Back reaction of vacuum and the renormalization group flow from the conformal fixed point

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Abstract. We consider the GUT-like model with two scalar fields which has infinitesimal deviation from the conformal invariant fixed point at high energy region. In this case the dominating quantum effect is the conformal trace anomaly and the interaction between the anomaly-generated propagating conformal factor of the metric and the usual dimensional scalar field. This interaction leads to the renormalization group flow from the conformal point. In the supersymmetric conformal invariant model such an effect produces a very weak violation of supersymmetry at lower energies.

1 Introduction

The formulation of consistent quantum field theory in curved space-time requires two new elements: the action of vacuum $S_v[g_{\mu\nu}]$ and also the nonminimal scalar-curvature interaction $\frac{\xi}{2}R\phi^2$ for each scalar field. The theory which lacks one of those elements is nonrenormalizable, because the corresponding counterterms appear anyway, and already at the one-loop level (see, for example, [1] for the introduction). One can suppose that such a theory with an action $S = S_{\text{min}} + S_{\text{non-min}} + S_v$ should be valid until the effects of quantum gravity become important, that is for the energies well below the Planck scale. It is naturally to suppose that at the very high energies below the Planck scale the theory doesn’t depend on the massive parameter and is conformal invariant. The condition of local conformal invariance is the special choice of the nonminimal parameter $\xi = \frac{1}{6}$.

The necessary action of vacuum has, in the conformal case, the restricted form:

$$S_v = \int d^4x \sqrt{-g} \left( a_1 C^2 + a_2 E + a_3 R \right),$$

where $\sqrt{-g} C^2 = \sqrt{-g} C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ is the conformal invariant square of the Weyl tensor and $E = R_{\mu\nu\alpha\beta}^2 - 4R_{\mu\nu}^2 + R^2$ is the Gauss-Bonnet integrand. On quantum level the conformal invariance is spoiled by the trace anomaly which takes place in both vacuum [2] and matter-fields [3] sectors of the theory. In the vacuum sector the trace anomaly arises (in the framework of dimensional regularization and the minimal subtraction scheme) because the counterterms, which must be added

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to the one-loop effective action, are local expressions in \( n \)-dimensional space-time. In general, the origin of the anomaly is the lack of regularization preserving both general covariance and local conformal invariance (see \[1\] for the review of conformal anomaly and related issues).

The interest to the conformal invariant theories in four dimensions (4\( \text{d} \)) is partially inspired by the important role which they play in the two-dimensional (2\( \text{d} \)) theories which are traditionally considered as a toy models for the realistic higher dimensional theories. There are, however, a serious differences between 2\( \text{d} \) and 4\( \text{d} \) theories. First of all, in 2\( \text{d} \) the fields with a different spin contribute to the anomaly (which comes from the \( \int d^2x\sqrt{-g}R \)-type counterterm) with a different sign, that leads to the existence of the critical dimension for the sigma-models and thus in 2\( \text{d} \) the anomaly may be canceled. In 4\( \text{d} \) there are three possible types of the one-loop counterterms (1) and the fields with the different spin give contributions with the same sign to the renormalization constants of the vacuum parameters \( a_1 \) and \( a_2 \) in (1). Therefore in 4\( \text{d} \) the critical dimension doesn’t exist. The situation remains the same, even if the supersymmetry and quantum effects of gravity use to be incorporated \[3\]. Another possibility which may be realized only in 2\( \text{d} \) is to start from the non-linear sigma-model in a background fields (defined as a geometric objects in target space) and thus provide the cancelation of anomaly in the noncritical dimension \[1, 6\]. In 4\( \text{d} \) this scheme doesn’t work either, because the nonlinear sigma-model is non-renormalizable and its possible higher derivative version \[8\] presumably contains unphysical massive ghosts. Thus, the quantum effects inevitably lead to the violation of conformal invariance through the trace anomaly with the consequent propagation of the conformal factor \[3, 10\]. The study of the quantum theory of the conformal factor as a low-energy version of quantum gravity has been started in \[3\] (See also \[14\]).

In the matter fields sector of the classically conformal invariant theory the one-loop divergences are also invariant but the finite one-loop contributions to the one-loop effective action are not. This can be, in particular, seen through the renormalization of the composite operators in the expression for the energy-momentum tensor \[3\]. This renormalization leads to the violation of the conformal invariance in the higher-loop divergences \[13\]. Thus, in 4\( \text{d} \), the classically conformal invariant theory suffers from two kinds of deceases: the conformal anomaly arises at one loop and produces the propagation of the conformal factor and also the nonconformal divergences take place at higher loops and break the renormalizability. Therefore the local conformal symmetry in 4\( \text{d} \) can not be exact, and can be realized only as an approximate high energy phenomena which may be called asymptotic conformal invariance. The asymptotic conformal invariance has been originally discovered in \[14, 17\] as a consequence of the conformal invariance of the one-loop divergences. This kind of the asymptotic conformal invariance doesn’t follow from the asymptotic freedom and it puts some extra constraints on the multiplet composition and on the values of the coupling constants of the gauge model \[13\]. The shortcoming of this approach is that the higher order corrections to the \( \beta \)-functions spoil the asymptotic conformal invariance and hence \( \xi = \frac{1}{6} \) is not the renormalization group fixed point beyond the one-loop level. Here we adopt another point of view on the asymptotic conformal invariance. Let us suppose that the asymptotically free or finite gauge theory in curved space-time, which is generated at Planck energy scale as an effective low-energy theory, is originally conformal invariant and thus the conformal invariant theory is the initial condition for the renormalization group flow in far UV rather than its fixed point.

For the theories with a weak coupling the one-loop effects are always dominating and hence the leading quantum effect is the conformal (trace) anomaly

\[
< T_\mu^\mu > = k_1 C^2 + k_2 E + k_3 \Box R.
\]

\footnote{The derivation of the anomaly-generated effective action of the conformal factor has been also performed in \[11\] for the case of the theory with torsion and in \[12\] for the supersymmetric matter on the background of simple supergravity.}
The values of $k_{1,2,3}$ depend only on the number of fields of different spin in a GUT model [3]. The anomaly leads to the equation for the effective action $2g_{\mu\nu} \delta W / \delta g_{\mu\nu} = - \sqrt{-g} T_{\mu}^{\nu}$. The solution of this equation is the 4d analog of the Polyakov action. It can be written in a local form [3] as:

$$W[g_{\mu\nu}, \sigma] = S_c[g_{\mu\nu}] + \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \sigma \Delta \sigma + \sigma \left[ k_1' C^2 + k_2' \left( E - \frac{2}{3} \Box R \right) \right] - \frac{1}{12} k_3' R^2 \right\}. \tag{2}$$

The values of the coefficients $k_{1,2,3}'$ are related to the ones of the trace anomaly in a well known way [3]. The propagator of the conformal factor $\sigma$ is an inverse to the fourth derivative conformal invariant operator [19, 9, 10]:

$$\Delta = \Box^2 + 2R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \Box + \frac{1}{3} (\nabla^\mu R) \nabla_\mu \tag{3}$$

and hence in the flat space limit $\sigma$ is massless field which can be essential at long distances. The solution (2) includes also an arbitrary conformal invariant functional $S_c$. In four dimensional quantum field theory this functional can not be calculated exactly, and one can only establish its high energy asymptotic or derive some lower order terms in its expansion. However this functional doesn’t depend on the conformal factor and thus it is irrelevant for our purposes [3].

From the physical point of view anomaly means the propagating conformal factor. Simultaneously the two-loop effects produce the violation of the conformal constraint $\xi = \frac{1}{6}$ and as a result the conformal factor starts to interact with the usual scalar field. The investigation of the physical consequences of this interaction has been started in [24]. The interaction with the propagating conformal factor modifies the renormalization group equation for the scalar coupling $f$ and $\xi$. These equations have minimal IR-stable fixed point $\xi = f = 0$, and in the vicinity of this point one meets the first order phase transition, as a result the Einstein gravity is induced [24]. In [24] the theory with one scalar field was considered. Here we are going to perform the detailed study of the renormalization group equations in the framework of more complicated models with two scalar multiplets. Our purpose is to estimate the running of the couplings due to the interaction with the anomaly-generated conformal factor and, in particular, explore the renormalization group flow from the conformal fixed point in the models which are finite and supersymmetric in flat space time.

Starting from the conformal initial point in far UV limit one can trace the renormalization group flows for the coupling constants and $\xi$ backward to the lower energies. In this way one can predict the value of $\xi$ at the lower energy scales. Thus we arrive at the necessity to study the renormalization group behavior in the theory which has only infinitesimal deviation from the conformal fixed point in far UV. To perform this in a consistent way one has to take into account the effect of conformal anomaly and the consequent quantum effects of the propagating conformal factor [24]. The detailed study of the renormalization group behavior is the purpose of the present paper.

The paper is organized in a following way. In the next section we formulate the model of GUT coupled to the propagating conformal factor. In section 3 the derivation of the one-loop counterterms is performed. Section 4 is devoted to the renormalization group equations for two different models, and in the last section we draw our conclusions.

\[2\] The action of gravity induced by conformal anomaly has interest for the construction of quantum gravity [14, 20]. It looks possible that the proper choice of the functional $S_c$ may revoke the discrepancy between the above effective action and direct calculations of $W[g_{\mu\nu}, \sigma]$ in [3] and with the results of the study of the two- and three-point functions [22, 23].
2 Interaction of matter fields with conformal factor

We suppose that the effects of conformal factor are relevant below the Planck scale where the nonminimal parameters \( \xi \) become slightly different from \( \frac{1}{6} \) due to the higher loop effects. In curved space-time the transition to the low energies (or long distances) corresponds to conformal transformation in the induced gravity action [1], after that classical fields and induced gravity appear in a different conformal points [24] (see also [13]). In order to take this into account one has to make a conformal transformation of the metric in [2] and then consider the unified theory. However it is more useful to perform the conformal transformation in the action of the matter fields. Such a transformation corresponds to some change of variables in the path integral for the unified theory, and doesn’t modify the results of quantum calculations in the matter field sector, which we are interested in.

The only source of conformal noninvariance in the action of the massless GUT model is the nonminimal term in the scalar sector. As far as the values of the parameters \( \xi \) are not equal to \( \frac{1}{6} \) the conformal factor starts to interact with the scalar fields. Consider the general \( SU(N) \) model with two types of scalar fields: real ones \( \Phi^a \) in the adjoint representation and complex ones \( \varphi^i \) in vector representation of the gauge group. In curved space-time one has to introduce two nonminimal parameters \( \xi_1 \) and \( \xi_2 \) – one for each type of the scalar fields. This model has been investigated in [23] without taking into account the back reaction of vacuum, and one can find more complete information including the full set of the \( \beta \)-functions in this paper. Below we write down only that parts of the \( \beta \)-functions which we shall actually use.

