Quantum Steering

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Quantum correlations between two parties are essential for the argument of Einstein, Podolsky, and Rosen in favour of the incompleteness of quantum mechanics. Schrödinger noted that an essential point is the fact that one party can influence the wave function of the other party by performing suitable measurements. He called this phenomenon quantum steering and studied its properties, but only in the last years this kind of quantum correlation attracted significant interest in quantum information theory. In this paper the theory of quantum steering is reviewed. First, the basic concepts of steering and local hidden state models are presented and their relation to entanglement and Bell nonlocality is explained. Then various criteria for characterizing steerability and structural results on the phenomenon are described. A detailed discussion is given on the connections between steering and incompatibility of quantum measurements. Finally, applications of steering in quantum information processing and further related topics are reviewed.

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I. INTRODUCTION

A. Overview

In 1935, Einstein, Podolsky and Rosen (EPR) presented their famous argument against the completeness of quantum mechanics (Einstein et al., 1935). In this argument, a two-particle state is considered, where one party can measure the position or momentum and the correlations of the state allow one to predict the results of these measurements on the other party if the same measurement is performed there. The EPR argument led to long-lasting discussions, but already directly after its publication, Schrödinger noted an interesting phenomenon in the argument: The first party can, by choosing the measurement, steer the state on the other side into an eigenstate of position or momentum. This cannot be used to transmit information, but still Schrödinger considered it to be magic.

The early works of Schrödinger (Schrödinger, 1935; Schrödinger, 1936) did not receive much attention (see also Section V.M). This changed in 2007, when a formulation in the language of quantum information processing was given and systematic criteria were developed (Wiseman et al., 2007). In the modern view, steering denotes the impossibility to describe the conditional states at one party by a local hidden state model. As such, steering denotes a quantum correlation situated between entanglement and Bell nonlocality. In the following years, the theory of steering developed rapidly. It was noted that steering provides the natural formulation for discussing quantum information processing, if for some of the parties the measurement devices are not well characterized. Also, the concept of steering helped to understand and answer open questions in quantum information theory. An important example here is the construction of counterexamples to the Peres conjecture, which states that certain weakly entangled states do not violate any Bell inequality. Finally, steering turned out to be closely related to the concept of joint measurability of generalized measurements in quantum mechanics. More precisely, measurements that are not jointly measurable are exactly the measurements that are useful to reveal the steering phenomenon. This has sparked interest in the question in which sense measurements in quantum mechanics can be considered as a resource.

This review article aims to give an introduction to the concept and applications of quantum steering. Starting from the basic definitions, we explain steering criteria and structural results on quantum steering. We also discuss in some detail related concepts, such as quantum entanglement or the joint measurability of observables. We focus on the conceptual and theoretical issues and on the finite-dimensional case and mention experiments only very shortly. For discussing quantum steering, the tool of semidefinite programming has turned out to be useful. Concerning this, we only discuss the main formulations, concrete examples and algorithms can be found in different review articles (Cavalini and Skrzypczyk, 2017).

As mentioned, quantum steering is related to several other central concepts in quantum theory, so it may be useful to the reader to mention related relevant literature here. First, a review on the quantitative aspects of the EPR argument can be found in (Reid et al., 2009). The phenomenon of entanglement is extensively discussed in (Horodecki et al., 2009) and methods to characterize it in (Gühne and Tóth, 2009). A detailed overview on Bell inequalities and their applications is given in (Brummer et al., 2014). Finally, the theory of quantum measurements is in depth developed in (Busch et al., 2016).

The structure of the current article is the following: In the remainder of this introduction we explain the idea of quantum steering and the main definitions. We also provide a short comparison with quantum entanglement and Bell nonlocality, as this is central for the further discussion.

Section II presents different methods for the detection of quantum steering. We discuss in detail how steerability can be inferred, if some correlations or the complete quantum state is known. These methods are then used in Section III, where we describe key conceptual aspects of steering. This includes the discussion of one-way steering, the superactivation of steering, the steerability of bound entangled states and the construction of steering maps. In addition, we can then present the relation to other types of quantum correlations in detail.

Section IV deals with the connections between steering and the joint measurability of observables. We explain the concept of joint measurability and its various connections to steering. These connections allow one to transfer results from one topic to the other. Section V describes different applications of steering as well as further topics. This includes applications in quantum key distribution and randomness certification, but also topics like multiparticle steering, steering of Gaussian states, the steering ellipsoid and historical aspects of steering. Finally, Section VI presents the conclusion and some open questions.

B. Steering as a formalization of the EPR argument

Let us start by recalling the EPR argument. Originally, EPR used the position and momentum of two particles to explain their line of reasoning (Einstein et al., 1935), but in the simplest setting, the argument can be explained with two spin-$\frac{1}{2}$ particles or qubits (Bohm, 1951). Consider two particles that are in different lo-
FIG. 1 Schematic description of the steering phenomenon: A state $\rho_{AB}$ is distributed between two parties. Alice performs a measurement (labeled by $x \in \{1, 2\}$) on her particle and obtains the result $a = \pm$. Bob receives the corresponding unnormalized conditional states $\rho_{a|x}$. If Bob cannot explain this assemblage of states by assuming pre-existing states at his location, he has to believe that Alice can influence his state from a distance.

They are in the so-called singlet state,

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

where $|0\rangle = |z^+\rangle$ and $|1\rangle = |z^-\rangle$ denote the two possible spin orientations in the $z$-direction. If Alice measures the spin of her particle in the $z$-direction and obtains the result $+1$ (or $-1$) then, due to the perfect anti-correlations of the singlet state, Bob’s state will be either in the state $|1\rangle$ (or $|0\rangle$). Similarly, if Alice measures the spin in the $x$-direction, Bob’s conditional states are given by $|x^+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, if Alice’s result is $-1$ and $|x^-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ for the result $+1$.

So, by choosing her measurement setting, Alice can predict with certainty the values of a $z$- or $x$-measurement on Bob’s side. According to EPR, this means that both observables must correspond to “elements of reality”, as each of them can be predicted in principle with certainty and without disturbing the system. This raises problems if one assumes that the wave function is a complete description of the physical situation, since the corresponding observables do not commute and the quantum mechanical formalism does not allow one to assign simultaneously definite values to both of them. Consequently, EPR concluded that quantum mechanics is incomplete.

Alice cannot transfer any information to Bob by choosing her measurement directions since Bob’s reduced state is independent of this choice. But, as Schrödinger noted, she can determine whether the wave function on his side is in an eigenstate of the Pauli matrix $\sigma_z$ or $\sigma_x$. This steering of the wave function is, in Schrödinger’s own words, “magic”, as it forces Bob to believe that Alice can influence his particle from a distance, see also Section II.A.

The situation for general quantum states other than the singlet state can be formalized as follows: Alice and Bob share a bipartite quantum state $\rho_{AB}$ and Alice performs different measurements, which do not need to be projective. For each of Alice’s measurement setting $x$ and result $a$, Bob remains with an unnormalized conditional state $\rho_{a|x}$. The set of these states is called the steering assemblage, and the conditional states obey the condition $\sum_a \rho_{a|x} = \rho_B$, meaning that the reduced state $\rho_B = \text{Tr}_A(\rho_{AB})$ on Bob’s side is independent of Alice’s choice of measurements.

After characterizing the states $\rho_{a|x}$, Bob may try to explain their appearance as follows: He assumes that initially his particle was in some hidden state $\sigma_\lambda$ with probability $p(\lambda)$, parametrized by some parameter (or hidden variable) $\lambda$. Then, Alice’s measurement and result just gave him additional information on the probability of the states. This leads to states of the form (Wiseman et al., 2007)

$$\rho_{a|x} = p(a|x) \int d\lambda p(\lambda|a,x)\sigma_\lambda$$

$$= \int d\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda.$$  

The equivalence between these two expressions is easy to verify if the setting $x$ can be chosen freely and does not depend on the parameter $\lambda$, i.e., $p(x,\lambda) = p(x)p(\lambda)$. The two representations, however, point at different interpretations.

The first representation can be interpreted as if the probability distribution $p(\lambda)$ is just updated to $p(\lambda|a,x)$, depending on the classical information about the result $a$ and setting $x$. Here, Bob does not need to believe that Alice has control over his state, her measurements and results just gave him additional information about the distribution of the states $\sigma_\lambda$.

The second representation can be interpreted as a simulation task. Here, Alice can simulate the state $\rho_{a|x}$ by drawing the states $\sigma_\lambda$ according to the distribution $p(\lambda)$ and, at the same time, announcing the result $a$ depending on the known setting $x$ and the parameter $\lambda$. Consequently, Bob does not need to believe that the initial state shared by him with Alice was entangled.

Generally, if a representation as in Eq. (2) exists, Bob does not need to assume any kind of action at a distance to explain the post-measurement states $\rho_{a|x}$. Consequently, he does not need to believe that Alice can steer his state by her measurements and one also says that the state $\rho_{AB}$ is unsteerable or has a local hidden state (LHS) model. If such a model does not exist, Bob is required to believe that Alice can steer the state in his laboratory by some “action at a distance”. In this case, the state is said to be steering. Note that steerability is an inherently asymmetric correlation, there are states where Alice can steer Bob but not the other way round, see also Section III.D.

For the wave function in Eq. (1) the corresponding assemblage is formed by the states $|0\rangle/|0\rangle/2$, $|1\rangle/|1\rangle/2$, $|x^+\rangle|x^+\rangle/2$, and $|x^-\rangle|x^-\rangle/2$ and one can directly see that no LHS model exists: The four conditional states
are, up to normalization, pure and thus cannot be mixtures of other states. Thus, the occurring normalized $\sigma_\lambda$ have to be proportional to the four conditional states. So Eq. (2) implies that $|\eta(\eta)/2 = \int d\lambda p(\lambda)p(a|x, \lambda)\sigma_\lambda$ for all four $|\eta(\eta$ and $\sigma_\lambda$ coming from the set $\{0, 0\}, \{1, 1\}, \{x^+, x^+, x^-\}, \{x^-, x^-\}$. As mixtures are excluded, one must have $p(a|x, \lambda) = 1$ if the $\sigma_\lambda$ corresponds to $\rho_{a|x}$ and therefore $p(\lambda) = 1/2$ for all $\lambda$. But then the probability distribution $p(\lambda)$ cannot be normalized.

For general states and measurements, however, the existence of an LHS model is not straightforward to decide. This leads to the question of how one can decide for a given state $\rho_{AB}$ or a given assemblage $\{\rho_{a|x}\}$ whether it is steerable or not, and this is one of the central questions of the present review article.

C. Steering, Bell nonlocality and entanglement

There is another way to motivate the definition of steering and steerable correlations as in Eq. (2). For that, we shortly have to explain the notions of local hidden variable (LHV) models and entanglement.

In a general Bell experiment, Alice and Bob perform measurements on their particles, denoted by $A_x$ and $B_y$ and labeled by $x$ and $y$. For the obtained results $a, b$ one asks whether their probabilities can be written as

$$p(a, b|x, y) = \int d\lambda p(\lambda)p(a|x, \lambda)p(b|y, \lambda).$$

Such a description is known as an LHV model: The hidden variable $\lambda$ occurs with probability $p(\lambda)$ and Alice and Bob can compute the occurring joint probabilities with local response functions $p(a|x, \lambda)$ and $p(b|y, \lambda)$. For a given finite number of settings $x, y$ and outcomes $a, b$ the probabilities that can be written as in Eq. (3) form a high-dimensional polytope. The facets of the polytope are described by linear inequalities, the so-called Bell inequalities. Quantum states can result in probabilities that violate the Bell inequalities, but deciding, whether a given state violates a Bell inequality or not is not straightforward and subject of an entire field of research [Brunner et al. 2014].

Let us now describe the notion of entanglement. In general, a state on a two-particle system is called separable, if it can be written as a convex combination of product states,

$$\rho_{AB} = \sum_k p_k \rho^A_k \otimes \rho^B_k,$$

otherwise it is called entangled. The separability of a quantum state is not easy to decide, except for systems consisting of two qubits or one qubit and a qutrit, where the method of the partial transposition gives a necessary and sufficient criterion, see also Section III.B.

For our discussion, it is important that the measurements on separable states clearly can be explained by an LHV model. A general measurement $M_x$ is given by a positive operator-valued measure (POVM). This means that one considers a set of effects $E_{a|x}$ that are positive operators, $E_{a|x} \geq 0$, summing up to the identity $\sum_a E_{a|x} = \mathbb{1}$. The probability of the result $a$ in a state $\rho$ is computed according to $p(a) = \text{Tr}(\rho E_{a|x})$. Applying this to a separable state, one directly sees that the probabilities of distributed measurements can be written as

$$p(a, b|x, y) = \sum_k p_k \text{Tr}(E^A_{a|x} \rho^A_k) \text{Tr}(E^B_{b|y} \rho^B_k).$$

This is clearly an LHV model as in Eq. (3), with the extra condition that the response functions $p(a|x, \lambda)$ and $p(b|y, \lambda)$ are coming from the quantum mechanical description of measurements.

Having Eqs. (3) and (5) in mind, one may ask whether the probabilities can also be described by a hybrid model, where Alice has a general response function, while Bob’s function is derived from the quantum mechanical measurement rule. That is, one considers probabilities of the form

$$p(a, b|x, y) = \int d\lambda p(\lambda)p(a|x, \lambda)\text{Tr}(E^B_{b|y} \sigma^B_\lambda).$$

The point is that such probabilities are exactly the ones that occur in the steering scenario. By linearity we can rewrite Eq. (6) as

$$p(a, b|x, y) = \text{Tr}(E^B_{b|y} \rho_{a|x}),$$

where $\rho_{a|x} = \int d\lambda p(\lambda)p(a|x, \lambda)\sigma^B_\lambda$ are the conditional states, allowing for an LHS model as in Eq. (2).

We can conclude that the steering phenomenon relies on quantum correlations which are between entanglement and violation of a Bell inequality. In fact, any state that violates a Bell inequality can be used for steering, and

![FIG. 2 Inclusion relation between entanglement, steering, and Bell inequality violations. The set of all states is convex. The states which have an LHS model and therefore do not violate any Bell inequality form a convex subset. The states with an LHS model are not-steerable and form a convex subset of the LHV states. Finally, the separable states are a convex subset of the LHV states.](image)
any steerable state is entangled (see also Fig. 2). These inclusions are strict in the sense that there are entangled states that cannot be used for steering, and there are steerable states that do not violate any Bell inequality. In Section III.A we will discuss in detail the relation and the known examples of states in the various subsets.

It is important to note that the indicated hierarchy represents also different levels of trust in the measurement devices in entanglement verification. In general, in quantum information processing tasks such as cryptography, it makes a difference whether or not one assumes that the measurement devices are well characterized. Completely uncharacterized devices can be seen as a black box, giving just some measurement results without any knowledge about the quantum description. There can also be situations where the devices are partly characterized, e.g., if the dimension of the quantum system is known, but not the precise form of the measurement operators.

Entanglement, steering, and Bell nonlocality correspond to different levels of trust in the following sense. The usual schemes of entanglement verification, such as quantum state tomography or entanglement witnesses, require well-characterized measurement devices. The violation of a Bell inequality, however, certifies the presence of entanglement without any assumption on the measurements or dimension of the system. Steering is between the two scenarios: If a state is steerable, its entanglement can be verified in a one-sided device-independent scenario, where Bob’s measurements are characterized, but Alice’s not. In some cases, the assumptions on Bob’s system can even be relaxed, see Section III.F for an example.

II. DETECTION OF STEERING

In this part of the review, we discuss how steering can be verified in different scenarios. Mainly three cases can be distinguished. First, given some expectation values of the form $\langle A_i \otimes B_j \rangle$, one can ask whether these correlations can prove steerability. Second, one can consider the case where Bob’s assemblage $\{g_{a|x}\}$ is given, and ask whether or not it can be explained by an LHS model. Finally, one can take a complete state $\rho_{AB}$ and ask whether this state allows seeing the phenomenon of steering if Alice makes appropriate measurements.

A. Steering detection from correlations

The simplest way to detect steering is to formulate criteria for the correlations between Alice’s and Bob’s measurement statistics. There can then be directly evaluated in experiments, without the need of reconstructing the whole assemblage.

This approach of detecting steerability has a natural connection to the task of entanglement verification (Gühne and Tóth 2009) and many concepts are similar to the case of entanglement detection. This includes linear criteria that are similar to entanglement witnesses, criteria based on variances or entropic uncertainty relations, and criteria similar to Bell inequalities.

1. Linear and nonlinear steering criteria

Some of the typical ideas for deriving steering criteria are best explained with an example. Consider two qubits and the operator

$$Q = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z.$$  \hspace{1cm} (8)

The question is which values $\langle Q \rangle$ can have for separable states. If one tries to maximize or minimize $\langle Q \rangle$ over separable states, it suffices to consider product states of the form $\rho_A \otimes \rho_B$, as these are the extreme points of the separable states. But for product states the single expectation values factorize and one has (Tóth 2005)

$$|\langle Q \rangle| = |\langle \sigma_x \rangle_A \langle \sigma_x \rangle_B + \langle \sigma_y \rangle_A \langle \sigma_y \rangle_B + \langle \sigma_z \rangle_A \langle \sigma_z \rangle_B| \leq \|a\| \|b\| \leq 1,$$  \hspace{1cm} (9)

with $a = (\langle \sigma_x \rangle_A, \langle \sigma_y \rangle_A, \langle \sigma_z \rangle_A)$ and $b$ defined analogously. Here, first the Cauchy-Schwarz inequality was used, and then the fact that for single qubit states $\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 \leq 1$ holds. For the singlet state, however, $\langle Q \rangle = -3$. So the operator $W = 1 + Q$ is an entanglement witness, as it has a positive mean value on all separable states, but is negative on some entangled states.

If one wishes to estimate $\langle Q \rangle$ for unsteerable states, then, in view of Eq. (9), it suffices to consider product distributions again. This time, however, only Bob’s results are described by quantum mechanics, so only the norm $\|b\| \leq 1$ is bounded, while $\|a\| = \sqrt{3}$ is possible. So, $|\langle Q \rangle| \leq \sqrt{3}$ is a valid steering inequality that allows detecting the steerability of the singlet state (Cavalcanti et al. 2009).

A possible modification and generalization is the following: Consider $N$ measurements $A_k$ on Alice’s side which can take the two values $\pm 1$ and arbitrary observables $B_k$ on Bob’s side. Then, for unsteerable states (Saunders et al. 2010)

$$\sum_{k=1}^{N} |\langle A_k \otimes B_k \rangle| \leq \max_{\{a_k\}} \lambda_{\text{max}} \left( \sum_{k=1}^{N} a_k B_k \right),$$  \hspace{1cm} (10)

where $\lambda_{\text{max}}(X)$ denotes the largest eigenvalue of $X$ and $a_k = \pm 1$. To prove this bound, it suffices to consider a product distribution as above, then each $A_k$ can just change the signs of the $B_k$, and the mean value of the resulting sum is bounded by the maximal eigenvalue.
A different kind of generalization uses expectation values on Bob’s side that are conditioned on Alice’s outcome. If Alice makes a measurement $A_k$ with possible results labeled by $a$, one can denote with $\langle B \rangle|_a$ the mean value of a measurement on Bob’s side, conditioned on the outcome $a$. Then one can consider the nonlinear expression

$$T_x^{(k)} = \sum_a p(a|k) \langle \sigma_x \rangle|_a^2$$

and summing this up for three Pauli measurements gives the bound for unsteerable states \cite{Wittmann2012}

$$T_x^{(1)} + T_y^{(2)} + T_z^{(3)} \leq 1.$$ (12)

This follows by considering product distributions and $\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 \leq 1$. Note that similar bounds on sums of squared mean values are known for many cases of anticommuting observables or mutually unbiased bases \cite{Toth2005,Wehner2010,Wei2009}, so one can directly generalize the criteria from above to broader classes of observables on Bob’s side \cite{Evans2013}. With mutually unbiased bases as a generalization of the Pauli matrices one can even find steering inequalities with an unbounded violation \cite{Marciniak2015,Rutkowski2017}. For all the criteria from above the question arises, which are the best measurement directions for a given state in order to detect steering. For the criterion in Eq. (10) this has been studied in \cite{EvansWiseman2014}. For criteria using Pauli matrices, one can still ask for the best orientation of the coordinate system. In \cite{McCloskey2017} it has been observed that often, but not always, the measurements that correspond to the semiaxes of the so-called steering ellipsoid (i.e., the ellipsoid of the potential conditional states in Bob’s Bloch sphere considering all possible measurements for Alice, see Section VII B) give strong criteria. For higher-dimensional systems, a systematic study of optimal measurements in restrictive scenarios, i.e., in the context of $N$ measurements of $k$ outcomes, has been performed in \cite{Bavareco2017}. So far, we have considered criteria that were motivated by concepts in entanglement theory. A different method to design steering inequalities for a given special scenario comes from the theory of semidefinite programs (SDPs). As we will see in Section IIB.3 the question of whether a given assemblage $\{\omega_{ijk}\}$ is steerable can be decided via an SDP. The corresponding dual problem can then be considered as a linear steering inequality. Further details are given in Section IIB.2.

The discussed criteria or small variations thereof have been used in several experiments \cite{Bennet2012,Saunders2010,Smith2012,Weston2018,Wittmann2012}. In the experimental works, it is also important to close loopholes, such as the one arising from inefficient detectors. Theoretical aspects of this issue are in detail discussed in \cite{Evans2013,Jeon2019,Vallone2013} and experimentally studied in \cite{Bennet2012,Smith2012,Weston2018,Wittmann2012}.

