Poncelet’s Theorem in the four non-isomorphic finite projective planes of order 9

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Abstract
We study Poncelet’s Theorem in the four non-isomorphic finite projective planes of order 9. Among these planes, only the Desarguesian plane turns out to be a Poncelet plane, while the other three planes which are constructed over the miniquaternion near-field of order 9, are not. This gives a complete discussion of Poncelet’s Theorem in finite projective planes of order 9.

Introduction
In 1813 Jean-Victor Poncelet [11] showed one of the most beautiful results in projective geometry, known as Poncelet’s Porism. One version reads as follows.

Theorem (Poncelet’s Porism). Let $C$ and $C'$ be two conics. If there exists an $m$-sided polygon, $m \geq 3$, such that the vertices lie on $C'$ and the sides are tangent to $C$, then there are infinitely many other such $m$-sided polygons. Moreover, for $m \neq m'$, one cannot find such an $m'$-sided polygon for the same pair of conics $C$ and $C'$.

There are numerous proofs of Poncelet’s Theorem in classical geometry arising from different areas of mathematics: Synthetic proofs [1], combinatorial proofs [7] and purely geometric proofs using properties of the Euclidean plane. Moreover, a deep connection between Poncelet’s Porism and the theory of elliptic curves has been established [4]. See the recent book by V. Dragović and M. Radnović [3] for an overview.

In the present paper, we consider Poncelet’s Theorem in finite geometries. In particular, we introduce the notion of a Poncelet plane in order to restate Poncelet’s Theorem for finite projective planes. For $q := p^k$, $p$ a prime and $k \geq 1$, it is well-known how to construct a finite projective plane over the finite field $GF(q)$ of order $q$, denoted by $PG(2, q)$ and also known as finite projective Desarguesian plane. Many properties of the real projective plane carry over to $PG(2, q)$. In [1, Section 16.6], Berger presented a proof of the general form of Poncelet’s Theorem, the Great Poncelet Theorem, for projective planes over general fields with more than five elements. But there exist finite projective planes which are not isomorphic to a projective Desarguesian plane $PG(2, q)$. The smallest order where one can find such examples is the order 9. In particular, there are exactly four non-isomorphic finite projective planes of order 9, as proved by Lam et al. in [9]. Besides $PG(2, 9)$ there are three non-Desarguesian finite projective planes of order 9, all of them constructed over a near-field of conics $C$ and $C'$. And our main result reads as follows.

Theorem. The only Poncelet plane of order 9 is the finite projective Desarguesian plane $PG(2, 9)$.

The main objects when studying Poncelet’s Theorem in finite projective planes are ovals, which are a generalization of conics. One difficulty when working in planes not constructed over a finite field, such as the three planes of order 9 we consider in this paper, is to find such ovals. Since order 9 is the smallest one where the question of finding ovals in non-Desarguesian planes becomes important, some work has been done on ovals (and generalizations thereof like unitals and arcs) in planes of this order. For some recent work, see for example [2, 5, 8, 10, 12, 16].
1 Preliminaries

We briefly recall the basic definitions concerning finite projective planes (see e.g. [6]). The triple \((\mathbb{P}, \mathbb{B}, \mathbb{I})\) with \(I \subset \mathbb{P} \times \mathbb{B}\) is called projective plane, if the following axioms are satisfied.

(A1) For any two elements \(P, Q \in \mathbb{P}\), \(P \neq Q\), there exists a unique element \(g \in \mathbb{B}\) with \((P, g) \in I\) and \((Q, g) \in I\).

(A2) For any two elements \(g, h \in \mathbb{B}\), \(g \neq h\), there exists a unique element \(P \in \mathbb{P}\) with \((P, g) \in I\) and \((P, h) \in I\).

(A3) There are four elements \(P_1, \ldots, P_4 \in \mathbb{P}\) such that \(\forall g \in \mathbb{B}\) we have \((P_i, g) \in I\) and \((P_j, g) \in I\) with \(i \neq j\) implies \((P_k, g) \notin I\) for \(k \neq i, j\).

Elements of \(\mathbb{P}\) are called points and elements of \(\mathbb{B}\) are called lines. By \((P, g) \in I\) we denote that the point \(P\) is incident with the line \(g\). A more convenient notation of this incidence relation is \(P \in g\). Three points are said to be collinear if they are incident with the same line and three lines are said to be concurrent if they are incident with the same point. A projective plane is called finite, if the sets \(\mathbb{P}\) and \(\mathbb{B}\) are finite. In that case, it turns out that each line is incident with \(n + 1\) points and each point is incident with \(n + 1\) lines, for some \(n \geq 1\). According to that, a finite projective plane \((\mathbb{P}, \mathbb{B}, I)\) is said to be of order \(n\), if \(|\mathbb{P}| = |\mathbb{B}| = n^2 + n + 1\), and denoted by \(\mathcal{P}_n\).

For any finite projective plane \(\mathcal{P}_n = (\mathbb{P}, \mathbb{B}, I)\) of order \(n\), we define the dual projective plane of \(\mathcal{P}_n\) by \(\mathcal{P}^D_n := (\mathbb{B}, \mathbb{P}, I^*)\), with \((P, g) \in I \Leftrightarrow (g, P) \in I^*\). Then \(\mathcal{P}_n\) is called self-dual, if \(\mathcal{P}_n \cong \mathcal{P}^D_n\), i.e. if there exists a bijective map \(\phi : (\mathbb{P}, \mathbb{B}) \to (\mathbb{B}, \mathbb{P})\) such that \((P, g) \in I \iff \phi(P, g) \in I^*\). Incidence statements where the sets \(\mathbb{P}\) and \(\mathbb{B}\) are interchanged are said to be dual to each other.

In this paper, we are mainly interested in finite projective planes of order 9, which means that we have \(9^2 + 9 + 1 = 91\) points and 91 lines. Each line is incident with 10 points and each point is incident with 10 lines.

In order to generalize conics to finite projective planes, the notion of ovals has been introduced: An oval in \(\mathcal{P}_n\) is a set of \(n + 1\) points, no three of which are collinear. Every conic is an oval, and for \(p\) odd, every oval in \(\text{PG}(2, p^k)\) is a conic.

A line which intersects an oval \(O\) in two points is called secant, a line which intersects \(O\) in one point is called tangent, and a line which is disjoint to \(O\) is called external line of \(O\).

