Mapping functions and critical behavior of percolation on rectangular domains

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Abstract

The existence probability $E_p$ and the percolation probability $P$ of the bond percolation on rectangular domains with different aspect ratios $R$ are studied via the mapping functions between systems with different aspect ratios. The superscaling behavior of $E_p$ and $P$ for such systems with exponents $a$ and $b$, respectively, found by Watanabe, Yukawa, Ito, and Hu in [Phys. Rev. Lett. 93, 190601 (2004)] can be understood from the lower order approximation of the mapping functions $f_R$ and $g_R$ for $E_p$ and $P$, respectively; the exponents $a$ and $b$ can be obtained from numerically determined mapping functions $f_R$ and $g_R$, respectively.

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I. INTRODUCTION

Universality and scaling in critical systems have attracted much attention in recent decades [1, 2, 3, 4]. It has been found that critical systems can be classified into different universality classes so that the systems in the same class have the same set of critical exponents [1]. According to the theory of finite-size scaling [2, 3, 4], if the dependence of a physical quantity $M$ of a thermodynamic system on a parameter $t$, which vanishes at the critical point $t = 0$, is of the form $M(t) \sim t^\alpha$ near the critical point, then for a finite system of linear dimension $L$, the corresponding quantity $M(L, t)$ is of the form

$$M(L, t) \sim L^{-\alpha y} F(tL^{y_t}) ,$$

where $y_t (\equiv \nu^{-1})$ is the thermal scaling power, $F(x)$ is the scaling function, and $x \equiv tL^{y_t}$ is the scaling variable. It follows from (1) that the scaled data $M(L, t)L^{y_t}$ for different values of $L$ and $t$ are described by a single finite-size scaling function (FSSF) $F(x)$. Thus it is important to know general features of the scaling function under various conditions. On the basis of the renormalization group arguments, Privman and Fisher [3, 4] proposed that systems in the same universality classes can have universal finite-size scaling functions (UFSSFs).

Percolation models [5, 6, 7, 8, 9, 10, 11, 12, 13] are ideal systems for studying universal and scaling behaviors near the critical point. The system is defined to be percolated when there is at least one cluster extending from the top to the bottom of the system. The average fraction of lattice sites in the percolating clusters is called the percolation probability [5, 6] (also known as percolation strength) and will be denoted by $P$ in the present paper. The probability that at least one percolating cluster exists in the system is called the existence probability [9] or crossing probability [13], and will be denoted by $E_p$ in the present paper [14].

Using a histogram Monte Carlo method [9] and other Monte Carlo methods, Hu and collaborators found that the FSSFs depend sensitively on the boundary conditions [15] and shapes [16] of the lattices and many two-dimensional percolation models (including bond and site percolation models on square, honeycomb, and triangular lattices, continuum percolation of soft disk and hard disks, bond percolation on random lattices, etc) [17] can have universal FSSFs (UFSSFs) for their $E_p$ and $P$, and many three-dimensional percolation
models (including site percolation on sc, bcc, and fcc lattices and bond percolation on sc lattice) have universal critical exponents and UFSSFs for their $E_p$ and $P$.

In the present paper, we will study the bond percolation model on $L_1 \times L_2$ rectangular lattices (domains), especially the dependence of $E_p$ and $P$ on the aspect ratio $R$, which is the ratio of the horizontal length $L_1$ to the vertical length $L_2 \equiv L$ of the system, i.e., $R = L_1/L_2$. For this system, it is well known that the critical bond density $\rho_c$ (also called bond occupation probability) is $1/2$, the critical exponent $\beta$ for the percolation probability $P$ is $5/36$, $y_t = 1/\nu = 3/4$, and the critical exponent for the existence probability $E_p$ is $0$, which and Eq. (1) imply that the dependence of $E_p$ on the linear dimension $L$, the bond density $\rho$, and the aspect ratio $R$ can be written as

$$E_p(L, \rho, R) = F_R(\varepsilon L^{y_t}), \quad (\varepsilon \equiv |\rho - \rho_c|/\rho_c) \quad (2)$$

