Stationary Gravitation Field Equation With Constant Ricci Scalar in Four Dimension

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Abstract. The solution for vacuum Einstein field equation for stationary spherical black hole was first published by Kerr. In this paper, the modified Kerr metric that satisfies Einstein’s condition, is considered. Furthermore, the metric can be modified to obtain stationary black hole metric with arbitrary constant Ricci scalar as generalization from Einstein condition. This condition implies a non-zero Einstein field equation and could be interpreted generally as a stationary black hole placed on a non-vacuum curved space-time background. Characterization of the geometry is to analyze singularity of the manifold, including the physical singularity and the formed horizon. The critical condition of the black hole is also considered. Moreover, surface gravity and surface area of the black hole are also calculated for further research.

1. Introduction

Another solutions of Einstein’s field equation was published 1963 by Roy Kerr. The solution describing the curvature around a rotating black hole. But these solution was done without cosmological constant involved. The cosmological term was added by Einstein to compensate the matters’ gravitational pull in static universe model. After the discovery of expanding universe, the interpretation of cosmological constant changed to explain the repelling effect of matters in the universe.

Considering the cosmological constant, Hassanudin had solved the corresponding Einstein equation for stationary black hole in his thesis [1]. This solution is called Kerr-de Sitter or Kerr-Anti-de Sitter depending on the sign of cosmological constant.

In this paper the solution stationary axisymmetric spacetime on a constant Ricci scalar condition will be calculated by modifying the solution from Einstein condition. The calculation and analysis of curvature singularity by considering some possibilities of the formed horizon, and energy density of the geometry are included. Besides, in order to consider the black hole thermodynamics, the surface gravity and surface area of the black hole are also calculated.

1.1. Axisymmetric stationary spacetime

In a general space, we can consider a point A in a coordinate system $x^\mu$ and another point B that has coordinate interval $dx^\mu$ with A. so from these two points it can be known the infinitesimal distance

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu,$$ (1)
with $g_{\mu\nu}$ is called metric tensor. Metric tensor is determined by the geometry of the considered spacetime and the coordinate system, The metric related to stationary axisymmetric spacetime can be represented as follow [2]

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\varphi}(d\varphi - \alpha dt)^2 + e^{2\rho_\varphi}(d\varphi)^2 + e^{2\rho_\vartheta}(dr)^2.$$ \hspace{1cm} (2)

1.2. Einstein’s field equation and considered conditions

Based on general covariance principle and tensor algebra, Einstein formulated his field equation as follows [3],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu},$$ \hspace{1cm} (3)

where $T_{\mu\nu}$ is energy-momentum distribution.

Einstein added the cosmological term to his equation. Einstein’s field equation with an added cosmological constant is written as follows [3],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$ \hspace{1cm} (4)

We consider calculation in the vacuum region, so the energy momentum momentum, $T_{\mu\nu}$, is zero. The last equation can be written to

$$R_{\mu\nu} = \Lambda g_{\mu\nu}.$$ \hspace{1cm} (5)

The term $\Lambda$ was interpreted as repulsive effect that compensates the gravitational pull. Thus preventing the spacetime to fall. Although Einstein later stated that it as no longer needed to use this term, it was still used to obtain the cosmological model more generally. Equation (5) is called Einstein’s condition. The solutions obtained by Hassanudin [2] are

$$g_{00} = -\frac{\Delta_r - a^2\Delta_\vartheta}{\rho^2},$$ \hspace{1cm} (6)

$$g_{01} = \frac{a}{\rho^2}\left(\sin^2(\theta)\Delta_r - (a^2 + r^2)\Delta_\vartheta\right),$$ \hspace{1cm} (7)

$$g_{11} = \frac{(a^2 + r^2)^2\Delta_\vartheta - a^2\sin^4(\theta)\Delta_r}{\rho^2},$$ \hspace{1cm} (8)

$$g_{22} = \frac{\rho^2}{\Delta_r},$$ \hspace{1cm} (9)

$$g_{33} = \frac{\rho^2\sin^2(\theta)}{\Delta_\vartheta},$$ \hspace{1cm} (10)

