An efficient quantum algorithm for lattice problems achieving subexponential approximation factor

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Abstract

We give a quantum algorithm for solving the Bounded Distance Decoding (BDD) problem with a subexponential approximation factor on a class of integer lattices. The quantum algorithm uses a well-known but challenging-to-use quantum state on lattices as a type of approximate quantum eigenvector to randomly self-reduce the BDD instance to a random BDD instance which is solvable classically. The running time of the quantum algorithm is polynomial for one range of approximation factors and subexponential time for a second range of approximation factors.

The subclass of lattices we study has a natural description in terms of the lattice’s periodicity and finite abelian group rank. This view makes for a clean quantum algorithm in terms of finite abelian groups, uses very relatively little from lattice theory, and suggests exploring approximation algorithms for lattice problems in parameters other than dimension alone.

A talk on this paper sparked many lively discussions and resulted in a new classical algorithm matching part of our result. We leave it as a challenge to give a classical algorithm matching the general case.

1 Introduction

We give an efficient quantum algorithm for a special case of the closest lattice vector problem in a new range of approximation factors, namely the subexponential range. In this type of problem a basis $B \in \mathbb{Z}^{n \times n}$ and a target vector $t \in \mathbb{Z}^n$ are given and the goal is to compute the closest lattice vector $Bc$ to $t$ for integer coefficients. This paper is about the important special case called Bounded Distance Decoding (BDD) with parameter $\varepsilon_1$. It has the extra promise that the distance is bounded in the sense that there exists $Bc \in L$ such that $\|Bc - t\| < \varepsilon_1 \lambda_1$, where $\lambda_1$ is the shortest nonzero vector length in $L$, and $\varepsilon_1 \leq 1/2$, making the answer unique. The term $\varepsilon_1$ is the approximaton factor and it is typically a function of $n$, the lattice dimension. A lattice can be specified by an infinite number of bases, making the problem difficult.

There are three broad ranges of approximation factors where lattice problems are unlikely to be NP-complete. Starting at the right end, the exponential range typically has the form...
$\varepsilon_1^{-1} = 2^n$ and was solved efficiently in the 80’s. Lentra, Lenstra, and Lovász [LLL82] gave an algorithm to compute an approximate shortest lattice vector, Babai [Bab86] gave algorithms for computing an approximate closest lattice vector to a target point, Kannan [Kan87] gave an exponential time enumeration algorithm, and Schnorr [Sch87, Sch94] extended these three by trading off running time in exchange for better approximation factors.

Adjacent to the exponential range is the subexponential range, which is the focus of this paper. Despite several decades of big advances in lattices, this region appears to either be very difficult, or has been neglected. In special-case lattices that allow more more efficient cryptography because of additional algebraic structure, a sequence of papers work led to an efficient quantum algorithm for approximating the shortest vector [EHKS14, BS16, CGS14, CDPR16]. Lattices with small determinant have also been examined [CL15]. The subexponential region has played a crucial role in recent advances in fully homomorphic encryption (FHE) [BV11a, BV11b, GSW13], where it is assumed that certain parameters cannot be solved efficiently.

At the hardest end of the range for algorithms and cryptography are polynomial approximation factors and very important questions about how well existing algorithms work and can be optimized for concrete security for the NIST standardization process.

Cryptography built on the LWE problem, which is as hard as worst-case lattice problems for good theoretical security, also allows many new primitives such as FHE and testing whether or not machines are quantum [BCM+18, Mah18b], and quantum FHE [Mah18a].

Therefore, the most important and pressing question is whether or not efficient algorithms exist for the polynomial approximation factor range. The realistic approach is to start with problems in the range adjacent to the ones that are already solvable, which is the subexponential range. Even if an efficient algorithm exists for the polynomial range it may be too difficult to find in one step, but the hope is that the techniques here will be applicable to a broader range of cases.

We propose a partition of lattices into finer blocks than just by dimension alone. The partition consists of sets of lattices $\mathcal{L}(n, q, r)$ indexed by lattice dimension $n$, periodicity $q$, and finite group rank $r$. The periodicity $q$ of a lattice is the minimum integer $q$ such that lattice contains the subgroup $q \cdot \mathbb{Z}^n \subseteq L$. Therefore $L \mod q$ is a finite abelian subgroup of $\mathbb{Z}_q^n$ and can be decomposed as $\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_r}$.

Through this lens we give a quantum algorithm on a subset of lattices with parameter $r \log q$ achieving a subexponential approximation factor of the form $2^{-\sqrt{r \log q}}$ and running in time $\text{poly}(n, \log q)$ (Theorem 21). To give a simplified comparison to existing algorithms analyzed in terms of dimension, for example, lattices with $\sqrt{r \log q} < n$ we get an improvement
over Babai’s algorithm, and BDD on lattices with finite group rank \( r = n^{1/4} \) and periodicity \( q = 2^{\sqrt{n}} \) can be solved for approximation factor \( 2^{-n^{3/8}} \). The periodicity \( q \) cannot be smaller than \( r \) or BDD becomes trivial. More generally, the new parameter range can be seen in the figures for \( r = 1 \) and general \( r \). Changing \( r \) changes the trivial region. The left axis has the parameter \( r \log q \) and the bottom axis plots the log of the approximation factor.

Moving beyond polynomial time, Schnorr’s hierarchy can also be applied to the output of the quantum algorithm as a black box to achieve an approximation factor of \( \exp(-r \log q \log \beta) \) in time \( \beta \cdot \text{poly}(n, \log q) \). For example, with \( \varepsilon \leq 1/2 \), there is a quantum algorithm achieving approximation factor \( 2^{n^\varepsilon \log n} \) in time \( 2^{n^{1-2\varepsilon}} \), while applying Schnorr to the original lattice gives time \( 2^{n^{1-\varepsilon}} \), i.e., with an extra \( n^\varepsilon \) in the time exponent.

Idea of proof. In the context of the partition, we reduce BDD on worst-case lattices in \( \mathcal{L}(n, q, r) \), to a problem we’ll call \( \tilde{\varepsilon}_1 \)-random-BDD where a random matrix \( \tilde{B} \in [q]^{m \times r} \) is chosen and \( \tilde{L} \in \mathcal{L}(m, q, r) \) has as generators the columns of \([B|q \cdot I]\), and a target vector \( \tilde{t} \) is given with distance at most \( \varepsilon_1 \lambda_1 \) to \( \tilde{L} \). From lattice theory we need that these types of lattices have a long shortest vector with high probability [Mica]. The problem can be solved for approximation factor \( \varepsilon_1 = 2^{-\sqrt{r \log q}} \) when the dimension is \( m = \sqrt{r \log q} \) by running LLL and Babai’s algorithm. Different from the worst-case BDD problem, the random-BDD problem can be solved for higher dimensions too because deleting rows until \( \sqrt{r \log q} \) is reached is still a random instance.

To randomly reduce the BDD instance to a random-BDD instance in a lower dimension, we revisit an old state with a “phase problem” and find a way to use it. Given a lattice basis and a radius, a goal for the last 20+ years has been to compute the quantum state \( |\psi_0\rangle = \sum_{v \in \mathcal{L}} \sum_{z \in C} |v + z\rangle \), where \( C \) is some shape with the prescribed radius such as a cube, Gaussian, or sphere (also see [AR05]). We use a cube for simplicity.

The main approach for computing \( |\psi_0\rangle \) is to compute a superposition over coefficients \( c \) in the first register and over the cube in the second register to get \( \sum_c |c\rangle \otimes \sum_{z \in C} |z\rangle \), then to entangle the registers by adding the corresponding lattice vector into the second register to get \( \sum_c \sum_{z \in C} |c, Bc + z\rangle \). The last step would be to “uncompute” in the first register and this is where the state becomes difficult to use. If an algorithm could solve BDD at this point then it could use \( Bc + z \) to compute \( c \) and uncompute the first register, but this is circular.

This challenge of uncomputing a register appears in lattice problems, graph isomorphism approaches, as well as other problems [ATS07]. Nevertheless, Regev [Reg09] found a way to use this in a constructive way by using an LWE oracle to erase the coefficient, and as a result
of his construction, reduce worst-case lattice problems to LWE. The more typical approach for algorithms might be to compute the quantum Fourier transform of the first register, because it is possible, and measure it. The resulting state is $|\psi_a\rangle = \sum_c \omega_q^{c a} \sum_{z \in \mathbb{C}} |Bc + z\rangle$, for a random vector $a$. This changes the “uncomputing” problem into a “phase” problem because $|\psi_a\rangle$ is similar to $|\psi_0\rangle$, the desired $a = 0$ case, but has the phase $\omega_q^{c a}$ mixed in in a problematic way. The difficulty is that it is no longer clear how to use $|\psi_a\rangle$. In [BKSW18] states related to this but with Gaussians were used to show an equivalence between LWE and an extension of a certain nonabelian hidden subgroup problem. In [CLZ21] the Arora-Ge algorithm [AG10] for LWE was used to uncompute projections.

