Some Remarks on the Total CR $Q$ and $Q'$-Curvatures

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Received November 09, 2017, in final form February 12, 2018; Published online February 14, 2018
https://doi.org/10.3842/SIGMA.2018.010

Abstract. We prove that the total CR $Q$-curvature vanishes for any compact strictly pseudoconvex CR manifold. We also prove the formal self-adjointness of the $P'$-operator and the CR invariance of the total $Q'$-curvature for any pseudo-Einstein manifold without the assumption that it bounds a Stein manifold.

Key words: CR manifolds; $Q$-curvature; $P'$-operator; $Q'$-curvature

2010 Mathematics Subject Classification: 32V05; 52T15

1 Introduction

The $Q$-curvature, which was introduced by T. Branson [3], is a fundamental curvature quantity on even dimensional conformal manifolds. It satisfies a simple conformal transformation formula and its integral is shown to be a global conformal invariant. The ambient metric construction of the $Q$-curvature [9] also works for a CR manifold $M$ of dimension $2n+1$, and we can define the CR $Q$-curvature, which we denote by $Q$. The CR $Q$-curvature is a CR density of weight $-n-1$ defined for a fixed contact form $\theta$ and is expressed in terms of the associated pseudo-hermitian structure. If we take another contact form $\hat{\theta} = e^\Upsilon \theta$, $\Upsilon \in \mathcal{C}^\infty(M)$, it transforms as

$$\hat{Q} = Q + P \Upsilon,$$

where $P$ is a CR invariant linear differential operator, called the (critical) CR GJMS operator. Since $P$ is formally self-adjoint and kills constant functions, the integral

$$\overline{Q} = \int_M Q,$$

called the total CR $Q$-curvature, is invariant under rescaling of the contact form and gives a global CR invariant of $M$. However, it follows readily from the definition of the CR $Q$-curvature that $Q$ vanishes identically for an important class of contact forms, namely the pseudo-Einstein contact forms. Since the boundary of a Stein manifold admits a pseudo-Einstein contact form [5], the CR invariant $\overline{Q}$ vanishes for such a CR manifold. Moreover, it has been shown that on a Sasakian manifold the CR $Q$-curvature is expressed as a divergence [1], and hence $\overline{Q}$ also vanishes in this case. Thus, it is reasonable to conjecture that the total CR $Q$-curvature vanishes for any CR manifold, and our first result is the confirmation of this conjecture:

Theorem 1.1. Let $M$ be a compact strictly pseudoconvex CR manifold. Then the total CR $Q$-curvature of $M$ vanishes: $\overline{Q} = 0$.

For three dimensional CR manifolds, Theorem 1.1 follows from the explicit formula of the CR $Q$-curvature; see [9]. In higher dimensions, we make use of the fact that a compact strictly pseudoconvex CR manifold $M$ of dimension greater than three can be realized as the boundary
of a complex variety with at most isolated singularities \([2, 10, 11]\). By resolution of singularities, we can realize \(M\) as the boundary of a complex manifold \(X\) which may not be Stein. In this setting, the total CR \(Q\)-curvature is characterized as the logarithmic coefficient of the volume expansion of the asymptotically Kähler–Einstein metric on \(X\) \([15]\). By a simple argument using Stokes’ theorem, we prove that there is no logarithmic term in the expansion.

Although the vanishing of \(\bar{Q}\) is disappointing, there is an alternative \(Q\)-like object on a CR manifold which admits pseudo-Einstein contact forms. Generalizing the operator of Branson–Fontana–Morpurgo \([4]\) on the CR sphere, Case–Yang \([7]\) (in dimension three) and Hirachi \([12]\) (in general dimensions) introduced the \(P'\)-operator and the \(Q'\)-curvature for pseudo-Einstein CR manifolds. Let us denote the set of pseudo-Einstein contact forms by \(\mathcal{PE}\) and the space of CR pluriharmonic functions by \(\mathcal{P}\). Two pseudo-Einstein contact forms \(\theta, \hat{\theta} \in \mathcal{PE}\) are related by \(\hat{\theta} = e^{\Upsilon} \theta\) for some \(\Upsilon \in \mathcal{P}\). For a fixed \(\theta \in \mathcal{PE}\), the \(P'\)-operator is defined to be a linear differential operator on \(\mathcal{P}\) which kills constant functions and satisfies the transformation formula

