EQUIVARIANT POINTWISE CLUTCHING MAPS

MIN KYU KIM

Abstract. In the paper, we introduce the terminology equivariant pointwise clutching map. By using this, we give details on how to glue an equivariant vector bundle over a finite set so as to obtain a new Lie group representation such that the quotient map from the bundle to the representation is equivariant. Then, we investigate the topology of the set of all equivariant pointwise clutching maps with respect to an equivariant vector bundle over a finite set. Results of the paper play a key role in classifying equivariant vector bundles over two-surfaces in other papers.

1. Introduction

Though equivariant vector bundle is a usual object in topology and geometry, there are only a few classification results, for example in the following extreme cases:

• a group action on a base space is free, see [At, p. 36], [S, p. 132],
• a group action on a base space is transitive, see [S, p. 130], [B, Proposition II.3.2],
• a base space is $S^1$, see [CKMS].

So, it would be meaningful even if we develop a systematic approach to classification of equivariant vector bundles only over low-dimensional manifolds. In oncoming papers, the author will do so through a slight generalization of clutching construction, which he calls equivariant clutching construction, see [At, p. 23-24] for clutching construction. And in this paper, we will investigate its technical essence (or its pointwise version) by using the terminology equivariant pointwise clutching map. So, we need simply explain the construction as a motivation. Before we give an explanation by an example, we introduce a useful notation. Let $G$ be a compact Lie group acting continuously on a topological space $X$, and let $E$ be an equivariant topological complex vector bundle over $X$. For a subset $A$ in $X$, let $G_A$ be the maximal subgroup of $G$ preserving $A$. Then, $E|_A$ is preserved by the group action restricted to $G_A$, where $E|_A$ is the pull-back bundle of $E$ by the inclusion from $A$ to $X$. Denote the bundle $E|_A$ equipped with the $G_A$-vector bundle structure by $E_A$. For a one point subset $\{x\} \subset X$, we denote $E_{\{x\}}$ simply by $E_x$, which is just a $G_x$-representation.

![Figure 1.1. The disjoint union of four faces of a regular tetrahedron](image-url)
Example 1.1. Denote by $T$ the order 24 full isometry group of a regular tetrahedron $X$. Then, $T$ acts naturally on $X$. In addition to the regular tetrahedron, we also consider the disjoint union $\overline{X}$ of four faces of $X$, and the quotient map $\pi : \overline{X} \rightarrow X$ which is the identity map whenever restricted to each face, see Figure [13]. Then, the $T$-action on $X$ is lifted to $\overline{X}$, i.e. $\overline{X}$ is equipped with a unique group action such that $\pi$ is equivariant. Here, $X$ can be considered to be obtained from $\overline{X}$ by gluing such that the quotient map $\pi$ is equivariant. We can say similar things to an equivariant topological complex vector bundle $E$ over $X$. The $T$-action on $E$ is lifted to the pull-back bundle $\pi^*E$ of $E$ by $\pi$. And, $E$ can be considered to be obtained from $\pi^*E$ by gluing such that the quotient map from $\pi^*E$ to $E$ is equivariant. The author calls this way of constructing $E$ equivariant clutching construction. Here, it is conceivable that we could obtain various equivariant vector bundles over $X$ by different ways of gluing. From this viewpoint, we only have to investigate equivariant vector bundles over $\overline{X}$ and their possible ways of gluing so as to classify equivariant vector bundles over $X$. However since equivariant vector bundles over $\overline{X}$ are easily understood, we only have to focus on possible ways of gluing. In doing so, the first thing to do is fiberwise gluing. To explain this, assume for the moment that we have already succeeded in obtaining a new equivariant vector bundle $K$ over $X$ from an equivariant vector bundle $J$ over $\overline{X}$ by gluing such that the quotient map from $J$ to $K$ is equivariant. Then, this implies that for each $x$ in some edge of $X$, we have obtained a new $G_x$-representation $K_x$ from the $G_x$-vector bundle $J_{\pi^{-1}(x)}$ by gluing such that the restricted quotient map from $J_{\pi^{-1}(x)}$ to $K_x$ is equivariant. We might call such a gluing of an equivariant vector bundle over a finite set fiberwise gluing temporarily. □

In the paper, we are concerned about fiberwise gluing. Our investigation of it is divided into two steps in a general setting as follows: given a compact Lie group $G$ acting continuously on a finite set $x$ and an equivariant topological complex vector bundle $F$ over $x$, we investigate (1) how to glue actually $F$ so as to obtain a new $G$-representation $w$ such that the quotient map from $F$ to $w$ is equivariant, and then (2) the topology of the set of all possible ways to glue $F$.

First, we deal with (1). To glue $F$, it is natural to consider a map in

$$\prod_{(x,x') \in x \times x} \text{iso}(F_x, F_{x'}),$$

i.e. a map $\psi$ contained therein is defined on $x \times x$ and the image $\psi(\bar{x}, \bar{x}')$ of $(\bar{x}, \bar{x}') \in x \times x$ is contained in $\text{iso}(F_x, F_{x'})$, where the notation $\text{iso}(\cdot, \cdot)$ denotes the set of inequivariant (i.e. not considering a group action) isomorphisms between two vector spaces. Such a map $\psi$ is called a pointwise clutching map with respect to $F$ if it satisfies the following:

(a) reflexivity : $\psi(\bar{x}, \bar{x}) = \text{id}_{F_x}$,

(b) symmetry : $\psi(\bar{x}, \bar{x}') = \psi(\bar{x}', \bar{x})^{-1}$,

(c) transitivity : $\psi(\bar{x}, \bar{x}'') = \psi(\bar{x}', \bar{x}'')\psi(\bar{x}, \bar{x}')$

for any $\bar{x}, \bar{x}', \bar{x}'' \in x$, where $\text{id}_{V}$ for a vector space $V$ is the identity map on $V$. By using a pointwise clutching map $\psi$, we would glue $F$ by defining an equivalence relation $\sim$ on $F$ as follows:

$$\bar{u} \sim \bar{u}' \iff \psi(\bar{x}, \bar{x}')\bar{u} = \bar{u}' \quad \text{for any } \bar{x}, \bar{x}' \in x \text{ and } \bar{u} \in F_x, \bar{u}' \in F_{x'},$$

i.e. $\bar{u}$ and $\bar{u}'$ are glued if and only if $\bar{u} \sim \bar{u}'$. Denote by $F/\psi$ the quotient space of $F$ by the equivalence relation, and let $p_\psi : F \rightarrow F/\psi$ be the quotient map. Since $\psi$ is isomorphism-valued, the quotient space $F/\psi$ inherits a unique vector space structure from $F$ such that $p_\psi$ is a fiberwise isomorphism, i.e. an equivariant isomorphism between vector spaces whenever restricted to each fiber. So, we have
obtained a vector space $F/\psi$ through gluing $F$ by using a pointwise clutching map $\psi$. To equip the vector space $F/\psi$ with an additional $G$-representation structure to match our goal (1), the pointwise clutching map $\psi$ should satisfy an additional equivariance condition.

We introduce a $G$-action on the set of pointwise clutching maps with respect to $F$. For a pointwise clutching map $\psi$ with respect to $F$, if we define a new map $g \cdot \psi$ for $g \in G$ in the product (1.1) as follows:

$$(g \cdot \psi)(\bar{x}, \bar{x}') \bar{u} := g\psi(g^{-1}\bar{x}, g^{-1}\bar{x}')g^{-1}\bar{u} \quad \text{for} \quad \bar{x}, \bar{x}' \in \bar{x} \quad \text{and} \quad \bar{u} \in F_{\bar{x}},$$

then $g \cdot \psi$ is also a pointwise clutching map with respect to $F$. If $\psi$ is fixed by the action, i.e. $g \cdot \psi = \psi$ for each $g \in G$, then $\psi$ is called equivariant. In Lemma 2.13 it is proved that a pointwise clutching map $\psi$ with respect to $F$ is equivariant if and only if the vector space $F/\psi$ can be equipped with a unique $G$-representation structure such that $p_{\psi}$ is equivariant. The unique $G$-representation structure is called a glued representation, and denoted simply by the same notation $F/\psi$. In summary, we glue $F$ by using an equivariant pointwise clutching map $\psi$ so as to obtain the glued representation $F/\psi$ so that the quotient map $p_{\psi}$ is equivariant. This is our answer to (1).

Next, we deal with (2). Let $\Psi_F$ be the set of all equivariant pointwise clutching maps with respect to $F$. We topologize $\Psi_F$ with the subspace topology inherited from the product topology on the product space (1.1). In [K], the author classified equivariant vector bundles over two-sphere through equivariant clutching construction, and from this he guesses affirmatively that a similar result holds for other two-surfaces. Technical difficulty therein is to calculate the zeroth homotopy group of $\Psi$ and the first homotopy group of each path-component of $\Psi$. Technical difficulty therein is to calculate the zeroth homotopy group of $\Psi$ and the first homotopy group of each path-component of $\Psi$. For example, the glued representation $F/\psi$ is an equivariant fiberwise isomorphism. So, we can define the following maps:

$$\text{gl} : \Psi_F \longrightarrow \text{ext} F, \quad \psi \mapsto F/\psi \quad \text{and} \quad \pi_0(\text{gl}) : \pi_0(\Psi_F) \longrightarrow \text{ext} F, \quad \bar{\psi} \mapsto \text{ext} F/\psi,$$

where the notation gl is the first two letters of glued representation and the second map is well-defined, see arguments below Lemma 2.13 for detail.

**Theorem 1.2.** Let a compact Lie group $G$ act continuously on a finite set $\bar{x}$, and let $F$ be an equivariant topological complex vector bundle over $\bar{x}$. Then, the map $\pi_0(\text{gl})$ is bijective. Especially, $\Psi_F$ is nonempty if and only if $\text{ext} F$ is nonempty.

