Analytic evaluation of the small–world effect on a model approaching the Erdős–Rényi random graph

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The famous Watts–Strogatz small–world network model does not approach the Erdős–Rényi random graph model in the limit of total randomization which can lead to confusion and complicates certain analyses. In this paper we revisit a simple alternative by Song and Wang, which, instead of rewiring, draws edges between pairs of nodes with a distance–based connection probability. We show that this model is simpler to analyze and approaches the true Erdős–Rényi random graph model in the corresponding limit. Analytic results regarding the degree distribution, degree variance, number of two-stars per node, number of triangles per node, clustering coefficient, and random walk mixing time are presented. Consequently, the small–world effect is illustrated by showing that the clustering coefficient decreases much slower than an upper bound on the message delivery time with increasing long-range connection probability. We propose that this modified model should be used as an alternative reference model for studying the influence of small–world topology on dynamic systems as well as a simple model to introduce numerous topics when teaching network science.

I. INTRODUCTION

When Watts and Strogatz published their 1998 paper “Collective dynamics of ’small-world’ networks” [1], it had a phenomenal influence on the field of complex systems and was one of the defining studies for the following success of network science to emerge as an interdisciplinary field. Not only was it the first of a succession of studies [2–5] trying to explain the small–world effect as based on Milgrim’s “six degrees of separation” experiment [6], it introduced a simple and intuitive network model which had, at its core, the defining properties to obtain a “complex” system. Starting with a regular, locally connected structure, a rewiring process introduces long–range edges until it ends in a completely randomized state, thus interpolating between two well-studied physical systems: a crystal and disorder. For even small amounts of rewired contacts, the probability that two neighbors are connected (typically large in social networks [7]) merely changes, while almost immediately, short paths between individual nodes appear, explaining how social networks can be both: highly clustered but with a small amount of necessary steps to reach one node from another.

This network model, based on rewiring edges, is widely used in the network literature, often to explore the influence of the small–world effect on the outcome of dynamic processes taking their course on the network. One important feature of the model is that the mean degree is constant, which is a first-order control parameter for a variety of dynamic systems based on, e.g. random walks or epidemic spreading [8]. Within this kind of research, people sometimes argue that one limit of the rewiring process reproduces the random graph, i.e. the Erdős–Rényi random graph model (as, for instance, in [9] [10]). However, due to the model’s definition, the maximally randomized Watts–Strogatz model does not actually equal the Erdős–Rényi model. More often, references to the disordered limit do not directly mention the Erdős–Rényi model as a limit but are ambiguous in their wording and thus easily misinterpretable, see e.g. [11][12]. Likewise, the original rewiring procedure, where each node rewire each of its edges to its rightmost neighbors with probability \( p_r \), is similarly misinterpreted, to mean just rewiring any edge [13]. While the differences or slight variations of the model might not be influential for some dynamics, in others they can cause clear deviations from expected results in random graphs (see e.g. Fig. 5), thus potentially leading to confusion or faulty interpretations. While the model is part of virtually every network science curriculum, actually calculating the clustering coefficient, the degree distribution or the small–world effect with pen and paper [19] is often omitted since these observables or effects are complicated to evaluate. We argue that a model where those properties can be easily evaluated without the aid of a computer and actually reproduce formerly derived results from the Erdős–Rényi model might keep students more engaged and trained to calculate properties of other network models.

A model which solves the problems discussed above has been introduced by Song and Wang [20]. Within their study, they showed that sampling edges from a distance-based connection probability eases the evaluations of the degree distribution and the clustering coefficient. In this paper, we reformulate and discuss this modified model, extend the evaluation of degree distribution and clustering coefficient to other network properties and show how it can be used to explain the small–world effect analytically by comparing the clustering coefficient to an upper bound of the message delivery time.