To reformulate Poncelet’s Theorem for finite projective planes, we take a closer look at pairs of ovals \(O_t\) and \(O_s\) in \(\mathcal{P}_n\). An \(m\)-sided Poncelet polygon is a polygon with \(m\) sides, \(3 \leq m \leq n + 1\), such that the vertices are on \(O_s\) and the sides are tangent to \(O_t\). According to that, we call \(O_t\) the tangent oval and \(O_s\) the secant oval of the Poncelet polygon. For \(3 \leq m \leq n + 1\) fixed, \((O_t, O_s)\) is said to form a Poncelet \(m\)-pair, if there exists at least one \(m\)-sided Poncelet polygon for \(O_t\) and \(O_s\), but no \(m'\)-sided Poncelet polygon, \(m' \neq m\), \(3 \leq m' \leq n + 1\), for the same pair can be constructed. We say that \((O_t, O_s)\) forms a Poncelet 0-pair, if no secant of \(O_s\) is a tangent of \(O_t\). We say that \((O_t, O_s)\) forms a Poncelet \(\infty\)-pair, if there exists at least one secant of \(O_s\), which is a tangent of \(O_t\), but no \(m\)-sided Poncelet polygon for \(3 \leq m \leq n + 1\) can be constructed.

Note that finite projective planes may exhibit rather unintuitive phenomena compared to the real projective plane. For example, a pair of ovals can be located such that no point of one oval is incident with a tangent of the other one and vice versa. Or, in finite projective planes of even order it may happen that two ovals have all their tangents in common.

With the terminology above, we call a finite projective plane \(\mathcal{P}_n\) a Poncelet plane, if every pair of ovals \((O_t, O_s)\) is a Poncelet \(m\)-pair, for \(3 \leq m \leq n + 1\), \(m = 0\) or \(m = \infty\).

Note that in planes of even order all tangents of an oval meet in one point, the so-called nucleus of the oval. Because of that, only Poncelet 0-pairs and Poncelet \(\infty\)-pairs can be constructed in such planes. In this sense, all planes of even order are Poncelet planes.
2 Poncelet's Theorem in $PG(2,9)$

The main goal in this section is to show that $PG(2,9)$ is a Poncelet plane. As mentioned earlier, Berger presented in [1] a synthetic proof of the Great Poncelet Theorem. His proof is formulated for projective planes over an arbitrary field with at least five elements. However, a number of additional thoughts are necessary to ensure that all steps in the proof work out over fields which are not algebraically closed. Since Berger shows a more general version of Poncelet’s Theorem we want to avoid this discussion, and, in order to make the paper self-contained, we present a shorter proof for Poncelet's Theorem in $PG(2,9)$ which is based upon Pascal’s Theorem, similar to [7]. However, as we work in a finite plane, we employ combinatorial arguments in a completely different way compared to [7]. In particular, we show the following statement.

**Theorem 2.1.** The finite projective Desarguesian plane of order 9 is a Poncelet plane.

In planes of order 9, ovals consist of 10 points. To prove that $PG(2,9)$ is a Poncelet plane, it is therefore enough to show that if a 3-sided or a 4-sided Poncelet polygon exists for a pair of ovals $(O_1, O_2)$, then this pair is a Poncelet 3-pair or 4-pair, respectively.

Let us quickly recall the construction of the finite projective Desarguesian plane $PG(2,q)$ constructed over $GF(q)$, $q := p^k$, $p$ an odd prime and $k \geq 1$. The points of $PG(2,q)$ are given by non-zero column vectors $[x, y, z]^T$ for $x, y, z \in GF(q)$, where $[\lambda x, \lambda y, \lambda z] = [x, y, z]$ for all $\lambda \in GF(q) \setminus \{0\}$. Similarly, all lines are denoted by row vectors $[x, y, z]$. A point $[x, y, z]^T$ is incident with a line $[a, b, c]$ if $ax + by + cz = 0$ in $GF(q)$.

The following facts are a collection of some elementary properties we will use later on (see e.g. [6] for proofs).

**Lemma 2.2.** Let $g = [g_1, g_2, g_3]$ and $h = [h_1, h_2, h_3]$ be two different lines in $PG(2,q)$. The unique intersection point $P$ of $g$ and $h$ is given by the vector product of $g$ and $h$, i.e.

$$P = [g_2h_3 - g_3h_2, g_3h_1 - g_1h_3, g_1h_2 - g_2h_1]^T.$$  

Similarly, for two points $P = [P_1, P_2, P_3]^T$ and $Q = [Q_1, Q_2, Q_3]^T$ in $PG(2,q)$, the unique line $g$ through $P$ and $Q$ is given by

$$g = [P_2Q_3 - P_3Q_2, P_3Q_1 - P_1Q_3, P_1Q_2 - P_2Q_1].$$

**Lemma 2.3.** Let $P = [P_1, P_2, P_3]^T$, $Q = [Q_1, Q_2, Q_3]^T$ and $R = [R_1, R_2, R_3]^T$ be three points in $PG(2,q)$. Then $P, Q, R$ are collinear if and only if

$$\det \begin{pmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{pmatrix} = 0.$$  

In finite projective Desarguesian planes $PG(2,q)$ over a field of odd characteristic, ovals coincide with conics (see, e.g., [13]). Thus, an oval can be described as the solutions of

$$O : ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0,$$  

where $a, b, c, d, e, f \in GF(q)$, $(a, b, c, d, e, f) \neq (0, 0, 0, 0, 0, 0)$ and the matrix $M_O$ associated to this quadratic form,

$$M_O = \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix}$$

is non-singular for ovals. Otherwise, for $M_O$ singular, the equation (1) describes a line, a pair of lines or a point.

The next step is to show the following result for Poncelet triangles.

**Theorem 2.4.** Let $(O_1, O_2)$ be a pair of ovals in $PG(2,q)$ such that a Poncelet triangle can be constructed. Then no $m$-sided Poncelet polygon for $4 \leq m \leq q+1$ for the same pair of ovals exists.
To see this, we need some preliminary results.

**Theorem 2.5.** Let $A, B, C, D, E$ and $F$ be the six vertices of a hexagon such that no three of them are collinear. Then, the intersection points of opposite sides

$$P = AB \cap DE, \quad Q = BC \cap EF, \quad R = CD \cap AF$$

are collinear if and only if the points $A, B, C, D, E$ and $F$ lie on an oval.

The if-statement is Pascal’s Theorem, the converse is known as the Braikenridge–Maclaurin Theorem. The line through $P, Q, R$ is called Pascal’s line (see Figure 1).

