IDEOAL ZETA FUNCTIONS ASSOCIATED TO A FAMILY OF
CLASS-2-NILPOTENT LIE RINGS

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Abstract. We produce explicit formulae for various ideal zeta functions associated to the members of an infinite family of class-2-nilpotent Lie rings, introduced in [1], in terms of Igusa functions. As corollaries we obtain information about analytic properties of global ideal zeta functions, local functional equations, topological, reduced, and graded ideal zeta functions, as well as representation zeta functions for the unipotent group schemes associated to the Lie rings in question.

1. Introduction and main result

The main object of this paper is the computation of various ideal zeta functions associated to the class-2-nilpotent Lie rings $L_{m,n}$ defined in [1].

1.1. Ideal zeta functions of Lie rings and algebras. Let $R$ be the ring of integers $O$ of a number field or a compact discrete valuation ring, such as the completion $O_p$ of $O$ at a nonzero prime ideal $p$ or a formal power series ring of the form $\mathbb{F}_q[[T]]$. Let $L$ be a nilpotent $R$-Lie algebra which is free of finite rank over $R$. The ideal zeta function of $L$ is the Dirichlet generating series

$$\zeta_L^R(s) = \sum_{I \triangleleft L} |L : I|^{-s},$$

enumerating $R$-ideals in $L$ of finite index in $L$. Here $s$ is a complex variable.

Assume now that $R = O$ is the ring of integers of a number field $K$. For a nonzero prime ideal $p \in \text{Spec}(O)$ we write $O_p$ for the completion of $O$ at $p$, a complete discrete valuation ring of characteristic zero and residue field $O/p$ of cardinality $q = q_p$, say. We consider $L(O_p) := L \otimes_O O_p$ as an $O_p$-Lie algebra. Little more than the Chinese Reminder Theorem is necessary to obtain the Euler product

$$\zeta_L^O(s) = \prod_{p \in \text{Spec}(O)} \zeta_{L(O_p)}^O(s);$$

cf. [7 § 3]. A deep theorem, in contrast, asserts that all the Euler factors $\zeta_{L(O_p)}^O(s)$ are rational functions in $q^{-s}$; cf. [7 Theorem 3.5]. Computing these rational functions is, in general, a hard problem.

1.2. The Lie rings $L_{m,n}$ and their ideal zeta functions. In the current paper we compute explicitly, for any given $m, n \in \mathbb{N}$, the ideal zeta functions of the $\mathfrak{o}$-Lie algebras $L_{m,n}(\mathfrak{o}) := L_{m,n} \otimes \mathfrak{o}$, where $L_{m,n}$ is the class-2-nilpotent Lie ring introduced in [1] and $\mathfrak{o}$ is a compact discrete valuation ring of arbitrary characteristic. Our main Theorem 1.1 expresses the ideal zeta functions $\zeta_{L_{m,n}(\mathfrak{o})}^\mathfrak{o}(s)$ as rational functions in $q$ and $q^{-s}$, where $q$ is the residue field.

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cardinality of \( \mathfrak{o} \), in terms of Igusa functions. To recall the definition of the Lie ring \( L_{m,n} \), put

\[
E = E(m,n) = \{ e \mid e = (e_1, \ldots, e_n) \in \mathbb{N}_0^n, e_1 + \cdots + e_n = m-1 \},
\]

\[
F = F(m,n) = \{ f \mid f = (f_1, \ldots, f_n) \in \mathbb{N}_0^n, f_1 + \cdots + f_n = m \}.
\]

The Lie ring \( L_{m,n} \) has generators

\[\{ x_e \mid e \in E \} \cup \{ y_f \mid f \in F \} \cup \{ z_j \mid j \in [n] \},\]

subject to the defining relations

\[(1.1) \quad [x_e, x_{e'}] = [y_f, y_{f'}] = [x_e, z_j] = [y_f, z_j] = [z_j, z_{j'}] = 0\]

for all \( e, e' \in E \), \( f, f' \in F \), \( j, j' \in [n] \) and, for all \( e \in E \) and \( f \in F \),

\[(1.2) \quad [x_e, y_f] = \begin{cases} z_i & \text{if } f - e \text{ is the } i\text{th standard basis vector of } \mathbb{Z}^n, \\ 0 & \text{else.} \end{cases}\]

Clearly \( Z(L_{m,n}) = L'_{m,n} := [L_{m,n}, L_{m,n}] = \langle z_1, \ldots, z_n \rangle \). In particular, \( L_{m,n} \) is nilpotent of class 2. Setting

\[e(m,n) = \#E = \binom{n+m-2}{n-1}, \quad f(m,n) = \#F = \binom{n+m-1}{n-1},\]

and further

\[d(m,n) = f(m,n) + e(m,n), \quad h(m,n) = d(m,n) + n,\]

we find that

\[\text{rk}_Z(L_{m,n}/L'_{m,n}) = d(m,n), \quad \text{rk}_Z(L'_{m,n}) = \text{rk}_Z(Z(L_{m,n})) = n, \quad \text{rk}_Z(L_{m,n}) = h(m,n).\]

We recall further from [12] the definition of the Igusa zeta function of degree \( n \)

\[
I_n(Y; X) = \sum_{I \subseteq \{1, \ldots, n\}} \binom{n}{|I|} \prod_{i \in I} X_i = \frac{1}{1 - X_i} \sum_{I \subseteq \{1, \ldots, n-1\}} \binom{n}{|I|} \prod_{i \in I} X_i
\]

subject to the defining relations

\[(1.1) \quad [x_e, x_{e'}] = [y_f, y_{f'}] = [x_e, z_j] = [y_f, z_j] = [z_j, z_{j'}] = 0\]

for all \( e, e' \in E \), \( f, f' \in F \), \( j, j' \in [n] \) and, for all \( e \in E \) and \( f \in F \),

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\]

where \( \ell \) denotes the classical Coxeter length function on the Coxeter group \( S_n \) and the descent statistic \( \text{Des} \) is defined via \( \text{Des}(w) = \{ i \in \{1, \ldots, n-1\} \mid w(i+1) < w(i) \} \). Moreover, for \( I = \{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\} \), the associated Gaussian multinomial is the polynomial

\[\binom{n}{I}_{Y} = \binom{n}{i_1}_{Y} \binom{n}{i_2}_{Y} \cdots \binom{n}{i_t}_{Y} \in \mathbb{Z}[Y],\]

defined in terms of the Gaussian binomials

\[\binom{a}{b}_{Y} = \frac{\prod_{i=a-b+1}^{a}(1-Y^i)}{\prod_{i=1}^{b}(1-Y^i)} \in \mathbb{Z}[Y]\]

for \( a, b \in \mathbb{N}_0 \) with \( a \geq b \). It is well known that the ideal zeta function \( \zeta_{\mathfrak{o}^d}(s) = \zeta^{\mathfrak{o}^d}_{\mathfrak{o}^d}(s) \) of the abelian \( \mathfrak{o} \)-Lie algebra \( \mathfrak{o}^d \), enumerating all \( \mathfrak{o} \)-submodules of finite index in \( \mathfrak{o}^d \), is given by

\[(1.4) \quad \zeta_{\mathfrak{o}^d}(s) = \frac{1}{\prod_{i=0}^{d-1}(1-q^{i-s})} = I_d(q^{-1}; (q^{(d-i)(i-s)})_{i=d-1}^{0});\]

cf., for instance, the first entry in the table on p. 218 of [7] (with \( m = 1 \) and \( r = 0 \)). The main result of this paper is the following.
Theorem 1.1. For any compact discrete valuation ring \( \mathfrak{o} \), with residue field cardinality \( q \),
\[
\zeta_{L,m,n}(\mathfrak{o})(s) = \zeta_{G,(m,n)}(s) \cdot I_n \left( q^{-1}; \left( q^{2j(m,n) - s} b^d_i(m,n) \right)_{i=0}^{n} \right),
\]
where, for \( i \in \{0, 1, \ldots, n - 1\} \),
\[
a^d_i(m,n) = (n - i)(i + d(m,n)),
\]
\[
b^d_i(m,n) = n - i + e(m,n) + \sum_{j=i+1}^{n} e(m,j).
\]

Example 1.2. For \( (m,n) = (2,3) \) we obtain
\[
\zeta_{L,2,3}(\mathfrak{o})(s) = \frac{1 + q^{9-7s} + q^{10-7s} + q^{18-10s} + q^{19-10s} + q^{28-17s}}{(1 - q^{1-s})(1 - q^{27-12s})(1 - q^{20-10s})(1 - q^{11-7s})}.
\]
This formula exemplifies what seems to be a general phenomenon: cancellations as observed in (1.3) do not appear to affect the Igusa function of degree \( n \) occurring in (1.3).

