On the Lefschetz and Hodge-Riemann theorems
Tien-Cuong Dinh and Việt-Anh Nguyên
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Abstract
We give an abstract version of the hard Lefschetz theorem, the Lefschetz decomposition and the Hodge-Riemann theorem for compact Kähler manifolds.

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1 Introduction

Let $X$ be a compact Kähler manifold of dimension $n$ and let $\omega$ be a Kähler form on $X$. Denote by $H^{p,q}(X, \mathbb{C})$ the Hodge cohomology group of bidegree $(p, q)$ of $X$ with the convention that $H^{p,q}(X, \mathbb{C}) = 0$ unless $0 \leq p, q \leq n$. When $p, q \geq 0$ and $p + q \leq n$, put $\Omega := \omega^{n-p-q}$ and define a Hermitian form $Q$ on $H^{p,q}(X, \mathbb{C})$ by

$$Q(\{\alpha\}, \{\beta\}) := i^{q-p}(-1)^{\frac{(p+q)(p+q+1)}{2}} \int_X \alpha \wedge \overline{\beta} \wedge \Omega$$

for smooth closed $(p, q)$-forms $\alpha$ and $\beta$. The last integral depends only on the classes $\{\alpha\}, \{\beta\}$ of $\alpha, \beta$ in $H^{p,q}(X, \mathbb{C})$.

The classical Hodge-Riemann theorem asserts that $Q$ is positive-definite on the primitive subspace $H^{p,q}(X, \mathbb{C})_{\text{prim}}$ of $H^{p,q}(X, \mathbb{C})$ which is given by

$$H^{p,q}(X, \mathbb{C})_{\text{prim}} := \{\{\alpha\} \in H^{p,q}(X, \mathbb{C}), \{\alpha\} \sim \{\Omega\} \sim \{\omega\} = 0\},$$

where $\sim$ denotes the cup-product on the cohomology ring $\oplus H^*(X, \mathbb{C})$, see e.g. Demailly [2], Griffiths-Harris [8] and Voisin [16].

The Hodge-Riemann theorem implies the hard Lefschetz theorem which says that the linear map $\{\alpha\} \mapsto \{\alpha\} \sim \{\Omega\}$ defines an isomorphism between $H^{p,q}(X, \mathbb{C})$ and $H^{n-q,n-p}(X, \mathbb{C})$. It also implies the following Lefschetz decomposition

$$H^{p,q}(X, \mathbb{C}) = \{\omega\} \sim H^{p-1,q-1}(X, \mathbb{C}) \oplus H^{p,q}(X, \mathbb{C})_{\text{prim}}$$
which is orthogonal with respect to the Hermitian form $Q$. Moreover, we easily obtain from the above theorems the signature of $Q$ in term of the Hodge numbers $h^{p,q} := \dim H^{p,q}(X, \mathbb{C})$. For example, when $p = q = 1$ the signature of $Q$ is equal to $(h^{1,1} - 1, 1)$.

The above three theorems are not true if we replace $\{\Omega\}$ with an arbitrary class in $H^{n-p-q,n-p-q}(X, \mathbb{R})$, even when the class contains a strictly positive form, see e.g. Berndtsson-Sibony [11, §9]. Our aim here is to give sufficient conditions on $\{\Omega\}$ for which these theorems still hold. We will say that such a class $\{\Omega\}$ satisfies the Hodge-Riemann theorem, the hard Lefschetz theorem and the Lefschetz decomposition theorem for the bidegree $(p, q)$.

If $E$ is a complex vector space of dimension $n$ and $\overline{E}$ its complex conjugate, we will introduce in the next section the notion of Hodge-Riemann cone in the exterior product $\bigwedge^k E \otimes \bigwedge^k \overline{E}$ with $0 \leq k \leq n$, see Definition 2.1 below. In practice, $E$ is the complex cotangent space at a point $x$ of $X$ and we obtain a Hodge-Riemann cone associated with $X$. Here is our main result.

**Theorem 1.1.** Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$. Let $p, q$ be non-negative integers such that $p + q \leq n$ and $\Omega$ a closed smooth form of bidegree $(n-p-q, n-p-q)$ on $X$. Assume that $\Omega$ takes values only in the Hodge-Riemann cones associated with $X$. Then $\{\Omega\}$ satisfies the Hodge-Riemann theorem, the hard Lefschetz theorem and the Lefschetz decomposition theorem for the bidegree $(p, q)$.

Roughly speaking, the hypothesis of Theorem 1.1 says that at every point $x$ of $X$, we can deform continuously $\Omega$ to $\omega^{n-p-q}$ in a “nice way”. However, we do not need that the deformation depends continuously on $x$ and a priori the deformation does not preserve the closedness nor the smoothness of the form.

We deduce from Theorem 1.1 the following corollary using a result due to Timorin [15], see Proposition 2.2 below.

