On the QCD result for the hyperfine splitting $M_{Y(1S)} - M_{\eta_b}$ and the value of $\alpha_s$

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Abstract

The measurement of the $\eta_b$ mass, together with a QCD result for the hyperfine splitting $E_{HFS} = M_{Y(1S)} - M_{\eta_b}$, allows us to determine the strong coupling constant $\alpha_s$ at a low energy scale. The result

$$\alpha_s(M_{Y(1S)}) = 0.197 \pm 0.002 |\Delta E_{HFS}^{exp} \pm 0.002|_{scheme} \pm 0.002 |\delta_{G^2} \pm 0.006|_{\delta m_b} \pm 0.005 |_{ho},$$

$$\alpha_s(M_{Z^0}) = 0.124 \pm 0.001 |\Delta E_{HFS}^{exp} \pm 0.001|_{scheme} \pm 0.001 |\delta_{G^2} \pm 0.003|_{\delta m_b} \pm 0.002 |_{ho}$$

is compatible with the current world average of $\alpha_s$ reported by the Particle Data Group, and shows that the experimental lowest-lying $\bar{b}b$ hyperfine splitting can be reproduced in terms of a perturbative and nonperturbative QCD contribution.

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The observation of the $\eta_b$ by the BaBar \cite{1, 2} and CLEO Collaborations \cite{3} comes after three decades of searches of the lightest pseudoscalar $b\bar{b}$ meson. The spin-singlet state $\eta_b$ has been detected in the radiative $\Upsilon(3S) \rightarrow \eta_b\gamma$ and $\Upsilon(2S) \rightarrow \eta_b\gamma$ decay modes, studying the spectrum of the final photon. The measured mass reported by the BaBar Collaboration is

$$M_{\eta_b} = 9388.9^{+3.1}_{-2.3} \text{ (stat)} \pm 2.7 \text{ (syst)} \text{ MeV}$$ \hspace{1cm} (1)

from $\Upsilon(3S) \rightarrow \eta_b\gamma$ \cite{1}, and

$$M_{\eta_b} = 9394.2^{+4.6}_{-4.8} \text{ (stat)} \pm 2.0 \text{ (syst)} \text{ MeV}$$ \hspace{1cm} (2)

from $\Upsilon(2S) \rightarrow \eta_b\gamma$ \cite{2}.

The signal of $\eta_b$ in the $\Upsilon(3S) \rightarrow \eta_b\gamma$ radiative decay has been confirmed by the CLEO Collaboration, which quotes \cite{3}

$$M_{\eta_b} = 9391.8 \pm 6.6 \text{ (stat)} \pm 2.0 \text{ (syst)} \text{ MeV} \ .$$ \hspace{1cm} (3)

The three mass measurements, combining in quadrature the statistic and systematic uncertainties, produce the average value \cite{4}

$$M_{\eta_b} = 9390.9 \pm 2.8 \text{ MeV}$$ \hspace{1cm} (4)

which is a remarkable result, since it provides us with a measurement of the hyperfine splitting (HFS) of the lowest-lying $b\bar{b}$ doublet,

$$E_{HFS}^{\text{exp}} = M_{\Upsilon(1S)} - M_{\eta_b} = 69.3 \pm 2.8 \text{ MeV} \ ,$$ \hspace{1cm} (5)

where we used the Particle Data Group value $M_{\Upsilon(1S)} = 9460.30 \pm 0.26 \text{ MeV}$ for the mass of $\Upsilon(1S)$ \cite{4}.

The experimental result \cite{5} can be compared to the predictions of quark models \cite{5}, lattice QCD \cite{6}, and QCD sum rules \cite{7}. Moreover, it is particularly interesting since it can be compared to the result of evaluations based on perturbative QCD with the inclusion of the leading nonperturbative contribution, an expression involving fundamental QCD parameters such as the strong coupling constant $\alpha_s$ at a low energy scale. The obtained value of $\alpha_s$ can be compared to other determinations, and considered when the average is carried out. In this way, one can also investigate if there is room, in the experimental result, for contributions not related to QCD, such as that from the mixing effect envisaged in \cite{8} under the assumption

2
of the existence of a light CP-odd pseudoscalar Higgs. The determination of $\alpha_s$ from the result (5) is the purpose of the present study.