near the critical point $\rho_c$. The function $F_R$ is universal for various percolation models considered in Ref. [17]. The aspect ratio dependence of the function $F_R$ is not clear; e.g., the systems with close values of $R$ should have similar forms of $F_R$, however, Eq. (2) do not tell us how similar they are. While the value of the existence probability at the critical point $E_p(\rho_c)$ was obtained by Cardy, such critical $E_p$ approaches quickly to 1 as the aspect ratio is increased, e.g., it is difficult to distinguish the value for $R = 16$ from that for $R = 32$. Additionally, the region for the rapid increase of $E_p$ shifts to the smaller value of $\rho$ with larger $R$ and fixed $L$, which cannot be described quantitatively. Therefore, we need other kinds of approach to study the aspect ratio dependence of the UFSSFs.

Recently, we found “superscaling” of the existence probability $E_p$ and the percolation probability $P$ for bond percolation on rectangular domains with different aspect ratios: $E_p(L, \rho, R) \sim F(\varepsilon' L^{y' R^a})$ and $P(L, \rho, R) \sim (L^{y R^b})^{-\beta} F(\varepsilon' L^{y R^b})$, with new exponents $a$ and $b$, where $\varepsilon' \equiv (\rho - \rho'_c)$ with $\rho'_c$ being the effective critical point; $a$ and $b$ were determined by a fitting procedure to be $a = 0.14(1)$ and $b = 0.05(1)$, respectively. In the present paper, we define mapping functions to connect UFSSFs for percolation on domains with different aspect ratios and find that $a$ and $b$ can be obtained from numerically determined mapping functions $f_R$ and $g_R$ for $E_p$ and $P$, respectively.
II. MAPPING FUNCTION AND FINITE-SIZE SCALING

In order to make the meaning of the mapping function becomes clear, we first introduce the mapping function for the percolation model on the square lattices with the same aspect ratio. Consider two different square systems $A$ and $B$ with the linear sizes $L_A$ and $L_B (> L_A)$, respectively. The existence probability of the system $B$ at density (also called occupation probability) $\rho$ is denoted by $E_p(L_B, \rho)$. The existence probability is a monotonic increasing function whose value changes from 0 to 1 as the density increases from 0 to 1. Therefore, for every $E_p(L_B, \rho)$ of system $B$, there is a corresponding value $\rho'$ of system $A$ so that the existence probability of system $A$ at $\rho'$, denoted as $E_p(L_A, \rho')$, is equal to $E_p(L_B, \rho)$ and we have

$$E_p(L_B, \rho) = E_p(L_A, \rho') \equiv E_p(L_A, f_Q(\rho)),$$

which defines the renormalization group transformation from $\rho$ to $\rho'$ in Ref. [9]. The mapping from $\rho$ to $\rho'$ is represented as the mapping function

$$\rho' = f_{A\rightarrow B}(\rho) \equiv f_Q(\rho).$$

The mapping function depends only on the ratio of the system size $Q = \frac{L_B}{L_A}$, and therefore, we denote the mapping function by $f_Q$. The critical point of the system, $\rho_c$, can be determined from the fixed point of Eq. (3):

$$E_p(L_B, \rho_c) = E_p(L_A, \rho_c),$$

which is equivalent to

$$f_Q(\rho_c) = \rho_c.$$

For fixed $Q$, $\rho_c$ of Eq. (5) approaches the exact critical point $1/2$ for the bond percolation on the square lattice as $L_A \to \infty$. Table 1 of the first reference in [9] shows that $\rho_c$ of Eq. (5) is already very accurate for $L_B = 16 - 20$ and $L_A = 8 - 16$.

