DYNAMICAL MASS GENERATION FOR NON-ABELIAN
GAUGE FIELDS WITHOUT THE HIGGS

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ABSTRACT

We present an alternative to the Higgs mechanism to generate masses for non-abelian gauge fields in (3+1)-dimensions. The initial Lagrangian is composed of a fermion with current-current and dipole-dipole type self-interactions minimally coupled to non-abelian gauge fields. The mass generation occurs upon the fermionic functional integration. We show that by fine-tuning the coupling constants the effective theory contains massive non-abelian gauge fields without any residual scalars or other degrees of freedom.
I. Introduction.

The standard model has been widely accepted as the theory of electro-weak interactions. It has successfully accounted for all experiments to date, making it perhaps one of the greatest successes of modern theoretical physics. However, apart from the unknown value of the top quark mass, one of the present mysteries in the standard model is the absence of the Higgs particle in present experiments. This fundamental particle is a scalar, which relegates it as the only one in this category. Perhaps, we should see these arguments as indications for looking for alternatives to the Higgs mechanism for generating masses for the elementary fermions and vector bosons.

Over the last twenty years, other mechanisms to account for mass generation in the standard model have been proposed such as technicolor theories\(^1\), and dynamical symmetry breaking via a top-quark condensate\(^2\) in analogy with BCS theory of superconductivity and the Nambu-Jona-Lasinio mechanism in nuclear structure\(^3\). The latter mechanism generates the Higgs particle (not as a fundamental particle) and its consequences with a four-fermion interaction.

In \((2+1)\)-dimensions, it is well-known that the addition of a topological Chern-Simons term to the gauge field kinetic part provides a mechanism for gauge fields mass generation without spoiling gauge invariance\(^4\). In a relatively similar spirit, in \((3+1)\)-dimensions, attempts to reproduce the Chern-Simons term involves a product of field strength \(\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\). However, since this expression can be written as a total derivative, it cannot bring any modifications to physics. It is therefore necessary to introduce another field if we want to mimic the Chern-Simons mechanism in \((3+1)\)-dimensions. For instance, Freedman and Townsend (F-T) and others have developed theories containing an antisymmetric tensor field coupled to an abelian gauge field\(^5,6,7,8\)

\[
\mathcal{L} = -\frac{1}{8} H^\mu H_\mu - \frac{1}{4} F^2_{\mu\nu} + \frac{1}{4} m \epsilon^{\mu\nu\alpha\beta} b_{\mu\nu} F_{\alpha\beta} \tag{1}
\]

where \(H^\mu = \epsilon^{\mu\nu\alpha\beta} \partial_\nu b_{\nu\alpha}\), \(b_{\mu\nu}\) stands for an antisymmetric tensor field and \(F_{\mu\nu}\) is the
field strength for the abelian gauge field. A generalization containing non-abelian
gauge fields and non-abelian antisymmetric tensor fields is provided by F-T

$$\mathcal{L} = -\frac{1}{8} A^a_{\mu} K^{\mu\nu,ab} A^b_{\nu} - \frac{1}{4} (F^a_{\mu\nu})^2 - \frac{1}{8} e \epsilon^{\mu\nu\alpha\beta} b^a_{\mu\nu} F^a_{\alpha\beta}$$

(2)

where $A^a_{\mu} = (K^{ab}_{\mu\nu})^{-1} \epsilon^{\nu\alpha\beta\gamma}(\partial_{\gamma} b^b_{\alpha\beta} + e f^{bcd} A^c_{\gamma} b^d_{\alpha\beta})$, $K^{\mu\nu ab} \equiv g^{\mu\nu} \delta^{ab} - g f^{abc} \epsilon^{\mu\nu\rho\sigma} b^c_{\rho\sigma}$

and $f^{abc}$ are the gauge group structure constants. The theory (1) describes a
massive photon of mass $m$ while the model (2) is equivalent to the classical model
of massive non-abelian gauge fields of mass $\frac{e^2}{2g}$ where $e$ and $g$ are the coupling
constants relative to gauge fields and antisymmetric tensor fields respectively. Both
theories respect gauge invariance and have interesting vector gauge symmetries.

