SWITCHING EQUILIBRIA. The Present Value Model for Stock Prices Revisited

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ABSTRACT

This paper analyzes the different dynamic features displayed by alternative RE equilibria in the context of the present value for stock prices with feedback. In particular, it is shown that there exists a unique (bubble-free) RE equilibrium implying cointegration between stock prices and dividends and this unique equilibrium is characterized either by the fundamental or, alternatively, by the backward equilibrium solution depending on the size of the feedback parameter. Based on this result, it is illustrated a simple mechanism of switching equilibria caused by small changes in the dividend process parameters. Using historical US data and structural estimation, the hypothesis of feedback from stock prices to dividends is tested. In addition, we test for the presence of switching equilibria as suggested by Timmermann (1994). The empirical results provide evidence of a small but very significant presence of feedback from stock prices to dividends. Moreover, when analyzing different sub-samples we find evidence supporting the hypothesis of switching equilibria.
1. INTRODUCTION

The present value model of stock prices assuming rational expectations (RE) was extensively tested during the 1980’s (Campbell and Shiller (1987), Chow (1989), West (1988), among others). Many of these studies find US stock prices to be more volatile than implied by the present value model. These studies share two common assumptions. First, they assume a unique RE equilibrium for stock prices. Second, they consider that the dividend process has remained unchanged over the whole sample period. The relaxation of either of these assumptions provides a potentially good approach to explaining the excess volatility found in the literature.

Following the first approach, Froot and Obstfeld (1991) propose the first parsimonious present value model of stock prices that has found empirical support. Froot and Obstfeld characterize stock prices using a rational \textit{intrinsic} bubble which depends exclusively on dividends. Recently, Ackert and Hunter (1999) have shown that Froot and Obstfeld’s model is observationally equivalent to a present value model of stock prices that in addition explicitly includes control over dividends by managers. Also following the first approach, Timmermann (1994) shows that the existence of feedback in a present value model generates multiple (bubble-free) RE solutions. Moreover, Timmermann provides evidence that stock prices appear to Granger-cause dividends, which he interprets as evidence of feedback from stock prices to dividends.\footnote{Timmermann (1994, p.1109) recognizes that Granger-causality does not necessarily constitute proof of a feedback relation. However, in the light of a model with imperfectly and heterogenously informed agents, Timmermann argues that the evidence of stock prices Granger-causing dividends can be interpreted as evidence of feedback from stock prices to dividends.} As pointed out by Timmermann (1994), there are many ways of rationalizing this feedback. One possibility is that the feedback reflects the effect of stock prices on dividends through the cost of capital restriction faced by the firm. Another possibility is that stock prices may summarize private information in a context of asymmetric information. In this context, the dividend policy followed by a firm may be conditional on stock prices. Furthermore, Timmermann (1994, p.1114) suggests that the excess volatility observed in stock prices may be explained by switches among the set of RE equilibria. However, Timmermann neither explains which type of mechanisms may lead the economy to switch between RE equilibria nor empirically studies the existence of switching equilibria.

The second approach is followed by Driffill and Sola (1998) and Evans (1998). Using the present value model of stock prices model, Driffill and Sola provide evidence that a regime-switching model describing the evolution of dividends accounts for much of the variation of U.S. stock prices. In particular, Driffill and
Sola (1998) find evidence that the period from 1910 to 1955 was characterized by a high growth-low variance state for the dividend process whereas the period from 1955 to 1975 was characterized by a low growth-high variance state. However, neither Driffill and Sola (1998) nor Evans (1998) provide an answer to the question of what type of mechanisms are involved in the regime-switching of the dividend process which occurred around 1955.

This paper builds upon these two approaches. On the one hand, this paper builds on Timmermann’s work by assuming the presence of a feedback mechanism from stock prices to dividends. We show that the presence of the feedback mechanism produces three alternative RE equilibria that we call \( \alpha_1 \)-fundamental, \( \alpha_2 \)-fundamental and backward equilibria, respectively. Each equilibrium displays different dynamic properties that may change for small perturbations of the dividend process parameters. The rationale for these variations in dividend parameters can easily be understood. For instance, changes in the feedback parameter can be caused by either changes in the cost of capital restriction faced by the firm or changes in the way dividend policy is conditional on stock price, which may depend on stock price volatility relative to dividend volatility. The hypothesis that the weight given to a variable to forecast another variable depends on the relative volatility of the two variables was postulated by Muth (1961a) and Lucas (1973) in different contexts. Therefore, according to the Muth-Lucas hypothesis, one expects that the higher (lower) the stock price volatility is relative to dividend volatility, the lower (higher) the informational content given to stock prices must be when deciding on dividends.

A key contribution of this paper is the proposition that there exists a unique (bubble-free) RE equilibrium implying cointegration between stock prices and dividends and this unique equilibrium is characterized either by the \( \alpha_2 \)-fundamental or, alternatively, by the backward equilibrium solution depending on the size of the feedback parameter. In particular, this proposition implies that the variance of the spread between stock prices and dividends, as defined by Campbell and Shiller (1987), is not fully characterized by the cointegration relationship until the dividend process is fully specified. This proposition together with Muth-Lucas hypothesis may illustrate a simple mechanism of switching equilibria as follows. According to the Muth-Lucas hypothesis a variation in stock price volatility (relative to dividend volatility) may induce a change in the
size of the feedback parameter that may lead to a switch between the backward and the \( \alpha_2 \)-fundamental equilibria.

This paper also builds on Drifill and Sola (1998) by taking into account the regime-switches and the resulting sub-samples detected by Drifill and Sola. We argue that any dividend process assumed in empirical studies in order to test the stock price model should be viewed as a reduced form that summarizes both behavioral relationships and economic policy rules. Therefore, the parameters characterizing the reduced form of dividends are likely to vary over time. These considerations suggest that a fair test of the RE stock prices model should be carried out by taking into account the possibility of changes in the dividend process parameters.

Given the evidence on cointegration between stock prices and dividends found and the error-correction structure implied by the present value model for stock prices under cointegration, we estimate the model in two steps. First, we estimate the error-correction representation of the cointegrated system in order to summarize the dynamic features exhibited by stock prices and dividends. Thus, the error-correction representation is the auxiliary model used to capture the empirical regularities displayed by the data. Second, we apply the simulated moments estimator (SME) suggested by Lee and Ingram (1991) and Duffie and Singleton (1993) to estimate the parameters of the stock prices model and to test the hypothesis of switching equilibria in the stock price-dividend relationship. Using annual data for the US, the empirical results show a good fit of the present value model for the sub-samples 1910-1955 and 1955-1975. Moreover, the empirical results provide evidence supporting the hypothesis of switching equilibria that explains the different stock price-dividend relationship observed in the sub-samples 1910-1955 and 1955-1975. Furthermore, we find evidence of a small but very significant presence of feedback from stock prices to dividends. This feedback is smaller in those periods in which the sample stock price volatility relative to dividend volatility is higher. All the evidence supporting the hypothesis of switching equilibria together with the evidence of an increase in the sample volatility of stock prices and the decrease of the feedback parameter, all changes distinguishing the sub-samples 1910-1955 and 1955-1975, are consistent with the simple mechanism of switching equilibria described above.

Another important contribution of this paper is to bring together the two recent approaches followed in the literature to explain the excess volatility of stock prices. By assuming that there is a feedback relationship from stock prices to dividends, our paper formally shows how switches between alternative RE equilibria for stock prices may also lead to regime-switches in the autoregressive-moving average (ARMA) representation characterizing the dividends. These results point out the theoretical possibility that the regime-switching in the
ARMA representation of the dividend process detected by Drifill and Sola (1998) and Evans (1998) may be due to switches between alternative RE equilibria. Although we focus our attention on switching equilibria, we do not view regime-switching in the dividend process and switching equilibria as two competing hypotheses for explaining stock price volatility. In fact, the analysis carried out in Sections 3 and 4 brings up the theoretical possibility that regime-switching in the dividend process and switching equilibria may interact in characterizing stock price volatility.

The rest of the paper is organized as follows. Section 2 introduces the present value model for stock prices and obtains the alternative RE solutions. Section 3 characterizes the dynamic properties displayed by the alternative RE equilibria in the space of the parameters describing the dividend process. Section 4 shows that switches between alternative RE equilibria lead to regime-switching in the ARMA representation of dividend process. Section 5 describes the SME and the empirical evidence found. Section 6 concludes.

2. THE PRESENT VALUE MODEL FOR STOCK PRICES

The RE present value model for stock prices states the following relationship between stock prices and dividends:

\[ p_t = \delta E_t (p_{t+1} + d_t) + u_t, \]  

where \( p_t \) is the real stock price at the beginning of time \( t \), \( d_t \) is the real dividend obtained from the stock during period \( t \), \( 0 < \delta < 1 \) is the constant discount factor and \( E_t \) denotes the conditional expectation operator given the information set, \( I_t \), available to the economic agents at the beginning of time \( t \). \( I_t \) includes current and past values of all random variables of the model, but the current value of dividends, \( d_t \), whose realization occurs during the period.\(^4\) \( u_t \) is a random measurement error term. We assume that \( u_t \) is an i.i.d. random error with mean zero and variance \( \sigma_u^2 \).

The present value model is completely characterized by specifying the process followed by the dividends. We assume a rather general process for dividends,

\[ d_t = \rho_0 + \rho_1 p_t + \rho_2 d_{t-1} + v_t, \]  

where \( \rho_1 \) and \( \rho_2 \) are both included in the interval \([0, 1]\), and \( v_t \) is an i.i.d. random variable with mean zero and variance \( \sigma_v^2 \). By assuming that \( 0 \leq \rho_2 \leq 1 \), the

\(^4\)In some papers, e.g. West (1988), dividends are dated at \( t + 1 \) instead of \( t \) in (1). It must be clear that this distinction is meaningless whenever it is assumed that \( d_t \) (or equivalently \( d_{t+1} \)) does not belong to the information set at the start of period \( t \), but does belong to the information set at the start of period \( t + 1 \).
dividend process postulates some inertia, since high (low) past dividends will generally lead to high (low) current dividends. \( v_t \) is not included in \( I_t \) because \( d_t \) is not included either. Moreover, equation (2.2) allows for the presence of a positive feedback from stock prices to dividends. As discussed at length by Timmermann (1994, p.1094), there are many ways of rationalizing this feedback.\(^5\)

One possibility is that the feedback may reflect the effect of stock prices on dividends via the cost of capital restriction faced by the firm. Another possibility is that stock prices may summarize private information in a context of asymmetric information. In this context, the dividend policy followed by a firm may be conditional on stock prices. Obviously, the feedback parameter \( \rho_1 \) may vary over time because the cost of capital restriction faced by a firm changes and/or because the manner in which dividend decisions are conditional on stock price changes depending on stock price volatility. As stated above, the Muth-Lucas hypothesis suggests that the higher (lower) the volatility of stock prices is, the lower (higher) the informational content given to stock prices must be when dividend decisions are made.

