Control of \((x, u)\)-flat systems by quasi-static feedback of classical states

Conrad Gstöttner\textsuperscript{a}, Bernd Kolar\textsuperscript{b} and Markus Schöberl\textsuperscript{a}

\textsuperscript{a}Institute of Automatic Control and Control Systems Technology, Johannes Kepler University, Linz, Austria;
\textsuperscript{b}Magna Powertrain Engineering Center Steyr GmbH & Co KG, Steyrer Str. 32, 4300 St. Valentin, Austria;

\textbf{ARTICLE HISTORY}
Compiled October 26, 2021

\textbf{ABSTRACT}
It is well-known that flat systems can be linearized by quasi-static feedback of generalized states. However, even though it is of considerable practical relevance for the controller design, the question whether they can also be linearized by quasi-static feedback of classical states is still open. In the present paper, we prove that for \((x, u)\)-flat systems such a linearizing feedback always exists, and provide a systematic procedure for its construction. Since the exact linearization is typically only an intermediate step in the design of a flatness-based tracking control, we also show how the tracking error dynamics can be stabilized by feedback of the classical state instead of a generalized one.

\textbf{KEYWORDS}
flatness, tracking control, nonlinear control

1. Introduction

The concept of flatness has been introduced in control theory by Fliess, Lévine, Martin and Rouchon in the 1990s, see e.g. Fliess, Lévine, Martin, and Rouchon (1992, 1995). Roughly speaking, a nonlinear control system of the form

\[ \dot{x} = f(x, u) \quad (1) \]

with an \(n\)-dimensional state \(x\) and an \(m\)-dimensional input \(u\) is flat, if there exist \(m\) differentially independent functions \(y^j = \varphi^j(x, u, u_1, \ldots, u_q)\), where \(u_k\) denotes the \(k\)-th time derivative of \(u\), such that \(x\) and \(u\) can locally be expressed by \(y\) and its time derivatives, i.e.,

\[ x = F_x(y, y_1, \ldots, y_{r-1}) , \quad u = F_u(y, y_1, \ldots, y_r) . \quad (2) \]

The parameterization \((2)\) of the system variables by the flat output allows for an elegant and systematic solution of both feed-forward and feedback problems, which is the reason for the ongoing popularity of flat systems, see e.g. Delaleau and Rudolph (1998); Fliess et al. (1995). The computation of flat outputs, however, is known to be a difficult problem. Recent research in this field can be found e.g. in Gstöttner, Kolar, and Schöberl (2020b, 2021); Nicolau and Respondek (2017, 2019).

A typical intermediate step in the design of a flatness-based control is an exact linearization of the system \((1)\) by a suitable feedback. The objective is that the closed-loop system possesses a
linear input-output behavior in the form of integrator chains between a new closed-loop input $v$ and the considered flat output $y$. A standard approach is an exact linearization by an endogenous dynamic feedback

$$\dot{z} = g(x, z, v)$$
$$u = \alpha(x, z, v),$$

where the new input $v$ is given by the highest time derivatives $y_r$ that are present in the parameterization (2). Thus, the closed-loop system

$$\dot{x} = f(x, \alpha(x, z, v))$$
$$\dot{z} = g(x, z, v)$$

(4)

has a linear input-output behavior $y_j^r = v_j$, $j = 1, \ldots, m$ between the new input $v$ and the flat output $y$. In Fließ, Lévine, Martin, and Rouchon (1999) or Kolar (2017) it is shown how such a dynamic feedback (3) can be constructed systematically from the parameterization (2). However, a drawback of this standard approach is that it is a dynamic feedback and that the order of the error dynamics of a subsequently designed tracking control is given by $\dim(x) + \dim(z)$, which is higher than the order $\dim(x) = n$ of the original system (1). A well-established alternative, which circumvents these drawbacks, is the exact linearization by a quasi-static feedback of generalized states proposed in Delaleau and Rudolph (1995, 1998); Rudolph and Delaleau (1998). In contrast to the more classical static or dynamic feedbacks, a quasi-static feedback is a feedback which also depends on time derivatives of the closed-loop input. In Delaleau and Rudolph (1998), it has been shown that every flat system (1) can be exactly linearized by a quasi-static feedback

$$u = \alpha(\tilde{x}_B, v, \dot{v}, \ldots)$$

(5)

of a so-called generalized Brunovsky state $\tilde{x}_B = (y, y_1, \ldots, y_{\kappa-1})$, which consists of suitably chosen time derivatives of the components $y_j$ of the flat output up to the orders $\kappa_j - 1$ and meets $\dim(\tilde{x}_B) = \dim(x)$, i.e., $\kappa_1 + \ldots + \kappa_m = n$. The new input $v$ is given by the time derivatives $y_\kappa$, and hence the closed-loop system

$$\dot{x} = f(x, \alpha(\tilde{x}_B, v, \dot{v}, \ldots))$$

possesses the linear input-output behavior $y_j^\omega = v_j$, $j = 1, \ldots, m$. The practical superiority of this approach over the before-mentioned one has been demonstrated e.g. in Kugi and Kiefer (2005). However, the feedback (5) is not a feedback of the original state $x$ of (1). Since except for static feedback linearizable systems a generalized Brunovsky state $\tilde{x}_B$ is not equivalent to the original state $x$ via a state transformation $\tilde{x}_B = \Phi_x(x)$, determining the required time derivatives of the flat output from available measurements is not a straightforward task. Furthermore, even though it is ensured by construction that the possible trajectories $v(t)$ of the new input $v = y_\kappa$ are not restricted by the initial value $\tilde{x}_B(0)$ of the generalized Brunovsky state, there is no general guarantee that this is also true with respect to the initial value $x(0)$ of the original state of the system (1). In other words, there is no guarantee that for every initial state $x(0)$ a desired input trajectory $v(t)$ requested e.g. by a controller for the exactly linearized system can be realized by a suitable trajectory $u(t)$ of the original control input. Thus, for a practical implementation also the usage of additional underlying control loops might be necessary, which ensure that every desired trajectory $v(t)$ is at least approximately realized.

The aim of the present paper is to propose a method which combines the advantages and avoids
the drawbacks of both above-mentioned approaches. More precisely, the idea is to perform an exact linearization of the system (1) by a quasi-static feedback

$$u = \alpha(x, v, \dot{v}, \ldots)$$ (6)

of the state $x$ instead of a generalized Brunovsky state $\tilde{x}_B$. Despite its obvious practical relevance, the question whether this is possible for flat systems (1) in general has not been addressed in the literature so far. The contribution of the paper in regard of this open question is twofold: First, we derive conditions under which an exact linearization with respect to a given flat output by a feedback (6) is possible, and provide a systematic procedure for its construction. Second, we show that for $(x, u)$-flat systems, i.e., systems with a flat output

$$y = \varphi(x, u)$$ (7)

which does not depend on time derivatives of the input, such an exact linearization is possible in general. Moreover, we show that the proposed method for the exact linearization can be used as a basis for the design of a flatness-based tracking control which also only requires measurements of the state $x$. Thus, in contrast to the standard approach described in Delaleau and Rudolph (1998), the usage of a generalized Brunovsky state can again be avoided. Preliminary results addressing this topic can be found in Kolar, Rams, and Schlacher (2017) and Gstöttner, Kolar, and Schöberl (2020a).

The paper is organized as follows: In Section 2 we introduce some notation and the differential-geometric framework. Our main results are contained in Section 3 which addresses the exact linearization of flat systems. First, we derive conditions to check which time derivatives of a given flat output are an appropriate choice for a new input. Subsequently, we show how to introduce this new input by a suitable feedback. Finally, we prove that for $(x, u)$-flat systems the new input can always be chosen in such a way that the required feedback is a quasi-static feedback of the state $x$. The results of Section 3 are then illustrated by two examples in Section 4. As an application, Section 5 deals with the design of a flatness-based tracking control on the basis of the exact linearization derived in Section 3. Finally, in Section 6 the tracking control design is illustrated with the continued examples of Section 4.

2. Preliminaries

In the following, some notation and the utilized differential-geometric framework is introduced.

2.1. Notation

Let $\mathcal{X}$ be an $n$-dimensional smooth manifold, equipped with local coordinates $x^i$, $i = 1, \ldots, n$. The tangent bundle and the cotangent bundle of $\mathcal{X}$ are denoted by $(T(\mathcal{X}), \tau_\mathcal{X}, \mathcal{X})$ and $(T^*(\mathcal{X}), \tau^*_\mathcal{X}, \mathcal{X})$. For these bundles we have the induced local coordinates $(x^i, \dot{x}^i)$ and $(x^i, \dot{x}_i)$ with respect to the holonomic bases $\{\partial_{x^i}\}$ and $\{dx^i\}$, respectively. We also make use of the Einstein summation convention. A vector field is a section of the tangent bundle, i.e., a map $v : \mathcal{X} \rightarrow T(\mathcal{X})$. In local coordinates, a vector field reads $v = v^i(x)\partial_{x^i}$. Likewise, a covector field or (differential) 1-form is a section of the cotangent bundle, i.e., a map $\omega : \mathcal{X} \rightarrow T^*(\mathcal{X})$ which in local coordinates reads $\omega = \omega_i(x)dx^i$. A $k$-dimensional codistribution on $\mathcal{X}$ is a map which assigns to each $p \in \mathcal{X}$ a $k$-dimensional linear subspace $P_p \subset T^*_p(\mathcal{X})$ of the cotangent space at $p$. Let $\omega^1, \ldots, \omega^k$ be covector fields such that $\omega_p^1, \ldots, \omega_p^k$ form a basis for $P_p$ at each $p$. We say that the codistribution $P$ is spanned
by the covector fields $\omega^1, \ldots, \omega^k$, which form a basis for $P$, i.e., $P = \text{span}\{\omega^1, \ldots, \omega^k\}$ with the span over the ring $C^\infty(\mathcal{X})$ of smooth functions $h : \mathcal{X} \to \mathbb{R}$. The $k$-fold Lie derivative of a function $\varphi$ along a vector field $v$ is denoted by $L^k_v \varphi$. By $\partial_x h$ we denote the $m \times n$ Jacobian matrix of $h = (h^1, \ldots, h^m)$ with respect to $x = (x^1, \ldots, x^n)$. The symbols $\subset$ and $\supset$ are used in the sense that they also include equality. We write $y^j$ for the $\alpha$-th time derivative of $y^j$ and $y^j = (y^j_1, \ldots, y^j_m)$. To keep expressions involving time derivatives short we use multi-indices. Let $A = (a^1, \ldots, a^m)$ and $B = (b^1, \ldots, b^m)$ be two multi-indices with $a^j \leq b^j$, $j = 1, \ldots, m$, which we abbreviate by $A \leq B$. Then

$$y^j_A = (y^j_1, \ldots, y^j_m), \quad y^j[A] = (y^j_{[a^1]}, \ldots, y^j_{[a^m]}), \quad y^j[A,B] = (y^j_{[a^1,b^1]}, \ldots, y^j_{[a^m,b^m]}),$$

where $y^j_{[a^j,b]} = (y^j_{a^j}, \ldots, y^j_{b^j})$ and $y^j_{[a^j]} = y^j_{[0,a^j]}$. Addition and subtraction of multi-indices is done componentwise and we define the addition and subtraction of a multi-index $A$ with an integer $c$ by $A \pm c = (a^1 \pm c, \ldots, a^m \pm c)$. Furthermore, we define $\#A = \sum_{j=1}^m a^j$. When e.g. $y$ is decomposed into blocks of several variables, the first subscript refers to the block and a second subscript is used for denoting time derivatives. For instance, $y^j_{A_1} = (y^j_{a_1^1}, \ldots, y^j_{a_1^m})$ with some multi-index $A_1 = (a_1^1, \ldots, a_1^m)$.

**Example 1.** Consider the tuple $y = (y^1, y^2)$, the integer $c = 2$ and the multi-indices $A = (1, 3)$, $B = (2, 3)$. We then have $y_c = (y_1^1, y_2^2)$, $y_A = (y_1^1, y_3^2)$, $y[A] = (y_1^1, y_2^2, y_1^2, y_2^2, y_3^3)$ and $y[A,B] = (y_1^1, y_2^2, y_3^3)$, as well as $y_{A+c} = (y_3^3, y_2^3)$.

