On the Ergodic theory of the Generalized incompressible flow

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Abstract To study the variation problem related to the incompressible fluid mechanics, Brenier brings the concept of generalized flow and shows that the generalized incompressible flow (GIF) is deeply related to the classical solution of the incompressible Euler equations. In this paper, we will study the ergodic theory of the GIF which may help us understand the dynamic property of the classical solution of the incompressible Euler equations. First, we show that the GIF has the weak recurrent property rather than the classical one. Then, we define the ergodicity of the GIF and discuss its relation with the classical ergodic flow. Next, we prove some ergodic theorems of the GIF. Finally, we give a theorem about the structure of the set of all GIFs.

1 Introduction

In his paper [1] published in 1966, V.I.Arnold studied the geometric approach to the incompressible Euler equations. This work brings many new ideas into the research of the ideal incompressible fluid, see [2][8] for more details of the group-theoretic and geometric research of hydrodynamics.

Let us recall that

Definition 1.1. (The incompressible Euler equations on M) Let M be a compact, oriented Riemannian manifold without boundary, we denote the Riemannian metric by ( , ) and the volume form by μ. The Euler equations of incompressible flows of homogenous fluid on the manifold M are

\[
\begin{align*}
\text{div}_\mu v &= 0, \\
\frac{Dv}{Dt} &= -\nabla p, \\
v|_{t=0} &= v_0,
\end{align*}
\]

(1.1)

here, \( \frac{Dv}{Dt} = \frac{dv}{dt} + \nabla_v v \), \( \nabla_v v \) is the covariant derivative of the velocity field \( v(x,t) \) along itself, \( p(x,t) \) is the scalar pressure field which can be determined by the condition \( \text{div}_\mu(\frac{Dv}{Dt}) = 0 \) up to a time-dependent constant. The first equation shows that \( v \) is divergence-free vector field, i.e. the flow of it preserve the volume form \( \mu \).

For the study of these equations via partial differential equation methods, one can refer to [3][10] for more details. Besides studying the velocity fields of the flow, there is an alternate way to study these equations. One usually calls it Lagrangian formulation, but it was first introduced by Euler. In this point of view, we focus on the trajectories of particles. Let \( x_0 \) be the initial position and \( x = g(t,x_0) \) be the position at time \( t \) of a trajectory that issues from \( x_0 \), then we have

\[
\begin{align*}
v(t,g(t,x_0)) &= \frac{d}{dt}g(t,x_0), \\
g(0,x_0) &= x_0.
\end{align*}
\]

(1.2)
Hence, the second equation of (1.1) turns into

\[ \frac{d^2}{dt^2}g(t, x_0) - \frac{\partial v(t, x_0)}{\partial t} + \nabla_v v(t, x_0) = - \nabla p(t, x_0). \]  

(1.3)

For fixed \( t \geq 0 \), assuming the velocity \( v \) belongs to \( C^1 \), then \( g(t, \cdot) \) (or \( g(\cdot) \)) is a diffeomorphism from \( M \) to \( M \).

And by the first equation of (1.4), we know that it has unit determinant (\( \det(\frac{\partial g}{\partial x}) \equiv 1 \)). To see a good introduction of these ideas, one can refer to chapter 3 \([12]\) for more details. Thus, in the Lagrangian formulation, the incompressible fluid can be considered as a map \( t \rightarrow g(t, \cdot) \), here \( g(t, \cdot) \) is volume preserving diffeomorphism. So we can say that the configuration space is \( S \text{Diff}(M) \), here \( S \text{Diff}(M) \) denote the group of volume preserving diffeomorphisms.

The group \( S \text{Diff}(M) \subseteq L^2(M; \mathbb{R}^n) \) can be seen as an infinite-dimensional Hilbert manifold and the energy

\[ E = \frac{1}{2} \int_M (v, v) \mu \]  

(1.4)

introduce a right-invariant weak-Riemannian metric on \( S \text{Diff}(M) \). Actually, this Riemannian structure is inherit from the Hilbert space \( L^2(M; \mathbb{R}^n) \). Arnold \([1]\) pointed out that the incompressible fluid is a geodesic on \( S \text{Diff}(M) \). To explain this, first, we must figure out what is the tangent space of \( S \text{Diff}(M) \). Let \( g_t : (a, b) \rightarrow S \text{Diff}(M) \) be a curve on \( S \text{Diff}(M) \), then \( g_t \in T_g S \text{Diff}(M) \), \( \forall t \in (a, b) \). Combining the first equation of (1.2) and the fact \( \text{div}_v v = 0 \), it is easy to get \( v(t, \cdot) = \frac{\partial g_t}{\partial t}(g_t^{-1}(\cdot)) \) and \( \text{div}_v v(t, \cdot) = 0 \), \( \forall t \in (a, b) \). Hence, the tangent space at \( g \in S \text{Diff}(M) \) is

\[ T_g S \text{Diff}(M) = \{ v \in C^1(M; \mathbb{R}^n) | \text{div}_v(v \circ g^{-1}) = 0 \}^{L^2(M; \mathbb{R}^n)} \]

\[ = \{ v \circ g | \text{div}_v v = 0, v \in C^1(M; \mathbb{R}^n) \}^{L^2(M; \mathbb{R}^n)}. \]

The group of volume preserving diffeomorphisms \( S \text{Diff}(M) \) is an infinite-dimensional Lie group, by the discussion above, the corresponding Lie algebra is the \( L^2 \) closure of the linear space of all the divergence-free vector fields. According to the Helmholtz decomposition Theorem, the orthogonal complement of \( T_g S \text{Diff}(M) \) in \( L^2 \) is

\[ T_g^1 S \text{Diff}(M) = \{ - \nabla p \circ g | \forall p : M \rightarrow \mathbb{R} \text{ is } C^1 \}^{L^2(M; \mathbb{R}^n)}. \]

From (1.3), the Euler fluid can be seen as the geodesic on \( S \text{Diff}(M) \). Therefore, the study of Euler equation can be converted into the following variation problem:

**Problem 1.** Given \( g, h \in S \text{Diff}(M) \), finding a smooth curve \( \{ g_t \}_{t \geq 0} \subset S \text{Diff}(M) \) minimizing the energy

\[ E(g) = \int_0^1 \frac{1}{2} ||g_t||_{L^2(M; \mathbb{R}^n)}^2 dt = \int_0^1 \int_M \frac{1}{2} \left( \frac{\partial g(t, x)}{\partial t}, \frac{\partial g(t, x)}{\partial t} \right) dt dx, \]  

(1.5)

among all curves satisfying \( g_0 = g, g_1 = h \).

An local existence and uniqueness theorem for this problem has been proved by Ebin and Marsden \([5]\) if \( g \) and \( h \) are sufficiently close to a sufficiently high order Sobolev norm.

In 1985, however, A.Shnirelman \([11]\) introduced a diffeomorphism \( g \in S \text{Diff}(K^3) \) such that there is no minimizing curves connecting the identity map to \( g \), here \( K^3 \) is the cube \([0, 1]^3\). Shnirelman’s counterexample was constructed via a special class of diffeomorphism \( g(x, z) = (h(x), z) \), here \( g \in S \text{Diff}(K^3), h \in S \text{Diff}(K^2) \). Because the fluid movements in two dimensional plane have more constraints, a minimizing path connecting \( \text{Id} \) to \( g \) would have to use the third dimension. Then we compress the fluid movements in the third dimension, which leads to the new movement having less kinetic energy than before.

