Automorphisms and a Cartography of the Solution Space for Vacuum Bianchi Cosmologies: The Type III Case

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Abstract

The theory of symmetries of systems of coupled, ordinary differential equations (ODE’s) is used to develop a concise algorithm for cartographing the solution space of vacuum Bianchi Einstein’s Field Equations (EFE). The symmetries used are the well known automorphisms of the Lie algebra for the corresponding isometry group of each Bianchi Type, as well as the scaling and the time reparameterization symmetry. Application of the method to Type III results in: a) the recovery of all known solutions without prior assumption of any extra symmetry, b) the enclosure of the entire unknown part of the solution space into a single, second order ODE in terms of one dependent variable and c) a partial solution to this ODE. It is also worth-mentioning the fact that the solution space is seen to be naturally partitioned into three distinct, disconnected pieces: one consisting of the known Siklos (pp-wave) solution, another occupied by the Type III member of the known Ellis-MacCallum family and the third described by the aforementioned ODE in which an one parameter subfamily of the known Kinnersley geometries resides. Lastly, preliminary results reported show that the unknown part of the solution space for other Bianchi Types is described by a strikingly similar ODE, pointing to a natural operational unification as far as the problem of solving the cosmological EFE’s is concerned.

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1 Introduction

Since the early times of cosmology, Automorphisms have been identified as possible key elements for a unified treatment of spatially homogeneous Bianchi Geometries [1]. Harvey has found the automorphisms of all 3-dimensional Lie Algebras [2], while the corresponding results for the 4-dimensional Lie Algebras have been reported in [3]. Jantzen’s tangent space approach sees the automorphic matrices as the means for achieving a convenient parametrization of a full scale-factor matrix in terms of a, desired, diagonal matrix [4]. Samuel and Ashtekar were the first to look upon Automorphisms from a space viewpoint [5]. The notion of Time-Dependent Automorphism Inducing Diffeomorphisms (A.I.D.’s), i.e. coordinate transformations mixing space and time in the new spatial coordinates and inducing automorphic motions on the scale-factor matrix, the lapse and the shift has been developed in [6].

In this communication we revisit the problem of solving the EFE’s for vacuum Bianchi Geometries. We begin with a full metric, i.e. we make no assumption for the lapse function $N^2$, the shift vector $N^\alpha$ and the spatial metric $\gamma_{\alpha\beta}$. Then we use the Time-Dependent A.I.D.’s to put the shift vector to zero. At this point the idea is to exploit, in a systematic way, the remaining symmetries of the field equations –sometimes called ”rigid” [7]– to transform them to the most simple form possible, without loss of generality. These are the well known symmetries following from the constant Automorphism group within each Bianchi Type, as well as the scaling of the metric by a constant and the time reparameterization symmetry (see e.g. [8]). Applying this analysis to Bianchi Type III Vacuum Cosmology we produce an exhaustive cartography of the entire space of its solutions.

The paper is organized as follows: in section 2, we present our method. In section 3, after a brief description of Bianchi Type III Cosmology we apply the method. We thus recover all known solutions, describe the unknown part of the solution space with a single, second order ODE in terms of one dependent variable and present a new solution. A brief preview of the corresponding results obtained for other Bianchi Types is also included. Finally some discussion and concluding remarks are given in section 4.

2 The Method

As it is well known, for spatially homogeneous spacetimes with a simply transitive action of the corresponding isometry group [10], [8], the line element, assumes the form

$$ds^2 = (N^\alpha N_\alpha - N^2) dt^2 + 2N_\alpha \sigma^\alpha_i dx^i dt + \gamma_{\alpha\beta} \sigma^\alpha_i \sigma^\beta_j dx^i dx^j$$

(2.1)

where the 1-forms $\sigma^\alpha_i$, are defined from:

$$d\sigma^\alpha = C_{\beta\gamma}^\alpha \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma^\alpha_{i,j} - \sigma^\alpha_{j,i} = 2C_{\beta\gamma}^\alpha \sigma^\gamma_i \sigma^\beta_j.$$  

(2.2)

Then the field equations are (e.g. [8]):

$$E_\alpha = K^\alpha\beta K_{\alpha\beta} - K^2 - R = 0$$

(2.3)
\[ E_\alpha = K_\alpha^\mu C_{\mu \epsilon} - K_\epsilon^\mu C_{\alpha \mu} = 0 \] (2.4)

\[ E_{\alpha \beta} = \dot{K}_{\alpha \beta} + N (2K_{\alpha}^\tau K_{\tau \beta} - KK_{\alpha \beta}) + 2N^\rho (K_{\alpha \nu} C_{\beta \rho}^\nu + K_{\beta \nu} C_{\alpha \rho}^\nu) - NR_{\alpha \beta} = 0 \] (2.5)

where

\[ K_{\alpha \beta} = -\frac{1}{2N} (\dot{\gamma}_{\alpha \beta} + 2\gamma_{\alpha \nu} C_{\beta \rho}^\nu N^\rho + 2\gamma_{\beta \nu} C_{\alpha \rho}^\nu N^\rho) \] (2.6)

is the extrinsic curvature and

\[ R_{\alpha \beta} = C^\lambda_{\sigma \tau} C_{\mu \nu} C_{\alpha \gamma} \gamma_{\tau \lambda \gamma} + 2C^\lambda_{\beta \lambda} C_{\alpha \nu} + 2C^\mu_{\alpha \nu} C_{\beta \lambda} \gamma_{\mu \nu} \gamma_{\lambda} + 2P_{\beta \mu} C_{\mu \nu} \gamma_{\lambda \gamma} + 2C^\lambda C_{\beta \mu} \gamma_{\lambda \nu} \] (2.7)

the Ricci tensor of the hyper-surface.

In particular spacetime coordinate transformations have been found, which reveal as symmetries of (2.3), (2.4), (2.5) the following transformations of the dependent variables \( N, N_\alpha, \gamma_{\alpha \beta} \):

\[ \tilde{N} = N, \quad \tilde{N}_\alpha = \Lambda^\rho_\alpha (N_\rho + \gamma_{\rho \sigma} P^\sigma), \quad \tilde{\gamma}_{\mu \nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \gamma_{\alpha \beta} \] (2.8)

where the matrix \( \Lambda \) and the triplet \( P^\alpha \) must satisfy:

\[ \Lambda^\rho_\alpha \Lambda^\beta_\mu = C^\alpha_{\mu \nu} \Lambda^\beta_\nu \Lambda^\gamma_\gamma \] (2.9)

\[ 2P^\mu C^\alpha_{\mu \nu} \Lambda^\beta_\beta = \tilde{\Lambda}^\alpha_\beta \] (2.10)

For all Bianchi Types, this system of equations admits solutions which contain three arbitrary functions of time plus several constants depending on the Automorphism group of each type. The three functions of time, are distributed among \( \Lambda \) and \( P \) (which also contains derivatives of these functions). So one can use this freedom either to simplify the form of the scale factor matrix or to set the shift vector to zero. The second action can always be taken, since, for every Bianchi type, all three functions appear in \( P^\alpha \).