When estimating the model below in Section 5, we consider the possibility of more lags characterizing the dividend process as in the following expression

\[
d_t = \rho_0 + \rho_1 p_t + \rho_2 d_{t-1} + \rho_3 p_{t-1} + \rho_4 d_{t-2} + v_t. \tag{2'}
\]

We also assume that \( u_t \) and \( v_t \) follow AR(1) processes given by

\[
u_t = \tau_0 u_{t-1} + z_t, \quad v_t = \tau_1 v_{t-1} + s_t, \text{ respectively; where } \tau_0 \text{ and } \tau_1 \text{ are both included in the interval } [-1, 1], \text{ and } z_t \text{ and } s_t \text{ are i.i.d. random errors with mean zero and variance } \sigma_z^2 \text{ and } \sigma_s^2, \text{ respectively.}
\]

Equations (2.1) and (2.2) form a bivariate system of difference equations. Using the undetermined coefficient method (Muth (1961b), McCallum (1983) among others) we begin by writing \( p_t \) as a linear function of \( u_t \) and the predetermined state variable \( d_{t-1} \), plus a constant,

\[
p_t = \pi_0 + \pi_1 d_{t-1} + \pi_2 u_t. \tag{2.3}
\]

For appropriate real values of \( \pi_0, \pi_1 \) and \( \pi_2 \), the expectational variable \( E_t p_{t+1} \) will then be given by

\[
E_t p_{t+1} = \pi_0 + \pi_1 E_t d_t = \pi_0 + \pi_1 \rho_0 + \pi_1 (1 + \rho_1) + \pi_1 \rho_1 + \pi_2 d_{t-1} + \pi_2 v_{t-1} + \pi_2 u_t. \tag{2.4}
\]

To evaluate the \( \pi \)'s, we substitute (2.2), (2.3) and (2.4) into (2.1), which gives

\[
\pi_0 + \pi_1 d_{t-1} + \pi_2 u_t = \delta \left[ \pi_0 (1 + \rho_1) + \pi_1 (1 + \rho_1) + \rho_0 (1 + \rho_1) \right] + \]

\(^5\)As is made clear throughout the paper, the dynamics of stock prices depend crucially on the presence of feedback from stock prices to dividends. Timmermann (1994, pp. 1103-1106) provides two examples of optimizing models of the stock market which exhibit feedback.
\[ \delta[\pi_1(\pi_1\rho_1 + \rho_2 + \rho_1) + \rho_2]dt_{t-1} + [1 + \delta\rho_1\pi_2(1 + \pi_1)]u_t. \]

This equation implies identities in the constant term, \(dt_{t-1}\) and \(u_t\) as follows:

\begin{align*}
\pi_0 & = \delta[\pi_0(1 + \rho_1 + \pi_1\rho_1) + \rho_0(1 + \pi_1)], \\
\pi_1 & = \delta[\pi_1(\pi_1\rho_1 + \rho_2 + \rho_1) + \rho_2], \\
\pi_2 & = 1 + \delta\rho_1\pi_2(1 + \pi_1).
\end{align*}

(2.5)

After some algebra, we can show that there are two solutions to the system of equations (2.5),

\begin{align*}
\pi^1 & = (\pi^1_0, \pi^1_1, \pi^1_2) = [\tau_1, \alpha_1, \varphi_1], \\
\pi^2 & = (\pi^2_0, \pi^2_1, \pi^2_2) = [\tau_2, \alpha_2, \varphi_2],
\end{align*}

(2.6) (2.7)

where

\begin{align*}
\alpha_1 & = \frac{1}{2\delta\rho_1} \left(1 - \delta(\rho_1 + \rho_2) + ([1 - \delta(\rho_1 + \rho_2)]^2 - 4\delta^2\rho_1\rho_2)^{1/2}\right), \\
\alpha_2 & = \frac{1}{2\delta\rho_1} \left(1 - \delta(\rho_1 + \rho_2) - ([1 - \delta(\rho_1 + \rho_2)]^2 - 4\delta^2\rho_1\rho_2)^{1/2}\right), \\
\tau_i & = \frac{\delta\rho_0(1 + \alpha_i)}{1 - \delta - \delta\rho_1(1 + \alpha_i)}, \\
\varphi_i & = \frac{1}{1 - \delta\rho_1(1 + \alpha_i)}, \\
& \quad i = 1, 2.
\end{align*}

In addition to RE solutions (2.6) and (2.7), the present value model for stock prices, equation (2.1), exhibits another RE equilibrium solution. This alternative solution is obtained by following the backward approach for solving linear RE models (see Broze and Szafarz (1991, ch.2)). This approach starts by using the premise that the RE of stock prices and dividends at period \(t\) are given by

\begin{align*}
E_t p_{t+1} & = p_{t+1} - \epsilon_{t+1}, \\
E_t d_t & = d_t - v_t,
\end{align*}

(2.8) (2.9)

respectively; where \(\epsilon_{t+1}\) (the rational prediction error) is an arbitrary martingale difference with respect to agents’ information set at period \(t\), \(I_t\). By using (2.8), (2.9) and rearranging, the present value model (2.1) can be written as

\[ p_t = \delta^{-1}p_{t-1} - d_{t-1} - \delta^{-1}u_{t-1} + v_{t-1} + \epsilon_t. \]

(2.10)

We refer to solution (2.10) as the backward solution to the present value model. Solutions (2.6) and (2.7) are called fundamental solutions in the sense that the
two solutions are only linear functions of a minimal set of state variables: $d_{t-1}$ and $u_t$. Notice that the fundamental solutions do not include variables such as $p_{t-1}$, $v_{t-1}$ and $\epsilon_t$, which enter into the backward solution. From now on we refer to solutions (2.6) and (2.7) as the $\alpha_1$-fundamental and $\alpha_2$-fundamental solutions, respectively.

At this point of the analysis, it is worth making some remarks on the set of RE solutions (equations (2.6), (2.7) and (2.10)):

**Remark 1.** The backward solution, (2.10), is a general RE solution for stock prices because it does not depend on the process followed by the dividends. Moreover, any particular solution of the present value model (2.1) (for instance, the fundamental solutions) satisfies (2.10).6

**Remark 2.** In spite of the previous remark, it must be clear that the time series obtained from the three alternative RE solutions display very different dynamic properties. In particular, as shown below in Propositions 2 and 3, the variance of stock prices differs considerably depending on which RE equilibrium characterizes stock price dynamics.

**Remark 3.** Fundamental solutions (2.6) and (2.7) satisfy condition (2.8) even though it was not imposed to derive them. That is, $\alpha_1$-fundamental and $\alpha_2$-fundamental solutions are particular RE solutions with the martingale difference term being characterized by a specific linear function of the measurement error $u_t$ (see footnote #6).

**Remark 4.** Fundamental solutions (2.6) and (2.7) only exist when7

$$[1 - \delta(\rho_1 + \rho_2)]^2 - 4\delta^2 \rho_1 \rho_2 \geq 0,$$

(2.11)

that is, when $\alpha_1$ and $\alpha_2$ are both real numbers.

**Remark 5.** Even without considering the fundamental solutions, the non-uniqueness issue still remains, since the backward solution represents an infinite number of RE equilibria indexed by the arbitrary martingale difference $\epsilon_t$.8

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6It can be shown that the fundamental solutions, (2.6) and (2.7), satisfy the linear difference equation (2.10) with the difference martingale given by

$$\epsilon_t = \frac{1}{1 - \delta \rho_i (1 + \alpha_i)} u_t,$$

for $i = 1, 2$, respectively.

7Following McCallum (1983, p.146, footnote #9), we implicitly believe that if some of the equilibrium solutions are real and the others are complex, then the latter are irrelevant because they are not economically sensible solutions.

8In order to simplify the discussion below, we focus our attention on the backward equilibrium solution characterized by $\epsilon_t = 0$ for all $t$. 

9
Remark 6. In the particular case where $\rho_1 = 0$ (that is, the dividends are exogenous), the $\alpha_1$-fundamental solution does not exist and the $\alpha_2$-fundamental solution can be expressed as

$$p_t = \frac{\delta \rho_0}{(1 - \delta \rho_2)(1 - \delta)} + \frac{\delta \rho_2}{1 - \delta \rho_2} d_{t-1} + u_t.$$ 

3. A CHARACTERIZATION OF THE DYNAMIC PROPERTIES OF ALTERNATIVE EQUILIBRIA

Although we estimate the present value model for stock prices in Section 5 assuming that stock prices and dividends are cointegrated process, we do not impose the cointegration restriction in this section, in order to provide a complete characterization of how the dynamics properties displayed by alternative equilibria may change for small perturbations of the dividend process parameters. This characterization allows us to illustrate mechanisms of switching equilibria induced by changes in the dividend process parameters that lead to small departures from the relationship imposed by the cointegration restriction.

The following proposition establishes the region in which only the backward solution exists because the others are complex.

**Proposition 1.** For all combinations of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ such that inequality (2.11) holds the three solutions considered exist, otherwise only the backward solution exists. Moreover, the set of values of $\rho_1$ and $\rho_2$ for which only the backward solution exists is empty for $\delta \leq 1/4$. The set of values of $\rho_1$ and $\rho_2$ for which the three solutions exist is non-empty for any $0 < \delta < 1$.

**Proof.** : See Appendix 1.

Figure 1 summarizes the results stated in Proposition 1. For a given $0 < \delta < 1$, the shaded region displays the combinations of values for $\rho_1$ and $\rho_2$ for which the RE equilibrium is characterized only by the backward solution (2.10) (that is, when inequality (2.11) is not satisfied). Notice that the higher (lower) the discount factor, the larger (smaller) is the region in which only the backward solution exists.

The following proposition states when the two fundamental solutions are stationary. Moreover, it is shown that the $\alpha_2$-fundamental solution exhibits a lower variance than the $\alpha_1$-fundamental solution.

**Proposition 2.** Let us assume that $\mu_i = \rho_2 + \alpha_i \rho_1$ is a real variable for $i = 1, 2$. Then fundamental solutions (2.6) and (2.7) are stationary if $\mu_i < 1$ for $i = 1, 2$. 

10
Figure 3.1: Existence of alternative equilibria

\[ \rho_1 = \rho_2 \]

\[ [1 - \delta (\rho_1 + \rho_2)] - 4 \delta \rho_1 \rho_2 = 0 \]

Figure 3.2: Features of alternative equilibria

\[ \rho_2 = a_2 \rho_1 = 1 \]

\[ [1 - \delta (\rho_1 + \rho_2)] - 4 \delta \rho_1 \rho_2 = 0 \]
respectively. If this is the case, the variance of stock prices under the fundamental solutions, (2.6) and (2.7), is given by

$$\lambda^i_0 = \frac{\alpha^2_i}{1 - \mu^2_i} \sigma^2_v + \frac{1 - \mu^2_i + \alpha^2_i \rho^2_i}{1 - \mu^2_i} \varphi^2_1 \sigma^2_u,$$  \hspace{1cm} (3.1)

for $i = 1, 2$, respectively. Furthermore, the variance of stock prices for the $\alpha_2$-fundamental solution, (2.7), is always lower than the variance of stock prices for the $\alpha_1$-fundamental solution, (2.6).

**Proof.** : See Appendix 1.

Figure 2 illustrates the regions in which the $\alpha_1$-fundamental and $\alpha_2$-fundamental solutions are stationary. Curve $\mu_1 = \mu_2$ displays combinations of $\rho_1$ and $\rho_2$ such that $[1 - \delta (\rho_1 + \rho_2)]^2 - 4\delta \rho_1 \rho_2 = 0$. Therefore, region $B$ is the area where only the backward solution is defined. The segment connecting the points $(\rho_1, \rho_2) = \left(\frac{1 - \delta}{\delta}, 0\right)$ and $(\rho_1, \rho_2) = \left(\frac{(1 - \delta)^2}{\delta}, \delta\right)$ displays the pairs $(\rho_1, \rho_2)$ such that $\mu_1 = 1$. Points located to the right of this segment imply combinations of $\rho_1$ and $\rho_2$ for which the $\alpha_1$-fundamental solution is stationary. Otherwise the $\alpha_1$-fundamental solution is not stationary. The segment from $(\rho_1, \rho_2) = (0, 1)$ to $(\rho_1, \rho_2) = \left(\frac{1 - \delta^2}{\delta}, \delta\right)$ displays combinations of $\rho_1$ and $\rho_2$ such that $\mu_2 = 1$. Points above this segment result in combinations of $\rho_1$ and $\rho_2$ for which the $\alpha_2$-fundamental solution is not stationary; otherwise, the $\alpha_2$-fundamental solution is stationary. All these statements are proved in Appendix 2.\footnote{Notice that Figure 2 represents a case where $\delta > 1/2$, otherwise point $\left(\frac{1 - \delta}{\delta}, 0\right)$ would lie on the right of (1,0).}

Summarizing, in the area in which the two fundamental solutions exist (all regions but $B$), we can distinguish three different regions ($A$, $C$, and $D$) depending upon the stationary characteristics of these solutions. In particular, the two fundamental solutions are stationary in region $C$, whereas only the $\alpha_2$-fundamental solution is stationary in region $A$. However, region $D$ shows the pairs $(\rho_1, \rho_2)$ for which none of the fundamental solutions is stationary.

The following proposition establishes the conditions under which the backward equilibrium solution (2.10) is stationary.

