On Interpolating Sesqui-Harmonic Legendre Curves in Sasakian Space Forms

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Abstract. We consider interpolating sesqui-harmonic Legendre curves in Sasakian space forms. We find the necessary and sufficient conditions for Legendre curves in Sasakian space forms to be interpolating sesqui-harmonic. Finally, we obtain an example for an interpolating sesqui-harmonic Legendre curve in a Sasakian space form.

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1. Introduction

A map \( \varphi : (M, g) \rightarrow (N, h) \) between Riemannian manifolds is called a harmonic map and a biharmonic map, respectively if it is a critical point of the \( E(\varphi) \) and \( E_2(\varphi) \)

\[
E(\varphi) = \int_{\Omega} \|d\varphi\|^2 d\nu_g,
\]

\[
E_2(\varphi) = \int_{\Omega} \|\tau(\varphi)\|^2 d\nu_g,
\]

where \( \Omega \) is a compact domain of \( M \). The harmonic map equation is

\[
\tau(\varphi) = tr(\nabla d\varphi) = 0, \quad (1.1)
\]

and it is called the tension field of \( \varphi \) [5]. The Euler-Lagrange equation of \( E_2(\varphi) \) is

\[
\tau_2(\varphi) = tr(\nabla^2 \varphi - \nabla \varphi) \tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi)) d\varphi) = 0, \quad (1.2)
\]

and it is called the bitension field of \( \varphi \) [11].

In [3], Branding defined and considered interpolating sesqui-harmonic maps between Riemannian manifolds. The author introduced an action functional for maps between Riemannian manifolds that interpolated between the actions for harmonic and biharmonic maps. The map \( \varphi \) is said to be interpolating sesqui-harmonic if it is a critical point of \( E_{\delta_1, \delta_2}(\varphi) \)

\[
E_{\delta_1, \delta_2}(\varphi) = \delta_1 \int_{\Omega} \|d\varphi\|^2 d\nu_g + \delta_2 \int_{\Omega} \|\tau(\varphi)\|^2 d\nu_g, \quad (1.3)
\]

where \( \Omega \) is a compact domain of \( M \) and \( \delta_1, \delta_2 \in \mathbb{R} \) [3]. The interpolating sesqui-harmonic map equation is

\[
\tau_{\delta_1, \delta_2}(\varphi) = \delta_2 \tau_2(\varphi) - \delta_1 \tau(\varphi) = 0 \quad (1.4)
\]

for \( \delta_1, \delta_2 \in \mathbb{R} \). An interpolating sesqui-harmonic map is biminimal if variations of \( (1.3) \) that are normal to the image \( \varphi(M) \subset N \) and \( \delta_2 = 1, \delta_1 > 0 \) [13]. For some recent study of biminimal immersions see [8], [13], [14] and [15].
Interpolating sesqui-harmonic curves in a 3-dimensional sphere were studied in [3]. In [6] and [7], Fetcu and Oniciuc considered biharmonic Legendre curves in Sasakian space forms. In [4], Cho, Inoguchi and Lee studied affine biharmonic curves in 3-dimensional pseudo-Hermitian geometry. In [10], Inoguchi and Lee studied affine biharmonic curves in 3-dimensional homogeneous geometries. In [16], the second author and Güvenç studied biharmonic Legendre curves in generalized Sasakian space forms. In [9], Güvenç and the second author studied $f$-biharmonic Legendre curves in Sasakian space forms. Motivated by the above studies, in the present paper, we consider interpolating sesqui-harmonic Legendre curves in Sasakian space forms. We obtain the necessary and sufficient conditions for Legendre curves in Sasakian space forms to be interpolating sesqui-harmonic. We also give an example for an interpolating sesqui-harmonic Legendre curve in a Sasakian space form.