As recognized since the early studies of quantum chromodynamics applied to mesons comprising heavy quarks [9], a quantitative description in QCD of the $Q\bar{Q}$ bound state is possible if the average distance between the quark pair is smaller than the typical QCD length scale $r \simeq 1/\Lambda_{QCD}$; the description is given in terms of a perturbative QCD expression and nonperturbative corrections.

The perturbative contribution to HFS comes from diagrams with external heavy quark-antiquark lines and the exchange of gluons and light-quark loops. At the leading order in the $\alpha_s$ expansion, such diagrams produce, in the static limit, the Coulombic $Q\bar{Q}$ potential

$$V_{QQ}^{(0)} = -C_F \frac{\alpha_s}{r}$$

with $C_F = (N_c^2 - 1)/(2N_c)$ the eigenvalue of the quadratic Casimir operator of the fundamental representation of the group $SU(N_c)$, $N_c$ being the number of colors. The expression of the potential at one- [10] and two-loop [11] orders has been recently enlarged to three loops [12]; the meson masses are obtained as energy levels of a Schrödinger equation.

At the leading order in $\alpha_s$, the perturbative contribution to the hyperfine splitting is proportional to the beauty quark mass and to the fourth power of $\alpha_s(\mu)$:

$$E_{HFS}^{LO} = \frac{C_F^4\alpha_s^4(\mu)m_b}{3}.$$  \hspace{1cm} (7)

In this expression, the dependence of $\alpha_s$ on the renormalization scale $\mu$ requires a proper choice of this parameter in order to apply the formula to the physical case. A milder $\mu$ dependence can be achieved including higher order corrections. The $O(\alpha_s)$ corrections to the $E_{HFS}^{LO}$ leading order term have been computed in Refs. [13, 14], and the result includes a logarithmically enhanced $\alpha_s \log \alpha_s$ term. Such kinds of terms can be resummed to all orders through a renormalization group analysis carried out in the framework of the (potential) nonrelativistic QCD effective theory, and indeed in Ref. [15] a Next-to-Leading Log (NLL) expression of the hyperfine splitting has been derived which includes a resummation of terms of the form $\alpha_s^n \log^{n-1} \alpha_s$, with $\alpha_s$ renormalized in the $\overline{\text{MS}}$ scheme. A discussion can be found in [16].

In the following we use the formula for $E_{HFS}^{NLL}$ in [15] (corrected in version 2 of the preprint...
in the arXiv), together with the expression of $\alpha_s$ to four loops \[17\]:

$$\alpha_s^{(4)}(\mu) = \frac{1}{\beta_0 L} \left\{ 1 - \frac{\beta_1}{\beta_0} \ln L + \frac{1}{\beta_0^2} \left[ \frac{\beta_1^2}{\beta_0} (\ln^2 L - \ln L - 1) + \frac{\beta_2}{\beta_0} \right] + \frac{1}{\beta_0^3 L^2} \left[ \frac{\beta_1^3}{\beta_0^2} \left( - \ln^3 L + \frac{5}{2} \ln^2 L + \ln L - \frac{1}{2} \right) - \frac{3\beta_1 \beta_2}{\beta_0^2} \ln L + \frac{\beta_3}{2\beta_0} \right] \right\}, \quad (8)$$

where $L = \ln (\mu^2/\Lambda^2)$ and $\beta_i$ are given by \[18\]

$$\beta_0 = \frac{1}{4\pi} \left[ \frac{11}{3} - \frac{2}{3} n_f \right]$$

$$\beta_1 = \frac{1}{(4\pi)^2} \left[ 102 - \frac{38}{3} n_f \right]$$

$$\beta_2 = \frac{1}{(4\pi)^3} \left[ \frac{2857}{18} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right]$$

$$\beta_3 = \frac{1}{(4\pi)^4} \left[ \left( \frac{149753}{6} + 3564 \zeta_3 \right) - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f \right.$$  

$$+ \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 + \frac{1093}{729} n_f^3 \right]; \quad (9)$$