**Corollary 1.2.** Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$. Let $p, q$ be non-negative integers such that $p + q \leq n$ and $\omega_1, \ldots, \omega_{n-p-q}$ be Kähler forms on $X$. Then the class $\{\omega_1 \wedge \ldots \wedge \omega_{n-p-q}\}$ satisfies the Hodge-Riemann theorem, the hard Lefschetz theorem and the Lefschetz decomposition theorem for the bidegree $(p, q)$.

The last result was obtained by the authors in [5], see also Cattani [3] for a proof using the theory of variations of Hodge structures and for related results. It solves a problem which has been considered in some important cases by Kho-vanskii [11, 12], Teissier [13, 14], Gromov [9] and Timorin [15]. The reader will find some applications of the above corollary in Gromov [9], Dinh-Sibony [6] and Keum-Oguiso-Zhang [10, 18].
2 Hodge-Riemann forms

In this section, we introduce the notion of Hodge-Riemann form in the linear setting and we will discuss some basic properties of these forms.

Let $E$ be a complex vector space of dimension $n$ and $\overline{E}$ its conjugate space. Denote by $V_{p,q}$ the space $\bigwedge^p E \otimes \bigwedge^q \overline{E}$ of $(p,q)$-forms with the convention that $V_{p,q} := 0$ unless $0 \leq p, q \leq n$. Recall that a form $\omega$ in $V^{1,1}$ is a Kähler form if it can be written as

$$\omega = idz_1 \wedge d\overline{z}_1 + \cdots + idz_n \wedge d\overline{z}_n$$

for some coordinate system $(z_1, \ldots, z_n)$ of $E$, where $z_i \otimes \overline{z}_j$ is identified with $dz_i \wedge d\overline{z}_j$.

Recall also that a form $\Omega$ in $V_{k,k}$ with $0 \leq k \leq n$, is real if $\Omega = \overline{\Omega}$. Let $V_{\mathbb{R}}^{k,k}$ denote the space of real $(k,k)$-forms. A form $\Omega$ in $V_{\mathbb{R}}^{k,k}$ is positive if it is a combination with positive coefficients of forms of type $i^k \alpha \wedge \overline{\alpha}$ with $\alpha \in V^{k,0}$. So, positive forms are real. If $\Omega$ is positive its restriction to any subspace of $E$ is positive. A positive $(k,k)$-form $\Omega$ is strictly positive, if its restriction to any subspace of dimension $k$ of $E$ does not vanish. The powers of a Kähler form are strictly positive forms. Fix a Kähler form $\omega$ as above.

Definition 2.1. A real $(k,k)$-form $\Omega$ in $V_{\mathbb{R}}^{k,k}$ is said to be a Hodge-Riemann form for the bidegree $(p,q)$ if $k = n - p - q$ and if there is a continuous deformation $\Omega_t \in V_{\mathbb{R}}^{k,k}$ with $0 \leq t \leq 1$, $\Omega_0 = \Omega$ and $\Omega_1 = \omega^k$ such that

(*) the map $\alpha \mapsto \Omega_t \wedge \omega^{2r} \wedge \alpha$ is an isomorphism from $V_{p-r,q-r}$ to $V_{n-q+r,n-p+r}$ for every $0 \leq r \leq \min\{p,q\}$ and $0 \leq t \leq 1$. The cone of such forms $\Omega$ is called the Hodge-Riemann cone for the bidegree $(p,q)$. We say that $\Omega$ is Hodge-Riemann if it is a Hodge-Riemann form for any bidegree $(p,q)$ with $p + q = n - k$.

Note that the property (*) for $t = 1$ is a consequence of the linear version of the classical hard Lefschetz theorem. The Hodge-Riemann cone is open in $V_{\mathbb{R}}^{k,k}$ and a priori depends on the choice of $\omega$. In practice, to check that a form is Hodge-Riemann is usually not a simple matter. We have the following result due to Timorin in [15].

Proposition 2.2. Let $k$ be an integer such that $0 \leq k \leq n$. Let $\omega_1, \ldots, \omega_k$ be Kähler forms. Then $\Omega := \omega_1 \wedge \ldots \wedge \omega_k$ is a Hodge-Riemann form.

Consider a square matrix $M = (\alpha_{ij})_{1 \leq i,j \leq k}$ with entries in $V^{1,1}$. Assume that $M$ is Hermitian, i.e. $\alpha_{ij} = \overline{\alpha}_{ji}$, for all $i, j$. We say that $M$ is Griffiths positive if for any row vector $\theta = (\theta_1, \ldots, \theta_k)$ in $\mathbb{C}^k \setminus \{0\}$ and its transposed $\overline{\theta}$, $\theta M \overline{\theta}$ is a Kähler form. We call Griffiths cone the set of $(k,k)$-forms in $V_{\mathbb{R}}^{k,k}$ which can be obtained as the determinant of a Griffiths positive matrix $M$ as above. We are still unable to answer the following question.

\[\text{\footnotesize{There are two other notions of positivity but we will not use here.}}\]
Problem 2.3. Is the Griffiths cone contained in the Hodge-Riemann cone?