**Overcoming the difficulty.** In this paper we revisit the state $|\psi_a\rangle = \sum_c \omega_q^{c a} \sum_{z \in \mathbb{C}} |Bc + z\rangle$ which we call a Phased Cube State (PCS), because there is a cube around each lattice point, and each cube has single phase across it. The value $q$ is chosen as the periodicity of the lattice, and $L \mod q$ is a subgroup $\mathbb{Z}_n^q$ where computations are done. This state can be efficiently created for any side length, and with a uniformly random and known $a$. The main idea is to see that $|\psi_a\rangle$ is almost an eigenvector of shifts by vectors close to the lattice, as in BDD instances, and that quantum phase estimation can compute an approximation of that information. More specifically, for $x \in \mathbb{Z}_n^q$, let $U_x|y\rangle = |y + x\rangle$ be a shift operator inside $\mathbb{Z}_n^q$. Then for $Bc \in L$, $U_{Bc}|\psi_a\rangle = \omega^{a c}_q|\psi_a\rangle$. If desired, the quantum phase estimation algorithm can be used to compute the inner product $a \cdot c$, and repeating the process results in these inner products for different random $a$, and the coefficients $c$ can be computed.

For a BDD instance with $t = Bc + \Delta$ and $\Delta$ controlled by the BDD promise, shifting by $t$ results in $U_t|\psi_a\rangle = \omega^{c a}_q U_\Delta|\psi_a\rangle \approx \omega^{c a}_q|\psi_a\rangle$ and we show that quantum phase estimation still returns an approximation of $c \cdot a$. Quantum phase estimation exponentiates the operator to the power of the precision requested, and because of degradation of $|\psi_a\rangle$ for higher powers, because $U_t^k|\psi_a\rangle = \omega^{k a s}_q U_k \Delta|\psi_a\rangle$, this limits how much information can be extracted. To accommodate the set of possible BDD target vectors we define the notation of having a set of operators together with a single approximate eigenvector. With the worst-case lattice problem BDD as input, this quantum subroutine is used to sample noisy inner products and construct a random BDD instance in a lower solvable dimension.

To summarize, we give a new quantum algorithm solving BDD on a range of subexponential approximation factors.

### 1.1 Comparison to classical algorithms and open problems

The case of $r = 1$ and exponential $q$ includes the well-studied Hidden Number Problem [BV96, Aka09] which appears to need structure beyond the worst-case to solve until this work.

**Remark.** A talk given on this work in September 2021 at the Simons Institute for the Theory of Computing sparked much productive discussion. In particular, the paper [DeW21] was posted with a classical algorithm solving a part of what we solve that had not appeared in the literature before. After going through existing literature, having discussions with the community, and waiting for responses, we leave it as a challenge to provide a classical algorithm matching the subexponential-time quantum algorithm.

For the polynomial-time range, we show...
Theorem 21. There is a poly$(n, \log q)$-time quantum algorithm solving $2^{-\Omega(\sqrt{r \log q})}$-BDD on lattices of dimension $n$, periodicity $q$, and finite group rank $r$.

This inspired the posting of a classical algorithm for this problem. In that paper [DvW21], the $r = 1$ case matches Theorem 20, but details are left out about how to match the $r > 1$ case. In Section 7 we complete the analysis and also generalize it to rectangle-periodic lattices, rather than just cubes. Our quantum algorithm already handles this type of lattice because it works for any finite group, so classical and quantum have the same performance, even though the ideas are completely different. The paper does not address our next theorem, which is exponentially faster than the best available classical algorithm:

Theorem 23. Let $L(B)$ be an $n$-dimensional $q$-periodic lattice with finite group rank $r$. Given an instance of $\varepsilon_1$-BDD $(B, t)$, with $2 \leq \beta \leq r \log q$, and

$$\varepsilon_1 = \left( \exp \left( -4 \sqrt{\frac{r \log q \log \beta}{\beta}} \right) \cdot 2^2 m_{p20}(n, \log q)^2 \right).$$

Algorithm 22 runs in quantum time $\approx \beta^2 \text{poly}(n, \log(q))$ and returns the closest vector to $t$ with probability at least $0.9$.

One other suggestion for a classical approach for this problem was using [GMPW20, Theorem 5.3]. It is not clear how the details would work, and in particular, how to handle the non-primitive case.

It is still open if the Schnorr trade-off we give in Section 6 can be done classically. These questions are out-of-scope for this paper and we leave them as open problems. The parameter $k \log q$ for a matrix of dimension $k \times n$ has appeared as a boundary for LWE where algorithms more carefully use the Gaussian error [MR09, LP11, BLP13, BCM+18].

There are many open problems and possible extensions. The most interesting is to try variations of the quantum algorithm. It is relatively clean and is easy to experiment with. For example, solving approximate arithmetic progressions is a possibility. Another is analyzing different worst-case lattice problems such as uSVP, using reductions between instances of LWE, which is a type of random BDD where the errors for each coordinate are i.i.d., to map between different dimensions and $q$ values, and use groups [BLP13, GINX16]. These new BDD algorithms can also be used to sample vectors of length $q/(\varepsilon_1 \lambda_1) \geq \eta(L^\perp)2^{\sqrt{r \log q}}$ in $L^\perp$. This can be done via quantum ([Reg09, Theorem 1.3]) or classical sampling ([GPV08]).

2 Background

2.1 Lattices, finite abelian groups, and distances

Every integer lattice in $\mathbb{Z}^n$ has minimum $q$ such that $q\mathbb{Z}^n \subseteq L$, and $L$ is called $q$-periodic, or $q$-ary. Because $q\mathbb{Z}^n$ is a subgroup of $L$, $L \mod q := L/q\mathbb{Z}^n$ has all information about the lattice in the sense that distances are preserved mod $q$ and $L = (L \mod \{q\}) + q\mathbb{Z}^n$. Computing the closest vector to a lattice over $\mathbb{Z}^n$ can be reduced to this case by reducing the lattice and vector mod $q$, solving the problem in $L \mod q$, and then mapping back to the integer solution of the original problem. Starting from the finite group the associated lattice is constructed by reintroducing $q\mathbb{Z}^n$, so the columns of $[B, q \cdot I]$ generate $L$. 
A finite abelian group can be decomposed as $L \mod q \cong \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k}$. The representation is the finite group decomposition $(G, q, r) \in \mathbb{Z}_q^{nxr} \times \mathbb{Z}^r$, where the columns of $G \in \mathbb{Z}_q^{nxr}$ span $G := L \mod q \leq \mathbb{Z}^n_q$, the vector $q = (q_1, \ldots, q_r) \in \mathbb{Z}^r$ gives the orders of each column in $G$ in the decomposition, and $r$ makes the rank visible in the notation. The specific decomposition can be chosen but in this case the unique one will be used where $q_i | q_{i+1}$. The set $\tilde{\mathcal{C}} := [q_1] \times \cdots \times [q_r]$ will be viewed a the set of coefficients of the group elements $v \in G$, where each has a unique coefficient vector $c \in \tilde{\mathcal{C}}$ such that $v = Gc$.

The setup is similar to the matrix $A \in \mathbb{Z}_q^{nxm}$ used in lattice-cryptography, but not exactly the same. In the worst-case to average-case reduction the input lattice $L$ has dimension $n$ and $q$ is arbitrary subject to sampling in the dual. The matrix $A$ is typically chosen randomly and as a result has finite group decomposition $\mathbb{Z}_q^n$ with high probability. Here we also are using $q$-ary lattices, but we start with a worst-case lattice $L$, use the specific periodicity $q$ of $L$, and decompose it mod $q$.

An arbitrary full-dimensional integer lattice $L \subseteq \mathbb{Z}^n$ is a $q$-periodic lattice for $q = \text{det}(L)$. This can be seen from the fact that $qB^{-1} \in \mathbb{Z}^{nxn}$, because using Cramer’s rule for inverting $B$ results in each entry having det$(L)$ in the denominator an integer in the numerator. Then using the integer vectors from the columns of $qB^{-1}$ takes $B$ to $qI = B(qB^{-1})$, which is $0 \mod q$. The parameters set this way may not always work in the quantum algorithms, for example, when det$(L)$ is too large relative to $n$.

Given a lattice $L \subseteq \mathbb{Z}^n$ the finite group decomposition can be efficiently computed. The quantum Fourier transform over the cyclic group $\mathbb{Z}_q$ maps $|c\rangle$ to $\frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} e^{2\pi i ac/q} |d\rangle$. In general for a finite group $G$ the Fourier transform maps vectors over the group to vectors over the character group $\hat{G} = \{\chi_a : a \in G\}$.

There will be a reindexing step for the eigenvector/eigenvalue calculation when a register holding a superposition of coefficients $\sum_{c \in \mathcal{C}} |c\rangle$ is transformed by the Fourier transform over $\mathbb{Z}_q^n$. This uses the subgroup embedding of $\mathcal{C}$ into $\mathcal{Z}_q^n$. Concretely this means that $(c_1, \ldots, c_r) \in \mathcal{C}$ maps to the element $(\frac{c_1}{q}, \ldots, \frac{c_r}{q}) \in \mathbb{Z}_q^n$, and $\chi_a(c)$ has phase $\frac{\sum_{i=1}^{r} \frac{c_i}{q}}{\sum_{i=1}^{r} \frac{a}{q}} = \frac{\sum_{i=1}^{r} c_i}{\sum_{i=1}^{r} a_i}$. Then $\sum_{c \in \mathcal{C}} \chi_a(c)|G(c + d)\rangle = \sum_{c \in \mathcal{C}} \chi_a(e - d)|Ge\rangle = \chi_a(-d) \sum_{c \in \mathcal{C}} \chi_a(e) |Ge\rangle$.