The expansion of the mapping function at the critical point can be written as

$$\rho' = f_Q(\rho) = f_Q(\rho_c) + \frac{df_Q}{d\rho}\bigg|_{\rho_c} (\rho - \rho_c) + \cdots.$$

It is well known that the thermal scaling power $y_t$ can be obtained from the equation

$$y_t = \frac{\ln \frac{df_Q}{d\rho}\bigg|_{\rho_c}}{\ln Q},$$

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which implies that the differential coefficient of the first order in Eq. (7) is given by

$$\left. \frac{df_Q}{d\rho} \right|_{\rho_c} = Q^\nu. \quad (9)$$

We can also have the above relation from the viewpoint of the mapping [24].

Using Eqs. (7), (5) and (9), we can rewrite Eq. (3) as

$$E_p(L_B, \rho) = E_p(L_A, \rho_c + Q^\nu (\rho - \rho_c))$$

$$= E_p(L_A, \rho_c + \varepsilon L_A^{-\nu}) \quad (10)$$

with the scaling variable $\varepsilon \equiv (\rho - \rho_c) L_B^{\nu}$. Since the system sizes $L_A$ and $L_B$ are arbitrary, we can choose $L_A$ as unity and rewrite $L_B$ as $L$. Then we have

$$E_p(L, \rho) = E_p(1, \rho_c + \varepsilon) \equiv F(\varepsilon), \quad (11)$$

with the scaling function $F$. Equation (11) implies that the finite-size scaling theory corresponds to the first order approximation of the expansion in Eq. (7).

To see the behavior of the mapping function $f_Q$, we determine $f_Q$ from numerically obtained existence probabilities. The procedures to determine $f_Q$ is as follows. (i) Prepare graphs of the existence probabilities vs. density. (ii) Draw a line parallel to the horizontal axis. (iii) Determine the two intersection point $\rho$ and $\rho'$ (see the inset of Fig. 1(a)). (iv) Plot all the pairs $(\rho, \rho')$ by sweeping from $E_p = 0$ to 1, then the function $\rho' = f_Q(\rho)$ is obtained.

The calculated mapping functions of the bond percolation on square lattices are shown in Fig. 1(a) with aspect ratio $R = 1$ and system sizes $L = 64, 128$ and 256. Free boundary conditions are taken for this and other systems in the present paper. All of these functions have linear forms and have the single intersection point, namely the critical point. The ratio of the slope of these lines gives the value of the critical exponent $\nu = 1/y_t$.

Similar arguments can be applied for the percolation probability $P$. A mapping function $g_Q$ is defined by,

$$L_B^{\nu} P(L_B, \rho, R) = L_A^{\nu} P(L_A, g_Q(\rho), R) \quad (12)$$

with a system size ratio $Q = L_B/L_A$. Here, we defined the mapping function with the additional factors $L_B^{\nu}$ and $L_A^{\nu}$ on the left-hand and right-hand sides of Eq. (12). With such factors, both sides of Eq. (12) are proportional to the FSSF for $P$ as can be seen from Eq. (11). For $E_p$, the critical exponent is 0 [21] and we need not add such factors in Eq. (3).
to define the mapping function. The functions $g_Q$ are shown in Fig. 1(b). Figure 1 shows that two kinds of functions $f_Q$ and $g_Q$ are identical near the critical point.

The UFSSFs corresponding to mapping functions in Fig. 1 are shown in Fig. 2. The goodness of the scaling in these figures come from the fact that the mapping functions are well approximated by the linear function when the systems have the identical aspect ratio. For the systems with different aspect ratios, the mapping functions for $E_p$ becomes highly non-linear function as shown in Fig. 3(b) below. Therefore, the scaling with low-order approximation does not show good data collapse for large values of the scaling variables (see Fig. 4 in Ref. [22]).

