Singularity of Generalized Grey Brownian Motion and Time-Changed Brownian Motion

José Luís da Silva,
CIMA, University of Madeira, Campus da Penteada, 9020-105 Funchal, Portugal.
Email: joses@staff.uma.pt

Mohamed Erraoui
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, BP 2390, Marrakech, Maroc
Email: erraoui@uca.ma

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Abstract

The generalized grey Brownian motion is a time continuous self-similar with stationary increments stochastic process whose one dimensional distributions are the fundamental solutions of a stretched time fractional differential equation. Moreover, the distribution of the time-changed Brownian motion by an inverse stable process solves the same equation, hence both processes have the same one dimensional distribution. In this paper we show the mutual singularity of the probability measures on the path space which are induced by generalized grey Brownian motion and the time-changed Brownian motion though they have the same one dimensional distribution. This singularity property propagates to the probability measures of the processes which are solutions to the stochastic differential equations driven by these processes.

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1 Introduction

Over the past decades, the physical and mathematical community has shown a growing interest in modeling anomalous diffusion processes. The label anomalous diffusion is assigned to processes whose particle displacement variance does not grow linearly in time, in opposition to the standard Brownian motion that is mainly characterized by a linear law. In this respect, the terms subdiffusion and superdiffusion are used for those processes whose variance grows in time is slower or faster than linear, respectively.

The generalized grey Brownian motion (ggBm for short) $B_{\beta,\alpha}$, $0 < \beta \leq 1$, $0 < \alpha < 2$ was introduced by [MP08] to model both subdiffusion and superdiffusion, see also [MM09]. This family of stochastic processes are $\alpha$-self-similar with stationary increments and includes in particular the class of fractional Brownian motion (fBm) $B^{H} = B_{1,\frac{H}{2}}$ with Hurst parameter $H = \frac{\alpha}{2}$ and $\beta = 1$ was well as standard Brownian motion (Bm) $W = B_{1,1}$ with $\alpha = \beta = 1$. For $\alpha = \beta \in (0,1)$ we obtain grey Brownian motion (gBm) $B_{\beta} = B_{\beta,\beta}$, $0 < \beta \leq 1$ introduced by [Sch90] to study time-fractional diffusion equation. Moreover, the probability density function (PDF) $f_{\beta,\alpha}$ of ggBm is the fundamental solution of the stretched time-fractional standard diffusion equation

$$u(x,t) = u_{0}(x) + \frac{1}{\Gamma(\beta)} \frac{\alpha}{\beta} \int_{0}^{t} s^{\alpha-1} \left(t^{\alpha/\beta} - s^{\alpha/\beta}\right)^{\beta-1} \frac{\partial^{2}}{\partial x^{2}} u(x,s) \, ds, \quad t \geq 0, \quad (1)$$

with initial condition $u_{0}(x) = \delta(x)$, see [MMP10] and references therein. Therefore, the Fourier transform of $f_{\beta,\alpha}$ is

$$\mathbb{E}(e^{i\theta B_{\beta,\alpha}(t)}) = E_{\beta} \left(-\frac{\theta^{2}}{2} \tau^{\alpha}\right), \quad \theta \in \mathbb{R}, \ t \geq 0,$$

where $E_{\beta}$ is the Mittag-Leffler function, see Section 1 for the definition and properties the this function.

It is important to observe that, starting from a master equation (1) which describes the dynamic evolution of a probability density function $f_{\beta,\alpha}$, it is always possible to define an equivalence class of stochastic processes with the same PDF function $f_{\beta,\alpha}$. All these processes provide suitable stochastic models for the starting equation (1). One way to do this is to use a subordination technique. Let us recall that heuristically, the ggBm $B_{\beta,\alpha}$ cannot
be a subordinated process (for example if \( \beta = 1 \) it reduces to a fractional Brownian motion).

Indeed, let \( S_\beta \) denotes a strictly increasing \( \beta \)-stable subordinator and \( U_\beta \) its inverse, that is

\[
U_\beta(t) := \inf \{ s \geq 0 | S_\beta(t) > s \}, \quad t \in \mathbb{R}^+ := [0, \infty).
\]

We consider the increasing, right continuous process given by \( t \mapsto U_{\beta,\alpha}(t) := U_\beta(t^{\alpha/\beta}) \). Now let us introduce the process \( X_{\beta,\alpha} \) defined by evaluating Brownian motion (Bm) \( W \), independent of the subordinator \( S_\beta \) and its inverse \( U_\beta \), at the random time \( U_{\beta,\alpha} \), this is

\[
X_{\beta,\alpha}(t) := W(U_{\beta,\alpha}(t)), \quad t \geq 0.
\]

We will termed it as continuous time-changed Brownian motion. Since \( U_\beta \) has a Mittag-Leffler distribution

\[
E(e^{-uU_\beta(s)}) = E_\beta(-us^\beta), \quad u > 0, \ s \in \mathbb{R}^+.
\]  

(2)

cf. \[0x0]Bin71, then \( X_{\beta,\alpha} \) has Laplace transform given by

\[
E(e^{-uX_{\beta,\alpha}(t)}) = E_\beta\left(-\frac{u^2}{2}t^\alpha\right), \quad u > 0, \ t \geq 0.
\]

Then it follows that the PDF function of the process \( X_{\beta,\alpha} \) is the fundamental solution of the stretched time-fractional equation (1). But this equation describes only the evolution in time of one-dimensional distributions; thus it is mathematical incomplete for the identification of the process. Clearly we need to know all finite dimensional distributions.

