefore, we assume that the retirement time $\tau$ is given in the first step. The individual’s objective is to maximize the overall utility of the consumption:

$$
V(\tau) = \max_{\pi, C, \tau} E \left\{ \int_0^T e^{-\rho s} u(C_s, h_s) ds \right\},
$$

(2.5)

where $\rho$ refers to the rate of time preference along with the mortality rate. Actually, the integral in (2.5) is divided into two parts. Before retirement, the individual earns wage and contributes to the social endowment insurance. After retirement, the individual receives benefit. At the same time, the level and the sensitivity of the habitual consumption both decline in this period. In the second step, we try to establish $\max_{\tau} \{ V(\tau) \}$ satisfying

$$
V = \max_{\tau} \{ V(\tau) \},
$$

(2.6)

which is the solution of the static optimization problem.

Furthermore, we study the restrictions on the admissible domain of the control variables. Although we set up the non-addictive consumption, the actual consumption should be above some threshold to maintain the minimal standard of living. We assume that there exists $L \geq 0$ such that $h_s - C_s \leq L$ holds for $s \in [0, T]$. Particularly, when $L = 0$, the actual consumption below the habitual consumption is not allowed and it depicts the addictive consumption model. In addition, the individual may initiate some loans to consume the labor capital in advance. And he/she is required to repay the loans at the maximal survival time. As such, the triple control $a \triangleq (\tau, \pi, C)$ is called admissible if the individual’s wealth $X_T^a$, under the $a$, remains nonnegative at time $T$, i.e.,

$$
X_T^a \geq 0
$$

almost surely. We denote the family of admissible triple controls $a \triangleq (\tau, \pi, C)$ by $\mathcal{A}$.

Our goal is to solve the individual’s optimization problem (2.5)-(2.6) within the admissible domain $\mathcal{A}$. The optimal asset allocation and consumption policies will be established correspondingly. In addition, the optimal retirement time will be obtained by the pre-commitment optimization after the value function $V(\tau)$ being established.
3. Solution of the stochastic optimization problem

In this section, we assume that the retirement time is given in the first step of optimization. Using martingale and duality methods, we derive the semi-analytical value function of the individual and establish the optimal asset allocation and consumption policies correspondingly. In the second step of optimization, we treat the optimal retirement time as a pre-commitment policy and establish the optimal solution by numerical methods. Interestingly, we can quantitatively analyze the relationship between the optimal retirement time and the exogenous parameters by introducing an innovative analytical method.

First, we solve the individual’s stochastic optimization problem with the given retirement time \( \tau \). The state price density process \( H_t = \{ H_t, 0 \leq t \leq T \} \) is as follows:

\[
H_t = \exp\{-rt - \frac{1}{2}\theta^2 t - \theta B_t\}, \ t \in [0, T],
\]

where \( \theta = \frac{\mu - r}{\sigma} \) is the market price of risk. Using Itô’s lemma, we have

\[
H_t X_t = \int_0^t [(1 - k)H_s W_s 1_{s \leq \tau} + g(\tau)De^{\epsilon t}H_s 1_{s > \tau} - C_s H_s] \, ds
\]
\[
+ \int_0^t [\sigma \pi_s H_s - \theta X_s H_s] \, dB_s, \ t \in [0, T].
\]  

(3.1)

For any \((\pi, C) \in A_\tau \triangleq \{ (\pi, C) : (\tau, \pi, C) \in A \}\) for given \( \tau \), \( X_t \geq X_t \) is valid, and \( X_t \) is the lower bound of \( X_t \). The details to obtain the explicit form of \( X_t \) is given in Appendix A.1. Thus, \( \{ \int_0^t [\sigma \pi_s H_s - \theta X_s H_s] \, dB_s, \ 0 \leq t \leq T \} \) is a supermartingale. Using the restriction that \( X_T \geq 0 \), we know that the consumption level satisfies the following restrictions:

\[
E \left[ \int_0^T C_s H_s ds \right] \leq E \left\{ \int_0^T \left[ (1 - k)H_s W_s 1_{s \leq \tau} + g(\tau)De^{\epsilon t}H_s 1_{s > \tau} \right] ds \right\}.
\]  

(3.2)

However, the habitual consumption process is introduced in the utility function and only the difference between the actual consumption and the habitual consumption produces utility. As such, \( c_s \triangleq C_s - h_s \) is the actual control variable, and the state price density process should be adjusted accordingly to solve the optimization problem.
The adjusted state price density process $\Gamma = \{ \Gamma_t, 0 \leq t \leq T \}$ is defined by

$$\Gamma_t \triangleq H_t + \psi(t) E_t \left\{ \int_t^T e^{-\int_u^t \eta(u) - \psi(u)du} H_s ds \right\}, \quad t \in [0, T].$$

And $\Gamma$ is the solution of the following recursive linear stochastic equation (cf. Detemple and Zapatero (1992)):

$$\Gamma_t = H_t + \psi(t) E_t \left\{ \int_t^T e^{-\int_u^t \eta(u)du} \Gamma_s ds \right\}, \quad t \in [0, T],$$

where $E_t(\cdot) \triangleq E[\cdot | F_t]$. And its exact value is derived in Appendix A.2. Particularly, when $\psi = 0$, $\Gamma_t$ degenerates to $H_t$.

Substituting (2.2), we have

$$E \left[ \int_0^T c_s \Gamma_s ds \right] = E \left\{ \int_0^T \left[ C_s - h_0 e^{-\int_0^s \eta(u)du} - \psi(s) \int_s^T e^{-\int_u^s \eta(v)dv} C_u du \right] \Gamma_s ds \right\}$$

$$= E \left\{ \int_0^T \left[ \Gamma_s - \psi(s) E_s \left( \int_s^T e^{\int_u^s \eta(v)dv} \Gamma_u du \right) \right] C_s ds \right\}$$

$$- E \left[ \int_0^T h_0 e^{-\int_0^s \eta(u)du} \Gamma_s ds \right]$$

$$= E \left[ \int_0^T H_s C_s ds \right] - h_0 E \left[ \int_0^T e^{-\int_0^s \eta(u)du} \Gamma_s ds \right]. \quad (3.3)$$

Fortunately, Fubini theorem could be applied to the generalized function in the second equation. This helps to simplify the form of the equation when we decompose the integral into the pre-retirement and the post-retirement parts.

