Multipartite fully-nonlocal quantum states

Mafalda L. Almeida,1,2 Daniel Cavalcanti,1,2 Valerio Scarani,2,3 and Antonio Acín1,4

1ICFO-Institut de Ciencies Fotoniques, E-08860 Castelldefels, Barcelona, Spain
2Centre for Quantum Technologies, National University of Singapore, Singapore
3Department of Physics, National University of Singapore, Singapore
4ICREA-Institució Catalana de Recerca i Estudis Avançats, Lluis Companys 23, 08010 Barcelona, Spain

We present a general method to characterize the quantum correlations obtained after local measurements on multipartite systems. Sufficient conditions for a quantum system to be fully-nonlocal according to a given partition as well as being (genuinely) multipartite fully-nonlocal are derived. These conditions allow us to identify all completely-connected graph states as multipartite fully-nonlocal quantum states. Moreover, we show that this feature can also be observed in mixed states: the tensor product of five copies of the Smolin state, a biseparable and bound entangled state, is multipartite fully-nonlocal.

I. INTRODUCTION

Correlations among the results of space-like separated measurements on composite quantum systems can be incompatible with a local model [1]. Such phenomenon, known as quantum nonlocality, is an intrinsic quantum feature and lies behind several applications in quantum information theory [2]. The majority of known results on quantum nonlocality refer to the bipartite scenario and, even though multipartite quantum correlations are a potential valuable resource for multiparty quantum information tasks, their characterization remains a general unsolved problem.

The most common method to detect nonlocal correlations is through the violation of a Bell inequality. It is however unclear whether the amount of violation quantifies nonlocality in a meaningful way [3]. For the bipartite scenario, Elitzur, Popescu and Rohrlich introduced a formalism for the study of nonlocality, named in what follows EPR-2, which naturally leads to a quantitative notion of nonlocality [4]. Consider a quantum state, ρ, and a set of local measurements. The obtained joint probability distribution of outcomes reads

\[ P_\rho(ab|xy) = \text{tr}(\rho M_x^a \otimes M_y^b) \]

where \( x \) and \( y \) label the measurement settings, \( a \) and \( b \) the measurement outcomes, \( M_x^a \) and \( M_y^b \) the corresponding measurement operators. The main idea is to consider all possible decompositions of \( P_\rho(ab|xy) \) into a local and a nonlocal part [5]:

\[ P_\rho(ab|xy) = p_L P_L(ab|xy) + (1 - p_L) P_{NS}(ab|xy). \] (1)

The nonlocal distribution \( P_{NS}(ab|xy) \) is in principle arbitrary, but it has to be no-signaling [6] since \( P_\rho(ab|xy) \) and \( P_L(ab|xy) \) have this property. For the given state \( \rho \), the goal is to identify the decomposition which maximizes the weight of the local part, \( p_L \), among all possible local measurements. The solution to this optimization problem, \( p_L(\rho) \), is clearly a function of the state only and can be interpreted as a measure of its nonlocal correlations.

In the bipartite case, some of the most basic questions on quantum nonlocality have been answered. In what follows it will be useful to express these findings in terms of the properties of the EPR-2 decomposition [7], namely of the weight \( p_L \). For instance, it is known that there exist mixed entangled states which are local or, equivalently, have \( p_L = 1 \) [7]. On the other hand, Gisin’s theorem proves that every pure entangled bipartite state violates a Bell inequality, which means that they have \( p_L < 1 \) [8]. For families of two-qubit and two-qutrit pure states, non-trivial bounds on the value of \( p_L \) have been provided [9, 10]. The same idea has been generalized to mixed states in Ref. [10]. Moreover, it is known that fully-nonlocal states exist, as maximally entangled states have \( p_L = 0 \) [4, 11].

