Cloud-Based Centralized/Decentralized Multi-Agent Optimization with Communication Delays

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Abstract—We present and analyze a computational hybrid architecture for performing multi-agent optimization. The optimization problems under consideration have convex objective and constraint functions with mild smoothness conditions imposed on them. For such problems, we provide a primal-dual algorithm implemented in the hybrid architecture, which consists of a decentralized network of agents into which centralized information is occasionally injected, and we establish its convergence properties. To accomplish this, a central cloud computer aggregates global information, carries out computations of the dual variables based on this information, and then distributes the updated dual variables to the agents. The agents update their (primal) state variables and also communicate among themselves with each agent sharing and receiving state information with some number of its neighbors. Throughout, communications with the cloud are not assumed to be synchronous or instantaneous, and communication delays are explicitly accounted for in the modeling and analysis of the system. Experimental results are presented to support the theoretical developments made.

I. INTRODUCTION

Algorithms for multi-agent and distributed optimization have been considered for a variety of problem formulations in part because of the varied collection of application domains in which such problems arise. Applications of multi-agent optimization can be found in robotics [4], [2], [18], power systems [6], sensor networks [20], [8], [17], and communications [1], [23], [7], [13].

These diverse applications lead to optimization problems of many different formulations. Correspondingly, algorithms have been developed that allow for a broad range of problem characteristics. For example, in [15] distributed linear programs with constraints on network connectivity and memory are considered. In [3] the authors consider a distributed method for minimizing a sum of convex functions over a digraph. The authors of [14] devise an algorithm for minimizing a sum of possibly non-differentiable convex functions over a time-varying digraph. Problems with time-varying graphs, non-differentiable objective functions, and noisy communication links are considered in [19], while [12] considers similar problems in which nodes in the network are expected to fail over time.

The development of decentralized methods in optimization has been motivated in part by the fact that centralized methods may not scale well for very large networks of agents [21, Section 1.1]. At the same time, centralized methods can more efficiently solve some problems, like the Credit Assignment Problem in multi-agent robotics, than decentralized methods [16, Section 3.1]. Such examples indicate that centralized information may be rich in a way that could be useful in networks which would otherwise be purely decentralized. In adding a centralized component to a decentralized network, it seems likely that the centralized component would operate slower than the decentralized components. Nevertheless, one may ask whether it would be useful to occasionally inject global information into a multi-agent network where such information would otherwise be absent.

Towards answering this question, we present here a multi-agent optimization architecture in which a cloud computer is used to occasionally provide centralized information to a network of agents solving a nonlinear programming problem. This cloud-based optimization architecture was introduced in [5], though here we substantially broaden the class of problems to be solved and allow for communications delays when communicating with the cloud. The cloud carries out computations based on information sent to it by agents in the network, and intermittently disseminates these results to the agents for use in their own computations. At the same time, each agent shares its state with some number of neighboring agents at each time. In [5], the assumption was made that all information in the network was synchronized at each time, so that all computations were relying on the same information. Here we eliminate this assumption and, as a consequence, delays occur which give rise to various kinds of errors. We present explicit bounds on these errors in terms of network constants and algorithm parameters.

To solve nonlinear program in a distributed manner across many agents, we cast such problems as a variational inequality and then use Tikhonov regularization, which endows the resulting variational inequality with certain properties that let us draw from existing convergence results. In particular we consider a fixed regularization as was done in [10]. A fixed regularization is desirable for multi-agent problems because it may be difficult to synchronize the timing of the changes in regularization parameters across large networks. Accordingly, we use the approach of [10] as a starting point, though the problem and approach there are quite different from the current paper. In [10], the need for these results stems from reducing computation times in Lagrangian subproblems associated with a dual optimization scheme by allowing inexactness in some computations. Here we use a different architecture and different model for delays.
to operate as fast as possible by using the most recent information available to the agents. Doing so results in delays in communication, and computations that rely on information of different ages. The resulting structure of delays will be detailed below.

The rest of the paper is organized as follows. Section II will cover the background concerning the problem to be solved and a centralized method for solving it. Then, Section III will cover the cloud architecture and modify the centralized solution method to fit with the hybrid centralized/decentralized system. Section IV will present the central solution method to fit with the hybrid centralized/decentralized system. Section V will present experimental results from an implementation of this algorithm on a team of mobile robots. Section VI will conclude the paper.

II. PROBLEM FORMULATION AND CENTRALIZED SOLUTION

In this section we formulate the problem to be solved. This section treats everything globally and the results here will be modified later in Section III to fit with a hybrid architecture described therein.

A. Variational Inequality Setup

Consider a multi-agent optimization problem comprised of $N$ agents indexed by $i \in I := \{1, \ldots, N\}$. Suppose that agent $i$ has state $x_i \in \mathbb{R}^{n_i}$ with $n_i \in \mathbb{N}$, and let the vector $x$ denote the column vector of all states, namely

$$
x = \begin{pmatrix} x_1 \\
x_2 \\
 \vdots \\
x_n 
\end{pmatrix} \in \mathbb{R}^n,
$$

where $n = \sum_{i=1}^N n_i$. Let each agent have a local objective function defined only on its own state, $f_i(x_i)$. We assume that $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$ is $C^1$ and convex. We also consider a global cost that is not necessarily separable, $c(x)$, and assume that $c : \mathbb{R}^n \to \mathbb{R}$ is both $C^1$ and convex as well. We assume that agent $i$ knows $c$ and $f_i$, but not $f_j$ for any $\ell \neq i$; that is, each agent knows the non-separable cost and its own local cost, but not the local cost function of any other agent. We assume further that the cloud does not know $f_i$ for any $i \in I$.

