On the Area of Pedal and Antipedal Triangles

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Abstract

We give a new proof of the formula expressing the area of the triangle whose vertices are the projections of an arbitrary point in the plane onto the sides of a given triangle, in terms of the geometry of the given triangle and the location of the projection point. Other related geometrical constructions and formulas are also presented.

1 Introduction

The setting in which all results of this paper are stated is that of a two-dimensional Euclidean plane. Given a point \( P \) in the plane of a given triangle \( \triangle ABC \), denoted by \( \Delta ABC \), a triangle \( \triangle A'B'C' \) is called the pedal triangle of \( P \) with respect to \( \Delta ABC \) if \( A', B' \) and \( C' \) are the projections of \( P \) onto the lines \( BC, AC, \) and \( AB \), respectively. The point \( P \) will be called the pedal point of \( \triangle A'B'C' \) with respect to \( \Delta ABC \) (see e.g., [1], [3]). Figures 1 and 2 depict two examples of pedal triangles, corresponding to the point \( P \) being inside and outside of \( \Delta ABC \), respectively.

![Figure 1](image1.png)  
![Figure 2](image2.png)

A familiar instance of a pedal triangle is the orthic triangle of a given triangle, the pedal triangle of the orthocenter of the given triangle, the meeting point of its three altitudes.

The main goal of this note is to give a new proof to the formula for the area of a pedal triangle of a point, relative to a fixed triangle. This formula takes into

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consideration, besides the geometrical characteristics of this fixed triangle, only the location of the pedal point. With the convention that $|\triangle XYZ|$ denotes the area of the triangle $XYZ$, the following holds:

**Theorem 1.1.** Let $\triangle ABC$ be a given triangle in the plane, and denote by $O$ and $R$ the center and the radius of the circumcircle, respectively. Let $P$ be an arbitrary point in the plane of $\triangle ABC$, and let $\triangle A'B'C'$ be the pedal triangle of $P$ with respect to $\triangle ABC$. Then

$$\frac{|\triangle A'B'C'|}{|\triangle ABC|} = \frac{|R^2 - OP^2|}{4R^2}. \quad (1.1)$$

This is a classical result. However, the proofs existing in the literature (we are aware of [2] and [4]) are rather complex and involved. Here we present an approach of algebraic nature, which is considerably more economical and direct. In addition, as consequences of Theorem 1.1 we note a couple of results of independent interest.

**Corollary 1.2.** Given a fixed triangle $ABC$, denote by $\triangle A'B'C'$ the pedal triangle of an arbitrary point $P$ with respect to $\triangle ABC$. Then the locus of all points $P$ such that the ratio of the area of $\triangle A'B'C'$ to that of $\triangle ABC$ is a fixed, given constant, is a circle concentric with the circumcircle of $\triangle ABC$.

**Corollary 1.3.** The locus of all points with the property that their projections onto the sides of a given triangle $ABC$ are three collinear points is the circumcircle of $\triangle ABC$.

These are both obvious from (1.1). Corollary 1.3 is usually attributed to Simson, and our contribution is to provide a conceptually new proof of this well-known fact.

$\triangle TUV$ is called the antipedal triangle of a point $K$ with respect to $\triangle ABC$ if the lines $KA, KB, KC$ are perpendicular to $VU, TV$ and $TU$, at points $A, B, C$, respectively. Consequently, it follows that $\triangle TUV$ is the antipedal triangle of a point $K$ with respect to $\triangle ABC$ if and only if $\triangle ABC$ is the pedal triangle of the point $K$ with respect to $\triangle TUV$. In the examples from Figure 1 and Figure 2, $\triangle ABC$ is the antipedal triangle of point $P$ with respect to $\triangle A'B'C'$.

For a given triangle $ABC$, line $AD'$ is isogonal to line $AD$ in $\angle BAC$ if the bisector of $\angle BAC$ is also the bisector of $\angle DAD'$; that is, if $AD'$ is the reflection of $AD$ in the bisector of $\angle BAC$. If lines $AD, BE, CF$ meet at a point $K$ then it is known that their isogonals $AD', BE', CF'$ in their respective angles $BAC, BCA, and BCA$ concur in a point $K'$. The point $K'$ is called the isogonal of the point $K$ for the triangle $ABC$. As a natural counterpart to Theorem 1.1 we also derive a similar formula for the area of an antipedal triangle.

