ON EXISTENCE AND NONEXISTENCE OF NONNEGATIVE SOLUTIONS TO SEMILINEAR DIFFERENTIAL EQUATION ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we give a clear cut relation between the volume growth $V(r)$ and the existence of nonnegative solutions to parabolic semilinear problem

$$\begin{cases}
\Delta u - \partial_t u + u^p = 0, \\
u(x, 0) = u_0(x),
\end{cases} \quad (*)$$

on a large class of Riemannian manifolds. We prove that for parameter $p > 1$, if

$$\int_{+\infty}^{+\infty} \frac{t}{V(t)^{p-1}} dt = \infty,$$

then $(*)$ has no nonnegative solution. If

$$\int_{+\infty}^{+\infty} \frac{t}{V(t)^{p-1}} dt < \infty,$$

then $(*)$ has positive solutions for small $u_0$.

1. Introduction

Let $(M, g)$ be a noncompact complete connected $n$-dimensional Riemannian manifold and consider nonnegative solutions to differential equations

$$\begin{cases}
\Delta u - \partial_t u + u^p = 0 \quad \text{in } M \times (0, \infty), \\
u(x, 0) = u_0(x) \quad \text{in } M,
\end{cases} \quad (1)$$

where parameter $p > 1$, $\Delta$ is the Laplace-Betrami operator of $M$, and $u_0(x) \geq 0$ not identically zero.

Fujita in [5] proved the following results in the case of $M = \mathbb{R}^n$:

(a) if $1 < p < 1 + \frac{2}{n}$ and $u_0 > 0$, equation $(1)$ has no solution;

(b) if $p > 1 + \frac{2}{n}$ and $u_0 > 0$ is smaller than a small Gaussian, then equation $(1)$ possesses positive solutions.

Later, several authors [1] [6] [8] showed that the critical case for problem $(1)$, that is $p = 1 + \frac{2}{n}$, belongs to blow-up case (a).

In [23], Zhang proved several nonexistence results for semilinear and quasilinear parabolic problems on manifolds provided the following setting:

(i) $V_{x_0}(r) \leq Cr^\alpha$;

(ii) Condition (G) (see Definition [12]).

Especially, he proved that: Assume conditions (i) and (ii) on manifold are satisfied, and $\alpha \geq 1$. If $1 < p \leq 1 + \frac{2}{n}$, then problem $(1)$ possesses no global positive solution.

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The celebrated idea of studying the nonnegative solutions in terms of the volume of the geodesic ball was due to Cheng and Yau [4]. They proved that if the inequality
\[ V_{x_0}(r) \leq Cr^2 \]
holds for some point \( x_0 \in M \) with all large enough \( r \), then any positive solution to \( \Delta u \leq 0 \) is identically constant, here \( V_{x_0}(r) \) is the volume of geodesic ball of radius \( r \) centered at \( x_0 \in M \).

Recently, this idea was used and developed for solutions to both elliptic equations (see, e.g. [11, 12, 18, 19]) and parabolic equations (see [15]). Particularly in [15], Mastrolia, Monticelli and Punzo proved that if
\[ V_{x_0}(r) \leq Cr^\frac{2}{p-1}(\ln r)^\frac{1}{p-1}, \]
then (1) has no weak supersolutions.

**Definition 1.1.** We say the volume condition
\[ \int_{r_0}^{+\infty} \frac{t}{V(t)^{p-1}} dt = (or <) \infty \]
is satisfied if there exists constant \( r_0 > 0 \) and \( x_0 \in M \) such that
\[ \int_{r_0}^{+\infty} \frac{t}{V_{x_0}(t)^{p-1}} dt = (or <) \infty \]

Noting that the choice of \( r_0 \) and \( x_0 \) doesn’t affect this condition, we can omit them in (3) and specify proper value if needed.

