Postulating an objective, spontaneous collapse process for the wave function is a way to overcome the quantum measurement problem and to explain the fundamental absence of spatial superpositions on the macroscopic scale [1]. This idea deems quantum mechanics incomplete and complements it with a fundamental stochastic modification that bridges the gap between the micro-cosmos of quantum phenomena and the classical world.

A prime example is the model of continuous spontaneous localization (CSL) [1] [2], which predicts a mass-amplified spatial decoherence effect. It reinstates macro-realism [3] and can be motivated from natural consistency requirements on generic ‘classicalizing’ modifications of quantum mechanics [4]. The spontaneous collapse is accompanied by a tiny amount of diffusive heating, impacting also classical states of motion, which could however be mitigated by adding colored noise [5] [6] and friction [7] to the model.

The CSL hypothesis has sparked numerous efforts to conceive [8–18] and perform [19–23] experiments that rule out a significant portion of its two-dimensional parameter space comprised of the CSL localization rate $\lambda$ and the localization length scale $r_C$. Each experimental test falsifies a certain set of parameters marked by an exclusion curve $\lambda(r_C)$. The best experimental bounds so far are surveyed in Fig. 1 (solid lines). They do not yet reach the critical regime of nano- to micrometer localization (CSL) [1, 2], which predicts a mass-length scales and CSL rates as low as the historic value $\lambda = 10^{-16}$ Hz for the reference mass 1 u at $r_C = 100$ nm (grey dot) [2] [24]. Accessing this regime with a quantum experiment could therefore deal the ultimate blow to the CSL hypothesis.

The most macroscopic matter-wave experiments to date (thin solid lines) are still many orders of magnitude away. A purpose-built space mission would have requirements on generic ‘classicalizing’ modifications of quantum mechanics predicting an objective quantum-to-classical transition. Here we show that precision interferometry with Bose-Einstein condensed atoms can serve to lower the current empirical bound on the localization rate parameter by six orders of magnitude. This works by focusing on the atom count distributions rather than just mean population imbalances in the interferometric signal of squeezed BECs, without the need for preparing highly entangled states. We discuss experimentally realistic measurement schemes which could probe and potentially rule out the entire relevant parameter space of CSL, including the historic values proposed by Ghirardi, Rimini, and Weber, below which CSL is no longer deemed a viable solution to the measurement problem of quantum mechanics.

Here we show that two-mode interference with Bose-Einstein condensed atoms can probe the critical CSL parameter regime with a few ten thousand atoms in less than a second of interference time and a few thousand repetitions. Our proposed scheme is based on sampling the atom count distribution in the output ports of the interferometer rather than determining a mean interference visibility upon varying the phase. Surprisingly, CSL can induce a broadening of the distribution that scales with the square of the atom number in the case of interacting atoms and overlapping modes. Even less demanding setups can readily improve the best current bounds. We first consider a standard Mach-Zehnder interferometer (MZI) operating a dilute, phase-squeezed BEC, and then a single-well interferometer (SWI) with controllable atom interactions. To demonstrate the feasibility of both schemes we estimate the measurement effort by means of the Cramer-Rao bound.

**MZI with dilute BEC.**—We start with the standard Mach-Zehnder interferometer setting where a BEC with $N$ atoms of mass $m$ is coherently split into two spatially separate arms and recombined after an effective interrogation time $t$ at a second beam splitter. The recombination maps the interferometric phase $\varphi$ to a population imbalance $n$ between the two output ports. Experimental realizations include double-well trapping of condensates on a chip [31], atoms suspended in optical standing-wave antinodes [28], and free-falling momentum-split condensates [20].

