THE HOMOTOPY CATEGORY OF FLAT FUNCTORS

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Abstract. Let $C$ be a small category and $G$ be a tensor Grothendieck category. We define a notion of flatness in the category $\text{Fun}(C,G)$ of all covariant functors from $C$ to $G$ and show that the inclusion $\mathbb{K}(\text{Flat} \mathcal{A}) \rightarrow \mathbb{K}(\mathcal{A})$ has a right adjoint where $\mathbb{K}(\mathcal{A})$ is the homotopy category of $\mathcal{A}$ and $\mathbb{K}(\text{Flat} \mathcal{A})$ its subcategory consisting of complexes of flat functors. In addition, we find a replacement for the quotient $\mathbb{D}_{\text{pur}}(\text{Flat} \mathcal{A}) = \mathbb{K}(\text{Flat} \mathcal{A})/\mathbb{K}_p(\text{Flat} \mathcal{A})$ of triangulated categories where $\mathbb{K}_p(\text{Flat} \mathcal{A})$ is the homotopy category of all pure acyclic complexes of flat functors.

1. Introduction

Functor categories are used as a potent tool for solving some important problems in representation theory. They are introduced by Maurice Auslander in [A66] and used in his proof of the first Brauer-Thrall conjecture (see [A78], [AR74], [AR72]).

In this work, we assume that $C$ is a small category, $(G,-\otimes -)$ is a closed symmetric tensor Grothendieck category and $A = \text{Fun}(C,G)$ is the category of all covariant functors from $C$ to $G$. The homotopy category of $\mathcal{A}$ has some interesting triangulated subcategories. One of them is the homotopy category $\mathbb{K}(\text{Flat} \mathcal{A})$ of flat functors which is studied by Amnon Neeman in [Ne08] and [Ne10]. He invented that, when $C$ is a category with a single object and $G$ is the category of modules over a ring $R$ with $1 \neq 0$, the homotopy category $\mathbb{K}(\text{Proj} R)$ of projective modules can be replaced by a quotient of $\mathbb{K}(\text{Flat} \mathcal{A})$ modulo its thick subcategory consisting of all pure acyclic complexes. This quotient is called the pure derived category of flat $R$-modules and denoted by $\mathbb{D}_{\text{pur}}(\text{Flat} R)$ (see [Mu07], [AS12], [HS13], [MS11] for more details). This is an important future of Neeman’s work, because, there are some closed symmetric tensor Grothendieck categories with no non-zero projective object (see [Ha97, III. Ex. 6.2]).

We know that there are different methods to define flatness in $G$. One of them is defined by the categorical purity and the other one is defined by the tensor purity. But, categorical flats are trivial in some situations. For example, if $X = \mathbb{P}^n(R) = \text{Proj}(R[x_1,\ldots,x_n])$ is the projective $n$-space over a commutative ring $R$, then the category $\mathcal{O}_X$ of all quasi-coherent $\mathcal{O}_X$-modules is a closed symmetric tensor Grothendieck category. By [ES15, Corollary 4.6], $\mathcal{O}_X$ does not have non-zero categorical flat objects. This shows that categorical flats are rare in $\text{Fun}(\mathcal{C},\mathcal{O}_X)$. Due to this lack of data the study needed to employ another way to define flatness. This is motivated us to define another notion of flatness in $\mathcal{A}$.

This paper is organized as follows. In Section 2, we define our concept of flatness in $\mathcal{A}$ and show that any object in $\mathcal{A}$ has a flat precover and a cotorsion preenvelope. In Section 3, we show that the category of all complexes in $\mathcal{A}$ admits flat covers and

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cotorsion envelopes. In Section 4, we prove that the pure derived category of flat functors can be replaced by the homotopy category of dg-cotorsion complexes of flat functors.

Before starting, let us fix some notations and definitions.

