2+1d Compact Lifshitz Theory, Tensor Gauge Theory, and Fractons

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Abstract

The 2+1d continuum Lifshitz theory of a free compact scalar field plays a prominent role in a variety of quantum systems in condensed matter physics and high energy physics. It is known that in compact space, it has an infinite ground state degeneracy. In order to understand this theory better, we consider two candidate lattice regularizations of it using the modified Villain formalism. We show that these two lattice theories have significantly different global symmetries (including a dipole global symmetry), anomalies, ground state degeneracies, and dualities. In particular, one of them is self-dual. Given these theories and their global symmetries, we can couple them to corresponding gauge theories. These are two different $U(1)$ tensor gauge theories. The resulting models have excitations with restricted mobility, i.e., fractons. Finally, we give an exact lattice realization of the fracton/lineon-elasticity dualities for the Lifshitz theory, scalar and vector charge gauge theories.
# Contents

1 Introduction 2

2 2+1d compact Lifshitz theory 8
   2.1 Laplacian $\phi$-theory ........................................ 10
      2.1.1 Self-duality ............................................. 11
      2.1.2 Global symmetry and ground state degeneracy .......... 12
      2.1.3 Robustness ............................................. 13
   2.2 Dipole $\phi$-theory ........................................... 14
      2.2.1 Lineon-elasticity duality .................................. 15
      2.2.2 Global symmetry, ground state degeneracy and mobility of defects . 16
      2.2.3 Robustness ............................................. 19

3 2+1d $U(1)$ tensor gauge theories 20
   3.1 Laplacian gauge theory .......................................... 20
      3.1.1 Global symmetry and mobility of defects ............... 22
   3.2 Scalar charge theory ........................................... 23
      3.2.1 Fracton-elasticity duality .................................. 24
      3.2.2 Global symmetry, ground state degeneracy, and mobility of defects . 25
      3.2.3 Robustness ............................................. 27

4 Discussion, Conclusion, and Outlook 27

A More on global symmetries of 2+1d dipole theories 30
   A.1 Global symmetry of 2+1d dipole $\phi$-theory ..................... 30
   A.2 Global symmetry of 2+1d scalar charge theory ................. 34

B Polynomial representation of functions on square lattice 38
   B.1 Naturalness of 2+1d Laplacian $\phi$-theory ..................... 40
   B.2 Mobility of defects in 2+1d $U(1)$ Laplacian gauge theory .... 44
1 Introduction

In recent years, a large class of models with strange features have been discovered with important applications to condensed matter physics and quantum information. The most striking of these are the fracton models [1–3] which host particle-like excitations with restricted mobility, such as those that cannot move (fractons), or can move only in a line (lineons) or a plane (planeons). (See [4–9] for reviews on this subject.) These models defy a standard continuum quantum field theory description at low energies. Instead, their peculiar features are captured by non-standard field theories with exotic global symmetries. One important consequence of such symmetries is a large ground state degeneracy [10], which is infinite in the continuum limit.

The challenge to find a standard continuum low-energy description of these theories points to a missing deep insight in our understanding of continuum quantum field theory. As we will demonstrate below, innocent-looking continuum models can be quite subtle and need a careful definition. And their physical consequences depend sensitively on that definition. In the opposite direction, some innocent-looking lattice models might not have any continuum low-energy field theory description. It is expected that a detailed study of lattice models, continuum models, and the relations between them will enhance our understanding of these important issues.

Perhaps the simplest theory that exhibits this behavior is the 2+1d compact Lifshitz field theory [11–20] described by the Lagrangian

\[ \mathcal{L} = \frac{\mu_0}{2} (\partial_\tau \phi)^2 + \frac{1}{2\mu} (\nabla^2 \phi)^2, \]

where \( \phi \sim \phi + 2\pi \) is a compact scalar, and \( \nabla^2 = \partial_x^2 + \partial_y^2 \) is the 2d spatial Laplacian operator.

(Throughout this paper, we work in the Euclidean signature, and \( \tau \) denotes Euclidean time.) The compact Lifshitz field theory has appeared in many different physical contexts, including deconfined quantum criticality [13].

It is well-known that, on a 2d spatial torus with periodic boundary conditions, this theory has infinite ground state degeneracy, where the ground states are labelled by the winding numbers in the \( x \) and \( y \) directions [11,12]. This is a consequence of the dipole symmetry [22–24,20,21,35,36] which shifts \( \phi \) by linear functions in \( x \) and \( y \). On the other hand, on the plane \( \mathbb{R}^2 \), the symmetry is much larger and includes shifts of \( \phi \) by harmonic functions of \( x \) and \( y \), of which the linear functions form only a tiny subset. To regularize this infinity and make sense of the exotic symmetry, we wish to place the theory on the lattice.

\[ ^1 \text{An even simpler version of this theory in 1+1d is analyzed in [21].} \]
It is commonly the case that given a continuum theory, there are several ways to regularize it on the lattice. The difference between them is in irrelevant operators in the continuum theory and therefore, it is not important. As we will see, this is not the case here.

How should we regularize the 2+1d Lifshitz Lagrangian (1.1)? One could “discretize” the Laplacian operator $\nabla^2$ as $\Delta_x^2 + \Delta_y^2$, which is the discrete Laplacian operator on the 2d spatial torus lattice. This discretization has the advantage of being well-defined on other spatial lattices, including general graphs. (See [37] for a discussion of exotic lattice models on graphs based on the discrete Laplacian operator.) Alternatively, one could first integrate by parts and replace $(\nabla^2 \phi)^2$ with

$$(\partial_x^2 \phi)^2 + (\partial_y^2 \phi)^2 + 2(\partial_x \partial_y \phi)^2,$$

and then discretize the three terms separately. We will see that these two discretizations are not the same and they do not defer merely by irrelevant operators. Instead, the discretization following from (1.2) is very close to the continuum theory (1.1). But the other one is an interesting peculiar lattice model, whose relation to the continuum theory is unclear.

Since $\phi$ is compact, the discretization has to take that into account. One possibility is to use trigonometric functions in the lattice action. We prefer to use a Villain-type formalism [38]. Here, we introduce on the lattice integer-valued gauge fields. Then, the two different discretizations above lead to the following schematic spatial kinetic terms:

Laplacian $\phi$-theory: $[(\Delta_x^2 + \Delta_y^2)\phi - 2\pi n]^2 + \cdots$

Dipole $\phi$-theory: $(\Delta_x^2 \phi - 2\pi n_{xx})^2 + (\Delta_y^2 \phi - 2\pi n_{yy})^2 + 2(\Delta_x \Delta_y \phi - 2\pi n_{xy})^2 + \cdots$

(We defer a detailed discussion of the Villain integer gauge fields and the additional terms in the ellipses to Section 2.)

We refer to the first model as the Laplacian $\phi$-theory because it makes use of the discrete Laplacian operator, and the second model as the dipole $\phi$-theory because it has dipole global symmetries, which will be discussed in Section 2.2.

Following [28][39][47][21][37], we focus on the global symmetries and other global aspects. To this end, we analyze the modified Villain formulation [48][45] of these two lattice models.

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1. We label the sites on the lattice as $(x, y)$, where $x, y$ are integers. (At the risk of confusing the reader, we use the same characters $(x, y)$ for the continuum coordinates and the discrete lattice coordinates.) We define $\Delta_x f(x + \frac{1}{2}, y) \equiv f(x + 1, y) - f(x, y)$ for a function on the sites, $\Delta_x f_x(x, y) \equiv f_x(x + \frac{1}{2}, y) - f_x(x - \frac{1}{2}, y)$ for a function on the $x$-links, $\Delta_x f_y(x + \frac{1}{2}, y + \frac{1}{2}) \equiv f_y(x + 1, y + \frac{1}{2}) - f_y(x, y + \frac{1}{2})$ for a function on $y$-links, and $\Delta_{xy} f(x, y + \frac{1}{2}) \equiv f_{xy}(x + \frac{1}{2}, y + \frac{1}{2}) - f_{xy}(x - \frac{1}{2}, y + \frac{1}{2})$ for a function on plaquettes. $\Delta_n$ is defined in a similar way. Using these definitions, if $f$ is a function on the sites, then $\Delta_x f$ is a function on the $x$-links, and so $\Delta_x^2 f(x, y) = f(x + 1, y) - 2f(x, y) + f(x - 1, y)$ and $\Delta_x \Delta_y f(x + \frac{1}{2}, y + \frac{1}{2}) = f(x + 1, y + 1) - f(x, y + 1) - f(x + 1, y) + f(x, y)$. We follow similar notation for functions that further depend on $\tau$.  

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2. We refer to the first model as the Laplacian $\phi$-theory because it makes use of the discrete Laplacian operator, and the second model as the dipole $\phi$-theory because it has dipole global symmetries, which will be discussed in Section 2.2.
The advantage of this formulation is that the symmetries, anomalies, and dualities of the continuum theory are already manifest on the lattice.

While naively the two lattice models in $\text{(1.3)}$ are regularizations of the same continuum Lifshitz Lagrangian $\text{(1.1)}$, they have very different properties:

- Their global symmetries are different. For example, the Laplacian model is invariant under the schematic transformation $\phi \rightarrow \phi + cxy$ with a constant $c$, while the dipole model does not enjoy such a symmetry.

- The ground state degeneracies (GSD) of the two models, when placed on a 2d spatial torus lattice with $L_i$ sites in the $i = x, y$ direction, are drastically different:

  \[
  \text{Laplacian } \phi\text{-theory: } \text{GSD} \sim \exp \left( \frac{4G}{\pi} L_x L_y \right) , \quad L_i \rightarrow \infty ,
  \]

  \[
  \text{Dipole } \phi\text{-theory: } \text{GSD} = L_x L_y ,
  \]

where $G \sim 0.916$ is the Catalan constant, for the Laplacian model.

- The Laplacian model is not robust against perturbations of (momentum) symmetry-preserving local operators, while the dipole model is robust.

- They have different dualities. The Laplacian lattice model is self-dual \[3\] but the dipole lattice model is dual to the lattice version of the vector charge theory $\text{[49, 24, 25, 50, 26, 51, 52]}$. In the continuum, this tensor gauge theory has gauge fields $(\hat{A}_\tau i, \hat{A}_{ij})$ and a spatial vector gauge parameter $\hat{\alpha}_i$ with gauge transformations

  \[
  \hat{A}_\tau i \sim \hat{A}_\tau i + \partial_\tau \hat{\alpha}_i , \quad \hat{A}_{xy} \sim \hat{A}_{xy} + \partial_x \hat{\alpha}_y + \partial_y \hat{\alpha}_x , \quad \hat{A}_{ii} \sim \hat{A}_{ii} + \partial_i \hat{\alpha}_i , \quad i = x, y .
  \]

This theory has defects that represent the worldline of particles that are allowed to move in one direction in space, i.e., it is a theory of lineons. The duality between the dipole $\phi$-theory and the vector charge theory had been discussed in the continuum in

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\[3\]The global aspects of such global symmetries will be discussed in details in Section 2. See, in particular, the discussion around equation \text{(2.16)}.

\[4\]Observe that the GSD of the Laplacian $\phi$-theory grows exponentially in the number of sites $L_x L_y$. A similar behavior is already present in a trivial system with decoupled spins. As explained in \text{[37]} and reviewed in Section 2.1.2, the origins of these two exponential behaviors are not the same. In the Laplacian $\phi$-theory, it is the large orders of some of the symmetry generators, whereas in the decoupled spin system, it is the extensive number of symmetry generators.

\[5\]The authors of \text{[13]} argued that the continuum Lifshitz theory is self-dual. The corresponding lattice Laplacian theory has such a self-duality \text{[37]}, which will be discussed in Section 2.1.1
the context of elasticity theory [53]. We refer to this duality as the lineon-elasticity duality. We will discuss its lattice version in Section 2.2.1.

Other differences between the two models are discussed in Section 2. The fact these two different regularizations of the same continuum Lifshitz Lagrangian (1.1) have completely different global symmetries, GSD, and dualities highlights the ambiguity of working just with the naive continuum Lagrangian. Having said that, it is clear that the dipole lattice theory is closer to the continuum Lifshitz theory than the more exotic Laplacian lattice theory.

The pure U(1) gauge theories associated with the global symmetries of the two models of (1.3) are also strikingly different. Schematically, the one associated with the Laplacian model has only two gauge fields, $A_\tau$ and $A$, with Laplacian gauge transformations

\[ A_\tau \sim A_\tau + \partial_\tau \alpha , \quad A \sim A + \nabla^2 \alpha . \tag{1.6} \]

On the other hand, the one associated with the dipole model is a rank-2 tensor gauge theory which has been discussed extensively in the literature [24, 25, 54, 55, 20, 56, 29, 51, 52]. It has four gauge fields, $A_\tau$ and $A_{xx}, A_{yy}, A_{xy}$, with rank-2 tensor gauge transformations

\[ A_\tau \sim A_\tau + \partial_\tau \alpha , \quad A_{ij} \sim A_{ij} + \partial_i \partial_j \alpha , \quad i, j = x, y . \tag{1.7} \]

The gauge theory of $(A_\tau, A_{ij})$ (possibly coupled to matter fields) is referred to in [24, 25, 50, 26] as the scalar charge theory to emphasize the fact that the gauge parameter $\alpha$ is a scalar. Similar to the Lifshitz theory, the continuum gauge transformations and Lagrangians do not specify unambiguously all the global aspects (such as GSD discussed below) of these gauge theories. In Section 3, we regularize these two gauge theories using the (modified) Villain lattice formulation, while preserving all the global symmetries, dualities, and anomalies. From this point on, we refer to the (modified) Villain lattice versions of these two gauge theories as the Laplacian gauge theory and the scalar charge theory.

Both the 2+1d Laplacian and the scalar charge theories are natural generalizations of the 1+1d tensor gauge theory studied in [21]. Both gauge theories have line defects that describe the worldline of an immobile particle, i.e., they are fracton models.

However, these models do not have GSD that grows sub-extensively in the system size.
Figure 1: **Left:** Dualities and relations between the two matter theories, the dipole \( \phi \)-theory and the \( \hat{\phi}_i \)-theory, and the two gauge theories, the scalar charge theory \((A_\tau, A_{ij})\) and the vector charge theory \((\hat{A}_\tau, \hat{A}_{ij})\). The dipole \( \phi \)-theory is the Higgs field of the \((A_\tau, A_{ij})\) theory, or conversely, the \((A_\tau, A_{ij})\) theory gauges the momentum dipole symmetry of \( \phi \). The relation between \( \hat{\phi}_i \) and \((\hat{A}_\tau, \hat{A}_{ij})\) is similar. The duality in Section 3.2.1 between \( \hat{\phi}_i \) and \((A_\tau, A_{ij})\) is the lattice version of the fracton-elasticity duality of [54,56,57], while the duality in Section 2.2.1 between the dipole \( \phi \)-theory and \((\hat{A}_\tau, \hat{A}_{ij})\) is the lattice version of the lineon-elasticity duality of [53]. **Right:** The Laplacian \( \phi \)-theory is self-dual [13,37] (see Section 2.1.1). The Laplacian \( \phi \)-theory Higgses the Laplacian gauge theory \((A_\tau, \hat{A})\), which has no duality. Note that while the Laplacian \( \phi \)-theory (denoted as \( \phi_{\text{Lap}} \)) and the dipole \( \phi \)-theory (denoted as \( \phi_{\text{dip}} \)) are both naive discretizations of the same continuum Lifshitz Lagrangian (1.1), their dualities are drastically different. Even though we use the continuum notations for these fields, our dualities are established as exact lattice dualities.

Their GSDs on a 2d spatial torus lattice are\(^6\)

\[
\begin{align*}
\text{Laplacian gauge theory:} & \quad \text{GSD} = 1, \\
\text{Scalar charge theory:} & \quad \text{GSD} = \gcd(L_x,L_y).
\end{align*}
\] (1.8)

Interestingly, the GSD of the scalar charge theory of \((A_\tau, A_{ij})\) depends on \( \gcd(L_x,L_y) \), which is a manifestation of UV/IR mixing in these exotic models [39,46].

The two models also differ in other ways. The scalar charge theory is exactly dual to the elasticity theory with displacement fields \((\hat{\phi}_x, \hat{\phi}_y)\). The continuum version of this exact lattice duality between the scalar charge theory and the \( \hat{\phi}_i \)-theory, known as the fracton-elasticity duality, has been discussed in [54,56,57]. See Figure 1 for some of the dualities and relations between these models. In contrast, the Laplacian gauge theory does not enjoy any duality. Other differences between the two theories are discussed in Section 3.

The Laplacian \( \phi \)-theory and gauge theory were recently introduced in [37] on a general

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\( ^6 \)To be precise, one can write a \( \theta \)-term in the Laplacian gauge theory, where \( \theta \sim \theta + 2\pi \). When \( \theta \neq \pi \), the ground state is non-degenerate, whereas there is a two-fold degeneracy at \( \theta = \pi \).
spatial graph. In this paper, we will place them on a 2d torus, and compare them with their dipole counterparts: the dipole $\phi$-theory and the scalar charge theory. We will make use of the results from [37] throughout.

