Localization of Energy-Momentum for a Black Hole Spacetime Geometry with Constant Topological Euler Density

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Abstract

The evaluation of the energy-momentum distribution for a new four-dimensional, spherically symmetric, static and charged black hole spacetime geometry with constant non-zero topological Euler density is performed by using the energy-momentum complexes of Einstein and Møller. This black hole solution was recently developed in the context of the coupled Einstein–non-linear electrodynamics of the Born-Infeld type. The energy is found to depend on the mass $M$ and the charge $q$ of the black hole, the cosmological constant $\Lambda$ and the radial coordinate $r$, while in both prescriptions all the momenta vanish. Some limiting and particular cases are
analyzed, illustrating the rather extraordinary character of the spacetime geometry considered.

**Keywords**: Energy-Momentum Complexes, Black Holes, Topological Euler Density

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1 Introduction

The issue of energy-momentum localization systematised researchers’ work in a special way. Looking deeply into the problem, it was clear that the main difficulty consists in the lack of a proper definition for the energy density of gravitational backgrounds. In this light, much research work has been done over the last years concerning the best tools used for the energy-momentum localization. A brief survey points out the leading role played notably by super-energy tensors [1]-[4], quasi-local expressions [5]-[9] and the famous energy-momentum complexes of Einstein [10]-[11], Landau-Lifshitz [12], Papapetrou [13], Bergmann-Thomson [14], Møller [15], Weinberg [16], and Qadir-Sharif [17]. Among the aforementioned computational tools, the energy-momentum complexes have been proven to be interesting and useful as well due to the diverse and numerous reasonable expressions that can be obtained by their application. Some observations are in order here. First, according to their underlying mathematical mechanism, their construction involves the use of two parts, one for the matter and one for the gravitational field. Second, despite the fact that the energy-momentum complexes allow one to obtain many interesting and physically meaningful results for different space-time geometries, their construction is connected to an inherent central problem, namely their coordinate dependence. As it is well-known from the relevant literature, this problem has been solved only in the case of the Møller prescription. The calculations for the energy-momentum of a given gravitational background in the Møller prescription enable the use of any system of coordinates. As for the other energy-momentum complexes, the Schwarzschild Cartesian coordinates and the Kerr-Schild Cartesian coordinates have to be utilised for the calculations providing physically reasonable results for the cases of space-time geometries in (3 + 1), (2 + 1) and (1 + 1) dimensions (see, e.g., [18]-[34] and references therein). At this point it should be noticed that, in order to avoid the coordinate dependence, an alternative method for the computation of the energy and momentum distributions is provided by the teleparallel equivalent of general relativity and certain modified versions of the teleparallel theory (see, e.g., [35]-[40] and references therein).

Regarding the Einstein, Landau-Lifshitz, Papapetrou, Bergmann-Thomson, Weinberg and Møller energy-momentum complexes there is an agreement with the definition of the quasi-local mass introduced by Penrose [11] and developed by Tod [42] for some gravitational backgrounds. We point out that some rather recent works show that several energy-momentum complexes “provide the same results” for any metric of the Kerr-Schild class and indeed even for solutions that are more general than those of the
Kerr-Schild class (see, e.g., [43] and [44], and the interesting article [45] on the subject). Further, the entire historical development of the energy-momentum complexes that started with the formulation of their definitions also includes the attempts made for their rehabilitation [46]-[49]. In this sense, perhaps the most interesting issue was the fact that different energy-momentum complexes yield the same results for the energy-momentum distribution in the case of various gravitating systems.

The present paper has the following structure: in Section 2 we describe the new class of four-dimensional spherically symmetric, static and charged black hole solutions with constant non-zero topological Euler density which we will consider. Section 3 focuses on the presentation of the Einstein and Møller prescriptions used for performing the calculations. In Section 4 we present the calculations and the results obtained for the energy and momentum distributions. Finally, in the Discussion provided in Section 5, we give a brief description of the results obtained as well as some limiting and particular cases. Throughout we use geometrized units (c = G = 1) and the signature chosen is (+, −, −, −). Further, the calculations are performed by using the Schwarzschild Cartesian coordinates \( \{t, x, y, z\} \) for the Einstein energy-momentum complex and the Schwarzschild coordinates \( \{t, r, \theta, \varphi\} \) for the Møller energy-momentum complex. Finally, Greek indices run from 0 to 3, while Latin indices range from 1 to 3.