**Theorem 1.4.** If $\triangle TUV$ is the antipedal triangle of the point $K$ with respect to $\triangle ABC$ then

$$\frac{|\triangle TUV|}{|\triangle ABC|} = \frac{4R^2}{|R^2 - OK_1^2|}. \quad (1.2)$$

with $O$ being the center and $R$ the radius of the circumcircle of $\triangle ABC$, and $K_1$ being the isogonal of $K$ (see Figure 6).
2 The Proof of Theorem 1.1

The idea of this proof is to use analytic geometry in order to recast (1.1) as a quadratic equation in the coordinates of the variable point $P$. Note that, in this scheme, it is not necessary to carefully keep track of the specific values of the coefficients of the quadratic equation; indeed, it is only the very nature of the algebraic equation which plays a role. This is a general principle which could be useful for other types of problems as well.

Turning to specifics, consider the lines $AB$, $BC$, $AC$, given by the equations

$$
\alpha_C x + \beta_C y + \gamma_C = 0,
$$

$$
\alpha_A x + \beta_A y + \gamma_A = 0,
$$

$$
\alpha_B x + \beta_B y + \gamma_B = 0,
$$

respectively, where the signs of the corresponding coefficients for each line are selected such that if a point $P(x, y)$ is inside $\triangle ABC$, then $\alpha_C x + \beta_C y + \gamma_C > 0$, $\alpha_A x + \beta_A y + \gamma_A > 0$, $\alpha_B x + \beta_B y + \gamma_B > 0$. Also, for a point $P(x_1, y_1)$, we denote by $d_C$, $d_A$, and $d_B$ the directed distance from $P$ to $AB$, $BC$, and $AC$, respectively. As a result, we have explicit formulas for $d_C$, $d_A$, and $d_B$. For example,

$$
d_C = \frac{\alpha_C x_1 + \beta_C y_1 + \gamma_C}{\sqrt{\alpha_C^2 + \beta_C^2}}, \quad (2.3)
$$

and similar expressions hold for $d_A$ and $d_B$. We note that the area of $\triangle A'B'C'$ can be written as a linear combination of the areas of $\triangle A'PB'$, $\triangle A'PC'$, and $\triangle C'PB'$. More specifically, one has

$$
|\triangle A'B'C'| = \pm |\triangle A'PB'| \pm |\triangle A'PC'| \pm |\triangle C'PB'|
$$

(2.4)

where the selection of $+$ or $-$ is dictated by the location of the point $P$. For example, if $P$ is inside $\triangle ABC$ (as it is the case in Figure 1), then the area of $\triangle A'B'C'$ is equal to the sum of the areas of $\triangle A'PB'$, $\triangle A'PC'$, and $\triangle C'PB'$, which further implies that the signs of the each of the terms in (2.4) should be $+$. Another example, corresponding to the point $P$ as in Figure 2, leads to the formula $|\triangle A'B'C'| = |\triangle A'PB'| - |\triangle A'PC'| + |\triangle C'PB'|$, so the corresponding signs are $(+, -, +)$. A specific choice of the three signs in the right-hand side of (2.4) turns out to depend on the location of the point $P$ relative to the three lines $AB$, $BC$, and $AC$, as well as the circumcircle of $\triangle ABC$. Figure 3 shows all possible combinations of signs associated with the various sub-regions in which the plane is partitioned by the aforementioned lines and circle.
The directed distances introduced can be used to express the areas of each of the triangles in the right hand side of (2.4) as follows:

\[ |\Delta P'B'C'| = \left| \frac{d_B d_C \sin(\hat{A})}{2} \right|, \quad |\Delta A'PB'| = \left| \frac{d_A d_B \sin(\hat{C})}{2} \right|, \tag{2.5} \]

\[ |\Delta A'PC'| = \left| \frac{d_A d_C \sin(\hat{B})}{2} \right|. \tag{2.6} \]

The absolute values from the numerators in these formulas can be dropped by keeping careful track of the signs of the directed distances based on the position of \( P \). Figure 4 shows the signs of each component in the triplet \((d_B d_C, d_A d_C, d_A d_B)\) associated with the various sub-regions in which the plane is partitioned by the lines \( AB \), \( AC \), and \( BC \).