The various 2+1d theories analyzed in this paper, and their schematic continuum Lagrangians, are summarized in Table 1.

| Theory                              | Continuum Lagrangian |
|-------------------------------------|----------------------|
| Compact Lifshitz theory $\phi$      | $L = (\partial_\tau \phi)^2 + (\nabla^2 \phi)^2$ |
| Elasticity theory of $\hat{\phi}_i$ | $L = \sum_i (\partial_\tau \hat{\phi}_i)^2 + \sum_{i,j} (\partial_i \hat{\phi}_j + \partial_j \hat{\phi}_i)^2$ |
| Scalar charge theory                | $L = \sum_{i,j} E_{ij}^2 + \sum_i B_i^2$ |
| Vector charge theory                | $L = \sum_{i,j} \hat{E}_{ij}^2 + \hat{B}^2$ |
| Laplacian gauge theory [37]         | $L = E^2$ |

| Theory                              | Continuum Lagrangian |
|-------------------------------------|----------------------|
| $A_\tau \sim A_\tau + \partial_\tau \alpha$ | $E_{ij} = \partial_\tau A_{ij} - \partial_i \partial_j A_\tau$ |
| $A_{ij} \sim A_{ij} + \partial_i \partial_j \alpha$ | $B_i = \partial_i A_{jj} - \partial_j A_{ij}$, $i \neq j$ |
| $\hat{A}_{ri} \sim \hat{A}_{ri} + \partial_r \hat{\alpha}_i$ | $\hat{E}_{ii} = \partial_r \hat{A}_{ii} - \partial_i \hat{A}_{ri}$ |
| $\hat{A}_{ii} \sim \hat{A}_{ii} + \partial_i \hat{\alpha}_i$ | $\hat{E}_{xy} = \partial_r \hat{A}_{xy} - \partial_y \hat{A}_{rx} - \partial_x \hat{A}_{ty}$ |
| $\hat{A}_{xy} \sim \hat{A}_{xy} + \partial_x \hat{\alpha}_y + \partial_y \hat{\alpha}_x$ | $\hat{B} = \partial_x^2 \hat{A}_{yy} + \partial_y^2 \hat{A}_{xx} - \partial_x \partial_y \hat{A}_{xy}$ |

Table 1: Summary of various 2+1d theories analyzed in this paper, and their naive continuum Lagrangians. Here, $i, j = x, y$, and repeated indices are not summed over.

The rest of this paper is organized as follows. In Section 2, we study the modified Villain versions of the Laplacian and dipole models of (1.3). One of the main results here is the stark contrast between the scalings of their GSDs with the system size. While the GSD of the dipole model grows linearly in the system size, the GSD of the Laplacian model grows exponentially. This can be understood as a consequence of the distinct exotic global symmetries of the two models. These differences are summarized in Table 2. Interestingly, the modified Villain version of the dipole model realizes the lineon-elasticity duality exactly on the lattice.

In Section 3, we discuss the modified Villain versions of the Laplacian gauge theory and
the scalar charge (gauge) theory. Once again, these two theories differ in several global aspects. For example, while both theories have fracton defects, a dipole of fractons is mobile in the scalar charge theory, but it is immobile in the Laplacian gauge theory. This can also be understood as a consequence of the distinct exotic global (time-like) symmetries of the two theories. These differences are summarized in Table 4. It is interesting to note that the modified Villain version of the scalar charge theory realizes the fracton-elasticity duality exactly on the lattice.

In Appendix A, we give more details about several aspects of the dipole $\phi$-theory and the scalar charge theory. These include the global symmetries, symmetry operators, charged operators, and defects in both theories.

In Appendix B we explore further properties of the the Laplacian theories on the square lattice. Specifically, we discuss the naturalness of the Laplacian $\phi$-theory, and the immobility of any finite set of defects in the Laplacian gauge theory on the square lattice.

2 2+1d compact Lifshitz theory

We begin with a naive analysis of the global symmetries of the continuum 2+1d compact Lifshitz theory \[11–20\] on a Euclidean spacetime 3-torus with lengths $\ell_\tau$, $\ell_x$, and $\ell_y$. As we will see, the dipole global symmetry has infinite order, which leads to infinite ground state degeneracy. This calls for a lattice regularization.

The Lagrangian is

$$L = \frac{\mu_0}{2} (\partial_\tau \phi)^2 + \frac{1}{2\mu} (\nabla^2 \phi)^2, \quad (2.1)$$

where $\phi$ is a compact scalar, i.e., $\phi \sim \phi + 2\pi$, and $\mu_0$ and $\mu$ are coupling constants of mass dimension 1.

The global symmetry of this theory includes:

- a $U(1)$ momentum (shift) symmetry, $\phi \to \phi + c$, where $c \sim c + 2\pi$ is a circle-valued constant\footnote{More precisely, here and throughout, by saying that $c$ is circle-valued we mean $c \in \mathbb{R}/2\pi\mathbb{Z}$.}

- a $\mathbb{Z} \times \mathbb{Z}$ momentum dipole symmetry,

  $$\phi \to \phi + \frac{2\pi m_x x}{\ell_x} + \frac{2\pi m_y y}{\ell_y}, \quad (2.2)$$

  More precisely, here and throughout, by saying that $c$ is circle-valued we mean $c \in \mathbb{R}/2\pi\mathbb{Z}$. 

where $m_x, m_y \in \mathbb{Z}$, and $x, y$ are continuum coordinates (rather than integers labelling the sites on a lattice), and

- a $U(1)$ one-form symmetry \[62\], which will be referred to as the “winding dipole” symmetry, with Noether currents\[\]

\[
J_{[rx]} = \frac{1}{2\pi} \partial_y \phi , \quad J_{[ry]} = -\frac{1}{2\pi} \partial_x \phi , \quad J_{[xy]} = \frac{1}{2\pi} \partial \tau \phi ,
\]  
(2.3)

that obey the current conservation equation

\[
\partial^\mu J_{[\mu \nu]} = 0 .
\]  
(2.4)

The conserved current leads to two conserved, integer-valued charges:

\[
Q_x = \oint dy J_{[rx]} , \quad Q_y = \oint dx J_{[ry]} .
\]  
(2.5)

Note that this symmetry is not a subsystem symmetry because $Q_x$ is independent of $x$ and $Q_y$ is independent of $y$. In addition, as with all one-form global symmetries, it leads to a time-like symmetry.\[\]

This theory is also invariant under translations, spatial rotations, and the Lifshitz scale transformation $\tau \to \lambda^2 \tau, x \to \lambda x, y \to \lambda y$.

The $\mathbb{Z} \times \mathbb{Z}$ momentum dipole symmetry operators do not commute with the winding dipole charges because the momentum dipole symmetry shifts the field $\phi$ by a configuration charged under the winding dipole symmetry.\[\] This leads to an infinite ground state degeneracy. To regularize this infinity, we wish to place the theory on the lattice and study its modified Villain version \[48, 45\].

As mentioned in the introduction, there are two natural ways to “discretize” the continuum Lagrangian (2.1). In the next two subsections, we study the modified Villain model associated with these two discretizations in detail, and note important distinctions between them. These differences are summarized in Table 2.

\[\]

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8We refer to this ordinary one-form global symmetry as a “winding dipole” symmetry because it is the continuum limit of the $\mathbb{Z}_{L_x} \times \mathbb{Z}_{L_y}$ winding dipole symmetry (2.27) of the dipole $\phi$-theory on a torus. See \[21\] for a similar phenomenon in 1+1d.

9Time-like global symmetries are symmetries that do not act on the ordinary Hilbert space, but do act on the Hilbert space of defects. They provide a convenient and powerful way to relate the restricted mobility of defects to a global symmetry. Consequently, they control the restricted mobility of charged excitations. See \[21\] for more details.

10This lack of commutativity between the momentum dipole symmetry and the winding dipole symmetry means that the full symmetry group is realized projectively. This fact can be thought of as a mixed anomaly between these two symmetries.
Table 2: Comparison of the two “discretizations” of the 2+1d continuum compact Lifshitz Lagrangian (2.1). Here, “Jac” is short-hand for the Jacobian group Jac($C_{L_x} \times C_{L_y}$) of the 2d torus graph $C_{L_x} \times C_{L_y}$, “mom.” stands for momentum, “wind.” stands for winding, and $G$ is the Catalan constant. The last row refers to robustness of the winding symmetry and ground state degeneracy after imposing the momentum symmetry—if the momentum symmetry is not imposed, then both models are not robust.

### 2.1 Laplacian $\phi$-theory

In this subsection, we discuss the modified Villain model associated with the first discretization of $(\nabla^2 \phi)^2$. We refer to it as the 2+1d Laplacian $\phi$-theory. (See [37] for the discussion of this model on a general spatial graph.)

Let us place the theory on a periodic 3d Euclidean spacetime lattice with $L_\tau$, $L_x$, and $L_y$ sites. The modified Villain action of the 2+1d Laplacian $\phi$-theory is

$$S = \frac{\beta_0}{2} \sum_{\tau\text{-link}} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{\text{site}} [((\Delta_x^2 + \Delta_y^2)\phi - 2\pi n)^2$$

$$+ i \sum_{\tau\text{-link}} \tilde{\phi} [\Delta_\tau n - (\Delta_x^2 + \Delta_y^2)n_\tau] ,$$

where $\phi$ is a real-valued field, $(n_\tau, n)$ are integer gauge fields, and $\tilde{\phi}$ is a real-valued Lagrange multiplier that makes the integer gauge fields flat. The locations of these fields are
| Location     | Laplacian \(\phi\)-theory | Dipole \(\tilde{\phi}\)-theory |
|--------------|-----------------------------|-------------------------------|
| site         | \(\phi, n, \tilde{n}_\tau\) | \(\phi, n_{ii}, \hat{n}_{ii}\) |
| \(\tau\)-link| \(\tilde{\phi}, n_\tau, \tilde{n}\) | \(\hat{A}_{ii}, n_\tau, \hat{n}\) |
| \(i\)-link   | \(-\)                           | \(\hat{A}_{ri}\)              |
| \(\tau i\)-plaq | \(-\)                           | \(-\)                        |
| \(x y\)-plaq | \(-\)                           | \(n_{xy}, \hat{n}_{xy}\)     |
| cube         | \(-\)                           | \(\hat{A}_{xy}\)              |

Table 3: Locations of various fields of the Laplacian and dipole \(\phi\)-theories of Section 2. Here, \(i = x, y\) and repeated indices are not summed over.

summarized in Table 3. There is a gauge symmetry

\[
\phi \sim \phi + 2\pi k, \quad n_\tau \sim n_\tau + \Delta_\tau k, \\
\tilde{\phi} \sim \tilde{\phi} + 2\pi \tilde{k}, \quad n \sim n + (\Delta_x^2 + \Delta_y^2)k,
\]

where \(k\) and \(\tilde{k}\) are integer gauge parameters. This integer gauge symmetry makes the scalar fields \(\phi\) and \(\tilde{\phi}\) compact.

### 2.1.1 Self-duality

The 2+1d Laplacian \(\phi\)-theory (2.6) is self-dual with \(\phi \leftrightarrow \tilde{\phi}\) and \(\beta_0 \leftrightarrow \frac{1}{(2\pi)^2\beta_0}\). Indeed, using the Poisson resummation formula for the integers \(n_\tau, n\), the dual action is

\[
S = \frac{1}{2(2\pi)^2\beta} \sum_{\text{site}} (\Delta_\tau \tilde{\phi} - 2\pi \tilde{n}_\tau)^2 + \frac{1}{2(2\pi)^2\beta_0} \sum_{\tau\text{-link}} [((\Delta_x^2 + \Delta_y^2)\tilde{\phi} - 2\pi \tilde{n})^2 \\
- i \sum_{\text{site}} \phi [\Delta_\tau \tilde{n} - (\Delta_x^2 + \Delta_y^2)\tilde{n}_\tau],
\]

where \((\tilde{n}_\tau, \tilde{n})\) are integer gauge fields that make \(\tilde{\phi}\) compact. Under the gauge symmetry (2.7), they transform as

\[
\tilde{n}_\tau \sim \tilde{n}_\tau + \Delta_\tau \tilde{k}, \quad \tilde{n} \sim \tilde{n} + (\Delta_x^2 + \Delta_y^2)\tilde{k}.
\]

This self-duality was suggested in the continuum in [13] and formulated, more recently, as an exact duality on the lattice in [37].
2.1.2 Global symmetry and ground state degeneracy

Let us discuss the global symmetry of the 2+1d Laplacian $\phi$-theory (2.6) [37].

- There is a $U(1)$ momentum symmetry that acts as $\phi \rightarrow \phi + c$, where $c \sim c + 2\pi$ is a circle-valued constant. A typical charged operator is $e^{i\phi}$.

- There is a discrete momentum symmetry that acts as
  \[
  \phi \rightarrow \phi + f(x, y),
  \]
  \[
  n \rightarrow n + \frac{1}{2\pi} (\Delta_x^2 + \Delta_y^2)f(x, y),
  \]
  with $f(x, y)$ any function satisfying $(\Delta_x^2 + \Delta_y^2)f(x, y) \in 2\pi\mathbb{Z}$. (We do not refer to it as a dipole symmetry, because the shift function $f(x, y)$ can have a more general form than for a dipole symmetry, where it is linear.) In other words, $f(x, y)$ is a circle-valued discrete harmonic function on the 2d spatial torus lattice. The symmetry group formed by such functions is the Jacobian group, $\text{Jac}(C_{L_x} \times C_{L_y})$, of the 2d torus lattice $C_{L_x} \times C_{L_y}$ with $L_i$ number of sites in the $i$ direction. See [63, 64] for the definition of this group and [37] for its relation to the Laplacian $\phi$-theory. There is no simple closed form formula for $\text{Jac}(C_{L_x} \times C_{L_y})$; for example, $\text{Jac}(C_2 \times C_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$ and $\text{Jac}(C_3 \times C_3) = \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_{18} \times \mathbb{Z}_{18}$.

- There is a $U(1)$ winding symmetry that acts as $\tilde{\phi} \rightarrow \tilde{\phi} + \tilde{c}$, where $\tilde{c} \sim \tilde{c} + 2\pi$ is a circle-valued constant. A typical charged operator is $e^{i\tilde{\phi}}$.

- There is a $\text{Jac}(C_{L_x} \times C_{L_y})$ discrete winding symmetry that acts as
  \[
  \tilde{\phi} \rightarrow \tilde{\phi} + \tilde{f}(x, y),
  \]
  where $\tilde{f}(x, y)$ is a circle-valued discrete harmonic function on the 2d spatial torus lattice.

The discrete momentum and winding symmetries do not commute with each other, which leads to a large ground state degeneracy equal to the order of the Jacobian group, $|\text{Jac}(C_{L_x} \times C_{L_y})|$. As shown in [37], the logarithm of GSD grows as $L_x L_y$. More concretely,

\[
\log \text{GSD} \approx \frac{4G}{\pi} L_x L_y,
\]

where $G$ is the Catalan constant. More generally, when this model is placed on a graph, the ground state degeneracy is equal to the number of spanning trees of the graph, which is a common measure of complexity of the graph.
As explained in [37], the origin of the exponential behaviour of GSD is different from a similar phenomenon in a system of decoupled spins. Let us review it here for concreteness. In a system of decoupled spins, there is a symmetry generator associated with each spin, and all of them have the same order. So, on a 2d spatial torus lattice with \( L_x = L_y = L \) sites in each direction, the number of generators is \( L^2 \), which leads to \( \log \text{GSD} \sim L^2 \). On the other hand, in the Laplacian \( \phi \)-theory, there are only \( O(L) \) generators of the \( \text{Jac}(C_L \times C_L) \) but some of them have very large orders, which leads to \( \log \text{GSD} \sim L^2 \).

Another consequence of this large symmetry is that the spatially separated two-point functions of monopole operators \( e^{i\phi} \) and dipole operators \( e^{i\Delta \phi} \) vanish. First, consider the spatially separated two-point function of the monopole operator

\[
\langle e^{i\phi(\tau,x,y)} e^{-i\phi(0,0,0)} \rangle, \quad (x, y) \neq (0, 0).
\]

The discrete momentum symmetry of the 2+1d Laplacian \( \phi \)-theory includes linear shifts in \( x \) and \( y \):

\[
\phi \rightarrow \phi + \frac{2\pi m_xx}{L_x} + \frac{2\pi m_yy}{L_y},
\]

\[
n \rightarrow n + m_x (\delta_{x,0} - \delta_{x,L_x-1}) + m_y (\delta_{y,0} - \delta_{y,L_y-1}),
\]

where \( m_i = 0, \ldots, L_i - 1 \). Under such shifts, the two-point function of the monopole operator acquires a nontrivial \((x, y)\)-dependent phase, and hence it vanishes.

More interestingly, consider the spatially separated two-point function of the dipole operator

\[
\langle e^{i\Delta \phi(\tau,x+\frac{1}{2}y)} e^{-i\Delta \phi(0,\frac{1}{2}y,0)} \rangle, \quad (x, y) \neq (0, 0).
\]

It is clearly invariant under the linear shifts (2.14). However, the discrete momentum symmetry of the 2+1d Laplacian \( \phi \)-theory also includes quadratic shifts in \( x \) and \( y \), such as

\[
\phi \rightarrow \phi + \frac{2\pi m_xy}{\gcd(L_x, L_y)} + \frac{2\pi m'(x^2 - y^2 - L_xx + L_yy)}{2\gcd(L_x, L_y)},
\]

\[
n \rightarrow n + \frac{m}{\gcd(L_x, L_y)} [L_y(\delta_{x,0} - \delta_{x,L_x-1}) + L_x(\delta_{y,0} - \delta_{y,L_y-1})] - \frac{m'}{\gcd(L_x, L_y)} (L_x\delta_{x,0} - L_y\delta_{y,0}),
\]

where \( m = 0, \ldots, \gcd(L_x, L_y) \), and \( m' = 0, \ldots, 2\gcd(L_x, L_y) \). Setting \( L_x = L_y = L \) for simplicity, we see that the two-point function of the dipole operator is not invariant under

---

\(^{11}\)The minimal number of generators of \( \text{Jac}(C_L \times C_L) \) is at most the number of nontrivial spatial integer gauge fields \( n \)'s after gauge fixing. One can gauge fix the \( n \)'s so that they are zero everywhere except at \( x = 0, 1 \), or at \( y = 0, 1 \). This means that the minimal number of generators of \( \text{Jac}(C_L \times C_L) \) is at most \( \min(2L_x, 2L_y) \).
these shifts, and hence it vanishes as well. Similar conclusion holds for the other dipole operator \( e^{i\Delta y \phi} \).

