Frugal and Truthful Auctions for Vertex Covers, Flows, and Cuts*

David Kempe  
Department of Computer Science,  
University of Southern California, CA 90089-0781, USA  
dkempe@usc.edu

Mahyar Salek  
Department of Computer Science,  
University of Southern California, CA 90089-0781, USA  
salek@usc.edu

Cristopher Moore  
Computer Science Department and Department of Physics and Astronomy,  
University of New Mexico, Albuquerque, NM 87131-0001, USA  
and Santa Fe Institute, Santa Fe NM 87501, USA  
moore@cs.unm.edu

June 14, 2011

Abstract

We study truthful mechanisms for hiring a team of agents in three classes of set systems: Vertex Cover auctions, $k$-flow auctions, and cut auctions. For Vertex Cover auctions, the vertices are owned by selfish and rational agents, and the auctioneer wants to purchase a vertex cover from them. For $k$-flow auctions, the edges are owned by the agents, and the auctioneer wants to purchase $k$ edge-disjoint $s$-$t$ paths, for given $s$ and $t$. In the same setting, for cut auctions, the auctioneer wants to purchase an $s$-$t$ cut. Only the agents know their costs, and the auctioneer needs to select a feasible set and payments based on bids made by the agents.

We present constant-competitive truthful mechanisms for all three set systems. That is, the maximum overpayment of the mechanism is within a constant factor of the maximum overpayment of any truthful mechanism, for every set system in the

*A preliminary version of this article appeared in the Proceedings of FOCS 2010 [19].
class. The mechanism for Vertex Cover is based on scaling each bid by a multiplier derived from the dominant eigenvector of a certain matrix. The mechanism for $k$-flows prunes the graph to be minimally $(k+1)$-connected, and then applies the Vertex Cover mechanism. Similarly, the mechanism for cuts contracts the graph until all $s$-$t$ paths have length exactly 2, and then applies the Vertex Cover mechanism.

1 Introduction

Many tasks require the joint allocation of multiple resources belonging to different bidders. For instance, consider the task of routing a packet through a network whose edges are owned by different agents. In this setting, it is necessary to obtain usage rights for multiple edges simultaneously from the agents. Similarly, if the agents own the vertices of a graph, and we want to monitor all edges, we need the right to install monitoring devices on nodes, and again obtain these rights from distinct agents.

Providing access to edges or nodes in such settings makes the agents incur a cost $c_e$, which the agents should be paid for. A convenient way to determine “appropriate” prices to pay the agents is by way of auctions, wherein the agents $e$ submit bids $b_e$ to an auctioneer, who selects a feasible subset $S$ of agents to use, and determines prices $p_e$ to pay the agents. The most basic case is a single-item auction. The auctioneer requires the service of any one of the agents, and their services are interchangeable. Single-item auctions have a long history of study, and are fairly well understood [20, 21]. Motivated by applications in computer networks and electronic commerce, several recent papers have considered the extension to a setup termed hiring a team of agents [3, 11, 12, 18, 28]. In this setting, there is a collection of feasible sets, each consisting of one or more agent. The auctioneer, based on the agents’ bids $b_e$, selects one feasible set $S$, and pays each agent $e \in S$ a price $p_e$.

Some of the well-studied special cases of set systems are path auctions [3, 12, 18, 24, 30], in which the feasible sets are paths from a given source $s$ to a given sink $t$, and spanning tree auctions [4, 14, 18, 28], in which the feasible sets are spanning trees of a connected graph. In both cases, the agents are the edges of the graph. In this paper, we extend the study to more complex examples of set systems, namely:

1. Vertex Covers: The agents are the vertices of the graph $G$, and the auctioneer needs to select a vertex cover [5, 11, 28]. Not only are vertex covers of interest in their own right, but they give a key primitive for many other set systems as well, an approach we explore in depth in this paper.

2. Flows: The agents are the edges of $G$, and the auctioneer wants to select $k$ edge-disjoint paths from $s$ to $t$. Thus, this scenario generalizes path auctions; the generalization turns out to require significant new techniques in the design and analysis of mechanisms.

3. Cuts: In the same setting as for flows, the auctioneer wants to purchase an $s$-$t$ cut.

In choosing an auction mechanism for a set system, the auctioneer needs to take into account that the agents are selfish. Ideally, the auctioneer would like to know the agents’
true costs $c_e$. However, the costs are private information, and a rational and selfish agent will submit a bid $b_e \neq c_e$ if doing so leads to a higher profit. The area of mechanism design \cite{23, 24, 25} studies the design of auctions for selfish and rational agents.

We are interested in designing truthful (or incentive-compatible) auction mechanisms: auctions under which it is always optimal for selfish agents to reveal their private costs $c_e$ to the auctioneer. Such mechanisms are societally desirable, because they make the computation of strategies a trivial task for the agents, and obviate the need for gathering information about the costs or strategies of competitors. They are also desirable from the point of view of analysis, as they allow us to identify bids with costs, and let us dispense with any kinds of assumptions about the distribution of agents’ costs. Thus, the outcomes of truthful mechanisms are stable in a stronger sense than Nash equilibria, and may give bidders more confidence that the right outcome will be reached. For this reason, truthful mechanism design has been a mainstay of game theory for a long time.

It is well known that any truthful mechanism will have to pay agents more than their costs at times; in this paper, we study mechanisms approximately minimizing the “overpayment.” The ratio between the payments of the “best” truthful mechanism and natural lower bounds has been termed the “Price of Truth” by Talwar \cite{28}, and studied in a number of recent papers \cite{3, 4, 11, 12, 14, 18, 28, 30}. In particular, \cite{18} and \cite{11} define and analyze different natural measures of lower bounds on payments, and define the notions of frugality ratio and competitiveness. The frugality ratio of a mechanism is the worst-case ratio of payments to a natural lower bound (formally defined in Section \protect\ref{sec:Notation}), maximized over all cost vectors of the agents. A mechanism is competitive for a class of set systems if its frugality ratio is within a constant factor of the frugality ratio of the best truthful mechanism, for all set systems in the class.

1.1 Our Contributions

In this paper, we present novel frugal mechanisms for three general classes of set systems: Vertex Covers, $k$-Flows, and Cuts. Vertex Cover auctions can be considered a very natural primitive for more complicated set systems. Under the natural assumption that there are no isolated vertices, they capture set systems with “minimal competition”: if the auction mechanism decides to exclude an agent $v$ from the selected set, this immediately forces the mechanism to include all of $v$’s neighbors, thus giving these neighbors a monopoly. Thus, a different interpretation of Vertex Cover auctions is that they capture any set system whose feasible sets can be characterized by positive 2SAT formulas: each edge $(i, j)$ corresponds to a clause $(x_i \lor x_j)$, stating that any feasible set must include at least one of agents $i$ and $j$.

Our mechanism for Vertex Cover works as follows: based solely on the structure of the graph $G$, we define an appropriate matrix $K$ and compute its dominant eigenvector $q$. After agents submit their bids $b_v$, the mechanism first scales each bid to $c'_e = b_e / q_v$, and then simply runs the VCG mechanism \cite{29, 9, 15} with these modified bids. We prove that this mechanism has a frugality ratio equal to the largest eigenvalue $\alpha$ of $K$, and that this is within a factor of 2 of the frugality ratio of any mechanism. The lower bound is based on pairwise competition between adjacent bidders for any truthful mechanism, and in a sense
can be considered the natural culmination of the lower bound techniques of [12, 18]. The upper bound is based on carefully balancing all possible worst cases of a single non-zero cost against each other, and showing that the worst case is indeed one of these cost vectors. We stress here that the mechanism does not in general run in polynomial time: the entries of $K$ are derived from fractional clique sizes in $G$, which are known to be hard to compute, even approximately. We discuss the issue of polynomial time briefly in Section 6.

Based on our Vertex Cover mechanism, we present a general methodology for designing frugal truthful mechanisms. The idea is to take the original set system, and prune agents from it until it has “minimal competition” in the above sense; subsequently, the Vertex Cover auction can be invoked. So long as the pruning is “composable” in the sense of [1] (see Section 3), the resulting auction is truthful. The crux is then to prove that the pruning step (which removes a significant amount of competition) does not increase the lower bound on payments too much. We illustrate the power of this approach with two examples.

1. For the $k$-flow problem, we show that pruning the graph to a minimum-cost $(k + 1)$ $s$-$t$-connected graph $H$ is composable, and increases the lower bound at most by a factor of $k + 1$. Hence, we obtain a $2(k + 1)$-competitive mechanism. Establishing the bound of $k + 1$ requires significant technical effort.

2. For the cut problem, we show that pruning the graph to a minimum-cost set of edges such that each $s$-$t$ path is cut at least twice gives a composable selection rule. Furthermore, it increases the lower bound by at most a factor of 2, leading to a 4-competitive mechanism. For the pruning step, we develop a primal-dual algorithm generalizing the Ford-Fulkerson Minimum-Cut algorithm.

We note that while the Vertex Cover mechanism is in general not polynomial, for both special cases derived here, the running time is in fact polynomial.

1.2 Relationship to Past and Parallel Work

As discussed above, a line of recent papers [3, 4, 11, 12, 14, 18, 28, 30] analyze frugality of auctions in the “hiring a team” setting, where the auctioneer wants to obtain a feasible set of agents, while paying not much more than necessary. In this context, the papers by Karlin, Kempe, and Tamir [18] and Elkind, Goldberg, and Goldberg [11] are particularly related to our work.

Karlin et al. [18] introduce the definitions of frugality and competitiveness which we use here. They also give competitive mechanisms for path auctions, and for so-called $r$-out-of-$k$ systems, in which the auctioneer can select any $r$ out of $k$ disjoint sets of agents. At the heart of both mechanisms is a mechanism for $r$-out-of-$(r + 1)$ systems. Our mechanism for Vertex Covers can be considered a natural generalization of this mechanism. Furthermore, both $r$-out-of-$k$ systems and path auctions are special cases of $r$-flows, since choosing an $r$-flow in a graph consisting of $k$ vertex-disjoint $s$-$t$ paths is equivalent to an $r$-out-of-$k$ system. Our approach of pruning the graph is similar in spirit to the approach in [18], where graphs were
also first pruned to be minimally 2-connected, and set systems were reduced to \( r \)-out-of-(\( r + 1 \)) systems. However, the combinatorial structure of \( k \)-flows makes this pruning (and its analysis) much more involved in our case.

Elkind et al. \cite{Elkind2011} study truthful mechanisms for Vertex Cover. They present a polynomial-time mechanism with frugality ratio bounded by \( 2\Delta \), where \( \Delta \) is the maximum degree of the graph, and also show that there exist graphs where the best truthful mechanism must have frugality ratio at least \( \Delta/2 \). Notice, however, that this does not guarantee that the mechanism is competitive. Indeed, there are graphs where the best truthful mechanism has frugality ratio significantly smaller than \( \Delta/2 \), and our goal is to have a mechanism which is within a constant factor of best possible for every graph.