In 1989, Y.Brenier \([4]\) considered the concept of generalized flow, this concept is closely related to L.C.Young’s idea of Young’s measure. Then he studied the generalized variation problem, the relation between this variation problem and problem \([1]\) is like the Kantorovitch problem and Monge problem in optimal transport. Next, we introduce Brenier’s work briefly.
Before giving the definition of the generalized flow, we may give a broader set than $S \text{Diff}(M)$

$$MP(M) = \{ \gamma : M \rightarrow M \mid \text{if } Y \text{ is a measurable subset of } M,$$

$$\text{then } \gamma^{-1}(Y) \text{ is measurable and } \text{meas}(\gamma^{-1}(Y)) = \text{meas}(Y) \}.$$ 

Then we can study the variation problem on $MP(M)$

**Problem 2.** Given $g, h \in MP(M)$, finding a smooth curve $\{g_t\}_{t \geq 0} \subset MP(M)$ minimizing the energy

$$E(g) = \int_{0}^{1} \frac{1}{2} \|g_t\|_{L^2(M,\mathbb{R}^n)}^2 dt = \int_{0}^{1} \int_{M} \int_{M} \frac{1}{2} \left( \frac{\partial g(t,x)}{\partial t}, \frac{\partial g(t,x)}{\partial t} \right) dt dx,$$  \hspace{1cm} (1.6)

among all curves satisfying $g_0 = g, g_1 = h$.

But because of the nonlinear constraint $\int_M f(g_t(x)) dx = \int_M f(y) dy, \forall f \in C(M)$, this problem is still very hard to answer. In the spirit of Young’s work in calculus of variations, Brenier brings the concept of generalized flow which is a probability measure on the product space $\Omega = M^{0,T}$, $\Omega$ can be view as the space of all the paths $t \in [0, T] \rightarrow z(t) \in M$ on $\Omega$. Then he proved the existence of the minimizing solution to the variation problem.

**Theorem 1.1. (Theorem 3.2 in [4])** For any generalized incompressible flow $q$, the generalized kinetic energy

$$E(q) = \int \rho(z(0)) \int \frac{1}{2} \| \dot{z}(t) \|^2 dq(dz), \hspace{1cm} (1.7)$$

is well defined in $[0, +\infty]$. If $E(q)$ is finite, then one can define the generalized Action

$$A(q) = \int \rho(z(0)) \left( \int \frac{1}{2} \| \dot{z}(t) \|^2 dt - \int U(t, z(t)) dt \right) dq(dz), \hspace{1cm} (1.8)$$

For any generalized final configuration $\eta$, if there is one generalized incompressible flow that can reach $\eta$ at time $T$ with a finite kinetic energy, then there is such a flow that minimizes the Action.

Here, a final configuration is a doubly stochastic probability measure $\eta$ on $M \times M$ which is a positive Borel measure satisfies

$$\int_{M \times M} f(x) \eta(dx,dy) = \int_{M \times M} f(y) \eta(dx,dy) = \int_{M} f(x) dx, \forall f \in C(M), \hspace{1cm} (1.9)$$

for each volume preserving map $h \in S \text{Diff}(M)$, we can associate a unique doubly stochastic probability measure $\eta_h$ defined by

$$\eta_h(dx,dy) = \delta(y-h(x)) dx. \hspace{1cm} (1.10)$$

Hence, Brenier’s theorem can answer the following variation problem

**Problem 3.** Given $h \in MP(M)$, finding a generalized incompressible flow $q$ minimizing the energy

$$E(q) = \int \rho(z(0)) \int \frac{1}{2} \| \dot{z}(t) \|^2 dt dq(dz), \hspace{1cm} (1.11)$$

among all the generalized incompressible flows reach the final configuration $\eta_h$, i.e. satisfies

$$\int_{\Omega} f(z(0),z(t)) dq(dz) = \int_{M \times M} f(x,y) \eta(dx,dy), \forall f \in C(M \times M). \hspace{1cm} (1.12)$$

Then, Brenier proved that under a natural restriction on the time scale, any classical solution to the Euler equations satisfies the generalized variation problem and is the unique minimizing solution. So the new concept of generalized flow did not miss the classical solution. Brenier also gives a formal relation between his generalized flow and DiPerna and Majda’s measure valued solution [7].

So we can see the concept of generalized incompressible flow (GIF) is deeply related to the study of incompressible Euler equation. Its dynamic property may give some lights to the research of incompressible
fluid mechanics. As we will mention in section 1, the GIF is indeed a larger class than the invariant measure on $\Omega = M^{[−∞, +∞]}$. So we may ask following questions: what are the recurrent and ergodic property of GIF? Is there any ergodic theorems of the GIF? What is the structure of the set of all the GIF? We will answer these questions in our paper.

The remainder of the paper is organized as follows:

In section 2, after giving the concept of the GIF, we will study the recurrent property of it. We will show that the GIF only has the recurrent property in a weak sense, and we are going to construct a counterexample which is a GIF, but for $q$ almost all $z(t) \in \Omega$, the flow $z(t)$ does not have the recurrence property! This is a big difference between classical and generalized incompressible flow.

In section 3, we will study the concept of the ergodicity of GIF. We will prove that a GIF induced by a classical ergodic flow on $M$ is an ergodic GIF we defined. Also, we will give some equivalent definition of the ergodicity.

In section 4, we will study several ergodic theorems of GIF. First, we will prove a maximal ergodic theorem by the harmonic analysis method, then we will prove the $L^1$ ergodic theorem. Finally, combining these two theorems we will prove a pointwise ergodic theorem of GIF.

In section 5, we will study the set of all GIFs, we denote it by $GIF(\Omega)$. Like the set of invariant measure on a compact metric space, the $GIF(\Omega)$ is a convex and compact set, but the difference is that the extreme points of the set is not the ergodic GIF. We will prove that the extreme points of the set is weak ergodic GIF which will be defined in this section.

## 2 Generalized incompressible flow and its recurrence property

First, we give the explicit definitions of the generalized incompressible flow. Let $X$ be a compact metric space, $\Omega = X^\mathbb{R}$ is the path space on $X$.

**Definition 2.1.** *(Generalized incompressible flow)* A probability measure $q$ on $\Omega$ is called a generalized incompressible flow (GIF) if it satisfies following incompressible condition:

$$\int_{\Omega} f(z(t_0))q(dz) = \int_{\Omega} f(z(0))q(dz), \forall f(x) \in C(X), \forall t_0 \in \mathbb{R}, \quad (2.1)$$

**Remark 2.1.** *(a)* By the definition, we can not say that $q$ is an invariant measure of the flow $\Gamma^s$ on $\Omega$:

$$\Gamma^s : \Omega \rightarrow \Omega$$

$$z(t) \mapsto z(t + s),$$

but for an invariant measure of $\Gamma^s$, it must be a GIF. The set of GIFs on $\Omega$ which we study in this paper is larger than the set of invariant measures on $\Omega$.