The affirmative answer to the question will allow us to obtain a transcendental version of the hyperplane Lefschetz theorem which is known for the last Chern class associated with a Griffiths positive vector bundle, see Voisin [16, p.312]. The Griffiths cone contains the wedge-products of Kähler forms (case where $M$ is diagonal) and Proposition 2.2 gives the affirmative answer to this case.

Note also that for the above problem it is enough to check the condition $(\ast)$ for $t = 0$ and $r = 0$. Indeed, we can consider $\Omega_t$ the determinant of the Griffiths positive matrix $M_t := (1 - t)M + tI\omega$, where $I$ is the identity matrix. It is enough to observe that $\Omega_t \wedge \omega^{2r}$ is the determinant of the Griffiths positive $(k + 2r) \times (k + 2r)$ matrix which is obtained by adding to $M_t$ a square block equal to $\omega$ times the identity $2r \times 2r$ matrix.

The following question is also open.

Problem 2.4. Let $\Omega_t$, $0 \leq t \leq 1$, be a continuous family of strictly positive $(k, k)$-forms in $V_{\mathbb{R}}^{k,k}$ with $\Omega_0 = \Omega$ and $\Omega_1 = \omega^k$. Assume the property $(\ast)$ in Definition 2.1 for $r = 0$ and for this family $\Omega_t$. Is $\Omega$ always a Hodge-Riemann form for the bidegree $(p, q)$?

Note that the strict positivity of $\Omega_t$ implies the property $(\ast)$ for $r = \min\{p, q\}$. This is perhaps a reason to believe that the answer to the above problem is affirmative. An interesting point here is that the cone of all forms $\Omega$ as in Problem 2.4 does not depend on $\omega$. The following result gives a partial answer to the question.

Proposition 2.5. Let $\Omega_t$ be as in Problem 2.4. Assume moreover that $\min\{p, q\} \leq 2$. Then $\Omega$ is a Hodge-Riemann form for the bidegree $(p, q)$.

Fix a coordinate system $(z_1, \ldots, z_n)$ of $E$ such that $\omega = idz_1 \wedge d\overline{z}_1 + \cdots + idz_n \wedge d\overline{z}_n$. So, this Kähler form is invariant under the natural action of the unitary group $U(n)$. We will need the following lemma.

Lemma 2.6. Let $\alpha$ be a form in $V^{p,q-1}$ with $q \geq 2$ and $p + q \leq n$. Assume that for every $\varphi \in V^{0,1}$ we can write $\alpha \wedge \varphi = \omega \wedge \beta$ for some $\beta \in V^{p-1,q-1}$. Then we can write $\alpha = \omega \wedge \gamma$ for some $\gamma \in V^{p-1,q-2}$.

Proof. Let $M$ denote the set of all forms $\alpha \in V_{\mathbb{R}}^{p,q-1}$ satisfying the hypothesis of the lemma. Observe that $M$ is invariant under the action of $U(n)$. So, it is a linear representation of this group. Let $P_j$ denote the primitive subspace of $V^{p-j,q-1-j}$, i.e. the set of $\phi \in V^{p-j,q-1-j}$ such that $\phi \wedge \omega^{n-p-q+2j} = 0$. It is well-known that the $P_j$ are irreducible representations of $U(n)$ and they are not isomorphic one another, see e.g. Fujiki [7, Prop. 2.2]. Moreover, we have the Lefschetz decomposition

$V_{\mathbb{R}}^{p,q-1} = \bigoplus_{0 \leq j \leq \min\{p,q-1\}} \omega^j \wedge P_j$. 


The space $\omega^j \wedge P_j$ is also a representation of $U(n)$ which is isomorphic to $P_j$. Therefore, it is enough to show that $M$ does not contain $P_0$.

Consider the form

$$\alpha := d\bar{z}_2 \wedge \ldots \wedge d\bar{z}_q \wedge dz_{q+1} \wedge \ldots \wedge dz_{p+q}.$$ 

A direct computation shows that $\alpha$ is a form in $P_0$. Observe that $\alpha \wedge d\bar{z}_1$ does not contain any factor $dz_j \wedge d\bar{z}_j$. Therefore, $\alpha \not\in M$ because $\alpha \wedge d\bar{z}_1$ does not belong to $\omega \wedge V^{p-1,q-1}$. The lemma follows. \qed

Given non-negative integers $p, q$ such that $p + q \leq n$ and a real form $\Omega$ of bidegree $(n - p - q, n - p - q)$, define the Hermitian form $Q$ by

$$Q(\alpha, \beta) := i^{q-p}(-1)^{\frac{(p+q)(p+q+1)}{2}} \star (\alpha \wedge \beta \wedge \Omega) \quad \text{for} \quad \alpha, \beta \in V^{p,q},$$

where $\star$ is the Hodge star operator. Define also the primitive subspace

$$P^{p,q} := \{ \alpha \in V^{p,q} : \alpha \wedge \Omega \wedge \omega = 0 \}.$$ 

The classical Lefschetz theorem asserts that the wedge-product with $\omega$ defines a surjective map from $V^{n-q,n-p}$ to $V^{n-q+1,n-p+1}$. Its kernel is of dimension $\dim V^{p,q} - \dim V^{p-1,q-1}$. Therefore, if the map $\alpha \mapsto \Omega \wedge \alpha$ is injective on $V^{p,q}$, the above primitive space has dimension $\dim V^{p,q} - \dim V^{p-1,q-1}$ which does not depend on $\Omega$.

