A new design criterion for spherically-shaped division algebra-based space-time codes

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Abstract—This work considers normalized inverse determinant sums as a tool for analyzing the performance of division algebra based space-time codes for multiple antenna wireless systems. A general union bound based code design criterion is obtained as a main result.

In our previous work, the behavior of inverse determinant sums was analyzed using point counting techniques for Lie groups; it was shown that the asymptotic growth exponents of these sums correctly describe the diversity-multiplexing gain trade-off of the space-time code for some multiplexing gain ranges. This paper focuses on the constant terms of the inverse determinant sums, which capture the coding gain behavior. Pursuing the Lie group approach, a tighter asymptotic bound is derived, allowing to compute the constant terms for several classes of space-time codes appearing in the literature. The resulting design criterion suggests that the performance of division algebra based codes depends on several fundamental algebraic invariants of the underlying algebra.

I. INTRODUCTION

In the last decade the problem of designing optimal space-time codes for the multiple-input multiple-output (MIMO) Rayleigh fading channel has attracted much attention from the coding community. Maximizing the normalized minimum determinant of a space-time code has been widely used as a design criterion. However, this approach concentrates on minimizing the worst case pairwise error probability (PEP), and does not consider its overall distribution. The diversity-multiplexing gain trade-off (DMT), on the other hand, describes the asymptotic overall error probability as the signal-to-noise ratio and codebook size grow to infinity. These two criteria are independent. Codes with the same DMT can have dramatically different normalized minimum determinants and vice versa.

In [11] we proposed a new criterion based on the inverse determinant sum of the code, which arises from the union bound for the PEP [9]. This approach forms a middle ground between DMT and normalized minimum determinant based criteria. We also proved that in many cases the growth of the inverse determinant sums describes the DMT of a given code for multiplexing gains $r \in [0, 1]$.

This study evidenced how the multiplicative structure of the unit group of the code comes into play; by considering the classical embedding of the unit group into a Lie group, we provided a classification of division algebra based codes according to the growth exponent of their inverse determinant sums.

In this paper we consider a normalized version of the inverse determinant sum, which allows us to compare the coding gains of different division algebra based codes with the same growth exponent. This approach takes into account both the number of occurrences of the worst case error probability and the overall distribution. As a main result we will get a new design criterion for division algebra based space-time codes.

Our method follows the lines presented in [11] combining information of the zeta-function and of the unit group of a maximal order of a division algebra. However, we tighten the previous bound and use an explicit version of Lie point counting from [12]. A central role in the analysis is played by the Tamagawa volume formula, which allows us to give a detailed description of the growth of the unit group.

II. PRELIMINARIES

We consider a slow fading channel with $n_t$ transmit and $n_r$ receive antennas, where the decoding delay is $T$ time units. The channel equation is $Y = \sqrt{\rho/n_t} H X + N$, where $H \in M_{n_r \times n_t} (\mathbb{C})$ is the channel matrix and $N \in M_{n_r \times T} (\mathbb{C})$ is the noise matrix. The entries of $H$ and $N$ are assumed to be independent identically distributed (i.i.d.) zero-mean complex circular symmetric Gaussian random variables with variance 1. $X \in M_{n_t \times T} (\mathbb{C})$ is the transmitted codeword, and $\rho$ represents the signal to noise ratio.

A. Matrix Lattices and spherically shaped coding schemes

We now suppose that $n_t = T = n$.

Definition 2.1: A space-time lattice code $L \subseteq M_{n} (\mathbb{C})$ has the form $\mathbb{Z} B_1 \oplus \mathbb{Z} B_2 \oplus \cdots \oplus \mathbb{Z} B_k$, where the matrices $B_1, \ldots, B_k$ are linearly independent over $\mathbb{R}$, i.e., form a lattice basis, and $k$ is called the rank or the dimension of the lattice.

Definition 2.2: If the minimum determinant of the lattice $L \subseteq M_{n} (\mathbb{C})$ is non-zero, i.e. it satisfies

$$\inf_{0 \neq X \in L} |\det(X)| > 0,$$

we say that the code has a non-vanishing determinant (NVD).