1.1. Category of complexes. Let $C(A)$ be the category of all complexes in $A$ (complexes are write cohomologically). A complex in $A$ is called acyclic if all cohomological groups are trivial. Let $X = (X^i, \partial_X^i)_{i \in \mathbb{Z}}$ be a complex in $A$, for any integer $n$, $X[n]$ denotes the complex $X$ shifted $n$ degrees to the left. The left truncation of $X$ at $n$ is denoted by $X_{\leq n}$ and defined by the complex

$$
\cdots \to X^{n-2} \to X^{n-1} \to \text{Ker} \partial_X^n \to 0.
$$

Let $f: X \to Y$ be a morphism of complexes. We say that $f$ is a quasi-isomorphism if for any $n \in \mathbb{Z}$, $H^n(f): H^n(X) \to H^n(Y)$ of cohomological groups is an isomorphism. The mapping cone of $f$ is denoted by $C_f$ and defined by the complex $(X[1] \oplus Y, \partial_{C_f})$ where

$$
\partial_{C_f} = \begin{pmatrix} \partial_{X[1]} & 0 \\ f[1] & \partial_Y \end{pmatrix}.
$$

We know that a morphism $f: X \to Y$ of complexes is a quasi-isomorphism if and only if $C_f$ is acyclic.

1.2. Cotorsion theories. Let $\mathcal{A}$ be a Grothendieck category. The right orthogonal of a class $S$ in $\mathcal{A}$ is defined by

$$
S^\perp := \{ B \in \mathcal{A} \mid \text{Ext}_\mathcal{A}^1(S, B) = 0, \text{ for all } S \in S \}.
$$

The left orthogonal $^\perp S$ is defined dually. The pair $(X, Y)$ of classes in $\mathcal{A}$ is called a cotorsion theory if $X^\perp = Y$ and $X = ^\perp Y$. An object $A \in \mathcal{A}$ has a special $Y$-preenvelope if there is an exact sequence $0 \to A \to Y' \to X' \to 0$, where $X' \in X^\perp$ and $Y' \in Y$. The special $X$-precover is defined dually. A cotorsion theory $(X, Y)$ in $\mathcal{A}$ is called complete if any object in $\mathcal{A}$ has a special $X$-precover and a special $Y$-preenvelope.

1.3. Orthogonality in triangulated categories. Let $S$ be a thick subcategory of a triangulated category $T$ and $S^\perp = \{ C \in T \mid \text{Hom}(S, C) = 0, \text{ for all } S \in S \}$. If we have a distinguished triangle $X \to C \to S \to X[1]$ such that $C \in S^\perp$ and $S \in S$ then, $S \to T$ has a right adjoint (see [B90, Lemma 3]). So, by [Kr10, Proposition 4.9.1], we have a triangle equivalence $S^\perp \to T/S$ of triangulated categories.

Setup: Let Flat$\mathcal{G}$ be the class of all tensor flat objects in $\mathcal{G}$ and

$$
\text{Cot} \mathcal{G} = \{ C \in \mathcal{G} \mid \forall F \in \text{Flat} \mathcal{G}, \text{ Ext}_\mathcal{G}^1(F, C) = 0 \}
$$

be the class of all cotorsion objects. In this paper, we assume that $(\text{Flat} \mathcal{G}, \text{Cot} \mathcal{G})$ is a complete cotorsion theory.

2. Flatness in functor categories

In this section, we present our notion of a flat functor and prove that the corresponding cotorsion theory is complete. Let $\text{Hom}\mathcal{G}(\cdot, \cdot): \mathcal{G} \times \mathcal{G}^{\text{op}} \to \mathcal{G}$ be the right adjoint of $- \otimes -$, $\mathcal{J}$ be an injective cogenerator in $\mathcal{G}$ and $(-)^+ = \text{Hom}\mathcal{G}(\cdot, \mathcal{J})$. Suppose that $\mathcal{B} = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{G})$ is the category of all contravariant functors from $\mathcal{C}^{\text{op}}$ into $\mathcal{G}$. Therefore, $(-)^+ = \text{Hom}\mathcal{G}(\cdot, \mathcal{J})$ is a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$. In the following definition, we use $(-)^+$ to define a notion of purity in $\mathcal{A}$. This method was first used by Bo Stenstr"om in [Sten68].
Definition 2.1. A short exact sequence $\mathcal{E}$ in $\mathcal{A}$ is said to be pure if $\mathcal{E}^+$ splits.

This definition encouraged us to look at the class of all pure injective functors. Recall that, a functor $\mathcal{P}$ is said to be pure injective if it is injective with respect to pure exact sequences in $\mathcal{A}$. In the following result, we show that the class of all pure injective functors is preenveloping. The proof is similar to the proof of [Ho13, Proposition 2.7] which we will prove it for further clarity.

Proposition 2.2. The class of all pure injective functors is preenveloping.

