Absence of bilinear condensate in three-dimensional QED

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There are plausibility arguments that QED in three dimensions has a critical number of flavors of massless two-component fermions, below which scale invariance is broken by the presence of bilinear condensate. We present numerical evidences from our lattice simulations using dynamical overlap as well as Wilson-Dirac fermions for the absence of bilinear condensate for any even number of flavors of two-component fermions. Instead, we find evidences for the scale-invariant nature of three-dimensional QED.
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1. Introduction

Parity-invariant QED$_3$ with 2$N_f$ flavors of massless two-component fermions coupled to three-dimensional non-compact Abelian gauge-fields has been studied in the past as a quantum field theory which can be tuned to be conformal or to have a mass-gap by changing $N_f$. The question is the following – is there a critical number of flavors of two-component fermions 2$N_f$ below which massless QED$_3$ in a finite box of length $\ell$ generates other low-energy length scales which are independent of $\ell$ as $\ell \to \infty$? One such low-energy length scale that is of interest is the bilinear condensate $\Sigma$ which, if non-zero, governs the following scaling of the low-lying eigenvalues $\lambda_i$ of the massless Dirac operator:

$$\lambda_i = \frac{z_i}{\ell^3} \frac{1}{\Sigma},$$

where $z_i$ are universal numbers depending only on the symmetries of the Dirac operator, and can be obtained from a random matrix model with the same symmetries (refer [1] for such a model corresponding to QED$_3$). In this talk, based on our publications [2, 3], we primarily address the existence of $\Sigma$ for small $N_f$ ($= 1, 2, 3, 4$) by asking if $\lambda \sim \ell^{-(1+p)}$ with $p = 2$. We summarize the status of the understanding of the critical $N_f$ before our studies in Figure 1 (see [2] and references therein, for a complete literature survey). The analytical computations, each with their own limitations, suggested that the critical $N_f$ lie between 0 and 4. The previous lattice studies suggested that it could be 1 or 2.

2. Lattice details

We regulated QED$_3$ in a finite box of physical volume $\ell^3$ using $L^3$ lattices. The lattice coupling
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Figure 2: On the left panel, the \( \ell \) dependence of the six low-lying, continuum extrapolated, eigenvalues of the overlap operator is shown. The Padé approximations to their \( \ell \) dependence with \( p = 1 \) are shown as the solid curves. On the right panel, the likelihood of different values of the exponent \( p \), measured using the \( \chi^2/\text{DOF} \) for the best fit of the Padé approximation with various values of \( p \) to the finite \( \ell \) data, is shown.

appearing in the gauge action is \( \beta = L/\ell \); the continuum limit at a fixed physical length \( \ell \) is taken by extrapolating to \( L \to \infty \). We regulated the two flavors of massless two-component fermions in a parity-invariant way using the Wilson-Dirac as well as overlap fermions. The fermion propagator \( G \) for the parity-preserving Wilson-Dirac fermion is

\[
G^{-1} = \begin{bmatrix}
0 & X \\
-X^\dagger & 0
\end{bmatrix}; \quad X = C_n + B - m_t.
\] (2.1)

\( C_n \) is the two-component naive Dirac operator, \( B \) is the Wilson term and \( m_t \) is tuned such that the lowest eigenvalue \( \lambda_1 \) of \( iG^{-1} \) is minimum. We further improved it by adding a Sheikholeslami-Wohlert term and by using HYP smeared links in the Dirac operator. The fermion propagator \( G \) for the overlap fermion, which has the full \( U(2N_f) \) symmetry even at finite lattice spacing, is given in terms of a unitary matrix \( V = (X^\dagger X)^{-1}X \) as

\[
G^{-1} = \begin{bmatrix}
0 & \frac{1-V}{1+V} \\
\frac{1-V}{1+V} & 0
\end{bmatrix}.
\] (2.2)

We define the “eigenvalues of the Dirac operator” in either case to be the eigenvalues \( \lambda_i \) of \( iG^{-1} \) which are real. We used standard HMC for generating \( \sim 500 \) – \( 1000 \) independent gauge configurations at all the simulation points \( (4 \leq \ell \leq 250) \). Using Wilson-Dirac fermions we studied \( N_f = 1, 2, 3 \) and \( 4 \). With the overlap fermion, we studied \( N_f = 1 \). At each \( \ell \), we used multiple \( L^3 \) lattices \( (12 \leq L \leq 24) \) in order to take the continuum limits.

