The sharp exponent in the study of the nonlocal Hénon equation in $\mathbb{R}^N$: a Liouville theorem and an existence result

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Abstract
We consider the nonlocal Hénon equation
\[ (-\Delta)^s u = |x|^\alpha u^p, \quad \mathbb{R}^N, \]
where $(-\Delta)^s$ is the fractional Laplacian operator with $0 < s < 1$, $-2s < \alpha$, $p > 1$ and $N > 2s$. We prove a nonexistence result for positive solutions in the optimal range of the nonlinearity, that is, when
\[ 1 < p < p^*_{\alpha,s} := \frac{N + 2\alpha + 2s}{N - 2s}. \]
Moreover, we prove that a bubble solution, that is a fast decay positive radially symmetric solution, exists when $p = p^*_{\alpha,s}$.

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1 Introduction
Problems with nonlocal diffusion that involve the fractional Laplacian operator, and other integro-differential operators, have been intensively studied in the last years since they appear when we try to model different physical situations as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics and relativistic quantum mechanics of stars (see [10,20] and references). They also appear in mathematical finance
(cf. [2, 21]), elasticity problems [58], obstacle problems [5, 6], phase transition [1] and crystal dislocation [27, 64] among others.

The aim of this work is to study the existence of positive solutions of the nonlocal Hénon equation

$$(-\Delta)^s u = |x|^\alpha u^p, \quad \mathbb{R}^N, \quad N > 2s.$$  \hspace{1cm} (1.1)

specified We will always consider $\alpha > -2s$ because in the case $\alpha \leq -2s$ it can be proved that there are no solutions of (1.1) (see Remark 2.2 ii) below).

The operator $(-\Delta)^s, 0 < s < 1$, is the well-known fractional Laplacian, which is defined on smooth functions as

$$(-\Delta)^s u(x) = a_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$  \hspace{1cm} (1.2)

where $a_{N,s}$ is a normalization constant that is usually omitted for brevity. The integral in (1.2) has to be understood in the principal value sense, that is, as the limit as $\epsilon \to 0$ of the same integral taken in $\mathbb{R}^N \setminus B_\epsilon(x)$, i.e., the complementary of the ball of center $x$ and radius $\epsilon$. It is clear that the fractional Laplacian operator is well defined for functions that belong, for instance, to $L^2_s \cap C^{1,1}_{loc}$ where

$$L^2_s := \left\{ u : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} < \infty \right\}.$$ 

Throughout the work we will call these solutions classical or pointwise. Of course other kinds of solutions can be considered. The weakest are the distributional solutions $u \in L^2_s$ introduced in [59] for which the equation is satisfied using the duality product

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^N} u(-\Delta)^s \varphi, \quad \varphi \in \mathcal{D}.$$ 

We will specify along the work for which kind of solutions our results can be proved (see Theorems 1.1–1.3). We suggest to the reader to see, for instance, [26, 59, 61] to read more about the basic properties of the nonlocal operator and the normalization constant that appears in (1.2).

In the local case, that is, when $s = 1$, the Hénon equation has been introduced in models of astrophysics (see for instance [41]). A close related model was introduced by Matukuma in 1930 with a different weight function $K(|x|)$ instead of $|x|^\alpha$. The case $\alpha = 0$ was studied by Emden–Fowler–Lane and, thus, in this case, (1.1) is sometimes called the Emden–Fowler–Lane equation. We notice that the solutions of the Hénon equation, and also Emden–Fowler–Lane equation, in the critical case, have a relationship with the best constants in Sobolev–Hardy inequality (see for example [47]). Moreover these kinds of problems are also connected with the mass transport and Yamabe problem (see for instance [22, 57, 66]). Since the previous one are cornerstone problems in Analysis it is hard to give a exhaustive list of references on the topic, we just mention, for example, [3, 13, 15, 17, 38, 39, 52, 62, 63].

In the nonlocal framework, the Hénon equation has been studied in, for instance, [30, 31, 45, 49, 60, 67].

Returning to the local case ($s = 1$), the nonexistence of positive solutions of (1.1) when $p$ is subcritical, that is,
1 < p < \frac{N + 2\alpha + 2}{N - 2},
and \( \alpha > -2 \) has been established by Wang et al. in [65]. The strategy used in [65], that can be also applied to more general weights \( K(x) \neq |x|^\alpha \), is based on the moving spheres method (see for example [46]) and the conclusion is obtained by analyzing the properties of the eigenvalues of the Laplacian. This idea is the one that we will follow to prove our nonexistence theorem in the nonlocal framework.

We also mention that in the recent work [37], García-Melián, by using also the method of moving spheres, deduces that, by proving a monotonicity property, the solutions are stables. This fact allows him to apply the Liouville theorem proved in [24] to conclude. See also [8,51,54] for previous results in which other kinds of Liouville theorems were proved in the non-optimal range or for special dimensions.

Regarding the critical case
\[
p = \frac{N + 2\alpha + 2}{N - 2},
\]
the situation is completely different when \( \alpha > 0 \) or \(-2 < \alpha < 0\). In fact, for positive values of the parameter, in [40] it was proved that there exist positive solutions that are not radially symmetric. When \( \alpha \) is negative the complete classification of all the possible solutions was also established in [37] using some partial results previously done in [28].

In the nonlocal case, as far as we know, it is not known a general Liouville result in all the expected range. There are some partial results in [30] where the authors obtained a non-existence theorem when \( \alpha > 0 \) and
\[
\frac{N + \alpha}{N - 2s} < p < \frac{N + \alpha + 2s}{N - 2s} := \hat{p}.
\]
We observe that the range is not optimal and, in our opinion, the authors in [30] have a gap in the proof of their non-existence result [30, Theorem 1.2]. More precisely, in a technical lemma, they used a local inequality, proved in [50] in the local framework, that should be checked for the solutions in the nonlocal case. Moreover, when \( p = \hat{p} \), by using the work of [49], they affirm in [30, Theorem 1.3] that if there exists a nonnegative solution then it has to be radially symmetric around the origin. However the result of Lu and Zhu that they cited can be applied only when \( \alpha \) is negative, not when \( \alpha \) is positive, and \( p = p_{\alpha,s}^* \) (not \( p = \hat{p} \)), where
\[
p_{\alpha,s}^* := \frac{N + 2\alpha + 2s}{N - 2s} \neq \hat{p}.
\]

As is happening in the local case we guess that the existence of solutions in the case \( p = \hat{p} \) is not awaited when \( \alpha > 0 \). For that reason our first objective is to obtain the following

**Theorem 1.1** If we consider \( 0 < s < 1, \alpha > -2s \) and
\[
1 < p < p_{\alpha,s}^*,
\]
then the problem (1.1) does not have any locally bounded positive distributional solution, if in the case \( N < 4s \) we additionally assume \( \alpha > -N/2 \).

Since to prove Theorem 1.1 we will follow, mainly, the strategy developed in [65], that can be applied to more general weights \( K(x) \neq |x|^\alpha \), our nonexistence result can be also proved for some more general equations than (1.1). However, for simplicity of the exposition, we will consider the particular case \( K(x) = |x|^\alpha \).
Remark 1.2 Just before submitting this version of the manuscript we found a very recent preprint where the conclusion of our Theorem 1.1 can be deduced (see [23]). Nevertheless, although the techniques used by Dai and Qin are similar as the one we develop here (mainly the moving sphere by using the Kelvin transform for the fractional Laplacian) our proof is different and we consider that our arguments and steps are much more simple and direct than the used by Dai and Qin. Moreover, we think that the technique used by the authors to prove the Liouville’s result given in [23, Theorem 1.2] cannot be applied if \(-2s < \alpha < 0\) because a fundamental inequality (see [23, (2.28)]), needed to apply the Maximum principle for Narrow Domain stated in [23, Theorem 2.7], fails for this range of \(\alpha\).

