Stringy Effects at Low-Energy Limit and Double Field Theory

Chen-Te Ma\textsuperscript{a,b,c} and Franco Pezzella\textsuperscript{d}

\textsuperscript{a} Institute of Quantum Matter, School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou 510006, Guangdong, China.
\textsuperscript{b} The Laboratory for Quantum Gravity and Strings, Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag, Rondebosch 7700, South Africa.
\textsuperscript{c} Department of Physics and Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan, R.O.C..
\textsuperscript{d} Istituto Nazionale di Fisica Nucleare - Sezione di Napoli, Complesso Universitario di Monte S. Angelo ed. 6, via Cintia, 80126 Napoli, Italy.

Abstract

In the simple case of bosonic closed string theory with one of the twenty-five spatial directions being a circle of radius $R$, one has to take the field-theory limit $R \gg \sqrt{\alpha'}$ to study the low-energy physics. The corresponding massless spectrum with zero winding number and internal momenta excitation is obtained when the left-moving and right-moving oscillator numbers equal to one, i.e. $N_L = N_R = 1$. When one considers instead $N_L \neq N_R$, then the strong constraint solution is not enough to take into account the massless states with nontrivial internal momentum and winding number. This stringy effect in the low-energy limit should be expected from a perturbation in the parameter $\sqrt{\alpha'}/R$. We use a linear gauge transformation to construct the quadratic action with a new mass term for the state $(N_L, N_R) = (2, 0), (0, 2)$, and the action is T-dual invariant. By defining a gauge transformation through a non-linear extension, a constraint acting on fields can be uniquely defined in triple-products. Finally, we show that the stringy effects only appear in a number of compact dimensions $d > 1$.

\textsuperscript{1}e-mail address: yefgst@gmail.com
\textsuperscript{2}e-mail address: franco.pezzella@na.infn.it
1 Introduction

It is well-known that the low-energy limit of String Theory \cite{1, 2} is defined by sending to zero the fundamental string length \( l = \sqrt{\alpha'} \) (\( \alpha' \) is the so-called slope parameter), i.e. \( \alpha' \to 0 \). In this way, the corresponding point particle description, containing Einstein gravity theory, is recovered in the \( D \)-dimensional Riemannian manifold - the target-space - in which the two-dimensional worldsheet swept out by the string in its motion is embedded. In particular, in the case of compactification of one of the spatial directions on a circle of radius \( R \), the gravitational theory can be obtained from the fluctuations of the target space when the radius \( R \) is much larger than the square root of the slope parameter (\( R \gg \sqrt{\alpha'} \)).

When compactified on a \( d \)-dimensional torus \( T^d \), the theory exhibits the peculiar stringy symmetry \( O(d, d; \mathbb{Z}) \) \cite{3, 4} for all the \( d \) compact directions \cite{5}: the target space duality (T-duality) \cite{3, 6}. It is a distinctive symmetry of strings since, differently from particles, one-dimensional objects can wrap \( d \) non-contractible cycles. The number of times a string winds around a circle in the target space is called winding number. The winding modes \( w^a (a = 1, 2, \cdots, d) \) have therefore to be added, in describing the string dynamics, to the Kaluza-Klein momentum modes \( p_a = n/R^a \) taking integer values along the compact circle dimension \( x^a \) of radius \( R^a \). On \( T^d \), the \( O(d, d; \mathbb{Z}) \) T-duality is a symmetry under, roughly speaking, the mapping of the radii of the compact dimensions into their inverse, together with the exchange of momentum and winding modes: in this way it establishes a connection between two apparently different but dual target spacetimes. This justifies the interest into a T-dual invariant formulation of the string world-sheet action that should generate a T-dual invariant target space theory. Such formulation could shed light on aspects of stringy geometry and hence of stringy gravity, unexplored so far.

