SCALAR–TENSOR THEORIES AND QUANTUM GRAVITY

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Abstract

Recently, it was shown that the quantum effects of the matter, could be used to

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determine the conformal degree of freedom of the space–time metric. So both gravity and quantum are geometrical features. Gravity determines the causal structure of the space–time, while quantum determines the scale of the space–time. In this article, it is shown that it is possible to use the scalar–tensor framework to build a unified theory in which both quantum and gravitational effects are present.

1 Quantum Force and Geometry

As it was discussed in references [1, 2], in the de-Broglie–Bohm quantum theory, the motion of a hypothetical ensemble of quantum particles would be given by the Hamilton–Jacobi equation:

\[ g^{\mu\nu} \nabla_\mu S \nabla_\nu S = \mathcal{M}^2 c^2 \] (1)

in which the mass field is defined as:

\[ \mathcal{M}^2 = m^2 + \frac{\hbar^2}{c^2} \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} \] (2)

and \( \rho \) is the ensemble density of the system satisfying the continuity equation:

\[ \nabla_\mu (\rho \nabla^\mu S) = 0 \] (3)

The path of the particle could be determined according to the guidance relation:

\[ P^\mu = \mathcal{M} u^\mu = \nabla^\mu S \] (4)

It is a simple task to show [3] that the Hamilton–Jacobi equation and the guidance relation lead to the following geodesic equation:

\[ \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\kappa} u^\nu u^\kappa = \mathcal{F}_Q^\mu \] (5)
The right hand side of this equation is the quantum force given by:

\[ \mathcal{F}_Q^\mu = \frac{1}{\mathcal{M}} (g^{\mu\nu} - u^\mu u^\nu) \nabla_\nu \mathcal{M} \] (6)

In reference [3], it was shown that by the conformal transformation:

\[ g_{\mu\nu} \rightarrow \phi g_{\mu\nu} \] (7)

\[ \phi = \frac{\mathcal{M}}{m} \] (8)

the equation of motion would be transformed to an equation in which there is no quantum effects. As a result, the geodesic equation would be changed to the one without the quantum force. This means that it is possible to have two identical pictures for investigating the quantal effects of matter in the curved spacetime background. According to the first picture, the spacetime metric is \( g_{\mu\nu} \) which contains only the gravitational effects of matter. The quantum effects affect the path of the particles via the quantum force \( \mathcal{F}_Q^\mu \).

In the second picture, the spacetime metric is given by \( \phi g_{\mu\nu} \) in which \( g_{\mu\nu} \) contains the gravitational and \( \phi \) (the conformal factor) contains the quantal effects of matter.

This shows that the quantum as well as the gravitational effects of matter have geometrical nature. The second picture mentioned above provides a unified geometrical framework for understanding the gravitational and quantum forces. Accordingly, we call the metric \( \phi g_{\mu\nu} \), the physical metric (containing both gravity and quantum) and the metric \( g_{\mu\nu} \), the background metric (including only gravity).

In order to have a theory which deals with both gravitational and quantum aspects of matter, one must write an action principle for such a system. Before proceeding, it
must be pointed out here that the aforementioned matter can be interpreted in two ways. According to the first interpretation, to bring in quantum effects one needs only to make a scale transformation (i.e. changing only the metric as is given by (7)). In the second interpretation one thinks of the relation (7) as representing a conformal transformation (i.e. transformation any physical quantity with an appropriate power of \(\phi\)). An action principle appropriate for the first way is written in [4] and for the second way in [3]. The equations of motion in both cases are obtained and the physical properties are investigated. The important point about both these works is that in order to fix the conformal degree of freedom of the space–time metric to the one given by the relation (7), the method of lagrange multiplier is used and in this way they are a little artificial. Here we shall show that in the framework of the scalar–tensor theories, it is possible to write an action principle, in which both gravitational and quantum contributions to the geometry are included and that the conformal degree of freedom of the space–time metric is fixed at the level of the equations of motion not needing the method of lagrange multiplier.