In the next theorem, we express each path-component of $\Psi_F$ as the quotient space of a Lie group by a closed subgroup, and this enables us to calculate homotopy groups of each path-component of $\Psi_F$. For $w \in \text{ext} F$, denote by $\Psi_F(w)$ the preimage $\text{gl}^{-1}(w)$. Then, each $\Psi_F(w)$ is a nonempty path-component of $\Psi_F$ by Proposition 2.13.
Theorem 1.3. Under the assumption of Theorem 1.2, let \( \bar{s} \subset \bar{x} \) be a subset containing exactly one element in each orbit of \( \bar{x} \). Then, the path-component \( \Psi_F(w) \) for each \( w \in \text{ext} F \) is homeomorphic to the quotient space

\[
\left( \prod_{s \in \bar{s}} \text{iso}_G(s) \right) / \text{iso}_G(w),
\]

where \( \text{iso}_K(w) \) for a closed subgroup \( K \subset G \) is the group of all \( K \)-isomorphisms of \( w \) and the group \( \text{iso}_G(w) \) is diagonally imbedded into the product group.

Through the homeomorphism of Theorem 1.3, we can show that \( \Psi_F(w) \) is simply connected in some important cases.

Proposition 1.4. Under the assumption of Theorem 1.2, we assume additionally that the \( G \)-action on \( \bar{x} \) is transitive. Given a representation extension \( w \) of \( F \) and a point \( \bar{x} \in \bar{x} \), if \( \text{res}^G_U \) is irreducible for each irreducible \( G \)-representation \( U \) contained in \( w \), then \( \Psi_F(w) \) is simply connected.

Corollary 1.5. Under the assumption of Theorem 1.2, if the \( G \)-action on \( \bar{x} \) is transitive and \( G \) is abelian, \( \Psi_F(w) \) is simply connected for each \( w \in \text{ext} F \).

This paper is organized as follows. In Section 2, we prove Theorem 1.2, 1.3. Section 3 deals with the fundamental group of the set of equivariant pointwise clutching maps, and we prove Proposition 1.4 and Corollary 1.5. In Section 4, we prove two useful lemmas on evaluation and restriction of an equivariant pointwise clutching map.

2. Proofs

Hereafter, let a compact Lie group \( G \) act continuously on a finite set \( \bar{x} \), and let \( F \) be an equivariant topological complex vector bundle over \( \bar{x} \) as before. Also, we abbreviate finite-dimensional representation of a Lie group as representation.

In this section, we prove main theorems. We begin this section by introducing a terminology. For a complex \( G \)-representation \( w \), we denote by \( \text{fiso}_G(F,w) \) the set of all equivariant fiberwise isomorphisms from \( F \) to \( w \). Below Lemma 2.5, we will see that the set \( \Psi_F(w) \) for \( w \in \text{ext} F \) is highly related to the set \( \text{fiso}_G(F,w) \).

Since \( \text{fiso}_G(F,w) \) is relatively easy to deal with, our strategy for proofs is to investigate \( \text{fiso}_G(F,w) \) first and then to relate \( \text{fiso}_G(F,w) \) to \( \Psi_F(w) \).

To begin with, we prove a basic lemma on equivariant pointwise clutching maps.

Lemma 2.1. A pointwise clutching map \( \psi \) with respect to \( F \) is equivariant if and only if the vector space \( F/\psi \) can be equipped with a complex \( G \)-representation structure such that \( p_\psi \) is equivariant. If exists, such a \( G \)-representation structure is unique.

Proof. The unique possible group action on \( F/\psi \) to guarantee equivariance of \( p_\psi \) is as follows:

\[
g \cdot u := p_\psi(g\bar{u}) \quad \text{for } g \in G, \ u \in F/\psi, \ \text{and any } \bar{u} \in F \text{ such that } u = p_\psi(\bar{u}).
\]

This is because if there exists such an action, it should satisfy the following:

\[
p_\psi(g\bar{u}) = g \cdot p_\psi(\bar{u}) \quad \text{by equivariance of } p_\psi
\]

\[
= g \cdot u. \quad \text{by definition of } \bar{u}
\]

It is easy that if the action is well-defined, then it is actually an action. By definition, the possible group action is well-defined if and only if

\[
(*) \quad p_\psi(\bar{u}) = p_\psi(\bar{u}') \implies p_\psi(g\bar{u}) = p_\psi(g\bar{u}')
\]
for any \( g \in G \) and \( \bar{u}, \bar{u}' \in F \). By definition of \( p_{\psi} \), each side of (*) is rewritten as follows:

\[
\begin{align*}
p_{\psi}(\bar{u}) = p_{\psi}(\bar{u}') & \iff \psi(\bar{x}, \bar{x}')\bar{u} = \bar{u}' \quad \text{and} \\
p_{\psi}(g\bar{u}) = p_{\psi}(g\bar{u}') & \iff \psi(g\bar{x}, g\bar{x}')g\bar{u} = g\bar{u}', \quad \text{i.e.} \quad (g^{-1} \cdot \psi)(\bar{x}, \bar{x}')\bar{u} = \bar{u}',
\end{align*}
\]

when \( \bar{u} \in F_{\bar{x}} \) and \( \bar{u}' \in F_{\bar{x}'} \) for some \( \bar{x}, \bar{x}' \in \bar{x} \). So, (*) is rewritten as

\[
\psi(\bar{x}, \bar{x}')\bar{u} = \bar{u}' \quad \implies \quad (g^{-1} \cdot \psi)(\bar{x}, \bar{x}')\bar{u} = \bar{u}'
\]

for any \( g \in G \) and \( \bar{u}, \bar{u}' \in F \). In summary, the unique possible action is well-defined if and only if \( \psi = g^{-1} \cdot \psi \) for any \( g \in G \), i.e. \( \psi \) is equivariant. Therefore, we obtain a proof. \( \square \)

When \( \psi \) is equivariant, the unique \( G \)-representation structure on \( F/\psi \) such that \( p_{\psi} \) is equivariant is denoted simply by the same notation \( \psi \). In this case, \( p_{\psi} \) is contained in \( \text{fiso}_G(F,F/\psi) \), and is the prototype of equivariant fiberwise isomorphisms.

For a time being, we investigate \( \text{fiso}_G(F,w) \). We start by an easy lemma.

**Lemma 2.2.** If there exists an equivariant fiberwise isomorphism \( p : F \rightarrow w \) for a complex \( G \)-representation \( w \), then \( w \) is contained in \( \text{ext}_G \).

Next, we factorize \( \text{fiso}_G(F,w) \) into a product of isomorphism groups. By using this, we can prove the converse of Lemma 2.2 i.e. for any \( w \) in \( \text{ext}_G \), there exists an equivariant fiberwise isomorphism from \( F \) to \( w \).

**Proposition 2.3.** Let \( \bar{s} \subset \bar{x} \) be a subset containing exactly one element in each orbit of \( \bar{x} \). For a complex \( G \)-representation \( w \), the following map is homeomorphic:

\[
i : \text{fiso}_G(F,w) \longrightarrow \prod_{\bar{s} \subset \bar{x}} \text{iso}_G(F_{\bar{s}},w), \quad p \mapsto \prod_{\bar{s} \subset \bar{x}} p|_{F_{\bar{s}}},
\]

And, the set \( \text{fiso}_G(F,w) \) is nonempty if and only if \( w \in \text{ext}_G \).

**Proof.** First, we describe \( F \) precisely. Since

\[
\bar{x} = \bigcup_{\bar{s} \subset \bar{x}} G \cdot \bar{s},
\]

the equivariant topological vector bundle \( F \) can be regarded as the disjoint union of \( F \) restricted to each orbit as follows:

\[
F = \bigsqcup_{\bar{s} \subset \bar{x}} F_{G,s},
\]

where \( \bigsqcup \) is the notation for disjoint union. By these expressions, we can define the following map:

\[
\text{fiso}_G(F,w) \longrightarrow \prod_{\bar{s} \subset \bar{x}} \text{fiso}_G(F_{G,s},w), \quad p \mapsto \prod_{\bar{s} \subset \bar{x}} p|_{F_{G,s}}.
\]

It is easy that the map is homeomorphic. To obtain a proof for the first statement, we will show that the following map is a homeomorphism:

\[
(\ast) \quad \text{fiso}_G(F_{G,s},w) \longrightarrow \text{iso}_G(F_{\bar{s}},w), \quad q \mapsto q|_{F_{\bar{s}}}
\]

for each \( \bar{s} \in \bar{x} \). For this, we would decompose the map (\ast) into a composition of two homeomorphisms. For \( \bar{s} \in \bar{x} \), the equivariant vector bundle \( F_{G,s} \) is \( G \)-isomorphic to \( G \times_{G} F_{\bar{s}} \) through the inverse of the following \( G \)-isomorphism:

\[
G \times_{G} F_{\bar{s}} \rightarrow F_{G,s}, \quad [g, \bar{u}] \mapsto g\bar{u},
\]
see [3] Proposition II.3.2. By this, we obtain a homeomorphism from \( \text{fiso}_G(F_{G,s}, w) \) to \( \text{fiso}_G(G \times_{G_s} F_s, w) \). Next, we consider the continuous map

\[
(\ast) \quad \text{fiso}_G(G \times_{G_s} F_s, w) \to \text{iso}_{G_s}(F_s, w), \quad q \mapsto q|_{F_s},
\]

where the fiber \([\text{id}, F_s]\) of \( G \times_{G_s} F_s \) is identified with \( F_s \). To show that the map is homeomorphic, we construct its inverse. For any \( G_s \)-isomorphism \( r \) from \( F_s \) to \( w \), define the following equivariant fiberwise isomorphism:

\[
G \times_{G_s} F_s \longrightarrow w, \quad [g, \bar{u}] \mapsto g \cdot r(\bar{u}).
\]

In this way, we obtain a map from \( \text{iso}_{G_s}(F_s, w) \) to \( \text{fiso}_G(G \times_{G_s} F_s, w) \). Easily, the map is the continuous inverse of the map \((\ast)\). So, we have obtained the following decomposition of \((\ast)\) into a composition of two homeomorphisms:

\[
\text{fiso}_G(F_{G,s}, w) \longrightarrow \text{fiso}_G(G \times_{G_s} F_s, w) \longrightarrow \text{iso}_{G_s}(F_s, w).
\]

As a result, \( \text{fiso}_G(F_{G,s}, w) \) is homeomorphic to \( \text{iso}_{G_s}(F_s, w) \). Therefore, we obtain a proof for the first statement.