Proof. Let $F$ be a functor. The canonical monomorphism $\varphi_F : F \to F^{+++}$ is pure, because $F^+ \to F^{+++}$ is the section of $F^{+++} \to F^+$. In addition, for a given diagram

$$
\begin{array}{c}
0 \to G \xrightarrow{i} K \\
\downarrow f \\
F^{++}
\end{array}
$$

we have the following commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
G \\
\downarrow f \\
F^{++}
\end{array} \\
\begin{array}{c}
i \\
\downarrow h \\
K
\end{array}
\end{array}
$$

such that $s_1 : K^{++} \to G^{++}$ is the section of $i^{++}$ and $s_2 : F^{+++} \to F^{++}$ is the section of $s$. Then, $s_2f^{++}s_1h : K \to F^{++}$ completes (2.1) to a commutative diagram. So, $F^{++}$ is pure injective.

Corollary 2.3. For a given functor $F$, $F^+$ is pure injective.

Proof. Since $F^+$ is a direct summand of $F^{+++}$ then by Proposition 2.2, $F^+$ is pure injective.

In [Her03, Theorem 6], the author used the categorical notion of purity and proved that $\mathcal{A}$ has enough categorical pure injective objects. This class of pure injectives is different from objects which are characterized in Proposition 2.2. Moreover, if $\mathcal{E}$ is an exact sequence in $\mathcal{A}$ then we deduce a degree-wise pure exact sequence $0 \to \mathcal{E} \to \mathcal{E}^{++}$ of complexes. This is the most important future of Definition 2.1.

Next, we use Definition 2.1 and extend the $\otimes$-purity to $\mathcal{A}$.

Definition 2.4. A functor $F$ in $\mathcal{A}$ is called flat if any short exact sequence ending in $F$ is pure.

It is necessary to say that, if $\mathcal{A}$ has enough projective objects, then Definition 2.4 and the categorical notion of flatness are equivalent (see [Sten68, Theorem 3]). But, in general case, $\mathcal{A}$ does not have non-zero categorical flat objects and Definition 2.4 is the only notion of flatness in $\mathcal{A}$. 

In the next result, we prove the well-known relation between flat and injective functors i.e., when \( \text{Flat}\mathcal{A} \) is the class of all flat objects in \( \mathcal{A} \) and \( \text{Inj}\mathcal{B} \) is the class of all injective objects in \( \mathcal{B} \), we show that \((\cdot)^+ : \text{Flat}\mathcal{A} \to \text{Inj}\mathcal{B}\) is a contravariant functor (see the proof of [Ho13, Theorem 2.13]).

**Proposition 2.5.** A functor \( F \) in \( \mathcal{A} \) is flat if and only if \( F^+ \) is injective in \( \mathcal{B} \).

**Proof.** If \( F^+ \) is injective, then any short exact sequence ending in \( F \) is pure in \( \mathcal{A} \), because \( \mathcal{E}^+ \) splits. So, \( F \) is flat.

Conversely, assume that \( F \) is flat and

\[
0 \to F^+ \to G \to H \to 0 \tag{2.3}
\]

be an arbitrary exact sequence in \( \mathcal{B} \). It is enough to show that (2.3) splits in \( \mathcal{B} \). The top row of the following pullback diagram

\[
\begin{array}{ccc}
0 & \to & H^+ \\
\downarrow & & \downarrow h \\
0 & \to & G^+ \\
\downarrow & & \downarrow j \\
0 & \to & F^+ \to F^{++} \\
\end{array}
\]

is pure by assumption. Hence, it is split \((H^+ \text{ is pure injective})\). So, there is a morphism \( h' : F \to P \) such that \( hh' = 1_F \). Therefore, if \( g_1 = gh' : F \to G^+ \), then \( f^+ g_1 = f^+ gh' = ihh' = i \). Furthermore, the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & F^+ \\
\downarrow j & & \downarrow k \\
0 & \to & G^+ \\
\downarrow & & \downarrow \\
0 & \to & H^+ \\
\end{array}
\]

implies that \( g_1^+ k f = g_1^+ f^++ j = i^+ j = 1_{F^+} \). So, (2.3) splits. \( \Box \)

