On irreducibility of certain Schur polynomials over fields of finite characteristic

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Abstract

We present an elementary proof that the Schur polynomial corresponding to an increasing sequence of exponents $(c_0, \ldots, c_{n-1})$ with $c_0 = 0$ is irreducible over every field of characteristic $p$ whenever the numbers $d_i = c_{i+1} - c_i$ are all greater than 1, not divisible by $p$, and satisfy $\gcd(d_i, d_{i+1}) = 1$ for every $i$.

1 Introduction

In this note, we investigate the classical Schur polynomials of the form

$$S_c(x) = \frac{V_c(x)}{V_{(0,1,\ldots,n-1)}(x)},$$

where $x = (x_0, \ldots, x_{n-1})$ is a sequence of indeterminates, $c = (c_0, \ldots, c_{n-1})$ — a (strictly) increasing sequence of non-negative integer exponents, and $V_c(x) = \det [x_j^{c_i}]$ is the generalized Vandermonde determinant. These polynomials have been extensively studied; see [4] and [5] for general reference. Surprisingly, little has been known for a long time about their irreducibility. In 1958, Farahat [3] proved that Schur polynomials are irreducible when the elementary symmetric polynomials $h_i(x)$ (on which Schur polynomials do depend polynomially) play the role of indeterminates, which did not clarify when the same happens over the basic indeterminates $x_i$. This question has been answered (over the field of complex numbers) only in recent years, by different methods, both in [2] and [6]:

**Theorem A** ([2], [6]). Let $K = \mathbb{C}$. Then, $S_c(x)$ is irreducible in $K[x]$ if and only if $c_0 = 0$ and $\gcd(c) = 1$.

As stated in [2], there seem to have been no prior research in this direction, apart from certain special cases ([1], [7]).

Our goal is to generalize Theorem A, in a possible wide range of cases, for fields of finite characteristic. Since the “only if” part of the theorem is easily seen to hold over any field $K$, we focus on the opposite direction. As observed by Prof. Andrzej Schinzel, the fact that Theorem A is satisfied over $\mathbb{C}$ forces it to hold over any field of sufficiently large characteristic $p$, by means of elimination theory [8,
Theorem 32]. However, this method leads to estimates for $p$ which depend multiply exponentially on the numbers $c_i$, which is not desirable in practice. On the other hand, from [7, Theorem 1] it follows that $S_{c}(x)$ is irreducible over any field $K$ of characteristic $p$ follows provided that

$$n = 3, \quad c_2 > 5, \quad p \not| c_1c_2(c_2 - c_1).$$

The main result of this paper requires significantly different preconditions:

**Theorem 1.** Denote $d_i = c_{i+1} - c_i$ for $0 \leq i \leq n - 2$. Assume that

$$c_0 = 0, \quad d_i > 1 \quad \text{for} \quad 0 \leq i \leq n - 2 \quad \text{and} \quad \gcd(d_i, d_{i+1}) = 1 \quad \text{for} \quad 0 \leq i \leq n - 3.$$

Then, $S_{c}(x)$ is irreducible in $K[x]$ whenever $K$ is a field of characteristic $p$ such that

$$p \not| d_i \quad \text{for every} \quad 0 \leq i \leq n - 2.$$

The conditions imposed on $d_i$ are strictly stronger than the condition $\gcd(c) = 1$ from Theorem A. Nevertheless, Theorem 1 seems to cover a significant range of non-trivial cases.

The proof of Theorem 1 will proceed by induction on $n$. For $n = 1, 2$, the theorem is trivially true since there are no sequences $c$ satisfying all the assumptions. Thus, the base of our induction will be the case $n = 3$. In this setting, our assumptions are similar to (1) and hence the case $n = 3$ of Theorem 1 follows almost entirely from [7, Theorem 1], with the only exception being the case when $c_2 = 5$ or $p \not| c_2$. Nevertheless, in Section 2 we present independently a general proof, which seems to be much more elementary than that of [7]. Then, the inductive reasoning given in Section 3 proves the theorem for $n \geq 4$, which is, to author’s knowledge, a new result except for some very special cases considered in [1].

Among the existing reasonings given in [7], [2] and [6], our proof mostly resembles that of [6], particularly in its elementary spirit of polynomial arithmetic. However, we cannot see any direct links between Rajan’s argument and the present method.

