IWAHORI COMPONENT OF BESSEL MODEL SPACES

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ABSTRACT. Let \( k_0 \) be a \( p \)-adic field of odd residual characteristic, and \( G \) a special orthogonal group defined as acting on a split \( 2n+1 \)-dimensional orthogonal space \( V \) over \( k_0 \). Let \( H \) be the Iwahori Hecke algebra of \( G \). A purpose of this short article is to compute the Iwahori component of a Bessel model space and identify it with an explicit projective \( H \)-module.

1. Introduction

The Bessel models considered in this paper have a rather long history. They were introduced by Novodvorski and Piatetski-Shapiro in the case \( n = 2 \) [NPS], extended by Bump, Friedberg and Furusawa [BFF] and naturally appear in theta correspondences [F]. Last but not least, these models appear as the second in a sequence of (generalized) Bessel models introduced by Gan, Gross and Prasad [GGP], where the first model is the Whittaker model space, also known as the Gelfand-Graev representation. A more precise definition is as follows. Let \( V \) be a split \( 2n+1 \)-dimensional orthogonal space over \( k_0 \), a \( p \)-adic field of odd residual characteristic. In particular, we can write \( V = X + X^\perp + k_0 \), where \( X \) is a totally isotropic subspace of dimension \( n \). Let \( v_1, \ldots, v_n \) be a basis of \( X \). Let \( P = LN \) be a parabolic subgroup of \( G = \text{SO}(V) \) defined as the stabilizer of the partial flag \( \langle v_1 \rangle \subset \ldots \subset \langle v_1, \ldots, v_{n-1} \rangle \). Here \( N \) is the unipotent radial and \( L = (k_0^\times)^{n-1} \times \text{PGL}_2(k_0) \) is the Levi factor. The Levi factor \( L \) acts naturally on characters of \( N \), and the stabilizer of a generic character is a one dimensional torus \( T_k \subset \text{PGL}_2(k_0) \), where \( k \) is a quadratic separable extension of \( k_0 \) such that \( T_k \cong k^\times/k_0^\times \). We assume henceforth that \( k \) is a field, so \( T_k \) is compact. Let \( \chi \) be the character of \( T_k \) obtained by extending the generic character of \( N \) trivially to \( T_k \). The (special) Bessel model space is \( \text{ind}_{T_k}^G \chi \). In fact, if \( k \) is ramified, then \( T_k \) is a group scheme with two connected components over \( O \), the ring of integers in \( k_0 \), so one can consider two characters \( \chi^\pm \) of \( T_k \) trivially to \( T_k \). The (special) Bessel model space is \( \text{ind}_{T_k}^G \chi^\pm \).

Let \( I \) be an Iwahori subgroup of \( G \) and \( H \) the Hecke algebra of compactly supported, \( I \)-bi-invariant complex functions on \( G \). As an abstract algebra, \( H \) corresponds to the Coxeter diagram of type \( \tilde{C}_n \), but with unequal parameters:

\[
\begin{array}{ccccccc}
0 & 1 & \cdots & n
\end{array}
\]

More precisely, \( H \) is generated by elements \( t_0, t_1, \ldots, t_n \), one corresponding to each vertex of the diagram, satisfying the braid relations as prescribed by the Coxeter diagram. These elements
satisfy the quadratic relations \((t_i + 1)(t_i - q) = 0\) for all \(i \neq 0\) and \((t_0 + 1)(t_0 - 1) = 0\), where \(q\) is the order of the residual field of \(k_0\). The algebra \(H\) has two finite dimensional subalgebras \(H_0\) and \(H_n\) obtained by removing the special vertices 0 and \(n\), respectively, of the Coxeter diagram. That is, \(H_0\) is generated by \(t_1, \ldots, t_n\) and \(H_n\) is generated by \(t_0, \ldots, t_{n-1}\). The algebra \(H_0\) is the subalgebra of \(H\) of functions supported on a hyper-special maximal compact subgroup of \(G\) i.e. whose reduction mod \(p\) is \(SO_{2n+1}\), while \(H_n\) consists of functions supported on a maximal subgroup whose reduction mod \(p\) is the disconnected group \(O_{2n}\). Let \(sgn\) and \(sgn'\) be two one-dimensional characters (types) of \(H_0\) defined by \(t_i \mapsto -1\) for \(i = 1, \ldots, n-1\) and \(t_n \mapsto -1\) and \(t_n \mapsto q\), respectively. It is well known ([BM] and [R]) that for irreducible \(G\)-modules generated by \(I\)-fixed vectors the presence of the type \(sgn\) is equivalent to existence of a non-zero Whittaker functional. A similar relationship between the type \(sgn'\) and existence of a Bessel model was discovered Brubaker, Bump and Friedberg [BBF]. Following a suggestion of Sol Friedberg, as the first result of this paper, we prove that there exist isomorphisms of \(H\)-modules

\[
\begin{align*}
\langle \text{ind}^G_{I_0N}(\chi) \rangle^I & \cong H \otimes_{H_0} sgn' \\
\langle \text{ind}^G_{I_0N}(\chi^\pm) \rangle^I & \cong H \otimes_{H_n} \epsilon^\pm,
\end{align*}
\]

where \(k\) is an unramified and ramified, respectively, quadratic extension of \(k_0\).

