Multi-Component Bell Inequality and its Violation for Continuous Variable Systems

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Abstract

Multi-component correlation functions are developed by utilizing d-outcome measurements. Based on the multi-component correlation functions, we propose a Bell inequality for bipartite d-dimensional systems. Violation of the Bell inequality for continuous variable (CV) systems is investigated. The violation of the original Einstein-Podolsky-Rosen state can exceed the Cirel’son bound, the maximal violation is 2.96981. For finite value of squeezing parameter, violation strength of CV states increases with dimension d. Numerical results show that the violation strength of CV states with finite squeezing parameter is stronger than that of original EPR state.

PACS numbers: 03.65.Ud, 03.65.Ta, 03.67.-a

I. INTRODUCTION

In their famous paper of 1935, Einstein, Podolsky and Rosen (EPR) questioned the completeness of quantum mechanics, based on a gedanken experiment involving the position and momentum of two entangled particles. Einstein believed that there must be elements of reality that quantum mechanics ignores. It is argued that the incomplete description could be avoided by postulating the presence of hidden variables that permit deterministic predictions for microscopic events. Furthermore, hidden variables could eliminate concerns for nonlocality. For a long time, EPR argument remained a philosophical debate on the foundation of quantum mechanics. In 1964, John Bell made an important step forward in this direction by considering a version based on the entanglement of spin-1/2 particles introduced by Bohm. Bell showed that the assumption of local realism had experimental consequences, and was not simply an appealing world view. In particular, local realism implies constraints on the statistics of two or more physically separated systems. These constraints, called Bell inequalities, can be violated by the statistical predictions of quantum mechanics.

Although most of the concepts in quantum information theory were initially developed for quantum systems with discrete variables, many quantum information protocols of continuous variables have also been proposed. In recent years, quantum nonlocality for position-momentum variables associated with original EPR states has attracted much attention. The original EPR state is a common eigenstate of the relative position \( \hat{x}_1 - \hat{x}_2 \) and the total linear momentum \( \hat{p}_1 + \hat{p}_2 \) and can be expressed as a \( \delta \)-function:

\[
\Psi(x_1, x_2) = \int_{-\infty}^{\infty} e^{(2\pi i / \hbar)(x_1 - x_2 + x_0)} dp.
\]

It is important to choose the appropriate type of observables for testing nonlocality for a given state. Bell has presented a local realistic model for position and momentum measurements based on spin-1/2 particles. He has also argued that the original EPR state would not exhibit nonlocal effects since the Wigner function representation of the original EPR state is positive everywhere and therefore admits a local hidden variable model. Recently, Banaszek and Wódkiewicz invoked the notion of parity as the measurement operator and interpreted the Wigner function as a correlation function for these parity measurements. The starting point of the demonstration is that the two-mode Wigner function can be interpreted as a correlation function for the joint measurement of the parity operator. In the limit \( r \to \infty \), when the original EPR state is recovered, a significant violation of Bell inequality takes place, however, the violation is not very strong. To avoid the unsatisfactory feature, Chen et al. introduced “pseudospin” operators based on parity, due to the fact that the degree of quantum nonlocality that we can uncover crucially depends not only on the given quantum state but also on the “Bell operator”. From reference, the violation of CHSH inequality for the original EPR state can reach the Cirel’son bound \( 2\sqrt{2} \).

In this paper, we propose a Bell inequality, which is based on multi-component correlation functions, for bipartite systems by utilizing d-outcome measurements. We then investigate violation of the inequality for continuous variable systems. Due to the considered d-outcome measurements, violation of original EPR state can exceed Cirel’son’s bound, the maximal violation is 2.96981. The CV case with finite value of squeezing parameter is also studied. The violation strength of CV states with finite squeezing parameter is stronger than that of original EPR state.