1.3. Mal’cev’s correspondence. Nilpotent Lie rings play an important role in the theory of finitely generated, torsion-free nilpotent groups. Indeed, by the Mal’cev correspondence, each such group \( G \) has an associated nilpotent Lie ring \( L_G \) such that, for almost all rational primes \( p \), the ideal zeta function \( \zeta_{L_G(\mathbb{Z}_p)}(s) \) is equal to the local normal subgroup zeta function \( \zeta_{G,p}(s) := \sum_{H \in G} |H|^{-s} \), enumerating the normal subgroups of \( G \) of finite \( p \)-power index; cf. [7, Theorem 4.1]. In nilpotency class two, one may turn instead to the Lie ring \( L_G := Z(G) \oplus G/(Z(G)) \). It is not hard to show that the identity \( \zeta_{L_G(\mathbb{Z}_p)}(s) = \zeta_{G,p}(s) \) holds for all primes \( p \).
Every class-2-nilpotent Lie ring arises in this way. Theorem 1.1 thus yields, as a corollary, all local normal subgroup zeta functions of the nilpotent groups \( \Delta_{m,n} \) associated to the Lie rings \( L_{m,n} = L_{\Delta_{m,n}} \), and thus the (global) normal zeta function
\[
\zeta_{\Delta_{m,n}}(s) := \sum_{H \in \Delta_{m,n}} |H|^{-s} = \prod_{p \text{ prime}} \zeta_{\Delta_{m,n,p}}(s) = \prod_{p \text{ prime}} \zeta_{L_{m,n,p}}(s) = \zeta_{L_{m,n}}(s).
\]
It was in this context of subgroup growth of finitely generated nilpotent groups that ideal zeta functions of Lie rings were first studied systematically.

We are not aware of any subgroup-growth-theoretic interpretation of the ideal zeta functions \( \zeta_{L_G(\mathfrak{o})}(s) \) if \( \mathfrak{o} \) is not of characteristic zero with prime residue field, i.e. of the form \( \mathfrak{o} = \mathbb{Z}_p \).

1.4. Previous work. Theorem 1.1 was previously known in a number of extremal cases.

Example 1.3. For \( n = 1 \), the Lie rings \( L_{m,1} \) are all isomorphic to the Heisenberg Lie ring
\[
\mathfrak{h} = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 0 \rangle_{\mathbb{Z}}.
\]
Theorem 1.1 thus confirms and extends the well-known, prototypical formula
\[
\zeta_{\mathfrak{h}(\mathfrak{o})}(s) = \frac{1}{(1 - q^{-s})(1 - q^{1-s})(1 - q^{2-3s})}.
\]
cf. [7, Proposition 8.1]. For \( n > 1 \), however, the Lie rings \( L_{m,n} \) are pairwise non-isomorphic. For practical purposes, we may thus restrict attention to \( n \geq 2 \) as in [1].

Example 1.4. For \( n = 2 \), the Lie rings \( L_{m,2} \) are the Lie rings associated, by the Mal’cev correspondence, to the indecomposable \( \mathfrak{D}^* \)-groups of odd Hirsch length \( 2m + 3 \) featuring in the classification up to commensurability of the finitely generated class-2-nilpotent groups with 2-dimensional centre, developed in [6]. Their local normal zeta functions, and thus the ideal zeta functions of the Lie rings \( L_{m,2}(\mathbb{Z}_p) \), are essentially computed in [16, Proposition 2], in agreement with the relevant special cases of Theorem 1.1.
Example 1.5. For $m = 1$, the Lie rings $L_{1,n}$ are the Lie rings associated to the so-called Grenham groups $G_{n+1}$. These class-2-nilpotent groups have presentations

$$G_{n+1} = \langle x, y_1, \ldots, y_n, z_1, \ldots, z_n \mid \forall i \in \{1, \ldots, n\} : [x, y_i] = z_i, \ z_i \text{ central} \rangle.$$

The normal zeta functions of these groups, and thus the ideal zeta functions of the Lie rings $L_{1,n}(\mathbb{Z}_p)$, are computed in [17, Theorem 5].

1.5. Related work. The paper [1], which introduced the groups $\Delta_{m,n}$, computes their pro-isomorphic zeta functions $\zeta_{\Delta_{m,n}}(s)$, enumerating the subgroups of finite index in $\Delta_{m,n}$ whose profinite completions are isomorphic to that of $\Delta_{m,n}$. By general principles, these zeta functions also satisfy Euler product decompositions indexed by the rational primes, whose factors are rational functions in $p^{-s}$. In the notation of the current paper, [1, Theorem 1.4] establishes that the Euler factor $\zeta_{\Delta_{m,n},p}(s)$ of $\zeta_{\Delta_{m,n}}(s)$ at a rational prime $p$, enumerating the relevant subgroups of $\Delta_{m,n}$ of $p$-power index, is of the form

$$\zeta_{\Delta_{m,n},p}(s) = I_1(p^{-1}; z_{n+1}) \cdot I_n(p^{-1}; (z_i)_{i=1}^n)$$

for explicitly given “numerical data” $z_i = p^{\alpha_i(m,n) - s \beta_i(m,n)}$, for integers $\alpha_i(m,n), \beta_i(m,n)$, comparable to (but different from) those given in [1.6].

The subgroup zeta functions $\zeta_{\Delta_{m,n}}(s) = \sum_{H \leq \Delta_{m,n}} |\Delta_{m,n}: H|^{-s}$, enumerating all finite index subgroups of the groups $\Delta_{m,n}$, are known explicitly only for $m = 1$, i.e. the Grenham groups from Example [1.5] cf. [18].

For a nonzero prime ideal $p$ in a number ring $\mathcal{O}$, the $\mathcal{O}_p$-ideal zeta functions $\zeta_{\mathcal{O}_p}^{\mathcal{O}_p}(s) = \sum_{H \leq \mathcal{O}_p} |\mathcal{O}_p: H|^{-s}$, the latter have been computed for primes $p$ which are unramified in $\mathcal{O}$ in [12] and for primes $p$ which are non-split in $\mathcal{O}$ in [13]. In the paper [2] we generalize these computations to cover the ideal zeta functions of algebras arising from a large class of Lie rings, including the Grenham Lie rings $L_{1,n}$, via base extensions with various compact discrete valuation rings. The paper gives a survey of applications of Igusa functions in the area of zeta functions of groups and rings, and is built on a generalization of Igusa functions.

1.6. Organization and notation. We prove Theorem 1.1 in Section 2 using the general method introduced in [17]. In Section 3 we collect a number of corollaries and porisms, notably pertaining to global analytic properties of ideal zeta functions, functional equations satisfied by local ideal zeta functions, their behaviour at zero, topological and reduced ideal zeta functions, graded ideal zeta functions, and the representation zeta functions associated to the groups $\Delta_{m,n}$.

We write $\mathbb{N} = \{1, 2, \ldots\}$ and $X_0 = X \cup \{0\}$ for a subset $X \subseteq \mathbb{N}$. Given $n \in \mathbb{N}_0$, we write $[n] = \{1, 2, \ldots, n\}$. The notation $I = \{i_1, \ldots, i_r\} \subseteq \mathbb{N}_0$ indicates that $i_1 < i_2 < \cdots < i_r$.

We write $\text{Mat}_{a,b}(R)$ for the set of $a \times b$ matrices over a ring $R$. The ring’s units are denoted by $R^\times$. We write $\text{Mat}_a(R)$ instead of $\text{Mat}_{a,a}(R)$. Given matrices $A_1, \ldots, A_n$ with the same number of rows, we write $(A_1 | \cdots | A_n)$ for their juxtaposition (or concatenation). We write $\text{Id}_n$ for the $n \times n$-identity matrix and $\text{0}_{a,b}$ for the zero matrix $(0)_{ij} \in \text{Mat}_{a,b}(R)$. Sometimes we write $\text{0}$ for a zero matrix whose dimensions are clear from the context.

