On the Baum-Connes conjecture in the real case

by Paul Baum and Max Karoubi

The classical Baum-Connes conjecture (for a given discrete countable group \( \Gamma \)) states that

the index map \( \mu(\Gamma) : K_j(\mathcal{E}\Gamma) \to K_j(C^*_r(\Gamma)) \)

is

an isomorphism (where \( j = 0, 1 \ mod 2 \)).

In this statement, \( K_j(C^*_r(\Gamma)) \) is the \( K \)-theory of the reduced \( C^* \)-algebra \( C^*_r(\Gamma) \)
(also denoted \( C^*_r(\Gamma; \mathbb{C}) \) in [9]) and \( K_j^c(\mathcal{E}\Gamma) \) is the complex equivariant Kasparov \( K \)-homology (with \( \Gamma \)-compact supports) of the space \( \mathcal{E}\Gamma \). This index map may be also defined in the real context, by using real Kasparov theory. In other words, there is an index map

\[
\mu_R(\Gamma) : KO_j^c(\mathcal{E}\Gamma) \to K_j(C^*_r(\Gamma; \mathbb{R}))
\]

where \( j \) takes its values in \( \mathbb{Z} \ mod 8 \). We may now ask whether \( \mu_R(\Gamma) \) is also an isomorphism.

One source of interest in this question (for a given group \( \Gamma \)) is the result of S. Stolz (with contributions from J. Rosenberg, P. Gilkey and others): the injectivity of \( \mu_R(\Gamma) \) implies the stable Gromov-Lawson-Rosenberg conjecture [2] about the existence of a Riemannian metric of positive scalar curvature on compact connected spin manifolds with \( \Gamma \) as fundamental group [10].

The purpose of this paper is to show that the Baum-Connes conjecture in the real case follows from the usual (i.e. complex) case. More precisely, our theorem is the following:

**Theorem.** Let \( \Gamma \) be a discrete countable group. If \( \mu(\Gamma) \) is an isomorphism then \( \mu_R(\Gamma) \) is also an isomorphism.

The proof relies on an interpretation of the index maps \( \mu(\Gamma) \) and \( \mu_R(\Gamma) \) as \( K \)-theory connecting homomorphisms associated to exact sequences of (real or
complex) $C^*$-algebras [3][8] and also on a general theorem for Banach algebras which follows directly from a “descent theorem” in topological $K$-theory:

**THEOREM** [4]. Let $A$ be a Banach algebra over the real numbers and let $A' = A \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. If $K_i(A') = 0$ for all $i \in \mathbb{Z}$ mod 2, then $K_j(A) = 0$ for all $j \in \mathbb{Z}$ mod 8.

1. Definition of $\mu(\Gamma)$ and $\mu_{\mathbb{R}}(\Gamma)$

1.1. In this section, we recall the basic definitions of [1] and observe that these definitions extend quite immediately to the real case.

The universal proper $\Gamma$-space is denoted by $E \Gamma$ and $K^\Gamma_j(E \Gamma)$ denotes the following colimit

$$\text{colim} \ K K^\Gamma_j(C_0(\Delta), \mathbb{C})$$

where $\Delta$ runs over all $\Gamma$-compact subspaces of $E \Gamma$ (by definition, a $\Gamma$-subspace $\Delta$ is called $\Gamma$-compact if the quotient space $\Delta/\Gamma$ is compact). The composition of the two homomorphisms

$$ KK^\Gamma_j(C_0(\Delta), \mathbb{C}) \rightarrow KK^\Gamma_j(C_0(\Delta) \rtimes \Gamma, C^*_r(\Gamma)) \rightarrow KK^j(\mathbb{C}, C^*_r(\Gamma)) $$

induces (by taking the colimit) the map $\mu$ referred to in the introduction. Here the first homomorphism is Kasparov’s descent map [5] and the second one is induced by the Kasparov product with

$$ 1 \in KK^0(\mathbb{C}, C_0(\Delta) \rtimes \Gamma) = K_0(C_0(\Delta) \rtimes \Gamma). $$

1.2 Remark. In this definition of $\mu$, the specific space $E \Gamma$ does not play a particular role. In other words, if $X$ is any proper $\Gamma$-space, we could define in the same way an “index map”

$$ \mu(X, \Gamma) : K^\Gamma_j(X) \rightarrow K^j(C^*_r(\Gamma)). $$

1.3 Remark. These definitions extend immediately to the real case. Hence, there is a real index map

$$ \mu_{\mathbb{R}}(X, \Gamma) : KO^\Gamma_j(X) \rightarrow K^j(C^*_r(\Gamma; \mathbb{R})). $$

The *real* Baum-Connes conjecture for the group states that $\mu_{\mathbb{R}}(E \Gamma, \Gamma) = \mu_{\mathbb{R}}(\Gamma)$ is an isomorphism for all $j \in \mathbb{Z}$ mod 8.

