THE FIRST EIGENVALUE OF DIRAC AND LAPLACE OPERATORS ON SURFACES

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Abstract. Let \((M, g, \sigma)\) be a compact Riemannian surface equipped with a spin structure \(\sigma\). For any metric \(\tilde{g}\) on \(M\), we denote by \(\mu_1(\tilde{g})\) (resp. \(\lambda_1(\tilde{g})\)) the first positive eigenvalue of the Laplacian (resp. the Dirac operator) with respect to the metric \(\tilde{g}\). In this paper, we show that
\[
\inf \frac{\lambda_1(\tilde{g})^2}{\mu_1(\tilde{g})} \leq \frac{1}{2},
\]
where the infimum is taken over the metrics \(\tilde{g}\) conformal to \(g\). This answers a question asked by Agricola, Ammann and Friedrich in [AAF99].

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1. Introduction

Let \((M, g, \sigma)\) be a compact Riemannian surface equipped with a spin structure \(\sigma\). For any metric \(\tilde{g}\) on \(M\), we denote by \(\Sigma_{\tilde{g}}M\) the spinor bundle associated to \(\tilde{g}\). We let \(\Delta_{\tilde{g}}\) be the Laplace-Beltrami operator acting on smooth functions of \(M\) and \(D_{\tilde{g}}\) be the Dirac operator acting on smooth spinor fields with respect to the metric \(\tilde{g}\). We also denote by \(\mu_1(\tilde{g})\) (resp. \(\lambda_1(\tilde{g})\)) the smallest positive eigenvalue of \(\Delta_{\tilde{g}}\) (resp. \(D_{\tilde{g}}\)).

Agricola, Ammann and Friedrich asked the following question in [AAF99]:

When \(M\) is a two dimensional torus, can we find on \(M\) a Riemannian metric \(\tilde{g}\) for which \(\lambda_1(\tilde{g})^2 < \mu_1(\tilde{g})\) ?

The main goal of this article is to answer this question. We prove the

Theorem 1.1. There exists a family of metrics \((g_{\varepsilon})_{\varepsilon}\) conformal to \(g\) for which
\[
\limsup_{\varepsilon \to 0} \lambda_1(g_{\varepsilon})^2 \text{Vol}_{g_{\varepsilon}}(M) \leq 4\pi
\]
\[
\liminf_{\varepsilon \to 0} \mu_1(g_{\varepsilon}) \text{Vol}_{g_{\varepsilon}}(M) \geq 8\pi.
\]
Theorem 1.1 clearly answers the question of [AAF99] but says much more: first, the result is true on any compact Riemannian surface equipped with a spin structure and not only when $M$ is a two-dimensional torus. In addition, the metric $\bar{g}$ can be chosen in a given conformal class. Finally, this metric $\bar{g}$ can be chosen such that $(2-\delta)\lambda_1(g)^2 < \mu_1(g)$ where $\delta > 0$ is arbitrary small. More precisely Theorem 1.1 shows

**Corollary 1.2.** On any compact Riemannian surface $(M,g)$, we have

$$\inf \frac{\lambda_1(\bar{g})^2}{\mu_1(\bar{g})} \leq \frac{1}{2}$$

where the infimum is taken over the metric $\bar{g}$ conformal to $g$.

Theorem 1.1 has other interesting consequences. Indeed, it proves

**Corollary 1.3.** For any compact surface $(M,g)$ equipped with a spin structure $\sigma$, we let

$$\lambda_{\text{min}}^+(M,g,\sigma) = \inf \lambda_1(\bar{g}) \operatorname{Vol}_{\bar{g}}^+(M)$$

where the infimum is taken over the metrics $\bar{g}$ conformal to $g$. Then, we have $\lambda_{\text{min}}^+(M,g,\sigma) \leq \lambda_{\text{min}}^+(S^2)$ where $\lambda_{\text{min}}^+(S^2)$ is the same invariant computed on the standard sphere $S^2$.

This corollary is an immediate consequence of the fact that $\lambda_{\text{min}}^+(S^2) = 2\sqrt{2}$ (see [AHM03]). This result was announced in [AHM03]. The conformal invariant $\lambda_{\text{min}}^+$ has been studied in many papers (see for example [HijNo, LatNo, Bär92, Amm03, AHM03, AM06]). Indeed, it has many relations with Yamabe problem (see [LP87]). Corollary 1.3 has been proved in all dimensions by Ammann in [Amm03] if either $n \geq 3$ or is $D$ is invertible. Corollary 1.3 extends the result to the remaining case: $n = 2$ and $\text{Ker}(D) \neq \{0\}$. In [AHM03], an alternative proof of the case $n \geq 3$ is given and the proof of the case $n = 2$ is sketched.

In the same spirit, a consequence of Theorem 1.1 is

**Corollary 1.4.** For any compact surface $(M,g)$, we let

$$\mu_{\text{sup}}(M,g) = \sup \mu_1(\bar{g}) \operatorname{Vol}_{\bar{g}}^+(M)$$

where the infimum is taken over the metrics $\bar{g}$ conformal to $g$. Then, we have $\mu_{\text{sup}}(M,g) \geq \mu_{\text{sup}}(S^2)$ where $\mu_{\text{sup}}(S^2)$ is the same invariant computed on the standard sphere $S^2$.