What is remarkable is the fact that after combining the signs of the fractions in (2.4), as depicted by Figure 3, with the signs of the products of the directed distance,
as depicted by Figure 4, formula (2.3), in junction with (2.5)-(2.6), becomes

$$\pm |\Delta A'B'C'| = \frac{d_B d_C \sin(\Delta A)}{2} + \frac{d_A d_B \sin(\Delta C)}{2} + \frac{d_A d_C \sin(\Delta B)}{2},$$

(2.7)

where + corresponds to the case when $P$ is contained in the circumcircle of $\Delta ABC$, with center $O$ (denoted by $(O)$), and $-\text{ corresponds to the case when } P$ is outside $(O)$. Making now use of (2.3) and the corresponding formulas for $d_A$, $d_B$, we can re-write (2.7) as

$$\pm |\Delta A'B'C'| = \frac{\alpha_A x_1 + \beta_A y_1 + \gamma_A}{\sqrt{\alpha_A^2 + \beta_A^2}} \cdot \frac{\alpha_C x_1 + \beta_C y_1 + \gamma_C}{\sqrt{\alpha_C^2 + \beta_C^2}} \cdot \sin(\Delta A)$$

$$+ \frac{\alpha_B x_1 + \beta_B y_1 + \gamma_B}{\sqrt{\alpha_B^2 + \beta_B^2}} \cdot \frac{\alpha_C x_1 + \beta_C y_1 + \gamma_C}{\sqrt{\alpha_C^2 + \beta_C^2}} \cdot \sin(\Delta B)$$

$$+ \frac{\alpha_A x_1 + \beta_A y_1 + \gamma_A}{\sqrt{\alpha_A^2 + \beta_A^2}} \cdot \frac{\alpha_B x_1 + \beta_B y_1 + \gamma_B}{\sqrt{\alpha_B^2 + \beta_B^2}} \cdot \sin(\Delta C).$$

(2.8)

In addition, using the fact that $|\Delta ABC|$ is a real constant that depends only on $A$, $B$ and $C$, (2.8) yields

$$\pm \frac{|\Delta A'B'C'|}{|\Delta ABC|} = \frac{(ax_1 + by_1 + c)(dx_1 + ey_1 + f)}{R^2 - OP^2}$$

$$+ \frac{(gx_1 + hy_1 + i)(jx_1 + ky_1 + l)}{R^2 - OP^2}$$

$$+ \frac{(mx_1 + ny_1 + o)(px_1 + qy_1 + r)}{R^2 - OP^2},$$

(2.9)

where $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q$ and $r$ are real constants that depend only on $A, B$ and $C$.

At this point, we observe that (1.1) becomes

$$\pm \frac{|\Delta A'B'C'|}{|\Delta ABC|} = \frac{R^2 - OP^2}{4R^2},$$

(2.10)

provided we select $+$ when $P$ is in $(O)$ and $-$ when $P$ is outside $(O)$. Clearly, the right hand side of (2.10) is a quadratic expression in $x_1$ and $y_1$. Hence, if we now take into account (2.10) and (2.9), we obtain that (1.1) is equivalent with

$$\lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 x_1 y_1 + \lambda_4 x_1 + \lambda_5 y_1 + \lambda_6 = 0,$$

(2.11)

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5,$ and $\lambda_6$ are real constants that depend only on $A, B$, and $C$. Any points that satisfy (1.1) satisfy (2.11), and vice versa. Furthermore, it is fairly easy to see that the point $O$ satisfies (1.1), as both sides of (1.1) will be $\frac{1}{4}$. In addition, six points that satisfy (1.1) are as follows: the vertices $A, B, C$, and the points diametrically opposed to the vertices, $A'', B'',$ and $C''$, which all lie on the circumcircle of $\Delta ABC$. These six points are distinct unless $\Delta ABC$ is a right triangle, in which case, they reduce to four distinct points. To locate an additional point in this
case that satisfies (1.1), we reason as follows; suppose that \( \triangle ABC \) is a right triangle with right angle at \( C \) (thus \( A = B'' \) and \( B = A'' \)). Select the point \( D \), located on the line \( BC \), with \( C \) being the midpoint of the segment \( DB \). Let \( C' \) be the projection of \( D \) onto \( AB \). Hence, \( \triangle DCC' \) is the pedal triangle of point \( D \) with respect to \( \triangle ABC \). For simplicity of notation, if we set \( BC = a \) and \( AC = b \), then \( a^2 + b^2 = 4R^2 \), where \( R \) is the circumradius of \( \triangle ABC \). Clearly, \( \triangle ABC \) is similar to \( \triangle DBC' \). From this, it is easy to deduce that \( DC' = \frac{ab}{R} \) and that \( BC' = \frac{ab}{2R} \). Consequently, \( OC' = \frac{a^2}{R} - R \). Since \( \triangle DCC' \) is a right triangle, \( OD^2 = OC'^2 + DC'^2 = 2a^2 + R^2 \). Therefore, \( \frac{OD^2 - R^2}{4R^2} = \frac{a^2}{2R^2} \). As for the area of the pedal triangle, we get

\[
|\triangle DCC'| = \frac{DC' \cdot DC \cdot \sin(\angle A)}{2} = \frac{ab \cdot a}{2} \cdot \frac{a}{2R} = \frac{a^2b}{4R^2},
\]

which in combination with the fact that \( |\triangle ABC| = \frac{ab}{2} \) implies \( \frac{|\triangle DCC'|}{|\triangle ABC|} = \frac{a^2}{2R^2} \). From this computation we can see that \( D \) satisfies (1.1).

