CROSSCAP STABILITY

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Abstract. We provide an alternative proof that Crosscaps are diffeomorphically stable.

1. Introduction

Perelman’s celebrated Stability Theorem in particular implies the following.

Topological Stability Theorem: Let \( \{M_\alpha\}_\alpha \) be a sequence of closed Riemannian \( n \)-manifolds with sectional curvature \( \geq k \). If the Gromov-Hausdorff limit of \( \{M_\alpha\}_\alpha \) is \( X \) and \( \dim (X) = n \), then all but finitely many of the \( M_\alpha \)'s are homeomorphic to \( X \). [17, 29]

Given this, it is at least natural to ask the

Diffeomorphism Stability Question. Let \( \{M_\alpha\}_\alpha \) be a sequence of closed Riemannian \( n \)-manifolds with sectional curvature \( \geq k \). If the Gromov-Hausdorff limit of \( \{M_\alpha\}_\alpha \) is \( X \) and \( \dim (X) = n \), then are all but finitely many of the \( M_\alpha \)'s diffeomorphic to each other? [16]

An affirmative answer to the Diffeomorphism Stability Theorem would provide a generalizations of Cheeger’s Finiteness Theorem and the Diameter Sphere Theorem [5], [6], [14], [16].

Definition 1.1. Let \( \mathcal{M}_k(n) \) be the class of closed Riemannian \( n \)-manifolds with sectional curvature \( \geq k \). A compact, \( n \)-dimensional \( X \in \text{closure} (\mathcal{M}_k(n)) \) is called diffeomorphically stable if for any sequence \( \{M_\alpha\}_{\alpha=1}^\infty \subset \mathcal{M}_k(n) \) with \( M_\alpha \rightarrow X \), in the Gromov–Hausdorff topology, all but finitely many of the \( M_\alpha \)'s are diffeomorphic to each other.

It follows from Theorem 6.1 in [22] that \( X \) is diffeomorphically stable if all of its points are \( (n, \delta) \)-strained in the sense of [11]. Examples of such spaces are curvature \( k \) crosscaps.

Example 1.2. (Crosscap) Let \( \mathcal{D}_k^n(r) \) be an \( r \)-ball in the \( n \)-dimensional, simply connected space form of constant curvature \( k \).

The constant curvature \( k \) Crosscap, \( C^n_{k,r} \), is the quotient of \( \mathcal{D}_k^n(r) \) obtained by identifying antipodal points on the boundary. Thus \( C^n_{k,r} \) is homeomorphic to \( \mathbb{R}P^n \). There is a canonical metric on \( C^n_{k,r} \) that makes this quotient map a submetry. The universal cover of

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$C_{k,r}^n$ is the double of $D^n_k(r)$. If we write this double as $\mathbb{D}^n_k(r) := D^n_k(r)^+ \cup_{\partial D^n_k(r)} D^n_k(r)^-$, then the free involution

$$A : \mathbb{D}^n_k(r) \longrightarrow \mathbb{D}^n_k(r)$$

that gives the covering map $\mathbb{D}^n_k(r) \longrightarrow C_{k,r}^n$ is

$$A : (x,+) \longmapsto (-x,-),$$

where the sign in the second entry indicates whether the point is in $D^n_k(r)^+$ or $D^n_k(r)^-$. Crosscaps are of particular interest since they are one of the three types of limit spaces with maximal volume in the sense of [13]. Here we provide an alternative proof that Crosscaps are diffeomorphically stable.

**Theorem 1.3.** All Crosscaps are diffeomorphically stable.

In other words, let $\{M_i\}_{i=1}^\infty$ be a sequence of closed Riemannian $n$–manifolds with $\sec M_i \geq k$ so that

$$M_i \longrightarrow C_{k,r}^n$$

in the Gromov-Hausdorff topology. Then all but finitely many of the $M_i$s are diffeomorphic to $\mathbb{R}P^n$.

Since all points of Crosscaps are $(n,0)$–strained, this follows from Theorem 6.1 in [22]. We present a different proof here that is of independent interest because of its relationship to the proofs of the main theorems in [26] and [31].

**Remark 1.4.** Theorem 1.3 when $k = 1$ and $r = \frac{\pi}{2}$ follows from the main theorem in [38] and the fact that $C_{1,\frac{\pi}{2}}^n$ is $\mathbb{R}P^n$ with constant curvature 1.

Section 2 introduces notations and conventions. Section 3 is review of necessary tools from Alexandrov geometry. Section 4 develops machinery and proves Theorem 1.3 in the case when $n \neq 4$. Theorem 1.3 in dimension 4 is proven in Section 5.

Throughout the remainder of the paper, we assume without loss of generality, by rescaling if necessary, that $k = -1, 0$ or 1.

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We are grateful to a referee of this paper for making us aware of the results in [22].

2. Conventions and Notations

We assume a basic familiarity with Alexandrov spaces, including but not limited to [1]. Let $X$ be an $n$–dimensional Alexandrov space and $x, p, y \in X$.

1. We call minimal geodesics in $X$ segments. We denote by $px$ a segment in $X$ with endpoints $p$ and $x$.
2. We let $\Sigma_p$ and $T_pX$ denote the space of directions and tangent cone at $p$, respectively.
3. For $v \in T_pX$ we let $\gamma_v$ be the segment whose initial direction is $v$.
4. Following [30], $\uparrow^p_x \subset \Sigma_x$ will denote the set of directions of segments from $x$ to $p$, and $\uparrow^p_x \in \uparrow^p_x$ denotes the direction of a single segment from $x$ to $p$. 
(5) We let $\angle(x, p, y)$ denote the angle of a hinge formed by $px$ and $py$ and $\tilde{\angle}(x, p, y)$ denote the corresponding comparison angle.

(6) Following [26], we let $\tau : \mathbb{R}^k \to \mathbb{R}_+$ be any function that satisfies

$$\lim_{x_1, \ldots, x_k \to 0} \tau(x_1, \ldots, x_k) = 0,$$

and abusing notation we let $\tau : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ be any function that satisfies

$$\lim_{x_1, \ldots, x_k \to 0} \tau(x_1, \ldots, x_k|y_1, \ldots, y_n) = 0,$$

provided that $y_1, \ldots, y_n$ remain fixed.

When making an estimate with a function $\tau$ we implicitly assert the existence of such a function for which the estimate holds.

(7) We denote by $\mathbb{R}^{1,n}$ the Minkowski space $(\mathbb{R}^{n+1}, g)$, where $g$ is the semi-Riemannian metric defined by

$$g = -dx_0^2 + dx_1^2 + \cdots + dx_n^2$$

for coordinates $(x_0, x_1, \ldots, x_n)$ on $\mathbb{R}^{n+1}$.

(8) We reserve $\{e_j\}_{j=0}^n$ for the standard orthonormal basis in both euclidean and Minkowski space.

(9) We use two isometric models for hyperbolic space,

$$H^n_+ := \{ (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} | - (x_0)^2 + (x_1)^2 + \cdots + (x_n)^2 = -1, \ x_0 > 0 \}$$
and

$$H^n_- := \{ (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} | - (x_0)^2 + (x_1)^2 + \cdots + (x_n)^2 = -1, \ x_0 < 0 \}.$$

(10) We obtain explicit double disks, $D^n_k(r) := D^n_k(r)^+ \cup_{\partial D^n_k(r)^+} D^n_k(r)^-$, by viewing $D^n_k(r)^+$ and $D^n_k(r)^-$ explicitly as

$$D^n_k(r)^+ := \begin{cases} \{ z \in H^n_+ \subset \mathbb{R}^{1,n} | \text{dist}_{H^n_+}(e_0, z) \leq r \} & \text{if } k = -1 \\ \{ z \in \{e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} | \text{dist}_{\mathbb{R}^{n+1}}(e_0, z) \leq r \} & \text{if } k = 0 \\ \{ z \in S^n \subset \mathbb{R}^{n+1} | \text{dist}_{S^n}(e_0, z) \leq r \} & \text{if } k = 1, \end{cases}$$

and

$$D^n_k(r)^- := \begin{cases} \{ z \in H^n_- \subset \mathbb{R}^{1,n} | \text{dist}_{H^n_-}(-e_0, z) \leq r \} & \text{if } k = -1 \\ \{ z \in \{-e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} | \text{dist}_{\mathbb{R}^{n+1}}(-e_0, z) \leq r \} & \text{if } k = 0 \\ \{ z \in S^n \subset \mathbb{R}^{n+1} | \text{dist}_{S^n}(-e_0, z) \leq r \} & \text{if } k = 1. \end{cases}$$

Since $r < \frac{\pi}{2}$ when $k = 1$, $D^n_k(r)^+$ and $D^n_k(r)^-$ are disjoint in all three cases.