![Pascal’s Theorem](image_url)

**Proof.** We choose coordinates such that

$$A = [1, 0, 0]^T, \quad C = [0, 1, 0]^T, \quad E = [0, 0, 1]^T.$$  

Since no three of the points $A, B, C, D, E$ and $F$ are collinear, all coordinates of the remaining three points are non-zero, so we have

$$B = [1, B_2, B_3]^T, \quad D = [1, D_2, D_3]^T, \quad F = [1, F_2, F_3]^T$$

with $B_2, B_3, D_2, D_3, F_2, F_3 \neq 0$. Moreover, we have $B_2 \neq D_2, \quad D_2 \neq F_2, \quad B_2 \neq F_2, \quad B_3 \neq D_3, \quad D_3 \neq F_3$ and $B_3 \neq F_3$. To see this, assume $B_2 = D_2$. In this case, the points

$$B = [1, B_2, B_3]^T, \quad D = [1, B_2, D_3]^T, \quad E = [0, 0, 1]^T$$

would be collinear, since they are all incident with the line $g = [-B_2, 1, 0]$. The same can be shown analogously for the other coordinates. Using Lemma 2.2 we get the connecting lines

$$AF = [0, F_3, -F_2], \quad AB = [0, B_3, -B_2], \quad BC = [B_3, 0, -1],$$

$$CD = [D_3, 0, -1], \quad DE = [D_2, -1, 0], \quad EF = [F_2, -1, 0].$$

Using Lemma 2.2 once more, we obtain

$$P = [B_2, B_2D_2, B_3D_2]^T, \quad Q = [1, F_2, B_3]^T, \quad R = [F_3, D_3F_2, D_3F_3]^T.$$
Obviously, \( F \) lies on the unique conic through the points \( A, B, C, D, E \) iff

\[
\begin{vmatrix}
A_1^2 & A_2^2 & A_3^2 & A_1A_2 & A_1A_3 & A_2A_3 \\
C_1^2 & C_2^2 & C_3^2 & C_1C_2 & C_1C_3 & C_2C_3 \\
E_1^2 & E_2^2 & E_3^2 & E_1E_2 & E_1E_3 & E_2E_3 \\
B_1^2 & B_2^2 & B_3^2 & B_1B_2 & B_1B_3 & B_2B_3 \\
D_1^2 & D_2^2 & D_3^2 & D_1D_2 & D_1D_3 & D_2D_3 \\
F_1^2 & F_2^2 & F_3^2 & F_1F_2 & F_1F_3 & F_2F_3
\end{vmatrix}
\]

\[
= \det \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & B_2^2 & B_3^2 & B_2 & B_3 & B_2B_3 \\
1 & D_2^2 & D_3^2 & D_2 & D_3 & D_2D_3 \\
1 & F_2^2 & F_3^2 & F_2 & F_3 & F_2F_3
\end{pmatrix}
\]

\[
= \det \begin{pmatrix}
B_2 & B_3 & B_2B_3 \\
D_2 & D_3 & D_2D_3 \\
F_2 & F_3 & F_2F_3
\end{pmatrix} = \det(P, Q, R) = 0
\]

But according to Lemma 2.3, this is precisely the case for \( P, Q \) and \( R \) being collinear. \( \square \)

Note that all finite projective Desarguean planes are self-dual (see [6]) and hence, we may consider the dual form of Theorem 2.5.

**Corollary 2.6.** Let \( A, B, C, D, E, F \) be the six vertices of a hexagon such that no three of them are collinear. Then, the diagonals \( AD, BE \) and \( CF \) meet in one point, the Brianchon point, if and only if the sides \( AB, BC, CD, DE, EF \) of the oval are tangents of an oval.

**Lemma 2.7.** Let \( O_3 \) be an oval with two inscribed triangles \( \triangle ACE \) and \( \triangle BDF \), such that no three of the vertices are collinear. Then the sides of the two triangles are tangents of an oval \( O_3 \).

This result was used in [14] to prove Poncelet’s Theorem in the real projective plane for triangles. Since the arguments used there cannot be applied to the finite projective plane, we have to give an alternative proof here.

**Proof of Lemma 2.7.** Let \( A = [1, 0, 0]^T \), \( C = [0, 1, 0]^T \) and \( E = [0, 0, 1]^T \) be on \( O_3 \), which leads to the oval equation \( xy + exz + fy^2 = 0 \), with \( e \neq 0 \) and \( f \neq 0 \). For every other point of this oval, all three coordinates are non-zero. In particular, we have (by scaling if necessary)

\[
B = [1, B_2, B_3]^T, \quad D = [1, D_2, D_3]^T, \quad F = [1, F_2, F_3]^T
\]

with \( B_2, B_3, D_2, D_3, F_2 \) and \( F_3 \) non-zero. The sides of \( \triangle ACE \) and \( \triangle BDF \) are denoted by

\[
\begin{align*}
\triangle ACE & : g_1 = \overline{AC}, \quad g_3 = \overline{CE}, \quad g_5 = \overline{EA}, \\
\triangle BDF & : g_2 = \overline{BD}, \quad g_4 = \overline{DF}, \quad g_6 = \overline{FB}.
\end{align*}
\]

Explicitly, we have

\[
\begin{align*}
g_1 &= [0, 0, 1], \quad g_2 = [-B_3D_2 + B_2D_3, B_3 - D_3, -B_2 + D_2] \\
g_3 &= [1, 0, 0], \quad g_4 = [-D_3F_2 + D_2F_3, D_3 - F_3, -D_2 + F_2] \\
g_5 &= [0, 1, 0], \quad g_6 = [B_3F_2 - B_2F_3, -B_3 + F_3, B_2 - F_2].
\end{align*}
\]

The intersection points of these lines are given by

\[
\begin{align*}
A_1 &= g_6 \cap g_1, \quad A_2 = g_1 \cap g_2, \quad A_3 = g_2 \cap g_3, \\
A_4 &= g_3 \cap g_4, \quad A_5 = g_4 \cap g_5, \quad A_6 = g_5 \cap g_6.
\end{align*}
\]
This leads to

\[
\begin{align*}
A_1 &= [B_3 - F_3, B_3 F_2 - B_2 F_3, 0]^T, \\
A_2 &= [B_3 - D_3, B_3 D_2 - B_2 D_3, 0]^T, \\
A_3 &= [0, B_2 - D_2, B_3 - D_3]^T, \\
A_4 &= [0, D_2 - F_2, D_3 - F_3]^T, \\
A_5 &= [D_2 - F_2, 0, -D_3 F_2 + D_2 F_3]^T, \\
A_6 &= [B_2 - F_2, 0, -B_3 F_2 + B_2 F_3]^T.
\end{align*}
\]