Moving to the multipartite scenario, we can find parallel results to those on bipartite quantum nonlocality. For instance, every pure entangled multipartite state violates a Bell inequality [12]. Also, no fraction of the statistics obtained by measuring any state manifesting a GHZ-like paradox can be described by a completely local model [5, 11, 13]. However, these results say nothing about the presence of genuine multipartite nonlocal correlations, i.e. those nonlocal correlations established between all the \( m \)-parties of a \( m \)-partite quantum state. In fact, very few steps were made in the characterization of genuine multipartite nonlocality. In 1987, Svetlichny provided the first Bell inequality able to identify genuine tripartite nonlocal correlations [14]. Much later, this result has been extended to \( m \)-partite genuine correlations [15]. In Ref. [16] the classification of nonlocal correlations according to various hybrid local-nonlocal models has been studied.

In this work, we present a general method to study multipartite nonlocality, including genuine multiparty, in the no-signaling scenario [6]. It is based on a multipartite version of the EPR-2 decomposition [1] and on the results of measurements held on a subset of the parties sharing a multipartite quantum state. Our framework provides sufficient conditions to detect genuine multipartite full-nonlocal correlations, which we simply designate by multipartite fully-nonlocal, in opposition to the (bipartite) full-nonlocality mentioned previously. We are able to identify the completely-connected graph states [17] as the first example of multipartite fully-nonlocal states, proving the existence of these states for any num-


ber of parties. Furthermore, we prove that multipartite full-nonlocality can also be observed in the mixed case: the tensor product of five copies of the Smolin state is a multipartite fully-nonlocal mixed quantum state.

II. CHARACTERIZING MULTIPARTITE NONLOCALITY

In order to introduce the complex structure of correlations in a multipartite scenario, we start by considering a possible extension of the EPR-2 decomposition for the tripartite case,

$$P_p(ab|xyz) = p_L P_L^{A:B:C} + p_{NL}^{A:B:C} + p_{NL}^{B:A:C} + p_{NL}^{C:A:B} + p_{NS} P_{NS},$$

where $p_L + p_{NL}^{A:B:C} + p_{NL}^{B:A:C} + p_{NL}^{C:A:B} + p_{NS} = 1$. Here, the distribution $P^L_{NL}$ is completely local and therefore it strictly contains classical correlations among the outcomes of the local measurements. On the other hand $P_{NL}^{A:B:C}$ (equivalently $P_{NL}^{B:A:C}$ and $P_{NL}^{C:A:B}$) represents a local-nonlocal hybrid model

$$P_{NL}^{A:B:C} = \int d\lambda \omega(\lambda) P(a|x,\lambda) P(b|y,\lambda),$$

which was originally introduced by Svetlichny. We see that measurement results in $A$ are classically correlated to the results in $B$ and $C$, but nonlocal correlations are allowed between $B$ and $C$. Note that contrary to what happens in the bipartite case, the distributions appearing in the different nonlocal terms of the EPR-2 decomposition are arbitrary and may in principle allow signaling between the corresponding parties. However, here we work in the no-signaling scenario and, thus, all the terms appearing in the decomposition are assumed to be compatible with this principle. The intuition is that the parties cannot signal even if they have access to the hidden-variable $\lambda$ in $n$. Finally, the component $P_{NS}$ is the only to contain genuine tripartite nonlocal correlations. This decomposition can easily be extended to an arbitrary number of parties, $m$. The richness of multipartite correlations expresses itself by the rapid growth of hybrid local-nonlocal terms with $m$.

Observe how this multipartite version of the EPR-2 decomposition clearly distinguishes bipartite from genuine $m$-partite quantum nonlocality. In order for a quantum state to violate a standard Bell inequality it is sufficient that it has $p_L < 1$, while it violates a Svetlichny inequality if and only if $p_{NS} > 0$. Analogously, bipartite full-nonlocality is present when $p_L = 0$ but multipartite full-nonlocality is synonymous of the much stronger condition $p_{NS} = 1$. Thus, the parameter $p_{NS}$ is the relevant quantity when studying genuine multipartite nonlocality. In what follows, we focus our analysis on this quantity and provide a sufficient criterion to detect multipartite fully-nonlocal correlations.

