Magnetic Strings In Five Dimensional Gauged Supergravity Theories

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ABSTRACT

Magnetic BPS string solutions preserving quarter of supersymmetry are obtained for all abelian gauged $d = 5 \ N = 2$ supergravity theories coupled to vector supermultiplets. Due to a “generalised Dirac quantization” condition satisfied by the minimized magnetic central charge, the string metric takes a universal form for all five dimensional gauged theories.

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1 Introduction

Black hole solutions of gauged extended supergravities have recently been the subject of intense research activities [1, 2, 3, 4, 5, 6]. This is largely motivated by the newly proposed correspondence between anti-de Sitter space, the ground state of gauged supergravity, and conformal field theories on its boundary [7]. Anti-de Sitter black hole solutions which break some or all of supersymmetries may be of relevance to the proposed anti-de Sitter/Conformal Field Theory correspondence.

BPS-saturated black holes constitute a large class of non-trivial gravitational backgrounds preserving some part of the supersymmetry of the theory. Supersymmetric BPS solutions for the theory of $N = 2$ gauged four dimensional pure supergravity were first considered in [1]. Also, BPS electrically charged solutions were found for the four dimensional $N = 8$ and $N = 2$ supergravity theories with vector multiplets in [1, 8]. In five dimensional gauged $N = 2$ supergravity coupled to vector supermultiplets [8], static spherically symmetric electrically charged BPS-saturated black holes were considered in [2]. A common feature of all these solutions is that they are supersymmetric and have naked singularities. BPS-saturated topological black holes in gauged supergravity, also with naked singularities, were obtained in [4]. Non-extreme electrically charged static black hole solutions of $N = 2$ five dimensional gauged supergravities were studied in [3]. The geometry of these solutions, in particular their singularity structure, and the domain of the parameters of the ADM mass for which horizons exist as well as their thermodynamic features were also studied. Such features could potentially provide an insight into dynamics of Yang-Mills theories with broken supersymmetry.

BPS magnetic string solutions which break half of supersymmetry in the theory of ungauged $N = 2$ five-dimensional supergravities were constructed in [9]. These solutions correspond, in models which can be obtained as compactifications of M-theory on a Calabi-Yau manifold, to five branes wrapping around the homology cycles of the Calabi-Yau compact space. Near the horizon, the supersymmetry of these solutions gets enhanced and fully restored. The horizon geometry of these solutions is identified with the space $AdS_3 \times S^2$.

Our purpose in this paper is to study spherically symmetric magnetically charged string solutions of five dimensional $N = 2$ gauged supergravity theory coupled to vector supermultiplets [10]. We organize this work as follows.
In Section 2, a brief review of five dimensional $N = 2$ supergravity is given within the context of very special geometry. In Section 3, magnetic string solution which preserves $1/4$ of the $N = 2$ supersymmetry are explicitly derived. It is shown that for any choice of the prepotential $V$ which defines the five dimensional theory, the string metric takes a universal form independent of the charge configuration of the solution. This universality is a consequence of the fact that the solutions depend on the magnetic central charge which, for the supersymmetric configuration, has to satisfy a generalised “Dirac quantization condition”. As discussed for the ungauged cases, a subclass of solutions of the $N = 2$ models are also solutions for models with more supersymmetry; i.e., $N = 4$ and $N = 8$ supergravity. This is the three-charge configuration with no self-intersections. The last section includes a summary and a discussion.