with coordinate $(t, \varphi, r, \theta)$, where

$$\rho^2 = r^2 + a^2\cos^2\theta,$$ \hspace{1cm} (11)

$$\Delta_r = \frac{1}{3}\Lambda r^2\left(r^2 + a^2\right) + r^2 - 2Mr + a^2,$$ \hspace{1cm} (12)

$$\Delta_\vartheta = \left(1 - \frac{1}{3}\Lambda a^2\cos^2\theta\right)\sin^2\theta.$$ \hspace{1cm} (13)
The corresponding inverse metric components are

\[ g^{00} = \frac{a^2 \sin^2(\theta)}{\rho^2 \Delta_\theta} - \frac{(a^2 + r^2)^2}{\rho^2 \Delta_r}, \] (14)

\[ g^{01} = \frac{a}{\rho^2} \left( \frac{\sin^2(\theta)}{\Delta_\theta} - \frac{a^2 + r^2}{\Delta_r} \right), \] (15)

\[ g^{11} = -\frac{1}{\rho^2 \Delta_\theta} - \frac{a^2}{\rho^2 \Delta_r}, \] (16)

\[ g^{22} = \frac{\Delta_\theta}{\rho^2}, \] (17)

\[ g^{33} = \frac{\Delta_\theta \csc^2(\theta)}{\rho^2}. \] (18)

The Kerr solution is asymptotically Minkowski, but the solution obtained by Hassanudin is asymptotically de Sitter. This implies a rotating black hole on de Sitter background.

Beside setting condition for Ricci tensor, we can also consider the some other conditions. One of them is to set the condition for Ricci scalar to be a constant

\[ R = g^{\mu\nu} R_{\mu\nu} = k, \] (19)

where \( k \) is a constant. This condition is more general than Einstein’s. This can be interpreted as a stationary black hole placed on a non-vacuum curved background. The non-vacuity could be caused by the presence of any kind of matter, energy, or field.

2. Calculation on Constant Ricci Scalar Condition

To find the solution for this condition, The solution for Einstein’s condition will be used and modified. The modification occurs in \( \Delta_r \) and \( \Delta_\theta \) shown in equation (12) and (13) in the following form

\[ \Delta_r = \frac{1}{3} \Lambda r^2 \left( r^2 + a^2 \right) + r^2 - 2Mr + a^2 + A(r), \] (20)

\[ \Delta_\theta = \left( 1 - \frac{1}{3} \Lambda a^2 \cos^2 \theta \right) \sin^2 \theta + B(\theta). \]

Using the solution for Einstein’s condition to calculate Ricci scalar yields

\[ k \rho^2 = 2 \left[ A^{\frac{1}{2}} \left( \Delta_r^2 \right) \right] + \frac{2}{\sin \theta} \left[ \frac{\Delta_\theta^2}{\sin^2 \theta} \left( \frac{1}{\Delta_\theta} \right) \right]. \] (21)

The notation \( A_{\mu\nu} \) is used to represent partial differentiation,

\[ A_{\mu\nu} = \partial_{\mu} A = \frac{\partial A}{\partial x^\nu}. \] (22)

Substituting the modified solution in equation (20), equation (21) becomes

\[ (k - 4\Lambda) \rho^2 = 2 \left[ A^{\frac{1}{2}} \left( A^{\frac{1}{2}} \right) \right] + \frac{2}{\sin \theta} \left[ \frac{B^{\frac{1}{2}}}{\sin \theta} \left( B^{\frac{1}{2}} \right) \right]. \] (23)