We would like to find an oval \(O_s\), such that the lines \(g_1, \ldots, g_6\) are tangents of it. By Brianchon’s Theorem (Corollary 2.6), we know that this is equivalent to showing that \(\overline{A_1 A_4}, \overline{A_2 A_5}\) and \(\overline{A_3 A_6}\) meet in one point. Using Lemma 2.2 we obtain

\[
\overline{A_1 A_4} = [B_3 D_3 F_2 - B_2 D_3 F_3 - B_3 F_2 F_3 + B_2 F_3^2, \\
- B_3 D_3 + B_3 F_3 + D_3 F_3 - F_3^2, B_3 D_2 - B_3 F_2 - D_2 F_3 + F_2 F_3]
\]

\[
\overline{A_2 A_5} = [B_3 D_3 F_2 - B_2 D_3 F_3 - B_3 D_2 F_3 + B_2 D_2 F_3, \\
- B_3 D_3 F_2 + D_2 D_3 F_3 + B_2 D_2 D_3 F_3, \\
- B_3 D_3 F_2 - B_3 D_2 F_3 - B_3 D_2 D_3 - B_3 D_2 F_2 + B_3 D_2 F_2 + B_2 D_3 F_2]
\]

\[
\overline{A_3 A_6} = [B_2 B_3 F_2 - B_3 D_2 F_2 - B_2 D_3 F_3 + B_2 D_2 F_3, \\
- B_2 B_3 + B_2 D_3 + B_3 F_2 - D_3 F_2, B_2^2 - B_2 D_2 - B_2 F_2 + D_2 F_2].
\]

Observe that the points \(B, D\) and \(F\) lie on the original oval, which means that they satisfy

\[
B_2 = -\frac{eB_3}{1 + fB_3}, \quad D_2 = -\frac{eD_3}{1 + fD_3}, \quad F_2 = -\frac{eF_3}{1 + fF_3}.
\]

Note that we have \(1 + fB_3 \neq 0, 1 + fD_3 \neq 0\) and \(1 + fF_3 \neq 0\). To see this, assume \(1 + fB_3 = 0\). It follows \(B_3 = -\frac{1}{f}\) and using the oval equation \(xy + exz + fyz = 0\) once more, we obtain \(-\frac{1}{f} = 0\), contradicting the fact that \(e \neq 0\) and \(f \neq 0\). Finally it follows that the three lines are concurrent, since

\[
\det(\overline{A_1 A_4}, \overline{A_2 A_5}, \overline{A_3 A_6}) = 0. \quad \Box
\]

**Proof of Theorem 2.4.** Let \((O_t, O_s)\) be a pair of ovals, such that there exists a triangle which consists of tangents of \(O_t\) and vertices on \(O_s\). Let the sides of this triangle be \(t_1, t_2\) and \(t_3\). Assume that there exists another closed polygon using tangents of \(O_t\) with vertices on \(O_s\). Since we need at least three vertices on \(O_s\), such that the lines connecting these are tangent to \(O_t\) for the new polygon, we can assume the existence of at least three such lines. Hence, we start with \(s_1\), which is a tangent of \(O_t\) and joins two points of \(O_s\), denoted by \(S_1\) and \(S_2\). By assumption, there exists another line \(s_2\), which joins \(S_2\) with another point of \(O_s\) and which is a tangent of \(O_t\). Let \(s_2\) intersect \(O_s\) in \(S_2\) and \(S_3\). The claim is now, that the line connecting \(S_1\) and \(S_3\), denoted by \(s_3\), is a tangent of \(O_t\) as well. Assume the contrary (Figure 2), i.e. assume that \(s_3\) is not a tangent of \(O_t\). By Lemma 2.7 there exists an oval \(O\) such that \(t_1, t_2, t_3, s_1, s_2\) and \(s_3\) are tangents of \(O\). But \(t_1, t_2, t_3, s_1\) and \(s_2\) are tangents of \(O_t\) as well. Since an oval is uniquely determined by five of its tangents, we have \(O_t = O\). Hence, every other polygon which can be closed is a Poncelet triangle as well. This shows that \((O_t, O_s)\) is a Poncelet 3-pair. \(\Box\)

Note that we did not restrict ourselves to \(PG(2, 9)\) in the above proof, i.e. Theorem 2.4 applies to all finite projective Desarguesian planes \(PG(2, q)\).

Now we turn our attention to 4-sided Poncelet polygons.

**Theorem 2.8.** Let \((O_t, O_s)\) be a pair of ovals in \(PG(2, 9)\) which carries a Poncelet quadrilateral. Then no \(m\)-sided Poncelet polygon, for \(3 \leq m \leq 10, m \neq 4\), for \((O_t, O_s)\) can be constructed.

We need to show that the existence of a Poncelet quadrilateral for a pair of ovals \((O_t, O_s)\) excludes the existence of a 5-sided Poncelet polygon as well as the existence of a 6-sided Poncelet polygon for the same pair. To see this, we start with a pair \((O_t, O_s)\) which carries a Poncelet quadrilateral. Recall the following fundamental theorem for \(PG(2, q)\) (see [6]).
Figure 2: Opposite assumption: Assume that the line connecting $S_1$ and $S_3$ is not tangent to $O_t$.

**Theorem 2.9.** Let $\{P_1, P_2, P_3, P_4\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$ be sets of four points, such that no three points of the same set are collinear. Then there exists a unique projective map $T$, such that $T(P_i) = Q_i$, for $1 \leq i \leq 4$.

**Proof of Theorem 2.9** We may assume that the pair of ovals $(O_t, O_s)$ carries the Poncelet quadrilateral

$$A = [1, -1, 0]^T, \quad B = [1, 0, -1]^T, \quad C = [1, 1, 0]^T, \quad D = [1, 0, 1]^T.$$ 

The equation of $O_s$ is of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0,$$

for $a, b, c, d, e, f \in GF(9)$ and the associated matrix is non-singular. We want the points $A, B, C, D$ to lie on the oval $O_s$, which gives four conditions for (2), namely

$$a + b - d = 0, \quad a + c - e = 0, \quad a + b + d = 0, \quad a + c + e = 0.$$ 