Several recent papers have extended the problem of hiring a team of agents in various directions. Cary, Flaxman, Hartline, and Karlin \cite{Cary2009} combine truthful auctions for hiring a team with revenue-maximizing auctions for selling items. Du, Sami, and Shi \cite{Du2013} and Iwasaki, Kempe, Saito, Salek, and Yokoo \cite{Iwasaki2012} study path auctions under the additional requirement that not only should they be truthful, but false-name proof: agents owning multiple edges have no incentive to claim that these edges belong to different agents. Du et al. show that there are no false-name proof mechanisms that are also Pareto-efficient, and Iwasaki et al. analyze the frugality ratio of false-name proof mechanisms, showing exponential lower bounds.

Results very similar to ours have been derived independently and simultaneously by Chen, Elkind, Gravin, and Petrov \cite{Chen2012}. Both papers first derive mechanisms for Vertex Cover auctions. Our mechanism is based on scaling the agents’ bids by the entries of the dominant eigenvector of a scaled adjacency matrix. It has constant competitive ratio for all graphs, but may not run in polynomial time. The mechanism of Chen et al., on the other hand, uses eigenvectors of the unscaled adjacency matrix. It may not be constant competitive on some inputs, but it always runs in polynomial time.

Chen et al. also propose the approach of reducing other set systems to Vertex Cover instances, called “Pruning-Lifting Mechanisms” there. In particular, they derive the same mechanism as the present paper for \( k \)-flows, with similar key lemmas in the proof. While their Vertex Cover mechanism is different from ours in general, on inputs derived from flow and cut problems, the scaling factor in our matrix is the same for all entries, and the mechanisms therefore coincide. In particular, the mechanisms in both papers are thus competitive and run in polynomial time.

While the mechanism of Chen et al. \cite{Chen2012} may not always be competitive due to the lack of scaling factors in the matrix, their proof of a lower bound involves a clever application of Young’s Inequality, and thus avoids losing the factor of 2 in our lower bound. Thus, whenever their mechanism coincides with ours, both mechanisms are optimal. In particular, this also implies that the \( k \)-flow mechanism of the present paper is \((k + 1)\)-competitive and our mechanism for \( s-t \) cuts is 2-competitive. Moreover, they prove stronger bounds on the \( k \)-flow mechanism: when compared against the lower bound from \cite{Elkind2011} (used in this paper, and defined formally in Section 2), the mechanism is in fact optimal.

Finally, in collaboration with the authors of \cite{Chen2012}, we recently showed that Young’s In-
equality can be applied to the analysis of our Vertex Cover mechanism, removing the factor of 2 from the lower bound. In other words, we show that our Vertex Cover mechanism is indeed optimal for all Vertex Cover instances. This result will be included in a joint full version of both papers.

2 Preliminaries

A set system \((E, \mathcal{F})\) has \(n\) agents (or elements), and a collection \(\mathcal{F} \subseteq 2^E\) of feasible sets. We call a set system monopoly-free if no element is in all feasible sets, i.e., if \(\bigcap_{S \in \mathcal{F}} S = \emptyset\). The three classes of set systems studied in this paper are:

1. Vertex Covers: here, the agents are the vertices of a graph \(G\), and \(\mathcal{F}\) is the collection of all vertex covers of \(G\). To avoid confusion, we will denote the agents by \(u, v\) instead of \(e\) in this case. Notice that every Vertex Cover set system is monopoly-free.

2. \(k\)-flows: here, we are given a graph \(G\) with source \(s\) and sink \(t\). The agents are the edges of \(G\). A set of edges is feasible if it contains at least \(k\) edge-disjoint \(s\)-\(t\) paths. A \(k\)-flow set system is monopoly-free if and only if the minimum \(s\)-\(t\) cut cuts at least \(k + 1\) edges.

3. Cuts: With the same setup as for \(k\)-flows, a set of edges is feasible if it contains an \(s\)-\(t\) cut. Thus, the set system is monopoly-free if and only if \(G\) contains no edge from \(s\) to \(t\).

The set system \((E, \mathcal{F})\) is common knowledge to the auctioneer and all agents. Each agent \(e \in E\) has a cost \(c_e\), which is private, i.e., known only to \(e\). We write \(c(S) = \sum_{e \in S} c_e\) for the total cost of a set \(S\) of agents, and also extend this notation to other quantities (such as bids or payments). A mechanism for a set system proceeds as follows:

1. Each agent submits a sealed bid \(b_e\).

2. Based on the bids \(b_e\), the auctioneer selects a feasible set \(S \in \mathcal{F}\) as the winner, and computes a payment \(p_e \geq b_e\) for each agent \(e \in S\). The agents \(e \in S\) are said to win, while all other agents lose.

Each agent, knowing the algorithm for computing the winning set and the payments, will choose a bid \(b_e\) maximizing her own profit, which is \(p_e - c_e\) if the agent wins, and 0 otherwise. We are interested in mechanisms where self-interested agents will bid \(b_e = c_e\). More precisely, a mechanism is truthful if, for any fixed vector \(b_{-e}\) of bids by all other agents, \(e\) maximizes her profit by bidding \(b_e = c_e\). If a mechanism is known to be truthful, we can use \(b_e\) and \(c_e\) interchangeably. It is well-known \([3, 21]\) that a mechanism is truthful only if its selection rule is monotone in the following sense: if all other agents’ bids stay the same, then a losing agent cannot become a winner by raising her bid. Once the selection rule is fixed, there is a unique payment scheme to make the mechanism truthful. Namely, each agent is paid her threshold bid: the supremum of all winning bids she could have made given the bids of all other agents.
2.1 Nash Equilibria and Frugality Ratios

To measure how much a truthful mechanism "overpays," we need a natural bound to compare the payments to. Karlin et al. [18] proposed using as a bound the solution of a natural minimization problem. Let $S$ be the cheapest feasible set with respect to the true costs $c_e$; ties are broken lexicographically.

\[
\begin{align*}
\text{Minimize} & \quad \nu^- (c) := \sum_{e \in S} x_e \\
\text{subject to} & \quad x_e \geq c_e \quad \text{for all } e \in S \\
& \quad x_e = c_e \quad \text{for all } e \notin S \\
& \quad \sum_{e \in S} x_e \leq \sum_{e \in T} x_e \quad \text{for all } T \in \mathcal{F} \\
& \quad \sum_{e \in S} x_e = \sum_{e \in T_e} x_e \\
\end{align*}
\]

(1)

The intuition for this optimization problem is that it captures the bids of agents in the cheapest "Nash Equilibrium" of a first-price auction with full information, under the assumption that the actual cheapest set $S$ wins, and the losing agents all bid their costs. That is, the mechanism selects the cheapest set with respect to the bids $x_e$, and pays each winning agent her bid $x_e$. The first constraint captures individual rationality. The third constraint states that the bids $x_e$ are such that $S$ still wins, and the final constraint states that for each winning agent, there is a tight set preventing her from bidding higher. That is, if $e$ increases her bid, the buyer will select a set $T$ excluding $e$ instead of $S$. We say that a vector $x$ is feasible if it satisfies all these constraints.

While this optimization problem is inspired by the analogy of Nash Equilibria, it should be noted that first-price auctions do not in general have Nash Equilibria due to tie-breaking issues (see a more detailed discussion in [16, 18]).

Elkind et al. [11] and Chen and Karlin [8] observed that the quantity $\nu^- (c)$ has several undesirable non-monotonicity properties. For instance, adding new feasible sets to the set system, and thus increasing the amount of competition between agents, can sometimes lead to higher values of $\nu^- (c)$. Similarly, lowering the costs of losing agents, or increasing the costs of winning agents, can sometimes increase $\nu^- (c)$. Furthermore, $\nu^- (c)$ is NP-hard to compute even if the set system is the set of all s-t paths [8].

Instead, Elkind et al. [11] propose replacing the minimization by a maximization in the above optimization problem. An important advantage of this optimization problem is that the maximization objective ensures that for every $e \in S$, there is a tight set $T$. Thus, the maximization objective removes the need for the final constraint, and turns the optimization problem into an instance of Linear Programming, which can be solved in many cases. We thus obtain the following definition (which [11] refers to as NTU$_{\text{max}}$). Intuitively, this definition captures the bids in the most expensive Nash Equilibrium of a first-price auction, with the same caveat as before about the non-existence of equilibria.
Maximize \( \nu(c) := \sum_{e \in S} x_e \) subject to (i) \( x_e \geq c_e \) for all \( e \) (ii) \( x_e = c_e \) for all \( e \notin S \) (iii) \( \sum_{e \in S} x_e \leq \sum_{e \in T} x_e \) for all \( T \in \mathcal{F} \) \hspace{1cm} (2)

As stated above, a consequence of this maximization is that, for every \( e \) in the winning set, there is a tight set \( T \) excluding \( e \) that prevents \( e \) from bidding higher:

\[ \forall e \in S : \exists T \in \mathcal{F} : e \notin T \text{ and } \sum_{e' \in S} x_{e'} = \sum_{e' \in T} x_{e'}. \] \hspace{1cm} (3)

We will refer to the bounds \( \nu^{-}(c) \) and \( \nu(c) \) as buyer-optimal and buyer-pessimal, respectively, throughout the paper. Moreover, due to the advantages discussed above, we will use the quantity \( \nu(c) \) as a natural lower bound for this paper. Despite the preceding discussion, in order to emphasize the intuition behind the bounds, we will refer to the \( x_e \) values of the LP \((2)\) as the Nash Equilibrium bids of agents \( e \), or simply the Nash bids of \( e \).

Notice that \( \nu(c) \) is defined for all monopoly-free set systems. We now formally define the frugality ratio of a mechanism \( \mathcal{M} \) for a set system \((E, \mathcal{F})\), and the notion of a competitive mechanism.

**Definition 2.1 (Frugality Ratio, Competitive Mechanism)** Let \( \mathcal{M} \) be a truthful mechanism for the set system \((E, \mathcal{F})\), and let \( P_M(c) \) denote the total payments of \( \mathcal{M} \) when the vector of actual costs is \( c \).

1. The frugality ratio of \( \mathcal{M} \) is

\[ \phi_{\mathcal{M}} = \sup c \frac{P_M(c)}{\nu(c)}. \]

2. The frugality ratio of the set system \((E, \mathcal{F})\) is

\[ \Phi_{(E, \mathcal{F})} = \inf_{\mathcal{M}} \phi_{\mathcal{M}}, \]

where the infimum is taken over all truthful mechanisms \( \mathcal{M} \) for \((E, \mathcal{F})\).

3. A mechanism \( \mathcal{M} \) is \( \kappa \)-competitive for a class of set systems \( \{(E_1, \mathcal{F}_1), (E_2, \mathcal{F}_2), \ldots\} \) if \( \phi_{\mathcal{M}} \) is within a factor \( \kappa \) of \( \Phi_{(E_i, \mathcal{F}_i)} \) for all \( i \).

**Remark 2.2** The frugality ratio of a mechanism is defined as instance-based. The frugality ratio of a set system captures the inherent structural complexity of that instance, which can be “exploited” with careful worst-case choices of costs.

Competitiveness, on the other hand, is defined over a class of set systems. If a single mechanism, such as the ones defined in this paper, is competitive, it does as well on each set system in the class as the best mechanism, which could possibly be tailored to this
specific instance. The nomenclature “competitive” is motivated by the analogy with online algorithms.

