Local Quadratic Convergence of Stochastic Gradient Descent with Adaptive Step Size

Adityanarayanan Radhakrishnan 1  Mikhail Belkin 2  Caroline Uhler 1

Abstract
Establishing a fast rate of convergence for optimization methods is crucial to their applicability in practice. With the increasing popularity of deep learning over the past decade, stochastic gradient descent and its adaptive variants (e.g., Adagrad, Adam, etc.) have become prominent methods of choice for machine learning practitioners. While a large number of works have demonstrated that these first order optimization methods can achieve sub-linear or linear convergence, we establish local quadratic convergence for stochastic gradient descent with adaptive step size for problems such as matrix inversion.

1. Introduction
Over the past decade, gradient descent and adaptive variants such as RMSprop (Tieleman & Hinton, 2012) and Adam (Kingma & Ba, 2015) have become standard choices of optimization method due to their ability to scale to large machine learning models such as deep neural networks. Establishing fast convergence rates and providing explicit learning rate schedules for these optimization methods has been recognized as an important problem for reducing the complexity of hyper-parameter tuning for practitioners. Thus, analyzing the convergence of these optimization methods is an active area of research in machine learning.

While the convergence of vanilla stochastic gradient descent has been studied extensively in the convex and non-convex setting (Bubeck, 2015; Karimi et al., 2016), understanding whether there exist adaptive step sizes that yield higher orders of convergence remains an open problem. In this work, we show that by a careful selection of an adaptive step size, gradient descent and stochastic gradient descent can converge at a quadratic rate locally. To build intuition, consider the following example of inverting a real number.

Example (Number Inversion). Suppose we wish to invert a real number $x$ using only additions and multiplications. One way to solve this problem is to find $w$ such that $wx = 1$ using gradient descent on the loss $\mathcal{L} = \frac{1}{2}(1 - wx)^2$. This method will yield a linear convergence rate if a fixed step size is used. However, using an adaptive step size of $w^{(t)2}$ yields a quadratic convergence rate. The update equations from gradient descent are as follows:

$$w^{(t+1)} = w^{(t)} + \gamma^{(t)}x(1 - w^{(t)}x)$$

$$= w^{(t)} + w^{(t)2}x(1 - w^{(t)}x),$$

where $\gamma^{(t)} = w^{(t)2}$ is the adaptive learning rate. Let $w^*$ be the solution to $w^*x = 1$ and let $u^{(t)} = w^{(t)} - w^*$. Consider the convergence of $|u^{(t)}|:

$$|u^{(t+1)}| = |u^{(t)} + (u^{(t)} + w^*)2x(1 - w^{(t)}x - w^*x)|$$

$$= |u^{(t)} - u^{(t)}(u^{(t)} + w^*)2x^2| \quad \text{as } w^*x = 1$$

$$= |(1 - w^{(t)2}x^2) - 2u^{(t)2}w^*x^2 - u^{(t)3}x^2|$$

$$= | - 2u^{(t)2}w^*x^2 - u^{(t)3}x^2|$$

$$= |2u^{(t)2}x + u^{(t)3}x^2|,$$

yielding a quadratic convergence rate for sufficiently small $u^{(0)}$.

For this problem, Newton’s method (Nocedal & Wright, 2006, Chapter 3) also gives a quadratic convergence rate; but interestingly, the update for Newton’s method is different, namely $u^{(t+1)} = 2u^{(t)} - u^{(t)2}x$.