2 **Scalar–Tensor Theories of Gravity**

Scalar–Tensor theories of gravity are purely metric theories describing the gravitational interactions by the space–time metric and some scalar field[7]. In the Kalutza–Klein theory, first developed to incorporate both the gravitational and electromagnetic fields, this scalar field naturally appears as a component of the five–dimensional metric.
The vacuum lagrangian of the scalar–tensor theory is given by:

\[ \mathcal{L} = \sqrt{-g} \left[ \phi R - \frac{\omega(\phi)}{\phi} \nabla_\mu \phi \nabla^\mu \phi \right] \]  

in which \( \omega(\phi) \) is a function of the scalar field \( \phi \). The scalar field is coupled to the scalar curvature in a non–minimal manner. This field is an additional degree of freedom of the gravitational field. So gravity consists of two parts, a spin–2 field (the metric) and a spin–0 field \( \phi \). These parts represent the Jordan frame. The scalar field is affected by matter distribution in the universe and \( \phi^{-1} \) determines the strength of the gravitational interaction (Satisfaction of Mach’s principle).

One can always make some conformal transformation and redefine the fields. So, the theory can be expressed in terms of an infinite class of conformally related frames. One of these transformations is the Dicke transformation\(^5\), which allows us to change the scalar–tensor lagrangian to the Einstein–Hilbert one:

\[ \mathcal{L} = \sqrt{-\tilde{g}} \left[ \tilde{R} - \nabla_\mu \Phi \nabla^\mu \Phi \right] \]  

in which \( \tilde{g}_{\mu\nu} = \phi g_{\mu\nu}, \ d\Phi = \sqrt{\omega + 3/2} d\phi / \phi, \ (\omega > -3/2) \) and the massless field \( \Phi \) is coupled to gravity minimally. These fields, \( \tilde{g}_{\mu\nu} \) and \( \Phi \), represent the Einstein’s conformal frame. Therefore in vacuum any scalar–tensor theory is equivalent to general relativity plus a massless scalar field. The standard minimal interaction between the gravitational matter \( (\psi) \) and the space–time metric can distinguish the Jordan and Einstein conformal frames.

In addition, as the scalar field has not yet been observed experimentally, one can assume any arbitrary interaction between this field and the gravitational matter\(^4\). If the ordinary matter is coupled to gravity minimally in Jordan conformal frame and there is not any
interaction between scalar field and matter, we have:

\[ \mathcal{L} = \sqrt{-g} \left[ \phi \mathcal{R} - \frac{\omega}{\phi} \nabla_\mu \phi \nabla^\mu \phi + \mathcal{L}_m(g, \phi) \right] \]  

(11)

Now, in the Einstein conformal frame this reads as:

\[ \mathcal{L}' = \sqrt{-\tilde{g}} \left[ \tilde{\mathcal{R}} - \tilde{\nabla}_\mu \Phi \tilde{\nabla}^\mu \Phi + \frac{1}{\phi^2(\Phi)} \mathcal{L}_m(\tilde{g}/\phi(\Phi), \psi) \right] \]  

(12)

Thus the scalar field interacts with matter.

Similarly the minimal interaction in Einstein conformal frame leads to the interaction between scalar field and matter in Jordan conformal frame. Therefore, by considering the matter coupling, the Einstein and Jordan conformal frames are two physically different frames. Now, the important question is: "which frame contains the space–time metric of the physical world?" Many arguments are presented in the literature for both alternatives [6].

In general, the important aims in constructing a scalar–tensor theory are the satisfaction of Mach’s principle, the investigation of the effects of the scalar field on the early universe singularity, the importance of the scalar field for constructing inflationary models, and so on. Finally, it must be noted that the scalar–tensor theory is one of the most important candidates for quantum gravity. This property is important for us here.

### 3 Quantum Force and Scalar–Tensor Theories

In the present work we use an appropriate scalar–tensor action along with matter action to derive all the equations of quantum gravity. These equations are the generalized Einstein’s
equations, the equation of motion of the conformal factor (which relates this factor and the quantum potential), and the Bohm’s equations of motion including the continuity equation and the quantum Hamilton–Jacobi equations.

Now, we start from the most general scalar–tensor action:

\[
\mathcal{A} = \int d^4x \left\{ \phi R - \frac{\omega}{\phi} \nabla^\mu \phi \nabla_\mu \phi + 2\Lambda \phi + \mathcal{L}_m \right\}
\]  (13)

in which \(\omega\) is a constant independent of the scalar field, and \(\Lambda\) is the so-called cosmological constant. Also, it is assumed that the matter lagrangian is coupled to the scalar field. This coupling is also present in the generalized Brans–Dicke[8] or Kalutza–Klein theories[9].