For clarity, let us rephrase some known results on multipartite nonlocality and stress the aim of the present work in terms of the generalized EPR-2 decomposition. The fact that every multipartite pure entangled state violates Bell inequality means that all of them have $p_L < 1$. As commented before, any state exhibiting a GHZ-like paradox has $p_L = 0$. Here we provide a sufficient criterion for a multipartite state to have $p_{NS} = 1$. This criterion identifies several multipartite pure quantum states, as well as a mixed one, as multipartite fully-nonlocal.

III. CRITERIA TO DETECT FULL-NONLOCALITY IN THE MULTIPARTITE SCENARIO

To start, we introduce a different version of multipartite EPR-2 decomposition, which focuses on the correlations across a specific bipartition of the composite system. Consider a $m$-partite state $\rho$ and a bipartition $A:B$, where $A$ contains $k$ parties and $B$ the remaining $m-k$. To simplify the notation, assume that $A$ contains the first $k$ parties and $B$ the $m-k$ remaining ones. Measurement settings in each partition are labeled by $X = (x_1, \ldots, x_k)$ and $Y = (x_{k+1}, \ldots, x_m)$, and the respective outcomes are $A = (a_1, \ldots, a_k)$ and $B = (a_{k+1}, \ldots, a_m)$. The new version of the multipartite EPR-2 decomposition is then given by

$$P_p(AB|XY) = P^L_{AB} P^L_{AB} (AB|XY) + (1 - P^L_{AB}) P^L_{NS} (AB|XY).$$

We use the subscript $L$ to indicate locality in the partition $A:B$, although the distribution $P^L_{AB}(AB|XY)$ is hybrid, i.e.

$$P^L_{AB}(AB|XY) = \int d\lambda \omega(\lambda) P(a_1 \cdots a_k | x_1 \cdots x_k, \lambda) P(a_{k+1} \cdots a_m | x_{k+1} \cdots x_m, \lambda),$$

and allows any no-signaling correlations among members of the same partition. The nonlocal component $P^L_{NS} (AB|XY)$ in contains all correlations not modeled by $P^L_{AB}$. Among all possible decompositions, we focus on the one maximizing the weight of the local part, $P^L_{AB}$. We then define full nonlocality with respect to the partition $A:B$ by $P^L_{AB} = 0$. It is evident that this multipartite version strongly resembles the original bipartite EPR-2 decomposition and we will see that this generalized bipartite decomposition form is essential in the following derivation.

Before showing how to detect multipartite fully-nonlocality, we must present a method to identify full-nonlocal correlations across a given bipartition of a $m$-partite quantum state. In fact, the following theorem constitutes the core of our results.
Theorem 1. An $m$-partite state $\rho$ is fully-nonlocal across a given bipartition $A : B$ ($p_{L}^{A:B} = 0$), in the no-signaling scenario, if it is possible to create a maximally entangled state between one party in each partition, for all outcomes of suitable local measurements on the remaining parties [19].

Proof: We are interested in showing that the outcome distribution of $\rho$, in the EPR-2 decomposition [1], has $p_{L}^{A:B} = 0$. From [1] we know that any bipartite maximally entangled state is fully nonlocal, i.e., $p_{L} = 0$. The proof of Theorem 1 will then follow by contradiction: $p_{L}^{A:B} > 0$ would imply $p_{L} > 0$ for the maximally entangled state.

Indeed, assume there is a positive local weight $p_{L}^{A:B} > 0$. For ease of notation, consider the case in which a maximally entangled state can be created between party $A_0$. For $A$ belonging to $A$, and $A_m$, belonging to $B$, by local measurements in the remaining parties, the classical maximally entangled state $\psi^{\alpha\beta}_{A_0}$ is fully nonlocal, i.e., the statistics of the state $\rho$ is modeled by the hybrid model in the EPR-2 decomposition (1).

$$P(a_1a_m, \tilde{a}\tilde{b}|x_1x_m, \tilde{x}\tilde{y}) = P(a_1a_m|x_1x_m, \tilde{a}\tilde{b}\tilde{x}\tilde{y})P(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}).$$

(6)

This simply expresses the usual no-signaling condition: the outcome of measurements on parties $A_2 \ldots A_{m-1}$ cannot depend on the choice of measurements by distant parties $A_1$ and $A_m$.