Now, we need to introduce some notations. Let \( f : c \to d \) be a morphism in \( \mathcal{C} \). We write \( s(f) = c \) and \( t(f) = d \). A path in \( \mathcal{C} \) is a sequence of morphisms. For a given \( c \in \mathcal{C} \), the functor \( E_c : \mathcal{A} \to \mathcal{G} \) is defined by \( E_c(F) = F(c) \). This is an exact functor with an exact right adjoint \( S^c : \mathcal{G} \to \mathcal{A} \) defined by \( S^c(F)(c) = \prod_{\text{Hom}_{\mathcal{C}}(c,e)} F \) (see [EEG09, pp. 317] for more details). It can be shown that, if \( \mathcal{E} \) is an injective cogenerator for \( \mathcal{G} \) then \( S^c(\mathcal{E}) \) is an injective cogenerator for \( \mathcal{A} \). We say that \( \mathcal{C} \) is left (right) rooted if there exists no path of the form \( \cdots \to c \to c \to \cdots \). Clearly, \( \mathcal{C} \) is left rooted if and only if \( \mathcal{C}^{\text{op}} \) is right rooted.

**Proposition 2.6.** Let \( I \) be an injective object in \( \mathcal{A} \). Then,

i) For any \( c \in \mathcal{C} \), \( I(c) \) is injective in \( \mathcal{G} \).

ii) For any \( c \in \mathcal{C} \), the canonical morphism \( I(c) \to \prod_{s(f) = c} I(t(f)) \) is a split epimorphism.

**Proof.** Let \( \mathcal{E} \) be an injective cogenerator in \( \mathcal{G} \). Then, for any \( c \in \mathcal{C} \), \( S^c(\mathcal{E}) \) satisfies in (i) and (ii). In addition, (i) and (ii) are preserved under products and direct summands. Since \( S^c(\mathcal{E}) \) (varying \( c \in \mathcal{C} \) and \( \mathcal{E} \)) cogenerate \( \mathcal{A} \) and any injective functor is a direct summand of a product of various \( S^c(\mathcal{E}) \). So, any injective object satisfies in (i) and (ii). \( \Box \)
The converse of this proposition holds if $\mathcal{C}$ is right rooted (see [EEG09, Theorem 4.2]).

**Proposition 2.7.** Let $F$ be a flat object in $\mathcal{A}$. Then,

1) For any $c \in \mathcal{C}$, $F(c)$ is flat in $\mathcal{G}$.

2) For any $c \in \mathcal{C}$, the canonical morphism $\coprod_{(f)=c} F(s(f)) \to F(c)$ is a pure monomorphism.

**Proof.** We know that for a given flat functor $F$ in $\mathcal{A}$, $F^+$ is injective in $\mathcal{B}$. So, by Propositions 2.5 and 2.6, $F$ has the desired properties. □

Here, we introduce another adjoint pair of functors. For a given $c \in \mathcal{C}$, $E_c$ has an exact left adjoint $S_c: \mathcal{G} \to \mathcal{A}$ which is defined by $S_c(F)(e) = \bigoplus_{\mathcal{Hom}_{\mathcal{C}}(c,e)} F$ (see [Mit81] for more details).

**Proposition 2.8.** The category $\mathcal{A}$ is Grothendieck.

**Proof.** A sequence in $\mathcal{A}$ is exact if it is exact in any $c \in \mathcal{C}$. This causes that $\mathcal{A}$ is an abelian category. In addition, coproducts and direct limits may be computed componentwise and so they are exact. If $\mathcal{S}$ is a set of generators in $\mathcal{G}$ then, $\{S_c(\mathcal{X})|c \in \mathcal{C}, \mathcal{X} \in \mathcal{S}\}$ is a set of generators for $\mathcal{A}$. Therefore, $\mathcal{A}$ is Grothendieck. □

**Proposition 2.9.** Let $\mathcal{E}: 0 \to F \to G \to K \to 0$ be a pure exact sequence in $\mathcal{A}$. Then $G$ is flat if and only if $F$ and $K$ are flats.

