Decoherence and Dynamical Entropy Generation in Quantum Field Theory

Jurjen F. Koksmajorsicap and Tomislav Prokopecicap
Institute for Theoretical Physics (ITP) & Spinoza Institute,
Utrecht University, Postbus 80195, 3508 TD Utrecht, The Netherlands

Michael G. Schmidticap
Institut für Theoretische Physik, Heidelberg University,
Philosophenweg 16, D-69120 Heidelberg, Germany

We formulate a novel approach to decoherence based on neglecting observationally inaccessible correlators. We apply our formalism to a renormalised interacting quantum field theoretical model. Using out-of-equilibrium field theory techniques we show that the Gaussian von Neumann entropy for a pure quantum state increases to the interacting thermal entropy. This quantifies decoherence and thus measures how classical our pure state has become. The decoherence rate is equal to the single particle decay rate in our model. We also compare our approach to existing approaches to decoherence in a simple quantum mechanical model. We show that the entropy following from the perturbative master equation suffers from physically unacceptable secular growth.

PACS numbers: 03.65.Yz, 03.70.+k, 03.67.-a, 98.80.-k, 03.65.-w, 03.67.Mn

I. INTRODUCTION

Decoherence is a quantum phenomenon that describes how a quantum system turns into a classical stochastic system. The theory of decoherence is widely used, e.g. in quantum computing †, black hole physics ‡, inflationary perturbation theory § and in elementary particle physics such as electroweak baryogenesis models ‡. It is hard to study decoherence in quantum field theory (QFT) as it requires out-of-equilibrium, finite temperature and interacting quantum field theoretical computations. Let us first discuss the conventional approach to decoherence ‡ and its shortcomings. We consider a system $S$ in interaction with an environment $E$. The observer’s inability to detect the environmental degrees of freedom allows us to trace over $E$ to construct a reduced density operator from the full density operator $\hat{\rho}_\text{red} = \text{Tr}_E[\hat{\rho}]$. The unitary von Neumann equation for the density operator $i\partial_t \hat{\rho} = [\hat{H}, \hat{\rho}]$ is replaced by a perturbative “master equation” for $\hat{\rho}_\text{red}$ that is no longer unitary:

$$i\partial_t \hat{\rho}_\text{red} = [\hat{H}_S, \hat{\rho}_\text{red}] + \mathcal{D}[\hat{\rho}_\text{red}],$$

(1)

where $\mathcal{D}[\hat{\rho}_\text{red}]$ collectively refers to all non-unitary dissipative terms ‡. If $\hat{\rho}_\text{red}$ is approximately diagonal in the pointer basis, our quantum system effectively becomes a classical stochastic system. In reality, however, systems do not fully decohere so in order to quantify the amount of decoherence, one is interested in the entropy increase due to the loss of quantum coherence.

The master equation (1) suffers from both theoretical and practical shortcomings. It is unsatisfactory that $\hat{\rho}_\text{red}$ evolves non-unitarily while the underlying QFT is unitary. On the practical side, the master equation is so complex that basic field theoretical questions have so far never been properly addressed: no well-established treatment to take perturbative corrections to $\hat{\rho}_\text{red}$ into account exists (for cases in quantum mechanics (QM) see however §), nor has any $\hat{\rho}_\text{red}$ ever been renormalised. Moreover, as we will show, the perturbative master equation leads even in very simple QM situations to physically unacceptable secular growth of the entropy, which is caused by the perturbative approximations used in specifying (1).

The goal of this letter is twofold. Firstly, we present our novel approach to decoherence that does not suffer from the shortcomings mentioned above and apply it to a very simple QM toy model of $N+1$ coupled simple harmonic oscillators (SHO’s). Secondly, we apply our approach to an interacting QFT, using non-equilibrium field theory techniques to calculate the time evolution of the Gaussian von Neumann entropy. The focus and novelty of this letter is not on calculational results, but instead on a clear and succinct presentation of our approach to decoherence and its relation to entropy generation in relativistic quantum field theories. The results of various calculations we discuss here serve mainly to test and illustrate our approach to decoherence. We refer the reader for further details to ‡.