We denote by $\wp$ a compact discrete valuation ring of arbitrary characteristic, with maximal ideal $p$, uniformizer $\pi \in p \setminus p^2$, and residue field cardinality $q$. The $p$-adic valuation on $\wp$ will be denoted by $\text{val}_p$.

Given a property $P$, the “Kronecker delta” $\delta_P$ is equal to 1 if $P$ holds and equal to 0 otherwise.
2. Proof of Theorem 1.1

We maintain, to a large extent, the notation of [17]. Throughout, \( m, n \in \mathbb{N} \) with \( n \geq 2 \) are arbitrary but fixed.

2.1. Commutator matrix. A key object in the computation of various zeta functions associated to the Lie ring \( L_{m,n} \) is its commutator matrix with respect to a \( \mathbb{Z} \)-basis. Consider the ordered (!) \( \mathbb{Z} \)-basis

\[
\mathcal{B}_{m,n} = (x_e, y_f, z_1, \ldots, z_n)_{e \in \mathbb{E}, f \in \mathbb{F}}
\]

of \( L_{m,n} \), where both the elements \( x_e \) and \( y_f \) are given, respectively, in reverse lexicographical ordering, viz. the ordering obtained from the usual lexicographical orderings on \( \mathbb{E} \) resp. \( \mathbb{F} \) but read backwards; cf. Example 2.4 (3).

Recall that the commutator matrix \( M_{m,n} \) of \( L_{m,n} \) with respect to \( \mathcal{B}_{m,n} \) is given as follows. For \( i, j \in [d(m,n)] \) and \( w_i, w_j \in \mathcal{B}_{m,n} \), write \( [w_i, w_j] = \sum_{k=1}^{n} \lambda_{ij}^k z_k \), for structure constants \( \lambda_{ij}^k \in \mathbb{Z} \). Then set \( Y = (Y_1, \ldots, Y_n) \) and

\[
M_{m,n}(Y) = \begin{pmatrix}
\sum_{k=1}^{n} \lambda_{ij}^k Y_k
\end{pmatrix}_{ij} \in \text{Mat}_{d(m,n)}(\mathbb{Z}[Y]).
\]

To give a general, explicit description of \( M_{m,n} \) we introduce the following notation. For \( e \in \mathbb{N} \) and a variable \( Y \), write

\[ S_e(Y) = Y \cdot \text{Id}_e \in \text{Mat}_e(\mathbb{Z}[Y]) \]

for the generic \((e \times e)\)-scalar matrix. Set \( Y' = (Y_2, \ldots, Y_n) \). We set \( B_{m,1}(Y) = Y \in \text{Mat}_1(\mathbb{Z}[Y]) \) and define recursively, for \( n \geq 2 \),

\[
B_{m,n}(Y) = \begin{pmatrix}
S_{e(1,n-1)}(Y_1) & S_{e(2,n-1)}(Y_1) \\
B_{1,n-1}(Y') & S_{e(2,n-1)}(Y_1) \\
& \ddots & \ddots & \ddots \\
& & B_{m-1,n-1}(Y') & S_{e(m,n-1)}(Y_1) \\
& & & B_{m,n-1}(Y') & S_{e(m,n-1)}(Y_1)
\end{pmatrix}.
\]

Note that \( B_{1,n}(Y) = (Y_1, \ldots, Y_n)^\text{tr} \in \text{Mat}_{d(n)}(\mathbb{Z}[Y]) \). Moreover, one checks easily that

\[ B_{m,n}(Y) \in \text{Mat}_{d(m,n),e(m,n)}(\mathbb{Z}[Y]), \]

say by using parts (1) and (2) of the following lemma.

**Lemma 2.1.**

1. \( \sum_{j=1}^{m} e(j, n-1) = e(m, n) \),
2. \( e(m, n) + f(m, n-1) = f(m, n) \),
3. \( \sum_{j=1}^{n} e(m, j) = f(m, n) \)

**Proof.** Trivial. (One may want to use parallel summation for (1) and (3); cf. [5] p. 174].) \( \square \)

Various examples of \( B_{m,n} \) are given in Example 2.4. The following is evident.

**Lemma 2.2.** If \( y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n \setminus \{0\} \), then \( B_{m,n}(y) \) has full rank \( e(m,n) \).

**Proposition 2.3.** The commutator matrix of \( L_{m,n} \) with respect to the \( \mathbb{Z} \)-basis \( \mathcal{B}_{m,n} \) is

\[
M_{m,n}(Y) = \begin{pmatrix}
B_{m,n}(Y) & -B_{m,n}(Y)^\text{tr}
\end{pmatrix} \in \text{Mat}_{d(m,n)}(\mathbb{Z}[Y]).
\]
Proof. Given the defining relations \([11]\) it is clear that the \((i, j)\)-entry of \(M_{m,n}(Y)\) vanishes if \(i\) and \(j\) are either both at most or both greater than \(c(m,n)\). The antisymmetry of \(M_{m,n}(Y)\) is also evident, as \(L_{m,n}\) is a Lie ring. To justify the specific shape of \(B_{m,n}(Y)\), recall that its columns are indexed by the generators \(x_e, e \in E\), whereas its rows are indexed by the generators \(y_f, f \in F,\) of \(L_{m,n}\). By definition of the commutator matrix \(M_{m,n}(Y)\), the entry in position \((x_e, y_f)\) is \(Y_i \delta_{[x_e,y_f]=z_i}\) for all \(i \in [n]\); cf. \([12]\).

Crucially, both sets of generators are ordered reverse-lexicographically. Therefore, for \(j = 1, \ldots, m\), the “\(j\)-th column block” \(B_{m,n}^{(j)}(Y)\) of \(B_{m,n}(Y)\), comprising columns numbered

\[
\sum_{s < j} e(s, n - 1) + 1, \ldots, \sum_{s \leq j} e(s, n - 1),
\]

echos the relations involving generators \(x_e\) indexed by elements \(e \in \mathbb{N}_0^n\) with first coordinate \(m - j\), i.e. of the form

\[
e = (m - j, e') \text{ for some } e' \in E(j, n - 1).
\]

Likewise, for \(i = 1, \ldots, m + 1\), the “\(i\)-th row block” of \(B_{m,n}(Y)\), comprising rows numbered

\[
\sum_{r < i} e(r, n - 1) + 1, \ldots, \sum_{r \leq i} e(r, n - 1),
\]

echos the relations involving generators \(y_f\) indexed by elements \(f \in \mathbb{N}_0^n\) with first coordinate \(m - i + 1\), i.e. of the form

\[
f = (m - i + 1, f') \text{ for some } f' \in F(i - 1, n - 1).
\]

We describe in detail the submatrices of the column block \(B_{m,n}^{(j)}(Y)\) defined by its intersection with the \(i\)-th row blocks of \(B_{m,n}(Y)\), thereby justifying the claim that

\[
B_{m,n}^{(j)}(Y) = \begin{pmatrix}
0_{e(j-1,n),e(j,n-1)} & S_{e(j,n-1)}(Y_1) & B_{j,n-1}(Y') \\
& 0_{f(m,n)-j,f(j,n),e(j,n-1)}
\end{pmatrix}.
\]

1. If \(i < j\), then the relevant rows of \(B_{m,n}^{(j)}(Y)\) comprise the relations between generators indexed by elements of the form

\[
e = (m - j, e'),
\]

\[
f = (m - i + 1, f').
\]

However, \((m - i + 1) - (m - j) = j - i + 1 \geq 2\), so \([x_e, y_f] = 0\) for the relevant elements, and hence the relevant submatrix of \(B_{m,n}^{(j)}(Y)\) is \(0_{e(j-1,n),e(j,n-1)}\).

2. If \(i = j\), then the relevant rows of \(B_{m,n}^{(j)}(Y)\) comprise the relations between generators indexed by elements of the form

\[
e = (m - j, e'),
\]

\[
f = (m - j + 1, f').
\]

As \([x_e, y_f] = z_1 \delta_{e'=f'}\), the relevant submatrix of \(B_{m,n}^{(j)}(Y)\) is \(S_{e(j,n-1)}(Y_1)\).