The path integral derivation of T-duality at all orders in \( \alpha' \) was first derived from the bosonic closed string sigma model \cite{7}. In the simplest case of compactification of only one spatial dimension, let us say \( x^a \), in Minkowski space in a circle of radius \( R \), the T-duality shows an equivalent physics \cite{8} for closed strings on the two different compactification scales, \( R \) and \( \alpha'/R \), and the equivalence is manifested by switching momentum and winding numbers along the compact dimension. In this case, the bosonic closed string coordinate along the compact dimension satisfies a quasi-periodic condition pro-
vided by the following

\[ X^a(\sigma^0, \sigma^1 + 2\pi) = X^a(\sigma^0, \sigma^1) + 2\pi\alpha' w^a, \]  

(1)

where

\[ w^a \equiv \frac{mR}{\alpha'} \quad m \in \mathbb{Z}. \]  

(2)

and where \( \sigma^0 \) and \( \sigma^1 \) are respectively the arbitrary time and space coordinates on the world-sheet. Under a T-duality transformation, the circle of radius \( R \) with coordinate \( x^a \) is mapped into the dual circle, with coordinate \( \tilde{x}_a \) and periodicity \( 2\pi\alpha'/R \) and, correspondingly, the string coordinate \( X^a \), sum of the left and right moving oscillators, is mapped into its dual \( \tilde{X}_a \), difference of the latter. It is also important to observe that while the momentum \( p_a \) is the conjugate variable to \( X^a \), the winding \( \omega^a \) results to be the conjugate variable of the coordinate \( \tilde{X}_a \). The equivalent physics, under the T-duality, between different compactification scales provides peculiar aspects of stringy geometry, which do not appear, of course, in the geometry of point particles.

As already stated above, the T-duality is stringy because it is a symmetry related to the winding. If interested in obtaining a low-energy gravitation theory of the bosonic closed string, one has to consider the field-theory limit \( (R \gg \sqrt{\alpha'}) \), which truncates windings. Therefore, compactification in such theory does not leave any track, any information deriving from windings. With a manifest T-duality invariant structure in the target space one could study stringy effects, due to windings, at low energy.

In order to have a manifest T-dual invariant formulation of the target space theory, the most intuitive realization is to introduce a manifest symmetry between windings \( \omega^a \) and momenta \( p_a \) \([9, 10]\) or, equivalently, a manifest symmetry between the coordinates \( X^a \) and their duals \( \tilde{X}_a \) already in the low-energy effective theory. This symmetry is geometrically achieved by substituting the tangent space in each point of the target space with the direct sum of the tangent and the cotangent spaces \([11]\), according to the spirit of Generalized Geometry. Such a construction makes the target space equipped with a useful structure, the so-called generalized metric \([12]\). This object was also previously obtained through a suitable string duality rotation \([13]\).

In the bosonic closed string theory, the symmetric relation between momenta and windings can be easily seen from the energy-mass spectrum of the string in the Minkowski
space, but it does not correspondingly appear in the Lagrangian. Hence the field-theory limit obtained from the latter cannot provide a low-energy effective description of windings, differently from a manifestly T-dual invariant Lagrangian \[14\] that should generate, in the low-energy limit, a manifestly T-dual invariant formulation of the target space in which the world-sheet is embedded \[15\]. In conclusion, the manifest T-dual invariant formulation of the bosonic sigma model should contain more complete and deep information about low-energy string physics.

The above development concerns the first-quantized string theory. One can think of making the same kind of considerations in the context of the second-quantized string theory, in particular of the closed string field theory \[16\]. It is well-known that for the classical theory of a point particle, the corresponding quantum mechanical formulation (first quantization) is described in terms of its coordinates becoming operators, while the corresponding quantum field theory (second quantization) elevates fields, i.e. functions of coordinates, to operators. The quantization of string theory with its world-sheet action is also similar. The first-quantized string theory is described by the quantization of string coordinates in the target space, while the second-quantized string theory is described by considering the quantization of a functional of such coordinates, i.e. the string field. In the low-energy limit that recovers the point-like description, one could obtain a more complete description while going from the first quantization to the second quantization.

The T-duality acting on a circle in the closed string field theory \[17\] is still a symmetry \[18\] with a suitable T-dual invariant string coupling constant \[19\]. It has been shown that the closed string field theory provides a consistent low-energy effective theory \[20\].

The closed string field theory on a \(T^d\) torus can be formulated in such a way that it can exhibit a manifest T-dual structure by introducing, in the 26-dimensional Minkowski spacetime with the \(d\) compact dimensions of \(T^d\), a \(2d\)-dimensional doubled torus, formed by the ordinary compact coordinates \(x^a (a = 1, 2, \ldots, d)\) and the dual compact coordinates \(\tilde{x}_a\). This is the core of the so-called Double Field Theory \[21\].