The standard measurement protocol varies the mean interferometric phase $\bar{\varphi}$ and extracts the interference visibility $V \in (0, 1)$ from the mean count difference $\bar{n} = NV\sin \bar{\varphi}$. Given that the measured visibility is lower than the ideal value $V = 1$ predicted by quantum theory, bounds on the CSL parameters could be obtained by attributing the uncontrolled visibility loss to spontaneous collapse. However, a thus defined CSL test offers no collective advantage over single-atom interferometry, regardless of any initial squeezing [32]; for non-interacting BEC, CSL predicts $V_{\text{CSL}} = \exp(-\Gamma t/2)$, with a single-atom dephasing rate $\Gamma v/2 = (m/u)^2 \lambda f_P(r_C)$. Here, the geometry factor $f_P$ assumes its maximum $f_P(r_C) = 1$.
Start by considering a uniformly split BEC with a well-defined phase around \( \bar{\varphi} \) at time \( t \). This phase could be read out directly by overlapping and imaging the phase-sheared condensates corresponding to the two modes [33]. More conventionally, it is mapped to \( n \) by recombination at the second beam splitter. CSL-induced dephasing causes a broadening of the phase distribution with time. For \( N \gg 1 \), the phase distribution is approximately Gaussian, with variance

\[
\sigma^2_\varphi(t) = \sigma^2_\varphi(0) + \Gamma_\varphi t + \zeta^2t^2\sigma^2_n(0),
\]

as follows from the theory in [33] (summarized in [34]). Here, \( \sigma_\varphi(0) \) and \( \sigma_n(0) \) are the initial phase and number uncertainties of the split BEC state, while the parameter \( \zeta \) accounts for phase dispersion due to atom-atom interactions [35], typically negligible in dilute condensates. Corrections to the Gaussian phase distribution underlying (1) are irrelevant here as we consider only \( \sigma_\varphi(t) \ll \pi \); they could be easily implemented via trigonometric mapping [33].

We now show that sampling the phase distribution for a fixed mean interferometric phase \( \bar{\varphi} \) is a more direct way to test CSL than probing visibility reduction. The role of the measurement precision and the atom number \( N \) becomes apparent for a two-mode state close to the minimum of the number-phase uncertainty relation, \( \sigma^2_n(0)\sigma^2_\varphi(0) \geq 1 \) [37]. A product state of two-mode superpositions, for example, corresponds to \( \sigma^2_\varphi(0) = 1/\sigma^2_n(0) = 1/N \), which yields a shot-noise limited phase distribution in the absence of dephasing and dispersion. Initial phase squeezing by the amount \( \sigma^2_\varphi(0) = \xi^2_0/N \) with \( \xi_0 < 1 \) can push this limit below shot noise [37], but the anti-squeezed conjugate variable will then be more detrimental in presence of phase dispersion (and minimum uncertainty might not be attainable by employing one-axis twisting [33]). Note that the advantage of large atom numbers \( N \) vanishes in a low-contrast measurement, where \( \sigma^2_\varphi(t) \sim 1 \gg \xi^2_0/N \), since the initial phase variance becomes irrelevant. In experiments this is typically due to uncontrollable phase noise and other sources of error.

Suppose we infer an effective squeezing \( \xi^2_t = N\sigma^2_\varphi(t) \) from a sample at known initial state parameters \((N, \xi_0, \sigma_n)\) and dispersion \( \zeta \). Then Eq. (1) implies that the data is consistent with

\[
\lambda \leq \frac{(u/m)^2 \xi^2_t - \xi^2_0 - \zeta^2t^2N\sigma^2_n(0)}{2Nt} f_\varphi(r_C).\tag{2}
\]