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II. BELL INEQUALITY FOR MULTI-COMPONENT CORRELATION FUNCTIONS

We consider a Bell-type scenario: two space-separated observers, denoted by Alice and Bob, measure two different local observables of $d$ outcomes, labelled by $0, 1, ..., N (= d - 1)$. We denote $X_i$ the observable measured by party $X$ and $x_i$ the outcome with $X = A, B (x = a, b)$. If the observers decide to measure $A_1, B_2$, the result is $(0, 4)$ with probability $P(a_1 = 0, b_2 = 4)$. Then let us introduce $d N$-dimensional unit vectors $v_0 = (1, 0, 0, 0, \cdots, 0, 0)$

$$v_1 = \left( -\frac{1}{N}, \frac{\sqrt{N^2 - 1}}{N}, 0, 0, \cdots, 0, 0 \right)$$

$$v_2 = \left( -\frac{1}{N}, -\frac{1}{N} \sqrt{\frac{(N+1)N}{N(N-1)}}, -\frac{1}{N} \sqrt{\frac{(N+1)N}{(N-1)(N-2)}}, 0, \cdots, 0 \right)$$

$$v_{N-1} = \left( -\frac{1}{N}, -\frac{1}{N} \sqrt{\frac{(N+1)N}{N(N-1)}}, -\frac{1}{N} \sqrt{\frac{(N+1)N}{(N-1)(N-2)}}, \cdots, -\frac{1}{N} \sqrt{\frac{(N+1)N}{3 \cdot 2}}, -\frac{1}{N} \sqrt{\frac{(N+1)N}{2 \cdot 1}} \right)$$

$$v_N = \left( -\frac{1}{N}, -\frac{1}{N} \sqrt{\frac{(N+1)N}{N(N-1)}}, -\frac{1}{N} \sqrt{\frac{(N+1)N}{(N-1)(N-2)}}, \cdots, -\frac{1}{N} \sqrt{\frac{(N+1)N}{3 \cdot 2}}, -\frac{1}{N} \sqrt{\frac{(N+1)N}{2 \cdot 1}} \right)$$

These $d$ vectors satisfy following properties:

(i) $\sum_{j=0}^{N} v_j = 0$

(ii) $v_j \cdot v_k = -\frac{1}{N}$ \hspace{1em} ($j \neq k$) \hspace{1em} (3)

For $d = 2$, it is just two valued variables (i.e., $v_0 = v_1 = -1$) obtained from a measurement. If the measure result of Alice is $m$, and Bob’s result is $n$ (where $m$ and $n$ are less than $N$), we then associate a vector $v_{m+n}$ for the correlation between Alice and Bob $|v_{m+n}|$ understood as $v_t$, where $t = (m + n)$, mod $d$. Based on which, we now construct multi-component correlation functions:

$$\bar{Q}_{ij} = \sum_{m,n} v_{m+n} P(a_i = m, b_j = n)$$

$$= \sum_{t=0}^{d} v_t P(m + n = t)$$

where $P(a_i = m, b_j = n)$ is the joint probability of $a_i$ obtain outcome $m$ and $b_j$ obtain outcome $n$ , and $\bar{Q}_{ij} = (Q_{ij}^{(0)}, Q_{ij}^{(1)}, Q_{ij}^{(2)}, \cdots, Q_{ij}^{(N-1)})$, $Q_{ij}^{(k)}$ represents the $k$-th component of the vector correlation function $\bar{Q}_{ij}$.

We now define a Bell quantity for the multi-component correlation functions,

$$B_d = B_d^{(0)} + \sqrt{\frac{N(N-1)}{(N+1)N} B_d^{(1)} + \frac{(N-1)(N-2)}{(N+1)N} B_d^{(2)}} + \cdots + \sqrt{\frac{2 \cdot 1}{(N+1)N} B_d^{(N-1)}}$$

$$= \sum_{k=0}^{N-1} \sqrt{\frac{(N+1-k)(N-k)}{(N+1)N} B_d^{(k)}}$$

where

$$B_d^{(0)} = Q_{11}^{(0)} + Q_{12}^{(0)} - Q_{21}^{(0)} + Q_{22}^{(0)}$$

$$B_d^{(k)} = Q_{11}^{(k)} - Q_{12}^{(k)} - Q_{21}^{(k)} + Q_{22}^{(k)}, \hspace{1em} (k \neq 0)$$