The agents are collectively subject to global inequality constraints of the form

$$
g(x) = \begin{pmatrix} g_1(x) \\
g_2(x) \\
 \vdots \\
g_m(x) 
\end{pmatrix} \leq 0,
$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq 1$. The constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are assumed to be convex and $C^1$ for all $j \in J := \{1, \ldots, m\}$. In addition to $g$, each agent’s state is also constrained to lie in a given set $X_i \subset \mathbb{R}^{n_i}$, i.e.,

$$
x_i \in X_i
$$

for every $i$, where $X_i$ is non-empty, compact, and convex. Letting

$$
X = \prod_{i=1}^N X_i \times \cdots \times \prod_{N} X_N,
$$

we encapsulate each set constraint by requiring

$$
x \in X.
$$

Regarding the class of optimization problems under consideration, we summarize the conditions that we have imposed in the following assumption.

Assumption 1: The set $X$ is non-empty, compact, and convex. The functions $\{g_j\}_{j \in J}$ and $c$ are convex and $C^1$ in $x$, and $f_i$ is convex and $C^1$ in $x_i$ for all $i \in I$.

For notational convenience, define the function

$$
f(x) = \sum_{i=1}^N f_i(x_i) + c(x).
$$

Let $\nabla_x f$ denote the operator $\frac{\partial}{\partial x}$, and define the map

$$
\nabla f(x) = \left( \nabla_{x_1}(f_1(x_1) + c(x)), \ldots, \nabla_{x_N}(f_N(x_N) + c(x)) \right).
$$

We enforce the following assumption on $\nabla f$.

Assumption 2: The map $\nabla f$ is Lipschitz continuous with Lipschitz constant $L_f$.

Note that any collection of functions $f_i$ and $c$ which are all $C^2$ comprise an $f$ that automatically satisfies Assumption 2 whenever $X$ is compact (cf. Assumption 1). Concerning the constraints $g$, we have the following assumptions.

Assumption 3: (Slater’s Condition) There exists a vector $\bar{x} \in X$ such that $g(\bar{x}) < 0$, i.e., the constraints are strictly feasible at $\bar{x}$.

Assumption 4: The gradient of each constraint, $\nabla g_j$, $j \in J$, is Lipschitz with constant $L_j$ and hence $\nabla g$ is Lipschitz with constant

$$
L_g = \sqrt{\sum_{j=1}^m L_j^2}.
$$

A global formulation of the multi-agent optimization problem under consideration is given by:

Problem 1:  

$$
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0 \\
& \quad x \in X.
\end{align*}
$$

Assumption 1 guarantees that Problem 1 has a solution and Assumption 3 guarantees that a dual solution exists with no duality gap. Denote an optimal primal-dual pair for Problem 1 by $(\hat{x}, \hat{\mu})$. We now define the Lagrangian associated with Problem 1 as

$$
L(x, \mu) = f(x) + \mu^T g(x),
$$

where $x \in X$ is as defined above and $\mu$ is a vector of Kuhn-Tucker multipliers in the non-negative orthant of $\mathbb{R}^m$, denoted $\mathbb{R}^m_+$. By definition, $L(\cdot, \mu)$ is convex for all $\mu \in \mathbb{R}^m_+$ and $L(x, \cdot)$ is concave for every $x$. Seminal work by Kuhn
and Tucker [11] showed that optimal primal-dual pairs for Problem [1] are saddle points of \( L \). This saddle point condition can be expressed concisely as: for all \( x \in X \) and \( \mu \in \mathbb{R}_+^m \),

\[
L(\hat{x}, \mu) \leq L(\hat{x}, \hat{\mu}) \leq L(x, \hat{\mu}).
\]

The problem of finding Lagrangian saddle points can be restated as a variational inequality (e.g., [9, Section 11.1]). Let \( \nabla_x \) and \( \nabla_\mu \) denote the operators \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial \mu} \), respectively. In the variational inequality setting, we wish to find a point \((\hat{x}, \hat{\mu}) \in X \times \mathbb{R}_+^m \) such that

\[
\left[ \begin{array}{c}
( x ) - ( \hat{x} ) \\
( \mu ) - ( \hat{\mu} )
\end{array} \right]^T \left[ \begin{array}{c}
\nabla_x L(\hat{x}, \hat{\mu}) \\
-\nabla_\mu L(\hat{x}, \hat{\mu})
\end{array} \right] \geq 0
\]

for all \((x, \mu) \in X \times \mathbb{R}_+^m \). In order to make use of certain established results concerning variational inequalities, we take two further theoretical steps: first we modify the map \((\nabla_x L - \nabla_\mu L)^T\) to make it strongly monotone, and second we find a (non-empty) compact, convex set containing the optimal primal-dual vectors. We describe both steps below.

### B. Tikhonov Regularization

The gradient map

\[
\left( \begin{array}{c}
\nabla_x L(x, \mu) \\
-\nabla_\mu L(x, \mu)
\end{array} \right)
\]

is monotone, and we use a fixed Tikhonov regularization in order to work with a strongly monotone map. This is done by regularizing the Lagrangian function as follows:

\[
L_{\nu, \epsilon}(x, \mu) = f(x) + \frac{\nu}{2} \| x \|^2 + \mu^T g(x) - \frac{\epsilon}{2} \| \mu \|^2,
\]

where \( \nu > 0 \) and \( \epsilon > 0 \), and these values are kept fixed to avoid the need to synchronize changes in parameter values across many agents.

Under this regularization, we see that \( L_{\nu, \epsilon}(\cdot, \cdot) \) is strongly convex for all \( \mu \in \mathbb{R}_+^m \) and \( L_{\nu, \epsilon}(x, \cdot) \) is strongly concave for all \( x \). These properties imply that \( \nabla_x L_{\nu, \epsilon} \) and \( -\nabla_\mu L_{\nu, \epsilon} \) are both strongly monotone; hence, the map \((\nabla_x L_{\nu, \epsilon} - \nabla_\mu L_{\nu, \epsilon})^T\) is also strongly monotone. In addition, the strongly convex-strongly concave property of \( L_{\nu, \epsilon} \), together with Assumption 1 and Assumption 2 guarantee the existence of a unique optimal primal-dual pair, \((\hat{x}_{\nu, \epsilon}, \hat{\mu}_{\nu, \epsilon}) \in X \times \mathbb{R}_+^m \). Using the regularized Lagrangian, we now state the variational inequality of interest.