The instance-based definition \cite{18,11} allows us a more fine-grained distinction between mechanisms than earlier work (e.g., \cite{3,24}), where a lower bound in terms of a worst case over all instances was used.

As discussed above, the motivation for the LPs \cite{1} and \cite{2} was that they provide “natural lower bounds” on the payments of any truthful mechanism. However, to the best of our knowledge, it was previously unknown whether the solutions do in fact provide lower bounds. Indeed, it is easy to define mechanisms that achieve arbitrarily lower payments for particular cost vectors, albeit at the cost of significantly higher payments on other cost vectors. Here, we establish that the objective value of the LP \cite{2} indeed does give a lower bound in terms of the frugality ratio. This resolves an open question from the preliminary version of this paper \cite{19}.

\textbf{Proposition 2.3} Let \((E,F)\) be an arbitrary set system and \(M\) a truthful and individually rational mechanism on \((E,F)\). Then, \(\phi_M \geq 1\).

\textbf{Proof.} Let \(c\) be an arbitrary cost vector. Let \(S \in F\) be the set minimizing \(c(S)\), and \(x\) the solution to the LP \cite{2}. Let \(S' \in F\) be the winning set for \(M\) with cost vector \(x\). Because \(M\) is truthful and individually rational, its payment \(P_M(x)\) is at least \(\sum_{e \in S'} x_e\). By the third constraint of the LP \cite{2}, \(\sum_{e \in S'} x_e \geq \sum_{e \in S} x_e\). Finally, by construction, we have that \(\nu(c) = \nu(x)\). Taken together, this implies that

\[ P_M(x) \geq \sum_{e \in S'} x_e \geq \sum_{e \in S} x_e = \nu(x). \]

By definition of the frugality ratio, this implies that \(\phi_M \geq 1\). \hfill \(\blacksquare\)

3 Vertex Cover Auctions

In this section, we describe and analyze a constant-competitive mechanism for Vertex Cover auctions. We then show how to use it as the basis for a methodology for designing frugal mechanisms for other set systems. The graph is denoted by \(G = (V,E)\), with \(n\) vertices. We write \(u \sim v\) to denote that \((u,v) \in E\).

Our mechanism is based on certain modifications to the well-known Vickrey-Clarke-Groves (VCG) mechanism \cite{29,0,15}. Recall that VCG always selects the cheapest feasible set \(S\) with respect to the submitted bids \(b_e\), and pays each agent her threshold bid.

3.1 Weighting the bids with an eigenvector

The important change to VCG in our mechanism is that each agent’s bid is scaled by an agent-specific multiplier. The multipliers capture “how important” an agent is for the solution, roughly in the sense of how many other agents can be omitted by including this
agent. They are computed as entries of the dominant eigenvector of a certain matrix $K$. As we will see, the computation of $K$ is NP-hard itself, so the mechanism will in general not run in polynomial time unless $P=NP$.

As a first step, our mechanism removes all isolated vertices. We assume that the resulting graph $G$ is connected. Let $1_v$ (for any vertex $v$) be the vector with 1 in coordinate $v$ and 0 in all other coordinates. We define $\nu_v = \nu(1_v) \geq 1$ to be the total “Nash Equilibrium” payment of the first-price auction in the sense of the LP (2) if agent $v$ has cost 1 and all other agents have cost 0. Notice that in this case, $v$ loses. We prove in Section 3.1 that $\nu_v$ is exactly the fractional clique number of the graph induced by the neighbors of $v$, without $v$ itself. This implies that unless ZPP=NP, $\nu_v$ cannot be approximated to within a factor $O(n^{1-\epsilon})$ in polynomial time, for any $\epsilon > 0$. Our inability to compute $\nu_v$ is the chief obstacle to a constant-competitive polynomial-time mechanism.

Let $A$ be the adjacency matrix of $G$ (with diagonal 0). Define $D = \text{diag}(1/\nu_1, 1/\nu_2, \ldots, 1/\nu_n)$, and $K = DA$. That is,

$$K_{u,v} = \begin{cases} 1/\nu_u & \text{if } u \sim v \\ 0 & \text{if } u \not\sim v. \end{cases}$$

If we define $K' = D^{-1/2}KD^{1/2} = D^{1/2}AD^{1/2}$, then $K$ and $K'$ have the same eigenvalues, and the eigenvectors of $K$ are of the form $D^{1/2} \cdot e$, where $e$ is an eigenvector of $K'$. Moreover,

$$K'_{u,v} = \begin{cases} 1/\sqrt{\nu_u \nu_v} & \text{if } u \sim v \\ 0 & \text{if } u \not\sim v, \end{cases}$$

so $K'$ is symmetric and has non-negative entries. By the Perron-Frobenius Theorem, the eigenvalues of $K'$ and $K$ are real. Since we assumed $G$ to be connected, the dominant eigenvector of $K'$ is unique and has positive entries, and the same holds for $K$.

Let $\alpha$ be the largest eigenvalue of $K$, and $q$ the corresponding eigenvector. Notice that given $K$ as input, $\alpha$ and $q$ can be computed efficiently and without knowledge of the agents’ bids or costs.

The mechanism $\mathcal{E} \mathcal{V}$ (which stands for “Eigenvector Mechanism”) is now as follows: after all nodes $v$ submit their bids $b_v$, the algorithm sets $c'_v = b_v/q_v$, and computes a minimum cost vertex cover $S$ with respect to the costs $c'_v$ (ties broken lexicographically). $S$ is chosen as the winning set, and each agent in $S$ is paid her threshold bid. Notice that the second step of the mechanism again requires the solution to an NP-hard problem.

$\mathcal{E} \mathcal{V}$ is truthful since the selection rule is clearly monotone, and the payments are the threshold bids. Thus, we can assume without loss of generality that bids and costs coincide. In the following, we analyze the frugality ratio of $\mathcal{E} \mathcal{V}$, and show that $\mathcal{E} \mathcal{V}$ is competitive.

**Lemma 3.1** $\mathcal{E} \mathcal{V}$ has frugality ratio at most $\alpha$.

**Proof.** We start by considering only cost vectors with exactly one non-zero entry, i.e., of the form $c = c_v \cdot 1_v$. For such a cost vector, the mechanism will choose a subset of $V \setminus \{v\}$ as
the winning set, and pay each \( u \) in that subset her threshold bid. We calculate the threshold bids of all these agents \( u \).

First, consider any agent \( u \sim v \). If \( u \) were to raise her bid above \((q_u/q_v) \cdot c_v\), while all agents besides \( u \) and \( v \) continued to bid 0, then the set \( V \setminus \{ u \} \) would be cheaper than \( \{ u \} \) with respect to the new bid vector \( c' \). Therefore, \( u \) would not be part of the winning vertex cover. Thus \( u \)'s threshold payment is at most \((q_u/q_v) \cdot c_v\).

Next, consider any agent \( u \not\sim v \). Because \( V \setminus \{ u, v \} \) is a vertex cover, \( u \) cannot raise her bid above zero without losing, so her threshold bid is 0. Hence, the total payment of \( EV \) is at most \( P(c) = (1/q_v) \cdot c_v \cdot \sum_{u\sim v} q_u \). On the other hand, by the definition of \( \nu_v \), and linearity of \( \nu \), we have that \( \nu(c) = c_v \nu_v \), so the frugality ratio for cost vectors of the form \( c_v \cdot 1_v \) is

\[
\frac{(1/q_v) \cdot c_v \cdot \sum_{u\sim v} q_u}{c_v \nu_v} = \frac{1}{q_v} \cdot \sum_{u\sim v} \frac{1}{\nu_v} \cdot q_u = \frac{1}{q_v} \cdot \alpha \cdot q_v = \alpha,
\]

where the second equality followed because the vector \( q \) is an eigenvector of \( K \) with eigenvalue \( \alpha \). Thus, for any cost vector with only one non-zero entry, the frugality ratio is at most \( \alpha \).

Now consider an arbitrary cost vector \( c \), and write it as \( c = \sum_v c_v 1_v \). We claim that \( P(c) \leq \sum_v c_v P(1_v) \). For consider any vertex \( u \in S \) winning with cost vector \( c \). If the cost vector were \( c_v 1_v \) instead, \( u \)'s payment would be \((q_u/q_v) \cdot c_v \) if \( u \sim v \) and 0 otherwise. On the other hand, when the cost vector is \( c \), if \( u \) bids strictly more than \( \sum_{u\sim v} q_u/q_v \cdot c_v \), then \( u \) cannot be in the winning set, as replacing \( u \) with all its neighbors would give a cheaper solution with respect to the costs \( c' \). Thus, each node \( u \) gets paid at most \( \sum_{v \sim u} q_u/q_v \cdot c_v \) with cost vector \( c \), and the total payment is at most

\[
P(c) = \sum_u \sum_{v \sim u} \frac{q_u}{q_v} \cdot c_v = \sum_v c_v \cdot \sum_{u \sim v} \frac{q_u}{q_v} = \sum_v c_v P(1_v).
\]

On the other hand, we have that

\[
\nu(c) \geq \sum_v c_v \nu(1_v) = \sum_v c_v \nu_v,
\]

because of the following argument: for each \( v \), let \( x^{(v)} \) be a an optimal solution for the LP (2) with cost vector \( 1_v \). Then, simply by linearity, the vector \( x = \sum_v c_v x^{(v)} \) is feasible for (2) with cost vector \( c \), and achieves the sum of the payments. Thus, the optimal solution to (2) with cost vector \( c \) can have no smaller total payments.

Combining the results of the previous two paragraphs, we have the following bound on the frugality ratio:

\[
\max_c \frac{P(c)}{\nu(c)} \leq \max_c \frac{\sum_v c_v P(1_v)}{\sum_v c_v \nu_v} \leq \max_v \frac{P(1_v)}{\nu_v} \leq \alpha.
\]

Next, we prove that no other mechanism can do asymptotically better.

**Lemma 3.2** Let \( M \) be any truthful vertex cover mechanism on \( G \). Then, \( M \) has frugality ratio at least \( \frac{\alpha}{2} \).
Proof. We construct a directed graph $G' = (V, E')$ from $G$ by directing each edge $e$ of $G$ in at least one direction. Consider any edge $e = (u, v)$ of $G$. Let $c$ be the cost vector in which $c_u = q_u$, $c_v = q_v$, and $c_i = 0$ for all $i \neq u, v$. When $M$ is run on the cost/bid vector $c$, at least one of $u$ and $v$ must be in the winning set $S$; otherwise, it would not be a vertex cover. If $u \in S$, then add the directed edge $(v, u)$ to $E'$. Similarly, if $v \in S$, then add $(u, v)$ to $E'$. If both $u, v \in S$, then add both directed edges. By doing this for all edges $e \in G$, we eventually obtain a graph $G'$.

Now give each node $v$ a weight $q_v$. Each node-weighted directed graph $(V, E')$ contains at least one node $v$ such that
\[
\sum_{u : (v, u) \in E'} q_u \geq \frac{1}{2} \sum_{u : u \sim v} q_u ,
\]
(see, e.g., the proof of Lemma 11 in [18]), and hence
\[
\sum_{u : (v, u) \in E'} q_u \geq \frac{1}{2} \sum_{u : u \sim v} q_u .
\]
Fix any such node $v$ in $G'$ with respect to the weights $q_v$.