Recently, a mechanism for photon mass generation in (3+1)-dimensions has
been suggested \cite{1}, which uses ideas present in the content of the abelian theory (1).
The mechanism consists of a functional integration over fermions minimally coupled
to a low-energy abelian gauge field. The fermions self-interacts via two types of
contact interactions: current-current and dipole-dipole terms. An antisymmetric
tensor field is introduced via the Hubbard-Stratonovich transformation to perform
the fermion’s integration. After imposing conditions on the coupling constants of
the theory, it is possible to write the low-energy effective action as the abelian
model discussed in Ref.\cite{5,6,7,8}, which reproduces a massive abelian gauge field.

Since the massive mediators of forces present in the weak interactions are known
to be of non-abelian nature, it is perhaps a good idea to reproduce the above
argument generalized to the case of non-abelian gauge fields. Thus we begin in
section II with the definition of our model at the high-energy scale $\Lambda$. In section
III, we integrate the fermions up to energy $\Lambda$ where new physics could occur. In
section IV, we collect the radiative corrections, find the effective current at low-
energies and write the low-energy effective theory for the gauge and tensor fields.
In the appendix, we analyze the physical content of the resulting gauge invariant
theory.

II. High-energy model at the scale $\Lambda$. 
We begin with the following non-renormalizable but SU(N) gauge invariant Lagrangian at the high-energy scale Λ in which a fermion field is minimally coupled to non-abelian gauge fields

\[ \mathcal{L} = -\frac{1}{2} tr F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(iD - m)\psi - g_1 tr j_\mu j^\mu - g_2 tr j_{\mu\nu} j^{\mu\nu} \]  

where \( D_\mu = \partial_\mu - ig A_\mu \) is the covariant derivative. The non-abelian gauge fields are defined by \( A_\mu = A^a_\mu T^a \) where \( T^a \) are the Lie algebra generators obeying the commutation relations \([T^a, T^b] = iC^{abc} T^c\) and the trace relation \( tr(T^a T^b) = \frac{1}{2} \delta^{ab} \).

The last two quantities in (3) are the current-current and dipole-dipole fermionic self-interactions. The four-vector current and the dipole current are given respectively by \( j_\mu = j_\mu^a T^a \) and \( j_{\mu\nu} = j_{\mu\nu}^a T^a \) with components

\[ j_\mu^a = \bar{\psi}\gamma_\mu T^a \psi \]
\[ j_{\mu\nu}^a = \bar{\psi}\sigma_{\mu\nu}\gamma_5 T^a \psi \]

both of which transform in the adjoint representation of the SU(N) gauge group, that is

\[ j_\mu^{a'} = U(\theta) j_\mu^a U^{-1}(\theta) \]
\[ j_{\mu\nu}^{a'} = U(\theta) j_{\mu\nu}^a U^{-1}(\theta) \]

with the usual transformations on fermion field and gauge fields

\[ \psi' = U(\theta)\psi \]
\[ A'_\mu = U(\theta) A_\mu U^{-1}(\theta) - \frac{i}{g} (\partial_\mu U(\theta)) U^{-1}(\theta) \]
\[ F'_{\mu\nu} = U(\theta) F_{\mu\nu} U^{-1}(\theta) \]

It is easy to see that the Lagrangian (3) is invariant under the transformations (5)-(6). The mass dimensions of the fields are \([\psi] = 3/2\), \([A_\mu] = 1\), and \([g] = 0\), \([g_1] = [g_2] = -2\) for the coupling constants.
In components, the Lagrangian (3) can be written as

\[ \mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F^{a,\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \]

\[ -\frac{g_1}{2} (\bar{\psi}\gamma^{\mu} T^a \psi) (\bar{\psi}\gamma^{\mu} T^a \psi) - \frac{g_2}{2} (\bar{\psi}\sigma^{\mu\nu}\gamma^5 T^a \psi) (\bar{\psi}\sigma^{\mu\nu}\gamma^5 T^a \psi) \]

where the field strength is given by

\[ F^{a\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc} A^b_\mu A^c_\nu \]

This model is interesting because of the form of the dipole-dipole interaction, which makes it different from the one studied in Ref.[2].

From now on, for definiteness and due to obvious potential applications, we will consider only the SU(2) gauge group with the usual su(2) Lie algebra given by the Pauli matrices \( T^a = \frac{\tau^a}{2}, \ a = 1, 2, 3 \) and structure constants given by \( C^{abc} = \epsilon^{abc} \).