3. If \(i = j + 1\), then the relevant rows of \(B_{m,n}^{(j)}(Y)\) comprise the relations between generators indexed by elements of the form

\[
e = (m - j, e') \text{ for some } e' \in E(j, n - 1),
\]

\[
f = (m - j, f') \text{ for some } f' \in F(j, n - 1).
\]

This justifies the claim that the relevant submatrix of \(B_{m,n}^{(j)}(Y)\) is \(B_{j,n-1}(Y')\).
(4) If $i > j + 1$, then the relevant rows of $B^{(j)}_{m,n}(Y)$ comprise the relations between generators indexed by elements of the form

$$
eq (m - j, e'),$$

$$f = (m - i + 1, f').$$

However, $(m - i + 1) - (m - j) = j - i + 1 < 0$, so $[x_e, y_f] = 0$ for the relevant elements, and hence the relevant submatrix of $B^{(j)}_{m,n}(Y)$ is $0_{(f(m,n) - f(j,n)), e(j,n-1)}$.  

\[Example 2.4.\]

(1) For $n = 2$, we obtain

$$B_{m,2}(Y_1, Y_2) = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & \ddots \\ \vdots & \ddots & Y_1 \\ Y_2 \end{pmatrix} \in \text{Mat}_{m+1,m}(\mathbb{Z}[Y_1, Y_2]).$$

Up to a simple reordering of rows and columns and swapping the variables $Y_1$ and $Y_2$, the matrix $M_{m,2}(Y_1, Y_2)$ is the commutator matrix described in [16, Theorem 4] (essentially [7, Theorem 6.3]) associated to the indecomposable $\mathcal{O}$-group $\Delta_{m,2}$; cf. Example [1.3].

(2) For $m = 1$, we obtain

$$M_{1,n}(Y_1, \ldots, Y_n) = \begin{pmatrix} Y_1 & -Y_1 & \ldots & -Y_n \\ Y_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Y_1 \\ Y_1 & \ldots & \ldots & Y_1 \end{pmatrix} \in \text{Mat}_{n+1}(\mathbb{Z}[Y_1, \ldots, Y_n]),$$

the commutator matrix of the Grenham Lie ring $L_{1,n}$ with respect to the $\mathbb{Z}$-basis $(x, y_1, \ldots, y_n, z_1, \ldots, z_n)$; cf. Example [1.5].

(3) For $m = 2, n = 3$,

$$B_{2,3} = \{e_{(1,0,0)}, e_{(0,1,0)}, e_{(0,0,1)}, f_{(1,0,0)}, f_{(1,1,0)}, f_{(1,0,1)}, f_{(0,2,0)}, f_{(0,1,1)}, f_{(0,0,2)}, z_1, z_2, z_3\},$$

yielding

$$B_{2,3}(Y_1, Y_2, Y_3) = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}.$$  

2.2. Informal overview of the proof. We use the general method introduced in [17]. The fact that there only the case $\mathfrak{o} = \mathbb{Z}_p$ is treated explicitly is inconsequential: all that is needed is the fact that $\mathbb{Z}_p$ is a compact discrete valuation ring.

According to [17, Lemma 1] (essentially [7, Lemma 6.1]) there exists a rational function $A^\mathfrak{o}_{m,n}$ in $q$ and $q^{-s}$ such that

$$\zeta^\mathfrak{o}_{L_{m,n}}(s) = \zeta^\mathfrak{o}_{d(m,n)}(s) \frac{1}{1 - q^{d(m,n)n - sh(m,n)}} A^\mathfrak{o}_{m,n}(q, q^{-s});$$

cf. Remark [2.5] below. The function $A^\mathfrak{o}_{m,n}$ may be viewed as a generating function enumerating the values of two integer-valued functions $w$ and $w'$ on the set $Y_n$ of homothety classes of lattices
in $Z(L_{m,n}(\mathfrak{o})) \cong \mathfrak{o}^n$, viz. vertices in the affine Bruhat-Tits building of type $\widetilde{A}_{n-1}$ associated to
the group $\mathrm{GL}_n(k)$, where $k = \mathrm{Frac}(\mathfrak{o})$ is the field of fractions of the local ring $\mathfrak{o}$:

\begin{equation}
A_{m,n}^q(q, q^{-s}) = \sum_{[\Lambda'] \in \mathcal{V}_n} q^{d(m,n)w([\Lambda'])-s w'([\Lambda'])}.
\end{equation}

The function $w$ captures the $(\log_q)$ of the index of the maximal integral element $\Lambda'_\text{max}$ of $[\Lambda']$ in $Z(L_{m,n}(\mathfrak{o}))$. It is a simple function of the elementary divisors of $\Lambda'_\text{max}$ relative to $\mathfrak{o}^n$; cf. (2.3).

The function $w'$ records, in addition, the index in $L_{m,n}(\mathfrak{o})$ of the lattice $X([\Lambda'])$ in $L_{m,n}(\mathfrak{o})$, defined by the condition

\begin{equation}
X([\Lambda'])/\Lambda'_\text{max} = Z(L_{m,n}(\mathfrak{o})/\Lambda'_\text{max});
\end{equation}

see (2.10). We will use the interpretation of this index as the index of the kernel of a system of linear congruences on $\mathfrak{o}^{d(m,n)}$ provided by [17, Theorem 6]; see Proposition 2.7.

The discussion so far applies, \textit{mutatis mutandis}, to (the $\mathfrak{o}$-points of) any class-2-nilpotent Lie ring. In general, the index of $X([\Lambda'])$ in the Lie ring’s abelianization will depend in an arithmetically subtle way on $[\Lambda']$. The key to the proof of Theorem 1.1 is the realization that, for the Lie rings $L_{m,n}$, the index of $X([\Lambda'])$ depends solely, and in a $(\log_q)$-linear fashion, on the elementary divisors of $\Lambda'_\text{max}$. Consequently, the rational function $A_{m,n}^q$ may be expressed in terms of an Igusa function of degree $n$. In the course of the proof of these facts we will compute the relevant $(\log_q)$-linear functionals explicitly, making heavy use of the combinatorial description (2.4) of the commutator description $([\Lambda'])$ of the center of $\Gamma_{m,n}$.

**Remark 2.5.** [17, Lemma 1] does not hold in the generality proclaimed in [17]. To apply for a prime $p$, the centre of the reduction of $G_p$ modulo $p$ needs to coincide with the reduction modulo $p$ of the centre of $G_p$. This condition is satisfied generically, but may fail for finitely many primes. (The main results of [17] are unaffected by this, as they are only stated (and known to hold only) for almost all primes $p$.) In the case of the groups $\Delta_{m,n}$, this set of exceptional primes is indeed empty, as one checks without difficulty, so the conclusion of [17, Lemma 1] applies for all primes $p$. I am grateful to the referee for pointing out these facts.

### 2.3. Parametrizing lattices

We recall, e.g. from [17], a parametrization of maximal integral lattices inside $\mathfrak{o}^n$. Let $\Lambda' \subseteq \mathfrak{o}^n$ be a maximal $\mathfrak{o}$-sublattice, i.e. $\pi^{-1}\Lambda' \not\subseteq \mathfrak{o}^n$, where $\pi \in p \setminus p^2$ is a uniformizer. The lattice $\Lambda'$ is said to be of \textit{type} $\nu(\Lambda') = (I, r_I)$ if

\[ I = \{i_1, \ldots, i_\ell\} < \subseteq [n - 1], \quad r_I = (r_i)_{i \in I} \in \mathbb{N}_I, \]

and $\Lambda'$ has elementary divisors

\begin{equation}
\left( (1)^{i_1}, (\pi r_1)^{i_2-i_1}, \ldots, (\pi^{\sum_{i \in I} r_i})^{(n-i_\ell)} \right) =: (\pi^{\nu}) \in \mathbb{N}_n.
\end{equation}