II. RESULTS

A. Model definitions

In the original model \( N \) nodes are positioned equidistantly on a ring and subsequently locally connected, i.e. connected to nodes in their vicinity with maximal lattice distance \( d \leq k/2 \) where \( k \) is an even positive integer. For the rewiring
Figure 1. Schematic representation of the alternative small–world model as introduced in [20] and discussed in this paper. Much like in the original model, we start with \( N \) nodes placed equidistantly on a ring. However, instead of rewiring, each pair of nodes is connected with distance-based probability \( p_d \) where \( d \) is their minimal distance on the ring. Within distance \( d \leq k/2 \), nodes are connected with short-range probability \( p_S \). For larger distances, nodes are connected with long-range probability \( p_L = \beta p_S \). With increasing redistribution parameter \( 0 \leq \beta \leq 1 \) connection probability is redistributed from the short-range regime to the long-range regime while the mean degree \( k \) is kept constant. Hence at \( \beta = 0 \) the short-range probability is unity while the long-range probability is zero which produces a \( k \)-nearest neighbor lattice. With increasing \( \beta \), long-range “shortcuts” become more probable until at \( \beta = 1 \) both connection probabilities are equal and thus the model becomes equal to the Erdős–Rényi model. A simple network generation algorithm is given as follows. Each node \( 0 \leq u \leq N - 1 \) connects to each of its \( k/2 \) rightmost short-range neighbors with probability \( p_S \). Afterwards, \( m_L \) long-range edges are drawn, where \( m_L \) follows \( B(N(1 - k)/2, p_L) \). For each long-range edge one chooses a source node \( u \) uniform at random from \( [0, N - 1] \). This node is then connected to a long-range neighbor \( v = (u + k/2 + z) \mod N \) where the integer \( z \) is drawn uniform at random from the interval \( [1, N - k - 1] \). If an already existing edge was chosen, repeat the procedure for this long-range edge. This algorithm has complexity \( O(Nk + \langle m_L \rangle) \) for sparse networks. Implementations of the algorithm are available as open source C++/Python packages [21][22].

Figure 2. The small–world effect as illustrated by the observables computed in this paper. The random walk message delivering mixing time Eq. (5) as an upper bound of the real message delivering mixing time decreases rapidly with the introduction of long-range links while clustering Eq. (6) preserves. Displayed here are results for \( N = 1001 \) and \( k = 4 \).

process, each node rewires its connections to its \( k/2 \) rightmost neighbors to any other node in the network with probability \( p_r \). It is easy to see that at \( p_r = 1 \), each node has minimum degree \( k/2 \). Furthermore, each original edge has been rewired and can only exist if the corresponding disconnected right-neighbor rewired one edge back to its original left-neighbor, meaning that an original edge exists with probability \( (k/2)/(N - 1) \) but then another short-range edge cannot exist. Both these properties lead to conceptual deviations from the Erdős–Rényi model in which each edge exists with probability \( p_{ER} = k/(N - 1) \).

In a variant of the modified model by Song and Wang [20] edges posses an inherent probability to exist, which varies for short-range (\( S \)) and long-range (\( L \)) contacts. A potential contact between nodes \( (i, j) \) is considered to be short-ranged if their distance in periodic boundary conditions is \( d(i, j) \leq k/2 \); it exists with probability \( p_S \). It is considered long-range if \( d(i, j) > k/2 \) and exists with probability \( p_L \). The distance is easily computed as \( d(i, j) = \min(|j - i|, N - |j - i|) \). In short, two nodes with lattice distance \( d \) are connected with probability

\[
P_d = \begin{cases} p_S, & \text{if } d \leq k/2, \\ p_L, & \text{otherwise.} \end{cases}
\]

Hence, if \( p_S = 1 \) and \( p_L = 0 \), the model produces a structure which is equal to the original model’s starting point, one-dimensional \( k \)-nearest neighbor lattice. On the other hand, if \( p_S = p_L = p \), each edge exists with probability \( p \) and hence the model reproduces the \( G(N, p) \) random graph. We can easily fix the mean degree by noticing that it is composed of a short-range degree \( \langle k_S \rangle \) and a long-range degree \( \langle k_L \rangle \). Each node has \( k \) potential short-range neighbors and \( N - 1 - k \) potential long-range neighbors. Thus, its expected degree is

\[
\langle k_S \rangle + p_L(N - 1 - k) = \langle k_S \rangle + \langle k_L \rangle = \langle k \rangle \equiv k.
\]

To keep the mean degree constant, we introduce a control parameter \( \beta \) which controls the trade-off of connection probability in the short- and long-range regimes such that \( p_L = \beta p_S \). Note that at \( \beta = 0 \), we have \( p_L = 0 \) and \( p_S = 1 \) while at \( \beta = 1 \) we find \( p_L = p_S \equiv p \). In order for the mean degree to be constant, the distance-based probabilities are evaluated to be

\[
\begin{align*}
\langle k_S \rangle &= \frac{1}{1 + \beta(N - 1 - k)/k}, \\
\langle k_L \rangle &= \frac{\beta}{1 + \beta(N - 1 - k)/k} = \beta p_S(\beta).
\end{align*}
\]

The short-range node degree \( k_S \) follows a binomial distribution \( B(k, p_S) \) and the long-range node degree \( k_L \) follows a binomial distribution \( B(N - 1 - k, p_L) \) where \( B(n, p) \) has probability mass function \( f(k) = \binom{n}{k}(1 - p)^{n-k}p^k \). A schematic explanation of the model as well as an efficient sampling algorithm are given in Fig. [1]. Open source implementations of the sampling algorithm are available for download [21][22].