In this work we adopt the latter point of view. When the shift has been set to zero, there is still a remaining “gauge” freedom consisting of all constant \( \Lambda^\alpha_\beta \) (Automorphism group matrices). Indeed the system (2.9), (2.10) accepts the solution \( \Lambda^\alpha_\beta = constant, \quad P^\alpha = 0 \). The generators of the corresponding motions, induced in the space of dependent variables spanned by \( \gamma_{\alpha \beta} \)’s (the lapse is given in terms of \( \gamma_{\alpha \beta}, \quad \dot{\gamma}_{\alpha \beta} \) by algebraically solving the quadratic constraint equation ) \( \tilde{\gamma}_{\mu \nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \gamma_{\alpha \beta} \) are [11] :

\[ X_{(I)} = \lambda^\rho_\beta \gamma_{\rho \beta} \frac{\partial}{\partial \gamma_{\alpha \beta}} \] (2.11)

with \( \lambda \) satisfying:

\[ \lambda^\rho_\beta C^\beta_{\gamma \rho} = \lambda^\rho_{(I) \beta} C^\alpha_{\gamma \rho} + \lambda^\rho_{(I) \gamma} C^\alpha_{\beta \rho} \] (2.12)
Now, these generators define a Lie algebra and each one of them induces, through its integral curves, a transformation on the configuration space spanned by the $\gamma_{\alpha\beta}$’s. If a generator is brought to its normal form (e.g. $\frac{\partial}{\partial z_i}$), then the Einstein equations, written in terms of the new dependent variables, will not explicitly involve $z_i$. They thus become a first order system in the function $\dot{z}_i$ \cite{12}. If the above Lie algebra happens to be abelian, then all generators can be brought, to their normal form simultaneously. If this is not the case, we can diagonalize in one step the generators corresponding to any eventual abelian subgroup. The rest of the generators (not brought in their normal form) continue to define a symmetry of the reduced system of EFE’s if the algebra of the $X_{(I)}$’s is solvable \cite{13}. One can thus repeat the previous step, by choosing one of these remaining generators. This choice will of course depend upon the simplifications brought to the system at the previous level. Finally if the algebra does not contain any abelian subgroup, one can always choose one of the generators, bring it to its normal form, reduce the system and search for its symmetries (if there are any). Lastly, two further symmetries of \cite{24}, \cite{21}, \cite{20} are also present and can be used in conjunction with the constant automorphisms: The time reparameterization $t \rightarrow f(t) + \alpha$, owing to the non-explicit appearance of time in these equations, and the scaling by a constant $\gamma_{\alpha\beta} \rightarrow \mu \gamma_{\alpha\beta}$ as can be straightforwardly verified. Their corresponding generators are:

$$Y_1 = \frac{1}{f} \frac{\partial}{\partial t} \tag{2.13}$$

$$Y_2 = \gamma_{\alpha\beta} \frac{\partial}{\partial \gamma_{\alpha\beta}} \tag{2.14}$$

These generators commute among themselves, as well as with the $X_{(I)}$’s, as it can be easily checked.

### 3 Application to Bianchi Type III

We are now going to apply the Method, previously discussed, to the case of Bianchi Type III. For this type the structures constants are \cite{14}

$$C^i_{13} = -C^i_{31} = 1$$

$$C^\alpha_{\beta\gamma} = 0 \quad \text{for all other values of } \alpha\beta\gamma \tag{3.1}$$

Using these values in the defining relation (2.2) of the 1-forms $\sigma^\alpha_i$ we obtain

$$\sigma^\alpha_i = \begin{pmatrix} 0 & e^{-x} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \tag{3.2}$$

The corresponding vector fields $\xi^i_\alpha$ (satisfying $[\xi_\alpha, \xi_\beta] = c^\gamma_{\alpha\beta} \xi_\gamma$) with respect to which the Lie Derivative of the above 1-forms is zero are:

$$\xi_1 = \partial_y \quad \xi_2 = \partial_z \quad \xi_3 = \partial_x + y \partial_y \tag{3.3}$$
The Time Depended A.I.D.’s are described by

\[
\Lambda^\alpha_\beta = \begin{pmatrix}
  e^{-2P(t)} & 0 & x(t) \\
  0 & c_{22} & c_{23} \\
  0 & 0 & 1
\end{pmatrix}
\] (3.4)

\[
P^\alpha = \left( x(t) \dot{P}(t) + \frac{1}{2} \ddot{x}(t), P^2(t), \dot{P}(t) \right)
\] (3.5)

where \( P(t), x(t) \) and \( P^2(t) \) are arbitrary functions of time. As we have already remarked the three arbitrary functions appear in \( P^\alpha \) and thus can be used to set the shift vector to zero.

The remaining symmetry of the EFE’s is, consequently, described by the constant matrix:

\[
M = \begin{pmatrix}
  e^{s_1} & 0 & s_4 \\
  0 & e^{s_2} & s_3 \\
  0 & 0 & 1
\end{pmatrix}
\] (3.6)

where the parametrization has been chosen so that the matrix becomes identity for the zero value of all parameters.