The second result of this paper concerns the restriction to \(G = SO_{2n+1}\) of the Steinberg representation \(St\) of the split orthogonal group \(SO_{2n+2}\). In fact, for the same price, one can consider two representations, the Steinberg representation \(St^+\), and its twist \(St^-\) by a quadratic character given as the composite of the spinor norm, and the unique non-trivial, unramified, quadratic character of \(k_0^2\). We prove that there exist isomorphisms of \(H\)-modules

\[
\langle St^\pm \rangle^I \cong H \otimes_{H_n} \epsilon^\pm.
\]

This result has several immediate consequences. If \(\pi\) is an irreducible representation of \(G\) generated by \(I\)-fixed vectors, then \(\pi\) is a quotient of \(St^+\) if and only if it has a non-zero (special) Bessel model. Furthermore, the Iwahori component of \(St\) is projective, hence \(\text{Ext}^1_G(St, \pi) = 0\) for all \(i > 0\). A similar result, projectivity of the Steinberg representation of \(GL_n\) when restricted to \(GL_n\), was obtained firstly in [CS1] for the Iwahori component, and then for all Bernstein components in [CS3]. See the article of Prasad [P] for a detailed discussion of ext-branching problems.

2. Basic case

Let \(k_0\) be a \(p\)-adic field of odd residual characteristic, \(O\) the ring of integers in \(k_0\), and \(\varpi\) a uniformizer. Let \(G = \text{PGL}_2(k_0)\) and, (ab)using the \(\text{GL}_2\)-terminology, let \(T\) be the torus of diagonal matrices and \(B = TU\) the Borel subgroup of upper triangular matrices. Let \(\bar{B} = TU\) be the Borel subgroup of lower triangular matrices. Then \(G/B = \mathbb{P}^1(k_0)\) is a projective line. Let \(k\) be a quadratic extension of \(k_0\) and \(T_k = k^\times / k_0^\times\). The torus \(T_k\) acts simply transitively on the projective line \(\mathbb{P}^1(k_0) \cong k^\times / k_0^\times\), hence we have exact decompositions

\[
G = T_k B = T_k \bar{B}.
\]

Let \(K\) be a maximal compact subgroup of integral matrices, and \(I\) the Iwahori subgroup of \(K\) such that \(I \cap U\) has the off-diagonal entry divisible by \(\varpi\). Then \(K = I \cup Is_1I\) where \(s_1\) is a permutation matrix. Let \(I = I \cup s_0 I\) be the normalizer of \(I\) in \(G\). The Iwahori Hecke algebra is generated by two elements \(t_0\) (supported and equal to 1 on \(s_0 I\)) and \(t_1\) (supported and equal to 1 on \(s_1 I\))
on $Is_1I$) satisfying relations $t_0^2 = 1$ and $(t_1 + 1)(t_1 - q) = 0$. Let $H_0$ and $H_1$ be the 2-dimensional subalgebras generated by $t_0$ and $t_1$, respectively. We have Bernstein decompositions

$$H \cong A \otimes H_1 \cong A \otimes H_0$$

where $A \cong \mathbb{C}[T/T(O)]$. This isomorphism satisfies the following properties. The group $T/T(O)$ is isomorphic to the co-character lattice of $T$, and for every $x \in T/T(O)$ let $\theta_x \in A$ be the element corresponding to $x$ under the isomorphism $A \cong \mathbb{C}[T/T(O)]$. If $(\pi, V)$ is a smooth $G$-module then we have a natural isomorphism $\pi^I \cong \pi^T \mathbb{C}_T$. This isomorphism intertwines the action $t_x$ on $\pi^I$ with the action of $x$ on $\pi^T \mathbb{C}_T$.

The algebra $H_0$ has 2 characters, denoted by $\epsilon^+(t_0) = 1$ and $\epsilon^-(t_0) = -1$, while $H_1$ has two characters $\text{sgn}(t_1) = -1$ and $\text{sgn}'(t_1) = q$. Inducing these characters to $H$ we get four projective $H$-modules.

Assume now that $k$ is unramified. The Bessel model space is

$$\Pi = \text{ind}_{T_k}^{G}(1) = C_t^\infty(T_k \backslash G).$$

Since $k$ is unramified, we can assume that $T_k$ sits in $K$. Thus the characteristic function of $K$ is contained in $\Pi$, and $t_1$ acts on it by the character $\text{sgn}'$. Hence, by the Frobenius reciprocity, we have a morphism of $H$-modules

$$H \otimes_{H_1} \text{sgn}' \to \Pi^I$$

where $1 \otimes 1$ is mapped to the characteristic function of $K$. We shall prove that this is an isomorphism by proving that it is so as $A$-modules. From the decomposition $G = T_k B$, it easily follows that the space of $T(O)$-fixed vectors in the Jacquet module $\Pi_T$ is isomorphic to $A \cong \mathbb{C}[T/T(O)]$. Furthermore, from the decomposition $K = T_k (B \cap K)$, it follows that the characteristic function of $K$ maps to $1 \in \mathbb{C}[T/T(O)]$ under the Jacquet module isomorphism. Now the claimed isomorphism follows from the Bernstein decomposition of $H$.