*(b)* Let $E \subset X$ be a Borel set of $X$. We have $\int_{\Omega} \chi_E(z(t_0))q(dz) = \int_{\Omega} \chi_E(z(0))q(dz)$, hence we can define $\mu(E) = \int_{\Omega} \chi_E(z(0))q(dz)$. This gives a probability measure on $\Omega$, such that

$$\int_{\Omega} f(z(t_0))q(dz) = \int_{\Omega} f(z(0))q(dz) = \int_X f(x)\mu(dx), \forall f(x) \in C(X), \forall t_0 \in \mathbb{R}. \quad (2.2)$$

Let $(\Omega, q)$ be a GIF; it has the recurrence property in the following sense:

**Theorem 2.1.** Let $(\Omega, q)$ be a GIF, $E \in B_X$, $\mu(E) > 0$, here $\mu$ is a probability measure on $X$ satisfies (2.2). Then for $\Phi = \{z(t) \in \Omega \mid z(0) \in E\}$, there exists $s \in \mathbb{R}^+$ such that $q(\Phi \cap \Gamma^{-s}\Phi) > 0$.

**Proof.** Let us consider the set $[\Gamma^{-s}\Phi \mid s > 0]$. If we have

$$q(\Gamma^{-s}\Phi \cap \Gamma^{-t}\Phi) = 0, \forall s \neq t > 0, \quad (2.3)$$
Therefore there exist $s < t$ such that $q(\Gamma^{-s} \Phi) > 0$, i.e. $q(\Phi \cap \Gamma^{-(t-t)} \Phi) = 0$.

\[ q(\Gamma^{-t} \Phi) = \int_\Omega \chi_{\Gamma^{-t} \Phi}(z(t))q(dz) = \int_\Omega \chi_{\Phi}(z(t+s))q(dz) \]

\[ = \int_\Omega \chi_{E}(z(s))q(dz) = \int_\Omega \chi_{E}(z(0))q(dz) = \mu(E) > 0. \]

Therefore there exist $s < t$ such that $q(\Gamma^{-s} \Phi \cap \Gamma^{-t} \Phi) > 0$, i.e. $q(\Phi \cap \Gamma^{-(t-s)} \Phi) = 0$.

\[ \square \]

As in the case of classical incompressible flow, we may guess:

**Conjecture.** Let $(\Omega, q)$ be a GIF, $E \in B_\Omega$, $\mu(E) > 0$, here $\mu$ is a probability measure on $X$ satisfying (2.2). Then for $\Phi = \{z(t) \in \Omega \mid z(0) \in E\}$, there exists $\Theta \subset \Phi$ such that $q(\Theta) = q(\Phi)$ and $\Theta = \{z(t) \in \Omega \mid \exists t_1 < \cdots < t_n < \cdots, z(t_j) \in E\}$.

Unfortunately, this conjecture is not true. Here, we will give a counter-example which is a GIF, but for $q$ almost every $z(t) \in \Omega$, the flow $z(t)$ does not have the recurrence property!

We can see that the lack of recurrence property is a big difference between the generalized and classical incompressible flow.

**Example 2.1.** Let $X$ be the unit circle $S^1$. Considering the following flow $z(t)$ on $S^1$:

\[ z_i(t) = \begin{cases} x + t (\text{mod } 2\pi), & t \in [0, \frac{\pi}{2}] \\ x + \frac{t}{\pi} (\text{mod } 2\pi), & t \in (\frac{\pi}{2}, +\infty) \end{cases} \]

We will study the generalized flow $q$ which equidistributes on the set of the flows defined above. It is easy to check that

\[ \int_\Omega \chi_{(x_1, x_2]}(z(t))q(dz) = \int_\Omega \chi_{(x_1, x_2]}(z(0))q(dz) = \int_X \chi_{(x_1, x_2]}(x)dx = \frac{x_2 - x_1}{2\pi}, \]

hence $q$ is a GIF. Because the flow we defined above stops at $t = \frac{\pi}{2}$, so for $q$ almost every $z(t)$, the flow does not have the recurrence property.

From this example we can see that the GIF only has the recurrence property in a very weak sense. Even so, in the following section, we will prove a pointwise ergodic theorem of the GIF.

### 3 Ergodicity

In this section, we will discuss the ergodicity of the GIF. For a good introduction to classical ergodic theory, one can refer to [6, 13]. First, we give the definition of ergodicity.

**Definition 3.1.** (Ergodic GIF) A GIF $q$ is called ergodic if for any $B \in B_\Omega$ which satisfies $\Gamma^{-t}(B) = B, \forall t \in \mathbb{R}$, we have $q(B) = 0$ or 1.

Let $j(t)$ be an ergodic flow on $(X, \mu)$. With the exception of a measure zero set, for all $x \in X$, we can find an unique $t_0$ such that $x = j(t_0)$, then we can define $G(t, x) = j(t_0 + t)$ for $\mu$ almost all $x \in X$. So we can define a generalized flow $q$ induced by $j(t)$ as follow:

\[ \int_X F(z(t))q(dz) = \int_X F(x \mapsto G(t, x))\mu(dx), \forall F(z) \in C(\Omega). \]

We claim that $q$ is a GIF. Actually, $\forall f(x) \in C(X)$, we have

\[ \int_X f(z(t_0))q(dz) = \int_X f(G(t_0, x))dx = \int_X f(x)\mu(dx), \forall t_0 \in \mathbb{R}, \]

the second equality is because $\mu$ is the invariant measure for the flow $j(t)$. Next, we prove that $q$ is ergodic.
Theorem 3.1. The GIF $q$ defined above is ergodic.

Proof. Let $Q = \{u(t) = j(t_0 + t), \forall t_0 \in \mathbb{R}\}$. Then, by the definition of $q$ (see (3.1)), we get $q(Q) = 1$. Let $B$ be a Borel set of $\Omega$ with $\Gamma^{-i}(B) = B, \forall t \in \mathbb{R}$.

If $B \cap Q = \emptyset$, then $q(B) = 0$. If $B \cap Q \neq \emptyset$, then we can find $j(t_0 + t) \in B \cap Q$. Since $\Gamma^{-b_0}(j(t_0 + t)) = j(t)$ and $B$ is invariant, we get $[\Gamma^{-b_0}(j(t))] \subset Q \cap B$, therefore $q(B) = 1$. We see that $q(B)$ is either 0 or 1, so $q$ is an ergodic GIF.

Before continuing our discussion, we first prove the following theorem. We will always assume that the map $(z(t), s) \mapsto \Gamma^s z(t)$ is measurable.

Theorem 3.2. Let $(\Omega, q)$ be a GIF. Then for $B \in \mathcal{B}_\Omega$, the following statements are equivalent:
1. $q(\Gamma^s B \Delta B) = 0, \forall s \in \mathbb{R}$.
2. There exists $F \in \mathcal{B}_\Omega$ such that $q(F \Delta B) = 0$ and $\Gamma^s F = F, \forall s \in \mathbb{R}$.

Proof. (2) $\Rightarrow$ (1) is obvious. Now we prove (1) $\Rightarrow$ (2). First taking the rational number set $Q$, define $F = \cap_{\xi \in Q} \Gamma^\xi B \subseteq B$. Then $\Gamma^s F = F, \forall s \in \mathbb{Q}$ and $q(B \Delta F) = q(B \setminus F) = q(B)\setminus F = 0$.

Suppose $B \in \mathcal{B}_\Omega$ satisfies (1). Then by the discussion above we can assume that $\Gamma^s B = B, \forall s \in \mathbb{Q}$. Define $B_z = \{s \in \mathbb{R} | \Gamma^s z \in B\} \subseteq \mathbb{R}$. By the assumption that $(z(t), s) \mapsto \Gamma^s z(t)$ is measurable, we know $B_z$ is measurable for each $z \in \Omega$.