It’s easily see that, if \( V_{x_0}(r) \) satisfy (2) or the following inequality
\[ V_{x_0}(r) \leq Cr^{\alpha_1}(\ln r)^{\alpha_2}(\ln \ln r)^{\alpha_3} \times \cdots \times (\ln \ln \cdots \ln r)^{\alpha_k}, \quad k \geq 3 \]
with parameter
\[ \alpha_1 = \frac{2}{p-1}, \quad \alpha_i = \frac{1}{p-1}, \quad (i = 2, 3, \cdots, k), \]
then \( V_{x_0}(r) \) satisfy
\[ \int_{r_0}^{+\infty} \frac{t}{V(t)^{p-1}} dt = \infty. \]

In this paper, we show that volume condition (4) is indeed the sufficient condition to the nonexistence of solutions to equation (1) on a large class of manifolds. Moreover, if we assume the initial value \( u_0 \) is small, then (4) is also necessary.

Let us introduce our setting. Unless specified, let \( M \) be a connected geodesically complete noncompact Riemannian manifold and satisfy condition (G) and (H).

**Definition 1.2** (Condition (G)). We say condition (G) is satisfied if there exists a constant \( C_0 \) such that
\[ \frac{\partial \log g^{1/2}}{\partial r} \leq \frac{C_0}{r}, \]
here \( g^{1/2} \) is the volume density of the manifold;

If the Ricci curvature of \( M \) is nonnegative, it is well known that condition (G) hold, in fact, \((\partial \log g^{1/2})/\partial r \leq 0\).
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**Definition 1.3** (Condition (H)). We say condition (H) is satisfied if \( P_t(x,y) \), the smallest fundamental solution of the classical heat equation
\[
\frac{\partial}{\partial t} u - \Delta u = 0, \quad \text{in } M
\]
exists and satisfies the following estimate
\[
P_t(x,y) \leq C \frac{1}{V_x(\sqrt{t})}, \quad (5)
\]
for all \( t > 0 \) and almost all \( x,y \in M \).

We know that \( P_t(x,y) \) is called heat kernel of \( \Delta \) and has the following properties

- Symmetry: \( P_t(x,y) = P_t(y,x) \), for all \( x,y \in M, t > 0 \).
- \( P_t(x,y) \geq 0 \), for all \( x, y \in M \) and \( t > 0 \), and
\[
\int_M P_t(x,y)dy \leq 1, \quad \text{for all } x \in M \text{ and } t > 0. \quad (6)
\]
- The semigroup identity: for all \( x, y \in M \) and \( t, s > 0 \),
\[
P_{t+s}(x,y) = \int_M P_t(x,z)P_s(z,y)dz. \quad (7)
\]

If the manifold has nonnegative Ricci curvature, by a famous result of Li and Yau in [13], we have
\[
P_t(x,y) \asymp C \frac{1}{V_x(\sqrt{t})} \exp \left( -\frac{d^2(x,y)}{ct} \right),
\]
where the sign \( \asymp \) means that both \( \leq \) and \( \geq \) are true but with different values of \( C, c \). It follows that (H) is satisfied if the manifold has nonnegative Ricci curvature.

The main results of the paper are the following.

**Theorem 1.4.** (a). If
\[
\int^{+\infty} \frac{t}{V(t)^{p-1}}dt = \infty
\]
then equation (1) has no nonnegative solutions.
(b). If
\[
\int^{+\infty} \frac{t}{V(t)^{p-1}}dt < \infty
\]
and \( u_0 \) is smaller than a small Gaussian, then (1) has positive solutions.

In the part (a) of Theorem 1.4 the “solution” can also be understood in a weak sense. Our proof are based on a method originating from [23] (also see [22]) and the proof of Theorem 11.14 in [10], that uses the Laplacian of the distance function and a carefully chosen test function. The arguments of part (b) are purely functional analytic.

We denote by \( W^2_c \) the subspace of \( W^2_{loc} \) of functions with compact support, and by \( B_x(r) \) the geodesic ball of radius \( r \) centered at \( x \in M \). \( V_x(r) \) is the volume of \( B_x(r) \). In all the proof, since the geodesic ball always centered at a fixed point \( x_0 \in M \), we may use the abbreviation \( B \equiv B_{x_0} \) and \( V \equiv V_{x_0} \). The symbol \( C, C_0, C_1, \cdots \) denote positive constants whose values are unimportant and may vary at different occurrences.
2. Proof of Theorem 1.4