3. Basic Tools From Alexandrov Geometry

The notion of strainers [11] in an Alexandrov space forms the core of the calculus arguments used to prove our main theorem. In this section, we review this notion and its relevant consequences. In some sense the idea can be traced back to [26], and some of the ideas that we review first appeared in other sources such as [37] and [39].
Definition 3.1. Let $X$ be an Alexandrov space. A point $x \in X$ is said to be $(n, \delta, r)$–strained by the strainer $\{(a_i, b_i)\}_{i=1}^n \subset X \times X$ provided that for all $i \neq j$ we have
\[
\begin{align*}
\angle (a_i, x, b_j) &> \frac{\pi}{2} - \delta, \quad \angle (a_i, x, b_i) > \pi - \delta, \\
\angle (a_i, x, a_j) &> \frac{\pi}{2} - \delta, \quad \angle (b_i, x, b_j) > \frac{\pi}{2} - \delta, \quad \text{and}
\end{align*}
\]
\[
\min_{i=1,\ldots,n} \{\text{dist}(\{a_i, b_i\}, x)\} > r.
\]

We say a metric ball $B \subset X$ is an $(n, \delta, r)$–strained neighborhood with strainer $\{a_i, b_i\}_{i=1}^n$ provided every point $x \in B$ is $(n, \delta, r)$–strained by $\{a_i, b_i\}_{i=1}^n$.

The following is observed in [39].

Proposition 3.2. Let $X$ be a compact $n$-dimensional Alexandrov space. Then the following are equivalent.

1: There is a (sufficiently small) $\eta > 0$ so that for every $p \in X$
\[
\text{dist}_{G-H}(\Sigma_p, S^{n-1}) < \eta.
\]

2: There is a (sufficiently small) $\delta > 0$ and an $r > 0$ such that $X$ is covered by finitely many $(n, \delta, r)$–strained neighborhoods.

Theorem 3.3. ([11] Theorem 9.4) Let $X$ be an $n$–dimensional Alexandrov space with curvature bounded from below. Let $p \in X$ be $(n, \delta, r)$–strained by $\{(a_i, b_i)\}_{i=1}^n$. Provided $\delta$ is small enough, there is a $\rho > 0$ such that the map $f : B(p, \rho) \to \mathbb{R}^n$ defined by
\[
f(x) = (\text{dist}(a_1, x), \text{dist}(a_2, x), \ldots, \text{dist}(a_n, x))
\]
is a bi-Lipschitz embedding with Lipschitz constants in $(1 - \tau(\delta, \rho), 1 + \tau(\delta, \rho))$.

If every point in $X$ is $(n, \delta, r)$–strained, we can equip $X$ with a $C^1$–differentiable structure defined by Otsu and Shioya in [27]. The charts will be smoothings of the map from the theorem above and are defined as follows: Let $x \in X$ and choose $\sigma > 0$ so that $B(x, \sigma)$ is $(n, \delta, r)$–strained by $\{a_i, b_i\}_{i=1}^n$. Define $d_{i,x}^n : B(x, \sigma) \to \mathbb{R}$ by
\[
d_{i,x}^n(y) = \frac{1}{\text{vol}(B(a_i, \eta))} \int_{z \in B(a_i, \eta)} \text{dist}(y, z).
\]
Then $\varphi_x^n : B(x, \sigma) \to \mathbb{R}^n$ is defined by
\[
\varphi_x^n(y) = (d_{1,x}^n(y), \ldots, d_{n,x}^n(y)).
\]

If $B$ is $(n, \delta, r)$–strained by $\{a_i, b_i\}_{i=1}^n$, any choice of $2n$–directions $\{(\uparrow a_i, \uparrow b_i)\}_{i=1}^n$ where $x \in B$ will be called a set of straining directions for $\Sigma_x$. As in, [11, 39], we say an Alexandrov space $\Sigma$ with curv $\Sigma \geq 1$ is globally $(m, \delta)$-strained by pairs of subsets $\{A_i, B_i\}_{i=1}^m$ provided
\[
|\text{dist}(a_i, b_j) - \frac{\pi}{2}| < \delta, \quad \text{dist}(a_i, b_i) > \pi - \delta,
\]
\[
|\text{dist}(a_i, a_j) - \frac{\pi}{2}| < \delta, \quad |\text{dist}(b_i, b_j) - \frac{\pi}{2}| < \delta
\]
for all $a_i \in A_i$, $b_i \in B_i$ and $i \neq j$. 
Remark 3.9. Note that without the strainer, apply the previous corollary with proof.

Theorem 3.4. (Theorem 9.5, cf also Section 3) Let $\Sigma$ be an $(n - 1)$-dimensional Alexandrov space with curvature $\geq 1$. Suppose $\Sigma$ is globally strained by $\{A_i, B_i\}$. There is a map $\tilde{\Psi} : \mathbb{R}^n \to S^{n-1}$ so that $\tilde{\Psi} : \Sigma \to S^{n-1}$ defined by

$$\tilde{\Psi}(x) = \tilde{\Psi} \circ (\text{dist}(A_1, x), \text{dist}(A_2, x), \ldots, \text{dist}(A_n, x))$$

is a bi-Lipschitz homeomorphisms with Lipshitz constants in $(1 - \tau(\delta), 1 + \tau(\delta))$.

Remark 3.5. The description of $\tilde{\Psi} : \mathbb{R}^n \to S^{n-1}$ in [1] is explicit but is geometric rather than via a formula. Combining the proof in [1] with a limiting argument, one can see that the map $\tilde{\Psi}$ can be given by

$$\tilde{\Psi}(x) = \left( \sum \cos^2(\text{dist}(A_i, x)) \right)^{-1/2}(\cos(\text{dist}(A_1, x)), \ldots, \cos(\text{dist}(A_n, x))).$$

In particular, the differentials of $\varphi^a : B(x, \sigma) \subset X \to \varphi(B(x, \sigma))$ are almost isometries.

Next we state a powerful lemma showing that for an $(n, \delta, r)$ strained neighborhood, angle and comparison angle almost coincide for geodesic hinges with one side in this neighborhood and the other reaching a strainer.

Lemma 3.6. (Lemma 5.6) Let $B \subset X$ be $(1, \delta, r)$-strained by $(y_1, y_2)$. For any $x, z \in B$

$$|\tilde{\alpha}(y_1, x, z) + \tilde{\alpha}(y_2, x, z) - \pi| < \tau(\delta, \text{dist}(x, z) | r)$$

In particular, for $i = 1, 2$,

$$|\alpha(y_i, x, z) - \tilde{\alpha}(y_i, x, z)| < \tau(\delta, \text{dist}(x, z) | r).$$

Corollary 3.7. Let $B \subset X$ be $(1, \delta, r)$-strained by $(a, b)$. Let $\{X^\alpha\}_{\alpha=1}^\infty$ be a sequence of Alexandrov spaces with curv$X^\alpha \geq k$ such that $X^\alpha \to X$. For $x, z \in B$, suppose that $a^\alpha, b^\alpha, x^\alpha, z^\alpha \in X^\alpha$ converge to $a, b, x, \text{ and } z$ respectively. Then

$$|\tilde{\alpha}(a^\alpha, x^\alpha, z^\alpha) - \tilde{\alpha}(a, x, z)| < \tau(\delta, \text{dist}(x, z), \tau(1/\alpha | \text{dist}(x, z)) | r).$$

Proof. The convergence $X^\alpha \to X$ implies that we have convergence of the corresponding comparison angles. The result follows from the previous lemma.

Lemma 3.8. Let $B \subset X$ be $(n, \delta, r)$-strained by $\{(a_i, b_i)\}_{i=1}^n$. Let $\{X^\alpha\}_{\alpha=1}^\infty$ have curv$X^\alpha \geq k$ and suppose that $X^\alpha \to X$. Let $\{(\gamma_{1,\alpha}, \gamma_{2,\alpha})\}_{\alpha=1}^\infty$ be a sequence of geodesic hinges in the $X^\alpha$ that converge to a geodesic hinge $(\gamma_1, \gamma_2)$ with vertex in $B$. Then

$$|\tilde{\alpha}(\gamma_{1,\alpha}(0), \gamma_{2,\alpha}(0)) - \tilde{\alpha}(\gamma_{1}(0), \gamma_{2}(0))| < \tau(\delta, \tau(1/\alpha | \text{len}(\gamma_1), \text{len}(\gamma_2)) | r).$$

Remark 3.9. Note that without the strainer, $\liminf_{\alpha \to \infty} <(\gamma_{1,\alpha}(0), \gamma_{2,\alpha}(0)) \geq <(\gamma_{1}(0), \gamma_{2}(0))$ [12], [1].

