CAN A LATTICE STRING HAVE A VANISHING COSMOLOGICAL CONSTANT?

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Abstract

We prove that a class of one-loop partition functions found by Dienes, giving rise to a vanishing cosmological constant to one-loop, cannot be realized by a consistent lattice string. The construction of non-supersymmetric string with a vanishing cosmological constant therefore remains as elusive as ever. We also discuss a new test that any one-loop partition function for a lattice string must satisfy.
1. Introduction

One of the most serious problems of the current theory of particle physics is its inability to account in a natural way for the large size of the universe and the smallness of the cosmological constant $\Lambda_{\text{cos}}$. If we ignore the condensates, then the cosmological constant is computed from vacuum loops. In ordinary quantum field theory, vacuum loops are infinite so this computation cannot be carried out reliably. Even if an ultraviolet cutoff is put in to render them finite, a fine-tuning of the cutoff to many many orders of magnitude will be needed to obtain the observed cosmological constant. Baby universes [1] have been suggested as a mechanism to give a small cosmological constant, but we will not discuss that scenario here.

Since fermion loops give an opposite sign than boson loops, vacuum loops can be rendered small, or even zero, by a suitable combination of fermionic and bosonic contributions. It is this classical mechanism to one-loop order which we will discuss in this paper. The combined contribution is zero for a supersymmetric theory, but unfortunately our world is not supersymmetric — not to the required accuracy of millivolts anyway. Nevertheless, there are still infinitely many other ways to arrange such a cancellation between the fermionic and the bosonic contributions.

In a superstring, the vacuum loops are finite. Moreover, the bosonic and fermionic mass spectra are highly constrained, so it becomes possible and interesting to ask whether a non-supersymmetric string theory can give rise to a zero cosmological constant. The cosmological constant in a string theory can be computed by integrating (over the modular-parameter $\tau$ in the fundamental region of the modular group) the partition function of the string, so the problem becomes that of finding the right partition functions with a zero integral.

There have been various approaches towards resolving the cosmological constant problem within the context of string theory. Taylor and Itoyama [2] found that $\Lambda_{\text{cos}}$ is exponentially suppressed if the string spectrum has equal numbers of massless bosons and fermions; however, the vast majority of self-consistent string models do not have this property. Ginsparg and Vafa [3] examined the case of toroidally compactified strings, and found that $\Lambda_{\text{cos}}$ is extremized when these compactified theories have enhanced gauge symmetries (i.e., at special background-field expectation values). Unfortunately, no cases were found for which the extremized value was zero. A more interesting suggestion was that of Moore [4], who proposed a certain modular-form symmetry known as Atkin-Lehner symmetry as a mechanism for obtaining a vanishing cosmological constant in the absence of spacetime supersymmetry. The basic idea is that any string theory giving rise to an Atkin-Lehner-symmetric partition function would have a vanishing one-loop cosmological
constant. Unfortunately, Moore [5], Taylor [6], and Schellekens [7] were unable to construct string models in $D > 2$ spacetime dimensions which give rise to such partition functions, and Balog and Tuite [8] succeeded in proving that no such string models can exist. Dienes [9] then was able to generalize the Atkin-Lehner mechanism in a variety of ways such that the conclusions of the Balog-Tuite no-go theorem are avoided. It therefore remains an open question as to whether string models exist with partition functions displaying these generalized Atkin-Lehner symmetries.

Another promising approach was obtained by Dienes [10] recently, who found a class of modular-invariant partition functions, given in eqs. (1a,b) below, which gives rise to a zero cosmological constant to one-loop order, though this class of partition functions does not exhibit an Atkin-Lehner or generalized Atkin-Lehner symmetry. Moreover, his partition functions satisfy a number of additional constraints (e.g. they have no on-shell tachyons) which physically acceptable strings are expected to obey. The partition functions he found are the kind that one would obtain from a lattice string [11], but after looking over more than 120,000 such strings with the help of a computer, he was unable to find a consistent string with such a partition function [10]. We will show in this paper that a string within this class of partition functions does not exist. As a result, the construction of a non-supersymmetric string with a zero cosmological constant remains as elusive as ever. Dienes’ proposal is briefly summarized in Sec. 2 below.

A lattice string [11,15] is defined by a self-dual lattice $\Lambda$ of signature $(22,10)$, and a fermionic vector $v$, and its partition function is modular invariant. On the other hand, modular invariant partition functions such as Dienes’ do not have to come from a single lattice $\Lambda$, since for instance linear combinations of modular-invariant functions are modular-invariant. In this paper we apply lattice techniques to investigate the question whether any consistent lattice string can be found corresponding to one of Dienes’ partition functions. We will be able to quickly rule out in Sec. 3 half of his class. In Sec. 4 we show that strings cannot be constructed from the remaining half of his class either, provided that the strings satisfy the half-norm property (given in eq. (9)). This property is extremely natural given the class of possible partition functions, and indeed is consistent with the type of string with which Dienes was concerned. In Ref. [12] we shall consider what can be proved when we drop that restriction and consider instead all conceivable strings.