Now, we prove the second statement. Since we have already obtained sufficiency by Lemma 2.2 we only have to show necessity. By definition of \( \text{ext}_F \), the space \( \text{iso}_{G_s}(F_s, w) \) for each \( s \in \mathfrak{s} \) is nonempty. So, \( \text{fiso}_G(F, w) \) is nonempty by the homeomorphism of the first statement, and necessity holds. Therefore, we obtain a proof for the second statement.

\[\square\]

**Corollary 2.4.** The set \( \text{fiso}_G(F, w) \) is nonempty and path-connected for any \( w \in \text{ext}_F \).

**Proof.** By Proposition 2.3 we only have to show that \( \text{iso}_{G_s}(F_s, w) \) is nonempty and path-connected for each \( s \in \mathfrak{s} \). By definition of \( \text{ext}_F \), the representation \( F_s \) is \( G_s \)-isomorphic to \( \text{res}^{G_{s'}}_{G_s} w \) for each \( s \in \mathfrak{s} \). So, \( \text{iso}_{G_s}(F_s, w) \) for each \( s \in \mathfrak{s} \) is nonempty, and is homeomorphic to \( \text{iso}_{G_{s'}}(w) \). By Schur’s lemma, \( \text{iso}_{G_s}(w) \) is homeomorphically group isomorphic to a product of general linear groups over \( \mathbb{C} \), see Section 3 for detail. Therefore, it is path-connected, and we obtain a proof. Here, note that we use the assumption that the scalar field is \( \mathbb{C} \).

\[\square\]

Similar to \( \Psi_F(w) \), we hereafter consider \( \text{fiso}_G(F, w) \) just for \( w \in \text{ext}_F \) by Lemma 2.2. Now, we start relating \( \text{fiso}_G(F, w) \) to \( \Psi_F(w) \). By Lemma 2.3 an equivariant pointwise clutching map \( \psi \) determines an equivariant fiberwise isomorphism \( p_\psi \).

In the next useful lemma, we prove a kind of converse of this, i.e. an equivariant fiberwise isomorphism determines an equivariant pointwise clutching map.

**Lemma 2.5.** For any \( p \in \text{fiso}_G(F, w) \) for \( w \in \text{ext}_F \), there exists a unique map \( \psi \) in \( \Psi_F(w) \) such that the following diagram commutes for some \( G \)-isomorphism from \( F/\psi \) to \( w \):

\[
\begin{array}{ccc}
F & \xrightarrow{p} & w \\
\downarrow & & \uparrow_{\cong} \\
F/\psi & \xrightarrow{p_\psi} & w \\
\end{array}
\]

And, the set \( \Psi_F(w) \) is nonempty for any \( w \in \text{ext}_F \).

**Proof.** First, we construct the unique pointwise clutching map \( \psi \) with respect to \( F \) satisfying the diagram for some inequivariant isomorphism from \( F/\psi \) to \( w \). If there exists such a map \( \psi \), then it should satisfy the following:

\[(\ast) \quad p_\psi(\bar{u}) = p_\psi(\bar{u}') \implies p(\bar{u}) = p(\bar{u}') \quad \text{for any } \bar{u}, \bar{u}' \in F\]
by the diagram. When \( \bar{u} \in F_x \) and \( \bar{u}' \in F_{x'} \), for some \( \bar{x}, \bar{x}' \in \mathfrak{X} \), each side of (*) is rewritten as follows:

\[
p_\psi(\bar{u}) = p_\psi(\bar{u}') \iff \bar{u}' = \psi(\bar{x}, \bar{x}') \bar{u}
\]

and

\[
p(\bar{u}) = p(\bar{u}') \iff p|_{F_z}(\bar{u}) = p|_{F_{z'}}(\bar{u}').
\]

Substituting these into (**), the formula (*) is rewritten as

\[
p|_{F_z}(\bar{u}) = p|_{F_{z'}}(\psi(\bar{x}, \bar{x}')\bar{u})
\]

for any \( \bar{u} \in F_x \) and \( \bar{x}' \in \mathfrak{X} \).

So, we obtain

(**) \[
\psi(\bar{x}, \bar{x}') = (p|_{F_{z'}})^{-1} \circ (p|_{F_z}) \quad \text{for} \quad \bar{x}, \bar{x}' \in \mathfrak{X}
\]

because \( p \) is a fiberwise isomorphism. The unique bijective map \( \alpha : F/\psi \to w \) satisfying the diagram is expressed as follows:

\[
\alpha(u) := p(\bar{u}) \quad \text{for any} \quad u \in F/\psi \quad \text{and} \quad \bar{u} \in F \quad \text{such that} \quad p_\psi(\bar{u}) = u.
\]

This is because

\[
p(\bar{u}) = (\alpha \circ p_\psi)(\bar{u}) = \alpha(p_\psi(\bar{u})) = \alpha(u).
\]

It is easy that \( \alpha \) is linear, so \( \alpha \) is an inequivalent isomorphism satisfying the diagram. And, \( \psi \) is the wanted unique pointwise clutching map.

Now, we will show that \( \psi \) and \( \alpha \) are equivariant. Equivariance of \( p \) can be expressed as follows:

\[
p(g^{-1} \bar{u}) = g^{-1} \cdot p(\bar{u}) \quad \text{for each} \quad g \in G \quad \text{and} \quad \bar{u} \in F.
\]

If we restrict the expression on each fiber \( F_x \) for \( \bar{x} \in \mathfrak{X} \), we obtain the following:

(***)

\[
(p|_{F_{x^{-1}}})g^{-1} = g^{-1} \cdot (p|_{F_x}),
\]

and by taking inverses of both terms of (**), we obtain

(****)

\[
g(p|_{F_{x^{-1}}})^{-1} = (p|_{F_x})^{-1} g \quad \text{on} \quad w.
\]

Then, we obtain

\[
\begin{align*}
(g \cdot \psi)(\bar{x}, \bar{x}') &= g\psi(g^{-1} \bar{x}, g^{-1} \bar{x}')g^{-1} \quad \text{by definition of} \quad g \cdot \psi \\
&= g(p|_{F_{x^{-1}}})^{-1} \circ (p|_{F_{x^{-1}}})g^{-1} \quad \text{by (**)} \\
&= [p|_{F_{x'}}]^{-1} g \circ [g^{-1}(p|_{F_x})] \quad \text{by (***) and (****)} \\
&= (p|_{F_{x'}}^{-1}) \circ (p|_{F_x}) \\
&= \psi(\bar{x}, \bar{x}') \quad \text{by (**)}
\end{align*}
\]

for any \( g \in G \) and \( \bar{x}, \bar{x}' \in \mathfrak{X} \), so \( \psi \) is equivariant. And, this implies that \( p_\psi \) is equivariant by Lemma 2.4. Since the diagram commutes and two maps \( p, p_\psi \) are equivariant, the isomorphism \( \alpha \) is also equivariant. Therefore, we obtain a proof for the first statement.

Now, we prove the second statement. For \( w \in \text{ext} F \), there exists an element \( p \) in \( \text{fiso}(F, w) \) by the second statement of Proposition 2.3. And, existence of \( p \) guarantees nonemptiness of \( \Psi_F(w) \) by the first statement of the lemma. Therefore, we obtain a proof.

By using Lemma 2.5, we can define a map \( \text{Cl} : \text{fiso}_G(F, w) \to \Psi_F(w) \) for \( w \in \text{ext} F \) sending an equivariant fiberwise isomorphism \( p \) to the equivariant pointwise clutching map \( \psi \) determined by \( p \), where \( \text{Cl} \) is the first two letters of clutching map. It is easy that the map \( \text{Cl} \) is continuous by (** in the proof of Lemma 2.6. Though the map \( \text{Cl} \) relates \( \text{fiso}_G(F, w) \) to \( \Psi_F(w) \), readers would not be satisfied
with it. To obtain a more refined relation, we consider a continuous iso<sub>𝔩</sub>(w)-action on fiso<sub>𝔩</sub>(F, w) as follows:

\[ \alpha \circ p := g \circ p \quad \text{for } \alpha \in \text{iso}_\mathfrak{𝔩}(w) \text{ and } p \in \text{fiso}_\mathfrak{𝔩}(F, w). \]

Then, the map

\[ \text{cl} : \text{fiso}_\mathfrak{𝔩}(F, w)/\text{iso}_\mathfrak{𝔩}(w) \rightarrow \Psi_F(w), \quad [p] \rightarrow \text{Cl}(p) \]

is well-defined by definition of Cl, and also continuous by the universal property of the quotient map from fiso<sub>𝔩</sub>(F, w) to fiso<sub>𝔩</sub>(F, w)/iso<sub>𝔩</sub>(w). Moreover, the following holds:

**Lemma 2.6.** For each \( w \in \text{ext} F \), the map \( \text{cl} \) is bijective.

*Proof. First, we prove surjectivity. We only have to show that \( \text{Cl} \) is surjective. Recalling that the domain and codomain of \( \text{cl} \) are nonempty by Proposition 2.3 and Lemma 2.5, we will find a preimage of an arbitrary element \( \psi \in \Psi_F(w) \) under Cl. Note that \( \alpha \circ p_\psi \) for any \( \alpha \in \text{iso}_\mathfrak{𝔩}(F/\psi, w) \) is an equivariant fiberwise isomorphism in fiso<sub>𝔩</sub>(F, w), where nonemptiness of iso<sub>𝔩</sub>(F/ψ, w) is guaranteed by definition of \( \Psi_F(w) \). By definition of Cl, it is easy that \( \text{Cl}(\alpha \circ p_\psi) = \psi \). So, Cl is surjective. Therefore, cl is surjective.