2 $D = 5 \ N = 2$ Gauged Supergravity

The theory of five-dimensional $N = 2$ supergravity coupled to abelian vector supermultiplets can be obtained by compactifying eleven-dimensional supergravity, the low-energy theory of M-theory, on a Calabi-Yau three-folds [11]. The massless spectrum of the theory contains $(h_{(1,1)} - 1)$ vector multiplets with real scalar components, and thus $h_{(1,1)}$ vector bosons (the additional vector boson is the graviphoton). The theory also contains $h_{(2,1)} + 1$ hypermultiplets, where $h_{(1,1)}$ and $h_{(2,1)}$ are the Calabi-Yau Hodge numbers. Gauged supergravity theories are obtained by gauging a subgroup of the R-symmetry group; the automorphism group of the supersymmetry algebra. The gauged $d = 5, \ N = 2$ supergravity theories are obtained by gauging the $U(1)$ subgroup of the $SU(2)$ automorphism group of the supersymmetry algebra [10]. This is achieved by introducing a linear combination of the abelian vector fields already present in the ungauged theory, i.e. $A_\mu = V_I A^I_\mu$, with a coupling constant $g$. The coupling of the fermi-fields to the $U(1)$ vector field breaks supersymmetry, and therefore gauge-invariant $g$-dependent terms must be introduced in order to preserve $N = 2$ supersymmetry. In a bosonic background, this amounts to the addition of a $g^2$-dependent scalar potential $V$ [10, 2].

The bosonic part of the gauged supersymmetric $N = 2$ Lagrangian which
describes the coupling of vector multiplets to supergravity is given by
\[
e^{-1} \mathcal{L} = \frac{1}{2} R + g^2 V - \frac{1}{4} G_{IJ} F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \frac{e^{-1}}{48} \epsilon^{\mu\nu\rho\sigma\lambda} C_{IJK} F_{\mu\nu}^I F_{\rho\sigma}^K A^k
\]  
(1)

with the space-time indices \((\mu, \nu) = 0, 1, \cdots, 4\), \(R\) is the scalar curvature, \(F_{\mu\nu}^I\) are the abelian field-strength tensors and \(e = \sqrt{-g}\) is the determinant of the Fünbein \(e_m^a\), \(V\) is the potential given by
\[
V(X) = V_I V_J \left(6X^I X^J - \frac{9}{2} g_{ij} \partial_i X^I \partial_j X^J\right),
\]  
(2)

where \(X^I\) represent the real scalar fields which satisfy the following condition
\[
\mathcal{V} = \frac{1}{6} C_{IJK} X^I X^J X^K = 1.
\]  
(3)

The physical quantities in (1) can all be expressed in terms of the homogeneous cubic polynomial \(\mathcal{V}\) which defines “very special geometry” [12]. We also have the relations
\[
G_{IJ} = -\frac{1}{2} \partial_I \partial_J \log V\bigg|_{V=1},
\]  
\[
g_{ij} = \partial_i X^I \partial_j X^J G_{IJ}\bigg|_{V=1},
\]  
(4)

where \(\partial_i\) and \(\partial_I\) refer, respectively, to a partial derivative with respect to the scalar field \(\phi^i\) and \(X^I = X^I(\phi^i)\).

It is worth pointing out that for Calabi-Yau compactification, \(\mathcal{V}\) represents the intersection form, \(X^I\) and \(X_I = \frac{1}{6} C_{IJK} X^J X^K\) correspond, respectively, to the size of the two- and four-cycles and \(C_{IJK}\) are the intersection numbers of the Calabi-Yau threefold.

### 3 Magnetic String Solutions

In this section the BPS extended magnetic solutions preserving 1/4 of the \(N = 2\) supersymmetry are constructed. This is achieved by solving for the
vanishing of supersymmetry transformation of the gravitino and gauginos
fields in a bosonic background. The supersymmetry transformation of these
fermionic fields in a bosonic background are given by [2]

\[ \delta \psi_\mu = \left( D_\mu + \frac{i}{8} X_I (\Gamma_{\mu}^{\nu}) F_{\nu I}^J + \frac{1}{2} g \Gamma_{\mu} X^I V_I - \frac{3i}{2} g V_I A_\mu^I \right) \epsilon, \]

\[ \delta \lambda_i = \left( \frac{3}{8} \Gamma_{\mu \nu} F_{\mu \nu}^I \partial_i X_I - \frac{i}{2} g_{ij} \Gamma_{\mu}^{\nu} \phi_j^j + \frac{3i}{2} g V_I \partial_i X_I \right) \epsilon \tag{5} \]

where \( \epsilon \) is the supersymmetry parameter and \( D_\mu \) is the covariant derivative.