2. Steering criteria from uncertainty relations

Steering inequalities based on uncertainty relations have been proposed already long before the formal definition of steerability in the context of the EPR argument \cite{Reid1989,Reid2009}. Also the criterion in Eqs. (11) \cite{Wittmann2012} can be seen as a criterion in terms of conditional variances. A systematic approach using entropic uncertainty relations (EURs) has been proposed in \cite{Walborn2011} for continuous variable systems and tailored to discrete systems by \cite{Schneeloch2013}. Here, we focus on the discrete version. In general, a measurement $M$ results in a probability distribution $\mathbf{P} = (p_1, \ldots, p_n)$ of the outcomes, for which one can consider the Shannon entropy $S(\mathbf{P}) = -\sum_i p_i \log(p_i)$ as the entropy of the measurement $S(M)$. For two projective measurements given by the corresponding hermitian operators $B_1 = \sum_i \lambda_i |v_i\rangle \langle v_i|$ and $B_2 = \sum_i \mu_i |w_i\rangle \langle w_i|$ on Bob’s side, one has the general EUR \cite{Maassen1988}

$$S(B_1) + S(B_2) \geq - \ln(\Omega_B),$$ (13)

where $\Omega_B = \max_{i,j} |\langle v_i | w_j \rangle|^2$ is the maximal overlap between the eigenstates. This and similar EURs are central to quantum information theory and quantum cryptography \cite{Coles2017}.

For product measurements $A \otimes B$ on two particles, one can consider the joint distribution and the conditional Shannon entropy $S(B|A) = S(A, B) - S(A)$. Then, for unsteerable states the relation

$$S(B_1|A_1) + S(B_2|A_2) \geq - \ln(\Omega_B)$$ (14)

holds. The intuition behind this criterion is that if Alice can predict from her measurement data Bob’s measurement results better than the EUR allows, then there cannot be local quantum states for Bob that reproduce such measurement results.

A generalization of this criterion to other entropies has been developed \cite{Costa2018a}. The general approach works for any entropy with the following properties: (i) the entropy is (pseudo-)additive for independent distributions; (ii) one has a state independent EUR; and (iii) the corresponding relative entropy is jointly convex. The resulting criteria based on Tsallis entropy are typically stronger than the ones from Shannon entropy. In addition, \cite{Krivvacky2018} obtained tight steering inequalities in terms of the Rényi entropy for scenarios with two measurements per party, and \cite{Jia2018}.
developed methods of detecting entanglement and steering based on universal uncertainty relations and fine-grained uncertainty relations using majorization. In the case of continuous variable systems, the entropic criteria proposed in [Walborn et al., 2011] are connected to one of the first criteria mentioned above. In [Reid, 1989] the question is addressed to which extent Alice can infer the value of Bob’s position $X_B$ or momentum $P_B$ by measuring her own canonical variables. The best estimator of $X_B$ has as uncertainty the minimal variance $\delta_{\text{min}}^2(X_B) = \int dx A_P(x_A) \delta^2(x_B|x_A)$, where $\delta^2(x_B|x_A)$ is the variance of the conditional probability distribution. Then, for quantum states that do not give rise to an EPR argument, the condition

$$\delta_{\text{min}}^2(X_B) \delta_{\text{min}}^2(P_B) \geq \frac{1}{4}$$

holds. Later, [Walborn et al., 2011] showed the criterion $S(X_B|X_A) + S(P_B|P_A) \geq \ln(\pi e)$ and demonstrated that this implies Eq. (15). In addition, [Walborn et al., 2011] reports the experimental observation of states, which cannot be detected by the entropic criterion, but not by Eq. (15).

Other experiments involving steering criteria from uncertainty relations have been reported in the case of continuous variable systems in [Bowen et al., 2003], and recently in the case of discrete systems [Wollmann et al., 2019] [Yang et al., 2019].

3. Steering and the CHSH inequality

Given a certain set of measurements, one can ask for the optimal inequalities for detecting quantum correlations. For Bell nonlocality and the case of two measurements with two outcomes each, it is known that the probabilities allow an LHV description, if and only if the Clauser-Horne-Shimony-Holt (CHSH) inequality

$$\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2$$  

holds [Fine, 1982], where also permutations of the measurements and outcomes have to be taken into account. More precisely, the inequality implies that the probabilities of all outcomes for the measurements $A_i B_j$ can be explained by a LHV model. Note that these probabilities include more information than the full correlations $\langle A_i B_j \rangle$ only, as the marginals $\langle A_i \rangle$ and $\langle B_j \rangle$ are independent of the full correlations. In other words, if the CHSH inequality is fulfilled, there is also no two-setting Bell inequality with marginal terms that is violated.

Similar statements are known from entanglement theory. For instance, one can consider the situation of two qubits, where Alice and Bob perform each the two measurements $\sigma_X$ and $\sigma_Z$ only, and not full tomography. For this scenario, all relevant entanglement witnesses have been characterized [Curti et al., 2004].

For quantum steering one can consider also two measurements with two outcomes per party, where only the measurements of Bob may be characterized. First, one can consider the case that Bob has a qubit and performs two mutually unbiased measurements (e.g., two Pauli measurements). For this scenario, it was shown by [Cavalcanti et al., 2015b] that the full correlations $\langle A_i B_j \rangle$ admit an LHS model, iff the inequality

$$\sqrt{(\langle A_1 + A_2 \rangle B_1)^2 + (\langle A_1 + A_2 \rangle B_2)^2} + \sqrt{(\langle A_1 - A_2 \rangle B_1)^2 + (\langle A_1 - A_2 \rangle B_2)^2} \leq 2$$

holds. Note that the resulting inequality and the underlying problem has some similarity to Bell inequalities for orthogonal measurement directions for one or both parties [Uffink and Seevinck, 2008].

For the more general scenario, one has to distinguish carefully whether the LHS model should explain the full correlations $\langle A_i B_j \rangle$ only, or in addition the marginal distributions $\langle A_i \rangle$ and $\langle B_j \rangle$.

Concerning full correlations, in [Girdhar and Cavalcanti, 2016] the case of uncharacterized projective measurements on Bob’s qubit has been considered. First, two projective measurements $B_1$ and $B_2$ on a qubit define a plane on the Bloch sphere, and in this plane one can always find a third measurement $B_3$ such that $B_1$ and $B_3$ are mutually unbiased; moreover, the mean values of $B_1$ and $B_2$ can be obtained from the mean values of $B_1$ and $B_3$ and vice versa. Then, it was shown that $B_1$ and $B_2$ allow an LHS model iff Eq. (17) holds for $B_1$ and $B_3$. In addition, it was shown that if Eq. (17) is violated, then the state violates also the original CHSH inequality and is thus nonlocal, but possibly for a different set of measurements (see also [Quan et al., 2017] for an independent proof). Finally, also a characterization of POVMs with two outcomes has been given in [Girdhar and Cavalcanti, 2016].

At the same time, [Quan et al., 2017] considered the question of whether full correlations and marginals of two dichotomic measurements can be explained via an LHS model. For this case, the equivalence is not true anymore. There are two-qubit states, which do not violate any CHSH inequality, nevertheless no LHS model can explain the full correlations and marginals of certain $A_1, A_2, B_1, B_2$. One can also find two-qubit states, for which steerability from Alice to Bob can be proved by two measurements on each side, but steering from Bob to Alice is not possible (see also Section III.D). Finally, the interplay between steering and Bell inequality violation for specific families of states has been discussed in [Costa and Angelo, 2016] [Quan et al., 2016].

4. Moment matrix approach

Another method that can be used for the characterization of quantum correlations consists of moment matrices.
or expectation value matrices. In general, one considers a set of operators of the form \( M_k = \{ A_{i_k} \otimes B_{j_k} \} \) and builds the matrix of expectation values

\[
\Gamma_{kl} = \langle M_k^\dagger M_l \rangle.
\]  

(18)

The remaining task is to characterize the possible matrices \( \Gamma \) that originate from unsteerable or separable states. Clearly, \( \Gamma \geq 0 \), i.e., it has no negative eigenvalues.

This approach of moment matrices is a well-known tool in entanglement theory (Háseler et al., 2008; Miranowicz et al., 2009; Moroder et al., 2008; Shchukin and Vogel, 2005). There one can argue that the matrix \( \Gamma \) inherits a separable structure so that approaches using the partial transposition can be applied. This also allows characterization of entanglement if some of the entries \( \Gamma_{kl} \) are not known or if the measurement devices are not trusted (Moroder et al., 2013).

Concerning steering, it follows from Eqs. (3, 4) that the correlations of unsteerable states can be explained by an underlying separable state, where the measurements of Alice are commuting (Kogias et al., 2015b; Moroder et al., 2016). The commutativity of Alice’s measurements together with possible exploitation of the structure of Bob’s characterized measurements (e.g., an algebraic structure such as the one of the Pauli spin operators) results in constraints on the moment matrix. In the end, for a given set of product operators \( \{ M_k \} \), one needs to check whether there exist (complex) parameters for the unknown entries of the moment matrix (such as squares of Alice’s measurements) that make the matrix positive. As any moment matrix is positive, proving that such an assignment of parameters is not possible implies that the underlying state is steerable. The main result of (Kogias et al., 2015b) can be then formulated as follows. For any unsteerable correlation experiment

\[
\Gamma_R \geq 0 \text{ for some } R,
\]  

(19)

where \( \Gamma_R \) is the moment matrix \( \Gamma \) for a set of parameters \( R \) (fulfilling the requirements inherited from commutativity on Alice’s side and possible further structure on Bob’s side) as the unknown entries. In (Kogias et al., 2015b) the authors further pointed out that checking the existence of such parameters forms a semidefinite program and provided various examples. Note that this approach can still be augmented by using the separable structure of \( \Gamma \).

In (Chen et al., 2016a) the concept of a moment matrix has been used to characterize steerability in a more refined way. Namely, one can also consider the moment matrices \( \Gamma_{ijk} \) for each state in the assemblage \( \{ \varrho_{ijk} \} \). Using the methods from (Moroder et al., 2013) this allows then to characterize and quantify steerability in a device independent way.

5. Steering criteria based on local uncertainty relations

Local uncertainty relations (LURs) are a common tool for entanglement detection and the underlying idea can directly be generalized to steering detection. For the case of entanglement, the idea is the following: Consider observables \( A_k \) on Alice’s side, obeying an uncertainty relation \( \sum_k \delta^2(A_k) \geq C_A \), where \( \delta^2(X) = \langle X^2 \rangle - \langle X \rangle^2 \) denotes the variance. An example of such a relation is \( \sum_{i=x,y,z} \delta^2(\sigma_i) \geq 2 \), for general observables such bounds can be computed systematically (Huang, 2012; Maccone and Pati, 2014; Schwonnek et al., 2017). Similarly, one can consider observables \( B_k \) for Bob, fulfilling \( \sum_k \delta^2(B_k) \geq C_B \), and the global observables \( M_k = A_k \otimes I + I \otimes B_k \). Then, for separable states, the bound \( \sum_k \delta^2(M_k) \geq C_A + C_B \) holds (Hofmann and Takeuchi, 2003). This is a very strong entanglement criterion and its properties have been studied in detail (Gittsovich et al., 2008; Gühne et al., 2006; Zhang et al., 2008).

For steering detection, we can use the same construction, the only difference is that Alice’s measurements are not characterized, so no uncertainty relation for them is available (Hi et al., 2015b; Zhen et al., 2016). Consequently, unsteerable states obey

\[
\sum_k \delta^2(M_k) \geq C_B.
\]  

(20)

The criterion of the LURs can be formulated in terms of covariance matrices (Gühne et al., 2007), and this also works for steering (Hi et al., 2015b). For a given quantum state \( \varrho \) and observables \( \{ X_k \} \), the symmetric covariance matrix \( \gamma \) is defined by its elements \( \gamma_{ij} = \langle (X_i X_j) + \langle X_j X_i \rangle / 2 - \langle X_i \rangle \langle X_j \rangle \rangle \). If one considers in a composite system the set of observables \( \{ X_k \} = \{ A_{ik} \otimes I, I \otimes B_{j_k} \} \) then the covariance matrix has a block structure

\[
\gamma_{AB} = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix},
\]  

(21)

where \( A = \gamma(\varrho_A, \{ A_i \}) \) and \( B = \gamma(\varrho_B, \{ B_j \}) \) are covariance matrices for the reduced states and \( C \) is the correlation matrix with elements \( C_{ij} = \langle A_i \otimes B_j \rangle - \langle A_i \rangle \langle B_j \rangle \).

Given this type of covariance matrices for unsteerable states it holds that

\[
\gamma_{AB} \succeq 0_A \oplus \kappa_B,
\]  

(22)

with \( \kappa_B = \sum_k p_k \gamma(\varrho_{i_k}) \) being a convex combination of covariance matrices of pure states on Bob’s system. Here, \( 0_A \) is a \( m \times m \) null matrix, where \( m \) is the number of observables on Alice’s side. The characterization of the possible \( \kappa_B \) has been discussed for typical cases, such as qubit states or local orthogonal observables (Gittsovich et al., 2008).

Finally, it should be noted that the criterion in Eq. (22) is the discrete analog to a criterion for the continuous variable case, see also Section V.C.
B. Steering detection from state assemblages

When full knowledge of the unnormalized conditional ensembles \( \rho_{a|x} \) on Bob’s side is available, steerability can be detected more efficiently. As already mentioned, a set of conditional ensembles on Bob’s side corresponding to certain measurement settings from Alice is called a steering assemblage. As Alice’s choice of measurement cannot be detected on Bob’s side, such assemblages are non-signalling in that \( \sum_a \rho_{a|x} = \sum_a \rho_{a|x'} \) for different settings \( x, x' \). It is interesting to note that, for the bipartite case, any non-signalling assemblage can be prepared with some shared state and some measurements on Alice’s side (Schrödinger 1936). See also Sections VI and VIII. Also, any unsteerable assemblage can be prepared using a separable state and commuting measurements on Alice’s side (Kogias et al. 2015b; Moroder et al. 2016). The main point for steering detection is that for finite steering assemblages, the question whether there exists an LHS model described by Eq. (2) can be decided via the so-called semidefinite programming (SDP) technique (Pusey 2013). The SDP approach also allows one to derive steering inequalities and to quantify the steerability of finite assemblages.

1. Formulation of the semidefinite program

The crucial idea is that for a finite steering assemblage \( \{\rho_{a|x}\}_{a,x} \), it is sufficient to consider a finite LHS ensemble \( \sigma_\lambda \) (Ali et al. 2009; Pusey 2013). Moreover, the response functions in Eq. (2) can also be chosen to be fixed. So the remaining problem is to construct a finite number of positive operators \( \sigma_\lambda \). We focus on the conceptual aspects of the SDP formulation, a detailed review on computational aspects can be found in (Cavalcanti and Skrzypczyk 2017).

Consider a set of \( m \) measurement settings on Alice’s side, \( x \in \{1, 2, \ldots, m\} \), each has \( q \) outcomes \( a \in \{1, 2, \ldots, q\} \). Given the shared state \( \rho_0 \), this gives rise to an assemblage \( \{\rho_{a|x}\}_{a,x} \) of \( m \) ensembles, each consisting of \( q \) conditional states. The space of the hidden variables \( \lambda \) can be constructed as follows. The variable \( \lambda \) can take \( q^m \) values, each can be thought of as a string of outcomes ordered according to the measurements, \((a_{x=1}, a_{x=2}, \ldots, a_{x=m})\). For such a string \( \lambda \), we denote by \( \lambda(x) \) the value of the outcome at position \( x \). Then \( D(a|x, \lambda) \) denotes the deterministic response function defined by \( D(a|x, \lambda) = \delta_{a, \lambda(x)} \). This means that \( D(a|x, \lambda) \) equals one for strings \( \lambda \) which predict the outcome \( a \) for the measurement \( x \) and zero otherwise.

The crucial statement is the following: a finite steering assemblage admits an LHS model described by Eq. (2) if and only if it also admits an LHS model with the constructed set of strings \( \lambda \) as the LHV and the fixed deterministic functions \( D(a|x, \lambda) \) as the response functions.

The latter means that there exists a set of (unnormalized) operators \( \sigma_\lambda \) satisfying

\[
\rho_{a|x} = \sum_\lambda D(a|x, \lambda) \sigma_\lambda \quad \text{for all } a, x,
\]

s.t.: \( \sigma_\lambda \geq 0 \) for all \( \lambda \). (23)

Writing with the explicit definitions of the hidden variable \( \lambda \) and the deterministic response function \( D(a|x, \lambda) \), the equality in the above equation is simply

\[
\rho_{a|x} = \sum_{\{a\}} \delta_{a,a_x} \sigma_{a_1, a_2, \ldots, a_m}.
\]

Intuitively, one can think of the hidden states \( \sigma_{a_1, a_2, \ldots, a_m} \) as being indexed by \( m \) variables. The conditional state \( \rho_{a|x} \) is obtained by summing the function over the values of all variables except for the \( x \)-th one, which is fixed \( a_x = a \).

To give an example, if one considers the case where Alice performs two measurements \( x \in \{1, 2\} \) with two possible outcomes \( a \in \{1\} \), the steering assemblage \( \{\rho_{a|x}\} \) is unsteerable if and only if it is possible to find four positive semidefinite operators \( \omega_{ij} \), with \( i, j = \pm 1 \) such that

\[
\rho_{+1} = \omega_{++} + \omega_{+-}, \quad \rho_{+2} = \omega_{++} + \omega_{-+},
\]

\[
\rho_{-1} = \omega_{-+} + \omega_{- -}, \quad \rho_{-2} = \omega_{++} + \omega_{- -}.
\]

It is remarkable that in passing from Eq. (2) to Eq. (23) we have passed from an arbitrary hidden variable to a finite discrete hidden variable and, at the same time, fixed the response functions. One notices that the finiteness of the set of measurements plays a crucial role in this approach.

Given an assemblage \( \{\rho_{a|x}\}_{a,x} \), determining the existence of \( \sigma_\lambda \) satisfying Eq. (23) is in fact a well-known problem in convex optimization. More precisely, it is known as a feasibility problem in semidefinite programming (SDP) (Boyd and Vandenberghe 2004), which can be solved straightforwardly by an appropriate ready-to-use software. Furthermore, it has been shown that using the so-called order-monotonic functions, the SDPs can be approximated by simpler ones (Zhu et al. 2016).

2. Steering inequalities from the SDP

The feasibility SDP (23) can be used to construct steering inequalities. First, one can convert such a feasibility problem to an explicit convex maximization,

\[
\max \mu \quad \text{w.r.t. } \mu, \{\sigma_\lambda\}
\]

s.t.: \( \rho_{a|x} = \sum_\lambda D(a|x, \lambda) \sigma_\lambda \quad \text{for all } a, x \)

\[
\sigma_\lambda \geq \mu 1 \quad \text{for all } \lambda.
\]
If the optimal value of \( \mu \) turns to be negative, then the problem in Eq. (23) is infeasible, indicating that the assemblage is steerable.

To analyze this maximization, there is a powerful tool in convex optimization known as duality theory. In a nutshell, the maximization problem in Eq. (26) is coupled to a so-called dual minimization problem,

\[
\min \quad \text{Tr} \sum_{a,x} F_{a|x} \varrho_{a|x} \\
\text{w.r.t.} \quad \{F_{a|x}\} \\
\text{s.t.} \quad \sum_{a,x} F_{a|x} D(a|x, \lambda) \geq 0 \quad \text{for all } \lambda \\
\text{Tr} \sum_{a,x,\lambda} F_{a|x} D(a|x, \lambda) = 1. 
\tag{27}
\]

The two problems are dual in the sense that the optimal value of the minimization in Eq. (27) is an upper bound for the maximization in Eq. (26). This is known as weak duality. Under weak additional conditions, strong duality also holds: the two optimal values are equal (Boyd and Vandenberghe 2004).

The duality theory implies that if there exists a collection of observables \( \{F_{a|x}\} \) satisfying the constraints in problem (27) and if \( \text{Tr} \sum_{a,x} F_{a|x} \varrho_{a|x} \leq 0 \) then the assemblage is steerable. So, the dual problem naturally defines a steering inequality. The minimizer of the problem (27) thus yields optimal steering inequalities for a steerable assemblage. Such steering inequalities found applications in several scenarios, see also Sections III.E and V.F.

### 3. Quantification of steerability with SDPs

The SDP approach also allows one to quantify the steerability of an assemblage \( \{\varrho_{a|x}\} \). There are several different quantification schemes. We selectively discuss some of those; for an extensive discussion we refer to (Cavalcanti and Skrzypczyk 2017).

The idea of quantifying the steerability of an assemblage is as follows. Let us fix the number of measurements \( m \) and the number of outcomes \( q \) per measurement at Alice’s side. Then the space of all assemblages, i.e., all different \( m \) decompositions of Bob’s reduced states to \( q \)-component ensembles, admits a natural convex structure. To be more precise, let \( \{\varrho_{a|x}\} \) and \( \{\tilde{\varrho}_{a|x}\} \) be two assemblages, then for \( 0 \leq p \leq 1 \), the set \( \{p\varrho_{a|x} + (1-p)\tilde{\varrho}_{a|x}\} \) is also an assemblage. Within the set of all assemblages, the unsteerable assemblages form a subset, which is clearly convex. Now, how much steerable an assemblage is can be measured by some kind of relative distance to the set of unsteerable assemblages. More precisely, as long as only the linear structure of the state assemblages is concerned, the absolute distance is not meaningful, and one can only consider relative ratios of distances on a line. In practice, one therefore compares the distances between the considered assemblage, the boundary of unsteerable assemblages and the boundary of all assemblages through a certain line. Different ratios constructed from these distances give rise to different steerability quantifiers.