Introducing the scale parameter \( \alpha \) we arrive at the following action for the conformal factor coupled to scalar fields:

\[
S_{sc} = W[g_{\mu\nu}, \alpha] + \int d^4x \sqrt{-g} \left\{ \frac{1}{2} (1 - 6\xi_1) \Phi^a \Phi^a + (1 - 6\xi_2) \varphi^+_i \varphi^i \right\} \left( \alpha^2 (\nabla \alpha)^2 + \alpha \Box \alpha \right)
+ \frac{1}{2} g^{\mu\nu} (D_{\mu} \Phi^a D_{\nu} \Phi^a) + g^{\mu\nu} (D_{\mu} \varphi^+_i (D_{\nu} \varphi^i) + \frac{1}{2} \xi_1 R \Phi^a \Phi^a + \xi_2 R \varphi^+_i \varphi^i - V(\Phi^a, \varphi^i)),
\]

where \( W[g_{\mu\nu}, \alpha] \) has been defined in [2] and we use the notation \((\nabla \alpha)^2 = g^{\mu\nu} \partial_{\mu} \alpha \partial_{\nu} \alpha\). In all the expressions \( D \) are the derivatives of the matter fields which are covariant with respect to both gauge and gravitational fields. The notations for the \( SU(N) \) group including the relations between symmetric \( D_{rab} \) and antisymmetric \( f_{abc} \) structure constants, generators \( \left( \frac{1}{2} \right)^j_i \), traces etc. can be found in [1] (see also second reference in [26]). The flat-space part of the potential has the form

\[
V(\Phi^a, \varphi^i) = \frac{1}{8} f_1 (\Phi^a \Phi^a)^2 + \frac{1}{8} f_2 (\Phi^a D_{rab} \Phi^b)^2 + \frac{1}{2} f_3 (\Phi^a \Phi^a) (\varphi^+_i \varphi^i)
+ \frac{1}{2} f_4 (\Phi^a D_{rab} \Phi^b)^2 \varphi^+_i \left( \frac{\lambda^j}{2} \right)_i \varphi^j + \frac{1}{2} f_5 (\varphi^+_i \varphi^i)^2.
\]

The action of scalar fields [1] must be supplemented by the action of spinors and gauge fields which are part of the GUT model. The corresponding Lagrangian has the form [25]

\[
\mathcal{L} = -\frac{1}{4} (G_{\mu\nu}^a)^2 + i \sum_{k=1}^m \bar{\psi}^a_{(k)} \left( \gamma^\mu D^a_{\mu} - h_1 f_{abc} \Phi^c \right) \psi^a_{(k)} + i \sum_{k=1}^m \bar{\psi}^i_{(k)} \left[ \gamma^\mu D_{\mu}^ij - h_2 \left( \frac{\lambda^j}{2} \right)_i \Phi^a \right] \psi_{i,(k)} +
+ i \sum_{s=1}^n \bar{\chi}_{i(s)} \gamma^\mu D^i_{\mu} \chi_{s} + ih_3 \sum_{k=1}^m \left[ \bar{\psi}^i_{(k)} \left( \frac{\lambda^a}{2} \right)_j \varphi^a \right] \varphi^{+j} \psi_{i,(k)} - \bar{\psi}^a_{(k)} \left( \frac{\lambda^a}{2} \right)_i \varphi^a \psi_{i,(k)}.
\]
We are interested in the quantum theory of matter fields and conformal factor $\sigma$ on the background of the classical metric. The interaction between matter fields and conformal factor arises as a result of the conformal transformation of the metric $g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} \exp(2\sigma)$ and the matter field $w \rightarrow w' = w \exp(d_w \sigma)$, where $d_w$ is the conformal weight of the field $w$. One can see that the massless spin-$\frac{1}{2}$ and spin-1 fields decouple from the conformal factor. The only kind of fields which takes part in such an interaction are scalars, for which the interaction with conformal factor appears as a result of the violation of the condition $\xi = 1/6$ at low energies. The contributions of other matter fields to the $\beta$-functions and effective potential of the scalar fields do not depend on the conformal factor, they are given by the usual expressions [25, 1].

3 Calculation of the one-loop divergences

The calculation of the one-loop divergences of the theory [1] is not a trivial problem because the classical action contains second derivative terms as well as fourth derivative ones. Here we shall use the approach of Ref. [26], where the one-loop divergences have been calculated for higher derivative quantum gravity coupled to the matter fields. Following [26], we shall use the background field method and generalized Schwinger-DeWitt technique of [27]. Thus we start from the separation of fields into background $\sigma, \phi$ and quantum $\tau, \eta$ ones by changing $(\sigma, \Phi^a, \varphi^i, \varphi^i_+ \rightarrow (\sigma', \Phi^{a'}, \varphi^{i'}, \varphi^{i'}_+)$ where

$$\sigma' = \sigma + \tau, \quad \Phi^{a'} = \Phi^a + i\eta^a, \quad \varphi^{i'} = \varphi^i + i\chi^i, \quad \varphi^{i'}_+ = \varphi^+_i + i\chi^+_i \quad (7)$$

and the imaginary units are introduced for convenience. The one-loop effective action is defined as

$$\Gamma = \frac{i}{2} \text{Tr} \ln \hat{H}, \quad (8)$$

where $\hat{H}$ is the bilinear (with respect to the quantum fields $\tau, \eta, \chi, \chi^+$) form of the classical action (4). After some algebra we get the following self-adjoint bilinear form

$$\hat{H} = \begin{pmatrix} H_{\tau\tau} & H_{\tau\eta} & H_{\tau\chi} & H_{\tau\chi^+} \\ H_{\eta\tau} & H_{\eta\eta} & H_{\eta\chi} & H_{\eta\chi^+} \\ H_{\chi^+\tau} & H_{\chi^+\eta} & H_{\chi^+\chi} & H_{\chi^+\chi^+} \\ H_{\chi^+\chi^+} & H_{\chi^+\chi} & H_{\chi\chi} & H_{\chi\chi^+} \end{pmatrix}, \quad (9)$$

with the following derivative structure of the operator $\hat{H}$:

$$\hat{H} = \begin{pmatrix} \Box^2 + 2V^\mu \nabla_\mu \nabla_\nu + N^\mu \nabla_\mu + U & \hat{Q}_1 \Box + \hat{Q}_2 \nabla_\mu + \hat{Q}_3 \\ P_1 \Box + P_2 \nabla_\mu + P_3 & 1 \Box + E^\mu \nabla_\mu + D \end{pmatrix}. \quad (10)$$

The bilinear form of the action is matrix differential operator with fourth derivatives in the $H_{\tau\tau}$ sector and with second derivatives in the sector of usual scalar fields and in the mixed pieces. This structure of the bilinear form is similar to the one which is known from the theory of multiscalar GUT coupled to higher derivative gravity [26]. This is indeed natural, because what we are doing now is nothing but the study of induced quantum gravity [4] unified with the same GUT model. This analogy facilitates the calculations considerably, because of the following three reasons:

i) The general expression for the divergent part of $\frac{i}{2} \text{Tr} \ln \hat{H}$ (10) is known from [26]

$$\Gamma_{div} = \frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \text{tr} \left\{ \frac{1}{4} P_2^2 \hat{Q}_2 + \frac{1}{4} P_1 \hat{Q}_3 - \frac{1}{4} V^\mu \hat{P}_1 \hat{Q}_1 - \hat{D} \hat{P}_1 \hat{Q}_1 + \frac{1}{2} (\hat{P}_1 \hat{Q}_1)^2 \right\}
+ \frac{1}{2} \hat{Q}_2 \nabla_\mu \hat{P}_1 + \frac{1}{6} R \hat{P}_1 \hat{Q}_1 + \frac{1}{24} V^\mu V_\mu + \frac{1}{48} (V^\mu)^2 - \frac{1}{6} V^\mu R_{\mu\nu} + \frac{1}{12} V^\mu R - U$$

5
\[
\frac{1}{20} \left( R_{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right) - \frac{1}{36} R^2 + i \frac{1}{2} \text{Tr} \ln \left\{ i \Box + \hat{E}^\lambda \nabla_\lambda + \hat{D} \right\}_{\text{div}} + \text{(surface terms)}.
\] (11)

The above formula includes the standard contribution of the second order operator

\[
\frac{i}{2} \text{Tr} \ln \left\{ i \Box + \hat{E}^\lambda \nabla_\lambda + \hat{D} \right\}_{\text{div}}
\]

(see, for example, [1]), the contribution of the minimal fourth order operator first calculated in [19] and also the contributions from the mixed sector derived in Ref. [20] for the case of higher derivative gravity coupled to matter fields. The formula (11) shows, in particular, that we do not need explicit expression for \( N^\lambda \).

ii) Calculating the divergences with the use of the formula (11) one can learn the form of the beta-functions. However, since the formula (11) is the same as for the higher derivative gravity, it is easy to see that the general structure of the renormalization group equations in our theory is also the same as the one established in [20]. In particular, the quantum gravity corrections to the beta-functions of the nonminimal parameters \( \xi_j \) are completely universal, they do not depend on the model and have the same form as for the one-scalar model [24]. Therefore all that we need is the form of the quantum gravity corrections to the beta-functions of the constants \( f_i \). These corrections can be calculated on flat background and hence in what follows we put \( g_{\mu\nu} = \eta_{\mu\nu} \) and switch off all the curvature dependent terms.

iii) The structure of the expression (11) is direct generalization of the one we have already studied in [24] for the 1-scalar model. One can easily check that here, just as in the 1-scalar model, the contributions of \( \text{tr} \hat{P}_1 \hat{Q}_1 \), \( \text{tr} (\hat{P}_1 \hat{Q}_2 + \hat{P}_2 \hat{Q}_1) \), \( \text{tr} (\hat{P}_1 \hat{Q}_3 + \hat{P}_3 \hat{Q}_1) \), and \( \text{tr} \hat{P}_2 \hat{Q}_2 \) cancel, and therefore we need only the expressions for \( V^{\mu\nu} \), \( U \), \( \hat{P}_1 \), \( \hat{Q}_1 \), \( \hat{E}^\lambda \), \( \hat{D} \). Disregarding the curvature dependent terms one can obtain the following expressions for \( V, U, \hat{Q}_1, \hat{P}_1, \hat{D} \):

\[
V^{\mu\nu} = -\alpha^2 (\zeta_1 \Phi^a \Phi^a + \zeta_2 \varphi^j \varphi^j) g^{\mu\nu}, \quad U = 0,
\]

\[
\hat{P}_1 = -i \alpha \left( \begin{array}{c} \zeta_2 \varphi^j \\ \zeta_2 \varphi^j \end{array} \right), \quad \hat{Q}_1 = -i \alpha \left( \begin{array}{cc} \zeta_1 \Phi^a & \zeta_2 \varphi^j \\ \zeta_2 \varphi^j & \zeta_2 \varphi^j \end{array} \right),
\]