To complement the previous nonexistence result we will study the existence of the solutions when \(p = p^*_{\alpha,s}\) for some range of \(\alpha\) and, in addition we will establish the fast decay estimates for the solutions that were not known until now. To explain the type of result that we have obtained in this context we want to highlight that, in the local case, writing the Laplacian in radial coordinates, it is very easy to check that the “bubble” functions
\[
b(x) = C(N, \alpha) \left( \frac{1}{1 + |x|^{2+s}} \right)^{\frac{N-2}{s}},
\]
are solutions of (1.1) when \(s = 1\). However in the nonlocal case this computation is not at all straightforward because the formula of the fractional Laplacian for radial functions has not a simple form (see [34]). Notice also that, in the nonlocal case when \(\alpha = 0\) the bubble is given by
\[
b(x) = C(N, s) \left( \frac{\lambda}{\lambda^2 + |x|^2} \right)^{\frac{N-2s}{2}},
\]
for some \(C(N, s), \lambda > 0\), (see [47]). We show up here that the computations in [47] cannot be adapted for the case \(\alpha \neq 0\) since the Fourier transform of
\[
u_{\mu}(x) = (1 + |x|^{\gamma})^{-\mu}, \quad \mu > 0, \quad 0 < \gamma \neq 2,
\]
does not have the desired expression as occurs when \(\gamma = 2\) (see Sect. 4 below). Thus, in order to prove the existence of the “bubble” solution of (1.1) in the critical case, that is, a radial solution when \(p = p^*_{\alpha,s}\), we have to use an alternative approach passing through an Emden-Fowler change of variables that forces us to study an alternative problem similar as the one that appears in [25]. For that we make a one-dimensional reduction argument and we work with a new problem of the form
\[
\mathcal{T}\bar{v}(\kappa) + A_{x,N} \bar{v}(\kappa) = \bar{v}^{p^*_{\alpha,s}}(\kappa), \quad \kappa \in \mathbb{R}.
\]
Here the operator \(\mathcal{T}\) is an integral operator with a kernel whose singularity is similar to the one of the fractional Laplacian and that has exponential decay at infinity (see Sect. 4 below). To get the existence of a solution of the new problem, some restriction in case \(s < 1/2\) appears by the fact that we need to assume that \(p^*_{\alpha,s}\) is subcritical in dimension one. For the proof of qualitative properties, such as the fast-decay of the bubble solution, i.e., decay as \(r^{-N+2s}\), we obtain a suitable and sharp estimate for the function \(\bar{v}\).

Summarizing, we have the following second main result.

Theorem 1.3 Let us consider \(p = p^*_{\alpha,s}\) defined in (1.3).

(a) If
\[
1/2 \leq s < 1, \quad 0 < \alpha,
\]
or

$$0 < s < 1/2, \ 0 < \alpha < 2s(N - 1)/(1 - 2s),$$

then the problem (1.1) admits a classical positive radially symmetric solution $u \in L^\infty(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$.

(b) If $0 < s < 1, \ -2s < \alpha < 0$, then there exists a weak (variational) positive radially symmetric solution $u \in L^\infty(\mathbb{R}^N) \cap H^s_{loc}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$ of (1.1).

c) All radial solutions $u$ of (1.1) are fast decay, that is, there exist constants $c_i > 0$, $i = 1, 2$, such that

$$c_1 r^{-N+2s} \leq u(r) \leq c_2 r^{-N+2s}, \quad r \geq 1.$$

To finish this introduction we give some comments in relation to the previous existence theorem:

(i) Another existence result related with the nonlocal Hénon equation can be found in, for instance, [67] where Yang, by using some ideas of [53], proves the existence of a positive, radially symmetric solution when $-2s < \alpha < 0$ and $p = p_{\alpha,s}^*$ establishing a refinement of a Sobolev-Hardy inequality in terms of a Morrey–Campanato space. In addition the author shows that the positive energy solution has to be radially symmetric and strictly decreasing. Since the uniqueness for equations like (1.1) is still nowadays an interesting open problem, even though in [67] an existence result like (b) was obtained, it is not clear if our solution, obtained with a different method, and that coincide.

(ii) We expect that

$$\lim_{r \to \infty} u(r) r^{N-2s},$$

exists and that $u$ has an explicit formula as in the local case (see (1.4)). But this is also wide-open. Notice that for $p \neq 2$ there is an argument done for $\alpha = 0$ to establish the exact behavior of the solutions for the $p$-fractional Laplacian equation with the associated critical exponent (see [11]). However we cannot see clearly how the method developed in [11] can be applied to the case $p = 2$, and, much less, when $\alpha \neq 0$.

(iii) We emphasize again that for $0 < s < 1/2$ and $\alpha > 2s(N - 1)/(1 - 2s)$ the problem in the Emden-Fowler variables is super-critical so it is not clear if a solution with fast decay can exist (see Sect. 4 below).

The rest of the paper is organized as follows: in Sect. 2 we introduce the notion of solutions that will be used along the work and a helping result. Section 3 deals with the proof of Theorem 1.1. Finally in Sect. 4 we obtain the proof of Theorem 1.3.

We remark here that along the work we will denote by $C$ a positive constant that may change from line to line.

## 2 Weak solution and integral form

In this section we revise the notion of solution and we present a key lemma that will be used in the next section to prove the Liouville’s result.

As we have commented in the Introduction, different kind of solutions of (1.1) can be considered:

- **classical (or pointwise)** solutions $u \in L_{2s} \cap C^{1,1}_{loc}$,
- **weak (or variational)** solutions $u \in H^s(\mathbb{R}^N)$, and
distributional solutions $u \in \mathcal{L}_{2s}$.

The first objective of the work on hands is to prove a nonexistence result for distributional solutions (see Theorem 1.1). Thus, the following auxiliary lemma, whose proof will be based on ideas of [69, Section 2] (see also [68]), will be developed for the weaker notion of solutions. However the conclusion is also true for those solutions that admit maximum and comparison principles.

**Lemma 2.1** If $u \in \mathcal{L}_{2s} \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$ is a nonnegative distributional solution of (1.1) then

$$u(x) = \int_{\mathbb{R}^N} \frac{|y|^{\alpha} u^p(y)}{|x - y|^{N-2s}} dy.$$  \hspace{1cm} (2.1)

In addition $u \in C^\infty(\mathbb{R}^N \setminus \{0\})$.

**Remark 2.2** (i) By the previous result, if $\alpha > 0$, locally bounded distributional solutions of (1.1) are classical. Moreover when $\alpha \leq -2s$ there is no a nonnegative locally bounded solution of (2.1) because the integral of the right is divergent at zero. Notice that the nonexistence for $\alpha \leq -2$ in the case $s = 1$ is based on a ODE argument, (see [24, Theorem 2.3]). This argument is not valid for the nonlocal case.

(ii) Close related results for the case $-2s < \alpha < 0$ can be found in [30], including a different regularity results at the singularity.

