Output Feedback Stabilization of Unstable Heat Equations with Time Delay in Boundary Observation

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Received 30 November 2021; Accepted 18 January 2022; Published 22 March 2022

1. Introduction

In the past decades, much effort has been made for the problems of unstable or antistable infinite-dimensional systems which can be applied to practical engineering (see [1–13]). For example, reference [14] relates the loss of the heat to the surrounding medium and the destabilizing heat generation inside the rod. The backstepping method introduced into the PDE systems in a few years has systematically been applied to stabilize some unstable or antistable hyperbolic and parabolic equations (see [6–8, 11, 13, 15, 16]). For example, Kang and Guo [6] consider boundary stabilization for a cascade of unstable heat PDE systems where an output feedback is designed by the backstepping transformation. Moreover, reference [13] has applied the backstepping transformation to overcome the destabilizing boundary condition for a wave equation. The predictor-based feedback control may be an effective method to make the unstable reaction-diffusion equation compensate the delay (see [7, 11]). There are some other references related to stability of parabolic equations, for example [17]. In this paper, we consider a heat equation which is controlled from one end and contains instability at the other end. The output feedback controllers designed by the backstepping method use the Volterra transformation to map an unstable PDE into a stable target PDE. It is shown that the controller based on the observer and predictor is also available for the unstable parabolic equation. In this sense, it is meaningful for this paper to consider the unstable heat equation with time delay in boundary observation.

It is a common phenomenon of the practical systems with time delay which means that the observation signals need time to achieve the controller ([18, 19]). Unfortunately, even a small amount of time delay may make the originally stable system unstable [20]. Actually, for distributed parameter control systems, stabilization of the system with observation or control suffering from time delay represents difficult mathematical challenges, as was mentioned in [21]. In recent years, references [22–24] have introduced the separation principle to make the wave, beam, and Schrödinger equations where observation signals suffer from the given time delay stabilized. This paper has been devoted to the parabolic heat equation with the delayed observation signal.

In this paper, we consider stabilization of the heat equation with the observation subject to a given time delay where an unstable boundary condition is coupled in the boundary $x = 0$, under the Neumann boundary control:
2. Stabilization with Neumann Boundary Control

In this section, we apply the backstepping transformation to obtain the stabilizing feedback for system (1) under the Neumann boundary control. And then, the observer system has been constructed while the predictor system has been designed. At last, the output feedback control based on the estimated state has been shown to stabilize system (1) for any given time delay.

2.1. Feedback Control via Backstepping Transform. System (1) is considered in the energy space $\mathcal{H} = L^2(0, 1)$ and the input (or output) space $U = Y = \mathbb{C}$. The energy of system (1) is

$$E(t) = \frac{1}{2} \int_0^1 w^2(x, t)dx. \quad (2)$$

For the original system (1), make the invertible change of variable

$$v(x, t) = w(x, t) + (c_0 + q) \int_0^x e^{q(x-\eta)} w(\eta, t)d\eta, \quad 0 \leq x \leq 1, t \geq 0, c_0 > 0, q > 0, \quad (3)$$

whose equivalent transformation is

$$w(x, t) = v(x, t) - (c_0 + q) \int_0^x e^{-c_0(x-\eta)} v(\eta, t)d\eta, \quad 0 \leq x \leq 1, t \geq 0, c_0 > 0, q > 0. \quad (4)$$

In order to convert (1) into the exponentially stable system [25]

$$\begin{align*}
\begin{cases}
\dot{v}_x(x, t) = v_{xx}(x, t), & 0 < x < 1, t > 0, \\
\dot{v}_x(0, t) = c_0 v(0, t), & t \geq 0, c_0 > 0, \\
\dot{v}_x(1, t) = 0, & t \geq 0,
\end{cases}
\end{align*} \quad (5)$$

the feedback control in (1) may be proposed as that

$$u(t) = -(c_0 + q) w(1, t) - q(c_0 + q) \int_0^1 e^{q(1-\eta)} w(\eta, t)d\eta, \quad t \geq 0, c_0 > 0, q > 0, \quad (6)$$