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**1.1 Preliminaries**

Before proceeding with the proof, we provide some auxiliary notations and facts.

By bold letters we denote sequences of integers or indeterminates from $K$; the elements of such sequences will be numbered starting from zero. The sequence $(0, \ldots, n - 1)$ will be denoted by $e_n$. Element removal will be marked by a hat: for example, $\hat{c}_{i,j}$ stands for $c$ with $c_i$ and $c_j$ removed. If $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$, we will denote by $a - b$ the sequence $(a_0 - b, \ldots, a_{n-1} - b)$.

For any $0 \leq i \leq n - 1$, we define the $x_i$-maximal (resp. minimal) part of a polynomial $P \in K[x]$, denoted $\max_{x_i} P$ (resp. $\min_{x_i} P$), to be the sum of all monomials in $P$ with the maximal (resp. minimal) $x_i$-degree; this degree will be denoted by $\deg_{x_i} P$ (resp. $\min \deg_{x_i} P$). The $x_i$-width of $P$, denoted $\Lambda_{x_i} P$, is the difference between the maximal and minimal $x_i$-degrees of $P$. 
Fact 1. The operators $\max_{x_i}$ and $\min_{x_i}$ are multiplicative.

Corollary 2. The operators $\deg_{x_i}$, $\min \deg_{x_i}$, $\Lambda_{x_i}$ are additive under polynomial multiplication.

For a sequence $x = (x_0, \ldots, x_{n-1})$ of indeterminates and $s = (s_0, \ldots, s_{n-1}) \in \mathbb{Z}^n$, we denote
$$x^s = x_0^{s_0} \cdots x_{n-1}^{s_{n-1}} \in K(x).$$

We will also write $x^k$ to denote $(x_0 \ldots x_{n-1})^k$. For any two rational functions $F, G \in K(x)$, we will write $F \sim G$ if there is some $s \in \mathbb{Z}^n$ and $a \in K \setminus \{0\}$ for which $F = a \cdot x^s \cdot G$. This is an equivalence relation.

2 The case $n = 3$

In this chapter, we provide a proof of Theorem 1 for $n = 3$. For simplicity, we denote $c = (0, a, b)$. Then, the assumptions of the theorem can be stated as follows:
$$1 < a < b - 1, \quad \gcd(a, b) = \gcd(a, b - a) = 1, \quad p \nmid a(b - a).$$

Suppose that
$$S_c(x) = A(x) \cdot B(x) \tag{2}$$
for some non-constant $A, B \in K[x]$. We will derive a contradiction. In what follows we assume, without losing generality, that the field $K$ is algebraically closed.

2.1 The structure of $S_c(x)$

In order to proceed with the proof, we will first analyze the expansion of $S_c(x)$ with respect to a single variable $x_i$, and investigate the divisibility properties (in $K[x_i]$) of the coefficients of this expansion.

For any $k \geq 1$, we denote
$$C_k(x) = \frac{x^k - 1}{x - 1} = 1 + x + x^2 + \ldots + x^{k-1}.$$

Fact 3. If $p \nmid k$, then $C_k(x)$ has $k - 1$ pairwise distinct roots in $K$, which are all the $k$-th roots of unity distinct from 1.

Proof. If $p \nmid k$, then the product $C_k(x) \cdot (x - 1) = x^k - 1$ is coprime to its derivative, $k x^{k-1}$ and hence has no multiple roots. On the other hand, all its roots must be $k$-th roots of unity, not equal to 1 since $C_k(1) = k \mod p \neq 0$. By Bézout’s Theorem, there can be at most $k - 1$ such elements, so they all must be roots of $C_k$. \qed

We may also define a two-variable version of $C_k$, and it follows from the above fact that it decomposes as follows:
$$C_k(x, y) := \frac{x^k - y^k}{x - y} = \sum_{i=0}^{k-1} x^i y^{k-1-i} = \prod_{\alpha \neq 1, \alpha^k = 1} (x - \alpha y). \tag{3}$$
Note also that the following holds in the ring of rational functions $K(x, y)$:

$$C_k(x, y) = \frac{x^k - y^k}{x - y} = (xy)^{k-1} \cdot \frac{y^{-k} - x^{-k}}{y^{-1} - x^{-1}} = (xy)^{k-1} \cdot C_k(\frac{1}{x}, \frac{1}{y}).$$