**Proof.** By assumption, $\mathcal{E}^+$ splits. Then $G^+$ is injective if and only if $F^+$ and $K^+$ are injectives. Consequently, by Proposition 2.5, $G$ is flat if and only if $F$ and $K$ are flats. □

**Lemma 2.10.** Any direct limit of pure exact sequences is pure.

**Proof.** Let $(\mathcal{E}_i: 0 \to F_i \to G_i \to H_i \to 0)_{i \in I}$ be a direct system of pure exact sequences in $\mathcal{A}$. Then, $(\mathcal{E}_i^+)^+ \equiv \lim_{\leftarrow i} \mathcal{E}_i^+$ splits. Therefore, $\lim_{\leftarrow i} \mathcal{E}_i^+$ is a pure exact sequence. □

**Remark 2.11.** It is known that in a $\lambda$-presentable Grothendieck category any $\lambda$-pure exact sequence is a direct limit of split exact sequences (see [AR94], [Cr94]) and hence by Lemma 2.10, it is pure in the sense of Definition 2.1. Particularly, for any family $\{X_i\}_{i \in I}$ in $\mathcal{A}$, the categorical pure exact sequence

\[
0 \to K \to \bigoplus_{i \in I} X_i \to \lim_{\leftarrow i} X_i \to 0
\]

is pure in the sense of Definition 2.1.

**Proposition 2.12.** The class $\text{Flat} \mathcal{A}$ is closed under pure subobjects, pure quotients and direct limits.

**Proof.** By Proposition 2.9, $\text{Flat} \mathcal{A}$ is closed under pure subobjects and pure quotients. If $\{X_i\}_{i \in I}$ is a family of flat functors, then $(\bigoplus_{i \in I} X_i)^+ = \prod_{i \in I} X_i^+$ is injective in $\mathcal{B}$ and so by Proposition 2.5, $\bigoplus_{i \in I} X_i$ is flat. Hence, by Remark 2.11 and Proposition 2.9, $\text{Flat} \mathcal{A}$ is closed under direct limits. □

**Proposition 2.13.** The category $\mathcal{A}$ has enough flat objects.

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Proof. Let \( c \in C \) and \( F \) be a flat object in \( C \). Then, we can consider \( S_c(F) \) as a functor over a left rooted category and hence by Proposition 2.6, \( (S_c(F))^+ \) is injective. It follows that, \( S_c(F) \) is flat (Proposition 2.5). For a given \( G \in A \) and \( c \in C \), let \( F(c) \to G(c) \to 0 \) be a flat precover of \( G(c) \). Then, we have an exact sequence \( \oplus_{c \in C} S_c(F(c)) \to G \to 0 \) where \( \oplus_{c \in C} S_c(F(c)) \) is flat by Proposition 2.12. \( \square \)

**Proposition 2.14.** The pair \((\text{Flat}_A, \text{Cot}_A)\) is a complete cotorsion theory in \( A \).

**Proof.** By Proposition 2.12, \( \text{Flat}_A \) is closed under direct limits. So, by [E06, Theorem 3.3], it is a covering class. Combine this with Proposition 2.13 and deduce that any object in \( A \) has a special flat precover ([EJ00, Corollary 7.2.3.]). Let \( G \) be a given functor and \( 0 \to G \to I \to K \to 0 \) be its injective envelope. Then the pullback diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
C & \to & C \\
\downarrow & & \downarrow \\
0 & \to & G & \to & I & \to & K & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & C & \to & F & \to & K & \to & 0 \\
\end{array}
\]

completes the proof where \( 0 \to C \to F \to K \to 0 \) is a special flat precover of \( K \). \( \square \)

3. Flat cotorsion theory in \( \mathbf{C}(A) \)

This section is devoted to the study of flat cotorsion theory in \( \mathbf{C}(A) \). An acyclic complex \( \cdots \to X^{n-1} \xrightarrow{\partial^{n-1}} X^n \xrightarrow{\partial^n} X^{n+1} \to \cdots \) in \( A \) is called pure acyclic if for any \( n \in \mathbb{Z} \), the exact sequence

\[
0 \to \text{Ker} \partial^n \to X^n \to \text{Im} \partial^n \to 0
\]

is pure. A complex

\[
F : \cdots \to F^{n-1} \xrightarrow{\partial^{n-1}} F^n \xrightarrow{\partial^n} F^{n+1} \to \cdots
\]

in \( A \) is called flat if it is a pure acyclic complex of flats or equivalently, it is an acyclic complex such that for any \( n \in \mathbb{Z} \), \( \text{Ker} \partial^n_F \) is flat (Proposition 2.5). Let \( \mathcal{C}_{\text{pac}}(\text{Flat}_A) \) be the class of all flat complexes in \( A \). A complex \( \mathcal{C} \) in \( A \) is called dg-cotorsion if \( \mathcal{C} \in \mathcal{C}_{\text{pac}}(\text{Flat}_A) \). An acyclic complex \( \mathcal{C} = (C^n, \partial^n_C) \) of cotorsion objects in \( A \) is called cotorsion if for any \( n \in \mathbb{Z} \), \( \text{Ker} \partial^n_C \) is cotorsion. Clearly, dg-cotorsion complexes are not necessarily cotorsion. But, it can be shown that any cotorsion complex is dg-cotorsion.