The first quantification of steerability of an assemblage was proposed by (Skrzypczyk et al. 2014), known as steering weight. An assemblage \( \{\varrho_{a|x}\} \) is first written as a convex combination of an unsteerable assemblage \( \varrho_{a|x}^{\text{LHS}} \) and a general assemblage \( \{\gamma_{a|x}\} \),

\[
\varrho_{a|x} = p\gamma_{a|x} + (1-p)\varrho_{a|x}^{\text{LHS}} \quad \text{for all } a, x. \tag{28}
\]

The steering weight of \( \{\varrho_{a|x}\} \), denoted by \( \text{SW}(\{\varrho_{a|x}\}) \), is the minimal weight \( p \) in such decomposition with respect to all possible choices of the general assemblage \( \{\gamma_{a|x}\} \) and the unsteerable assemblage \( \{\varrho_{a|x}^{\text{LHS}}\} \). A geometrical illustration of the steering weight is given in Fig. 3 (left). As the set of all assemblages and unsteerable assemblages can be characterized via SDPs, the steering weight can also be determined by an SDP. More precisely, \( \text{SW}(\{\varrho_{a|x}\}) \) is given by

\[
\min \quad 1 - \text{Tr} \sum_{\lambda} \sigma_{\lambda} \\
\text{w.r.t.} \quad \{\sigma_{\lambda}\} \\
\text{s.t.} \quad \varrho_{a|x} - \sum_{\lambda} D(a|x, \lambda) \sigma_{\lambda} \geq 0 \quad \text{for all } a, x \\
\sigma_{\lambda} \geq 0 \quad \text{for all } \lambda. \tag{29}
\]

A similar quantification of steerability is steering robustness, first defined in the context of subchannel discrimination (Piani and Watrous 2015); see also Section V.G. Here, the steering robustness \( \text{SR}(\{\varrho_{a|x}\}) \) is given by the minimal weight on a general assemblage \( \{\gamma_{a|x}\} \) considered as noise one needs to mix to the assemblage \( \{\varrho_{a|x}\} \) so that it becomes unsteerable. The geometrical illustration is given in Fig. 3 (right). Like the steering weight, the steering robustness \( \text{SR}(\{\varrho_{a|x}\}) \) can be computed via a simple SDP (Cavalcanti and Skrzypczyk 2016).
the set of noisy measurements will shrink to fit inside the noisy measurements. For a certain level of noise added, through a depolarizing channel and obtains a new set of noise to the set of measurements, e.g., by sending them ically limited to projective ones. One can add certain convex hull.

The idea is as follows. One starts with a finite set of measurements (Cavalcanti et al., 2016). Thus one can conclude an LHS model for the set of all measurements, but for a noisier version of the considered state. This construction works similarly for Bell nonlocality (Cavalcanti et al., 2016).

While the SDP approach has proven to be useful in algorithmically constructing certain LHS and LHV models (Cavalcanti et al., 2016; Cavalcanti and Skrzypczyk, 2017; Fillettaz et al., 2018; Hirsch et al., 2016b), it has a significant computational drawback. To reduce the noise needed to add to the state, the original finite set of measurements needs to be sufficiently large. However, the size of the SDP, as one observes, increases exponentially with respect to the number of measurement settings. As clearly illustrated in a systematic study (Fillettaz et al., 2015), this often imposes a significant computational difficulty on the problem of deciding the steerability with high accuracy even for two-qubit states.

C. Steering detection from full information

When the complete density matrix \( \varrho_{AB} \) is exploited, one might expect to have a more complete characterization, i.e., a necessary and sufficient condition for steerability. Like entanglement detection or Bell nonlocality detection, this is a difficult question. There were exact results only for entanglement detection of low-dimensional or special states. It is thus very encouraging that some results can also be derived for quantum steering. It was recognized already by Wiseman et al. (2007) that for certain highly symmetric states, the problem of determining steerability with projective measurements can be solved completely, see Section III.B. Recently, a complete characterization of quantum steering has been also achieved for two-qubit states and projective measurements (Jevtic et al., 2015; Nguyen et al., 2019; Nguyen and Vu, 2016b).

1. Two-qubit states and projective measurements

From Eq. 1, one sees that in order to determine the steerability of a given state one has to consider all possible LHS ensembles \( \{ p(\lambda), \sigma_\lambda \} \), and for each measurement, solve for the response functions \( p(a|x, \lambda) \). The source of difficulty is that the possible choice of the index \( x \) seems to be arbitrary: it can be a discrete variable, a real-valued variable or a multi-dimensional variable, etc. It is now worth re-examining how the SDP approach discussed in Section II.B works:

\[
\min \text{ Tr } \sum_{\lambda} \sigma_\lambda - 1
\]

w.r.t. \( \{ \sigma_\lambda \} \)

\[
\text{s.t. } \sum_{\lambda} D(\sigma|x, \lambda) \sigma_\lambda - \varrho_{a|x} \geq 0 \quad \text{for all } a, x
\]

\[
\sigma_\lambda \geq 0 \quad \text{for all } \lambda.
\]
one assumes that Alice can only make a finite number of measurements, which implies the finiteness of a necessary LHS ensemble — a unique choice of the hidden variable is thus singled out. When Alice’s set of measurements is not finite, this approach breaks down. Fortunately, one can show that (Nguyen et al., 2018, 2019) Eq. (31) has a so-called principal radius (32) over all LHS ensembles, which implies the finiteness of a necessary and sufficient condition for the critical radius, namely 1 —

\[ R(q_{\text{AB}}) = \frac{d}{L_{\text{QAB}}} \]

\[ 1 - R(q_{\text{AB}}) = \frac{\lambda}{L_{\text{QAB}}} \]

FIG. 4. The operational meaning of the critical radius: 1 — R(q_{AB}) measures the distance from \( g \) to the surface of unsteerable/steerable states relatively to \((\mathbb{I}_A \otimes g_B)/2\). Figure taken from (Nguyen et al., 2019).

\[ R(q_{AB}) = \max\{\alpha \geq 0 : q^{(\alpha)}_{AB} \text{ is unsteerable}\}, \]

where unsteerability is considered with respect to projective measurements. We will see that this definition can be naturally generalized to generalized measurements and higher-dimensional systems.

The definition of the critical radius by Eq. (33) also allows for its evaluation. It is shown that (Nguyen et al., 2018) by replacing Bob’s Bloch sphere by polytopes from inside and from outside, one obtains rigorous upper and lower bounds for the critical radius. Both the computation of the upper and lower bounds are linear programs, of which the sizes scale cubically with respect to the numbers of vertices of the polytopes. Upon increasing the numbers of vertices, the two bounds quickly converge to the actual value of the critical radius. This approach has been used to access the geometry of the set of unsteerable states via its two-dimensional random cross-sections (Nguyen et al., 2019); see Fig. 3.

Notably, for the so-called Bell-diagonal states, or T-states, an explicit formula for the critical radius has been obtained,

\[ R(q_T) = 2\pi N_T |\det(T)|, \]

where \( T \) is the correlation matrix of the T-state, \( T_{ij} = \text{Tr}(q_T \sigma_i \otimes \sigma_j) \) for \( i, j = 1, 2, 3 \), and the normalization factor \( N_T \) is given by an integration over the Bloch sphere \( N_T^{-1} = \int dS(\vec{m})|\vec{m}^T - \bar{m}|^{-2} \) (Jevtic et al., 2015; Nguyen and Vu, 2016b). Based on this solution for T-states, analytical bounds for the critical radius of a general state can also be derived (Nguyen et al., 2019).

For further discussions on LHS models for two-qubit states in special cases, see (Miller et al., 2018; Yu et al., 2018; Zhang and Zhang, 2019).
2. Steering of higher-dimensional systems and with generalized measurements

Characterizing quantum steering of higher-dimensional systems and with generalized measurements (POVMs) is difficult. Most of the results on quantum steering, in this case, rely on the idea of adding sufficient noise to the state such that an LHS for simpler dimensions and with generalized measurements (see Section III.A), superactivation of nonlocality by local filtering (see Section III.C) and one-way steering with POVMs (see Section III.D).

Remarkably, the critical radius approach explained in the previous subsection gives a promising framework to generally analyze quantum steering with POVMs and higher-dimensional systems. In fact, in any dimension, one can define the critical radius with respect to a certain class of measurements in the same way as Eq. (31). In particular, considering the set of generalized measurements (POVMs) of $n$ outcomes, one can define the critical radius for a bipartite state $\rho_{AB}$ of dimension $d_A \times d_B$ by

$$R_n(\rho_{AB}) = \max\{\alpha \geq 0 : \rho_{AB}^{(\alpha)} \text{ is unsteerable}\},$$

with $\rho_{AB}^{(\alpha)} = \alpha \rho_{AB} + (1 - \alpha)(I_A \otimes \rho_B)/d_A$, $\rho_B = \text{Tr}_A(\rho_{AB})$ and unsteerability being considered with respect to POVMs of $n$ outcomes on Alice’s side. Defined in this way, $1 - R_n(\rho_{AB})$ can still be interpreted as measuring the distance from $\rho_{AB}$ to the surface separating unsteerable/steerable states, here defined with respect to POVMs of $n$ outcomes on Alice’s side; see again Fig. 4.

However, direct evaluation of the critical radius from the definition Eq. (37) is clearly not possible.

Interestingly, an alternative formula for the critical radius in similarity to Eq. (32) and Eq. (33) (Nguyen et al., 2018, 2019) can also be found for high dimensional systems. To this end, for a finite dimensional bipartite state $\rho_{AB}$, one can define the principal radius for a given LHS ensemble $\mu$ by

$$r_n^{-1}(\rho_{AB}, \mu) = \sup_{Z,E} F^{-1}(\rho_{AB}, \mu, Z, E),$$

with $F^{-1}(\rho_{AB}, \mu, Z, E)$ defined to be

$$\sum_{i=1}^n \text{Tr}[\rho_{AB}(E_i \otimes Z_i)] - \frac{1}{d_A} \sum_{i=1}^n \text{Tr}(E_i)\text{Tr}(\rho_B Z_i)$$

$$\int d\mu(\sigma) \max\{\langle Z_i, \sigma \rangle\} - \frac{1}{d_A} \sum_{i=1}^n \text{Tr}(E_i)\text{Tr}(\rho_B Z_i),$$

where the supremum is taken over all possible $n$-POVMs $E = (E_1, E_2, \ldots, E_n)$ on Alice’s side and all possible $n$ observables $Z = (Z_1, Z_2, \ldots, Z_n)$ on Bob’s side. The critical radius as defined by Eq. (37) can be computed as

$$R_n^{-1}(\rho_{AB}) = \min_{\mu} r_n^{-1}(\rho_{AB}, \mu).$$

In this way, the problem of computing the critical radius and the principal radius is in principle an optimization problem. Unfortunately, even in this form, a deterministic algorithm to compute the principal radius and the critical radius with $n \geq 3$ is still unknown, and one has to invoke heuristic techniques in practice (Nguyen et al., 2018, 2019).

One observes that to study quantum steering, the set of generalized measurements was stratified according to their number of outcomes. This calls for an investigation of the relation between them. Since POVMs of $n$ outcomes form a natural subset of POVMs of $n + 1$ outcomes, one has a decreasing chain $R_2(\rho_{AB}) \geq R_3(\rho_{AB}) \geq R_4(\rho_{AB}) \geq \cdots$. As extreme POVMs have at most $d_A^2$ outcomes with $d_A$ being Alice’s dimension, this chain turns to equality at $n = d_A^2$. Using the evidence from a heuristic computation for the principal radius, it has been conjectured (Nguyen et al., 2019) that for two-qubit
states ($d_A^2 = 4$), the chain in fact consists of a single number, namely $R_2(\varrho_{AB}) = R_3(\varrho_{AB}) = R_4(\varrho_{AB})$. In words, this conjecture implies that measurements of two outcomes (dichotomic measurements) are sufficient to fully demonstrate the quantum steerability of a two-qubit system; measurements of more outcomes are not necessary. Unfortunately, extrapolating this conjecture to higher-dimensional systems fails; it is later shown (Nguyen and Gühne 2020) that this equality breaks down already for a system of two qutrits (see also Section II.A).

3. Full information steering inequality

As we discussed, for high-dimensional systems, even when full information about a state is available, a computable necessary and sufficient condition for quantum steerability is not available. In this case, the detection of steerability still relies on steering inequalities. An example of steering inequalities based on full information of the state is given by Zhen et al. (2016) in terms of the so-called local orthogonal observables. Embedding in a higher-dimensional space if necessary, we can assume that Alice and Bob have the same local dimension $d$. One can choose a set of $d^2$ orthogonal operators $\{G_k\}$ which serves as a basis for the local observable space, i.e., $\text{Tr}(G_i G_j) = \delta_{ij}$, and $\{G_k\}$ spans the space of operators (Yu and Liu, 2005). The Pauli matrices are a familiar example of such orthogonal operators for a qubit system. By means of the Schmidt decomposition in the operator space, one can choose the orthonormal observables for the local spaces at Alice and Bob, $\{G_k^A\}$ and $\{G_k^B\}$, such that the joint state $\varrho_{AB}$ can be written as

$$\varrho_{AB} = \sum_{k=1}^{d^2} \lambda_k G_k^A \otimes G_k^B,$$

(41)

where $\lambda_k \geq 0$. Then using the local uncertainty relations (see Section II.A), Zhen et al. (2016) showed that the state $\varrho_{AB}$ is steerable from $A$ to $B$ if

$$\sum_{k=1}^{d^2} \delta^2 (g_k G_k^A + G_k^B) < d - 1$$

(42)

for some choice of $g_k$, where $\delta^2 (X)$ denotes the variance of operator $X$. By a particular choice of $g_k$, one can easily show that if

$$\sum_k \lambda_k > \sqrt{d},$$

(43)

the state is steerable (Zhen et al., 2016). This elegant inequality resembles the familiar computable cross norm or realignment (CCNR) entanglement criterion (Chen and Wu, 2003; Rudolph, 2005), where $\sum_k \lambda_k > 1$ implies that the state is entangled.

Note that the steering inequalities (42) and (43) are different from various inequalities discussed in Section II.A in the sense that they exploit the full information about the state.

III. CONCEPTUAL ASPECTS OF STEERING

In this section we review results on the general properties and structures of quantum steering. We start with a detailed discussion on the connection between steering, entanglement, and Bell nonlocality. We also present in some detail LHS models for different families of states. Then, we explain properties like one-way steering, steering of bound entangled states, steering maps and the superactivation of steering.

A. Hierarchy of correlations

We explained already in Section II.C that there is a hierarchy between Bell nonlocality, steering, and entanglement in the sense that one implies the other, but not the other ways around. In this section, we first discuss in some detail the known examples of states where the notions differ. Then we explain how the relations between the three concepts can be exploited to characterize one via another. Detailed LHS models are discussed in Section III.B.

When discussing the existence of an LHV or LHS model for a given quantum state, one has to distinguish whether the model should explain the results for all projective measurements, or, more generally, for all POVMs. Let us start our discussion with projective measurements. The inequivalence between the notion of entanglement and Bell nonlocality was, in fact, one of the starting points of entanglement theory (Werner, 1989). For that, one may consider the so-called two-qubit Werner state

$$\psi(p) = p |\psi^+\rangle \langle \psi^+ | + (1 - p) \frac{1}{4}$$

(44)

where $|\psi^+\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the singlet state. Using the PPT criterion (see Section III.E) one can directly verify that this state is entangled iff $p > 1/3$. In (Werner, 1989), however, an LHS model for projective measurements was constructed for all values $p \leq 1/2$. Moreover, in (Acín et al., 2006; Hirsch et al., 2017) it was shown that an LHV model exists up to $p \leq 1/K_G(3) \approx 0.6829$, where $K_G(3)$ is the Grothendieck constant of order three, so up to this value no Bell inequality can be violated. These results demonstrate that there are entangled states which do not show Bell nonlocality. Using the fact that the Werner states is steerable for $p > 1/2$ (Wiseman et al., 2007), this also proves that steering and Bell nonlocality are inequivalent for projective measurements. States of this type have been prepared experimentally and their
steerability has been demonstrated in (Saunders et al., 2010).

It remains to discuss the more general case of POVMs. First, in (Barrett, 2002) an LHV model for Werner states with \( \eta \leq 5/12 \) has been constructed which explains all the measurement probabilities for arbitrary POVMs. In fact, this model can directly be converted into an LHS model (Quintino et al., 2015). Consequently, there are entangled states for which all correlations for POVMs can be explained by an LHV model. In addition, (Quintino et al., 2015) presented examples of states in a 3 × 3 system which are steerable in both directions, but nevertheless an LHV model for all POVMs can be found. This proves the inequivalence of Bell nonlocality and steering for POVMs.

As mentioned, a local model that explains the results of all projective measurements does not necessarily explain all the correlations for POVMs. It is not clear, however, that POVMs give an advantage in the detection of steering or Bell nonlocality. As discussed in Section II.C.2 there is numerical evidence that two-qubit states which are unsteerable for projective measurements are also unsteerable for POVMs (Nguyen et al., 2019). Concerning Bell nonlocality, in (Gomez et al., 2016; Vertesi and Bene, 2010) Bell inequalities have been presented, for which the maximal violation requires POVMs, but this does not imply that the states leading to this violation do not violate also some Bell inequality for projective measurements.

Given the similarity in the definitions of Bell nonlocality, quantum steerability and nonseparability, one may expect that some methods to characterize the different notions can be related to each other. Specifically, given a state that admits an LHV model as in Eq. (3), one may expect that by adding suitable separable noise to the state, one can obtain a state that admits an LHS model (2). This has been shown to be the case (Chen et al., 2018a). The authors showed that if a bipartite qubit-qubit state \( \varrho_{AB} \) admits an LHV model, then the state

\[
\tilde{\varrho}_{AB} = \mu \varrho_{AB} + (1 - \mu) \frac{1_A}{2} \otimes \frac{1_B}{2},
\]

with \( \mu = 1/\sqrt{3} \) is unsteerable from Alice to Bob. Turning the logic around, if \( \tilde{\varrho}_{AB} \) is steerable, then \( \varrho_{AB} \) must be Bell nonlocal. A similar statement between steerability and nonseparability has also been obtained (Chen et al., 2018a; Das et al., 2019). Namely, if a bipartite qubit-qubit state \( \varrho_{AB} \) is unsteerable from Alice to Bob, then the state

\[
\tilde{\varrho}_{AB} = \mu \varrho_{AB} + (1 - \mu) \frac{1_A}{2} \otimes \varrho_B,
\]

with \( \mu = 1/\sqrt{3} \) is separable. Or, if the latter is entangled, the former is steerable. Detailed applications of this approach to detecting different nonlocality notions from the others can be found in (Chen et al., 2018a, 2016; Das et al., 2019).

B. Special states and their local hidden variable models

As already highlighted in Section II.C, the fact that for quantum steering, there is a canonical choice for the hidden variable, namely Bob’s pure states, turns out to have far-reaching consequences. The point is that, having simplified the LHS model from Eq. (2) to the form of Eq. (31), the symmetry of the state has stronger implications on the choice of the LHS ensemble (Nguyen et al., 2018). For certain highly symmetric states such as the Werner states and the isotropic states, the symmetry is then enough to uniquely single out an optimal choice of the LHS ensemble, rendering their exact characterizations of quantum steering with projective measurements possible (Jones et al., 2007; Wiseman et al., 2007). This is in contrast to Bell nonlocality: in Eq. (3), no canonical choice of the LHV is possible. Thus even for highly symmetric states such as the isotropic states and the Werner states, no exact characterization of Bell nonlocality is known.

1. Werner states

Suppose Alice and Bob share the Werner state of dimension \( d \times d \) (Werner, 1989), defined by

\[
W_d^\eta = \frac{d - 1 + \eta}{d - 1} \frac{I_d}{d^2} - \frac{\eta}{d - 1} V,
\]

where \( I_d \) is the bipartite identity operator and \( V \) is the flip operator given by \( V|\phi,\psi\rangle = |\psi,\phi\rangle \). Here we follow the parameterization by (Wiseman et al., 2007) so that \( W_d^\eta \) is a product state if the mixing parameter \( \eta = 0 \) and is a state at all only if \( \eta \leq 1 \). The Werner state is entangled if and only if \( \eta > \frac{1}{d^2-1} \) (Werner, 1989).

In fact, the Werner state was constructed in a way such that it is invariant under the same local unitary transformation at Alice’s and Bob’s side (Werner, 1989), that is, for any unitary operator \( U \) acting in dimension \( d \), \( W_d^\eta = (U \otimes U)W_d^{\eta}(U^\dagger \otimes U^\dagger) \). This implies that the optimal LHS ensemble on Bob’s Bloch sphere can be chosen to be symmetric under the unitary group \( U(d) \), i.e., the Haar measure (Nguyen et al., 2018; Wiseman et al., 2007).

Identifying the hidden variable \( \lambda \) indexing the LHS with Bob’s pure states \( |\lambda\rangle \), it remains to construct the response function \( p(a|x, \lambda) \) for a projective measurement \( \{E_{a|x}\} = \{P_a|x\} \) to complete an LHS model. Note that for a projection outcome \( P_{a|x} = |a\rangle \langle a| \) at Alice’s side, Bob’s conditional state is

\[
\varrho_{a|x} = \frac{d - 1 + \eta}{d(d - 1)} \frac{I}{d} - \frac{\eta}{d(d - 1)} |a\rangle \langle a|.
\]

The minus sign in front of the last term indicates that the two parties in the Werner state are anti-correlated. To construct the response function, it is natural then to
old Eq. (50) is strictly stronger than that of Eq. (51).

Interestingly, for dimension $d$ is still unknown even in dimension $d = 2$, the threshold for steerability has an analytical expression in all dimensions. Wiseman et al. (2007) also noted that the construction given above was actually the original construction by Werner (1989) to show that Bell nonlocality and entanglement are distinct notions.

Projective measurements are however not the only case where the quantum steerability of the Werner states can be exactly characterised. Specifically, it was shown that (Nguyen and Gühne 2020) when Alice is limited to making dichotomic measurements, the threshold upto which the Werner state is unsteerable can also be derived in practically closed form,

$$\eta \leq (d - 1)^2[1 - (1 - 1/d)^{(d-1)}],$$  \hfill (51)

for $d \leq 10^5$; see Fig. 6 (left). The threshold is also conjectured to hold for all dimensions (Nguyen and Gühne 2020). Interestingly, for dimension $d \geq 3$, the threshold Eq. (51) is strictly stronger than that of Eq. (51).

These are thus concrete examples illustrating that quantum steering with dichotomic measurements is strictly weaker than that of measurements with more outcomes for higher-dimensional systems, contrasting with the conjecture on their equivalence for two-qubit systems (see Section II.C).