**Fact 4.** Let $x, y, z$ be any permutation of the variables $x_0, x_1, x_2$ and let

$$S_c(x) = \sum_{i=0}^{b-2} P_i(y, z) \cdot x^i$$

be the expansion of $S_c(x)$ with respect to $x$. Then

(5a) \hspace{1cm} P_0 \sim C_{b-a}(y, z), \quad P_{b-2} \sim C_a(y, z),

(5b) \hspace{1cm} C_{b-a}(y, z) \mid P_i \hspace{1cm} \text{for} \hspace{1cm} 0 \leq i \leq a - 1, \quad C_a(y, z) \mid P_i \hspace{1cm} \text{for} \hspace{1cm} a - 1 \leq i \leq b - 2.

(Here, divisibility is regarded in $K[y, z]$).

**Proof.** 1. For convenience, we will start with a proof under additional assumption that $a \leq b - a$. Denote $D = a + b - 2$. Evaluating $S_c$ from its definition leads to

$$S_c = \left| \begin{array}{ccc} x^a & y^b & x^b \\ y^a & y^b & z^b \\ z^a & z^b & z^b \end{array} \right| \cdot \frac{(y - x)(z - x)(z - y)}{(y - x)(z - x)(z - y)} = \left| \begin{array}{ccc} y^a - x^a & y^b - x^b \\ z^a - x^a & z^b - x^b \end{array} \right| \cdot \frac{(y - x)(z - x)(z - y)}{(y - x)(z - x)(z - y)} =$$

$$= \left| \begin{array}{ccc} C_a(x, y) & C_b(x, y) \\ C_a(x, z) & C_b(x, z) \end{array} \right| \cdot \frac{(z - y)}{(z - y)} = \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} x^{D-i-j} (y^i z^j - z^i y^j).$$

This implies that for every $0 \leq d \leq b - 2$ we have

$$P_d = \frac{1}{z - y} \sum_{0 \leq i \leq a-1, \atop 0 \leq j \leq b-1, \atop i+j=D-d} (y^i z^j - z^i y^j) = 0 + \frac{1}{z - y} \sum_{0 \leq i \leq a-1, \atop 0 \leq j \leq b-1, \atop i+j=D-d} (y^i z^j - z^i y^j) =$$

$$= \frac{1}{z - y} \left( \sum_{i=\max(0, D-d-b+1)}^{\min(a-1, D-d-a)} y^i z^{D-d-i} - \sum_{j=\max(a, D-d-a+1)}^{\min(b-1, D-d)} y^j z^{D-d-j} \right)$$

Both sums appearing in the last formula must have the same length, which we denote by $l$. Let $i_0, j_0$ denote the starting values for $i$ and $j$ in the corresponding sums; then,

$$P_d = \frac{1}{z - y} \left( \sum_{i=i_0}^{i_0+l-1} y^i z^{D-d-i} - \sum_{j=j_0}^{j_0+l-1} y^j z^{D-d-j} \right) = C_i(y, z) \cdot C_{j_0-i_0}(y, z) \cdot y^{i_0} z^{D-d-j_0-l+1}.$$

(7)

(In fact, $D - d - j_0 - l + 1$ equals $i_0$, but we will not need this). Coming back to (6), we have

$$j_0 - i_0 = \begin{cases} b - a & \text{if} \ 0 \leq d \leq a - 1, \\ b - 1 - d & \text{if} \ a - 1 \leq d \leq b - a - 1, \\ a & \text{if} \ b - a - 1 \leq d \leq b - 2, \end{cases} \quad l = \begin{cases} d + 1 & \text{if} \ 0 \leq d \leq a - 1, \\ a & \text{if} \ a - 1 \leq d \leq b - a - 1, \\ b - 1 - d & \text{if} \ b - a - 1 \leq d \leq b - 2, \end{cases}$$
whence
\[
C_i \cdot C_{j_0 - i_0} = \begin{cases} 
C_{b-a} \cdot C_{d+1} & \text{for } 0 \leq d \leq a - 1, \\
C_a \cdot C_{b-1-d} & \text{for } a - 1 \leq d \leq b - 2.
\end{cases}
\]

In view of the above conditions, the claim follows straightforwardly from (7). This finishes the proof in the case when \( a \leq b - a \).