We next apply the Hubbard-Stratonovich transformation by introducing auxiliary non-abelian antisymmetric tensor fields \( b^{a\mu\nu} \), which belong also in the su(2)-Lie algebra, transform according to the adjoint representation and have mass dimension \( [b^{a\mu\nu}] = 1 \). Their introduction permit us to rewrite the dipole-dipole interaction as

\[ -\frac{g_2}{2} (\bar{\psi}\sigma^{\mu\nu}\gamma^5 \frac{\tau^a}{2} \psi)^2 \to -\frac{1}{2g_2} b^{a\mu\nu}_\mu b^{a\mu\nu}_\nu + i b^{a\mu}_\mu (\bar{\psi}\sigma^{\mu\nu}\gamma^5 \frac{\tau^a}{2} \psi) \]

since one can regain the original Lagrangian by solving the equation of motion for \( b^{a\mu\nu}_\mu \) and substituting the result in the Lagrangian. As noted in Ref.[10], we could also transform, in a similar way, the current-current interaction by introducing other auxiliary gauge fields \( a^{a}_\mu \). In what follows, we choose instead to consider only the introduction of auxiliary antisymmetric tensor fields and treat perturbatively the remaining four-fermion term.

III. Fermionic functional integration.
We calculate the effective action for non-abelian gauge fields and antisymmetric tensor fields, which we treat as low-energy external fields, by integrating out the fermions

\[ e^{i \Gamma_{\text{eff}}[A^a, b^a]} = \int \mathcal{D} \bar{\psi} \mathcal{D} \psi e^{i \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^{a\mu\nu} - \frac{1}{2g_2} b^a_{\mu\nu} b^{a\mu\nu} + \bar{\psi} (\iota \slashed{D} - m + g A^a \tau^a + \gamma^a \gamma^5 b^a) \psi - \frac{g_1}{2} (\bar{\psi} \gamma^\mu \gamma^5 \frac{\tau^a}{2} \psi)^2 \right\} } \]

(10)

where \( \gamma^a \equiv i \sigma^{\mu\nu} \gamma_5 b^a_{\mu\nu} \). The first two terms, being external fields, can be put out of the functional integral. The last term makes the integration difficult since it is not quadratic in the fermionic fields. Instead, we expand it in a power series in \( g_1 \)

\[ e^{i \Gamma_{\text{eff}}[A^a, b^a]} = e^{i \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^{a\mu\nu} - \frac{1}{2g_2} b^a_{\mu\nu} b^{a\mu\nu} \right\} } \int \mathcal{D} \bar{\psi} \mathcal{D} \psi e^{i S_0} \left( 1 - i \frac{g_1}{2} \langle \bar{\psi} \gamma^\mu \gamma^5 \frac{\tau^a}{2} \psi \rangle^2 + \ldots \right) \]

(11)

and the effective action can be rewritten to order \( g_1 \) by reconstructing the exponential as

\[ e^{i \Gamma_{\text{eff}}[A^a, b^a]} = e^{i \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^{a\mu\nu} - \frac{1}{2g_2} b^a_{\mu\nu} b^{a\mu\nu} - \frac{g_1}{2} \langle j^a_\mu \rangle \right\} } e^{i \Gamma_0^{(f)}} \]

(13)

where \( \langle j^a_\mu \rangle \) is the current expectation value and where the fermionic contribution

\[ \Gamma_0^{(f)} = -i \text{Tr} \log(i \iota \slashed{D} - m + g A^a \tau^a + \gamma^a \gamma^5 \frac{\tau^a}{2}) \]

(14)

will be evaluated in a derivative expansion up to two derivatives on fields. In the last equality the trace is taken over spinor indices, group indices and momenta.
Let us start with a global explanation of our strategy. In contrast to the abelian case where an effective theory for photons was derived only up to two-point functions [that is \( n = 2 \) in eq.(14)], in the non-abelian effective theory, we will need to go up to four-point functions. The reasons for this will become clear in the course of the calculations. In part this is guided by the presence of the SU(2) gauge invariance and also by noticing that the self-coupling of the non-abelian gauge fields already contains terms with four fields due to the nonlinearity of the theory. Competition against those expressions will result in our model.