A connection to the original model can be drawn by noting that for small \( \beta \) the short-range connection probability \( p_S \) should be approximately equal to the probability that an edge has not been rewired \( 1 - p_r \) in the original model. To ensure
that $p_r = 1$ for $\beta = 1$ we set $p_r = [1 - p_S]/[1 - p_{ER}]$. This enables us to draw a comparison between the difference of both models with respect to observables of dynamic systems. In Fig. [3] we evaluate the pair-averaged first passage time of a discrete-time random walk process on both the modified and the original model to show that, indeed, the difference between both can be of significance.

**B. Network properties**

Since the node degree is given as the superposition of short-range and long-range degree, the degree variance can be easily computed as $\text{Var}[k] = \text{Var}[k_{S}] + \text{Var}[k_{L}] = k p_S (1 - p_S) + (N - 1 - k) p_L (1 - p_L)$. For increasing $\beta$ both short-range and long-range variances increase, as well, such that the degree variance is an increasing function of $\beta$, as shown in Fig. [4]. The full degree distribution is computable by noting that any node degree is $k = k_{S,i} + k_{L,i}$, such that its distribution is

$$p_{k} = \sum_{k_S = 0}^{\min(k',k)} \binom{k}{k_S} \binom{N - 1 - k}{k' - k_S} (1 - p_S)^{k - k_S} \times$$

$$\times p_S^k (1 - p_L)^{N - 1 - k' + k_S} p_L^{k' - k_S},$$

(2)

which is similar to the result derived in [20] and is shown in Fig. [4].

Given the network’s $(N \times N)$-sized adjacency matrix $A_{ij} = 1$ if nodes $i$ and $j$ are connected and $A_{ij} = 0$ otherwise, a clustering coefficient is computable by introducing the conditional probability that any two-star a node is the center of is also a triangle

$$C = P[A_{iu} A_{uv} A_{vi} = 1 | A_{iu} A_{iv} = 1] = \frac{\langle A_{iu} A_{uv} A_{vi} \rangle}{\langle A_{iu} A_{iv} \rangle} = \frac{\Delta}{\Lambda},$$

similar to the definition in [20]. We will, however, derive the final result in a slightly different way in the following. To evaluate the expected number of two-stars per node $\Lambda = \langle A_{iu} A_{iv} \rangle$ one observes that a node of degree $k_v$ is part of $(1/2)k_v(k_v - 1)$ two-stars such that the mean number of two-stars per node is given as $\Lambda = (1/2) [\text{Var}[k] + k(k - 1)]$ (illustrated in Fig. [4b]).

In order to find the the expected number of triangles per node we recognize that every node is statistically equivalent. Thus, without loss of generality, we compute the number of triangles that node 1 is part of as the sum over all possible remaining node pairs considering their connection probability based on their distance as

$$\Delta = \sum_{u = 2}^{N-1} \sum_{v = u+1}^{N} \sum_{(i,j) \neq (u,v)} p_{u,v} A_{iu} A_{uv} A_{vj}.$$

Here, $F, G, H,$ and $I$ are the areas of summation where three, two, one and no node pairs are of short-range distance, respectively. Those areas are evaluated in Fig. [5] for odd numbers of $N$. The expected number of triangles decreases with increasing $\beta$, as expected and as shown in Fig. [4b]. Consider-

![Figure 3](image-url) The pair-averaged first passage time (PAFPT) of a discrete-time random walk process $\tau = (N(N - 1))^{-1} \sum_{uv} N^2 \tau_{uv}$ (where $\tau_{uv}$ is the mean first passage time between target node $u$ and source node $v$) is an example of a network observable differing significantly (relative error $\approx 7\%$) from the corresponding result of the Erdős–Rényi model in the limit of $\beta = 1$. In contrast, the result from the modified model described in Sec. II A approaches the desired limit. Inset: A minimum in the PAFPT emerges at $\beta = 10^{-4}$.