Any local realistic description of the previous Gedanken experiment imposes the following inequality:

$$-2 \left( \delta_{2d} + (1 - \delta_{2d}) \frac{d + 1}{d - 1} \right) \leq B_d \leq 2$$

Obviously, this inequality reduces to the usual CHSH inequality for $d = 2$.

The quantum prediction for the joint probability reads $P^{QM}(a_i = m, b_j = n) = \langle \psi | \hat{P}(a_i = m) \otimes \hat{P}(b_j = n) | \psi \rangle$ (8)

where $i, j = 1, 2; \hspace{1em} m, n = 0, ..., N, \hspace{1em} \hat{P}(a_i = m) = \hat{U}_A^{\dagger} |m\rangle \langle m| \hat{U}_A$ is the projector of Alice for the $i$-th measurement and similar definition for $\hat{P}(b_j = n)$. 
It is well known that the two-mode squeezed vacuum state can be generated in the nondegenerate optical parametric amplifier (NOPA) \(^8\) is

\[
|\text{NOPA}\rangle = e^{r(a_1^\dagger a_2^\dagger - a_1a_2)}|00\rangle = \sum_{n=0}^{\infty} (\tanh r)^n r^n \cosh r |mn\rangle, \tag{9}
\]

where \(r > 0\) is the squeezing parameter and \(|mn\rangle \equiv |n\rangle_1 \otimes |n\rangle_2\). The NOPA states \(|\text{NOPA}\rangle\) can also be written as \(^3\):

\[
|\text{NOPA}\rangle = \sqrt{1 - \tanh^2 r} \int dq \int dq' g(q, q'; \tanh r)|qq\rangle,
\]

where \(g(q, q'; x) = \frac{1}{\sqrt{\pi(1-x^2)}} \exp \left[ -\frac{q^2 + q'^2 - 2qq'}{2(1-x^2)} \right]\) and \(|qq\rangle \equiv |q\rangle_1 \otimes |q\rangle_2\), with \(|q\rangle\) being the usual eigenstate of the position operator. Since \(\lim_{x \to 1} g(q, q'; x) = \delta(q - q')\), one has \(\lim_{x \to \infty} \int dq \int dq' g(q, q'; \tanh r)|qq\rangle = \int \tanh r)|qq\rangle = |\text{EPR}\rangle\), which is just the original EPR state. Thus, in the infinite squeezing limit, \(|\text{NOPA}\rangle\) becomes the original, normalized EPR state. Following Brukner et al. \(^10\), we can map the two-mode squeezed state onto a \(d\)-dimensional pure state:

\[
|\psi_d\rangle = \frac{\text{sech} r}{\sqrt{1 - \tanh^2 2^{d-1}}} \sum_{n=0}^{d-1} (\tanh r)^n |nn\rangle. \tag{10}
\]

If the measurement result of Alice is \(m\) photons, and Bob’s result is \(n\) photons, we then ascribe a vector \(v_{m+n}\) for the correlation between Alice and Bob. And \(\hat{P}(a_i = m, b_j = n)\) is the joint probability of \(a_i\) obtain \(m\) photons and \(b_j\) obtain \(n\) photons. More precisely, for the two-mode squeezed state one obtains following joint probability

\[
P^{QM}(a_i = m, b_j = n) = \langle \psi_d|\hat{P}(a_i = m) \otimes \hat{P}(b_j = n)|\psi_d\rangle. \tag{11}
\]