The first term of Eq.(3.3) is estimated in (3.2), and we denote the second term by

$$z \triangleq E \left[ \int_0^T e^{-\int_0^s \eta(u)du} \Gamma_s ds \right],$$

whose exact value is derived in Appendix A.3.

Thus, we establish the subsidiary problem of the original problem, that is, maximizing the expected utility:

$$E \left[ \int_0^T u_s(c_s) ds \right], \quad (3.4)$$

where $u_s(c_s) = e^{-\rho s} u(c_s)$, $s \in [0, T]$, with the restrictions:

$$E \left[ \int_0^T c_s \Gamma_s ds \right] \leq A - h_0 z, \quad (3.5)$$
where
\[ A = \mathbb{E} \left\{ \int_0^T \left[ (1-k)H_s W_s 1_{\{s \leq \tau\}} + g(\tau) D e^{\xi t H_s 1_{\{s > \tau\}}} \right] \, ds \right\}. \]

The exact value of \( A \) is derived in Appendix A.3. \( A \) is the discounted present value of the incomes of the wages (excluding social insurance contribution) and the benefits from the social insurance under the risk neutral probability \( Q \). The risk neutral probability \( Q \) on \((\Omega, \mathcal{F})\) is defined by
\[ \left. \frac{dQ}{dP} \right|_{\mathcal{F}_T} = \exp \left\{ -\frac{1}{2} \theta^2 T - \theta BT \right\}. \]

Remarkably, if the individual’s initial wealth is \( A \) and has no subsequent incomes, the corresponding optimal asset allocation and consumption policies will be equivalent to the original optimization problem.

In order to make the problem (3.4) well defined, a prerequisite is added:
\[ \mathbb{E} \left[ \int_0^T -L \Gamma_s ds \right] \leq A - h_0 z. \tag{3.6} \]
Thus, the incomes of the wages and the benefits are adequate to support the minimal consumption level.

**Lemma 3.1.** The subsidiary problem (3.4) obtains its maximum when
\[ c_s = c^*_s \triangleq \mathcal{Y}_s(\nu \Gamma_s), \ s \in [0, T], \tag{3.7} \]
where
\[ \mathcal{Y}_s(y) = \arg \max_{x \geq -L} \{u_s(x) - xy\}, \ s \in [0, T]. \tag{3.8} \]
And \( \nu > 0 \) (the value can be \(+\infty\)) satisfies the following equation:
\[ \mathbb{E} \left[ \int_0^T \mathcal{Y}_s(\nu \Gamma_s) \Gamma_s ds \right] = A - h_0 z. \tag{3.9} \]

**Proof.** Define \( f(x) \triangleq \mathbb{E} \left[ \int_0^T \mathcal{Y}_s(x \Gamma_s) \Gamma_s ds \right] \). Then \( f \) is a non-increasing continuous function based on the property of \( \mathcal{Y}_s \) and monotone convergence theorem. In addition, as \( \mathcal{Y}_s(0+) = \infty \) and \( \mathcal{Y}_s(\infty) = -L \), we have the estimations: \( f(0+) = \infty \) and \( f(\infty) = \mathbb{E} \left[ \int_0^T -L \Gamma_s ds \right] \). According to the assumption that \( \mathbb{E} \left[ \int_0^T -L \Gamma_s ds \right] \leq A - h_0 z \), we know that there exists a \( \nu \) satisfying (3.9).

Furthermore, the maximal expected utility of (3.4) can be obtained by the following implicit form:
\[ \mathbb{E} \left[ \int_0^T V_s(\nu \Gamma_s) ds \right] + \nu(A - h_0 z), \]
where
\[ V_s(y) \triangleq \max_{x \geq -L} \{u_s(x) - xy\}, \quad s \in [0, T]. \]

Based on the definitions of \( Y \) and \( V \), it is easy to deduce that
\[
E \left[ \int_0^T u_s(c_s) ds \right] \leq E \left[ \int_0^T (V_s(\nu \Gamma_s) + c_s \nu \Gamma_s) ds \right] = E \left[ \int_0^T V_s(\nu \Gamma_s) ds \right] + \nu (A - h_0 \xi). \text{ And the equality holds when } c_s = Y_s(\nu \Gamma_s). \]

We have established the necessary condition of the existence of \( c^* \triangleq \{ c_t^*, \ 0 \leq t \leq T \} \). We still need to establish the corresponding optimal consumption policy \( C^* \triangleq \{ C_t^*, \ 0 \leq t \leq T \} \) and optimal risky investment policy \( \pi^* \triangleq \{ \pi_t^*, \ 0 \leq t \leq T \} \) to make the process \( c^* \) attainable.

As \( h \) satisfies the ordinary differential equation (abbr. ODE):
\[
dh_t = [\psi(t)c_t - \eta(t)h_t] dt = [\psi(t)(c_t + h_t) - \eta(t)h_t] dt = [\psi(t)c_t - (\eta(t) - \psi(t))h_t] dt, \quad 0 \leq t \leq T,
\]
solving this ODE, we have, for \( t \in [0, T] \),
\[
h_t = e^{-\int_0^t (\eta(s) - \psi(s)) ds} h_0 + \psi(t) \int_0^t e^{-\int_0^u (\eta(u) - \psi(u)) du} c_s ds. \quad (3.10)
\]

By combining (3.7) and (3.10), the explicit form of the consumption level \( C^* \) is given as follows:
\[
C_t^* = Y_t(\nu \Gamma_t) + e^{-\int_0^t (\eta(s) - \psi(s)) ds} h_0 + \psi(t) \int_0^t e^{-\int_0^u (\eta(u) - \psi(u)) du} Y_s(\nu \Gamma_s) ds, \quad t \leq T.
\]

However, the corresponding asset allocation policy \( \pi^* \) is not easy to derive. We need to reformulate the wealth process \( X^* \) skillfully and construct the corresponding \( \pi^* \) to make the wealth process attainable.