2. Isotropic states

Another important family of states that allows for exact characterization of quantum steering is that of isotropic states (Wiseman et al. 2007). The isotropic state of dimension $d \times d$ at mixing parameter $\eta$, $0 \leq \eta \leq 1$, is defined by

$$S^\eta_d = (1 - \eta) \mathbb{I}^{d^2} + \eta |\psi^+\rangle\langle\psi^+|,$$ \hfill (52)

where $|\psi^+\rangle = 1/\sqrt{d} \sum_{i=0}^{d-1} |i, i\rangle$. From the definition, one notes that the isotropic state is defined with respect to a particular choice of basis. The isotropic state is entangled if and only if $\eta > 1/(d + 1)$ (Horodecki and Horodecki 1999). In similarity to the Werner state, the isotropic state also has a unitary symmetry, namely, $S^\eta_d = (U \otimes U^\dagger) S^\eta_d (U \otimes U^\dagger)$ for any $d \times d$ unitary matrix $U$, with $U$ being its complex conjugate. This again implies that the optimal choice of LHS ensemble is the uniform Haar measure over Bob’s Bloch sphere.

For a projection outcome $P_{a|x} = |a\rangle \langle a|$ on her side, with an isotropic state, Alice steers Bob’s system to the conditional state

$$\rho_{a|x} = \frac{1 - \eta}{d^2} + \frac{\eta}{d} |\bar{a}\rangle \langle \bar{a}|,$$ \hfill (53)

where $|\bar{a}\rangle$ is the complex conjugate of state $|a\rangle$. In contrast to Eq. (51), the plus sign in front of the last term in the above equation indicates that parties sharing an isotropic state are correlated upto a complex conjugation. This motivates the following choice of the response function

$$p(a|x, \lambda) = \begin{cases} 1 \text{ if } |(\lambda |a\rangle| > |(\lambda |a\rangle|) \forall a' \neq a \\ 0 \text{ otherwise.} \end{cases}$$ \hfill (54)

where we have also again identified the hidden variable $\lambda$ indexing the LHS ensemble with Bob’s pure states $|\lambda\rangle$. This construction leads to an LHS model for the isotropic state with

$$\eta \leq H_d - \frac{1}{d - 1},$$ \hfill (55)

where $H_d = 1 + 1/2 + 1/3 + \cdots + 1/d$. It can again be shown that this threshold is optimal; for $\eta > (H_d - 1)/(d - 1)$ no construction for the response function is possible (Wiseman et al. 2007). Remarkably, Almeida et al. (2007) also obtained this threshold in
an attempt to construct an LHV model for the isotropic states before learning of the definition of quantum steering. This threshold is presented in Fig. 6 (right) together with that for separability.

Like the Werner state, the quantum steerability of the isotropic states with dichotomic measurements can also be exactly characterized. It was shown in [Nguyen and Gühne 2020] that for \( d \leq 10^2 \), if
\[
\eta \leq 1 - d^{-1/(d-1)},
\]
the isotropic state is unsteerable when Alice’s measurements are limited to dichotomic ones; otherwise, it is steerable (see Fig. 6 (right)). The threshold is also conjectured to hold for all dimensions [Nguyen and Gühne 2020].

3. LHS model for POVMs

As quantum steering with projective and dichotomic measurements is well-understood for the Werner states and the isotropic states, one may hope that certain LHS models with general POVMs for them can also be constructed. This is indeed the case. By an explicit construction, Barrett [2002] demonstrated that sufficiently weakly entangled Werner states do admit an LHV model for all POVMs. Under the light of the formal definition of quantum steering [Wiseman et al. 2007], the LHV model turns out to be an LHS model [Quintino et al. 2015]. The model was revised recently [Nguyen and Gühne 2020], and it can be shown that Barrett’s original construction works best for the isotropic states; for the Werner states, a better model can be constructed. Further, for two-qubit systems, the construction can also be extended to Bell-diagonal states [Nguyen and Gühne 2019].

To construct the model, it is sufficient to consider only POVMs with rank-1 effects, \( \{E_{a|x}\} = \{\alpha_{a|x}|a\rangle\langle a|\} \), where \( |a\rangle\langle a| \) are rank-1 projections and \( 0 \leq \alpha_{a|x} \leq 1 \) [Barrett 2002]. This is because other POVMs can be post-processed from these (see also Section IV). The optimal choice for the LHS ensemble is again the uniform distribution over Bob’s Bloch sphere [Nguyen et al. 2018; Wiseman et al. 2007]. It is then left to construct the response functions \( p(a|x, \lambda) \) for the mentioned measurements.

For the isotropic state, the response function can be given as [Almeida et al. 2007; Barrett 2002]
\[
p(a|x, \lambda) = \frac{\alpha_{a|x}}{d} |\lambda| \Theta(|\lambda|) \frac{1}{2} - \frac{1}{d}
+ \frac{\alpha_{a|x}}{d} \left( 1 - \sum_a \alpha_{a|x} |\lambda| \Theta(|\lambda|) \frac{1}{2} - \frac{1}{d} \right).
\]

With this choice of the response function, direct computation shows that the isotropic state is unsteerable for arbitrary POVMs on Alice’s side if [Almeida et al. 2007; Barrett 2002]
\[
\eta \leq \frac{3d - 1}{d + 1} (d - 1)^{d-1} d^{-d}.
\]

As we mentioned, this construction was originally suggested as an LHS model for Werner states and the same threshold Eq. (58) was found [Barrett 2002; Quintino et al. 2015]. However, it was shown [Nguyen and Gühne 2020] that for the Werner state, a better choice of the response functions is possible, namely
\[
p(a|x, \lambda) = \frac{\alpha_{a|x}}{d} (1 - |\langle \lambda | a \rangle|^2) \Theta(1/d - |\langle \lambda | a \rangle|^2)
+ \frac{\alpha_{a|x}}{d} \left( 1 - \sum_a \alpha_{a|x} |\lambda| \Theta(1/d - |\langle \lambda | a \rangle|^2) \right).
\]

The Werner state was then shown to be unsteerable for arbitrary POVMs on Alice’s side if [Nguyen and Gühne 2020]
\[
\eta \leq \frac{1 + (d - 1)^{d+1} d^{-d}}{d + 1}.
\]

The two bounds Eq. (58) and Eq. (60) are also presented in Fig. 6. For the Werner states, the bound Eq. (60) is strictly better than the bound given by Eq. (58) for \( d \geq 3 \). However both bounds Eq. (58) and Eq. (60) are strictly within the respective thresholds for the isotropic states and the Werner states to be unsteerable with projective measurements, Eq. (55) and Eq. (50). On the other hand, the constructions Eq. (57) and Eq. (59) are by no means optimal; in fact, it is expected to be not optimal [Nguyen and Gühne 2019]. Thus it is still unclear at the moment if steering with projective measurements is equivalent to steering with generalized measurements even for these highly symmetric states.

The case of two-qubit Werner states \( (d = 2) \) is slightly better understood. In this case, both the bounds Eq. (55) and Eq. (60) show that for \( \frac{1}{17} \leq \eta \leq \frac{1}{12} \), the Werner state is unsteerable for arbitrary POVMs on Alice’s side [Barrett 2002; Quintino et al. 2015]. In the range \( \frac{1}{17} \leq p \leq \frac{1}{12} \), the state is also known to be unsteerable if the POVMs are limited to those with 3 outcomes [Werner 2014]. For most general POVMs, numerical evidences based on the critical radius approach are available, which indicate that the state is also unsteerable in this range [Nguyen et al. 2018; 2019].

To conclude this section, we refer the interested readers to [Augusiak et al. 2014] for further constructions of LHV and LHS models.
C. Steering and local filtering

For characterizing steerability and other correlations in quantum states, it is relevant to study their behaviour under local operations. Given a general quantum state $\varrho_{AB}$ one can consider states of the type

$$\tilde{\varrho}_{AB} = \frac{1}{N} (T_A \otimes T_B) \varrho_{AB} (T_A^\dagger \otimes T_B^\dagger) ,$$

(61)

where $T_A/B$ are some transformation matrices and $N$ denotes a potential renormalization.

Then, one can ask whether the correlations in the state $\varrho_{AB}$ are related to those of the state $\tilde{\varrho}_{AB}$. Clearly, this depends on the properties of the matrices $T_A/B$. For them there are mainly two possible choices: Either one restricts them to be unitary $T_A/B = U_{A/B}$ and therefore one considers local unitary transformations. Or one considers general invertible matrices $T_A/B = F_{A/B}$, which are the so-called local filtering operations and are more general than local unitaries.

For the case of entanglement one can directly see from the definition in Eq. 41 that local filtering operations keep the property of a state being separable or entangled. Filtering operations can, nevertheless, change the amount of entanglement. Any full rank state can be brought into a normal form under filtering operations, where the reduced states $\tilde{\varrho}_{A/B}$ are maximally mixed (Leinaas et al. 2006; Verstraete et al. 2003). In this form, certain entanglement measures are maximized (Verstraete et al. 2003) and bringing a state in this normal form can improve many entanglement criteria (Gittsovich et al. 2008). For Bell nonlocality, it can be seen that local unitary transformations keep the property of a state having an LHV model. But local filtering operations are already too general, there are two-qubit states that do not violate the CHSH inequality, but after local filtering, they do (Gisin 1996; Popescu 1995).

Steering is a notion between entanglement and nonlocality, so a mixed behaviour under local nonlocality can be expected. In fact, it was noted in (Galego and Aolita 2015; Quintino et al. 2015; Uola et al. 2014) that local unitaries on Alice’s side and local filtering on Bob’s side,

$$\tilde{\varrho}_{AB} = \frac{1}{N} (U_A \otimes F_B) \varrho_{AB} (U_A^\dagger \otimes F_B^\dagger) ,$$

(62)

do not change the steerability of a state. Also the critical radius as a steering parameter (see Section III.C.1) is not affected. With these transformations one can achieve that $\tilde{\varrho}_B$ is maximally mixed on its support. If a state is in this form, this can simplify calculations and therefore it is good starting point to study the steerability of a state (Nguyen et al. 2019).

D. One-way steerable states

The asymmetry between the two parties in the definition of quantum steering immediately strikes one with the question whether there is a state where Alice can steer Bob, but not the other way around (Wiseman et al. 2007). Such one-way steerable states were first constructed for continuous variable systems (Midgley et al. 2010; Olsen 2013). One-way steerable states for discrete systems were studied later by Bowles et al. (2014); Evans and Wiseman (2014); Skrzypczyk et al. (2014). More recently a simpler family of one-way steerable two-qubit states was identified in (Bowles et al. 2016). This family of states is given by

$$\varrho_{AB}(\alpha, \theta) = \alpha |\psi_\theta\rangle \langle \psi_\theta | + (1 - \alpha) \mathbb{I} \otimes \varrho_B ,$$

(63)

where $|\psi_\theta\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle$ and $\varrho_B = \text{Tr}_A |\psi_\theta\rangle \langle \psi_\theta |$, with $0 \leq \alpha \leq 1$ and $0 < \theta \leq \pi/4$. The states can be brought into the two-qubit Werner states with the same mixing probability $\alpha$ by a local filtering on Bob’s side and a local unitary on Alice’s side. Therefore it is steerable from Alice to Bob if and only if $\alpha > 1/2$ (see Section III.C). Using the uniform distribution as an ansatz for the LHS ensemble, Bowles et al. (2016) showed that the state is unsteerable from Bob to Alice for $\cos^2(2\theta) \geq \frac{2^{2\alpha} - 1}{2^{2\alpha}}$. With the complete characterization of steerability for two-qubit states described in Section III.C.1 the boundary of the set of unsteerable states from Bob to Alice has been obtained with high accuracy (Nguyen et al. 2019; see Fig. 7). It is clearly visible from the figure that $\varrho_{AB}(\alpha, \theta)$ is one-way steerable for a large range of parameters.

The one-way steering phenomenon can also be shown to persist when the measurements are extended to...
POVMs (Quintino et al., 2015). The idea to construct an example is as follows. One first embeds a state which is unsteerable from Alice to Bob with respect to projective measurements, but steerable for the other direction, into a higher dimension on Alice’s side. One then constructs a state that admits an LHS model for all POVMs performed on Alice’s side using Eq. (36) with the state $\sigma_A$ chosen to be supported only in the extended dimension on Alice’s side. With this choice of $\sigma_A$, it is easy to show that the state is still steerable from Bob to Alice (Quintino et al., 2015). The constructed state is thus one-way steerable also when one considers all POVMs.

The one-way steering phenomenon also attracts attention from the experimental side. Early experiments demonstrating one-way steering were carried out for continuous variable systems and Gaussian measurements (Händchen et al., 2012). The effects of various types of noise on the direction of steering were later analyzed and probed experimentally by Qin et al. (2017). Experiments demonstrating one-way steering for discrete systems were performed by Sun et al. (2016), Xiao et al. (2017), and Xiao et al. (2017) concentrated on demonstrating one-way steering when measurements are limited to two and three settings. Wollmann et al. (2016) also demonstrated the persistence of the phenomena for POVMs. Most recently it was realized (Baker et al., 2018) that existing experiments demonstrating one-way steering committed certain assumptions on the states or the measurements and were therefore inconclusive. A conclusive experiment (Fischler et al., 2018) was then performed shortly after.

E. Steering with bound entangled states

Now we discuss the steerability of so-called bound entangled states. This provides a relevant example for the fact that the characterization of steering gives new insights into old problems in entanglement theory.

Before presenting the result, we have to recall some facts about the entanglement criterion of the positivity of the partial transpose (PPT criterion) and entanglement distillation. Let us start with the PPT criterion (Horodecki et al., 1996; Peres, 1996b). Generally, for a two-particle state $\rho = \sum_{ij,kl} \rho_{ij,kl} |i\rangle \langle j| \otimes |k\rangle \langle l|$ the partial transposition with respect to Bob is defined as

$$\rho^{T_B} = \sum_{ij,kl} \rho_{ij,kl} |i\rangle \langle j| \otimes |k\rangle \langle l|.$$  

Similarly, one can define a partial transposition with respect to Alice which obeys $\rho^{T_A} = (\rho^{T_B})^T$. Note that the partial transposition may change the eigenvalues of a matrix, contrary to the full transposition.

The PPT criterion states that for separable states the partial transposition has no negative eigenvalues, $\rho^{T_B} \succeq 0$, such states are also called PPT states. It was further proven that for systems consisting of two qubits ($2 \times 2$-systems) or one qubit and one qutrit ($2 \times 3$-system) this criterion is sufficient for separability and all PPT states are separable. For all other dimensions, PPT entangled states exist, these states are, in some sense, weakly entangled, as they cannot be used for certain quantum information tasks.

The main quantum information task where PPT entangled states are useless is the task of entanglement distillation. Entanglement distillation is the process where many copies of some noisy entangled state are distilled to few highly entangled pure states via local operations and classical communication (Horodecki et al., 1998). Surprisingly, not all entangled states can be used for distillation, and these undistillable states are called bound entangled. It was shown that PPT entangled states are bound entangled, but there are some NPT states, for which it has been conjectured that they are also bound entangled (Pankowski et al., 2010).

In 1999, Peres formulated the conjecture that bound entangled states do not violate any Bell inequality (Peres, 1999). This conjecture was based on an analogy between a general distillation protocol and Bell inequalities for many observers, but for a long time no proof could be found. In 2013 the conjecture was made that bound entangled states are also useless for steering (Pusey, 2013; Skrzypczyk et al., 2014). This so-called stronger Peres conjecture could potentially open a way to prove the original Peres conjecture, especially as the PPT criterion and the question of steerability are closely related to SDPs.

It was shown by Moroder et al. (2014), however, that some bound entangled states can be used for steering, and an explicit example for two qutrits was given. The idea to find the counterexample is the following. For a given state of two qutrits and two measurements with three outcomes each one can decide the steerability of the assemblage $\{\rho_{ij|kl}\}$ with an SDP (see also Section IV.B.1). Considering the dual formulation of the SDP, one finds that the operator

$$W = A_{11} \otimes Z_{13} + A_{12} \otimes Z_{23} + A_{12} \otimes Z_{31} + A_{12} \otimes Z_{32} + (A_{31} + A_{31} - 1) \otimes Z_{33}$$

defines a steering inequality, that is, $\text{Tr}(\rho W) \geq 0$ for unsteerable states. Here, the $A_{ij|kl}$ are arbitrary measurement operators for Alice and the set $\{Z_{13}, Z_{23}, Z_{31}, Z_{32}, Z_{33}\}$ consists of five positive operators, obeying the four semidefinite constraints $Z_{13} + Z_{23} + Z_{31} + Z_{32} + Z_{33} \geq 0$ for $i, j \in \{1, 2\}$.

Given this steering inequality, one can look for steerable PPT states by an iteration of SDPs: One starts with a random initial steerable state $\rho$ and fixes Alice’s measurements $A_{ij|kl}$ to be measurements in two mutually unbiased bases. Then, by optimizing the $Z_{ij} \otimes \rho$ via an SDP one can minimize the mean value $\text{Tr}(\rho W)$ and find the
optimal steering inequality $W$. Given this $W$, one can ask for the minimal expectation value of it with respect to PPT states, this is again an SDP. Having found the PPT state with the smallest $\text{Tr}(\rho W)$ one can optimize over the $Z_{ij}$ again and then iterate. In practice this procedure converges quickly towards PPT states which are steerable, delivering the desired counterexamples to the stronger Peres conjecture.

Having found the counterexamples, it is a natural question whether these states also violate a Bell inequality. Indeed, as has been shown by [Vértesi and Brunner 2014], these states are also counterexamples to the original Peres conjecture. Finally, [Yu and Oh 2017] presented an analytical approach, giving explicit families of PPT entangled states in any dimension $d \geq 3$ which, for appropriate parameters, violate Bell inequalities or can be used for steering, see Fig. 8.

**F. Steering maps and dimension-bounded steering**

In this section, we describe how the steering problem can be viewed as a certain kind of separability problem [Moroder et al., 2016]. This allows to apply the powerful techniques of entanglement theory [Gühne and Tóth, 2009] [Horodecki et al., 2009], and study problems such as the detection of steering, if Bob’s system is not well characterized and only its dimension is known.

To formulate the main idea it suffices to consider the case of two measurements ($x \in \{1,2\}$) with two outcomes ($\pm$) on Alice’s side. As discussed in Section L.B.1 steerable of the assemblage $\{g_{ij}\}$ can be detected by an SDP. More precisely, Eqs. (25) states that the assemblage is unsteerable, if one limits four positive semidefinite operators $\omega_{ij}$ with $i,j = \pm$ such that $\omega_{+1} = \omega_{++} + \omega_{+-}$, $\omega_{+2} = \omega_{++} + \omega_{-+}$, $\omega_{-1} = \omega_{-+} + \omega_{--}$, and $\omega_{-2} = \omega_{++} + \omega_{--}$. These equations are not independent. If one takes $\omega_{++}$ as a free variable, one has the relations $\omega_{+-} = \theta_{+1} - \omega_{++}$, $\omega_{-+} = \theta_{+2} - \omega_{++}$, and $\omega_{--} = \theta_{+1} - \theta_{+2} + \omega_{++}$. Of course, this is only a valid solution if all $\omega_{ij}$ are positive semidefinite.

Then, one takes four positive definite operators $Z_{ij}$ with $i,j = \pm$, which obey the relation $Z_{++} = Z_{++} + Z_{+-} - Z_{-+} - Z_{--}$, and considers the bipartite operator

$$\Sigma_{AB} = \sum_{ij} Z_{ij} \otimes \omega_{ij}. \quad (66)$$

This is, after appropriate normalization, a separable state as in Eq. (4). The point is that with the given relations on the $\omega_{ij}$ and $Z_{ij}$ this state can be written as

$$\Sigma_{AB} = Z_{++} \otimes \theta_{+1} + Z_{+-} \otimes \theta_{+2} + Z_{-+} \otimes (\theta_{B} - \theta_{+1} - \theta_{+2}) \quad (67)$$

as can be verified by direct inspection. Here, all the dependencies on the $\omega_{ij}$ dropped out, so $\Sigma_{AB}$ is uniquely determined by the assemblage and the $Z_{ij}$ only. Also the required normalization follows directly from Eq. (67).

From this the desired connection to the separability problem follows: Given an unsteerable assemblage and operators $Z_{ij}$ obeying the conditions from above, the state $\Sigma_{AB}$ in Eq. (67) is separable. Moreover, one can show the opposite direction: If the assemblage is steerable, then there exist a set of operators $Z_{ij}$ such that the state $\Sigma_{AB}$ is entangled. In this case, the entanglement of $\Sigma_{AB}$ can even be detected by a special entanglement witness, namely the flip operator.

The statement can be generalized to an arbitrary number of measurements and outcomes [Moroder et al., 2016]. In fact, it is related to the dual of the original SDP.

This reformulation of the steerability problem can give insights in the detection of steering, if the measurements on Bob’s side are not fully characterized, but only the dimension of the space where the measurements act on is known. The core idea is the following: For any bipartite state $\rho_{AB}$ and sets of local orthonormal observables $G_{A}^{X}$ and $G_{B}^{P}$ [that is, $\text{Tr}(G_{A}^{X}G_{B}^{P}) = \delta_{ij}$ for $X \in \{A,B\}$] one can build the matrix $A_{kl} = \text{Tr}(\rho_{AB}G_{A}^{X} \otimes G_{B}^{P})$. Then, the computable cross-norm or realignment (CCNR) criterion states that if $\rho_{AB}$ is separable, then the trace norm is bounded by one, $\|A\|_{1} \leq 1$ [Chen and Wu, 2003] [Rudolph, 2005], see also Section II.C.3. This criterion has already been used to detect entanglement with uncharacterized devices, if the dimension is known: If Alice and Bob make uncharacterized measurements $A_{k}$ and $B_{l}$ they can build the expectation value matrix $A_{kl} = \text{Tr}(\rho_{AB}A_{k} \otimes B_{l})$ and, using the dimension assumption, from that estimate the trace norm $\|A\|_{1}$ [Moroder and Gittsovich, 2012].

