Ginzburg–Landau Relaxation for Harmonic Maps on Planar Domains into a General Compact Vacuum Manifold

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Abstract

We study the asymptotic behaviour, as a small parameter $\varepsilon$ tends to zero, of minimisers of a Ginzburg–Landau type energy with a nonlinear penalisation potential vanishing on a compact submanifold $\mathcal{N}$ and with a given $\mathcal{N}$-valued Dirichlet boundary datum. We show that minimisers converge up to a subsequence to a singular $\mathcal{N}$-valued harmonic map, which is smooth outside a finite number of points around which the energy concentrates and whose singularities’ location minimises a renormalised energy, generalising known results by Bethuel, Brezis and Hélein for the circle $S^1$. We also obtain $\Gamma$-convergence results and uniform Marcinkiewicz weak $L^2$ or Lorentz $L^2$ estimates on the derivatives. We prove that solutions to the corresponding Euler–Lagrange equation converge uniformly to the constraint and converge to harmonic maps away from singularities.

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1. Introduction

Given a smooth compact connected manifold \( \mathcal{N} \) which can be assumed, thanks to Nash’s embedding theorem [43], to be isometrically embedded into the Euclidean space \( \mathbb{R}^\nu \) for some \( \nu \in \mathbb{N}_* \), given a bounded domain \( \Omega \subset \mathbb{R}^2 \) with Lipschitz boundary and given \( g \in W^{1/2,2}(\partial\Omega, \mathcal{N}) \), a minimising harmonic map \( u : \Omega \rightarrow \mathcal{N} \) with boundary condition \( g \) is a map which minimises the Dirichlet energy

\[
\int_\Omega \frac{|Du|^2}{2}
\]  

on the nonlinear subspace

\[
W_g^{1,2}(\Omega, \mathcal{N}) := \{ u \in W^{1,2}(\Omega, \mathbb{R}^\nu) : u \in \mathcal{N} \text{ almost everywhere in } \Omega \text{ and } \text{tr}_{\partial\Omega} u = g \}
\]  

of the Sobolev space \( W^{1,2}(\Omega, \mathbb{R}^\nu) \) of functions having a square-summable weak derivative, where \( \text{tr}_{\partial\Omega} \) denotes the trace operator on \( W^{1,2}(\Omega, \mathbb{R}^\nu) \). It is known since Morrey’s work that, when the domain \( \Omega \) is two-dimensional, any minimising harmonic map is smooth [41].

Because of topological obstructions, the space \( W_g^{1,2}(\Omega, \mathcal{N}) \) can happen to be empty; if \( g \in C(\partial\Omega, \mathcal{N}) \), this will be the case if and only if the map \( g \) cannot be extended to a continuous map from \( \Omega \) to \( \mathcal{N} \) (see [49]). This occurs for example when the domain \( \Omega \) is simply-connected while the manifold \( \mathcal{N} \) is not simply-connected and the map \( g \) is not homotopic to a constant map.

The Ginzburg–Landau relaxation strategy consists in replacing the constraint that \( u \in \mathcal{N} \) almost everywhere in \( \Omega \) by an additional penalisation term to the Dirichlet energy (1.1). Fixing a nonnegative function \( F \in C(\mathbb{R}^\nu, [0, +\infty)) \) such
that $F^{-1}(\{0\}) = \mathcal{N}$, one defines for every $\varepsilon \in (0, +\infty)$, the Ginzburg–Landau energy as

$$E_F^\varepsilon(u) := \int_\Omega \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2}. \quad (1.3)$$

In the present work, we will require the function $F$ to satisfy the following non-degeneracy condition:

there exist $\delta_F, m_F, M_F \in (0, +\infty)$ such that for every $z \in \mathbb{R}^n$ with $\text{dist}(z, \mathcal{N}) < \delta_F$,

$$m_F \text{dist}(z, \mathcal{N})^2 \leq F(z) \leq M_F \text{dist}(z, \mathcal{N})^2.$$  \quad (1.4)

The existence of minimisers $u_\varepsilon$ of $E_F^\varepsilon$ under the Dirichlet boundary condition $\text{tr}_\partial \Omega u_\varepsilon = g$ follows from a classical result in the direct method of calculus of variations (see for example [22, Corollary 3.24]). When $\varepsilon \to 0$, one expects the function $u_\varepsilon$ to eventually take its values into the manifold $\mathcal{N}$ except in some small singular regions; the limiting map can then play a role of generalised solution of the Dirichlet problem for harmonic maps into the manifold $\mathcal{N}$.

Our first main result (Theorem 7.3) describes this asymptotic behaviour of minimisers of the Ginzburg–Landau energy when $\varepsilon \to 0$: if for each $\varepsilon > 0$, the function $u_\varepsilon$ is a minimiser of $E_F^\varepsilon$ under the boundary condition $\text{tr}_\partial \Omega u_\varepsilon = g$, then there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ converging to 0, a finite set of points $\{a_1, \ldots, a_k\} \subset \Omega$ and a map $u_\ast \in W^{1,2}_\text{loc}(\Omega \setminus \{a_1, \ldots, a_k\})$ such that $u_{\varepsilon_n} \rightharpoonup u_\ast$ strongly in $W^{1,2}_\text{loc}(\Omega \setminus \{a_1, \ldots, a_k\})$, where $u_\ast$ is an $\mathcal{N}$-valued harmonic map in $\Omega \setminus \{a_1, \ldots, a_k\}$ and the configuration of points $\{a_1, \ldots, a_k\}$ minimises a renormalised energy. This renormalised energy is defined as the sum of a renormalised energy for harmonic maps that we have defined in [38] and that we present in Section 3, and a term defined in Section 4 depending on the singularities and on the penalisation nonlinearity $F$.

When $\mathcal{N} = S^1 \subset \mathbb{R}^2$ and $F(z) = (1 - |z|^2)^2$, we recover the seminal results of Bethuel, Brezis & Hélein [9], for the original Ginzburg–Landau functional used to model the behaviour of type II superconductors for a star-shaped domain $\Omega$; the results were later extended to simply-connected domains in [51]; here we do not assume that $\Omega$ is simply-connected and provide thus new results for the original Ginzburg–Landau functional in the multiply-connected case. In the case of a general target manifold $\mathcal{N}$, the leading-order asymptotics and the topological charges of singularities in our results (Theorem 7.3 (ii) and (vi) at the $o(\log 1/\varepsilon)$ level) are due to Canevari [15].

Functionals of the form (1.3) appear in various other physical models besides the Ginzburg–Landau model in superconductivity. The Landau–de Gennes theory describes the state of a nematic liquid crystal via a field of symmetric traceless $3 \times 3$ matrices which minimises an energy of the form (1.3) with $\mathcal{N} \simeq \mathbb{R}P^2$; the study of such minimisers has been the object of many works [4,5,15,29]. Energies of the form (1.3) also appear in physics in Chern-Simon-Higgs theory [5] with $\mathcal{N} = S^1 \times \{0\} \simeq S^1$ and other phase transitions problems like biaxial molecules in nematic phase ($\mathcal{N} \simeq SU(2)/Q$, where $Q$ is the quaternion group), superfluid $^3$He in dipole-free phase with $\mathcal{N} \simeq SU(2) \times SU(2)/H$ where $H$ is a subgroup of
$SU(2) \times SU(2)$ isomorphic to four copies of $S^1$ and superfluid $^3He$ in dipole-locked phase with $\mathcal{N} \simeq \mathbb{R}P^3$ [36].

Minimisation of Ginzburg–Landau type energies also appears as a strategy in meshing algorithms for numerical analysis and computer graphics: in order to generate a quadrangular meshing of a surface or a hexahedral meshing of a three-dimensional domain, one constructs first a guiding cross-field or frame-field which is mathematically a map taking its values into $SO(2)/C_4$ and $SO(3)/O$, where $C_4$ is the cyclic group of order 4 of direct symmetries of a square, and $O$ is the octahedral group of direct symmetries of the cube [6,18,30,35,52]. Mathematically, in the latter case $\pi_1(SO(3)/O) = 2O$ is the nonabelian binary octahedral group. Since one would like these cross-fields or frame-fields to minimise a Dirichlet energy and since one can face topological obstructions as described earlier in this introduction, the strategy consists in constructing these fields using a Ginzburg–Landau relaxation. The cross-fields and frame-fields will necessarily have singularities and one hopes to place these singularities in an optimal way using this procedure.

The asymptotics that we obtain imply in particular that when the domain $\Omega$ is a disk and the boundary datum $g$ is an atomic minimising geodesic in $\mathcal{N}$ (see Section 3.2), then the asymptotic profile is of the form $u_\ast(x) = g(x/|x|)$ (Theorem 8.1). This generalises the answer of Bethuel, Brezis & Hélein to Matano’s original problem on the Ginzburg–Landau equation [23].

As another consequence of our results, the stress-energy tensor of the limit $u_\ast$ has vanishing flux around the singularities—equivalently, the residue of the Hopf differential of $u_\ast$ vanishes at each singularity.

The results presented above are not confined to minimisers of the Ginzburg–Landau energy, and imply in particular $\Gamma$-convergence results at first and second order similar to the classical case, generalising $\Gamma$-convergence results at first order [31,34,47] and $\Gamma$-convergence results at second order [1]. All our results also come with Marcinkiewicz weak $L^2$ estimates—or equivalently estimates in the endpoint Lorentz space $L^{2,\infty}$—on the gradient as for the original Ginzburg–Landau functional [50].

We consider next the improvements in the asymptotics that can be obtained when $u_\varepsilon$ is a weak solution to the equation

$$\Delta u_\varepsilon = \frac{\nabla F(u_\varepsilon)}{\varepsilon^2} \quad \text{in } \Omega,$$

that we refer to (1.5) as the generalised Ginzburg–Landau equation. Minimisers of the Ginzburg–Landau energy $\mathcal{E}_F^\varepsilon$ satisfy the corresponding Euler–Lagrange equation, i.e., (1.5) is satisfied under reasonable assumptions (see Section 9.1). We prove in Theorem 9.3 that under a boundedness assumption on $\nabla F(u_\varepsilon)$, the distance to the manifold $\text{dist}(u_\varepsilon n, \mathcal{N})$ converges uniformly to 0 up to the boundary and away from singularities for any boundary datum $g \in W^{1/2,2}(\partial \Omega, \mathcal{N})$—which is not continuous in general. We next prove in Theorem 9.6 that weakly converging solutions of (1.5) converge to harmonic maps. Finally, we obtain higher-order convergence up to the boundary under a higher regularity assumption on the boundary datum (Theorem 9.10).
Another strategy to study phase-transition problems where one deals with manifold-valued order-parameters has been implemented in [16, 17] by constructing a substitute to the Jacobian determinant used in the classical $S^1$-valued Ginzburg-Landau theory to obtain first-order $\Gamma$-convergence results; this substitute is obtained by using flat chains in the setting of manifolds with abelian fundamental groups. Other types of topological obstructions have been analysed via a Ginzburg–Landau relaxation in the case of two-dimensional Riemannian manifolds [32, 33]: the authors prove the convergence of vector fields minimising some Ginzburg–Landau type energy to a canonical unit-length harmonic tangent field with a finite number of singularities; the singularities arise from a non-vanishing Euler-Poincaré characteristic, their number is determined by the Poincaré–Hopf index theorem and their position is governed by a renormalised energy.

We continue the present work with a preliminary section on the projection onto the manifold and on non-degeneracy conditions on $F$ (Section 2). We next recall in Section 3 the definitions and properties of singular energy, geometric renormalised energy, renormalisable singular mappings and synharmony from [38]. In Section 4, we introduce a quantity measuring the energy of a vortex with a given boundary condition at infinity. We combine then the different tools to obtain an upper bound on the energy of minimisers in Section 5.

In Section 6, we obtain by Sandier’s vortex-ball method [47] a first lower-bound on the energy and then following Jerrard’s strategy [34] we obtain localised estimates. We apply then these estimates to energy convergence results, implying convergence of minimisers and $\Gamma$-convergence results (Section 7). We also explain how our results locate singularities on a disk with an atomic minimising geodesic as boundary datum (Section 8).

In the last section Section 9 we give sufficient conditions for minimisers to be solutions of the Ginzburg–Landau equation. Then we study solutions to this equation and we prove uniform convergence of these solutions to the constraint manifold $\mathcal{N}$, weak convergence to harmonic maps and higher-order convergence away from singularities.

2. Retraction on the Manifold and Non-degeneracy of the Relaxation Potential

2.1. Embedding and nearest point retraction

The Ginzburg–Landau relaxation procedure requires an isometric embedding of the vacuum manifold $\mathcal{N}$ into $\mathbb{R}^\nu$. The classical Nash embedding theorem [43] provides such an embedding. When $\mathcal{N} = G/H$ where $G$ is a Lie group and $H \subset G$ is a closed subgroup, it can be relevant to use an equivariant isometric embedding due to Moore [39] (see also [40]): there exists an isometric embedding $\Psi : G/H \rightarrow \mathbb{R}^\nu$ and a representation $R : G \rightarrow \text{Lin}(\mathbb{R}^\nu)$ such that for every $g \in G$ and $y \in G/H$, $\Psi(gy) = R(g)(\Psi(y))$; in contrast with Nash’s embedding theorem, the dimension $\nu$ of the target space $\mathbb{R}^\nu$ depends on the metric on $G$ and on the choice of the subgroup $H$, and the compactness of $G/H$ is essential (there is no such embedding if $G/H$ is the hyperbolic space $\mathbb{H}^n \simeq O(1, n)/O(1) \times O(n)$).
We define the function \( \text{dist}_\mathcal{N} : \mathbb{R}^v \to [0, +\infty) \) by setting for each \( y \in \mathbb{R}^v \),
\[
\text{dist}_\mathcal{N}(y) := \text{dist}(y, \mathcal{N}) := \inf \{|y - z| : z \in \mathcal{N}\}
\]
and for each \( \delta \in (0, +\infty) \), the set
\[
\mathcal{N}_\delta := \{y \in \mathbb{R}^v : \text{dist}(y, \mathcal{N}) < \delta\}.
\]
The next lemma describes the nearest point retraction of a neighbourhood of \( \mathcal{N} \) on \( \mathcal{N} \).

**Lemma 2.1.** There exists \( \delta_\mathcal{N} > 0 \) such that the nearest point retraction \( \Pi_\mathcal{N} : \mathcal{N}_{\delta_\mathcal{N}} \to \mathcal{N} \) characterized by
\[
|y - \Pi_\mathcal{N}(y)| = \text{dist}(y, \mathcal{N})
\]
is well-defined and smooth. Moreover, if the mappings \( P_\mathcal{N}^\top : \mathcal{N} \to \text{Lin}(\mathbb{R}^v, \mathbb{R}^v) \) and \( P_\mathcal{N} : \mathcal{N} \to \text{Lin}(\mathbb{R}^v, \mathbb{R}^v) \) are defined for each \( y \in \mathcal{N} \) by setting \( P_\mathcal{N}^\top(y) \) and \( P_\mathcal{N}(y) \) as the orthogonal projections on \( T_y \mathcal{N} \) and \( (T_y \mathcal{N})^\perp \), identified as linear subspaces of \( \mathbb{R}^v \), then for every \( y \in \mathcal{N}_{\delta_\mathcal{N}} \) and \( v \in \mathbb{R}^v \),
\[
|D \text{dist}_\mathcal{N}(y)[v]|^2 \leq |P_\mathcal{N}^\top(\Pi_\mathcal{N}(y))[v]|^2 \tag{2.1}
\]
and
\[
\left(1 - \frac{\text{dist}_\mathcal{N}(y)}{\delta_\mathcal{N}}\right)|D \Pi_\mathcal{N}(y)[v]|^2 \leq |P_\mathcal{N}^\top(\Pi_\mathcal{N}(y))[v]|^2 \leq C|D \Pi_\mathcal{N}(y)[v]|^2, \tag{2.2}
\]
for some constant \( C \in (0, +\infty) \) depending on \( \mathcal{N} \) and \( v \) only.

In the particular case of the sphere \( \mathcal{N} = S^n \subseteq \mathbb{R}^{n+1} \), one has \( \Pi_\mathcal{N}(y) = y/|y| \) if \( y \in \mathbb{R}^{n+1}\setminus\{0\} \), \( D \Pi_\mathcal{N}(y)[v] = (v|y|^2 - y(y \cdot v))/|y|^3 \), and thus \( |D \Pi_\mathcal{N}(y)[v]|^2 = |v|^2/|y|^2 - (y \cdot v)^2/|y|^4 \) for \( v \in \mathbb{R}^{n+1} \). Moreover \( \text{dist}_{S^n}(y) = ||y| - 1| \) and \( |D \text{dist}_{S^n}(y)[v]| = |v \cdot y|/|y| \) for \( y \in \mathbb{R}^{n+1}\setminus\{0\} \) and \( v \in \mathbb{R}^{n+1} \). Besides, if \( z \in S^n \) and \( v \in \mathbb{R}^{n+1} \); \( P_{S^n}^\perp(z)[v] = z(z \cdot v) \) and \( P_{S^n}(z)[v] = v - z(z \cdot v) \), so that in this case Lemma 2.1 is a consequence of the formulae
\[
|D \text{dist}_{S^n}(y)[v]|^2 = |P_{S^n}^\perp(\Pi_{S^n}(y))[v]|^2 \tag{2.3}
\]
and
\[
|y|^2|D \Pi_{S^n}(y)[v]|^2 = |P_{S^n}^\top(\Pi_{S^n}(y))[v]|^2,
\]
for \( y \in \mathbb{R}^{n+1}\setminus\{0\} \) and \( v \in \mathbb{R}^{n+1} \).

The smoothness of the nearest point retraction is classical [25]. For related computations on the distance function to embedded manifolds, we refer the reader to [3, 24]. For every \( y \in \mathcal{N}_{\delta_\mathcal{N}} \) and \( v \in \mathbb{R}^v \), we have, by orthogonality \( |P_{\mathcal{N}}^\perp(\Pi_\mathcal{N}(y))[v]|^2 + |P_{\mathcal{N}}^\top(\Pi_\mathcal{N}(y))[v]|^2 = |v|^2 \), and thus, by Lemma 2.1,
\[
|D \text{dist}_\mathcal{N}(y)[v]|^2 + \left(1 - \frac{\text{dist}_\mathcal{N}(y)}{\delta_\mathcal{N}}\right)|D \Pi_\mathcal{N}(y)[v]|^2 \leq |v|^2. \tag{2.3}
\]
In the proof of Lemma 2.1 and throughout this work we will use the following facts about the nearest point projection:

for all $y \in \mathcal{N}_\delta$, $y - \Pi_{\mathcal{N}}(y) \in (T_{\Pi_{\mathcal{N}}(y)}\mathcal{N})^\bot$, \hspace{1cm}(2.4)
for all $y \in \mathcal{N}$, $D\Pi_{\mathcal{N}}(y)$ is the orthogonal projection onto $T_y\mathcal{N}$ i.e., $D\Pi_{\mathcal{N}}(y) = P^\top_{\mathcal{N}}(y)$, \hspace{1cm}(2.5)
for all $y \in \mathcal{N}$, $-D^2\Pi_{\mathcal{N}}(y) : T_y\mathcal{N} \otimes T_y\mathcal{N} \to (T_y\mathcal{N})^\bot$ is the second fundamental form of $\mathcal{N} \subset \mathbb{R}^v$ at $y$. \hspace{1cm}(2.6)

Point (2.4) follows from the characterization of the map $\Pi_{\mathcal{N}}$. For (2.5) we refer to [42, Lemma 3.1]. We denote by $B_x : T_x\mathcal{N} \otimes T_x\mathcal{N} \to (T_x\mathcal{N})^\bot$ the second fundamental form of $\mathcal{N}$ at $x \in \mathcal{N}$ and we refer to [20, Definition 6.2.2] for the definition.

We observe that, for $y \in \mathcal{N}$, $D^2\Pi_{\mathcal{N}}(y)|_{T_y\mathcal{N} \otimes T_y\mathcal{N}} = DP^\top_{\mathcal{N}}(y)|_{T_y\mathcal{N} \otimes T_y\mathcal{N}}$, and we refer to [42, Lemma 3.2] for (2.6).

Proof of Lemma 2.1. It is well-known that when $\delta > 0$ is small enough, the nearest point retraction $\Pi_{\mathcal{N}}$ is well-defined on $\mathcal{N}_\delta$. For every $y \in \mathcal{N}_\delta$, by using (2.4) we find that

$$P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[\Pi_{\mathcal{N}}(y) - y] = 0.$$ 

Differentiating this identity with respect to $y$ by using the chain rule and the Leibniz rule, we find for every $y \in \mathcal{N}_\delta$ and $v \in \mathbb{R}^v$,

$$P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[D\Pi_{\mathcal{N}}(y)[v] - v] + (DP^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[D\Pi_{\mathcal{N}}(y)[v]])[\Pi_{\mathcal{N}}(y) - y] = 0. \hspace{1cm}(2.7)$$

Noting that $D\Pi_{\mathcal{N}}(y)[v] \in T_{\Pi_{\mathcal{N}}(y)}\mathcal{N}$, that for every $z \in \mathcal{N}$, $P^\top_{\mathcal{N}}(z) + P^\bot_{\mathcal{N}}(z) = \text{id}$ so that $DP^\top_{\mathcal{N}}(z)[w] = -DP^\bot_{\mathcal{N}}(z)[w]$ whenever $w \in T_z\mathcal{N}$, we infer from (2.7) that

$$P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[D\Pi_{\mathcal{N}}(y)[v] - v] - (DP^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[D\Pi_{\mathcal{N}}(y)[v]])[\Pi_{\mathcal{N}}(y) - y] = 0. \hspace{1cm}(2.8)$$

We observe that for every $w \in \mathbb{R}^v$, $x \in \mathcal{N} \mapsto P^\bot_{\mathcal{N}}(x)[w] \in T_x\mathcal{N}$ is a smooth map, and therefore we have [20, Proposition 6.2.3] if $x \in \mathcal{N}$, $w, z \in T_x\mathcal{N}$ and $u \in (T_x\mathcal{N})^\bot$, \hspace{1cm}(2.9)

$$z \cdot (DP^\bot_{\mathcal{N}}(x)[w])[u] = -u \cdot B_x(z, w).$$

Moreover, since for every $y \in \mathcal{N}_\delta$, $v \in \mathbb{R}^v$, $D\Pi_{\mathcal{N}}(y)[v] \in T_{\Pi_{\mathcal{N}}(y)}\mathcal{N}$, we have

$$D\Pi_{\mathcal{N}}(y)[v] \cdot P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[D\Pi_{\mathcal{N}}(y)[v] - v] = |D\Pi_{\mathcal{N}}(y)[v]|^2 - D\Pi_{\mathcal{N}}(y)[v] \cdot P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[v]. \hspace{1cm}(2.10)$$

Therefore, we have, by taking the inner product of (2.8) with the vector $D\Pi_{\mathcal{N}}(y)[v]$, in view of (2.9) and (2.10),

$$|D\Pi_{\mathcal{N}}(y)[v]|^2 + (\Pi_{\mathcal{N}}(y) - y) \cdot B_{\Pi_{\mathcal{N}}(y)}[D\Pi_{\mathcal{N}}(y)[v], D\Pi_{\mathcal{N}}(y)[v]] = P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[v] \cdot D\Pi_{\mathcal{N}}(y)[v]. \hspace{1cm}(2.11)$$
Hence, if \( \delta_{\mathcal{N}} \in (0, \delta) \) satisfies \( \frac{1}{\delta_{\mathcal{N}}} \geq \sup \{|B_y(z, w)| : y \in \mathcal{N}, z, w \in T_y\mathcal{N}, |z| \leq 1, |w| \leq 1\} \), we have for every \( y \in \mathcal{N}_{\delta_{\mathcal{N}}} \) and \( v \in \mathbb{R}^v \),

\[
\left(1 - \frac{1}{\delta_{\mathcal{N}}} |\Pi_{\mathcal{N}}(y) - y|\right)|D\Pi_{\mathcal{N}}(y)[v]| \leq |P_{\mathcal{N}}^T(\Pi_{\mathcal{N}}(y))[v]|,
\]

which is the first inequality in (2.2). In particular, \( \ker P_{\mathcal{N}}^T(\Pi_{\mathcal{N}}(y)) \subset \ker D\Pi_{\mathcal{N}}(y) \) and moreover \( \ker D\Pi_{\mathcal{N}}(y) = \ker P_{\mathcal{N}}^T(\Pi_{\mathcal{N}}(y)) \) since \( D\Pi_{\mathcal{N}}(y) \) and \( P_{\mathcal{N}}^T(\Pi_{\mathcal{N}}(y)) \) are onto from \( \mathbb{R}^v \) to \( T\Pi_{\mathcal{N}}(y)\mathcal{N} \). This yields the second inequality in (2.2).

The first estimate (2.1), follows from the fact that for every \( y \in \mathcal{N}_{\delta_{\mathcal{N}}} \setminus \mathcal{N} \) and \( v \in \mathbb{R}^v \),

\[
D \text{dist}_{\mathcal{N}}(y)[v] = \frac{v \cdot (y - \Pi_{\mathcal{N}}(y))}{|y - \Pi_{\mathcal{N}}(y)|} = \frac{P_{\mathcal{N}}^T(\Pi_{\mathcal{N}}(y))[v] \cdot (y - \Pi_{\mathcal{N}}(y))}{|y - \Pi_{\mathcal{N}}(y)|}.
\]

\[\square\]

### 2.2. Non-degeneracy of the penalising potential

If the function \( F \) satisfies the first order non-degeneracy condition

\[
F \in C^1(\mathbb{R}^v, [0, +\infty)) \text{ and there exist } \delta_F \in (0, \delta_{\mathcal{N}}) \text{ and } m_F, M_F \in (0, +\infty) \text{ such that}
\]

\[
m_F \text{dist}(z, \mathcal{N})^2 \leq DF(z)|z - \Pi_{\mathcal{N}}(z)| \leq M_F \text{dist}(z, \mathcal{N})^2 \quad \text{for every } z \in \mathcal{N}_{\delta_F},
\]

then it satisfies our zero order non-degeneracy assumption (1.4). This fact will be useful in Section 9.4.

**Lemma 2.2.** If \( F \in C^1(\mathbb{R}^v, [0, +\infty)) \) with \( F = 0 \) on \( \mathcal{N} \) and if (2.13) holds, then (1.4) holds.

**Proof.** By the assumption (2.13), we have for every \( z \in \mathcal{N}_{\delta_F} \) and \( t \in [0, 1] \),

\[
m_F t \text{dist}(z, \mathcal{N})^2 \leq DF((1 - t)\Pi_{\mathcal{N}}(z) + tz)[z - \Pi_{\mathcal{N}}(z)] \leq M_F t \text{dist}(z, \mathcal{N})^2;
\]

the conclusion follows by integration with respect to \( t \) over \([0, 1]\) since \( F = 0 \) on \( \mathcal{N} \). \[\square\]

A more explicit condition on \( F \) that implies (2.13) is given by the second order condition

\[
F \in C^2(\mathbb{R}^v, [0, +\infty)) \text{ and for every } y \in \mathcal{N}
\]

and \( v \in (T_y\mathcal{N})^\perp \setminus \{0\} \), \( D^2 F(y)[v, v] > 0 \).