**Proposition 3. Let us assume that $\mu_i = \rho_2 + \alpha_i \rho_1$ is a real variable for $i = 1, 2$. If $\mu_i < 1$ for $i = 1, 2$, then the backward solution, (2.10), is stationary. In this case, the variance of stock prices characterized by the backward solution, $\lambda^b_0$, is given by

$$\lambda^b_0 = \frac{(1 + \mu_1 \mu_2) \rho^2_2}{(1 - \mu^2_1)(1 - \mu^2_2)(1 - \mu_1 \mu_2)} \sigma^2_v + \frac{(1 + \mu_1 \mu_2)(1 + \rho^2_2) - 2\rho_2(1 + \mu_2)(1 - \mu_1 \mu_2) \delta^2}{(1 - \mu^2_1)(1 - \mu^2_2)(1 - \mu_1 \mu_2)} \sigma^2_u.$$

$$\hspace{1.5cm} (3.2)$$
Proof. See Appendix 1.

Figure 2 can also be interpreted in terms of the stationary conditions of the backward solution. We show in Appendix 2 that, considering the area where the three solutions exist (all regions but \( B \)), the backward solution is only stationary in region \( C \).\(^{10}\)

Taking into account the results stated in Propositions 1-3, we can classify the combinations of the \( 0 < \rho_1 < 1 \) and \( 0 < \rho_2 < 1 \) in the following regions displayed in Figure 2: in region \( D \) all three solutions (the two fundamentals and the backward solution) exist but none of them is stationary. In region \( A \), only the \( \alpha_2 \)-fundamental solution is stationary. In region \( C \), the three solutions are stationary, but the \( \alpha_1 \)-fundamental solution always has a larger variance than the \( \alpha_2 \)-fundamental solution. Finally, in region \( B \), only the backward solution exists because the fundamental solutions are complex solutions.

Parameters \( \mu_i \) defined on the above propositions can be interpreted by looking at the difference equation system formed by (2.1) and (2.2). System (2.1) and (2.2) can be expressed in matrix form as

\[
A_0 + A_1 Y_t + A_2 Y_{t+1} + A_3 U_{t+1} = 0,
\]

where

\[
Y_t = \begin{pmatrix} p_t \\ d_{t-1} \end{pmatrix}, \quad U_{t+1} = \begin{pmatrix} u_t \\ p_{t+1} - E_t p_{t+1} \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 \\ \rho_0 \end{pmatrix},
A_1 = \begin{pmatrix} 1 \\ -\rho_1 \\ -\rho_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\delta & -\delta \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & \delta & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Operating and rearranging, this matrix system can be written as

\[
Y_{t+1} = BY_t + CU_{t+1} + D, \quad (3.3)
\]

where \( B = -A_2^{-1} A_1 \), \( C = -A_2^{-1} A_3 \) and \( D = -A_2^{-1} A_0 \).

It is easy to show that \( \mu_i \) for \( i = 1, 2 \) are the eigenvalues of transition matrix \( B \). In this context the areas illustrated in Figure 2 can be interpreted in terms of the stationary properties of the difference equation system (3.3): \( i \) If no equilibrium solution is stationary (that is, in area \( D \)), difference equation system (3.3) is a source; \( ii \) when only the \( \alpha_2 \)-fundamental solution is stationary (that is, the parameters of \( \rho_1 \) and \( \rho_2 \) lie in area \( A \)), difference equation system (3.3) is a saddle path; \( iii \) if all three equilibrium solutions are stationary (that is, in area \( C \)),

\(^{10}\)The backward solution is not stationary in region \( D \) because \( \mu_i > 1 \), for \( i = 1, 2 \) and it is not stationary in region \( A \) since \( \mu_1 > 1 \).
The difference equation system (3.3) is a *sink*. Moreover if either $\mu_1$ or $\mu_2$ is equal to one, then stock prices and dividends are cointegrated because in this case $|B - I| = 0$. It is easy to show that if $\mu_i = \rho_2 + \alpha_i \rho_1 = 1$, then $\rho_2 = \frac{1}{1 - \delta} \rho_1$ and the cointegrated vector is $(1, -\delta / (1 - \delta))$. The latter result simply says that the *spread* between stock prices and dividends (as defined by Campbell and Shiller (1987)) $p_t - (\delta / (1 - \delta)) d_{t-1}$ is stationary when $p_t$ and $d_{t-1}$ are cointegrated.

The following proposition states the properties of the solutions when prices and dividends are cointegrated.

**Proposition 4.** Let us define $\mu_i = \rho_2 + \alpha_i \rho_1$ for $i = 1, 2$. If $\mu_i = 1$ for some $i = 1, 2$ then the $\alpha_1$-fundamental solution does not exist and the $\alpha_2$-fundamental solution takes the form

$$
\pi^2 = (\pi_0^2, \pi_1^2, \pi_2^2) = \left( \frac{-\delta \rho_0}{(1 - \delta)(\delta - \rho_2)}, \frac{\delta}{1 - \delta}, \frac{1}{\rho_2} \right).
$$

Furthermore among all solutions there exists a unique equilibrium implying cointegration between stock prices and dividends with two possible scenarios:

i) If $\mu_1 > \mu_2$, the backward equilibrium solution implies that stock prices and dividends are cointegrated whereas the $\alpha_2$-fundamental solution is stationary.

ii) If $\mu_1 > \mu_2 = 1$, the $\alpha_2$-fundamental solution implies that stock prices and dividends are cointegrated whereas the backward equilibrium solution is explosive.

**Proof.** See Appendix 1.

Notice that the first case characterized in this proposition corresponds to all combinations of $\rho_1$ and $\rho_2$ which belong to the segment connecting the points $(\rho_1, \rho_2) = (1 - \delta / \delta, 0)$ and $(\rho_1, \rho_2) = (1 - \delta^2 / \delta, \delta)$. The second case collects all combinations of $\rho_1$ and $\rho_2$ in the segment $(\rho_1, \rho_2) = (1 - \delta^2 / \delta, \delta)$ and $(\rho_1, \rho_2) = (0, 1)$.

The following corollary summarizes the results of all propositions:

**Corollary 1.** Let us define $\mu_i = \rho_2 + \alpha_i \rho_1$. Given that $\mu_1 \geq \mu_2$, any combination of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ can be classified in one of the following cases depending on whether or not the alternative solutions are stationary:

i) If $\mu_1$ and $\mu_2$ are complex, the backward solution is the only real equilibrium solution.
ii) If $\mu_1$ and $\mu_2$ are real and such that $\mu_1 > 1$, and $\mu_2 > 1$, all three equilibrium solutions exist, but they are explosive.

iii) If $\mu_1$ and $\mu_2$ are real and such that $\mu_1 < 1$, and $\mu_2 < 1$, all three alternative equilibrium solutions are stationary.

iv) If $\mu_1$ and $\mu_2$ are real and such that $\mu_1 > 1$, and $\mu_2 < 1$, only the $\alpha_2$-fundamental equilibrium solution is stationary.

v) If $\mu_1 = 1 > \mu_2$, the backward equilibrium solution implies that stock prices and dividends are cointegrated whereas the $\alpha_2$-fundamental solution is stationary and the $\alpha_1$-fundamental solution does not exist.

vi) If $\mu_1 > \mu_2 = 1$, the $\alpha_2$-fundamental solution implies that stock prices and dividends are cointegrated whereas the backward equilibrium solution is explosive and the $\alpha_1$-fundamental solution does not exist.

Proof. This is straightforward from Propositions 1, 2, 3 and 4.

Notice that statements v) and vi) in Corollary 1 establish that there is a unique equilibrium solution implying cointegration between stock prices and dividends (either the backward solution when $\mu_1 = 1$ or the $\alpha_2$-fundamental solution when $\mu_2 = 1$). Given the evidence on cointegration between stock prices and dividends found in Section 5 we show a simple mechanism that may lead to switches between the backward and the $\alpha_2$-fundamental equilibria triggered by changes in stock price volatility. First, consider that the economy is initially located at a point such as $X$ in Figure 2. At this point, as previously shown in Corollary 1, the only equilibrium implying cointegration between stock prices and dividends is characterized by the backward equilibrium solution.\(^{11}\) Now assume that there is an increase in stock price volatility relative to dividend volatility. Based on the Muth-Lucas hypothesis this increase may induce a sufficient decrease in the value of the feedback parameter, $\rho_1$, such that the economy is now located at a point such as $Y$ in Figure 2 where the only equilibrium implying cointegration is characterized by the $\alpha_2$-fundamental solution. A second switching equilibria mechanism is just the opposite. If the economy is initially at a point such as $Y$ a reduction of stock price volatility may induce a sufficient increase in the value of $\rho_1$, leading the economy to a point such as $X$ which induces a switching

\(^{11}\)Recall that the $\alpha_2$-fundamental solution at a point such as $X$ does not imply cointegration between stock prices and dividends.
The existence of switching equilibria provides an additional source of variation in stock prices. In addition to fluctuations in stock prices caused by innovations in the dividend process, stock price movements can be determined by changes in the parameters characterizing the dividend process that may lead to switches between alternative RE equilibria.

In order to understand how switching equilibria induced by small changes in the feedback parameter $\rho_1$ may imply large stock market swings under the hypothesis of cointegration between stock prices and dividends is useful to obtain the expressions of the first-differences of stock prices under the $\alpha_2$-fundamental and the backward equilibrium solutions, denoted by $\Delta p_t^f$ and $\Delta p_t^b$, respectively. Formally, it is easy to show that

$$\Delta p_t^f = \frac{\delta \rho_0(\delta - \rho_2 - 1)}{(1 - \delta)(\delta - \rho_2)} - \rho_2(p_{t-1} - \frac{\delta}{(1 - \delta)}d_{t-2}) + \frac{1}{\rho_2}u_t + \frac{\delta}{(1 - \delta)}v_{t-1},$$

$$\Delta p_t^b = -\rho_0 + \rho_2 \frac{(1 - \delta)}{\delta}(p_{t-1} - \frac{\delta}{(1 - \delta)}d_{t-2}) - \frac{1}{\delta}u_t + v_{t-1}.$$

By comparing these two expressions, we observe that the response of $\Delta p_t$ to changes in the spread differs widely depending on which equilibrium solution characterizes the unique (bubble-free) equilibrium under the hypothesis of cointegration. Thus, an increase of the spread decreases $\Delta p_t$ under the $\alpha_2$-fundamental equilibrium, whereas it increases $\Delta p_t$ under the backward solution. Therefore, a small change in the feedback parameter, $\rho_1$ (recall that this change implies a change in $\rho_2$ under the hypothesis of cointegration since $\rho_2 = 1 - \frac{\delta}{(1 - \delta)}\rho_1$), if it induces a switching equilibria, implies a large change in stock prices. To see this more precisely, calculate the difference between the coefficients associated with the spread under the backward and $\alpha_2$-fundamental equilibrium solutions.

---

12 Given the evidence on cointegration, we have focused only on switches between equilibria characterized by the cointegration restriction. However, when the cointegration restriction does not hold, one can illustrate many examples of mechanisms leading to switching equilibria. For instance, consider that the economy is located at a point such as $W$ in Figure 2 where the equilibrium is characterized by the $\alpha_2$-fundamental solution since the backward solution and the $\alpha_1$-fundamental solution are explosive at $W$. Now assume that there is a change in the value(s) of $\rho_1$ or/and $\rho_2$ such that condition (2.11) does not hold (for instance, a switch from $W$ to $Z$ in Figure 2). This change in the value(s) of the parameter(s) characterizing the dividend process triggers a jump from the equilibrium described by the $\alpha_2$-fundamental solution to the equilibrium characterized by the backward solution since, as stated in Remark 4, the $\alpha_2$-fundamental solution only exists when condition (2.11) holds.

13 Notice that the cointegration restriction imposes a close relationship between the three structural parameters. This restriction implies that $\rho_2$ has to adjust in a particular manner when the feedback parameter, $\rho_1$, changes.
denoted by $\tau$:
\[
\tau = \rho_2 \left( \frac{1 - \delta}{\delta} \right) - (-\rho_2) = \frac{\rho_2^2 - \delta}{\delta} > 0.
\]

Using the cointegration restriction, this difference can be written in terms of $\rho_1$ as
\[
\tau = \frac{1}{\delta} - \frac{\rho_1}{1 - \delta}
\]

Then,
\[
\frac{\partial \tau}{\partial \rho_1} \bigg|_{\rho_2=1-\delta/\rho_1} = -\frac{1}{(1 - \delta)} < 0.
\]

Since the discount factor, $\delta$, takes a value smaller than (but close to) one the latter expression shows that a small change in $\rho_1$, if it induces switching equilibria, implies a large change in the coefficient associated with the spread, strongly affecting how stock prices respond to changes in the spread. In particular, if $\rho_1$ decreases, inducing a switching from the backward solution to the $\alpha_2$-fundamental solution, this reduction implies a large fall in the way stock prices respond to a change in the spread (that is, an increase in $\tau$).