## 2 Description of the New Black Hole Solution with Constant Topological Euler Density

This section deals with the presentation of the new four-dimensional spherically symmetric, static and charged black hole solution with constant topological Euler density \(^{[50]}\) examined in the present study. Connecting geometry with topology, from the generalised Gauss–Bonnet theorem (see, e.g. \([51]\)) applied to four dimensions the Euler–Poincaré characteristic is obtained by the integral of the Euler density

\[
\mathcal{G} = \frac{1}{32\pi^2} (R^\kappa_{\lambda \mu \nu} R^{\kappa \lambda \mu \nu} - 4 R^\kappa_{\lambda \mu} R_{\kappa \lambda} + R^2),
\]

where \(R^\kappa_{\lambda \mu \nu}\) is the Riemann curvature tensor, \(R_{\kappa \lambda}\) is the Ricci tensor, and \(R\) is the Ricci scalar\(^{[1]}\). For a general, spherically symmetric and static geometry described by the line element

\[
ds^2 = f(r) dt^2 - f(r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)
\]

eq. (1) becomes

\[
\mathcal{G} = \frac{4}{r^2} \{ f'(r)^2 + [f(r) - 1] f''(r) \}, \tag{3}
\]

\(^{[1]}\)Often the topological Euler density is given without the factor 1/32\(\pi^2\) (see, e.g. \([52]\)). In fact, the terms in the parenthesis constitute the so-called “quadratic Gauss-Bonnet term” in the Lovelock gravity Lagrangian.
where the constant $32\pi^2$ has been absorbed in $G$. For constant topological density $G = \alpha \neq 0$, Eq. (3) gives for the metric function

$$f(r) = 1 \pm \left(1 - 2A + Br + \frac{\alpha r^4}{24}\right)^{1/2}, \quad (4)$$

with $A$, $B$ arbitrary constants. In what follows, we will keep only the negative sign of Eq. (4), as it is the one leading to black hole solutions.

The new solution derived in [50] is based on the coupling of gravity to non-linear electromagnetic fields as described by the non-linear generalisation of Maxwell’s electrodynamics according to the Born–Infeld theory. Thus, in the chosen case of electrovacuum, the radial electric field

$$E(r) = \frac{r^2}{4q}(4R^{\mu\nu}R_{\mu\nu} - R^2)^{1/2} \quad (5)$$

where $q$ the electric charge, solves the Einstein–non-linear electrodynamics coupled system with the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R$ calculated by using the line element (2) and the metric function (4). In fact, with the values $A = \frac{1}{2} + \frac{r^2\Lambda}{3}$, $B = \frac{4MA}{3}$ and $\alpha = \frac{8\Lambda^2}{3}$, with $\Lambda$ is the cosmological constant and $M$ is the mass of the black hole, the line element (2) with the metric function (4) becomes

$$ds^2 = \left[1 - \sqrt{\frac{4MAr}{3} + \frac{\Lambda^2r^4}{9} - \frac{2q^2\Lambda}{3}}\right]dt^2 - \left[1 - \sqrt{\frac{4MAr}{3} + \frac{\Lambda^2r^4}{9} - \frac{2q^2\Lambda}{3}}\right]^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (6)$$

describing a Reissner–Nordström–de Sitter black hole spacetime geometry.

Now one can distinguish between two different cases, namely the massive case ($M \neq 0$) and the massless case ($M = 0$). In the first case, Eq. (6), when $q = 0$ and $\Lambda > 0$, shows that the geometry is regular everywhere except at the origin $r = 0$. However, this case has no particular interest from the electrodynamic viewpoint. Black hole solutions with zero mass were proposed as a conjecture by A. Strominger [53] in order to explain conifold singularities in the context of string theory. In particular, the ten-dimensional IIA (resp. IIB) string theory admits black D2- (resp. D3-) brane solutions with a mass proportional to their area. After applying a Calabi-Yau compactification these solutions may wrap around minimal 2-surfaces (resp. 3-surfaces) in the Calabi-Yau space and they appear as four-dimensional black holes. As the area of the surface around which they wrap is let to go to zero, the corresponding extremal black holes become massless, topologically stable, structures. In fact, the existence of stable black hole solutions with zero ADM mass was shown in [54] although their relation to the massless solutions suggested in [53] is still not clarified. Indeed, since then there has been an increasing interest in massless black hole solutions (see, e.g. [55]-[56] and references therein). Recently, massless black hole solutions were obtained from the dyonic black hole solution of the Einstein–Maxwell–dilaton theory [57]. Although the black hole solution examined here does not originate
from string theory we will proceed to the consideration of the massless ($M = 0$) case despite its physically dubious character.