**Proof** As we commented before, we follow closely the ideas given in [69] (see also [68]) that use the Green’s function $G_R(x, y)$ for the fractional Laplacian in a ball. Since it is known (see for example [16,44]) that $|G_R(x, y)| \leq C|x - y|^{-(N-2s)}$, we notice that the solution of (1.1) in $B_R(0)$ with zero boundary condition given by

$$\int_{B_R(0)} G_R(x, y)|y|^{\alpha} u^p(y) dy, \quad x \in B_R(0),$$

is well define. Therefore using the fact that

$$\int_{\mathbb{R}^N} \frac{|y|^{\alpha}}{|x - y|^{N-2s}} dy = \infty, \quad \alpha > -2s,$$

by applying the same argument, based on a maximum principle, that can be found in [68, Theorem 3], we obtain

$$u(x) = \int_{\mathbb{R}^N} \frac{|y|^{\alpha} u^p(y)}{|x - y|^{N-2s}} dy.$$  \hspace{1cm} (2.1)

The regularity is nowadays quite standard and it is based on a bootstrapping argument. See for example a localized version of the regularity results of Schauder type in [18, Theorem 2.1] or in [12,59].

\[ \square \]

### 3 Liouville theorem

The aim of this section is to prove Theorem 1.1, i.e. that there exists no positive solution of the nonlocal Hénon equation when $p$ is subcritical. For that we will follow the ideas developed in [46,65] for the local case taking advantage on the fact that the fractional Kelvin transform, that is fundamental to apply the method of moving spheres, has been studied thoroughly in the...
nonlocal framework in [9] (see also [56]). First of all, using the fact that \((-\Delta)^s |x|^{2s-N} = 0\) when \(x \neq 0\), it is easy to check that

\[
(-\Delta)^s \overline{u}(x) = \frac{1}{|x|^{N+2s}} (-\Delta)^s u \left( \frac{x}{|x|^2} \right),
\]

with

\[
\overline{u}(x) := \frac{1}{|x|^{N-2s}} u \left( \frac{x}{|x|^2} \right), \quad x \in \mathbb{R}^N \setminus \{0\}.
\]

Then, for \(\lambda > 0\), using the homogeneity property of the fractional Laplacian we get that

\[
(-\Delta)^s u_\lambda(x) = \left( \frac{\lambda}{|x|} \right)^{N+2s} (-\Delta)^s \left( \frac{\lambda^2 x}{|x|^2} \right),
\]

(3.1)

where

\[
u_\lambda(x) := \frac{1}{\lambda^{N-2s}} \overline{u} \left( \frac{x}{\lambda^2} \right) = \left( \frac{\lambda}{|x|} \right)^{N-2s} u \left( \frac{\lambda^2 x}{|x|^2} \right), \quad x \in \mathbb{R}^N \setminus \{0\},
\]

(3.2)

is the Kelvin transform of \(u\). Let us now define

\[
w_\lambda(x) := u_\lambda(x) - u(x), \quad x \in \mathbb{R}^N \setminus \{0\}.
\]

(3.3)

Before proving the auxiliary results needed to obtain the Liouville Theorem, for the convenience of the reader, we introduce now a Maximum principle for narrow domains for our problem (see [7,19]). That is

**Lemma 3.1** Let \(v \in L^2_{2s} \cap C^1_{loc}\) and \(\lambda > 0\). If \(v(x) = -v_\lambda(x)\) for \(x \in B_\lambda(0) \setminus \{0\}\) and satisfies

\[
\begin{align*}
(-\Delta)^s v - C(x)|x|^\alpha v &\geq 0 \quad \text{in } \Omega \subseteq B_\lambda \setminus \{0\}, \alpha > -2s, \\
v &\geq 0 \quad \text{in } (B_\lambda \setminus \{0\}) \setminus \Omega.
\end{align*}
\]

where \(\overline{C}(x) > 0\) is bounded from above in the points where \(v\) attains a negative minimum, then there exists a small \(\delta(\lambda) > 0\) such that \(\inf_{\Omega} v \geq 0\) as long as

\[
\Omega \subseteq \{ x \in \mathbb{R}^N : \lambda - \delta(\lambda) < |x| < \lambda \}.
\]

In addition, for \(\lambda\) small we can take \(\lambda - \delta(\lambda) = \lambda^{2-\mu}, \mu < 1\), with

\[
\frac{2s + 2\alpha}{\alpha} < \mu \text{ if } \alpha < 0.
\]

**Proof** Suppose by contradiction that \(\inf_{\Omega} v = v(x_{\min}) < 0\). Following [19, Theorem 2.2] we get that

\[
(-\Delta)^s \overline{v}(x_{\min}) \leq v(x_{\min}) F(\lambda, x_{\min}),
\]

where \(\overline{v}(x) := v(x) - v(x_{\min})\) and

\[
F(\lambda, x_{\min}) \geq \int_{\mathbb{R}^N \setminus B_\lambda} dy \frac{dy}{|x_{\min} - y|^{n+2s}} \geq C(\lambda - |x_{\min}|)^{-2s}.
\]

Since on the other hand, by the fact that \(v(x_{\min}) < 0\),

\[
(-\Delta)^s \overline{v}(x_{\min}) = (-\Delta)^s v(x_{\min}) \geq \overline{C}|x_{\min}|^\alpha v(x_{\min}),
\]

we get that there exists a positive constant \(c\) such that

\[
|x_{\min}|^\alpha (\lambda - |x_{\min}|)^{2s} > c, \quad c > 0,
\]

(3.4)
and therefore $\lambda^{\alpha+2s} (\lambda/|x_{\min}| - 1)^{2s} > c$. From here using that $\alpha > -2s$ we get a contradiction if $\delta(\lambda)$ is small. Observe that, for $0 < \lambda < 1$, we can consider, without loss of generality that $\lambda - \delta(\lambda) := \lambda^{2-\mu}$, $\mu < 1$. Thus (3.4) implies the existence of $c > 0$ such that

$$\lambda^{2s}\lambda^{\alpha} > c \quad \text{for} \quad \alpha \geq 0 \quad (\text{resp.} \quad \lambda^{2s}\lambda^{(2-\mu)\alpha} > c \quad \text{for} \quad -2s < \alpha < 0).$$

Since $\alpha + 2s > 0$, the previous inequalities clearly imply a contradiction if we choose $\lambda$ small enough as long as $\alpha < \mu < \frac{N-2s}{\alpha + N}$ when $-2s < \alpha < 0$.

By the previous lemma we have the next

**Proposition 3.2** Let $u \in L^{2s} \cap C^{1,1}_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ be a nonnegative solution of (1.1) with $1 < p < p^{*}_{\alpha,s}$. If in the case $N < 4s$ we assume that $-N/2 < \alpha < 0$, then there exists $\lambda_0 > 0$ such that

$$w_\lambda(x) \geq 0 \quad \text{for every} \quad x \in B^{2s} \setminus \{0\}, \lambda < \lambda_0,$$

(3.5)

where $w_\lambda$ and $p^{*}_{\alpha,s}$ were given in (3.3) and (1.3) respectively.

**Proof** To prove (3.5) let $\lambda > 0$ be a small parameter fixed but arbitrary. Since by Lemma 2.1

$$u(x) = \int_{\mathbb{R}^N} \frac{|y|^\alpha u^p(y)}{|x-y|^{N-2s}} \, dy,$$

then, for $|x| \geq \lambda^{\alpha}$ we get that

$$u(x) \geq \int_{B^{2s}_{\lambda^{\alpha}}} \frac{|y|^\alpha u^p(y)}{|x-y|^{N-2s}} \, dy \geq \frac{c}{|x|^{N-2s}} \int_{B^{2s}_{\lambda^{\alpha}}} |y|^\alpha \, dy \geq \frac{C_0 \lambda^{\mu(\alpha+N)}}{|x|^{N-2s}}.$$ 

Because the previous inequality implies that

$$u_\lambda(x) \geq C_0 \lambda^{\mu(\alpha+N)} \lambda^{(N-2s)} x \in B^{2s}_{\lambda^{\alpha}} \setminus \{0\},$$

then if

$$\mu < (N-2s)/(\alpha + N)(< 1),$$

we clearly get

$$w_\lambda(x) > 0, \quad x \in B^{2s}_{\lambda^{\alpha}} \setminus \{0\},$$

(3.6)

choosing $\lambda < \lambda_1$ for some $\lambda_1$ small enough. We observe that in the case $-2s < \alpha \leq 0$, we have

$$\frac{2s + 2\alpha}{\alpha} \leq \frac{N-2s}{\alpha + N}$$

as long as $N \geq 4s$. Thus, if $N < 4s$ the assumption $\alpha > -N/2$ is needed to be able to choose $\mu$ satisfying the hypothesis of Lemma 3.1.