Clearly $|\mathfrak{o}^d : \Lambda'| = q^{\sum_{i \in I} r_i(n-i)}$, whence, in the notation of [16, Definition 2],

\begin{equation}
w([\Lambda']) = \sum_{i \in I} r_i(n-i).
\end{equation}

It is well known and not hard to show (cf., for instance, [17, Lemma 2]) that

\begin{equation}
f_{I, r_I}(q) := \#\{\Lambda' \leq \mathfrak{o}^d | \Lambda' \text{ maximal and of type } (I, r_I)\} = \binom{n}{I} q^{\sum_{i \in I} r_i(n-i)}.
\end{equation}

Here $\binom{n}{I} q^{-1}$ is the value of the Gaussian multinomial $\binom{n}{I} Y \in \mathbb{Z}[Y]$ at $Y = q^{-1}$. Of central importance in the following is the elementary fact that the group $\Gamma_n := \mathrm{GL}_n(\mathfrak{o})$ acts transitively on the set of maximal lattices of given type $\nu$. Denoting, for $i \in [n]$, by $\varepsilon_i$ the $i$-th standard basis vector of $\mathfrak{o}^n$, the lattice

\[ \Lambda' = \bigoplus_{i=1}^n (\pi^{\nu})_i \mathfrak{o} \varepsilon_i \]
is evidently of type $\nu$. The stabilizer subgroup $\Gamma_{\nu}$ of $\Lambda'$ in $\Gamma_n$ is easily described explicitly, but we will not need such a description. What we will need are two facts. First, $\Gamma_{\nu}$ contains the (Borel) subgroup $B_n$ of lower-triangular matrices in $\Gamma_n$. (Note that the matrix description of $\Gamma_{\nu}$ on p. 1203 of [19] in terms of block matrices which are block upper-triangular modulo $p$ is given with respect to the reverse ordering of the elementary divisors (2.4).) Second, the orbit-stabilizer theorem gives us a bijection between maximal lattices of type $(\nabla, \nu)$.

Fix a coset $\alpha \Gamma_{\nu}$. We claim that, after a permutation of the rows if necessary (corresponding to a monomial change of $\mathfrak{a}$-basis for $\mathfrak{s}^n$), it contains a representative of the form

$$
\alpha_0 = \begin{pmatrix}
\alpha_{1n} & \cdots & \alpha_{2n-1} \\
\vdots & \ddots & \vdots \\
\alpha_{n1} & \cdots & \alpha_{nn}
\end{pmatrix} \in \Gamma_n.
$$

Indeed, if the $(1,n)$-entry of $\alpha_0$ is a unit (as we may assume without loss of generality), we may use it to "clear" the remaining entries in the first row of $\alpha_0$ by right-multiplication by a suitable element of $B_n \leq \Gamma_{\nu}$. The claim follows inductively.

We write $\alpha_0 = (\alpha^{(1)} | \cdots | \alpha^{(d)})$, i.e. $\alpha^{(j)}$ denotes the $j$-th column of $\alpha_0$. Note that the antidiagonal entries of $\alpha_0$ are all units: $\alpha_{j,n+1-j} \in \mathfrak{s}$ for all $j \in [n]$.

2.4. Solving linear congruences. Let $\Lambda' \leq \mathfrak{s}^n$ be a maximal lattice of type $(I, r_I)$, corresponding to a coset $\alpha \Gamma_{\nu}$ as described in Section 2.3. By [17] Theorem 6 the index $[L_{m,n}(\mathfrak{o}) : X([\Lambda'])]$ equals the index in $\mathfrak{o}^{d(m,n)}$ of the following system of linear congruences, where we write $g = (g^{(1)}, g^{(2)}) \in \mathfrak{o}^{e(m,n)} \times \mathfrak{o}^{f(m,n)} \cong \mathfrak{o}^{d(m,n)}$:

$$
\forall j \in [n] : g_{m,n}(\alpha^{(j)}) \equiv 0 \mod (\pi^\nu)_j.
$$

Set $r := \sum_{i \in I} r_i$.

Lemma 2.6. (2.7) holds if and only if $g^{(1)} \equiv 0 \mod (\pi^r)$ and

$$
g^{(2)} (\pi^r B_{m,n}(\alpha^{(1)}) | \cdots | \pi^{\sum_{i \in I} r_i} B_{m,n}(\alpha^{(j)}) | \cdots | B_{m,n}(\alpha^{(n)})) \equiv 0 \mod (\pi^r).
$$

Proof. By multiplying each of the congruences in (2.7) by the appropriate power of the uniformizer $\pi$ we may consider them all as congruences modulo $(\pi^r)$. By concatenating the relevant matrices, we obtain that (2.7) is equivalent to the single congruence

$$
g \left( \pi^r M_{m,n}(\alpha^{(1)}) | \cdots | \pi^{\sum_{i \in I} r_i} M(\alpha^{(j)}) | \cdots | M_{m,n}(\alpha^{(n)}) \right) \equiv 0 \mod (\pi^r).
$$

Note that the vector $\alpha^{(n)} \in \mathfrak{o}^n$ is nonzero modulo $p$. By Proposition 2.3 and Lemma 2.2

$$
g_{m,n}(\alpha^{(n)}) = g \left( B_{m,n}(\alpha^{(n)}) - B_{m,n}(\alpha^{(n)})^\text{tr} \right) \equiv 0 \mod (\pi^r)
$$

thus only holds if $g^{(1)} \equiv 0 \mod (\pi^r)$. Deleting the first $e(m,n)$ rows from the matrix in (2.9) one sees that, in this case, (2.7) is equivalent to (2.8).

Note that, in the $f(m,n) \times (e(m,n) \cdot n)$ matrix in (2.8), the first $i_1$ blocks $\pi^r B_{m,n}(\alpha^{(j)})$, $j \in [s_1]$, i.e. the first $e(m,n) \cdot i_1$ columns, are of course redundant. Recall from [16, Def. 2] that, with $X([\Lambda'])$ defined as in (2.3),

$$
\nu'([\Lambda']) = \log_q([\mathfrak{o}^n : \Lambda']) + \log_q([L_{m,n}(\mathfrak{o}) : X([\Lambda'])]).
$$

Proposition 2.7. The index of the lattice of elements $g \in \mathfrak{o}^{d(m,n)}$ satisfying the simultaneous congruences (2.7) equals

$$
q^{\sum_{i \in I} r_i (e(m,n)+\sum_{j=1}^n e(m,j))}.
$$
In other words,
\[ w'([\Lambda']) = \sum_{i \in I} r_i \left( n - \ell + e(m, n) + \sum_{j=\ell+1}^{n} e(m, j) \right). \]

Proof. For \( j = 1, \ldots, n \), write
\[ B_j := \pi \sum_{i \geq j} r_i B_{m,n}(\alpha(j)) \in \text{Mat}_{f(m,n), e(m,n)}(\mathfrak{o}) \]
for the \( j \)-th column block of the matrix in (2.8). Note that \( \text{val}_\pi(B_j) = \sum_{i \geq j} r_i \in \mathbb{N}_0 \). Set
\[ B := (B_1 | \cdots | B_n) \in \text{Mat}_{f(m,n), ne(m,n)}(\mathfrak{o}). \]

In the light of Lemma 2.6 we need to prove that the index in \( \mathfrak{o}^{f(m,n)} \) of the solutions of the congruence
\[ g^{(2)} B \equiv 0 \mod (\pi^r) \]
equals \( q \sum_{i \in I} r_i (\sum_{j=\ell+1}^{n} e(m,j)) \). For this it suffices to show that
\[ (2.11) \quad B \text{ is equivalent to } (0_{f(m,n), ne(m,n) - f(m,n)} | D_{f(m,n)}) \in \text{Mat}_{f(m,n), ne(m,n)}(\mathfrak{o}), \]
where
\[ D_{f(m,n)} := \begin{pmatrix} \pi \sum_{i \geq n-1} r_i \text{Id}_{e(m,n-1)} & \text{Id}_{e(m,n)} \\ \vdots \\ \pi^r \text{Id}_{e(m,1)} \end{pmatrix} \in \text{Mat}_{f(m,n)}(\mathfrak{o}). \]

We proceed inductively, replacing \( B \) successively by equivalent matrices.

We first note that the top \( e(m,n) \) rows of \( B_n \) form a matrix \( \tilde{B} \in \text{GL}_{e(m,n)}(\mathfrak{o}) \), as \( \alpha_{1n} \in \mathfrak{o}^* \).