Let $\|\cdot\|_F$ be the Frobenius norm. For $M > 0$ we define the finite code

$$L(M) = \{ a | a \in L, \|a\|_F \leq M \}.$$
and the sphere with radius \( M \)
\[
B(M) = \{ a | a \in M_n(\mathbb{C}) , \|a\|_F \leq M \}.
\]

Let \( L \subseteq M_n(\mathbb{C}) \) be a \( k \)-dimensional lattice. For any fixed \( m \in \mathbb{Z}^+ \) we define
\[
S^m_L(M) := \sum_{X \in L(M) \setminus \{0\}} \frac{1}{|\text{det}(X)|^m}.
\]

Our main goal is to study the growth of this sum as \( M \) increases. Note, however, that in order to have a fair comparison between two different space-time codes, these should be normalized to have the same average energy. Namely, the volume \( \text{Vol}(L) \) of the fundamental parallelotope
\[
P(L) = \{ \alpha_1 B_1 + \alpha_2 B_2 + \ldots + \alpha_k B_k | \alpha_i \in [0,1) \ \forall i \}
\]
should be normalized to 1. The normalized version of the inverse determinent sums problem is then to consider the growth of the sum \( \hat{S}^m_L(M) = S^m_L(M) \) over the lattice \( \hat{L} = \text{Vol}(L)^{-1/k}L \). Since \( \hat{L}(M) = \text{Vol}(L)^{-1/k}(M \text{Vol}(L)^{1/k}) \), we have
\[
\hat{S}^m_L(M) = \text{Vol}(L)^{mn/k}S^m_L(M \text{Vol}(L)^{1/k}). \tag{1}
\]

B. Cyclic division algebras, maximal orders and zeta functions

Let us now consider the mathematical theory that most easily gives us high dimensional NVD lattices.

Let \( E/K \) be a cyclic field extension of degree \( n \) with Galois group \( \text{Gal}(E/K) = \langle \sigma \rangle \). Define a cyclic algebra
\[
D = (E/K, \sigma, \gamma) = E \oplus uE \oplus u^2E \oplus \ldots \oplus u^{n-1}E,
\]
where \( u \in D \) is an auxiliary generating element subject to the relations \( xu = u\sigma(x) \) for all \( x \in E \) and \( u^n = \gamma \in K^* \). We assume that \( D \) is a division algebra.

Every element \( x = x_0 + u x_1 + \ldots + u^{n-1}x_{n-1} \in D \) has the following left regular representation as a matrix \( \psi(x) \):
\[
\begin{pmatrix}
  x_0 & \gamma x_1 & \ldots & \gamma^{n-1}x_{n-1} \\
  x_1 & \sigma(x_0) & \gamma x_1 & \ldots & \gamma^{n-2}x_{n-2} \\
  x_2 & \sigma(x_1) & \sigma(x_0) & \gamma x_1 & \ldots & \gamma^{n-3}x_{n-3} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_{n-1} & \sigma(x_{n-2}) & \ldots & \sigma(x_{n-3}) & \ldots & \gamma x_0 \\
end{pmatrix}
\]

The mapping \( \psi \) is an injective \( K \)-algebra homomorphism that allows us to identify \( D \) with its image in \( M_n(\mathbb{C}) \). Note that for \( x \in D \), \( \text{det}(\psi(x)) = n \gamma(x) \), the reduced norm of \( x \).

We recall here some concepts concerning the theory of orders in division algebras. Due to lack of space, we have reduced the exposition to a minimum; we refer the reader to [3].

**Definition 2.3:** Let \( \mathcal{O}_K \) be the ring of integers of \( K \). An \( \mathcal{O}_K \)-order \( \Lambda \) in \( D \) is a subring of \( D \), having the same identity element as \( D \), and such that \( \Lambda \) is a finitely generated module over \( \mathcal{O}_K \) and generates \( D \) as a linear space over \( K \).

We say that \( \Lambda \) is a maximal order if it is not properly contained into any other \( \mathcal{O}_K \)-order of \( D \).

