Stability analysis of Single-Brane with Gauss-Bonnet Term in a Bulk

Irsan Rahman1,*, Agussalim Agussalim1, Agus Suroso2,3, Freddy P. Zen2,3

1 Department of Physics, Faculty of Mathematics and Natural Sciences, Universitas Muslim Maros, Maros, Indonesia
2 Department of Physics, Institut Teknologi Bandung, Bandung 40132, Indonesia
3 Indonesia Center for Theoretical and Mathematical Physics (ICTMP) Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, 40132, Indonesia

*irsan@umma.ac.id

Abstract. We study single-brane 4 + n dimensions which is embedded in bulk 5 + n dimensions with scalar field and Gauss Bonnet terms in bulk. The brane field equation is obtained by performing a bulk field projection by using the Gauss-Codazzi equation. The Einstein brane field equation is formed into the standard Einstein field equation in the Theory of general relativity with additional terms Gauss Bonnet and extra terms. Furthermore cosmological application obtained by reviewing brane’s spacetime is homogeneous and isotropic. FRW metric is taken with two scale factors that are internal dimensions and external dimensions which have a relationship $b(t)=a(t)^\gamma$. A dynamic analysis is performed to determine the stability of this model by taking the case of the absence of extra terms. Finally, we get a stable solution for this model occurs when the extra dimension n=2.

1. Introduction

The theory of quantum mechanics and the general relativity are two successful theories explaining physical phenomena in their respective areas. The problem is that there is no theory that can combine the two theories into a consistent theory. One candidate theory that is able to combine these theories is string theory. Brane’s idea emerged from string theory which was later adopted by physicists into the theory of Braneworld gravity [1]. Arkani Hamed, et al. in 1998 was first developed the braneworld model, which was used to solve hierarchy problem in particle physics [2]. Then two Braneworld models were introduced by Randall and Sundrum in 1999, known as Randall-Sundrum I (RS I) and Randall-Sundrum II (RS II) [3,4].

In the Brane-world model, there are two approaches used to obtain Einstein’s field equations in the brane. The first approach is the formulation of the curvature covariance as done by Shiromizu in 2010. In this approach, the 5-dimensional bulk quantities are projected onto the brane quantities [5,6,7]. The second approach is to expand the gradient that is known as gradient expansion method as Kanno and Soda did in 2002. In this approach, a low energy limit is defined as a limit where the energy density of matter in the brane is smaller than the brane tension [8,9,10].

In this study, we studied the braneworld model by adding the Gauss-Bonnet term to the 5 + n dimension bulk. The 4 + n dimensional Einstein brane field equation is taken from a previous study [11].
which showed the existence of an extra term. Furthermore, dynamic analysis is carried out to investigate the stability of this model.

2. Methods
The method used in this research is an analytical study by analyzing the stability of this model. Stability analysis is obtained through a dynamical system approach. Dynamical system approach is done by forming autonomous equations and determining critical points. To see a solution, perturbation is carried out around the critical points. Stable condition occurs when both eigenvalues are negative.

3. Setup Model
The model in this study can be explained through the action of equation

\[
S = \int d^{5+n}x \sqrt{-g} \left[ \frac{\mathcal{R}}{2\kappa^2} - \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right] + \int d^{4+n}x \sqrt{-h} [ -\sigma(\phi) + L_m ] \delta(y),
\]

(1)

The second term of the action of equation above is the Gauss-Bonnet equation that is in the 5 + n dimension bulk while the material term is localized in the 4 + n dimensional brane.

The variation of the action of equation above for the $g_{ab}$ metric is then carried out by projection of the bulk field to the Brane field using the Gauss-Codazzi equation obtained by the Einstein brane field equation

\[
G_{\nu}^\mu = \kappa^2 \left( \nabla_\nu \phi \nabla_\mu \phi - \delta_\nu^\mu \left( \frac{1}{2} \nabla^a \phi \nabla_{(a)} \phi - V_{eff} \right) \right) + 4\kappa^2 \left( (\nabla^\mu \nabla_\nu f(\phi)) R - (\nabla^\rho \nabla_\rho f(\phi)) R - 2 \left( \nabla^\rho \nabla_\rho f(\phi) \right) R_\rho^\nu - 2 \left( \nabla_\rho \nabla^\rho f(\phi) \right) R_\rho^\nu + 2 \delta^\nu_\sigma (\nabla_\sigma \nabla^\rho f(\phi)) R_\rho^\mu - 2 \delta^\mu_\rho (\nabla^\rho f(\phi)) R_\rho^\nu \right)
\]

(2)

Where $X_{\mu}^\nu$ is an extra term containing bulk quantities. The existence of this extra term causes the Einstein field equation in the brane is not closed. To solve this brane field equation, we must first know the geometry of the bulk [10].