Proof. Apply the previous corollary with $x^\alpha = \gamma_{1,\alpha}(0)$, $z^\alpha = \gamma_{1,\alpha}(\varepsilon)$, $x^\alpha \to x$, and $z^\alpha \to z$ to conclude

$$<((\gamma_{1,\alpha}^\alpha, \gamma_{1,\alpha}^\alpha(0)) - <((\gamma_{1}^\alpha, \gamma_{1}(0))) < \tau(\delta, \text{dist}(x, z), \tau(1/\alpha | \text{dist}(x, z)) | r).$$
Similar reasoning with \( x^\alpha = \gamma_{2,\alpha}(0) \), \( z^\alpha = \gamma_{2,\alpha}(\varepsilon) \), \( x = \lim_{\alpha \to \infty} x^\alpha \), and \( z = \lim_{\alpha \to \infty} z^\alpha \) gives

\[
\angle(\uparrow x^\alpha, \gamma_{2,\alpha}(0)) - \angle(\uparrow z^\alpha, \gamma_{2}(0)) < \tau(\delta, \text{dist}(x,z), \tau(1/\alpha|\text{dist}(x,z)|) \mid r).
\]

Since \( \text{dist}(x,z) \) may be as small as we please, the result then follows from Theorem 3.4.

\[\square\]

**Lemma 3.10.** [39] Lemma 1.8.2) Let \( \{(a_i, b_i)\}_{i=1}^n \) be an \((n, \delta, r)\)-strainer for \( B \subset X \). For any \( x \in B \) and \( \mu > 0 \), let \( \Sigma^\mu_x \) be the set of directions \( v \in \Sigma_x \) so that \( \gamma_{[0,\mu]}(v) \) is a segment. For any sufficiently small \( \mu > 0 \), \( \Sigma^\mu_x \) is \( \tau(\delta, \mu) \)-dense in \( \Sigma_x \).

**Corollary 3.11.** Suppose \( X^\alpha \to X \), \( \{(a_i, b_i)\}_{i=1}^n \) is an \((n, \delta, r)\)-strainer for \( B \subset X \), and \((n, \delta, r)\)-strainers \( \{(a_i^\alpha, b_i^\alpha)\}_{i=1}^n \) for \( B^\alpha \subset X^\alpha \) satisfy

\[
(\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^n, B^\alpha) \to (\{(a_i, b_i)\}_{i=1}^n, B).
\]

For any fixed \( \mu > 0 \) and any sequence of directions \( \{v^\alpha\}_{\alpha=1}^\infty \subset \Sigma_x \) with \( x^\alpha \in B^\alpha \), there is a sequence \( \{w^\alpha\}_{\alpha=1}^\infty \subset \Sigma^\mu_x \) with

\[
\angle(w^\alpha, v^\alpha) < \tau(\delta, \mu)
\]

so that a subsequence of \( \{\gamma_{w^\alpha}\}_{\alpha=1}^\infty \) converges to a geodesic \( \gamma : [0, \mu] \to X \).

From Arzela-Ascoli and Hopf-Rinow, we conclude

**Proposition 3.12.** Let \( X \) be an Alexandrov space and \( p, q \in X \). For any \( \varepsilon > 0 \), there is a \( \delta > 0 \) so that for all \( x \in B(p, \delta) \) and all \( y \in B(q, \delta) \) and any segment \( xy \), there is a segment \( pq \) so that

\[
\text{dist}(xy, pq) < \varepsilon.
\]

We end this section by showing that convergence to a compact Alexandrov space \( X \) without collapse implies the convergence of the corresponding universal covers, provided \( |\pi_1(X)| < \infty \). For our purposes, when \( X = C^n_k \), it would be enough to use \[34\] or \[9\].

The key tools are Perelman’s Stability and Local Structure Theorems and the notion of first systole, which is the length of the shortest closed non-contractible curve. Perelman’s proof of the Local Structure Theorem can be found in \[29\], this result is also a corollary to his Stability Theorem, whose proof is published in \[17\].

**Theorem 3.13.** Let \( \{X_i\}_{i=1}^\infty \) be a sequence of \( n \)-dimensional Alexandrov spaces with a uniform lower curvature bound converging to a compact, \( n \)-dimensional Alexandrov space \( X \). If the fundamental group \( \pi_1(X) \) is finite, then

1. A subsequence of the universal covers, \( \{\tilde{X}_i\}_{i=1}^\infty \), of \( \{X_i\}_{i=1}^\infty \) converges to the universal cover, \( \tilde{X} \), of \( X \).
2. A subsequence of the deck action by \( \pi_1(X_i) \) on \( \{\tilde{X}_i\}_{i=1}^\infty \) converges to the deck action of \( \pi_1(X) \) on \( \tilde{X} \).

**Proof.** In \[29\], Perelman shows \( X \) is locally contractible. Let \( \{U_j\}_{j=1}^n \) be an open cover of \( X \) by contractible sets and let \( \mu \) be a Lebesgue number of this cover. By Perelman’s Stability Theorem, there are \( \tau(\frac{1}{\mu}) \)-Hausdorff approximations

\[
h_i : X \to X_i
\]
that are also homeomorphisms. Therefore, if \( i \) is sufficiently large, \( \{ h_i(U_j) \}_{j=1}^n \) is an open cover for \( X_i \) by contractible sets with Lebesgue number \( \mu/2 \). It follows that the first systoles of the \( X_i \)'s are uniformly bounded from below by \( \mu \). Since the minimal displacement of the deck transformations by \( \pi_1(X_i) \) on \( \tilde{X}_i \to X_i \) is equal to the first systole of \( X_i \), this displacement is also uniformly bounded from below by \( \mu \). By pre-compactness, a subsequence of \( \{ \tilde{X}_i \} \) converges to a length space \( Y \). From Proposition 3.6 of [9], a subsequence of the actions \( (\tilde{X}_i, \pi_1(X_i)) \) converges to an isometric action by some group \( G \) on \( Y \). By Theorem 2.1 in [8], \( X = Y/G \). Since the displacements of the (nontrivial) deck transformations by \( \pi_1(X_i) \) on \( \tilde{X}_i \to X_i \) are uniformly bounded from below, the action by \( G \) on \( Y \) is properly discontinuous. Hence \( Y \to Y/G = X \) is a covering space of \( X \). By the Stability Theorem, \( Y \) is simply connected, so \( Y \) is the universal cover of \( X \).

\[ \square \]

**Remark 3.14.** When the \( X_i \) are Riemannian manifolds, one can get the uniform lower bound for the systoles of the \( X_i \)'s from the generalized Butterfly Lemma in [11]. The same argument also works in the Alexandrov case but requires Perelman’s critical point theory, and hence is no simpler than what we presented above.

Lens spaces show that without the noncollapsing hypothesis this result is false even in constant curvature.

4. Cross Cap Stability

The main step to prove Theorem 1.3 is the following.

**Theorem 4.1.** Let \( \{ M^\alpha \}_{\alpha=1}^\infty \) be a sequence of closed Riemannian \( n \)-manifolds with \( \sec M^\alpha \geq k \) so that

\[ M^\alpha \to C^r_{k,r} \]

in the Gromov-Hausdorff topology. Let \( \tilde{M}^\alpha \) be the universal cover of \( M^\alpha \). Then for all but finitely many \( \alpha \), there is a \( C^1 \) embedding

\[ \tilde{M}^\alpha \to \mathbb{R}^{n+1} \setminus \{0\} \]

that is equivariant with respect to the deck transformations of \( \tilde{M}^\alpha \to M^\alpha \) and the \( \mathbb{Z}_2 \)-action on \( \mathbb{R}^{n+1} \) generated by \(-id\).

Two and three manifolds have unique differential structures up to diffeomorphism; so in dimensions two and three Theorems 1.3 and 4.1 follow from the main result of [13]. We give the proof in dimension 4 in section 6. Until then, we assume that \( n \geq 5 \).

**Proof of Theorem 1.3 modulo Theorem 4.1.** By Perelman’s Stability Theorem all but finitely many \( \{ M^\alpha \}_{\alpha=1}^\infty \) are homeomorphic to \( S^n \) (cf [13]). Combining this with Theorem 4.1 and Brown’s Theorem 9.7 in [24] gives an H–cobordism between the embedded image of \( \tilde{M}^\alpha \subset \mathbb{R}^{n+1} \) and the standard \( S^n \). Modding out by \( \mathbb{Z}_2 \), we see that \( M^\alpha \) and \( \mathbb{R}P^n \) are H–cobordant. Since the Whitehead group of \( \mathbb{Z}_2 \) is trivial ( [19], [25], p. 373), any H–cobordism between \( M_\alpha \) and \( \mathbb{R}P^n \) is an S–cobordism and hence a product, which completes the proof. [2, 23, 35] \[ \square \]
The proof of Theorem 1.3 does not exploit any a priori differential structure on the Crosscap. Instead we exploit a model embedding of the double disk
\[ \mathbb{D}_k^n (r) \hookrightarrow \mathbb{R}^{n+1}, \]
whose restriction to either half, \( \mathbb{D}_k^n (r)^\pm \) or \( \mathbb{D}_k^n (r)^- \), is the identity on the last \( n \)-coordinates. By describing the identity \( \mathbb{D}_k^n (r) \rightarrow \mathbb{D}_k^n (r) \) in terms of distance functions, we then argue that this embedding can be lifted to all but finitely many of a sequence \( \{ M^a \} \) converging to \( \mathbb{D}_k^n (r) \).