2. Preliminaries

Let $M = (M^{2n+1}, \phi, \xi, \eta, g)$ be an almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$. A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called a Sasakian manifold if it is normal, that is,

$$N_{\phi} = -2d\eta \otimes \xi$$

where $N_{\phi}$ is the Nijenhuis tensor field of $\phi$ [1]. It is well-known that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y) X$$

and

$$\nabla_X \xi = -\phi X$$

[2]. The sectional curvature of a $\phi$-section is called a $\phi$-sectional curvature. When the $\phi$-sectional curvature is a constant, then the Sasakian manifold is called a Sasakian space form and it is denoted by $M(c)$ [2]. The curvature tensor $R$ of a Sasakian space form $M(c)$ is given by

$$R(X, Y) Z = \frac{c + 3}{4} \{g(Y, Z)X - g(X, Z)Y\}$$

$$+ \frac{c - 1}{4} \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z$$

$$+ \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X + g(X, Z)\eta(Y) \xi - g(Y, Z)\eta(X) \xi\}$$

(2.1)

for all $X, Y, Z \in TM$ [2].

A submanifold of a Sasakian manifold $M$ is called an integral submanifold if $\eta(X) = 0$, for every tangent vector $X$. An integral curve of a Sasakian manifold $M$ is called a Legendre curve [2].
3. Interpolating sesqui-harmonic Legendre curves in Sasakian space forms

Let \( \gamma : I \subset \mathbb{R} \rightarrow (M^n, g) \) be a curve parametrized by arc length in a Riemannian manifold \((M^n, g)\). Then \( \gamma \) is called a Frenet curve of osculating order \( r \), \( 1 \leq r \leq n \), if there exists orthonormal vector fields \( \{ E_i \} \) along \( \gamma \) such that

\[
E_1 = T = \gamma', \\
\nabla_T E_1 = k_1 E_2, \\
\nabla_T E_i = -k_{i-1} E_{i-1} + k_i E_{i+1}, \quad 2 \leq i \leq n-1, \\
\nabla_T E_n = -k_{n-1} E_{n-1},
\]

where the function \( \{ k_1, k_2, \tau, k_3, \ldots, k_n \} \) are called the curvatures of \( \gamma \) \[12\].

Firstly, we have the following theorem for an interpolating sesqui-harmonic Legendre curve in a Sasakian space form:

**Theorem 3.1.** Let \( M(c) = (M^{2n+1}, \phi, \xi, \eta, g) \) be a Sasakian space form with constant \( \phi \)-sectional curvature \( c \) and \( \gamma : I \subset \mathbb{R} \rightarrow M(c) \) be a Legendre curve of osculating order \( r \) and \( m = \min\{r, 4\} \). Then \( \gamma \) is interpolating sesqui-harmonic if and only if there exists real numbers \( \delta_1, \delta_2 \) such that

(1) \( c = 1 \) or \( \phi T \perp E_2 \) or \( \phi T \in \{ E_2, \ldots, E_m \} \); and

(2) the first \( m \) of the following equations are satisfied:

\[
-3\delta_2 k_1 k_1' = 0, \\
\delta_2 \left[ k_1'' - k_1^3 - k_1 k_2^2 - \left( \frac{c+3}{4} \right) k_1 \right] \\
+3 \left( \frac{c-1}{4} \right) k_1 [g(\phi T, E_2)]^2 \left( \frac{c-1}{4} \right) k_1 [\eta(E_2)]^2 - \delta_1 k_1 = 0, \\
\delta_2 \left[ 2k_1 k_2 + k_1 k_2' + 3 \left( \frac{c-1}{4} \right) k_1 g(\phi T, E_2) g(\phi T, E_3) \right] \\
- \left( \frac{c-1}{4} \right) k_1 \eta(E_2) \eta(E_3) = 0, \\
\delta_2 \left[ k_1 k_2 k_3 + 3 \left( \frac{c-1}{4} \right) k_1 g(\phi T, E_2) g(\phi T, E_4) \right] \\
- \left( \frac{c-1}{4} \right) k_1 \eta(E_2) \eta(E_4) = 0.
\]

**Proof.** Let \( \gamma : I \rightarrow M \) be a Legendre curve of osculating order \( r \) in \( M(c) \). By the use of (1.1) and (3.1), we have

\[
\tau(\gamma) = \nabla_T T = k_1 E_2.
\]

