Diffraction of Cherenkov Radiation at the Open End of a Shallow Corrugated Waveguide

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A problem of diffraction of a symmetrical transverse magnetic mode $TM_0$ by an open-ended corrugated cylindrical waveguide is considered. A depth and a period of corrugations are supposed to be much less than the wavelength and the waveguide radius. Therefore a corrugated waveguide wall can be described in terms of equivalent boundary conditions, i.e. a corresponding impedance boundary condition can be applied. The diffraction problem is solved using the modified tayloring technique in Jones formulation. Solution of the Wiener-Hopf-Fock equation of the problem is used to obtain an infinite linear system for reflection coefficients, the latter is solved numerically using the reduction technique.

I. INTRODUCTION

In recent years, the different types of corrugated structures have proved themselves as a promising terahertz devices [1–3]. One of the proposed radiation mechanism is the Smith-Purcell radiation [4–6]. The alternative scheme is the use of small corrugations when a generated wavelength is sufficiently larger than the corrugation parameters [7–9]. Wakefields generated by bunches moving through an infinite metallic waveguide with small corrugations were also analyzed [10, 11].

However, for practice it is important to investigate the diffraction processes occuring when the generated wakefield incidents an open aperture of the waveguide and exits the open space. This problem is much more complicated (compared to the problem of wakefield calculation) and is of fundamental importance. It is worth noting that mentioned problem for the case of an open-ended waveguide with smooth perfectly conducting walls can be solved rigorously by a number of methods, with the solution having the closed form [12–13]. But for applications to radiation sources this case is marginally suitable because no wakefield is generated. For wakefield excitation, certain slow-wave structure should be added into a waveguide: wall corrugation and dielectric layer are most typical examples. However, such a modification complicates the procedure of rigorous solution significantly. For example, it is know that this solution can not be obtained in the closed form [14]. Therefore, problems with open-ended waveguides are typically solved using some approximate techniques [15–16]. To estimate the accuracy of these methods it is extremely useful to have a rigorous solution for similar problems.

One possible way here is to solve the corresponding “embedded” structure first (see Refs. [17, 18] and then perform the limiting procedure [13], but this way is rather cumbersome. An elegant method, called the generalized tayloring technique, has been proposed several decades ago for parallel-plate waveguides with dielectric filling [19]. Recently, a problem of $TM_{0l}$ mode radiation from an open-ended circular waveguide with uniform dielectric filling has been solved using the generalization of this approach to the cylindrical geometry. In the present paper, we also utilize this approach to describe the transformation of $TM_{0l}$ mode at the open end of a waveguide with shallow corrugation, both inside and outside.

Note that the mode under consideration is directly related to the Cherenkov radiation generated in the shallow corrugated waveguide. In conformity with the work [10], the wave field of a bunch moving along the corrugated waveguide axis is a single-frequency wakefield. Besides, the connection between the electromagnetic field of a transverse magnetic eigenmode and the wave field is shown in [11]. It follows from the paper [11] that $H_W^\phi = A \cdot \text{Re} \left( H_{\phi \omega} \right)$, where $H_W^\phi$ is the magnetic component of the wave field, $A$ is some amplitude constant, $H_{\phi \omega}$ is the harmonic magnetic component of the $TM_{0l}$ diffracted mode, $\omega$ means the wave field frequency. Thus, the same structure of the Cherenkov radiation field and $TM_{0l}$ mode field can be achived by appropriate choosing the mode frequency and amplitude.

The term “shallow corrugations” means that the considered wavelength and waveguide radius are much larger than the corrugation periods. The solution is based on using the equivalent boundary conditions (EBC) [20–21]. The EBC means the substitution of the corrugated surface by the smooth one with impedance boundary condition. This approach has been successfully used for the investigation of the wavefield generated by the bunches moving through corrugated waveguides [10, 11, 22]. Next, to apply the aforementioned generalized tayloring technique, the reflected field in the waveguide is decomposed into a series of corrugated waveguide eigenmodes. The field in the external area (free space) is presented, in turn, by Fourier-type integral transforms, and corresponding boundary conditions are applied to these functions (the so-called Jones formulation). Solution of the Wiener-Hopf-Fock equation is obtained using the factorization method [12, 14] and then utilized to construct an infinite linear system for reflection coefficients of waveguide modes.