2. Now suppose that \( a > b - a \); then we may apply the above argument for the expansion
\[
S_{\overline{c}}(x, y, z) = \sum_{i=0}^{b-2} \overline{P}_i(y, z) \cdot x^i, \quad \text{where } \overline{c} = (0, b - a, b),
\]
obtaining
\[
\overline{P}_0 \sim C_a(y, z), \quad \overline{P}_{b-2} \sim C_{b-a}(y, z), \\
C_a(y, z) \mid \overline{P}_i \quad \text{for } 0 \leq i \leq b - a - 1, \quad C_{b-a}(y, z) \mid \overline{P}_i \quad \text{for } b - a - 1 \leq i \leq b - 2.
\]

Now, in the field of rational functions \( K(x) \), we have
\[
S_c(x, y, z) = \frac{V(0, a, b)(x)}{V(0, 1, 2)(x)} = (xyz)^{b-2} \cdot \frac{V(-b, a-b, 0)(x)}{V(-2, -1, 0)(x)} = (xyz)^{b-2} \cdot S_{\overline{c}}(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}),
\]
whence it follows that
\[
P_i(y, z) = (yz)^{b-2} \cdot \overline{P}_{b-2-i}(\frac{1}{y}, \frac{1}{z}) \quad \text{for } 0 \leq i \leq b - 2.
\]
This means that the conditions (5) for the expansion \( S_c = \sum_i P_i \cdot x^i \) follow from (8) and (4). \( \Box \)

2.2 Proof of the Theorem

In the following steps, we derive contradiction from the assumption (2), relying on Fact 4. We retain the notation used in this fact, in particular, \( (x, y, z) \) denotes an arbitrary permutation of the variables \((x_0, x_1, x_2)\).

1. Since the numbers \( a, b - a \) are coprime and not divisible by \( p \), it follows from Fact 3 that the polynomials \( C_a(x) \) and \( C_{b-a}(x) \) have no common roots in \( K \). Let \( \alpha \) be any root of \( C_a(x) \); then \( y - \alpha z \) is a factor of \( C_a(y, z) \) but not of \( C_{b-a}(y, z) \). Let \( \pi : K[x, y, z] \rightarrow K[x, z] \) denote the homomorphism resulting from substituting \( \alpha z \) for \( y \), which may be viewed as a reduction modulo \( y - \alpha z \).

In what follows, we will technically operate on the images \( \pi(A) \) and \( \pi(B) \), but perhaps the more intuitive way to understand the proof is to think of the \( \pi \)-surviving part of \( A \), understood as the sum of those summands in the expansion \( A = \sum_i A_i \cdot x^i \) which do not vanish under \( \pi \) (and likewise for \( B \)). This polynomial is not equal to \( \pi(A) \), but it clearly has the same minimal and maximal \( x \)-degree.

For \( X = A, B \), we will say that the \( \pi \)-remaining part of \( X \) is aligned to the left (resp. right) if its minimal (resp. maximal) \( x \)-degree is the same as of the whole \( X \).

2. Applying \( \pi \) to both sides of (2) gives
\[
\pi(A) \cdot \pi(B) = \sum_{i=0}^{b-2} \pi(P_i) \cdot x^i,
\]
where Fact 4 guarantees that

$$\pi(P_0) \neq 0, \quad \pi(P_i) = 0 \quad \text{for } i \geq a - 1.$$  

This implies that the \( \pi \)-surviving parts in \( A \) and \( B \) are aligned to the left:

\[ (9) \quad \min \deg_x \pi(A) = \min \deg_x A = \min \deg_x \pi(B) = \min \deg_x B = 0 \]

and that we can control the sum of their widths:

\[ (10) \quad \Lambda_\pi \pi(A) + \Lambda_\pi \pi(B) = \Lambda_\pi \pi(S_c) \leq a - 2. \]