As a consequence, we get that the Steinberg representation $(t_0 \mapsto -1, t_1 \mapsto -1)$ does not have the (unramified) Bessel model, observed by Waldspurger many years ago.

Assume now that $k$ is ramified. The image of valuation on $k$ is $\frac{1}{2} \mathbb{Z}$, hence $T_k$ has two unramified characters $\chi^+$, the trivial, and $\chi^-$ which takes value $-1$ on any uniformizing element in $k$. A (twisted) Bessel model space is

$$\Pi^\pm = \text{ind}_{T_k}^{G}(\chi^\pm),$$

i.e. $\Pi^+$ is the Bessel model, while $\Pi^-$ is the twisted. In this case $T_k \subset \tilde{I}$ and $\chi^\pm$ can be extended to characters of $\tilde{I}$, trivial on $I$. These functions, supported on $I$ are elements in $\Pi^\pm$ and we have isomorphisms of $H$-modules

$$(\Pi^\pm)^I \cong H \otimes_{H_0} \text{sgn}^\pm.$$

**Remark:** Now assume that $G = D^\times/k_0^\times$ where $D$ is a quaternion algebra over $k_0$. If we normalize the valuation on $D^\times$ so that the image is $\mathbb{Z}$, then the restriction of valuation to $k_0$ has the image $2\mathbb{Z}$. Thus the valuation gives a homomorphism of $G$ on $\mathbb{Z}/2\mathbb{Z}$. The kernel of this map is the Iwahori subgroup, hence the Iwahori Hecke algebra is the group algebra of $\mathbb{Z}/2\mathbb{Z}$. Its representations correspond to the Steinberg and twisted Steinberg representation by the Jacquet-Langlands correspondence. The image of valuation on $k$ is $2\mathbb{Z}$ or $\mathbb{Z}$ if $k$ is unramified and ramified, respectively. Thus only the trivial representation has the Bessel model if $k$ is ramified. By Gross-Prasad conjectures (a theorem), the unramified twist of of the Steinberg representation of $\text{PGL}_2(k_0)$ ($t_0 \mapsto 1, t_1 \mapsto -1$) has a (ramified) Bessel model, which fits with all of the above.
3. HECKE ALGEBRA OF ODD SPLIT ORTHOGONAL GROUPS

This is mostly taken from [GS]. Let $V$ be a split $2n+1$-dimensional orthogonal space over $k_0$. In particular, we can write $V = X + X^\vee + k_0$, where $X$ is a totally isotropic subspace of dimension $n$. A basis $v_1, \ldots, v_n$ of $X$ gives a maximal split torus $T = (k_0)^\times$ in $SO(V)$. Let $\Sigma$ be the corresponding root system of type $B_n$. We realize the root system in $E = \mathbb{R} e_1 + \ldots + \mathbb{R} e_n$, so that

$$\Sigma = \{ \pm e_i \pm e_j \} \cup \{ \pm e_i \}.$$

Fix a set $\Delta$ of simple roots consisting of roots $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_n$. The choice of simple roots gives us a standard Borel subgroup $B = TU$ and its opposite $\bar{B} = T \bar{U}$. One can think of $B$ as stabilizing the partial flag $\langle v_1 \rangle \subset \ldots \subset \langle v_1, \ldots, v_n \rangle$.

Roots can be viewed as functionals on $E$, using the standard dot product on $E$. Affine roots are functionals $\alpha + m$ where $\alpha \in \Sigma$ and $m \in \mathbb{Z}$. We have a set of simple affine roots $\Delta_a = \Delta \cup \{ \alpha_0 \}$ where $\alpha_0 = 1 - e_1 - e_2$. The affine Weyl group $W_a$ (of type $B_n$) is generated by reflections about the root hyperplanes. The connected components of the complement in $E$ of the union of the root hyperplanes are called chambers. Let $C$ be the chamber consisting of all $x \in E$ such that $\alpha_i(x) > 0$ for all $i = 0, \ldots, n$. The group $W_a$ is generated by reflections $s'_0$ and $s_1, \ldots, s_n$ corresponding to the simple affine roots satisfying the braid relations given by the following Coxeter diagram:

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.figure
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The affine Weyl group is a semi direct product of the Weyl group $W$ (generated by $s_1, \ldots, s_n$) and a normal subgroup consisting of translations by $\lambda = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n m_i$ is even. The extended affine Weyl group $W_e$, relevant to the adjoint group $SO(V)$, is defined by enlarging the group of translations to full $\mathbb{Z}^n \cong T/T(O)$. It turns out that $W_e$ is also a Coxeter group. Indeed, $W_e = W_a \cup s_0 W_a$ where $s_0$ is the reflection about the hyperplane $x_1 = \frac{1}{2}$. One checks that

$$s'_0 = s_0 s_1 s_0.$$

Now it is easy to see that $W_e$ is isomorphic to the affine Weyl group of type $C_n$, since it is generated by reflections $s_0, \ldots, s_n$ and braid relations corresponding to the following Coxeter diagram:

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.figure
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Let $I$ be the Iwahori subgroup of $SO(V)$ corresponding to the chamber $C$ i.e. it is generated by $T(O)$ and the affine root spaces $U_\alpha$ for all affine roots $\alpha$ such that $\alpha > 0$ on $C$. The $I$-double
cosets in $\text{SO}(V)$ are parameterized by $W_e$. Let $l_0$ be the weighted length function on $W_e$ such that $l_0(s_0) = 0$ and it is 1 on other simple reflections. Then $[lwI : f] = q^l(w)$ for every $w \in W_e$.