Thus the induced transformation on the scale factor matrix is \( \tilde{\gamma}_{\alpha\beta} = M^\mu_\alpha M^\nu_\beta \gamma_{\mu\nu} \), which explicitly reads:

\[
\begin{align*}
\tilde{\gamma}_{11} &= e^{2s_1} \gamma_{11} \\
\tilde{\gamma}_{12} &= e^{s_1+s_2} \gamma_{12} \\
\tilde{\gamma}_{13} &= e^{s_1} \left( s_3 \gamma_{11} + s_4 \gamma_{12} + \gamma_{13} \right) \\
\tilde{\gamma}_{22} &= e^{s_2} \gamma_{22} \\
\tilde{\gamma}_{23} &= e^{s_2} \left( s_3 \gamma_{12} + s_4 \gamma_{22} + \gamma_{23} \right) \\
\tilde{\gamma}_{33} &= s_3^2 \gamma_{11} + 2 s_3 \left( s_4 \gamma_{12} + \gamma_{13} \right) + s_4^2 \gamma_{22} + 2 s_4 \gamma_{23} + \gamma_{33}
\end{align*}
\] (3.7)

The previous equations, define a group of transformations \( G_r \) of dimension \( r = \text{dim}(\text{Aut}(III)) = 4 \). The four generators of the group, can be evaluated from the relation:

\[
X_A = \left( \frac{\partial \tilde{\gamma}_{\alpha\beta}}{\partial s_A} \right)_{s=0} \frac{\partial}{\partial \gamma_{\alpha\beta}}
\] (3.8)

where \( A = \{1, 2, 3, 4\} \). Applying this definition to (3.7) we have the generators:

\[
X_1 = 2\gamma_{11} \frac{\partial}{\partial \gamma_{11}} + \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + \gamma_{13} \frac{\partial}{\partial \gamma_{13}}
\] (3.9)

\[
X_2 = \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + 2\gamma_{22} \frac{\partial}{\partial \gamma_{22}} + \gamma_{23} \frac{\partial}{\partial \gamma_{23}}
\] (3.10)

5
\[ X_3 = \gamma_{12} \frac{\partial}{\partial \gamma_{13}} + \gamma_{22} \frac{\partial}{\partial \gamma_{23}} + 2\gamma_{23} \frac{\partial}{\partial \gamma_{33}} \]  
(3.11)

\[ X_4 = \gamma_{11} \frac{\partial}{\partial \gamma_{13}} + \gamma_{12} \frac{\partial}{\partial \gamma_{23}} + 2\gamma_{13} \frac{\partial}{\partial \gamma_{33}} \]  
(3.12)

The algebra \( g_r \) that corresponds to the group \( G_r \) has the following table of commutators:

\[
\begin{align*}
[X_1, X_2] &= 0, \quad [X_1, X_3] = 0, \quad [X_1, X_4] = X_4, \\
[X_2, X_3] &= X_3, \quad [X_2, X_4] = 0, \quad [X_3, X_4] = 0
\end{align*}
\]  
(3.13)

As it is evident from the above commutators (3.13) the group is non-abelian, so we cannot diagonalize at the same time all the generators. However, if we calculate the derived algebra of \( g_r \), we have

\[ g_r' = \{ [X_A, X_B] : X_A, X_B \in g_r \} \Rightarrow g_r' = \{ X_3, X_4 \} \]  
(3.14)

and furthermore, it’s second derived algebra reads:

\[ g_r'' = \{ [X_A, X_B] : X_A, X_B \in g_r' \} \Rightarrow g_r'' = \{ 0 \} \]  
(3.15)

Thus, the group \( G_r \) is solvable since the \( g_r'' \) is zero. As it is evident \( X_3, X_4, Y_2 \) generate an Abelian subgroup, and we can, therefore, bring them to their normal form simultaneously. The appropriate transformation of the dependent variables is:

\[
\begin{align*}
\gamma_{11} &= e^{u_1+2u_6} \\
\gamma_{12} &= e^{u_1+u_2+u_4+u_6} \\
\gamma_{13} &= e^{u_1+u_6} (e^{u_6} u_3 + e^{u_2+u_4} u_5) \\
\gamma_{22} &= e^{u_1+2u_4} \\
\gamma_{23} &= e^{u_1+u_4} (e^{u_2+u_6} u_3 + e^{u_4} u_5) \\
\gamma_{33} &= e^{u_1} (1 + e^{2u_6} u_3^2 + 2e^{u_2+u_4+u_6} u_3 u_5 + e^{2u_4} u_5^2)
\end{align*}
\]  
(3.16)

In these coordinates the generators \( Y_2, X_A \) assume the form:

\[
\begin{align*}
Y_2 &= \frac{\partial}{\partial u_1} \quad X_3 = \frac{\partial}{\partial u_3} \quad X_4 = \frac{\partial}{\partial u_5} \\
X_2 &= \frac{\partial}{\partial u_4} - u_5 \frac{\partial}{\partial u_5} \quad X_1 = \frac{\partial}{\partial u_6} - u_3 \frac{\partial}{\partial u_3}
\end{align*}
\]  
(3.17)

Except of the parametrization (3.16) there is also another one achieving the same result (3.17), which simply attributes a - sign to \( \gamma_{12} \) and therefore any solution later described will remain valid under this change.

Evidently, a first look at (3.16) gives the feeling that it would be hopeless even to write down the Einstein equation. However, the simple form of the generators (3.17) ensures us that these equations will be of first order in the functions \( \dot{u}_1, \dot{u}_3 \) and \( \dot{u}_5 \).
3.1 Description of the Solution Space

Before we begin solving the Einstein equations, a few comments for the possible values of the functions \( u_i, i = 1, \ldots, 6 \) will prove very useful.

The determinant of \( \gamma_{\alpha \beta} \), is

\[
det[\gamma_{\alpha \beta}] = e^{3u_1+2(u_4+u_6)} (1 - e^{2u_2})
\]  

(3.18)

so we must have \( u_2 < 0 \).

The transformation from the \( \gamma' \) s to the \( u' \) s, becomes singular when \( \gamma_{12} = 0 \), since the function \( u_2 \) equals to

\[
u_2 = \ln(|\gamma_{12}|) - \frac{\ln(\gamma_{11}\gamma_{22})}{2}.
\]  

(3.19)

So two cases are naturally arising, according to whether \( \gamma_{12} \) is different or equal to zero. If \( \gamma_{12} \neq 0 \) the two linear constraint equations, written in the new variables \( \gamma_{12} \), give

\[
E_1 = 0 \Rightarrow -e^{u_6} (e^{u_6} \dot{u}_3 + e^{u_2+u_4} \dot{u}_5) = 0
\]  

(3.20)

\[
E_2 = 0 \Rightarrow -\frac{1}{2} e^{u_4} (e^{u_2+u_6} \dot{u}_3 + e^{u_4} \dot{u}_5) = 0
\]  

(3.21)

This system admits only the trivial solution, since the determinant of the 2x2 matrix formed by the coefficients of \( \dot{u}_3, \dot{u}_5 \) becomes zero only for the forbidden value \( u_2 = 0 \). We thus have

\[
u_3 = k_3, \quad u_5 = k_5
\]  

(3.22)

Now, these values of \( u_3, u_5 \) make \( \gamma_{13}, \gamma_{23} \) functionally dependent upon \( \gamma_{11}, \gamma_{12}, \gamma_{22} \) (see \( \gamma_{12} \)). It is thus possible to set these two components to zero by means of an appropriate constant automorphism.