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II. GENERAL EXPRESSIONS

A. Field components

Geometry of the problem is shown in Fig. 1. The harmonic exp(−iωt) axially symmetrical transverse magnetic mode TM_{0l} falling on the open end of the corrugated cylindrical waveguide. We assume that waveguide walls are perfectly conductive and the following conditions are fulfilled

\[ d \ll a, \quad d_3 \ll a, \quad d \ll \lambda, \quad d_3 \ll \lambda, \]  

(1)

where \( \lambda \) is the mode wavelength, \( a \) is the waveguide radius, \( d \) and \( d_3 \) mean corrugation period and depth respectively.

Conditions (1) allow us to replace complicated boundary conditions on the corrugated walls with equivalent boundary conditions on the smooth surface [23].

\[ E_{2\omega m}\bigg|_{r=a} = \eta_m H_{\varphi \omega m}\bigg|_{r=a}, \]  

(2)

where \( m \) is the mode number, \( \eta_m \) means the impedance. Impedance is determined by the waveguide and mode characteristics

\[ \eta_m(k_{zm}) = \frac{i \omega}{c} \left( \frac{d_2 d_3}{d} - \delta \frac{c^2 k_{zm}^2}{\omega^2} \right). \]  

(3)

Here \( c \) is the speed of light in free space, \( \omega \) is the mode frequency, \( k_{zm} \) is the longitudinal wavenumber and \( \delta \) parameters has the form

\[ \delta = \frac{d_3 + \frac{td}{2\pi}}{2\pi} \int_0^{1/\sigma} \frac{du}{\sqrt{(1 - u)(1 - \sigma u)}} \left( \sqrt{1 - tu + 1} \right) + \frac{d}{2\pi} \ln \left( \frac{\sigma - 1}{\sigma} \right). \]  

(4)

Parameters \( t \) and \( \sigma \) should be found from the following equations:

\[ \int_0^t \frac{\sqrt{-u} du}{\sqrt{u(1-u)(\sigma - u)}} = \frac{d_1}{d}, \]  

\[ \int_1^t \frac{\sqrt{u - t} du}{\sqrt{u(1-u)(\sigma - u)}} = 2\pi \frac{d_3}{d}. \]  

(5)

The mode structure of corrugated waveguide is considered in [11] in detail. The incident field components can be written in form

\[ H_{r \varphi}^{(1)} = J_1(\chi r) \exp(ik_{z1} z), \]

\[ E_{r \varphi}^{(1)} = \frac{c}{\omega} k_{z1} J_1(\chi r) \exp(ik_{z1} z), \]

\[ E_{z \varphi}^{(1)} = \frac{i c}{\omega} \chi J_0(\chi r) \exp(ik_{z1} z), \]  

(6)

where \( J_{0,1}(\chi r) \) are Bessel functions, the longitudinal wavenumber is equal to

\[ k_{z1} = \sqrt{\omega^2/c^2 - \chi_1^2}, \quad \text{Im}(k_{z1}) > 0, \]  

(7)

the transverse wavenumbers \( \chi_1 \) are determined by the dispersion equation

\[ \frac{\omega}{c} \eta_1 J_1(\chi_1 a) - i \chi_1 J_0(\chi_1 a) = 0. \]  

(8)

We use the following approach to solving the problem. The reflected field is represented as a series of corrugated waveguide eigenmodes

\[ H_{r \varphi}^{(r)} = \sum_{m=1}^\infty R_m J_1(\chi_m r) \exp(-ik_{zm} z), \]

\[ E_{r \varphi}^{(r)} = -\frac{c}{\omega} \sum_{m=1}^\infty R_m k_{zm} J_1(\chi_m r) \exp(-ik_{zm} z), \]

\[ E_{z \varphi}^{(r)} = \frac{i c}{\omega} \sum_{m=1}^\infty R_m \chi_m J_0(\chi_m r) \exp(-ik_{zm} z), \]  

(9)

Mode excitation coefficients \( R_m \) are unknown. Note that expressions (6) and (9) are written as a solution of equation for magnetic field component \( H_{\varphi \omega} \)

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) H_{\varphi \omega} = 0. \]  

(10)

In order to specify the field in the area external to the waveguide we divide area \( r > a \) into two parts denoted as “1” and “2” (see Fig. 2). The unknown fields in areas “1” and “2” can be written using the following integral transform:

\[ \Psi_{-(1,2)}(r, \alpha) = \frac{1}{2\pi} \int_0^{\infty} H_{r \varphi}^{(2)}(r, z) \exp(i az) \, dz, \]  

(11)

\[ \Psi_{+(1,2)}(r, \alpha) = \frac{1}{2\pi} \int_0^{\infty} H_{r \varphi}^{(1,2)}(r, z) \exp(i az) \, dz. \]  

(12)
The result of the transform (11) or (12) is the function regular in areas Im \( \alpha < 0 \) and Im \( \alpha > 0 \) respectively [14]. Note that function \( \Psi_{\pm}^{(1)}(r, \alpha) \) is not defined since the area “1” is determined only for \( z > 0 \).