Because $\Gamma^s B = B, \forall s \in \mathbb{Q}$, we get $s + B_z = B_z, \forall s \in \mathbb{Q}$. Using Proposition 8.6 in [6], we know that either $m(B_z) = 0$ or $m(\mathbb{R} \setminus B_z) = 0$, here $m$ is the Lebesgue measure on $\mathbb{R}$. Hence we can define $F = \{z \in \Omega | m(B_z) > 0\} = \{z \in \Omega | m(\mathbb{R} \setminus B_z) = 0\}$. If $z \in F$ and $s \in \mathbb{R}$, then $B_{z+s} = \{r \in \mathbb{R} | \Gamma^r z \in B\} = B_z - s$. Using Proposition 8.6 in [6] again, we see that $m(B_z) > 0$ implies $m(B_{z+s}) > 0$, i.e. $\Gamma^s z \in F$. So $\Gamma^s F = F, \forall s \in \mathbb{R}$.

Let $F$ be measurable and $q(B \Delta F) = 0$. Let $I \subseteq \mathbb{R}$ be a measurable set of positive measure and define

$$f(z) = \frac{1}{m(I)} \int_I \chi_B(\Gamma^s z) dm(s).$$ (3.3)

We see that $f(z) = \chi_F$. Hence by Fubini’s theorem, $F$ is measurable. Noticing that $\chi_B(\Gamma^s z) = \chi_{\Gamma^{-s}B(z)}$, we get

$$q(B \Delta F) = \int_\Omega (\chi_B + \chi_F - 2 \chi_B \chi_F) q(dz) = \frac{1}{m(I)} \int_I \int_\Omega (\chi_B + \chi_{\Gamma^{-s}B} - 2 \chi_B \chi_{\Gamma^{-s}B}) q(dz) m(ds) = \frac{1}{m(I)} \int_I q(B \setminus \Gamma^{-s}B) m(ds) = 0.$$

Now we could give several equivalent way in defining the ergodicity.

Theorem 3.3. Let $(\Omega, q)$ be a GIF, then the following statements are equivalent:
1. $q$ is ergodic.
2. If $\Phi \in \mathcal{B}_\Omega$ with $q(\Gamma^s \Phi \Delta \Phi) = 0, \forall s \in \mathbb{R}$, we have $q(\Phi) = 0$ or 1.
3. If $f$ is $q$–measurable and $f(\Gamma^s) = f(z), \forall s \in \mathbb{R}, \forall z \in \Omega$, then for $q$ almost every $z$, $f$ is constant.

Proof. (1) $\Rightarrow$ (2). If $\Phi \in \mathcal{B}_\Omega$ satisfies $q(\Gamma^s \Phi \Delta \Phi) = 0, \forall s \in \mathbb{R}$, then by theorem 3.2 we get $\Psi \in \mathcal{B}_\Omega$ such that $q(\Phi \Delta \Psi) = 0$ and $\Gamma^s \Psi = \Psi, \forall s \in \mathbb{R}$. By definition 3.1 we know that $q(\Psi) = 0$ or 1, i.e. $q(\Phi) = 0$ or 1.

(2) $\Rightarrow$ (1). If $\Gamma^s \Phi = \Phi, \forall s \in \mathbb{R}$, so $q(\Gamma^s \Phi \Delta \Phi) = 0, \forall s \in \mathbb{R}$. By (2), we have $q(\Phi) = 0$ or 1 which proves the ergodicity.

(1) $\Rightarrow$ (3). We can assume that $f$ is the real-valued function. If $q$ is ergodic, we define

$$Z(k, n) = \{z \in \Omega | \frac{k}{2^n} \leq f(z) < \frac{k+1}{2^n}\} = f^{-1}(\frac{k}{2^n}, \frac{k+1}{2^n}).$$ (3.4)
Proof. For any maximal function $q(3)$ (Maximal ergodic theorem) Theorem 4.1. is measurable. First, we will prove the maximal ergodic theorem.

Then we have

Lemma 4.1. (Finite Vitali covering lemma) Before we prove this theorem, we will introduce the finite Vitali covering lemma on $\mathbb{R}$.

Let $X$ be a compact metric space, $\Omega = X^{[0, +\infty)}$. In our following discussion, we will assume that the map

$$\Gamma : (\Omega \times \mathbb{R}, q \times m) \rightarrow (\Omega, q)$$

$$(z(t), t_0) \rightarrow z(t + t_0)$$

is measurable. First, we will prove the maximal ergodic theorem.

**Theorem 4.1. (Maximal ergodic theorem)** Let $(\Omega, q)$ be a GIF, for $f(x) \in L^1(X)$ and $\alpha > 0$, we define the maximal function

$$f^*(z) = \sup_{T > 0} \frac{1}{T} \int_0^T f(z(t))dt,$$

and the set

$$E^f_\alpha = \{z \in \Omega | f^*(z) > \alpha\},$$

then we have

$$\alpha q(E^f_\alpha) \leq \|f\|_{L^1(X)}.$$

Before we prove this theorem, we will introduce the finite Vitali covering lemma on $\mathbb{R}$.

**Lemma 4.1. (Finite Vitali covering lemma)** For any collection of intervals $I_1 = [a_1, a_1 + l_1], \ldots, I_k = [a_k, a_k + l_k]$ in $\mathbb{R}$, there is a disjoint subcollection $I_{j_1}, \ldots, I_{j_k}$ such that

$$I_1 \cup \cdots \cup I_k \subseteq \bigcup_{m=1}^k [a_{jm} - l_{jm}, a_{jm} + 2l_{jm}].$$

Using finite Vitali covering lemma, we will prove the following lemma.

**Lemma 4.2.** $\forall g \in L^1(\mathbb{R})$ and $\alpha > 0$, define the maximal function

$$g^*(a) = \sup_{T > 0} \frac{1}{T} \int_0^T g(a + t)dt,$$

and the set

$$E^g_\alpha = \{a \in \mathbb{R} | g^*(a) > \alpha\}.$$n

Then we have

$$\alpha |E^g_\alpha| \leq 3 \|g\|_{L^1(\mathbb{R})} = 3 \int_{-\infty}^{+\infty} g(t)dt.$$

**Proof.** For any $a \in E^g_\alpha$, choosing $l(a)$ such that \(\frac{1}{l(a)} \int_0^{l(a)} g(a + t)dt > \alpha\). Since $\mathbb{R}$ is second countable, we can write the set $O = \bigcup_{a \in E^g_\alpha} [a, a + l(a)]$ as a union of countable many sets $O = \bigcup_{i=1}^{+\infty} [a_i, a_i + l(a_i)]$. Fix some $K \geq 1$, and using the covering lemma, we get the subcollection $I_{j_1}, \ldots, I_{j_k}$. Since they are disjoint, it follows that

$$\sum_{j=1}^k \int_{I_{j}} g(t)dt \leq \|g\|_{L^1(\mathbb{R})},$$

where $k \geq 1$.
where the left hand side equals to
\[
\sum_{j=1}^{k} l(a_{ij}) \frac{1}{l(a_{ij})} \int_{0}^{l(a_{ij})} g(a_{ij} + t) dt > \sum_{j=1}^{k} l(a_{ij}) \alpha, \tag{4.9}
\]
by the choice of \(l(a)\). However, since
\[
I_1 \cup \cdots \cup I_K \subseteq \bigcup_{j=1}^{k} [a_{ij} - l(a_{ij}), a_{ij} + 2l(a_{ij})], \tag{4.10}
\]
we have
\[
| I_1 \cup \cdots \cup I_K | \leq 3 \sum_{j=1}^{k} l(a_{ij}). \tag{4.11}
\]
By (4.9) and (4.11), we have
\[
\alpha | I_1 \cup \cdots \cup I_K | \leq 3 \|g\|_{L^1(\mathbb{R})}, \tag{4.12}
\]
for any \(K \geq 1\), which gives
\[
\alpha m(O) \leq 3 \|g\|_{L^1(\mathbb{R})}. \tag{4.13}
\]
Then by \(E_{\alpha} \subseteq O\), we get the lemma.