In the case \( \gamma_{12} = 0 \) we can again bring simultaneously into normal form the corresponding \( X_3, X_4, Y_2 \). The appropriate change of dependent variables is given by:

\[
\gamma_{\alpha \beta} = \begin{pmatrix}
  e^{u_1+2u_6} & 0 & e^{u_1+2u_6}u_3 \\
  0 & e^{u_1+2u_5} & e^{u_1-u_4+u_5} \\
  e^{u_1+2u_6}u_3 & e^{u_1-u_4+u_5} & e^{u_1} (1 + e^{-2u_4} + e^{2u_6} u_3^2)
\end{pmatrix}
\]  

(3.23)

In these variables all three linear constraint equations can be integrated, yielding:

\[
E_1 = 0 \Rightarrow -e^{2u_6} \dot{u}_3 = 0 \Rightarrow u_3 = k_3
\]  

(3.24)

\[
E_2 = 0 \Rightarrow -\frac{1}{2} e^{-u_4+u_5} (\dot{u}_4 + \dot{u}_5) = 0 \Rightarrow u_5 = k_5 - u_4
\]  

(3.25)

\[
E_3 = 0 \Rightarrow -2 e^{2u_4+2u_6} u_3 \dot{u}_3 + \dot{u}_4 + \dot{u}_5 + 2 e^{2u_4} \dot{u}_6 = 0 \Rightarrow u_6 = k_6
\]  

(3.26)
Again, these values imply that a constant automorphism suffices to set the (13) and (23) components of the scale-factor matrix to zero, i.e. to put it into diagonal form. We have thus reached a first important conclusion, that is:

*Without loss of generality, we can start our investigation of the solution space for Type III vacuum Bianchi Cosmology from a block-diagonal form of the scale-factor matrix (and, of course, zero shift)*

\[
\gamma_{\alpha\beta} = \begin{pmatrix}
\gamma_{11} & \gamma_{12} & 0 \\
\gamma_{12} & \gamma_{22} & 0 \\
0 & 0 & \gamma_{33}
\end{pmatrix}
\] (3.27)

Note that this conclusion could have not been reached off mass-shell, due to the fact that the time-dependent Automorphism (3.4) does not contain the necessary two arbitrary functions of time in the (13) and (23) components (besides the fact that all the freedom in arbitrary functions of time has been used to set the shift to zero). As we have earlier remarked, since the algebra (3.13) is solvable, the remaining (reduced) generators \(X_1, X_2\) (corresponding to diagonal constant automorphisms) as well as \(Y_2\) continue to define a Lie-Point symmetry of the reduced EFE’s and can thus be used for further integration of this system of equations.

### 3.1.1 Case I: \(\gamma_{12} = 0\)

The remaining (reduced) automorphism generators are

\[
X_1 = 2\gamma_{11} \frac{\partial}{\partial \gamma_{11}}, \quad X_2 = 2\gamma_{22} \frac{\partial}{\partial \gamma_{22}}
\]

The appropriate change of dependent variables which brings these generators -along with \(Y_2\)- into normal form, is described by the following scale-factor matrix :

\[
\gamma_{\alpha\beta} = \begin{pmatrix}
e^{u_1+u_3} & 0 & 0 \\
0 & e^{u_2+u_3} & 0 \\
0 & 0 & e^{u_3}
\end{pmatrix}
\] (3.28)

In these variables the first two linear constraint equations are identically satisfied, while the third reads \(E_3 = 0 \Rightarrow -2 \dot{u}_1 = 0 \Rightarrow u_1 = k_1\). Substituting this value of \(u_1\) into the quadratic constraint equation \(E_0\) we obtain the lapse function

\[
N^2 = \frac{1}{16} e^{u_3} \dot{u}_3 (2\ddot{u}_2 + 3\dot{u}_3)
\] (3.29)

Now, substitution of \(u_1 = k_1\) and the above value for the lapse \(N^2\) into the spatial EFE’s results in the single, independent equation :

\[
(\dot{u}_2 + \dot{u}_3)(2\dot{u}_3\ddot{u}_2 - 2\dot{u}_2\ddot{u}_3 + 2\dot{u}_2^2\dot{u}_3 + 3\dot{u}_3^2 + 5\ddot{u}_2\dot{u}_3^2)
\] (3.30)

This equation is, as expected from the theory, of the first order in \(\dot{u}_2, \dot{u}_3\). Notice that this result could have not been reached had we chosen any particular time gauge, such
as $N^2 = F(u_2, u_3, t)$: Not only $u_2, u_3, t$ would appear in the Spatial EFE’s, but also the number of independent such equations would have been increased to 2. This remark should not be taken as a negative view for complete gauge fixing, but rather as pointing to the fact that keeping the gauge freedom into the game helps manifesting the symmetries of the system and eventually solving the equations.

Equation (3.30) is readily integrated, leading to two different space-times according to which parenthesis is set to zero. If the first is made to vanish, i.e. $u_2 = k_2 - u_3$, the ensuing line-element is the known (Type III) cosmological disguise of Minkowski space-time [15]:

$$ds^2 = -\frac{1}{16}e^{u_3}u_3 dt^2 + \frac{1}{4}e^{u_3} dx^2 + e^{k_1+u_3-2x} dy^2 + e^{k_2} dz^2 \quad (3.31)$$

the constants being of course absorbable by the constant automorphisms and a shift in $u_3$.