Since we assumed that $p_{L}^{A:B} > 0$, a fraction of the statistics of the state $\rho$ is modeled by the hybrid model [5]. Then, for every measurements choice $(\tilde{x}, \tilde{y})$ on the $N-2$ parties, there is at least one outcome $(\tilde{a}, \tilde{b})$ which the hybrid model predicts with non-zero probability: $p_{L}^{A:B}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}) > 0$. The post-measurement state associated to this outcome, $\psi^{\tilde{a}\tilde{b}}_{\tilde{x}\tilde{y}}$, is maximally entangled by assumption and has correlations according to the induced EPR-2 decomposition

$$P_{\psi^\mathcal{L}}(a_1a_m|x_1x_m) = p_{L}P_{L}^{A:B}(a_1a_m|x_1x_m, \tilde{a}\tilde{b}\tilde{x}\tilde{y}) +$$

$$+ (1 - p_{L})P_{N}^{A:B}(a_1a_m|x_1x_m, \tilde{a}\tilde{b}\tilde{x}\tilde{y}),$$

(7)

with local weight

$$p_{L} = \frac{p_{L}^{A:B}}{P_{L}^{A:B}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})}P_{\rho}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}).$$

(8)

Notice that the induced nonlocal distribution is well-defined because its only constraint, the no-signaling condition, is not affected [20]. Also, the fact that we can apply the no-signaling condition to every component of the hybrid model [6], guarantees that we obtain the valid induced hybrid distribution

$$P_{\psi^\mathcal{L}}(a_1b_1|x_1y_1, \tilde{a}\tilde{b}\tilde{x}\tilde{y}) =$$

$$\int d\lambda \omega(\lambda)P(a_1|x, \tilde{a}\tilde{x}\lambda)P(b_1|y, \tilde{b}\tilde{y}\lambda).$$

(9)

where the induced probability density is

$$\omega(\lambda) = \frac{\omega(\lambda)P(\tilde{a}\tilde{x}\lambda)P(\tilde{b}\tilde{y}\lambda)}{P_{L}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})}.$$ 

(10)

According to [11], the local weight $p_{L}$ is necessarily positive. This is known to be impossible since it corresponds to a maximally entangled state $\psi_{2}^{\tilde{a}\tilde{b}}$ [11]. Therefore, the distribution for the state $\rho$ must be fully nonlocal on the bipartition $A : B$, $p_{L}^{A:B} = 0$. □

An important remark on the previous result is that the proof is presented as a sequence of measurements only by clarity reasons. In fact, given that all measurements are performed on spatially-separated parties, the results of measurements on parties $A_2 \ldots A_{m-1}$ are guaranteed to be independent from the measurement choices of parties $A_1$ and $A_m$ by the no-signaling principle. Therefore, there is no need to impose any time-ordering on the measuring events and we are in the most standard framework of nonlocality, which considers single local measurements in space-like separated systems.

We are now ready to finally present the result which, combined with Theorem 1, provides the sufficient criterion to detect multipartite full-nonlocality.

Theorem 2. A probability distribution is multipartite fully-nonlocal ($p_{NS} = 1$) if and only if it is fully nonlocal ($p_{L}^{A:B} = 0$) in every bipartition $A : B$.

Proof: The proof proceeds again by contradiction. Assume that $p_{NS} < 1$. Then, there is at least one local/hybrid model in the EPR-2 decomposition [2] with positive weight. However, there is always a bipartite splitting of the parties such that this term contributes to the corresponding local part in Eq. [1]. But this is not in contradiction with the fact that $p_{L}^{A:B} = 0$ for every bipartition. Then, $p_{NS}$ is equal to one for the initial distribution. □

IV. MULTIPARTITE FULLY-NONLOCAL STATES

A. Completely-connected graph states

From Theorems 1 and 2 we can immediately identify the completely-connected graph states as being multipartite fully-nonlocal states. This comes from the fact that these states fulfill all the necessary requirements: for any pair of qubits, there are local Pauli measurements on the remaining $N-2$ parties which project the pair of particles into a maximally entangled state for every measurement outcome [17][21].
FIG. 1: (Color online) Local measurements are performed on parties $A_2 \cdots A_{m-1}$ of a $m$-partite state $\rho$. If for every outcome of this partial measurement, $A_1$ and $A_m$ share a maximally entangled state, $\rho$ has $p_{AB}^2 = 0$ (Theorem 1). If it holds for any possible bipartition, the state $\rho$ is multipartite fully-nonlocal, $p_{NS} = 1$. (Theorem 2)