Thus for $M = 0$, the line element (6) becomes

$$ds^2 = \left[ 1 - \sqrt{\frac{\Lambda^2 r^4}{9} - \frac{2q^2 \Lambda}{3}} \right] dt^2 - \left[ 1 - \sqrt{\frac{\Lambda^2 r^4}{9} - \frac{2q^2 \Lambda}{3}} \right]^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2). \quad (7)$$

Here, when $q = 0$ and $\Lambda \neq 0$ the de Sitter solution is obtained, while for $q \neq 0$ but $\Lambda = 0$ one gets the Minkowski solution. In the case $q \neq 0$ and $\Lambda < 0$, an event horizon exists and the spacetime is singular at the origin $r = 0$, while the electric field is everywhere regularisable. An overall detailed study of the black hole’s behaviour in the massive as well as in the massless case is presented in [50].

3 Einstein and Møller Energy-Momentum Complexes

In this section we outline the definitions of the Einstein and Møller energy-momentum complexes.

The expression for the Einstein energy-momentum complex [10] in the case of a (3+1)-dimensional gravitational background is given by

$$\theta^\mu_\nu = \frac{1}{16\pi} h^\mu_{\nu,\lambda}. \quad (8)$$

The Freud superpotentials $h^\mu_{\nu,\lambda}$ in (8) are

$$h^\mu_{\nu,\lambda} = \frac{1}{\sqrt{-g}} g_{\nu\sigma} \left[ -g (g^{\mu\sigma} g^{\lambda\kappa} - g^{\lambda\sigma} g^{\mu\kappa}) \right]_{,\kappa} \quad (9)$$

and satisfy the antisymmetric property

$$h^\mu_{\nu,\lambda} = -h^\lambda_{\nu,\mu}. \quad (10)$$

We notice that in the Einstein prescription the local conservation law is respected:

$$\theta^\mu_{\nu,\mu} = 0. \quad (11)$$

Consequently, the energy and momentum can be calculated in Einstein’s prescription by

$$P^\nu_\mu = \iint \theta^\nu_\mu \, dx^1 \, dx^2 \, dx^3. \quad (12)$$

Here, $\theta^\nu_0$ and $\theta^\nu_i$ represent the energy and momentum density components, respectively.

Applying Gauss’ theorem, the energy-momentum reads

$$P^\nu_\mu = \frac{1}{16\pi} \iint h^\nu_\mu n_i \, dS, \quad (13)$$
with \( n_i \) the outward unit normal vector over the surface \( dS \). In Eq. (13) the component \( P_0 \) represents the energy.

According to [15], the Møller energy-momentum complex is

\[
\mathcal{J}_\nu = \frac{1}{8\pi} M_{\nu,\lambda}^\mu,
\]

(14)

with the Møller superpotentials \( M_{\nu,\lambda}^\mu \) given by

\[
M_{\nu,\lambda}^\mu = \sqrt{-g} \left( \frac{\partial g_{\nu\sigma}}{\partial x^\kappa} - \frac{\partial g_{\nu\kappa}}{\partial x^\sigma} \right) g^{\mu\kappa} g^{\lambda\sigma}.
\]

(15)

The Møller superpotentials \( M_{\nu,\lambda}^\mu \) satisfy the antisymmetric property

\[
M_{\nu,\lambda}^\mu = -M_{\lambda,\nu}^\mu.
\]

(16)

As in the case of the Einstein prescription, in the Møller prescription the local conservation law is also satisfied:

\[
\frac{\partial \mathcal{J}_\nu}{\partial x^\mu} = 0.
\]

(17)

In (17) the component \( \mathcal{J}_0^0 \) represents the energy density and the \( \mathcal{J}_i^0 \) give the momentum density components.

For the Møller energy-momentum complex, the energy-momentum distributions are given by

\[
P_\nu = \iiint \mathcal{J}_\nu dx^1 dx^2 dx^3.
\]

(18)

In particular, the energy distribution can be computed by

\[
E = \iiint \mathcal{J}_0^0 dx^1 dx^2 dx^3.
\]

(19)

Using Gauss’ theorem one gets

\[
P_\nu = \frac{1}{8\pi} \iint M_{\nu}^{0i} n_i dS,
\]

(20)

where, again, \( n_i \) is the outward unit normal vector over the surface \( dS \).