**B. Equation for functions \( \Psi_{\pm}^{(2)}(r, \alpha) \)**

Let consider the area “2” and apply the integral transform (11)-[12] to the equation for the magnetic field component \([10]\)

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\omega^2}{c^2} \right) H_{r}\omega e^{i\alpha z} dz + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\partial^2 H_{r}\omega}{\partial z^2} e^{i\alpha z} dz = 0. \tag{13}
\]

By virtue of (11)-[12] the first integral has the simple meaning

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\omega^2}{c^2} \right) H_{r}\omega e^{i\alpha z} dz = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\omega^2}{c^2} \right) (\Psi_{+}^{(2)} + \Psi_{-}^{(2)}). \tag{14}
\]

Integrating the second term in (14) by parts twice we obtain the following expression

\[
\int_{-\infty}^{+\infty} \frac{\partial^2 H_{r}\omega}{\partial z^2} e^{i\alpha z} dz = \left[ \frac{\partial H_{r}\omega}{\partial z} e^{i\alpha z} - \frac{i\alpha H_{r}\omega}{c} e^{i\alpha z} \right]_0^0 - \left[ \frac{\partial H_{r}\omega}{\partial z} e^{i\alpha z} - \frac{i\alpha H_{r}\omega}{c} e^{i\alpha z} \right]_0^0 + \left[ \frac{\partial H_{r}\omega}{\partial z} e^{i\alpha z} - \frac{i\alpha H_{r}\omega}{c} e^{i\alpha z} \right]_0^0. \tag{15}
\]

It is assumed that \( |H_{r}\omega^{(2)}| \to 0 \) and \( \left| \partial H_{r}\omega^{(2)}/\partial z \right| \to 0 \) at \( |z| \to \infty \). Furthermore, area “2” doesn’t contain any boundary at \( z = 0 \) so functions \( H_{r}\omega^{(2)} \) and \( \partial H_{r}\omega^{(2)}/\partial z \) are continuous at \( z = 0 \). Thus, the second integral in (14) is equal to

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\partial^2 H_{r}\omega}{\partial z^2} e^{i\alpha z} dz = -\alpha^2 (\Psi_{-}^{(2)} + \Psi_{+}^{(2)}). \tag{16}
\] Combining expressions (16) and (14) we finally obtain the differential equation for function \( \Psi_{\pm}^{(2)}(r, \alpha) \)

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\kappa^2}{c^2} \right) \Psi_{\pm}^{(2)}(r, \alpha) = 0, \tag{17}
\]

where \( \kappa^2(\alpha) = \omega^2/c^2 - \alpha^2 \). The equation (17) can be easily reduced to the first order Bessel equation and its solution in area \( r > a \) is well-known

\[
\Psi_{\pm}^{(2)}(r, \alpha) = C_2 H_1^{(1)}(\kappa r). \tag{18}
\]

Coefficient \( C_2 \) will be defined later using the corresponding continuity conditions.

**C. Equation for function \( \Psi_{\pm}^{(1)}(r, \alpha) \)**

Now let consider the area “1” and apply the integral transform (12) to the equation for the magnetic field component \([10]\)

\[
\frac{1}{2\pi} \int_{0}^{+\infty} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\omega^2}{c^2} \right) H_{r}\omega e^{i\alpha z} dz + \frac{1}{2\pi} \int_{0}^{+\infty} \frac{\partial^2 H_{r}\omega}{\partial z^2} e^{i\alpha z} dz = 0. \tag{19}
\]

Performing calculations similar to those for function \( \Psi_{+}^{(2)}(r, \alpha) \) we obtain

\[
\frac{1}{2\pi} \int_{0}^{+\infty} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\omega^2}{c^2} \right) H_{r}\omega e^{i\alpha z} dz = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\omega^2}{c^2} \right) \Psi_{+}^{(1)}, \tag{20}
\]

\[
\frac{1}{2\pi} \int_{0}^{+\infty} \frac{\partial^2 H_{r}\omega^{(1)}}{\partial z^2} e^{i\alpha z} dz = -\alpha^2 \Psi_{+}^{(1)} + \frac{1}{2\pi} \left( i\alpha H_{r}\omega^{(1)} \right) \bigg|_{z=0}^{0} - \frac{\partial H_{r}\omega^{(1)}}{\partial z} \bigg|_{z=0}^{0}. \tag{21}
\]
As a result, the following inhomogeneous differential equation can be written for function $\Psi_+^{(1)}$