**Lemma 2.3.** If \( F \in C^2(\mathbb{R}^v, [0, +\infty)) \) with \( F = 0 \) on \( \mathcal{N} \) and if (2.14) holds, then (2.13) holds.
Proof. By compactness of \( \mathcal{N} \), by continuity of \( D^2 F \) and by (2.14), there exist \( \delta_F \in (0, \delta_N) \) and \( m_F, M_F \in (0, +\infty) \) such that for every \( z \in \mathcal{N}_{\delta_F} \) and \( v \in (T_{\Pi_{\mathcal{N}(z)} \mathcal{N}})^\perp \),

\[
m_F |v|^2 \leq D^2 F(z)[v, v] \leq M_F |v|^2.
\]

In particular, since \( z - \Pi_{\mathcal{N}'}(z) \in (T_{\Pi_{\mathcal{N}'}(z)} \mathcal{N})^\perp \), we have for every \( t \in [0, 1] \),

\[
m_F \text{ dist}(z, \mathcal{N})^2 \leq D^2 F((1 - t)\Pi_{\mathcal{N}'}(z) + tz)[z - \Pi_{\mathcal{N}'}(z), z - \Pi_{\mathcal{N}'}(z)] \leq M_F \text{ dist}(z, \mathcal{N})^2,
\]

and the conclusion follows by integration over \([0, 1]\) since \( DF \equiv 0 \) on \( \mathcal{N} \). \( \square \)

Remark 2.4. Many potentials \( F \) satisfy the condition (2.14), the most canonical being \( F(z) := \text{ dist}(z, \mathcal{N})^2 \) in a neighbourhood of \( \mathcal{N} \): we have for every \( z \in \mathcal{N}_{\delta_N} \) and \( v \in \mathbb{R}^v \),

\[
DF(z)[v] = 2(z - \Pi_{\mathcal{N}'}(z)) \cdot v,
\]

and for every \( v_1, v_2 \in \mathbb{R}^v \),

\[
D^2 F(z)[v_1, v_2] = 2(v_1 - D\Pi_{\mathcal{N}'}(z)[v_1]) \cdot v_2,
\]

so that, in particular, \( D^2 F(z)[v, v] = 2|v|^2 \) if \( z \in \mathcal{N} \) and \( v \in (T_z \mathcal{N})^\perp \), since then \( D\Pi_{\mathcal{N}'}(z) \) is the orthogonal projection on \( T_z \mathcal{N} \).

Remark 2.5. In the previous example of the squared distance function, we have \( |\nabla F|^2 = 4F \). In general, if \( F \in C^3(\mathbb{R}^v, [0, +\infty)) \) vanishes on \( \mathcal{N} \) and satisfies (2.14), then the function \( G \), defined by \( G(y) := |\nabla F(y)|^2 \), vanishes on \( \mathcal{N} \) and satisfies (2.14). Indeed, if \( y \in \mathcal{N} \) and \( v \in T_y \mathcal{N}^\perp \setminus \{0\} \), then \( DG(y)[v] = 2(D^2 F(y)[v])|\nabla F(y)| \) and so \( D^2 G(y)[v, v] = 2|D^2 F(y)[v]|^2 > 0 \) as \( DF \equiv 0 \) on \( \mathcal{N} \).

3. Renormalised Energies and Renormalisable Harmonic Maps

3.1. Topological resolution of the boundary datum

Following our previous work [38], we describe here the resolution of obstructions of the boundary datum that are responsible for asymptotical singularities for Ginzburg–Landau type functionals.

Given an open set \( \Omega \subset \mathbb{R}^2 \), an integer \( k \in \mathbb{N} \) and a family of distinct points \( a_1, \ldots, a_k \in \Omega \), we define the quantity

\[
\tilde{\rho}(a_1, \ldots, a_k) := \sup\{\rho > 0 : \bar{B}_\rho(a_i) \cap \bar{B}_\rho(a_j) = \emptyset \text{ for each } i, j \in \{1, \ldots, k\} \text{ such that } i \neq j \text{ and } \bar{B}_\rho(a_i) \subset \Omega \text{ for each } i \in \{1, \ldots, k\}\}
\]

(3.1)

and the notion of topological resolution [38, Definitions 2.1 and 2.2]
Definition 3.1. Given $\Omega \subset \mathbb{R}^2$ a domain with a Lipschitz boundary, $k \in \mathbb{N}_+$, $k$ maps $\gamma_1, \ldots, \gamma_k \in \text{VMO}(\mathbb{S}^1, \mathcal{N})$ and a map $g \in \text{VMO}(\partial \Omega, \mathcal{N})$, we say that $(\gamma_1, \ldots, \gamma_k)$ is a topological resolution of $g$ whenever there exist points $a_1, \ldots, a_k \in \Omega$, a radius $\rho \in (0, \rho(a_1, \ldots, a_k))$, and a continuous map $u \in C(\bar{\Omega}\setminus \bigcup_{i=1}^{k} B_{\rho}(a_i), \mathcal{N})$ such that $u|_{\partial \Omega}$ is homotopic to $g$ in $\text{VMO}(\partial \Omega, \mathcal{N})$ and for each $i \in \{1, \ldots, k\}$, $u(a_i + \rho \cdot)_{|\mathbb{S}^1}$ is homotopic to $\gamma_i$ in $\text{VMO}(\mathbb{S}^1, \mathcal{N})$.

Definition 3.1 is invariant under changes of the positions of points and of the radius, and under homotopies of $g$ in $\text{VMO}(\partial \Omega, \mathcal{N})$ and of $\gamma_1, \ldots, \gamma_k$ in $\text{VMO}(\mathbb{S}^1, \mathcal{N})$. If the maps $g, \gamma_1, \ldots, \gamma_k$ are continuous, then we can assume, without loss of generality in the definition, that $g = u|_{\partial \Omega}$ and $u(a_i + \rho \cdot)_{|\mathbb{S}^1} = \gamma_i$ everywhere [13, 14]. Topological resolutions can be characterised algebraically in the fundamental group $\pi_1(\mathcal{N})$ by conjugacy classes [38, Proposition 2.4].

3.2. Singular energy

The minimal length in the homotopy class of $\gamma \in \text{VMO}(\mathbb{S}^1, \mathcal{N})$ is defined as

$$\lambda(\gamma) := \inf \left\{ \int_{\mathbb{S}^1} |\tilde{\gamma}'| : \tilde{\gamma} \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ and } \gamma \text{ are homotopic in } \text{VMO}(\mathbb{S}^1, \mathcal{N}) \right\}. \quad (3.2)$$

We have then that

$$\inf \left\{ \int_{\mathbb{S}^1} |\tilde{\gamma}'|^2 : \tilde{\gamma} \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ and } \gamma \text{ are homotopic in } \text{VMO}(\mathbb{S}^1, \mathcal{N}) \right\} = \frac{\lambda(\gamma)^2}{2\pi}, \quad (3.3)$$

and equality is achieved if and only if $\gamma$ is a minimising geodesic. The quantity $\lambda(\gamma)$ is invariant under homotopy in $\text{VMO}(\mathbb{S}^1, \mathcal{N})$.

The systole of the manifold $\mathcal{N}$ is the length of the shortest closed non-trivial geodesic on $\mathcal{N}$:

$$\text{sys}(\mathcal{N}) = \inf \left\{ \lambda(\gamma) : \gamma \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ is not homotopic to a constant} \right\}. \quad (3.4)$$

In particular, for every $\gamma \in \text{VMO}(\mathbb{S}^1, \mathcal{N})$, we have $\lambda(\gamma) \in \{0\} \cup [\text{sys}(\mathcal{N}), +\infty)$. When $\mathcal{N}$ is compact, $\text{sys}(\mathcal{N}) > 0$.

Proposition 3.2. If $\mathcal{N}$ is compact, then the set $\{\lambda(\gamma) : \gamma \in \text{VMO}(\mathbb{S}^1, \mathcal{N})\}$ is discrete.

Proof. By homotopy invariance of $\lambda(\gamma)$ and thanks to the existence of geodesics in each homotopy class, we can assume that the maps $\gamma$ are taken to be minimising geodesics. We consider thus a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in $C^1(\mathbb{S}^1, \mathcal{N})$ of minimising closed geodesics such that the sequence of numbers $(\lambda(\gamma_n))_{n \in \mathbb{N}}$ converges. In view of (3.3) and the Ascoli–Arzelá compactness criterion, there is a subsequence of $(\gamma_n)_{n \in \mathbb{N}}$ that converges uniformly and hence up to a further subsequence all the maps in the sequence $(\gamma_n)_{n \in \mathbb{N}}$ are homotopic and thus $(\lambda(\gamma_n))_{n \in \mathbb{N}}$ is constant, which implies that the set $\{\lambda(\gamma) : \gamma \in \text{VMO}(\mathbb{S}^1, \mathcal{N})\}$ is discrete. □
The first key quantity in the asymptotics for Ginzburg–Landau type functionals is the following [38, Section 2.2]:

**Definition 3.3.** If \( \Omega \subset \mathbb{R}^2 \) is a Lipschitz bounded domain and \( g \in \text{VMO}(\partial \Omega, \mathcal{N}) \), we define its **singular energy** to be

\[
E_{\text{sg}}(g) := \inf \left\{ \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} : k \in \mathbb{N}_+ \text{ and } (\gamma_1, \ldots, \gamma_k) \text{ is a topological resolution of } g \right\}.
\]

The singular energy \( E_{\text{sg}} \) is invariant under homotopies in \( \text{VMO}(\partial \Omega, \mathcal{N}) \). For every \( \gamma \in \text{VMO}(S^1, \mathcal{N}) \), we have \( E_{\text{sg}}(\gamma) \leq \frac{\lambda(\gamma)^2}{4\pi} \) (where in the definition of \( E_{\text{sg}}(\gamma) \), the circle \( S^1 \) is thought as the boundary of \( \Omega = B_1 \)) and for every \( g \in \text{VMO}(\partial \Omega, \mathcal{N}) \),

\[
E_{\text{sg}}(g) \in \{0\} \cup \left[ \frac{\text{sys}(\mathcal{N})^2}{4\pi}, +\infty \right).
\]

We say that \( (\gamma_1, \ldots, \gamma_k) \) is a **minimal topological resolution** of \( g \) whenever it is a topological resolution of \( g \) such that \( E_{\text{sg}}(g) = \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} \) and for every \( i \in \{1, \ldots, k\} \), \( \lambda(\gamma_i) > 0 \) [38, Definition 2.7]. For example, if \( g \in \text{VMO}(\partial \Omega, S^1) \) and \( \text{deg}(g) = d \in \mathbb{Z} \) then \( E_{\text{sg}}(g) = \pi |d|^2 \), and a minimal topological resolution is given by \( |d| \) maps of degree 1 if \( d > 0 \), and \( |d| \) maps of degree \(-1\) if \( d < 0 \). However, in general, minimal topological resolutions are not necessarily unique.

A closed curve \( \gamma \in \mathcal{C}(S^1, \mathcal{N}) \) is said to be **atomic** whenever \( (\gamma) \) is a minimal topological resolution of \( \gamma \) [38, Definition 2.8]. In particular, if \( \lambda(\gamma) = \text{sys}(\mathcal{N}) \), then \( \gamma \) is atomic. Atomicity does not exclude the existence of an alternative minimal topological resolution into several maps, this is the case for the manifold \( \mathcal{N} \) arising as quotient of \( SU(2) \times SU(2) \) in models of superfluid \( ^3\text{He} \) [38, Section 9.3.5].

### 3.3. Synharmony between geodesics

The notion of synharmony between geodesics quantifies how homotopic mappings can be connected almost through a minimising harmonic map [38, Section 3.2].

**Definition 3.4.** The **synharmonicity** between two given maps \( \gamma, \beta \in W^{1/2,2}(S^1, \mathcal{N}) \), is defined as

\[
d_{\text{synh}}(\gamma, \beta) := \inf \left\{ \int_{S^1 \times [0, L]} \frac{|Du|^2}{2} - \frac{L}{4\pi} \lambda(\gamma)^2 : L \in (0, +\infty), \right. \\
\left. u \in W^{1,2}(S^1 \times [0, L], \mathcal{N}), \right. \\
\left. \text{tr}_{S^1 \times \{0\}} u = \gamma \text{ and } \text{tr}_{S^1 \times \{L\}} u = \beta \right\}.
\]
The synharmonicity is an extended pseudo-distance which is continuous with respect to the strong topology in $W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ [38, Proposition 3.3]. Bounded sets in $W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ which contain only homotopic maps have bounded synharmonicity [38, Proposition 3.5].

Two maps $\gamma, \beta \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ are synharmonic whenever $d_{\text{synh}}(\gamma, \beta) = 0$ [38, Definition 3.6]. The synharmony between minimising geodesics is an equivalence relation, partitioning each homotopy class of minimising geodesics into synharmony classes. If $\gamma, \beta \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ and $d_{\text{synh}}(\gamma, \beta) = 0$, then either $\gamma = \beta$ almost everywhere in $\mathbb{S}^1$ or both $\beta$ and $\gamma$ are minimising geodesics [38, Proposition 3.7]. Minimising geodesics that are homotopic through minimising geodesics are synharmonic [38, Proposition 3.8]; this covers in particular $\gamma \circ R$ and $\gamma$ where $R \in SO(2)$. Although homotopic minimising closed geodesics on a manifold are not synharmonic in general, this is the case on examples that motivate the use of Ginzburg–Landau type energies in physics and geometry.

### 3.4. Renormalised energies of configurations of points

Given a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial \Omega$, a map $g \in W^{1/2,2}(\partial \Omega, \mathcal{N}) \subset \text{VMO}(\partial \Omega, \mathcal{N})$, $k \in \mathbb{N}$, and $k$ closed minimising geodesics $\gamma_1, \ldots, \gamma_k \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ that form a topological resolution of $g$, we consider the geometrical renormalised energy [38, Section 2.3] defined on the configuration space of $\Omega$,

$$
\text{Conf}_k \Omega := \{(a_1, \ldots, a_k) : a_i \neq a_j \text{ if } i \neq j\},
$$

by setting for every $(a_1, \ldots, a_k) \in \text{Conf}_k \Omega$,

$$
\mathcal{E}_{g; \gamma_1, \ldots, \gamma_k}^{\text{geom}}(a_1, \ldots, a_k) := \lim_{\rho \to 0} \mathcal{E}_{g; \gamma_1, \ldots, \gamma_k}^{\text{geom}, \rho}(a_1, \ldots, a_k) - \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{1}{\rho}
$$

$$
= \inf_{\rho \in (0, \rho(a_1, \ldots, a_k))} \mathcal{E}_{g; \gamma_1, \ldots, \gamma_k}^{\text{geom}, \rho}(a_1, \ldots, a_k) - \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{1}{\rho},
$$

(3.6)

where for a radius $\rho \in (0, \tilde{\rho}(a_1, \ldots, a_k))$, we have set

$$
\mathcal{E}_{g; \gamma_1, \ldots, \gamma_k}^{\text{geom}, \rho}(a_1, \ldots, a_k) = \inf \left\{ \int_{\Omega \setminus \bigcup_{i=1}^{k} \tilde{B}_\rho(a_i)} \frac{|Du|^2}{2} : u \in W^{1/2}(\Omega \setminus \bigcup_{i=1}^{k} \tilde{B}_\rho(a_i), \mathcal{N}) \right\}.
$$

(3.7)

The function $\mathcal{E}_{g; \gamma_1, \ldots, \gamma_k}^{\text{geom}} : \text{Conf}_k \Omega \to \mathbb{R}$ is locally Lipschitz-continuous [38, Proposition 4.1]. If $(\gamma_1, \ldots, \gamma_k)$ is a minimal topological resolution of $g$, then the function $\mathcal{E}_{g; \gamma_1, \ldots, \gamma_k}^{\text{geom}}$ is bounded from below on $\text{Conf}_k \Omega$ [38, Proposition 5.6]; moreover if $\limsup_{n \to \infty} \mathcal{E}_{g; \gamma_1, \ldots, \gamma_k}^{\text{geom}}(a_1^n, \ldots, a_k^n) < +\infty$, then the singularities $a_1^n, \ldots, a_k^n$
always stay away from the boundary and from each other unless their recombination yields another minimal topological resolution of the boundary datum \( g \) [38, Proposition 6.1]. (In many examples of interest this does not happen; it occurs for instance for the torus \( S^1 \times S^1 \) and in models of superfluid \( ^3 \)He in dipole-free phase.)

The quantity \( \mathcal{E}^{\text{geom}}_{g, \gamma_1, \ldots, \gamma_k}(a_1, \ldots, a_k) \) depends on the curves \( \gamma_i \) only up to synharmonicity: if for each \( i \), the curves \( \gamma_i \) and \( \tilde{\gamma}_i \) are synharmonic, then [38, Proposition 3.10]

\[
\mathcal{E}^{\text{geom}}_{g, \gamma_1, \ldots, \gamma_k}(a_1, \ldots, a_k) = \mathcal{E}^{\text{geom}}_{g, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_k}(a_1, \ldots, a_k). \tag{3.8}
\]

3.5. Renormalised energy of renormalisable maps

A last notion from [38] that we will be using is the notion of renormalisable singular mappings and their renormalised energies [38, Definition 7.1].

**Definition 3.5.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz domain. A mapping \( u : \Omega \to \mathcal{N} \) is renormalisable whenever there exists a finite set \( \{a_1, \ldots, a_k\} \subset \Omega \) such that if \( \rho > 0 \) is small enough, \( u \in W^{1,2}(\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i), \mathcal{N}) \) and its renormalised energy is finite:

\[
\mathcal{E}^{\text{ren}}(u) := \liminf_{\rho \to 0} \int_{\Omega \setminus \bigcup_{i=1}^k \bar{B}_\rho(a_i)} \frac{|Du|^2}{2} - \sum_{i=1}^k \frac{\lambda(\text{tr} \, B_\rho(a_i) \, u)^2}{4\pi} \log \frac{1}{\rho} < +\infty.
\]

The set of renormalisable mappings is denoted by \( W^{1,2}_{\text{ren}}(\Omega, \mathcal{N}) \). For every \( u \in W^{1,2}_{\text{ren}}(\Omega, \mathcal{N}) \) one has

\[
\mathcal{E}^{\text{ren}}(u) = \lim_{\rho \to 0} \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du|^2}{2} - \sum_{i=1}^k \frac{\lambda(\text{tr} \, B_\rho(a_i) \, u)^2}{4\pi} \log \frac{1}{\rho}
\]

\[
= \sup_{\rho \in (0, \rho(a_1, \ldots, a_k))} \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du|^2}{2} - \sum_{i=1}^k \frac{\lambda(\text{tr} \, B_\rho(a_i) \, u)^2}{4\pi} \log \frac{1}{\rho}. \tag{3.9}
\]

The structure of renormalisable mappings is described in the following [38, Proposition 7.2]:

**Proposition 3.6.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz domain. If \( u \in W^{1,2}_{\text{ren}}(\Omega, \mathcal{N}) \), then either one has \( u \in W^{1,2}(\Omega, \mathcal{N}) \) or there exist \( k \in \mathbb{N}_+ \), \( (a_1, \ldots, a_k) \in \text{Conf}_k \, \Omega \) and \( \gamma_1, \ldots, \gamma_k \in C^1(S^1, \mathcal{N}) \) such that

(i) \( (\gamma_1, \ldots, \gamma_k) \) is a topological resolution of \( \text{tr} \, a \, \Omega \),
(ii) for each \( i \in \{1, \ldots, k\} \), \( \gamma_i \) is a non-trivial minimising closed geodesic,
(iii) for each \( i \in \{1, \ldots, k\} \), there exists a sequence \( (\rho_\ell)_{\ell \in \mathbb{N}} \) converging to 0 such that the sequence \( (\text{tr} \, B_\rho(u(a_i + \rho_\ell \cdot)), \gamma_i) \) converges strongly to \( \gamma_i \) in \( W^{1,2}(S^1, \mathcal{N}) \),
(iv) for each \( i \in \{1, \ldots, k\} \), \( \lim_{\rho \to 0} d_{\text{synh}}(\text{tr} \, B_\rho(u(a_i + \rho \cdot)), \gamma_i) = 0 \),
(v) \( \mathcal{E}^{\text{ren}}(u) \geq \mathcal{E}^{\text{geom}}_{g, \gamma_1, \ldots, \gamma_k}(a_1, \ldots, a_k) \).

In this case, we denote the set of singularities by \( \text{sing}(u) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\} \); in the case where \( u \in W^{1,2}(\Omega, \mathcal{N}) \), we set \( \text{sing}(u) = \emptyset \).
Stricly speaking, the set \( \text{sing}(u) \) is only defined up to synharmony of the mappings \( \gamma_1, \ldots, \gamma_k \). Given \( u \in W^{1,2}_{\text{ren}}(\Omega, \mathcal{N}) \), \( a \in \Omega \) and a minimising geodesics \( \gamma \), we have that \( (a, \gamma) \in \text{sing}(u) \) if and only if \( Du \) is not square-integrable near \( a \) and if \( \lim_{\rho \to 0} d_{\text{synh}}(\text{tr}_{S^1} u(a_i + \rho \cdot), \gamma) = 0 \). In particular, the set \( \text{sing}(u) \) is well-defined up to synharmony of minimising geodesics.

4. Minimal Energy on Disks with Boundary Conditions

We recall that \( F \) denotes the Ginzburg–Landau penalisation which satisfies \( F \in C(\mathbb{R}, [0, +\infty)) \) and \( F^{-1}([0]) = \mathcal{N} \). For every radius \( R \in (0, +\infty) \) and every curve \( \gamma \in W^{1,2}(S^1, \mathbb{R}^v) \), we set

\[
Q^R_{F, \gamma} := \inf \left\{ \int_{B_R} \frac{|Du|^2}{2} + F(u) : u \in W^{1,2}(B_R, \mathbb{R}^v) \text{ s.t. } \text{tr}_{S^1} u(R \cdot) = \gamma \right\}.
\]

(4.1)

where \( B_R \subset \mathbb{R}^2 \) is the disk of radius \( R \) centred at the origin \( 0 \in \mathbb{R}^2 \). By scaling, we have for every \( \varepsilon, R \in (0, +\infty) \)

\[
\inf \left\{ \int_{B_R} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} : u \in W^{1,2}(B_R, \mathbb{R}^v) \text{ and } \text{tr}_{S^1} u(R \cdot) = \gamma \right\} = Q^R_{F, \gamma}/\varepsilon.
\]

(4.2)

**Proposition 4.1.** If \( \gamma \in C^1(S^1, \mathcal{N}) \) is a minimising geodesic, then the map

\[
R \in (0, +\infty) \mapsto Q^R_{F, \gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log R
\]

is non-increasing.

By Proposition 4.1, for every minimising closed geodesic \( \gamma \in C^1(S^1, \mathcal{N}) \), we can define

\[
Q_{F, \gamma} := \lim_{R \to +\infty} \left( Q^R_{F, \gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log R \right) \in [-\infty, +\infty).
\]

(4.3)

When \( \mathcal{N} = S^1 \), Proposition 4.1 is due to Bethuel, Brezis and Hélein [9, Lemma III.1].

**Remark 4.2.** We shall see in Section 6 that \( Q_{F, \gamma} > -\infty \) if \( \gamma \) is an atomic minimising geodesic, i.e. \( E^{sg}(\gamma) = \frac{\lambda(\gamma)^2}{4\pi} \) (see Corollary 6.9).
Proof of Proposition 4.1. Given $0 < R < S < +\infty$, we consider a map $u \in W^{1,2}(B_R, \mathbb{R}^v)$ such that $\text{tr}_{\mathbb{S}^1} u(R \cdot) = \gamma$ on $\mathbb{S}^1$ and we define the map $v \in W^{1,2}(B_S, \mathbb{R}^v)$ for $x \in B_R$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \in B_R, \\ \gamma \left( \frac{x}{|x|} \right) & \text{if } x \in B_S \setminus B_R. \end{cases}$$

Since $\gamma$ is by assumption a minimising geodesic, we have

$$\int_{B_S} \frac{|Dv|^2}{2} + F(v) \leq \int_{B_R} \frac{|Du|^2}{2} + F(u) + \int_{\mathbb{S}^1} \frac{|\gamma'|}{2} \int_R^S \frac{dr}{r}$$

$$= \int_{B_R} \frac{|Du|^2}{2} + F(u) + \frac{\lambda(\gamma)^2}{4\pi} \log \frac{S}{R}.$$

By minimising over $u$ and by definition (4.3) of $Q_{F,\gamma}^R$, we get

$$Q_{F,\gamma}^S - \frac{\lambda(\gamma)^2}{4\pi} \log S \leq Q_{F,\gamma}^R - \frac{\lambda(\gamma)^2}{4\pi} \log R. \quad \square$$

Proposition 4.3. If $\gamma, \tilde{\gamma} \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$, then for every $R \in (0, +\infty)$, we have

$$\inf_{S \geq R} \left( Q_{F,\gamma}^S - \frac{(\tilde{\gamma})^2}{4\pi} \log S \right) \leq Q_{F,\gamma}^R - \frac{(\gamma)^2}{4\pi} \log R + d_{\text{synh}}(\gamma, \tilde{\gamma}).$$

In particular, if $\gamma$ and $\tilde{\gamma}$ are minimising geodesics, then $Q_{F,\gamma} \leq Q_{F,\gamma} + d_{\text{synh}}(\gamma, \tilde{\gamma})$. If moreover the maps $\gamma$ and $\tilde{\gamma}$ are synharmonic, then $Q_{F,\gamma} = Q_{F,\tilde{\gamma}}$.

Proof of Proposition 4.3. We can assume that the maps $\gamma$ and $\tilde{\gamma}$ are homotopic and, in particular, that $\lambda(\gamma) = \lambda(\tilde{\gamma})$ since otherwise $d_{\text{synh}}(\gamma, \tilde{\gamma}) = +\infty$.

We take $R \in (0, +\infty)$, $u \in W^{1,2}(B_R, \mathbb{R}^v)$ such that $\text{tr}_{\mathbb{S}^1} u(R \cdot) = \gamma$, $L > 0$ and $H \in W^{1,2}(\mathbb{S}^1 \times [0, L], \mathcal{N})$ such that $H(\cdot, 0) = \gamma$, $H(\cdot, L) = \tilde{\gamma}$. We define $v \in W^{1,2}(B_{e^L R}, \mathbb{R}^v)$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \in B_R, \\ H \left( \frac{x}{|x|}, \log \frac{|x|}{R} \right) & \text{if } x \in B_{e^L R} \setminus B_R. \end{cases}$$

By taking the infimum with respect to $u$ in the energy of $v$ we obtain

$$Q_{F,\tilde{\gamma}}^{e^L R} - \frac{(\tilde{\gamma})^2}{4\pi} \log(e^L R) \leq Q_{F,\gamma}^R - \frac{(\gamma)^2}{4\pi} \log R + \int_{\mathbb{S}^1 \times [0, L]} \frac{|DH|^2}{2} + \frac{L}{4\pi} \lambda(\gamma)^2,$$

and thus, by definition of synharmonicity (Definition 3.4),

$$\inf_{S \geq R} \left( Q_{F,\gamma}^S - \frac{(\gamma)^2}{4\pi} \log S \right) \leq Q_{F,\gamma}^R - \frac{(\gamma)^2}{4\pi} \log R + d_{\text{synh}}(\gamma, \tilde{\gamma}). \quad \square$$
Let \( u \in W^{1,2}_{\text{loc}}(\Omega, \mathcal{N}) \) and let \( \text{sing}(u) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\} \) be given by Proposition 3.6. We define
\[
Q_F(u) := \sum_{i=1}^k Q_{F, \gamma_i},
\]
where the quantity \( Q_{F, \gamma} \) is defined in (4.3). Finally if \( \gamma \) does not take its values in \( \mathcal{N} \) but is still close to it, the difference between \( Q^R_{F, \gamma} \) and \( Q^R_{F, \Pi_N(\gamma)} \) can be estimated as follows.

