On the Approximation of Isolated Eigenvalues of Ordinary Differential Operators

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Abstract. We extend a result of Stolz and Weidmann on the approximation of isolated eigenvalues of singular Sturm–Liouville and Dirac operators by the eigenvalues of regular operators.

1. Introduction

The approximation of isolated eigenvalues of singular ordinary differential operators by the eigenvalues of regular operators is an important and well studied topic since the latter ones can be computed numerically with arbitrary precision. See the recent monograph by Zettl [8] or in particular the recent survey [7] by Weidmann.

While the case of eigenvalues below the essential spectrum is well understood, the case of eigenvalues in essential spectral gaps was only recently solved by Stolz and Weidmann in [3] (see also [4]).

Let \( \tau \) be either a Sturm–Liouville expression

\[
\tau = \frac{1}{r} \left( -\frac{d}{dx} p \frac{d}{dx} + q \right),
\]

or a Dirac system

\[
\tau = \frac{1}{r} \left( \sigma_2 \frac{d}{dx} + q \right),
\]

where \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) is the Pauli matrix, on the interval \( I = (a, b) \).

As usual, we will assume the coefficients \( p^{-1}, q, r \) are real-valued locally integrable functions with \( p, r > 0 \) in the Sturm-Liouville case and \( q, r \) are real, symmetric \( 2 \times 2 \) matrices with \( r > 0 \) in the Dirac case.

An endpoint \( a \) or \( b \) is called regular if it is finite and the coefficients are integrable near this endpoint. If both endpoints are regular, we will call \( \tau \) regular.

Let \( \mathcal{D}(\tau) \) be the maximal domain of definition of \( \tau \). Then \( \tau \) is self-adjoint on \( \mathcal{D}(\tau) \) if it is limit point (l.p.) near both \( a \) and \( b \). Otherwise we will impose an additional boundary condition at every endpoint where \( \tau \) is limit circle (l.c.). In this way we obtain a self-adjoint operator \( H \) associated with \( \tau \).

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A solution \( \psi_a(z, x) (\psi_b(z, x)) \) of \( \tau \psi = z\psi \) which is square integrable near \( a \) (b) and satisfies the boundary condition at \( a \) (b) (if any) is called a Weyl solution. Such a solution is unique up to a constant and it exists at least for \( z \in \mathbb{C} \setminus \sigma_{ess}(H) \).

Our aim is to approximate \( H \) by regular operators \( H_n \) obtained by restricting \( \tau \) to a finite interval \((a_n, b_n) \subseteq (a, b)\) ([3, Chap. 14]). The case \( a_n = a \) or \( b_n = b \) is allowed.

Fix functions \( u, v \in \mathcal{D}(\tau) \). Pick \( a_n \downarrow a \), \( b_n \uparrow b \). Define \( H_n \)

\[
H_n : \mathcal{D}(H_n) \to L^2((a_n, b_n); r \, dx),
\]

where

\[
\mathcal{D}(H_n) = \{ f \in \mathcal{D}(\tau_n)) | W_{a_n}(u, f) = W_{b_n}(v, f) = 0 \}.
\]

Then Stolz and Weidmann prove the following

**Theorem 1.** Define \( H_n \) as above with \( u = \psi_a(\lambda_a) \) and \( v = \psi_b(\lambda_b) \) with \( \lambda_a, \lambda_b \in [\lambda_0, \lambda_1] \) (in particular, assume that the corresponding Weyl solutions exist). Let \( P_\lambda(H) \) be the spectral projection of \( H \) corresponding to the Borel set \( \Omega \subseteq \mathbb{R} \).

If \( \dim \ker P_\lambda(H) \) is finite, then the eigenvalues of \( H \) in \( (\lambda_0, \lambda_1) \) are exactly the limits of eigenvalues of \( H_n \) which lie in \( (\lambda_0, \lambda_1) \). The corresponding (one-dimensional) eigenprojections converge in norm.

If \( \dim \ker P_\lambda(H) \) is infinite, then the eigenvalues of \( H_n \) accumulate in \( (\lambda_0, \lambda_1) \) as \( n \to \infty \).

In fact, in [3] this result is only proven in the case \([\lambda_0, \lambda_1] \cap \sigma_{ess}(H) = \emptyset\) and \( \lambda_0, \lambda_1 \) are not eigenvalues of \( H \). Hence we will provide a proof of this slightly generalized version below.

As pointed out in [3], this result has of course one practical drawback: The Weyl solutions used to generate the boundary conditions of \( H_n \) will not be known explicitly in general. To evade this obstacle they show that their result still holds if the Weyl solution of a nearby operator is chosen instead.