2. Index maps and connecting homomorphisms in $K$-theory
2.1. The strategy for proving our theorem is as follows. We will describe (in this section) a $C^*$-algebra whose $K$-theory (real or complex) vanishes precisely when the corresponding version of the Baum–Connes conjecture is true. In the next section we will apply to this $C^*$-algebra the result of [4], according to which the $K$-theory of a real $C^*$-algebra vanishes if and only if the $K$-theory of its complexification vanishes.

2.2. To construct the required $C^*$-algebra we have chosen to use the method of [3], [7], [8]. Let $X$ be a locally compact space and let $\Gamma$ be a countable discrete group $\Gamma$ acting properly on $X$. Choose a separable Hilbert space $H$ with a representation of $C^*$-algebras $\psi : C_0(X) \to B(H)$ and a unitary group representation $\tau : \Gamma \to U(H)$ which are compatible in the sense that $\psi(\gamma.f) = \tau(\gamma).\psi(f).\tau(\gamma)^*$, where $\gamma.f$ is the function $x \mapsto f(\gamma^{-1}x)$. (Note that these conditions imply that we have in fact a representation of the crossed product $C^*$-algebra $C_0(X) \rtimes \Gamma$ in $B(H)$.) It is also required that $H$ be a ‘large’ representation in a certain technical sense; it is sufficient to take $H = L^2(X; \mu) \otimes \ell^2(\Gamma) \otimes H'$, where $H'$ is an auxiliary infinite-dimensional Hilbert space and $\mu$ is a Borel measure on $X$ whose support is all of $X$.

Within this setting, we define the support in $X \times X$ of an operator $T$, denoted $\text{Supp}(T)$, as the complement of the points $(x,y)$ such that there exists a neighborhood $U \times V$ of $(x,y)$ such that $\psi(f)T\psi(g) = 0$, for $f$ supported in $U$ and $g$ supported in $V$.

2.3. Following [3] and [7], we define now the $C^*$-algebra $D_T^*(X)$ and a closed ideal $C_T^*(X)$. Thus there is an exact sequence of $C^*$-algebras

$$0 \to C_T^*(X) \to D_T^*(X) \to D_T^*(X)/C_T^*(X) \to 0. \quad (\mathcal{E})$$

By definition, $D_T^*(X)$ is the closure of the algebra in $B(H)$ consisting of all the (bounded) operators $T$ such that

1) $T$ is $\Gamma$-invariant, i.e. $T.\tau(\gamma) = \tau(\gamma)T$ for all $\gamma$ in $\Gamma$.

2) $\text{Supp}(T)$ is $\Gamma$-compact, i.e. its quotient by $\Gamma$ is compact in $(X \times X)/\Gamma$.

3) For all $f$ in $C_0(X)$, $T\psi(f) - \psi(f)T$ is a compact operator on $H$.

The ideal $C_T^*(X)$ is the closure of the algebra in $B(H)$ consisting of the (bounded) operators $T$ which satisfy (1), (2), and a stronger condition:

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1In the analogous context of surgery theory, the $K$-theory of this $C^*$-algebra would be the ‘fiber of assembly’ or ‘structure set’ term in the surgery exact sequence.

2We assume $X$ to be also second countable in order to get separable Hilbert spaces.

3Here $\Gamma$ is acting on $X \times X$ by the diagonal action.
3′) For all \( f \) in \( C_0(X) \), \( T\psi(f) \) and \( \psi(f)T \) are compact operators on \( H \).

2.4 Example. If \( \Gamma \) is a finite group and \( X \) is compact, it is well known that the \( K \)-theory of the \( C^* \)-algebra \( D^*_\Gamma(X)/C^*_\Gamma(X) \) is the \( K \)-homology, with a shift of dimension, of the cross-product algebra \( C(X) \rtimes \Gamma \) (this is “Paschke duality” [6]). In the simplest case when \( X \) is a point, the exact sequence above is essentially equivalent to a direct sum of exact sequences of the form

\[
0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{B}(H) \longrightarrow \mathcal{B}(H)/\mathcal{K} \longrightarrow 0
\]
as many as the number of conjugacy classes in \( \Gamma \).

2.5 THEOREM [8]. For any proper cocompact \( \Gamma \)-space \( X \), there is a canonical Morita equivalence between the \( C^* \)-algebra \( D^*_\Gamma(X)/C^*_\Gamma(X) \) and the \( C^* \)-algebra \( C^*_\Gamma(X) \).