From (3.1), we get

\[
\nabla_T \nabla_T T = -k_1^2 E_1 + k_1' E_2 + k_1 k_2 E_3,
\]
\[ \nabla_T \nabla_T \nabla_T T = -3k_1k'_1E_1 + (k''_1 - k^3_1 - k_1k^2_2) E_2 + (2k'_1k_2 + k'_2k_1) E_3 + (k_1k_2k_3) E_4, \]  
(3.7)

\[ R(T, \nabla_T \nabla_T T) = - \left( \frac{c+3}{4} \right) k_1 E_2 - 3 \left( \frac{c-1}{4} \right) k_1g(\phi T, E_2) \phi T + \left( \frac{c-1}{4} \right) k_1E_2, \]  
(3.8)

Using the equations (3.6), (3.7) and (3.8) into the equation (4.1) in [3], we find

\[ \tau_{\delta_1, \delta_2}(\gamma) = (-3\delta_2k_1k'_1) E_1 + \left[ \delta_2 \left( k''_1 - k^3_1 - k_1k^2_2 + \left( \frac{c+3}{4} \right) k_1 \right) - \delta_1k_1 \right] E_2 + \delta_2(2k'_1k_2 + k'_2k_1) E_3 + \delta_2(k_1k_2k_3) E_4 \]
\[ + 3 \left( \frac{c-1}{4} \right) \delta_2k_1g(\phi T, E_2) \phi T - \left( \frac{c-1}{4} \right) \delta_2k_1E_2, \]  
(3.9)

Taking the scalar product of equation (3.9) with \( E_2, E_3 \) and \( E_4 \) respectively, then we obtain the desired result. \( \square \)

Now we shall discuss some special cases of Theorem 3.1:

**Case I.** \( c = 1 \).

From Theorem 3.1, we have:

**Proposition 3.1.** Let \( M(1) = (M^{2n+1}, \phi, \xi, \eta, g) \) be a Sasakian space form with \( c = 1 \) and \( \gamma : I \subset \mathbb{R} \rightarrow M(1) \) be a Legendre curve of osculating order \( r \) such that \( \frac{\delta_1}{\delta_2} \neq 0 \). Then \( \gamma \) is interpolating sesqui-harmonic if and only if

\[ k_1 = \text{constant} > 0, \quad k_2 = \text{constant}, \]
\[ k_1^2 + k_2^2 = 1 - \frac{\delta_1}{\delta_2}, \]
\[ k_2k_3 = 0 \]

where \( 1 - \frac{\delta_1}{\delta_2} > 0, \delta_1, \delta_2 \) is a constant.

**Proof.** Assume that \( \gamma \) is an interpolating sesqui-harmonic Legendre curve of osculating order \( r \) in \( M(1) \) such that \( \frac{\delta_1}{\delta_2} \neq 0 \) and \( c = 1 \). From Theorem 3.1 we obtain the result. \( \square \)

Using Proposition 3.1 we have:

**Theorem 3.2.** Let \( M(1) = (M^{2n+1}, \phi, \xi, \eta, g) \) be a Sasakian space form with \( c = 1 \) and \( \gamma : I \subset \mathbb{R} \rightarrow M(1) \) be a non geodesic Legendre curve of osculating order \( r \). Then

1. **It is a Legendre geodesic or**
2. \( \gamma \) is interpolating sesqui-harmonic with \( \frac{\delta_1}{\delta_2} \neq 0 \) if and only if it is a Legendre circle with \( k_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}} \) where \( 1 - \frac{\delta_1}{\delta_2} > 0 \) is a constant or
3. \( \gamma \) is interpolating sesqui-harmonic with \( \frac{\delta_1}{\delta_2} \neq 0 \) if and only if it is a Legendre helix with \( k_1^2 + k_2^2 = 1 - \frac{\delta_1}{\delta_2} \) where \( 1 - \frac{\delta_1}{\delta_2} > 0 \), \( \delta_1, \delta_2 \) is a constant.
In both cases, if \(1 - \frac{\delta_1}{\delta_2} < 0\), then such an interpolating sesqui-harmonic Legendre curve does not exist.