The invariant $\mu_{\text{sup}}$ has been studied in [CGPS03] and Corollary 1.4 is a particular case of Theorem A in this paper. We obtain here another proof.

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2. **Generalized metrics**

Let $f$ be a smooth positive function and set $\bar{g} = f^2 g$. Let also for $u \in C^\infty(M)$

$$I_\bar{g}(u) = \frac{\int_M |\nabla u|_{\bar{g}}^2 \, dv_{\bar{g}}}{\int_M u^2 \, dv_{\bar{g}}}.$$  

It is well known that $\mu_1(\bar{g}) = \inf I_\bar{g}(u)$ where the infimum is taken over the smooth non-zero functions $u$ for which $\int_M u \, dv_{\bar{g}} = 0$. We now can write all these expressions in the metric $g$. We then see that for $u \in C^\infty(M)$, we have

$$I_\bar{g}(u) = \frac{\int_M |\nabla u|^2_{\bar{g}} \, dv_{\bar{g}}}{\int u^2 f^2 \, dv_{\bar{g}}}$$

and $\mu_1(\bar{g}) = \inf I_\bar{g}(u)$ where the infimum is taken over the smooth non-zero functions $u$ for which $\int_M u f^2 \, dv_{\bar{g}} = 0$. Now if $f$ is only of class $C^{0,a}(M)$ for some $a > 0$, we can define $\bar{g} = f^2 g$. The 2-form $\bar{g}$ is not really a metric since $f$ is not smooth. We then say that $g$ is a *generalized metric*. We can
We also have proves Lemma 2.1. □

It is easy to see that \( \lim_{n \to \infty} \) and set \( v = u - \frac{u_n f_n^2 dv_g}{f_n dv_g} \). We have \( \int_M v^2 f_n^2 dv_g = 0 \) and hence

\[
|\int_M u_n f^2 dv_g| = |\int_M u_n (f^2 - f_n^2) dv_g| \leq C \int_M |u_n| (f + f_n)^2 \| f - f_n \|_{\infty}.
\]

Since the sequence \((f_n)_n\) tends uniformly to \( f \) and since \( \int_M f_n^2 u_n^2 dv_g = 1 \), we get that \( \lim_n \int_M u_n f^2 dv_g = 0 \). In the same way,

\[
\int_M f^2 u_n^2 dv_g = \int_M f_n^2 u_n^2 dv_g + o(1) = 1 + o(1).
\]

Finally, we obtain

\[
\int_M f^2 v^2 dv_g = 1 + o(1).
\]

Together with (2) and (3), we obtain that \( \mu_1(\bar{g}) \leq \liminf_n \mu_1(f_n^2 g) \). Now, let \( u \) be associated to \( \mu_1(\bar{g}) \) and set \( v = u - \frac{u_n f_n^2 dv_g}{f_n dv_g} \). We have \( \int_M v^2 f_n^2 dv_g = 0 \) and hence

\[
\mu_1(f_n^2 g) \leq I_{\bar{g}}(v).
\]

It is easy to see that \( \lim_n I_{f_n^2 g}(v) = I_{\bar{g}}(u) = \mu_1(\bar{g}) \). We then obtain that \( \mu_1(\bar{g}) \geq \limsup_n \mu_1(f_n^2 g) \). This proves Lemma 2.1. □
In the same way, if $\tilde{g} = f^2g$ is a metric conformal to $g$ where $f$ is positive and smooth, we define

$$J_\tilde{g}(\psi) = \int_M |D\tilde{g}\psi|^2 f^{-2} \tilde{g} \, dv_{\tilde{g}} / \int_M \langle D\tilde{g}\psi, \psi \rangle_{\tilde{g}} \tilde{g} \, dv_{\tilde{g}}.$$  

The first eigenvalue of the Dirac operator $D\tilde{g}$ is then given by $\lambda_1^+(\tilde{g}) = \inf J_\tilde{g}(\psi)$ where the infimum is taken over the smooth spinor fields $\psi$ for which $\int_M \langle D\psi, \psi \rangle_{\tilde{g}} \tilde{g} \, dv_{\tilde{g}} > 0$. Now, it is well known (see [Hit74, Hij01]) that we can identify isometrically on each fiber spinor fields for the metric $g$ and spinor fields for the metric $\tilde{g}$. Moreover, we have for all smooth spinor fields $\phi$ such that $\lambda_1(\tilde{g}) = J_{\tilde{g}}(\phi)$ and such that $D\tilde{g}\phi \equiv \lambda_1(\tilde{g}) f\phi$.  

We then have a result similar to Lemma 2.1:

**Lemma 2.2.** If $(f_n)$ is a sequence of smooth positive functions which converges uniformly to $f$, then $\lambda_1(f_n^2g)$ tends to $\lambda_1(\tilde{g})$.

The proof is similar to the one of Lemma 2.1 and we omit it here.