A general spherically symmetric string solution can be written in the
following form

\[ ds^2 = -e^{2v} dt^2 + e^{2t} dz^2 + e^{2U} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \] \hspace{1cm} \tag{6}

and for the gauge fields we take

\[ A^I_{\phi} = -q^I \cos \theta, \]

\[ F^{I}_{\theta \phi} = q^I \sin \theta, \] \hspace{1cm} \tag{7}

where the functions \( (U, V, T) \) are functions of \( r \), and \( (\theta, \phi) \) are the polar
coordinates of the 2-sphere.

The Fubnbein of the above metric are given by

\[ e^0_t = e^V, \quad e^0_e = e^{-V}, \]

\[ e^1_z = e^T, \quad e^1_z = e^{-T}, \]

\[ e^2_r = e^U, \quad e^2_r = e^{-U}, \]

\[ e^3_\theta = r, \quad e^3_\theta = \frac{1}{r}, \]

\[ e^4_\phi = r \sin \theta, \quad e^4_\phi = \frac{1}{r \sin \theta}. \]

The spin connections for the above metric are given by

\[ \omega_{\mu \nu}^{ab} = (-, +, +, +), \{ \Gamma^a, \Gamma^b \} = 2\eta^{ab}, \quad D_\mu = \partial_\mu + \frac{1}{2} \omega_{\mu ab} \Gamma^a, \quad \omega_{\mu ab} \]

is the spin connection, and \( \Gamma^a \) are Dirac matrices and \( \Gamma^{a_1 a_2 \cdots a_n} = \frac{1}{n!} \Gamma^{[a_1} \Gamma^{a_2} \cdots \Gamma^{a_n]}. \)
\[
\begin{align*}
    w_{02} &= -V'e^{V-U}, \\
    w_{12} &= T'e^{T-U}, \\
    w_{23} &= -e^{-U}, \\
    w_{24} &= -e^{-U}\sin \theta, \\
    w_{34} &= -\cos \theta.
\end{align*}
\] (8)

where \((0, 1, 2, 3, 4)\) represent the flat indices.

Then from the supersymmetry transformation of the fermionic fields we obtain

\[
\begin{align*}
    \delta \psi_t &= \left( \partial_t + \frac{1}{2} e^{V-V'} \Gamma_{02} + \frac{i}{4} Z e^{V} \Gamma_{034} + \frac{1}{2} g e^{V} X^I V_I \Gamma_0 \right) \epsilon, \\
    \delta \psi_z &= \left( \partial_z + \frac{1}{2} e^{T-T'} \Gamma_{12} + \frac{i}{4} Z e^{T} \Gamma_{134} + \frac{1}{2} g e^{T} X^I V_I \Gamma_1 \right) \epsilon, \\
    \delta \psi_\theta &= \left( \partial_\theta - \frac{1}{2} e^{-U} \Gamma_{23} - \frac{i}{2} Z \Gamma_2 - \frac{1}{2} g r X^I V_I \Gamma_3 \right) \epsilon, \\
    \delta \psi_\phi &= \left( \partial_\phi - \left( \frac{e^{-U}}{2} \Gamma_{24} - \frac{i}{2} Z \Gamma_3 - \frac{g r}{2} \Gamma_4 \right) \sin \theta + \frac{1}{2} \left( 3 i g q^I V_I - \Gamma_{34} \right) \cos \theta \right) \epsilon, \\
    \delta \psi_r &= \left( \partial_r + \frac{i e U}{4} \frac{Z}{r^2} \Gamma_{234} + \frac{1}{2} g e^{U} X^I V_I \Gamma_2 \right) \epsilon
\end{align*}
\] (9)

where \(Z = q^I X_I\) is the magnetic central charge. As supersymmetric breaking conditions we take the following conditions

\[
\begin{align*}
    \Gamma_3 \Gamma_4 \epsilon &= i \epsilon, \\
    \Gamma_2 \epsilon &= -\epsilon.
\end{align*}
\] (10)