In Double Field Theory, the definition of the squared mass of a physical state in all the \(D = 26\) (non-compact and compact) dimensions of the target space is given by putting at the same level the momentum \(k^2\) along the non-compact dimensions, the
Kaluza-Klein and the winding momenta according to the following equation:

\[
M^2 \equiv -(k^2 + p^2 + \omega^2) = \frac{2}{\alpha'}(N_L + N_R - 2) \tag{3}
\]

where \( N_L \) is the number of left-moving oscillators, \( N_R \) is the number of right-moving oscillators. In particular for \( N_L = N_R = 1 \), the eq. (3) defines a set of fields with \( M^2 = 0 \) in \( D \) dimensions and with the same index structure as for the uncompactified theory but keeping their full dependence on the coordinates of the doubled torus and the ones of the uncompactified space: \( h_{jk}(x^{\mu}, x^a, \tilde{x}_a) \), \( b_{jk}(x^{\mu}, x^a, \tilde{x}_a) \), \( d(x^{\mu}, x^a, \tilde{x}_a) \) with \( j, k = 1, \cdots, D \), \( \mu, \nu = 0, 1, \cdots, D-d \) and \( a = 1, 2, \cdots, d \). On such fields, which are components \( \psi' \)'s of the string field, the so-called weak constraint \( \partial_a \partial^{\dot{a}} \psi = 0 \) has to be imposed. The weak constraint is reminiscent of the level matching condition \( N_L - N_R - \alpha' p_a w^a = 0 \) with zero winding numbers \( \omega^a \) or zero Kaluza-Klein momenta \( p_a \) that have to be imposed on the closed string physical states, at the presence of compact dimensions. Let us remind, indeed, that this is the modified version of the usual level matching condition \( N_L = N_R \) in non-compact Minkowski space for bosonic closed states. The partial derivative with an \( O(d, d) \) index \( A = 1, 2, \cdots, 2d \), is defined by \( \partial_A \equiv \left( \tilde{\partial}^a \partial_a \right)^T \), where \( \partial_a \equiv \partial/\partial x^a \) and \( \tilde{\partial}^a \equiv \partial/\partial \tilde{x}_a \), and the \( O(d, d) \) indices are raised or lowered by the \( O(d, d) \) invariant metric

\[
\eta_{AB} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\tag{4}
\]

The operator \( \partial_A \partial^A \) has to annihilate both fields also at higher levels and gauge parameters, let us denote the first or the latter generically by \( f \), hence \( \partial_A \partial^A f = 0 \). The weak constraint applied to a product of fields gives rise to the definition of the so-called strong constraint.

When subject to the weak constraint, i.e. when annihilated by the operator \( \partial_A \partial^A \), the above solutions \( h_{jk}, b_{jk}, d \) which result to be independent of \( \tilde{x}_a \) provide respectively the symmetric tensor (metric), antisymmetric tensor (Kalb-Ramond), and dilaton fields in \( D \) dimensions, while the solutions independent on \( x^a \) give their duals. In particular \( h_{jk} \) contains \( h_{\mu\nu} \equiv g_{\mu\nu}, h_{\mu a}, h_{ab} \).

In the context of Double Field Theory, one can obtain a low-energy cubic theory for the above defined states with \( N_L = N_R = 1 \) with vanishing Kaluza-Klein momenta and/or vanishing windings.
The cubic theory can be obtained from perturbations around the constant background fields, but the most non-trivial point is the first evidence, in the context of the closed string field theory, for a theory gauge-invariant under standard linearised diffeomorphisms and antisymmetric tensor gauge transformations (and subject to the weak constraint). This motivates studying a theory that could put in evidence stringy aspects at the low-energy limit.

Let us remind here that the definition in eq. (3) is considerably different from the one of the squared mass $\mathcal{M}^2$ in the uncompactified $(26-d)$-dimensional Minkowski space generated by compactification and given by (let us take $d = 1$):

$$
\mathcal{M}^2 \equiv -k^2 = p^2 + w^2 + \frac{2}{\alpha'}(N_L + N_R - 2)
$$

$$
= \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N_L + N_R - 2),
$$

The choice $N_L = N_R = 1$ with $M^2 = 2(N_L + N_R - 2)/\alpha' = 0$ excludes any stringy excitation $\sim 1/\alpha'$ when $d = 0$. For $n = m = 0$, the field content for $N_L = N_R = 1$ is given by the metric field $g_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$, and the scalar dilaton field $\Phi$ in terms of which the dilaton field $\phi$ with $\exp(-2\Phi) \equiv \exp(-2\phi)\sqrt{-\det g_{\mu\nu}}$ can be defined. The uncompact spacetime indices are labeled by $\mu, \nu$. In addition to those states which were already present in the uncompactified theory there are also states coming from the compactification: for $d = 1$ there are two massless vectors originating from the Kaluza-Klein compactification of the bosonic string on the circle, and that are just part of the originally 26-dimensional graviton and antisymmetric tensor field and, furthermore, a massless scalar field which is also a compactified degree of freedom of the 26-dimensional metric, and whose vacuum expectation value corresponds to the radius $R$ of the circle.