$n_f$ is the number of active flavors ($n_f = 4$ in the case of the $\bar{b}b$ system) and $\zeta_3 = \zeta(3)$. The renormalization group improved expression of $E_{HFS}^{NLL}$ involves the beauty quark mass $m_b$ as an overall factor. It also involves the strong coupling $\alpha_s$ evaluated at a low energy renormalization scale $\mu$ and at a matching scale $m_b$, as well as on the QCD parameter $\Lambda$. The dependence on the two different scales $\mu$ and $m_b$ allows us, through an error analysis, to bound the value of $\Lambda$.

For a heavy $Q\bar{Q}$ pair the nonperturbative contribution to the hyperfine splitting is related to the dynamics of the colored quarks in the gluon background. If the size of the quarkonium system is smaller than the fluctuations of the background gluon field, this background field can be considered homogeneous and constant, and parametrized by gluon condensates. The lowest dimensional nonperturbative contribution to the hyperfine splitting of the lightest $\bar{b}b$ doublet has been evaluated in \[19, 20\] and involves the dimension-four gluon condensate,
\[ <G^2> = \langle 0 | \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{\mu\nu a} | 0 \rangle : \]

\[ E_{HFS}^{NP} = \frac{m_b}{3} \left( C_F^4 \alpha_s^3 \tilde{\alpha}_s^3 \right) \frac{18.3 \pi^2 < G^2 >}{m_b^4 (C_F \alpha_s)^6}. \]  

(10)

In this expression the coupling \( \tilde{\alpha}_s \) is defined as

\[ \tilde{\alpha}_s(\mu) = \alpha_s(\mu) \left\{ 1 + \left( a_1 + \gamma_E \frac{\tilde{\beta}_0}{2} \right) \frac{\alpha_s(\mu)}{\pi} + \right. \]

\[ \left. \left( \gamma_E \left( a_1 \tilde{\beta}_0 + \frac{\tilde{\beta}_1}{8} \right) + \left( \frac{\pi^2}{12} + \gamma_E \right) \frac{\tilde{\beta}_0^2}{4} + a_2 \right) \frac{\alpha_s^2(\mu)}{\pi^2} \right\}, \]  

(11)

with \( \tilde{\beta}_i = (4\pi)^{i+1} \tilde{\beta}_i \), since the effective Coulombic potential \( V_{eff} = -C_F \alpha_s \tilde{\beta}_0 \) has been considered with the inclusion of two-loop corrections [13, 21]. The parameters \( a_1 \) and \( a_2 \) are given by

\[ a_1 = \frac{31 C_A - 20 T_F n_f}{36} \]

\[ a_2 = \frac{1}{16} \left\{ \left[ \frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3} \zeta(3) \right] C_A^2 \right. \]

\[ - \left[ \frac{1798}{81} + \frac{56}{3} \zeta(3) \right] C_A T_F n_f \]

\[ - \left[ \frac{55}{3} - 16 \zeta(3) \right] C_F T_F n_f + \frac{400}{81} T_F^2 n_f^2 \} \],  

(12)

with \( \gamma_E \) the Euler constant, \( C_A = N_c \) the eigenvalue of the quadratic Casimir operator of the adjoint representation of \( SU(N_c) \), and \( T_F = \frac{1}{2} \). This contribution must be added to the perturbative one, and represents the first term, in the vacuum condensate expansion, of a series involving condensates of higher dimension and higher powers of the inverse heavy quark mass [22].