Then, the above transformations reduce to

\[
\begin{align*}
    \delta \psi_t &= \left( \partial_t - \frac{1}{2} (V'e^{V-U} + Z e^{V} \frac{1}{2 r^2} - g e^{V} X^I V_I) \Gamma_0 \right) \epsilon, \\
    \delta \psi_z &= \left( \partial_z - \frac{1}{2} (T'e^{T-U} + Z e^{T} \frac{1}{2 r^2} - g e^{T} X^I V_I) \Gamma_1 \right) \epsilon,
\end{align*}
\]

6
\[ \delta \psi_\theta = \left( \partial_\theta - \frac{1}{2} (e^{-U} - Z \frac{1}{r} - gr X^I V_I) \Gamma_3 \right) \epsilon, \]
\[ \delta \psi_\phi = \left( \partial_\phi - \frac{i}{2} (1 - 3g V_I q') \cos \theta - \frac{1}{2} (e^{-U} - Z \frac{1}{r} - gr X^I V_I) \sin \theta \Gamma_4 \right) \epsilon, \]
\[ \delta \psi_r = \left( \partial_r + \frac{e^U}{2} \left( \frac{Z}{2r^2} - g X^I V_I \right) \right) \epsilon. \]  

(11)

The vanishing of the above equations implies the following conditions on the supersymmetry spinor \( \epsilon \),

\[ \partial_t \epsilon = 0, \]
\[ \partial_\theta \epsilon = 0, \]
\[ \partial_\phi \epsilon = 0, \]
\[ 3g q' V_I = 1, \]
\[ -e^{-U} + \frac{Z}{r} + gr X^I V_I = 0, \]
\[ -e^{-U} T' - \frac{Z}{2r^2} + g X^I V_I = 0, \]
\[ -e^{-U} V' - \frac{Z}{2r^2} + g X^I V_I = 0. \]  

(12)

The last two equations in (12) implies that one should set \( T = V \). Moreover, we make the following choice\(^3\)

\[ X^I V_I = 1, \]  

(13)

Then one immediately obtain from the fifth equation of (12), the following expression for \( U \)

\[ e^{-U} = \frac{Z}{r} + gr, \]  

(14)

Using the last equation of (12), we obtain the following differential equation for \( V \),

\[ V' e^{-U} = g - \frac{Z}{2r^2}. \]  

(15)

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\(^3\)one could have set \( X^I V_I \) to an arbitrary constant but the net effect would be a rescaling of the coupling constant \( g \)
The above differential equation can be easily solved by noticing that it can be rewritten in the following form
\[
\frac{dV}{dr} = \frac{d}{dr} \log (e^{-U}) - \frac{1}{4} \frac{d}{dr} \log \left( \frac{e^{-U}}{gr} \right)
\]
where we have implicitly assumed that Z takes a constant value to be determined and therefore one obtains the following solution for V
\[
e^V = e^{-\frac{3}{4}U}(gr)^{\frac{1}{4}}.
\] (16)

The scalar fields are chosen to minimize the magnetic central charge, as in the case of the double extreme solutions in the ungauged theory,\[13, 9\] i.e.,
\[
\partial_i Z = \partial_i (q^I X_I) = \frac{1}{3} C_{IJK} X^J \partial_i (X^K) q^I = 0.
\] (17)

With the above ansatz, the gaugino transformations given by
\[
\delta \lambda^i = \left( \frac{3}{8} \partial_i X_I \Gamma^{\mu\nu} F_{\mu\nu}^{I} - \frac{i}{2} g_{ij} \Gamma^{\mu} \partial_{\mu} \phi^j + \frac{3}{2} i g V_I \partial_i X^I \right) \epsilon
\]
\[
= \left( \frac{3}{8} \partial_i X_I \Gamma^{\mu\nu} F_{\mu\nu}^{I} + \frac{3i}{4} \Gamma^{\mu} \partial_{\mu} X^I \partial_i X_I + \frac{3}{2} i g V_I \partial_i X^I \right) \epsilon
\] (18)
can be easily seen to vanish identically.