\[
\square
\]

**Proof of Theorem 4.1.** We define
\[
f_M^*(z) = \sup_{0 \leq T \leq M} \frac{1}{T} \int_{0}^{T} f(z(t)) dt, \tag{4.14}
\]
and the set
\[
E_{\alpha,M}^f = \{z \in \Omega \mid f_M^*(z) > \alpha\}. \tag{4.15}
\]
Given \(J > 0\), let
\[
g(t) = \begin{cases} f(z(t)), & t \in [0, J] \\ 0, & \text{otherwise}. \end{cases} \tag{4.16}
\]
Similarly, we can define
\[
g_M^*(a) = \sup_{0 \leq T \leq M} \frac{1}{T} \int_{0}^{T} g(a + t) dt. \tag{4.17}
\]
If \(0 \leq a \leq J - M\) and \(0 \leq t \leq M\), we have
\[
g(a + t) = f(z(a + t)), \tag{4.18}
\]
so
\[
g_M^*(a) = f_M^*(z(a + t)), \quad 0 \leq a \leq J - M. \tag{4.19}
\]
According to Lemma 4.2 \(\forall z \in \Omega, \alpha > 0\), we have
\[
\alpha | \{a \in [0, J - M] \mid g_M^*(a) > \alpha\} | \leq \alpha | \{a \in [0, J] \mid g_M^*(a) > \alpha\} |
\leq \alpha | \{a \in \mathbb{R} \mid g^*(a) > \alpha\} |
\leq 3 \|g\|_{L^1(\mathbb{R})},
\]
or
\[
\alpha \int_{0}^{J-M} \chi_{E_{\alpha,M}^f}(z(a + t)) da = \alpha | \{a \in [0, J - M] \mid f_M^*(z(a + t)) > \alpha\} |
\leq 3 \int_{0}^{J} f(z(t)) dt.
\]
Letting \( J \to \infty \) gives
\[
\alpha q(E_{\alpha,M}) \leq 3 \| f \|_{L^1(\Omega)}.
\] (4.21)

Finally, letting \( M \to \infty \), we get theorem 4.1.

Recall that we have defined flow \( \Gamma^s \) on \( \Omega \) in section 2:
\[
\Gamma^s : \quad \Omega \longrightarrow \Omega
\]
\[
\text{z}(t) \mapsto z(t+s).
\]

Then we can define an operator \( U_s \) as follows
\[
U_s : \quad L^2_q(\Omega) \longrightarrow L^2_q(\Omega)
\]
\[
f(z(t)) \longrightarrow f(\Gamma^s(z(t))).
\]

Let \( f(x) \in L^2_q(X) \), then
\[
\int_{\Omega} | f(z(t_0)) |^2 q(dz) = \int_{\Omega} | f(z(t_0 + s)) |^2 q(dz).
\] (4.22)

Actually, if \( f(x) \in L^2_q(X) \), then \( f(z(t_0)) \in L^2_q(\Omega) \). Hence \( \{ f(z(t_0)) \mid f(x) \in L^2_q(X), t_0 \in \mathbb{R} \} \) can generate a linear subspace of Hilbert space \( L^2_q(\Omega) \). We will denote the \( L^2_q \)- closure of the subspace by \( H \), this is a Hilbert space. And because
\[
\int_{\Omega} | af(z(t_0)) + bg(z(t_1)) |^2 q(dz) = \int_X | af(x) + bg(x) |^2 \mu(dx)
\]
\[
= \int_{\Omega} | af(z(t_0 + s)) + bg(z_1 + s) |^2 q(dz).
\]

and
\[
\int_{\Omega} | f(z(t_0)) |^2 q(dz) = \lim_{n \to \infty} \int_{\Omega} | f_n(z(t_0)) |^2 q(dz)
\]
\[
= \lim_{n \to \infty} \int_{\Omega} | f_n(z(t_0 + s)) |^2 q(dz)
\]
\[
= \int_{\Omega} | f(z(t_0 + s)) |^2 q(dz),
\]

hence \( \{ U_s \mid s \in (R) \} \) is the unitary operator group on the Hilbert space \( H \). In order to proving the following \( L^1 \) ergodic theorem, we need the mean ergodic theorem:

**Theorem 4.2. (Mean ergodic theorem)** Let \( \{ U_t \mid t \in (R) \} \) be the strong continuous one-parameter unitary operator group on the Hilbert space \( H \). \( P \) is the orthogonal project operator from \( H \) to \( \{ f \in H \mid U_t f = f, t \in \mathbb{R} \} \), then
\[
\frac{1}{T} \int_0^T U_t dt = P.
\] (4.23)

By which, we can prove the \( L^1 \) ergodic theorem

**Theorem 4.3. (\( L^1 \) ergodic theorem)** Let \( (\Omega, \mathcal{F}, \mu) \) be a GIF, then \( \forall f(x) \in L^1(X) \), we have
\[
\frac{1}{T} \int_0^T f(z(t+s)) ds \overset{L^1_q}{\longrightarrow} f'(z),
\] (4.24)

here \( f' \in L^1_q(\Omega) \) and \( \int_{\Omega} f'(z) q(dz) = \int_{\Omega} f'(\Gamma^z) q(dz) \).
Proof of Theorem 4.3. \( \forall g(x) \in L^\infty_q(X) \subseteq L^1_q(X) \), by the mean ergodic theorem, we have

\[
\frac{1}{T} \int_0^T g(z(t+s))ds \xrightarrow{T \to \infty} g'(z) \in L^2_q(\Omega). \tag{4.25}
\]

Because

\[
\| \frac{1}{T} \int_0^T g(z(t+s))ds \|_{L^\infty_q(\Omega)} \leq \| g(x) \|_{L^\infty_q(X)}, \tag{4.26}
\]

so we have

\[
\| A_T^q \cdot x \|_q \leq \| g(x) \|_{L^\infty_q(X)} q(B), \forall B \in \mathcal{B}_Q. \tag{4.27}
\]

Here \( A_T^q = \frac{1}{T} \int_0^T g(z(t+s))ds \). Hence \( \| g' \|_{L^\infty_q(\Omega)} \leq \| g \|_{L^\infty_q(X)} \), i.e. \( g' \in L^\infty_q(\Omega) \). Because \( \| \cdot \|_{L\Omega} \leq \| \cdot \|_{L^q_q} \), we have

\[
\frac{1}{T} \int_0^T g(z(t+s))ds \xrightarrow{T \to \infty} g'(z) \in L^\infty_q(\Omega), \tag{4.28}
\]

so the theorem holds for the dense set \( L^\infty_q(X) \subseteq L^1_q(X) \).