Let \( \{ w_1, \ldots, w_{n^2} \} \) be a basis of a maximal order \( \Lambda \) over \( \mathcal{O}_K \). The relative discriminant of \( \Lambda \) over \( \mathcal{O}_K \) is defined by
\[
d(\Lambda|\mathcal{O}_K) = \text{det}\left( \text{tr}(w_i w_j) \right)_{i,j=1}^{n^2},
\]
and doesn’t depend on the choice of maximal order. We denote by \( \text{Ram}(D) \) the set of primes of \( \mathcal{O}_K \) which divide \( d(\Lambda|\mathcal{O}_K) \), which are also called the ramified primes [8]. Moreover, for each \( p \in \text{Ram}(D) \), one can define a notion of ramification index \( 1 < m_p \leq n \) such that \( m_p|n \) and
\[
d(\Lambda|\mathcal{O}_K) = \prod_{p \in \text{Ram}(D)} p^{(m_p-1)\frac{n}{m_p}}. \tag{2}
\]

Given an order \( \Lambda \), we define its Hecke zeta function as
\[
\zeta_{\Lambda}(s) = \sum_{I} \frac{1}{|\Lambda : I|^s}, \tag{3}
\]
where the sum is taken over all right ideals \( I \) of \( \Lambda \). A more explicit formula for \( \zeta_{\Lambda}(s) \) is given in [2] p. 175:
\[
\zeta_{\Lambda}(s) = \prod_{i=0}^{n-1} \prod_{p \in \text{Ram}(D)} \prod_{|j| \leq n-1 \backslash j \not= 0 \mod m_p} (1 - N(p)^{-ns}). \tag{4}
\]

Here \( \zeta_{\Lambda}(s) \) is the Dedekind zeta function of the center \( K \), and \( N(p) = |\mathcal{O}_K/p| \) is an imaginary quadratic number field, \( N(p) = |p|^2 \), and if \( K = \mathbb{Q} \), \( N(p) = p \).

The function \( \zeta_{\Lambda}(s) \) is well-defined for \( \Re(s) > 1 \), but diverges for \( s \to 1 \).

In the following we will suppose that the center \( K \) of our algebra is either \( \mathbb{Q} \) or a complex quadratic field \( \mathbb{Q}(\sqrt{-d}) \). Then \( L = \psi(\Lambda) \) is a lattice in \( M_n(\mathbb{C}) \), of dimension \( k = n^2 \) if \( K = \mathbb{Q} \) and \( k = 2n^2 \) if \( K = \mathbb{Q}(\sqrt{-d}) \), and we can consider the corresponding inverse determinant sums.

C. Inverse Determinant Sums and the Unit Group

The unit group \( \Lambda^* \) of an order \( \Lambda \) consists of elements \( x \in \Lambda \) such that there exists an \( y \in \Lambda \) with \( xy = 1 \).

If \( K = \mathbb{Q} \) or \( K = \mathbb{Q}(\sqrt{-d}) \), the units of reduced norm 1 form a subgroup of finite index in \( \Lambda^* \) [6] p. 221:

**Lemma 2.1:** The unit group \( \Lambda^* \) has a subgroup
\[
\Lambda^1 = \{ x | x \in \Lambda^*, \text{nr}(x) = 1 \},
\]
and we have \( [\Lambda^* : \Lambda^1] < \infty \).

**Remark 2.1:** When \( D \) is a quaternion algebra with no real ramified places, \( \text{nr} : \Lambda^* \to \mathcal{O}_K^* \) is surjective [7] Theorem 11.6.1] and therefore \( [\Lambda^* : \Lambda^1] = |\mathcal{O}_K^*| \). The cardinality \( |\mathcal{O}_K^*| \) is equal to 2 if \( K = \mathbb{Q} \) and \( K = \mathbb{Q}(\sqrt{-d}) \), except for the special cases \( K = \mathbb{Q}(i) \) (\( |\mathcal{O}_K^*| = 4 \)) and \( K = \mathbb{Q}(e^{\pi i / 3}) \) (\( |\mathcal{O}_K| = 6 \)).