The scaling with \( 1/Nt \) highlights the trade-off between measurement resolution and interference time: a short-time precision measurement, ideally at vanishing \( \zeta \) [39], can be on par with a conventional long-time interferometer at low contrast. For example, already a setup with \( N = 3 \times 10^9 \) rubidium atoms and \( t = 0.8 \text{ s} \) at \( \Delta_\varphi = 10 \mu\text{m}, \) \( w_x = 100 \text{ nm} \) would extend state-of-the-art non-interferometric CSL tests, assuming one prepares a moderate initial squeezing of \( \xi_0 = 0.9 \) and detects an effective broadening of \( \xi_t = 1.1 \) [dotted curve in Fig. (1b)]. Interference of similarly sized condensates over seconds was already demonstrated [29]. For an ultimate CSL test ruling out the original GRW parameters [24], one would need e.g. \( N = 10^9 \) cesium atoms over \( t = 20 \text{ s} \), and a measured broadening from \( \xi_0 = 0.3 \) to not more than \( \xi_t = 1.3\xi_0 \). This is an ambitious experiment, but the requirements have to be put into perspective with other high-mass interference proposals [9, 17, 25, 26].

Figure 1. CSL bounds from different experiments. The blue-shaded region in (a) represents the falsified parameters from the three most macroscopic interferometric tests so far: (1) near-field interferometry [22], (2) MZI with atoms [28], and (3) MZI with BEC [29]. The yellow-shaded region marks the best non-interferometric bounds from (4) x-ray emission [21], and from force-noise sensing in (5) the LISA pathfinder mission [20, 21], and (6) with layered micro-cantilevers [23]. The here proposed Mach-Zehnder (dotted) and single-well (dashed) interferometer setups are plotted for two scenarios: one that matches the state of the art, shown in the magnified detail (b), and an ultimate test reaching the historic CSL values (grey dot) in (a).
In a measurement with interacting BEC at sufficiently phasing contribution \( \Gamma_P \) CSL bound. Assuming, as before, that one infers an effective interaction-induced phase dispersion, the collapse also causes atom loss and thus depletes the two-mode condensate \([41]\). We shall omit this heating effect and assume that lost atoms are not detected, which only underestimates the impact of CSL on phase stability.

The interplay between interaction-induced phase dispersion and CSL-induced diffusion results in an \( N^2 \)-amplified overall impact of CSL on the phase distribution of the two-mode state: Interactions cause a phase spread that grows with the conjugate atom number uncertainty, which in turn increases by virtue of diffusion. The combined effect on the relevant phase distribution is

\[
\sigma_p^2(t) = \sigma_p^2(0) + \Gamma_P t + \zeta^2 t^2 \left( \sigma_n^2(0) + \frac{\Gamma_s N^2 t}{6} \right). \tag{3}
\]

In a measurement with interacting BEC at sufficiently large \( N \) and \( \zeta \), the diffusion term quickly exceeds the dephasing contribution \( \Gamma_P t \), which results in an improved CSL bound. Assuming, as before, that one infers an effective \( \zeta \) from a sample and conservatively attributes all incoherent broadening to CSL, the data would be consistent with

\[
\lambda \leq \frac{(u/m)^2 \xi_t^2 - \xi_0^2 - \zeta^2 t^2 N \sigma_n^2(0)}{2Nt/d(r_{C}) + (N^2/6)\zeta^2 t^2 f_S(r_{C})}. \tag{4}
\]

Here one would have to detect diffusion-induced broadenings on top of a potentially large systematic broadening caused by dispersion alone, which requires more measurement data (see below).

To this end we propose an echo-like interference protocol in which the dispersion broadening cancels: If one has control over the atom-atom interaction, for example via a Feshbach resonance \([11, 12]\), one may switch it from attractive to repulsive \((\zeta \rightarrow -\zeta)\) halfway at time \( t/2 \). (Alternatively, one could perform a \( \pi \)-rotation to flip the two-mode state around its mean spin direction on the generalized Bloch sphere.) A straightforward calculation \([34]\) reveals that this cancels the pure dispersion term in \( \frac{N^2 \sigma_n^2(0)}{6} \), while reducing the diffusion term by merely the factor four. We arrive at the dispersionless CSL bound

\[
\lambda < \frac{(u/m)^2}{N^3 \xi_t^2 C_2 f_S(r_{C})} \frac{\xi_0^2}{f_0(r_{C})}, \tag{5}
\]

Here we have omitted the dephasing term, which only underestimates the CSL sensitivity.