### 2.1.3 Robustness

We now discuss whether the GSD and the global symmetry of the low-energy limit of our modified Villain lattice is robust or not. (See [39] for general discussions on robustness and naturalness.) More specifically, we impose the momentum symmetry in the microscopic lattice model, and ask whether there are local, relevant operators in the low-energy limit that violate the winding symmetry and lift the GSD. Typically, the notion of robustness requires a certain scaling symmetry to determine which operators are relevant and which ones are not. Furthermore, the notion of a local operator only makes sense as we take the number of lattice sites to infinite, \( L_i \to \infty \). Our Laplacian \( \phi \)-theory does not admit a conventional continuum limit, so the above notions need to be appropriately generalized. Nonetheless, we will see that there are local operators (supported at a single site on the lattice) that lift the GSD and violate the winding symmetry. Since they act nontrivially on the space of ground states, they should be considered relevant in this sense.

Let us impose the \( U(1) \) and discrete momentum symmetries. This symmetry excludes local operators such as \( e^{i\Delta x \phi} \), \( e^{i\Delta y \phi} \) in the Lagrangian. In fact, this symmetry further excludes local operators of the form \( \prod_{i=1}^{m} e^{iq_i \phi(0,x_i,y_i)} \), \( q_i \in \mathbb{Z} \), that cannot be written as \( \prod_{j=1}^{n} e^{ir_j (\Delta x_2 + \Delta y_2) \phi(0,x_j,y_j)} \), \( r_j \in \mathbb{Z} \). We can think of these operators as being higher order than the operators that are already present in the action. See Appendix B.1 for more details.

However, we can add the winding operator \( e^{i\tilde{\phi}} \) to the Lagrangian (2.6) because it is invariant under the momentum symmetries. Such a perturbation breaks the winding symmetry and hence it lifts the ground states. Therefore, if we impose only the momentum symmetries, the ground state degeneracy is not robust.

### 2.2 Dipole \( \phi \)-theory

In this subsection, we study the modified Villain model associated with the second discretization of \((\nabla^2 \phi)^2\). We refer to it as the 2+1d dipole \( \phi \)-theory.

Let us place the theory on a periodic 3d Euclidean spacetime lattice with \( L_{\tau}, L_x, \) and \( L_y \).
The modified Villain action of the 2+1d dipole $\phi$-theory is

$$S = \frac{\beta_0}{2} \sum_{\tau\text{-link}} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta'}{2} \sum_{\text{site}} \left[ (\Delta_x^2 \phi - 2\pi n_{xx})^2 + (\Delta_y^2 \phi - 2\pi n_{yy})^2 \right]$$

$$+ \frac{\beta}{2} \sum_{xy\text{-plaq}} (\Delta_x \Delta_y \phi - 2\pi n_{xy})^2 + i \sum_{x\text{-link}} \hat{A}_{rx}(\Delta_x n_{yy} - \Delta_y n_{xy}) + i \sum_{y\text{-link}} \hat{A}_{ry}(\Delta_y n_{xx} - \Delta_x n_{xy})$$

$$+ i \sum_{\tau\text{-link}} \hat{A}_{xy}(\Delta_x n_{xy} - \Delta_x \Delta_y n_\tau) - i \sum_{\tau\text{-link}} \hat{A}_{yy}(\Delta_y n_{xx} - \Delta_x^2 n_\tau) - i \sum_{\tau\text{-link}} \hat{A}_{xx}(\Delta_y n_{yy} - \Delta_y^2 n_\tau),$$

where $\beta$ and $\beta'$ are not a priori related to each other, whereas $\beta = 2\beta'$ when related to \cite{1}. Here, $\phi$ is a real-valued scalar field, $(n_\tau, n_{ij})$ are integer gauge fields, and $(\hat{A}_{\tau i}, \hat{A}_{ij})$ are real-valued Lagrange multipliers that make the integer gauge fields flat. The locations of these fields on the lattice are summarized in Table 3. They have a gauge symmetry

$$\phi \sim \phi + 2\pi k, \quad \hat{A}_{\tau i} \sim \hat{A}_{\tau i} + \Delta_\tau \hat{\alpha}_i + 2\pi \hat{k}_i,$$

$$n_\tau \sim n_\tau + \Delta_\tau k, \quad \hat{A}_{ii} \sim \hat{A}_{ii} + \Delta_i \hat{\alpha}_i + 2\pi \hat{k}_i,$$

$$n_{ij} \sim n_{ij} + \Delta_i \Delta_j k, \quad \hat{A}_{xy} \sim \hat{A}_{xy} + \Delta_x \hat{\alpha}_y + \Delta_y \hat{\alpha}_x + 2\pi \hat{k}_{xy},$$

where $k$ and $\hat{k}$'s are integer gauge parameters, and $\hat{\alpha}_i$ is a real-valued gauge parameter along the $\tau i$-plaquette. The integer gauge symmetry makes $\phi$ and $(\hat{A}_{\tau i}, \hat{A}_{ij})$ compact.

### 2.2.1 Lineon-elasticity duality

Using the Poisson resummation formula for the integers $(n_\tau, n_{ij})$, the action \eqref{2.17} is dualized to

$$S = \frac{\beta_0}{2} \sum_{xy\text{-plaq}} \hat{\mathcal{E}}_{xy}^2 + \frac{\beta'}{2} \sum_{\text{site}} (\hat{\mathcal{E}}_{xx}^2 + \hat{\mathcal{E}}_{yy}^2) + \frac{\gamma}{2} \sum_{\tau\text{-link}} \hat{\mathcal{B}}^2$$

$$- i \sum_{\text{site}} \phi (\Delta_x \hat{n} - \Delta_x \Delta_y \hat{n}_{xy} + \Delta_x^2 \hat{n}_{yy} + \Delta_x^2 \hat{n}_{xx}) ,$$

where

$$\beta_0 = \frac{1}{(2\pi)^2 \beta}, \quad \beta' = \frac{1}{(2\pi)^2 \beta'}, \quad \gamma = \frac{1}{(2\pi)^2 \beta_0},$$

and the field strengths are

$$\hat{\mathcal{E}}_{xy} = \Delta_x \hat{A}_{xy} - \Delta_x \hat{A}_{rx} - \Delta_y \hat{A}_{ry} - 2\pi \hat{n}_{rxy},$$

$$\hat{\mathcal{E}}_{ii} = \Delta_\tau \hat{A}_{ii} - \Delta_i \hat{A}_{\tau i} - 2\pi \hat{n}_{r\tau i},$$

$$\hat{\mathcal{B}} = \Delta_x \Delta_y \hat{A}_{xy} - \Delta_x^2 \hat{A}_{yy} - \Delta_y^2 \hat{A}_{xx} - 2\pi \hat{n}.$$

15
The integer gauge fields \((\hat{n}_{\tau ij}, \hat{n})\) have the gauge symmetry

\[
\begin{aligned}
\hat{n}_{\tau xy} &\sim \hat{n}_{\tau xy} + \Delta_\tau \hat{k}_{xy} - \Delta_x \hat{k}_{xy} - \Delta_y \hat{k}_{\tau x}, \\
\hat{n}_{\tau ii} &\sim \hat{n}_{\tau ii} + \Delta_\tau \hat{k}_{ii} - \Delta_i \hat{k}_{\tau i}, \\
\hat{n} &\sim \hat{n} + \Delta_x \Delta_y \hat{k}_{xy} - \Delta^2_x \hat{k}_{yy} - \Delta^2_y \hat{k}_{xx}, \\
\end{aligned}
\]  

(2.22)

This modified Villain model is a lattice discretization of the 2+1d vector charge theory of [49–52] in the continuum (see also [65,24,25,66,26] for a similar gauge theory in 3+1d). See Table 1 for information about the continuum theory.

In the continuum, the duality is schematically given by the map,

\[
\begin{aligned}
\partial_\tau \phi &\leftrightarrow B = \partial_x \partial_y \hat{A}_{xy} - \partial_x^2 \hat{A}_{yy} - \partial_y^2 \hat{A}_{xx}, \\
\partial_x \partial_y \phi &\leftrightarrow \hat{E}_{xy} = \partial_\tau \hat{A}_{xy} - \partial_x \hat{A}_{\tau y} - \partial_y \hat{A}_{\tau x}, \\
\partial_i^2 \phi &\leftrightarrow \hat{E}_{ij} = \partial_\tau \hat{A}_{ij} - \partial_j \hat{A}_{\tau j}, \\
\end{aligned}
\]  

(2.23)

where \((\hat{A}_{\tau i}, \hat{A}_{ij})\) are gauge fields with gauge transformations

\[
\begin{aligned}
\hat{A}_{\tau i} &\sim \hat{A}_{\tau i} + \partial_\tau \hat{\alpha}_i, \\
\hat{A}_{xy} &\sim \hat{A}_{xy} + \partial_x \hat{\alpha}_y + \partial_y \hat{\alpha}_x, \\
\hat{A}_{ii} &\sim \hat{A}_{ii} + \partial_i \hat{\alpha}_i,
\end{aligned}
\]  

(2.24)

and \(\hat{\alpha}_i\) are gauge parameters.

The continuum version of the duality between the dipole \(\phi\)-theory and the vector charge theory was discussed in [53] in the context of elasticity theory. Since this model has defects representing the worldline of lineons (see Section 2.2.2 below), we dub this duality as the lineon-elasticity duality.

### 2.2.2 Global symmetry, ground state degeneracy and mobility of defects

Let us discuss the global symmetry of the 2+1d dipole \(\phi\)-theory \((2.17)\). (See Appendix A.1 for more details.)

- The \(U(1)\) momentum (dual magnetic) symmetry acts as \(\phi \rightarrow \phi + c\), where \(c \sim c + 2\pi\) is a circle-valued constant. It is a magnetic global symmetry from the dual gauge theory of point of view.
- The \(\mathbb{Z}_{L_x}\) momentum (dual magnetic) dipole symmetry acts as

\[
\begin{aligned}
\phi &\rightarrow \phi + 2\pi m_{xx} \frac{x}{L_x}, \\
n_{xx} &\rightarrow n_{xx} + m_{xx} (\delta_{x,0} - \delta_{x,L_x-1})
\end{aligned}
\]  

(2.25)
where \( m_{xx} = 0, 1, \ldots, L_x - 1 \). Note that the \( m_{xx} = L_x \) transformation is trivial because it can be undone by an integer gauge transformation \((2.18)\) with \( k = x \). More intuitively, the \( m_{xx} = L_x \) transformation is trivial because \( \phi \) is compact. Also note that this transformation is compatible with the periodicity of the spatial torus lattice, i.e., \( x \sim x + L_x \).

- The \( \mathbb{Z}_{L_y} \) momentum (dual magnetic) dipole symmetry acts similarly with \( x \) and \( y \) exchanged.

- The \( U(1)^3 \) winding (dual electric) symmetry acts as

\[
\hat{A}_{ii} \to \hat{A}_{ii} + \frac{\hat{c}_{ii}}{L_i}, \quad \hat{A}_{xy} \to \hat{A}_{xy} + \frac{\hat{c}_{xy}}{\text{lcm}(L_x, L_y)} ,
\]

(2.26)

where \( \hat{c}_{ij} \sim \hat{c}_{ij} + 2\pi \) are circle-valued constants.\(^{12}\)

- The \( \mathbb{Z}_{L_x} \) winding (dual electric) dipole symmetry acts as

\[
\hat{A}_{yy} \to \hat{A}_{yy} + 2\pi \hat{m}_{xx} \frac{x}{L_x} \delta_{y,0} ,
\]

\[
\hat{n} \to \hat{n} - \hat{m}_{xx} \delta_{y,0} (\delta_{x,0} - \delta_{x,L_y-1}) .
\]

(2.27)

where \( \hat{m}_{xx} = 0, \ldots, L_x - 1 \).

- The \( \mathbb{Z}_{L_y} \) winding (dual electric) dipole symmetry acts similarly with \( x \) and \( y \) exchanged.

The momentum and winding dipole symmetries do not commute with each other, which leads to a ground state degeneracy of \( L_x L_y \). See the discussion around \((A.16)\) for more details. In fact, every state in the Hilbert space is \( L_x L_y \)-fold degenerate. More abstractly, the momentum and winding dipole symmetries are realized projectively. This can be viewed as a mixed anomaly between them.

Let us briefly discuss the fate of these symmetries in a specific continuum limit. (Recall that, as in the 1+1d version of this theory \([21]\), there are several distinct continuum limits.) We introduce the lattice spacings \( a_x, a \) and take the limit \( a_x, a \to 0 \) and \( L_i \to \infty \) with \( \ell_x = a_x L_x \) and \( \ell_x = a L_x \) fixed, while scaling the lattice coupling constants as

\[
\beta_0 = \frac{\mu_0 a^2}{a_x} , \quad \beta = 2\beta' = \frac{2a_x}{\mu a^2},
\]

(2.28)

\(^{12}\)In particular, the transformations with \( \hat{c}_{ij} = 2\pi \) are trivial and can be undone by gauge transformations of the form \((2.18)\). Similar comments apply to various other electric global symmetries discussed below. See \([21]\) for a simpler example in 1+1d.
where $\mu, \mu_0$ are fixed continuum coupling constants with mass dimensions 1. In this continuum limit, the above global symmetries reduce to the global symmetries of the 2+1d continuum compact Lifshitz theory discussed at the beginning of Section 2.13

Space-like symmetries:

\begin{align*}
U(1) \text{ momentum} & \rightarrow U(1) \text{ momentum}, \\
\mathbb{Z}_{L_x} \times \mathbb{Z}_{L_y} \text{ momentum dipole} & \rightarrow \mathbb{Z} \times \mathbb{Z} \text{ momentum dipole}, \\
U(1)^3 \text{ winding} & \rightarrow \text{ does not exist}, \\
\mathbb{Z}_{L_x} \times \mathbb{Z}_{L_y} \text{ winding dipole} & \rightarrow \text{ part of } U(1) \text{ “winding dipole” one-form}. \quad (2.29)
\end{align*}

Time-like symmetries:

\begin{align*}
U(1)^2 \text{ winding} & \rightarrow \text{ does not exist}, \\
\mathbb{Z}_{\gcd(L_x,L_y)} \text{ winding dipole} & \rightarrow \text{ part of } U(1) \text{ “winding dipole” one-form}.
\end{align*}

In particular, the two integer-valued charges \((2.5)\) of the \(U(1)\) “winding dipole” one-form symmetry generate two space-like \(U(1)\) symmetries that are the continuum limits of the \(\mathbb{Z}_{L_x}\) and \(\mathbb{Z}_{L_y}\) winding dipole symmetries on the lattice.

Note that the \(\mathbb{Z}_{L_x} \times \mathbb{Z}_{L_y}\) momentum dipole symmetry of the 2+1d dipole \(\phi\)-theory is smaller than the discrete momentum symmetry of the 2+1d Laplacian \(\phi\)-theory (see Section 2.1.2). For example, in addition to the linear shifts of \((2.25)\) (or equivalently, \((2.14)\)), \((2.10)\) includes the quadratic shifts \((2.16)\). Consequently, while the two-point function of the monopole operator \(e^{i\phi}\) vanishes in both Laplacian and dipole \(\phi\)-theories, the two-point function of the dipole operator \(e^{i\Delta t \phi}\) vanishes only in the Laplacian \(\phi\)-theory.

There are also time-like global winding symmetries that act on defects that extend in the Euclidean time direction such as

\begin{equation}
\exp \left( i \sum_{x \text{-link: fixed } x,y} \hat{A}_{\tau x} \right), \quad (2.30)
\end{equation}

which represents the worldline of a static particle. We have:

- The \(U(1)^2\) time-like winding (dual electric) symmetry acts as

\begin{equation}
\hat{A}_{\tau i} \rightarrow \hat{A}_{\tau i} + \frac{\hat{c}_{\tau i}}{L_\tau}, \quad (2.31)
\end{equation}

where \(\hat{c}_{\tau i} \sim \hat{c}_{\tau i} + 2\pi\) are circle-valued constants. The charge, which is commonly referred

\(^{13}\text{In taking the continuum limit of the } \mathbb{Z}_{\gcd(L_x,L_y)} \text{ time-like winding dipole symmetry, we assume that } L_x/L_y \text{ is fixed.}\)
to as the “gauge charge”, associated with this time-like global symmetry generated by 
\( \hat{c}_\tau \) is a vector in space, and hence the name vector charge theory.