2.6 THEOREM [3][6]. For any proper \( \Gamma \)-space \( X \), there is a natural isomorphism

\[
K^\Gamma_j(X) := \colim_{\Delta} \ K K^\Gamma_1(C_0(\Delta), \mathbb{C}) \xrightarrow{\cong} K^\Gamma_{j+1}(D^*_\Gamma(X)/C^*_\Gamma(X))
\]
where \( \Delta \) runs over all the \( \Gamma \)-compact subspaces of \( X \).

2.7 THEOREM [8]. For any proper \( \Gamma \)-space \( X \), we have a commutative diagram

\[
\begin{array}{ccc}
K^\Gamma_j(X) & \xrightarrow{\mu} & K_j(C^*_\Gamma(X)) \\
\cong \downarrow & & \cong \downarrow \\
K_{j+1}(D^*_\Gamma(X)/C^*_\Gamma(X)) & \xrightarrow{\delta} & K_j(C^*_\Gamma(X))
\end{array}
\]
where \( \mu \) is the Baum-Connes map and where \( \delta \) is the \( K \)-theory connecting homomorphism associated to the exact sequence \( (\mathcal{E}) \) above.

2.8 Remark. It is important to notice that the three theorems above are also true in the real case (see [9] for a detailed account of this “real Paschke duality”). In this case, \( C^*_\Gamma(X) \) has to be replaced by \( C^*_\Gamma(X; \mathbb{R}) \). The real analogs of the \( C^* \)-algebras \( D^*_\Gamma(X) \) and \( C^*_\Gamma(X) \) shall be denoted \( D^*_\Gamma(X; \mathbb{R}) \) and \( C^*_\Gamma(X; \mathbb{R}) \).

2.9 COROLLARY. The Baum-Connes map \( \mu : K^\Gamma_j \longrightarrow K_j(C^*_\Gamma(X)) \) is an isomorphism for all \( j \in \mathbb{Z} \) mod 2 if and only if the \( K \)-groups \( K_j(D^*_\Gamma(X)) = 0 \) for all \( j \). In the same way, the real Baum-Connes map \( \mu_R : KO^\Gamma_j(X) \longrightarrow K_j(C^*_\Gamma(X; \mathbb{R})) \) is an isomorphism for all \( j \in \mathbb{Z} \) mod 8 if and only if the \( K \)-groups \( K_j(D^*_\Gamma(X; \mathbb{R})) = 0 \) for all \( j \).

3. Proof of the Baum-Connes conjecture in the real case for a given group \( \Gamma \) (assuming its validity for \( \Gamma \) in the complex case)
3.1. As we have shown in the second section, the complex (resp. real) Baum- Connes conjecture is equivalent to the vanishing of the $K$-groups $K_j(D^*_\Gamma(X))$ (resp. $K_j(D^*_\Gamma(X;\mathbb{R}))$) for $X = \mathbb{E}\Gamma$. If we put $A = D^*_\Gamma(X;\mathbb{R})$, its complexification $A' = A \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $D^*_\Gamma(X)$. The scheme of the argument is then the following, where $BC(\Gamma)$ (resp. $BC_R(\Gamma)$) stands for the Baum-Connes conjecture (resp. the real Baum-Connes conjecture) for a given discrete group $\Gamma$:

$$BC(\Gamma) \iff K_\ast(A') = 0 \implies K_\ast(A) = 0 \iff BC_R(\Gamma).$$

3.2. The only point to show is the implication $K_\ast(A') = 0 \implies K_\ast(A) = 0$, which follows from the descent theorem stated in [4] in the general framework of Banach algebras. More precisely, let $A$ be any Banach algebra over the real numbers and $A'$ denote its complexification $A \otimes_{\mathbb{R}} \mathbb{C}$. There is then a cohomology spectral sequence with $E_2^{pq} = H^p(\mathbb{RP}_2; K_{-q}(A'))$ converging to $K_{-q-p}(A) \oplus K_{-q-p+4}(A)$, where $\mathbb{RP}_2$ is the real projective plane and $H^p$ means usual singular cohomology with local coefficients\(^4\). The hypothesis $K_\ast(A') = 0$ implies that the $E_2$ term of the spectral sequence is 0. Therefore the $E_\infty$ term is also 0. Since moreover the filtration is finite (because $\mathbb{RP}_2$ is finite dimensional), $K_\ast(A)$ must be also 0.

\(^4\)In fact, there is at most one non zero differential, therefore $E^3 = E^\infty$. 

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