Regularity of Stationary Boltzmann Equation in Convex Domains

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Abstract

The higher regularity estimate has been a challenging question regarding the Boltzmann equation in bounded domains. Indeed, it is well-known to have "the non-existence of a second order derivative at the boundary" in Guo et al. (Invent Math 207:115–290, 2017) even for symmetric convex domains such as a disk or sphere. In this paper, we answer this question in the affirmative by constructing the $C^{1,\beta}$ solutions away from the grazing boundary, for any $\beta < 1$, to the stationary Boltzmann equation with the non-isothermal diffuse boundary condition in strictly convex domains, as long as a smooth wall temperature has small fluctuation pointwisely.

1. Introduction

An interesting physical application of the kinetic theory is its mesoscopic description of the heat transfer of rarefied gas. The quantitative description of the stationary state and a derivation of macroscopic models (as the Knudsen number $\kappa_n \to \infty$) can be achieved through the famous steady Boltzmann equation

$$v \cdot \nabla_x F = \frac{1}{\kappa_n} Q(F, F), \quad (x, v) \in \Omega \times \mathbb{R}^3,$$

where the hard sphere collision operator $Q(F, F)$ takes the form

$$Q(F_1, F_2) := Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2)$$

$$= \int_{\mathbb{R}^3} \int_{S^2} |(v - u) \cdot \omega|[F_1(u')F_2(v') - F_1(u)F_2(v)]d\omega du, \quad (1.2)$$

with $u' = u + [(v - u) \cdot \omega] \omega$, $v' = v - [(v - u) \cdot \omega] \omega$ with $\omega \in S^2$.

In particular, when the gas interacts with a non-isothermal boundary it is well-known that the non-equilibrium steady states can be constructed by the Boltzmann
equation (1.1). The kinetic description of the boundary interaction with the gas particles has been extensively investigated in various aspects (see [3–7,16,17,21,22] and the references therein). In this paper we are interested in one of the basic and physical conditions, the so-called diffuse reflection boundary condition, which takes into account an instantaneous thermal equilibrating with the non-constant wall temperature of a reflecting gas particle:

$$F(x,v)_{|n(x)\cdot v<0} = M_W(x,v) \int_{n(x)\cdot u>0} F(x,u)[n(x) \cdot u]du, \quad x \in \partial \Omega.$$  (1.3)

Here the outward normal at the boundary $\partial \Omega$ is denoted by $n(x)$, and we define the wall Maxwellian associated with the described wall temperature $T_W(x)$ at $x \in \partial \Omega$:

$$M_W(x,v) = \frac{2\pi}{T_W} M_{1,0,T_W} := \frac{1}{2\pi [T_W(x)]^2} e^{-\frac{|v|^2}{2T_W(x)}}.$$  (1.4)

Recently, a unique stationary solution of (1.1) with (1.3) in general bounded domains has been constructed in an $L^\infty$-space when the non-constant wall temperature is a small fluctuation around any constant temperature $T_0$ in [9] (see [15] for the construction in convex domains). Moreover, the authors prove that such non-equilibrium solutions are dynamically and asymptotically stable. We also refer to relevant literatures [11] and the references therein for the PDE aspects of non-equilibrium steady states. As an important application of such construction the authors further derive the Fourier law (Navier-Stokes-Fourier system, more precisely) rigorously as $Kn \to \infty$ in [10]. On the other hand, for each fixed finite Knudsen number $Kn$, they formulate a criterion of the Fourier law in mesoscopic level in [9]. Utilizing the available numeric results, they illustrate the violation of such a criterion, which demonstrates a deviation from the Fourier law for each fixed finite Knudsen number $Kn$.

Qualitatively the kinetic and macroscopic descriptions of heat transfer are remarkably different in the presence of boundaries in particular. In the absence of fluid velocity flow, a macroscopic description via the Fourier law is given by the Laplace equation with suitable boundary condition, which enjoys analytic smoothness of the solutions. On the other hand, the kinetic description from the Boltzmann equation (1.1) possesses a boundary singularity intrinsically ([18]). Such a drastic discrepancy comes from the convection effect $Kn v \cdot \nabla_x F$, which has small factor but non-zero for any finite Knudsen number $Kn > 0$. Indeed, it is very interesting to study the quantitative effect of such a convection term $Kn v \cdot \nabla_x F$ in the interaction of the boundary and collision process in the limiting process $Kn \to \infty$. Our work in this paper originates from this motivation.

As the first step toward this goal, in this paper we are looking for the smoothness of the stationary Boltzmann equation for fixed $Kn \sim 1$, comparable to the regularity of the corresponding (in a sense of $Kn \to \infty$) elliptic equation, for which the Schauder estimates are available. More precisely the main purpose of this paper is to develop a robust and unified higher regularity estimate in $C^{1,\beta}_x$ with the aid of weights for the stationary Boltzmann solutions to (1.1) with the diffuse reflection boundary condition (1.3) in the convex domains. For this purpose we focus on the
convex domain as a discontinuous singularity appears in the non-convex domain in general [13, 18].

In general convex domains, regularity estimates at most up to the first derivatives away from the so-called grazing set

\[ \gamma_0 := \{ (x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v = 0 \} \]

has been established in [1, 2, 14] for the nonlinear dynamical Boltzmann equation. The key idea of the approach is based on the so-called kinetic distance, which is almost invariant along the characteristics. With the aid of such weight a generic singularity of the first order derivatives can be controlled. We refer to [8] for the regularity of the stationary linear equation up to the first derivatives. However, any higher regularity beyond the first order derivatives away from the boundary has been a challenging question. Apparently any second order derivatives estimate seems impossible due to the well-known "non-existence of second order spatial normal derivative at the boundary" in [14] even in the convex domain, or even in symmetric domains. We note that the mechanism of such phenomenon is against the conventional effect of the collision in some sense, which will be described in Section 1.2. Throughout this paper we will use the following notations:

**Notations:**

- \( f \lesssim g \) ⇔ there exists \( 0 < C < \infty \) such that \( 0 \leq f \leq Cg \);
- \( f \sim g \) ⇔ there exists \( 0 < C < \infty \) such that \( 0 \leq f \leq Cg \leq Cf \);
- \( f \ll g \) ⇔ there exists a small constant \( c > 0 \) such that \( 0 \leq f \leq cg \);
- \( f = O(g) \) ⇔ \( |f| \lesssim g \);
- \( f = o(g) \) ⇔ \( |f| \ll g \).

### 1.1. Main theorem

Throughout this paper we assume the domain is defined as \( \Omega = \{ x \in \mathbb{R}^3 : \xi(x) < 0 \} \) via a \( C^3 \) function \( \xi : \mathbb{R}^3 \to \mathbb{R} \). Equivalently we assume that for all \( q \in \partial \Omega \), there exists a \( C^3 \) function \( \eta_q \) and \( 0 < \delta_1 \ll 1 \), such that

\[ \eta_q : B_+(0; \delta_1) \ni x_q := (x_{q,1}, x_{q,2}, x_{q,3}) \to \mathbb{R}^3, \]

where the map is one-to-one and onto to the image \( \mathcal{O}_q := \eta_q(B_+(0; \delta_1)) \) when \( \delta_1 \) is sufficiently small. Moreover, \( \eta_q(x_q) \in \partial \Omega \) if and only if \( x_{q,3} = 0 \) within the image of \( \eta_q \). We refer to [10] for the construction of such \( \xi \) and \( \eta_q \). We further assume that the domain is strictly convex in the following sense:

\[ \sum_{i,j=1}^3 \xi_i \xi_j \partial_i \partial_j \xi(x) \gtrsim |\xi|^2 \quad \text{for all} \quad x \in \tilde{\Omega} \quad \text{and} \quad \xi \in \mathbb{R}^3. \]

Without loss of generality we may assume that \( \nabla \xi \neq 0 \) near \( \partial \Omega \).

In order to control the generic singularity at the boundary we adopt the following weight of [14]:

**Definition 1.** For sufficiently small \( 0 < \varepsilon \ll \| \xi \|_{C^2} \), we define a kinetic distance

\[ \alpha(x, v) := \chi_{\varepsilon} (\tilde{\alpha}(x, v)), \quad (x, v) \in \tilde{\Omega} \times \mathbb{R}^3 \]

\[ \tilde{\alpha}(x, v) := \sqrt{|v \cdot \nabla_x \xi(x)|^2 - 2\xi(x)(v \cdot \nabla_x^2 \xi(x) \cdot v)}, \quad (x, v) \in \tilde{\Omega} \times \mathbb{R}^3. \]
where \( \chi_a : [0, \infty) \rightarrow [0, \infty) \) stands for a non-decreasing smooth function such that
\[
\chi_a(s) = s \text{ for } s \in [0, a], \quad \chi_a(s) = 2a \text{ for } s \in [4a, \infty],
\]
and \(|\chi_a'(\tau)| \leq 1\) for \( \tau \in [0, \infty) \).

We note that \( \alpha \equiv 0 \) on the grazing set \( \gamma_0 \). From a computation, we have
\[
|v \cdot \nabla_x \alpha(x, v)| \leq |v| \alpha(x, v),
\]
and together with \( \tau \chi_\varepsilon'(\tau) \leq \chi_\varepsilon(\tau) \), this implies
\[
e^{-|v|s} \alpha(x - sv, v) \leq \alpha(x, v) \leq e^{|v|s} \alpha(x - sv, v) \text{ as long as } x - sv \in \tilde{\Omega}.
\]
(1.10)

The definition of \( \alpha, \tilde{\alpha} \) in (1.8) implies
\[
\tilde{\alpha}(x, v) \geq \alpha(x, v).
\]
(1.11)

We extend the outward normal in the domain:
\[
n(x) := \chi'_\varepsilon/2(\text{dist}(x, \partial \Omega))\nabla \xi(x)/|\nabla \xi(x)| \text{ for all } x \in \tilde{\Omega}.
\]
(1.12)

In particular, we note that \( n(x) \equiv 0 \) when \( \text{dist}(x, \partial \Omega) \geq 2\varepsilon \). In order to explore the “better” behavior of the tangential derivative versus the normal derivative we define a \( G \)-derivative (which is a matrix)
\[
\nabla \parallel f(x) = G(x) \nabla_x f(x),
\]
(1.13)

where
\[
G(x) := (I - n(x) \otimes n(x)).
\]
(1.14)

Note that near the boundary, from (1.12) we have
\[
G(x)n(x) = 0 \text{ for } \text{dist}(x, \partial \Omega) \leq \varepsilon/2.
\]
(1.15)

From the definition of \( n \) in (1.12), the \( G \)-derivative is actually a full derivative away from the boundary: if \( \text{dist}(x, \partial \Omega) \geq 2\varepsilon \), then \( G(x) \nabla_x = \nabla_x \).

**Main Theorem.** Fix \( \kappa_\partial > 0 \). Assume the domain is convex (1.7) and the boundary is \( C^3 \). Suppose \( \sup_{x \in \Omega} |TW(x) - T_0| < 1 \) for some constant \( T_0 > 0 \) and \( TW(x) \in C^1(\partial \Omega) \). For given \( m > 0 \) we construct a unique solution
\[
F(x, v) = mM_{1,0,T_0}(v) + \sqrt{M_{1,0,T_0}(v)} f(x, v) \geq 0,
\]
(1.16)
to the stationary Boltzmann equation (1.1) and the diffuse reflection boundary condition (1.3) such that \( \int_{\Omega \times \mathbb{R}^3} f \sqrt{M_{1,0,T_0}(v)} = 0 \), and
\[
|wf|_{\infty} \lesssim T_W - T_0 |L^\infty(\beta \Omega), \quad w(v) := e^{\rho |v|^2} \text{ for some } 0 < \rho < 1/4.(1.17)
\]
Moreover, \( f \) (and \( F \)) belongs to \( C^1(\tilde{\Omega} \times \mathbb{R}^3 \setminus \gamma_0) \) locally and satisfies
\[
||w_{\partial}(v)\alpha(x, v)\nabla_x f(x, v)||_{L^\infty(\Omega \times \mathbb{R}^3)} \lesssim ||TW - T_0||_{C^1(\partial \Omega)},
\]
(1.18)
\[
||w_{\partial/2}(v)|v|\nabla f(x, v)||_{L^\infty(\Omega \times \mathbb{R}^3)} \lesssim ||TW - T_0||_{C^1(\partial \Omega)},
\]
(1.19)
\[
|||v|^2\nabla v f(x, v)||_{L^\infty(\Omega \times \mathbb{R}^3)} \lesssim ||TW - T_0||_{C^1(\partial \Omega)},
\]
(1.20)
where \( w_{\partial}(v) = e^{\tilde{\rho}|v|^2} \) with \( 0 < \tilde{\rho} \ll \rho \).
If we further assume $T_W(x) \in C^2(\partial \Omega)$, then for any $0 \leq \beta < 1$, the solution $F(x, v)$ belongs to $C^{1,\beta}(\bar{\Omega} \times \mathbb{R}^3 \setminus \gamma_0)$ locally. Moreover,

\begin{align}
\sup_{x,y \in \Omega} \| w_\beta(v)|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{2+\beta} \frac{\nabla_x f(x, v) - \nabla_x f(y, v)}{|x-y|^\beta} \|_{L^\infty(\mathbb{R}^3_v)} \\
\lesssim \| T_W - T_0\|_{C^2(\partial \Omega)}, \quad (1.21)
\end{align}

\begin{align}
\sup_{x,y \in \Omega} \| w_{\beta/2}(v)|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta} \frac{\nabla_v f(x, v) - \nabla_v f(y, v)}{|x-y|^\beta} \|_{L^\infty(\mathbb{R}^3_v)} \\
\lesssim \| T_W - T_0\|_{C^2(\partial \Omega)}, \quad (1.22)
\end{align}

\begin{align}
\sup_{x,y \in \Omega} \| w_{\beta/2}(v)|v|^3 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta} \frac{\nabla_v f(x, v) - \nabla_v f(y, v)}{|x-y|^\beta} \|_{L^\infty(\mathbb{R}^3_v)} \\
\lesssim \| T_W - T_0\|_{C^2(\partial \Omega)}. \quad (1.23)
\end{align}

**Remark 1.** The unique solvability and the pointwise estimate has been established in [9]. We record the statement of the theorem in Section 2 for the sake of readers’ convenience.

**Remark 2.** The second estimate (1.19) implies that any tangential spatial derivatives of $f(x, v)$ does not blow up near the grazing set. Also comparing the $C^{1,\beta}$ estimates (1.21) and (1.22), the weight in the semi-norm of the tangential spatial derivative has a lower power in terms of $\alpha$ than the one for the normal derivative.

**Remark 3.** Estimating differential quotient with respect to $v$ has some subtle (probably technical) issue, since the trajectory is not stable at $v = 0$ near the boundary. Since our motivation of the paper is investigating the regularity in space we omit to discuss them. This issue (instability of the trajectory at $v = 0$ in the Hölder norm estimate) will be discussed in a forthcoming paper, [20].

### 1.2. Major difficulties

In this section we illustrate the major difficulties, and in the next sections we will explain the key ideas and analytical development to overcome such obstacles. A generic feature of the boundary problem of the Boltzmann equation is a singularity of solutions, which originates mainly from 1) characteristics feature of the phase boundary $\partial \Omega \times \mathbb{R}^3$ with respect to the transport operator (i.e. the phase boundary is always characteristic but not uniformly characteristic at the grazing set $\gamma_0$ of (1.5)), and 2) the mixing effect of the collision operator.

The effect of characteristics phase boundary can appear in several ways. Depending on the shape of the domain, the generic boundary singularity at $\gamma_0$ can propagate inside the domain and affect the global dynamics. Indeed, it has been proved in [18] that any general non-convex domains admit smooth initial datum which will produce the discontinuity for the Boltzmann solution in a stable manner, which propagates along the trajectory. Although such discontinuity-type singularities may stay near the boundary for the convex domains, its derivatives blow up near the grazing set. Actually it is not merely the effect of characteristics phase boundary but also the mixing effect of the collision operator: the mixing immediately produces a singular source term for the normal derivative at the boundary. In
[14], the authors quantify the rate of the blow-up with respect to the kinetic distance of (1.8) and study the mixing effect by the collision operator in terms of the kinetic distance. As a result they establish the first order derivatives estimate for the dynamical Boltzmann equation. On the other hand, the kinetic distance produces a loss of moment and they utilize a fast decay weight $e^{-C(x)v_t}$ to recover such a loss. In other words, the success of the approach in [14] to the dynamic problem can be achieved in the space losing its exponential moment quickly (exponentially). Evidently utilizing such functional spaces is not possible in the stationary problem, which is one of the major difficulties to establish the main theorem.

The effect of the nonlinear collision operator is complex, in particular, within the interaction of the transport operator, which eventually restricts our regularity strictly below two derivatives in any $L^p$-space: the boundary singularity of Boltzmann solutions appears as $\frac{\partial F}{\partial n} \sim Q(F, F) \not\in L^1_{loc}$, while the leading order term of any second order derivatives $\nabla_x(x, v)\partial n$ contains a factor of $Q(\nabla_x F, F)(x_b(x, v), v)$ at a backward exit position $x_b(x, v)$ which is defined through a backward exit time $t_b$:

$$t_b(x, v) := \sup\{s > 0 : x - sv \in \Omega\}, \quad x_b(x, v) := x - t_b(x, v)v. \quad (1.24)$$

Due to a lack of symmetry of $\frac{\partial F}{\partial n}$, in particular for the diffuse reflection boundary condition, any possibility of cancellation in the integration formula $Q(\frac{\partial F}{\partial n}, F)$ can be expelled generically in [14]. Then it follows that $|\frac{\partial^2 F}{\partial n^2}(x, v)| = \infty$ for all $x \in \partial\Omega$. This singularity likely appears at all boundary points with all velocities then propagates along the trajectory inside the domain, and masses up all directional derivatives. Even strictly below the second derivatives estimate, at first glance it is not obvious that the similar failure is avoidable in our weighted $C^{1,\beta}$. Moreover, we encounter similar type of, but much more geometrically involved, terms associated with the diffuse reflection boundary condition intertwined with the transport operator. Such non-integrable singularities could barge in the higher order estimates, which is the other major difficulty of the proof.

### 1.3. Regularizing via the mixing of the binary collision, transport, and diffuse reflection

To overcome such difficulties described in Section 1.2., we establish a novel and robust quantitative estimate of regularization effect (in space and velocity) of the velocity mixing via the diffuse reflection boundary condition (1.3) or/and the binary collision (1.1) intertwined with the transport operator.

We demonstrate the scheme first for $\nabla_x F$, of which the most singular term comes from the boundary contribution such as

$$\nabla_x \partial n(x, v) \int_{n(x, v)^1 > 0} \nabla_x F(x_b(x, v), v^1)|n(x_b(x, v)) \cdot v^1|dv^1. \quad (1.25)$$

Upon using the transport operator once again, the contribution of the collision operator (ignoring the singularity of $Q$ for simplicity) can be viewed as

$$\nabla_x \partial n(x, v) \int_{n(x, v)^1 > 0} \int_0^{t_b(x_b, v)} \int_{\mathbb{R}^3} \nabla_x F(x_b - sv^1, u)|n(x_b) \cdot v^1|dudsv^1. \quad (1.26)$$
A key observation is that the \( x \)-derivative has a natural relation with the \( v^1 \)-derivative as

\[
\nabla_x F (x_b - (t^1 - s) v^1, u) = \frac{\nabla_{v^1} [F(x_b - (t^1 - s) v^1, u)]}{-(t^1 - s)}.
\tag{1.27}
\]

When \( t^1 - s \) has a positive lower bound, thanks to the \( v^1 \)-integral from the diffuse reflection boundary condition, we are able to remove such a \( v^1 \)-derivative completely from \( F \). As a result of the integration by parts, the singularity of \( \nabla_{v^1} f_b(x_b, v^1) \) occurs, which will be compensated by the boundary measure and thus we obtain a bound like \( \nabla_x x_b(x, v) \times \|F\|_\infty \). When \( t^1 - s \) is small we use the so-called the nonlocal-to-local estimate and derive \( O(|t^1 - s|) \alpha(x, v)^{-1} \|\alpha \nabla_x F\|_\infty \). We will describe the nonlocal-to-local estimate and its application in detail at the next subsection.

On the other hand, the boundary contribution of (1.25) upon applying the transport operator appears as

\[
\nabla_x x_b(x, v) \int_{n(x_b) \cdot v^1 > 0} \nabla_x x_b F(x_b(x^1, v^1), v^1) n(x^1) \cdot v^1 |dv^1|, \tag{1.28}
\]

where \( x^1 = x_b(x, v) \). The key idea is to convert \( v^1 \)-integration to the integration in \((x^2, t_b(x^1, v^1)) = (x_b(x^1, v^1), t_b(x^1, v^1))\), while the change of variables produces a factor of the Jacobian as \( \frac{|n(x^2) \cdot v^1|}{t_b(x^1, v^1)} \). Then we are able to move \( \nabla_x x_b \)-derivative from \( F \) via the integration by parts, while the derivative to the geometric components arise. Using the convexity and boundary measure crucially we are able to bound this amount by \( \nabla_x x_b(x, v) \times \|F\|_\infty \).

### 1.4. Higher regularity

For the higher regularity estimate in the weighted \( C^{1,\beta} \)-space, we 1) adopt the idea of Section 1.3 with stronger weight in \( \alpha \), 2) crucially establish a “better” estimate for the tangential derivatives, 3) use the full range of the nonlocal-to-local estimate, and 4) carefully study the possibly harmful (which has been explained in the last paragraph of Section 1.2.) term

\[
\frac{1}{|x - y|^{\beta}} \int_{t_b(x, y)}^{t_b(x, v)} Q(\nabla_x F, F)(x - sv, v) \, ds.
\]

By expressing \( \nabla_x F(x, v) - \nabla_x F(y, v) \) along the trajectories (see (7.36)–(7.48) for the details), we notice that the difference is singular at least as

\[
\frac{\nabla_x x_b(x, v) - \nabla_x x_b(y, v)}{|x - y|^{\beta}} \int_{n(x_b) \cdot v^1 > 0} \nabla_x x_b F(x_b, v^1) n(x_b) \cdot v^1 |dv^1|, \tag{1.29}
\]

where the integration is bounded using the weighted \( C^1 \)-estimate. By the mean value type estimate and the computation of \( \nabla^2_x x_b \), for \( x(\tau) = \tau x + (1 - \tau)y \), we derive that the difference quotient of \( \nabla_x x_b \) is bounded by

\[
|x - y|^{1-\beta} \int_{0}^{1} \frac{|v|^3}{\alpha^3(x(\tau), v)} \, d\tau. \tag{1.30}
\]
We prove that $\alpha(x(\tau), v) \geq \min\{\alpha(x, v), \alpha(y, v)\}$ for $|x - y| < \min\{\frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|}\}$ in the convex domains, for which we use the weight of $\min\{\frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|}\}^{2+\beta}$ for $\nabla_x F(x, v) - \nabla_x F(y, v)$. The convexity of the domain is crucial since any similar type of the bound is false for the non-convex domains in general.

Unfortunately this estimate with the weight of the power $2 + \beta$ is too singular! In particular the difference quotient of $\nabla \parallel$ of the scheme of Section 1.3 when the trajectories from two different points hit the boundary. Then we adopt the idea examined through (1.29), which turns out to be $1\alpha(\text{weight min})\parallel$. We prove that $\parallel 1106$ Hongxu Chen & Chanwoo Kim

In this paper we elaborate the so-called nonlocal-to-local estimate, which consists of analytical and geometrical arguments: first we study the integrand and derive a gain of power such as, for $1, \beta < 3$

$$\|\min\left\{\frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|}\right\}^\beta \nabla F\|_\infty \times \int_{\text{small interval}} \frac{1}{\min\left\{\frac{\alpha(x-sv, u)}{|u|}, \frac{\alpha(y-sv, u)}{|u|}\right\}^\beta} \, du \, ds. \quad (1.31)$$

The second author and collaborators studied a similar estimate of (1.31) in [14]. In this paper we elaborate the so-called nonlocal-to-local estimate, which consists of analytical and geometrical arguments: first we study the $u$-integration of the integrand and derive a gain of power such as, for $1 < \beta < 3$

$$\frac{1}{\min\{\xi(x - sv, u), \xi(y - sv, u)\}^{\frac{\beta-1}{2}}}, \quad (1.32)$$

where $\xi(x)$ can be understood as the distance from $x$ to the boundary. Second we employ $s \mapsto \xi(x - sv, u)$ with the Jacobian $ds = \frac{1}{|u, \nabla \xi|} d\xi(x - sv, u)$ and recover a power of $\alpha$ as in the bound of $\xi$ through the geometric velocity lemma. We crucially utilize such a gain of $\alpha$ to extract a smallness in (1.31).

Lastly we discuss the possible harmful term $\frac{1}{|x - y|^\beta} \int_{t_b(x, v)}^{t_b(y, v)} Q(\nabla_x F, F)(x - sv, v) \, ds$. First we apply the $\alpha$-weighted bound for $\nabla_x F$ and then establish $Q(\nabla_x F, F)(x - sv, v) \sim \ln|\xi(x - sv)|$. Upon the time integration on $[t_b(y, v), t_b(x, v)]$ we derive a bound

$$\min\{\frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|}\} \ln\left(\min\{\frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|}\}\right).$$

For $|x - y| < \min\{\frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|}\}$, we realize the difference quotient is bounded. Of course such bound blows up if $\beta = 1$. 

Now we state the outline for our paper. In Section 2 we prove several lemmas which serve as preliminary. Section 3 and Section 4 are devoted to establish the ideas in Section 1.3 as well as the nonlocal-to-local estimate and (1.18). In Section 5 and Section 6 we establish the rest of weighted $C^1$ estimates. Finally, in Section 7 we prove the weighted $C^{1,\beta}$ estimate.

2. Preliminaries

2.1. Basic notions

We first record the unique existence theorem of [9].

**Existence Theorem of [9]** Assume the domain is open bounded and the boundary is smooth. For $m > 0$ and $0 < \rho < 1/4$, if $\sup_{x \in \partial \Omega} |T_W(x) - T_0| \ll 1$, then there exists a unique mild solution

$$F(x, v) = m M_{1,0,T_0}(v) + \sqrt{M_{1,0,T_0}(v)} f(x, v) \geq 0,$$

with $\int_{\Omega \times \mathbb{R}^3} f \sqrt{M_{1,0,T_0}(v)} = 0$ to (1.1) and (1.3) such that

$$\| w f \|_{L^\infty(\hat{\Omega} \times \mathbb{R}^3)} := \| w f \|_{L^\infty(\partial \Omega)} \lesssim \| T_W - T_0 \|_{L^\infty(\partial \Omega)},$$

$$w(v) := e^{\rho |v|^2} \text{ with } 0 < \rho < 1/4.$$

Without loss of generality, we assume $m = 1$, $T_0 = 1$ in (1.16). Then we define the reference global Maxwellian and its perturbation:

$$\mu := M_{1,0,1}, \quad F(x, v) = \mu(v) + \sqrt{\mu} f(x, v).$$

Plugging (1.16) into (1.1) and (1.3), we obtain the equation and boundary condition for $f$:

$$v \cdot \nabla_x f + v f = K(f) + \Gamma(f, f),$$

$$f(x, v)|_{n(x) \cdot v < 0} = \frac{M_W(x, v)}{\sqrt{\mu(v)}} \int_{n(x) \cdot u > 0} f(x, u) \sqrt{\mu(u)}|n(x) \cdot u| du + r(x, v).$$

Here $v(v), K(f), \Gamma(f, f)$ are the linear Boltzmann operator(see [12]) given by

$$v(v) f = - \frac{Q(\mu, \sqrt{\mu} f)}{\sqrt{\mu}}, \quad K(f) = \frac{Q(\sqrt{\mu} f, \mu)}{\sqrt{\mu}},$$

$$\Gamma(f, f) = \frac{Q(\sqrt{\mu} f, \sqrt{\mu} f)}{\sqrt{\mu}}.$$}

The $r(x, v)$ is the remainder term. By $\sqrt{2\pi} \int_{n(x) \cdot u > 0} \sqrt{\mu(u)}|n(x) \cdot u| du = 1$, this term is given by

$$r(x, v) := \frac{M_W(x, v)}{\sqrt{2\pi} - \mu(v)}.$$
Consider a linear transport equation with the inflow boundary condition
\[
v \cdot \nabla_x f + v(x)f = h(x, v), \quad (x, v) \in \Omega \times \mathbb{R}^3,
\]
\[
f(x, v) = g(x, v), \quad (x, v) \in \gamma_-.
\]
(2.7)
(2.8)
As we can not rely on the Gronwall-type estimate, we will use the Duhamel's formula to express the equation along the trajectory:
\[
f(x, v) = \mathbf{1}_{t \geq t_b} e^{-v(x)v(t)} f(x_b(x, v), v) + \mathbf{1}_{t < t_b} e^{-v(x)v(t)} f(x - tv, v)
\]
\[
\int_{\max\{0, t - t_b\}}^{t} e^{-v(x)v(t-s)} h(x - (t-s)v, v)dv.
\]
(2.9)
Here we fix \(t \gg 1\).

In order to obtain \(C^1\) estimate we take the spatial derivative to (2.9) to get
\[
\partial_{x_j} f(x, v) = \mathbf{1}_{t \geq t_b} e^{-v(x)v(t)} \partial_{x_j} [f(x_b(x, v), v)]
\]
\[
- \mathbf{1}_{t \geq t_b} v(x) \partial_{x_j} f(x_b(x, v), v) e^{-v(x)v(t)} f(x_b(x, v), v)
\]
\[
+ \mathbf{1}_{t < t_b} e^{-v(x)v(t)} \partial_{x_j} [f(x - tv, v)]
\]
\[
+ \int_{\max\{0, t - t_b\}}^{t} e^{-v(x)v(t-s)} \partial_{x_j} [h(x - (t-s)v, v)]dv
\]
\[
- \mathbf{1}_{t \geq t_b} \partial_{x_j} t_b e^{-v(x)v(t)} h(x - t_bv, v),
\]
(2.10)
(2.11)
(2.12)
(2.13)
(2.14)
where \(x_b(x, v)\) and \(t_b(x, v)\) represent the backward exit position and time which are defined in (1.24). The derivative of \(t_b(x, v)\) and \(x_b(x, v)\) has singular behavior as stated in (2.32), such singularity will be cancelled by our weight \(\omega\) defined in (1.8). With a compatibility condition it is standard to check the piecewise formula (2.10)–(2.14) is actually a weak derivative of \(f\) and continuous across \(\{t = t_b(x, v)\}\) (see [13]) for the details.

**Definition 2.** Recall the backward exit position \(x_b\) and backward exit time \(t_b\) in (1.24), we define a stochastic cycles as \((x^0, v^0) = (x, v) \in \bar{\Omega} \times \mathbb{R}^3\) and inductively
\[
x^1 := x_b(x, v), \quad v^1 \in \{v^1 \in \mathbb{R}^3 : n(x^1) \cdot v^1 > 0\},
\]
\[
v^k \in \{v^k \in \mathbb{R}^3 : n(x^k) \cdot v^k > 0\}, \quad \text{for } k \geq 1,
\]
\[
x^{k+1} := x_b(x^k, v^k), \quad v_b := t_b(x^k, v^k) \text{ for } n(x^k) \cdot v^k \geq 0.
\]
(2.15)
(2.16)
(2.17)
Choose \(t \geq 0\). We define \(t^0 = t\) and
\[
t^k = t - \{t_b + t^1 + \cdots + t^{k-1}\}, \quad \text{for } k \geq 1.
\]
(2.18)

**Remark 4.** Here \(x^{k+1}\) depends on \((x, v, x^1, v^1, \ldots, x^k, v^k)\), while \(v^k\) is a free parameter whose domain (2.16) only depends on \(x^k\).
Recall (1.6). Since the boundary is compact and $C^3$, for fixed $0 < \delta_1 \ll 1$ we may choose a finite number of $p \in \bar{P} \subset \partial \Omega$ and $0 < \delta_2 \ll 1$ such that $\mathcal{O}_p = \eta_p (B_+ (0; \delta_1)) \subset B (p; \delta_2) \cap \Omega$ and $\{ \mathcal{O}_p \}$ forms a finite covering of $\partial \Omega$. We further choose an interior covering $\mathcal{O}_0 \subset \Omega$ such that $\{ \mathcal{O}_p \}_{p \in \mathcal{P}}$ forms an open covering of $\tilde{\Omega}$. We define a partition of unity

$$1_{\tilde{\Omega}} (x) = \sum_{p \in \mathcal{P}} t_p (x) \text{ such that } 0 \leq t_p (x) \leq 1, \quad t_p (x) \equiv 0 \text{ for } x \notin \mathcal{O}_p. \quad (2.19)$$

Without loss of generality (see [19]) we can always reparametrize $\eta_p$ such that $\partial x_{p,i} \eta_p \neq 0$ for $i = 1, 2, 3$ at $x_{p,3} = 0$, and an orthogonality holds as

$$\partial x_{p,i} \eta_p \cdot \partial x_{p,j} \eta_p = 0 \text{ at } x_{p,3} = 0 \text{ for } i \neq j \text{ and } i, j \in \{1, 2, 3\}. \quad (2.20)$$

At $x_{p,3} = 0$, the $x_{p,3}$ derivative gives the outward normal

$$n_p (x_p) = \frac{\partial x_{p,3} \eta_p}{\langle \partial x_{p,3} \eta_p, \partial x_{p,3} \eta_p \rangle}. \quad (2.21)$$

For simplicity, we denote

$$\partial_i \eta_p (x_p) := \partial x_{p,i} \eta_p. \quad (2.22)$$

**Definition 3.** For $x \in \tilde{\Omega}$, we choose $p \in \mathcal{P}$ as in (1.6). We define

$$T_{x_p} = \left( \begin{array}{c} \frac{\partial_1 \eta_p (x_p)}{\sqrt{g_{p,11}} (x_p)} \frac{\partial_2 \eta_p (x_p)}{\sqrt{g_{p,22}} (x_p)} \frac{\partial_3 \eta_p (x_p)}{\sqrt{g_{p,33}} (x_p)} \end{array} \right)^T, \quad (2.23)$$

with $g_{p,ij} (x_p) = \langle \partial_i \eta_p (x_p), \partial_j \eta_p (x_p) \rangle$ for $i, j \in \{1, 2, 3\}$. Here $A^t$ stands the transpose of a matrix $A$. Note that when $x_{p,3} = 0$, $T_{x_p} \frac{\partial_3 \eta_p (x_p)}{\sqrt{g_{p,33}} (x_p)} = e_i$ for $i = 1, 2, 3$ where $\{e_i\}$ is a standard basis of $\mathbb{R}^3$.

We define

$$v_j (x_p) = \frac{\partial_j \eta_p (x_p)}{\sqrt{g_{p,jj}} (x_p)} \cdot v. \quad (2.24)$$

We note that from (2.20), the map $T_{x_p}$ is an orthonormal matrix when $x_{p,3} = 0$. Therefore both maps $v \rightarrow v (x_p)$ and $v (x_p) \rightarrow v$ have a unit Jacobian at $x_{p,3} = 0$. This fact induces a new representation of boundary integration of diffuse boundary condition in (2.4): For $x \in \partial \Omega$ and $p \in \mathcal{P}$ as in (1.6),

$$\int_{n (x) \cdot v > 0} f (x, v) \sqrt{\mu (v)} \{n (x) \cdot v\} dv = \int_{x_{p,3} > 0} f (\eta_p (x_p), T_{x_p}^t v (x_p)) \sqrt{\mu (v (x_p))} v_3 (x_p) dv (x_p). \quad (2.25)$$

We have used the fact of $\mu (v) = \mu (|v|) = \mu (|T_{x_p}^t v (x_p)|) = \mu (|v (x_p)|) = \mu (v (x_p))$ and $x_{p,3} = 0$.

Now we reparametrize the stochastic cycle using the local chart defined in Definition 2.
Definition 4. Recall the stochastic cycles (2.16). For each cycle \( x^k \) let us choose \( p^k \in \mathcal{P} \) in (1.6). Then we denote

\[
x^k_{p^k} := (x^k_{p^k,1}, x^k_{p^k,2}, 0) \quad \text{such that} \quad \eta_{p^k}(x^k_{p^k}) = x^k, \quad \text{for } k \geq 1,
\]

\[
v^k_{p^k,i} := \frac{\partial_j \eta_{p^k}(x^k_{p^k})}{\sqrt{g_{p^k,jj}(x^k_{p^k})}} \cdot v^k, \quad \text{for } k \geq 1.
\] (2.26)

From (2.21) we denote the outward normal at \( x^k \) as

\[
n(x^k) = n_{p^k}(x^k_{p^k}).
\] (2.27)

Conventionally, we denote

\[
x^0_{p^0} := x^0 = x, \quad v^0_{p^0} := v^0 = v.
\] (2.28)

We define

\[
\partial_{x^k_{p^k,i}} \left[ a(\eta_{p^k}(x^k_{p^k}), v^k) \right] := \frac{\partial \eta_{p^k}(x^k_{p^k})}{\partial x^k_{p^k,i}} \cdot \nabla_x a(\eta_{p^k}(x^k_{p^k}), v^k), \quad i = 1, 2.
\] (2.29)

Conventionally we denote

\[
\nabla_x a(x^k, v^k) = (\partial_{x^k_{p^k,1}} \left[ a(\eta_{p^k}(x^k_{p^k}), v^k) \right], \partial_{x^k_{p^k,2}} \left[ a(\eta_{p^k}(x^k_{p^k}), v^k) \right]).
\]

2.2. Properties of stochastic cycle

In this subsection we list useful properties of the stochastic cycle defined in Definitions 2 and 4.

Lemma 2.1. For the \( t_b \) and \( x_b \) defined in (2.16) and (2.17), the derivative reads

\[
\frac{\partial t^{k+1}_b}{\partial x^{k+1}_j} = \frac{1}{v^3(x^{p^k+2})} \frac{\partial_3 \eta_{p^{k+2}}(x^{k+2})}{\sqrt{g_{p^{k+2},33}(x^{k+2})}} \cdot e_j,
\] (2.30)

\[
\frac{\partial t^{k+1}_b}{\partial v^{k+1}_j} = \frac{i^{k+1}_b}{v^3(x^{p^k+2})} - \frac{\partial_3 \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}(x^{k+2})}} |_{x^{k+2}} \cdot e_j.
\] (2.31)

And thus

\[
\nabla_x t_b = \frac{n(x_b)}{n(x_b) \cdot v} \cdot \nabla_v t_b = -\frac{i_b n(x_b)}{n(x_b) \cdot v}.
\]

\[
\nabla_x x_b = I d_{3 \times 3} - \frac{n(x_b) \otimes v}{n(x_b) \cdot v} \cdot \nabla_v x_b = -i_b Id + \frac{i_b n(x_b) \otimes v}{n(x_b) \cdot v}.
\] (2.32)

For \( i = 1, 2 \),
\[
\frac{\partial x_{j}^{k+1}}{\partial x_{j}^{k+1}} = \frac{1}{\sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \left[ \frac{\partial \eta_{p_{k+2}}(x_{p_{k+2}}^{k+2})}{\partial x_{j}^{p_{k+2},i}} \frac{v_{k+2}^{p_{k+2},i}}{\sqrt{g_{p_{k+2},i}^{p_{k+2}}}} - \frac{\partial \eta_{p_{k+2}}(x_{p_{k+2}}^{k+2})}{\partial x_{j}^{p_{k+2},i}} \frac{v_{k+2}^{p_{k+2},i}}{\sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \right] \cdot e_{j},
\]

(2.33)

\[
\frac{\partial x_{j}^{k+2}}{\partial v_{j}^{p_{k+2},i}} = \frac{v_{k+2}^{p_{k+2},i}}{v_{k+2}^{p_{k+2},i} \sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \frac{\partial \eta_{p_{k+2}}(x_{p_{k+2}}^{k+2})}{\partial x_{j}^{p_{k+2},i}} \frac{v_{k+2}^{p_{k+2},i}}{\sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \cdot \eta_{p_{k+2}}(x_{p_{k+2}}^{k+2}) - \frac{\partial \eta_{p_{k+2}}(x_{p_{k+2}}^{k+2})}{\partial x_{j}^{p_{k+2},i}} \frac{v_{k+2}^{p_{k+2},i}}{v_{k+2}^{p_{k+2},i} \sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \cdot \eta_{p_{k+2}}(x_{p_{k+2}}^{k+2})
\]

(2.34)

Proof. First of all we have

\[
x_{j}^{k+2} = \eta_{p_{k+2}}(x_{p_{k+2}}^{k+2}) = x_{j}^{k+1} - t_{b}^{k+1} v_{j}^{k+1} = \eta_{p_{k+2}}(x_{p_{k+2}}^{k+2}) - t_{b}^{k+1} v_{j}^{k+1}.
\]

(2.36)

Proof of (2.30) We take \( \frac{\partial}{\partial x_{j}^{k+2}} \) to (2.36) to get

\[
\sum_{l=1,2} \frac{\partial x_{j}^{k+2}}{\partial x_{j}^{k+1}} \frac{\partial \eta_{p_{k+2}}}{\partial x_{j}^{k+2, l}} \bigg|_{x_{j}^{k+2}} = -t_{b}^{k+1} \frac{\partial v_{j}^{k+1}}{\partial x_{j}^{k+1}} - \frac{\partial t_{b}^{k+1}}{\partial x_{j}^{k+1}} v_{j}^{k+1} + e_{j}
\]

(2.37)

Then we take an inner product with \( \frac{\partial \eta_{p_{k+2}}}{\sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \bigg|_{x_{j}^{k+2}} \) to (2.37) to have

\[
\sum_{l=1,2} \frac{\partial x_{j}^{k+2}}{\partial x_{j}^{k+1}} \frac{\partial \eta_{p_{k+2}}}{\partial x_{j}^{k+2, l}} \bigg|_{x_{j}^{k+2}} \cdot \frac{\partial \eta_{p_{k+2}}}{\sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \bigg|_{x_{j}^{k+2}} = -t_{b}^{k+1} \frac{\partial v_{j}^{k+1}}{\partial x_{j}^{k+1}} \frac{\partial \eta_{p_{k+2}}}{\sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \bigg|_{x_{j}^{k+2}} + e_{j} \cdot \frac{\partial \eta_{p_{k+2}}}{\sqrt{g_{p_{k+2},i}^{p_{k+2}}}} \bigg|_{x_{j}^{k+2}}.
\]

(2.38)
Due to (2.20) the LHS equals zero. Now we consider the RHS. From (2.24)
\[ v^{k+1} \cdot \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}} |_{x^{k+2}} = v_3(x_{p^{k+2}}). \]

From (2.38), we conclude (2.30). \[ \square \]

**Proof of (2.31)** We apply \( \partial v_j^{k+1} \) to (2.36) and take \( \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}} |_{x^{k+2}} \) to have
\[ \frac{\partial x^{k+2}}{\partial v_j^{k+1}} \cdot \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}} |_{x^{k+2}} = \sum_{l=1}^{2} \frac{\partial x_j^{k+2}}{\partial x_j^{k+1}} \frac{\partial x_j^{k+2}}{\partial v_j^{k+1}} |_{x^{k+2}} \cdot \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}} |_{x^{k+2}} = - \left\{ \partial x_j^{k+1} e_j + v_j^{k+1} \frac{\partial \eta_{p^{k+2}}}{\partial v_j^{k+1}} \right\} \cdot \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}} |_{x^{k+2}}, \]
Thus we apply (2.20) and (2.26) and use (2.24) to obtain (2.31). \( \square \)

**Proof of (2.32)** The first line of (2.32) follows directly from (2.30) and (2.31). For the second line we take \( \partial x_j^{k+1} \) and \( \partial v_j^{k+1} \) to (2.36). Again using (2.30) and (2.31) we conclude (2.32). \( \square \)

**Proof of (2.33)** We take inner product with \( \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}} |_{x^{k+2}} \) to (2.37) to have
\[ \sum_{l=1}^{2} \frac{\partial x_j^{k+2}}{\partial x_j^{k+1}} \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}} |_{x^{k+2}} \cdot \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}} |_{x^{k+2}} = \frac{\partial x_j^{k+1} e_j + v_j^{k+1} \frac{\partial \eta_{p^{k+2}}}{\partial v_j^{k+1}}}{\sqrt{g_{p^{k+2},ii}}} |_{x^{k+2}}. \]
By (2.24),
\[ v_j^{k+1} \cdot \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}} |_{x^{k+2}} = v_j(x_{p^{k+2}}) / \sqrt{g_{p^{k+2},ii}}. \]
Then, from (2.30), we conclude (2.33). \( \square \)

**Proof of (2.34)** Since
\[ \frac{\partial x_j^{k+2}}{\partial x_j^{k+1}} \frac{\partial x_j^{k+2}}{\partial v_j^{k+1}} = \nabla_{x^{k+1}} x_j^{k+2} \cdot \frac{\partial x_j^{k+1} \eta_{p^{k+1}}}{\partial v_j^{k+1}} |_{x^{k+1}, p^{k+1} = i}, \]
by (2.33) we conclude (2.34). \( \square \)

**Proof of (2.35)** For \( i = 1, 2, j = 1, 2, 3 \), we apply \( \partial v_j^{k+1} \) to (2.36) and take
\[ \frac{\partial \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}} |_{x^{k+2}} \]
to obtain
\[ \quad \]
Proof. Clearly, (2.39) follows from Lemma 2.2. and thus
\[ \text{where we have used (2.39) in the last inequality.} \]

The following two lemmas are immediate consequences of Lemma 2.1.

Lemma 2.2.
\[ t_b(x, v) \lesssim \frac{|n(x_b(x, v)) \cdot v|}{|v|^2}, \]  
(2.39)

and thus
\[ |\nabla v t_b| \lesssim \frac{1}{|v|^2}, \quad |\nabla v x_b| \lesssim \frac{1}{|v|}. \]  
(2.40)

Proof. Clearly, (2.39) follows from \( \frac{n(x_b(x, v)) \cdot v}{|v|^2} \gtrsim x - x_b = t_b |v| \).

By (2.32) and (2.39) we have
\[
|\nabla v t_b| \lesssim \frac{|n(x_b(x, v)) \cdot v|}{|v|^2} \leq \frac{1}{|v|^2},
\]
\[
|\nabla v x_b| \lesssim \frac{|n(x_b(x, v)) \cdot v|}{|v|^2} + \frac{|n(x_b(x, v)) \cdot v|}{|v|^2} \leq \frac{1}{|v|}.
\]

For (2.41) by the definition of \( T_{x_b} \) in (2.23), and using (2.35), we have
\[
|\nabla v T_{x_b}^i| \lesssim \frac{|\eta|}{|v|} \times |\nabla v [x_b^1, 1 + x_b^1]| \lesssim \frac{|\eta|}{|v|} \frac{|v| t_b}{|n(x_b(x_p^1))) \cdot v|} \lesssim \frac{|\eta|}{|v|},
\]
where we have used (2.39) in the last inequality.

Then the lemma follows.

Lemma 2.3. The following map is one-to-one:
\[
\begin{align*}
\nu^{k+1} &\in \{n(x^{k+1}) \cdot \nu^{k+1} > 0 : x_b(x^{k+1}, \nu^{k+1}) \in B(p^{k+2}, \delta_2) \} \\
&\mapsto (x^{k+2}_{p^{k+2}, 1}, x^{k+2}_{p^{k+2}, 2}, t_b^{k+1}) \quad \text{for} \quad k = 0, 1, \ldots, m - 1.
\end{align*}
\]  
(2.42)
with 
\[
\frac{\partial (x^{k+2}_{p^k+2,1}, x^{k+2}_{p^k+2,2}, t^{k+1}_b)}{\partial v^{k+1}} \cdot |t^{k+1}_b|^3
\]

\[
= \frac{1}{\sqrt{g_{p^k+2,11}(x^{k+2}_{p^k+2}) g_{p^k+2,22}(x^{k+2}_{p^k+2})}} |n(x^{k+2}) \cdot v^{k+1}| \quad (2.43)
\]

Proof. Combining (2.31) and (2.35) we conclude
\[
\det \left( \frac{\partial (x^{k+2}_{p^k+2,1}, x^{k+2}_{p^k+2,2}, t^{k+1}_b)}{\partial v^{k+1}} \right) \]

\[
= |t^{k+1}_b|^3 \det \left( \begin{array}{c}
\frac{1}{\sqrt{g_{p^k+2,11}(x^{k+2}_{p^k+2}) g_{p^k+2,22}(x^{k+2}_{p^k+2})}} |n(x^{k+2}) \cdot v^{k+1}|
\end{array} \right)
\]

\[
= -|t^{k+1}_b|^3 \frac{1}{v^{k+1}_{p^k+2,3}} \frac{1}{\sqrt{g_{p^k+2,11}(x^{k+2}_{p^k+2}) g_{p^k+2,22}(x^{k+2}_{p^k+2})}} \frac{\partial^3 n_{p^k+2}}{\partial x^{k+1}_{p^k+1,1} \partial x^{k+1}_{p^k+1,2} \partial n_{p^k+2}} |n(x^{k+2}) \cdot v^{k+1}|
\]

\[
\cdot \left( \begin{array}{c}
\frac{\partial^3 n_{p^k+2}}{\partial x^{k+1}_{p^k+1,1} \partial x^{k+1}_{p^k+1,2} \partial n_{p^k+2}} \bigg|_{x^{k+2}}
\end{array} \right)
\]

\[
\times \left( \begin{array}{c}
\frac{\partial^3 n_{p^k+2}}{\partial x^{k+1}_{p^k+1,1} \partial x^{k+1}_{p^k+1,2} \partial n_{p^k+2}} \bigg|_{x^{k+2}}
\end{array} \right)
\]

\[
= \frac{1}{\sqrt{g_{p^k+2,11}(x^{k+2}_{p^k+2}) g_{p^k+2,22}(x^{k+2}_{p^k+2})}} |n^{k+1}_b|^3 \quad (2.43),
\]

where we have used (2.20).