The contribution to \( \Gamma_{\text{eff}} \) quadratic in non-abelian gauge fields and in antisymmetric tensor fields is obtain when we put \( n = 2 \) in the last expression

\[
\frac{i}{2} \text{tr} \left( \frac{\tau^a \tau^b}{2} \right) \text{tr} \int \frac{d^4 p}{(2\pi)^4} \langle p | \frac{1}{i \partial^2 - m} (gA^a + \phi^a) \frac{1}{i \partial^2 - m} (gA^b + \phi^b) | p \rangle. \tag{15}
\]

We evaluate it using the cutoff parameter \( \Lambda \) at which our model (3) is defined in order to control the infinite logarithmic divergence. To ensure the SU(2) gauge invariance with respect to \( A_\mu \), we use a Pauli-Villars regularization in conjunction with the cutoff regularization method when computing its kinetic part. Three different quantities are obtained:

\[
\Gamma_0^{(f)}[A^a, A^b] = -\frac{1}{48\pi^2} g^2 \delta^{ab} \log(\Lambda^2/m^2) \int d^4 x A^a_\mu (g^{\mu\nu} (i\partial^2 - i\partial^\mu i\partial^\nu)) A^b_\nu,
\]

\[
\Gamma_0^{(f)}[A^a, b^b] = \frac{m}{8\pi^2} g \delta^{ab} \log(\Lambda^2/m^2) \int d^4 x \epsilon^{\alpha\beta\mu\nu} b^b_{\alpha\beta} \partial_\mu A^a_\nu,
\]

\[
\Gamma_0^{(f)}[b^a, b^b] = \frac{1}{8\pi^2} \delta^{ab} \log(\Lambda^2/m^2) \int d^4 x \left( m^2 b^a_{\mu\nu} b^b_{\mu\nu} + \frac{1}{6} b^a_{\rho\nu} [g^{\mu\nu} (i\partial^2 - 4(i\partial^\mu)(i\partial^\rho)) b^b_{\mu\nu}] \right). \tag{16}
\]

Note that we have kept only the leading contributions since we have assumed that \( \Lambda \gg m \). The calculations to this order do not differ from the abelian case (c.f. Ref.[10]) except for the overall group theoretic factor \( \text{tr} \left( \frac{\tau^a \tau^b}{2} \right) \).

Next we compute the three and four-point functions involving only gauge fields. This is done by putting \( n = 3 \) and \( n = 4 \) respectively. Since gauge invariance needs
to be respected with respect to $A^a_\mu$, we continue to use a gauge invariant regulator i.e. the Pauli-Villars method in conjunction with the cutoff. The result then completes the first term of Eq. (16) to form the kinetic energy and self-coupling $F^a_\mu F^{a,\mu}$ with its logarithmic divergence.

The rest of the calculation is performed with at least one antisymmetric tensor field. We compute the three-point amplitudes. We obtain three different contributions. The most interesting one is given by

$$\Gamma^{(f)}_0[A^a, A^b, b^c] = \frac{m}{16\pi^2} g^2 \epsilon^{abc} \log(\Lambda^2/m^2) \int d^4x \epsilon^{\alpha\beta\mu\nu} A^a_\mu A^b_\nu b^c_{\alpha\beta}, \quad (17)$$

because it completes the second expression of Eq. (16) in order to make it SU(2) gauge invariant. This is the desired $B \wedge F$-term. The second less important but still present contribution is

$$\Gamma^{(f)}_0[A^a, b^b, b^c] = \frac{i}{48\pi^2} g \epsilon^{abc} \log(\Lambda^2/m^2) \int d^4x \left( 2A^a_\mu [2b^b_\mu (i\partial^\nu)b^{c\nu}_\alpha + (i\partial^\alpha)b^{b\nu}_\mu b^{c\nu}_\alpha] 
+ 2A^{a\alpha} [2b^b_\mu (i\partial^\nu)b^{c\nu}_\alpha + (i\partial^\nu)b^{b\mu}_\nu b^{c\nu}_\alpha] - A^{a\alpha} [b^{b\mu}_\nu (i\partial^\alpha)b^{c\nu} + (i\partial^\alpha)b^{b\mu}_\nu b^{c\nu}] \right) \quad (18)$$

The last three-point contribution contains three antisymmetric tensor fields $\Gamma^{(f)}_0[b^a, b^b, b^c]$ and carries also a logarithmic divergence. Fortunately, it is not necessary to compute it here since our scaling argument (see below) will help us to remove it from the effective theory. We request the reader to keep this in mind.