\( \forall f \in L^1_q(X) \), fix \( \varepsilon > 0 \), we can choose \( g \in L^\infty_q(X) \) with \( \| f - g \|_{L^\infty_q(\Omega)} \leq \varepsilon \), then

\[
\| \frac{1}{T} \int_0^T f(z(t+s))ds - \frac{1}{T} \int_0^T g(z(t+s))ds \|_{L^1_q(\Omega)} \leq \varepsilon. \tag{4.29}
\]

We have proved that there exists \( g'(z) \in L^\infty_q(\Omega) \) with

\[
\| \frac{1}{T} \int_0^T g(z(t+s))ds - g'(z) \|_{L^1_q(\Omega)} \leq \varepsilon, \text{ for } T > T_0. \tag{4.30}
\]

Hence we get

\[
\| \frac{1}{T} \int_0^T f(z(t+s))ds - \frac{1}{T} \int_0^T f(z(t+s))ds \|_{L^1_q(\Omega)} \leq 4\varepsilon, \text{ for } T, T' > T_0, \tag{4.31}
\]

which means the ergodic averages form a Cauchy sequence in \( L^1_q(\Omega) \). So they have a limit \( f'(z) \in L^1_q(\Omega) \). Because

\[
\| \frac{1}{T} \int_0^T f(z(t+s))dt - \frac{1}{T} \int_0^T f(z(t))dt \|_{L^1_q(\Omega)} \leq \frac{2}{T} \| f(x) \|_{L^1_q(X)}, \tag{4.32}
\]

we know that \( f'(z) \) satisfies \( \int_Q f'(z)q(dz) = \int_Q f'(\Gamma^z z)q(dz) \).

\[ \square \]

Now we can prove the pointwise ergodic theorem of the GIF

**Theorem 4.4. (Pointwise ergodic theorem)** Let \( (\Omega, q) \) be a GIF, \( \phi(x) \in L^1(X) \). Then for \( q \)-almost all \( z(t) \in \Omega \), we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(z(t))dt = \Phi(z), \tag{4.33}
\]

where \( \Phi \in L^1_q(\Omega) \) and \( \int_Q \Phi(z)q(dz) = \int_X \phi(x)\mu(dx) \).

**Proof of Theorem 4.4.** Assume that \( \phi_0 \in L^\infty(X) \). Then by theorem 4.3 we know that

\[
A_T(\phi_0) = \frac{1}{T} \int_0^T \phi_0(z(t))dt \xrightarrow{T \to \infty} \Phi_0 \in L^1_q(\Omega), \text{ as } T \to \infty, \tag{4.34}
\]

where \( \Phi_0 \) satisfies \( \int_Q \Phi_0(z)q(dz) = \int_Q \Phi_0(\Gamma^z z)q(dz) \).

Then, for given \( \varepsilon > 0 \), choose some \( M \), such that

\[
\| \Phi_0 - A_M(\phi_0) \|_{L^1_q(\Omega)} < \varepsilon^2. \tag{4.35}
\]

Apply the following Maximal ergodic theorem (which is more general than theorem 4.1, but the proof is same) to the function \( \psi(z) = \Phi_0 - A_M(\phi_0) \),
Theorem 4.5. (Maximal ergodic theorem) Let \((\Omega, q)\) be a GIF, for \(f(z) \in L^1_q(\Omega)\) satisfies \(\int_\Omega f(z)q(dz) = \int_\Omega f(t^z)zq(dz) = \int_\Omega f(t)q(dz)\) and \(\alpha > 0\), we define the maximal function

\[
f^*(z) = \sup_{T > 0} \frac{1}{T} \int_0^T f(t^z)dt,
\]

and the set

\[
E^f_\alpha = \{z \in \Omega \mid f^*(z) > \alpha\},
\]

then we have

\[
\alpha q(E^f_\alpha) \lesssim \|f\|_{L^1_q(\Omega)}.
\]

Then we get

\[
eq q(\{z \in \Omega \mid \sup_{T > 0} (A_T(\Phi_0 - A_M(\Phi_0))) (\|\cdot\|_q) > \varepsilon\}) < \varepsilon^2.
\]

Clearly, \(A_T(\Phi_0) = \Phi_0\). And we have

\[
A_T(A_M(\Phi_0)) = \frac{1}{TM} \int_0^T \int_0^M \phi_0(\zeta(t + s))dtds = A_T(\Phi_0) + O_M(\frac{\|\phi_0\|_\infty}{T}).
\]

If \(M\) is fixed and \(T \to \infty\), we see that

\[
q(\{z \in \Omega \mid \limsup_{T \to \infty} |\Phi_0 - A_T(\Phi_0) - (\Phi_0)| > \varepsilon\}) = q(\{z \in \Omega \mid \limsup_{T \to \infty} |\Phi_0 - A_M(\Phi_0)| (\|\cdot\|_q) > \varepsilon\})
\]

\[
\leq q(\{z \in \Omega \mid \sup_{T \to \infty} (A_T(\Phi_0 - A_M(\Phi_0))) (\|\cdot\|_q) > \varepsilon\}) < \varepsilon,
\]

which shows that \(A_T(\Phi_0) \to \Phi_0\) for \(q\)-almost all \(z\).

\[
\forall \phi \in L^1_q(\Omega), \text{ fix } \varepsilon > 0 \text{ and choose } \phi_0 \in L^\infty_q(\Omega) \text{ with } \|\phi - \phi_0\|_{L^\infty_q(\Omega)} \leq \varepsilon^2. \text{ Let } \Phi \in L^1_q(\Omega) \text{ be the } L^1_q-\text{limit of } A_T(\phi) = \frac{1}{T} \int_0^T \phi(z(t))dt \text{ and } \Phi_0 \in L^1_q(\Omega) \text{ be the } L^1_q-\text{limit of } A_T(\phi_0) = \frac{1}{T} \int_0^T \phi_0(z(t))dt. \text{ Because } \|A_T(\phi) - A_T(\phi_0)\|_{L^1_q(\Omega)} \leq \|\phi - \phi_0\|_{L^1_q(\Omega)}, \text{ we have } \|\Phi - \Phi_0\|_{L^1_q(\Omega)} \leq \varepsilon^2. \text{ From this and the maximal ergodic theorem [4.1] we get}
\]

\[
q(\{z \in \Omega \mid \limsup_{T \to \infty} |\Phi - A_T(\phi)| (\|\cdot\|_q) > 2\varepsilon\})
\]

\[
\leq q(\{z \in \Omega \mid (|\Phi - \Phi_0| + \limsup_{T \to \infty} |\Phi_0 - A_T(\phi_0)| + \sup_{T \to \infty} |A_T(\phi - \phi_0)|) > 2\varepsilon\})
\]

\[
\leq q(\{z \in \Omega \mid (|\Phi - \Phi_0| > \varepsilon)\} + q(\{z \in \Omega \mid \sup_{T \to \infty} (A_T(\phi_0 - \phi_0))) (\|\cdot\|_q) > \varepsilon\})
\]

\[
\leq \varepsilon^{-1}\|\Phi - \Phi_0\|_{L^1_q(\Omega)} + \varepsilon^{-1}\|\phi - \phi_0\|_{L^1_q(\Omega)} \leq 2\varepsilon,
\]

which shows that \(A_T(\phi) \to \Phi\) for \(q\)-almost all \(z\). It is clearly \(\int_\Omega \Phi(z)q(dz) = \int_\Omega \Phi(x)\mu(dz)\) by the Fubini’s theorem.