Here we would like to mention a closed similar result associated to the fractional Poisson process (fPp) \( N_\beta, 0 < \beta \leq 1 \), see [MGS04, MNV11, BO09]. More precisely, the fPp is a natural generalization of the standard Poisson process \( N \) with intensity \( \lambda > 0 \). The corresponding iid waiting times \( J_n \) are Mittag-Leffler distributed, that is

\[
P(J_n > t) = E_\beta(-\lambda t^\beta).
\]

Then the fPp

\[
N_\beta(t) = \max \{ n \geq 0 | J_1 + \ldots + J_n \leq t \}
\]
is a renewal process with Mittag-Leffler waiting times and its distribution is given by

\[ p_n(t) := P(N_\beta(t) = n) = \frac{(\lambda t^\beta)^n}{n!} E_\beta^{(n)}(-\lambda t^\beta), \quad n \geq 0, \ t > 0, \]

with \( E_\beta^{(n)}(-\lambda t^\beta) \) := \( \frac{d^n}{dz^n} E_\beta(z)|_{z=-\lambda t^\beta} \). Moreover the probability generating function \( G_\beta \) of \( N_\beta \) is

\[ G_\beta(z, t) := \mathbb{E}(z^{N_\beta(t)}) = E_\beta(\lambda t^\beta(z - 1)). \]

Consider now the time-changed process \( N(U_\beta(t)) \), called in [MNV11] fractal time Poisson process (ftPp). [BO09] showed that both processes \( N_\beta \) and \( N(U_\beta) \) have the same one-dimensional distributions since they solved the same time-fractional analogue of the forward Kolmogorov equation with the same point source. So far, we have a similar scenario as ggBm \( B_{\beta,\alpha} \) and the time-changed process \( W(U_{\beta,\alpha}) \), that is they have the same one dimensional density functions. The surprising fact here is indeed that the ftPp \( N_\beta \) and the time-changed process \( N(U_\beta) \) are the same process. The reason for this is: \( N_{\beta} \) and \( N(U_\beta) \) are both pure jump processes with iid Mittag-Leffler distributed waiting times between jumps, see [MNV11, Theorem 2.2]. So it is natural to ask whether the processes \( B_{\beta,\alpha} \) and \( W(U_{\beta,\alpha}) \) are same or not. Unfortunately this is not the case. This difference with the Poissonian case is essentially a consequence of the fact that \( B_{\beta,\alpha} \) and \( W(U_{\beta,\alpha}) \) have not the same \( p \)-variation index. Indeed they are not both semimartingales, specifically \( B_{\beta,\alpha} \) is not a semimartingale. Moreover, their probability laws \( P_{B_{\beta,\alpha}} \) and \( P_{W(U_{\beta,\alpha})} \), defined on \( C([0, 1], \mathbb{R}) \) (the space of continuous functions \( w : [0, 1] \rightarrow \mathbb{R} \) such that \( w(0) = 0 \)), are mutually singular, see Theorem 9 below.

In Section 1 we define the families of processes used in the paper, their main properties and canonical realizations. In Section 2 we prove the mutual singularity of the probability measures \( P_{B_{\beta,\alpha}} \) and \( P_{W(U_{\beta,\alpha})} \). We will determine explicit events which distinguish between the two measures. The key tool is the \( p \)-variation of the paths of the corresponding processes. In Section 3 we apply the same idea to show that the singularity property propagates to the solutions of stochastic differential equations (SDEs) driven by ggBm and time-changed process \( W(U_{\beta,\alpha}) \) of the form

\[ X(t) = X(0) + \int_0^t g(X(s)) dB_{\beta,\alpha}(s), \ 0 \leq t \leq 1, \]
and

\[ Y(t) = Y(0) + \int_0^t f(Y(s)) dW(U_{\beta,\alpha}(s)), \quad 0 \leq t \leq 1. \]

In this section we introduce the required path spaces in order to realize the stochastic process we deal with as canonical processes. These spaces are subsets of \( C(\mathbb{T}, \mathbb{R}) \), the space of continuous functions \( w : \mathbb{T} \to \mathbb{R} \) such that \( w(0) = 0 \). The time parameter set \( \mathbb{T} \) is either \( \mathbb{R}^+ \) or \([0, 1]\) depending on the process in question, we have in mind namely the ggBm \( B_{\beta,\alpha} \) and the subordination of the Bm \( W \) by a family of time-change \( U_{\beta,\alpha} \), see details below. The functions \( Y(t), t \in \mathbb{T} \) taking values in \( \mathbb{R} \), defined by \( Y(t)(w) := w(t) \) are called coordinate mappings.