under which system (1) becomes that [8]

$$\begin{align*}
\begin{cases}
\dot{w}_x(x, t) = w_{xx}(x, t), & 0 < x < 1, t > 0, \\
\dot{w}_x(0, t) = -q w(0, t), & t \geq 0, q > 0, \\
\dot{w}_x(1, t) = -(c_0 + q) w(1, t) - q(c_0 + q) \int_0^1 e^{q(1-\eta)} w(\eta, t)d\eta, & t \geq 0, c_0 > 0, q > 0.
\end{cases}
\end{align*} \quad (7)$$
Based on the exponential stability of system (5) and the equivalence between systems (5) and (7), we have the following theorem [8].

**Theorem 1.** For any initial value \( w_0 \in \mathcal{H} \), under feedback (6), the closed-loop system (7) is exponentially stable.

\[
\begin{align*}
\dot{w}_x(x,s) &= \dot{\bar{w}}_x(x,s), \quad 0 < x < 1, 0 < s < t - \tau, t > \tau, \\
\dot{w}_x(0,s) &= -q y(s + \tau) + k_1 [\hat{w}(0,s) - y(s + \tau)], \quad 0 \leq s \leq t - \tau, t > \tau, k_1 > 0, \\
\dot{w}_x(1,s) &= u(s), \quad 0 \leq s \leq t - \tau, t > \tau, \\
\dot{w}(x,0) &= \bar{w}_0(x), \quad 0 \leq x \leq 1,
\end{align*}
\]

where \( \bar{w}_0 \) is the arbitrarily assigned initial value of the observer.

Set the error for the observer system

\[
\begin{align*}
\epsilon_x(x,s) &= \epsilon_{xx}(x,s), \quad 0 < x < 1, 0 < s < t - \tau, t > \tau, \\
\epsilon_x(0,s) &= k_1 \epsilon(0,s), \quad 0 \leq s \leq t - \tau, t > \tau, k_1 > 0, \\
\epsilon_x(1,s) &= 0, \quad 0 \leq s \leq t - \tau, t > \tau, \\
\epsilon(x,0) &= \epsilon_0(x), \quad 0 \leq x \leq 1.
\end{align*}
\]

System (10) can be written as follows:

\[
\frac{d}{ds} \epsilon(\cdot,s) = \mathcal{A} \epsilon(\cdot,s),
\]

where the operator \( \mathcal{A} \) is defined as follows:

\[
\mathcal{A} \varphi = \varphi'' \quad \forall \varphi \in D(\mathcal{A}), \\
D(\mathcal{A}) = \{ \varphi \in H^2(0,1) | \varphi'(0) = k_1 \varphi(0)(k_1 > 0), \varphi'(1) = 0 \}.
\]

It is known that \( \mathcal{A} \) generates an exponentially stable \( C_0 \)-semigroup on \( \mathcal{H} \) such that

\[
\| \epsilon(\cdot,s) \|_\mathcal{H} \leq Me^{-\omega s}, \quad \forall s > 0,
\]

for some positive constants \( M, \omega \) ([25]). Thus, for any \( w_0 \in \mathcal{H} \) and \( \bar{w}_0 \in \mathcal{H} \), there exists a unique solution of (12) such that

\[
\begin{align*}
\tilde{w}_x(x,s,t) &= \tilde{w}_{xx}(x,s,t), \quad 0 < x < 1, 0 < t - \tau < s < t, t > \tau, \\
\tilde{w}_x(0,s,t) &= -q \tilde{w}(0,s,t), \quad t - \tau \leq s \leq t, t > \tau, q > 0, \\
\tilde{w}_x(1,s,t) &= u(s), \quad t - \tau \leq s \leq t, t > \tau, \\
\tilde{w}(x,t - \tau,t) &= \tilde{w}(x,t - \tau), \quad 0 \leq x \leq 1, t > \tau,
\end{align*}
\]