Next, we prove injectivity. Assume that \( \text{cl}([p]) = \text{cl}([p']) \) for \( p, p' \in \text{fiso}_\mathfrak{𝔩}(F, w) \). Put \( \psi = \text{Cl}(p) = \text{Cl}(p') \). By definition of Cl,

\[ \psi(x, x') = (p|_{F_{x'}})^{-1} \circ p|_{F_x} = (p'|_{F_{x'}})^{-1} \circ p'|_{F_x} \]

for any \( x, x' \in \overline{x} \). Then, we have

\[ p'|_{F_{x'}} \circ (p|_{F_{x'}})^{-1} = p'|_{F_x} \circ (p|_{F_x})^{-1}, \]

i.e. \( (p'|_{F_{x'}}) \circ (p|_{F_{x'}})^{-1} \) is constant regardless of \( x \), and is contained in iso<sub>𝔩</sub>(w) because \( p, p' \) are fiberwise isomorphisms. If we denote by \( \beta \) this inequivariant isomorphism, then we have \( p' = \beta \circ p \). Easily, \( \beta \in \text{iso}_\mathfrak{𝔩}(w) \) because \( p, p' \) are equivariant. So, \( p' = \beta \circ p \), and hence \( \text{Cl}([p]) = \text{Cl}([p']) \). Therefore, we obtain a proof. □

**Remark 2.7.** In the proof of Lemma 2.6, we constructed the inverse \( \text{cl}^{-1} \) as follows:

\[ \text{cl}^{-1} : \Psi_F(w) \rightarrow \text{fiso}_\mathfrak{𝔩}(F, w)/\text{iso}_\mathfrak{𝔩}(w), \quad \psi \rightarrow [\alpha \circ p_\psi] \]

for any map \( \alpha \in \text{iso}_\mathfrak{𝔩}(F/\psi, w) \). Note that \( \alpha \) is dependent of \( \psi \) because the domain \( F/\psi \) of \( \alpha \) is dependent of \( \psi \). To stress this dependence, we will denote \( \alpha \) by \( \alpha_\psi \).

Since continuity of the inverse is our next issue, we will refer to the remark several times.

For a time being, we will prove that \( \text{cl}^{-1} \) is also continuous for each \( w \in \text{ext} F \). For this, we should construct \( p_\psi \) and \( \alpha_\psi \) which vary continuously with \( \psi \in \Psi_F \) by Remark 2.7. At the present stage, this is not easy because the representation space (i.e. the vector space structure) of \( F/\psi \) varies with \( \psi \). To solve this problem, we need give a new definition of \( F/\psi \). As a candidate of it, we define a new \( G \)-representation. Pick a point \( \bar{x}_0 \in \overline{x} \).

**Definition 2.8.** For \( \psi \in \Psi_F \), we define the operation \( \psi_\mathfrak{𝔩} \) of \( G \) on \( F_{x_0} \) as follows:

\[ g \psi_\mathfrak{𝔩} u := \psi(g\bar{x}_0, \bar{x}_0)gu \quad \text{for } g \in G \text{ and } u \in F_{x_0}. \]

In particular, the operation restricted to \( G_{x_0} \) is identical with the representation \( F_{x_0} \), i.e. \( g \psi_\mathfrak{𝔩} u = gu \) for \( g \in G_{x_0} \).

We show that the operation becomes a \( G \)-representation.

**Lemma 2.9.** For any \( \psi \in \Psi_F \), the operation \( \psi_\mathfrak{𝔩} \) of \( G \) on \( F_{x_0} \) is a complex \( G \)-representation.
Proof. Since \( \text{id} \ast \bar{u} = \bar{u} \) for any \( \bar{u} \in F_{x_0} \) by definition, we only have to show that
\[
g \ast_\psi (h \ast_\psi \bar{u}) = (gh) \ast_\psi \bar{u}
\]
for any \( g, h \in G \) and \( \bar{u} \in F_{x_0} \).

By calculation, we have
\[
g \ast_\psi (h \ast_\psi \bar{u}) = \psi(gx_0, \bar{x}_0)ghx_0, \bar{x}_0)h\bar{u}
\]
\[
= \psi(gx_0, \bar{x}_0)g\psi(ghx_0, g^{-1}g\bar{x}_0)g^{-1}gh\bar{u}
\]
\[
= \psi(gx_0, \bar{x}_0)\psi(ghx_0, g\bar{x}_0)gh\bar{u}
\]
\[
= \psi(ghx_0, \bar{x}_0)gh\bar{u}
\]
\[
= (gh) \ast_\psi \bar{u}.
\]

Note that the representation space of \((F_{x_0}, \ast_\psi)\) does not change, and that its \(G\)-representation structure varies continuously with \(\psi\). In Lemma 2.11, we will see that \((F_{x_0}, \ast_\psi)\) replaces 
\(F/\psi\). If definition of \(F/\psi\) is changed, definition of \(p_\psi\) should be also changed so as to be in accordance with \((F_{x_0}, \ast_\psi)\). The following \(p'_\psi\) is a candidate for a new definition of \(p_\psi\).

Lemma 2.10. For any \(\psi \in \Psi_F\), the map
\[
p'_\psi : F \to (F_{x_0}, \ast_\psi), \quad \bar{u} \mapsto \psi(\bar{x}, \bar{x}_0)\bar{u}
\]
is an equivariant fiberwise isomorphism.

Proof. Since it is easy to show that \(p'_\psi\) is a fiberwise isomorphism, we only have to show that \(p'_\psi\) is equivariant, i.e.
\[
p'_\psi(g\bar{u}) = g \ast_\psi p'_\psi(\bar{u})
\]
for any \(g \in G\) and \(\bar{u} \in F_{x_0}\). By calculation, we have
\[
p'_\psi(g\bar{u}) = \psi(g\bar{x}, \bar{x}_0)g\bar{u}
\]
\[
= \psi(g\bar{x}, \bar{x}_0)\psi(g\bar{x}, g\bar{x}_0)g\bar{u}
\]
\[
= \psi(g\bar{x}, \bar{x}_0)\psi(g^{-1}g\bar{x}, g^{-1}g\bar{x}_0)g^{-1}g\bar{u}
\]
\[
= \psi(g\bar{x}, \bar{x}_0)\psi(\bar{x}, \bar{x}_0)\bar{u}
\]
\[
= g \ast_\psi \psi(\bar{x}, \bar{x}_0)\bar{u}
\]
\[
= g \ast_\psi p'_\psi(\bar{u})
\]
by definition of \(\ast_\psi\).

for \(g \in G\) and \(\bar{u} \in F_{x_0}\). Therefore, we obtain a proof.

Easily, the codomain \(F_{x_0}\) of \(p'_\psi\) does not change, and the map \(p'_\psi\) varies continuously with \(\psi\). Now, we can replace \(F/\psi\) and \(p_\psi\) with \((F_{x_0}, \ast_\psi)\) and \(p'_\psi\), respectively.

Lemma 2.11. For any \(\psi \in \Psi_F\), the following diagram commutes for some \(G\)-isomorphism from \(F/\psi\) to \((F_{x_0}, \ast_\psi)\):

\[
\begin{array}{ccc}
F & \xrightarrow{p'_\psi} & (F_{x_0}, \ast_\psi) \\
\downarrow{p_\psi} & & \downarrow{=} \\
F/\psi & & \\
\end{array}
\]

Proof. By Lemma 2.10, \(p'_\psi\) is an equivariant fiberwise isomorphism from \(F\) to the \(G\)-representation \((F_{x_0}, \ast_\psi)\). By definition of \(\text{Cl}\), we can check \(\text{Cl}(p'_\psi) = \psi\). By Lemma 2.5 we obtain the diagram.
By Lemma 2.11, we can henceforward put
\[ F/\psi := (F_{w}, \ast_{\psi}) \quad \text{and} \quad p_{\psi} := p'_{\psi}. \]

We can observe that previous results stated by using previous \( F/\psi \) and \( p_{\psi} \) still hold for new \( F/\psi \) and \( p_{\psi} \). Now, we return to our arguments below Remark 2.7 to show that \( c_{1}^{-1} \) is continuous. Since we constructed \( p_{\psi} \) in Lemma 2.10 which varies continuously with \( \psi \in \Psi_{F}(w) \), the remaining is to construct \( \alpha_{\psi} \) in \( \text{iso}(F/\psi, w) \) which varies continuously with \( \psi \in \Psi_{F}(w) \).

**Lemma 2.12.** For arbitrary \( \psi_{0} \in \Psi_{F} \), put \( w = F/\psi_{0} \). Then, there exists a continuous map
\[ \alpha : U \subset \Psi_{F}(w) \rightarrow \text{iso}(F_{w_{0}}) \]
for a small neighborhood \( U \) of \( \psi_{0} \) such that \( \alpha(\psi) \) is contained in \( \text{iso}_{G}(F/\psi, w) \) for each \( \psi \in U \).

**Proof.** Note that \( w \in \text{ext} F \) and \( \psi_{0} \in \Psi_{F}(w) \) by definition of gl. For \( \psi \in \Psi_{F}(w) \), let \( \rho_{\psi} : G \rightarrow \text{iso}(F_{w_{0}}) \) be the group isomorphism corresponding to the \( G \)-representation \( F/\psi = (F_{w_{0}}, \ast_{\psi}) \), i.e. \( \rho_{\psi}(g)u = g \ast_{\psi} u \) for \( g \in G \) and \( u \in F_{w_{0}} \). Note that \( \rho_{\psi} \) varies continuously with \( \psi \) because the \( G \)-representation structure of \( F/\psi \) varies continuously with \( \psi \). By using \( \rho_{\psi} \), we define a map \( \alpha \) as follows:
\[ \alpha : \Psi_{F}(w) \rightarrow \text{end}(F_{w_{0}}), \quad \psi \mapsto \int_{G} \rho_{\psi_{0}}(g)\rho_{\psi}(g)^{-1} \, dg, \]
where \( \text{end}(\cdot) \) is the set of inequvariant endomorphisms of a vector space and \( dg \) is a Haar measure. Then, it is trivial that \( \alpha(\psi_{0}) = \text{constant} \cdot \text{id}_{F_{w_{0}}} \) by definition.

And, it is easy that \( \alpha \) is continuous because \( \rho_{\psi} \) varies continuously with \( \psi \). So, the remaining is to show that \( \alpha(\psi) \) is contained in \( \text{iso}_{G}(F/\psi, w) \) for each \( \psi \) in a small neighborhood \( U \) of \( \psi_{0} \). First, we show that \( \alpha(\psi) \) is contained in \( \text{end}_{G}(F/\psi, w) \) for each \( \psi \in \Psi_{F}(w) \), i.e.
\[ \rho_{\psi_{0}}(g')\alpha(\psi)\rho_{\psi}(g')^{-1} = \alpha(\psi) \]
for each \( \psi \in \Psi_{F}(w) \) and \( g' \in G \) because group isomorphisms corresponding to \( F/\psi \) and \( w \) are \( \rho_{\psi} \) and \( \rho_{\psi_{0}} \), respectively. By calculation, we have
\[ \rho_{\psi_{0}}(g')\alpha(\psi)\rho_{\psi}(g')^{-1} = \int_{G} \rho_{\psi_{0}}(g')\rho_{\psi_{0}}(g)\rho_{\psi}(g)^{-1}\rho_{\psi}(g')^{-1} \, dg \]
\[ = \int_{G} \rho_{\psi_{0}}(g')\rho_{\psi_{0}}(g)^{-1} \, dg \]
\[ = \int_{G} \rho_{\psi_{0}}(g)\rho_{\psi}(g)^{-1} \, dg \quad \text{by invariance of Haar measure} \]
\[ = \alpha(\psi). \]

So, we have shown that \( \alpha(\psi) \) is contained in \( \text{end}_{G}(F/\psi, w) \) for each \( \psi \in \Psi_{F}(w) \). Furthermore since \( \alpha(\psi_{0}) = \text{constant} \cdot \text{id}_{F_{w_{0}}} \), the endomorphism \( \alpha(\psi) \) is an isomorphism for each \( \psi \) in a small neighborhood \( U \) of \( \psi_{0} \). So, the restriction of \( \alpha \) to \( U \) is a wanted map.