\[
\hat{P}' f = P' f + P(f \Upsilon)
\]

under the rescaling \(\hat{\theta} = e^{\Upsilon} \theta\). The \(Q'\)-curvature is a CR density of weight \(-n - 1\) defined for \(\theta \in \mathcal{PE}\), and satisfies

\[
\hat{Q}' = Q' + 2P' \Upsilon + P(\Upsilon^2)
\]

for the rescaling. Thus, if \(P'\) is formally self-adjoint on \(\mathcal{P}\), the total \(Q'\)-curvature

\[
\bar{Q}' = \int_M Q'
\]

gives a CR invariant of \(M\). In dimension three and five, the formal self-adjointness of \(P'\) follows from the explicit formulas \([6, 7]\). In higher dimensions, Hirachi \([12, \text{Theorem 4.5}]\) proved the formal self-adjointness under the assumption that \(M\) is the boundary of a Stein manifold \(X\); in the proof he used Green’s formula for the asymptotically Kähler–Einstein metric \(g\) on \(X\), and the global Kählerness of \(g\) was needed to assure that a pluriharmonic function is harmonic with respect to \(g\). In this paper, we slightly modify his proof and prove the self-adjointness of \(P'\) for general pseudo-Einstein manifolds:

**Theorem 1.2.** Let \(M\) be a compact strictly pseudoconvex CR manifold. Then the \(P'\)-operator for a pseudo-Einstein contact form satisfies

\[
\int_M (f_1 P' f_2 - f_2 P' f_1) = 0
\]

for any \(f_1, f_2 \in \mathcal{P}\).

Consequently, the CR invariance of \(\bar{Q}'\) holds for any CR manifold which admits a pseudo-Einstein contact form:

**Theorem 1.3.** Let \(M\) be a compact strictly pseudoconvex CR manifold which admits a pseudo-Einstein contact form. Then the total \(Q'\)-curvature is independent of the choice of \(\theta \in \mathcal{PE}\).

We note that \(\bar{Q}'\) is a nontrivial CR invariant since it has a nontrivial variational formula; see \([13]\). We also give an alternative proof of Theorem 1.3 by using the characterization \([12, \text{Theorem 5.6}]\) of \(\bar{Q}'\) as the logarithmic coefficient in the expansion of some integral over a complex manifold with boundary \(M\).
2 Proof of Theorem 1.1

We briefly review the ambient metric construction of the CR $Q$-curvature; we refer the reader to [9, 12, 13] for detail.

Let $\tilde{X}$ be an $(n+1)$-dimensional complex manifold with strictly pseudoconvex CR boundary $M$, and let $r \in C^\infty(\tilde{X})$ be a boundary defining function which is positive in the interior $X$. The restriction of the canonical bundle $K_{\tilde{X}}$ to $M$ is naturally isomorphic to the CR canonical bundle $K_M := \wedge^{n+1}(T^{0,1}M)\bot \subset \wedge^{n+1}(CT^*M)$. We define the ambient space by $\tilde{X} = K_{\tilde{X}} \setminus \{0\}$, and set $\mathcal{N} = K_M \setminus \{0\} \cong \tilde{X}|_M$. The density bundles over $\tilde{X}$ and $M$ are defined by

$$\mathcal{E}(w) = (K_{\tilde{X}} \otimes K_\tilde{X})^{-w/(n+2)}, \quad \mathcal{E}(w) = (K_M \otimes K_M)^{-w/(n+2)} \cong \mathcal{E}(w)|_M$$

for each $w \in \mathbb{R}$. We call $\mathcal{E}(w)$ the CR density bundle of weight $w$. The space of sections of $\mathcal{E}(w)$ and $\mathcal{E}(w)$ are also denoted by the same symbols. We define a $\mathbb{C}^*$-action on $\tilde{X}$ by $\delta_\lambda u = \lambda^{n+2}u$ for $\lambda \in \mathbb{C}^*$ and $u \in \tilde{X}$.