From (17), it follows that the critical values of $X^I$ and its dual are given by\[9\]
\[
X^I = \frac{q^I}{Z}, \quad X_I = \frac{1}{6} C_{IJK} q^J q^K Z^2
\] (19)
and thus the critical value of the magnetic central charge is
\[
Z^3 = \frac{1}{6} C_{IJK} q^I q^J q^K.
\] (20)

Using the conditions $X^I V_I = 1$ and the fourth relation of (12), one obtains a generalised Dirac quantization condition
\[
3 \sqrt{\frac{1}{6} C_{IJK} q^I q^J q^K} = \frac{1}{3g}.
\] (21)
The above condition reduces in the case of pure supergravity, i.e., for the case where there are no vector supermultiplets and one graviphoton charge $q^0$, to the Dirac quantization of the magnetic charge

$$q^0 = \frac{1}{3g}.$$  \hspace{1cm} (22)

A similar condition was obtained by [1].

To summarize, the magnetic string solution to $D = 5$, $N = 2$ gauged supergravity coupled to vector multiplets is given by

$$ds^2 = (gr)^\frac{1}{2}e^{-\frac{3}{2}U}(-dt^2 + dz^2) + e^{2U}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$e^{-U} = \frac{1}{3gr} + gr.$$  \hspace{1cm} (23)

and the gauge fields and the scalars are given by

$$A_I^\phi = -q^I \cos \theta,$$

$$X_I = 3gq^I.$$  \hspace{1cm} (24)

$$X_I = 3gq^I.$$  \hspace{1cm} (25)

The Killing spinor is independent of the angular variables and the radial dependence of the Killing spinor is obtained by solving for its radial differential equation given by

$$\left( \partial_r - \frac{e^U}{2}(g - \frac{1}{6gr^2}) \right) \epsilon = 0.$$  \hspace{1cm} (26)

Using the relation (15), the above differential equation can be written in the following simple form

$$\left( \partial_r - \frac{1}{2}V' \right) \epsilon = 0$$

and therefore

$$\epsilon(r) = e^{\frac{1}{2}V} \epsilon_0.$$  \hspace{1cm} (27)

where $\epsilon_0$ is a constant spinor satisfying the constraints

$$\Gamma_3\Gamma_4\epsilon_0 = i\epsilon_0, \quad \Gamma_2\epsilon_0 = -\epsilon_0.$$  \hspace{1cm} (29)
4 Discussions

In this paper we have obtained explicit magnetic string solutions for all $N = 2$ supergravity models in five dimensions. The magnetic charges satisfy a generalised Dirac quantization which for the case of pure supergravity implies that the magnetic charge is fixed in terms of the inverse of the coupling constant $g$. A subclass of solutions of $N = 2$ supergravity are also solutions of supergravity theories with $N = 4$ and $N = 8$ supersymmetries. Those are the gauged versions of the “toroidal”-type compactifications.

The fact that the magnetic central charge, for all charge configurations, is given in terms of the coupling constant $g$ implies that the metric solution takes a universal form for all gauged theories in five dimensions.

Clearly like the BPS spherical electric solutions in four and five dimensional gauged supergravity theories, our magnetic string solution represent a naked singularity. However, it was observed in [4] that for four dimensional purely magnetic solutions one can get extremal genuine black holes with event horizons if the two sphere is replaced with the quotients of the hyperbolic two-space $H^2$. In our case, this would result in the following solitonic solution.

$$\begin{align*}
    ds^2 &= (gr)^{\frac{1}{2}} e^{-\frac{3}{2}U} (-dt^2 + dz^2) + e^{2U} dr^2 + r^2 \left(d\theta^2 + \sinh^2 \theta d\phi^2 \right) \\
    A_I^\phi &= -q_I \cosh \theta, \quad X^I = 3gq_I, \quad e^{-U} = gr - \frac{1}{3gr} \quad (30)
\end{align*}$$

Details of the above solution and its supersymmetric properties will be given elsewhere.

Acknowledgments

W. Sabra would like to thank the physics department of Rockefeller university for hospitality during which some of this work was completed. W. Sabra would also like to thank Dietmar Klemm for a useful discussion.
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