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \kappa^2 \right) \Psi_+^{(1)} = F^{(1)},
\]

\[
F^{(1)} = \frac{1}{2\pi} \left( \frac{\partial H_{\phi\omega}^{(1)}}{\partial z} \right)_{z=+0} - i\alpha H_{\phi\omega}^{(1)} \bigg|_{z=+0}. \tag{23}
\]

Explicit form of function $F^{(1)}$ can be found using the continuity condition for field components $H_{\phi\omega}$ and $E_{r\omega}$ at surface $z = 0$, $r < a$. The continuity of function $H_{\phi\omega}$ means

\[
(H_{\phi\omega}^{(i)} + H_{\phi\omega}^{(r)}) \bigg|_{z=0} = H_{\phi\omega}^{(1)} \bigg|_{z=+0}. \tag{24}
\]

The substitution of the incident and reflected field by expression (6) and (9) leads to the following equality

\[
H_{\phi\omega}^{(1)} \bigg|_{z=+0} = J_1(\chi r) + \sum_{m=1}^{\infty} R_m J_1(\chi_m r). \tag{25}
\]

In order to use the continuity of function $E_{r\omega}$ we rewrite this component in accordance with Maxwell equations

\[
E_{r\omega} = -\frac{i\kappa}{\omega} \frac{\partial (H_{\phi\omega})}{\partial z}. \tag{26}
\]

Thus, the continuity of function $E_{r\omega}$ leads to the following:

\[
\left( \frac{\partial H_{\phi\omega}^{(i)}}{\partial z} + \frac{\partial H_{\phi\omega}^{(r)}}{\partial z} \right) \bigg|_{z=-0} = \frac{\partial H_{\phi\omega}^{(1)}}{\partial z} \bigg|_{z=+0}. \tag{27}
\]

Using (6)-(9), one can obtain

\[
\frac{\partial H_{\phi\omega}^{(1)}}{\partial z} \bigg|_{z=+0} = ik_z J_1(\chi r) - i \sum_{m=1}^{\infty} R_m k_{zm} J_1(\chi_m r). \tag{28}
\]

The final form of function $F^{(1)}$ is obtained by substitution of (25) and (28) in (29)

\[
F^{(1)}(r, \alpha) = i \frac{R_m k_{zm} + \alpha}{2\pi} J_1(\chi_m r) - \frac{i}{2\pi} \sum_{m=1}^{\infty} R_m(k_{zm} + \alpha) J_1(\chi_m r). \tag{30}
\]

Based on the form of the equation (22) right part, we present the particular solution as a series of Bessel functions

\[
\Psi_+^{(1)}(r, \alpha) = A J_1(\chi r) + \sum_{m=1}^{\infty} B_m J_1(\chi_m r). \tag{31}
\]

Substituting this form in (22), one can obtain

\[
\left( \kappa^2 - \chi_1^2 \right) A J_1(\chi r) + \sum_{m=1}^{\infty} B_m(k_{zm}^2 - \chi_m^2) J_1(\chi_m r) = \frac{i}{2\pi}(k_{zl} - \alpha) J_1(\chi r) - \frac{i}{2\pi} \sum_{m=1}^{\infty} R_m(k_{zm} + \alpha) J_1(\chi_m r). \tag{32}
\]

To determine coefficients $A, \{B_m\}$ we multiply the equation (32) by function $r J_1(\chi_m r)$ (the bar means the complex conjugation) and integrate over the radial variable $r$. Then the orthogonal property of Bessel functions (11) is used

\[
\int_0^a f(r)J_1(\chi_m r)r dr = \delta_{np} \left[ \frac{a^2}{2} \left( J_1^2(\chi_n a) + J_0^2(\chi_n a) \right) - \frac{\alpha}{\chi_n} J_0(\chi_n a) J_1(\chi_n a) \right]. \tag{33}
\]

where $\delta_{np}$ means the Kronecker symbol. These mathematical transformations lead to the following result

\[
A = \frac{i}{2\pi} \frac{k_{zl} - \alpha}{k_{zl}^2 - \alpha^2} = \frac{i}{2\pi} \frac{1}{k_{zl} + \alpha},
\]

\[
B_m = -\frac{i}{2\pi} \frac{R_m k_{zm} + \alpha}{k_{zm}^2 - \alpha^2} = -\frac{i}{2\pi} \frac{R_m}{k_{zm} - \alpha}, \tag{34}
\]

where $k_{zm}$ is defined by (7).