As a result of this study, we have also come up with a new test any partition function coming from a lattice string must satisfy. This is contained in the Corollary in Sec. 4. This test can be used to check whether an interesting looking partition function could come from a lattice string.

Some of the terminology of lattices used here can be found in the Appendix. For an
introduction to the basic theory of lattices, see e.g. Ref. 13 and particularly Ref. 14. A discussion of the lattice string can be found in Ref. 15.

2. Dienes’ class of partition functions

Dienes’ one-loop partition functions are $\text{Im}(\tau)^{-1} \eta(\tau)^{-24} \eta(\tau)^{-12} T(\tau, \bar{\tau})$, for $T$ given by

$$ T(\tau, \bar{\tau}) = cQ(\tau, \bar{\tau}) + I(\tau, \bar{\tau}), $$

(1a)

where $Q$ is given by

$$ Q(\tau, \bar{\tau}) = \bar{\theta}_2^2 \theta_2^2 \{\theta_2^4 \theta_4^4 \bar{\theta}_2^4 \theta_4^4 - \theta_2^8 \bar{\theta}_2^8 - \bar{\theta}_2^8 \theta_2^8 + \theta_2^{12} [4 \theta_2^8 \bar{\theta}_2^4 + 13 \theta_2^4 \bar{\theta}_2^8 \theta_2^4] \} $$

$$ + \bar{\theta}_4^2 \theta_4^2 \{\theta_4^4 \theta_4^4 \bar{\theta}_4^4 \theta_4^4 - \theta_4^8 \bar{\theta}_4^8 - \bar{\theta}_4^8 \theta_4^8 + \theta_4^{12} [4 \theta_4^8 \bar{\theta}_4^4 + 13 \theta_4^4 \bar{\theta}_4^8 \theta_4^4] \} $$

$$ + \bar{\theta}_2^4 \theta_2^4 \theta_4^4 \bar{\theta}_4^4 \theta_4^4 - \theta_2^8 \bar{\theta}_2^8 - \bar{\theta}_2^8 \theta_2^8 + \theta_2^{12} [4 \theta_2^8 \bar{\theta}_2^4 + 13 \theta_2^4 \bar{\theta}_2^8 \theta_2^4] \} $$

(1b)

and where $I$ is any arbitrary modular-invariant function of $\tau$ and $\bar{\tau}$ with the property that the Taylor expansion $\sum m,n a_{mn} q^m \bar{q}^n$ of $\eta(\tau)^{-24} \eta(\tau)^{-12} I(\tau, \bar{\tau})$ satisfies $a_{mn} = -a_{nm}$. Here and throughout this paper, $q \defeq \exp(\pi i \tau)$ and $\bar{q} \defeq \exp(-\pi i \bar{\tau})$. The functions $\theta_i \defeq \theta_i(\tau)$ and $\bar{\theta}_i \defeq \theta_i(-\bar{\tau}) = \bar{\theta}_i(\tau)$ are Jacobi theta functions defined in eqs. (A.1), and $\eta$ denotes the Dedekind eta function. Hence $I$ is of the form $(\theta_2 \theta_3 \theta_4)^4 [X(\tau, \bar{\tau}) - X(-\bar{\tau}, -\tau)]$. Functions $I$ of this type will not contribute to the cosmological constant, and hence can be left arbitrary, but they certainly could affect whether a lattice corresponding to the partition function $T(\tau, \bar{\tau})$ of (1a) could be found.

Note that eq. (1b) implies the string has 22 left-moving bosonic degrees of freedom and 10 right-moving ones.

The constant $c$ in (1a) is limited by the off-shell tachyon constraint of Ref. [10] to be a rational number of the form $n/32$, with $1 \leq n \leq 10$. We can derive this [12,16] simply by counting unit vectors without using the off-shell tachyon constraint, but this restriction on $c$ is not needed in this paper save for a convenient classification of Dienes’ partition functions into two classes which we will consider in Secs. 3 and 4 respectively. In Sec. 3, we shall show that a lattice string cannot be constructed if $|c| > 5/32$. In Sec. 4, we will show that a lattice string cannot be constructed for any $c$ if the half norm property (discussed in eq. (9) below) is satisfied.

3. The $|c| > 5/32$ case

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Before proceeding to the proof, let us first define a few things to establish notation. Also see the Appendix for further details.

A lattice \( \Lambda \) shall be called \( v \)-even for some vector \( v \) (not necessarily in \( \Lambda \)) if

\[
r^2 + 2r \cdot v \equiv 0 \pmod{2} \ \forall r \in \Lambda.
\]

(2)

It is easy to show that a lattice \( \Lambda \) can be \( v \)-even for some \( v \) only if \( \Lambda \) is integral; whenever \( \Lambda \) is integral, such a \( v \) can always be found, and must lie in \( \frac{1}{2} \Lambda^* \).