### 1.1 Canonical Realization of Generalized Grey Brownian Motion

Let \( 0 < \beta < 1 \) and \( 1 < \alpha < 2 \) be given. A continuous stochastic process defined on a complete probability space \((\Omega, \mathcal{F}, P)\) is a generalized grey Brownian motion, denoted by \( B_{\beta,\alpha} = \{ B_{\beta,\alpha}(t), t \geq 0 \} \), see [MM09], if:

1. \( B_{\beta,\alpha}(0) = 0, P \) almost surely.

2. Any collection \( \{ B_{\beta,\alpha}(t_1), \ldots, B_{\beta,\alpha}(t_n) \} \) with \( 0 \leq t_1 < t_2 < \ldots < t_n < \infty \) has characteristic function given, for any \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \), by

\[
E \left( \exp \left( i \sum_{k=1}^n \theta_k B_{\beta,\alpha}(t_k) \right) \right) = E_\beta \left( -\frac{1}{2} \theta^\top \Sigma_\alpha \theta \right),
\]

where

\[
\Sigma_\alpha = \left( t_k^\alpha + t_j^\alpha - |t_k - t_j|^\alpha \right)_{k,j=1}^n
\]

and the joint probability density function is equal to

\[
f_\beta(\theta, \Sigma_\alpha) = \frac{(2\pi)^{-\frac{n}{2}}}{\sqrt{\det \Sigma_\alpha}} \int_0^\infty \tau^{-\frac{n}{2}} e^{-\frac{1}{2} \theta^\top \Sigma_\alpha^{-1} \theta} M_\beta(\tau) d\tau.
\]

Here \( E_\beta \) is the Mittag-Leffler (entire) function

\[
E_\beta(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C},
\]
and where $M_\beta$ is the so-called $M$-Wright probability density function with Laplace transform
\[
\int_0^\infty e^{-s\tau} M_\beta(\tau) \, d\tau = E_\beta(-s).
\]
(4)
The density $M_\beta$ is a particular case of the Wright function $W_{\lambda,\mu}$, $\lambda > -1$, $\mu \in \mathbb{C}$ and its series representation, which converges in the whole complex $z$-plane, is given by
\[
M_\beta(z) := W_{-\beta,1-\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}.
\]
For the choices $\beta = 1/2$ and $\beta = 1/3$ the corresponding $M$-Wright functions are
\[
M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right),
\]
(5)
\[
M_{1/3}(x) = 3^{2/3} \text{Ai}\left(\frac{x}{3^{1/3}}\right),
\]
(6)
where Ai is the Airy function see [OLBC10] for more details and properties.

The absolute moments of order $\delta > -1$ in $\mathbb{R}^+$ of the density $M_\beta$ are finite and given by
\[
\int_0^\infty \tau^{\delta} M_\beta(\tau) \, d\tau = \frac{\Gamma(\delta + 1)}{\Gamma(\beta\delta + 1)}.
\]
(7)

The grey Brownian motion has the following properties:

1. For each $t \geq 0$, the moments of any order are given by
\[
\left\{
\begin{array}{l}
E(B_{\beta,\alpha}^{2n+1}(t)) = 0, \\
E(B_{\beta,\alpha}^{2n}(t)) = \frac{(2n)!}{2^n\Gamma(\beta n + 1)} t^{\alpha n}.
\end{array}
\right.
\]

2. The covariance function has the form
\[
E(B_{\beta,\alpha}(t)B_{\beta,\alpha}(s)) = \frac{1}{2\Gamma(\beta + 1)} (t^{\alpha} + s^{\alpha} - |t - s|^{\alpha}), \quad t, s \geq 0.
\]
(8)

3. For each $t, s \geq 0$, the characteristic function of the increments is
\[
E(e^{i\theta(B_{\beta}(t) - B_{\beta}(s))}) = E_\beta\left(-\frac{\theta^2}{2}|t - s|^{\alpha}\right), \quad \theta \in \mathbb{R}.
\]
(9)
4. The process \( B_{\beta,\alpha} \) is non Gaussian, \( \alpha/2 \)-self-similar with stationary increments.

5. The \( \alpha/2 \)-self-similarity of the ggBm and the ergodic theorem imply that, with probability 1, we have

\[
\lim_{n \to +\infty} \sum_{i=1}^{2^n} \left| B_{\beta,\alpha} \left( \frac{i}{2^n} \right) - B_{\beta,\alpha} \left( \frac{i-1}{2^n} \right) \right|^{2/\alpha} = \mathbb{E} \left( |B_{\beta,\alpha}(1)|^{2/\alpha} \right) =: \mu_{\beta,\alpha},
\]

see [DSE18].

6. The ggBm is not a semimartingale. In addition, \( B_{\alpha,\beta} \) cannot be of finite variation on \([0, 1]\) and by scaling and stationarity of the increment on any interval.