It can be separated into 2 equations:

\[ (k - 4\Lambda) r^2 = 2 \left[ A^{\frac{1}{2}} \left( A^{\frac{1}{2}} \right) \right] - C_1, \] (24)
\[
(k - 4\Lambda)a^2 \cos^2 \theta = \frac{2}{\sin \theta} \left[ \frac{B^2}{\sin \theta} \left( \frac{B^3}{B} \right)_\theta \right] + C_1,
\]
which have solutions
\[
A(r) = \frac{1}{12} (k - 4\Lambda) r^4 + \frac{1}{2} C_1 r^2 + C_2 r + C_3,
\]
\[
B(\theta) = \frac{1}{12} (k - 4\Lambda) a^2 \cos^2 \theta - \frac{1}{2} C_1 \cos^2 \theta - C_4 \cos \theta + C_5.
\]
Thus the value of \(\Delta_r\) and \(\Delta_\theta\) for constant Ricci scalar condition is written as follows.
\[
\Delta_r = \frac{1}{3} A r^2 a^2 + r^2 - 2mr + a^2 + \frac{1}{12} kr^4 + \frac{1}{2} C_1 r^2 + C_2 r + C_3,
\]
\[
\Delta_\theta = 1 - \frac{A}{3} a^2 \cos^2 \theta - \cos^2 \theta + \frac{1}{12} ka^2 \cos^2 \theta - \frac{1}{2} C_1 \cos^2 \theta - C_4 \cos \theta + C_5,
\]
where \(C_1, C_2, C_3, C_4, C_5\) are constants.

3. The Singularity
There are two kinds of black hole singularity including coordinate singularity and the real one. In a certain coordinate, the singularity can be related to the event horizon, but the real singularity is still the same whatever the coordinate is and can be observed through its curvature.

3.1. Event Horizon
This singularity is observed in Boyer-Lindquist coordinates when the \(g_{rr}\) is singular, that is
\[
\Delta_r = r^4 \left( \frac{C_1}{2} + 1 - \frac{a^2 A}{3} \right) + a^2 + r(C_2 - 2M) + C_3 - \frac{kr^4}{12} = 0,
\]
or in other form
\[
-\frac{12}{k} \Delta_r = r^4 + Ar^2 + Br + C = 0,
\]
where
\[
A = -\frac{12}{k} \left( \frac{C_1}{2} + 1 - \frac{a^2 A}{3} \right),
\]
\[
B = -\frac{12}{k} (C_2 - 2M), \quad \text{and}
\]
\[
C = -\frac{12}{k} (a^2 + C_3).
\]
The solutions of this polynomial are
\[
r_{1,2,3,4} = \pm \sqrt[\pm(i)]{\frac{Q}{2}} \pm \left( \frac{1}{2} \right) \sqrt{\left( 2A + Q \right) \pm \left( \frac{2B}{\sqrt{Q}} \right)},
\]
where
\[
P = 2A^3 - 72AC + 27B^2 + \sqrt{(2A^3 - 72AC + 27B^2)^2 - 4\left( A^2 + 12C \right)^3}, \quad \text{and}
\]
\[
Q = \frac{\sqrt{2}(A^2 + 12C)}{3\sqrt{P}} - \frac{2A}{3} + \frac{\sqrt{P}}{3\sqrt{2}}.
\]
These solutions represent \( r_+ \) and \( r_- \) which is associated with black hole event horizon, and \( r_c \) and \( r_{--} \) which is associated with cosmological event horizon \([5]\). If \( k \) is very small and positive, then \( r_c, r_+ \), and \( r_- \) are positive, but the \( r_{--} \) is negative, e.g. Kerr-de Sitter black hole.

For more information about the critical value of these solutions, we consider discriminant of the polynomial

\[
D = 16A^4C - 4A^3B^2 - 128A^2C^2 + 144AB^2C - 27B^4 + 256C^3. \tag{36}
\]

The equation \( D = 0 \) gives three conditions, those are polynomial with quadruple root, triple root, and double root(s). These conditions are related to critical values of the black hole.

1. Quadruple root \( A = B = C = 0 \)
2. Triple root \( A = 2\sqrt{3}\sqrt{C}, \quad B = \pm \frac{8\sqrt{3}C^{3/4}}{3^{3/4}}, \quad C < 0 \)
3. Two double roots
   - A double and two real roots \( A < -2\sqrt{C}, \quad -\infty < C < \infty \)
     \[
     B = \pm \frac{1}{3} \sqrt{-A^3 \pm \sqrt{(A^2 + 12C)^3} + 36AC}.
     \]
   - A double and two complex roots \( A > -2\sqrt{C}, \quad -\infty < C < \infty \)
     \[
     B = \pm \frac{1}{3} \sqrt{-A^3 + \sqrt{(A^2 + 12C)^3} + 36AC}.
     \]

If the value of \( A, B, \) and \( C \) are returned, then we will get explicit form of the critical value.