\[
\square
\]

5 More on the Generalized incompressible flow

Let \(X\) be a compact metric space. Although \(\Omega = X^\mathbb{R}\) is compact with respect to the product topology, i.e. the pointwise convergence topology, we cannot find a metric compatible to this topology. On the other hand, we may define a metric on \(\Omega\) as follows

\[
d(z(t), z'(t)) = \begin{cases} \sup_{t \in \mathbb{R}} |z(t) - z'(t)|, & \text{if } |z(t) - z'(t)| < 1, \forall t, \\ 1, & \text{otherwise.} \end{cases}
\]

(5.1)
Hence, we get a uniformly convergence topology, and $\Omega$ is a metric space. Unfortunately, we cannot make $\Omega$ compact and metrizable at the same time. Fortunately, because the product space of Hausdorff spaces is still Hausdorff with respect to the product topology, so we can get a compact Hausdorff space.

Then using the Riesz representation theorem on compact Hausdorff space and the Krylov and Bogolioubov theorem we can prove that

**Theorem 5.1.** The set of all GIFs, denoted by $GIF(\Omega)$, is non-empty.

**Proof.** Actually, map $\Gamma^s : \Omega \rightarrow \Omega$ introduces a map on the set of generalized flows $GF(\Omega)$,

$$ (\Gamma^s)_* : GF(\Omega) \rightarrow GF(\Omega) \quad q \mapsto (\Gamma^s)_* q, $$

here $(\Gamma^s)_* q(B) = q((\Gamma^s)^{-1} B), \forall B \in \mathcal{B}_\Omega$. If $(\omega_n)_{n=1}^{\infty}$ is a sequence of GF, we can construct a new sequence $[q_n]_{n=1}^{\infty}$ by

$$ q_n = \frac{1}{n} \int_0^n (\Gamma^s)_* \omega_n ds. \quad (5.2) $$

By the weak−$^*$ compactness of GF, there exists $q$ such that a subsequence $q_{n_j}$ weak−$^*$ converge to $q$. we can prove that $q \in GIF(\Omega)$, even more, $q$ is an invariant measure of $\Gamma^s$ on $\Omega$, which is a smaller subset of $GIF(\Omega)$ as we mentioned before.

$$ \lim_{j \to \infty} \int_{\Omega} f(\Gamma^s z) q(dz) = \int_{\Omega} f(z) q(dz), $$

so we have $\int_{\Omega} f(\Gamma^s z) q(dz) = \int_{\Omega} f(z) q(dz)$, i.e. $q \in GIF(\Omega)$.

Before continuing our discussion, we define the weak ergodicity

**Definition 5.1.** (Weak Ergodic GIF) Let $(\Omega, q)$ be a GIF, for $E \in \mathcal{B}_X$, define $\Phi = \{z \in \Omega \mid z(0) \in E\}$. We call $q$ is weak ergodic, if $\Gamma^s \Phi = \Phi$, then we have $q(\Phi) = 0$ or 1.

In section 3, we have proved that the GIF $q$ defined from a classical ergodic flow is ergodic. Actually, we can prove that all the ergodic GIF can be defined in a similar way.

**Theorem 5.2.** If the GIF $q$ is ergodic, then $q$− almost all $z \in \Omega$ is ergodic.

**Proof.** Let $z \in \Omega$, we can see that $\Psi = \{\Gamma^s z \mid s \in \mathbb{R}\}$ is invariant under $\Gamma^s$. Hence by the ergodicity, $q(\Psi) = 0$ or 1. If $q(\Psi) = 1$, i.e. $q$ is supported on $\Psi$, $\forall z \in \Omega$ define a flow $T^t$ on $\{z(t) \mid t \in \mathbb{R}\} \subseteq X$, here $T^t : z(t_0) \rightarrow z(t_0 + t)$.

Let $B \subseteq X$ be an invariant set of $T^t$. Define $\Phi = \{z \in \Omega \mid z(0) \in B\}$. Because $B$ is invariant under $T^t$, we get $\Gamma^s \Phi = \Phi, \forall s \in \mathbb{R}$, so $q(\Phi) = 0$ or 1. Then by the incompressible condition, we have

$$ \mu(B) = \int_X \chi_B \mu(dx) = \int_\Omega \chi_B(z(0)) q(dz) = \int_\Omega \chi_\Phi(z) q(dz) = q(\Phi) = 0 \text{ or } 1, $$

which shows that the flow $z(t)$ is ergodic.
Remark 5.1. (a) choose $z(t) \in \text{supp } q$, by Theorem 5.1, $z(t)$ is ergodic. So $T^t$ is defined on $X$ except a zero measure set. Hence
\[
\int_{\Omega} f(z(t)) q(dz) = \int_{X} f(x \mapsto T^t(x)) \mu(dx), \forall f(x) \in C(X),
\]
which is similar to (5.1).

(b) Weak ergodic GIFs is a larger class than ergodic GIFs. We can see that for ergodic GIF, it can only support on $\{z(t+s) \mid s \in \mathbb{R}\}$, here $z$ is a classical ergodic flow. However, for weak ergodic GIF, it can support on $\{x(t+s) \mid s \in \mathbb{R}\} \cup \{y(t+s) \mid s \in \mathbb{R}\}$, if $x(t)$ and $y(t)$ are ergodic flows with respect to the same measure $\mu$ on $X$.

Before giving a theorem about some properties of $GIF(\Omega)$, we prove a useful lemma

Lemma 5.1. Let $(\Omega, q)$ be a GIF, $\forall E \in \mathcal{B}_X, \Phi = \{z \in \Omega \mid z(0) \in E\}$, then $q(\Gamma^{-s}\Phi) = q(\Phi)$.

Proof. we have
\[
q(\Gamma^{-s}\Phi) = \int_{\Omega} \chi_{\Gamma^{-s}\Phi}(z) q(dz) \\
= \int_{\Omega} \chi_E(z(s)) q(dz) \\
= \int_{\Omega} \chi_E(z(0)) q(dz) \\
= \int_{\Omega} \chi_{\Phi}(z) q(dz) = q(\Phi),
\]
which proves the lemma.

Now we will study some properties of the set $GIF(\Omega)$.

Theorem 5.3. (1) $GIF(\Omega)$ is a compact subset of $GF(\Omega)$.
(2) $GIF(\Omega)$ is convex.
(3) $q$ is an extreme point of $GIF(\Omega)$ iff $q$ is a weak ergodic GIF.

Proof.(1) Suppose $\{q_n\}_1^\infty$ is a sequence of members of $GIF(\Omega)$ and $q_n \rightarrow q$ in $GF(\Omega)$. Then
\[
\int_{\Omega} f(z(t_0)) q(dz) = \lim_{n \rightarrow \infty} \int_{\Omega} f(z(t_0)) q_n(dz) \\
= \lim_{n \rightarrow \infty} \int_{\Omega} f(z(0)) q_n(dz) \\
= \int_{\Omega} f(z(0)) q(dz),
\]
we have $q \in GIF(\Omega)$. Hence $GIF(\Omega)$ is a closed subset of compact set $GF(\Omega)$. So it must be compact.