In the theoretical expression of \( E_{HFS} \) the coupling constant \( \alpha_s \) appears both in the perturbative and nonperturbative terms, and the formula

\[ E_{HFS} = E_{HFS}^{NLL} + E_{HFS}^{NP} \]  

(13)

involves the factor \( m_b \), the renormalization and the matching scales, together with the QCD parameter \( \Lambda(\alpha_s, 4) \), the QCD scale in our problem with four active flavors. We fix the
gluon condensate to the commonly accepted value
\[ \langle 0| \frac{\alpha_s}{\pi} G_{\mu\nu} G^{\mu\nu} |0 \rangle = (0.012 \pm 0.004) \text{ GeV}^4 \]
and we include the uncertainty on the value of \( m_b \) in the denominator of \( \langle \rangle \) in the uncertainty of the condensate. In the same uncertainty, which is of about 33\%, we can also include the effect of a possible difference in the scale of \( \alpha_s \) in the non perturbative contribution with respect to the perturbative one. At odds with other analyses, we keep the nonperturbative contribution using its face value, instead of, e.g., fixing it from the charmonium hyperfine splitting and then rescaling to the bottomonium case [15].

Proceeding in the numerical analysis, we divide \( E_{HFS}^{exp} \) and \( E_{HFS} \) by \( M_\Upsilon/2 \), obtaining \( \tilde{E}_{HFS}^{exp} \) and \( \tilde{E}_{HFS} \). In the QCD expression, this amounts to dividing by the mass \( m_b \) defined in the 1S bottom quark mass scheme. The change of the scheme induces a higher order \( \alpha_s \) correction in the theoretical formula, which is beyond the chosen level of accuracy; in \( \tilde{E}_{HFS} \) the remaining \( m_b \) dependence is only logarithmic, and encodes the dependence of the matching scale. Hence, the procedure is to evolve \( \alpha_s \) from \( M_{Z^0} \) to \( m_b \) with four-loop accuracy, and below \( m_b \) to use \( \alpha_s \) according to the logarithmic precision of the expression for \( E_{HFS} \). To check the uncertainty in this procedure, we have also used full four-loop accuracy for \( \alpha_s \) at all scales; the difference in the numerical result is included in the final error budget.

We define the function
\[ \xi(\Lambda, \mu, m_b) = \frac{(\tilde{E}_{HFS}^{exp} - \tilde{E}_{HFS}^{NLL} - \tilde{E}_{HFS}^{NP})^2}{(\Delta E_{HFS}^{exp})^2} \]
where \( \lambda = \Lambda^{(n_f=4)} \), and search its minima in the space of parameters \( \mu, \lambda \) and \( m_b \). To fix the range of \( \lambda \) allowed by the comparison between the experimental datum and the theoretical expression we proceed in the following way. The minima of \( \xi(\lambda, \mu, m_b) \) determine an implicit relation between its variables. Once we have obtained a set of three values \( (\lambda^*, \mu^*, m_b^*) \) corresponding to the minimum of \( \xi \), we fix \( m_b \) to the value \( m_b^* \) and study \( \xi(\lambda, \mu, m_b^*) \) as a function of \( \lambda \) and \( \mu \) (the central band depicted in Fig.1). The procedure is repeated using in the expression of \( \xi \), as the experimental datum, the values \( \tilde{E}_{HFS}^{exp} - \Delta \tilde{E}_{HFS}^{exp} \) and \( \tilde{E}_{HFS}^{exp} + \Delta \tilde{E}_{HFS}^{exp} \) respectively, obtaining the left and right bands in Fig. 1 hence bounding the parameters \( \lambda \) and \( \mu \) to a region of the parameter plane depicted in Fig.1.

Along the curves of minima, there are ranges of \( \lambda^{(n_f=4)} \) where the dependence on the renormalization scale \( \mu \) is minimized. We bound \( \lambda^{(n_f=4)} \) in these ranges, imposing that the condition \( \frac{\partial \lambda^{(n_f=4)}}{\partial \mu} = 0 \) is satisfied: the corresponding band is depicted in Fig.1. In this way we bound \( \mu \) within \( 1800 \text{ MeV} \leq \mu \leq 2200 \text{ MeV} \) (slightly larger values than the scale
Figure 1: Correlation between the renormalization scale $\mu$ and $\Lambda^{(n_f=4)}$ from the $\bar{b}b$ hyperfine splitting at a fixed matching scale. The central (blue) curve refers to the central value of $\hat{E}_{HFS}^{\exp}$, the left and right (gray) curves to $\hat{E}_{HFS}^{\exp} - \Delta \hat{E}_{HFS}^{\exp}$ and $\hat{E}_{HFS}^{\exp} + \Delta \hat{E}_{HFS}^{\exp}$, respectively, and the matching scales are $m_b = 4724$, 4746 and 4691 MeV, respectively. To minimize the dependence of $\Lambda^{(n_f=4)}$ on $\mu$, a vertical band is found through the condition $\frac{\partial \Lambda^{(n_f=4)}}{\partial \mu} = 0$.