**Theorem 3.2.** If Assumption (3.6) is satisfied, the corresponding risky investment policy \( \pi^* \) is given by
\[
\pi_t^* = \frac{1}{\sigma} \left( \frac{\varphi_t}{H_t} + \theta X_t^* \right), \quad t \leq T,
\]
where the process \( \varphi \) is defined in Eq. (3.13).

**Proof.** Based on SDE (3.1), the optimal wealth process \( X^* \) has the boundary constraint that \( X_T^* = 0 \), that is, the optimum is attained when the individual consumes all the incomes and has nothing at time \( T \). Otherwise, he/she could
consume more at time \( T \) to improve the overall utility. By combing (3.1) with \( X^*_T = 0 \), the wealth process \( X^* \) is reformulated as follows:

\[
X^*_t = \frac{1}{H_t} \mathbb{E}_t \left\{ \int_t^T \left[ C^*_sH_s - (1 - k)H_sW_s1_{\{s \leq \tau\}} - g(\tau)De^{\xi s}H_s1_{\{s > \tau\}} \right] ds \right\} \tag{3.11}
\]

for \( t \in [0, T] \). Let \( M \) be an \( \mathbb{F} \)-adapted, right continuous martingale and satisfy:

\[
M_t = \mathbb{E}_t \left\{ \int_T^0 \left[ C^*_sH_s - (1 - k)H_sW_s1_{\{s \leq \tau\}} - g(\tau)De^{\xi s}H_s1_{\{s > \tau\}} \right] ds \right\}, \text{a.s.,} \tag{3.12}
\]

for \( t \in [0, T] \). By martingale representation theorem, there exists a unique square-integrable process \( \phi = \{ \phi_t, 0 \leq t \leq T \} \) satisfying

\[
M_t = \int_0^t \phi_s dB_s, \quad t \leq T. \tag{3.13}
\]

Thus, using (3.11), (3.12) and (3.1), we have

\[
M_t = \mathbb{E}_t \left\{ \int_T^0 \left[ C^*_sH_s - (1 - k)H_sW_s1_{\{s \leq \tau\}} - g(\tau)De^{\xi s}H_s1_{\{s > \tau\}} \right] ds \right\}
+ \int_0^t \left[ C^*_sH_s - (1 - k)H_sW_s1_{\{s \leq \tau\}} - g(\tau)De^{\xi s}H_s1_{\{s > \tau\}} \right] ds
= X^*_tH_t - \int_0^t \left[ (1 - k)H_sW_s1_{\{s \leq \tau\}} + g(\tau)De^{\xi s}H_s1_{\{s > \tau\}} - C^*_sH_s \right] ds
= \int_0^t \left[ \sigma \pi^*_sH_s - \theta X^*_sH_s \right] dB_s. \tag{3.14}
\]

Comparing the diffusion terms of \( M_t \) in (3.13) and (3.14), we have

\[
\pi^*_t = \frac{1}{\sigma} \left( \frac{\varphi_t}{H_t} + \theta X^*_t \right).
\]

The detailed calculation of \( C^* \), \( X^* \) and \( \pi^* \) is derived in Appendix A.4. Thus, the optimal consumption and asset allocation policies are established with the given retirement time.

At last, we treat the optimal retirement time as a pre-commitment policy and establish the optimum at time 0. Because the retirement time is variable in the second step of optimization, we rewrite the individual’s value function as follows:

\[
V(\tau) = \mathbb{E} \left[ \int_0^T u_s(\mathcal{V}_s(\nu_\tau \Gamma_{\tau,s}))ds \right],
\]

where \( \Gamma_s, A, z \) and \( \nu \) are rewritten as \( \Gamma_{\tau,s}, A(\tau), z(\tau) \) and \( \nu_\tau \), and \( \tau \) is the control variable. \( \nu_\tau \) is determined by \( \mathbb{E} \left[ \int_0^T \mathcal{V}_s(\nu_\tau \Gamma_{\tau,s})\Gamma_{\tau,s}ds \right] = A(\tau) - h_0z(\tau) \).
The remained work is to solve $\arg\max_{\tau}\{V(\tau)\}$. The optimum is attained when the value of $\tau$ is at the boundary points $\tau_{\text{min}}$, $\tau_{\text{max}}$ or the extreme point satisfying $\frac{\partial V(\tau)}{\partial \tau} = 0$. It is difficult to derive the explicit form of $\tau$ and we solve it numerically. Fortunately, we can study the influence of the exogenous parameters on the optimal retirement time by an innovative analytical method. The specific analysis is given in the next section.

4. Theoretical and numerical implications

In this section, we first introduce the certainty equivalence of the actual consumption and the habitual consumption processes. Under the reasonable parameter settings, we study the evolution of the optimal consumption over time and try to explain the “retirement consumption puzzle”. Then, we study the influence of the exogenous parameters on the optimal consumption policies. Theoretically, exogenous variables can be divided into two categories that affect the amount of wealth and the habitual level, and they influence the optimal consumption ultimately through these two intermediate variables. The last, we use an innovative idea to study the influence of the parameters on the optimal retirement time analytically.

4.1. Certainty equivalence of the actual and habitual consumption processes. Because the actual and habitual consumption processes are both stochastic, we usually explore $\mathbb{E}[C^*_t]$ and $\mathbb{E}[h^*_t]$ to study the evolutions of the two processes over time. However, the traditional explored method is not accurate, especially under the S-shaped utility and the long time horizon. There is a very small probability that the amounts of the actual and habitual consumption are magnificent because of excellent wealth appreciation or radical investment policies. In this circumstance, the magnificent consumption levels of some tracks will greatly raise the level of the expectations $\mathbb{E}[C^*_t]$ and $\mathbb{E}[h^*_t]$ over the long time horizon. Thus, the traditional method loses its representativeness.