Equations \(4\) and \(5\) demonstrate that BEC interference can exhibit a favourable scaling with the atom number \( N \). While the atom-atom interaction may lead to a transient buildup of correlations, our scheme does not require preparing or detecting many-atom correlations, least of all genuine multipartite entanglement.

**SWI with interacting BEC.** We propose to realize the amplified CSL test scheme in a single-well interferometer (SWI) configuration, in which the BEC is split between the lowest two eigenmodes of a harmonic trapping potential. Such a setup was demonstrated in Ref. \([43]\). For simplicity, we assume an elongated condensate along the \( z \)-direction and set the \( y \)-trap frequency to \( 6\omega \) \([34]\), so that

\[
\begin{align*}
fs(r_C) &= \frac{r_C^2 x_0^2}{\sqrt{(r_C^2 + x_0^2/6)(r_C^2 + x_0^2)^3}} \\
fr(r_C) &= 3x_0^2 f_S(r_C)/8(r_C^2 + x_0^2) \tag{3},
\end{align*}
\]

Given the trap frequency \( \omega \) and the associated ground-state width \( x_0 = \sqrt{2\hbar\omega/m} \), we find that CSL diffusion maximizes at the length scale \( r_C = \sqrt{2/3}x_0 \), with \( \Gamma_S = 120\Gamma_p/27 = 288/625 (m/u)^2 \lambda \). This is where the experiment yields the strongest CSL bounds.

For an initially number-squeezed state of \( N = 3 \cdot 10^5 \) atoms at \( \xi_0 = 5 \) (chosen large to suppress the impact of phase dispersion) and an estimated broadening to no more than \( \xi_t = 200 \) after \( t = 0.5 \) s at \( \zeta = 6 \) mHz (no echo), the CSL bound \([41]\) results in the dashed line in Fig. \(1b\). It is comparable to the best bounds from non-interferometric tests in this region. We have assumed a trap frequency tuned to \( x_0 = 0.5 \mu m \) in order to cover the respective \( r_C \)-window. The example would rule out a CSL contribution to the phase broadening that is roughly one third of the interaction-induced dispersion.

The dashed line in Fig. \(1a\) corresponds to a realistic scenario for an ultimate CSL test with rubidium atoms based on Eq. \(5\) - we consider a trap frequency \( \omega = 146\) kHz amounting to \( x_0 = 100 \) nm, \( N = 5 \times 10^4 \), and \( t = 200 \) ms in an echo scheme with an initially unsqueezed state \( \xi_0 = 1 \), \( \zeta = 4 \) Hz, and a measured increase of \( \xi_0 \) by no more than 15%.

**Measurement uncertainties.** When proposing experimentally challenging tests of ever weaker collapse models, it is not enough to ensure that the proposed setup parameters be within reach of current or future technology. One must realistically assess the minimum measurement time and effort for a conclusive outcome as well.

The relevant figure of merit is the number of measurement repetitions \( k \) and the corresponding total integration time \( kt \), which for a viable proposal must not be unreasonably long. The minimum \( k \) is given by the number of sample points one needs in order to determine the width of the atom count distribution and extract a lower bound of falsified CSL rates \( \lambda \) at the desired precision. We can estimate \( k \) with help of the Cramer-Rao
Table I. Parameters for the proposed CSL tests in single-well (SWI) and Mach-Zehnder interferometer (MZI) setups. We list the atom number $N$, the phase squeezing parameter $\xi_0$, the interference time $t$, and the number of measurement runs $k$ needed to exclude CSL rates greater than $\lambda_{\text{min}}$ with a precision of 10%. More conservatively, $k_{1.5}$ runs are required if the phase spread without CSL is increased by 50% due to additional noise.