Let $H$ denote the Iwahori Hecke algebra of $\text{SO}(V)$. For any $w \in W_a$ let $t_w \in H^+$ be the characteristic function of the double coset parameterized by $w$. For simplicity, let $t_i$ denote $t_{s_i}$. Then $t_i$ satisfy quadratic relations $t_i^2 = 1$ and $(t_i - q)(t_i + 1) = 0$, if $i \neq 0$, and braid relations given by the above Coxeter diagram. In fact, as an abstract algebra, $H$ is generated by $t_0, \ldots, t_n$ modulo these quadratic and braid relations. Let $H_0$ and $H_n$ be the finite dimensional subalgebras generated by all $t_i$ except $t_n$ and $t_0$, respectively. We have Bernstein decompositions

$$H \cong A \otimes H_0 \cong A \otimes H_n.$$ 

where $A$ is isomorphic to $\mathbb{C}[T/T(O)]$ and the isomorphism has the same property with respect to the Jacquet functor as in the rank one case ($G \cong \text{PGL}_2(k_0)$).

Then $H_0$ and $H_n$ each have 2 characters $\epsilon^+, \epsilon^-$ and sgn, sgn', respectively, such that $t_i \mapsto -1$ for all $i \neq 0, n$, with the rest of definition as in the rank one case. We also have four projective $H$-modules obtained by inducing these characters to $H$.

4. Special Bessel Models

Let $P = LN$ be a standard parabolic subgroup corresponding to the short root $e_n$, with the Levi $L = (k_0^\times)^{n-1} \times \text{PGL}_2$, where $\text{PGL}_2$ corresponds to the short root $e_n$. In other words, the group $P$ is defined as the stabilizer of the partial flag $\langle v_1 \rangle \subset \cdots \subset \langle v_1, \ldots, v_{n-1} \rangle$. In particular, $N/[N,N]$ is a vector space on which $L$ acts. The action of the $\text{PGL}_2$ factor is trivial on the subspace spanned by the lines corresponding to the roots $e_1 - e_2, \ldots, e_{n-2} - e_{n-1}$, and an irreducible representation on the 3-dimensional subspace spanned by the lines corresponding to the roots $e_{n-1} - e_n, e_{n-1}, e_{n-1} + e_n$. Let $x_i$ be the pinning coordinates for the roots $e_i - e_{i+1}$, and $x_n$ for the root $e_{n-1} + e_n$. (The pinnings are chosen so that the subgroup generated by $T(O)$ and the affine root groups $U_\alpha$ for all $\alpha$ such that $\alpha(0) \geq 0$ is a hyper-special maximal compact subgroup.) For every $a \in F^\times$ define a character

$$\psi_a : N \to \mathbb{C}^\times$$

by

$$\psi_a(n) = \psi(x_1 + \cdots + x_{n-1} + ax_n)$$

where $\psi : k_0 \to \mathbb{C}^\times$ has conductor $\varpi O$. The stabilizer of $\psi_a$ in $L$ is a torus $T_k \subset \text{PGL}_2$. We choose $a = u \in O^\times$ so that $k$ is the unramified extension of $k_0$. If $a = \tau$ is a uniformizing element in $O$, then $k$ is ramified. Abusing notation, we extend $\psi_a$ to a character $\chi$ of $NT_k$, trivial on $T_k$. We also have two characters $\chi^\pm$ of $NT_k$ in the ramified case, due to disconnectedness of $T_k$.

For sake of uniform exposition, let $\chi$ denote any of these characters of $T_k N$, for a moment. The Bessel space is

$$\Pi = \text{ind}_{NT_k}^G(\chi).$$

Lemma 1. As $T$-modules, $\Pi_T(U) \cong \mathbb{C}[T/T(O)]$.

Proof. Let $W_n \subset W$ consist of $w \in W$ such that $w^{-1}(e_n) > 0$. We have a Bruhat decomposition

$$G = PW_n \tilde{B} = NLW_n \tilde{B} = NT_k W_n \tilde{B}$$

where for the second equality we used that $\text{PGL}_2 = T_k \cdot (\text{PGL}_2 \cap \tilde{B})$. It follows that $\Pi$, as a $\tilde{B}$-module, has a filtration arising from the geometry of the Bruhat decomposition with subquotients $\Pi_w$ parameterized by $w \in W_n$. If $w \neq 1$, then there exists a simple root $\alpha$, necessarily different.
from $e_n$, such that $w^{-1}(\alpha) < 0$. Now it is easy to check \cite{CS2} that $(\Pi_w)_U = 0$. The bottom piece of the filtration is isomorphic to $C^\infty_c(T_k \backslash \overline{P})$. It is clear that

$$(C^\infty_c(T_k \backslash \overline{P}))_N \cong C^\infty_c(T_k \backslash L) \cong C^\infty_c((k_0^*)^{n-1}) \otimes C^\infty_c(T_k \backslash \text{PGL}_2) .$$

Now lemma follows from the PGL$_2$-computation. \hfill $\square$

In view of the isomorphism of $A \cong C[T/T(O)]$-modules $\Pi^I \cong \Pi^{T(O)}_U$, it follows that $\Pi^I$ is a free $A$-module of rank one. In order to determine the $H$-structure, we need to compute the action of the finite subalgebra $H_0$ or $H_n$ on a generator of the $A$-module. We shall do this by carefully picking the Iwahori subgroup. Recall that the chamber $C$ was defined by $\alpha_i > 0$ for all simple roots $\alpha_i$, $i = 0, \ldots, n$.