We also need the following lemma.

**Lemma 2.7.** Let $\Omega_t$ be a continuous family of real $(k,k)$-forms in $V_{\mathbb{R}}^{k,k}$ with $\Omega_0 = \Omega$, $\Omega_1 = \omega^k$ and $0 \leq t \leq 1$. Assume that $\alpha \mapsto \Omega_t \wedge \alpha$ is an isomorphism from $V^{p,q}$ to $V^{n-q,n-p}$ for every $0 \leq t \leq 1$ and $\alpha \mapsto \Omega_t \wedge \omega^2 \wedge \alpha$ is an isomorphism from $V^{p-1,q-1}$ to $V^{n-q+1,n-p+1}$ for every $0 < t \leq 1$. If a form $\alpha$ in $V^{p,q-1}$ (resp. $V^{p-1,q}$) satisfies $\alpha \wedge \Omega \wedge \omega = 0$, then $\alpha$ belongs to $\omega \wedge V^{p-1,q-2}$ (resp. $\omega \wedge V^{p-2,q-1}$).

**Proof.** Let $V$ denote the space of forms $\beta \in V^{p,q}$ such that $Q(\beta, \phi) = 0$ for every $\phi$ in $\omega \wedge V^{p-1,q-1} + P^{p,q}$. The hypothesis implies that $Q$ is non-degenerate. Therefore, we obtain

$$\dim \omega \wedge V^{p-1,q-1} + \dim P^{p,q} = \dim V^{p-1,q-1} + \dim V^{p,q} - \dim V^{p-1,q-1} = \dim V^{p,q},$$

and hence

$$\dim V = \dim V^{p,q} - \dim (\omega \wedge V^{p-1,q-1} + P^{p,q}) = \dim (\omega \wedge V^{p-1,q-1} \cap P^{p,q}).$$

On the other hand, by definition of $P^{p,q}$, the space $\omega \wedge V^{p-1,q-1} \cap P^{p,q}$ is contained in $V$. We deduce that these two spaces coincide.

Let $\alpha \in V^{p,q-1}$ such that $\alpha \wedge \Omega \wedge \omega = 0$ (the case $\alpha \in V^{p-1,q}$ can be treated in the same way). Fix a form $\varphi$ in $V^{0,1}$. By Lemma 2.6 we only need to show
that \( \alpha \wedge \varphi \) belongs to \( V \). It is clear that \( Q(\alpha \wedge \varphi, \phi) = 0 \) for \( \phi \in \omega \wedge V^{p-1,q-1} \). It remains to show that \( Q(\alpha \wedge \varphi, \phi) = 0 \) for \( \phi \in P^{p,q} \). For this purpose, it is enough to consider the case where \( \varphi = d \zeta \) since \( \{d \zeta_1, \ldots, d \zeta_n\} \) is a basis of \( V^{0,1} \).

Using the continuous deformation of \( \Omega \) in the hypothesis, we obtain as in Proposition 2.8 below that the restriction of \( Q \) to \( P^{p,q} \) is semi-positive. Observe that \( \alpha \wedge d \zeta_j \) is in \( P^{p,q} \). Hence,

\[
Q(\alpha \wedge d \zeta_j, \alpha \wedge d \zeta_j) \geq 0.
\]

The sum over \( j \) of \( Q(\alpha \wedge d \zeta_j, \alpha \wedge d \zeta_j) \) vanishes since \( \alpha \wedge \Omega \wedge \omega = 0 \). We deduce that all the above inequalities are in fact equalities. Now, since \( Q \) is semi-positive on \( P^{p,q} \), by Cauchy-Schwarz’s inequality, \( Q(\alpha \wedge d \zeta_j, \phi) = 0 \) for \( \phi \in P^{p,q} \). This completes the proof.

**Proof of Proposition 2.5.** Assume without loss of generality that \( q \leq p \). Observe that for every \( \alpha \) non-zero in \( V^{n-k-s,0} \) we have \( i^{(n-k-s)2} \alpha \wedge \overline{\gamma} \wedge \Omega_t \wedge \omega^s > 0 \). So, we only have to consider the case \( q = 2 \) and to check the property (*) for \( r = 1 \). We will show that the map \( \alpha \mapsto \Omega_t \wedge \omega \wedge \alpha \) is injective on \( V^{p,1} \) and the map \( \alpha \mapsto \Omega_t \wedge \omega^2 \wedge \alpha \) is injective on \( V^{p-1,1} \). The result will follow easily.