### III. SOME EXAMPLES

For \(d = 3\), we have three outcomes \(v_0 = (1, 0), v_1 = (-1/2, \sqrt{3}/2), v_2 = (-1/2, -\sqrt{3}/2)\). Accordingly the NOPA state is divided into three groups, namely

\[
|\text{NOPA}\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \left( \tanh^{3n} r |3n\rangle \langle 3n| + \tanh^{3n+1} r |3n+1\rangle \langle 3n+1| + \tanh^{3n+2} r |3n+2\rangle \langle 3n+2| \right). \tag{12}
\]

The two-component correlation functions depend on quantum version of joint probabilities

\[
P^{QM}(a_i = m, b_j = n) = \langle \text{NOPA}|\hat{P}(a_i = m) \otimes \hat{P}(b_j = n)|\text{NOPA}\rangle. \tag{13}
\]

Local realistic description imposes \(-4 \leq B \leq 2\). Numerical results show that \(B_{d = 3}(r = 1.4068) \approx 2.90638\); \(B_{d = 3}(r \to \infty) = 4/(6\sqrt{3} - 9) \approx 2.87293\). For \(B_{d = 3}(r \to \infty)\), the four optimal two-component quantum correlations read:

\[
\tilde{Q}_{11} = \tilde{Q}_{22} = \tilde{Q}_{12} = \left( \frac{2\sqrt{3} + 1}{6}, -\frac{2 - \sqrt{3}}{6} \right),
\]

\[
\tilde{Q}_{21} = \left( -\frac{1}{3}, -\frac{2}{3} \right),
\]

\[
|\tilde{Q}_{ij}| = \sqrt{(Q_{ij}^x)^2 + (Q_{ij}^y)^2} = \frac{\sqrt{5}}{3}. \tag{14}
\]

One thing worth to note that one can treat those three-dimensional vectors in terms of complex numbers, namely, \(v_0 = 1, v_1 = \omega, v_2 = \omega^2\) for simplicity, where \(\omega = \exp(2\pi/3)\). Now the Bell inequality becomes \(^11\)

\[
-4 \leq \text{Re}[Q_{11} + Q_{12} - Q_{21} + Q_{22}] + \sqrt{3} \text{Im}[Q_{11} - Q_{12} - Q_{21} + Q_{22}] \leq 2. \tag{15}
\]

In this sense, we can check the generalized “parity” operator \((\omega)^n\) other than usual parity operator \((-1)^n\), the two-component correlation function also reads

\[
Q_{ij} = \langle \text{NOPA}|\hat{U}_A^i \otimes \hat{U}_B^j (\omega)^n \hat{U}_A \otimes \hat{U}_B|\text{NOPA}\rangle, \tag{16}
\]

where \(\hat{U}_{AB}\) are generally \(U(3)\) transformations and we can sufficiently take them as the products of three spin-coherent operators:

\[
\hat{U}_A = e^{i\xi z\hat{U}_+ - \xi^* \hat{U}_-} e^{i\xi z\hat{U}_- - \xi^* \hat{U}_+} e^{i\xi z \hat{U}_+ - \xi^* \hat{U}_-} e^{-i\xi z \hat{U}_- - \xi^* \hat{U}_+},
\]

where \(\xi_j = \theta_j e^{-i\varphi_j}\); actually, the phases \(\varphi_j\) can be set to be zero, since they do not affect the maximal violation. Hence \(\hat{U}_{AB}\) is a \(SO(3)\) rotation. \(\hat{I}_\pm, \hat{U}_\pm\) and \(\hat{V}_\pm\) are pseudo-su(3)-spin which can be realized by the Fock states as

\[
\hat{I}_+ = \sum_{n=0}^{\infty} |3n\rangle \langle 3n+1|, \quad \hat{I}_- = \sum_{n=0}^{\infty} |3n+1\rangle \langle 3n|,
\]

\[
\hat{I}_z = \frac{1}{2}(|3n\rangle \langle 3n| - |3n+1\rangle \langle 3n+1|),
\]