Proof. Let \(\gamma : I \to M(1)\) be an interpolating sesqui-harmonic curve with \(\frac{\delta_1}{\delta_2} \neq 0\). From Theorem 3.1, if we consider the osculating order \(r = 2\), then \(\gamma\) is a Legendre circle with \(k_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}}\) where \(1 - \frac{\delta_1}{\delta_2} > 0\) is a constant. Similarly, if we consider the osculating order \(r = 3\), then we obtain that \(k_2\) is a non-zero constant. Thus, \(\gamma\) is a Legendre helix with \(k_1^2 + k_2^2 = 1 - \frac{\delta_1}{\delta_2}\) where \(1 - \frac{\delta_1}{\delta_2} > 0\) is a constant. On the other hand, assume that \(\gamma\) is a Legendre circle with \(k_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}}\) or a Legendre helix with \(k_1^2 + k_2^2 = 1 - \frac{\delta_1}{\delta_2}\) where \(1 - \frac{\delta_1}{\delta_2} > 0\) is a constant. Obviously, \(\gamma\) satisfies Theorem 3.1 respectively. It is trivial that \(1 - \frac{\delta_1}{\delta_2} < 0\) cannot be possible. If \(1 - \frac{\delta_1}{\delta_2} = 0\), we obtain a geodesic. This proves the theorem. \(\square\)

Case II. \(c \neq 1\) and \(\phi T \perp E_2\).

From Theorem 3.1, we can state:

Proposition 3.2. Let \(M(c) = (M^{2n+1}, \phi, \xi, \eta, g)\) be a Sasakian space form with \(c \neq 1\), \(\phi T \perp E_2\) and \(\gamma : I \subset \mathbb{R} \to M(c)\) be a Legendre curve of osculating order \(r\) such that \(\frac{\delta_1}{\delta_2} \neq 0\). Then \(\gamma\) is interpolating sesqui-harmonic if and only if

\[
k_1 = \text{constant} > 0, \quad k_2 = \text{constant}, \quad k_1^2 + k_2^2 = \frac{c + 3}{4} - \frac{\delta_1}{\delta_2}, \quad k_2k_3 = 0
\]

where \(\delta_1, \delta_2\) is a constant.

Proof. Let \(\gamma\) be an interpolating sesqui-harmonic Legendre curve of osculating order \(r\) in \(M(c)\) such that \(c \neq 1\), \(\phi T \perp E_2\) and \(\frac{\delta_1}{\delta_2} \neq 0\). From Theorem 3.1, we get the result. \(\square\)

From [7], we have the following lemma:

Lemma 3.1. [7] Let \(\gamma\) be a Legendre Frenet curve of osculating order 3 in a Sasakian space form \(M(c)\) and \(\phi T \perp E_2\). Then \(\{T = E_1, E_2, E_3, \phi T, \nabla_T \phi T, \xi\}\) is linearly independent at any point of \(\gamma\) and therefore \(n \geq 3\).

Hence we can state:

Theorem 3.3. Let \(M(c) = (M^{2n+1}, \phi, \xi, \eta, g)\) be a Sasakian space form with \(c \neq 1\), \(\phi T \perp E_2\) and \(\gamma : I \subset \mathbb{R} \to M(c)\) a Legendre curve of osculating order \(r\).

(1) If \(c \leq -3\) and \(\frac{\delta_1}{\delta_2} \geq 0\), then \(\gamma\) is interpolating sesqui-harmonic if and only if it is a geodesic.

(2) If \(c > -3\) and \(\frac{\delta_1}{\delta_2} < 0\), then \(\gamma\) is interpolating sesqui-harmonic if and only if either
(a) $\gamma$ is of osculating order $r = 2$, $n \geq 2$ and $\gamma$ is a circle with $k_1^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$, in which case $\{T, E_2, \phi T, \nabla_T \phi T, \xi\}$ are linearly independent, or

(b) $\gamma$ is of osculating order $r = 3$, $n \geq 3$ and $\gamma$ is a helix with $k_1^2 + k_2^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$, in which case $\{T, E_2, E_3, \phi T, \nabla_T \phi T, \xi\}$ are linearly independent, where $\delta_1, \delta_2 \in \mathbb{R}$.

Proof. (1) From Proposition 3.2 if we take $c \leq -3$ and $\frac{\delta_1}{\delta_2} \geq 0$, it is easy to see that $\gamma$ is interpolating sesqui-harmonic if and only if it is a geodesic.