Let \( \Sigma \) denote the set of \( t \) satisfying the above property. By continuity, \( \Sigma \) is open in \([0,1]\). Moreover, by Lefschetz theorem, it contains the point \( 1 \). Assume that \( \Sigma \) is not equal to \([0,1]\). Let \( t_0 < 1 \) be the minimal number such that \([t_0,1] \subset \Sigma \). We will show that \( t_0 \in \Sigma \) which is a contradiction. Up to a re-parametrization of the family \( \Omega_t \), we can assume for simplicity that \( t_0 = 0 \).

Consider a form \( \alpha \in V^{p,1} \) such that \( \Omega \wedge \omega \wedge \alpha = 0 \). We deduce from Lemma 2.7 that \( \alpha = \omega \wedge \gamma \) with \( \gamma \in V^{p-1,0} \). We have \( \gamma \wedge \overline{\gamma} \wedge \Omega \wedge \omega^2 = 0 \). The positivity of \( \Omega \) implies that \( \gamma = 0 \) and then \( \alpha = 0 \). So, the map \( \alpha \mapsto \Omega \wedge \omega \wedge \alpha \) is injective on \( V^{p,1} \). By dimension reason, this map is bijective from \( V^{p,1} \) to \( V^{n-1,n-p} \). Now, we apply again Lemma 2.7 but to \( \Omega_t \wedge \omega \) instead of \( \Omega_t \) and \((p,1)\) instead of \((p,q)\). We obtain as above that the map \( \alpha \mapsto \Omega \wedge \omega^2 \wedge \alpha \) is injective on \( V^{p-1,1} \). Therefore, \( 0 \) is a point in \( \Sigma \). This completes the proof.

We give now fundamental properties of Hodge-Riemann forms that we will use in the next section. We fix a norm on each space \( V^{*,*} \).

**Proposition 2.8.** Let \( \Omega \) be a form satisfying the condition (*) in Definition 2.7 for \( r = 0,1 \). Then the space \( V^{p,q} \) splits into the \( Q \)-orthogonal direct sum

\[
V^{p,q} = P^{p,q} \oplus \omega \wedge V^{p-1,q-1}
\]

and the Hermitian form \( Q \) is positive-definite on \( P^{p,q} \). Moreover, for any constant \( c_1 > 0 \) large enough, there is a constant \( c_2 > 0 \) such that

\[
\|\alpha\|^2 \leq c_1 Q(\alpha, \alpha) + c_2 \|\alpha \wedge \Omega \wedge \omega\|^2 \quad \text{for} \quad \alpha \in V^{p,q}.
\]
Proof. The $Q$-orthogonality is obvious. By classical Lefschetz theorem, the wedge-product with $\omega$ defines an injective map from $V^{p-1,q-1}$ to $V^{p,q}$. Therefore, we have

$$\dim V^{p,q} = \dim P^{p,q} + \dim V^{p-1,q-1} = \dim P^{p,q} + \dim \omega \wedge V^{p-1,q-1}.$$ 

On the other hand, the property $(\ast)$ for $r = 1$ implies that the intersection of $P^{p,q}$ and $\omega \wedge V^{p-1,q-1}$ is reduced to 0. We then deduce the above decomposition of $V^{p,q}$. Of course, this property still holds if we replace $\Omega$ with $\Omega_t$.

Denote by $Q_t$ and $P^{p,q}_t$ the Hermitian form and the primitive space associated with $\Omega_t$ which are defined as above. Since the dimension of $P^{p,q}_t$ is constant, this space depends continuously on $t$. By classical Hodge-Riemann theorem, $Q_1$ is positive-definite on $P^{p,q}_t$. If $Q$ is not positive-definite on $P^{p,q}$, there is a maximal number $t$ such that $Q_t$ is not positive-definite. The maximality of $t$ implies that $Q_s$ is positive-definite on $P^{p,q}_s$ when $s > t$. It follows by continuity that there is an element $\alpha \in P^{p,q}_s$, $\alpha \neq 0$, such that $Q_t(\alpha, \beta) = 0$ for $\beta \in P^{p,q}_s$. By definition of $P^{p,q}_t$, this identity holds also for $\beta \in \omega \wedge V^{p-1,q-1}$. We then deduce that the identity holds for all $\beta \in V^{p,q}$. It follows that $\alpha \wedge \Omega_t = 0$. This is a contradiction. So, $Q$ is positive-definite on $P^{p,q}$.