On the other hand by (3.1) and the fact that $p$ is subcritical, we have that
\[
(-\Delta)^s w_\lambda(x) = |x|^\alpha \left( \left( \frac{\lambda}{|x|} \right)^{N+2s+2\alpha-p(N-2s)} u^p_\lambda(x) - u^p(x) \right)
\geq |x|^\alpha (u^p_\lambda(x) - u^p(x))
= p |x|^\alpha \varphi(x) w_\lambda(x), \quad x \in B_\lambda \setminus \{0\},
\]
where \( u^{p-1}_\lambda \leq \varphi \leq u^{p-1} \), if \( w_\lambda \) is negative. Since \( 0 \leq \varphi(x) \leq C \) then the Lemma 3.1 can be applied with
\[
\Omega = \{ x \in B_\lambda : w_\lambda(x) \leq 0 \} \setminus B_{\lambda-\mu}
\]
(3.7)
obtaining that
\[
w_\lambda(x) \geq 0, \quad x \in B_\lambda \setminus B_{\lambda-\mu}.
\]
(3.8)
if \( \lambda < \lambda_2 \) small enough. Thus, the desired conclusion follows from (3.6) and (3.8) for \( \lambda_0 = \min\{\lambda_1, \lambda_2\} \). \( \square \)

By the previous proposition
\[
0 < \tilde{\lambda} := \sup\{\mu > 0 : w_\lambda \geq 0 \text{ in } B_\lambda \setminus \{0\} \text{ with } 0 < \lambda < \mu\},
\]
is well define and \( \tilde{\lambda} \leq \infty \). We analyze now if \( \tilde{\lambda} \) can be equal to infinite.

**Proposition 3.3** Let \( u \in L_{2s} \cap C^{1,1}_{loc}(\mathbb{R}^N \setminus \{0\}) \) be a nonnegative solution of (1.1) with \( 1 < p < p^{\ast}_{a.s.} \). If \( \tilde{\lambda} < \infty \) then \( w_{\tilde{\lambda}} = 0 \) in \( B_{\tilde{\lambda}} \setminus \{0\} \).

**Proof** Here we follow the classical moving type argument (see [19]). First of all we notice that, by continuity, \( w_{\tilde{\lambda}} \geq 0 \) in \( B_{\tilde{\lambda}} \setminus \{0\} \). Thus, by the strong maximum principle, \( w_{\tilde{\lambda}} > 0 \) or \( w_{\tilde{\lambda}} = 0 \) in \( B_{\tilde{\lambda}} \setminus \{0\} \). We assume by contradiction that the first possibility holds.

Let \( 0 < \delta < \delta_0 \) be a small parameter, fixed but arbitrary such that
\[
0 < \delta_0 \leq \delta(\lambda) \text{ given in Lemma 3.1, we could apply this Maximum Principle for narrow domains for}
\]
\[
\Omega = \{ x \in B_\lambda : w_\lambda(x) \leq 0 \} \setminus B_{\lambda-\delta},
\]
(3.9)
obtaining that \( w_\lambda \geq 0 \) in \( B_\lambda \setminus \{0\} \) for \( \lambda \in (\tilde{\lambda}, \tilde{\lambda} + (\delta_0 - \delta)) \) that contradicts the definition of \( \tilde{\lambda} > 0 \). \( \square \)

The previous proposition implies that \( \tilde{\lambda} \) has to be infinite because using (3.7), we know that \( w_{\tilde{\lambda}}(x) = u(x) \) cannot be possible if \( p < p_{a.s.}^{\ast} \). Thus our last step will be to reject the possibility that \( \tilde{\lambda} = \infty \) as the next result shows. Observe that, once we rule this option, the nonexistence of a positive solution of (1.1) and, therefore, the proof of Theorem 1.1 will be done.

**Proposition 3.4** Let \( \tilde{\lambda} \) be defined in (3.9). Then \( \tilde{\lambda} \neq \infty \). 

\( \square \) Springer
Proof Following the ideas used in [48] for the local case, let us suppose by contradiction that \( \widetilde{\lambda} = \infty \). It is clear that (3.2) and (3.9) imply

\[
u_\lambda(x) \leq u(x) \quad \text{if } |x| \geq \lambda \text{ with } 0 < \lambda < \widetilde{\lambda}.
\]

Then, for every \( |x| \geq 1 \), we can consider \( \lambda := |x|^{|1/2|} \) obtaining that

\[
u(x) \geq \nu_{|x|^{1/2}}(x) \geq \frac{1}{|x|} \min_{|z| \leq 1} u(z) := \frac{c_0}{|x|^{N-2s}}.
\]

Since \( 1 < p < p^*_{\alpha,s} \), this clearly implies that

\[
\lim_{|z| \to \infty} |x|^{\alpha+2s}u^{p-1}(x) = \infty.
\]

(3.10)

Thus the function \( v(z) := u(x + |x||z|), x \in \mathbb{R}^N \), satisfies

\[
\begin{cases}
(-\Delta)^sv(z) = |x|^{2s}c(x + |x||z|)v(z), & |z| \leq 1, \\
v \geq 0, & \mathbb{R}^N,
\end{cases}
\]

where \( c(x) := |x|^{\alpha}u^{p-1}(x) \). Therefore by [55, Theorema 1] we get that

\[
\lambda_1(B_1) > \sup_{|z| \leq 1} |x|^{2s}c(x + |x||z|), \quad x \in \mathbb{R}^N,
\]

where \( \lambda_1(B_1) \) is the first eigenvalue of the fractional Laplacian. The previous inequality implies a contradiction with (3.10) because \( |x| \) can be taken arbitrarily big. Thus the desired conclusion follows.

\( \square \)

4 Critical exponent and existence of the bubble

The aim of this section is to prove the existence result announced in Theorem 1.3, that is, establishes the existence of radially symmetric solutions of the nonlocal critical Hénon equation and the fast decay of these.

Before introducing an additional functional setting that will be used along this section to get the previous two objectives, we summarize briefly how the form of the, commonly called, “bubble” can be obtained when \( \alpha = 0 \). As we have commented in the Introduction of the present work the strategy followed in the case \( \alpha \neq 0 \) and \( s = 1 \) cannot be applied in the nonlocal framework because the expression of the fractional Laplacian for radial functions cannot be simplified as it occurs with the Laplacian. However when \( s \neq 1 \) but \( \alpha = 0 \) we can still do an explicit computation to get the form of the solution (see [47]). The key point when \( \alpha = 0 \) is that the Fourier transform of the radial function

\[
u_\mu(x) = (1 + |x|^2)^{-\mu}, \ \mu > 0,
\]

is well known, and is given by

\[
\mathcal{F}u_\mu(\xi) = |\xi|^{-\frac{N}{2}} K_{\mu - \frac{N}{2}}(|\xi|),
\]

where \( K \) is a Bessel function that satisfies \( K_a = C K_{-a}, \ a \in \mathbb{R} \) (see [47]). Using this property of \( K \) it follows that

\[
|\xi|^{2s} \mathcal{F}u_{\frac{N}{2s}}(\xi) = |\xi|^s K_{-s}(|\xi|) = C |\xi|^s K_s(|\xi|) = C \mathcal{F}u_{\frac{N+2s}{2}}(\xi).
\]
that is,
\[ |x|^{-(N+2s)} \ast u_{\frac{N-2s}{2}}(x) = C u_{\frac{N+2s}{2}}(x) = C \left( u_{\frac{N-2s}{2}} \right)^{\frac{N+2s}{2}}(x). \]

This gives that all radially symmetric solutions when \( \alpha = 0 \) must be the standard bubbles in the nonlocal case given by
\[ b(x) = C(N, s) \left( \frac{\lambda}{\lambda^2 + |x|^2} \right)^{\frac{N-2s}{2}}, \]
for some \( C(N, s), \lambda > 0 \). The computations we have just done before show up that the strategy used when \( \alpha = 0 \) and \( s \neq 1 \) cannot be adapted to our case because the Fourier transform of
\[ u_\mu(x) = (1 + |x|^{\gamma})^{-\mu}, \mu > 0, 0 < \gamma \neq 2, \]
has not the desired expression as occurs when \( \gamma = 2 \). Therefore to find the shape of the bubble when \( 0 < s < 1, \alpha \neq 0 \), we have to use a completely different strategy.