If the second parenthesis of (3.30) is set to zero, i.e. $u_2 = k_3 - \frac{3u_3}{2} + \ln(1 + k_2 e^{u_3})$, we obtain an equivalent form of the Type III member of the known Ellis-MacCallum family of solutions [8],[15]:

$$ds^2 = \kappa^2 \left(-\frac{e^{\frac{3u_3}{2}}}{4(e^{\frac{u_3}{2}} - 1)} dt^2 + e^{u_3} dx^2 + e^{u_3-2x} dy^2 + e^{\frac{u_3}{2}}(e^{\frac{u_3}{2}} - 1) dz^2 \right) \quad (3.32)$$

where again we have used constant automorphisms and a shift of $u_3$ to take outside of the metric an overall constant. We can be assured that the constant is essential either by checking that indeed the metric inside the parenthesis does not admit a homothetic Killing vector field or, more primarily, by finding an invariant relation between curvature and higher derivative curvature scalars which explicitly involves $\kappa$. For metric (3.32) one such invariant relation is:

$$\frac{18 Q_1^4}{(Q_2 - g^{AB}Q_1 A Q_1 B)^3} = \kappa^2, \quad Q_1 = R^{KLMN}R_{KLMN}, \quad Q_2 = \Box R^{KLMN}R_{KLMN} \quad (3.33)$$

where capital Latin letters denote space-time indices ranging in the interval (0-3), the semicolon stands for covariant differentiation, and the $\Box$ for the covariant D’Alebertian. This relation, being a constant scalar constructed out of the intrinsic geometry (the Riemmann tensor and its covariant derivatives), characterizes, along with many others that can be found, this metric: It will be valid for any equivalent, under general coordinate transformations, form of (3.32). It is also noteworthy to observe that in both the above line-elements the arbitrary function of time $u_3$ appears; This is because the number of symmetry generators matches the number of scale-factors (both are 3), so that the system of spatial EFE’s is reduced to first order without any choice of time. In the case of a block-diagonal scale-factor matrix one of the four scale factors will have to play the role of time before the corresponding system can be reduced. Lastly, metric (3.32) admits, except of (3.33), a fourth Killing vector field acting on the surfaces of simultaneity, namely

$$\xi_4 = -2y \partial_x + (e^{2x} - y^2) \partial_y \quad (3.34)$$
There is thus a $G_4$ symmetry group acting (of course, multiply transitively) on each $V_3$ of this metric, with an algebra having the following table of (non-vanishing) commutators:

$$[\xi_1, \xi_3] = \xi_1, \quad [\xi_1, \xi_4] = -2\xi_3, \quad [\xi_3, \xi_4] = \xi_4$$

(3.35)

However, it is interesting to note that we have not imposed the extra symmetry from the beginning, but rather it emerged as a result of the investigation process.

3.1.2 Case II: $\gamma_{12} \neq 0$

The remaining (reduced) automorphism generators are

$$X_1 = 2\gamma_{11} \frac{\partial}{\partial \gamma_{11}} + \gamma_{12} \frac{\partial}{\partial \gamma_{12}}, \quad X_2 = \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + 2\gamma_{22} \frac{\partial}{\partial \gamma_{22}}$$

The appropriate change of dependent variables which brings these generators along with $Y_2$ into normal form, is now given by:

$$\gamma_{\alpha\beta} = \begin{pmatrix}
e^{u_1 + 2u_4} & e^{u_1 + u_2 + u_4} & 0 \\
e^{u_1 + u_2 + u_4} & e^{u_1 + 2u_2} & u_3 \\
0 & 0 & e^{u_1}
\end{pmatrix}$$

(3.36)

The generators are now reduced to

$$Y_2 = \frac{\partial}{\partial u_1}, \quad X_2 = \frac{\partial}{\partial u_2}, \quad X_1 = \frac{\partial}{\partial u_4}$$

(3.37)

indicating that the system will be of first order in the derivatives of these variables. The remaining variable $u_3$ will enter, (along with $\dot{u}_3$, $\ddot{u}_3$) explicitly in the system and is therefore advisable (if not mandatory) to be used as the time parameter, i.e. to effect the change of time coordinate

$$t \to u_3(t) = s, \quad u_1(t) \to u_1(t(s)), \quad u_2(t) \to u_2(t(s)), \quad u_4(t) \to u_4(t(s)).$$

(3.38)

This choice of time will of course be valid only if $u_3$ is not a constant. We are thus led to consider two cases according to the constancy or non-constancy of this variable.

The case $u_3 = k_3$

The determinant of the scale-factor matrix becomes $\text{det}[\gamma_{\alpha\beta}] = e^{3u_1 + 2(u_2 + u_4)}(-1 + k_3)$. We thus have $k_3 > 1$. The two linear constraint equations are identically satisfied, while the third yields

$$E_3 = 0 \Rightarrow \frac{\dot{u}_2 + (1 - 2k_3) \dot{u}_4}{2(1 - k_3)} = 0 \Rightarrow u_4 = k_4 + \frac{u_2}{2k_3 - 1}$$

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Inserting these values of $u_3, u_4$ into the quadratic constraint equation we obtain the following lapse

$$(N)^2 = \frac{e^{u_1} (-1 + k_3) (3 (1 - 2 k_3)^2 \dot{u}_1^2 + 8 k_3 (-1 + 2 k_3) \dot{u}_1 \dot{u}_2 + 4 k_3 \dot{u}_2^2)}{4 (1 - 2 k_3)^2 (-3 + 4 k_3)} \quad (3.39)$$

Use of these values of $u_3, u_4$, $(N)^2$ in the spatial EFE’s results, as expected, in a system which is of the first order in the unknown variables $\dot{u}_1, \dot{u}_2$. The coefficient of $\ddot{u}_2$ in $E_{33} = 0$ is

$$2 e^{u_1} k_3 \dot{u}_1 ((-1 + 2 k_3) \dot{u}_1 + \dot{u}_2)$$

and can be safely regarded different from zero, since the possibilities $\dot{u}_1 = 0, \dot{u}_2 = (1 - 2 k_3) \dot{u}_1$ easily lead (through $E_{33} = 0$ itself) to zero and negative lapse, respectively. We can thus solve $E_{33} = 0$ for $\ddot{u}_2$ and substitute into $E_{12} = 0$ which becomes

$$e^{k_3 + u_1 + \frac{2 k_3 u_2}{1 + 2 k_3} k_3 (\dot{u}_1 + 2 \dot{u}_2) ((-3 + 6 k_3) \dot{u}_1 + 2 (3 - 2 k_3) \dot{u}_2)} (6 - 20 k_3 + 16 k_3^2) = 0$$

Again, the second parenthesis in the numerator leads to zero lapse, leaving us with the only alternative $\dot{u}_1 = -2 \omega_2 \Rightarrow u_1 = 2 k_1 - 2 \omega_2$ which indeed satisfies all spatial EFE’s.