**Proposition 3.1.** Let \( G = (G^i, \partial^i_G) \in \mathcal{C}_{\text{pac}}(\text{Flat}_A) \). Then, for any \( i \in \mathbb{Z} \), \( G^i \) is a cotorsion functor.
Proof. For $i \in \mathbb{Z}$, assume that

$$0 \to G^i \xrightarrow{f} C \xrightarrow{g} F \to 0$$

be a cotorsion preenvelope of $G^i$. By the pushout of $f$ and $\partial_iG$, we deduce the following exact sequence

$$\begin{array}{ccc}
0 & \to & G^{i-1} \\
\downarrow & & \downarrow \\
0 & \to & G^i \\
\downarrow \partial^i & \downarrow f & \downarrow f\partial^i \\
0 & \to & C \\
\downarrow \partial^i & \downarrow s & \\
0 & \to & F \\
\downarrow & & \\
0 & \to & 0 \\
\end{array}$$

of complexes which is split by the choice of $G$ (the last column is a flat complex). Then, $G^i$ is cotorsion.

Let $\mathcal{C}(\text{dg-Cot}A)$ be the class of all dg-cotorsion complexes in $\mathcal{A}$. We prove that the pair $(\mathcal{C}_{\text{pac}}(\text{Flat}A), \mathcal{C}(\text{dg-Cot}A))$ is a complete cotorsion theory in $\mathcal{C}(\mathcal{A})$. First of all, we prove the assertion in the category of all short exact sequences in $\mathcal{A}$.

Lemma 3.2. Any short exact sequence in $\mathcal{A}$ has a special flat precover and a special cotorsion preenvelope.

Proof. Let $G : 0 \to G' \to G \to G'' \to 0$ be an exact sequence in $\mathcal{A}$. We find an exact sequence $0 \to C \to F \to G \to 0$ in $\mathcal{C}(\mathcal{A})$ where $F$ is a short exact sequence of flat functors and $C$ is a short exact sequence of cotorsion functors. Let

$$0 \to C'' \to F'' \to G'' \to 0$$

be a special flat precover of $G''$. Consider the following pullback diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
C'' & \to & C'' \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}$$

Let $\mathcal{C}(\text{dg-Cot}A)$ be the class of all dg-cotorsion complexes in $\mathcal{A}$. We prove that the pair $(\mathcal{C}_{\text{pac}}(\text{Flat}A), \mathcal{C}(\text{dg-Cot}A))$ is a complete cotorsion theory in $\mathcal{C}(\mathcal{A})$. First of all, we prove the assertion in the category of all short exact sequences in $\mathcal{A}$.
and let $0 \to C' \to F \xrightarrow{h} P \to 0$ be a special flat precover of $P$. The pullback of $j$ and $h$ completes the proof.

**Proposition 3.3.** Any bounded above acyclic complex in $\mathcal{A}$ has a special flat precover and a special cotorsion preenvelope.

**Proof.** Let $Y$ be a bounded above acyclic complex in $\mathcal{A}$. We construct an exact sequence $0 \to C \to F \to Y \to 0$ in $\mathcal{C}(\mathcal{A})$ where $F$ is a bounded above flat complex and $C$ is a bounded above acyclic complex of cotorsions. Without loss of generality, we can assume that $Y = (Y^i, \partial_X^i)_{i \leq 0}$. By Lemma 3.2, we have the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & T^{-1} & & C^{-1} & & C^0 & & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & K^{-1} & & F^{-1} & & F^0 & & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & \text{Ker}(\partial^{-1}) & & Y^{-1} & & Y^0 & & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
\]

in $\mathcal{A}$ where the middle row is an exact sequence of flats and the top row is an exact sequence of cotorsions. For any $i < 0$, we use Lemma 3.2 and an inductive procedure to obtain pure exact sequences $0 \to T^{-1} \to C^{-1} \to C^0 \to 0$ of flats and exact sequences $0 \to T^{-1} \to C^{-1} \to T^i \to 0$ of cotorsions. For any $i > 0$, set $C^i := 0$, $F^i := 0$. Then, $F = (F^i)$ and $C = (C^i)$ have the desired property.