An outline of our work is as follows. In Section 2, we discuss related work. In Section 3 we provide an adaptive learning rate for local quadratic convergence of gradient descent in the multi-dimensional setting. In Section 4, we extend this adaptive learning rate to establish quadratic convergence of gradient descent in the stochastic setting. In Section 5, we provide experiments to demonstrate our learning rate working in practice. In Section 6, we provide a class of linear systems under which linear convergence
Local Quadratic Convergence of Stochastic Gradient Descent with Adaptive Step Size

2. Related Work

Sub-linear and linear convergence rates for stochastic gradient descent (SGD) have been provided across a number of settings. A survey of traditional results for the convex setting is presented in Bubeck (2015). Recent works establish linear convergence of SGD in the non-convex setting (Bassily et al., 2018; Vaswani et al., 2019; Karimi et al., 2016) by using the Polyak-Lojasiewicz inequality (Lojasiewicz, 1963; Polyak, 1963). Recent work has also focused on providing convergence rates for adaptive methods such as Adagrad (Duchi et al., 2011). In particular, Radhakrishnan et al. (2020); Xie et al. (2020) provide sufficient conditions for linear convergence of Adagrad and Adagrad-Norm (a normed version of Adagrad).

Several works have also developed pre-conditioner methods for gradient descent in order to improve convergence in practice, but currently, the established rates of convergence are sub-linear or linear. For example, RMSprop (Tieleman & Hinton, 2012) and Adam (Kingma & Ba, 2015) are popular adaptive gradient methods for training deep neural networks, and these methods can be interpreted as pre-conditioner methods (Staib et al., 2020). Sufficient conditions for sub-linear convergence of these methods are provided in Bernstein et al. (2018); Zaheer et al. (2018). Ma & Belkin (2017) introduce EigenPro, which is a pre-conditioner for kernel machines that incorporates approximate second order information. Recent work by Anil et al. (2020); Yao et al. (2020) introduce second order pre-conditioner methods for neural networks, and Yao et al. (2020) establish a linear convergence rate for their method applied to strongly convex functions.

Newton’s method (Nocedal & Wright, 2006, Chapter 3) serves as a classical example of an algorithm that achieves quadratic convergence locally. Recent work presents an extension of Newton’s method to compute the \( p^{th} \) root (and the inverse \( p^{th} \) root of a positive definite matrix) using only matrix multiplications (Iannazzo, 2005). Under sufficient conditions on the spectrum of the matrix, Guo & Higham (2006) give an initialization under which this method converges at a quadratic rate.

The updates in Newton’s method rely on second-order information and can be expensive to compute on large datasets. Several quasi-Newton methods such as L-BFGS (Nocedal & Wright, 2006, Chapter 3) have been developed to avoid full construction of the Hessian, and recently, Mokhtari et al. (2018) developed a stochastic quasi-Newton method with a super-linear rate of convergence. The work by Mokhtari et al. (2018) builds upon several prior works by Rodomanov & Proezd (2016); Sohl-Dickstein et al. (2014); Wang et al. (2017); Byrd et al. (2016), which also introduce stochastic quasi-Newton methods. While the work by Mokhtari et al. (2018) uses only matrix-vector multiplications like the stochastic version of our method, ours has a simpler formulation and is history-free, i.e. it does not need to store any additional terms other than the parameters.

3. Gradient Descent with Adaptive Step Size

In the following theorem (with proof in Appendix A), we extend the analysis of number inversion from the introduction to the problem of inverting an (invertible) square matrix \( X \in \mathbb{R}^{n \times n} \).

**Theorem 1.** Let \( X, I \in \mathbb{R}^{n \times n} \) with \( I \) denoting the identity matrix, and let \( X \) be invertible. Gradient descent with right-multiplicative step size \( \gamma(t) = W(t)^TXW(t) \) to solve \( \arg \min_{W} \frac{1}{2} \| I - WX \|_F^2 \) converges at quadratic rate locally.