| setup   | $N$  | $\xi_0$ | $t$  | $\lambda_{\text{min}}$ | $k$ | $k_{1.5}$ |
|---------|------|----------|------|------------------------|-----|----------|
| Rb MZI  | $3 \times 10^5$ | 0.9 | 0.8 s $10^{-10}$ Hz | 2086 | 3775 |
| Rb SWI  | $3 \times 10^5$ | 5   | 0.5 s $10^{-10}$ Hz | 3381 | 6423 |
| Cs MZI  | $1 \times 10^9$ | 0.3 | 20 s $10^{-10}$ Hz | 1033 | 1692 |
| Rb SWI  | $5 \times 10^4$ | 1   | 0.2 s $10^{-10}$ Hz | 3065 | 5771 |

bound, or Bernstein-von Mises theorem [44]: In the presence of CSL, theory predicts a probability distribution of atom count differences $p(n|\lambda; r_C, I)$ conditioned on the CSL rate $\lambda$ at a given CSL length $r_C$ and background information $I$, which subsumes all relevant experimental parameters (including e.g. $N, t, \varphi$). The corresponding Fisher Information (FI) [45] then bounds the precision of the $\lambda$ estimate from the data by $\Delta \lambda \geq 1/\sqrt{k I(\lambda|r_C, I)}$, in the limit of large $k$.

The outcome distributions $p(n|\lambda; r_C, I)$ for the proposed scenarios are very well approximated by Gaussians, and the required consistent (unbiased) $\lambda$-estimators are simple linear functions of the estimated variances $\xi^2$, as given in Eqs. (2), (4), and (5). Moreover, by separating the CSL and the conventional term in the variance, $\sigma^2(t) = \sigma^2_{\text{conv}}(t) + \alpha^2_{\text{CSL}}(t)\lambda$, the FI can be written explicitly as $I(\lambda|r_C, I) = 1/2[\sigma^2_{\text{conv}}(t)/\alpha^2_{\text{CSL}}(t)+\lambda]^2$. Hence a CSL test at a fixed relative uncertainty $\delta = \Delta \lambda/\lambda$ requires at least

$$k \geq \frac{2}{\delta^2} \left(1 + \frac{\sigma^2_{\text{conv}}(t)}{\lambda \alpha^2_{\text{CSL}}(t)}\right)^2$$

measurement repetitions. Clearly, the required number grows with $\lambda^{-2}$ if one probes $\lambda$-values at which the CSL-induced broadening is dominated by conventional broadening. This also implies that any additional known source of decoherence or phase noise in the experiment will rapidly increase $k$, calling for precision measurements. For a more conservative estimate, suppose one aims for an experiment in which conventional (and well characterized) sources of noise would result in a reduced interference visibility $V = \exp(-\gamma t)$. We can then account for the corresponding phase broadening by replacing $\sigma^2_{\text{conv}} \rightarrow \sigma^2_{\text{conv}} + 2\gamma t$ in (6).

Table I lists the here proposed experimental scenarios including their key parameters and compares the required number of measurement repetitions $k$ to rule out CSL rates $\lambda \geq \lambda_{\text{min}}$ with precision $\delta = 0.1$. We also include values $k_{1.5}$ accounting for a 50 percent increased conventional phase spread, $2\gamma t = 0.5\sigma^2_{\text{conv}}$. We ignore the chance of CSL-induced atom loss, which is less than $10^{-6}$ per atom. All considered cases require a few thousand repetitions. For an ultimate CSL test in an MZI, this translates into more than 5 hours of net interference time, as opposed to about 10 minutes for the equivalent SWI.

Conclusion—We have presented experimental scenarios based on two-mode atom BEC interference with either spatially separate or overlapping modes that are capable of ruling out spontaneous collapse as a solution to the measurement problem. They solely rely on standard techniques such as squeezing and the manipulation of the interaction strength via e.g. Feshbach resonances. Importantly, they do not require the preparation of highly entangled states.

