Lorentz-violating graviton masses: getting around ghosts, low strong coupling scale and VDVZ discontinuity

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A theory with the action combining the Einstein–Hilbert term and graviton mass terms violating Lorentz invariance is considered at linearized level about Minkowskian background. It is shown that with one of the masses set equal to zero, the theory has the following properties: (i) there is a gap of order $m$ in the spectrum, where $m$ is the graviton mass scale; (ii) the dispersion relations at $p^2 \gg m^2$ are $\omega \propto p^2$, the spectrum of tensor modes being relativistic, while other modes having unconventional maximum velocity; (iii) the VDVZ discontinuity is absent; (iv) the strong coupling scale is $(mM_P)^{1/2}$. The latter two properties are in sharp contrast to the Lorentz-invariant gravity with the Pauli–Fierz mass term.

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I. INTRODUCTION AND SUMMARY

Massive, Lorentz-invariant gravity about flat background has a number of peculiar properties. For general graviton mass terms, the theory possesses ghosts — fields with wrong sign of the kinetic term. If the graviton mass terms are of the Fierz–Pauli form [1], the ghosts are absent, but in the limit of vanishing graviton mass $m$, the graviton propagator exhibits the van Dam–Veltman–Zakharov (VDVZ) discontinuity [2, 3] originating from a scalar degree of freedom which does not decouple in the massless limit. At the classical level this scalar may not be a problem due to non-linear effects [4, 5, 6]. However, at the quantum level the theory becomes strongly coupled [7] at energy scale $(m^4 M_P)^{1/5}$, which has been confirmed by explicit calculations [8]. By adding higher order operators, this scale can be raised, but in any case it is at best of order $(m^2 M_P)^{1/3}$, which is well below the naive expectation $(mM_P)^{1/2}$. This story repeats itself in brane-world models with gravity modified in the infrared [9, 10, 11, 12]: either there are ghosts [13, 14, 15, 16] or strong coupling occurs at energies well below a naive estimate [17, 18] (see, however, Refs. [19, 20, 21]).

In Minkowskian background, most of the models considered so far exhibit Lorentz invariance. It has been pointed out recently [22], however, that the Higgs mechanism for gravity would most likely involve the violation of Lorentz invariance, and, indeed, a model has been constructed [22] in which excitations about flat space-time have Lorentz-violating spectrum. This is not yet a model with massive graviton, since in the large range of spatial momenta $p$, the dispersion law is $\omega \propto p^2$, while at very low momenta, the frequency $\omega(p)$ is imaginary and the excitations grow. Still the question arises whether the violation of Lorentz invariance may help to obtain a theory of massive gravitons which does not have ghosts and VDVZ discontinuity, and whose strong coupling scale is $(mM_P)^{1/2}$.

In this paper we adopt bottom-up approach, and simply consider a deformation of GR by Lorentz-violating graviton mass terms, about flat space-time. We show that with one of the masses set equal to zero, the theory possesses desirable properties (provided that some combinations of other masses obey positivity conditions): there is a gap of order $m$ in the spectrum, the dispersion relations at $p^2 \gg m^2$ are $\omega \propto p^2$ (the spectrum of tensor modes is relativistic, while other modes have unconventional maximum velocity), the VDVZ discontinuity is absent and the strong coupling scale is $(mM_P)^{1/2}$.

Technically, both VDVZ discontinuity and low strong coupling scale occur due to normalisation factors relating the original fields to canonically normalised ones [8]: some of these factors are of order $(m^2 M_P)^{-1}$ while a naive expectation would be $(mM_P)^{-1}$. Thus, after introducing notations in Section 2, we consider linear excitations in Section 3, emphasising the normalisation issue. We find that the normalisation factors are indeed at most of order $(m^2 M_P)^{-1}$ for all kinds of modes, which immediately tells that the strong coupling scale is $(mM_P)^{1/2}$. We also study the spectrum and find that it has a gap, again for all kinds of modes. In Section 4 we analyse interaction between conserved sources, at linearised level, and show explicitly that in the massless limit it reduces to the GR form, i.e., there is no VDVZ discontinuity.