We have shown in [11] proof of Proposition 6.7] that the growth of the inverse determinant sum for \( L = \psi(\Lambda) \) is completely characterized by the growth of the unit group:
\[
S^2_{\psi(\Lambda)}(M) = \sum_{x \in \mathcal{X}(M)} |\psi(x \Lambda^*) \cap B(M)| / |\text{det}(\psi(x))|^{2n^2}, \tag{5}
\]
where \( X(M) \) is some collection of elements \( x \in \Lambda \) such that \( \|\psi(x)\|_F \leq M \), each generating a different right ideal. Let \( j = [\Lambda^*: \Lambda^1] \). By choosing a set \( \{a_1, \ldots, a_j\} \) of coset leaders of \( \Lambda^1 \) in \( \Lambda^* \), we have

\[
S_{\psi(\Lambda)}^{2n_r}(M) = \sum_{x \in X(M)} \sum_{i=1}^j \frac{|\psi(xa_i\Lambda^1) \cap B(M)|}{|\det(\psi(x))|^{2n_r}}.
\] (6)

To obtain a good estimate of the inverse determinant sum bound, we need to study the behavior of the terms \( |\psi(xa_i\Lambda^1) \cap B(M)| \). This will be done in the next section using some tools from Lie group theory.

### III. Lie Groups, Lattices and Volumes of Spheres

In this section we will consider a Lie group \( G \), where \( G = SL_n(\mathbb{R}) \), \( SL_n(\mathbb{C}) \) or \( SL_n(\mathbb{H}) \), and its arithmetic lattice subgroups, that are discrete subgroups having finite covolume. In the following we will discuss the problem of counting the number of points of these subgroups that lie inside the sphere \( B(M) \). We refer the reader to [4] for the relevant definitions and an introduction to the subject. Here we consider \( SL_n(\mathbb{H}) \) as embedded in \( M_{2n}(\mathbb{C}) \) by replacing each quaternion element by its common \( 2 \times 2 \) matrix representation.

Each of these groups admits a multiplicative Haar measure that gives us a natural concept of volume \( \text{Vol}_G \). In particular we can consider the volumes of the balls \( \text{Vol}_G(B(M)) \), where \( B(M) \) here refers to all the matrices in \( G \) that have Frobenius norm smaller than \( M \).

Let us now concentrate on lattice subgroups \( H \) that are cocompact, meaning that the factor group \( G/H \) is compact. In the following two results we suppose that \( G \) is one of the previously mentioned Lie groups.

**Theorem 3.1 (Corollary 1.11 and Remark 1.12, [3]):** Consider a Lie group \( G \), a discrete cocompact lattice \( H \subset G \) and \( x \in G \). We then have that

\[
\lim_{M \to \infty} \frac{|xH \cap B(M)|}{\text{Vol}_G(B(M))} = \frac{1}{\text{Vol}_G(G/H)}
\]

The limit is approached uniformly for all \( x \in G \). The asymptotic growth of the arithmetic lattice is thus completely determined by the volume of the ball \( \text{Vol}_G(B(M)) \). The following estimate holds:

**Lemma 3.2:** We have that

\[
\text{Vol}_G(B(M)) \sim C_G M^T,
\]

where the growth exponent is

- \( T = n^2 - n \) if \( G = SL_n(\mathbb{R}) \),
- \( T = 2n^2 - 2n \) if \( G = SL_n(\mathbb{C}) \),
- \( T = 4n^2 - 4n \) if \( G = SL_n(\mathbb{H}) \).

This result is a consequence of a general theorem of [3]. The computation of these exponents in the cases \( SL_n(\mathbb{R}) \), \( SL_n(\mathbb{C}) \), \( SL_n(\mathbb{H}) \) can be found in our previous work [11, Appendix A].

By rescaling both the discrete set and the ball, recalling that \( |\det(\psi(a_i))| = 1 \), we have

\[
|\psi(xa_i\Lambda^1) \cap B(M)| \leq \frac{\psi(xa_i\Lambda^1)}{|\det(\psi(x))|^{2n_r}} \cap B \left( \frac{M}{|\det(\psi(x))|^{2n_r}} \right)
\]