We prove now the last assertion in the proposition for a fixed constant $c_1$ large enough. Consider a form $\alpha \in V^{p,q}$. The first assertion implies that we can write

$$\alpha = \beta + \omega \wedge \gamma \quad \text{with} \quad \beta \in P^{p,q} \quad \text{and} \quad \gamma \in V^{p-1,q-1}$$

and we have

$$Q(\alpha, \alpha) = Q(\beta, \beta) + Q(\omega \wedge \gamma, \omega \wedge \gamma).$$

Since the wedge-product with $\Omega_1 \wedge \omega^2$ defines an isomorphism between $V^{p-1,q-1}$ and $V^{n-q+1,n-p+1}$, there is a constant $c > 0$ such that

$$c^{-1}||\gamma \wedge \Omega \wedge \omega^2|| \leq ||\gamma|| \leq c||\gamma \wedge \Omega \wedge \omega^2|| = c||\alpha \wedge \Omega \wedge \omega||.$$ 

Therefore, there is a constant $c' > 0$ such that

$$||\alpha||^2 \leq c'(||\beta||^2 + ||\gamma||^2) \leq c'\alpha^2 + c'c^2||\alpha \wedge \Omega \wedge \omega||^2.$$ 

Finally, since $Q$ is positive-definite on $P^{p,q}$ and since $c_1 > 0$ is large enough, we obtain

$$c'\alpha^2 \leq c_1 Q(\beta, \beta) = c_1 (Q(\alpha, \alpha) - Q(\omega \wedge \gamma, \omega \wedge \gamma)) \leq c_1 Q(\alpha, \alpha) + c_1 c'\alpha^2 \leq c_1 Q(\alpha, \alpha) + c_1 c^2||\gamma \wedge \Omega \wedge \omega^2||^2 \leq c_1 Q(\alpha, \alpha) + c_1 c^3||\alpha \wedge \Omega \wedge \omega||^2.$$ 

We then deduce the estimate in the proposition by taking $c_2 := c'c^2 + c_1c^3$. \qed


3 Lefschetz and Hodge-Riemann theorems

In this section, we prove Theorem 1.1. Corollary 1.2 is then deduced from that theorem and Proposition 2.2. We will use the results of the last section for \( E \) the complex cotangent space of \( X \) at a point and \( \omega \) the Kähler form on \( X \). So, we can define at every point of \( X \) a Hodge-Riemann cone for bidegree \((p, q)\). We now use the notation in Theorem 1.1. Let \( \mathcal{E}^{p,q} \) (resp. \( L^2_{p,q} \)) denote the spaces of smooth (resp. \( L^2 \)) forms on \( X \) of bidegree \((p, q)\).

**Proposition 3.1.** Assume that \( p, q \geq 1 \). Then, for every closed form \( f \in \mathcal{E}^{p,q}(X) \) such that \( \{f\} \in H^{p,q}(X, \mathbb{C})_{\text{prim}} \), there is a form \( u \in L^2_{p-1,q-1}(X) \) such that

\[
\dd^c u \wedge \Omega \wedge \omega = f \wedge \Omega \wedge \omega.
\]

**Proof.** Consider the subspace \( H \) of \( L^2_{n-p+1,n-q+1}(X) \) defined by

\[
H := \{ \dd^c \alpha \wedge \Omega \wedge \omega : \alpha \in \mathcal{E}^{q-1,p-1}(X) \}
\]

and the linear form \( h \) on \( H \) given by

\[
h(\dd^c \alpha \wedge \Omega \wedge \omega) := (-1)^{p+q+1} \int_X \alpha \wedge f \wedge \Omega \wedge \omega.
\]

We prove that \( h \) is a well-defined bounded linear form with respect to the \( L^2 \)-norm restricted to \( H \).

We claim that there is a constant \( c > 0 \) such that

\[
\|\dd^c \alpha\|_{L^2} \leq c \|\dd^c \alpha \wedge \Omega \wedge \omega\|_{L^2}.
\]

Indeed, we use the inequality in Proposition 2.8 applied to \( \dd^c \alpha \) instead of \( \alpha \) and the complex cotangent spaces of \( X \) instead of \( E \). Since \( X \) is compact, we can find common constants \( c_1 \) and \( c_2 \) for all cotangent spaces. We then integrate over \( X \) and obtain

\[
\|\dd^c \alpha\|_{L^2} \leq c_1 Q(\dd^c \alpha, \dd^c \alpha) + c_2 \|\dd^c \alpha \wedge \Omega \wedge \omega\|_{L^2}^2,
\]

where \( Q \) is defined in Section 1. Using Stokes’ formula, we obtain

\[
Q(\dd^c \alpha, \dd^c \alpha) = i^{n-p} (-1)^{(p+q)(p+q+1)/2} \int_X \dd^c \alpha \wedge \dd^c \alpha \wedge \Omega = 0.
\]

We then deduce easily the claim.

Now, by hypothesis the smooth form \( f \wedge \Omega \wedge \omega \) is exact. Therefore, there is a form \( g \in \mathcal{E}^{n-q,n-p}(X) \) such that

\[
\dd^c g = f \wedge \Omega \wedge \omega.
\]
see e.g. [2, p.41]. Using again Stokes’ formula and the above claim, we obtain

\[
\left| \int_X \alpha \wedge f \wedge \Omega \wedge \omega \right| = \left| \int_X \alpha \wedge d\partial^c g \right| = \left| \int_X d\partial^c \alpha \wedge g \right| \\
\leq \|g\|_{L^2} \|d\partial^c \alpha\|_{L^2} \leq c \|g\|_{L^2} \|d\partial^c \alpha \wedge \Omega \wedge \omega\|_{L^2}.
\]

It follows that \( h \) is a well-defined form whose norm in \( L^2 \) is bounded by \( c \|g\|_{L^2} \).