### 2.2. Geometric framework and flatness

Throughout this contribution, we use a finite-dimensional differential-geometric framework like e.g. in [Kolar, Schöberl, and Schlacher (2016)](Kolar, Schöberl, and Schlacher (2016)). In order to compute time derivatives of functions of the system variables and their derivatives along trajectories of (1), a manifold $\mathcal{X} \times U_{l_u}$ with coordinates $(x, u, u_1, \ldots, u_{l_u})$ is introduced, where $u_\alpha$ denotes the $\alpha$-th time derivative of the input $u$ and $l_u$ is some large enough but finite integer. The time derivative of a function $\varphi(x, u, u_1, \ldots, u_{l_u-1})$, which does not explicitly depend on $u_{l_u}$, is then given by the Lie derivative $L_{f_u} \varphi$ along the vector field

$$f_u = f^i(x, u) \partial_{x^i} + \sum_{\alpha=0}^{l_u-1} u_{\alpha+1}^j \partial_{u^j_\alpha}. \quad (8)$$

In the remainder of the paper, we assume that $l_u$ is chosen big enough such that $f_u$ acts as time derivative on all functions considered. Within this differential-geometric framework, flatness can be defined as follows.

**Definition 1.** The system (1) is called flat if there exists an $m$-tuple of smooth functions

$$y^j = \varphi^j(x, u, u_1, \ldots), \quad j = 1, \ldots, m$$

\[(9)\]
defined on $\mathcal{X} \times \mathcal{U}$ and smooth functions $F_x^i$ and $F_u^j$ such that locally

$$
\begin{align*}
x^i &= F_x^i(\varphi, L_{f_x} \varphi, \ldots, L_{f_x}^{R-1} \varphi), \quad i = 1, \ldots, n \\
u^j &= F_u^j(\varphi, L_{f_u} \varphi, \ldots, L_{f_u}^R \varphi), \quad j = 1, \ldots, m
\end{align*}
$$

with some multi-index $R = (r^1, \ldots, r^m)$. The $m$-tuple (9) is called a flat output.

By taking the exterior derivative of the expressions (10), we find that flatness implies

$$
dx \in \text{span}\{d\varphi, dL_{f_x} \varphi, \ldots, dL_{f_x}^{R-1} \varphi\} \quad \text{and} \quad du \in \text{span}\{d\varphi, dL_{f_u} \varphi, \ldots, dL_{f_u}^R \varphi\}.
$$

In fact, the converse is also true. This is a consequence of the following more general result relating the functional dependence of functions and the linear dependence of their differentials.

**Lemma 1.** Consider a set of smooth functions $g^1, \ldots, g^k$ as well as another smooth function $h$ which are all defined on the same manifold. Then

$$
dh \in \text{span}\{dg^1, \ldots, dg^k\}
$$

is equivalent to the existence of a smooth function $\psi : \mathbb{R}^k \to \mathbb{R}$ such that locally

$$
h = \psi(g^1, \ldots, g^k)
$$

holds identically. If the differentials $dg^1, \ldots, dg^k$ are linearly independent, then the function $\psi$ is unique.

The proof is straightforward and follows immediately by introducing a maximal number of independent functions of the set $g^1, \ldots, g^k$ as coordinates on the underlying manifold, see e.g. Nijmeijer and van der Schaft (1990). Another well-known important implication of Definition 1 is that the differentials $d\varphi, dL_{f_x} \varphi, \ldots, dL_{f_x}^R \varphi$ of derivatives of a flat output up to arbitrary order $\beta$ are linearly independent. Because of Lemma 1, this linear independence in turn implies that the maps $F_x$ and $F_u$ in (10) are unique.

### 3. Exact linearization of flat systems

In this section, we discuss the exact linearization of a system (1) with respect to a given flat output (9) by a suitable feedback. The feedback-modified system shall possess a linear input-output behavior of the form $y^j_{a^j} = v^j, j = 1, \ldots, m$ – with suitable integers $a^j$ – between a newly introduced input $v$ and the flat output $y$. Such an exact linearization is used e.g. as an intermediate step in the design of a flatness-based tracking control.

In principle, it is tempting to choose the orders $a^j$ of the time derivatives of the components of the flat output which are used as new (closed-loop) input $v$ as low as possible. However, it is important to ensure that like for the original control input $u$ the system (1) permits arbitrary (sufficiently differentiable) trajectories $v(t)$ independently of its initial state $x(0)$. Otherwise, desired input trajectories $v(t)$ requested e.g. by a controller for the exactly linearized system could only

---

1 The integer $l_u$ needs to be chosen large enough such that the Lie derivative $L_{f_u}^{\beta} \varphi$ indeed yields the correct expression for the $\beta$-th time derivative of the functions $\varphi$. The linear independence of the differentials $d\varphi, dL_{f_x} \varphi, \ldots, dL_{f_x}^R \varphi$ implies that the time evolution of the flat output is not constrained by any autonomous differential equation $\chi(y, y_1, \ldots, y_{\beta}) = 0$. 

---
be realized by a suitable trajectory \( u(t) \) of the original control input in case of compatible initial states \( x(0) \). This is of course not acceptable. For instance, as a particularly obvious example, it is clear that the possible trajectories of the components of an \( x \)-flat output depend on \( x(0) \) and are hence not a suitable choice for a new input. We thus need an appropriately chosen \( m \)-tuple of time derivatives \( v = y_A \), \( A = (a^1, \ldots, a^m) \) of the flat output \( 9 \) which possesses this property. Simple geometric conditions for the choice of a feasible input are given in Section 3.1. In Section 3.2 we show how to systematically construct a (possibly dynamic) feedback which introduces such a new input.

**Remark 1.** Consider the special case of \( y = \varphi(x, u) \) being a linearizing output in the sense of static feedback linearization \( 4 \) i.e., an output with a vector relative degree of \( n \) (see e.g. Isidori (1995)). In this case, the \( R \)-th time derivatives of the linearizing output are of the form \( y_R = \varphi_R(x, u) \) with \( \text{rank}(\partial_u \varphi_R) = m \). Thus, it is obvious that with the choice \( v = y_R \) arbitrary trajectories \( v(t) \) can be realized independently of the initial state \( x(0) \) by a suitable choice of the trajectory of the control input \( u(t) \). Solving \( v = \varphi_R(x, u) \) for the control input \( u \) yields a static feedback \( u = \alpha(x, v) \) which introduces \( v \) as new input. The closed-loop system has the linear input-output behavior \( y_R = v \).

### 3.1. Geometric conditions for the choice of an input

In the following, we state easily verifiable conditions to check whether a selection of time derivatives of a flat output \( 9 \) is a feasible choice for an input \( v \), where feasible means that arbitrary (sufficiently differentiable) trajectories \( v(t), t \geq 0 \) can be realized regardless of the initial state \( x(0) \) of the system \( 1 \) by applying an appropriate trajectory for the original control input \( u(t), t \geq 0 \). Let \( A = (a^1, \ldots, a^m) \geq 0 \) be a multi-index and \( v = y_A \) the corresponding \( a^j \)-th time derivatives of the components of the considered flat output. In order to be able to realize arbitrary (sufficiently often differentiable) trajectories \( v(t) = \partial^A y(t) \) regardless of the initial state \( x(0) \), there must not exist any nontrivial relation of the form

\[
\psi(x, y_A, y_{A+1}, \ldots) = 0. \tag{11}
\]

Otherwise, an evaluation of (11) at \( t = 0 \) would imply that at least one of the time derivatives of one component of \( v(t) \) at \( t = 0 \) is determined by \( x(0) \) and the other time derivatives of \( v(t) \), which is a contradiction to the requirement that arbitrary trajectories \( v(t) \) shall be possible. In the geometric framework of Section 2.2, the time derivatives of the flat output correspond to Lie derivatives along the vector field \( 8 \). Thus, the non-existence of any non-trivial relation (11) corresponds to the non-existence of any non-trivial relation

\[
\psi(x, L^A_{f^1} \varphi, L^{A+1}_{f^2} \varphi, \ldots) = 0.
\]

By using Lemma 1 this condition can be reformulated as linear independence of the differentials \( dx, dL^A_{f^1} \varphi, dL^{A+1}_{f^2} \varphi, \ldots \) of the above functions. However, it is not necessary to actually check the linear independence of this infinite set of differentials. Because of the linear independence of differentials of time derivatives of a flat output up to arbitrary order and \( dx \in \text{span}\{d \varphi_{[R-1]}\} \), it is

---

2Note that for systems 4 with rank(\( \partial_u f = m \)) such a linearizing output would only depend on the state \( x \).

3Since \( v \) consists of time derivatives of a flat output 9, for every desired trajectory \( v(t) \) an integration with arbitrary integration constants (initial values at \( t = 0 \) of the lower-order time derivatives) certainly yields matching trajectories \( y(t) \). However, the question is whether for every initial state \( x(0) \) the integration constants can be chosen such that the trajectory \( x(t) \) determined from \( y(t) \) by the parameterization (10) is consistent with \( x(0) \). The desired trajectory \( v(t) \) can then be realized by applying the trajectory \( u(t) \) of the original control input determined by (10).
sufficient to consider only time derivatives up to the order $R - 1$.

Consequently, we immediately obtain the following proposition.

**Proposition 1.** The system (1) permits arbitrary trajectories $v(t)$, $t \geq 0$ for the time derivatives $v = L_{f_x}^A \varphi$ of a flat output (9) regardless of its initial state $x(0)$ if and only if the differentials

$$
dx, 
\frac{dL}{d_x A}^A \varphi, 
\frac{dL}{d_x A + 1}^A \varphi, 
\vdots, 
\frac{dL}{d_x R - 1}^A \varphi
$$

are linearly independent.

Note that for $A \geq R$, the condition of Proposition 1 is always met. Furthermore, it can be shown that the condition of Proposition 1 can only be satisfied for multi-indices $A$ with $\#A \geq n$.

### 3.2. Construction of the linearizing feedback

Once a suitable selection $v = y_A$ of time derivatives of the flat output has been chosen in accordance with Proposition 1, the next step is to construct the required feedback needed to actually introduce these quantities as new inputs. In the following, we show how to derive such a feedback systematically.

For simplicity, we assume $A \leq R$ since the choice $A = R$ is always possible.

Let $A \leq R$ be a multi-index such that the condition of Proposition 1 is met. From $x = F_x(\varphi_{[R-1]})$ it follows that

$$
dx \in \text{span}\{d\varphi_{[R-1]}\} \subset \text{span}\{d\varphi_{[R]}\} = \text{span}\{d\varphi_{[A-1]}, d\varphi_{[A,R]}\}.
$$

Because of the linear independence of the differentials $dx$, $d\varphi_A$, $d\varphi_{A+1}, \ldots$, we can construct another basis for $\text{span}\{d\varphi_{[R]}\}$ by replacing an appropriate selection of $n$ of the $\#A$ differentials $d\varphi_{[A-1]}$ by the $n$ differentials $dx$. That is,

$$
\text{span}\{d\varphi_{[R]}\} = \text{span}\{d\varphi_{[A-1]}, d\varphi_{[A,R]}\} = \text{span}\{dx, d\varphi_c, d\varphi_{[A,R]}\},
$$

where $d\varphi_c$ is a suitable selection of $\#A - n$ complementary differentials out of the set $d\varphi_{[A-1]}$. With Lemma 1, it then follows that all the functions $\varphi_{[R]}$ can be expressed as functions of $x$, $\varphi_c$ and $\varphi_{[A,R]}$, i.e., there exist $\#R + m$ smooth functions $\psi_{i,j}^j : \mathbb{R}^{\#R + m} \rightarrow \mathbb{R}$ such that locally

$$
\varphi_{i,j}^j = \psi_{i,j}^j(x, \varphi_c, \varphi_{[A,R]}), \quad j = 1, \ldots, m, \quad l_j = 0, \ldots, r^j.
$$

It immediately follows that all these functions together, i.e. $\Psi = (\psi_0^1, \ldots, \psi_1^1, \ldots, \psi_m^m, \ldots, \psi_{r^m}^m)$, define a diffeomorphism $\Psi : \mathbb{R}^{\#R + m} \rightarrow \mathbb{R}^{\#R + m}$ such that locally

$$
\varphi_{[R]} = \Psi(x, \varphi_c, \varphi_{[A,R]}).
$$

4Since differentials $d\varphi_{i,j}^j$ with $\beta \geq r^j$ are linearly independent of the differentials $d\varphi_{[R-1]}$, it follows from $dx \in \text{span}\{d\varphi_{[R-1]}\}$ that they are linearly independent of the differentials $dx$ anyway.
The inverse of \( (12) \) is given by

\[
x = F_x(\varphi_{[R-1]})
\]

\[
\varphi_c = \varphi_c
\]

\[
\varphi_{[A,R]} = \varphi_{[A,R]},
\]

so in fact we have simply extended the flat parameterization \( x = F_x(\varphi_{[R-1]}) \) to the diffeomorphism \( (13) \). Now define \( \varphi_{c,1} = L_{f_c} \varphi_c \) as the first time derivatives of the functions \( \varphi_c \). Since the functions \( \varphi_c \) belong to \( \varphi_{[A-1]} \), their derivatives belong to \( \varphi_{[A]} \) and in turn to \( \varphi_{[R]} \). They can thus be expressed as functions of \( x, \varphi_c \) and \( \varphi_{[A,R]} \) in the form \( \varphi_{c,1} = \psi_{c,1}(x, \varphi_c, \varphi_{[A,R]}), \) where \( \psi_{c,1} \) denotes the corresponding components of \( (12) \). Let us introduce now the abbreviations \( z = \varphi_c, v = \varphi_A, v_1 = \varphi_{A+1} \), \ldots, \( v_{R-A} = \varphi_R, \) or \( v_{[R-A]} = \varphi_{[A,R]} \) for short. With these abbreviations we have

\[
\dot{z} = L_{f_c} z = \psi_{c,1}(x, z, v_{[R-A]})
\]

\[
\dot{v}_{[R-A-1]} = L_{f_c} v_{[R-A-1]} = v_{[1,R-A]},
\]

and with \( u = F_u(\varphi_{[R]}) \) and \( (12) \) the control input \( u \) can be expressed as

\[
u = F_u \circ \Psi(x, z, v_{[R-A]}).
\]