Suppose that \( H = \psi(\Lambda^1) \) is a cocompact lattice subgroup of \( G \), where \( G = SL_n(\mathbb{C}), SL_n(\mathbb{R}) \) or \( SL_{n/2}(\mathbb{H}) \). Note that the scaled set

\[
\frac{\psi(xa_i\Lambda^1)}{|\det(\psi(x))|^{2n_r}} \cap B(M) \sim \frac{\text{Vol}_G(B(M)\text{det}(\psi(x))^{-\frac{n_r}{2}})}{\text{Vol}_G(G/\psi(\Lambda^1))}
\]

is of the form \( y_iH \) with \( y_i = \psi(xa_i)/|\det(\psi(xa_i))^{\frac{n_r}{2}} \in G \). Using Lemma 3.2 we then have the asymptotic estimate

\[
|\psi(xa_i\Lambda^1) \cap B(M)| \sim \frac{C_G[M^T]}{\text{Vol}_G(G/\psi(\Lambda^1))}|\det(\psi(x))|^{\frac{n_r}{2}}.
\]

Combining equations (6) and (7), we obtain

\[
S_{\psi(\Lambda)}^{2n_r}(M) \sim \frac{C_G[\Lambda^*: \Lambda^1]M^T}{\text{Vol}_G(G/\psi(\Lambda^1))} \sum_{x \in X(M)} \frac{1}{|\det(\psi(x))|^{2n_r+T/n}}
\]

Let \( K = \mathbb{Q} \) or \( \mathbb{Q}(\sqrt{-d}) \). Since the index of a principal right ideal \( x\Lambda \) of \( \Lambda \) is given by \( [\Lambda : x\Lambda] = N_{\overline{D}/D}(x) = |\det(\psi(x))|^{\frac{1}{2}[K:\mathbb{Q}]} \), recalling the definition of the Hey zeta function (3), we have

\[
\sum_{x \in X(M)} \frac{1}{|\det(x)|^m} \leq \sum_{x \in X(M)} \frac{1}{[\Lambda : x\Lambda]^{\frac{m}{m-K}}} \leq \zeta_{\Lambda} \left( \frac{m}{n[K:\mathbb{Q}]} \right).
\]

Note that if all right ideals of \( \Lambda \) are principal, then this bound is asymptotically tight. We can now state the following lemma:

**Lemma 3.3:** Let \( \Lambda \) be a maximal order in a division algebra \( D \) of degree \( n \) over \( K \), where \( K = \mathbb{Q} \) or \( \mathbb{Q}(\sqrt{-d}) \), such that all right ideals of \( \Lambda \) are principal. Suppose that \( \psi(\Lambda^1) \) is a cocompact lattice subgroup of \( G \), where \( G = SL_n(\mathbb{C}), SL_n(\mathbb{R}) \) or \( SL_{n/2}(\mathbb{H}) \). Let \( a = \frac{2n_r+T/n}{n-K} > 1 \). Then the normalized inverse determinant sum is asymptotically given by:

\[
\tilde{S}_{\psi(\Lambda)}^{2n_r}(M) \sim \frac{C_G[\Lambda^*: \Lambda^1]\text{Vol}_G(\psi(\Lambda^1))^{2n_r+T/n}}{\text{Vol}_G(G/\psi(\Lambda^1))}\zeta_{\Lambda}(a) M^T.
\]

### IV. Inverse Determinant Sums of Central Division Algebras over Complex Quadratic Fields

Consider the case where \( D \) is an index \( n \) \( K \)-central division algebra, where \( K = \mathbb{Q}(\sqrt{-d}) \) is a complex quadratic field such that \( O_K \) is a principal ideal domain (PID). The dimension of the lattice \( \Lambda \) is then \( k = 2n^2 \), and the volume of its fundamental parallelepiped is [10]

\[
\text{Vol}(\psi(\Lambda)) = 2^{-n^2 \sqrt{|d(\Lambda:Z)|}}.
\]

In the following we will denote \( SL_n(\mathbb{C}) \) with \( G \). Note that \( \psi(\Lambda^1) \subset SL_n(\mathbb{C}) \) and that it is a cocompact lattice subgroup...
Then, all right ideals of \( \Lambda \) are principal \( \text{[49]} \) so that Lemma \( \text{[54]} \) holds. Specializing the Lemma to the complex quadratic case, we obtain for \( n_r > 1 \)

\[
\hat{S}_{\psi(\Lambda)}^{2n_r}(m) \sim \frac{C_G}{2^{n_r^2}} \frac{|O_{\Lambda}| |d(\Lambda[Z])|^{\frac{1}{2}}}{\text{Vol}(G/\psi(\Lambda))} \zeta(t) M^{2n^2 - 2n} \tag{8}
\]

where \( t = \frac{n_r}{n} + 1 - \frac{1}{n} \). Here we have used the fact that \([\Lambda^* : \Lambda] = |O_{\Lambda^*}| \) (Remark \[2.1\]).