Then a section of $\mathcal{E}(w)$ can be identified with a function on $\tilde{X}$ which is homogeneous with respect to this action:

$$\tilde{\mathcal{E}}(w) \cong \{ f \in C^\infty(\tilde{X}) \mid \delta_\lambda f = |\lambda|^{2w}f \text{ for } \lambda \in \mathbb{C}^* \}.$$

Similarly, sections of $\mathcal{E}(w)$ are identified with homogeneous functions on $\mathcal{N}$.

Let $\rho \in \tilde{\mathcal{E}}(1)$ be a density on $\tilde{X}$ and $(z^1, \ldots, z^{n+1})$ local holomorphic coordinates. We set $\rho = |dz^1 \wedge \cdots \wedge dz^{n+1}|^{2/(n+2)} \rho \in \tilde{\mathcal{E}}(0)$ and define

$$\mathcal{J}[\rho] := (-1)^{n+1} \det \left( \begin{array}{cc} \rho & \partial_\tau \rho \\ \partial_\tau \rho & \partial_\tau \partial_\tau \rho \end{array} \right).$$

Since $\mathcal{J}[\rho]$ is invariant under changes of holomorphic coordinates, $\mathcal{J}$ defines a global differential operator, called the Monge–Ampère operator. Fefferman [8] showed that there exists $\rho \in \tilde{\mathcal{E}}(1)$ unique modulo $O(r^{n+3})$ which satisfies $\mathcal{J}[\rho] = 1 + O(r^{n+2})$ and is a defining function of $\mathcal{N}$. We fix such $\rho$ and define the ambient metric $\tilde{g}$ by the Lorentz–Kähler metric on a neighborhood of $\mathcal{N}$ in $\tilde{X}$ which has the Kähler form $-i\partial\bar{\partial}\rho$.

Recall that there exists a canonical weighted contact form $\theta \in \Gamma(T^*M \otimes \mathcal{E}(1))$ on $M$, and the choice of a contact form $\theta$ is equivalent to the choice of a positive section $\tau \in \mathcal{E}(1)$, called a CR scale; they are related by the equation $\theta = \tau \theta$. For a CR scale $\tau \in \mathcal{E}(1)$, we define the CR $Q$-curvature by

$$Q = \Delta^{n+1} \log \tau |_{\mathcal{N}} \in \mathcal{E}(-n-1),$$

where $\Delta = -\tilde{\nabla}^f \tilde{\nabla}^f$ is the Kähler Laplacian of $\tilde{g}$ and $\tau \in \tilde{\mathcal{E}}(1)$ is an arbitrary extension of $\tau$. It can be shown that $Q$ is independent of the choice of an extension of $\tau$, and the total CR $Q$-curvature $Q$ is invariant by rescaling of $\tau$.

The total CR $Q$-curvature has a characterization in terms of a complete metric on $X$. We note that the $(1,1)$-form $-i\partial\bar{\partial}\log \rho$ descends to a Kähler form on $X$ near the boundary. We extend this Kähler metric to a hermitian metric $g$ on $X$. The Kähler Laplacian $\Delta = -g^{ij} \nabla_i \nabla_j$ of $g$ is related to $\tilde{\Delta}$ by the equation

$$\rho \tilde{\Delta} f = \Delta f, \quad f \in \tilde{\mathcal{E}}(0) \quad (2.1)$$

near $\mathcal{N}$ in $\tilde{X} \setminus \mathcal{N}$. In the right-hand side, we have regarded $f$ as a function on $X$.