The wave equation of the scalar field and the generalized Einstein’s equations can be written as:

\[
R + 2\omega \nabla^\phi \nabla_\phi - \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla_\mu \phi + 2\Lambda + \frac{\partial \mathcal{L}_m}{\partial \phi} = 0
\]  (14)

\[
G^\mu_\nu - \Lambda g^\mu_\nu = -\frac{1}{\phi} T^\mu_\nu - \frac{1}{\phi^2} \left[ \nabla^\mu \nabla_\nu - g^\mu_\nu \nabla^2 \right] \phi + \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla_\nu \phi - \frac{1}{2} \frac{\omega}{\phi^3} g^\mu_\nu \nabla^\alpha \phi \nabla_\alpha \phi
\]  (15)

where \(G^\mu_\nu = R^\mu_\nu - 1/2 R g^\mu_\nu\) is Einstein’s tensor.

The scalar curvature can be evaluated from the contracted form of the latter equation, and it can be substituted in the relation (14). Then we have:

\[
\frac{2\omega - 3}{\phi} \square \phi = -\frac{T}{\phi} + 2\Lambda - \frac{\partial \mathcal{L}_m}{\partial \phi}
\]  (16)

The matter lagrangian for an ensemble of relativistic particles of mass \(m\) is (without any quantum contribution):

\[
\mathcal{L}_{m\text{(no–quantum)}} = \frac{\rho}{m} \nabla^\mu S \nabla_\mu S - \rho m
\]  (17)
This lagrangian can be generalized if one assumes that there is some interaction between the scalar field and the matter field. Here, for simplicity, it is assumed that this interaction is in the form of powers of $\phi$. In order to bring in the quantum effects, one needs to add terms containing the quantum potential. Physical intuition leads us to the fact that it is necessary to assume some interaction between cosmological constant and matter quantum potential. This suggestion will be confirmed after obtaining all of the equations of motion. These arguments lead us to consider the matter lagrangian as:

$$L_m = \frac{\rho_m}{m} \phi^a \nabla^\mu S \nabla_\mu S - m \rho_b \phi^b - \Lambda (1 + Q)^c$$

(18)

in which the $a$, $b$, and $c$ constants are fixed later. Therefore the energy–momentum tensor is:

$$T^{\mu \nu} = -\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu \nu}} \int d^4x \sqrt{-g} L_m = -\frac{1}{2} g^{\mu \nu} L_m + \frac{\rho}{m} \phi^a \nabla^\mu S \nabla_\nu S - \frac{1}{2} \Lambda c Q (1 + Q)^{c-1} g^{\mu \nu}$$

$$- \frac{1}{2} \alpha \Lambda c \nabla_\alpha \sqrt{\rho} \nabla_\beta \left( \frac{(1 + Q)^{c-1}}{\sqrt{\rho}} \right) \left[ g^{\mu \nu} g^{\alpha \beta} - g^{\alpha \mu} g^{\beta \nu} - g^{\beta \mu} g^{\alpha \nu} \right]$$

(19)

Using the matter lagrangian and contracting the above tensor, one can calculate the first and third terms in the relation (16). The other equations, the continuity equation and the quantum Hamilton–Jacobi equation, are expressed respectively as:

$$\nabla^\mu (\rho \phi^a \nabla_\mu S) = 0$$

(20)

$$\nabla^\mu S \nabla_\mu S = m^2 \phi^{b-a} - \frac{1}{2} \Lambda m \frac{Q}{\rho \phi^a} (1 + Q)^{c-1} + \frac{1}{2} \Lambda m \alpha \frac{1}{\sqrt{\rho \phi^a}} \square \left( \frac{(1 + Q)^{c-1}}{\sqrt{\rho}} \right)$$

(21)

To simplify the calculations, with due attention to the equation (16), one can choose $\omega$ to be $\frac{3}{2}$. Then a perturbative expansion for the scalar field and matter distribution density
can be used as:

\[ \phi = \phi_0 + \alpha \phi_1 + \cdots \]  (22)

\[ \sqrt{\rho} = \sqrt{\rho_0} + \alpha \sqrt{\rho_1} \cdots \]  (23)

In the zeroth order approximation, the scalar field equation gives:

\[ b = a + 1; \quad \phi_0 = 1 \]  (24)