So, the solution of equation (22) has the form

\[
\Psi_+^{(1)}(r, \alpha) = C_1 J_1(\kappa r) + \frac{i}{2\pi} \frac{J_1(\chi r)}{k_{zl} + \alpha} - \frac{i}{2\pi} \sum_{m=1}^{\infty} R_m \frac{J_1(\chi_m r)}{k_{zm} - \alpha}. \tag{35}
\]

III. COEFFICIENT $C_1$

Introduce a new function

\[
\Phi_+^{(1)}(r, \alpha) = \frac{1}{2\pi} \int_0^{+\infty} \left[ E_{z\omega}^{(1)}(r, z) - \eta(\alpha) H_{\phi\omega}^{(1)}(r, z) \right] \exp(i\alpha z) dz, \tag{36}
\]

\[
\Phi^{(2)}(r, \alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ E_{z\omega}^{(2)}(r, z) - \eta(\alpha) H_{\phi\omega}^{(2)}(r, z) \right] \exp(i\alpha z) dz. \tag{37}
\]
where coefficient $\eta(\alpha)$ is defined by analogy of the impedance \( (3) \)
\[
\eta(\alpha) = \frac{i \omega}{c} \left( \frac{d_2 d_3}{d} - \frac{c^2 \alpha^2}{\omega^2} \right) \tag{38}
\]
and $E_{2\omega}$ can be expressed through the $H_{\phi \omega}$
\[
E_{2\omega} = \frac{i c}{\omega r} \frac{\partial (r H_{\phi \omega})}{\partial r}. \tag{39}
\]