3. Evidence from \( \ell \)-scaling of the low-lying eigenvalues of Dirac operator

In a finite physical box, the spectrum of the Dirac operator is discrete. Thus, one can talk about the \( \ell \)-dependence of the individual low-lying eigenvalues. As we noted in the introduction, the \( i \)-th

\footnote{The Wilson mass \( m_t = 1 \) in overlap simulations
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Figure 3: The plot compares the $\ell$-dependence of the first three low-lying eigenvalues, after taking the continuum limit, using Wilson fermions (open symbols) and overlap fermions (filled symbols) for the $N_f = 1$ case.

low-lying eigenvalue $\lambda_i$ will scale as $\ell^{-3}$ when there is a condensate $\Sigma$. If $\ell^{-3}$ scaling is not found, we can conclude that a bilinear condensate is absent and instead we can obtain the mass anomalous dimension of the scale-invariant theory; since $\lambda$ has an engineering dimension of mass, the mass anomalous dimension $\gamma_m$ is $p$ if $\lambda \sim \ell^{-p-1}$ and $p < 1$.

In the left panel of Figure 2, we show the dependence of the continuum extrapolated values of $\lambda_i \ell$ as a function of $1/\ell$ for the six low-lying eigenvalues of the overlap operator in a log-log plot. At any finite $\ell$ that we studied, the slope $d\log(\lambda \ell)/d\log(1/\ell)$ is less than 2, the value that is expected if $\Sigma \neq 0$. In fact, it is less than 1. We estimate the exponent of the power-law that would be seen as $\ell \to \infty$ by describing the $\ell$-dependence of our data by

$$\lambda \ell = \ell^{-p} F(1/\ell),$$

with an unknown scaling correction $F$. We approximate $F$ by a $[1/1]$ Padé approximant. We find it numerically stable to write the Padé approximant in terms of $\tanh(1/\ell)$. The best fits of the above ansatz with $p = 1$ to the data are shown by the solid curves in the left panel of Figure 2. In the right panel, we show the $\chi^2$/DOF for such fits to the six low-lying eigenvalues as a function of the exponent $p$. The value $p = 2$ is clearly ruled out, which implies the absence of a condensate. Assuming the theory does not generate other length scales as well, we can estimate the mass anomalous dimension $\gamma_m = p$ of the theory to be $1.0(2)$ from the same plot. Further, we support the correctness of our result by comparing the $\ell$-dependence of the continuum extrapolated low-lying eigenvalues of the two different lattice Dirac operators in Figure 3. A perfect agreement between the Wilson-Dirac and the overlap formalisms is seen. Due to such an agreement, we study the $N_f = 2, 3, 4$ cases using only the Wilson-Dirac fermion.

In Figure 4, we show the $\ell$-dependence of the continuum extrapolated smallest eigenvalue for different number of flavors $N_f = 1, 2, 3$ and 4. The eigenvalues scale with a smaller exponent $p$ as $N_f$ increases, consistent with the expectation that if $N_f = 1$ does not have a bilinear condensate, the $N_f = 2, 3, 4$ also would not. Thus QED$_3$ does not have a bilinear condensate for all non-zero $N_f$. Again, assuming this means that QED$_3$ is scale-invariant for all $N_f$, we estimate the mass anomalous dimension to be $\gamma_m = 1.0(2), 0.6(2), 0.37(6)$ and $0.28(6)$ for $N_f = 1, 2, 3, 4$ respectively. Surprisingly, this agrees with an analytical calculation [10] of $\gamma_m$ to $\mathcal{O}(1/N_f^2)$ where no assumption
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Figure 4: The $\ell$-dependence of the smallest eigenvalue of the Wilson-Dirac operator for $N_f = 1, 2, 3$ and 4. The expected scaling when a bilinear condensate is present, $\lambda \ell \sim \ell^{-2}$, is shown by the black straight line in this log-log plot. The exponent $p$ for the asymptotic $\ell$-scaling seems to decrease as $1/N_f$.