Finally we have to compute the four-point amplitudes by taking $n = 4$ in Eq. (14). As stated before we have computed the quartic term in $A^a_\mu$. The two expressions involving one or three $A^a_\mu$ fields ($\Gamma^{(f)}_0[A^a, b^b, b^c, b^d]$ and $\Gamma^{(f)}_0[A^a, A^b, A^c, b^d]$) vanish since they involve a trace over an odd number of gamma matrices. The quartic contribution involving antisymmetric tensor fields does not need to be evaluated (again see below). Then, the only remaining quantity to be computed is the one involving two $A^a_\mu$ and two $b^b_\mu$ fields. Evaluation of the leading ultraviolet divergence
for \( \Gamma_0^{(f)}[A^a, A^b, b^c, b^d] \) gives

\[
\Gamma_0^{(f)}[A^a, A^b, b^c, b^d] = \frac{1}{16\pi^2} g^2 \Delta^{abcd} \log(\Lambda^2/m^2)
\int d^4x \left( 2A^{a\mu} A^{b\nu} b^{c}_{\nu\alpha} b^{d\alpha}_{\mu} + 2A^{a\mu} A^{b\nu} b^{c}_{\mu\alpha} b^{d\alpha}_{\nu} - A^{a\mu} A^{b\nu} b^{c}_{\nu\alpha} b^{d\alpha}_{\mu} \right)
\]

(19)

where \( \Delta^{abcd} = \left[ \text{tr} \left( \frac{x^a}{2} \frac{x^b}{2} \frac{x^c}{2} \frac{x^d}{2} \right) - \frac{1}{3} \delta^{ad} \delta^{bc} \right] \), which is a sum of products of two delta.

This completes the computation of the radiative corrections to one-loop order.

IV. Low-energy effective action.

We are now in a position to evaluate the current, which will allow us to find the first order correction in \( g_1 \) to the effective action. From Eqs. (16)-(19), the effective current arising from the presence of matter coupled to the non-abelian gauge fields is given by

\[
\langle j^{a\lambda} \rangle = \frac{\delta \Gamma_0^{(f)}}{\delta A^a_\lambda} = \frac{m}{8\pi^2} \log(\Lambda^2/m^2) \epsilon^{\lambda \mu \nu \alpha} (\partial_\alpha b^{a}_{\mu \nu} + \epsilon^{abc} b^{b}_{\mu \nu} A^{c}_{\alpha})
\]

\[+ \frac{i}{48\pi^2} g \epsilon^{abc} \log(\Lambda^2/m^2) [2b^{b}_{\nu} (i\partial_\nu) \delta^{c\alpha} + \ldots]
\]

\[+ \frac{1}{8\pi^2} g^2 \Delta^{abcd} \log(\Lambda^2/m^2) [2A^{b\nu} b^{c}_{\nu\alpha} b^{d\lambda\alpha} + \ldots]
\]

(20)

where the ellipsis represents contributions to the current of the same form as the term displayed in the corresponding square bracket but with shuffled indices. Note that the first expression above can be rewritten as

\[
\frac{m}{24\pi^2} \log(\Lambda^2/m^2) \epsilon^{\lambda \mu \nu \alpha} H^{e}_{\alpha \mu \nu}
\]

where \( H^{e}_{\alpha \mu \nu} = (D_\alpha b_{\mu \nu} + \text{cyclic})^e \) and \( D_\mu \) is the covariant derivative written in the adjoint representation of the Lie algebra.
Now is time for serious recollection of the results. The effective current enters the effective action to order $g_1$ upon squaring Eq. (20). The current-current interaction therefore contains a logarithmic divergence squared. In order, to write an effective theory with antisymmetric tensor fields and non-abelian gauge fields treated on the same foot, and for which the kinetic term of the antisymmetric tensor fields has no scale dependence at the tree level, we need to rescale $b^a_{\mu\nu}$ as

$$\sqrt{g_1} \frac{m}{4\pi^2} \log(\Lambda^2/m^2) \ b^a_{\mu\nu} \rightarrow b^a_{\mu\nu}$$  \hspace{1cm} (21)

Without any further assumptions, we obtain for the low-energy effective Lagrangian density,