Proof of part (a). Suppose $u$ is a solution of $u$ and $\psi \geq 0$ is a function in $W^2_c(M \times [0, \infty))$. Multiplying $\psi$ to both side of $u$ and integrating by part, we have

$$
\int_0^\infty \int_M u^p \psi dxdt = - \int_0^\infty \int_M u \Delta \psi dxdt - \int_0^\infty \int_M u \partial_t \psi dxdt - \int_M u_0 \psi dx
$$

We use the following test function:

$$
\psi(x, t) = \varphi^q(r(x), t)
$$

where $q = p/(p-1)$, and $r(x) = d(x_0, x)$ for some fixed point $x_0 \in M$. It follows that

$$
\Delta \psi = q \varphi^{q-1} \Delta \varphi + q(q-1) \varphi^{q-2} |\nabla \varphi|^2,
$$

$$
\partial_t \psi = q \varphi^{q-1} \partial_t \varphi
$$

Substituting above into (8) we have

$$
\int_0^\infty \int_M u^p \varphi^q dxdt \leq C \left( \int_M u \varphi^{q-1} (-\Delta \varphi) dx + \int_M u \varphi^{q-1} (-\partial_t \varphi) dx \right)
$$

Let $h \in W^2_c([0, \infty)$ be a function satisfying

$$
h(r) = 1, r \in [0, 1]; h(r) = 0, r \in [2, \infty); -C_1 \leq h' \leq 0, |h''| \leq C_1, r \in (1, 2).
$$

Fix a finite increasing sequence $\{r_k\}, \{k = 0, 1, \cdots, i\}$. Let $Q_k = B(r_k) \times [0, r_k^2)$. Define $\varphi$ by

$$
\varphi(x, t) = \begin{cases} 1, & (x, t) \in Q_0, \\
\alpha \frac{(r_k - r_{k-1})^2}{(V(r_k)r^l)} h\left(\frac{r(x)}{r_k}\right) h\left(\frac{t(x)}{T_k}\right) + T_k, & (x, t) \in Q_k \backslash Q_{k-1}, k = 1, \cdots, i \\
0, & (x, t) \in M \backslash Q_i
\end{cases}
$$

where

$$
\alpha = \left( \sum_{k=1}^i \frac{(r_k - r_{k-1})^2}{V(r_k)r^l} \right)^{-1}
$$

and

$$
T_k = \left\{ \begin{array}{ll}
\alpha \sum_{j=k+1}^i \frac{(r_j - r_{j-1})^2}{V(r_j)r^l}, & k = 1, \cdots, i - 1 \\
0, & k = i
\end{array} \right.
$$

Clearly $\varphi \in W^2_c(M \times [0, \infty))$. Noting $\varphi(\cdot, t)$ is radial on $M$, we have when $(x, t) \in Q_k \backslash Q_{k-1},$

$$
\Delta \varphi = \frac{\partial^2 \varphi(r, t)}{\partial r^2} + \left( \frac{n - 1}{r} + \frac{\partial \log g^{1/2}}{\partial r} \right) \frac{\partial \varphi(r, t)}{\partial r}
$$

Specifying $\{r_i\}$ to be a geometric sequence with $r_k = 2r_{k-1}$, we have

$$
-C \alpha \frac{(r_k - r_{k-1})}{V(r_k)r^l} \leq \frac{\partial \varphi(r, t)}{\partial r} \leq 0,
$$

where $a = \left( \sum_{k=1}^i \frac{(r_k - r_{k-1})^2}{V(r_k)r^l} \right)^{-1}.$
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\[ \left| \frac{\partial^2 \varphi(r, t)}{\partial r^2} \right| \leq \frac{Ca}{V(r_k)^{p-1}}. \]  \hspace{1cm} (12)

Combining (10), (11), (12) and condition (G), we have

\[-\Delta \varphi \leq \frac{Ca}{V(r_k)^{p-1}}, \quad (x, t) \in Q_k \setminus Q_k-1. \] \hspace{1cm} (13)

Also by the definition of \( \varphi \), we have

\[-\partial_t \varphi \leq \frac{Ca}{V(r_k)^{p-1}}, \quad (x, t) \in Q_k \setminus Q_k-1. \] \hspace{1cm} (14)

Substituting (13) and (14) into (9), we have

\[ \int_0^\infty \int_M u^p \varphi^q \, dx \, dt \leq C \int_0^\infty \int_M \sum_{k=1}^i \frac{u\varphi^{q-1}}{V(r_k)^{p-1}} \chi_{Q_k \setminus Q_k-1} \, dx \, dt \]