We use \( \tilde{B} \) to clear—by suitable column operations—all other entries in the top \( e(m,n) \) rows of \( B \). Note that this does not affect the last \( e(m,j) \) columns in either of the matrices \( B_j, j = 1, \ldots, n - 1 \), nor the valuations of these matrices. We now use \( \tilde{B} \) to clear—by suitable row operations—all entries of \( B \) below \( \tilde{B} \), leaving the other columns unaffected. We may then also assume that \( \tilde{B} = \text{Id}_{e(m,n)} \).

We have thus replaced \( B \) by an equivalent matrix of the form
\[ \begin{pmatrix} B'_1 & \cdots & B'_{n-1} & \text{Id}_{e(m,n)} \end{pmatrix}, \]
where, for each \( j = 1, \ldots, n - 1 \), the matrix \( B'_j \) has valuation \( \sum_{i \geq j} r_i \) and the matrices \( \begin{pmatrix} 0 \\ B'_j \end{pmatrix} \) and \( B_j \in \text{Mat}_{f(m,n), e(m,n)}(\mathfrak{o}) \) coincide in their last \( e(m,j) \) columns. Set
\[ B' = (B'_1 | \cdots | B'_{n-1}) \in \text{Mat}_{f(m,n-1), (n-1)e(m,n)}(\mathfrak{o}). \]

The top \( e(m,n-1) \) rows and last \( e(m,n-1) \) columns of \( B'_{n-1} \) form a matrix \( \pi \sum_{i \geq n-1} r_i \) \( \tilde{B} \) for \( \tilde{B} \in \text{GL}_{e(m,n-1)}(\mathfrak{o}) \). We may use it to clear all other entries in the top \( e(m,n-1) \) rows of \( B' \). Note that this does not affect the last \( e(m,j) \) columns in either of the matrices \( B'_j, j = 1, \ldots, n - 2 \), nor the valuations of these matrices. We now use \( \pi \sum_{i \geq n-1} r_i \) \( \tilde{B} \) to clear all entries of \( B' \) below \( \tilde{B} \), leaving the other columns unaffected. We may then also assume that \( \tilde{B} = \text{Id}_{e(m,n-1)} \).

We have thus replaced \( B' \) by an equivalent matrix of the form
\[ \begin{pmatrix} B''_1 & \cdots & B''_{n-2} & \pi \sum_{i \geq n-1} r_i \text{Id}_{e(m,n-1)} \end{pmatrix}, \]
where, for \( j = 1, \ldots, n - 2 \), the matrices \( B_j' \in \text{Mat}_{f(m,n-2),e(m,n)}(\mathfrak{o}) \) each have valuation 
\[ \sum_{i \geq j} r_i \text{ and the matrices } \left( \begin{array}{c} 0 \\ B_j' \end{array} \right), \]
and \( B_j \in \text{Mat}_{f(m,n),e(m,n)}(\mathfrak{o}) \) coincide in their last \( e(m,j) \) columns.

The claim \((2.11)\) follows by continuing inductively in this manner. \( \Box \)

2.5. **Completion of the proof of Theorem 1.1** We are now ready to complete the computation of the rational function \( A^q_{m,n}(q,q^{-s}) \) featuring in \((2.2)\). Indeed, using \((2.2)\), \((2.5)\), \((2.6)\), and Proposition 2.7, we obtain

\[
A^q_{m,n}(q,q^{-s}) = \sum_{[\Lambda'] \in V_n} q^{d(m,n)w([\Lambda']) - s \ell w'(\Lambda')}
= \sum_{I \subseteq [n-1]} \sum_{r_j \in \mathbb{N}} f_I, r_I(q) \sum_{s \in I} r_s \left( (n-\ell)d(m,n) - s(n-\ell+e(m,n)+\sum_{j=1}^n e(m,j)) \right)
= \sum_{I \subseteq [n-1]} \left( \begin{array}{c} n \\ I \end{array} \right) \prod_{s \in I} q^{a^z_{s}(m,n)-s b^z_{s}(m,n)}
\]

with \( a^z_s(m,n) \) and \( b^z_s(m,n) \) defined as in \((1.6)\). Using Lemma 2.1 (3) one easily computes
\[
(a^q_0(m,n), b^q_0(m,n)) = (d(m,n), n, h(m,n)),
\]
whence, using \((1.3)\), we obtain that
\[
\frac{1}{1 - q^{d(m,n)n-s h(m,n)}} A^q_{m,n}(q,q^{-s}) = I_n \left( q^{-1}; \left( q^{a^q_i(m,n)-s b^q_i(m,n)} \right)_{i=0}^{n-1} \right).
\]

Theorem \(1.1\) follows now from \((2.2)\).

3. **Corollaries and porisms**

We record a few consequences of Theorem \(1.1\) and its proof. Throughout, \( \mathfrak{o} \) denotes, as before, a compact discrete valuation ring. Let \( \mathcal{O} \) be the ring of integers of a number field \( K \), with Dedekind zeta function \( \zeta_K(s) \). We set \( L_{m,n}(\mathcal{O}) := L_{m,n} \otimes \mathbb{Z} \mathcal{O} \).

3.1. **Global analytic properties.**

**Corollary 3.1.** The ideal zeta function \( \zeta^q_{L_{m,n}(\mathcal{O})}(s) \) has abscissa of convergence \( \alpha^q(m,n) = d(m,n) \) and allows for meromorphic continuation to (at least) the complex half-plane
\[
\{ s \in \mathbb{C} \mid \Re(s) > \beta^q(m,n) \},
\]
where
\[
\beta^q(m,n) := \max \left\{ \frac{a^q_i(m,n) - 1}{b^q_i(m,n)} \mid i = 0, \ldots, n - 1 \right\},
\]
and even the whole complex plane if \( n \leq 2 \). In any case, the continued function has a simple pole at \( s = \alpha^q(m,n) \).

**Proof.** It is well known (see \((1.3)\)) that \( \zeta_{\mathcal{O}_d}(s) = \prod_{i=0}^{d-1} \zeta_K(s-i) \), has abscissa of convergence \( s = d \), and admits meromorphic continuation to the whole complex plane to a function that has a simple pole at \( s = d \). It thus suffices to note that the Euler product
\[
(3.1) \quad \prod_{p \in \text{Spec}(\mathcal{O}) \setminus \{0\}} I_n \left( q^{-1}; \left( q^{a^q_i(m,n)-s b^q_i(m,n)} \right)_{i=n-1}^0 \right)
\]
has
\( (A) \) abscissa of convergence \( \max \left\{ \frac{a^q_i(m,n)+1}{b^q_i(m,n)} \mid i = 0, \ldots, n - 1 \right\} < d(m,n) \) and
\( (B) \) meromorphic continuation to \( \{ s \in \mathbb{C} \mid \Re(s) > \beta^q(m,n) \} \).
To verify (A) we observe that the Euler factors of (3.1) may be written in the form

$$\sum_{w \in S_n} \frac{-\ell(w)}{q_p^{\ell(w)}} \prod_{j \in \text{Des}(w)} a_j^{\varepsilon(m,n)-s} b_j^{\varepsilon(m,n)} \prod_{i=0}^{n-1} \left(1 - q_p^{a_i^{\varepsilon(m,n)-s} b_i^{\varepsilon(m,n)}}\right).$$

Both numerator and denominator of this expression are given by bivariate polynomial expressions in $q_p$ and $q_p^{-s}$ with integer coefficients. The abscissa of convergence of the Euler product

$$\prod_{p \in \text{Spec}(\mathcal{O}) \setminus \{0\}} \frac{1}{\prod_{i=0}^{n-1} \left(1 - q_p^{a_i^{\varepsilon(m,n)-s} b_i^{\varepsilon(m,n)}}\right)}$$

arising from the denominators of (3.2) is $\max \left\{ \frac{a_i^{\varepsilon(m,n)+1}}{b_i^{\varepsilon(m,n)}} \mid i = 0, \ldots, n-1 \right\}$. We omit the elementary proof of the fact that this quantity is dominated by $d(m,n)$. It is a simple exercise to check that it dominates the abscissa of convergence of the Euler product

$$\prod_{p \in \text{Spec}(\mathcal{O}) \setminus \{0\}} \sum_{w \in S_n} q_p^{-\ell(w)} \prod_{j \in \text{Des}(w)} a_j^{\varepsilon(m,n)-s} b_j^{\varepsilon(m,n)}$$

over the numerators of (3.2). The latter is given, for instance, by the formula in [3, Lemma 5.4].