3. The metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$

In this paragraph, we construct the metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$ which will satisfy:

$$\limsup_{\varepsilon \to 0} \lambda_1(g_{\alpha,\varepsilon})^2 \text{Vol}_{g_{\alpha,\varepsilon}}(M) \leq 4\pi + C(\alpha).$$  

where $C(\alpha)$ is a positive constant which goes to 0 with $\alpha$ and

$$\liminf_{\varepsilon \to 0} \mu_1(g_{\alpha,\varepsilon}) \text{Vol}_{g_{\alpha,\varepsilon}}(M) \geq 8\pi.$$  

Clearly this implies Theorem 1.1. By Lemmas 2.1 and 2.2, one can assume that the metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$ are generalized metrics. We just have to define the volume of $M$ for generalized metric by $\text{Vol}_{f_2g}(M) = \int_M f^2 dv_g$. At first, without loss of generality, we can assume that $g$ is flat near a point $p \in M$. Let $\alpha > 0$ be a small number to be fixed later such that $g$ is flat on $B_p(\alpha)$. We set for all $x \in M$ and $\varepsilon > 0$,

$$f_{\alpha,\varepsilon}(x) = \begin{cases} \varepsilon^2 \frac{e^{2\alpha^2 r^2}}{r^2} & \text{if } r \leq \alpha \\ \varepsilon^2 + \alpha^2 & \text{if } r > \alpha \end{cases}$$

where $r = d_g(.,p)$. The function $f_{\alpha,\varepsilon}$ is of class $C^{0,\alpha}$ for all $\alpha \in [0,1]$ and is positive on $M$. We then define for all $\varepsilon > 0$, $g_{\alpha,\varepsilon} = f_{\alpha,\varepsilon}^2g$. The 2-forms $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$ will be the desired generalized metrics. For these metrics, we have

$$\text{Vol}_{g_{\alpha,\varepsilon}}(M) = \int_M f_{\alpha,\varepsilon}^2 dv_g = \int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g + \int_{M \setminus B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g.$$
Since $g$ is flat on $B_p(\alpha)$, we have
\[ \int_{B_p(\alpha)} f_{\alpha,\epsilon}^2 dv_g = \int_{0}^{2\pi} \int_{0}^{\epsilon r} \frac{r}{(\epsilon^2 + r^2)^2} dr d\Theta. \]
Setting $y = \frac{\alpha}{\epsilon}$ we obtain:
\[ \int_{B_p(\alpha)} f_{\alpha,\epsilon}^2 dv_g = 2\pi \epsilon^2 \int_{0}^{\frac{\epsilon}{\alpha}} \frac{r}{(1 + r^2)^2} dr = 2\pi \epsilon^2 \left( \int_{0}^{\epsilon} \frac{r}{(1 + r^2)^2} dr + o(1) \right) = \pi \epsilon^2 + o(\epsilon^2). \]
Since $f_{\alpha,\epsilon}^2 \leq \frac{\epsilon^4}{\alpha^2}$ on $M \setminus B_p(\alpha)$, we have $\int_{M \setminus B_p(\alpha)} f_{\alpha,\epsilon}^2 dv_g = o(\epsilon^2)$. We obtain
\[ \text{Vol}_g(M) = \pi \epsilon^2 + o(\epsilon^2). \] (7)
In the whole paper, the notation "$o(\cdot)$" must be understood as $\epsilon$ tends to 0.

4. PROOF OF RELATION (5)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x) = \frac{2}{1 + |x|^2}$. Let $\psi_0$ be a non-zero parallel spinor field on $\mathbb{R}^2$ such that $|\psi_0|^2 = 1$. As in [AHM03], we set on $\mathbb{R}^2$
\[ \psi(x) = f \left( \frac{x}{\epsilon} \right) (1 - x) \cdot \psi_0. \]
As easily computed, we have on $\mathbb{R}^2$
\[ D\psi = f\psi \text{ and } |\psi| = f \frac{x}{\epsilon}. \] (8)
Now, we fix a small number $\delta > 0$ such that $g$ is flat on $B_p(\delta)$. Then, we take $\epsilon \leq \alpha \leq \delta$. We will let $\epsilon$ go to 0. We let also $\eta$ be a smooth cut-off function defined on $M$ such that $0 \leq \eta \leq 1$, $\eta(B_p(\delta)) = \{1\}$, $\eta(M \setminus B_p(2\delta)) = \{0\}$. Identifying $B_p(\delta)$ in $M$ with $B_0(\delta)$ in $\mathbb{R}^2$, we can define a smooth spinor field on $M$ by $\psi_\epsilon = \eta(x) \psi \left( \frac{x}{\epsilon} \right)$. Using (8), we have
\[ Dg(\psi_\epsilon) = \nabla \eta \cdot \psi \left( \frac{x}{\epsilon} \right) + \frac{\eta}{\epsilon} f \left( \frac{x}{\epsilon} \right) \psi \left( \frac{x}{\epsilon} \right). \] (9)
Since $\langle \nabla \eta \cdot \psi \left( \frac{x}{\epsilon} \right), \psi \left( \frac{x}{\epsilon} \right) \rangle \in i \mathbb{R}$ and since $|Dg \psi_\epsilon|^2 \in \mathbb{R}$, we have
\[ \int_{M} |Dg \psi_\epsilon|^2 f_{\alpha,\epsilon}^{-1} dv_g = I_1 + I_2 \] (10)
where
\[ I_1 = \int_{M} |\nabla \eta|^2 \left| \psi \left( \frac{x}{\epsilon} \right) \right|^2 dx \quad \text{and} \quad I_2 = \int_{M} \frac{\eta^2}{\epsilon^2} f^2 \left( \frac{x}{\epsilon} \right) \left| \psi \left( \frac{x}{\epsilon} \right) \right|^2 f_{\alpha,\epsilon}^{-1} dx. \]
At first, let us deal with $I_1$. By (8),
\[ I_1 \leq C \int_{M} f \left( \frac{x}{\epsilon} \right) f_{\alpha,\epsilon}^{-1} dx = C \int_{B_p(\alpha)} f \left( \frac{x}{\epsilon} \right) f_{\alpha,\epsilon}^{-1} dx + C \int_{B_p(2\delta) \setminus B_p(\alpha)} f \left( \frac{x}{\epsilon} \right) f_{\alpha,\epsilon}^{-1} dx \]
where, as in the following, $C$ denotes a constant independent of $\alpha$ and $\epsilon$. On $B_p(\alpha)$, $f \left( \frac{x}{\epsilon} \right) f_{\alpha,\epsilon}^{-1} = 2$. Hence,
\[ \int_{B_p(\alpha)} f \left( \frac{x}{\epsilon} \right) f_{\alpha,\epsilon}^{-1} dx \leq C \alpha^2. \]
On $B_p(2\delta) \setminus B_p(\alpha)$, since $\epsilon \leq \alpha$,
\[ f \left( \frac{x}{\epsilon} \right) f_{\alpha,\epsilon}^{-1} \leq \frac{4\alpha^2}{\epsilon^2 + r^2} = \frac{4\alpha^2}{\epsilon^2 \left(1 + \left( \frac{\alpha}{\epsilon} \right)^2 \right)} \]
Hence,