Now consider the cost vector $c$ with $c_v = q_v$ and $c_i = 0$ for all $i \neq v$. By monotonicity of the selection rule of $M$ (which follows from the truthfulness of $M$), at least all nodes $u$ such that $(v, u) \in G'$ must be part of the selected set $S$ of $M$, and must be paid at least $q_u$. Therefore, the total payment of $M$ is at least
\[
\sum_{u : (v, u) \in G'} q_u \geq \frac{1}{2} \sum_{u \sim v} q_u = \frac{1}{2} \nu_v \sum_{u \sim v} \frac{1}{\nu_v} q_u = \frac{1}{2} \nu_v \cdot \alpha q_v ,
\]
where the last equality followed from the fact that $q$ is an eigenvector of the matrix $K$.

On the other hand, as in the proof of Lemma 3.1, $\nu(c) = \nu_v q_v$ for our cost vector $c$, so the frugality ratio is at least $\frac{1}{2} \alpha$, when the cost vector is $c$. \hfill \blacksquare

Combining Lemma 3.1 and Lemma 3.2, we have proved the following theorem:

**Theorem 3.3** $\mathcal{E}V$ is 2-competitive for Vertex Cover auctions.

**Remark 3.4** The lower bound of $\frac{1}{2} \alpha$ on the frugality ratio of any mechanism can potentially be large. For instance, for a complete bipartite graph $K_{n,n}$, we have $\alpha = \Theta(n)$. Thus, such large overpayments are inherent in truthful mechanisms in general. However, truthful mechanisms may be much more frugal on specific classes of graphs.

**Remark 3.5** $\mathcal{E}V$ in general does not run in polynomial time. For the final step, computing a minimum-cost vertex cover with respect to the scaled costs, we could use a monotone 2-approximation, as suggested by Elkind et al. [11]. The hardness of computing $K$ is more severe. However, notice that for specific classes of graphs, such as degree-bounded or triangle-free graphs, $K$ can be computed efficiently, giving us non-trivial polynomial-time mechanisms for Vertex Cover on those classes. This issue is discussed more in Section 6.
3.2 Nash Equilibria and the Fractional Clique Problem

In this section, we show that the Nash Equilibrium values $\nu_v$ used for scaling of the matrix actually have a natural interpretation. To state the result, recall that the fractional clique number is the solution to the linear program

$$\begin{align*}
\text{Maximize} & \quad \sum_{u} x_u \\
\text{subject to} & \quad \sum_{u \in I} x_u \leq 1 \quad \text{for all independent sets } I \\
& \quad x_u \geq 0 \quad \text{for all } u
\end{align*}$$

(4)

The fractional chromatic number is the solution of the dual problem, where we have a variable $y_I$ for each independent set $I$ and a constraint $\sum_{I \ni u} y_I \geq 1$ for each vertex $u$, and we minimize $\sum_{I} y_I$. By LP duality, the fractional clique number and the fractional chromatic number are equal.

**Proposition 3.6** Let $G_v$ be the subgraph induced by the neighborhood of $v$ but without $v$ itself. Then, $\nu_v$ is exactly the fractional clique number, and thus the fractional chromatic number, of $G_v$.

**Proof.** Let $x$ be any bid vector feasible for the LP (2). First, for all vertices $u$ that do not share an edge with $v$, we must have $x_u = 0$, because $V \setminus \{u, v\}$ is a feasible set. So we can restrict our attention to $G_v$.

For a set $I$, we write $x(I) = \sum_{u \in I} x_u$ for the total bids of the vertices in $I$. If $I$ is an independent set in $G_v$, then $x(I) \leq 1$. The reason is that the set $V \setminus I$ is also feasible, and would cost less than $V \setminus \{v\}$ if $x(I)$ exceeded 1. Thus, any feasible bid vector $x$ induces a feasible solution to the LP (4), of the same total cost.

Conversely, if we have a feasible solution to the LP (4), we can extend it to a bid vector for all agents by setting $x_v = 1$, and $x_u = 0$ for all vertices $u$ outside $v$’s neighborhood. We need to show that each feasible set $T$, i.e., each vertex cover, has total bid $x(T)$ at least as large as the set $V \setminus \{v\}$. If $T$ does not contain $v$, it must contain all of $v$’s neighbors; it thus has the same bid as $V \setminus \{v\}$ by definition. Otherwise, because $V \setminus T$ is an independent set, the feasibility of $x$ for the LP (4) implies that $x(V \setminus T) \leq 1$. Thus, $x(T) \geq x(V) - 1 = x(V \setminus \{v\})$, and the two LPs (2) and (4) have the same value. 

Standard randomized rounding arguments (see, e.g., [22]) imply that for any graph, the chromatic number and the fractional chromatic number are within a factor $O(\log n)$ of each other. Therefore, any approximation hardness results for Graph Coloring also apply to the Fractional Clique Problem with at most a loss of logarithmic factors. In particular, the result of Feige and Kilian [13] implies that unless ZPP=NP, $\nu_v$ cannot be approximated to within a factor $O(n^{1-\epsilon})$ in polynomial time, for any $\epsilon > 0$. 

13
3.3 Composability and a General Design Approach

Vertex Cover auctions can be used naturally as a way to deal with other types of set systems. First, pre-process the set system by removing a subset of agents, turning the remaining set system into a Vertex Cover instance; then, run $\mathcal{EV}$ on that instance.

The important part is to choose the pre-processing rule to ensure that the overall mechanism is both truthful and competitive. A condition termed **composability** in [1] Definition 5.2 is sufficient to ensure truthfulness. We show here that a comparison between lower bounds is sufficient to show competitiveness.

**Definition 3.7 (Composability [1])** Let $\sigma$ be a selection rule mapping bid vectors to subsets of (remaining) agents. We say that $\sigma$ is composable if $\sigma(b) = T$ implies that $\sigma(b'_e, b_{-e}) = T$ for any $e \in T$ and $b'_e \leq b_e$. In other words, not only can a winning agent not become a loser by bidding lower; she cannot even change which set containing her wins.

Formally, when we talk about “removing” a set of agents from a set system, we are replacing $(E, \mathcal{F})$ with $(T, \mathcal{F}|_T)$, where $T = \sigma(b)$, and $\mathcal{F}|_T := \{ S \in \mathcal{F} \mid S \subseteq T \}$.

**Theorem 3.8** Let $\sigma$ be a composable selection rule with the following additional property: For all monopoly-free set systems $(E, \mathcal{F})$ in the class, and all cost vectors $c$, writing $(E', \mathcal{F}') := (\sigma(c), \mathcal{F}|_{\sigma(c)})$:

1. $(E', \mathcal{F}')$ is a Vertex Cover instance, and
2. $\nu_{(E', \mathcal{F}')} (c) \leq \kappa \cdot \nu_{(E, \mathcal{F})} (c)$.

Let the Remove-Cover Mechanism $\mathcal{RCM}$ consist of running $\mathcal{EV}$ on $(E', \mathcal{F}')$. Then $\mathcal{RCM}$ is a truthful $2\kappa$-competitive mechanism.

**Proof.** Truthfulness is proved in [1 Lemma 5.3]. The proof is short, and we include a version here for completeness. Consider any agent $e$, and a bid vector $b_{-e}$ for agents other than $e$. Because $\sigma$ is composable, and thus also monotone, there is a threshold bid $\tau_e$ such that $e$ wins iff her bid is at most $\tau_e$. Furthermore, whenever $b_e \leq \tau_e$, the set $\sigma(b)$ is uniquely determined, and independent of $b_e$. Thus, whenever $b_e \leq \tau_e$, $\mathcal{EV}$ will be run on the same set system $(\sigma(b), \mathcal{F}|_{\sigma(b)})$, and the selection rule of $\mathcal{EV}$ on this set system is monotone. Hence, the overall selection rule of $\mathcal{RCM}$ is monotone for $e$, implying directly that $\mathcal{RCM}$ is truthful.

The upper bound on the frugality ratio of $\mathcal{RCM}$ follows simply from Lemma 3.1 and the assumption of the theorem:

$$P_{\mathcal{RCM}}(c) \leq \alpha((E', \mathcal{F}')) \cdot \nu_{(E', \mathcal{F}')} (c) \leq \alpha((E', \mathcal{F}')) \cdot \kappa \cdot \nu_{(E, \mathcal{F})} (c).$$

To prove the lower bound, let $\mathcal{M}$ be any truthful mechanism for $(E, \mathcal{F})$, and let $(E', \mathcal{F}')$ be the Vertex Cover set system maximizing $\alpha((E', \mathcal{F}'))$. We consider cost vectors $c$ with $c_e = \infty$ (or some very large finite values) for $e \notin E'$. For such cost vectors, we can safely disregard all elements $e \notin E'$ altogether, as they will not affect the solutions to the LP (2), nor be part of any solution selected by $\mathcal{M}$.
But then, \( \mathcal{M} \) is exactly a mechanism selecting a feasible solution to the Vertex Cover instance \((E', \mathcal{F}')\). By Lemma 3.2, \( \mathcal{M} \) thus has frugality ratio at least \( \alpha((E', \mathcal{F}'))/2 \), completing the proof.

A simple general way to obtain a composable rule is to choose the set with the minimum total cost, from some subset of the feasible sets:

**Lemma 3.9** Let \( \sigma \) be any rule with consistent tie breaking selecting a set \( S \) minimizing \( b(S) \) over all sets \( S \) with a certain property \( P \). Then \( \sigma \) is composable.

**Proof.** Consider any agent \( e \) who is part of the winning set \( S \) with respect to \( b \). If \( e \)‘s bid decreases by \( \epsilon \), the cost of \( S \) decreases by \( \epsilon \), while the costs of all other sets decrease by at most \( \epsilon \). Thus, because ties are broken consistently, \( S \) will still be selected.

### 4 A Mechanism for Flows

We apply the methodology of Theorem 3.8 to design a mechanism \( \mathcal{FM} \) for purchasing \( k \) edge-disjoint \( s-t \) paths. We are given a (directed) graph \( G = (V, E) \), source \( s \), sink \( t \), and target number \( k \). As discussed earlier, the agents are edges of \( G \). We assume that \( G \) is monopoly-free, which is equivalent to saying that the minimum \( s-t \) cut contains at least \( k + 1 \) edges. For convenience, we will refer to a set of \( k \) edge-disjoint \( s-t \) paths simply as a \( k \)-flow, and omit \( s \) and \( t \).