**Problem 2:** Find the point \((\hat{x}_{\nu, \epsilon}, \hat{\mu}_{\nu, \epsilon}) \in X \times \mathbb{R}_+^m \) such that for all \((x, \mu) \in X \times \mathbb{R}_+^m \),

\[
\left[ \begin{array}{c}
( x ) - ( \hat{x}_{\nu, \epsilon} ) \\
( \mu ) - ( \hat{\mu}_{\nu, \epsilon} )
\end{array} \right]^T \left[ \begin{array}{c}
\nabla_x L_{\nu, \epsilon}(\hat{x}_{\nu, \epsilon}, \hat{\mu}_{\nu, \epsilon}) \\
-\nabla_\mu L_{\nu, \epsilon}(\hat{x}_{\nu, \epsilon}, \hat{\mu}_{\nu, \epsilon})
\end{array} \right] \geq 0.
\]

We note that determining the solution to the above variational inequality is the same as determining the saddle-point of the regularized Lagrangian function \( L_{\nu, \epsilon} \), i.e., \((\hat{x}_{\nu, \epsilon}, \hat{\mu}_{\nu, \epsilon}) \in X \times \mathbb{R}_+^m \) solves the above variational inequality problem if and only if for all \((x, \mu) \in X \times \mathbb{R}_+^m \),

\[
L_{\nu, \epsilon}(\hat{x}_{\nu, \epsilon}, \mu) \leq L_{\nu, \epsilon}(\hat{x}_{\nu, \epsilon}, \hat{\mu}_{\nu, \epsilon}) \leq L_{\nu, \epsilon}(x, \hat{\mu}_{\nu, \epsilon}).
\]

**C. Bounds on Dual Variables**

We proceed along the lines of [22] and derive a bound on \( \hat{\mu}_{\nu, \epsilon} \). Letting \( \hat{x} \) denote a Slater point for the constraints \( g \), we define the dual function associated with \( L_{\nu, \epsilon} \) as

\[
q_{\nu, \epsilon}(\mu) := \min_{x \in X} L_{\nu, \epsilon}(x, \mu).
\]

Now consider an arbitrary multiplier \( \hat{\mu} \in \mathbb{R}_+^m \) and let the point \( \hat{x} \in X \) be such that \( \hat{x} = \min_{x \in X} L_{\nu, \epsilon}(x, \hat{\mu}) \). By the definition of \( q_{\nu, \epsilon} \), we then have

\[
q_{\nu, \epsilon}(\hat{\mu}) \leq L_{\nu, \epsilon}(\hat{x}, \hat{\mu}) \leq L_{\nu, \epsilon}(\hat{x}, \hat{\mu}_{\nu, \epsilon}).
\]

In view of the saddle-point property of \((\hat{x}_{\nu, \epsilon}, \hat{\mu}_{\nu, \epsilon}) \) (cf. Equation 1), it further follows that

\[
q_{\nu, \epsilon}(\mu) \leq L_{\nu, \epsilon}(\hat{x}_{\nu, \epsilon}, \hat{\mu}_{\nu, \epsilon}) \leq L_{\nu, \epsilon}(\hat{x}, \hat{\mu}_{\nu, \epsilon}).
\]

Using the regularized Lagrangian expression, we have

\[
q_{\nu, \epsilon}(\mu) \leq f(\hat{x}) + \hat{\mu}_{\nu, \epsilon}^T g(\hat{x}) + \frac{\nu}{2} \| x \|^2 - \frac{\epsilon}{2} \| \mu \|^2 \\
\leq f(\hat{x}) + \hat{\mu}_{\nu, \epsilon}^T g(\hat{x}) + \frac{\nu}{2} \| x \|^2.
\]

Rearranging terms then gives

\[
- \sum_{j=1}^{m} j \hat{\mu}_{\nu, \epsilon,j} g_j(\hat{x}) \leq f(\hat{x}) + \frac{\nu}{2} \| x \|^2 - q_{\nu, \epsilon}(\hat{\mu}),
\]

from which we conclude that for any \( \hat{\mu} \in \mathbb{R}_+^m \),

\[
\sum_{j=1}^{m} j \hat{\mu}_{\nu, \epsilon,j} \leq \frac{f(\hat{x}) + \frac{\nu}{2} \| x \|^2 - q_{\nu, \epsilon}(\hat{\mu})}{\min_{1 \leq j \leq m} \{ -g_j(\hat{x}) \}}.
\]

For any \( \nu > 0 \) and \( \epsilon > 0 \), we certainly have

\[
q_{\nu, \epsilon}(\hat{\mu}) \leq L_{\nu, \epsilon}(\hat{x}, \hat{\mu}) \geq L_{0, \epsilon}(\hat{x}, \hat{\mu}) = f(\hat{x}) + \hat{\mu}_{\epsilon}^T g(\hat{x}) - \frac{\epsilon}{2} \| \hat{\mu} \|^2.
\]

Selecting \( \hat{\mu} = 0 \), observe that

\[
L_{0, \epsilon}(\hat{x}, 0) = f(\hat{x}).
\]

Letting \( f_\epsilon^* = \min_{x \in X} f(x) \), the bound in Equation 3 can be simplified to

\[
\sum_{j=1}^{m} j \hat{\mu}_{\nu, \epsilon,j} \leq \frac{f(\hat{x}) + \frac{\nu}{2} \| x \|^2 - f_\epsilon^*}{\min_{1 \leq j \leq m} \{ -g_j(\hat{x}) \}}.
\]

Using that \( \hat{\mu}_{\nu, \epsilon,j} \geq 0 \) for all \( j \in J \) and defining

\[
D_\nu = \left\{ \mu \in \mathbb{R}_+^m : \| \mu \|_1 \leq \min_{1 \leq j \leq m} \{ -g_j(\hat{x}) \} \right\},
\]

we see that \( D_\nu \) is non-empty, compact, and convex, and we are guaranteed that \( \hat{\mu}_{\nu, \epsilon} \in D_\nu \). Using \( D_\nu \), we can now define the algorithm used to solve Problem 2.