![Figure 4](image-url) (a) Analytic results for a degree distribution Eq. (2) and (b), expected number of two-stars per node $\Lambda$, expected number of triangles per node $\Delta$ and node degree variance as given in the main text. While both degree variance and number of two-stars increase with increasing long-range redistribution parameter $\beta$, the number of triangles decreases. The results shown were computed for $N = 1001$ and $k = 4$.
In the respective limits we find $G = (3k/8)(k+2)$ for the region where two pairs are short-ranged (pink), $H = (k/8)(12N-26 - 11k)$ for the region where one pair of nodes is short-ranged (green), and $I = (1/8)[5k^2 - k(12N-26) + 4(N^2 - 3N + 2)]$ for the remaining area (orange). Note that here, the sum has been shifted to be $\sum_{u=(N-1)/2+1}^{(N-1)/2} \sum_{v=u+1}^{(N-1)/2} \cdot$ such that $u$ and $v$ are equal to their lattice distance to a focal node 0.

In this case, the clustering coefficient is then given by

$$C(\beta) = p_3^3 \times \frac{F + GB^2 + H\beta + 1B^3}{(1/2)\Var[k] + k(k-1)}. \quad (3)$$

In the respective limits we find

$$C(\beta = 0) = \frac{3(k-2)}{4(k-1)}$$

$$C(\beta = 1) = \frac{\sum_{u=2}^{N-1} \sum_{v=u+1}^{N} p_3^3}{\sum_{u=2}^{N-1} \sum_{v=u+1}^{N} p_2^2} = p_n,$$

which are the expected results for both the $k$-nearest neighbor lattice as well as the Erdős–Rényi graph. Further considering Eqs. (1) and (3) as well as noting that $\Var[k](\beta \to 0) = 0$, in the limit of small long-range redistribution one finds

$$C(\beta \ll 1) \approx p_3^3 = 1 - 3\beta^2 N^2 - k - 1/k + O(\beta^3) \quad (4)$$

which will be of importance for quantifying the small–world effect in the following.

C. Small–world effect

In the original model, the small–world effect was illustrated by comparing the clustering coefficient to the average shortest path length of the networks. While random networks have short path lengths, they possess low clustering, on the other hand regular networks are highly clustered, while nodes are, on average, quite distant from one another. With rewiring only a short amount of edges it was shown that shorter paths appear immediately while high clustering preservers, explaining the small-world effect. However, in the following we will take a different approach.

One of the purposes of the original model was to explain the Milgram small–world experiment [6] where participants had to mail letters to strangers by mailing them to a person they did know and instruct them to pass the letter further. In the following we will illustrate the small–world effect by showing that an upper bound for the delivery time of those messages decreases much faster than the clustering coefficient with increasing probability of long-range edges.

Considering completely uninformed individuals, the mailing process can be upper bounded by assuming it follows a random walk process where the random walkers correspond to the letters to be sent to recipients. At each integer time step $t$, the letter resides on a node $u$. Subsequently, one of $u$'s neighbors $v$ is chosen uniform at random as the next recipient of the message. At the next time step $t+1$ the letter then resides at node $v$. This process is repeated indefinitely and is governed by the master equation $\varphi_v(t) = \sum_{i=1}^{N} \langle A_{uv}/k_u \rangle \varphi_u(t-1)$ where $\varphi_v(t)$ is the probability that the letter is on node $v$ at time $t$ and $W_{uv} = A_{uv}/k_u$ is the probability that the letter is sent from node $u$ to node $v$. Instead of generating adjacency matrices and averaging over the results of their corresponding transition matrices we will compute an average medium matrix where each edge in the network is replaced by the probability distribution is approached on this average medium network is given by the eigenvalue gap of the transition matrix $W_{uv}$ as $m_{mix}^{-1} = 1 - \omega_1$ where $\omega_0 = 1$ is the largest eigenvalue and $\omega_1$ is the second largest eigenvalue [24]. The average medium transition matrix is circulant based on the vector

$$w = k^{-1} (0, p_{S1}, \ldots, p_{SN}, p_{PL}, \ldots, p_{PL}, p_{PS}, \ldots, p_{PS}).$$

In this case, the $j$-th eigenvalue of $W_{uv}$ is given as $\omega_j = \sum_{i=1}^{N} w_i \exp(i2\pi v/N)$ such that the second largest eigenvalue can be easily computed as $\omega_1 = p_S\Gamma/k - p_L(1+\Gamma)/k$ where $\Gamma = 2k/\sum_{j=1}^{N} \cos(2\pi j/N) = k - (N^2/2)(k/(2k+1)\Gamma + 1) + O(N^{-4})$ which yields the mixing time

$$t_{mix}(\beta) = \left[ 1 - \frac{\Gamma - \beta(1+\Gamma)}{k + \beta(N-1-k)} \right]^{-1}. \quad (5)$$