To specify \( \mathcal{FM} \), all we need to do is describe a composable pre-processing rule \( \sigma \). Our rule is simple: Choose \((k+1)\) edge-disjoint \( s-t \) paths, of minimum total bid with respect to \( b \), breaking ties lexicographically. We call such a subgraph a \((k+1)\)-flow, where it is implicit that we are only interested in integer flows, and identify the flow with its edge set. Call the minimum-cost \((k+1)\)-flow \( H \). (In Section 3.3, we generically referred to this set system as \((E', \mathcal{F}').\))

**Theorem 4.1** The mechanism \( \mathcal{FM} \) is truthful and \( 2(k+1) \)-competitive and runs in polynomial time.

We show this theorem in three parts. First, we establish that the \( k \)-flow problem on \( H \) indeed forms a Vertex Cover instance (Lemma 4.2). By far the most difficult step is showing that the lower bound satisfies \( \nu_H(c) \leq (k+1) \cdot \nu_G(c) \) for all cost vectors \( c \) (Lemma 4.4). The composability of \( \sigma \) follows from Lemma 3.9. Together, these three facts allow us to apply Theorem 3.8 and conclude that \( \mathcal{FM} \) is a truthful \( 2(k+1) \)-competitive mechanism. Finally, we verify that \( \mathcal{FM} \) runs in polynomial time (Lemma 4.7).

**Lemma 4.2** The instance \((E', \mathcal{F}')\) whose feasible sets are all \( k \)-flows on \( H \) is a Vertex Cover set system.
Proof. Recall that $H$ is a minimal $(k+1)$-flow, a fact that we exploit repeatedly in this proof. The edges of $H$ are the vertices in the Vertex Cover instance. For clarity, consider explicitly the graph $R$, which contains a vertex $u_e$ for each edge $e \in H$, and an edge between $u_e, u_{e'}$ if and only if removing $e$ would create a monopoly for $e'$. This is the case iff there exists at least one minimum $s$-$t$ cut in $H$ containing both $e$ and $e'$; in particular, $R$ is symmetric. The construction is illustrated for the case $k = 2$ in Figure 1. An alternative characterization of the edges in $R$ is given in Proposition 4.3 below, and will be used as part of this proof.

For any set of edges $E'$ in $H$, let $N(E')$ be the corresponding set of nodes in $R$. Thus, for any minimum $s$-$t$ cut $E'$, the set $N(E')$ forms a clique in $R$.

Figure 1: A minimally 3-connected graph (left) and the resulting vertex cover instance for $k = 2$ (right).

If $E'$ is a $k$-flow, then for any pair of edges $e, e'$ that lie on a minimum $s$-$t$ cut, $E'$ must contain at least one of $e, e'$. Thus, $N(E')$ is a vertex cover of $R$.

Conversely, let $E'$ be a set of edges in $H$ such that $N(E')$ is a vertex cover of $R$. We will show that for every $s$-$t$ cut $F \subseteq E$, at least $k$ edges of $E'$ cross $F$, i.e., $|E' \cap F| \geq k$. This will imply that $E'$ is a $k$-flow. Assume for contradiction that $|E' \cap F| < k$. Because $N(E')$ is a vertex cover of $R$, there can be no edge between any pair of vertices in $N(F \setminus E')$ in $R$. By definition, this means that for any pair $e, e' \in F \setminus E'$, there is no minimum $s$-$t$ cut containing both $e$ and $e'$. By Proposition 4.3 below, this is equivalent to saying that for each pair $e, e' \in F \setminus E'$, the graph $H$ contains a path from $e$ to $e'$ or a path from $e'$ to $e$.

Consider a directed graph whose vertices are the edges $F \setminus E'$, with an edge from $e$ to $e'$ whenever $H$ contains a path from $e$ to $e'$. By the above argument, this graph is a tournament graph, and thus contains a Hamiltonian path. That is, there is an ordering $e_1, \ldots, e_t$ of the edges in $F \setminus E'$ such that each $e_{i+1}$ is reachable from $e_i$ in $H$. By adding a path from $s$ to $e_1$ and from $e_t$ to $t$, we thus obtain an $s$-$t$ path $P$ containing all edges in $F \setminus E'$. The graph $H \setminus P$ is a $k$-flow, so the set $E' \cap F$, having size less than $k$, cannot be an $s$-$t$ cut in $H \setminus P$. Let $P'$ be an $s$-$t$ path in $H \setminus P$ disjoint from $E' \cap F$. By construction, $P'$ is also disjoint from $F \setminus E'$. Thus, we have found an $s$-$t$ path $P'$ in $H$ disjoint from $F$, contradicting the assumption that $F$ is an $s$-$t$ cut.

Proposition 4.3 Let $H$ be a graph consisting of $k + 1$ edge-disjoint $s$-$t$ paths, and let $e = (u, v), e' = (u', v')$ be two edges of $H$. Then, there is a minimum $s$-$t$ cut containing both $e$ and $e'$ if and only if there is no path from $v$ to $u'$ and no path from $v'$ to $u$.

Proof. Assume that there is a path from $v$ to $u'$. Let $P$ be a concatenation of an $s$-$v$ path using $e$, the path from $v$ to $u'$, and a path from $u'$ to $t$ using $e'$. Then, $H \setminus P$ is a $k$-flow,
and therefore has \( k \) edge-disjoint paths. Any \( s-t \) cut in \( H \) must thus contain at least \( k \) edges from \( H \setminus P \), and no \( s-t \) cut with fewer than \( k + 2 \) edges can contain both \( e \) and \( e' \).

Conversely, if there is no minimum cut containing both \( e, e' \), then every minimum cut in \( H \setminus \{e, e'\} \) must contain \( k \) edges. Thus, \( H \setminus \{e, e'\} \) contains \( k \) edge-disjoint \( s-t \) paths. Removing these paths from \( H \) leaves us with a 1-flow, i.e., one \( s-t \) path. By construction, this path must contain \( e \) and \( e' \); thus, at least one is reachable from the other.

**Lemma 4.4** \( \nu_H(c) \leq (k + 1) \cdot \nu_G(c) \) for all \( c \).

**Proof.** Let \( S \) be the cheapest \( k \)-flow in \( G \) with respect to the costs \( c \). Because \( H \) is a \((k + 1)\)-flow, Corollary 4.6 below implies that \( \nu_H(c) = k \cdot \pi_H(c) \), where \( \pi_H(c) \) is the cost of the most expensive \( s-t \) path in \( H \).

Let \( x \) be a solution to the LP (2) with cost vector \( c \) on the graph \( G \). Define a graph \( G' \) consisting of all edges in \( S \), as well as all edges that are in at least one tight feasible set \( T \) (i.e., a set \( T \) for which the constraint (iii) is tight, meaning that \( x(T) = x(S) \)).

By definition, \( G' \) contains at least \( k + 1 \) edge-disjoint \( s-t \) paths. Lemma 4.5 below (the key step) implies that all \( s-t \) paths in \( G' \) have the same total bid with respect to \( x \). Let \( P \) be an \( s-t \) path in \( G' \) of maximum total cost \( c(P) \). By individual rationality (Constraint (i) in the LP (2)), we have that \( x(P) \geq c(P) \), and hence \( x(P') \geq c(P) \) for all \( s-t \) paths \( P' \). In particular, \( \nu_{G'}(c) = x(S) \geq k \cdot c(P) \). Since \( H \) is a minimum-cost \((k + 1)\)-flow, and because \( G' \) contains at least \( k + 1 \) edge-disjoint \( s-t \) paths, we have \( \pi_H(c) \leq (k + 1) c(P) \). Thus,

\[
\nu_{G}(c) \geq k \cdot c(P) \geq \frac{k}{k + 1} \cdot \pi_H(c) = \frac{1}{k + 1} \cdot \nu_H(c),
\]

which completes the proof.

**Lemma 4.5** Let \( x \) be a solution to the LP (2), and \( G' \) as in the proof of Lemma 4.4. Let \( v \) be an arbitrary node in \( G \), and \( P_1, P_2 \) two paths from \( v \) to \( t \). Then, \( x(P_1) = x(P_2) \).

**Proof.** Let \( \mathcal{E} \) be the collection of all tight \( k \)-flows from \( s \) to \( t \) except \( S \), i.e., the set of all \( F \) such that \( F \neq S \), \( F \) consists of exactly \( k \) edge-disjoint \( s-t \) paths, and \( x(F) = x(S) \). We define a directed multigraph \( \tilde{G} \) as follows: for each \( F \in \mathcal{E} \), we add to \( \tilde{G} \) a copy of each edge \( e \in F \) (creating duplicate copies of edges \( e \) which are in multiple flows \( F \)). We call these edges **forward edges**. In addition, for each edge \( e = (u, v) \in S \), we add \( |\mathcal{E}| \) copies of the **backward edge** \((v, u)\) to \( \tilde{G} \), i.e., we direct \( e \) the other way.

In the resulting multigraph, each node \( v \) has an in-degree equal to its out-degree. For \( v \neq s, t \) this follows since each edge set we added constitutes a flow. For \( v = s, t \), it follows since each \( F \in \mathcal{E} \) adds \( k \) edges out of \( s \) and into \( t \), while the \( |\mathcal{E}| \) copies of \( S \) add \( k|\mathcal{E}| \) edges into \( s \) and out of \( t \). As a result, \( \tilde{G} \) is Eulerian, a fact we use below.

We define a mapping \( \gamma(e) \), which assigns to each edge \( e \in \tilde{G} \) its “original” edge in \( G \). As usual, we extend notation and write \( \gamma(R) = \{\gamma(e) \mid e \in R\} \) for any set \( R \) of edges.

We will be particularly interested in analyzing collections of cycles in \( \tilde{G} \). We say that two cycles \( C_1, C_2 \) are **image-disjoint** if \( \gamma(C_1) \cap \gamma(C_2) = \emptyset \). A **cycle set** is any set of zero or
more image-disjoint cycles in \( \tilde{G} \) (which we identify with its edge set), and \( \Gamma \) denotes the collection of all cycle sets. For a cycle set \( \mathcal{C} \in \Gamma \), let \( \mathcal{C}^{+} \) and \( \mathcal{C}^{-} \) denote the set of forward and backward edges in \( \mathcal{C} \), respectively. Then, we define \( \phi(\mathcal{C}) = S \cup \gamma(\mathcal{C}^{+}) \setminus \gamma(\mathcal{C}^{-}) \). It is easy to see that for each cycle set \( \mathcal{C} \), \( \phi(\mathcal{C}) \) is a \( k \)-flow in \( G' \). Conversely, for every \( k \)-flow \( F \) in \( G' \), there is a cycle set \( \mathcal{C} \in \Gamma \) with \( \phi(\mathcal{C}) = F \).

We assign each edge \( e \in \tilde{G} \) a weight \( w_e \). For forward edges \( e \), we set \( w_e = x_{\gamma(e)} \), while for backward edges \( e = (u, v) \), we set \( w_e = -x_{\gamma(e)} \). Notice that because each copy of \( S \) contributes weight \(-x(S)\), and each set \( F \in \mathcal{E} \) contributes \( x(F) = x(S) \), the sum of all weights in \( \tilde{G} \) is 0.

Now, let \( \mathcal{C} \) be any cycle set, and \( F = \phi(\mathcal{C}) \) its corresponding \( k \)-flow. We have

\[
\sum_{e \in \mathcal{C}} w_e = x(F \setminus S) - x(S \setminus F) = x(F) - x(S),
\]

Thus \( F \) is tight, i.e., \( x(F) = x(S) \), if and only if \( \sum_{e \in \mathcal{C}} w_e = 0 \).