We now construct a canonical version of \( B_{\beta,\alpha} \). Let \( (C([0, 1], \mathbb{R}), \mathcal{B}(C([0, 1], \mathbb{R}))) \) be the measurable space of all real-valued continuous functions vanishing at zero with the \( \sigma \)-algebra \( \mathcal{B}(C([0, 1], \mathbb{R})) \) generated by the cylinder sets. By using the Kolmogorov extension theorem, the probability measure \( P_{B_{\beta,\alpha}} \) induced by \( B_{\beta,\alpha} \) on \( C([0, 1], \mathbb{R}) \) is then characterized by the probability measure of the cylinder sets. That is, for any \( 0 \leq t_1 < t_2 < \ldots < t_n < \infty, A_1, \ldots, A_n \) Borel measurable sets and \( n \geq 1 \) we have

\[
P_{B_{\beta,\alpha}} \{ w \in C([0, 1], \mathbb{R}) \mid w(t_1) \in A_1, \ldots, w(t_n) \in A_n \} = \int_{A_1 \times \ldots \times A_n} f_{\beta}(\theta, \Sigma_{\alpha}) \, d\theta_1 \ldots d\theta_n.
\]

So that the coordinate process

\[ B_{\beta,\alpha}(t)(w) = w(t), \quad w \in C([0, 1], \mathbb{R}), \ t \in [0, 1], \]

is a ggBm.

### 1.2 Canonical Realization of Time-Changed Brownian Motion

In what follows we consider a filtered probability space \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})\). The filtration \((\mathcal{F}_t)_{t \geq 0}\) is assumed to satisfy the usual conditions, that is increasing, right continuous and \( \mathcal{F}_0 \) contains all \( P \)-negligible events in \( \mathcal{F} \) and supporting a standard Brownian motion \( W \) and an independent \( \beta \)-stable process \( S_{\beta} \), \( 0 < \beta < 1 \).
Let $S_\beta = \{S_\beta(t), \ t \in [0, 1]\}$ be a strictly increasing $\beta$-stable subordinator, with Laplace transform given by, see [Ber96, Ch. III]

$$E(e^{-\theta S_\beta(t)}) = e^{-\theta^\beta}, \ \theta > 0, \ t \in [0, 1].$$

Define its inverse process by

$$U_\beta(t) := \inf \{s \geq 0 | S_\beta(t) > s\}, \ t \in \mathbb{R}^+.$$

Remark 1. 1. The process $U_\beta$ has continuous and nondecreasing paths. In addition, it is a $\beta$-self-similar process but has neither stationary nor independent increments, so it not a Lévy process, see [MS04].

2. It follows from (4) that the distribution of $U_\beta(t)$, $t > 0$ is absolutely continuous with respect to the Lebesgue measure and its density is $\mathbb{M}_\beta(\tau, t) := t^{-\beta} M_\beta(t^{-\beta}\tau), \ \tau \geq 0$. $\mathbb{M}_\beta$ is called $\mathbb{M}$-Wright function in two variables which appears as the fundamental solution of the time fractional drift equation, see [MMP10] and references therein for more details. As a consequence $U_\beta(t)$, $t \geq 0$ is not a bounded random variable.

We define the time-change process

$$U_{\beta, \alpha}(t) := U_\beta(f(t)),$$

where $f(t) := t^{\alpha/\beta}$, $t \in [0, 1]$. The maps $t \mapsto U_{\beta, \alpha}(t)$, $t \in [0, 1]$ are almost surely increasing and continuous.

Lemma 2. The family of time-change processes $U_{\beta, \alpha} = \{U_{\beta, \alpha}(t), t \in [0, 1]\}$ is a nondecreasing family of $\mathcal{F}_t$-stopping times such that $E(U_{\beta, \alpha}(t)) = T_\alpha^{\beta/\beta+1}$.

Proof. Indeed, for any $s \geq 0$, we have

$$[U_{\alpha, \beta}(t) \leq s] = [t^{\alpha/\beta} \leq S_\beta(s)] \in \mathcal{F}_s.$$ 

Moreover we have from assertion 3 of Remark 1 and equality (7) that $E(U_{\beta, \alpha}(t)) = T_\alpha^{\beta/\beta+1}$. \hfill $\square$

We introduce the subordination of $W$ by the time-change process $U_{\beta, \alpha}$, that is

$$X_{\beta, \alpha} := \{X_{\beta, \alpha}(t) := W(U_{\beta, \alpha}(t)), t \in [0, 1]\}.$$

and denote by $P_{X_{\beta, \alpha}}$ the measure induced by $X_{\beta, \alpha}$ on $(C([0, 1], \mathbb{R}), \mathcal{B}(C([0, 1], \mathbb{R})))$. The process $X_{\beta, \alpha}$ possesses the following properties.
1. The time-change process $X_{\beta,\alpha}$ is a non Gaussian and $\alpha/2$-self-similar. In fact, the characteristic function of $X_{\beta,\alpha}(t) = W(U_{\beta,\alpha}(t))$ is given by

$$E(e^{i\theta X_{\beta,\alpha}(t)}) = \int_0^\infty E(e^{i\theta W(\tau)}) dP_{U_{\beta,\alpha}(t)}(\tau) = \int_0^\infty e^{-\frac{\theta^2}{2}\tau} dP_{U_{\beta,\alpha}(t)}(\tau) = E_{\beta} \left( -\frac{\theta^2}{2} t^\alpha \right),$$

(11)

where in the last equality we used (2) and the fact that $(f(t))^\beta = t^\alpha$. Since $U_\beta$ is $\beta$-self-similar, then $U_{\beta,\alpha}$ is $\alpha$-self-similar. Using the fact that $W$ is $1/2$-self-similar, it follows that $X_{\beta,\alpha}$ is $\alpha/2$-self-similar.