Let the error for the predictor system

\[
\epsilon(x,s,t) = \tilde{w}(x,s,t) - w(x,s), \quad 0 \leq x \leq 1, t - \tau \leq s \leq t, t > \tau.
\]
Then, by (1), (15), and (16), \( \varepsilon(x,s,t) \) satisfies the following system:

\[
\begin{cases}
\varepsilon_x(x,s,t) = \varepsilon_{xx}(x,s,t), & 0 < x < 1, t - \tau < s < t, t > \tau, \\
\varepsilon_x(0,s,t) = -q\varepsilon(0,s,t), & t - \tau \leq s \leq t, t > \tau, q > 0, \\
\varepsilon_x(1,s,t) = 0, & t - \tau \leq s \leq t, t > \tau, \\
\varepsilon(x,t - \tau,t) = \varepsilon(x,t), & 0 \leq x \leq 1, t > \tau.
\end{cases}
\]  

(17)

It can be expressed as that

\[
\frac{d}{ds} \varepsilon(\cdot,s,t) = \mathcal{B} \varepsilon(\cdot,s,t),
\]

(18)

where

\[
\begin{cases}
\mathcal{B} \varphi = \varphi'', & \forall \varphi \in D(\mathcal{B}), \\
D(\mathcal{B}) = \{ \varphi \in H^2(0,1) | \varphi(0) = -q\varphi(0)(q > 0), \varphi'(1) = 0 \}.
\end{cases}
\]

(19)

Next, we define a new inner product equivalent as the energy in (2)

\[
\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}}^{(1)} = \int_0^1 \cosh(\alpha(1-x))\varphi_1(x)\varphi_2(x)dx,
\]

(20)

for some given \( \alpha > 0 \) satisfying that \( 2q\cosh(\alpha) - \text{asinh}(\alpha) < 0. \)

Thus, for any \( \varphi \in D(\mathcal{B}) \), we have that

\[
\langle \varphi, \varphi \rangle_{\mathcal{H}}^{(1)} = \int_0^1 \cosh(\alpha(1-x))\varphi^2(x)dx.
\]

(21)

By simple computations of integration by parts, it results that

\[
\langle \mathcal{B} \varphi, \varphi \rangle_{\mathcal{H}}^{(1)} = \langle \varphi'', \varphi \rangle_{\mathcal{H}}^{(1)} = \int_0^1 \cosh(\alpha(1-x))\varphi''(x)\varphi(x)dx,
\]

\[
= \left[ \frac{q\cosh(\alpha) - \text{asinh}(\alpha)}{2} \right] \varphi(0)^2 + \int_0^1 \cosh(\alpha(1-x))\left[ \alpha^2\varphi(x)^2/2 - \varphi''(x)^2 \right]dx,
\]

(22)

\[
\leq \alpha^2/2 \cdot \langle \varphi, \varphi \rangle_{\mathcal{H}}^{(1)}.
\]

Thus, system (17) generates a strongly continuous semigroup \( e^{\mathcal{B}t} \) which indicates that

\[
\| e^{\mathcal{B}t} \|_{\mathcal{H}} \leq M_1 e^{\omega_1 t}, \quad \forall s \geq 0,
\]

(23)

for some positive constants \( M_1 \) and \( \omega_1 \) [26]. Therefore, the solution of system (17) can be represented as follows:

\[
\varepsilon(\cdot,s,t) = e^{\mathcal{B}(t-s-t)\tau} \varepsilon(\cdot,t-\tau), \quad t - \tau \leq s \leq t, t > \tau,
\]

(24)

which together with (14) and (23) result that

\[
\| \varepsilon(\cdot,s,t) \|_{\mathcal{H}} \leq MM_1 e^{(\omega+\omega_1)\tau} \| \varepsilon(\cdot,0) \|_{\mathcal{H}} e^{-\omega t}, \quad \forall t \geq \tau.
\]

(25)

The estimated state has naturally been chosen as follows:

\[
\bar{w}(x,t) = \bar{w}(x,t,t), \quad \forall t \geq \tau.
\]

(26)

assured by the theorem below.