We naturally explore the certainty equivalence of the actual and habitual consumption processes to study the optimal policies. Under this perspective, the process $\hat{C}$ and its corresponding habitual process $\hat{h}$ satisfying $u_t(\hat{C}_t - \hat{h}_t) = \mathbb{E}[u_t(C^*_t - h^*_t)]$ are defined as the certainty equivalence of the
consumption processes. That is, when the individual’s deterministic actual
and habitual consumption amounts are \( \{\hat{C}_t, \ t \in [0,T]\} \) and \( \{\hat{h}_t, \ t \in [0,T]\} \),
he/she obtains the same utility as the one under the stochastic consumption
processes. Particularly, \( \hat{c}_t = \hat{C}_t - \hat{h}_t \) is the excess consumption under certainty
equivalence. It directly measures the magnitude of the utility obtained at time
\( t \).

In the next subsection, we study the evolution of the optimal consumption
over time under the certainty equivalence perspective.

4.2. Parameter settings and the baseline model. In this subsection, we
first set up the reasonable parameter settings according to the real data in the
financial market, the labor market and the social insurance market.

For the financial market, the risk-free interest rate is \( r = 0.02 \), and the
expected return and the volatility of the risky asset are \( \mu = 0.08, \sigma = 0.4, \)
respectively. As such, the market price of risk is \( \theta = 0.15 \). For the rules of the
labor market, we assume that the parameters of the individual’s wage income
are \( \alpha = 0.028, \beta = 0.02 \) and \( W_0 = 10 \). The individual begins to work at the
age of 25 and his/her maximal survival age is 100. Besides, the individual can
choose the actual retirement age within \([50, 80]\) and the statutory retirement
age is 65. Retiring at the statutory retirement age, the individual can obtain
the full benefit. As such, we have \( t_0 = 0, \ T = 75, \tau_{\text{min}} = 25, \tau_{\text{max}} = 55 \)
and \( \tau_{\text{st}} = 40 \). Furthermore, the parameters related to the social insurance are given
as follows. The contribution rate is \( k = 0.2 \) and the full benefit is \( D = 6 \) at
time 0. The elastic parameter in the penalty variable is \( \zeta = 0.015 \) and the
growth rate of the benefit is \( \xi = 0.018 \). In general, if the individual retires at
the minimal retirement age, he/she will obtain nearly 70% of the full benefit.

For the settings of the habitual consumption level, we assume that \( \psi = \eta \)
usually holds. As such, the habitual level is the arithmetic average of the past
consumption levels. And it is realistic to assume that \( \eta = \psi = 0.05 \). After
retirement, both the level and the sensitivity of the habitual consumption
decline, and we have \( l = 0.6 \) and \( m = 0.3 \). These two parameters are originally
used to depict the changes of the habitual consumption after retirement. Thus,
we choose the logical values and change their values to study the impacts on
the optimal consumption and retirement policies. Besides, we assume that
the initial habitual level is $h_0 = 6$ and the minimal consumption constraint requires that $L = 0.5$. For the parameters in the S-shaped utility function, $\rho = 0.04$, $\gamma = 0.8$ and $\kappa = 2.25$ are realistic settings.

Then, based on the baseline model, we exhibit the certainty equivalence of the optimal consumption $\hat{C}$ and the habitual consumption $\hat{h}$. First, we assume that the retirement time is given at the statutory retirement time. That is, $\tau = \tau_{st} = 40$. Figure 1 shows the evolution of the optimal consumption and the habitual consumption over time. We observe a sharp decline in the actual and habitual consumption levels at retirement. This is consistent with the empirical evidence observed in the “retirement consumption puzzle”. Because the incomes of the baseline model are relatively abundant, the actual consumption is higher than the habitual consumption, and this leads to the spiral rise of the two consumption levels. However, the growth rates of the actual and the habitual consumption levels decline after retirement, which is caused by the reduced sensitivity of the habitual formulation. In addition, we find that the initial consumption is higher than the wage income. In this circumstance, the individual initiates some loans to improve the early consumption level and obtains higher utility.

![Figure 1. The optimal consumption $\hat{C}$ and the habitual consumption $\hat{h}$.](image-url)
In Figure 2 and Figure 3, we select three typical retirement times $\tau = 35, 40, 45$ to study its influence on the excess consumption level $\hat{c}$ and the optimal consumption level $\hat{C}$.

Theoretically, the optimal excess consumption level $\hat{c}$ is balanced between two effects. The first is the wealth effect. If the individual obtains more incomes, he/she will naturally consume more. The second is the habitual effect. Larger excess consumption rises the habitual consumption level even more and leads to more pressure on the follow-up consumption. This prevents the individual from consuming too much in the early time. If the habitual effect weakens, time preference effect will be the dominance. That is, the early consumption produces higher utility due to the human nature of impatience. In this circumstance, the individual will consume more at the moment, and vice versa. Thus, we observe a sharp rise in the excess consumption level after retirement because of the reduced sensitivity of the habitual level and the weakened habitual effect.

![Fig. 2. The optimal excess consumption $\hat{c}$ with respect to different retirement time $\tau$.](image-url)
In Figure 2, the excess consumption levels are both low before and after retirement in the $\tau = 35$ case due to the lower total incomes. Interestingly, the excess consumption is lower before retirement and higher after retirement in the $\tau = 45$ case than in the $\tau = 40$ case. Longer working time also means using the larger habitual consumption benchmark for a longer time. Thus, the habitual effect strengthens. The individual decreases the former consumption and increases the latter consumption to balance the two effects. Furthermore, although the excess consumption rises a lot at retirement, it cannot offset the sharp decline in the habitual level. Thus, we observe a decline in the actual consumption $\hat{C}$ at retirement in Figure 3.