**Proposition 4.4.** If \( F \in C(\mathbb{R}^v, [0, +\infty)) \) satisfies \( F^{-1}(\{0\}) = \mathcal{N} \) and (1.4), and if \( \gamma \in W^{1,2}(S^1, \mathbb{R}^v) \) satisfies \( \text{dist}_N(\gamma(\cdot)) < \delta_N/2 \) on \( N \), then for every \( R \geq 2 \),
\[
|Q^R_{F, \gamma} - Q^R_{F, \Pi_N \circ \gamma}| \leq C \int_{S^1} \left( |\gamma'|^2 + RF(\gamma) \right).
\]

**Proof.** Given \( u \in W^{1,2}(B_R, \mathbb{R}^v) \) such that \( \text{tr}_{S^1} u(R \cdot) = \gamma \), we define \( v : B_R \to \mathbb{R}^v \) by setting for each \( x \in B_R \),
\[
v(x) = \begin{cases} u(R x) & \text{if } |x| \leq R - 1, \\
(R - |x|)\gamma(\frac{x}{|x|}) + (|x| - (R - 1))\Pi_N(\gamma(\frac{x}{|x|})) & \text{if } R - 1 \leq |x| \leq R. \end{cases}
\]
We compute that \( Dv(x) = \frac{R}{R-1} Du \left( \frac{R}{R-1} x \right) \) if \( |x| \leq R - 1 \), and if \( R - 1 \leq |x| \leq R \),
\[
|Dv(x)|^2 = |\Pi_N(\gamma(\frac{x}{|x|})) - \gamma(\frac{x}{|x|})|^2 + \frac{1}{|x|^2} |(R - |x|)\gamma(\frac{x}{|x|}) - (|x| - (R - 1))D\Pi_N(\gamma(\frac{x}{|x|}))\gamma'(\frac{x}{|x|})|^2.
\]
By smoothness and compactness the derivatives of \( \Pi_N \) are bounded in \( \mathcal{N}_{\delta_N/2} \) and we have
\[
|\gamma'(\frac{x}{|x|}) - (|x| - (R - 1))D\Pi_N(\gamma(\frac{x}{|x|}))\gamma'(\frac{x}{|x|})|^2 \leq C|\gamma'(\frac{x}{|x|})|^2.
\]
Moreover, in view of (1.4), we estimate
\[
|\Pi_N(\gamma(\frac{x}{|x|})) - \gamma(\frac{x}{|x|})|^2 \leq \text{dist}(\gamma(\frac{x}{|x|}), \mathcal{N})^2 \leq C_1 F(\gamma(\frac{x}{|x|}))
\]
and
\[
F\left( \Pi_N(\gamma(\frac{x}{|x|})) + (R - |x|)\left( \gamma(\frac{x}{|x|}) - \Pi_N(\gamma(\frac{x}{|x|})) \right) \right) \leq C_2 \text{dist}\left( \Pi_N(\gamma(\frac{x}{|x|})) + (R - |x|)\left( \gamma(\frac{x}{|x|}) - \Pi_N(\gamma(\frac{x}{|x|})) \right) \right)^2
\]
\[
\leq C_3|\Pi_N(\gamma(\frac{x}{|x|})) - \gamma(\frac{x}{|x|})|^2 \leq C_4 F(\gamma(\frac{x}{|x|})).
\]
By using a change of variables and integration in polar coordinates we arrive at
\[
\int_{B_R} \frac{|Dv|^2}{2} + F(v) \leq \int_{B_R} \frac{|Du|^2}{2} + F(u) + C_5 \left( \int_{S^1} \frac{|\gamma'|^2}{R} + \int_{S^1} RF(\gamma) \right).
\]
It follows thus that
\[ Q_{F,N_\gamma}^R \leq C_2 \left( \int_{\mathbb{S}^1} |\gamma|^2 R + RF(\gamma) \right). \]

The proof of the converse inequality is similar. \hfill \Box

### 5. Upper Bound on the Energy of Minimisers

Thanks to the singular and renormalised energies presented in Section 3 and the minimal energy on disks developed in Section 4, we establish an upper bound on the Ginzburg–Landau energy \( E^\varepsilon(u) \), defined in (1.3). In this section, \( \Omega \) is a Lipschitz bounded domain and \( F \in C(\mathbb{R}^v, [0, +\infty)) \) satisfies \( F^{-1}([0]) = \mathcal{N} \) and (1.4).

We first give an upper bound on the energy of minimisers with given Dirichlet boundary datum in terms of the infimum of the geometric renormalised energy.

**Proposition 5.1.** Let \( g \in W^{1/2,2}(\partial \Omega, \mathcal{N}) \), \( k \in \mathbb{N}_a \), \( a_1, \ldots, a_k \) be distinct points in \( \Omega \), and let \((\gamma_1, \ldots, \gamma_k)\) be a minimal topological resolution of \( g \). Then, as \( \varepsilon \to 0 \),

\[
\inf \{ E^\varepsilon_F(u) \mid u \in W^{1,2}(\Omega, \mathbb{R}^v) \text{ and } \text{tr}_{\partial \Omega} u = g \} \leq E^{\text{geom}}(g) \log \frac{1}{\varepsilon} + E^{\text{geom}}_{\gamma_1, \ldots, \gamma_k}(a_1, \ldots, a_k) + \sum_{i=1}^k Q_{F,\gamma_i} + o(1).
\]

When \( \mathcal{N} = \mathbb{S}^1 \), Proposition 5.1 is due to Bethuel, Brezis and Hélein [9, Lemma VIII.1].

**Proof of Proposition 5.1.** For every \( \rho \in (0, \bar{\rho}(a_1, \ldots, a_k)) \), we consider a map \( u_* \in W^{1,2}(\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i), \mathcal{N}) \) such that \( \text{tr}_{\partial \Omega} u_* = g \) and \( \text{tr}_{\mathbb{S}^1} u_*(a_i + \rho \cdot) = \gamma_i \) for every \( i \in \{1, \ldots, k\} \) and maps \( u_1, \ldots, u_k \in W^{1,2}(B_\rho, \mathbb{R}^v) \) such that \( \text{tr}_{\mathbb{S}^1} u_i(\rho \cdot) = \gamma_i \). We then set

\[
u(x) := \begin{cases} u_*(x) & \text{if } x \in \Omega \setminus \bigcup_{i=1}^k B_\rho(a_i), \\ u_i(x - a_i) & \text{if } x \in B_\rho(a_i) \text{ for some } i \in \{1, \ldots, k\}, \end{cases}
\]

and we have, since \( F(u_*) = 0 \) in \( \Omega \setminus \bigcup_{i=1}^k B_\rho(a_i) \),

\[
E^\varepsilon_F(u) = \int_{\Omega} \frac{|Du|^2}{2 \varepsilon^2} + \frac{F(u)}{\varepsilon^2} = \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_*|^2}{2} + \sum_{i=1}^n \int_{B_\rho} \frac{|Du_i|^2}{2 \varepsilon^2} + \frac{F(u_i)}{\varepsilon^2}.
\]

By taking the infimum over \( u_*, u_1, \ldots, u_k \), we obtain by (3.7) and (4.2),

\[
\inf \{ E^\varepsilon_F(u) \mid u \in W^{1,2}(\Omega, \mathbb{R}^v) \text{ and } \text{tr}_{\partial \Omega} u = g \} \leq E^{\text{geom}, \rho}_{\gamma_1, \ldots, \gamma_k}(a_1, \ldots, a_k) + \sum_{i=1}^k Q_{F,\gamma_i}^{\rho/\varepsilon}.
\]
By choosing now $\rho = \sqrt{\epsilon}$, we obtain

$$\inf \{ E^\epsilon_F (u) : u \in W^{1,2}(\Omega, \mathbb{R}^v) \text{ and } \text{tr}_{\rho \Omega} u = g \} - \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{1}{\epsilon}$$

$$\leq E^\text{geom, }\sqrt{\epsilon}_\rho(a_1, \ldots, a_k) - \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{1}{\sqrt{\epsilon}} + \sum_{i=1}^k Q_{F,\gamma_i}^\rho - \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{1}{\sqrt{\epsilon}},$$

and the conclusion follows by letting $\epsilon \to 0$ from the definition (3.6) of $E^\text{geom}_\rho(a_1, \ldots, a_k)$ and the definition (4.3) of $Q_{F,\gamma_i}^\rho$. $\square$

We also have an upper bound around singularities for renormalisable maps.

**Proposition 5.2.** For every $u \in W^{1,2}_\text{ren}(\Omega, \mathcal{N})$, if $\text{sing}(u) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\}$, then for every $\rho \in (0, \tilde{\rho}(a_1, \ldots, a_k))$, as $\epsilon \to 0$,

$$\inf \{ E^\epsilon_F (v) : v \in W^{1,2}(\Omega, \mathbb{R}^v) \text{ and } v = u \text{ in } \Omega \setminus \bigcup_{i=1}^k B_\rho(a_i) \}$$

$$\leq \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{1}{\epsilon} + \epsilon E^\text{ren}(u) + Q_F(u) + o(1).$$

The quantity $Q_F(u)$ has been defined in (4.4).

**Proof of Proposition 5.2.** For every $u_1, \ldots, u_k \in W^{1,2}(B_\rho, \mathbb{R}^v)$ such that $\text{tr}_{\partial \Omega} u_i (\rho \cdot) = \text{tr}_{\partial \Omega} u_i (a_i + \rho \cdot)$ for each $i \in \{1, \ldots, k\}$, if we define the function $v : \Omega \to \mathbb{R}^v$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega \setminus \bigcup_{i=1}^k B_\rho(a_i), \\ u_i(x - a_i) & \text{if for some } i \in \{1, \ldots, k\}, x \in B_\rho(a_i), \end{cases}$$

then we have

$$E^\epsilon_F(v) = \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du|^2}{2} + \sum_{i=1}^k \int_{B_\rho} \frac{|Du_i|^2}{2} + \frac{F(u_i)}{\epsilon^2},$$

and thus, by taking the infimum over $u_1, \ldots, u_k$, we obtain by (4.2),

$$\inf \{ E^\epsilon_F (v) : v \in W^{1,2}(\Omega, \mathbb{R}^v) \text{ and } v = u \text{ in } \Omega \setminus \bigcup_{i=1}^k B_\rho(a_i) \} - \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{1}{\rho}$$

$$\leq \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du|^2}{2} - \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{1}{\rho} + \sum_{i=1}^k \left( Q_{F,\gamma_i}^{\rho/\epsilon}(u_{a_i} + \rho \cdot) - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{\rho}{\epsilon} \right).$$

We conclude by the definition (3.9) of $E^\text{ren}(u)$, by the definitions (4.3) and (4.4) of the quantities $Q_{F,\gamma_i}(u)$ and $Q_F(u)$, and by Proposition 4.3 and (iv) in Proposition 3.6. $\square$
6. Lower Bounds on the Energy

We derive a lower bound for the Ginzburg–Landau energy $E_{F}^{\varepsilon}(u)$, defined in (1.3), of maps $u$ in $W^{1,2}(\Omega, \mathbb{R}^{\nu})$ with given boundary datum $\text{tr}_{\partial \Omega} u = g$ that matches the upper bound of Proposition 5.2. We first prove in Section 6.1 a lower bound of the form $E_{sg}^{g}(g) \log \frac{1}{\varepsilon} - C$ for maps in $W^{1,2}(\Omega, \mathbb{R}^{\nu})$ and for the Ginzburg–Landau energy. This lower bound along with a localisation of the energy argument allows us to prove boundedness of sequences which have their energies bounded by $E_{sg}^{g}(g) \log \frac{1}{\varepsilon} + C$ in Section 6.2. We have seen in the previous section that such a bound is satisfied by minimisers of (1.3). With the help of the compactness of minimisers we are able to improve the lower bound and obtain the desired result in Section 7.1.

In this section, $\Omega$ is a Lipschitz bounded domain and $F \in C(\mathbb{R}^{\nu}, [0, +\infty))$ satisfies $F^{-1}(\{0\}) = N$ and (1.4).

6.1. Global lower bound

The global lower bound depends on the tubular neighbourhood extension energy.

**Definition 6.1.** Let $\Omega \subset \mathbb{R}^{2}$ be a bounded Lipschitz domain and $g \in W^{1/2,2}(\partial \Omega, N)$. We define the *tubular neighbourhood extension energy* of $g$ to be

$$E_{\text{ext}}^{g} := \inf \left\{ \int_{\partial \Omega \times [0,1]} \frac{|Dv|^{2}}{2} : v \in W^{1,2}(\partial \Omega \times [0,1], N) \text{ and } \text{tr}_{\partial \Omega \times \{0\}} v = g \right\}.$$

**Proposition 6.2.** There exists a constant $C \in (0, +\infty)$, depending only on $\Omega$ and $F$, such that for every $\varepsilon > 0$ and every $u \in W^{1,2}(\Omega, \mathbb{R}^{\nu})$ with $g := \text{tr}_{\partial \Omega} u$ satisfying $g \in N$ almost everywhere on $\partial \Omega$, we have

$$E_{F}^{\varepsilon}(u) + C E_{\text{ext}}^{g} - E_{sg}^{g}(g) \log \frac{1}{C E_{sg}^{g}(g)}$$

$$\geq \frac{1}{C} \left( \frac{|D(\text{dist}_N \circ u)|^{2}}{2} + \frac{F(u)}{\varepsilon^{2}} + \sup_{t>0} t^{2} L^{2}(\{|Du|^{-1}(\{t, +\infty\})\}) \right),$$

where the last term on the left-hand side is understood to vanish when $E_{sg}^{g}(g) = 0$.

When $N = S^{1}$, Proposition 6.2, without the weak estimate on the gradient, is due to Sandier [47, Theorem 2], the corresponding weak estimate being due to Serfaty and Tice [50, Theorem 2]. In the general case, the fact that the left-hand side is non-negative is due to Canevari [15].

Proposition 6.2 will follow from a slightly refined result for smooth maps (see Lemma 6.7) together with an approximation argument. The proof of Lemma 6.7 follows Sandier’s strategy [47, Proof of Theorem 2] by an application of the coarea formula and the lower estimates for the Dirichlet energy outside a compact set of maps into a manifold, which depends on the one-dimensional Hausdorff content, whose definition and properties we recall now.
Definition 6.3. The one-dimensional Hausdorff content of a compact set \( K \subset \mathbb{R}^2 \) is defined as

\[
\mathcal{H}^1_\infty(K) := \inf \left\{ \sum_{B \in \mathcal{B}} \text{diam}(B) : K \subset \bigcup_{B \in \mathcal{B}} B \text{ and } \mathcal{B} \text{ is a finite collection of closed disks} \right\}.
\]

The one-dimensional Hausdorff content is an outer measure and is bounded from above by the Hausdorff measure

\[
\mathcal{H}^1_\infty(K) \leq \mathcal{H}^1(K) \tag{6.1}
\]

We also recall the following lemma which will be used repeatedly to transform a covering of some set by disks into a covering by closed disks with disjoint closure (see Lemma 4.1 in [48]):

Lemma 6.4. For every finite set \( \mathcal{B} \) of disks of \( \mathbb{R}^2 \), there exists a finite set \( \mathcal{B}' \) of disjoint non-empty closed disks of \( \mathbb{R}^2 \) such that

\[
\mathcal{B} = \bigcup_{B' \in \mathcal{B}'} \{ B \in \mathcal{B} : B \subseteq B' \},
\]

and

\[
\sum_{B' \in \mathcal{B}'} \text{diam}(B') = \sum_{B \in \mathcal{B}} \text{diam}(B).
\]

We finally rely on the equality between the one-dimensional Hausdorff content of a compact set and of its boundary.

Lemma 6.5. If \( K \subset \mathbb{R}^2 \) is compact, then \( \mathcal{H}^1_\infty(K) = \mathcal{H}^1_\infty(\partial K) \).

Lemma 6.5 does not hold for the Hausdorff measure; the proof of Lemma 6.5 can be seen to work when \( K \subset \mathbb{R}^n \) is compact and \( n \geq 2 \); the equality fails when \( n = 1 \) and \( K = [0, 1] \subset \mathbb{R} \).

Proof of Lemma 6.5. By monotonicity of the Hausdorff content, we have \( \mathcal{H}^1_\infty(K) \geq \mathcal{H}^1_\infty(\partial K) \). It remains thus to establish the converse inequality.

We fix \( \eta > 0 \). By definition of the Hausdorff content, there exist points \( a_1, \ldots, a_k \in \mathbb{R}^2 \) and radii \( \rho_1, \ldots, \rho_k \in (0, +\infty) \) such that \( \partial K \subseteq \bigcup_{i=1}^k B_{\rho_i}(a_i) \) and \( \sum_{i=1}^k 2\rho_i \leq \mathcal{H}^1_\infty(\partial K) + \eta \). By Lemma 6.4, we can assume that \( \bar{B}_{\rho_i}(a_i) \cap \bar{B}_{\rho_j}(a_j) = \emptyset \) if \( i, j \in \{1, \ldots, k\} \) with \( i \neq j \). We claim that \( K \subset \bigcup_{i=1}^k B_{\rho_i}(a_i) \). Indeed, assume by contradiction that there exists a point \( x \in K \setminus \bigcup_{i=1}^k B_{\rho_i}(a_i) \). Since the disks \( \bar{B}_{\rho_1}(a_1), \ldots, \bar{B}_{\rho_k}(a_k) \) are pairwise disjoint, the set \( \mathbb{R}^2 \setminus \bigcup_{i=1}^k B_{\rho_i}(a_i) \) is path-connected. Since the set \( K \) is compact, we have \( \mathbb{R}^2 \setminus (K \cup \bigcup_{i=1}^k B_{\rho_i}(a_i)) \neq \emptyset \) and there exists thus a continuous map \( \gamma \in C([0, 1], \mathbb{R}^2 \setminus \bigcup_{i=1}^k \bar{B}_{\rho_i}(a_i)) \) such that \( \gamma(0) = x \) and \( \gamma(1) \notin K \). Since the map \( \gamma \) is continuous, there exists some
$t_* \in [0, 1]$ such that $\gamma(t_*) \in \partial K$ and we would thus have $\partial K \setminus \bigcup_{i=1}^{k} B_{\rho_i}(a_i) \neq \emptyset$, which is a contradiction. We thus have

$$\mathcal{H}^1_\infty(K) \leq 2 \sum_{i=1}^{k} \rho_i \leq \mathcal{H}^1_\infty(\partial K) + \eta;$$

we conclude by letting $\eta \to 0$. \hfill $\square$

We will use the lower estimate on the Dirichlet energy of maps into a manifold proved in \cite[Theorem 5.1]{38}.

**Theorem 6.6.** For every Lipschitz bounded domain $\Omega \subset \mathbb{R}^2$, every compact set $K \subset \Omega$ such that $\mathcal{H}^1_\infty(K) > 0$ and every map $v \in W^{1,2}(\Omega \setminus K, \mathcal{N})$, we have

$$\int_{\Omega \setminus K} \frac{|Dv|^2}{2} \geq \mathcal{E}^{sg}(tr_{\partial \Omega} v) \log \frac{\text{dist}(K, \partial \Omega)}{2\mathcal{H}^1_\infty(K)}. \quad (6.2)$$

More precisely, there exists a constant $C > 0$ such that

$$\sup_{t > 0} t^2 \mathcal{L}^2\left(\{x \in \Omega \setminus K : |Dv| \geq t\}\right) \leq C \left( \int_{\Omega \setminus K} \frac{|Dv|^2}{2} - \mathcal{E}^{sg}(tr_{\partial \Omega} v) \log \frac{\text{dist}(K, \partial \Omega)}{2\mathcal{H}^1_\infty(K)} \right). \quad (6.3)$$

The left-hand side of (6.3) is the weak-$L^2$ quasi-norm of $|Dv|$. Theorem 6.6 has its roots in a corresponding estimate for maps outside a finite collection of disks \cite[Corollary II.1]{9}.

We are now ready to state a slightly refined version of Proposition 6.2 in the smooth setting.

**Lemma 6.7.** There exist constants $C \in (0, +\infty)$ and $\delta \in (0, +\infty)$ depending only on $\Omega$ and $F$, such that for every $\varepsilon > 0$ and every map $u \in C^2(\bar{\Omega}, \mathbb{R}^v)$ with $g := \text{tr}_{\partial \Omega} u$ satisfying $g(\partial \Omega) \subseteq \mathcal{N}$ and $\mathcal{E}^{sg}(g) > 0$, we have

$$\mathcal{E}^\varepsilon_F(u) + C \mathcal{E}^{ext}(g) - \mathcal{E}^{sg}(g) \log \frac{1}{C \mathcal{E}^{sg}(g)}$$

$$\geq \frac{1}{C} \left( \int_{\Omega} \frac{|D(\text{dist}_N \circ u)|^2}{2} + \frac{F(u)}{\varepsilon^2} \right) + \sup_{t > 0} t^2 \mathcal{L}^2\left(\{x \in \Omega : |Du(x)| \geq t\}\right)$$

$$+ \mathcal{E}^{sg}(g) \frac{1}{\delta} \int_0^{\delta} \Psi \left( \frac{\mathcal{H}_{\infty}^1(K_s)}{C \mathcal{E}^{sg}(g)} \right) \text{d}s,$$

where the function $\Psi : (0, +\infty) \rightarrow \mathbb{R}_+$ is defined by $\Psi(\tau) := \tau - 1 - \log \tau$ for each $\tau \in (0, +\infty)$, and where the sets $K_s$ are defined for every $s \in (0, +\infty)$ by

$$K_s := \{x \in \Omega : \text{dist}(u(x), \mathcal{N}) \geq s\}.$$
Before proving Lemma 6.7 we extend maps in \( u \in W^{1,2}(\Omega, \mathbb{R}^v) \) in the following way. In view of Definition 6.1, there exists \( \delta_{\partial\Omega} > 0 \) such that if we set
\[
\Omega_{\delta_{\partial\Omega}} := \{ x \in \mathbb{R}^2 : \text{dist}(x, \partial\Omega) < \delta_{\partial\Omega} \},
\]
then, we can extend the function \( u \in W^{1,2}(\Omega, \mathbb{R}^v) \) to a function \( u \in W^{1,2}(\Omega_{\delta_{\partial\Omega}}, \mathbb{R}^v) \) in such a way that \( u \in \mathcal{N} \) almost everywhere in \( \Omega_{\delta_{\partial\Omega}} \setminus \Omega \) and
\[
\int_{\Omega_{\delta_{\partial\Omega}} \setminus \Omega} \frac{|Du|^2}{2} \leq C_1 \mathcal{E}^{\text{ext}}(g),
\]
for some constant \( C_1 \) depending only on \( \partial\Omega \) (see [38, Lemma 5.5]).

In the rest of the present work we will always assume that maps \( u \in W^{1,2}(\Omega, \mathbb{R}^v) \) are extended to the larger domain \( \Omega_{\delta_{\partial\Omega}} \) as explained above.