The equations \( (14) \) together with \( (15) \) may be considered as a \( \dim(z) + \dim(v_{[R-A-1]}) = (#R - n) \)-dimensional endogenous dynamic feedback

\[
\dot{z} = \psi_{c,1}(x, z, v_{[R-A]})
\]

\[
\dot{v}_{[R-A-1]} = v_{[1,R-A]}
\]

\[
u = F_u \circ \Psi(x, z, v_{[R-A]}),
\]

which applied to \( (1) \) yields the closed-loop system

\[
\ddot{x} = f(x, F_u \circ \Psi(x, z, v_{[R-A]}))
\]

\[
\ddot{z} = \psi_{c,1}(x, z, v_{[R-A]})
\]

\[
\dot{v}_{R-A-1} = v_{[1,R-A]}
\]

with the new input \( v_{R-A} \). Note that the feedback \( (16) \) is indeed endogenous, since \( z \) and \( v_{[R-A]} \) are just certain time derivatives of the components of the flat output and thus functions of \( x, u \) and time derivatives of \( u \). The feedback \( (16) \) is a special case of the linearizing endogenous dynamic feedback \( (3) \). The difference is that a part of the controller dynamics is given by integrator chains of length \( R - A \). The input \( v_{R-A} \) of the closed-loop system \( (17) \) corresponds to the input \( v \) of the closed-loop system \( (4) \). The state transformation

\[
y_{[A-1]} = \psi_{[A-1]}(x, z, v_{[R-A-1]})
\]

\[
v_{[R-A-1]} = v_{[R-A-1]},
\]

with \( \psi_{[A-1]} \) composed of the corresponding components of \( (12) \), which replaces \( x \) and \( z \) by the derivatives of the flat output up to the orders \( A - 1 \) and the quantities \( v_{[R-A-1]} \) are kept as states,
would transform the closed-loop system (17) into Brunovsky normal form
\[
\begin{align*}
\dot{y}_1 &= y_1 \\
\vdots \\
\dot{y}_{m-1} &= v_1 \\
\dot{v}_1 &= v_1 \\
\vdots \\
\dot{v}_{m-a} &= v_{m-a}.
\end{align*}
\]
For the more general case with the feedback (3), this is shown in detail in Kolar (2017). In particular, the \(a_j\)-th time derivatives of the components of the flat output along trajectories of (17) are given by \(y_{a_j} = v_j\). Now consider the feedback
\[
\dot{z} = \psi_{c,1}(x, z, v_{[R-A]}) \\
u = F_u \circ \Psi(x, z, v_{[R-A]})
\]
obtained from the dynamic feedback (16) by omitting the integrator chains \(\dot{v}_{[R-A-1]} = v_{[1,R-A]}\). Applied to (11), this feedback yields the closed-loop system
\[
\begin{align*}
\dot{x} &= f(x, F_u \circ \Psi(x, z, v_{[R-A]})) \\
\dot{z} &= \psi_{c,1}(x, z, v_{[R-A]})
\end{align*}
\]
with the new input \(v\). The feedback (18) is a \(\dim(z) = (\#A - n)\)-dimensional dynamic feedback which additionally depends, like a quasi-static feedback, on time derivatives of the closed-loop input \(v\). The closed-loop systems (17) and (19) generated by (16) and (18), respectively, coincide except for the integrator chains \(\dot{v}_{[R-A-1]} = v_{[1,R-A]}\). Computing the time derivative of a function along trajectories of (17) or along trajectories of (19) yields exactly the same result. For the closed-loop system (19), we thus also have \(y_{a_j} = v_j\), i.e., the desired linear input-output behavior. We summarize these observations in the following theorem.

**Theorem 1.** An \(m\)-tuple of time derivatives \(v = L^A_f \varphi\) of a flat output (9) which satisfies the condition of Proposition 1 can be introduced as a new input by a feedback (18) with \(\dim(z) = \#A - n\). In the case \(\#A = n\), \(z\) is empty and the feedback degenerates to a quasi-static feedback \(u = F_u \circ \Psi(x, v_{[R-A]})\) of the state \(x\).

**Remark 2.** For simplicity we have assumed \(A \leq R\), since for \(A = R\) the condition of Proposition 1 is always met. With \(A = R\), the feedback (16) is independent of time derivatives of \(v\) and coincides with the classical endogenous dynamic feedback (3) mentioned in the introduction. Although not particularly useful, deriving a linearizing feedback for the general case with an arbitrary (larger) multi-index \(A\) satisfying Proposition 1 would only require minor modifications in the above derivation. However, since the order of the resulting dynamic feedback would be higher, such a choice seems to be undesirable.

For the design of a flatness-based tracking control, the special case \(\#A = n\) is particularly interesting, since it ensures a tracking error dynamics of minimal order. Furthermore, it has the advantage that according to Theorem 1 the required feedback is a quasi-static one and no controller states need to be initialized. Even though it has been shown in Delaleau and Rudolph (1998) that
every flat system can be exactly linearized by a quasi-static feedback of a generalized state, the question whether every flat system can be exactly linearized by a quasi-static feedback of a classical state is still open. In the following subsection, we show that for the practically most relevant case of \((x,u)\)-flat systems this is indeed always possible with respect to every \((x,u)\)-flat output \((\vec{7})\).

### 3.3. Exact linearization of \((x,u)\)-flat systems

In this section, we prove that every system \((\vec{1})\) with a flat output of the form \((\vec{7})\) can be exactly linearized by a quasi-static feedback of its original state \(x\). Because of Theorem \((\vec{1})\) we only have to show that there exists a multi-index \(A\) with \(#A = n\) that meets the condition of Proposition \((\vec{1})\) In the remainder of the paper, we refer to such a special multi-index by \(\kappa = (\kappa_1, \ldots, \kappa_m)\) in order to distinguish it from the general case\(^5\).

In the following, we show that for every flat output \((\vec{7})\) of a system \((\vec{1})\) a multi-index \(\kappa\) with these properties can be constructed systematically. In each step of the procedure, certain time derivatives of the flat output are introduced as new coordinates on \(X \times U_{[0,1]}\), such that finally the coordinates \((x,u,u_1,\ldots)\) are replaced by \((x,v,v_1,\ldots)\) with \(v = y_\kappa\). Roughly speaking, we differentiate each component of the flat output until it depends explicitly on the input \(u\). By means of a coordinate transformation, we then replace as many components of the input \(u\) as possible by these derivatives. In the next step, each of the remaining components of the flat output is further differentiated until again an explicit dependence on the remaining components of the original input \(u\) occurs, and again as many of its components as possible are replaced by these derivatives. This procedure is continued until all components of the original input \(u\) have been replaced by time derivatives of the flat output.

**Step 1:** Define the multi-index \(K_1 = (k_1^1, \ldots, k_1^m)\) such that

\[
L_{f_u}^{k_1^j - 1} \varphi^j = \varphi^j_{k_1^j}(x), \quad L_{f_u}^{k_1^j} \varphi^j = \varphi^j_{k_1^j}(x,u),
\]

i.e., \(k_1^j\) denotes the relative degree of the component \(\varphi^j\) of the flat output. Note that for an \((x,u)\)-flat output where \(y^j = \varphi^j(x,u)\) explicitly depends on \(u\), we have \(k_1^j = 0\). Introduce \(m_1 = \text{rank}(\partial_u \varphi_{K_1})\) of the \(m\) functions \(\varphi_{K_1}\) and their time derivatives as new coordinates. By reordering the components of the flat output we can always achieve that \(\text{rank}(\partial_u \varphi_{1,\kappa_1}) = m_1\), where \(\varphi_1 = (\varphi^1, \ldots, \varphi^{m_1})\) and \(\kappa_1 = (k_1^1, \ldots, k_1^{m_1})\) consists of the first \(m_1\) integers in \(K_1\), enabling us to apply the coordinate transformation

\[
\begin{align*}
v_1 &= \varphi_{1,\kappa_1}(x,u) \\
u_{\text{rest}_1} &= (u_{m_1+1}, \ldots, u^m) \\
v_{1,1} &= \varphi_{1,\kappa_1+1}(x,u,u_1) \\
u_{\text{rest}_{1,1}} &= (u_{m_1+1}, \ldots, u_{m_1}^m) \\
v_{1,2} &= \varphi_{1,\kappa_1+2}(x,u,u_1,u_2) \\
u_{\text{rest}_{1,2}} &= (u_{m_1+1}, \ldots, u_{m_2}^m) \\
&\vdots
\end{align*}
\]

That is, we replace the inputs \(u^1, \ldots, u^{m_1}\) and their time derivatives by \(v_1\) and its time derivatives

\(^5\)Even though we use the same symbol \(\kappa\) as in Delaleau and Rudolph [1998], the set \(y_{[\kappa-1]}\) does not necessarily form a generalized Brunovsky state in the sense of Delaleau and Rudolph [1998]. Conversely, an input \(v = y_\kappa\) with \(\kappa\) defined by a generalized Brunovsky state of Delaleau and Rudolph [1998] does not necessarily meet the condition of Proposition \((\vec{1})\)
(which may require a renumbering of the inputs). In these coordinates we have

\[
y_{1,[\kappa_{1}-1]} = \varphi_{1,[\kappa_{1}-1]}(x) \\
y_{1,\kappa_{1}} = v_{1}
\]

\[
y_{rest_{1},[K_{rest_{1}}-1]} = \varphi_{rest_{1},[K_{rest_{1}}-1]}(x) \\
y_{rest_{1},K_{rest_{1}}} = \varphi_{rest_{1},K_{rest_{1}}}(x, v_{1})
\]

where \(y_{1} = (y^{1}, \ldots, y^{m_{1}})\), \(y_{rest_{1}} = (y^{m_{1}+1}, \ldots, y^{m})\), \(\varphi_{rest_{1},\beta} = (\varphi_{\beta}^{m_{1}+1}, \ldots, \varphi_{\beta}^{m})\) o \(\hat{\Phi}\) with the inverse \(\Phi\) of the transformation \(20\), and \(K_{rest_{1}} = (k_{1}^{m_{1}+1}, \ldots, k_{1}^{m})\). Note that \(\varphi_{rest_{1},K_{rest_{1}}}\) is independent of \(u_{rest_{1}}\), since otherwise \(\text{rank}(\partial_{u}\varphi_{K_{1}})\) would have been larger than \(m_{1}\).