Since $a = 0$ would lead to a singular matrix we may scale $a = 1$ and the equation for $O_s$ is

$$O_s(f) : x^2 - y^2 - z^2 + 2fyz = 0,$$

for $f \neq \pm 1$, which ensures that the associated matrix is non-singular. It is enough to consider ovals of the above form for $O_s$. Now, the four lines

$$\overline{AB} = [1, 1, 1], \quad \overline{BC} = [1, -1, 1], \quad \overline{CD} = [1, -1, -1], \quad \overline{DA} = [1, 1, -1]$$

need to be tangents of $O_t$. To find the corresponding oval equation, we first determine the equations of ovals which contain the four points $[1, 1, 1]^T$, $[1, -1, 1]^T$, $[1, -1, -1]^T$ and $[1, 1, -1]^T$. We have to solve the system of equations for its coefficients

$$a + b + c + d + e + 2f = 0, \quad a + b + c - d + e - 2f = 0,$$

$$a + b + c - d - e + 2f = 0, \quad a + b + c + d - e - 2f = 0.$$ 

We immediately obtain $d = e = f = 0$ and after scaling $a = 1$, we end up with the equation

$$x^2 + by^2 - (1 + b)z^2 = 0.$$ 

Since we need an oval with tangents $[1, 1, 1], [1, -1, 1], [1, -1, -1]$ and $[1, 1, -1]$ rather than points, we have to take the equation which corresponds to the inverse matrix of the matrix associated to the equation $x^2 + by^2 - (1 + b)z^2 = 0$ which is

$$O_t(b) : x^2 + \frac{1}{b}y^2 - \frac{1}{1+b}z^2 = 0,$$

for $b \neq 0, -1$. To exclude the simultaneous existence of a 4-sided Poncelet polygon and a 5-sided or 6-sided Poncelet polygon, respectively, it is enough to consider pairs of the ovals described above.
Moreover, when calculating the coefficients \( y \) because changing the sign of the \( H \) hence it is enough to consider

\[
a^2 + a = 1, \quad a^4 a^j = a^{i+j}, \quad a^8 = 1.
\]

Therefore, we consider pairs of ovals of the form \( (O_t(b), O_s(f)) \) for

\[
b \in \{1, a, a^2, a^3, a^5, a^6, a^7\} \quad \text{and} \quad f \in \{0, a, a^2, a^3, a^5, a^6, a^7\}.
\]

By inspecting the pair \( (O_t(b), O_s(f)) \) we obtain exactly the same results as for \( (O_t(b), O_s(-f)) \), because changing the sign of the \( y \)-coordinate has the effect

\[
[x, y, z]^T \in O_s(f) \iff [x, -y, z]^T \in O_s(-f)
\]

and

\[
[x, y, z]^T \in O_t(b) \iff [x, -y, z]^T \in O_t(b).
\]

Hence, it is enough to consider \( f \in \{0, a, a^2, a^3\} \).

Moreover, when calculating the coefficients \( \frac{1}{T} \) and \( \frac{1}{T+8} \) for all values of \( b \) above, we obtain

\[
O_t(1) : x^2 + y^2 + z^2 = 0 \quad \quad O_t(a^5) : x^2 + a^3 y^2 + a^2 z^2 = 0
\]

\[
O_t(a) : x^2 + a^7 y^2 + a^5 z^2 = 0 \quad \quad O_t(a^6) : x^2 + a^2 y^2 + a^3 z^2 = 0
\]

\[
O_t(a^2) : x^2 + a^9 y^2 + a z^2 = 0 \quad \quad O_t(a^7) : x^2 + a y^2 + a^6 z^2 = 0.
\]

\[
O_t(a^3) : x^2 + a^5 y^2 + a^7 z^2 = 0
\]

Note that interchanging the \( y \) and \( z \) coordinate does not change the incidence relations for both ovals, as both equations are symmetric, i.e.

\[
[x, y, z]^T \in O_t(b) \iff [x, z, y]^T \in O_t(b)
\]

and

\[
[x, y, z]^T \in O_s(f) \iff [x, z, y]^T \in O_s(f).
\]

Therefore, it is enough to consider \( b \in \{1, a, a^2, a^5\} \).

All we have to do is to exclude the existence of a 5-sided and a 6-sided Poncelet polygon for the following 16 oval pairs

\[
(O_t(b), O_s(f)), \ b \in \{1, a, a^2, a^5\}, \ f \in \{0, a, a^2, a^3\}.
\]

By direct inspection, we count the number of points on \( O_s(f) \) that are incident with a tangent of \( O_t(b) \). Table 1 contains these numbers.

|        | \( O_t(1) \) | \( O_t(a) \) | \( O_t(a^2) \) | \( O_t(a^5) \) |
|--------|--------------|--------------|--------------|--------------|
| \( O_s(0) \) | 8            | 6            | 4            | 4            |
| \( O_s(a) \) | 4            | 6            | 4            | 8            |
| \( O_s(a^2) \) | 8            | 4            | 5            | 5            |
| \( O_s(a^5) \) | 4            | 6            | 8            | 4            |

Since by construction all of these pairs already form one 4-sided Poncelet polygon, the condition of 9 or 10 exterior points of \( O_t(b) \) on \( O_s(f) \) is necessary to find a 5-sided or 6-sided Poncelet polygon, respectively. Since there are at most eight exterior points of \( O_t(b) \) on \( O_s(f) \), we can exclude their existence. This completes the proof of \( PG(2, 9) \) being a Poncelet plane. \( \square \)
3 Poncelet’s Theorem in the finite projective planes over $\mathfrak{S}$

3.1 The miniquaternion near-field $\mathfrak{S}$

We describe the near-field we use to construct the three non-Desarguesian finite projective planes, denoted by $\Omega$, $\Omega^D$, and $\Psi$. All notations and well-known properties are based on [15].

A \textit{finite near-field} is a system $(\mathfrak{S}, +, \circ)$, such that

(i) $\mathfrak{S}$ is finite,

(ii) $(\mathfrak{S}, +)$ is a commutative group with identity 0,

(ii) the multiplication is a group operation on $\mathfrak{S}\setminus \{0\}$ with identity 1 and

(iv) the multiplication is right distributive over the addition, i.e.

$$(m + n)l = ml + nl, \forall m, n, l \in \mathfrak{S}.$$ 

Note that we do not necessarily need the multiplication to be commutative, hence the left distribution law does not have to be valid for all elements in the near-field. This is exactly the property used in the construction of the non-isomorphic planes of order 9. We need to describe addition and multiplication for a near-field with nine elements. For this, consider

$$\mathfrak{S} = \{0, 1, -1, i, -i, j, -j, k, -k\}$$

where we define

$$j := 1 + i, \quad k := 1 - i.$$ 

We can view the nine elements as elements over the basis $\{1, i\}$ and call $\mathfrak{D} := \{0, 1, -1\}$ the real elements and $\mathfrak{S}^* := \{i, -i, j, -j, k, -k\}$ the complex elements. By the definition of $j$ and $k$ above and taking the coefficients of 1 and $i$ modulo 3, we are able to add any two elements in the near-field (Table 2). For the multiplication in $\mathfrak{S}$, we want to end up with a non-commutative operation. We use the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik$$

which again enables us to multiply any two elements in $\mathfrak{S}$ (Table 2).