A distance on $\mathbb{Z}_q^n$ will be needed to define and solve BDD on subgroups $G$ of $\mathbb{Z}_q^n$, and also for the phase estimation statement. Following Cassels [Cas97] specialized to finite groups, with $\Lambda = q\mathbb{Z}^n$, the modular distance on the quotient $\mathbb{Z}_q^n = \mathbb{Z}^n / \Lambda = \mathbb{Z}^n / (q\mathbb{Z}^n)$ is defined from the Euclidean distance on $\mathbb{Z}^n$ by $\|y\|_q = \min_{a \in y + q\mathbb{Z}^n} \|a\|$. For any $y \in Z_q^n$, $\| \cdot \|_q$ satisfies (1) $\|kx\|_q \leq k\|x\|_q$ for integers $k \geq 0$, (2) $\|y + z\|_q \leq \|y\|_q + \|z\|_q$ for $z \in \mathbb{Z}_q^n$, and there exists $a \in y + q\mathbb{Z}^n$ such that $\|y\|_q = \|a\|$. In one dimension we may write $|y|_q$ for $\|y\|_q$. The distance between points in $\mathbb{Z}_q^n$ matches the Euclidean distance as long as it is at most $q/2$. This definition also allows any choice of coset representatives for $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. It is equal to the zero-centered set for $\mathbb{Z}_q$ where the class $y \mod q$ is represented by an integer $x$ so that $-q/2 \leq x \leq q/2$, then it holds that $x \in \mathbb{Z}_q^n$, $\|x\|_q = \|x\|$ when $\|x\| \leq q/2$, and $\|x\|_q = q - \|x\|$ when $\|x\| \geq q/2$.

For phase estimation we will also use a distance mod 1. In this case take the Euclidean distance on $\mathbb{R}$ and define the distance on $\mathbb{R}/\mathbb{Z}$ by $|y|_1 = \min_{a \in y + \mathbb{Z}} |a|$. This has the same properties listed above, but to 1/2 instead of $q/2$. 

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Given a subgroup $G$ of $\mathbb{Z}_q^n$, define a shortest (nonzero) element length to be

$$\lambda_1(G) = \min_{v \in G \setminus \{0\}} \|v\|_q.$$  

Also define $\text{dist}_q(y, G) = \min_{v \in G} \|y - v\|_q$ for any $y \in \mathbb{Z}_q^n$. Note that all group elements have length at most $q/2$. In particular, for the trivial case when $\Lambda_q = q\mathbb{Z}^n$, $\lambda_1(\Lambda_q) = 0$.

The main tool with $q$-ary lattices in dimension $m$ is that for a randomly chosen one, the shortest vector length is known within a constant with high probability. The following can be found in [Mica].

**Claim 1.** There exists a constant $\delta > 0$ such that if $\tilde{G}$ is a uniformly chosen matrix from $\mathbb{Z}_q^{m \times r}$, and let $L(\tilde{G})$ denote the corresponding $q$-ary lattice. Then

1. $\Pr_{\tilde{G}}(\lambda_1(L(\tilde{G}))) < \delta \sqrt{mq^{1-r/m}} \leq 1/2^m$,
2. $\Pr_{\tilde{G}}(\mathbb{Z}_q^m \tilde{G} = \mathbb{Z}_q^n) \geq 1 - 1/q^{m-r}$.

To distinguish the underlying operations, ranges for integers will be denoted by $[q] := \{0, 1, \ldots, q-1\}$. In this case, for example, addition and multiplication of numbers from $[q]$ are over $\mathbb{Z}$. If an element $g \in \mathbb{Z}_q^n$, for example, then addition is mod $q$. An integer times a group element represents the number of operations to perform, for example, for $g \in \mathbb{Z}_q^n$, $3 \in [q]$, $3g = g + g + g$.

The following basis reduction algorithm will be used:

**Lemma 2** ([MG02, Lemma 7.1]). There is a polynomial time algorithm that on input a lattice basis $B$ and linearly independent lattice vectors $S \subset L(B)$ such that $\|s_1\| \leq \|s_2\| \leq \cdots \leq \|s_n\|$, outputs a basis $R$ equivalent to $B$ such that $\|r_k\| \leq \max\{(\sqrt{k}/2)\|s_k\|, \|s_k\|\}$ for all $k = 1, \ldots, n$. Moreover, the new basis satisfies $\text{span}(r_1, \ldots, r_k) = \text{span}(s_1, \ldots, s_k)$ and $\|r_k\| \leq \|s_k\|$ for all $k = 1, \ldots, n$.

**Definition 3** ($\varepsilon_1$-BDD). Given a lattice $L \subseteq \mathbb{Z}^n$ and a vector $t$ such that $\text{dist}(t, L) < \varepsilon_1 \lambda_1$, with $\varepsilon_1 \leq 1/2$, output the closest vector.

The nearest plane algorithm due to Babai is an algorithm that given $L \subseteq \mathbb{Z}^n$, and $t \in \mathbb{Z}^n$ returns a vector $v \in L$ such that $\|v - t\| \leq \text{dist}(t, L) \cdot 2^{n/2}$. BDD can be solved with this algorithm when $2^{n/2}\text{dist}(t, L) \leq \lambda_1/2$ because the answer is unique. The BDD problem can be solved with an approximation factor/time tradeoff with an approximate CVP algorithm based on the following two theorems.

**Theorem 4** ([Sch94, Theorem 8]). Let $b_1, \ldots, b_m \in \mathbb{R}^n$ be a $\beta$-reduced basis and let $x = \sum_{i=1}^m x_i b_i^\ast$. Suppose that $\|b_k^\ast\| = \max(\|b_{k+1}^\ast\|, \ldots, \|b_m^\ast\|)$, $m - \beta + 1 \leq k \leq m$. Let $v = \sum_{i=1}^m v_i b_i$ be a lattice point such that $\sum_{i=1}^m v_i|x_j - \sum_{i=1}^m v_i \mu_{i,j}|^2\|b_i^\ast\|^2$ is minimal for all $v_1, \ldots, v_m \in \mathbb{Z}$, and $|x_j - \sum_{i=1}^m v_i \mu_{i,j}| \leq 1/2$ for $j = k - 1, \ldots, 1$, then $\|t - v\|^2 \leq m2^{(m-1)/(\beta-1)} \min_{u \in L} \|t - u\|^2$.

**Theorem 5** ($\beta^{n/\beta}$-Approximate CVP in time $\beta^3 \log B$). There is an algorithm that on input a CVP instance $(L, t)$ for an $m$-dimensional lattice and a vector $t$ in the span of $L$, returns a vector $v$ such that

$$\|t - v\| \leq \sqrt{m} \beta^{(m-1)/(\beta-1)} \min_{u \in L} \|t - u\|.$$
The running time is \( O(nm(\beta^{O(\beta)} + m^2) \log B) \), where \( B \) is the maximal length of the given basis vectors.

**Proof.** To compute the approximate closest vector, following Page 516 of [Sch94], use [Sch87] to compute an “approximate” \( \beta \)-reduced basis as in Theorem 4 using \( O(nm(\beta^{O(\beta)} + m^2) \log B) \) steps, then use Kannan’s algorithm to compute the closest vector using enumeration, and then use Theorem 4 for the bound. By the statement on page 511, \( \gamma_{\beta} \leq (2/3)\beta \) for \( m \geq 2. \)

### 2.2 Quantum computation

For a positive integer \( q \), let \( F_q \) denote the Fourier transform over \( \mathbb{Z}_q \). On a basis state with \( 0 \leq x < q \), this operation maps \( |x\rangle \mapsto 1/\sqrt{q} \sum_{i=0}^{q-1} \omega_q^{ix} |i\rangle \) and can be computed in time \( \text{poly}(\log(q)) \). The Fourier transform over a direct product \( \mathbb{Z}_q \times \mathbb{Z}_r \) is \( F_q \otimes F_r = (F_q \otimes I)(I \otimes F_r) \), and can be computed on one register at a time.

**Claim 6.** For quantum states \( |\phi\rangle \) and \( |\psi\rangle \), \( |||\phi\rangle - |\psi\rangle|| = \sqrt{2(1 - \text{Re}(\langle \phi | \psi \rangle))} \).

**Lemma 7 ([BV97, Lemma 3.2.6]).** For quantum states \( |\phi\rangle \) and \( |\phi'\rangle \), if \( |||\phi\rangle - |\phi'\rangle|| \leq \epsilon \), then the total variation distance between the probability distributions resulting from measurements of the two states is at most \( 4\epsilon \).

For superpositions the representatives \( \mathbb{Z}_q = \{0, \ldots, q - 1\} \) will be used. It is convenient because of the typical quantum Fourier transform definition. Note that the norm \( \|\cdot\|_q \) defined earlier is independent of the choice of representatives.

Two main subroutines are for computing the quantum Fourier transform and computing the phase of an eigenvalue of a unitary. Given a unitary \( U \) and an eigenvector \( |\psi\rangle \) with eigenvalue \( \omega^\theta \in \mathbb{C}^* \), and a power \( T \), the phase estimation algorithm approximates the phase \( \theta \) of the eigenvalue. The first step of the algorithm computes the Hadamard transform on \( \log m \) qubits and then computes the controlled-\( k \)-\( U \) in superposition, resulting in the phase state \( 1/\sqrt{T} \sum_{k=0}^{T-1} |k\rangle \otimes U^k |\psi\rangle = 1/\sqrt{T} \sum_{k=0}^{T-1} \omega^{k\theta} |k\rangle \otimes |\psi\rangle \). The Fourier transform over \( \mathbb{Z}_T \) is computed in the first register and it is measured, resulting in a value \( h \in [T] \), where \( h/T \) approximates \( \theta \).