II. ENTROPY AND CORRELATORS

It is of course widely appreciated that entropy can be generated as a result of an incomplete knowledge of a system. The infinite hierarchy of irreducible $n$-point functions completely captures all properties of a system, however only a finite subset can be probed experimentally. This leads us to propose ‡: neglecting observationally inaccessible correlators will give rise to an increase in entropy of the system as perceived by the observer.
So both \( S \) and \( E \) evolve unitarily, however the observer would say that \( S \) evolves to a mixed state with positive entropy as information about \( S \) is dispersed in inaccessible correlation functions. Giraud and Serreau\cite{13} advocate similar ideas which they illustrate by an analysis in \( \lambda \phi^4 \), building on earlier work\cite{3,14}. The total von Neumann entropy can be subdivided as follows:

\[
S_{\text{VN}} = S^g(t) + S^\text{ng}(t) = S^g_S + S^g_E + S^\text{ng}_S + S^\text{ng}_E.
\] (2)

Here, \( S^g \) is the total Gaussian von Neumann entropy, that contains information about both \( S \) and \( E \) and their correlations \( S_E \), and \( S^\text{ng} \) is the total non-Gaussian von Neumann entropy (consisting of \( S \), \( E \) and \( S_E \) contributions). In the following, we assume that the relevant properties of quantum systems are encoded in the Gaussian part \( \hat{\rho}_g \) of the full density operator \( \hat{\rho} \). This is justified as higher order non-Gaussian \( n \)-point functions are perturbatively suppressed. Our approach can be improved if e.g. three- or four-point correlators are experimentally accessible such that knowledge of these correlators can be included in the definition of the entropy\cite{10}. Although \( S_{\text{VN}} \) is conserved in unitary theories, \( S^g_S(t) \) can increase at the expense of other decreasing contributions to the total von Neumann entropy, such as \( S^\text{ng} \). Calzetta and Hu\cite{13} prove an \( H \)-theorem for a quantum mechanical \( O(N) \)-model and refer to "correlation entropy" what we would call "Gaussian von Neumann entropy". For e.g. the Gaussian system density matrix for a real scalar field, one can straightforwardly find its associated von Neumann entropy\cite{10,14}:

\[
S^g_S = -\text{Tr}[\hat{\rho}_g \ln \hat{\rho}_g] = \frac{\Delta + 1}{2} \ln \left[ \frac{\Delta + 1}{2} \right] - \frac{\Delta - 1}{2} \ln \left[ \frac{\Delta - 1}{2} \right].
\] (3)

Here, \( \Delta \) is the phase space area that the system’s state occupies in Wigner space \((h=1, k=||\vec{K}||)\):

\[
\Delta^2(k,t) = 4\langle \{ \hat{\phi}^2 \} \{ \hat{\pi}^2 \} - \{ \langle \hat{\phi} \rangle, \langle \hat{\pi} \rangle \}^2 \rangle.
\] (4)

The most efficient way of solving for the three Gaussian correlators in\cite{10} is to solve for the statistical propagator \( F_\phi(x,y) = \text{Tr}[\hat{\rho}(t_0) \{ \hat{\phi}(x), \hat{\phi}(y) \}/2] \), from which the correlators can be straightforwardly extracted:

\[
\begin{align*}
\langle \hat{\phi}(\vec{x},t)\hat{\phi}(\vec{y},t') \rangle &= F_\phi(\vec{x},t;\vec{y},t')|_{t=t'} \quad \text{(5a)} \\
\langle \hat{\pi}(\vec{x},t)\hat{\pi}(\vec{y},t') \rangle &= \partial_t \partial_{t'} F_\phi(\vec{x},t;\vec{y},t')|_{t=t'} \quad \text{(5b)} \\
\langle \{ \hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t') \} \rangle &= \partial_t \partial_{t'} \{ \hat{F}_\phi(\vec{x},t;\vec{y},t') \}|_{t=t'} \quad \text{(5c)}
\end{align*}
\]