Note that this is the multiplication law of the quaternion group, which explains the name ‘miniquaternion near-field’. Note also, that in this near-field the left distribution law does not hold in general, as for example $i(j + k) = i(-1) = -i$ but $ij + ik = k - j = i$.

3.2 The plane $\Omega$

In order to construct the finite projective plane $\Omega$ of order 9 using the near-field $\mathfrak{S}$ we start with an affine plane and extend it to a projective plane. We distinguish between so-called \textit{proper points} on $\Omega$, which are in affine form $(x, y)$, and \textit{ideal points}, which connect the parallel lines. More precisely, the points of $\Omega$ are given by

- 81 proper points of the form $(x, y)$, for $x, y \in \mathfrak{S}$,
- 9 ideal points of the form $(1, y, 0)$ for $y \in \mathfrak{S}$ and
- one ideal point of the form $(0, 1, 0)$.

The lines of $\Omega$ are given by

- 81 proper lines of the form $y = x\mu + \nu$ for $\mu, \nu \in \mathfrak{S}$, denoted by $(\mu, \nu)$,
- 9 proper lines of the form $x = \lambda$ for $\lambda \in \mathfrak{S}$, denoted by $(\lambda)$, and
- one ideal line, denoted by $\mathfrak{I}$.  

On the line $y = x\mu + \nu$ there are nine proper points $(x, y)$ and the ideal point $(1, \mu, 0)$. On the line $x = \lambda$ there are nine proper points $(\lambda, y)$ and the ideal point $(0, 1, 0)$. All 10 ideal points are on the ideal line $\mathfrak{l}$.

It is crucial to consider $y = x\mu + \nu$ instead of $y = \mu x + \nu$, since multiplication is not commutative. It can be shown that the above defined points and lines with the incidence relation give indeed a finite projective plane of order 9 (see [15]).

Now we want to find ovals in the plane $\Omega$, i.e. we want to find sets of 10 points, no three of which are collinear. Compared to finite projective coordinate planes, it is much harder to find ovals in this plane, since ovals cannot be described by quadratic forms. Hence, for a set of 10 points, we have to search all lines connecting them to be sure that no three of them are collinear. For any point on the oval, we end up with nine lines connecting this point to the other points on the oval, and all these secants have to be different. Similarly it is much harder to find the tangent in a point of the oval. Nevertheless, the set $O_1$ given by

$$\left\{ \left( \begin{array}{c} -1 \\ i \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} -i \\ -j \\ 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ j \\ i \\ -1 \end{array} \right), \left( \begin{array}{c} -k \\ 1 \\ 0 \\ -1 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \right\}$$

is an example of an oval in $\Omega$. To see this, we have to calculate all secants and check, whether they are different. Table 2 shows all secants and tangents of $O_1$.

Recall that in $PG(2, 9)$, Pascal’s Theorem plays a central rôle in the proof of Poncelet’s Closure Theorem. We will see that Pascal’s Theorem is not true in general in the plane $\Omega$. For this, take for example the six points

$$A = (-1, i), \ B = (0, 1), \ C = (1, 0), \ D = (i, j), \ E = (-k, -i), \ F = (0, 1, 0).$$

These points all lie on $O_1$, hence they lie indeed on a non-degenerate hexagon. We have

$$AB : y = xk + 1, \quad BC : y = -x + 1, \quad CD : y = -xi + 1$$
$$DE : y = xk - j, \quad EF : x = -k, \quad FA : x = -1.$$
Table 3: The diagonal entries are the tangents of $O_1$ and the other entries are the secants of the points of $O_1$ listed in the first row and column.

$$
\begin{array}{cccccccc}
O_1 & -1 & i & 0 & 1 & -i & j & k \\
-1 & 0 & 1 & -i & j & k & 0 & i \\
0 & 1 & -i & j & k & 0 & i & -1 \\
1 & 0 & -i & j & k & 0 & i & 1 \\
-1 & j & k & 0 & i & -1 & j & -1 \\
0 & 1 & -i & j & k & 0 & i & 1 \\
1 & 0 & -i & j & k & 0 & i & 1 \\
-1 & j & k & 0 & i & -1 & j & -1 \\
0 & 1 & -i & j & k & 0 & i & 1 \\
1 & 0 & -i & j & k & 0 & i & 1 \\
\end{array}
$$

The intersection points we need in Pascal’s Theorem are given by

$$P = (1, k, 0), \quad Q = (-k, -j), \quad R = (-1, -i).$$

These are not collinear, as the line through $P$ and $Q$ is given by $y = xk + k$ and the line through $P$ and $R$ is $y = xk + j$.

**Theorem 3.1.** The finite projective plane $\Omega$ of order 9 is not a Poncelet plane.

**Proof.** We have to find a pair of ovals $(O_t, O_s)$ which carries at the same time an $n$-sided and an $m$-sided Poncelet polygon with $m \neq n$ and $m, n \geq 3$. For $O_t$ we take $O_1$, and for $O_s$ we choose the oval

$$\left\{ \left( \begin{array}{c} 0 \\ j \end{array} \right), \left( \begin{array}{c} i \\ i \end{array} \right), \left( \begin{array}{c} i \\ -k \end{array} \right), \left( \begin{array}{c} -j \\ j \end{array} \right), \left( \begin{array}{c} j \\ -j \end{array} \right), \left( \begin{array}{c} -k \\ k \end{array} \right) \right\}$$

Again, we have to ensure that all secants of this set are different. For this, see Table 3.

For this oval pair $(O_t, O_s)$, we can now find simultaneously a 5-sided Poncelet polygon and a 4-sided Poncelet polygon. To see this, start with the point $(0, j)$ on $O_s$. The line joining $(0, j)$ and $(1, -j, 0)$ is the line $y = -xj + j$, as given in Table 3. This line is a tangent of oval $O_t$, namely the tangent in the point $(1, 0)$. Continuing with $(1, -j, 0)$, we see that the line joining $(1, -j, 0)$ and $(-k, i)$ is $y = -xj - i$, which is the tangent of $O_t$ in the point $(j, k)$. Moreover, the line joining $(-k, i)$ and $(j, 0)$ is $y = xi + k$, which is the tangent of $O_t$ in $(-i, -j)$. The line joining $(j, 0)$ and $(i, i)$ is $y = -xi - k$, which is again a tangent of $O_t$, namely in the point $(-k, -i)$. For the last step, we see that the line joining $(i, i)$ and $(0, j)$ is $y = xi + j$, which is the tangent of $O_t$ in the point $(-j, -1)$. This gives a 5-sided Poncelet polygon for this pair. Similarly, by starting with $(i, -k)$ on $O_s$, a 4-sided Poncelet polygon occurs. To summarize the result, we have the 5-sided
any tangent of $\Omega$ is not a Poncelet plane. The remaining point $(i, j)$ on $O_s$ is an inner point of $O_t$, which means that it is not incident with any tangent of $O_t$. This pair $(O_t, O_s)$ is therefore no Poncelet $m$-pair for any possible value of $m$, which shows that $\Omega$ is not a Poncelet plane.