The eq. (3) shows that the states with $M^2 = 0$ in double field theory have to satisfy the condition $N_L + N_R = 2$. Effective field theory in the $(26-d)$-dimensional Minkowski space made by the $(26-d)$ non-compact dimensions is obtained by considering states for which $\mathcal{M}^2$ is zero or small with the leading terms given in an expansion in $\mathcal{M}^2$. Instead, in double field theory, an effective theory will be given therefore by those with $M^2 = 0$ or small. This implies that one could consider even states that are heavy from the point of view of the lower non-compact dimensions. This is the case of the states with $(N_L = 2, N_R = 0)$ and $(N_L = 0, N_R = 2)$, which are instead considered in this letter, and whose contribution has not been studied yet. The stringy effects due to the
simultaneous presence of momenta and windings only appear in the low-energy effective theory when \( N_L \neq N_R \), and these effects should be probed through a solution beyond the weak constraint.

The central question that we would like to address in this letter is the following: *What is the low-energy effective theory that highlights the stringy features in Double Field Theory?* As already claimed, this is essentially based on the case \( N_L = N_R = 1 \) with no Kaluza-Klein momenta and/or windings. One could think of its action as corresponding to the leading term in a suitable expansion in the perturbation parameter \( \alpha'/R^2 \) providing a particle description order-by-order with the leading order term corresponding to the case of \( N_L = N_R = 1 \) and with mere stringy effects appearing in the next-order term corresponding to the \((N_L, N_R) = (2, 0), (0, 2)\) case.

A problem related to non-associativity [22] is circumvented in Double Field Theory at the cubic level, allowing one to determine uniquely the quadratic level by a linear gauge transformation for the \((N_L, N_R) = (1, 1)\) [21]. We use the same way to obtain the quadratic theory with a new mass term \( \sim (N_L - N_R) \), and the action is T-dual invariant. Since it is impossible to generate a quadratic mass term from the \( O(26, 26) \) structure, the gauge transformation of the \( O(26, 26) \) fields should be independent on the new mass parameter. We discuss how to construct a gauge transformation with the \( O(26, 26) \) spacetime indices. This gauge transformation uniquely requires to know how the constraint acts on fields in the triple-products. Finally, we will show that the stringy effects cannot exist in the \( d = 1 \) case by solving the solution of the constraint as in the weak constraint [23].

2 Quadratic Theory

In this section, the procedure adopted in ref. [21] for getting the quadratic term of the action will be followed in order to write the analogous term for \((N_L, N_R) = (2, 0), (0, 2)\). In the case \((N_L, N_R) = (1, 1)\), the quadratic term in the low-energy effective action is obtained from the expansion around a constant background and can be uniquely constructed through the linear gauge transformation \( \delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu + \tilde{\partial}_\mu \tilde{\epsilon}_\nu + \tilde{\partial}_\nu \tilde{\epsilon}_\mu \), \( \delta b_{\mu\nu} = -(\tilde{\partial}_\mu \epsilon_\nu - \tilde{\partial}_\nu \epsilon_\mu) - (\partial_\mu \tilde{\epsilon}_\nu - \partial_\nu \tilde{\epsilon}_\mu) \), \( \delta \Phi = -(\partial_\mu \epsilon^\mu - \tilde{\partial}_\mu \tilde{\epsilon}^\mu)/2 \), starting from Einstein gravity theory [21]. We can also use the same procedure to construct the quadratic theory for the states \((N_L, N_R) = (2, 0), (0, 2)\). For the case \((N_L, N_R) = (1, 1)\), the
quadratic theory is [21]:

\[ S_{Q11} = \frac{1}{16\pi G} \int dx d\tilde{x} \left( \frac{1}{4} h^{\mu\nu} \partial_\mu \partial_\nu h_{\mu\nu} + \frac{1}{2} \tilde{\partial}^{\mu} h_{\mu\rho} \partial_\nu h^{\nu\rho} - 2\Phi \partial^\mu \partial^\nu h_{\mu\nu} - 4\Phi \partial^\mu \partial_\mu \Phi \\
+ \frac{1}{4} h^{\mu\nu} \partial_\mu \partial_\nu h_{\mu\nu} + \frac{1}{2} \tilde{\partial}^{\mu} h_{\mu\rho} \partial_\nu h^{\nu\rho} + 2\Phi \tilde{\partial}^{\mu} \tilde{\partial}^{\nu} h_{\mu\nu} - 4\Phi \tilde{\partial}^{\mu} \partial_\mu \Phi \\
+ \frac{1}{4} b^{\mu\nu} \partial_\mu \partial_\nu b_{\mu\nu} + \frac{1}{2} \partial^{\nu} b_{\mu\nu} \partial_\mu \partial_\nu b_{\mu\nu} \\
+ \frac{1}{4} b^{\mu\nu} \partial_\mu \partial_\nu b_{\mu\nu} + \frac{1}{2} \partial^{\nu} b_{\mu\nu} \partial_\mu \partial_\nu b_{\mu\nu} \\
+(\partial_\rho h^{\mu\nu})\partial^\nu b_{\mu\nu} + (\tilde{\partial}^{\rho} h_{\mu\nu})\partial_\nu b_{\mu\nu} - 4\Phi \partial^{\mu} \tilde{\partial}^{\nu} b_{\mu\nu} \right), \] (6)

where \( G \) is the gravitational constant. The spacetime indices in the above quadratic theory are raised or lowered by a constant metric. This theory is gauge invariant under the weak constraint [21].

In the case \((N_L, N_R) = (2, 0), (0, 2)\), the only modification to do concerns the weak constraint that now becomes [21]:

\[ \partial J \partial^J f = -2(N_L - N_R)/\alpha' f \equiv -\lambda f. \] (7)

By using the same linear gauge transformation with this constraint, one has

\[ \delta S_{Q2} = \frac{1}{16\pi G} \int dx d\tilde{x} \left( \frac{\lambda}{4} \delta (b^{\mu\nu} b_{\mu\nu}) + \lambda h^{\mu\nu} (\partial_\nu \epsilon_\mu + \tilde{\partial}_\nu \epsilon_\mu) + 4\lambda \Phi (\partial^\nu \epsilon_\nu - \tilde{\partial}^\nu \epsilon_\nu) \right). \] (8)

Since it is hard to introduce additional terms to cancel the non-invariant term, we assume \( \epsilon_\mu = \bar{\epsilon}_\mu \), obtaining:

\[ \delta S_{Q2} = \frac{1}{16\pi G} \int dx d\tilde{x} \left( \frac{\lambda}{4} \delta (b^{\mu\nu} b_{\mu\nu}) + \lambda h^{\mu\nu} (\partial_\nu \bar{\epsilon}_\mu + \tilde{\partial}_\nu \epsilon_\mu) + 4\lambda \Phi (\partial^\nu \bar{\epsilon}_\nu - \tilde{\partial}^\nu \epsilon_\nu) \right) \\
= \frac{1}{16\pi G} \int dx d\tilde{x} \left( \frac{\lambda}{4} \delta (b^{\mu\nu} b_{\mu\nu}) + \frac{\lambda}{4} \delta (h^{\mu\nu} h_{\mu\nu}) - 4\Phi \delta (d^2) \right). \] (9)

Hence we can introduce the new additional term to cancel the non-gauge invariant terms in the quadratic theory

\[ S_{Qn2} = \frac{1}{16\pi G} \int dx d\tilde{x} \left( -\frac{\lambda}{4} b^{\mu\nu} b_{\mu\nu} - \frac{\lambda}{4} h^{\mu\nu} h_{\mu\nu} + 4\lambda \Phi^2 \right). \] (10)
The choice $\epsilon_\mu = \tilde{\epsilon}_\mu$ is made because this theory cannot be described by a strong constraint solution. In other words, the fields in the target space and the fields in the dual target space do not constitute independent physical degrees of freedom.

When the dilaton field $\phi$ vanishes, the scalar dilaton at the quadratic order is $-h^\mu_\mu/4$. Therefore, we obtain the expected massive graviton term $h^{\mu\nu}h_{\mu\nu} - (h^\mu_\mu)^2$.