We are ready to prove that \( \text{cl} : \text{fiso}(F, w)/\text{iso}(F, w) \rightarrow \Psi_{F}(w) \) is a homeomorphism for \( w \in \text{ext} F \).

**Proposition 2.13.** For each \( w \in \text{ext} F \), the map \( \text{cl} \) is homeomorphic. And, \( \Psi_{F}(w) \) is nonempty and path-connected.

**Proof.** As we have seen before, \( \text{cl} \) is bijective and continuous. So, we only have to show that \( \text{cl}^{-1} \) is continuous. By Remark 2.7, the inverse \( \text{cl}^{-1} \) is as follows:
\[ \text{cl}^{-1} : \Psi_{F}(w) \rightarrow \text{fiso}(F, w)/\text{iso}(F, w), \quad \psi \mapsto \left[ \alpha_{\psi} \circ p_{\psi} \right] \]
for any $G$-isomorphism $\alpha_\psi$ in $\text{iso}_G(F/\psi, w)$. To show that $\text{cl}^{-1}$ is continuous, we will show that $\text{cl}^{-1}$ is continuous at an arbitrary $\psi_0 \in \Psi_F(w)$. Instead of $\alpha_\psi$, we consider the continuous map $\alpha$ of Lemma 2.12 defined on a small neighborhood $U$ of $\psi_0$ whose image $\alpha(\psi)$ is contained in $\text{iso}_G(F/\psi, w)$ for each $\psi \in U$. Then, the inverse $\text{cl}^{-1}$ restricted to $U$ is expressed as follows:

$\text{cl}^{-1}|_U : U \rightarrow \text{fiso}_G(F, w)/\text{iso}_G(w), \quad \psi \rightarrow [\alpha(\psi) \circ p_\psi].$

Since both $\alpha(\psi)$ and $p_\psi$ vary continuously with $\psi$, the map $\text{cl}^{-1}$ on $U$ is continuous. In particular, the map $\text{cl}^{-1}$ is continuous at $\psi_0$. Therefore, we obtain a proof for the first statement.

Now, we prove the second statement. By Corollary 2.4 $\text{fiso}_G(F, w)$ is nonempty and path-connected. So, the quotient space $\text{fiso}_G(F, w)/\text{iso}_G(w)$ is nonempty and path-connected. Since the map $\text{cl}$ is homeomorphic by the first statement, we obtain that $\Psi_F(w)$ is nonempty and path-connected. Therefore, we obtain a proof. □

Now, we can prove Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.2**. By arguments below Lemma 2.13 the representation space of $F/\psi$ is constant regardless of $\psi$, and the $G$-representation structure of $F/\psi$ varies continuously with $\psi$. By definition of $\text{gl}$, the map $\text{gl}$ is continuous. By Proposition 2.13 each preimage of $\text{gl}$ is nonempty, i.e. $\text{gl}$ is surjective. Again by Proposition 2.13 each preimage of $\text{gl}$ is path-connected, i.e. the map $[\text{gl}]$ is injective. Therefore, we obtain a proof. □

**Proof of Theorem 1.3**. Let $\mathfrak{s} \subset \mathfrak{X}$ be a subset containing exactly one element in each orbit of $\mathfrak{X}$. By Proposition 2.13 we have the following homeomorphism:

$i : \text{fiso}_G(F, w) \rightarrow \prod_{s \in \mathfrak{s}} \text{iso}_{G_s}(F_s, w), \quad p \mapsto \prod_{s \in \mathfrak{s}} p|_{F_s}.$

If we define an $\text{iso}_G(w)$-action on the product $\prod_{s \in \mathfrak{s}} \text{iso}_{G_s}(F_s, w)$ as follows:

$\alpha \cdot \prod_{s \in \mathfrak{s}} p_s := \prod_{s \in \mathfrak{s}} \alpha \circ p_s$

for $\alpha \in \text{iso}_G(w)$ and $p_s \in \text{iso}_{G_s}(F_s, w)$, then $i$ is $\text{iso}_G(w)$-homeomorphic. So, $\Psi_F(w)$ is homeomorphic to

$$\left( \prod_{s \in \mathfrak{s}} \text{iso}_{G_s}(F_s, w) \right)/\text{iso}_G(w)$$

because $\Psi_F(w)$ is homeomorphic to $\text{fiso}_G(F, w)/\text{iso}_G(w)$ by Proposition 2.13. Since $w \in \operatorname{ext} F$ and hence we have $\operatorname{res}_F^G w \cong F_s$ by definition, the topological space $\text{iso}_{G_s}(F_s, w)$ is $\text{iso}_G(w)$-homeomorphic to $\text{iso}_{G_s}(w)$. Therefore, $\Psi_F(w)$ is homeomorphic to

$$\left( \prod_{s \in \mathfrak{s}} \text{iso}_{G_s}(w) \right)/\text{iso}_G(w),$$

and we obtain a proof. □

Applying Theorem 1.3 to two extreme cases, we obtain the following corollary:

**Corollary 2.14**. For each $w \in \operatorname{ext} F$, the set $\Psi_F(w)$ is homeomorphic to

\[
\begin{cases}
\text{iso}_G(w)/\text{iso}_G(w) & \text{for any } \bar{x} \in \mathfrak{X} \quad \text{if the } G\text{-action on } \mathfrak{X} \text{ is transitive}, \\
\text{iso}_G(w)^{|\mathfrak{X}|} & \text{if the } G\text{-action on } \mathfrak{X} \text{ is trivial}.
\end{cases}
\]
Example 3.1. Assume that the $G$-action on $\bar{x}$ is trivial. Denote $|\bar{x}|$ by $N$. By Theorem 1.3, $\Psi_F(w)$ is homeomorphic to $\text{iso}_G(w)^N/\text{iso}_G(w)$. We consider the map

$$j : \text{iso}_G(w)^{N-1} \to \text{iso}_G(w)^N/\text{iso}_G(w), \quad (\alpha_1, \cdots, \alpha_{N-1}) \mapsto [(\alpha_1, \cdots, \alpha_{N-1}, \text{id})].$$

It is easy that $j$ is bijective. And if we consider smooth structures of two spaces, it is easy that $j$ is locally diffeomorphic. Therefore, $j$ is a homeomorphism. \hfill $\Box$

3. The Fundamental group of the set of equivariant pointwise clutching maps

In this section, we investigate the fundamental group of $\Psi_F(w)$ for $w \in \text{ext} F$ when the $G$-action on $\bar{x}$ is transitive, and prove Proposition 1.4 and Corollary 1.5.

When the $G$-action on $\bar{x}$ is transitive, $\Psi_F(w)$ is homeomorphic to $\text{iso}_{G_x}(w)/\text{iso}_G(w)$ for any $x \in \bar{x}$ by Corollary 2.11. Hence to calculate $\pi_1(\Psi_F(w))$, we need understand the following long exact sequence of homotopy groups for the fibration $\text{iso}_{G_x}(w) \to \text{iso}_{G_1}(w)/\text{iso}_{G}(w)$ with the fiber $\text{iso}_{G}(w)$:

$$\pi_1(\text{iso}_G(w)) \xrightarrow{\pi_1(k)} \pi_1(\text{iso}_{G_x}(w)) \to \pi_1(\text{iso}_{G_1}(w)/\text{iso}_{G}(w)) \to \pi_0(\text{iso}_{G_{\bar{x}}}(w)),$$

where $\pi_1(k)$ is a map induced by the inclusion $k : \text{iso}_G(w) \to \text{iso}_{G_x}(w)$. Since $\pi_0$'s in the long exact sequence are trivial, we only have to understand $\pi_1(k)$ to calculate $\pi_1(\text{iso}_{G_x}(w)/\text{iso}_{G}(w))$, see arguments below Example 3.1 for triviality of $\pi_0$'s. For notational simplicity, we generalize the situation slightly. Instead of $\pi_1(k)$ in (3.1), we will investigate

$$\pi_1(\text{iso}_G(w)) \to \pi_1(\text{iso}_K(w))$$

for a complex $G$-representation $w$ and a closed subgroup $K \subset G$, which is a map induced by the inclusion $\text{iso}_G(w) \hookrightarrow \text{iso}_K(w)$. So in this section, we often refer to the map (3.2).

Let $w$ be a complex $G$-representation, and $K$ be a closed subgroup of $G$. The $G$-representation $w$ is decomposed into a direct sum of irreducible $G$-representations as follows:

$$w \cong l_1 \cdot U_1 \oplus \cdots \oplus l_m \cdot U_m$$

for some natural numbers $l_i$'s and irreducible $G$-representations $U_i$'s such that $U_i \not\cong U_{i'}$ whenever $i \neq i'$. Then by Schur’s lemma, we have a homeomorphic group isomorphism

$$\text{iso}_G(w) \cong \text{GL}(l_1, \mathbb{C}) \times \cdots \times \text{GL}(l_m, \mathbb{C}),$$

see [BD] p. 69] for Schur’s lemma and see [BD] Exercise 9 of p. 72 for the group isomorphism. For explanation of the group isomorphism (3.4), we give an example.