4.1. **Unramified case.** We let $C_u = -C$ and, abusing notation, let $I$ be the Iwahori subgroup generated by $T(O)$ and affine root groups $U_a$ for all $\alpha > 0$ on $C_u$. Note that previously defined simple reflections $s_1, \ldots, s_{n-1}, s_n$, corresponding to the roots, $e_1 - e_2, \ldots, e_{n-1} - e_n, e_n$ are still simple for this choice of the chamber. In particular, we have a finite subalgebra $H_n$ generated by $t_i$, the characteristic functions of $I s_i I$, for $i = 1, \ldots, n$. Let $\text{sgn}'$ be a character of $H_n$ such that $t_n \mapsto q$ and $t_i \mapsto -1$ for $i \neq n$.

**Theorem 2.** Let $\Pi$ be the unramified Bessel model space. Then $\Pi^I \cong H \otimes_{H_n} \text{sgn}'$ as $H$-modules.

**Proof.** Let $K = I \cup I s_n I$. Then $T_k \subset K$, in fact, note that $K = T_k I$. Hence we have a decomposition

$$NT_k I = NK = N(K \cap L)(K \cap \bar{N})$$

where the last factorization is a direct product as sets. Since the choice of $I$ guarantees that $\psi_u$, when restricted to $I \cap N$ is trivial, we have a unique function $f_0 \in \Pi^I$ supported on $NT_k I$ such that $f_0(nl\bar{n}) = \psi_u(n)$ for all $n \in N$, $l \in K \cap L$ and $\bar{n} \in K \cap \bar{N}$. Note that $f_0$ sits in the bottom piece of the $B$-filtration of $\Pi$ and under the isomorphism $\Pi^I \cong C[T/T(O)]$ it maps to the characteristic function of $T(O)$. We now need the following lemma:

**Lemma 3.** For every $i = 1, \ldots, n$, let $t_i \ast f_0$ denote the action of $t_i \in H_n$ on $f_0 \in \Pi^I$. Then $t_n \ast f_0 = q f_0$ and $t_i \ast f_0 = -f_0$ for $i \neq n$.

**Proof.** Since $f_0$ is right $K$-invariant, for $K = I \cup I s_n I$, it follows at once that $t_n \ast f_0 = q f_0$. We now use the decomposition $G = UW_c I$ and observe that $NK \subset UI \cup Us_n I$. It follows that $f_0(s_i) = 0$ for all $i \neq n$. This will be of use to compute the action of $t_i$ on $f_0$, for $s_i$ the reflection corresponding to $\alpha_i = e_i - e_{i+1}$. We shall simplify notation, in order to avoid writing the subscript $i$ many times, and will write $s = s_i$ and $\alpha = \alpha_i$. Note that the affine root space $U_\alpha$ is not contained in $I$, but $U_{-\alpha}$ and $U_{\alpha+1}$ are. Let $x_\beta : O \rightarrow U_\beta$ denote a pinning for any affine root $\beta$. Then

$$IsI = \cup_{t \in O/\omega O} x_\alpha(t)sI.$$ 

Recall that

$$(t_i \ast f_0)(g) = \int_{IsI} f_0(gh) \, dh.$$ 

Thus, if $(t_i \ast f_0)(g) \neq 0$, then there exists $h \in IsI$ such that $gh \in NT_k I$. Since $h^{-1} \in IsI$, it follows that $g \in NT_k IsI = \cup_{t \in O/\omega O} NT_k x_{-\alpha}(t)sI.$
There are two cases to discuss. If \( t \in \varpi O \) then \( NT_k x_{-\alpha}(t)sI = NT_k sI \). If \( t \in (O/\varpi O)^{\times} \), then we have an identity \( x_\alpha (1/t) x_{-\alpha}(-t) = x_{-\alpha}(t) \) \((\text{mod } T(O))\), and
\[
NT_k x_{-\alpha}(t)sI = NT_k x_\alpha(1/t) x_{-\alpha}(t)I = NT_k I.
\]
We have shown that \( t_i * f_0 \) is supported on \( NT_k I \) and \( NT_k sI \) so it suffices to compute the value of \( t_i * f_0(g) \) for \( g = 1 \) and \( s \). To that end,
\[
(t_i * f_0(s)) = \sum_{t \in O/\varpi O} f_0(s x_{-\alpha}(t)s) = \sum_{t \in O/\varpi O} f_0(x_\alpha(t)) = \sum_{t \in O/\varpi O} \psi(t) = 0,
\]
\[
(t_i * f_0(1)) = \sum_{t \in O/\varpi O} f_0(x_{-\alpha}(t)s) = \sum_{t \in (O/\varpi O)^{\times}} f_0(x_\alpha(1/t) x_{-\alpha}(-t)) = \sum_{t \in (O/\varpi O)^{\times}} \psi(1/t) = -1.
\]
We note that the second identity follows from the fact that \( f_0(s) = 0 \). The lemma is proved. \( \square \)

Now it is easy to prove the theorem. The Frobenius reciprocity implies that we have a natural map from \( H \otimes_{H_n} sgn' \) to \( \Pi^f \) that sends \( 1 \otimes 1 \) to \( f_0 \). This map is an isomorphism since \( H \otimes_{H_n} sgn' \cong \Pi^f \) as \( A \)-modules.