\[
\hat{D} = \hat{H} = \begin{pmatrix}
-\zeta_1 Z \delta^{ab} + V^{ab} & V^a_j & V^a i \\
V^i_b & -\zeta_2 Z \delta^j_i + V^i_j & V^{ij} \\
V^b_i & V^i_j & \zeta_2 Z \delta^j_i + V^j_i
\end{pmatrix},
\] (12)

where \( Z = \alpha^2 (\nabla \sigma)^2 + \alpha (\Box \sigma) \) and

\[
V_{ab} = \frac{f_1}{2} \Phi^2 \delta^{ab} + f_1 \Phi^a \Phi^b + f_2 \left( \frac{1}{2} D_{\mu
u} D_{\rho\sigma} + D_{\mu\rho} D_{\nu\sigma} \right) \Phi^c \Phi^d + f_3 \delta^{ab} \varphi^2 + f_4 D_{\mu\nu} \varphi^i \left( \frac{\lambda^r}{2} \right)^i_j \varphi^j,
\]

\[
V_{aj} = f_3 \Phi^a \varphi^j + f_4 D_{\nu\alpha} \Phi^c \varphi^k \left( \frac{\lambda^r}{2} \right)^k_j \varphi^j,
\]

\[
V_{ai} = f_3 \Phi^a \varphi^i + f_4 D_{\nu\alpha} \Phi^c \varphi^k \left( \frac{\lambda^r}{2} \right)^k_i \varphi^i,
\]

\[
V_j^i = \frac{1}{2} f_3 \Phi^c \Phi^d \delta^j_i + \frac{1}{2} f_4 D_{\nu\alpha} \Phi^d \left( \frac{\lambda^r}{2} \right)^i_j \varphi^j + f_5 \varphi^2 \delta^j_i + f_5 \varphi^j \varphi^i,
\]

\[
V^{ij} = f_5 \varphi^i \varphi^j.
\]
Substituting these expressions into (11) we arrive at the explicit form of the corrections to the matter sector of $\Gamma_{\text{div}}$ from the quantum conformal factor\footnote{The renormalization in the vacuum sector was discussed in [24] for the case of the one-scalar model. Its form doesn’t depend on the number of scalar fields and that is why we do not discuss it here.}

\begin{equation}
\Gamma^{(1)}_{\text{div}} = \frac{\mu^{n-4}}{\varepsilon} 2 \int d^nx \sqrt{-g} \left\{ \frac{1}{2} \alpha^4 (1 - \zeta_1)^2 (\Phi^a \Phi^a)^2 + 2\alpha^4 \zeta_2^2 (1 - \zeta_2)(\varphi_i^+ \varphi^i)^2 + \frac{3}{2} f_1 \alpha^2 \zeta_1^2 (\Phi^a \Phi^a)^2 + 2 \alpha^4 \zeta_2 (1 - \zeta_1 - \zeta_2 + \zeta_1 \zeta_2) \Phi^2 \varphi^+ \varphi + \frac{3}{2} f_2 \alpha^2 \zeta_1^2 (\Phi^a D_{abc} \Phi^b)^2 + f_3 \alpha^2 [\zeta_1^2 + 4 \zeta_1 \zeta_2 + \zeta_2^2] \Phi^2 \varphi^+ \varphi + f_4 \alpha^2 [\zeta_1^2 + 4 \zeta_1 \zeta_2 + \zeta_2^2] (D_{abc} \Phi^b) \varphi_i^+ \left( \frac{\lambda^c}{2} \right)_i \varphi^j + 6 f_5 \alpha^2 \zeta_2^2 (\varphi^+ \varphi)^2 \right\} ,
\end{equation}

where $\varepsilon = (4\pi)^2 (n - 4)$ is the parameter of dimensional regularization, and we disregarded all surface and matter-independent terms.

One can see that there are no any divergences which lead to renormalization of scalar fields $\Phi, \varphi$. Thus the only modifications due to the contributions of the conformal factor are in the renormalization of couplings $f_{1,2,3,5}$ and parameters $\xi_{1,2}$, and the renormalization group equations include only $\beta-$functions, but not $\gamma$-functions. The contribution of the conformal factor to the effective potential of the scalar fields depends on the $\beta$-functions only.

### 4 Renormalization group equations

Now we are in a position to perform the renormalization group study of the back-reaction of vacuum to the matter fields. For our purposes it is convenient to use the formulation of the renormalization group in curved space-time, given in [17]. The renormalization group equation for the effective action in curved space-time has the form

\begin{equation}
\left\{ \mu \frac{d}{d\mu} + \beta_p \frac{d}{dp} - \int d^4 \sqrt{-g} \gamma_w \frac{\delta}{\delta w(x)} \right\} \Gamma[w, p, g_{\mu\nu}, \mu] = 0 ,
\end{equation}

where $w$ is the full set of the quantum fields (gauge, spinor, Higgs and conformal factor $\sigma$) and $p$ – complete set of parameters including $\xi$’s. The solution corresponding to the desirable scaling behavior has the form

\begin{equation}
\Gamma[e^{-2t}g_{\mu\nu}, w, p, \mu] = \Gamma[g_{\mu\nu}, w(t), p(t), \mu] ,
\end{equation}

where $\mu$ is the dimensional parameter of renormalization. Effective fields and coupling constants obey the equations

\begin{equation}
(4\pi)^2 \frac{dw(t)}{dt} = (\gamma_w + d_w) \phi ,
\end{equation}

\begin{equation}
(4\pi)^2 \frac{dp(t)}{dt} = \beta_p + p \frac{dp}{dt} ,
\end{equation}

where $\gamma$ and $\beta$ functions are defined as usual. According to the results of the previous section the renormalization of the fields and all couplings except $f_i$ and $\xi_j$ are not modified by the contributions of the quantum field $\sigma$. Thus the $\beta$-functions for the gauge and Yukawa coupling constants are just the ones derived in [23]. On the other hand, these contributions to the $\beta f_i$ and $\beta \xi_j$ all have
universal form and do not depend on the gauge group of the theory. Thus we find for our theory (from this moment we will be using the variables \(\zeta_i = 1 - 6\xi_j\) for convenience):

\[
\beta_{f_i} = \beta_{f_i}^{(0)} + \Delta \beta_{f_i}, \quad \beta_{\zeta_j} = \beta_{\zeta_j}^{(0)} + \Delta \beta_{\zeta_j}, \quad \beta_{h_k} = \beta_{h_k}^{(0)}, \quad \beta_y = \beta_y^{(0)},
\]

(18)

where \(\beta_{p}^{(0)}\) is the \(\beta\)-function for the effective parameter \(p\) in curved space-time without back reaction of vacuum (or other form of quantum gravity) and \(\Delta \beta_p\) are the quantum gravity corrections. In our case, contrary to the high derivative gravity [26] those \(\Delta \beta_p\)'s are nonzero only for the scalar and the nonminimal parameters.

According to recent communications (see, for example, [28]) the values of \(\xi_j\) are very important at the energies between the Planck scale \(M_{Pl} = 10^{19} GeV\) and the unification scale \(M_X = 10^{14} GeV\), because there is a hope to meet natural inflation for this rate of character energies. Thus our purpose is to evaluate the running of \(\xi_j\) (or \(\zeta_j\)) backward from the high energy Planck scale. As a lower end of the energy interval one can take the unification point \(M_X = 10^{14} GeV\) but, since it doesn’t lead to the serious changes in the calculations we shall take, as a lower limit, the Fermi scale \(M_F = 100 GeV\).

4.1 SU(N) model: running away from the finite theory

As an examples of the effect of the quantum conformal factor we shall consider two toy models with two scalar fields: one which possesses the one-loop finiteness without supersymmetry [25], and another one which admits \(N = 2\) supersymmetry but is not finite.

Let us start with the generic model [4]. The counterterms [13] lead to the following expressions for \(\Delta \beta_p\):

\[
(4\pi)^2 \Delta \beta_{f_1} = 12\alpha^2 f_1 \zeta_1^2 + 4\alpha^4 \zeta_1^2 (\zeta_1 - 1)^2, \quad f_1(0) = f_1; \quad (19)
\]

\[
(4\pi)^2 \Delta \beta_{f_2} = 12\alpha^2 f_2 \zeta_1^2, \quad f_2(0) = f_2; \quad (20)
\]

\[
(4\pi)^2 \Delta \beta_{f_3} = 2\alpha^2 f_3 (\zeta_1^2 + 4\zeta_1 \zeta_2 + \zeta_2^2) + 4\alpha^4 \zeta_1 \zeta_2 (1 - \zeta_1 - \zeta_2 + \zeta_1 \zeta_2), \quad f_3(0) = f_3; \quad (21)
\]

\[
(4\pi)^2 \Delta \beta_{f_4} = 2\alpha^2 f_4 (\zeta_1^2 + 4\zeta_1 \zeta_2 + \zeta_2^2), \quad f_4(0) = f_4; \quad (22)
\]

\[
(4\pi)^2 \Delta \beta_{f_5} = 12\alpha^2 f_5 \zeta_2^2 + 4\alpha^4 \zeta_2^2 (\zeta_2 - 1)^2, \quad f_5(0) = f_5 \quad (23)
\]

and

\[
(4\pi)^2 \Delta \beta_{\zeta_j} = 2\alpha^2 \zeta_j^2 (\zeta_j - 1), \quad \zeta_j(0) = \zeta_j. \quad (24)
\]

Let us now consider the asymptotic behavior of the effective couplings \(f_i(t), \zeta_j(t)\). As it was already mentioned above, all the \(\beta\)-functions (corresponding to gauge, Yukawa, scalar couplings and to the nonminimal parameters \(\xi_j\)) vanish in the conformal fixed point

\[
h_k(t) = h_k^*, \quad f_i(t) = f_i^*, \quad \zeta_j = 0, \quad (25)
\]

where \(f_i^*, h_k^*\) are the values corresponding to the fixed point in flat space-time. The corrections \(\Delta \beta\) indicate that there is also a second "minimal" fixed point with the same (24) solutions for \(h_k(t)\) and \(f_i(t)\) but with \(\zeta_j = 1\) (this corresponds to the \(\xi_j = 0\) in [4], that is why we call this fixed
point minimal). The behavior of the effective charges and effective potential in the vicinity of the minimal fixed point has been studied in [24] for the one-scalar model. Now we are interested in the behavior of the effective charges close to the conformal fixed point.

As far as the renormalization group equations for \( g(t), h_k(t), f_i(t), \zeta_j(t) \) are very cumbersome, it is reasonable to take particular models in which the study of these equations performs easier. As a first example, let us take a particular model of (4) with \( N = 8 \) and one scalar multiplet in the adjoint representation. When the number of spinor multiplets is \( m = 84 \) the theory is one-loop finite in the flat space-time (see Ref. [25] for a detailed discussion of such a model without taking the quantum effects of the conformal factor into account). To perform the numerical analysis of the renormalization group equations one has to impose the initial conditions at far UV. We suppose that the running from the conformal fixed point happens because of the two-loop contributions which violate the \( \xi = \frac{1}{6} \) condition and thus switch on the interaction with the conformal factor. Thus, to impose the initial conditions one has to evaluate the two-loop effects. The two-loop contributions are, generally speaking, proportional to \( \frac{g^4}{(4\pi)^4} \) multiplied by the combinatorial factor. Let us take the typical value of the parameter \( \zeta \) as \( 10^{-3} \), while the coupling constant is \( g^2 = 0.1 \). Then the numerical analysis enables one to trace the behavior of the effective couplings \( \zeta(t), f_1, f_2(t) \) for \( 0 > t = \ln(\mu/M_{Pl}) > -39 \) that corresponds to the running between \( \mu = M_{Pl} = 10^{19} \text{GeV} \) and \( \mu = M_P = 10^2 \text{GeV} \).