Note that continuity of functions $E_{2\omega}$ and $H_{\phi \omega}$ at $r = a$, $z > 0$ leads to the equality \( (4) \)
\[
\Phi_\pm^{(1)}(a, \alpha) = \Phi_\pm^{(2)}(a, \alpha). \tag{40}
\]
Direct calculations with using \( (45) \) result in the following:
\[
\Phi_\pm^{(1)}(r, \alpha) = C_1 \left( \frac{i k_0}{k_0} J_0(\kappa r) - \eta(\alpha) J_1(\kappa r) \right) + \frac{i}{2\pi} \left[ \frac{i k_0}{k_0} J_0(\chi r) - \eta(\alpha) J_1(\chi r) \right] \tag{41}
\]
\[
- \sum_{m=1}^{\infty} R_m \frac{i k_0}{k_0} J_0(\chi_m r) - \eta(\alpha) J_1(\chi_m r) \right] k_{zm} - \alpha
\]
where $k_0 = \omega/c$. Using the function $\Psi^{(2)}(r, \alpha)$ \( (18) \) it is also obtained
\[
\Phi_\pm^{(2)}(r, \alpha) = C_2 \left( \frac{i k_0}{k_0} H_0^{(1)}(\kappa r) - \eta(\alpha) H_1^{(1)}(\kappa r) \right) \tag{42}
\]
Assuming $r = a$ in expression \( (40) \) one can obtain
\[
\Phi_\pm^{(1)}(a, \alpha) = \Phi_\pm^{(2)}(a, \alpha) = C_1 D_f(\alpha) + \frac{i}{2\pi} \left[ \frac{i k_0}{k_0} J_0(\chi a) - \eta(\alpha) J_1(\chi a) \right] \tag{43}
\]
\[
- \sum_{m=1}^{\infty} R_m \frac{i k_0}{k_0} J_0(\chi_m a) - \eta(\alpha) J_1(\chi_m a) \right] k_{zm} - \alpha
\]
where
\[
D_f(\alpha) = \frac{i k_0}{k_0} J_0(\kappa a) - \eta(\alpha) J_1(\kappa a) \tag{44}
\]
Assuming $r = a$ in expression \( (40) \) one can obtain
\[
\Phi_\pm^{(1)}(a, \alpha) = \Phi_\pm^{(2)}(a, \alpha) = C_1 D_f(\alpha)
\]
\[
+ \frac{i}{2\pi} \left[ \frac{i k_0}{k_0} J_0(\kappa a) - \eta(\alpha) J_1(\kappa a) \right] \tag{45}
\]
\[
- \sum_{m=1}^{\infty} R_m \frac{i k_0}{k_0} J_0(\chi_m a) - \eta(\alpha) J_1(\chi_m a) \right] k_{zm} - \alpha
\]
where
\[
D_f(\alpha) = \frac{i k_0}{k_0} J_0(\kappa a) - \eta(\alpha) J_1(\kappa a). \tag{46}
\]
Taking into account \( (44) \) one can obtain from \( (45) \) :\[
\Psi_\pm^{(1)}(a, \alpha) = \Psi_\pm^{(2)}(a, \alpha)
\]
\[
\frac{J_0(\kappa a)}{D_f(\alpha)} + \frac{i}{2\pi} \frac{J_1(\kappa a)}{D_f(\alpha)} - \frac{1}{2\pi} \frac{J_0(\chi a)}{D_f(\alpha)} - \eta(\alpha) J_1(\chi a) \tag{47}
\]
\[
- \frac{i}{2\pi} \sum_{m=1}^{\infty} \frac{R_m}{k_m - \alpha} \left[ \frac{J_0(\chi_m a)}{D_f(\alpha)} - \eta(\alpha) J_1(\chi_m a) \right]
\]
Since the equality $\kappa(k_{zp}) = \chi_p, p = 1, 2, \ldots$ is complied, we conclude according to \( (8) \) that $D_f(\pm k_{zp}) = 0$. Expanding the function $D_f(\alpha)$ in a Taylor series near $\alpha = k_{zp}$ we obtain
\[
D_f(\alpha) \approx D_f'(\alpha) (\alpha - k_{zp}) \quad \alpha \rightarrow k_{zp}, \tag{48}
\]
where
\[
D_f'(\alpha) = \frac{dD_f(\alpha)}{d\alpha} \mid_{\alpha=k_{zp}} = k_{zp} J_1(\chi_p a)
\]
\[
\times \left[ 2i \delta \chi_p k_0 - 2 \eta_0 \frac{\chi_p}{k_0} \left( 1 - \frac{k_{zp}^2}{\chi_p^2} \right) \right]. \tag{49}
\]
Thus, it can be concluded that the expression \( (45) \) right part has pole singularity in the area $\text{Im} \ \alpha > 0$ at $\alpha = k_{zp}$. The residues of terms with such poles in the right part of the expression \( (45) \) should cancel each other since the left part is the regular function in the area $\text{Im} \ \alpha > 0$. For the residues calculation it is necessary to find the following limit:
\[
\lim_{\alpha \rightarrow k_{zp}} \frac{i k_0}{k_0} J_0(\chi_p a) - \eta(\alpha) J_1(\chi_p a) \quad k_{zp} - \alpha
\]
\[
= -2i \delta J_1(\chi_p a) \frac{k_{zp}}{k_0}, \tag{50}
\]
where $\delta$ is determined by \( (44) \). Using the continuity condition $\Phi_\pm^{(1)}(a, \alpha) = \Phi_\pm^{(2)}(a, \alpha)$ we finally obtain from \( (45) \) the following requirement:
\[
\Phi_\pm^{(1)}(a, k_{zp}) = \frac{i}{2\pi} \left[ \frac{J_1(\chi a)(\eta(\alpha) - \eta_p)}{k_{zp} + k_{zp}} \right.
\]
\[
- \sum_{m=1}^{\infty} R_m \frac{J_1(\chi a)(\eta(\alpha) - \eta_p)}{k_{zm} - \alpha}
\]
\[
+ R_p \left( 2i \delta J_1(\chi_p a) \frac{k_{zp}}{k_0} - D_f' \right) \tag{51}
\]
Note that we have utilized the following relations valid for any $m = 1, 2, \ldots$ and $p = 1, 2, \ldots$
\[
\frac{i k_0}{k_0} J_0(\chi_m a) - \eta(\alpha) J_1(\chi_m a) = J_1(\chi_m a)(\eta_m - \eta(\alpha)),
\]
\[
\frac{i k_0}{k_0} J_0(\chi_m a) - \eta(\alpha) J_1(\chi_m a) = J_1(\chi_m a)(\eta_m - \eta). \tag{52}
\]
IV. COEFFICIENT $C_2$

In this paper, it is assumed that the external surface is corrugated as well as the internal waveguide surface. Then the equivalent boundary condition $E_{z\omega} = \eta^m H_{\omega \omega}$ is fulfilled at $r = a + 0$, $z < 0$ that leads to the

$$
\left. \frac{i}{k_0} \left( \frac{\Psi_{-}^{(2)}(r, \alpha)}{r} + \frac{\partial \Psi_{-}^{(2)}(r, \alpha)}{\partial r} \right) \right|_{r=a} = \eta(\alpha) \Psi_{-}^{(2)}(a, \alpha). 
$$