**Theorem 3.4.** Any acyclic complex in $\mathcal{A}$ has a special flat precover and a special cotorsion preenvelope.

**Proof.** Let $X = (X^i, \partial_X^i)$ be an acyclic complex in $\mathcal{A}$. We find an exact sequence $0 \to X \to C \to F \to 0$ of complexes, where $F$ is flat and $C$ is cotorsion. Let $0 \to X^0 \to I^0 \xrightarrow{g} T^0 \to 0$ be the injective envelope of $X^0$ where $I^0$ and $T^0$ are bounded above acyclic complexes. By Lemma 3.3, we have a short exact sequence $0 \to C' \to F' \xrightarrow{f} T \to 0$ of complexes where $F'$ is a flat complex and $C'$ is cotorsion. By the pullback of $f$ and $g$, we deduce the exact sequence $0 \to X^0 \to C_1 \xrightarrow{f} F' \to 0$ where $C_1$ is a bounded above acyclic complex of cotorsion objects ($\forall i > 0, C_1^i = 0$) and $F'$ is a bounded above flat complex ($\forall i > 0, F'^i = 0$). We use Lemma 3.2 and an inductive procedure to construct a cotorsion complex $C$ from $C_1$ and a flat complex $F$ from $F'$ which satisfy in the following exact sequence $0 \to X \to C \to F \to 0$ of complexes. Consequently, any acyclic complex in $\mathcal{A}$ has a special flat precover and a special cotorsion preenvelope.
Corollary 3.5. The pair \((C_{pac}(Flat,A), C(dg-Cot,A))\) is a complete cotorsion theory in \(C(A)\).

Proof. Let \(X\) be a complex in \(A\). There is a quasi-isomorphism \(f : X[-1] \to I\) where \(I\) is a dg-injective complex in \(A\) ([AF90]). By Theorem 3.4, there is an exact sequence \(0 \to C' \to F \to \text{Cone}(f) \to 0\) of complexes where \(F\) is flat and \(C\) is cotorsion. Then, the pullback of \(F \to \text{Cone}(f)\) and \(I \to \text{Cone}(f)\) completes the proof. \(\square\)

4. The Pure derived category of flat functors

Let \(K(A)\) be the homotopy category of \(A\) and \(K(Flat,A)\) be its full subcategory consisting of all complexes of flat functors. Let \(K_p(Flat,A)\) be the full subcategory of \(K(Flat,A)\) consisting of all flat complexes, \(K(dg-Cof,A)\) be the essential image of dg-cotorsion complexes of flats in \(A\). Indeed, \(K(dg-Cof,A)\) is a full triangulated subcategory of \(K(Flat,A)\) and it is closed under isomorphisms. In this section, we prove that \(K(dg-Cof,A)\) and the pure derived category \(D_{pac}(Flat,A)\) of flats in \(A\) are equivalent as triangulated categories.

Theorem 4.1. The inclusion \(K(Flat,A) \to K(A)\) has a right adjoint.

Proof. Let \(\alpha\) be an infinite cardinal. By Proposition 2.12, \(Flat,A\) is closed under \(\alpha\)-direct limits and so it is closed under \(\alpha\)-pure subobject. So, by [Kr12, Theorem 5], the inclusion \(K(Flat,A) \to K(A)\) has a right adjoint. \(\square\)

Let \(K_p(Flat,A)\) be the thick subcategory of \(K(Flat,A)\) consisting of pure acyclic complexes and \(D_{pac}(Flat,A) = K(Flat,A)/K_p(Flat,A)\) be the pure derived category of flat functors. In the following, we prove the main result of this work.

Theorem 4.2. There is an equivalence \(K(dg-Cof,A) \to D_{pac}(Flat,A)\) of triangulated categories.