Next, it is shown that switching equilibria not only alters the way stock prices react to changes in the spread, but also implies changes in the variance of the spread. When stock prices and dividends are cointegrated these two variables follow integrated processes and there no second moments of these variables. In this context, an alternative to studying the size of fluctuations in the stock market by looking at $\Delta p_t$ is to analyze the spread between stock prices and dividends since, as pointed out above, the spread is stationary when stock prices and dividends are cointegrated. The following proposition characterizes the spread for the two alternative equilibrium solutions when stock prices and dividends are cointegrated. Denote the spread at period $t$ by $sp_t$.

**Proposition 5.** Assuming that stock prices and dividends are cointegrated:

i) the unconditional mean and variance of the spread under the $\alpha_2$-fundamental equilibrium, $sp_t^f$, are given by
\[
E(sp_t^f) = \frac{-\delta \rho_0}{(1 - \delta)(\delta - \rho_2)},
\]
\[
var(sp_t^f) = \frac{1}{\rho_2^2} \sigma_u^2,
\]
respectively.
ii) the unconditional mean and variance of the spread under the backward equilibrium, $sp_t^b$, are given by

$$E(sp_t^b) = \frac{-\delta \rho_0}{(1-\delta)(1-\delta^2)},$$

$$\text{var}(sp_t^b) = \frac{1}{(\delta^2 - \rho_2^2)}\sigma_u^2 + \frac{\delta^4}{(1-\delta)(1-\delta^2)(\delta^2 - \rho_2^2)}\sigma_v^2,$$

respectively.

**Proof.** See Appendix 1.

Notice that the variance of the spread under the backward solution is only well defined when $\delta > \rho_2$ (that is, along the line connecting the points $(\frac{1-\delta}{\delta}, 0)$ and $(\frac{(1-\delta)^2}{\delta}, \delta)$ in Figure 2. Proposition 5 shows that switching equilibria induces only changes in the functional form of the variance of the spread since the mean of the spread is the same under the two alternative solutions. In particular, the variance of the spread under the $\alpha_2$-fundamental equilibrium does not depend on the volatility of the innovations associated with the dividend process, $\sigma_v^2$. However, the variance of the spread under backward equilibrium is a linear combination of the variances of $u_t$ and $v_t$.

4. REGIME-SWITCHING IN THE DIVIDEND PROCESS

By assuming that there is a feedback relationship from stock prices to dividends, this section shows how switches between alternative RE equilibria for stock prices may also lead to regime-switches in the ARMA representation characterizing the dividend process.

By plugging fundamental solutions (2.6) and (2.7) into equation (2.2), we obtain ARMA representations for the dividend process under the two alternative fundamental solutions

$$d_t = \frac{\rho_0(1-\delta)}{(1-\delta) - \delta \rho_1 (1+\alpha_i)} + \mu_i d_{t-1} + \frac{\rho_1}{1-\delta \rho_1 (1+\alpha_i)} u_t + v_t,$$

where $\mu_i = \rho_2 + \alpha_i \rho_1$, for $i = 1, 2$. Notice that the dividend process associated with each stock price fundamental solution follows a first-order autoregressive process. Moreover, any such process is stationary if and only if the associated stock price equilibrium solution is itself stationary (that is, $\mu_i < 1$). After some algebra, we can show from this expression that the mean of the dividend process
is an increasing function of $\alpha_i$. Thus, the mean dividend is higher for the $\alpha_1$-fundamental solution than for the $\alpha_2$-fundamental solution.

In order to obtain the dividend process under the backward solution, we first write the backward solution (2.10) as an ARMA(2,1) process (see the Proof of Proposition 3 in Appendix 1):

$$q(L)p_t = -\rho_0 + m(L)v_t + n(L)u_t,$$

where $L$ is the lag operator, $q(L) = 1 - (1 - \delta - \rho_1 + \rho_2)L + \rho_2\delta^{-1}L^2$, $m(L) = -\rho_2L^2$ and $n(L) = -\delta^{-1}(1 - \rho_2L)L$. By substituting equation (4.1) into equation (2.2) and after some algebra, we obtain that the dividend process under the backward equilibrium solution is characterized by the following ARMA(3,2):

$$(1 - \rho_2 L)(1 - \mu_1 L)(1 - \mu_2 L)d_t = \rho_0(1 - \mu_1)(1 - \mu_2)$$

$$+ (1 - (\mu_1 + \mu_2)L + (\mu_1\mu_2 - \rho_2)L^2)v_t - \frac{\rho_1}{\delta}(1 - \rho_2L)u_{t-1}.$$  

If $\rho_2 < 1$, the dividend process characterized by the stock price backward solution is stationary if and only if the backward solution is itself stationary (that is, $\mu_2 < \mu_1 < 1$). An extension of these results for the cointegration case is straightforward.

Using US stock market data, Driffill and Sola (1998) have recently found evidence of regime-switching in the ARMA representation of the dividend process without explaining what forces are causing this regime switching. Our results point out the theoretical possibility that the regime-switching in the ARMA representation found by Driffill and Sola may be explained by switches between alternative RE equilibria. In short, the results in Section 3 and 4 have shown that changes in the dividend process parameters may lead to switching equilibria and vice-versa.

5. EMPIRICAL EVIDENCE

We use data on US stock prices and dividends. More precisely, these data come from the Standard and Poor’s stock price and dividend indexes taken from the Securities Price Index Record. We study the period 1871-1989. Stock prices are January values whereas dividends are annual averages for the calendar year. Nominal stock prices and dividends are deflated by the producer price index (1982=100) for January of each year in order to get real figures.$^{14}$

$^{14}$The producer price index is taken from Table 26.2, series 5, in Shiller (1989, chapter 26).
5.1. Estimation Procedure

As shown in Table 1, standard unit root and cointegration tests suggest that stock price and dividend processes are I(1) and the two variables are cointegrated.\[^{15}\]

In order to analyze the implications of cointegration, we firstly rewrite the $\alpha_1$-fundamental and $\alpha_2$-fundamental equilibrium solutions and the dividend process (2.2) in matrix form

$$
\begin{align*}
\begin{pmatrix}
\Delta p_t \\
\Delta d_{t-1}
\end{pmatrix}
&= 
\begin{pmatrix}
\pi_0 + \pi_1 \rho_0 \\
\rho_0 + \rho_1 \pi_0
\end{pmatrix}
+ 
\begin{pmatrix}
\pi_1 \rho_1 - 1 \\
0
\end{pmatrix}
\begin{pmatrix}
\pi_1 \rho_2 \\
\pi_1 \rho_1 + \rho_2 - 1
\end{pmatrix}
\begin{pmatrix}
p_{t-1} \\
d_{t-2}
\end{pmatrix} \\
&+ 
\begin{pmatrix}
\pi_2 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
\pi_2 \rho_1
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\pi_1
\end{pmatrix}
\begin{pmatrix}
u_t \\
u_{t-1}
\end{pmatrix},
\end{align*}
$$
\[(5.1)\]

where $\Delta p_t = p_t - p_{t-1}$, and $\Delta d_{t-1} = d_{t-1} - d_{t-2}$. As discussed above, $p_t$ and $d_t$ are not measured contemporaneously since the stock price is known at the beginning of the period whereas dividends are paid over the period. Stock prices and dividends are cointegrated if the error-correction companion matrix is singular. Formally, this condition imposes the following restriction:

$$
\pi_1 \rho_1 + \rho_2 = 1.
$$

By using the second equation of (2.5), as shown in Proposition 4 above, we can easily show that the above cointegration restriction implies that the $\alpha_2$-fundamental solution is the unique fundamental RE solution given by \(\pi = (\pi_0, \pi_1, \pi_2) = (\frac{-\delta \rho_0}{(1-\delta)(\delta-\rho_2)}, \frac{\delta}{1-\delta}, \frac{1}{\rho_2})\) and that $\rho_2 = 1 - \delta \rho_1 / (1 - \delta)$. Moreover, the cointegrating vector is $(1, -\delta / (1 - \delta))$.\[^{16}\]

Similarly, we can write the backward solution (2.10) and the dividend process (2.2) in matrix form

$$
\begin{align*}
\begin{pmatrix}
\Delta p_t \\
\Delta d_{t-1}
\end{pmatrix}
&= 
\begin{pmatrix}
-\rho_0 \\
\rho_0
\end{pmatrix}
+ 
\begin{pmatrix}
\delta^{-1} - \rho_1 - 1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
-\rho_2 \\
\rho_2 - 1
\end{pmatrix}
\begin{pmatrix}
p_{t-1} \\
d_{t-2}
\end{pmatrix} \\
&+ 
\begin{pmatrix}
\delta^{-1} - \rho_1 - 1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
u_t \\
u_{t-1}
\end{pmatrix}.
\end{align*}
$$

\[^{15}\text{Augmented Dickey-Fuller tests and Phillips-Perron tests (Phillips and Perron, 1988) were carried out for the whole sample and alternative subsamples. Table 1 shows the Phillips-Perron } Z_{\rho}\text{ statistics. For the whole sample the test results are qualitatively similar to those provided by Campbell and Shiller (1987). For the alternative subsamples the empirical evidence on cointegration is much weaker than that found for the whole sample. In spite of this weak evidence on cointegration when dealing with alternative subsamples, we estimate the model below imposing the cointegration restriction for each subsample. The reason is that cointegration is usually viewed as a long-run equilibrium relation. Thus, the fact that } Z_{\rho}\text{ statistics show weaker evidence of cointegration when dealing with some subsamples than with the whole sample is interpreted as a lack of power of the cointegration test in small samples.}\]

\[^{16}\text{Alternatively, the singularity of the companion matrix may imply the condition } \pi_1 \rho_1 = 1. \text{ However, this alternative restriction implies a trivial cointegration vector given by } (1, 0).\]
Stock prices and dividends are cointegrated if the error-correction companion matrix is singular. One can easily show that the cointegration restriction for the backward solution is the same as the one found for the fundamental solution (that is, \( \rho_2 = 1 - \delta \rho_1 / (1 - \delta) \)) and that the cointegrating vector is \((1, -\delta / (1 - \delta))\) as before.

The alternative rational expectations solutions when the dividend process is characterized by \((2')\) are obtained in Appendix 3. Moreover, it is shown that the cointegration restriction associated with \((2')\) is given by \(\rho_2 + \rho_4 = 1 - \delta (\rho_1 + \rho_3) / (1 - \delta)\). Notice that this cointegration restriction is similar to the cointegration restriction associated with \((2.2)\) in the sense that the sum of the parameters associated with the dividend lags \((\rho_2 + \rho_4)\) is linked to the sum of parameters associated with stock price variables \((\rho_1 + \rho_3)\) that appears in equation \((2')\).

When writing and estimating alternative matrix solution systems \((5.1)\) and \((5.2)\) with the cointegration restriction imposed, we care that the left-hand side variables belong to the information set known to the market at the start of period \(t, I_t\). As pointed out by Campbell and Shiller (1987) and West (1988), in analyzing the stock price-dividend relationship both variables are required to belong to the same information set, since there might otherwise be spurious results.

Given the evidence of cointegration between stock prices and dividends and the error-correction representation implied by the present value model for stock prices, as shown in \((5.1)\) or \((5.2)\), we estimate the model in two steps. First, we estimate the error-correction representation of the cointegrated system in order to summarize the dynamics characterizing stock prices and dividends. Thus, the error-correction representation is the auxiliary model used to capture the empirical regularities displayed by the data. Second, we apply the simulated moments estimator (SME) suggested by Lee and Ingram (1991) and Duffie and Singleton (1993) to estimate the structural parameters of the stock prices model under the two alternative RE solutions \((5.1)\) and \((5.2)\) imposing the cointegration restriction.