We have performed the numerical analysis of the full system of the renormalization group equations for the effective gauge, Yukawa, scalar couplings and \( \zeta(t) \) with taking into account the corrections (19) – (24). As an initial point of the renormalization group flow in far UV we choose the values (see [25] for the details)

\[
\begin{align*}
&h_2 = 0; \quad f_1^* \approx 0.43621, \quad f_2^* \approx 0.30277
\end{align*}
\]

that provides the one-loop finiteness in the theory without back reaction of vacuum. The results for the contribution to the couplings due to back-reaction, that is the differences \( Df_j \) and \( D\zeta_j \) between the couplings in the with and without the back-reaction of vacuum, are plotted in Fig. 1. It is easy to see that the plots for the deviations look like a plane lines. The reason for this is that the effect of the back reaction is very week and in such a “small” interval on the logarithmic scale the non-linear function looks like a linear. Also the numerical values presented at the Fig. 1 show that the effect is very small. For instance, the deviation of \( f_1, f_2 \) from the finite fixed point is about six orders smaller than the value of the coupling itself in this fixed point. One has to notice that since the effect is almost linear, it is quite easy to construct the same plots for other initial deviations \( D\zeta_1 \).

The similar numerical analysis has been performed for the more interesting \( SU(5) \) model (4) - (6) with two kinds of scalar fields and with

\[
\begin{align*}
m = 1, \quad n = 15; \quad h^2_{1,2,3}(t) = h^*_{1,2,3}, \quad f^2_{1,\ldots,5}(t) = f^*_{1,\ldots,5};
\end{align*}
\]

where

\[
\begin{align*}
h^*_{1,2,3} &= (1.421, 1.681, 2.361); \quad f^*_{1,2,3,4,5} = (0.659, 1.293, 0.324, 1.677, 1.039).
\end{align*}
\]

The results of the numerical solution are presented at the Fig. 2. The parameter \( D\zeta_1(0) \) here is taken 0.01 as in the previous example, and we kept the constraint \( D\zeta_1 = D\zeta_2 \) for the sake of simplicity. Qualitatively the results are the same as in the one-scalar case – the effect exists but it is tiny.
4.2 Model with the rigid breaking of supersymmetry

Let us now consider the $SU(2)$ gauge model with an action \cite{29,16,1}:

$$S = \int d^4\sqrt{-g} \left\{ -\frac{1}{4} (G_{\mu\nu}^a)^2 + \frac{1}{2} g^\mu\nu \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} g^\mu\nu \partial_\mu \Lambda \partial_\nu \Lambda + \frac{1}{2} \xi_1 R \varphi^2 + \frac{1}{2} \xi_2 R \Lambda^2 + i \bar{\psi}_a \gamma^\mu \nabla_\mu \psi_b - h_{1} \bar{\psi}_a F_{abc} \varphi_b \psi_c - i h_{2} \bar{\psi}_a \gamma^5 F_{abc} \Lambda_b \psi_c + \frac{1}{8} f_1 (\varphi^2)^2 - \frac{1}{8} f_2 (\Lambda^2)^2 - \frac{1}{4} f_3 \varphi^2 \Lambda^2 - \frac{1}{2} f_4 \left[ \varphi^2 \Lambda^2 - (\varphi \Lambda)^2 \right] \right\}$$ \hfill (26)

where $\varphi$ and $\Lambda$ are scalar and pseudoscalar fields in the adjoint representation of the $SU(2)$ gauge group. Other notations are obvious. In flat space, for the particular values of the couplings

$$f_1 = f_2 = f_3 = 0, \quad f_4 = h_{1}^2 = h_{2}^2 = g^2$$ \hfill (27)

the model \cite{28} possesses the $N = 2$ supersymmetry. For other values of the couplings one meets the renormalizable theory which has also second non-supersymmetric renormalization group fixed point \cite{29}. In curved space time this model is also renormalizable (if only the nonminimal terms and the action of vacuum are introduced), and admits the asymptotic conformal invariance for (27) and $\xi_1 + \xi_2 = 1/6$. 

Figure 1: Contribution to the running coupling constants due to back-reaction. $SU[8]$ model with one scalar field and 84 spinor multiplets.

Figure 2: Contribution to the running coupling constants due to back-reaction. $SU[5]$ model with two scalar fields and with the spinor content $m = 1, n = 15$. 
Taking into account our previous results we arrive at the renormalization group equations for the above model with the back reaction of vacuum. The gauge coupling behaves as

$$g^2(t) = g^2 \left(1 + \frac{8g^2t}{(4\pi)^2}\right)^{-1}$$

and supersymmetry fixes Yukawa constants to be $h_1^2 = h_2^2 = g^2$. In flat space-time those relations hold under renormalization. The same happens when we put our theory in curved space-time and take the back reaction of vacuum into account, because the quantum conformal factor contributes to the $\beta$-functions of the scalar couplings and nonminimal parameters. We are interested in the renormalization group flow from the supersymmetric and conformal invariant UV fixed point, that is why we have to impose the constraints (27) on the initial data. As far as the back reaction doesn’t concern the behavior of the Yukawa couplings, the relations (27) for the Yukawa couplings hold for all scales and they can be indeed used in the RG equations directly, while $f_{1,2,3,4}$ should be regarded as arbitrary quantities. After we use the relations between gauge and Yukawa coupling constants, the RG equations for $f_{1,2,3,4}(t)$ become

$$
\begin{align*}
(4\pi)^2 \frac{df_1}{dt} &= 11f_1^2 + 3f_3^2 + 8f_3f_4 + 8f_4^2 - 8f_1g^2 - 8g^4 + 12\alpha^2 f_1\zeta_1^2 + 4\alpha^4 \zeta_1^2 (\zeta_1 - 1)^2, \\
(4\pi)^2 \frac{df_2}{dt} &= 11f_2^2 + 3f_3^2 + 8f_3f_4 + 8f_4^2 - 8f_2g^2 - 8g^4 + 12\alpha^2 f_2\zeta_2^2 + 4\alpha^4 \zeta_2^2 (\zeta_2 - 1)^2, \\
(4\pi)^2 \frac{df_3}{dt} &= 4f_3^2 + (5f_3 + 4f_1)(f_1 + f_2) + 8f_3^2 - 8f_3g^2 - 8g^4 + 2\alpha^2 f_3(\zeta_1^2 + 4\zeta_1\zeta_2 + \zeta_2^2) + 8\alpha^4 \zeta_1\zeta_2 (1 - \zeta_1 - \zeta_2 + \zeta_1\zeta_2), \\
(4\pi)^2 \frac{df_4}{dt} &= 6f_4^2 - 8f_4g^2 - 6g^4 + 8f_3f_4 + 2f_1(f_1 + f_2) + 2\alpha^2 f_4(\zeta_1^2 + 4\zeta_1\zeta_2 + \zeta_2^2),
\end{align*}
$$

while for $\zeta_1, \zeta_2(t)$ we meet the equations:

$$
\begin{align*}
(4\pi)^2 \frac{d\zeta_1}{dt} &= \zeta_1 (5f_1 - 4g^2) + \zeta_2(3f_3 + 4f_4) + 2\alpha^2 \zeta_1^2 (\zeta_1 - 1), \\
(4\pi)^2 \frac{d\zeta_2}{dt} &= \zeta_2 (5f_2 - 4g^2) + \zeta_1(3f_3 + 4f_4) + 2\alpha^2 \zeta_2^2 (\zeta_2 - 1).
\end{align*}
$$

In order to study the renormalization group behaviour of the effective couplings $f(t)$, $\zeta(t)$ we have to choose the initial value of $g^2$ and the initial variation $D\zeta_1 = -D\zeta_2 \neq 0$. As in the previous case we take in the far UV $g^2 = 0.1$ and $\zeta_1 = 0.001$. The results of the numerical analysis, concerning the contributions due to back-reaction, are plotted in Figs. 3 for the case $\zeta_1 = 0.001$.

One can see that in this case the effect of quantum conformal factor is essentially the same as in the previous case of the $SU(8)$ and $SU(5)$ models. Because of those effects one meets the nonzero $f_{1,2,3}$ couplings with the first two having the values about $10^{-8}$ at the $M_X$ unification scale and about $10^{-6}$ at the $M_F$ Fermi scale, while they remain identically zero without the back reaction of the conformal factor.

In order to give some speculation on the strongly coupled theories we also investigated the case with initial values of $g^2 = 0.5$ and $\zeta = 0.1$. The results are plotted at Fig. 4.
Figure 3: Contribution of back-reaction to the running couplings $f_i(t)$ and $\zeta_j(t)$. First example with $\zeta_1(0) = -\zeta_2(0) = 0.001$.

Figure 4: Contribution of back-reaction to the running couplings $f_j(t)$ and $\zeta_j(t)$. $g^2 = 0.5$ and $\zeta_1(0) = -\zeta_2(0) = 0.1$.

5 Conclusion

The arguments were presented that in four-dimensional quantum field theory in curved space-time the local conformal symmetry can not be exact, and that it can be only approximate symmetry which serves as the initial point for the renormalization group equations in the far UV limit. Starting from the conformal invariant theory one meets the trace anomaly and consequent propagation of the conformal factor of the metric. We have considered the interaction between the quantized conformal factor and the matter fields in a theory with approximate conformal symmetry, in a region close to the scale of asymptotic freedom and asymptotic conformal invariance.