4 Energy and Momentum Distribution for the Black Hole Solution with Constant Topological Euler Density

To compute the energy and momenta with the Einstein energy-momentum complex, we have to transform the metric given by the line element (6) in Schwarzschild Cartesian
coordinates by using the coordinate transformation \( x = r \sin \theta \cos \varphi, \) \( y = r \sin \theta \sin \varphi, \) \( z = r \cos \theta. \) Thus, we obtain a new form for the line element:

\[
ds^2 = f(r)dt^2 - (dx^2 + dy^2 + dz^2) - \frac{f^{-1}(r) - 1}{r^2}(x dx + y dy + z dz)^2.
\]  

(21)

In Schwarzschild Cartesian coordinates for \( \nu = 1, 2, 3 \) and \( i = 1, 2, 3 \) we find the following components of the superpotential \( h_{i}^{\nu} \)

\[
\begin{align*}
  h_{1}^{01} &= h_{1}^{02} = h_{1}^{03} = 0, \\
  h_{2}^{01} &= h_{2}^{02} = h_{2}^{03} = 0, \\
  h_{3}^{01} &= h_{3}^{02} = h_{3}^{03} = 0.
\end{align*}
\]

(22)

In order to compute the non-vanishing components of the superpotentials in the Einstein prescription we use (9) and we obtain the following expressions:

\[
\begin{align*}
  h_{0}^{01} &= \frac{2x}{r^2} \sqrt{\frac{4MAr}{3} + \frac{\Lambda^2 r^4}{9}} - \frac{2q^2 \Lambda}{3}, \\
  h_{0}^{02} &= \frac{2y}{r^2} \sqrt{\frac{4MAr}{3} + \frac{\Lambda^2 r^4}{9}} - \frac{2q^2 \Lambda}{3}, \\
  h_{0}^{03} &= \frac{2z}{r^2} \sqrt{\frac{4MAr}{3} + \frac{\Lambda^2 r^4}{9}} - \frac{2q^2 \Lambda}{3}.
\end{align*}
\]

(23)-(25)

With the aid of the line element (21), the expression for the energy given by (13) and the expressions (23)-(25) for the superpotentials, we get the energy distribution for the examined black hole in the Einstein prescription:

\[
E_E = \frac{r}{2} \sqrt{\frac{4MAr}{3} + \frac{\Lambda^2 r^4}{9}} - \frac{2q^2 \Lambda}{3}.
\]

(26)

In order to calculate the momentum components we employ (13) and (22) and performing the calculations we find that all the momenta vanish:

\[
P_x = P_y = P_z = 0.
\]

(27)

In the Møller prescription we perform the calculations in Schwarzschild coordinates \( \{t, r, \theta, \varphi\} \) with the aid of the line element (6) and we find only one non-vanishing superpotential:

\[
M_{0}^{01} = -\frac{1}{6} \frac{12MA + 4\Lambda^2 r^3}{\sqrt{12MAr + \Lambda^2 r^4} - 6q^2 \Lambda} r^2 \sin \theta.
\]

(28)

Using the above expression for the superpotential and the expression for the energy obtained from (20), we get the energy in the Møller prescription:

\[
E_M = -\frac{r^2}{12} \frac{12MA + 4\Lambda^2 r^3}{\sqrt{12MAr + \Lambda^2 r^4} - 6q^2 \Lambda}.
\]

(29)

Finally, all the momenta are found to be zero:

\[
P_r = P_\theta = P_\varphi = 0.
\]

(30)
5 Discussion

Our paper focuses on the analysis of the energy-momentum localization for a new four-dimensional, spherically symmetric, static and charged black hole spacetime geometry with constant non-zero topological Euler density, given by the line element (6). The solution describes a Reissner–Nordström–de Sitter spacetime geometry as the result of the coupling of Einstein gravity with non-linear electrodynamics of the Born–Infeld type. For \( q = 0 \) the solution has a near-de Sitter behavior, while for \( \Lambda > 0 \) the solution is regular everywhere except at the origin \( r = 0 \). To perform our study we use the Einstein and M\ øller energy-momentum complexes. The calculations provide the well-defined expressions (26) and (29) for the energy distribution in both prescriptions. These energy distributions depend on the mass \( M \) and the charge \( q \) of the black hole, the cosmological constant \( \Lambda \) and the radial coordinate \( r \), while in both energy-momentum complexes all the momenta vanish.