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu} \left( 1 + \frac{1}{12\pi^2} g^2 \log \frac{\Lambda^2}{m^2} \right) + \frac{1}{12} H^{a}_{\mu\nu\rho} H^{a\mu\nu\rho} + \frac{g}{4\sqrt{g_1}} \epsilon^{\alpha\beta\mu\nu} b^a_{\alpha\beta} F^a_{\mu\nu}$$

$$+ \frac{1}{2g_2} \left( \frac{16\pi^4}{g_1 m^2 \log(\Lambda^2/m^2)} \right) \left( \frac{m^2}{4\pi^2 g_2 \log \frac{\Lambda^2}{m^2}} - 1 \right) b^a_{\mu\nu} b^{a\mu\nu}$$

$$- \frac{1}{3} \left( \frac{g_1 m^2 \log(\Lambda^2/m^2)}{g_1 m^2 \log(\Lambda^2/m^2)} \right) b^a_{\rho} [g^{\rho\mu}\partial^2 - 4\partial^\mu \partial^\rho] b^a_{\mu\nu}$$

$$- \frac{1}{3} \left( \frac{g^2}{g_1 m^2 \log(\Lambda^2/m^2)} \right) \epsilon^{abc} \{2A^{a\mu}[2b^b_{\mu\nu}(\partial^a b^c_{\alpha}) + (\partial^a b^b_{\mu\nu})b^c_{\alpha\nu}] + \ldots \}$$

$$+ \frac{1}{6} \left( \frac{g^2}{\sqrt{g_1 m^2 \log(\Lambda^2/m^2)}} \right) \epsilon^{\lambda\mu\nu\alpha} \epsilon^{abc} (D_{\alpha} b^a_{\mu\nu}) [2b^b_{\lambda\rho}(\partial^\sigma b^c_{\rho}) + \ldots]$$

$$- \left( \frac{g^2}{\sqrt{g_1 m^2 \log(\Lambda^2/m^2)}} \right) \Delta^{abcd} \epsilon^{\lambda\mu\nu\alpha} (D_{\alpha} b^a_{\mu\nu}) [2A^{b\rho} b^c_{\rho\lambda} b^d_{\sigma} + \ldots]$$  \hspace{1cm} (22)

where the first and third to the sixth term in Eq. (22) were obtained by direct computations. The second and the last two are instead obtained by squaring the effective current. Incidentally, we left uncomputed the $b^3$ and $b^4$ expressions because these are suppressed by more than one logarithm of the scale $\Lambda$. Eq. (22) corresponds to the effective action we wanted to order four in fields.

We now proceed with approximations that will help us to recover the claims stated in the abstract. In order to eliminate the mass term for the antisymmetric
tensor fields, we impose that the cut-off $\Lambda^2$ satisfy the gap equation

$$m^2 = \Lambda^2 e^{-4\pi^2/m^2 g^2} \quad (23)$$

which goes within the perturbative expansion argument that the coupling $g_2$ is small for sufficiently large $\Lambda$ and fixed fermion mass. Upon using the gap equation, we next assume that the ratios

$$\frac{\pi^2}{g_1 m^2 \log(\Lambda^2/m^2)} = \frac{g_2}{4g_1} \quad (24)$$

and

$$\frac{\pi^2 g}{\sqrt{g_1} m^2 \log(\Lambda^2/m^2)} = \frac{gg_2}{4\sqrt{g_1}} \quad (25)$$

are small, which permits us to smoothly get ride of the unwanted remaining expressions in Eq. (22). The second ratio (25) is a necessary extra condition present only in the non-abelian theory.

With the assumptions given above, and upon renormalizing the coupling constant $g$ by absorbing the logarithmic divergence present in the first quantity, we obtain a low-energy effective Lagrangian of the form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{12} H^a_{\mu\nu\rho} H^{a\mu\nu\rho} + \frac{1}{4\sqrt{g_1}} g\epsilon^{\alpha\beta\mu\nu} b^a_{\alpha\beta} F^a_{\mu\nu} \quad (26)$$

valid up to energy $m$ and which describes massive SU(2) non-abelian gauge fields with mass $\frac{g}{\sqrt{g_1}}$ [see appendix].

V. Conclusions.

We have succeeded in functionally integrating the four-Fermi theory to end up with an effective $b \wedge F$-theory in agreement with the model proposed by Lahiri but different than the one proposed by Freedman-Townsend. It is interesting
to note that we did not reproduce Freedman-Townsend’s model (2) because the nonlinearities in the antisymmetric tensor fields are suppressed by the cutoff $\Lambda$. This is perhaps unfortunate because the F-T model has higher reducible vector gauge invariance and behaves properly for renormalization purpose$^{11}$. However, it has been shown to have unitarity problems in tree-level scattering$^{12}$.