In the first order approximation one gets:

\[ \alpha \phi_1 = \frac{c}{2}(1 - a)Q + \frac{a}{2} \tilde{Q} \]  (25)

in which:

\[ \tilde{Q} = \alpha \nabla \mu \sqrt{\rho} \nabla ^\mu \sqrt{\rho} \rho \]  (26)

Since the scalar field is the conformal factor of the space–time metric, and because of some arguments\[4, 3\] show that this field is a function of matter quantum potential, one might choose the constant \( a \) equall to zero. Then, the scalar field is independent of \( \tilde{Q} \) and we have:

\[ \alpha \phi_1 = \frac{c}{2} Q \]  (27)

Also the Bohmian equations of motion give:

\[ \nabla _\mu S \nabla ^\mu S = m^2 (1 + cQ/2) - \Lambda mc \frac{Q - \tilde{Q}}{\rho_0} \]  (28)

It is necessary to choose \( c = 2 \) in order that the first term on the right hand side be the same as the quantum mass \( \mathcal{M} \). These choises for parameters \( a, b \) and \( c \) lead to the non–perturbative quantum gravity equations as follows:

\[ \phi = 1 + Q - \frac{\alpha}{2} \Box Q \]  (29)
\[ \nabla^\mu S \nabla_\mu S = m^2 \phi \frac{2\Lambda m}{\rho} (1 + Q)(Q - \bar{Q}) + \frac{\alpha \Lambda m}{\rho} \left( \Box Q - 2\nabla_\mu Q \frac{\nabla^\mu \sqrt{\rho}}{\sqrt{\rho}} \right) \] (30)

\[ \nabla^\mu (\rho \nabla_\mu S) = 0 \] (31)

\[ G^{\mu\nu} - \Lambda g^{\mu\nu} = -\frac{1}{\phi} T^{\mu\nu} - \frac{1}{\phi^3} [\nabla^\mu \nabla^\nu - g^{\mu\nu} \Box] \phi + \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{2} \frac{\omega}{\phi^2} g^{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \] (32)

We conclude this section by pointing out some important hints:

- It is very interesting that in the framework of the scalar–tensor theories, one is able to derive all of the quantum gravity equations of motion without using the method of lagrange multipliers as it is done in the previous works[3, 4].

- In the suggested quantum gravity theory, the causal structure of the space–time \((g_{\mu\nu})\) is determined via equation (32). This shows that except for back–reaction terms of the quantum effects on \(g_{\mu\nu}\), the causal structure of the space–time is determined by the gravitational effects of matter. Quantum effects, determine directly the scale factor of the space–time, from the relation (29).

- It must be noted that the mass field given by the right hand side of the relation (30), consists of two parts. The first part which is proportionnal to \(\alpha\), is a purely quantum effect, and the second part which is proportional to \(\alpha \Lambda\), is a mixture of the quantum effects and the large scale structure introduced via the cosmological constant.
4 Concluding Remarks

This formulation of quantum gravity leads us to the physical frame naturally. This frame is determined by matter distribution uniquely. Thus, one may conclude this point is confirmed that the physical frame can not be specified without matter fields. Therefore the matter distribution determines the local curvature of the space–time (in conformity with Mach’s principle). Furthermore, from the matter equation of motion one can see that the cosmological constant (a large scale structure constant) and the quantum potential are coupled together. This is another manifestation of the Mach’s principle.

The guiding equation $P^\mu = \mathcal{M}u^\mu = \nabla^\mu S$, leads us to the geodesic equation. From relation (30) we have:

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\kappa} u^\nu u^\kappa = \frac{1}{\mathcal{M}} (g^\mu\nu - u^\mu u^\nu) \nabla_\nu \mathcal{M}$$

(33)

where

$$\mathcal{M}^2 = m^2 \phi - \frac{2\Lambda m}{\rho} (1 + Q)(Q - \tilde{Q}) + \frac{\alpha \Lambda m}{\rho} \left( \Box Q - 2\nabla_\mu Q \frac{\nabla^\mu \sqrt{\rho}}{\sqrt{\rho}} \right)$$

(34)

Thus, in the present theory, the scalar field produces quantum force that appears on right hand side and violates the equivalance principle. Similarly, in Kalutza–Klein theory, the scalar field (dilaton) produces some fifth force leading to the violation of the equivalance principle[10].

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