THE IONESCU–WAINGER MULTIPLIER THEOREM
AND THE ADELES

TERENCE TAO

Abstract. The Ionescu–Wainger multiplier theorem establishes good $L^p$ bounds for Fourier multiplier operators localized to major arcs; it has become an indispensable tool in discrete harmonic analysis. We give a simplified proof of this theorem with more explicit constants (removing logarithmic losses that were present in previous versions of the theorem), and give a more general variant involving adelic Fourier multipliers. We also establish a closely related adelic sampling theorem that shows that $\ell^p(\mathbb{Z}^d)$ norms of functions with Fourier transform supported on major arcs are comparable to the $L^p(\mathbb{A}_\mathbb{Z}^d)$ norm of their adelic counterparts.

1. Introduction

This paper will be concerned with the $L^p$ theory of Fourier multiplier operators on various locally compact abelian groups, such as $\mathbb{Z}^d$, $\mathbb{R}^d$, and $\mathbb{A}_\mathbb{Z}^d$. In order to treat these groups in a unified fashion we adopt the following abstract harmonic analysis notation.

Definition 1.1 (Pontryagin duality). An LCA group is a locally compact abelian group $G = (G, +)$ equipped with a Haar measure $\mu_G$. A Pontryagin dual of an LCA group $G$ is an LCA group $G^* = (G^*, +)$ with a Haar measure $\mu_{G^*}$ and a continuous bihomomorphism $(x, \xi) \mapsto x \cdot \xi$ (which we call a pairing) from $G \times G^*$ to the unit circle $T = \mathbb{R}/\mathbb{Z}$, such that the Fourier transform $\mathcal{F}_G : L^1(G) \to C(G^*)$ defined by

$$\mathcal{F}_G f(\xi) := \int_G f(x) e(x \cdot \xi) \, d\mu_G(x),$$

where $e : T \to \mathbb{C}$ is the standard character $e(\theta) := e^{2\pi i \theta}$, extends to a unitary map from $L^2(G)$ to $L^2(G^*)$; in particular we have the Plancherel identity

$$\int_G |f(x)|^2 \, d\mu_G(x) = \int_{G^*} |\mathcal{F}_G f(\xi)|^2 \, d\mu_{G^*}(\xi).$$

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for all \( f \in L^2(G) \), as well as the inversion formula
\[
\mathcal{F}_G^{-1} F(x) = \int_{G^*} F(\xi) e(-x \cdot \xi) \, d\mu_{G^*}(\xi)
\]
for all \( F \in L^1(G^*) \cap L^2(G^*) \).

If \( \Omega \subset G^* \) is measurable, we say that \( f \in L^2(G) \) is Fourier supported in \( \Omega \) if \( \mathcal{F}_G f \) vanishes outside of \( \Omega \) (modulo null sets). The space of such functions will be denoted \( L^2(G)^\Omega \).

If \( m \in L^\infty(G^*) \), we define the associated Fourier multiplier operator \( T_m : L^2(G) \to L^2(G) \) by the formula
\[
\mathcal{F}_G T_m f := m \mathcal{F}_G f
\]
for all \( f \in L^2(G) \), thus \( T_m = \mathcal{F}_G^{-1} m \mathcal{F}_G \). We refer to \( m \) as the symbol of \( T_m \).

For any finite-dimensional normed vector space \( V \), we extend \( T_m \) to an operator on \( L^2(G;V) \) in the obvious fashion.

We will focus in particular on the Pontryagin dual pairs
\[
(G,G^*) = (\mathbb{Z}^d, \mathbb{T}^d), (\mathbb{R}^d, \mathbb{R}^d), (A_{\mathbb{Z}}^d, \mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d)
\]
where \( A_{\mathbb{Z}} = \mathbb{R} \times \hat{\mathbb{Z}} \) denotes the adelic integers and \( d \geq 1 \) is an integer; see Appendix A for a more precise description of these pairs. To avoid technicalities we shall largely restrict attention to smooth symbols \( m \), although rougher symbols can also be treated by applying suitable limiting arguments, as our estimates will not depend on any smooth norms of \( m \). We view the adelic space \( A_{\mathbb{Z}}^d = \mathbb{R}^d \times \mathbb{Z}^d \) as a simplified model of the lattice \( \mathbb{Z}^d \) that captures both the “continuous” aspects of this lattice (via the factor \( \mathbb{R}^d \)) and the “arithmetic” aspects of this lattice (via the factor \( \hat{\mathbb{Z}}^d \)). The reader may wish to restrict attention to the one-dimensional case \( d = 1 \) as it already captures all of the key ideas, but the extension to higher values of \( d \) requires only minor notational changes and is also useful in some applications (e.g., [13]), so we work with general \( d \) in this paper.