Table 1: Phillips-Perron Z_\rho tests
| Period      | Variable       | With Trend | Without Trend |
|------------|----------------|------------|---------------|
| Sample     | $p_t$          | -13.32     | -3.59         |
| 1871-1989  | $d_t$          | -17.36     | -1.17         |
|            | $\Delta p_t$  | -107.09    | -107.11       |
|            | $\Delta d_t$  | -92.66     | -92.71        |
|            | $p_t - (\delta/(1 - \delta))d_{t-1}$ | -40.72 | -40.28 |
| Subsample  | $p_t$          | 10.02      | 9.33          |
| 1871-1910  | $d_t$          | 10.15      | 8.81          |
|            | $\Delta p_t$  | -34.72     | -34.72        |
|            | $\Delta d_t$  | -31.96     | -32.59        |
|            | $p_t - (\delta/(1 - \delta))d_{t-1}$ | -24.47 | -20.91 |
| Subsample  | $p_t$          | 9.91       | 9.31          |
| 1910-1955  | $d_t$          | 9.99       | 8.77          |
|            | $\Delta p_t$  | -36.33     | -36.30        |
|            | $\Delta d_t$  | -36.17     | -36.27        |
|            | $p_t - (\delta/(1 - \delta))d_{t-1}$ | -23.64 | -23.79 |
| Subsample  | $p_t$          | 10.10      | 9.51          |
| 1955-1975  | $d_t$          | 10.28      | 8.99          |
|            | $\Delta p_t$  | -15.29     | -14.78        |
|            | $\Delta d_t$  | -4.41      | -2.75         |
|            | $p_t - (\delta/(1 - \delta))d_{t-1}$ | -4.87 | -7.70 |

Note: The Phillips-Perron $Z_\rho$ statistics are corrected for fourth-order serial correlation. The results are qualitatively similar to those obtained when considering Phillips-Perron $Z_\rho$ and augmented Dickey-Fuller tests, or when considering alternative orders of the serial correlation correction in computing Phillips-Perron statistics. For a sample size of 500 observations, the critical values for the Phillips-Perron $Z_\rho$ test are: with trend: 10%, -18.1; 5%, -21.5; 1%, -28.9; without trend: 10%, -11.2; 5%, -14.0; 1%, -20.5. A table displaying the critical values for the Phillips-Perron $Z_\rho$ test is reported in Fuller (1976, p. 371).

The use of the SME estimator is especially appropriate in this context for several reasons. First, the alternative RE equilibrium solutions of the present value model of stock prices follow an error correction model. Therefore, the model of stock prices can be tested by comparing the restrictions imposed on the error correction model parameters with the best estimates of these parameters from time series data of stock prices and dividends. Second, since the SME belongs to the class of the generalized method of moment estimators, this method delivers consistent estimates even in the presence of autocorrelated and heteroskedastic disturbances. In other words, the SME allows us to estimate consistently a parsimonious present value model of stock prices. Finally, we aim to study how well the standard present value model of stock prices and its extensions...
(alternative RE solutions, alternative specifications of the dividend process, alternative specifications of the disturbance processes) accommodate features of the data. The SME allows us to carry out this comprehensive specification analysis. By using the SME estimator we can easily analyze many possible extensions subject to an auxiliary model that summarizes the data dynamics.

The SME makes use of a set of statistics computed from the data set used and from a number of different simulated data sets generated by the model being estimated. Since a sufficient condition for the SME to be consistent and asymptotically normal is that the time series used in the estimation should be covariance stationary, we work with first-differences of stock prices and dividends and with the spread at time \( t \) defined by \( p_t - (\delta/(1-\delta))d_{t-1} \). More specifically, the statistics used to carry out the SME are the coefficients from the error-correction representation of the two-variable cointegrated system formed by first differences of \( p_t \) and \( d_{t-1} \) with eight-lags. To find the appropriate lag of the error-correction representation, the likelihood ratio test was used. The null hypothesis tested was \( s \) lags versus \( s + 1 \) lags. The lowest number of lags \( s \) associated with the non-rejection of the null was chosen. For the whole sample we find \( s = 8 \) whereas for the sub-samples 1871-1910, 1910-1955 and 1955-1975, we find \( s \) equal to 3, 1 and 1, respectively.\(^{17}\)

The systematic approach followed in this paper in order to find the appropriate lag of the error-correction representation that summarizes the data dynamics implies that the SME implemented can be viewed as belonging to the “class” of efficient method of moments estimator developed by Gallant and Tauchen (1996).\(^{18}\) Moreover, since the error-correction representation nests the structural model, based on Gallant and Tauchen’ results, we can conclude that the SME used in this paper is as efficient as maximum likelihood.

To implement the method, we construct a \( p \times 1 \) vector with the coefficients of the error-correction representation obtained from real data, denoted by \( H_T(\theta_0) \), where \( p \) in this application is 37,\(^{19}\) \( T \) denotes the length of the time series data, and \( \theta \) is a \( k \times 1 \) vector whose components are the parameters of the model.

\(^{17}\)As shown by Hosking (1981), a portmanteau test for checking whether residuals from the error-correction model are white noise may be viewed as a sequence of Lagrange ratio tests for zero restrictions on the coefficients of the error-correction representation as the one carried out in this paper.

\(^{18}\)Related papers using a parametric model to define the auxiliary model are Smith(1993) and Gourieroux, Monfort and Renault (1993). See also Bansal, Gallant, Hussey and Tauchen (1995) and Gallant, Hsieh and Tauchen (1997) for applications of the efficient method of moments estimator using a nonparametric auxiliary model.

\(^{19}\)We have 32 coefficients from an eight-lag, two variable system, two coefficients for the error correction term since the cointegration vector is known, and three more coefficients from the non-redundant elements of the covariance matrix of the residuals. Notice that \( p \) becomes 17, 9 and 9 for the first, second and third subsamples, respectively.
being estimated. The true parameter values are denoted by \( \theta_0 \). In our model, the parameters are \( \theta = (\rho_1, \sigma_z, \sigma_s, \delta, \tau_0, \tau_1) \) or \( \theta = (\rho_1, \sigma_z, \sigma_s, \delta, \tau_0, \tau_1, \rho_3, \rho_4) \) depending on how many lags are considered when specifying the dividend process. As shown in Appendix 3, the cointegration restriction imposes that 

\[
\rho_2 = 1 - \delta \rho_1 / (1 - \delta)
\]

in the former case and 

\[
\rho_2 + \rho_4 = 1 - \delta (\rho_1 + \rho_3) / (1 - \delta)
\]

in the latter case.

Given that the real data are by assumption a realization of a stochastic process, we decrease the randomness in the estimator by simulating the model \( n \) times. Time series of simulated data are obtained recursively using the matrix system (5.1) (alternatively, the matrix system (5.2)), imposing the cointegration restriction 

\[
\rho_2 = 1 - \delta \rho_1 / (1 - \delta)
\]

Since we estimate the model many times (we analyze three alternative rational expectations solutions and four data sets: the whole sample and three sub-samples), as a compromise, we make \( n = 5 \) in this application. For each simulation a \( p \times 1 \) vector of error-correction representation coefficients, denoted by \( H_{N_i}(\theta) \), is obtained from the simulated time series of \( p_t \) and \( d_{t-1} \) generated from the model being estimated, where \( N = nT \) is the length of the simulated data. Averaging the \( n \) realizations of the simulated error-correction model coefficients, i.e., 

\[
H_N(\theta) = \frac{1}{n} \sum_{i=1}^{n} H_{N_i}(\theta),
\]

we obtain a measure of the expected value of the simulated coefficients of the error-correction representation, \( E(H_{N_i}(\theta)) \). To generate simulated values of \( \Delta p_t \) and \( \Delta d_{t-1} \) we need the starting values of the first-differences of stock prices and dividends. In the estimation, we have arbitrarily computed these starting values by using the first observed values of stock prices and dividends in our sample. For the SME to be consistent, the initial values must have been drawn from a stationary distribution. In practice, to avoid the influence of the starting values we follow Lee and Ingram’s suggestion of generating a realization from the stochastic processes of \( \Delta p_t \) and \( \Delta d_{t-1} \) of length \( 2N \), discard the first \( N \)-simulated observations, and use only the remaining \( N \) observations to carry out the estimation. After \( N \) observations have been simulated, the influence of the initial conditions must have disappeared.

The SME of \( \theta_0 \) is obtained from the minimization of a distance function of error-correction representation coefficients from real and simulated data. Formally,

\[
\min_{\theta} J_T = (H_T(\theta_0) - H_N(\theta))^TW(H_T(\theta_0) - H_N(\theta)),
\]

where the weighting matrix \( W^{-1} \) is the covariance matrix of \( H_T(\theta_0) \).

Denoting the solution of the minimization problem by \( \hat{\theta} \), Lee and Ingram (1991) and Duffie and Singleton (1993) prove the following results:

\[
\sqrt{T}(\hat{\theta} - \theta_0) \to N(0, (B'WB)^{-1}),
\]

\[
TJ_T \to \chi^2(p - k),
\]

24
where $B$ is a full rank matrix given by $B = E(\frac{\partial H_{N_i}(\theta)}{\partial \theta})$. For small values of $n$ the variance of the estimated parameter vector is $(1 + \frac{1}{n})(B'WB)^{-1}$; and the statistic in the latter expression should be $(1 + \frac{1}{n})TJ_T$.

The objective function $J_T$ was minimized using the optimization package OPTMUM programmed in GAUSS language. The Broyden-Fletcher-Goldfard-Shanno algorithm was applied. To compute the covariance matrix we need to obtain $B$. Computation of $B$ requires two steps: first, obtaining the numerical first derivatives of the coefficients of the error-correction model with respect to the estimates of the parameters $\theta$ for each of the $n$ simulations; second, averaging the $n$-numerical first derivatives to get $B$.

5.2. Empirical Results

We estimate the alternative RE equilibrium solutions of the present value model. The estimation results are displayed in Table 2. These results show that the fundamental solution provides the best fit even estimating a more restricted dividend process as (2.2) than the dividend process (2’) considered when estimating the backward solution. Moreover, the estimated value of $\rho_1$ is small but statistically significant, supporting the hypothesis of a feedback relationship from stock prices to dividends. As shown above, the presence of feedback implies the existence of multiple RE equilibria. The value of the goodness-of-fit statistic for the fundamental solution $(1 + \frac{1}{n})TJ_T = 125.46$, which is distributed as a $\chi^2(31)$ for the whole sample, clearly shows that the cross-equation restrictions imposed by the RE in this equilibrium solution (and thus, in any equilibrium solution considered) are not supported by the data.

In order to detect the presence of switching equilibria, we estimate the alternative solutions of the model for three sub-samples. The first sub-sample considers the period 1871-1910, the second sub-sample goes from 1910 to 1955 and the third sub-sample studies the period from 1955 to 1975. The second and third sub-samples were identified by Driffill and Sola (1998) and Evans (1998) as being periods characterized by different dividend processes. The split of the whole sample in these three sub-samples can be further motivated by looking at Figure 3. Figure 3 shows the evolution of the spread between stock prices and dividends. The value of $\delta$ used to compute the spread is 0.93. This value is associated with the sample-average of the U.S. gross real return for the period 1910-1975. This period covers the two sub-samples for which the present value model for stock prices is not rejected by the data. By looking at Figure 3, we observe three different periods, the spread behaves remarkably differently depending on which period is considered. A similar conclusion is reached in regard to Table 3 when comparing the descriptive statistics of the spread for the alternative
sub-samples. In particular, the mean and the standard deviation of the spread are much larger for the third sub-sample than for the other sub-samples. The autocorrelation functions of the spread for the first and third sub-samples show more persistence than for the second sub-sample. A similar conclusion is obtained from the estimates of the first-order autoregression.\textsuperscript{20}

It is well known that time series characterized by alternative regimes are highly persistent. As shown in Table 3, the fact that the spread for the whole sample exhibits a lot more persistence than any sub-sample can be viewed as a signal that alternative equilibria are characterizing the different sub-samples.

By estimating the alternative sub-samples separately, we can analyze whether feedback from stock prices to dividends and switching equilibria is still detected by allowing for changes in the dividend process parameters (that is, by allowing for regime-switching in the dividend process).