**Algorithm 1:** Given an initial point \((x(0), \mu(0)) \in X \times D_\nu \), execute the update law

\[
x(k + 1) = \Pi_X \left[ x(k) - \alpha \nabla_x L_{\nu, \epsilon}(x(k), \mu(k)) \right]
\]

\[
\mu(k + 1) = \Pi_{D_\nu} \left[ \mu(k) + \tau \nabla_\mu L_{\nu, \epsilon}(x(k), \mu(k)) \right],
\]

until some stopping criterion is reached.
Here $\Pi_X[\cdot]$ and $\Pi_{D_\nu}[\cdot]$ are the projections onto the sets $X$ and $D_\nu$, respectively, with respect to the standard Euclidean norm. In the next section we will explicitly reformulate Algorithm 1 for the cloud-based multi-agent case.

III. CLOUD ARCHITECTURE AND HYBRID SOLUTION

We now cover the architecture that will be used to implement a hybrid form of Algorithm 1. Then we cover how the centralized means of solving this problem will be modified and deployed on this architecture.

A. Overview

Let agent $i$ have neighborhood set $N_i$ containing the indices of all agents it is directly coupled to by $g$ and $e$. That is, if the computation of

$$\frac{\partial L_{\nu,e}}{\partial x_i}$$

requires $x_j$, then $j \in N_i$, and agent $j$ sends its state to agent $i$ at each time. This structure of communications necessitates that $i \in N_j$ if and only if $j \in N_i$.

Let the agents share their states with each other at each timestep via wireless communication links. In this framework, each agent stores and manipulates a local copy of Problem 2 onboard and updates its own state within that local copy based on computations it performs onboard. Within each timestep, agent $i$ computes an updated value of its own state, $x_i$, and then shares the new value of $x_i$ with agent $j$ for all $j \in N_i$. In many cases, including in very large networks of agents, we expect that the neighborhood set of each agent will be a small subset of total collection of agents so that $|N_i| < N$ for all $i$, where $|\cdot|$ denotes cardinality. Computing values of dual variables using Algorithm 1 will require all states in the system (see Equation (4)), and given that $|N_i| < N$, we see that no agent will be able to perform these computations. Furthermore, there is no assumption that the communication graph of the system is connected, nor is it even assumed that each agent is coupled to any other agent. In such cases, there is not any way to aggregate all states in the network onboard a single agent, even after long periods of time.

To fill this gap, we use a cloud computer as was done in [5]. The cloud computer is assumed to be capable of executing computationally intensive calculations quickly as would be the case with a computer cluster or server farm. Occasionally every agent sends its state to the cloud and after some time each agent receives back an updated dual vector that it stores onboard and incorporates into its own local calculations of state updates. Because the cloud must aggregate all states in the network, it is assumed that there are delays in communicating with the cloud and, due to these delays, Algorithm 1 will be modified.

The precise update law used by each agent will be detailed below, though for the current discussion it is sufficient to note that each agent executes some onboard update law using its own state information, information it receives from its neighbors, and the most recent dual vector it has received from the cloud, regardless of how long ago that dual vector was received. Suppose the agents send their states to the cloud at some timestep $k_0$ and suppose they all have some dual vector $\mu_0$ onboard which was received just prior to sending their states to the cloud (the states sent at time $k_0$ were not computed using $\mu_0$). While the agents are waiting to receive an updated dual vector, $\mu_1$, they continue to communicate with each other as before and continue to use $\mu_0$, which is held constant onboard each agent, in their computations of state updates.

Suppose the agents’ states from $k_0$ arrive at the cloud at time $k_0 + p_0$ for some $p_0 \in \mathbb{N}$. With all states received, the cloud can compute the next dual update using a rule similar to that in Algorithm 1. Suppose that computing the next dual vector takes some number of timesteps $q_0 \in \mathbb{N}$ so that $\mu_1$ has been computed at time $k_0 + p_0 + q_0$. Then the cloud sends $\mu_1$ to each agent and it takes $r_0 \in \mathbb{N}$ timesteps to reach the agents, arriving at time $k_0 + p_0 + q_0 + r_0$. Before agent $i$ computes a primal update of $x_i$ using $\mu_1$, it again sends its state to the cloud and then uses $\mu_1$ in its subsequent computations. Then this process of the agents sharing states with their neighbors, receiving a delayed dual update, and sending their states to the cloud is repeated. Note that the delays $p_0$, $q_0$, and $r_0$ are not assumed to be constant but instead are associated with $k_0$ and are allowed to vary with each communications cycle, i.e., if the agents again send their states to the cloud at time $k_1$, there is no need for $p_1 = p_0$, $q_1 = q_0$, or $r_1 = r_0$.

B. Update Law Derivation

To derive the per-agent update law based on the above communications scheme, we consider the transfer of information through the network starting from the initial point $(x(0), \mu(0))$ and generalize it to an arbitrary communications cycle.