For any contact form $\theta$ on $M$, there exists a boundary defining function $\rho$ such that

$$\partial|_{TM} = \theta, \quad |\partial \log \rho|_g = 1 \text{ near } M \text{ in } X, \quad (2.2)$$
where \( \vartheta := \text{Re}(i\partial \rho) \) ([15, Lemma 3.1]). Let \( \xi \) be the \((1, 0)\)-vector filed on \( X \) near \( M \) characterized by
\[
\xi \rho = 1, \quad \xi \perp g \mathcal{H},
\]
where \( \mathcal{H} := \ker \partial \rho \subset T^{1,0}X \). Then, \( N := \text{Re} \xi \) is smooth up to the boundary and satisfies \( N \rho = 1, \vartheta(N) = 0 \). Moreover, \( \nu := \rho N \) is \((\sqrt{2})^{-1}\) times the unit outward normal vector filed along the level sets of \( \rho \). By Green’s formula, for any function \( f \) on \( X \) we have
\[
\int_{\rho > \epsilon} \Delta f \, \text{vol}_g = \int_{\rho = \epsilon} \nu f \, \nu_\bot \text{vol}_g. \tag{2.3}
\]
Since the Monge–Ampère equation implies that \( g \) satisfies \( \text{vol}_g = -\left(\frac{n!}{2}\right)^{-1}(1 + O(\rho))\rho^{-n-2} d\rho \wedge \vartheta \wedge (d\vartheta)^n \), the formula (2.3) is rewritten as
\[
\int_{\rho > \epsilon} \Delta f \, \text{vol}_g = -\left(\frac{n!}{2}\right)^{-1} \int_{\rho = \epsilon} N f \cdot (1 + O(\epsilon))\epsilon^{-n} \vartheta \wedge (d\vartheta)^n. \tag{2.4}
\]
With this formula, we prove the following characterization of \( \mathcal{Q} \).

**Lemma 2.1** ([15, Proposition A.3]). For an arbitrary defining function \( \rho \), we have
\[
lp \int_{\rho > \epsilon} \text{vol}_g = \frac{(-1)^n}{(n!)^2(n+1)!} \mathcal{Q},
\]
where \( lp \) denotes the coefficient of \( \log \epsilon \) in the asymptotic expansion in \( \epsilon \).

**Proof.** Since the coefficient of \( \log \epsilon \) in the volume expansion is independent of the choice of \( \rho \) [15, Proposition 4.1], we may assume that \( \rho \) satisfies (2.2) for a fixed contact \( \theta \) on \( M \). We take \( \tilde{\tau} \in \mathcal{E}(1) \) such that \( \rho = \tilde{\tau} \rho \). Then, \( \vartheta \) is the contact form corresponding to the CR scale \( \tilde{\tau}|_N \). By the same argument as in the proof of [12, Lemma 3.1], we can take \( F \in \mathcal{E}(0), \, G \in \mathcal{E}(-n - 1) \) which satisfy
\[
\tilde{\Delta}(\log \tilde{\tau} + F + G \rho^{n+1} \log \rho) = O(\rho^\infty), \quad F = O(\rho), \quad G|_N = \frac{(-1)^n}{n!(n+1)!} \mathcal{Q}.
\]
We set \( G := \tilde{\tau}^{n+1} G \in \mathcal{E}(0) \). By (2.1) and the equation \( \rho \tilde{\Delta} \log \rho = n + 1 \), we have
\[
\Delta(\log \rho - F - G \rho^{n+1} \log \rho) = n + 1 + O(\rho^\infty).
\]
Then, by using (2.4), we compute as
\[
(n + 1) \, lp \int_{\rho > \epsilon} \text{vol}_g = lp \int_{\rho > \epsilon} \Delta(\log \rho - F - G \rho^{n+1} \log \rho) \, \text{vol}_g
\]
\[
= -\left(\frac{n!}{2}\right)^{-1} \int_{\rho = \epsilon} N(\log \rho - F - G \rho^{n+1} \log \rho) \cdot (1 + O(\epsilon))\epsilon^{-n} \vartheta \wedge (d\vartheta)^n
\]
\[
= \frac{n + 1}{n!} \int_M G \theta \wedge (d\theta)^n
\]
\[
= \frac{(-1)^n}{(n!)^2} \mathcal{Q}.
\]
Thus we complete the proof. \( \blacksquare \)
Thus, by Lemma 2.1 we obtain