We next show that for any cycle \( C \) in \( \tilde{G} \), the \( k \)-flow \( \phi(C) \) is tight, and therefore that \( C \) has total weight zero, i.e., \( \sum_{e \in C} w_e = 0 \). Assume for contradiction that this is not the case, and let \( C \) be a cycle with \( \sum_{e \in C} w_e \neq 0 \). Let \( F = \phi(C) \) be the corresponding \( k \)-flow. Because we showed above that \( \sum_{e \in C} w_e = x(F) - x(S) \), we can rule out the possibility that \( \sum_{e \in C} w_e < 0 \); otherwise, \( x(F) < x(S) \), which would violate Constraint (iii) of the LP (2).

If \( \sum_{e \in C} w_e > 0 \), consider the multigraph \( \tilde{G}' \) obtained by removing \( C \) from \( \tilde{G} \). Its total weight is \( \sum_{e \in \mathcal{C}} w_e < 0 \), because the sum of all weights in \( \tilde{G} \) is 0 (as shown above). Since \( \tilde{G} \) is Eulerian, so is \( \tilde{G}' \), and its edges can be partitioned into a collection of edge-disjoint cycles \( \{C_1, \ldots, C_t\} \). By the Pigeonhole Principle, at least one of the \( C_i \) must have negative total weight. But then \( x(F_i) < x(S) \), where \( F_i = \phi(C_i) \), violating Constraint (iii) of (2) as in the previous case. This completes the proof that \( \phi(C) \) is tight for any cycle \( C \). By our observation above, the total weight of any cycle \( C \) is zero.

Finally, we prove the statement of the lemma by induction on a reverse topological sorting of the vertices \( v \)—that is, an ordering in which the index of \( v \) is at least as large as the index of any \( u \) such that \((v, u) \in G' \). Because \( G' \) is acyclic, such a sorting exists. The base case \( v = t \) is trivial. For \( v \neq t \), let \( P_1, P_2 \) be two \( v \)-\( t \) paths. We distinguish three cases, based on the first edges \( e_1 = (v, u_1) \) and \( e_2 = (v, u_2) \) of the paths \( P_1, P_2 \).

1. If \( \tilde{G} \) contains a forward edge \((v, u_1)\) and a backward edge \((u_2, v)\) (or vice versa), then since every set of edges added to \( \tilde{G} \) is a flow, \( \tilde{G} \) must contain a \( v \)-\( t \) path \( P'_1 \) entirely consisting of forward edges and starting with \( e_1 \), and a \( t \)-\( v \) path \( P'_2 \) entirely consisting of backward edges and ending with \( e_2 \) (backward). Applying the induction hypothesis to \( u_1 \) and \( u_2 \), since \( P_1 \) and \( P'_1 \) share their first edges and similarly for \( P_2 \) and \( P'_2 \), we have \( x(\gamma(P'_1)) = x(P_1) \) and \( x(\gamma(P'_2)) = x(P_2) \). Because \( P'_1 \cup P'_2 \) forms a cycle, it has total weight 0. Then \( x(\gamma(P'_1)) = -w_{P_2} = w_{P'_1} = x(\gamma(P'_1)) \), and so \( x(P_1) = x(P_2) \).

2. If \( \tilde{G} \) contains forward edges \((v, u_1)\) and \((v, u_2)\), then it contains \( v \)-\( t \) paths \( P'_1, P'_2 \) starting with \((v, u_1)\) and \((v, u_2)\), respectively, and consisting entirely of forward edges. Applying
the induction hypothesis to $u_1$ and $u_2$, we have $x(\gamma(P'_1)) = x(P_1)$ and $x(\gamma(P'_2)) = x(P_2)$. Since every set of edges added to $\tilde{G}$ is a flow, $\tilde{G}$ must contain an $s$-$v$ path $P$ consisting entirely of forward edges, and $\tilde{G}$ must also contain a $t$-$s$ path $P'$ consisting entirely of backward edges. Because $P \cup P' \cup P'_i$ forms a cycle for each $i \in \{1, 2\}$ and has total weight zero, we obtain $x(P_i) = x(\gamma(P'_i)) = -w_{P \cup P'}$ for each $i$. In particular, $x(P_1) = x(P_2)$.

3. Finally, if $\tilde{G}$ contains backward edges $(u_1, v)$ and $(u_2, v)$, we apply an argument similar to the previous case. By induction, $x(\gamma(P'_1)) = x(P_1)$, and $x(\gamma(P'_2)) = x(P_2)$. Again using the fact that $\tilde{G}$ consists of flows, it contains $t$-$v$ paths $P'_1, P'_2$ with respective last edges $(u_1, v)$ and $(u_2, v)$. In addition, $\tilde{G}$ contains a $v$-$s$ path $P$ consisting entirely of backward edges, and an $s$-$v$ path $P'$ consisting entirely of forward edges. Then for each $i \in \{1, 2\}$, $P \cup P' \cup P'_i$ forms a cycle with total weight zero, so $x(P_i) = x(P_2)$.

As a corollary, we can derive a characterization of Nash Equilibria in $(k + 1)$-flows.

**Corollary 4.6** If $G$ is a $(k + 1)$-flow, then a bid vector $x$ is a Nash Equilibrium if and only if $x(P) = \pi_G(c)$ for all $s$-$t$ paths $P$. In particular, all Nash Equilibria have the same total cost $x(S) = k \cdot \pi_G(c)$, where $S$ is the winning set.

**Proof.** First, because $G$ is a $(k + 1)$-flow, the graph $G'$ defined in the proof of Lemma 4.4 actually equals $G$, since it must contain $k + 1$ edge-disjoint $s$-$t$ paths. If $x$ is a Nash Equilibrium, then by Lemma 4.5 all $s$-$t$ paths $P$ have the same total bid $x(P)$. Let $\hat{P}$ be an $s$-$t$ path maximizing $c(P)$, i.e., $c(\hat{P}) = \pi_G(c)$. $G \setminus \hat{P}$ is a $k$-flow, and clearly the cheapest $k$-flow by definition of $\hat{P}$. Therefore, all agents in $\hat{P}$ lose, and $x(\hat{P}) = c(\hat{P})$ by Constraint (ii) of the LP (2).

Finally, we show that the mechanism $\mathcal{EV}$ runs in polynomial time for the special case of graphs derived from $k$-flows.

**Lemma 4.7** For the Vertex Cover instance derived from computing a $k$-flow on a $(k + 1)$-flow, the mechanism $\mathcal{EV}$ runs in polynomial time.

**Proof.** There are two steps which are of concern: computing the values $\nu_v$, and finding the cheapest vertex cover with respect to the scaled bids. The latter is exactly a Minimum Cost Flow problem by Proposition 4.2 and thus solvable in polynomial time with standard algorithms [2].

For the former, we claim that $\nu_{u_e} = k$ for all $u_e \in R$. By Proposition 3.6 and LP duality, $\nu_{u_e}$ is upper bounded by the chromatic number of $u_e$’s neighborhood, and lower bounded by its clique number. Since each edge $e \in H$ is part of a minimum cut of size $k + 1$, and the edges of the minimum cut form a clique in $R$, the clique number of $u_e$’s neighborhood is at least $k$. On the other hand, we can decompose $H$ into $k + 1$ edge-disjoint paths, and color the vertices corresponding to each path with its own color in $R$. By Proposition 4.3 this is a valid coloring, and shows that $u_e$’s neighborhood is $k$-colorable.
Remark 4.8  The factor 2 in the result of Theorem 4.1 comes from the factor \( \frac{1}{2} \) in the lower bound in Lemma 3.2. Using a more refined lower bound based on Young’s Inequality, Chen et al. [7] showed that for an unscaled version of the Vertex Cover mechanism, the factor \( \frac{1}{2} \) in the lower bound is unnecessary. For the instances of Vertex Cover produced as a result of the pruning in this section, the mechanism from [7] coincides with \( \mathcal{E} \mathcal{V} \), and hence \( \mathcal{F} \mathcal{M} \) is the same as the flow mechanism [7].

Chen et al. also showed that while \( \mathcal{F} \mathcal{M} \) is \((k + 1)\)-competitive when compared against the buyer-optimal lower bound [18], it is in fact optimal compared to the buyer-pessimal version [11].

5  A Mechanism for Cuts

As a second application of our methodology, we give a competitive mechanism \( \mathcal{C} \mathcal{M} \) for purchasing an \( s-t \) cut, given a (directed) graph \( G = (V, E) \), source \( s \), and sink \( t \). Again, the agents are edges. Here, the necessary monopoly-freeness is equivalent to \( G \) not containing the edge \((s, t)\).

As before, it suffices to specify and analyze a composable pre-processing rule \( \sigma \). Our pre-processing rule is to compute a minimum-cost set \( E' \) of edges (with respect to the submitted bids \( b \)), such that \( E' \) contains at least two edges from each \( s-t \) path. We call such an edge set a double cut. We show below restricting the set system to \( E' \) gives a Vertex Cover instance, and at most increases the cost of the winning set by a factor of 2.

5.1  Restricting to a double cut

To restrict the set system to \( E' \), we contract all edges in \( E \setminus E' \). Since no such edge will be cut, contracting it ensures that its endpoints will always lie on the same side of the cut. Let \( H \) denote the resulting graph. We begin with a simple structural lemma about \( H \).

Lemma 5.1  In \( H \), all \( s-t \) paths have length exactly 2.

Proof.  If there were an \( s-t \) path of length 1 in \( H \), i.e., an edge \((s, t)\), then consider the edge \((u, v)\) in the original graph corresponding to \((s, t)\). Because \( u \) was contracted with \( s \), and \( v \) with \( t \), there must be an \( s-u \) path and a \( v-t \) path in \( G \) using only edges from \( E \setminus E' \). In that case, \((u, v)\) is the only edge on this path contained in \( E' \), so \( E' \) cannot have been a double cut. Similarly, if there were an \( s-t \) path \( P \) of length at least 3, then at least one edge \((u, v)\) of \( P \) has neither \( s \) nor \( t \) as an endpoint. This edge could be safely contracted, i.e., removed from \( E' \), in which case \( E' \) was not a minimum-cost double cut.

Theorem 5.2  The double cut selection rule is composable and produces a Vertex Cover instance with \( \nu_H(c) \leq 2 \nu_G(c) \). Furthermore, both the selection rule and the subsequent Vertex Cover mechanism can be computed in polynomial time. Thus, \( \mathcal{C} \mathcal{M} \) is a polynomial-time 4-competitive mechanism.
Composability follows from Lemma 3.9, and the final conclusion then follows from Theorem 3.8 once we establish the other claims.

We can obtain a Vertex Cover instance by imposing a graph structure on H, treating each edge as a vertex and adding an edge \((e, e')\) between any two edges that form an \(s-t\) path. A set of edges is an \(s-t\) cut if and only if it contains at least one of \(e, e'\) in each such pair, so it is a vertex cover of the resulting graph.