**Step 2:** Define the multi-index \(K_{2} = (k_{2}^{1}, \ldots, k_{2}^{m-m_{1}})\) such that

\[
L_{f_{u}}^{k_{2}-1} \varphi_{rest_{1}}^{j} = \varphi_{rest_{1},k_{2}}^{j}(x, v_{1}, v_{1}, \ldots, u_{rest_{1}}) \\
L_{f_{u}}^{k_{2}} \varphi_{rest_{1}}^{j} = \varphi_{rest_{1},k_{2}+1}^{j}(x, v_{1}, v_{1}, \ldots, u_{rest_{1}}).
\]

Similar as before, we introduce \(m_{2} = \text{rank}(\partial_{u_{rest_{1}}} \varphi_{rest_{1},K_{2}})\) of the \(m_{1}\) functions \(\varphi_{rest_{1},K_{2}}\) and their time derivatives as new coordinates. By reordering the components of the flat output belonging to \(y_{rest_{1}}\) we can always achieve that \(\text{rank}(\partial_{u_{rest_{1}}} \varphi_{2,k_{2}}) = m_{2}\), where \(\varphi_{2} = (\varphi_{rest_{1}1}, \ldots, \varphi_{rest_{1}}^{m_{2}})\) consists of the first \(m_{2}\) integers in \(K_{2}\), enabling us to apply the coordinate transformation\(^{7}\)

\[
\begin{align*}
v_{2} &= \varphi_{2,k_{2}}(x, v_{1}, v_{1}, \ldots, u_{rest_{1}}) \\
u_{rest_{2}} &= (u_{1}^{m_{1}+m_{2}+1}, \ldots, u_{m}) \\
v_{2,1} &= \varphi_{2,k_{2}+1}(x, v_{1}, v_{1}, \ldots, u_{rest_{1}}, u_{rest_{1},1}) \\
u_{rest_{2},1} &= (u_{1}^{m_{1}+m_{2}+1}, \ldots, u_{1}^{m}) \\
v_{2,2} &= \varphi_{2,k_{2}+2}(x, v_{1}, v_{1}, \ldots, u_{rest_{1}}, u_{rest_{1},1}, u_{rest_{1},2}) \\
u_{rest_{2},2} &= (u_{2}^{m_{1}+m_{2}+1}, \ldots, u_{2}^{m}) \\
&\vdots
\end{align*}
\]

That is, we replace the inputs \(u_{1}^{m_{1}+1}, \ldots, u_{1}^{m_{1}+m_{2}}\) and their time derivatives by \(v_{2}\) and its time derivatives (which may again require a renumbering of the inputs belonging to \(u_{rest_{1}}\)). In these

\(^{6}\)Introducing not only \(v_{1}\) but also its time derivatives as new coordinates is crucial for preserving the simple structure of the vector field \(5\). Concerning the regularity of the transformation \(20\) it should be noted that \(\text{rank}(\partial_{u_{\varphi_{1,n}}} \varphi_{1,n}) = m_{1}\) implies \(\text{rank}(\partial_{u_{\varphi_{1,n+1}}} \varphi_{1,n+1}) = m_{1}\) for \(\alpha \geq 1\). In the new coordinates, the vector field \(6\) has the form \(f_{u} = f_{u} \circ \Phi_{u_{\varphi_{1,n}}} + \sum_{\alpha=0}^{n-1} (v_{1,n+1}^{\alpha} \theta_{u_{\varphi_{1,n+1}}}^{\alpha} + u_{rest_{1},n+1}^{\alpha+1} \theta_{u_{rest_{1},n+1},n+1}^{\alpha} + \ldots + \theta_{u_{rest_{1},n},n+1}^{\alpha})\) with the inverse \(\Phi\) of the transformation \(20\). The in general non-zero components in the \(\theta_{u_{\varphi_{1,n}}}\)-directions do not bother us as long as \(l_{u}\) is chosen large enough.

\(^{7}\)The functions \(\varphi_{2,k_{2}}\) depend on derivatives of \(v_{1}\), which are only available up to the order \(l_{u}\). Thus, some higher-order time derivatives of \(u_{rest_{1}}\) must be kept as coordinates on the finite-dimensional manifold \(X \times U_{l_{u}}\) and cannot be replaced by time derivatives of \(v_{2}\). However, this is no problem as long as \(l_{u}\) is chosen large enough.
coordinates we have

\[ y_{1,[\kappa_1-1]} = \varphi_{1,[\kappa_1-1]}(x) \]
\[ y_{1,\kappa_1} = v_1 \]
\[ y_{2,[\kappa_2-1]} = \varphi_{2,[\kappa_2-1]}(x, v_1, v_{1,1}, \ldots) \]
\[ y_{2,\kappa_2} = v_2 \]
\[ y_{\text{rest} 2,[K_{\text{rest} 2}-1]} = \varphi_{\text{rest} 2,[K_{\text{rest} 2}-1]}(x, v_1, v_{1,1}, \ldots) \]
\[ y_{\text{rest} 2,K_{\text{rest} 2}} = \varphi_{\text{rest} 2,K_{\text{rest} 2}}(x, v_1, v_{1,1}, \ldots, v_2) \],

where \( y = (y^{m_1+1}, \ldots, y^{m_1+m_2}) \), \( y_{\text{rest} 2} = (y^{m_1+m_2+1}, \ldots, y^m) \), \( \varphi_{\text{rest} 2,\beta} = (\varphi_{\text{rest} 1,\beta}, \ldots, \varphi_{\text{rest} i,\beta}) \circ \hat{\Phi} \) with the inverse \( \hat{\Phi} \) of the transformation \([21]\) and \( K_{\text{rest} 2} = (k_{2}^{m_2+1}, \ldots, k_{2}^{m-1}) \).

**Step i:** In the \( i \)-th step we are concerned with the dependence of the time derivatives of the functions \( \varphi_{\text{rest} i-1} = (\varphi_{\text{rest} i-2}, \ldots, \varphi_{\text{rest} 1}) \) on the inputs \( u_{\text{rest} i-1} = (u^{m_1+\ldots+m_{i-1}+1}, \ldots, u^m) \). Define the multi-index \( K_i = (k_1^1, \ldots, k_i^{m-m_i-\ldots-m_{i-1}}) \) such that

\[ \left[L^j_{f_{u}} \varphi_{\text{rest} i-1} \right]_{k^{j-1}_{1}, k^{j}_{2}}(x, v_1, v_{1,1}, \ldots, v_{i-1,1}, \ldots) \]
\[ \left[L^j_{f_{u}} \varphi_{\text{rest} i-1} \right]_{k^{j}_{1}, k^{j}_{2}}(x, v_1, v_{1,1}, \ldots, v_{i-1,1}, \ldots, u_{\text{rest} i-1}) \].

Introduce \( m_i = \text{rank}(\partial_{u_{\text{rest} i-1}} \varphi_{\text{rest} i-1}, K_i) \) of the \( m - m_1 - \ldots - m_{i-1} \) functions \( \varphi_{\text{rest} i-1}, K_i \) and their time derivatives as new coordinates. By reordering the components of the flat output belonging to \( \varphi_{\text{rest} i-1} \) we can always achieve that \( \text{rank}(\partial_{u_{\text{rest} i-1}} \varphi_{i,\kappa_i}) = m_i \), where \( \varphi_i = (\varphi_{\text{rest} i-1}^{1}, \ldots, \varphi_{\text{rest} i-1}^{m_i}) \) and \( \kappa_i = (k_i^1, \ldots, k_i^{m_i}) \) consists of the first \( m_i \) integers in \( K_i \), enabling us to apply the coordinate transformation

\[ v_{i} = \varphi_{i,\kappa_i}(x, v_1, v_{1,1}, \ldots, v_{i-1,1}, \ldots, u_{\text{rest} i-1}) \]
\[ u_{\text{rest} i} = (u^{m_1+\ldots+m_{i-1}+1}, \ldots, u^m) \]
\[ v_{i,1} = \varphi_{i,\kappa_i+1}(x, v_1, v_{1,1}, \ldots, v_{i-1,1}, \ldots, u_{\text{rest} i-1}, u_{\text{rest} i-1}) \]
\[ u_{\text{rest} i,1} = (u^{m_1+\ldots+m_{i-1}+1}, \ldots, u^m) \]
\[ v_{i,2} = \varphi_{i,\kappa_i+2}(x, v_1, v_{1,1}, \ldots, v_{i-1,1}, \ldots, u_{\text{rest} i-1}, u_{\text{rest} i-1}, u_{\text{rest} i-1}, u_{\text{rest} i-1}) \]
\[ u_{\text{rest} i} = (u^{m_1+\ldots+m_{i-1}+1}, \ldots, u^m) \]
\[ \vdots \]

That is, we replace the inputs \( u^{m_1+\ldots+m_{i-1}+1}, \ldots, u^{m_1+\ldots+m_i} \) and their time derivatives by \( v_i \) and its time derivatives (which may require a renumbering of the inputs belonging to \( u_{\text{rest} i-1} \)). In these
coordinates we have

\[ y_{1,[\kappa_1-1]} = \varphi_{1,[\kappa_1-1]}(x) \]
\[ y_{1,\kappa_1} = v_1 \]

\[ y_{2,[\kappa_2-1]} = \varphi_{1,[\kappa_2-1]}(x, v_1, v_1, \ldots) \]
\[ y_{2,\kappa_2} = v_2 \]

\[ \vdots \]

\[ y_{i,[\kappa_i-1]} = \varphi_{i,[\kappa_i-1]}(x, v_1, v_1, \ldots, v_{i-1}, v_{i-1}, \ldots) \]
\[ y_{i,\kappa_i} = v_i \]

\[ y_{\text{rest}_i,[\text{rest}_i-1]} = \varphi_{\text{rest}_i,[\text{rest}_i-1]}(x, v_1, v_1, \ldots, v_{i-1}, v_{i-1}, \ldots) \]
\[ y_{\text{rest}_i,\text{rest}_i} = \varphi_{\text{rest}_i,\text{rest}_i}(x, v_1, v_1, \ldots, v_{i-1}, v_{i-1}, \ldots, v_i), \]

where \( y_i = (y^{m_1+\ldots+m_{i-1}+1}, \ldots, y^{m_1+\ldots+m_i}), \ y_{\text{rest}_i} = (y^{m_1+\ldots+m_i+1}, \ldots, y^m), \ \varphi_{\text{rest}_i,\beta} = (\varphi_{\text{rest}_{i-1},\beta}, \ldots, \varphi_{\text{rest}_{i-1},\beta}) \circ \Phi \) with the inverse \( \hat{\Phi} \) of the transformation \( (22) \) and \( K_{\text{rest}_i} = (k^{m_1+1}_1, \ldots, k^{m_{i-1}-\ldots-m_{i-1}}_i) \).

**Last Step:** The procedure terminates when in some step, let us call it the \( e \)-th step, the Jacobian matrix \( \partial_{u_{\text{rest}_{e-1}}}[\varphi_{\text{rest}_{e-1}},K_e] \) has full rank and thus no components \( y_{\text{rest}_e} \) remain. At this point, i.e., after the \( (e-1) \)-th step, we already have

\[ y_{1,[\kappa_1-1]} = \varphi_{1,[\kappa_1-1]}(x) \]
\[ y_{1,\kappa_1} = v_1 \]

\[ y_{2,[\kappa_2-1]} = \varphi_{2,[\kappa_2-1]}(x, v_1, v_1, \ldots) \]
\[ y_{2,\kappa_2} = v_2 \]

\[ \vdots \]

\[ y_{e-1,[\kappa_{e-1}-1]} = \varphi_{e-1,[\kappa_{e-1}-1]}(x, v_1, v_1, \ldots, v_{e-2}, v_{e-2}, \ldots) \]
\[ y_{e-1,\kappa_{e-1}} = v_{e-1} \]

\[ y_{\text{rest}_{e-1},[\text{rest}_{e-1}-1]} = \varphi_{\text{rest}_{e-1},[\text{rest}_{e-1}-1]}(x, v_1, v_1, \ldots, v_{e-2}, v_{e-2}, \ldots) \]
\[ y_{\text{rest}_{e-1},\text{rest}_{e-1}} = \varphi_{\text{rest}_{e-1},\text{rest}_{e-1}}(x, v_1, v_1, \ldots, v_{e-2}, v_{e-2}, \ldots, v_{e-1}). \]

The multi-index \( K_e = (k^1_e, \ldots, k^{m_{e-1}-\ldots-m_{e-1}}_e) \) is again defined such that

\[ L^{k^{j-1}}_{f_e}\varphi_{\text{rest}_{e-1}} = \varphi_{\text{rest}_{e-1},k^j_e}^j(x, v_1, v_1, \ldots, v_{e-1}, v_{e-1}, \ldots) \]
\[ L^{k^j_e}\varphi_{\text{rest}_{e-1}} = \varphi_{\text{rest}_{e-1},k^j_e}^j(x, v_1, v_1, \ldots, v_{e-1}, v_{e-1}, \ldots, u_{\text{rest}_{e-1}}), \]

where by assumption we have now a regular Jacobian matrix \( \partial_{u_{\text{rest}_{e-1}}}[\varphi_{\text{rest}_{e-1}},K_e] \) (and thus \( \varphi_e = \varphi_{\text{rest}_{e-1}} \) and \( \kappa_e = K_e \)). In conclusion, with the new coordinates successively constructed by this
procedure, the flat output and its derivatives up to the orders \( \kappa \) are given by

\[
\begin{align*}
y_{1,\kappa_{1}-1} & = \varphi_{1,\kappa_{1}-1}(x) \\
y_{1,\kappa_{1}} & = v_1 \\
y_{2,\kappa_{2}-1} & = \varphi_{2,\kappa_{2}-1}(x, v_1, v_{1,1}, \ldots) \\
y_{2,\kappa_{2}} & = v_2 \\
\vdots \\
y_{e-1,\kappa_{e-1}-1} & = \varphi_{e-1,\kappa_{e-1}-1}(x, v_1, v_{1,1}, \ldots, v_{e-2}, v_{e-2,1}, \ldots) \\
y_{e-1,\kappa_{e-1}} & = v_{e-1} \\
y_{e,\kappa_{e}-1} & = \varphi_{e,\kappa_{e}-1}(x, v_1, v_{1,1}, \ldots, v_{e-1}, v_{e-1,1}, \ldots) \\
y_{e,\kappa_{e}} & = \varphi_{e,\kappa_{e}}(x, v_1, v_{1,1}, \ldots, v_{e-1}, v_{e-1,1}, \ldots, u_{\text{rest}_{e-1}})(= v_e).
\end{align*}
\] (23)

**Remark 3.** If the procedure is applied to a linearizing output in the sense of static feedback linearization, then due to the regularity of \( \partial_u \varphi_R(x, u) \) the first step is already the last step and we have \( \kappa_1 = K_1 = R \).