For a pure state we have \( \Delta = 1 \) and \( S^g_S = 0 \), whereas for a mixed state \( \Delta > 1 \) and \( S^g_S > 0 \). The phase space area is conserved by the evolution \( d/dt[\Delta^2] = 0 \) in free theories. As \( S^g_S \) is the only invariant measure of the entropy of a Gaussian state\cite{13}, we argue that \( S^g_S \) should be taken as the quantitative measure for decoherence. This is to be contrasted with most of the literature where different, non-invariant measures are proposed\cite{3,13}.

Finally, let us connect the standard approach to ours. The expectation values in\cite{13} can in principle be calculated from the reduced density matrix, consider e.g.

\[
\langle \phi^2 \rangle = \text{Tr}_{S+E}[\hat{\rho}_g \hat{\phi}^2] = \text{Tr}[\hat{\rho}_{\text{red}} \hat{\phi}^2].
\]

Suppose one has been able to solve for \( \hat{\rho} \) in full generality, one would then be interested in evaluating a "reduced von Neumann entropy": \( S^\text{VN}_S(t) = -\text{Tr}[\hat{\rho}_{\text{red}}(t) \ln \hat{\rho}_{\text{red}}(t)] \). We thus conclude that in the Gaussian approach we also advocate, one would find \( S^\text{VN}_S = S^g_S \). The main point of this letter is that solving for a renormalised \( F_\phi \) in an interacting QFT is more tractable than solving for a full \( \hat{\rho} \), as we have recently witnessed a further development of sophisticated techniques in out-of-equilibrium QFT\cite{17}.

\[\text{III. QUANTUM MECHANICS}\]

Let us consider the well-known SHO model of \( N+1 \) coupled oscillators:

\[
L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega_n^2 x^2 + \sum_{n=1}^{N} \left( \frac{1}{2} \omega_n^2 - \frac{1}{2} \omega_n^2 q_n^2 - \lambda_n q_n x \right),
\] (6)

where \( x \) plays the role of \( S \), and the \( \{ q_n \} \) of \( E \). This model provides us with a level playing ground to compare the two approaches to decoherence as QM toy models like these are free from the drawbacks previously mentioned: we can actually solve\cite{10}, and we do not have to worry about renormalisation. Let us start by considering our approach. Equation\cite{2} simplifies to \( S_{\text{VN}} = S^g_S + S^g_E + S^\text{ng}_S + S^\text{ng}_E \) as \( S^\text{ng} = 0 \). The role of the neglected non-Gaussianities \( S^\text{ng} \) in the QFT case is played in the quadratic QM model by the correlation entropy \( S^\text{ng}_S \). Let us consider \( N = 50 \) oscillators, where all \( E \) oscillators with frequencies \( \{ \omega_n \} \) act as a thermal bath at temperature \( \beta^{-1}, k_B = 1 \). Initially, we require that \( S \) is in a pure state \( S^g_S = 0 \). Following our approach to decoherence, we can numerically study the exact evolution of the three Gaussian QM analogues of equation\cite{13}: \( \langle \hat{x}^2 \rangle, \langle \hat{p}_x^2 \rangle \) and \( \langle \{ \hat{x}, \hat{p}_x \} \rangle \). In the resonant regime where one or more \( \omega_n \sim \omega_0 \), the \( \{ q_n \} \) couple effectively to \( x \) such that we expect to observe swift
thermatisation and decoherence. In figure [14] we show the resulting evolution of the entropy (solid line). The oscillations are a manifestation of Poincaré’s recurrence theorem [11]. The dot-dashed line is the thermal entropy corresponding to a temperature $\beta \omega_0 = 2$ indicating perfect thermatisation, which at high temperatures, $\beta \omega_0 \ll 1$, would correspond to perfect decoherence. Given the weak coupling $\lambda/\omega_0^2=3/40$ we observe imperfect decoherence (incomplete thermisation). If we increase $\lambda$, we observe full thermisation of $S$ [11]. We conclude that a finite amount of entropy is generated at the expense of obtaining a negative contribution to the correlation entropy.