Thus, GR with Lorentz-violating graviton masses is a healthy theory. This suggests that there may exist a Higgs phase of gravity which has Minkowskian background, violates Lorentz invariance, describes massive (and/or unstable) gravitons and has intrinsic energy scale $(mM_P)^{-1}$. 

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II. LORENTZ-VIOLATING MASS TERMS

Let us consider the action

\[ S = S_{EH} + S_m \]  

where \( S_{EH} \) is the Einstein–Hilbert term and \( S_m \) is the graviton mass term that explicitly violates the Lorentz symmetry, but does not violate the Euclidean symmetry of the three-dimensional space. The corresponding Lagrangian is

\[ L_m = \frac{M_p^2}{2}[m_0^2 h_{00} h_{00} + 2m_1^2 h_{0i} h_{0i} - m_2^2 h_{ij} h_{ij} + m_3^2 h_{ii} h_{jj} - 2m_4^2 h_{00} h_{ii}] \]  

Here \( h_{\mu\nu} \) are perturbations about Minkowski metric. In what follows we will make a comparison to the Fierz–Pauli case, the corresponding Lagrangian being

\[ L_{FP} = \frac{M_p^2}{2}[-m_0^2 h_{\mu\nu} h^{\mu\nu} + m_2^2 (h_{\mu}^\mu)^2] \]

Thus, the Fierz–Pauli Lagrangian is recovered when all masses in eq. (2), except for \( m_0 \), are equal, \( FP : \quad m_0^2 = 0, \quad m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2 \)

The latter property explains the conventions used in eq. (2).

Throughout this paper we study the case

\[ m_0 = 0 \]  

In this case the field \( h_{00} \) enters the action linearly, i.e., it acts as the Lagrange multiplier. We will see that this property (plus inequalities involving other masses) is sufficient to ensure that the theory is free of ghosts. We will also assume that all other masses are proportional to a single scale which we generically denote by \( m \).

The properties of perturbations depend on the representation of the Euclidean symmetry group. It is thus convenient to express \( h_{\mu\nu} \) in terms of irreducible fields,

\[
\begin{align*}
h_{00} &= \psi \\
h_{0i} &= u_i + \partial_0 v \\
h_{ij} &= \chi_{ij} + (\partial_i s_j + \partial_j s_i) + \partial_i \partial_j \sigma + \delta_{ij} \tau
\end{align*}
\]

Here \( \chi_{ij} \) is tranverse-traceless (tensor modes); \( u_i \) and \( s_i \) are transverse (vectors), while other fields are three-dimensional scalars. Under the gauge transformations of GR, \( h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \), the tensor modes are invariant, while vectors and scalars transform in the following way,

\[
\begin{align*}
u_i &\rightarrow u_i + \partial_0 \xi_i^T \\
s_i &\rightarrow s_i + \xi_i^T
\end{align*}
\]

where \( \xi_i^T \) is the transverse part of \( \xi_i \), and

\[
\begin{align*}
\psi &\rightarrow \psi + 2\partial_0 \xi_0 \\
v &\rightarrow v + \partial_0 \eta + \xi_0 \\
\sigma &\rightarrow \sigma + 2\eta \\
\tau &\rightarrow \tau
\end{align*}
\]

where \( \eta \) is the longitudinal part of \( \xi \), i.e., \( \xi^L_i = \partial_i \eta \). There is one gauge-invariant combination in the vector sector,

\[ w_i = u_i - \partial_0 s_i \]  

and two gauge-invariants in the scalar sector, namely, \( \tau \) and \( \Phi = \psi - 2\partial_0 v + \partial_0^2 \sigma \)
Up to total derivatives, the quadratic part of the Einstein–Hilbert Lagrangian is expressed in terms of these gauge-invariant combinations,

\[ L_{EH} = \frac{M_{Pl}^2}{2} \left( -\chi_{ij} \Box \chi_{ij} - 2w_i \Delta w_i \right) + 4\Phi \Delta \tau + 6\tau \partial_0^2 \tau - 2\tau \Delta \tau \]  