**Intuition for the Learning Rate.** We arrived at our choice of adaptive learning rate by analyzing the gradient descent update in two-layer linear neural networks. In particular, when using gradient descent to solve the following optimization problem

\[
\arg \min_{W_1 \in \mathbb{R}^{d \times k}, W_2 \in \mathbb{R}^{k \times d}} \frac{1}{2} \| I - W_1W_2X \|_F^2,
\]

the update rule for \( W_1 \) proceeds as follows:

\[
W_1^{(t+1)} = W_1^{(t)} + (I - W_1^{(t)}W_2^{(t)}X)X^TW_2^{(t)}\gamma.
\]

Note that unlike gradient descent in the single layer setting, gradient descent in this setting involves terms of higher orders in \( W_1 \) and \( W_2 \). Thus, we chose the learning rate appropriately to ensure that only these higher order terms remained.

**Remarks.** When the target matrix is no longer the identity but rather an invertible matrix \( Y \in \mathbb{R}^{n \times n} \), we can simply transform the system such that the target is the identity. For example, when \( Y \) is an orthogonal matrix, we simply transform \( X \) to \( XY^T \) to reduce to the case above. Interestingly, the update for Newton’s method in this setting is given by \( W^{(t+1)} = 2W^{(t)} - W^{(t)}XW^{(t)} \), which again differs from our update rule. Importantly, note that it is non-trivial to extend this formulation of Newton’s method to the stochastic setting due to the non-commutativity of matrices.

**Computing the \( d^{th} \) Root of Symmetric Positive Definite Matrices.** Just as Newton’s method can be extended to compute \( d^{th} \) roots of matrices (Iannazzo, 2005), we can extend our method to do the same for symmetric positive definite matrices.
Proposition 1. Let \( X \in \mathbb{R}^{n \times n} \) be a symmetric positive definite matrix and let \( I \in \mathbb{R}^{n \times n} \) denote the identity matrix. Assuming \( W^{(0)} \) commutes with \( X \), gradient descent with right multiplicative step size \( \gamma(t) = \frac{1}{\| W^{(t)} \|^2} \) to solve \( \arg \min_{W \in \mathbb{R}^n} \frac{1}{2} \| I - W^d X \|^2_F \) converges at a quadratic rate locally.

The proof is presented in Appendix B. Thus far, we have demonstrated that adaptive gradient descent achieves quadratic convergence for a class of problems in the full batch setting. In the next section, we demonstrate that gradient descent with adaptive step size converges quadratically in the stochastic setting.

4. SGD with Adaptive Step Size

We return to the setting of matrix inversion and now demonstrate that stochastic gradient descent with right-multiplicative learning rate \( W^{(t)^T} W^{(t)} \) yields quadratic convergence.

Theorem 2. Let \( X, I \in \mathbb{R}^{n \times n} \) with \( I \in \mathbb{R}^{n \times n} \) denoting the identity matrix. Assuming all examples are seen per epoch, stochastic gradient descent with step size \( \gamma(t) = W^{(t)^T} W^{(t)} \) used to solve \( \arg \min_{W \in \mathbb{R}^n} \frac{1}{2} \| I - W X \|^2_F \) converges at a quadratic rate locally.

The proof is presented in Appendix C. The proof relies on the following proposition (proof in Appendix D), which demonstrates that using all examples per epoch eliminates the linear term in the error term, \( U^{(t)} \), defined below.

Proposition 2. Let \( W^* \) denote the solution to \( W X = I \) and let \( U^{(t)} = W^{(t)} - W^* \). For positive constants \( C_1, C_2 \), \( \| U^{(n)} \| \leq C_1 \| U^{(n(-1))} \|^4 + C_2 \| U^{(n(-1))} \|^6 \).

Remarks. The benefit of the stochastic setting is that it involves only matrix-vector multiplications. From the analysis above, the quadratic convergence rate in the stochastic setting is achieved by iterating through each example the same number of times per epoch. In fact, unless all examples are iterated over the same number of times per epoch, the rate will always have a non-vanishing linear term. While this analysis matches implementations of SGD in practice, this is different from the usual analysis of SGD where the rate is analyzed by considering the expectation over the randomness of the choice of example per step of the update (Vaswani et al., 2019; Bassily et al., 2018). Proceeding according to this usual style of analysis will only lead to an expected linear convergence rate.