To verify claim (B), we employ [3, Lemma 5.5] and note that

$$\max \left\{ -\ell(w) + \sum_{j \in \text{Des}(w)} a_j^{\varepsilon(m,n)} \mid w \in S_n \setminus \{e\} \right\}$$

is attained at one of the elements $w \in S_n$ with $\# \text{Des}(w) = 1$.

The stronger claim for $n = 2$ follows from the observation that the Euler product (3.1) is

$$\prod_{p \in \text{Spec}(\mathcal{O}) \setminus \{0\}} \frac{1 + q_p^{1-s}(2m+1)}{(1 - q_p^{2m+2-s(2m+1)})(1 - q_p^{2(2m+1)-s(2m+3)})} = \frac{\zeta_K((2m+1)s - 2m - 2)\zeta_K((2m+3)s - 2(2m+1))\zeta_K((s-1)(2m+1))}{\zeta_K((s-1)(4m+2))}.$$

For $n = 1$ it follows from the fact that $\zeta_{L_{m,1}(\mathcal{O})}^q(s) = \zeta_K(s)\zeta_K(s-1)\zeta_K(3s-2)$; see (1.7).

Remark 3.2. It remains an interesting challenge to determine the maximal domain of meromorphy of the global ideal zeta functions $\zeta_{L_{m,n}(\mathcal{O})}^q(s)$ for general $m$ and $n$. The good analytic properties for $n \leq 2$ are, in any case, exceptional: for $n > 2$, the numerator of an Igusa function of degree $n$ will not, in general, factor nicely; see, for instance, Example 1.2 (where we obtain $\beta^2(2,3) = \max\{11-1, \frac{20}{10}-1\} = \frac{19}{10} < 9 = \alpha^2(2, 3)$).

3.2. Local functional equations.

Corollary 3.3.

$$\zeta_{L_{m,n}(\mathcal{O})}^q(s) \mid_{q \to q^{-1}} = (-1)^{h(m,n)} q^{(h(m,n)) - s(d(m,n)+h(m,n))} \zeta_{L_{m,n}(\mathcal{O})}^q(s).$$

Proof. Cf. [17, Theorem 4].

For almost all residue field characteristics, these functional equations had been established, in greater generality, in [19, Theorem C]; see also [20, Theorem 1.2 and Corollary 1.3].
3.3. p-Adic behaviour at zero. Rossmann has put forward the remarkable expectation that quite general local zeta functions associated with nilpotent algebras of endomorphisms should have predictable behaviour at \( s = 0 \). The following consequence of Theorem 1.1 establishes [9] Conjecture IV ([\( p \)-adic form]) in the relevant special cases.

Corollary 3.4.

\[
\frac{\zeta^a_{L_{m,n}(\phi)}(s)}{\zeta_{\varphi(h(m,n))}(s)} \bigg|_{s=0} = 1
\]

Proof. Note that both \( \zeta^a_{L_{m,n}(\phi)}(s) \) and \( \zeta_{\varphi(h(m,n))}(s) \) have a simple pole at \( s = 0 \). By [1,4] it suffices to observe that

\[
I_n \left( q^{-1}; \left( q^{n-i(i+d(m,n))} \right)_{i=n-1}^0 \right) = \zeta_0^n(-d(m,n)) = \frac{1}{\prod_{i=0}^{n-1} (1 - q^{d(m,n)+i})}. \quad \Box
\]

3.4. Topological and reduced ideal zeta functions. The next corollaries concern the topological and reduced ideal zeta functions associated to the Lie rings \( L_{m,n} \). Informally, these are two related (but distinct) limiting objects capturing the behaviour of \( \zeta^a_{L_{m,n}(\phi)}(s) \) as \( \phi \rightarrow 1 \); see [9] and [3], respectively, for details and precise definitions.

For our purposes, the following ad hoc definitions may suffice. Let \( Z(s) = I_n(q^{-1}; (x_i)_{i=1}^n) \) for numerical data \( x_i = q^{a_i-b_is} \), for integers \( a_i \in \mathbb{N}_0, b_i \in \mathbb{N} \). Define the topological zeta function \( Z_{\text{top}}(s) \in \mathbb{Q}(s) \) via

\[
Z(s) = Z_{\text{top}}(s)(q - 1)^{-n} + O((q - 1)^{-n+1})
\]

and the reduced zeta function

\[
Z_{\text{red}}(Y) := I_n(1; (Y^b_i)_{i=1}^n) \in \mathbb{Q}(Y).
\]

We omit the proofs of the following simple calculations.

Lemma 3.5.

1. \( Z_{\text{top}}(s) = \prod_{i=1}^{n!} (b_is-a_i)! \).
2. \( Z_{\text{red}}(Y) = \sum_{w \in S_n} \prod_{j \in \text{Des}(w)} Y^{b_j}. \)

Corollary 3.6.

\[
Z_{\text{red}}(Y)(1 - Y)^n|_{Y=1} = s^{-n} Z_{\text{top}}(s^{-1})|_{s=0} = \frac{n!}{\prod_{i=1}^{b_i}} \in \mathbb{Q}_{>0}.
\]

Corollary 3.7.

1. The topological ideal zeta function of \( L_{m,n} \) is given by

\[
\zeta^a_{L_{m,n},\text{top}}(s) = \frac{n!}{\prod_{j=0}^{d(m,n)-1} (s-j) \prod_{i=0}^{n-1} (b_i^a(m,n)s-a_i^a(m,n))} \in \mathbb{Q}(s).
\]

It has degree \( -h(m,n) \) in \( s \), a simple pole at \( s = 0 \) with residue \( \left( -1 \right)^{h(m,n)-1} \binom{n}{h(m,n)-1} ! \) and satisfies

\[
s^{-h(m,n)} \zeta^a_{L_{m,n},\text{top}}(s^{-1})|_{s=0} = \frac{n!}{\prod_{i=0}^{n-1} b_i^a(m,n)} =: \mu^a_{m,n} \in \mathbb{Q}_{>0},
\]

a nonzero rational number satisfying \( \mu^a_{m,n} h(m,n)! \in \mathbb{N} \).
2. The reduced ideal zeta function of \( L_{m,n} \) is given by

\[
\zeta^a_{L_{m,n},\text{red}}(Y) = \sum_{w \in S_n} \prod_{j \in \text{Des}(w)} Y^{b_{n-j}(m,n)}. \frac{b_j^a(m,n)}{(1 - Y)^{d(m,n)} \prod_{i=0}^{n-1} (1 - Y^{b_i^a(m,n)})}.
\]

It has degree \( -d(m,n) - h(m,n) \) in \( Y \), a pole of order \( h(m,n) \) at \( Y = 1 \), and satisfies

\[
\zeta^a_{L_{m,n},\text{red}}(Y)(1 - Y)^{h(m,n)}|_{Y=1} = \mu^a_{m,n} \in \mathbb{Q}_{>0}.
\]
Example 3.8. For $(m, n) = (2, 3)$ (see Example 1.2) we obtain
\[
\zeta^\mathfrak{gr}_{L_{2,3,\text{top}}}(s) = 1 \quad \text{for } s \neq 0,
\]
and
\[
\zeta^\mathfrak{gr}_{L_{2,3,\text{red}}}(Y) = \frac{1 + 2Y^7 + 2Y^{10} + Y^{17}}{(1 - Y)^9(1 - Y^7)(1 - Y^{10})(1 - Y^{12})},
\]
whence
\[
\mu_{2,3}^q = \frac{1}{140}.
\]

Remark 3.9. Together, corollaries 3.4 and 3.7 confirm the conjectures in [9] Sections 8.1 and 8.2 in the relevant special cases.

It remains an interesting challenge to give an intrinsic, algebraic interpretation of the “multiplicities” $\mu_{m,n}^q$. That they occur as invariants of both the reduced and the topological zeta functions seems remarkable.