Now we prove the map (2.42) is one to one. Assume that there exists \(v\) and \(\tilde{v}\) satisfy
\[
\chi_b(x^{k+1}, v) = \chi_b(x^{k+1}, \tilde{v}) \text{ and } \iota_b(x^{k+1}, v) = \iota_b(x^{k+1}, \tilde{v}).
\]
We choose \(p \in \partial \Omega\) near \(x_b(x^{k+1}, v)\) and use the same parametrization. Then, by an expansion, for some \(\bar{v} \in \{a\bar{v} + (1-a)v : a \in [0,1]\},
\]

\[
0 = \left( \begin{array}{c}
\chi_{b,1}(x^{k+1}, \bar{v})
\chi_{b,2}(x^{k+1}, \bar{v})
\iota_{b,3}(x^{k+1}, \bar{v})
\end{array} \right) - \left( \begin{array}{c}
\chi_{b,1}(x^{k+1}, v)
\chi_{b,2}(x^{k+1}, v)
\iota_{b,3}(x^{k+1}, v)
\end{array} \right) = \left( \begin{array}{c}
\nabla_{\bar{v}} \chi_{b,1}(x^{k+1}, \bar{v})
\nabla_{\bar{v}} \chi_{b,2}(x^{k+1}, \bar{v})
\nabla_{\bar{v}} \iota_{b,3}(x^{k+1}, \bar{v})
\end{array} \right)(\bar{v} - v).
\]

This equality can be true only if the determinant of the Jacobian matrix equals zero. Then (2.43) implies that \(\iota_b(x^{k+1}, \tilde{v}) = 0\). But this implies \(x^{k+1} = x_b(x^{k+1}, \tilde{v})\) and hence \(n(x^{k+1}) \cdot \tilde{v} = 0\) which is out of our domain. \(\square\)
The next lemma describe the properties of a convex domain.

**Lemma 2.4.** Given a $C^2$ convex domain defined in (1.7),

$$|n_{p^{k+j}}(x_{pk+j}) \cdot (x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2}))| \sim |x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2})|^2, \ j = 1, 2, \quad (2.44)$$

For $j' = 1, 2$,

$$\left| \frac{\partial [n_{p^{k+j}}(x_{pk+j}) \cdot (x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2}))]}{\partial x_{p^{k+2}, j'}} \right| \lesssim \|\eta\|_{C^2} |x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2})|, \ j = 1, 2. \quad (2.45)$$

**Proof.** First we prove (2.44). By Taylor’s expansion, for $x, y \in \partial \Omega$ and some $0 \leq t \leq 1$,

$$\xi(y) - \xi(x) = 0 - 0 = \nabla \xi(x) \cdot (y-x) + \frac{1}{2} (y-x)^T \nabla^2 \xi(x + t(y-x))(y-x).$$

Thus, from (1.12),

$$|n(x) \cdot (x - y)| \sim (y-x)^T \nabla^2 \xi(x + t(y-x))(y-x).$$

From the convexity (1.7), we have

$$|n_{p^{k+j}}(x_{pk+j}) \cdot (x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2}))| \geq C\|x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2})|^2.$$ 

Since $\xi$ is $C^2$ at least,

$$|\{x^{k+1} - y\} \cdot n(x^{k+1})| \leq \|\xi\|_{C^2} |x^{k+1} - y|^2.$$

Also notice that

$$|n_{p^{k+1}}(x_{pk+1}^{k+1}) \cdot (x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2}))| = |v_{p^{k+1}, j}^{k+1}|(t^{k+2} - t^{k+1}),$$

and thus

$$\frac{|v_{p^{k+1}, j}^{k+1}|}{|v_{p^{k+1}}^{k+1}|} \geq \frac{C\|x^{k+1} - x^{k+2}|^2}{|v_{p^{k+1}}^{k+1}| |t^{k+1} - t^{k+2}|} \cdot \frac{|t^{k+1} - t^{k+2}|}{|x^{k+1} - x^{k+2}|} \cdot |v_{p^{k+1}, j}^{k+1}|^2 = C\|x^{k+1} - x^{k+2} = C\|x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2})|.$$

By the same computation we can easily conclude

$$\frac{|v_{p^{k+1}, j}^{k+1}|}{|v_{p^{k+1}}^{k+1}|} \leq C\|x^{k+1} - \eta_{p^{k+2}}(x_{pk+2}^{k+2})|.$$
Then we prove (2.45). For \( j = 1, j' = 1, 2 \) we have

\[
\left| \frac{\partial [n_{\alpha p+1}(x_{p+1}^{k+1}) \cdot (x^{k+1} - \eta_{\alpha p+2}(x_{p+2}^{k+2}))]}{\partial x_{p+2}^{k+2}, j'} \right| \\
\leq \left| n_{\alpha p+1}(x_{p+1}^{k+1}) \cdot \partial_j' \eta_{\alpha p+2}(x_{p+2}^{k+2}) \right| \\
= \left| n_{\alpha p+1}(x_{p+1}^{k+1}) \cdot \partial_j' \eta_{\alpha p+1}(x_{p+1}^{k+1}) \right| \\
+ n_{\alpha p+1}(x_{p+1}^{k+1}) \cdot \left[ \partial_j' \eta_{\alpha p+2}(x_{p+2}^{k+2}) - \partial_j' \eta_{\alpha p+1}(x_{p+1}^{k+1}) \right] \\
\leq 0 + \| \eta \|_{C^2} |x^{k+1} - \eta_{\alpha p+2}(x_{p+2}^{k+2})|,
\]

where we applied (2.20) and (2.27).

For \( j = 2 \), we have

\[
\left| \frac{\partial [n_{\alpha p+2}(x_{p+2}^{k+2}) \cdot (x^{k+1} - \eta_{\alpha p+2}(x_{p+2}^{k+2}))]}{\partial x_{p+2}^{k+2}, j'} \right| \\
\leq \left| n_{\alpha p+2}(x_{p+2}^{k+2}) \cdot \partial_j' \eta_{\alpha p+2}(x_{p+2}^{k+2}) \right| + \| \eta \|_{C^2} |x^{k+1} - \eta_{\alpha p+2}(x_{p+2}^{k+2})| \\
= \| \eta \|_{C^2} |x^{k+1} - \eta_{\alpha p+2}(x_{p+2}^{k+2})|,
\]

where we applied (2.20) and (2.27). \( \square \)

### 2.3. Properties of tangential derivative

Aiming to the regularity estimate of (1.19) without the \( \alpha \)–weight, we establish several properties of the tangential derivative. We summarize them in Lemma 2.5–2.7.

**Lemma 2.5.** For \( x = \eta_p(x_p) \in \partial \Omega \), we have the following equivalence:

\[
|G(x) \nabla_x f(x, v)| \sim \sum_{j=1,2} \partial_{x_p, j} f(\eta_p(x_p), v).
\]

**Proof.** By (2.23) we have

\[
\partial_i \eta_p(x_p) = \sqrt{g_{p,ii}(x_p)} T^i_{x_p} e_i.
\]

Denote \( \mathcal{F}(x) = \nabla_x f(x, v) T^i_{x_p} \), we have

\[
\sum_{j=1,2} \partial_{x_p, j} f(\eta_p(x_p), v) \\
= \sqrt{g_{p,11}(x_p)} \nabla_x f(x, v) T^i_{x_p} e_1 + \sqrt{g_{p,22}(x_p)} \nabla_x f(x, v) T^i_{x_p} e_2 \\
= \sqrt{g_{p,11}(x_p)} \mathcal{F} e_1 + \sqrt{g_{p,22}(x_p)} \mathcal{F} e_2.
\]
We also have
\[ G(x) \nabla_x f(x, v) = \nabla_x f \left( T^i \big|_{x_p} T_{x_p} - T^i \big|_{x_p} e_3 e_3^i T_{x_p} \right) \]
\[ = 3 \left( T_{x_p} - e_3 e_3^i T_{x_p} \right) = 3 \left( I - e_3 \otimes e_3 \right) T_{x_p} \]
\[ = (3 e_1 3 e_2 0) T_{x_p} = 3 e_1 \frac{\partial_1 \eta_p(x_p)}{\sqrt{g_{p,11}(x_p)}} + 3 e_2 \frac{\partial_2 \eta_p(x_p)}{\sqrt{g_{p,22}(x_p)}}. \]

Since \( \partial_1 \eta_p(x_p) \perp \partial_2 \eta_p(x_p) \),
\[ |G(x) \nabla_x f(x, v)| \approx \left| \sqrt{g_{p,11}(x_p)} 3 e_1 + \sqrt{g_{p,22}(x_p)} 3 e_2 \right| \approx \left| \sum_{j=1,2} \partial_{x_p, j} f(\eta_p(x_p), v) \right|. \]

**Lemma 2.6.** For any \( s \in [0, t_b] \), we have
\[ |G(x) - G(x - sv)| \lesssim \frac{\tilde{\alpha}(x, v)}{|v|}. \tag{2.48} \]

Thus
\[ |G(x) \nabla_x f(x - sv)| \lesssim \frac{\|w_{\tilde{\eta}/2}|v| \nabla f\| + \|w_{\tilde{\eta}/2} \alpha \nabla_x f\|_\infty}{|v| |w_{\tilde{\eta}/2}(v)|}. \tag{2.49} \]

**Proof.** By the definition (1.14), we have
\[ |G(x) - G(x - sv)| \lesssim |n(x - sv) \otimes (n(x - sv) - n(x))| \]
\[ + |(n(x - sv) - n(x)) \otimes n(x)|. \]

Then by (1.12) we have
\[ |\nabla x n(x)| \lesssim |\nabla_x \left[ \chi_{\tilde{\xi}/2}(\text{dist}(x, \partial \Omega)) \right] + |\chi_{\tilde{\xi}/2}(\text{dist}(x, \partial \Omega)) \nabla_x \frac{\nabla x \xi(x)}{|\nabla x \xi(x)|}| \]
\[ \lesssim |\chi'' \times \nabla x \text{dist}(x, \partial \Omega)| \]
\[ + |\chi_{\tilde{\xi}/2}(\text{dist}(x, \partial \Omega)) \frac{|\nabla x \xi(x)| |\nabla^2 \xi(x) - \nabla \xi(x) \otimes \nabla^2 \xi(x) / |\nabla \xi(x)|^2|}{|\nabla \xi(x)|^2}| \]
\[ \lesssim 1 + |\chi_{\tilde{\xi}/2}(\text{dist}(x, \partial \Omega)) | \frac{|\nabla^2 \xi(x)|}{|\nabla \xi(x)|}. \]

From (1.9) and (1.12) we have \( |\nabla \xi(x)| \gtrsim 1 \) when \( \text{dist}(x, \partial \Omega) \ll 1 \). When \( \text{dist}(x, \partial \Omega) \gtrsim 1 \) we take \( \varepsilon \) to be small enough such that \( \chi_{\varepsilon}(\text{dist}(x, \partial \Omega)) = 0 \). Hence
\[ |\nabla x n(x)| \lesssim \| \xi \|_{C^2}. \tag{2.50} \]

Then we use (2.39) to have
\[ |n(x - sv) - n(x)| \lesssim t_b |v| \| \xi \|_{C^2} \lesssim \frac{\tilde{\alpha}(x, v)}{|v|}. \]

Thus we conclude (2.48).
Last we prove (2.49). We rewrite

\[ |G(x)\nabla_x f(x - sv, v)| \]

\[ = |G(x - sv)\nabla_x f(x - sv, v) + [G(x) - G(x - sv)]\nabla_x f(x - sv, v)| \]

\[ \lesssim \frac{|w_{\tilde{\theta}/2}||\nabla f||_\infty}{|v||w_{\tilde{\theta}/2}(v)|} + \frac{\tilde{\alpha}(x, v)}{|v|} |\nabla_x f(x - sv, v)| \]

\[ \lesssim \frac{|w_{\tilde{\theta}/2}||\nabla f||_\infty}{|v||w_{\tilde{\theta}/2}(v)|} + \frac{|w_{\tilde{\theta}/2}\tilde{\alpha}\nabla_x f||_\infty}{|v||w_{\tilde{\theta}/2}(v)|} \]

\[ \lesssim \frac{|w_{\tilde{\theta}/2}||\nabla f||_\infty + |w_{\tilde{\theta}}\alpha\nabla_x f||_\infty}{|v||w_{\tilde{\theta}/2}(v)|}, \]

where we applied (1.10), and we used (1.8) to have \( w_{\tilde{\theta}/2}(v)\tilde{\alpha}(x, v) \lesssim w_{\tilde{\theta}}(v)\alpha(x, v) \).

Then we conclude the lemma.

Lemma 2.7. For \( x_b(x, v) = \eta_{p^1}(x_{p^1}) \) and \( i = 1, 2 \), we have

\[ \left| G(x)\nabla_x x_{p^1,i} \right| \lesssim 1, \quad (2.51) \]

\[ \left| G(x)\nabla_x x_b(x, v) \right| \lesssim \frac{1}{|v|}. \quad (2.52) \]

Proof. By (2.48) in Lemma 2.6 we have

\[ \left| G(x)\nabla_x x_{p^1,i} \right| \lesssim \left| G(x_b)\nabla_x x_{p^1,i} \right| + \tilde{\alpha}(x, v) \left| \nabla_x x_{p^1,i} \right|. \quad (2.53) \]

By (2.33), the definition of \( x^1_{p^1,3}, n(x_b) \) in (2.26) and (1.15), we have

\[ (2.53)_1 \lesssim 1 + |v|G(x_b) \frac{n(x_b)}{|n(x_b) \cdot v|} = 1. \]

Again by (2.33) and using definition of \( \tilde{\alpha} \) in (1.8), we have

\[ (2.53)_2 \lesssim \frac{\tilde{\alpha}(x, v)}{|v|} + \frac{\tilde{\alpha}(x, v)}{|v|} \frac{|v|}{|n(x_b) \cdot v|} \lesssim 1. \]

We conclude (2.51).

For (2.52) by (2.30) we have

\[ (2.53) \lesssim 0 + \frac{1}{|v||n(x_b) \cdot v|} \lesssim \frac{1}{|v|}. \]

We conclude (2.52). \qed
2.4. Properties of Hölder’s estimate

To prove the $C^{1,\beta}$ estimate (1.21)–(1.23) we need several $C^{1,\beta}$ estimate for $x_b$ and $t_b$. We summarize them in Lemma 2.8 and Lemma 2.10. Lemma 2.9 serves as a key ingredient to prove Lemma 2.8.

**Lemma 2.8.** We have the following estimates:

\[
\left| \frac{x_b(x, v) - x_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^\beta},
\]

(2.54)

\[
\left| \frac{t_b(x, v) - t_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{1}{|v| \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^\beta},
\]

(2.55)

\[
\left| \frac{e^{-C v_b(x,v) - e^{-C v_b(y,v)}}}{|x - y|^\beta} \right| \lesssim \left| \frac{e^{-C v_b(x,v) - e^{-C v_b(y,v)}}}{|x - y|^\beta} \right|
\]

(2.56)

\[
\left| \frac{n(x_b(x, v)) - n(x_b(y, v))}{|x - y|^\beta} \right| \lesssim \|\xi\|_{C^2} \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^\beta},
\]

(2.57)

\[
\left| \frac{\nabla_x y_b(x, v) - \nabla_x x_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{2+\beta}},
\]

(2.58)

\[
\left| \frac{\nabla_x y_b(x, v) - \nabla_x t_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{1}{|v| \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{2+\beta}},
\]

(2.59)

\[
\left| \frac{\nabla_v x_b(x, v) - \nabla_v x_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}},
\]

(2.60)

\[
\left| \frac{\nabla_v y_b(x, v) - \nabla_v y_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{1}{|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}},
\]

(2.61)

\[
\left| \frac{G(y) \nabla_x x_b(x, v) - \nabla_x x_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}},
\]

(2.62)

\[
\left| \frac{G(y) \nabla_x y_b(x, v) - \nabla_x y_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}},
\]

(2.63)

\[
\left| \frac{f_b(x, v) - f_b(y, v)}{|x - y|^\beta} \right| \lesssim \frac{\|w f_b\|_\infty^{1-\beta} \|w \alpha \nabla_x f_b\|_\infty^\beta}{w_\beta(v) \min \{\alpha(x, v), \alpha(y, v)\}^\beta}.
\]

(2.64)
For $x_b(x, v) = \eta_{p(x)}(x_{p(x)}^1), x_b(y, v) = \eta_{p(y)}(x_{p(y)}^1)$ (see the definition (7.4) in Section 7), $i \in \{1, 2\}$, we have

$$\frac{|x_{p(x)}^1 - x_{p(y)}^1|}{|x - y|^\beta} \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^\beta},$$

(2.65)

$$\frac{|T_{x_{p(x)}^1} - T_{x_{p(y)}^1}|}{|x - y|^\beta} \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^\beta},$$

(2.66)

$$\frac{|\nabla_x x_{p(x)}^1 - \nabla_x x_{p(y)}^1|}{|x - y|^\beta} \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^\beta},$$

(2.67)

$$\frac{|\nabla_v x_{p(x)}^1 - \nabla_v x_{p(y)}^1|}{|x - y|^\beta} \lesssim \frac{1}{|v| \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}},$$

(2.68)

$$\frac{|\nabla_v T_{x_{p(x)}^1} - \nabla_v T_{x_{p(y)}^1}|}{|x - y|^\beta} \lesssim \frac{1}{|v| \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}},$$

(2.69)

When $x, y \in \partial \Omega$,

$$\frac{|x_b(x, v) - x_b(y, v)|}{|x - y|} \lesssim 1,$$

(2.70)

for $\tilde{\theta} \ll 1$,

$$\frac{|M_w(x, v) - M_w(y, v)|}{\sqrt{\mu(v)|x - y|^\beta}} \lesssim \omega_{\tilde{\theta}}^{-1}(v) \|T_w - T_0\|_{C^1}.$$  

(2.71)

For $x = \eta_{p(x)}(x_{p(x)}), y = \eta_{p(y)}(x_{p(y)}) \in \partial \Omega$, and $x_b(x, v) = \eta_{p(x)}(x_{p(x)}^1), x_b(y, v) = \eta_{p(y)}(x_{p(y)}^1)$. For $i, j \in \{1, 2\}$ we have

$$\frac{|\partial_{x_{p(x)}, j} x_{p(x)}^1 - \partial_{x_{p(y)}, j} x_{p(y)}^1|}{|x - y|^\beta} \lesssim \frac{1}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^3}.$$  

(2.72)

We need the next lemma to prove this

**Lemma 2.9.** Define

$$x(\tau) := (1 - \tau)x + \tau y, \quad |\dot{x}(\tau)| = |x - y|.$$  

(2.73)

If $|x - y| \lesssim \varepsilon \min\left\{ \frac{\tilde{\alpha}(x, v)}{|v|}, \frac{\tilde{\alpha}(y, v)}{|v|} \right\} \ll 1$, then

$$\tilde{\alpha}(x(\tau), v) \gtrsim \min\left\{ \tilde{\alpha}(x, v), \tilde{\alpha}(y, v) \right\}.$$  

(2.74)
Proof. By the definition (1.8) we have
\[ \alpha^2(x(\tau), v) = |\nabla \xi(x(\tau)) \cdot v|^2 - 2\xi(x(\tau))(v \cdot \nabla^2 \xi(x(\tau)) \cdot v). \]  
(2.75)
We expand $|\nabla \xi(x(\tau)) \cdot v|^2$ and $-2\xi(x(\tau))(v \cdot \nabla^2 \xi(x(\tau)) \cdot v)$ separately: we expand in $\tau$ as
\[ |\nabla \xi(x(\tau)) \cdot v|^2 = |\nabla \xi(x(0)) \cdot v|^2 + \int_0^\tau d\tau' 2(\nabla \xi(x(\tau')) \cdot v)\dot{\xi}(\tau) \cdot \nabla^2 \xi(x(\tau')) \cdot v, \]  
(2.76)
\[ = -2\xi(x(\tau))(v \cdot \nabla^2 \xi(x(\tau)) \cdot v) + O(|x - y|)||\xi||_{C^3}|v|^2, \]  
(2.77)
where we have used (2.73).
For (2.76), we further expand in $\tau'$ and obtain
\[ (2.75) = |\nabla \xi(x) \cdot v|^2 + 2(\nabla \xi(x) \cdot v)O(|x - y|)|v||\xi|_{C^2} + \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \dot{x}(\tau'') \cdot \nabla^2 \xi(x(\tau'')) \cdot v \dot{x}(\tau'') \cdot \nabla^2 \xi(x(\tau'')) \cdot v \]  
(2.78)
\[ + \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' 2(\nabla \xi(x(\tau'')) \cdot v)\dot{x}(\tau'')\dot{x}(\tau'')\nabla^3 \xi(x(\tau'')) \cdot v \]  
(2.79)
From the convexity (1.7) we have
\[ (2.78) + (2.79) = O(1)|\dot{x}|^2||\xi||_{C^3}|v|^2 = O(\varepsilon^2) \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^2 |v|^2. \]  
(2.80)
From (2.80) we have
\[ (2.75) + 2\xi(x(\tau))O(|v|^2) = |\nabla \xi(x) \cdot v|^2 + O(\varepsilon)\alpha(x, v) \min \{\alpha(x, v), \alpha(y, v)\} \]  
(2.81)
\[ + O(\varepsilon^2) \min \{\alpha(x, v), \alpha(y, v)\}^2. \]
Now we claim
\[ -\xi(x(\tau)) \geq \min \{-\xi(x), -\xi(y)\}. \]  
(2.82)
From \( \frac{d}{d\tau}(-\xi(x(\tau))) = -\dot{x}(\tau) \cdot \nabla_x \xi(x(\tau)) \) and convexity (1.7),
\[ \frac{d^2}{d\tau^2}(-\xi(x(\tau))) = -\dot{x}(\tau) \cdot \nabla^2_x \xi(x(\tau)) \cdot \dot{x}(\tau) \leq -|\dot{x}(\tau)|^2 \leq 0. \]
Thus $-\xi(x(\tau))$ is a concave function of $\tau$. From $0 \leq \tau \leq 1$, we prove our claim (2.82) as
\(-\xi(x(t)) = -\xi(x((1-t) \cdot 0 + t \cdot 1)) \geq - (1-t)\xi(x(0)) - t\xi(x(1)) = - (1-t)\xi(x) - t\xi(y) \geq \min \{-\xi(x), -\xi(y)\} \).

Now combining (2.81) and (2.82) we conclude that
\[
(2.75) \gtrsim |\nabla \xi(x) \cdot v|^2 + \min \{-\xi(x), -\xi(y)\} |v|^2 \\
+ O(\varepsilon)\alpha(x, v) \min \{\tilde{\alpha}(x, v), \tilde{\alpha}(y, v)\} + O(\varepsilon^2) \min \{\tilde{\alpha}(x, v), \tilde{\alpha}(y, v)\}^2.
\]

Similarly we can set \(x(t) = (1-t)y + tx\). From \(x(0) = y\), following the same argument we derive
\[
(2.75) \gtrsim |\nabla \xi(y) \cdot v|^2 + \min \{-\xi(x), -\xi(y)\} |v|^2 \\
+ O(\varepsilon)\tilde{\alpha}(x, v) \min \{\tilde{\alpha}(x, v), \tilde{\alpha}(y, v)\} + O(\varepsilon^2) \min \{\tilde{\alpha}(x, v), \tilde{\alpha}(y, v)\}^2.
\]

From the definition of (1.8) using (2.83) and (2.84) we have
\[
(2.75) \geq \min \left\{ \tilde{\alpha}(x, v), \tilde{\alpha}(y, v) \right\}^2 - O(\varepsilon) \min \left\{ \tilde{\alpha}(x, v), \tilde{\alpha}(y, v) \right\}^2 .
\]

Hence from \(\varepsilon \ll 1\) we conclude (2.74). \(\square\)

Now we start the proof of Lemma 2.8.

Proof of Lemma 2.8. For all estimates we assume \(|x - y| \leq \varepsilon \min \{\frac{\tilde{\alpha}(x, v)}{|v|}, \frac{\tilde{\alpha}(y, v)}{|v|}\}\), otherwise the lemma follows immediately by (2.32). Thus we can apply (2.74) during the whole proof. We will use the \(x(t)\) defined in (2.73).

Proof of (2.54) We have
\[
\frac{|x_b(x, v) - x_b(y, v)|}{|x - y|^\beta} = \frac{1}{|x - y|^\beta} \int_0^1 d\tau \frac{d}{d\tau} \nabla x_b(x(t), v) \\
= \frac{1}{|x - y|^\beta} \int_0^1 |\dot{x}(t)| |\nabla x_b(x(t), v)| d\tau \\
\lesssim \frac{1}{|x - y|^\beta - 1} \int_0^1 \frac{|v|}{\tilde{\alpha}(x(t), v)} \lesssim \frac{|v|^\beta}{\min \{\alpha(x, v), \alpha(y, v)\}^{\beta}},
\]
where we have used Lemma 2.9, (1.11), (2.32) and \(\beta < 1\) in the last line. \(\square\)

Proof of (2.55) We have
\[
\frac{|t_b(x, v) - t_b(y, v)|}{|x - y|^\beta} = \frac{1}{|x - y|^\beta} \int_0^1 d\tau \frac{d}{d\tau} \nabla t_b(x(t), v) \\
= \frac{1}{|x - y|^\beta} \int_0^1 |\dot{x}(t)| |\nabla t_b(x(t), v)| d\tau \\
= |x - y|^{1-\beta} \int_0^1 \frac{1}{\tilde{\alpha}(x(t), v)} \lesssim \frac{|v|^\beta}{|v| \min \{\alpha(x, v), \alpha(y, v)\}^{\beta}},
\]
where we have used Lemma 2.9, (1.11), (2.32) and \(\beta < 1\) in the last line.
Proof of (2.56) The first inequality is clear since \(|e^{-CvB(x,v)} - e^{-CvB(y,v)}| \lesssim 1\). To prove the second inequality we have

\[
\frac{|e^{-CvB(x,v)} - e^{-CvB(y,v)}|}{|x - y|^{\beta}} = \frac{1}{|x - y|^{\beta}} \int_0^1 d\tau \frac{d}{d\tau} e^{-CvB(x(\tau),v)} \\
\lesssim \frac{1}{|x - y|^{\beta}} \int_0^1 d\tau (vB(x(\tau), v))e^{-CvB(x(\tau),v)}|x - y| \frac{1}{|n(B(x(\tau),v)) \cdot v|} \\
\lesssim |x - y|^{1-\beta} \frac{1}{\min \{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\}} \lesssim \frac{1}{|v| \min \left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{\beta}},
\]

where we have used (2.32) in the second line, Lemma 2.9 and (1.11) in the last line. □

Proof of (2.57) Since

\[
\frac{|n(xB(x,v)) - n(xB(y,v))|}{|x - y|^{\beta}} = \frac{|n(xB(x,v)) - n(xB(y,v))| |xB(x,v) - xB(y,v)|}{|xB(x,v) - xB(y,v)| |x - y|^{\beta}} \\
\lesssim \|\tilde{\xi}\|_{C^2} \frac{|xB(x,v) - xB(y,v)|}{|x - y|^{\beta}}.
\]

By (2.54) we derive (2.57). □

Proof of (2.58) We have

\[
\frac{|\nabla xB(x,v) - \nabla xB(y,v)|}{|x - y|^{\beta}} = \frac{1}{|x - y|^{\beta}} \int_0^1 d\tau \frac{d}{d\tau} \nabla xB(x(\tau),v) \\
= \frac{1}{|x - y|^{\beta}} \int_0^1 |\dot{x}(\tau)| |\nabla xB(x(\tau),v)| d\tau \\
\lesssim |x - y|^{1-\beta} \int_0^1 \frac{|v|^3}{|\tilde{\alpha}(x(\tau),v)|^3}.
\]

Here we have used (2.32) to have

\[
\frac{|\nabla xB(x(\tau),v)|}{|\nabla xB(x(\tau),v)|} \\
\lesssim \left[\frac{\|\eta\|_{C^2} |v| |n(xB(x(\tau),v)) \cdot v|}{|n(xB(x(\tau),v)) \cdot v|^2} + \|\eta\|_{C^2} \frac{|v||n(xB(x(\tau),v)) \otimes v|}{|n(xB(x(\tau),v)) \cdot v|^2}\right] \\
\times |\nabla xB(x(\tau),v)| \\
\lesssim \frac{|v|^2}{|n(xB(x(\tau),v)) \cdot v|^2} + \frac{|v|^3}{|n(xB(x(\tau),v)) \cdot v|^3} \lesssim \frac{|v|^3}{|n(xB(x(\tau),v)) \cdot v|^3},
\]

where we have used \(|n(xB(x(\tau),v)) \cdot v| \leq |v|\) in the last inequality. Then by Lemma 2.9 and (1.11) we obtain (2.58). □
Proof of (2.59) We have

\[
\frac{|\nabla_{x} f_{b}(x, v) - \nabla_{x} f_{b}(y, v)|}{|x - y|\beta} = \frac{1}{|x - y|\beta} \int_{0}^{1} d\tau \frac{d}{d\tau} \nabla_{x} f_{b}(x(\tau), v) \\
= \frac{1}{|x - y|\beta} \int_{0}^{1} |\dot{x}(\tau)||\nabla_{x} f_{b}(x(\tau), v)| d\tau \\
\lesssim |x - y|^{1 - \beta} \int_{0}^{1} \frac{|v|}{|\tilde{\alpha}(x(\tau), v)|^{3}},
\]

where we have used (2.32) to conclude

\[
|\nabla_{x}(\nabla_{x} f_{b}(x(\tau), v))| \lesssim \frac{||\eta||_{C^{2}}(|n(x_{b}(x(\tau), v))||\cdot v|}{|n(x_{b}(x(\tau), v))|^{2}} \nabla_{x} x_{b}(x(\tau), v) \\
+ \frac{n(x_{b}(x(\tau), v)||\eta||_{C^{2}}|v|}{|n(x_{b}(x(\tau), v))|^{2}} \nabla_{x} x_{b}(x(\tau), v) \\
\lesssim \frac{|v|}{|n(x_{b}(x(\tau), v))|^{3}}. \tag{2.88}
\]

Thus, by Lemma 2.9 and (1.11), we obtain (2.59). \hfill \Box

Proof of (2.60) We have

\[
\frac{|\nabla_{v} f_{b}(x, v) - \nabla_{v} x_{b}(y, v)|}{|x - y|\beta} = \frac{1}{|x - y|\beta} \int_{0}^{1} d\tau \frac{d}{d\tau} \nabla_{v} f_{b}(x(\tau), v) \\
= \frac{1}{|x - y|\beta} \int_{0}^{1} |\dot{x}(\tau)||\nabla_{v} f_{b}(x(\tau), v)| d\tau \\
\lesssim |x - y|^{1 - \beta} \int_{0}^{1} \frac{|v|}{|\tilde{\alpha}(x(\tau), v)|^{2}},
\]

where we have used (2.32) and (2.39) to conclude that

\[
|\nabla_{v} f_{b}(x(\tau), v)| \lesssim |\nabla_{x} f_{b}(x(\tau), v)| + |\nabla_{x} f_{b}(x(\tau), v)| \frac{|v|}{|n(x_{b}(x(\tau), v))|^{3}} \\
+ |f_{b}(x(\tau), v)| \frac{|\nabla_{x} x_{b}(x(\tau), v)|||\eta||_{C^{2}}|v|^{3} + |v|^{2}}{|n(x_{b}(x(\tau), v))|^{2}} \\
\lesssim \frac{|v|}{|n(x_{b}(x(\tau), v))|^{2}} + \frac{|n(x_{b}(x(\tau), v))|^{3}}{|v|^{2}} \frac{|n(x_{b}(x(\tau), v))|^{3}}{|n(x_{b}(x(\tau), v))|^{3}} \\
\lesssim \frac{|v|}{|n(x_{b}(x(\tau), v))|^{3}} \\
\lesssim \frac{|v|}{|n(x_{b}(x(\tau), v))|^{3}} \\
\lesssim \frac{|v|}{|n(x_{b}(x(\tau), v))|^{3}}. 
\]

Thus, by Lemma 2.9 and (1.11), we conclude (2.60). \hfill \Box

Proof of (2.61) We have

\[
\frac{|\nabla_{v} f_{b}(x, v) - \nabla_{v} x_{b}(y, v)|}{|x - y|\beta} = \frac{1}{|x - y|\beta} \int_{0}^{1} d\tau \frac{d}{d\tau} \nabla_{v} f_{b}(x(\tau), v) \\
= \frac{1}{|x - y|\beta} \int_{0}^{1} |\dot{x}(\tau)||\nabla_{v} x_{b}(x(\tau), v)| d\tau \\
\lesssim |x - y|^{1 - \beta} \int_{0}^{1} \frac{1}{|\tilde{\alpha}(x(\tau), v)|^{2}},
\]
where we have used (2.32) and (2.39) to conclude that
\[
|\nabla_x \nabla_v t_b(x(\tau), v)|
\]
\[
\lesssim |\nabla_x t_b(x(\tau), v)| \frac{1}{|n(x_b(x(\tau), v)) \cdot v|}
\]
\[
+ |t_b(x(\tau), v)| \frac{1}{|n(x_b(x(\tau), v)) \cdot v|^2} |\nabla_x x_b(x(\tau), v)| |\eta|_{C^2} |v| |v| + |v|
\]
\[
\lesssim \frac{1}{|v||n(x_b(x(\tau), v)) \cdot v|} + \frac{|n(x_b(x(\tau), v)) \cdot v|}{|v|^2} \frac{|v|^2}{|n(x_b(x(\tau), v)) \cdot v|^3}
\]
\[
\lesssim \frac{1}{|n(x_b(x(\tau), v)) \cdot v|^2}.
\]
Thus by Lemma 2.9 and (1.11) we conclude (2.61).

Proof of (2.62) From (2.85) and (2.86), we bound
\[
|G(y) \nabla_x (\nabla_x x_b(x(\tau), v))|
\]
\[
\lesssim \left[ \frac{\|\eta\|_{C^2} |v|}{|n(x_b(x(\tau), v)) \cdot v|} + \frac{\|\eta\|_{C^2} |v| |G(y)n(x_b(x(\tau), v)) \otimes v|}{|n(x_b(x(\tau), v)) \cdot v|^2} \right] |\nabla_x x_b(x(\tau), v)|
\]
\[
\lesssim \frac{|v|^2}{|n(x_b(x(\tau), v)) \cdot v|^2},
\]
where we have used
\[
|G(y)n(x_b(x(\tau), v))| \lesssim |G(y)n(y)| + |n(x_b(x(\tau), v)) - n(y)|
\]
\[
\lesssim |x_b(x(\tau), v) - y| \lesssim |x_b(x(\tau), v) - x(\tau)| + |x(\tau) - y|
\]
\[
\lesssim \min \left\{ \frac{\tilde{\alpha}(x, v)}{|v|}, \frac{\tilde{\alpha}(y, v)}{|v|} \right\}.
\]
Thus
\[
|G(y) \nabla_x x_b(x, v) - \nabla_x x_b(y, v)| \lesssim |x - y|^{1-\beta} \int_0^1 \frac{|v|^2}{|\tilde{\alpha}(x(\tau), v)|^2},
\]
and we conclude (2.62) from (1.11).

Proof of (2.63) From (2.88) we have
\[
|G(y)[\nabla_x t_b(x, v) - \nabla_x t_b(y, v)]|
\]
\[
\lesssim \frac{1}{|x - y|^\beta} \times \int_0^1 d\tau \left| \frac{\|\eta\|_{C^2}}{|n(x_b(x(\tau), v)) \cdot v|} \nabla_x x_b(x(\tau), v) \right|
\]
\[
+ \|\eta\|_{C^2} |v| \frac{G(y)n(x_b(x(\tau), v))}{|n(x_b(x(\tau), v)) \cdot v|^2} \left| \nabla_x x_b(x(\tau), v) \right|
\]
\[
\lesssim \frac{1}{|x - y|^\beta} \int_0^1 d\tau \frac{|v|}{|n(x_b(x(\tau), v)) \cdot v|^2} \lesssim \frac{1}{|v|} \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta},
\]
where we have used (2.89) to $G(y)n(x_b(x(\tau), v))$ and (1.11).
Proof of (2.64) By (2.74) in Lemma 2.9 we have
\[
|f_s(x, v) - f_s(y, v)| \lesssim w^{-1}(v)\|w f_s\|_\infty |x - y|^{-\beta} |f_s(x, v) - f_s(y, v)|^{\beta} \\
\lesssim w^{-1}(v)\|w f_s\|_\infty |x - y|^{-\beta} \int_0^1 \frac{1}{|x - y|^{\beta}} \left| \frac{\partial x}{\partial \tau} \right| f_s(x(\tau), v)|^\beta \\
\lesssim w^{-1}(v)\|w f_s\|_\infty |x - y|^{-\beta} w^{-\beta}(v)\|w f_s\|_\infty |x - y|^{-\beta} \int_0^1 \frac{1}{|x - y|^{\beta}} \left| \frac{\partial x}{\partial \tau} \right| f_s(x(\tau), v)|^\beta \\
\lesssim w^{-1}(v)\|w f_s\|_\infty |x - y|^{-\beta} w^{-\beta}(v)\|w f_s\|_\infty |x - y|^{-\beta} \min \left\{ \alpha(x, v), \alpha(y, v) \right\}^{\beta},
\]
where we have used \( \tilde{\vartheta} \ll \vartheta \) to have
\[
w^{-\beta}(v)w^{-1}(v) = e^{\gamma |v|^2 (\beta - 1) \vartheta - \beta \tilde{\vartheta}} \leq e^{-\beta |v|^2}.
\]
\(\square\)

Proof of (2.65) From (2.33), (2.26) and (2.32),
\[
\nabla_x x_{p^1(x), i} \sim v \cdot \frac{\partial_x p^1(x) n(x_b(x, v))}{\|n(x_b(x, v)) \cdot v|}.
\]
Then we apply the same computation as the proof of (2.54) to conclude (2.65). \(\square\)

Proof of (2.66) For this estimate we can assume \(|x_b(x, v) - x_b(y, v)| \ll 1\), otherwise, for \(|x_b(x, v) - x_b(y, v)| \gtrsim \delta \) we use (2.54) and (2.23) to have
\[
\frac{|T_{x_{p^1(x), i}} - T_{x_{p^1(y), i}}|}{|x - y|^{\beta}} \lesssim \frac{|T_{x_{p^1(x), i}} - T_{x_{p^1(y), i}}|}{|x_b(x, v) - x_b(y, v)|^{\beta}} \frac{|x_b(x, v) - x_b(y, v)|^{\beta}}{|x - y|^{\beta}} \lesssim \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{\beta},
\]
with \(|x_b(x, v) - x_b(y, v)| \ll 1\), we can assume that \(x_b(x, v)\) and \(x_b(y, v)\) correspond to the same \( p \) in (2.19). Then we drop the dependency on \( p \) and write \( p^1(x) = p^1(y) = p \). The variable depend on \( x \) are \( x_{p^1(x)} \) and \( x_{p^1(y)} \). Thus
\[
\frac{|T_{x_{p^1(x)}, i} - T_{x_{p^1(y)}, i}|}{|x - y|^{\beta}} \lesssim \frac{|T_{x_{p^1(x)}, i} - T_{x_{p^1(y)}, i}|}{|x_{p^1(x)} - x_{p^1(y)}|^{\beta}} \frac{|x_{p^1(x)} - x_{p^1(y)}|^{\beta}}{|x - y|^{\beta}} \lesssim \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{\beta},
\]
where we have used intermediate value theorem to \( T \) in (2.23) and (2.65). \(\square\)

Proof of (2.67) Following the proof of (2.65), it is straightforward to verify that
\[
\nabla_x [\nabla_x x_{p^1(x), i}] \lesssim \frac{|v|^3}{|n(x_b(x, v)) \cdot v|^3}.
\]
Then we follow the proof of (2.58) to conclude (2.67). \(\square\)

Proof of (2.68) Following the proof of (2.65), it is straightforward to verify that
\[
\nabla_v [\nabla_x x_{p^1(x), i}] \lesssim \frac{|v|}{|n(x_b(x, v)) \cdot v|^2}.
\]
Then we follow the proof of (2.60) to conclude (2.68).

Proof of (2.69) The \( v \)-derivative reads as

\[
\nabla_v \frac{\partial_i \eta_p(x^1_{p,l}(x))}{\sqrt{g_{p,ii}(x^1_{p,l}(x))}} = \sum_{j=1,2} \nabla_v x^1_{p,l}(x), j \frac{\partial_j \eta \sqrt{g_{p,ii}} - \partial_j \eta \partial_i \eta}{g_{p,ii}}.
\]

Similar to the proof of (2.66), we can assume \( x_b(x, v) \) and \( x_b(y, v) \) correspond to the same \( p \). Then we apply (2.40), (2.65) and (2.68) to have

\[
\begin{aligned}
&\frac{|\nabla_v \frac{\partial_i \eta_p(x^1_{p,l}(x))}{\sqrt{g_{p,ii}(x^1_{p,l}(x))}} - \nabla_v \frac{\partial_i \eta_p(x^1_{p,l}(y))}{\sqrt{g_{p,ii}(x^1_{p,l}(y))}}|}{|x - y|^{\beta}} \\
&+ \frac{1}{|v|} \left| \frac{x^1_{p,l}(x) - x^1_{p,l}(y)}{|x - y|^{\beta}} \right| \\
&\times \left[ \frac{\partial_{ij} \eta_p(x^1_{p,l}(x)) \sqrt{g_{p,ii}(x^1_{p,l}(x))}}{g_{p,ii}(x^1_{p,l}(x))} - \frac{\partial_{ij} \eta_p(x^1_{p,l}(y)) \sqrt{g_{p,ii}(x^1_{p,l}(y))}}{g_{p,ii}(x^1_{p,l}(y))} \right] \\
&\lesssim \frac{\|\eta\|_{C^3}}{|v| \min \left\{ \frac{c(x, v)}{|v|}, \frac{c(y, v)}{|v|} \right\}^{1+\beta}}.
\end{aligned}
\]

In the last line we used intermediate value theorem for \( \partial_{ij} \eta \). By definition of \( T \) in (2.23) we conclude (2.69).

Proof of (2.70) Since \(|x - y| < 1\), we can assume that \( x, y \in B(p; \delta_2) \), where \( B(p; \delta_2) \) is defined in (2.19). Then both \( x, y \) correspond to the same \( p \). Only for proof of this estimate we denote

\[
x = \eta_p(x_p(x)), \quad y = \eta_p(x_p(y)),
\]

\[
\eta_p(x_p(\tau)) := \tau \eta_p(x_p(x)) + (1 - \tau) \eta_p(x_p(y)).
\]

By mean value theorem, there exists \( c \in [0, 1] \) such that

\[
x - y = \eta_p(x_p(x)) - \eta_p(x_p(y)) = \nabla \eta_p(c x_p(x) + (1 - c)x_p(y))(x_p(x) - x_p(y)).
\]

Thus

\[
\frac{|x_b(\eta_p(x_p(x)), v) - x_b(\eta_p(x_p(y)), v)|}{|x - y|}
= \frac{1}{|x - y|} \left| \int_0^1 \frac{d}{d\tau} x_b(\eta_p(x_p(\tau)), v) \right|.
\]
\[
\begin{align*}
&= \frac{1}{|x-y|} \left| \int_0^1 d\tau \nabla_x x_b(\eta_p(x_p(\tau)),v) \frac{d}{d\tau} \eta_p(x_p(\tau)) \right| \\
&= \frac{1}{|x-y|} \left| \int_0^1 d\tau \nabla_x x_b(\eta_p(x_p(\tau)),v) \nabla \eta_p(c x_p(x) + (1-c)x_p(y))(x_p(x) - x_p(y)) \right| \\
&\lesssim \frac{|x_p(x) - x_p(y)|}{|x-y|} \int_0^1 d\tau \nabla_x \eta_p(c x_p(x) + (1-c)x_p(y)) \\
&\quad - \frac{n(x_b(\eta_p(x_p(\tau)),v)) \otimes v}{|n(x_b(\eta_p(x_p(\tau)),v)) \cdot v|} \nabla \eta_p(c x_p(x) + (1-c)x_p(y)) \\
&\lesssim \|\eta\|_{C^1} + \frac{n(\eta_p(c x_p(x) + (1-c)x_p(y))) \cdot v}{|n(x_b(\eta_p(x_p(\tau)),v)) \cdot v|} \\
&\quad + |v| \frac{\|\eta_p(c x_p(x) + (1-c)x_p(y)) - n(x_b(\eta_p(x_p(\tau)),v))\|}{|n(x_b(\eta_p(x_p(\tau)),v)) \cdot v|} \\
&\lesssim 1 + \frac{\|\xi\|_{C^2} |x_b(\eta_p(x_p(\tau)),v) - \eta_p(x_p(\tau))|}{|n(x_b(\eta_p(x_p(\tau)),v)) \cdot v|} \\
&\quad + \frac{|\eta_p(x_p(\tau)) - \eta_p(c x_p(x) + (1-c)x_p(y))|}{|n(x_b(\eta_p(x_p(\tau)),v)) \cdot v|} \\
&\lesssim 1 + \frac{\alpha(n(x_b(\eta_p(x_p(\tau)),v)) + |v||x-y|}{|n(x_b(\eta_p(x_p(\tau)),v)) \cdot v|} \lesssim 1.
\end{align*}
\]

In the fourth line we have used (2.32). In the last three lines we have used (2.48) and \(|x-y| \lesssim O(\varepsilon) \min\{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\} \). \(\square\)

**Proof of (2.71)** Since \(\|T_w - T_0\|_{L^\infty} \ll 1\) from **Existence Theorem**, by the definition of \(M_w\) in (1.4) we apply the mean value theorem to have

\[
\frac{|M_w(x, v) - M_w(y, v)|}{\sqrt{\mu(v)|x-y|^\beta}} \lesssim \frac{|M_w(x, v) - M_w(y, v)|}{\sqrt{\mu(v)|x-y|}} \lesssim \left\| \frac{\nabla_x M_w(x, v)}{\sqrt{\mu(v)}} \right\|_{L^\infty} \\
\lesssim T_0 \left\| \nabla_x T_w |v| \sqrt{\mu(v)} \right\|_{L^\infty} \lesssim w_0^{-1}(v) \|T_w - T_0\|_{C^1}.
\]

\(\square\)

**Proof of (2.72)** From (1.6) it is equivalent to compute

\[
\frac{|\partial_j \eta_p(x_p(x)) \nabla_i x_p^1(x) - \partial_j \eta_p(y)(x_p(y)) \nabla_i x_p^1(y)|}{|x-y|^\beta} \lesssim \frac{|\partial_j \eta_p(x_p(x)) - \partial_j \eta_p(y)(x_p(y))||\nabla_i x_p^1(x)|}{|x-y|^\beta} \\
+ \frac{|\partial_j \eta_p(x_p(x))||\nabla_i x_p^1(x) - \nabla_i x_p^1(y)||}{|x-y|^\beta} \\
\lesssim \frac{\|\eta\|_{C^2}}{\alpha(x,v)} \|\eta\|_{C^1} \frac{|\nabla_i x_p^1(x) - \nabla_i x_p^1(y)|}{|x-y|^\beta},
\]

where we have used (2.34). Denote \(x_b(x(\tau), v) = \eta_{p^1(x(\tau))}(x_p^1(x(\tau)))\). Then
Lemma 2.10. For any $s \in [0, \min(t_\theta(x, v), t_\theta(y, v))]$, we have

\[
\frac{\partial}{\partial \tau} \left( \nabla_x \psi^1_p(x(\tau), i) \right) = \frac{\partial}{\partial \tau} x(\tau) \nabla_x \psi^1_p(x(\tau), i)
\]

Applying (2.33), we further bound

\[
\frac{|\nabla_x \psi^1_p(x, i) - \nabla_x \psi^1_p(y, i)|}{|x - y|^{\beta}} = \frac{1}{|x - y|^{\beta}} \left| \int_0^1 d\tau \frac{d}{d\tau} \left( \nabla_x \psi^1_p(x(\tau), i) \right) \right|
\]

In the third line we have applied the derivative to (2.33). In the fourth line we have used (1.11) and $|\nabla_x \psi^1_p| \lesssim \frac{|v|}{|n(x_b(x(\tau), v)) - v|^2 g_{p_1(x(\tau)), ii}(\psi^1_p(x(\tau)))}$ from (2.33). \hfill \Box

Lemma 2.10. For any $s \in [0, \min(t_\theta(x, v), t_\theta(y, v))]$, we have

\[
\left| \frac{G(x) \nabla_x f_s(x - sv, v) - G(y) \nabla_x f_s(y - sv, v)}{|x - y|^{\beta}} \right| \lesssim \frac{\nabla_{|s|} f_s(x - sv, v) - \nabla_{|s|} f_s(y - sv, v)}{|x - y|^{\beta}} + \frac{\alpha(x, v)}{|v|} \left\| \nabla_x f_s(x - sv, v) - \nabla_x f_s(y - sv, v) \right\| \lesssim \frac{\alpha(x, v)}{|v|} \left\| \nabla_x f_s \right\| \lesssim \frac{1}{w_\theta(v)} \alpha(y - sv, v) \cdot \frac{w_\theta(v) \alpha(y - sv, v)}{w_\theta(v)}.
\]

\[
(2.90)
\]

Proof. First we rewrite

\[
G(x) \nabla_x f_s(x - sv, v) - G(y) \nabla_x f_s(y - sv, v)
\]

\[
= G(x - sv) \nabla_x f_s(x - sv, v) - G(y - sv) \nabla_x f_s(y - sv, v) \quad (2.91)
\]

\[
+ \left( G(x) - G(x - sv) \right) \nabla_x f_s(x - sv, v) \quad (2.92)
\]

\[
+ \left( G(y - sv) - G(y) \right) \nabla_x f_s(y - sv, v) \quad (2.93)
\]

Note that from (1.13) the contribution of (2.91) appears in (2.90). For (2.92) and (2.93) we apply (2.48) and rearrange terms to derive that
we conclude the lemma. □

2.11. The property of the collision operator is summarized in Lemma 2.12 and operators. We summarize the property of diffuse boundary condition in Lemma Lemma 2.13.