A central problem in harmonic analysis is to understand the operator norm \( \|T_m\|_{B(L^p(G))} \) of a Fourier multiplier operator \( T_m \) on a Lebesgue space \( L^p(G) \) (restricting \( T_m \) initially to some dense subclass such as the Schwartz-Bruhat space \( S(G) \) to avoid technicalities). For \( p = 2 \) this norm is just the \( L^\infty(G^*) \) norm of \( m \), but for other choices of \( p \) the situation is considerably more complicated. Our initial focus here will be on understanding this problem in the case where \( G = \mathbb{Z}^d \) and \( m \) is supported on “major arcs”.
If \( m \in C^\infty_c(\mathbb{R}^d) \) is a smooth symbol, then \( T_m \) is a Fourier multiplier operator on \( L^2(\mathbb{R}^d) \), but we can also define associated Fourier multiplier operators \( T_{m;\alpha} \) on \( \ell^2(\mathbb{Z}^d) \) for various shifts \( \alpha \in \mathbb{T}^d \) by the formula

\[
T_{m;\alpha} := T_{m_{\alpha}}
\]

where \( m_{\alpha} \in C^\infty(\mathbb{T}^d) \) is the symbol

\[
m_{\alpha}(\xi) := \sum_{\theta \in \mathbb{R}^d: \xi = \alpha + \theta \mod \mathbb{Z}^d} m(\theta).
\]

Equivalently, one has

\[
T_{m;\alpha} f(n) = \int_{\mathbb{R}^d} m(\theta)e(-n \cdot (\alpha + \theta)) \mathcal{F}_{\mathbb{Z}^d} f(\alpha + \theta) \, d\theta.
\]

More generally, for any finite set \( \Sigma \subset \mathbb{R}^d \), define

\[
T_{m;\Sigma} := \sum_{\alpha \in \Sigma} T_{m;\alpha},
\]

thus

\[
T_{m;\Sigma} f(n) = \sum_{\alpha \in \Sigma} \int_{\mathbb{R}^d} m(\theta)e(-n \cdot (\alpha + \theta)) \mathcal{F}_{\mathbb{Z}^d} f(\alpha + \theta) \, d\theta.
\]

If the support of \( m \) is suitably restricted, then the \( \ell^p(\mathbb{R}^d) \) multiplier theory of \( T_{m;\alpha} \) or \( T_{m;\Sigma} \) is closely tied to the \( L^p(\mathbb{Z}^d) \) multiplier theory of \( T_m \). One basic manifestation of this is via the following sampling principle of Maygar, Stein, and Wainger [9]. For any \( \xi_0 \in \mathbb{R}^d \), let \( \xi_0 + [-r, r]^d \) denote the closed cube of sidelength \( 2r \) centred at \( \xi_0 \). We also define analogous balls (or cubes or “arcs”) \( \alpha + [-r, r]^d \subset \mathbb{T}^d \) for \( \alpha \in \mathbb{T}^d \). For any positive integer \( Q \), let

\[
\mathbb{T}^d[Q] := \{ x \in \mathbb{T}^d : Qx = 0 \} = \left( \frac{1}{Q} \mathbb{Z}/\mathbb{Z} \right)^d
\]

denote the subgroup of \( Q \)-torsion points of the torus \( \mathbb{T}^d \).

**Proposition 1.2** (Maygar–Stein–Wainger sampling principle). Let \( d \geq 1 \) be an integer, let \( 1 \leq p \leq \infty \), and let \( V \) be a finite-dimensional Banach space.