The contributions of the conformal factor to the $\beta$-functions of the scalar coupling constants have been calculated and it was shown that these contributions drive the theory out from the conformal point and also modify the values of the scalar coupling constants. If the starting model in far UV possesses supersymmetry and (or) finiteness, the interaction with the conformal factor leads to the violation of those properties at lower energies. For the initial violations about $D\zeta = 0.001$ the numerical values of the deviations of the scalar couplings from the symmetric state range from $10^{-8}$ to $10^{-6}$ at the Fermi scale and about one order less at the Unification scale. Indeed the effect is very weak and therefore all the dependencies are very close to be linear. In particular the above deviations may be considered as proportional to the squares of the initial violations $(D\zeta)^2$ of the conformal invariance. At higher loops the supersymmetry breaking in the scalar sector should violate supersymmetry in the Yukawa and gauge interactions. As a result the relations between the masses of the particles at low energies may differ from the ones which are expected
Figure 5: Contribution of back-reaction to the running couplings $f_j(t)$ and $\zeta_j(t)$. $g^2 = 0.5$ and $\zeta_1(0) = -\zeta_2(0) = 0.5$. This example was included only to illustrate the nonrealistic order of magnitude for $g^2$ and initial deviations $D\zeta_i$ for which the nonlinear behavior shows up.

in supersymmetric GUT’s and the supersymmetry is not seen at low energies. However, since the numerical effect of such a violation will be extremely small, it is very difficult to see how one can observe it. One can say that if the local conformal invariance exists as a high energy symmetry, it holds as a very good approximation at lower energies.
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Abstract. We consider the GUT-like model with two scalar fields which has infinitesimal deviation from the conformal invariant fixed point at high energy region. In this case the dominating quantum effect is the conformal trace anomaly and the interaction between the anomaly-generated propagating conformal factor of the metric and the usual dimensional scalar field. This interaction leads to the renormalization group flow from the conformal point. In the supersymmetric conformal invariant model such an effect produces a very weak violation of supersymmetry at lower energies.

1 Introduction

The formulation of consistent quantum field theory in curved space-time requires two new elements: the action of vacuum $S_v[g_{\mu\nu}]$ and also the nonminimal scalar-curvature interaction $\frac{1}{2}\xi R\phi^2$ for each scalar field. The theory which lacks one of those elements is nonrenormalizable, because the corresponding counterterms appear anyway, and already at the one-loop level (see, for example, [1] for the introduction). One can suppose that such a theory with an action $S = S_{\text{min}} + S_{\text{non-min}} + S_v$ should be valid until the effects of quantum gravity become important, that is for the energies well below the Planck scale. It is naturally to suppose that at the very high energies below the Planck scale the theory doesn’t depend on the massive parameter and is conformal invariant. The condition of local conformal invariance is the special choice of the nonminimal parameter $\xi = \frac{1}{6}$. The necessary action of vacuum has, in the conformal case, the restricted form:

$$S_v = \int d^4x\sqrt{-g}\left(a_1 C^2 + a_2 E + a_3 \Box R\right),$$

where $\sqrt{-g} C^2 = \sqrt{-g} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}$ is the conformal invariant square of the Weyl tensor and $E = R_{\mu\nu\alpha\beta}^2 - 4R_{\mu\nu}^2 + R^2$ is the Gauss-Bonnet integrand. On quantum level the conformal invariance is spoiled by the trace anomaly which takes place in both vacuum [2] and matter-fields [3] sectors of the theory. In the vacuum sector the trace anomaly arises (in the framework of dimensional regularization and the minimal subtraction scheme) because the counterterms, which must be added to the one-loop effective action, are local expressions in $n$-dimensional space-time. In general, the origin of the anomaly is the lack of regularization preserving both general covariance and local conformal invariance (see [4] for the review of conformal anomaly and related issues).

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The interest to the conformal invariant theories in four dimensions \((4d)\) is partially inspired by the important role which they play in the two-dimensional \((2d)\) theories which are traditionally considered as toy models for the realistic higher dimensional theories. There are, however, a serious differences between \(2d\) and \(4d\) theories. First of all, in \(2d\) the fields with a different spin contribute to the anomaly (which comes from the \(\int d^2 x \sqrt{-g} R\)-type counterterm) with a different sign, that leads to the existence of the critical dimension for the sigma-models and thus in \(2d\) the anomaly may be canceled. In \(4d\) there are three possible types of the one-loop counterterms \((1)\) and the fields with the different spin give contributions with the same sign to the renormalization constants of the vacuum parameters \(a_1\) and \(a_2\) in \((1)\). Therefore in \(4d\) the critical dimension doesn’t exist. The situation remains the same, even if the supersymmetry and quantum effects of gravity use to be incorporated \([5]\). Another possibility which may be realized only in \(2d\) is to start from the non-linear sigma-model in a background fields (defined as a geometric objects in target space) and thus provide the cancelation of anomaly in the noncritical dimension \([6, 7]\). In \(4d\) this scheme doesn’t work either, because the nonlinear sigma-model is non-renormalizable and its possible higher derivative version \([8]\) presumably contains unphysical massive ghosts. Thus, the quantum effects inevitably lead to the violation of conformal invariance through the trace anomaly with the consequent propagation of the conformal factor \([9, 10]^{1}\). The study of the quantum theory of the conformal factor as a low-energy version of quantum gravity has been started in \([13]\) (See also \([14]\)).

In the matter fields sector of the classically conformal invariant theory the one-loop divergences are also invariant but the finite one-loop contributions to the one-loop effective action are not. This can be, in particular, seen through the renormalization of the composite operators in the expression for the energy-momentum tensor \([3]\). This renormalization leads to the violation of the conformal invariance in the higher-loop divergences \([15]\). Thus, in \(4d\), the classically conformal invariant theory suffers from two kinds of deceases: the conformal anomaly arises at one loop and produces the propagation of the conformal factor and also the nonconformal divergences take place at higher loops and break the renormalizability. Therefore the local conformal symmetry in \(4d\) can not be exact, and can be realized only as an approximate high energy phenomena which may be called asymptotic conformal invariance.

The asymptotic conformal invariance has been originally discovered in \([16, 17]\) as a consequence of the conformal invariance of the one-loop divergences. This kind of the asymptotic conformal invariance doesn’t follow from the asymptotic freedom and it puts some extra constraints on the multiplet composition and on the values of the coupling constants of the gauge model \([18]\). The shortcoming of this approach is that the higher order corrections to the \(\beta\)-functions spoil the asymptotic conformal invariance and hence \(\xi = \frac{1}{6}\) is not the renormalization group fixed point beyond the one-loop level. Here we adopt another point of view on the asymptotic conformal invariance. Let us suppose that the asymptotically free or finite gauge theory in curved space-time, which is generated at Planck energy scale as an effective low-energy theory, is originally conformal invariant and thus the conformal invariant theory is the initial condition for the renormalization group flow in far UV rather than its fixed point.

For the theories with a weak coupling the one-loop effects are always dominating and hence the leading quantum effect is the conformal (trace) anomaly

\[
<T_{\mu}^{\mu} > = k_1 C^2 + k_2 E + k_3 \Box R.
\]

The values of \(k_{1,2,3}\) depend only on the number of fields of different spin in a GUT model \([2]\). The anomaly leads to the equation for the effective action \(2g_{\mu\nu} \delta W / \delta g_{\mu\nu} = -\sqrt{-g} T_{\mu}^{\mu}\). The solution of this equation is the \(4d\) analog of the Polyakov action. It can be written in a local form \([9]\) as:

\[
W[g_{\mu\nu}, \sigma] = S_{[g_{\mu\nu}]} + \int d^4 x \sqrt{-g} \left\{ \frac{1}{2} \sigma \Delta \sigma + \sigma \left[ k'_1 C^2 + k'_2 \left( E - \frac{2}{3} \Box R \right) \right] - \frac{1}{12} k'_3 R^2 \right\}.
\]  

\(^{1}\)The derivation of the anomaly-generated effective action of the conformal factor has been also performed in \([11]\) for the case of the theory with torsion and in \([12]\) for the supersymmetric matter on the background of simple supergravity.
The values of the coefficients $k'_1, k'_2, k'_3$ are related to the ones of the trace anomaly in a well known way [9]. The propagator of the conformal factor $\sigma$ is an inverse to the fourth derivative conformal invariant operator [19, 9, 10]:

\[
\Delta = \Box^2 + 2R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \Box + \frac{1}{3} (\nabla^\mu R) \nabla_\mu
\]  

(3)

and hence in the flat space limit $\sigma$ is massless field which can be essential at long distances. The solution (2) includes also an arbitrary conformal invariant functional $S_c$. In four dimensional quantum field theory this functional can not be calculated exactly, and one can only establish its high energy asymptotic or derive some lower order terms in its expansion. However this functional doesn’t depend on the conformal factor and thus it is irrelevant for our purposes.

From the physical point of view anomaly means the propagating conformal factor. Simultaneously the two-loop effects produce the violation of the conformal constraint $\xi = \frac{1}{6}$ and as a result the conformal factor starts to interact with the usual scalar field. The investigation of the physical consequences of this interaction has been started in [24]. The interaction with the propagating conformal factor modifies the renormalization group equation for the scalar coupling $f$ and $\xi$. These equations have minimal IR-stable fixed point $\xi = f = 0$, and in the vicinity of this point one meets the first order phase transition, as a result the Einstein gravity is induced [24]. In [24] the theory with one scalar field was considered. Here we are going to perform the detailed study of the renormalization group equations in the framework of more complicated models with two scalar multiplets. Our purpose is to estimate the running of the couplings due to the interaction with the anomaly-generated conformal factor and, in particular, explore the renormalization group flow from the conformal fixed point in the models which are finite and supersymmetric in flat space time.

Starting from the conformal initial point in far UV limit one can trace the renormalization group flows for the coupling constants and $\xi$ backward to the lower energies. In this way one can predict the value of $\xi$ at the lower energy scales. Thus we arrive at the necessity to study the renormalization group behavior in the theory which has only infinitesimal deviation from the conformal fixed point in far UV. To perform this in a consistent way one has to take into account the effect of conformal anomaly and the consequent quantum effects of the propagating conformal factor [24]. The detailed study of the renormalization group behavior is the purpose of the present paper.

The paper is organized in a following way. In the next section we formulate the model of GUT coupled to the propagating conformal factor. In section 3 the derivation of the one-loop counterterms is performed. Section 4 is devoted to the renormalization group equations for two different models, and in the last section we draw our conclusions.

2 Interaction of matter fields with conformal factor

We suppose that the effects of conformal factor are relevant below the Planck scale where the non-minimal parameters $\xi$ become slightly different from $1/6$ due to the higher loop effects. In curved space-time the transition to the low energies (or long distances) corresponds to conformal transformation in the induced gravity action [1], after that classical fields and induced gravity appear in a different conformal points [24] (see also [13]). In order to take this into account one has to make a conformal transformation of the metric in (2) and then consider the unified theory. However it is more useful to perform the conformal transformation in the action of the matter fields. Such a transformation corresponds to some change of variables in the path integral for the unified theory, and

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2 The action of gravity induced by conformal anomaly has interest for the construction of quantum gravity [14, 20]. It looks possible that the proper choice of the functional $S_c$ may revoke the discrepancy between the above effective action and direct calculations of $W[\eta_{\mu\nu}, \sigma]$ in [21] and with the results of the study of the two- and three-point functions [22, 23].
doesn’t modify the results of quantum calculations in the matter field sector, which we are interested in.

The only source of conformal noninvariance in the action of the massless GUT model is the nonminimal term in the scalar sector. As far as the values of the parameters $\xi$ are not equal to 0, the conformal factor starts to interact with the scalar fields. Consider the general $SU(N)$ model with two types of scalar fields: real ones $\Phi^a$ in the adjoint representation and complex ones $\varphi^i$ in vector representation of the gauge group. In curved space-time one has to introduce two nonminimal parameters $\xi_1$ and $\xi_2$ – one for each type of the scalar fields. This model has been investigated in [25] without taking into account the back reaction of vacuum, and one can find more complete information including the full set of the $\beta$-functions in this paper. Below we write down only that parts of the $\beta$-functions which we shall actually use.