**A. Quaternion division algebras with complex quadratic center**

Let us now concentrate on the case where we have a quaternion division algebra \((n = 2)\). Note that we have \( d(\Lambda[Z]) = |d(\Lambda[O_{\Lambda}])|^2 d(O_{\Lambda}[Z])^4 \).

From the discriminant formula \( \text{[2]} \), remarking that \( m_p = 2 \) for the ramified primes, we get

\[
|d(\Lambda[O_{\Lambda}])| = \prod_{p \in \text{Ram}_f(D)} |p|^2 = \prod_{p \in \text{Ram}_f(D)} N(p)
\]

In the quaternion case, the covolume of the unit group \( \Lambda^1 \) in \( \text{SL}_2(\mathbb{C}) \) can be computed explicitly and is given by the Tamagawa volume formula (see \[7\] equation (11.2)), and \[12\] Chapitre IV, Corollaire 1.8):

\[
\text{Vol}(G/\psi(\Lambda^1)) = |d(O_{\Lambda}[Z])|^{\frac{1}{2}} \zeta_K(2) \prod_{p \in \text{Ram}_f(D)} (N(p) - 1)
\]

Let \( s = \frac{n_r}{n} \), so that \( t = s + \frac{1}{2} \). From equation \( \text{[4]} \) we have

\[
\zeta_{\Lambda}(t) = \zeta_K(2s + 1) \zeta_K(2s) \prod_{p \in \text{Ram}_f(D)} (1 - N(p)^{-2s})
\]

After simplifying the expression \( \text{[8]} \), we obtain

\[
\hat{S}_{\psi(\Lambda)}^{2n_r}(m) \sim C_G |O_{\Lambda^*}^r| |d(O_{\Lambda}[Z])|^{2s - \frac{1}{2}} \frac{\zeta_K(2s + 1) \zeta_K(2s)}{2^{n_r^2}} \quad \cdot \prod_{p \in \text{Ram}_f(D)} \frac{N(p)^{1/2} (1 - N(p)^{-2s})}{N(p)}
\]

For the symmetric case \( n_r = n = 2 \), we finally get:

\[
\hat{S}_{\psi(\Lambda)}^{4}(m) \sim \zeta_K(3) C_G |O_{\Lambda^*}^r| |d(O_{\Lambda}[Z])|^{\frac{3}{2}} \quad \cdot \prod_{p \in \text{Ram}_f(D)} \left( N(p)^{1/2} + N(p)^{-1/2} \right) M^4 \tag{9}
\]

**Example 4.1:** Suppose \( K = \mathbb{Q}(i) \). To find the best maximal order code according to equation \( \text{[9]} \), we need to minimize the product \( \prod_{p \in \text{Ram}_f(D)} (N(p)^{1/2} + N(p)^{-1/2}) \). The function \( x \mapsto \sqrt{x} + 1/\sqrt{x} \) being increasing for \( x \geq 1 \), this can be done by choosing the smallest possible number of ramified primes, which is two, and the two primes with the smallest possible norm. This design criterion coincides with the one proposed in \[10\] based on the normalized minimum determinant.

**V. INVERSE DETERMINANT SUMS OF \( \mathbb{Q} \)-CENTRAL DIVISION ALGEBRAS**

We now suppose that \( D \) is a division algebra with center \( \mathbb{Q} \). We distinguish two main cases, depending on the ramification of the algebra at infinity.