In Figure 4, we study the optimal asset allocation policies over time to ensure the integrity of the problem. As usual, we explore the expectation of the amount allocated to the risky asset. We observe a downward curve due to the life cycle phenomenon. Thus, the special variation assumption in habitual persistence after retirement does not affect the trend of investment.

Interestingly, there is an unsmooth rise around the time $\tau_{min} = 25$. Before the time $\tau_{min}$, the individual’s wage income is fluctuating. However, after the time $\tau_{min}$, the individual receives static wage and benefit incomes. In addition, the random sources of the wage and the risky investment are the same. As such, the former risky investment should be smaller to avoid bearing too much
risk. The latter risky investment can be larger as the risk bearing ability is improved. The result could also be drawn from Eq. (A.5).

![Fig. 4. The optimal risky investment amount $E\pi_t^\ast$.](image)

4.3. **Impacts of the parameters on the optimal consumption.** In this subsection, we study the impacts of the exogenous parameters on the certainty equivalence of the optimal consumption $\hat{C}$ under the assumption that the individual retires at the statutory retirement time.

In Figure 5, we study the impacts of total wealth $A$ (the discounted present value of the incomes) on the optimal consumption $\hat{C}$. In fact, the value of $A$ is determined by a family of parameters, such as $W_0$, $D$, $r$, $\mu$, $\sigma$, $k$, $\xi$ and $\zeta$, etc. The result confirms the positive correlation between the total wealth and the optimal consumption. When the total wealth $A$ is relatively large, the trends of the optimal consumption are almost the same. Interestingly, we observe that the percentage of decline in consumption at retirement is negatively correlated with the total wealth. This result is consistent with the empirical evidence that the consumption drop is negatively correlated with income replacement rate after retirement (cf. Schwerdt (2005)). However, when $A$ is relatively small, the optimal consumption shows a declining trend. In the case of $A = 200$, which could be interpreted as $W_0 = 4$ and $D = 2$, the total incomes are not enough to support the consumption above the habitual level. Thus, the actual
consumption is lower than the habitual consumption. This leads to the spiral decline of the both consumption levels and produces negative utilities.

Fig. 5. The impacts of total wealth $A$ on the optimal consumption $\hat{C}$.

In Figure 6, we study the impacts of the habitual parameters $\psi$ and $\eta$ on the optimal consumption $\hat{C}$. In order to make the model well defined, we assume that $\psi = \eta$. As discussed in Subsection 4.2, smaller $\psi$ and $\eta$ represent lower sensitivity of the habitual level. Because the habitual effect weakens, the time preference effect becomes the dominance. As such, the individual increases former consumption and reduces latter consumption. In the extreme case $\psi = \eta = 0$, we observe a completely declining trend of the optimal consumption. In this circumstance, the habitual consumption level is not affected by the former consumption. Thus, larger consumption at the moment will produce higher utility according to the time preference effect.
In Figure 7, we study the impacts of the shrinking habitual proportion $l$ on the optimal consumption $\hat{C}$. $l$ measures the magnitude of the decline in the level of habitual consumption after retirement. Different $l$ leads to the redistribution of the consumption levels before and after retirement. When $l$ is smaller, the lower consumption makes the individual feel the same satisfactory as before retirement. Thus, he/she can consume more before retirement. Although the consumption level after retirement is reduced, the excess consumption is still considerable because of the sharp decline of the habitual effect.
In Figure 8, we study the impacts of the shrinking sensitivity parameter $m$ on the optimal consumption $\hat{C}$. Because $m$ only influences the sensitivity of the habitual level after retirement, its variation hardly has impacts on the consumption levels before retirement. When $m$ is smaller, the habitual effect weakens. In this circumstance, the individual prefers to consume more in the early time because of the time preference effect. Similar to the result in Figure 6, the optimal consumption exhibits a declining trend after retirement in the extreme case $m = 0$.

![Figure 8](image)

**Fig. 8.** The impacts of shrinking sensitivity parameter $m$ on the optimal consumption $\hat{C}$.

In Figure 9, we study the impacts of the initial habitual consumption level $h_0$ on the optimal consumption $\hat{C}$. The result shows that the lower initial consumption increases the admissible domain of the follow-up consumption. And the inertia of the larger excess consumption in the former time leads to the higher rise of the latter consumption. On the contrary, when $h_0$ is large, the actual and habitual consumption levels in the latter time are relatively small because of the small excess consumption in the former time.
Fig. 9. The impacts of initial habitual consumption $h_0$ on the optimal consumption $\hat{C}$.

4.4. Optimal retirement time. In this subsection, we treat the optimal retirement time as a pre-commitment control variable and establish the optimal retirement time to maximize the individual’s value function. Then, we establish an analytical method, which can be used to analyze the impacts of the parameters on the optimal retirement time quantitatively.

Before establishing the optimal retirement time, we first analyze the two impacts of the retirement time on the overall utility. The first is the wealth effect. The wealth effect can be accurately estimated by the function $A(\tau)$. In Figure 10, we observe that $A(\tau)$ is an increasing function with respect to $\tau$. From this perspective, the later an individual retires, the more wealth and higher utility he/she can obtain. The second is the habitual effect. The habitual level increases with the extension of the working time according to the baseline model. From this perspective, the later an individual retires, he/she suffers from higher habitual consumption level and obtains lower utility. In Figure 11, we modify the parameter settings to preserve the validation of $A = 500$, as such, the influence of the wealth effect is excluded. We observe a negative relationship between the retirement time and the overall utility simply based on the habitual effect. Overall, the optimal retirement time is the balance between the two effects.
Next, we study the impacts of the retirement time on the overall utility based on the two effects and establish the optimal retirement time numerically. In Figure 12, the individual’s utility function first increases and then decreases with respect to the extension of retirement time. In the former time, the wealth effect dominates the habitual effect. And the habitual effect becomes the dominance in the latter time. Interestingly, the optimal retirement time is
around $\tau_{st} = 40$, which is exactly the statutory retirement age of 65, according to the baseline model.