- The \( \mathbb{Z}_{\text{gcd}(L_x, L_y)} \) time-like winding (dual electric) dipole symmetry acts as

\[
\hat{A}_{\tau x}(\tau, x + \frac{1}{2}, y) \rightarrow \hat{A}_{\tau x}(\tau, x + \frac{1}{2}, y) - 2\pi \hat{m}_{\tau xy} \delta_{\tau,0} \frac{y}{\text{gcd}(L_x, L_y)},
\]

\[
\hat{A}_{\tau y}(\tau, x, y + \frac{1}{2}) \rightarrow \hat{A}_{\tau y}(\tau, x, y + \frac{1}{2}) + 2\pi \hat{m}_{\tau xy} \delta_{\tau,0} \frac{x}{\text{gcd}(L_x, L_y)},
\]

\[
\hat{n}_{\tau xy}(\tau, x + \frac{1}{2}, y + \frac{1}{2}) \rightarrow \hat{n}_{\tau xy}(\tau, x + \frac{1}{2}, y + \frac{1}{2}) + \frac{\hat{m}_{\tau xy} \delta_{\tau,0}}{\text{gcd}(L_x, L_y)} (L_x \delta_{x,L_x-1} - L_y \delta_{y,L_y-1}),
\]

where \( \hat{m}_{\tau xy} = 0, \ldots, \text{gcd}(L_x, L_y) - 1 \).

As a consequence of the time-like dipole symmetry, the particle associated with the static
defect (2.30) can move in the \( x \)-direction, but it can only hop by \( \text{gcd}(L_x, L_y) \) sites in the
\( y \)-direction. Hence, we call it an \( x \)-lineon. Similarly, there is a \( y \)-lineon associated with the
defect of \( \hat{A}_{\tau y} \). Therefore, the dipole \( \phi \)-theory is a model of lineons.

2.2.3 Robustness

Let us impose the \( U(1) \) momentum and \( \mathbb{Z}_{L_x} \times \mathbb{Z}_{L_y} \) momentum dipole symmetries. In this
case, there are no lower order difference terms that one can add to the action (2.17) because
\( e^{i\Delta_x \phi} \) and \( e^{i\Delta_y \phi} \) are not invariant under the momentum dipole symmetry. In other words,
the action (2.17) includes the most relevant terms that preserve the momentum symmetries.
Moreover, the winding symmetries are robust because the operators charged under them are
extended in space. In particular, the \( L_x L_y \)-fold ground state degeneracy is robust once we
impose the momentum symmetries.

Alternatively, we can start with the dual gauge theory and impose the dual electric
(winding) symmetries. In this case, the ground state degeneracy is not robust because we can
always perturb the dual action (2.19) by the monopole operator \( e^{i\phi} \) which acts nontrivially
on the ground states.

This situation is reminiscent of the robustness of an ordinary \( U(1) \) gauge theory in 2+1d
or its dual compact scalar theory. Imposing the momentum symmetry of the scalar, which
is the magnetic zero-form symmetry of its dual gauge theory, makes the theory robust. Its
other symmetry is a winding one-form symmetry, or its dual a one-form electric symmetry,
but there is no local operator charged under it. Conversely, imposing only the latter
symmetry, the theory is not robust. Now, there are local operators that are charged under
the momentum symmetry of the scalar or equivalently, the magnetic symmetry of the gauge
3 2+1d $U(1)$ tensor gauge theories

Since the momentum symmetries of the 2+1d dipole and Laplacian $\phi$-theories are very different, the corresponding $U(1)$ gauge theories are also very different. These are, respectively, the scalar charge theory and the Laplacian gauge theory in Table 1. In this section, we discuss the modified Villain formulation of the two $U(1)$ tensor gauge theories. The differences between these two theories are summarized in Table 4.

3.1 Laplacian gauge theory

We can gauge the momentum symmetry of the Laplacian $\phi$-theory of Section 2.1 by coupling it to the gauge fields $(A_\tau, A; m_\tau)$, where $A_\tau, A$ are real-valued and $m_\tau$ is integer-valued. In

| Theory | Laplacian gauge theory $(A_\tau, A)$ | Scalar charge theory $(A_\tau, A_{ij})$ |
|--------|------------------------------------|-----------------------------------------|
| Duality | No duality | Dual to the matter theory $\hat{\phi}_i$ |
| Space-like global symmetry | $U(1)$ electric | $U(1)^3 \times \mathbb{Z}_{\gcd(L_x, L_y)}$ electric |
| Time-like global symmetry | $U(1) \times \text{Jac}$ | $U(1) \times \mathbb{Z}_{L_x} \times \mathbb{Z}_{L_y}$ magnetic |
| Ground state degeneracy | 1 | gcd($L_x, L_y$) |
| Robustness | Yes | No |

Table 4: Comparison of the two $U(1)$ tensor gauge theories associated with the momentum symmetries of the two models in Section 2. Here, “Jac” is short-hand for the Jacobian group Jac($C_{L_x} \times C_{L_y}$) of the 2d torus graph $C_{L_x} \times C_{L_y}$. The last row refers to robustness of the magnetic symmetry and ground state degeneracy after imposing the electric symmetry. Since the Laplacian gauge theory does not have a magnetic symmetry or nontrivial ground state degeneracy, it is trivially robust.
Table 5: Locations on the lattice of various fields of the Laplacian and the scalar charge theories of Section 3. Here, $i,j = x,y$, $i \neq j$, and repeated indices are not summed over.

In this section, we study the pure gauge theory of $\left( A_\tau, A; m_\tau \right)$ described by the action

$$ S = \frac{\gamma}{2} \sum_{\tau\text{-link}} E^2, \quad (3.1) $$

where $E = \Delta_\tau A - \left( \Delta_x^2 + \Delta_y^2 \right) A_\tau - 2\pi m_\tau$ is the electric field of $\left( A_\tau, A; m_\tau \right)$. It is invariant under the gauge symmetry

$$ A_\tau \sim A_\tau + \Delta_\tau \alpha + 2\pi q_\tau, $$

$$ A \sim A + \left( \Delta_x^2 + \Delta_y^2 \right) \alpha + 2\pi q, $$

$$ m_\tau \sim m_\tau + \Delta_\tau q - \left( \Delta_x^2 + \Delta_y^2 \right) q_\tau, \quad (3.2) $$

where $\left( q_\tau, q \right)$ are integer gauge parameters, and $\alpha$ is a real gauge parameter. The locations of these fields on the lattice are summarized in Table 5. This integer gauge symmetry makes the gauge fields $\left( A_\tau, A \right)$ compact.

We can add a $\theta$-term to the action (3.1):

$$ \frac{i\theta}{2\pi} \sum_{\tau\text{-link}} E. \quad (3.3) $$

Here, $\theta \sim \theta + 2\pi$ because $\sum_{\tau\text{-link}} E = -2\pi \sum_{\tau\text{-link}} m_\tau \in 2\pi \mathbb{Z}$.

Furthermore, this theory is robust because all the local operators are polynomials in the electric field strength and lattice derivatives. Hence, they are higher order than the terms in the action.
As in Section 2.1, the 2+1d $U(1)$ Laplacian gauge theory can be placed on a general spatial graph \[37\].

### 3.1.1 Global symmetry and mobility of defects

Let us discuss the space-like and time-like global symmetries of the 2+1d $U(1)$ Laplacian gauge theory.

- There is a $U(1)$ electric space-like symmetry that acts as $A \rightarrow A + \frac{c}{L_x L_y}$, where $c \sim c + 2\pi$ is a circle-valued constant. The charged operator is a surface operator supported on the whole space

$$
\exp \left( i \sum_{\text{site: fixed } \tau} A \right).
$$

- There is a $U(1)$ electric time-like symmetry that acts as $A_\tau \rightarrow A_\tau + \delta_{\tau,0} c_\tau$, where $c_\tau \sim c_\tau + 2\pi$ is a circle-valued constant. The charged object is a line defect that extends in the Euclidean time direction

$$
\exp \left( i \sum_{\text{link: fixed } x,y} A_\tau \right).
$$

- There is a discrete electric time-like symmetry that acts as

$$
A_\tau \left( \tau + \frac{1}{2}, x, y \right) \rightarrow A_\tau \left( \tau + \frac{1}{2}, x, y \right) + \delta_{\tau,0} f_\tau(x, y),
$$

$$
m_\tau \left( \tau + \frac{1}{2}, x, y \right) \rightarrow m_\tau \left( \tau + \frac{1}{2}, x, y \right) - \frac{1}{2\pi} \delta_{\tau,0} (\Delta_x^2 + \Delta_y^2) f_\tau(x, y),
$$

where $f_\tau(x, y)$ is a circle-valued discrete harmonic function on the 2d spatial torus lattice. The corresponding symmetry group is the Jacobian group, $\text{Jac}(C_{L_x} \times C_{L_y})$, of the 2d torus lattice.

It was proven in \[37\] that the discrete electric time-like symmetry implies that the particle described by the static defect (3.5) is immobile, i.e., it is a fracton. In fact, if we place the theory on an infinite square lattice $\mathbb{Z}^2$, then we show, in Appendix B.2, that any finite set of particles is immobile unless they are in the trivial superselection sector, i.e., they can be “annihilated locally.”
3.2 Scalar charge theory

We can gauge the momentum symmetry of the 2+1d dipole $\phi$-theory of Section 2.2 by coupling it to the rank-2 $U(1)$ tensor gauge fields $(A_\tau, A_{ij}; n_{ri}, n_x, n_y)$. In this section, we will study the pure gauge theory of these gauge fields. Our lattice model is a discretization of the continuum scalar charge theory of [24, 25, 54, 50, 55, 56, 29, 51, 52]. The action is

$$S = \frac{\gamma_0}{2} \sum_{\text{cube}} \mathcal{E}_{xy}^2 + \frac{\gamma'_0}{2} \sum_{\tau-\text{link}} (\mathcal{E}_{xx}^2 + \mathcal{E}_{yy}^2) + \frac{\gamma}{2} \left( \sum_{x-\text{link}} B_x^2 + \sum_{y-\text{link}} B_y^2 \right)$$

$$+ i \sum_{\tau_x-\text{plaq}} \hat{\phi}_x (\Delta_x n_x - \Delta_x n_{xy} + \Delta_y n_{xy}) + i \sum_{\tau_y-\text{plaq}} \hat{\phi}_y (\Delta_y n_y - \Delta_y n_{xx} + \Delta_x n_{xy}) ,$$

(3.7)

where

$$\mathcal{E}_{ij} = \Delta_{i} A_{ij} - \Delta_{i} A_{j} - 2 \pi n_{ri} ,$$

$$B_x = \Delta_x A_{yy} - \Delta_y A_{xy} - 2 \pi n_x ,$$

$$B_y = \Delta_y A_{xx} - \Delta_x A_{xy} - 2 \pi n_y ,$$

(3.8)

are the gauge invariant field strengths of $(A_\tau, A_{ij}; n_{ri}, n_i)$. Here, $(A_\tau, A_{ij})$ are real gauge fields, $(n_{ri}, n_i)$ are integer gauge fields, and $\hat{\phi}_i$ are real Lagrange multipliers that make the integer gauge fields flat. The locations on the lattice of these fields are summarized in Table 5. The gauge symmetry is

$$A_\tau \sim A_\tau + \Delta_{\tau} \alpha + 2 \pi k_\tau ,$$

$$A_{ij} \sim A_{ij} + \Delta_i \Delta_j A_{ij} - 2 \pi n_{ri} ,$$

$$n_{ri} \sim n_{ri} + \Delta_{i} k_{ij} - \Delta_{i} k_{ij} ,$$

$$n_x \sim n_x + \Delta_x k_{yy} - \Delta_y k_{xy} ,$$

$$n_y \sim n_y + \Delta_y k_{xx} - \Delta_x k_{xy} ,$$

(3.9)

where $\alpha$ is a real gauge parameter, and $(k_{\tau}, k_{ij})$ and $\hat{k}_i$ are integer gauge parameters.
3.2.1 Fracton-elasticity duality

Using the Poisson resummation formula for the integers \((n_{\tau ij}, n_x, n_y)\), the action (3.7) is dualized to

\[
S = \frac{\hat{\beta}_0}{2} \left[ \sum_{x\text{-link}} (\Delta_x \hat{\phi}_x - 2\pi \hat{n}_{\tau x})^2 + \sum_{y\text{-link}} (\Delta_y \hat{\phi}_y - 2\pi \hat{n}_{\tau y})^2 \right] + \frac{\hat{\beta}}{2} \sum_{\text{cube}} (\Delta_x \hat{\phi}_y + \Delta_y \hat{\phi}_x - 2\pi \hat{n}_{xy})^2 \\
+ \frac{\hat{\beta}'}{2} \sum_{\tau\text{-link}} \sum_i (\Delta_i \hat{\phi}_i - 2\pi \hat{n}_{\tau i})^2 - i \sum_{\tau\text{-link}} A_\tau (\Delta_x \Delta_y \hat{n}_{xy} - \Delta_x^2 \hat{n}_{yy} - \Delta_y^2 \hat{n}_{xx}) \\
+ i \sum_{\text{site \(i \neq j\)}} A_{ij} (\Delta_r \hat{n}_{jj} - \Delta_j \hat{n}_{rj}) - i \sum_{xy\text{-plaq}} A_{xy} (\Delta_r \hat{n}_{xy} - \Delta_x \hat{n}_{ry} - \Delta_y \hat{n}_{rx}) ,
\]

(3.10)

where

\[
\hat{\beta}_0 = \frac{1}{(2\pi)^2 \gamma} , \quad \hat{\beta} = \frac{1}{(2\pi)^2 \gamma_0} , \quad \hat{\beta}' = \frac{1}{(2\pi)^2 \gamma_0} ,
\]

(3.11)

and the integer gauge fields \((\hat{n}_{\tau i}, \hat{n}_{ij})\) have a gauge symmetry

\[
\hat{n}_{\tau i} \sim \hat{n}_{\tau i} + \Delta_x \hat{k}_i , \quad \hat{n}_{ij} \sim \hat{n}_{ij} + \Delta_i \hat{k}_i , \quad \hat{n}_{xy} \sim \hat{n}_{xy} + \Delta_x \hat{k}_y + \Delta_y \hat{k}_x .
\]

(3.12)

The theory (3.10) can be thought of as a “matter theory” with the matter fields \((\hat{\phi}_x, \hat{\phi}_y)\). It is closely related to the gauge theory (2.19). Gauging the momentum symmetry of the matter theory (3.10) couples the theory to the gauge fields \((\hat{A}_{\tau i}, \hat{A}_{ij})\) of (2.19). The relations between the dipole \(\phi\)-theory, the vector charge theory \((\hat{A}_{\tau i}, \hat{A}_{ij})\), the scalar charge theory \((A_\tau, A_{ij})\), and the matter theory \(\hat{\phi}_i\) are summarized in Figure 1.

The continuum Lagrangian for the matter fields \((\hat{\phi}_x, \hat{\phi}_y)\) takes the form

\[
\mathcal{L} = \frac{\hat{\mu}_0}{2} \sum_i (\partial_i \hat{\phi}_i)^2 + \frac{\hat{\mu}}{2} \sum_{i,j} (\partial_i \hat{\phi}_j + \partial_j \hat{\phi}_i)^2 .
\]

(3.13)

The fields \((\hat{\phi}_x, \hat{\phi}_y)\) have figured as displacement fields in an effective field theory for elasticity, and its relation to the scalar charge theory was explored in \([54, 56, 57]\). Specifically, disclinations in the elasticity theory correspond to fracton defects in the scalar charge theory. See also \([32]\) for another appearance of this theory. The duality is schematically given by the map,

\[
\partial_\tau \hat{\phi}_i \leftrightarrow B_i = \partial_i A_{jj} - \partial_j A_{ij} , \quad i, j = x, y , \quad i \neq j , \\
\partial_i \hat{\phi}_i \leftrightarrow E_{jj} = \partial_r A_{jj} - \partial_r^2 A_\tau , \quad i, j = x, y , \quad i \neq j , \\
\partial_x \hat{\phi}_y + \partial_y \hat{\phi}_x \leftrightarrow E_{xy} = \partial_r A_{xy} - \partial_x \partial_y A_\tau .
\]

(3.14)

\[\text{Their relations are similar to the four theories (with similar notations) studied in [40].}\]
Our exact duality above is a lattice version of the fracton-elasticity duality in the continuum.

3.2.2 Global symmetry, ground state degeneracy, and mobility of defects

Let us discuss the global symmetry of the modified Villain version of the 2+1d scalar charge

\[ \mathcal{A}_{ii} \rightarrow \mathcal{A}_{ii} + \frac{c_{ii}}{L_i}, \quad \mathcal{A}_{xy} \rightarrow \mathcal{A}_{xy} + \frac{c_{xy}}{\gcd(L_x, L_y)}, \tag{3.15} \]

where \( c_{ij} \sim c_{ij} + 2\pi \) are circle-valued constants.