\[
\int_{B_p(2\delta) \setminus B_p(\alpha)} f \left( \frac{x}{\varepsilon} \right) f_{\alpha,\varepsilon}^{-1} dx \leq \frac{4\alpha^2}{\varepsilon^2} \int_0^{2\pi} \int_{\alpha}^\delta \frac{r}{(1 + (\frac{r}{\varepsilon})^2)} dr d\Theta
\]

\[
\leq 8\pi\alpha^2 \int_{2\delta}^\infty \frac{r}{(1 + r^2)} dr
\]

\[
\leq 8\pi\alpha^2 \ln \left( \frac{\varepsilon^2 + \delta^2}{\varepsilon^2 + \alpha^2} \right).
\]

We get

\[
\int_{B_p(2\delta) \setminus B_p(\alpha)} f \left( \frac{x}{\varepsilon} \right) f_{\alpha,\varepsilon}^{-1} dx \leq C \alpha^2 \ln \left( \frac{2\delta^2}{\alpha^2} \right).
\]

Finally, we obtain

\[
I_1 \leq C \alpha^2 + C \ln \left( \frac{2\delta^2}{\alpha^2} \right) = a(\alpha) \quad (11)
\]

where \(a(\alpha)\) goes to 0 with \(\alpha\). Now, by (8),

\[
I_2 \leq 8\pi \alpha^2 \int_{2\delta}^\infty \frac{r}{(1 + r^2)} dr.
\]

Mimicking what we did to get (7), we obtain that

\[
I_2 \leq 8\pi \alpha^2 + o(1)
\]

when \(\varepsilon\) tends to 0. Together with (10) and (11), we obtain

\[
\int_M |D_g \psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1} dv_g \leq 8\pi + a(\alpha) + o(1).
\]

In the same way, by (9), since \(\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g \in \mathbb{R}\) and since \((\nabla \eta \cdot \psi(\frac{x}{\varepsilon}), \psi(\frac{x}{\varepsilon})) \in i\mathbb{R}\), we have

\[
\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = \int_M \eta^2 f \left( \frac{x}{\varepsilon} \right) \left| \psi \left( \frac{x}{\varepsilon} \right) \right|^2 dv_g.
\]

By (8), this gives

\[
\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = \int_M \eta^2 f \left( \frac{x}{\varepsilon} \right) dv_g.
\]

With the computations made above, it follows that

\[
\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = 4\pi \varepsilon + o(\varepsilon).
\]

Together with (12) and (11), we obtain

\[
\lambda_1(g_{\alpha,\varepsilon})^2 \text{Vol}_{g_{\alpha,\varepsilon}}(M) \leq (J_{g_{\alpha,\varepsilon}}(\psi_\varepsilon))^2 \text{Vol}_{g_{\alpha,\varepsilon}}(M) \leq \left( \frac{8\pi + a(\alpha) + o(1)}{4\pi \varepsilon + o(\varepsilon)} \right)^2 \left( 4\pi \varepsilon^2 + o(\varepsilon^2) \right) = \frac{1}{\varepsilon} \left( 4\pi + a(\alpha) + o(1) \right).
\]

Relation (5) immediately follows.
First we need the following estimate

**Lemma 5.1.** For any \( \varepsilon > 0 \) and \( u \in C^\infty_c(B_p(\alpha)) \), then

\[
\int_M u^2 f^2_{\alpha, \varepsilon} dv_g \leq \frac{\varepsilon^2}{8} \int_M |\nabla u|^2 dv_g + \frac{1}{\pi \varepsilon^2} \left( \int_M u f^2_{\alpha, \varepsilon} dv_g \right)^2.
\]