**Theorem 8 (Phase estimation).** Let \( |\psi\rangle \) denote a quantum state on \( n \) qubits and \( U \) unitary on \( n \) qubits for which \( |\psi\rangle \) is an eigenstate with eigenvalue \( \theta \). Let \( t \) be an integer multiple of \( 1/2^r \) closest to \( \theta \). The phase estimation algorithm returns \( \hat{x} \) with probability at least \( 4/2^{2r} \). If \( t = m + r + 1 \) and \( T = 2^t \), then \( h \in [T] \) is returned such that \( h/T \) satisfies \( |h/T - \theta|_1 \leq 1/2^r \) with probability at least \( 1 - 1/2^{2r} \). The running time of the algorithm is \( \text{poly}(n, t) \) times the time to compute \( U^T \).

### 3 Approximate eigenvector of many operators

In this section we define the notion of an approximate eigenvector and show how well the phase estimation algorithm works compared to the exact eigenvector case.
Definition 9. For a unitary $U$, an $\varepsilon_{ev}$-approximate eigenvector is a vector $|\psi\rangle$ with associated eigenvalue $\lambda \in \mathbb{C}^*$ satisfies $\|U|\psi\rangle - \lambda|\psi\rangle\| \leq \varepsilon_{ev}$. This may be denoted $(U, |\psi\rangle, \lambda, \varepsilon_{ev})$, and where $U$ and $|\psi\rangle$ are given as input.

This notion will be used where one vector is used as an approximate eigenvector of a set of unitaries. From that point of view, it may be helpful to say that $U$ approximates the unitary $V = \lambda I$ on the subspace spanned by $|\psi\rangle$ because $V|\psi\rangle = \lambda I|\psi\rangle = \lambda|\psi\rangle$. This may be denoted $(\lambda I, |\psi\rangle, \lambda, 0)$. This also means that $|\lambda| = 1$, as in the definition.

Lemma 10. Let $U$ be a unitary and $|\psi\rangle$ be an $\varepsilon_{ev}$-approximate eigenvector with eigenvalue $\lambda$. Then $\forall k, \|U^k|\psi\rangle - \lambda^k|\psi\rangle\| \leq k\varepsilon_{ev}$.

Proof. Proof by induction on $k$. The base case is $k = 1$ where $\|U^1|\psi\rangle - \lambda^1|\psi\rangle\| \leq 1 \cdot \varepsilon_{ev}$ by assumption. Assume the claim is true for $k - 1$, that is, $\|U^{k-1}|\psi\rangle - \lambda^{k-1}|\psi\rangle\| \leq (k - 1)\varepsilon_{ev}$. Let $U^{k-1}|\psi\rangle = \lambda^{k-1}|\psi\rangle + \varepsilon_{k-1}|E_{k-1}\rangle$, where $\varepsilon_{k-1} \leq (k - 1)\varepsilon_{ev}$. Then

$$\|U^k|\psi\rangle - \lambda^k|\psi\rangle\| = \|U \cdot U^{k-1}|\psi\rangle - \lambda^k|\psi\rangle\|$$

(1)

$$= \|U(\lambda^{k-1}|\psi\rangle + \varepsilon_{k-1}|E_{k-1}\rangle) - \lambda^k|\psi\rangle\|$$

induction hypothesis (2)

$$= \|U|\psi\rangle + \lambda^{-(k-1)}\varepsilon_{k-1}U|E_{k-1}\rangle - \lambda|\psi\rangle\|$$

multiply by $\lambda^{-(k-1)}$ (3)

$$\leq \|U|\psi\rangle - \lambda|\psi\rangle\| + \|\lambda^{-(k-1)}\varepsilon_{k-1}U|E_{k-1}\rangle\|$$

(4)

$$\leq \varepsilon_{ev} + \varepsilon_{k-1} \leq k\varepsilon_{ev}.$$ 

(5)

The fact that $\lambda$ has norm one was used, and the last inequality is by definition of $\varepsilon_{ev}$-approximate eigenvector.

Lemma 11. Consider the step before measuring in the phase estimation routine run up to power $T$ on an approximate eigenvector instance $(U, |\psi\rangle, \lambda, \varepsilon_{ev})$ versus an exact instance $(\lambda I, |\psi\rangle, \lambda, 0)$. Then the distance between these two states is at most $T\varepsilon_{ev}$.

Proof. If $|\psi\rangle$ is an eigenvector with eigenvalue $\lambda$ for some operator $V$, the first step of the eigenvalue estimation algorithm when given power $T$ would be to create the phase state

$$\frac{1}{\sqrt{T}} \sum_{i=0}^{T-1} |k\rangle \otimes (V^i|\psi\rangle) = \frac{1}{\sqrt{T}} \sum_{i=0}^{T-1} \lambda^i |k\rangle \otimes |\psi\rangle.$$ 

Instead, the approximate eigenvector instance $(U, |\psi\rangle, \lambda, \varepsilon_{ev})$ is given, and the state computed is $\frac{1}{\sqrt{T}} \sum_{i=0}^{T-1} |k\rangle \otimes (U^i|\psi\rangle)$.

By Lemma 10, for each $k$, let the difference vector be $\varepsilon_k|E_k\rangle = U^k|\psi\rangle - \lambda^k|\psi\rangle$ with $|\varepsilon_k| \leq k\varepsilon_{ev}$. Comparing the distance between the approximate eigenvector and the exact
eigenvector state before measurement gives

\[
\left\| \left( F_T \otimes I \right) \frac{1}{\sqrt{T}} \sum_{k=0}^{T-1} |k\rangle \otimes (U^k|\psi\rangle) - \left( F_T \otimes I \right) \frac{1}{\sqrt{T}} \sum_{k=0}^{T-1} |k\rangle \otimes (\lambda^k|\psi\rangle) \right\| \quad (6)
\]

\[
= \left\| \frac{1}{\sqrt{T}} \sum_{k=0}^{T-1} |k\rangle \otimes (U^k|\psi\rangle) - \frac{1}{\sqrt{T}} \sum_{k=0}^{T-1} |k\rangle \otimes (\lambda^k|\psi\rangle) \right\| \quad F_T \text{ is unitary} 
\]

\[
= \left\| \frac{1}{\sqrt{T}} \sum_{k=0}^{T-1} |k\rangle \otimes (U^k|\psi\rangle - \lambda^k|\psi\rangle) \right\| = \left\| \frac{1}{\sqrt{T}} \sum_{k=0}^{T-1} |k\rangle \otimes (\varepsilon_k|E_k\rangle) \right\| 
\]

\[
\leq \sqrt{\frac{1}{T} \sum_{k=0}^{T-1} (k\varepsilon_{ev})^2} \leq \varepsilon_{ev} \sqrt{\frac{1}{T} \sum_{k=0}^{T-1} k^2} 
\]

\[
= \varepsilon_{ev} \sqrt{\frac{1}{T} \frac{1}{6}(T-1)T(2(T-1)+1)} \leq \varepsilon_{ev} \sqrt{\frac{1}{6}(2T^2 - 3T + 1)} \leq \varepsilon_{ev} T/\sqrt{3}. \quad (11)
\]

Lemma 12 (Phase Estimation on an approximate eigenvector).

There exists a quantum algorithm that on input

\[ \mathcal{T}_{in} = (U, |\psi\rangle, q, \varepsilon_{ev}, p_{err}) \]

where \( p_{err}^2/(128\varepsilon_{ev}) \geq 1 \), \( \text{etime}(U, T) \) is the time it takes to compute \( U^T \), \( |\psi\rangle \) is supported on vectors in \( \mathbb{Z}_q^n \), where \( (U, |\psi\rangle, \omega_q, \varepsilon_{ev}) \) is an approximate eigenvector for some \( s \in \mathbb{Z}_q \), returns \( \mathcal{O} \in \mathbb{Z}_q \) such that

\[ \Pr \left( |\mathcal{O} - s|_q \leq 129q\varepsilon_{ev}/p_{err}^2 \right) \geq 1 - p_{err}. \]

The running time of the algorithm is \( \text{poly}(n \log(q), \text{etime}(U, p_{err}^2/(8\varepsilon_{ev}))) \).

Proof. Let

\[ b = \lceil \log(2/p_{err}) \rceil \quad a = \lceil \log(p_{err}^2/(2^7\varepsilon_{ev})) \rceil \quad T = 2^{a+b+1}, \]

and then \( T \leq 2^{(\log(2/p_{err})+1)+\log(p_{err}^2/(2^7\varepsilon_{ev})+1)} = \frac{2}{p_{err}} p_{err}^2 2^{7\varepsilon_{ev}} = p_{err}^2/(2^7\varepsilon_{ev}) \). First consider running phase estimation on \( V = \omega_q^s I \), eigenvector \( |\psi\rangle \) and power \( T \) returns \( h \) such that

\[ \Pr \left( \left| \frac{h}{T} - \frac{s}{q} \right|_1 \leq \frac{1}{2^a} \right) \geq 1 - \frac{1}{2^a} \]

by Theorem 8. By the choice of \( a \), \( \frac{1}{2^a} = 2^{-\lceil \log(p_{err}^2/(2^7\varepsilon_{ev})) \rceil} \leq 2^7\varepsilon_{ev}/p_{err}^2 \), and by choice of \( b \), \( \frac{1}{2^a} = 2^{-\lceil \log(2/p_{err}) \rceil} \leq p_{err}/2. \) Therefore

\[ \Pr \left( \left| \frac{h}{T} - \frac{s}{q} \right|_1 \leq 2^7\varepsilon_{ev}/p_{err}^2 \right) \geq 1 - p_{err}/2. \]
Scaling by $q$, the condition is equivalent to $|q^{\frac{h}{T}} - s_q| \leq 27q\varepsilon_{ev}/p_{err}^2$. Let $O = |q^{\frac{h}{T}}|$. Then

$$|O - s_q| = |q^{\frac{h}{T}} - q^{\frac{h}{T}} + q^{\frac{h}{T}} - s_q| \leq |q^{\frac{h}{T}} - q^{\frac{h}{T}}| + |q^{\frac{h}{T}} - s_q| \leq \frac{1}{2} + 27q\varepsilon_{ev}/p_{err}^2 \leq 29q\varepsilon_{ev}/p_{err}^2.$$