- The \( U(1)^3 \) electric (dual winding) symmetry acts as

\[ A_{xy}(\tau, x + \frac{1}{2}, y + \frac{1}{2}) \rightarrow A_{xy}(\tau, x + \frac{1}{2}, y + \frac{1}{2}) + \frac{2\pi m_{xy}}{\gcd(L_x, L_y)} (\tilde{L}_y \delta_{x,L_x-1} - \tilde{L}_x \delta_{y,L_y-1}) \],

\[ A_{xx}(\tau, x, y) \rightarrow A_{xx}(\tau, x, y) - \frac{2\pi m_{xy}}{\gcd(L_x, L_y)} \tilde{L}_y y (\delta_{x,0} - \delta_{x,L_x-1}) \],

\[ A_{yy}(\tau, x, y) \rightarrow A_{yy}(\tau, x, y) + \frac{2\pi m_{xy}}{\gcd(L_x, L_y)} \tilde{L}_x x (\delta_{y,0} - \delta_{y,L_y-1}) \],

\[ n_x(\tau, x + \frac{1}{2}, y) \rightarrow n_x(\tau, x + \frac{1}{2}, y) - \frac{m_{xy}}{\gcd(L_x, L_y)} \tilde{L}_x L_x (\tilde{L}_y \delta_{x,0} - \delta_{y,L_y-1}) \delta_{x,L_x-1} \],

\[ n_y(\tau, x, y + \frac{1}{2}) \rightarrow n_y(\tau, x, y + \frac{1}{2}) + \frac{m_{xy}}{\gcd(L_x, L_y)} \tilde{L}_y L_y (\delta_{x,0} - \delta_{x,L_x-1}) \delta_{y,L_y-1} \],

where \( m_{xy} = 0, \ldots, \gcd(L_x, L_y) - 1 \), and \( \tilde{L}_i \) are integer solutions of the equation \( \tilde{L}_x L_x + \tilde{L}_y L_y = \gcd(L_x, L_y) \).

- The \( U(1)^2 \) magnetic (dual momentum) symmetry acts as \( \hat{\phi}_i \rightarrow \hat{\phi}_i + \hat{c}_i \), where \( \hat{c}_i \sim \hat{c}_i + 2\pi \) are circle-valued constants.

- The \( \mathbb{Z}_{\gcd(L_x,L_y)} \) magnetic (dual momentum) dipole symmetry acts as

\[ \hat{n}_{xy}(\tau + \frac{1}{2}, x + \frac{1}{2}, y) \rightarrow \hat{n}_{xy}(\tau + \frac{1}{2}, x + \frac{1}{2}, y) - \frac{m_{xy}}{\gcd(L_x, L_y)} (L_x \delta_{x,L_x-1} - L_y \delta_{y,L_y-1}) \].

(3.17)
\[ \hat{m}_{xy} = 0, 1, \ldots, \gcd(L_x, L_y) - 1. \]

Note that the \( \hat{m}_{xy} = \gcd(L_x, L_y) \) transformation is trivial because it can be undone by an integer gauge transformation with \( \hat{k}_x = -y, \hat{k}_y = x. \) More intuitively, the \( \hat{m}_{xy} = \gcd(L_x, L_y) \) transformation is trivial because \( \hat{\phi}_i \) are compact.

The electric and magnetic dipole symmetries do not commute with each other, which leads to a ground state degeneracy of \( \gcd(L_x, L_y) \). See the discussion around (A.37) for more details. In fact, every state in the Hilbert space is \( \gcd(L_x, L_y) \)-fold degenerate.

There are also time-like electric symmetries:

- The \( U(1) \) time-like electric (dual winding) symmetry acts as
  \[ \mathcal{A}_\tau \rightarrow \mathcal{A}_\tau + \frac{c_\tau}{L_\tau}, \]
  where \( c_\tau \sim c_\tau + 2\pi \) is a circle-valued constant. The charge, which is commonly referred to as the “gauge charge”, associated with this time-like global symmetry generated by \( c_\tau \) is a scalar in space, and hence the name scalar charge theory.

- The \( \mathbb{Z}_{L_x} \) time-like electric (dual winding) dipole symmetry acts as
  \[ \mathcal{A}_\tau (\tau + \frac{1}{2}, x, y) \rightarrow \mathcal{A}_\tau (\tau + \frac{1}{2}, x, y) + 2\pi m_{\tau x} \delta_{\tau,0} \frac{x}{L_x}, \]
  \[ n_{\tau xx}(\tau + \frac{1}{2}, x, y) \rightarrow n_{\tau xx}(\tau + \frac{1}{2}, x, y) - m_{\tau x} \delta_{\tau,0} (\delta_{x,0} - \delta_{x,L_x-1}), \]
  where \( m_{\tau x} = 0, \ldots, L_x - 1. \)

- The \( \mathbb{Z}_{L_y} \) time-like electric (dual winding) dipole symmetry acts in a similar way with \( x \) and \( y \) exchanged.

As a consequence of the time-like dipole symmetry, the particle described by the defect

\[ \exp \left( i \sum_{\tau \text{-link: fixed } x,y} \mathcal{A}_\tau \right), \]

cannot move, i.e., it is a fracton. The immobility of the fracton in the scalar charge theory is usually attributed to “dipole moment conservation” as discussed in [24, 26]. Here, we give a more precise explanation in terms of the selection rules imposed by the time-like global symmetries.
3.2.3 Robustness

Let us impose the $U(1)^3$ electric and $\mathbb{Z}_{\gcd(L_x,L_y)}$ electric dipole symmetries. Then, the magnetic symmetry and the ground state degeneracy are not robust because the monopole operators $e^{i\hat{\phi}_i}$ acts nontrivially on the ground states.

Alternatively, we can start with the dual matter theory and impose the dual momentum (magnetic) symmetries. In this case, the dual winding (electric) symmetry is robust because the operators charged under it are extended. It follows that the ground state degeneracy of the matter theory is also robust once we impose the dual momentum symmetries.

4 Discussion, Conclusion, and Outlook

In this work, we studied two lattice regularizations of the 2+1d compact Lifshitz theory [11–20] using the modified Villain formulation of [48,45]. The two models have the same naive continuum Lagrangians given by (1.1).

Surprisingly, the differences between the two lattice models are significant enough that their low-energy limits are actually distinct and the naive conclusion that both are described by (1.1) is imprecise. In particular, the two models have different global symmetries, anomalies, ground state degeneracies, dualities, etc.

For example, one of the models, referred to as the Laplacian $\phi$-theory, is self-dual, whereas the other one, referred to as the dipole $\phi$-theory, is dual to a lattice version of a gauge theory known as the vector charge theory [49–52]. The latter contains defects that capture the worldlines of particles that can move only along a line, i.e., lineons. As the name suggests, the dipole $\phi$-theory has a dipole global symmetry which has received a lot of attention in the recent years in the context of fractons. Furthermore, the duality between the dipole $\phi$-theory and the vector charge theory is a rigorous lattice version of the lineon-elasticity duality in the continuum [53].

We also studied the pure gauge theories in 2+1d associated with the global symmetries of the Laplacian and dipole $\phi$-theories. In the former case, we find the Laplacian gauge theory, which hosts defects that describe immobile particles, i.e., fractons. On the other hand, the latter gives a lattice version of the scalar charge theory [24,25,50,26], which is also known to host defects that describe fractons. However, the two gauge theories differ in several aspects. For example, while the Laplacian gauge theory has no duality, the scalar charge theory is dual to an elasticity theory of displacements. The latter duality is a rigorous lattice version of the fracton-elasticity duality in the continuum [54,56,57].

We emphasize that most of our discussion, including the “fractonic” nature of the defects
in the gauge theories, is not an artifact of the “discretization” of the time direction. The relation between lattice models with discrete spacetime and lattice models with discrete space and continuous time is well understood. One way to see that is to consider the Hamiltonian version of these modified Villain models, as in \[68\]–\[70\].

Alternatively, we can take the continuum limit in the time direction as follows. Consider the scalar charge theory of Section 3.2. Integrating out the scalar fields $\hat{\phi}_i$ in the action (3.7) imposes the flatness of the integer gauge fields $(n_{\tau ij}, n_i)$. We can gauge fix $n_{\tau ij} = 0$ everywhere except at $\tau = \tau_0$, using the integer gauge parameters $k_{ij}$. In other words, after gauge fixing,

$$n_{\tau ij}(\tau, x, y) = \delta_{\tau,\tau_0} \bar{n}_{\tau ij}(x, y).$$

(4.1)

Flatness then implies that $n_i$’s are independent of $\tau$. Now, we introduce the lattice spacing $a_\tau$ in the time direction and take the limit $a_\tau \to 0$, while keeping $\frac{1}{g_0^2} = \gamma_0$, $\frac{1}{g_0'^2} = \gamma_0'$, $\frac{1}{g^2} = \frac{\gamma}{a_\tau}$, fixed. We also scale the field $A_\tau$ so that $A_\tau \equiv \frac{1}{a_\tau} A_\tau$ is fixed. In this limit, the action (3.7) becomes\[^{15}\]

$$S = \oint d\tau \left[ \frac{1}{2g_0^2} \sum_{\text{plaq}} E_{xy}^2 + \frac{1}{2g_0'^2} \sum_{\text{site}} (E_{xx}^2 + E_{yy}^2) + \frac{1}{2g^2} \left( \sum_{\text{x-link}} B_x^2 + \sum_{\text{y-link}} B_y^2 \right) \right],$$

(4.3)

where the sums inside the brackets are over the spatial lattice and we defined

$$E_{ij} = \partial_\tau A_{ij} - \Delta_i \Delta_j A_\tau - 2\pi \delta(\tau - \tau_0) \bar{n}_{\tau ij}(x, y).$$

(4.4)

Note that the magnetic fields $B_x$ and $B_y$ in (3.8) are well-defined with continuous $\tau$ because $n_i$’s are independent of $\tau$. The defect (3.20) then becomes

$$\exp \left( i \oint d\tau A_\tau(\tau, x, y) \right).$$

(4.5)

It still describes the worldline of a fracton, i.e., an immobile particle, because of the time-like

\[^{15}\]We use $\tau$ to denote the continuum time in this action, but $(x, y)$ still labels the sites on the spatial lattice.
Electric dipole symmetry:

\[
A_\tau(\tau, x, y) \rightarrow A_\tau(\tau, x, y) + \delta(\tau - \tau_0) \left( 2\pi m_{\tau x} \frac{x}{L_x} + 2\pi m_{\tau y} \frac{y}{L_y} \right),
\]

\[
\bar{n}_{\tau xx}(x, y) \rightarrow \bar{n}_{\tau xx}(x, y) - m_{\tau x} \left( \delta x, 0 - \delta x, L_x - 1 \right),
\]

\[
\bar{n}_{\tau yy}(x, y) \rightarrow \bar{n}_{\tau yy}(x, y) - m_{\tau y} \left( \delta y, 0 - \delta y, L_y - 1 \right),
\]

where \(m_{\tau i} = 0, \ldots, L_i - 1\).

One can take a similar continuum limit in the time direction for the Laplacian gauge theory of Section 3.1 or the vector charge theory of Section 2.2.1 without changing any of our conclusions about the mobility of the defects.

In an upcoming paper [58], we will discuss the \(\mathbb{Z}_N\) version of the \(U(1)\) Laplacian gauge theory, and compare it with the 2+1d rank-2 \(\mathbb{Z}_N\) tensor gauge theory [50, 59, 51, 52, 60]. We will then consider an anisotropic generalization of the \(\mathbb{Z}_N\) Laplacian model that can be defined on any spatial lattice of the form \(\Gamma \times \mathbb{Z}_{L_z}\), where \(\Gamma\) is a general graph. We will present this lineon model both in terms of a modified Villain lattice action (or more precisely, an integer \(BF\) model [45]), and as the low-energy limit of a stabilizer code. The stabilizer code is

\[
H = -\gamma_1 \sum_{i,z} G(i, z) - \gamma_2 \sum_{i,z} F(i, z + \frac{1}{2}) + \text{c.c.},
\]

where

\[
G(i, z) = V_z(i, z + \frac{1}{2})^\dagger V_z(i, z - \frac{1}{2}) \prod_{j: (i,j) \in \Gamma} V(i, z)V(j, z)^\dagger,
\]

\[
F(i, z + \frac{1}{2}) = U(i, z + 1)^\dagger U(i, z) \prod_{j: (i,j) \in \Gamma} U_z(i, z + \frac{1}{2})U_z(j, z + \frac{1}{2})^\dagger.
\]

where \(i, j\) label the sites on \(\Gamma\), and \((i, j) \in \Gamma\) means that \(i\) and \(j\) are connected by an edge. Here \(U(i, z), V(i, z)\) are conjugate \(\mathbb{Z}_N\) variables on the sites, i.e., \(U(i, z)V(i, z) = e^{2\pi i/N} V(i, z)U(i, z)\). Similarly, \(U_z(i, z + \frac{1}{2}), V_z(i, z + \frac{1}{2})\) are conjugate variables living on the \(z\)-links, i.e., \(U_z(i, z + \frac{1}{2})V_z(i, z + \frac{1}{2}) = e^{2\pi i/N} V_z(i, z + \frac{1}{2})U_z(i, z + \frac{1}{2})\). This anisotropic \(\mathbb{Z}_N\) lattice model is gapped, robust, and has lineons. Its GSD on a general graph is given by \(|\text{Jac}(\Gamma, N)|^2\), where \(\text{Jac}(\Gamma, N)\) is a mod \(N\) reduction of the Jacobian group \(\text{Jac}(\Gamma)\) of \(\Gamma\). A special case of \(\Gamma\) corresponds to a square lattice with sizes \(L_x, L_y\). When \(N\) is prime, the logarithm of the GSD is given by the dimension of a quotient ring (as a vector space):

\[
\log_N \text{GSD} = 2 \dim_{\mathbb{Z}_N} \frac{\mathbb{Z}_N[X, Y]}{(Y(X - 1)^2 + X(Y - 1)^2, X^{L_x - 1} - 1, Y^{L_y - 1})}.
\]

In this case, as in the celebrated Haah’s code [2], the logarithm of the GSD depends in a
complicated, non-monotonic way on \( L_x, L_y \) and grows at most linearly in the system size. Furthermore, there are sequences of \( L_x, L_y \) going to infinity such that the GSD stays finite.

Notes added: As we were finishing this paper, [61] appeared on the arXiv. It overlaps with some of the findings in our upcoming paper [58]. In particular, the stabilizer code (4.7) and the GSD formula on a general graph appear in [61]. In [58], we will investigate this model also on a cubic lattice and derive (4.9).

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A More on global symmetries of 2+1d dipole theories

In this appendix, we will analyze the global symmetries of the modified Villain versions of the 2+1d dipole \( \phi \)-theory of Section 2.2 and the 2+1d scalar charge theory of Section 3.2 in more detail. We will discuss the charges/symmetry operators, charged operators/defects. We will also discuss the ground state degeneracy as a consequence of space-like symmetries and restricted mobility of defects as a consequence of time-like symmetries.

A.1 Global symmetry of 2+1d dipole \( \phi \)-theory

The global symmetries of the modified Villain model (2.17) are listed below:

- The \( U(1) \) momentum (dual magnetic) symmetry acts as \( \phi \to \phi + c \), where \( c \sim c + 2\pi \). The Noether currents are

\[
J_\tau = i\beta_0(\Delta_\tau \phi - 2\pi n_\tau) , \quad J_{xy} = i\beta(\Delta_x \Delta_y \phi - 2\pi n_{xy}) , \quad J_{ii} = i\beta'(\Delta_i^2 \phi - 2\pi n_{ii}) ,
\]

(A.1)

and they satisfy

\[
\Delta_\tau J_\tau = \Delta_i \Delta_j J_{ij} , \quad \Delta_\tau J_{xy} = \Delta_i n_{xy} , \quad \Delta_\tau J_{ii} = \Delta_i n_{ii} ,
\]

(A.2)
which follows from the equation of motion of $\phi$. The charge is

$$Q = \sum_{\tau\text{-link: fixed } \tau} J_\tau, \quad (A.3)$$

and the charged operator is $e^{i\phi}$.

- The $\mathbb{Z}_{L_x}$ momentum (dual magnetic) dipole symmetry acts as

$$\phi \rightarrow \phi + 2\pi m_{xx} \frac{x}{L_x},$$

$$n_{xx} \rightarrow n_{xx} + m_{xx} (\delta_{x,0} - \delta_{x,L_x-1}), \quad (A.4)$$

where $m_{xx} = 0, 1, \ldots, L_x - 1$. The symmetry operator is

$$U^{(x)}_{m_{xx}} = \exp \left( \frac{2\pi i m_{xx}}{L_x} \sum_{\tau\text{-link: fixed } \tau} x J_\tau - i m_{xx} \sum_{\tau\text{-link: fixed } \tau} \left[ \hat{A}_{yy}(x = 0) - \hat{A}_{yy}(x = L_x - 1) \right] \right),$$

$$= \exp \left( -\frac{2\pi i m_{xx}}{L_x} \sum_{\tau\text{-link: fixed } \tau} x \hat{n} \right), \quad (A.5)$$

The charged operators are $e^{i\phi}$, and the dipole operator $e^{i\Delta_x \phi}$.

- The $\mathbb{Z}_{L_y}$ momentum (dual magnetic) dipole symmetry acts in a similar way with $x$ and $y$ exchanged.

- There is a $U(1)^3$ winding (dual electric) symmetry that shifts

$$\hat{A}_{ii} \rightarrow \hat{A}_{ii} + \frac{\hat{c}_{ii}}{L_i}, \quad \hat{A}_{xy} \rightarrow \hat{A}_{xy} + \frac{\hat{c}_{xy}}{\text{lcm}(L_x, L_y)}, \quad (A.6)$$

where $\hat{c}_{ij} \sim \hat{c}_{ij} + 2\pi$. The Noether currents are

$$\hat{J}_{\tau ij} = \frac{1}{2\pi} (\Delta_i \Delta_j \phi - 2\pi n_{ij}), \quad \hat{J} = \frac{1}{2\pi} (\Delta_\tau \phi - 2\pi n_\tau), \quad (A.7)$$

and they satisfy

$$\Delta_{\tau_1} \hat{J}_{\tau_1 ij} = \Delta_i \Delta_j \hat{J}, \quad \Delta_x \hat{J}_{rxy} = \Delta_y \hat{J}_{rxx}, \quad \Delta_y \hat{J}_{rxy} = \Delta_x \hat{J}_{ryy}, \quad (A.8)$$

---

16 These operators and their correlation functions have been discussed extensively in the context of spontaneous symmetry breaking of the ordinary and dipole symmetries in [34,20,36].