For Kerr-Newman-de Sitter case, Dehghani and KhajehAzad \([4]\) suggest that there are exist two possible critical conditions. First condition, \( r_- \) coincides with \( r_+ \) which is related to black hole rotation and charges, and the other one, \( r_+ \) coincides with \( r_c \) which is related to the curvature of the background space time.

4. Black Hole Thermodynamics

Hawking showed that the dynamics of a black hole are similar to the thermodynamics processes \([6]\).

For a rotating black hole, the first law can be written as

\[
dM = \frac{1}{8\pi} \kappa dA + \Omega_H dJ. \tag{37}
\]

The change of energy is correlated with the change of mass and the change of entropy is correlated with the change of surface area. The work term is correlated with the rotation of the black hole and the temperature is correlated with the surface gravity. The more general form of the first law is given by Gibbons \([7]\) as

\[
dM = \frac{1}{8\pi} \kappa dA + \Omega_H dJ + \psi^\alpha d\psi^\lambda + \chi_\lambda d\phi^\lambda + \left( \frac{\partial M}{\partial \phi^a} \right) d\phi^a. \tag{38}
\]

This equation also includes the possibility of any other works due to the presence of any charges or any fields, e.g. electric or magnetic charge in electric or magnetic field.

4.1. Surface gravity

Killing vector for a axisymmetric black hole is given by

\[
\chi = \partial_t + \Omega_H \partial_\phi
\]

with \( \Omega_H \) is the black hole's rotation speed at surface,

\[
\Omega_H = -\frac{g_{\phi\phi}}{g_{\theta\theta}} = \frac{a(\Delta_t^2 + r^2) \Delta_t - \sin^2(\theta) \Delta_r}{(a^2 + r^2)^2 \Delta_t - a^2 \sin^2(\theta) \Delta_r}. \tag{40}
\]
Norm of the Killing vector is given by

\[ V = \sqrt{-\mathcal{X}_a \mathcal{X}^a} = \sqrt{\frac{\Delta_a \Delta_\mu \left(a^2 \left(-\sin^2(\theta) + a^2 + r^2\right) + \left(a^2 \cos^2(\theta) + r^2\right) \left(a^2 + r^2\right)^2 \Delta_a - a^2 \sin^4(\theta) \Delta_\mu}{(a^2 + r^2)^2 \left(a^2 + r^2\right)^2 \Delta_a - a^2 \sin^4(\theta) \Delta_\mu}}. \] (41)

Finally, we can calculate the surface gravity as

\[ \kappa = \frac{1}{6} \sqrt{\frac{r \left(2a^2 \Delta - 3C_1 + kr^2 - 6\right) - 3C_2 + 6M}{(a^2 + r^2)^2}}. \] (42)

Surface gravity is evaluated at the surface, therefore we take \( r = r_+ \) as the result.

4.2. Surface area

A black hole's surface area is formed at the outer event horizon. It can be calculated as follows.

\[ A = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sqrt{\gamma} \, d\theta \, d\phi, \] (43)

where

\[ \gamma = \left(a^2 + r_+^2\right)^2 \sin^2(\theta). \] (44)

If the integral in equation (43) is evaluated, we obtain

\[ A = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(a^2 + r_+^2\right) \sin(\theta) \, d\theta \, d\phi = 4\pi \left(a^2 + r_+^2\right). \] (45)

5. Conclusion

The solution for stationary black hole under constant Ricci scalar condition has been calculated. It yields more general solution than Kerr-dS or Kerr-AdS with several constants that have not yet been interpreted physically. The horizon is calculated to be the roots of a polynomial. The surface gravity, and its surface area have also been calculated.

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