This behaviour is to be contrasted with the dashed line, the master equation (11). The entropy resulting from evolving $\rho_{\text{eq}}$ suffers from physically unacceptable unbounded secular growth. In cases where the entropy obtained from the master equation settles to a constant value before it breaks down, it generically overshoots. Let us stress that we are in the deep perturbative (and resonant) regime and that all eigenfrequencies following from [13] are positive. The perturbative master equation incorrectly resums self-mass contributions, unlike the 2PI (two particle irreducible) scheme available in QFT. Away from the resonant regime, where all $\{\omega_n\}$ significantly differ from $\omega_0$, the evolution of the entropy from the master equation agrees up to the expected perturbative corrections with our approach, but this is the regime where only a negligible amount of entropy is generated (no decoherence). We conclude that decoherence rates at early times can be calculated accurately using the standard approach to decoherence however the perturbative master equation breaks down at late times such that the total entropy increase, i.e.: the total amount of decoherence, cannot be calculated reliably.

### IV. QUANTUM FIELD THEORY

We will consider the following interacting scalar QFT:

$$S[\phi, \chi] = \int d^3 x \{ L_0[\phi] + L_0[\chi] + L_{\text{int}}[\phi, \chi] \} ,$$

where:

$$L_0[\phi] = \frac{1}{2} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x) - \frac{1}{2} m_{\phi}^2 \phi^2(x) \tag{8a}$$

$$L_{\text{int}}[\phi, \chi] = - \frac{1}{2} \lambda \chi^4(x) - \frac{1}{2} \hbar \chi^2(x) \phi(x) \tag{8b}.$$

Here, $\phi$ will play the role of $S$ and $\chi$ of $E$. We assume that $\langle \hat{\phi} \rangle = 0 = \langle \hat{\chi} \rangle$, which can be realised by suitably renormalising the tadpoles. Equation (2) reduces to $S_{\text{VN}} = S^\phi_S + S^\chi_S + S^\chi_E = 0$. The analogy with the simple QM model is clear: in the QM case $S^\phi_S$ increases as information is neglected in the Gaussian $SE$ correlations, whereas in the QFT model $S^\phi_S$ increases as non-Gaussian information is neglected, mainly in the $SE$ correlations. Note that in [13] non-Gaussian $S$ correlations are neglected.

We work in the Schwinger-Keldysh or in-in formalism. The 2PI effective action captures the effect of perturbative loop corrections to the various propagators $i \Delta^{ab}_\phi$ and $i \Delta^{ab}_\chi$, $\{a,b\} = \pm$. Varying the 2PI effective action yields the Kadanoff-Baym (KB) equations for the Wightman propagators:

$$(\partial_x^2 - m_{\phi}^2) i \Delta^{\pm}_{\phi}(x;y) - \sum_{c=\pm} c \int d^3 x \hat{M}_{\phi}^{c}(x;x) i \Delta^{\mp}_{\phi}(x;y) = 0,$$

where the self-masses for $\phi$ at one loop read:

$$i M^{ab}_{\phi}(x;y) = - \frac{i h^2}{2} \left( i \Delta^{ab}_{\phi}(x;y) \right)^2 .$$

Using dimensional regularisation, we find that a local mass counterterm renormalises our theory [3]:

$$i M^{\pm}_{\phi,ct}(x;y) = \mp \frac{i h^2 \Gamma(\frac{D}{2} - 1) \mu^{D-4}}{16 \pi^2 (D - 3)(D - 4)} \delta^0(x-y).$$