Example 3.1. Assume that $m = 2$ and $l_1 = 2$, $l_2 = 3$ for the decomposition (3.3), i.e.

$$w \cong (U_1 \oplus U_1) \oplus (U_2 \oplus U_2 \oplus U_2).$$

If we express an element in $\text{GL}(2, \mathbb{C}) \times \text{GL}(3, \mathbb{C})$ of (3.4) as

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

```
for $A \in \text{GL}(2, \mathbb{C})$ and $B \in \text{GL}(3, \mathbb{C})$, the element is corresponding to the following element in $\text{iso}_G(w)$ through the isomorphism \eqref{eq:3.4}:
\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix},
\begin{pmatrix}
  v_1 \\
  v_2 \\
  v_3
\end{pmatrix}
\mapsto
\begin{pmatrix}
  A \cdot 
  \begin{pmatrix}
    u_1 \\
    u_2
  \end{pmatrix},
  B \cdot 
  \begin{pmatrix}
    v_1 \\
    v_2 \\
    v_3
  \end{pmatrix}
\end{pmatrix}
\]
for \(\left( (u_1, u_2), (v_1, v_2, v_3) \right) \in (U_1 \oplus U_1) \oplus (U_2 \oplus U_2 \oplus U_2)\), where $0$ is a zero matrix with a suitable size.
\[\square\]

Now, we digress briefly into the zeroth and first homotopy groups of $\text{iso}_K(w)$ for a complex $G$-representation $w$ and a closed subgroup $K \subset G$. First, the space $\text{iso}_K(w)$ is homeomorphically group isomorphic to a product of general linear groups over $\mathbb{C}$ in the exactly same way with $\text{iso}_G(w)$. It is well known that general linear groups over $\mathbb{C}$ are path-connected, and that their fundamental groups are $\mathbb{Z}$, more precisely the determinant map $\det : \text{GL}(l, \mathbb{C}) \to \mathbb{C}^*$ for $l \in \mathbb{N}$ induces the group isomorphism
\[
\pi_1(\det) : \pi_1\left( \text{GL}(l, \mathbb{C}) \right) \to \pi_1(\mathbb{C}^*) \cong \mathbb{Z}.
\]
So, the zeroth homotopy group of $\text{iso}_K(w)$ for any closed subgroup $K \subset G$ is trivial as mentioned earlier. And, if $\text{iso}_K(w)$ is homeomorphically group isomorphic to a product of $m$ general linear groups over $\mathbb{C}$, then the fundamental group of $\text{iso}_K(w)$ is $\mathbb{Z}^m$.

We return to our investigation of the map \eqref{eq:5.2} induced by the inclusion. In this time, we will apply Schur’s lemma to the restricted $K$-representation $\text{res}_K^G w$ for a closed subgroup $K$ of $G$ so as to investigate $\text{iso}_K(w)$. In understanding the restricted representation, it is helpful to investigate whether irreducible $G$-representations $U_i$’s contained in $w$ satisfy the following condition on irreducibility:

Condition (I). $\text{res}_K^G U_i$’s are all irreducible.

For a while, we assume Condition (I). Under the condition, our investigation of $\text{res}_K^G w$ is divided into two steps: first, we additionally assume that $\text{res}_K^G U_i$’s are all isomorphic, and then we get rid of the assumption later.

**The first step: Condition (I) holds, and $\text{res}_K^G U_i$’s are all isomorphic.**

By the assumption of this step, $\text{res}_K^G w$ is decomposed into a direct sum of irreducible $K$-representations as follows:
\[
\text{res}_K^G w \cong (l_1 + \cdots + l_m) \cdot \text{res}_K^G U_1.
\]
By Schur’s lemma, we have a homeomorphic group isomorphism
\[
\text{iso}_K(w) \cong \text{GL}(l_1 + \cdots + l_m, \mathbb{C}).
\]
And, the inclusion $\text{iso}_G(w) \hookrightarrow \text{iso}_K(w)$ is regarded as the injection
\[
I_{l_1, \ldots, l_m} : \text{GL}(l_1, \mathbb{C}) \times \cdots \times \text{GL}(l_m, \mathbb{C}) \longrightarrow \text{GL}(l_1 + \cdots + l_m, \mathbb{C}),
\[
(A_1, \ldots, A_m) \longmapsto
\begin{pmatrix}
  A_1 & 0 & 0 & 0 \\
  0 & A_2 & 0 & 0 \\
  0 & 0 & \ddots & 0 \\
  0 & 0 & 0 & A_m
\end{pmatrix}
\]
Lemma 3.2. By the diagram, we can prove the following lemma: assume that Condition (I) holds, and that

\[ \text{We define the following diagonal matrices:} \]

\[ (3.7) \]

\[
\begin{array}{ccc}
\text{iso}_G(w) & \xrightarrow{\text{inclusion}} & \text{iso}_K(w) \\
\cong & & \cong \\
\text{GL}(l_1, \mathbb{C}) \times \cdots \times \text{GL}(l_m, \mathbb{C}) & \xrightarrow{\mathbf{I}_{1, \ldots, im}} & \text{GL}(l_1 + \cdots + l_m, \mathbb{C}).
\end{array}
\]

By the diagram, we can prove the following lemma:

**Lemma 3.2.** For a complex \( G \)-representation \( w \) and a closed subgroup \( K \subset G \), assume that Condition (I) holds, and that \( \text{res}_K^G U_i \)'s are all isomorphic. Then, the map \( (3.8) \)

\[ \pi_1 \left( \text{iso}_G(w) \right) \longrightarrow \pi_1 \left( \text{iso}_K(w) \right) \]

induced by the inclusion is surjective.

**Proof.** By the diagram \( (3.7) \), we only have to show that \( \pi_1 (\mathbf{I}_{1, \ldots, im}) \) is surjective. We define the following diagonal matrices:

\[ A_1(t) = \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & \text{Id}_{l_1-1} \end{pmatrix} \in \text{GL}(l_1, \mathbb{C}), \quad A_2(t) = \text{Id}_{l_2}, \quad \ldots, \quad A_m(t) = \text{Id}_{l_m} \]

for \( t \in [0, 1] \), where \( \text{Id}_l \) is the identity matrix of size \( l \). Then, the loop

\[ I_{1, \ldots, im} (A_1(t), A_2(t), \ldots, A_m(t)) \]

for \( t \in [0, 1] \) becomes a generator in \( \pi_1 \left( \text{GL}(l_1 + \cdots + l_m, \mathbb{C}) \right) \cong \mathbb{Z} \). Therefore, we obtain a proof. \( \square \)

**The second step: Condition (I) holds, but \( \text{res}_K^G U_i \)'s are not necessarily isomorphic.**

We relabel the decomposition \( (3.3) \) of \( w \) as follows:

\[ (3.8) \]

\[ w \cong \bigoplus_{1 \leq i \leq a} \left( \bigoplus_{1 \leq j_i \leq b_i} l_{i,j_i} \cdot U_{i,j_i} \right) \]

for some \( a, b_i, l_{i,j_i} \in \mathbb{N} \) and irreducible \( G \)-representations \( U_{i,j_i} \), so that

1. \( i = i' \) and \( j_i = j_{i'} \) if and only if \( U_{i,j_i} \cong U_{i',j_{i'}} \),
2. \( i = i' \) if and only if \( \text{res}_K^G U_{i,j_i} \cong \text{res}_K^G U_{i',j_{i'}} \).

In the relabeled decomposition \( (3.8) \), if we put

\[ w_i := \bigoplus_{1 \leq j_i \leq b_i} l_{i,j_i} \cdot U_{i,j_i}, \]

then \( w \cong \oplus_i w_i \). And, \( w_i \)'s satisfy the following:

1. for each \( i \), all irreducible representations contained in \( w_i \) satisfy the assumption of the first step,
2. whenever \( i \neq i' \), there is no irreducible \( K \)-representation contained in both \( \text{res}_K^G w_i \) and \( \text{res}_K^G w_{i'} \).

By (a), we can apply the first step to each \( w_i \) to obtain

\[ \text{iso}_G(w_i) \cong \prod_{1 \leq j_i \leq b_i} \text{GL}(l_{i,j_i}, \mathbb{C}) \quad \text{and} \quad \text{iso}_K(w_i) \cong \text{GL} \left( \sum_{1 \leq j_i \leq b_i} l_{i,j_i}, \mathbb{C} \right). \]
Then by Schur’s lemma and (b), we have

\[
(3.9) \quad \text{iso}_{G}(w) \cong \prod_{1 \leq i \leq a} \text{iso}_{G}(w_i) \cong \prod_{1 \leq i \leq a} \left( \prod_{1 \leq j_i \leq b_i} \text{GL}(l_{i,j_i}, \mathbb{C}) \right)
\]

and

\[
(3.10) \quad \text{iso}_{K}(w) \cong \prod_{1 \leq i \leq a} \text{iso}_{K}(w_i) \cong \prod_{1 \leq i \leq a} \text{GL} \left( \sum_{1 \leq j_i \leq b_i} l_{i,j_i}, \mathbb{C} \right).
\]

Next, we would find out a matrix expression for the inclusion \( \text{iso}_{G}(w) \hookrightarrow \text{iso}_{K}(w) \) through (3.9) and (3.10). Since (a) holds, the inclusion \( \text{iso}_{G}(w_i) \hookrightarrow \text{iso}_{K}(w_i) \) is regarded as the injection

\[
I_{i_1, \ldots, i_{b_i}} : \prod_{1 \leq j_i \leq b_i} \text{GL}(l_{i,j_i}, \mathbb{C}) \hookrightarrow \text{GL} \left( \sum_{1 \leq j_i \leq b_i} l_{i,j_i}, \mathbb{C} \right),
\]

\[
\prod_{1 \leq j_i \leq b_i} A_{i,j_i} \mapsto \begin{pmatrix}
A_{i,1} & 0 & 0 & 0 \\
0 & A_{i,2} & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & A_{i,b_i}
\end{pmatrix}
\]

by (3.10). We can piece together these injections so that the inclusion \( \text{iso}_{G}(w) \hookrightarrow \text{iso}_{K}(w) \) is regarded as the injection

\[