On the other hand, we have

$$\Lambda_\pi A + \Lambda_\pi B = \Lambda_\pi S_c = b - 2 > a - 2,$$

hence the \( \pi \)-surviving parts cannot be aligned to the right in both \( A \) and \( B \). However, this must happen for one of them: otherwise both terms \( \max_x A \), \( \max_x B \) would vanish under \( \pi \) and hence be divisible by \( y - \alpha z \), while we know on the other hand that

$$\max_x A \cdot \max_x B = \max_x S_c = P_{b-2} \sim C_\alpha(y, z)$$

which is not divisible by \( (y - \alpha z)^2 \) because \( \alpha \) is not a multiple root of \( C_\alpha(x) \). Concluding, the \( \pi \)-surviving part must be aligned to the right in exactly one of the factors \( A, B \); assume without loss of generality that this happens for \( A \):

\[ (11) \quad \Lambda_\pi \pi(A) = \Lambda_\pi A, \quad \Lambda_\pi \pi(B) < \Lambda_\pi B. \]

**3.** Now, let \( \beta \) be any root of \( C_{b-a}(x) \), and consider the homomorphism \( \rho : K[x, y, z] \rightarrow K[x, z] \) related to substituting \( \beta z \) for \( y \); this can be seen as the reduction modulo \( y - \beta z \) which divides \( C_{b-a}(y, z) \) but not \( C_\alpha(y, z) \). Since the factors \( C_\alpha(y, z) \) and \( C_{b-a}(y, z) \) play a symmetric role in \( (5) \), the reasoning from step 2 can be repeated with \( \rho \) in place of \( \pi \), with some modifications which include interchanging the concepts of left and right alignment. By this method, we obtain as an analogue of \( (9) \) that the \( \rho \)-surviving parts of \( A, B \) are aligned to the right:

\[ (9') \quad \deg_x \rho(A) = \deg_x A, \quad \deg_x \rho(B) = \deg_x B; \]

then — as an analogue of \( (10) \) — that

\[ (10') \quad \Lambda_\rho \rho(A) + \Lambda_\rho \rho(B) = \Lambda_\rho \rho(S_c) \leq b - a - 2 \]

and finally — as an analogue of \( (11) \) — that the \( \rho \)-surviving part is aligned to the left in exactly one of \( A, B \). However, since we have already distinguished \( A \) from \( B \) in \( (11) \), we must now consider both cases:

\[ (11a') \quad \Lambda_\rho \rho(A) = \Lambda_\rho A, \quad \Lambda_\rho \rho(B) < \Lambda_\rho B, \]

\[ (11b') \quad \Lambda_\rho \rho(A) < \Lambda_\rho A, \quad \Lambda_\rho \rho(B) = \Lambda_\rho B. \]

Assuming that \( (11b') \) holds, we obtain a contradiction:

\[ b - 2 = \Lambda_\rho S_c = \Lambda_\rho A + \Lambda_\rho B \quad (11),(11b') \quad \Lambda_\rho \pi(A) + \Lambda_\rho \rho(B) \leq \]

\[ \leq \Lambda_\rho \pi(A) + \Lambda_\rho \rho(A) + \Lambda_\rho \pi(B) + \Lambda_\rho \rho(B) \quad (10),(10') \quad \leq (b - a - 2) + (a - 2) = b - 4. \]
Hence (11a') must hold, which intuitively means that both the \(\pi\)-surviving part and the \(\rho\)-surviving part in \(A\) are aligned to both sides in \(A\), while in \(B\) one of them is aligned only to the left and the other only to the right.

4. Our assumption that \(a, b - a > 1\) implies that the polynomials \(C_a(x), C_{b-a}(x)\) are not constant and hence there exist some \(\alpha\) and \(\beta\) as above. Then, using the conditions derived in steps 2 and 3, we obtain:

\[
\Lambda_x \pi(B) \leq a - 2 - \Lambda_x \pi(A) \leq a - 2 - \Lambda_x A, \tag{10}
\]

\[
\Lambda_x \rho(B) \leq b - a - 2 - \Lambda_x \rho(A) \leq b - a - 2 - \Lambda_x A, \tag{11}
\]

\[
\Lambda_x B = \Lambda_x S_c - \Lambda_x A = b - 2 - \Lambda_x A, \tag{11a'}
\]

from which we deduce that \(\Lambda_x B \geq \Lambda_x \pi(B) + \Lambda_x \rho(B) + \Lambda_x A + 2\), in particular,

\[
\Lambda_x B > \Lambda_x A. \tag{12}
\]