In order to study the limiting behavior of the energy distributions obtained by the Einstein and M\ øller prescriptions, we consider the energy for \( r \to \infty \) in the uncharged case \( q = 0 \) for the massive \( (M \neq 0) \) black hole, and for \( r \to \infty, \Lambda = 0 \) and \( q = 0 \) for the massless \( (M = 0) \) black hole.

Starting with the massive black hole \( (M \neq 0) \) the results for the limiting cases \( r \to \infty \) and \( q = 0 \) are presented in Table 1.

| Energy | \( r \to \infty \) | \( q = 0 \) |
|--------|-----------------|--------------|
| \( E_E \) | \( \infty \) | \( \frac{r}{2} \sqrt{\frac{4M\Lambda r}{3} + \frac{\Lambda^2 r^4}{9}} \) |
| \( E_M \) | \( -\infty \) | \( -\frac{r^2}{12} \frac{12M\Lambda + 4\Lambda^2 r^3}{\sqrt{12M\Lambda r + \Lambda^2 r^4}} \) |

Table 1: Limiting behaviour of the energy of the massive \( (M \neq 0) \) black hole solution in the Einstein and M\ øller prescriptions.

For the charged \( (q \neq 0) \) black hole without cosmological constant \( (\Lambda = 0) \), the spacetime geometry becomes the Minkowski geometry. If the black hole is uncharged \( (q = 0) \) and the cosmological constant is non-zero \( (\Lambda > 0) \), then the spacetime is regular everywhere except at the origin \( r = 0 \), as it is inferred from the calculation of the curvature invariants (Kretschmann and Ricci) in [50]. The evolution of the Einstein energy and of the M\ øller energy with respect to the \( r \) coordinate, as given by Eqns. (26) and (29), is presented for \( \Lambda > 0 \) and \( \Lambda < 0 \) in Fig. 1.

In the massless case \( (M = 0) \), when the black hole is charged \( (q \neq 0) \) and the cosmological constant is non-zero \( (\Lambda \neq 0) \), the spacetime geometry is described by the line element (7). In fact, an event horizon appears at \( r_{eh} = \left[ \frac{q}{\Lambda} + \frac{6q^2}{\Lambda} \right]^{1/2} \) for a negative cosmological constant \( \Lambda < -3/(2q^2) \). Further, when \( \Lambda > 0 \), at the position \( r_s = \left( \frac{6q^2}{\Lambda} \right)^{1/4} \)
a singularity appears, while, when $\Lambda < 0$, this singularity can be avoided but there appears another singularity at $r = 0$ (see [50]). If this massless black hole has no charge ($q = 0$) but the cosmological constant in non-zero ($\Lambda \neq 0$), then a de Sitter spacetime geometry is obtained:

$$ds^2 = \left(1 - \sqrt{\frac{\Lambda^2 r^4}{9}}\right) dt^2 - \left(1 - \sqrt{\frac{\Lambda^2 r^4}{9}}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

with a cosmological horizon appearing at $r_{ch} = \sqrt{\frac{2}{\Lambda}}$. The energy of the massless black hole for the charged and the uncharged case, computed in the Einstein and Møller prescriptions, is presented in Table 2.

In Table 3 we present the limiting behaviour of the Einstein energy and the Møller energy for the charged and the uncharged massless ($M = 0$) black hole as $r \to \infty$. In both cases the cosmological constant is non-zero.

According to the obtained results, it can be inferred that the Einstein and Møller energy-momentum complexes provide a powerful tool for the study of the energy-momentum localization of gravitating systems. As a future perspective, the investigation of the problem of the energy-momentum localization, in the context of the black hole solution considered here, by applying other energy-momentum complexes as well as the teleparallel equivalent of general relativity (TEGR), is planned.
Energy \( q \neq 0, \Lambda \neq 0 \) \begin{align*}
E_E &= r \frac{\Lambda^2 r^4}{9} - \frac{2q^2 \Lambda}{3} \\
E_M &= -\frac{1}{3} \frac{\Lambda^2 r^5}{\sqrt{\Lambda^2 r^4 - 6q^2 \Lambda}}
\end{align*}

Energy \( q = 0, \Lambda \neq 0 \) \begin{align*}
E_E &= r \frac{\Lambda^2 r^4}{9} \\
E_M &= -\frac{1}{3} \frac{\Lambda^2 r^5}{\sqrt{\Lambda^2 r^4}}
\end{align*}

Table 2: Energy of the massless \((M = 0)\) black hole solution in the Einstein and Møller prescriptions.

Table 3: Limiting behaviour of the energy of the massless black hole solution in the Einstein and Møller prescriptions.

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