Now, in general, it is known that any quadratic equation in terms of \( x \) and \( y \) either has no solutions, or has as its graph a conic section. However, since there are points satisfying (2.11), the latter must be true. Thus, the shape of the locus of points satisfying (2.11) is a conic, meaning that the locus of points satisfying (1.1) is either a point, two intersecting lines, a parabola, a hyperbola, a circle, an ellipse, or the whole plane (if all the lambdas are zero). Using the seven points, \( A, B, C, A'', B'', C'', O \), when \( \triangle ABC \) is not right, and the six points \( A, B, C, D, C'', O \), when \( \triangle ABC \) is right, identified earlier as belonging to the locus, one can eliminate all of the possible types of conics except for the whole plane. This means that for every \( P \) in the plane (1.1) holds.

3 The Area of an Antipedal Triangle

Theorem (1.1) provides us with an efficient formula to compute the area of a pedal triangle given the geometry of the reference triangle and the location of the pedal point. This is also useful for other purposes, such as computing the area of an antipedal triangle in terms of the geometry of the reference triangle and the location of the antipedal point. Before proceeding with the proof of Theorem (1.4), we prove a useful result on homotopic triangles. We do so by making use of a few rudiments of vector calculus. For a reference, we refer the reader to any multi-variable calculus textbook.

Given a triangle \( A_1A_2A_3 \) along with a triangle \( B_1B_2B_3 \) inscribed in it, we describe a procedure for obtaining a triangle, \( C_1C_2C_3 \), that is inscribed in \( \triangle B_1B_2B_3 \) and is homotopic to \( \triangle A_1A_2A_3 \). By definition, two triangles are called homotopic if their sides are parallel.

**Proposition 3.1.** Let \( \triangle A_1A_2A_3 \) be arbitrary and assume that \( B_1 \in A_2A_3 \), \( B_3 \in A_2A_1 \), \( B_2 \in A_1A_3 \) (see Figure 5). Take \( C_1 \in B_2B_3 \), \( C_2 \in B_1B_3 \), \( C_3 \in B_1B_2 \) such that

\[
\frac{A_2B_3}{B_3A_1} = \frac{B_2C_3}{C_3B_1}, \quad \frac{A_3B_1}{B_1A_2} = \frac{B_3C_1}{C_1B_2}, \quad \frac{A_1B_2}{B_2A_3} = \frac{B_1C_2}{C_2B_3}. \quad (3.13)
\]
Then $\Delta A_1A_2A_3$ and $\Delta C_1C_2C_3$ are homotopic and, in addition, $|\Delta B_1B_2B_3|$ is the geometric mean of $|\Delta A_1A_2A_3|$ and $|\Delta C_1C_2C_3|$, i.e.

$$|\Delta B_1B_2B_3|^2 = |\Delta A_1A_2A_3| \cdot |\Delta C_1C_2C_3|. \quad (3.14)$$

Conversely, if $\Delta A_1A_2A_3$ and $B_1 \in A_2A_3$, $B_3 \in A_2A_1$, $B_2 \in A_1A_3$ are given and $C_1 \in B_2B_3$, $C_2 \in B_1B_3$, $C_3 \in B_1B_2$ are such that $\Delta A_1A_2A_3$ and $\Delta C_1C_2C_3$ are homotopic, then (3.13) and (3.14) hold.

**Proof.** An affine transformation of the plane into itself consists of a linear transformation followed by a translation and it has the following properties: maps lines into lines, parallel lines into parallel lines, and preserves the ratio of line segments determined by points on a line.

