SO(2) symmetry of a Maxwell $p$-form theory

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Abstract

We find a universal SO(2) symmetry of a $p$-form Maxwell theory for both odd and even $p$. For odd $p$ it corresponds to the duality rotations but for even $p$ it defines a new set of transformations which is not related to duality rotations. In both cases a symmetry group defines a subgroup of the $O(2,1)$ group of R-linear canonical transformations which has also a natural representation on the level of quantization condition for $p$-brane dyons.

1. Introduction

The old idea of electric-magnetic duality plays in recent years very prominent role (see e.g. [1]). In the present Letter we investigate this idea in the context of a Maxwell $p$-form theory in $D = 2p + 2$ dimensional Minkowski space-time $\mathcal{M}^{2p+2}$ (we choose a signature of the Minkowski metric $(-,+,\ldots,+)$. The motivation to study such type of theories comes e.g. from a string theory where one considers higher dimensional objects (so called $p$-branes) interacting with a gauge field.

Let $A$ denote a $p$-form potential and $F = dA$ be a corresponding $(p+1)$-form field strength. Then the generalized Maxwell equations are given by:

$$d \star F = 0,$$

where “$\star$” denotes a Hodge operation in $\mathcal{M}^{2p+2}$. This operation satisfies

$$\star \star = (-1)^p,$$

which implies a fundamental difference between duality transformations defined for different parities of $p$. Introducing a complex field

$$X := F + i \star F$$

the duality rotations have the following form:

$$X \rightarrow e^{-i\alpha} X,$$  \hspace{1cm} \text{for } p \text{ odd}, \hspace{1cm} (4)
$$X \rightarrow \cosh \alpha X + i \sinh \alpha \overline{X},$$  \hspace{1cm} \text{for } p \text{ even}, \hspace{1cm} (5)

with $\overline{X}$ denoting a complex conjugation of $X$. Note, that there is an essential difference between analytical properties of $SO(2)$ rotations \([4]\) and hyperbolic $SO(1,1)$ rotations \([5]\).

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Namely, the former are $\mathbb{C}$-linear whereas the latter are only $\mathbb{R}$-linear. This mathematical difference has even stronger physical consequences. It is well known that Maxwell $p$-form electrodynamics is duality invariant only for odd $p$ \cite{2,3}, i.e. only (4) defines a symmetry of a theory.

In the present Letter we show that there is a universal $SO(2)$ symmetry of a $p$-form Maxwell theory which is valid for any $p$. When $p$ is odd this symmetry is equivalent to (4) but when $p$ is even it does not correspond to the duality rotation (5).

In order to find this new symmetry we look more closely into the hamiltonian structure of a $p$-form theory. Decomposing $\mathcal{M}^{2p+2}$ into a time-line and a $(2p+1)$ dimensional space-like hyperplane $\Sigma$ the Maxwell equations (1) supplemented by the Bianchi identity $dF = 0$ have the following form:

$$\dot{E} = dB , \quad \dot{B} = (-1)^p dE ,$$

(6)

together with the Gauss constraints

$$dE = 0 , \quad dB = 0 ,$$

(7)

where $E$ and $B$ are electric and magnetic $p$-forms on $\Sigma$ defined in the standard way via $F$ (see e.g. \cite{2}). The dual $(p+1)$-forms $\mathcal{E} = *E$ and $\mathcal{B} = *B$ are defined via the Hodge star $*$ on $\Sigma$ induced form $\mathcal{M}^{2p+2}$ (we shall use the same symbol for both operations). Note the presence of a $p$-dependent sign in (6) which plays a crucial role in what follows.

Now, our strategy looks as follows: in the next section we define new variables $Q$’s and $\Pi$’s which are more convenient for our purpose. We call them reduced variables because they already solve the Gauss constraints (7). In Section 3 we reformulate the canonical structure of a $p$-form theory in terms of the reduced variables. It allows us to make an observation that theories with different parities of $p$ are related by a simple transformation of variables. It turns out that the canonical structure possesses a natural $O(2,1)$ invariance group. For odd $p$ there is a $SO(2)$ subgroup of $O(2,1)$ corresponding to duality rotations (4). However, for even $p$ no such a subgroup corresponding to $SO(1,1)$ rotations (5) exists. Nevertheless, also in this case, there is a $SO(2)$ subgroup defining a symmetry of a Maxwell theory. This subgroup is a true counterpart of (4) for a theory with $p$ even. Finally, we show that there is a natural realization of the above $O(2,1)$ symmetry on a level of the generalized Dirac quantization condition for $p$-brane dyons \cite{6}.