**The Model Embedding.** Let \( A : \mathbb{D}_k^n (r) \rightarrow \mathbb{D}_k^n (r) \) be the free involution mentioned in Example 1.2. For \( z \in \mathbb{D}_k^n (r) \), we define \( f_z : \mathbb{D}_k^n (r) \rightarrow \mathbb{R} \) by
\[
f_z(x) = h_k \circ \text{dist} (A(z), x) - h_k \circ \text{dist} (z, x) \tag{4.1.1}
\]
where \( h_k : \mathbb{R} \rightarrow \mathbb{R} \) is defined as
\[
h_k(x) = \begin{cases} \frac{1}{2 \sinh r} \cosh(x) & \text{if } k = -1 \\ \frac{x^2}{4r} & \text{if } k = 0 \\ \frac{1}{2 \sin r} \cos(x) & \text{if } k = 1. \end{cases}
\]
Recall that we view \( \mathbb{D}_k^n (r)^\pm \) as metric \( r \)-balls centered at \( p_0 = e_0 \) and \( A(p_0) = -e_0 \) in either \( H_\pm \), \( \{ \pm e_0 \} \times \mathbb{R}^n \), or \( S^n \). For \( i = 1, 2, \ldots, n \) we set
\[
p_i := \begin{cases} \cosh(r)e_0 + \sinh(r)e_i & \text{if } k = -1 \\ e_0 + re_i & \text{if } k = 0 \\ \cos(r)e_0 - \sin(r)e_i & \text{if } k = 1. \end{cases} \tag{4.1.2}
\]
The functions \( \{ f_i \}_{i=1}^n := \{ f_{p_i} \}_{i=1}^n \) are then restrictions of the last \( n \)-coordinate functions of \( \mathbb{R}^{n+1} \) to \( \mathbb{D}_k^n (r)^\pm \). We set \( f_0 := f_{p_0} \). In contrast to \( f_1, \ldots, f_n \), our \( f_0 \) is not a coordinate function. On the other hand its gradient is well defined everywhere on \( \mathbb{D}_k^n (r) \setminus \{ p_0, A(p_0) \} \), even on \( \partial \mathbb{D}_k^n (r)^+ = \partial \mathbb{D}_k^n (r)^- \) where it is normal to \( \partial \mathbb{D}_k^n (r)^+ = \partial \mathbb{D}_k^n (r)^- \).

Define \( \Phi : \mathbb{D}_k^n (r) \rightarrow \mathbb{R}^{n+1} \), by
\[
\Phi = (f_0, f_1, f_2, \cdots, f_n),
\]
and observe that

**Proposition 4.2.** \( \Phi \) is a continuous, \( \mathbb{Z}_2 \)-equivariant embedding.

**Proof.** Write \( \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n \) and let \( \pi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be projection. Since \( f_1, f_2, \cdots, f_n \) are coordinate functions, the restrictions
\[
\pi \circ \Phi|_{\mathbb{D}_k^n (r)^\pm} : \mathbb{D}_k^n (r)^\pm \rightarrow \mathbb{R}^n
\]
are both the identity. From this and the definition of \( f_0 \), we conclude that \( \Phi \) is one-to-one. Since \( \mathbb{D}_k^n (r) \) is compact, it follows that \( \Phi \) is an embedding. The \( \mathbb{Z}_2 \)-equivariance is immediate from definition 4.1.1. \( \Box \)
Lifting the Model Embedding. To start the proof of Theorem 4.1 let \( \{M^\alpha\}_{\alpha=1}^\infty \) be a sequence of closed Riemannian \( n \)-manifolds with \( \sec M^\alpha \geq k \) so that
\[
M^\alpha \rightarrow C_{k,r}^n
\]
and we let \( \{\tilde{M}^\alpha\}_{\alpha=1}^\infty \) denote the corresponding sequence of universal covers. From Theorem 3.13 a subsequence of \( \{\tilde{M}^\alpha\}_{\alpha=1}^\infty \) together with the deck transformations \( \tilde{M}^\alpha \rightarrow M^\alpha \) converge to \((D^n_k(r), A)\). For all but finitely many \( \alpha \), \( \pi_1(M^\alpha) \) is isomorphic to \( \mathbb{Z}_2 \). We abuse notation and call the nontrivial deck transformation of \( \tilde{M}^\alpha \rightarrow M^\alpha \), \( A \).

First we extend definition 4.1.1 by letting \( f^\alpha_z : \tilde{M}^\alpha \rightarrow \mathbb{R} \) be defined by
\[
f^\alpha_z(x) = h_k \circ \text{dist}(A(z), x) - h_k \circ \text{dist}(z, x).
\]
(4.2.1)

Let \( p^\alpha_i \in \tilde{M}^\alpha \) converge to \( p^\alpha_i \in D^n_k(r) \), and for some \( d > 0 \) define \( f^\alpha_{i,d} : \tilde{M}^\alpha \rightarrow \mathbb{R} \) by
\[
f^\alpha_{i,d}(x) = \frac{1}{\text{vol}B(p^\alpha_i, d)} \int_{q^\alpha \in B(p^\alpha_i, d)} f^\alpha_{q}(x).
\]
(4.2.2)

Differentiation under the integral gives

**Proposition 4.3.** The \( f^\alpha_{i,d} \) are \( C^1 \) and \( |\nabla f^\alpha_{i,d}| \leq 2 \).

We now define \( \Phi^\alpha_d : \tilde{M}^\alpha \rightarrow \mathbb{R}^{n+1} \) by
\[
\Phi^\alpha_d = (f^\alpha_0_d, f^\alpha_1_d, f^\alpha_2_d, \ldots, f^\alpha_n_d).
\]
As \( \alpha \rightarrow \infty \) and \( d \rightarrow 0 \), \( \Phi^\alpha_d \) converges to \( \Phi \) in the Gromov–Hausdorff sense. Since \( \Phi \) is an embedding it follows that \( \Phi^\alpha_d \) is one–to–one in the large. More precisely,

**Proposition 4.4.** For any \( \nu > 0 \), if \( \alpha \) is sufficiently large and \( d \) is sufficiently small, then
\[
\Phi^\alpha_d(x) \neq \Phi^\alpha_d(y),
\]
provided \( \text{dist}(x, y) > \nu \).

Since the \( \mathbb{Z}_2 \)-equivariance of \( \Phi^\alpha_d \) immediately follows from definition 4.2.2 all that remains to prove Theorem 4.1 is the following proposition:

**Proposition 4.5.** There is a \( \rho > 0 \) so that \( \Phi^\alpha_d \) is one to one on all \( \rho \)-balls, provided that \( \alpha \) is sufficiently large and \( d \) is sufficiently small.

This is a consequence of Key Lemma 4.7 (stated below), whose statement and proof occupy the remainder of this section.

**Uniform Immersion.** The proof of the Inverse Function Theorem in [32] gives

**Theorem 4.6.** *(Quantitative Immersion Theorem)* Let
\[
\mathbb{R}^n_i := \{(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n+1})\} \subset \mathbb{R}^{n+1}
\]
and let
\[
P_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n_i
\]
be orthogonal projection.
Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be a $C^1$ map so that for some $a \in \mathbb{R}^n$, $\lambda > 0$, and $\rho > 0$, there is an $i \in \{1, \ldots, n + 1\}$ so that

$$|d(P_i \circ F)_a (v)| \geq \lambda |v|$$

and

$$|d(P_i \circ F)_a (v) - d(P_i \circ F)_x (v)| < \frac{\lambda}{2} |v|$$

for all $x \in B(a, \rho)$ and $v \in \mathbb{R}^n$, then $(P_i \circ F)|_{B(a, \rho)}$ is a one-to-one, open map.

We note that every space of directions to $D^n_k(r)$ is isometric to $S^{n-1}$. By proposition [3.2] there are $r, \delta > 0$ so that every point in the double disk has a neighborhood $B$ that is $(n, \delta, r)$–strained. If $B \subset D^n_k(r)$ is $(n, \delta, r)$–strained by $\{a_i, b_i\}_{i=1}^n$, by continuity of comparison angles, we may assume there are sets $B^\alpha \subset M^\alpha (n, \delta, r)$–strained by $\{a_i^\alpha, b_i^\alpha\}_{i=1}^n$ such that

$$(\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^n, B^\alpha) \rightarrow ((a_i, b_i)_{i=1}^n, B).$$

Given $x^\alpha \in B^\alpha$, we let $\varphi_{x^\alpha}$ be as in [3.3.1].

To prove Proposition [4.5] it suffices to prove the following.

**Key Lemma 4.7.** There is a $\lambda > 0$ and $\rho > 0$ so that for all $x^\alpha \in M^\alpha$ there is an $i_{x^\alpha} \in \{0, 1, \ldots, n\}$ such that the function $F := \Phi_d^\alpha \circ (\varphi_{x^\alpha})^{-1}$ satisfies

(1) $$\left|d(P_{i_{x^\alpha}} \circ F)_{\varphi_{x^\alpha} (x^\alpha)} (v)\right| > \lambda |v|$$

and

(2) $$\left|d(P_{i_{x^\alpha}} \circ F)_{\varphi_{x^\alpha} (y)} (v) - d(P_{i_{x^\alpha}} \circ F)_{\varphi_{x^\alpha} (x^\alpha)} (v)\right| < \frac{\lambda}{2} |v|$$

for all $y \in B(x^\alpha, \rho)$ and $v \in \mathbb{R}^n$, provided that $\alpha$ is sufficiently large and $d$ and $\eta$ are sufficiently small.

We show in the next subsection that part 1 of Key Lemma 4.7 holds, and in the following subsection we show that part 2 holds.