For any Euclidean lattice \( \Lambda \) define the shifted theta constant \( \Theta_\Lambda(vu|\tau) \) to be

\[
\Theta_\Lambda(vu|\tau) = \sum_{r \in \Lambda} \exp\left[\pi i \tau (r + u)^2 + 2\pi i (r + u) \cdot v\right].
\]

(3a)

For an indefinite lattice \( \Lambda \), the shifted theta constant is defined by

\[
\Theta_\Lambda(vu|\tau, \bar{\tau}) = \sum_{r \in \Lambda} \exp\left[\pi i \tau (r_L + u_L)^2 - \pi i \bar{\tau} (r_R + u_R)^2 + 2\pi i (r + u) \cdot v\right],
\]

(3b)

where we write \( r \in \Lambda \) in the usual way as \( r = (r_L; r_R) \) (so dot products in \( \Lambda \) are given by \( r \cdot r' = r_L \cdot r'_L - r_R \cdot r'_R \)). We will also use the short-hand \( \Theta(\Lambda)(\tau, \bar{\tau}) \) for \( \Theta_\Lambda(00|\tau, \bar{\tau}) \) and \( \Theta(\Lambda)(\tau) \) for \( \Theta_\Lambda(00|\tau) \). They will sometimes be called pure theta constants.

We know that if Dienes’ partition function corresponds to a consistent lattice string [11,15], then we must have

\[
T(\tau, \bar{\tau}) = \Theta_\Lambda(vv|\tau, \bar{\tau}),
\]

(4)

for some odd indefinite \( v \)-even self-dual lattice \( \Lambda \) of signature \((22,10)\), and some fermionic vector \( v = (0; v_R) \) satisfying \( v^2 = -v_R^2 = -1 \). The fermionic vector is there to provide fermionic quantum numbers for the string; it is given by the projection of an internal momentum along the fermionic vector.

Given any indefinite lattice \( \Lambda \) of signature \((m, n)\), define the \( n \)-dimensional Euclidean lattice \( \Lambda_R \), called the RHS of \( \Lambda \), by:

\[
\Lambda_R \overset{\text{def}}{=} \{r_R|(0; r_R) \in \Lambda\},
\]

(5)

where the Euclidean dot products in \( \Lambda_R \) are given by \( r_R \cdot r'_R = -(0; r_R) \cdot (0; r'_R) \). Define similarly the \( m \)-dimensional Euclidean lattice \( \Lambda_L \) (the LHS of \( \Lambda \)).
partition function is not precisely known because of the arbitrary nature of $I(\tau, \bar{\tau})$ in (1). We will now discuss two methods of overcoming that difficulty.

The first method involves looking at the shifted theta constant for $\Lambda_R$, which can be obtained from the shifted theta constant for $\Lambda$ by putting $q$ equal to zero (i.e. considering the limit $\tau \to +\infty i$) in eq. (1a), because any acceptable $I$ in eq. (1) vanishes in this limit. From (1a) and (4), and using the Jacobi identity, we obtain

$$\Theta_{\Lambda_R}(v_R v_R|\tau) = 4c[\bar{\theta}_3^8 \bar{\theta}_4^2 + \bar{\theta}_3^6 \bar{\theta}_4^4 - \bar{\theta}_3^4 \bar{\theta}_4^6 - \bar{\theta}_3^2 \bar{\theta}_4^8]$$

$$= 16c\{8\bar{q} - 896\bar{q}^5 + 5184\bar{q}^9 + \cdots\}. \tag{6}$$

Next, consider the symmetrization

$$\tilde{\Theta}_{\Lambda}(vv|\tau, \bar{\tau}) \equiv \theta_2^4(\bar{\tau})\theta_3^4(-\bar{\tau})\theta_4^4(-\bar{\tau})\Theta_{\Lambda}(vv|\tau, \bar{\tau}) + \bar{\theta}_2^4(\tau)\bar{\theta}_3^2(\tau)\Theta_{\Lambda}(vv|\tau, \bar{\tau}). \tag{7a}$$

By definition the $I$ in eq. (1) satisfy $\tilde{I} = 0$, so we can similarly write down explicitly an expression for $\tilde{\Theta}_{\Lambda}(vv|\tau, \bar{\tau})$ in terms of the Jacobi functions (we use the Jacobi identity in the Appendix to rewrite this polynomial so that it is of degree $\leq 2$ in both $\theta_2$ and $\bar{\theta}_2$. See also the remark in the second paragraph of Sec. 4). The result is very messy and is given in Ref. [16], but fortunately we only need two of the terms:

$$\tilde{\Theta}_{\Lambda}(vv|\tau, \bar{\tau}) = -13c\theta_3^{18}\bar{\theta}_3^{14}\bar{\theta}_4^8 - 11c\theta_3^{18}\theta_4^4\bar{\theta}_3^6\bar{\theta}_4^{16} + \cdots. \tag{7b}$$

In particular, note that the coefficients in eq. (7b) of $\theta_3^{18}\bar{\theta}_3^{14}\bar{\theta}_4^8$ and $\theta_3^{18}\theta_4^4\bar{\theta}_3^6\bar{\theta}_4^{16}$ are not equal. We will show in Sec. 4 that those coefficients must be equal for any lattice satisfying the half-norm property eq. (9).