\textsuperscript{20}The features of the spread remain unchanged for other reasonable values of \( \delta \).
Table 4 displays the estimation results for the period 1871-1910. The best fit is obtained with the $\pi_4^+$-fundamental solution obtained in Appendix 3 when considering the dividend process (2'). The estimation results show that $\rho_1$ and $\rho_3$ are small but statistically significant, again showing evidence of the presence of feedback from stock prices to dividends. For this fundamental solution the value of the goodness-of-fit test $(1 + \frac{1}{n})TJ_T = 38.06$, which is distributed as a $\chi^2(9)$ for this sub-sample, shows that the cross-equation restrictions imposed by this solution are also rejected at standard critical values.

The estimation results for the period 1910-1955 are displayed in Table 5. For this sub-sample, the backward solution fits the data better than any other solution. The significant values of $\rho_1$ and $\rho_3$ also show evidence of feedback from stock prices to dividends for this sub-sample. The goodness-of-fit statistic $(1 + \frac{1}{n})TJ_T = 1.630$, which is distributed as a $\chi^2(1)$ for this sub-sample, shows that the cross-equation restrictions imposed by this equilibrium are not rejected at any standard critical value.

Table 6 shows the estimation results for the period 1955-1975. The $\pi_4^+$-fundamental solution fits the data better than the backward solution for this sub-sample. The significant values of $\rho_1$ and $\rho_3$ show evidence of feedback from stock prices to dividends in this sub-sample. Moreover, the goodness-of-fit statistic for this solution $(1 + \frac{1}{n})TJ_T = 3.09$, which is distributed as a $\chi^2(2)$ for this sub-sample, clearly shows that the data do not reject the cross-equation restrictions imposed by this equilibrium at any standard critical value.

Comparing the estimation results we observe that each sub-sample is characterized by a different RE equilibrium. These estimation results provide some evidence supporting the hypothesis of switching equilibria. Moreover, the parameter values of the dividend process change when considering alternative sub-samples and the corresponding best equilibrium solution in terms of the goodness-of-fit statistic. In particular, the parameter values significantly change from 1910-1955 to 1955-1975. Since the sample variance of stock prices relative to the sample variance of dividends for the period 1955-1975 is one and half times larger than for the sub-sample 1910-1955 (more precisely, these values are 1095.7848 and 702.826, respectively), we observe that the estimated values for the sum of the feedback parameters $(\rho_1 + \rho_3)$ for these two sub-samples support

\[ \delta^2(1 - \delta)^2\rho_4^2 + 2\delta(1 - \delta)[\delta(\delta\rho_3 + \rho_1) + 1 - \delta]\rho_4 + [\delta(\delta\rho_3 + \rho_1) - (1 - \delta)]^2 = 0. \]
the Muth-Lucas hypothesis that the higher the volatility of stock prices is relative to dividend volatility, the lower the informational content given to stock prices (measured by the size of $\rho_1 + \rho_3$) must be when dividend decisions are made.\textsuperscript{22,23} By assuming that $\delta = 0.93$ we have that $(1 - \delta)^2 / \delta = 0.0052688$ in Figure 2. The point estimates of $(\rho_1 + \rho_3)$ for the sub-samples 1910-1955 and 1955-1975 are 0.00598 and 0.00510, respectively. These point estimates suggest, on the one hand, that the equilibrium for the sub-sample 1910-1955 is described by a point such as $X$ in Figure 2 where the backward solution characterized the unique equilibrium implying cointegration between stock prices and dividends. On the other hand, the point estimates suggest that the equilibrium for the sub-sample 1955-1975 is described by a point such as $Y$ where the backward solution characterized the unique equilibrium implying cointegration between stock prices and dividends. Obviously, these conclusions are based on point estimates that are subject to sample variability. However, finding evidence on whether $(\rho_1 + \rho_3)$ changes significantly between sub-samples is rather problematic. The reason is that for reasonable values of $\delta$, the value of $(1 - \delta)^2 / \delta$ has to be quite small, so to have evidence of switching equilibria caused by small changes in the feedback parameters the changes in the estimates of $(\rho_1 + \rho_3)$ have to be very small, as our estimates show.

The hypothesis of switching equilibria provides an explanation of why the present value model fits the data poorly in terms of the goodness-of-fit statistic when analyzing the whole sample, but the fit is much better when considering the alternative sub-samples.

Our estimation results can be summarized as follows. First, our empirical results, using structural estimation, provide additional evidence supporting the hypothesis of a feedback mechanism from stock prices to dividends found by Timmermann (1994) using (OLS) reduced form estimation. Second, our empirical evidence supports the hypothesis of switching equilibria: each of the sub-samples analyzed is characterized by an alternative RE equilibrium. Third, based on the Muth-Lucas hypothesis our estimation results suggest that switching equilibria are caused by changes in the dividend process parameters induced by changes in

\textsuperscript{22}Recall that when stock prices and dividends are cointegrated it is the size of the sum of the feedback parameters that determines which solution characterizes the equilibrium when the dividend process is $(2')$. Therefore, the results for the cointegration case can still be represented by the line connecting the points $[(1 - \delta)/\delta, 0]$ and $(0,1)$ in Figure 2 with $\rho_1$ and $\rho_2$ being replaced by $\rho_1 + \rho_3$ and $\rho_2 + \rho_4$, respectively.

\textsuperscript{23}A similar conclusion is reached if instead of the sample variance of stock prices relative to dividend variance we look at the relative volatility of the innovations entering the stock price and dividend equations (that is, $\sigma_z/\sigma_s$). We observe that this relative volatility measure during the subsample 1955-1975 is twice as high as that for the subsample 1910-1955 (more precisely, these values are 84.57 and 43.14, respectively).
the relative volatility of stock prices to dividends. Finally, taking into account the presence of feedback and allowing for changes in the parameters characterizing the dividend process, we still find evidence supporting the hypothesis of switching equilibria.

6. CONCLUSIONS

Many studies have found that US stock prices are more volatile than is implied by the present value model. This paper shows that the observed excess volatility can be attributed in principle to switches between alternative (bubble-free) RE equilibrium solutions of the present value model. We use annual US data and the method of simulated moments to estimate the present value model for stock prices under the assumption that stock prices and dividends are cointegrated. When analyzing different sub-samples, the empirical results provide evidence supporting the hypothesis of switching equilibria and a good fit of the present value model for the sub-samples 1910-1955 and 1955-1975. Moreover, we find evidence of a small but very significant presence of feedback from stock prices to dividends. This feedback is smaller in those periods in which the sample relative stock price volatility is higher. The latter empirical result supports the Muth-Lucas hypothesis which, in this context, postulates that the informational content given to stock prices when deciding on dividends is inversely related to stock price volatility. Thus, this evidence can be used to illustrate a simple mechanism of switching equilibria in the context of cointegration between stock prices and dividends. This mechanism works as follows. The small change in the dividend process parameters induced by variations in the relative volatility of stock prices, can imply that the equilibrium becomes infeasible under the hypothesis of cointegration, which forces the economy to switch to another equilibrium. The evidence found supporting the hypothesis of switching equilibria together with the evidence of an increase in the sample volatility of stock prices and a decrease of the feedback parameters, all changes distinguishing the sub-samples 1910-1955 and 1955-1975, is consistent with the simple mechanism of switching equilibria described above. Summing up, this paper shows theory and evidence that small changes in the dividend process parameters may induce switching equilibria and, thus, these small changes help to explain the large US stock market swings observed between subsequent sub-samples.
Table 2. Empirical results for the US stock market. Period 1871-1989.

| SOLUTION | FUNDAMENTAL | BACKWARD |
|----------|-------------|----------|
| \((1 + \frac{r}{n})TJ_T\) | 125.460 | 125.816 |
| \(\rho_1\) | 0.00087 | 0.01302 |
| (0.000418) | (0.00202) |
| \(\sigma_z\) | 0.20470 | 0.18990 |
| (0.00239) | (0.00973) |
| \(\sigma_s\) | 0.00392 | 0.00385 |
| (0.00004) | (0.00004) |
| \(\delta\) | 0.79289 | 0.94632 |
| (0.02800) | (0.04727) |
| \(\tau_0\) | 0.96033 | 0.02108 |
| (0.02177) | (0.11217) |
| \(\tau_1\) | 0.08207 | 0.20249 |
| (0.08459) | (0.22110) |
| \(\rho_3\) | 0.79289 | 0.94632 |
| (0.02800) | (0.04727) |
| \(\rho_4\) | 0.00392 | 0.00385 |
| (0.00004) | (0.00004) |

Standard errors are in parentheses.

Table 3. Descriptive statistics of the spread for the alternative samples (\(\delta = 0.93\))

| Period | 1871-1989 | 1871-1910 | 1910-1955 | 1955-1975 |
|--------|-----------|-----------|-----------|-----------|
| Mean   | 0.45659   | 0.22442   | 0.20038   | 1.29307   |
| Std. Deviation | 0.46568 | 0.12118 | 0.21215 | 0.36457 |

| Autocorrelations |
|------------------|
| First | 0.87314 | 0.59264 | 0.51421 | 0.57076 |
| Second | 0.75369 | 0.37214 | 0.15799 | 0.39854 |
| Third | 0.72163 | 0.41842 | -0.01884 | 0.33715 |
| Forth | 0.67453 | 0.36488 | -0.15929 | 0.27937 |

| Autoregression |
|----------------|
| Constant | 0.05839 | 0.08816 | 0.09762 | 0.51831 |
| (0.02848) | (0.031730) | (0.04152) | (0.25437) |
| \(sp_{t-1}\) | 0.89443 | 0.62772 | 0.51193 | 0.60283 |
| (0.04449) | (0.12718) | (0.14315) | (0.19040) |
| \(DW\) | 2.02360 | 2.14799 | 1.80188 | 1.82682 |
| \(R^2\) | 0.78756 | 0.41742 | 0.26211 | 0.38518 |

Standard errors associated with the autoregression coefficients are in parentheses. DW denotes the Durbin-Watson statistic.
Table 4. Empirical results for the US stock market. Period 1871-1910.

| SOLUTION | FUNDAMENTAL | BACKWARD |
|----------|-------------|----------|
| $(1 + \frac{1}{n})TJ_T$ | 38.056 | 36.842 |
| $\rho_1$ | 0.00618 | 0.00607 |
| (0.00121) | (0.00123) |
| $\sigma_z$ | 0.00236 | 0.06320 |
| (0.00500) | (0.00377) |
| $\sigma_s$ | 0.00211 | 0.00211 |
| (0.00005) | (0.00005) |
| $\delta$ | 0.95215 | 0.95182 |
| (0.06648) | (0.04808) |
| $\tau_0$ | −0.21098 | −0.21725 |
| (0.18489) | (0.15308) |
| $\tau_1$ | 0.17956 | 0.46373 |
| (0.28053) | (0.19497) |
| $\rho_3$ | 0.01479 | 0.02369 |
| (0.00586) | (0.00565) |
| $\rho_4$ | −0.02173 | 0.12209 |
| (0.04935) | (0.26174) |

Standard errors are in parentheses in this and all subsequent tables.

Table 5. Empirical results for the US stock market. Period 1910-1955.

| SOLUTION | FUNDAMENTAL | BACKWARD |
|----------|-------------|----------|
| $(1 + \frac{1}{n})TJ_T$ | 2.291 | 1.630 |
| $\rho_1$ | 0.02991 | 0.02643 |
| (0.00565) | (0.00525) |
| $\sigma_z$ | 0.16462 | 0.17342 |
| (0.00486) | (0.00444) |
| $\sigma_s$ | 0.00548 | 0.00402 |
| (0.00018) | (0.00235) |
| $\delta$ | 0.69949 | 0.97924 |
| (0.06633) | (0.01032) |
| $\tau_0$ | 0.78211 | 0.08320 |
| (0.12063) | (0.18960) |
| $\tau_1$ | −0.18605 | −0.76106 |
| (0.27267) | (0.43867) |
| $\rho_3$ | −0.02854 | −0.02045 |
| (0.00590) | (0.00557) |
| $\rho_4$ | 0.00584 | −0.42536 |
| (0.0106) | (0.22793) |
Table 6. Empirical results for the US stock market. Period 1955-1975.

| SOLUTION | FUNDAMENTAL | BACKWARD |
|----------|-------------|-----------|
| $(1 + \frac{1}{n}) T J_T$ | 3.086 | 106.237 |
| $\rho_1$ | 0.00864 | 0.00951 |
| (0.00168) | (0.02229) |
| $\sigma_2$ | 0.14377 | 0.25288 |
| (0.04443) | (0.02805) |
| $\sigma_s$ | 0.00170 | 0.00034 |
| (0.00012) | (0.00739) |
| $\delta$ | 0.96737 | 0.96604 |
| (0.01689) | (0.06237) |
| $\tau_0$ | 0.47299 | −0.11582 |
| (0.25194) | (0.30601) |
| $\tau_1$ | 0.55585 | 0.11210 |
| (0.46782) | (0.50342) |
| $\rho_3$ | −0.00313 | 0.00352 |
| (0.00198) | (0.00293) |
| $\rho_4$ | −0.15867 | |
| (0.48295) | |

APPENDIX 1

Proof of Proposition 1:

The proof is straightforward. According to the definition criterion, we consider that an equilibrium solution exists when the solution is real rather than complex. When inequality (2.11) holds, all three solutions considered exist. However when inequality (2.11) does not hold, $\alpha_i$ is a complex number for $i = 1, 2$. Therefore, only the backward solution exists because the two fundamental solutions are complex.