And the equation of motion of the scalar field on the brane

\[
-\nabla^a \nabla_a \phi + V_{,\phi} + f_{,\phi} (R^2 - 4R_{ab} R^{ab} + (4 + n) R_{\alpha\beta\gamma} R^{\alpha\beta\gamma}) = J_{n},
\]

(3)

From the above equation, it can be seen that the scalar field in the brane is not conserved with $J_n$ defined as the current density of the scalar field from brane to bulk. The metric used in this study is the Friedmann-Robertson-Walker $4 + n$ dimension.

\[
ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j + b^2(\tau) \delta_{ab} dz^a dz^b,
\]

(4)

Where $a$ is the scale factor for the 3 common space dimensions and $b$ is the scale factor for $n$ external dimensions.

By assuming that $\phi$ is only a function of $t$, space-time is flat ($k_b = 0$ dan $k_a = 0$) and take the scale factor relationship between the external and internal dimensions $b(t) = a(t)^\gamma$, then $H_b = \gamma H_a = \gamma H$ the Friedmann equation is obtained

\[
k^2 \left( \frac{1}{2} f(\phi)^2 + V_{eff} \right) + 24(1 + \beta_0) H^3 f(\phi) + X_0^0 = 3(1 + \alpha_0) H^2
\]

(5)

as;

\[
\alpha_0 = n \left( \frac{n-1}{6} + \gamma \right)
\]

(6)
β_0 = \frac{1}{6} n\gamma (9\gamma (n - 1) + 18 + \gamma^2 (-3n + n^2 + 2)). 
(7)

And the equation of motion of the scalar field
\ddot{\phi} + 3(1 + \alpha_1)H\dot{\phi} + V_{eff}\phi + 24(1 + \alpha_2)H^2 = fn.
(8)

as;
\alpha_1 = \frac{ny}{3} 
(9)
\alpha_2 = ny \left(3 - \gamma + \frac{3}{2} ny\right) + \frac{ny^4}{24} (-n - 2n^2 + n^3 + 2)
(10)
\alpha_3 = 3ny - \frac{7}{6} nny^2 + \frac{3}{2} (ny)^2 - \frac{1}{2} (ny)^2 + \frac{1}{6} (ny)^3
(11)

4. Stability Analysis
Furthermore, dynamic analysis is carried out to determine the stability of this model. This analysis is taken for the absence of extra terms (material dominance) \(X_i^0 = 0\). Taking the case of the absence of extra terms and to simplify writing \(k = 1\). Then we take the form of effective potential \(V_{eff}\) and \(f(\phi)\) in,

\[ V_{eff} = V_0 e^{-\lambda \phi} \; ; \; f(\phi) = \left(\frac{f_0}{\mu}\right) e^{\mu \phi} \]

as \(\lambda > 0\). Then, a new variable is defined

\[ x_1 = \frac{\dot{\phi}}{\sqrt{6(1 + \alpha_0)}H} ; \; x_2 = \frac{\sqrt{V_{eff}}}{\sqrt{3(1 + \alpha_0)}H} ; \; x_3 = f_{\phi}H^2 ; \; N = \ln a \]

By taking the variables above, equation (5) becomes:

\[ 1 = x_1^2 + x_2^2 + \frac{8(1 + \beta_0)\sqrt{6}}{\sqrt{1 + \alpha_0}} x_1 x_3. \]
(12)

Define the density parameter

\[ \Omega_\phi = x_1^2 + x_2^2 ; \; \Omega_{GB} = \frac{8(1 + \beta_0)\sqrt{6}}{\sqrt{1 + \alpha_0}} x_1 x_3. \]