**Lemma 2.** For every \((x, u)\)-flat output \([7]\) of the system \([1]\), the stated procedure terminates after at most \( m \) steps.

**Proof.** The existence of a multi-index \( K_i = (k^1_i, \ldots, k^{m-m_{i-1}-\ldots-m_{i-1}}_i) \) such that \( L^j_{f_e \varphi_{\text{rest}_{e-1}}} \) explicitly depends on at least one of the inputs belonging to \( u_{\text{rest}_{e-1}} \) in each step (set \( \varphi_{\text{rest}_0} = \varphi \) and \( u_{\text{rest}_0} = u \)) is a direct consequence of the linear independence of the differentials \( d\varphi, dL_{f_e \varphi}, \ldots, dL^j_{f_e \varphi} \) of time derivatives of a flat output for arbitrary \( \beta \). Indeed, assume that \( L^j_{f_e \varphi} \) for arbitrarily large \( l \) only depends on \( x \) and \( v_1, v_{1,1}, \ldots, v_{l-1}, v_{l-1,1}, \ldots \) but not on \( u_{\text{rest}_{e-1}} \). Then because of \( dL^j_{f_e \varphi} \) \( \in \text{span}\{dx, dv_1, dv_{1,1}, \ldots, dv_{l-1}, dv_{l-1,1}, \ldots\} \subset \text{span}\{d\varphi_{[R-1]}, d\varphi_1, d\varphi_{1,1}, \ldots, d\varphi_{l-1}, d\varphi_{l-1,1}, \ldots\} \), the differentials \( dL^j_{f_e \varphi} \) could be expressed as a linear combination of the differentials \( d\varphi_{[R-1]}, d\varphi_1, d\varphi_{1,1}, \ldots, d\varphi_{l-1}, d\varphi_{l-1,1}, \ldots \). However, for \( l \geq r \), with \( r \) being the order of the highest time derivative of \( y_{\text{rest}_{e-1}} \) needed in the flat parameterization \([10]\), this would be a contradiction to the linear independence of the differentials \( d\varphi, dL_{f_e \varphi}, \ldots, dL^j_{f_e \varphi} \) for arbitrary \( \beta \). Consequently we have \( m_i = \text{rank}(\partial_{u_{\text{rest}_{e-1}}} \varphi_{\text{rest}_{e-1}, \kappa_i}) \geq 1 \) in every step, and since there are only \( m \) inputs the procedure terminates after at most \( m \) steps. \( \square \)

**Theorem 2.** For every \((x, u)\)-flat output \([7]\) of the system \([1]\), the multi-index \( \kappa = (\kappa_1, \ldots, \kappa_e) \) formed by the multi-indices \( \kappa_i \) constructed in the above procedure satisfies \( \#\kappa = n \) and \( \kappa \leq R \). Furthermore, the differentials \( dx, d\varphi_{\kappa}, d\varphi_{\kappa+1}, \ldots \) are linearly independent, i.e., the condition of Proposition \([7]\) is met.

**Proof.** The existence of the flat parameterization \( x = F_x(\varphi_{[R-1]}) \) implies

\[
\text{span}\{dx\} \subset \text{span}\{d\varphi_{[R-1]}\}.
\]

In other words, there exist exactly \( n \) independent linear combinations of the differentials of the flat output and its time derivatives which are contained in \( \text{span}\{dx\} \). Now consider the expressions for the flat output and its time derivatives \([23]\) in the coordinates successively constructed during the procedure. Because of the independence of the time derivatives of a flat output, it is possible to
construct exactly $\#\kappa$ independent linear combinations of the differentials of the functions $\varphi_{[\kappa-1]}$ of (23) and the differentials
\[ dv_1, dv_{1,1}, dv_{1,2}, \ldots, dv_2, dv_{2,1}, dv_{2,2}, \ldots, dv_{e-1}, dv_{e-1,1}, dv_{e-1,2}, \ldots \]
which are contained in span$\{dx\}$. Consequently, the condition $\#\kappa = n$ follows.

The property $\kappa \leq R$ follows from the fact that every linear combination of differentials of time derivatives of the flat output which is contained in span$\{dx\}$ can only consist of differentials of time derivatives up to the order $R - 1$. Thus, $\kappa - 1 \leq R - 1$ and accordingly $\kappa \leq R$.

The linear independence of the differentials $dx, d\varphi_\kappa, d\varphi_{\kappa+1}, \ldots$ can easily be verified in the constructed coordinates, where they are given by $dx, dv, dv_1, \ldots$.

Remark 4. In fact, it can be shown that the time derivatives of $v$ that appear in the functions $\varphi_{i,[\kappa-1]}$, $i = 1, \ldots, e$ of (23) correspond to time derivatives of the flat output (7) up to the order $R - 1$.

Combining Theorem 2 with Theorem 1 immediately yields our main result.

Corollary 1. Every $(x,u)$-flat system (1) can be exactly linearized with respect to every $(x,u)$-flat output (7) by a quasi-static feedback of the form
\[ u = F_u \circ \Psi(x, v_{[R-\kappa]}). \] (24)
The input-output behavior of the closed-loop system is given by $y_{\kappa} = v$.

Remark 5. In fact, the linearizing feedback (24) follows directly from the above procedure for constructing the multi-index $\kappa$. Indeed, in the coordinates successively introduced during the procedure, the flat output and its derivatives read as (23). The feedback (24) is thus simply obtained by substituting the expressions for $y_{[\kappa-1]}$ from (23) and $y_{[\kappa,R]} = v_{[R-\kappa]}$ into the flat parameterization $u = F_u(y_{[R]})$ of the control input.

4. Examples

In this section, we demonstrate the procedure of Section 3.3 for the construction of linearizing quasi-static feedbacks of classical states by two $(x,u)$-flat examples.

4.1. Academic example

Consider the four-input system
\[ \begin{align*}
\dot{x}^1 &= u^1 \\
\dot{x}^2 &= x^9 \\
\dot{x}^3 &= u^2 - u^1 u^3 \\
\dot{x}^4 &= u^3 \\
\dot{x}^5 &= x^3 + x^4 u^1 \\
\dot{x}^6 &= x^7 (u^1 u^3 - u^2 - 1) + u^1 x^4 (u^1 + x^4) - x^8 u^1 \\
\dot{x}^7 &= x^4 + u^1 \\
\dot{x}^8 &= x^4 x^7 u^1 - x^6 \\
\dot{x}^9 &= x^{10} + u^2 + u^3 \\
\dot{x}^{10} &= u^4
\end{align*} \] (25)
with the $(x,u)$-flat (actually $x$-flat) output $y = (x^1, x^2, x^5, x^8)$. The multi-index $R$ containing the highest orders of the derivatives of the components of the flat output in the flat parameterization
is given by \( R = (4, 3, 5, 5) \). The vector field \( \mathbf{K} \) for this system reads
\[
\begin{align*}
    f_u & = u^1 \partial_{x^1} + x^9 \partial_{u_2} + (u^2 - u^1 u^3) \partial_{x^3} + u^3 \partial_{x^4} + (x^3 + x^4 u^1) \partial_{x^5} + (x^7 (u^1 u^3 - u^2) - 1) + u^1 x^4 (u^1 + x^4) - x^7 u^1 \partial_{x^6} + (x^4 + u^1) \partial_{x^7} + (x^4 x^7 u^1 - x^6) \partial_{x^8} + (x^{10} + u^2 + u^3) \partial_{x^9} + u^4 \partial_{x^{10}} + \sum_{\alpha=0}^{k-1} (u_{\alpha+1} \partial_{u_{\alpha+1}}^1 + u_{\alpha+1} \partial_{u_{\alpha+1}^2} + u_{\alpha+1} \partial_{u_{\alpha+1}^3} + u_{\alpha+1} \partial_{u_{\alpha+1}^4}).
\end{align*}
\]

In the following, we apply the procedure stated in Section 3.3 to obtain a multi-index \( \kappa \) with the properties stated in Theorem 2. Subsequently, we derive a quasi-static feedback \( (24) \) which introduces the derivatives \( \nu = y_\kappa \) of the flat output as new inputs, and thus exactly linearizes the system.

**Step 1:** Differentiating the components of the flat output along the vector field \( (26) \), we find that \( K_1 = (1, 2, 1, 1) \). The corresponding \( k_1^i \)-th derivatives of the components of the flat output are given by
\[
\varphi_{K_1} = \begin{bmatrix}
    u^1 \\
    x^{10} + u^2 + u^3 \\
    x^3 + x^4 u^1 \\
    x^4 x^7 u^1 - x^6
\end{bmatrix},
\]
and we have \( m_1 = \text{rank}(\partial_u \varphi_{K_1}) = 2 \). We obviously have \( \text{rank}(\partial_u \varphi_{1, K_1}) = m_1 \) with \( \varphi_1 = (\varphi^1, \varphi^2) \), \( \kappa_1 = (k_1^1, k_1^2) = (1, 2) \), and accordingly \( \varphi_{\text{rest}_1} = (\varphi^3, \varphi^4) \), \( K_{\text{rest}_1} = (k_1^3, k_1^4) = (1, 1) \). Applying the change of coordinates
\[
v_1 = \begin{bmatrix}
    \varphi^1_{k_1^1} \\
    \varphi^2_{k_1^1}
\end{bmatrix} = \begin{bmatrix}
    u^1 \\
    x^{10} + u^2 + u^3
\end{bmatrix}, \quad v_{1,1} = \begin{bmatrix}
    \varphi^1_{k_1^1+1} \\
    \varphi^2_{k_1^1+1}
\end{bmatrix} = \begin{bmatrix}
    u^1 \\
    u^1 + u^2 + u^3
\end{bmatrix}, \quad \ldots,
\]
by which we replace \( u^1 \) and \( u^2 \) and their derivatives by \( v_1 \) and its derivatives and keep \( u_{\text{rest}_1} = (u^3, u^4) \) and its derivatives as coordinates, yields
\[
\varphi_{1, \kappa_1} = \begin{bmatrix}
    \varphi^1_2 \\
    \varphi^2_2
\end{bmatrix} = \begin{bmatrix}
    v^1_2 \\
    v^2_2
\end{bmatrix}, \quad \varphi_{\text{rest}_1, K_{\text{rest}_1}} = \begin{bmatrix}
    \varphi^3_2 \\
    \varphi^4_2
\end{bmatrix} = \begin{bmatrix}
    x^3 + x^4 v^1_1 \\
    x^4 x^7 v^1_1 - x^6
\end{bmatrix}.
\]

**Step 2:** We proceed by differentiating \( \varphi_{\text{rest}_1} = (\varphi^3, \varphi^4) \) until an explicit dependence on \( u_{\text{rest}_1} = (u^3, u^4) \) occurs. We have
\[
\begin{align*}
    L^2_{f_u} \varphi^3 & = v^2_1 - x^{10} - u^3 + x^4 v^1_{1,1} \\
    L^2_{f_u} \varphi^4 & = x^7 (v^2_1 - x^{10} - u^3 + x^4 v^1_{1,1} + 1) + x^8 v^1_1
\end{align*}
\]
and thus \( K_2 = (2, 2) \). Since only \( u^3 \) occurs explicitly, we have \( m_2 = \text{rank}(\partial_{u_{\text{rest}_1}} \varphi_{\text{rest}_1, K_2}) = 1 \). We obviously have \( \text{rank}(\partial_{u_{\text{rest}_1}} \varphi_{2, \kappa_2}) = m_2 \) with \( \varphi_2 = \varphi^3 \), \( \kappa_2 = k_2^1 = 2 \), and accordingly \( \varphi_{\text{rest}_2} = \varphi^4 \), \( K_{\text{rest}_2} = k_2^2 = 2 \). Applying the change of coordinates
\[
v^1_2 = \varphi^3_{k_2^1} = v^1_1 - x^{10} - u^3 + x^4 v^1_{1,1}, \quad v^1_{2,1} = \varphi^3_{k_2^1+1} = v^2_{1,1} - u^4 - u^1 + u^3 v^1_{1,1} + v^1_{1,2} x^4, \quad \ldots,
\]
by which we replace \( u^3 \) and its derivatives by \( v_2 \) and its derivatives and keep \( u_{\text{rest}_2} = u^4 \) and its
derivatives as coordinates, yields
\[ \varphi_{3,\kappa_2} = \varphi_2^3 = v_2^1, \quad \varphi_{\text{rest}_2,K_{\text{rest}_2}} = \varphi_2^4 = x^8 v_1^1 + x^7(1 + v_2^1). \]

**Step 3:** We proceed by differentiating \( \varphi_{\text{rest}_2} = \varphi^4 \) until an explicit dependence on \( u_{\text{rest}_2} = u^4 \) occurs. Because of
\[
\begin{align*}
I_{f_1}^3 \varphi^4 &= x^4 x^7(v_1^1)^2 + v_1^1(v_2^1 - x^6 + 1) + x^4(v_2^1 + 1) + x^8 v_1^1 + x^7 v_2^1, \\
I_{f_1}^4 \varphi^4 &= h^1(x^4, x^6, x^7, x^8, x^{10}, v_1^1, v_1^1, v_1^1, v_1^1, v_1^1, v_2^1, v_2^1, v_2^1, v_2^1, v_2^1, v_2^2), \\
I_{f_1}^5 \varphi^4 &= h^2(x^4, x^6, x^7, x^8, x^{10}, v_1^1, v_1^1, v_1^1, v_1^1, v_1^1, v_2^1, v_2^1, v_2^1, v_2^1, v_2^1, v_2^2, v_2^2, v_2^3, u^4)
\end{align*}
\]
this is the case for its 5-th derivative, and thus \( \kappa_3 = 5 \).