### 4.1 An alternative problem

Following the ideas developed in [25] we will apply the Emden Fowler change of variables in (1.1) when \( p = p_{\alpha, s}^a, \alpha > -2s \). That is, we look for radial solutions of the form
\[ w(x) = |x|^{\frac{N-2s}{2}} v(|x|), \]  
(4.1)

where \( v \) is some radial function to be determinate. Denoting by
\[ \beta := -\frac{N - 2s}{2}, \]  
(4.2)

and writing \( |x| = r \), it is clear that
\[ (-\Delta)^s w(x) = \int_0^\infty \int_{S^{N-1}} \frac{r^\beta v(r) - \rho^\beta v(\rho)}{r^2 + \rho^2 - 2r\rho \langle \theta, \sigma \rangle} \rho^{N-1} d\sigma d\rho \]
\[ = r^{-2s+\beta} \int_0^\infty \int_{S^{N-1}} \tilde{\rho}^{N-1} \frac{v(r) - \tilde{\rho}^\beta v(\tilde{\rho})}{|1 + \tilde{\rho}^2 - 2\tilde{\rho} \langle \theta, \sigma \rangle|^{\frac{N+2s}{2}}} d\sigma d\tilde{\rho}, \]

where \( \tilde{\rho} = \rho/r \). Then
\[ (-\Delta)^s w(x) = r^{-2s+\beta} (L v(r) + A_{s, N} v(r)), \quad r > 0, \]  
(4.3)

with
\[ L v(r) := \int_0^\infty \int_{S^{N-1}} \tilde{\rho}^{N-1+\beta} \frac{v(r) - v(\tilde{\rho})}{|1 + \tilde{\rho}^2 - 2\tilde{\rho} \langle \theta, \sigma \rangle|^{\frac{N+2s}{2}}} d\sigma d\tilde{\rho}, \]  
(4.4)

and
\[ A_{s, N} := \int_0^\infty \int_{S^{N-1}} \tilde{\rho}^{N-1} \frac{1 - \tilde{\rho}^\beta}{|1 + \tilde{\rho}^2 - 2\tilde{\rho} \langle \theta, \sigma \rangle|^{\frac{N+2s}{2}}} d\sigma d\tilde{\rho}. \]  
(4.5)
Thus from (4.1), (4.3) and the fact that $\alpha + \beta p_{a,s}^* = -2s + \beta$, we conclude that

$$L v(r) + A_{s,N} v(r) = p_{a,s}^* (r), \quad r > 0.$$  \hspace{1cm} (4.6)

Finally doing the Emden Fowler change of variable $r = e^\kappa$, $\tilde{\rho} = e^\tau - \kappa$ the function $\tilde{v}(\kappa) = v(e^\kappa), \kappa \in \mathbb{R}$, satisfies

$$\mathcal{T} \tilde{v}(\kappa) + A_{s,N} \tilde{v}(\kappa) = \tilde{p}_{a,s}^* (\kappa), \quad \kappa \in \mathbb{R},$$  \hspace{1cm} (4.7)

where

$$\mathcal{T} \tilde{v}(\kappa) = \int_{\mathbb{R}} \int_{S^{N-1}} e^{(\tau - \kappa)N+\beta} \frac{v(e^\kappa) - v(e^\tau)}{|1 + e^{-2(\kappa - \tau)} - 2e^{-(\kappa - \tau)} \langle \theta, \sigma \rangle|^{\frac{N+2s}{2}}} d\sigma d\tau.$$

being

$$K(t) = e^{-t \frac{N+2s}{2}} \int_{S^{N-1}} \frac{1}{|1 + e^{-2t} - 2e^{-t} \langle \theta, \sigma \rangle|^{\frac{N+2s}{2}}} d\sigma, \quad t \in \mathbb{R}.$$  \hspace{1cm} (4.9)

We notice here that, in fact, the operators $L$, $\mathcal{T}$ and also the constant $A_{s,N}$ depend of $\beta$, but, since along all this section, $\beta$ is fixed and defined in (4.2), for simplicity, we omit it. We show now some useful properties, proved in [25] (see also Remark 4.7 below), that we will use later.

**Proposition 4.1** It is true that

- the constant $A_{s,N}$ given in (4.5) is positive,
- $K(t) = c(N, s) \int_0^\pi \sin y^{N-2s}(\cosh t - \cos y)^{-\frac{N+2s}{2}} dy$, for some $c(N, s) > 0$,
- the kernel $K$ is even, strictly positive and satisfies

$$K(t) \sim \frac{1}{t^{1+2s}}, \quad t \to 0, \quad K(t) \sim e^{-t \frac{N+2s}{2}}, \quad t \to \infty.$$  \hspace{1cm} (4.10)

That is, around the singularity of the origin, the kernel behaves like the ones of the fractional Laplacian operator in dimension one.

### 4.2 Existence of solutions

Our objective now is to obtain the existence of solutions of (4.7) that will imply the existence of radially symmetric solutions of (1.1) when $p = p_{a,s}^*$. Observe that, to be consistent with [49, Theorem 1] (and [67]), the existence has to be proved, at least, for every $-2s < \alpha < 0$. For that, we will introduce the functional framework needed to work with. First of all we consider the Sobolev space

$$H^s_K(\mathbb{R}) := \left\{ u : \mathbb{R} \to \mathbb{R} : u \in L^2(\mathbb{R}) \text{ and } [u]_{H^s_K(\mathbb{R})} < \infty \right\},$$

where

$$[u]_{H^s_K(\mathbb{R})} := \int_{\mathbb{R}} \int_{\mathbb{R}} (u(\kappa) - u(\tau))^2 K(\kappa - \tau) d\tau d\kappa.$$
with $K$ given in (4.9). By Proposition 4.1 we deduce that $K(t)$ is an even, decreasing monotone function that satisfies $K \in L^1(\mathbb{R})$, $\min(|x|^2, 1) \, dx$. Moreover it can be checked that
\[
\sup_{\bar{r} \geq 0} \left\{ \lim_{r \to 0} r^{2\bar{s}} \int_{B_1(0) \setminus B_r(0)} K(t) \, dt \right\} = s,
\]
and
\[
\lim_{r \to 0} r^{2s} \int_{B_1(0) \setminus B_r(0)} K(t) \, dt = \frac{1}{2s} > 0.
\]
Thus, by [14, Theorem 3.1] and a density argument we get the next

**Proposition 4.2** (Sobolev inequalities) Let $0 < s < 1/2$. If $u \in H^s_K(\mathbb{R})$ then
\[
||u||^2_{L^{2s}_t(\mathbb{R})} \leq c[u]_{H^s_K(\mathbb{R})},
\]
for some $c > 0$. Therefore the space $H^s_K(\mathbb{R})$ is continuously embedded in $L^q(\mathbb{R})$ for every $1 \leq q \leq 2/(1-2s) := 2^*_s$.