2. The process $X_{\beta,\alpha}$ has no stationary increments, it is not a Lévy process. Indeed, it follows from Corollary 2.46 in [MP10] we have

$$E \left[ (X_{\beta,\alpha}(t) - X_{\beta,\alpha}(s))^2 \right] = E \left[ (X_{\beta,\alpha}(t))^2 \right] - E \left[ (X_{\beta,\alpha}(s))^2 \right].$$

A simple computation shows that

$$E \left[ (X_{\beta,\alpha}(t))^2 \right] = \frac{t^\alpha}{\Gamma(\beta + 1)} \quad \text{and} \quad E \left[ (X_{\beta,\alpha}(s))^2 \right] = \frac{s^\alpha}{\Gamma(\beta + 1)}.$$

Therefore, we obtain

$$E \left[ (X_{\beta,\alpha}(t) - X_{\beta,\alpha}(s))^2 \right] = \frac{t^\alpha}{\Gamma(\beta + 1)} - \frac{s^\alpha}{\Gamma(\beta + 1)}. \quad (12)$$

3. The process $X_{\beta,\alpha}$ is an ($G_t$)-semimartingale, where $G_t := F_{U_{\beta,\alpha}(t)}$, see [Jac79, Corollary 10.12].

Let $(C(\mathbb{R}^+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}^+, \mathbb{R})), P_W)$ be the classical Wiener space, that is the space of all continuous functions $w : \mathbb{R}^+ \rightarrow \mathbb{R}$ with $w(0) = 0$, endowed with the locally uniform convergence topology. $P_W$ is the Wiener measure so that the coordinate process

$$W(t)(w) := w(t), \quad \forall t \geq 0, \ w \in C(\mathbb{R}^+, \mathbb{R})$$

is a standard Brownian motion.

Let $\mathbb{U}$ be the space of all continuous nondecreasing functions $l : [0, 1] \rightarrow \mathbb{R}^+$ with $l(0) = 0$. The space $\mathbb{U}$ is equipped with uniform convergence topology. Denote by $P_{U_{\beta,\alpha}}$ the probability measure induced by $U_{\beta,\alpha}$ on $\mathbb{U}$. Then the time-change process $U_{\beta,\alpha}$ can be realized as a canonical process on $(\mathbb{U}, \mathcal{B}(\mathbb{U}), P_{U_{\beta,\alpha}})$ by

$$U_{\beta,\alpha}(t)(l) := l(f(t)), \ t \in [0, 1], \ l \in \mathbb{U}. \quad (9)$$
Moreover, the processes $S_\beta$ and $W$ are assumed to be mutually independent, then we have also the independence between $W$ and the time-change process $U_{\beta,\alpha}$.

Since $U_{\beta,\alpha}$ and $W$ are independent, $X_{\beta,\alpha}$ is the canonical process on the product space $(C(\mathbb{R}^+, \mathbb{R}) \times \mathbb{U}, B(C(\mathbb{R}^+, \mathbb{R})) \otimes B(\mathbb{U}), P_W \otimes P_{U_{\beta,\alpha}})$ such that

$$X_{\beta,\alpha}(t)(w, l) := W(U_{\beta,\alpha}(t)(l))(w) = w(l(f(t))), \ t \in [0, 1], \ w \in C(\mathbb{R}^+, \mathbb{R}), \ l \in \mathbb{U}. $$

Notice that we need to use the space $C(\mathbb{R}^+, \mathbb{R})$ for the realization of $X_{\beta,\alpha}$ due to the fact that $U_{\beta}(t), \ t \geq 0$ is not a bounded random variable, cf. Remark 1-3. Moreover, $P_{X_{\beta,\alpha}}$ is a probability measure on the path space

$$\mathcal{X} = \{w \circ l \circ f : [0, 1] \rightarrow \mathbb{R} | w \in C(\mathbb{R}^+, \mathbb{R}), \ l \in \mathbb{U}\}, \quad (13)$$

equipped with the uniform convergence topology.

## 2 Singularity of Generalized Grey Brownian Motion and Time-changed Brownian Motion

In this section we establish the mutual singularity of the probability measures $P_{B_{\beta,\alpha}}$ and $P_{X_{\beta,\alpha}}$ on $C([0, 1], \mathbb{R})$, see Theorem 9 below. Let us first show that both processes $B_{\beta,\alpha}$ and $X_{\beta,\alpha}$ have only the same one dimensional distributions.

**Proposition 3.** The processes $B_{\beta,\alpha}$ and $X_{\beta,\alpha}$ have only the same one dimensional distribution.