III. MAPPING FUNCTION AND SUPERSCALING

As shown in Fig. 3(a), we define the mapping function $f_R(\rho)$ to map the existence probability of the bond percolation model on $RL \times L$ lattice, $E_p(L, \rho, R)$, into the existence probability of the bond percolation model on $L \times L$ lattice, $E_p(L, f_R(\rho), 1)$, with following equation

$$E_p(L, \rho, R) = E_p(L, f_R(\rho), 1).$$  

(13)

The curves of the existence probabilities for the same $L$ and different $R$ do not intersect, since their values at the point $\rho_c$ of the thermodynamic system are different [20]. Therefore, we introduce the effective critical point $\rho'_c$, which satisfies the equation

$$f_R(\rho'_c) = \rho_c,$$

and consider the expansion of the mapping function $f_R$ at $\rho'_c$ as

$$f_R(\rho) = f_R(\rho'_c) + \sum_{n=1}^{\infty} \frac{\frac{d^n f_R}{d\rho^n}}{n!} \frac{(\rho - \rho'_c)^n}{n!}$$

(14)

$$= \rho_c + \frac{d f_R}{d \rho} \bigg|_{\rho'_c} (\rho - \rho'_c) + \frac{\frac{d^2 f_R}{d\rho^2}}{2} \bigg|_{\rho'_c} \frac{(\rho - \rho'_c)^2}{2} + \cdots.$$  

(15)

The expansion coefficients depend only on $R$. With similar arguments in Ref. [24], the aspect ratio dependence of the differential coefficient of the first order can be assumed to be

$$\frac{d f_R}{d \rho} \bigg|_{\rho'_c} \sim R^a,$$  

(16)
and taking into account up to the first order derivative of the expansion, we obtain the approximated mapping function as,

\[ f_R(\rho) - \rho_c \propto R^a (\rho - \rho'_c) \]

From the finite-size scaling, Eq. (13), we can derive the superscaling form to be,

\[ E_p(L, f_R(\rho), 1) \sim F((f_R(\rho) - \rho_c) L^{\eta c}) \] (17)

and Eq. (13), we can derive the superscaling form to be,

\[ E_p(L, \rho, R) = E_p(L, f_R(\rho), 1) \]

\[ \sim F((f_R(\rho) - \rho_c) L^{\eta c}) \] (18)

\[ \sim F((\rho - \rho'_c) L^{\eta c} R^b). \] (19)

\[ \sim F((\rho - \rho'_c) L^{\eta c} R^b). \] (20)

The mapping functions \( f_R \) of existence probabilities of the bond percolation model on the square lattices are shown in Fig. 3(b) for the system size \( L = 256 \) and aspect ratios \( R = 1, 2, 4, 8 \) and 16. It shows that these functions do not have any intersection points, since \( E_p(L, \rho, R) > E_p(L, \rho, 1) \). Additionally, while the mapping functions of \( f_Q \) in Fig. 1 have linear forms, the curves of \( f_R \) in Fig. 3(b) for \( R > 1 \) are not linear.

Consider the mapping of the percolation probabilities with different aspect ratio as,

\[ P(L, \rho, R) = P(L, g_R(\rho), 1), \] (21)

with a mapping function \( g_R \). Similar to the case of the existence probability, the behavior of the differential coefficient of \( g_R \) can be assumed as

\[ \left. \frac{dg_R}{d\rho} \right|_{\rho'_c} \sim R^b. \] (22)

with the effective critical point \( \rho'_c \) defined by \( g_R(\rho'_c) = \rho_c \) as shown in Fig. 4(a). Note that, the value of effective critical point is different from the one defined by \( f_R(\rho'_c) = \rho_c \). We can obtain the superscaling formula as,

\[ P(L, \rho, R) \sim L^{-\beta y} F ((\rho - \rho'_c) L^{\eta y} R^b). \] (23)

This scaling form is different from the form

\[ P(L, \rho, R) \sim (L^{\eta y} R^b)^{-\beta} F ((\rho - \rho'_c) L^{\eta y} R^b) \] (24)

obtained by a heuristic argument in [22]. The mapping functions of percolation probabilities are determined and shown in Fig. 4(b). While they also do not have any intersection points,
they have almost linear forms. The scaling plot based on Eq. (23) is shown in Fig. 5. It shows good scaling behavior especially around at the effective critical point. Please note that the right-hand sides of Eqs. (23) and (24) differ only by the factor $R^{-b\beta}$ which is very close to 1.