Introducing the scalar parameter $\alpha$ we arrive at the following action for the conformal factor coupled to scalar fields:

$$S_{sc} = W[g_{\mu\nu}, \sigma] + \int d^4x \sqrt{-g} \left\{ \left[ \frac{1}{2} (1 - 6\xi_1) \Phi^a \Phi_a + (1 - 6\xi_2) \varphi^i \varphi^i \right] (\alpha^2 (\nabla \sigma)^2 + \alpha \Box \sigma) \right. + \frac{1}{2} g^{\mu\nu} (D_\mu \Phi)^a (D_\nu \Phi)^a + g^{\mu\nu} (D_\mu \varphi^i) (D_\nu \varphi^i) + \frac{1}{2} \xi_1 R \Phi^a \Phi_a + \xi_2 R \varphi^i \varphi^i - V(\Phi^a, \varphi^i) \right\},$$

(4)

where $W[g_{\mu\nu}, \sigma]$ has been defined in (2) and we use the notation $(\nabla \sigma)^2 = g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma$. In all the expressions $D$ are the derivatives of the matter fields which are covariant with respect to both gauge and gravitational fields. The notations for the $SU(N)$ group including the relations between symmetric $D_{rabc}$ and antisymmetric $f_{abc}$ structure constants, generators $(\frac{1}{2})^i$, traces etc. can be found in [1] (see also second reference in [26]). The flat-space part of the potential has the form

$$V(\Phi^a, \varphi^i) = \frac{1}{8} f_1 (\Phi^a \Phi^a)^2 + \frac{1}{8} f_2 (\Phi^a D_{rabc} \Phi^b)^2 + \frac{1}{2} f_3 (\Phi^a \Phi^a) (\varphi^i \varphi^i)$$

$$+ \frac{1}{2} f_4 (\Phi^a D_{rabc} \Phi^b)^2 \varphi^i \varphi^j \left( \frac{\lambda}{2} \right)^i_j + \frac{1}{2} f_5 (\varphi^i \varphi^i)^2.$$  

(5)

The action of scalar fields (4) must be supplemented by the action of spinors and gauge fields which are part of the GUT model. The corresponding Lagrangian has the form [25]

$$\mathcal{L} = -\frac{1}{4} (G_{\mu\nu}^a)^2 + i \sum_{k=1}^m \tilde{\psi}_k^a \left( \gamma^\mu D_\mu - h_1 f^{abc} \Phi^c \right) \psi_k^b + i \sum_{k=1}^m \tilde{\varphi}_k^i \left[ \gamma^\mu D_\mu - h_2 \left( \frac{\lambda}{2} \right)^i_j \Phi^a \right] \varphi^j_k +$$

$$+ i \sum_{i=1}^n \tilde{\chi}_i^a \gamma^\mu D_\mu \chi_i^a + i h_3 \sum_{k=1}^m \left[ \tilde{\psi}_k^a \left( \frac{\lambda}{2} \right)^i_j \varphi^+ \psi^+_k \right] - \tilde{\psi}_k^a \left( \frac{\lambda}{2} \right)^i_j \varphi^i \psi^+_k.$$

(6)

We are interested in the quantum theory of matter fields and conformal factor $\sigma$ on the background of the classical metric. The interaction between matter fields and conformal factor arises as a result of the conformal transformation of the metric $g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} \exp(2\alpha \sigma)$ and the matter field $w \rightarrow w' = w \exp(d_w \alpha \sigma)$, where $d_w$ is the conformal weight of the field $w$. One can see that the massless spin-1 and spin-1 fields decouple from the conformal factor. The only kind of fields which takes part in such an interaction are scalars, for which the interaction with conformal factor appears as a result of the violation of the condition $\xi = 1/6$ at low energies. The contributions of other matter fields to the $\beta$-functions and effective potential of the scalar fields do not depend on the conformal factor, they are given by the usual expressions [25, 1].
3 Calculation of the one-loop divergences

The calculation of the one-loop divergences of the theory (4) is not a trivial problem because the classical action contains second derivative terms as well as fourth derivative ones. Here we shall use the approach of Ref. [26], where the one-loop divergences have been calculated for higher derivative quantum gravity coupled to the matter fields. Following [26], we shall use the background field method and generalized Schwinger-DeWitt technique of [27]. Thus we start from the separation of fields into background $\sigma, \phi$ and quantum $\tau, \eta$ ones by changing $(\sigma, \Phi^a, \varphi^i, \varphi^+_i) \rightarrow (\sigma', \Phi^a', \varphi^i', \varphi^+_{i'})$ where

$$\sigma' = \sigma + \tau, \quad \Phi^a' = \Phi^a + i\eta^a, \quad \varphi^i' = \varphi^i + i\chi^i, \quad \varphi^+_{i'} = \varphi^+_i + i\chi^+_i$$
and the imaginary units are introduced for convenience. The one-loop effective action is defined as

$$\Gamma = \frac{i}{2} \text{Tr} \ln \hat{H},$$

where $\hat{H}$ is the bilinear (with respect to the quantum fields $\tau, \eta, \chi, \chi^+$) form of the classical action (4). After some algebra we get the following self-adjoint bilinear form

$$\hat{H} = \begin{pmatrix}
H_{\tau \tau} & H_{\tau \eta} & H_{\tau \chi} & H_{\tau \chi^+} \\
H_{\eta \tau} & H_{\eta \eta} & H_{\eta \chi} & H_{\eta \chi^+} \\
H_{\chi \tau} & H_{\chi \eta} & H_{\chi \chi} & H_{\chi \chi^+} \\
H_{\chi^+ \tau} & H_{\chi^+ \eta} & H_{\chi^+ \chi} & H_{\chi^+ \chi^+}
\end{pmatrix},$$

with the following derivative structure of the operator $\hat{H}$:

$$\hat{H} = \left( \Box^2 + 2V^{\mu\nu}\nabla_\mu \nabla_\nu + N^\mu \nabla_\mu + U \right) \hat{Q}_1 + \frac{1}{4} \hat{P}_2 + \hat{\nabla}^\mu + \hat{P}_3$$

The bilinear form of the action is matrix differential operator with fourth derivatives in the $H_{\tau \tau}$ sector and with second derivatives in the sector of usual scalar fields and in the mixed pieces. This structure of the bilinear form is similar to the one which is known from the theory of multiscalar GUT coupled to higher derivative gravity [26]. This is indeed natural, because what we are doing now is nothing but the study of induced quantum gravity (2) unified with the same GUT model. This analogy facilitates the calculations considerably, because of the following three reasons:

i) The general expression for the divergent part of $\frac{\mu^{n-4}}{\varepsilon} 2 \int d^n x \sqrt{-g} \text{Tr} \left\{ \frac{1}{4} \hat{P}_2 \hat{Q}_2 - \frac{1}{4} \hat{P}_1 \hat{Q}_3 + \hat{V}^{\mu\nu} \hat{P}_1 \hat{Q}_1 - \hat{D} \hat{P}_1 \hat{Q}_1 + \frac{1}{2} (\hat{P}_1 \hat{Q}_1)^2 
+ \frac{1}{2} \hat{Q}_2 \nabla_\mu \hat{P}_1 - \frac{1}{6} \hat{R} \hat{P}_1 \hat{Q}_1 + \frac{1}{24} \hat{V}^{\mu\nu} \hat{V}_{\mu\nu} + \frac{1}{48} (\hat{V}^\mu)^2 - \frac{1}{6} \hat{V}^{\mu\nu} \hat{R}_{\mu\nu} + \frac{1}{12} \hat{V}^\mu \hat{R} - \hat{U}
+ \frac{1}{20} \left( \hat{R}^{\mu\nu} \hat{R}_{\mu\nu} - \frac{1}{3} \hat{R}^2 \right) - \frac{1}{36} \hat{R}^2 \right\} + \frac{i}{2} \text{Tr} \ln \left\{ \hat{I} + \hat{E}^\lambda \nabla_\lambda + \hat{D} \right\}_{\text{div}} + \text{(surface terms)} \right).$

The above formula includes the standard contribution of the second order operator

$$\frac{i}{2} \text{Tr} \ln \left\{ \hat{I} + \hat{E}^\lambda \nabla_\lambda + \hat{D} \right\}_{\text{div}}$$
(see, for example, [1]), the contribution of the minimal fourth order operator first calculated in [10] and also the contributions from the mixed sector derived in Ref. [26] for the case of higher derivative...
gravity coupled to matter fields. The formula (11) shows, in particular, that we do not need explicit expression for \( N^\lambda \).

ii) Calculating the divergences with the use of the formula (11) one can learn the form of the beta-functions. However, since the formula (11) is the same as for the higher derivative gravity, it is easy to see that the general structure of the renormalization group equations in our theory is also the same as the one established in [26]. In particular, the quantum gravity corrections to the beta-functions of the nonminimal parameters \( \xi_j \) are completely universal, they do not depend on the model and have the same form as for the one-scalar model [24]. Therefore all that we need is the form of the quantum gravity corrections to the beta-functions of the constants \( f_i \). These corrections can be calculated on flat background and hence in what follows we put \( g_{\mu \nu} = \eta_{\mu \nu} \) and switch off all the curvature dependent terms.