\[
\hat{U}_+ = \sum_{n=0}^{\infty} |3n+1\rangle \langle 3n+2|, \quad \hat{U}_- = \sum_{n=0}^{\infty} |3n+2\rangle \langle 3n+1|,
\]

\[
\hat{U}_z = \frac{1}{2}(|3n+1\rangle \langle 3n+1| - |3n+2\rangle \langle 3n+2|),
\]

\[
\hat{V}_+ = \sum_{n=0}^{\infty} |3n\rangle \langle 3n+2|, \quad \hat{V}_- = \sum_{n=0}^{\infty} |3n+2\rangle \langle 3n|,
\]

\[
\hat{V}_z = \frac{1}{2}(|3n\rangle \langle 3n| - |3n+2\rangle \langle 3n+2|).
\]

Operators \(\{\hat{I}_\pm, \hat{I}_z\}, \{\hat{U}_\pm, \hat{U}_z\}\) and \(\{\hat{V}_\pm, \hat{V}_z\}\) form three \(SU(2)\) groups, respectively, and \(\{\hat{I}_\pm, \hat{U}_\pm, \hat{V}_\pm, \hat{I}_z, \hat{U}_z + \hat{V}_z)/\sqrt{3}\) forms a \(SU(3)\) group.

We can similarly get \(B_d(r = \text{finite value})\) and \(B_d(r \to \infty)\) with different \(d\). We list them partly in Table \(^\|\).
Obviously, the degree of the violation increases with dimension \( d \), and the violation strength of CV states with finite squeezing parameter is stronger than that of original EPR state. When squeezing parameter goes to infinity, the violation strength points we get are larger with the dimension, see Fig. 1. We calculate the maximal quantum violation (B) goes to 3.12885, which exceeds the Cirel’son bound. The reason for this is due to the fact that we consider \( d \) (greater than 2)-outcome measurements. Numerical results show that the violation strength of CV states with finite squeezing parameter is stronger than that of original EPR state.

It is interesting to note that for \( B_d(r \to \infty) \), the four optimal multi-component quantum correlations share the same module: \( |\vec{Q}_{ij}| = \sqrt{2/3} \). When \( d \) tends to infinity, \( |\vec{Q}_{ij}| = \sqrt{2/3} \). However, we do not have an analytical way to find a bound for the violation with finite squeezing parameter. For this case, what we do is draw a graph to see the variation of \( B_d(r = \text{finite \ value}) \) with increasing dimension, see Fig. 1. We calculate the maximal quantum violation for CV states with different \( d \). The more the value of dimension, the more difficult to find a maximal violation. Hence the violation strength points we get are for \( d \leq 330 \). With these values, it is easy to see that the violation increases from slowly to slowly with increasing of \( d \). Which means that there exists a limit for quantum violation when \( d \) goes to infinity. Until now, we do not have an exact value of the limit. We use a software, which can give experiment expression given enough points, to find an expression that describes the curve in Fig. 1.

\[
B = 3.12885 - 1.06535/d + 2.13122/d^2 - 2.19262e^{-\frac{d}{4}}
\]

When \( d \to \infty \), quantum violation (B) goes to 3.12885. Hence, such value can be thought as an approximate violation limit for CV states with finite squeezing parameter.

### IV. CONCLUSION

In summary, we construct multi-component correlation functions based on \( d \)-outcome measurements. A Bell inequality for continuous variables case. The degree of the violation increases with dimension \( d \), and the limit of the violation for the original EPR state is found to be 2.96981, which exceeds the Cirel’son bound. The reason for this is due to the fact that we consider \( d \) (> 2)-outcome measurements. Numerical results show that the violation strength of CV states with finite squeezing parameter is stronger than that of original EPR state.

This work is supported by NUS academic research grant WBS: R-144-000-089-112. J.L.C acknowledges financial support from Singapore Millennium Foundation and (in part) by NSF of China (No. 10201015).

Note added: While completing this work we learn a similar result obtained in Ref. 13, which based on the CGLMP inequality 12.
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