**Proof** First of all we notice that in [14, Theorem 3.1] the authors obtained (4.11) when $u \in H^s_K(\mathbb{R})$ has compact support. Thus, adapting to our setting [35, Lemma 2.2 and Lemma 2.3], since we get that $C_c^{\infty}(\mathbb{R})$ is dense in $H^s_K(\mathbb{R})$, by a standard density argument, we conclude that (4.11) is true for all $u \in H^s_K(\mathbb{R})$, as wanted.

Further, by [14, Theorem 3.2], we get

**Proposition 4.3** (Compact embeddings) Let $0 < s < 1/2$, $\Omega \subseteq \mathbb{R}$ a bounded domain and $1 \leq q < 2/(1-2s)$. Then every bounded sequence in $H^s_K(\mathbb{R})$ has a convergent subsequence in $L^q(\Omega)$. That is, the embedding $H^s_K(\mathbb{R}) \hookrightarrow L^q_{loc}(\mathbb{R})$ is compact.

We highlight the important fact that $p^*_{a,s}$, given in (1.3), is subcritical for the Sobolev space $H^s_K(\mathbb{R})$ in the sense that
\[
p^*_{a,s} = \frac{N + 2s + 2\alpha}{N - 2s} < \frac{1 + 2s}{1 - 2s} = 2^*_s - 1, \text{ when } s < 1/2,
\]
for every $\alpha < 0$ and with some restrictions in the case that $\alpha > 0$. In fact, since $N > 2s > 1$,
\[
2s + \alpha(1 - 2s) < 2s < 2sN, \text{ when } \alpha < 0.
\]
In the case $\alpha > 0$ we have to add the hypothesis
\[
\alpha < \frac{2s(N - 1)}{1 - 2s} \left( \lim_{s \to 1/2^-} \right) \to \infty.
\]
There is nothing to prove in the case $s \geq 1/2$ because there is no critical Sobolev exponent in dimension one. We also notice that $p^*_{a,s} > 1$ because $-2s < \alpha$. Therefore, by (4.10) and the previous observation, using Propositions 4.2–4.3, following verbatim the proof of [33, Theorem 1.3] based on variational method together with concentration compactness principle we conclude the next existence result.

**Theorem 4.4** If
(i) $-2s < \alpha$ when $1/2 \leq s < 1$, or
(ii) $-2s < \alpha < \frac{2s(N-1)}{1-2s}$ when $0 < s < 1/2$. 
then the problem
\[ \mathcal{T}\bar{v} + A_{s,N} \bar{v} = \bar{v}^{p^*_\alpha,s} \text{ in } \mathbb{R}, \]
where \( \mathcal{T} \) and \( K \) were given in (4.8) and (4.9) respectively, has a nonnegative variational solution. Moreover, since \( f(t) = t^{p^*_\alpha,s} \) is a Hölder function, the solution is classical and, in fact, is positive.

The regularity is obtained again by a bootstrapping argument based on [25, Proposition 3.10 and Proposition 3.11] (see also [29]) and on a \( L^\infty \) bound of Proposition 4.5 below.

As we commented before, the condition (ii) of the previous theorem is trivially true for every \( 0 < s < 1/2 \) and \(-2s < \alpha < 0\) as we expected. Moreover we emphasize that the previous result implies the existence of positive and radially symmetric solutions of the critical non-local Hénon equation even when \( \alpha > 0 \), which, as far as we know, was not proved until now.

4.3 Qualitative properties of solutions: the “bubble”

We prove now a qualitative property of the solutions of (4.7) that will be the key step to find, later, the shape of the “bubble” for the nonlocal Hénon equation. More precisely we want to obtain that the solutions decay to zero in (plus and minus) infinity. Before going to the statement and the proof of this result, we observe that reverse the operator \( \mathcal{T} \) seems not simple at all so we cannot follow closely the proof of [33, Theorem 3.4] in order to get the desired qualitative property. Thus, instead of using this well known approach we will prove directly that the solutions are bounded and regular enough to conclude the desired decay. That is, we have the following.

**Proposition 4.5** If \( u \in H^s_K(\mathbb{R}) \) is a positive variational solution of (4.7) given by Theorem 4.4 then \( u \in L^\infty(\mathbb{R}) \) and \( u(t) \to 0 \) when \( |t| \to \infty \).

**Proof** The key point of the proof is that, since for every convex function \( \varphi \),
\[ \mathcal{T}(\varphi(u)) \leq \varphi'(u)\mathcal{T}(u), \tag{4.12} \]
we can adapt the ideas in [4, Proposition 2.2] for the unbounded space \( \mathbb{R} \) in order to obtain that
\[ u \in L^\infty(\mathbb{R}). \tag{4.13} \]

In fact let us define, for \( \beta \geq 1 \) and \( M > 0 \) large,
\[ \varphi(x) = \varphi_{M,\beta}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x^\beta, & \text{if } 0 < x < M, \\ \beta M^{\beta-1}(x-M) + M^\beta, & \text{if } x \geq M. \end{cases} \]

Since \( \varphi \) is Lipschitz, with constant \( K = \beta M^{\beta-1} \), it is clear that \( \varphi(u) \in H^s_K(\mathbb{R}) \). Moreover by (4.12), since \( \varphi(u)\varphi'(u) \geq 0 \) and \( u > 0 \),
\[ \int_{\mathbb{R}} \varphi(u)\mathcal{T}(\varphi(u)) \, d\tau \leq \int_{\mathbb{R}} \varphi(u)\varphi'(u)u^{p^*_\alpha,s}. \]

Using now that \( u\varphi'(u) \leq \beta \varphi(u) \) by Proposition 4.2 and the fact that \( p^{s_\alpha,s} < 2s_\alpha - 1 \), the above estimate becomes
\[ \left( \int_{\mathbb{R}} (\varphi(u))^{2s_\alpha} \right)^{\frac{2}{2s_\alpha}} \leq C \beta \int_{\mathbb{R}} (\varphi(u))^2 u^{2s_\alpha-2}. \tag{4.14} \]
Since $\beta \geq 1$ and $\varphi(u)$ is linear when $u \geq M$, it can be checked that both sides of (4.14) are finite.

Let $\beta = \beta_1$ be such that $2\beta_1 = 2^* + 1$ and $R$ large to be determined later. Then, Hölder’s inequality with $p = (2\beta_1 - 1)/2 = 2^*/2$ and $p' = 2^*/(2^* - 2)$ gives

$$\int_{\mathbb{R}} (\varphi(u))^2 u^{2^* - 2} \leq \int_{\{u \leq R\}} \frac{(\varphi(u))^2}{u} R^{2^*_u - 1}$$

$$+ \left( \int_{\mathbb{R}} (\varphi(u))^{2^*_u} \right)^{2^*_u - 2} \left( \int_{\{u > R\}} u^{2^*_u} \right)^{2^*_u - 2}.$$

By the Monotone Convergence Theorem, we may take $R$ so that

$$\left( \int_{\{u > R\}} u^{2^*_u} \right)^{2^*_u - 2} \leq \frac{1}{2 C \beta_1}.$$

In this way, the second term above is absorbed by the left hand side of (4.14) to get

$$\left( \int_{\mathbb{R}} (\varphi(u))^{2^*_u} \right)^{2^*_u / 2} \leq 2 C \beta_1 \left( \int_{\{u \leq R\}} \frac{(\varphi(u))^2}{u} R^{2^*_u - 1} \right).$$

Letting $M \to \infty$ in the left hand side, it follows that

$$\left( \int_{\mathbb{R}} u^{2^*_u \beta_1} \right)^{2^*_u / 2} \leq 2 C \beta_1 \left( R^{2^*_u - 1} \int_{\mathbb{R}} u^{2^*_u} \right) < \infty.$$