(i) If \( m \in C^\infty_c(\mathbb{R}^d) \) is supported in \( [-\frac{1}{2}, \frac{1}{2}]^d \), then

\[
\|T_{m;0}\|_{B(\ell^p(\mathbb{Z}^d;V))} \leq O(1)^d \|T_m\|_{B(L^p(\mathbb{R}^d;V))}.
\]

(See Section 1.2 for our conventions on asymptotic notation, as well as our notation \( B(W) \) for the operator norm on a normed vector space \( W \).)
(ii) More generally, if $Q \geq 1$ is an integer, and $m \in C_c^\infty(\mathbb{R}^d)$ is supported in $[-\frac{1}{2Q}, \frac{1}{2Q}]^d$, then
\[
\|T_m \cdot T_d^d\|_{B(L^p(\mathbb{Z}^d; \mathcal{V}))} \leq O(1)^d \|T_m\|_{B(L^p(\mathbb{R}^d; \mathcal{V}))}.
\]

Proof. Part (i) is [9, Proposition 2.1] and part (ii) is [9, Corollary 2.1], after noting that all implied constants in the proof are at most exponential in the dimension $d$. \qed

In the remarks after [9, Proposition 2.1] the question is posed as to whether the $O(1)^d$ constant in (1.2) can be made independent of $d$, or even replaced with 1. Although not the main focus of this paper, in Appendix [3] we show that the answer to these questions is negative if $p$ is sufficiently close to 1 or $\infty$. Most likely the answer is negative for all $p \neq 2$, but we were unable to demonstrate this. (For $p = 2$ it is easy to see from Plancherel’s theorem that the $O(1)^d$ factor may be deleted.)

This sampling principle lets us control certain Fourier multiplier operators whose symbol is supported in sets of the form
\[
T_d^d[Q] + [-\varepsilon, \varepsilon]^d
\]
for $\varepsilon > 0$ small enough (in particular, the above proposition applies when $\varepsilon \leq 1/2Q$). For applications to discrete harmonic analysis (particularly involving averaging over “arithmetic” sets such as polynomial sequences or primes), it would be desirable to have a similar estimate that could handle symbols supported on the “classical major arcs”
\[
\bigcup_{q=1}^N T_d^d[q] + [-\varepsilon, \varepsilon]^d
\]
for some $N \geq 1$ and $\varepsilon > 0$. As it turns out, these classical arcs are inconvenient to work with directly for the purposes of $\ell^p$ multiplier theory. However, in the remarkable work of Ionescu and Wainger [7], a more complicated major arc set
\[
\mathcal{M} = \Sigma_{\leq k} + [-\varepsilon, \varepsilon]^d
\]
was introduced for which (for suitable choices of parameters) contained the classical major arc set (1.3) while simultaneously enjoying a satisfactory $\ell^p$ multiplier theory for relatively large values of $\varepsilon$. The Ionescu–Wainger multiplier theorem has since been indispensable in many results in discrete harmonic analysis, in particular providing an analogue of Littlewood-Paley theory adapted to major arcs; see e.g., [18], [9], [12], [13], [14], [8].

In this note we give a general version of the Ionescu–Wainger theorem which avoids some logarithmic loss factors present in earlier treatments,
and quantifies the dependence on various parameters. To describe the result we need some notation.

**Definition 1.3 (Generalized Ionescu–Wainger major arcs).** A major arc parameter set is a quadruplet $(d, k, S, \varepsilon)$ where $d, k \geq 1$ are integers, $S$ is a finite collection of pairwise coprime integers, and $\varepsilon > 0$ is a real number. For any $A \subseteq S$, we let $Q_A := \prod_{q \in A} q$, and let $\Sigma_{\subseteq A} \subseteq \mathbb{T}^d$ denote the subgroup

$$
\Sigma_{\subseteq A} := \mathbb{T}^d[Q_A].
$$

We let $\Sigma_A$ denote the set

$$
\Sigma_A := \Sigma_{\subseteq A} \setminus \bigcup_{B \subseteq A} \Sigma_{\subseteq A}.
$$