4.2. Ramified case. Let \( w \) be the permutation defined by \( i \mapsto n + 1 - i \) for all \( i = 1, \ldots, n \). We let \( C_r = w(C) \) and, abusing notation, let \( I \) be the Iwahori subgroup generated by \( T(O) \) and affine root groups \( U_\alpha \) for all \( \alpha > 0 \) on \( C_r \). Note that previously defined simple reflections \( s_1, \ldots, s_{n-1} \), corresponding to the roots, \( e_1 - e_2, \ldots, e_{n-1} - e_n \) are still simple for this choice of the chamber. Let \( s_0 \) be the reflection corresponding to the affine functional \( e_n - 1/2 \). Observe that \( s_0 \) is not a root reflection, however, it normalizes the chamber \( C_r \). Let \( H_0 \) be the finite algebra generated by \( t_i \), the characteristic functions of \( I s_i I \), for \( i = 0, \ldots, n - 1 \). Note that \( t_0^2 = 1 \). Let \( \epsilon^{pm} \) be two characters of \( H_0 \) such that \( t_0 \mapsto \pm 1 \) and \( t_i \mapsto -1 \) for \( i \neq 0 \).

**Theorem 4.** Let \( \Pi^\pm \) be the two ramified Bessel model spaces. Then \((\Pi^\pm)^f \cong H \otimes_{H_0} \epsilon^\pm \) as \( H \)-modules.

**Proof.** We just give a sketch of the proof. In this case \( s_0 \) normalizes \( I \) and let \( K = I \cup s_0 I \). Then \( T_k \subset K, K = T_k I \) and we have a decomposition
\[
NT_k I = NK = N(K \cap L)(K \cap \bar{N})
\]
where the last factorization is a direct product as sets. The choice of \( I \) guarantees that \( \psi_r \), when restricted to \( I \cap N \) is trivial. Let \( f_0^+ \in (\Pi^+)^f \) supported on \( NT_k I \) such that \( f_0^+(nl\bar{n}) = \psi_u(n) \) for all \( n \in N, l \in K \cap L \) and \( \bar{n} \in K \cap \bar{N} \), and \( f_0^- \in (\Pi^-)^f \) such that \( f_0^-(nl\bar{n}) = \pm \psi_u(n) \) where the sign depends whether \( l \in K \cap L \) is in \( I \cap L \) or not. The proof now proceeds in the same way as in the unramified case, by showing that \( f_0^\pm \) are eigenfunctions for \( H_0 \). We leave details to the reader. \( \square \)

5. Steinberg representation

Let \( G \) be a split reductive group over \( k_0 \). Let \( St \) be the Steinberg representation of \( G \). Let \( B \) be a Borel subgroup of \( G, \bar{B} \) the unipotent radical of \( \bar{B} \), the Borel opposite to \( B \), and \( X_w = Bw\bar{B} \) are the Bruhat cells. Write \( X = B\bar{U} \) for the open cell. For any subset \( J \) of simple roots \( \Pi \), let \( P_J \) be the standard parabolic subgroup associated to \( J \) (and containing \( B \)). In particular, \( P_0 = B \). Let \( C_c^{\infty}(P_J \setminus G) \) be the space of compactly supported smooth \( P_J \)-invariant functions on
G. Let St be the Steinberg representation of G. We use the following realization of the Steinberg representation:

$$\text{St} = C_c^\infty(B \setminus G)/ \sum_{\emptyset \neq J \subset \Pi} C_c^\infty(P_J \setminus G).$$

Thus we have a $\tilde{B}$-equivariant map $\Omega : C_c^\infty(B \setminus X) \rightarrow \text{St}$ given as the composition of natural maps

$$(1) \quad C_c^\infty(B \setminus X) \rightarrow C_c^\infty(B \setminus G) \rightarrow \text{St}.$$

**Proposition 5.** The map $\Omega$ is a $\tilde{B}$-equivariant isomorphism of $C_c^\infty(B \setminus X)$ and St.

**Proof.** Let $\mathbb{C}[W]$ denote the space of functions on $W$. Consider it as a $W$-module for the action by right translations. For every simple root $\alpha$, let $W_\alpha = \{1, s_\alpha\}$. Then $\mathbb{C}[W_\alpha \setminus W]$ is a submodule of $\mathbb{C}[W]$ consisting of left $W_\alpha$-invariant functions. For injectivity we need the following lemma.