In this subsection we list some properties of the boundary condition and collision operators. We summarize the property of diffuse boundary condition in Lemma 2.11. The property of the collision operator is summarized in Lemma 2.12 and Lemma 2.13.

Lemma 2.11. For the diffuse boundary condition of f in (2.4), let \( x_b(x, v) = \eta_{p_1(x)}(x_{p_1}^1(x)) \in \partial \Omega \) (see (7.4)), we have

\[
\begin{align*}
\|r\|_\infty &< \infty, \quad |\partial_{x_{p_1}^1, i} r(\eta_{p_1}(x_{p_1}^1), v)| \lesssim \|T_W - T_0\|_{C^1}, \\
|v|^2 \nabla_v [r(x_b(x, v), v)] &\lesssim \|T_W - T_0\|_{C^1}, \\
\left| \frac{\partial_{x_{p_1}^1, i, i} \eta_{p_1}(x_{p_1}^1), v - \partial_{x_{p_1}^1, (i), i} \eta_{p_1}(x_{p_1}^1), v)}{|x_b(x, v) - x_b(y, v)|^\beta} \right| &\lesssim \|T_W - T_0\|_{C^2}, \\
\left| \frac{\partial_{x_{p_1}^1, i} M_W(\eta_{p_1}(x_{p_1}^1), v) - \partial_{x_{p_1}^1, (i), i} M_W(\eta_{p_1}(x_{p_1}^1), v)}{\sqrt{\mu(v)}|x_b(x, v) - x_b(y, v)|^\beta} \right| &\lesssim \|T_W - T_0\|_{C^2},
\end{align*}
\]

Applying mean value theorem to \( n(x) - n(y) \), \( n(x - sv) - n(x - sv) \) with (2.50), we conclude the lemma.

2.5. Properties of boundary condition and collision operators

Lemma 2.11. For the diffuse boundary condition of f in (2.4), let \( x_b(x, v) = \eta_{p_1(x)}(x_{p_1}^1(x)) \in \partial \Omega \) (see (7.4)), we have

\[
\begin{align*}
\|r\|_\infty &< \infty, \quad |\partial_{x_{p_1}^1, i} r(\eta_{p_1}(x_{p_1}^1), v)| \lesssim \|T_W - T_0\|_{C^1}, \\
|v|^2 \nabla_v [r(x_b(x, v), v)] &\lesssim \|T_W - T_0\|_{C^1}, \\
\left| \frac{\partial_{x_{p_1}^1, i, i} \eta_{p_1}(x_{p_1}^1), v - \partial_{x_{p_1}^1, (i), i} \eta_{p_1}(x_{p_1}^1), v)}{|x_b(x, v) - x_b(y, v)|^\beta} \right| &\lesssim \|T_W - T_0\|_{C^2}, \\
\left| \frac{\partial_{x_{p_1}^1, i} M_W(\eta_{p_1}(x_{p_1}^1), v) - \partial_{x_{p_1}^1, (i), i} M_W(\eta_{p_1}(x_{p_1}^1), v)}{\sqrt{\mu(v)}|x_b(x, v) - x_b(y, v)|^\beta} \right| &\lesssim \|T_W - T_0\|_{C^2},
\end{align*}
\]

\[
\begin{align*}
\frac{1}{|x_b(x, v) - x_b(y, v)|^\beta} \\
\times \left| \int_{n(x_b(x, v)) \cdot v^1 > 0} f(x_b(x, v), v^1) \sqrt{\mu(v^1)} \{n(x_b(x, v)) \cdot v^1\} dv^1 \\
- \int_{n(x_b(y, v)) \cdot v^1 > 0} f(x_b(y, v), v^1) \sqrt{\mu(v^1)} \{n(x_b(y, v)) \cdot v^1\} dv^1 \right| &\lesssim \|w_\beta \alpha \nabla_x f\|_\infty.
\end{align*}
\]

Proof. From (2.6), it is straightforward to derive the estimate for \( \|r\|_\infty \). We take derivative to r to obtain

\[
|\partial_{x_{p_1}^1, i} r(\eta_{p_1}(x_{p_1}^1), v) |
\]
\[
\begin{align*}
&\partial_t \eta_p \left( x^1, p^1 \right) \\
&= \frac{\partial_t \eta_p \left( x^1, p^1 \right) \nabla_x \left( \frac{1}{2\pi [TW(x_0, v)]^2} e^{-\frac{|v|^2}{2TW(x_0, v)}} \right)}{\sqrt{2\pi \mu(v)}} \\
&= \frac{\partial_t \eta_p \left( x^1, p^1 \right) \nabla_x TW(x_b(x, v))}{\sqrt{2\pi \mu}} \left( \frac{-1}{\pi [TW(x_b(x, v))]^3} + \frac{|v|^2}{4\pi [TW(x_b(x, v))]^4} \right) \\
&\times e^{-\frac{|v|^2}{2TW(x_b(x, v))}} \\
&\lesssim T_0 \|TW - T_0\|_C^1,
\end{align*}
\]

where we have used \(\|TW - T_0\|_\infty \ll 1\) from Existence Theorem. Here we note that the \(\nabla_x\) above represents the partial derivative.

Then we take \(v\) derivative to have

\[
\|v \partial_v r(x_b(x, v), v)\| \\
= |\nabla_v \frac{M_W}{\sqrt{2\pi \mu}} - \nabla_v \sqrt{\mu}| \lesssim |\nabla_v \frac{e^{-\frac{|v|^2}{2TW(c_b(x, v))}}}{\sqrt{\mu(v)}}| \\
\lesssim \|TW - T_0\|_1 C^1 \frac{1}{\mu(v)} \times |\nabla_v e^{-\frac{|v|^2}{2TW(c_b(x, v))}} \sqrt{\mu(v)}| \\
+ |\nabla_v \sqrt{\mu(v)} TW^2(x_b(x, v)) e^{-\frac{|v|^2}{2TW(c_b(x, v))}}| \\
\lesssim \|TW - T_0\|_1 C^1 \frac{1}{\mu(v)} \times e^{-\frac{|v|^2}{2TW(c_b(x, v))}} \sqrt{\mu(v)} |v|^2 |\nabla_v x_b(x, v)| \\
\lesssim \|TW - T_0\|_1 C^1 \frac{e^{-\frac{|v|^2}{2TW(c_b(x, v))}} |v|}{\sqrt{\mu(v)}},
\]

where we have used (2.40) in the last line. Since the coefficient for \(|v|^2\) in exponent is negative, we conclude (2.95).

For (2.96) from (2.99) we apply the mean value theorem to bound

\[
\|\partial_t \eta_{p^1}^1(x(x_b(x, v), \partial_t \eta_{p^1}^1(y(x_b(x, v))| \\
\|\nabla_x TW(x_b(x, v)) - \nabla_x TW(x_b(y, v))| \lesssim \|\nabla_x TW\|_\infty, \\
-1 \frac{\pi [TW(x_b(x, v))]^3 + \frac{1}{\pi [TW(x_b(x, v))]^3} + \frac{|v|^2}{4\pi [TW(x_b(x, v))]^4}}{\pi [TW(x_b(x, v))]^3 + \frac{1}{\pi [TW(x_b(x, v))]^3} + \frac{|v|^2}{4\pi [TW(x_b(x, v))]^4}} - \frac{|v|^2}{4\pi [TW(x_b(x, v))]^4} \\
\times e^{-\frac{|v|^2}{2TW(c_b(x, v))}} \\
\lesssim \|TW\|_{C^1} |v|^4 e^{-\frac{|v|^2}{2TW(c_b(x, v))}} \lesssim \|\nabla_x^2 TW\|_\infty,
\]

\[
\|\nabla_x^2 TW\|_\infty.
\]

\[
\|\nabla_x TW\|_\infty.
\]

\[
\|\nabla_x^2 TW\|_\infty.
\]
and thus (2.96) follows from \( \frac{1}{\sqrt{W(x)}} \).

Since \( \partial_{x_p^1} r(\eta_p^1(x_p^1), v) = \frac{1}{\sqrt{2\pi}} \partial_{x_p^1} M_W(\eta_p^1(x_p^1), v) \), (2.97) also follows.

Last we prove (2.98). We rewrite the LHS of (2.98) as

\[
\int_{n(x_b(x,v)) \cdot v^1 > 0} \frac{1}{|x_b(x,v) - x_b(y,v)|^\beta} \left| \int \left( \int_{\frac{n(x_b(x,v)) \cdot v^1}{|v^1|} > 0} f(x_b(x,v), v^1) \sqrt{\mu(v^1)} n(x_b(x,v)) \cdot v^1 \right) \right| \right. \\
+ \left. \frac{\int |n(x_b(x,v)) - n(x_b(y,v))| \geq \frac{n(x_b(x,v)) \cdot v^1}{|v^1|} \geq 0 f(x_b(x,v), v^1) \sqrt{\mu(v^1)} n(x_b(x,v)) \cdot v^1 |}{|x_b(x,v) - x_b(y,v)|^\beta} \right) \\
+ \frac{\int |n(x_b(x,v)) - n(x_b(y,v))| \geq \frac{n(x_b(x,v)) \cdot v^1}{|v^1|} \geq 0 f(x_b(x,v), v^1) \sqrt{\mu(v^1)} n(x_b(x,v)) \cdot v^1 |}{|x_b(x,v) - x_b(y,v)|^\beta}.
\]

(2.101)

(2.102)

Clearly from (2.64) and (2.57), we have

\[
\left| (2.101) \right| \lesssim \int \left( \int_{\frac{n(x_b(x,v)) \cdot v^1}{|v^1|} > 0} \left( \frac{\| w f \|_{1}^{1-\beta} \| w \partial_{\beta} \nabla_x f \|_{\infty}^{\beta} \sqrt{\mu(v^1)} \right) \right. \\
+ \left. \frac{\min \left\{ \alpha(x_b(x,v), v^1), \alpha(x_b(y,v), v^1) \right\}^{\beta}}{\| w \partial_{\beta} \nabla_x f \|_{\infty} + \| w f \|_{\infty}} \right) \\
\lesssim \| w \partial_{\beta} \nabla_x f \|_{\infty} + \| w f \|_{\infty},
\]

where we have used Young’s inequality with \( 1 - \beta + \beta = 1 \) and definition of \( \alpha \) in (1.8) with \( \beta < 1 \).

For (2.102), from (2.57) we bound

\[
\int |n(x_b(x,v)) - n(x_b(y,v))| \geq \frac{n(x_b(x,v)) \cdot v^1}{|v^1|} \geq 0 f(x_b(x,v), v^1) \sqrt{\mu(v^1)} n(x) \cdot v^1 | \\
\lesssim \begin{array}{c}
\frac{|n(x_b(x,v)) - n(x_b(y,v))|}{|x_b(x,v) - x_b(y,v)|^\beta} \\
\frac{\int f(x_b(x,v), v^1) \sqrt{\mu(v^1)} v^1 \| w f \|_{\infty}}{\| w f \|_{\infty}}.
\end{array}
\]

Then we conclude the lemma.

Aside from the boundary condition, we also need to estimate the collision operator. The next two lemmas describe the properties of the collision operator \( K \) and \( \Gamma \).

**Lemma 2.12.** The linear Boltzmann operator \( K(f) \) in (2.5) is given by

\[
K f(x,v) = \int_{\mathbb{R}^3} k(v,u) f(x,u) du.
\]

The kernel \( k(v,u) \) satisfies:

\[
|k(v,u)| \lesssim k_{e}(v,u), \quad |\nabla_{v} k(v,u)|
\]
\( \langle u \rangle k_2(v, u)/|v - u|, \ k_2(v, u) := e^{-e|v-u|^2}/|v - u|. \) (2.103)

And for \( 3 > c \geq 0, \)
\[
\int_{\mathbb{R}^3} k_\nu(v, u) \frac{1}{|u|^c} du \lesssim \frac{1}{|v|^c}. \tag{2.104}
\]

Moreover, for the operator \( \nu \) and \( \Gamma \) in (2.5), we have
\[
|K(f) + \Gamma(f, f)| = O(1) \| w f \|_\infty, \tag{2.105}
\]
\[
v = O(\sqrt{|v|^2 + 1}) \gtrsim 1, \ |\nabla v| \lesssim 1, \tag{2.106}
\]
\[
|\nabla v \Gamma(f, f)| \lesssim \frac{\| w f \|_\infty^2}{|v|^2} + \frac{\| w f \|_\infty \| v \|^2 \nabla v f \|_\infty}{|v|^2}, \tag{2.107}
\]
\[
|\nabla_x \Gamma(f, f)(v)| = O(|w f|_\infty) \left\{ \| \nabla f(v) \| + \int_{\mathbb{R}^3} k_\nu(v, u) |\nabla_x f(u)| du \right\}, \tag{2.108}
\]
\[
|G(x) \nabla x \Gamma(f, f)(x, v)| = O(|w f|_\infty) \{ |G(x) \nabla_x f(x, v) + \int_{\mathbb{R}^3} k_\nu(v, u) |G(x) \nabla_x f(x, u)| \}. \tag{2.109}
\]

Proof of (2.103) We define
\[
\Gamma_{\text{gain}}(f_1, f_2) = \frac{Q_{\text{gain}}(\sqrt{\mu} f_1, \sqrt{\mu} f_2)}{\sqrt{\mu}}, \quad \Gamma_{\text{loss}}(f_1, f_2) = \frac{Q_{\text{loss}}(\sqrt{\mu} f_1, \sqrt{\mu} f_2)}{\sqrt{\mu}}, \tag{2.110}
\]
where \( Q_{\text{gain}}, Q_{\text{loss}} \) are defined in (1.2).

By the Grad estimate in [12],
\[
\Gamma_{\text{gain}}(\sqrt{\mu}, f) + \Gamma_{\text{gain}}(f, \sqrt{\mu}) = \int_{\mathbb{R}^3} k_2(v, u) f(u) du, \tag{2.111}
\]
\[
v(\sqrt{\mu} f) = \int_{\mathbb{R}^3} k_1(v, u) f(u) du,
\]
where
\[
k_1(v, u) = C_k |u - v| e^{-\frac{|u|^2 + |v|^2}{2}}, \tag{2.112}
\]
\[
k_2(v, u) = C_k \frac{1}{|u - v|} e^{-\frac{1}{4} |u-v|^2 - \frac{1}{4} \frac{(u^2 - |v|^2)^2}{|u-v|^2}}. \tag{2.113}
\]

We compute the derivative:
\[
|\nabla u k_1(v, u)| \lesssim e^{-\frac{|u|^2 + |v|^2}{2}} + |u||u - v| e^{-\frac{|u|^2 + |v|^2}{2}} \lesssim e^{-\frac{|v-u|^2}{4}} \lesssim e^{-e|v-u|^2}/|v - u|^2.
\]

And
\[
|\nabla u k_2(v, u)| \lesssim \frac{1}{|v - u|^2} e^{-\frac{1}{4} |v-u|^2} + \frac{1}{|v - u|} e^{-\frac{1}{4} |v-u|^2 - \frac{1}{4} \frac{(u^2 - |v|^2)^2}{|v-u|^2}}.
\]
where we have used

\[
\begin{align*}
&\leq \frac{1}{|v-u|^4} \quad \\
&\leq \frac{|v-u|^2}{|u|^4} \\
&\leq \frac{|v-u|^2}{|u-u|^4}.
\end{align*}
\]

**Proof of (2.104)** We consider two cases. When \(|u| > \frac{|v|}{2}\), we have

\[
\int_{|u| > \frac{|v|}{2}} k_\rho(v, u) \frac{1}{|u|^c} \, du \lesssim \int_{|u| > \frac{|v|}{2}} \frac{e^{-e|v-u|^2}}{|v-u|} \frac{1}{|u|^c} \, du
\]

\[
\lesssim \frac{1}{|v|^c} \int_{\mathbb{R}^3} \frac{e^{-e|v-u|^2}}{|v-u|} \, du \lesssim \frac{1}{|v|^c}.
\]

When \(|u| \leq \frac{|v|}{2}\) we bound \(|v-u| \geq \frac{|v|}{2}\), and thus

\[
\int_{|u| \leq \frac{|v|}{2}} k(v, u) \frac{1}{|u|^c} \, du \lesssim \frac{e^{-e|v|^2/2}}{|v|} \int_{|u| \leq \frac{|v|}{2}} \frac{1}{|u|^c} \\
\lesssim \frac{e^{-e|v|^2/2}}{|v|} \int_{0 \leq r \leq \frac{|v|}{2}} \int_{\partial B(0, r)} dS \frac{1}{|r|^c} \\
\lesssim \frac{e^{-e|v|^2/2}}{|v|^3} \lesssim \frac{e^{-e|v|^2/2}|v|^2}{|v|^c} \lesssim \frac{1}{|v|^c}.
\]

In the second line we used the polar coordinate with \(|u|= r\). In the third line we used \(c < 3\) to compute the \(r\) integral.

Then we conclude (2.104). \(\square\)

**Proof of (2.105)** For \(K(f)\) we bound

\[
K(f) \lesssim \|f\|_\infty \int_{\mathbb{R}^3} |k(v, u)| \, du \lesssim \|f\|_\infty \lesssim \|wf\|_\infty,
\]

where we have used \(|k(v, u)| \lesssim k_\rho(v, u) \in L^1_u\).

For \(\Gamma\), clearly

\[
|\Gamma_{\text{gain}}(f, f)| \lesssim |\Gamma_{\text{gain}}(e^{-e|v|^2}, |f|)| \times \|wf\|_\infty.
\]

By (2.111) we bound \(|\Gamma_{\text{gain}}(e^{-e|v|^2}, |f|)|\) using different exponent of \(k_2(v, u)\), we conclude that

\[
\Gamma_{\text{gain}}(f, f) \lesssim \|wf\|_\infty^2 \lesssim \|wf\|_\infty.
\]
For the other term we bound
\[
|v(\sqrt{\mu} f)(v)| \lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} |v - u| e^{-\varrho|v|^2} \sqrt{\mu(u)}|f(u)|
\]
\[
\lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} |v - u| e^{-C|v-u|^2} \lesssim \|wf\|_{\infty},
\]
(2.115)
where we have used
\[
e^{-\varrho|v|^2} e^{-\varrho|u|^2} \lesssim e^{-\frac{C}{2}|v-u|^2}.
\]

The proof for (2.106) is standard (see Chapter 3 in [12]).

**Proof of (2.107)** The velocity derivative for the nonlinear Boltzmann operator reads
\[
\nabla_v \Gamma(f, f) = \nabla_v \left( \Gamma_{\text{gain}}(f, f) - \Gamma_{\text{loss}}(f, f) \right)
\]
\[
= \Gamma_{\text{gain}}(\nabla_v f, f) + \Gamma_{\text{gain}}(f, \nabla_v f) - \Gamma_{\text{loss}}(\nabla_v f, f) - \Gamma_{\text{loss}}(f, \nabla_v f) - \Gamma_{v, \text{gain}}(f, f) - \Gamma_{v, \text{loss}}(f, f).
\]
(2.116)
(2.117)
Here we have defined
\[
\Gamma_{v, \text{gain}}(f, f) - \Gamma_{v, \text{loss}}(f, f)
\]
\[
:= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u \cdot \omega| f(v + u \perp) f(v + u \|) \nabla_v \sqrt{\mu(v + u)} dv du 
\]
\[
- \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u \cdot \omega| f(v + u) f(v) \nabla_v \sqrt{\mu(v + u)} dv du.
\]
(2.118)

From (2.114) we have
\[
|\Gamma_{\text{gain}}(f, \partial_v f) + \Gamma_{\text{gain}}(\partial_v f, f)| \lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v - u|} |\partial_v f| du.
\]
(2.119)
For \(|v(\sqrt{\mu} \partial_v f)(v)|\) we have
\[
|v(\sqrt{\mu} \partial_v f)(v)| \lesssim \|wf\|_{\infty} e^{-\varrho|v|^2} \sqrt{\mu} \partial_v f(v)
\]
\[
\lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} |v - u| e^{-\varrho|v|^2} \sqrt{\mu(u)} |\partial_v f(u)|
\]
\[
\lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v - u|} |\partial_v f(u)| du,
\]
(2.120)
where we have used \(e^{-\varrho|v|^2} |v - u| \sqrt{\mu(u)} \lesssim \frac{e^{-\varrho|v-u|^2}}{|v - u|} \).

Then we combine (2.119) and (2.120), and use (2.104) with \(c = 2\) to conclude
\[
(2.116) \lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v - u|} |\nabla_v f|
\]
\[
\lesssim \|wf\|_{\infty} \|v\|^2 \nabla_v f \|_{\infty} \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v - u|} \frac{1}{|u|^2}.
\]
Lemma 2.13. If

\[ \text{Lemma 2.13.} \quad \text{if} \quad 1136 \quad \text{Hongxu Chen & Chanwoo Kim} \]

where we have used

\[ \text{Proof of (2.108)} \quad \text{Replacing the } \nabla_v \text{ by } \nabla_x \text{ in (2.119) and (2.120), we have} \]

\[ |\Gamma_{\text{gain}}(f, \partial_x f) + \Gamma_{\text{gain}}(\partial_x f, f)| \lesssim \|w f\|_\infty \int \frac{e^{-\theta|v-u|^2}}{|v-u|} |\partial_x f| du, \]

\[ |v(\sqrt{\mu} \partial_x f)(v)| \lesssim \|w f\|_\infty \int \frac{e^{-\theta|v-u|^2}}{|v-u|} |\partial_x f(u)| du. \]

Proof of (2.109) Since

\[ G(x) \nabla_x \Gamma(f, f)(x, v) = G(x) \Gamma(\nabla_x f, f) + G(x) \Gamma(f, \nabla_x f) \]

\[ = \Gamma(G(x) \nabla_x f, f) + \Gamma(f, G(x) \nabla_x f), \quad (2.121) \]

from (2.108), we conclude (2.109). ∎

Lemma 2.13. If 0 < \( \tilde{\theta} \) < \( \theta \) and if 0 < \( \tilde{\theta} \) < \( \theta - \frac{\tilde{\theta}}{4} \),

\[ k \frac{e^{\tilde{\theta}|v|^2}}{e^{\theta|u|^2}} \lesssim k_{\tilde{\theta}}(v, u), \quad (2.22) \]

where \( k \) is defined in (2.103). Also,

\[ |\nabla_v k(v, u)| \frac{e^{\tilde{\theta}|v|^2}}{e^{\theta|u|^2}} \lesssim \frac{\langle v \rangle k_{\tilde{\theta}}(v, u)}{|v-u|}. \quad (2.23) \]

As a consequence, when \( \tilde{\theta} \ll \theta \), we have

\[ w_{\tilde{\theta}}(v) |K(f) + \Gamma(f, f)| \lesssim \|w_{\tilde{\theta}} f\|_\infty \lesssim \|w f\|_\infty. \quad (2.24) \]
Proof of (2.122) Note that the \( k \) in (2.103) equals to \( k_1(v,u) + k_2(v,u) \) in (2.112), (2.113), then

\[
k(v,u) = \exp \left\{ -\varrho |v - u|^2 - \varrho \frac{|v| - |u|^2}{|v - u|^2} + \tilde{\varrho} |v|^2 - \tilde{\varrho} |u|^2 \right\}.
\]

Let \( v - u = \eta \) and \( u = v - \eta \). Then the exponent equals

\[
-\varrho |\eta|^2 - \varrho \frac{||\eta|^2 - 2v \cdot \eta|^2}{|\eta|^2} - \tilde{\varrho} |v - \eta|^2 - |v|^2
\]
\[
= -2\varrho |\eta|^2 + 4\varrho v \cdot \eta - 4\varrho \frac{v \cdot \eta |^2}{|\eta|^2} - \tilde{\varrho} |\eta|^2 - 2v \cdot \eta
\]
\[
= (-2\varrho - \tilde{\varrho}) |\eta|^2 + (4\varrho + \tilde{\varrho}) v \cdot \eta - 4\varrho \frac{v \cdot \eta |^2}{|\eta|^2}.
\]

If \( 0 < \tilde{\varrho} < 4\varrho \) then the discriminant of the above quadratic form of \(|\eta|\) and \( \frac{v \cdot \eta |}{|\eta|^2} \) is

\[
(4\varrho + \tilde{\varrho})^2 - 4(-2\varrho - \tilde{\varrho})(-4\varrho) = 4\tilde{\varrho}^2 - 16\varrho \tilde{\varrho} < 0.
\]

Hence, the quadratic form is negative definite. We thus have, for \( 0 < \tilde{\varrho} < \varrho - \frac{\tilde{\varrho}}{4} \), the following perturbed quadratic form is still negative definite: \(-(\varrho - \tilde{\varrho}) |\eta|^2 - (\varrho - \tilde{\varrho}) \frac{|\eta|^2 - 2v \cdot \eta|^2}{|\eta|^2} - \tilde{\varrho} |\eta|^2 - 2v \cdot \eta \leq 0 \)

Proof of (2.123) Taking the derivative to (2.112) and (2.113) we have

\[
\nabla_v k_1(v,u) = \frac{v - u}{|v - u|} k_1(v,u) - v k_1(v,u),
\]
\[
\nabla_v k_2(v,u) = \frac{v - u}{|v - u|^2} k_2(v,u)
\]
\[
- k_2(v,u) \left[ \frac{v - u}{2} + \frac{v(|u|^2 - |v|^2)|v - u|^2 - (|u| - |v|^2)^2 |v - u|}{2|v - u|^4} \right].
\]

Thus

\[
|\nabla_v k(v,u)| e^{\tilde{\varrho}|v|^2} e^{\varrho |u|^2} \leq \left[ \frac{1}{|v - u|} + |v - u| + \langle v \rangle \right] k_1(v,u) + k_2(v,u) \frac{e^{\tilde{\varrho}|v|^2}}{|v - u|^2}
\]
\[
\leq \left[ \frac{1}{|v - u|} + |v - u| + \langle v \rangle \right] k_{\tilde{\varrho}}(v,u) \lesssim \frac{(v) k_{\varrho}(v,u)}{|v - u|}
\]

for \( c < \tilde{\varrho} \). In the last line we have applied (2.22).

Proof of (2.124) By Lemma 2.12, we have

\[
|w_{\tilde{\varrho}}(v) K f| \lesssim \int_{\mathbb{R}^3} k_{\tilde{\varrho}}(v,u) \frac{w_{\tilde{\varrho}}(v)}{w_{\tilde{\varrho}}(u)} w_{\tilde{\varrho}}(u)|f(u)| du
\]
\[
\lesssim \|w_{\tilde{\varrho}} f\|_{\infty} \int_{\mathbb{R}^3} k_{\tilde{\varrho}}(v,u) du \lesssim \|w_{\tilde{\varrho}} f\|_{\infty}, \tag{2.125}
\]

where we have used \( k_{\varrho}(v,u) \in L_u^1 \).
For $\Gamma$, we follow (2.114) and have
\[ w_\theta(v) |_{/Gamma1} \Gamma_{\text{gain}}(f, f) | \lesssim \| w f \|_\infty w_\theta(v) |_{/Gamma1} \Gamma_{\text{gain}}(e^{-\theta|v|^2}, f) | \lesssim \| w_\theta f \|_\infty, \]
here we bound $|\Gamma_{\text{gain}}(e^{-\theta|v|^2}, f)|$ using different exponent of $k_2(v, u)$ in (2.111), and we apply the same computation as (2.125) to have
\[ w_\theta(v) |_{/Gamma1} \Gamma_{\text{gain}}(e^{-\theta|v|^2}, f) | \lesssim w_\theta(v) \int_{\mathbb{R}^3} k_\theta(v, u) | f(u) | \lesssim \| w_\theta f \|_\infty. \]

For the other term we follow (2.115) to have
\[ w_\theta(v) |_{/Gamma1} \Gamma_{\text{gain}}(e^{-\theta|v|^2}, f) | \lesssim w_\theta(v) \int_{\mathbb{R}^3} \| v - u | e^{-(e-\theta)|v|^2} \sqrt{\mu(u)} | f(u) | \lesssim \| w_\theta f \|_\infty. \]

\[ \square \]

3. Differentiation Along the Stochastic Cycles: Mixing via Diffuse Reflection and Transport

The main purpose of this section is to provide crucial differentiation form of the transport equation with the diffuse reflection boundary condition, which will be stated in Proposition 1. Several geometric integration by parts will be employed as being described in Section 1.3.

Consider a sequence of linear transport equation for $\ell \geq 1$ with the inflow boundary condition
\[ v \cdot \nabla_x f^\ell + \nu(v) f^\ell = h^\ell(x, v), \quad (x, v) \in \Omega \times \mathbb{R}^3, \quad (3.1) \]
\[ f^\ell(x, v) = g^\ell(x, v), \quad (x, v) \in \gamma_. \quad (3.2) \]

Here we set $f^0 = 0$.

Later we will substitute the $h^\ell$ by the sequence of collision operator defined as
\[ h^\ell(x, v) := K(f^{\ell-1}) + \Gamma(f^{\ell-1}, f^{\ell-1}), \quad (3.3) \]
and $g^\ell$ by the sequence of boundary condition:
\[ f^\ell(x, v)|_{\gamma_\perp} = \frac{M_{\nu}(x, v)}{\sqrt{\mu(v)}} \int n(x) \cdot v^1 > 0 f^{\ell-1}(x, v^1) \sqrt{\mu(v^1)}(n(x) \cdot v^1) dv^1 + r(x, v), \quad (3.4) \]
where $r(x, v)$ is defined in (2.6).

Note that from the collision operator (3.3) and boundary condition (3.4) and $f^0 = 0$,
\[ h^1(x, v) = 0, \quad g^1(x, v) = r(x, v). \]

We have the following expansion:
**Proposition 1.** Suppose $f$ solves inhomogeneous steady transport equation (3.1) with the diffuse BC (3.4). Then

$$w_{\tilde{\theta}}(v)\partial_{x_i} f^\ell(x, v)$$

$$= O(1)w_{\tilde{\theta}}(v) \frac{n_j(x^1)}{\alpha(x, v)}$$

$$\times \left\{ \frac{v(v)}{w_{\tilde{\theta}}(v)} \|w_{\tilde{\theta}} f^\ell\|_{L^\infty(\partial\Omega)} + |v| \frac{M_W(x^1, v)}{\sqrt{\mu(v)}} \|w_{\tilde{\theta}} f^{\ell-1}\|_{L^\infty(\partial\Omega)} \right\}$$

$$+ \frac{O(1)}{\alpha(x, v)} e^{-v(v)t} (w_{\tilde{\theta}}\alpha\partial_{x_j} f^\ell)(x - tv, v)$$

$$+ \int_{\max\{0, t - \beta\}}^t e^{-v(v)(t-s)} w_{\tilde{\theta}}(v)\partial_{x_j} h^\ell(x - (t-s)v, v) \, ds$$

$$+ O(1)e^{-v(v)n} \frac{n_j(x^1)}{\alpha(x, v)} \frac{|w_{\tilde{\theta}}(v) M_W(x^1, v)|}{\sqrt{\mu(v)}} \int_{n(x^1), v > 0} dv^1(n^1 \cdot v^1)\sqrt{\mu(v^1)}$$

$$\times \left\{ e^{-v(v)t} \frac{1}{\alpha(x^1, v^1)} (\alpha\nabla_{x^1} f^{\ell-1})(x^1 - t\cdot v^1, v^1) \right\}$$

$$+ \int_{\max\{0, t - \beta\}}^t e^{-v(v)(t-s)} \nabla_{x^1} h^{\ell-1}(x^1 - (t-s)v^1, v^1) \, ds^1.$$ 

Here $\tilde{\theta} > \tilde{\theta}$, $\nabla_{x^1} a(x^1 + \cdot)$ stands the tangential derivative $\nabla_{x^1_p}[a(\eta_{p^1}(x^1_{p^1}) + \cdot)]$ in a local coordinate of (1.6) as in (2.29).

To estimate the trace of $f^\ell$ in (3.5), we need the following lemma:

**Lemma 3.1.** (Theorem (9.2.1) in [5]) For $f^\ell$ satisfying (3.1), we have the following property for the trace of $f^\ell$:

$$\|f^\ell\|_{L^\infty(\partial\Omega)} \lesssim \|f^\ell\|_\infty + \|h^\ell\|_\infty. \tag{3.11}$$

The following lemma is a direct consequence of Lemma 3.1

**Lemma 3.2.** Let $h^\ell(x, v) := K(f^{\ell-1}) + \Gamma(f^{\ell-1}, f^{\ell-1})$, then the trace of $f^\ell$ is well-defined. Moreover, for some $\tilde{\theta} > \tilde{\theta}$, we have

$$\|f^\ell\|_{L^\infty(\partial\Omega)} \leq \|w_{\tilde{\theta}} f^\ell\|_{L^\infty(\partial\Omega)} \lesssim \|w f^\ell\|_\infty + \|w f^{\ell-1}\|_\infty. \tag{3.12}$$

**Proof.** $w_{\tilde{\theta}} f^\ell$ satisfies

$$v \cdot \nabla_x (w_{\tilde{\theta}} f^\ell) + v(v)(w_{\tilde{\theta}} f^\ell) = w_{\tilde{\theta}} h^\ell(x, v).$$
Applying Lemma 3.1 to the above equation, we have
\[
\|w_\theta f^\ell\|_{L^\infty(\partial \Omega)} \lesssim \|w_\theta f^\ell\|_\infty + \|w_\theta h^\ell\|_\infty \\
\lesssim \|fw^\ell\|_\infty + \|fw^{\ell-1}\|_\infty,
\]
where we have applied (2.124) to \(h^\ell\).

\[\square\]

**Proof of Proposition 1.** Consider \(f^\ell\) solves (3.1) and (3.4). Choose \(t \gg 1\). Recall (2.29). Same as (2.10)–(2.13), for \(k \geq 1, n(x_{p^k}) \cdot v^k > 0\), and \(i = 1, 2, \) or \(k = 0\) with \(i = 1, 2, 3,\)

\[
w_\theta(v^k)\partial_{x_{p^k},i} [f^\ell(n_{p^k}(x_{p^k}), v^k)]
\]

\[= 1_{k \geq t_b}e^{-\nu_t b^k} w_\theta(v^k)\partial_{x_{p^k},i} [f^\ell(x_b(n_{p^k}(x_{p^k}), v^k), v^k)]
\] (3.13)

\[-1_{k \geq t_b}v_b^k \partial_{x_{p^k},i} t_b^k e^{-\nu_t b^k} w_\theta(v^k) f^\ell(x_b(n_{p^k}(x_{p^k}), v^k), v^k)\]

\[+ 1_{k < t_b}e^{-\nu_t b^k} w_\theta(v^k)\partial_{x_{p^k},i} [f^\ell(n_{p^k}(x_{p^k}) - t_b^k v^k, v^k)]
\] (3.14)

\[+ \int_{\max[0,t^k-t_b^k]} e^{-\nu_t (t^k-s^k)} w_\theta(v^k)\partial_{x_{p^k},i} [h^\ell(n_{p^k}(x_{p^k}) - (t^k-s^k)v^k, v^k)]ds^k
\] (3.15)

\[+ \partial_{x_{p^k},i} b^k e^{-\nu_t b^k} w_\theta(v^k)h^\ell(x_b(n_{p^k}(x_{p^k}), v^k), v^k),
\] (3.16)

where we denoted \(v^k = v(v^k)\).

**Estimate of (3.13).** From (3.4) and (2.25) with replacing \(f\) by \(f^\ell\), for \(k \geq 1\) with \(i = 1, 2, \) or \(k = 0\) with \(i = 1, 2, 3,\) if \(x_b(n_{p^k}(x_{p^k}), v^k) \in \mathcal{O}_{p^k+1}\) then

\[
w_\theta(v^k)\partial_{x_{p^k},i} [f^\ell(x_b(n_{p^k}(x_{p^k}), v^k), v^k)]
\]

\[= \sum_{j=1,2} \frac{\partial x_{p^k+1, j}}{\partial x_{p^k, i}} w_\theta(v^k) \partial_{x_{p^k+1, j}} [f^\ell(n_{p^k+1}(x_{p^k+1}), v^k)]
\] (3.18)

\[= \sum_{j=1,2} \frac{\partial x_{p^k+1, j}}{\partial x_{p^k, i}} \left[ w_\theta(v^k) M_w(n_{p^k+1}(x_{p^k+1}), v^k) \right]
\]

\[\times \int_{\mathcal{V}_{p^k+1}^k, p^k+1, 3 > 0} \sqrt{\mu(v^k)} v^k_{p^k+1, 3} dv^k_{p^k+1, 3}
\] (3.19)

\[\times \partial_{x_{p^k+1, j}} [f^{\ell-1}(n_{p^k+1}(x_{p^k+1}), T_{p^k+1}^k v^k_{p^k+1})]
\]

\[+ O(1) \sum_{j=1,2} w_\theta(v^k)\partial_{x_{p^k+1, j}} r(n_{p^k+1}(x_{p^k+1}), v^k)\{1 + \|f^{\ell-1}\|_{L^\infty(\partial \Omega)}\}\]. (3.20)
Note that the above equalities for $k = 0$ gives an identity of $\partial_{x_i} [ f^\ell (x_0(x, v), v)]$.

It is relatively simple to derive that, from (2.34),

$$\tag{3.20} \frac{O(1)}{w_{\ell} r(x^{k+1})} [1 + \| w_{\ell} f^{\ell-1} \|_{L^\infty(\partial\Omega)}].$$ \hspace{1cm} (3.21)

Now we consider (3.19). We compute $(3.19)_{(k,i,a)} = (2.29)_{(k+1,j,f^{\ell-1})} + (3.22)$. Here (3.22) is given by

$$\left( \frac{\partial x_{p+1, i}}{\partial x_{p+1, j}} T_{x_{p+1, i}}^{k+1} v_{p+1}^{k+1} \right) \cdot \nabla_v f^{\ell-1} (\eta_{p+1}(x_{p+1}^{k+1}), T_{x_{p+1}^{k+1}}^{k+1} v_{p+1}^{k+1})$$

$$= \sum_{l,m} \frac{\partial}{\partial x_{p+1, i}} \left( \frac{\partial m \eta_{p+1, l}(x_{p+1}^{k+1})}{\sqrt{g_{p+1, mm}(x_{p+1}^{k+1})}} \right) v_{p+1}^{k+1} \times \partial m \eta_{p+1, l}(x_{p+1}^{k+1}), T_{x_{p+1}^{k+1}}^{k+1} v_{p+1}^{k+1}$$

$$= \sum_{m,n} (3.23)_{mn} v_{p+1, m}^{k+1} \partial v_{k+1, n}^{k+1} [ f^{\ell-1} (\eta_{p+1}(x_{p+1}^{k+1}), T_{x_{p+1}^{k+1}}^{k+1} v_{p+1}^{k+1})].$$ \hspace{1cm} (3.22)

where

$$(3.23)_{mn} := \sum_{l} \frac{\partial}{\partial x_{p+1, i}} \left( \frac{\partial m \eta_{p+1, l}(x_{p+1}^{k+1})}{\sqrt{g_{p+1, mm}(x_{p+1}^{k+1})}} \right) \frac{\partial n \eta_{p+1, l}(x_{p+1}^{k+1})}{\sqrt{g_{p+1, nn}(x_{p+1}^{k+1})}}.$$ \hspace{1cm} (3.23)

Here we have used (2.23) and (2.24).

First we consider the contribution of (3.22) in (3.19). We substitute (3.22)–(3.23) for $(3.19)_{*}$ and then apply the integration by parts with respect to $\partial v_{k+1}^{k+1}$ to derive that

$$\int_{\psi_{p+1, 3} > 0} f^{\ell-1} (\eta_{p+1}(x_{p+1}^{k+1}), T_{x_{p+1}^{k+1}}^{k+1} v_{p+1}^{k+1}$$

$$\times \sum_{m,n} (3.23)_{mn} \partial v_{k+1, n}^{k+1} [ v_{p+1, m}^{k+1} v_{p+1, 3} \sqrt{\mu(v_{p+1}^{k+1})}]dv_{p+1}^{k+1}$$

$$= O(1) \| \eta \|_{C^2} \| w_{\ell} f^{\ell-1} \|_{L^\infty(\partial\Omega)}.$$ \hspace{1cm} (3.24)

Here we have used

$$f^{\ell-1} (\eta_{p+1}(x_{p+1}^{k+1}), T_{x_{p+1}^{k+1}}^{k+1} v_{p+1}^{k+1}) \sum_{m,n} (3.23)_{mn} v_{p+1, m}^{k+1} v_{p+1, 3} \sqrt{\mu(v_{p+1}^{k+1})}$$

$$\equiv 0 \text{ when } v_{p+1, 3}^{k+1} = 0$$

for $\| f^{\ell-1} \|_{L^\infty(\partial\Omega)} < \infty$ by Lemma 3.2.

*Estimate of the contribution of $(2.29)_{(k,i,a)} \to (k+1,j,f^{\ell-1})$ in (3.19).* Since the velocity variables of

$(2.29)_{(k,i,a)} \to (k+1,j,f^{\ell-1})$ is written in Cartesian coordinate as $v_{p+1}^{k+1}$ (not $v_{p+1}^{k+1}$), we rewrite the $v_{p+1}^{k+1}$-integration of (3.19) in $v_{p+1}^{k+1}$-integration. Then, along the trajectory, $(2.29)_{(k,i,a)} \to (k+1,j,f^{\ell-1})$ can be represented by (3.13)–(3.16) with $(k, i, \ell) \to$
\((k + 1, j, \ell - 1)\). Here we further replace \((3.13)_{(k, i, \ell) \rightarrow (k + 1, j, \ell - 1)}\) by \(1_{r_{k+1} \geq b} \times e^{-v_{k+1}^{k+1}}\) \((3.18)_{(k, i, \ell) \rightarrow (k + 1, j, \ell - 1)}\). We note that we do not use a further expansion of \((3.19)-(3.20)\). Throughout the process, we derive an identity

\[
(3.19) \text{ with } [(3.19)_{*} = (2.29)_{(k, i, a) \rightarrow (k+1, j, f^{\ell-1})}]
\]

\[
= \int_{n_{k+1}, v_{k+1} > 0} 1_{r_{k+1} \geq b} e^{-v_{k+1}^{k+1}} \sum_{p_{k+2} \in \mathcal{P}} t_{p_{k+2}}(x_{b}(x^{k+1}, v^{k+1}))
\]

\[
\times \sum_{j' = 1, 2} \frac{\partial x^{k+2}_{p_{k+2}, j'}}{\partial x^{k+1}_{p_{k+1}, j}} \partial_{x^{k+2}_{p_{k+2}, j'}} \left[ f^{\ell-1}(\eta_{p_{k+2}}(x^{k+2}_{p_{k+2}}, v^{k+1})) \right]
\]

\[
\times \sqrt{\mu(v^{k+1})} \{n^{k+1} \cdot v^{k+1}\} dv^{k+1}
\]

\[
+ \int_{n_{k+1}, v_{k+1} > 0} \sum_{p_{k+2} \in \mathcal{P}} t_{p_{k+2}}(x_{b}(x^{k+1}, v^{k+1}))
\]

\[
\times [(3.14) + (3.15) + (3.16)]_{(k, i, \ell) \rightarrow (k + 1, j, \ell - 1)}
\]

\[
\times \sqrt{\mu(v^{k+1})} \{n^{k+1} \cdot v^{k+1}\} dv^{k+1}.
\]

Here we have denoted \(n^{k+1} = n(x^{k+1})\). It is relatively easy to derive

\[
(3.26) = O(1) \|w_{b} x^{\ell-1}\|_{L_{\infty}(\partial \Omega)}
\]

\[
+ O(1) [(3.9) + (3.10)]_{(t^{1}, x^{1}, v^{1}, f) \rightarrow (t^{k+1}, x^{k+1}, v^{k+1}, f^{\ell-1})},
\]

where we have used \(|\partial x^{k+1}_{p_{k+1}}| \leq \frac{1}{n(x^{k+1}) \cdot v^{k+1}}\) from \((2.32)\).

In order to take off \(\partial x^{k+2}_{p_{k+2}, j'}\) from \(f^{\ell-1}\) in \((3.25)\) we use the change of variables of \((2.42)\). Note that

\[
v^{k+1} = (x^{k+1} - \eta_{p_{k+2}}(x^{k+2}_{p_{k+2}})) / t_{b}^{k+1}.
\]

Now we apply the change of variables of \((2.42)\) and derive that

\[
= \sum_{p_{k+2} \in \mathcal{P}} \int_{|x^{k+2}_{p_{k+2}}| < \delta_{1}} \int_{0}^{t_{b}^{k+1}} e^{-v(u^{k+1})} t_{p_{k+2}}(\eta_{p_{k+2}}(x^{k+2}_{p_{k+2}}))
\]

\[
\times \sum_{j' = 1, 2} \frac{\partial x^{k+2}_{p_{k+2}, j'}}{\partial x^{k+1}_{p_{k+1}, j}} \partial_{x^{k+2}_{p_{k+2}, j'}} \left[ f^{\ell-1}(\eta_{p_{k+2}}(x^{k+2}_{p_{k+2}}, v^{k+1})) \right]
\]

\[
\times n_{p_{k+2}}(x^{k+1}_{p_{k+2}}) \cdot (x^{k+1} - \eta_{p_{k+2}}(x^{k+2}_{p_{k+2}})) / t_{b}^{k+1}
\]

\[
\times \sqrt{\mu(v^{k+1})} dx_{b}^{k+1} \sqrt{g_{p_{k+2}, 11} g_{p_{k+2}, 22} dx_{p_{k+2}, 1} dx_{p_{k+2}, 2}}.
\]
Here we read $g_{p^{k+2},ii}$ at $x^{k+2}_{p^{k+2}}$.

We apply the integration by parts with respect to $p^{k+2, j'}$ for $j' = 1, 2$. For $t_{p^{k+2}}(\eta_{p^{k+2}}(x^{k+2}_{p^{k+2}})) = 0$ when $|x^{k+2}_{p^{k+2}}| = \delta_1$ from (2.19), such contribution of $|x^{k+2}_{p^{k+2}}| = \delta_1$ vanishes. Then we derive

$$
(3.25) = \sum_{p^{k+2} \in \mathcal{D}} \int \int \int_0^{x^{k+1}} \partial x^{k+2}_{p^{k+2}, j'} \times \left[ e^{-v(v^{k+1})b^{k+1}} \sqrt{\mu(v^{k+1})t_{p^{k+2}}(\eta_{p^{k+2}}(x^{k+2}_{p^{k+2}}))} \right] \cdots
$$
$$
+ \sum_{p^{k+2} \in \mathcal{D}} \int \int \int_0^{x^{k+1}} \partial x^{k+2}_{p^{k+2}, j'} \left[ \sum_{j'=1, 2} \frac{\partial x^{k+2}_{p^{k+2}, j'} \cdot (x^{k+1} - \eta_{p^{k+2}}(x^{k+2}_{p^{k+2}}))}{|t_b^{k+1}|^4} \right] \cdots
$$

From (2.34) and (2.44), (2.45), we derive that

$$
\left| \frac{\partial}{\partial x^{k+2}_{p^{k+2}, j'}} \left( \sum_{j'=1, 2} \frac{\partial x^{k+2}_{p^{k+2}, j'} \cdot (x^{k+1} - \eta_{p^{k+2}}(x^{k+2}_{p^{k+2}}))}{|t_b^{k+1}|^4} \right) \right| \lesssim \|\eta\|_{C^2} \left\{ 1 + \frac{|v^{k+2}_{p^{k+2}, 3}|^2}{|v^{k+2}_{p^{k+2}, 3}|^2} |x^{k+1} - \eta_{p^{k+2}}(x^{k+2}_{p^{k+2}})| \right\}
$$
$$
\lesssim O(\|\eta\|_{C^2}) \left\{ 1 + \frac{|v^{k+2}_{p^{k+2}, 3}|^2}{|v^{k+2}_{p^{k+2}, 3}|^2} |x^{k+1} - \eta_{p^{k+2}}(x^{k+2}_{p^{k+2}})| \right\}
$$
$$
\lesssim O(\|\eta\|_{C^2}) \frac{1}{|v^{k+2}_{p^{k+2}, 3}|^2} = O(\|\eta\|_{C^2}) \frac{|t_b^{k+1}|}{|n_{p^{k+2}}(x^{k+1}_{p^{k+2}}) \cdot (x^{k+1} - \eta_{p^{k+2}}(x^{k+2}_{p^{k+2}}))|}.
$$

Now using (3.28) for (3.30), (3.33) for (3.31), and (2.45) for (3.32), we derive that

$$
|\langle 3.25 \rangle | \lesssim \|\eta\|_{C^2} \|f^{\ell-1}\|_{L^\infty(\partial \Omega)} \int \int \int_0^{x^{k+1}} e^{-v_0 t_b^{k+1}} \left[ \frac{|x^{k+1} - \eta_{p^{k+2}}(x^{k+2}_{p^{k+2}})|^3}{|t_b^{k+1}|^5} + \frac{|x^{k+1} - \eta_{p^{k+2}}(x^{k+2}_{p^{k+2}})|^2}{|t_b^{k+1}|^4} \right] e^{-\frac{|x^{k+1} - \eta_{p^{k+2}}(x^{k+2}_{p^{k+2}})|^2}{4|t_b^{k+1}|^2}}
$$
where we have used Lemma 3.2 and

\[
\begin{align*}
\left[ \frac{|x^{k+1} - \eta_{p_k^2}(x_{p_k+2}^{k+2})|^3}{|t_b^{k+1}|^5} + \frac{|x^{k+1} - \eta_{p_k^2}(x_{p_k+2}^{k+2})|^2}{|t_b^{k+1}|^4} \right] e^{-\frac{|x^{k+1} - \eta_{p_k^2}(x_{p_k+2}^{k+2})|^2}{4|t_b^{k+1}|^2}} &
\leq \\
\left[ \frac{|x^{k+1} - \eta_{p_k^2}(x_{p_k+2}^{k+2})|^9/2}{|t_b^{k+1}|^9/2} + \frac{|x^{k+1} - \eta_{p_k^2}(x_{p_k+2}^{k+2})|^{7/2}}{|t_b^{k+1}|^{7/2}} \right] e^{-\frac{|x^{k+1} - \eta_{p_k^2}(x_{p_k+2}^{k+2})|^2}{4|t_b^{k+1}|^2}} &
\leq \frac{1}{|t_b^{k+1}|^{1/2}} |x^{k+1} - \eta_{p_k^2}(x_{p_k+2}^{k+2})|^{3/2}.
\end{align*}
\]

Finally collecting terms (3.13)–(3.16), and (3.21), (3.34) and setting \( k = 0 \), we prove the Proposition 1. \( \Box \)

4. Mixing via the Binary Collision and Transport

In this section we mainly establish the integration by parts technique mentioned in Section 1.3 using the mixing of the binary collision and the transport operator. In particular, we will prove Proposition 2. As direct consequence of Proposition 1 and Proposition 2 we will give a proof of the (1.18) in Main Theorem.