The main purpose of this note is to propose an alternate way of approximating \( H \) which only involves one Weyl solution at one endpoint. More precisely, we show that if \( \tau \) is l.p. at one end point, the Weyl solution of the other endpoint can also be chosen instead:

**Theorem 2.** Suppose \( \tau \) is l.p. at \( b \). Define \( H_n \) as above with \( u = \psi_a(\lambda_a) \), \( \lambda_a \in [\lambda_0, \lambda_1] \), and \( v = \psi_a(\lambda_0) \) or \( v = \psi_a(\lambda_1) \) (in particular, we assume that the corresponding Weyl solutions exist).

If \( \dim \ker P_\lambda(H) \) is finite, then the eigenvalues of \( H \) in \( (\lambda_0, \lambda_1) \) are exactly the limits of eigenvalues of \( H_n \) which lie in \( (\lambda_0, \lambda_1) \). The corresponding (one-dimensional) eigenprojections converge in norm.

If \( \dim \ker P_\lambda(H) \) is infinite, then the eigenvalues of \( H_n \) accumulate in \( (\lambda_0, \lambda_1) \) as \( n \to \infty \).

In particular, if one endpoint is regular, a solution satisfying the boundary condition at this endpoint can be taken in this case. Clearly the result cannot hold if \( \tau \) is l.c. at \( b \), since our assumptions contain no information on the boundary condition of \( H \) at \( b \) in this case.

**Remark 3.** (i). As shown in [3], if \( \tau \) is l.c. at \( a \), then \( u \) can be choosen to be any function in \( \mathcal{D}(\tau) \) generating the boundary condition of \( H \) at \( a \).
(ii). The same result (with the same proof) holds for Jacobi operators (see \[5\]).

2. Approximation by regular operators

We begin by recalling that $H_n$ converges to $H$ strongly (\[7\]). Strictly speaking this statement makes no sense since $H_n$ and $H$ live in different Hilbert spaces. This can be easily fixed by using $\alpha I \oplus H_n \oplus \alpha I$ on $L^2((a, b); r \, dx) = L^2((a, b_n); r \, dx) \oplus L^2((a_n, b_n); r \, dx) \oplus L^2([a_n, b_n], r \, dx)$ where $\alpha$ is a fixed real constant outside $[\lambda_0, \lambda_1]$. Alternatively, one can also use generalized strong convergence as introduced in \[9\].

Lemma 4. Suppose that either $H$ is limit point at $a$ or that $u = \psi_-(\lambda_0)$ for some $\lambda_0$ and similarly, that either $H$ is limit point at $b$ or $v = \psi_+(\lambda_1)$ for some $\lambda_1$. Then $H_n$ converges to $H$ in strong resolvent sense as $n \to \infty$.

In addition, we need the following abstract result. In this respect we remark that for a self-adjoint projector $P$ we have

\[ \dim \text{Ran}(P) = \text{tr}(P) = \|P\|_1, \]

where $\|\cdot\|_1$ denotes the trace class norm. If $P$ is not finite rank, then it is of course not trace class and all three numbers are equal $\infty$.

Lemma 5. Let $A_n, A$ be self-adjoint operators such that $A_n \to A$ in strong resolvent sense. Then

\[ \text{tr}(P_{(\lambda_0, \lambda_1})(A)) \leq \liminf_{n \to \infty} \text{tr}(P_{(\lambda_0, \lambda_1)}(A_n)). \]

If in addition, $\text{tr}(P_{(\lambda_0, \lambda_1)}(A_n)) \leq \text{tr}(P_{(\lambda_0, \lambda_1)}(A))$, then

\[ \lim_{n \to \infty} \text{tr}(P_{(\lambda_0, \lambda_1)}(A_n)) = \text{tr}(P_{(\lambda_0, \lambda_1)}(A)) \]

and if $\text{tr}(P_{(\lambda_0, \lambda_1)}(A)) < \infty$ we even have

\[ \lim_{n \to \infty} \|P_{(\lambda_0, \lambda_1)}(A_n) - P_{(\lambda_0, \lambda_1)}(A)\|_1 = 0. \]

Proof. The first part is just Lemma 5.2 from \[10\]. This also implies the second if $\text{tr}(P_{(\lambda_0, \lambda_1)}(A)) = \infty$. Otherwise, if $\text{tr}(P_{(\lambda_0, \lambda_1)}(A)) < \infty$, we have

\[ \limsup_{n \to \infty} \text{tr}(P_{(\lambda_0, \lambda_1)}(A_n)) \leq \text{tr}(P_{(\lambda_0, \lambda_1)}(A)) \]

and the first claim follows. The second is then a consequence of Grümm’s theorem (\[2\] Thm. 2.19).}

Now it remains to show that this result is applicable in our situation.