**Proof of Lemma 6.7.** We proceed in several steps:

**Step 1. Splitting normal and tangential derivatives.** We set for \( x \in \Omega \setminus K_{\delta_N} \),
\[
D^\top u(x) := P_N^\top(\Pi_N^N(u(x))) \circ Du(x) \quad \text{and} \quad D^\bot u(x) := P_N^\bot(\Pi_N^N(u(x))) \circ Du(x),
\]
with the nearest point retraction \( \Pi_N^N \) and the projections \( P_N^\top \) and \( P_N^\bot \) being defined in Lemma 2.1; it holds in particular, within the set \( \Omega \setminus K_{\delta_N} \), that
\[
\left( 1 - \frac{\text{dist}_N \circ u}{\delta_N} \right) |D(\Pi_N^N \circ u)|^2 \leq |D^\top u|^2 \leq C_2 |D(\Pi_N^N \circ u)|^2,
\]
and
\[
|D(\text{dist}_N \circ u)|^2 \leq |D^\bot u|^2.
\]
We also let \( \delta_F \in (0, \delta_N) \) be a constant as in Lemma 2.3 so that for all \( y \in \mathcal{N}_{\delta_F} \), we have
\[
F(y) \geq \frac{m_F}{2} \text{dist}_N(y)^2.
\]

By orthogonality between \( P_N^\bot \) and \( P_N^\top \), we have, for every \( \delta \in (0, \delta_F] \),
\[
\mathcal{E}_F^e(u) = \int_{\Omega \setminus K_{\delta}} \left( \frac{|D^\bot u|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right) + \int_{\Omega \setminus K_{\delta}} \left( \frac{|Du|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right) =: (\text{I}) + (\text{II}) + (\text{III}).
\]

**Step 2. Estimate of (I) from below.** Since \( u \in C^2(\Omega, \mathbb{R}^v) \), by Sard’s lemma and by the implicit function theorem, for almost every \( s \in (0, +\infty) \), the set \( K_s \subset \Omega \) has a \( C^2 \) boundary and
\[
\partial K_s = \Sigma_s := \{ x \in \Omega : \text{dist}(u(x), \mathcal{N}) = s \}.
\]
Hence, using successively (6.7), Young's inequality, (6.8) and the coarea formula, we obtain

\[
(I) = \int_{\Omega \setminus K_\delta} \left( \frac{|D^+ u|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right) \geq \int_{\Omega \setminus K_\delta} \left( \frac{|D(\text{dist}_N \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right) \\
\geq \int_{\Omega \setminus K_\delta} \frac{1}{\varepsilon} |D(\text{dist}_N \circ u)| \sqrt{2F \circ u} \\
\geq \sqrt{m_F} \int_{\Omega \setminus K_\delta} \frac{1}{\varepsilon} |D(\text{dist}_N \circ u)| + (\text{dist}_N \circ u) \\
= \sqrt{m_F} \int_{0}^{\delta} \mathcal{H}^1(\Sigma_s) \frac{s}{\varepsilon} \, ds.
\]

However, by Lemma 6.5 and (6.1), we have for almost every \( s > 0 \), \( \mathcal{H}^1_{\infty}(K_s) = \mathcal{H}^1_{\infty}(\Sigma_s) \leq \mathcal{H}^1(\Sigma_s) \); hence

\[
(I) \geq \sqrt{m_F} \int_{0}^{\delta} \mathcal{H}^1_{\infty}(K_s) \frac{s}{\varepsilon} \, ds.
\]

Moreover, by Chebyshev's inequality, we have also

\[
(I) \geq \frac{1}{C_3} \left( \int_{0}^{\delta} \mathcal{H}^1_{\infty}(K_s) \frac{s}{\varepsilon} \, ds + \int_{\Omega \setminus K_\delta} \frac{|D(\text{dist}_N \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2} \\
+ \sup_{t > 0} t^2 \mathcal{L}^2(\{x \in \Omega \setminus K_\delta : |D^+ u(x)| \geq t/2\}) \right).
\]

\[(6.10)\]

**Step 3. Estimate of (II) from below.** By (6.6) and Fubini's theorem, we have

\[
(II) = \int_{\Omega \setminus K_\delta} \frac{|D^\top u|^2}{2} \geq \int_{\Omega \setminus K_\delta} \left( \frac{1}{\delta} \int_{\text{dist}_N \circ u} \right) \frac{|D(\Pi_N \circ u)|^2}{2} \\
= \int_{\Omega \setminus K_\delta} \left( \frac{1}{\delta} \int_{\text{dist}_N \circ u} \left( \int_{\Omega \setminus K_s} \frac{|D(\Pi_N \circ u)|^2}{2} \right) \, ds \right) \\
= \frac{1}{\delta} \int_{0}^{\delta} \left( \int_{\Omega \setminus K_s} \frac{|D(\Pi_N \circ u)|^2}{2} \right) \, ds.
\]

\[(6.11)\]

Then, by (6.5), we have for every \( s \in (0, \delta) \),

\[
\int_{\Omega \setminus K_s} \frac{|D(\Pi_N \circ u)|^2}{2} \geq \int_{\Omega \setminus K_s} \frac{|D(\Pi_N \circ u)|^2}{2} - C_1 \mathcal{E}^{\text{ext}}(g),
\]
while by the lower estimate on the Dirichlet energy of mappings Theorem 6.6, since \( \Pi_{\mathcal{N}} \circ u \in W^{1,2}(\Omega_{\delta_{\Omega}} \setminus K_\delta, \mathcal{N}) \), for every \( t > 0 \),

\[
\int_{\Omega_{\delta_{\Omega}} \setminus K_\delta} \frac{|D(\Pi_{\mathcal{N}} \circ u)|^2}{2} \geq \mathcal{E}^{sg}(g) \log \frac{\delta_{\Omega}}{2\mathcal{H}^1_{cc}(K_\delta)} + \frac{1}{C_4} t^2 \mathcal{L}^2((x \in \Omega \setminus K_\delta : |D(\Pi_{\mathcal{N}} \circ u)(x)| \geq t)),
\]

for some constant \( C_4 > 0 \). Since by (6.6), we have \(|D^\top u| \leq \sqrt{C_2}|D(\Pi_{\mathcal{N}} \circ u)|\), we have also

\[
\mathcal{L}^2((x \in \Omega \setminus K_\delta : |D(\Pi_{\mathcal{N}} \circ u)(x)| \geq t)) \geq \mathcal{L}^2((x \in \Omega \setminus K_\delta : |D^\top u(x)| \geq \sqrt{C_2} t)).
\]

We thus arrive at

\[
\int_{\Omega \setminus K_\delta} \frac{|D(\Pi_{\mathcal{N}} \circ u)(x)|^2}{2} \geq \mathcal{E}^{sg}(g) \frac{1}{\delta} \int_0^\delta \log \frac{\delta_{\Omega}}{2\mathcal{H}^1_{cc}(K_\delta)} \, ds - C_1 \mathcal{E}^{ext}(g)
\]

\[
+ \frac{1}{C_4} t^2 \mathcal{L}^2((x \in \Omega \setminus K_\delta : |D^\top u(x)| \geq \sqrt{C_2} t)) \, ds.
\]

By integration with respect to \( s \) over \((0, \delta)\), we obtain in view of (6.11),

\[
(\mathbf{II}) \geq \mathcal{E}^{sg}(g) \frac{1}{\delta} \int_0^\delta \log \frac{\delta_{\Omega}}{2\mathcal{H}^1_{cc}(K_\delta)} \, ds - C_1 \mathcal{E}^{ext}(g)
\]

\[
+ \frac{1}{C_4} t^2 \frac{1}{\delta} \int_0^\delta \mathcal{L}^2((x \in \Omega \setminus K_\delta : |D^\top u(x)| \geq \sqrt{C_2} t)) \, ds.
\]

By Fubini’s theorem, we compute that

\[
\frac{1}{\delta} \int_0^\delta \mathcal{L}^2((x \in \Omega \setminus K_\delta : |D^\top u| \geq \sqrt{C_2} t)) \, ds
\]

\[
= \int_{\{x \in \Omega \setminus K_\delta : |D^\top u| \geq \sqrt{C_2} t\}} \left(1 - \frac{\text{dist}_{\mathcal{N}} \circ u}{\delta}\right)
\]

\[
\geq \frac{1}{2} \mathcal{L}^2((x \in \Omega \setminus K_{\delta/2} : |D^\top u(x)| \geq \sqrt{C_2} t))
\]

and by the change of variable \( s = 2\sqrt{C_2} t \),

\[
\sup_{t > 0} t^2 \mathcal{L}^2((x \in \Omega \setminus K_{\delta/2} : |D^\top u| \geq \sqrt{C_2} t))
\]

\[
\geq \sup_{s > 0} \frac{s^2}{4C_2} \mathcal{L}^2((x \in \Omega \setminus K_{\delta/2} : |D^\top u(x)| \geq s/2)).
\]

Hence, we have proved

\[
(\mathbf{II}) \geq \mathcal{E}^{sg}(g) \frac{1}{\delta} \int_0^\delta \log \frac{\delta_{\Omega}}{2\mathcal{H}^1_{cc}(K_\delta)} \, ds - C_1 \mathcal{E}^{ext}(g)
\]

\[
+ \frac{1}{8C_2C_4} \sup_{t > 0} t^2 \mathcal{L}^2((x \in \Omega \setminus K_{\delta/2} : |D^\top u(x)| \geq t/2)).
\]

(6.12)
Step 4. Estimate of (III) from below. By (6.7) and Chebyshev’s inequality, we have

\[
(III) = \int_{K_\delta} \left( \frac{|Du|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right) \geq \frac{1}{2} \left( \int_{K_\delta} \left( \frac{|D(\text{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right) + \int_{K_\delta} \frac{|Du|^2}{2} \right)
\]
\[
\geq \frac{1}{8} \left( \int_{K_\delta} \left( \frac{|D(\text{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right) + \sup_{t>0} t^2 \mathcal{L}^2 \left( \{ x \in K_\delta : |Du(x)| \geq t \} \right) \right).
\]

(6.13)

Step 5. Putting things together. We first observe that for every \( t > 0 \),

\[
t^2 \mathcal{L}^2 \left( \{ x \in K_\delta \cup \Omega \setminus K_{\delta/2} : |Du(x)| \geq t \} \right)
\]
\[
\leq t^2 \mathcal{L}^2 \left( \{ x \in K_\delta : |Du(x)| \geq t \} \right)
\]
\[
+ t^2 \mathcal{L}^2 \left( \{ x \in \Omega \setminus K_{\delta/2} : |D^2 u(x)| \geq t/2 \} \right)
\]
\[
+ t^2 \mathcal{L}^2 \left( \{ x \in \Omega \setminus K_{\delta/2} : |D^\perp u(x)| \geq t/2 \} \right),
\]

which in view of (6.9), by adding (6.10), (6.12) and (6.13), gives the existence of a constant \( C_5 > 0 \) such that

\[
\mathcal{E}_F^e(u) \geq \mathcal{E}^\text{sg}(g) \frac{1}{\delta} \int_0^\delta \left( \frac{\delta H_\infty^1(K_s) s}{C_3 \mathcal{E}^\text{sg}(g)} + \log \frac{\delta \Omega}{2\mathcal{H}_\infty^1(K_s)} \right) ds - C_1 \mathcal{E}^\text{ext}(g)
\]
\[
+ \frac{1}{C_5} \left( \int_\Omega \frac{|D(\text{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right)
\]
\[
+ \sup_{t>0} t^2 \mathcal{L}^2 \left( \{ x \in K_\delta \cup \Omega \setminus K_{\delta/2} : |Du(x)| \geq t \} \right).
\]

(6.14)

Applying the identity \( \tau = 1 + \log \tau + \Psi(\tau) \) to \( \tau = \frac{\delta H_\infty^1(K_s) s}{C_3 \mathcal{E}^\text{sg}(g)} \), we obtain

\[
\frac{\delta H_\infty^1(K_s) s}{C_3 \mathcal{E}^\text{sg}(g)} + \log \frac{\delta \Omega}{2\mathcal{H}_\infty^1(K_s)} = 1 + \log \frac{\delta \Omega}{2C_3 \mathcal{E}^\text{sg}(g)} + \Psi \left( \frac{\delta H_\infty^1(K_s) s}{C_3 \mathcal{E}^\text{sg}(g)} \right),
\]

and we compute that

\[
\frac{1}{\delta} \int_0^\delta \left( 1 + \log \frac{\delta \Omega}{2C_3 \mathcal{E}^\text{sg}(g)} \right) ds = \log \frac{\delta^2 \delta \Omega}{2C_3 \mathcal{E}^\text{sg}(g)}.
\]

Hence, there exists a constant \( C > 0 \) such that

\[
\mathcal{E}_F^e(u) + C \mathcal{E}^\text{ext}(g) - \mathcal{E}^\text{sg}(g) \log \frac{1}{C \mathcal{E}^\text{sg}(g)}
\]
\[
\geq \frac{1}{C} \left( \int_\Omega \frac{|D(\text{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2} \right)
\]
\[
+ \sup_{t>0} t^2 \mathcal{L}^2 \left( \{ x \in K_\delta \cup \Omega \setminus K_{\delta/2} : |Du(x)| \geq t \} \right)
\]
\[
+ \mathcal{E}^\text{sg}(g) \frac{1}{\delta} \int_0^\delta \Psi \left( \frac{\mathcal{H}_\infty^1(K_s) s}{C \mathcal{E}^\text{sg}(g)} \right) ds \right).
\]

(6.15)
Since we have
\[
L^2\left(\{x \in \Omega : |Du(x)| \geq t\}\right) \leq L^2\left(\{x \in K_{\delta_F} \cup \Omega \setminus K_{\delta_F/2} : |Du(x)| \geq t\}\right) + L^2\left(\{x \in K_{\delta_F/2} \cup \Omega \setminus K_{\delta_F/4} : |Du(x)| \geq t\}\right),
\]
the desired estimate follows by taking the average of (6.15) for \(\delta \in \{\delta_F, \frac{\delta_F}{2}\}\).

Proof of Proposition 6.2. If the Ginzburg–Landau functional is continuous with respect to the \(W^{1,2}\) strong convergence, the conclusion follows from Lemma 6.7 and an approximation argument.

If the Ginzburg–Landau functional is not continuous, we consider a non-decreasing sequence \((F^\ell)_{\ell \in \mathbb{N}}\) of bounded and continuous functions coinciding with \(F\) in a neighbourhood of \(\mathcal{N}\) and converging to \(F\) almost everywhere. The proposition holds for each of these functions, since \(E^\varepsilon_F\) is continuous for the \(W^{1,2}\) strong convergence by Lebesgue’s dominated convergence. The constants appearing only depend on \(\delta_{F^\ell}\) and \(m_{F^\ell}\), which by construction of \(F^\ell\) are independent on \(\ell\). Hence the conclusion holds with the same constant for each \(F^\ell\), and the conclusion then holds by Lebesgue’s monotone convergence theorem.

As a consequence of Proposition 6.2, we obtain a lower estimate on Ginzburg–Landau energy in terms of the Ginzburg–Landau energy on the boundary.

Corollary 6.8. There exists a constant \(C \in (0, +\infty)\), depending only on \(\Omega\) and \(F\), such that for every \(\varepsilon > 0\) and every \(u \in W^{1,2}(\Omega, \mathbb{R}^v)\) with \(g := \text{tr}_{\partial\Omega} u \in W^{1,2}(\partial\Omega, \mathbb{R}^v)\) and \(\text{dist}(g, \mathcal{N}) \leq \delta_N\), we have
\[
E^\varepsilon_F(u) + CE^\varepsilon_{F,\partial\Omega}(g) \geq E^{sg}(\Pi_N \circ g) \frac{1}{C\varepsilon E^{sg}(\Pi_N \circ g)}.
\]

Here, \(E^\varepsilon_{F,\partial\Omega}\) is the boundary Ginzburg–Landau functional, defined for \(g \in W^{1,2}(\partial\Omega, \mathbb{R}^v)\) by
\[
E^\varepsilon_{F,\partial\Omega}(g) := \int_{\Omega} \frac{|g'|^2}{2} + \frac{F(g)}{\varepsilon^2}.
\]

Proof of Corollary 6.8. We define \(\tilde{\Omega} := \Omega_\delta\) and \(\tilde{u} : \tilde{\Omega} \to \mathbb{R}^v\) by
\[
\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ (1 - \frac{\text{dist}(x, \partial\Omega)}{\delta})u(\Pi_{\partial\Omega}(x)) + \frac{\text{dist}(x, \partial\Omega)}{\delta} \Pi_{\partial\Omega}u(\Pi_{\partial\Omega}(x)) & \text{otherwise.} \end{cases}
\]

where \(\Pi_{\partial\Omega}\) is a retraction of \(\Omega_\delta \setminus \Omega\) on \(\partial\Omega\). By assumption (1.4) on \(F\), we have
\[
E^\varepsilon(\tilde{u}) \leq E^\varepsilon_F(u) + C_1 E^\varepsilon_{F,\partial\Omega}(g). \tag{6.16}
\]
Moreover, if \(\tilde{g} := \text{tr}_{\partial\Omega} \tilde{u}\), we have
\[
E^\varepsilon(\tilde{g}) \leq C_2 E^\varepsilon_F(g). \tag{6.17}
\]

The conclusion then follows from Proposition 6.2 applied to the function \(\tilde{u}\) and from the estimates (6.16) and (6.17).
As a consequence of Proposition 6.2, we obtain the finiteness of the quantity $Q_{F, \gamma}$ defined in (4.3), when $\gamma$ is an atomic minimising geodesic, i.e. when $E^{\text{sg}}(\gamma) = \lambda(\gamma)^2/(4\pi)$.

**Corollary 6.9.** Let $F \in C(\mathbb{R}^v, [0, +\infty))$. If $F^{-1}([0]) = \mathcal{N}$, if $F$ satisfies (1.4) and if $\gamma \in C^1(S^1, \mathcal{N})$ is an atomic minimising geodesic, then $Q_{F, \gamma} > -\infty$.

**Proof.** By Proposition 6.2 applied to $\Omega_1 = B_1$, the unit disk with centre 0 in $\mathbb{R}^2$, there exists a constant $C \in (0, +\infty)$ such that for every $\varepsilon > 0$ and every $u \in W^{1,2}(B_1, \mathbb{R}^v)$ we have that

$$\int_{B_1} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq E^{\text{sg}}(\text{tr}_{B_1} u) \log \frac{1}{C_1 E^{\text{sg}}(\text{tr}_{B_1} u)} - C_1 E^{\text{ext}}(\text{tr}_{B_1} u).$$

By taking the infimum over $u$ such that $\text{tr}_{B_1} u = \gamma$, we obtain, in view of (4.2), with $\rho = \frac{1}{\varepsilon}$,

$$Q_{F, \gamma}^\rho - E^{\text{sg}}(\gamma) \log \rho \geq E^{\text{sg}}(\gamma) \log \frac{1}{C_1 E^{\text{sg}}(\gamma)} - C_1 E^{\text{ext}}(\gamma).$$

The claim follows from (4.3) since, by assumption, $E^{\text{sg}}(\gamma) = \frac{\lambda(\gamma)^2}{4\pi}$.

**6.2. Localised lower bound on the energy**

The next proposition provides some information on the localisation of the energy of mappings satisfying a logarithmic bound.

**Proposition 6.10.** There exists a constant $C \in (0, +\infty)$ such that for every $\kappa \in (0, +\infty)$, $\eta \in (0, 1/C)$, $\gamma \in (0, 1)$, $\varepsilon \in (0, +\infty)$ and $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ such that

$$E^{\text{sg}}(g) > 0, \quad Ce^{C\gamma \kappa (E^{\text{sg}}(g)\varepsilon)^{1-\gamma}} \leq \gamma \eta,$$

$$CE^{\text{sg}}(g)\varepsilon \leq \gamma \eta, \quad C\varepsilon \kappa \leq 1 \quad \text{and} \quad E^{\text{ext}}(g) \leq \kappa,$$

if $u \in W^{1,2}(\Omega, \mathbb{R}^v)$ satisfies $\text{tr}_{\partial\Omega} u = g$ and

$$\int_{\Omega} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \leq E^{\text{sg}}(g) \log \frac{1}{\varepsilon E^{\text{sg}}(g)} + \kappa,$$

then, if we still denote by $u$ the extension to $\Omega_{\delta\Omega}$ satisfying (6.5), there exists a collection of disks $B$ in $\mathbb{R}^2$ with

(i) for every $B \in B$, $\text{diam}(B) \leq 2\eta$ and $\bar{B} \subset \Omega_{\delta\Omega}$,

(ii) for every $B \in B$, $\text{dist}_{\mathcal{N}} \circ \text{tr}_{B} u < \delta_{\mathcal{N}}$, the map $\Pi_{\mathcal{N}} \circ \text{tr}_{B} u$ is not homotopic to a constant and the maps $(\Pi_{\mathcal{N}} \circ \text{tr}_{B} u)_{B \in B}$ are a topological resolution of $g = \text{tr}_{\partial\Omega} u$.  

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(iii) for every subset $B' \subset B$,
\[
\int_{\Omega \cap \bigcup_{B \in B'} B} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \sum_{B \in B'} \mathcal{E}^\text{sg}(\Pi_N \circ \partial_B u) \log \frac{\gamma \eta}{C \mathcal{E}^\text{sg}(g) \varepsilon} - C(\kappa + \mathcal{E}^\text{sg}(g)),
\]

(iv) one has
\[
\frac{\text{sys}(\mathcal{N})^2}{4\pi} \# B \leq \sum_{B \in B} \mathcal{E}^\text{sg}(\Pi_N \circ \partial_B u) \leq \mathcal{E}^\text{sg}(g) + \frac{(\log \frac{C}{\gamma \eta} + C\varepsilon)\mathcal{E}^\text{sg}(g) + (1 + C\varepsilon)\kappa}{\log \frac{\gamma \eta}{C\mathcal{E}^\text{sg}(g)}}.
\]

In the statement of Proposition 6.10, $\text{sys}(\mathcal{N})$ denotes the systole of the manifold $\mathcal{N}$ defined in (3.4) as the shortest length of a closed geodesic which is not homotopic to a constant.

Proposition 6.10 has its roots in lower bounds for minimisers of the Ginzburg–Landau energy when $\mathcal{N} = S^1$ [9, Theorem V.2]; localised lower bounds for $\mathcal{N} = S^1$ are originally due to Sandier [47, Theorem 3'] and Jerrard [34, Theorem 1.2].

We follow in our proof the Jerrard’s strategy [34] (see also the recent work by Ignat and Jerrard [32]). As a first tool to prove Proposition 6.10, we have a Sobolev type embedding theorem with dependence on $\varepsilon$ for maps defined on $S^1_r := \partial B_r$, the circle of radius $r$ centred at the origin in $\mathbb{R}^2$ (see also [34, Lemma 2.3]).

**Lemma 6.11.** There exists a constant $C > 0$ such that for every $r > 0$, every $h \in W^{1,2}(S^1_r, \mathbb{R})$ and every $\varepsilon \in (0, r]$,
\[
\|h\|_{L^{\infty}(S^1_r)}^2 \leq C \int_{S^1_r} \varepsilon |h'|^2 + \frac{1}{\varepsilon} h^2.
\]

**Proof.** By Morrey–Sobolev embedding, the function $h$ is continuous on $S^1_r$. By the mean value theorem, there exists a point $a \in S^1_r$ such that $h(a)^2 = \frac{1}{2\pi r} \int_{S^1_r} h^2$. By the fundamental theorem of calculus we can write
\[
h(x)^2 = h(a)^2 + \int_{t_a}^{t_x} \left((h \circ \gamma)^2\right)' \, dt
\]
where $\gamma$ is a smooth path on $S^1_r$ such that $\gamma(t_a) = a$, $\gamma(t_x) = x$ and $|\gamma'| \equiv 1$.

Thus, for any $C > 0$, by using Young’s inequality $2|h'| \leq C\varepsilon|h'|^2 + \frac{|h|^2}{C\varepsilon}$ and by recalling that $\varepsilon \leq r$ we find
\[
\|h^2\|_{L^{\infty}(S^1_r)} \leq \frac{1}{2\pi r} \int_{S^1_r} h^2 + \int_{S^1_r} |(h^2)'| \leq \int_{S^1_r} C\varepsilon|h|^2 + \left(\frac{1}{2\pi} + \frac{1}{C}\right) \frac{1}{\varepsilon} h^2.
\]

The conclusion follows by taking $C = \frac{1}{4\pi}$, which solves $C = \frac{1}{2\pi} + \frac{1}{\varepsilon}$. □
The next tool for the proof of Proposition 6.10, is a lower bound on the Ginzburg–Landau energy on circles at scales larger than $\varepsilon$. (When $\mathcal{N} = S^1$, see [34, Proof of Proposition 3.1, Claim 1]).

**Lemma 6.12.** There exists a constant $c_1 > 0$, such that for every $r > 0$, for every $u \in W^{1,2}(S^1_r, \mathbb{R}^n)$ such that $\text{dist}(u, \mathcal{N}) < \delta_{\mathcal{N}}$ almost everywhere in $S^1_r$, and for every $\varepsilon < r$, one has

$$\int_{S^1_r} \frac{|u'|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \frac{1}{c_1} \frac{\varepsilon}{\lambda(\Pi_{\mathcal{N}} \circ u)^2} + \frac{4\pi r}{\delta_{\mathcal{N}}^2}.$$

We remark that the right-hand side in the inequality of Lemma 6.12 is an increasing function of $c_1$. The proof of Lemma 6.12 relies on the following elementary inequality:

**Lemma 6.13.** For every $z \in [0, 1]$ and $\alpha \in (0, +\infty)$, one has

$$1 - \frac{z}{\alpha} + z^2 \geq \frac{1}{\alpha + 1}.$$

**Proof.** If $\alpha \geq \frac{1}{2}$, then the left-hand side in the desired inequality is minimal for $z = \frac{1}{2\alpha} \in [0, 1]$ and we thus obtain that for every $z \in [0, 1], \frac{1-z}{\alpha} + z^2 \geq \frac{1}{\alpha} - \frac{1}{4\alpha^2} \geq \frac{1}{\alpha + 1}$; if $\alpha < \frac{1}{2}$, we have $\frac{1-z}{\alpha} + z^2 \geq 2(1 - z) + z^2 = 1 + (1 - z)^2 \geq 1 \geq \frac{1}{\alpha + 1}$. $\square$

**Proof of Lemma 6.12.** Since by assumption $\text{dist}(u, \mathcal{N}) < \delta_{\mathcal{N}}$ almost everywhere on $S^1_r$ and since the function $F$ satisfies the non-degeneracy assumption (1.4), we have by Lemma 2.1 and by (2.3),

$$\int_{S^1_r} \frac{|u'|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \left(1 - \frac{\text{dist}_{\mathcal{N}} \circ u}{\delta_{\mathcal{N}}}ight) \frac{|(\Pi_{\mathcal{N}} \circ u)'|^2}{2} + \frac{|(\text{dist}_{\mathcal{N}} \circ u)'|^2}{2} + \frac{m_F}{2\varepsilon^2} (\text{dist}_{\mathcal{N}} \circ u)^2,$$

almost everywhere on $S^1_r$. If we set $\theta := \|\text{dist}_{\mathcal{N}} \circ u\|_{L^\infty(S^1_r)} \in [0, \delta_{\mathcal{N}}]$, we have on the one hand, by definition of $\theta$ and by the characterisation of $\lambda(\Pi_{\mathcal{N}} \circ u)$ (see (3.3)),

$$\int_{S^1_r} \left(1 - \frac{\text{dist}_{\mathcal{N}} \circ u}{\delta_{\mathcal{N}}}ight) \frac{|(\Pi_{\mathcal{N}} \circ u)'|^2}{2} \geq \left(1 - \frac{\theta}{\delta_{\mathcal{N}}}ight) \frac{\lambda(\Pi_{\mathcal{N}} \circ u)^2}{4\pi r}, \quad (6.19)$$

and on the other hand, by Lemma 6.11,

$$\int_{S^1_r} \frac{|(\text{dist}_{\mathcal{N}} \circ u)'|^2}{2} + \frac{m_F}{2\varepsilon^2} (\text{dist}_{\mathcal{N}} \circ u)^2 \geq \frac{\theta^2}{C_1 \varepsilon}, \quad (6.20)$$
for some constant $C_1 > 0$. It follows thus from (6.19) and (6.20) that if $c_1 \leq \delta_N^2 / C_1$, by applying Lemma 6.13 with $z = \theta / \delta_N$, since $\theta \leq \delta_N$

$$\int_{\Omega} |u|^2 / 2 + F(u) / \varepsilon^2 \geq \left( 1 - \frac{\theta}{\delta_N} \right) \frac{\lambda(\Pi_{\Omega} \circ u)^2}{4\pi r} + \frac{\theta^2}{C_1\varepsilon} \geq \frac{c_1}{\varepsilon} \left( \left( 1 - \frac{\theta}{\delta_N} \right) \frac{\lambda(\Pi_{\Omega} \circ u)^2\varepsilon}{4\pi rc_1} + \left( \frac{\theta}{\delta_N} \right)^2 \right)$$

$$\geq \frac{c_1}{\varepsilon} \left( \frac{4\pi rc_1}{\lambda(\Pi_{\Omega} \circ u)^2} + 1 \right) = \frac{1}{\varepsilon c_1} + \frac{4\pi r}{\lambda(\Pi_{\Omega} \circ u)^2}. \quad \square$$

A last tool is the following lower bound on the energy inside the disk $B_r$ of radius $r$ centered at the origin, with non-trivial boundary conditions:

**Lemma 6.14.** There exists a constant $c_2 > 0$, such that if $r > 0$, if $u \in W^{1,2}(B_r, \mathbb{R}^v)$ satisfies $\|\text{dist}(\text{tr}_{\partial B_r} u(\cdot), \mathbb{N})\|_{L^\infty(\partial B_r)} < \delta_N$ and if $\int_{B_r} |Du|^2 \leq c_2$, then the map $\Pi_{\mathbb{N}} \circ \text{tr}_{\partial B_r} u$ is homotopic to a constant map.