We consider a solution of the Boltzmann equation (3.1) with \( h^\ell(x, v) \) given by (3.3), and the diffuse BC (3.4). The main result is an estimate of (3.3)-contribution in (3.7) and (3.10).

Proposition 2. We bound (3.7) \((3.7)_{x=K_f^\ell-1}\), (3.7) \((3.7)_{x=\Gamma(f^{\ell-1}, f^{\ell-1})}\), and (3.8) · (3.10) \((3.10)_{x=K_f^\ell-2+\Gamma(f^{\ell-2}, f^{\ell-2})}\) respectively as

\[
\begin{align*}
\int_{\max[0, t - t_b]} e^{-v(t-s)} w_\tilde{\vartheta}(v) \partial_x K f^{\ell-1}(x - (t - s)v, v) \, ds &
\leq \frac{O(1)}{\alpha(x, v)} \left\{ \left( \varepsilon + \sup_{0 \leq i \leq \ell - 1} \| w f^{\ell-1-i} \|_\infty \right) \sup_{0 \leq i \leq \ell - 1} \| \alpha \nabla_x f^{\ell-1-i} \|_\infty \right. \\
&\left. + \varepsilon^{-1} \sup_{0 \leq i \leq \ell - 1} \| w f^{\ell-1-i} \|_\infty \right\},
\end{align*}
\]

(4.1)
\[
\int_{t_{0}}^{t} e^{-v(t-s)} w_\beta \partial_1 \Gamma (f^{\ell}, f^{\ell})(x - (t - s)v, v) \, ds
\]
\[
= \frac{O(1)}{\alpha(x, v)} \times \left\{ \epsilon \sup_{0 \leq i \leq \ell - 1} \| w_\beta \alpha \nabla_x f^{\ell - 1 - i} \|_\infty + \epsilon^{-1} \sup_{0 \leq i \leq \ell - 1} \| w f^{\ell - 1 - i} \|_\infty \right\},
\]
(4.2)
\[
\| w_\beta \alpha \nabla_x f^{\ell} \|_\infty \lesssim \sup_{i \geq 0} \| w f^i \|_\infty + \sup_{i \geq 1} \| w f^i \|_\infty.
\]

Proof of (1.18) in Main Theorem By Lemma 3.2, we have

\[
\sup_{i \geq 0} \| w_\beta f^i \|_{L^\infty(\partial \Omega)} \lesssim \sup_{i \geq 0} \| w f^i \|_\infty + \sup_{i \geq 1} \| w f^i \|_\infty.
\]

Combining Proposition 1 and Proposition 2 we obtain that for \( t \gg 1 \) and \( \varepsilon \ll 1 \),

\[
\| w_\beta \alpha \nabla_x f^{\ell} \|_\infty \lesssim o(1) \sup_{i \leq \ell - 1} \| w_\beta \alpha \nabla_x f^i \|_\infty + C(\varepsilon) \| T_W - T_0 \|_{C^1} \sup_{i \geq 0} \| w f^i \|_\infty,
\]

where the \( \| T_W - T_0 \|_{C^1} \) comes from \( |\nabla_x r(x^1)| \) in (3.5).

By a standard argument we pass the limit and conclude that the unique solution in Existence Theorem satisfies the weighted \( C^1 \) estimate (1.18).

4.2.4.1 Nonlocal-to-local estimate and small time contributions

The key lemma to prove Proposition 2 is the following Nonlocal-to-Local estimate:

**Lemma 4.1.** Denote \( x' = x - (t - s)v, \ y' = y - (t - s)v \). Assume \( (t, x, v) \in [0, \infty) \times \tilde{\Omega} \times \mathbb{R}^3 \) and \( t - t_0(x, v) \leq t - t_1 \leq t - t_2 \leq t \). Then for \( 0 < \beta < 1 \) and some \( C > 0 \),

\[
\int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^3} \frac{e^{-C(v)(t-s)} e^{-\varepsilon |v-u|^2}}{|v-u| \alpha(x', u)} \, du \, ds
\]
\[
\lesssim |e^{-C_1 v_{t_1}} - e^{-C_1 v_{t_2}}| \beta \left[ \ln |v| + \ln \alpha(x, v) \right] + \frac{1}{|v|^{1-\beta}}
\]
Thus
\[
\int_{t-t_b(x,v)}^t \int_{\mathbb{R}^3} \frac{e^{-C(v)(t-s)} e^{-e[v-u]^2}}{|v-u||\alpha(x',u)|} \, du \, ds \lesssim \frac{1}{\alpha(x,v)},
\]
(4.5)
and for \( \varepsilon \ll 1 \),
\[
\int_{t-t_b(x,v)}^t \int_{\mathbb{R}^3} \frac{1_{s \geq t-\varepsilon} e^{-C(v)(t-s)} e^{-e[v-u]^2}}{|v-u||\alpha(x',u)|} \, du \, ds \lesssim \frac{O(\varepsilon)}{\alpha(x,v)}.
\]
(4.6)
For \( 1 < p < 3 \), we have
\[
\int_{t-t_b(x,v)}^t \int_{\mathbb{R}^3} \frac{k(v,u)}{|u|^2 \min \left\{ \frac{\alpha(x',u)}{|u|}, \frac{\alpha(y',u)}{|u|} \right\}^p} \, du \, ds \lesssim \frac{\min\{1, O(t)\}}{|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^p},
\]
(4.7)
and
\[
\int_{t-t_b(x,v)}^t \int_{\mathbb{R}^3} \frac{k(v,u)}{|u|^2 \min \left\{ \frac{\alpha(x',u)}{|u|}, \frac{\alpha(y',u)}{|u|} \right\}^p} \, du \, ds \lesssim \frac{\varepsilon}{|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^p}.
\]
(4.8)
For \( \beta < 1 \),
\[
\int_{t-t_b(x,v)}^t \frac{e^{-C(v)(t-s)}}{|v| \min\{\xi(x'), \xi(y')\}^{\beta/2}} \, ds \lesssim \frac{1}{|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^\beta}.
\]
(4.9)
\[
\int_{\mathbb{R}^3} k(v,u) \frac{1}{|\alpha(x,u)|^\beta} \, du \lesssim 1.
\]
(4.10)
\[
\int_{t-t_b(x,v)}^t \int_{\mathbb{R}^3} \frac{e^{-C(v)(t-s)} e^{-e[v-u]^2}}{|v-u|^2 \min \{\alpha(x',u), \alpha(y',u)\}} \, du \, ds \lesssim \frac{1}{\min \{\alpha(x,v), \alpha(y,v)\}^\beta}.
\]
(4.11)
Remark 5. We note that (4.6) can be considered as a boarderline case of Lemma 10 in [14] in which the integral \( 1/\alpha^\beta \) is considered for \( \beta \gtrless 1 \).

The proof of Proposition 2 only require (4.5) and (4.6). But in the Section 7, in the proof of the weighted \( C^{1,\beta} \) estimate (1.21), the nonlocal-to-local type estimate will be involved with different power of \( \alpha \). We summarize all these estimates in this single lemma.
Proof. During the whole proof we assume \( \alpha = \tilde{\alpha} \). For the other case, when \( \alpha \gtrsim 1 \), the lemma follows from \( k(v, u) \in L^1 \).

**Proof of (4.4) (4.5) and (4.6)** We only prove (4.4). (4.5) and (4.6) follow directly from (4.4).

**Step 1.** We claim that, for \( y \in \bar{\Omega} \) and \( \varrho > 0 \),

\[
\int_{\mathbb{R}^3} \frac{e^{-e|v-u|^2}}{|v-u|} \frac{1}{\alpha(y, u)} du \lesssim 1 + |\ln |\xi(y)|| + |\ln |v||. \tag{4.12}
\]

Recall (2.24) and set \( v = v(y) \) and \( u = u(y) \). For \( |u| \gtrsim O(1)|v| \),

\[
\left[ |u_3(y)|^2 + |\xi(y)||u|^2 \right]^{1/2} \gtrsim \left[ |u_3(y)|^2 + |\xi(y)||u|^2 \right]^{1/2}. \tag{4.13}
\]

Thus

\[
\int_{|u| \leq |u| \leq 4|v|} \lesssim \int \int e^{-\varrho|v| - |u|^2} \frac{1}{|v| - |u|} \int_{0}^{4|v|} \left[ |u_3|^2 + |\xi(y)||u|^2 \right]^{1/2} du_3
\]

\[
\lesssim \int \int \left[ |u_3|^2 + |\xi(y)||u|^2 \right]^{1/2} du_3
\]

\[
= \ln \left( \sqrt{|u_3|^2 + |\xi(y)||u|^2} + |u_3| \right)_{0}^{4|v|}
\]

\[
= \ln \left( \sqrt{16|v|^2 + |\xi(y)||u|^2} + 16|v|^2 \right) = \ln \left( \sqrt{16|v|^2 + |\xi(y)||u|^2} \right)
\]

\[
\lesssim \ln |v| + \ln |\xi(y)|. \tag{4.14}
\]

If \( |u| \gtrsim 4|v| \) then \( |u - v|^2 \gtrsim \frac{|v|^2}{4} + \frac{|u|^2}{4} \) and hence \( e^{-e|v-u|^2} \leq e^{-\varrho|v|^2} e^{-\varrho|u|^2} e^{-\varrho|v-u|^2} \). This, together with (4.13), implies

\[
\int_{|u| \geq 4|v|} \lesssim e^{-\varrho|v|^2} \int \int \frac{e^{-\varrho|v|}}{|v| - |u|} \int_{0}^{1} \frac{e^{-\varrho|u_3|^2}}{\left[ |u_3|^2 + |\xi(y)||u|^2 \right]^{1/2}} du_3
\]

\[
\lesssim e^{-\varrho|v|^2} + e^{-\varrho|v|^2} \int_{0}^{1} \frac{du_3}{\left[ |u_3|^2 + |\xi(y)||u|^2 \right]^{1/2}}
\]

\[
eq e^{-\varrho|v|^2} + e^{-\varrho|v|^2} \ln \left( \sqrt{|u_3|^2 + |\xi(y)||u|^2} + |u_3| \right)_{0}^{1}
\]

\[
eq e^{-\varrho|v|^2} \left\{ 1 + \ln \left( \sqrt{1 + |\xi(y)||u|^2} + 1 \right) - \ln \left( \sqrt{1 + |\xi(y)||u|^2} \right) \right\}
\]

\[
\lesssim e^{-\varrho|v|^2} \left\{ \ln |v| + \ln |\xi(y)| \right\}. \tag{4.15}
\]

For \( |u| \leq \frac{|v|}{4} \), we have \( |v - u| \geq |v| - |u| \geq |v| - \frac{|v|}{4} \geq \frac{|v|}{2} \). We have

\[
\int_{|u| \leq \frac{|v|}{4}} \lesssim e^{-\varrho|v|^2} |v| \int \int_{u_3 + |u| \leq \frac{|v|}{2}} \frac{du_3}{\left[ |u_3|^2 + |\xi(y)||u|^2 \right]^{1/2}}
\]
\[ \lesssim |v|e^{-\frac{\theta}{4}|v|^2} \int_{|\mathbf{u}_3| \leq \frac{1}{2}} \int_{|\mathbf{\tilde{u}}| \leq \frac{1}{2}} \frac{d\mathbf{\tilde{u}}d\mathbf{\tilde{u}_3}}{[|\mathbf{\tilde{u}_3}|^2 + |\xi(y)||\mathbf{\tilde{u}}|]^{1/2}}, \]

where we have used \(|v|\tilde{u} = u\). Using the polar coordinate \(\mathbf{\tilde{u}_1} = |\mathbf{\tilde{u}}| \cos \rho, \mathbf{\tilde{u}_2} = |\mathbf{\tilde{u}}| \sin \rho\), we have

\[
\lesssim |v|e^{-\frac{\theta}{4}|v|^2} \int_0^{\frac{1}{2}} d\mathbf{\tilde{u}_3} \int_0^{\frac{\pi}{2}} d|\mathbf{\tilde{u}}| \left[ \frac{|\mathbf{\tilde{u}_3}|^2 + |\xi(y)||\mathbf{\tilde{u}}|]^{1/2} \right. \\
= |v|e^{-\frac{\theta}{4}|v|^2} \int_0^{\frac{1}{2}} d\mathbf{\tilde{u}_3} \frac{1}{|\xi(y)|} \left( \sqrt{\frac{|\mathbf{\tilde{u}_3}|^2 + |\xi(y)|}{2}} - |\mathbf{\tilde{u}_3}| \right) \\
= |v|e^{-\frac{\theta}{4}|v|^2} \ \times \left\{ \frac{1}{2} |\mathbf{\tilde{u}_3}| \sqrt{\sqrt{\frac{|\mathbf{\tilde{u}_3}|^2 + |\xi(y)|}{2}} + \sqrt{\frac{|\mathbf{\tilde{u}_3}|^2 + |\xi(y)|}{2}} + |\mathbf{\tilde{u}_3}|} - \frac{1}{2} |\mathbf{\tilde{u}_3}|^{1/2} \right\}
\] \\
\[ = |v|e^{-\frac{|v|}{4}|v|^2} \ \times \left\{ \frac{1}{4} \sqrt{1 + \frac{|\xi(y)|}{2}} + \frac{|\xi(y)|}{4} \log \left( \sqrt{1 + \frac{|\xi(y)|}{2}} + \frac{1}{2} \right) - \frac{|\xi(y)|}{4} \log \left( \frac{1}{2} \right) - \frac{1}{8} \right\}
\]
\[ \lesssim 1 + \log(|\xi(y)|). \quad (4.16) \]

Collecting terms from (4.14), (4.15), and (4.16), we prove (4.12).

**Step 2.** We prove the following statement: for \(x \in \partial \Omega\), we can choose \(0 < \tilde{\delta} \ll \Omega\) such that

\[ \delta^{1/2} |v \cdot \nabla \xi(x - (t - s)v)| \geq \Omega |v|\sqrt{-\xi(x - (t - s)v)}, \]

for \(s \in \left[ t - t_b(x, v), t - t_b(x, v) + \tilde{t} \right] \cup \left[ t - \tilde{t}, t \right], \quad (4.17) \)

\[ \delta^{1/2} \times \alpha(x, v) \geq \Omega |v|\sqrt{-\xi(x - (t - s)v)}, \]

for \(s \in \left[ t - t_b(x, v) + \tilde{t}, t - \tilde{t} \right], \quad (4.18) \)

here \(\tilde{t} = \min \left\{ \frac{t_b(x, v)}{2}, \frac{\delta \alpha(x, v)}{|v|^2} \right\}. \) We note that when \(\tilde{t} < \delta \alpha(x, v) / |v|^2\), (4.18) vanishes.

If \(v = 0\) or \(v \cdot \nabla \xi(x) \leq 0\), since \(x \in \partial \Omega\), then (4.17) and (4.18) hold clearly with \(t_b(x, v) = \tilde{t} = 0\). We may assume \(v \neq 0\) and \(v \cdot \nabla \xi(x) > 0\). Since \(v \cdot \nabla \xi(x) > 0\), from (1.10), we have \(|v \cdot \nabla \xi(x_b(x, v))| > 0\). Then we must have \(v \cdot \nabla \xi(x_b(x, v)) < 0\), otherwise \(v \cdot \nabla \xi(x_b(x, v)) > 0\) implies \(x_b\) is not the backward exit position defined in (1.24). By the mean value theorem there exists at least one \(t^* \in (t - t_b(x, v), t)\) such that \(v \cdot \nabla \xi(x - t^*v) = 0\). Moreover due to the convexity
in (1.7) we have \( \frac{d}{ds}(v \cdot \nabla \xi(x - (t - s)v)) = v \cdot \nabla^2 \xi(x - (t - s)v) \cdot v \geq C_\xi |v|^2 \), and therefore \( t^* \in (t - t_b(x, v), t) \) is unique.

Let \( s \in [t - \tilde{t}, t] \) for \( 0 < \tilde{\delta} \ll 1 \) and \( t \leq \tilde{\delta} \frac{\alpha(x, v)}{|v|^2} \). Then from the fact that \( v \cdot \nabla \xi(x - (t - \tau)v) \) is non-decreasing in \( \tau \in [t - t^*, t] \) and \( x \in \partial \Omega \),

\[
|v|^2(1)\xi(x - (t - s)v) = \int_s^t |v|^2 v \cdot \nabla \xi(x - (t - \tau)v) d\tau \leq \tilde{\delta} \alpha(x, v)|v| \cdot \nabla \xi(x). \tag{4.19}
\]

Since \( |v \cdot \nabla \xi(x)| \leq \alpha(x, v) \leq C_\Omega \alpha(x - (t - s)v) \) \( v \leq C_\Omega \{ |v \cdot \nabla \xi(x - (t - s)v)\| + \| \nabla^2 \xi \|_\infty |v| \sqrt{-\xi(x - (t - s)v)} \} \), we choose \( \tilde{\delta} \ll (C_\Omega \| \nabla^2 \xi \|_\infty^{-2} \) and absorb

\[
\tilde{\delta} \alpha(x, v) \times C_\Omega \{ |v \cdot \nabla \xi(x - (t - s)v)| + \| \nabla^2 \xi \|_\infty |v| \sqrt{-\xi(x - (t - s)v)} \}
\]

\[
\leq \tilde{\delta} \times (C_\Omega \| \nabla^2 \xi \|_\infty |v| \sqrt{-\xi(x - (t - s)v)})^2 \]

by the left hand side of (4.19). This gives (4.17) for \( s \in [t - \tilde{t}, \tilde{t}] \). The proof for \( s \in [t - t_b(x, v), t - t_b(x, v) + \tilde{t}] \) is same.

For (4.18), we assume \( \tilde{t} = \tilde{\delta} \frac{\alpha(x, v)}{|v|^2} \), otherwise (4.18) vanishes. For \( \xi(x - (t - s)v) \) is non-increasing in \( s \in [t - t^*, t] \), we have \( |v|^2(1)\xi(x - (t - s)v) \geq |v|^2(1)\xi(x - \alpha(x, v)|v|^2) \) for \( s \in [t - t^*, t - \tilde{\delta} \alpha(x, v)|v|^2] \). By an expansion, for \( s^* := t - \tilde{\delta} \frac{\alpha(x, v)}{|v|^2} \),

\[
|v|^2(1)\xi(x - \tilde{\delta} \frac{\alpha(x, v)}{|v|^2} - v)
\]

\[
= |v|^2(2 \cdot \nabla \xi(x))\tilde{\delta} \frac{\alpha(x, v)}{|v|^2} + \int_{s^*}^{t} \int_{s^*}^{\tau} |v|^2 v \cdot \nabla^2 \xi(x - (t - \tau')v) \cdot v d\tau' d\tau. \tag{4.20}
\]

The last term of (4.20) is bounded by \( \| \nabla^2 \xi \|_\infty \tilde{\delta}^2 \frac{\alpha(x, v)}{|v|^2} |v|^4 \leq \| \nabla^2 \xi \|_\infty \tilde{\delta}^2 \alpha(x, v) \). Since \( v \cdot \nabla \xi(x) \leq \alpha(x, v) \), for \( \tilde{\delta} \ll \| \nabla^2 \xi \|_\infty^{-1/2} \), the right hand side of (4.20) is bounded below by \( \tilde{\delta} \alpha(x, v) \). This completes the proof of (4.18) when \( s \in [t - t^*, t - \tilde{\delta} \frac{\alpha(x, v)}{|v|^2}] \). The proof for the case of \( s \in [t - t_b(x, v) + \tilde{\delta} \alpha(x, v)|v|^2, t - t^*] \) is same.

Step 3. From (4.12), for the proof of (4.4), it suffices to estimate

\[
\int_{t - t_b(x, v)}^{t} 1_{t - t_2 \geq s \geq t - t_1} e^{-C(v)(t - s)} |\ln |\xi(x - (t - s)v)|||ds \tag{4.21}
\]

\[
+ \int_{t - t_b(x, v)}^{t} 1_{t - t_2 \geq s \geq t - t_1} e^{-C(v)(t - s)} (1 + |\ln |v||)|ds.
\]

We simply bound the second term of (4.21) as

\[
(1 + |\ln |v||) \int_{t - t_1}^{t - t_2} e^{-C(v)(t - s)} \lesssim (1 + |\ln |v||)(v)^{-1} e^{-C(v)t_2} - e^{-C(v)t_1}. \tag{4.22}
\]
For the first term of (4.21), we first assume \( x \in \partial \Omega \). For utilizing (4.17) and (4.18), we split the first term of (4.21) as

\[
\begin{align*}
&\frac{\int_{t-h}^{t} + \int_{t-h}^{t-\tilde{t}} e^{-C(v)(t-s)/\beta} ds}{(4.23)_1} + \frac{\int_{t-h}^{t-\tilde{t}} e^{-C(v)(t-s)/\beta} ds}{(4.23)_2}.
\end{align*}
\]

(4.23)

Without loss of generality, we assume \( t-t_2 \in [t-\tilde{t}, t] \), \( t-t_1 \in [t-h(x, v)+\tilde{t}, t-\tilde{t}] \). For the first term \((4.23)_1\) we use a change of variables \( s \leftrightarrow -\xi(x-(t-s)v)\) in \( s \in [t-h(x, v), t-t^*] \) and \( s \in [t-t^*, t] \) separately with \( ds = |v \cdot \nabla_x \xi(x-(t-s)v)|^{-1} d|\xi| \). From (4.19) we have \(|\xi(x-(t-s)v)| \leq \delta e^{2(x,v)}\). Then applying Hölder inequality with \( \beta + (1-\beta) = 1 \) and using (4.17), we get

\[
(4.23)_1 \leq \left( \int_{t-h}^{t} e^{-C(v)(t-s)/\beta} ds \right)^{\beta} \left( \int_{t-h}^{t-\tilde{t}} e^{-C(v)(t-s)/\beta} ds \right)^{1-\beta}
\]

\[
\times \left[ \int_0^{\delta e^{2(x,v)}} |\ln |\xi||^{1/(1-\beta)} \left( \frac{d|\xi|}{\delta^{1/2}|v|\sqrt{|\xi|}} \right)^{1-\beta} \right]
\]

\[
\leq |e^{-C(v)t_2/\beta} - e^{-C(v)t_1/\beta}| \frac{1}{|v|^{1-\beta}},
\]

(4.24)

where we have used \( t-\tilde{t} > t-t_1, t-t_2 > t-h(x, v) + \tilde{t} \) and \( |\ln |\xi||^{1/(1-\beta)} \) \( \in L^1_{loc}(0, \infty) \) for \( \beta < 1 \) in the last line.

On the other hand, from (4.18),

\[
(4.23)_2 \leq \left( \int_{t-h}^{t} e^{-C(v)(t-s)} \left| \ln \left( \frac{\delta e^{2(x,v)}}{|v|^2} \right) \right| ds \right)
\]

\[
\leq 2 \int_{t-h}^{t} e^{-C(v)(t-s)} \left( |\ln \delta| + |\ln \alpha(x, v)| + |\ln |v|| \right) ds
\]

\[
\leq 2 |e^{-C(v)t_2} - e^{-C(v)t_1}| \langle v \rangle^{-1/2} \left( |\ln \delta| + |\ln \alpha(x, v)| + |\ln |v|| \right),
\]

(4.25)

where we have used a similar estimate of (4.22).

Now as assume \( x \notin \partial \Omega \). We find \( \tilde{x} \in \partial \Omega \) and \( \tilde{t} \) so that

\[
x = \tilde{x} - (\tilde{t} - t)v \text{ and } \tilde{t} > t.
\]

Then clearly, \( x-(t-s)v = \tilde{x}-(\tilde{t}-s)v \). Since \( \tilde{x} \in \partial \Omega \), applying the same computation as (4.24) and (4.25), the first term of (4.21) is bounded by

\[
\int_{t-h}^{\tilde{t}} 1_{t-h(x,v)+\tilde{t} \geq t} e^{-C(v)(\tilde{t}-s)} |\ln |\xi-(\tilde{t}-s)v|||ds
\]

\[
\leq (4.24) + |e^{-C(v)t_2} - e^{-C(v)t_1}| \langle v \rangle^{-1/2} \left( |\ln \delta| + |\ln \alpha(x, v)| + |\ln |v|| \right),
\]

(4.26)
Here we used $\alpha(\tilde{x}, v) \sim \alpha(x, v)$ from Lemma 1.10 to obtain the same upper bound as (4.24). Again using $\alpha(\tilde{x}, v) \sim \alpha(x, v)$, we conclude

\[(4.26) \lesssim (4.24) + (4.25).\]

Then we conclude the first inequality in (4.4) using $\beta < 1$.

For the second inequality, from (1.8) and (1.9) we bound a term of the upper bound of second line of (4.4) as

\[
\{v\}^{-1/2} \left(1 + |\ln |v|| + |\ln \alpha(x, v)|\right) \frac{1}{|v|^{1-\beta}} \times \frac{\alpha(x, v)}{\alpha(x, v)} \lesssim \frac{1}{\alpha(x, v)},
\]

where we have used $\alpha(x, v) \leq \min\{1, |v|\}$ and $1 - \beta < 1$.

\[\Box\]

Proof of (4.7) and (4.8) Again we only prove (4.7). Clearly, we have

\[
\int_0^t e^{-v(t-s)} ds \int_{\mathbb{R}^3} du \frac{k(v, u)}{|u|^2} \min \left\{ \frac{\alpha(x - (t-s)v, u)}{|u|}, \frac{\alpha(y - (t-s)v, u)}{|u|} \right\}^p
\]

\[
\leq \int_0^t e^{-v(t-s)} ds \int_{\mathbb{R}^3} du \frac{k(v, u)}{|u|^2} \left( \frac{\alpha(x - (t-s)v, u)}{|u|} \right)^p
\]

\[
+ \int_0^t e^{-v(t-s)} ds \int_{\mathbb{R}^3} du \frac{k(v, u)}{|u|^2} \left( \frac{\alpha(y - (t-s)v, u)}{|u|} \right)^p.
\]

By Lemma 1 in [14], we have

\[
\int_0^t e^{-v(t-s)} ds \int_{\mathbb{R}^3} du \frac{k(v, u)}{\alpha^p(x - (t-s)v, u)} \frac{|u|^{p-2}}{|v|^{p-2}} \frac{|v|^{p-2}}{p-2}
\]

\[
\lesssim |v|^{p-2} \left[ \min\{1, t\} \times \min\left\{ \frac{1}{|v|^2 \alpha^{p-2}(x, v)}, \frac{\alpha^{1/2-p/2}(x, v)}{|v|^{p-1}} \right\}
\]

\[+ \frac{1}{\alpha^{p-1}(x, v)} \int_0^t e^{-\frac{c}{|v|}(t-s)} ds \right].
\]

We bound

\[
\frac{1}{|v|^2 \alpha^{p-2}(x, v)} \leq \frac{\alpha^2(x, v)}{|v|^2} \frac{1}{\alpha^p(x, v)} \leq \frac{1}{\alpha^p(x, v)},
\]

\[
\frac{\alpha^{1/2-p/2}(x, v)}{|v|^{p-1}} \leq \frac{\alpha^{p-1}(x, v)}{|v|^{p-1}} \frac{1}{\alpha^{p/2-1/2+p-1}(x, v)}
\]

\[
\leq \frac{1}{\alpha^{3p/2-3/2}(x, v)} \lesssim \frac{1}{\alpha^p(x, v)},
\]

\[
\frac{1}{\alpha^{p-1}(x, v)} \int_0^t e^{-\frac{c}{|v|}(t-s)} ds \lesssim \min\{1, O(t)\} \frac{1}{\alpha^p(x, v)},
\]

where we have used $\alpha \leq 1$ and $p < 3$.

Then we conclude (4.7). \[\Box\]
Proof of (4.9) Since \[ \frac{1}{\min[\xi(x'), \xi(y')]^{\beta/2}} \lesssim \frac{1}{\xi^{\beta/2}(x')} + \frac{1}{\xi^{\beta/2}(y')} , \] we only need to prove
\[ \int_{t-\tau(x,v)}^t \frac{e^{-C(v)(t-s)}}{|v|\xi(x')^{\beta/2}} \lesssim \frac{1}{|v|^2 \min \left\{ \frac{\alpha(v,v)}{|v|}, \frac{\alpha(v,v)}{|v|} \right\}^{\beta}. \]

We split the integral as
\[ \int_{t-\tau(x,v)}^t + \int_{t-\tau(x,v)}^t + \int_{t-\tau(x,v)}^t. \]

Similarly to (4.24), the first two terms are bounded by
\[ 2 \int_0^{\delta} \frac{1}{|v||\xi|^{\beta/2} \delta^{-1/2} |v|\sqrt{|\xi|}} \lesssim \frac{1}{|v|^2}. \]

Similarly to (4.25), the third term is bounded by
\[ \int_0^t \frac{e^{-C(v)(t-s)}}{|v|} \frac{1}{|\delta|^{\beta/2} \alpha^\beta (x,v)} ds \lesssim \frac{1}{|v|(v)} \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(v,v)}{|v|} \right\}^{\beta}. \]

Proof of (4.10) Similarly to the proof of (4.6) we consider three cases: \[ \frac{|v|}{4} \leq |u| \leq 4|v|, \quad |u| \geq 4|v|, \quad |u| \leq \frac{|v|}{4}. \]

By a similar computation as to that for (4.14),(4.15) and (4.16), we obtain
\[ \int_{|v|}^{\frac{|v|}{4}} \int_{|u|}^{4|v|} \lesssim \int_0^{4|v|} \frac{du_3}{|u_3|^2 + |\xi(y)||v|^2} \lesssim 1, \]
\[ \int_{|u|}^{4|v|} \lesssim e^{-\frac{\beta}{8} |v|^2} \int_0^{4|v|} \frac{du_3}{|u_3|^2 + |\xi(y)||v|^2} \lesssim 1, \]
\[ \int_{|u|}^{4|v|} \lesssim \int_{|u_3|}^{4|v|} \int_{|u|}^{4|v|} \frac{du_3}{|u_3|^2 + |\xi(y)||v|^2} \lesssim 1. \]

Here we used \[ \frac{1}{|u|^\rho} \in L^1_u. \]

Proof of (4.11) By Hölder inequality with \( \frac{1}{3} + \frac{1}{3/2} = 1 \) and split \( |v - u|^2 = |v - u|^{4/3+\varepsilon} |v - u|^{2/3-\varepsilon} \) we have
\[ \int_{t-\tau(x,v)}^t ds e^{-C(v)(t-s)} \times \int \frac{e^{-\frac{\beta}{8} |v-u|^2}}{|v-u|^2} \min \left\{ \frac{\alpha(x-(t-s)v, u), \alpha(y-(t-s)v, u)}{|v-u|^2} \right\}^{\beta} du \]
\[ \lesssim \int_{t-\tau(x,v)}^t e^{-C(v)(t-s)} ds \left( \int \frac{e^{-\frac{\beta}{8} |v-u|^2}}{|v-u|^{2+\varepsilon}} du \right)^{2/3} \]
\[
\times \left( \int_{\mathbb{R}^3} \frac{e^{-\vartheta |v-u|^2}}{|v-u|^{2-\varepsilon}} \min \{\alpha(x - (t-s)v, u), \alpha(y - (t-s)v, u)\}^{3\beta} du \right)^{1/3}
\]
\[
\lesssim \left( \int_{t-t_h(x,v)} e^{-C(v)(t-s)} \times \int_{\mathbb{R}^3} \frac{e^{-\vartheta |v-u|^2}}{|v-u|^{2-\varepsilon}} \min \{\alpha(x - (t-s)v, u), \alpha(y - (t-s)v, u)\}^{3\beta} du \right)^{1/3}
\]
\[
\times \left( \int_{t-t_h(x,v)} e^{-C(v)(t-s)} \right)^{2/3}.
\]

Since \(3\beta < 3\) by (4.7) we have
\[
(4.27) \lesssim \left( \frac{1}{\min \{\alpha(x, v), \alpha(y, v)\}^{3\beta}} \right)^{1/3},
\]
then we finish the proof. \(\square\)

4.2. Proof of Proposition 2

**Step 1. Convert \(\nabla_x\) for \(\nabla_v\) along the trajectory using binary collision**

Expansion of (3.7) with (3.7) for \(K f^{\ell-1}(x - (t-s)v, v)\). First we consider (3.7) with (3.7) for \(K f^{\ell-1}(x - (t-s)v, u)\). Temporarily denote \(y = x - (t-s)v\). Proposition 1 gives a formula of \(w_{\tilde{\vartheta}}(u) \partial \xi_i f^{\ell-1}(y, u)\) by (3.5)–(3.10) with \(h^\ell = (3.3)\) and \((x, v, \ell) \to (y, u, \ell - 1)\). We split the contribution of (3.7) with (3.7) for \(K f^{\ell-2}(y - (s-s^0)u, u)\), which is (4.34), and the rest. The rest is given as

\[
\int_{\max (0, t-t_h)}^{t} ds e^{-v(y)(t-s)} \int_{\mathbb{R}^3} du \ k(v, u) \left\{ \frac{O(1)}{\alpha(y, u)} n_i(x^1) w_{\tilde{\vartheta}}(u) \frac{w_{\tilde{\vartheta}}(v)}{w_{\tilde{\vartheta}}(u)} \ O(1) n_i(x^1) w_{\tilde{\vartheta}}(u) \right\} \frac{M_W(x^1, u)}{\sqrt{\mu(u)}} \| w_{\tilde{\vartheta}} f^{\ell-2} \|_{L^\infty(\partial \Omega)}
\]

\[
+ |u| \| \nabla_x r(x^1) \| (1 + \| w_{\tilde{\vartheta}} f^{\ell-2} \|_{L^\infty(\partial \Omega)})
\]

\[
+ \frac{O(1)}{\alpha(y, u)} e^{-v(y)\xi_i} w_{\tilde{\vartheta}}(u) \partial \xi_i f^{\ell-1}(y - s, u)
\]

\[
+ \int_{\max (0, s-t_h)}^{s} e^{-v(u)(t-s)} w_{\tilde{\vartheta}}(u) \partial \xi_i \Gamma(f^{\ell-2}, f^{\ell-2})(y - (s-s^0)v, v) ds'
\]

\[
+ O(1) e^{-v(u)s} n_i(x^1) |u| w_{\tilde{\vartheta}}(u) M_W(x^1, u) \frac{1}{\alpha(y, u)} \frac{1}{\sqrt{\mu(u)}} \int_{n(x^1), v^1 > 0} du \{n^1 \cdot v^1 \} \sqrt{\mu(v^1)}
\]

\[
\times \left( e^{-v(u)v} \right) \frac{1}{\alpha(x^1, v^1)} (\alpha \nabla_x f^{\ell-2})(x^1 - t^1 v^1, v^1)
\]
\[ + \int_{\max(0, t^1 - \tau^0_k)}^{t_1} e^{-v(t^1 - s^1)} \frac{\partial}{\partial x^i} \frac{h^{\ell - 2}(x^1 - (t^1 - s^1)\nu, v^1, v^1)}{h^{\ell - 2}} ds^1 \right\}, \quad (4.33) \]

where we intentionally have abused the notations as \( x^1 = x^1(y, u), t^1 = s - \tau_k(y, u) \) for the sake of simplicity (see (2.16) and (2.17)). We will estimate (4.28)–(4.33) later together with the other expansions.

Now we focus on the contribution of (3.7) of (3.7) \( s = K f^{\ell - 2}(y - (s - s^0)u, u) \).

We split the time integration in \( s^0 \) as

\[
\int_{\max(0, t - \tau_k)}^{t} ds e^{-v(t - s)} \int_{\mathbb{R}^3} du \, k(v, u) w_{\beta}(v) \\
\times \int_{\max(0, s - \tau_k(y, u))}^{s} ds^0 e^{-v(s - s^0)} \mathbf{1}_{s^0 \leq s - \epsilon} + \mathbf{1}_{s^0 > s - \epsilon} \]

\[
\times \int_{\mathbb{R}^3} du' \, k(u, u') \partial_{x_j} f^{\ell - 2}(x - (t - s)v - (s - s^0)u, u'). \quad (4.34)\]

Note that \( \partial_{x_j} f^{\ell - 2}(x - (t - s)v - (s - s^0)u, u') = \frac{-1}{s - s^0} \partial_{u_i} [f^{\ell - 2}(x - (t - s)v - (s - s^0)u, u')] \). Applying an integration by parts with respect to \( \partial_{u_i} \), we derive an identity of the contribution of \( \{ s^0 \leq s - \epsilon \} \) in (4.34) as

\[
\int_{\max(0, t - \tau_k)}^{t} ds e^{-v(t - s)} w_{\beta}(v) \int_{\mathbb{R}^3} du \\
\times \int_{\max(0, s - \tau_k(y, u))}^{s} ds^0 e^{-v(s - s^0)} \mathbf{1}_{s^0 \leq s - \epsilon} \]

\[
\times \int_{\mathbb{R}^3} du' \, \partial_{u_i} [k(v, u)k(u, u')] f^{\ell - 2}(x - (t - s)v - (s - s^0)u, u') \\
- \int_{\max(0, t - \tau_k)}^{t} ds e^{-v(t - s)} w_{\beta}(v) \int_{\mathbb{R}^3} du \\
\times \int_{\max(0, s - \tau_k(y, u))}^{s} ds^0 \partial_{u_i} v(u) e^{-v(s - s^0)} \mathbf{1}_{s^0 \leq s - \epsilon} \]

\[
\times \int_{\mathbb{R}^3} du' \, k(v, u)k(u, u') f^{\ell - 2}(x - (t - s)v - (s - s^0)u, u') \\
+ \int_{\max(0, t - \tau_k)}^{t} ds e^{-v(t - s)} w_{\beta}(v) \int_{\mathbb{R}^3} du \\
\times \mathbf{1}_{s \geq \tau_k(y, u)} e^{-v(u)\tau_k(y, u)} \mathbf{1}_{\tau_k(y, u) \geq \epsilon} \partial_{u_i} \tau_k(y, u) \]

\[
\times \int_{\mathbb{R}^3} du' \, k(v, u)k(u, u') f^{\ell - 2}(x - (t - s)v - \tau_k(y, u)u, u'). \quad (4.35)\]

From (2.103) and Lemma 2.13, for the first term in (4.35) we have

\[ w_{\beta}(v) \partial_{u_i} [k(v, u)k(u, u')] f^{\ell - 2}(x - (t - s)v - (s - s^0)u, u') \]
Thus the second term is bounded by

$$\lesssim \frac{w_\beta(v)}{w_\beta(u)} \left( \frac{\partial_n k(v, u)k(u, u') + \partial_n k(u, u')k(v, u)}{w(u')} \right) \frac{w_\beta(u')}{w(u')} \| w f^{\ell-2} \|_{\infty}$$

$$\lesssim k_\beta(v, u)k_\beta(u, u') \left( \frac{1}{|v-u|} + \frac{1}{|u-u'|} \right) \| u \| \frac{w_\beta(u')}{w(u')} \| w f^{\ell-2} \|_{\infty}$$

$$\lesssim k_\beta(v, u)k_\beta(u, u') \left( \frac{1}{|v-u|} + \frac{1}{|u-u'|} \right) \| w f^{\ell-2} \|_{\infty}. \quad (4.36)$$

Since \( \frac{k_\beta(v, u)}{|v-u|}, \frac{k_\beta(u, u')}{|u-u'|} \in L^1_u \), the first term of (4.35) is bounded by

$$O(\epsilon^{-1}) \| w f^{\ell-2} \|_{\infty}. \quad (4.37)$$

For the second term in (4.35), similarly to (4.36) we have

$$w_\beta(v)k(v, u)k(u, u') f^{\ell-2}(x - (t-s)v - (s-s^0)u, u')$$

$$\lesssim k_\beta(v, u)k_\beta(u, u') \| w f^{\ell-2} \|_{\infty}.$$  

Thus the second term is bounded by

$$O(\epsilon^{-1}) \| w f^{\ell-2} \|_{\infty}. \quad (4.38)$$

From (2.103) and (2.32), we conclude the third term in (4.35) is bounded by

$$\| f^{\ell-2} \|_{L^\infty(\partial\Omega)} \int_{\max\{0, t-t_b\}}^t ds \ e^{-v(t-s)}$$

$$\times \int_{\mathbb{R}^3} du k_\beta(v, u) \mathbf{1}_{t_b(y, u) \geq \epsilon} \frac{\partial t_b(y, u)}{\partial u_i} \int_{\mathbb{R}^3} du' k_\beta(u, u')$$

$$\lesssim \| w_\beta f^{\ell-2} \|_{L^\infty(\partial\Omega)} \int_{\max\{0, t-t_b\}}^t ds \ e^{-v(t-s)} \int_{\mathbb{R}^3} du \frac{k_\beta(v, u)}{\alpha(y, u)}. \quad (4.39)$$

This term will be estimated later using Lemma 4.1.

On the other hand by Lemma 2.13 the contribution of \( \{ s^0 > s - \epsilon \} \) in (4.34) is controlled by

$$\| w_\beta \alpha \nabla_x f^{\ell-2} \|_{\infty} \int_{\max\{0, t-t_b\}}^t ds \ e^{-v(t-s)} \int_{\mathbb{R}^3} du k_\beta(v, u) \frac{w_\beta(v)}{w_\beta(u)}$$

$$\times \int_{\mathbb{R}^3} 1_{s^0 > s - \epsilon}$$

$$\times e^{-v_0(u)(s-s^0)} \frac{w_\beta(u)}{w_\beta(u') \alpha(y - (s-s^0)u, u')} \ |y - (s-s^0)u, u'| \ d u' d s^0$$

$$\lesssim \| w_\beta \alpha \nabla_x f^{\ell-2} \|_{\infty} \int_{\max\{0, t-t_b\}}^t ds \ e^{-v(t-s)} \int_{\mathbb{R}^3} du k_\beta(v, u)$$

$$\times \int_{\mathbb{R}^3} 1_{s^0 > s - \epsilon}$$

$$\times e^{-v_0(u)(s-s^0)} k_\beta(u, u') \frac{1}{\alpha(y - (s-s^0)u, u')} \ d u' d s^0. \quad (4.40)$$
This term will be estimate later using Lemma 4.1.

Expansion of (3.10)$_{\alpha=Kf_{2-\alpha}}$ and (4.33)$_{h_{2}=Kf_{3-\alpha}}$. We split

\[
(3.10)_{\alpha=Kf_{2-\alpha}} = \int_{t}^{1} \mathbf{1}_{s^{1} \geq r_{1} - \epsilon} \cdots + \int_{t}^{1} \mathbf{1}_{s^{1} \leq r_{1} - \epsilon} \cdots,
\]

(4.41)

\[
(4.33)_{h_{2}=Kf_{3-\alpha}} = \int_{t}^{1} \mathbf{1}_{s^{1} \geq r_{1} - \epsilon} \cdots + \int_{t}^{1} \mathbf{1}_{s^{1} \leq r_{1} - \epsilon} \cdots.
\]

(4.42)

We simply derive an intermediate estimate (see (2.29))

\[
| (4.41)_{2} | + | (4.42)_{2} |
\]

\[
\leq \sum_{i=0,1} \int_{r^{2},t^{1} - \epsilon} e^{-\nu_{0}(v_{1})(t^{1} - s^{1})}
\]

\[
\times \int_{\mathbb{R}^{3}} k_{\alpha}(v_{1}, u') | \nabla_{x}^{1} \eta_{p^{1}} | \frac{\alpha | \nabla_{x} f^{\ell_{2} - i}(x^{1} - (t^{1} - s^{1})v_{1}, u') |}{\alpha (x^{1} - (t^{1} - s^{1})v_{1}, u')} \text{d}u' \text{d}s^{1}
\]

\[
\leq \| \eta \|_{C^{1}} \sum_{i=0,1} \| \alpha | \nabla_{x} f^{\ell_{2} - i} |_{\infty} \sup_{t^{1},x^{1},v^{1}} \int_{t_{1},t^{2}}^{t_{1}} e^{-\nu_{0}(v_{1})(t^{1} - s^{1})}
\]

\[
\times \int_{\mathbb{R}^{3}} k_{\alpha}(v_{1}, u') \frac{1}{\alpha (x^{1} - (t^{1} - s^{1})v_{1}, u')} \text{d}u' \text{d}s^{1}.
\]

(4.43)

This term, together with (4.40), will be estimate later using Lemma 4.1.

Now we consider (4.41)$_{1}$ and (4.42)$_{1}$. Recall (2.29). The key observation is the following interchange of spatial derivatives and velocity derivatives: for $t^{1} \neq s^{1}$ and $i = 0, 1$,

\[
\frac{\partial}{\partial p^{1}_{i,j}} \left[ \int_{\mathbb{R}^{3}} k(v^{1}, u') f^{\ell_{2} - i}(\eta_{p^{1}}(x^{1}_{p^{1}}) - (t^{1} - s^{1})v_{1}, u') \text{d}u' \right]
\]

\[
= \sum_{\ell=1}^{3} \frac{\partial}{\partial x^{1}_{p^{1},j}} \left[ \int_{\mathbb{R}^{3}} k(v^{1}, u') \partial_{x_{\ell}} f^{\ell_{2} - i}(\eta_{p^{1}}(x^{1}_{p^{1}}) - (t^{1} - s^{1})v_{1}, u') \text{d}u' \right]
\]

\[
= -\frac{1}{t^{1} - s^{1}} \sum_{\ell=1}^{3} \frac{\partial}{\partial x^{1}_{p^{1},j}} \left[ \int_{\mathbb{R}^{3}} k(v^{1}, u') \partial_{x_{\ell}} f^{\ell_{2} - i}(\eta_{p^{1}}(x^{1}_{p^{1}}) - (t^{1} - s^{1})v_{1}, u') \text{d}u' \right]
\]

\[
= -\frac{1}{t^{1} - s^{1}} \sum_{\ell=1}^{3} \frac{\partial}{\partial x^{1}_{p^{1},j}} \left[ \int_{\mathbb{R}^{3}} k(v^{1}, u') \partial_{x_{\ell}} f^{\ell_{2} - i}(\eta_{p^{1}}(x^{1}_{p^{1}}) - (t^{1} - s^{1})v_{1}, u') \text{d}u' \right]
\]

\[
= -\frac{1}{t^{1} - s^{1}} \sum_{\ell=1}^{3} \frac{\partial}{\partial x^{1}_{p^{1},j}} \left[ \int_{\mathbb{R}^{3}} k(v^{1}, u') \partial_{x_{\ell}} f^{\ell_{2} - i}(\eta_{p^{1}}(x^{1}_{p^{1}}) - (t^{1} - s^{1})v_{1}, u') \text{d}u' \right]
\]
\[ \int_{n^1 \cdot v^1 > 0} \left[ \text{[the contribution of (4.44) in (4.41)_1 and (4.42)_1]} \right] \times \sqrt{\mu(v^1)} \{n^1 \cdot v^1 \} dv^1 = \int_{n^1 \cdot v^1 > 0} \int_0^{t^1} \left[ \sum_{\ell = 1}^{3} \frac{\partial \eta_{p^1, \ell}(x^1_{p^1})}{\partial x^1_{p^1, j}} \right] \frac{1}{t^1 - s^1} \sum_{\ell = 1}^{3} \frac{\partial \eta_{p^1, \ell}(x^1_{p^1})}{\partial x^1_{p^1, j}} \left[ \int_{\mathbb{R}^3} k(v^1, u^1) f^{\ell-2-i}(\eta_{p^1}(x^1_{p^1}) - (t^1 - s^1)v^1, u^1) du^1 \right] ds^1 \times \sqrt{\mu(v^1)} \{n^1 \cdot v^1 \} dv^1 + \int_{n^1 \cdot v^1 = 0} \int_0^{t^1} \left[ \sum_{\ell = 1}^{3} \frac{\partial \eta_{p^1, \ell}(x^1_{p^1})}{\partial x^1_{p^1, j}} \right] \frac{1}{t^1 - s^1} \sum_{\ell = 1}^{3} \frac{\partial \eta_{p^1, \ell}(x^1_{p^1})}{\partial x^1_{p^1, j}} \left[ \int_{\mathbb{R}^3} k(v^1, u^1) f^{\ell-2-i}(\cdot, u^1) du^1 \right] ds^1 \times \sqrt{\mu(v^1)} \{n^1 \cdot v^1 \} dv^1 + \int_{n^1 \cdot v^1 > 0} \int_{\mathbb{R}^3} k(v^1, u^1) f^{\ell-2-i}(x^2, u^1) du^1 \sqrt{\mu(v^1)} \{n^1 \cdot v^1 \} dv^1 = O(\delta^{-1}) \|\eta\|_{C^1} \|w f^{\ell-2-i}\|_{\infty} + \|f^{\ell-2-i}\|_{L^\infty(\partial \Omega)} \times \left\{ 1 + \int_{n^1 \cdot v^1 > 0} \frac{\|n^1 \cdot v^1\|^2}{\|n^2 \cdot v^1\|^2} \sqrt{\mu(v^1)} dv^1 \right\} = O(\delta^{-1}) \|\eta\|_{C^1} \|w f^{\ell-2-i}\|_{\infty} + \|w f^{\ell-2-i}\|_{L^\infty(\partial \Omega)}. \]
From (2.103), (1.6) and $t^1 - s^1 \geq \epsilon$, we bound the contribution of (4.45) in (4.41) and (4.42) by
\[
\int_{n(x^1) \cdot v^1 > 0} \left[ \text{the contribution of (4.45) of (4.41) and (4.42)} \right] \times \sqrt{\mu(v^1)} \{n^1 \cdot v^1\} \, dv^1 = O(\epsilon^{-1}) \| \eta \|_{C^1} \sup_{i=0,1} \| w f^{\ell - 2 - i} \|_{\infty}. \tag{4.47}
\]

Now we consider contributions of $\Gamma$ in (3.7), (3.10), (4.30), and (4.33). From (2.108)
\[
\begin{align*}
| (3.7) | & = |\Gamma(f^{\ell - 1}, f^{\ell - 1}) | + | (3.10) | + | (4.30) | + | (4.33) | \\
& \leq \sup_i O(\| w f^{\ell - 1 - i} \|_{\infty}) \\
& \times \sup_i \sum_{j=0,1} \left\{ \int_{\max[0,t^j - t^j_b]}^{t^j} e^{-\nu_0(v^j)(t^j - s^j)} w_0(v^j) \right. \\
& \times | \nabla_x f^{\ell - 1 - i} (x^j - (t^j - s^j)v^j, v^j) | \, ds \\
& + \int_{\max[0,t^j - t^j_b]}^{t^j} e^{-\nu_0(v^j)(t^j - s^j)} \\
& \times \int_{\mathbb{R}^3} k(v^j, u) \frac{w_0(v)}{w_0(u)} | \nabla_x f^{\ell - 1 - i} (x^j - (t^j - s^j)v^j, u) | \, ds \\
& \leq \sup_i O(\| w f^{\ell - 1 - i} \|_{\infty}) \sup_i \| w_0 \|_\infty \| \nabla_x f^{\ell - 1 - i} \|_{\infty} \\
& \times \left\{ \sum_{j=0,1} \int_{\max[0,t^j - t^j_b]}^{t^j} e^{-\nu_0(v^j)(t^j - s^j)} \frac{1}{\alpha(x^j - (t^j - s^j)v^j, v^j)} \, ds \right. \\
& + \int_{\mathbb{R}^3} k(v^j, u) \frac{1}{\alpha(x^j - (t^j - s^j)v^j, u)} \, ds \\
& \times \int_{\mathbb{R}^3} k(v^j, u) \frac{1}{\alpha(x^j - (t^j - s^j)v^j, u)} \, ds \\
& \right\}, \tag{4.48}
\end{align*}
\]

where we have used Lemma 2.13 in the last line.