2. Reduced variables

Let us choose spherical coordinates on $\Sigma$ centered at some arbitrary point and let $S^{2p}(r)$ denote a $2p$ dimensional sphere of radius $r$. Using the canonical embedding

$$\phi_r : S^{2p}(r) \rightarrow \Sigma \quad (8)$$

let us define the following $(p-1)$-forms on each $S^{2p}(r)$:

$$Q_1 := \phi_r^*(\partial_r \int E) , \quad (9)$$

$$Q_2 := \phi_r^*(\partial_r \int B) , \quad (10)$$

and

$$\Pi_1 := r \Delta_{p-1}^{-1} d * \phi_r^* B , \quad (11)$$

$$\Pi_2 := -r \Delta_{p-1}^{-1} d * \phi_r^* E , \quad (12)$$
where \( \Delta_{p-1} \) denotes a Laplacian on \((p - 1)\)-forms on \( S^{2p}(1) \). In the above formulae “\( d \)” and “\( \star \)” are operations on \( S^{2p}(r) \) induced from \( \Sigma \) via \( \phi_r \). Note, that \( \Delta_{p-1} \) is invertible (cf. [4]), therefore, \( \Pi \)'s are well defined.

The crucial property of the above defined variables consists in the following

**Theorem 1** The quantities \((Q_{\alpha}, \Pi_{\alpha}); \alpha = 1, 2\) contain the entire gauge-invariant information about the fields \( E \) and \( B \). Moreover, they already solve Gauss constraints (7).

For proof see [4]. Note, that \( \Pi \)'s are highly nonlocal functions of \( E \) and \( B \). However, it was observed long ago [5] that in ordinary \((p = 1)\) electrodynamics duality rotations (4) are generated by a nonlocal operator. Therefore, as we show, \( Q \)'s and \( \Pi \)'s are well suited to study the duality invariance of a \( p \)-form theory. More detailed analysis of these variables may be found in [4].

### 3. Canonical structure

Having any two \( k \)-forms \( \alpha \) and \( \beta \) on \( S^{2p}(r) \) let us define a standard scalar product:

\[
(\alpha, \beta)_r := \int_{S^{2p}(r)} \alpha \wedge \star \beta .
\]

Moreover, let

\[
(\alpha, \beta) := \int_0^\infty dr \ (\alpha, \beta)_r .
\]

With this notation one has

**Theorem 2** The phase space of a \( p \)-form theory is endowed with the canonical symplectic structure \( \Omega_p \) given by:

\[
\Omega_p = (\delta \Pi_1, \wedge \delta Q_1) + (-1)^{p+1} (\delta \Pi_2, \wedge \delta Q_2) .
\]

(13)

For proof see [4]. Now, let us define complex forms:

\[
Q := Q_1 + iQ_2 ,
\]

(14)

\[
\Pi := \Pi_1 + i\Pi_2 .
\]

(15)

Denoting by \( \Omega_- \) (\( \Omega_+ \)) a symplectic form \( \Omega_p \) for odd (even) \( p \) one has:

\[
\Omega_- = \text{Re} (\delta \Pi, \wedge \delta Q) ,
\]

(16)

\[
\Omega_+ = \text{Re} (\delta \Pi, \wedge \delta Q) ,
\]

(17)

where “\( \text{Re} \)” stands for a real part. Finally, Maxwell equations rewritten in terms of reduced variables read (cf. [4]):

\[
\dot{Q} = -\Delta_{p-1} \Pi ,
\]

\[
\dot{\Pi} = -\Delta_{p-1}^{-1} \left[ r^{-1} \partial_r^2 (rQ) + r^{-2} \Delta_{p-1} Q \right] ,
\]

(18)

for odd \( p \), and

\[
\dot{Q} = -\Delta_{p-1} \Pi ,
\]

\[
\dot{\Pi} = -\Delta_{p-1}^{-1} \left[ r^{-1} \partial_r^2 (rQ) + r^{-2} \Delta_{p-1} Q \right] ,
\]