A similar approach can be used for steering [Moroder et al., 2016]. For a choice of $Z_{ij}$ one considers the state $\Sigma_{AB}$. Then, on Alice’s side one takes a set of local orthogonal observables $G_{A}^{X}$ and for Bob’s side uncharacterized measurements $B_{l}$ and builds an expectation

![FIG. 8 Inclusion relation between the PPT states and entanglement, steering, and Bell inequality violations. Separable states are PPT, but some entangled states are PPT as well. PPT entangled states are bound entangled, as no pure state entanglement can be distilled from them. There exist, however, PPT states that can be used for steering and also PPT states that violate Bell inequalities. These states are counterexamples to the Peres conjecture.](image)
value matrix, which can be used to estimate whether $\Sigma_{AB}$ violates the CCNR criterion. If this is the case, then the original assemblage was steerable. The resulting criteria are strong: For two-qubit Werner states $\varrho(p) = p|\psi\rangle\langle\psi| + (1 - p) \mathbb{1}/4$ and Pauli measurements $\sigma_x, \sigma_y$ and $\sigma_z$, one can evaluate from the data the steering inequality in Eq. (12). It detects steerability for $p > 1/\sqrt{3}$, which is the same threshold as the steering inequality. So, for this case the approach allows to draw the same conclusion from the resulting data, but without assuming that the measurements were correct Pauli measurements, the only assumption that is made is that Bob’s space is a qubit.

G. Superactivation of steering

Let us come back to the formulation of quantum steering as a simulation task where Alice tries to convince Bob that she can steer his system from a distance as discussed in Section I. Note that in this protocol, Alice has to prepare a large number of pairs of particles in a certain state. One of the particles in each pair is then sent to Bob. Note that it is crucial for Alice to prepare many copies of the state, so that Bob later on can do tomography to verify the steered states on his side. Alice then declares the set of measurements she can make, or equivalently the assemblage she can steer Bob’s system to. To maximize her steering ability, Alice clearly should choose the largest set of measurements. Most often, Alice’s measurements are assumed to be projective measurements (or POVMs) on separated particles on her side. This, however, is not yet the maximal set of measurements she can do. As Alice has prepared a large number of bipartite states, she can actually make collective measurements on several particles on her side. We will see that when such collective measurements are considered, the steerability of a state may change. More precisely, for an unsteerable (but entangled) state $\varrho_{AB}$, one asks whether there exists a finite number $n$ such that $\varrho_{AB}^n$ is steerable. In this case, we say that the quantum steerability of $\varrho_{AB}$ can be superactivated.

For nonseparable, a similar question is answered trivially negative for any states, but for Bell nonlocality, it has been extensively investigated since (Peres 1996a). For Bell nonlocality, the confirmative answer was first obtained by Palazuelos (2012) and later refined by Cavalcanti et al. (2013a). The authors showed that indeed for a certain state $\varrho_{AB}$ which admits an LHV model, for sufficiently large $n$, $\varrho_{AB}^n$ can violate some Bell inequality. Note that this is distinct from the notion of superactivation of bound entanglement (Shor et al. 2003). While the superactivation of Bell nonlocality investigated by the mentioned authors also implies the ability to superactivate quantum steering, exact characterizations of quantum steering also significantly simplify the understanding of the phenomenon (Hsieh et al. 2016). In fact, Quintino et al. (2016) extended the results of Cavalcanti et al. (2013a) to show that the steerability of all unsteerable states $\varrho_{AB}$ that satisfy the so-called reduction criterion for entanglement (Horodecki and Horodecki 1999) can be superactivated. The reduction criterion states that if $\mathbb{1}_A \otimes \varrho_B - \varrho_{AB}$ is not positive, then the state $\varrho_{AB}$ is nonseparable. As satisfying the reduction criterion is a necessary and sufficient condition for a two-qubit or a qubit-qutrit state to be nonseparable (Horodecki and Horodecki 1999), the steerability of all entangled states of dimension $2 \times 2$ or $2 \times 3$ can be superactivated.

Their idea is based on the exact threshold for quantum steering of the isotropic state in Eq. (55). For convenience, here the isotropic state is reparametrized as

$$S_f^d = f|\phi_+\rangle\langle\phi_+| + (1 - f) \frac{1 - |\phi_+\rangle\langle\phi_+|}{d^2 - 1},$$

with the same notation as defined in Eq. (55) and $f = 1 - (1 - 1/d^2)(1 - \eta)$. According to Eq. (55), the isotropic state in Eq. (68) is unsteerable if only if $f \leq (1 + d)H_d/d^2$. It is known that a state that violates the reduction criterion can be brought into an entangled isotropic state with $f > 1/d$ by local filtering on Bob’s side and the so-called isotropic twirling operation. As we mentioned in Section II.C local filtering on Bob’s side does not change the steerability of the state. The isotropic twirling operation consists of averaging the state under certain random local unitary transformations, thus also does not increase the steerability. So it is sufficient to show that the steerability of the isotropic state $S_f^d$ with $f > 1/d$ can be superactivated. This again can be shown by observing that the isotropic twirling on $(S_f^d)^{\otimes n}$ yields the isotropic state $S_{d^n}$ of dimension $d^n \times d^n$. Thus $(S_f^d)^{\otimes n}$ is steerable if

$$f > \frac{|(1 + d^n)H_{d^n} - d^n|^{1/n}}{d^2}.$$  

At large $n$, the right-hand side asymptotically approaches $1/d$. Therefore, whenever $f > 1/d$, there exists $n$ such that the inequality is satisfied, or equivalently the steerability of $S_f^d$ can be superactivated.

Beyond states that violate the reduction criterion, one may ask if quantum steerability, or more generally, Bell nonlocality can always be superactivated for arbitrary entangled states. This question remains as a challenge for future research. If this were the case, the hierarchy of quantum nonlocality would be unified into a single concept (Cavalcanti et al. 2013a).

Besides the notion of superactivation of quantum nonlocality via collective measurements on multiple copies of the state as described above, there is also the notion of superactivation of quantum nonlocality via local filtering (on both sides). The phenomenon is dated back
exists a third measurement whose statistics can be classified simultaneously (or jointly), i.e. whether there is the possibility of deducing the variances of observables places certain restrictions on the variances of the measured observables. Such restrictions do not, however, give any further operational insight into the incompatibility breaking quantum channels (Choi isomorphism). Moreover, we discuss in detail how the statistics of several measurements from the statistics of two POVMs, i.e. for POVMs $\{A_\lambda\}_{\lambda}$, where $0 < \mu \leq 1$. The question is now how to find candidates for a joint measurement. For the above pair, an educated guess [i.e. a candidate with similar symmetry as the pair $(S_{\pm|x|}^\mu, S_{\pm|x|}^\nu)$] gives

$$G_{i,j}^\mu := \frac{1}{4}(1 \pm i\mu \sigma_x + j\mu \sigma_z),$$

where $i, j \in \{-, +\}$. One notices straight away that

$$S_{\pm|x|}^\mu = G_{\pm, \pm}^\mu + G_{\pm, -}^\mu$$

i.e. there exist (deterministic) post-processings that give the original measurements. The last thing to check is that $\{G_{i,j}^\mu\}_{i,j}$ forms a POVM. As the normalisation follows from the definition, one is left with checking the positivity of the elements, which is equivalent to $\mu \leq 1/\sqrt{2}$. It can be shown that this is indeed the optimal threshold for joint measurability in our example, i.e. beyond this

### IV. JOINT MEASURABILITY AND STEERING

In this section we discuss the problem of joint measurability to which steering is related in a many-to-one manner [Kinkas et al. 2017] [Quintino et al. 2014] [Uola et al. 2015, 2014]. Joint measurability is a natural extension of commutativity for general measurements. Operationally it corresponds to the possibility of deducing the statistics of several measurements from the statistics of a single one. The connection between the concepts of joint measurability and steering unlocks the technical machinery developed within the framework of quantum measurement theory to be used in the context of quantum correlations. More precisely, joint measurability has been studied extensively already a few decades before steering was formulated in its modern form. We review the connection on three levels: joint measurability on Alice’s side (pure states), on Bob’s side (mixed states), and on the level of the incompatibility breaking quantum channels (Choi isomorphism). Moreover, we discuss in detail how known results on one field can be mapped to new ones on the other.

#### A. Measurement incompatibility

Measurement incompatibility manifests itself in various operationally motivated forms in quantum theory. Maybe the best-known notion is that of non-commutativity. Here by non-commutativity we mean the mutual non-commutativity of the POVM elements of two POVMs, i.e. for POVMs $\{A_\lambda\}_\lambda$ and $\{B_\lambda\}_\lambda$, we ask whether $[A_\lambda, B_\lambda] = 0$ for all $\lambda$.

One possible operationally motivated extension of commutativity is that of joint measurability. Namely, one can ask whether two measurements can be performed simultaneously (or jointly), i.e. whether there exists a third measurement whose statistics can be classically processed to match those of the original pair. Further fine-tunings of measurement incompatibility have been presented in the literature, e.g. coexistence, broadcastability, and non-disturbance [Busch et al. 2016] [Heinosaari 2016] [Heinosaari and Wolf 2010], see also Section IV.E. Typically all the incompatibility related extensions of non-commutativity (on a single system) coincide with non-commutativity for projective measurements, but for the case of POVMs they form a strict hierarchy [Heinosaari and Wolf 2010]. It is worth mentioning that in the process matrix formulation of POVMs, even commuting process POVMs can be incompatible [Sedlák et al. 2016].

For investigating steering from the measurement perspective, the notion of joint measurability appears most fitting. A set $\{A_{a|x}\}_{a,x}$ of POVMs (i.e. positive operators summing up to the identity for every $x$) is said to be jointly measurable if there exists a POVM $\{G_\lambda\}_\lambda$ together with classical post-processings $\{p(a|x, \lambda)\}_{a,x,\lambda}$ such that

$$A_{a|x} = \sum_\lambda p(a|x, \lambda)G_\lambda.$$  \hspace{1cm} (70)

The POVM $\{G_\lambda\}_\lambda$ is called a joint observable or a joint measurement of the set $\{A_{a|x}\}_{a,x}$.

To give an example of a set of jointly measurable POVMs, one could use a mutually commuting pair of POVMs in which case a joint measurement is given by a POVM whose elements are products of the original ones. For a more insightful example, we take a pair of noisy Pauli measurements defined as

$$S^\mu_{\pm|x|} := \frac{1}{2}(1 \pm i\mu \sigma_x)$$

$$S^\nu_{\pm|x|} := \frac{1}{2}(1 \pm \mu \sigma_z),$$

where $0 < \mu \leq 1$. The question is now how to find candidates for a joint measurement. For the above pair, an educated guess [i.e. a candidate with similar symmetry as the pair $(S^\mu_{\pm|x|}, S^\nu_{\pm|x|})$] gives

$$G_{i,j}^\mu := \frac{1}{4}(1 \pm i\mu \sigma_x + j\mu \sigma_z),$$

where $i, j \in \{-, +\}$. One notices straight away that

$$S^\mu_{\pm|x|} = G^\mu_{\pm, \pm} + G^\mu_{\pm, -}$$

$$S^\nu_{\pm|x|} = G^\nu_{\pm, \pm} + G^\nu_{\pm, -},$$

i.e. there exist (deterministic) post-processings that give the original measurements. The last thing to check is that $\{G_{i,j}^\mu\}_{i,j}$ forms a POVM. As the normalisation follows from the definition, one is left with checking the positivity of the elements, which is equivalent to $\mu \leq 1/\sqrt{2}$. It can be shown that this is indeed the optimal threshold for joint measurability in our example, i.e. beyond this
threshold the POVMs \( \{ S^u_{\pm|x} \} \) and \( \{ S_{\pm|x}^u \} \) do not admit a joint measurement (Busch 1986).

The above example shows that joint measurability is indeed a proper generalization of commutativity. In the literature, many such examples have been discussed in finite and continuous variable quantum systems (Busch et al., 2016). A typical question is as above: how much noise can be added until measurements become jointly measurable. For small numbers of measurements and outcomes, this can be efficiently checked with SDP (Uola et al., 2009). For more complicated scenarios various optimal and semi-optimal analytical and numerical techniques have been developed (Bavaresco et al., 2017; Designolle et al., 2019; Heinosaari et al., 2016; Kunjwal et al., 2014; Uola et al., 2016).

B. Joint measurability on Alice’s side

Comparing the definition of joint measurability with that of unsteerability, one recognises similarities. Indeed, joint measurability is a question about the existence of suitable post-processings and a common POVM, whereas unsteerability asks the existence of suitable response functions and a common state ensemble. To make the connection exact, we recall the main result of (Quintino et al., 2014; Uola et al., 2014):

A set of measurements \( \{ A_{a|x} \}_{a,x} \) is not jointly measurable if and only if it can be used to demonstrate steering with some shared state.

To be more precise, using a jointly measurable set of observables on Alice’s side, i.e. \( A_{a|x} = \sum_{\lambda} \rho(a|x,\lambda) G_{\lambda} \) and a shared state \( \varrho_{AB} \) results in a state assemblage

\[
\varrho_{a|x} = \sum_{\lambda} \rho(a|x,\lambda) \text{tr}(G_{\lambda} \otimes I) \varrho_{AB}
\]

(67)

\[
= \sum_{\lambda} \rho(a|x,\lambda) \sigma_{\lambda}
\]

(68)

where we have written \( \sigma_{\lambda} = \text{tr}_A[(G_{\lambda} \otimes I) \varrho_{AB}] \). Hence, the existence of a joint observable for Alice’s measurements implies the existence of a local hidden state model. For the other direction, using a full (finite) Schmidt rank state \( |\psi\rangle = \sum_i \lambda_i |i\rangle \) one has

\[
\varrho_{a|x} := \text{tr}_A[(A_{a|x} \otimes I)|\psi\rangle\langle\psi|] = C A_{a|x}^T C,
\]

(69)

where \( C = \sum_j \lambda_{j} |j\rangle \langle j| \) and \( X^T \) is the transpose of the operator \( X \) in the basis \( \{|j\rangle\}_{j} \). Assuming that the assemblage \( \{ \varrho_{a|x} \}_{a,x} \) has a local hidden state model one gets

\[
A_{a|x} = \sum_{\lambda} p(a|x,\lambda) C^{-1} \sigma_{\lambda} C^{-1}
\]

(70)

from which it is clear that \( \{ C^{-1} \sigma_{\lambda} C^{-1} \}_\lambda \) forms the desired joint measurement of \( \{ A_{a|x} \}_{a,x} \).

To demonstrate a possible use of this result, one can consider a steering scenario where Alice performs measurements on a noisy isotropic state. This noise in the state can be translated to Alice’s measurements by writing

\[
\text{tr}[(A_{a|x} \otimes I)\varrho_{AB}] = \text{tr}[(A_{a|x}^\mu \otimes I)\varrho_{AB}],
\]

(71)

where \( \varrho_{AB} = \mu |\psi^+\rangle\langle\psi^+| + (1-\mu) I \) and \( A_{a|x}^\mu = \mu A_{a|x} + (1-\mu) \text{tr}[(A_{a|x} \otimes I)\varrho_{AB}] \) with \( \mu \in [0,1] \). For different sets of measurements on Alice’s side one can either solve the steerability by using known incompatibility results or vice versa. To give an example, consider the known (Wiseman et al., 2007) steerability threshold for the noisy isotropic state (with projective measurements) \( \mu^* = (d^2 - 1)/(d - 1) \). Using Eq. (80) one sees that for any \( \mu > \mu^* \) there exists a set of projective measurements that remains incompatible with the amount \( \mu \) of white noise. On the contrary, any set of projective measurements with an amount \( \mu \leq \mu^* \) of white noise results in an unsteerable state assemblage (with the isotropic state) and, hence, such a set is jointly measurable.

It is worth noting that the connection between incompatibility of Alice’s measurements and steerability of the resulting assemblage is strongly motivated by a similar work on non-locality. In (Wolf et al., 2009) incompatibility of Alice’s measurements was proven to be equivalent to the ability of violating the CHSH inequality (when optimizing over Bob’s measurements and the shared state). This connection, however, is known not to be true in general. In (Bene and Vértesi, 2018; Hirsch et al., 2018) counterexamples for the non-locality connection in scenarios with more measurement settings are presented, i.e. there exist sets of measurements that are not jointly measurable but always lead to local correlations. In contrast, joint measurability and steering can both be described in terms of operational contextuality (Tavakoli and Uola, 2019). The CHSH inequality is a criterion for this type of contextuality. It is an open question whether there are other contextuality inequalities that fully characterise incompatibility. In (Tavakoli and Uola, 2019) numerical evidence is provided that a specific family of contextuality criteria generalising the CHSH inequality characterise the incompatibility of sets of binary qubit measurements. Such characterisation, among any other, is directly applicable to steerability of state assemblages by the use of the techniques presented in the following subsection.

C. Joint measurability on Bob’s side

The connection between steering and joint measurements presented in the above section is based on the use of full Schmidt rank states (on finite-dimensional systems). To loosen the assumption on purity of the state,
we recall the main result of Uola et al. (2015):

The question of steerability (of a state assemblage) is a non-normalised version of the joint measurement problem.

More precisely, by normalising a state assemblage \( \{g_{a|x}\}_{a,x} \) one gets abstract POVMs \( \tilde{B}_{a|x} := g_{a|x} g_B^{-1/2} \), where \( g_B = \sum_a g_{a|x} \) and a pseudo-inverse is used when necessary. Note that we use tilde to distinguish between Bob’s actual measurements and the normalised state assemblage (which consists of abstract POVMs on a possibly smaller dimensional system than the one Bob’s measurements act on).

It is straightforward to show (Uola et al., 2015) that the state assemblage \( \{g_{a|x}\}_{a,x} \) is steerable if and only if the abstract POVMs \( \{\tilde{B}_{a|x}\}_{a,x} \) are not jointly measurable. Namely, as the normalisation keeps the post-processing functions fixed, the local hidden states map to joint measurements of the normalised assemblage and joint measurements map to local hidden states.

Such connection broadens the set of techniques that are translatable between the fields of joint measurability and steering. In general, joint measurability criteria map into steering criteria and vice versa. To give an example, we take a well-known joint measurability characterisation of two qubit POVMs (Busch, 1986). Namely, take two qubit POVMs of the form

\[
A_{\pm|x} := \frac{1}{2}(\mathbb{1} \pm \tilde{a}_x \cdot \vec{\sigma}),
\]

where \( x = 1,2 \). This pair is jointly measurable if and only if

\[
\|\tilde{a}_1 + \tilde{a}_2\| + \|\tilde{a}_1 - \tilde{a}_2\| \leq 2.
\]

Note that this criterion is necessary for joint measurability in the more general case, i.e. for pairs of POVMs given as

\[
A_{x|x} := \frac{1}{2}((1 + \alpha_x)\mathbb{1} \pm \tilde{a}_x \cdot \vec{\sigma}), \quad A_{-x|x} = \mathbb{1} - A_{+x|x},
\]

where \( \alpha_x \in [-1,1] \) and \( \|\tilde{a}_x\| \leq 1 + \alpha_x \). In Busch and Schmidt (2010), Stano et al. (2008), Yu et al. (2010) a necessary and sufficient criterion for joint measurability of such pairs is given as

\[
(1 - F_1^2 - F_2^2) \left( 1 - \frac{\alpha_1^2}{F_1^2} \frac{\alpha_2^2}{F_2^2} \right) \leq (\tilde{a}_1 \cdot \tilde{a}_2 - \alpha_1 \alpha_2)^2,
\]

with \( F_i = \frac{1}{2}(\sqrt{1 + \alpha_i} - \|\tilde{a}_i\| + \sqrt{1 - \alpha_i} - \|\tilde{a}_i\|), \) for \( i = 1,2 \).

The criteria in Eq. (82) and Eq. (84) are both steering inequalities. The latter of them characterises all pairs of binary unsteerable assemblages in the qubit case. Labeling members of such assemblages by \( g_{\pm|x} = \frac{1}{2}(\mathbb{1} \pm \lambda \sigma_z) \) and \( g_{\pm|\pm} = \beta \mathbb{1} \pm r^\mp \sigma_z \) we present a comparison of these criteria in Fig. 9 (Uola et al., 2015).

As another example, we demonstrate how steering robustness defined in Eq. (30) translates to an incompatibility robustness (Uola et al., 2015). Recall that the steering robustness \( SR(g_{a|x}) \) of an assemblage \( \{g_{a|x}\}_{a,x} \) can be written as

\[
\min \ t \geq 0 \\
\text{s.t.} \quad \frac{\partial g_{a|x} + t \gamma_{a|x}}{1 + t} \text{ unsteerable for all } a,x.
\]

Here the optimization is over assemblages \( \{\gamma_{a|x}\}_{a,x} \) and positive numbers \( t \), see also Fig. 3. Mapping a state assemblage into a set of POVMs, one can define an incompatibility robustness IR(\( \tilde{B}_{a|x} \)) for a set \( \{\tilde{B}_{a|x}\}_{a,x} \) as (Uola et al., 2015)

\[
\min \ t \geq 0 \\
\text{s.t.} \quad \frac{\tilde{B}_{a|x} + t T_{a|x}}{1 + t} \text{ jointly measurable for all } a,x.
\]

Here the optimisation is performed over POVMs \( \{T_{a|x}\}_{a,x} \) and positive numbers \( t \). Note that this definition works also for a generic set of POVMs, i.e. the POVMs do not need to originate explicitly from a steering problem. To give the incompatibility robustness IR(A_{a|x}) of a finite set of POVMs \( \{A_{a|x}\}_{a,x} \) in the stan-
where the normalised state assemblages appear as abstract processings. Joint measurements to infinite-dimensional systems and incompatibility breaking quantum channels are closely related to the connection presented here. 

To demonstrate the power of the above result, we list some of its implications [Kiukas et al. 2017]. First, the connection is quantitative in the sense that the incompatibility robustness of \( \{A^i(A_{a|x})\}_{a,x} \) coincides with the so-called consistent steering robustness (i.e. a special case of steering robustness: one allows mixing only with assemblages that have the same total state as the original assemblage) of \( \{\varrho_{a|x}\}_{a,x} \). Second, for pure states the corresponding channel \( \Lambda^i \) is unitary, hence, extending the main results of [Quintino et al. 2014; Uola et al. 2014] presented in Section [IV.B] to the infinite-dimensional case. Third, the result characterises unsteerable states as those whose corresponding Choi-Jamiołkowski channel is incompatibility breaking (i.e. outputs only jointly measurable observables). Finally, seemingly different steering problems (such as NOON states subjected to photon loss and systems with amplitude damping dynamics) can have the same channel \( \Lambda \), hence, making it possible to solve many steering problems in one go.

E. Further topics on incompatibility

As mentioned in the beginning of this section, measurement incompatibility manifests itself in various ways in quantum theory. In this section we review briefly two well-known fine-tunings of commutativity and discuss their relation to quantum correlations.