5. Experiments

In the following, we demonstrate that our gradient descent and stochastic gradient descent algorithms also achieve the quadratic rate in practice. We begin by showing that when initialized sufficiently close to the true solution \( W^* \) (i.e., satisfying the condition on \( U^{(0)} \)), our algorithms achieve quadratic convergence when inverting large random matrices.

Construction of Invertible Matrix \( X \). The singular values of \( X \) are created by sampling \( n \) values from a standard normal distribution and taking absolute value. Then the left and right orthonormal matrices \( U, V \) of \( X \) are generated by taking \( U, V \) from the SVD of a random matrix with i.i.d. standard normal entries.

Quadratic Convergence of GD with Adaptive Step Size. In Figure 1a, we demonstrate our algorithms on inverting a random \( 100 \times 100 \) matrix \( X \). In Figure 1a, we have initialized \( W^{(0)} \) for gradient descent and stochastic gradient descent to be \( .4 X^{-1} \) and \( .5 X^{-1} \) respectively. When initializing close to the true solution \( X^{-1} \), we see that both algorithms achieve quadratic convergence rates, in line with our theoretical results.

Of course in practice, one cannot initialize in this manner as it requires computing \( X^{-1} \). We demonstrate in Figure 1b that we can accelerate matrix inversion by using usual gradient descent (or alternatively randomized Kaczmarz iteration (Strohmer & Vershynin, 2009)) with initialization at 0 until the loss is sufficiently low, and then substitute our adaptive algorithm to accelerate convergence. In Figure 1b, we invert \( X \) by first using gradient descent with learning rate 0.1 and then after the loss is below \( 10^{-4} \), we switch to our adaptive method, which converges quadratically.

Quadratic Convergence of SGD with Adaptive Step Size. We now demonstrate that for our Adaptive SGD method, each of the \( n \) examples must be used the same number of times per epoch in order to achieve quadratic convergence. Instead of visualizing the log-loss per epoch, we visualize the loss per iteration in Figure 2a. In this figure the stars indicate the loss after an epoch. There are sharp drops in the loss function due to individual examples, as seen in the example used near iteration 1200. Without the inclusion of this example, the loss would decrease linearly instead of quadratically. In fact, if we randomly sample examples at each iteration instead of cycling through all examples per epoch, we can only guarantee a linear convergence rate. This is demonstrated in Figure 2b in which we invert a random \( 1000 \times 1000 \) matrix using adaptive SGD with randomly selected samples per epoch versus using all examples per epoch. Both methods are initialized close to the true solution (at \( .1 X^{-1} \)), but randomly sampling per epoch can lead to a linear rate of convergence.

\(^1\)Initializing both with \( .5 X^{-1} \) yields overlapping curves.
Local Quadratic Convergence of Stochastic Gradient Descent with Adaptive Step Size

Figure 1. Using gradient descent with adaptive step size leads to locally quadratic convergence rate. In (a), we demonstrate quadratic convergence when initialization is close to the true solution. In (b), we demonstrate that for well conditioned matrices, standard gradient descent (with linear rate) can be run until the learned solution is close enough to $X^{-1}$ (indicated by the vertical, dashed line), at which point the quadratic rate can be substituted to accelerate convergence.

Figure 2. Stochastic gradient descent with adaptive step size converges quadratically if and only if an epoch consists of an equal count of each of the training examples. In (a), we demonstrate that within a single epoch our adaptive SGD algorithm has linear convergence rate until the last few examples are seen. In (b), we demonstrate that using randomly sampled data instead of using all examples per epoch can result in linear convergence.

6. What is the Optimal Rate of Convergence?

While the above examples demonstrate that gradient descent with adaptive step size can converge at a quadratic rate locally, is there an alternate learning rate schedule under which we can establish an even faster rate of convergence? In the following theorem (proof in Appendix E), we prove that for linear systems, gradient descent with matrix polynomial learning can achieve at most linear convergence when the solution $W^* \in \mathbb{R}^{k \times d}$ has rank less than $d$.