Now we discuss the field contents of the level $(N_L, N_R) = (2, 0), (0, 2)$. For the $(N_L, N_R) = (2, 0)$ case, we have $(n, m) = (1, 2), (2, 1), (-1, -2), (-2, -1)$ or $(n, m) = (-1, 2), (-2, 1), (1, -2), (2, -1)$. Each field content can be generated by the creation operators, $(a_{\tilde{j}1})^\dagger (a_{\tilde{j}2})^\dagger$ and $(a_{\tilde{j}2})^\dagger$, acting on the vacuum state. The creation operator $(a_{\tilde{j}1})^\dagger$ acting on the vacuum state generates the first-excited state. The indices of transverse light-cone directions are labeled by $\tilde{j}, \tilde{j}_1, \tilde{j}_2 = 2, 3, \cdots, 25$. For each choice of the $(N_L, N_R)$, the first creation operators generate the graviton field with $(D - 2)(D - 1)/2 - 1$ physical degrees of freedom, the dilaton field with one physical degree of freedom, and the one-form gauge field with $(D - 2)$ physical degrees of freedom. Here $D = 26$. Since the massive theory still has the gauge symmetry, the gauge parameters annihilate some degrees of freedom for getting the physical degrees of freedom in the quadratic theory.

We find that the field contents for $(N_L, N_R) = (2, 0), (0, 2)$ do not have an anti-symmetric field, but a one-form gauge field can be used to replace the anti-symmetric field for obtaining the gauge invariant quadratic theory $b_{\mu\nu} \rightarrow -((\tilde{\partial}_\mu A_\nu - \tilde{\partial}_\nu A_\mu) - (\partial_\mu A_\nu - \partial_\nu A_\mu)$. The gauge transformation of the $A_\mu$ is $\delta A_\mu \equiv \epsilon_\mu$. After doing that, the quadratic theory for the $(N_L, N_R) = (2, 0), (0, 2)$ is obtained. Therefore, we can find that the stringy effect mainly appears in the mass parameter $\lambda$. This implies that the stringy effect at the low-energy limit should come from a solution going beyond a strong constraint solution, and this must provide a massive theory to the target space.

Finally, we give some comments on T-duality [6, 7] for the quadratic action. Since what we constructed is just a deformation of $\lambda$, the only possible deformation in the T-duality also comes from $\lambda$. It is easy to show that the quadratic action is T-dual invariant for the Minkowski background field, vanishing Kalb-Ramond background field, and the $\Phi = \phi - h^\mu_\mu/4 + \cdots$. This means that the T-duality rule at the quadratic level does not yield a deformation from $\lambda$. In double field theory, we introduce a dual target space to have momentum and winding modes simultaneously. Therefore, the action in
the target space theory must be T-dual invariant. The T-duality provides a strongly restricted constraint to the quadratic theory.

3 Gauge Transformation

The O(D, D) generalized metric \[13\] cannot have the massive term because we have the equality \(\lambda \mathcal{H}_{M\bar{N}}^M \mathcal{H}_{\bar{M}\bar{N}} = \lambda\), where

\[
\mathcal{H}_{M\bar{N}} \equiv \begin{pmatrix} g^{-1} & -g^{-1}B \\Bg^{-1} & g - Bg^{-1}B \end{pmatrix},
\]

(11)

The O(D, D) spacetime indices are labeled by \(\bar{M}, \bar{N} = 0, 1, \cdots, 2D - 1\). Since we cannot generate the non-trivial term with \(\lambda\), the suitable gauge transformation should not depend on it. Here we use the same gauge transformation of the generalized metric \(\mathcal{H}_{\bar{M}\bar{N}}\) and the scalar dilaton field \(\Phi\):

\[
\delta_\xi \mathcal{H}^{\bar{M}\bar{N}} = \xi^\bar{P} \partial_{\bar{P}} \mathcal{H}^{\bar{M}\bar{N}} + (\partial^{\bar{M}} \xi_{\bar{P}} - \partial_{\bar{P}} \xi^{\bar{M}}) * \mathcal{H}^{\bar{P}\bar{N}} + (\partial^{\bar{N}} \xi_{\bar{P}} - \partial_{\bar{P}} \xi^{\bar{N}}) * \mathcal{H}^{\bar{M}\bar{P}},
\]

\[
\delta_\Phi d = -\frac{1}{2} \partial_{\bar{M}} \xi^{\bar{M}} + \xi^{\bar{M}} * \partial_{\bar{M}} \Phi,
\]

(12)

in which the gauge parameters \(\xi^\bar{P}\) and \(\xi_{\bar{P}}\) are defined by: \(\xi^\bar{P} \equiv \left(\bar{\xi}_\nu \xi^\mu\right)^T\) and \(\xi_{\bar{P}} \equiv \eta_{PQ}\bar{\xi}^Q\), and the partial derivatives \(\partial^{\bar{M}}\) and \(\partial_{\bar{M}}\) are defined by: \(\partial_{\bar{P}} \equiv \left(\bar{\partial}^\nu \partial_\mu\right)^T\) and \(\partial^\bar{P} \equiv \eta^{PQ}\partial_Q\). When we consider a non-zero \(\lambda\), we choose \(\tilde{\xi}_\nu = \xi_\nu\). The star product (*) provides the constraint to the fields and multi-products of the fields.