The KB equations can be solved when written in terms of the causal propagator $i \Delta^{\pm}_{\phi}(x;y) = \text{Tr} \left( \rho(t_0) | \hat{\phi}(x), \hat{\phi}(y) \rangle \langle \hat{\phi}(x), \hat{\phi}(y) | \right) = i \Delta^{\pm +} - i \Delta^{+ \pm}$ and the statistical propagator $F_{\phi} = (i \Delta^{++} + i \Delta^{--})/2$. Initially, at $t = t_0$, $\phi$ is in a pure state and $\chi$ is in thermal equilibrium at temperature $\beta^{-1}$. We assume that $\lambda \gg h$ such that we can neglect the backreaction from $S$ on $E$. Perturbatively, this argument is justified as the backreaction contributes only at $\mathcal{O}(h^4/\omega_0^5)$:

Here, a single (double) solid black line corresponds to a free (resummed) $\phi$ propagator. Appreciate that the leading order self-masses yield Gaussian corrections to $i \Delta^{ab}_\phi$. 

![Figure 2: $S^\phi_S$ as a function of time (solid), which quickly settles to $S_{\text{ms}}$ (dashed), where “ms” is an abbreviation for mixed state. We use $\beta m_{\phi} = 1/2$, $k/m_{\phi} = 1$, $\hbar/m_{\phi} = 3$ and $N = 2000$ up to $tm_{\phi} = 100$.](image-url)
so all non-Gaussianities are indeed perturbatively suppressed. In our approximation scheme χ influences φ thus via the 1PI self-masses which are calculated elsewhere [9, 12]. Memory effects play an important role in non-equilibrium QFT and stem from the time integrals \( \int_{-\infty}^{\infty} dt \) over the self-masses in [9] with some cancellations such that the final result is causal. We deal with the initial time divergences by including memory effects at times \( t < t_0 \), and an appropriate truncation method of these integrals (for alternative approaches see [18], e.g. Garny and Müller impose non-Gaussian initial conditions).

The resulting evolution for the entropy is shown in figure 2 which follows from \( F_\phi \) and equations (3–5). Initially, we have \( S_{ms}^0(t_0) = 0 \) which then rapidly increases and settles to its asymptotic value \( S_{ms} \). We conclude that although a pure state remains pure under unitary evolution, the observer perceives this state over time as a mixed state with positive entropy \( S_{ms} \) as non-Gaussianities are dynamically generated. In other words, a realistic observer cannot probe all information about \( S \) and thus discerns a loss of coherence of our pure state.

We can extract two relevant quantities: the maximal amount of decoherence and the decoherence rate. The maximal amount of decoherence \( S_{ms} \) can be calculated by independent means and offers a powerful check of our approximation scheme [9, 12]. At late times, \( F_\phi \) is time translationary invariant such that the entropy is time independent, which can be appreciated from equation (5), e.g. \( F_\phi(k,0) \equiv \int dk'/(2\pi) F_\phi(k') \) is time independent. \( F_\phi(k') \) follows directly from the KB equations and the self-masses in Fourier space [9, 12]. We show in figure 3 that \( S_{ms} \) reduces to the (free) thermal entropy when \( h \to 0 \) given by \( \Delta_{free} = \coth(\beta\omega_e)/2 \) and equation (3). We thus see that \( S_{ms} \) is the interacting thermal entropy.

![Figure 3: The interacting thermal entropy \( S_{ms} \) as a function of \( h/m_\phi \). The dashed black line indicates the free thermal entropy. We use \( \beta m_\phi = 1/2 \) and \( k/m_\phi = 1 \).](image)

The decoherence rate is the rate at which the phase space area (or entropy) changes. We observe as leading order effect an exponential approach to \( \Delta_{ms} \) in figure 4:

\[
\delta \Delta_k(t) = \Delta_{ms} - \Delta_k(t) \simeq (\Delta_{ms} - \Delta_k(0)) \exp[-\Gamma_{dec} t]
\]