Since the mapping functions $f_R$ for $E_p$ have finite curvature, the scaling formula $E_p(L, \rho, R) \sim F(c' L^a R^b)$ will be improved if we take into account higher order. This nonlinear scaling form was shown in [22], and it corresponds to approximate mapping function with a quadratic form as $f_R(\rho) = c_2 \rho^2 + c_1 \rho + c_0$, which is the expression obtained by taking into account up to the second order derivatives in Eq. (15). With the coefficients $c_2, c_1$ and $c_0$ listed in [22], we calculate the differential coefficients and show the results in Fig. 6(a). The mapping functions of the percolation probabilities have almost linear form, we just fit them with linear function to obtain the differentiations. The obtained differential coefficients of the percolation probabilities are shown in Fig. 6(b). Figures 6(a) and (b) show power law behavior $R^a$ and $R^b$ with values of exponents $a = 0.14$ and $b = 0.05$ which are consistent with the results in [22]. Thus $a$ and $b$ can be obtained from mapping functions $f_R$ and $g_R$ for $E_p$ and $P$, respectively.

IV. SUMMARY AND DISCUSSION

In summary, we have studied the aspect ratio dependence of FSSFs of existence and percolation probabilities based on the idea of the mapping functions. From the analysis on the numerically obtained mapping functions, the superscaling behaviors are found to be the first order approximation of the mapping functions at the effective critical point. Since the mapping regime used here is pretty general, we can apply it for any other systems and geometric conditions.

Many lattice phase transition models, such as the Ising model, the Potts model, the dilute Potts model, the hydrogen bonding model for water, lattice hard-core particle models, etc, have been shown to be corresponding to some correlated percolation models [25]. Some of such correlated percolation models have been found to have good finite-size scaling behaviors [26]. It is of interest to use mapping functions to study super-scaling in such correlated percolation models and other critical systems [27].

Another interesting development for critical systems is finite-size corrections for lattice
phase transition models \[28, 29, 30, 31, 32, 33\]. Using exact partition functions of the Ising model on finite lattices \[34\] and exact finite-size corrections for the free energy, the internal energy, and the specific heat of the Ising model \[32\], Wu, Hu and Izmailian \[35\] obtained UFSSFs for the free energy, the internal energy, and the specific heat with analytic equations, which are free of simulation errors. Based on finite-size corrections for the dimer model on finite lattices \[36\], Izmailian, Oganesyan, Wu, and Hu \[37\] obtained UFSSFs for the dimer model with analytic equations, which are also free of simulation errors. It is of interest to extend above results for the Ising and dimer models to rectangular domains with various aspect ratios. We can then study superscaling behavior of physical quantities of the Ising and dimer models with the help of mapping functions.

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[1] H. E. Stanley, 1971. *Introduction to Phase Transitions and Critical Phenomena* (Oxford Univ. Press, New York 1971).
[2] M. E. Fisher, in *Proc. 1970 E. Fermi Int. School of Physics*, M. S. Green ed. (Academic, NY, 1971) Vol. 51, p. 1.
[3] V. Privman and M. E. Fisher, Phys. Rev. B 30, 322 (1984).
[4] *Finite-size Scaling and Numerical Simulation of Statistical Systems*, V. Privman ed. (World Scientific, Singapore, 1990).
[5] F. Y. Wu, J. Stat. Phys. 18, 115 (1978).
[6] J. W. Essam, Rep. Prog. Phys. 43, 53 (1980).
[7] P. J. Reynolds, H. E. Stanley and W. Klein, Phys. Rev. B 21, 1223 (1980).
[8] S. B. Lee and S. Torquato, Phys. Rev. A 41, 5338 (1990).
[9] C.-K. Hu, Phys. Rev. B 46, 6592 (1992); Phys. Rev. Lett. 69, 2739 (1992).
[10] R. M. Ziff, Phys. Rev. Lett 69, 2670 (1992).
[11] D. Stauffer and A. Aharony, Introduction to Percolation Theory, Revised 2nd. ed. (Taylor and Francis, London, 1994).
[12] M. Sahimi: Applications of Percolation Theory (Taylor and Francis, London, 1994).
[13] R. M. Ziff, Phys. Rev. E 54, 2547 (1996).
[14] Other names for $E_p$ include cumulative distribution function used in the sentence above Eq. (4.7) of [7], connectivity function used in page 5341 of [8], crossing probability used in [10], etc.