In conclusion, in the constructed coordinates the time derivatives of the flat output up to the orders \( \kappa = (\kappa_1, \kappa_2, \kappa_3) \) with \( \kappa_1 = (1, 2), \kappa_2 = 2, \kappa_3 = 5 \) are given by
\[
\begin{align*}
y_{1, [\kappa_1 - 1]} &= \begin{bmatrix} \varphi_1^1 \\ \varphi_2^1 \\ \varphi_1^2 \\ \varphi_2^2 \end{bmatrix} = \begin{bmatrix} x^1 \\ x^2 \\ x^9 \end{bmatrix}, \\
y_{1, \kappa_1} &= \begin{bmatrix} \varphi_1^3 \\ \varphi_2^3 \\ \varphi_1^4 \\ \varphi_2^4 \end{bmatrix} = \begin{bmatrix} v_1^1 \\ v_1^1 \\ v_1^1 \end{bmatrix} \\
y_{2, [\kappa_2 - 1]} &= \begin{bmatrix} \varphi_3^1 \\ \varphi_3^2 \\ \varphi_3^3 \\ \varphi_3^4 \end{bmatrix} = \begin{bmatrix} x^5 \\ x^3 + x^4 v_1^1 \end{bmatrix}, \\
y_{2, \kappa_2} &= \varphi_3^3 = v_2^2 \\
y_{3, [\kappa_3 - 1]} &= \begin{bmatrix} \varphi_4^1 \\ \varphi_4^2 \\ \varphi_4^3 \\ \varphi_4^4 \end{bmatrix} = \begin{bmatrix} x^8 \\ x^4 x^7 v_1^1 - x^6 \\ x^8 v_1^1 + x^7(1 + v_2^1) \\ x^4 x^7(v_1^1)^2 + v_1^1(v_2^1 - x^6 + 1) + x^4(v_2^1 + 1) + x^8 v_1^1 + x^7 v_2^1 \\ h^1(x^4, x^6, x^7, x^8, x^{10}, v_1^1, v_1^1, v_1^1, v_1^1, v_1^1, v_2^1, v_2^1, v_2^1, v_2^1, v_2^2, v_2^2, v_2^2, v_2^3, u^4) \end{bmatrix} \\
y_{3, \kappa_3} &= \varphi_5^3 = h^2(x^4, x^6, x^7, x^8, x^{10}, v_1^1, v_1^1, v_1^1, v_1^1, v_1^1, v_2^1, v_2^1, v_2^1, v_2^1, v_2^2, v_2^2, v_2^2, v_2^3, u^4) = v_3^1.
\end{align*}
\]

The linearizing quasi-static feedback which introduces the derivatives \( v = y_n \) as inputs is obtained by substituting the above expressions (27) for \( y_{[\kappa - 1]} \) and \( y_{[\kappa, R]} = u_{[R - \kappa]} \) into the flat parameterization \( u = F_u(y_{[R]}) \) of the control inputs (not stated here explicitly), which yields
\[
\begin{align*}
u_1^1 &= v_1^1, \\
u_2^1 &= v_2^1 - x^4 v_1^1, \\
u_3^1 &= v_3^1 - v_2^1 - x^{10} + x^4 v_1^1, \\
u_4^1 &= F_u(x^4, x^6, x^7, x^{10}, v_1^1, v_1^1, v_1^1, v_1^1, v_1^1, v_1^1, v_1^1, v_1^1, v_2^1, v_2^1, v_2^2, v_2^2, v_2^2, v_2^2, v_3^1). \\
\end{align*}
\]

The linearizing feedback (28) is of the desired form (6), i.e., it is a quasi-static feedback of the state \( x \) of the system.

### 4.2. 3D gantry crane
Consider a gantry crane as in Fig. 1. The trolley position is denoted by \( x_T \) and \( y_T \). The length of
A state representation of these equations of motion is given by

\begin{align*}
(m_T + m_L)\ddot{x}_T + m_L R \sin(\beta) \ddot{\phi} + \phi \cos(\beta) \ddot{\beta} + m_L R \ddot{\beta}(2 \cos(\beta) \dot{\phi} - \phi \sin(\beta) \ddot{\beta}) &= u^1 \\
(m_B + m_T + m_L)\ddot{y}_T + m_L R \sin(\alpha) \cos(\beta) \ddot{\phi} + m_L R \phi(\cos(\alpha) \cos(\beta) \ddot{\phi} - \sin(\alpha) \sin(\beta) \ddot{\beta}) - m_L R \left( \sin(\alpha)(\phi \cos(\beta)(\dddot{\phi}^2 + \dddot{\beta}^2) + 2 \sin(\beta) \dddot{\phi} \dddot{\beta}) + 2 \cos(\alpha) \dddot{\phi}(\phi \sin(\beta) \dddot{\phi} - \cos(\beta) \dddot{\beta}) \right) &= u^2 \\
m_L R \ddot{\phi}(\dddot{x}_T + m_L R \sin(\alpha) \cos(\beta) \dddot{y}_T + (J + m_L R^2) \dddot{\phi} - m_L R \left( R \phi(\dddot{\beta}^2 + \cos(\beta) \dddot{\beta}^2) + g \cos(\alpha) \cos(\beta) \right) &= u^3 \\
R \phi \cos(\alpha) \cos(\beta) \dddot{y}_T + (R \phi \cos(\beta))^2 \dddot{\phi} + R \phi \cos(\beta) \left( 2 R \dddot{\phi}(\cos(\beta) \dddot{\phi} - \phi \sin(\beta) \dddot{\beta}) + g \sin(\alpha) \right) &= 0 \\
R \phi \cos(\beta) \dddot{x}_T - R \phi \sin(\alpha) \sin(\beta) \dddot{y}_T + (R \phi)^2 \dddot{\beta} + R \phi \left( 2 R \dddot{\phi} + R \phi \sin(\beta) \cos(\beta) \dddot{\phi}^2 + \cos(\alpha) \sin(\beta) g \right) &= 0.
\end{align*}

A state representation of these equations of motion is given by

\begin{align*}
\dot{x}_T &= v_{x_T} \\
\dot{v}_{x_T} &= f_{v_{x_T}}(\phi, \alpha, \beta, \omega_\alpha, \omega_\beta, \omega_\phi, \omega_\beta, u^1, u^2, u^3) \\
\dot{y}_T &= v_{y_T} \\
\dot{v}_{y_T} &= f_{v_{y_T}}(\phi, \alpha, \beta, \omega_\alpha, \omega_\beta, \omega_\phi, \omega_\beta, u^1, u^2, u^3) \\
\dot{\phi} &= \omega_\phi \\
\dot{\omega}_\phi &= f_{\omega_\phi}(\phi, \alpha, \beta, \omega_\alpha, \omega_\beta, \omega_\phi, \omega_\beta, u^1, u^2, u^3) \\
\dot{\alpha} &= \omega_\alpha \\
\dot{\omega}_\alpha &= f_{\omega_\alpha}(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, u^1, u^2, u^3) \\
\dot{\beta} &= \omega_\beta \\
\dot{\omega}_\beta &= f_{\omega_\beta}(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, u^1, u^2, u^3).
\end{align*}

Figure 1. Schematic diagram of a 3D gantry crane.
The load position given by $y = (x_T + R\phi \sin(\beta), y_T + R\phi \sin(\alpha) \cos(\beta), R\phi \cos(\alpha) \cos(\beta))$ forms an $(x,u)$-flat (actually $x$-flat) output of the system. In the following, we apply the procedure of Section 3.3 to obtain a multi-index $\kappa$ with the properties stated in Theorem 2. Subsequently, we derive a quasi-static feedback (24) which introduces the derivatives $v = y_\kappa$ of the flat output as new inputs, and thus exactly linearizes the system.

**Step 1:** Differentiating the components of the flat output along the corresponding vector field (8), we find that $K_1 = (2, 2, 2)$ with the corresponding time derivatives being of the form

$$
\varphi_{K_1} = \begin{bmatrix}
\varphi_1^1(\phi, \alpha, \beta, \omega_\alpha, \omega_\beta, u^1, u^2, u^3) \\
\varphi_2^2(\phi, \alpha, \beta, \omega_\alpha, \omega_\beta, u^1, u^2, u^3) \\
\varphi_3^3(\phi, \alpha, \beta, \omega_\alpha, \omega_\beta, u^1, u^2, u^3)
\end{bmatrix}
$$

and $\text{rank}(\partial_u \varphi_{K_1}) = 1$. We can choose e.g. $\varphi_1 = \varphi^3$ and thus $\varphi_{\text{rest}_1} = (\varphi^1, \varphi^2)$, and apply the change of coordinates

$$
v_1^1 = \varphi_3^3, \quad v_1^{1,1} = \varphi_3^3, \quad \ldots,
$$

by which we replace the input $u^3$ and its derivatives by $v_1$ and its derivatives, and keep $u_{\text{rest}_1} = (u^1, u^2)$ and its derivatives as coordinates. This results in

$$
\varphi_{1, \kappa_1} = \varphi_3^3 = v_1^1, \quad \varphi_{\text{rest}_1, \kappa_{\text{rest}_1}} = \begin{bmatrix}
\varphi_1^1 \\
\varphi_2^2
\end{bmatrix} = \begin{bmatrix}
\varphi_2^1(\alpha, \beta, v_1^1) \\
\varphi_2^2(\alpha, v_1^1)
\end{bmatrix}.
$$

Note that the choice $\varphi_1 = \varphi^3$ is not unique but the most practical one, since the other two possible choices $\varphi_1 = \varphi^1$ and $\varphi_1 = \varphi^2$ would lead to feedback laws with singularities for $\alpha = 0$ or $\beta = 0$.

**Step 2:** We proceed by differentiating $\varphi_{\text{rest}_1} = (\varphi^1, \varphi^2)$ until an explicit dependence on $u_{\text{rest}_1} = (u^1, u^2)$ occurs. This is the case for the 4-th derivatives

$$
L^4_{f_\alpha} \varphi^1 = \varphi_4^1(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, v_1^1, v_{1,1}^1, v_{1,2}^1, u^1),
$$

$$
L^4_{f_\alpha} \varphi^2 = \varphi_4^2(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, v_1^1, v_{1,1}^1, v_{1,2}^1, u^2),
$$

i.e., $K_2 = (4, 4)$. Because of $m_2 = \text{rank}(\partial_u \varphi_{\text{rest}_1, K_{\text{rest}_1}}) = 2$, the procedure terminates at this point and we have $\kappa_2 = K_2$.