**Theorem 3.** Let $X \in \mathbb{R}^{d \times n}$ be generic, $Y \in \mathbb{R}^{k \times n}$, and suppose that there exists $W^* \in \mathbb{R}^{k \times d}$ such that $W^*X = Y$ and $W^*$ has rank less than $d$. Gradient descent with matrix polynomial learning rate $\gamma(t) = P[W(t)^T W(t)]$ used to solve $\arg \min_{W \in \mathbb{R}^{k \times d}} \frac{1}{2} \| Y - WX \|_F^2$ converges with order at most 1.

**Remarks.** We note that Theorem 3 does not rule out the possibility that there may be adaptive learning rates involving polynomials of sufficiently high degree in $X$, $Y$, and $W^{(t)}$ that can provide higher order convergence. However, the cost of computing such polynomials must also be taken into account when considering the practical application of such a method. For example, if the $c_i$ terms in the proof of Theorem 3 are matrices, and $\gamma(t)$ is a matrix polynomial of sufficiently high degree $r$, then we can apply linear regression to find the coefficients $c_i$ to provide as fast a convergence rate as desired. However, solving a regression problem to find these coefficients is impractical.

7. Discussion

In this work, we first demonstrated that gradient descent with adaptive step size can converge at a quadratic rate locally. We then extended our result to the stochastic setting and verified our results empirically. Lastly, we demonstrated that if the learning rate is a matrix polynomial in the
parameters and the number of regression targets is fewer than the number of features, the best possible rate of convergence is linear. We now briefly discuss future directions of interest.

While our quadratic convergence results are local, it would be interesting to understand whether there is a simple initialization scheme (as in Guo & Higham (2006)) under which our method achieves this rate. Another avenue of interest would be to understand whether there are other computationally tractable learning rates that could provide even higher orders of convergence locally.

Acknowledgements

A. Radhakrishnan and C. Uhler were partially supported by NSF (DMS-1651995), ONR (N00014-17-1-2147 and N00014-18-1-2765), the MIT-IBM Watson AI Lab, and a Simons Investigator Award to C. Uhler. The Titan Xp used for this research was donated by the NVIDIA Corporation. M. Belkin acknowledges support from NSF (IIS-1815697 and IIS-1631460) and a Google Faculty Research Award.

References

Anil, R., Gupta, V., Koren, T., Regan, K., and Singer, Y. Second order optimization made practical. *arXiv preprint arXiv:2002.09018*, 2020.

Bassily, R., Belkin, M., and Ma, S. On exponential convergence of SGD in non-convex over-parametrized learning. *arXiv preprint arXiv:1811.02564*, 2018.

Bernstein, J., Wang, Y.-X., Azizzade-nesheli, K., and Anandkumar, A. signSGD: Compressed optimisation for non-convex problems. In *International Conference on Machine Learning*, 2018.

Bubeck, S. Convex optimization: Algorithms and complexity. *Foundations and Trends in Machine Learning*, 8(3-4):231–357, 2015.

Byrd, R., Hansen, S., Nocedal, J., and Singer, Y. Stochastic quasi-Newton methods for nonconvex stochastic optimization. *SIAM Journal on Optimization*, 26(2):1008–1031, 2016.

Duchi, J., Hazan, E., and Singer, Y. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of Machine Learning Research*, 12:2121–2159, 2011.

Guo, C.-H. and Higham, N. J. A Schur-Newton method for the matrix \( p \)th root and its inverse. *SIAM Journal on Matrix Analysis and Applications*, 28(3):788–804, 2006.

Iannazzo, B. On the Newton method for the matrix \( P \)th root. *SIAM Journal on Matrix Analysis and Applications*, 28(2):503–523, 2005.