**Definition 5.1:** Let \( D \) be an index \( n \) \( \mathbb{Q} \)-central division algebra. If

\[
D \otimes_\mathbb{Q} \mathbb{R} \cong M_n(\mathbb{R}),
\]

we say that \( D \) is not ramified at the infinite place (or split). If \( 2 | n \) and

\[
D \otimes_\mathbb{Q} \mathbb{R} \cong M_{n/2}(\mathbb{H}),
\]

we say that \( D \) is ramified at the infinite place.

We will refer to the isomorphism given in the previous definition as \( \psi_1 \). The mapping \( \psi_1 \) has similar properties to the mapping \( \psi \) obtained by the left regular representation (in particular the results about norms and lattice structure of \( \psi_1(A) \) are true; see \[\text{[11]}\] for more details). Note that every right ideal of \( \Lambda \) is principal except possibly when \( D \) is a quaternion algebra which is ramified at the infinite place \( \text{[8]} \).

**A. Split division algebras with center \( \mathbb{Q} \)**

Suppose \( K = \mathbb{Q} \) and \( D \otimes_\mathbb{Q} \mathbb{R} = M_n(\mathbb{R}) \), and let \( \Lambda \) be a maximal \( \mathbb{Z} \)-order of \( D \). The dimension of the lattice \( \Lambda \) is \( k = n^2 \), and the fundamental parallelotope has volume \( \text{Vol}(\psi_1(A)) = |d(\Lambda[Z])|^{\frac{1}{2}} \).

In the following we will denote \( \text{SL}_2(\mathbb{R}) \) with \( G \). Just as before we have that \( \psi_1(\Lambda) \subseteq G \) and that it is a cocompact lattice subgroup \( \text{[6]} \) Theorem 1]. Specializing Lemma \( \text{[5.5]} \) to the split rational case, we obtain for \( s = \frac{n_r}{n}, t = s + 1 - \frac{1}{n}\)

\[
\hat{S}_{\psi(\Lambda)}^{2n_r}(m) \sim \frac{2 C_G \text{Vol}(\psi_1(A)^t)}{2^{n_r^2}} \zeta(t) M^{n^2 - n}
\]

Here we have used the fact that \([\Lambda^* : \Lambda] = 2 \) (Remark \[2.1\]). If \( D \) is a quaternion algebra \((n = 2)\), we have the following Tamagawa volume formula for the unit group \( \text{[7]} \) \[12\]:

\[
\text{Vol}(\text{SL}_2(\mathbb{R})/\psi_1(\Lambda^1)) = \zeta(2) \prod_{p \in \text{Ram}_f(D)} (p - 1),
\]

where \( \zeta \) denotes the Riemann zeta function. Using the formula \( \text{[4]} \) for the Hey zeta function, we obtain

\[
\hat{S}_{\psi(\Lambda)}^{2n_r}(m) \sim \frac{2 C_G \zeta(2s + 1) \zeta(2s)}{\zeta(2)} \prod_{p \in \text{Ram}_f(D)} \frac{p^{s + \frac{1}{2}} (1 - p^{-2s})}{p - 1} M^2.
\]

When \( s = 1 \), corresponding to \( n_r = n/2 \), we get

\[
\hat{S}_{\psi(\Lambda)}^{2n_r}(m) \sim 2 C_G \zeta(3) \prod_{p \in \text{Ram}_f(D)} (p^{1/2} + p^{-1/2}) M^2.
\]
B. Ramified division algebras with center \( \mathbb{Q} \)

Suppose \( K = \mathbb{Q} \) and \( D \otimes \mathbb{Q} \mathbb{R} = M_{n/2}(\mathbb{H}) \), and let \( \Lambda \) be a maximal \( \mathbb{Z} \)-order of \( D \). The dimension of the lattice \( \Lambda \) is \( k = n^2 \), and its fundamental parallelohedron has again volume \( \text{Vol}(\psi_1(\Lambda)) = [d(\Lambda)]^{\frac{k}{2}} \). \( \Box \)