On the other hand, from (12) we obtain that

\[
\deg_x \pi(B) \leq a - 2, \tag{9}
\]

\[
\min \deg_x \rho(B) \geq a. \tag{12}
\]

Denoting \(B = \sum_i B_i \cdot x^i\), we deduce that \(\pi(B_{a-1}) = \rho(B_{a-1}) = 0\), whence

\[
(y - \alpha z) \cdot (y - \beta z) \mid B_{a-1}. \tag{14}
\]

5. Observe that the property (13) can be used to distinguish between \(A\) and \(B\) instead of (11), and its important advantage over (11) is that it does not depend on the choice of \(\alpha\). Hence it follows that, with \(A\) and \(B\) chosen to satisfy (13), the conditions (9–11) hold for every \(\alpha\) being a root of \(C_a(x)\), and the conditions (9'), (10'), (11a') hold for every \(\beta\) being a root of \(C_{b-a}(x)\). It then follows that (14) holds for every suitable \(\alpha\) and \(\beta\), which by (3) means that

\[
C_a(y, z) \cdot C_{b-a}(y, z) \mid B_{a-1}. \tag{15}
\]

However, then we have

\[
\deg_y B \geq \deg_y C_a(y, z) + \deg_y C_{b-a}(y, z) = (a - 1) + (b - a - 1) = \deg_y S_c,
\]

which implies that \(\deg_y A = 0\), i.e. \(A\) does not depend on \(y\). Similarly, \(A\) must not depend on \(z\).

6. Recall that \(x, y, z\) were chosen as any permutation of the variables \(x_0, x_1, x_2\). Therefore it follows from the above argument that if \(S_c(x)\) decomposes as \(X \cdot Y\), then for every \(i = 0, 1, 2\), one of the factors \(X, Y\) (call it \(A_i\)) must depend only on \(x_i\). It remains to consider two cases: if \(A_0 \neq A_1\), then \(S_c = A_0 \cdot A_1\) cannot at all depend on \(x_2\), a contradiction; otherwise the factor \(A_0 = A_1\) must be constant. This finishes the proof.

3 The inductive step

We will now prove Theorem 1 for given \(n \geq 4\), assuming its validity for \(n - 1\). Let the numbers \(c_i, d_i, p\) and the field \(K\) satisfy the assumptions of the theorem and suppose that \(S_c(x) = A(x) \cdot B(x)\); we will show that one of the factors \(A, B\) must be a constant.
The main idea of the proof is to consider the shape of \( S_c \), by which we mean the set of these \( s \in \mathbb{Z}_{\geq 0}^n \) for which \( S_c \) contains a monomial of the form \( a \cdot x^s, \ a \in K \). Intuitively, this set is a discretized version of a highly regular polytope. The key observation (see Fact 5 below) is that certain its faces are Schur polynomials in \( n - 1 \) variables, which (up to multiplication by a monomial) satisfy our inductive assumption. This will give us strong limitations on the shape of \( A \) and \( B \), enabling to derive a contradiction.

### 3.1 Auxiliary facts

**Fact 5.** We have

\[
\begin{align*}
\min_{x_i} S_c(x) &= \hat{x}^{c_1-1}_i \cdot S_{c_0-c_1}(\hat{x}_i), \\
\max_{x_i} S_c(x) &= S_{c_{n-1}}(\hat{x}_i).
\end{align*}
\]

*Proof.* First, observe that by Laplace expansion we have (recall that we assume \( c_0 = 0 \)):

\[
\begin{align*}
\min_{x_i} V_c(x) &= V_{c_0}(\hat{x}_i) = \hat{x}^{c_1}_i \cdot V_{c_0-c_1}(\hat{x}_i), \\
\max_{x_i} V_c(x) &= V_{c_{n-1}}(\hat{x}_i),
\end{align*}
\]

which proves the claim since the operators \( \min_{x_i}, \max_{x_i} \) are multiplicative. For the minimal part, it remains to notice that

\[
\begin{align*}
\min_{x_i} S_c(x) &= \frac{\min_{x_i} V_c(x)}{\min_{x_i} V_{c_0}(x)} = \frac{\hat{x}^{c_1}_i \cdot V_{c_0-c_1}(\hat{x}_i)}{\hat{x}^{1}_i \cdot V_{(1,2,...,n)-1}(\hat{x}_i)} = \hat{x}^{c_1-1}_i \cdot V_{c_0-c_1}(\hat{x}_i),
\end{align*}
\]

\[
\square
\]