$\mu \simeq 1500$ MeV chosen in [15] to compute the central value of $E_{HFS}^{NLL}$, and we obtain for $\Lambda^{(n_f=4)}$,

$$\Lambda^{(n_f=4)} = 398^{+12}_{-13} \text{ MeV} \ .$$

(15)

At $\mu = 2000$ MeV the values of $\Lambda^{(n_f=4)}$ where the function $\xi$ vanishes are depicted in Fig.2. In this region of the parameter space, the perturbative contribution to the hyperfine splitting amounts to $E_{HFS}^{NLL} = 65.84$ MeV, while the nonperturbative contribution is $E_{HFS}^{NP} = 3.58$ MeV; therefore, for the lowest-lying beauty doublet the splitting is mainly of perturbative origin. It is interesting to consider the case where the non perturbative term is forced to be zero. In this condition, the experimental value of $E_{HFS}^{\exp}$ is reproduced through a larger value of the QCD parameter: $\Lambda^{(n_f=4)} = 414^{+10}_{-13}$ MeV, so that an uncertainty of 16 MeV can be attributed to $\Lambda^{(n_f=4)}$ from the $D = 4$ gluon condensate. Moreover, using four-loop accuracy in $\alpha_s$ at all scales or the procedure of using $\alpha_s$ with different accuracies described above induces an error of $\pm 20$ MeV on $\Lambda^{(n_f=4)}$, quoted as $\Delta \Lambda^{(n_f=4)}|_{\text{scheme}}$ in the final result.

In the $m_b - \Lambda$ parameter plane, a simple correlation is found between the parameters, which can be easily understood due to the logarithmic dependence of $E_{HFS}^{NLL}$ on $m_b/\Lambda$. The uncertainty on $\Lambda$ is linked to the variation of the matching scale, which cannot be sensibly larger than $M_\Upsilon/2$. A variation of 250 MeV of this scale induces a shift of $\Lambda^{(n_f=4)}$ of about 40 MeV, an uncertainty dominating the error of $\Lambda^{(n_f=4)}$. The central value of the result for $\Lambda^{(n_f=4)}$ corresponds to $m_b = M_\Upsilon/2$. 

7
The last source of uncertainty comes from the neglect of (uncalculated) higher order contributions to $E_{\text{HFS}}$. The size of these contributions has been estimated considering the difference $E_{\text{HFS}}^{\text{NLL}} - E_{\text{HFS}}^{\text{LL}}$, with the conclusion that it is about 20.5\% of the central value of $E_{\text{HFS}}^{\text{NLL}}$ \cite{15}. This effect produces an uncertainty, quoted as ho (higher orders), of ±33 MeV to $\Lambda^{(n_f=4)}$. To be conservative, we include this uncertainty in the final error, even though some higher order effects (for example in the accuracy of $\alpha_s$) have been considered separately.

Following all the steps in the outlined procedure, we obtain a result for $\Lambda^{(n_f=4)}$ from the experimental $\bar{b}b$ hyperfine splitting:

$$\Lambda^{(n_f=4)} = 398^{+12}_{-13} \left| E_{\text{HFS}}^{\text{exp}} \right|_{\text{scheme}} \pm 16 \left| \delta^{<G^2>} \right|_{\delta m_b} \pm 33 \left| \delta m_b \right|_{\text{ho}} \text{ MeV} . \quad (16)$$