**Theorem 2.** For \( \forall t \geq \tau \), we have that

\[
\| \bar{w}(\cdot,t) - \bar{w}(\cdot,t) \|_{\mathcal{H}} \leq MM_1 e^{(\omega+\omega_1)\tau} \| \bar{w}_0 \|_{\mathcal{H}} e^{-\omega t},
\]

(27)

where \( \bar{w}_0 \) is the initial value of the observer (8), \( \omega_0 \) is the initial value of the original system (1), and \( M_1, \omega, \) and \( M_1, \omega_1 \) are constants in (13) and (23), respectively.

2.3. Stabilization of the Closed-Loop System. For the estimated state (24) and (7), we naturally obtain the estimated output feedback control law as follows:
Theorem 3. For \( \forall t > \tau \), any \( w_0 \in \mathcal{H} \) and \( \tilde{w}_0 \in \mathcal{H} \), there exists a unique solution of systems (32)–(34) such that \( \omega(\cdot,t) \in C(\tau,\infty; \mathcal{H}) \), \( \epsilon(\cdot,s) \in C(0, t-\tau; \mathcal{H}) \), \( \epsilon(\cdot,s,t) \in C([t-\tau,t] \times [\tau,\infty); \mathcal{H}) \), and for any \( \epsilon(\cdot,0) \in D(\mathcal{A}) \) where \( \mathcal{A} \) is defined in (11), system (32) decays exponentially in the meaning that
where $L_0$ is defined in (54) and (55) and $M_0$ and $M$ are defined in (13) and (57), respectively.

**Proof.** Due to (21) and (23), for any $\epsilon(\cdot, t - \tau) \in \mathcal{H}$ determined by (33), there exists a unique solution of (34) such that

$$
\|w(\cdot, t)\|_{\mathcal{H}} \leq \|w_0\|_{\mathcal{H}} + (c_0 + q)L_0M_0Me^{\omega t}
$$

(35)

$$
\sqrt{C_1 + \frac{d^2}{\omega}C_2} \|\epsilon(\cdot, 0)\|_{\mathcal{H}} \cdot \left(e^{-\omega t/2} + e^{\omega t/2} \cdot e^{\omega t}\right),
$$

(36)

Thus, $\epsilon(\cdot, s, t) \in C([-\tau, t] \times [\tau, \infty); \mathcal{H})$.

Now for system (34), because of (18) and (19), the solution of (34) can be represented as [23]

$$
\epsilon(\cdot, s, t) = \sum_{n=0}^{\infty} a_n(t)e^{\lambda_n t} \varphi_n(\cdot),
$$

(38)

where

$$
\begin{align*}
\lambda_n &= -(n\pi)^2 + 2q + O(n^{-1}), \\
\varphi_n(x) &= \cos(n\pi x) + O(n^{-1}).
\end{align*}
$$

(39)

Thus, by Cauchy–Schwartz inequality,

As a result, we have that

$$
\{ \epsilon(1, t, t) = \sum_{n=0}^{\infty} a_n(t)e^{\lambda_n t} \varphi_n(1), \forall t > \tau, \}
$$

(40)

$$
\{ \epsilon(x, t, t) = \sum_{n=0}^{\infty} a_n(t)e^{\lambda_n t} \varphi_n(x), \forall 0 \leq x \leq 1, t > \tau. \}
$$

(41)

Similarly, we obtain that

$$
|\epsilon(1, t, t)| \leq \left(\sum_{n=0}^{\infty} |a_n(t)e^{\lambda_n t}|^2\right)^{1/2} \cdot \left(\sum_{n=0}^{\infty} |\lambda_n^{-1} e^{\lambda_n t}|^2\right)^{1/2}, \forall t > \tau.
$$

(42)