(6)

where \( \Delta \) is the three-dimensional Laplacian, while the mass term is

\[ L_m = \frac{M_{Pl}^2}{2} \left[ 2m_1^2 (u_i u_i + \partial_i v \partial_i v) - m_2^2 (\chi_{ij} \chi_{ij} + 2\partial_i s_j \partial_i \partial_j s_j + \partial_i \partial_j \sigma \partial_i \partial_j \sigma + 2\tau \Delta \sigma + 3\tau^2) + m_3^2 (\Delta \sigma + 3\tau)^2 - 2m_4^2 \psi (\Delta \sigma + 3\tau) \right] \]  

(7)

We are now ready to study the physical excitations.

III. PHYSICAL EXCITATIONS AND THEIR NORMALISATION

A. Tensors

The field equation for tensors is

\[ \Box \chi_{ij} + m_2^2 \chi_{ij} = 0 \]  

(8)

Thus, these modes have relativistic spectrum with mass \( m_t = m_2 \). From the action (6) one observes that the normalisation factor is \( M_{Pl}^{-1} \), which is usual for gravitons. This means that strong coupling for tensors at low energies is only due to their interactions with other modes. We will not consider tensors any longer in this paper.

B. Vectors

It is clear from eqs. (6) and (4) that the field \( u_i \) enters the action without time derivatives, i.e., it is a non-dynamical field. We integrate it out by using the field equation obtained by varying the action with respect to \( u_i \) itself,

\[ \Delta (u_i - \partial_0 s_i) - m_1^2 u_i \equiv \Delta w_i - m_1^2 u_i = 0 \]  

(9)

We get

\[ u_i = \frac{\Delta}{\Delta - m_1^2} \partial_0 s_i \]

Plugging this expression back into the action, we obtain for vector part

\[ L_v = M_{Pl}^2 \left[ m_1^2 \partial_0 s_i \frac{\Delta}{\Delta - m_1^2} \partial_0 s_i + m_2^2 s_i \Delta s_i \right] \]  

(10)

Note that in momentum space

\[ \frac{\Delta - m_1^2}{\Delta} = \frac{p^2 + m_1^2}{p^2} \]

is positive for positive \( m_1^2 \), so the term with time derivatives has correct sign (the term with spatial gradient also has correct sign).

To obtain the canonical action, we define a new field \( s_i^c \) such that

\[ s_i = \frac{1}{m_1 M_{Pl}} \sqrt{\frac{\Delta - m_1^2}{2\Delta}} s_i^c \]
and get

\[ L_v = \frac{1}{2} \left[ \partial_0 s^*_i \partial_0 s^c_i - \frac{m^2}{m^2_1} \partial_j s^*_i \partial_j s^c_i - m^2_2 s^*_i s^c_i \right] \]

Thus, the spectrum is

\[ \omega^2 = \frac{m^2}{m^2_1} \mathbf{p}^2 + m^2_2 \]

In terms of the canonically normalised field \( s^*_i \), the original fields \( s_i \) and \( u_i \) are proportional to \((mM_{Pl})^{-1}\). The only gauge-invariant combination in the vector sector, \( w_i \), written in terms of the canonically normalised field is of order

\[ w_i \propto \frac{m}{M_{Pl}} s^*_i \]  

This follows, e.g., from eq. (10). This behavior of the vector modes at small \( m \) is precisely the same as in the Fierz–Pauli case; in particular, the analysis of Ref. [7] suggests that strong coupling occurs at

\[ E \sim \sqrt{mM_{Pl}} \]  

which is a relatively high scale.

Thus, for tensor and vector fields nothing changes, as compared to the Fierz–Pauli case, except for the “speed of light” for vectors. The only constraints we have up to now are

\[ m^2_1 > 0, \quad m^2_2 > 0 \]

Then both vectors and tensors are massive positive energy fields.