Rather than directly trying to solve eqs. (1) and (4) for $\Lambda$, it will be simpler first to try to solve eqs. (6) and (7b) for $v_R$, $\Lambda_R$, and ultimately $\Lambda$ (or show that no such solution exists).

One obvious difficulty one encounters is that eqs. (6) and (7b) involve ‘shifted theta constants’ $\Theta_{\Lambda_R}(v_R v_R|\tau)$ etc., rather than pure theta constants. This makes it much harder to read off information about $\Lambda_R$ and $\Lambda$ because the fermionic vector $v$ or $v_R$ is not a priori known. Eqs. (8) below are designed to overcome this complication. With this in mind, make the following definitions.

Let $\Lambda_B$ be the sublattice of $\Lambda$ consisting only of the even norm vectors (i.e. the bosons). Then it can be shown that its determinant is $|\Lambda_B| = 4$, and its signature is still $(22,10)$. Let $\Lambda_{BR} \equiv (\Lambda_B)_R$. Let $u = (0; u_R) \in \Lambda$ be any odd-normed vector living entirely in the right-hand side of $\Lambda$ (such vectors will always exist, by eq. (6)). Then $\Lambda = \Lambda_B[u]$ and
\[ \Lambda_R = \Lambda_{BR}[u_R] \] (see the Appendix for this gluing notation). Note also that \(2v, 2u \in \Lambda_B\) and \(2v_R, 2u_R \in \Lambda_{BR}\).

Now define lattices \(\Lambda'_R \overset{\text{def}}{=} \Lambda_{BR}[v_R + u_R]\) and \(\Lambda''_R \overset{\text{def}}{=} \Lambda_{BR}[v_R]\); define \(\Lambda'\) and \(\Lambda''\) similarly. Note that \(\Lambda_R, \Lambda'_R\) and \(\Lambda''_R\) are all integral (in fact, odd) and have equal determinant. Both \(\Lambda_R\) and \(\Lambda'_R\) are \(v_R\)-even, and \(\Lambda''_R\) is \(u_R\)-even.

An explicit calculation, using eqs. (3) and some formulas in Ref. [10], yields:

\[ \Theta_{\Lambda_R}(v_Rv_R|\bar{\tau}) = \Theta(\Lambda''_R)(\bar{\tau}) - \Theta(\Lambda'_R)(\bar{\tau}), \quad (8a) \]
\[ \Theta_{\Lambda}(vu|\tau, \bar{\tau}) = \Theta(\Lambda'')(\tau, \bar{\tau}) - \Theta(\Lambda')(\tau, \bar{\tau}). \quad (8b) \]

Note that eqs. (6) and (8a) imply that the number of norm 1 vectors in \(\Lambda''_R\) minus the number in \(\Lambda'_R\) must equal \(128c\). However it is easy to see that \(\Lambda''_R\) and \(\Lambda'_R\), being integral, Euclidean and 10-dimensional, can both have at most 20 norm 1 vectors. Therefore \(-20 \leq 128c \leq 20\), i.e. \(0 \neq |c| \leq 5/32\).

This immediately gives us the first main result of this paper:

**Theorem A:** No lattice string exists having a partition function \(T\) in Dienes’ class (eq. (1)) if \(|c| > 5/32\).

4. The half-norm case

We shall now proceed to consider the case of a general \(c\). One way is to enumerate all solutions \(\Lambda_R, v_R, c\) to eq. (6), and then proceed to study whether they can satisfy (1). The enumeration can be done if (the determinant) \(|\Lambda_R| < 64\) — there are exactly three of these up to integral equivalence, and they are listed in Ref. [16]. However, a complete enumeration of all \(\Lambda_R\) satisfying eq. (6) could be unfeasible. Nevertheless a large class of solutions \(\Lambda_R\) to eq. (6), which includes all such solutions we have ever found, satisfies the following property, which we shall call the half-norm property:

\[ g \in \Lambda^*_R \Rightarrow g^2 \in \frac{1}{2}\mathbb{Z}. \quad (9) \]

Indeed, it may turn out that any solution of eq. (6) must satisfy this additional property, since the contributions to expressions such as eq. (7b) of glue vectors \((g_L; g_R) \in \Lambda\) with \(g_R\) violating eq. (9) (i.e. with \(g^2_R \notin \frac{1}{2}\mathbb{Z}\)) would otherwise have to conveniently cancel out.
In any case, strings with the spin structures considered in Ref. [10] all seem to satisfy this property. We shall therefore assume the half-norm property to be valid from now on; this assumption is dropped in Ref. [12].