**Proof.** We have by chain rule and the trace theorem

$$\|\Pi_{\mathbb{N}} \circ \text{tr}_{\partial B_r} u\|_{\tilde{W}^{1/2,2} (\partial B_r)} \leq C_1 \|\text{tr}_{\partial B_r} u\|_{\tilde{W}^{1/2,2} (\partial B_r)} \leq C_2 \|Du\|_{L^2(B_r)}.$$ 

On the other hand, if $\|\Pi_{\mathbb{N}} \circ \text{tr}_{\partial B_r} u\|_{\tilde{W}^{1/2,2} (\partial B_r)}$ is small enough, then $\Pi_{\mathbb{N}} \circ \text{tr}_{\partial B_r} u$ is homotopic in $\text{VMO}(\partial B_r, \mathbb{N})$ to a constant map see [13, Lemma A.19]. \quad \square

**Proof of Proposition 6.10.** We first consider the case where $u \in C^2(\bar{\Omega}, \mathbb{R}^v)$ with $\text{tr}_{\partial\Omega} u = g$. We recall that, in view of Definition 6.1, we have assumed that the function $u$ is extended to a function $u \in W^{1,2}(\Omega_{\delta\Omega}, \mathbb{R}^v)$ in such a way that $u \in \mathbb{N}$ almost everywhere in $\Omega_{\delta\Omega} \setminus \Omega$ and

$$\int_{\Omega_{\delta\Omega} \setminus \Omega} |Du|^2 / 2 \leq C_1 \mathcal{E}^{\text{ext}}(g).$$

By Lemma 6.7, there exist constants $C_2$ and $\delta \in (0, \delta_N)$, depending on $F$ and $\Omega$ only, such that

$$\mathcal{E}^{\text{sg}}(g) \frac{1}{\delta} \int_0^\delta \Psi \left( \frac{\mathcal{H}^1_\infty(K_\delta)}{2C_2\varepsilon\mathcal{E}^{\text{sg}}(g)} \right) ds \leq C_2(\kappa + \mathcal{E}^{\text{sg}}(g)).$$

Since for every $\tau \in (0, +\infty)$, one has $\Psi(\tau) = \tau - 1 - \log \tau \geq \frac{\tau}{2} - \log 2$, we deduce that

$$\frac{1}{\delta} \int_0^\delta \frac{\mathcal{H}^1_\infty(K_\delta)}{2C_2\varepsilon} ds \leq C_2(\kappa + \mathcal{E}^{\text{sg}}(g)) + \mathcal{E}^{\text{sg}}(g) \log 2 \leq (C_2 + \log 2)(\kappa + \mathcal{E}^{\text{sg}}(g)) \quad (6.21)$$

and then, by monotonicity of the Hausdorff content, that

$$\mathcal{H}^1_\infty(K_\delta) \leq \frac{2}{\delta^2} \int_0^\delta \mathcal{H}^1_\infty(K_\delta) s ds \leq C_3\varepsilon(\kappa + \mathcal{E}^{\text{sg}}(g)). \quad (6.22)$$
Since the set \( K_\delta \subset \Omega \subset \Omega_{\delta_{0}\Omega} \) is compact, by definition of the Hausdorff content (Definition 6.3) and by Lemma 6.4, there exists a family of disks \( B_0 \) with disjoint closures such that
\[
K_\delta \subset \bigcup_{B_\rho(a) \in B_0} \bar{B}_\rho(a),
\]
and
\[
\sum_{B_\rho(a) \in B_0} 2\rho \leq 2 \mathcal{H}^{1}_{\infty}(K_\delta) \leq 2C_3\varepsilon(\kappa + \mathcal{E}^{sg}(g)). \tag{6.23}
\]
In particular, if we assume that
\[
2C_3\varepsilon(\kappa + \mathcal{E}^{sg}(g)) \leq \delta_{\partial \Omega}/2, \tag{6.24}
\]
and if, without loss of generality, all the disks of \( B_0 \) intersect \( K_\delta \), then the disks of \( B_0 \) are all contained in \( \Omega_{\delta_{0}\Omega}/2 \). We define
\[
\bar{s} := \sup \left\{ s \in [0, +\infty) : \frac{s}{\log(1 + s)} \varepsilon \left( \frac{\mathcal{E}^{sg}(g)}{c_0} \log \frac{C_4}{\mathcal{E}^{sg}(g)} \varepsilon + C_5\kappa \right) \leq \frac{\delta_{\partial \Omega}}{4} \right\},
\]
where \( c_0 = \min\{c_1, c_2\} \) with \( c_1, c_2 \) defined in Lemma 6.12, Lemma 6.14,
\[
C_4 = e^{2C_3c_0} \quad \text{and} \quad C_5 := \frac{1}{c_0} + 2C_3. \tag{6.25}
\]
We claim that for every \( s \in [0, \bar{s}) \), there exists a collection of disks \( B(s) \) such that
(a) the closure of the disks in \( B(s) \) are disjoint and contained in \( \Omega_{\delta_{0}\Omega} \),
(b) if \( t \in [0, s) \), then \( \bigcup_{B_\rho(b) \in B(t)} B_\rho(b) \subset \bigcup_{B_\rho(a) \in B(s)} B_\rho(a) \),
(c) for every \( B_\rho(a) \in B(s) \), \( \text{dist}_{\mathcal{N}} \circ \text{tr}_{\partial B_\rho(a)} u < \delta_{\mathcal{N}} \) and
\[
\rho \geq \frac{\varepsilon \mathcal{E}^{sg}(\text{tr}_{\partial B_\rho(a)} u)}{c_0},
\]
(d) for every \( B_\rho(a) \in B(s) \),
\[
\int_{B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \frac{c_0}{\varepsilon} \left( \frac{\rho}{s} \log(1 + s) \right) - \sum_{B_\rho(b) \in B_0 \atop B_\rho(b) \subset B_\rho(a)} \sigma.
\]
In order to construct this collection of disks \( B(s) \) for \( s \in [0, \bar{s}) \) we first set \( B(0) := B_0 \). We have showed that (a) holds for \( s = 0 \) provided (6.24) holds; the assertion (b) holds trivially when \( s = 0 \) and (c) holds since every connected component of \( K_\delta \) is contained in a unique disk of \( B_0 \). Finally for (d), we observe that when \( s \to 0 \) the limit of the right-hand side vanishes. By continuity, we can take \( B(s) = B(0) \) for \( s > 0 \) close enough from 0. We assume now that the assertions (a), (b), (c) and (d) are satisfied for some \( s_* \in (0, \bar{s}) \). We define then the set of disks
\[
B_* := \{ B_\rho(a) \in B(s_*) : \text{equality holds in (c)} \}.\]
These disks are referred to as minimising disks.

The first step consists in an expansion phase: we let the radii of the minimising disks grow by defining, for $s \geq s_*$,

$$\mathcal{B}(s) := \{ B_{ps/s_*}(a) : B_\rho(a) \in \mathcal{B}_s \} \cup (\mathcal{B}(s) \setminus \mathcal{B}_s) .$$

and the number

$$s^* := \sup \{ \sigma \in [s_*, \tilde{s}] : \text{for each } s \in [s_*, \sigma) \text{ (a) holds,}$$

strict inequality holds in (c) for each $B_\rho(a) \in \mathcal{B}(s) \setminus \mathcal{B}_s$

and $\varepsilon \notin (\rho, \tilde{s}/s_*)$ for each $B_\rho(a) \in \mathcal{B}(s)$.}

We check that for $s_* \leq s \leq \tilde{s}$ the families of disks $\mathcal{B}(s)$ satisfy (a), (b), (c), (d). By construction, the assertions (a) and (c) hold for every $s \in [s_*, s^*)$. Property (b) is also satisfied. We now prove (d). If $B_\rho(a) \in \mathcal{B}(s) \setminus \mathcal{B}_s$, (d) is true by assumption. If $B_\rho(a) \in \mathcal{B}_s$, we first consider the case where $\rho s^*/s_* \leq \varepsilon$. Then since equality holds in (c) and since maps homotopic to a constant have zero singular energy $\mathcal{E}^s$, the map $\text{tr}_B \rho(a) : \Pi_\mathcal{N} \circ u$ is not homotopic to a constant. Hence for every $s \in [s_*, s^*)$, the map $\text{tr}_B \rho(a) : \Pi_\mathcal{N} \circ u$ is not homotopic to a constant and satisfies by Lemma 6.14, since $\rho s^*/s_* \leq \varepsilon$,

$$\int_{B_{ps/s_*}(a)} |Du|^2 + \frac{F(u)}{\varepsilon^2} \geq \int_{B_{ps/s_*}(a)} |Du|^2 \geq c_0 \geq \frac{c_0 \rho s^*}{\varepsilon s_*}$$

$$\geq \frac{c_0 \rho \log(1 + s)}{s_*} - \sum_{B_\sigma(b) \subset B_{ps/s_*}(a)} \sigma .$$

In the last inequality we have used the inequality $\log(1 + s) \leq s$. If $B_\rho(a) \in \mathcal{B}_s$ and $\rho s^*/s_* > \varepsilon$ we have, from the definition of $s^*$, that $\rho > \varepsilon$. Then, if $B_\rho(a) \in \mathcal{B}_s$, we apply Lemma 6.12, since $\text{dist}_\mathcal{N} \circ u < \delta_\mathcal{N}$ on $B_{ps/s_*}(a) \setminus B_\rho(a) \subset \Omega_{\delta_\mathcal{N}} \setminus K_{\delta_\mathcal{N}}$ and since $\mathcal{E}^s(\Pi_\mathcal{N} \circ \text{tr}_B \rho(a) \circ u) = \frac{\rho c_0}{s_*}$ for $t \in (\rho, \rho s^*/s_*)$

$$\int_{B_{ps/s_*}(a) \setminus B_\rho(a)} |Du|^2 + \frac{F(u)}{\varepsilon^2} \geq \frac{c_0 \rho s^*}{\varepsilon s_*} \int_{\rho}^{\rho s^*/s_*} \frac{1}{\varepsilon^2} \frac{1}{\rho} \frac{1}{s_*} + t \, dt \geq \frac{c_0 \rho \log(1 + s)}{s_*} - \sum_{B_\sigma(b) \subset B_\rho(a)} \sigma .$$

We use that from (d), for any $B_\rho(a) \in \mathcal{B}(s_*)$ we have

$$\rho \frac{\log(1 + s_*)}{s_*} \leq \sum_{B_\sigma(b) \subset B_\rho(a)} \sigma + \varepsilon \frac{\int_{B_\rho(a)} |Du|^2 + F(u)}{c_0} .$$

(6.26)
and therefore by (6.26) and (6.27)
\[ \int_{B_{\rho s/s}(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \frac{c_0}{\varepsilon} \left( \rho \frac{\log(1+s)}{s} - \sum_{B_\rho(b) \subseteq B_\rho(a)} \sigma \right). \]

Moreover, we deduce from (d), from our assumption (6.18) and from (6.23) that
\[ \sum_{B_\rho(a) \in \mathcal{B}(s)} \rho \leq \frac{s}{\log(1+s)} \left( \frac{\varepsilon}{c_0} \int_{\Omega} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} + \sum_{B_\rho(b) \subseteq B_\rho(a)} \sigma \right) \leq \frac{s}{\log(1+s)} \left( \frac{\varepsilon s^*}{c_0} \log \left( \frac{C_4}{\varepsilon s^*} \log \left( \frac{C_5 \varepsilon}{s^*} \right) \right) \right), \tag{6.28} \]
in view of the definition of \( C_4 \) and \( C_5 \) in (6.25). Thus we find a collection the desired collection of disks \( B(s) \) for \( 0 \leq s < s^* \). In order to define \( B(s^*) \), we set
\[ B^* := \{ B_{\rho s/s}(a) : B_\rho(a) \in \mathcal{B}_s \} \cup \mathcal{B}(s^*) \setminus \mathcal{B}_s. \]

We first note that by (6.28), since \( s < \tilde{s} \), we have \( \bigcup_{B_\rho(a) \in B^*} B_\rho(a) \subseteq \Omega_{\delta \kappa}/2 \). We also note that the family \( B^* \) satisfies all the desired properties except that some disks can have boundaries that intersect each other. If this is the case we perform then a disk merging procedure by Lemma 6.4 and we define \( B(s^*) \) to be the resulting disk collection. By (c), for every \( B_\rho(a) \in \mathcal{B}(s^*) \), we have
\[ \rho = \sum_{B_\rho(b) \subseteq B_\rho(a)} \sigma \geq \sum_{B_\rho(b) \subseteq B_\rho(a)} \frac{\varepsilon s^*}{c_0} \mathcal{E}^{\text{sg}}(\text{tr}_\partial B_\sigma(b) v) \geq \frac{\varepsilon s^*}{c_0} \mathcal{E}^{\text{sg}}(\text{tr}_\partial B_\rho(b) v), \]
so that assertion (c) still holds for the modified collection of disks. We also have, since \( B^* \) satisfies (d),
\[ \int_{B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \sum_{B_\rho(b) \subseteq B_\rho(a)} \int_{B_\rho(b)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \]
\[ \geq \sum_{B_\rho(b) \subseteq B_\rho(a)} \frac{c_0}{\varepsilon} \left( \rho \frac{\log(1+s)}{s} - \sum_{B_\tau(c) \subseteq B_\rho(a)} \tau \right) \]
\[ = \frac{c_0}{\varepsilon} \left( \rho \frac{\log(1+s)}{s} - \sum_{B_\tau(c) \subseteq B_\rho(a)} \tau \right), \]
and hence assertion (d) also holds for the modified collection of disks. We can then continue alternatively with expansion phases and merging steps. Since at each step either the number of disks decreases or the number of disks with equality in (c) increases, we fill the full announced interval of \([0, \tilde{s}]\) in a finite number of steps.

In order to conclude if
\[ \eta \geq 2\varepsilon \mathcal{E}^{\text{sg}}(g)/(\gamma c_0), \tag{6.29} \]
we set
\[ \tilde{s} := \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} - 1 \geq 1 \]
so that
\[ \frac{\tilde{s}}{\log(1 + \tilde{s})} \geq \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} \]
\[ \leq \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} - 1 \geq \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} + C_5 \kappa \]
so that
\[ \log \left( \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} + C_4 \kappa \right) \]
\[ \leq \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} - 1 \geq \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} + C_5 \kappa \]
provided that
\[ (C_4 \varepsilon^{C_5 \kappa})^{\gamma} (E^{sg}(g) \varepsilon)^{1 - \gamma} \leq c_0 \gamma\eta. \] (6.30)
It follows that if \( \eta \leq \delta_{\partial\Omega}/\sqrt{4}, \tilde{s} \in [0, \tilde{s}] \). Moreover, we have, by (6.28),
\[ \sum_{B_{\rho}(a) \in B(\tilde{s})} \rho \leq \eta. \] (6.31)
We now define the collection \( B := \{ B_{\rho}(a) \in B(\tilde{s}) : E^{sg}(\Pi_{N'} \circ \text{tr}_{\partial B_{\rho}(a)} u) > 0 \} \).
We then have for every \( B_{\rho}(a) \in B \), by (c) and by (d),
\[ \int_{B_{\rho}(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq E^{sg}(\Pi_{N'} \circ \text{tr}_{\partial B_{\rho}(a)} u) \log \left( \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} \right) - \sum_{B_{\sigma}(b) \in B_{\sigma}(a)} \sigma. \]
Hence, for every subcollection of disks \( B' \subset B \), by summing and by (6.23), we obtain
\[ \int_{\bigcup_{B_{\rho}(a) \in B'} B_{\rho}(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \sum_{B_{\rho}(a) \in B'} E^{sg}(\Pi_{N'} \circ \text{tr}_{\partial B_{\rho}(a)} u) \log \left( \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)} \right) - 2C_3 (E^{sg}(g) + \kappa) \varepsilon. \] (6.32)
By (3.5), our assumption (6.18) and by (6.32), we deduce that
\[ \frac{\text{sys}(N')^2}{4\pi} \#B \leq \sum_{B_{\rho}(a) \in B} E^{sg}(\Pi_{N'} \circ \text{tr}_{\partial B_{\rho}(a)} u) \]
\[ \leq E^{sg}(g) + \frac{(2C_3 \varepsilon + \log \frac{1}{c_0 \gamma\eta}) E^{sg}(g) + (2C_3 \varepsilon + 1) \kappa}{\log \frac{c_0 \gamma\eta}{\varepsilon E^{sg}(g)}}. \] (6.33)
The proposition is proved when \( u \in C^2(\tilde{\Omega}) \), with (i) following from (6.31), with a constant \( C \) in the conditions coming from the conditions (6.24), (6.29) and (6.30); the conclusion (iii) follows from (6.32) and (iv) from (6.33).
In the general case where \( u \in W^{1,2}(\Omega, \mathbb{R}^v) \), we first consider the case where the function \( F \) is bounded and continuous, so that the Ginzburg–Landau functional is continuous for the strong convergence in \( W^{1,2} \). We consider a sequence \((u_n)_{n \in \mathbb{N}}\) in \( C^2(\Omega) \) converging strongly to \( u \) in \( W^{1,2}(\Omega, \mathbb{R}^v) \). We apply the proposition to \( u_n \) and let \( B_n \) be the associated disks. By (iv), up to a subsequence, 

\[
(\Pi_{\Omega} \circ \text{tr}_{\partial B_n(a)} u_n)_{B_n(a) \in B_n} \text{ can be chosen to remain in the same homotopy class and } \# B_n \text{ can be chosen to be constant.}
\]

If \( F \) is not bounded, then we apply the proposition to a sequence of bounded functions \( \tilde{F}_l \in C(\mathbb{R}^v, [0, +\infty)) \) such that \( \tilde{F}_l \leq F \), \( \tilde{F}_l \) converges to \( F \) everywhere in \( \mathbb{R}^v \) and \( \tilde{F}_l = F \) on a neighbourhood of \( N \). Since the constants only depend on the behaviour of \( F \) in a neighbourhood of \( N \), the constants can be taken independently on \( l \in \mathbb{N} \) and the conclusion follows by Lebesgue’s monotone convergence theorem.

7. Energy Convergence

We investigate first in § 7.1 the convergence of sequences whose Ginzburg–Landau energy satisfies a logarithmic bound. This bound being satisfied for minimisers in view of Proposition 5.1, we apply this result to minimisers and get additional properties in Section 7.3.

In this section \( \Omega \) is a Lipschitz bounded domain and \( F \in C(\mathbb{R}^v, [0, +\infty)) \) satisfies \( F^{-1}([0]) = N \) and (1.4).

7.1. Convergence of bounded sequences

The main result about convergence of sequences whose Ginzburg–Landau energy satisfies a logarithmic bound is

**Theorem 7.1.** Let \( g \in W^{1,2}(\partial \Omega, N) \), \((u_n)_{n \in \mathbb{N}}\) be a sequence in \( W^{1,2}((\Omega, \mathbb{R}^v)) \) with \( \text{tr}_{\partial \Omega} u_n = g \) and \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence in \((0, +\infty)\) converging to \( 0 \) such that

\[
\sup_{n \in \mathbb{N}} \int_{\Omega} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - E_{sg}(g) \log \frac{1}{\varepsilon_n} < +\infty.
\]

Then up to a subsequence, there exists a map \( u^* \in W^{1,2}_{\text{ren}}(\Omega, N) \), such that if we write \( \text{sing}(u^*) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\} \), we have

(i) the sequence \((u_n)_{n \in \mathbb{N}}\) converges to \( u^* \) weakly in \( W^{1,2}_{\text{loc}}(\Omega \setminus \{a_1, \ldots, a_k\}, \mathbb{R}^v) \) and almost everywhere in \( \Omega \),

(ii) \( E_{sg}(g) = \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi}, \)

(iii) \( \sup_{n \in \mathbb{N}} \int_{\Omega} \frac{|D(\text{dist}(N \circ u_n))|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} + \sup_{t>0} t^2 \mathcal{L}^2(\{|Du_n|^{-1}([t, +\infty))\}) < +\infty, \)

(iv) one has, narrowly as measures on \( \Omega \),

\[
\frac{|Du_n|^2}{2} \log \frac{1}{\varepsilon_n} \longrightarrow \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi} \delta_{a_i},
\]
(v) $\mathcal{E}^{\text{ren}}(u_*) + Q_F(u_*) \leq \liminf_{n \to \infty} \int_{\Omega} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - \mathcal{E}^{sg}(g) \log \frac{1}{\varepsilon_n}$.

(vi) For every $\rho \in (0, \bar{\rho}(a_1, \ldots, a_k))$,

$$
\mathcal{E}^{\text{ren}}(u_*) + Q_F(u_*) \leq \int_{\Omega} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - \mathcal{E}^{sg}(g) \log \frac{1}{\varepsilon_n}.
$$

Theorem 7.1 follows from Proposition 6.10 as in [34,47].

**Proof of Proposition 7.1.** The boundedness assertion (iii) follows immediately from the lower bound for the Ginzburg-Landau energy of Proposition 6.2. Since $g \in W^{1/2,2}(\partial \Omega, \mathcal{N})$, there exists a map $w \in W^{1,2}(\Omega_{\delta_\Omega}, \mathcal{N})$, with $\Omega_{\delta_\Omega}$ defined in (6.4), such that $\text{tr}_{\partial \Omega} w = g$. For each $n \in \mathbb{N}$, we define the function $\bar{u}_n \in W^{1,2}(\Omega_{\delta_\Omega}, \mathbb{R}^v)$ in such a way that $\bar{u}_n|_{\Omega} = u_n$ and $\bar{u}_n|_{\partial \Omega} = w$.

We let $C_1 \in (0, +\infty)$ be a constant as in Proposition 6.10 and we consider a sequence $(\eta_p)_{p \in \mathbb{N}}$ in $(0, +\infty)$ converging to 0. Since whatever the constants $\kappa \in (0, +\infty)$, $\gamma \in (0, 1)$, and for each $p \in \mathbb{N}$, there exists $n_p \in \mathbb{N}$ such that, for every $n \geq n_p$,

$$
C_1 e^{\gamma C_1 \kappa} (\mathcal{E}^{sg}(g) \varepsilon_n)^{1-\gamma} \leq C_1 \gamma \eta_p, \quad C_1 \mathcal{E}^{sg}(g) \varepsilon_n \leq \frac{1}{2} \gamma \eta_p, \quad \text{and} \quad C_1 \kappa \varepsilon_n \leq 1,
$$

we have by Proposition 6.10 and by the energy bound (7.1) a finite collection $\mathcal{B}_{n,p}$ of disjoint disks of radii less than $\eta_p$ such that for every $n \geq n_p$ and every $B \in \mathcal{B}_{n,p}$, we have $\bar{B} \subset \Omega_{\delta_\Omega}$ and $\text{dist}_{\mathcal{N}} \circ \text{tr}_{\partial \Omega} \bar{u}_n < \delta_{\mathcal{N}}$, such that for some constant $C_2 > -\infty$ independent of $p$, $n$ (which may depend on $\gamma$, $\kappa$ and $\mathcal{E}^{sg}(g)$), we have in view of (iii) and (iv) in Proposition 6.10, for every $B' \subseteq \mathcal{B}_{n,p}$

$$
\int_{\bigcup_{B \in \mathcal{B}'} B} \frac{|D\bar{u}_n|^2}{2} + \frac{F(\bar{u}_n)}{\varepsilon_n^2} \geq \sum_{B \in \mathcal{B}'} \mathcal{E}^{sg}(\Pi_{\mathcal{N}} \circ \text{tr}_{\partial \Omega} \bar{u}_n) \log \frac{\eta_p}{\varepsilon_n} - C_2, \quad (7.2)
$$

with $\mathcal{E}^{sg}(\Pi_{\mathcal{N}} \circ \text{tr}_{\partial \Omega} \bar{u}_n)$ and such that the maps $(\Pi_{\mathcal{N}} \circ \text{tr}_{\partial \Omega} \bar{u}_n)_{B \in \mathcal{B}_{n,p}}$ form a topological resolution of $g$.

Since the manifold $\mathcal{N}$ is compact, in view of Proposition 3.2, the set $\{\lambda(\gamma) : \gamma \in \text{VMO}(S^1, \mathcal{N})\}$ is discrete, and thus there exists $\delta > 0$ such that if $(\gamma_1, \ldots, \gamma_\ell)$ is a topological resolution of $g$, and $\sum_{i=1}^{\ell} \mathcal{E}^{sg}(\gamma_i) \leq \mathcal{E}^{sg}(g) + \delta$, then $\sum_{i=1}^{\ell} \mathcal{E}^{sg}(\gamma_i) = \mathcal{E}^{sg}(g)$. By Proposition 6.10 (iv), and taking $n_p$ larger if necessary, we can thus assume that

$$
\#_{\mathcal{B}_{n,p}} \frac{\text{sys}(\mathcal{N})^2}{4\pi} \leq \sum_{B \in \mathcal{B}_{n,p}} \mathcal{E}^{sg}(\Pi_{\mathcal{N}} \circ \text{tr}_{\partial \Omega} \bar{u}_n) = \mathcal{E}^{sg}(g),
$$

so that $(\Pi_{\mathcal{N}} \circ \text{tr}_{\partial \Omega} \bar{u}_n)_{B \in \mathcal{B}_{n,p}}$ is a minimal topological resolution of $\text{tr}_{\partial \Omega} u_n = g$.

By (7.2), it follows that

$$
\int_{\bigcup_{B \in \mathcal{B}_{n,p}} B} \frac{|D\bar{u}_n|^2}{2} + \frac{F(\bar{u}_n)}{\varepsilon_n^2} \geq \mathcal{E}^{sg}(g) \log \frac{\eta_p}{\varepsilon_n} - C_2, \quad (7.3)
$$
which with our assumption (7.1), yields

$$
\int_{\Omega_{\delta}\setminus \bigcup_{B \in \mathcal{B}} B} \frac{|D\tilde{u}_n|^2}{2} + \frac{F(\tilde{u}_n)}{\varepsilon_n^2} \leq \mathcal{E}^{\text{sg}}(g) \log \frac{1}{\eta_p} + C_3. \quad (7.4)
$$

Let $\mathcal{C}_{n,p}$ denote the set of centres of the disks in $\mathcal{B}_{n,p}$. Up to a subsequence in $n$ and by a diagonal argument, we can assume that for each $p \in \mathbb{N}$, the sequence $(\mathcal{C}_{n,p})_{n \in \mathbb{N}}$ converges in Hausdorff distance to a finite set $\mathcal{C}_p$ in $\hat{\Omega}$ of cardinality at most $4\pi \mathcal{E}^{\text{sg}}(g)/\text{sys}(\mathcal{N})^2$. Taking a subsequence, we can assume further that $(\mathcal{C}_p)_{p \in \mathbb{N}}$ converges in Hausdorff distance to a finite set $\mathcal{C} = \{a_1, \ldots, a_k\} \subset \hat{\Omega}$, with $k \leq 4\pi \mathcal{E}^{\text{sg}}(g)/\text{sys}(\mathcal{N})^2$. (The sets $\mathcal{C}_{p,n}$, $\mathcal{C}_p$ and $\mathcal{C}$ being possibly empty, we understand that a sequence converges to the empty set in Hausdorff distance whenever it is eventually a constant sequence of empty sets.)