With the value of $\Lambda^{(n_f=4)}$ in (16) it is possible to evolve $\alpha_s$ to the $\Upsilon(1S)$ and to the $Z^0$ mass scale, implementing the proper matching condition at $\mu = M_f$ to include the fifth flavor, the beauty \cite{17,24}:

$$\alpha_s^{(n_f-1)}(M_f) = \alpha_s^{(n_f)}(M_f) \left[ 1 + k_2 \left( \frac{\alpha_s^{(n_f)}(M_f)}{\pi} \right)^2 + k_3 \left( \frac{\alpha_s^{(n_f)}(M_f)}{\pi} \right)^3 \right] \quad (17)$$

with $k_2 = \frac{11}{72}$ and $k_3 = \frac{564731}{124416} - \frac{82043}{27648} \zeta_3 - \frac{2633}{31104} (n_f - 1)$. We find

$$\alpha_s(M_{\Upsilon(1S)}) = 0.197 \pm 0.002 \left| \Delta E_{\text{HFS}}^{\text{exp}} \right|_{\text{scheme}} \pm 0.002 \left| \delta^{<G^2>} \right|_{\delta m_b} \pm 0.005 \left| \delta m_b \right|_{\text{ho}} . \quad (18)$$

and

$$\alpha_s(M_{Z^0}) = 0.124 \pm 0.001 \left| \Delta E_{\text{HFS}}^{\text{exp}} \right|_{\text{scheme}} \pm 0.001 \left| \delta^{<G^2>} \right|_{\delta m_b} \pm 0.003 \left| \delta m_b \right|_{\text{ho}} . \quad (19)$$
Figure 3: Measurements of $\alpha_s(M_{Z^0})$ used in [26] to compute the 2009 world average, together with the determination of $\alpha_s(M_{Z^0})$ obtained in this paper. The continuous vertical line corresponds to the world average value in [26], and the dashed lines take the error into account: $\alpha_s(M_{Z^0}) = 0.1184 \pm 0.0007$. This result is dominated by the HPQCD determination (indicated as $Q\bar{Q}$ states) [27].

Equations (18) and (19) show the quality of the determination of the strong coupling constant from the $b\bar{b}$ hyperfine splitting: for comparison, the determination of $\alpha_s$ from the ratio $R_\gamma = \Gamma(\Upsilon \to \gamma gg)/\Gamma(\Upsilon \to ggg)$ of radiative/hadronic decay widths of $\Upsilon(1S)$ corresponds to $\alpha_s(M_\Upsilon) = 0.184_{-0.014}^{+0.015}$ and $\alpha_s(M_{Z^0}) = 0.119_{-0.005}^{+0.006}$ [25].

The result in Eq. (19) can be compared to the world average of $\alpha_s$. The 2009 average computed in [26] is obtained considering, together with the result from the ratio of radiative/hadronic $\Upsilon$ decay widths, the determinations of $\alpha_s$ from $\tau$-lepton decays, deep inelastic scattering processes (in particular, from nonsinglet structure functions and jet production rates), $e^+e^-$ processes (event shapes and jet production rates), and from electroweak precision fits. Moreover, a determination of the HPQCD Collaboration, based on the analysis of the $Q\bar{Q}$ system on the lattice, is included: $\alpha_s(M_{Z^0}) = 0.1183 \pm 0.0008$ [27]. As one can see by looking at Fig. 3 this last value dominates the present average, $\alpha_s(M_{Z^0}) = 0.1184 \pm 0.0007$ [26], and compared to this average, the value quoted in (19) is less than 2$\sigma$ higher. The result (19) from the $b\bar{b}$ hyperfine splitting, $\alpha_s(M_{Z^0}) = 0.124 \pm 0.004$ (with the error obtained by combining in quadrature the various uncertainties), together with the 2009 world average of $\alpha_s$ obtained in [26], slightly increases the value:

$$\alpha_s(M_{Z^0}) = 0.1186 \pm 0.0007.$$ (20)

Our conclusion is that there is the possibility to accommodate the experimental datum on the hyperfine splitting of the lowest-lying $b\bar{b}$ doublet with the QCD, perturbative and
nonperturbative, description of it. The resulting value of $\alpha_s$ is compatible with the world average within less than 2 standard deviations. The inclusion of the value of $\alpha_s$ determined in this paper slightly increases the average.

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