Now we need to use a mathematical theorem: Cor. 10.2 in Ref. [18]. Together with the half-norm assumption, this corollary implies that the theta constants of all glue classes in \( \Lambda_R^*/\Lambda'_R \) and \( \Lambda''_R^*/\Lambda''_R \) can be expressed as polynomials in \( \theta_2^2, \theta_3^2, \theta_4^2 \), and that there is only one such form for each such polynomial subject to the additional condition that it be of degree \( \leq 2 \) in \( \theta_2 \). It can be shown, using for example Thm. 2.5 of Ref. [19], that any integral lattice \( \Lambda \) satisfying eq. (9) also has the property that any glue \( g \in \Lambda^* \) must be of order 1, 2 or 4. This immediately implies that the determinant \( |\Lambda| \) must be a power of 2; if it is to also be a solution of eq. (6) it can be shown that the determinant must be a power of 4. Also note that if one of \( \Lambda_R, \Lambda'_R, \Lambda''_R \) satisfies eq. (9), all do (see eq. (10)).

There are several simultaneous solutions [16] \( \Lambda_R \) to eqs. (6) and (9). Their determinants range from \( 4^2 = 16 \) (for which there are 3 solutions) to \( 4^8 = 16384 \) (with 4 solutions). It is unnecessary to explicitly find any of these, however. In the following paragraphs we will show that the partition function eq. (4) of any self-dual \( v \)-even \( \Lambda \) of signature (22,10) whose RHS \( \Lambda_R \) satisfies eq. (9), cannot be in the class eq. (1c). In particular, we will show that in the expansion of \( \tilde{\Theta}_\Lambda(vv|\tau\bar{\tau}) \), the coefficients of \( \theta_3^{18}\theta_4^4\theta_4^4\theta_4^4 \) and \( \theta_3^{18}\theta_4^4\theta_3^6\theta_4^{16} \) must be equal, thus violating eq. (7b).

The motivation for this approach was obtained by explicitly computing the partition functions corresponding to some explicit solutions \( \Lambda \) to eq. (6), and comparing with eq. (7b). An example of such a calculation was included in Ref. [16].

Let \( \Lambda \) be any self-dual, \( v \)-even lattice of signature (22,10) whose RHS \( \Lambda_R \) satisfies eq. (9). Let \( D = |\Lambda_R| \). Enumerate the \( D \) glue classes \( [g_i]_{\Lambda_R} \) of \( \Lambda_R^*/\Lambda_R \). Without loss of generality (by replacing \( g_i \) if necessary with \( g_i + u_R \)) we may choose the representatives so that \( g_i \cdot v_R \in \mathbb{Z} \). Define \( g'_i = g_i + v_R + u_R, \ g''_i = g_i + v_R; \) then \( [g'_i]_{\Lambda'_R} \) and \( [g''_i]_{\Lambda''_R} \) for \( i = 1, \ldots, D \) exhaust the \( D \) glue classes of \( \Lambda_R^*/\Lambda'_R \) and \( \Lambda_R^*/\Lambda''_R \). Note that the correspondences \( g_i \leftrightarrow g'_i \leftrightarrow g''_i \) define group isomorphisms \( \Lambda_R^*/\Lambda_R \cong \Lambda'_R^*/\Lambda'_R \cong \Lambda''_R^*/\Lambda''_R \).

Now consider the LHS \( \Lambda_L \) of \( \Lambda \) — it too has \( D \) glue classes in \( \Lambda_L^*/\Lambda_L \), by e.g. Thm.2.4 of Ref. [15]. They can be enumerated in such a way that

\[
\Lambda = \bigcup_{i=1}^{D} ([h_i]_{\Lambda_L}; [g_i]_{\Lambda_R}).
\]

Then \( h_i \leftrightarrow g_i \) allows us to extend the above glue group isomorphisms. We also get the following congruences:
\[ h_i \cdot h_j \equiv g_i \cdot g_j \equiv g_i' \cdot g_j' \equiv g_i'' \cdot g_j'' \pmod{1}, \quad 1 \leq i, j \leq D. \]  

(10)

A simple calculation establishes the gluings

\[ \Lambda' = \bigcup_{i=1}^{D} ([h_i] \Lambda_L; [g_i] \Lambda'_R) \quad \text{and} \quad \Lambda'' = \bigcup_{i=1}^{D} ([h_i] \Lambda_L; [g_i''] \Lambda''_R). \]