Define the delay between the agents receiving $\mu(t)$ and $\mu(t + 1)$ as $d(t) := p_t + q_t + r_t$. When the system is initialized, agent $i$ has onboard its own state, $x_i(0)$, the primal stepsize $\alpha$, and the initial dual vector, $\mu(0)$. The cloud is initialized with the same value of $\mu(0)$ and stepsize $\tau$. At time $k = 0$, each agent sends its state to the cloud and then the optimization process begins. Upon receiving all agents’ states (which comprise the full vector $x(0)$), the cloud will compute $\mu(1)$ according to

$$\mu(1) = \Pi_{D_\nu}[\mu(0) + \tau \nabla_x L_{\nu,e}(x(0), \mu(0))].$$

Simultaneously, agent $i$ will use the update law

$$x_i(k + 1) = \Pi_X[x_i(k) - \nabla_x L_{\nu,e}(x(k), \mu(0))]$$

until it receives $\mu(1)$ from the cloud, and updates of this form occur synchronously across the agents. When each agent

1. The agents’ states can arrive at the cloud at different times in which case the cloud can wait to compute an updated dual vector until it has received all agents’ states. Here, we assume all states arrive simultaneously for simplicity.
2. For consistency of notation, $\nabla_x L_{\nu,e}$ is written as an argument of the full state vector $x(0)$ here, though $\nabla_x L_{\nu,e}$ will only depend upon the states of agents with indices in $N_i$. 
receives $\mu(1)$, it sends the state $x_i(d(0))$ to the cloud before performing any computations with $\mu(1)$. After every state of the form $x_i(d(0))$ arrives at the cloud (which is not assumed to happen instantly), the cloud will compute $\mu(2)$ according to

$$\mu(2) = \Pi_{D_{\nu}} \left[ \mu(1) + \tau \nabla \mu L_{\nu,e}(x(d(0)), \mu(1)) \right].$$

Equation (6) shows that Equation (5) in Algorithm 2 can be distributed among the agents in a natural way by having each agent update its own state (recall that $X = X_1 \times \cdots \times X_N$). However, Equation (6) reveals that Equation (4) must be modified to account for the communication delays present in the system. In particular, the arguments of $\nabla \mu L_{\nu,e}$ are no longer aligned in time because $\mu(2)$ is computed using $\mu(1)$ and values of $x$ that are based on $\mu(0)$.

C. Multi-agent Implementation

To compactly express this cycle of communication and computation, we implement a change in notation. Let $\mu$ be indexed by the time variable $t \in \mathbb{N}$. In this notation, the state of each agent will be indexed both over timesteps at which the agents compute primal updates and also over which dual vector is currently being used in its computations. The time index of the agents’ computations will be $k \in \mathbb{N}$ and agent $i$’s state will have a superscript to denote the time index of the dual variable agent $i$ currently has onboard. The results of agent $i$’s $k^{th}$ state update using $\mu(t)$ will be denoted $x_i^k(k)$. Using this notation, we restate Algorithm 1 to explicitly specify the update law for agent $i$ and to account for the delays in dual vectors seen above.

Algorithm 2: Let agent $i$ have initial state $x_i^0(0)$, stepsize $\alpha$, and initial dual vector $\mu(0)$. Let the cloud have initial multiplier vector $\mu(0)$ and stepsize $\tau$. Execute for each agent $i$, the following two steps: for all $k = 0, \ldots, d(t) - 1$, $t \geq 0$,

$$x_i^k(k + 1) = \Pi_{X_i} \left[ x_i^k(k) - \alpha \nabla x_i L_{\nu,e}(x_i^k(t), \mu(t)) \right]$$

and for all $t \geq 0$,

$$\mu(t + 1) = \Pi_{D_{\nu}} \left[ \mu(t) + \tau \nabla \mu L_{\nu,e}(x^{t-1}(d(t - 1)), \mu(t)) \right]$$

until a certain stopping criterion is reached by each agent and the cloud.

In the setting of Algorithm 2 we define $x^{-1}(d(-1)) = x^0(0)$.

IV. CONVERGENCE ANALYSIS

We now show that Algorithm 2 “nearly” converges to the saddle point of $L_{\nu,e}$ and bound the quantities $||x_i^k(k) - \hat{x}_{\nu,e}||$ and $||\mu(t) - \hat{\mu}_{\nu,e}||$, where $x_i^k(k)$ is obtained by stacking the vectors $x_i^k(k)$, $i = 1, \ldots, N$. To do so we first establish the following lemma concerning convergence when $\mu(t)$ is fixed.

Lemma 1: Let Assumptions 1-4 hold. Let each agent use the primal regularization parameter $\nu$. Define the constants $M_\nu = \max_{\mu \in D_{\nu}} ||\mu||$ and $C_f = L_f + \nu + M_\nu L_p$. Enforce that the primal stepsize $\alpha$ satisfies $0 < \alpha < 2/C_f$. Let $x_i^*, k$ denote the fixed point of Equation (7) for agent $i$ with $\mu(t)$ fixed, and define $x_i^* = (x_i^k(k) \ldots x_i^k(k))^T$. Define also $x_i^k(k) = (x_i^k(k) \ldots x_i^k(k))^T$. For a fixed dual vector $\mu(t)$ the sequence $\{x_i^k(k)\}, k = 1, \ldots, d(t)$ satisfies

$$||x_i^k(k) - x_i^*|| \leq q_{\mu}^{k/2} ||x_i^0(0) - x_i^*||,$$

where $(0, 1) \ni q_{\mu} := 1 - \alpha (2 - \alpha C_f)$.

Proof: See [10], Lemma 4.4.

Next we establish an inequality that will be used to prove Theorem 1 below.

Lemma 2: Let Assumption II hold. Then the inequality

$$\mu(2) - \mu(1)^T (-g(x_2) + g(x_1)) \geq \frac{\nu}{M_g} ||x_2 - x_1||^2$$

is satisfied by all $x_1, x_2 \in X$ and $\mu_1, \mu_2 \in \mathbb{R}_+^n$, where $M_g = \max_{x \in X} \|\nabla g(x)\|.$

Proof: See [10], Lemma 4.1.