(2) It is clear that if $q_1, q_2 \in GIF(\Omega)$, then $(1-s)q_1 + sq_2$ is a GIF.

(3) ($\Rightarrow$) Suppose $q \in GIF(\Omega)$ and it is not weak ergodic, then there exists a Borel set $E \subset X$. We have $\Phi = \{z \in \Omega \mid z(0) \in E\}$ such that $\Gamma^{-s}\Phi = \Phi, \forall s \in \mathbb{R}$ with $0 < q(\Phi) < 1$.
Define $q_1$ and $q_2$ by
\[
q_1(B) = \frac{q(B \cap \Phi)}{q(\Phi)}, \quad (5.4)
\]
\[
q_2(B) = \frac{q(B \cap (\Omega \setminus \Phi))}{q(\Phi)}. \quad (5.5)
\]
Next we prove that $q_1, q_2 \in GIF(\Omega)$. Let $Q \subset X$, we only need to prove
\begin{align*}
\int_{\Omega} \chi_Q(z(t_0))q_1(dz) &= \int_{\Omega} \chi_Q(z(0))q_1(dz), \quad (5.6) \\
\int_{\Omega} \chi_Q(z(t_0))q_2(dz) &= \int_{\Omega} \chi_Q(z(0))q_2(dz). \quad (5.7)
\end{align*}
Also, notice that
\begin{equation}
q_1(\Psi(t_0)) = \int_{\Omega} \chi_Q(z(t_0))q_1(dz) = \int_{\Omega} \chi_Q(z(0))q_1(dz), \quad (5.8)
\end{equation}
here $\Psi(t_0) = \{z \in \Omega \mid z(t_0) \in Q\}$.
Because $\Gamma^{-t}\Phi = \Phi$, we get
\begin{equation}
\Phi = \Phi(t_0) = \{z \in \Omega \mid z(t_0) \in E\}, \forall t_0 \in \mathbb{R}. \quad (5.9)
\end{equation}
Hence,
\begin{equation}
q_1(\Psi(t_0)) = \frac{q(\Psi(t_0) \cap \Phi)}{q(\Phi)} = \frac{q(\Psi(t_0) \cap \Phi(t_0))}{q(\Phi)}. \quad (5.10)
\end{equation}
Because $\Psi(t_0) \cap \Phi(t_0) = \{z \in \Omega \mid z(t_0) \in Q \cap E = \Gamma^{-t}(\Psi(0) \cap \Phi(0))$, and by lemma 5.1 we have
\begin{equation}
\frac{q(\Psi(t_0) \cap \Phi(t_0))}{q(\Phi)} = \frac{q(\Psi(0) \cap \Phi(0))}{q(\Phi)} = q(\Psi(0)). \quad (5.11)
\end{equation}
Combining (5.8), (5.10), (5.11), we get (5.6) which proves $q_1 \in GIF(\Omega)$. Similarly, we can prove $q_2 \in GIF(\Omega)$. So we have $q_1 \neq q_2 \in GIF(\Omega)$ such that
\begin{equation}
q(B) = q(\Phi)q_1(B) + (1 - q(\Phi))q_2(B). \quad (5.12)
\end{equation}
($\Leftarrow$) Suppose GIF $q$ is weak ergodic, and
\begin{equation}
q(B) = pq_1(B) + (1 - p)q_2(B), \quad (5.13)
\end{equation}
here $q_1, q_2 \in GIF(\Omega)$ and $0 < p < 1$. We will prove $q_1 = q_2$. As we showed in remark 2.1, we can find $\mu_1, \mu_2$ such that
\begin{equation}
\int_{\Omega} f(z(t_0))q_1(dz) = \int_{\Omega} f(z(0))q_1(dz) = \int_{\chi} f(x)\mu_i(dx), \forall f(x) \in C(X), \forall t_0 \in \mathbb{R}, i = 1, 2. \quad (5.14)
\end{equation}
Clearly, $q_1 \ll q$ and $\mu_1 \ll \mu$. Let $E \in \mathcal{B}_X$, $\Phi = \{z \in \Omega \mid z(0) \in E\}$. We have
\begin{align*}
q_1(\Phi) &= \mu_1(E) = \int_{E} \frac{d\mu_1}{d\mu}(x)\mu(dx) \\
&= \int_{\Omega} \chi_E(z(0))\frac{d\mu_1}{d\mu}(z(0))q(dz) \\
&= \int_{\Omega} \chi_\Phi(z)\frac{d\mu_1}{d\mu}(z(0))q(dz) \\
&= \int_{\Phi} \frac{d\mu_1}{d\mu}(z(0))q(dz) = \int_{\Phi} \frac{dq_1}{dq}(z)q(dz).
\end{align*}
By Radon-Nikodym theorem, $\frac{d\mu_1}{dq}(z(0)) = \frac{dq_1}{dq}(z)$, $q$– almost everywhere on $\Phi$. Let $E = \{x \mid \frac{d\mu_1}{dq}(x) < 1\}$, then $\Phi = \{z \in \Omega \mid z(0) \in E\} = \{z \in \Omega \mid \frac{d\mu_1}{dq}(z) < 1\}$. We have
\begin{align*}
\int_{\Phi \cap \Gamma^{-t}\Phi} \frac{dq_1}{dq}(z)q(dz) + \int_{\Phi \cap \Gamma^{-t}\Phi} \frac{dq_1}{dq}(z)q(dz) \\
= q_1(\Phi) = q_1(\Gamma^{-t}\Phi) \\
= \int_{\Phi \cap \Gamma^{-t}\Phi} \frac{dq_1}{dq}(z)q(dz) + \int_{\Gamma^{-t}\Phi \cap \Phi} \frac{dq_1}{dq}(z)q(dz),
\end{align*}
here we used lemma 5.1. Since \( \frac{dq}{dq} < 1 \) on \( \Phi \setminus \Gamma^{\le} \Phi \) and \( \frac{dq}{dq} \ge 1 \) on \( \Gamma^{\le} \Phi \setminus \Phi \), also we have

\[
q(\Gamma^{\le} \Phi \setminus \Phi) = q(\Gamma^{\le} \Phi) - q(\Phi \cap \Gamma^{\le} \Phi) = q(\Phi) - q(\Phi \cap \Gamma^{\le} \Phi) = q(\Phi \setminus \Gamma^{\le} \Phi),
\]

so \( q(\Gamma^{\le} \Phi \setminus \Phi) = q(\Phi \setminus \Gamma^{\le} \Phi) = 0 \), i.e. \( q(\Gamma^{\le} \Phi \Delta \Phi) = 0 \). Therefore, by the weak ergodicity of \( q \), we know \( q(\Phi) = 0 \) or 1. But if \( q(\Phi) = 1 \), then \( q(\Omega) = \int_{\Phi} \frac{dq}{dq}(z)q(dz) < q(\Phi) = 1 \), this is contradict to \( q(\Omega) = 1 \), hence \( q(\Phi) = 0 \).

Similarly, we can prove that \( q(\{ z \in \Omega \mid \frac{dq}{dq}(z) > 1 \}) = 0 \). So \( q \) is almost everywhere we have \( q_1 = q \). Therefore we prove that \( q \) is an extreme point of \( GIF(\Omega) \).

\[\square\]

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