From (1.10)
\[
(4.48) \leq e^{C\Omega} \sum_{j=0,1} \int_0^{t^j} e^{-\nu_0(v^j)(t^j - s^j)} \, ds \times \alpha(x^j, v^j) \leq \frac{C\Omega}{\nu_0(v^j)} \alpha(x^j, v^j). \tag{4.49}
\]

**Step 2. Estimate of (4.49), (4.43), (4.40) and (4.39) using Lemma 4.1.**

Clearly $\| \alpha \nabla_x f^{\ell} \|_{\infty} \leq \| w_0^{\alpha} \nabla_x f^{\ell} \|_{\infty}$, applying Lemma 4.1 we bound
\[
|(4.39)| \leq O(\| f^{\ell - 2 \infty} \|_{L^\infty(\partial\Omega)}) \frac{\alpha(x, v)}{\alpha(x, v)}. \tag{4.50}
\]
\[
\begin{align*}
\frac{|(4.40)| + |(4.43)|}{|w_\theta^i \alpha \nabla_x f^\ell_{-i}|_\infty} & \leq \frac{O(\varepsilon)}{\alpha(x, v)} \sup_{0 \leq i \leq \ell - 1} \|w_\theta^i \alpha \nabla_x f^\ell_{-i}|_\infty, \quad (4.51) \\
\frac{|(4.49)|}{|w_\theta^i \alpha \nabla_x f^\ell_{-i}|_\infty} & \leq \frac{O(\sup_{0 \leq i \leq \ell - 1} \|w f^\ell_{-i}|_\infty)}{\alpha(x, v)} \sup_{0 \leq i \leq \ell - 1} \|w_\theta^i \alpha \nabla_x f^\ell_{-i}|_\infty.
\end{align*}
\]

First we prove (4.1). From (4.6)

\[
\int_{\max\{0, t - \eta\}}^t ds \ e^{-v(t-s)} \int_{\mathbb{R}^3} du \ k(v, u) (4.28)
\]

\[
\leq \frac{1 + \|\theta\|_{C^1}}{\alpha(x, v)} \left(1 + \sup_{i=0, 1} \|w_\theta^i f^\ell_{-i}|_L^\infty(\partial \Omega)\right),
\]

\[
\int_{\max\{0, t - \eta\}}^t ds \ e^{-v(t-s)} \int_{\mathbb{R}^3} du \ k(v, u) \{(4.29) + (4.31)(4.32)\}
\]

\[
\leq \frac{e^{-v_0 t}}{\alpha(x, v)} \|w_\theta^i \alpha \nabla_x f^\ell_{-i}|_\infty.
\]

From (4.48)–(4.50) and (4.51)

\[
\int_{\max\{0, t - \eta\}}^t ds \ e^{-v(t-s)}
\]

\[
\times \int_{\mathbb{R}^3} du \ k(v, u) \{(4.30) + (4.31)(4.33)\}_{h^\ell_{-2} = \Gamma(f^\ell_{-3}, f^\ell_{-3})}
\]

\[
\leq \frac{O(\sup_{0 \leq i \leq \ell - 1} \|w f^\ell_{-i}|_\infty)}{\alpha(x, v)} \sup_{0 \leq i \leq \ell - 1} \|w_\theta^i \alpha \nabla_x f^\ell_{-i}|_\infty.
\]

From (4.42) \leq (4.42)_1 + (4.43) and (4.46), (4.47), (4.51)

\[
\int_{\max\{0, t - \eta\}}^t ds \ e^{-v(t-s)} \int_{\mathbb{R}^3} du \ k(v, u) \{(4.33)\}_{h^\ell_{-2} = Kf^\ell_{-3}(x^1_{l-1}(s_0), y^1)}
\]

\[
\leq \frac{1}{\alpha(x, v)} \times \left\{ O(\varepsilon) \sup_{0 \leq i \leq \ell - 1} \|\alpha \nabla_x f^\ell_{-i}|_\infty \\
+ O(\varepsilon^{-1}) \left(\sup_{0 \leq i \leq \ell - 1} \|w f^\ell_{-i}|_\infty + \sup_{0 \leq i \leq \ell - 1} \|w_\theta^i f^\ell_{-i}|_L^\infty(\partial \Omega)\right)\right\}.
\]

From |(4.34)| \leq |(4.35)| + |(4.40)| and (4.51),

\[
|(4.34)| \leq \frac{1}{\alpha(x, v)} \times \left\{ O(\varepsilon^{-1}) \left[\|w f^\ell_{-2}|_\infty + \sup_{0 \leq i \leq \ell - 1} \|w_\theta^i f^\ell_{-i}|_L^\infty(\partial \Omega)\right] \\
+ O(\varepsilon) \sup_{0 \leq i \leq \ell - 1} \|w_\theta^i \alpha \nabla_x f^\ell_{-i}|_\infty\right\}.
\]

By Lemma 3.2, we have

\[
\sup_{i \geq 0} \|w_\theta^i f^i|_L^\infty(\partial \Omega) \leq \sup_{i \geq 0} \|w f^i|_\infty + \sup_{i \geq 1} \|w f^{i-1}|_\infty. \quad (4.52)
\]
Collecting the terms and applying (4.52), we complete the proof of (4.1).

The proof of (4.2) comes from (4.41), (4.43), (4.46), (4.47) and (4.52). We prove (4.3) from (4.48)–(4.50) and (4.2).

5. $C^1$ Estimate of Tangential Derivative and Continuity of $C^1$ Solution

5.1. $C^1$ estimate of tangential derivative

In this subsection we prove (1.19) in the Main Theorem. Section 3 and 4 already conclude the estimate (1.18), from now on we will drop the super index in $f^\ell$ and only analyze the property of $\nabla_x f$.

Proof of (1.19) For $x \in \Omega$ we use (2.10)–(2.14) to have

$$G(x)\nabla_x f(x, v) = 1_{t \geq t_b} e^{-v(t)} b G(x) \sum_{i=1,2} \nabla_x x^1_{p^1; i} \partial \eta^1_{p^1} f(\eta^1_{p^1}(x^1_{p^1}), v)$$

$$- 1_{t \geq t_b} v(t) \nabla_x h e^{-v(t)} b G(x) f(x, v), v)$$

$$+ 1_{t < t_b} e^{-v(t)} G(x) \nabla_x f(x - tv, v)$$

$$- 1_{t \geq t_b} G(x) \nabla_x t b e^{-v(t)} h(x - t_b v, v)$$

$$+ \int_{t \geq t_b} G(x) e^{-v(t-s)} \nabla_x h(x - (t-s)v, v) dv$$

where $h = K(f) + \Gamma(f, f)$.

We focus on the estimate of (5.5). (5.1)-(5.4) will be estimated with (5.5) together.

Estimate of (5.5) with $h = K(f)$. Let $y = x - (t-s)v$. Rewriting $G(x) = G(x) - G(y) + G(y)$ and applying (2.48) in Lemma 2.6 to $G(x) - G(y)$ we have

$$\left|\left| G(x) - G(y) \right|\right|_{t = t_b} \leq \left|\left| G(x) f(y, u) \nabla_x f(y, u) du \right|\right|_{t = t_b}$$

Then applying (4.5) in Lemma 4.1 with $y = x - (t-s)v$ and (2.122) we obtain

$$\left|\left| G(x) f(y, u) \nabla_x f(y, u) du \right|\right|_{t = t_b} \leq \left|\left| w_{\bar{\alpha}} \nabla_x f \right|\right|_{t = t_b}$$

We focus on (5.6). We further expand $G(y) \nabla_x f(y, u)$ along $u$:

$$\leq \left|\left| w_{\bar{\alpha}} \nabla_x f \right|\right|_{t = t_b}$$
Thus by (2.51)

First we estimate (5.1) and (5.8). We start from (5.1). From Section 3 we have

In (5.8) we denoted $x_b(x - (t - s)v, u) = \eta_{p^1(u)}(x^1_{p^1(u)})$.

Then we estimate (5.1)–(5.4) together with (5.8)–(5.11).

First we estimate (5.1) and (5.8). We start from (5.1). From Section 3 we have

Then by (2.51)

For (5.8) similarly by (2.51) we apply (2.122) and (2.104) with $c = 1$ to have

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Thus by (2.51)
Then we estimate (5.2) and (5.9). We start from (5.2). By (2.52) we conclude

$$\| w_{\tilde{\theta}/2} f \|_{L^\infty(\partial\Omega)} \lesssim \frac{\| w_{\tilde{\theta}} f \|_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}. \quad (5.16)$$

For (5.9) similarly by (2.52) we apply (2.104) with $c = 1$ to have

$$\| w_{\tilde{\theta}/2} f \|_{L^\infty(\partial\Omega)} \lesssim \frac{\| w_{\tilde{\theta}} f \|_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}.$$  

Then we estimate (5.3) and (5.10). For (5.3) we apply (2.49) to have

$$\| w_{\tilde{\theta}/2} f \|_{L^\infty(\partial\Omega)} \lesssim e^{-t} \left[ \frac{\| w_{\tilde{\theta}/2} |v| \| f(x,v) \|_\infty}{w_{\tilde{\theta}/2}(v)|v|} + \frac{\| w_{\tilde{\theta}} \nabla_x f \|_\infty}{w_{\tilde{\theta}/2}(v)|v|} \right]. \quad (5.18)$$

Similarly for (5.10) applying (2.49) and (2.104) with $c = 1$ we have

$$\| w_{\tilde{\theta}/2} f \|_{L^\infty(\partial\Omega)} \lesssim e^{-t} \left[ \frac{\| w_{\tilde{\theta}/2} |v| \| f(x,v) \|_\infty}{w_{\tilde{\theta}/2}(v)|v|} + \frac{\| w_{\tilde{\theta}} \nabla_x f \|_\infty}{w_{\tilde{\theta}/2}(v)|v|} \right]. \quad (5.19)$$

Then we estimate (5.4) and (5.11). For (5.4), by (2.52) and (3.3) we have

$$\| w_{\tilde{\theta}} f \|_{L^\infty(\partial\Omega)} \lesssim \frac{K(f) + \Gamma(f,f)}{|v|} \lesssim \frac{\| f \|_{L^\infty(\partial\Omega)} + 1}{|v|} \int_{\mathbb{R}^3} |k(v,u)| f(x - t_0 v, u) |du|$$

$$\lesssim \frac{\| f \|_{L^\infty(\partial\Omega)} + 1}{w_{\tilde{\theta}/2}(v)|v|} \int_{\mathbb{R}^3} |k_{\tilde{\theta}}(v,u)| \frac{w_{\tilde{\theta}/2}(v)}{w_{\tilde{\theta}/2}(u)} du$$

$$\lesssim \frac{\| w_{\tilde{\theta}} f \|_{L^\infty(\partial\Omega)} + \| w_{\tilde{\theta}} f \|_{L^2(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}. \quad (5.20)$$

For (5.11) similarly by (2.52) and (2.104) with $c = 1$ we have

$$\| w_{\tilde{\theta}} f \|_{L^\infty(\partial\Omega)} \lesssim \frac{\| w_{\tilde{\theta}} f \|_{L^\infty(\partial\Omega)} + \| w_{\tilde{\theta}} f \|_{L^2(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}$$

$$\times \int_{\max(0,t-t_0)}^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |k(v,u)| \frac{w_{\tilde{\theta}/2}(v)}{w_{\tilde{\theta}/2}(u)} \frac{1}{|u|} du$$

$$\lesssim \frac{\| w_{\tilde{\theta}} f \|_{L^\infty(\partial\Omega)} + \| w_{\tilde{\theta}} f \|_{L^2(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}. \quad (5.21)$$

Last we estimate (5.12), this estimate is the most delicate one. We apply the decomposition (4.34) to $d's$. 
When \( s' > s - \varepsilon \), we apply (2.49) in Lemma 2.6 to have

\[
(5.12) \quad 1_{s' > s - \varepsilon} \quad \leq \quad \frac{\|v\| \|\nabla f\|_\infty + \|w_{\tilde{\theta}} \alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)} \times \int^t_{\max(0,t-\delta b)} \int_{\mathbb{R}^3} \mathbf{k}(v,u) \frac{w_{\tilde{\theta}/2}(v)}{w_{\tilde{\theta}/2}(u)} \, dv \, ds \int^s_{s-\varepsilon} \int_{\mathbb{R}^3} \mathbf{k}(u,u') \frac{w_{\tilde{\theta}/2}(u)}{w_{\tilde{\theta}/2}(u')} \, |u| \, |v| \|w_{\tilde{\theta}/2}(v)||v| \|v\|_{\infty} + \|w_{\tilde{\theta}/2}(v)||\nabla f||_\infty}{w_{\tilde{\theta}/2}(v)|v|},
\]

(5.22)

where we have applied (2.104) twice with \( c = 1 \).

On the other hand when \( s' \leq s - \varepsilon \), we exchange \( \nabla_x \) for \( \nabla_{u'} \):

\[
\nabla_x f (x - (t - s)v - (s - s')u, u') = \frac{-1}{s - s'} \nabla_{u'} [f (x - (t - s)v - (s - s')u, u')].
\]

Then we perform an integration by parts with respect to \(du\) and obtain

\[
(5.12) \quad 1_{s' \leq s - \varepsilon} \quad = \quad \left| \int^t_{\max(0,t-\delta b)} ds \, e^{v(t-s)} \int_{\mathbb{R}^3} du \right| \quad \times \quad \int^s_{\max(0,s-\delta b)} ds' \, e^{-v(u)(s-s')} 1_{s' \leq s - \varepsilon} \quad \times \quad \int_{\mathbb{R}^3} du' G(y) \nabla_{u'} [\mathbf{k}(v,u) \mathbf{k}(u,u')] f (y - (s - s')u, u') \quad - \quad \left| \int^t_{\max(0,t-\delta b)} ds \, e^{v(t-s)} \int_{\mathbb{R}^3} du \right| \quad \times \quad \int^s_{\max(0,s-\delta b)} ds' \, \nabla_{u'} e^{-v(u)(s-s')} 1_{s' \leq s - \varepsilon} \quad \times \quad \int_{\mathbb{R}^3} du' G(y) \mathbf{k}(v,u) \mathbf{k}(u,u') f (y - (s - s')u, u') \quad + \quad \left| \int^t_{\max(0,t-\delta b)} ds \, e^{v(t-s)} \int_{\mathbb{R}^3} du \right| \quad \times \quad \int_{\mathbb{R}^3} du' G(y) \mathbf{k}(v,u) \mathbf{k}(u,u') f (y - \delta b(y,u)u, u') \quad \times \quad \left| \int_{\mathbb{R}^3} du' G(y) \mathbf{k}(v,u) \mathbf{k}(u,u') f (y - \delta b(y,u)u, u') \right|. \quad (5.24)
\]

We bound \(|G(y)| \leq 1\). Then applying (2.103) and (2.106) the first and second terms of (5.24) are bounded by

\[
O(\varepsilon^{-1}) \frac{\|w_{\tilde{\theta}} f\|_\infty}{w_{\tilde{\theta}/2}(v)} \int^t_{\max(0,t-\delta b)} ds \, e^{-v(t-s)} \int_{\mathbb{R}^3} \int^s_{\max(0,s-\delta b(y,u))} ds' \, e^{-v(u)(s-s')}.
\]
Thus applying (2.104) with $w \theta/2(u) \approx O(e^{-t} \|w f\|_\infty)$

\[
\int_{\mathbb{R}^3} \frac{k(v, u) (u)^2}{|v - u|} \frac{w_{\theta/2}(u)}{w_{\theta}(u)} |u| k(u, u') \frac{w_{\theta}(u')}{w_{\theta}(u)} \int_{\max[0, t - h]}^{t} ds e^{-v(t-s)} \int_{\mathbb{R}^3} \int_{\max[0, s-t(h, u)]}^{s} ds' e^{-v(s-s')} \approx \frac{O(e^{-1} \|w f\|_\infty)}{w_{\theta/2}(v) |v|}.
\]

where we have used (2.122), $(u)^2 |u| w_{\theta/2}^{-1}(u) \lesssim w_{\theta/2}^{-1}(u)$ and (2.104) with $c = 1$.

For the third term we apply (2.32) and (2.48) to have

\[
\left| G(y) \frac{\nabla u h(y, u)}{h(y, u)} \right| = \left| G(y)n(x_b(y, u)) \right|
\]

\[
= \left| \frac{[G(y) - G(x_b(y, u))] n(x_b(y, u))}{n(x_b(y, u)) \cdot u} + G(x_b(y, u)) n(x_b(y, u)) \right|
\]

\[
\lesssim \frac{\tilde{\alpha}(y, u)}{n(x_b(y, u)) \cdot u |\alpha|} \lesssim \frac{1}{|\alpha|}.
\]

Thus applying (2.104) with $c = 1$ the third term is bounded by

\[
\frac{\|w_{\theta/2} f\|_{L^\infty(\partial \Omega)}}{w_{\theta/2}(v)} \int_{\max[0, t - h]}^{t} e^{-v(t-s)} ds \int_{\mathbb{R}^3} |u| k(v, u) \frac{w_{\theta/2}(u)}{w_{\theta}(u)} \int_{\mathbb{R}^3} k(u, u') \lesssim \frac{\|w_{\theta} f\|_{L^\infty(\partial \Omega)}}{w_{\theta/2}(v) |v|}.
\]

Therefore, we conclude

\[
(5.12) 1_{s' < s - \varepsilon} \lesssim \frac{O(e^{-1}) \|w f\|_\infty + \|w_{\theta} f\|_{L^\infty(\partial \Omega)}}{w_{\theta/2}(v) |v|}.
\]

We estimate (5.13) together with (5.5) $1_{h=\Gamma}$. We apply (2.109), (2.49) and (4.11) with $c = 1$ to have

\[
\| |(5.5) 1_{h=\Gamma}| \|
\lesssim \int_{\max[t - h]}^{t} e^{-v(t-s)} \|w f\| |w_{\theta} f|\|_\infty
\]

\[
\times \left| G(x) \nabla x f(x - (t - s)v) + \int_{\mathbb{R}^3} k_\theta(v, u) |G(x) \nabla x f(x - (t - s)v, u)| \right|
\]

\[
\lesssim \frac{\|w f\|_\infty [\|w_{\theta/2} |v| \nabla f|\|_\infty + \|w_{\theta} \alpha \nabla x f|\|_\infty]}{w_{\theta/2}(v)}
\]

\[
\times \left[ \frac{1}{v} + \int_{\max[0, t - h]}^{t} e^{-v(t-s)} \int_{\mathbb{R}^3} k_\theta(v, u) \frac{w_{\theta/2}(v)}{w_{\theta}(u)} |u| du ds \right]
\]

\[
\lesssim \|w f\|_\infty \frac{\|w_{\theta/2} |v| \nabla f|\|_\infty}{|v|} + \frac{\|w f\|_\infty \|w_{\theta} \alpha \nabla x f|\|_\infty}{|v|}.
\]
\[
\|w_f\|_\infty \lesssim \frac{\|w_{\tilde{\phi}/2} v|\nabla f\|_\infty + \|w_{\tilde{\phi}} \alpha \nabla_x f\|_\infty}{w_{\tilde{\phi}/2}(v)|v|}.
\] (5.27)

Then, similarly, we have

\[
(5.13)
\begin{align*}
\lesssim & \int_\mathbb{R}^3 \kappa_q(v, u) \frac{\|w_f\|_\infty [\|w_{\tilde{\phi}/2} v|\nabla f\|_\infty + \|w_{\tilde{\phi}} \alpha \nabla_x f\|_\infty]}{w_{\tilde{\phi}/2}(v)|v|} \\
\times & \int_\mathbb{R}^3 \kappa_q(v, u) \frac{w_{\tilde{\phi}/2}(v)}{w_{\tilde{\phi}/2}(u)|u|} \frac{1}{w_{\tilde{\phi}/2}(v)|v|} \\
\lesssim & \int_\mathbb{R}^3 \kappa_q(v, u) \frac{\|w_f\|_\infty [\|w_{\tilde{\phi}/2} v|\nabla f\|_\infty + \|w_{\tilde{\phi}} \alpha \nabla_x f\|_\infty]}{w_{\tilde{\phi}/2}(v)|v|}.
\end{align*}
\] (5.28)

Now we estimate \(\|w_{\tilde{\phi}} f\|_{L^\infty(\partial \Omega)}\). Similar to Lemma 3.2, we let \(h = K(f) + \Gamma(f, f)\), following the same proof as to that of Lemma 3.2 and Existence Theorem, we have

\[
\|w_{\tilde{\phi}} f\|_{L^\infty(\partial \Omega)} \lesssim \|w_f\|_\infty = o(1).
\] (5.29)

With the above estimate, we combine (5.15), (5.17), (5.19), (5.21), (5.23), (5.26), (5.27) and (5.28), then we conclude

\[
(5.5) \lesssim \left[ o(1) + e^{-t} \left[ \frac{\|w_f\|_\infty}{w_{\tilde{\phi}/2}(v)|v|} + \frac{\|w_{\tilde{\phi}/2} v|\nabla f\|_\infty}{w_{\tilde{\phi}/2}(v)|v|} \right] + \frac{\|w_{\tilde{\phi}} \alpha \nabla_x f\|_\infty}{w_{\tilde{\phi}/2}(v)|v|} \right].
\] (5.30)

Combining (5.18), (5.14), (5.30) and (5.29) we conclude

\[
|\nabla f| \lesssim O(\epsilon^{-1}) \left[ \frac{\|w_f\|_\infty}{w_{\tilde{\phi}/2}(v)|v|} + \frac{\|w_{\tilde{\phi}/2} v|\nabla f\|_\infty}{w_{\tilde{\phi}/2}(v)|v|} \right] + \left[ \frac{\|w_{\tilde{\phi}/2} v|\nabla f\|_\infty}{w_{\tilde{\phi}/2}(v)|v|} \right] + o(1).
\] (5.31)

Then from \(t \gg 1\) and \(\|w_f\|_\infty \ll 1\) in the Existence Theorem we conclude (1.19).

\[\square\]

### 5.2. Continuity of \(C^1\) solution

In this section we prove the continuity of \(\nabla_x f\). The continuity of \(G(x)\nabla_x f\) will follow directly from the continuity of \(\nabla_x f\). We only need to prove \(\nabla_x f\) is continuous at \(t = t_b(x, v)\). We consider (5.1) - (5.5) without \(G(x)\). When \(t = t_b(x, v)\), (5.3) reads

\[e^{-v_b} \nabla_x f(x - t_b v, v).\]
At the boundary $x - r_b v = x_b(x, v) = \eta_{p^1}(x_{p^1})$, we use the notation (2.28),(2.29) and decompose the spatial derivative as

$$
v \cdot \nabla_x f = \sum_{i=1}^{2} v^1_{p^1,i} \frac{\partial x^1_{p^1,i}}{\sqrt{g^1_{p^1,ii}(x_{p^1})}} + v^1_{p^1,3} \frac{\partial x^1_{p^1,3}}{\sqrt{g^1_{p^1,33}(x_{p^1})}}.
$$

Then from the equation (2.3), we derive

$$\frac{\partial x^1_{p^1,3}}{\sqrt{g^1_{p^1,33}(x_{p^1})}} = - \sum_{i=1}^{2} v^1_{p^1,i} \frac{\partial x^1_{p^1,i}}{\sqrt{g^1_{p^1,ii}(x_{p^1})}} - \frac{v(v)f}{v^1_{p^1,3}} + \frac{K(f) + \Gamma(f, f)}{v^1_{p^1,3}}\quad(5.32)_1$$

Plugging these terms back to the spatial derivative $\nabla_x f$, we derive that

$$\nabla_x f = \nabla_x f T T^t$$

$$= \left(\frac{\partial x^1_{p^1,1}}{\sqrt{g^1_{p^1,11}(x_{p^1})}}, \frac{\partial x^1_{p^1,2}}{\sqrt{g^1_{p^1,22}(x_{p^1})}}\right) \left(\begin{array}{c} \partial_1 \eta_{p^1}(x_{p^1}) \\ \partial_2 \eta_{p^1}(x_{p^1}) \\ \partial_3 \eta_{p^1}(x_{p^1}) \\ \partial_1 \eta_{p^1}(x_{p^1}) \\ \partial_2 \eta_{p^1}(x_{p^1}) \\ \partial_3 \eta_{p^1}(x_{p^1}) \\ \partial_1 \eta_{p^1}(x_{p^1}) \\ \partial_2 \eta_{p^1}(x_{p^1}) \\ \partial_3 \eta_{p^1}(x_{p^1}) \end{array}\right)\right)\cdot \left(\begin{array}{c} \partial_1 \eta_{p^1}(x_{p^1}) \\ \partial_2 \eta_{p^1}(x_{p^1}) \\ \partial_3 \eta_{p^1}(x_{p^1}) \end{array}\right) = \left(\begin{array}{c} \sum_{i=1}^{2} \frac{\partial x^1_{p^1,i}}{\sqrt{g^1_{p^1,ii}(x_{p^1})}} \partial_1 \eta_{p^1}(x_{p^1}) \\ \sum_{i=1}^{2} \frac{\partial x^1_{p^1,i}}{\sqrt{g^1_{p^1,ii}(x_{p^1})}} \partial_2 \eta_{p^1}(x_{p^1}) \\ \sum_{i=1}^{2} \frac{\partial x^1_{p^1,i}}{\sqrt{g^1_{p^1,ii}(x_{p^1})}} \partial_3 \eta_{p^1}(x_{p^1}) \end{array}\right)\cdot e_1 + \left(\begin{array}{c} -\frac{v^1_{p^1,3}}{\sqrt{g^1_{p^1,33}(x_{p^1})}} \partial_1 \eta_{p^1}(x_{p^1}) \\ \frac{v^1_{p^1,3}}{\sqrt{g^1_{p^1,33}(x_{p^1})}} \partial_2 \eta_{p^1}(x_{p^1}) \\ \frac{v^1_{p^1,3}}{\sqrt{g^1_{p^1,33}(x_{p^1})}} \partial_3 \eta_{p^1}(x_{p^1}) \end{array}\right)\cdot e_2$$

which is exactly the same as $\sum_{i=1,2} \nabla_x x^1_{p^1,i} \partial_1 x^1_{p^1,i} f(\eta_{p^1}(x^1_{p^1}), v)$ in (5.1) by applying (2.33).

The contribution of $(5.32)_2$ and $(5.32)_3$ in (5.33) are

$$\left((5.32)_2 + (5.32)_3\right) \cdot \frac{\partial_3 \eta_{p^1}(x^1_{p^1})}{\sqrt{g^1_{p^1,33}(x^1_{p^1})}}$$
which is exactly the same as \(-v(\nu)\nabla_x f\) in (5.3) and \(\nabla_x h\) in (5.4) by applying (2.32).

Thus \(\nabla_x f\) is continuous at \(t = t_b\).

\[6. C^1_v\text{ Estimate}\]

In this section we prove the \(C^1_v\) estimate, which is (1.20) in Main Theorem.  

\textbf{Proof of (1.20)} We take the \(v\) derivative to (2.9) and have

\[
\begin{align*}
\nabla_v f(x, v) &= 1_{t \geq t_b} e^{-v t_b} \nabla_v [f(x_b, v)] \\
&\quad - 1_{t \geq t_b} v \nabla_v t_b(x, v) e^{-v t_b} f(x_b, v) \\
&\quad - 1_{t \geq t_b} \nabla_v v(x_b) e^{-v t_b} f(x_b, v) \\
&\quad + 1_{t \leq t_b} e^{-v t} \nabla_v [f(x - t v, v)] \\
&\quad - 1_{t \leq t_b} \nabla_v v(x) e^{-v t} f(x - t v, v) \\
&\quad - \int_{\max(0, t - t_b)}^t \nabla_v v(v) e^{-v(t-r)} (t-s) h(x - (t-s) v, v) ds \\
&\quad + \int_{\max(0, t - t_b)}^t e^{-v(t-r)} \nabla_v [h(x - (t-s) v, v)] ds \\
&\quad - 1_{t \geq t_b} \nabla_v t_b(x, v) e^{-v t_b} h(x - t_b v, v).
\end{align*}
\]

Note that from Section 5.2, we have \(\|w f\|_{L^\infty(\partial \Omega)} \leq \|w f\|_{\infty}\), then in the following estimate we will not specify the norm on \(\partial \Omega\).

First we estimate (6.1) and (6.7), which are the most delicate.

For (6.1), we apply the boundary condition (2.25) to obtain

\[
|6.1| \lesssim \nabla_v \left[ \frac{M_W(x_b, v)}{\sqrt{\mu(v)}} \int_{\nu_3 > 0} f(x_b, T_{x_p}^I v^1) \sqrt{\mu(v)} v_3^1 dv^1 + r(x_b, v) \right]
\]

\[
\lesssim \|w f\|_{\infty} \|T_W - T_0\|_{C^1} + \|v\|_{L^2} \nabla_v [r(x_b, v)]_{L^\infty} \|v\|^2
\]

\[
+ \frac{|v|^2 M_W(x_b, v)}{\sqrt{\mu(v)}|v|^2} \left| \int_{\nu_3 > 0} \nabla_v [f(x_b, T_{x_p}^I v^1)] \sqrt{\mu(v)} v_3^1 dv^1 \right|. \quad (6.10)
\]

\textbf{Main Theorem}
In the second line we used the same computation as (2.100) to have

\[
\left| \nabla_v \frac{M_W(x_b, v)}{\sqrt{\mu(v)}} \right| \lesssim T_W e^{-\frac{2T_W|v|^2}{\sqrt{\mu(v)}}} \lesssim \frac{1}{|v|^2}.
\]

Applying (2.95) have

\[ (6.9) \lesssim \frac{\|w f\|_\infty + 1}{|v|^2}. \tag{6.11} \]

Using chain rule we have

\[
(6.10)_* = \nabla_v x_b \nabla_x f(x_b, T_{x_p} f_v^1) + \nabla_v T_{x_p} T_{x_p} f_v^1 \nabla_v f(x_b, T_{x_p} f_v^1).
\]

Then we estimate the contribution of both terms above in (6.10). Applying (2.40), the contribution of (6.10)_1 is bounded by

\[
\frac{1}{|v|^2} \int_{n(x_b) - v^1 > 0} \frac{\alpha(x_b, v^1)}{|v^1|} |\nabla_x f(x_b, v^1)| \mu^{1/4}(v^1) dv^1 \leq \frac{\|\alpha \nabla_x f\|_\infty \mu^{1/4}(v^1)}{|v|^2}, \tag{6.12}
\]

where we have used (1.8) to have \(n(x_b) - v^1 |\sqrt{\mu(v^1)} \lesssim \alpha(x_b, v^1) \mu^{1/4}(v^1)\).

For the contribution of (6.10)_2, we exchange the \(v\) derivative into \(v^1\) derivative:

\[
\nabla_v f(x_b, T_{x_p} f_v^1) = \nabla_v^1 [f(x_b, T_{x_p} f_v^1)] T_{x_p}.
\]

Then the contribution of (6.10)_2 in (6.12) can be written as

\[
\frac{|O(1)|}{|v|} \int_{v^1_3 > 0} \nabla_v T_{x_p} T_{x_p} f_v^1 \nabla_v^1 [f(x_b, T_{x_p} f_v^1)] T_{x_p} \nabla_v^1 \frac{1}{\sqrt{\mu(v^1)}} dv^1 \\
= \frac{|O(1)|}{|v|} \int_{v^1_3 > 0} \nabla_v T_{x_p} f(x_b, T_{x_p} f_v^1) T_{x_p} \nabla_v^1 [v^1_3 \sqrt{\mu(v^1)}] dv^1 \\
\lesssim \frac{|O(1)|}{|v|} \int_{v^1_3 > 0} \frac{1}{|v|} \|\eta\|_{C^2} \|w f\|_\infty \mu^{1/4}(v^1) dv^1 \lesssim \frac{\|w f\|_\infty}{|v|^2}, \tag{6.13}
\]

where we applied an integration by parts to \(dv^1\) in the second line, and used (2.41) in the third line.

Combining (6.12), (6.13) and (6.11) we conclude

\[
|\text{(6.1)}| \lesssim \frac{\|w f\|_\infty + \|\alpha \nabla_x f\|_\infty}{|v|^2}. \tag{6.14}
\]

Then we estimate (6.7). For \(h = K(f)\), we compute

\[
\int_{\max\{0, t - \tau\}}^{t'} e^{-v(t-s)} \int_{\mathbb{R}^3} |\nabla_v [k(v, u) f(x - (t-s) v, u)]| ds
\]
\begin{align*}
&\int_{\max\{0, t-t_h\}}^t e^{-v(t-s)} \\
&\times \int_{\mathbb{R}^3} \left| \nabla_v k(v, u) f(x - (t-s)v, u) + k(v, u) \nabla_v [f(x - (t-s)v, u)] \right| \\
&\lesssim \int_{[0, t-t_h]} e^{-v(t-s)} \\
&\times \int_{\mathbb{R}^3} \left[ \|w f\|_{\infty} \frac{w^{-1}(u) \langle v \rangle k_\theta(v, u)}{|v-u|} + \|k(v, u)(t-s) \nabla_x f(x - (t-s)v, u)| \right] \\
&\lesssim \int_{[0, t-t_h]} e^{-v(t-s)} \int_{\mathbb{R}^3} \left[ \|w f\|_{\infty} \frac{e^{-|v-u|^2/2}}{|v-u|^2} \frac{1}{|v|^2} \\
&\quad + t_h \|w \bar{\theta} \alpha \nabla_x f\|_{\infty} \frac{w \bar{\theta}(v) k(v, u)}{|w \bar{\theta}(u) \alpha(x - (t-s)v, u)|} \frac{1}{w \bar{\theta}(v)} \right] \\
&\lesssim \frac{\|w f\|_{\infty}}{|v|^2} + \frac{\tilde{\alpha}(x, v)}{w \bar{\theta}(v)} \frac{\|w \bar{\theta} \alpha \nabla_x f\|_{\infty}}{|v|^2} \lesssim \frac{\|w f\|_{\infty}}{|v|^2} + \frac{\|w \bar{\theta} \alpha \nabla_x f\|_{\infty}}{|v|^2}. \tag{6.15}
\end{align*}

In the sixth line we have used
\[ w^{-1}(u)e^{-|v-u|^2}(v) = e^{-|v-u|^2/2}e^{-|v-u|^2/2}e^{-|v|^2}(v) \lesssim e^{-|v-u|^2/2}e^{-C|v|^2(|v|^2)} \lesssim e^{-|v-u|^2/2}. \]

In the last line we have applied \( e^{-|v-u|^2/2} \in L^1_v, (4.5) \) in Lemma 4.1 and (2.39).

For \( h = \Gamma(f, f) \) we apply (2.107) to have
\begin{align*}
\|\mathbf{(6.7)}\|_{h=\Gamma(f, f)} &\lesssim \frac{1}{|v|^2} \int_{\max\{0, t-t_h\}}^t e^{-v(t-s)}[\|w f\|_{\infty}^2 + \|w f\|_{\infty} \|v^2 \nabla_v f\|_{\infty}] \\
&\lesssim \frac{\|w f\|_{\infty}^2 + \|w f\|_{\infty} \|v^2 \nabla_v f\|_{\infty}}{|v|^2}. \tag{6.16}
\end{align*}

Then we estimate all the other terms, which follow from more direct computation. For (6.2) we apply (2.40) and (2.106); for (6.3) we apply (2.106); for (6.4) we apply (2.40) and \( t \leq t_h \leq \tilde{\alpha}(x, v)/|v| \); for (6.5) we apply (2.106); for (6.6) we apply (2.106) and (2.124); for (6.8) we apply (2.40) and (2.105), then we obtain the following bound:
\begin{align*}
\|\mathbf{(6.2)}\| &\lesssim \frac{\|w f\|_{\infty}}{|v|^2} \lesssim \frac{\|w f\|_{\infty}}{|v|^2}, \tag{6.17} \\
\|\mathbf{(6.3)}\| &\lesssim \frac{\|v^2 f\|_{\infty}}{|v|^2} \lesssim \frac{\|w f\|_{\infty}}{|v|^2}, \tag{6.18} \\
\|\mathbf{(6.4)}\| &\lesssim e^{-t \nabla_x f(x - tv, v)} + e^{-t \nabla_v f(x - tv, v)} \lesssim e^{-t \frac{\|\alpha \nabla_x f\|_{\infty}}{|v|^2}} + e^{-t \frac{\|v^2 \nabla_v f\|_{\infty}}{|v|^2}}, \tag{6.19}
\end{align*}
\[ |(6.5)| \lesssim t e^{-t} \frac{\|v^2 f\|_\infty}{|v|^2} \lesssim \frac{\|w_f\|_\infty}{|v|^2}, \]  
\[ |(6.6)| \lesssim \frac{\|v^2 h\|_\infty}{|v|^2} \int_{\max\{0,t-t_b\}}^{t} (t-s)e^{-v(t-s)} \, ds \lesssim \frac{\|w_\theta h\|_\infty}{|v|^2}, \]  
\[ |(6.8)| \lesssim \frac{\|w_f\|_\infty}{|v|^2}. \]  

Combining (6.14)–(6.22) we conclude
\[ |\nabla v_f| \lesssim \|TW - T_0\|_{C^1} \frac{(e^{-t} + \|w_f\|_\infty)\|v^2 \nabla v_f\|_\infty + \|w_f\|_\infty + \|\alpha \nabla x f\|_\infty}{|v|^2}. \]

Since \( e^{-t} \ll 1 \) from \( t \gg 1 \) and \( \|w_f\|_\infty \ll 1 \) from **Existence Theorem**, we conclude the proof.

\[ \square \]

### 7. \( C^{1,\beta} \) Solutions in Convex Domains

#### 7.1. Proof of (1.21) and (1.22)

Given the continuity of the \( C^1 \) solution in Section 5.2, in this section we prove the Hölder regularity, which are (1.21) and (1.22) in the **Main Theorem**.

For simplicity we denote
\[ [\nabla_x f(\cdot,v)]_{C^{1,\beta}} \]
\[ := \sup_{x,y \in \Omega} \|w_\theta(v)\|_v^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{2+\beta} \frac{\nabla_x f(x,v) - \nabla_x f(y,v)}{|x-y|^{1+\beta}} \|L^\infty\|, \]  
\[ [\nabla f(\cdot,v)]_{C^{1,\beta}} \]
\[ := \sup_{x,y \in \Omega} \|w_\theta(v)\|_v^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta} \frac{|\nabla f(x,v) - \nabla f(y,v)|}{|x-y|^{1+\beta}} \|L^\infty\|. \]

Here \( \nabla = G(x) \nabla_x \) and \( G \) is defined in (1.14). We note that the weight in (7.1) and (7.2) are different in terms of the power.

To prove the weighted \( C^{1,\beta} \) we will estimate the characteristics starting from two different positions \( x \) and \( y \). In result we define the backward exit time and position that correspond to \( x \) and \( y \).

The first backward exit position and time are denoted using the same notation as
\[ x_b(x,v), \quad x_b(y,v), \quad t_b(x,v), \quad t_b(y,v). \]

For simplicity we denote the second backward exit position and time as
\[ \begin{align*}
x_b^2(x) &= x_b(x_b(x,v), v^1), & x_b^2(y) &= x_b(x_b(y,v), v^1), \\
t_b^2(x) &= t_b(x_b(x,v), v^1), & t_b^2(y) &= t_b(x_b(y,v), v^1).
\end{align*} \]
Similarly to Definition 4, we choose $p^1(x), p^2(x), p^1(y), p^2(y) \in \mathcal{P}$ such that

$$\begin{align*}
  x^i_{p^i(x)} := (x^i_{p^i(x), 1}, x^i_{p^i(x), 2}, 0) \quad &\text{ such that } \eta_{p^i(x)}(x^i_{p^i(x)}) = \begin{cases} x_b(x, v), & i = 1; \\ x_b(x), & i = 2. \end{cases} \\
  x^i_{p^i(y)} := (x^i_{p^i(y), 1}, x^i_{p^i(y), 2}, 0) \quad &\text{ such that } \eta_{p^i(y)}(x^i_{p^i(y)}) = \begin{cases} x_b(y, v), & i = 1; \\ x_b(y), & i = 2. \end{cases}
\end{align*}$$

(7.4) (7.5)

Since $\|wf\|_\infty, \|v|\nabla f\|_\infty, \|v|^2 \nabla_v f\|_\infty, \|w_\beta \alpha \nabla f\|_\infty$ are all bounded from previous sections, for simplicity, throughout this section we assume

$$\max\{\|wf\|_\infty, \|v|\nabla f\|_\infty, \|v|^2 \nabla_v f\|_\infty\} \leq \|\alpha \nabla f\|_\infty \leq \|\alpha \nabla f\|_\infty^2 \leq \|w_\beta \alpha \nabla f\|_\infty^2.$$  

(7.6)

Note that the above assumption implies $\|w_\beta \alpha \nabla f\|_\infty \geq 1$. Then the bound of (7.1) and (7.2) are given by the following proposition.

**Proposition 3.** Suppose $F = \mu + \sqrt{\mu} f$ solves the steady Boltzmann equation (1.1) with boundary condition (1.3), then

$$\begin{align*}
  [\nabla f (\cdot, v)]_{C^{0, \beta}_{x:1+\beta}} \lesssim o(1) [\nabla f (\cdot, v)]_{C^{0, \beta}_{x:1+\beta}} + C_\varepsilon \|TW - T_0\|_{C^2} \|w_\beta \alpha \nabla f\|_\infty^2,
\end{align*}$$

(7.7)

and

$$\begin{align*}
  [\nabla f (\cdot, v)]_{C^{0, \beta}_{x:2+\beta}} \lesssim o(1) [\nabla f (\cdot, v)]_{C^{0, \beta}_{x:2+\beta}} + C_\varepsilon \|TW - T_0\|_{C^2} \|w_\beta \alpha \nabla f\|_\infty^2,
\end{align*}$$

(7.8)

where $C_\varepsilon \gg 1$.

These two estimates together conclude (1.21) and (1.22).

Below we present two lemmas regarding the collision operators. These two lemmas will be used to estimate the difference of the collision operators.

**Lemma 7.1.** For $h(x, v) = Kf(x, v) + \Gamma(f, f)(x, v)$, we have

$$\begin{align*}
  \frac{|h(x, v) - h(y, v)|}{|x - y|^\beta} \lesssim \|w_\beta \alpha \nabla f\|_\infty \quad &\text{ and } \quad \|\nabla h(x, v)| \lesssim \|w_\beta \alpha \nabla f\|_\infty \int_{\mathbb{R}^1} k_\beta(v, u) |\alpha(x, u)| \, du.
\end{align*}$$

(7.9) (7.10)

**Proof.** Since

$$\begin{align*}
  |\Gamma(f, f)(x, v) - \Gamma(f, f)(y, v)|
  &= \Gamma(f(x) - f(y), f(x))(v) + \Gamma(f(y), f(x) - f(y))(v) \\
  \lesssim \|wf\|_\infty (\int_{\mathbb{R}^1} k(v, u) |f(x, u) - f(y, u)| \, du + |f(x, v) - f(y, v)|),
\end{align*}$$

(7.11)
we have
\[
\frac{|h(x, v) - h(y, v)|}{|x - y|^\beta}
\]
\[= \frac{|K[f(x, v) - f(y, v)] + \Gamma(f, f)(x, v) - \Gamma(f, f)(y, v)|}{|x - y|^\beta}
\]
\[\lesssim \int_{\mathbb{R}^3} k(v, u) \frac{f(x, u) - f(y, u)}{|x - y|^\beta} du + \|w_f\|_\infty \int_{\mathbb{R}^3} k(v, u) \frac{f(x, u) - f(y, u)}{|x - y|^\beta} du + \frac{|f(x, v) - f(y, v)|}{|x - y|^\beta}
\]
\[\lesssim \|w_f\|_\infty \frac{\|w_\tilde{\varphi} \alpha \nabla_x f\|_\infty}{w_\tilde{\varphi}(v)} \times \left[ \int_{\mathbb{R}^3} \frac{w_\tilde{\varphi}(u) k_\varphi(v, u)}{w_\tilde{\varphi}(v) \min \{\alpha(x, u), \alpha(y, u)\}^\beta} + \frac{1}{w_\tilde{\varphi}(v) \min \{\alpha(x, v), \alpha(y, v)\}^\beta} \right]
\]
where we have applied (2.64) and (4.10) in Lemma 4.1.

Then we prove (7.10). Clearly from Lemma 2.13,
\[
|\nabla_x Kf(x, v)| = |\int_{\mathbb{R}^3} k(v, u) \nabla_x f(x, u)| \lesssim \frac{\|w_\tilde{\varphi} \alpha \nabla_x f\|_\infty}{w_\tilde{\varphi}(v)} \int_{\mathbb{R}^3} \frac{k(v, u) w_\tilde{\varphi}(v)}{\alpha(x, u) w_\tilde{\varphi}(u)}
\]
For \(\Gamma\), we bound
\[
|\nabla_x \Gamma(f, f)(x, v)| = |\Gamma(\nabla_x f, f) + \Gamma(f, \nabla_x f)|
\]
\[\lesssim \|w_f\|_\infty \left[ |\nabla_x f(x, v)| + \int_{\mathbb{R}^3} |k(v, u) \nabla_x f(x, u)| \right]
\]
\[\lesssim \frac{\|w_\tilde{\varphi} \alpha \nabla_x f\|_\infty}{w_\tilde{\varphi}(v) \alpha(x, v)} + \frac{\|w_\tilde{\varphi} \alpha \nabla_x f\|_\infty}{w_\tilde{\varphi}(v)} \int_{\mathbb{R}^3} \frac{k(v, u) w_\tilde{\varphi}(v)}{\alpha(x, u) w_\tilde{\varphi}(u)} du,
\]
again by Lemma 2.13 we conclude the lemma.

