Completely and Partially Integrable Systems of Total Differential Equations

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Abstract

Constrained Hamiltonian systems are investigated by using the Hamilton-Jacobi method. Integration of a set of equations of motion and the action function is discussed. It is shown that we have two types of integrable systems: a) Partially integrable systems, where the set of equations of motion is only integrable. b) Completely integrable systems, where the set of equations of motion and the action function is integrable. Two examples are studied.

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1 Introduction

Recently the Hamilton-Jacobi method was initiated [1-5] to investigate singular system. The equivalent Lagrangian method [6] is used to obtain the equations of motion as total differential equations in many variables, which require the investigation of integrability conditions. If the system is integrable, one can solve the equations of motion without using any gauge fixing conditions [3,4]. In order to obtain the path integral quantization of constrained systems [7-11], we have to discuss the integrability conditions in terms of the action. In previous works [1-5] the integrability conditions on the set of equations of motion is discussed without considering the integrability conditions of the action function. Also in reference [7], we discuss the integrability conditions of a set of total differential equations and the action function as well.

The aim of this paper is to show that it is not possible to obtain the quantization for partially integrable systems and one should consider the integrability conditions for the whole set of equations of motion and the action function.

The plan of our paper is the following:

In section 2, we present the Hamilton-Jacobi method. In section 3, integration of constrained systems is discussed. In section 4 two examples are worked out and finally in section 5 we present our conclusions.

2 The Hamilton-Jacobi method

In the Hamilton-Jacobi method if we start with a singular Lagrangian $L = L(q, \dot{q}, t)$, $i = 1, 2, ..., n$, with Hessian matrix of rank $(n - r)$, $r < n$, then the generalized momenta can be written as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, ..., n - r,$$

$$p_{\mu} = \frac{\partial L}{\partial t_{\mu}}, \quad \mu = n - r + 1, ..., n,$$

where $q_i$ are divided into two sets, $q_a$ and $t_{\mu}$. Since the rank of the Hessian matrix is $(n - r)$, one can solve the expressible velocities from (1) and after
substituting in (2), we get

\[ p_\mu = -H_\mu(q_\nu, t_\mu, p_\sigma; t). \]  

(3)

The canonical Hamiltonian \( H_0 \) reads

\[ H_0 = p_\alpha \dot{q}_\alpha + p_\mu t_\mu | t_\nu = -H_\nu - L(t, q_\nu, t_\nu, \dot{q}_\alpha), \quad \mu, \nu = n - r + 1, \ldots, n. \]  

(4)

The set of Hamilton-Jacobi partial differential equations [HJPDE] is expressed as [1, 2]

\[ H'_\alpha \left( t_\beta, q_\alpha, \frac{\partial S}{\partial q_\alpha}, \frac{\partial S}{\partial t_\alpha} \right) = 0, \quad \alpha, \beta = 0, n - r + 1, \ldots, n, \]  

(5)

where \( p_\beta = \frac{\partial S[q_\alpha; t_\alpha]}{\partial t_\beta} \) and \( p_\alpha = \frac{\partial S[q_\alpha; t_\alpha]}{\partial q_\alpha} \) with \( t_0 = t \) and \( S \) being the action.

The equations of motion are obtained as total differential equations in many variables as follows [1,2]:

\[ dq_\alpha = \frac{\partial H'_\alpha}{\partial p_\alpha} dt_\alpha, \quad dp_\alpha = \frac{\partial H'_\alpha}{\partial q_\alpha} dt_\alpha, \quad dp_\beta = -\frac{\partial H'_\beta}{\partial t_\beta} dt_\alpha. \]  

(6)

\[ dz = \left( -H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_\alpha} \right) dt_\alpha; \]  

(7)

\[ \alpha, \beta = 0, n - r + 1, \ldots, n, a = 1, \ldots, n - r \]

where \( z = S(t_\alpha; q_\alpha) \).

## 3 Completely and Partially Integrable Systems

As was clarified, that the equations (6,7) are obtained as total differential equations in many variables, which require the investigation of integrability conditions. To achieve this goal we define the linear operator \( X_\alpha \) which corresponds to total differential equations (6,7) as

\[ X_\alpha f(t_\beta, q_\alpha, p_\sigma, z) = \frac{\partial f}{\partial t_\alpha} + \frac{\partial H'_\alpha}{\partial p_\alpha} \frac{\partial f}{\partial q_\alpha} - \frac{\partial H'_\alpha}{\partial q_\alpha} \frac{\partial f}{\partial p_\alpha}. \]
\[ +(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}) \frac{\partial f}{\partial z}, \]
\[ = [H'_\alpha, f] - \frac{\partial f}{\partial z} H'_\alpha, \]
\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r, \]
where the commutator \([,]\) is the square bracket (for details, see the appendix).