**Proof.** Let \( g_\varepsilon = f^2_{\alpha, \varepsilon} g \). Then \((B_p(\alpha), g_\varepsilon)\) is embedded in a canonical sphere of volume \( \int_{R^2} \left( \frac{\varepsilon^2}{\varepsilon^2 + r^2} \right)^2 dx = \pi \varepsilon^2 \). Then from the Poincaré-Sobolev inequality, we have

\[
\int_M u^2 dv_{g_\varepsilon} \leq \frac{1}{\mu_{1, \varepsilon}} \int_M |\nabla^\varepsilon u|^2_{g_\varepsilon} dv_{g_\varepsilon} + \frac{1}{V_{g_\varepsilon}} \left( \int_M u dv_{g_\varepsilon} \right)^2
\]

where \( \mu_{1, \varepsilon} = \frac{8}{\varepsilon^2} \) is the first nonzero eigenvalue of the Laplacian on the sphere of volume \( V_{\varepsilon} = \pi \varepsilon^2 \) and \( \nabla^\varepsilon u \) denotes the gradient of \( u \) with respect to the metric \( g_\varepsilon \). Now since \( |\nabla^\varepsilon u|_{g_\varepsilon}^2 = f^2_{\alpha, \varepsilon} |\nabla u|_{g}^2 \) and \( dv_{g_\varepsilon} = f^2_{\alpha, \varepsilon} dv_g \), we get the desired result.

\[ \square \]

**Lemma 5.2.** For any \( u, v \in C^\infty(M) \), we have

\[
\int_M (\Delta u) v^2 dv_g = \int_M |\nabla (uv)|_{g}^2 dv_g - \int_M u^2 |\nabla v|_{g}^2 dv_g.
\]

**Proof.** The proof is an elementary calculation.

\[ \square \]

Because of the relation \( 9 \), the inequality \( 9 \) is equivalent to the following

\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \mu_1 (g_\varepsilon) \geq 8
\]

In order to prove this inequality, we assume that for any \( \varepsilon \) small enough, there exists \( k, 0 < k < 1 \) so that

\[
\mu_1 (g_\varepsilon) < \frac{8}{\varepsilon^2} k.
\]

Let \( u_\varepsilon \) be an eigenfunction associated to \( \mu_1 (g_\varepsilon) \). Then \( u_\varepsilon \in C^2(M) \) and \( \Delta_{g_\varepsilon} u_\varepsilon = \mu_1 (g_\varepsilon) u_\varepsilon \) where \( \Delta_{g_\varepsilon} \) denotes the Laplacian associated to the metric \( g_\varepsilon \). Since the dimension is 2, \( \Delta_{g_\varepsilon} = \frac{1}{f^2_{\alpha, \varepsilon}} \Delta \) and

\[
\Delta u_\varepsilon = \mu_1 (g_\varepsilon) f^2_{\alpha, \varepsilon} u_\varepsilon.
\]

We normalize \( u_\varepsilon \) so that \( \| u_\varepsilon \|_{H^1_1} = 1 \). Up to a subsequence we can assume that \( \int_M |\nabla u_\varepsilon|^2 dv_g \to l \) and \( \int_M u_\varepsilon^2 dv_g \to l' \) with \( l + l' = 1 \). Since \( (u_\varepsilon) \) is bounded in \( H^2_1 \), there exists a subsequence so that \( u_\varepsilon \to u \) weakly in \( H^1_1 \). In the following, all the convergences are up to subsequence. We sometimes omit to recall this fact.

**Lemma 5.3.** There exists a constant \( c_0 \) such that \( u = c_0 \).
Proof. Let \( \varphi \in C^\infty(M) \) and

\[
\eta_\rho := \begin{cases} 
1 & \text{on } B_\rho(\rho) \\
0 & \text{on } M \setminus B_\rho(2\rho) 
\end{cases}
\]
satisfying \( 0 \leq \eta_\rho \leq 1 \) and \( |\nabla \eta_\rho| \leq \frac{1}{\rho} \). We have

\[
\int_M \langle \nabla u, \nabla \varphi \rangle = \int_M \langle \nabla u, \nabla (\eta_\rho \varphi) \rangle dv_g + \int_M \langle \nabla u, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g. \tag{16}
\]

Now we have

\[
\int_M \langle \nabla u, \nabla (\eta_\rho \varphi) \rangle dv_g = \int_M \langle \nabla u, \nabla \eta_\rho \rangle \varphi dv_g + \int_M \langle \nabla u, \nabla \eta_\rho \rangle \eta_\rho \varphi dv_g
\leq C \left( \int_{B_\rho(2\rho)} |\nabla u|^2 dv_g \right)^{1/2} \left( \int_{B_\rho(2\rho)} |\nabla \eta_\rho|^2 dv_g \right)^{1/2}
+ \left( \int_{B_\rho(2\rho)} |\nabla u|^2 dv_g \right)^{1/2} \left( \int_{B_\rho(2\rho)} |\nabla \varphi|^2 dv_g \right)^{1/2}.
\]