**Lemma 7.2.** Denote \(x' = x - (t - s)v, y' = y - (t - s)v\). For the difference of \(\nabla_x \Gamma\), we have
\[
\int_0^t e^{-v(t-s)} \left| \nabla_x \Gamma(f, f)(x', v) - \nabla_x \Gamma(f, f)(y', v) \right| \frac{\alpha(1)[\nabla_x f(\cdot, v)]^2}{\varphi_{x,2+\beta}} + \|w_\tilde{\varphi} \alpha \nabla_x f\|_\infty^2
\]
\[\lesssim \frac{\|w_\tilde{\varphi} \alpha \nabla_x f\|_\infty^2}{w_\tilde{\varphi}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{2+\beta}}, \quad (7.12)
\]
and
\[ \int_0^t e^{-y(t-s)} |G(x') \nabla_x \Gamma(f, f)(x', v) - G(y') \nabla_x \Gamma(f, f)(y', v)| |x-y|^\beta \]
\[ \lesssim \frac{o(1)[\|f(\cdot, v)\|_{L_x^\infty}^{\alpha, \beta} + \|w_\beta \alpha \nabla_x f\|_\infty^2}{w_\beta^2 |v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}}. \] (7.13)

Proof. We rewrite the difference of \( \nabla_x \Gamma \) as
\[
\nabla_x \Gamma(f, f)(x, v) - \nabla_x \Gamma(f, f)(y, v) = \Gamma(\nabla_x f(x) - \nabla_x f(y), f(x))(v) + \Gamma(f(x) - f(y), \nabla_x f(x))(v) \\
+ \Gamma(\nabla_x f(y), f(x) - f(y))(v) + \Gamma(f(y), \nabla_x f(x) - \nabla_x f(y))(v).
\]
Then by (2.114) we bound
\[
\|\Gamma(\nabla_x f(x) - \nabla_x f(y), f(x))(v)\| \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} k_\varphi(v, u) |\nabla_x f(x, u) - \nabla_x f(y, u)| du,
\]
\[
\|\Gamma(f(y), \nabla_x f(x) - \nabla_x f(y))(v)\| \lesssim \|wf\|_\infty (\nabla_x f(x, v) - \nabla_x f(y, v)) \\
+ \|wf\|_\infty \int_{\mathbb{R}^3} k_\varphi(v, u) |\nabla_x f(x, u) - \nabla_x f(y, u)|.
\]
For \( \Gamma(f(x) - f(y), \nabla_x f(x))(v) \), we bound \( \Gamma_{\text{loss}} \) in (2.110) as
\[
\|\Gamma_{\text{loss}}(f(x) - f(y), \nabla_x f(x))(x)\| \lesssim |\nabla_x f(x, v)| v (|\nabla_x f(x) - f(y)|) \\
\lesssim |\nabla_x f(x, v)| \int_{\mathbb{R}^3} k_\varphi(v, u) |f(x, u) - f(y, u)|.
\]
Then we use Carleman’s representation (see [14]) to write the \( \Gamma_{\text{gain}} \) in (2.110) as
\[
\|\Gamma_{\text{gain}}(f(x) - f(y), \nabla_x f(x))(v)\| \\
\lesssim \int_{\mathbb{R}^3} du \frac{|\nabla_x f(x, u)|}{|v-u|} \int_{(u-v), \omega=0} \frac{|f(x, v + \omega) - f(y, v + \omega)|}{|x-y|^\beta} \\
\lesssim \int_{\mathbb{R}^3} du \kappa(v, u) |\nabla_x f(x, u)| \int_{S^2} \frac{w_\beta^{-1} (v + \omega) \|w_\beta \alpha \nabla_x f\|_\infty}{\min \{\alpha(x, v + \omega), \alpha(y, v + \omega)\}^\beta} \\
\lesssim \frac{\|w_\beta \alpha \nabla_x f\|_\infty^2}{w_\beta(v)} \int_{\mathbb{R}^3} du \frac{w_\beta(u) \kappa(v, u) \min \{\xi(x), \xi(y)\}^{\beta/2}}{w_\beta(u) \alpha(x, u) \min \{\xi(x), \xi(y)\}^{\beta/2}} \\
\times \int_{S^2} d\omega e^{-\tilde{\theta}|v|^2/2} \frac{w_\beta^{-1} (v + \omega)}{|v + \omega|^\beta} \\
\lesssim \frac{\|w_\beta \alpha \nabla_x f\|_\infty^2}{w_\beta^{-3/2}(v)} \int_{\mathbb{R}^3} du \frac{\kappa(v, u) \alpha(x, u) \min \{\xi(x), \xi(y)\}^{\beta/2}}{\min \{\xi(x), \xi(y)\}^{\beta/2}[1 + \log |\xi(y)| + |\log |v||.} \] (7.14)
Here we applied (2.64) in the third line. In the fourth line we have used that for $|\omega| \leq 1$,
\[
w^{-1}_\delta(v + \omega) = w^{-1}_\delta(v + \omega)w^{-1}_\delta(v + \omega) \lesssim w^{-1}_\delta(v + \omega)e^{-\tilde{\theta}|v|^2}e^{\tilde{\theta}v \cdot \omega} \lesssim w^{-1}_\delta(v + \omega)e^{-\tilde{\theta}|v|^2}e^{\tilde{\theta}v \cdot \omega} \lesssim w^{-1}_\delta(v + \omega)e^{-\tilde{\theta}|v|^2/2}.
\]
and $\alpha(x, v) \geq \sqrt{-\overline{\xi}(x)}|v|$. In the sixth line we use Lemma 2.13 and in the last line we use (4.12).

Thus by (2.64) with (7.6) we obtain
\[
\frac{|\nabla_x \Gamma(f, f)(x', v) - \nabla_x \Gamma(f, f)(y', v)|}{|x - y|^\beta} \lesssim \left[ O\left(\|w_\delta f\|_\infty\right) \frac{|\nabla_x f(x', v) - \nabla_x f(y', v)|}{|x - y|^\beta}
+ O\left(\|w_\delta f\|_\infty\right) \int_{\mathbb{R}^3} k(v, u) \frac{|\nabla_x f(x', u) - \nabla_x f(y', u)|}{|x - y|^\beta} \right] \tag{7.15}
+ \left[ |\nabla_x f(x', v)| \int_{\mathbb{R}^3} k(v, u) \frac{|f(x', u) - f(y', u)|}{|x - y|^\beta} du \right.
+ |f(x', v) - f(y', v)| \int_{\mathbb{R}^3} k(v, u) |\nabla_x f(x', u)| du
\right]
+ \frac{\|w_\delta \alpha \nabla_x f_\infty \|_{w_\delta}^{3/2}(v)}{w_\delta^{3/2}(v)} \left[ \frac{|\log |\tilde{\xi}(x')||}{\min \{\tilde{\xi}(x'), \tilde{\xi}(y')\}^{\beta/2}} + \frac{|\log |v||}{\min \{\tilde{\xi}(x'), \tilde{\xi}(y')\}^{\beta/2}} \right]. \tag{7.17}
\]

We bound $w^{-3/2}_\delta(v)|v| \leq w^{-1}_\delta(v)$. By (4.7) and (4.10) in Lemma 4.1 we have
\[
\int_0^t e^{-\nu(v)(t-s)} \tag{7.15}
\lesssim \left[ \frac{\|w_\delta f_\infty \|_{w_\delta} \left[ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right]^{2+\beta}}{\|w_\delta f_\infty \|_{\mathbb{R}^3} \frac{\|w_\delta(f_\infty(\cdot, v))\|_{w_\delta}^{0, \beta}}{x, 2+\beta} \left[ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right]^{2+\beta}} \right]
\times \left[ \frac{\|w_\delta f_\infty \|_{w_\delta} \left[ \frac{\alpha(x, u)}{|u|}, \frac{\alpha(y, u)}{|u|} \right]^{2+\beta}}{\|w_\delta f_\infty \|_{\mathbb{R}^3} \frac{\|w_\delta(f_\infty(\cdot, u))\|_{w_\delta}^{0, \beta}}{x, 2+\beta} \left[ \frac{\alpha(x, u)}{|u|}, \frac{\alpha(y, u)}{|u|} \right]^{2+\beta}} \right] \tag{7.18}
\]
By (2.64),(4.7) and (4.10) in Lemma 4.1 we have
\[
\int_0^t e^{-\nu(v)(t-s)} \tag{7.16}
\]
Then we have
\[
\frac{w_{\tilde{\theta}}^{-2}(v)|v| |w_{\tilde{\theta}}\alpha \nabla_x f|^2}{|v| |\alpha(x, v)|} \leq \int_0^t e^{-v(t-s)} \int_{\mathbb{R}^3} \frac{w_{\tilde{\theta}}(v)k_{\tilde{\theta}}(v, u)}{w_{\tilde{\theta}}(u) \min \{\alpha(x', u), \alpha(y', u)\}^\beta} \, du \, dv
\]
\[
+ \frac{w_{\tilde{\theta}}^{-2}(v)|v| |w_{\tilde{\theta}}\alpha \nabla_x f|^2}{|v| \min \{\alpha(x, v), \alpha(y, v)\}^\beta} \leq \int_0^t e^{-v(t-s)} \int_{\mathbb{R}^3} \frac{w_{\tilde{\theta}}(v)k_{\tilde{\theta}}(v, u)}{w_{\tilde{\theta}}(u) \alpha(x', u)} \, du \, dv
\]
\[
\leq \frac{\|w_{\tilde{\theta}}\alpha \nabla_x f\|_{L_2}^2}{w_{\tilde{\theta}}(v)|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}}
\]
\[
\leq \frac{\|w_{\tilde{\theta}}\alpha \nabla_x f\|_{L_2}^2}{w_{\tilde{\theta}}(v)|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{2+\beta}}.
\] (7.19)

For (7.17), we bound \(|\log(\xi(x'))| \leq \frac{1}{\min(\xi(x), \xi'(y'))}\) and \(w_{\tilde{\theta}}^{-3/2}(v)|v| \leq w_{\tilde{\theta}}^{-1}(v)\). Then we have
\[
\int_0^t e^{-v(t-s)} \, ds \quad (7.17)
\]
\[
\leq \|w_{\tilde{\theta}}\alpha \nabla_x f\|_{L_2}^2 \left[ \int_0^t \frac{w_{\tilde{\theta}}^{-3/2}(v)e^{-v(t-s)}}{\min \{\xi(x'), \xi(y')\}^{\beta/2+\delta}} \, ds + \int_0^t \frac{w_{\tilde{\theta}}^{-3/2}(v)e^{-v(t-s)}}{\log |v||} \, ds \right]
\]
\[
\leq \|w_{\tilde{\theta}}\alpha \nabla_x f\|_{L_2}^2 \left[ \int_0^t \frac{w_{\tilde{\theta}}^{-1}(v)e^{-v(t-s)}}{\min \{\xi(x'), \xi(y')\}^{\beta/2+\delta}} \, ds + \int_0^t \frac{w_{\tilde{\theta}}^{-1}(v)e^{-v(t-s)}}{|v| \min \{\xi(x'), \xi(y')\}^{\beta/2}} \, ds \right]
\]
\[
\leq \frac{\|w_{\tilde{\theta}}\alpha \nabla_x f\|_{L_2}^2}{w_{\tilde{\theta}}(v)|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^\beta}.
\] (7.20)

where we have applied Lemma 4.1 in the last line with small enough \(\delta\) such that \(\beta/2 + \delta \ll 1\).

Using \(\|wf\|_{L_\infty} \ll 1\) from Existence Theorem we conclude (7.12).

Then we prove (7.13). From (7.15)–(7.17) we can rewrite
\[
\left|G(x') \nabla_x \Gamma(f, f)(x', v) - G(y') \nabla_x \Gamma(f, f)(y', v)\right|\]
\[
\leq \left| \frac{G(y') - G(x')}{|x - y|^\beta} \nabla_x \Gamma(f, f)(y', v) + G(x') \nabla_x \Gamma(f, f)(x', v) - \nabla_x \Gamma(f, f)(y', v) \right|\]
\[
\leq \left| \frac{G(y') - G(x')}{|x' - y'|^{\beta}} \right| \|wf\|_{L_\infty} \|w_{\tilde{\theta}}\alpha \nabla_x f\|_{L_\infty}
\]
\[
\times \left[ \frac{w_{\tilde{\theta}}^{-1}(v)|v|}{|v| |\alpha(y, v)|} + \frac{w_{\tilde{\theta}}^{-1}(v)}{w_{\tilde{\theta}}(u)} \int_{\mathbb{R}^3} \frac{w_{\tilde{\theta}}(v)|k(v, u)|}{w_{\tilde{\theta}}(u) |\alpha(y - (t-s)v, u)|} \, du \right]
\] (7.21)
\[
+ |G(x')| \times \left[ (7.15) + (7.16) + (7.17) \right].
\] (7.22)
We bound $w^{-1}_\tilde{\theta}(v)|v| \lesssim w^{-1}_{\tilde{\theta}/2}(v)$. Then we have

$$
\int_0^t e^{-\nu(v)(t-s)} (7.21)
\lesssim \frac{\|w_\tilde{\theta} \alpha \nabla_x f\|_\infty^2}{w_{\tilde{\theta}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}} \\
+ \frac{\|w_\tilde{\theta} \alpha \nabla_x f\|_\infty^2}{w_{\tilde{\theta}/2}(v)} \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |u|^2 \min \left\{ \frac{\alpha(x,u)}{|u|}, \frac{\alpha(y,u)}{|u|} \right\} du
\lesssim \frac{\|w_\tilde{\theta} \alpha \nabla_x f\|_\infty^2}{w_{\tilde{\theta}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}}^{1+\beta}
$$

where we have used (2.57) in the first line, (2.122) in the second line and (4.5) in Lemma 4.1 with $\frac{\alpha(x,v)}{|v|} \lesssim 1$ in the last line.

For (7.22), since $|G| \lesssim 1$, from (7.20) and (7.19) the contribution of (7.16)-(7.17) is already included in (7.13). Then we consider the contribution of (7.15), which reads

$$
\|w_f\|_\infty \frac{|G(x')[\nabla_x f(x',v) - \nabla_x f(y',v)]|}{|x-y|^\beta} \\
+ \|w_f\|_\infty \int_{\mathbb{R}^3} k_{\tilde{\theta}}(v,u) \frac{|G(x')[\nabla_x f(x',u) - \nabla_x f(y',u)]|}{|x-y|^\beta}.
$$

Since $w^{-1}_\tilde{\theta}(v)|v| \lesssim w^{-1}_{\tilde{\theta}/2}(v)$, we rewrite

$$
\left| G(x') \frac{\nabla_x f(x',v) - \nabla_x f(y',v)}{|x-y|^\beta} \right| \\
= \left| \frac{\nabla_x f(x',v) - \nabla_x f(y',v)}{|x-y|^\beta} + \left[ G(x') - G(y') \right] \frac{\nabla_x f(y',v)}{|x-y|^\beta} \right| \\
\lesssim \left| \frac{\nabla_x f(x',v) - \nabla_x f(y',v)}{|x-y|^\beta} \right| + \frac{w^{-1}_\tilde{\theta}(v)|v| \|w_\tilde{\theta} \alpha \nabla_x f\|_\infty}{|v|\alpha(y',v)} \\
\lesssim \frac{w_{\tilde{\theta}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x',v)}{|v|}, \frac{\alpha(y',v)}{|v|} \right\}^{1+\beta}}{w_{\tilde{\theta}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}}
$$

Thus applying (2.122) and (4.7) in Lemma 4.1 with $p = 1 + \beta$ we obtain

$$
\int_0^t e^{-\nu(v)(t-s)} |G(x')(7.15)|
$$
Finally by \( \| w f \|_{\infty} \ll 1 \) from the Existence Theorem we conclude the lemma. \( \square \)

We need one more lemma before the proof. In the following lemma we express the difference quotient in (7.1),(7.2) along the characteristics. This lemma will significantly simplify the proof of Proposition 3.

**Lemma 7.3.** Suppose \( f \) solves inhomogeneous steady transport equation with the diffuse BC. Then

\[
\frac{\partial_x f(x, v) - \partial_x f(y, v)}{|x - y|^\beta} = \frac{O(1) \| w_0 \alpha \xi f \|_{\infty} + o(1)\| \partial_x f(\cdot, v) \|_{c, 0, \beta}}{w_0(v)|v|^{2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{2 + \beta}} + \frac{O(1)}{w_0(v)|v|^{2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}} \frac{|\partial_{x_1} f(x_b(x, v), v) - \partial_{x_1} f(x_b(y, v), v)|}{|x_b(x, v) - x_b(y, v)|^\beta}.
\]  

(7.23)
\[
+ \int_{\max\{0,t-\min[t_b(x,v),t_b(y,v)]\}}^{t} e^{-v(t-s)} \left[ \partial_{x_i} h(x - (t-s)v, v) - \partial_{x_i} h(y - (t-s)v, v) \right] \frac{[x - y]^\beta}{|x - y|}\]
(7.25)

We denote \([G(x)\nabla_x f(x, v)]_i\) as the \(i\)-th component of \(G(x)\nabla_x f\), then

\[
[G(x)\nabla_x f(x, v)]_i - [G(y)\nabla_x f(y, v)]_i = \tilde{\alpha}(x, v) \times (7.25)
\]
(7.26)

\[
+ \frac{O(1)\|w_{\tilde{g}}\alpha \nabla_x f\|_\infty + o(1)[\nabla_{x_i} f(\cdot, v)]_{C^{0,\beta}_{x;1+\beta}} + o(1)\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x;2+\beta}}}{w_{\tilde{g}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}
\]
(7.27)

\[
+ \int_{\max\{0,t-\min[t_b(x,v),t_b(y,v)]\}}^{t} e^{-v(t-s)} \left[ \frac{G(x - (t-s)v)\nabla_x h(x - (t-s)v, v)}{[x - y]^\beta} - \frac{G(y - (t-s)v)\nabla_x h(y - (t-s)v, v)}{[x - y]^\beta} \right] \]
(7.28)

\[
+ \frac{O(1) |x_{b}(x, v) - x_{b}(y, v)|}{[x - y]^\beta} \frac{\left| \partial_{x_i} \partial_{p_i(\cdot),i}^{1} f(x_{b}(x, v), v) - \partial_{x_i} \partial_{p_i(\cdot),i}^{1} f(x_{b}(y, v), v) \right|}{|x_{b}(x, v) - x_{b}(y, v)|^\beta}.
\]
(7.29)

**Proof.** For \(f\) satisfying (2.7), different to (2.10)–(2.14), we use \(\min\{t_b(x, v), t_b(y, v)\}\) to split the cases. For simplicity, denote \(t_m(v) = \min\{t_b(x, v), t_b(y, v)\}\. We express \(\nabla_x f(x, v)\) along the trajectory as:

\[
\partial_{x_i} f(x, v) = \mathbf{1}_{i \geq t_m(v)} \sum_{i=1,2} e^{-\nu h} \partial_{x_i} \partial_{p_i^{\perp}(x),i}^{1} f(x_{b}(x, v), v)
\]
(7.30)

\[
- \mathbf{1}_{i \geq t_m(v)} \nu \nabla_x h(x, v) e^{-\nu h(x,v)} f(x_{b}(x, v), v)
\]
(7.31)

\[
+ \mathbf{1}_{i < t_m(v)} e^{-\nu t} \partial_{x_i} f(x - t v, v)
\]
(7.32)

\[
+ \mathbf{1}_{i \geq t_m(v)} \int_{0}^{t} e^{-\nu(t-s)} \partial_{x_i} [h(x - (t-s)v, v)] ds
\]
(7.33)

\[
+ \mathbf{1}_{i \geq t_m(v)} \int_{t-t_m(v)}^{t} e^{-\nu(t-s)} \partial_{x_i} [h(x - (t-s)v, v)] ds
\]
(7.34)

\[
- \mathbf{1}_{i \geq t_m(v)} \partial_{x_i} \partial_{t_m(v)} e^{-\nu t_m(v)} h(x - t_m(v)v, v) ds.
\]
(7.35)

Here we note that when \(t \geq t_m(v)\), we evaluate \(\partial_{x_i} f(x, v)\) along the characteristics at \(x_{b}(x, v)\) regardless the relationship between \(t_b(x, v)\) and \(t_b(y, v)\).
Taking the difference of \( \partial_{x_i} f(x, v) \) and \( \partial_{y_i} f(y, v) \) using (7.30)–(7.35) we obtain

\[
\frac{\partial_{x_i} f(x, v) - \partial_{x_i} f(y, v)}{|x - y|^{\beta}} = 1_{t \geq t_m(v)} \, O(1) \, \frac{\partial_{x_i} h(x, v) - \partial_{x_i} h(y, v)}{|x - y|^{\beta}} e^{-v b(x, v)} f(x_b(x, v), v) \tag{7.36}
\]

\[
+ 1_{t \geq t_m(v)} \frac{\partial_{x_i} h(y, v) (x_b(x, v) - x_b(y, v))}{|x - y|^{\beta}} e^{-v b f(x_b(x, v), v) - e^{-v b f(x_b(y, v), v)}} \tag{7.37}
\]

\[
+ 1_{t \geq t_m(v)} \frac{\partial_{x_i} h(y, v) [e^{-v b f(x_b(y, v), v)} - e^{-v b f(x_b(y, v), v)}]}{|x - y|^{\beta}} \tag{7.38}
\]

\[
+ 1_{t \geq t_m(v)} \sum_{i=1,2} \frac{\partial_{x_i} x^{1}_{p^{1}(x), i} - \partial_{x_i} x^{1}_{p^{1}(y), i}}{|x - y|^{\beta}} e^{-v b f(x_b(x, v), v)} \partial_{p^{1}(x), i} \tag{7.39}
\]

\[
+ 1_{t \geq t_m(v)} \sum_{i=1,2} \frac{\partial_{x_i} x^{1}_{p^{1}(y), i}}{|x - y|^{\beta}} e^{-v b f(x_b(y, v), v)} \partial_{p^{1}(y), i} f(x_b(x, v), v) \tag{7.40}
\]

\[
+ 1_{t < t_m(v)} \frac{e^{-v [\partial_{x_i} f(x - t v, v) - \partial_{x_i} f(y - t v, v)]}}{|x - t v - (y - t v)|^{\beta}} \tag{7.41}
\]

\[
+ 1_{t \geq t_m(v)} \sum_{i=1,2} e^{-v b f(x_b(x, v), v)} \frac{\partial_{x_i} x^{1}_{p^{1}(x), i}}{|x - y|^{\beta}} \frac{|x_b(x, v) - x_b(y, v)|^{\beta}}{|x - y|^{\beta}} \tag{7.42}
\]

\[
\times \frac{[\partial_{x_i} x^{1}_{p^{1}, i} f(x_b(x, v), v) - \partial_{x_i} x^{1}_{p^{1}, i} f(x_b(y, v), v)]}{|x_b(x, v) - x_b(y, v)|^{\beta}} \tag{7.43}
\]

\[
+ 1_{t \leq t_m(v)} \int_0^t \frac{e^{-v (t - s)} [\partial_{x_i} h(x - (t - s) v, v) - \partial_{x_i} h(y - (t - s) v, v)]}{|x - y|^{\beta}} \tag{7.44}
\]

\[
+ 1_{t \geq t_m(v)} \int_{t - t_m(v)}^t \frac{e^{-v (t - s)} [\partial_{x_i} h(x - (t - s) v, v) - \partial_{x_i} h(y - (t - s) v, v)]}{|x - y|^{\beta}} \tag{7.45}
\]

\[
+ 1_{t \geq t_m(v)} \int_{\min\{t - t_h(x, v), t - t_h(y, v)\}}^{t - t_m(v)} \frac{e^{-v (t - s)} \partial_{x_i} h(x - (t - s) v, v)}{|x - y|^{\beta}} \tag{7.46}
\]

\[
+ 1_{t \geq t_m(v)} \int_{\min\{t - t_h(x, v), t - t_h(y, v)\}}^{t - t_m(v)} \frac{e^{-v (t - s)} \partial_{x_i} h(y - (t - s) v, v)}{|x - y|^{\beta}} \tag{7.47}
\]
\[ + t_{\geq t_m(v)} \partial_x \eta_b(x, v) e^{-v(x,v)} h(x - \eta_b(x,v), v) - h(y - \eta_b(y,v), v) \]

\[ - t_{\geq t_m(v)} \partial_x \eta_b(x, v) e^{-v(x,v)} h(y - \eta_b(y,v), v) \]

\[ - t_{\geq t_m(v)} \partial_x \eta_b(y, v) e^{-v(x,v)} h(y - \eta_b(y,v), v) \]

First we estimate (7.36)–(7.41). For (7.36) we apply (2.59); for (7.37) we apply (2.32), (2.64), (2.54) and (7.6); for (7.38) we apply (2.32) and (2.56), then we obtain

\[ (7.36) = \frac{O(1)w_{\tilde{\theta}}(v) |v| f(x_b(x,v), v)}{w_{\tilde{\theta}}(v)|v|^2} = \frac{O(1)w_{\tilde{\theta}}(v)|v|^{2+\beta}}{w_{\tilde{\theta}}(v)|v|^2} \]

\[ (7.37) = \frac{O(1)\partial_x \eta_b(x,v) f(x_b(x,v), v) - f(x_b(y,v), v) |x_b(x,v) - x_b(y,v)|}{|x_b(x,v) - x_b(y,v)|} \]

\[ = \frac{O(1)\|w_{\tilde{\theta}}(v)\|_{\infty}}{\|w_{\tilde{\theta}}(v)\|_{\infty}} \frac{1}{\min \{\alpha(x,v), \alpha(y,v)\}} \frac{\|\alpha \nabla f\|_{\infty}}{\|\alpha \nabla f\|_{\infty}} \]

\[ (7.38) = \frac{O(1)\|w_{\tilde{\theta}}(v)\|_{\infty}}{w_{\tilde{\theta}}(v)\|\alpha \nabla f\|_{\infty}} \frac{1}{\|\alpha \nabla f\|_{\infty}} \frac{\|\alpha \nabla f\|_{\infty}}{\|\alpha \nabla f\|_{\infty}} \]

\[ = \frac{O(1)\|w_{\tilde{\theta}}(v)\|_{\infty}}{w_{\tilde{\theta}}(v)|v|^2} \frac{\|\alpha \nabla f\|_{\infty}}{\|\alpha \nabla f\|_{\infty}} \]

From Section 4,

\[ \partial_{x_1 p_{l_1}(x,v)} f(\eta_{p_l}(x_{p_l}(x), v)) = (3.19) \]

\[ + (3.20) = O(1) \frac{M_{W}(\eta_{p_l}(x_{p_l}(x), v))}{\sqrt{\mu(v)}} \frac{\|\alpha \nabla f\|_{\infty}}{\|\alpha \nabla f\|_{\infty}} \]

thus with \( \tilde{\theta} \ll 1 \),

\[ \max \{ |v| \partial_{x_1 p_{l_1}(x,v)} f(\eta_{p_l}(x_{p_l}(x), v)), |v|^2 \partial_{x_1 p_{l_1}(x,v)} f(\eta_{p_l}(x_{p_l}(x), v)) \} \]

\[ = O(1) \frac{\|\alpha \nabla f\|_{\infty}}{w_{\tilde{\theta}}(v)}. \]
Then for (7.39) we apply (2.58); for (7.40) we apply (2.32) and (2.56), then we obtain

\[
(7.39) = \frac{O(1)|v|^2 \partial_{\alpha} f(x, v)}{|v|^2} \frac{O(1)}{w_\beta(v)|v|^2} = \frac{O(1)}{w_\beta(v)|v|^2} \left\| \alpha \nabla_x f \right\|_\infty^{2+\beta},
\]

\[
(7.40) = \frac{O(1)|v|^2 \partial_{\alpha} f(x, v)}{|v|^2} \frac{O(1)}{w_\beta(v)|v|^2} = \frac{O(1)}{w_\beta(v)|v|^2} \left\| \alpha \nabla_x f \right\|_\infty^{1+\beta},
\]

\[
(7.41) = O(1) \frac{e^{-vt}}{w_\beta(v)|v|^2} \left\| \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\|_2^{2+\beta} \frac{\nabla_x f(x, v)}{e^{0, \beta}}.
\]

Therefore, from (7.6) and \(t \gg 1\), we conclude

\[
(7.36) + \cdots + (7.41) = (7.23).
\]

For (7.42), from (2.33), such contribution is included in (7.24).
The contribution of (7.43) and (7.44) are already included in (7.25).
Then we estimate (7.45)–(7.48). We apply (7.10) to (7.45) to have

\[
(7.45) = O(1) \frac{w_\beta \alpha \nabla_x f}{w_\beta(v)} \int_{\max(t-h(x, v), t-h(y, v))}^{\min(t-h(x, v), t-h(y, v))} \int_{\mathbb{R}^3} \frac{e^{-v(t-s)}\kappa_\beta(v, u)}{\alpha(x - (t-s) u, v)} |x - y|^\beta
\]

\[
= \frac{O(1)}{w_\beta(v) \alpha \nabla_x f} \int_{\max(t-h(x, v), t-h(y, v))}^{\min(t-h(x, v), t-h(y, v))} \int_{\mathbb{R}^3} \frac{e^{-v(t-s)}\kappa_\beta(v, u)}{\alpha(x - (t-s) u, v)} |x - y|^\beta
\]

\[
= \frac{O(1)}{w_\beta(v) \alpha \nabla_x f} \left\| \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\|_2^{1+\beta},
\]

\[
(7.57)
\]

\[
(7.58)
\]

where we have applied (4.4) of Lemma 4.1 in second line, (2.56) in the third line.

Then for (7.46) we apply (2.32),(2.54) and (7.9); for (7.47) we apply (2.59) and (2.105); for (7.48) we apply (2.56) and (2.105), then we obtain

\[
(7.46) = O(1) \frac{1}{\alpha(x, v)} \frac{h(x, v) - h(y, v)}{|x - y|^\beta}
\]

\[
O(1) \frac{w_\beta \alpha \nabla_x f}{w_\beta(v) \min \{\alpha(x, v), \alpha(y, v)\}^{1/\beta}}
\]

\[
(7.59)
\]

where we have applied (4.4) of Lemma 4.1 in second line, (2.56) in the third line.
we apply (2.62) and (2.47); for (2.52), (2.54), (2.64) and (7.6); for \( \nabla G \) presentation to express the vector consists of the element (7.36):

\[
(7.47) = O(1) \frac{\|w_{\beta}v\|}{\|w_{\beta}(v)\|} \frac{\|h\|}{\|w_{\beta}(v)\|} \frac{\nabla_x h(x, v) - \nabla_x h(y, v)}{|x - y|^\beta},
\]

\[
(7.48) = O(1) \frac{\|w_{\beta}h\|}{\|w_{\beta}(v)\|} \frac{\|e^{-v\theta(x, v)} - e^{-v\theta(y, v)}\|}{\|w_{\beta}(v)\|} \frac{|x - y|^\beta}{\|w_{\beta}(v)\| |v|^2 \min \{\alpha(x, v), \alpha(y, v)\}}.
\]

Therefore, by (7.6), collecting (7.58), (7.59), (7.60), (7.62) we conclude

\[
(7.45) + \cdots + (7.48) = (7.23).
\]

Then we prove the estimate (7.26)-(7.29).

We rewrite

\[
G(x)\nabla_x f(x, v) - G(y)\nabla_x f(y, v) = \frac{G(y) - G(x)}{|x - y|^\beta} \nabla_x f(x, v) + G(y) \frac{\nabla_x f(x, v) - \nabla_x f(y, v)}{|x - y|^\beta}.
\]

By (2.57) we conclude that

\[
(7.63)_i = O(1) \frac{\|w_{\beta}v\|}{\|w_{\beta}(v)\|} \frac{\|\nabla_x f\|}{|x - y|^\beta}.
\]

Then we consider (7.64). Note that (7.36)-(7.48) represent the \( i \)-th component of \( \nabla_x f(x, v) - \nabla_x f(y, v) \), for convenience we define a notation that represents the vector consists of the element (7.36):

\[
[\{7.36\}] = [(7.36)_{i=1}, (7.36)_{i=2}, (7.36)_{i=3}].
\]

Similarly we can define the same notation for (7.37)-(7.48). We can use this representation to express \( G\frac{\nabla_x f(x, v) - \nabla_x f(y, v)}{|x - y|^\beta} \).

Then for \( G\frac{\{7.36\}}{\{7.36\}} \) we apply (2.63); for \( G\frac{\{7.37\}}{\{7.37\}} \) we apply (2.52), (2.54), (2.64) and (7.6); for \( G(y)[\{7.38\}] \) we apply \|G\| = O(1); for \( G(y)[\{7.39\}] \) we apply (2.62) and (2.47); for \( G(y)[\{7.40\}] \) we apply \|G\| = O(1); for \( G(y)[\{7.42\}] \) we apply (2.51), then we obtain

\[
[\{G(y)[7.36]\}]_i = O(1) \frac{\|w_{\beta}v\|}{\|w_{\beta}(v)\|} \frac{\|f\|}{|v|^2 \min \{\alpha(x, v), \alpha(y, v)\}} \frac{1+\beta}{|v|^\beta},
\]

\[
(7.65) = O(1) \frac{\|w_{\beta}v\|}{\|w_{\beta}(v)\|} \frac{\|\nabla_x f\|}{|x - y|^\beta}.
\]
\[ [G(y)(7.37)]_i = O(1) \frac{\| w_{\tilde{y}} \alpha \nabla_x f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}, \]  

(7.67)

\[ [G(y)(7.38)]_i = O(1)(7.51) = O(1) \frac{\| w f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}, \]

(7.68)

\[ [G(y)(7.39)]_i = O(1) \frac{\| w_{\tilde{y}}(v)|v|^2 \frac{\partial x_{\rho_i}}{\partial x_{\rho_i}} f \|_{\infty}}{|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}} \]

\[ = O(1) \frac{\| w f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}} \]

(7.69)

\[ [G(y)(7.40)]_i = O(1)(7.54) = O(1) \frac{\| w f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}}, \]

(7.70)

For \( [G(y)(7.41)]_i \), we rewrite

\[ G(y) \frac{\nabla_x f(x - tv, v) - \nabla_x f(y - tv, v)}{|x - y|^\beta} = \frac{G(x)\nabla_x f(x - tv, v) - G(y)\nabla_x f(y - tv, v)}{|x - y|^\beta} \overset{\text{(7.71)}_1}{=} [G(y) - G(x)] \frac{\nabla_x f(x - tv, v)}{|x - y|^\beta} \overset{\text{(7.71)}_2}{=} \]

\[ = \frac{\| |v|^2 w_{\tilde{y}}(v) \alpha \nabla_x f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}} \]

(7.71)

We apply (2.57) to have

\[ [(7.71)_2]_i = O(1) \frac{\| |v|^2 w_{\tilde{y}}(v) \alpha \nabla_x f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}}, \]

For (7.71) we apply (2.90) and conclude

\[ |(7.71)_1| = O(1) \frac{\left\| \nabla_x f(\cdot,v) \right\|_{C^{0,\beta}_{x:1+\beta}}}{\| |v|^2 w_{\tilde{y}}(v) \alpha \nabla_x f \|_{\infty}} \]

\[ + \frac{\tilde{a}(x,v)|\nabla_x f(\cdot,v)|}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{2+\beta}} \]

\[ + \frac{|v| O(1) \| w_{\tilde{y}} \alpha \nabla_x f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}} \]

\[ + \frac{\left\| [\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x:1+\beta}} + \left\| [\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x:2+\beta}} \right\| + \| w_{\tilde{y}} \alpha \nabla_x f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}}, \]

\[ = O(1) \frac{\left\| |v|^2 w_{\tilde{y}}(v) \alpha \nabla_x f \|_{\infty}}{w_{\tilde{y}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|\alpha|}, \frac{\alpha(y,v)}{|\alpha|} \right\}^{1+\beta}} \]
where we have used \( \tilde{\alpha}(v) w_{\tilde{\phi}}^{-1}(v) \lesssim \alpha(v) w_{\tilde{\phi}/2}^{-1}(v) \), \(|v| w_{\tilde{\phi}/2}^{-1}(v) \lesssim w_{\tilde{\phi}/2}^{-1}(v) \). Thus with \( e^{-v t} \ll 1 \) when \( t \gg 1 \),

\[
[G(y)[(7.41)]_i = o(1) \frac{\|w_{\tilde{\phi}}\alpha \nabla_x f\|_{\infty} + \|\nabla_x f(\cdot, v)\|_{C^{0,\beta}_{x:1+\beta}} + \|\nabla_x f(\cdot, v)\|_{C^{0,\beta}_{x:2+\beta}}}{w_{\tilde{\phi}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}},
\]

(7.72)

which is already included in (7.27).

Then we estimate \([G(y)[(7.43)]_i \). We rewrite

\[
G(y)[\nabla_x h(x -(t-s)v) - \nabla_x h(y -(t-s)v)]
\]

\[
= G(x) \nabla_x h(x -(t-s)v) - G(y) \nabla_x h(y -(t-s)v)
\]

\[
\tag{7.73}_1
\]

\[
+ \frac{G(y) - G(x)}{|x - y|^\beta} \nabla_x h(x -(t-s)v).
\]

(7.73)

We bound \( w_{\tilde{\phi}}^{-1}(v)|v| \lesssim w_{\tilde{\phi}/2}^{-1}(v) \). From (2.57) and (7.10) the contribution of (7.73)_2 in \([G(y)[(7.43)]_i\) is

\[
O(1)\|\xi\|_{C^2} \frac{\|w_{\tilde{\phi}}\alpha \nabla_x f\|_{\infty} |v| w_{\tilde{\phi}}^{-1}(v)}{w_{\tilde{\phi}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}},
\]

which is included in (7.27).

For (7.73)_1 we apply (2.90) in Lemma 2.10. Then such contribution in \([G(y)[(7.43)]_i\) equals to

\[
\frac{[\nabla \nabla h(x -(t-s)v) - \nabla \nabla h(y -(t-s)v)]}{|x - y|^\beta}
\]

\[
+ \tilde{\alpha}(x, v) \partial_{x_i} h(x -(t-s)v) v - \partial_{x_i} h(y -(t-s)v) v
\]

\[
+ \frac{\|w_{\tilde{\phi}}\alpha \nabla_x f\|_{\infty}}{w_{\tilde{\phi}/2}(v) |v|} \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta},
\]

which are included in (7.26),(7.27) and (7.29) respectively.
Last we estimate (7.45)–(7.48). For (7.45) we apply \(|G(y)| = O(1)|\); for (7.46) we apply (2.54),(7.9) and (2.63); for (7.47) we apply (2.105),(2.32) and (2.63) ; for (7.48) we apply \(|G| = O(1)|\), then we obtain

\[
G(y)[(7.45)] = O(1)((7.57) \equiv O(1)(7.57) = \frac{O(1)\|w_\beta \alpha \nabla_x f\|_\infty}{w_\beta/2(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta},
\]

\[
G(y)[(7.46)] = \frac{O(1)(h(x_b(x,v),v) - h(x_b(y,v),v)|x_b(x,v) - x_b(y,v)|^\beta}{|x - y|^\beta} \leq \frac{1}{w_\beta(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta},
\]

\[
G(y)[(7.47)] = \frac{O(1)(|h||h|_\infty)}{w_\beta(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta},
\]

\[
G(y)[(7.48)] = O(1)(7.61) = \frac{O(1)(|w_f||w_f|_\infty)}{w_\beta/2(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}.
\]

These four estimates are all included in (7.27).

We conclude the lemma.

\[\square\]

Now we are ready to prove Proposition 3.

**Proof of Proposition 3.** We will use 3 steps to prove this proposition. Since we already expand the difference quotient in Lemma 7.3, we mainly estimate (7.24),(7.25) and (7.26),(7.28),(7.29). The estimate of (7.24) is put in Step 1 and the estimate of (7.25) is put in Step 2. Thus Step 1 and Step 2 together conclude (7.7). In Step 3 we estimate (7.26),(7.28),(7.29) and conclude (7.8).

Before going into these steps we first list some estimates for the \(\alpha\)-weight. We will heavily rely on these estimates to make the computation more precise. For \(0 \leq s \leq t_b(x_b(x,v),v^1)|\), we have

\[
\alpha(x_b(x,v),v^1) \sim \alpha(x_b(x,v) - sv^1,v^1) \sim \alpha(x_b(x_b(x,v),v^1),v^1),
\]

and

\[
|n(x_b(x,v)) \cdot v^1|e^{-C|v^1|^2} = \left[ 1 |n(x_b(x,v)) \cdot v^1| \geq 1 + 1 |n(x_b(x,v)) \cdot v^1| < 1 \right] |n(x_b(x,v)) \cdot v^1| e^{-C|v^1|^2}
\]
\[ \alpha(x_b(x, v), v^1) \frac{\|n(x_b(x, v)) \cdot v^1\|}{\alpha(x_b(x, v), v^1)} e^{-C|v^1|^2} + \alpha(x_b(x, v), v^1)e^{-C|v^1|^2} \leq \alpha(x_b(x, v), v^1)e^{-C|v^1|^2/2}, \]

(7.79)

where we have used (1.8) and (1.10) in the derivation.

Suppose \( \alpha(x_b(x, v)) \leq \alpha(x_b(y, v)) \). We let \( \varepsilon \ll 1 \) such that \( \beta + \varepsilon < 1 \) and \( \frac{1-\beta}{1-\beta-\varepsilon} < 2 \). Then we apply the Hölder’s inequality with \( (\beta + \varepsilon) + (1 - \beta - \varepsilon) = 1 \) to have

\[
\int_{n(x_b(x,v)) \cdot v^1 > 0} e^{-C|v^1|^2} |n(x_b(x, v)) \cdot v^1| \frac{e^{-C|v^1|^2}}{\alpha(x_b(x, v), v^1)^{1+\beta}} dv^1
= \int_{n(x_b(x,v)) \cdot v^1 > 0} e^{-C|v^1|^2} \frac{e^{-C|v^1|^2}}{|v^1|^{1-\beta}} \alpha^\beta(x_b(x, v), v^1) dv^1
\leq \left( \int_{n(x_b(x,v)) \cdot v^1 > 0} e^{-C|v^1|^2} \right)^{1-\beta-\varepsilon} \left( \int_{n(x_b(x,v)) \cdot v^1 > 0} e^{-C|v^1|^2} \alpha^\beta(x_b(x, v), v^1) dv^1 \right)^{\beta+\varepsilon} \lesssim 1,
\]

(7.80)

where we have used

\[
\int_{n(x_b(x,v)) \cdot v^1 > 0} \frac{e^{-C|v^1|^2}}{\alpha^\beta(x_b(x, v), v^1)} dv^1
\leq \int_{n(x_b(x,v)) \cdot v^1 > 0} \frac{1}{\alpha^\beta(x_b(x, v), v^1)} dv^1 + \int_{n(x_b(x,v)) \cdot v^1 > 0} 1 \frac{e^{-C|v^1|^2}}{|n(x_b(x, v)) \cdot v^1|^{\beta}} dv^1 \lesssim 1.
\]

Then we start the proof. \( \square \)

**Step 1: estimate of (7.24).**

We focus on

\[
w_\beta(v) |v|^2 \frac{|\partial_{x_1} f(x_b(x, v), v) - \partial_{x_1} f(x_b(y, v), v)|}{|x_b(x, v) - x_b(y, v)|^\beta}.
\]

(7.81)

Applying the diffuse boundary condition (2.4) we get

\[
\partial_{x_1} f(x_b(x, v), v) = \frac{M_W(x_b(x, v), v)}{\sqrt{\mu(v)}} \int_{v^1_{p_1(x), 3 > 0}}\left( \partial_{p_1(x), i} T_{p_1(x)}^i |p_1(x)^i v^1| \right) \cdot \nabla_v f(\eta_{p_1(x)}(x^1_{p_1(x)}), T_{x^1_{p_1(x)}}^i v^1_{p_1(x)})
\]

(7.82)
From Lemma 2.11 the contribution of the last two terms of (7.82) in (7.81) is bounded by
\[
\|T_W - T_0\|_{C^2} \leq \|w_\beta |\nabla_x f|\|_{\infty} + 1.
\]

**Velocity derivative:** first we consider the contribution of \((7.82)_1\) in (7.24). From (3.22), we rewrite \((7.82)_1\) as
\[
\int_{v_{p,1}^1>0} \sum_{m,n}(3.23)_{mn,k+1 \to 1}(x)v_{p,m}^1 \\
\times \partial_{v_{p,n}^1} \left[ f(\eta_{p,x}^1(x), T_{x_{p}^1}^t v_{p,1}^1) \sqrt{\mu(v)p_{m}^1} \right] v_{p,1}^1 \ dx_{p}^1 \\
= \int_{v_{p,1}^1>0} \sum_{m,n}(3.23)_{mn,k+1 \to 1}(x) \\
\times f(\eta_{p,x}^1(x), T_{x_{p}^1}^t v_{p,1}^1) \partial_{v_{p,n}^1} \left[ \sqrt{\mu(v)p_{m}^1} v_{p,1}^1 v_{p,1}^1 v_{p,1}^1 \right] \ dx_{p}^1.
\] (7.83)

Here we dropped the \(x\) dependence in \(v_{p,1}^1\) since it becomes a dummy variable.

From Section 5.2, \(\|u f\|_{L^\infty(\partial \Omega)} \leq \|w f\|_{\infty}\), for the rest estimate we will not specify the norm on \(\partial \Omega\). We apply (3.24) to have
\[
(7.83) = O(1)\|\eta\|_{C^2} ||w f||_{\infty}.
\] (7.84)

Then the contribution of \((7.82)_1\) in (7.24) can be written as
\[
\frac{w_\beta (v)|v|^2}{|x_0(x,v) - x_0(y,v)|^\beta} \left[ M_W(x_0(x,v), v) - M_W(x_0(y,v), v) \right] \times (7.83) \\
+ \frac{M_W(x_0(y,v), v)}{\sqrt{\mu(v)}} \\
\times \int_{v_{p,1}^1>0} \left[ f(\eta_{p,x}^1(x), T_{x_{p}^1}^t v_{p,1}^1) - f(\eta_{p,x}^1(y), T_{x_{p}^1}^t v_{p,1}^1) \right] \\
\times \sum_{m,n}(3.23)_{mn,k+1 \to 1}(x) \partial_{v_{p,n}^1} \left[ \sqrt{\mu(v)p_{m}^1} v_{p,1}^1 v_{p,1}^1 v_{p,1}^1 \right] \ dx_{p}^1 \\
+ \frac{M_W(x_0(y,v), v)}{\sqrt{\mu(v)}} \int_{v_{p,1}^1>0} \sum_{m,n} \partial_{v_{p,n}^1} \left[ \sqrt{\mu(v)p_{m}^1} v_{p,1}^1 v_{p,1}^1 v_{p,1}^1 \right] \\
\times \left[ M_W(x_0(x,v), v) - M_W(x_0(y,v), v) \right] \times (7.85)
\] (8.6)
\[ x f(\eta p^1(x) x^1 p^1, T^t \eta p^1(x) x^1 p^1) \left[ (3.23)_{mn,k+1 \rightarrow 1} - (3.23)_{mn,k+1 \rightarrow 1} \right] dv^1. \]  

(7.87)

For (7.85), since \( \tilde{\theta} \ll 1 \), from (7.84) we derive that

\[ (7.85) \lesssim \|\eta\|_{C^2} \|wf\|_{\infty} \frac{w_\theta(v)}{\sqrt{\mu(v)}} [M_W(x_b(x, v), v) - M_W(x_b(y, v), v)] \]

\[ \lesssim \|\eta\|_{C^2} \|wf\|_{\infty} \|T_W\|_{C^1}. \]  

(7.88)

For (7.86), from (2.64) with (7.6) and (2.54) we compute

\[ f_s(\eta p^1(x) x^1 p^1, T^t \eta p^1(x) x^1 p^1) - f_s(\eta p^1(y) x^1 p^1, T^t \eta p^1(y) x^1 p^1) \]

\[ \lesssim \left| \eta p^1(x) x^1 p^1 - \eta p^1(y) x^1 p^1 \right| \|
\begin{align*}
&\left| T^t x^1 p^1 - T^t x^1 p^1 \right| \\
&\left| \eta p^1(x) x^1 p^1 - \eta p^1(y) x^1 p^1 \right| \\
&\min \left\{ \alpha(\eta p^1(x) x^1 p^1), T^t x^1 p^1, \alpha(\eta p^1(y) x^1 p^1), T^t x^1 p^1 \right\} \|\nabla_v f\|_{\infty} \|
\begin{align*}
&\left| T^t x^1 p^1 - T^t x^1 p^1 \right| \\
&\left| \eta p^1(x) x^1 p^1 - \eta p^1(y) x^1 p^1 \right| \\
&\min \left\{ \alpha(\eta p^1(x) x^1 p^1), T^t x^1 p^1, \alpha(\eta p^1(y) x^1 p^1), T^t x^1 p^1 \right\} \|\nabla_v f\|_{\infty} \|
\end{align*}
\end{align*}
\]

(7.89)

where we have used the definition of \( T_{x^1} \) in (2.23) and the mean value theorem regarding \( \nabla_v f \) in the last line.

Since \( \sum_{m,n} (3.23)_{mn,k+1 \rightarrow 1} (x) \partial_{x^1 p^1, m} [\sqrt{\mu(v)} x^1 p^1, x^1 p^1, x^1 p^1, m] \lesssim 1 \), we have

\[ (7.86) \lesssim \|\nabla_x f\|_{\infty} \frac{w_\theta(v)|v|^2 M_W(x_b(y, v), v)}{\sqrt{\mu(v)}} \]

\[ \times \int_{v^1 p^1 > 0} M_{\mu^{1/4}(v)} \min \left\{ \alpha(\eta p^1(x) x^1 p^1), T^t x^1 p^1, \alpha(\eta p^1(y) x^1 p^1), T^t x^1 p^1 \right\} \|
\]

\[ \lesssim (\|\nabla_x f\|_{\infty} + \|v|^2 \nabla_v f\|_{\infty}) \frac{M_W(x_b(y, v), v)}{\sqrt{\mu(v)}} \]
\[
\times \left[ \int_{\mathbb{R}^3} \left( \frac{\mu^{1/4}(v^1_{p_1})}{|n(x_b(x, v)) \cdot T_{x^i}^1 v^1_{p_1}|^\beta} + \frac{\mu^{1/4}(v^1_{p_1})}{|n(x_b(y, v)) \cdot T_{x^i}^1 v^1_{p_1}|^\beta} \right) \right] \nonumber
\]
\[
+ \int_{\mathbb{R}^3} \frac{\mu^{1/4}(v^1_{p_1})}{|v^1_{p_1}|} \right] \nonumber
\]
\[
\lesssim \|\alpha \nabla_x f\|_\infty + \|v\|^2 \nabla_v f\|_\infty, \quad (7.90)
\]
where we have used \( \beta < 1 \) in the last line.