Let us consider the zero-level curve of the term inside the square-root in the definition of the $\alpha_i$:

$$(1 - \delta(\rho_1 + \rho_2))^2 - 4\delta^2\rho_1\rho_2 = 0. \quad (A.1)$$

By differentiating this curve, we obtain

$$\frac{\partial \rho_2}{\partial \rho_1} = -\frac{1 - \delta(\rho_1 - \rho_2)}{1 + \delta(\rho_1 - \rho_2)},$$

which is negative for all $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. Therefore, by the implicit function theorem we know that there exists a unique continuously differentiable
function $\rho_2 = \phi(\rho_1)$, which has negative slope, characterizing the zero-level curve, \((A.1)\). It is easy to see that $\phi$ is a convex function whose intercepts are \((\rho_1, \rho_2) = (0, \delta^{-1})\) and \((\rho_1, \rho_2) = (\delta^{-1}, 0)\). Furthermore, pairs of \((\rho_1, \rho_2)\) in the lower (upper) contour set of \((A.1)\) do (not) satisfy inequality \((2.11)\). (See Figure 1).

It is easy to prove that along the zero-level curve, \((A.1)\)

$$\frac{\partial \rho_1}{\partial \delta} \bigg|_{\rho_2 \text{ constant}} = -\frac{2\rho_1 (1 - 2\delta \rho_2) + \rho_1 (2 - \delta)}{\delta (1 + \delta (\rho_1 - \rho_2))} < 0.$$  

This means that the smaller the discount factor is the farther from the origin the level curve \((A.1)\) is. Furthermore, for $\delta = 1/4$, \((A.1)\) crosses the pair \((\rho_1, \rho_2) = (1, 1)\); this means that the upper contour set of \((A.1)\) does not intersect the region in which $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. Therefore, for $\delta \leq 1/4$, the region in which the backward solution is the unique equilibrium solution is an empty set.

On the other hand, the combination \((\rho_1, \rho_2) = \left(\frac{1}{4\delta}, \frac{1}{4\delta}\right)\) is in level curve \((A.1)\). Therefore, at least any combination \((\rho_1, \rho_2) = \left(\frac{1}{4\delta} - \epsilon, \frac{1}{4\delta} - \epsilon\right)\) where $0 < \epsilon < 1/4\delta$ always belongs to the lower contour set of \((A.1)\) for any $0 < \delta < 1$. This means that the region in which the three solutions considered are defined is always a non empty set.

**Proof of Proposition 2:**

Using \((2.2)\), after recursive substitutions and rearranging, fundamental solutions \((2.6)\) and \((2.7)\) can be written as follows

$$a(L)p_i = k + b(L)v_{t-1} + c(L)u_t, \quad (A.2)$$

where $L$ denotes the lag operator and $a(L) = 1 - \mu_i L$, $b(L) = \alpha_i$, $c(L) = \varphi_i$ and $k = \alpha_i p_0 + \varphi_i$ for $i = 1, 2$, respectively.

As is well known, the process characterized by \((A.2)\) is stationary if $|\mu_i| < 1$. When $\mu_i$ is real, $\mu_i = \rho_2 + \alpha_i \rho_1 \geq 0$ for $i = 1, 2$, and therefore the condition for fundamental solutions to be stationary is simply $\mu_i < 1$.

Using standard results (for instance, see Granger and Newbold (1977, pp.26-27)), the autocovariance generating function for the stock price process \((A.2)\), when the process is stationary, can be written as follows

$$\lambda(L)^i = \frac{b(L)b(L^{-1})}{a(L)a(L^{-1})} \sigma_u^2 + \frac{c(L)c(L^{-1})}{a(L)a(L^{-1})} \sigma_u^2.$$  

The variance of the stock price processes characterized by fundamental solutions \((2.6)\) and \((2.7)\) ($\lambda_0^i$, for $i = 1, 2$; respectively) is equal to the coefficient associated
with $L^0$ in the power series expansion of the autocovariance generating function $\lambda(L)^i$, which, after some algebra, can be written as expression (3.1)

$$\lambda^i_0 = \frac{\alpha^2_i \sigma^2_v}{1 - \mu^2_i} + \frac{1 - \mu^2_i + \alpha^2_i \rho^2_1 \varphi^2_i \sigma^2_u}{1 - \mu^2_i}.$$  

Notice that $\mu_i = \rho_2 + \alpha_i \rho_1$ and $\varphi_i = 1/(1 - \delta \rho_1 (1 + \alpha_i))$. Therefore, in order to show that the variance of the $\alpha_2$-fundamental solution, (2.7), is lower than the variance of the $\alpha_1$-fundamental solution, (2.6), it is sufficient to show that $\lambda^i_0$ is increasing in $\alpha_i$ since $\alpha_1 \geq \alpha_2$. Let us denote the first and second terms in (3.1) by $A$ and $B$, respectively. Then

$$\frac{\partial \lambda^i_0}{\partial \alpha_i} = \frac{\partial A}{\partial \alpha_i} + \frac{\partial B}{\partial \alpha_i}.$$  

Operating, we can obtain that

$$\frac{\partial A}{\partial \alpha_i} = \frac{2 \sigma^2_v \alpha_i (1 - \mu_i \rho_2)}{(1 - \mu^2_i)} > 0,$$

$$\frac{\partial B}{\partial \alpha_i} = \frac{2 \alpha_i \rho^2_1 (1 - \mu_i \rho_2)}{1 - \mu^2_i} + 2 \varphi^2_i \rho_1 \delta (1 - \mu^2_i + \alpha^2_1 \rho^2_1) > 0.$$  

Notice that these partial derivatives are positive under the stationary condition, $\mu_i < 1$. This implies that the variance of stock prices is increasing in $\alpha_i$. This completes the proof.

**Proof of Proposition 3:**

The backward equilibrium solution, (2.10), with $\epsilon_t = 0$ can be written as an ARMA process as follows. By taking into account the dividend process, equation (2.2), the backward solution can be written as

$$p_t = -\rho_0 + (\delta^{-1} - \rho_1)p_{t-1} - \rho_2 \delta u_{t-2} - \delta^{-1} u_{t-1}.$$  

Adding and subtracting $\rho_2 p_{t-1}$ from this and using (2.10), we have that

$$p_t = -\rho_0 + (\delta^{-1} - \rho_1 + \rho_2)p_{t-1} - \rho_2 \delta^{-1} p_{t-2} - \rho_2 \delta^{-1} u_{t-2} - \rho_2 v_{t-1} - \delta^{-1} u_{t-1},$$

or alternatively

$$q(L)p_t = -\rho_0 + m(L)v_t + n(L)u_t, \quad (A.3)$$

where $q(L) = 1 - (\delta^{-1} - \rho_1 + \rho_2)L + \rho_2 \delta^{-1} L^2$, $m(L) = -\rho_2 L^2$ and $n(L) = -\delta^{-1} (1 - \rho_2 L) L$.  

34
Since the backward solution, (2.10), can be expressed as the ARMA(2,1) process (A.3), this equilibrium solution is stationary whenever all roots of \( q(L) = 0 \) lie outside the unit circle. Let us denote by \( q_1 \) and \( q_2 \) the roots of \( q(L) = 0 \). Thus,

\[
q(L) = \frac{\rho_2}{\delta}(L - q_1)(L - q_2) = 0,
\]

where

\[
q_1 = \frac{(\delta^{-1} - \rho_1 + \rho_2) + \sqrt{(\delta^{-1} - \rho_1 + \rho_2)^2 - 4\rho_2\delta^{-1}}}{2\rho_2\delta^{-1}},
\]

\[
q_2 = \frac{(\delta^{-1} - \rho_1 + \rho_2) - \sqrt{(\delta^{-1} - \rho_1 + \rho_2)^2 - 4\rho_2\delta^{-1}}}{2\rho_2\delta^{-1}}.
\]

After some algebra, it is easy to show that the square root in the definition of the \( q \)'s is the same as that in the definition of the \( \alpha \)'s. Therefore, we can establish the following relationship between these \( q \)'s and the \( \alpha \)'s that define the fundamental solutions:

\[
q_i = \frac{\delta}{\rho_2}(\rho_2 + \alpha_i \rho_1),
\]

for \( i = 1, 2 \). Moreover, it is easy to see that \( q_1q_2 = \frac{\delta}{\rho_2} \). Therefore, for \( i, j = 1, 2 \) and \( i \neq j \),

\[
q_i = \frac{1}{\rho_2 + \alpha_j \rho_1} = \frac{1}{\mu_j},
\]

for \( i \neq j \). Therefore, \( q_1 \) and \( q_2 \) are both outside the unit circle if and only if \( \mu_i < 1 \), for \( i = 1, 2 \).

In order to obtain the variance of the backward solution when the process is stationary, let us rewrite \( q(L) \) in (A.3) as follows:

\[
q(L) = (1 - \mu_1 L)(1 - \mu_2 L),
\]

where \( 0 < \mu_i = q_i^{-1} < 1 \). The autocovariance generating function for the stock prices process under the backward solution, \( \lambda(L)^b \), can be written as follows

\[
\lambda(L)^b = \left(\frac{\sigma^2 m(L)m(L^{-1})}{q(L)q(L^{-1})} + \frac{\sigma^2 n(L)n(L^{-1})}{q(L)q(L^{-1})}\right)
+ \frac{\sigma^2 \rho_2^2 + \sigma^2 \delta^{-2} (1 - \rho_2 L)(1 - \rho_2 L^{-1})}{(1 - \mu_1 L)(1 - \mu_2 L)(1 - \mu_1 L^{-1})(1 - \mu_2 L^{-1})}.
\]

The variance of the stock price process characterized by the backward solution is derived from the coefficient associated with \( L^0 \) in the power series expansion of
the autocovariance generating function $\lambda(L)^b$. After simple, but tedious, algebra, one can show that the variance of stock prices process is given by equation (3.2). This completes the proof.

**Proof of Proposition 4:**

In Appendix 2 we show that when $\mu_i = \rho_2 + \alpha_i \rho_1 = 1$ for $i = 1$ or $i = 2$, we are along the line $\rho_2 = 1 - \frac{\delta}{1-\delta} \rho_1$. After some calculation it is easy to see that along this line

$$\alpha_i = \begin{cases} \frac{1-\delta}{\delta \rho_1} - 1, & \text{for } i = 1, \\ \frac{1}{1-\delta}, & \text{for } i = 2. \end{cases}$$

Taking this result into account we can see that the $\alpha_1$-fundamental solution does not exist because $\pi_0^1$ is not defined and the $\alpha_2$-fundamental solution is given by

$$\pi^2 = (\pi_0^2, \pi_1^2, \pi_2^2) = \left( \frac{-\delta \rho_0}{(1-\delta)(\delta - \rho_2)}, \frac{\delta}{1-\delta}, \frac{1}{\rho_2} \right).$$

On the other hand, taking into account that the $\alpha_2$-fundamental solution can be expressed as the ARMA process (A.2), it is immediate to see that when $\mu_1 = 1 > \mu_2$, the $\alpha_2$-fundamental solution is stationary because $\mu_2$ is inside the unit circle. However if $\mu_1 > \mu_2 = 1$, the $\alpha_2$-fundamental solution implies that stock prices and dividends are cointegrated.