For the error bound, consider using $U$ instead on the $\varepsilon_{ev}$-approximate eigenvector $|\psi\rangle$. By Lemma 11 and Lemma 7 the error increases by at most $4T\varepsilon_{ev} \leq p_{err}/2$. The union bound on the phase estimation error and the approximate eigenvector error gives a total error at most $p_{err}/2 + p_{err}/2 \leq p_{err}$. ■

4 An approximate eigenvector of shift operators close to group elements

In this section a quantum state with a random phase is defined that is an approximate eigenvector of shift operators whose shifts are close to points in the lattice $L$. As described in Section 2.1, formally the setting will be in the finite abelian group $G = L \mod q$, which is a subgroup of $\mathbb{Z}_n^q$, together with a distance $\|\cdot\|_q$ on $\mathbb{Z}_n^q$. This setup makes it possible to take a $q$-periodic lattice and target vector, reduce them mod $q$, define and solve BDD over $\mathbb{Z}_n^q$, and to map the solution back to the integers. For clarity this section will be restricted to finite groups.

4.1 Phased Cube States and BDD on Subgroups of $\mathbb{Z}_q^n$

The approximate eigenvector is a superposition of lattice points with a phased cube around each point. The cube’s side length controls how much cubes around two nearby $\mathbb{Z}_q^n$ points overlap.

For $\sigma \in \mathbb{N}$ define the zero-centered set of “radius” $\sigma$ as $[\pm \sigma] = \{q - \sigma + 1, \ldots, \sigma\} \subseteq \mathbb{Z}_q$. The set $[\pm \sigma]$ has $2\sigma$ elements.

**Definition 13.** Let $2\sigma \in \mathbb{N}$ be a side length.

1. Define the cube state around a point $y \in \mathbb{Z}_q^n$ by

$$|C(y)\rangle = \frac{1}{(2\sigma)^n/2} \sum_{z \in [\pm \sigma]^n} |y + z\rangle.$$

2. Let $G$ be a subgroup of $\mathbb{Z}_q^n$ with $r = \text{finite-group-rank}(G)$, generator matrix $G$ and coefficient space $\tilde{C}$. Define the phased cube state with label $a \in \mathbb{Z}_q^r$ to be

$$|\psi_a\rangle = \frac{1}{\sqrt{|G|}} \sum_{c \in \tilde{C}} \chi_a(c)|C(Gc)\rangle = \frac{1}{\sqrt{|G|(2\sigma)^n}} \sum_{c \in \tilde{C}} \chi_a(c) \sum_{z \in [\pm \sigma]^n} |Gc + z\rangle.$$

**Lemma 14** (Cube state properties).

1. Given $y \in \mathbb{Z}_q^n$, $|C(y)\rangle|y\rangle$ is computable in time $\text{poly}(n \log q)$.
2. Define the shift operator $U_x$ by $U_x|y\rangle = |x + y\rangle$.
\[
\forall x, y \in \mathbb{Z}_q^n, U_x|C(y)\rangle = |C(x + y)\rangle,
\]
and the transformation $|C(y)\rangle|x\rangle$ to $|C(x + y)\rangle|x\rangle$ is computable in time $\text{poly}(n \log q)$.

3. Let $|C(y)\rangle$ be a cube state of side length $2\sigma$ and let $\Delta \in \mathbb{Z}_q^n$.
\[
(a) \text{ Then } \| |C(y)\rangle - |C(y + \Delta)\rangle \| \leq \sqrt{n \| \Delta \|_2}.
\]
\[
(b) \text{ If } \| \Delta \|_q \geq \sqrt{n}2\sigma + 1 \text{ then } \langle C(y)|C(y + \Delta)\rangle = 0.
\]

Proof.
\[
|0, y\rangle \xrightarrow{P_{2\sigma}^\otimes n} \frac{1}{(2\sigma)^{n/2}} \sum_{z \in \mathbb{Z}_q^n} |z, y\rangle \tag{12}
\]
\[
(\sigma) \xrightarrow{1} \frac{1}{(2\sigma)^{n/2}} \sum_{z \in [\pm \sigma]^n} |z, y\rangle, \quad z \rightarrow z - (\sigma, \ldots, \sigma), \text{ and reindex} \tag{13}
\]
\[
(\sigma) \xrightarrow{1} \frac{1}{(2\sigma)^{n/2}} \sum_{z \in [\pm \sigma]^n} |y + z, y\rangle \tag{14}
\]
\[
= |C(y)\rangle|y\rangle. \tag{15}
\]

For $x \in \mathbb{Z}_q^n$, $U_x|C(y)\rangle = \frac{1}{(2\sigma)^{n/2}} \sum_{z \in [\pm \sigma]^n} U_x|y + z\rangle = \frac{1}{(2\sigma)^{n/2}} \sum_{z \in [\pm \sigma]^n} |x + y + z\rangle = |C(x + y)\rangle$. Therefore, given $|C(y)\rangle|x\rangle$, one addition from the second register into the first register results in $|C(x + y)\rangle|x\rangle$.

For (3a), start with
\[
\| |C(y)\rangle - |C(y + \Delta)\rangle \|^2 = 2 \cdot (1 - \Re(\langle C(y)|C(y + \Delta)\rangle)).
\]

Since
\[
\langle C(y)|C(y + \Delta)\rangle = \langle C(0)|C(\Delta)\rangle \text{ \quad \quad \quad \quad \quad \quad \text{invariant under shift by } -y \tag{16}
\]
\[
= \frac{1}{(2\sigma)^n} \sum_{z \in [\pm \sigma]^n} \sum_{z' \in [\pm \sigma]^n} \langle z|z' + \Delta \rangle \tag{17}
\]
\[
= \frac{\# \text{ common points in } [\pm \sigma]^n \text{ and } [\pm \sigma]^n + \Delta}{(2\sigma)^n} \tag{18}
\]
\[
\geq \frac{(2\sigma - \| \Delta \|_q)^n}{(2\sigma)^n} = (1 - \frac{\| \Delta \|_q}{2\sigma})^n \quad \text{Bernoulli's inequality} \tag{19}
\]

Therefore
\[
\| |C(y)\rangle - |C(y + \Delta)\rangle \| \leq \sqrt{n \| \Delta \|_2}.
\]

For (3b) assume $\| \Delta \|_q \geq \sqrt{n}2\sigma + 1$. To have a common point, there must exist $z, z' \in [\pm \sigma]^n$ such that $z = z' + \Delta \in \mathbb{Z}_q^n$. For this to happen, $z - z' = \Delta$, and so $\sqrt{n}2\sigma + 1 \leq \| \Delta \|_q = \| z - z' \|_q \leq \| z \|_q + \| z' \|_q \leq 2\sqrt{n}\sigma$, which is a contradiction, so no points are in common. \hfill \blacksquare
Algorithm 15 (Computing a PCS state).
Input: A decomposed subgroup $(G, q, r)$ of $\mathbb{Z}_q^n$ and a cube side length $2\sigma$.
Output: $|\psi_a\rangle$ and $a \in \mathbb{Z}_q^r$.

\[
|0, 0\rangle \rightarrow |C(0))|0\rangle \quad \text{by Lemma 14(1)} \tag{20}
\]
\[
\frac{(F_{\tilde{C}})}{2} \frac{1}{|G|} \sum_{c \in \tilde{C}} |C(0))|c\rangle \tag{21}
\]
\[
\frac{(Gc)}{2} \frac{1}{|G|} \sum_{c \in \tilde{C}} |C(Gc))|c\rangle \tag{22}
\]
\[
\frac{(F_{\tilde{C}})}{2} \frac{1}{|G|} \sum_{c \in \tilde{C}} |C(Gc))| \left( \frac{1}{|Z_q^r|} \sum_{a \in Z_q^r} \chi_a(c)|a\rangle \right) \tag{23}
\]
\[
M_2 \frac{1}{|G|} \sum_{c \in \tilde{C}} \chi_a(c)|C(Gc))|a\rangle \tag{24}
\]
\[
= |\psi_a\rangle|a\rangle. \tag{25}
\]

Lemma 16 (Phased Cube State Approximate Eigenvector Properties).
Let $G$ be a subgroup of $\mathbb{Z}_q^n$ with decomposition $(G, q, r) \in \mathbb{Z}_q^n \times \mathbb{Z}_r^r$, with shortest (nonzero) element length $\lambda_1 = \min_{v \in G \setminus \{0\}} \|v\|_q$, and a phased cube state $|\psi_a\rangle$ for $a \in \mathbb{Z}_q^r$ and side length $1/\sqrt{\lambda_1 n} \leq 2\sigma \leq 1/\sqrt{\lambda_1 n}$. Then

1. If $c \in \tilde{C}$, let $Gc = v \in G$, and then $U_v|\psi_a\rangle = \chi_a(-c)|\psi_a\rangle$.

2. $\forall \Delta \in \mathbb{Z}_q^n$ with $\|\Delta\|_q \leq \frac{\lambda_1}{2}$, $\|U_\Delta|\psi_a\rangle - |\psi_a\rangle\| \leq 4n^{3/4} \sqrt{\frac{\|\Delta\|_q}{\lambda_1}}$.