Finally, inserting these values of $u_3, u_4, u_1$ in the lapse (3.39) and the scale-factor matrix (3.36) we obtain the following line-element (after using the constant automorphisms and a shift in $u_2$ to purify the metric from the absorbable constants):

$$ds^2 = -\lambda^2 d\xi^2 + \frac{\xi^2}{4} dx^2 + e^{-2x}\xi^{4\lambda} dy^2 + \frac{\lambda - 1}{2\lambda - 1} dz^2 + 2 e^{-x}\xi^{2\lambda} dy dz \quad (3.40)$$

where the constant $\lambda$ is related to $k_3$ by $k_3 = \frac{\lambda - 1}{2\lambda - 1} \Rightarrow 0 < \lambda < \frac{1}{2}$ and we have adopted the time gauge $e^{-u_2} = \xi$ for simplicity.

This metric is an equivalent form of a solution originally given by Siklos [16] and reproduced in [13]. An overall multiplicative constant has been omitted from (3.40) since it admits the following Homothetic Killing vector field ($\mathcal{L}_H g_{AB} = \mu g_{AB}$)

$$H^A = \xi \frac{\partial}{\partial \xi} + (1 - 2\lambda) y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$$

It also admits three more Killing vector fields (except (3.3)) acting on space-time, namely

$$\xi_4 = e^{-\frac{\lambda}{2\lambda - 1}} \frac{2 \lambda}{\xi} e^{-\frac{x}{2\lambda - 1}} \partial_x$$

$$\xi_5 = e^{-\frac{x}{2\lambda - 1}} y \frac{\partial}{\partial x} + \frac{2 \lambda y}{\xi} e^{-\frac{x}{2\lambda - 1}} \partial_x + \frac{\lambda (\lambda - 1)}{4\lambda - 1} e^{4\lambda - 1} x \xi^{-4\lambda + 1} \partial_y - \lambda e^{2\lambda - 1} x \xi^{-2\lambda + 1} \partial_z$$

$$\xi_6 = e^{-\frac{x}{2\lambda - 1}} z \frac{\partial}{\partial x} + \frac{2 \lambda z}{\xi} e^{-\frac{x}{2\lambda - 1}} \partial_x - \lambda e^{2\lambda - 1} x \xi^{-2\lambda + 1} \partial_y - \lambda (2\lambda - 1) e^{\frac{x}{2\lambda - 1}} \xi \partial_z$$

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The $\xi$ field breaks down for $\lambda = 1$, in which case the valid expression is

$$\xi' = \left\{ \frac{y}{e^{2x}}, \frac{y}{2e^{2x}} \xi, \frac{-2x + \log(\xi)}{16}, \frac{-\sqrt{\xi}}{4e^x} \right\}$$  \hspace{1cm} (3.41)

The first of these is null $\xi^A \xi_B g_{AB} = 0$ and covariantly constant $\xi^A_{;B} = 0$, signaling that the metric is a pp-wave. Consequently all scalar curvatures, constructed by forming scalar contractions of tensor product of the Riemmann tensor and its covariant derivatives of any order (such as $Q_1, Q_2$ in (3.33)), vanish identically (see e.g. [17]). This raises the interesting question of how can we be certain that the constant $\lambda$ is essential. An answer can be found in terms of equalities between tensors–constructed out of the Riemann tensor and its covariant derivatives–that hold true in these space-times [18],[19]. For metric (3.40) such a relation is:

$$R^A_B C_D R_{AEFC;GH} = \frac{4\lambda^2 - 2\lambda + 3}{-4\lambda^2 + 2\lambda + 2} R^A_B C_D E R_{AFCG;H}$$  \hspace{1cm} (3.42)

By the quotient law the expression of $\lambda$ in the right-hand side of this relation is a scalar function, and being a constant it can not change value under any coordinate transformation; thus $\lambda$ can not be altered by such a transformation and is, therefore, essential.

The algebra of the six killing fields, (3.3), (3.41) has the following table of non vanishing commutators:

$$[\xi_1, \xi_3] = \xi_1 \quad [\xi_1, \xi_5] = \xi_4 \quad [\xi_3, \xi_4] = -\frac{\xi_4}{2\lambda} \quad [\xi_3, \xi_5] = \frac{2\lambda - 1}{2\lambda} \xi_5 \quad [\xi_3, \xi_6] = -\frac{1}{2\lambda} \xi_6$$  \hspace{1cm} (3.43)

There is an isotropy group $G_2$ of null rotations emanating from this algebra, which is easily seen by taking a linear combination of these fields:

$$Y_1 = \xi_1 - 2\lambda \xi_3 \quad Y_2 = \xi_4 \quad Y_3 = -\xi_6$$
$$Y_4 = \xi_2 \quad Y_5 = \xi_3 \quad Y_6 = \xi_5$$  \hspace{1cm} (3.44)

e.g. $[Y_1, Y_2] = Y_2$ and $[Y_1, Y_3] = Y_3$

The space (being pp-wave) does not obviously have curvature singularities, it thus seems to be geodesically complete and is of Petrov Type N.

**The case** $u_3 \neq k_3$

The function $u_3$ is now a valid choice of time and $det[\gamma_{\alpha\beta}] = e^{3u_1 + 2(u_2 + u_6)} (-1 + s)$ implies the range $(1, +\infty)$ for the new time $s$. The only non-vanishing linear constraint equation $E_3 = 0$ yields

$$u_4 = \int \frac{\dot{u}_2}{2s - 1} ds + k_4$$  \hspace{1cm} (3.45)

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while the quadratic constraint equation $E_0 = 0$ gives the lapse

$$(N)^2 = \frac{e^{u_1}}{4 (1 - 2 s)^2 (-3 + 4 s)} \left[ 2 (2 s - 1)^2 \dot{u}_1 + 3 (2 s - 1)^2 (s - 1) \dot{u}_1^2 + (4 s - 2) \dot{u}_2 + 8 s (s - 1) (2 s - 1) \ddot{u}_1 \dot{u}_2 + 4 s (s - 1) \ddot{u}_2^2 \right]$$

(3.46)