![Fig. 12. The impacts of retirement time $\tau$ on overall utility $V(\tau)$.](image)

The last, the above discussions on the two effects are helpful to study the impacts of the exogenous parameters on the optimal retirement time. Besides, we establish a quantitative analytical method as follows. For any exogenous parameter $y$, we have the denotation that $\tau^*(y) \triangleq \mathop{\arg\sup}_{\tau} \{V(y, \tau)\}$. As such, $V_{\tau}(y, \tau^*(y)) = 0$ is valid. Differentiating with respect to $y$ at both sides of this equality, $\frac{d\tau^*(y)}{dy} = -\frac{V_{\tau\tau}(y, \tau^*(y))}{V_{\tau\tau}(y, \tau^*(y))}$ where $V_{\tau\tau}(y, \tau^*(y)) < 0$. As such, $\frac{d\tau^*(y)}{dy}$ has the same sign with $V_{\tau\tau}(y, \tau^*(y))$. Furthermore, the second-order derivative can be approximated by

$$
\frac{1}{\delta y \delta \tau} [V(y + \delta y, \tau^*(y) + \delta \tau) - V(y, \tau^*(y) + \delta \tau) - V(y + \delta y, \tau^*(y)) + V(y, \tau^*(y))].
$$

(4.1)

Even if $V$ is not smooth, judging the sign of (4.1) is still useful. The sign is determined by comparing the size of the two values $V(y + \delta y, \tau^*(y) + \delta \tau) - V(y, \tau^*(y) + \delta \tau)$ and $V(y + \delta y, \tau^*(y)) - V(y, \tau^*(y))$, i.e., we can check whether the disturbance of $y$ will produce higher utility in the case of earlier retirement or later retirement. Remarkably, we only exhibit the parameters of high relevance and sensitivity in Table 1. Table 1 shows the relationship between the exogenous parameters and the optimal retirement time. The upward arrow indicates that the larger parameter leads to later retirement, and vice versa.
Echoing the discussions on the wealth effect and the habitual effect, $W_0$, $\alpha$, $k$, $D$ and $\xi$ are the parameters reflecting wealth effect, and $\psi$, $m$, $l$ and $h_0$ are the parameters reflecting habitual effect. Naturally, larger initial level $W_0$ and wage growth rate $\alpha$ will lead to more wealth if the retirement is postponed. On the contrary, larger $k$ leads to more contribution during the working period. And larger $D$ and $\xi$ lead to more benefit during the retirement period. Thus, these parameters make the individual retire early.

For the habitual effect, larger $\psi$ and $m$ reflect higher sensitivity of the habitual level. As such, earlier retirement is required to prevent the habitual level from rising too high and reducing overall utility. Besides, if the shrinking habitual proportion $l$ is larger, the utility improvement by the shrinking habitual level after retirement is less effective. Thus, later retirement better balances the wealth effect and the habitual effect in this circumstance. Naturally, early retirement is required to control the habitual level when the initial habitual consumption $h_0$ is large.

5. Conclusion

In this paper, we assume that the level and the sensitivity of the habitual consumption both decline at the time of retirement. Under the variation assumption in habitual persistence, we establish the theoretical results consistent with the empirical evidence. That is, the optimal consumption experiences a sharp decline at retirement. In fact, the optimal decisions are balanced between the wealth effect and the habitual effect. Smaller growth inertia (sensitivity) of the habitual level weakens the habitual effect and leads to larger excess consumption after retirement. However, this effect cannot offset the sharp decline in the habitual level. Furthermore, the individual with higher habitual sensitivity retires early to prevent the habitual level from rising too high. Moreover, if the shrinking habitual parameter at retirement is larger, delaying retirement is optimal because of that the utility improvement by the shrinking

| $W_0$ | $\alpha$ | $k$ | $D$ | $\xi$ | $\psi$ | $m$ | $l$ | $h_0$ |
|-------|----------|-----|-----|------|-------|-----|-----|-------|
| ↑     | ↑        | ↓   | ↓   | ↓    | ↓     | ↓   | ↑   | ↓     |
habitual level after retirement is less effective and the wealth effect becomes the dominance.

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APPENDIX A. DETAILS OF THE CALCULATIONS

A.1. The calculation of $C_t$ and $X_t$. Before deriving the exact form of $X_t$, we claim that the consumption level $C_t$ has the lower boundary $C_t$, which is the required minimal consumption:

$$C_t = e^{-\int_t^s (\eta(u)-\psi(u))du} h_0 - \psi(t) \int_0^t e^{-\int_u^s (\eta(v)-\psi(v))dv} L ds - L.$$  

In fact, $C_t$ is obtained when the individual keeps the consumption level with the gap $-L$ to the habitual consumption level. As such, $X_t$ can be written as

$$X_t = \int_t^T \left[ e^{-r(s-t)} C_s - g(t) De^{\xi s} e^{-r(s-t)} 1_{\{s>\tau\}} \right] ds.$$  

Remarkably, the condition $X_t \geq X_t$ indicates that the total wealth must be adequate to support the required minimal consumption in the worst scenario that the wage income and the risky investment return decay to 0, i.e., Brownian motion $B$ experiences a sharp decline.
A.2. The calculation of $\Gamma$. As $\frac{H_s}{H_t} = \exp \left\{ -r(s-t) - \frac{1}{2}\theta^2(s-t) - \theta(B_s-B_t) \right\}$, $s \geq t$, is independent of $\mathcal{F}_t$, we have

$$E_t \left\{ \frac{H_s}{H_t} \right\} = E \left\{ \frac{H_s}{H_t} \right\} = e^{-r(s-t)}.$$ 