First, in the case of joint measurability one asks for the existence of a common POVM and a set of post-processings. One can relax this concept by dropping the assumption on post-processings. Namely, we say that a set of POVMs \( \{A_{a|x}\}_{a,x} \) is coexistent if there exists a POVM \( \{C_\lambda\}_\lambda \) such that

\[
A_{X|x} = \sum_{\lambda \in \mathcal{T}_{X|x}} C_\lambda, \quad \text{(90)}
\]

where \( \mathcal{T}_{X|x} \) is a subset of outcomes of \( \{C_\lambda\}_\lambda \) for every pair \((X,x)\). Here we have used the notation \( X \) to emphasize that the definition is not only required to hold for all POVM elements of \( \{A_{a|x}\}_{a,x} \), but also for sums of outcomes, e.g. for \( X = \{a_1,a_2\} \) one has \( A_{X|x} = A_{a_1|x} + A_{a_2|x} \). To give the concept a physical interpretation, in [Heinosaari et al. 2016] the authors noted that the definition is equivalent to the joint measurability of
the set of all binarizations (i.e. coarse-grainings to two-valued ones) of the involved measurements. Whereas it is clear that joint measurability implies coexistence (by the use of deterministic post-processing), the other direction does not hold in general (Heinosaari and Wolf, 2010).

As coexistence is closely related to joint measurability, one can ask if anything can be learned from using this concept in the realm of steering. As pointed out in (Uola et al., 2014) one can reach steering with coexistent POVMs on the uncharacterised side (provided that the measurements are not jointly measurable).

What has not appeared in the literature so far, but we wish to point out here, is that when using this concept on the characterized side, one can find examples of steerable assemblages that nevertheless form one ensemble. Consider the example of coexistent but not jointly measurable POVMs given in (Reeb et al., 2013) by defining a vector $|\varphi\rangle = \frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |3\rangle)$ and the POVMs

$$A_{\varphi 1} := \frac{1}{2} (|1\rangle - i \langle 1 |) (i \langle 1 |), \quad i = 1, 2, 3$$

$$A_{\varphi 2} := \frac{1}{2} |\varphi\rangle \langle \varphi |, \quad A_{\varphi 3} := 1 - A_{\varphi 2}. \quad (92)$$

To see that these POVMs are coexistent, one can define a POVM $\{C_\lambda\}_\lambda$ through the elements

$$\left( \frac{1}{2} |1\rangle \langle 1 | + \frac{1}{2} |2\rangle \langle 2 | + \frac{1}{2} |3\rangle \langle 3 |, \frac{1}{2} |\varphi\rangle \langle \varphi |, \frac{1}{2} 1 - |\varphi\rangle \langle \varphi | \right) \quad \text{for all states } \varphi. \quad (93)$$

For the proof that these POVMs are not jointly measurable we refer to (Reeb et al., 2013). Applying the mapping between measurement assemblages and state assemblages (with a full rank state $\rho_B$, see also Section IV.C) to the above (or any similar) example, one ends up with a steerable state assemblage that nevertheless fits into a single ensemble.

As another example, we consider the concept of measurement disturbance. A POVM $\{A_a\}_a$ is called non-disturbing with respect to a POVM $\{B_b\}_b$ if there exists an instrument $\{J_a\}_a$ implementing $\{A_a\}_a$ (i.e. $\text{tr}[J_a(\rho)] = \text{tr}[A_a \rho]$ for all states $\rho$) such that

$$\sum_a \text{tr}[J_a(\rho) B_b] = \text{tr}[\rho B_b] \quad (94)$$

holds for all states $\rho$ and all outcomes $b$.

Non-disturbance is located in between commutativity and joint measurability. Clearly commutativity implies non-disturbance by the use of the Lüders rule and non-disturbance implies joint measurability by defining for a non-disturbing scenario $G_{a,b} = J^*_a(B_b)$, where the dagger refers to the Heisenberg picture. For a proof that the implications can not be reversed in general, and for more detailed analysis on when the implications are reversible, we refer to (Heinosaari and Wolf, 2010).

As some disturbing measurements can be jointly measurable, measurement disturbance is only necessary but not sufficient for steering. One could, however, ask if there exist other types of quantum correlations or tasks for which disturbance is necessary and sufficient. It turns out that the question can be answered in positive and one answer is given by violations of typical (i.e. choose between measuring or not measuring) models of macrorealism (Uola et al., 2019). More precisely, the authors of (Uola et al., 2019) have shown that when all classical disturbance (i.e. clumsy measurement implementation) is isolated from a quantum system, the system can violate macrorealism with some initial state if and only if the involved measurements do not fulfil the definition of non-disturbance.

Motivated by the strong connections between quantum measurement theory and quantum correlations presented in this section (see also Section V.D and Section V.H), it will be an interesting question for future research to isolate the measurement resources behind other quantum tasks. Conversely, it will be of interest to see if other concepts of incompatibility such as broad-castability (Heinosaari, 2016), incompatibility on many copies (Carmeli et al., 2016), and measurement simulability (Oszmaniec et al., 2017) will find counterparts in the realm of quantum correlations. To conclude, we note that whereas further connections between measurement theory and correlations remain unknown, jointly measurable sets (Carmeli et al., 2019; Skrzypczyk et al., 2019), or more generally all convex subsets of measurements (Uola et al., 2019a), can be characterized through state discrimination tasks.

V. FURTHER TOPICS AND APPLICATIONS OF STEERING

In this section we discuss further aspects and applications of steering. We start with multipartite steering, steering of Gaussian states and temporal steering. Then we discuss applications of steering such as quantum key distribution, randomness certification or channel discrimination. Finally, we review the resource theory of steering and the phenomenon of post-quantum steering.

A. Multipartite steering

The extension of steering to multipartite systems is an emerging field of research and different approaches for defining multipartite steering exist. Before explaining them, we would like to point out some peculiarities of the multipartite scenario, if one considers steering across a bipartition.
1. Steering across a bipartition

In order to discuss the different effects that play a role for steering in the multipartite scenario, consider a tripartite state $\rho_{ABC}$ and investigate steering across a given bipartition, say $AB|C$ for definiteness. Then, there are different scenarios that have to be distinguished, where in all of them Alice and Bob want to steer Charlie.

- **Global steering**: In the simplest case, Alice and Bob make global measurements on their two particles and steer Charlie. This reduces to a bipartite steering problem for $\rho_{AB|C}$ and all the usual methods can be applied.

- **Reduced steering**: Another simple case arises, if Alice (or Bob) try to steer Charlie, without needing the help of the other. If the action of one of them is not required, this reduces to the bipartite steerability of the reduced state $\rho_{AC}$ or $\rho_{BC}$. Again, all the bipartite steering theory can be applied.

- **Local steering**: The interesting case arises, if Alice and Bob try to steer Charlie by local measurements on their respective parties. In this case, one has to consider the assemblage $\rho_{ab|xy}$ and ask whether its elements can be written as

$$\rho_{ab|xy} = \int d\lambda p(\lambda)p(ab|x,y,\lambda)\sigma^C. \quad (95)$$

Here, we can distinguish among several cases, depending on the properties of $p(ab|A,B,\lambda)$. It may be a general probability distribution, it may obey the non-signaling constraint, or it may factorize,

$$p(ab|x,y,\lambda) = p(a|x,\lambda)p(b|y,\lambda). \quad (96)$$

As $p(ab|x,y,\lambda)$ has the interpretation of a simulation strategy [see Eq. (2)] the latter means that Alice and Bob play an independent strategy.

A simple example of the difference between global and local steering can be constructed from the phenomenon of superactivation of steering [Quintino et al. 2016] (see also Section III.G). For certain states, one copy of the state is unsteerable, but many copies of the same state may become steerable. So one can consider a state $\rho_{ABCC'} = \rho_{AC} \otimes \rho_{BC}$, where $\rho_{BC}$ is a copy of $\rho_{AC}$, and $\rho_{AC}$ is unsteerable, but its steerability can be superactivated [where only two copies are already enough (Quintino et al. 2016)]. For this state, local measurements give an unsteerable state assemblage as

$$\rho_{ab|xy}^{CC'} = \text{Tr}_{AB}[(A_{ab} \otimes B_{by} \otimes 1_{CC'})\rho_{ABCC'}] = \text{Tr}_{AB}[(A_{ab} \otimes 1_{C})\rho_{AC} \otimes (B_{by} \otimes 1_{CC'})\rho_{BC}] = \int d\lambda d\mu p(\lambda)p(\mu)p(ab|x,y,\lambda)p(by|y,\mu)\sigma^C \otimes \sigma_{\mu}^{CC'} = \int d\nu p(ab|x,y,\nu)\sigma_{\nu}^{CC'}, \quad (97)$$

and the state is locally unsteerable even with the restriction to a factorizing $p(ab|x,y,\lambda)$. However, because of the superactivation phenomenon, this state is steerable with global measurements.

Another possibility is to consider the bipartition $A|BC$, where Alice wants to steer Bob and Charlie. Here, one has to consider the ensemble $\rho_{AC}^{BC}$ and ask whether it can be written as $\rho_{AC}^{BC} = \int d\lambda p(\lambda)p(ab|x,\lambda)\sigma^B \otimes \sigma^C$. In this case, two possible scenarios emerge, one where the state of Bob and Charlie is a single system, reducing to a bipartite steering problem, and another one where the local hidden state of Bob and Charlie factorizes, i.e. $\sigma^A = \hat{\sigma}^B \otimes \hat{\sigma}^C$.

From the above picture, one can see that the extension of steering to multipartite systems leads to different scenarios, making its characterization even more difficult than the ones for entanglement and nonlocality.

2. Different approaches towards multipartite steering

The existing works on multipartite steering can be divided into two different approaches. The first approach sees steering as an one-sided device-independent entanglement verification and translates this to the multipartite scenario. The second approach asks for a multipartite system whether steering is possible for a given bipartition.

To discuss the first approach, we need to recall the basic definitions of the different entanglement classes for multipartite systems [Gühne and Töth 2009]. For a three-party system $\rho_{ABC}$ one calls the state fully separable, if it can be written as

$$\rho_{ABC}^{fs} = \sum_k p_k \rho^{A}_k \otimes \rho^{B}_k \otimes \rho^{C}_k, \quad (98)$$

where the $p_k$ form a probability distribution. If a state is not of this form, it is entangled, but not all particles are necessarily entangled. For instance, a state of the form $\rho_{ABC}^{bs} = \sum_k p_k \rho^{A}_k \otimes \rho^{BC}_k$ may contain entanglement between $B$ and $C$, but it is separable for the bipartition $A|BC$ and therefore called biseparable. More generally, mixtures of biseparable states for the different partitions are also biseparable

$$\rho_{ABC}^{bs} = p_{A|BC} \rho^{A}_{A|BC} + p_{B|AC} \rho^{B}_{B|AC} + p_{C|AB} \rho^{C}_{C|AB}, \quad (99)$$

and states which are not biseparable are genuine multipartite entangled. These definitions can straightforwardly be extended to more than three particles.

One-sided device-independent entanglement detection in the multipartite scenario has first been discussed by [Cavalcanti et al. 2011]. There criteria for full separability in the form of Mermin-type inequalities have been given, which hold for $k$ trusted sites and $N-k$ untrusted sites. These criteria can be violated in quantum
mechanics, and the possible violation increases exponentially with the number of parties. Inequalities for higher-dimensional systems have been derived in [He et al., 2011].

In general, one if has a quantum network of $N$ parties where some of the parties perform untrusted measurements, the parties which trust their measurement apparatus can perform quantum state tomography, and reconstruct the conditional state after the untrusted parties announce their measurement choices and outcomes. For three parties there are two one-sided device-independent scenarios: when only one party’s device is untrusted, with state assemblage

$$\varrho_{a|\chi}^{BC} = \text{Tr}_A(A_{a|\chi} \otimes I_B \otimes I_C \varrho_{ABC}).$$  \hspace{1cm} (100)

and when two of them are untrusted

$$\varrho_{abc|xy}^C = \text{Tr}_{AB}(A_{a|x} \otimes B_{b|y} \otimes I_C \varrho_{ABC}).$$  \hspace{1cm} (101)

If $\varrho_{ABC}$ is biseparable, this condition imposes constraints on the assemblages. Then, for a given state, to test whether the assemblages of the form [100] or [101] obey the conditions, one can use SDPs [Cavalcanti et al., 2015a]. Also entropic conditions for this scenario have been studied [Costa et al., 2018a; Riccardi et al., 2018].

The second approach uses steering between the bipartitions to define genuine multipartite steering [He and Reid, 2013]. First, for two parties one can say that they share steering if the first one can steer the other or vice versa. Then, for three parties one can define genuine multipartite steerability as the impossibility of describing a state with a model where steering is shared between two parties only. This means that the state cannot be described by mixtures of bipartitions as in Eq. (99), where for each partition (e.g., $A|BC$) the two-party state (e.g., $BC$) is allowed to be steerable.

One can then directly see that for checking this criterion, it is sufficient to consider the bipartitions $AB|C$, $AC|B$, and $BC|A$, where the two-party sites are uncharacterized and the single-party site obeys quantum mechanics. In addition, on the two-party site only local measurements are allowed, but the results only have to obey the non-signaling condition. For proving genuine multipartite steering in this sense, several methods are possible. If the state is pure, it suffices to check the steerability for the mentioned bipartitions, as a pure state convex combinations into different bipartitions are not possible [He and Reid, 2013]. Otherwise, one may derive a linear (or convex) inequality that holds for unstearable states of all relevant bipartitions. Due to linearity, it also holds for convex combinations and violation rules out the model mentioned above. This approach has been experimentally used in [Armstrong et al., 2015; Li et al., 2015].

B. The steering ellipsoid

Note that the definition of quantum steering Eq. (2) requires considering the ensembles of unnormalized conditional states at Bob’s side. However, one can expect that important insights can be gained by simply studying the normalized version of these conditional states. Note that in doing so, two things are lost: the steering ensemble to which a conditional state belongs, and the probability with which the conditional state is steered to.

For two-qubit states, the normalized conditional states Alice can steer Bob’s system to form an ellipsoid inside Bob’s Bloch sphere, referred to as the steering ellipsoid [Jevtic et al., 2014; Shi et al., 2011, 2012; Verstraete, 2002]. Detailed analysis of their geometry has led to the proposal to use them as a tool to represent two-qubit quantum states, in a way similar to the Bloch representation of states of a single qubit. In particular, given the reduced states of both parties, a steering ellipsoid on one side allows recovering of the density operator up to a certain local unitary or anti-unitary operation on the other side [Jevtic et al., 2014]. Special attention later on was attracted to the volumes of the steering ellipsoids [Cheng et al., 2014; McCloskey et al., 2017; Milne et al., 2014; Zhang et al., 2019]. In particular, Milne et al. [2014, 2015] showed that the volumes of the steering ellipsoids give upper bounds for the entanglement of the state in terms of its concurrence. Even more interestingly, it is shown that the volumes of the steering ellipsoids obey certain monogamy relations [Cheng et al., 2016; Milne et al., 2014, 2015], which will be discussed in the following.

Consider a system of three qubits $ABC$. Denote the volume of the steering ellipsoids for steering from $A$ to $B$ and $A$ to $C$ by $V_{B|A}$ and $V_{C|A}$, respectively. Milne et al. [2014, 2015] showed that for all pure states of the system of three qubits $ABC$, one has

$$\sqrt{V_{B|A} + V_{C|A}} \leq \sqrt{4\pi/3}.$$  \hspace{1cm} (102)

The authors also showed that the famous Coffman-Kundu-Wootters monogamous inequality for entanglement [Coffman et al., 2000] can be derived from this inequality. However, Cheng et al. [2016] showed that the monogamy relation Eq. (102) is violated when the three qubits are in certain mixed states. Instead, the authors showed that a weaker monogamy relation can be derived for all possible states over the three qubits,

$$V_{B|A}^{2/3} + V_{C|A}^{2/3} \leq (4\pi/3)^{2/3}.$$  \hspace{1cm} (103)

Recently, both the monogamy relation Eq. (103) and the violation of Eq. (102) were illustrated experimentally [Zhang et al., 2019].

It is in fact the analysis of the geometry of the steering ellipsoids for Bell-diagonal states that leads to the
exact characterization of quantum steering for this family of states [Jevtic et al. 2015] [Nguyen and Vu 2016a]; see also Section 11.4C. Beyond the Bell-diagonal states, little is known about the extent to which the steering ellipsoids, or their volumes, can characterize the quantum steerability of the state. In particular, an interesting question for future research might be whether the monogamy relations Eq. (102) and Eq. (103) can induce certain monogamy relations between some measures of steering such as the critical radii as defined in Section 11.4C.

C. Gaussian steering

1. A criterion for steering of Gaussian states

Before discussing Gaussian steering some preliminary notions are needed. First, Gaussian systems refer to a special class of continuous variable scenarios. Hence, one deals with infinite-dimensional Hilbert spaces $\mathcal{H}_N^\mathbb{C} = L^2(\mathbb{R})$, where the index $N$ refers to the number of modes. Every Gaussian state is described by a real symmetric matrix, the so-called covariance matrix $V$ satisfying

$$V + i\Omega \geq 0,$$

(104)

where $\Omega = \oplus_{j=1}^N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. More precisely, the covariance matrix of a quantum state $\rho$ is given as $(V)_{ij} = \text{Tr}[\rho [R_i - r_i, R_j - r_j]]$, where $R = (Q_1, P_1, \ldots, Q_n, P_n)^T$ with quadrature operators $Q_i$ and $P_i$ satisfying $[Q_i, P_j] = i\delta_{ij} \mathbb{1}$ and $[Q_i, Q_j] = [P_i, P_j] = 0$, and $r_j = \text{tr}[\rho R_j]$. Moreover, every real symmetric matrix satisfying Eq. (104) defines a Gaussian state. The use of the word Gaussian in this context originates from the fact that the above described states correspond to the ones whose characteristic function $\hat{\rho}(x) := \text{tr}[W(x)\rho]$ is Gaussian. Here

$$W(x) = e^{-ix^T R} \text{ with } x = (q_1, p_1, \ldots, q_n, p_n)^T$$

and

$$\hat{\rho}(x) = e^{-\frac{1}{2} x^T V x - i r^T x}.$$  

(105)

Second, a Gaussian measurement is a POVM $M_a$ (with values in $a \in \mathbb{R}^d$) whose outcome distribution for any Gaussian state is Gaussian. Such POVMs correspond to triples $(K, L, m)$ satisfying

$$L - iK^T \Omega K \geq 0,$$

(106)

where $K$ is an $N \times d$ matrix, $L$ is an $d \times d$ matrix, and $m$ is a displacement vector. The correspondence between the POVM $M_a$ and the triple $(K, L, m)$ is given through the operator-valued characteristic function as $M(p) := \int d\alpha e^{ip^T a} M_a$.

With these definitions we are ready to state the characterisation of steerable states in Gaussian systems originally given in [Wiseman et al. 2007].

A bipartite Gaussian state with covariance matrix $V_{AB}$ and displacement $r_{AB}$ is unsteerable with Gaussian measurements if and only if

$$V_{AB} + i(0_A \oplus \Omega_B) \geq 0.$$

(108)

Here $0_A$ is a zero matrix on Alice’s side and $\Omega_B$ is the matrix $\oplus_{j=1}^N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on Bob’s side.

In contrast to other steering scenarios, the Gaussian case appears special in that the steerability of a state can be characterized through an easy to evaluate inequality. This is, however, not the only special feature for Gaussian steering. Namely, within the Gaussian regime one can also prove monogamy relations for steering with more than two parties [Adesso and Simon 2016] [Ji et al. 2015a] [Lami et al. 2016] [Reid 2013]. One should note that the monogamy can break when one is allowed to perform non-Gaussian measurements [Ji et al. 2016].

2. Refining Gaussian steering with EPR-type observables

As a special case of interest in the Gaussian regime we discuss steering with canonical quadratures. It was shown in [Kiukas et al. 2017] that steerability of a given state in the Gaussian scenario can be readily detected by a pair of quadrature observables.

To be more concrete, we sketch the construction of the quadratures from [Kiukas et al. 2017]. First, a channel is called Gaussian if it maps Gaussian states to Gaussian states. Gaussian channels between systems of $n$ and $m$ degrees of freedom correspond to triples $(M, N, c)$ with $M$ being a real $2n \times 2n$ matrix, $N$ being a real $2m \times 2m$ matrix, and $c$ being the displacement, that satisfy

$$N - i M^T \Omega M + i\Omega \geq 0.$$  

(109)

The transformation of Gaussian states on the level of covariance matrices is given as

$$V \mapsto M^T V M + N, \quad r \mapsto M^T r + c.$$  

(110)

Given that a bipartite Gaussian state has a corresponding Choi-Jamiolkowski channel with parameters $(M, N, c)$, one first notes that the state is unsteerable with Gaussian measurements if and only if the channel parameters define also a Gaussian measurement [Kiukas et al. 2017]. Hence, for a steerable state there exists two vectors $x$ and $y$ such that $(y^T - ia^T)(N - iM^T \Omega M)(y + ix) < 0$. As the triple $(M, N, c)$ also fulfils Eq. (109), we have $r := x^T \Omega y > 0$ and

$$(M\tilde{x})^T \Omega M\tilde{y} > \frac{1}{2}(\tilde{x}^T N\tilde{x} + \tilde{y}^T N\tilde{y}),$$

(111)
where $\tilde{x} = r^{-1/2}x$ and $\tilde{y} = r^{-1/2}y$. From here one can construct two canonical quadratures as $Q_\tilde{x} = \tilde{x}^T R$ and $P_\tilde{y} = \tilde{y}^T R$. These are canonical as by definition $\tilde{x}^T \Omega \tilde{y} = 1$. To see that the state is indeed steerable with these measurements we refer to [Kiukas et al., 2017].

To summarise, we state the following refined characterisation of Gaussian steering [Kiukas et al., 2017]:

For a bipartite Gaussian state $\varrho_{AB}$ with a covariance matrix $V_{AB}$ and displacement $r_{AB}$ the following are equivalent: (i) $\varrho_{AB}$ is steerable with Gaussian measurements (ii) $\varrho_{AB}$ is steerable with some pair of canonical quadratures (iii) $V_{AB} + i(0 \otimes \Omega_B)$ is not positive semi-definite.