If we insert these values $(N)^2$, $u_4$ into the spatial EFE’s they become the following polynomial system of first order in $\dot{u}_1$, $\dot{u}_2$

$$\ddot{u}_1 = \left( \begin{array}{c} 1 \\ \dot{u}_1 \\ \dot{u}_1^2 \\ \dot{u}_1^3 \\ \end{array} \right) A_1 \left( \begin{array}{c} 1 \\ \ddot{u}_2 \\ \ddot{u}_2^2 \\ \ddot{u}_2^3 \\ \end{array} \right), \quad \ddot{u}_2 = \left( \begin{array}{c} 1 \\ \dot{u}_1 \dot{u}_1^2 \dot{u}_1^3 \end{array} \right) A_2 \left( \begin{array}{c} 1 \\ \ddot{u}_2 \\ \ddot{u}_2^2 \\ \ddot{u}_2^3 \end{array} \right)$$

(3.47)

$$A_1 = \left( \begin{array}{cccc} 0 & 2 & 4 & 0 \\ 4 & \frac{8 s (2 s - 3) (s - 1)}{4 s^2 - 10 s + 13} & 0 & 0 \\ \frac{1}{4 s^2 - 7 s + 3} & 4 & \frac{8 s (2 s - 3) (s - 1)}{8 s^2 - 10 s + 13} & 0 \\ -\frac{6 s (s - 1)}{4 s - 3} & 0 & 0 & 0 \end{array} \right)$$

(3.48)

$$A_2 = \left( \begin{array}{cccc} 0 & -\frac{8 s + 5}{8 s^3 - 18 s^2 + 13 s - 3} & 24 s^2 - 50 s + 18 & 8 s (2 s - 3) (s - 1) \\ -\frac{12 s}{4 s^2 - 7 s + 3} & 2 s - 3 & -\frac{16 s^2 (s - 1)}{8 s^2 - 10 s + 13} & 0 \\ 24 s^2 - 50 s + 18 & 8 s (2 s - 3) (s - 1) & \frac{8 s (2 s - 3) (s - 1)}{8 s^2 - 10 s + 13} & 0 \\ -\frac{6 s + 3}{4 s - 3} & -\frac{6 s (s - 1)}{4 s - 3} & 0 & 0 \end{array} \right)$$

(3.49)

Due to the form of $A_1$, $A_2$ (their components are rational functions of the time $s$), system (3.47) can be partially integrated with the help of the following Lie-Bäklund transformation

$$\dot{u}_1(s) = \frac{(2 s - 3) \tan r(s) - 2 s (8 s^2 - 10 s + 3) \dot{r}(s)}{4 s \sqrt{s - 1} (4 s - 3)}$$

$$\dot{u}_2(s) = \frac{2 s - 1}{8 s (4 s - 3) \sqrt{(s - 1)^3}} \left( 2 (-4 s + 3) \sqrt{s - 1} + 3 (s - 1) \tan r(s) + 2 s (s - 1) (4 s - 3) \dot{r}(s) \right)$$

(3.50)

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resulting in the single, second order ODE for the variable $r(s)$

$$
\ddot{r} = \left( \tan r - \frac{\sqrt{s - 1}}{2} \right) \dot{r}^2 + \frac{(-16 s + 6) \sqrt{s - 1} + (5 s - 3) \tan r}{2 s (4 s - 3) \sqrt{(s - 1)}} \dot{r} \\
+ \frac{-9 (s - 1)^2 \tan^2 r + 18 (s - 1)^{3/2} \tan r + 4 s (4 s - 3)}{8 s^2 (4 s - 3)^2 (s - 1)^{3/2}}
$$

(3.51)

This equation contains all the information concerning the unknown part of the solution space of the Type III vacuum Cosmology. Unfortunately, it does not posses any Lie-point symmetries that can be used to reduce its order and ultimately solve it. However, its form can be substantially simplified through the use of new dependent and independent variable $(\rho, u(\rho))$ according to

$$
r(s) = \pm \arcsin \frac{u(\rho)}{\sqrt{\rho^2 - 1}}, \quad s = \frac{3(\rho - 1)}{3\rho - 5}, \quad \rho > \frac{5}{3}
$$

thereby obtaining the equation

$$
\ddot{u} = \pm \frac{1 - \dot{u}^2}{\sqrt{(6 \rho - 10) (\rho^2 - u^2 - 1)}} \Rightarrow \dot{u}^2 = \frac{(1 - \dot{u}^2)^2}{(6 \rho - 10) (\rho^2 - u^2 - 1)}
$$

(3.52)

with the corresponding lapse

$$
(N)^2 = \frac{\dot{u}^2 - 1}{8 (3 \rho - 5) (\rho^2 - u^2 - 1)} e^{u_1}
$$

(3.53)

$(\dot{u} = \frac{du}{d\rho})$ and the scale-factor matrix is given by (3.36) after insertion of (3.45), $u_3 = s = \frac{3(\rho - 1)}{3\rho - 5}$ and the transformations of $u_1, u_2$ that led to $u$. Independently of the way we have reached this result, one can check (through an algebraic computing facility such as Mathematica) that the line element thus described is indeed a solution of all the EFE’s, provided of course (3.52) is satisfied. One can also check that it does not admit any Homothetic or null, covariantly constant vector field. Therefore, the two independent constants of the general solution to (3.52) along with a multiplicative constant will comprise the expected three essential constants of the general Type III vacuum Cosmology: The general algorithm for calculating this number when a space time gauge has been chosen (usually zero-shift and unit lapse), in which case the constraints must be viewed as restrictions on the initial data, reads as [15]:

12 (for the six components of $\gamma_{\alpha\beta}$) -1 (for the time reparameterization covariance) -number of independent constraints -dimension of Automorphism Group.

When a space-time gauge has not been fixed, i.e. when constraints are being viewed as symmetry generators, the relevant counting is given by [20]:

$$
D = 2 \times \text{(number of } \gamma_{\alpha\beta}) \\
- 2 \times \text{(number of linear constraints)} \\
- 2 \times \text{(Quadratic constraint)} \\
- \text{(number of parameters of outer-Aut)} \\
- (n)
$$
where \( n \equiv \dim(\text{inner-Aut}) \) - number of independent linear constraints.

In our case the number of independent linear constraints is 3, and the dimension of the inner-Aut is 2, so \( n = -1 \). The constants that appear at the outer-Aut are 2 and obviously the number of \( \gamma_{\alpha\beta} \) is 6. Thus, the expected maximal number of essential constants is indeed 3, by both ways of counting.