(51)

The equation (51) is fulfilled because the variable $\alpha$ in the impedance $\eta(\alpha)$ plays the role of the longitudinal wavenumber for the components $E_{z\omega}$ and $H_{\omega \omega}$ integral transforms. The equality (51) leads to the following condition according to (37):

$$
\Phi_{-}^{(2)}(a, \alpha) = 0. 
$$

(52)

Using the equation (41) at $r = a$ we finally obtain

$$
C_2 = \frac{\Phi_{+}^{(2)}(a, \alpha)}{D_H(\alpha)},
$$

(53)

where

$$
D_H(\alpha) = \frac{i\kappa}{k_0} H_0^{(1)}(\kappa a) - \eta(\alpha) H_1^{(1)}(\kappa a). 
$$

(54)

V. WIENER-HOPF-FOCK EQUATION.

The following step of the solution is the derivation of the Wiener-Hopf-Fock equation for functions that regular in upper and lower half-plane of the complex variable $\alpha$. After some cumbersome mathematical transformations the combination of the expression (18), boundary condition $\Psi_{-}^{(2)}(a, \alpha) = \Phi_{-}^{(2)}(a, \alpha)$, expressions (45) and (53) can be reduced to the equality

$$
\frac{\Phi_{+}^{(2)}(a, \alpha)}{D_J(\alpha)} J_1(\kappa a) + \frac{i}{2\pi} \frac{J_1(\chi a)}{k_0} + \frac{i}{2\pi} \sum_{m=1}^{\infty} \frac{R_m J_1(\chi m a)}{k_m - \alpha} \\
- \frac{i \kappa}{2\pi} \frac{J_1(\kappa a)}{\kappa_0} \frac{J_1(\chi a)(\eta - \eta(\alpha))}{(\kappa_0 + \alpha) D_J(\alpha)} \\
+ \frac{i \kappa}{2\pi} \sum_{m=1}^{\infty} \frac{J_1(\chi m a)(\eta_m - \eta(\alpha))}{(k_m - \alpha) D_J(\alpha)} \\
+ \Psi_{-}^{(2)}(a, \alpha) = \frac{\Phi_{+}^{(2)}(a, \alpha) H_1^{(1)}(\kappa a)}{D_H(\alpha)}. 
$$

(55)

Combining the similar terms one can obtain

$$
2k_0 \Phi_{+}^{(2)}(a, \alpha) \\
\delta^2 \kappa^3(\alpha) G(\alpha) K(\alpha) \\
+ \Psi_{-}^{(2)}(\alpha) + U^l(\alpha) - \sum_{m=1}^{\infty} R_m V^m(\alpha) = 0,
$$

(56)

where

$$
U^l(\alpha) = \frac{i}{2\pi} \frac{J_1(\chi a)}{D_J(\alpha)} J_f(\alpha) - J_1(\kappa a)(\eta - \eta(\alpha)) \frac{k_0 + \alpha}{k_0}, \\
V^m(\alpha) = \frac{i}{2\pi} \frac{J_1(\chi m a)}{D_J(\alpha)} J_f(\alpha) - J_1(\kappa a)(\eta_m - \eta(\alpha)) \frac{k_m - \alpha}{k_m - \alpha},
$$

(57)

$$
G(\alpha) = -\pi a k_0 \delta \eta(\alpha) \kappa^3(\alpha) J_1(\kappa a) H_1^{(1)}(\kappa a), \\
K(\alpha) = 1 - i \frac{k_0 J_0(\kappa a) H_1^{(1)}(\kappa a)}{k_0 \eta} J_1(\kappa a) H_1^{(1)}(\kappa a) \\
- \frac{k^2}{k_0^2 \eta^2} J_0(\kappa a) H_1^{(1)}(\kappa a), 
$$

(58)

with $G(\alpha) \to 1$, $K(\alpha) \to 1$ for $|\alpha| \to \infty$, $-\text{Im} k_0 < \text{Im} \alpha < \text{Im} k_0$.