We can think of this in turn as a flow problem as follows. Lemma 5.1 implies that \(H\) is of the following form: in addition to \(s\) and \(t\), there are vertices \(v_1, \ldots, v_\ell\), and for each \(i = 1, \ldots, \ell\), a set of parallel edges \(E_i\) from \(s\) to \(v_i\), and a set of parallel edges \(E'_i\) from \(v_i\) to \(t\). Any \(s-t\) cut has to include, for each \(i\), all of \(E_i\) or all of \(E'_i\). Thus, if we define a minimally 2-connected graph consisting of a series of vertices \(u_0, u_1, \ldots, u_\ell\) with two vertex-disjoint paths of length \(|E_i|\) and \(|E'_i|\) between \(u_{i-1}\) and \(u_i\) for each \(i\), an \(s-t\) cut in \(H\) is a 1-flow from \(u_0\) to \(u_\ell\). We can then apply Lemma 4.2. Notice that this equivalence also establishes that \(\mathcal{E}\mathcal{V}\) runs in polynomial time on the instances produced by this selection rule.

As before, the key part is to analyze the increase in the lower bound.

**Lemma 5.3** For all cost vectors \(c\), \(\nu_H(c) \leq 2\nu_G(c)\).

**Proof.** Let \((S, \overline{S})\) be the cheapest \(s-t\) cut in \(G\) with respect to the costs \(c\), and \(x\) a solution to the LP (2) with cost vector \(c\) on the graph \(G\). Let \(C\) be the set of all minimum \(s-t\) cuts \((T, \overline{T})\) with respect to the costs \(x\); thus, each of these cuts has cost \(x(E(S, \overline{S}))\). Define \(T^- = \bigcap_{(T, \overline{T}) \in C} T\), and \(T^+ = \bigcup_{(T, \overline{T}) \in C} T\). Then, both \((T^-, \overline{T^-})\) and \((T^+, \overline{T^+})\) are minimum \(s-t\) cuts as well (see, e.g., [2, Exercise 6.39]).

Furthermore, the edge sets \(E(T^-, \overline{T^-})\) and \(E(T^+, \overline{T^+})\) are disjoint. For assume that there is an edge \(e = (u, v)\) in common between these sets. Then, \(u \in \bigcap_{(T, \overline{T}) \in C} T\) and \(v \in \bigcap_{(T, \overline{T}) \in C} \overline{T}\).

In particular, this implies that \(u \in S\) and \(v \in \overline{S}\). As stated above in equation (3), since \(x\) maximizes the LP (2), Constraint (iii) must be tight for some feasible set excluding \(e\), since otherwise the bid \(x_e\) could be increased. Let \((T, \overline{T})\) be the corresponding cut. Then \(x(E(T, \overline{T})) = x(E(S, \overline{S}))\), and \(e\) does not cross \((T, \overline{T})\). Thus, either both \(u\) and \(v\) are in \(T\), or both are in \(\overline{T}\). Since \((T, \overline{T}) \in C\), this gives a contradiction.

Now define \(G' := E(T^-, \overline{T^-}) \cup E(T^+, \overline{T^+})\). Because \(G'\) consists of two disjoint \(s-t\) cuts, it is a double cut, and the cost-minimality of \(H\) implies that \(c(G') \geq c(H)\). These two cuts both have minimal cost, so \(\nu_G(c) = x(G') / 2\). By the “individual rationality” LP constraint (i), \(x(G') \geq c(G')\), and hence

\[
\nu_G(c) = \frac{x(G')}{2} \geq \frac{c(G')}{2} \geq \frac{c(H)}{2} \geq \frac{\nu_H(c)}{2}.
\]

For the last inequality, notice that in the “Nash Equilibrium” on \(H\), for each \(i\), the cheaper of \(E_i\) and \(E'_i\) will collectively raise their bids to the cost of the more expensive one, so the total bid of the winning set will be \(\nu_H(c) = \sum_i \max(c(E_i), c(E'_i)) \leq c(H)\). \IEEEQED
5.2 A Primal-Dual Algorithm for Minimum Double-Cuts

Finally, we present a polynomial time algorithm to compute a minimum-cost double cut. The minimum-cost double cut is characterized by integer solutions to the following LP, where $\mathcal{P}$ denotes the set of all $s$-$t$ paths in $G$.

\[
\begin{align*}
\text{Minimize} & \quad \sum_{e \in E} c_e x_e \\
\text{subject to} & \quad \sum_{e \in P} x_e \geq 2 \quad \text{for all } P \in \mathcal{P} \\
 & \quad x_e \leq 1 \quad \text{for all edges } e \in E \\
 & \quad x_e \geq 0 \quad \text{for all } e, \\
\end{align*}
\]  

(5)

Remark 5.4 It is not difficult to show that the constraint matrix for this LP is totally unimodular. By a well-known theorem [27], because the right-hand sides of the constraints are integral, total unimodularity implies that all the vertices of the LP’s polytope are integral. Since there is a separation oracle for the LP (as well as an equivalent polynomial-sized LP formulation), an integer solution can be found in polynomial time, giving us a polynomial-time algorithm. However, the resulting algorithm is rather inefficient.

Here, we present a more efficient primal-dual algorithm generalizing the Ford-Fulkerson Max-Flow algorithm. The dual of the LP is

\[
\begin{align*}
\text{Maximize} & \quad 2 \sum_{P \in \mathcal{P}} f_P - \sum_{e} r_e \\
\text{subject to} & \quad \sum_{P: e \in P} f_P \leq c_e + r_e \quad \text{for all } e \in E \\
 & \quad f_P, r_e \geq 0 \quad \text{for all } P \in \mathcal{P} \text{ and all } e \in E. \\
\end{align*}
\]  

(6)

We interpret the dual variables $f_P$ as describing a flow in the usual way. That is, the flow along each edge $e$ is

\[f_e = \sum_{P: e \in P} f_P.\]

We say that $e$ is saturated if $f_e = c_e + r_e$. We call $r_e$ the relief on $e$: in order to send more flow on an edge $e$, we can increase its capacity, but we pay for it in the objective function. It is worth augmenting the flow along a path so long as at most one edge on the path is saturated, since increasing the flow and the relief of the saturated edge at the same time increases the dual objective.

Our primal-dual algorithm is similar to the Ford-Fulkerson algorithm, and is based on the same concept of a residual graph. The residual graph contains forward edges for all edges $e$ in the original graph, even when they are saturated, because it is possible to send more flow by adding relief. In addition, if $e = (u, v)$ carries flow $f_e$, then the residual graph, as usual, contains the backward edge $(v, u)$ with capacity $f_e$. To capture how much relief would have to be added to augment the flow along a path, we define, for each edge $e$ in the residual graph, a length $\ell_e$ as follows:

1. If $e$ is a saturated forward edge, then $\ell_e = 1$.  

22
2. If \((v, u)\) is a backward edge such that \((u, v)\) has positive relief, then \(\ell_{(v,u)} = -1\).

3. The lengths of all other edges are \(\ell_e = 0\).

For paths \(P\), we define \(\ell(P) = \sum_{e \in P} \ell_e\). We give our primal-dual algorithm as Algorithm 1.

**Algorithm 1** Flow computation for Minimum Double Cut

1: Flow Computation:
2: Let \(f\) be an arbitrary maximum flow on \(G\).
3: while there is an \(s-t\) path \(P\) with \(\ell(P) \leq 1\) in the residual graph \(G_f\) do
4: Let \(P\) be such a path with minimum length \(\ell(P)\).
5: Augment the flow on \(P\) by \(\delta\), while simultaneously increasing the relief of any saturated edge by \(\delta\), and decreasing the relief of any backward edge by \(\delta\), for the smallest value of \(\delta\) such that this action increases \(\ell(P)\), i.e., the smallest \(\delta\) such that either a new forward edge becomes saturated or the relief on a backward edge becomes zero.

Notice that for any path \(P\) of length at most 1, augmenting the flow increases the dual objective. This follows since the total number of saturated edges is at most one greater than the total number of backward edges with relief; for the latter, each unit of flow reduces the total relief by one unit, while for the former, each unit of flow increases the total relief by one unit. Thus, the total increase in relief for sending \(\delta\) units of flow is at most \(\delta\), while the first term of the objective function, i.e., the value of the flow, increases by \(2\delta\).

As with the Ford-Fulkerson algorithm, the running time could be pseudo-polynomial with a poor choice of the augmenting path \(P\). But breaking ties for the smallest total number of edges in \(P\) gives strongly polynomial running time, as with the Edmonds-Karp algorithm.

When the algorithm terminates, we have a set \(T\) of saturated edges, and a subset \(R \subseteq T\) of relief edges \(e\) with \(r_e > 0\). We will pick two edge-disjoint \(s-t\) cuts \((S_1, \overline{S}_1), (S_2, \overline{S}_2)\) with the properties that:

1. Only saturated edges cross either of the cuts.
2. \(S_1 \subseteq S_2\).
3. Each relief edge crosses one of the two chosen cuts.

This will naturally satisfy all complementary slackness conditions for the two LPs, and thus prove optimality of the cuts.

To define and compute the two cuts, we focus on the graph \(G'\) obtained from the residual graph by removing all forward edges \(e\) with \(f_e = 0\). Importantly, we use the same notion of length defined above. From now on, all references to reachability, distances, etc. are with respect to \(G'\).

For each node \(v\), let \(d_v\) denote the minimum distance from \(s\) to \(v\) in \(G'\). We show next that \(G'\) has no negative cycles, so these distances are well-defined.

**Lemma 5.5** \(G'\) has no path from the sink \(t\) to the source \(s\) of length strictly less than \(-1\). In particular, \(G'\) contains no negative cycles.
Proof. We show by induction that these properties hold for the residual graph in each iteration. Since $G'$ is obtained from the residual graph only by deleting edges with zero flow (and thus length 0 or 1 only), distances can only increase in $G'$.

Initially, all edges have length 0 or 1, and there are no backward edges, so the claim clearly holds. If the residual graph contained a negative-length cycle $C$, then $C$ would have to contain at least one flow-carrying edge $e = (u, v)$. Since $e$ has incoming flow from $s$ and outgoing flow to $t$, the residual graph would contain a path of backward edges from $u$ to $s$ and one from $t$ to $v$. Thus, any negative cycle would give arbitrarily negative-length paths from $t$ to $s$. It is therefore enough to establish the first claim.

Consider an iteration when flow is augmented along a path $P$. Suppose that this generates a $t$-$s$ path $P'$ in the residual graph of length strictly less than $-1$. $P \cup P'$ gives a cycle. If we assign edges in $P$ their length prior to the augmentation, and edges in $P'$ their length after the augmentation, then the total length of the cycle $P \cup P'$ is negative. The only edges in $P'$ whose length can have decreased through the augmentation are the backward versions of edges to which relief was added by $P$. They were saturated before the augmentation, so their forward length was 1, and their backward length after augmentation is $-1$.