(3.11) \quad \prod_{1 \leq i \leq a} \left( \prod_{1 \leq j_i \leq b_i} \text{GL}(l_{i,j_i}, \mathbb{C}) \right) \hookrightarrow \prod_{1 \leq i \leq a} \text{GL} \left( \sum_{1 \leq j_i \leq b_i} l_{i,j_i}, \mathbb{C} \right),
\]

\[
\prod_{1 \leq i \leq a} \left( \prod_{1 \leq j_i \leq b_i} A_{i,j_i} \right) \mapsto \prod_{1 \leq i \leq a} I_{i_1, \ldots, i_{b_i}} \left( A_{i_1, 1}, \ldots, A_{i_{b_i}, i} \right)
\]

through two group isomorphisms (3.9), (3.10). We give an example of (3.11).

**Example 3.3.** Assume that \( a = 2 \) and \( b_1 = 2, b_2 = 3 \) for the relabeled decomposition (3.8), i.e.

\[
w \cong \left( l_{1,1} \cdot U_{1,1} \oplus l_{1,2} \cdot U_{1,2} \right) \oplus \left( l_{2,1} \cdot U_{2,1} \oplus l_{2,2} \cdot U_{2,2} \oplus l_{2,3} \cdot U_{2,3} \right).
\]

Then, \( \text{iso}_{G}(w) \cong \text{GL}(l_{1,1}, \mathbb{C}) \times \text{GL}(l_{1,2}, \mathbb{C}) \times \text{GL}(l_{2,1}, \mathbb{C}) \times \text{GL}(l_{2,2}, \mathbb{C}) \times \text{GL}(l_{2,3}, \mathbb{C}) \) and

\[
\text{iso}_{K}(w) \cong \text{GL}(l_{1,1} + l_{1,2}, \mathbb{C}) \times \text{GL}(l_{2,1} + l_{2,2} + l_{2,3}, \mathbb{C})
\]

by Schur’s lemma. And, the inclusion \( \text{iso}_{G}(w) \hookrightarrow \text{iso}_{K}(w) \) is regarded as the injection

\[
\prod_{1 \leq i \leq a} \left( \prod_{1 \leq j_i \leq b_i} \text{GL}(l_{i,j_i}, \mathbb{C}) \right) \hookrightarrow \prod_{1 \leq i \leq a} \text{GL} \left( \sum_{j_i} l_{i,j_i}, \mathbb{C} \right),
\]

\[
\left( A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2} \right) \left( A_{2,3} \right),
\]

\[
\begin{pmatrix}
A_{1,1} & 0 & 0 & 0 \\
0 & A_{1,2} & 0 & 0 \\
0 & 0 & A_{2,2} & 0 \\
0 & 0 & 0 & A_{2,3}
\end{pmatrix}
\]

for \( A_{i,j_i} \in \text{GL}(l_{i,j_i}, \mathbb{C}) \).

By the inclusion (3.11), we can get rid of an assumption in Lemma 3.2.

**Lemma 3.4.** For a complex \( G \)-representation \( w \) and a closed subgroup \( K \) of \( G \), assume that Condition (I) holds. Then, the map (3.2)

\[
\pi_1 \left( \text{iso}_{G}(w) \right) \rightarrow \pi_1 \left( \text{iso}_{K}(w) \right)
\]

induced by the inclusion is surjective.
Proof. As in the proof of Lemma 3.2, we only have to show that the \( \pi_1 \) map induced by \( \bar{G} \) is surjective. For this, it suffices to show that \( \pi_1(\bar{G}, x, \bar{G}) \) is surjective for each \( x \). But, this holds by the proof of Lemma 3.2. Therefore, we obtain a proof.

Now, we can prove Proposition 1.4 and Corollary 1.5.

Proof of Proposition 1.4. By Corollary 2.14, \( \Psi \) for each \( i \).

Proof of Corollary 1.5. If Condition (I) does not hold, then the map (3.2) is not necessarily surjective. However, this holds by the proof of Lemma 3.2. Therefore, we obtain a proof. □

Proof of Corollary 1.5. If \( G \) is abelian, complex irreducible \( G \)-representations are all one-dimensional. So, their restrictions to a closed subgroup are still irreducible. So, the Condition (I) holds. Therefore, we obtain a proof by Proposition 1.4. □

If Condition (I) does not hold, then the map (3.2) is not necessarily surjective. Though we do not indulge into such a situation any more in the paper, interested readers might need the following lemma to investigate the map:

Lemma 3.5. For a complex irreducible \( G \)-representation \( w \) and a closed subgroup \( K \) of \( G \), assume that \( w \) is decomposed into a direct sum of irreducible \( K \)-representations as follows:

\[
l_1 \cdot V_1 \oplus \cdots \oplus l_m \cdot V_m
\]

for some \( l_i \in \mathbb{N} \) and irreducible \( K \)-representations \( V_i \) such that \( V_i \not\cong V_j \) whenever \( i \neq j \). Then, the map (3.2) induced by the inclusion is as follows:

\[
\pi_1(\text{Iso}_G(w)) \cong \mathbb{Z} \longrightarrow \pi_1(\text{Iso}_K(w)) \cong \mathbb{Z}^m, \quad n \mapsto (l_1 n, \cdots, l_m n)
\]

for \( n \in \mathbb{Z} \) up to sign.

Proof. By Schur’s lemma, we have

\[
\text{Iso}_G(w) \cong \text{GL}(1, \mathbb{C}) \quad \text{and} \quad \text{Iso}_K(w) \cong \text{GL}(l_1, \mathbb{C}) \times \cdots \times \text{GL}(l_m, \mathbb{C}).
\]

So, we obtain \( \pi_1(\text{Iso}_G(w)) \cong \mathbb{Z} \) and \( \pi_1(\text{Iso}_K(w)) \cong \mathbb{Z}^m \). Since the loop \( e^{2\pi t i} \in \text{GL}(1, \mathbb{C}) = \mathbb{C}^* \) for \( t \in [0, 1] \) gives a generator of \( \pi_1(\text{Iso}_G(w)) \), it suffices to find its image to obtain a proof. The loop considered as contained in \( \text{Iso}_K(w) \) is expressed as

\[
(e^{2\pi t i} \cdot \text{Id}_{l_1}, \cdots, e^{2\pi t i} \cdot \text{Id}_{l_m}) \in \text{GL}(l_1, \mathbb{C}) \times \cdots \times \text{GL}(l_m, \mathbb{C})
\]

for \( t \in [0, 1] \). Since the loop \( e^{2\pi t i} \cdot \text{Id}_l \in \text{GL}(l, \mathbb{C}) \) for \( t \in [0, 1] \) and \( l \in \mathbb{N} \) gives \( t \) times a generator of \( \pi_1(\text{GL}(l, \mathbb{C})) \), we obtain a proof. □

4. LEMMAS ON EVALUATION AND RESTRICTION OF AN EQUIVARIANT POINTWISE CLUTCHING MAP

To explain motivation for evaluation of an equivariant pointwise clutching map by an example, we recall Example 1.1.
Example 4.1 (Continued from Exercise 1.1). For a vertex $x \in X$, we consider an arbitrary equivariant pointwise clutching map $\psi$ with respect to $(\pi^*E)_{\pi^{-1}(x)}$. Put $\bar{x} := \pi^{-1}(x) = \{ \bar{x}_i \mid i \in \mathbb{Z}_3 \}$, see Figure 4.1. Then, the map $\psi$ is determined by the evaluation $\psi(\bar{x}_0, \bar{x}_1)$, i.e. the following are determined by $\psi(\bar{x}_0, \bar{x}_1)$:

$$
\psi(\bar{x}_i, \bar{x}_i), \quad \psi(\bar{x}_i, \bar{x}_{i+1}), \quad \psi(\bar{x}_{i+1}, \bar{x}_i)
$$

for $i \in \mathbb{Z}_3$.

We give details on this. The isotropy subgroup $T_x$ at $x$ of the tetrahedral group $T$ is the dihedral group of order 6. Let $a$ and $b$ be elements of $T_x$ such that

$$
a \cdot \bar{x}_i = \bar{x}_{i+1} \quad \text{and} \quad b \cdot \bar{x}_i = \bar{x}_{i-1}
$$

for $i \in \mathbb{Z}_3$. By reflexivity of pointwise clutching map, we have

$$
\psi(\bar{x}_i, \bar{x}_i) = \text{id} \quad \text{for any } i \in \mathbb{Z}_3.
$$

By equivariance of $\psi$, we have

$$
\psi(\bar{x}_i, \bar{x}_{i+1}) = a^i \psi(\bar{x}_0, \bar{x}_1) a^{-i} \quad \text{for any } i \in \mathbb{Z}_3.
$$

By symmetry of pointwise clutching map, we have

$$
\psi(\bar{x}_{i+1}, \bar{x}_i) = \psi(\bar{x}_i, \bar{x}_{i+1})^{-1} = (a^i \psi(\bar{x}_0, \bar{x}_1) a^{-i})^{-1} = a^i \psi(\bar{x}_0, \bar{x}_1)^{-1} a^{-i}
$$

for any $i \in \mathbb{Z}_3$. So, we have shown that $\psi$ is determined by its evaluation at $(\bar{x}_0, \bar{x}_1)$. \hfill \Box

We will prove a result generalizing Example 4.1. Prior to this, we introduce some terminologies. A subset $B \subset X \times X$ is called a binary relation on $X$. For binary relations $B, B'$ on $X$, we define the following three operations:

$$
B^{-1} = \{ (\bar{x}', \bar{x}) \mid (\bar{x}, \bar{x}') \in B \},
$$

$$
B' \circ B = \{ (\bar{x}, \bar{x}'') \mid \text{there exists } \bar{x}' \text{ such that } (\bar{x}, \bar{x}') \in B \text{ and } (\bar{x}', \bar{x}'') \in B' \},
$$

$$
G \cdot B = \{ (g\bar{x}, g\bar{x}') \mid g \in G \text{ and } (\bar{x}, \bar{x}') \in B \}.
$$

A binary relation $B$ on $X$ is called an equivariant equivalence relation if the following four hold:

(a) reflexivity : $\Delta \subset B$,
(b) symmetry : $B^{-1} \subset B$,
(c) transitivity : $B \circ B \subset B$,
(d) equivariance : $G \cdot B \subset B$. 

\begin{figure}
\centering
\includegraphics[width=0.7\textwidth]{figure4.1}
\caption{The preimage of a vertex}
\end{figure}
where $\Delta$ is the diagonal subset of $\mathfrak{x} \times \mathfrak{x}$. Then, there exists the smallest equivariant equivalent relation containing $B$ for any binary relation $B$ on $\mathfrak{x}$. More precisely, if we put $B' = (G \cdot B) \cup (G \cdot B^{-1})$, then it can be checked that the following binary relation on $\mathfrak{x}$ is the smallest equivariant equivalence relation containing $B$:

$$\Delta \cup B' \cup (B' \circ B') \cup (B' \circ B' \circ B') \cup \cdots .$$

For a binary relation $B$ on $\mathfrak{x}$, we define the following evaluation map:

$$ev_B : \Psi_F \rightarrow \prod_{(x,x') \in B} \text{iso} \left( F_x, F_{x'} \right), \quad \psi \mapsto \prod_{(x,x') \in B} \psi(x, x').$$

**Lemma 4.2.** For a binary relation $B$ on $\mathfrak{x}$, if the smallest equivariant equivalence relation containing $B$ is equal to $\mathfrak{x} \times \mathfrak{x}$, then the map $ev_B$ is homeomorphic to its image.

**Proof.** First, we prove that $ev_B$ is injective. The map $ev_B$ is injective if and only if any two maps $\psi, \psi' \in \Psi_F$ identical on $B$ are the same map. Hence to show that the map $ev_B$ is injective, we only have to show that each $\psi$ is determined by $\psi(x, x')$'s for $(x, x') \in B$. However, this is basically the same with the proof of Example 4.1 because the smallest equivariant equivalence relation containing $B$ is equal to the whole set $\mathfrak{x} \times \mathfrak{x}$ by the assumption. So, we obtain a proof for injectivity.

Second, we show that $ev_B$ is continuous. By definition, the map $ev_B$ is the composition of an inclusion and an projection as follows:

$$\Psi_F \hookrightarrow \prod_{(x,x') \in \mathfrak{x} \times \mathfrak{x}} \text{iso} \left( F_x, F_{x'} \right) \rightarrow \prod_{(x,x') \in B} \text{iso} \left( F_x, F_{x'} \right).$$

Since the inclusion and the projection are continuous, the map $ev_B$ is continuous.

Last, we prove that $ev_B^{-1}$ defined on the image of $ev_B$ is continuous. For simplicity, we will give a proof only for the case in Example 4.1. In the case, the binary relation $B = \{(x_0, x_1)\}$ satisfies the assumption of the lemma, and the image of $ev_B$, denoted by $\text{im } ev_B$, is contained in $\text{iso}(F_{x_0}, F_{x_1})$. We will express $ev_B^{-1}$ as the following composition of an inclusion and some map $s$:

$$\text{im } ev_B \hookrightarrow \text{iso} \left( F_{x_0}, F_{x_1} \right) \xrightarrow{s} \prod_{(x,x') \in \mathfrak{x} \times \mathfrak{x}} \text{iso} \left( F_x, F_{x'} \right).$$

Denoting $\text{iso}(F_{x_i}, F_{x_i})$ by $\text{iso}_{i,j}$, we define $s$ as follows:

$$\text{iso}_{0,0,1} \rightarrow \text{iso}_{0,0,1} \times \text{iso}_{0,2} \times \text{iso}_{1,0} \times \text{iso}_{1,2} \times \text{iso}_{2,0} \times \text{iso}_{2,1} \times \text{iso}_{2,2},$$

$$A \mapsto \left( \text{id}, A, a^{-1}A^{-1}a, A^{-1}, \text{id}, aAa^{-1}, a^2Aa^{-2}, aA^{-1}a^{-1}, \text{id} \right).$$

This is nothing but piecing together formulas on $\psi(x_i, x_i), \psi(x_i, x_{i+1}), \psi(x_{i+1}, x_i)$ for $i \in \mathbb{Z}_2$ in Example 4.1 and we can check that the above composition is equal to the function $ev_B^{-1}$. So, $ev_B^{-1}$ is continuous because $s$ is continuous. Therefore, we obtain a proof. \hfill \Box

We give some examples of this lemma.

**Example 4.3.**

1. Assume that $\mathbb{Z}_m = \langle a \rangle$ acts on $\mathfrak{x} = \{\bar{x}_i | i \in \mathbb{Z}_m\}$ as follows:

$$a \cdot \bar{x}_i = \bar{x}_{i+1} \quad \text{for each } i \in \mathbb{Z}_m.$$

Then, each $\psi \in \Psi_F$ is determined by $\psi(\bar{x}_0, \bar{x}_1)$.

2. Assume that the order $2m$ dihedral group

$$D_m = \langle a, b \mid a^m = \text{id}, b^2 = \text{id}, bab^{-1} = a^{-1} \rangle$$

acts on $\mathfrak{x} = \{\bar{x}_i | i \in \mathbb{Z}_m\}$ as follows:

$$a \cdot \bar{x}_i = \bar{x}_{i+1} \quad \text{and} \quad b \cdot \bar{x}_i = \bar{x}_{-i} \quad \text{for each } i \in \mathbb{Z}_m.$$
Then, each $\psi \in \Psi_F$ is determined by $\psi(\bar{x}_0, \bar{x}_1)$.

(3) Assume that the order 4 dihedral group $D_4$ acts on $\bar{x} = \{\bar{x}_i | i \in \mathbb{Z}_4\}$ as follows:

$$a \cdot \bar{x}_i = \bar{x}_{i+2} \quad \text{and} \quad b \cdot \bar{x}_i = \bar{x}_{-i+1} \quad \text{for each} \ i \in \mathbb{Z}_4.$$ 

Then, each $\psi \in \Psi_F$ is determined by its evaluation on

$$B = \{(\bar{x}_0, \bar{x}_1), (\bar{x}_0, \bar{x}_3)\}.$$ 

Next, we recall Example 4.4 once again to explain motivation for restriction of an equivariant pointwise clutching map by an example.

**Example 4.4** (Continued from Exercise 3). To glue the pullback vector bundle $\pi^*E$, we focused on how to glue $(\pi^*E)_{\pi^{-1}(x)}$ for any point $x$ in an edge of a regular tetrahedron $X$. If we have an understanding of all equivariant pointwise clutching maps with respect to $(\pi^*E)_{\pi^{-1}(x)}$ for each $x$, then we could pick one in each $\Psi_{(\pi^*E)_{\pi^{-1}(x)}}$ and collect them to form a continuous system of equivariant pointwise clutching maps. But the meaning of ‘continuous’ is not clear at the present stage because we can not compare two equivariant pointwise clutching maps contained in $\Psi_{(\pi^*E)_{\pi^{-1}(x')}}$’s for two different $x'$s. Let us explain this. For two nearby points $x \neq x'$ in an edge of $X$, let $\psi$ and $\psi'$ be equivariant pointwise clutching maps with respect to $(\pi^*E)_{\pi^{-1}(x)}$ and $(\pi^*E)_{\pi^{-1}(x')}$, respectively. If both $x$ and $x'$ are in the interior of the edge, it would be conceivable to compare $\psi$ with $\psi'$ because $(\pi^*E)_{\pi^{-1}(x)}$ and $(\pi^*E)_{\pi^{-1}(x')}$ are $G$-isomorphic and we can identify them through a suitable $G$-isomorphism. But if $x$ is a vertex and $x'$ is not, the situation is different because cardinalities of their preimages under $\pi$ are different, see Figure 4.2. To solve this problem, we restrict the map $\psi$ to the smaller subset $\{\bar{x}_0, \bar{x}_1\} \times \{\bar{x}_0, \bar{x}_1\}$. Then, the restricted map will turn out to be an equivariant pointwise clutching map with respect to $(\pi^*E)_{\{\bar{x}_0, \bar{x}_1\}}$. And then we could compare the map so obtained with $\psi'$ because $(\pi^*E)_{\{\bar{x}_0, \bar{x}_1\}}$ and $(\pi^*E)_{\pi^{-1}(x')}$ are $G$-isomorphic and we can identify them through a suitable $G$-isomorphism. 

We prove a lemma on restriction of an equivariant pointwise clutching map.

**Lemma 4.5.** For a subset $\mathcal{X}' \subset \mathcal{X}$ and the maximal subgroup $K \subset G$ preserving $\mathcal{X}'$, the map

$$\text{ev}_{\mathcal{X}' \times \mathcal{X}'} : \Psi_F \to \Psi_{F_{\mathcal{X}'},} \quad \psi \mapsto \text{ev}_{\mathcal{X}' \times \mathcal{X}'}(\psi)$$

is well-defined, and we have a $K$-isomorphism

$$\text{res}^G_K F/\psi \cong F_{\mathcal{X}'}/\text{ev}_{\mathcal{X}' \times \mathcal{X}'}(\psi)$$

for any $\psi \in \Psi_F$. 

---

**Figure 4.2.** Preimages of a vertex and its nearby non-vertex point in an edge.
Proof. Note that $F_{\mathcal{X}}$ is a $K$-vector bundle. For any $\psi \in \Psi_{F}$, we consider the map $p_{\psi} : F \to F/\psi$. Restricting $p_{\psi}$ to $F_{\mathcal{X}}$, we obtain an equivariant fiberwise isomorphism

$$p_{\psi}|_{F_{\mathcal{X}}} : F_{\mathcal{X}} \to \text{res}^{G}_{K} F/\psi.$$

By Lemma 2.5, we can construct an equivariant pointwise clutching map $\psi'$ in $\Psi_{F_{\mathcal{X}}}$ such that the following diagram commutes:

$$\begin{array}{ccc}
F_{\mathcal{X}} & \xrightarrow{p_{\psi}|_{F_{\mathcal{X}}}} & \text{res}^{G}_{K} F/\psi \\
p_{\psi'} & \searrow & \\
F_{\mathcal{X}}/\psi' & \sim & 
\end{array}$$

By definition of $\psi'$, the map $\psi'$ is equal to $\text{ev}_{\mathcal{X} \times \mathcal{X}}(\psi)$, so the map $\text{ev}_{\mathcal{X} \times \mathcal{X}}(\psi)$ is contained in $\Psi_{F_{\mathcal{X}}}$, and the map $\text{ev}_{\mathcal{X} \times \mathcal{X}}$ is well-defined. Also, we obtain the $K$-isomorphism by the diagram. Therefore, we obtain a proof. □

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Department of Mathematics Education, Gyeongin National University of Education, 45 Gyodaegil, Gyeyang-gu, Incheon, 407-753, Republic of Korea

E-mail address: mkkim@kias.re.kr