Using the H"older's inequality, we obtain

\[ \int_0^\infty \int_M u^p \varphi^q \, dx \, dt \leq C \left( \int_0^\infty \int_M u^p \varphi^q \, dx \, dt \right)^{1/p} \left( \int_0^\infty \int_M \left( \sum_{k=1}^i \frac{\chi_{Q_k \setminus Q_k-1}}{V(r_k)^{p-1}} \right)^q \, dx \, dt \right)^{1/q} \]

\[ = C \left( \int_0^\infty \int_M u^p \varphi^q \, dx \, dt \right)^{1/p} \left( \sum_{k=1}^i \int_{Q_k \setminus Q_{k-1}} \frac{1}{V(r_k)^{p-1}} \, dx \, dt \right)^{1/q} \]

It follows that

\[ \left( \int_0^\infty \int_M u^p \varphi^q \, dx \, dt \right)^{(p-1)/p} \leq C \left( \sum_{k=1}^i \int_{Q_k \setminus Q_{k-1}} \frac{1}{V(r_k)^{p-1}} \, dx \, dt \right)^{1/q} \]

\[ \leq C \left( \sum_{k=1}^i \frac{p_k^2}{V(r_k)^{p-1}} \right)^{1/q} \]

\[ \leq C \left( \sum_{k=1}^i \frac{(r_k - r_{k-1})^2}{V(r_k)^{p-1}} \right)^{1/q} \]

\[ = C_0 (q-1)/q \] \hspace{1cm} (15)
On the other hands

\[ a^{-1} = \sum_{k=1}^{i} \frac{(r_k - r_{k-1})^2}{V(r_k)^{p-1}} = C \sum_{k=1}^{i} \frac{r_k^2 - r_{k-1}^2}{V(r_k)^{p-1}} \geq C \sum_{k=2}^{i} \int_{r_{k-1}}^{r_k} \frac{r}{V(r)^{p-1}} \, dx = C \int_{2r_0}^{r_1} \frac{r}{V(r)^{p-1}} \, dx \quad (16) \]

Using the volume condition

\[ \int_{r_0}^\infty \frac{r}{V(r)^{p-1}} \, dx = \infty, \]

for every \( r_0 > 0 \), we have form (16), if \( i \to +\infty \), then \( a \to 0 \). Thus we can choose sufficient large \( i \), such that

\[ a \leq \frac{1}{r_0} \]

Substituting above into (15), we have

\[ \int_0^{r_0} \int_{B(r_0)} u^p \, dx \, dt \leq \frac{C}{r_0^{p-1}} \]

Let \( r_0 \to +\infty \), we obtain

\[ \int_0^{\infty} \int_{M} u^p \, dx \, dt \to 0 \]

which implies that

\[ u \equiv 0. \]

But \( u_0 \geq 0 \) is not identically zero, by Maximum principle, we know that \( u \) is positive almost everywhere, which leads a contradiction, and then there’s no nonnegative solution to problem (1).

**Proof of part (b).** Fix a point \( x_0 \in M \). Let us define the operator

\[ Tu(x, t) = \int_{M} P_t(x, y)u_0(y) \, dy + \int_0^t \int_{M} P_{t-s}(x, y)u^p(y, s) \, dx(y) \, ds. \quad (17) \]

acting on the space \( S_M \) defined by

\[ S_M = \{ u \in L^\infty(M \times [0, \infty)) | 0 \leq u(x, t) \leq \lambda P_{t+\delta}(x, x_0) \} . \quad (18) \]

where \( \lambda > 0 \) is a constant to be chosen later, and \( \delta > 1 \) is a fixed constant. It is easy to see that \( S_M \) is a close set of \( L^\infty(M \times [0, \infty)) \).