17 Naively, it might appear that $\hat{A}_{xy}$ can be shifted by $\hat{c}^x(x) + \hat{c}^y(y)$, but in fact these shifts can be gauged away except for the zero mode.
which follow from the equations of motion of $\hat{A}_{ij}$ and $\hat{A}_{\tau i}$. The first equation is the conservation equation, and the last two equations are difference conditions (Gauss laws). The charges are

\[
\hat{Q}_{xx}(y) = \sum_{\text{site: fixed } \tau, y} \hat{j}_{\tau xx} = - \sum_{\text{site: fixed } \tau, y} n_{xx},
\]

\[
\hat{Q}_{yy}(x) = \sum_{\text{site: fixed } \tau, x} \hat{j}_{\tau yy} = - \sum_{\text{site: fixed } \tau, x} n_{yy},
\]

\[
\hat{Q}_{xy}^x(x) = \sum_{\text{xy-plaq: fixed } \tau, x} \hat{j}_{\tau xy} = - \sum_{\text{xy-plaq: fixed } \tau, x} n_{xy},
\]

\[
\hat{Q}_{xy}^y(y) = \sum_{\text{xy-plaq: fixed } \tau, y} \hat{j}_{\tau xy} = - \sum_{\text{xy-plaq: fixed } \tau, y} n_{xy}.
\]

The difference conditions (Gauss laws) imply that the four charges are independent of their arguments. Moreover, the last two charges are related as

\[
- \sum_{\text{xy-plaq: fixed } \tau} n_{xy} = L_x \hat{Q}_{xy}^x = L_y \hat{Q}_{xy}^y = \text{lcm}(L_x, L_y) \hat{Q}_{xy},
\]

where $\hat{Q}_{xy}$ is an integer. So there are only three independent $U(1)$ charges, $\hat{Q}_{ij}$.

The charged operators are

\[
\hat{W}_{yy}(\tau + \frac{1}{2}, x) = \exp \left( i \sum_{\text{\tau-link: fixed } \tau, x} \hat{A}_{yy} \right),
\]

\[
\hat{W}_{xx}(\tau + \frac{1}{2}, y) = \exp \left( i \sum_{\text{\tau-link: fixed } \tau, y} \hat{A}_{xx} \right),
\]

\[
\hat{W}_{xy}(\tau + \frac{1}{2}) = \exp \left( i \sum_{\text{cube: fixed } \tau} \hat{A}_{xy} \right),
\]

respectively. Actually, the third operator in (A.11) is not minimally charged under the $U(1)$ winding symmetry generated by $\hat{c}_{xy}$. Instead, the minimally charged operator is the “diagonal” operator

\[
\exp \left( i \sum_{s=0}^{\text{lcm}(L_x, L_y)-1} \left[ \hat{A}_{xy}(\tau + \frac{1}{2}, x + s + \frac{1}{2}, y + s + \frac{1}{2}) + \hat{A}_{xx}(\tau + \frac{1}{2}, x + s, y + s) + \hat{A}_{yy}(\tau + \frac{1}{2}, x + s, y + s) \right] \right),
\]

32
which, however, is also charged under the other $U(1)$ winding symmetries generated by $\hat{c}_{ii}$. A different minimally charged operator, which is not charged under the other $U(1)$ winding symmetries is

$$\exp \left[ \frac{i}{\gcd(L_x, L_y)} \sum_{x,y} \left( \hat{A}_{xy}(\tau + \frac{1}{2}, x + \frac{1}{2}, y + \frac{1}{2}) - 2\pi xy \hat{n}(\tau + \frac{1}{2}, x, y) \right) \right]. \quad (A.13)$$

- There is a $\mathbb{Z}_{L_x}$ winding (dual electric) dipole symmetry that shifts

$$\hat{A}_{yy} \to \hat{A}_{yy} + 2\pi \hat{m}_{xx} \frac{x}{L_x} \delta_{y,0},
\hat{n} \to \hat{n} - \hat{m}_{xx} \delta_{y,0} (\delta_{x,0} - \delta_{x,L_x-1}). \quad (A.14)$$

where $\hat{m}_{xx} = 0, \ldots, L_x - 1$. The symmetry operator is

$$\hat{U}^{(x)}_{\hat{m}_{xx}} = \exp \left( -\frac{2\pi i \hat{m}_{xx}}{L_x} \sum_{\text{site: fixed } \tau} \hat{n}_{xx}(y = 0) \right). \quad (A.15)$$

The charged operators are $\hat{W}_{yy}(\tau + \frac{1}{2}, x)$, and $\hat{W}_{yy}(\tau + \frac{1}{2}, x + 1)\hat{W}_{yy}(\tau + \frac{1}{2}, x)^{-1}$. The diagonal operator (A.12) is also charged under this symmetry.

- There is also a $\mathbb{Z}_{L_y}$ winding (dual electric) dipole symmetry, which acts in a similar way with $x$ and $y$ exchanged.

The momentum (dual magnetic) and winding (dual electric) dipole symmetries do not commute:

$$U^{(x)}_{\hat{m}_{xx}} \hat{U}^{(x)}_{\hat{m}_{xx}} = e^{-\frac{2\pi i}{L_x} \hat{m}_{xx} \hat{m}_{xx} \hat{U}^{(x)}_{\hat{m}_{xx}} U^{(x)}_{\hat{m}_{xx}}}, \quad (A.16)$$

and similarly in the $y$-direction. This signals a mixed 't Hooft anomaly between them and leads to a large $L_xL_y$-fold ground state degeneracy.

There are also time-like winding (dual electric) symmetries that act on the defects of $(\hat{A}_{ri}, \hat{A}_{ij})$:

- The $U(1)^2$ time-like winding (dual electric) symmetry acts as

$$\hat{A}_{ri} \to \hat{A}_{ri} + \frac{\hat{c}_{ri}}{L_r}, \quad (A.17)$$
where $\hat{c}_\tau \sim \hat{c}_\tau + 2\pi$ is circle-valued. The charged defects are

$$\hat{W}_{\tau x}(x + \frac{1}{2}, y) = \exp \left( i \sum_{x\text{-link; fixed } x,y} \hat{A}_{\tau x} \right), \quad (A.18)$$

and similarly $\hat{W}_{\tau y}(x, y + \frac{1}{2})$. The defect $\hat{W}_{\tau x}(x + \frac{1}{2}, y)$ describes the world-line of a particle on the $x$-link $(x + \frac{1}{2}, y)$. Since it can move in the $x$-direction via $\hat{A}_{\tau x}$, we call it the $x$-lineon. Similarly, the defect $\hat{W}_{\tau y}(x, y + \frac{1}{2})$ describes the world-line of a $y$-lineon.

- The $\mathbb{Z}_{\gcd(L_x,L_y)}$ time-like winding (dual electric) dipole symmetry acts as

$$\hat{A}_{\tau x}(\tau, x + \frac{1}{2}, y) \to \hat{A}_{\tau x}(\tau, x + \frac{1}{2}, y) - 2\pi \hat{m}_{\tau xy} \delta_{\tau,0} \frac{y}{\gcd(L_x,L_y)},$$
$$\hat{A}_{\tau y}(\tau, x, y + \frac{1}{2}) \to \hat{A}_{\tau y}(\tau, x, y + \frac{1}{2}) + 2\pi \hat{m}_{\tau xy} \delta_{\tau,0} \frac{x}{\gcd(L_x,L_y)},$$
$$\hat{n}_{\tau xy}(\tau, x + \frac{1}{2}, y + \frac{1}{2}) \to \hat{n}_{\tau xy}(\tau, x + \frac{1}{2}, y + \frac{1}{2}) + \frac{\hat{m}_{\tau xy} \delta_{\tau,0}}{\gcd(L_x,L_y)} (L_x \delta_{x,L_x-1} - L_y \delta_{y,L_y-1}),$$

(A.19)

where $\hat{m}_{\tau xy} = 0, \ldots, \gcd(L_x,L_y) - 1$.

These time-like symmetries imply that the defects $\hat{W}_{\tau x}(x + \frac{1}{2}, y)$ and $\hat{W}_{\tau x}(x' + \frac{1}{2}, y')$ have the same time-like charges if and only if

$$(x', y') = (x + s_x, y + \gcd(L_x,L_y)s_y),$$

(A.20)

where $s_x, s_y$ are integers. In other words, an $x$-lineon can move anywhere in the $x$-direction, but it can hop only by $\gcd(L_x,L_y)$ sites in the $y$-direction. On the other hand, a dipole of $x$-lineons separated in the $y$-direction, described by the defect $\hat{W}_{\tau x}(x + \frac{1}{2}, y')\hat{W}_{\tau x}(x + \frac{1}{2}, y)^{-1}$, is fully mobile. Similar mobility restrictions apply to the $y$-lineons. Interestingly, a pair of $x$- and $y$-lineons on the links $(x + \frac{1}{2}, y)$ and $(x, y + \frac{1}{2})$ respectively can move “diagonally” together to $(x + s + \frac{1}{2}, y + s)$ and $(x + s, y + s + \frac{1}{2})$ respectively, for any $s \in \mathbb{Z}$.

### A.2 Global symmetry of 2+1d scalar charge theory

The global symmetries of the modified Villain model (3.7) are listed below:

- The $U(1)^3$ electric (dual winding) symmetry acts as

$$\mathcal{A}_{ii} \to \mathcal{A}_{ii} + \frac{c_{ii}}{L_i}, \quad \mathcal{A}_{xy} \to \mathcal{A}_{xy} + \frac{c_{xy}}{\gcd(L_x,L_y)},$$

(A.21)
where $c_{ij} \sim c_{ij} + 2\pi$ are circle-valued constants. The Noether currents are

$$
J_{\tau ii} = -i\gamma_0 c_{ii}, \quad J_{\tau xy} = i\gamma_0 c_{xy}, \quad J_x = i\gamma_0 E_{xx}, \quad J_y = i\gamma_0 E_{xy},
$$

(A.22)

and they satisfy

$$
\Delta_{\tau} J_{\tau xx} = \Delta_y J_y, \quad \Delta_{\tau} J_{\tau xy} = \Delta_x J_x + \Delta_x J_y, \quad \Delta_{\tau} J_{\tau yy} = \Delta_x J_x, \quad \Delta_{\tau} J_{\tau xy} - \Delta_y^2 J_{\tau yy} - \Delta_x^2 J_{\tau xx} = 0.
$$

(A.23)

They follow from the equations of motion of $A_{ij}$ and $A_{\tau}$. The first three equations are the conservation equations, and the last equation is the difference condition (Gauss law). The charges are

$$
Q_{ii}(x^i) = \sum_{\tau\text{-link}: \text{fixed } \tau} J_{\tau ii}, \quad Q_{xy} = \sum_{\text{cube: fixed } \tau} J_{\tau xy}.
$$

(A.24)

The difference condition (Gauss law) implies that $Q_{ii}$ is independent of $x^i$, while $Q_{xy}$ is a multiple of gcd$(L_x, L_y)$\footnote{This can be seen easily in the dual frame (3.10), where the charge is $Q_{xy} = -\sum_{\text{cube: fixed } \tau} \hat{A}_{xy}$.} The charged operators are

$$
W_{xx}(\tau, y) = \exp \left( i \sum_{\text{site: fixed } \tau, y} A_{xx} \right), \quad W_{yy}(\tau, x) = \exp \left( i \sum_{\text{site: fixed } \tau, x} A_{yy} \right),
$$

$$
W_{xy}(\tau, x + \frac{1}{2}) = \exp \left( i \sum_{\text{xy-plaq: fixed } \tau, x} A_{xy} \right), \quad W_{xy}(\tau, y + \frac{1}{2}) = \exp \left( i \sum_{\text{xy-plaq: fixed } \tau, y} A_{xy} \right),
$$

(A.25)

respectively. Actually, the two operators in the second line of (A.25) are not minimally charged under the $U(1)$ electric symmetry generated by $c_{xy}$. Instead, the minimally charged operator is

$$
\exp \left[ i \frac{1}{\text{lcm}(L_x, L_y)} \sum_{x,y} \left( A_{xy}(\tau,x + \frac{1}{2}, y + \frac{1}{2}) \right. \right.
$$

$$
- \frac{2\pi \tilde{L}_y L_y x}{\gcd(L_x, L_y)} n_y(\tau, x, y + \frac{1}{2}) - \frac{2\pi \tilde{L}_x L_x y}{\gcd(L_x, L_y)} n_x(\tau, x + \frac{1}{2}, y) \bigg) \right],
$$

(A.26)

where $\tilde{L}_i$ are integer solutions of the equation $\tilde{L}_x L_x + \tilde{L}_y L_y = \gcd(L_x, L_y)$. Another
minimally charged operator is

\[ W_{xy}^x(\tau, x + \frac{1}{2}) L_y W_{xy}^y(\tau, y + \frac{1}{2}) L_z . \]  

(A.27)

• The \( \mathbb{Z}_{\gcd(L_x, L_y)} \) electric (dual winding) dipole symmetry acts as

\[
\begin{align*}
A_{xy}(\tau, x + \frac{1}{2}, y + \frac{1}{2}) &\to A_{xy}(\tau, x + \frac{1}{2}, y + \frac{1}{2}) + \frac{2\pi m_{xy}}{\gcd(L_x, L_y)} \left( L_y \delta_{x, L_x - 1} - L_x \delta_{y, L_y - 1} \right), \\
A_{xx}(\tau, x, y) &\to A_{xx}(\tau, x, y) - \frac{2\pi m_{xy}}{\gcd(L_x, L_y)} \tilde{L}_y y (\delta_{x,0} - \delta_{x, L_x - 1}), \\
A_{yy}(\tau, x, y) &\to A_{yy}(\tau, x, y) + \frac{2\pi m_{xy}}{\gcd(L_x, L_y)} \tilde{L}_x x (\delta_{y,0} - \delta_{y, L_y - 1}), \\
n_x(\tau, x + \frac{1}{2}, y) &\to n_x(\tau, x + \frac{1}{2}, y) - \frac{m_{xy}}{\gcd(L_x, L_y)} \tilde{L}_x x (\delta_{y,0} - \delta_{y, L_y - 1}) \delta_{x, L_x - 1}, \\
n_y(\tau, x, y + \frac{1}{2}) &\to n_y(\tau, x, y + \frac{1}{2}) + \frac{m_{xy}}{\gcd(L_x, L_y)} \tilde{L}_y y (\delta_{x,0} - \delta_{x, L_x - 1}) \delta_{y, L_y - 1},
\end{align*}
\]

(A.28)

where \( m_{xy} = 0, \ldots, \gcd(L_x, L_y) - 1 \). The symmetry operator is

\[
U_{m_{xy}} = \exp \left[ \frac{2\pi i m_{xy}}{\gcd(L_x, L_y)} \left( \tilde{L}_x \sum_{y=L_y} \hat{n}_{xy}(\tau + \frac{1}{2}, x + \frac{1}{2}, y + \frac{1}{2}) + x \Delta_y \hat{n}_{xx}(\tau + \frac{1}{2}, x, y + \frac{1}{2}) \right) \right. \\
- \left. \tilde{L}_y \sum_{x=L_x-1} \hat{n}_{xy}(\tau + \frac{1}{2}, x + \frac{1}{2}, y + \frac{1}{2}) + y \Delta_x \hat{n}_{yy}(\tau + \frac{1}{2}, x + \frac{1}{2}, y) \right].
\]

(A.29)

The charged operators are \( W_{xy}^i(\tau, x^i + \frac{1}{2}) \). The minimally charged operator is

\[ W_{xy}^y(\tau, y + \frac{1}{2}) L_y W_{xy}^x(\tau, x + \frac{1}{2}) L_x . \]  

(A.30)

\[
\prod_{s_y=1}^{L_y \, \gcd(L_x, L_y)} W_{xy}^y(\tau, y + \gcd(L_x, L_y) s_y + \frac{1}{2}) \prod_{s_x=1}^{L_x \, \gcd(L_x, L_y)} W_{xy}^x(\tau, x + \gcd(L_x, L_y) s_x + \frac{1}{2})^{-1}.
\]

(A.31)

• The \( U(1)^2 \) magnetic (dual momentum) symmetry acts as \( \hat{\phi}_i \to \hat{\phi}_i + \hat{\xi}_i \), where \( \hat{\xi}_i \sim \hat{\xi}_i + 2\pi \).
are circle-valued constants. The Noether currents are
\[ \hat{J}_{\tau x} = \frac{1}{2\pi} B_x, \quad \hat{J}_{\tau y} = \frac{1}{2\pi} B_y, \]
\[ \hat{J}_{xx} = -\frac{1}{2\pi} E_{yy}, \quad \hat{J}_{yy} = -\frac{1}{2\pi} E_{xx}, \quad \hat{J}_{xy} = \frac{1}{2\pi} E_{xy}, \]
and they satisfy
\[ \Delta_{\tau} \hat{J}_{\tau x} + \Delta_x \hat{J}_{xx} + \Delta_y \hat{J}_{xy} = 0, \]
\[ \Delta_{\tau} \hat{J}_{\tau y} + \Delta_y \hat{J}_{yy} + \Delta_x \hat{J}_{xy} = 0, \]
which follow from the equations of motion of \( \hat{\phi}_i \). The charges are
\[ Q_i = \sum_{i-\text{link; fixed } \tau} \hat{J}_i, \]
and the charged operators are \( e^{i\hat{\phi}_i} \).