**Corollary 6.** For every \( i \neq j \), we have

\[
\min_{x_i} \max_{x_j} S_c(x) = \hat{x}^{c_1-1}_i \cdot S_{c_0,n-1-c_1}(\hat{x}_{i,j}) = \max_{x_j} \min_{x_i} S_c(x).
\]

**Corollary 7.** For every \( i \neq j \), we have

\[
\begin{align*}
\min_{x_j} \max_{x_j} S_c(x) &= \min_{x_j} S_c(x), \\
\max_{x_j} \min_{x_j} S_c(x) &= \max_{x_j} S_c(x).
\end{align*}
\]

**Lemma 8.** Let \( C(x) \) be a factor of \( S_c(x) \) such that

\[
\begin{align*}
\min_{x_i} C &\sim \min_{x_i} S_c, & \max_{x_j} C &\sim \max_{x_j} S_c.
\end{align*}
\]

Then, for every \( k \neq i, j \), we have

\[
\Lambda_{x_k} C = \Lambda_{x_k} S_c.
\]

*Proof.* Assume that such \( i, j, C \) exist and let

\[
\begin{align*}
\min_{x_i} C &= a \cdot x^s \cdot \min_{x_i} S_c, \\
\max_{x_j} C &= a' \cdot x^{s'} \cdot \max_{x_j} S_c.
\end{align*}
\]

Denote \( r = s' = s \). We will show that \( r = (0, \ldots, 0) \).
First, note that since $S_c$ is homogeneous, $C$ must also be, whence in particular

$$ r \circ (1, \ldots, 1) = 0. \tag{16} $$

Then, we notice that, by Corollary 6,

$$ \min_{x_i} \max_{x_j} C = a' \cdot x^s \cdot \min_{x_i} \max_{x_j} A = a' \cdot x^s \cdot \max_{x_i} \min_{x_j} A = \frac{a'}{a} \cdot x^r \cdot \max_{x_i} \min_{x_j} C, $$

whence

$$ \deg_{x_k} \min_{x_i} \max_{x_j} C = r_k + \deg_{x_k} \max_{x_i} \min_{x_j} C \quad \text{for all } k. \tag{17} $$

On the other hand, by the definition of the minimal and maximal part, we have

$$ \deg_{x_i} \min_{x_i} \max_{x_j} C = \min \deg_{x_i} \max_{x_j} C \geq \min \deg_{x_i} \max_{x_j} C, $$
$$ \deg_{x_j} \min_{x_i} \max_{x_j} C = \deg_{x_j} \max_{x_i} \min_{x_j} C \geq \deg_{x_j} \min_{x_i} \max_{x_j} C, $$

which in view of (17) implies that

$$ r_i, r_j \geq 0. \tag{18} $$

Finally, for every $k \neq i, j$, using Corollary 7 together with (15) we obtain

$$ \min \deg_{x_k} C \leq \min \deg_{x_k} \max_{x_j} C = s'_k + \min \deg_{x_k} \max_{x_j} S_c = s'_k + \min \deg_{x_k} S_c, $$
$$ \deg_{x_k} C \geq \deg_{x_k} \min_{x_i} \max_{x_j} C = s_k + \deg_{x_k} \min_{x_i} S_c = s_k + \deg_{x_k} S_c, $$

whence

$$ \Lambda_{x_k} S_c \geq \Lambda_{x_k} \max_{x_i} C \geq \Lambda_{x_k} S_c + s_k - s'_k = \Lambda_{x_k} S_c - r_k, \tag{19} $$

which implies that

$$ r_k \geq 0. $$

Together with (18) and (16), this proves the claim that $r = (0, \ldots, 0)$. Putting this knowledge back into (19) for every $k \neq i, j$ finishes the proof. $\square$

### 3.2 The inductive step

We will now give the inductive step for the proof of Theorem 1.