From the gauge transformation, we know that the constraint [21] \(\partial_{\bar{M}} \partial^{\bar{M}} \mathcal{A} = -\lambda \mathcal{A}\) implies \(\partial_{\bar{M}} \partial^{\bar{M}}(\mathcal{A} * \mathcal{B}) = -\lambda(\mathcal{A} * \mathcal{B})\). In order to study the gauge transformation, we first let the Lie derivative act on a scalar field \(\mathcal{L}_\xi \tilde{\Phi} = \xi^\mu \partial_\mu * \tilde{\Phi}\). By direct computation, one can explicitly show that \([\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}]_* \tilde{\Phi} = \mathcal{L}_{[\xi_1, \xi_2]}_* \tilde{\Phi}\), where \([\xi_1, \xi_2]_* \equiv \xi_1^\mu * \partial_\mu \xi_2^\nu - \xi_2^\mu * \partial_\mu \xi_1^\nu\). This implies that the gauge transformation does not depend on the \(\lambda\). The closure property can appear because the star product first acts on the gauge parameters and then on the scalar field.

We first consider the gauge transformation of the scalar dilaton field:

\[
[\delta_{\xi_1}, \delta_{\xi_2}]_* \Phi = \frac{1}{2} \partial_{\bar{M}}[\xi_1, \xi_2]_{\bar{C}}^{\bar{M}} - [\xi_1, \xi_2]_{\bar{C}}^{\bar{M}} * \partial_{\bar{M}} \Phi
\]

\[
= -\delta_{[\xi_1, \xi_2]_{\bar{C}}} * \Phi,
\]

(13)
We also find that the above gauge transformation does not depend on $\lambda$. Furthermore, the result also yields the constraint in the triple-products $\partial_{\bar{M}}\partial_{\bar{N}}(\mathcal{A} \ast \mathcal{B} \ast \mathcal{C}) = -\lambda(\mathcal{A} \ast \mathcal{B}) \ast \mathcal{C}$. This condition implies $(\mathcal{A} \ast \partial^M \mathcal{B}) \ast \partial_M \mathcal{C} = \frac{1}{4}(\mathcal{A} \ast \mathcal{B}) \ast \mathcal{C}$. The above is already restricted by how the constraint acts on the fields in the triple-products. When $\lambda = 0$, the constraint is equivalent to the strong constraint but going beyond this latter for $\lambda \neq 0$. Let us observe that the fields do not depend on the doubled non-compact directions.

We also show that the generalized metric has the same property as in the scalar dilaton $[\delta_{\xi_1}, \delta_{\xi_2}]_* \mathcal{H}^\bar{M}\bar{N} = -[\delta_{\xi_1}, \delta_{\xi_2}]_C \mathcal{H}^\bar{M}\bar{N}$. Although string theory only requires the constraint acting on a single field, requiring the suitable gauge transformation provides the uniquely additional constraints to the triple-products.

Finally, we comment on how to apply the gauge transformation to the non-vanishing $\lambda$. For a simple extension to the non-zero $\lambda$, we want to retain the $O(D, D)$ space-time indices. When $\mathcal{H}_{\bar{M}\bar{N}}$ is promoted to an $O(D, D)$ matrix, one needs to integrate out the auxiliary field $\lambda$ from the term $\lambda_{\bar{M}\bar{N}}(\mathcal{H} \mathcal{H} - \eta)^{\bar{M}\bar{N}}$ in the action. After turning on $\lambda \neq 0$, the simplest way is to deform the relation that we obtained by integrating out $\lambda$ as $\mathcal{H} \mathcal{H} = \cdots$. When the constraint acts on the $\mathcal{H} \mathcal{H}$, we obtain $\partial_{\bar{M}}\partial_{\bar{N}}(\mathcal{H} \mathcal{H}) = -\lambda(\mathcal{H} \mathcal{H})$. For consistency, we should do a deformation to obtain $\partial_{\bar{M}}\partial_{\bar{N}}(\cdots) = -\lambda(\cdots)$. If we choose $(\cdots) = \eta$, the $\lambda$ must vanish, and a suitable deformation provides a consistent relation compatible with the constraint. Retaining the $O(D, D)$ indices (even losing an $O(D, D)$ structure) is quite useful because the gauge transformation of the generalized metric can be directly applied to $\lambda \neq 0$, but the matrix elements of the $\mathcal{H}_{\bar{M}\bar{N}}$ are changed. We apply the gauge transformation without using the relation $\mathcal{H} \mathcal{H} = \eta$.