In the present paper, we call $E_p$ the existence probability as in our earlier papers.
[15] C.-K. Hu, J. Phys. A: Math. Gen. 27, L813 (1994); Phys. Rev. Lett. 76, 3875 (1996).
[16] C.-K. Hu and J.-A. Chen, J. Phys. A: Math. Gen. 28, L73 (1995).
[17] C.-K. Hu, C.-Y. Lin, and J.-A. Chen, Phys. Rev. Lett. 75, 193 (1995) and 75, 2786(E) (1995), Physica A 221, 80 (1995); C.-K. Hu and C.-Y. Lin, Phys. Rev. Lett. 77, 8 (1996); C.-K. Hu and F.-G. Wang, J. Korean Physical Soc. 31, S271 (1997); H. P. Hsu, S. C. Lin and C.-K. Hu, Phys. Rev. E 64, 016127 (2001); H. Watanabe, et al., J. Phys. Soc. Japan. 70, 1537 (2001).
[18] C.-Y. Lin and C.-K. Hu, Phys. Rev. E, 58, 1521 (1998); C.-Y. Lin, C.-K. Hu and J.-A. Chen, J. Phys. A: Math. Gen. 31, L111 (1998).
[19] R. A. Monetti and E. V. Albano, Z. Phys. B: Condens. Matter, 82, 129 (1991).
[20] J. L. Cardy, J. Phys. A 25, L201 (1992).
[21] In the limit $L \to \infty$, $E_p = 0$ for the bond density $\rho$ smaller than $\rho_c$ and $E_p = 1$ for bond density $\rho$ larger than $\rho_c$. If we write $E_p \sim (\rho - \rho_c)^c$ for $\rho > \rho_c$, then the critical exponent $c$ is 0.
[22] H. Watanabe, S. Yukawa, N. Ito, and C.-K. Hu, Phys. Rev. Lett. 93, 19601 (2004). This paper contains some typos, see Ref. [23] for details; see also G. Pruessner and N. R. Moloney, Phys. Rev. Lett. 95, 258901 (2005).
[23] H. Watanabe and C.-K. Hu, Phys. Rev. Lett. 95, 258902 (2005).
[24] The identity $f_{Q-1}(f_Q(\rho)) = \rho$ directly leads to the relation $f_{Q-1}'|_{\rho_c} \times f_Q'|_{\rho_c} = 1$ ($f_Q'$ representing the derivative of $f_Q$ with respect to $\rho$), because $f_Q(\rho_c) = \rho_c$ regardless of the value of $Q$.

Therefore the $Q$-dependence of the coefficient should have the form $f_Q'|_{\rho_c} = Q^\alpha$.
[25] C.-K. Hu, Physica A 116, 265 (1982); ibid. 119, 609 (1983); J. Phys. A: Math. Gen. 16, L321
(1983); Phys. Rev. B 29, 5103 (1984); ibid. 29, 5109 (1984); C.-K. Hu and K.-S. Mak, ibid. 39, 2948 (1989); ibid. 42, 965 (1990); ibid. 40, 5007 (1989).

[26] Y. Okabe, K. Kaneda, M. Kikuchi, and C.-K. Hu, Phys. Rev. E 59, 1585 (1999); Y. Tomita, Y. Okabe, and C.-K. Hu, ibid. 60, 2716 (1999); C.-K. Hu, J.-A. Chen, and C.-Y. Lin, Physica A 266, 27 (1999).

[27] One possible example is the system with nice universal dynamic scaling behavior, see e.g. F. G. Wang and C.-K. Hu, Phys. Rev. E 56, 2310 (1997).