Note that the function $U^l(\alpha)$ has pole singularities at $\alpha = -k_{zp}$, $(p = 1, 2, \ldots)$, where

$$
\text{Res} \ U^l(\xi) = \frac{ij_1(\chi a) J_1(\chi a)(\eta - \eta(\alpha))}{2\pi i D_J(\kappa - k_{zp})},
$$

(59)

The function $V^m(\alpha)$ has pole singularities at $\alpha = \pm k_{zp}$, $(p = 1, 2, \ldots)$ except points $\alpha = \pm k_{zm}$.

$$
\text{Res} \ V^m(\xi) = \frac{i}{2\pi} \frac{J_1(\chi m a) J_1(\chi m a)(\eta_m - \eta(\alpha))}{D_J(\kappa_m + k_{zp})}. 
$$

(60)

Factorizing the functions using standart formulas (14)

$$
\kappa(\alpha) = \kappa_+(\alpha) \kappa_-(\alpha), \\
G(\alpha) = G_+(\alpha) G_-(\alpha), \\
K(\alpha) = K_+(\alpha) K_-(\alpha),
$$

(61)

and multiplying (56) by $\kappa_3(\alpha) G_-(\alpha) K_-\alpha(\alpha)$ we obtain

$$
2k_0 \delta^2 \Phi_{+}^{(2)}(a, \alpha) \\
\kappa_3(\alpha) G_+(\alpha) K_+(\alpha) + \Psi_{-}^{(2)}(a, \alpha) \kappa_3(\alpha) G_-(\alpha) K_-\alpha(\alpha) \\
+ S'(\alpha) - \sum_{m=1}^{\infty} T^m(\alpha) = 0,
$$

(62)

where

$$
S'(\alpha) = U^l(\alpha) \kappa_3(\alpha) G_-(\alpha) K_-\alpha(\alpha), \\
T^m(\alpha) = V^m(\alpha) \kappa_3(\alpha) G_-(\alpha) K_-\alpha(\alpha).
$$

(63)

General factorization formulas can be written in the following form (14):

$$
S_+^{(2)}(\alpha) = \frac{1}{2\pi i} \int_{-\infty + i \alpha}^{\infty + i \alpha} S'(\xi) d\xi,
$$

(64)
where $-\text{Im}k_0 < h < \text{Im}\alpha < \text{Im}k_0$. It is convenient to enclose the integration path in (64) to the lower half-plane $\xi$. Then the integral is determined by contributions of the poles $\xi = -k_{zq}, q = 1, 2, \ldots$. Similarly, the function $T^m_\pm(\alpha)$ is determined by the contributions of the poles $\xi = -k_{zq}, q = 1, 2, \ldots, q \neq m$. Then, the functions $S^\pm_+(\alpha)$, $T^m_\pm(\alpha)$ which are regular in area $\text{Im}\alpha > -\text{Im}k_0$ can be calculated using (59), (60):

$$S^\pm_+(\alpha) = \sum_{q=1}^{\infty} \frac{\kappa^3_+(k_{zq})G_+(k_{zq})K_+(k_{zq})}{k_{zq} + \alpha} \text{Res} U^l(\xi),$$

$$T^m_\pm(\alpha) = \sum_{q=1}^{\infty} \frac{\kappa^3_+(k_{zq})G_+(k_{zq})K_+(k_{zq})}{k_{zq} + \alpha} \xi = -k_{zq}. \quad \text{(65)}$$

Now we can rewrite the Wiener-Hopf-Fock equation so that the left part contains only functions regular in area $\text{Im}\alpha > 0$ and the right part — functions regular in area $\text{Im}\alpha < 0$

$$\frac{2k_0\delta^{-2} \Phi^2_+ (a, \alpha)}{\kappa^3_+(a)G_+(a)K_+(a)} + S^\pm_+(\alpha) - \sum_{m=1}^{\infty} R_m T^m_\pm(\alpha) =$$

$$= -\Psi^2_+(a, \alpha)k^3_+(a)G_-(a)K_-(a) - S^\pm_+(\alpha) \quad \text{(66)}$$

$$+ \sum_{m=1}^{\infty} R_m T^m_\pm(\alpha) = P(\alpha),$$

$$W_{pm} = \frac{\kappa_+(k_{zp})G_+(k_{zp})K_+(k_{zp})}{2k_0} \sum_{q=1}^{\infty} \frac{\kappa_+(k_{zq})G_+(k_{zq})K_+(k_{zq})}{q \neq m} \text{Res} V^m(\xi) + \frac{i}{2\pi} \left[ \frac{\eta_m J_0(\chi_m a) - \eta_p J_1(\chi_p a)}{k_{zm} - k_{zp}} \right].$$

As a result, the unknown coefficients $R_m$ can be calculated using the reduction method.

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