Eliminate \(\ddot{\phi}\) in equation (8) use equation (12) so:

\[ \left(1 + \frac{96(1 + \alpha_2)(1 + \beta_0)}{(1 + \alpha_0)} x_3^2 - \frac{4\sqrt{6(1 + \alpha_0)}}{(1 + \alpha_0)} (3(1 + \beta_0) - (1 + \alpha_2))x_1 x_3\right) \frac{H}{H^2} \]

\[ = -3(1 + \alpha_3) x_1^2 - 4\left(\frac{\sqrt{6(1 + \alpha_0)}}{1 + \alpha_0}\right) ((1 + \alpha_1) + 3(1 + \alpha_3)(1 + \beta_0))x_1 x_3 + 24(1 + \beta_0)\mu x_1^2 x_3 \]

\[ + 12x_2^2 x_3 - \frac{96(1 + \beta_0)(1 + \alpha_3)}{(1 + \alpha_0)} x_3^2. \]
(13)

We take a special case to see a stable point for perturbation by setting \(x_3 = 0\). For \(x_3 = 0\), the Autonomous equation is obtained.

\[ \frac{dx_1}{dN} = -3(1 + \alpha_3)x_1 + \frac{\sqrt{6(1 + \alpha_0)}}{2}\lambda x_2^2 + 3(1 + \alpha_3)x_1^3 \]
(14)
\[ \frac{dx_2}{dN} = x_2 \left(3(1 + \alpha_3)x_1^2 - \frac{\sqrt{6(1+\alpha_0)}}{2} \lambda x_1 \right), \quad (15) \]

with critical point:
\[ (x_1, x_2) = \left(\frac{\sqrt{6}(1+\alpha_0)}{6(1+\alpha_3)}, \pm \frac{\sqrt{1-(1+\alpha_0)^2}}{6(1+\alpha_3)^2} \right). \quad (16) \]

To see stability, perturbation is carried out around the critical point
\[ \frac{dx_1}{dN} = \frac{dx_1^0}{dN} + \frac{d\delta x_1}{dN} \quad (17) \]
\[ \frac{dx_2}{dN} = \frac{dx_2^0}{dN} + \frac{d\delta x_2}{dN} \quad (18) \]

Eigenvalue from above variable obtained:
\[ \lambda_{1,2} = \frac{1}{2} \left[-3(1 + \alpha_3) + \frac{3}{2} \frac{1+\alpha_0}{1+\alpha_3} \lambda^2 \pm \sqrt{\left(-3(1 + \alpha_3) + \frac{3}{2} \frac{1+\alpha_0}{1+\alpha_3} \lambda^2 \right)^2 + 12(1 + \alpha_0) \left(1 - \frac{1+\alpha_0}{6(1+\alpha_3)^2} \lambda^2 \right) \lambda^2} \right]. \quad (19) \]

A stable solution occurs if the eigenvalue \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \).

To see a stable solution, review the model with no extra dimensions \( n = 0 \). Eigenvalues for the absence of a system of extra dimensions
\[ \lambda_{1,2} = \frac{1}{2} \left[-3 + \frac{3}{2} \lambda^2 \pm \sqrt{\left(-3 + \frac{3}{2} \lambda^2 \right)^2 + 12 \left(1 - \frac{1}{6} \lambda^2 \right)} \right]. \quad (20) \]

In the eigenvalues equation above, there is no stable solution for all parameter values \( \lambda \) \( (\lambda_1 < 0 \) and \( \lambda_2 < 0 \)), in other words a solution in the absence of extra dimensions is an unstable solution.