### C. Scalars

As we already pointed out, we will consider ghost-free case (3). In this case the field \( \psi \equiv h_{00} \) is the Lagrange multiplier. The corresponding constraint is

\[ \frac{\delta S}{\delta \psi} \propto 2\Delta \tau + m^2_0 \psi - m^2_4 (\Delta \sigma + 3\tau) = 0 \]  

We use this constraint to express \( \sigma \) through \( \tau \),

\[ \sigma = \frac{2}{m^2_4} \tau - \frac{3}{\Delta} \tau \]

Now, the field \( v \) enters the action without time derivatives (the term proptional to \( \partial_0 v \Delta \tau \) in (6) may be written as \( v \partial_0 \Delta \tau \)). Thus, we eliminate this field by making use of its field equation,

\[ \frac{\delta S}{\delta v} \propto 2\partial_0 \tau - m^2_4 v = 0 \]  

and find

\[ v = \frac{2}{m^2_4} \partial_0 \tau \]

We plug the expressions (14) and (16) back into the action, and obtain the action in terms of the only remaining dynamical field \( \tau \). The corresponding Lagrangian is

\[ L_\tau = \frac{M^2_{Pl}}{2} \left[ \frac{8}{m^2_4} - \frac{8}{m^2_1} \right] \Delta \tau \partial^2_0 \tau - 4 \frac{m^2_2 - m^2_4}{m^2_1} (\Delta \tau)^2 
- 6 \tau \partial^2_0 \tau + \left( \frac{8 m^2_2}{m^2_4} - 2 \right) \tau \Delta \tau - 6 m^2_2 \tau^2 \]  

(17)
This is the central formula of this section. It enables one to immediately derive both the spectrum and normalisation factor relating $\tau$ and canonically normalised field $\tau^c$,

$$
\tau = \frac{1}{M_{Pl}} \left[ - \left( \frac{8}{m_4^2} - \frac{8}{m_1^2} \right) \Delta + 6 \right]^{-\frac{1}{2}} \cdot \tau^c \propto \frac{m}{M_{Pl}} \tau^c
$$

In the general case the Lagrangian (17) contains terms enhanced by $1/m^2$, which explains why the normalisation factor for $\tau$ is proportional to $m/M_{Pl}$. The largest fields $\sigma$ and $v$ are proportional to $1/(M_{Pl}m)$, in a complete analogy to vector modes. To understand the properties of another gauge-invariant field $\Phi$, we make use of a linear combination of the two remaining field equations,

$$
\Phi - \tau + m_2^2 \sigma = 0
$$

From eq. (14) we deduce that

$$
\Phi \propto \frac{m}{M_{Pl}} \tau^c
$$

Thus, both gauge-invariant fields $\tau$ and $\Phi$, expressed through the canonically normalised field $\tau^c$, are of order $m/M_{Pl}$, again in analogy to the vector case, eq. (11). All this ensures that the strong coupling scale in the scalar sector is the same as in the vector sector, and is given by eq. (12), and suggests that there is no VDVZ discontinuity.

For

$$
m_1^2 > m_4^2 > 0, \quad m_2^2 > m_3^2
$$

and

$$
4m_2^2 > m_4^2
$$

all terms in the action have correct signs (recall that $\Delta$ is negative-definite). Thus, there are no ghosts or tachyons. The spectrum is

$$
\omega^2 = \frac{p^2 + z\mu_1^2}{p^2 + \mu_0^2} \cdot \frac{\mu_0^2}{\mu_1^2} \cdot p^2 + \frac{m_2^2 \mu_0^2}{p^2 + \mu_0^2}
$$

where the parameters are all positive and are defined as follows,

$$
4 \frac{m_4^2}{m_1^2} - 4 = \frac{3}{\mu_0^2}, \quad 2 \frac{m_2^2 - m_3^2}{m_1^2} = \frac{3}{\mu_1^2}, \quad 4 \frac{m_2^2}{m_4^2} - 1 = 3z
$$

At high momenta one has

$$
\omega^2 = \frac{\mu_0^2}{\mu_1^2} \cdot p^2, \quad p^2 \gg m^2
$$

while at low momenta there is a gap

$$
\omega^2 = m_2^2, \quad p = 0
$$

In this sense the scalar mode is also massive.