By Hahn-Banach theorem, we can extend \( h \) to a bounded linear form on \( L^2_{n-p+1,n-q+1}(X) \). Let \( u \) be a form in \( L^2_{p-1,q-1}(X) \) that represents \( h \). It follows from the definition of \( h \) that

\[
\int_X u \wedge d\partial^c \alpha \wedge \Omega \wedge \omega = (-1)^{p+q+1} \int_X \alpha \wedge f \wedge \Omega \wedge \omega = - \int_X f \wedge \alpha \wedge \Omega \wedge \omega
\]

for all test forms \( \alpha \in \mathcal{E}^{q-1,p-1}(X) \). The form \( u \) satisfies the proposition. \( \square \)

We have the following result.

**Proposition 3.2.** Let \( u \) be as in Proposition [2, p.41] Then there is a form \( v \in \mathcal{E}^{q-1,p-1}(X) \) such that \( d\partial^c v = d\partial^c u \).

**Proof.** We can assume without loss of generality that \( p \leq q \). The idea is to use the ellipticity of the Laplacian operator associated with \( \overline{\partial} \) and a special inner product on \( \mathcal{E}^{p,q}(X) \). We first construct this inner product. Fix an arbitrary Hermitian metric on the vector bundle \( \Lambda^{r,s}(X) \) of differential \((r,s)\)-forms on \( X \) with \((r,s) \neq (p,q)\) and denote by \( \langle \cdot, \cdot \rangle \) the associated inner product on \( \mathcal{E}^{r,s}(X) \).

Using the first assertion in Proposition [2, p.41], for any \( \alpha, \alpha' \in \mathcal{E}^{p,q}(X) \), we can write in a unique way

\[
\alpha = \beta + \omega \wedge \gamma \quad \text{and} \quad \alpha' = \beta' + \omega \wedge \gamma'
\]

with \( \beta, \beta' \in \mathcal{E}^{p,q}(X) \) and \( \gamma, \gamma' \in \mathcal{E}^{q-1,p-1}(X) \) such that \( \beta \wedge \Omega \wedge \omega = 0 \) and \( \beta' \wedge \Omega \wedge \omega = 0 \). Define an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{E}^{p,q}(X) \) by setting

\[
\langle \alpha, \alpha' \rangle := Q(\beta, \beta') + \langle \gamma, \gamma' \rangle = Q(\alpha, \beta') + \langle \gamma, \gamma' \rangle.
\]

This inner product is associated with a Hermitian metric on \( \Lambda^{p,q}(X) \).

Using the positivity of \( Q \) given in Proposition [2, p.41] we see that \( \langle \cdot, \cdot \rangle \) defines a Hermitian metric on \( \mathcal{E}^{p,q}(X) \). Consider now the norm \( \|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle} \). Then there is a constant \( c > 0 \) such that

\[
c^{-1}(\|\beta\|_{L^2} + \|\gamma\|_{L^2}) \leq \|\alpha\| \leq c(\|\beta\|_{L^2} + \|\gamma\|_{L^2}).
\]

Consider the \((p,q)\)-current \( h := d\partial^c u - f \) which belongs to a Sobolev space. We have

\[
\overline{\partial} h = 0, \quad \partial h = 0 \quad \text{and} \quad h \wedge \Omega \wedge \omega = 0.
\]

9
The last identity says that if we decompose \( h \) as we did above for \( \alpha, \alpha' \), the second component in the decomposition vanishes. Therefore, \( \langle \partial \alpha, h \rangle = Q(\partial \alpha, h) \) for any form \( \alpha \in \mathcal{E}^{p,q-1}(X) \). Using Stokes' formula, we obtain
\[
\langle \partial \alpha, h \rangle = Q(\partial \alpha, h) = i^{q-p}(-1)^{p+q-1+\frac{(p+q)(p+q+1)}{2}} \int_X \alpha \wedge \partial h \wedge \Omega = 0.
\]
If \( \overline{\partial} \) is the adjoint of \( \partial \) with respect to the considered inner products, we deduce that \( \overline{\partial} h = 0 \). On the other hand, \( \partial h = 0 \). Therefore, \( h \) is a harmonic current with respect to the Laplacian operator \( \partial \overline{\partial} + \overline{\partial} \partial \), see Section 5 in [17, Chap. IV]. Consequently, by elliptic regularity, \( h \) is smooth, see e.g. Theorem 4.9 in [17, Chap. IV]). Hence, \( dd^c u \) is smooth. We deduce the existence of \( v \in \mathcal{E}^{p-1,q-1}(X) \) such that \( dd^c v = dd^c u \), see e.g. [2, p.41].