The unprecedented sensitivity scaling with the third power of the atom number in the single-well interferometer is an effect of the interplay between CSL-induced atom diffusion and interaction-induced phase dispersion. This should facilitate macroscopic quantum tests with precision atom interferometry, in laboratory or space-based experiments [46].

In comparison to classical heating experiments, which currently provide the best CSL bounds, interferometric test schemes are robust against conceivable modifications of collapse models with colored noise or friction. The latter confine their heating effect to a finite frequency window and temperature, while leaving the intended decoherence effect on macroscopic superpositions largely intact.

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For $N$ indistinguishable bosonic particles in two modes, represented by annihilation operators $c_{a,b}$, we can make use of the collective spin representation \cite{44}, with $J = N/2$ and spin operators $J_x = (c_a^\dagger c_a + c_b^\dagger c_b)/2$, $J_y = (c_a^\dagger c_b - c_b^\dagger c_a)/2i$, and $J_z = (c_a^\dagger c_a - c_b^\dagger c_b)/2$. In the presence of CSL, the time evolution of the collective spin state representing the condensate can be described by the master equation \cite{33,34}:

$$\frac{\partial}{\partial t} \rho = \frac{1}{\hbar^2} \left[ J_x + h\zeta J_z^2, \rho \right] + \mathcal{L}\rho \tag{A1}$$

The first term in the Hamiltonian describes the free rotation around the $z$-axis of the generalized Bloch sphere at an angular frequency given by the energy difference $\epsilon$ between the two involved modes. The second term accounts for atom-atom interactions in the condensate to lowest order close to the equator, $\hbar\zeta = 2(\partial \mu/\partial n)_{n=0}$, where $n$ is the population difference (i.e. two times the $J_z$-eigenvalue) and $\mu$ is the chemical potential \cite{34}. CSL in second quantization contributes the Lindblad generator

$$\mathcal{L}\rho = \lambda \frac{m^2 r_C^3}{u^2} \int d^3 q \left[ A(q)\rho A^\dagger(q) - \frac{1}{2} \{ A^\dagger(q)A(q), \rho \} \right], \tag{A2}$$

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with $A(q) = \int d^3p \, a^\dagger(p) a(p - hq)$ and $a(p)$ the particle annihilation operator in momentum representation. Restricting the dynamics to the two relevant modes of the interferometer, we can expand the Lindblad operator as

$$\begin{align*}
A(q) &= \sum_{j,k} \langle \psi_j | e^{iq \cdot \mathbf{r}} | \psi_k \rangle \, C_j^\dagger C_k
\quad (A3) \\
&= [W_{aa}(q) + W_{bb}(q)] \frac{N}{2} + [W_{aa}(q) - W_{bb}(q)] \, J_z
\quad + [W_{ab}(q) + W_{ba}(q)] \, J_x + i [W_{ab}(q) - W_{ba}(q)] \, J_y.
\end{align*}$$

Here, $|\psi_{a,b} \rangle$ denote the single-particle wave functions of the two modes, $r$ the respective position operator, and $W_{jk}(q) = W_{kj}^*(-q)$ the overlap matrix elements of the momentum displacement operator. Note that the $J_y$-term vanishes for bound states with real-valued wave functions. The terms we have omitted here would describe incoherent hopping of atoms between the condensate and other undetected modes, which causes atom loss. Formally, we can account for this CSL-induced heating effect in the data analysis of an experiment by introducing conditional outcome probabilities, given that the detected $N$ atoms have remained in the condensate at all times. This way, the CSL bounds would not depend on the additional classical observation of depletion \[33\]. Moreover, since particles lost from the condensate (or, less likely, regained ones) will always increase the phase uncertainty $\sigma_{\varphi}$, our omission of the depletion effect only underestimates the influence of CSL. Additionally, we will ignore the $z$-coordinate as the condensate is elongated in this direction and integrated out upon readout \[31\] [33].