We now state one of the key theoretical results of the paper.

Theorem 1: Let Assumptions II hold. Suppose that each agent uses regularization parameter $\nu > 0$ and the cloud uses regularization parameter $\epsilon > 0$. Let $\alpha$ be bounded as in Lemma II and let the dual stepsize $\tau$ be bounded according to

$$\tau < \min \left\{ \frac{2\nu}{M_g^2 + 2\epsilon \nu}, \frac{2\epsilon}{1 + \epsilon^2} \right\}.$$

Define the constant

$$q_{d} := (1 - \tau \epsilon)^2 + \tau^2,$$

which is in the set $(0, 1)$ by the definition of $\tau$. Then for all $t \in \mathbb{N}$ we have

$$||\mu(t + 1) - \hat{\mu}_{\nu,e}||^2 \leq q_{d}^{t+1} ||\mu(0) - \hat{\mu}_{\nu,e}||^2$$

and

$$\sum_{i=1}^{t} q_{d}^{-i} \left( q_{d} M_g^2 \|x_i^0(0) - \hat{x}_{\nu,e}\|^2 + 2\tau^2 M_g^2 M_x^2 q_{p}^\frac{d(t - i)}{2} \right),$$

where $M_x = \max_{x \in X} ||x - y||$ and $q_{p}$ is as defined in Lemma II.

Proof: For economy of notation we define

$$g^{-1} = g(x^{-1}(d(t - 1))).$$

We see that

$$\mu(t + 1) = \Pi_{D_{\nu}} \left[ \mu(t) + \tau (g^{-1} - \epsilon \mu(t)) \right]$$

and

$$\hat{\mu}_{\nu,e} = \Pi_{D_{\nu}} \left[ \hat{\mu}_{\nu,e} + \tau (g(\hat{x}_{\nu,e}) - \epsilon \hat{\mu}_{\nu,e}) \right].$$

The projection operator $\Pi_{D_{\nu}}[\cdot]$ is non-expansive and hence using Equations (10) and (11) gives

$$||\mu(t + 1) - \hat{\mu}_{\nu,e}||^2 \leq$$

$$(1 - \tau \epsilon)^2 ||\mu(t) - \hat{\mu}_{\nu,e}||^2 + \tau^2 \|g(\hat{x}_{\nu,e}) - g^{-1}\|^2$$

$$- 2\tau (1 - \tau \epsilon) \langle \mu(t) - \hat{\mu}_{\nu,e}, T(g(\hat{x}_{\nu,e}) - g^{-1}) \rangle, (12)$$
We now place a bound on the term $\tau^2 \|g(\hat{x}_{\nu,\epsilon}) - g(t_{t-1})\|^2$.
Defining $x_{t-1}^* = (x_{t-1}^{1, T}, \ldots, x_{t-1}^{N, T})^T$, we see that
\[
\|g(\hat{x}_{\nu,\epsilon}) - g_{t-1}\|^2 = \|g(x_{t-1}^{*}) - g(x_{t-1}^{*}) - g_{t-1}\|^2
\leq \|g(x_{t-1}^{*}) - g(x_{t-1}^{*})\|^2 + \|g(x_{t-1}^{*}) - g_{t-1}\|^2
+ 2\|g(x_{t-1}^{*}) - g_{t-1}\|\|g(x_{t-1}^{*}) - g_{t-1}\|.
\]
From the Lipschitz property of $g$, we see that
\[
\|g(\hat{x}_{\nu,\epsilon}) - g(x_{t-1}^{*})\| \leq M_g \|x_{t-1}^{*} - x_{t-1}^{*}(d(t-1))\|
\]
and that
\[
\|g(x_{t-1}^{*}) - g_{t-1}\|^2 \leq M_g \|x_{t-1}^{*} - x_{t-1}^{*}(d(t-1))\|^2,
\]
which now together give
\[
\|g(\hat{x}_{\nu,\epsilon}) - g_{t-1}\|^2 \leq M_g \|x_{t-1}^{*} - x_{t-1}^{*}(d(t-1))\|^2 + 2\|g(x_{t-1}^{*}) - g_{t-1}\|\|g(x_{t-1}^{*}) - g_{t-1}\|.
\]
Examining the term $(\mu(t) - \tilde{\mu}_{\nu,\epsilon})^T g(\hat{x}_{\nu,\epsilon})$, in Equation (12), we add and subtract the same term inside the second set of parentheses, giving
\[
(\mu(t) - \tilde{\mu}_{\nu,\epsilon})^T g(\hat{x}_{\nu,\epsilon}) - g_{t-1}
= (\mu(t) - \tilde{\mu}_{\nu,\epsilon})^T g(x_{t-1}^{*}) + g(x_{t-1}^{*}) - g_{t-1}
\geq \frac{\nu}{M_g} \|g(x_{t-1}^{*}) - g(x_{t-1}^{*})\|^2 + (\mu(t) - \tilde{\mu}_{\nu,\epsilon})^T g(x_{t-1}^{*}) - g_{t-1},
\]
where the last inequality follows from Lemma 2. Multiplying Equation (14) by $-2\tau (1 - \tau)$ then gives
\[
-2\tau (1 - \tau) (\mu(t) - \tilde{\mu}_{\nu,\epsilon})^T g(\hat{x}_{\nu,\epsilon}) - g_{t-1}
\leq -2\tau (1 - \tau) \frac{\nu}{M_g} \|g(x_{t-1}^{*}) - g(x_{t-1}^{*})\|^2 + 2\tau^2 \|\mu(t) - \tilde{\mu}_{\nu,\epsilon}\|^2
+ (1 - \tau)^2 \|g(x_{t-1}^{*}) - g(x_{t-1}^{*})\|^2.
\]
Substituting Equations (13) and (15) into Equation (12) and combining like terms gives
\[
\|\mu(t+1) - \tilde{\mu}_{\nu,\epsilon}\|^2 \leq \left((1 - \tau)^2 + \tau^2\right) \|\mu(t) - \tilde{\mu}_{\nu,\epsilon}\|^2
+ \left(\tau^2 - 2\tau (1 - \tau) \frac{\nu}{M_g}\right) \|g(x_{t-1}^{*}) - g(x_{t-1}^{*})\|^2
+ \left((1 - \tau)^2 + \tau^2\right) \|g(x_{t-1}^{*}) - g(x_{t-1}^{*}) - (d(t-1) - 1)\|.
\]
By Lemma 1 we see that
\[
\|x_{t-1}^{*}(d(t-1)) - x_{t-1}^{*}\| \leq q_{t-1}^d M_g M_x M_x M_x.
\]
Also, by definition of $M_g$ and $M_x$ we have
\[
\|g(x_{t-1}) - g(x_{t-1}^{*})\|^2 \leq M_g^2 \|x_{t-1} - x_{t-1}^{*}\|^2 \leq M_g^2 M_x^2.
\]
while using $M_g$ and Lemma 1 gives
\[
\|g(x_{t-1}^{*}) - g_{t-1}\|^2 \leq M_g^2 q_{t-1}^d \|x_{t-1}^{*}(0) - x_{t-1}^{*}\|^2 \leq M_g^2 M_x^2 q_{t-1}^d.
\]
Along the lines of Equation (17), we also see that
\[
\|\hat{x}_{\nu,\epsilon} - x_{t-1}^{*}\| \|x_{t-1}^{*} - x_{t-1}^{*}(d(t-1))\| \leq M_g^2 q_{t-1}^d / 2.
\]
Using Equations (17), (18), (19), and (20) in Equation (16) gives
\[
\|\mu(t+1) - \tilde{\mu}_{\nu,\epsilon}\|^2 \leq
\left((1 - \tau)^2 + \tau^2\right) \|\mu(t) - \tilde{\mu}_{\nu,\epsilon}\|^2 + \left(\tau^2 - 2\tau (1 - \tau) \frac{\nu}{M_g}\right) M_g^2 M_x^2
+ \left((1 - \tau)^2 + \tau^2\right) M_g^2 M_x^2 q_{t-1}^d / 2 + 2\tau^2 M_g^2 M_x^2 q_{t-1}^d / 2 - 1 / e^2 \leq 0.
\]
Using that
\[
\tau \leq \min\left\{\frac{2\nu}{M_g + 2\epsilon\nu}, \frac{2\epsilon}{1 + \epsilon^2}\right\},
\]
we have
\[
(1 - \tau)^2 + \tau^2 \leq 0.
\]
Setting $q_d = (1 - \tau)^2 + \tau^2$, Equation (21) gives
\[
\|\mu(t+1) - \tilde{\mu}_{\nu,\epsilon}\|^2 \leq
q_d \|\mu(t) - \tilde{\mu}_{\nu,\epsilon}\|^2 + M_g^2 M_x^2 q_{t-1}^d + 2\tau^2 M_g^2 M_x^2 q_{t-1}^d / 2.
\]
Applying Equation (22) recursively then gives the desired result.