(19)
for even \( p \). One easily shows that (18) and (19) define hamiltonian equations with respect to \( \Omega^- \) and \( \Omega^+ \) respectively generated by the following Hamiltonian:

\[
H_p = \frac{1}{2(p-1)!} \left[ (r^{-1}Q, r^{-1}Q) - (r^{-1}\partial_r(rQ), \Delta_{p-1}^{-1}r^{-1}\partial_r(rQ)) - (\Pi, \Delta_{p-1}\Pi) \right]
\] (20)

Moreover, one may easily prove

Lemma 1 **Numerically** \( H_p \) **equals to** the standard Maxwell Hamiltonian obtained via the symmetric energy-momentum tensor

\[
H_p = \frac{1}{2p!} \int_{\Sigma} (E \wedge E + B \wedge B)
\]

Note, that the difference between theories with different parities of \( p \) is now very transparent. Namely, they are related by a simple replacement \( \Pi \rightarrow \Pi \). This way one obtains (17) from (16) and (19) from (18). Note, that the Hamiltonian (20) is invariant under \( \Pi \rightarrow \Pi \).

4. Canonical transformations and duality rotations

Now, let us look for the canonical transformation with respect to \( \Omega^- \) and \( \Omega^+ \). In the class of \( \mathbb{R} \)-linear transformations we have

Lemma 2 The following generators:

\[
G_1 = \text{Im} \ (Q, \Pi) , \quad G_2 = \text{Im} \ (Q, \Pi) , \quad G_3 = \text{Re} \ (Q, \Pi) , \quad G_4 = \text{Re} \ (Q, \Pi) ,
\]
generates both with respect to \( \Omega^- \) and \( \Omega^+ \) the \( O(2,1) \) group of \( \mathbb{R} \)-linear canonical transformations.

Note, that the set \( (G_1, G_2, G_3, G_4) \) is closed with respect to \( \Pi \rightarrow \Pi \). Moreover, there is \( O(2) \) subgroup of \( \mathbb{C} \)-linear transformations generated by \( (G_1, G_3) \) and \( (G_2, G_4) \) for odd and even \( p \) respectively (actually, \( G_3 \) and \( G_4 \) generate the corresponding centers of \( O(2,1) \)).

Now, it is easy to prove the following

Theorem 3 The Maxwell Hamiltonian (20) is invariant under the action of \( G_1 \) and \( G_2 \) for odd and even \( p \) respectively.

Let us observe that the duality rotations (1) and (3) may be expressed as follows:

\[
Q \rightarrow e^{-ia} Q , \quad \Pi \rightarrow e^{-ia} \Pi ,
\] (21)

for \( p \) odd, and

\[
Q \rightarrow \cosh \alpha \ Q - i \sinh \alpha \overline{Q} , \quad \Pi \rightarrow \cosh \alpha \ \Pi + i \sinh \alpha \overline{\Pi} ,
\] (22)

for even \( p \). One immediately sees that (21) are generated by \( G_1 \) but none of \( G_k \) does correspond to (22). This fact that the hyperbolic \( SO(1,1) \) rotations are not even implementable as canonical transformations was observed in a slightly different context in [3].
Therefore, the true counterpart of (21) for $p$ even is not (22) (it is obvious because they are not related via $\Pi \rightarrow \Pi$) but

$$
Q \rightarrow e^{-i\alpha} Q , \\
\Pi \rightarrow e^{i\alpha} \Pi ,
$$

(23)

which is generated by $G_2$ via $\Omega_+$. Finally, let us note that quantum mechanics applied to a $p$-form theory implies the following quantization condition for $p$-brane dyons [6]:

$$
e_1g_2 + (-1)^pe_2g_1 = nh ,
$$

(24)

with an integer $n$ ($h$ denotes the Planck constant). For odd $p$ the above condition is a generalization of the famous Dirac condition [7] but for even $p$ it was observed only recently [6]. Again, a parity of $p$ plays a crucial role in (24). Introducing a complex charge $q := e + ig$

the formula (24) may be rewritten as follows:

$$
\text{Im} (q_1q_2) = nh , \quad p \text{ odd} ,
$$

(25)

$$
\text{Im} (q_1q_2) = nh , \quad p \text{ even} .
$$

(26)

Let us observe that there is a direct correspondence between formulae (16) and (25) and formulae (17) and (26). Therefore, making the following replacements: $\Pi \rightarrow q_1$ and $Q \rightarrow q_2$ we obtain the natural action of $O(2,1)$ on the level of charges.

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