Let us now compare our results to the Lorentz-invariant case. In that case, the relation $m_0 = 0$ implies the Fierz–Pauli form of the mass terms. In the Fierz–Pauli case the terms in the first line in eq. (17) vanish, so $\tau \propto 1/M_{Pl}$ in terms of canonically normalised field, and $v, \sigma \propto 1/(M_{Pl}m^2)$. This is in agreement with Ref. [7], and implies the VDVZ discontinuity, as well as the low energy scale of strong coupling. Needless to say, the action for $\tau$ takes the Lorentz-invariant form.

D. High-energy limit

To end up this section, let us mention an alternative way of obtaining the above properties of the three-vector and scalar modes in the high-energy regime $E \gg m$. To analyse the quadratic action directly in the high-energy limit, one
makes use of the formalism of the Stückelberg type. The relevant part of the metric perturbations is “pure gauge”,

\[ h_{\mu\nu} = \partial_\mu \pi_\nu + \partial_\nu \pi_\mu \]  

(19)

and the relevant part of the Lagrangian is the mass term \[ L_m = \frac{M_{Pl}^2}{2} [2m_1^2(\partial_0 \pi_i)^2 + 2m_1^2(\partial_i \pi_0)^2 + (8m_4^2 - 4m_3^2)\pi_0 \partial_0 \partial_i \pi_i - 2m_2^2(\partial_i \pi_j)^2 - (2m_2^2 - 4m_3^2)(\partial_i \pi_i)^2] \]  

(20)

For the transverse part, \( \pi^T_i = s_i \), one immediately obtains that this Lagrangian coincides with the high-energy limit of the Lagrangian (10). In the scalar part, the field \( \pi_0 \) is non-dynamical, and may be eliminated by making use of its field equation, \( \pi_0 = \frac{2m_2^2 - m_1^2}{m_1^2 \Delta} \partial_0 \partial_i \pi_i \). Then the longitudinal part of the Lagrangian (20) becomes

\[ L^L = \frac{M_{Pl}^2}{2} \left[ 8m_4^4 \left( \frac{1}{m_4^2} - \frac{1}{m_3^2} \right) (\partial_0 \pi^L_i)^2 - 4(m_2^2 - m_3^2)(\partial_i \pi^L_i)^2 \right] \]

This coincides with the Lagrangian (17), if one identifies \( \pi^L_i = \frac{1}{2} \partial_\iota \sigma \) and recalls eq. (14), which in the high-energy limit reduces to \( \sigma = 2\tau/m_2^2 \).

Thus, the Lorentz-violating mass terms give rise to healthy kinetic terms for all components of \( \pi \)'s. Repeating the analysis of Ref. \[7\], it is straightforward to see that the strong-coupling scale is indeed the same in the transverse and longitudinal sectors, and is given by eq. (12). This is in sharp contrast to the Fierz–Pauli case, in which the mass terms per se do not produce the kinetic term for the longitudinal part of \( \pi \). \[7\].

Finally, let us briefly discuss the case \( m_0 \neq 0 \). In that case, the term \( 4m_0^2(\partial_0 \pi_0)^2 \) is added to the Lagrangian (20), so the field \( \pi_0 \) becomes dynamical. Depending on the sign of \( m_1^2 \), either the kinetic term for \( \pi_i \) (the first term in eq. (20)) or the gradient term for \( \pi_0 \) (the second term in eq. (20)) has wrong sign, so the energy is unbounded from below. The case \( m_0 \neq 0, m_1 = 0 \) may be interesting. In that case the fields \( \pi_i \) are non-dynamical; after integrating them out at the level of the effective Lagrangian (20), no terms with spatial gradient of \( \pi_0 \) appear. By analysing the complete linearised theory, one finds that there are no physical excitations in the scalar and vector sectors, yet tensor modes obey eq. (8) and have physical excitations of mass \( m_t = m_2 \). We think that the theory with \( m_0 \neq 0 \) and \( m_1 = 0 \) deserves further study.