The limit of the last term is 0 when \( \rho \to 0 \). Moreover from the definition of \( \eta_\rho \) and from the fact that \( M \) is a 2-dimensional locally flat domain, the limit of \( \left( \int_{B_\rho(2\rho)} |\nabla \eta_\rho|^2 dv_g \right)^{1/2} \) is bounded in a neighborhood of 0. Then we deduce that

\[
\int_M \langle \nabla u, \nabla (\eta_\rho \varphi) \rangle dv_g \to 0 \tag{17}
\]
when \( \rho \to 0 \). On the other hand

\[
\left| \int_M \langle \nabla u, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right| = \lim_{\varepsilon \to 0} \left| \int_M \langle \nabla u_\varepsilon, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right|
= \lim_{\varepsilon \to 0} \left| \int_M \langle \Delta u_\varepsilon, (1 - \eta_\rho) \varphi \rangle dv_g \right|
= \lim_{\varepsilon \to 0} \left| \mu_1(g_\varepsilon) \int_M f_{\alpha,\varepsilon} u_\varepsilon (1 - \eta_\rho) \varphi dv_g \right|.
\]

Now from the definition of \( f_{\alpha,\varepsilon} \) and from \( \varepsilon_4 \) we get

\[
\left| \mu_1(g_\varepsilon) \int_M f_{\alpha,\varepsilon} u_\varepsilon (1 - \eta_\rho) \varphi dv_g \right| \leq \frac{8}{\varepsilon^2} k\varepsilon^4 \left( \int_M u_\varepsilon^2 dv_g \right)^{1/2} \left( \int_M (1 - \eta_\rho) \varphi^2 dv_g \right)^{1/2}
\]
where \( C \) is a constant depending on the compact support of \( (1 - \eta_\rho) \varphi \). Then making \( \varepsilon \to 0 \), we deduce that

\[
\int_M \langle \nabla u, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g = 0.
\]
Now, reporting this and \ref{lem:17} in \ref{lem:16} we obtain that \( \int_M (\nabla u, \nabla \varphi) dv_g = 0 \) and \( \Delta u = 0 \) on \( M \) in the sense of distributions. This implies that \( u \equiv c_0 \) on \( M \) for a constant \( c_0 \).

\[
\text{Lemma 5.4. Let } (c_\varepsilon)_\varepsilon \text{ be a bounded sequence of real numbers. Then}
\]

\[
\int_M f_{\alpha, \varepsilon}^2 u^2 dv_g \leq O(\varepsilon^2 \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \varepsilon^3).
\]

**Proof.** Let \( \eta \) be a \( C^\infty \) function defined on \( M \) so that

\[
\eta := \begin{cases} 
1 & \text{on } B_p(\alpha/2) \\
0 & \text{on } M \setminus B_p(\alpha)
\end{cases}
\]

satisfying \( 0 \leq \eta \leq 1 \) and \( |\nabla \eta| \leq 1 \).

From the lemma \ref{lem:5.1} we have

\[
\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g \leq \frac{\varepsilon^2}{8} \int_M |\nabla((u_\varepsilon - c_\varepsilon) \eta)|^2 dv_g + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon}^2 dv_g \right)^2
\]

and applying the lemma \ref{lem:5.2} to the first term of the right hand side, we get

\[
\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g \leq
\]

\[
\frac{\varepsilon^2}{8} \int_M (\Delta(u_\varepsilon - c_\varepsilon))(u_\varepsilon - c_\varepsilon) \eta^2 dv_g + \frac{\varepsilon^2}{8} \int_M (u_\varepsilon - c_\varepsilon)^2 |\nabla \eta|^2 dv_g + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon}^2 dv_g \right)^2.
\]

From \ref{lem:16} we deduce that

\[
\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g \leq
\]

\[
\frac{\varepsilon^2}{8} \mu_1(g_\varepsilon) \int_M u_\varepsilon(u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon}^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon}^2 dv_g \right)^2.
\]

**First case:** assume that \( \int_M u_\varepsilon(u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon}^2 dv_g \geq 0 \).

The relation \ref{lem:14} implies

\[
\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g \leq k \int_M u_\varepsilon(u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon}^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon}^2 dv_g \right)^2.
\]

A straightforward computation shows that

\[
(1 - k) \int_M u_\varepsilon^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha, \varepsilon}^2 \eta^2 dv_g \leq
\]

\[
(2 - k)c_\varepsilon \int_M u_\varepsilon f_{\alpha, \varepsilon}^2 \eta^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon}^2 dv_g \right)^2.
\] (18)

Now note that
\[
\int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g = \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g + \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 dv_g = \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g + \frac{1}{\mu_1(g_\varepsilon)} \int_M \Delta u_\varepsilon dv_g = \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g \leq \int_{M \setminus B_{p(\alpha/2)}} u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g
\]

and from the definition of \(f_{\alpha,\varepsilon}\) and \(\eta\) and from the fact that \(u_\varepsilon\) is bounded in \(L^2\), we deduce that

\[
\int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g = O(\varepsilon^4).
\]

Since \(c_\varepsilon\) is bounded, (18) becomes

\[
(1 - k) \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2
\]

\[
= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g + \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon dv_g - c_\varepsilon \int_M f_{\alpha,\varepsilon}^2 \eta \right)^2
\]

\[
= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g - c_\varepsilon \int_M f_{\alpha,\varepsilon}^2 \eta \right)^2 \quad (19)
\]

where in the last equality we have used the fact that \(\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon dv_g = \frac{1}{\mu_1(g_\varepsilon)} \int_M \Delta u_\varepsilon dv_g = 0\).