Moreover, we have, for every $p, q \in \mathbb{N}$, for $n \in \mathbb{N}$ large enough by (7.2),

$$
\int_{\Omega_{\delta}\setminus \Omega} \frac{|D\tilde{u}_n|^2}{2} + \frac{F(\tilde{u}_n)}{\varepsilon_n^2} \geq \sum_{B \in \mathcal{B}} \int_B \frac{|D\tilde{u}_n|^2}{2} + \frac{F(\tilde{u}_n)}{\varepsilon_n^2} \\
+ \sum_{B \in \mathcal{B}_{n,q}} \int_B \frac{|D\tilde{u}_n|^2}{2} + \frac{F(\tilde{u}_n)}{\varepsilon_n^2} \\
\geq \sum_{B \in \mathcal{B}} \mathcal{E}^{\text{sg}}(\Pi_N \circ \text{tr}_{\delta_B} \bar{u}_n) \log \frac{\eta_p}{\varepsilon_n} \\
+ \sum_{B \in \mathcal{B}_{n,q}} \mathcal{E}^{\text{sg}}(\Pi_N \circ \text{tr}_{\delta_B} \bar{u}_n) \log \frac{\eta_q}{\varepsilon_n} - 2C_2 \\
\geq \mathcal{E}^{\text{sg}}(g) \log \frac{\eta_p}{\varepsilon_n} \\
+ \#\{B \in \mathcal{B}_{n,q} : B \cap \bigcup B' \in \mathcal{B} \} B' = \emptyset \} \\
\frac{\text{sys}(\mathcal{N})^2}{4\pi} \log \frac{\eta_q}{\varepsilon_n} - 2C_2. \\
(7.5)
$$

from which it follows that if $n$ is large enough, we have $\text{dist}_\mathcal{H}(\mathcal{C}_{n,p}, \mathcal{C}_{n,q}) \leq \eta_p + \eta_q$. Passing to the limit, we have $\text{dist}_\mathcal{H}(\mathcal{C}_p, \mathcal{C}_q) \leq \eta_p + \eta_q$ and then $\text{dist}_\mathcal{H}(\mathcal{C}_p, \mathcal{C}) \leq \eta_p$. In particular, for every $p \in \mathbb{N}$, for $n \in \mathbb{N}$ large enough, we have $\text{dist}_\mathcal{H}(\mathcal{C}_{n,p}, \mathcal{C}) \leq 2\eta_p$, and thus

$$
\int_{\Omega_{\delta}\setminus \bigcup_{i=1}^k B_{3\eta_p}(a_i)} \frac{|D\tilde{u}_n|^2}{2} + \frac{F(\tilde{u}_n)}{\varepsilon_n^2} \leq \mathcal{E}^{\text{sg}}(g) \log \frac{1}{3\eta_p} + C_4, \quad (7.6)
$$

with $C_4 := C_3 + \mathcal{E}^{\text{sg}}(g) \log 3$. By weak compactness, Rellich’s compactness theorem and a diagonal argument, the sequence $(\tilde{u}_n)_{n \in \mathbb{N}}$ converges almost everywhere to some $\tilde{u}_* : \Omega_{\delta\Omega} \to \mathbb{R}^\nu$ and weakly to $\tilde{u}_*$ in $W^{1,2}(\Omega_{\delta\Omega} \setminus \bigcup_{i=1}^k \tilde{B}_p(a_i), \mathbb{R}^\nu)$ for
every \( \rho > 0 \). We have \( \tilde{u}_* = w \) on \( \Omega \setminus \Omega_{3\delta} \). We define \( u_* = \tilde{u}_*|_\Omega \). Since, by Fatou’s lemma,
\[
\int_{\Omega \setminus \bigcup_{i=1}^k B_{3\eta_p}(a_i)} F(u_*) \leq \lim_{n \to \infty} \frac{\varepsilon_n^2}{2} \int_{\Omega \setminus \bigcup_{i=1}^k B_{3\eta_p}(a_i)} F(u_n) \leq 0,
\]
we have \( u_* \in \mathcal{N} \) almost everywhere in \( \Omega \). Moreover, for every \( p \in \mathbb{N} \), we have by (7.6) and by lower semicontinuity
\[
\int_{\Omega_{3\delta} \setminus \bigcup_{i=1}^k B_{3\eta_p}(a_i)} \frac{|D\tilde{u}_*|^2}{2} \leq \liminf_{n \to \infty} \int_{\Omega_{3\delta} \setminus \bigcup_{i=1}^k B_{3\eta_p}(a_i)} \frac{|D\tilde{u}_n|^2}{2} + \frac{F(\tilde{u}_n)}{\varepsilon_n^2} \leq \mathcal{E}^{sg}(g) + C_4.
\]
By [38, Lemma 6.2], for \( p \) large enough so that
\[
3\eta_p < \tilde{\rho} := \sup\{r > 0 \mid \text{ for each } i \in \{1, \ldots, k\}, B_r(a_i) \subset \Omega_{3\delta} \text{ and for each } j \in \{1, \ldots, k\}\setminus\{i\}, B_r(a_i) \cap B_r(a_j) = \emptyset\},
\]
we have
\[
\int_{\Omega_{3\delta} \setminus \bigcup_{i=1}^k B_{3\eta_p}(a_i)} \frac{|D\tilde{u}_*|^2}{2} \geq \frac{1}{4\pi} \frac{\lambda(\tr_{\partial B_{3\eta_p}(a_i)} \bar{u}_*)}{3\eta_p} \log \frac{\tilde{\rho}}{3\eta_p} \left(1 - \left(\frac{2\pi C_5 \mathcal{E}^{ext}(\tr_{\partial B_{3\eta_p} u})}{\lambda(\tr_{\partial B_{3\eta_p}} u)^2 \log \frac{\tilde{\rho}}{3\eta_p}}\right)^{1/2}\right)^2,
\]
where
\[
v_{\tilde{\rho},3\eta_p}(a) := \frac{1}{2\pi} \frac{1}{\log \frac{\tilde{\rho}}{3\eta_p}} \int_{(B_{\tilde{\rho}}(a) \setminus \bar{B}_{3\eta_p}(a)) \cap \Omega} \frac{1}{|x-a|^2} \, dx \leq 1.
\]
Since \( (\tr_{\partial B_{3\eta_p}(a_i)} \bar{u}_*)\}_{1 \leq i \leq k} \) is a topological resolution of \( \tr_{\partial \Omega_{3\delta}} \bar{u}_* \), and thus of \( g \), we have
\[
\sum_{i=1}^k \frac{\lambda(\tr_{\partial B_{3\eta_p}(a_i)} \bar{u}_*)}{4\pi v_{\tilde{\rho},3\eta_p}(a_i)} \geq \sum_{i=1}^k \frac{\lambda(\tr_{\partial B_{3\eta_p}(a_i)} \bar{u}_*)}{4\pi} \geq \mathcal{E}^{sg}(g).
\]
It thus follows by (7.7) and (7.8) that \( \lim_{p \to \infty} v_{\tilde{\rho},3\eta_p}(a_i) = 1 \) which implies that \( a_i \in \Omega \) (since \( \Omega \) has Lipschitz boundary). It also follows that \( (\tr_{\partial B_{3\eta_p}(a_i)} \bar{u}_*)\}_{1 \leq i \leq k} \) is a minimal topological resolution of \( g \). Hence, in view of Definition 3.5 and (7.7), the map \( u_* \) is renormalisable. Thus, if we let \( \text{sing}(u_*) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\} \) we obtain (i) and (ii).

We now prove (iv). By (i) and (7.1), we deduce that, up to a subsequence,
\[
\frac{|Du_n|^2}{2|\log \varepsilon_n|} \to \sum_{i=1}^k \alpha_i \delta_{a_i} \text{ in the narrow topology of measures}
\]
for some constants $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Thanks to the upper bound (7.1) and the lower bound Proposition 6.2, we have that

$$\left( \int_{\Omega} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n} - E^{sg}(g) \log \frac{1}{\varepsilon_n} \right)_{n \in \mathbb{N}}$$

is bounded. Using this and the fact that, by (iii), $\frac{1}{|\log \varepsilon_n|} \int_{\Omega} \frac{F(u_n)}{\varepsilon_n^2} \to 0$ we obtain that

$$\sum_{i=1}^{k} \alpha_i = E^{sg}(g).$$  \hspace{1cm} (7.10)

By (7.6) and by Fatou’s lemma, for almost every $r \in (0, \bar{\rho})$, we have for each $i \in \{1, \ldots, k\}$,

$$\liminf_{n \to \infty} \int_{\partial B_r(a_i)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} < +\infty. \hspace{1cm} (7.11)$$

By Sobolev embedding and Arzela-Ascoli criterion, we obtain that for $n$ large enough $\Pi_{N} \circ \text{tr}_{\partial B_r(a_i)} u_n$ and $\text{tr}_{\partial B_r(a_i)} u_*$ are homotopic, and thus in particular

$$E^{sg}(\text{tr}_{\partial B_r(a_i)} u_n) = E^{sg}(\text{tr}_{\partial B_r(a_i)} u) = E^{sg}(\gamma_i) = \frac{\lambda(\gamma_i)}{4\pi}.$$

Hence, taking such an $r \in (\bar{\rho}/2, \bar{\rho})$, we get by Corollary 6.8

$$\int_{\partial B_r(a_i)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} \geq \left( \log \frac{1}{\varepsilon_n} \right) \frac{\lambda(\gamma_i)}{4\pi} - C_6. \hspace{1cm} (7.12)$$

Thus we have $\alpha_i \geq \frac{\lambda(\gamma_i)^2}{4\pi}$, and in view of (7.10), we obtain (iv).

The rest of the proof is devoted to assertions (v) and (vi). For almost every $r \in (0, \bar{\rho})$, for each $i \in \{1, \ldots, k\}$ and $n \in \mathbb{N}$, $\text{tr}_{\partial B_r(a_i)} u_n = u_n|_{\partial B_r(a_i)}$. Hence if we define $\gamma_{r,n}^i : \mathbb{S}^1 \to \mathbb{R}^v$ by $\gamma_{r,n}^i(x) := u_n(a_i + r x)$, we have by (7.11), for almost every $r \in (0, \bar{\rho})$

$$\liminf_{n \to \infty} \int_{\mathbb{S}^1} \frac{|(\gamma_{r,n}^i)'|^2}{2} + \frac{r^2}{\varepsilon_n^2} F(\gamma_{r,n}^i) < +\infty. \hspace{1cm} (7.13)$$

There exists thus a subsequence $(n_{\ell})_{\ell \in \mathbb{N}}$ (depending on $r$) such that

$$\sup_{\ell \in \mathbb{N}} \int_{\mathbb{S}^1} \frac{|(\gamma_{r,n_{\ell}}^i)'|^2}{2} + \frac{r^2}{\varepsilon_{n_{\ell}}^2} F(\gamma_{r,n_{\ell}}^i) < +\infty. \hspace{1cm} (7.14)$$

In particular, $F(\gamma_{r,n_{\ell}}^i) \to 0$ a.e. up to extraction of a subsequence if necessary. Hence, by (1.4), we have $\text{dist}_{N} \circ \gamma_{r,n_{\ell}}^i \to 0$ a.e. so that for $\ell$ large enough, we have $\text{dist}_{N} \circ \gamma_{r,n_{\ell}}^i < \delta_N/2$. Thus, by Proposition 4.4 and by (7.14), we have

$$\lim_{\ell \to \infty} \left| Q_{F,\Pi_{N} \circ \gamma_{r,n_{\ell}}}^{r/\varepsilon_{n_{\ell}}} - Q_{F,\gamma_{r,n_{\ell}}}^{r/\varepsilon_{n_{\ell}}} \right| \leq \lim_{\ell \to \infty} C_7 \frac{\varepsilon_{n_{\ell}}}{r} \int_{\mathbb{S}^1} \frac{|(\gamma_{r,n_{\ell}}^i)'|^2}{2} + \frac{r^2}{\varepsilon_{n_{\ell}}^2} F(\gamma_{r,n_{\ell}}^i) = 0. \hspace{1cm} (7.15)$$
so that
\[
\liminf_{\ell \to \infty} Q^{r/\varepsilon_{n\ell}}_{F, \gamma_{i,n\ell}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n\ell}} = \liminf_{\ell \to \infty} Q^{r/\varepsilon_{n\ell}}_{F, \Pi_{N^t} \circ \gamma_{i,n\ell}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n\ell}}.
\] (7.16)

On the other hand, by (7.14) and Sobolev embeddings, the sequence \((\gamma_{i,n\ell})_{\ell \in \mathbb{N}}\) converges strongly to \(u_*(a_i + r \cdot)\) in \(W^{1/2,2}(S^1, \mathbb{R}^v)\), and thus \((\Pi_{N^t} \circ \gamma_{i,n\ell})_{\ell \in \mathbb{N}}\) converges to \(u_*(a_i + r \cdot)\) in \(W^{1/2,2}(S^1, \mathbb{N})\). Hence, by [38, Proposition 3.3 (vi)], \(\lim_{\ell \to \infty} d_{\text{synh}}(\Pi_{N^t} \circ \gamma_{i,n\ell}, u_*(a_i + r \cdot)) = 0\). Thus by (7.16) and Proposition 4.3
\[
\liminf_{\ell \to \infty} Q^{r/\varepsilon_{n\ell}}_{F, \gamma_{i,n\ell}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n\ell}} = \liminf_{\ell \to \infty} Q^{r/\varepsilon_{n\ell}}_{F, u_*(a_i + r \cdot)} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n\ell}}.
\] (7.17)

Finally by Proposition 4.3 again, we have in view of (7.17)
\[
\liminf_{\ell \to \infty} Q^{r/\varepsilon_{n\ell}}_{F, \gamma_{i,n\ell}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n\ell}} \geq Q_{F, \gamma_i} - d_{\text{synh}}(u_*(a_i + r \cdot), \gamma_i).
\]

It follows thus that
\[
\lim_{\ell \to \infty} \int_{\Omega} \frac{|Du_{n\ell}|^2}{2} + \frac{F(u_{n\ell})}{\varepsilon_{n\ell}^2} - \mathcal{E}_{\text{sg}}(g) \log \frac{1}{\varepsilon_{n\ell}}
\geq \liminf_{\ell \to \infty} \int_{\Omega \setminus \bigcup_{i=1}^k B_r(a_i)} \frac{|Du_{n\ell}|^2}{2} + \frac{F(u_{n\ell})}{\varepsilon_{n\ell}^2} + \sum_{i=1}^k \liminf_{\ell \to \infty} \int_{B_r(a_i)} \frac{|Du_{n\ell}|^2}{2} + \frac{F(u_{n\ell})}{\varepsilon_{n\ell}^2}
\geq \mathcal{E}_{\text{sg}}(g) \log \frac{1}{r} + \sum_{i=1}^k \liminf_{\ell \to \infty} Q^{r/\varepsilon_{n\ell}}_{F, \gamma_{i,n\ell}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n\ell}}
\geq \int_{\Omega \setminus \bigcup_{i=1}^k B_r(a_i)} \frac{|Du_{*}|^2}{2} - \mathcal{E}_{\text{sg}}(g) \log \frac{1}{r} + Q_{F, u_*} - \sum_{i=1}^k d_{\text{synh}}(u_*(a_i + r \cdot), \gamma_i).
\]

We reach the conclusion (v) by letting \(r \to 0\).

The proof of (vi) proceeds as the proof of (v) in order to reach
\[
\liminf_{\ell \to \infty} \sum_{i=1}^k \int_{B_r(a_i) \setminus B_r(a_i)} \frac{|Du_{n\ell}|^2}{2} + \frac{F(u_{n\ell})}{\varepsilon_{n\ell}^2} - \mathcal{E}_{\text{sg}}(g) \log \frac{1}{\varepsilon_{n\ell}}
\geq \sum_{i=1}^k \int_{B_r(a_i) \setminus B_r(a_i)} \frac{|Du_{*}|^2}{2} - \mathcal{E}_{\text{sg}}(g) \log \frac{1}{r} + Q_{F, u_*} - \sum_{i=1}^k d_{\text{synh}}(u_*(a_i + r \cdot), \gamma_i),
\]
the conclusion follows then by letting \(r \to 0\) and additivity of integrals. \(\square\)
Remark 7.2. (Γ-convergence) For each \( g \in W^{1/2,2}(\partial \Omega, \mathcal{N}) \) and \( \varepsilon \in (0, +\infty) \), we define \( E^\varepsilon_g \) on set of measurable functions by setting

\[
E^\varepsilon_g(u) = \begin{cases} 
\int_\Omega \left( \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \right) - E^\varepsilon(g) \log \frac{1}{\varepsilon} & \text{if } u \in W^{1,2}(\Omega, \mathbb{R}^v) \text{ and } \text{tr}_{\partial\Omega} u = g \text{ on } \partial\Omega, \\
+\infty & \text{otherwise},
\end{cases}
\]

and we define the limit functional \( E^g_0 \) on the set of measurable functions by setting

\[
E^g_0(u) = \begin{cases} 
E^{\text{ren}}_g(u) + Q_F(u) & \text{if } u \in A_g(\Omega, \mathcal{N}), \\
+\infty & \text{otherwise},
\end{cases}
\]

(7.18)

where \( A_g(\Omega, \mathcal{N}) \) is the set of maps \( u \in W^{1,2}_{\text{ren}}(\Omega, \mathcal{N}) \) such that \( \text{tr}_{\partial\Omega} u = g \) and \((\gamma_1, \ldots, \gamma_k)\) is a minimal topological resolution of \( g \), where \( \text{sing}(u) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\} \).

The family of functionals \( (E^\varepsilon_g)_{\varepsilon > 0} \) Γ-converges as \( \varepsilon \to 0 \) to \( E^g_0 \) in \( L^p(\Omega, \mathbb{R}^v) \) endowed with the strong topology for every \( p \in [1, +\infty) \), and in \( W^{1,p}(\Omega, \mathbb{R}^v) \) endowed with the weak or strong topology for every \( p \in [1, 2) \). The upper bound follows from the upper bound Proposition 5.2 and the lower bound from Theorem 7.1. For \( \mathcal{N} = S^1 \) and for the strong convergence in \( W^{1,1} \), a Γ-convergence result at leading order, i.e. the Γ-convergence of \( E^\varepsilon/g \log 1/\varepsilon \), is due to Jerrard and Soner [31, Theorem 4.1]. For \( \mathcal{N} = S^1 \), a Γ-convergence type result at next order can be found in [1]; if \( Ju = \det \nabla u \) denotes the Jacobian of \( u \), the authors show the Γ-convergence of the energy \( \inf_{J_u = J} E^\varepsilon_F(u) - E^g(\log 1/\varepsilon) \log 1/\varepsilon \) in the Jacobian variable \( J \in C^{0,1}(\Omega) \) endowed with the convergence in the flat norm. Our framework allows us to state the Γ-convergence of \( E^\varepsilon_F(u) - E^g(\log 1/\varepsilon) \) in the variable \( u \); this in particular requires to introduce the renormalised energy \( E^{\text{ren}}_g \) of renormalisable maps (not necessarily harmonic away from singularities). In the case of the circle \( \mathcal{N} = S^1 \), this approach is reminiscent of [28].

7.2. Convergence of minimisers

We are now ready to fully state and prove our result about the convergence of minimisers.

Theorem 7.3. Let \( g \in W^{1/2,2}(\partial \Omega, \mathcal{N}) \), let \( (\varepsilon_n)_{n \in \mathbb{N}} \) be a sequence in \((0, +\infty)\) converging to 0 and for each \( n \in \mathbb{N} \) let \( u_n \in W^{1,2}(\Omega, \mathbb{R}^v) \) be a minimiser of the Ginzburg–Landau energy \( E^\varepsilon_F \) under the Dirichlet boundary condition \( \text{tr}_{\partial\Omega} u_n = g \).

Then, up to a subsequence, there exists a map \( u_* \in W^{1,2}_{\text{ren}}(\mathcal{N}, \mathcal{N}) \) such that if we write \( \text{sing}(u_*) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\} \), we have

(i) the sequence \((u_n)_{n \in \mathbb{N}}\) converges almost everywhere to \( u_* \) and strongly in \( W^{1,2}_{\text{loc}}(\hat{\Omega}\setminus\{a_1, \ldots, a_k\}) \) and \( F(u_n)/\varepsilon_n^2 \to 0 \) in \( L^1_{\text{loc}}(\hat{\Omega}\setminus\{a_1, \ldots, a_k\}) \),

(ii) \( E^g(\lambda) = \sum_{i=1}^k \frac{\lambda_i (\gamma_i)^2}{4\pi} \),

(iii) \( \sup_{n \in \mathbb{N}} \int_\Omega \frac{|D(\text{dist}_{\mathcal{N}} \circ u_n)|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} + \sup_{r>0} r^2 E^2(|Du_n|^{-1}[0, +\infty)) < +\infty \),

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(iv) one has, narrowly as measures on $\Omega$,
\[
\frac{|Du_n|^2}{2 \log \frac{1}{\varepsilon_n}} \rightharpoonup \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} \delta_{a_i},
\]

(v) $\text{tr}_{\partial \Omega} u_* = g$ and $u_*$ is a minimising renormalisable singular harmonic map (see Remark 7.5) so that in particular, for every $\rho \in \bar{B}(a_1, \ldots, a_k)$, $u_* \in \mathcal{C}^\infty(\Omega \setminus \{a_1, \ldots, a_k\})$, $N$ is harmonic minimising in $\Omega \setminus \bigcup_{i=1}^{k} \bar{B}_\rho(a_i)$ with respect to its own boundary conditions,

(vi) we have the equalities
\[
\lim_{n \to \infty} \int_{\Omega} \frac{|Du_n|^2}{2} + \frac{F(u_n) - E^{\text{sg}}(g) \log \frac{1}{\varepsilon_n}}{\varepsilon_n^2} = \mathcal{E}^{\text{ren}}(u_*) + Q_F(u_*)
\]
\[
= \inf \{ \mathcal{E}^{\text{ren}}(u) + Q_F(u) : u \in W^{1,2}_{\text{ren}}(\Omega, N) \text{ and } \text{tr}_{\partial \Omega} u = g \}
\]
\[
= \mathcal{E}^{\text{geom}}_{g, \gamma_1, \ldots, \gamma_k}(a_1, \ldots, a_k) + \sum_{i=1}^{k} Q_{F, \gamma_i} = W_{\text{min}},
\]

where
\[
W_{\text{min}} := \inf \left\{ \mathcal{E}^{\text{geom}}_{g, \eta_1, \ldots, \eta_\ell}(b_1, \ldots, b_\ell) + \sum_{i=1}^{\ell} Q_{F, \eta_i} : b_1, \ldots, b_\ell \in \Omega \text{ are distinct and } (\eta_1, \ldots, \eta_\ell) \text{ is a minimal topological resolution of } g \right\}.
\] (7.19)

When $N = S^1$, the weak $L^2$ estimate (iii) on the gradient is due to Serfaty and Tice [50, Proposition 1.3].

Remark 7.4. By (i) and (iii), for every $p \in [1, 2)$, the sequence $(u_n)_{n \in \mathbb{N}}$ converges to $u$ strongly in $W^{1,p}(\Omega)$. When $N = S^1$, such a convergence was known for smooth data [9, Lemma X.11] and $W^{1/2,2}$ data [10].

Remark 7.5. Following [38, Definition 7.8], the map $u_* \in W^{1,2}_{\text{ren}}(\Omega, N)$ being a minimising renormalisable singular harmonic map means that if we set $\text{sing}(u_*) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\}$, then for every map $v \in W^{1,2}_{\text{ren}}(\Omega, N)$ with $\text{sing}(v) = \{(b_1, \gamma_1), \ldots, (b_k, \gamma_k)\}$ (that is, $\text{sing}(v)$ differs from $\text{sing}(u_*)$ only by the position of the points, but not by the $\gamma_i$), one has
\[
\mathcal{E}^{\text{ren}}(u_*) \leq \mathcal{E}^{\text{ren}}(v).
\]

In particular $u_*$ is a stationary renormalisable harmonic map, which means that $u_*$ is harmonic away from its singularities and that its stress-energy energy tensor has vanishing flux around every singularity, or equivalently the residue of its Hopf differential vanishes at every singularity [38, Lemma 7.7 and Proposition 7.9].
When $\mathcal{N} = \mathbb{S}^1$, Theorem 7.3 is essentially due to Bethuel, Brezis and Hélein [9] for star-shaped domains, and to Struwe for simply-connected domains [51]. The existence of finitely many singularities and the strong convergence in the case of a general compact manifold $\mathcal{N}$ was proved by Canevari [15] for general smooth bounded domains.

**Proof of Theorem 7.3.** Since $(u_n)_{n \in \mathbb{N}}$ is a sequence of minimisers, it follows from Proposition 5.1 that we have

$$
\limsup_{n \to \infty} \int_{\Omega} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - E_{\text{sg}}(g) \log \frac{1}{\varepsilon_n} \leq W_{\text{min}}.
$$

(7.20)

By (7.20) and by Theorem 7.1, up to a subsequence, there exists a family of points $(a_1, \ldots, a_k)$ in $\Omega$ such that $(u_n)_{n \in \mathbb{N}}$ converges weakly in $W^{1,2}_{\text{loc}}(\Omega \setminus \{a_1, \ldots, a_k\}, \mathbb{R}^n)$ to some limit $u_* \in W^{1,2}_{\text{ren}}(\Omega, \mathcal{N})$ and

$$
E_{\text{ren}}(u_*) + Q_F(u_*) \leq W_{\text{min}}.
$$

(7.21)

Note that from Theorem 7.1, (ii), (iii) and (iv) hold. Furthermore (i) also holds if the strong convergence is replaced by the weak convergence.

Since the map $u_*$ is renormalisable, by Proposition 3.6 and by (4.4),

$$
E_{\text{ren}}(u_*) + Q_F(u_*) \geq E_{\text{geom}}^{\text{ren}}(a_1, \ldots, a_k) + \sum_{i=1}^k Q_{F,\gamma_i},
$$

(7.22)

where we have set $\text{sing}(u_*) = \{(a_1, \gamma_1), \ldots, (a_k, \gamma_k)\}$. It follows thus from (7.19), (7.21) and (7.22) that

$$
E_{\text{ren}}(u_*) + Q_F(u_*) = W_{\text{min}} = E_{\text{geom}}^{\text{ren}}(a_1, \ldots, a_k) + \sum_{i=1}^k Q_{F,\gamma_i}.
$$

(7.23)

By Proposition 5.2, since $(u_n)_{n \in \mathbb{N}}$ is a sequence of almost-minimisers and since $E_{\text{sg}}(g) = \sum_{i=1}^k \lambda_i(\gamma_i)^2/4\pi$, we have also that

$$
\limsup_{n \to \infty} \int_\Omega \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - E_{\text{sg}}(g) \log \frac{1}{\varepsilon_n}
\leq \inf \{E_{\text{ren}}(u) + Q_F(u) : u \in W^{1,2}_{\text{ren}}(\Omega, \mathcal{N}) \text{ and } \text{tr}_{\partial \Omega} u = g\},
$$

(7.24)

which, together with (v) in Theorem 7.1, yields

$$
E_{\text{ren}}(u_*) + Q_F(u_*) = \lim_{n \to \infty} \int_\Omega \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - E_{\text{sg}}(g) \log \frac{1}{\varepsilon_n}
= \inf \{E_{\text{ren}}(u) + Q_F(u) : u \in W^{1,2}_{\text{ren}}(\Omega, \mathcal{N}) \text{ and } \text{tr}_{\partial \Omega} u = g\}.
$$

(7.25)

Thus we have proved (vi). In particular, $u_*$ is a minimising renormalisable singular harmonic map. It follows that if $\rho \in \mathcal{P}(a_1, \ldots, a_k)$ and if the map $u \in$
$W^{1,2}_{\text{loc}}(\Omega \setminus \{a_1, \ldots, a_k\}, N)$ satisfies $\text{tr}_\Omega u = g$ on $\partial \Omega$ and $u = u_*$ in $\bigcup_{i=1}^k B_\rho(a_i)$, then $\text{sing}(u) = \text{sing}(u_*)$ and $\mathcal{E}^{\text{ren}}(u) \geq \mathcal{E}^{\text{ren}}(u_*)$; hence,
\[ \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du|^2}{2} - \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_*|^2}{2} = \mathcal{E}^{\text{ren}}(u) - \mathcal{E}^{\text{ren}}(u_*) \geq 0, \]
so that $u_*$ is harmonic minimising in $\Omega \setminus \bigcup_{i=1}^k \bar{B}_\rho(a_i)$ with respect to its own boundary conditions and, in particular, $u_* \in C^\infty(\Omega \setminus \{a_1, \ldots, a_k\}, N)$ by the result of [41]; this proves (v).