3. $\forall \mathbf{y} \in \mathbb{Z}_q^n$, let $G\mathbf{s} \in G$ with coefficients $\mathbf{s} \in \tilde{C}$ be such that $\Delta = \mathbf{y} - G\mathbf{s}$ satisfies $\|\Delta\|_q \leq \frac{\lambda_1}{2}$. Then $\|U_\Delta|\psi_a\rangle - \chi_a(-s)|\psi_a\rangle\| \leq 4n^{3/4} \sqrt{\frac{\|\Delta\|_q}{\lambda_1}} =: \varepsilon_{ev}$. In particular, for all $a \in \mathbb{Z}_q^r$, the state $|\psi_a\rangle$ is an $\varepsilon_{ev}$-approximate eigenvector of $U_\mathbf{y}$ for any element $\mathbf{y}$ where $\text{dist}_q(\mathbf{y}, G) \leq \frac{\lambda_1}{2}$.

4. Given a decomposed group $(G, q, r)$, Algorithm 15 with side length $2\sigma$ computes the state $|\psi_a\rangle|a\rangle$ in time $\text{poly}(n \log q)$, where $a$ is a uniformly chosen element from $\mathbb{Z}_q^r$. 

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Proof. For (1), let \( \mathbf{v} = \mathbf{Gc} \in G \) and \( \mathbf{a} \in \mathbb{Z}_q^n \), then

\[
U_\mathbf{v}|\psi_a\rangle = \frac{1}{\sqrt{|G|}} \sum_{\mathbf{d} \in \hat{G}} \chi_a(\mathbf{d}) U_\mathbf{v}|C(\mathbf{Gd})\rangle
\]

(26)

\[
= \frac{1}{\sqrt{|G|}} \sum_{\mathbf{d} \in \hat{G}} \chi_a(\mathbf{d})|C(\mathbf{Gc + Gd})\rangle
\]

(27)

\[
= \frac{1}{\sqrt{|G|}} \sum_{\mathbf{d'} \in \hat{G}} \chi_a(\mathbf{d'} - \mathbf{c})|C(\mathbf{Gd'})\rangle \quad \text{reindex coeff with } \mathbf{d'} = \mathbf{c} + \mathbf{d}
\]

(28)

\[
= \frac{1}{\sqrt{|G|}} \sum_{\mathbf{d'} \in \hat{G}} \chi_a(\mathbf{d'}) \chi_a(-\mathbf{c})|C(\mathbf{Gd'})\rangle
\]

(29)

\[
= \chi_a(-\mathbf{c})|\psi_a\rangle.
\]

(30)

For (2), let \( \Delta \in \mathbb{Z}_q^n \), and then

\[
\left\| U_\Delta|\psi_a\rangle - |\psi_a\rangle \right\|
\]

(31)

\[
= \left\| U_\Delta \frac{1}{\sqrt{|G|}} \sum_{\mathbf{c} \in \hat{G}} \chi_a(\mathbf{c})|C(\mathbf{Gc})\rangle - \frac{1}{\sqrt{|G|}} \sum_{\mathbf{c} \in \hat{G}} \chi_a(\mathbf{c})|C(\mathbf{Gc})\rangle \right\|
\]

(32)

\[
= \left\| \frac{1}{\sqrt{|G|}} \sum_{\mathbf{c} \in \hat{G}} \chi_a(\mathbf{c})|C(\mathbf{Gc + \Delta})\rangle - \frac{1}{\sqrt{|G|}} \sum_{\mathbf{c} \in \hat{G}} \chi_a(\mathbf{c})|C(\mathbf{Gc})\rangle \right\| \quad \text{apply } U_\Delta
\]

(33)

\[
= \left\| \frac{1}{\sqrt{|G|}} \sum_{\mathbf{c} \in \hat{G}} \chi_a(\mathbf{c}) (|C(\mathbf{Gc + \Delta})\rangle - |C(\mathbf{Gc})\rangle) \right\| \quad \text{group terms}
\]

(34)

\[
= \left\| \frac{1}{\sqrt{|G|}} \sum_{\mathbf{c} \in \hat{G}} \chi_a(\mathbf{c}) \epsilon_c|E_c\rangle \right\| \quad \text{by Lemma 14(3a),}
\]

(35)

\[
\text{let } \epsilon_c|E_c\rangle \text{ be the difference, with } |\epsilon_c| \leq \sqrt{n \frac{||\Delta||_\sigma}{\lambda_1}}
\]

(36)

\[
\leq \max_{\mathbf{c} \in \hat{G}} |\epsilon_c| \leq \sqrt{n \frac{||\Delta||_\sigma}{\lambda_1}} \leq \sqrt{8n^{3/2} \frac{||\Delta||_\sigma}{\lambda_1}} \leq 4n^{3/4} \sqrt{\frac{||\Delta||_\sigma}{\lambda_1}}.
\]

(37)

For (3), let \( \mathbf{y} \in \mathbb{Z}_q^n \) satisfy \( \text{dist}_q(\mathbf{y}, G) \leq \epsilon_1 \lambda_1 \), let \( \mathbf{Gs} \in G \) be the closest element to \( \mathbf{y} \), and let \( \Delta = \mathbf{y} - \mathbf{Gs} \). Then

\[
\| U_\mathbf{y}|\psi_a\rangle - \chi_a(-\mathbf{s})|\psi_a\rangle \| = \| U_{\mathbf{Gs} + \Delta}|\psi_a\rangle - \chi_a(-\mathbf{s})|\psi_a\rangle \| = |\chi_a(-\mathbf{s})| \cdot \| U_\Delta|\psi_a\rangle - |\psi_a\rangle \| \leq 4n^{3/4} \sqrt{\frac{||\Delta||_\sigma}{\lambda_1}}
\]

by part (1) and then part (2).

Finally, for (4), the algorithm computes the Fourier transform, addition in \( \mathbb{Z}_q^n \), and measurement, which are all polynomial time in \( n \) and \( \log q \). By Lemma 14 the \( |\hat{C}(\mathbf{v})\rangle \)'s form an orthogonal set of states. The probability of measuring \( \mathbf{a} \) is hence proportional to

\[
\| \sum_{\mathbf{c} \in \hat{G}} \chi_a(\mathbf{c})|\hat{C}(\mathbf{Gc})\rangle \|^2 = \sum_{\mathbf{v} \in G} \| C(\mathbf{v}) \|^2
\]

which is independent of \( \mathbf{a} \).  

\[ \blacksquare \]

Using PCS states and the phase estimation algorithm on approximate eigenvectors we define the following algorithm which outputs LWE samples but with error different than the typical Gaussian error. In order keep the terminology more closely related to lattices we phrase it as sampling inner products.
Algorithm 17. SampleHIP((G, q, r), 2σ, ε1, t, perr)
Input: finite group decomposition (G, q, r), side length 2σ, target vector t ∈ Z^n.

1. Compute a PCS state using Algorithm 15 on (G, q) and 2σ to get a random label a and state \(|ψ_a⟩\) on \((n + r)\log q\) qubits.

2. Let \(ε_{ev} = \sqrt{ε_1} \cdot 4n^{3/4}\). Run phase estimation on \(T_{in} = (U_t, |ψ_a⟩, q, ε_{ev}, p_{err})\) to get output \(O\) and return \((a, O) \in \mathbb{Z}^r_q \times \mathbb{Z}_q\).

Lemma 18 (Sampling Hidden Inner Products). If \(2σ \in \left[\frac{1}{4} \frac{λ_1(G)}{\sqrt{m}}, \frac{1}{2} \frac{λ_1(G)}{\sqrt{m}}\right]\), \(0 < ε_1 < \frac{1}{2}\) and \(\text{dist}_q(t, G) \leq ε_1 λ_1\) then Algorithm 17 runs in time \(\text{poly}(n \log q, \log(p_{err}/ε_1))\) and returns a uniformly random \(a \in \mathbb{Z}^r_q\), and an \(O \in \mathbb{Z}_q\) satisfying

\[
\Pr\left(|O - (−s) \cdot a|_q \leq 129 \cdot q \cdot \sqrt{ε_1} \cdot 4n^{3/4} \cdot p_{err}^{-2}\right) \geq 1 - p_{err}.
\]

where the element \(G_s\) is the closest group element to \(t\) in \(G\).

Proof. In the first step of Algorithm 17 on input \((G, q, r)\) with given side length \(2σ\) Algorithm 15 returns a state \(|ψ_a⟩\) on \((n + r)\log q\) qubits, where \(a \in \mathbb{Z}^r_q\) is uniformly random by Lemma 16. Since by assumption \(\text{dist}_q(t, G) \leq ε_1 λ_1 < \frac{λ_1}{2}\) the element \(G_s\) is the closest group element to \(G\).