The sum in the statement of Theorem 1 contains terms that will eventually become small by virtue of the leading term being $q_{t-1}^d$ and getting large. The interpretation of Theorem 1 is that vectors $\mu(t)$ computed by the cloud will eventually become close to $\tilde{\mu}_{\nu,\epsilon}$, though the distance between them will never become zero. In fact, after a long time this distance is dominated by the first few terms of the summation in Equation (9) precisely because the term containing $q_{t-1}^d$ goes to zero. We now bound the distance between primal vectors and their optima.

**Theorem 2:** Let Assumptions 1-4 hold. Then for the sequence of primal vectors $\{x^d(t)\}_{t \in [N]}$ generated by Algorithm 2, we have
\[
\|x^d(t) - \hat{x}_{t, \epsilon}\| \leq q_{t-1}^d / 2 M_x + M_g \nu \frac{\|\mu(t) - \tilde{\mu}_{\nu,\epsilon}\|}{\nu}.
\]
along with
\[
\max\{0, g_j(x^d(t))\} \leq M_g \left(q_{t-1}^d / 2 M_x + M_g \frac{\|\mu(t) - \tilde{\mu}_{\nu,\epsilon}\|}{\nu}\right). \tag{22}
\]
**Proof:** Using Lemmas 1 and 2, we see that
\[
\|x^d(t) - \hat{x}_{t, \epsilon}\| \leq \|x^d(t) - x_{t}^*\| + \|x_{t}^* - \hat{x}_{t, \epsilon}\| \leq q_{t-1}^d / 2 M_x + M_g \frac{\|\mu(t) - \tilde{\mu}_{\nu,\epsilon}\|}{\nu}. \tag{23}
\]
Using the convexity of each constraint function $g_j$, we see that
\[
g_j(x^d(t)) \leq g_j(\hat{x}_{t, \epsilon}) + \nabla g_j(x^d(t))^T (x^d(t) - \hat{x}_{t, \epsilon}) \leq \|\nabla g_j(\hat{x}_{t, \epsilon})\| \|x^d(t) - \hat{x}_{t, \epsilon}\| \leq M_g \left(q_{t-1}^d / 2 M_x + M_g \frac{\|\mu(t) - \tilde{\mu}_{\nu,\epsilon}\|}{\nu}\right), \tag{24}
\]
for all $j \in J$, where we have used the bound on $\|x^j(d(t)) - \hat{x}_{\nu,\epsilon}\|$ from Equation (23).

The first half of Theorem 2 says that $x^j_t$ will eventually become close to $\hat{x}_{\nu,\epsilon}$, with the degree of closeness depending in part by the distance between $\mu(t)$ and $\hat{\mu}_{\nu,\epsilon}$.