Finally, by (7.25) and Theorem 7.1 (vi), we have for every $\rho \in (0, \bar{\rho}(a_1, \ldots, a_k))$,

\[ \limsup_{n \to \infty} \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon^2_n} \leq \limsup_{n \to \infty} \left( \int_{\Omega} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon^2_n} - \mathcal{E}^{\text{sg}}(g) \log \frac{1}{\varepsilon_n} \right) \]
\[ - \liminf_{n \to \infty} \left( \int_{\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon^2_n} - \mathcal{E}^{\text{sg}}(g) \log \frac{1}{\varepsilon_n} \right) \leq \int_{\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_*|^2}{2}, \]
which implies the announced strong convergence in (i).

\[ \square \]

8. An Explicit Computation of the Renormalised Energy

Although the geometric renormalised energy of singularities and the renormalised energy of renormalisable maps are defined via a shrinking holes approach and are thus quite implicit, if $\Omega$ is simply-connected and $g$ is a reparametrisation of a minimising atomic geodesic in $N$, the geometric renormalised energy of a single singularity coincides strikingly with $N = S^1$ [38, Theorem 10.1]. When $\Omega = B_1$ this geometric renormalised energy can be explicitly computed and this allows one to locate asymptotic singularities for strictly atomic minimising geodesics as boundary conditions (see (a) and (b) in Theorem 8.1), as Bethuel, Brezis and Hélein did for $N = S^1$ [9, Theorem 0.4] in response to a question of Matano.

**Theorem 8.1.** Let $\Omega$ be a Lipschitz bounded domain and let $F \in \mathcal{C}(\mathbb{R}^\nu, [0, +\infty))$ satisfy $F^{-1}(\{0\}) = N$ and (1.4). Assume that

(a) $g : S^1 \to N$ is a minimising geodesic,
(b) if $(\gamma_1, \ldots, \gamma_k)$ is a minimal topological resolution of $g$ then $k = 1$ and $\gamma_1$ is homotopic to $g$,
(c) every map homotopic to $g$ is synharmonic to $g$.

If for each $\varepsilon \in (0, +\infty)$, $u_\varepsilon$ is a minimiser of $\mathcal{E}^\varepsilon_F$ in $W^{1,2}(B_1, \mathbb{R}^\nu)$ under the condition $\text{tr}_\Omega u_\varepsilon = g$, then

\[ u_\varepsilon \xrightarrow{\varepsilon \to 0} u_* \text{ in } W^{1,2}_{\text{loc}}(B_1 \setminus \{0\}, \mathbb{R}^\nu), \]
with $u_*(x) = g(x / |x|)$. 

Proof. By our assumptions and Theorem 7.3, given any sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) in \((0, +\infty)\) converging to 0, up to extraction of a subsequence, there exists a map \(u_* \in W^{1,2}_{\text{ren}}(\Omega, \mathcal{N})\) such that \(\text{sing}(u_*) = \{(a, \gamma)\}\) for some \(a \in \Omega\) and some minimising geodesic \(\gamma: S^1 \to \mathcal{N}\) homotopic to \(g\), and such that \((u_{\varepsilon_n})_{n \in \mathbb{N}}\) converges to \(u_*\) in \(W^{1,2}_{\text{loc}}(B \setminus \{a\}, \mathbb{R}^v)\) with
\[
\mathcal{E}^{\text{ren}}(u_*) + Q_{F, \gamma} = \inf \{ \mathcal{E}^{\text{geom}}_{g, \gamma}(x) + Q_{F, \gamma} : x \in B_1\} = \mathcal{E}^{\text{geom}}_{g, \gamma}(a) + Q_{F, \gamma}.
\]
By (c), \(\gamma\) and \(g\) are synharmonic, which implies that \(\mathcal{E}^{\text{geom}}_{g, \gamma} = \mathcal{E}^{\text{geom}}_{g, g}\) in view of (3.8). But, from [38, Theorem 10.1], we know that \(\mathcal{E}^{\text{geom}}_{g, g}\) has a unique minimiser at \(x = 0\), and thus \(a = 0\). By the characterisation (3.9) of the renormalised energy of the renormalisable map \(u_*\),
\[
0 = \mathcal{E}^{\text{ren}}(u_*) = \sup_{\rho \in (0,1)} \int_{B_1 \setminus B_\rho} \frac{|Du_*|^2}{2} - \frac{(\lambda(g))^2}{4\pi} \log \frac{1}{\rho}.
\]
In view of the elementary lower bound
\[
\int_{B_1 \setminus B_\rho} \frac{|Du_*|^2}{2} \geq \frac{(\lambda(g))^2}{4\pi} \log \frac{1}{\rho} + \frac{1}{2} \int_{B_1 \setminus B_\rho} |\frac{\partial u_*}{\partial r}|^2,
\]
this implies that for almost every \(x \in B_1\), \(u_*(x) = g\left(\frac{x}{|x|}\right)\). Since the limit is independent of the subsequence, the convergence holds for the whole family.

9. Convergence of Solutions to the Ginzburg–Landau Equation

We consider now solutions to the Ginzburg–Landau equation (1.5) arising at least formally as the Euler–Lagrange equation of the Ginzburg–Landau energy (1.3).

9.1. Boundedness and Euler–Lagrange equation for minimisers

We first show that under fairly general and reasonable conditions, minimisers of the Ginzburg–Landau energy \(\mathcal{E}^\varepsilon_F\) are weak solutions to the Ginzburg–Landau equation (1.5).

Proposition 9.1. Let \(\Omega\) be a Lipschitz bounded domain and let \(F \in C^1(\mathbb{R}^v, [0, +\infty))\) such that \(F^{-1}(\{0\}) = \mathcal{N}\). If \(u \in W^{1,2}(\Omega, \mathbb{R}^v)\) is a minimiser for the Ginzburg–Landau energy \(\mathcal{E}^\varepsilon_F\) under its own Dirichlet boundary conditions, and if \(\nabla F(u) \in L^1_{\text{loc}}(\Omega)\), then for every \(\varphi \in C^1_c(\Omega, \mathbb{R}^v)\),
\[
\int_{\Omega} Du \cdot D\varphi + \frac{\nabla F(u)}{\varepsilon^2} \cdot \varphi = 0.
\]

Proposition 9.1 does not make any assumption on \(F\) beyond continuous differentiability; in particular \(F\) needs not satisfy (1.4).

The proof of Proposition 9.1 follows a truncation argument due to BOUSQUET [12,45, proof of Theorem 4.23].
Proof of Proposition 9.1. Let $\theta \in C^1(\mathbb{R}_+)$ such that $\theta = 1$ on $[0, 1]$ and $\theta = 0$ on $[2, +\infty)$. For every $R > 0$, we consider the function $\eta_R := \theta(|u|/R)$. We have, since $\eta_R \in W^{1,2}(\Omega)$ and since $u$ is bounded on the set $\{\eta_R \neq 0\}$,

$$0 = \lim_{t \to 0} \frac{E_F^i(u + t\eta_R \varphi) - E_F^i(u)}{t} = \int_{\Omega} \eta_R Du \cdot D\varphi + Du \cdot (D\eta_R)\varphi + \frac{\nabla F(u)}{\varepsilon^2} \cdot \varphi \eta_R$$

$$= \int_{\Omega} \left( Du \cdot D\varphi + \frac{\nabla F(u)}{\varepsilon^2} \cdot \varphi \right) \theta\left( \frac{|u|}{R} \right)$$

$$+ \sum_{i=1}^2 \int_{\Omega} (\partial_i u \cdot \varphi)(\partial_i u \cdot \frac{u}{|u|}) \frac{\nabla F(u)}{R^2} \cdot \varphi \eta_R.$$

Letting $R \to +\infty$, we conclude in view of Lebesgue’s dominated convergence theorem. $\square$

The condition $\nabla F(u) \in L^1(\Omega)$ in Proposition 9.1 can be obtained by establishing an a priori bound on the minimiser.

Proposition 9.2. Let $\Omega$ be a Lipschitz bounded domain and let $F \in C^1(\mathbb{R}^v, [0, +\infty))$ such that $F^{-1}([0]) = \mathcal{N}$. If there exists a map $\Psi : C^{0,1}(\mathbb{R}^v, \mathbb{R}^v)$ such that

(a) $\Psi$ is non-expansive in $\mathbb{R}^v$, i.e., $|\Psi(x) - \Psi(y)| \leq |x - y|$,

(b) $F \circ \Psi \leq F$ in $\mathbb{R}^v$,

(c) $\Psi = \text{id}$ on $\mathcal{N}$,

then for every $g \in W^{1/2,2}(\partial \Omega, \mathcal{N})$, if $u$ is a minimiser for the Ginzburg–Landau energy under the boundary condition $\text{tr}_{\partial \Omega} u = g$, then $u \in \tilde{K}_\Psi$ almost everywhere in $\Omega$, where

$$K_\Psi := \left\{ x \in \mathbb{R}^v : \limsup_{h \to 0} \frac{|\Psi(x + h) - \Psi(x)|}{|h|} = 1 \right\}.$$

In particular, if $F(Rz/|z|) \leq F(z)$ for some $R > 0$ and every $z \in \mathbb{R}^v \setminus B_R$, taking

$$\Psi(z) := \begin{cases} z & \text{if } |z| < R \\ Rz/|z| & \text{if } |z| \geq R \end{cases},$$

we conclude that any minimiser of the Ginzburg–Landau energy (1.3) satisfies $\|u\|_{L^\infty(\Omega)} \leq R$. This is the case in particular when $\mathcal{N} = \mathbb{S}^1$ and $F(z) = (1 - |z|^2)^2/4$ [8, Proposition 2].

When the set $K_\Psi$ is bounded, Proposition 9.1 also implies that $u$ is a weak solution of the Ginzburg–Landau equation.

Proof of Proposition 9.2. If $u$ is a minimiser, we set $v = \Psi \circ u$. By (c), we have $\text{tr}_{\partial \Omega} v = \Psi \circ \text{tr}_{\partial \Omega} u = g$ on $\partial \Omega$. Now, by (b) and since $u$ is a minimiser, we have

$$\int_{\Omega} \frac{|Du|^2}{2} \leq \int_{\Omega} \frac{|Dv|^2}{2} + \int_{\Omega} \frac{F(v)}{\varepsilon^2} - \int_{\Omega} \frac{F(u)}{\varepsilon^2} \leq \int_{\Omega} \frac{|Dv|^2}{2}.$$
By (a) and by the chain rule for distributional derivatives [2], we have $|Dv|^2 \leq |Du|^2$ almost everywhere in $\Omega$, and either $|Dv|^2 = |Du|^2 = 0$ or $|Dv|^2 < |Du|^2$ on $u^{-1}(\mathbb{R}^n \setminus K_\psi)$. By optimality, this means that $Du = 0$ a.e. on $u^{-1}(\mathbb{R}^n \setminus K_\psi)$; hence, by the chain rule, $D\left(\text{dist}(u(x), K_\psi)\right) = 0$ a.e. on $u^{-1}(\mathbb{R}^n \setminus K_\psi)$. Since the weak derivative of $\text{dist}(u(\cdot), K_\psi)$ also vanishes a.e. on the zero level set, i.e. on $u^{-1}(\bar{K}_\psi)$, this implies that $D\left(\text{dist}(u(x), K_\psi)\right) = 0$ a.e. in $\Omega$. By (c), $\mathcal{N} \subset K_\psi$ and thus by the trace condition we find $\text{dist}(u, K_\psi) = 0$ almost everywhere in $\Omega$ which implies the conclusion. \hfill $\Box$

9.2. Uniform convergence to the manifold

Given a boundary datum $g \in W^{1/2, 2}(\partial \Omega, \mathcal{N})$, we show that the asymptotic vanishing of the penalisation term in the Ginzburg–Landau equation for a sequence of solutions implies that the distance to the manifold vanishes asymptotically uniformly.

**Theorem 9.3.** Let $\Omega$ be a Lipschitz bounded domain, let $g \in W^{1/2, 2}(\partial \Omega, \mathcal{N})$, and assume that $F \in C^1(\mathbb{R}^n, [0, +\infty))$ satisfies $F^{-1}(\{0\}) = \mathcal{N}$ and (1.4). If $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence in $(0, +\infty)$ converging to 0, if for every $n \in \mathbb{N}$, $u_n \in W^{1, 2}(\Omega, \mathbb{R}^n)$ is a solution to the Ginzburg–Landau equation (1.5) such that $\text{tr}_{\partial \Omega} u_n = g$, and if for some $a \in \Omega$ and $\rho > 0$, we have

$$\lim_{n \to \infty} \int_{\Omega \cap B_\rho(a)} \frac{F(u_n)}{\varepsilon_n^2} = 0,$$

and

$$\sup_{n \in \mathbb{N}} \|Du_n\|_{L^2(\Omega \cap B_\rho(a))} + \varepsilon_n^2 \|\nabla u_n\|_{L^\infty(\Omega \cap B_\rho(a))} < +\infty,$$

then

$$\lim_{n \to \infty} \|\text{dist}(u_n, \mathcal{N})\|_{L^\infty(\Omega \cap B_{\rho/2}(a))} = 0.$$

The assumptions of boundedness on $Du_n$ in $L^2$ and of convergence of $F(u_n)/\varepsilon_n^2$ in $L^1$ hold for sequences of minimisers away from singularities (Theorem 7.3). The uniform bound on $\nabla u_n$ follows from an a priori bound on $u_n$ (see Proposition 9.2) and the local boundedness of $\nabla F$ in view of the Ginzburg–Landau equation (1.5); it could also follow from the global boundedness of $\nabla F$.

The uniform convergence to the vacuum manifold $\mathcal{N}$ away from singularities was known for $\mathcal{N} = \mathbb{S}^1$ [8, Step B.2]. The result is reminiscent of uniform convergence of the modulus of $W^{1, 2}$-converging sequences of functions whose Laplacian and whose modulus on the boundary are controlled [11, Lemma 2.13].

The next lemma states that harmonic functions tend uniformly to the image of their trace when we approach the boundary.

**Lemma 9.4.** If $\Omega \subset \mathbb{R}^2$ has a Lipschitz boundary, if $v \in W^{1, 2}(\Omega, \mathbb{R}^n)$ satisfies $-\Delta v = 0$ in $\Omega$ and if $\text{tr}_{\partial \Omega} v \in \mathcal{N}$ almost everywhere in $\partial \Omega$, then

$$\lim_{x \in \Omega \atop \text{dist}(x, \partial \Omega) \to 0} \text{dist}(v(x), \mathcal{N}) = 0.$$
Lemma 9.4 follows from the corresponding property for harmonic extensions of functions of vanishing mean oscillation (VMO) [14, Theorem A3.2] and the embedding of $W^{1/2,2}(\partial \Omega, \mathbb{R}^v)$ in VMO$(\partial \Omega, \mathbb{R}^v)$ (see [11, Lemma 2.12] for $\mathcal{N} = S^1$). We give a direct proof when $v$ is in $W^{1,2}(\Omega, \mathbb{R}^v)$.

**Proof of Lemma 9.4.** Since the function $v$ is harmonic, by the maximum principle, $v$ is bounded. There exists a constant $C_1 < +\infty$ such that for every $y \in \Omega$ (see for example [27, Theorem 2.10]),

$$|Dv(y)| \leq C_1 \|v\|_{L^\infty(\Omega)} \operatorname{dist}(y, \partial \Omega). \quad (9.1)$$

For every $x \in \Omega$, let $r := \operatorname{dist}(x, \partial \Omega)$. If $0 < \eta < 1$, we have

$$\operatorname{dist}(v(x), \mathcal{N}) \leq \int_{B_{2r}(x) \cap \Omega} |v(y) - v(x)| \, dy + \int_{B_{2r}(x) \cap \Omega} \operatorname{dist}(v(y), \mathcal{N}) \, dy. \quad (9.2)$$

In view of (9.1), we have

$$\int_{B_{2r}(x) \cap \Omega} |v(y) - v(x)| \, dy \leq C_1 \|v\|_{L^\infty(\Omega)} \frac{1}{\eta} - 1. \quad (9.3)$$

Next, since the set $\Omega$ has a Lipschitz boundary and $\operatorname{tr}_{\partial \Omega} \operatorname{dist}(v, \mathcal{N}) = 0$, we have the following Poincaré inequality

$$\int_{B_{2r}(x) \cap \Omega} \operatorname{dist}(v(y), \partial \Omega)^2 \leq C_2 r^2 \int_{B_{2r}(x) \cap \Omega} |Dv|^2. \quad (9.4)$$

It follows from (9.4) that

$$\int_{B_{2r}(x) \cap \Omega} \operatorname{dist}(v(y), \mathcal{N}) \, dy \leq \left( \int_{B_{2r}(x) \cap \Omega} \operatorname{dist}(v(y), \mathcal{N})^2 \, dy \right)^{1/2} \leq \frac{C_3}{\eta} \left( \int_{B_{2r}(x) \cap \Omega} |Dv|^2 \right)^{1/2}. \quad (9.5)$$

In order to conclude we observe that when $\eta$ is small enough, the first-term in the right-hand side of (9.2) can be made arbitrarily small by (9.3), while for any given $\eta > 0$ the second term in the right-hand side of (9.2) goes to 0 as $\operatorname{dist}(x, \partial \Omega) \to 0$ in view of (9.5) and Lebesgue’s dominated convergence theorem since $v \in W^{1,2}(\Omega, \mathbb{R}^v)$. \qed

**Lemma 9.5.** (Regularity estimate) If $\Omega$ is a Lipschitz bounded domain and if $w \in W^{1,2}(\Omega, \mathbb{R}^v)$ is such that $\Delta w \in L^\infty(\Omega, \mathbb{R}^v)$ and $\operatorname{tr}_{\partial \Omega} w = 0$, then for every $\rho > 0$, and for every $a \in \Omega$,

$$\|Dw\|_{L^\infty(\Omega \cap B_{r/2}(a))} \leq C(\rho) \left( \|Dw\|_{L^2(\Omega \cap B_r(a))} \right. + \left. \|\Delta w\|_{L^\infty(\Omega \cap B_r(a))} \right)^{1/2} \|Dw\|_{L^2(\Omega \cap B_r(a))}^{1/2}. \quad (9.6)$$

Lemma 9.5 is reminiscent of the $L^\infty$ estimates [8, Lemma A.1].
**Proof of Lemma 9.5.** We fix $p > 2$. Since $\partial \Omega$ has a Lipschitz boundary, there exists $\rho_0 > 0$ such that if $2 \operatorname{dist}(x, \partial \Omega) \leq r \leq \rho_0$, then $B_r(x) \cap \Omega$ is homeomorphic to a half-disk and has uniformly Lipschitz boundary. By a finite covering argument, we can assume that $\rho \leq \rho_0$.

If $x \in \Omega$ and $r \in (0, \rho) \setminus \{ \operatorname{dist}(x, \partial \Omega), 2 \operatorname{dist}(x, \partial \Omega) \}$, we have, by classical Calderón–Zygmund estimates and a scaling argument,

$$
\| D^2 w \|_{L^p(\Omega \cap B_r^2(x))} \leq C_1 \left( \frac{\| w \|_{L^p(\Omega \cap B_r(x))}}{r^2} + \| \Delta w \|_{L^p(\Omega \cap B_r(x))} \right); \quad (9.6)
$$

combining this with the two-dimensional Sobolev inequality

$$
\| w \|_{L^p(\Omega \cap B_r(x))} \leq C_2 r^{2/p} (r^{-1} \| w \|_{L^2(\Omega \cap B_r(x))} + \| D w \|_{L^2(\Omega \cap B_r(x))})
$$

and the Poincaré inequality

$$
\| w \|_{L^2(\Omega \cap B_r(x))} \leq C_3 r \| D w \|_{L^2(\Omega \cap B_r(x))}
$$

(which is true when $r \geq 2 \operatorname{dist}(x, \partial \Omega)$, since $w = 0$ on $\partial \Omega$, and also when $r \leq \operatorname{dist}(x, \partial \Omega)$ if we assume without loss of generality that $\int_{\Omega \cap B_r(x)} w = 0$), we obtain

$$
\| D^2 w \|_{L^p(\Omega \cap B_r^2(x))} \leq C_4 \left( \frac{\| D w \|_{L^2(\Omega \cap B_r(x))}}{r^{2 - \frac{2}{p}}} + \| \Delta w \|_{L^p(\Omega \cap B_r(x))} \right); \quad (9.7)
$$

moreover, by the Morrey–Sobolev embedding $W^{1, p} \subset C^{0, 1 - 2/p}$ and the Cauchy–Schwarz inequality,

$$
|D w(x)| \leq \int_{\Omega \cap B_r(x)} |D w(x) - D w(y)| \, dy + \int_{\Omega \cap B_r(x)} |D w| \leq C_5 \left( \| D^2 w \|_{L^p(\Omega \cap B_r(x))} I^{1 - \frac{2}{p}} + \frac{\| D w \|_{L^2(\Omega \cap B_r(x))}}{r} \right), \quad (9.8)
$$

and it follows thus from (9.7), (9.8) and $\| \Delta w \|_{L^p(\Omega \cap B_r(x))} \leq (\pi r^2)^{1/p} \| \Delta w \|_{L^\infty(\Omega \cap B_r(x))}$ that

$$
|D w(x)| \leq C_6 \left( r \| \Delta w \|_{L^\infty(\Omega \cap B_r(x))} + \frac{\| D w \|_{L^2(\Omega \cap B_r(x))}}{r} \right). \quad (9.9)
$$

Now, if $x \in \Omega \cap B_{\rho/2}(a)$ then, for every $r \in (0, \rho) \setminus \{ \operatorname{dist}(x, \partial \Omega), 2 \operatorname{dist}(x, \partial \Omega) \}$, we have the inclusion $\Omega \cap B_{r/2}(x) \subset \Omega \cap B_{\rho}(a)$, and we deduce from (9.9) that

$$
|D w(x)| \leq C_6 \left( r \| \Delta w \|_{L^\infty(\Omega \cap B_{\rho}(a))} + \frac{\| D w \|_{L^2(\Omega \cap B_{\rho}(a))}}{r} \right). \quad (9.10)
$$

We observe now that (9.9) holds also for $r \in (0, \rho) \cap \{ \operatorname{dist}(x, \partial \Omega), 2 \operatorname{dist}(x, \partial \Omega) \}$ with $2C_6$ instead of $C_6$ and we conclude by taking

$$
r := \min \left( \rho/2, \sqrt{\frac{\| D w \|_{L^2(\Omega \cap B_{\rho}(a))}}{\| \Delta w \|_{L^\infty(\Omega \cap B_{\rho}(a))}}} \right)
$$

if $\| D w \|_{L^2(\Omega \cap B_{\rho}(a))} \| \Delta w \|_{L^\infty(\Omega \cap B_{\rho}(a))} > 0$, and $r := \rho/2$ otherwise. \(\boxdot\)
Proof of Theorem 9.3. Following [8, Proof of Step B.1], let $v \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ be a solution to the Dirichlet problem
\[
\begin{align*}
\Delta v &= 0 \quad \text{in } \Omega, \\
v &= g \quad \text{on } \partial \Omega.
\end{align*}
\]
For each $n \in \mathbb{N}$, we define the function $w_n := u_n - v$, which satisfies by assumption on $u_n$ and by construction of $v$,
\[
\begin{align*}
\Delta w_n &= \nabla F(u_n) \quad \text{in } \Omega, \\
w_n &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
By Lemma 9.5 and by assumption, we have
\[
\|Dw_n\|_{L^\infty(\Omega \cap B_{\rho/2}(a))} \leq C_1 \left( \|Dw_n\|_{L^2(\Omega \cap B_{\rho}(a))} + \|\Delta w_n\|_{L^\infty(\Omega \cap B_{\rho}(a))} \right)^{1/2} \|Dw_n\|_{L^2(\Omega \cap B_{\rho}(a))}^{1/2}.
\]
(9.11)
Now let $\delta \in (0, \frac{\delta}{2})$. By Lemma 9.4, there exists $r > 0$, such that if $\text{dist}(x, \partial \Omega) \leq r$, then $\text{dist}(v(x), \mathcal{N}) \leq \delta/2$. If moreover $x \in \Omega \cap B_{\rho/2}(a)$ and $\text{dist}(x, \partial \Omega) \leq \varepsilon_n \delta/(4C_2) < r$, then, thanks to (9.11), we have $|w_n(x)| \leq \delta/2$. Hence for $n$ large enough, as $u_n = w_n + v$,
for all $x \in \Omega \cap B_{\rho/2}(a)$ such that $\text{dist}(x, \partial \Omega) \leq \varepsilon_n \delta/(2C_2), \text{ dist}(u_n(x), \mathcal{N}) \leq \delta$.
(9.12)
We consider now a point $x \in \Omega \cap B_{\rho/2}(a)$ such that $\text{dist}(x, \partial \Omega) > \varepsilon_n \delta/(4C_2)$; we have, by classical estimates on harmonic extensions (see (9.1)) and by (9.11), that
\[
|Du_n(x)| \leq \frac{C_3}{\varepsilon_n \delta}.
\]
(9.13)
We assume now by contradiction that there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $B_{\rho/2}(a) \cap \Omega$ such that $\text{dist}(u_n(a_n), \mathcal{N}) \geq 2\delta$. By continuity, we can assume that $\text{dist}(u_n(a_n), \mathcal{N}) = 2\delta$. By (9.12), we have in particular $\text{dist}(a_n, \partial \Omega) > \varepsilon_n \delta/(2C_2)$ and so, for $n$ large enough,

$B_{\varepsilon_n \delta/(4C_2)}(a_n) \subset \{ x \in \Omega \cap B_{\rho}(a) : \text{dist}(x, \partial \Omega) > \varepsilon_n \delta/(4C_2) \}$.

Since the distance to a closed set is non-expansive, using (9.13), we have if $x \in B_{\varepsilon_n \delta/(4C_2)}(a_n)$,
\[
|\text{dist}(u_n(x), \mathcal{N}) - \text{dist}(u_n(a_n), \mathcal{N})| \leq |u_n(x) - u_n(a_n)| \leq \frac{C_3}{\varepsilon_n \delta} |x - a_n|.
\]
Hence, for every $x \in B_{C_4 \varepsilon n \delta^2}(a_n)$ with $C_4 := \inf\{1/(\delta_N C_2); 1/C_3\}$, we have
\[
\delta \leq \text{dist}(u_n(x), \mathcal{N}) \leq 3\delta < \delta_N.
\]
Hence, we have if $n$ is large enough, using (1.4),
\[
\frac{m_F}{2} C_4^2 \pi \delta^6 \leq \frac{m_F}{2} \int_{B_{C_4 \varepsilon n \delta^2}(a_n)} \frac{\text{dist}(u_n, \mathcal{N})^2}{\varepsilon_n^2} \leq \int_{\Omega \cap B_\rho(a)} \frac{F(u_n)}{\varepsilon_n^2},
\]
which cannot hold by assumption if $n \in \mathbb{N}$ is large enough since the right-hand side goes to zero.