Plugging the Lindblad operator \[A3\] into the master equation \[A2\] reduced to the $xy$-plane, we identify two $r_C$-dependent geometry factors that will be relevant for the two configurations discussed here,

$$\begin{align*}
f_P(r_C) &:= \frac{r_C^2}{2m} \int d^2q \, e^{-q^2r_C^2} |W_{aa}(q) - W_{bb}(q)|^2, \\
f_S(r_C) &:= \frac{r_C^2}{2m} \int d^2q \, e^{-q^2r_C^2} |W_{ab}(q) + W_{ba}(q)|^2.
\end{align*}$$

\[A4\] \[A5\]

For the MZI scenario in the main text, we consider two spatially distinct, but otherwise identical modes with no overlap. $|r| |\psi_b \rangle = (r - \Delta x) |\psi_a \rangle$ such that $W_{ab}(q) \approx 0$. The CSL generator \[A2\] then reduces to pure dephasing between the modes, i.e. a phase noise channel in the collective spin representation \[37\],

$$\begin{align*}
\mathcal{L} \rho &= \Gamma_P \left[ J_z \rho J_z - \frac{1}{2} \{J^2_z, \rho\} \right] + \Gamma_S \left[ J_x \rho J_x - \frac{1}{2} \{J^2_x, \rho\} \right].
\end{align*}$$

\[A6\]

The typical scaling behavior of the geometry factor $f_P(r_C)$ for the dephasing rate becomes evident for Gaussian modes of width $w_x = w_y \ll \Delta_x$.

$$f_P(r_C) = \frac{1 - \exp \left[ -\Delta_x^2 / 4(w_x^2 + r_C^2) \right]}{1 + w_x^2 / r_C^2}. \quad \text{(A7)}$$

If the two interfering modes overlap spatially, then $W_{ab} \neq 0$ and CSL will not only induce dephasing, but also diffusion. For the SWI configuration in the main text, we consider a condensate split between the ground and first excited state of a harmonic potential in $x$-direction, with trapping frequency $\omega$ and ground-state width $w_0 = \sqrt{\hbar/2m\omega}$. Moreover, we assume both modes to be Gaussian along the $y$-direction, with a width $w_y$, and we omit the elongated $z$-profile as argued above. This leads to

$$\begin{align*}
W_{aa}(q) &= e^{-q_x^2 w_y^2 / 2 - q_z^2 w_0^2 / 2}, \\
W_{bb}(q) &= (1 - q_z^2 w_0^2) \, W_{aa}(q), \\
W_{ab}(q) &= i q_x w_y W_{aa}(q) = W_{ba}(q), \\
f_P(r_C) &= \frac{3 \sqrt{2} \pi x_0^4}{8 \sqrt{r_C^2 + \omega^2 (r_C^2 + x_0^2)^2}}, \\
f_S(r_C) &= \frac{\sqrt{2} \pi x_0^2}{\sqrt{r_C^2 + \omega^2 (r_C^2 + x_0^2)^2}}. \quad \text{(A8)}
\end{align*}$$

We have set $w_y = x_0 / \sqrt{6}$ for the proposed setups in the main text.

**Appendix B: Phase dispersion together with diffusion in phase and number difference**

Here we recall the phase-space method of Ref. \[33\] to solve the time evolution of two-mode BEC states according to the master equation \[A1\] in the presence of dephasing and diffusion,

$$\begin{align*}
\mathcal{L} \rho &= \Gamma_P \left[ J_z \rho J_z - \frac{1}{2} \{J^2_z, \rho\} \right] + \Gamma_S \left[ J_x \rho J_x - \frac{1}{2} \{J^2_x, \rho\} \right].
\end{align*}$$

\[B1\]