In conclusion, in the constructed coordinates the time derivatives of the flat output up to the
orders $\kappa = (\kappa_1, \kappa_2)$ with $\kappa_1 = 2$, $\kappa_2 = (4, 4)$ are of the form

$$y_{1,[\kappa_1]} = \begin{bmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{bmatrix} = \begin{bmatrix} \varphi^1(\phi, \alpha, \beta) \\ \varphi^1(v_T, \phi, \beta, \omega_\phi, \omega_\beta) \\ \varphi^2(\alpha, \beta, v_1) \\ \varphi^3(\alpha, \beta, \omega_\alpha, \omega_\beta, v_1, v_{1,1}) \\ \varphi^2(\phi, \alpha, \beta, v_{1T}, \omega_\phi, \omega_\beta) \\ \varphi^3(\phi, \alpha, \beta, \omega_\alpha, \omega_\beta, v_1, v_{1,1}) \end{bmatrix}$$

$$y_{1,\kappa_1} = \varphi^2 = v_1$$

$$y_{2,[\kappa_2]} = \begin{bmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \\ \varphi^4 \\ \varphi^5 \\ \varphi^6 \end{bmatrix} = \begin{bmatrix} \varphi^1(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, v_1, v_{1,1}, v_{1,2}, u_1^1) \\ \varphi^2(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, v_1, v_{1,1}, v_{1,2}, u_2^1) \end{bmatrix} = \begin{bmatrix} v_2^1 \\ v_2^2 \end{bmatrix}$$

The quasi-static feedback which introduces the derivatives $v = y_\kappa$ as inputs is obtained by substituting the above expressions (30) for $y_{[\kappa_1]}$ and $y_{[\kappa,R]} = v_{[R-\kappa]}$ into the flat parameterization $u = F_u(y_{[R]})$ of the control inputs (not stated here explicitly), which yields

$$u^1 = \tilde{F}_u^1(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, v_1, v_{1,1}, v_{1,2}, v_2^1)$$

$$u^2 = \tilde{F}_u^2(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, v_1, v_{1,1}, v_{1,2}, v_2^2)$$

$$u^3 = \tilde{F}_u^3(\phi, \alpha, \beta, \omega_\phi, \omega_\alpha, \omega_\beta, v_1, v_{1,1}, v_{1,2}, v_2^1)$$

(31)

The linearizing feedback (31) is again of the desired form (6).

5. Tracking control design for $(x, u)$-flat systems

In Subsection 3.3, we have shown that every $(x, u)$-flat system (1) can be exactly linearized with respect to every $(x, u)$-flat output (17) by a quasi-static feedback of the state $x$. Thus, for an exact linearization, the usage of a generalized Brunovsky state can be avoided. However, as already mentioned, the exact linearization is often just an intermediate step in the design of a flatness-based tracking control. The standard approach for the stabilization of the tracking error dynamics that can be found in e.g., Delaleau and Rudolph (1998) makes again use of a generalized Brunovsky state. Therefore, in the following, we how to derive tracking control laws of the form $u = \eta(x, y_{[R]}^d(t))$, which only involve the classical state $x$ and the reference trajectory $y^d(t)$, on the basis of the exact linearization of Corollary 1.

The exact linearization of Corollary 1 was achieved by a quasi-static feedback

$$u = F_u \circ \Psi(x, v_{[R-\kappa]}) = \tilde{F}_u(x, v_{[R-\kappa]})$$

(32)

which results in the linear input-output behavior $y_\kappa = v$ between the flat output $y$ and the new input $v$. Recall that the multi-index $\kappa$ is obtained by applying the procedure stated in Section 3.3 and that the linearizing feedback (32) then follows directly by substituting the expressions for $y_{[\kappa_1]}$ from (23) and $y_{[\kappa,R]} = v_{[R-\kappa]}$ into the flat parameterization $u = F_u(y_{[R]})$. In the following, the special structure (23) of the derivatives of the flat output in the coordinates successively introduced during
the procedure will again be crucial. Given a sufficiently often differentiable reference trajectory \( y^d(t) \), the control law

\[
v^j_i = y^j_i, d_i - \sum_{\beta=0}^{\kappa^j_i - 1} a^j_i, \beta (y^j_i, d_i - y^j_i, d_i) , \quad i = 1, \ldots, e, \quad j_i = 1, \ldots, m_i \quad (33)
\]

for the input \( v \) results in the linear tracking error dynamics

\[
e^j_i = \sum_{\beta=0}^{\kappa^j_i - 1} a^j_i, \beta e^j_i = 0 ,
\]

where \( e^j_i = y^j_i - y^j_i, d_i \). The roots of the characteristic polynomials of the tracking error dynamics can be adjusted by the coefficients \( a^j_i, \beta \in \mathbb{R} \). The time derivatives \( y^{[\kappa_i-1]} \) occurring in \((33)\) can be expressed in terms of \( x \) and \( v^{[R-\kappa_i-1]} \), see \((23)\). Substituting the corresponding expressions for \( y^{[\kappa_i-1]} \) of \((23)\) into \((33)\) yields

\[
v^j_1 = y^j_1, d_1 - \sum_{\beta=0}^{\kappa^j_1 - 1} a^j_1, \beta (\varphi^j_1, \beta (x) - y^j_1, d_1) , \quad j_1 = 1, \ldots, m_1 \\
v^j_2 = y^j_2, d_2 - \sum_{\beta=0}^{\kappa^j_2 - 1} a^j_2, \beta (\varphi^j_2, \beta (x, v_1, v_1, \ldots) - y^j_2, d_2) , \quad j_2 = 1, \ldots, m_2 \\
v^j_3 = y^j_3, d_3 - \sum_{\beta=0}^{\kappa^j_3 - 1} a^j_3, \beta (\varphi^j_3, \beta (x, v_1, v_1, \ldots, v_2, v_2, \ldots) - y^j_3, d_3) , \quad j_3 = 1, \ldots, m_3 \\
\vdots \\
v^j_e = y^j_e, d_e - \sum_{\beta=0}^{\kappa^j_e - 1} a^j_e, \beta (\varphi^j_e, \beta (x, v_1, v_1, \ldots, v_{e-1}, v_{e-1}, \ldots) - y^j_e, d_e) , \quad j_e = 1, \ldots, m_e .
\]

Due to the special structure of the time derivatives \( y^{[\kappa_i-1]} \) of the flat output in \((23)\), the equations \((34)\) depend on the time derivatives \( v^{[R-\kappa_i]} \) in a triangular manner. The right-hand sides of the equations for \( v_1 \) do not depend on \( v \) at all, the right-hand sides of the equations for \( v_2 \) only depend on \( v_1 \) and its time derivatives, and so on. The complete set of time derivatives \( v^{[R-\kappa]} \), which is needed for practically realizing the control law \((33)\) for the input \( v \) via the actual control input \( u \), i.e., via the feedback \((32)\), can be determined systematically by differentiating the equations \((34)\)
with respect to time and solving the resulting system of equations

\[ v_1^{j_1} = y_{1,\kappa_1}^{j_1,d} - \sum_{\beta=0}^{\kappa_1^{j_1}-1} a_1^{j_1,\beta}(\varphi_{1,\beta}^{j_1}(x) - y_{1,\beta}^{j_1,d}), \quad j_1 = 1, \ldots, m_1 \]

\[ v_{1,1}^{j_1} = y_{1,\kappa_1+1}^{j_1,d} - a_1^{j_1,\kappa_1^{j_1}-1}(v_1^{j_1} - y_{1,\kappa_1}^{j_1,d}) - \sum_{\beta=0}^{\kappa_1^{j_1}-2} a_1^{j_1,\beta}(\varphi_{1,\beta+1}^{j_1}(x) - y_{1,\beta+1}^{j_1,d}) \]

\[ : \]

\[ v_2^{j_2} = y_{2,\kappa_2}^{j_2,d} - \sum_{\beta=0}^{\kappa_2^{j_2}-1} a_2^{j_2,\beta}(\varphi_{2,\beta}^{j_2}(x, v_1, v_1, \ldots) - y_{2,\beta}^{j_2,d}), \quad j_2 = 1, \ldots, m_2 \]

\[ v_{2,1}^{j_2} = y_{2,\kappa_2+1}^{j_2,d} - a_2^{j_2,\kappa_2^{j_2}-1}(v_2^{j_2} - y_{2,\kappa_2}^{j_2,d}) - \sum_{\beta=0}^{\kappa_2^{j_2}-2} a_2^{j_2,\beta}(\varphi_{2,\beta+1}^{j_2}(x, v_1, v_1, \ldots) - y_{2,\beta+1}^{j_2,d}) \]

\[ : \]

\[ v_3^{j_3} = y_{3,\kappa_3}^{j_3,d} - \sum_{\beta=0}^{\kappa_3^{j_3}-1} a_3^{j_3,\beta}(\varphi_{3,\beta}^{j_3}(x, v_1, v_1, \ldots, v_2, v_2, \ldots) - y_{3,\beta}^{j_3,d}), \quad j_3 = 1, \ldots, m_3 \]

\[ v_{3,1}^{j_3} = y_{3,\kappa_3+1}^{j_3,d} - a_3^{j_3,\kappa_3^{j_3}-1}(v_3^{j_3} - y_{3,\kappa_3}^{j_3,d}) - \sum_{\beta=0}^{\kappa_3^{j_3}-2} a_3^{j_3,\beta}(\varphi_{3,\beta+1}^{j_3}(x, v_1, v_1, \ldots, v_2, v_2, \ldots) - y_{3,\beta+1}^{j_3,d}) \]

\[ : \]

\[ v_e^{j_e} = y_{e,\kappa_e}^{j_e,d} - \sum_{\beta=0}^{\kappa_e^{j_e}-1} a_e^{j_e,\beta}(\varphi_{e,\beta}^{j_e}(x, v_1, v_1, \ldots, v_{e-1}, v_{e-1}, \ldots) - y_{e,\beta}^{j_e,d}), \quad j_e = 1, \ldots, m_e \]

from top to bottom for the time derivatives \( v_{i,\beta}^{j_i,\beta} \) as a function of \( x \) and \( y_{i,R}^{d} \); i.e., \( v_{i,R-\kappa} = \phi(x, y_{i,R}^{d}) \).

Substituting this solution into the feedback (32) yields a tracking control law of the desired form

\[ u = \eta(x, y_{R}^{d}) \]

which only depends on the state \( x \) and (time derivatives of) the reference trajectory.

6. Examples continued

In this section, we derive tracking control laws for the two systems that were already exactly linearized in Section 4.

6.1. Academic example

Consider again the system (25) with the flat output \( y = (y_1, y_2, y_3) \) where \( y_1 = (y_1, y_2) = (x_1, x_2^2) \), \( y_2 = y_3^2 = x_5 \), \( y_3 = y_4 = x_8 \). In Section 4 we have derived the linearizing feedback (28), which introduces \( v = y_5 \) with \( \kappa = (\kappa_1, \kappa_2, \kappa_3) \) and \( \kappa_1 = (1, 2), \kappa_2 = 2, \kappa_3 = 5 \) as new input. The control
law \(33\) for \(v\) is thus given by

\[
\begin{align*}
v_1^1 &= y_1^{1d} - a_1^{1.0}(y_1^{1} - y_1^{1.d}) \\
v_1^2 &= y_2^{2d} - a_2^{0.0}(y_2^2 - y_2^{2.d}) - a_2^{1.1}(y_2^1 - y_2^{1.d}) \\
v_2^1 &= y_2^{3d} - a_2^{1.0}(y_3 - y_3^{d.3}) - a_2^{1.1}(y_1 - y_1^{d.3}) \\
v_2^2 &= y_3^{4d} - a_3^{1.0}(y_4 - y_4^{d.4}) - a_3^{1.1}(y_1^4 - y_1^{4.d}) - a_3^{1.2}(y_2^4 - y_2^{4.d}) - a_3^{1.3}(y_3^4 - y_3^{4.d}) - a_3^{1.4}(y_4^4 - y_4^{4.d}).
\end{align*}
\]

Substituting the corresponding expressions \(27\) for \(y_{[\kappa]}\) into this control law yields

\[
\begin{align*}
v_1^1 &= y_1^{1d} - a_1^{1.0}(\varphi_1^1(x) - y_1^{d.1}) \\
v_1^2 &= y_2^{2d} - a_2^{2.0}(\varphi_2^2(x) - y_2^{d.2}) - a_2^{2.1}(\varphi_2^1(x) - y_1^{d.2}) \\
v_2^1 &= y_2^{3d} - a_2^{1.0}(\varphi_3^3(x) - y_3^{d.3}) - a_2^{1.1}(\varphi_1^3(x,v_1^1) - y_1^{d.3}) \\
v_2^2 &= y_3^{4d} - a_3^{1.0}(\varphi_4^4(x) - y_4^{d.4}) - a_3^{1.1}(\varphi_1^4(x,v_1^1) - y_1^{d.4}) - a_3^{1.2}(\varphi_2^4(x,v_1^1,v_2^1) - y_2^{d.4}) \\
&
\quad - a_3^{1.3}(\varphi_3^4(x,v_1^1,v_1^1,v_2^1,v_2^1,v_2^1,v_2^1,v_2^1) - y_3^{d.4}) - a_3^{1.4}(\varphi_4^4(x,v_1^1,v_1^1,v_1^1,v_2^1,v_2^1,v_2^1,v_2^1,v_2^1) - y_4^{d.4}).
\end{align*}
\]