Moreover, from the fact that

$$
\|\delta\epsilon(\cdot, t - \tau)\|_{\mathcal{H}} = \left(\sum_{n=0}^{\infty} |\lambda_n a_n(t)e^{\lambda_n t}|^2\right)^{1/2}, \forall t > \tau,
$$

(43)

where $\delta$ is defined in (11), there exists some positive constant $C$ independent of $t$ such that

$$
\|\epsilon(1, t, t)\|_{\mathcal{H}} \leq C\|\delta\epsilon(\cdot, t - \tau)\|_{\mathcal{H}}, \forall t > \tau,
$$

(44)

$$
\|\epsilon(x, t, t)\|_{\mathcal{H}} \leq C\|\delta\epsilon(\cdot, t - \tau)\|_{\mathcal{H}}, \forall 0 \leq x \leq 1, t > \tau,
$$

where

$$
C = \left(\sum_{n=0}^{\infty} |\lambda_n^{-1} e^{\lambda_n t}|^2\right)^{1/2}.
$$

(45)

Moreover, by (14) and $C_q$-semigroup theory, the following inequality holds:
for any \( t > \tau \), where \( M \) and \( \omega \) are given in (13).

In conclusion, based on the equations from (37) to (43), we have that

\[
\| \mathcal{A} \varepsilon (\cdot, t) \|_X \leq M e^{-\omega (t-\tau)} \| \mathcal{A} \varepsilon (\cdot, 0) \|_X, \tag{46}
\]

\[
| \mathcal{E} (1, t, t) | \leq C M e^{-\omega (t-\tau)} \| \mathcal{A} \varepsilon (\cdot, 0) \|_X, \forall t > \tau,
\]

\[
| \mathcal{E} (x, t, t) | \leq C M e^{-\omega (t-\tau)} \| \mathcal{A} \varepsilon (\cdot, 0) \|_X, \quad \forall 0 \leq x \leq 1, t > \tau. \tag{47}
\]

From the second inequality of (44), it is shown that

\[
\left| \int_0^1 e^{\eta (1-\eta)} \varepsilon (\eta, t, t) d\eta \right| \leq e^q C M e^{-\omega (t-\tau)} \| \mathcal{A} \varepsilon (\cdot, 0) \|_X, \quad \forall t > \tau, q > 0. \tag{48}
\]

On the other hand, the equivalent system of (32) can be illustrated easily by the transformation (2) as follows:

\[
\begin{align*}
\dot{v}_x (x, t) &= v_x (x, t), \quad 0 < x < 1, t > \tau, \\
v_x (0, t) &= c_0 v (0, t), \quad t > \tau, c_0 > 0, \\
v_x (1, t) &= -(c_0 + q) \varepsilon (1, t, t) - q (c_0 + q) \int_0^1 e^{\eta (1-\eta)} \varepsilon (\eta, t, t) d\eta, \quad t > \tau, q > 0.
\end{align*}
\]

It can be written as follows:

\[
\frac{d}{dt} v(\cdot, t) = \mathcal{A}_0 v(\cdot, t) + \mathcal{B}_0 \left( (c_0 + q) \left( \varepsilon (1, t, t) + q \int_0^1 e^{\eta (1-\eta)} \varepsilon (\eta, t, t) d\eta \right) \right), \tag{50}
\]

where

\[
\begin{align*}
\mathcal{A}_0 \varphi &= \varphi'' , \quad \forall \varphi \in D(\mathcal{A}_0), \\
D(\mathcal{A}_0) &= \left\{ \varphi \in H^1 (0, 1) | \varphi (0) = \varphi (1) = 0 \right\}, \\
\mathcal{B}_0 &= -\delta (x - 1). \tag{51}
\end{align*}
\]

A direct computation shows

\[
\mathcal{B}_0 \mathcal{A}_0^{-1} f = -\int_0^1 \left( \frac{1}{c_0 + \eta} \right) f (\eta) d\eta, \quad \forall f \in \mathcal{H}. \tag{52}
\]

which means that \( \mathcal{B}_0 \mathcal{A}_0^{-1} \) is bounded.