In Fig. we show how both clustering coefficient and mixing time decrease with increasing long-range redistribution.
parameter $\beta$. In the limits we find the expected results from $k$-regular networks and an average medium approximation of the Erdős–Rényi graph

\[
\begin{align*}
t_{\text{mix}}(\beta = 0) &= \left[ 1 - \frac{\Gamma}{k} \right]^{-1} \frac{N^2}{\pi^2 (k/2 + 1)(k + 1)}^3, \\
t_{\text{mix}}(\beta = 1) &= \left[ 1 - \frac{1}{N-1} \right]^{-1} = 1 - \frac{1}{N}. 
\end{align*}
\]

This implies that for small long-range redistributions the relative mixing time decreases as

\[
t_{\text{mix}}(\beta) / t_{\text{mix}}(0) = 1 - \beta \left( \frac{3N^3}{\pi^2 (k/2 + 1)(k + 1)} k - \frac{N + 1}{k} \right) + O(\beta^2). (6)
\]

Comparing Eqs. (4) and (6), one can easily see that for small $\beta$ the rate with which the mixing time decreases is of order $N^3$ while the rate with which the clustering coefficient decreases is of order $N$, which is a difference of two orders of magnitude. This shows that even with a small amount of long-range connection probability, the delivery time of randomly passed messages declines rapidly while clustering is still preserved.

### III. DISCUSSION

We discussed an alternative small–world network model which approaches the Erdős–Rényi random graph model in the limit of maximum disorder, since the original small–world network model does not. Within this model, instead of rewiring edges, long-range contacts are introduced by redistributing connection probability from short-range to long-range potential neighbors while keeping the mean degree constant. Constructing small–world networks in this way allows for a thorough analytical analysis of network properties such as the degree distribution, the degree variance, the average number of two-stars, the average number of triangles, and the clustering coefficient. While analytically evaluating the average shortest-path length stills seems to be rather tedious, an upper bound of the random walk mixing time can be computed using an average medium approximation. We showed that for a small amount of redistributed long-range connection probability the clustering coefficient decreases with a rate proportional to the number of nodes $N$ while the upper bound of the delivery time decreases with a rate of order $N^3$, hence illustrating how social networks can have both high clustering as well as a favorable topology to efficiently forward messages to unknown recipients.

In the following we will discuss the modified model’s applicability to teach network concepts. As network theory curricula typically introduce Erdős–Rényi random graphs early on as one of the first network models, the concept of drawing edges with a certain probability is known to students. We argue that extending this concept to draw edges from two categories (short-range and long-range) with two connection probabilities is a natural way to extend this formalism on a path to more complicated models. Based on the derivation of the degree distribution of the random graph one can easily comment on the distribution of random variables’ superposition and derive the degree distribution of the small-world model. Subsequently, similarly to the clustering coefficient computable in the Erdős–Rényi model, the clustering coefficient of the modified small–world model can be computed as the conditional probability that two nodes are connected given that they are neighbors of a focal node, in contrast to the local clustering coefficient in the original model. This further allows for the introduction of an average medium where each edge is replaced by the probability that it exists. Consequently using this average medium approximation one can use the modified model to introduce the random walk formalism and show how to evaluate its mixing time to arrive at the small–world effect based on the message delivery time and the Milgram small–world experiment (with a careful discussion of its flaws). We furthermore argue that the more simplistic picture of drawn instead of rewired edges is more intuitive. Instead of an individual explicitly deciding to change one of its short-range contacts to a long-range contact, there is an inherent probability to be connected to “near” nodes as well as a smaller probability to be connected to nodes “further away”.

Finally, we suggest the modified model to be used as an alternative to the original model when studying the influence of the small–world effect on dynamic systems, since the modified model truly interpolates between two well-studied systems, a nearest-neighbor lattice and the Erdős–Rényi model and thus allows for simpler and more reliable comparisons of results.

### MATERIALS

The network sampling algorithm described in Fig. 2 is implemented for Python and C++ and available for download [21, 22]. Additionally, several Python functions to compute the model’s network properties as well as the average medium mixing time are implemented in [23].

### DATA AVAILABILITY

No datasets were generated or analysed during the current study.

### COMPETING INTERESTS

The author declares no competing interests.

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