These immediately imply

\[ \Theta_{\Lambda'}(\tau \bar{\tau}) = \sum_{i=1}^{D} \Theta([h_i] \Lambda_L)(\tau) \cdot \Theta([g_i'] \Lambda'_R)(\bar{\tau}), \quad (11a) \]

with a similar expression for \( \Theta_{\Lambda''} \). Hence

\[ \Theta_A(\nu \nu|\tau \bar{\tau}) = \sum_{i=1}^{D} \Theta([h_i] \Lambda_L)(\tau) \cdot \{ \Theta([g_i''] \Lambda''_R)(\bar{\tau}) - \Theta([g_i'] \Lambda'_R)(\bar{\tau}) \} \]

\[ \overset{\text{def}}{=} \sum_{i=1}^{D} \Theta([h_i] \Lambda_L)(\tau) \cdot \Delta_i(\bar{\tau}), \quad (11b) \]

\[ \tilde{\Theta}_A(\nu \nu|\tau \bar{\tau}) = \sum_{i=1}^{D} \{ \Theta([h_i] \Lambda_L)(\tau) \cdot \Delta_i(\bar{\tau}) \cdot \bar{\theta}^4_2 \bar{\theta}^4_3 \bar{\theta}^4_4 + \Theta([h_i] \Lambda_L)(\bar{\tau}) \cdot \Delta_i(\tau) \cdot \theta^4_2 \theta^4_3 \theta^4_4 \}. \quad (11c) \]

As mentioned above, Cor.10.2 in Ref. [18] tells us we can uniquely write eq. (11c) as a polynomial in the Jacobi functions so that it is of degree \( \leq 2 \) in both \( \theta_2 \) and \( \bar{\theta}_2 \). We are trying to show that the coefficients of \( \theta^3_3 \theta^4_4 \bar{\theta}^2_3 \bar{\theta}^4_3 \) and \( \theta^3_3 \theta^4_4 \bar{\theta}^6_3 \bar{\theta}^{10}_3 \) in eq. (11c) — call them \( A \) and \( B \) — are equal.

Note that \( A = A_1 - A_2 + A_3 \), where \( A_1, A_2, A_3 \) are, respectively, the coefficients of \( \theta^3_3 \theta^4_4 \bar{\theta}^2_3 \bar{\theta}^4_3 \), \( \theta^3_3 \theta^4_4 \bar{\theta}^{10}_3 \) and \( \theta^3_3 \theta^4_4 \bar{\theta}^6_3 \bar{\theta}^{10}_3 \) in \( \Theta_A(\nu \nu|\tau \bar{\tau}) \); similarly, \( B = -B_1 + B_2 \) where \( B_1, B_2 \) are, respectively, the coefficients of \( \theta^3_3 \theta^4_4 \bar{\theta}^2_3 \bar{\theta}^4_3 \) and \( \theta^3_3 \theta^4_4 \bar{\theta}^6_3 \bar{\theta}^{10}_3 \) in \( \Theta_A(\nu \nu|\tau \bar{\tau}) \).

Now, consider any \( \Delta_k = \Theta([g_k'] \Lambda'_R) - \Theta([g_k] \Lambda'_R) \) for which \( g_k^2 \in \mathbb{Z} \). Then each \( \Delta_k \) can be expressed as a polynomial in \( \bar{\theta}^2_3 \) and \( \bar{\theta}^4_3 \). Let \( \Delta_k' \) consist of those terms in \( \Delta_k \) in which \( \bar{\theta}^2_3 \) occurs to odd power. For example, \( \Theta_{\Lambda'_R} = 4c\bar{\theta}^6_3 \bar{\theta}^4_3 - 4c\bar{\theta}^8_3 \bar{\theta}_4^8 \), by eq. (6). \textit{A priori} one would expect these \( \Delta_k' \) to look like \( r_k \bar{\theta}^{10}_3 + s_k \bar{\theta}^8_3 \bar{\theta}_4 + t_k \bar{\theta}^8_3 \bar{\theta}_4^8 \), for arbitrary \( r_k, s_k, t_k \in \mathbb{R} \). However, if we can show that for each of these \( k \) there exists an \( \ell_k \in \mathbb{R} \) such that

\[ \Delta_k'(\bar{\tau}) = \ell_k(\bar{\theta}^8_3 \bar{\theta}_4^4 - \bar{\theta}^8_3 \bar{\theta}_4^4), \quad (12) \]
then it is easy to see that \( A_1 = -B_1 \) and \( A_2 = A_3 = B_2 = 0 \), i.e. that \( A = B \). Hence it suffices to show that eq. (12) holds for each \( k \) for which \( g_k^2 \in \mathbb{Z} \).

This leads us to the second main result of this paper.

**Theorem B:** There is no string theory with partition function of the type given in eq. (1a), based on a lattice \( \Lambda \) whose RHS \( \Lambda_R \) satisfies the half-norm property, i.e. eq. (9).

**Proof** We will begin by making some general observations about the theta constants of lattices satisfying eq. (9). Only in the final paragraph of the proof will it be applied to \( \Lambda' \) and \( \Lambda'' \).