Lemma 6. Let $H_n$ be defined as in \[9\] with

(i) $u = \psi_a(\lambda_a), v = \psi_b(\lambda_b)$ with $\lambda_a, \lambda_b \in [\lambda_0, \lambda_1]$ or

(ii) $u = \psi_a(\lambda_a)$ with $\lambda_a \in [\lambda_0, \lambda_1]$ and $v = \psi_b(\lambda_b)$ with $\lambda_b \in \{\lambda_0, \lambda_1\}$.

Then,

\[ \text{tr}(P_{(\lambda_0, \lambda_1)}(H_n)) \leq \text{tr}(P_{(\lambda_0, \lambda_1)}(H)). \]

Proof. Abbreviate $P = P_{(\lambda_0, \lambda_1)}(H), P_n = P_{(\lambda_0, \lambda_1)}(H_n)$.

(i). Since this part is identical to the proof in \[14\], we just give an outline. Let $\hat{\psi}_1, \ldots, \hat{\psi}_k \in \text{Ran}P_n$ be the normalized eigenfunctions of $H_n$, construct

\[ \hat{\psi}_j(x) = \begin{cases} \gamma_{a,j} u(x), & x < a_n, \\ \hat{\psi}_j(x), & a_n \leq x \leq b_n, \\ \gamma_{b,j} v(x), & x > b_n, \end{cases} \]

and similarly, for $\psi_j$.

(ii). Then $\hat{\psi}_j \in \text{Ran}P_n$ and $\hat{\psi}_j \to \psi_j$ strongly. The proof is then the same as in \[14\].


where $\gamma_{a,j}, \gamma_{b,j}$ are chosen such that $\psi_j \in \mathcal{D}(\tau)$. A computation now shows that

$$\|(H - \frac{\lambda_1 + \lambda_0}{2})\psi\| < \frac{\lambda_1 - \lambda_0}{2} \|\psi\|$$

for any $\psi$ in the linear span of the $\psi_j$'s, which yields the first result.

(ii) By considering two steps from $(a_n, b_n)$ to $(a, b_n)$ and from $(a, b_n)$ to $(a, b)$, we see that the first step is covered by (i) and hence it is no restriction to assume $a_n = a$. Now proceed as in the previous case but use

$$\psi_j(x) = \begin{cases} \tilde{\psi}_j(x) - \gamma_j v(x), & x \leq b_n, \\ 0, & x > b_n, \end{cases}$$

where $\gamma_j$ are chosen such that $\psi_j \in \mathcal{D}(\tau)$. Now let $\psi = \sum_j c_j \psi_j$ be in the linear span of the $\psi_j$'s. Then, since $v$ is also an eigenvector of $H_n$ and hence orthogonal to the $\psi_j$'s, we have

$$\|(H - \frac{\lambda_1 + \lambda_0}{2})\psi\|^2 = \sum_j |c_j|^2 \left( \frac{2\lambda_j - \lambda_1 - \lambda_0}{2} \tilde{\psi}_j(x) - \frac{2\lambda_0 - \lambda_1 - \lambda_0}{2} \gamma_j v(x) \right)^2$$

$$= \sum_j |c_j|^2 \left( \frac{2\lambda_j - \lambda_1 - \lambda_0}{2} \right)^2 + |\gamma_j|^2 \left( \frac{\lambda_1 - \lambda_0}{2} \right)^2 \|v\|_{(a,b_n)}^2$$

$$< \left( \frac{\lambda_1 - \lambda_0}{2} \right)^2 \|\psi\|^2,$$

where $\gamma = \sum_j c_j \gamma_j$. Hence the second result follows. \hfill $\square$

Theorem 1 and Theorem 2 now follow by combining the last two lemmas.

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REFERENCES

[1] F. Gesztesy, B. Simon, and G. Teschl, Zeros of the Wronskian and renormalized oscillation Theory, Am. J. Math. 118, 571–594 (1996).
[2] B. Simon, Trace Ideals and Their Applications, 2nd ed., Amer. Math. Soc., Providence, 2005.
[3] G. Stolz and J. Weidmann, Approximation of isolated eigenvalues of ordinary differential operators, J. Reine und Angew. Math. 445, 31–44 (1993).
[4] G. Stolz and J. Weidmann, Approximation of isolated eigenvalues of general singular ordinary differential operators, Results Math. 28, no. 3-4, 345–358 (1995).
[5] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, Math. Surv. and Mon. 72, Amer. Math. Soc., Rhode Island, 2000.
[6] J. Weidmann, Spectral theory of ordinary differential operators, Lecture Notes in Mathematics, 1258, Springer, Berlin, 1987.
[7] J. Weidmann, Spectral theory of Sturm–Liouville operators; approximation by regular problems, in Sturm–Liouville Theory: Past and Present (eds. W. Amrein, A. Hinz and D. Pearson), 29–43, Birkhäuser, Basel, 2005.
[8] A. Zettl, Sturm–Liouville Theory, Mathematical Surveys and Monograph 121, Amer. Math. Soc., Rhode Island, 2005.