iii) The structure of the expression (11) is direct generalization of the one we have already studied in [24] for the 1-scalar model. One can easily check that here, just as in the 1-scalar model, the contributions of \( \text{tr} \, \bar{P}_1 Q_1 \), \( \text{tr} \, (\bar{P}_1 Q_2 + P_2 \bar{Q}_1) \), \( \text{tr} \, (\bar{P}_1 \bar{Q}_3 + P_3 \bar{Q}_1) \), and \( \text{tr} \, P_2 \bar{Q}_2 \) cancel, and therefore we need only the expressions for \( V^{\mu \nu}, U, P_1, Q_1, E^\lambda, D \). Disregarding the curvature dependent terms one can obtain the following expressions for \( V, U, \bar{Q}_1, P_1, D \): 

\[
V^{\mu \nu} = -\alpha^2 (\zeta_1 \Phi \Phi^a + \zeta_2 \varphi_i^+ \varphi_i) \, g^{\mu \nu}, \quad U = 0, \\
\bar{P}_1 = -i\alpha \left( \zeta_2 \Phi^a \right), \\
\bar{Q}_1 = -i\alpha \left( \zeta_1 \Phi \zeta_2 \varphi^j \zeta_2 \varphi^+_j \right), \\
\bar{D} = \bar{H} = \begin{pmatrix} -\zeta_1 Z \delta^{ab} + V^{ab} & V^a & V^{a \dagger} \\ V^b & -\zeta_2 Z \delta^i_j + V^i_j & V^{i \dagger}_j \\ V^b_i & V^i_j & \zeta_2 Z \delta^i_j + V^i_j \end{pmatrix},
\]

where \( Z = \alpha^2 (\nabla \sigma)^2 + \alpha (\Box \sigma) \) and

\[
V_{ab} = \frac{f_1}{2} \Phi^2 \delta^{ab} + f_1 \Phi^a \Phi^b + f_2 \left( \frac{1}{2} D_{ab} D_{cd} + D_{rac} D_{rbd} \right) \Phi^c \Phi^d + \frac{f_3}{2} \delta^{ab} \varphi^2 + f_4 D_{ab} \varphi^+_i \left( \frac{\lambda^i}{2} \right)^{\dagger} \varphi^j, \\
V_{ai} = f_3 \Phi^a \varphi^+_i + f_4 D_{rac} \Phi^c \varphi^+_i \left( \frac{\lambda^i}{2} \right)^{\dagger}, \\
V_{ib} = f_3 \Phi^b \varphi^+_i + f_4 D_{rbd} \Phi^c \varphi^+_i \left( \frac{\lambda^i}{2} \right)^{\dagger}, \\
V^a_j = \frac{1}{2} f_5 \Phi^a \varphi^j + \frac{1}{2} f_4 D_{rac} \Phi^c \left( \frac{\lambda^a}{2} \right)^{\dagger} \varphi^b, \\
V^b_j = \frac{1}{2} f_5 \Phi^b \varphi^j + \frac{1}{2} f_4 D_{rbd} \Phi^c \left( \frac{\lambda^b}{2} \right)^{\dagger} \varphi^c, \\
V^i_j = \frac{1}{2} f_5 \Phi^i \varphi^j + \frac{1}{2} f_4 D_{rac} \Phi^c \left( \frac{\lambda^i}{2} \right)^{\dagger} \varphi^c, \\
V^{ij} = f_5 \delta^i_j \varphi^j, \\
V^{ij} = f_5 \varphi^+_i \varphi^+_j.
\]

Substituting these expressions into (11) we arrive at the explicit form of the corrections to the matter sector of \( \Gamma_{\text{div}} \) from the quantum conformal factor\(^3\).

\[
\Gamma_{\text{div}}^{(1)} = \frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \left( \frac{1}{2} \alpha^4 (1 - \zeta_1)^2 (\Phi^a \Phi^a)^2 + 2 \alpha^4 \zeta_2^2 (1 - \zeta_2) (\varphi^+_i \varphi^i)^2 + \frac{3}{2} f_1 \alpha^2 \zeta_1^2 (\Phi^a \Phi^a)^2 \right)
\]

\(^3\)The renormalization in the vacuum sector was discussed in [24] for the case of the one-scalar model. Its form doesn’t depend on the number of scalar fields and that is why we do not discuss it here.
\[ +2\alpha^4 \zeta_1 \zeta_2 (1 - \zeta_1 - \zeta_2 + \zeta_1 \zeta_2) \Phi^2 \varphi^+ \varphi + \frac{3}{2} f_2 \alpha^2 \zeta_1^2 (\Phi^2 D_{abc} \Phi^b)^2 + f_3 \alpha^2 \left[ \zeta_1^2 + 4 \zeta_1 \zeta_2 + \zeta_2^2 \right] \Phi^2 \varphi^+ \varphi \]
\[ + f_4 \alpha^2 \left[ \zeta_1^2 + 4 \zeta_1 \zeta_2 + \zeta_2^2 \right] (\Phi^2 D_{abc} \Phi^b) \varphi_i^+ \left( \frac{\lambda^i}{2} \right) \varphi_j^+ + 6 f_5 \alpha^2 (\varphi^+ \varphi)^2 \right) , \]

where \( \varepsilon = (4\pi)^2(\eta - 4) \) is the parameter of dimensional regularization, and we disregarded all surface and matter-independent terms.

One can see that there are no any divergences which lead to renormalization of scalar fields \( \Phi \), \( \varphi \). Thus the only modifications due to the contributions of the conformal factor are in the renormalization of couplings \( f_{1,2,3,4,5} \) and parameters \( \zeta_{1,2} \), and the renormalization group equations include only \( \beta \)-functions, but not \( \gamma \)-functions. The contribution of the conformal factor to the effective potential of the scalar fields depends on the \( \beta \)-functions only.

### 4 Renormalization group equations

Now we are in a position to perform the renormalization group study of the back-reaction of vacuum to the matter fields. For our purposes it is convenient to use the formulation of the renormalization group in curved space-time, given in [17, 1]. The renormalization group equation for the effective action in curved space-time has the form

\[ \left\{ \mu \frac{d}{d\mu} + \beta_{\nu} \frac{d}{dp} - \int d^4 \sqrt{-g} \gamma_{\nu} \frac{\delta}{\delta w(x)} \right\} \Gamma[w, p, g_{\mu\nu}, \mu] = 0 , \]

where \( w \) is the full set of the quantum fields (gauge, spinor, Higgs and conformal factor \( \sigma \)) and \( p \) – complete set of parameters including \( \xi \)'s. The solution corresponding to the desirable scaling behavior has the form

\[ \Gamma[e^{-2t}g_{\mu\nu}, w, p, \mu] = \Gamma[g_{\mu\nu}, w(t), p(t), \mu] , \]

where \( \mu \) is the dimensional parameter of renormalization. Effective fields and coupling constants obey the equations

\[ (4\pi)^2 \frac{dw(t)}{dt} = (\gamma_{\nu} + d_{\nu}) \psi , \quad w(0) = w , \quad \]

\[ (4\pi)^2 \frac{dp(t)}{dt} = \beta_{\nu} + p d_{\nu} , \quad p(0) = p , \]

where \( \gamma \) and \( \beta \) functions are defined as usual. According to the results of the previous section the renormalization of the fields and all couplings except \( f_{\nu} \) and \( \xi \) are not modified by the contributions of the quantum field \( \sigma \). Thus the \( \beta \)-functions for the gauge and Yukawa coupling constants are just the ones derived in [25]. On the other hand, these contributions to the \( \beta_{f_{\nu}} \) and \( \beta_{\xi_{\nu}} \) all have universal form and do not depend on the gauge group of the theory. Thus we find for our theory (from this moment we will be using the variables \( \zeta_{\nu} = 1 - 6\xi \) for convenience):

\[ \beta_{f_{\nu}} = \beta_{f_{\nu}}^{(0)} + \Delta \beta_{f_{\nu}} , \quad \beta_{\xi_{\nu}} = \beta_{\xi_{\nu}}^{(0)} + \Delta \beta_{\xi_{\nu}} , \quad \beta_{h_{\nu}} = \beta_{h_{\nu}}^{(0)} , \quad \beta_{\varphi} = \beta_{\varphi}^{(0)} , \]

where \( \beta_{\nu}^{(0)} \) is the \( \beta \)-function for the effective parameter \( \nu \) in curved space-time without back reaction of vacuum (or other form of quantum gravity) and \( \Delta \beta_{\nu} \) are the quantum gravity corrections. In our case, contrary to the high derivative gravity [26] those \( \Delta \beta_{\nu} \)'s are nonzero only for the scalar and the nonminimal parameters.
According to recent communications (see, for example, [28]) the values of $\xi_j$ are very important at the energies between the Planck scale $M_P = 10^{19}$ GeV and the unification scale $M_X = 10^{14}$ GeV, because there is a hope to meet natural inflation for this rate of character energies. Thus our purpose is to evaluate the running of $\xi_j$ (or $\zeta_j$) backward from the high energy Planck scale. As a lower end of the energy interval one can take the unification point $M_X = 10^{14}$ GeV but, since it doesn’t lead to the serious changes in the calculations we shall take, as a lower limit, the Fermi scale $M_F = 100$ GeV.

4.1 SU(N) model: running away from the finite theory

As an examples of the effect of the quantum conformal factor we shall consider two toy models with two scalar fields: one which possesses the one-loop finiteness without supersymmetry [25], and another one which admits $N = 2$ supersymmetry but is not finite.

Let us start with the generic model (4). The counterterms (13) lead to the following expressions for $\Delta \beta_j$:

$$(4\pi)^2 \Delta \beta_{f_1} = 12\alpha^2 f_1 \zeta_1^2 + 4\alpha^4 \zeta_1^2 (\zeta_1 - 1)^2,$$

$$f_1(0) = f_1; \quad (19)$$

$$(4\pi)^2 \Delta \beta_{f_2} = 12\alpha^2 f_2 \zeta_1^2,$$

$$f_2(0) = f_2; \quad (20)$$

$$(4\pi)^2 \Delta \beta_{f_3} = 2\alpha^2 f_3 (\zeta_1^2 + 4\zeta_1 - 1),$$

$$f_3(0) = f_3; \quad (21)$$

$$(4\pi)^2 \Delta \beta_{f_4} = 2\alpha^2 f_4 (\zeta_1^2 + 4\zeta_1 - 1),$$

$$f_4(0) = f_4; \quad (22)$$

$$(4\pi)^2 \Delta \beta_{f_5} = 12\alpha^2 f_5 \zeta_1^2 + 4\alpha^4 \zeta_1^2 (\zeta_1 - 1)^2,$$

$$f_5(0) = f_5 \quad (23)$$

and

$$(4\pi)^2 \Delta \beta_{\zeta_j} = 2\alpha^2 \zeta_j^2 (\zeta_j - 1),$$

$$\zeta_j(0) = \zeta_j. \quad (24)$$

Let us now consider the asymptotic behavior of the effective couplings $f_i(t), \zeta_j(t)$. As it was already mentioned above, all the $\beta$-functions (corresponding to gauge, Yukawa, scalar couplings and to the nonminimal parameters $\xi$) vanish in the conformal fixed point

$$h_k(t) = h_k^*, \quad f_i(t) = f_i^*, \quad \zeta_j = 0, \quad (25)$$

where $f_i^*, h_k^*$ are the values corresponding to the fixed point in flat space-time. The corrections $\Delta \beta$ indicate that there is also a second "minimal" fixed point with the same (25) solutions for $h_k(t)$ and $f_i(t)$ but with $\zeta_j = 1$ (this corresponds to the $\xi_j = 0$ in (4), that is why we call this fixed point minimal). The behavior of the effective charges and effective potential in the vicinity of the minimal fixed point has been studied in [24] for the one-scalar model. Now we are interested in the behavior of the effective charges close to the conformal fixed point.