Let \( \Lambda \) be any 10-dimensional (integral) lattice satisfying eq. (9). Let \( D = |\Lambda| \). We can write

\[
\Theta(\Lambda) = a\theta_3^{10} + b\theta_3^8\theta_2^2 + c\theta_3^4\theta_4^4 + d\theta_3^4\theta_4^6 + e\theta_3^2\theta_4^8 + f\theta_4^{10},
\]

(13a)

where \( a, b, c, d, e, f \) are real, and \( f = 1 - a - b - c - d - e \). Then

\[
\Theta(\Lambda^\ast) = Da\theta_3^{10} + Db\theta_3^8\theta_2^2 + Dc\theta_3^6(\theta_3^4 - \theta_4^4)
\]

\[
+ Dd\theta_3^4\theta_2^6 + De\theta_3^2(\theta_3^4 - \theta_4^4)^2 + Df\theta_2^{10}
\]

(13b)

\[
= (Da + Dc + De)\theta_3^{10} + Db\theta_3^8\theta_2^2 + (-Dc - 2De)\theta_3^6\theta_4^4
\]

\[
+ Dd\theta_3^4\theta_2^6 + De\theta_3^2\theta_4^8 + Df\theta_2^{10}
\]

(13c)

(see Ref. [17]). Because \( \Lambda^\ast \) only has one zero vector, eq. (13b) implies \( a = 1/D \).

Now let \( g \in \Lambda^\ast, g^2 \in \mathbb{Z} \). Then \( g \) will be order 1, 2 or 4, and \( \Theta([g]\Lambda) \) will be of the same form as eq. (13a): i.e. \( \Theta([g]\Lambda) = a_g\theta_3^{10} + b_g\theta_3^8\theta_2^2 + \cdots \). Of course, \( \Lambda_1[g] \) will also satisfy eq. (9). Consider first the case where \( g \) is of order 2. Then the previous paragraph applied to both \( \Lambda \) and \( \Lambda_1[g] \) immediately implies that \( a_g = 1/D \). Hence the same conclusion must apply to \( g \) of order 4 (and trivially to order 1 glues) — i.e. the leading coefficient for any integral-normed glue \( g \) of \( \Lambda \) is \( a_g = 1/D \). Moreover, note that the coefficient of \( \theta_3^{10} \) in the theta constant of a glue class of non-integral norm must be 0.

Of course, we can rewrite eq. (13c) as the sum of \( \Theta([g]\Lambda) \) for all glue classes \( [g]\Lambda_1 \in \Lambda^\ast/\Lambda_1 \). We then get \( N/D \) as the coefficient of \( \theta_3^{10} \) there, where \( N \) is the number of integral-normed glue classes of \( \Lambda_1 \). Hence \( c + e = (N - D)/D^2 \). The same technique as in the previous paragraph allows us to find a similar formula for \( c_g + e_g \), for glues \( g \) of \( \Lambda_1 \) with integral norm.

Finally, consider the two lattices \( \Lambda' \) and \( \Lambda'' \) corresponding to an integral \( \Lambda_R \) satisfying eq. (9). Their glues \( g'_k, g''_k \) can be paired as was done around eq. (10). Consider integral-normed glues \( g'_k \leftrightarrow g''_k \). From the previous two paragraphs (and the dot product-preserving
glue group isomorphism analogous to that defined around eq. (10)), two things should be clear: 

\[ a_{g_k'} = a_{g_k''} \quad \text{and} \quad e_{g_k'} = e_{g_k''} \]

Hence

\[ \Delta'_k = (a_{g_k''} - a_{g_k'})B_3^{10} + (c_{g_k''} - c_{g_k'})B_4^6B_4^4 + (e_{g_k''} - e_{g_k'})B_3^2B_4^8 \]

necessarily satisfies eq. (12). QED

The argument considered in this section can be formulated into a general test. In particular, let \( Q_0(\tau, \bar{\tau}) \) be any given function, and consider the class of partition functions

\[ T_0(\tau, \bar{\tau}) = cQ_0(\tau, \bar{\tau}) + I(\tau, \bar{\tau}), \]

where \( I = (\theta_2\theta_3\theta_4)^4[X(\tau, \bar{\tau}) - X(-\bar{\tau}, -\tau)] \) is any skew symmetric function as defined before, and \( c \) is any nonzero real constant. Then:

**Corollary:** If such a partition function \( T_0 \) is to be realizable by a lattice string defined by a lattice \( \Lambda \) whose RHS satisfies eq. (9), the following two conditions must be satisfied:

(i) \( \tilde{Q}_0 \) can be written as a polynomial in \( \theta_i \) and \( \bar{\theta}_i \); and

(ii) making this polynomial of degree \( \leq 2 \) in both \( \theta_2 \) and \( \bar{\theta}_2 \), the coefficients of \( \theta_3^{18}\theta_4^{14}\bar{\theta}_3^{14}\bar{\theta}_4^8 \) and \( \theta_3^{18}\theta_4^{14}\bar{\theta}_3^{14}\bar{\theta}_4^{16} \) must be equal.