IV. INTERACTION WITH CONSERVED SOURCES

To see explicitly that the VDVZ discontinuity is absent in the interesting case (18), let us study interactions between conserved sources. In GR, the field induced by conserved energy-momentum \( T_{\mu\nu} \) is

GR: \( h_{\mu\nu} = -\frac{1}{\Box} \left( t_{\mu\nu} - \frac{1}{2} t_\lambda^\lambda \right) + \ldots \)

where dots denote longitudinal terms irrelevant for the interaction between conserved sources, and

\[ t_{\mu\nu} = \frac{1}{M_{Pl}^2} T_{\mu\nu} \]

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1 Even more special case is \( m_0 \neq 0, m_1 = 0 \) and \( m_2^2 = m_3^2 - m_4^2 \), in which the quadratic action possesses an accidental (?) gauge symmetry.
On the other hand, in the Fierz–Pauli case in the limit \( m \to 0 \), one has

\[
\text{FP: } h_{\mu\nu} = -\frac{1}{\Box} \left( t_{\mu\nu} - \frac{1}{3} t_{\lambda}^{\lambda} \right) + \ldots
\]

The difference between the two expressions is precisely the VDVZ discontinuity.

In terms of the gauge-invariant three-vector and scalar fields, the corresponding expressions are

\[
\text{GR and FP: } w_i = \frac{1}{\Delta} t_{0i} \quad (21)
\]

\[
\text{GR: } \tau = \frac{1}{2\Delta} t_{00} \quad (22)
\]

\[
\Phi = \frac{1}{2\Delta} \left( t_{ii} + t_{00} - \frac{3}{\Delta} \partial_0^2 t_{00} \right) \quad (23)
\]

\[
\text{FP: } \tau = \frac{1}{2\Delta} t_{00} + \frac{1}{6\Box} (t_{00} - t_{ii}) \quad (24)
\]

\[
\Phi = \frac{1}{2\Delta} \left( t_{ii} + t_{00} - \frac{3}{\Delta} \partial_0^2 t_{00} \right) - \frac{1}{6\Box} (t_{00} - t_{ii}) \quad (25)
\]

Let us see that the massless limit of the theory with Lorentz-violating mass terms reproduces the GR expressions (22) and (23), as well as eq. (21) for the vector part. By massless limit we mean

\[
m_i^2 \to 0 \, , \quad \frac{m_i^2}{m_j^2} = \text{fixed} \, , \quad i, j = 1, \ldots, 4
\]

We still consider the ghost-free case (3).

The reason for emphasising the gauge-invariant fields is as follows. At the linearised level the source term is

\[
S_{\text{int}} = - \int d^4x \, T_{\mu\nu} h^{\mu\nu}
\]

Making use of the conservation equations, \( \partial_{\mu} T_{\mu}^\nu = 0 \), one expresses this action through gauge-invariant variables (1) and (2),

\[
S_{\text{int}} = - \int d^4x \left( T_{ij} \chi_{ij} - 2 T_{0iw_i} + T_{00} \Phi + T_{ii} \tau \right) \quad (26)
\]

The interaction between conserved sources is thus determined by the gauge-invariant fields produced. The tensor part is trivial, so our purpose is to calculate the fields \( w_i, \Phi \) and \( \tau \) generated by the conserved source \( T_{\mu\nu} \).

### A. Vectors

Making use of eqs. (1) and (2), one varies the action (1), with the source term (26) added, with respect to \( u_i \) and \( s_i \), and obtains the following equations,

\[
\Delta (u_i - \partial_0 s_i) - m_1^2 u_i = t_{0i} \quad (27)
\]

\[
\partial_0 u_i - \partial_0^2 s_i - m_2^2 s_i = \frac{1}{\Delta} \partial_0 t_{0i} \quad (28)
\]

From eq. (27) we get

\[
u_i = \frac{\Delta}{\Delta - m_1^2} \partial_0 s_i + \frac{1}{\Delta - m_1^2} t_{0i}
\]

Plugging this into eq. (28) we obtain

\[
s_i = - \frac{1}{\Delta (\partial_0^2 - m_1^2 \Delta + m_2^2)} \partial_0 t_{0i}
\]
which is finite in the massless limit. The gauge-invariant combination is

\[ w_i = \frac{m_i^2}{\Delta - m_i^2} \partial_0 s_i + \frac{1}{\Delta - m_i^2} t_{0i} \]

It has smooth massless limit, which coincides with the GR expression \[21\].