Using the same arguments as above we see that \(\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g = O(\varepsilon^4)\). Reporting this in (19) we get

\[
(1 - k) \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{O(\varepsilon^4)}{\varepsilon^2} \int_M f_{\alpha,\varepsilon}^2 \eta dv_g + \frac{\varepsilon^2}{\pi \varepsilon^2} \left( \int_M f_{\alpha,\varepsilon}^2 \eta dv_g \right)^2.
\]

Now

\[
\int_M f_{\alpha,\varepsilon}^2 \eta dv_g = \int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = \int_0^{2\pi} \int_0^{\alpha/\varepsilon} \int_0^t \frac{\varepsilon^4 r}{(\varepsilon^2 + r^2)^2} dr d\Theta
\]

\[
= 2\pi \varepsilon^2 \int_0^{\alpha/\varepsilon} \frac{t}{(1 + t^2)^2} dt \leq 2\pi \varepsilon^2 \int_0^{+\infty} \frac{t}{(1 + t^2)^2} dt = \pi \varepsilon^2.
\]

This gives
(1 - k) \int_M u_\varepsilon^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha, \varepsilon}^2 \eta^2 dv_g \leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|^2_{L^2} + \frac{c_\varepsilon^2}{\pi \varepsilon^2} \left( \int_M f_{\alpha, \varepsilon}^2 \eta dv_g \right)^2
\leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|^2_{L^2} + c_\varepsilon^2 \int_M f_{\alpha, \varepsilon}^2 \eta dv_g
= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|^2_{L^2}.

Finally we have

(1 - k) \int_M u_\varepsilon^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g \leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|^2_{L^2} + c_\varepsilon^2 \int_M f_{\alpha, \varepsilon}^2 (\eta - \eta^2) dv_g
\leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|^2_{L^2} + c_\varepsilon^2 \int_{B_\varepsilon(\alpha)} f_{\alpha, \varepsilon}^2 dv_g
= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|^2_{L^2}.

(20)

Second case: Assume that \( \int_M u_\varepsilon (u_\varepsilon - c_\varepsilon) \eta^2 f_{\alpha, \varepsilon} dv_g \leq 0. \)

In this case, we have

\[ \int_M u_\varepsilon^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g - 2c_\varepsilon \int_M u_\varepsilon f_{\alpha, \varepsilon}^2 \eta^2 dv_g + c_\varepsilon \int_M f_{\alpha, \varepsilon}^2 \eta^2 dv_g \leq \]
\[ O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|^2_{L^2} + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha, \varepsilon} dv_g \right)^2 \]
and we conclude as in the previous case.

Then we have proved that

\[ \int_M u_\varepsilon^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g = O(\varepsilon^4 + \varepsilon^2 \|u_\varepsilon - c_\varepsilon\|^2_{L^2}). \]

To finish the proof, we write

\[ \int_M u_\varepsilon^2 f_{\alpha, \varepsilon}^2 dv_g = \int_M u_\varepsilon^2 f_{\alpha, \varepsilon}^2 \eta^2 dv_g + \int_M u_\varepsilon^2 f_{\alpha, \varepsilon}^2 (1 - \eta^2) dv_g \]

and the last term is \( O(\varepsilon^4) \) which completes the proof.

\( \square \)

**Proof of Relation (13).** First we apply the lemma 5.4 to \( c_\varepsilon = c_0 \) and we see that \( c_0 \neq 0. \) Indeed, let us compute the \( L^2 \)-norm of the gradient of \( u_\varepsilon. \)

\[ \int_M |\nabla u_\varepsilon|^2 dv_g = \int_M (\Delta u_\varepsilon) u_\varepsilon dv_g \leq \frac{8k}{\varepsilon^2} \int_M f_{\alpha, \varepsilon}^2 u_\varepsilon^2 dv_g
= \frac{8k}{\varepsilon^2} O(\varepsilon^2 \|u_\varepsilon - c_0\|^2_{L^2} + \varepsilon^4)
= o(1). \]

Then we deduce that up to a subsequence
\[ \int_M |\nabla u_\varepsilon|^2dv_g \rightarrow 0. \]

But we have chosen \( u_\varepsilon \) so that \( \|u_\varepsilon\|_{H^2}^2 = 1 \). Then \( \|u_\varepsilon\|_{L^2} \rightarrow 1 \) and \( c_0 \neq 0 \).

Now let us consider \( \overline{u_\varepsilon} = \frac{1}{\text{vol}(M)} \int_M u_\varepsilon dv_g \) and \( a_\varepsilon = \|u_\varepsilon - \overline{u_\varepsilon}\|_{H^2}^2 \). Then \( u_\varepsilon \rightarrow c_0 \) and \( a_\varepsilon \rightarrow 0 \). It follows that the function \( v_\varepsilon = \frac{u_\varepsilon - \overline{u_\varepsilon}}{a_\varepsilon} \) satisfies \( \|v_\varepsilon\|_{H^2}^2 = 1 \) and there exists \( v \in H^2 \) so that \( v_\varepsilon \rightarrow v \) weakly in \( H^2 \) and strongly in \( L^2 \).

To prove (13) we will consider two cases.