Karimi, H., Nutini, J., and Schmidt, M. Linear convergence of gradient and proximal-gradient methods under the Polyak-Lojasiewicz condition. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pp. 795–811. Springer, 2016.

Kingma, D. P. and Ba, J. Adam: A method for stochastic optimization. In *International Conference on Learning Representations*, 2015.

Lojasiewicz, S. A topological property of real analytic subsets (in French). *Les équations aux dérivées partielles*, 117:87–89, 1963.

Ma, S. and Belkin, M. Diving into the shallows: a computational perspective on large-scale shallow learning. In *Advances in Neural Information Processing Systems*, 2017.

Mokhtari, A., Eisen, M., and Ribeiro, A. IQN: An incremental quasi-Newton method with local superlinear convergence rate. *SIAM Journal on Optimization*, 28(2):1670–1698, 2018.

Nocedal, J. and Wright, S. J. *Numerical Optimization*, volume 2. Springer, 2006.

Polyak, B. Gradient methods for minimizing functionals (in Russian). *Zh. Vychisl. Mat. Mat. Fiz.*, 3(4):643–653, 1963.

Radhakrishnan, A., Belkin, M., and Uhler, C. Linear convergence and implicit regularization of generalized mirror descent with time-dependent mirrors. *arXiv preprint arXiv:2009.08574*, 2020.

Rodomanov, A. and Proezd, K. A superlinearly-convergent proximal Newton-type method for the optimization of finite sums. In *International Conference on Machine Learning*, 2016.

Sohl-Dickstein, J., Poole, B., and Ganguli, S. Fast large-scale optimization by unifying stochastic gradient and quasi-Newton methods. In *International Conference on Machine Learning*, 2014.

Staib, M., Reddi, S., Kale, S., Kumar, S., and Sra, S. Escaping saddle points with adaptive gradient methods. *arXiv preprint arXiv:1901.09149*, 2020.

Strohmer, T. and Vershynin, R. A randomized Kaczmarz algorithm with exponential convergence. *Journal of Fourier Analysis and Applications*, 15(292), 2009.

Tieleman, T. and Hinton, G. Lecture 6.5—RmsProp: Divide the gradient by a running average of its recent mag-
Vaswani, S., Bach, F., and Schmidt, M. Fast and faster convergence of SGD for over-parameterized models and an accelerated perceptron. In *International Conference on Artificial Intelligence and Statistics*, 2019.

Wang, X., Ma, S., Goldfarb, D., and Liu, W. Stochastic quasi-Newton methods for nonconvex stochastic optimization. *SIAM Journal on Optimization*, 27(2):927–956, 2017.

Xie, Y., Wu, X., and Ward, R. Linear convergence of adaptive stochastic gradient descent. In *International Conference on Artificial Intelligence and Statistics*, 2020.

Yao, Z., Gholami, A., Shen, S., Keutzer, K., and Mahoney, M. W. ADAHESSIAN: An adaptive second order optimizer for machine learning. *arXiv preprint arXiv:2006.00719*, 2020.

Zaheer, M., Reddi, S., Sachan, D., Kale, S., and Kumar, S. Adaptive methods for nonconvex optimization. In *Advances in Neural Information Processing Systems*, 2018.
A. Proof of Theorem 1

Proof. Gradient descent proceeds as follows:

\[ W^{(t+1)} = W^{(t)} + (I - W^{(t)}X)X^T \gamma^{(t)} \]
\[ = W^{(t)} + (I - W^{(t)}X)X^T W^{(t)} W^{(t)}. \]

Note that \( \gamma^{(t)} \) needs to be multiplied on the right hand side of the gradient, i.e., it is a non-commutative learning rate. Let \( U^{(t)} = W^{(t)} - W^* \) where \( W^* \) is the solution satisfying \( W^*X = I \), and let \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \) be the singular values of \( X \).