In the appendix, we have performed an analysis to count the degrees of freedom of the theory (26). We have shown that it does indeed contain, for each color, two massive transverse modes combined with a massive longitudinal mode of the same mass, which is necessarily interpreted as the third degree of freedom for the non-abelian gauge fields.

Of course, our presentation of the counting of degrees of freedom here is restricted to the study of the Lagrangian’s bilinear part and more has to be said in order to understand the interactions of the model in this basis. Renormalizability and unitarity of model (26) deserve an other study and generalization to include $SU(2)_L \times U(1)_Y$ are presently under investigation.

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Appendix: Physical Content.

It remains to show that the model in Eq. (26) has the proper number of degrees of freedom necessary to insure a description of massive non-abelian gauge fields. We follow Ref.[8] and absorb the gauge field coupling constant into the gauge fields for convenience. It is well-known that the pure gauge field part of the the Lagrangian density (26) can be written in terms of the canonical variable $A^a$ and the electric field (its conjugate) $E^a$, in the following way

$$L_F = -\frac{1}{g^2} \left\{ \partial_0 A^a \cdot E^a + \frac{1}{2} (E^a \cdot E^a + B^a \cdot B^a) - A_0^a ((D \cdot E)^a) \right\}$$

(27)

where the electric and magnetic fields are

$$E^a = F_a^{i0} = -\nabla A_0^0 - \partial_0 A^a + \epsilon^{abc} A^b A_c^0$$

$$B^a = -\frac{1}{2} \epsilon^{ijk} F_{a}^{j} = \nabla \times A^a - \frac{1}{2} \epsilon^{abc} A^b \times A^c$$

(28)

and the magnetic field can be read as a function of the variable $A^a$. In this way, $A_0^a$ is then easily identified as the Lagrange multiplier.

Upon integration by parts, the second term of the Lagrangian (26) can be written as

$$L_{bF} = \frac{1}{\sqrt{g_1}} v^{a,\mu} A_0^a + \frac{1}{2\sqrt{g_1}} \epsilon^{abc} [2A_0^a b^b \cdot A^c - b^a \cdot (A^b \times A^c)]$$

(29)

and the kinetic energy (and three and four bosons couplings) for the antisymmetric tensor fields as

$$L_{H^2} = \frac{1}{2} v^{a,\mu} v_{a,\mu} + \epsilon^{abc} [2A_0^a b^b \cdot A^c - v_0^a A^b \cdot A^c + (v^a \times A^b) \cdot b^c]$$

$$- \frac{1}{4} \left[ A_\mu^a A_\nu^b b_\tau^c b^{\nu \tau} - A_\mu^a A_\nu^c b_\tau^b b^{\nu \tau} \right] + \frac{1}{2} \left[ A_\mu^a A_\nu^c b_\tau^b b^{\nu \tau} - A_\mu^a A_\nu^b b_\tau^c b^{\nu \tau} \right]$$

(30)
where we introduced the convenient notation

\[ v_\alpha^\beta = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \partial_\nu b^\mu_{\alpha} \]
\[ a^a = \frac{1}{2} \epsilon^{ijk} b_{jk}^a \]
\[ b^a = -b_{0i}^a \]  

(31)

The dynamical content of the theory is located in the bilinear part in fields. In order to exhibit the physical content of the theory, we use the non-abelian gauge invariance of the theory together with a change of variable motivated by the vector gauge invariance of the Lagrangian (26) up to the bilinear part in fields. In order to proceed, we define vectors in space by the known decomposition into longitudinal and transverse part: \( J = J_T + J_L \) such that \( \nabla \cdot J_T = 0 \) and \( \nabla \times J_L = 0 \). In a non-abelian theory such as the present case, it is always possible to choose for a fixed vector \( n_\mu \), non-abelian gauge fields \( n_\mu A^a_\mu = 0 \). We choose this gauge fixing condition such that \( A_L = 0 \). In this gauge, the Lagrangian’s bilinear part in fields is

\[
\mathcal{L}^{\text{bilinear}} = -\frac{1}{2g^2} \left[ A_T^a \cdot \Box A_T^a + A^a_0 \nabla^2 A_0^a \right] + \frac{1}{\sqrt{g_4}} \left[ v_0^a A_0^a - v^a \cdot A_T^a \right] + \frac{1}{2} v_0^a v_0^a - \frac{1}{2} v^a \cdot v^a
\]

(32)

where \( v_0^a = -\nabla \cdot a^a \) and \( v^a = \nabla \times b^a + \partial_0 a^a \).