The symmetrized function \( \tilde{Q}_0 \) is obtained from \( Q_0 \) as in (7a). These conditions are necessary but not sufficient, though they are enough to rule out the class in eq. (1). As we showed in Sections 3 and 4, Dienes’ choice of \( Q_0 = Q \) satisfies condition (i) but not (ii). See Ref. [10] for other conditions to be satisfied.

**5. Summary and Conclusions**

We have been concerned with the question whether a non-supersymmetric lattice string can yield a zero cosmological constant to one-loop order. The cosmological constant is given by the integral of a partition function. Of those partition functions whose integrals vanish, we have examined a class written down by Dienes in [10] (eq. (1)). These partition functions are of the form that might come from a lattice string.

In Sec. 3, we have ruled out any possibility that such partition functions do come from a lattice string if \( |c| > 5/32 \). In Sec. 4, we ruled out all other \( c \) values with the technical restriction of the ‘half norm property’ (eq. (9)). The half-norm property characterizes the
most natural class of possibilities for the lattice Λ in eq. (4), since otherwise eq. (7b) (and the equation for Θ(Λ′ R∗) − Θ(Λ′ R∗) that results by replacing r → −1/τ in eq. (6)) could hold only if the contributions to their power series made by the glues (h_ι; g_ι) (and g_ι, resp.) with norms h_ι^2, g_ι^2 \notin \frac{1}{2} \mathbb{Z}, were all to conveniently cancel.

We have also devised a new test (the Corollary in Sec. 4) for a general class of partition functions to check whether they can come from any lattice string. It is using this test that we ruled out above the class in eq. (1). The same test however can be applied to other partition functions as well.

Clearly, our aim now is to extend this proof beyond the half-norm assumption. Unfortunately the analysis then becomes more complicated, and will be included in a subsequent paper [12]. The goal of that paper is to determine to what extent an acceptable solution to eq. (6) must satisfy eq. (9), and thus cannot be extended to a solution of eq. (7b).

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Appendix

Below we define and discuss the various theta functions used in the text. We will also state some relevant definitions and results on lattices.

The Jacobi \( \theta \)-functions which we need are defined by:

\[
\begin{align*}
\theta_2(\tau) &\equiv \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2}, \quad (A.1a) \\
\theta_3(\tau) &\equiv \sum_{m=-\infty}^{\infty} q^{m^2}, \quad (A.1b) \\
\theta_4(\tau) &\equiv \sum_{m=-\infty}^{\infty} (-q)^{m^2}, \quad (A.1c)
\end{align*}
\]

where \( q = \exp(\pi i \tau) \).
Relations abound between $\theta_2$, $\theta_3$ and $\theta_4$ — see e.g. Ref. [17]. In particular:

\[ \begin{align*}
\theta_2(\tau) &= \theta_3(\tau/4) - \theta_3(\tau), \\
\theta_4(\tau) &= 2\theta_3(4\tau) - \theta_3(\tau), \\
\theta_3(\tau)^4 &= \theta_2(\tau)^4 + \theta_4(\tau)^4.
\end{align*} \] (A.2a, b, c)

We will find it simpler not to avail ourselves of the linear relations eqs. (A.2a, b). However, eq. (A.2c) (called the Jacobi identity) will be used. The arguments of the $\theta$-functions used in this paper will always be $\tau$, unscaled, and so we will often suppress the argument.

The glue classes of lattices are discussed for example in Ref. [14]. In short, $[g]\Lambda_0 \overset{\text{def}}{=} g + \Lambda_0$ is called a glue class of a lattice $\Lambda_0$ if $g \in \mathbb{Q} \otimes \Lambda_0$ ($\mathbb{Q}$ is the set of all rational numbers). For an integral lattice $\Lambda$, the group $\Lambda^*/\Lambda$ consists of $D$ glue classes of $\Lambda$, where $D = |\Lambda|$ is the determinant of $\Lambda$ (defined as $\det(e_i \cdot e_j)$, where $\{e_i\}$ is a set of basis vectors of $\Lambda$), and $\Lambda^*$ is the dual lattice of $\Lambda$.

Given a glue class $[g]\Lambda_0$ of a Euclidean lattice $\Lambda_0$, its theta constant is defined to be

\[ \Theta([g]\Lambda_0)(\tau) \overset{\text{def}}{=} \sum_{x \in \Lambda_0} \exp[\pi i \tau (g + x)^2] \]

\[ \Theta(\Lambda_0) \overset{\text{def}}{=} \Theta([0]\Lambda_0), \] (A.3)

where as usual $\text{Im}(\tau) > 0$. See Ref. [17] for properties of these functions.

Theta constants of integral lattices and their glue classes are examples of modular forms. A discussion of general results concerning lattice theta constants as modular forms can be found on pp.382-8 of Ref. [13].
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