**Lower bound on the differential.** We begin by illustrating that, in a sense, the first part of the key lemma holds for the model embedding.

**Lemma 4.8.** There is a $\lambda > 0$ so that for all $v \in T^n_k(r)$ there is a $j (v) \in \{0, 1, \ldots, n\}$ so that

$$|D_v f_{j(v)}| > \lambda |v|.$$

**Proof.** Recall that the double disk $D^n_k(r)$ is the union of two copies of $D^n_k(r)$ that we call $D^n_k(r)^+$ and $D^n_k(r)^{−}$—glued along their common boundary—that throughout this section we call $S := \partial D^n_k(r)^{±}$.

If $x \in D^n_k(r) \setminus S$, then for $i \neq 0$, $\nabla f_i$ is unambiguously defined; moreover,

$$\{\nabla f_i (x)\}_{i=1}^n$$

is an orthonormal basis. Thus the lemma certainly holds on $D^n_k(r) \setminus S$. 

For \( x \in S \) and \( i \in \{1, \ldots, n\} \), we can think of the gradient of \( f_i \) as multivalued. More precisely, for \( x \in S \), we view
\[
S \subset D^n_k(r)^\pm \subset \left\{ \begin{array}{l l}
H^n_{\pm} & \text{if } k = -1 \\
\{ \pm e_0 \} \times \mathbb{R}^n & \text{if } k = 0 \\
S^n & \text{if } k = 1
\end{array} \right.
\]
and define \( \nabla f_i^\pm \) to be the gradient at \( x \) of the coordinate function that extends \( f_i \) to either \( H_{\pm} \), \( \{ \pm e_0 \} \times \mathbb{R}^n \), or \( S^n \).

From definition 4.1 for any \( v \in T_xD^n_k(r) \)
\[
D_v f_i = \left\{ \begin{array}{l l}
\langle \nabla f_i^+, v \rangle & \text{if } v \text{ is inward to } D^n_k(r)^+ \\
\langle \nabla f_i^-, v \rangle & \text{if } v \text{ is inward to } D^n_k(r)^-
\end{array} \right.
\]
Notice that the projections of \( \nabla f_i^+ \) and \( \nabla f_i^- \) onto \( T_xS \) coincide, so for \( v \in T_xS \) we have
\[
D_v f_i = \langle \nabla f_i^+, v \rangle = \langle \nabla f_i^-, v \rangle.
\]
As \( \{ \nabla f_i^+ \}_{i=1}^n \) is an orthonormal basis, the lemma holds for \( v \in TS \) and hence also for \( v \) in a neighborhood \( U \) of \( TS \subset T D^n_k(r)|_S \). Since \( \nabla f_0 \) is well defined on \( S \) and normal to \( S \), for any unit \( v \in T D^n_k(r)|_S \setminus U \), we have \( |D_v f_0| > 0 \). The lemma follows from the compactness of the set of unit vectors in \( T D^n_k(r)|_S \setminus U \). □

Notice that at \( p_k \) and \( A(p_k) \) the gradients of \( f_k \) and \( f_0 \) are colinear. Using this we conclude

**Addendum 4.9.** Let \( p_k \) be any of \( p_1, \ldots, p_n \). There is an \( \varepsilon > 0 \) so that for all \( x \in B(p_k, \varepsilon) \cup B(A(p_k), \varepsilon) \) and all \( v \in T_xD^n_k(r) \), the index \( j(v) \) in the previous lemma can be chosen to be different from \( k \).

**Lemma 4.10.** There is a \( \lambda > 0 \) so that for all \( v \in T_xD^n_k(r) \) there is a \( j(v) \in \{0, 1, \ldots, n\} \) so that
\[
|D_v f_j| > \lambda |v|
\]
for all \( z \in B(p_{j(v)}, d) \), provided \( d \) is sufficiently small.

**Proof.** If not then for each \( i = 0, 1, \ldots, n \) there is a sequence \( \{ z_i^j \}_{i=1}^\infty \subset D^n_k(r) \) with \( \text{dist}(z_i^j, p_i) < \frac{1}{j} \) and a sequence of unit \( v^j \in T_{x^j}D^n_k(r) \) so that
\[
|D_{v^j} f_j| < \frac{1}{j}.
\]
Choose the segments \( x^jz_i^j \) and \( x^jA(z_i^j) \) so that
\[
\angle \left( \left\uparrow_{x^j} z_i^j, v^j \right) \right) = \angle \left( \left\uparrow_{x^j} z_i^j, v^j \right) \right) \text{ and } \]
\[
\angle \left( \left\uparrow_{x^j} z_i^j, v^j \right) \right) = \angle \left( \left\uparrow_{x^j} z_i^j, v^j \right) \right).
\]
After passing to subsequences, we have \( v^j \to v \), \( x^j \to x \) and
\[
x^jz_i^j \to xp_i \]
\[
x^jA(z_i^j) \to xA(p_i).
\]
Lemma 4.12. There is an

for some choice of segments $xp_i$ and $xA(p_i)$. Using Lemma 3.8 and Corollary 3.11 we conclude

\[
\angle \left( \uparrow_{x^j}^{\alpha}, v^j \right) - \angle \left( \uparrow_{x^j}^{\alpha}, v \right) < \tau \left( \delta, \tau \left( \frac{1}{j} \right) \text{dist} \left( x, p_i \right) \right),
\]

\[
\angle \left( \uparrow_{x^j}^{A(\alpha)}, v^j \right) - \angle \left( \uparrow_{x^j}^{A(\alpha)}, v \right) < \tau \left( \delta, \tau \left( \frac{1}{j} \right) \text{dist} \left( x, A(p_i) \right) \right).
\]

(4.10.1)

If $x \notin S$, then the segments $xp_i$ and $xA(p_i)$ are unambiguously defined, and so the previous inequality and the hypothesis $\left| D_{v, f_{x}}^{\alpha} \right| < \frac{1}{j}$, contradict the previous lemma and its addendum.

If $x \in S$ and $v \in T_xS$, then

\[
\angle \left( \uparrow_{x}^{\alpha}, v \right) \text{ and } \angle \left( \uparrow_{x}^{A(\alpha)}, v \right)
\]

are independent of the choice of the segments $xp_i$ and $xA(p_i)$, so the hypothesis $\left| D_{v, f_{x}}^{\alpha} \right| < \frac{1}{j}$ together with the Inequalities [4.10.1] contradict the previous lemma and its addendum. Thus our result holds for $v \in TS$ and hence also for $v$ in a neighborhood $U$ of $TS \subset T\mathbb{D}^n_k(r)|_S$.

For a unit vector $v \in T\mathbb{D}^n_k(r)|_S \setminus U$, we saw in the proof of the previous lemma that for some $\lambda > 0$

\[
\left| D_{v, f_{0}} \right| > \lambda.
\]

(4.10.2)

For $x \in S$, we have unique segments $xp_0$ and $xA(p_0)$, so the hypothesis $\left| D_{v, f_{x}}^{\alpha} \right| < \frac{1}{j}$ and inequalities [4.10.1] contradict Inequality [4.10.2].

Combining the proof of the previous lemma with Addendum [4.9] we get

**Addendum 4.11.** Let $p_k$ be any of $p_1, \ldots, p_n$. There is an $\varepsilon > 0$ so that for all $x \in B(p_k, \varepsilon) \cup B(A(p_k), \varepsilon)$ and all $v \in T_x\mathbb{D}^n_k(r)$, the index $j(v)$ in the previous lemma can be chosen to be different from $k$.

**Lemma 4.12.** There is a $\lambda > 0$ so that for all $v \in T\bar{M}^\alpha$ there is a $j(v) \in \{0, 1, \ldots, n\}$ so that

\[
D_{v, f_{j(v), d}}^{\alpha} \lambda \left| v \right|,
\]

provided $\alpha$ is sufficiently large and $d$ is sufficiently small.