In the next step the tuple \(T_0 = (U_t, |ψ_a⟩, χ_a(−s), ε_{ev} = \sqrt{ε_1} \cdot 4n^{3/4})\) is an approximate eigenvector instance by Lemma 16. In addition, since \(U_t\) is the shift operator, \(U_t^l\) can be computed in time \(\text{poly}(n, \log(t))\) by repeated squaring. It follows by Lemma 12 that running phase estimation on the tuple \(T_{in} = (U_t, |ψ_a⟩, q, ε_{ev}, p_{err})\) returns a value \(O \in \mathbb{Z}_q\) that satisfies

\[
\Pr\left(|O - (−s) \cdot a|_q \leq 129 \cdot q \cdot ε_{ev} \cdot p_{err}^{-2} = 129 \cdot q \cdot \sqrt{ε_1} \cdot 4n^{3/4} \cdot p_{err}^{-2}\right) \geq 1 - p_{err}.
\]

The running time requires \(\text{poly}(n \log q)\) time for the PCS generation Algorithm 15 and an additional \(\text{poly}(n \log q, \log(p_{err}/ε_{ev}))\) for the phase estimation algorithm on approximate eigenvectors by Lemma 12, for a total of \(\text{poly}(n \log q, \log(p_{err}/ε_1))\).

5 Random Self Reducibility for subgroups \(G\) of \(\mathbb{Z}_q^n\) of finite group rank \(r\)

This section establishes RSR for instances of BDD over subgroups of \(\mathbb{Z}_q^n\) using a quantum algorithm.

Algorithm 19. SampleBDD((G, q, r), λ1, t, ε1, m, perr)
Input: A decomposed subgroup \((G, q, r)\) of \(\mathbb{Z}_q^n\), a target vector \(t \in \mathbb{Z}_q^n\), an estimate of the length of the shortest vector \(λ_1\), required error probability \(p_{err}\), and a target dimension \(m\).

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1. Let \( p_{err}^{PE} = p_{err}/(2m) \), \( 2\sigma = \frac{1}{2} \sqrt{\lambda_1/n} \).

2. For each \( i \in [m] \) run
\[
(a_i \in \mathbb{Z}_q^r, O_i \in \mathbb{Z}_q) = \text{SampleHIP}((G, q, r), 2\sigma, p_{err}^{PE}, \varepsilon_1, t).
\]

3. Let \( \tilde{G} \in \mathbb{Z}_q^{m \times r} \) have rows formed by the \( a_i \), let \( \tilde{g}_i \) denote the \( i \)th column of \( \tilde{G} \), and let \( \tilde{G} = \langle \tilde{g}_1, \ldots, \tilde{g}_r \rangle \subseteq \mathbb{Z}_q^m \).

4. Define the target element \( \tilde{t} = (O_1, \ldots, O_m) \).

5. Return \( \tilde{G}, \tilde{t} \).

**Lemma 20** (Coefficient-preserving random group sampling). Let \( (G, q, r) \) be a decomposed subgroup of \( \mathbb{Z}_q^n \), \( t \in \mathbb{Z}_q^n \), and \( \varepsilon_1 > 0 \). Let \( \lambda_1 \) be an estimate of \( \lambda_1 \) such that \( \lambda_1 \in [\lambda_1, 2\lambda_1] \).

If \( \text{dist}_q(t, G) \leq \varepsilon_1\lambda_1(G) \), then \( \text{SampleBDD}((G, q, r), \lambda_1, t, \varepsilon_1, m, p_{err}) \) returns a random subgroup \( \tilde{G} = \langle \tilde{g}_1, \ldots, \tilde{g}_r \rangle \subseteq \mathbb{Z}_q^m \), a vector \( t \in \mathbb{Z}_q^m \) such that w.p. at least \( 1 - p_{err} \), when \( p_{err}/2 \geq 1/2^m + 1/q^{m-r} \),

1. **Preservation of bounded distance to lattice:**
\[
\text{dist}_q(\tilde{t}, \tilde{G}) \leq \sqrt{\varepsilon_1} \cdot q^r/m \lambda_1(\tilde{G}) \cdot 260n^{3/4}m^{2.5}p_{err}^{-2}.
\]

2. **Preservation of coefficients:**
If \( s \in \mathbb{Z}_q^r \) is such that \( \| t - Gs \|_q \leq \varepsilon_1\lambda_1(G) \), then \( \| \tilde{t} - G\tilde{s} \|_q \leq \varepsilon_1\lambda_1(\tilde{G}) \).

Let \( s \) be such that \( Gs \in G \), denote the closest group element to \( t \) in \( G \). Then \( s \) is the unique element such that \( Gs \) is the closest element to \( \tilde{t} \) in \( \tilde{G} \).

The running time of the procedure is \( \text{poly}(n \log q, m, \log(m \cdot p_{err}/\varepsilon_1)) \).

**Proof.** For the distance, by Lemma 18 and the union bound over \( m \) samples,
\[
\Pr \left( \forall i \in [m], \| O_i - (-s) \cdot a_i \|_q \leq 129 \cdot q^{3/4} \sqrt{\varepsilon_1}n^{3/4}(p_{err}^{PE})^{-2} \right) \geq 1 - m \cdot p_{err}^{PE} \geq 1 - p_{err}/2.
\]

When this condition holds,
\[
\text{dist}_q(\tilde{t}, \tilde{G}) \leq \| \tilde{t} - G\tilde{s} \|_q
\]
\[
= \| (O_1, \ldots, O_m) - (s \cdot a_1, \ldots, s \cdot a_m) \|_q
\]
\[
\leq \sqrt{m} \max_i \| O_i - s \cdot a_i \|_q \quad \text{by Lemma 18}
\]
\[
\leq 129\sqrt{mq} \sqrt{\varepsilon_1}n^{3/4}(p_{err}^{PE})^{-2} \quad \text{by Lemma 18}
\]
\[
\leq \delta 129 \cdot q^{3/4} \sqrt{\varepsilon_1}n^{3/4} \cdot m \cdot (p_{err}/(2m))^{-2} \quad \text{w.p. } \geq 1 - 1/2^m \text{ by Claim 1}
\]
\[
= \delta 129 \cdot \sqrt{\varepsilon_1}n^{3/4} (p_{err}^{PE})^{-2}.
\]

Hence by the union bound, the distance is bounded w.p. at least \( 1 - p_{err}/2 - 1/2^m \).
For the secret coefficients \( s \), let \( \tilde{G} \) denote the \( m \times r \) matrix whose rows are \( a_i \). Observe that by definition we have that \( \tilde{G} \cdot s \) is at distance at most \( \text{dist}_q(\tilde{t}, \tilde{G}) \) from \( \tilde{t} \). Hence for \( s \) to be the unique vector that satisfies this equation we need to additionally require that \( \tilde{G} \) is primitive. By Claim 1 we have \( \Pr(\tilde{G} \text{ is primitive}) \geq 1 - \frac{1}{q^{m-r}} \).

It follows that both conditions of the lemma are satisfied w.p. at least \( 1 - p_{\text{err}}/2 - 1/2^m - 1/q^{m-r} \geq 1 - p_{\text{err}} \).

By Lemma 18 the running time is \( \text{poly}(n \log q, \log(p_{\text{err}}/\varepsilon_1), m) \).

\[ \square \]

**Theorem 21.** There is a \( \text{poly}(n, \log q) \)-time quantum algorithm solving \( 2^{-\Omega(\sqrt{r \log q})} \)-BDD on lattices of dimension \( n \), periodicity \( q \), and finite group rank \( r \).

**Proof.** Compute the finite abelian group decomposition \((G, q, r)\) of \( L(B) \mod q \). Call SampleBDD with \((G, q, r), \hat{\lambda}_1, t \mod q, \varepsilon_1, m = \sqrt{r \log q}, p_{\text{err}} = 1/10 \). By Lemma 20, \( \tilde{G} \) and \( \tilde{t} \) are returned satisfying

\[
2^m \text{dist}_q(\tilde{t}, \tilde{G}) \leq 2^m \sqrt{\varepsilon_1 \hat{\lambda}_1(\tilde{G})} q^{r/m} \delta 129 \cdot 4^2 n^{3/4} m^{2.5} 100 \leq \sqrt{\varepsilon_1} 2^{m+r \log q/m} \hat{\lambda}_1(\tilde{G}) \delta 129 \cdot 4^2 n^{3/4} m^{2.5} 100 \leq c \sqrt{\varepsilon_1} 2^{r \log q} \hat{\lambda}_1(\tilde{G}) n^\varepsilon \leq \hat{\lambda}_1(\tilde{G})/2
\]

Choose \( \varepsilon_1 = (c 2^{-4 \sqrt{r \log q} n^{-2\varepsilon'}}) \). Mapping back up to the integers, Babai’s nearest plane algorithm computes the closest vector \( \mod q \) in polynomial time, and that can be mapped to the original problem. See the steps of the algorithm in the next section for details. \[ \square \]

As discussed in the introduction, \( r \log q < n^2 \), \( 2^m \)-approximation algorithm. For example, subexponential approximation factors such as \( \varepsilon_1 = 2^{-n^\delta} \) are possible when \( \sqrt{r \log q} \leq n^\delta \), or \( r \log q \leq n^{2\delta} \).

### 6 Trading off running time for approximation factor

The algorithm in this section applies Schnorr’s hierarchy theorem as a black box, rather than applying Babai’s algorithm. The algorithm will first compute the finite abelian group decomposition of the lattice, solve BDD in that group, then map the solution back to the integers.

**Algorithm 22** (Algorithm Q: \( \alpha \)-BDD on \( n \)-dimensional \( q \)-periodic lattices).

*Input:* A lattice basis \( B \in \mathbb{Z}^{n \times n} \), for a \( q \)-periodic lattice with finite group rank \( r \), a target vector \( t \in \mathbb{Z}^n \), a trade-off parameter \( \beta \).