The solutions for the existence of extra dimensions \( n \neq 0 \) and the value of the parameter \( \gamma = -1 \). In this case the eigenvalue equation
\[ \lambda_{1,2} = \frac{1}{2} \left[b \pm \sqrt{b^2 + d} \right], \quad (21) \]
as
\[ b = -3 \left(1 - \frac{n}{3}\right) + \frac{3}{2} \left(1 + \frac{n(n-7)}{6}\right) \lambda^2 \quad (22) \]
\[ d = 12 \left(1 + \frac{n(n-7)}{6}\right) \left(1 - \frac{1-n}{6} \lambda^2 \right) \lambda^2. \quad (23) \]

Here's a stable solution for some extra dimensional values \( n \).

| \( n \) | \( b \) | \( d \) | \( M_1 < 0 \) | \( M_2 < 0 \) |
|---|---|---|---|---|
| 1 | \(-2\) | \(\infty\) | | |
| 2 | \(-3\lambda^2 - 1\) | \(-8\lambda^2(1 - \lambda^2/8)\) | \(0 < \lambda \sqrt{8}\) | All value \( \lambda \) |
| 3 | | | | |
| 4 | \(9\lambda^2/2 + 1\) | \(-12\lambda^2(1 + \lambda^2/18)\) | No solution | No solution |

From the table above, it can be seen that for extra dimensions \( n \) values 1, 3, and 4, there is no parameter value \( \lambda \) which gives a negative eigen value solution so that is an unstable solution. For an
extra dimension $n = 2$ both eigenvalues will be negative when the parameter value is $0 < \lambda \sqrt{8}$ so that in this condition the system is stable.

For all values of extra dimensions $n$ and parameter $\lambda$, the area of negative eigenvalues are given as in Figure (1) below.

**Figure 1.** The negative eigenvalues solution for each extra dimension value $n$ and the parameter $\lambda$

In Figure (1) it can be seen that for extra dimensions from $n = 0$ to $n = 10$ the extra dimensional value gives both negative eigenvalues or a stable solution only exists for the extra dimension $n = 2$.

To analyze other areas we will look for the extra dimensions $n$ and the very large parameter values $\lambda$, $n \to \infty$ and $\lambda \to \infty$ as follows. The eigenvalues for $n \to \infty$ can be written as

$$M_{1,2} = 4 - 3\lambda^2 \pm \sqrt{16 + 8\lambda^2 + 9\lambda^4}.$$  \hspace{1cm} (24)

The eigenvalues above can be described for each parameter value $\lambda$ as shown in Figure (2).  

**Figure 2.** Eigenvalues for extra dimensions $n \to \infty$

From Figure (2) above, it can be seen that for the extra dimension $n \to \infty$ all eigenvalues are positive for each parameter value $\lambda$, so that in the case of very large extra dimension $n \to \infty$ none of them satisfies the stable solution.

Furthermore, for the parameter value $\lambda$ is very large ($\lambda \to \infty$) The eigenvalues can be written as

$$M_{1,2} = \frac{-3(6-7n+n^2)}{4(-3+n)} \pm \frac{4(-3+n)}{\sqrt{(6-7n+n^2)} + \frac{9(6-7n+n^2)^2}{16(-3+n)^2}}.$$  \hspace{1cm} (25)

The eigenvalues above can be drawn for each extra dimension $n$ as in figure (3).
From Figure (3), it can be seen that for the parameter value $\lambda \to \infty$ the eigenvalues are positive for all the values of the extra n dimensions, so that in the case of the parameter $\lambda$ is very large $\lambda \to \infty$ none of which satisfies the stable solution.

To see the stability of system can also be seen by looking at the trajectory of the system in phase space $(x_1, x_2)$. Figure (4) is the system trajectory in phase space$(x_1, x_2)$ for extra dimensions $n = 2$ and the parameter value $\lambda = 1$. In this trajectory, it can be seen that the two critical points are stable.

5. Conclusion
In this study, a dynamic analysis was carried out to determine the stability of this model. The stability analysis carried out is the absence of extra dimensions ($x_0^2 = 0$). In this case, the dynamic analysis is performed by setting $x_3 = 0$ or the case of scalar field dominance. Further analysis was carried out for the absence of extra dimensions ($n = 0$) and the existence of extra dimensions ($n \neq 0$). In the first case, assuming the absence of extra dimensions, there is no statistical solution for each parameter value $\lambda$. So it can be concluded that a solution in the absence of extra dimensions is an unstable solution. For the second case, in the presence of extra dimensions and taking the parameter value $\gamma = -1$, a stable solution is obtained for the extra dimension $n = 2$ with parameter value $0 < \lambda \sqrt{B}$.

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