**Lemma.** A system of total differential equations (6,7) is integrable if and only if
\[ [H'_\alpha, H'_\beta] = 0, \quad \forall \alpha, \beta. \]

**Proof.** Suppose that (9) is satisfied, then
\[ (X_\alpha, X_\beta)f = (X_\alpha X_\beta - X_\beta X_\alpha)f, \]
\[ = [H'_\alpha, [H'_\beta, f]] - [H'_\beta, [H'_\alpha, f]] - 2 \frac{\partial f}{\partial z} [H'_\alpha, H'_\beta]. \]

Now we apply the Jacobi relation
\[ [f, [g, h]] = [g, [h, f]] + [h, [f, g]], \]
to right side of formula (10), we find
\[ (X_\alpha, X_\beta)f = [[H'_\alpha, H'_\beta], f] - \frac{\partial f}{\partial z} [H'_\alpha, H'_\beta]. \]

From (9), we conclude that
\[ (X_\alpha, X_\beta)f = 0. \]
Conversely, if the system is Jacobi (integrable), then (13) is satisfied for any \(\alpha\) and \(\beta\) and we get
\[ [H'_\alpha, H'_\beta] = 0. \]

Now the total differential for any function \(F(t_\beta, q_a, p_a)\) can be written as
\[ dF = \frac{\partial F}{\partial q_a} dq_a + \frac{\partial F}{\partial p_a} dp_a + \frac{\partial F}{\partial t_\alpha} dt_\alpha, \]
\[ = \left( \frac{\partial F}{\partial q_a} \frac{\partial H'_\alpha}{\partial q_a} - \frac{\partial F}{\partial p_a} \frac{\partial H'_\alpha}{\partial p_a} + \frac{\partial F}{\partial t_\alpha} \right) dt_\alpha, \]
\[ = \{F, H'_\alpha\} dt_\alpha, \]
where the commutator \{,\} is the Poisson bracket. Now, using this result, we have
\[ dH'_\beta = \{H'_\beta, H'_\alpha\} dt_\alpha, \]
and, consequently, the integrability condition (14) reduces to
\[ dH'_\alpha = 0, \forall \alpha. \]
This is the necessary and sufficient condition that the system (6,7) of total differential equations be completely integrable and we call this system as *Completely Integrable Model*. However, equations (6) form here by themselves a completely integrable system of total differential equations. If these are integrated, then only simple quadrature has to be carried out in order to obtain the action [1,2].

On the other hand, we must emphasis that the total differential equations can be very well be completely integrable without (14) holding and therefore without the total system (6) and (7) being integrable and we call this system as *Partially Integrable Model*. In fact, if \[ \{H'_\beta, H'_\alpha\} = F_m(t, t_\mu), \]
where \(F_m\) are functions of \(t_\alpha\) and \(m\) is integer, then the total differential equations (6), will be integrable.

4 Examples

To clarify the ideas given in the previous sections, we shall consider two examples: The first one is a partially integrable system and the second one is a completely integrable system.

4.1 A partially integrable system

Let us consider, the following singular Lagrangian
\[ L = \frac{1}{2} \left( \dot{q}_1^2 + \frac{1}{q_2^2} \right), \]
where \(q_2 \neq 0\). The generalized momenta are calculated as
\[ p_1 = \dot{q}_1, \quad p_2 = 0. \]
Following the Hamilton-Jacobi method, we obtain the set of Hamilton-Jacobi partial differential equations as follows,

\[ H'_{0} = p_{0} + \frac{1}{2} \left( p_{1}^{2} - \frac{1}{q_{2}^{2}} \right) = 0, \quad H'_{0} = p_{2} = 0. \] (20)

This set leads us to obtain the following total differential equations

\[ dq_{1} = p_{1} dt, \quad dp_{1} = 0, \quad dp_{2} = \frac{1}{q_{2}} dt, \] (21)

\[ dz = \frac{1}{2} \left( p_{1}^{2} + \frac{1}{q_{2}^{2}} \right) dt. \] (22)

Now, equations of motion (21) are integrable and have the following solutions,

\[ q_{1} = c_{1} t + c_{2}, \quad p_{1} = c_{2}, \] (23)

\[ p_{2} = 0, \quad q_{2} = c_{3} \neq 0, \] (24)

where \( c_{1}, c_{2} \) and \( c_{3} \) are arbitrary constants.