We consider high atom numbers $N \gg 1$ and coherent superpositions with a well-defined interferometric phase, i.e. collective spin states that are sharply localized on the equator of the generalized Bloch sphere, $\langle J_z \rangle \approx 0$ and $\Delta J_{x,y,z}^2 \ll N^2$. To a very good approximation, the state is then represented by a continuous Wigner function $w(t, \varphi, n)$ in the flat phase space of the conjugate variables $n$ (population difference) and $\varphi$ (interferometric phase angle), provided the support of $w(t)$ is limited to $|n| \ll N$ and stretches over $\varphi$-windows $\ll 2\pi$. The latter constraint can be alleviated with help of a trigonometric mapping of the Wigner function onto a periodic function over the equator, $\varphi \in [0, 2\pi)$, but this will not be necessary for the cases studied here. In a rotating frame that absorbs the free linear evolution of the interferometric phase, $\varphi \rightarrow \varphi + \epsilon t / \hbar$, the master equation \[A1\] translates into the Fokker-Planck equation

$$\begin{align*}
\partial_t w(t, \varphi, n) &\approx - \varsigma n \partial_\varphi w(t, \varphi, n) + \frac{N^2 \Gamma_S}{4} \partial^2_\varphi w(t, \varphi, n) \\
&+ \frac{\Gamma_P}{2} \partial^2_n w(t, \varphi, n). \quad \text{(B2)}
\end{align*}$$
It further approximates the spin diffusion term, which would normally have an additional $\varphi$-dependence, by its angular average, resulting in an additional factor $1/2$ for the $\Gamma_S$-dependent term. The approximation is valid as long as the diffusion rate is small, $\Gamma_S \ll \epsilon/\hbar$, and the interference time extends over at least one free oscillation period.

Equation (B2) constitutes a Gaussian channel, and every initially Gaussian Wigner function will thus remain Gaussian. In particular, the $\varphi$-marginal of the Wigner function, which represents the phase distribution of the interfering state after time $t$, will be of the form

$$
\int dn w_t(\varphi, n) = \frac{1}{\sqrt{2\pi} \sigma_\varphi(t)} \exp \left[ -\frac{\varphi^2}{2\sigma_\varphi^2(t)} \right].
$$

More generally, we can obtain an exact solution for arbitrary initial states with help of the characteristic function in Fourier space,

$$
\chi_t(s, q) := \int d\varphi dn w_t(\varphi, n)e^{i\varphi + iqn}.
$$

It evolves according to

$$
\partial_t \chi_t(s, q) = \zeta s \partial_q \chi_t(s, q) - \left( \frac{\Gamma_P}{2} s^2 + \frac{N^2 \Gamma_S t}{4} q^2 \right) \chi_t(s, q),
$$

which yields the solution

$$
\chi_t(s, q) = \chi_0(s, q + \zeta ts) \times \exp \left[ -\frac{\Gamma_P t}{2} s^2 - \frac{N^2 \Gamma_S t}{4} \left( q^2 + \zeta tqs + \frac{\zeta^2 t^2}{3} s^2 \right) \right]. \tag{B6}
$$

The characteristic function generates the moments of the state’s phase distribution as $\langle \varphi^k \rangle_t = (-i \partial_k)^k \chi_t(0, 0)$. Plugging the above solution into $\sigma_\varphi^2(t) = \langle \varphi^2 \rangle_t - \langle \varphi \rangle_t^2$ leads to the formula (3) in the main text, if we further assume that initially $\langle n \rangle_0, \langle n \varphi \rangle_0 = 0$.

For the two-step echo protocol, the solution can be constructed by first computing $\chi_{t/2}(s, q)$ according to (B6) and then inserting the result as the new initial condition for (B6) with a sign-flipped $\zeta$. The result simplifies to

$$
\chi_t(s, q) = \chi_0(s, q) \exp \left[ -\frac{\Gamma_P t}{2} s^2 - \frac{N^2 \Gamma_S t}{4} \left( q^2 + \frac{\zeta^2 t^2}{12} s^2 \right) \right], \tag{B7}
$$

which produces the results discussed in the main text.