- The \( \mathbb{Z}_{\gcd(L_x, L_y)} \) magnetic (dual momentum) dipole symmetry acts as
\[ \hat{\phi}_x(\tau + \frac{1}{2}, x + \frac{1}{2}, y) \to \hat{\phi}_x(\tau + \frac{1}{2}, x + \frac{1}{2}, y) - 2\pi \hat{m}_{xy} \frac{y}{\gcd(L_x, L_y)}, \]
\[ \hat{\phi}_y(\tau + \frac{1}{2}, x, y + \frac{1}{2}) \to \hat{\phi}_y(\tau + \frac{1}{2}, x, y + \frac{1}{2}) + 2\pi \hat{m}_{xy} \frac{x}{\gcd(L_x, L_y)}, \]
\[ \hat{n}_{xy}(\tau + \frac{1}{2}, x + \frac{1}{2}, y + \frac{1}{2}) \to \hat{n}_{xy}(\tau + \frac{1}{2}, x + \frac{1}{2}, y + \frac{1}{2}) - \hat{m}_{xy} \frac{(L_x \delta_{x,L_x-1} - L_y \delta_{y,L_y-1})}{\gcd(L_x, L_y)}, \]
where \( \hat{m}_{xy} = 0, \ldots, \gcd(L_x, L_y) - 1 \). The symmetry operator is
\[ \hat{U}_{\hat{m}_{xy}} = \exp \left(-\frac{2\pi i \hat{m}_{xy}}{\gcd(L_x, L_y)} \sum_{x,y} \left[ x n_y(\tau, x, y + \frac{1}{2}) - y n_x(\tau, x + \frac{1}{2}, y) \right] \right). \]

The charged operators are \( e^{i\hat{\phi}_i} \), and the dipole operators \( e^{i\Delta_x \hat{\phi}_y} \) and \( e^{i\Delta_y \hat{\phi}_x} \).

The electric (dual winding) and magnetic (dual momentum) dipole symmetries do not commute:
\[ U_{\hat{m}_{xy}} \hat{U}_{\hat{m}_{xy}} = e^{-\frac{2\pi i}{\gcd(L_x, L_y)} \hat{m}_{xy} \hat{m}_{xy}} \hat{U}_{\hat{m}_{xy}} U_{\hat{m}_{xy}}. \]
This signals a mixed 't Hooft anomaly between them and leads to a ground state degeneracy of \( \gcd(L_x, L_y) \).

There are also time-like electric (dual winding) symmetries that act on the defects of \( (A_\tau, \hat{A}_{ij}) \):
• The $U(1)$ time-like electric (dual winding) symmetry acts as
\[ \mathcal{A}_\tau \to \mathcal{A}_\tau + \frac{c_\tau}{L_\tau}, \tag{A.38} \]
where $c_\tau \sim c_\tau + 2\pi$ is a circle-valued constant. The charged defect is
\[ W_\tau(x, y) = \exp \left( i \sum_{\tau \text{-link: fixed } x,y} \mathcal{A}_\tau \right). \tag{A.39} \]
It describes the world-line of a particle at $(x, y)$.

• The $\mathbb{Z}_{L_x}$ time-like electric (dual winding) dipole symmetry acts as
\[ \mathcal{A}_\tau(\tau + \frac{1}{2}, x, y) \to \mathcal{A}_\tau(\tau + \frac{1}{2}, x, y) + 2\pi m_{\tau x} \delta_{\tau, 0} \frac{x}{L_x}, \]
\[ n_{\tau xx}(\tau + \frac{1}{2}, x, y) \to n_{\tau xx}(\tau + \frac{1}{2}, x, y) - m_{\tau x} \delta_{\tau, 0} (\delta_x, 0 - \delta_{x,L_x-1}), \tag{A.40} \]
where $m_{\tau x} = 0, \ldots, L_x - 1$.

• The $\mathbb{Z}_{L_y}$ time-like electric (dual winding) dipole symmetry acts in a similar way with $x$ and $y$ exchanged.

These time-like symmetries imply that the defects $W_\tau(x, y)$ and $W_\tau(x', y')$ have the same time-like charges if and only if $(x', y') = (x, y)$. In other words, the particle described by the defect $W_\tau$ cannot move, i.e., it is a fracton. On the other hand, any dipole of fractons, described by the defect $W_\tau(x', y')W_\tau(x, y)^{-1}$, is fully mobile.

### B Polynomial representation of functions on square lattice

In this appendix, we develop a polynomial representation of functions on the infinite square lattice $\mathbb{Z}^2$, and use it to show the following:

1. The naturalness of the action (2.6) with respect to the global symmetry of the 2+1d Laplacian $\phi$-theory of Section 2.1. More precisely, we show that the local operator $\prod_{i=1}^n e^{iq_i\phi(0,x_i,y_i)}$ is invariant under the momentum symmetry if and only if it can be written as $\prod_{j=1}^m e^{ir_j\Delta_L\phi(0,x_j,y_j)}$, where $q_i, r_j \in \mathbb{Z}$ and $\Delta_L := \Delta_x^2 + \Delta_y^2$ is the discrete Laplacian operator. Of course the winding operator $e^{i\hat{\phi}}$ is invariant under the momen-
tum symmetry, and it is relevant because it acts nontrivially on the ground states, so the action (2.6) is not natural unless we impose the winding symmetry too.

2. The immobility of a finite set of defects with arbitrary charges, unless they can be “locally annihilated,” in the 2+1d $U(1)$ Laplacian gauge theory of Section 3.1.

The polynomial representation was originally developed in the context of translationally invariant Pauli stabilizer codes [71].

On an infinite square lattice $\mathbb{Z}^2$, any function $f$ can be associated with a formal Laurent power series in the variables $X,Y$:

$$\hat{f}(X,Y) = \sum_{(x,y) \in \mathbb{Z}^2} f(x,y) X^{-x} Y^{-y} .$$

We can think of $X = e^{ip_x}$ and $Y = e^{ip_y}$ as phases with $p_x$ and $p_y$ momenta conjugate to $x$ and $y$. Then, this definition of $\hat{f}(X,Y)$ is the discrete Fourier transform of $f(x,y)$. Related to that, $X$ and $Y$ are generators of lattice translations in the $x$ and $y$ directions:

$$X \hat{f}(X,Y) = \sum_{(x,y) \in \mathbb{Z}^2} f(x+1,y) X^{-x} Y^{-y} ,$$

and similarly for $Y$. Then, the difference operator $\Delta_x$ is associated with $X - 1$ because

$$(X - 1) \hat{f}(X,Y) = \sum_{(x,y) \in \mathbb{Z}^2} \Delta_x f(x + \frac{1}{2}, y) X^{-x} Y^{-y} .$$

Recall that $\Delta_x f(x + \frac{1}{2}, y) = f(x + 1, y) - f(x, y)$.

More generally, any local difference operator is associated with a Laurent polynomial $s(X,Y)$ with integer coefficients, i.e., an element of $\mathbb{Z}[X,X^{-1},Y,Y^{-1}]$ satisfying $s(1,1) = 0$. (Here, $\mathbb{Z}[X,Y,\ldots]$ is the set of polynomials in $X,Y,\ldots$ with integer coefficients, and therefore $\mathbb{Z}[X,X^{-1},Y,Y^{-1},\ldots]$ is the set of Laurent polynomials in $X,Y,\ldots$ with integer coefficients.)

For example, the discrete Laplacian operator $\Delta_L := \Delta_x^2 + \Delta_y^2$ corresponds to the Laurent polynomial

$$p(X,Y) = (X - 2 + X^{-1}) + (Y - 2 + Y^{-1}) .$$

We can equivalently work with

$$\tilde{p}(X,Y) = XY p(X,Y) = Y(X-1)^2 + X(Y-1)^2 ,$$

which is simply a translated version of $\Delta_L$. Note that $\tilde{p}(X,Y) \in \mathbb{Z}[X,Y]$, i.e., $\tilde{p}(X,Y)$ is a polynomial, whereas $p(X,Y)$ is a Laurent polynomial. Indeed, we can always translate a
difference operator so that the associated Laurent polynomial is a polynomial.\footnote{In the continuum, a differential operator in space becomes in momentum space a multiplication by a polynomial in the momenta. On the lattice, we follow the interpretation of $X$ and $Y$ as lattice translation generators, i.e., $X = e^{ip_x}$, $Y = e^{ip_y}$, and then a difference operator becomes a polynomial in $X$ and $Y$.}

Let us define a \textit{lexicographic monomial order}, $X \succ Y$, on $\mathbb{Z}[X,Y]$. This means we can compare any two monomials as follows: $X^m Y^n \succ X^k Y^l$ if $m > k$, or $m = k$ and $n > l$. Clearly, this is a total order on all monomials in $X,Y$. Given a nonzero polynomial, its \textit{leading term} is the term with the largest monomial among all its terms. The corresponding coefficient and monomial are called \textit{leading coefficient} and \textit{leading monomial} respectively. If the leading coefficient is $\pm 1$, the polynomial is said to be \textit{monic}.

We say a polynomial $a(X,Y)$ is \textit{reducible} by another polynomial $b(X,Y)$ if some term of $a(X,Y)$ is a multiple of the leading term of $b(X,Y)$. Furthermore, if $b(X,Y)$ is monic, then $a(X,Y)$ can be written uniquely as

$$a(X,Y) = c(X,Y)b(X,Y) + d(X,Y), \quad \text{(B.6)}$$

where $c(X,Y)$ is the \textit{quotient} and $d(X,Y)$ is the \textit{remainder}, which are uniquely determined by demanding that $d(X,Y)$ is not reducible by $b(X,Y)$. This operation is known as \textit{multivariate division} with respect to a given monomial order.

\textbf{B.1 Naturalness of 2+1d Laplacian $\phi$-theory}

In this appendix, we show that the action (2.6) is natural with respect to the global symmetry of the 2+1d Laplacian $\phi$-theory.

Usually, the notion of naturalness assumes that a certain global symmetry is imposed on the system and then all the relevant operators in the action respect this symmetry \cite{7}. (See \cite{39} for a more recent discussion comparing the notions of naturalness and robustness.) For this to make sense, we need some scaling property, which determines which terms in the action should be viewed as relevant, and which terms should be viewed as irrelevant. In the lattice system, without a continuum limit, there is no such obvious scaling. Instead, we show that every term that respects the symmetry can be expressed in terms of lattice derivatives acting on other terms that are already present in the action. More precisely, we will show that every term invariant under the momentum symmetry can be expressed in terms of gauge invariant functions of $\Delta L \phi$ and lattice derivatives of them. Then, we will exclude more terms using the winding symmetry. See more details below.

In the continuum, the conclusion of this appendix is the following trivial statement. A differential operator $D$ that annihilates every real harmonic function on $\mathbb{R}^2$, $f(x,y)$ includes...
the Laplacian as a factor. To see that, use holomorphic coordinates and write \( f = g(z) + \bar{g}(\bar{z}) \). Then, \( Df = 0 \) means that \( D \) must include a factor of \( \partial_z \) and using the reality, it should also have a factor of \( \partial_{\bar{z}} \). Therefore, \( D = D'\partial_z \partial_{\bar{z}} \) with a differential operator \( D' \).

The momentum symmetry of the 2+1d Laplacian \( \phi \)-theory on the square lattice includes shifts by real-valued discrete harmonic functions \( f(x, y) \) on \( \mathbb{Z}^2 \) (see Section 2.1.2). We would like to find other terms invariant under this symmetry. We look for terms depending on \( D\phi \) with some difference operator \( D \). Let \( \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) \) be the set of all real-valued discrete harmonic functions. By definition, any \( f \in \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) \) satisfies

\[
\Delta_L f(x, y) = 0 \iff p(X, Y) \hat{f}(X, Y) = 0 . \tag{B.7}
\]

We would like to find the condition that the difference operator \( D \) should satisfy such that \( Df(x, y) = 0 \) for all \( f \in \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) \).20

One trivial possibility is \( D = D' \circ \Delta_L \) because \((D' \circ \Delta_L)f(x, y) = 0 \) for any operator \( D' \). This means, we could add to the action (2.6) a term of the form

\[
-\sum_{\text{site}} \cos[(D' \circ \Delta_L)\phi] , \tag{B.8}
\]

and preserve the global symmetry. This is considered a higher-order term than \( \Delta_L \) and is always compatible with the momentum global symmetry.

A more interesting possibility would be a \( D \) that cannot be written as \( D' \circ \Delta_L \), and yet \( Df(x, y) = 0 \) for all \( f \in \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) \). Below, we will show that this is impossible. Equivalently, we show that any \( D \) that satisfies \( Df(x, y) = 0 \) for all \( f \in \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) \) is of higher order than \( \Delta_L \). This implies that the action (2.6) is natural with respect to the global momentum symmetry of the 2+1d Laplacian \( \phi \)-theory.

Let us rephrase the above problem in terms of polynomials. Let \( q(X, Y) \) be the Laurent polynomial associated with \( D \). By an appropriate translation, we can write \( X^m Y^n q(X, Y) = \tilde{q}(X, Y) \), where \( \tilde{q}(X, Y) \) is a polynomial. If there is a polynomial \( \tilde{r}(X, Y) \) such that \( \tilde{q}(X, Y) = \tilde{r}(X, Y)\tilde{p}(X, Y) \), then \( D \) is of higher order than \( \Delta_L \), i.e., \( D = D' \circ \Delta_L \), where \( D' \) is the operator associated with \( X^a Y^b \tilde{r}(X, Y) \) for some \( a, b \in \mathbb{Z} \).

With the above preparations, the central result of this appendix can be stated in terms of polynomials as follows: suppose \( \tilde{q}(X, Y) \) is a polynomial such that

\[
\tilde{q}(X, Y) \hat{f}(X, Y) = 0 , \quad \forall f \in \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) , \tag{B.9}
\]

We should impose \( Df = 0 \) rather than the weaker condition \( Df(x, y) = 0 \) mod \( 2\pi \) because it should be true for \( cf(x, y) \) for any \( c \in \mathbb{R} \).
then \( \tilde{q}(X, Y) = \tilde{r}(X, Y)\tilde{p}(X, Y) \) for some \( \tilde{r}(X, Y) \in \mathbb{Z}[X, Y] \), where \( \tilde{p}(X, Y) \) is the polynomial (B.5) associated with the discrete Laplacian operator \( \Delta_L \).

More specifically, since \( \tilde{p}(X, Y) \) is monic with leading term \( X^2Y \), by multivariate division with respect to lexicographic order, \( \tilde{q}(X, Y) \) can be written uniquely as

\[
\tilde{q}(X, Y) = X^2\alpha(X) + X\beta(Y) + \gamma(Y) + \tilde{r}(X, Y)\tilde{p}(X, Y),
\]

where \( \alpha(X) \in \mathbb{Z}[X], \beta(Y), \gamma(Y) \in \mathbb{Z}[Y], \) and \( \tilde{r}(X, Y) \in \mathbb{Z}[X, Y] \). The above statement is then equivalent to showing that \( \alpha(X) = \beta(Y) = \gamma(Y) = 0 \) if (B.9) is obeyed, which means that \( D \) is of higher order than \( \Delta_L \).

We parameterize the polynomials as

\[
\alpha(X) = \sum_{i=0}^{u} a_i X^i, \quad \beta(Y) = \sum_{j=0}^{v} b_j Y^j, \quad \gamma(Y) = \sum_{k=0}^{w} c_k Y^k,
\]

with nonnegative integers \( u, v, \) and \( w \). Then, we apply (B.10) to a specific set of discrete harmonic functions parameterized by \( t \):

\[
f_t(x, y) \equiv X_t^x Y_t^y,
\]

\[
X_t \equiv -t \left( \frac{1 + t}{1 - t} \right), \quad Y_t \equiv t \left( \frac{1 - t}{1 + t} \right).
\]

(It is easy to check that \( f_t \in \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) \) for any \( t \neq 0, \pm 1 \), i.e., \( \tilde{p}(X, Y) f_t(X, Y) = 0 \).) Then, using (B.10) and (B.9), we get

\[
X_t^2\alpha(X_t) + X_t\beta(Y_t) + \gamma(Y_t) = 0.
\]

Next, we turn this into a polynomial in \( t \) by multiplying it by \( (1 - t)^{u+2}(1 + t)^\nu \) with \( \nu = \max(v - 1, w) \)

\[
\sum_{i=0}^{u} a_i \alpha_i(t; u, \nu) + \sum_{j=0}^{v} b_j \beta_j(t; u, \nu) + \sum_{k=0}^{w} c_k \gamma_k(t; u, \nu) = 0,
\]

\[\text{It is related to the discrete exponential function on the square lattice } \mathbb{Z}^2 \text{, which is the discrete analog the exponential function on the complex plane } \mathbb{C} \cong \mathbb{R}^2.\]
where

\[
\alpha_i(t; u, \nu) = (1 - t)^{u+2}(1 + t)^{v}X_t^{i+2} = (-t)^{i+2}(1 + t)^{v+i+2}(1 - t)^{u-i}, \\
\beta_j(t; u, \nu) = (1 - t)^{u+2}(1 + t)^{v}X_t^{j+2} = -t^{j+1}(1 + t)^{v-j+1}(1 - t)^{u+j+1}, \\
\gamma_k(t; u, \nu) = (1 - t)^{u+2}(1 + t)^{v}Y_t^{k+2} = t^k(1 + t)^{v-k}(1 - t)^{u+k+2}.
\] (B.15)

Since the equation (B.14) holds for any \( t \neq 0, \pm 1 \), the polynomial in (B.14) must vanish identically, even at \( t = 0, \pm 1 \). What can we say about the coefficients \( a_i \)'s, \( b_j \)'s, and \( c_k \)'s then?