**Proof.** If the lemma were false, then there would be a sequence of unit vectors $\{v^\alpha\}_{\alpha=1}^{\infty}$ with $v^\alpha \in T_{x^\alpha}\bar{M}^\alpha$ such that for all $i$,

\[
\left| D_{v^\alpha, f_{i, d}}^{\alpha} \right| < \tau \left( \frac{1}{\alpha}, d \right).
\]

Let $\lim_{\alpha \to \infty} x^\alpha = x \in \mathbb{D}^n_k(r)$. By Corollary 3.11 for any $\mu > 0$ there is a sequence $\{w^\alpha\}_{\alpha=1}^{\infty}$ with $w^\alpha \in \Sigma^\mu_{x^\alpha}$ such that

\[
\angle (v^\alpha, w^\alpha) < \tau (\delta, \mu).
\]
Since $|\nabla f_{i,d}^\alpha| \leq 2$,
\[
|D_w f_{i,d}^\alpha| < \tau \left( \delta, \mu, \frac{1}{\alpha}, d \right)
\]
for all $i$. After passing to a subsequence, we conclude that $\{\gamma_{w}^\alpha|_{[0,\mu]}\}_{\alpha=1}^\infty$ converges to a segment $\gamma_w|_{[0,\mu]}$. By the previous lemma, there is a $\lambda > 0$ and a $j(w)$ so that for all $z \in B(p_j(w), d)$,
\[
|D_w f_z| > \lambda |w|
\]
provided $d$ is small enough. Moreover, by Addendum 4.11 we may assume that
\[
dist(x, p_j(w)) > 100d > \mu
\]
and
\[
dist(x, A(p_j(w))) > 100d > \mu.
\]
By the Mean Value Theorem, there is a $z_{j(w)}^\alpha \in B\left(p_j^\alpha(w), d\right)$ with
\[
D_w f_{z_{j(w)}}^\alpha = D_w f_{j(w)}^\alpha.
\]
Choose segments $x^\alpha z_{j(w)}^\alpha$ and $x^\alpha A(z_{j(w)}^\alpha)$ in $\tilde{M}^\alpha$ so that
\[
\langle (\uparrow x^\alpha, w^\alpha) \rangle = \langle (\uparrow z_{j(w)}^\alpha, w^\alpha) \rangle
\]
and
\[
\langle (\uparrow x^\alpha, w^\alpha) \rangle = \langle (\uparrow z_{j(w)}^\alpha, w^\alpha) \rangle.
\]
After passing to a subsequence, we may assume that for some $z_{j(w)} \in B(p_j(w), d)$, $x^\alpha z_{j(w)}^\alpha$ and $x^\alpha A(z_{j(w)}^\alpha)$ converge to segments $x z_{j(w)}$ and $x A(z_{j(w)}^\alpha)$, respectively. By Lemma 3.8,
\[
\langle (\uparrow x^\alpha, \gamma_{w}^\alpha(0)) \rangle - \langle (\uparrow x^\alpha, \gamma_{w}^\alpha(0)) \rangle < \tau \left( \delta, \tau (1/\alpha|\mu, \text{dist}(x, z_{j(w)})) \right)
\]
and
\[
\langle (\uparrow x^\alpha, \gamma_{w}^\alpha(0)) \rangle - \langle (\uparrow x^\alpha, \gamma_{w}^\alpha(0)) \rangle < \tau \left( \delta, \tau (1/\alpha|\mu, \text{dist}(x, A(z_{j(w)}))) \right).
\]
Combining the previous two sets of displays with 4.12.3,
\[
|D_w f_{z_{j(w)}}^\alpha - D_w f_{z_j(w)}| < \tau (\delta, \tau (1/\alpha|\mu)).
\]
So by Equation 4.12.4,
\[
|D_w f_{z_{j(w)}}^\alpha - D_w f_{z_j(w)}| < \tau (\delta, \tau (1/\alpha|\mu)),
\]
but this contradicts Inequalities 4.12.1 and 4.12.2.

The first claim of Key Lemma 4.7 follows by combining the previous lemma with the fact that the differentials of the $\varphi^{\eta}_{x^\alpha} \gamma_{w}^\alpha$ are almost isometries.

**Remark 4.13.** Note that when $x^\alpha$ is close to $p_k$ or $A(p_k)$, the desired estimate
\[
|d(P_{i,d} \circ F)\varphi^{\eta}_{x^\alpha}(x^\alpha)(v)| > \lambda |v|
\]
holds with $P_{i,d} = P_k$. This follows from Addendum 4.11 and the proof of the previous lemma.
Equicontinuity of Differentials. In this subsection, we establish the second part of the key lemma. If \( x^\alpha \) is not close to one of the \( p_k \)'s or \( A (p_k) \)'s we will show the stronger estimate

\[
\left| d (F) \phi_{x^\alpha}^\alpha (y) - d (F) \phi_{x^\alpha}^\alpha (x^\alpha) (v) \right| < \frac{\lambda}{2} |v| . \tag{4.13.1}
\]

So at such points, the second part of the key lemma holds with any choice of coordinate projection \( P_{i,x} \).

For \( x^\alpha \) close to \( p_k \) or \( A (p_k) \), we will show

\[
\left| d (P_k \circ F) \phi_{x^\alpha}^\alpha (y) - d (P_k \circ F) \phi_{x^\alpha}^\alpha (x^\alpha) (v) \right| < \frac{\lambda}{2} |v| , \tag{4.13.2}
\]

where \( \lambda \) is the constant whose existence was established in the previous section. Together with remark \( \text{[1.13]} \), this will establish the key lemma.

Suppose \( B \subset \mathbb{D}_k^n (r) \) is \((n, \delta, r)-strained\) by \( \{(a_i, b_i)\}_{i=1}^n \). Let \( x, y \in B \) and let

\[
\varphi : B \rightarrow \mathbb{R}^n
\]

be the map defined in \( \text{3.3.1} \) and \( \text{[27]} \). Set

\[
P_{x,y} := (d \varphi)_y^{-1} \circ (d \varphi)_x : T_x \mathbb{D}_k^n (r) \rightarrow T_y \mathbb{D}_k^n (r).
\]

It follows that \( P_{x,y} \) is a \((\tau, \eta)-\)isometry.

Lemma 4.14. Let \( B \subset \mathbb{D}_k^n (r) \) be \((n, \delta, r)-strained\) by \( \{(a_i, b_i)\}_{i=1}^n \). Given \( \varepsilon > 0 \) and \( x \in B \), there is a \( \rho (x, \varepsilon) > 0 \) so that the following holds.

For all \( k \in \{0, 1, \ldots, n\} \), there is a subset \( E_{k,x} \subset \{B (p_k, d) \cup B (A (p_k), d)\} \) with measure \( \mu (E_{k,x}) < \varepsilon \) so that for all \( z \in B (p_k, d) \setminus E_{k,x} \), all \( y \in B (x, \rho (x, \varepsilon)) \), and all \( v \in \Sigma_x \),

\[
\left| \angle (v, \uparrow_x^z) - \angle (P_{x,y} (v), \uparrow_y^z) \right| < \tau (\varepsilon, \delta, \eta \| \text{dist} (x, z)) \quad \text{and} \quad \left| \angle (v, \uparrow_x^z) - \angle (P_{x,y} (v), \uparrow_y^z) \right| < \tau (\varepsilon, \delta, \eta \| \text{dist} (x, A(z))).
\]

Proof. Let \( C_x = \{z \in \text{Cutlocus} (x) \} \) or \( A(z) \in \text{Cutlocus} (x) \) and set

\[
E_{k,x} = B (C_x, \nu) \cap \{B (p_k, d) \cup B (A (p_k), d)\}.
\]

Choose \( \nu > 0 \) so that \( \mu (E_{k,x}) < \varepsilon \).

By Proposition \( \text{3.12} \), for each \( z \in B (p_k, d) \setminus E_{k,x} \), there is a \( \rho (x, z, \varepsilon) \) so that for all \( y \in B (x, \rho (x, z, \varepsilon)) \) and any choice of segment \( zy \),

\[
\text{dist} (zx, zy) < \varepsilon,
\]

where \( zx \) is the unique segment from \( z \) to \( x \).

Making \( \rho (x, z, \varepsilon) \) smaller and using Corollary \( \text{3.7} \), it follows that for any \( \tilde{a}_i, \tilde{a}_i \in B (a_i, \eta) \),

\[
\left| \angle (\tilde{a}_x, \uparrow_x^z) - \angle (\tilde{a}_y, \uparrow_y^z) \right| < \tau (\delta, \varepsilon, \eta \| \text{dist} (x, z), \text{dist} (y, z)) = \tau (\delta, \varepsilon, \eta \| \text{dist} (x, z)).
\]

It follows that

\[
\left| (d \varphi^\alpha)_x (\uparrow_x^z) - (d \varphi^\alpha)_y (\uparrow_y^z) \right| < \tau (\delta, \varepsilon, \eta \| \text{dist} (x, z)),
\]
and hence
\[ \langle P_{x,y}(\uparrow_z), \uparrow_y \rangle = \langle (d\varphi^n)_y^{-1} \circ (d\varphi^n)_x(\uparrow_z), \uparrow_y \rangle < \tau(\delta, \varepsilon, \eta | \text{dist}(x, z)). \]
So for any \( v \in \Sigma_x \),
\[ |\langle v, \uparrow_x \rangle - \langle (P_{x,y}(v), \uparrow_y \rangle| \leq |\langle v, \uparrow_x \rangle - \langle (P_{x,y}(v), \uparrow_x \rangle| + |\langle (P_{x,y}(v), \uparrow_y \rangle - \langle (P_{x,y}(v), \uparrow_y \rangle| < \tau(\delta, \varepsilon, \eta | \text{dist}(x, z)) \]
\[ = \tau(\varepsilon, \delta, \eta | \text{dist}(x, z)). \]

Using Proposition 3.12 and the precompactness of \( B(p_k, d) \setminus E_{k, x} \), we can then choose \( \rho(x, z, \varepsilon) \) to be independent of \( z \in B(p_k, d) \setminus E_{k, x} \). A similar argument gives the second inequality. \[ \square \]

**Corollary 4.15.** Given any \( \varepsilon > 0 \), there is a \( \rho(\varepsilon) > 0 \) so that for any \( x \in D^n_k(r), y \in B(x, \rho(\varepsilon)) \), and \( z \in B(p_i, d) \setminus E_{i, x} \), we have
\[ |D_v f_z - D_{P_{x,y}(v)} f_z| < \tau(\varepsilon, \delta, \eta | \text{dist}(z, x), \text{dist}(A(z), x)) \]
for all unit vectors \( v \in \Sigma_x \).