**End of the proof of Theorem 1.1.** Let \( f \) be a closed form in \( \mathcal{E}^{p,q}(X) \) such that \( \{f\} \in H^{p,q}(X, \mathbb{C})_{\text{prim}} \). We first show that \( Q(\{f\}, \{f\}) \geq 0 \). Let \( v \) be the smooth \( (p-1,q-1) \)-form given by Proposition 3.2. Then we have
\[
(f - dd^c v) \wedge \Omega = 0.
\]
Here, we should replace \( dd^c v \) with 0 when either \( p = 0 \) or \( q = 0 \). Using Proposition 2.8 to each point of \( X \), after an integration on \( X \), we obtain
\[
i^{q-p}(-1)^{\frac{(p+q)(p+q+1)}{2}} \int_X (f - dd^c v) \wedge (\overline{f} - dd^c v) \wedge \Omega \geq 0.
\]
Using Stokes' formula and that \( f \) is closed, we obtain
\[
\int_X f \wedge \overline{f} \wedge \Omega = \int_X (f - dd^c v) \wedge (\overline{f} - dd^c v) \wedge \Omega.
\]
Therefore, \( Q(\{f\}, \{f\}) \geq 0 \). The equality occurs if and only if \( f = dd^c v \), i.e. \( \{f\} = 0 \). Hence, \( \{\Omega\} \) satisfies the Hodge-Riemann theorem for the bidegree \( (p,q) \).

We deduce that the map \( \{\alpha\} \mapsto \{\alpha\} \sim \{\Omega\} \) is injective on \( H^{p,q}(X, \mathbb{C})_{\text{prim}} \). If \( \{\alpha\} \) is a class in \( H^{p,q}(X, \mathbb{C}) \) such that \( \{\alpha\} \sim \{\Omega\} = 0 \), \( \{\alpha\} \) is a primitive class and hence \( \{\alpha\} = 0 \). Therefore, \( \{\Omega\} \) satisfies the hard Lefschetz theorem for the bidegree \( (p,q) \).

The classical hard Lefschetz theorem implies that \( \{\alpha\} \mapsto \{\alpha\} \sim \{\omega\} \) is an injective map from \( H^{p-1,q}(X, \mathbb{C}) \) to \( H^{p,q}(X, \mathbb{C}) \). Therefore,
\[
\dim\{\omega\} \sim H^{p-1,q-1}(X, \mathbb{C}) = \dim H^{p-1,q-1}(X, \mathbb{C}).
\]
This Lefschetz theorem also implies that \( \{\alpha\} \mapsto \{\omega\} \) is a surjective map from \( H^{n-q,n-p}(X, \mathbb{C}) \) to \( H^{n-q+1,n-p+1}(X, \mathbb{C}) \). This together with the hard Lefschetz theorem for \( \{\Omega\} \) yield
\[
\dim H^{p,q}(X, \mathbb{C})_{\text{prim}} = \dim H^{p,q}(X, \mathbb{C}) - \dim H^{n-q+1,n-p+1}(X, \mathbb{C})
= \dim H^{p,q}(X, \mathbb{C}) - \dim H^{p-1,q-1}(X, \mathbb{C})
= \dim H^{p,q}(X, \mathbb{C}) - \dim\{\omega\} \sim H^{p-1,q-1}(X, \mathbb{C}).
\]
The hard Lefschetz theorem can also be applied to \( \{ \Omega \wedge \omega^2 \} \) and to the bidegree \((p-1, q-1)\). We deduce that the intersection of \( \{ \omega \} \sim H^{p-1,q-1}(X, \mathbb{C}) \) and \( H^{p,q}(X, \mathbb{C})_{\text{prim}} \) is reduced to 0. This together with the above dimension computation gives us the following decomposition into a direct sum

\[
H^{p,q}(X, \mathbb{C}) = \{ \omega \} \sim H^{p-1,q-1}(X, \mathbb{C}) + H^{p,q}(X, \mathbb{C})_{\text{prim}}.
\]

Finally, the previous decomposition is orthogonal with respect to \( Q \) by definition of primitive space. So, \( \{ \Omega \} \) satisfies the Lefschetz decomposition theorem. \( \square \)

**Remark 3.3.** In order to obtain the Hodge-Riemann theorem and the hard Lefschetz theorem (resp. the Lefschetz decomposition), it is enough to assume the property \((\ast)\) in Definition 2.1 for \( r = 0, 1 \) (resp. \( r = 0, 1, 2 \)). When \((\ast)\) is satisfied for all \( r \), we can apply inductively these theorems to \( \Omega \wedge \omega^{2r} \) and then obtain the signature of \( Q \) on \( H^{p,q}(X, \mathbb{C}) \).

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T.-C. Dinh, UPMC Univ Paris 06, UMR 7586, Institut de Mathématiques de Jussieu, 4 place Jussieu, F-75005 Paris, France. 
dinh@math.jussieu.fr, [http://www.math.jussieu.fr/~dinh](http://www.math.jussieu.fr/~dinh)

V.-A. Nguyên, Mathématique - Bâtiment 425, Université Paris-Sud, 91405 Orsay, France. 
VietAnh.Nguyen@math.u-psud.fr.