Next, we note that the above Lagrangian’s bilinear part in fields is invariant under the vector gauge invariance

\[
\tilde{b}_{\mu\nu}^a = b_{\mu\nu}^a + \partial_\mu \Gamma_\nu^a - \partial_\nu \Gamma_\mu^a
\]

(33)

without changing the non-abelian gauge fields. Eq. (33) is not an invariance of the full Lagrangian. Perhaps this is unfortunate, but we can still use a field redefinition to uncover the physical degrees of freedom of the theory at the expense of introducing complications in the interactions. For instance, we can always redefine
the antisymmetric tensor fields such that
\[
\tilde{a}^a = a^a_L \\
\tilde{b}^a = (b^a)_T - \partial_0 \Gamma^a_T
\]
that is, the new vector field \(\tilde{a}^a\) has no transverse part and the \(\tilde{b}^a\)-field has no longitudinal part. Indeed, it is sufficient to take
\[
\Gamma^a_T = -\nabla \times a^a_L \\
\Gamma_0^a = \frac{\nabla \cdot (b^a)_L}{\nabla^2}
\]

Equipped with this field redefinition, we substitute in the Lagrangian
\[
a^a \to \tilde{a}^a + \nabla \times \Gamma^a_T \\
b^a \to \tilde{b}^a + \partial_0 \Gamma^a_T - \nabla \Gamma_0^a
\]

with \(\tilde{a}^a\) and \(\tilde{b}^a\) given by eq.(34), to obtain
\[
\mathcal{L}^{\text{bilinear}} = -\frac{1}{2g^2} \left[ A^a_T \cdot \Box A^a_T + A_0^a \nabla^2 A_0^a \right] \\
- \frac{1}{\sqrt{g_1}} \left[ A_0^a \nabla \cdot \tilde{a}^a + (\nabla \times \tilde{b}^a + \partial_0 \tilde{a}^a) \cdot A^a_T \right] \\
- \frac{1}{2} \tilde{a}^a \cdot \nabla^2 \tilde{a}^a - \frac{1}{2} (\nabla \times \tilde{b}^a + \partial_0 \tilde{a}^a) \cdot (\nabla \times \tilde{b}^a + \partial_0 \tilde{a}^a)
\]

showing clearly that the fields \(A_0^a\) and \(\tilde{b}^a\) do not propagate, as it is the case for \(a^a_T\) and \(b^a_T\).

We eliminate the fields \(A_0^a\) and \(\tilde{b}^a\) by using their equation of motion to the order considered \(A_0^a = -\frac{g^2}{\sqrt{g_1}} \nabla \cdot \tilde{a}^a\) and \(\tilde{b}^a = -\frac{1}{\sqrt{g_1}} \frac{\nabla \times A^a_T}{\nabla^2}\) and substitute these into the bilinear part of the Lagrangian (37), we get
\[
\mathcal{L}^{\text{bilinear}} = -\frac{1}{2g^2} A^a_T \cdot \left( \Box - \frac{g^2}{g_1} \right) A^a_T + \frac{1}{2} \tilde{a}^a \cdot \left( \Box - \frac{g^2}{g_1} \right) \tilde{a}^a
\]

exhibiting in this way the physical content of the theory, that is for each group direction, two transverse modes with mass \(\frac{g}{\sqrt{g_1}}\) and one longitudinal mode with the
same mass. Since the transverse modes have gained a mass from its coupling to the antisymmetric tensor fields, we are led to interpret the longitudinal mode $a_i^a_L$ as the third degree of freedom for each massive non-abelian gauge fields, $a = 1, 2, 3$, hence the Lagrangian (26) describes massive non-abelian gauge fields without any residual degrees of freedom.

Of course, the interactions have been modified by this choice of variables but writing the interactions in this basis would not be enlightening, so we do not pursue this further since we have achieved the desired result of computing the number of degrees of freedom.

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