**Proof.** Since \( D^n_k(r) \) is compact, the \( \rho(\varepsilon, x) \) from the previous lemma can be chosen to be independent of \( x \).

Given \( x \in D^n_k(r), y \in B(x, \rho(\varepsilon)) \), and \( v \in \Sigma_x \), choose segments \( yz \) and \( yA(z) \) so that
\[ \langle \uparrow_z, P_{x,y}(v) \rangle = \langle \uparrow_y, P_{x,y}(v) \rangle \quad \text{and} \]
\[ \langle \uparrow_{A(z)}, P_{x,y}(v) \rangle = \langle \uparrow_y, P_{x,y}(v) \rangle. \]
Since the segments \( xz \) and \( xA(z) \) are unique, the result follows from the formula for directional derivatives of distance functions, the previous lemma, and the chain rule. \[ \square \]

We can lift a strainer from \( D^n_k(r) \) to any \( \tilde{M}^\alpha \) if \( \text{dist}_{GH}(\tilde{M}^\alpha, D^n_k(r)) \) is sufficiently small. So if \( x^\alpha \) and \( y^\alpha \) are sufficiently close, we define
\[ P_{x^\alpha, y^\alpha} := (d\varphi^n)^{-1} \circ (d\varphi^n)_x : T_{x^\alpha} \tilde{M}^\alpha \to T_{y^\alpha} \tilde{M}^\alpha. \]

**Lemma 4.16.** Let \( i \in \{0, \ldots, n\} \). There is a \( \rho > 0 \) so that for any \( x^\alpha \in \tilde{M}^\alpha \), any \( y^\alpha \in B(x^\alpha, \rho) \), and any unit \( v^\alpha \in T_{x^\alpha} \tilde{M}^\alpha \) we have
\[ |D_{v^\alpha} f^\alpha_{i,d} - D_{P_{x^\alpha,y^\alpha}(v^\alpha)} f^\alpha_{i,d}| < \tau\left(\rho, \frac{1}{\alpha}, \delta, \eta | \text{dist}(x^\alpha, p^\alpha_i), \text{dist}(x^\alpha, A(p^\alpha_i))\right), \]
provided \( d \) is sufficiently small.

**Proof.** If not, then for any \( \rho > 0 \) and some \( i = 0, 1, \ldots, n \), there would be a sequence of points \( x^\alpha \to x \in D^n_k(r) \), a sequence of unit vectors \( \{v^\alpha\}_{\alpha=1}^\infty \) and a constant \( C > 0 \) that is independent of \( \alpha, \delta, \) and \( \eta \) so that
\[ |D_{v^\alpha} f^\alpha_{i,d} - D_{P_{x^\alpha,y^\alpha}(v^\alpha)} f^\alpha_{i,d}| \geq C, \]
\[ \text{dist}(x, p_i) \geq C, \] and
\[ \text{dist}(x, A(p_i)) \geq C \quad (4.16.1) \]
for some \( y^\alpha \in B(x^\alpha, \rho) \). Choose \( \varepsilon > 0 \) and take \( \rho < \rho(\varepsilon) \) where \( \rho(\varepsilon) \) is from the previous corollary. We assume \( B(x, \rho(\varepsilon)) \) is \((n, \delta, r)\)-strained. Let \( y = \lim y^\alpha \) and \( \mu > 0 \) be sufficiently small. By corollary 3.11 there are sequences \( \{w^\alpha\}_{\alpha=1}^\infty \in \Sigma^\mu_{x^\alpha} \) and \( \{\tilde{w}^\alpha\}_{\alpha=1}^\infty \in \Sigma^\mu_{y^\alpha} \) so that

\[
\nabla (\omega^\alpha, w^\alpha) < \tau (\delta, \mu) \quad \text{and} \quad \nabla (P_{x^\alpha, y^\alpha} (w^\alpha), \tilde{w}^\alpha) < \tau (\delta, \mu) \tag{4.16.2}
\]

and subsequences \( \{\gamma_{w^\alpha}\}_{\alpha=1}^\infty \) and \( \{\gamma_{\tilde{w}^\alpha}\}_{\alpha=1}^\infty \) converging to segments \( \gamma_w \) and \( \gamma_{\tilde{w}} \) that are parameterized on \([0, \mu] \). Since \( |\nabla f^\alpha_{i,x}| \leq 2 \), we may assume for a possibly smaller constant \( C \) that

\[
|D_{w^\alpha} f^\alpha_{i,x} - D_{\tilde{w}^\alpha} f^\alpha_{i,x}| \geq C.
\]

Thus for some \( z^\alpha \in B(p^\alpha_i, d) \) with \( \text{dist}_{\text{Haus}}(z^\alpha, E_{i,x}) > 2\nu \),

\[
|D_{w^\alpha} f^\alpha_{i,x} - D_{\tilde{w}^\alpha} f^\alpha_{i,x}| \geq \frac{C}{2}. \tag{4.16.3}
\]

Passing to a subsequence, we have \( z^\alpha \to z \in B(p_i, d) \setminus E_{i,x} \). As in the proof of Lemma 4.12 (Inequality 4.12.5), we have

\[
|D_{w^\alpha} f^\alpha_{i,x} - D_w f_z| < \tau (\delta, \tau (1/\alpha|\mu|)) \quad \text{and} \quad |D_{\tilde{w}^\alpha} f^\alpha_{i,x} - D_{\tilde{w}} f_z| < \tau (\delta, \tau (1/\alpha|\mu|)).
\]

Thus,

\[
|D_{w^\alpha} f^\alpha_{i,x} - D_{\tilde{w}^\alpha} f^\alpha_{i,x}| \leq |D_{w^\alpha} f^\alpha_{i,x} - D_w f_z| + |D_w f_z - D_{\tilde{w}} f_z| + |D_{\tilde{w}} f_z - D_{\tilde{w}^\alpha} f^\alpha_{i,x}| < \tau (\delta, \tau (1/\alpha|\mu|)) \leq \tau (\varepsilon, \delta, \mu, \eta, \tau (1/\alpha|\mu|))
\]

by the previous corollary and Inequalities 4.16.1 and 4.16.2. Choosing \( \varepsilon, \delta, \eta, \mu, \) and \( 1/\alpha \) small enough, we have a contradiction to 4.16.3.

The previous lemma, together with the definitions of \( \Phi^\alpha_d, (\varphi^\alpha)^{-1} \) and \( P_{x^\alpha, y^\alpha} \) establishes the estimates 4.13.1 and 4.13.2 and hence the second part of Key Lemma, completing the proof of Theorem 1.3 except in dimension 4.

5. Recognizing \( \mathbb{R}P^4 \)

To prove Theorem 1.3 in dimension 4, we exploit the following corollary of the fact that \( \text{Diff}_+ (S^3) \) is connected [3].

**Corollary 5.1.** Let \( M \) be a smooth 4-manifold obtained by smoothly gluing a 4-disk to the boundary of the nontrivial 1-disk bundle over \( \mathbb{R}P^3 \). Then \( M \) is diffeomorphic to \( \mathbb{R}P^4 \).

To see that our \( M^\alpha \)'s have this structure, we use standard triangle comparison and argue as we did in the part of Section 4 titled “Lower Bound on Differential” to conclude
**Proposition 5.2.** For any fixed \( \rho_0 > 0 \), \( f_{0,d}^{\alpha} \) does not have critical points on \( M^\alpha \setminus \{ B(p_0^\alpha, \rho_0) \cup B(A(p_0^\alpha), \rho_0) \} \), and \( \nabla f_{0,d}^{\alpha} \) is gradient-like for \( \text{dist} (A(p_0^\alpha), \cdot) \) and \( -\text{dist} (p_0^\alpha, \cdot) \), provided \( \alpha \) is sufficiently large and \( d \) is sufficiently small.

Finally, using Swiss Cheese Volume Comparison (see 1.1 in [13]) we will show

**Proposition 5.3.** There is a \( \rho_0 > 0 \) so that \( \text{dist} (p_0^\alpha, \cdot) \) does not have critical points in \( B(p_0^\alpha, \rho_0) \), provided \( \alpha \) is sufficiently large.