**D. Temporal and channel steering**

So far we have concentrated on steering in spatial scenarios, i.e. scenarios where Alice and Bob are space-like separated. Some efforts for defining similar concepts in temporal scenarios, i.e. scenarios where Alice and Bob form a prepare-and-measure type scenario, and on the level of quantum channels have also been pursued in the literature [Chen et al., 2017, 2016b, 2014; Piani, 2015]. In a temporal scenario (consisting of two measurement times), one can ask whether a state assemblage resulting from measurements at the first time step allows a local hidden state model on the second time step. Of course, in temporal scenarios signalling is possible and, hence, such models are sometimes trivially violated. Despite signalling, temporal steering has found applications in non-Markovianity [Chen et al., 2016b] and in QKD [Bartkiewicz et al., 2016], and some criteria [Chen et al., 2014] and quantifiers [Bartkiewicz et al., 2016] have been developed. As the criteria and quantifiers resemble strongly those of spatial steering presented in Sections II.A and II.B we don’t wish to go through them in detail.

In channel steering [Piani, 2015] one is interested in instrument assemblages instead of state assemblages. Namely, given a quantum channel $\Lambda^{C \rightarrow A \otimes B}$ from Charlie to Bob and its extension $\Lambda^{C \rightarrow A \otimes B}$, one asks if an assemblage defined as

$$J_{a|x}(\cdot) := \text{tr}_A[(A_{a|x} \otimes 1)\Lambda^{C \rightarrow A \otimes B}(\cdot)],$$

where $\{A_{a|x}\}_{a,x}$ is a set of POVMs, can be written as

$$J_{a|x}(\cdot) = \sum_{\lambda} p(a|x, \lambda) J_\lambda(\cdot)$$

for some instrument $\{J_\lambda\}_\lambda$ (i.e. a collection of CP maps summing up to a quantum channel) and classical post-processing $\{p(a|x, \lambda)\}_{a,x,\lambda}$. Whenever this is the case, the instrument assemblage $\{J_{a|x}\}_{a,x}$ is called unsteerable.

The concept of channel steering relates to the coherence of the channel extension. Namely, a channel extension $\Lambda^{C \rightarrow A \otimes B}$ is coherent if it can not be written as

$$\Lambda^{C \rightarrow A \otimes B}(\cdot) = \sum_{\lambda} J_\lambda(\cdot) \otimes \sigma_\lambda$$

for some instrument $\{J_\lambda\}_\lambda$ and states $\{\sigma_\lambda\}_\lambda$. Any extension that is of this form is called incoherent. One can show that incoherent extensions always lead to unsteerable instrument assemblages and any unsteerable instrument assemblage can be prepared through some incoherent extension [Piani, 2015]. Note that in spatial steering any separable state leads to an unsteerable assemblage and any unsteerable assemblage can be prepared with a separable state [Moroder et al., 2016].

In the original paper defining channel steering [Piani, 2015], the concept is mainly probed through channel extensions as above. This leads to some connections with state-based correlations. For example, an extension can lead to a steerable instrument assemblage if and only if its Choi state allows Alice to steer Bob, and an extension is incoherent if and only if the Choi state is separable in the cut $A|BC^*$, where $C^*$ is the extra input system from the isomorphism.

One can investigate the channel protocol by replacing the extension with a (minimal) dilation. For completeness, we note that a minimal dilation of a channel $\Lambda : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ can be written as $\Lambda(\cdot) = \text{tr}_A[V(\cdot)V^\dagger]$, where $V|\psi\rangle = \sum_k |\varphi_k\rangle \otimes (K_k|\psi\rangle)$ for all $|\psi\rangle \in \mathcal{H}$, $\{K_k\}_{k=1}^N$ form a linearly independent Kraus decomposition of $\Lambda$, and $\{|\varphi_k\rangle\}_{k}$ is an orthonormal basis of the ancillary system. In this case, the correspondence between instruments and POVMs on the dilation is one-to-one and is given through

$$J_{a|x}(\cdot) = \text{tr}_A[(A_{a|x} \otimes 1)V(\cdot)V^\dagger].$$

This generalises directly the connection between joint measurements and spatial steering to the level of channel steering [Uola et al., 2018]. Namely, a measurement assemblage $\{A_{a|x}\}_{a,x}$ on the minimal dilation is jointly measurable if and only if the corresponding instrument assemblage is unsteerable. By noticing, furthermore, that channel steering with trivial inputs (i.e. one-dimensional input system) corresponds to spatial steering, and that in this case a dilation corresponds to a purification of the total state of the assemblage, one recovers the connection between joint measurements and spatial steering.

The dilation technique can also be used to prove that any non-signalling state assemblage originates from a set of non-signalling instruments [Uola et al., 2018], hence, showing that channel steering captures non-trivial (i.e. non-signalling) instances of temporal steering (with two time steps), and that in this case a connection between temporal steering and joint measurements follows di-
rectly from the one between channel steering and incompatibility. Moreover, using the channel framework one can translate concepts from spatial to temporal scenarios. One example of this is given in [Uola et al., 2018] showing that temporal steering and violations of macrorealism respect a similar strict hierarchy as spatial steering and non-locality. Note that in [Ku et al., 2018b] the hierarchy was proven independently.

E. Quantum key distribution

In quantum key distribution (QKD) two main types of protocols can be distinguished [Scarani et al., 2009]. In prepare & measure (PM) schemes, such as the BB84 protocol, Alice prepares some quantum states and sends them to Bob, who performs measurements on them. Using classical communication, Alice and Bob can then try to generate a secret key from the measurement data. In entanglement-based (EB) schemes, such as the E91 protocol, an entangled quantum state is distributed to Alice and Bob, and both make measurements on their part of the state. The source of the state might be under control of an eavesdropper Eve. Again, the measurement data are then used to generate a secret key.

A central result concerns the role of entanglement for security. In [Curty et al., 2004] it was proved that entanglement is a necessary precondition for security. For EB schemes, this means that if the measurement data can be explained by a separable state, then no secret key can be distilled. For PM schemes, one can consider an equivalent EB scheme, then the same statement holds. It should be noted, however, that the provable presence of entanglement was not shown to be sufficient for secret key generation. The question of whether entanglement can be verified depends on the measurement data taken and the assumptions made on the measurements. In a device-independent scheme, where no assumptions about the measurements are made, only Bell inequalities can be used to test the presence of entanglement. Still, device-independent QKD can be proved to be secure against certain attacks [Acín et al., 2007].

One can also consider an asymmetric situation, where one party trusts its devices and the other one does not. This can be realistic, e.g. if Alice corresponds to a client of a bank having only a cheap device, while Bob represents the bank itself. Clearly, in such a situation QKD can only work if the underlying state is steerable. In [Branciard et al., 2012] this problem has been considered for the BBM92 protocol [Bennett et al., 1992]. In this protocol, Alice and Bob share a two-qubit Bell state and measure either $A_1 = B_1 = \sigma_z$ or $A_2 = B_2 = \sigma_y$. The correlations for the measurement $A_1 \otimes B_1$ are used for the key generation, while the correlations in the $A_2 \otimes B_2$ measurement are used to estimate Eve's information.

In [Branciard et al., 2012] the security of one-sided device-independent QKD using this protocol against attacks where Eve has no quantum memory has been studied (see also Fig. 10). It has been shown that only the detector efficiency of the untrusted party matters and that already for detector efficiencies $\eta \geq 0.659$ a secret key can be distilled, and a non-zero key rate proves that the underlying states are steerable. The obtainable key rates are higher and the required detector efficiencies are lower than in the fully device-independent case.

In addition, results on steering and PM schemes for QKD have been obtained in [Branciard et al., 2012, Ma and Lütkenhaus, 2012]. An analysis for finite key length can be found in [Wang et al., 2013, Zhou et al., 2017], and upper bounds on the key rate in one-sided device-independent QKD have been obtained in [Kaur et al., 2018]. Finally, it should be noted that similar ideas have also been studied and implemented for QKD with continuous variables [Gehring et al., 2015, Walk et al., 2016].

F. Randomness certification

The task of randomness certification can be defined as follows [Acín et al., 2012, Law et al., 2014]. On a quantum system $\varrho$ a measurement labeled by $z$ is made and the result $c$ is obtained. Depending on the situation, the measurement may be a joint measurement on two parties of an entangled state; then the labels for the measurement and result can be written as $z = (x, y)$ and $c = (a, b)$, as in the Bell scenario in Eq. (3). The task is to quantify the extent to which an external adversary Eve can predict the outcome $c$ of the probability distribution $p(c|z)$. Clearly, this depends on the assumptions made about Eve: For instance, one can distinguish the case

![FIG. 10 Key rates for the QKD based on steering. For different visibilities of the initial state ($V \in \{1, 0.99, 0.98, 0.95\}$) lower bounds on the key rate are shown. For perfect visibility (solid blue line) a key can be extracted for detector efficiencies of $\eta \geq 0.659$ for Alice. The dashed line shows a bound (obtained with the same methods) for the fully device-independent scenario for the case of perfect visibility. The figure is taken from [Branciard et al., 2012].](image-url)
where the state $\rho$ is fixed and only known to Eve from the case where Eve indeed provides the state. In the former case, one can furthermore distinguish the knowledge Eve has. She may have only classical information about the state or she may hold a purification of it, see (Law et al., 2014) for a detailed discussion.

In the simplest case, the state $\rho = \ket{\psi}\bra{\psi}$ is pure and the measurement $z$ is characterized. Then, the best strategy for Eve is to guess the $c$ with the maximal probability, and the probability of guessing correctly is given by

$$G(z, \psi) = \max_c p(c|z, \psi).$$

If the state $\rho$ is mixed then Eve may hold a purification of it and she may know the exact decomposition $\rho = \sum_k p_k |\phi_k\rangle\langle \phi_k|$. Consequently, the maximal guessing probability is

$$G(z, \rho) = \max_{p_k, \phi_k} \sum_k p_k G(z, \phi_k),$$

where the maximization runs over all decompositions of $\rho$. If the measurements $z$ are not characterized, one has to optimize over all possible quantum realizations of the classical probability distribution $p(c|z)$. So, the maximal guessing probability is

$$G(z, p(c|z)) = \max_{\rho, M_{c|z}} G(z, \rho),$$

where the maximization runs over all quantum realizations, described by a state $\rho$ and measurement operators $M_{c|z}$ with $p(c|z) = \text{Tr}(M_{c|z} \rho)$. Optimizations over this set can be carried out by hierarchies of SDPs (Navascues et al., 2007; 2008). In all cases, the number of random bits that one can extract from $p(c|z)$ is given by the min-entropy, $H_{\text{min}}(G) = -\log_2(G)$.

Initially, the task of randomness certification was mainly studied in the Bell scenario, where $z = (x, y)$ and $c = (a, b)$ describe measurements on an entangled state (Acín et al., 2012). Here, the devices are not characterized and as soon as a Bell inequality is violated, one can prove that the results of a fixed setting cannot be predicted, so the randomness is certified. In (Law et al., 2014) randomness certification has been studied for the steering scenario: Again, one makes local measurements on an entangled state, but this time the devices on Bob’s side are characterized. This leads to additional constraints in the SDP hierarchy (Navascues et al., 2007) and consequently more randomness can be extracted. Interestingly, also for states that are not steerable the randomness can be certified; so the violation of a steering inequality is not necessary for randomness certification in the one-sided device-independent scenario.

Also in (Passaro et al., 2013) the task of randomness certification in the distributed one-sided device-independent scenario between two parties was studied. But here mainly the randomness for a single measurement setting of Alice was considered. It has been shown that this can directly be computed with an SDP, without the need of a convergent hierarchy of SDPs. This is then shown to hold also for the scenario considered in (Law et al., 2014). In (Skrzypczyk and Cavalcanti, 2018) the problem has been considered for two $d$-dimensional systems. A steering inequality has been derived, such that the maximal violation guarantees $\log(d)$ random bits for Alice’s outcomes in the one-sided device-independent scenario. Furthermore, any pure entangled state with full Schmidt rank can be used to generate this amount of randomness.

Finally (Curchod et al., 2017) showed that if one considers the Bell scenario, then sequential measurements on one party can lead to an unbounded generation of randomness. The extension of this to the steering scenario is discussed in (Coyle et al., 2018).

G. Subchannel discrimination

In (Piani and Watrous, 2015) an operational characterisation of steerable quantum states is provided. The idea is similar to the main result of (Piani and Watrous, 2009) stating that every entangled state provides an advantage over separable ones in some channel discrimination task to the realm of steering. In the case of steering, the related task turns out to be that of subchannel discrimination, i.e. discriminating different branches of a quantum evolution. Namely, take an instrument $I = \{I_a\}_a$ i.e. a collection of completely positive maps summing up to a quantum channel), a POVM $B = \{B_k\}_k$, an input state $\rho$ and define the probability of correctly identifying the subchannel (i.e. instrument element) as

$$p_{\text{cor}}(I, B, \rho) := \sum_a \text{tr}[I_a(\rho) B_a].$$

To find the best strategy for the task, one maximises over input states and POVMs on the output.

As mentioned above, entanglement provides an advantage in channel discrimination tasks, i.e. tasks of discriminating between subchannels of the form $J_a = p(a) A_a$, where $\{A_a\}_a$ are quantum channels. To prove a similar result for general subchannels, the authors of (Piani and Watrous, 2015) limit the set of allowed measurements between the system (i.e. outputs of the instruments) and the ancilla to local measurements supported by forward communication from output toancilla (i.e. one-way LOCC measurements). Such measurements have POVM elements of the form $C_{a|\text{out-anc}} = \sum_x A_{a|x} \otimes B_x$, where $\{B_x\}_x$ is a POVM on the output system and $\{A_{a|x}\}_a$ is a POVM on the ancilla for every $x$. The probability of correctly identifying the branch of the evolution with such measurements and a shared state $\varrho_{AB}$ is given as $p_{\text{cor}}(I, 1-\text{LOCC}, \varrho_{AB}) = \sum_{a, x} \text{tr}_{\text{out}}[A_{a|x}^\dagger (B_x) A_{a|x}].$ Note
that any unsteerable state can perform at most as good as some single system state. One sees this by using an LHS model for the assemblage in the above equation and by choosing the best performing hidden state as the single system state.

To prove the main result of the paper, the authors define a quantity called steering robustness of a bipartite state $\varrho_{AB}$ by maximising the steering robustness of all possible assemblages the state allows. More formally

$$R_{\text{steer}}^{A\to B}(\varrho_{AB}) = \sup \{ R(A) | \{ A_{a|x} \}_{a,x} \}, \quad (120)$$

where $R(A)$ is the steering robustness of the assemblage $\varrho_{a|x} = \text{tr}_A([A_{a|x} \otimes 1] \varrho_{AB})$. Clearly the quantity $R_{\text{steer}}^{A\to B}(\varrho_{AB})$ is zero if and only if the state $\varrho_{AB}$ is unsteerable. The main result now reads:

For any steerable state there exists a subchannel discrimination task (with forward communication from the steerable) the assemblage in the above equation and as some single system state. One sees this by using an operational characterisation. Experimental demonstration of this result has been presented in (Sun et al., 2018).

Note that this result gives the set of steerable states an operational characterisation. Experimental demonstration of this result has been presented in (Sim et al., 2015). Here the supremum is taken over all instruments and one-sided device-independent and fully device-independent non-locality robustness. Hence, on top of the one-sided device-independent and fully device-independent lower bounds on incompatibility, one gets also a device-independent lower bound on a quantifier of steering.

We follow (Cavalcanti and Skrzypczyk, 2016) to make the aforementioned hierarchy more concrete. It is worth mentioning that the hierarchy presented here corresponds to one choice of quantifiers. Analogous results are possible for various fine-tuned quantifiers.

To write down the result, recall the definitions of incompatibility, steering and non-locality robustness. Steering robustness is defined in Eq. (30) and analogously to that one defines incompatibility robustness $IR(A_{a|x})$ of a set $\{A_{a|x}\}_{a,x}$ of POVMs as

$$\min \ t \quad \text{s.t.} \quad A_{a|x} + tN_{a|x} \geq 0 \quad \text{for all } a, x, \quad t \geq 0, \quad N_{a|x} \geq 0 \quad \text{for all } a, x, \quad \sum_a N_{a|x} = 1 \quad \text{for all } x, \quad G_\lambda \geq 0 \quad \text{for all } \lambda, \quad \sum_\lambda G_\lambda = 1. \quad (122)$$

Note that here $D(|x, \lambda) \in \{0, 1\}$ is a deterministic assignment for every $x$ and $\lambda$. The interpretation of this robustness is that one mixes the POVMs $\{M_{a|x}\}_{a,x}$ with $\{N_{a|x}\}_{a,x}$ until they become jointly measurable.

Now, if Alice’s measurements $\{A_{a|x}\}_{a,x}$ in a steering scenario with a state $\varrho_{AB}$ have incompatibility robustness $t$, then replacing Alice’s measurements with the jointly measurable POVMs $\sum_a \rho_{a|x} + tN_{a|x} \geq 0$ shows that $t$ is an upper bound for the steering robustness of $\varrho_{AB} := \text{tr}_A([A_{a|x} \otimes 1] \varrho_{AB})$. In other words, steering robustness of a given assemblage lower bounds the incompatibility robustness of the measurements on the steering party. This bound, moreover, is one-sided device-independent. Notice that with a fine-tuned steering quantifier called consistent steering robustness the aforementioned inequality is tight for full Schmidt rank states (Cavalcanti and Skrzypczyk, 2016), see also (Kuikas et al., 2017).

For the device-independent quantification of steering and incompatibility one can use the non-locality robustness $NR[p(a,b|x,y)]$ of a probability table $\{p(a,b|x,y)\}_{a,b,x,y}$ given as (Cavalcanti and Skrzypczyk, 2016)

$$\min \ r \quad \text{s.t.} \quad p(a,b|x,y) + rq(a,b|x,y) \geq 0 \quad \text{for all } a, b, x, y, \quad r \geq 0, \quad q(a,b|x,y) \in Q, \quad (123)$$

where $Q$ is the set of all possible quantum correlations defined as

$$Q = \{ \text{tr}_B([A_{a|x} \otimes B_{b|y}] \varrho_{AB}) | \{A_{a|x}\}_{a,x}, \{B_{b|y}\}_{b,y} \text{ POVMs , } \varrho_{AB} \text{ a state} \}. \quad (124)$$
Similarly to the one-sided device-independent quantification of incompatibility above, one sees that for a given state assemblage \( \{ \sigma_{a|x} \}_{a,x} \) the non-locality robustness of any probability table originating from this assemblage, i.e. \( p(a, b|x, y) = Tr[\sigma_{a|x} B_{b|y}] \) with \( \{ B_{b|y} \}_{b,y} \) being POVMs on Bob’s side, gives a lower bound for the steering robustness of the assemblage. In other words

\[
IR(A_{a|x}) \geq SR(\sigma_{a|x}) \geq NR[p(a, b|x, y)]. \tag{125}
\]

Hence, using the machinery of (Cavalcanti and Skrzypczyk 2016) one finds one-sided device-independent and fully device-independent lower bounds for quantifiers of measurement incompatibility and device-independent lower bounds for quantifiers of steering.

I. Secret sharing

Secret sharing is a cryptography protocol that allows a dealer (Alice) to send a message to players (Bob and Charlie) in a way such that the message can only be decoded if when the players work together—neither of them can decode it by himself. If Alice shared a secret key (see Section V.E) with Bob and another with Charlie, she can simply encode the message twice with the two keys to ensure that only Bob and Charlie together can decode the message. Thus, normal quantum key distribution protocols already provide one with protocols for quantum secret sharing. But one can do it more straightforwardly with multipartite entanglement (Hillery et al. 1999); see also (Zukowski et al. 1998) for a related protocol. Take the case where Alice prepares a large number of the Greenberger-Horne-Zeilinger (GHZ) states (Hillery et al. 1999).

\[
|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{126}
\]

Alice keeps one particle, and sends the other two to Bob and Charlie. Each then measures their particles in random directions, \( x \) or \( y \). After communicating via a classical public channel, they can identify the triplets where they have measured in directions \( xxx, xyy, yxy, yyx \); the other triplets are discarded. As one can check, the GHZ state is an eigenstate of the retained measurement operators. In these triplets, Bob and Charlie can use their outcomes to predict Alice’s outcomes. However, the outcomes at Bob’s or Charlie’s sides separately are not enough to infer her outcomes. Thus, Alice can use the series of outcomes of her measurements to encode the message in her secret sharing protocol.

The fact that Bob and Charlie have to collaborate to infer the measurement outcomes at Alice’s side resembles the distinction between the concepts of local and global steering in the multipartite steering scenarios (He and Reid 2013; Xiang et al. 2017; see also Section V.A). This similarity has been made precise in analyzing the security of secret sharing (He and Reid 2013; Kogias et al. 2017; Xiang et al. 2017). Specifically, Xiang et al (2017) computed the secret key-rate bound which guarantees unconditional security of the protocol against eavesdroppers and dishonest players (Kogias et al. 2017) for three-mode Gaussian states and found it to be essentially the quantification of the difference between collective steering and local steering from Bob and Charlie to Alice (Kogias et al. 2015a). To our knowledge, whether this quantitative relation between quantum steering and quantum secret sharing extends beyond Gaussian states is at the moment unknown.