Now consider removing edges that appear both forward and backward in $P \cup P'$. We obtain a union of edge-disjoint cycles, such that all edges in these cycles were present in the residual graph prior to the flow augmentation. Of these edge-disjoint cycles, by the argument of the previous paragraph, at least one cycle $C$ has negative length with respect to the previous paragraph’s definition. The edges in $P' \setminus P$ don’t change their length, so $C$ had negative length before the augmentation, contradicting the induction hypothesis. \qed

For two nodes $u, v$, we write $u \rightarrow_0 v$ if there is a path of length at most 0 from $u$ to $v$ in $G'$. We now define the cuts. Let $S_1 := \{ v \mid d_v \leq 0 \}$. Define $E' := \{ (u, v) \in R \mid d_u > 0 \}$. Now, let $U$ be the set of all vertices lying on a $v$-$t$ path for some edge $(u, v) \in E'$, and let $S_2$ be the set of all vertices $y$ such that $y \rightarrow_0 w$ for some $w \in U$. Clearly, $(S_1, \overline{S_1})$ and $(S_2, \overline{S_2})$ define two $s$-$t$ cuts using only saturated edges.

**Lemma 5.6** No edge crosses both cuts $(S_1, \overline{S_1})$ and $(S_2, \overline{S_2})$. Each relief edge $e \in R$ crosses one of the cuts $(S_1, \overline{S_1})$ or $(S_2, \overline{S_2})$.

**Proof.** To prove the first claim, suppose that $e = (u, v)$ crosses both cuts. By the definition of $\overline{S_2}$, that means that there is an edge $e' = (u', v') \in E' \subseteq R$ such that there is a path of length at most 0 from $v$ to some node $w$ on a $v'$-$t$ path. Consider the path from $s$ to $u$ (of length at most 0), followed by $e$ (of length at most 1), followed by the path from $v$ to $w$, and then the path to $u'$ backwards, followed by $e'$ backwards. This is a path of length at most 0 from $s$ to $u'$, meaning that $u'$ should have been in $S_1$, and contradicting that $e'$ was in $E'$.

To prove the second claim, suppose that a relief edge $e = (u, v)$ crosses neither of the cuts. We distinguish two cases:

1. If $u \in S_1$ then, since $e$ does not cross either cut, $v \in S_1$, so $d_v \leq 0$. But then the $s$-$v$ path of length at most 0, followed by $e$ backwards (of length $-1$), followed by the $u$-$s$ path backward (of length at most 0) gives a negative cycle, contradicting Lemma 5.5.
2. If $u \notin S_1$, then $d_u > 0$ and $e \in E'$. Thus $v \in U$, and since $v \rightarrow_0 v$, we have $v \in S_2$. Since $e$ does not cross either cut, we also have $u \in S_2$. This means that there is a $w \in U$ and $e' = (u', v') \in E'$ such that $w$ lies on a $v'$-$t$ path, and $u \rightarrow_0 w$. Now consider the path from $t$ to $v$ (of length at most 0), then using $e$ backwards (of length $-1$), then the length-0 path from $u$ to $w$ and the path from $w$ to $v$ backwards (of length at most 0), followed by $e'$ backwards, and the path from $u'$ to $s$ backwards (of length at most 0). This gives a $t$-$s$ path of total length at most $-2$, again contradicting Lemma 5.5.

Lemma 5.6 implies that the set $R$ of relief edges forms a double cut. Thus our algorithm finds a minimum-cost double cut in polynomial time.

Remark 5.7 As in Remark 4.8 for the flow mechanism $FM$, we can show that on instances derived from the pruning step, $EV$ coincides with the mechanism of [7]. Thus, the tighter analysis shows that $CM$ is in fact 2-competitive. We conjecture that $CM$ is indeed optimal when compared against the buyer-pessimal lower bound of [11].

6 Directions for Future Work

We have presented novel truthful and competitive mechanisms for three important combinatorial problems: Vertex Covers, $k$-flows, and $s$-$t$ cuts. The Vertex Cover mechanism was based on scaling the submitted bids by multipliers derived as components of the dominant eigenvector of a suitable matrix. Both the flow and cut mechanisms were based on pruning the input graph, and then applying the Vertex Cover mechanism to the pruned version. Besides the individual mechanisms, we believe that the methodology of reducing input instances to Vertex Cover problems may be of interest for future frugal mechanism design.

In general, the Vertex Cover mechanism does not run in polynomial time, due to two obstacles. First, computing the matrix $K$ requires computing the largest fractional clique size in the neighborhood of each node $v$. Subsequently, computing the solution with respect to scaled costs requires finding a cheapest vertex cover. For the second obstacle, it seems quite likely that monotone algorithms such as the one in [11] could be adapted to our setting, and yield constant-factor approximations. However, the difficulty of computing the entries of $K$ seems more severe. In fact, we conjecture that no polynomial-time truthful mechanism for Vertex Cover can be constant-competitive. This result would be quite interesting, in that it would show that the requirements of incentive-compatibility and computational tractability together can lead to significantly worse guarantees than either requirement alone. A positive resolution of this conjecture would thus be akin to the types of hardness results demonstrated recently for the Combinatorial Public Project Problem [26].

While our methodology of designing composable pre-processing algorithms will likely be useful for other problems as well, it does not apply to all set systems. It is fairly easy to construct set systems for which no such pruning algorithm is possible. Even when pruning is possible in principle, it may come with a large blowup in costs.
Thus, the following bigger question still stands: which classes of set systems admit constant-competitive mechanisms? The main obstacle is our inability to prove strong lower bounds on frugality ratios. To date, all lower bounds (here, as well as in [12, 18]) are based on pairwise comparisons between agents, which can then be used to show that certain agents, by virtue of losing, will cause large payments. This technique was exactly the motivation for our Vertex Cover approach. In order to move beyond Vertex Cover based mechanisms, it will be necessary to explore lower bound techniques beyond the one used in this paper.

In recent joint work with the authors of [7], we have shown that the factor $\frac{1}{2}$ in the lower bound of Lemma 3.2 can be removed, thus showing that $\mathcal{EV}$ is optimal. The proof of this result will be presented in a joint full version of the present paper with [7].

Acknowledgment

We would like to thank Edith Elkind, Uriel Feige, Nick Gravin, Anna Karlin, Tami Tamir, and Mihalis Yannakakis for useful discussions and pointers, and anonymous reviewers for useful feedback. D.K. is supported in part by an NSF CAREER Award, an ONR Young Investigator Award and an award from the Sloan Foundation. C.M. is supported in part by the McDonnell Foundation.

References

[1] Gagan Aggarwal and Jason D. Hartline. Knapsack auctions. In Proc. 17th ACM Symp. on Discrete Algorithms, pages 1083–1092, 2006.

[2] Ravindra K. Ahuja, Thomas L. Magnanti, and James B. Orlin. Network Flows. Prentice Hall, 1993.

[3] Aaron Archer and Eva Tardos. Frugal path mechanisms. In Proc. 13th ACM Symp. on Discrete Algorithms, pages 991–999, 2002.

[4] Sushil Bikhchandani, Sven de Vries, James Schummer, and Rakesh Vohra. Linear programming and Vickrey auctions. IMA Volume in Mathematics and its Applications, Mathematics of the Internet: E-auction and Markets, 127:75–116, 2001.

[5] Gruia Calinescu. Bounding the payment of approximate truthful mechanisms. In Proc. 15th Intl. Symp. on Algorithms and Computation, pages 221–233, 2004.

[6] Matthew Cary, Abraham D. Flaxman, Jason D. Hartline, and Anna Karlin. Auctions for structured procurement. In Proc. 19th ACM Symp. on Discrete Algorithms, pages 304–313, 2008.

[7] Ning Chen, Edith Elkind, Nick Gravin, and Fedor Petrov. Frugal mechanism design via spectral techniques. In Proc. 51st IEEE Symp. on Foundations of Computer Science, 2010.
[8] Ning Chen and Anna Karlin. Cheap labor can be expensive. In Proc. 18th ACM Symp. on Discrete Algorithms, pages 707–715, 2007.

[9] Edward H. Clarke. Multipart pricing of public goods. Public Choice, 11:17–33, 1971.

[10] Ye Du, Rahul Sami, and Yaoyun Shi. Path auctions when an agent can own multiple edges. In Proc. 1st Workshop on Economics of Networks Systems, 2006.

[11] Edith Elkind, Leslie Goldberg, and Paul Goldberg. Frugality ratios and improved truthful mechanisms for vertex cover. In Proc. 9th ACM Conf. on Electronic Commerce, pages 336–345, 2007.

[12] Edith Elkind, Amit Sahai, and Kenneth Steiglitz. Frugality in path auctions. In Proc. 15th ACM Symp. on Discrete Algorithms, pages 701–709, 2004.

[13] Uriel Feige and Joe Kilian. Zero knowledge and the chromatic number. Journal of Computer and System Sciences, 57(2):187–199, 1998.

[14] Rahul Garg, Vijay Kumar, Atri Rudra, and Akshat Verma. Coalitional games on graphs: core structures, substitutes and frugality. In Proc. 5th ACM Conf. on Electronic Commerce, pages 248–249, 2003.

[15] Theodore Groves. Incentives in teams. Econometrica, 41:617–631, 1973.

[16] Nicole Immorlica, David R. Karger, Evdokia Nikolova, and Rahul Sami. First-price path auctions. In Proc. 7th ACM Conf. on Electronic Commerce, pages 203–212, 2005.

[17] Atsushi Iwasaki, David Kempe, Yasumasa Saito, Mahyar Salek, and Makoto Yokoo. False-name proof mechanisms for hiring teams. In Proc. 3rd Workshop on Internet and Network Economics (WINE), pages 245–256, 2007.

[18] Anna Karlin, David Kempe, and Tami Tamir. Beyond VCG: Frugality of truthful mechanisms. In Proc. 46th IEEE Symp. on Foundations of Computer Science, pages 615–624, 2005.

[19] David Kempe, Mahyar Salek, and Cristopher Moore. Frugal and truthful mechanisms for vertex covers, flows, and cuts. In Proc. 51st IEEE Symp. on Foundations of Computer Science, pages 745–754, 2010.

[20] Paul Klemperer. Auction theory: A guide to the literature. Journal of Economic Surveys, 13(3):227–286, 1999.

[21] Vijay Krishna. Auction Theory. Academic Press, 2002.

[22] Carsten Lund and Mihalis Yannakakis. On the hardness of approximating minimization problems. Journal of the ACM, 41(5):960–981, 1994.
[23] Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. *Microeconomic Theory*. Oxford University Press, 1995.

[24] Noam Nisan and Amir Ronen. Algorithmic mechanism design. In *Proc. 31st ACM Symp. on Theory of Computing*, pages 129–140, 1999.

[25] Christos Papadimitriou. Algorithms, games and the Internet. In *Proc. 33rd ACM Symp. on Theory of Computing*, pages 749–752. ACM Press, 2001.

[26] Christos Papadimitriou, Michael Schapira, and Yaron Singer. On the hardness of being truthful. In *Proc. 49th IEEE Symp. on Foundations of Computer Science*, pages 250–259, 2008.

[27] Christos Papadimitriou and Kenneth Steiglitz. *Combinatorial Optimization*. Dover, 1982.

[28] Kunal Talwar. The price of truth: Frugality in truthful mechanisms. In *Proc. 21st Annual Symp. on Theoretical Aspects of Computer Science*, pages 608–619, 2003.

[29] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *J. of Finance*, 16:8–37, 1961.

[30] Qiqi Yan. On the price of truthfulness in path auctions. In *Proc. 3rd Workshop on Internet and Network Economics (WINE)*, pages 584–589, 2007.