Thus it suffices to prove Proposition 3.1 for the particular triangle $A_1A_2A_3$: $A_1 = (0,1), A_2 = (0,0), A_3 = (1,0)$, since any other triangle can be transformed via an affine transformation into this particular triangle while preserving the desired properties. In addition, let $B_1, B_2, B_3, C_1, C_2, C_3$ be as in Proposition 3.1. We set

$$k_1 := \frac{A_3B_1}{B_1A_2} = \frac{B_3C_1}{C_1B_2}, \quad k_2 := \frac{A_1B_2}{B_2A_3} = \frac{B_1C_2}{C_2B_3}, \quad k_3 := \frac{A_2B_3}{B_3A_1} = \frac{B_2C_3}{C_3B_1}. \quad (3.15)$$

If $M, N, P$ are three collinear points, with coordinates $M(m_1, m_2), P(p_1, p_2)$, and $N$ between $M$ and $P$, satisfying $\frac{MN}{NP} = k$, for some real, positive constant $k$, then $N$ has coordinates

$$N = \left(\frac{m_1 + kp_1}{1 + k}, \frac{m_2 + kp_2}{1 + k}\right). \quad (3.16)$$

This fact, in combination with (3.15) yields

$$B_1 = \left(\frac{1}{1 + k_1}, 0\right), \quad B_2 = \left(\frac{k_2}{1 + k_2}, \frac{1}{1 + k_2}\right), \quad B_3 = \left(0, \frac{k_3}{1 + k_3}\right). \quad (3.17)$$
Furthermore,

\[ C_1 = \left( \frac{k_1 k_2}{1 + k_1}, \frac{k_3}{1 + k_3} + \frac{1}{1 + k_1} \right), \quad C_2 = \left( \frac{k_2 k_3}{1 + k_2}, \frac{k_3}{1 + k_3} \right), \]

\[ C_3 = \left( \frac{k_2}{1 + k_2} + \frac{k_3}{1 + k_3}, \frac{1}{1 + k_3} \right). \quad (3.18) \]

It is obvious that

\[ |\Delta A_1 A_2 A_3| = \frac{1}{2}. \quad (3.19) \]

Next, using vector calculus, we will compute the areas of \( \Delta B_1 B_2 B_3 \) and \( \Delta C_1 C_2 C_3 \). More specifically, using the fact that the area of a triangle spanned by two vectors is equal to half the norm of their cross product, we can write,

\[ |\Delta B_1 B_2 B_3| = \frac{1}{2} \left\| \overrightarrow{B_1 B_2} \times \overrightarrow{B_1 B_3} \right\| = \frac{k_1 k_2 k_3 + 1}{2(1 + k_1)(1 + k_2)(1 + k_3)}. \quad (3.20) \]

A similar reasoning applies to \( \Delta C_1 C_2 C_3 \), namely

\[ |\Delta C_1 C_2 C_3| = \frac{1}{2} \left\| \overrightarrow{C_1 C_2} \times \overrightarrow{C_1 C_3} \right\| = \frac{(k_1 k_2 k_3 + 1)^2}{2(1 + k_1)^2(1 + k_2)^2(1 + k_3)^2}. \quad (3.21) \]

Identity (3.14) now follows by combining (3.19), (3.20), and (3.21), thus completing the proof of the first part of Proposition 3.1.

Finally, the converse statement (as recorded in the last part of the proposition) follows from the uniqueness of a triangle homotopic with \( \Delta A_1 A_2 A_3 \) and inscribed in \( \Delta B_1 B_2 B_3 \), plus what we have proved so far.

\[ \square \]

**Proof of Theorem 1.4** If \( \Delta DEF \) is the pedal triangle of the point \( K_1 \) with respect to \( \Delta ABC \), then

\[ \angle F K_1 A + \angle K_1 A F = \frac{\pi}{2}. \quad (3.22) \]
However, because $K$ is the isogonal of $K_1$, this means that

$$\angle F K_1 A + \angle K A E = \frac{\pi}{2}. \quad (3.23)$$

Keeping in mind that the quadrilateral $AFK_1E$ can be inscribed in a circle, (3.23) means that $AK \perp EF$, therefore $VU \parallel EF$. Similar reasoning can be done to show that $VT \parallel DF$ and $TU \parallel DE$. This implies that $\triangle TUV$ and $\triangle DEF$ are homotopic. From Proposition 3.1 we obtain

$$|\triangle DEF| \cdot |\triangle TUV| = |\triangle ABC|^2, \quad (3.24)$$

Theorem 1.1 implies

$$\frac{|\triangle DEF|}{|\triangle ABC|} = \frac{|R^2 - OK_1^2|}{4R^2}. \quad (3.25)$$

Therefore, $|\triangle TUV| = \frac{|\triangle ABC|}{|\triangle DEF|} = \frac{4R^2}{|R^2 - OK_1^2|}$, as claimed.

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