RATIONAL CURVES ON HYPERSURFACES OF A PROJECTIVE VARIETY

B. WANG
OCT 19, 2014

Abstract. In this paper, we extend our result in [3] to hypersurfaces of any smooth projective variety \( Y \). Precisely we let \( X_0 \) be a generic hypersurface of \( Y \) and \( c_0 : \mathbb{P}^1 \to X_0 \) be a generic birational morphism to its image, i.e. \( c_0 \in \text{Hom}_{\text{bir}}(\mathbb{P}^1, X_0) \) is generic, such that

1. \( \dim(X_0) \geq 3 \),
2. \( H^1(N_{c_0/Y}) = 0 \).

Then

\[
H^1(N_{c_0/X_0}) = 0.
\]

(0.1)

As an application we prove that the Clemens’ conjecture holds for Calabi-Yau complete intersections of dimension 3.

1 Statement

We work over complex numbers, \( \mathbb{C} \). A general or generic hypersurface is referred to a hypersurface as a point in a complement of a countable union of proper closed subsets of the space of all hypersurfaces. Let \( Y \) be a smooth projective variety with \( \dim(Y) \geq 4 \). Let \( \mathcal{L} \) be a very ample line bundle on \( Y \) such that \( \dim(H^0(\mathcal{L})) \geq 5 \). Let

\[
X_0 = \text{div}(f_0) \subset Y
\]

where \( f_0 \in H^0(\mathcal{L}^h) \) is generic. Let

\[
c_0 : \mathbb{P}^1 \to C_0 \subset X_0 \subset Y
\]

be a birational morphism onto an irreducible rational curve \( C_0 \). Because the induced sheaf morphism

\[
(c_0)_* : \mathcal{O}_Y \to \mathcal{O}_{\mathbb{P}^1}
\]

is surjective. We have well-defined normal sheaves \( N_{c_0/X_0} \) and \( N_{c_0/Y} \) both on \( \mathbb{P}^1 \).

Theorem 1.1. (Main Theorem). Let \( f_0 \) be generic. If \( c_0 \) is generic in an irreducible component of \( \text{Hom}_{\text{bir}}(\mathbb{P}^1, X_0) \), and

\[
H^1(N_{c_0/Y}) = 0,
\]

(1.2)

then

\[
H^1(N_{c_0/X_0}) = 0.
\]

(1.3)
2 Application

Our first application of the result is the proof of Clemens’ conjecture for Calabi-Yau complete intersections of dimension 3. More precisely we call the following statement of the corollary, Clemens’ conjecture for Calabi-Yau complete intersections.

Corollary 2.1. Let $X_0$ be a generic, Calabi-Yau complete intersection of dimension 3. Let $c_0 : \mathbb{P}^1 \to X_0$ be a morphism that is birational to its image. Then

1. $c_0$ is an immersion,
2. and $N_{c_0/X_0}$ is a vector bundle satisfying

\begin{equation}
N_{c_0/X_0} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).
\end{equation}

Remark

The name “Clemens’ conjecture for complete intersections” is known for the following reason. It follows from the corollary that if they exist, there are finitely many rational curves $C_0$ of each degree on $X_0$ such that they are all immersed with normal bundle (2.1) for its normalization $c_0$. The corollary does not prove the existence of rational curves. All these assertions for quintic threefolds $X_0$ are main parts of Clemens’ conjecture.

Proof. (following the main theorem). Let’s assume $X_0 \subset \mathbb{P}^n$ and $X_0$ is defined by the polynomials $f_1, \ldots, f_{n-3}$. Let $\deg(f_i) = h_i$. Let

\begin{equation}
Y_i = \cap_{j=1}^{i} \{ f_j = 0 \}.
\end{equation}

Let $S$ be the space of the complete intersections containing $X_0$. So $Y = Y_{n-3}$.

Then applying main theorem inductively on $Y_i$, we obtain that for generic $c_0 \in Hom_{bir}(\mathbb{P}^1, X_0)$, the normal sheaf $N_{c_0/X_0}$ exists and

\begin{equation}
H^1(N_{c_0/X_0}) = 0.
\end{equation}

Next we show $c_0$ is an immersion. Let $\Gamma$ be an irreducible component of the incidence scheme

\begin{equation}
\{(c, [f]) \in M \times S : f(c) = 0 \}
\end{equation}

that dominates $S$, where $M = \oplus^{n+1} \mathcal{O}_{\mathbb{P}^1}(d)$ for some positive integer $d$. Let $\Gamma_{X_0}$ be the fibre of $\Gamma$ over the generic $X_0$. Then as in lemma 2.8, [3], for a generic $c_0 \in \Gamma_{X_0}$,