(2) Assume that $c > -3$, $\frac{\delta_1}{\delta_2} < 0$ and $\gamma : I \rightarrow M(c)$ be an interpolating sesqui-harmonic curve. From Proposition 3.2 if we take $n \geq 2$ and $\gamma$ is of osculating order $r = 2$, then $\gamma$ is a circle with $k_1^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$. Using Lemma 3.1 we have that $\{T, E_2, \phi T, \nabla_T \phi T, \xi\}$ are linearly independent. Similarly, if we take $n \geq 3$ and $\gamma$ is of osculating order $r = 3$, then we obtain that $k_2$ is a non-zero constant. Thus, $\gamma$ is a helix with $k_1^2 + k_2^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$. Using Lemma 3.1 we have that $\{T, E_2, E_3, \phi T, \nabla_T \phi T, \xi\}$ are linearly independent. Conversely, assume that $\gamma$ is a Legendre circle with $k_1^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$ or a Legendre helix with $k_1^2 + k_2^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$. Obviously, $\gamma$ satisfies Theorem 3.4 respectively. Hence, we obtain the desired result. □

Case III. $c \neq 1$ and $\phi T \parallel E_2$.

From Theorem 3.1 we have:

Proposition 3.3. Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \subset \mathbb{R} \rightarrow M(c)$ be a Legendre curve of osculating order $r$ with $\phi T \parallel E_2$ and $\frac{\delta_1}{\delta_2} \neq 0$. Then $\gamma$ is interpolating sesqui-harmonic if and only if

$$k_1 = \text{constant} > 0, \ k_2 = \text{constant},$$

$$k_1^2 + k_2^2 = c - \frac{\delta_1}{\delta_2},$$

$$k_2k_3 = 0$$

where $\delta_1, \delta_2$ is a constant.

Proof. Assume $\gamma$ is an interpolating sesqui-harmonic Legendre curve in $M(c)$ such that $c \neq 1$, $\phi T \parallel E_2$ and $\frac{\delta_1}{\delta_2} \neq 0$. From Theorem 3.1 we get the result. □

Hence we can state:

Theorem 3.4. Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \subset \mathbb{R} \rightarrow M(c)$ a Legendre curve of osculating order $r$ such that $\phi T \parallel E_2$. Then $\{T, \phi T, \xi\}$ is the Frenet frame field of $\gamma$.

(1) If $c < 1$ and $\frac{\delta_1}{\delta_2} \geq 0$, then $\gamma$ is interpolating sesqui-harmonic if and only if it is a geodesic.

(2) If $c > 1$ and $\frac{\delta_1}{\delta_2} < 0$, then $\gamma$ is interpolating sesqui-harmonic if and only if it is a helix with $k_1^2 = c - 1 - \frac{\delta_1}{\delta_2}, \ (k_2 = 1)$ where $\delta_1, \delta_2 \in \mathbb{R}$. 

Proof. If we take $\phi T \parallel E_2$, we get $g(\phi T, E_2) = \pm 1$, $g(\phi T, E_3) = g(\phi T, E_4) = 0$.

(1) From Proposition 3.3 and the above equations and if we take $c \leq 1$ and $\frac{\delta_1}{\delta_2} \geq 0$, it is easy to see that $\gamma$ is interpolating sesqui-harmonic if and only if it is a geodesic.

(2) If $c > 1$, $\frac{\delta_1}{\delta_2} < 0$ from Proposition 3.3 and the above equations, we have $k_1 = \text{constant}$ and $k_1^2 = c - 1 - \frac{\delta_1}{\delta_2}$, and $k_2 = 1$. Conversely, assume that $\gamma$ is a Legendre helix with $k_1^2 = c - 1 - \frac{\delta_1}{\delta_2}$ and $k_2 = 1$. Then $\gamma$ satisfies Theorem 3.1 obviously. This completes the proof of the theorem. \hfill $\square$

Case IV. $c \neq 1$ and $g(\phi T, E_2) \neq 0, 1, -1$.

**Proposition 3.4.** Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c \neq 1$, $g(\phi T, E_2) \neq 0, 1, -1$ and $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ a Legendre curve of osculating order $r$ such that $4 \leq r \leq 2n+1$, $n \geq 2$. Then $\gamma$ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if

$$k_1 = \text{constant} > 0,$$

$$k_1^2 + k_2^2 = \frac{c + 3}{4} + \frac{3(c - 1)}{4} f^2 - \frac{\delta_1}{\delta_2},$$

$$k_2' = -\frac{3(c - 1)}{4} fg(E_3, \phi T),$$

$$k_2 k_3 = -\frac{3(c - 1)}{4} fg(E_4, \phi T).$$

**Proof.** Assume that $\gamma$ is an interpolating sesqui-harmonic Legendre Frenet curve such that $g(\phi T, E_2)$ is not a constant equal to 0, 1 or $-1$. In this case, we get $4 \leq r \leq 2n+1$, $n \geq 2$ and $\phi T \in \text{span} \{ E_2, E_3, E_4 \}$.