**Proof.** For any $\theta \in \mathbb{R}$, it follows from (3), with $n = 1$ and (11), that

$$E(e^{i\theta B_{\beta,\alpha}(t)}) = E(e^{i\theta X_{\beta,\alpha}(t)}) = E_{\beta}\left(-\frac{\theta^2}{2}t^{\alpha}\right).$$

This shows that the one dimensional distribution of $B_{\beta,\alpha}$ and $X_{\beta,\alpha}$ coincides. In order to show that the distributions of higher order do not coincide it is sufficient to prove that, for any $0 \leq s < t \leq 1$

$$E\left[(B_{\beta,\alpha}(t) - B_{\beta,\alpha}(s))^2\right] \neq E\left[(X_{\beta,\alpha}(t) - X_{\beta,\alpha}(s))^2\right]. \quad (14)$$

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On one hand, it is clear that
\[ E \left[ (B_{\beta,\alpha}(t) - B_{\beta,\alpha}(s))^2 \right] = \frac{|t - s|^\alpha}{\Gamma(\beta + 1)}. \]
On the other hand, from (11) we have
\[ E \left[ (X_{\beta,\alpha}(t) - X_{\beta,\alpha}(s))^2 \right] = \frac{t^\alpha}{\Gamma(\beta + 1)} - \frac{s^\alpha}{\Gamma(\beta + 1)}. \]
Therefore, the conclusion follows from the fact that for any \( \alpha \in (1, 2) \), \( |t - s|^\alpha \neq t^\alpha - s^\alpha \).

As a consequence of Proposition 3 the processes \( B_{\beta,\alpha} \) and \( X_{\beta,\alpha} \) do not induce the same probability measures on the path space \( C([0, 1], \mathbb{R}) \). The task of distinguishing between the two measures becomes a question of interest. One possibility is to investigate the variation of the paths under these measures.

We start by recalling the notion of \( p \)-variation, \( p \)-variation index of a function.

Let \( f \) be a function in \( C([0, T], \mathbb{R}) \). The \( p \)-variation, \( 0 < p < \infty \), of \( f \) is defined by
\[ v_p(f; [0,T]) := \sup_{\Pi_n} \sum_{k=1}^n |f(t_k) - f(t_{k-1})|^p, \quad (15) \]
where the supremum is taken over all partitions \( \Pi_n = \{t_0, t_1, \ldots, t_n\}, n \geq 1 \) of \([0, T]\) with \( 0 = t_0 < t_1 < \ldots < t_n = T \).

**Remark 4.** If all sample path of a stochastic process \( X \) have bounded \( p \)-variation on \([0, t]\) for certain \( 0 < p < \infty \) and \( 0 < t \leq 1 \), as the set of all partitions on \([0, t]\) is not measurable, then the function \( w \mapsto v_p(X(\cdot, w); [0, t]) \) need not be measurable. In order to overcome this difficulty, for a continuous stochastic process, the \( p \)-variation over an interval is indistinguishable from the \( p \)-variation over a countable and everywhere dense set, which is always measurable.

Let \( f \) be a continuous function on \([0,1]\) and \( \lambda = \{\lambda_m | m \geq 1\} \) a nested sequence of dyadic partitions \( \lambda_m = \{i2^{-m} : i = 0, \ldots, 2^m\}, m \geq 1 \), of \([0,1]\). For \( 0 < p < \infty \), \( m \geq 1 \), and \( t \in [0,1] \), let
\[ v_p(f; \lambda_m)(t) := \max \left\{ \sum_{j=1}^k |f(s_{ij}(j) \wedge t) - f(s_{ij}(j-1) \wedge t)|^p \bigg| 0 = i(0) < i(1) < \ldots < i(k) = 2^m \right\}, \]
where $s_i := i2^{-m}$, $i = 0, \ldots, 2^m$. Since $\lambda$ is a nested sequence of partitions, the sequence $v_p(f; \lambda_m)(t)$, $m \geq 1$ is non-decreasing for each $t \in [0, 1]$. For $t \in [0, 1]$ we define

$$v_p(f)(t) := \sup_{m \geq 1} v_p(f; \lambda_m)(t) = \lim_{m \to \infty} v_p(f; \lambda_m)(t). \quad (16)$$

For continuous functions $f$ the $v_p(f)(t)$ is equal to $v_p(f; [0, t])$ for any $t \in [0, 1]$. For a continuous stochastic process $X = \{X(t), \ 0 \leq t \leq 1\}$ and for each $t \in [0, 1]$

$$v_p(X)(t, w) := v_p(X(\cdot, w))(t)$$

is possibly an unbounded but a measurable function of $w \in \Omega$. It is clear from the definition (16) that if $X$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, then $v_p(X)$ is also adapted to $(\mathcal{F}_t)_{t \geq 0}$ whose almost all sample paths are non-decreasing. If $v_p(X)(1) < \infty$ almost surely, then $\{v_p(X; [0, t]), t \in [0, 1]\}$ is a stochastic process which is indistinguishable from $\{v_p(X)(t), t \in [0, 1]\}$, see Theorem 2, page 117 in [Nor18].

**Definition 5** (cf. [Nor18]). Let $0 < p < \infty$ and $X = \{X(t), \ 0 \leq t \leq 1\}$ be a given stochastic process. We say that $X$ is of bounded $p$-variation if $X$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$ and $v_p(X)(1) < \infty$ almost surely. We call $v_p(X)$ the $p$-variation process of $X$.