\begin{equation}
\frac{T_{c_0} \Gamma_{X_0}}{\ker} \simeq H^0(c_0^*(T_{X_0})),
\end{equation}

where $\ker$ is the line in $M$ going through $c_0 \neq 0$ and 0. By (2.3),

\begin{equation}
dim(T_{c_0} \Gamma_{X_0}) = 4.
\end{equation}

Now we consider it from a different point of view. Because $c_0$ is a birational map to its image, there are finitely many points $t_i \in \mathbb{P}^1$ where the differential map

\begin{equation}
(c_0)_* : T_{t_i} \mathbb{P}^1 \to T_{c_0(t_i)} X_0
\end{equation}
is not injective. Assume its vanishing order at \( t_i \) is \( m_i \). Let

\[(2.8) \quad m = \sum_i m_i.\]

Let \( s(t) \in H^0(\mathcal{O}_{\mathbb{P}^1}(m)) \) such that

\[\text{div}(s(t)) = \Sigma_i m_i t_i.\]

The sheaf morphism \((c_0)_*\) is injective and induces a composition morphism \( \xi_* \) of sheaves

\[(2.9) \quad T_{\mathbb{P}^1} \xrightarrow{(c_0)_*} c_0^*(T_{\mathcal{X}_0}) \otimes \mathcal{I}_{\text{div}(s)} \xrightarrow{\text{res}} c_0^*(T_{\mathcal{X}_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(-m),\]

where \( \mathcal{I}_{\text{div}(s)} \) is the ideal sheaf of \( \text{div}(s) \). It is easy to see that the induced bundle morphism \( \xi_b \) is injective. Let

\[(2.10) \quad N_m = \frac{c_0^*(T_{\mathcal{X}_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(-m)}{\xi_b(T_{\mathbb{P}^1})}.\]

Then

\[(2.11) \quad \dim(H^0(N_m)) = \dim(H^0(c_0^*(T_{\mathcal{X}_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(-m))) - 3.\]

On the other hand, three dimensional automorphism group of \( \mathbb{P}^1 \) gives a rise to a 3-dimensional subspace \( Au \) of

\[H^0(c_0^*(T_{\mathcal{X}_0})).\]

By \((2.6)\), \( Au = H^0(c_0^*(T_{\mathcal{X}_0})) \). Over each point \( t \in \mathbb{P}^1 \), \( Au \) spans a one dimensional subspace. Hence

\[(2.12) \quad c_0^*(T_{\mathcal{X}_0}) = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-k_2),\]

where \( k_1, k_2 \) are some positive integers. This implies that

\[(2.13) \quad \dim(H^0(c_0^*(T_{\mathcal{X}_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(-m))) = \dim(H^0(\mathcal{O}_{\mathbb{P}^1}(2 - m))).\]

Then

\[(2.14) \quad \dim(H^0(c_0^*(T_{\mathcal{X}_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(-m))) = 3 - m.\]

Since \( \dim(H^0(N_m)) \geq 0 \), by \((2.11)\), \(-m \geq 0\). By the definition of \( m \), \( m = 0 \). Hence \( c_0 \) is an immersion.

Next we prove \((2)\). Notice that \((c_0)_*(T_{\mathbb{P}^1})\) is a subbundle generated by global sections. It must be the \( \mathcal{O}_{\mathbb{P}^1}(2) \) summand in \((2.12)\) because \( k_1, k_2 \) are positive. Therefore

\[(2.15) \quad N_{c/\mathcal{X}_0} \cong \mathcal{O}_{\mathbb{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-k_2).\]

Since \( \deg(c_0^*(T_{\mathcal{X}_0})) = 0 \), \( k_1 = k_2 = 1 \). Part \((2)\) is proved.

Further applications can be expected. In the light of theorem 1.1, the Calabi-Yau condition is not responsible for the vanishing of the first cohomology. Let’s be more specific. The main theorem states the condition

\[H^1(N_{c_0/\mathcal{X}_0}) = 0\]
is descending to hypersurfaces as the dimension of the complete intersection is descending. Such descending will not stop until the dimension reaches 3. Therefore the vanishing property

\[ H^1(N_{c_0}/X_0) = 0 \]

holds for all generic complete intersections whose dimensions are larger than or equal to 3. The vanishing of the first cohomology of the normal bundle is originated from the projective space. Its disconnection with Calabi-Yau condition gives a wide range of applications. Those applications include the extensions of results in [1] to general Fano complete intersections, and extensions of related results and conjectures in [2] to general complete intersections of corresponding types. We will discuss these extensions separately elsewhere.