To simplify notation, we assume that the norms used are Frobenius norms unless otherwise specified. Then we obtain:

\[
\|U^{(t+1)}\|^2 = \|U^{(t)} + (I - U^{(t)}X - X^T(U^{(t)} + W^*)^T(U^{(t)} + W^*)\|^2 \\
= \|U^{(t)} - U^{(t)}X XT(U^{(t)} + W^*)^T(U^{(t)} + W^*)\|^2 + \|U^{(t)}X XT(U^{(t)} + W^*)^T(U^{(t)} + W^*)\|^2 \text{ as } W^*X = I \\
= \|U^{(t)}(I - XX^T W^*) - U^{(t)}X XT(U^{(t)} + W^*)^T(U^{(t)} + W^*)\|^2 \\
= \| - U^{(t)}X XT(U^{(t)} + W^*)^T(U^{(t)} + W^*)\|^2 \\
\leq \|XX^T\|^2 \left(\|U^{(t)}\|^2 + \|W^*\|^2\right)^2 \\
\leq \sigma_1^2 \|U^{(t)}\|^6 + n \sigma_1^2 \|U^{(t)}\|^4.
\]

Hence provided that \( \|U^{(0)}\|^2 \) is sufficiently close to zero, \( \|U^{(t)}\|^2 \) converges quadratically. \( \square \)

B. Proof of Proposition 1

Proof. By assumption, \( W^{(0)} \) commutes with \( X \) so that \( W^{(t)} \) is a matrix polynomial in \( X \) and hence commutes with \( X \) for all \( t \). In particular, the update equation can be written as

\[ W^{(t+1)} = W^{(t)} + d \gamma^{(t)} W^{(t)}^{d-1} (I - W^{(t)}^{d} X) X \\
= W^{(t)} + \frac{1}{d} W^{(t)}^{d+1} (I - W^{(t)}^{d} X) X.
\]

We let \( U^{(t)} = W^{(t)} - W^* \) where \( W^*dX = I \). As before, we obtain

\[
\|U^{(t+1)}\|^2 = \|U^{(t)} + \frac{1}{d} (U^{(t)} + W^*)^{d+1} (I - (U^{(t)} + W^*)^{d} X) X\|^2 \\
\leq \|U^{(t)}(I - W^*^{d+1} W^*^{d-1} X^2)\|^2 + \sum_{i=2}^{2d+1} C_i \|U^{(t)}\|^2 \text{ for constants } C_i \in \mathbb{R}_+ \\
= \sum_{i=2}^{2d+1} C_i \|U^{(t)}\|^2 \|f\| \leq \sum_{i=2}^{2d+1} C_i \|U^{(t)}\|^2. 
\]

Hence provided that \( \|U^{(0)}\|^2 \) is sufficiently small, gradient descent with adaptive step size achieves a quadratic convergence rate. \( \square \)

C. Proof of Theorem 2

Proof. Let \( X_i \) denote the \( i^{th} \) column vector of \( X \) and let \( e_i \) denote the \( i^{th} \) column vector of \( I \) (that is \( e_i \) is a vector with a 1 in the \( i^{th} \) coordinate and 0 elsewhere). We let an epoch be a pass through a random ordering of the \( n \) pairs \((X_i, e_i)\) where each pair corresponds to the update

\[ W^{(t+1)} = W^{(t)} + (e_i - W^{(t)}X_i)X_i^T \gamma^{(t)}, \tag{1} \]

where we set \( \gamma^{(t)} = W^{(t)}^T W^{(t)} \). Defining \( U^{(t)} = W^{(t)} - W^* \), we will show that the sequence \( \|U^{(nt)}\|^2 \) for \( t \in \mathbb{N} \) converges at a quadratic rate. We prove this by using Proposition 2 below, which states that the term of \( U^{(nt)} \) that is linear in \( U^{(n(t-1))} \) has coefficient 0. In particular, Proposition 2 implies that \( \|U^{(nt)}\|^2 \leq C_1 \|U^{(nt-1)}\|^4 + C_2 \|U^{(nt-1)}\|^6 \) for constants \( C_1, C_2 > 0 \). Hence, initializing such that \( \|U^{(0)}\|^2 \) is sufficiently close to 0 yields a quadratic rate of convergence. \( \square \)
D. Proof of Proposition 2