In order to express the required time derivatives \(v_{[R-\kappa]}\) in terms of \(x\) and \(y_{[R]}\), we differentiate the equations for \(v\) up to the order \(R - \kappa\), which yields

\[
\begin{align*}
v_1^1 &= y_1^{1d} - a_1^{1.0}(\varphi_1^1(x) - y_1^{d.1}) \\
v_1^1 &= y_1^{1d} - a_1^{1.0}(v_1^1 - y_1^{d.1}) \\
v_1^1 &= y_3^{1d} - a_1^{1.0}(v_1^{1,d.1} - y_1^{d.1}) \\
v_1^1 &= y_1^{1d} - a_1^{1.0}(y_1^1 - y_1^{d.1}) \\
v_1^2 &= y_2^{2d} - a_2^{2.0}(\varphi_2^2(x) - y_2^{d.2}) - a_2^{2.1}(\varphi_1^2(x) - y_1^{d.2}) \\
v_1^2 &= y_2^{2d} - a_2^{1.0}(y_2^1 - y_2^{d.2}) - a_2^{2.1}(y_1^1 - y_1^{d.2}) \\
v_1^2 &= y_3^{3d} - a_2^{1.0}(\varphi_3^3(x,v_1^1,v_1^2,v_1^2,v_1^2,v_1^2,v_1^1) - y_3^{d.3}) - a_2^{1.1}(\varphi_2^3(x,v_1^1,v_1^1,v_1^2,v_2^2,v_2^2,v_2^2) - y_2^{d.3}) \\
v_1^2 &= y_3^{3d} - a_2^{1.0}(v_2^1 - y_2^{d.3}) - a_2^{1.1}(v_2^1 - y_2^{d.2}) \\
v_1^2 &= y_4^{3d} - a_2^{1.0}(v_2^1 - y_2^{d.3}) - a_2^{1.1}(v_2^1 - y_2^{d.2}) \\
v_1^2 &= y_5^{3d} - a_2^{1.0}(v_2^1 - y_2^{d.3}) - a_2^{1.1}(v_2^1 - y_2^{d.2}) \\
v_1^2 &= y_5^{4d} - a_3^{1.0}(\varphi_4^4(x) - y_4^{d.4}) - a_3^{1.1}(\varphi_1^4(x,v_1^1) - y_1^{d.4}) - a_3^{1.2}(\varphi_2^4(x,v_1^1,v_1^1) - y_2^{d.4}) \\
&
\quad - a_3^{1.3}(\varphi_3^4(x,v_1^1,v_1^1,v_1^1,v_1^1) - y_3^{d.4}) - a_3^{1.4}(\varphi_4^4(x,v_1^1,v_1^1,v_1^1,v_1^1,v_1^1) - y_4^{d.4}).
\end{align*}
\]

This system of equations can easily be solved from top to bottom in order to obtain \(v_{[R-\kappa]}\) in terms of \(x\) and \(y_{[R]}\). Inserting the solution into the linearizing feedback \(28\) yields a tracking control law
of the form

\[ u^1 = y_1^{d,1} + a_1^{1,0}(y_1^{d,d} - x^1) \]
\[ u^2 = \eta^2(x^1, x^2, x^3, x^4, x^5, y_1^{d,1}, y_1^{d,d}, y_2^{d,1}, y_2^{d,d}) \]
\[ u^3 = \eta^3(x^1, x^2, x^3, x^4, x^5, x^6, x^7, y_1^{d,1}, y_1^{d,d}, y_2^{d,1}, y_2^{d,d}) \]
\[ u^4 = \eta^4(x^1, \ldots, x^10, y_1^{d,1}, y_1^{d,d}, \ldots, y_4^{d,1}, y_4^{d,d}, \ldots, y_4^{d,d}) \]

\[ v^1 = y_2^{d,3} - a_1^{1,0}(y_1^{d,3} - y_4^{d,3}) - a_1^{1,1}(y_3^{d,3} - y_4^{d,3}) \]
\[ v^2 = y_4^{d,1} - a_2^{1,0}(y_1^{d,1} - y_2^{d,1}) - a_2^{1,1}(y_2^{d,1} - y_2^{d,1}) - a_2^{1,3}(y_3^{d,1} - y_3^{d,1}) \]
\[ v^2 = y_4^{d,2} - a_2^{2,0}(y_2^{d,2} - y_2^{d,2}) - a_2^{2,1}(y_2^{d,2} - y_2^{d,2}) - a_2^{2,3}(y_3^{d,2} - y_3^{d,2}) \]

6.2. 3D gantry crane

Consider again the system (29) with the flat output \( y = (y_1, y_2) \) where \( y_1 = y^3 = (R\phi \cos(\alpha) \cos(\beta)) \), \( y_2 = (y^1, y^2) = (x_T + R\phi \sin(\beta), y_T + R\phi \sin(\alpha) \cos(\beta)) \). In Section 4 we have derived the linearizing feedback (31), which introduces \( v = y_\kappa \) with \( \kappa = (\kappa_1, \kappa_2) \) and \( \kappa_1 = 2, \kappa_2 = (4, 4) \) as new input. The control law (33) for \( v \) is thus given by

\[ v^1_1 = y_2^{3,3} - a_1^{1,0}(y_1^{3,3} - y_3^{3,3}) - a_1^{1,1}(y_3^{3,3} - y_3^{3,3}) \]
\[ v^2_1 = y_4^{3,3} - a_2^{1,0}(y_1^{3,3} - y_2^{3,3}) - a_2^{1,1}(y_2^{3,3} - y_2^{3,3}) - a_2^{1,3}(y_3^{3,3} - y_3^{3,3}) \]
\[ v^2_2 = y_4^{3,4} - a_2^{2,0}(y_2^{3,4} - y_2^{3,4}) - a_2^{2,1}(y_2^{3,4} - y_2^{3,4}) - a_2^{2,3}(y_3^{3,4} - y_3^{3,4}) \]

Substituting the corresponding expressions (30) for \( y_\kappa = 1 \) into this control law and differentiating it with respect to time yields

\[ v^1_1 = y_2^{3,3} - a_1^{1,0}(\varphi^3(\phi, x_\phi, x_\beta, \omega_\phi, \omega_\alpha, \omega_\beta) - y_1^{3,3}) \]
\[ v^2_1 = y_4^{3,3} - a_1^{1,0}(v_1^{3,3} - y_3^{3,3}) - a_1^{1,1}(v_1^{3,3} - y_3^{3,3}) \]
\[ v^2_2 = y_4^{3,4} - a_2^{1,0}(\varphi^3(\phi, x_\phi, x_\beta, \omega_\phi, \omega_\alpha, \omega_\beta) - y_1^{3,4}) \]
\[ v^2_2 = y_4^{3,4} - a_2^{2,0}(\varphi^2(\phi, x_\phi, x_\beta, \omega_\phi, \omega_\alpha, \omega_\beta) - y_1^{3,4}) \]
\[ v^2_2 = y_4^{3,4} - a_2^{2,0}(\varphi^2(\phi, x_\phi, x_\beta, \omega_\phi, \omega_\alpha, \omega_\beta) - y_1^{3,4}) \]
\[ v^2_2 = y_4^{3,4} - a_2^{2,0}(\varphi^2(\phi, x_\phi, x_\beta, \omega_\phi, \omega_\alpha, \omega_\beta) - y_1^{3,4}) \]

This system of equations can easily be solved from top to bottom in order to obtain \( v_{R-\kappa} \) in terms of \( x \) and \( y_{d,R} \). Inserting the solution into the linearizing feedback (31) yields a tracking control law of the form

\[ u^1 = \eta^1(x_T, \phi, x_\beta, v_\phi, \omega_\phi, \omega_\alpha, \omega_\beta, y_1^{1,1}, x_1^{1,1}, x_2^{1,1}, x_3^{1,1}, y_1^{3,3}, y_1^{3,3}, y_2^{3,3}, y_3^{3,3}, y_4^{3,3}) \]
\[ u^2 = \eta^2(y_T, \phi, x_\beta, v_\phi, \omega_\phi, \omega_\alpha, \omega_\beta, y_2^{2,2}, x_1^{2,2}, x_2^{2,2}, x_3^{2,2}, y_1^{3,3}, y_1^{3,3}, y_2^{3,3}, y_3^{3,3}, y_4^{3,3}) \]
\[ u^3 = \eta^3(x_T, y_T, \phi, x_\beta, v_\phi, \omega_\phi, \omega_\alpha, \omega_\beta, y_1^{1,1}, x_1^{1,1}, x_2^{1,1}, x_3^{1,1}, y_1^{3,3}, y_1^{3,3}, y_2^{3,3}, y_3^{3,3}, y_4^{3,3}) \]
7. Conclusions

We have approached the problem of the exact linearization of flat systems \((\text{1})\) by investigating which time derivatives of a flat output \((\text{9})\) are suitable as new (closed-loop) input. We have given easily verifiable conditions and have shown how to systematically construct the feedback required for actually introducing the new input. Subsequently, we have proven that for every \((x,u)\)-flat output \((\text{7})\) of a system \((\text{1})\) there exists an \(m\)-tuple \(\kappa\) with \(#\kappa = n\) such that \(v = y_\kappa\) is a feasible input, and that such an input can be introduced by a quasi-static feedback of the state \(x\). Compared to the well-known exact linearization by quasi-static feedback of a generalized Brunovsky state, our approach has the advantage that it requires only knowledge of the state \(x\) and not of time derivatives of the flat output up to a certain order. Furthermore, we have shown that on the basis of such an exact linearization it is even possible to systematically design a tracking control which only depends on the state \(x\) and the reference trajectory, i.e., again without using a generalized Brunovsky state. Future research will address the exact linearization with respect to general flat outputs \((\text{9})\), which may also depend on time derivatives of the input \(u\).

Acknowledgments

The authors would like to thank J. Rudolph and A. Irscheid for interesting discussions concerning the exact linearization by quasi-static feedback of generalized states.

References

Delaleau, E., & Rudolph, J. (1995). Decoupling and linearization by quasi-static feedback of generalized states. Proceedings 3rd European Control Conference (ECC), 1069–1074.
Delaleau, E., & Rudolph, J. (1998). Control of flat systems by quasi-static feedback of generalized states. International Journal of Control, 71(5), 745–756.
Fliess, M., Lévine, J., Martin, P., & Rouchon, P. (1992). Sur les systèmes non linéaires différentiellement plats. Comptes rendus de l’Académie des sciences. Série I, Mathématique, 315, 619–624.
Fliess, M., Lévine, J., Martin, P., & Rouchon, P. (1995). Flatness and defect of non-linear systems: introductory theory and examples. International Journal of Control, 61(6), 1327–1361.
Fliess, M., Lévine, J., Martin, P., & Rouchon, P. (1999). A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. IEEE Transactions on Automatic Control, 44(5), 922–937.
Göstöttner, C., Kolar, B., & Schöberl, M. (2020a). On the linearization of flat two-input systems by prolongations and applications to control design. IFAC-PapersOnLine, 53(2), 5479-5486. (21st IFAC World Congress)
Göstöttner, C., Kolar, B., & Schöberl, M. (2020b). A structurally flat triangular form based on the extended chained form. International Journal of Control.
Göstöttner, C., Kolar, B., & Schöberl, M. (2021). Necessary and sufficient conditions for the linearizability of two-input systems by a two-dimensional endogenous dynamic feedback. arXiv:2106.14722 [math.DS].
Isidori, A. (1995). Nonlinear control systems (3rd ed.). London: Springer.
Kolar, B. (2017). Contributions to the differential geometric analysis and control of flat systems. Aachen: Shaker Verlag.
Kolar, B., Rams, H., & Schlacher, K. (2017). Time-optimal flatness based control of a gantry crane. Control Engineering Practice, 60, 18-27.
Kolar, B., Schöberl, M., & Schlacher, K. (2016). Properties of flat systems with regard to the parameterization of the system variables by the flat output. IFAC-PapersOnLine, 49(18), 814-819. (10th IFAC Symposium on Nonlinear Control Systems)
Kugi, A., & Kiefer, T. (2005). Nichtlineare Trajektorienfolgeregelung für einen Laborhelikopter. e&i Elektrotechnik und Informationstechnik, 122(9), 300–307.
Nicolau, F., & Respondek, W. (2017). Flatness of multi-input control-affine systems linearizable via one-fold prolongation. *SIAM J. Control and Optimization, 55*, 3171-3203.

Nicolau, F., & Respondek, W. (2019). Normal forms for multi-input flat systems of minimal differential weight. *International Journal of Robust and Nonlinear Control, 29*(10), 3139-3162.

Nijmeijer, H., & van der Schaft, A. (1990). *Nonlinear dynamical control systems*. New York: Springer.

Rudolph, J., & Delaleau, E. (1998). Some examples and remarks on quasi-static feedback of generalized states. *Automatica, 34*(8), 993-999.