**First case:** Assume that up to a subsequence \( a_\varepsilon = O(\varepsilon) \).

We have

\[ \int_M (\Delta u_\varepsilon)^2 dv_g = \mu_1(g_\varepsilon)^2 \int_M f^{4}_{\alpha,\varepsilon} u_\varepsilon^2 dv_g \leq \mu_1(g_\varepsilon)^2 \int_M f^{2}_{\alpha,\varepsilon} u_\varepsilon^2 dv_g \leq \frac{64k}{\varepsilon^4} O(\varepsilon^2 \|u_\varepsilon - \overline{u_\varepsilon}\|_{L^2}^2 + \varepsilon^4) \leq \frac{64k}{\varepsilon^4} O(\varepsilon^2 a_\varepsilon^2 + \varepsilon^4) \leq M. \]

Then \( \|\Delta u_\varepsilon\|_{L^2}, \|\nabla u_\varepsilon\|_{L^2} \) and \( \|u_\varepsilon\|_{L^2} \) are bounded. It well known that the norms

\[ \|v\| = \|\Delta v\|_{L^2} + \|\nabla v\|_{L^2} + \|v\|_{L^2} \]

and \( \|v\|_{H^2} \) are equivalent (it is a direct consequence of Bochner formula). Hence, this implies that \( (u_\varepsilon)_\varepsilon \) is bounded in \( H^2 \) which is embedded in \( C^0 \). Then \( u_\varepsilon \rightarrow c_0 \) uniformly up to a subsequence. Since \( c_0 \neq 0 \) it follows that for \( \varepsilon \) small enough \( u_\varepsilon \) has a constant sign, which is not possible because \( u_\varepsilon \) is an eigenfunction in the metric \( g_\varepsilon \).

**Second case:** Assume that \( \varepsilon = a_\varepsilon o(1) \). In this case we have the

**Lemma 5.5.** \( v_\varepsilon \rightarrow c_1 \) in \( H^2 \) where \( c_1 \) is a constant.

**Proof.** The proof is similar to this of lemma 5.3. Indeed we consider \( \varphi \in C^\infty(M) \) and the function \( \eta_\rho \) defined in this previous proof. Then

\[ \int_M \langle \nabla v, \nabla \varphi \rangle = \int_M \langle \nabla v, \nabla (\eta_\rho \varphi) \rangle dv_g + \int_M \langle \nabla v, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g. \]

By the same arguments we have \( \int_M \langle \nabla v, \nabla (\eta_\rho \varphi) \rangle dv_g \rightarrow 0 \) when \( \rho \rightarrow 0 \). Moreover

\[ \left| \int_M \langle \nabla v, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right| = \lim_{\varepsilon \rightarrow 0} \left| \int_M \langle \nabla v_\varepsilon, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right| = \lim_{\varepsilon \rightarrow 0} \left| \int_M (\Delta v_\varepsilon)(1 - \eta_\rho) \varphi dv_g \right| = \lim_{\varepsilon \rightarrow 0} \frac{\mu_1(g_\varepsilon)}{a_\varepsilon} \int_M f^{2}_{\alpha,\varepsilon} v_\varepsilon(1 - \eta_\rho) \varphi dv_g. \]
Now let \( \varepsilon \) distributions and \( v \) Laplacian with respect to the metric \( g \). From the definition of \( a_{\varepsilon} \) and the definition of \( \mu(g) \), we have

\[
a_{\varepsilon}^2 \leq 2 \left( \int_M |\nabla u_{\varepsilon}|^2 dv_g + \int_M (u_{\varepsilon} - \pi_{\varepsilon})^2 dv_g \right) \leq 2 \left( 1 + \frac{1}{\mu(g)} \right) \int_M |\nabla u_{\varepsilon}|^2 dv_g
\]

\[
= 2 \left( 1 + \frac{1}{\mu(g)} \right) \int_M \Delta u_{\varepsilon} u_{\varepsilon} dv_g
\]

\[
= 2 \left( 1 + \frac{1}{\mu(g)} \right) \mu_1(g) \int_M f_{a_{\varepsilon}}^2 u_{\varepsilon}^2 dv_g. \tag{22}
\]

Applying lemma 5.4 we get

\[\int_M f_{a_{\varepsilon}}^2 u_{\varepsilon}^2 dv_g = O(\varepsilon^2 \|u_{\varepsilon} - c_{\varepsilon}\|_{L^2}^2 + \varepsilon^4)
\]

\[= O(\varepsilon^2 \|u_{\varepsilon} - \pi_{\varepsilon} - a_{\varepsilon} c_1\|_{L^2}^2 + \varepsilon^4)
\]

\[= O \left( a_{\varepsilon}^2 \varepsilon^2 \left\| \frac{u_{\varepsilon} - \pi_{\varepsilon}}{a_{\varepsilon}} - c_1 \right\|_{L^2}^2 + \varepsilon^4 \right)
\]

\[= O(\varepsilon^4) + o(a_{\varepsilon}^2 \varepsilon^2).
\]

Now reporting this in (22) with the estimate (14) we find

\[a_{\varepsilon}^2 \leq C \frac{8k}{\varepsilon^2} (O(\varepsilon^4) + o(a_{\varepsilon}^2 \varepsilon^2))
\]

\[= O(\varepsilon^2) + a_{\varepsilon}^2 o(1).
\]

But \( \varepsilon = a_{\varepsilon} o(1) \). Then \( a_{\varepsilon}^2 \leq C a_{\varepsilon}^2 o(1) \) and for \( \varepsilon \) small enough \( a_{\varepsilon} = 0 \) and \( u_{\varepsilon} \) is a constant which is impossible.

\[\square\]

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