This pair of ovals gives even one more proof of $\Omega$ not being a Poncelet plane, namely by changing the roles of $O_t$ and $O_s$. If we consider points on $O_t$ and tangents of $O_s$, we find simultaneously a 4-sided and a 3-sided Poncelet polygon, namely

Poncelet polygon

\[
\begin{pmatrix} 0 \\ j \end{pmatrix} \xrightarrow{(1, 0)} \begin{pmatrix} 1 \\ -j \end{pmatrix} \xrightarrow{(j, k)} \begin{pmatrix} -k \\ i \end{pmatrix} \xrightarrow{(-i, -j)} \begin{pmatrix} -j \\ 0 \end{pmatrix} \xrightarrow{(j, i)} \begin{pmatrix} 0 \\ j \end{pmatrix}
\]

and the 4-sided Poncelet polygon

\[
\begin{pmatrix} i \\ -k \end{pmatrix} \xrightarrow{(-1, i)} \begin{pmatrix} j \\ 0 \end{pmatrix} \xrightarrow{(i, j)} \begin{pmatrix} j \\ -j \end{pmatrix} \xrightarrow{(0, 1)} \begin{pmatrix} -k \\ -k \end{pmatrix} \xrightarrow{(1, 0, 0)} \begin{pmatrix} i \\ -k \end{pmatrix}.
\]

The remaining point $(-j, j)$ on $O_s$ is an inner point of $O_t$, which means that it is not incident with any tangent of $O_t$. This pair $(O_t, O_s)$ is therefore no Poncelet $m$-pair for any possible value of $m$, which shows that $\Omega$ is not a Poncelet plane.

This pair of ovals gives even one more proof of $\Omega$ not being a Poncelet plane, namely by changing the roles of $O_t$ and $O_s$. If we consider points on $O_t$ and tangents of $O_s$, we find simultaneously a 4-sided and a 3-sided Poncelet polygon, namely

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{(-k, i)} \begin{pmatrix} i \\ j \end{pmatrix} \xrightarrow{(-k, -k)} \begin{pmatrix} -j \\ -1 \end{pmatrix} \xrightarrow{(-j, j)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{(0, j)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{(i, -k)} \begin{pmatrix} -i \\ -j \end{pmatrix} \xrightarrow{(j, -j)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{(j, 0)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

This shows that in $\Omega$, Poncelet’s Theorem is not even partially true for only 3-sided polygons or only 4-sided polygons, as we could find counter examples in both cases.
3.3 The plane $\Omega^D$

Now, we want to look at the dual plane of $\Omega$, that is, we want to change the rôle of the points and lines constructed for $\Omega$ to obtain the plane $\Omega^D$. Note that $\Omega^D$ is indeed not isomorphic to $\Omega$ (see [15]).

Recall the incidence relation for $\Omega$, given by

$$(x, y) \in (\mu, \nu) \iff y = x\mu + \nu.$$  

By changing the rôles of points and lines, $(x, y)$ denotes a line in $\Omega^D$ and $(\mu, \nu)$ denotes a point in $\Omega^D$. Hence, the incidence relation becomes

$$(\mu, \nu) \in (x, y) \iff \nu = -x\mu + y.$$  

For the line $x = \lambda$, the incidence relation stays the same. Moreover, the ideal line is not changed either. By adjusting the notation above by taking $x$ instead of $-x$, we obtain the incidence relations for $\Omega^D$. Namely, on the line $y = \mu x + \nu$, there are nine proper points $(x, y)$ and the ideal point $(1, \mu, 0)$. On the line $x = \lambda$, there are nine proper points $(\lambda, y)$ and the ideal point $(0, 1, 0)$. All 10 ideal points are on the ideal line $\mathcal{I}$.

**Theorem 3.2.** The finite projective plane $\Omega^D$ of order 9 is not a Poncelet plane.

**Proof.** In order to prove that $\Omega^D$ is not a Poncelet plane either we can dualize the ovals from the previous section. For this, recall the oval $O_t$ given by

$$\left\{(\frac{-1}{i}, \frac{0}{1}, \frac{1}{0}), (\frac{-i}{-j}, \frac{i}{j}), (\frac{-j}{k}, \frac{j}{1}, \frac{-k}{0}, \frac{1}{0})\right\}$$

in $\Omega$. The set of tangents of this oval is given by the lines

$$\left\{(\frac{j}{k}, \frac{-i}{1}, \frac{-j}{j}, \frac{i}{k}, \frac{j}{1}, \frac{-i}{-j}, \frac{-j}{-i}, \frac{0}{k}, \frac{-k}{0})\right\}.$$  

Note that for the proper lines $(\mu, \nu)$, we have to take the minus sign for the $x$-coordinate. This gives the oval $O_t^D$

$$\left\{(\frac{-j}{k}, \frac{i}{1}, \frac{j}{k}, \frac{-i}{1}, \frac{-j}{j}, \frac{j}{1}, \frac{-i}{-k}, \frac{k}{0}, \frac{-k}{-k})\right\}$$

in $\Omega^D$.

Similarly, the dualization of the oval $O_s$ leads to another oval in $\Omega^D$, namely $O_s^D$ given by

$$\left\{(\frac{1}{0}, \frac{1}{-i}, \frac{-1}{-1}, \frac{1}{-j}, \frac{0}{0}, \frac{-1}{0}, \frac{1}{1}, \frac{j}{i}, \frac{-j}{-i})\right\}.$$  

Now we can dualize the $n$-sided Poncelet polygons as well. Recall that for $O_t$ and $O_s$ in $\Omega$, we had the 4-sided Poncelet polygon

$$(\frac{i}{j}, \frac{-1,i}{-k})$$

for vertices on $O_s$.