Let \( u_0 \) satisfy

\[ 0 \leq u_0(x) \leq \frac{\lambda}{2} P_{\delta}(x, x_0). \quad (19) \]
Now let us show \( TS_M \subset S_M \). Using (7) and (19), we have

\[
\int_M P_t(x, y)u_0(y)dx(y) \leq \frac{\lambda}{2} \int_M P_t(x, y)P_\delta(y, x_0)dx(y) = \frac{\lambda}{2}P_{t+\delta}(x, x_0).
\]  

(20)

Using (5), (7) and (18), we have

\[
\int_0^t \int_M P_{t-s}(x, y)u^p(y, s)dx(y)ds \leq \lambda^p \int_0^t \int_M P_{t-s}(x, y)P^p_{s+\delta}(y, x_0)dx(y)ds
\]

\[
\leq \lambda^p C_3^{p-1} \int_0^t \frac{1}{V(\sqrt{s+\delta})^p-1} \int_M P_{t-s}(x, y)P_{s+\delta}(y, x_0)dx(y)ds
\]

\[
\leq \lambda^p C_3^{p-1}P_{t+\delta}(x, x_0) \int_0^t \frac{1}{V(\sqrt{s+\delta})^p-1}ds
\]

(21)

By our volume condition

\[
\int_0^\infty \frac{s}{V(s)^{p-1}}ds < \infty,
\]

there exists a constant \( C_4 \) such that

\[
\int_0^t \frac{1}{V(\sqrt{s+\delta})^p-1}ds = \int_\delta^{t+\delta} \frac{1}{V(\sqrt{s})^p-1}ds
\]

\[
= C \int_\delta^{\sqrt{t+\delta}} \frac{s}{V(s)^{p-1}}ds
\]

\[
\leq C_4
\]

(22)

Substituting above into (21), we obtain, for small enough \( \lambda \),

\[
\int_0^t \int_M P_{t-s}(x, y)u^p(y, s)dx(y)ds \leq \lambda^p C_3^{p-1}C_4P_{t+\delta}(x, x_0)
\]

\[
\leq \frac{\lambda}{2}P_{t+\delta}(x, x_0).
\]

(23)

Combining (17), (20) with (23), we obtain

\[
0 \leq Tu \leq \lambda P_{t+\delta}(x, x_0).
\]

Hence

\( TS_M \subset S_M \).

Now let us show that \( T \) is a contraction map. For \( u_1, u_2 \in S_M \), we have

\[
|Tu_1(x, t) - Tu_2(x, t)| \leq \int_0^t \int_M P_{t-s}(x, y)|u_1^p(y, s) - u_2^p(y, s)| dx(y)ds
\]

(24)

Noting that

\[
|u_1^p(y, s) - u_2^p(y, s)| \leq p \max\{u_1^{p-1}(y, s), u_2^{p-1}(y, s)\} |u_1(y, s) - u_2(y, s)|
\]

\[
0 \leq Tu_1 - Tu_2 \leq \lambda |u_1 - u_2|.
\]
and combining with (5), (6), (18) and (22), we obtain that

\[
|Tu_1(x, t) - Tu_2(x, t)| \leq p\lambda^{p-1} \|u_1 - u_2\|_\infty \int_0^t \int_M P_{t-s}(x, y) \frac{1}{V(\sqrt{s + \delta})^{p-1}} dy ds
\]

\[
\leq p\lambda^{p-1} C_1^{p-1} \|u_1 - u_2\|_\infty \int_0^t \int_M \frac{1}{V(\sqrt{s + \delta})^{p-1}} ds
\]

\[
= p\lambda^{p-1} C_1^{p-1} \|u_1 - u_2\|_\infty \int_0^t \int_M P_{t-s}(x, y) dy ds
\]

\[
\leq p\lambda^{p-1} C_4 \|u_1 - u_2\|_\infty
\]

Redoing \( \lambda \) small enough such that \( p\lambda^{p-1} C_4^{p-1} < 1 \), we obtain that \( T \) is a contraction map. Applying fixed point theorem, we derive that there exists a fixed point \( u \in S_M \) satisfying

\[
u(x, t) = \int_M P_t(x, y) u_0(y) dy + \int_0^t \int_M P_{t-s}(x, y) u^p(y, s) dy ds. \quad (25)
\]

Since \( u_0 \geq 0 \), then \( u \) is positive on \( M \). Since \( u_0, u \in L^2(M) \), by a standard argument of regularity (see Theorem 7.6 and 7.7, [10]), we know the integrals in (25) are both smooth on \( M \times (0, \infty) \), hence we obtain that \( u \) is a global positive solution of problem (1).

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