For fixed \((u, v, w)\), we have a set \( P(u, v, w) \) of polynomials in \( t \), \( \{\alpha_i : i = 0, \ldots, u\} \cup \{\beta_j : j = 0, \ldots, v\} \cup \{\gamma_k : k = 0, \ldots, w\} \). We will show that these polynomials are linearly independent, and therefore, \( \alpha(X) = \beta(Y) = \gamma(Y) = 0 \) and \( \tilde{q}(X, Y) = \tilde{r}(X, Y) \tilde{p}(X, Y) \).

First, note that for \( v \leq v_0 \) and \( w \leq v_0 - 1 \), we have \( P(u, v_0, w) \subseteq P(u, v_0, v_0 - 1) \) and \( P(u, v, v_0 - 1) \subseteq P(u, v_0, v_0 - 1) \) because \( \max(v - 1, v_0 - 1) = \max(v_0 - 1, w) \). So it suffices to show that the polynomials in the set \( P(u, v_0, v_0 - 1) \) are linearly independent for all \( u \geq 0 \) and \( v_0 \geq 1 \). We proceed by induction:

- **Base case:** It is easy to check that the set \( P(0, 1, 0) \) is linearly independent, and hence \( P(0, 0, 0) \) is also linearly independent.

- **Induction step:** Assume that \( P(u, v_0, v_0 - 1) \) is linearly independent. Consider \( P(u + 1, v_0, v_0 - 1) \):

\[
\alpha_i(t; u + 1, v_0 - 1) = \begin{cases} 
(1 - t)\alpha_i(t; u, v_0 - 1), & \text{for } i = 0, \ldots, u, \\
(-t)^{u+3}(1 + t)^{v_0+u+2}, & \text{for } i = u + 1,
\end{cases} \\
\beta_j(t; u + 1, v_0 - 1) = (1 - t)\beta_j(t; u, v_0 - 1), & \text{for } j = 0, \ldots, v_0, \\
\gamma_k(t; u + 1, v_0 - 1) = (1 - t)\gamma_k(t; u, v_0 - 1), & \text{for } k = 0, \ldots, v_0 - 1.
\] (B.16)

The polynomials in the first, third, and fourth lines are linearly independent by the induction hypothesis. The second line is nonzero at \( t = 1 \), whereas the other three lines vanish at \( t = 1 \), so the second line is independent of the other polynomials. Thus, \( P(u + 1, v_0, v_0 - 1) \) is linearly independent.
Now consider $P(u, v_0 + 1, v_0)$:

$$
\alpha_i(t; u, v_0) = (1 + t)\alpha_i(t; u, v_0 - 1), \quad \text{for } i = 0, \ldots, u,
\beta_j(t; u, v_0) = \begin{cases} 
(1 + t)\beta_j(t; u, v_0 - 1), & \text{for } j = 0, \ldots, v_0, \\
-t^{v_0+2}(1-t)^{u+v_0+2}, & \text{for } j = v_0 + 1, 
\end{cases}
\gamma_k(t; u, v_0) = \begin{cases} 
(1 + t)\gamma_k(t; u, v_0 - 1), & \text{for } k = 0, \ldots, v_0 - 1, \\
t^{v_0}(1-t)^{u+v_0+2}, & \text{for } k = v_0, 
\end{cases}
$$

The polynomials in the first, second, and fourth lines are linearly independent by the induction hypothesis. Those in the third and fifth lines are linear independent of the other polynomials because they do not vanish at $t = -1$, and of each other because they have different degrees. Thus, $P(u, v_0 + 1, v_0)$ is linearly independent.

Therefore, $P(u, v, w)$ is linearly independent for any $(u, v, w)$. Since the polynomial in $t$ in (B.14) must vanish identically, it follows that $a_i = b_j = c_k = 0$, so $\alpha(X) = \beta(Y) = \gamma(Y) = 0$. Hence, $\tilde{q}(X,Y) = \tilde{r}(X,Y)\tilde{p}(X,Y)$, which is what we set out to show.

It follows that any difference operator $D$ (which is associated with the polynomial $\tilde{q}(X,Y)$) respecting the momentum global symmetry must be of higher order than $\Delta_L$, i.e., $D = D' \circ \Delta_L$.

Next, we impose also the winding symmetry. This excludes terms such as $\cos \tilde{\phi}$. Using an argument similar to the one above, it is easy to see that the model is also natural with respect to the winding symmetry. One way to see that is to first dualize the theory and apply the argument above with $\phi \leftrightarrow \tilde{\phi}$. We conclude that the action (2.6) is natural if we impose its entire global symmetry.

### B.2 Mobility of defects in 2+1d $U(1)$ Laplacian gauge theory

In this appendix, we prove the immobility of any finite set of defects with arbitrary charges (except in some trivial cases) in the 2+1d $U(1)$ Laplacian gauge theory on the infinite square lattice $\mathbb{Z}^2$.

Before proceeding, let us distinguish between two kinds of defects that capture the motion of a particle. Typically, when a particle can move between two points, there is an operator supported in a small region, e.g., a line joining the two points. However, there are also situations where the operator that moves the particle can have a support spanning $O(L_x)$ or $O(L_y)$ sites even though the two points are separated by a much smaller distance. (See [21] for a discussion and examples of both kinds of operators.) The existence of the latter kind
of operators depends on the number-theoretic properties of $L_i$, whereas the former kind of operators exist for all $L_i$. In particular, only the former make sense on the infinite square lattice.

Consider the defect
\[
\exp \left[ i \sum_{\tau} A_{\tau} (\tau + \frac{1}{2}, x, y) \right],
\]
which describes the worldline of a single particle with unit charge. Under the time-like symmetry that shifts
\[
A_{\tau}(\tau + \frac{1}{2}, x, y) \rightarrow A_{\tau}(\tau + \frac{1}{2}, x, y) + \delta_{\tau,0} \left( \frac{2\pi m_x x}{L_x} + \frac{2\pi m_y y}{L_y} \right), \quad m_x, m_y \in \mathbb{Z},
\]
the defect (B.18) acquires an $(x,y)$-dependent phase, so it cannot bend. In other words, the particle is completely immobile, i.e., it is a fracton.

Since the time-like symmetry in (B.19) is present also in the scalar charge theory, the same conclusion holds there. The selection rules from the time-like global symmetries give a more precise explanation of the intuitive “dipole moment conservation” discussed in [24–26]. (See [21] for a discussion.)

Next, consider the defect
\[
\exp \left( i \sum_{\tau} \left[ A_{\tau}(\tau + \frac{1}{2}, x_0, y_0) - A_{\tau}(\tau + \frac{1}{2}, x, y) \right] \right),
\]
which describes the worldline of a dipole of particles with charges $\pm 1$ with separation $(x_0, y_0)$. The shift (B.19) imposes the constraint that the defect cannot move unless the separation is held fixed. This is the only constraint in the scalar charge theory, and as long as it is met, the dipole can move. There are additional constraints in the Laplacian gauge theory. Indeed, under the time-like symmetry that shifts (for simplicity, we set $L_x = L_y = L$)
\[
A_{\tau}(\tau + \frac{1}{2}, x, y) \rightarrow A_{\tau}(\tau + \frac{1}{2}, x, y) + \delta_{\tau,0} \left[ 2\pi mxy \frac{L}{L} + 2\pi m'(x^2 - y^2 - Lx + Ly) \right],
\]
where $m, m' \in \mathbb{Z}$, the defect (B.20) acquires an $(x,y)$-dependent phase, so it cannot bend. In other words, the dipole is also completely immobile.

More generally, consider the defect
\[
\exp \left[ i \sum_{\tau} \sum_{i=1}^n q_i A_{\tau}(\tau + \frac{1}{2}, x_i, y_i) \right].
\]
which describes the world-lines of \( n \) particles labelled by \( i = 1, \ldots, n \), with positions \((x_i, y_i)\), and charges \( q_i \). It is difficult to analyze this case in full generality on a finite lattice, so we limit ourselves to an infinite square lattice.

Under the shift of \( A_\tau \) by a discrete harmonic function \( f \in \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) \) at a fixed time \( \tau = 0 \), the phase acquired by the defect is

\[
\exp \left[ i \sum_{i=1}^{n} q_i f(x_i, y_i) \right].
\]

The defect carries trivial time-like charges (i.e., it is in the trivial superselection sector) if and only if for all discrete harmonic functions \( f \in \mathcal{H}(\mathbb{Z}^2, \mathbb{R}) \)

\[
\sum_{i=1}^{n} q_i f(x_i, y_i) = 0 \iff q(X,Y)\hat{f}(X,Y) = 0.
\]

(Once again, we cannot impose the weaker condition \( \sum_{i=1}^{n} q_i f(x_i, y_i) = 0 \mod 2\pi \) because this equation should be true even for \( cf(x, y) \) for any \( c \in \mathbb{R} \).) As we showed in Section B.1 this is possible if and only if \( q(X,Y) = r(X,Y)p(X,Y) \) for some Laurent polynomial \( r(X,Y) \).

To see the physical meaning of being invariant under the time-like symmetry, assume that such an \( r(X,Y) \) exists. Then, we can construct the following defect

\[
\exp \left[ i \sum_{\tau < 0} \sum_{i=1}^{n} q_i A_\tau(\tau + \frac{1}{2}, x_i, y_i) \right] \times \exp \left[ -i \sum_{j=1}^{m} r_j A(0, x_j, y_j) \right].
\]

Here, \( r_j \) and \((x_j, y_j)\) are obtained from \( r(X,Y) = \sum_{j=1}^{m} r_j X^{x_j} Y^{y_j} \). This defect describes annihilation of the \( n \) particles at time \( \tau = 0 \). To see that this defect is gauge invariant, observe that, under a gauge transformation, the exponent transforms as

\[
\sum_{i=1}^{n} q_i \alpha(0, x_i, y_i) - \sum_{j=1}^{m} r_j \Delta L \alpha(0, x_j, y_j).
\]

This is the coefficient of \( X^0 Y^0 \) term in \([q(X,Y) - r(X,Y)p(X,Y)]\hat{\alpha}(X,Y)\), and so it vanishes\(^{22}\)

\(^{22}\)Here, \( \hat{\alpha}(X,Y) \) is the formal Laurent power series associated with the gauge parameter \( \alpha(x, y) \). It should not be confused with the gauge parameter \( \hat{\alpha}_i \) in Section 2.2.
In fact, we can write the defect (B.24) as
\[
\exp \left[ i \sum_{\tau < 0} \sum_{j=1}^{m} r_j \Delta L A_{\tau} \left( \tau + \frac{1}{2}, x_j, y_j \right) \right] \times \exp \left[ -i \sum_{j=1}^{m} r_j A(0, x_j, y_j) \right]
\]
\[= \prod_{j=1}^{m} \exp \left[ i r_j \sum_{\tau < 0} \Delta L A_{\tau} \left( \tau + \frac{1}{2}, x_j, y_j \right) - i r_j A(0, x_j, y_j) \right]. \tag{B.26}
\]

Each factor here describes particles being annihilated “locally” because the operator that annihilates them is local.

The result in (B.26) can be understood intuitively as follows. In this special case, the collection of defects coming from the past (B.22) can be expressed as a “total spatial derivative” using the Laplacian as in the first factor in (B.26). In this form, each term with the Laplacian can end using the local operator made out of \( A \) in the second factor in (B.26). The result, in this case, is that the collection of defects can be annihilated by an operator at time \( \tau = 0 \).

Let us now examine the mobility of the \( n \) particles described by the defect (B.22). We stress that we consider mobility only under the restriction that the charges of the particles and the separations between them remain fixed. (Relaxing these two restrictions can lead to more possibilities, which we will not discuss here.) Then, we say that the \( n \) particles can move by \((x_0, y_0) \neq (0, 0)\) if there is a defect of the form
\[
\exp \left[ i \sum_{\tau < 0} \sum_{i=1}^{n} q_i A_{\tau} \left( \tau + \frac{1}{2}, x_i, y_i \right) \right] \times \exp \left[ i \sum_{k=1}^{l} s_k A(0, x_k, y_k) \right]
\]
\[\times \exp \left[ i \sum_{\tau \geq 0} \sum_{i=1}^{n} q_i A_{\tau} \left( \tau + \frac{1}{2}, x_i + x_0, y_i + y_0 \right) \right]. \tag{B.27}
\]

This defect exists, i.e., it is gauge invariant, if and only if
\[
(X^{x_0}Y^{y_0} - 1)q(X, Y) = s(X, Y)p(X, Y), \tag{B.28}
\]
where \( s(X, Y) = \sum_{k=1}^{l} s_k X^{x_k}Y^{y_k} \). Equivalently, this is precisely the condition for which the time-like charges of the \( n \) particles remain unchanged after displacing them by \((x_0, y_0)\).

If \( q(X, Y) \) is a multiple of \( p(X, Y) \), i.e., \( q(X, Y) = r(X, Y)p(X, Y) \) for some Laurent

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This is analogous to the following very well known fact in standard \( U(1) \) gauge theories. A dipole of particles with charges \( \pm 1 \) is represented by the defect \( \exp \left[ i \int d\tau [A_{\tau}(\tau, x) - A_{\tau}(\tau, 0)] \right] = \exp \left[ i \int d\tau \int_0^\tau dx' \partial_x A_{\tau}(\tau, x') \right] \). It can end at \( \tau = 0 \), as described by \( \exp \left[ i \int_0^\infty d\tau [A_{\tau}(\tau, x) - A_{\tau}(\tau, 0)] \right] \times \exp \left[ -i \int_0^\infty dx' A_{\tau}(0, x') \right] \).
polynomial $r(X,Y)$, then we can always choose $s(X,Y) = r(X,Y)(X^{x_0}Y^{y_0} - 1)$ so that (B.28) is satisfied. However, this is not an interesting situation because, when $q(X,Y) = r(X,Y)p(X,Y)$, the defect (B.22) has trivial time-like charges as explained around (B.23). Consequently, similar to the discussion around (B.26), this situation can be interpreted as “locally annihilating” the particles and then “locally creating” them elsewhere. For example, when $r(X,Y) = 1$, the defect (B.27) is

$$\exp \left[ i \sum_{\tau < 0} \Delta_L A_{\tau}(\tau + \frac{1}{2}, 0, 0) \right] \times \exp \left[ iA(0, x_0, y_0) - iA(0, 0, 0) \right]$$

\times \exp \left[ i \sum_{\tau \geq 0} \Delta_L A_{\tau}(\tau + \frac{1}{2}, x_0, y_0) \right],

(B.29)

where the operator $e^{-iA(0,0,0)}$ annihilates the particles around $(0,0)$ and then the operator $e^{iA(0,x_0,y_0)}$ creates them around $(x_0,y_0)$. For more general $r(X,Y)$, the defect (B.27) is a product of defects of the form (B.29).

Can we have a defect like (B.27) when $q(X,Y)$ is not a multiple of $p(X,Y)$? Imposing (B.28), we see that this can happen if and only if $X^{x_0}Y^{y_0} - 1$ shares a nontrivial factor with $p(X,Y)$. Let us show that the latter cannot happen.

First, it is easy to check that $p(X,Y)$ is monic, non-constant, and irreducible up to a monomial.\(^2\) Let $d = \gcd(x_0,y_0)$, which is well defined because $(x_0,y_0) \neq (0,0)$. We can write

$$X^{x_0}Y^{y_0} - 1 = (X^{x_0'}Y^{y_0'})^d - 1 = (X^{x_0'}Y^{y_0'} - 1)t(X,Y),$$

(B.30)

where $x_0' = x_0/d$, $y_0' = y_0/d$, and $t(X,Y) = \sum_{c=0}^{d-1}(X^{x_0'}Y^{y_0'})^c$ is a Laurent polynomial with $t(1,1) = d \neq 0$. The last condition implies that $p(X,Y)$ cannot share a nontrivial factor with $t(X,Y)$. Since $\gcd(x_0',y_0') = 1$, the factor $X^{x_0'}Y^{y_0'} - 1$ is monic, non-constant, and irreducible up to a monomial\(^7\). So, $p(X,Y)$ cannot share a nontrivial factor with $X^{x_0'}Y^{y_0'} - 1$ as well. Therefore, $p(X,Y)$ does not share a nontrivial factor with $X^{x_0}Y^{y_0} - 1$.

To summarize, a finite set of charged particles cannot move in the 2+1d $U(1)$ Laplacian gauge theory on a square lattice unless they are in the trivial superselection sector, i.e., they can be “annihilated locally.” We remind the reader that when we say “move”, the particles retain their charges and move in a rigid way.

\(^2\)A polynomial is said to be \textit{irreducible} if it cannot be written as product of two polynomials, neither of which is $\pm 1$. We say a Laurent polynomial $g(X,Y)$ is \textit{irreducible up to a monomial} if $X^ag^b g(X,Y)$ is an irreducible polynomial for some $a, b \in \mathbb{Z}$. For example, $p(X,Y) = XYp(X,Y)$ is an irreducible polynomial, so $p(X,Y)$ is irreducible up to a monomial.
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