The second half makes a similar statement about the extent of any constraint violations, namely that the degree of any constraint violation depends upon the distance from $\mu(t)$ to $\hat{\mu}_{\nu,\epsilon}$.

Both statements in Theorem 2 say that the length of delays between dual updates can be beneficial when it is long, though naturally longer delays also require more time for convergence.

While the point $(\hat{x}_{\nu,\epsilon}, \hat{\mu}_{\nu,\epsilon})$ is not necessarily a saddle point of the original (unregularized) Lagragian, it is guaranteed to be sufficiently close when $\nu$ and $\epsilon$ are small enough; an extended discussion of this matter is in [10, Section 3.2]. In effect, Algorithm 2 lets the agents approach $\hat{x}_{\nu,\epsilon}$ which itself is not far from an optimal solution to Problem 1 in terms of the optimal function value and small feasibility violation of the functional constraints.

V. Experimental Results

Algorithm 2 was simulated for and then run with 8 planar agents. We outline the problem and cover the simulation results, and then present the experimental results.

All agents are planar so that $x_i \in \mathbb{R}^2$ for all $i$ and $x \in \mathbb{R}^4$.

The sum of the per-agent objective functions is

$$\sum_{i=1}^{8} f_i(x_i) = \|x_1\|^2 + \|x_2 - \left(-\frac{1}{1}\right)\|^2 + \|x_3 - \left(0, -\frac{1}{0.6}\right)\|^2 + \|x_4 - \left(-\frac{1.4}{1.4}\right)\|^2 + \|x_5 - \left(-0.1, 0.5\right)\|^2 + \|x_6 - \left(0.7, 0.7\right)\|^2 + (x_{7,1} - 0.5)^2 + x_{7,2} - 1.1 + (x_{8,1} + 0.3)^2 + x_{8,2}^4.$$ 

The non-separable term in the cost is

$$c(x) = \frac{1}{200} (\|x_1 - x_4\|^2 + \|x_1 - x_8\|^2 + \|x_4 - x_8\|^2).$$

As before the total cost used was $f(x) = \sum_{i=1}^{8} f_i(x) + c(x)$.

The constraints used were

$$g(x) = \begin{pmatrix} \|x_1 - x_2\|^2 - 0.6 \\ \|x_1 - x_5\|^2 - 1.2 \\ \|x_7 - x_8\|^2 - 1.8 \\ \|x_1 - x_3\|^2 - 0.4 \\ \|x_4 - x_6\|^2 - 0.9 \end{pmatrix} \leq 0,$$

and each agent was confined to the box $[-1.5, 1.5] \times [-1.0, 1.5].$

The constants needed to solve this problem were computed (approximately) numerically and are shown in Table I.

| Symbol | Value |
|--------|-------|
| $L_f$  | 26.9982 |
| $L_g$  | 4.2426 |
| $M_{\nu}$ | 84.7430 |
| $M_{\theta}$ | 13.5277 |
| $M_{x}$ | $\sqrt{122}$ |

TABLE I

CONSTANTS ASSOCIATED WITH PROBLEM 2

The regularization parameters were chosen to be $\nu = \epsilon = 0.1$, giving $C_f = 64.191$. The primal and dual stepsizes were chosen to be

$$\alpha = 0.9 \cdot \frac{2}{C_f} = 0.02804$$

and

$$\tau = 0.9 \cdot \min \left\{ \frac{2\nu}{M_{\nu}^2 + 2\nu}, \frac{2\epsilon}{1 + \epsilon^2} \right\} = 9.835 \cdot 10^{-4}.$$ 

All delays had length determined by a random integer drawn from a uniform distribution on the integers between 10 and 100, inclusive. Algorithm 2 was run until the agents had computed 250,000 state updates, during which time the cloud computed 4,539 dual updates.

In simulation, the initial total distance between the agents’ positions and their regularized optima, $\|x^0(0) - \hat{x}_{\nu,\epsilon}\|$, was 2.6024 and their final total distance, $\|x^{4539}(250000) - \hat{x}_{\nu,\epsilon}\|$, was 0.0843. In addition, after only 25,000 iterations the total
distance of the agents to \( \hat{x}_{\nu,e} \) was 0.1392, indicating that fewer steps can be taken while still achieving an acceptable ending state. In the dual space, the final distance to the regularized optimum was \( \|\mu(4.539) - \hat{\nu}_{\nu,e}\| = 0.0393 \), indicating close convergence in the dual space as well.

This problem was executed on a team of 8 Khepera III robots; these robots are pictured in Figure 2 where they are 14 seconds into the experimental run. Position data was gathered using an OptiTrack motion capture system and the cloud-based algorithm was used to generate position waypoints for the agents. The experiment was run until the robots and cloud completed 75,000 total updates; this point was reached in the middle of a communications cycle, and that cycle was allowed to finish, giving 73,720 total state updates by each agent and 1,340 dual updates by the cloud. The final error in the primal space was \( \|x(1340) - \hat{x}_{\nu,e}\| = 0.0689 \) and the final error in the dual space was \( \|\mu(1340) - \hat{\nu}_{\nu,e}\| = 0.0587 \), indicating close convergence of the robots to \( (\hat{x}_{\nu,e}, \hat{\nu}_{\nu,e}) \). A plot of the regularized optima, simulation results, and experimental results is shown in Figure 1 indicating close agreement among the three sets of data plotted there.

VI. CONCLUSION

We presented a hybrid centralized/decentralized algorithm for solving multi-agent nonlinear programs with inequality constraints. To do this, we used a Tikhonov regularization of the problem and a computing regime that spread computations across the agents and a cloud computer. The architectural model incorporated communications delays in the system and approximate convergence of the algorithm was proven. Experimental results were provided to show the applicability of these results.

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