Graph states are known to possess several peculiar features like being perfect quantum channels for quantum communication, or (some of them) universal resources for measurement-based quantum computation [17]. The fact that completely-connected graph states are multipartite fully-nonlocal is one more interesting feature of this important class of multipartite entangled states.

B. A multipartite fully-nonlocal mixed state

We now present the first known example of mixed state which contains multipartite fully-nonlocality. This example is based on the so-called Smolin state, a four-partite bound entangled state given by [18]

$$\rho_{S}^{ABCD} = \frac{1}{4} \sum_{i=0}^{3} |\psi_i\rangle\langle\psi_i|^A_B \otimes |\psi_i\rangle\langle\psi_i|^C_D,$$

where $|\psi_i\rangle$ denote the four Bell states.

The Smolin state is biseparable among all two-qubit versus two-qubit partitions, and consequently no entanglement can be distilled from it by any local operations and classical communication. Interestingly, it was shown in Ref. [22] that the combination of five of these states, namely

$$M^S = \rho^S_{ABCD} \otimes \rho^S_{ABCE} \otimes \rho^S_{ABDE} \otimes \rho^S_{ACDE} \otimes \rho^S_{BCDE},$$

is distillable. Actually, Ref. [22] presents a distillation protocol which deterministically transforms $M^S$ into a singlet state between any pair of parties. We can then apply our sufficient criterion to conclude that $M^S$ has $p_{NS} = 1$. This result proves the existence of mixed states which contain multipartite fully-nonlocal quantum correlations.

V. CONCLUSION

We have seen how generalizations of the EPR-2 decomposition for quantum probability distributions and outcomes of partial measurements on quantum systems can be used to study multipartite nonlocal correlations. Our formalism gives sufficient conditions to detect multipartite full-nonlocality and identifies all completely-connected graph states as examples of multipartite fully-nonlocal states. This result solves a fundamental question concerning the characterization of nonlocal quantum correlations: in the no-signalling scenario, multipartite fully-nonlocal states exist for any number of parties. In addition, we also provide an example of such extreme nonlocality for a multipartite mixed state.

Finally, our work opens new questions on the characterization of multipartite nonlocality. Would our conclusions be affected if the different terms appearing in the decomposition were not constrained by the no-signaling principle? Providing this extra resource could give the hybrid models the ability to reproduce a fraction of the quantum probability distribution, and then full-nonlocality would be lost. Another interesting open question is to characterize the set of all multipartite fully-nonlocal quantum states. Is the derived sufficient criterion also necessary to identify multipartite fully-nonlocal quantum correlations?

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A joint probability distribution is local whenever it can be written as \( P(ab|xy) = \int d\lambda \omega(\lambda) P(a|x, \lambda) P(b|y, \lambda) \) where the classical variable \( \lambda \) is distributed according to the probability distribution \( \omega(\lambda) \). This definition easily generalizes to an arbitrary number of parties. Those probability distributions which do not admit this decomposition are said to be nonlocal.

A distribution \( P(ab|xy) \) is no-signaling if the local distributions for one party do not depend on the choice of measurements by the other party, for instance \( P(a|x, \lambda) \equiv \sum_b P(ab|xy) = P(a|x) \). This definition easily generalizes to an arbitrary number of parties.

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Actually, our proof also applies for slightly more general protocols in which the local measurements among parties belonging to the same block in the bipartition can be correlated using classical communication. It also applies for situations where parties \( A_1 \) and \( A_m \) are projected into any fully nonlocal state (\( p_L = 0 \)).

If \( P_{N,S}^{A,H}(\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}) = 0 \), the nonlocal distribution is not well-defined but the induced state \( \psi_2^{AB} \) is local and our proof still holds.

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