As far as the renormalization group equations for $g(t), h_k(t), f_i(t), \zeta_j(t)$ are very cumbersome, it is reasonable to take particular models in which the study of these equations performs easier. As a first example, let us take a particular model of (4) with $N = 8$ and one scalar multiplet in the adjoint representation. When the number of spinor multiplets is $m = 84$ the theory is one-loop finite in the flat space-time (see Ref. [25] for a detailed discussion of such a model without taking the quantum effects of the conformal factor into account). To perform the numerical analysis of the renormalization group equations one has to impose the initial conditions at far UV. We suppose that
the running from the conformal fixed point happens because of the two-loop contributions which violate the $\xi = \frac{1}{6}$ condition and thus switch on the interaction with the conformal factor. Thus, to impose the initial conditions one has to evaluate the two-loop effects. The two-loop contributions are, generally speaking, proportional to $\frac{g^2}{6\pi^2}$ multiplied by the combinatorial factor. Let us take the typical value of the parameter $\zeta$ as $10^{-3}$, while the coupling constant is $g^2 = 0.1$. Then the numerical analysis enables one to trace the behavior of the effective couplings $\zeta(t), f_{i,2}(t)$ for $0 > t = \ln(\mu/M_P) > -39$ that corresponds to the running between $\mu = M_P = 10^{19} \text{GeV}$ and $\mu = M_F = 10^2 \text{GeV}$.

We have performed the numerical analysis of the full system of the renormalization group equations for the effective gauge, Yukawa, scalar couplings and $\zeta(t)$ with taking into account the corrections (19) - (24). As an initial point of the renormalization group flow in far UV we choose the values (see [25] for the details)

$$h_2 = 0; \quad f_1^* \approx 0.43621, \quad f_2^* \approx 0.30277$$

that provides the one-loop finiteness in the theory without back reaction of vacuum. The results for the contribution to the couplings due to back-reaction, that is the differences $DF_j$ and $DC_j$ between the couplings in the with and without the back-reaction of vacuum, are plotted in Fig. 1. It is easy to see that the plots for the deviations look like a plane lines. The reason for this is that the effect of the back reaction is very weak and in such a “small” interval on the logarithmic scale the non-linear function looks like a linear. Also the numerical values presented at the Fig. 1 show that the effect is very small. For instance, the deviation of $f_{1,2}$ from the finite fixed point is about six orders smaller than the value of the coupling itself in this fixed point. One has to notice that since the effect is almost linear, it is quite easy to construct the same plots for other initial deviations $D\zeta_1$.

The similar numerical analysis has been performed for the more interesting $SU(5)$ model (4) - (6) with two kinds of scalar fields and with

$$m = 1, \quad n = 15; \quad h_{1,2,3}^2(t) = h_{1,2,3}^*, \quad f_{1,...,5}^2(t) = f_{1,...,5}^*;$$

where

$$h_{1,2,3}^* = (1.421, 1.681, 2.361); \quad f_{1,2,3,4,5}^* = (0.659, 1.293, 0.324, 1.677, 1.039).$$

The results of the numerical solution are presented at the Fig. 2. The parameter $DC_1(0)$ here is taken $0.01$ as in the previous example, and we kept the constraint $D\zeta_1 = D\zeta_2$ for the sake of simplicity. Qualitatively the results are the same as in the one-scalar case – the effect exists but it is tiny.

Figure 1: Contribution to the running coupling constants due to back-reaction. $SU[8]$ model with one scalar field and 84 spinor multiplets.
4.2 Model with the rigid breaking of supersymmetry

Let us now consider the $SU(2)$ gauge model with an action [29, 16, 1]:

\[
S = \int d^4\sqrt{-g} \left\{ -\frac{1}{4} (G_{\mu\nu})^2 + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} g^{\mu\nu} \partial_\mu \Lambda \partial_\nu \Lambda + \frac{1}{2} \xi_1 R \varphi^2 + \frac{1}{2} \xi_2 R \Lambda^2 + \frac{i}{8} f_1 (\varphi^2)^2 - \frac{1}{8} f_2 (\Lambda^2)^2 - \frac{1}{4} f_3 \varphi^2 \Lambda^2 - \frac{1}{2} f_4 \left[ \varphi^2 \Lambda^2 - (\varphi \Lambda)^2 \right] \right\}
\]

where \(\varphi\) and \(\Lambda\) are scalar and pseudoscalar fields in the adjoint representation of the $SU(2)$ gauge group. Other notations are obvious. In flat space, for the particular values of the couplings

\[
f_1 = f_2 = f_3 = 0, \quad f_4 = h_1^2 = h_2^2 = g^2
\]

the model (26) possesses the $N = 2$ supersymmetry. For other values of the couplings one meets the renormalizable theory which has also second non-supersymmetric renormalization group fixed point [29]. In curved space-time this model is also renormalizable (if only the nonminimal terms and the action of vacuum are introduced), and admits the asymptotic conformal invariance for (27) and \(\xi_1 + \xi_2 = 1/6\).

Taking into account our previous results we arrive at the renormalization group equations for the above model with the back reaction of vacuum. The gauge coupling behaves as [29]

\[
g^2(t) = g^2 \left( 1 + \frac{8g^2t}{(4\pi)^2} \right)^{-1}
\]

and supersymmetry fixes Yukawa constants to be \(h_1^2 = h_2^2 = g^2\). In flat space-time these relations hold under renormalization [29]. The same happens when we put our theory in curved space-time and take the back reaction of vacuum into account, because the quantum conformal factor contributes to the \(\beta\)-functions of the scalar couplings and nonminimal parameters. We are interested in the renormalization group flow from the supersymmetric and conformal invariant UV fixed point, that is why we have to impose the constraints (27) on the initial data. As far as the back reaction doesn't concern the behavior of the Yukawa couplings, the relations (27) for the Yukawa couplings hold for all scales and they can be indeed used in the RG equations directly, while \(f_{1,2,3,4}\) should be regarded
as arbitrary quantities. After we use the relations between gauge and Yukawa coupling constants, the RG equations for $f_{1,2,3,4}(t)$ become

$$(4\pi)^2 \frac{df_1}{dt} = 11f_1^2 + 3f_2^2 + 8f_3f_4 + 8f_4^2 - 8f_1g^2 - 8g^4 + 12\alpha^2f_1\xi_1^2 + 4\alpha^4\xi_1^2(\zeta_1 - 1)^2,$$

$$(4\pi)^2 \frac{df_2}{dt} = 11f_2^2 + 3f_3^2 + 8f_3f_4 + 8f_4^2 - 8f_2g^2 - 8g^4 + 12\alpha^2f_2\xi_2^2 + 4\alpha^4\xi_2^2(\zeta_2 - 1)^2,$$

$$(4\pi)^2 \frac{df_3}{dt} = 4f_3^2 + (5f_3 + 4f_4)(f_1 + f_2) + 8f_4^2 - 8f_3g^2 - 8g^4 + 2\alpha^2f_3(\xi_1^2 + 4\xi_1\xi_2 + \xi_2^2) +
+ 8\alpha^4\xi_1\xi_2(1 - \zeta_1 - \zeta_2 + \zeta_1\zeta_2),$$

$$(4\pi)^2 \frac{df_4}{dt} = 6f_4^2 - 8f_4g^2 - 6g^4 + 8f_3f_4 + 2f_4(f_1 + f_2) + 2\alpha^2f_4(\xi_2^2 + 4\xi_1\xi_2 + \xi_2^2),$$

(28)

while for $\xi_{1,2}(t)$ we meet the equations:

$$(4\pi)^2 \frac{d\xi_1}{dt} = \zeta_1(5f_1 - 4g^2) + \zeta_2(3f_3 + 4f_4) + 2\alpha^2\xi_1^2(\zeta_1 - 1),$$

$$(4\pi)^2 \frac{d\xi_2}{dt} = \zeta_2(5f_2 - 4g^2) + \zeta_1(3f_3 + 4f_4) + 2\alpha^2\xi_2^2(\zeta_2 - 1).$$

(29)

In order to study the renormalization group behaviour of the effective couplings $f(t)$, $\xi(t)$ we have to choose the initial value of $g^2$ and the initial variation $D\zeta_1 = -D\zeta_2 \neq 0$. As in the previous case we take in the far UV $g^2 = 0.1$ and $\zeta_1 = 0.001$. The results of the numerical analysis, concerning the contributions due to back-reaction, are plotted in Figs. 3 for the case $\zeta_1 = 0.001$.

![Figure 3: Contribution of back-reaction to the running couplings $f_j(t)$ and $\zeta_j(t)$. First example with $\zeta_1(0) = -\zeta_2(0) = 0.001$.](image)

One can see that in this case the effect of quantum conformal factor is essentially the same as in the previous case of the $SU(8)$ and $SU(5)$ models. Because of those effects one meets the nonzero $f_{1,2,3}$ couplings with the first two having the values about $10^{-8}$ at the $M_X$ unification scale and about $10^{-8}$ at the $M_F$ Fermi scale, while they remain identically zero without the back reaction of the conformal factor.

In order to give some speculation on the strongly coupled theories we also investigated the case with initial values of $g^2 = 0.5$ and $\zeta = 0.1$. The results are plotted at Fig. 4.
Figure 4: Contribution of back-reaction to the running couplings $f_j(t)$ and $\zeta_j(t)$. $g^2 = 0.5$ and $\zeta_1(0) = -\zeta_2(0) = 0.1$.

Figure 5: Contribution of back-reaction to the running couplings $f_j(t)$ and $\zeta_j(t)$. $g^2 = 0.5$ and $\zeta_1(0) = -\zeta_2(0) = 0.5$. This example was included only to illustrate the nonrealistic order of magnitude for $g^2$ and initial deviations $D\zeta_i$ for which the nonlinear behavior shows up.

5 Conclusion

The arguments were presented that in four-dimensional quantum field theory in curved space-time the local conformal symmetry can not be exact, and that it can be only approximate symmetry which serves as the initial point for the renormalization group equations in the far UV limit. Starting from the conformal invariant theory one meets the trace anomaly and consequent propagation of the conformal factor of the metric. We have considered the interaction between the quantized conformal factor and the matter fields in a theory with approximate conformal symmetry, in a region close to the scale of asymptotic freedom and asymptotic conformal invariance.

The contributions of the conformal factor to the $\beta$-functions of the scalar coupling constants have been calculated and it was shown that these contributions drive the theory out from the conformal point and also modify values of the scalar coupling constants. If the starting model in far UV possesses supersymmetry and (or) finiteness, the interaction with the conformal factor leads to the violation of those properties at lower energies. For the initial violations about $D\zeta_i = 0.001$ the numerical values of the deviations of the scalar couplings from the symmetric state range from $10^{-8}$ to $10^{-6}$ at the Fermi scale and about one order less at the Unification scale. Indeed the effect is very weak and therefore all the dependencies are very close to be linear. In particular the above deviations may be considered as proportional to the squares of the initial violations $(D\zeta_i)^2$ of the conformal invariance. At higher loops the supersymmetry breaking in the scalar sector should violate supersymmetry in the Yukawa and gauge interactions. As a result the relations between the masses
of the particles at low energies may differ from the ones which are expected in supersymmetric GUT's and the supersymmetry is not seen at low energies. However, since the numerical effect of such a violation will be extremely small, it is very difficult to see how one can observe it. One can say that if the local conformal invariance exists as a high energy symmetry, it holds as a very good approximation at lower energies.
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