Here, \( \Gamma_{dec} \) is the decoherence rate of our model which is well described by the single particle decay rate of our interaction: \( \Gamma_\phi \to \chi \chi = \text{Im}(iM_\phi)/\omega_\phi \), where \( iM_\phi \) is the retarded self-mass projected on the quasi particle shell:

\[
\Gamma_{dec} \approx \Gamma_\phi \to \chi \chi = \frac{\hbar^2}{32\pi\omega_\phi} + \frac{\hbar^2}{16\pi k_b\beta\omega_\phi} \log \left[ \frac{1 - e^{-\frac{\hbar}{2}(\omega_\phi + k)}}{1 - e^{-\frac{\hbar}{2}(\omega_\phi - k)}} \right].
\]

**V. CONCLUSION**

We have studied the time evolution of the Gaussian von Neumann entropy in an interacting, out-of-equilibrium, finite temperature QFT in a renormalised and perturbative 2PI scheme. We have extracted two relevant quantitative measures of decoherence: the maximal amount of decoherence and the decoherence rate. This study builds the QFT framework for other decoherence studies in relevant situations where different types of fields and interactions are involved. In cosmology for example, the decoherence of scalar gravitational perturbations can be induced by e.g. fluctuating tensor modes (gravitons) [2], isocurvature modes [19] or even gauge fields. In quantum information physics it is very likely that future quantum computers will involve coherent light beams that interact with other parts of the quantum computer as well as with an environment [11]. For a complete understanding of decoherence in such complex systems it is clear that a QFT framework such as developed here is necessary.

[1] M. A. Nielsen, I. L. Chuang, Cambridge University Press (2000); E. Knill, R. Laflamme, and G. J. Milburn, Nature 409 (2001) 46.
[2] R. M. Wald, Living Rev. Rel. 4 (2001) 6.
[3] T. Prokopec, Class. Quant. Grav. 10 (1993) 2295; R. H. Brandenberger, T. Prokopec and V. F. Mukhanov, Phys. Rev. D 48 (1993) 2443; C. Kiefer, I. Lohmar, D. Polarski and A. Starobinsky, Class. Quant. Grav. 24 (2007) 1699-1718.
[4] G. R. Farrar, M. E. Shaposhnikov, Phys. Rev. Lett. 70 (1993) 2833-2836.
[5] H. D. Zeh, Found. Phys. 1 (1970); E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch and I. O. Stamatescu, Springer (2003).
[6] W. H. Zurek, Rev. Mod. Phys. 75 (2003) 715.
[7] A. O. Caldeira, A. J. Leggett, Physica 121A (1983) 587-616.
[8] B. L. Hu, J. P. Paz, Y. Zhang, Phys. Rev. D47 (1993) 1576-1594; B. L. Hu, A. Matacz, Phys. Rev. D49 (1994) 6612-6635.
[9] J. F. Koksma, T. Prokopec, M. G. Schmidt, Phys. Rev. D81 (2010) 065030.
[10] J. F. Koksma, T. Prokopec, M. G. Schmidt, Annals Phys. 325 (2010) 1277-1303.
[11] J. F. Koksma, T. Prokopec and M. G. Schmidt, Annals Phys. 326 (2011) 1548.
[12] J. F. Koksma, T. Prokopec and M. G. Schmidt, Phys. Rev. D 83 (2011) 085011.
[13] A. Giraud, J. Serreau, Phys. Rev. Lett. 104 (2010) 230405.
[14] D. Campo, R. Parentani, Phys. Rev. D78 (2008) 065044; Phys. Rev. D 78 (2008) 065045.
[15] E. A. Calzetta, B. L. Hu, Phys. Rev. D68 (2003) 065027.
[16] M. Sohma, A. S. Holevo and O. Hirota, Phys. Rev. A 59 (1999) 1820.
[17] J. Berges, AIP Conf. Proc. 739 (2005) 3.
[18] M. Garny, M. M. Muller, Phys. Rev. D80 (2009) 085011; S. Borsanyi and U. Reinosa, Renormalised Nonequilibrium Quantum Field Theory: Scalar Fields, Phys. Rev. D 80 (2009) 125029.
[19] T. Prokopec, G. I. Rigopoulos, JCAP 0711 (2007) 029.