Despite the relatively simple form of \((3.52)\), its general solution is, to the best of our knowledge, not known. However, we have managed to obtain a partial solution in the parametric form

\[
u(\xi) = \frac{4}{3} \left(1 + 2e^{2\xi}\right)^{3/2} \quad \rho = \frac{1}{3} (5 + \text{sech}^2 \xi)
\]

which makes the functions \( u_1, u_2, u_4 \) read as

\[
u_1(\xi) = k_1 + \xi + \ln \cosh \xi
\]
\[
u_2(\xi) = k_2 + \ln \text{sech} \xi - \frac{1}{2} \ln (\cosh 2\xi + 2)
\]
\[
u_4(\xi) = k_4 - \ln \cosh \xi + \frac{1}{2} \ln (\cosh 2\xi + 2)
\]

and the lapse \((N)^2 = \frac{e^{k_1 + 2\xi}}{4(2e^{2\xi} + 1)}\). The ensuing metric, after the usual purification with the constant automorphisms and a shift in \( \xi \), is given by:

\[
d s^2 = \kappa^2 \left(-\frac{e^{2\xi}(e^{2\xi} + 1)}{4(2e^{2\xi} + 1)} \, d\xi^2 + \frac{e^\xi}{4} \cosh \xi \, dx^2 + e^{-2x+\xi} (\cosh 2\xi + 2) \text{sech} \xi \, dy^2 + e^\xi \text{sech} \xi \, dz^2 + 2e^{-x+\xi} \text{sech} \xi \, dy \, dz\right)
\]

As we have already remarked, this metric does not admit a Homothety and therefore the constant \( \kappa \) is essential. It does not satisfy the invariant relation \((3.33)\), and it is not a pp-wave. Therefore we conclude that it is inequivalent to \((3.32)\) or \((3.40)\). This monoparametric family belongs to the Kinnersley vacuum solutions \[9\]. It is quite interesting that it also admits a fourth killing vector field

\[
\xi_4 = -16y \, \partial_x + (e^{2x} - 8y^2) \, \partial_y - 2e^x \, \partial_z
\]

which produces with \((3.3)\) the following table of (non-vanishing) commutators:

\[
[\xi_1, \xi_3] = \xi_1, \quad [\xi_1, \xi_4] = -16 \xi_3, \quad [\xi_3, \xi_4] = \xi_4
\]

The isotropy group inferred from the above algebra (see the last commutator) is a \( G_1 \) spatial rotation.

Curiously enough, this algebra is equivalent to \((3.35)\) as a simple scaling of \( \xi_1, \xi_4 \) by \( 2\sqrt{2} \) shows. Of course, the multiply transitive character of the action of the underlying group on the corresponding \( V_3 \)'s allows for these, and thus for the space-times in which they are embedded, to be inequivalent.

Again, the extra symmetry emerged in the course of the investigation of the solution space. Of course it must have something to do with the particular nature of the solution, but it was not set as a starting point.
3.2 Preview for other Bianchi Types

The method described in the previous sections can be applied to other Types as well. The general pattern is similar to that of Type III: The pp-wave solutions (for Types admitting such geometries) occupy one part of the solution space, the other known solutions reside on another part, and the unknown part of the solution space is always described by an ODE strikingly similar to (3.52), namely:

$$\ddot{u}^2 = \frac{(-1 + \dot{u}^2)^2}{(\kappa + \lambda \rho)(\rho^2 - u^2 - 1)}$$  \hspace{1cm} (3.57)

Details will be included in a forthcoming work. As indicative examples we give the form of the ODE for Types IV and VII$_h$:

**Type IV**

$$\ddot{u}^2 = \frac{(-1 + \dot{u}^2)^2}{(\kappa + \lambda \rho)(\rho^2 - u^2 - 1)} \quad \kappa = -6, \lambda = 6$$  \hspace{1cm} (3.58)

**Type VII$_h$**

$$\ddot{u}^2 = \frac{(-1 + \dot{u}^2)^2}{(\kappa + \lambda \rho)(\rho^2 - u^2 - 1)} \quad \kappa = -6 + \frac{4}{h^2}, \lambda = -6$$  \hspace{1cm} (3.59)

and of course

**Type III**

$$\ddot{u}^2 = \frac{(-1 + \dot{u}^2)^2}{(\kappa + \lambda \rho)(\rho^2 - u^2 - 1)} \quad \kappa = -10, \lambda = 6$$  \hspace{1cm} (3.60)

4 Discussion

When one is trying to solve Einstein’s Equations in cosmology, one has to deal with a nonlinear system of coupled, ordinary differential equations. The strategy that is frequently used, is to simplify the system by choosing, a convincing form for the scale factor matrix, usually obtained by an a priori assumption of extra symmetry (e.g. $\gamma_{\alpha\beta} = \text{diag}(a(t), b(t), c(t))$) and then try to solve it, hoping to find some solution. Clearly, this procedure can by no means guarantee access to the full space of solutions for the problem at hand. In this work we have presented a method for solving Einstein’s Field Equations in the case of vacuum Bianchi Geometries. The main idea is to consider the Group of constant Automorphisms, which emerges as the residual freedom left after the time dependent A.I.D.’s (2.8), (2.9) have been used to set the shift $N^\alpha$ to zero, as a Lie point-symmetry of the EFE’s. In a step-by-step procedure one can bring some of the generators of this group in normal form and simplify the rest, thereby reducing the
order of the system of equations. Which of the generators, and how, can be utilized in each step depends upon the characteristics of their Lie Algebra (abelian, solvable etc.). It is also important that the information gained at a particular level must be used and, in fact, may be vital for the implementation of the next step. The method is, by construction, sweeping out all possible solutions, since no ad-hoc assumption has been made. Therefore, if successfully applied to a given Bianchi Cosmology, it will result in the cartography of the entire space of solutions.

The successful application of the procedure to Bianchi Type III resulted in the recovery of all known solutions without prior assumption of any extra symmetry (3.32, 3.40), the enclosure of the entire unknown part of the solution space into a single, second order ODE in terms of one dependent variable (3.52), and a partial solution to this ODE. It is of interest that the solution space is naturally partitioned into three distinct disconnected pieces. Of great importance may be considered the fact that a strikingly similar ODE describes the unknown part of the solution space for other lower Bianchi Types. For Types VIII, IX there remain no rigid automorphisms after the shift has been set to zero and the constant rotations have been used to diagonalize the scale-factor matrix. However there is the scaling symmetry $Y_2$ that can serve as a starting point. This issue, along with the presentation of the detailed cartography for the lower Types is in our immediate scopes. Finally the method can be extended towards either the inclusion of matter content, or in 4+1 Spatially Homogeneous Cosmologies.

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