3.5. Graded ideal zeta functions. Let $R$ be a ring as in Section 1.1. The graded Lie algebra associated to $L_{m,n}(R) : = L_{m,n} \otimes \mathbb{Z}$ is the $R$-Lie algebra
\[
\text{gr } L_{m,n}(R) = \frac{L_{m,n}(R)}{L_{m,n}(R)'} \oplus \frac{L'_{m,n}(R)}{=:\langle L(1) \rangle} \oplus \frac{L''_{m,n}(R)}{=:\langle L(2) \rangle}.
\]

An $R$-ideal $I$ of $\text{gr } L_{m,n}(R)$ is graded if $I = (I \cap I^{(1)}) \oplus (I \cap I^{(2)})$. The graded ideal zeta function of $L_{m,n}(R)$ is the Dirichlet series
\[
\zeta^\mathfrak{gr}_{L_{m,n}(R)}(s) = \sum_{I_{\mathfrak{gr}} \supseteq L_{m,n}(R)} |\text{gr } L_{m,n}(R) : I|^{-s},
\]
enumerating the graded ideals of $\text{gr } L_{m,n}(s)$ of finite index; cf. [11] and [8]. One advantage of writing $\zeta^\mathfrak{gr}_{L_{m,n}(R)}(s)$ in terms of the generating function $A_{m,n}^q(q, q^{-s})$ defined in (2.2) (see (2.2)) is that a trivial modification yields a formula for the graded ideal zeta function. Indeed,
\[
\zeta^\mathfrak{gr}_{L_{m,n}(R)}(s) = \zeta_{\mathfrak{d}(m,n)}(s) \cdot \frac{1}{1 - q^{-s}h(m,n)} A_{m,n}^q(q, q^{-s}),
\]
where
\[
A_{m,n}^q(q, q^{-s}) = \sum_{[\Lambda] \in \mathcal{V}_n} q^{-w([\Lambda])};
\]
cf. [8] Example 1.6]. Modifying the computation in Section 2.5 yields the following result.

Theorem 3.10.
\[
\zeta^\mathfrak{gr}_{L_{m,n}(R)}(s) = \zeta_{\mathfrak{d}(m,n)}(s) \cdot I_n \left( q^{-1}; \left( q^{i(n-i)-sb^i(m,n)} \right)_{i=0}^{0} \right),
\]
where, for $i \in \{0, 1, \ldots, n - 1\}$, the numerical data $b^i(m, n)$ is as in (1.6) in Theorem 1.1.

All the results recorded in Section 3.1 to 3.4 have “graded analogues”. We only note here the behaviour of $\zeta^\mathfrak{gr}_{L_{m,n}(R)}(s)$ at $s = 0$, to be compared with Corollary 3.3.

Corollary 3.11.
\[
\left. \frac{\zeta^\mathfrak{gr}_{L_{m,n}(R)}(s)}{\zeta_{\mathfrak{d}(m,n)}(s) \zeta_{\mathfrak{d}^n}(s)} \right|_{s=0} = \frac{n}{h(m, n)}.
\]

This behaviour is analogous to that observed for some (and conjectured for all) free nilpotent Lie rings, but not universal; cf. [8] Conjecture 6.11 and Remark 6.13].
3.6. Representation zeta functions. Let $G_{m,n} = G_{L,m,n}$ be the unipotent group scheme associated to the nilpotent Lie ring $L_{m,n}$ as in [15, Section 2.4]. Given a ring of integers $\mathcal{O}$ of a number field $K$, the group $G = G_{m,n}(\mathcal{O})$ is a finitely generated torsion-free nilpotent group of nilpotency class 2 and Hirsch length $h(m,n) \cdot |K : \mathbb{Q}|$. (For $\mathcal{O} = \mathbb{Z}$ we recover the groups $\Delta_{m,n} = G_{m,n}(\mathbb{Z})$ from [1].) Denote by

$$\zeta_G(s) = \sum_{n=1}^{\infty} \tau_n(G) n^{-s}$$

the representation zeta function of $G$, encoding the numbers $\tau_n(G)$ of twist-isoclasses of irreducible complex $n$-dimensional representations of $G$; see, for instance, [15, Section 1.1], for background.

**Theorem 3.12.** For all $m, n \in \mathbb{N}$,

$$\zeta_{G_{m,n}(\mathcal{O})}(s) = \frac{\zeta_K(se(m,n) - n)}{\zeta_K(se(m,n))}.$$

**Proof.** By [15, (1.4)], the representation zeta function $\zeta_{G_{m,n}(\mathcal{O})}(s)$ is an Euler product of representation zeta functions of the form $\zeta_{G_{m,n}(\mathcal{O})_p}(s)$, where $\mathcal{O}_p$ is the completion of $\mathcal{O}$ at a nonzero prime ideal $p$. We fix such an ideal $p$ and write $\mathcal{O} = \mathcal{O}_p$ and $q = q_p$ for the residue field cardinality of $\mathcal{O}$. We use the notation and results of [15, Section 2], specifically those for nilpotency class 2 in [15, Section 2.4], to compute the rational function $\zeta_{G_{m,n}(\mathcal{O})_p}(s) \in \mathbb{Q}(q^{-s})$ (cf. [15, Corollary 2.19]) explicitly. The quantity $r$ is equal to $d(m,n)$. For $N > 0$ we find

$$W_N(\mathcal{O}) = (\mathcal{O}/p^N)^n \setminus (\mathcal{O}/p^N)^n,$$

where $W_0(\mathcal{O}) = \emptyset$, whence

$$\#W_N(\mathcal{O}) = \begin{cases} (1 - q^{-n})q^{nN} & \text{if } N > 0, \\ 1 & \text{otherwise.} \end{cases}$$

One checks immediately that, for all $N \in \mathbb{N}_0$,

$$N_{N,a}^p = \#W_N(\mathcal{O}) \delta_{a = (0)_{se(m,n)}, (N)(|d(m,n)/2| - e(m,n))}.$$

Indeed, by Lemma 2.22, for $N > 0$ and any $y \in W_N(\mathcal{O})$, the two matrices

$$M_{m,n}(y) \quad \text{and} \quad \left( \begin{array}{c} \text{Id}_{e(m,n)} \\ -\text{Id}_{e(m,n)} \end{array} \right) \in \text{Mat}_{d(m,n)}(\mathcal{O}/p^N),$$

are equivalent. By [15, Proposition 2.18], it follows that

$$\zeta_{G_{m,n}(\mathcal{O})_p}(s) = \sum_{N \in \mathbb{N}_0, a \in \mathbb{N}_0^{d(m,n)/2}} N_{N,a}^p q^{-s} \sum_{i=1}^{d(m,n)/2} (N-a_i)s \quad N \in \mathbb{N}$$

$$= 1 + \sum_{N \in \mathbb{N}} (1 - q^{-n})q^{nN}(q^{-s}N)_{e(m,n)}$$

$$= 1 + (1 - q^{-n}) q^{n-se(m,n)} \frac{1 - q^{-se(m,n)}}{1 - q^{-n-se(m,n)}} = \frac{1 - q^{-se(m,n)}}{1 - q^{-n-se(m,n)}}.$$

The result follows from the well-known Euler factorization $\zeta_K(s) = \prod_p (1 - q_p^{-s})^{-1}$. \hfill $\Box$

**Remark 3.13.** For $n = 1$, Theorem 3.12 generalizes the well-known formulae for the representation zeta functions of the Heisenberg groups $H(\mathcal{O}) = G_{m,1}(\mathcal{O})$; see Example 1.3 and [15, Theorem B]. For $m = 1$, we recover the representation zeta functions of the Grenham groups $G_{n+1}$ (see Example 1.5), computed by Snocken in his PhD-thesis (see [14, Example 6.2]).

We note an immediate consequence of Theorem 3.12 regarding the topological representation zeta function $\zeta_{G_{m,n},\text{top}}(s) \in \mathbb{Q}(s)$ of $G_{m,n}$; cf. [10, Definition 3.6].
Corollary 3.14.

\[ \zeta_{G_{m,n,\text{top}}}(s) = \frac{se(m,n)}{se(m,n) - n}. \]

Consequently, all questions raised in [10, Section 7]—except possibly Question 7.3—have positive answers for the group schemes \( G_{m,n} \).

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