Let us investigate the integrability conditions in terms of the action. From equation (14), we obtain

\[ \{ H'_{2}, H'_{0} \} = \frac{1}{q_{2}} \neq 0, \] (25)

One should notice that, the integrability conditions (25) are not satisfied. Hence, the action function is not integrable, and it has no unique solution.

**4.2 A completely integrable system**

As a second example, let us consider a two dimensional particle in a uniform circular motion, the Lagrangian of this system is given by [12]

\[ L = \frac{1}{2} m \omega (q^{a} \epsilon_{ab} \dot{q}^{b} - \omega q^{a} g_{ab} q^{b}). \] (26)

Here, \( m \) is a mass parameter, \( a, b = 1, 2 \), \( g_{ab} \) is the metric tensor of a two dimensional Euclidean space and \( \epsilon_{ab} \) is the completely antisymmetric tensor \( (\epsilon_{12} = +1) \).
The canonical momenta $p_a$ conjugated to the generalized coordinate $q^a$ are obtained as

$$p_a = -\frac{1}{2}m\omega\epsilon_{ab}q^b.$$  \hspace{1cm} (27)

The canonical Hamiltonian reads

$$H^0 = \frac{m\omega^2}{2}q^a q_a.$$  \hspace{1cm} (28)

Following the Hamilton-Jacobi method, we obtain the set of Hamilton-Jacobi partial differential equations as follows:

$$H'^0 = p_0 + \frac{m\omega^2}{2}q^a q_a = 0, \hspace{1cm} (29)$$

$$H'^a = p_a + \frac{1}{2} m\omega\epsilon_{ab} q^b = 0. \hspace{1cm} (30)$$

The equations of motion and the action function are obtained as total differential equations as follows:

$$dp_a = -(m\omega^2 q^a) dt + (\frac{m\omega}{2}\epsilon_{ab}) dq^b,$$  \hspace{1cm} (31)

$$dS = -(\frac{m\omega^2}{2} q^a q_a) dt + p_a dq^a.$$  \hspace{1cm} (32)

The next step is to check the integrability conditions. The total variation of the constraint, lead us to obtain $dq^a$ in terms of $dt$ as follows

$$dq^a = -(\omega \epsilon^{ab} q_b) dt.$$  \hspace{1cm} (33)

Using (30) and (32, 33), we found after some calculations that the integrable action has the form

$$S = \int dt \left[ \frac{m}{2}(\dot{q}_1)^2 - \frac{m\omega^2}{2}(q_1)^2 \right].$$  \hspace{1cm} (34)

This result coincide with the results obtained in reference [12], by using the Senjanovic [13] and the Batalin, Fradkin, Tyutin ($BFT$) [14] methods.

In fact the action in (34) is the integrable action function for the reduced system in one dimensional harmonic oscillator.
5 conclusion

In this work we have investigated the integrability conditions in terms of the action. In order to have a completely integrable system (6,7), the integrability conditions (14) must be satisfied. In the first example, even though the equation of motion (21) are integrable, the action function is not integrable and hence, it has no unique solution. In this case we could not obtain the path integral quantization for the model (18). For the system (26), the integrability conditions lead us to obtain the integrable action in terms of the canonical variables and this result coincide with the results obtained in reference [12]. The obtained reduced system is an one dimensional harmonic oscillator.

In order to obtain the path integral formulation of constrained systems, we propose to use the action and the integrability conditions on it, without any need to solve the equations of motion. In other words, we must have a completely integrable model in order to quantize the constrained system.

Completely integrable models are of particular interest from the quantum point of view and hopefully, to discuss the quantization subtleties and non-perturbative effects [15-17]. These results encourage us to investigate the integrable models in two-dimensional Dilaton Gravity by using the Hamilton – Jacobi method, and this study is now under investigation.

A Square brackets and Poisson brackets

In this appendix we shall give a brief review on two kinds of commutators: the square and the Poisson brackets.

The square bracket is defined as

\[
[F, G]_{q_i p_i z_i} = \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} + \frac{\partial F}{\partial p_i} (p_i \frac{\partial G}{\partial z_i} - p_i \frac{\partial F}{\partial z_i}).
\] (35)

The Poisson bracket is defined as

\[
\{f, g\}_{q_i p_i} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i}.
\] (36)

According to above definitions, the following relation holds

\[
[H'_\alpha, H'_\beta] = \{H'_\alpha, H'_\beta\}.
\] (37)
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