\[
\tau \in \mathcal{E}(1)\text{ is a pseudo-Einstein CR scale and \( \tau \in \mathcal{E}(1) \) such that } \partial \bar{\partial} \log \tau = 0 \text{ near } N \text{ in } \tilde{X}.
\]

The corresponding contact form \( \theta \) is called a pseudo-Einstein contact form and characterized in terms of associated pseudo-hermitian structure; see [12, 13, 14]. If \( \tau \) is a pseudo-Einstein CR scale, another \( \tau \) is pseudo-Einstein if and only if \( \tau = e^{-\mathcal{Y}} \tau \) for a CR pluriharmonic function \( \mathcal{Y} \in \mathcal{P} \). For any \( f \in \mathcal{P} \), we take an extension \( \tilde{f} \in \mathcal{E}(0) \) such that \( \partial \bar{\partial} \tilde{f} = 0 \) near \( M \) in \( \tilde{X} \) and define

\[
P'(f) = -\Delta^{n+1}(\tilde{f} \log \tau)|_N \in \mathcal{E}(-n - 1).
\]

We note that the germs of \( \tau \) and \( \tilde{f} \) along \( N \) is unique, and \( P'f \) is assured to be a density by \( \Delta \tilde{f} |_N = 0 \). The \( Q' \)-curvature is defined by

\[
Q' = \Delta^{n+1}(\log \tau)^2|_N \in \mathcal{E}(-n - 1).
\]

Here, the homogeneity of \( Q' \) follows from the fact \( \Delta \log \tau |_N = 0 \).

To prove the formal self-adjointness of \( P' \), we use its characterization in terms of the metric \( g \). We define a differential operator \( \Delta' \) by \( \Delta' f = -g^{ij} \partial_i \partial_j f \). Since \( g \) is Kähler near the boundary, \( \Delta' \) agrees with \( \Delta \) near \( M \) in \( X \).

Lemma 3.1 ([12, Lemma 4.4]). Let \( \tau \in \mathcal{E}(1) \) be a pseudo-Einstein CR scale and \( \tau \in \mathcal{E}(1) \) its extension such that \( \partial \bar{\partial} \log \tau = 0 \) near \( N \) in \( \tilde{X} \). Let \( \rho = \frac{\tau}{\rho} \) be the corresponding defining function. Then, for any \( f \in C^\infty(\tilde{X}) \) which is pluriharmonic in a neighborhood of \( M \) in \( \tilde{X} \), there exist \( F, G \in C^\infty(\tilde{X}) \) such that \( F = O(\rho) \) and

\[
\Delta'(f \log \rho - F - G\rho^{n+1} \log \rho) = (n+1)f + O(\rho^\infty).
\]

Moreover, \( \tau^{-n-1}G|_M = \frac{(-1)^{n+1}}{(n+1)!}P'f \) holds.
In the statement of [12, Lemma 4.4], the Laplacian $\Delta$ is used, but we may replace it by $\Delta'$ since they agree near the boundary in $X$.