In the following we will denote \( \text{SL}_{n/2}(\mathbb{H}) \) with \( G \). Just as before we have that \( \psi_1(\Lambda^1) \subseteq G \) and that it is a cocompact lattice subgroup [6, Theorem 1]. As discussed before, Lemma 3.3 holds if \( n > 2 \). In this case, we have for \( t = \frac{2n}{n} + 1 - \frac{2n}{n} \),

\[
\tilde{S}_{\psi_1}(\Lambda)(M) \sim \frac{2C_G \text{Vol}(\psi_1(\Lambda))^t}{\text{Vol}_G(G/\psi_1(\Lambda))} \zeta_4(t) M^{n^2-2n}.
\]

If \( n = 2 \) we have growth exponent \( T = 0 \). Indeed, the group of units \( \Lambda^1 \) is a finite subgroup of the compact group \( \text{SL}_1(\mathbb{H}) \cong \{a, b \in \mathbb{C} \mid |a|^2 + |b|^2 = 1\} \), which is a 4-dimensional sphere.

Let us now suppose that \( \Lambda \) has class number 1, so that every right ideal is principal. The finite unit group changes our analysis slightly and we can use directly equation (3) to get

\[
\tilde{S}_{\psi_1}(\Lambda)(M) \sim \zeta_4(s, |d(\Lambda[Z])^{\frac{k}{2}} M^4 |d(\Lambda[Z])^{\frac{k}{2}} |\Lambda^*|),
\]

where \( \zeta_4(s, M) \) denotes the truncated Hey zeta function (over the ideals with index smaller than \( M \)). The bound is asymptotically tight since \( |\det(\psi_1(x))| = ||\psi_1(x)||_F^2/2 \).

Let us now concentrate on the scenario where \( n_r = 1 \). The previous then transforms into

\[
\tilde{S}_{\psi_1}(\Lambda)(M) \sim \zeta(1, |d(\Lambda[Z])^{\frac{k}{2}} M^4 |\Lambda^*|) \quad (2).
\]

If \( \Lambda \) has class number 1, the Eichler mass formula gives

\[
\prod_{p \in \text{Ram}_f(D)} (p-1) |\Lambda^*| = 24.
\]

We also have that \( |d(\Lambda[Z])|^{1/2} = \prod_{p \in \text{Ram}_f(D)} p \). Equation (4) then implies the following:

**Proposition 5.1:** Let \( D \) be a \( \mathbb{Q} \)-central quaternion division algebra, which is ramified at infinity, and that \( \Lambda \) is a maximal order in \( D \). If \( \Lambda \) has class number 1, then

\[
\tilde{S}_{\psi_1}(\Lambda)(M) \sim 24 \zeta(1, |d(\Lambda[Z])^{\frac{k}{2}} M^4 |\Lambda^*|) \quad (2).
\]

Therefore, we expect all space-time codes carved from maximal orders of class number 1 in quaternion division algebras of this type to have asymptotically the same performance when using one receive antenna.

**Example 5.1:** Consider the cyclic division algebras \( \mathcal{H}_2 = (\mathbb{Q}(i)/\mathbb{Q}, \sigma, -1) \) and \( \mathcal{H}_7 = (\mathbb{Q}(\sqrt{-7})/\mathbb{Q}, \sigma, -1) \), where \( \sigma \) denotes complex conjugation. Note that \( \text{Ram}_f(\mathcal{H}_2) = \{2\} \), \( \text{Ram}_f(\mathcal{H}_7) = \{7\} \). The corresponding maximal orders \( \Lambda_2, \Lambda_7 \) have class number 1. Note that the Hurwitz order \( \Lambda_2 \) contains the order of the Alamouti code.

From Proposition 5.1, we expect similar performance for these codes for one receive antenna when using large signal constellations. For two receive antennas, we get

\[
\tilde{S}_{\psi_1}(\Lambda) \sim \zeta(3) \zeta(4) \prod_{p \in \text{Ram}_f(D)} (p + 1 + 1/p),
\]

so we expect better performance from the Hurwitz order \( \Lambda_2 \), which has a smaller ramified prime. Figure 1 shows that this is the case, and that the performance gap increases for \( n_r = 3 \). Note that for finite constellations \( \Lambda_2 \) is still slightly better than \( \Lambda_7 \) even for one receive antenna.

![Fig. 1. Simulation results for spherically-shaped codes based on quaternion algebras over \( \mathbb{Q} \) which are ramified at infinity, using 4-PAM constellations.](image)

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