### 3 Proof of the main theorem

By the assumption of the main theorem, we may regard

\[ Y \subset \mathbb{P}^n \]

as a smooth subvariety of dimension \( e \geq 4 \), and \( L = \mathcal{O}_{\mathbb{P}^n}(1)|_Y \). Let

\[ f_0 \in H^{0}(L^h) \]

be generic such that

\[ \text{div}(f_0) = X_0. \]

Next we choose subspaces \( V_1 \simeq \mathbb{P}^{n-e-1}, V_2 \simeq \mathbb{P}^e \) of \( \mathbb{P}^n \) such that

\[ V_1 \cap Y = \emptyset. \]

The projection \( Pr \)

\[ \mathbb{P}^n \setminus V_1 \xrightarrow{Pr} V_2 \]

is restricted to a generically finite-to-one map from \( Y \). The projection \( Pr(X_0) \) is a generic hypersurface of \( V_2 \) because \( f_0 \) is generic. We denote the section that defines \( Pr(X_0) \) by \( Pr(f_0) \). Now we consider the rational curves. Notice that \( Pr \) induces a rational map

\[ \eta : \Gamma_{X_0} \rightarrow \Gamma_{Pr(f_0)} \]

\[ c_0 \rightarrow Pr(c_0) = Pr \circ c_0. \]

We may assume that \( c_0 \) lies in the regular locus of \( \eta \), and \( \eta(c_0) \) is not a constant map for generic \( c_0 \in \Gamma_{X_0} \). The map is clearly surjective. Then if \( Pr(c_0) \in \Gamma_{Pr(f_0)} \) is generic, there is an inverse \( c_0 = \eta^{-1}(c_0) \) in \( \Gamma_{X_0} \) which is also generic in \( \text{Hom}(\mathbb{P}^1, X_0) \).

Let \( U \subset \mathbb{P}^1 \) be the open set of \( \mathbb{P}^1 \) such that the \( Pr|_Y \) at \( c_0(U) \) is smooth. Let \( c_0', Pr(c_0)' \) be the restriction of \( c_0, Pr(c_0) \) to \( U \). Then we have commutative diagram

\[ \begin{array}{ccc}
N_{c_0'/Y} & \rightarrow & N_{Pr(c_0)'/V_2} \\
\downarrow & & \downarrow \\
(c_0')^*(N_{X_0/Y}) & \rightarrow & (Pr(c_0)')^*(N_{Pr(X_0)/V_2}).
\end{array} \]
Now we apply theorem 1.1 in [3] to the generic hypersurface $Pr(X_0)$ of the projective space $V_2$. We obtain that $H^1(N_{Pr(c_0)/X_0}) = 0$. By the exact sequence

$$H^0(N_{Pr(c_0)/V_2}) \rightarrow H^0(Pr(c_0)^*(N_{Pr(X_0)/V_2})) \rightarrow H^1(N_{Pr(c_0)/X_0}) = 0,$$

we obtain that the map

$$H^0(N_{Pr(c_0)/V_2}) \rightarrow H^0(Pr(c_0)^*(N_{Pr(X_0)/V_2})) \quad (3.7)$$

is surjective. Applying this surjectivity to the diagram (3.6), we obtain that the natural map

$$H^0(N_{c_0/Y}) \xrightarrow{\pi_1} H^0((c_0)^*N_{X_0/Y}) \quad (3.8)$$

is also surjective. Considering the exact sequence

$$0 \rightarrow N_{c_0/X_0} \rightarrow N_{c_0/Y} \rightarrow c_0^*(N_{X_0/Y}) \rightarrow 0 \quad (3.9)$$

we obtain another exact sequence

$$H^0(N_{c_0/Y}) \xrightarrow{\pi_2} H^0(c_0^*(N_{X_0/Y})) \xrightarrow{\cong} H^1(N_{c_0/X_0}) \rightarrow H^1(N_{c_0/Y}) = 0.$$

Since $\pi_1$ is surjective, image of $\pi_2$ must be 0. Because of the assumption, $H^1(N_{c_0/Y}) = 0$, $H^1(N_{c_0/X_0})$ must be zero. The main theorem is proved.

REFERENCES

[1] J. Harris, M. Roth and J. Starr, Rational curves on Hypersurfaces of low degree, J. Reine Angew. Math., 571 (2004), pp. 45-96.
[2] C. Voisin, On some problems of Kobayashi and Lang: Algebraic approaches, International Press, Current developments in mathematics (2003), pp. 53–125.
[3] B. Wang, Rational curves on hypersurfaces, ArXiv 1410.3090, 2014