For the Lyapunov function of system (46),

\[
E_0 (t) = \frac{1}{2} \int_0^1 v^2 (x, t) dx. \tag{54}
\]

The direct computation shows that

\[
\dot{E}_0 (t) = -c_0 v^2 (0, t) - \int_0^1 v_x^2 (x, t) dx, \tag{55}
\]

which means that

\[
c_0 \int_0^T v^2 (0, t) dt \leq E_0 (0). \tag{56}
\]

for any \( T > 0 \). This inequality together with (50) illustrates that \( \mathcal{B}_0 \) is admissible for \( e^{\delta t} \). Therefore, there exists a unique solution of (46) such that \( v(\cdot, t) \in C (t, \infty; \mathcal{H}) \). Since the operator \( \mathcal{A}_0 \) generates an exponentially stable \( C_0 \)-semigroup [25], it follows from (44) and (45) that [27, 28]
\[
\begin{align*}
&\leq (c_0 + q) M \left[ \frac{1}{\tau} \int_0^{1/2} \left( e^{2q(1-\eta)} \eta e^2 (\eta, s, s) \right) ds \right]^{1/2} \\
&\leq (c_0 + q) M \sqrt{C_1^2 + \frac{t}{2} e^{2qC_2^2}}/\omega \| \mathcal{A} (\cdot, 0) \|_\mathcal{F}, \\
&\leq (c_0 + q) \left\| e^{\mathcal{A} (t-s)} \mathcal{B}_0 \left[ (c_0 + q) \left( \epsilon (1, s, s) + q \int_0^1 e^{q(1-\eta)} \epsilon (\eta, s, s) d\eta \right) \right] \right\|_\mathcal{F} \\
&\leq (c_0 + q) \left\| e^{\mathcal{A} (t-s)} \mathcal{B}_0 \left( 0 \right) \right\|_{L^2(t/2, t)} \\
&\leq (c_0 + q) \left\| \epsilon (1, \cdot, \cdot) + q \int_0^1 e^{q(1-\eta)} \epsilon (\eta, s, s) d\eta \right\|_{L^2(t/2, t)} \\
&\leq (c_0 + q) L_0 M \omega e^{\omega t} \sqrt{C_1^2 + \frac{t}{2} e^{2qC_2^2}} /\omega \| \mathcal{A} (\cdot, 0) \|_\mathcal{F} \cdot e^{-w t/2},
\end{align*}
\]

for some constants \( L_0 > 0 \) independent of
\[
\epsilon (1, t, t) + q \int_0^1 e^{q(1-\eta)} \epsilon (\eta, t, t) d\eta,
\]
where

\[
\begin{align*}
v(\cdot, t) &= e^{\mathcal{A} (t-s)} v(\cdot, \tau) \\
&\quad + (c_0 + q) \int_\tau^t e^{\mathcal{A} (t-s)} \mathcal{B}_0 \left( \epsilon (1, s, s) + q \int_0^1 e^{q(1-\eta)} \epsilon (\eta, s, s) d\eta \right) ds \\
&\quad + (c_0 + q) e^{\mathcal{A} (t/2 - \tau)} \\
&\quad \cdot \int_{\tau/2}^{t/2} e^{\mathcal{A} (t/2-s)} \mathcal{B}_0 \left( \epsilon (1, s, s) + q \int_0^1 e^{q(1-\eta)} \epsilon (\eta, s, s) d\eta \right) ds.
\end{align*}
\]