**Lemma 6.** Let $\delta \in \mathbb{C}[W]$ be the delta function corresponding to the identity element. Then $\delta$ cannot be written as a linear combination of elements in $\mathbb{C}[W_\alpha \setminus W]$ where $\alpha$ runs over all simple roots.

**Proof.** Functions in $\mathbb{C}[W_\alpha \setminus W]$ are perpendicular to the sign character. Hence any linear combination of such functions is also perpendicular to the sign character. But $\delta$ is not, hence lemma. □

We can now prove injectivity of $\Omega$. Let $f \in C_c^\infty(B \setminus X)$ be in the kernel of $\Omega$. Then there exist $f_\alpha \in C_c^\infty(P_\alpha \setminus G)$ such that $f = \sum_{\alpha \in \Pi} f_\alpha$. For every $\bar{u} \in \bar{U}$, the function $w \mapsto f_\alpha(\bar{u} \bar{w})$ is in $\mathbb{C}[W_\alpha \setminus W]$. On the other hand, $w \mapsto f(\bar{w} \bar{u})$ is a multiple of $\delta$. Lemma implies that $f(\bar{u}) = 0$.

For surjectivity, let $V_r \subseteq C_c^\infty(B \setminus G)$ be the subspace of functions supported on the union of the Bruhat cells $X_w$ for $w \in W$ such that $l(w) \leq r$. Let $V_w = C_c^\infty(B \setminus X_w)$. Then, if $r > 1$, we have an exact sequence

$$0 \rightarrow V_{r-1} \rightarrow V_r \rightarrow \bigoplus_{l(w) = r} V_w \rightarrow 0.$$

Let $v \in \text{St}$ be the image of $v_r$. We need to show that $v$ is the image of some $f' \in V_{r-1}$. For every $w$ such that $l(w) = r$, pick $f_w \in V_w$ supported on $X_{w'}$ for $l(w) < r$ and $X_{w'}$. Then $f - \sum_{l(w) = r} f_w \in V_{r-1}$. Since $r > 1$, for every $w$ such that $l(w) = r$, there exists a simple root $\alpha$ such that $l(s_\alpha w) = r - 1$. The group $G$ has a cell decomposition as a union of $Y_w = P_\alpha w \bar{U}$ where $w$ runs over all $w \in W$ such that $l(s_\alpha w) = l(w) - 1$. Note that $B \setminus X_w = P_\alpha \setminus Y_w$ for such $w$. Going back to our fixed $w$ such that $l(w) = r$, there exists a function $h_w \in C_c^\infty(P_\alpha \setminus G)$ such that the support of $h_w$ is on $Y_w$ and larger orbits, and $h_w = f_w$ on $B \setminus X_w = P_\alpha \setminus Y_w$. The support of $h_w$, viewed as an element of $C_c^\infty(B \setminus G)$, is contained in $X_w$ and the union of $X_{w'}$ such that $l(w) < l(w)$. Hence $f' = f - \sum_{l(w) = r} h_w \in V_{r-1}$ and $f'$ has the image $v$ in $\text{St}$. Hence $\Omega$ is surjective. □

Let $\text{ch}_I$ be the characteristic function of $B(\bar{U} \cap I)$. Since $I = (B \cap I)(\bar{U} \cap I)$, it is an $I$-fixed element in $C_c^\infty(B \setminus G)$. Hence $v_0 = \Omega(\text{ch}_I)$ spans the line of $I$-fixed vectors in $\text{St}$. In other words, after switching the roles of $B$ and $B$ we have the following:

**Corollary 7.** As a $B$-module, the Steinberg representation is isomorphic to $\text{ind}_B^G(1) \equiv C_c^\infty(U)$. This isomorphism maps a non-zero $I$-fixed vector to the characteristic function of $I \cap U$. 
6. Restricting Steinberg

Let $G'$ be a split orthogonal group of the type $D_{n+1}$, and $T'$ a maximal split torus of $G'$. Let $\Sigma'$ be the corresponding root system. The standard realization of corresponding root system is in $E' = \mathbb{R} e_1 + \ldots + \mathbb{R} e_{n+1}$, so that $\Sigma' = \{ \pm e_i \pm e_j \}$. Let $\Delta'$ be a set of simple roots consisting of roots $\alpha_1 = e_1 - e_2, \ldots, \alpha_n = e_n - e_{n+1}$ and $\alpha_{n+1} = e_n + e_{n+1}$. Let $s_1, \ldots, s_{n+1}$ be the corresponding simple reflections. The choice of simple roots gives us a standard Borel subgroup $B' = T'U'$ and its opposite $\bar{B}' = T'U'$. Let $W'$ denote the Weyl group. We have an involution $\sigma$ of $\Delta'$ that permutes $\alpha_n$ and $\alpha_{n+1}$. This involution lifts to an involution of $G'$ such that the group of fixed points is $G$, the split odd orthogonal group. Let $T \subset T'$ and $U \subset U'$ be the subgroups of $\sigma$-fixed elements. Since $T$ contains a strongly $G'$-regular element, the centralizer of $T$ in $G'$ is $T'$. This implies that the normalizer $N_G(T)$ is contained in $N_{G'}(T')$ and we have an identity $N_G(T) = (N_{G'}(T'))^\sigma$. The quotients $N_G(T)/T(O)$ and $N_{G'}(T')/T'(O)$ are isomorphic to extended affine Weyl groups $W_e$ and $W'_e$ and our next task is to describe the identity $N_G(T) = (N_{G'}(T'))^\sigma$ on the level of extended Weyl groups.