The tangent of $O_t$ in $(1,i)$ is $(j,k)$, hence we can start with the corresponding point $(-j,k)$ on $O_t^D$. The connection of two points of $O_s$ in $\Omega$ is now the same as the intersection of two tangents of $O_s^D$. Hence, in $\Omega^D$, we have to take vertices on $O_t^D$ and tangents of $O_s^D$. The next tangent we consider, namely the tangent in $(i,j)$, corresponds to $(-j,-i)$, a point on $O_s^D$. The line connecting
Similarly to the construction of the plane $\Omega$, we can define the plane $\Psi$ using again the fact that $3.4$ The plane $\Psi$

Similarly to the construction of the plane $\Omega$, we can define the plane $\Psi$ using again the fact that $\mathcal{S}$ is not left distributive. We make use of the homogeneous approach, unlike the affine approach before. A point is defined as the set of vectors $\{P\kappa, \kappa \in \mathcal{S}, \kappa \neq 0\}$, $P \in \mathbb{S}^3 \setminus \{(0,0,0)\}$. A point is called real, if there exists a non-zero $\kappa$ in $\mathcal{S}$, such that all coordinates of $P\kappa$ are in $\mathcal{D}$. Otherwise, the point is called complex. Note that there are 13 real points and 78 complex points.

The line through $P$ and $Q$ is defined by

$$\{P\} \cup \{P\kappa + Q, \kappa \in \mathcal{S}\}.$$  

A line is called real if at least two real points are on the line, otherwise complex. We can choose the line at infinity $z = 0$. All points not on this line can be parameterized by $P = (x, y, 1)$ and all points on the line $z = 0$ can be seen as $Q = (1, \kappa, 0)$. We get 13 real lines, namely

- 9 lines of the form $y = mx + c$, $m, c \in \mathcal{D}$, denoted by $(m, c, 1)$,
- 3 lines of the form $x = c$, $c \in \mathcal{D}$, denoted by $(c, 1, 0)$, and
- one line $z = 0$, denoted by $(0, 0, 0)$.

The 78 complex lines are given by

- 54 lines of the form $y - s = \kappa(x - r)$, $r, s \in \mathcal{D}$, $\kappa \in \mathcal{S}^*$, denoted by $(s, r, \kappa)$,
- 18 lines of the form $y = mx + \kappa$, $m \in \mathcal{D}$, $\kappa \in \mathcal{S}^*$, denoted by $(m, \kappa, 1)$, and
- 6 lines $x = \kappa$, $\kappa \in \mathcal{S}^*$, denoted by $(\kappa, 1, 0)$.

Note that we have parameterized the lines and points in a different way, since for example the vectors $(1, 1, i)$ and $(-1, -1, -i)$ do not represent the same line, but they do represent the same point. It can be shown that these points and lines together with the incidence relations form indeed a finite projective plane of order 9 which is not isomorphic to $\Omega$ or $\Omega^D$, and $\Psi$ is self-dual (see [15]).

**Theorem 3.3.** The finite projective plane $\Psi$ of order 9 is not a Poncelet plane.

**Proof.** Look at the two ovals $O_t$

$$\left\{ \begin{array}{c}
(-1) \\
-1 \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
0 \\
i \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
1 \\
0 \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
-1 \\
i \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
j \\
k \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
k \\
i \\
1
\end{array} \right\}$$

and $O_s$

$$\left\{ \begin{array}{c}
i \\
k \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
0 \\
-1 \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
1 \\
i \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
1 \\
k \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
i \\
k \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
k \\
i \\
1
\end{array} \right\}$$

Similarly to the approach before, in Table 5 and 6 we just list all secants and tangents to check that these sets are indeed ovals.
Table 5: Oval $O_t$ in $\Psi$

| $O_t$  | (-1) | (-1) | (0) | (0) | (1) | (0) | (j) | (1) | (j) | (0) |
|--------|------|------|-----|-----|-----|-----|-----|-----|-----|-----|
| (-1)   | 1    | 1    | 0   | 1   | 1   | 0   | 1   | 1   | 1   | 1   |
| (1)    | 0    | 1    | -1  | -1  | -1  | 0   | -1  | -1  | -1  | -1  |
| (-1)   | 1    | 1    | 0   | 1   | 1   | 0   | 1   | 1   | 1   | i   |
| (1)    | 0    | -1   | -1  | -k  | 0   | 1   | -1  | -1  | -1  | -1  |
| (-i)   | i    | i    | i   | -i  | -i  | 1   | i   | i   | i   | i   |
| (0)    | i    | 0    | -1  | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| (0)    | 0    | 0    | 1   | -k  | 1   | 0   | -j  | 1   | 1   | 0   |
| (0)    | 0    | 0    | 1   | 0   | 1   | 1   | 0   | 1   | 1   | 1   |
| (-k)   | 1    | 1    | 0   | 1   | 1   | 0   | 1   | 1   | 1   | 1   |
| (j)    | 0    | 0    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| (j)    | 0    | 0    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| (j)    | 0    | 0    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |

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Table 6: Oval $O_s$ in $\Psi$

| $O_s$ | $(-1)\ i$ | $(-1)\ k$ | $0\ i$ | $0\ j$ | $1\ i$ | $1\ j$ | $1\ k$ | $i\ j$ | $k\ i$ |
|-------|------------|------------|--------|--------|--------|--------|--------|--------|--------|
| $(-1)\ i$ | $(-1)\ i$ | $-1\ i$ | $0\ j$ | $0\ j$ | $1\ j$ | $1\ j$ | $1\ k$ | $i\ j$ | $k\ i$ |
| $(-1)\ k$ | $(-1)\ k$ | $0\ i$ | $0\ j$ | $0\ j$ | $1\ j$ | $1\ j$ | $1\ k$ | $i\ j$ | $k\ i$ |
| $0\ i$ | $0\ i$ | $0\ j$ | $0\ j$ | $0\ j$ | $0\ j$ | $0\ j$ | $0\ j$ | $0\ j$ | $0\ j$ |
| $1\ i$ | $1\ i$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ |
| $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ | $1\ j$ |
| $1\ k$ | $1\ k$ | $1\ k$ | $1\ k$ | $1\ k$ | $1\ k$ | $1\ k$ | $1\ k$ | $1\ k$ | $1\ k$ |
| $k\ i$ | $k\ i$ | $k\ i$ | $k\ i$ | $k\ i$ | $k\ i$ | $k\ i$ | $k\ i$ | $k\ i$ | $k\ i$ |
Using these tables, we will see that in $\Psi$ we can find pairs of ovals which carry an $n$-sided and an $m$-sided Poncelet polygon for $m \neq n$. Indeed, we are able to find the 5-sided Poncelet polygon

$$
\begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix} 
\rightarrow
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix} 
\rightarrow
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} 
\rightarrow
\begin{pmatrix}
-1 \\
1 \\
1
\end{pmatrix}
$$

and the 3-sided Poncelet polygon

$$
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix} 
\rightarrow
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix} 
\rightarrow
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix} 
\rightarrow
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
$$

This shows that $\Psi$ is not a Poncelet plane.

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