For a function $f : [0, 1] \to \mathbb{R}$, the $p$-variation index of $f$ is defined by

$$v(f) := v(f; [0, 1]) := \begin{cases} \inf \{p > 0 | v_p(f; [0, 1]) < +\infty\}, & \text{if the set is non empty,} \\ +\infty, & \text{otherwise.} \end{cases}$$

The $p$-variation index of a stochastic process $X$ is defined similarly and is denoted by $v(X; [0, 1])$. More precisely, the $p$-variation index $v(X; [0, 1])$ is defined provided $v(X(\cdot, w); [0, 1])$ is a constant for almost all $w \in \Omega$. We say that the $p$-variation index of $X$ is $q$ if with probability 1, the $p$-variation index $v(X; [0, 1]) = q$.

**Example 6.** 1. For a standard Brownian motion $W$ we have almost surely $v_p(W; [0, 1]) < +\infty$ for $p > 2$ and $v_2(W; [0, 1]) = +\infty$. Then $v(W; [0, 1]) = 2$, see [Lév40].

2. If $Y$ is a semimartingale then, for any $p > 2$, $Y$ has bounded $p$-variation, see [Nor18]. In addition, it is well known that it must have unbounded $p$-variation for every $p < 2$, see [Nor18, Thm. 5 page 120]. Then we conclude that $v(Y; [0, 1]) = 2$.  

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3. For a fractional Brownian motion $B^H$. We have almost surely $v_p(B^H; [0, 1]) < +\infty$ for $p > 1/H$ and $v_{1/H}(B^H; [0, 1]) = +\infty$. So $v(B^H, [0, 1]) = 1/H$, see [DN98, Sec. 5.3].

4. For ggBm $B_{\beta,\alpha}$ we have almost surely $v_p(B_{\beta,\alpha}; [0, 1]) < +\infty$ for $p > 2/\alpha$ and $v_{2/\alpha}(B_{\beta,\alpha}; [0, 1]) = +\infty$. So $v(B_{\beta,\alpha}; [0, 1]) = 2/\alpha$, see [DSE18].

Now we consider the sets

\[ V_{2/\alpha} := \left\{ w \in C([0, 1], \mathbb{R}) \bigg| v(w; [0, 1]) = \frac{2}{\alpha} \right\} \quad (17) \]

\[ V_{2,\text{loc}} := \left\{ w \in C([0, T], \mathbb{R}) \bigg| v(w, [0, T]) = 2, \forall T > 0 \right\}. \quad (18) \]

Note that $V_{2,\text{loc}}$ is a measurable set. In addition, we define the subset $\hat{V}$ of $X$ (cf. (13)) by

\[ \hat{V} := \{ \hat{w} := w \circ l \circ f : [0, 1] \rightarrow \mathbb{R} \mid w \in V_{2,\text{loc}}, l \in \mathbb{U} \}. \quad (19) \]

It is easy to see that for any $\hat{w} \in \hat{V}$, we have $v(\hat{w}; [0, 1]) = 2$.

**Lemma 7.** The sets $V_{2/\alpha}$ and $\hat{V}$ are disjoint.

**Proof.** Let $\hat{w} \in V_{2/\alpha} \cap \hat{V}$ be given. As $\hat{w} \in V_{2/\alpha}$, it follows from the definition of $V_{2/\alpha}$ that

\[ v(\hat{w}; [0, 1]) = \frac{2}{\alpha}. \]

On the other hand, $\hat{w} \in \hat{V}$ then $v(\hat{w}; [0, 1]) = 2$ which is absurd because $\frac{2}{\alpha} < 2$, for any $\alpha \in (1, 2)$. \qed

In order to prove the main theorem of this section we need the following result.

**Proposition 8** ([MP10, Thm. 1.35]). Let $T > 0$ be given and $\Pi_n = \{t_0, t_1, \ldots, t_n\}$, $n \geq 1$ with $0 = t_0 < t_1 < \ldots < t_n = T$ be a sequence of nested partitions of $[0, T]$, that is at each state one or more partition points are added, such that the mesh $\|\Pi_n\| := \max_{0 \leq i \leq n} |t_{i-1} - t_i| \rightarrow 0$, $n \rightarrow \infty$. Then, almost surely, we have

\[ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} |W(t_i) - W(t_{i-1})|^2 = T. \quad (20) \]
Theorem 9. The two probability measures \( P_{B,\beta,\alpha} \) and \( P_{X,\beta,\alpha} \) are mutually singular.

**Proof.** As a consequence of Proposition 8 we have \( P_W(\mathcal{V}_{2,\text{loc}}) = 1 \), then we obtain \( P_{X,\beta,\alpha}(\hat{\mathcal{V}}) = 1 \). On the other hand, (10) implies that \( P_{B,\beta,\alpha}(\mathcal{V}_{2/\alpha}) = 1 \). From Lemma 7 we have \( \mathcal{V}_{2/\alpha} \cap \hat{\mathcal{V}} = \emptyset \). This last requirement guarantees the mutual singularity of the probability measures \( P_{B,\beta,\alpha} \) and \( P_{X,\beta,\alpha} \). \( \square \)

Remark 10. The fact that \( P_{X,\beta,\alpha}(\hat{\mathcal{V}}) = 1 \) is also a consequence of the semimartingale property of \( X_{\beta,\alpha} \), see Example 6-2.