1. Compute the finite abelian group decomposition \((G, q, r)\) of \( L(B) \mod q \).

2. Iterate over powers of 2 for \( \hat{\lambda}_1 = 2, 4, 8 \ldots, q \):

   (a) Let \( m = (\beta - 1) \sqrt{\beta \log q} + 1 \leq \sqrt{\beta \log q} \) and sample random instance of BDD.
in $\mathbb{Z}_q^n$:
\[
(G, \hat{t}) = \text{SampleBDD}((G, q, r), \lambda_1, t \mod q, \varepsilon_1, m, p_{err} = 1/10).
\]

(b) With operations over the integers, compute $\text{HNF}([ \tilde{G} \mid q \cdot 1 ])$ to get a basis $\tilde{B} \in \mathbb{Z}^{m \times m}$. Treating $\tilde{t}$ as an integer vector, run the $\sqrt{m} \beta(m-1)/(\beta-1)$-approximate CVP algorithm (Theorem 5) on $(\tilde{B}, \tilde{t})$. Denote output by $\tilde{v} \in L(\tilde{B})$.

(c) Compute $s \in \mathbb{Z}^r$ such that $\tilde{v} = Bs$. If $\| t - Bs \|_q \leq \lambda_1/2$, then output $Bs + t - (t \mod q)$.

**Theorem 23.** Let $L(B)$ be an $n$-dimensional $q$-periodic lattice with finite group rank $r$. Given an instance of $\varepsilon_1$-BDD $(B, t)$, with $2 \leq \beta \leq r \log q$, and
\[
\varepsilon_1 = \left( \exp\left(-4\sqrt{\frac{\log q \log \beta}{\beta}}\right) \cdot 2^2mp_20(n, \log q)^2 \right).
\]
Algorithm 22 runs in quantum time $\approx \beta^2 \text{poly}(n, \log(q))$ and returns the closest vector to $t$ with probability at least $0.9$.

**Proof.** For a lattice of dimension $m$, Schnorr’s algorithm returns $\tilde{v}$ such that
\[
\| \tilde{v} - \tilde{t} \| \leq \sqrt{m} \beta^{m-1} \text{dist}(\tilde{t}, L(\tilde{B}))
\]
by Theorem 5 \( (48) \)
\[
\leq \sqrt{\varepsilon_1} \beta^{m-1} q^{r/m} \sqrt{m} p_{20}(n, \log q) \lambda_1(G)
\]
by Lemma 20, w.p. $0.9$ \( (49) \)
\[
= \sqrt{\varepsilon_1} \exp\left(\frac{(m-1) \log \beta + r \log q}{m}\right) \sqrt{m} p_{20}(n, \log q) \lambda_1(G)
\]
\( (50) \)
\[
\leq \sqrt{\varepsilon_1} \exp\left(2\sqrt{\frac{\log q \log \beta}{\beta}}\right) \sqrt{m} p_{20}(n, \log q) \lambda_1(G)
\]
Choose $m = \sqrt{\frac{\beta \log q}{\log \beta}}$ \( (51) \)
\[
\leq \lambda_1(G)/2.
\]
(52)

The dimension $m$ was chosen to maximize the decoding radius for a given $r \log q$ and $\beta$. ■

**Corollary 24.** If $r \log q = n \log n$, $0 \leq \varepsilon \leq 1/2$, then BDD can be solved with factor $\varepsilon_1 \approx \exp\left(-4\varepsilon \log n\right)$ in time $(n^{1-2\varepsilon})^{n^{1-2\varepsilon}} \text{poly}(n, \log q) \leq 2n^{1-2\varepsilon} \text{log}^n \text{poly}(n, \log q)$.

This running time has an $n^\varepsilon$ factor less than using Schnorr directly, without reanalyzing it for $q$-periodic lattices.

**Proof.** Choose $\beta = n^{1-2\varepsilon}$ and then $\exp\left(-4\sqrt{\frac{\log q \log \beta}{\beta}}\right) = \exp\left(-4\sqrt{\frac{n \log n \log n^{1-2\varepsilon}}{n^{1-2\varepsilon}}}\right) = \exp\left(-4\varepsilon \log n\right)$. The time is $(n^{1-2\varepsilon})^{n^{1-2\varepsilon}} \text{poly}(n, \log q) \leq 2n^{1-2\varepsilon} \text{log}^n \text{poly}(n, \log q)$. ■

**Corollary 25.** If $r \log q = (\log n^c)^3$ and $r = \log n^c$, choose $\beta = \log n^c$, and then BDD can be solved with factor $\varepsilon_1 \approx \exp\left(-4\sqrt{\frac{\log q \log \beta}{\beta}}\right) = \exp\left(-4\sqrt{\frac{(\log n^c)^2 \log \log n^c}{\log n^c}}\right) = \exp\left(-4\sqrt{\frac{\log \log n^c}{\log n^c}}\right)$ in time $(\log n^c)^{\log n^c} \text{poly}(n, \log q) \leq 2n^{c \log \log n^c} \text{poly}(n, \log q) = n^c \log n^c \text{poly}(n, \log q)$ because $\beta = \log n^c$. 

18
Using LLL for BDD and SIVP on rectangle-periodic lattices

In this section extend the algorithm in [DvW21] to meet the bounds of Theorem 21 for polynomial-time algorithms and generalize it further to handle rectangle-periodic lattices. We also record that it can be used to solve SIVP for related parameters. The quantum algorithm also handles rectangle-periodic lattices without change.

More details about [DvW21] can also be found in [ABCG21], and about LLL in [MG02].

Call a lattice \( L \subseteq \mathbb{Z}^n \) \( r \)-periodic for a vector \( \mathbf{r} = (r_1, \ldots, r_n) \) if \( H := r_1 \mathbb{Z} \times \cdots \times r_n \mathbb{Z} \) is a subgroup of \( L \). All information about the lattice is contained in the finite abelian group \( L/H \cong \mathbb{Z}^{r_1} \times \cdots \times \mathbb{Z}^{r_n} \). Let \((G, q, r)\) be the finite group decomposition of \( L/H \). This notation uses \( r \) and \( r_i \) for the rectangle side lengths, and is separate from the finite group rank \( r \).

Recall that the coefficient space is \( \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_r} \) with generators \( g_1, \ldots, g_r \in \mathbb{Z}^{r_1} \times \cdots \times \mathbb{Z}^{r_n} \). Let \( q = \max\{r_1, \ldots, r_n\} \).

Lemma 26. There is an algorithm running in time \( \text{poly}(r, \log q) \) that takes a lattice \( L \subseteq \mathbb{Z}^n \) and returns a basis with minimum Gram-Schmidt length at least \( \exp(-c \sqrt{r \log q}) \cdot \lambda_1 \), for a constant \( c \).

Proof. Given a basis \( B' \) of a lattice, compute the minimum axis-aligned rectangle in \( L \) computing a lattice vector on each axis, and dividing by the gcd of the coordinates in each axis. Next compute the finite group decomposition \((G, q, r)\), and lift \( g_1, \ldots, g_r \) to \( \mathbb{Z}^n \). Sort the lengths and label so that \( r_1 \leq \cdots \leq r_n \) and run the algorithm in Lemma 2 on a basis from \( \text{HNF}(g_1, \ldots, g_r, r_1 e_1, \ldots, r_n e_n) \) and the linearly independent set \( S = \{r_1 e_1, \ldots, r_n e_n\} \), resulting in basis \( B \). Then run LLL on with constant \( \Delta > 1 \) and let \( C = \text{LLL}(B, \Delta) \).

The Lovasz Condition (LC) in LLL implies for \( j \leq i \),
\[
\frac{\|c_i^*\|}{\Delta^{i-j}} \leq \cdots \leq \frac{\|c_{i-j}^*\|}{\Delta^{j}} \leq \frac{\|c_{i-j}^*\|}{\Delta^0} \leq \frac{\|c_{i-1}^*\|}{\Delta^1} \leq \cdots \leq \frac{\|c_{i-1}^*\|}{\Delta^{m-1}} \leq \frac{\|c_i^*\|}{\Delta^m},
\]
which by the choice of \( m \) below, satisfies the bound. The first inequality is because \( 0 \neq c_i^* = c_1 \in L \).

For case 2, assume \( m \leq i \).

1. Using LC again, multiply \( m \) consecutive vector lengths to get
\[
\frac{\|c_{i-m+1}^*\| \cdots \|c_i^*\|}{\Delta^{m(m-1)/2}} = \frac{\|c_{i-m+1}^*\| \cdots \|c_i^*\|}{\Delta^{\sum_{j=0}^{m-1} j}} \leq \|c_i^*\|^m.
\]
This will be used below.

2. \( r \)-periodic, block of \( m \) vectors. Let \( R \) contain the indices of the \( r \) largest values in \( \mathbf{r} = (r_1, \ldots, r_n) \). When sorted it will contain \( r_{n-r+1}, \ldots, r_n \). Then
There is an algorithm running in time 

\[
\frac{r_1 \cdots r_n}{\prod_{j \in R} r_j} = \frac{1}{r_1 \cdots r_{i-m}} \cdot \frac{r_1 \cdots r_n}{\prod_{j \in R} r_j} 
\]

is equal to the minimum basis length in the Gram-Schmidt vectors in the primal, which by the max basis length of the Gram-Schmidt vectors in the dual. This is equal to the minimum basis length in the Gram-Schmidt vectors in the dual, which by Lemma 26 gives a bound on the lengths of \(q \cdot \exp(O(\sqrt{r \ln q})) / \lambda_1(L)\).
Sampling short vectors can also be done using a quantum algorithm using BDD to create phaseless Gaussian superpositions so that computing the Fourier transform and measuring samples lattice points from a discrete Gaussian of a similar length.

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