J. Quantum teleportation

Steering shares a close conceptual similarity with state teleportation (Bennett 1993). We follow Cavalcanti et al. (2017) and consider the following abstract teleportation protocol. Alice and Bob share a bipartite quantum state \( \rho^{AB} \). Charlie, perceived as the verifier, draws a pure state \( \omega_x \) from a certain set of \( |x\rangle \) states indexed by \( x \), gives it to Alice and asks her to teleport it to Bob. Without knowing the state \( \omega_x \), Alice makes a measurement with POVM elements \( \{ E_n^{CA} \} \) jointly on the received state and her particle that is entangled with Bob’s system. Depending on Alice’s outcome \( a \), Bob’s system is then ‘steered’ to a conditional state,

\[
\theta_a^B(\omega_x) = \frac{Tr_{CA}[(E_n^{CA} \otimes 1^B)(\omega_x \otimes \rho^{AB})]}{p(a|\omega_x)}, \tag{127}
\]

where the normalization \( p(a|\omega_x) = Tr[(E_n^{CA} \otimes 1^B)(\omega_x \otimes \rho^{AB})] \) is the probability for Alice to get outcome \( a \) in her protocol given she received state \( \omega_x \) from Charlie. Alice communicates her measurement outcome \( a \) to Bob, who then makes some appropriate local unitary operation; by the end of the procedure, the state of his system is \( U_a^B \omega_x \). The design of Alice’s measurement and Bob’s local unitary operations is such that the final state at Bob’s side resembles Charlie’s original state \( \omega_x \) as much as possible. The quality of the teleportation protocol can be assessed by the so-called average fidelity,

\[
F_\text{tel} = \frac{1}{|x|} \sum_{a,x} p(a|\omega_x) Tr[\omega_x U_a^B \rho_{a|^B}(\omega_x) U_a^B]. \tag{128}
\]

If the teleportation is perfect, the average fidelity is 1 for pure states \( \omega_x \).

One can easily observe the similarity of the teleportation protocol with that of quantum steering: instead of receiving a classical input \( x \), Alice received a quantum state from Charlie \( \omega_x \), as an input; see Fig. 11. The idea of receiving quantum inputs from a verifier (Charlie) instead of a classical input has been previously considered...
for entanglement [Branciard et al., 2013; Buscemi, 2012], and later for quantum steering both theoretically and experimentally [Cavalcanti et al., 2013b; Kocsis et al., 2015]. The benefit of allowing for the quantum inputs is that the verifier can now verify that Alice and Bob share a quantum correlation (entanglement, quantum steering) without trusting their measurement devices or their actual measurements [Hall, 2018]; see also Section I and further discussion in [Hall and Rivas, 2019]. Utilizing the similarity, Cavalcanti et al. [2017] showed that all entangled states can demonstrate nonclassical teleportation in some sense. One should, however, note that the introduced notion of nonclassical teleportation does not imply high teleportation average fidelity, which has been a standard figure of merit.

There is another line of works which attempt to relate quantum steering with the security of quantum teleportation. In teleporting a state to Bob, Alice does not want an eavesdropper to also obtain some version of the state. Certain security is guaranteed when the average fidelity of teleportation in Eq. (128) is high enough [Pirandola et al., 2013b, 2015]. It is then shown that for certain family of bipartite states, to obtain the required fidelity, the state is not only entangled but necessarily two-way steerable [He et al., 2015].

Another way to investigate the security of teleportation is to study its sister protocol known as entanglement swapping. In this protocol, the state given to Alice by Charlie is priory entangled with another particle which Charlie keeps. By the end of the teleportation protocol performed by Alice and Bob, the entanglement between Charlie and Alice is transferred to that between Charlie and Bob. In this case, the teleportation can be secured by the monogamy of entanglement: if Charlie is sufficiently entangled with Bob, an eavesdropper cannot be entangled with Charlie. Instead of monogamy of entanglement, Reid [2013] then used the monogamy of a certain steering inequality to demonstrate the security of quantum teleportation.

K. Resource theory of steering

A resource theory is typically seen as consisting of two basic components: free states and free operations. Free states constitute a set which remains unchanged under the actions of free operations. In this sense, one could define a resource theory merely from the free operations. Consequently, any state that is not free has some resource in it, as it cannot be created from the set of free states with free operations. As an example, in the case of entanglement, free states are given by the set of separable states and free operations are local operations assisted by classical communication (LOCC). Another important aspect of a resource theory are resource measures or monotones. A proper measure should not increase under free operations, i.e. free operations can not create the resource, should be faithful, i.e. equal to zero only for free states, and should be convex, i.e. randomisation should not create resource either.

In the case of steering a resource theory has been proposed [Gallego and Aolita, 2015]. The free states in this theory are the unsteerable assemblages and free operations can be any operations on the assemblages that do not map unsteerable assemblages into steerable ones. In [Gallego and Aolita, 2015] one-way stochastic LOCC operations are shown to be free operations. In order to introduce one-way (stochastic) LOCC operations, we adapt the notation of [Gallego and Aolita, 2015] for state assemblages. Namely, any state ensemble \( \{ \rho_a \} \) can be embedded into larger space via the correspondence \( \{ \hat{\rho} \} : = \{ \sum_a |a\rangle \langle a| \otimes \rho_a \} \). To make a similar correspondence for state assemblages, one can define a map \( \hat{\varphi}_{A|X}(x) : = \sum_a |a\rangle \langle a| \otimes \hat{\rho}_{A|x} \) where \( X \) and \( A \) label the sets of Alice’s inputs and outputs respectively. Now, a one-way (stochastic) LOCC operation \( M \) is defined on the assemblage \( \hat{\varphi}_{A|X} \) as

\[
M(\hat{\varphi}_{A|X}) := \sum_\omega (1 \otimes K_\omega) W_\omega(\hat{\varphi}_{A|X})(1 \otimes K_\omega^\dagger),
\]

where \( \{ K_\omega \} \) are Kraus operators and \( \{ W_\omega \} \) are wiring maps defined pointwise as

\[
W_\omega(\hat{\varphi}_{A|X})(x_f) := \sum_x p(x|x_f, \omega) \sum_{a_f, a} p(a_f|a, x, x_f, \omega) (|a_f\rangle \langle a| \otimes 1) \hat{\varphi}_{A|X}(x)|a\rangle \langle a_f| \otimes 1.
\]
part, communicating the information about which operation (i.e., $\omega$) was performed, and the uncharacterized party applying the corresponding classical pre- and post-processing [i.e., $p(\cdot|x_f,\omega)$ and $p(\cdot[a,x,x_f,\omega])$] on their side. Note that the use of one-way (stochastic) LOCC operations as free operations has also an interesting physical motivation: they can be seen as safe operations in one-sided device-independent quantum key distribution (Gallego and Aolita, 2015).

A typical resource theory aims at quantifying the resource in hand. For this purpose, one wishes to find a mapping (or monotone) $f$ from the set of states of the resource theory to the set of non-negative real numbers that fulfills certain requirements. In the case of steering the following requirements are considered (Gallego and Aolita, 2015):

- $f(\hat{\vartheta}_{A|X}) = 0$ if and only if $\hat{\vartheta}_{A|X}$ is unsteerable
- $f$ is non-increasing on average under deterministic one-way LOCC

If in addition the mapping $f$ is convex, it is called a convex steering monotone. In (Gallego and Aolita, 2015) the typical steering quantifiers, i.e., steerable weight and convex steering monotone, are shown to the typical steering quantifiers, i.e. steerable weight and convex steering monotone. In (Gallego and Aolita, 2015) the following requirements are considered (Gallego and Aolita, 2015):

- $f(\hat{\vartheta}_{A|X}) = 0$ if and only if $\hat{\vartheta}_{A|X}$ is unsteerable
- $f$ is non-increasing on average under deterministic one-way LOCC

Post-quantum steering is the phenomenon that certain assemblages $\{\vartheta_{a|x}\}$ may not be realizable by quantum mechanics, although no signaling between the parties is possible. For the case of Bell inequalities, it is known that there are probability distributions which are non-signaling, but cannot come from a quantum state. The most prominent example is the Popescu-Rohrlich (PR) box (Popescu and Rohrlich, 1994), which is a non-signaling distribution for two parties with two measurements having two outcomes, which leads to a violation of the CHSH inequality with a value $\langle S_{\text{CHSH}} \rangle = 4$ while in quantum mechanics only values $\langle S_{\text{CHSH}} \rangle \leq 2\sqrt{2}$ can occur. The analogous question for steering highlights the difference between steering in the bipartite and the multipartite case.

For the bipartite case, one may consider an assemblage $\{\vartheta_{a|x}\}$, obeying the no-signaling constraint $\sum_a \vartheta_{a|x} = \sum_a \vartheta_{a|x'} = \vartheta_B$ for all $x, x'$. As already mentioned in Sections II.B and V.M, any such assemblage can be realized by quantum mechanics. This means that there is a state $\varrho_{AB}$ and measurements $E_{a|x}$ such that $\vartheta_{a|x} = \text{Tr}_A(E_{a|x} \varrho_{AB})$.

This is not the case for the tripartite scenario (Sainz et al., 2015). Here, one considers the scenario where Alice and Bob make local measurements in order to steer Charlie’s state. So, Charlie has an assemblage $\{\vartheta_{ab|xy}\}$, where $x$ and $a$ ($y$ and $b$) denote the measurement setting and outcome of Alice (Bob). Besides being positive and the normalization constraint $\text{Tr}(\sum_y \vartheta_{ab|xy}) = \text{Tr}(\varrho_C) = 1$ this assemblage should fulfill that neither Alice nor Bob can signal to the other parties, that is

$$\sum_a \vartheta_{ab|xy} = \sum_a \vartheta_{ab|x'y} \text{ for all } x, x', \sum_b \vartheta_{ab|xy} = \sum_b \vartheta_{ab|xy'} \text{ for all } y, y'. \quad (131)$$

One can directly check that these conditions imply that also Alice and Bob jointly cannot signal to Charlie by the choice of their measurements.

Contrary to the bipartite case, an assemblage obeying these constraints does not need to have a quantum realization. A simple counterexample can be derived from the PR box mentioned above: If the conditional states are of the form $\vartheta_{ab|xy} = p(ab|xy)|0\rangle\langle 0|_C$, where $p(ab|xy)$ is the probability table of the PR box, the assemblage is clearly non-signaling, but cannot be realized within quantum mechanics. In (Sainz et al., 2015) more interesting examples of this behavior were provided. Using iterations of SDPs the authors found an example of a qudit assemblage $\vartheta_{ab|xy}$ with the properties that for any possible measurement $E_{a|x}$ of Charlie, the resulting probability distribution can be explained by a fully local hidden variable model, which is even a stronger requirement than being non-signaling. Still, the assemblage has no quantum realization, so there is no state $\varrho_{ABC}$ such that $\vartheta_{ab|xy} = \text{Tr}_{AB}(E_{a|x} \otimes E_{b|y} \varrho_{ABC})$.

In further works the theory of post-quantum steering has been extended. (Sainz et al., 2018) provided general methods to construct examples of post-quantum steering and defined a quantifier of this phenomenon. (Hoban and Sainz, 2018) established a connection to the theory of quantum channels, and (Chen et al., 2018b) used moment matrices to characterize the phenomenon.

M. Historical aspects of steering

1. Discussions between Schrödinger and Einstein

As mentioned already in the introduction, the first observation of the steering phenomenon dates back to Schrödinger’s discussions of the EPR argument with Einstein. Schrödinger corresponded with several physicists on this problem, the letters have been edited by (von Meyenn, 2011).

To understand the origin of Schrödinger’s idea, it is important to note that Einstein did not like the way the
EPR paper was written and how the argument was formulated, for detailed discussions see (Kiefer 2015). Instead, Einstein preferred a somehow simpler version. He published this version much later (Einstein, 1948), but he also explained the basic idea in a letter to Schrödinger on June 19th, 1935.

The argument (in the formulation of (Einstein, 1948)) goes as follows. First, one considers position $X$ and momentum $P$ as non-commuting observables. Then there are, according to Einstein, two possibilities:

(i) One can assume that position and momentum have definite values before a measurement of them is carried out. Then one has to admit that the wave function $|\psi\rangle$ is not a complete description.

(ii) One can assume that the values of position or momentum are created during a measurement. This is compatible with the assumption that the wave function $|\psi\rangle$ is a complete description. If $|\psi\rangle$ is a complete description, it follows, according to Einstein, that two different wave functions describe two different physical situations.

These two ways of thinking cannot be distinguished without additional assumptions. Here, Einstein introduces a locality principle, stating that if one considers a bipartite system, the real physical situation at one side is independent of what happens on the other side.

In order to conclude the incompleteness of the quantum mechanical description, one can consider a pure entangled wave function, as in the usual EPR argument. Then, the conditional wave function $|\phi\rangle_B$ on Bob’s side depends on the choice of the measurement on Alice’s side. According to the locality principle, however, the physical reality on Bob’s side cannot change. So, one arrives at a contradiction to (ii) and the incompleteness follows. It is interesting to note that for this argument the (perfect) correlations between measurements on both sides are not relevant. As Einstein formulated it: “I couldn’t care less whether or not $|\phi\rangle_B$ and $|\phi\rangle_B'$ are eigenstates of some observables.”

In the direct reply to this letter (on July 13th, 1935) Schrödinger spelled out that the dependence of the conditional state $|\phi\rangle_B$ includes some “steering” from a distance. Although this phenomenon does not allow signaling between the parties, he considers it to be magic. Recalling discussions with Einstein and colleagues in Berlin during the 1920s, he writes:

“All the others told me that there is no incredible magic in the sense that the system in America gives $X = 6$ if I perform in the European system nothing or a certain action (you see, we put emphasis on spatial separation), while it gives $X = 5$ if I perform another action; but I only repeated myself: It does not have to be so bad in order to be silly. I can, by maltreating the European system, steer the American system deliberately into a state where either $X$ is sharp, or into a state which is certainly not of this class, for example where $P$ is sharp. This is also magic!”

It must be added, of course, that the view of the steering phenomenon as “nonlocal” is based on a certain interpretation of the wave function, which is not shared by everyone, see (Griffiths, 2019) for a discussion. In any case, the question remains for which states this phenomenon can be observed and which states on Bob’s system can be reached by performing measurements on Alice’s side. This was also part of the discussion between Schrödinger and other physicists (such as von Laue) and Schrödinger presented his results in two subsequent papers.

2. The two papers by Schrödinger

The first paper entitled “Discussion of probability relations between separated systems” (Schrödinger, 1935) was submitted in August 1935. Schrödinger states that he finds it “rather discomforting” that quantum mechanics allows a system to be steered by performing measurements in a different location. He then presents several results on this phenomenon.

First, he shows that every bipartite pure state can be written as

$$|\psi\rangle = \sum_k s_k |a_k\rangle |b_k\rangle$$

(132)

where the vectors $|a_k\rangle$ and $|b_k\rangle$ form orthogonal sets. This is nowadays called the Schmidt decomposition. He proves that this is unique if the coefficients $s_k$ are different. He also recognizes that this implies that for generic states there is one measurement for Alice (defined by the eigenvectors $|a_k\rangle$), which is perfectly correlated with a measurement on Bob’s side (defined by the $|b_k\rangle$).

Then, he discusses in some more detail the EPR state from the 1935 argument (Einstein et al., 1935), which is not a generic state, as all the Schmidt coefficients coincide. He proves that for any observable $F(X_2, P_2)$ on Bob’s side the value can be predicted by making a suitable measurement $F(X_1, P_1)$ on Alice’s side. This fact appeared already in a letter from Schrödinger to von Laue, and it demonstrates the surprising effect that one system seems to know the answers to all possible questions on the other system.

The second paper entitled “Probability relations between separated systems” was submitted in April 1936 (Schrödinger, 1936). Schrödinger first states that the essence of the previous work was the observation that in quantum mechanics one can not only determine the wave function at one party by making measurement on the other, but one can also control the state at one side by choosing the measurements on the other side. So the question arises, to which extent the wave function can be controlled.

In order to answer this, he first proves a statement on density matrices. A given density matrix may have
different decompositions into pure states
\[\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| = \sum_i q_i |\phi_i\rangle\langle\phi_i|,\]  
(133)
and the question arises, which conditions the \(|\phi_i\rangle\) and \(|\psi_k\rangle\) have to fulfill. Schrödinger proves, that two ensembles give the same density matrix, if and only if there is a unitary matrix \(U\) such that
\[\sqrt{p_k} |\psi_k\rangle = \sum_i U_{ki} \sqrt{q_i} |\phi_i\rangle.\]  
(134)
This implies that any \(|\phi_i\rangle\) in the range of the space spanned by the \(|\psi_k\rangle\) can be an element of a suitable ensemble.

Then, Schrödinger applies this to the bipartite state in Eq. (132). Here, the reduced state
\[\rho_B = \sum_k s_k^2 |b_k\rangle\langle b_k| = \sum_i q_i |\beta_i\rangle\langle\beta_i|\]  
(135)
has different decompositions. As already mentioned, the ensemble \(\{s_k^2,|b_k\rangle\}\) can be reached by making the measurement defined by the orthogonal states \(|a_k\rangle\) on Alice’s side, and the question arises, whether any other ensemble \(\{q_i,|\beta_i\rangle\}\) can be reached. Schrödinger proves that this is the case. Especially, if the state has full Schmidt rank and \(|b_k\rangle\) span the whole space, any state \(|\beta_i\rangle\) on Bob’s side can be prepared by making a suitable measurement on Alice’s side.

Finally, Schrödinger stresses again that he finds the phenomenon of controlling a distant state repugnant and suggests that quantum mechanics may be modified to avoid it. As a potential modification, he suggests that for a state as in Eq. (132) the phase relations between the \(s_k\) may be lost. This means that instead of taking the pure state \(\rho = |\psi\rangle\langle\psi|\) the two-particle system should be described by a diagonal mixed state
\[\rho = \sum_k s_k^2 |a_k b_k\rangle\langle a_k b_k|,\]  
(136)
which he considers to be a possible modification, not contradicting the experimental evidence at that time.

3. Impact of these papers

In the following years, Schrödinger’s ideas on steering were not further considered in the literature. His mathematical results from the second paper, however, were several times rederived without any reference to him. In the following, we give a short overview, a detailed discussion can be found in [Kirkpatrick 2006].

A first rediscovery was presented by Jaynes [Jaynes, 1957]. He derived the first statement in Eq. (134) while studying general properties of density matrices. Based on Jaynes’ paper, Hadjisavvas presented later a simplified proof and an extension to infinite-dimensional systems [Hadjisavvas 1981].

The second mathematical statement (below Eq. (135)) was derived by Gisin in the context of modifications of the Schrödinger dynamics [Gisin 1989]. Here, the question arises whether the modified dynamics for pure states extends uniquely to mixed states. If it were different for two decompositions such as the ones in Eq. (133), then ensembles will become distinguishable at some point. Given the fact that both ensembles can be prepared by measurements on a distant system, this would lead to a violation of the non-signaling condition, enforced by special relativity. Finally, both mathematical statements from Schrödinger have also been rederived independently by [Hughston et al. 1993].

Besides these mathematical results, the notion of steering as a kind of quantum correlation was not discussed for a long time. The situation changed in the eighties of the last century. Then, Bell inequalities started to attract more attention [Clauser and Shimony 1978] and the mathematical notions of entangled and separable states were studied [Primas 1983; Werner 1984; Werner 1989].

The paper [Vujčić and Herbut, 1988] was the first to give a clear summary of Schrödinger’s ideas and an extension of his results for continuous variable systems. Also, Vujčić and Herbut argue that steering is different from Bell nonlocality, as it is based on the formalism of quantum mechanics. Independently, Reid (1989) presented quantitative conditions for continuous variable systems to lead to an EPR-type argument. Verstraete noted in his dissertation [Verstraete 2002] the connection of Schrödinger’s ideas to quantum teleportation and entanglement transformations, as in both cases one aims at preparing a quantum state on one side by making measurements on the other. Also, the notion of the steering ellipsoid was introduced there. Shortly thereafter, steering was recognized to be relevant for foundational questions of quantum mechanics [Clifton et al. 2003; Spekkens 2007]. Finally, Wiseman and coworkers introduced the notion of local hidden state models [Wiseman et al. 2007], laying the foundation for the modern notion of quantum steering.

VI. CONCLUSION

The notion of quantum steering is motivated by the Einstein-Podolsky-Rosen argument and it took seven decades until a precise formulation was given. Since then, quantum steering has initiated a new surge of results in quantum information and the foundations of quantum mechanics: Old concepts were put into the new light; long-standing problems gained progress and some were resolved; connections between areas were established, and novel problems were formulated. In this review, we
have sketched the dynamic development of the field over the last ten years. Yet, future research is facing many challenges. So, to close the review, we summarize some of the open problems:

- As a complete characterization of quantum steerability has been obtained for two-qubit systems and projective measurements, it is desirable to extend such a characterization to higher-dimensional systems. Although there is an indication that such an extension is possible, much remains to be worked out in details.

- The question, whether there are states that are unsteerable with projective measurements, but are steerable with POVMs is also relevant. The analogous question in the context of Bell nonlocality has been a long-standing problem without any evidence whether such a state exists. With quantum steering, one now has evidence indicating that there might be no such state for a two-qubit system. Yet, to date there is no rigorous proof of their non-existence. In particular, one might still expect that such a state exists in high dimensions.

- We have discussed in Section [HLC] that the operational definition of steerability requires multiple copies of the considered state, which implies the possibility of making collective measurements on Alice’s side. Thus, apart from being fundamental, the question whether all entangled states become steerable upon making collective measurements on Alice’s side is also important to the intrinsic consistency of the concept.

- The connection between quantum steering and incompatibility has initiated further questions: are there other connections between the different notions of incompatibility and different forms of quantum correlations?

- A further open question asks whether there are other physically motivated properties of state assemblages than that of having a local hidden state model. For instance, one may want to deduce something more than entanglement and incompatibility from such properties. Examples are the preparability from states with a given Schmidt number or properties motivated by incompatibility, such as compatibility on many copies, coexistence or simulability.

- The study of multipartite steering is in its infancy. A systematic investigation and comparison between different definitions is necessary in the future.

- A closely related research direction is to study steering of parties who are connected in networks.

For a given directed network one may ask whether there is a quantum state allowing steering along the directed edges.

- For applications, it would be interesting to identify tasks in quantum information processing, where the assumptions that can be made are highly asymmetric. Then, the methods developed in steering theory may be useful to study the role of correlations therein.

- In experiments, quantum steering is verified by a finite number of measurement settings. There are only few works on optimizing the measurement settings (when having a fixed number of inputs/outputs) and clearly more research is needed to serve as inputs for experiments.

This is only a small list of problems, and clearly further interesting challenges remain. Given the current interest in the steering phenomenon, we expect that the old observations from Schrödinger can still influence current and future discussions on the foundations of quantum mechanics.

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