Consider the affine root $\alpha_0 = 1 - e_1 - e_2$. Then $\Delta'_a = \Delta' \cup \{ \alpha_0 \}$ is a set of simple affine roots, and let $C' \subset E'$ be the corresponding chamber. The extended affine Weyl group $W'_a$ is a semi-direct product of $W'$ and the group $\mathbb{Z}^{n+1}$ of translations of $E'$. Then $W'_a = W'_a \cup s_0 W'_a$ where $W'_a$ is the affine Weyl group and $s_0$ is a unique non-trivial element in $W'_e$ that stabilizes $C'$. More precisely, $s_0$ is a central symmetry about the point $(1/2, 0)$ in the two dimensional subspace spanned by $e_1$ and $e_{n+1}$. The extended affine Weyl group is generated by $s_0, s_1, \ldots, s_{n+1}$. The involution $\sigma$ descends to $W'_a$ and $(W'_a)^\sigma$ consists of elements commuting with the linear map on $E'$ defined by $e_{n+1} \mapsto -e_{n+1}$ and identity on $E \subset E'$. Note that $s_0, s_1, \ldots, s_{n-1} \in (W'_e)^\sigma$, and we have a natural isomorphism $(W'_e)^\sigma \cong W_e$ is obtained by restricting the action of $w \in (W'_e)^\sigma$ to $E$. Note that under this isomorphism $s_0, \ldots, s_{n-1} \in W'_e$ map to the elements of $W_e$ denoted by the same symbols.

Let $I' \subset G'$ be the Iwahori subgroup corresponding to $C'$ and $H'$ the corresponding Iwahori Hecke algebra. This algebra is generated by $t'_i$, the characteristic functions of $I's_iI$, for $i = 0, \ldots, n + 1$. The Steinberg representation of $G'$ is the unique irreducible representation $\text{St}$ such that $\text{St}_{I''} = C_0, t_i * v_0 = -v_0$ for $i = 1, \ldots, n + 1$, and $t_0 * v_0 = v_0$. Let $C = E \cap C'$, and $I$ the corresponding Iwahori subgroup, and $H$ the Hecke algebra, generated by $t_i, i = 0, \ldots, n$. Let $H_0 \subset H$ be the finite algebra generated by $t_i, i = 0, \ldots, n$, and $\epsilon^+$ the character of $H_0$ such that $t_0 \mapsto 1$ and $t_i \mapsto -1$, for $i = 1, \ldots, n - 1$.

**Lemma 8.** Let $\text{St}$ be the Steinberg representation of $G'$ and $v_0 \in \text{St}_{I'}$. Then $H_0$ acts on $v_0$ by the character $\epsilon^+$.

**Proof.** For $i = 0, \ldots, n - 1$ the Hecke algebra elements $t_i$ and $t'_i$ are the characteristic functions of $I's_iI$ and $I's_iI'$, respectively, where the two $s_i$ coincide. The statement of the lemma is that $t_i * v_0 = t_i' * v_0$ for these $i$. If $g * v$ denotes the action of $g \in G'$ on $v \in \text{St}$ then $t_i * v_0 = \sum_j g_j * v_0$ after writing $I's_iI = \bigcup_j g_j I$ (a disjoint sum). Thus it suffices to that $I's_iI' = \bigcup_j g_j I'$, with the same $g_j$. This is obvious for $i = 0$ since $s_0$ normalizes $I$ and $I'$. For $i = 1, \ldots, n - 1$ the root $\alpha_i = e_i - e_{i+1}$ is a simple for both groups. We have

$$I's_iI = \bigcup_{t \in O/\mathbb{Z}O} x_{\alpha_i}(t)s_iI$$

and a similar decomposition for $I's_iI'$.

$\square$
Theorem 9. Let $St$ be the Steinberg representation of $G'$, the split orthogonal group of type $D_{n+1}$. Let $v_0$ be a non-zero $I'$-fixed vector in $St$. Then the $H$-module $St^I$ is generated by $v_0$ and isomorphic to $H \otimes_{H_0} \epsilon^+$. In particular, it is projective.

Proof. By Lemma 8 and the Frobenius reciprocity, we have a natural map $H \otimes_{H_0} \epsilon^+ \to St^I$. We need to show that $St^I \cong A$, and generated by $v_0$. To that end, we use that $St^I \cong (St)^{T(O)}$. We can compute the right side using Corollary 7, which says that $St \cong C_\infty(U')$. Hence $St_U \cong C_\infty(U'/U)$. Now note that $U'/U \cong k_0^n$ as $T = (k_0^\times)^n$-module. Under the isomorphism $St_U \cong C_\infty(k_0^n)$ the vector $v_0$ is mapped to the characteristic function of $O^n$. Now observe that $T/T(O)$-translates of the characteristic function of $O^n$ form a basis of $C_\infty(k_0^n)^{T(O)}$ on which $T/T(O)$ acts freely. In other words, $C_\infty(k_0^n)^{T(O)}$ is isomorphic to $A$, as an $A$ module. □

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