3 Singularity of the Solutions of SDEs Driven by Generalized Grey Brownian Motion and Time-Changed Brownian motion

In this section we show that the singularity property from Section 2 propagates to the probability measures induced by the processes which are solutions to the SDEs driven by \( \text{ggBm } B_{\beta,\alpha} \) and the time-changed \( \text{Bm } W(U_{\beta,\alpha}) \). In order to show this result, at first we recall some results on the existence and uniqueness solutions of SDEs driven by processes with the bounded \( p \)-variation. In \([\text{Lyo94}]\) considered the integral equation

\[
y(t) = y(0) + \int_0^t g(y(s)) \, dh(s), \quad 0 \leq t \leq 1,
\]

where \( y(0) \in \mathbb{R} \) and \( h \) is continuous with bounded \( p \)-variation on \([0,1]\) for some \( p \in [1,2] \). He proved that this equation has a unique solution in the space \( C^\kappa_p([0,1]) \) of continuous functions of bounded \( p \)-variation if \( g \in C^{1+\kappa}(\mathbb{R}) \) for some \( \kappa > p - 1 \).

Since almost all sample paths of \( \text{ggBm } B_{\beta,\alpha} \) have bounded \( p \)-variation for \( p > 2/\alpha \), cf. \([\text{DSE18}]\), the SDE

\[
X(t) = x(0) + \int_0^t g(X(s)) \, dB_{\beta,\alpha}(s), \quad 0 \leq t \leq 1,
\]

can be solved pathwise. It has a unique solution if \( g \in C^{1+\kappa}(\mathbb{R}) \) for some \( \kappa > 2/\alpha - 1 \), see \([\text{Kub00}]\) and references therein. It should be noted that
the solution $X$ belongs $C^p_p([0, 1])$ for every $p \in ]2/\alpha, \varpi + 1[$. It follows that $v(X; [0, 1]) \leq 2/\alpha$. So, we have $P_X(\tilde{V}_{2/\alpha}) = 1$ where

$$\tilde{V}_{2/\alpha} := \left\{ w \in C([0, 1], \mathbb{R}) \left| v(w; [0, 1]) \leq \frac{2}{\alpha} \right. \right\}.$$

Let us consider the stochastic differential equation driven by the time-changed Bm

$$Y(t) = y(0) + \int_0^t f(Y(s)) dW(U_{\beta,\alpha}(s)), \quad 0 \leq t \leq 1,$$

where $f$ is a real-valued function, defined on $\mathbb{R}$ which satisfies the Lipschitz condition. Then there exists a unique $(G_t)$-semimartingale $Y$ for which (21) holds. Moreover this solution can be written in the form $Y = Z \circ U_{\beta,\alpha}$, where $Z$ satisfies the stochastic differential equation

$$Z(t) = y(0) + \int_0^t f(Z(s)) dW(s), \quad 0 \leq t \leq 1,$$

see [Kob11, Thm. 4.2]. Since $Y$ is a semimartingale then we conclude that $v(Y; [0, 1]) = 2$, see Example 6-2. It follows that $P_Y(\tilde{V}_2) = 1$ where

$$\tilde{V}_2 := \left\{ w \in C([0, 1], \mathbb{R}) \left| v(w; [0, 1]) = 2 \right. \right\}.$$

As $1 < \alpha < 2$, then $2/\alpha < 2$ form which follows that the sets $\tilde{V}_2$ and $\tilde{V}_{2/\alpha}$ are disjoint. This shows the following theorem.

**Theorem 11.** The two probability measures $P_X$ and $P_Y$ are mutually singular.

**Remark 12.** We would like to mention that the result of Theorem 11 can be extend to more general SDEs. Indeed, let us consider the following SDE

$$X(t) = x(0) + \int_0^t \sigma_1(X(s)) dB_{\beta,\alpha}(s) + \int_0^t b_1(X(s)) ds, \quad 0 \leq t \leq 1,$$

where $b_1$ is a Lipschitz continuous function and $\sigma_1 \in C^{1+\varpi}(\mathbb{R})$ for some $\varpi > 2/\alpha - 1$ with $\varpi > p - 1$. Since almost all sample paths of ggBm $B_{\beta,\alpha}$ have bounded $p$-variation for $p > 2/\alpha$, then this equation has a unique solution $X$ in
$C^\infty_p([0,1])$ for every $p \in ]2/\alpha, \infty+1[$, see [Kub02]. It follows that $\nu(X;[0,1]) \leq 2/\alpha$.

On the other hand let us consider the following SDE

$$Y(t) = y(0) + \int_0^t \sigma_2(Y(s)) dW(U_{\beta,\alpha}(s)) + \int_0^t b_2(Y(s)) dU_{\beta,\alpha}(s), \quad 0 \leq t \leq 1,$$

where $\sigma_2$ and $b_1$ is a Lipschitz continuous functions. Then there exists a unique $(\mathcal{G}_t)$-semimartingale $Y = Z \circ U_{\beta,\alpha}$ for which (22) holds, where $Z$ satisfies the stochastic differential equation

$$Z(t) = y(0) + \int_0^t \sigma_2(Z(s)) dW(s) + \int_0^t b_2(Z(s)) ds, \quad 0 \leq t \leq 1,$$

see [Kob11, Thm. 4.2]. As $Y$ is a semimartingale then we conclude that $\nu(Y;[0,1]) = 2$. It follows that the two probability measures are mutually singular.

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