Hence, we can take $f(t) = g(\phi T, E_2)$. So by a differentiation, we obtain

$$f'(t) = g(\nabla_T \phi T, E_2) + g(\phi T, \nabla_T E_2)$$

$$= -k_1 g(T, \phi T) + k_2 g(E_3, \phi T) + g(E_2, \xi) + k_1 g(E_2, \phi E_2).$$

Since $\gamma$ is a Legendre curve and $\phi$ is anti-symmetric, we have $\eta(E_2) = 0$, $g(T, \phi T) = 0$ and $g(E_2, \phi E_2) = 0$. Thus we obtain

$$f'(s) = k_2 g(E_3, \phi T). \quad (3.10)$$

Additionally, we can write

$$\phi T = g(\phi T, E_2) E_2 + g(\phi T, E_3) E_3 + g(\phi T, E_4) E_4. \quad (3.11)$$

From Theorem 3.3 the equations (3.10) and (3.11), the curve $\gamma$ is interpolating sesqui-harmonic if and only if

$$k_1 = \text{constant},$$

$$k_1^2 + k_2^2 = \frac{c + 3}{4} + \frac{3(c - 1)}{4} f^2 - \frac{\delta_1}{\delta_2},$$

$$k_2' = -\frac{3(c - 1)}{4} fg(E_3, \phi T).$$
If $\gamma : I \subset \mathbb{R} \rightarrow M(c)$ satisfies the converse statement, it is obvious that the first four of the equations in Theorem 3.1 are satisfied. Thus $\gamma$ is interpolating sesqui-harmonic. This proves the theorem.

Using the equation (3.10) and the third equation of Proposition 3.4, we obtain

$$k'_2 = -\frac{3(c-1)}{4}fg(E_3, \phi T) = -\frac{3(c-1)}{4}f' k_2$$

$$k_2 k'_2 = -\frac{3(c-1)}{4}f f'$$

$$k_2^2 = -\frac{3(c-1)}{4}f^2 + w_0$$

(3.12)

where $w_0 =$constant. Substituting the equation (3.12) in the second equation of Proposition 3.4, we get

$$k_1^2 = \frac{c+3}{4} + \frac{3(c-1)}{2}f^2 - \frac{\delta_1}{\delta_2} - w_0.$$ 

Then we have $f =$constant. Thus $k_2 =$constant $> 0$, $g(E_3, \phi T) = 0$ and then $\phi T = fE_2 + g(\phi T, E_4) E_4$. We obtain that there exists a unique constant $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ such that $f = \cos \alpha_0$ and $g(E_4, \phi T) = \sin \alpha_0$.

So we can state:

**Theorem 3.5.** Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c \neq 1$, $n \geq 2$ and $\gamma : I \subset \mathbb{R} \rightarrow M(c)$ a Legendre curve of osculating order $r$ such that $g(\phi T, E_2) \neq 0, 1, -1$.

1. If $c \leq -3$ and $\frac{\delta_1}{\delta_2} \geq 0$, then $\gamma$ is interpolating sesqui-harmonic if and only if it is a geodesic.

2. If $c > -3$ and $\frac{\delta_1}{\delta_2} < 0$, then $\gamma$ is interpolating sesqui-harmonic if and only if $\phi T = \cos \alpha_0 E_2 + \sin \alpha_0 E_4$,

$$k_1, k_2, k_3 = \text{constant} > 0,$$

$$k_1^2 + k_2^2 = \frac{c+3}{4} + \frac{3(c-1)}{4} \cos^2 \alpha_0 - \frac{\delta_1}{\delta_2},$$

$$k_2 k_3 = -\frac{3(c-1)}{8} \sin 2\alpha_0,$$

where $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ is constant such that $(c+3+3(c-1)\cos^2 \alpha_0)\delta_2 - 4\delta_1 > 0$ and $3(c-1)\sin 2\alpha_0 < 0$.