9.3. Weak convergence of solutions

The next result shows that, under some assumptions, the limit of weakly converging sequences of solutions to the Ginzburg–Landau equation are harmonic maps. For a related result when $\mathcal{N}$ is a compact manifold of dimension 1 we refer to [37].

**Theorem 9.6.** Let $\Omega$ be a Lipschitz bounded domain and assume that $F \in C^1(\mathbb{R}^\nu, [0, +\infty))$ satisfies $F^{-1}(0) = \mathcal{N}$ and (1.4). Assume also that $g \in W^{1,2}(\Omega, \mathbb{R}^\nu)$, that $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence in $(0, +\infty)$ converging to 0 and that for every $n \in \mathbb{N}$, $u_n \in W^{1,2}(\Omega \cap B_\rho(a), \mathbb{R}^\nu)$ is a solution to the Ginzburg–Landau equation (1.5) with $\text{tr}_\partial \Omega u_n = g$. If for some $a \in \Omega$ and $\rho > 0$, we have

(i) $(u_n|_{\Omega \cap B_\rho(a)})_{n \in \mathbb{N}}$ converges weakly to some limit $u$ in $W^{1,2}(\Omega \cap B_\rho(a), \mathbb{R}^\nu)$,

(ii) $\lim_{n \to \infty} \|\text{dist}(u_n, \mathcal{N})\|_{L^\infty(\Omega \cap B_\rho(a))} = 0$,

(iii) $\lim_{n \to \infty} \int_{\Omega \cap B_\rho(a)} \frac{|D\Pi_{\mathcal{N}}(u_n)\nabla F(u_n)|}{\varepsilon_n^2} = 0$,

then $u$ is a $\mathcal{N}$-valued harmonic map in $\Omega \cap B_\rho(a)$.

For the classical Ginzburg–Landau equation, which corresponds to (1.5) when $F(z) = (1-|z|^2)^2$, we have $\nabla F(z) = -4(1-|z|^2)z$, $\Pi_{\mathbb{S}^1}(z) = \frac{\text{id}}{|z|} - \frac{z \otimes z}{|z|^2}$; hence,
\[
D\Pi_{\mathbb{S}^1}(u_n)[\nabla F(u_n)] = -4(1-|u_n|^2) \frac{u_n}{|u_n|} + 4(1-|u_n|^2) \frac{(u_n \cdot u_n)u_n}{|u_n|^3} = 0
\]
which implies (iii). We recover from Theorem 7.1, Theorem 9.3 and Theorem 9.6 that solutions to the classical Ginzburg–Landau satisfying an a priori bound (see Proposition 9.2) and an upper-bound of the form (7.1) converge to a harmonic map with values into $\mathbb{S}^1$ outside a finite set of singularities when $\varepsilon$ goes to zero [9, Theorem X.1].

When $F \in C^3(\mathbb{R}^\nu)$, the condition (iii) in Theorem 9.6 follows from $\lim_{n \to \infty} \int_{\Omega \cap B_\rho(a)} F(u_n) \frac{u_n}{\varepsilon_n^2} = 0$, in view of the next lemma.
Lemma 9.7. Let \( F \in \mathcal{C}^3(\mathcal{N}_\delta, [0, +\infty)) \). If \( F^{-1}([0]) = \mathcal{N} \) and \( F \) satisfies (1.4), then there exist constants \( C \in (0, +\infty) \) and \( \delta \in (0, \delta_N) \) such that, for every \( z \in \mathcal{N}_\delta \),
\[
|D\Pi_{\mathcal{N}'}(z)[\nabla F(z)]| \leq CF(z), \tag{9.14}
\]

Proof of Lemma 9.7. By a second-order Taylor expansion of \( DF(z) \), we have for \( v \in \mathbb{R}^v \)
\[
DF(z)[D\Pi_{\mathcal{N}'}(z)[v]] = DF(\Pi_{\mathcal{N}'}(z))[D\Pi_{\mathcal{N}'}(z)[v]]
+ D^2F(\Pi_{\mathcal{N}'}(z))[z - \Pi_{\mathcal{N}'}(z), D\Pi_{\mathcal{N}'}(z)[v]]
+ O(|v| \text{dist}(z, \mathcal{N})^2). \tag{9.15}
\]
We first have, for every \( z \in \mathcal{N}_\delta \),
\[
DF(\Pi_{\mathcal{N}'}(z)) = 0, \tag{9.16}
\]
so that the first term in the right-hand side of (9.15) vanishes. Differentiating (9.16), we get for \( v, w \in \mathbb{R}^v \) and \( z \in \mathcal{N}_\delta \),
\[
D^2F(\Pi_{\mathcal{N}'}(z))[w, D\Pi_{\mathcal{N}'}(z)[v]] = 0, \tag{9.17}
\]
so that the second term in the right-hand side of (9.15) also vanishes. We deduce from (9.15), (9.16) and (9.17) that
\[
|D\Pi_{\mathcal{N}'}(z)^*[\nabla F(z)]| \leq C_1 \text{dist}(z, \mathcal{N})^2.
\]
Since \( D\Pi_{\mathcal{N}'}(z) \) is self-adjoint and \( \nabla F(z) = 0 \) when \( z \in \mathcal{N} \), we have
\[
|D\Pi_{\mathcal{N}'}(z)^*[\nabla F(z)] - D\Pi_{\mathcal{N}'}(z)[\nabla F(z)]| \leq C_2 \text{dist}(z, \mathcal{N})^2, \quad \text{for all } z \in \mathcal{N}_\delta \quad \square
\]
and the conclusion follows, in view of (1.4). \quad \square

We begin the proof of Theorem 9.6 with the following geometrical identity for the nearest-point projection:

Lemma 9.8. For every \( y \in \mathcal{N} \), \( h \in \mathbb{R}^v \) and \( w \in T_y\mathcal{N} \), we have
\[
w \cdot D^2\Pi_{\mathcal{N}'}(y)[h, h] = w \cdot D^2\Pi_{\mathcal{N}'}(y)[h, D\Pi_{\mathcal{N}'}(y)[h]]
+ D\Pi_{\mathcal{N}'}(y)[h] \cdot D^2\Pi_{\mathcal{N}'}(y)[h, w]. \tag{9.18}
\]

Proof. Setting \( h^\top := D\Pi_{\mathcal{N}'}(y)[h] \) and \( h^\perp := h - h^\top \), we need to prove that the following quantity vanishes:
\[
w \cdot D^2\Pi_{\mathcal{N}'}(y)[h, h] - w \cdot D^2\Pi_{\mathcal{N}'}(y)[h, h^\top] - h^\top \cdot D^2\Pi_{\mathcal{N}'}(y)[h, w]
= w \cdot D^2\Pi_{\mathcal{N}'}(y)[h^\top, h^\perp] + w \cdot D^2\Pi_{\mathcal{N}'}(y)[h^\perp, h^\perp]
- h^\top \cdot D^2\Pi_{\mathcal{N}'}(y)[h^\top, w] - h^\top \cdot D^2\Pi_{\mathcal{N}'}(y)[h^\perp, w]. \tag{9.19}
\]
Since \( h^\perp \in T_y\perp \mathcal{N} \), and since \( \Pi_{\mathcal{N}'}(y + th^\perp) = \Pi_{\mathcal{N}'}(y) \) for all \( t \) small enough, we have, by differentiating twice:
\[
D^2\Pi_{\mathcal{N}'}(y)[h^\perp, h^\perp] = 0. \tag{9.20}
\]
By the connection between the nearest-point projection and the second fundamental form \[42\text{, lemma } 3.2\], we have
\[ w \cdot D^2 \Pi_N(y)[h^\perp, h^\top] = -h^\perp \cdot B_y(w, h^\top) = -h^\top \cdot D^2 \Pi_N(y)[h^\perp, w]. \]

Finally, since \(w \in T_y \mathcal{N}, h^\top \in T_y \mathcal{N}\) and by using that \(D^2 \Pi_N(y) : T_y \mathcal{N} \otimes T_y \mathcal{N} \to T_y \mathcal{N}\) we have
\[ h^\top \cdot D^2 \Pi_N(y)[h^\top, w] = 0. \tag{9.22} \]

In view of (9.20), (9.21) and (9.22), the right-hand side of (9.19) vanishes and the conclusion follows. \(\Box\)

\textbf{Lemma 9.9.} For every \(y \in \mathcal{N}_{\delta \mathcal{N}}\), the map \(\alpha_{\mathcal{N}}(y) := D \Pi_N(y) D \Pi_N(y)^* : T_{\Pi_N(y)} \mathcal{N} \to T_{\Pi_N(y)} \mathcal{N}\) is invertible. Moreover, if a map \(u \in W^{2,1}_\text{loc}(\Omega, \mathbb{R}^v)\) satisfies \(\|\text{dist}(u, \mathcal{N})\|_{L^\infty(\Omega)} < \delta \mathcal{N}\), then we have
\[
\left| \alpha_{\mathcal{N}}(u)^{-1} D \Pi_N(u)[\Delta u] - \text{div}[\alpha_{\mathcal{N}}(u)^{-1} D(\Pi_N \circ u)] \right| \leq C|u - \Pi_N(u)||Du|^2.
\]

Here, we recall that \(D \Pi_N(y)^* : T_{\Pi_N(y)} \mathcal{N} \to T_{\Pi_N(y)} \mathcal{N} \subset \mathbb{R}^v\) stands for the adjoint of \(D \Pi_N(y)\) which is defined by
\[ D \Pi_N(y)[v] \cdot w = v \cdot D \Pi_N(y)^*[w] \] for all \(v \in \mathbb{R}^v\) and \(w \in T_{\Pi_N(y)} \mathcal{N}\).

Lemma 9.9 is a generalisation of the decomposition when \(\mathcal{N} = S^1\) of \(u\) into its modulus and argument [8, (51)–(52)], which is connected to the substitution in the Schrödinger equation to obtain Madelung equations; see e.g. \[19\] and references therein.

\textbf{Proof of Lemma 9.9.} First of all, we have that \(\alpha_{\mathcal{N}}(y)\) is invertible since \(D \Pi_N(y) : T_{\Pi_N(y)} \mathcal{N} \to T_{\Pi_N(y)} \mathcal{N}\) is onto.

Let \(i \in \{1, 2\}\). We have, on the one hand
\[
\partial_i^2(\Pi_N \circ u) = \partial_i \left( \alpha_{\mathcal{N}}(u) \alpha_{\mathcal{N}}(u)^{-1} \partial_i (\Pi_N \circ u) \right)
= \alpha_{\mathcal{N}}(u)^{-1} \partial_i (\alpha_{\mathcal{N}}(u)^{-1} \partial_i (\Pi_N \circ u))
+ \partial_i (\alpha_{\mathcal{N}}(u)) \alpha_{\mathcal{N}}(u)^{-1} \partial_i (\Pi_N \circ u), \tag{9.23}
\]

with
\[
\partial_i (\alpha_{\mathcal{N}}(u))
= D^2 \Pi_N(u)[\partial_i u] \circ D \Pi_N(u)^* + D \Pi_N(u) \circ \left( D^2 \Pi_N(u)[\partial_i u] \right)^*.
\tag{9.24}
\]

On the other hand, we have
\[
\partial_i^2(\Pi_N \circ u)
= \partial_i (D \Pi_N(u)[\partial_i u]) = D^2 \Pi_N[\partial_i u, \partial_i u] + D \Pi_N(u)[\partial_i^2 u], \tag{9.25}
\]
and therefore by (9.23), (9.24) and (9.25), we have

\[
D\Pi_N(u)[\partial^2_i u] - \alpha_N(u)\partial_i (\alpha_N(u)^{-1}\partial_i (\Pi_N \circ u)) \\
= (D^2\Pi_N(u)[\partial_i u]) \circ D\Pi_N(u)^* \\
+ D\Pi_N(u) \circ D^2\Pi_N(u)[\partial_i u^*][\alpha_N(u)^{-1}\partial_i (\Pi_N \circ u)] \\
- D^2\Pi_N(u)[\partial_i u, \partial_i u].
\]

(9.26)

Since the left-hand side of (9.26) lies in \( T_{\Pi_N(u)}N \), it suffices to estimate the projection of the right-hand side of (9.26) on \( T_{\Pi_N(u)}N \).

Since \( D\Pi_N(u^*) = D\Pi_N(u)^* \) is the orthogonal projection onto the tangent space \( T_{\Pi_N(u)}N \), we have that both \( D\Pi_N(u) = D\Pi_N(u)^* \) and \( \alpha_N(u^* \circ u) \) are the identity on \( T_{\Pi_N(u)}N \). Hence, by using a Taylor expansion, we have for every \( v \in T_{\Pi_N(u)}N \),

\[
v \cdot D^2\Pi_N(u)[\partial_i u] \left[ D\Pi_N(u)^*\alpha_N(u)^{-1}\partial_i (\Pi_N \circ u) \right] \\
= v \cdot D^2\Pi_N(u^*) \circ D\Pi_N(u)[\partial_i u, \partial_i u] \\
+ O((|v||u - \Pi_N(u)||\partial_i u|^2),
\]

(9.27)

and, in view of Lemma 9.8,

\[
v \cdot D^2\Pi_N(u)[\partial_i u, \partial_i u] \\
= v \cdot D^2\Pi_N(u^*) \circ D\Pi_N(u)[\partial_i u, \partial_i u] + O((|v||u - \Pi_N(u)||\partial_i u|^2) \\
= v \cdot D^2\Pi_N(u^*) \circ D\Pi_N(u)[\partial_i u, \partial_i u] \\
+ D^2\Pi_N(u)[\partial_i u, v] \cdot D\Pi_N(u^*) \circ D\Pi_N(u)[\partial_i u] \\
+ O((|v||u - \Pi_N(u)||\partial_i u|^2).
\]

(9.29)

Hence from (9.26), (9.27), (9.28) and (9.29) we arrive at

\[
|\alpha_N(u)^{-1}D\Pi_N(u)[\partial^2_i u] - \partial_i (\alpha_N(u)^{-1}\partial_i (\Pi_N \circ u))| \\
\leq C_1|u - \Pi_N(u)||\partial_i u|^2.
\]

(9.30)

The conclusion then follows by the triangle inequality and summing (9.30) over \( i \in \{1, 2\} \).

\[\square\]

**Proof of Theorem 9.6.** By classical regularity estimates, we have \( u_n \in W^{2,p}(\Omega, \mathbb{R}^n) \) for every \( p \in (1, +\infty) \). It follows from our assumption \( \lim_{n \to \infty} \|\text{dist}(u_n, N)\|_{L^\infty(\Omega \cap B_p(a))} = 0 \), that for \( n \in \mathbb{N} \) large enough we have \( \|\text{dist}(u_n, N)\|_{L^\infty(\Omega \cap B_p(a))} < \delta_N \) so that we can define \( v_n := \Pi_N \circ u_n|_{\Omega \cap B_p(a)} \).
By smoothness of $\Pi_{\mathcal{N}'}$ and the assumption (i), we know that the sequence $(v_n)_{n \in \mathbb{N}}$ converges to $u$ weakly in $W^{1,2}(\Omega \cap B_\rho(a), \mathbb{R}^v)$. Moreover, we have
\[
D\Pi_{\mathcal{N}'}(v_n)[\Delta v_n] = f_n + g_n, \tag{9.31}
\]
where, using the same notation $\alpha_{\mathcal{N}'}(y) := D\Pi_{\mathcal{N}'}(y)D\Pi_{\mathcal{N}'}(y)^*$ as in Lemma 9.9,
\[
f_n := D\Pi_{\mathcal{N}'}(v_n) \left[ \text{div} \left( \left( \text{id} - \alpha_{\mathcal{N}'}(u_n)^{-1} \right) Dv_n \right) \right]
\]
and
\[
g_n = D\Pi_{\mathcal{N}'}(v_n) \text{div} \left( \alpha_{\mathcal{N}'}(u_n)^{-1} Dv_n \right).
\]

By weak convergence, $(Dv_n)_{n \in \mathbb{N}}$ is bounded in $L^2$. Using the fact that $\alpha_{\mathcal{N}'}(y)$ depends smoothly on $y \in \mathcal{N}', \mathcal{N}''$, and that $\alpha_{\mathcal{N}'}(y) = \text{id}_{T_y\mathcal{N}'}$ when $y \in \mathcal{N}$, we find by our assumption (ii),
\[
\lim_{n \to \infty} \| (\text{id} - \alpha_{\mathcal{N}'}(u_n)^{-1}) Dv_n \|_{L^2(\Omega \cap B_\rho(a))} = 0,
\]
and we deduce that
\[
\| f_n \|_{H^{-1}(\Omega \cap B_\rho(a), \mathbb{R}^v)} \longrightarrow_{n \to \infty} 0. \tag{9.32}
\]

Now, we have from Lemma 9.9 and by smoothness of $D\Pi_{\mathcal{N}'}$,
\[
\| g_n \|_{L^1} \leq C_1 \left( \| \alpha_{\mathcal{N}'}^{-1}(u_n) D\Pi_{\mathcal{N}'}(u_n) \Delta u_n \|_{L^1} + \| u_n - \Pi_{\mathcal{N}'}(u_n) \|_{L^1} \right)
\]
and since $u_n$ satisfies the Ginzburg–Landau equation, we have by our assumption (iii),
\[
\| D\Pi_{\mathcal{N}'}(u_n)[\Delta u_n] \|_{L^1(\Omega \cap B_\rho(a))} \leq \frac{1}{\varepsilon^2_n} \| D\Pi_{\mathcal{N}'}(u_n)[\nabla F(u_n)] \|_{L^1(\Omega \cap B_\rho(a))} \longrightarrow_{n \to \infty} 0.
\]

We have also $\| u_n - \Pi_{\mathcal{N}'}(u_n) \|_{L^1} \to 0$ by the assumption (ii) and by boundedness of $(\| Du_n \|)_{n \in \mathbb{N}}$ in $L^2(\Omega \cap B_\rho(a))$. Hence
\[
\| g_n \|_{L^1(\Omega \cap B_\rho(a), \mathbb{R}^v)} \longrightarrow_{n \to \infty} 0. \tag{9.33}
\]

Since $D\Pi_{\mathcal{N}'}(v_n)$ is the orthogonal projection on $T_{v_n}\mathcal{N}'$, the conclusion follows from (9.31), (9.32), (9.33) and the result about weak limits of Palais-Smale sequences for the harmonic maps equation in [7] (see also [26] and [46]).
9.4. Higher-order convergence of solutions

Under regularity assumptions on the boundary, we improve the convergence away from singularities.

**Theorem 9.10.** Let $\Omega$ be a bounded open set with $C^2$ boundary and assume that $F \in C^1(\mathbb{R}^v, [0, +\infty))$ satisfies $F^{-1}(0) = \mathcal{N}$ and (2.13). Let also $g \in C^2(\partial \Omega, \mathcal{N})$, $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence in $(0, +\infty)$ converging to 0 and $(u_n)_{n \in \mathbb{N}}$ be a sequence of solutions to (1.5) with $u_n \in C^2(\tilde{\Omega}, \mathbb{R}^v)$ and $u_n|_{\partial \Omega} = g$. If $F \in C^4(\mathcal{N})$ for some $\delta \in (0, \delta_{\mathcal{N}})$, and if for some $a \in \tilde{\Omega}$ and $\rho \in (0, +\infty)$, we have

i) $(u_n)_{n \in \mathbb{N}}$ converges to some $\mathcal{N}$-valued harmonic map $u_\ast$ in $W^{1,2}(\Omega \cap B_\rho(a), \mathbb{R}^v)$, 

ii) $\lim_{n \to \infty} \|u_n - u_\ast\|_{L^\infty(\Omega \cap B_\rho(a))} = 0$,

iii) $\lim_{n \to \infty} \int_{\Omega \cap B_\rho(a)} F(u_n) \frac{|Du_n|^2}{\varepsilon_n^2} = 0$,

then $(u_n)_{n \in \mathbb{N}}$ is bounded in $W^{2,p}_{\text{loc}}(\Omega \cap B_r(a))$ for all $p \in [1, +\infty)$ and $r \in (0, \rho)$.

In particular, it follows by the Morrey–Sobolev embedding that $(u_n)_{n \in \mathbb{N}}$ converges to $u_\ast$ in $C^{1,\alpha}(\Omega \cap B_\rho/2(a))$ for all $0 < \alpha < 1$.

The first tool to prove Theorem 9.10 is the following proposition that was proved in [21] in dimension $n \geq 3$ and whose proof is the same for $n = 2$. It relies on the fact that when $\text{dist}(u_\ast, \mathcal{N})$ is small, a Böchner-type formula holds:

$-\Delta e_\varepsilon(u_n) \leq Ce_\varepsilon(u_n)^2$ if $u_n$ is a solution of (1.5) and where $e_\varepsilon(u) = \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2}$ and on boundary elliptic estimates on the gradient.

**Proposition 9.11.** [21, Proposition 3.1] Let $\Omega$ be a bounded open set with $C^2$ boundary, let $g \in C^2(\partial \Omega, \mathcal{N})$, and let $F \in C^1(\mathbb{R}^v, [0, +\infty))$ such that $F^{-1}(0) = \mathcal{N}$, $F \in C^3(\mathcal{N})$ for some $\delta \in (0, \delta_{\mathcal{N}})$ and $F$ satisfies (2.13). There exist $\varepsilon_0, \eta_0 \in (0, +\infty)$ and $C = C(F, \Omega, g) \in (0, +\infty)$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $\rho \in (0, 1)$ and $a \in \tilde{\Omega}$, if $u \in C^2(\tilde{\Omega}, \mathbb{R}^v)$ is a solution of (1.5) with $\text{tr}_{\partial \Omega} u = g$, $\|\text{dist}(u, \mathcal{N})\|_{L^\infty(\Omega \cap B_\rho(a))} < \delta_{\mathcal{N}}$ and

$$E := \int_{\Omega \cap B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \leq \eta_0,$$  \hspace{1cm} (9.34)

then

$$\rho^2 \sup_{B_\rho/2(a)} \left( \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \right) \leq C(E + \rho^2).$$  \hspace{1cm} (9.35)

**Proof of Theorem 9.10.** By a covering argument, we can restrict our attention to the case $r = \frac{\rho}{4}$ with $\rho > 0$ sufficiently small so that

$$\int_{\Omega \cap B_\rho(a)} \frac{|Du_n|^2}{2} \leq \eta_0/2,$$

with $\eta_0$ given by Proposition 9.11, and thus when $n \in \mathbb{N}$ is large enough

$$\int_{\Omega \cap B_\rho(a)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} \leq \eta_0.$$
It follows then, from Proposition 9.11, that
\[ \sup_{n \in \mathbb{N}} \| Du_n \|_{L^\infty(B_{\rho/2}(a))} < +\infty. \] (9.36)

Let \( Q(y) := \text{dist}_N(y, N)^2 \). A direct computation shows that
\[ \Delta(Q(u_n)) = DQ(u_n)[\Delta u_n] + \sum_{i=1}^{2} D^2 Q(u_n)[\partial_i u_n, \partial_i u_n]. \] (9.37)

Since \( u_n \) satisfies the Ginzburg–Landau equation (1.5) and by (2.13), we have
\[ DQ(u_n)[\Delta u_n] = DQ(u_n) \left[ \frac{\nabla F(u_n)}{\epsilon_n^2} \right] \]
\[ = 2 \frac{\nabla F(u_n)}{\epsilon_n^2} \cdot (u_n - \Pi_N(u_n)) \geq \frac{2m_F}{\epsilon_n^2} \text{dist}(u_n, N)^2. \]

Moreover, by the computation of the second derivatives of the squared distance given in Remark 2.4, using (2.2) and the inequality \( \frac{1}{\sqrt{1-x}} \leq 1 + x \) on \((0, \frac{1}{2})\), we have for every \( z \in N_{2\delta_N}/2 \) and \( v \in \mathbb{R}^v \),
\[ D^2 Q(z)[v, v] = 2|v|^2 - 2D\Pi_N(z)[v] \cdot v \geq 2|v|^2 - \frac{2|v|^2}{\sqrt{1 - \frac{\text{dist}(z, N)}{\delta_N}}} \]
\[ \geq -\frac{2 \text{dist}(z, N)|v|^2}{\delta_N}. \]

Hence, by (9.36), (9.37) and the two preceding estimates, we have for \( n \) large enough, in \( B_{\rho/2}(a) \),
\[ \Delta(Q(u_n)) \geq \frac{2m_F}{\epsilon_n^2} \text{dist}(u_n, N)^2 - \frac{2 \text{dist}(z, N)|Du_n|^2}{\delta_N} \geq \frac{C_2}{\epsilon_n^2} Q(u_n) - C_3 \sqrt{Q(u_n)}. \]

We have thus proved that the function \( Q \circ u_n \) satisfies for \( n \) large enough
\[ \begin{cases} -\epsilon_n^2 \Delta(Q \circ u_n) + C_2 Q \circ u_n \leq C_3 \epsilon_n^2 \sqrt{Q \circ u_n} & \text{in } B_{\rho/2}(a) \cap \Omega, \\ Q \circ u_n = 0 & \text{on } B_{\rho/2}(a) \cap \partial \Omega, \end{cases} \] (9.38)

where the boundary condition holds because \( u_n \in N \) on \( \partial \Omega \). As in [44, Lemma 6 and Lemma 7], we deduce from the maximum principle that
\[ Q \circ u_n = \text{dist}(u_n, N)^2 \leq C_4 \epsilon_n^2 \text{ in } B_{\rho/4}(a) \cap \Omega. \]

Since \( |\nabla F|^2 = 0 \) on \( N \), by minimality, we have also \( D(|\nabla F|^2) = 0 \) on \( N \); as \( F \in C^3(N_\delta) \), this means that there is a constant \( C_5 \in (0, +\infty) \) with \( |\nabla F(z)|^2 \leq C_5 \text{dist}(z, N)^2 \) for every \( z \in \overline{N}_{\delta_N} \). Hence,
\[ |\Delta u_n| = \frac{|\nabla F(u_n)|}{\epsilon_n^2} \leq \sqrt{C_4 C_5} \text{ in } B_{\rho/4}(a) \cap \Omega. \] (9.39)

By elliptic estimates we obtain that \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( W^{2,p}_{\text{loc}}(B_{\rho/4}(a) \cap \overline{\Omega}) \) for every \( p \in [1, +\infty) \).
The $C^{1,\alpha}$ convergence is the best we can hope for if we consider convergence up to the boundary, since if we had $C^2$ convergence up to the boundary we would have $\Delta u^* = 0$ on the boundary which is incompatible with $-\Delta u^* = B_{u^*}(\nabla u^*, \nabla u^*)$, where $B_{u^*}$ is the second fundamental form of $\mathcal{N}$ at $u^*$, see [8, Remark 1] when $\mathcal{N} = S^1$. However it is natural to address the question of higher convergence in the interior of $\Omega$ away from the singularities. Since this relies on a bootstrap argument such a result is not easy to obtain for a general potential $F$ and should be rather addressed for specific $F$. We refer to [9] and [44] for results in this direction in the Ginzburg–Landau and Landau–de Gennes models.

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