Figure 5: (Left) The zero spatial momentum scalar correlator $G(t) = \langle \Sigma(0)\Sigma(t) \rangle$ as a function of temporal separation $t$. The different lines are tangents to the correlator, with slope $k(t)$, at various $t$ on the log-log plot. (Right) The mass anomalous dimension given by $\gamma_m(t) = 1 - k(t)/2$ is plotted as a function of the scale $t$.

about bilinear condensate is made; the analytical values are $\gamma_m = 1.19, 0.56, 0.37$ and 0.28 for $N_f = 1, 2, 3, 4$ respectively.

The other way to obtain the mass anomalous dimension is to study the scalar correlator $G(t) = \langle \Sigma(0)\Sigma(t) \rangle$ projected to zero spatial momentum. The correlator is shown as a function of the temporal separation $t$ in the left panel of Figure 5. The first thing to notice is the concave-up nature of the correlator. This indicates the absence of a mass-gap, thereby ruling out the presence of another length scale in addition to a bilinear condensate. The slope on the log-log plot, $k(t) = \frac{d\log(G(t))}{d\log(t)}$, is related to a scale dependent mass anomalous dimension $\gamma_m(t)$ as $\gamma_m(t) = 1 - k(t)/2$. This is shown as a function of $1/t$ in the right panel of Figure 5. The mass anomalous dimension at the IR fixed point to which QED$_3$ with $N_f = 1$ flows to, is $\gamma^* = \lim_{t \to \infty} \gamma_m(t)$. We estimate by an extrapolation that $\gamma^* = 0.8(1)$. This is consistent with the estimate $1.0(2)$ from the eigenvalues.
The Inverse Participation Ratio (IPR) is defined as

\[ I_2 = \left\langle \int (\psi_\lambda^*(x) \psi_\lambda(x))^2 d^3x \right\rangle, \tag{4.1} \]

where \( \psi_\lambda \) is the normalized eigenvector corresponding to the eigenvalue \( \lambda \). In random matrix models, which are ergodic, \( I_2 \sim \ell^{-3} \). Thus, if the theory has a condensate, the low-lying eigensystem of the Dirac operator would be described by a random matrix model. Thus the IPR corresponding to the low-lying eigenvalues should show a \( \ell^{-3} \) scaling. This is another test for the presence of \( \Sigma \). Instead, if the theory is scale-invariant, the finite size scaling of IPR would be \( I_2 \sim \ell^{-3+\eta} \), where \( \eta \) is a critical exponent. The exponent \( \eta \) is related to a quantity called number variance \( \Sigma \) which measures correlations between the eigenvalues. The number variance \( \Sigma_2(n) \) is defined as the variance of the number of eigenvalues below a value \( \lambda \) which on the average contains \( n \) eigenvalues. In ergodic random matrix models, \( \Sigma_2(n) \sim \log(n) \). For a critical theory, \( \Sigma_2(n) \sim (\eta/6)n \), where \( \eta \) is the critical exponent from the IPR [11].

In the left panel of Figure 6, we have shown the \( \ell \)-scaling of IPR for \( N_f = 1 \). For large \( \ell \), the onset of scaling is clearly seen. The scaling is \( I_2 \sim \ell^{-2.62(1)} \). Firstly, this rules out the ergodic \( \ell^{-3} \) scaling. The theory has a non-zero critical exponent \( \eta = 0.38(1) \). As explained above, in a critical theory, \( \eta \) should satisfy a critical relation to the slope of number variance. In the right panel of Figure 6, we have shown \( \Sigma_2(n) \) as a function of \( n \). Again, clearly there is a disagreement with the expectation from the nonchiral random matrix theory thereby ruling out condensate in another way. We see a linear rise in \( \Sigma_2(n) \) indicating a critical behavior. As \( \ell \) is increased, the slope of the linear
rise seems to approach $\eta/6$, as shown by the black line in the figure. Thus, both the IPR and $\Sigma_2$ show critical behavior, and also they satisfy the critical relation between the two.

5. Conclusions

In this talk, we presented convincing numerical evidences for the absence of a bilinear condensate for all $N_f$. Instead, we found evidences for QED$_3$ to be scale-invariant, and we estimated the mass anomalous dimension at the infra-red fixed point at various $N_f$. In another work [12], we established the presence of a condensate in the 't Hooft limit using the same methods we described here. This suggests an interesting phase diagram in the $(N_f,N_c)$ plane whose one side is conformal while the other side has a mass-gap, providing a powerful system to understand the generation of mass in QFTs. We aim to present results on this in a future Lattice meeting.

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