As such, for $t \in [0, T]$, 

$$\Gamma_t = H_t + \psi(t)E_t \left\{ \int_t^T e^{-\int_t^s (\eta(u) - \psi(u))du} H_s ds \right\}$$

$$= H_t \left\{ 1 + \psi(t)E_t \left[ \int_t^T e^{-\int_t^s (\eta(u) - \psi(u))du} \frac{H_s}{H_t} ds \right] \right\}$$

$$= H_t \left\{ 1 + \psi(t) \int_t^T e^{-\int_t^s (\eta(u) - \psi(u))du} e^{-r(s-t)} ds \right\}$$

$$= H_t (1 + F_t),$$

where 

$$F_t = \psi(t) \int_t^T e^{-\int_t^s (\eta(u) - \psi(u))du} e^{-r(s-t)} ds$$

$$= \begin{cases} \frac{w e^{-(r+\eta-\psi)(r-t)}}{r+\eta-\psi} \left[ 1 - e^{-(r+\eta-\psi)(T-t)} \right] & 0 \leq t < \tau, \\
\frac{r + \eta - \psi}{r + \eta - \psi} \left[ 1 - e^{-(r+\eta-\psi)(T-t)} \right], & \tau \leq t < T. \end{cases}$$

A.3. The calculation of $A$ and $z$. Similar to Appendix A.2, for $t \leq s \leq \tau_{\text{min}}$,

$$E_t \left( \frac{H_s W_s}{H_t W_t} \right) = E \left( \frac{H_s W_s}{H_t W_t} \right)$$

$$= E \left[ e^{(\alpha - r + \beta s)(s-t) + \beta(B_s-B_t)e^{-r(s-t) - \frac{1}{2}\theta^2(s-t) - \theta(B_s-B_t)}} \right]$$

$$= e^{(\alpha - r + \beta\tau)z(s-t)}.$$ 

As such, we derive the exact value of the restrictions in (3.5) as follows:

$$A = E \left\{ \int_0^T \left[ (1 - k)H_s W_s 1_{\{s\leq\tau\}} + g(\tau)De^{\xi s}H_s 1_{\{s>\tau\}} \right] ds \right\}$$

$$= E_0 \left\{ \int_0^T \left[ (1 - k)H_s W_s 1_{\{s\leq\tau\}} + g(\tau)De^{\xi s}H_s 1_{\{s>\tau\}} \right] ds \right\}$$

$$= \int_0^T \left\{ (1 - k)E_0[H_s W_s] 1_{\{s\leq\tau\}} + g(\tau)De^{\xi s}E_0[H_s] 1_{\{s>\tau\}} \right\} ds$$

$$= \int_0^T \left\{ (1 - k)E_0 e^{(\alpha - r + \beta\tau)s} 1_{\{s\leq\tau\}} + \left[ g(\tau)De^{\xi s}e^{-r s} 1_{\{s>\tau\}} \right] \right\} ds,$$ 

(A.1)
and the value of \( z \) is

\[
z = E \left[ \int_0^T e^{-\int_0^s \eta(u) du} \Gamma_s ds \right]
= E \left[ \int_0^T e^{-\int_0^s \eta(u) du} H_s (1 + F_s) ds \right]
= \int_0^T e^{-\int_0^s \eta(u) du} e^{-r_s (1 + F_s)} ds.
\]  \( \text{(A.2)} \)

By combining (A.1) and (A.2), the exact value of \( A - h_0z \) is established.

A.4. **The calculation of \( C^* \), \( X^* \) and corresponding \( \pi^* \).** The function \( Y_s \) has the exact form that

\[
Y_s(y) = \begin{cases} 
-L, & y \geq y_0(s), \\
\frac{-y}{e^{-\rho s}} - \frac{1}{\gamma}, & 0 < y < y_0(s), 
\end{cases}
\]

where \( y_0(s) \) satisfies the following equation:

\[
e^{-\rho s} \left( \frac{y_0(s)}{e^{-\rho s}} \right)^{\frac{1-\gamma}{\gamma}} - (-\kappa) e^{-\rho s} \frac{L^{1-\gamma}}{1-\gamma} = y_0(s) \left( \frac{y_0(s)}{e^{-\rho s}} \right)^{\frac{1}{\gamma}} + L.
\]

In fact, \( y_0(s) \) is the gradient of the tangential line from \(-L\) to \( \left( \frac{y_0(s)}{e^{-\rho s}} \right)^{\frac{1}{\gamma}} \).

Based on the exact form of the function \( Y_s \), we have

\[
Y_s(\nu \Gamma_s) = -L 1_{\nu \Gamma_s \geq y_0(s)} + \left( \frac{\nu \Gamma_s}{e^{-\rho s}} \right)^{\frac{1}{\gamma}} 1_{\nu \Gamma_s < y_0(s)}
= -L 1_{H_s > \frac{y_0(s)}{e^{-\rho s}(1 + F_s)}} + H_s^{\frac{1}{\gamma}} \left( \nu (1 + F_s) e^{\rho s} \right)^{\frac{1}{\gamma}} 1_{H_s < \frac{y_0(s)}{e^{-\rho s}(1 + F_s)}}.
\]

As such, we calculate the process \( X^* \) as follows:

\[
X^*_t = \frac{1}{H_t} E_t \left\{ \int_t^T \left[ C^*_s H_s - (1 - k) H_s W_s 1_{\{s \leq \tau\}} - g(\tau) De^{\xi s} H_s 1_{\{s > \tau\}} \right] ds \right\}
\]

\[
= \frac{1}{H_t} E_t \left\{ \int_t^T C^*_s H_s ds \right\} - (1 - k) \frac{1}{H_t} E_t \left[ \int_t^T H_s W_s 1_{\{s \leq \tau\}} ds \right]
- \frac{1}{H_t} E_t \left[ \int_t^T g(\tau) De^{\xi s} H_s 1_{\{s > \tau\}} ds \right]
\]

\[
= \frac{1}{H_t} E_t \left[ \int_t^T C^*_s H_s ds \right] - (1 - k) W_t E_t \left[ \int_t^T H_s W_s \frac{1_{\{s \leq \tau\}}}{H_t W_t} ds \right]
- W_t E_t \left[ \int_t^T g(\tau) De^{\xi s} \frac{H_s}{H_t} 1_{\{s > \tau\}} ds \right].
\]  \( \text{(A.3)} \)