Proof. We prove this for the case when \( t = 1 \), from which the general result follows easily. Without loss of generality, let the order of updates for the first epoch be \((X_1, e_1), (X_2, e_2)\ldots (X_n, e_n)\). We first show that \( C_0 = \| \prod_{i=1}^{n} (I - X_iX_i^TW^*W^*) \|^2 \).

By equation 1, we have that
\[
U^{(i)} = U^{(i-1)} + (e_i - (U^{(i-1)} + W^*)X_i)X_i^T(U^{(i-1)} + W^*)
= U^{(i-1)} - U^{(i-1)}X_iX_i^T(U^{(i-1)} + W^*) + W^*U^{(i-1)} + W^*W^*
= U^{(i-1)}[I - X_iX_i^TW^*W^*] - U^{(i-1)}X_iX_i^TU^{(i-1)}W^* - U^{(i-1)}X_iX_i^TW^*U^{(i-1)}
- U^{(i-1)}X_iX_i^TU^{(i-1)}U^{(i-1)}
\]

Hence the coefficient of \( U^{(i-1)} \) in the expansion of \( U^{(i)} \) is \([I - X_iX_i^TW^*W^*]\). Proceeding recursively from \( i = n \) to \( i = 1 \) proves that the coefficient of \( U^{(0)} \) in the expansion of \( U^{(n)} \) is \( \prod_{i=1}^{n} (I - X_iX_i^TW^*W^*) \).

Now we show that this term is 0. First note that \( W^*X_i = e_i \) and \( X_i^TW^*W^* = e_i^T \). Hence, we obtain that for \( i \not= j \),
\[
X_iX_i^TW^*W^*X_jX_j^TW^*W^* = X_i(X_i^TW^*)(W^*X_j)X_j^TW^*W^*
= X_iX_i^T e_jX_j^TW^*W^*
= 0 \quad \text{as} \quad e_i^Te_j = 0 \text{ for } i \not= j,
\]
and thus that
\[
\prod_{i=1}^{n} (I - X_iX_i^TW^*W^*) = I - \sum_{i=1}^{n} X_iX_i^TW^*W^* = I - \sum_{i=1}^{n} X_iX_i^T W^* = I - XX^T W^* = 0,
\]
which completes the proof.

E. Proof of Theorem 3

Proof. Gradient descent proceeds according to the update:
\[
W^{(t+1)} = W^{(t)} + (Y - W^{(t)}X)X^T \gamma^{(t)}.
\]

Again, assuming that \( U^{(t)} = W^{(t)} - W^* \), we have:
\[
U^{(t+1)} = U^{(t)} - U^{(t)}XX^T \gamma^{(t)} = U^{(t)}[I_d - XX^T \gamma^{(t)}].
\]

Now letting \( \gamma^{(t)} = \sum_{i=0}^{r} c_i(W^{(t)T}W^{(t)})^i \), we obtain
\[
\gamma^{(t)} = \sum_{i=0}^{r} c_i((U^{(t)} + W^*)^T(U^{(t)} + W^*))^i.
\]

Note that the constant term of \( U^{(t)} \) in \( \gamma^{(t)} \) is given by \( C_0 = \sum_{i=0}^{r} c_i(W^{(t)T}W^{*})^i \) for \( \{c_i\}_{i=1}^{r} \subset \mathbb{R} \). Thus, if \( W^* \) has rank less than \( d \), then \( XX^T C_0 \) cannot equal \( I_d \). Hence, the term of \( U^{(t+1)} \) that is linear in \( U^{(t)} \) has a non-zero coefficient, which completes the proof.