**Remark 3.1.** For $c \neq 1$ and $g(\phi T, E_2) \neq 0, 1, -1$, there are also interpolating sesqui-harmonic curves which are not helices.
Now, we give brief information about the Sasakian space form $\mathbb{R}^{2n+1}(-3)$ \cite[2]{2}:

Let us take $M = \mathbb{R}^{2n+1}$ with the standard coordinate functions $(x_1, ..., x_n, y_1, ..., y_n, z)$, the contact structure $\eta = \frac{1}{2}(dz - \sum_{i=1}^{n}y_idx_i)$, the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$ and the tensor field $\phi$ given by

$$\phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$  

The Riemannian metric is $g = \eta \otimes \eta + \frac{1}{2} \sum_{i=1}^{n} ((dx_i)^2 + (dy_i)^2)$. Thus, $\mathbb{R}^{2n+1}(-3)$ is a Sasakian space form with constant $\phi$--sectional curvature $c = -3$. The vector fields

$$X_i = 2\frac{\partial}{\partial y_i}, \quad X_{i+n} = \phi X_i = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}), \quad 1 \leq i \leq n, \quad \xi = 2\frac{\partial}{\partial z}, \quad (3.13)$$

form a $g$-orthonormal basis and Levi-Civita connection is obtained as

$$\nabla_{X_i}X_j = \nabla_{X_i}X_{j+n} = 0, \quad \nabla_{X_i}X_{j+n} = \delta_{ij}\xi, \quad \nabla_{X_{i+n}}X_j = -\delta_{ij}\xi, \quad (3.14)$$

$$\nabla_{\xi}X_i = \nabla_{\xi}X_{i+n} = 0, \quad \nabla_{\xi}X_{i+n} = \nabla_{\xi}X_{i+n} = X_i \quad (3.15)$$

(see \cite[1]{1}).

Now, we give an example for interpolating sesqui-harmonic Legendre curves in $\mathbb{R}^{5}(-3)$:

**Example.** Let $\gamma = (\gamma_1, ..., \gamma_5)$ be a unit speed Legendre curve in $\mathbb{R}^{5}(-3)$. We can write the tangent vector field $T$ of

$$\gamma T = \frac{1}{2} \{\gamma_3'X_1 + \gamma_4'X_2 + \gamma_1'X_3 + \gamma_2'X_4 + (\gamma_5' - \gamma_1'\gamma_3 - \gamma_2'\gamma_4)\xi\}.$$  

Using the above equation, $\eta(T) = 0$ and $g(T, T) = 1$, we have

$$\gamma_5' = \gamma_1'\gamma_3 + \gamma_2'\gamma_4$$

and

$$(\gamma_1')^2 + ... + (\gamma_5')^2 = 4.$$  

So for a Legendre curve (3.14), (3.15) and (3.13) gives us

$$\nabla_{\gamma T} = \frac{1}{2} (\gamma_3''X_1 + \gamma_4''X_2 + \gamma_1''X_3 + \gamma_2''X_4), \quad (3.16)$$

and

$$\phi T = \frac{1}{2} (-\gamma_1'X_1 - \gamma_2'X_2 + \gamma_3'X_3 + \gamma_4'X_4). \quad (3.17)$$

From (3.16) and (3.17), $\phi T \perp E_2$ if and only if

$$\gamma_1'\gamma_3'' + \gamma_2'\gamma_4'' = \gamma_3'\gamma_1'' + \gamma_4'\gamma_2''.$$  

So we can state the following example:

Let us take $\gamma(t) = (\sin 2t, -\cos 2t, 0, 0, 1)$ in $\mathbb{R}^{5}(-3)$. By the use of Theorem 3.1 and the above equations, $\gamma$ is an interpolating sesqui-harmonic Legendre curve with
osculating order $r = 2$, $k_1 = 2$, $\delta_1 = -8$, $\delta_2 = 2$ and $\phi T \perp E_2$. We can see that Theorem 3.1 are verified. From the equations (3-1) in [7], the curve $\gamma$ is not biharmonic. Hence the biharmonicity and interpolating sesqui-harmonic of $\gamma$ are different.

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