Proof. The proof contains two parts. In part one, we show that \(K_p(Flat,A)\) and \(D_{pac}(Flat,A)\) are equivalent. Let \(X\) be a complex of flats. By Theorem 3.4, there is an exact sequence \(0 \to C \to F \to X \to 0\) (4.1) where \(F \in C_{pac}(Flat,A)\) and \(C \in C(dg-Cof,A)\). In addition, by Proposition 3.1, \(C\) is a complex of cotorsion objects. Then, (4.1) is degree-wise split and so there is a canonical morphism \(u : X \to \Sigma C\) such that \(C \to F \to X \xrightarrow{u} \Sigma C\) is a distinguished triangle in \(K(Flat,A)\). But, \(\text{Ext}^1_{C(A)}(F,X) = 0\) for all flat complex. Therefore, \(X \in K_p(Flat,A)^\perp\) and hence, \(K_p(Flat,A) \to K(Flat,A)\) has a right adjoint. This implies the equivalence \(K_p(Flat,A)^\perp = K(dg-Cof,A) \to D_{pac}(Flat,A)\) of triangulated categories.

In the second part, we show that \(K_p(Flat,A)^\perp = K(dg-Cof,A)\). Let \(X\) be a complex of flats in \(A\). If \(X \in K(dg-Cof,A)\), then \(X\) is isomorphic in \(K(Flat,A)\) to a dg-cotorsion complex \(X'\) of flats. But, \(\text{Ext}^1_{C(A)}(F,X') = 0\) for all flat complex. Then, for any flat complex \(F \in K_p(Flat,A)^\perp\), \(\text{Hom}_{K(A)}(F,X) = \text{Hom}_{K(A)}(F,X') = 0\). Then \(X \in K_p(Flat,A)^\perp\). Conversely, assume that \(X \in K_p(Flat,A)^\perp\). By part one, we a distinguished triangle \(X \to C \to F \xrightarrow{u} \Sigma X\) where \(C\) is a dg-cotorsion flat complex and \(F\) is a flat complex. Applying \(\text{Hom}_{K(A)}(\cdot, C)\) on the triangle and deduce the desired result. \(\square\)
In the reminder of this section, we give some examples of Grothendieck categories which can be replaced by $G$.

4.1. **Category of $R$-modules.** Let $R$ be an associative ring with $1 \neq 0$ and $G$ be the category of all left $R$-modules. Then $A = \text{Fun}(C, G)$ is a generalization of the category $C(R)$ of complexes of $R$-modules.

4.2. **Category of quasi-coherent sheaves of $\mathcal{O}_X$-modules.** Let $(X, \mathcal{O}_X)$ be a quasi-compact and semi-separated scheme and $\mathcal{M}\text{od}_X$ be the category of all sheaves of $\mathcal{O}_X$-modules and $\Omega\text{co}X$ be the category of all quasi-coherent sheaves of $\mathcal{O}_X$-modules. We know that $\Omega\text{co}X$ is the full subcategory of $\mathcal{M}\text{od}_X$ and the inclusion functor $i : \Omega\text{co}X \to \mathcal{M}\text{od}_X$ has a right adjoint $Q : \mathcal{M}\text{od}_X \to \Omega\text{co}X$ ([TT90]). So, $Q$ preserves injective cogenerators and by [MS11, pp. 1109] $\Omega\text{co}X$ admits arbitrary products. Also, for each pair $\mathcal{F}$ and $\mathcal{F}'$ of quasi-coherent sheaves of $\mathcal{O}_X$-modules, $\mathcal{H}\text{om}^q(\mathcal{F}, \mathcal{F}') = Q\mathcal{H}\text{om}(\mathcal{F}, \mathcal{F}')$ is an internal Hom structure on $\Omega\text{co}X$. If $\mathcal{J}$ is an injective cogenerator in $\Omega\text{co}X$ and $(\cdot)^+ = \mathcal{H}\text{om}^q(\cdot, \mathcal{J})$ then, by [Ho17a], an exact sequence $\mathcal{L} : 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $\Omega\text{co}X$ is pure if and only if $\mathcal{L}^+$ splits. Therefore, a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat if and only if $\mathcal{F}^+$ is injective. Consequently, flat quasi-coherent $\mathcal{O}_X$-modules are flat in the sense of Definition 2.4. By [Ho17a, §3], $\Omega\text{co}X$ does not have enough projective objects. But, any quasi-coherent sheaves of $\mathcal{O}_X$-module is a quotient of a flat quasi-coherent sheaf (see [Mu07, Corollary 3.21]).

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