Last we estimate (7.87). From (3.23) and (2.32) we compute
\[
\left| (3.23)_{mn}(x) - (3.23)_{mn}(y) \right| = \left| \eta_{p^1(x)}(x^1_{p^1(x)}) - \eta_{p^1(y)}(x^1_{p^1(y)}) \right| \|\eta\|_{C^3},
\]
where we have used \( \eta \in C^3 \) and mean value theorem in the last line. Thus we conclude
\[
(7.87) \lesssim \|\eta\|_{C^3} \|w f\|_\infty. \quad (7.91)
\]

Combining (7.88),(7.90) and (7.91), we conclude that the contribution of the velocity derivative (7.82) in (7.81) has an upper bound
\[
|\langle 7.81, 7.82 \rangle_1 | \lesssim \|\eta\|_{C^3} \|T_w\|_{C^1} \|\alpha \nabla_x f\|_\infty, \quad (7.92)
\]
where we used (7.6).

Spatial derivative: we consider the contribution of the spatial derivative (7.82) in (7.24). We rewrite the \( v^1_{p_1} \)-integration using \( v^1 \) integration and get
\[
(7.82)_2 = e^{-\nu_b(x, v)} \frac{M_{W}(x_b(x, v), v)}{\sqrt{\mu(v)}}
\]
\[
\times \int_{n(x_b(x, v)) \cdot v^1 > 0} \frac{\partial}{\partial x^i_{p^1(x, y)}} \left[ f(\eta_{p^1(x)}, x^1_{p^1(x), y}, v^1) \right] \sqrt{\mu(v^1)} n(x_b(x, v)) \cdot v^1 \, dv^1. \quad (7.93a)
\]

(7.93)
Then the contribution of (7.82) in (7.24) can be written as

\[
\left[ \frac{w_\eta(v)}{|v|^2} \left[ M_W(x_b(x, v), v) - M_W(x_b(y, v), v) \right] \right. \\
\left. + \frac{|v|^2 M_W(x_b(y, v), v)}{\sqrt{\mu(v)} \sqrt{\mu(v)} x_b(x, v) - x_b(y, v)} \right] \\
\times \left( \int_{n(x_b(x, v)) \cdot v^1 > 0} \frac{\partial}{\partial x^1_{p^1(x), i}} f(\eta_{p^1(x)}(x^1_{p^1(x)}, v^1)) \sqrt{\mu(v^1)} |\eta(x_b(x, v)) \cdot v^1| dv^1 \right. \\
\left. - \int_{n(x_b(y, v)) \cdot v^1 > 0} \frac{\partial}{\partial x^1_{p^1(y), i}} f(\eta_{p^1(y)}(x^1_{p^1(y)}, v^1)) \sqrt{\mu(v^1)} |\eta(x_b(y, v)) \cdot v^1| dv^1 \right) \\
\right]. 
\]  

(7.95)

From (2.47) in Lemma 2.5,

\[
(7.93) \lesssim \|v\| \|f\|_\infty \int \sqrt{\mu(v^1)} \frac{|\eta(x_b(x, v)) \cdot v^1|}{|v^1|} dv^1 \lesssim \|v\| \|\nabla f\|_\infty. 
\]

Thus applying (2.54) and (2.71) we derive that

\[
(7.94) \lesssim \|\eta\|_c^2 \|v\| \|\nabla f\|_\infty. 
\]

For (7.95), we express \[\sum_{i=1,2} x^1_{p^1(x), i} f(\eta_{p^1(x)}(x^1_{p^1(x)}, v^1)) \text{ and } \sum_{i=1,2} x^1_{p^1(y), i} f(\eta_{p^1(y)}(x^1_{p^1(y)}, v^1)\]

using (7.30)–(7.35) with the notation (7.3):

\[
1_{t^1 \geq \min\{t^2_b(x), t^2_b(y)\}} e^{-v^2_b(x)} \sum_{i=1,2} \frac{\partial x^1_{p^1(x), i}}{\partial x^1_{p^1(x), i}} \left[ f(x_b(\eta_{p^1(x)}(x^1_{p^1(x)}, v^1)), v^1) \right] \\
- 1_{t^1 \geq \min\{t^2_b(x), t^2_b(y)\}} v \sum_{i=1,2} \frac{\partial x^1_{p^1(x), i}}{\partial x^1_{p^1(x), i}} t^2_b(x) e^{-v^2_b(x)} f(x^2_b(x), v^1) \\
+ 1_{t^1 \leq \min\{t^2_b(x), t^2_b(y)\}} e^{-v^1} \sum_{i=1,2} \frac{\partial x^1_{p^1(x), i}}{\partial x^1_{p^1(x), i}} \left[ f(\eta_{p^1(x)}(x^1_{p^1(x)}), t^1 v^1, v^1) \right] \\
+ 1_{t^1 \leq \min\{t^2_b(x), t^2_b(y)\}} \times \int_0^{t^1} e^{-v(t^1 - s^1)} \sum_{i=1,2} \frac{\partial x^1_{p^1(x), i}}{\partial x^1_{p^1(x), i}} \left[ h(\eta_{p^1(x)}(x^1_{p^1(x)}), (t^1 - s^1) v^1, v^1) \right] ds^1 \\
+ 1_{t^1 \geq \min\{t^2_b(x), t^2_b(y)\}} \times \int_{t^1 - t^2_b(x)}^{t^1} e^{-v(t^1 - s^1)} \sum_{i=1,2} \frac{\partial x^1_{p^1(x), i}}{\partial x^1_{p^1(x), i}} \left[ h(\eta_{p^1(x)}(x^1_{p^1(x)}), (t^1 - s^1) v^1, v^1) \right] ds^1 \\
+ 1_{t^1 \geq \min\{t^2_b(x), t^2_b(y)\}} \sum_{i=1,2} \frac{\partial x^1_{p^1(x), i}}{\partial x^1_{p^1(x), i}} t^2_b(x) e^{-v^2_b(x)} h(x^2_b(x), v^1). 
\]  

(7.100)

(7.101)

(7.102)
We first estimate the boundary term (7.97). We split (7.97) into two cases using \(\min\{\alpha(x_b(x, v), v^1), \alpha(x_b(y, v), v^1)\}\). We put the discussion for
\[\min\{\alpha(x_b(x, v), v^1), \alpha(x_b(y, v), v^1)\} \leq \epsilon, \text{ or } |v| \geq \epsilon\]
together with the estimate of (7.98)–(7.102). Here we discuss the case that
\[\min\{\alpha(x_b(x, v), v^1), \alpha(x_b(y, v), v^1)\} \geq \epsilon \text{ and } |v| \leq \epsilon^{-1}.
For this case the difference quotient of (7.97) reads as

\[
\frac{1}{|x_b(x, v) - x_b(y, v)|^\beta} \left( \int_{n(x_b(x, v)) \cdot v^1 > 0} (7.97)(x) \sqrt{\mu(v^1)} |n(x_b(x, v)) \cdot v^1| dv^1 - \int_{n(x_b(y, v)) \cdot v^1 > 0} (7.97)(y) \sqrt{\mu(v^1)} |n(x_b(y, v)) \cdot v^1| dv^1 \right).
\tag{7.103}
\]

We perform the change of variable (2.42) and use (3.29) to rewrite this as

\[
\int_{n(x_b(x, v)) \cdot v^1 > 0} \frac{\partial}{\partial x^1_{p^1(i)}} [f(\eta_{p^2}(x^2_{p^2}), v^1)] \sqrt{\mu(v^1)} |n(x_b(x, v)) \cdot v^1| dv^1
\]

\[
= \sum_{p^2 \in \mathcal{P}} \int_{[x^2_{p^2}] < 1} \int_{0}^{t - \eta_{b_{x_b(x, v)}}} e^{-v^1(t)} |t_{p^2}^2(\eta_{p^2}(x^2_{p^2}))
\]

\[
\times \sum_{j = 1, 2} \frac{\partial x^2_{p^2,j}}{\partial x^1_{p^1(x), j}} \frac{\partial x^2_{p^2,j}}{\partial x^1_{p^1(y), j}} [f(\eta_{p^2}(x^2_{p^2}), v^1)]
\]

\[
\times \frac{n_{p^1}(x) (x^1_{p^1(x)}) \cdot (x_b(x, v) - \eta_{p^2}(x^2_{p^2})) n_{p^2}(x^2_{p^2}) \cdot (x_b(x, v) - \eta_{p^2}(x^2_{p^2}))}{|t_{p^2}^2|^4}
\]

\[
\times e^{-\frac{|x_{b_{x_b(x, v)}} - \eta_{p^2}(x^2_{p^2})|^4}{4|t_{p^2}^2|^4}} \frac{\sqrt{g_{p^2,11} g_{p^2,22}} dx^2_{p^2,1} dx^2_{p^2,2}}{dx^2_{p^2}}.
\tag{7.104}
\]

Here we dropped the \(x\) dependence on \(p^2(x)\) since \(x^2_{p^2}\) becomes dummy variable after the change of variable.

In (7.104) the variables that depend on \(x\) are \(t_{b_{x_b(x, v)}}, x_{b_{x_b(x, v)}}, x^1_{p^1(x)}\). Thus we have

\[
(7.103) = \frac{(7.104)(x) - (7.104)(y)}{|x_{b_{x_b(x, v)}} - x_{b_{x_b(y, v)}}|}\]

\[
= \sum_{p^2 \in \mathcal{P}} \int_{[x^2_{p^2}] < 1} \int_{0}^{t - \min\{t_{b_{x_b(x, v)}, t_{b_{x_b(y, v)}}}\}} \int_{t - \max\{t_{b_{x_b(x, v)}, t_{b_{x_b(y, v)}}}\}} \left[ \frac{\partial x^2_{p^2,j}}{\partial x^1_{p^1(x), j}} - \frac{\partial x^2_{p^2,j}}{\partial x^1_{p^1(y), j}} \right] \frac{n_{p^1}(x) (x^1_{p^1(x)}) \cdot (x_b(x, v) - \eta_{p^2}(x^2_{p^2}))}{|t_{p^2}^2|^4} \frac{\sqrt{g_{p^2,11} g_{p^2,22}} dx^2_{p^2,1} dx^2_{p^2,2}}{dx^2_{p^2}}.
\tag{7.105}
\]

\[
+ \sum_{p^2 \in \mathcal{P}} \int_{[x^2_{p^2}] < 1} \int_{0}^{t - \max\{t_{b_{x_b(x, v)}, t_{b_{x_b(y, v)}}}\}} \left[ \frac{\partial x^2_{p^2,j}}{\partial x^1_{p^1(x), j}} - \frac{\partial x^2_{p^2,j}}{\partial x^1_{p^1(y), j}} \right] \frac{n_{p^1}(x) (x^1_{p^1(x)}) \cdot (x_b(x, v) - \eta_{p^2}(x^2_{p^2}))}{|t_{p^2}^2|^4} \frac{\sqrt{g_{p^2,11} g_{p^2,22}} dx^2_{p^2,1} dx^2_{p^2,2}}{dx^2_{p^2}}.
\tag{7.106}
\]
\[
- \frac{n p_1(y)(x^1 p_1(y)) \cdot (x_b(y,v) - \eta p_2(x^2 p_2))}{t_b^2} \ldots
\]
\[
+ \left[ \frac{n p_2(x^2 p_2) \cdot (x_b(x,v) - \eta p_2(x^2 p_2))}{|t_b|^4} - \frac{n p_2(x^2 p_2) \cdot (x_b(y,v) - \eta p_2(x^2 p_2))}{|t_b|^4} \right] \ldots
\]
\[
+ \left[ e^{\frac{3|x_b(x,v) - \eta p_2(x^2 p_2)|}{4|t_b|}} - e^{\frac{|x_b(y,v) - \eta p_2(x^2 p_2)|}{4|t_b|}} \right] \ldots.
\]

Since \( \min(\alpha(x_b(x,v), v^1), \alpha(x_b(y,v), v^1)) \geq \varepsilon \), from (1.11), clearly we have
\(|n(x_b(x,v)) \cdot v^1| \gtrsim \varepsilon \). Moreover, due to \(|v^1| \leq \varepsilon^{-1} \), we have a lower bound for \( t_b^2 \) from (2.39):
\[
t_b^2 \gtrsim \frac{1}{|v^1|} \min \left\{ |n(x_b(x,v)) \cdot v^1|, |n(x_b(y,v)) \cdot v^1| \right\} \gtrsim \varepsilon^3.
\]

From (2.47) in Lemma 2.5 and Lemma 2.8 we obtain the following estimate:
\[
\left| \frac{\partial^2}{\partial p_j \partial p'_j} f(n p_2(x^2 p_2), v^1) \right| \leq \|v\| \|\nabla f\| \|f\| < \infty, \quad \left| \frac{\partial^2}{\partial x^1 p_1 \partial x^1 p'_1} f \right| \lesssim 1,
\]
\[
\left| \frac{n p_1(x)(x^1 p_1(x)) \cdot (x_b(x,v) - \eta p_2(x^2 p_2))}{t_b^2} \right| \lesssim \Omega \varepsilon^{-3},
\]
\[
\left| \frac{n p_2(x^2 p_2) \cdot (x_b(x,v) - \eta p_2(x^2 p_2))}{|t_b^2|^4} \right| \lesssim \Omega \varepsilon^{-12}.
\]

Now we estimate (7.105)–(7.109). By (2.56) we compute
\[
|\text{(7.105)}| \frac{|x_b(x,v) - x_b(y,v)|^\beta}{|x_b(x,v) - x_b(y,v)|^\beta}
\]
\[
O(\varepsilon^{-15}) \lesssim \frac{e^{-\nu t_b(x,v)}}{e^{-\nu t_b(y,v)}} \lesssim O(\varepsilon^{-15}) \frac{|x_b(x,v) - x_b(y,v)|^\beta}{|x_b(x,v) - x_b(y,v)|^\beta}
\]
\[
O(\varepsilon^{-15}) \frac{|x_b(x,v) - x_b(y,v)|^\beta}{|x_b(x,v) - x_b(y,v)|^\beta} = O(\varepsilon^{-15}) \frac{|x - y|^\beta}{|x - y|^\beta}
\]
\[
\lesssim \min \left\{ \alpha(x_b(x,v), v^1), \alpha(x_b(y,v), v^1) \right\} \frac{|x - y|^\beta}{|x - y|^\beta} \lesssim O(\varepsilon^{-16}) \frac{|x - y|^\beta}{|x - y|^\beta}.
\]

The extra term \( \frac{|x - y|^\beta}{|x_b(x,v) - x_b(y,v)|^\beta} \) will be cancelled by \( \frac{|x_b(x,v) - x_b(y,v)|^\beta}{|x - y|^\beta} \) in (7.24).
Then we estimate (7.106). By (2.72) we have
\[
\left| \frac{\partial x^2_{p_2,j'}}{\partial x^1_{p_1(x),j}} - \frac{\partial x^2_{p_2,j'}}{\partial x^1_{p_1(y),j}} \right| \lesssim \frac{1}{\min \{\alpha(x_b(x, v), v^1), \alpha(x_b(y, v), v^1)\}} \geq \varepsilon \\
\lesssim \frac{1}{\min \{\alpha(x_b(x, v), v^1), \alpha(x_b(y, v), v^1)\}} \lesssim O(\varepsilon^{-6}).
\]
Thus
\[
\frac{|(7.106)|}{|x_b(x, v) - x_b(y, v)|^\beta} \lesssim O(\varepsilon^{-15}) \int_0^\infty e^{-\frac{t}{\varepsilon^6}} \frac{|\frac{\partial x^2_{p_2,j'}}{\partial x^1_{p_1(x),j}} - \frac{\partial x^2_{p_2,j'}}{\partial x^1_{p_1(y),j}}|}{|x_b(x, v) - x_b(y, v)|^\beta} \lesssim O(\varepsilon^{-21}).
\]

(7.111)

Then we estimate (7.107). By (2.57) we compute
\[
\frac{|n_{p_1(x)}(x^1_{p_1(x)} \cdot (x_b(x, v) - \eta_{p_2}(x^2_{p_2})) - n_{p_1(y)}(x^1_{p_1(y)} \cdot (x_b(y, v) - \eta_{p_2}(x^2_{p_2})))|}{\varepsilon^2} \lesssim O(\varepsilon^{-3}) \frac{|n_{p_1(x)}(x^1_{p_1(x)} - n_{p_1(y)}(x^1_{p_1(y)}))|}{|x_b(x, v) - x_b(y, v)|^\beta} \lesssim O(\varepsilon^{-3}).
\]

Thus
\[
\frac{|(7.107)|}{|x_b(x, v) - x_b(y, v)|^\beta} \lesssim O(\varepsilon^{-15}).
\]

(7.112)

For (7.108) we compute the difference as
\[
\frac{|n_{p_2}(x^2_{p_2}) \cdot (x_b(x, v) - \eta_{p_2}(x^2_{p_2})) - n_{p_2}(x^2_{p_2}) \cdot (x_b(y, v) - \eta_{p_2}(x^2_{p_2}))|}{\varepsilon^2} \lesssim O(\varepsilon^{-12}) \frac{|n_{p_2}(x^2_{p_2})|^4}{|x_b(x, v) - x_b(y, v)|^\beta} \lesssim O(\varepsilon^{-12}).
\]

Thus
\[
\frac{|(7.108)|}{|x_b(x, v) - x_b(y, v)|^\beta} \lesssim O(\varepsilon^{-12}).
\]

(7.113)

Last we estimate (7.109). By mean value theorem,
\[
-\frac{|x_b(x, v) - \eta_{p_2}(x^2_{p_2})|^4}{4|\varepsilon|} - e^{-\frac{|x_b(y, v) - \eta_{p_2}(x^2_{p_2})|^4}{4|\varepsilon|}} \lesssim O(\varepsilon^{-12}).
\]
Thus
\[
\frac{|(7.109)|}{|x_b(x, v) - x_b(y, v)|^\beta} \lesssim O(\varepsilon^{3\beta}).
\] (7.114)

Therefore, from (7.6), we collect (7.110),(7.111),(7.112),(7.113) and (7.114) to conclude
\[
| (7.103) | \\
\lesssim \left[ O(\varepsilon^{-21}) + O(\varepsilon^{-16}) \frac{|x - y|^\beta}{|x_b(x, v) - x_b(y, v)|^\beta} \right] \| w_{\beta} \nabla_x f \|^2_{\infty}. \] (7.115)

Then we estimate the rest terms in (7.97)–(7.102). First we rewrite the contribution of these terms in (7.95) into
\[
\int_{n(x_b(x, v)) \cdot v^1 > 0} (7.97) \mathbf{1} \ldots (x) + \cdots + (7.102)(x) \\
\quad - \frac{|x_b(x, v) - x_b(y, v)|^\beta}{|x_b(x, v) - x_b(y, v)|^\beta} \\
\quad \frac{|n(x_b(x, v) - n(x_b(y, v)))| \geq \frac{n(x_b(x, v)) \cdot v^1}{|v^1|} > 0}{|x_b(x, v) - x_b(y, v)|^\beta} \\
\quad \frac{|n(x_b(x, v)) \cdot v^1| - |n(x_b(y, v)) \cdot v^1| \sqrt{\mu(v^1)}}{|x_b(x, v) - x_b(y, v)|^\beta} \\
\quad \times \left[ (7.97) \mathbf{1} \ldots (x) + \cdots + (7.102)(x) \right] \] (7.117)
\[
+ \frac{|n(x_b(y, v)) \cdot v^1| \sqrt{\mu(v^1)}}{|x_b(x, v) - x_b(y, v)|^\beta} \\
\quad \times \left[ (7.97) \mathbf{1} \ldots (x) + \cdots + (7.102)(x) - (7.97) \mathbf{1} \ldots (y) - \cdots - (7.102)(y) \right]. \] (7.118)

By (2.47) in Lemma 2.5,
\[
| (7.97) + \cdots + (7.102) | \lesssim (5.1) + \cdots (5.5) \\
\lesssim (5.31) \lesssim \frac{\| v \|_{\nabla f} \| f \|_{\infty}}{|v^1|}. \]

For (7.116), from (2.57) and (7.6) we have
\[
\frac{1}{|x_b(x, v) - x_b(y, v)|^\beta} \int_{n(x_b(x, v)) - n(x_b(y, v))] \geq \frac{n(x_b(x, v)) \cdot v^1}{|v^1|} > 0} (7.97) + \cdots + (7.102) | \\
\frac{|n(x_b(x, v) - n(x_b(y, v)))| \geq \frac{n(x_b(x, v)) \cdot v^1}{|v^1|} > 0}{|x_b(x, v) - x_b(y, v)|^\beta} \\
\frac{|n(x_b(x, v)) \cdot v^1| - |n(x_b(y, v)) \cdot v^1| \sqrt{\mu(v^1)}}{|x_b(x, v) - x_b(y, v)|^\beta} \\
\quad \times \left[ (7.97) \mathbf{1} \ldots (x) + \cdots + (7.102)(x) \right]. \]
\[ \left\| \eta \right\| _{C^2} \| \alpha \nabla_x f \|_\infty \]
\[ \right|_{x_b(x, v) - x_b(y, v)}^{\beta} \]
\[ \times \int_{[n(x_b(x, v)) - n(x_b(y, v))] \geq \frac{n(x_b(x, v)) + 1}{|v|^1} > 0} \frac{|n(x_b(x, v)) \cdot v^1|}{|v|^1} \sqrt{\mu(v^1)} \]
\[ \lesssim \left\| \eta \right\| _{C^2} \| \alpha \nabla_x f \|_\infty |n(x_b(x, v)) - n(x_b(y, v))| \]
\[ \lesssim \left\| \eta \right\| _{C^2} \| \alpha \nabla_x f \|_\infty. \]

Similarly
\[ \frac{1}{|x_b(x, v) - x_b(y, v)|^{\beta}} \int_{[n(x_b(x, v)) - n(x_b(y, v))] \geq \frac{n(x_b(y, v)) + 1}{|v|^1} > 0} \cdots \lesssim \left\| \eta \right\| _{C^2} \| \alpha \nabla_x f \|_\infty. \]

Thus
\[ \left| (7.116) \right| \lesssim \left\| \eta \right\| _{C^2} \| \alpha \nabla_x f \|_\infty. \]  \hspace{1cm} (7.119)

For (7.117), applying (2.57) we have
\[ \left| (7.117) \right| \lesssim \int \frac{|n(x_b(x, v)) \cdot v^1| - |n(x_b(y, v)) \cdot v^1|}{|x_b(x, v) - x_b(y, v)|^{\beta}} \sqrt{\mu(v^1)} \cdots \]
\[ \lesssim \left\| \eta \right\| _{C^2} \| \alpha \nabla_x f \|_\infty. \]  \hspace{1cm} (7.120)

Then we focus (7.118), this estimate is the most delicate one. First of all we bound
\[ |n(x_b(y, v) \cdot v^1)| \leq \underbrace{|n(x_b(x, v)) - n(x_b(y, v))|}_{(7.121)_1} \]
\[ + \min \left\{ n(x_b(x, v)) \cdot v^1, n(x_b(y, v)) \cdot v^1 \right\}. \]  \hspace{1cm} (7.121)

By (2.57) the contribution of (7.121)_1 in (7.118) is bounded by
\[ \frac{\left| (7.118) \right|_{(7.121)_1}}{|x_b(x, v) - x_b(y, v)|^{\beta}} \]
\[ \lesssim \int_{n(x_b(x, v)) \cdot v^1 > 0, n(x_b(y, v)) \cdot v^1 > 0} \frac{|n(x_b(x, v)) \cdot v^1| - |n(x_b(y, v)) \cdot v^1|}{|x_b(x, v) - x_b(y, v)|^{\beta}} \sqrt{\mu(v^1)} \cdots dv^1 \]
\[ \lesssim \left\| \eta \right\| _{C^2} \| \alpha \nabla_x f \|_\infty \int_{n(x_b(x, v)) \cdot v^1 > 0, n(x_b(y, v)) \cdot v^1 > 0} \sqrt{\mu(v^1)} \lesssim \left\| \eta \right\| _{C^2} \| \alpha \nabla_x f \|_\infty. \]  \hspace{1cm} (7.122)

We focus on the contribution of (7.121)_2 in (7.118). Then without loss generality, we can assume
\[ |n(x_b(y, v)) \cdot v^1| = \min \left\{ |n(x_b(x, v)) \cdot v^1|, |n(x_b(y, v)) \cdot v^1| \right\}. \]  \hspace{1cm} (7.123)
In result we can replace \(|n(x_b(y,v))\cdot v^1|\) or \(|n(x_b(y,v))\cdot v^1|\) by \(\min\{|n(x_b(x,v))\cdot v^1|, |n(x_b(y,v))\cdot v^1|\}\).

Note that from (2.47) in Lemma 2.5,
\[
\sum_{i=1,2} \partial_{p^1_{i(x),i}} f(\eta_{p^1_{i(x)}}(x_{p^1_{i(x),i}}), v^1) \sim G(x_b(x,v)) \nabla_x f(x_b(x,v), v^1),
\]
and we have an expression of
\[
G(x_b(x,v)) \nabla_x f(x_b(x,v), v^1) - G(x_b(y,v)) \nabla_x f(x_b(y,v), v^1)
\]
from Lemma 7.3. Thus the contribution of (7.97)–(7.102) in (7.118) can be expressed using (7.26)–(7.29), with replacing \(x \rightarrow x_b(x,v), y \rightarrow x_b(y,v), v \rightarrow v^1, x_{p^1,i} \rightarrow x^i_{p^2,i}, x_b(x,v) \rightarrow \eta_{p^2(x)}(x_{p^2(x)}^2), x_b(y,v) \rightarrow \eta_{p^2(y)}(x_{p^2(y)}^2)\).

From (7.80) we derive that the contribution of (7.27) is bounded by
\[
\int_{n(x_b(x,v))\cdot v^1>0, n(x_b(y,v))\cdot v^1>0} |n(x_b(x,v))\cdot v^1| \sqrt{\mu(v^1)}
\]
where we have used (7.80).

Then we estimate the contribution of (7.28), (7.29) and (7.26).

We begin with (7.29). Note that we only need to consider the case of \(\min\{|\alpha(x_b(x,v), v^1)|, |\alpha(x_b(y,v), v^1)|\} \leq \varepsilon, \) or \(|v^1| \geq \varepsilon^{-1}\). We derive
\[
\int 1_{\min\{|\alpha(x_b(x,v), v^1)|, |\alpha(x_b(y,v), v^1)|\} \leq \varepsilon, \text{ or } |v^1| \geq \varepsilon^{-1}} \sqrt{\mu(v^1)}|n(x_b(x,v))\cdot v^1| |
\]
where we have applied (2.70) and (2.47) in Lemma 2.5 to the fourth line, (7.123) and (7.80) to the integral in the last line.
Then we focus on the contribution of (7.25). First we consider \( h = K(f) \).

Denote
\[ x^s = x_b(x, v) - (t^1 - s^1)v^1, \quad y^s = x_b(y, v) - (t^1 - s^1)v^1, \]
we need to compute
\[ \int \left( \sqrt{\mu(v^1)}|n(x_b(y, v)) \cdot v^1| \right) \int_0^{t^1} ds^1 e^{-v(y)\langle t^1 - s^1 \rangle} \]
\[ \times \int_{\mathbb{R}^3} du k(u^1, u) \frac{G(x^s) \nabla_x f(x^s, u) - G(y^s) \nabla_x f(y^s, u)}{|x_b(x, v) - x_b(y, v)|^{\beta}}. \tag{7.126} \]

We use the decomposition (4.41) for the \( dx^1 \) integral. When \( t^1 - s^1 \leq \varepsilon \), we apply (4.8) in Lemma 4.1 with \( p = 1 + \beta \) and (7.123) to conclude that
\[ |(7.126)|_{t^1 - s^1 \leq \varepsilon} \]
\[ \lesssim |\nabla x|_f f(\cdot, v)|_{C^{0, \beta}_{x:1+\beta}} \int \sqrt{\mu(v^1)}|n(x_b(y, v)) \cdot v^1| dv^1 \]
\[ \times \int_{t^1 - \varepsilon}^{t^1} ds^1 e^{-v(y)\langle t^1 - s^1 \rangle} \int_{\mathbb{R}^3} du k(u^1, u) \frac{1}{|u|^2 \min \left\{ \frac{\alpha(x^s, u)}{|u|}, \frac{\alpha(y^s, u)}{|u|} \right\}^{1+\beta}} \]
\[ \lesssim O(\varepsilon)|\nabla x|_f f(\cdot, v)|_{C^{0, \beta}_{x:1+\beta}} \int \sqrt{\mu(v^1)} \frac{|n(x_b(y, v), v^1)|}{|v^1|^2 \min \left\{ \frac{\alpha(x_b(x, v), v^1)}{|v^1|}, \frac{\alpha(x_b(y, v), v^1)}{|v^1|} \right\}^{1+\beta}} dv^1 \]
\[ \lesssim O(\varepsilon)|\nabla x|_f f(\cdot, v)|_{C^{0, \beta}_{x:1+\beta}}. \]

When \( t^1 - s^1 \geq \varepsilon \). We rewrite things as
\[ (7.126)_a = \frac{G(x^s) \nabla_x f(x^s, u) - G(y^s) \nabla_x f(y^s, u)}{|x_b(x, v) - x_b(y, v)|^{\beta}} \]
\[ \quad + \frac{G(y^s) \nabla_x f(y^s, u) - \nabla_x f(y^s, u)}{|x_b(x, v) - x_b(y, v)|^{\beta}}. \tag{7.127} \]
\[ (7.127)_1 \]
\[ (7.127)_2 \]

By (2.57) we have \( (7.127)_1 \lesssim \frac{\|\alpha \nabla_x f \|_\infty}{\alpha(y^s, u)} \). Thus, the contribution in (7.126) is bounded by
\[ \|\alpha \nabla_x f \|_\infty \int \sqrt{\mu(v^1)}|n(x_b(y, v)) \cdot v^1| dv^1 \int_0^{t^1} ds^1 e^{-v(y)\langle t^1 - s^1 \rangle} \int_{\mathbb{R}^3} du \frac{k(u^1, u)}{\alpha(y^s, u)} \]
\[ \lesssim \|\alpha \nabla_x f \|_\infty \int \sqrt{\mu(v^1)} \frac{|n(x_b(y, v)) \cdot v^1|}{\min \left\{ \alpha(x_b(x, v), v^1), \alpha(x_b(y, v), v^1) \right\}} dv^1 \lesssim \|\alpha \nabla_x f \|_\infty, \]
where we have used (4.5) in Lemma 4.1 and (7.121).

Then we focus on the contribution of (7.127)_2. We exchange \( \nabla_x \) for \( \nabla_{x^1} \):
\[ \nabla_x f(x^s, u) = \nabla_x f(x_b(x, v) - (t^1 - s^1)v^1, u) = \frac{\nabla_{x^1} f(x_b(x, v) - (t^1 - s^1)v^1, u)}{(t^1 - s^1)}. \]
Then we perform an integration by parts for $d\nu^1$. The $d\nu^1$ integral in (7.126) $I_{t^1-s^1}\geq\varepsilon$ becomes

$$
\left| \int \nabla_{\nu^1} \left[ n(x_b(y, v)) \cdot v^1 |\sqrt{\mu(v^1)} \right] \int_{t^1}^{t^1} \left. \frac{e^{-v^1(t^1-s^1)}}{(t^1-s^1)^k} k_{\Phi}(v^1, u) G(y^s) \right| \frac{f(x^s, u) - f(y^s, u)}{|x_b(x, v) - x_b(y, v)|^\beta} \right| \\
\leq \left| \int \nabla_{\nu^1} \left[ n(x_b(y, v)) \cdot v^1 |\sqrt{\mu(v^1)}e^{-v^1(t^1-s^1)} \right] \right| 
$$

(7.128)

$$
+ \left| \int \nabla_{\nu^1} k_{\Phi}(v^1, u) \right| 
$$

(7.129)

$$
+ \left| \int \nabla_{\nu^1} G(x_b(y, v) - (t^1-s^1)v^1) \right| 
$$

(7.130)

$$
+ \left| \int \nabla_{\nu^1} \min \left\{ t_b^2(x), t_b^2(y) \right\} \frac{e^{-v^1\min\{t_b^2(x), t_b^2(y)\}}}{\min\{t_b^2(x), t_b^2(y)\}} \right| 
$$

(7.131)

For (7.128), since

$$
|\nabla_{\nu^1} [n(x_b(y, v) \cdot v^1) \sqrt{\mu(v^1)}e^{-v^1(t^1-s^1)}]| \lesssim \mu^{1/4}(v^1)e^{-v^1(t^1-s^1)/2},
$$

by (2.64) with (7.6) and (4.10) we have

$$
(7.128) \lesssim O(\varepsilon^{-1}) ||\alpha\nabla_x f||_{\infty} \int \mu^{1/4}(v^1) \int e^{-v^1(t^1-s^1)/2} ds^1 \\
x \int_{\mathbb{R}^3} \min \left\{ \alpha(x^s, v^1), \alpha(y^s, v^1) \right\}^\beta du \\
\lesssim O(\varepsilon^{-1}) ||\alpha\nabla_x f||_{\infty} \int \mu^{1/4}(v^1) \lesssim O(\varepsilon^{-1}) ||\alpha\nabla_x f||_{\infty}.
$$

For (7.129) from (2.103), we have $\nabla_{\nu^1} k(v^1, u) \lesssim \frac{(v^1)k(v^1, u)}{|v^1-u|}$. Then by (4.11) in Lemma 4.1 we have

$$
(7.129) \lesssim O(\varepsilon^{-1}) ||\alpha\nabla_x f||_{\infty} \int |n(x_b(y, v)) \cdot v^1 |\sqrt{\mu(v^1)}(v^1) du \\
x \int e^{-v^1(t^1-s^1)} ds^1 \int_{\mathbb{R}^3} \frac{k_{\Phi}(v^1, u)}{|v^1-u||\min\{\alpha(x^s, v^1), \alpha(y^s, v^1)\}|^\beta du \\
\lesssim O(\varepsilon^{-1}) ||\alpha\nabla_x f||_{\infty} \int \frac{|n(x_b(y, v)) \cdot v^1 |\mu^{1/4}(v^1)}{\min\{\alpha(x_b(x, v^1), \alpha(x_b(y, v), v^1)\}|^\beta \\
\lesssim O(\varepsilon^{-1}) ||\alpha\nabla_x f||_{\infty}.
$$
For (7.130), since $|\nabla_v t G(x_b(y, v) - (t^1 - s^1)v^1)| \lesssim \|\xi\|_{C^2}(t^1 - s^1)$, and $(t^1 - s^1)e^{-v(t^1)(t^1 - s^1)} \lesssim e^{-v(t^1)(t^1 - s^1)/2}$, we have

$$\tag{7.130} \lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty \int \mu^{1/4}(v^1)dv^1 \lesssim O(\varepsilon^{-1})\|\xi\|_{C^2}\|\alpha\nabla_x f\|_\infty.$$  

For (7.131), since we consider $t^1 - s^1 \geq \varepsilon$, min$\{t^2_b(x), t^2_b(y)\} \geq \varepsilon$. From (2.32) we have

$$\nabla_v \min \{t^2_b(x), t^2_b(y)\} e^{-v(t^1)} \min \{t^2_b(x), t^2_b(y)\} \lesssim O(\varepsilon^{-1}).$$

Denote

$$x^b = x_b(x, v) - \min \{t^2_b(x), t^2_b(y)\}v^1, \quad y^b = x_b(y, v) - \min \{t^2_b(x), t^2_b(y)\}v^1.$$  

Using (2.32) and from (4.10) in Lemma 4.1 we have

$$\tag{7.131} \lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty \int \frac{|n(x_b(y, v)) \cdot v^1|}{|n(x_b(y, v)) \cdot v^1|} \sqrt{\mu(v^1)}$$

$$\times \int_{\mathbb{R}^3} \min \{\alpha(x^b, u), \alpha(y^b, u)\} \leq O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty \int \sqrt{\mu(v^1)} \lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty.$$  

Thus the contribution of (7.127)$_2$ in (7.126) is bounded by

$$O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty.$$  

Then we obtain

$$|\tag{7.126}| \lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty + O(\varepsilon)\|\nabla_x f\|_\infty C_{\varepsilon:1+\beta}.$$  

Then we consider $h = \Gamma(f, f)$. We use (7.13) in Lemma 7.2 and (7.123) and (7.78) to obtain

$$\int |n(x_b(y, v)) \cdot v^1| \mu(v^1) \int_{t^1}^{t^1} ds^1 e^{-v(t^1)(t^1 - s^1)}$$

$$\times \frac{|G(x_b(x, v))\nabla_x \Gamma(f, f)(x^s, v^1) - G(x_b(y, v))\nabla_x \Gamma(f, f)(y^s, v^1)|}{|x_b(x, v) - x_b(y, v)|^\beta}$$

$$\lesssim (\|\alpha\nabla_x f\|_\infty^2 + \|\nabla f\|_\infty \|\nabla_x f\|_{C_{\varepsilon:1+\beta}}^2)$$

$$\times \int \sqrt{\mu(v^1)} \frac{|n(x_b(y, v)) \cdot v^1|}{|v^1|^2 \min \{\alpha(x_b(x, v), v^1), \alpha(x_b(y, v), v^1)\}^{1+\beta}} dv^1$$

$$\lesssim \|\alpha\nabla_x f\|_\infty^2 + O(1)\|\nabla_x f\|_{C_{\varepsilon:1+\beta}}^2.$$  

$$\tag{7.134}$$
The last term is (7.26). This estimate is similar to the contribution of (7.28). Note that \(\sqrt{\mu(v^1)}\alpha(x_b(x, v), v^1) \lesssim \mu^{1/4}(v^1)\alpha(x_b(x, v), v^1)\), we need to compute

\[
\int \mu^{1/4}(v^1)|n(x_b(y, v)) \cdot v^1| \int_0^{t_1} ds^1 e^{-v(s^1)(t^1-s^1)}
\]

\[
\times \int _{\mathbb{R}^3} du \alpha(x_b(x, v), v^1)|u|^{2} \min \left\{ \frac{\alpha(x^u,u)}{|u|}, \frac{\alpha(y^u,u)}{|u|} \right\}^{2+\beta} \left( \frac{|n(x_b(y, v), v^1)|}{|v^1|} \right)^{1+\beta}
\]

\[
\lesssim O(\varepsilon)[\nabla_x f(\cdot, v)]_{C_{x:2+\beta}}^{0,\beta} \int \mu^{1/4}(v^1) \int_{t^1-\varepsilon}^{t^1} ds^1 e^{-v(s^1)(t^1-s^1)} \int_{\mathbb{R}^3} du k(v^1, u) |u|^{-2} \min \left\{ \frac{\alpha(x^u,u)}{|u|}, \frac{\alpha(y^u,u)}{|u|} \right\}^{2+\beta} \left( \frac{|n(x_b(y, v), v^1)|}{|v^1|} \right)^{1+\beta}
\]

Again we first consider \(t^1-s^1 \leq \varepsilon\). We apply (4.8) in Lemma 4.1 with \(p = 2+\beta\) and (7.123) to obtain

\[
\left| (7.135) 1_{t^1-s^1 \leq \varepsilon} \right| \leq O(\varepsilon)[\nabla_x f(\cdot, v)]_{C_{x:2+\beta}}^{0,\beta} \int \mu^{1/4}(v^1) \int_{t^1-\varepsilon}^{t^1} ds^1 e^{-v(s^1)(t^1-s^1)} \int_{\mathbb{R}^3} du k(v^1, u) |u|^{-2} \min \left\{ \frac{\alpha(x^u,u)}{|u|}, \frac{\alpha(y^u,u)}{|u|} \right\}^{2+\beta} \left( \frac{|n(x_b(y, v), v^1)|}{|v^1|} \right)^{1+\beta}
\]

For \(t^1-s^1 \geq \varepsilon\), we apply the same integration by parts technique as in (7.128)--(7.131). The only difference is, here we do not have an extra term \(G(y^s)\). But this term does not play a role in the estimates of (7.128),(7.129) and (7.131). Thus for this case, we have the same upper bound as (7.132).

Combining (7.115),(7.119),(7.120),(7.122),(7.124), (7.133),(7.125), (7.134) and (7.135), we conclude that

\[
(7.95) \lesssim o(1) \left[ [\nabla_x f(\cdot, v)]_{C_{x:2+\beta}}^{0,\beta} + [\nabla_{\parallel} f(\cdot, v)]_{C_{x:1+\beta}}^{0,\beta} \right] + \left[ O(\varepsilon^{-2}) + O(\varepsilon^{-16}) \frac{|x-y|^\beta}{|n(x_b(x, v), v^1)|} \right] \|w^\beta_\alpha \nabla_x f\|_\infty.
\]

This, together with (7.96) and (7.92), leads to the conclusion:

\[
(7.24) \lesssim \frac{\|T_W - T_0\|_{C^2} \|T_W - T_0\|_{C^2}}{w^\beta_\alpha(v)|v|^{2} \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}} \times o(1) \left[ [\nabla_x f(\cdot, v)]_{C_{x:2+\beta}}^{0,\beta} + [\nabla_{\parallel} f(\cdot, v)]_{C_{x:1+\beta}}^{0,\beta} \right] + O(\varepsilon^{-2}) \|w^\beta_\alpha \nabla_x f\|_\infty,
\]

where we have applied (2.54) to \(\frac{|x_b(x, v) - x_b(y, v)|^\beta}{|x-y|^\beta}\).
Step 2: estimate of (7.25).

Now we estimate the contribution of the collision operator. First we consider $h = \Gamma(f, f)$. Applying (7.12) in Lemma 7.2 we have

$$\tag{7.25} \int_{h=\Gamma} \frac{o(1)[\nabla_x f(\cdot, v)]_{C^0, 1+\beta} + \|w_\beta \nabla_x f\|_\infty}{w_\beta(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{2+\beta}}. \quad (7.138)$$

Now we focus on the estimate for $h = K(f)$, which is

$$\int_{t-m(v)}^t ds e^{-v(t-s)} \int_{\mathbb{R}^3} du \frac{\partial_x f(x-(t-s)v, u) - \partial_x f(y-(t-s)v, u)}{|x-y|^\beta}. \quad (7.139)$$

Since $|x-y| = |x-(t-s)v - [y-(t-s)v]|$, we express (7.139) by (7.23)-(7.25). The contribution of (7.23) in (7.139) is bounded by

$$\frac{o(1)[\nabla_x f(\cdot, v)]_{C^0, 1+\beta} + \|w_\beta \nabla_x f\|_\infty}{w_\beta(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{2+\beta}} \times \int_{t-m(v)}^t ds e^{-v(t-s)} \int_{\mathbb{R}^3} du \frac{w_\beta(v)k(v, u)}{w_\beta(u)|u|^2 \min \left\{ \frac{\alpha(x-(t-s)v, u)}{|u|}, \frac{\alpha(y-(t-s)v, u)}{|u|} \right\}^{2+\beta}} \quad (7.140)$$

where we have applied Lemma 4.1.

Then we consider the contribution of (7.24) in (7.139). By (7.137), (2.122) and (4.7), such contribution is bounded by

$$\|T_W - T_0\|_{C^2} \frac{o(1)[\nabla_x f(\cdot, v)]_{C^0, 1+\beta} + \|\nabla f(\cdot, v)\|_{C^0, 1+\beta}}{w_\beta(v)} + O(e^{-21})\|\alpha \nabla f\|_\infty^2$$

$$\times \int_{t-m(v)}^t ds e^{-v(t-s)} \int_{\mathbb{R}^3} du \frac{w_\beta(v)k(v, u)}{w_\beta(u)|u|^2 \min \left\{ \frac{\alpha(x-(t-s)v, u)}{|u|}, \frac{\alpha(y-(t-s)v, u)}{|u|} \right\}^{1+\beta}}$$

$$\leq \|T_W - T_0\|_{C^2} \frac{o(1)[\nabla_x f(\cdot, v)]_{C^0, 1+\beta} + \|\nabla f(\cdot, v)\|_{C^0, 1+\beta}}{w_\beta(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}} + O(e^{-21})\|\alpha \nabla f\|_\infty^2. \quad (7.141)$$

Then we focus the contribution of the collision operator, the (7.25) in (7.139). We first estimate $h = \Gamma(f,f)$. By Lemma 7.2, such contribution in (7.139) is bounded by

$$\frac{1}{w_\beta(v)} \int_{t-m(v)}^t ds e^{-v(t-s)} \int_{\mathbb{R}^3} dw_\beta(v)k(v, u) \frac{o(1)[\nabla_x f(\cdot, v)]_{C^0, 1+\beta} + \|w_\beta \nabla_x f\|_\infty^2}{w_\beta(u)|u|^2 \min \left\{ \frac{\alpha(x,u)}{|u|}, \frac{\alpha(y,u)}{|u|} \right\}^{2+\beta}}$$
\[
\frac{o(1)[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,1+\beta}} + \|w_\beta \alpha \nabla_x f\|_\infty^2}{w_\beta (v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|y|} \right\}^{2+\beta}},
\]  
(7.142)

where we have used Lemma 4.1 and Lemma 2.13.

Then we estimate \( h = K(f) \), which is the most delicate one. We denote \( t_m^s(u) = \min \{t_h(x - (t - s)v, u), t_h(y - s(t - s)v, u) \} \). We need to compute

\[
\int_{t - t_m(v)}^{t} ds e^{-v(t-s)} \int_{\mathbb{R}^3} dw_k(v, u) \int_{s - t_m(u)}^{s} ds' e^{-v(s-s')} \int_{\mathbb{R}^3} du \times k(u, u') \nabla_x f(x - (t - s)v - (s - s')u, u') - \nabla_x f(y - (t - s)v - (s - s')u, u') 
\]

\[
\frac{|x - y|^\beta}{|x - y|^\beta} \]  
(7.143)

We first decompose the \( s' \) integration as

\[
\int_{s - \varepsilon}^{s} ds' sps + \int_{0}^{s - \varepsilon} ds'.
\]
(7.144)

Applying (4.8) in Lemma 4.1 with \( p = 2 + \beta \) we conclude that the contribution of (7.144)\(_1\) in (7.143) is bounded by

\[
\frac{[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,2+\beta}}}{w_\beta (v)} \int_{t_m(s)}^{t} ds e^{-v(t-s)} \int_{\mathbb{R}^3} dw_k(v, u) \int_{s - t_m(u)}^{s} ds' e^{-v(s-s')} \int_{\mathbb{R}^3} du \times k(u, u') w_\beta (u) 
\]

\[
\int_{\mathbb{R}^3} du' w_\beta (u')|u'|^2 \min \left\{ \frac{\alpha(x-(t-s)v-(s-s')u,u')}{|u'|}, \frac{\alpha(y-(t-s)v-(s-s')u,u')}{|u'|} \right\}^{2+\beta} O(\varepsilon)[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,2+\beta}}
\]

\[
\frac{1}{w_\beta (v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|y|} \right\}^{2+\beta}}.
\]

Then we consider contribution of (7.144)\(_2\). For simplicity we denote

\[
x'' = x - (t - s)v - (s - s')u, \quad y'' = y - (t - s)v - (s - s')u,
\]

\[
x'' - y'' = x - y.
\]
(7.145)

We exchange \( \nabla_x \) for \( \nabla_u \):

\[
\nabla_x f(x - (t - s)v - (s - s')u, u') - \nabla_x f(y - (t - s)v - (s - s')u, u') = \nabla_u[f(x'', u') - f(y'', u')] \frac{-1}{s - s'}.
\]

Since \( s - s' \geq \varepsilon \) the contribution of (7.144)\(_2\) in (7.143) is

\[
\int_{t - t_m(v)}^{t} ds e^{-v(t-s)} \int_{\mathbb{R}^3} dw_k(v, u) \int_{s - t_m(u)}^{s - \varepsilon} e^{-v(s-s')} ds' \frac{1}{s - s'}
\]

\[
\times \int_{\mathbb{R}^3} du' k(u, u') \nabla_u[f(x'', u') - f(y'', u')] \frac{-1}{|x - y|^\beta}.
\]
(7.146)
Then we integrate by part for $d\nu$ to have

\begin{equation}
\int_t^{t_m(v)} dt \int_{\mathbb{R}^3} d\nu e^{-\nu(t-s)} \int_{\mathbb{R}^3} d\nu \mathbf{1}_{s-s' \geq \epsilon} \times \left[ \nabla_\nu [k(v, u)k(u, u')] \int_{s-t_m(v)}^{s-e} e^{-\nu(u)(s-s')} ds' \int_{\mathbb{R}^3} d\nu' f(x'', u') - f(y'', u') \right. \left. \frac{|x-y|^{\beta}}{|x-y|^{\beta}} \right]
\end{equation}

(7.147)

\begin{align}
+ k(v, u) \int_{s-t_m(v)}^{s-e} \nabla_\nu e^{-\nu(u)(s-s')} ds' \int_{\mathbb{R}^3} d\nu' k(u, u') f(x'', u') - f(y'', u') \frac{|x-y|^{\beta}}{|x-y|^{\beta}} \right]
\end{align}

(7.148)

\begin{align}
+ k(v, u) \nabla_\nu t_m^2(u) e^{-\nu t_m^2(u)} \int_{\mathbb{R}^3} d\nu' k(u, u') f(x, u') - f(y, u') \right]
\end{align}

(7.149)

Here we denoted

\begin{equation}
x^b = x - (t-s)v - t_m^2(u)u, \quad y^b = y - (t-s)v - t_m^2(u)u.
\end{equation}

(7.150)

First we estimate (7.147). We begin with $\nabla_\nu k(u, u')$. Since $w_\theta^{-1}(u')|u'|^2 \lesssim 1$, from (2.64) with (7.6) and (2.103) we have

\begin{align}
&|\int_{\mathbb{R}^3} d\nu \mathbf{1}_{s-s' \geq \epsilon} \times \left[ \nabla_\nu [k(v, u)k(u, u')] \int_{s-t_m(v)}^{s-e} e^{-\nu(u)(s-s')} ds' \int_{\mathbb{R}^3} d\nu' f(x'', u') - f(y'', u') \right. \left. \frac{|x-y|^{\beta}}{|x-y|^{\beta}} \right]
\end{align}

(7.147)

\begin{align}
\lesssim \left[ \frac{\alpha(x'', u') \alpha(y'', u')}{\min \{\alpha(x'', u'), \alpha(y'', u')\}} \right]
\end{align}

(7.151)

where we have used (4.11), (2.122) and (2.123) in the fifth line, (4.10) in the last line.