**Proof of Theorem 1.2.** We extend $f_j$ to a function on $\overline{X}$ such that $\partial \overline{\partial} f_j = 0$ in a neighborhood of $M$ in $\overline{X}$. Let $\tau$ be a pseudo-Einstein CR scale and $\rho = \rho/\tau$ the corresponding defining function. Then we have $\omega = -i\partial \overline{\partial} \log \rho$ near $M$ in $X$. We take $F_j, G_j$ as in Lemma 3.1 so that $u_j := f_j \log \rho - F_j - G_j \rho^{n+1} \log \rho$ satisfies $\Delta' u_j = (n+1) f_j + O(\rho^\infty)$. We consider the coefficient of $\log \epsilon$ in the expansion of the integral

$$I_\epsilon = \Re \int_{\rho > \epsilon} (i\partial f_1 \wedge \overline{\partial} u_2 \wedge \omega^n + i\partial f_2 \wedge \overline{\partial} u_1 \wedge \omega^n - f_1 f_2 \omega^{n+1}),$$

which is symmetric in the indices 1 and 2. Since $d\omega = 0$, $\partial \overline{\partial} f_2 = 0$ near $M$ in $X$, we have

$$i\partial f_1 \wedge \overline{\partial} u_2 \wedge \omega^n = d\left( i f_1 \overline{\partial} u_2 \wedge \omega^n \right) - i f_1 \overline{\partial} u_2 \wedge \omega^n + in f_1 \overline{\partial} u_2 \wedge d\omega \wedge \omega^{n-1}$$

$$= d\left( i f_1 \overline{\partial} u_2 \wedge \omega^n \right) + \frac{1}{n+1} f_1 \Delta' u_2 \omega^{n+1} + (\text{cpt supp}),$$

$$i\partial f_2 \wedge \overline{\partial} u_1 \wedge \omega^n = -d\left( \overline{\partial} u_2 \wedge \omega^n \right) + (\text{cpt supp}),$$

where (cpt supp) stands for a compactly supported form on $X$. Thus,

$$I_\epsilon = \int_{\rho > \epsilon} \frac{1}{n+1} f_1 (\Delta' u_2 - (n+1) f_2) \omega^{n+1}$$

$$+ \Re \int_{\rho = \epsilon} i(f_1 \overline{\partial} u_2 - u_1 \partial f_2) \wedge \omega^n + \int_{\rho > \epsilon} (\text{cpt supp}).$$

The first and the third terms contain no log terms. Since $\omega = d(\partial/\rho)$ near $M$ in $X$, the second term is computed as

$$\Re \int_{\rho = \epsilon} i(f_1 \overline{\partial} u_2 - u_1 \partial f_2) \wedge \omega^n = \epsilon^{-n} \Re \int_{\rho = \epsilon} \left( i f_1 \overline{\partial} (f_2 \log \rho - F_2 - G_2 \rho^{n+1} log \rho) \wedge (d\theta)^n \right.$$

$$- i \left( f_1 \log \rho - F_1 - G_1 \rho^{n+1} log \rho \right) \wedge \partial f_2 \wedge (d\theta)^n) + O(\epsilon^\infty).$$

The logarithmic term in the right-hand side is

$$\log \epsilon \int_{\rho = \epsilon} (n+1) f_1 G_2 \theta \wedge (d\theta)^n + 2\epsilon^{-n} \log \epsilon \Re \int_{\rho = \epsilon} i f_1 \overline{\partial} f_2 \wedge (d\theta)^n + O(\epsilon \log \epsilon).$$

The coefficient of $\log \epsilon$ in the first term is

$$\frac{(-1)^{n+1}}{(n!)^2} \int_M f_1 P' f_2.$$  \hspace{1cm} (3.1)

The second term is equal to

$$2\epsilon^{-n} \log \epsilon \Re \int_{\rho > \epsilon} i\partial f_1 \wedge \overline{\partial} f_2 \wedge (d\theta)^n + \epsilon^{-n} \log \epsilon \int_{\rho > \epsilon} (\text{cpt supp}).$$

The first term in this formula is symmetric in the indices 1 and 2 while the second term gives no $\log \epsilon$ term. Therefore, (3.1) should also be symmetric in 1 and 2, which implies the formal self-adjointness of $P'$. \hfill \blacksquare
4 Proof of Theorem 1.3