4 $d = 1$

Now we consider a one-doubled compact direction $d = 1$ to solve the constraint. Then the star product is defined by the following [21]:

$$A \ast 1 \equiv \sum_K A_K \exp(iKX)\delta_{KK,\lambda}.$$ 

$$A \ast B = B \ast A \equiv \sum_{K_A,K_B} A_{K_A} B_{K_B} \exp(i(K_A + K_B)X)\delta_{KK_A\lambda} \delta_{KK_B\lambda} \delta_{KK_A\lambda} \cdot \frac{-1}{4}. \quad (14)$$
The generic solution of the equation $K_A K^A = \lambda$ is given by the $K_A = \left( a \ \lambda/(2a) \right)^T$ for each non-zero $a$. Therefore, we have the generic solution for the momenta $K_A = \left( a \ \lambda/(2a) \right)^T$ and $K_B = \left( b \ \lambda/(2b) \right)^T$. Then this provides $a/b + b/a = -1$ when $\lambda \neq 0$ from the equation $K_B K_C = -\lambda/2$. It is easy to show that the $a/b$ is an imaginary number. This implies no solution with $\lambda \neq 0$ in the $d = 1$. We emphasize that the result is quite general without considering the triple-products. Hence we need to go beyond the $d = 1$ case in order to obtain a non-trivial solution with $\lambda \neq 0$.

5 Outlook

Following what done in ref. [21] in getting the low-energy effective theory in double field theory in the case $(N_L, N_R) = (1, 1)$, we have constructed the quadratic theory for the case $(N_L, N_R) = (2, 0), (0, 2)$ level. From the spectrum, we should obtain the next non-trivial order effect in the low-energy effective theory. The main appearance of the stringy effect is the new mass term, which is proportional to the difference of $N_L - N_R$. The gauge transformation with the $O(D, D)$ indices was constructed. From the gauge transformation, it turns out that the $\lambda$ parameter only appears in the generalized metric $H_{MN}$, as expected. Hence this has simplified a non-linear construction aimed to find a suitable matrix element for $H_{MN}$. This suggests that a non-linear extension becomes a possible task. We have shown that in the $d = 1$ case it is impossible to have a solution which could exhibit a mere stringy effect. This also implies that a higher dimensional torus cannot have the new mass term. Therefore, we should try a solution, which cannot be the Cartesian products of a one-dimensional compact manifold. It would be interesting to find such a solution and explore a non-linear extension for the quadratic theory because all this could give information on stringy effects at the low-energy limit. It seems that this analysis can be done only in Double Field Theory because this latter does not loose tracks of the winding modes (or dual target space).

Acknowledgments

The authors would like to thank David S. Berman and Jeong-Hyuck Park for useful discussions and the Yukawa Institute for Theoretical Physics at the Kyoto University for their hospitality and partial support during the workshop “New Frontiers in String Theory 2018”.

Chen-Te Ma was supported by the Post-Doctoral International Exchange Program
and China Postdoctoral Science Foundation, Postdoctoral General Funding: Second Class (Grant No. 2019M652926) and is indebted to Nan-Peng Ma for his encouragement. He is also grateful to the Tohoku University, Okinawa Institute of Science and Technology Graduate University, Istituto Nazionale di Fisica Nucleare - Sezione di Napoli, Kadanoff Center for Theoretical Physics at the University of Chicago, Stanford Institute for Theoretical Physics at the Stanford University, Kavli Institute for Theoretical Physics at the University of California Santa Barbara, National Tsing Hua University, Israel Institute for Advanced Studies at the Hebrew University of Jerusalem, Jinan University, Institute of Physics at the University of Amsterdam, and Institute of Theoretical Physics at the Chinese Academy of Sciences for their kind hospitality in different stages of this work.

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