The term with $\nabla_\nu k(v, u)$ can be similarly bounded by

\begin{align}
\frac{\alpha(x, v)}{|v|} \quad \frac{\alpha(y, v)}{|v|}
\end{align}

(7.152)
Then we estimate (7.148). From (2.64) with (7.6) we have

\[
| (7.148) | \leq \frac{\| w_\tilde{\theta} \alpha \nabla_x f \|_\infty}{w_\tilde{\theta}(v)} \int_0^t ds e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} du \, w_\tilde{\theta}(u) k(v, u) \int_0^{s-\varepsilon} e^{-\nu(u)(s-s')} ds' \\
\times \int_{\mathbb{R}^3} du' |u'|^2 w_\tilde{\theta}^{-1}(u') \frac{w_\tilde{\theta}(u) k(u, u')}{w_\tilde{\theta}(u') |u'|^2} \min \{ \alpha(x'', u'), \alpha(y'', u') \}^\beta \\
\lesssim \frac{\| w_\tilde{\theta} \alpha \nabla_x f \|_\infty}{w_\tilde{\theta}(v) |v|^2} \int_0^t ds e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} du \, \frac{1}{|u|^2} k_\tilde{\theta}(v, u) \\
\lesssim \frac{\| w_\tilde{\theta} \alpha \nabla_x f \|_\infty}{w_\tilde{\theta}(v) |v|^2 \min \{ \alpha(x, v, |v|), \alpha(y, v, |v|) \}^{2+\beta}},
\]  

(7.153)

where we have used Lemma 2.13 in the second line and (4.10) in Lemma 4.1 in the fourth line.

Last we estimate (7.149). Since we are considering \( s-s' \geq \varepsilon \), we have \( t_b(x-(t-s)v, u) \geq \varepsilon \). From (2.64) with (7.6) and (2.32), we have

\[
| (7.149) | \leq \frac{\| w_\tilde{\theta} \alpha \nabla_x f \|_\infty}{w_\tilde{\theta}(v)} \int_{t-m(v)}^t ds e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} du \, w_\tilde{\theta}(u) k(v, u) \frac{e^{-\nu(u)t_m'(u)}}{t_m'(u)} \\
\times \nabla u t_m'(u) \int_{\mathbb{R}^3} du' |u'| w_\tilde{\theta}^{-1}(u') \frac{w_\tilde{\theta}(u) k(u, u')}{w_\tilde{\theta}(u') |u'|^2} \min \{ \alpha(x'^b, u'), \alpha(y'^b, u') \}^\beta \\
\lesssim \frac{O(\varepsilon^{-1}) \| w_\tilde{\theta} \alpha \nabla_x f \|_\infty}{w_\tilde{\theta}(v)} \int_{t-m(v)}^t ds e^{-\nu(v)(t-s)} \\
\times \int_{\mathbb{R}^3} du |u|^2 \min \{ \alpha(x-(t-s)v, u, |u|), \alpha(y-(t-s)v, u, |u|) \} \frac{1}{|u|^2} k_\tilde{\theta}(v, u) \\
\lesssim \frac{O(\varepsilon^{-1}) \| w_\tilde{\theta} \alpha \nabla_x f \|_\infty}{w_\tilde{\theta}(v) |v|^2 \min \{ \alpha(x, v, |v|), \alpha(y, v, |v|) \}^{2+\beta}},
\]  

(7.154)

where we have used Lemma 2.13 in the fourth line and (4.10) in Lemma 4.1 in the last line.

Then combining (7.151), (7.152), (7.153) and (7.154) we conclude

\[
| (7.143) | \lesssim \frac{o(1) \| \nabla_x f (\cdot, v) \|_{C^{0,\beta}} + O(\varepsilon^{-1}) \| w_\tilde{\theta} \alpha \nabla_x f \|_\infty}{w_\tilde{\theta}(v) |v|^2 \min \{ \alpha(x, v, |v|), \alpha(y, v, |v|) \}^{2+\beta}}.
\]  

(7.155)
Combining (7.155), (7.142), (7.141), (7.140) and (7.138) we conclude that

\[(7.25) \lesssim \frac{\|T_W - T_0\|_{C^2}}{w_{\bar{\theta}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{2+\beta}} \times \left[ o(1) \left[ \nabla_x f(\cdot,v) \right]_{C^{0,\beta},x,2+\beta} + \left[ \nabla_y f(\cdot,v) \right]_{C^{0,\beta},y,1+\beta} \right] + O(\varepsilon^{-21}) \|w_{\bar{\theta}}\alpha \nabla_x f\|_\infty^2 \right]. \tag{7.156} \]

Finally from (7.23)–(7.25) and the estimate (7.156), (7.137), we conclude the proof of (7.7).

**Step 3: proof of (7.8).**

Now we prove (7.8). From Lemma 7.3, (7.24) is already bounded from (7.137).

For (7.26), since $w_{\bar{\theta}}^{-1}(v)\bar{\alpha}(x,v) \lesssim w_{\bar{\theta}/2}^{-1}(v)\alpha(x,v)$, by (7.156) we conclude

\[(7.26) \lesssim \frac{\bar{\alpha}(x,v)}{|v|} \times (7.156) \]

\[= \frac{o(1) \left[ \nabla_x f(\cdot,v) \right]_{C^{0,\beta},x,2+\beta} + \left[ \nabla_y f(\cdot,v) \right]_{C^{0,\beta},y,1+\beta} \right] + O(\varepsilon^{-21}) \|w_{\bar{\theta}}\alpha \nabla_x f\|_\infty^2}{w_{\bar{\theta}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}. \tag{7.157} \]

Then we only need to estimate (7.28). First we consider $h = \Gamma(f, f)$. Such contribution is directly bounded using (7.13) in Lemma 7.2, thus

\[(7.28)_{h=\Gamma} \lesssim \frac{o(1) \left[ \nabla_x f(\cdot,v) \right]_{C^{0,\beta},x,1+\beta} + \left[ \nabla_y f(\cdot,v) \right]_{C^{0,\beta},y,1+\beta} \right] + O(\varepsilon^{-21}) \|w_{\bar{\theta}}\alpha \nabla_x f\|_\infty^2}{w_{\bar{\theta}/2}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}. \tag{7.158} \]

Then we consider $h = K(f)$, which reads

\[\int_{t-t_m(v)}^t ds e^{-v(t-s)} \int_{\mathbb{R}^3} d\mu(v,u) \times \frac{G(x - (t-s)v)\nabla_x f(x - (t-s)v,u) - G(y - (t-s)v)\nabla_x f(y - (t-s)v,u)}{|x - y|^\beta}. \tag{7.159} \]

We express (7.159) by (7.26)-(7.29) along $u$.

Note that

\[(7.26) \lesssim (7.157), \quad (7.29) \lesssim (7.137), \]

we conclude that the contribution of (7.26),(7.27) and (7.29) in (7.159) are bounded by
where we have used (4.7) in Lemma 4.1 with $p = 1 + \beta$.

Then we focus on the contribution of the double collision operator (7.28). By Lemma 7.2 the contribution of $h = \Gamma$ is bounded by

$$\frac{1}{w_{\tilde{\theta}/2}(v)} \int_{t-L_m(v)}^{t} ds e^{-v(t-s)} \int_{\mathbb{R}^3} du w_{\tilde{\theta}/2}(v) k(v, u) \times \frac{o(1)[\nabla f (\cdot, v)]_{C_{x,1+\beta}^{0,\beta}} + \|w_{\tilde{\theta}} \alpha \nabla f\|_\infty^2}{w_{\tilde{\theta}/2}(u)|u|^2 \min \left\{ \frac{\alpha(x - (t-s) v, u)}{|u|}, \frac{\alpha(y - (t-s) v, u)}{|u|} \right\}^{1+\beta}}.$$  \hspace{1cm} (7.160)

Last we focus on the contribution of $h = K(f)$. Recall the notation $x''$, $y''$ in (7.145). We need to compute

$$\int_{t-L_m(v)}^{t} ds e^{-v(t-s)} \int_{\mathbb{R}^3} du k(v, u) \int_{s-L_m(u)}^{s} ds' e^{-v(s-s')} \int_{\mathbb{R}^3} du' k(u', u') \times \frac{G(x'') \nabla f (x'', u') - G(y'') \nabla f (y'', u')}{|x - y|^{\beta}}.$$  \hspace{1cm} (7.162)

We apply the decomposition (7.144) for $ds'$.

When $s - s' \leq \varepsilon$, by (4.8) in Lemma 4.1 with $p = 1 + \beta$ we have

$$\int_{t-L_m(v)}^{t} ds e^{-v(t-s)} \int_{\mathbb{R}^3} du \frac{w_{\tilde{\theta}/2}(v)k(v, u)}{w_{\tilde{\theta}/2}(u)} \int_{s-L_m(u)}^{s} ds' e^{-v(s-s')} \times \int_{\mathbb{R}^3} du' w_{\tilde{\theta}/2}(u')k(u', u') \frac{[\nabla f (\cdot, v)]_{C_{x,1+\beta}^{0,\beta}}}{w_{\tilde{\theta}/2}(u')|u'|^2 \min \{\alpha(x'', u'), \alpha(y'', u')\}^{1+\beta}}.$$
\[
\lesssim O(\varepsilon) \frac{[\nabla \parallel f(\cdot, v)]_{C^{0, \beta}}_{x, 1+\beta}}{w_{\theta/2}(v)|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}}.
\]  

(7.163)

When \( s - s' \geq \varepsilon \), we rewrite

\[
(7.162)_s = \frac{[G(x'') - G(y'')] \nabla_x f(x'', u')}{|x - y|^\beta} + \frac{G(y'')[\nabla_x f(x'', u') - \nabla_x f(y'', u')]}{|x - y|^\beta}.
\]  

(7.164)

(7.165)

For (7.164), since \( |u'|w_{\theta}^{-1}(u') \lesssim w_{\theta/2}^{-1}(u') \), we apply (2.57) to conclude that the contribution of (7.164) in (7.162) is bounded by

\[
\lesssim \frac{\|w_{\theta/2}^{-1} \nabla_x f\|_{\infty}}{w_{\theta/2}(v)|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}},
\]  

(7.166)

where we have applied Lemma 2.13 and (4.5) in Lemma 4.1.

For (7.165), we exchange \( \nabla_x f(x'', u') \) and perform an integration by parts to \( du \). Since \( |G(y'')| \lesssim 1 \), the contribution of (7.165) in (7.162) is bounded by (7.147),(7.148),(7.149) and with an extra term that corresponds to the derivative of \( G(y'') \):

\[
\lesssim \frac{\|w_{\theta/2}^{-1} \nabla_x f\|_{\infty}}{w_{\theta/2}(v)|v|^2 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}},
\]
Here we have used (2.64) with (7.6) in the fourth line and Lemma 2.13 in the second last line.

Thus the contribution of (7.165) in (7.162) is bounded by

\[
(7.147) + (7.148) + (7.149) + (7.167) \\
\lesssim (7.151) + (7.153) + (7.154) + (7.167)
\]

\[
\lesssim \frac{O(\varepsilon^{-1})\|w_{\delta}^{\alpha}\nabla_{x}f\|_{\infty}}{w_{\delta/2}(v)|v|^{2} \min\left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}.
\]

(7.168)

This, together with (7.166) and (7.163), leads to the conclusion:

\[
(7.162) \lesssim \frac{o(1)\left[\left|\nabla_{x}f(\cdot,v)\right|_{C^{0,\beta}_{x,2+\beta}} + \left|\nabla_{x}f(\cdot,v)\right|_{C^{0,\beta}_{x,1+\beta}}\right] + O(\varepsilon^{-21})\|w_{\delta}^{\alpha}\nabla_{x}f\|_{\infty}^{2}}{w_{\delta/2}(v)|v|^{2} \min\left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}.
\]

(7.169)

Finally collecting (7.169),(7.161),(7.160),(7.158),(7.157) and Lemma 7.3 we conclude the proof of (7.8).

\[\square\]

7.2. Proof of (1.23)

In this section we prove the Hölder regularity (1.23). For simplicity, we denote

\[
\left[\nabla_{v}f_{S}(\cdot,v)\right]_{C^{0,\beta}_{x,1+\beta}} := \sup_{x,y \in \Omega} \left\|w_{\delta/2}(v)|v|^{3} \min\left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta} \left|\nabla_{v}f_{S}(x,v) - \nabla_{v}f_{S}(y,v)\right| \right\|_{L_{\infty}^{\infty}}.
\]

(7.170)

We will use (7.6) for simplifying the proof.

Proof of (1.23) Similaly to (7.30) - (7.35) and (6.1) - (6.8), we use

\[t_{m}(v) = \min\{t_{b}(x,v), t_{b}(y,v)\}\]

to express \(\nabla_{v}f_{S}(x,v)\) as

\[\nabla_{v}f_{S}(x,v) = 1_{t \geq t_{m}} e^{-\nu t} \nabla_{v}[f_{S}(x_{b}(x,v),v) - f_{S}(x_{b}(x,v),v)]\]

\[\quad \quad \quad - 1_{t \leq t_{m}} v \nabla_{v}t_{b}(x,v) e^{-\nu t} f_{S}(x_{b}(x,v),v)\]
\[\quad \quad \quad - 1_{t \leq t_{m}} \nabla_{v}v(t_{b}(x,v)) e^{-\nu t} f_{S}(x_{b}(x,v),v)\]
\[\quad \quad \quad + 1_{t \leq t_{m}} e^{-\nu t} \nabla_{v}[f_{S}(x-t,v,v)]\]
\[\quad \quad \quad - 1_{t \leq t_{m}} \nabla_{v}v(t)e^{-\nu t} f_{S}(x-t,v,v)\]
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\[
- \int_{t-t_b(x,v)}^{t} \nabla_v v(\nu) e^{-\nu(t-s)} (t-s) h(x-(t-s)v, v) ds \\
+ \int_{t-t_b(x,v)}^{t} e^{-\nu(t-s)} \nabla_v [h(x-(t-s)v, v)] ds \\
- \int_{0}^{t} \nabla_v v(\nu) e^{-\nu(t-s)} (t-s) h(x-(t-s)v, v) ds \\
+ \int_{0}^{t} e^{-\nu(t-s)} \nabla_v [h(x-(t-s)v, v)] ds \\
- \nabla_v h(x, v) e^{-\nu h(x, v)} h(x-t_b(x, v), v). 
\]

Taking the difference of \(\nabla_v f_x(x, v)\) and \(\nabla_v f_x(y, v)\) using the above equation we have

\[
\frac{\nabla_v f_x(x, v) - \nabla_v f_x(y, v)}{|x-y|^\beta} = \int_{t \geq t_m} \frac{e^{-\nu h(x, v)} - e^{-\nu h(y, v)}}{|x-y|^\beta} \nabla_v f_x(x_b(x, v), v) \quad (7.171)
\]

\[
+ \int_{t \geq t_m} e^{-\nu h(y, v)} \frac{\nabla_v [f_x(x_b(y, v), v)] - \nabla_v [f_x(x_b(x, v), v)]}{|x-y|^\beta} \quad (7.172)
\]

\[
- \int_{t \geq t_m} \frac{\nabla_v f_x(y, v) - \nabla_v f_x(x, v) + \nabla_v v(\nu)(y-x) - \nabla_v v(\nu)(x-y)}{|x-y|^\beta} \quad (7.173)
\]

\[
- \int_{t \geq t_m} \frac{\nabla_v f_x(y, v) + \nabla_v v(\nu)(y-x) - \nabla_v f_x(x, v)}{|x-y|^\beta} \quad (7.174)
\]

\[
+ \int_{t \leq t_m} e^{-\nu(t-s)} \frac{\nabla_v [f_x(x-tv, v)] - \nabla_v [f_x(y-tv, v)]}{|x-y|^\beta} \quad (7.175)
\]

\[
- \int_{t \leq t_m} \frac{\nabla_v v(\nu)e^{-\nu(t-s)} f_x(x, v) - f_x(y, v)}{|x-y|^\beta} \quad (7.176)
\]

\[
- \int_{t \leq t_m} \frac{\nabla_v v(\nu)e^{-\nu(t-s)} f_x(x, v) - f_x(y, v)}{|x-y|^\beta} \quad (7.177)
\]

\[
- \int_{t \geq t_m} \frac{1}{|x-y|^\beta} \int_{t-t_m}^{t-t_m} \frac{h(x-(t-s)v, v) - h(y-(t-s)v, v)}{|x-y|^\beta} ds \\
\quad \times \left[ 1_{t-b_h(x,v) \geq t-b_h(y,v)} \nabla_v v(\nu)e^{-\nu(t-s)} h(x-(t-s)v, v) \\
+ 1_{t-b_h(x,v) \leq t-b_h(y,v)} \nabla_v v(\nu)e^{-\nu(t-s)} h(y-(t-s)v, v) \right] ds \quad (7.178)
\]

\[
+ \int_{t \leq t_m} \frac{e^{-\nu(t-s)} \nabla_v [h(x-(t-s)v, v)] - \nabla_v [h(y-(t-s)v, v)]}{|x-y|^\beta} ds 
\quad (7.179)
\]
+ \frac{1}{t-m} \int_{t-m}^{t} \frac{1}{|x-y|^\beta} \left[ \sum_{i=1}^{r} \int_{\min_i}^{t} e^{-|t-s|} \nabla_v[h(x-(t-s)v,v)] \right] ds \\
+ \frac{1}{t-m} \int_{t-m}^{t} \frac{1}{|x-y|^\beta} \left[ \sum_{i=1}^{r} \int_{\min_i}^{t} e^{-|t-s|} \nabla_v[h(y-(t-s)v,v)] \right] ds \\
- \frac{1}{t-m} \int_{t-m}^{t} \frac{1}{|x-y|^\beta} \left[ \sum_{i=1}^{r} \int_{\min_i}^{t} e^{-|t-s|} h(x-(t-s)v,v) - h(y-(t-s)v,v) \right] ds \\
(7.181) \\
(7.182) \\
(7.183) \\
(7.184) \\
(7.185) \\

The estimate of (7.172), (7.180) and (7.183) are the most delicate, we will estimate them in Step 2 and Step 3. Now we estimate the rest terms in Step 1.

**Step 1: Estimate of the rest of the terms.**

We will use the estimate

$$ \frac{|v|^c}{w_{\tilde{\theta}}(v)} \lesssim \frac{1}{w_{\tilde{\theta}/2}(v)|v|^3}, $$

for $c \geq -2$. 

(7.186)

For (7.171) we apply (2.56), (2.40) and (7.186) to have

$$ |(7.171)| \lesssim \frac{1}{|v| \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}} \left[ |\nabla_v \cdot \chi_b(x,v)||\nabla_x f_s(\chi_b(x,v),v)| + |\nabla_v f_s(x_b(x,v),v)| \right] \\
\lesssim \frac{1}{w_{\tilde{\theta}/2}(v)|v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}} \frac{1+\beta}{\|w_{\tilde{\theta}}\nabla_x f_s\|_{\infty}} \\
+ \|w_{\tilde{\theta}/2}(v)|v|^2 \nabla_v f_s \|_{\infty}. $$

For (7.173) we apply (2.61) and (2.56) to have

$$ |(7.173)| \lesssim \|w f_s\|_{\infty} \frac{1}{w_{\tilde{\theta}}(v)|v| \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}}^{1+\beta}. $$

For (7.174) we apply (2.39), (2.40) and (2.56) to have

$$ |(7.174)| \lesssim \|w f_s\|_{\infty} \frac{1}{w_{\tilde{\theta}}(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}}^\beta. $$
For (7.175) we apply (2.39), (2.40), (2.54) and (2.64) to have

\[
| (7.175) | \lesssim \frac{1}{|v|} \frac{1}{\|w_f\|_{\infty}} \frac{1}{w_\theta(v) \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{2\beta}}.
\]

For (7.176) we use \( t \leq t_m \) and (2.39) to have

\[
| (7.176) | \lesssim e^{-vt} \left[ t[\nabla_x f_s(x - tv, v) - \nabla_x f_s(y - tv, v)] + \nabla_v f_s(x - tv, v) - \nabla_v f_s(y - tv, v) \right]
\]

\[
\lesssim \frac{1}{w_\theta(v) |v| \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}} |x - y|^{\beta} + \frac{1}{w_\theta/2(v) |v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}} |x - y|^{1+\beta}.
\]

For (7.177) we apply (2.64) and (7.6) to have

\[
| (7.177) | \lesssim \frac{\|w_\theta \alpha \nabla_x f_s\|_{\infty}^2}{w_\theta(v) \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{\beta}}.
\]

For (7.178) we apply (7.9) to have

\[
| (7.178) | \lesssim \frac{\|w_f\|_{\infty}}{w_\theta(v) \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{\beta}}.
\]

For (7.179) we apply (2.56) and (2.124) to have

\[
| (7.179) | \lesssim \frac{\|w_f\|_{\infty}}{w_\theta(v)} \frac{|t_h(x,v) - t_h(y,v)|}{|x - y|^{\beta}} \lesssim \frac{\|w_f\|_{\infty}}{w_\theta(v) |v| \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{\beta}}.
\]

For (7.181) we apply the same computation in (6.16) and (6.15), and use (7.6), (2.56) to have

\[
| (7.181) | \lesssim \frac{|t_h(x,v) - t_h(y,v)|}{|x - y|^{\beta}} \frac{\|w_f\|_{\infty}}{w_\theta/2(v) |v|^2} \lesssim \frac{\|w_\theta \alpha \nabla_x f_s\|_{\infty}^2}{w_\theta/2(v) |v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{\beta}}.
\]

For (7.182) we apply (7.9) to have

\[
| (7.182) | \lesssim \frac{\|w_f\|_{\infty}}{w_\theta(v) \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{\beta}}.
\]

For (7.184), we apply (2.61), (2.56), (2.40) and (2.124) to have

\[
\lesssim \frac{\|w_f\|_{\infty}}{w_\theta(v)} \times \left[ \frac{|\nabla_v t_h(x,v) - \nabla_v t_h(y,v)|}{|x - y|^{\beta}} + \frac{|\nabla_v t_h(y,v)||e^{-vt_h(x,v)} - e^{-vt_h(y,v)}|}{|x - y|^{\beta}} \right]
\]

\[
\lesssim \frac{\|w_f\|_{\infty}}{w_\theta(v)} \times \left[ \frac{1}{|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}}^{1+\beta} + \frac{1}{|v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{\beta}} \right].
\]
For (7.185) we apply (2.40), (7.9) and (7.6) to have

\[ |(7.185)| \lesssim \frac{\|w_\beta \alpha \Sigma_u f_s\|_\infty^2}{w_\beta(v)|v|^2 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{\beta}}. \]

Combining all the estimates above we use (7.6), (7.186) and Proposition 3 to conclude

\[ (7.171) - (7.185) \] except (7.172), (7.180), (7.183)

\[ \lesssim \frac{\|w_\beta \alpha \Sigma_u f_s\|_\infty^2 + o(1)[\Sigma_v f_s(\cdot, v)]}{w_\beta/2(v)|v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}. \] (7.187)

Step 2: estimate of (7.172). We apply the boundary condition (2.25) with the notation (7.4) to have

\[ \Sigma_v f_s(x_b(x, v), v) \]
\[ = \nabla_v \left[ \frac{M_w(x_b(x, v), v)}{\sqrt{\mu_0(v)}} \int_{v_3 > 0} f_s(x_b(x, v), T^{t}_{x_p(x)} v^1) \sqrt{\mu_0(v^1)}v^1_3 dv^1 \right. \]
\[ + \left. r(x_b(x, v), v) \right] \]
\[ = \nabla_v \left[ \frac{M_w(x_b(x, v), v)}{\sqrt{\mu_0(v)}} \int_{v_3 > 0} f_s(x_b(x, v), T^{t}_{x_p(x)} v^1) \sqrt{\mu_0(v^1)}v^1_3 dv^1 \right. \]
\[ + \nabla_v [r(x_b(x, v), v)] \]
\[ + \frac{M_w(x_b(x, v), v)}{\sqrt{\mu_0(v)}} \int_{v_3 > 0} \nabla_v [f_s(x_b(x, v), T^{t}_{x_p(x)} v^1)] \sqrt{\mu_0(v^1)}v^1_3 dv^1. \] (7.189)

First we consider the contribution of (7.188) in (7.172), which equals

\[ \frac{\nabla_v \left[ \frac{M_w(x_b(x, v), v)}{\sqrt{\mu_0(v)}} \right] - \nabla_v \left[ \frac{M_w(x_b(y, v), v)}{\sqrt{\mu_0(v)}} \right]}{|x - y|^{\beta}} \]
\[ \times \int_{v_3 > 0} f_s(x_b(x, v), T^{t}_{x_p(x)} v^1) \sqrt{\mu_0(v^1)}v^1_3 dv^1 \]
\[ + \nabla_v \left[ \frac{M_w(x_b(x, v), v)}{\sqrt{\mu_0(v)}} \right] \]
\[ \times \int_{v_3 > 0} f_s(x_b(x, v), T^{t}_{x_p(x)} v^1) - f_s(x_b(y, v), T^{t}_{x_p(y)} v^1) \frac{|x - y|^{\beta}}{\sqrt{\mu_0(v^1)}v^1_3 dv^1}. \] (7.191)

For (7.191) we apply the definition in (1.4) to have

\[ \nabla_v \left[ \frac{M_w(x_b(x, v), v)}{\sqrt{\mu_0(v)}} \right] = \nabla_v \frac{e^{-\frac{|v|^2}{2T_w(x_b(x,v))}} e^{\frac{|v|^2}{2T_0}} T_0}{\sqrt{2\pi (T_w(x_b(x,v)))^2}}. \]
Then we apply (2.60), (2.54) and (2.40) to bound
\[
|\nabla_v \left( \frac{|v|^2}{2T_w(x_b(x, v))} \right) - \nabla_v \left( \frac{|v|^2}{2T_w(x_b(y, v))} \right)|
\]
\[
\lesssim |x - y|^{\beta} \left[ \left| \frac{T_w(x_b(x, v))}{T_w(x_b(y, v))} - 1 \right| |x_b(x, v) - x_b(y, v)|^{\beta} + \left| \frac{\nabla_v x_b(x, v)}{T_w(x_b(x, v))} - \frac{\nabla_v x_b(y, v)}{T_w(x_b(y, v))} \right| |x_b(x, v) - x_b(y, v)|^{\beta} \right]
\]
\[
\lesssim |x - y|^{\beta} \left[ \frac{\|T_w\|_{C^1} |v|}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}} + \frac{1}{|v| \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}} \right]^{1+\beta}.
\]
Similarly, we have
\[
|\nabla_v \left( \frac{1}{2T_w(x_b(x, v))} \right) - \nabla_v \left( \frac{1}{2T_w(x_b(y, v))} \right)|
\]
\[
\lesssim \left| \frac{T_w(x_b(x, v))}{T_w(x_b(y, v))} - 1 \right| |x_b(x, v) - x_b(y, v)|^{\beta} + \left| \frac{\nabla_v x_b(x, v)}{T_w(x_b(x, v))} - \frac{\nabla_v x_b(y, v)}{T_w(x_b(y, v))} \right| |x_b(x, v) - x_b(y, v)|^{\beta}
\]
\[
\lesssim \left[ \frac{\|T_w\|_{C^1} |v|}{\min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}} + \frac{1}{|v| \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}} \right]^{1+\beta}.
\]
We combine the estimate above to bound (7.191) as
\[
(7.191) \lesssim \frac{\|w f_s\|_{L^\infty}}{w_\beta(v)|v|^3 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}}^{1+\beta} \int_{v_3 > 0} \sqrt{\mu_0(v_1^4 v_1^4} dv^4
\]
\[
\lesssim \frac{\|w f_s\|_{L^\infty}}{w_\beta(v)|v|^3 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}}^{1+\beta},
\]
where we have used that for \( c \in \{1, 0, -1\} \) and \|T_w - T_0\|_{L^\infty} \ll 1,
\[
e^{-\frac{|v|^2}{2T_0(x_b(x, v))}} e^{\frac{|v|^2}{2T_0}} |v|^c \lesssim \frac{1}{w_\beta(v)|v|^3}.
\]
For (7.192) we apply the above estimate, the same computation as (2.100) and (7.89), (7.90), then use (2.54) and (7.6) to obtain

\[
(7.192) \lesssim \frac{\|w_\theta^p \alpha \nabla_x f_s\|_2^2}{w_\theta(v)|v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}. \tag{7.194}
\]

Then we consider the contribution of (7.189) in (7.172). By the definition of \( r \) in (2.6), we apply the same bound in (7.193) to bound such contribution as

\[
\frac{1}{w_\theta(v)|v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}. \tag{7.195}
\]

Then we consider the contribution of (7.190) in (7.172), which equals

\[
\frac{1}{\sqrt{\mu_0(v)}} \frac{M_w(x_b(x,v)) - M_w(x_b(y,v))}{|x-y|^\beta} \int_{v_1 > 0} \nabla_v[f_s(x_b(x,v), T_{x_{p_1(x)}^1}^i v^1)] \sqrt{\mu_0(v^1)} v_3^1 dv^1 (7.196)
\]

\[
+ \frac{M_w(x_b(y,v))}{\sqrt{\mu_0(v)}} \int_{v_1 > 0} \nabla_v[f_s(x_b(x,v), T_{x_{p_1(x)}^1}^i v^1)] - \nabla_v[f_s(x_b(y,v), T_{x_{p_1(y)}^1}^i v^1)] |x-y|^\beta \times \sqrt{\mu_0(v^1)} v_3^1 dv^1. \tag{7.197}
\]

Applying (2.71) and (2.54) we have

\[
(7.196) \lesssim \frac{\|w_\theta f_s\|_\infty + \|\alpha \nabla_x f_s\|_\infty}{\sqrt{\mu_0(v)}} M_w(x_b(x,y)) - M_w(x_b(y,v)) \frac{|x_b(x,v) - x_b(y,v)|^\beta}{|x-y|^\beta} |x_b(x,v) - x_b(y,v)|^\beta
\]

\[
\lesssim \frac{\|w_\theta f_s\|_\infty + \|\alpha \nabla_x f_s\|_\infty}{w_\theta(v) \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}. \tag{7.198}
\]

Here for the \( v^1 \) integral we apply the bound (6.12), (6.13) for (6.10).

Then we focus on (7.197). We use the notation (7.4) and (7.5) the partial \( x \) derivative, then we have

\[
(7.197) = \frac{M_w(x_b(y,v))}{\sqrt{\mu_0(v)}} \int_{v_1 > 0} dv^1 \sqrt{\mu_0(v^1)} v_3^1 \times \left[ \sum_{i=1,2} \frac{\nabla_v x_{p_1(x)}^1, \partial x_{p_1(x)}^1, f_s(\eta_{p_1(x)}^1(x_{p_1(x)}^1), T_{x_{p_1(x)}^1}^i v^1)}{|x-y|^\beta} \right]
\]

\[
- \frac{\nabla_v x_{p_1(y)}^1, \partial x_{p_1(y)}^1, f_s(\eta_{p_1(y)}^1(x_{p_1(y)}^1), T_{x_{p_1(y)}^1}^i v^1)}{|x-y|^\beta} \right) \tag{7.199}
\]
\[
\n\n\n\n\int_{x_1}^{I_1(x_i)} v^1 \nabla_T f_z(x_b(x, v), T_{x_1}^{I_1(x_i)} v^1) - \nabla_T f_z(x_b(y, v), T_{x_1}^{I_1(y)} v^1) \frac{|x - y|^\beta}{\mu_0(v)} \nabla_{x_1} \phi \sum_{i=0}^{\infty} \alpha_1 \partial_{x_1}^{I_1(x_i)} \phi(z) |x(z) - y| \beta
\]

For (7.199) we have

\[
(7.199) = \frac{M_{\tilde{w}}(x_b(y, v)) w_\tilde{w}(v)|v|^2}{\sqrt{\mu_0(v) w_\tilde{w}(v)|v|^2}} \int_{v_3 > 0} \nabla_1 \nu \sqrt{\mu_0(v_1)v_3}
\]

\[
\times \sum_{i=1,2} \left[ \nabla_T x_1^{I_1(x_1)}, i - \nabla_T x_1^{I_1(y)}, i \right] \frac{|x - y|^\beta}{|x - y|^\beta} \partial_1 x_1^{I_1(x_1), i} f_z(\eta p^{I_1}(x_1), T_{x_1}^{I_1(x_1)} v^1)
\]

\[
+ \frac{|x_b(x, v) - x_b(y, v)|^\beta}{|x_b(x, v) - x_b(y, v)|^\beta} \nabla_T x_1^{I_1(y), i}
\]

\[
\times \partial_1 x_1^{I_1(y), i} f_z(\eta p^{I_1}(x_1), T_{x_1}^{I_1(y)} v^1) - \partial_1 x_1^{I_1(y), i} f_z(\eta p^{I_1}(y), T_{x_1}^{I_1(y)} v^1) \bigg] \bigg] \bigg]
\]

By changing the \( dv^1 \) integral back to \( dv^1 \) integral in (2.25) and applying (2.60) and (2.40), we have

\[
(7.199)
\]

\[
\int_1 \frac{v_3}{w_\tilde{w}(v)|v|^3} \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\} 1^\beta
\]

\[
\times \int_{n(x, y, v) > 0} \||\alpha \nabla f_z|| \nu \sqrt{\mu_0(v)} |n(x_b(y, v))| v^1| \nabla_1 x_1^{I_1(x_1)} f_z(x_b(x, v), v^1) d^v_1
\]

\[
+ \frac{|x_b(x, v) - x_b(y, v)|^\beta}{|x_b(x, v) - x_b(y, v)|^\beta}
\]

\[
\times \int_{n(x, y, v) > 0} \sqrt{\mu_0(v)|v|} |n(x_b(y, v))| v^1| \partial_1 x_1^{I_1(x_1), i} f_z(x_b(y, v), v^1) d^v_1
\]

\[
\left[ \int_{n(x, y, v) > 0} \sqrt{\mu_0(v)|v|} |n(x_b(y, v))| v^1| \partial_1 x_1^{I_1(x_1), i} f_z(x_b(y, v), v^1) d^v_1 \right]
\]

Clearly

\[
(7.201) \lesssim \frac{\||\alpha \nabla f_z||_\infty}{w_\tilde{w}(v)|v|^3} \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\} 1^\beta
\]

The other term is the same as (7.95), which is bounded by (7.136). Thus by Proposition 3,

\[
(7.202) \lesssim \frac{\||\alpha \nabla f_z||_\infty}{w_\tilde{w}(v)|v|^3} \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\} 1^\beta
\]
For (7.200) we have

$$(7.200) \lesssim \frac{M_w(x_b(y, v))w_\tilde{g}(v)|v|^2}{\sqrt{\mu_0(v)w_\tilde{g}(v)|v|^2}} \left[ \int_{V_1^0} dv_1 \sqrt{\mu_0(v^1)v_3^1} \right]$$

$$\times \left| \nabla \cdot T'_{x_{p^1(x)}} - \nabla \cdot T'_{x_{p^1(y)}} \right| \left| v_1^1 \right| \| v^1_2 \nabla T'_{x_{p^1(y)}} \|_2$$

$$+ \left| \int_{V_1^0} dv_1 \sqrt{\mu_0(v^1)v_3^1} \nabla \cdot T'_{x_{p^1(y)}} \right| \left| v_1^1 \right| \| v^1_2 \nabla T'_{x_{p^1(y)}} \|_2$$

$$\left[ x \right] - y \right|^{\beta}.$$ (7.205)

Applying (2.69) we have

$$(7.205) \lesssim \frac{\| v_1^1_2 \nabla T'_{x_{p^1(y)}} \|_2}{w_\tilde{g}(v)|v|^3 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^{1+\beta}}.$$ (7.207)

For the other term (7.206), we exchange the $v$-derivative to $v^1$-derivative, then

$$(7.206) = \frac{1}{w_\tilde{g}(v)|v|^3} \int_{V_1^0} dv_1 \sqrt{\mu_0(v^1)v_3^1} \nabla \cdot T'_{x_{p^1(y)}} \| v^1_2 \nabla T'_{x_{p^1(y)}} \|_2$$

$$\left[ x \right] - y \right|^{\beta}.$$
\[
\lesssim \frac{1}{\omega_\theta(v)|v|^3} \times \left[ \frac{1}{\min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}} \right]^{1+\beta} \|v\|^2 \nabla v f_s \|_\infty + \frac{|x_b(x,v) - x_b(y,v)|^\beta}{|x-y|^\beta}
\]
\[
\times \int_{v_1>0} \tilde{\nu}^{1/4}(v^1) \left| f_s(x_b(x,v), T'_{x_b(y,v)} v^1) - f_s(x_b(y,v), T'_{x_b(y,v)} v^1) \right| \frac{1}{|x_b(x,v) - x_b(y,v)|^\beta}.
\]
\[
\lesssim \frac{\|v\|^2 \nabla v f_s \|_\infty + \|\alpha \nabla x f_s \|_\infty}{\omega_\theta(v)|v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}.
\] (7.208)

In the sixth last line we have used (2.66), (2.54), (2.41) and applied an integration by part for \(dv^1\). In the third last line, we have used (2.41). In the last line we have used (2.54) and applied the same computation as (7.89), (7.90).

Combining (7.193), (7.194), (7.195), (7.198), (7.203), (7.204), (7.207) and (7.208), we use (6) and (7.186) to conclude
\[
(7.172) \lesssim \frac{\|w_\theta \alpha \nabla x f_s \|_\infty^2}{w_\theta/2(v)|v|^3 \min \left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta}}.
\] (7.209)

**Step 3: estimate of (7.180) and (7.183).** We focus on (7.180), the estimate of (7.183) is the same. First we consider \(h = K(f_s)\), we compute
\[
\int_{t-t_m}^t e^{-v(t-s)} \int_{\mathbb{R}^3} \nabla_v [k(v,u) \{ f(x-(t-s)v,u) - f(y-(t-s)v,u) \}] ds
\]
\[
= \int_{t-t_m}^t e^{-v(t-s)} \int_{\mathbb{R}^3} \nabla_v [k(v,u) \{ f(x-(t-s)v,u) - f(y-(t-s)v,u) \}] ds
\]
\[
+ \int_{t-t_m}^t e^{-v(t-s)} \int_{\mathbb{R}^3} k(v,u) \nabla_v [ f(x-(t-s)v,u) - f(y-(t-s)v,u) ] ds.
\] (7.210)

For (7.210) we have
\[
(7.210) \lesssim \int_{t-t_m}^t e^{-v(t-s)} \int_{\mathbb{R}^3} k_\theta(v,u) \frac{\|\alpha \nabla x f_s \|_\infty^2}{w_\theta(v)|v-u| \min \left\{ \alpha(x-(t-s)v,u), \alpha(y-(t-s)v,u) \right\}^{1+\beta}} ds
\]
\[
\lesssim \int_{t-t_m}^t e^{-v(t-s)} \frac{\|\alpha \nabla x f_s \|_\infty^2}{w_\theta(v)|v-u| \min \left\{ \alpha(x,v), \alpha(y,v) \right\}^{1+\beta}} ds
\]
\[
\lesssim \frac{\|\alpha \nabla x f_s \|_\infty^2}{w_\theta(v) \min \left\{ \alpha(x,v), \alpha(y,v) \right\}^{1+\beta}}.
\] (7.212)
\[
\begin{align*}
= \int_{t-t_m}^t e^{-v(t-s)} \int_{\mathbb{R}^3} du \\
\times \frac{(t-s)k(v, u)|\nabla_x f(x-(t-s)v, u) - \nabla_x f(y-(t-s)v, u)|}{|x-y|^\beta} ds \\
= \int_{t-t_m}^t e^{-v(t-s)} \int_{\mathbb{R}^3} du t_m k(v, u) \\
\times \frac{w_\theta(v)[\nabla_x f_\delta(\cdot, v)]_{C^{0, \beta}_{x, 2+\beta}}}{w_\theta(v)|u|^2 \min \{\frac{\alpha(x-(t-s)v, u)}{|u|}, \frac{\alpha(y-(t-s)v, u)}{|u|}\}^{2+\beta}} ds \\
\leq \frac{\|w_\theta \nabla_x f_\delta\|_\infty^2}{w_\theta(v)|v|^3 \min \{\frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|}\}^{1+\beta}}. \tag{7.213}
\end{align*}
\]

In the second line we have used \((t-s) \leq t_m\). In the last two lines we have used (4.7), (2.39) and Lemma 2.13.

Next we consider \(h = \Gamma(f_s, f_s)\). We use (2.117) and (2.116) to compute

\[
\begin{align*}
\int_{t-t_m}^t e^{-v(t-s)} \nabla_x [\Gamma(f_s, f_s)(x-(t-s)v, v) - \Gamma(f_s, f_s)(x-(t-s)v, v)] ds \\
= \int_{t-t_m}^t e^{-v(t-s)} \frac{\Gamma(\nabla_v f_s, f_s)(x-(t-s)v, v) - \Gamma(\nabla_v f_s, f_s)(y-(t-s)v, v)}{|x-y|^\beta} ds \tag{7.214} \\
+ \int_{t-t_m}^t e^{-v(t-s)} \Gamma(f_s, \nabla_v f_s)(x-(t-s)v, v) - \Gamma(f_s, \nabla_v f_s)(y-(t-s)v, v) ds \tag{7.215} \\
+ \int_{t-t_m}^t e^{-v(t-s)} (t-s) \frac{\Gamma(\nabla_x f_s, f_s)(x-(t-s)v, v) - \Gamma(\nabla_x f_s, f_s)(y-(t-s)v, v)}{|x-y|^\beta} ds \tag{7.216} \\
+ \int_{t-t_m}^t e^{-v(t-s)} (t-s) \frac{\Gamma(\nabla_v f_s, f_s)(x-(t-s)v, v) - \Gamma(\nabla_v f_s, f_s)(y-(t-s)v, v)}{|x-y|^\beta} ds \tag{7.217} \\
+ \int_{t-t_m}^t e^{-v(t-s)} \Gamma_v.gain(f_s, f_s)(x-(t-s)v, v) - \Gamma_v.gain(f_s, f_s)(y-(t-s)v, v) ds \tag{7.218} \\
- \int_{t-t_m}^t e^{-v(t-s)} \Gamma_v.loss(f_s, f_s)(x-(t-s)v, v) - \Gamma_v.loss(f_s, f_s)(y-(t-s)v, v) ds. \tag{7.219}
\end{align*}
\]

Applying (2.39) for \(t-s \leq t_m\), we use Proposition 3 to have
\((7.216) + (7.217)\) \(\lesssim \frac{\min\{\alpha(x, v), \alpha(y, v)\}}{|v|^2} \times (7.12) \)
\[
\lesssim \frac{\|w_{\tilde{\theta}}\alpha \nabla_x f_s\|_\infty^2}{w_{\tilde{\theta}}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|x|}, \frac{\alpha(y,v)}{|y|}\right\}^{1+\beta}}.
\]

For \((7.214)\) and \((7.215)\), we apply the same estimate as \((7.15)\) and \((7.16)\) with replacing \(\nabla_x\) derivative by \(\nabla_v\) derivative. By the same computation as \((7.14), (7.17)\) change to
\[
\int_{\mathbb{R}^3} d\mathbf{k}(v, u)|\nabla_v f_s(x, u)| \int_{\mathbb{R}^2} \frac{w_{\tilde{\theta}}^{-1}(v + \omega)\|w_{\tilde{\theta}}\alpha \nabla_x f_s\|}{w_{\tilde{\theta}}(v)^{1/2}} \min\{\alpha(x, v + \omega), \alpha(y, v + \omega)\}^\beta
d\omega e^{-\tilde{\theta}|v|^2/2} \frac{w_{\tilde{\theta}}^{-1}(v + \omega)}{|v + \omega|^\beta}
\int_{\mathbb{R}^3} d\mathbf{u} \frac{w_{\tilde{\theta}}^{1/2}(v)}{|u|^2 \min\{\xi(x), \xi(y)\}^{\beta/2}}
\int_{\mathbb{R}^3} d\mathbf{u} \frac{\|w_{\tilde{\theta}}\alpha \nabla_v f_s\|_\infty \|w_{\tilde{\theta}}^{1/2}v\|^2 \nabla_v f_s\|_\infty w_{\tilde{\theta}}^{-1}(v)}{|v|^2 \min\{\xi(x), \xi(y)\}^{\beta/2}}.
\]

Applying the same computation as \((7.20)\), the contribution of the above term is bounded by
\[
\frac{\|w_{\tilde{\theta}}\alpha \nabla_v f_s\|_\infty \|w_{\tilde{\theta}}^{1/2}v\|^2 \nabla_v f_s\|_\infty}{w_{\tilde{\theta}}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|x|}, \frac{\alpha(y,v)}{|y|}\right\}^{1+\beta}}.
\]

For the contribution of \((7.15)\) with replacing \(\nabla_x\) by \(\nabla_v\), we apply the same computation as \((7.18)\) to bound such contribution by
\[
\frac{\|w_{f_s}\|_\infty [\nabla_v f_s(\cdot, v)]_{C^0,\beta_{x,1+\beta}}}{w_{\tilde{\theta}}^{1/2}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|x|}, \frac{\alpha(y,v)}{|y|}\right\}^{1+\beta}}.
\]

For the contribution of \((7.16)\) with replacing \(\nabla_x\) by \(\nabla_v\), we apply the same computation as \((7.19)\) to bound such contribution by
\[
\frac{\|w_{\tilde{\theta}}\alpha \nabla_x f_s\|_\infty \|w_{\tilde{\theta}}^{1/2}v\|^2 \nabla_v f_s\|_\infty}{w_{\tilde{\theta}}^{1/2}(v)|v|^2} \int_0^\tau e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \frac{w_{\tilde{\theta}}^{-1}(v)\mathbf{k}(v, u)}{w_{\tilde{\theta}}(u)|u|^2 \min\{\alpha(x', u), \alpha(y', u)\}^\beta}
+ \frac{\|w_{\tilde{\theta}}\alpha \nabla_v f_s\|_\infty \|w_{\tilde{\theta}}^{1/2}v\|^2 \nabla_v f_s\|_\infty}{w_{\tilde{\theta}}^{1/2}(v)|v|^2 \min\{\alpha(x,v), \alpha(y,v)\}^\beta} \int_0^\tau e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \frac{w_{\tilde{\theta}}^{1/2}(v)\mathbf{k}(v, u)|v|^2}{w_{\tilde{\theta}}^{1/2}(u)|u|^2}.
\]
Combining (7.221), (7.222), (7.223) and using (7.6), (7.186), \(\|w_{f_s}\|_\infty \ll 1\) from the Existence Theorem, we conclude that

\[
(7.214) + (7.215) \lesssim \frac{\|w_{\bar{\theta}}\|_{\infty} \|\nabla x f_s\|_{\infty} + o(1)[\|\nabla_v f_s(\cdot, v)\|_2]_{C^{0,\beta}}}{w_{\bar{\theta}/2}(v)|v|^{3 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}}^{1+\beta}. \tag{7.224}
\]

For (7.218) and (7.219), we use (2.118) to have

\[
\Gamma_{v, gain}(f_s, f_s)(x, v) - \Gamma_{v, gain}(f_s, f_s)(y, v) = \int_{\mathbb{R}^3} \int_{S^2} d\omega u \cdot \omega \frac{|v + u|}{2T_w} \sqrt{\mu(v + u)}
\times \left( [f_s(x, v + u_\perp) - f_s(y, v + u_\perp)] f_s(x, v + u) \right.
\quad \left. + [f_s(x, v + u_\parallel) - f_s(y, v + u_\parallel)] f_s(y, v + u_\perp) \right).
\]

Clearly \(\frac{|v + u|}{2T_w} \sqrt{\mu(v + u)} \lesssim \mu^{1/4}(v + u)\), using such bound for \(\frac{|v + u|}{2T_w} \sqrt{\mu(v + u)}\) the above term has the same form as \(\Gamma_{gain}(f_s(x) - f_s(y), f_s(x)) + \Gamma_{gain}(f_s(y), f_s(x) - f_s(y))\) in (2.110). Thus we can use (7.9) to bound (7.218) as

\[
(7.218) \lesssim \frac{\|w_{\bar{\theta}}\|_{\infty} \|\nabla x f_s\|_{\infty}}{w_{\bar{\theta}}(v) \min \{\alpha(x, v), \alpha(y, v)\}}. \tag{7.225}
\]

Similarly for (7.219) we also have

\[
(7.219) \lesssim \frac{\|w_{\bar{\theta}}\|_{\infty} \|\nabla x f_s\|_{\infty}}{w_{\bar{\theta}}(v) \min \{\alpha(x, v), \alpha(y, v)\}}. \tag{7.226}
\]

Combining (7.212), (7.213), (7.220), (7.224), (7.225) and (7.226) we use (7.186) and (7.6) to conclude

\[
(7.180) \lesssim \frac{\|w_{\bar{\theta}}\|_{\infty} \|\nabla x f_s\|_{\infty}^2 + o(1)[\|\nabla_v f_s(\cdot, v)\|_2]_{C^{0,\beta}}}{w_{\bar{\theta}/2}(v)|v|^{3 \min \left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}}^{1+\beta}. \tag{7.227}
\]

Finally, we combine (7.187), (7.209) and (7.227), and conclude (1.23). \(\square\)

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