Since \( \mathcal{A}_0 \) generates an exponentially stable \( C_0 \)-semigroup, there exist two positive constants \( M_0, \omega_0 \) such that
\[
\begin{align*}
\| v(\cdot, t) \|_{\mathcal{F}} &\leq M_0 e^{-\omega_0 (t-s)} \| v(\cdot, \tau) \|_{\mathcal{F}} + (c_0 + q) L_0 M_0 M_0 e^{-\omega_0 t} \sqrt{C_1^2 + \frac{t}{2} e^{2qC_2^2}} /\omega \\
&\leq M_0 e^{-\omega_0 (t-s)} \| v(\cdot, \tau) \|_{\mathcal{F}} + (c_0 + q) L_0 M_0 e^{-\omega_0 t} \sqrt{C_1^2 + \frac{t}{2} e^{2qC_2^2}} /\omega \\
&\cdot \| \mathcal{A} (\cdot, 0) \|_\mathcal{F} \cdot \left( e^{-w t/2} + e^{-w t/2} \cdot e^{\omega t} \right), \forall t > \tau.
\end{align*}
\]
\[ \|w(\cdot,t)\|_{H} \leq M_{0}e^{-\omega_{0}(t-\tau)}\|w_{0}\|_{H} + (\epsilon_{0} + q)L_{0}M_{0}Me^{\omega_{0}\tau}\sqrt{C_{1}^{2} + q^{2}e^{2qC_{2}^{2}}/\omega} \]

\[ \cdot \|\partial\epsilon(\cdot,0)\|_{H} \cdot (e^{-\omega_{0}t/2} + e^{-\omega_{0}t/2} \cdot e^{\omega_{0}t}), \forall t > \tau, \]

where \( L_{0} \) is defined in (54) and (55), and \( M_{0} \) and M are defined in (55) and (13), respectively.

3. Appendix: Stabilization with Dirichlet Boundary Control

For the system under the Dirichlet boundary control,

\[
\begin{align*}
    w_{t}(x,t) &= w_{xx}(x,t), \quad 0 < x < 1, t > 0, \\
    w_{x}(0,t) &= -qw(0,t), \quad t \geq 0, \\
    w(1,t) &= u(t), \quad t \geq 0, \\
    y(t) &= w(0,t-\tau), \quad t > \tau, \\
    w(x,0) &= w_{0}(x), \quad 0 \leq x \leq 1.
\end{align*}
\]

By the similar design of estimated feedback control as that in the previous section, it can be stabilized for any given time delay. Firstly, we can apply the backstepping transformation to obtain the stabilizing feedback for system (62). And then, we design the observer and predictor systems. The output feedback control based on the estimated state can be shown to stabilize system (62) for any given time delay.

4. Numerical Simulation

In this section, numerical simulations have been given to show the effectiveness of the stabilizing output feedback controller. The space step is 0.025, and the time step is \( 10^{-4} \).

For the closed-loop system (29)–(31) with the Neumann boundary control, initial values have been chosen as that

\[ w_{0}(x) = x - 1, \]

\[ \epsilon_{0}(x) = x^{2} - 2x - 2, \]

and the parameters have been chosen as that...
\begin{equation}
\begin{aligned}
k_1 &= 1, \\
q &= 1, \\
c_0 &= 1.
\end{aligned}
\end{equation}

while the time delay is \( \tau = 0.1 \). Then, it is found that the state of the closed-loop system is almost rest after a certain time shown in Figure 1.

For the closed-loop system with the Dirichlet boundary control, we choose the initial values as follows:
\begin{equation}
\begin{aligned}
w_0(x) &= x^2 - 2x, \\
e_0(x) &= \cos\left(\frac{\pi x}{2}\right),
\end{aligned}
\end{equation}

and the parameter has been chosen as \( q = 1 \) while the time delay is \( \tau = 0.1 \). It is shown that the state of the closed-loop system is evidently stable shown in Figure 2.

5. Conclusion

In this paper, we consider an unstable heat equation where there exists instability in the Neumann or Dirichlet boundary conditions. For the unstable Neumann boundary condition, we stabilize the heat equation where the observation signal suffers from a given time delay by designing the estimated state feedback controller based on the construction of the observer and predictor systems. Numerical simulations show the effectiveness of the stabilized controller. For the unstable Dirichlet boundary condition, a similar controller may be given to make the originally unstable system stable exponentially which can be shown in the simulation results of this paper. The future research direction may be the stabilization of the high dimensional or abstract parameter distributed systems with unstable effects [29, 30].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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