Which implies that $u \in L^{2^*_u \beta_1}(\mathbb{R})$. Let us now consider $\beta > \beta_1$. Going back to inequality (4.14) and using, as before, that $\varphi_M, \beta(u) \leq u^\beta$ in the right hand side and taking $M \to \infty$ in the left hand side we obtain

$$\left( \int_{\mathbb{R}} u^{2^*_u \beta} \right)^{2^*_u / 2} \leq C \beta \left( \int_{\mathbb{R}} u^{2\beta + 2^*_u - 2} \right).$$

where $C > 0$ is independent of $\beta$. Thus

$$\left( \int_{\mathbb{R}} u^{2^*_u \beta} \right)^{1 / (2^*_u - 1)} \leq (C \beta)^{1 / (2^*_u - 1)} \left( \int_{\mathbb{R}} u^{2\beta + 2^*_u - 2} \right)^{1 / (2^*_u - 1)}.$$

Defining now the same iterative process as in [4] we get (4.13). In fact let define $\beta_{m+1}, m \geq 1$ such that so that

$$2\beta_{m+1} + 2^*_u - 2 = 2^*_u \beta_m.$$

Therefore

$$\beta_{m+1} - 1 = \left( \frac{2^*_u}{2} \right)^m (\beta_1 - 1), \quad m \geq 1,$$

and replacing it in (4.15) we get

$$\left( \int_{\mathbb{R}} u^{2^*_u \beta_{m+1}} \right)^{1 / (2^*_u \beta_{m+1} - 1)} \leq (C \beta_{m+1})^{1 / (2^*_u \beta_{m+1} - 1)} \left( \int_{\mathbb{R}} u^{2^*_u \beta_m} \right)^{1 / (2^*_u \beta_m - 1)}.$$
Then, defining for \( m \geq 1 \)
\[
A_m := \left( \int_{\mathbb{R}} u_m^{2s} \right)^{\frac{1}{2s(2m-1)}} \quad \text{and} \quad C_{m+1} := C \beta_{m+1},
\]
using a limiting argument, we conclude that there exists \( C_0 > 0 \), independent of \( m > 1 \), such that
\[
A_{m+1} \leq \prod_{k=2}^{m+1} C_k^{\frac{1}{2s(k-1)}} A_1 \leq C_0 A_1,
\]
which implies that \( \|u\|_{L^\infty(\mathbb{R})} \leq C_0 A_1 \).

Once we have proved (4.13) it is possible to apply [43, Theorem 1.1] (see [25, Proposition 3.10]) to obtain the Hölder regularity of the solution. More precisely we deduce that there exists \( 0 < \gamma < 1 \) and \( c > 0 \), depending on \( s \), such that for every \( R > 0 \)
\[
|u(\tau) - u(\kappa)| \leq c \frac{|\tau - \kappa|^\gamma}{R^\gamma}, \quad |\tau| \leq R, \quad |\kappa| \leq R.
\]
(4.16)

Namely, recovering \( \mathbb{R} \) with arbitrary balls \([-R, R] \), by (4.13) and (4.16) we have obtained that \( u \in L^\infty(\mathbb{R}) \cap C^\gamma(\mathbb{R}) \cap L^{2s}(\mathbb{R}) \) which implies that \( u(t) \to 0 \) when \( |t| \to \infty \) completing the proof.

We are able now to obtain good bounds for the solutions of (4.7), that is, the kind of estimate given in c) of Theorem 1.3. In fact, if we prove that the solutions of (4.7) satisfy
\[
\bar{v}(t) \leq C e^{-\left(\frac{N-2s}{2s}\right)|t|}, \quad t \in \mathbb{R}.
\]
(4.17)
then we get a bound for the solution of (1.1). Thus our next objective is the following

**Proposition 4.6** If \( u \in H_K^s(\mathbb{R}) \) is a nonnegative variational solution of (4.7) given by Theorem 4.4 then there exists \( C > 0 \) such that
\[
|u(t)| \leq C e^{-\left(\frac{N-2s}{2s}\right)|t|}, \quad t \in \mathbb{R}.
\]
(4.19)

**Proof** First of all we notice that by [32, Lemma 3.1, Corollary 3.1 and Remark 3.1], see also Remark 4.7, there exists a strictly concave function \( c \) define on \((-N, 2s)\) such that
\[
(\Delta)^s r^\mu = c(\mu) r^{-2s}, \quad -N < \mu < 2s.
\]
(4.20)
The function \( c(\mu) \) is positive in \((-N + 2s, 0)\), has two zeros \( c(0) = c(-N + 2s) = 0 \) and satisfies \( \lim_{\mu \to -N^+} c(\mu) = \lim_{\mu \to 2s^-} c(\mu) = -\infty \). Thus, considering \( v(r) := r^{\mu-\beta} \) in (4.3), from (4.20), it follows that
\[
\mathcal{L} r^{\mu-\beta} + A_{s,N} r^{\mu-\beta} = c(\mu) r^{\mu-\beta}, \quad r > 0, \quad -N < \mu < 2s,
\]
where \( \beta \) was given in (4.2). Doing the Emden Fowler change of variable, the previous identity is equivalent to

\[
\mathcal{T} v_\mu(t) + A_{s,N} v_\mu(t) = c(\mu) v_\mu(t), \quad \text{with} \quad v_\mu(t) = e^{t(\mu + \frac{N - 2s}{2})},
\]

(4.21)

for every \( t \in \mathbb{R} \), and \(-N < \mu < 2s\). Let fix now \( R > 0 \) big enough and let us consider \( u \geq 0 \) a weak solution of (4.7). By Theorem 4.4 we know that this solution is classical and therefore we can evaluate the operator pointwise. Using now Proposition 4.5 there exists \( \delta(R) > 0 \) such that

\[
\mathcal{T} u(t) + A_{s,N} u(t) = u^p_{a,s}(t) \leq \delta u(t), \quad |t| > R.
\]

Since for some \(-N < \mu_1 < \mu_2 < 0 \) and the properties of the function \( c(\mu) \), it is true that \( c(\mu_1) = \delta \), \( i = 1, 2 \), using comparison twice and the fact that \( u \) is bounded, from (4.21) we get that there exists \( C > 0 \) such that

\[
u(t) \leq C e^{t(\mu_i + \frac{N - 2s}{2})}, \quad \text{for some} \quad \mu_i \quad \text{and} \quad |t| > R.
\]

More precisely, writing \( \mu_1 := (-N + 2s) + \varepsilon_1 \) and \( \mu_2 := -\varepsilon_2 \), for some positive \( \varepsilon_1(\delta) \), \( \varepsilon_2(\delta) \) smaller than \( N \), we get

\[
u(t) \leq C_1 e^{t(-\frac{N - 2s + \varepsilon_1}{2})} \quad \text{and} \quad u(t) \leq C_2 e^{t(\frac{N - 2s}{2} - \varepsilon_2)}, \quad |t| > R.
\]

(4.22)

Without loss of generality we prove (4.19) for the case \( t > 0 \) and the proof can be easily adapted for \( t \leq 0 \) (see Remark 4.7 below). We define now

\[
\tilde{\mu}_1 := -N + 2s + \varepsilon_1 > 0 \quad \text{and} \quad \varepsilon_2(\delta) \quad \text{will be chosen later (see (4.26))}.
\]

Thus, for some \( \varepsilon > \varepsilon_1 \) that will be chosen later (see (4.26)). Since by (4.22) it is clear that

\[
\mathcal{T} u(t) + A_{s,N} u(t) \leq e^t \tilde{\mu}_1, \quad |t| > R,
\]