The formal self-adjointness of the $P'$-operator implies the CR invariance of the total $Q'$-curvature. When $n \geq 2$, the CR invariance can also be proved by the following characterization of $Q'$ in terms of the hermitian metric $g$ on $X$ whose fundamental 2-form $\omega = ig_{jk}\theta^j \wedge \theta^k$ agrees with $-i\partial\bar{\partial}\log \rho$ near $M$ in $X$:

**Theorem 4.1** ([12, Theorem 5.6]). Let $\tau \in \mathcal{E}(1)$ be a pseudo-Einstein CR scale and $\tilde{\tau} \in \tilde{\mathcal{E}}(1)$ its extension such that $\partial\bar{\partial}\log \tilde{\tau} = 0$ near $N$ in $\tilde{X}$. Let $\rho = \rho/\tilde{\tau}$ be the corresponding defining function. Then we have

$$\int_{\rho > \epsilon} i\partial\log \rho \wedge \bar{\partial}\log \rho \wedge \omega^n = \frac{(-1)^n}{2(n!)^2} Q'$$

(4.1)

for any defining function $r$.

In [12, Theorem 5.6], it is assumed that $X$ is Stein and $\omega = -i\partial\bar{\partial}\log \rho$ globally on $X$, but as the logarithmic term is determined by the boundary behavior, it is sufficient to assume $\omega = -i\partial\bar{\partial}\log \rho$ near $M$ in $X$ as above.

**Proof of Theorem 1.3.** Let $\tau$, $\rho$ be as in Theorem 4.1 and let $\tilde{\rho}$ be the defining function corresponding to another pseudo-Einstein CR scale $\tilde{\tau}$. Then we can write as $\tilde{\rho} = e^\Upsilon \rho$ with $\Upsilon \in C^\infty(X)$ such that $\partial\bar{\partial}\Upsilon = 0$ near $M$ in $X$.

Using the defining function $\rho$ for $r$ in the formula (4.1), we compute as

$$\int_{\rho > \epsilon} i\partial\log \tilde{\rho} \wedge \bar{\partial}\log \tilde{\rho} \wedge \omega^n = \int_{\rho > \epsilon} i(\partial\log \rho + \partial\Upsilon) \wedge (\bar{\partial}\log \rho + \bar{\partial}\Upsilon) \wedge \omega^n$$

$$= \int_{\rho > \epsilon} i\partial\log \rho \wedge \bar{\partial}\log \rho \wedge \omega^n + \int_{\rho > \epsilon} i\partial\Upsilon \wedge \bar{\partial}\Upsilon \wedge \omega^n$$

$$+ 2 \text{Re} \int_{\rho > \epsilon} i\partial\log \rho \wedge \bar{\partial}\Upsilon \wedge \omega^n.$$

The second term in the last line is

$$\int_{\rho > \epsilon} i\partial\Upsilon \wedge \bar{\partial}\Upsilon \wedge \omega^n = \int_{\rho = \epsilon} i\partial\Upsilon \wedge \bar{\partial}\Upsilon \wedge \omega^n + \int_{\rho > \epsilon} (\text{cpt supp}) = 0.$$

Since $\omega = d(\partial/\rho)$ near $M$ in $X$, we have

$$\int_{\rho > \epsilon} i\partial\log \rho \wedge \bar{\partial}\Upsilon \wedge \omega^n = \log \epsilon \int_{\rho = \epsilon} i\partial\Upsilon \wedge \omega^n + \int_{\rho > \epsilon} (\text{cpt supp})$$

$$= \epsilon^{-n} \log \epsilon \int_{\rho = \epsilon} i\partial\Upsilon \wedge (d\theta)^n + \int_{\rho > \epsilon} (\text{cpt supp})$$

$$= \epsilon^{-n} \log \epsilon \int_{\rho > \epsilon} (\text{cpt supp}) + \int_{\rho > \epsilon} (\text{cpt supp}),$$

which implies that the third term is also 0. Thus, $Q'$ is independent of the choice of a pseudo-Einstein CR scale $\tau$. ■

**Acknowledgements**

The author would like to thank the referees for their comments which were helpful for the improvement of the manuscript.
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