The last two terms in (A.3) are

\[
W_t E_t \left[ \int_t^T H_s W_s \frac{1_{\{s \leq \tau\}}}{H_t W_t} ds \right] = W_t O_t,
\]
where
\[
O_t = \int_t^T \left[ e^{(\alpha-r+\theta\beta)(s-t)}1_{\{s\leq t\cup s\leq \tau_{\text{min}}\}} + e^{(\alpha-r+\theta\beta)(\tau_{\text{min}}-t)}e^{-r(s-\tau_{\text{min}})}1_{\{t<\tau_{\text{min}}<s\leq t\}} \\
+ e^{-r(s-t)}1_{\{\tau_{\text{min}}\leq t<s\leq t\}} \right] ds,
\]
and
\[
E_t \left[ \int_t^T g(\tau)|D\xi_s\frac{H_s}{H_t}1_{\{s>\tau\}}| ds \right] = \int_t^T g(\tau)|D\xi_s e^{-r(s-t)}1_{\{s>\tau\}}| ds.
\]
The first term in (A.3) can be rewritten as follows:
\[
\frac{1}{H_t} E_t \left[ \int_t^T C_s^* H_s ds \right] = \frac{1}{H_t} E_t \left\{ \int_t^T \left[ \mathcal{Y}_s(\nu \Gamma_s) + e^{-\int_0^s (\eta(u)-\psi(u)) du} h_0 + \psi(s) \int_0^s e^{-\int_0^u (\eta(v)-\psi(v)) dv} Y_u(\nu \Gamma_u) du \right] H_s ds \right\}
\]
\[
= \int_t^T \left[ e^{-\int_0^s (\eta(u)-\psi(u)) du} h_0 + \psi(s) \int_0^s e^{-\int_0^u (\eta(v)-\psi(v)) dv} Y_u(\nu \Gamma_u) du \right] e^{-r(s-t)} ds \\
+ \frac{1}{H_t} E_t \left\{ \int_t^T \left[ \mathcal{Y}_s(\nu \Gamma_s) + \psi(s) \int_t^s e^{-\int_0^u (\eta(v)-\psi(v)) dv} Y_u(\nu \Gamma_u) du \right] H_s ds \right\}.
\]
The second equation holds as the integral can be divided into two parts: one before time \(t\) and the other after time \(t\), and the former part is \(\mathcal{F}_t\)-measurable.

We denote the latter part as \(f_t(H_t)\):
\[
f_t(H_t) \triangleq \frac{1}{H_t} E_t \left\{ \int_t^T \left[ \mathcal{Y}_s(\nu \Gamma_s) + \psi(s) \int_t^s e^{-\int_0^u (\eta(v)-\psi(v)) dv} Y_u(\nu \Gamma_u) du \right] H_s ds \right\}
\]
\[
= \frac{1}{H_t} E_t \left\{ \int_t^T \left[ \mathcal{Y}_s(\nu \Gamma_s) H_s + \psi(s) \int_t^s e^{-\int_0^u (\eta(v)-\psi(v)) dv} Y_u(\nu \Gamma_u) E[H_s|H_u] du \right] ds \right\}
\]
\[
= \frac{1}{H_t} E_t \left\{ \int_t^T \mathcal{Y}_s(\nu \Gamma_s) H_s \left[ 1 + e^{-\int_0^s (\eta(v)-\psi(v)) dv} \int_t^s \psi(u)e^{-r(u-s)} du \right] ds \right\}
\]
\[
= \frac{1}{H_t} E_t \left\{ \int_t^T \mathcal{Y}_s(\nu \Gamma_s) H_s N_s ds \right\}
\]
\[
= \int_t^T \frac{1}{H_t} E_t \left\{ \mathcal{Y}_s(\nu \Gamma_s) H_s \right\} N_s ds
\]
\[
= \int_t^T e^{-r(s-t)} N_s \left\{ (e^{\rho s} y_0(s)) - \frac{1}{\Phi(\rho s y_0(s))} \right\} \frac{dF'(d_{1,t,s}(\frac{y_0(s)}{\nu(1+F_s)H_t}))}{\Phi(d_{2,t,s}(\frac{y_0(s)}{\nu(1+F_s)H_t}))} \left[ 1 - \Phi(d_{2,t,s}(\frac{y_0(s)}{\nu(1+F_s)H_t})) \right]
\]
\[
- \Phi(d_{1,t,s}(\frac{y_0(s)}{\nu(1+F_s)H_t})) \right\} ds,
\]
where \(\Phi(\cdot)\) is cumulative distribution function of the standard normal distribution,
\[ N_s = 1 + e^{-\int_0^s (\eta(v)-\psi(v))dv} \int_s^T \psi(u)e^{-r(u-s)}du, \]

\[ d_{1,t,s}(x) = \frac{1}{-\theta \sqrt{s-t}} \left[ \log(x) + (r + \frac{\theta^2}{2})(s-t) \right], \]

\[ d_{2,t,s}(x) = \frac{1}{-\theta \sqrt{s-t}} \left[ \log(x) + (r + \frac{\theta^2}{2})(s-t) + (1-\gamma)\theta \sqrt{s-t} \right]. \]

Thus, the wealth process can be rewritten as follows:

\[ X_t^* = f_t(H_t) - (1-k)W_tO_t + \int_0^t \cdots ds. \]

Based on martingale representation theorem and a huge amount of detailed calculations, using the volatility term of \( X_t^* \), it follows that

\[ \pi_t^* = \frac{-f_t'(H_t)\theta H_t - (1-k)\beta W_tO_t1_{t<\tau_{\text{min}}}}{\sigma}. \] (A.5)