In the same way the backward solution can be written as the ARMA process (A.3). Therefore when $\mu_1 = 1 > \mu_2$, the backward solution implies cointegration between the stock prices and dividends. However if $\mu_1 > \mu_2 = 1$, the backward solution is explosive because one of the two roots is outside the unit circle.

**Proof of Proposition 5:**

From Proposition 4, it is easy to show that the spread under the $\alpha_2$-fundamental solution is given by

$$sp_f^t = \frac{-\delta \rho_0}{(1-\delta)(\delta - \rho_2)} + \frac{1}{\rho_2} u_t.$$ 

Therefore,

$$E(sp_f^t) = \frac{-\delta \rho_0}{(1-\delta)(\delta - \rho_2)},$$

$$var(sp_f^t) = \frac{1}{\rho_2^2} \sigma_u^2.$$ 

On the other hand, after some algebra and using the cointegration restriction, $\rho_2 = 1 - \delta \rho_1/(1-\delta)$, the backward equilibrium solution (2.10) can be written as follows

$$sp_b^t = -\frac{\rho_0}{1-\delta} + \frac{\rho_2}{\delta} sp_b^{t-1} - \delta^{-1} u_{t-1} - \frac{\delta}{1-\delta} v_{t-1}.$$
Then, it is straightforward to show that
\[ E(s_{\text{pt}}^b) = \frac{-\delta \rho_0}{(1 - \delta)(\delta - \rho_2)}. \]
\[ \text{var}(s_{\text{pt}}^b) = \frac{1}{(\delta^2 - \rho_2^2)} \sigma_u^2 + \frac{\delta^4}{(1 - \delta)^2(\delta^2 - \rho_2^2)} \sigma_v^2. \]

APPENDIX 2

In this appendix we prove the statements about Figure 2 made in Section 3.

Let us consider the level curve
\[ \mu_1 \equiv \rho_2 + \alpha_1 \rho_1 = 1. \quad (A.4) \]
By differentiating this curve, we obtain
\[ \frac{\partial \rho_2}{\partial \rho_1} = -\frac{1 - \delta (\rho_1 - \rho_2)}{1 + \delta (\rho_1 - \rho_2)} \sqrt{[1 - \delta (\rho_1 + \rho_2)]^2 - 4 \rho_1 \rho_2}. \quad (A.5) \]
Taking into account that throughout \((A.4)\),
\[ 2\delta(1 - \rho_2) - [1 - \delta (\rho_1 + \rho_2)] = \sqrt{[1 - \delta (\rho_1 + \rho_2)]^2 - 4 \rho_1 \rho_2}, \quad (A.6) \]
we can rewrite \((A.5)\), as
\[ \frac{\partial \rho_2}{\partial \rho_1} = -\frac{\delta}{1 - \delta}. \]
Therefore, the implicit function associated with \((A.4)\) is a linear function with negative slope. On the other hand, \((\rho_1, \rho_2) = (\frac{1 - \delta}{\delta}, 0)\) and \((\rho_1, \rho_2) = (\frac{(1 - \delta)^2}{\delta}, \delta)\) lies on \((A.4)\). Notice that the latter pair also satisfies the zero-level curve \((A.1)\) defined in Proposition 1
\[ [1 - \delta (\rho_1 + \rho_2)]^2 - 4 \delta^2 \rho_1 \rho_2 = 0, \quad \iff \quad \mu_1 = \mu_2. \]
Moreover, since the term on the right-hand side of \((A.6)\) is non negative, all pairs of \((\rho_1, \rho_2)\) in the level curve \((A.4)\) must satisfy
\[ \rho_2 \leq \frac{2\delta - 1}{\delta} + \rho_1. \]
Notice that the pair \((\rho_1, \rho_2) = (\frac{(1 - \delta)^2}{\delta}, \delta)\) satisfies the latter equation with equality.
All these results mean that the level curve (A.4) is equivalent to the set of all pairs \((\rho_1, \rho_2)\) on the line \(\rho_2 = 1 - \frac{\delta}{1+\rho_1}\), such that \(\rho_2 \leq \delta\).

Furthermore, it is easy to see that,

\[
\frac{\partial \mu_1}{\partial \rho_1} < 0. \tag{A.7}
\]

This means that all pairs satisfying inequality (2.11) and located to the right (left) of the level curve (A.4) satisfy \(\mu_1 < 1\) \((\mu_1 > 1)\).

On the other hand, for any combination satisfying inequality (2.11),

\[
\mu_1 = \rho_2 + \alpha_1 \rho_1 \geq \frac{1 + \delta \rho_2}{2\delta}.
\]

Since all pairs satisfying inequality (2.11) and located northwest of combination \(\left(\frac{(1-\delta)^2}{\delta}, \delta\right)\) are such that \(\rho_2 > \delta\),

\[
\mu_1 \geq \frac{1 + \delta \rho_2}{2\delta} > \frac{1 + \delta^2}{2\delta} > 1.
\]

This result and (A.7) imply that, only for those pairs satisfying inequality (2.11) and located to the right of the level curve (A.4), the \(\alpha_1\)-fundamental solution is stationary.

Following the same steps, we can see that the level curve \(\mu_2 \equiv \rho_2 + \alpha_2 \rho_1 = 1\) is equivalent to the set of all pairs \((\rho_1, \rho_2)\) on the line \(\rho_2 = 1 - \frac{\delta}{1+\rho_1}\), such that \(\rho_2 \leq \delta\). In other words, in Figure 2 the level curve \(\mu_2 = 1\) is equivalent to the segment connecting the points \((\rho_1, \rho_2) = \left(\frac{(1-\delta)^2}{\delta}, \delta\right)\) and \((\rho_1, \rho_2) = (0, 1)\).

Moreover, it is easy to prove that the \(\alpha_2\)-fundamental solution is not stationary only for those pairs satisfying inequality (2.11) and located on and above that level curve.
APPENDIX 3

In this appendix we obtain the cointegration restriction and the fundamental solutions when the dividend process is described by \( (2') \). In order to derive the cointegration restriction recall Remark 1, that is, the backward solution \((2.10)\) is an RE equilibrium solution no matter what process is followed by the dividends. Thus, equations \((2.10)\) and \((2')\) can be written after some algebra as follows

\[
\begin{pmatrix}
\Delta p_t \\
\Delta d_{t-1}
\end{pmatrix} = \begin{pmatrix}
-\rho_0 \\
\rho_0
\end{pmatrix} + \begin{pmatrix}
\delta^{-1} - \rho_1 - \rho_3 - 1 & -(\rho_2 + \rho_4) \\
\rho_1 + \rho_3 & \rho_2 + \rho_4 - 1
\end{pmatrix} \begin{pmatrix}
p_{t-1} \\
d_{t-2}
\end{pmatrix} + \begin{pmatrix}
\rho_3 \\
-\rho_3
\end{pmatrix} \begin{pmatrix}
\Delta p_{t-1} \\
\Delta d_{t-2}
\end{pmatrix} + \begin{pmatrix}
\delta^{-1} u_{t-1} \\
u_{t-1}
\end{pmatrix}.
\]

Stock prices and dividends are then cointegrated if the companion matrix

\[
\begin{pmatrix}
\delta^{-1} - \rho_1 - \rho_3 - 1 & -(\rho_2 + \rho_4) \\
\rho_1 + \rho_3 & \rho_2 + \rho_4 - 1
\end{pmatrix}
\]

is singular. This condition implies the following cointegration restriction: \( \rho_2 = 1 - \rho_4 - \delta(\rho_1 + \rho_3)/(1 - \delta). \)

We next obtain the fundamental solutions when the dividend process is described by \((2')\). We begin by writing \( p_t \) as a linear function of \( u_t \) and the predetermined state variables \( d_{t-1}, p_{t-1}, d_{t-2} \) plus a constant,

\[
p_t = \pi_0 + \pi_1 d_{t-1} + \pi_2 p_{t-1} + \pi_3 d_{t-2} + \pi_4 u_t.
\]

For appropriate real values of \( \pi_0, \pi_1, \pi_2, \pi_3 \) and \( \pi_4 \), the expectational variable \( E_t p_{t+1} \) will then be given by

\[
E_t p_{t+1} = \pi_0 (1 + \pi_1 \rho_1 + \pi_2) + \pi_1 \rho_0 + (\pi_1 \rho_1 + \pi_2 + \rho_2 + \pi_3) d_{t-1} + \\
(\pi_2 (\pi_1 \rho_1 + \pi_2) + \pi_1 \rho_3) p_{t-1} + (\pi_3 (\pi_1 \rho_1 + \pi_2) + \pi_1 \rho_4) d_{t-2} + \\
\pi_4 (\pi_1 \rho_1 + \pi_2) u_t.
\]

To evaluate the \( \pi \)'s, we substitute \((2'), (A.8)\) and \((A.9)\) into \((2.1)\). That equation implies identities in the constant term, \( d_{t-1}, p_{t-1}, d_{t-2} \) and \( u_t \) as follows:

\[
\begin{align*}
\pi_0 &= \delta(\pi_0 (1 + \pi_1 \rho_1 + \pi_2 + \rho_1) + \rho_0 (1 + \pi_1)), \\
\pi_1 &= \delta(\pi_1 (\pi_1 \rho_1 + \pi_2 + \rho_1 + \rho_2 + \pi_3), \\
\pi_2 &= \delta(\pi_2 (\pi_1 \rho_1 + \pi_2 + \rho_1) + \rho_3 (1 + \pi_1)), \\
\pi_3 &= \delta(\pi_3 (\pi_1 \rho_1 + \pi_2 + \rho_1) + \rho_4 (1 + \pi_1)), \\
\pi_4 &= 1 + \delta \pi_4 (\pi_1 \rho_1 + \rho_1 + \rho_2).
\end{align*}
\]

Taking into account the cointegration restriction, we solve this five equation system with five unknowns. One can show that \( \pi_4 \) has two solutions given by

\[
\pi_4 = \frac{\delta(\rho_3 + \rho_1) - (1 - \delta)(1 - \delta \rho_4)}{2(1 - \delta) \delta \rho_4}
\]
\[ \pm \frac{((\delta \rho_3 + \rho_4) - (1 - \delta)(1 - \delta \rho_4))^2 + 4(1 - \delta)^2 \delta \rho_4)^{1/2}}{2(1 - \delta) \delta \rho_4}. \]

Notice that \( \rho_2 \) does not appear in this expression because the cointegration restriction has been already imposed. We denote the two alternative solutions of \( \pi_4 \) by \( \pi_4^+ \) and \( \pi_4^- \), respectively. Once a solution of \( \pi_4 \) is obtained, the associated solution for \( \pi_0, \pi_1, \pi_2 \) and \( \pi_3 \) can be derived recursively. We do not show the expressions of the \( \pi \)’s because they are (cumbersome) non-linear functions of \( \pi_4 \) and the structural parameters of the model. These expressions are available upon request.
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Table A.1. Descriptive statistics of the *spread* for the alternative samples ($\delta = 0.95$)

| Period       | 1871-1989 | 1871-1910 | 1910-1955 | 1955-1975 |
|--------------|-----------|-----------|-----------|-----------|
| Mean         | 0.19114   | 0.07403   | -0.03347  | 0.86596   |
| Std. Deviation | 0.38457   | 0.11126   | 0.20989   | 0.31722   |
| Autocorrelations |          |           |           |           |
| First        | 0.81236   | 0.44871   | 0.39001   | 0.43880   |
| Second       | 0.65603   | 0.16830   | 0.00475   | 0.25718   |
| Third        | 0.63564   | 0.20510   | -0.12958  | 0.25467   |
| Forth        | 0.58813   | 0.14287   | -0.23323  | 0.24212   |
| Autoregression |         |           |           |           |
| Constant     | 0.04011   | 0.04042   | -0.02151  | 0.44864   |
| ($sp_{t-1}$) | (0.02326) | (0.01920) | (0.03223) | (0.20149) |
| $sp_{t-1}$   | 0.82481   | 0.48203   | 0.38433   | 0.48186   |
| ($sp_{t-1}$) | (0.05505) | (0.14473) | (0.15390) | (0.21919) |
| $DW$         | 2.03186   | 2.03215   | 1.80332   | 1.79374   |
| $R^2$        | 0.67315   | 0.24592   | 0.14766   | 0.23198   |

43