(4.23)

our next objective is to find a suitable function \( \tilde{\mu}_1 \) satisfying (4.19) and

\[
\mathcal{T} \tilde{\mu}_1(t) + A_{s,N} \tilde{\mu}_1 = e^t \tilde{\mu}_1, \quad |t| > R,
\]

(4.24)

that will allow us to apply a comparison principle and obtain the same bounded estimate (4.19) for the function \( u \). For that, on the first hand, we notice that by (4.21)

\[
\mathcal{T} e^t \tilde{\mu}_1 + A_{s,N} e^t \tilde{\mu}_1 = c \left( \frac{\tilde{\mu}_1 - N - 2s}{2} \right) e^t \tilde{\mu}_1,
\]

as long as

\[-N < \tilde{\mu}_1 - \frac{N - 2s}{2} < 2s.\]

Thus, for some \( \varepsilon > 0 \) chosen later (see (4.26)), the function

\[
\tilde{\nu}_1(t) := \frac{1}{c \left( \frac{\tilde{\mu}_1 - N - 2s}{2} \right)} e^t \tilde{\mu}_1, \quad t \in \mathbb{R},
\]
is a particular solution of \((4.24)\). On the other hand we know that, for every \(M > 0\),
\[
\tilde{v}_{2,M}(t) := Me^{-(N-2s)/2}, \quad t \in \mathbb{R}, \tag{4.25}
\]
is a solution of the homogeneous equation \(\mathcal{T} \cdot + A_s, N \cdot = 0\) in \(\mathbb{R}\). We choose now
\[
\max \left\{ \varepsilon_1, \frac{\alpha(N-2s)}{N+2s+2\alpha} \right\} < \varepsilon < \frac{\alpha + 2s(N-2s)}{N+2s+2\alpha} (< N), \tag{4.26}
\]
that in particular implies
\[
-N < \tilde{\mu}_1 - \frac{N-2s}{2} < -N + 2s (< 2s), \tag{4.27}
\]
and also
\[
c \left( \tilde{\mu}_1 - \frac{N-2s}{2} \right) < 0.
\]
Therefore, for every \(M > 0\), the function
\[
\tilde{u}_{1,M}(t) := \tilde{v}_{2,M}(t) + \tilde{v}_1(t), \quad t \in \mathbb{R},
\]
satisfies \((4.24)\). Moreover choosing \(M > 0\) such that, by Proposition 4.5, \(u(t) \leq \tilde{u}_{1,M}(t)\) for \(|t| < R\), by \((4.23)\) and comparison, it follows in particular that
\[
u(t) \leq \tilde{u}_{1,M} \leq Me^{-(N-2s)/2} \text{ when } t > 0,
\]
as wanted.

**Remark 4.7**

(i) To obtain the desirable bound for the solutions of \((4.7)\) in the case \(t \leq 0\), the proof follows as the one done before for \(t > 0\) by using now the fact that \(u(t) \leq e^{-t(\varepsilon + \beta)}\) for some \(\varepsilon > \varepsilon_2\) (see \((4.22)\)) and by considering \(\tilde{v}_{2,M}(t) := Me^{-\varepsilon t\beta}\) instead of \((4.25)\) with \(\beta\) defined in \((4.2)\) and some \(M > 0\). We highlight that the previous function also satisfies \(\mathcal{T} \cdot + A_s, N \cdot = 0\) in \(\mathbb{R}\) because the constants functions are, trivially, \(s\)-harmonic in all the space. The conclusion follows because we get
\[
u(t) \leq \tilde{u}_{1,M} = Me^{-t(\frac{N-2s}{2})} \text{ when } t > 0,
\]
with \(\tilde{\mu}_1\) verifying \((4.27)\).

(ii) We show up that, from the proof presented above, it can be deduced that the estimate \((4.19)\) is also true for every regular solution of \((4.7)\), that is, not only for the one given by Theorem 4.4.

(iii) We notice here that in [36] the explicit value
\[
c(\mu) = 4^s \frac{\Gamma \left( -\mu + 2s \frac{N+2s}{4} \right) \Gamma \left( N+\mu \frac{N+2s}{4} \right)}{\Gamma \left( -\mu \frac{2}{2} \right) \Gamma \left( N+\mu - 2s \frac{N+2s}{4} \right)}
\]
can be deduced for the range \(-N + 2s < \mu < 0\). From the previous expression it is clear that
\[
c(\beta) = 4^s \frac{\Gamma^2 \left( N+2s \frac{N+2s}{4} \right)}{\Gamma^2 \left( N-2s \frac{N+2s}{4} \right)} > 0.
\]
with $\beta \in (-N + 2s, 0)$ defined in (4.2). Thus, since $A_{s,N} = c(\beta)$, where $A_{s,N}$ was given in (4.5), the previous observation is an alternative way to prove Proposition 4.1 (i).

We are now in position to prove our second main Theorem.

**Proof Theorem 1.3** Let us consider

$$u(r) := r^{-\frac{N-2s}{2}} v(r),$$

where $v(r)$ was obtained in Theorem 4.4.

(a) The existence part is direct if $\alpha > 0$ since the previous function $u(r)$ is, clearly, a classical solution of (1.1) for $p = p_{\alpha,s}^*$. 

(b) If $0 > \alpha > -2s$ the solution is not regular due to the singularity at zero, so we need to argue in a different way. First of all we prove that $u$ is a strong solution of (1.1) with $p = p_{\alpha,s}^*$ where, here, the notion of strong solutions is the one that appears in [42, Definition 2.4], that is, the functions for which the principal value converges in $L^1_{\text{loc}}(\mathbb{R}^N)$. Let us consider $x \in K$, where $K$ is a compact subset. We define

$$g_\epsilon(x) := a_{N,s} \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

and

$$g(x) := u(x)^{p_{\alpha,s}^*} |x|^{\alpha}.$$

By the fact that $u \in C^1(\mathbb{R}^N \setminus \{0\}) \cap L^\infty(\mathbb{R}^N)$, $\gamma > 2s - 1$, (see Lemma 2.1 and (4.18)), it can be proved that

$$|g_\epsilon(x)| \leq C(K, \|u\|_{L^\infty}, \|u\|_{C^1,\gamma}) := h$$

for every $\epsilon > 0$ and $x \in K$.

Thus, since $h \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $g_\epsilon(x) \to g(x), x \in K$, we can apply the Dominate Convergent theorem to get that $g_\epsilon \to g$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ obtaining that $u$ is a strong solution of (1.1) when $p = p_{\alpha,s}^*$.

Finally we notice that $u(-\Delta)^s u = u^{p_{\alpha,s}^* + 1} |x|^{\alpha} \in L^1(\mathbb{R}^N)$ because $u$ has fast decay (see (4.18)) and $\alpha > -2s$. Then, by the Plancherel identity and [26, Proposition 3.4] we get

$$\int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dy < \infty.$$

Since $u$ is locally bounded we have that $u \in H^s_{\text{loc}}(\mathbb{R}^N)$ ($u \in H^s(\mathbb{R}^N)$ in the case $N \geq 4$) and, therefore, by [42, Corollary 2.7] we conclude that $u$ is also a weak solution of (1.1) with $p = p_{\alpha,s}^*$.

(c) For the lower bound of $u$ we observe that, since $(-\Delta)^s u \geq 0$ in $\mathbb{R}^N \setminus \{0\}$, by [11, Proposition 3.4 and Proposition 3.5] there exists a sub-solution with the desired lower bound. Thus by comparison we get that the estimate is also true for $u$, that is, $u(x) \geq c_1 |x|^{-N+2s}$.

Using the decay estimate obtained for $v$ in Proposition 4.6 clearly we also obtain the upper bound of $c$ (see (4.18)).

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