**1.** Since we assume that $S_c = A \cdot B$, by Facts 5 and 1 we have

$$ \min_{x_i} A \cdot \min_{x_i} B \sim S_{c_0 - c_1}(\tilde{x}_i), \quad \max_{x_i} A \cdot \max_{x_i} B = S_{c_{n-1}}(\tilde{x}_i) \quad \text{for all } i. $$

Both right-hand side polynomials are irreducible over $K$ by the inductive assumption. Therefore one of the polynomials $\max_{x_i} A$, $\max_{x_i} B$ must be a constant, and one of $\min_{x_i} A$, $\min_{x_i} B$ must be a monomial. Our first goal will be to prove that (up to switching between $A$ and $B$) we may assume that

$$ \min_{x_i} A \sim \text{const}, \quad \max_{x_i} B = \text{const} \quad \text{for all } i. \tag{20} $$
2. If (20) is false, then some of the two factors $A$, $B$ (let it be $A$) must satisfy the assumptions of Lemma 8 with some indices $i \neq j$; then, by the lemma we have

$$\Lambda_{x_k} B = \Lambda_{x_k} S_c - \Lambda_{x_k} A = 0$$

for every $k \neq i, j$.

Moreover, we have $\min \deg_{x_k} B \leq \min \deg_{x_k} S_c = 0$; hence $\deg_{x_k} B = 0$, which means that $B$ may depend only on $x_i$ and $x_j$.

Suppose that $B$ has a non-monomial irreducible factor $C$, and let $\sigma$ be any permutation of all the variables $x$ satisfying

$$\sigma(\{i, j\}) \cap \{i, j\} = \emptyset.$$ (Such a permutation must exist for $n \geq 4$ variables). Denoting by $P^\sigma$ the image of a polynomial $P$ under the action of $\sigma$, we have

$$C^\sigma | A^\sigma \cdot B^\sigma = S_c^\sigma = S_c = A \cdot B,$$

so $C^\sigma$ must divide either $A$ or $B$. However, since $C$ may depend only on $x_i$ and $x_j$, $C^\sigma$ may depend only on $x_{\sigma(i)}$ and $x_{\sigma(j)}$; in particular, it does not depend on $x_i$, $x_j$. Hence $C^\sigma \nmid B$. On the other hand, if $C^\sigma | A$, then, since $C^\sigma$ does not depend on $x_i$, we would have

$$C^\sigma | \min_{x_i} A \sim \min_{x_i} S_c,$$

with the latter polynomial being irreducible by Fact 5 and the inductive assumption. Hence, as $C^\sigma$ is not a monomial, we must have $C^\sigma \sim \min_{x_i} S_c$, which contradicts the fact that $C^\sigma$ depends on at most two variables. This shows that $B$ has no non-monomial irreducible factors and thus is a monomial. However, $S_c$ has no non-trivial monomial factors because its minimal $x_i$-degree is 0 for every $i$. Therefore $B = \text{const}$, contrary to our assumptions. This proves (20).

3. Having proven (20), we deduce by Fact 5 that

$$\Lambda_{x_i} A(x) \geq \Lambda_{x_i} \min_{x_j} A(x) = \Lambda_{x_i} \min_{x_j} S_c(x) = \Lambda_{x_i} S_{c_0 - c_1}(x) = c_{n-1} - c_1 - (n - 2),$$

$$\Lambda_{x_i} B(x) \geq \Lambda_{x_i} \max_{x_j} B(x) = \Lambda_{x_i} \max_{x_j} S_c(x) = \Lambda_{x_i} S_{c_{n-1}} = c_{n-2} - (n - 2),$$

which implies

$$c_{n-1} - (n - 1) = \Lambda_{x_i} S_c = \Lambda_{x_i} A + \Lambda_{x_i} B \geq c_{n-1} + c_{n-2} - c_1 - 2(n - 2),$$

that is,

$$n - 3 \geq c_{n-2} - c_1.$$ (21)

Taking into account that $c$ is strictly increasing, this condition can be fulfilled only when

$$c_2 - c_1 = c_3 - c_2 = \ldots = c_{n-2} - c_{n-3} = 1,$$

which contradicts our assumption that $c_{i+1} - c_i > 1$ for all suitable $i$. This finishes the proof Theorem 1.

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