**B. Scalars**

In the scalar sector, the field equations read

\[ 2\Delta \tau - m_1^2 (\Delta \sigma + 3\tau) = t_{00} \]
\[ 2\partial_0 \tau - m_1^2 v = \frac{1}{\Delta} \partial_0 t_{00} \]
\[ 2\partial_0^2 \tau - m_2^2 \Delta \sigma - m_3^2 \tau + m_2^2 \Delta \sigma + 3m_3^2 \tau - m_3^2 \psi = \frac{1}{\Delta} \partial_0^2 t_{00} \]
\[ 2\Delta (\psi - 2\partial_0 v + \partial_0^2 \sigma) - 2\Delta \tau + 2m_4^2 \Delta \sigma \]
\[ = 2\Delta \Phi - 2\Delta \tau + 2m_4^2 \Delta \sigma = t_{ii} - \frac{3}{\Delta} \partial_0^2 t_{00} \]

Equations \[29\] – \[32\] are straightforwardly solved. Equation \[29\] gives

\[ \sigma = \frac{2}{m_1^2} \tau - \frac{1}{m_1^2 \Delta} t_{00} - \frac{3}{\Delta} \tau \]

From eq. \[31\] we find

\[ v = \frac{2}{m_1^2} \partial_0 \tau - \frac{1}{m_1^2 \Delta} \partial_0 t_{00} \]

while eq. \[31\] yields

\[ \psi = \frac{1}{m_4^2} \left( 2\partial_0^2 \tau - 2 \frac{m_2^2 - m_3^2}{m_4^2} \Delta \tau + 2m_2^2 \tau + \frac{m_2^2 - m_3^2}{m_4^2} t_{00} - \frac{1}{\Delta} \partial_0^2 t_{00} \right) \]

Plugging these expressions into eq. \[32\] we obtain

\[ \left( \frac{8}{m_4^2} - \frac{8}{m_1^2} \right) \partial_0^2 \tau - 4 \frac{m_2^2 - m_3^2}{m_4^2} \Delta \tau - 6 \partial_0^2 \tau + \left( \frac{8}{m_4^2} \right) \Delta \tau - 2m_2^2 \tau \]
\[ = \left( \frac{4}{m_4^2} - \frac{4}{m_1^2} \right) \partial_0^2 t_{00} - 2 \frac{m_2^2 - m_3^2}{m_4^2} \Delta t_{00} + t_{ii} - \frac{3}{\Delta} \partial_0^2 t_{00} + 2 \frac{m_2^2}{m_4^2} t_{00} \]

Of course the left hand side of this equation matches the Lagrangian \[17\]. The point is that the coefficients in the right hand side are such that in the massless limit, the GR expression \[22\] is reproduced (provided that \( m_1^2 \neq m_4^2 \) and/or \( m_2^2 \neq m_4^2 \)), i.e., \( \tau = t_{00}/(2\Delta) + O(m^2) \). Now, from the latter expression and eq. \[33\] it follows that \( \sigma \) is finite in the massless limit. Then from eq. \[32\] one finds that \( \Phi \) also has massless limit which coincides with the GR expression \[23\]. Thus, there is no VDVZ discontinuity.

As a cross check, one recovers the Fierz–Pauli case by setting all masses in eq. \[34\] equal to each other. The resulting equation is straightforwardly solved; then using eqs. \[22\] and \[33\], one indeed finds that the expressions \[24\] and \[25\] are obtained in the massless limit.

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