*Proof.* Since \( \text{vol} M^\alpha \to \text{vol} \mathcal{D}_k^n (r) \), \( \text{vol} B(p_0^\alpha, r) \to \text{vol} \mathcal{D}_k^n (r) \). Via Swiss Cheese Volume Comparison (see 1.1 in [13]) we shall see that the presence of a critical point close to \( p_0^\alpha \) contradicts \( \text{vol} B(p_0^\alpha, r) \to \text{vol} \mathcal{D}_k^n (r) \). Suppose \( q_\alpha \) is critical for \( \text{dist} (p_0^\alpha, \cdot) \), and \( \text{dist} (p_0^\alpha, q_\alpha) = d_\alpha \to 0 \). Let \( x, y \) be points in \( \partial \mathcal{D}_k^n (d_\alpha) \) at maximal distance. By Swiss Cheese Comparison and 1.4 in [13],
\[
\text{vol} (B(q_\alpha, 2d_\alpha) \setminus B(p_0^\alpha, d_\alpha)) \leq \text{vol} (\mathcal{D}_k^n (2d_\alpha) \setminus \{ B(x, d_\alpha) \cup B(y, d_\alpha) \}) = \text{vol} (\mathcal{D}_k^n (2d_\alpha)) - 2\text{vol} (\mathcal{D}_k^n (d_\alpha)).
\]

Since
\[
\text{vol} B(p_0^\alpha, d_\alpha) \leq \text{vol} \mathcal{D}_k^n (d_\alpha),
\]
we conclude
\[
\text{vol} (B(q_\alpha, 2d_\alpha)) \leq \text{vol} (\mathcal{D}_k^n (2d_\alpha)) - \text{vol} (\mathcal{D}_k^n (d_\alpha)) < \kappa \cdot \text{vol} \mathcal{D}_k^n (2d_\alpha)
\]
for some \( \kappa \in (0, 1) \). By relative volume comparison for \( \rho \geq 2d_\alpha \),
\[
\kappa > \frac{\text{vol} B(q_\alpha, 2d_\alpha)}{\text{vol} \mathcal{D}_k^n (2d_\alpha)} \geq \frac{\text{vol} B(q_\alpha, \rho)}{\text{vol} \mathcal{D}_k^n (\rho)}
\]
or
\[
\kappa \cdot \text{vol} \mathcal{D}_k^n (\rho) > \text{vol} B(q_\alpha, \rho).
\]

Since
\[
B(p_0^\alpha, r) \subset B(q_\alpha, r + d_\alpha),
\]
\[
\text{vol} B(p_0^\alpha, r) < \kappa \cdot \text{vol} \mathcal{D}_k^n (r + d_\alpha).
\]
Letting \( d_\alpha \to 0 \), we conclude that
\[
\text{vol} B(p_0^\alpha, r) < \kappa \cdot \text{vol} \mathcal{D}_k^n (r),
\]
a contradiction. \( \square \)

An identical argument shows

**Proposition 5.4.** There is a \( \rho_0 > 0 \) so that \( \text{dist} (A(p_0^\alpha), \cdot) \) does not have critical points in \( B(A(p_0^\alpha), \rho) \), provided \( \alpha \) is sufficiently large.
Combining the previous three propositions, we see that \((f_{0,d}^{-1}(0), d)\) is diffeomorphic to \(S^3\). By Geometrization, \((f_{0,d}^{-1}(0)) / \{\text{id}, A\}\) is diffeomorphic to \(\mathbb{RP}^3\). If \(\rho_0\) is as in Proposition 5.2, it follows that \((f_{0,d}^{-1}([\rho_0, 0])) / \{\text{id}, A\}\) is the nontrivial 1–disk bundle over \(\mathbb{RP}^3\). Thus \(\tilde{M}^\alpha\) consists of two smooth 4–disks that get interchanged by \(A\). Thus \(\tilde{M}^\alpha\) has the structure of Corollary 5.1 and is hence diffeomorphic to \(\mathbb{RP}^4\).

Remark 5.5. The proof of Perelman’s Parameterized Stability Theorem [17] can substitute for Geometrization to allow us to conclude that \(f^{-1}(0) / \{\text{id}, A\}\) is homeomorphic and therefore diffeomorphic to \(\mathbb{RP}^3\). The need to cite the proof rather than the theorem stems from the fact that the definition of admissible functions in [17] excludes \(f_{0,d}\). It is straightforward (but tedious) to see that the proof goes through for an abstract class that includes \(f_{0,d}\).

The fact that \(\mathbb{RP}^4\) admits exotic differential structures can be seen by combining [18] with either [4] or [7].

REFERENCES

[1] Y. Burago, M. Gromov, G. Perelman, A.D. Alexandrov spaces with curvatures bounded from below, I, Uspechi Mat. Nauk. 47 (1992), 3–51.
[2] D. Barden, The structure of manifolds, Ph.D. Thesis, Cambridge University, Cambridge, England.
[3] J. Cerf, La stratification naturelle des espaces de fonctions diff´erentiables r´eelles et le th´eor`eme de la pseudo-isotopie, Publ. Math. I.H.E.S. 39 (1970), 5-173.
[4] S. E. Cappell and J. L. Shaneson, Some new four-manifolds. Ann. of Math. 104 (1976), 61-72.
[5] J. Cheeger, Comparison and finiteness theorems for Riemannian manifolds, Thesis, Princeton University, 1967.
[6] J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math. 92 (1970) 61-74.
[7] R. Fintushel and R. Stern, An exotic free involution on \(S^4\), Ann. of Math. 113 (1981), 357-365.
[8] K. Fukaya, Theory of convergence for Riemannian orbifolds. Japan. J. Math., 12 (1986), 121–160.
[9] K. Fukaya and T. Yamaguchi, Isometry groups of singular spaces. Math. Z. 216 (1994), 31–44.
[10] R. Greene and H. Wu, Integrals of subharmonic functions on manifolds of nonnegative curvature, Inventiones Math. 27(1974) 265-298.
[11] K. Grove and P. Petersen, Bounding homotopy types by geometry, Ann. of Math. 128 (1988), 195-206.
[12] K. Grove and P. Petersen, Manifolds near the boundary of existence, J. Diff. Geom. 33 (1991), 379-394.
[13] K. Grove and P. Petersen, Volume comparison à la Alexandrov, Acta. Math. 169 (1992), 131-151.
[14] K. Grove and K. Shiohama, A generalized sphere theorem, Ann. of Math. 106 (1977), 201-211.
[15] K. Grove and F. Wilhelm, Hard and soft packing radius theorems. Ann. of Math. 142 (1995), 213–237.
[16] K. Grove and F. Wilhelm, Metric constraints on exotic spheres via Alexandrov geometry. J. Reine Angew. Math. 487 (1997), 201–217.
[17] V. Kapovitch, Perelman’s stability theorem. Surveys in differential geometry. 11 (2007), 103-136.
[18] I. Hambleton, M. Kreck, and Teichner, Non-orientable 4–manifolds with fundamental group of order 2. Trans. Amer. Math. Soc. 344 (1994), 649-665.
[19] G. Higman, The units of group-rings, Proc. London Math. Soc. 46 (1940), 231–248.
[20] W. LaBach, On diffeomorphisms of the \(n\)–disk, Proc. Japan Acad. 43 (1967), 448-450.
[21] M. Kervaire and J. Milnor, Groups of homotopy spheres: I, Ann. of Math. 77 (1963), 504-537.
[22] Kuwae, K., Machigashira, Y., and Shioya T., Sobolev spaces, Laplacian, and heat kernel on Alexandrov spaces, Math. Z. 238 (2001), no. 2, 269–316.
[23] B. Mazur, Relative neighborhoods and the theorems of Smaie, Ann. of Math 77, (1963), 232-249.
[24] J. Milnor, Lectures on the H-Cobordism Theorem, Princeton University Press (1965).
[25] J. Milnor, Whitehead torsion Bull. Amer. Math. Soc. 72 (1966), 358–426.
[26] Y. Otsu, K. Shiohama and T. Yamaguchi, A new version of differentiable sphere theorem. Invent. Math. 98 (1989), 219–228.
[27] Y. Otsu, T. Shioya, The Riemannian Structure of Alexandrov Spaces J. Differential Geometry 39 (1994), 629–658.
[28] N. Li, X. Rong, Relative Volume Rigidity in Alexandrov Geometry, Pacific J. Math. 259 (2012), no. 2, 387–420.
[29] G. Perelman, Alexandrov spaces with curvature bounded from below II, preprint 1991.
[30] A. Petrunin, Semiconeave functions in Alexandrov’s Geometry Surv. in Diff. 11 (2007), 137-201.
[31] C. Pro, M. Sill., and F. Wilhelm, The diffeomorphism type of manifolds with almost maximal volume, preprint.
[32] W. Rudin, Principles of mathematical analysis. Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, (1976)
[33] K. Shiohama, T. Yamaguchi, Positively curved manifolds with restricted diameters, Perspectives in Math. 8 (1989), 345-350.
[34] C. Sormani, G. Wei, Universal covers for Hausdorff limits of noncompact spaces Trans. Amer. Math. Soc. 356 (2004), 1233 – 1270.
[35] J. Stallings, Projective class groups and Whitehead groups, (mimeographed) Rice University, Houston, Texas
[36] N. Steenrod, Topology of Fibre Bundles, Princeton U. Press, 1951.
[37] F. Wilhelm, Collapsing to almost Riemannian spaces. Indiana Univ. Math. J. 41 (1992), 1119–1142.
[38] T. Yamaguchi, Collapsing and pinching under a lower curvature bound. Ann. of Math. 133 (1991), 317–357.
[39] T. Yamaguchi, A convergence theorem in the geometry of Alexandrov spaces. Actes de la Table Ronde de Géométrie Différentielle. (1992), 601–642.

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