[28] A. E. Ferdinand, J. Math. Phys. 8, 2332 (1967); A. E. Ferdinand and M. E. Fisher, Phys. Rev. 185, 832 (1969).

[29] R. M. Ziff, S. R. Finch, and V. S. Adamchik, Phys. Rev. Lett. 79, 3447 (1997); P. Kleban and R. M. Ziff, Phys. Rev. B 57, R8075 (1998).

[30] C.-K. Hu, J.-A. Chen, N. Sh. Izmailian, and P. Kleban, Phys. Rev. E. 60, 6491-6499 (1999).

[31] J. Salas, J. Phys. A: Math. Gen. 34, 1311 (2001); 35, 1833 (2002); W. Janke and R. Kenna, Phys. Rev. B 65, 064110 (2002); N. Sh. Izmailian and C.-K. Hu, Phys. Rev. Lett. 86, 5160 (2001).

[32] E.V. Ivashkevich, N. Sh. Izmailian, and C.-K. Hu, J. Phys. A: Math. Gen. 35, 5543 (2002); N. Sh. Izmailian and C.-K. Hu, Phys. Rev. E 65, 036103 (2002); N. Sh. Izmailian, K.B. Oganesyan, and C.-K. Hu, ibid. 65, 056132 (2002).

[33] A brief review of finite-size corrections for critical lattice models can be found in N. Sh. Izmailian and C.-K. Hu, Phys. Rev. E 76, 041118 (2007).

[34] M.-C. Wu and C.-K. Hu, J. Phys. A: Math. Gen. 35, 5189 (2002).

[35] M.-C. Wu, C.-K. Hu, and N.Sh. Izmailian, Phys. Rev. E 67, 065103(R) (2003).

[36] N.Sh. Izmailian, K.B. Oganesyan, and C.-K. Hu, Phys. Rev. E 67, 066114 (2003); N. Sh. Izmailian, V. B. Priezzhev, P. Ruelle, and C.-K. Hu, Phys. Rev. Lett. 95, 260602 (2005).

[37] N. Sh. Izmailian, K. B. Oganesyan, M.-C. Wu, and C.-K. Hu, Phys. Rev. E 73, 016128 (2006).
FIG. 1: Mapping functions of the bond percolation model on square lattices with free boundary conditions (a) $f_Q$ for the existence probabilities $E_p$ as a function of $\rho$, and (b) $g_Q$ for the percolation probabilities $P$ as a function of $\rho$. The insets in (a) and (b) show how to construct the mapping functions for $E_p$ and $P$, respectively.
FIG. 2: UFSSFs of (a) existence probabilities and (b) percolation probabilities corresponding to Fig. 1. The aspect ratio is $R = 1$, and the system sizes are $L = 64(Q = 1)$, $128(Q = 2)$, and $256(Q = 4)$, respectively. The scaling variable is denoted by $\varepsilon \equiv (\rho - \rho_c)L^n$. 
FIG. 3: (a) Procedures to obtain the mapping functions of existence probabilities with different aspect ratios. (b) Obtained mapping functions of systems with $L = 256$ and $R = 1, 2, 4, 8$ and 16 (from bottom to top). The solid circles are the effective critical points $\rho' c$. 
FIG. 4: (a) Procedures to obtain the mapping functions of percolation probabilities with different aspect ratios. (b) Obtained mapping functions of systems with $L = 128$ and $R = 1, 2, 4, 8$ and 16 (from bottom to top).
FIG. 5: Superscaling plot of the percolation probabilities $P$ using Eq. (23) for systems with $R = 1, 2, 4, 8$ and 16, and $L = 128$ with $b = 0.05$.  

$P = L^{\gamma \beta} (\rho - \rho_c)^{\nu} R^b$
FIG. 6: Plots of (a) $u = \frac{df}{d\rho|_{\rho_e}}$ and (b) $v = \frac{dg}{d\rho|_{\rho_e}}$ as a function $R$. The solid lines are $R^a$ and $R^b$ with exponents $a = 0.14$ and $b = 0.05$, respectively. Decimal logarithms are taken for both axes.