We define the sets

$$
\binom{S}{k} := \{ A \subseteq S : |A| = k \}; \quad \binom{S}{\leq k} := \{ A \subseteq S : |A| \leq k \}
$$

and let $\Sigma_{\leq k}$ denote the set

$$
\Sigma_{\leq k} := \bigcup_{A \in \binom{S}{k}} \Sigma_A = \bigcup_{A \in \binom{S}{\leq k}} \Sigma_A.
$$

The major arc set $\mathcal{M} = \mathcal{M}(d, k, S, \varepsilon)$ associated to the parameter set $(d, k, S, \varepsilon)$ is defined as

$$
\mathcal{M} := \Sigma_{\leq k} + [-\varepsilon, \varepsilon]^d.
$$

A major arc parameter set is said to be $(r, c)$-good for some integer $r \geq 1$ and $0 < c < 1$ if one has the smallness condition

$$
\varepsilon < \frac{c}{2r q_{\text{max}} 2^r k}
$$

for some integer $q_{\text{max}}$ that is greater than or equal to all the elements of $S$.

Expanding out the definitions, we see that $\mathcal{M}$ consists of all elements of $\mathbb{T}^d$ of the form $\frac{a}{q} + \theta \mod \mathbb{Z}^d$, where $q$ is the product of at most $k$ elements of $S$, $a \in \mathbb{Z}^d$, and $\theta \in [-\varepsilon, \varepsilon]^d$. The major arc sets $\mathcal{M}$ considered here are more general than the ones constructed in [7], which involved a specific choice of $S$ involving a partition of all the primes up to a certain threshold. In Section 5 we explain how the major arcs in [7] become a special case of the ones considered here. The parameter $c$ is of minor technical importance and the reader may wish to fix it as an absolute constant (e.g., $c = 1/2$) for most of the following discussion. In typical applications one should think of the quantities $d, k, r$ as being bounded, $|S|$ and $q_{\text{max}}$ as being large, and $\varepsilon$ as being quite small.
We can now state our first form of the Ionescu–Wainger multiplier theorem. To simplify the bounds slightly we adopt the notation

\[ \log x := \log(2 + x). \]

**Theorem 1.4** (Ionescu–Wainger multiplier theorem, real form). Let \((d, k, S, \varepsilon)\) be a major arc parameter set, and let \(H\) be a finite-dimensional Hilbert space. Let \(m \in C^\infty_c(\mathbb{R}^d)\) be supported on \([-\varepsilon, \varepsilon]^d\). Then if \((d, k, S, \varepsilon)\) is \((r, c)\)-good for some integer \(r \geq 1\) and \(0 < c < 1\), one has

\[ \|T_m \mathcal{S}_{\leq k}\|_{B(\ell^{2r}(\mathbb{Z}^d; H))} \leq O_c(1)^d O(r \log^{1/2}(kr))^k \|T_m\|_{B(L^{2r}(\mathbb{R}^d))}; \]

more generally, one has

\[ \| \sum_{A \in \mathcal{S}_{\leq k}} \epsilon_A T_m \mathcal{S}_A\|_{B(\ell^{2r}(\mathbb{Z}^d; H))} \leq O_c(1)^d O(r \log^{1/2}(kr))^k \|T_m\|_{B(L^{2r}(\mathbb{R}^d))}. \]

whenever \(\epsilon_A\) is a complex number with \(|\epsilon_A| \leq 1\) for each \(A \in \mathcal{S}_{\leq k}\).

The factor of \(O_c(1)^d O(r \log^{1/2}(kr))^k\) looks somewhat messy, but the key point is that it is uniform in the parameters \(S, \varepsilon, H, m\). The original version of this result in [7], when adapted to this notion of major arc, gave instead a bound of the form \(O_{c,d,k,r}(\log |S|)\), which was later refined in [11] to \(O_{c,d,k,r}(\log |S|)\) (see also [12], [13], [14]). The dependence of \(c\) will be unimportant in applications as one can typically take \(c\) to equal a constant value such as \(c = 1/2\). The restriction to even integer exponents \(2r\) will be removed in Theorem 1.7 below (at the cost of worsening the bounds slightly). We work with finite dimensional Hilbert spaces \(H\) here to avoid technical complications, but one can extend this result to separable Hilbert spaces without difficulty by a standard limiting argument.

We prove Theorem 1.4 in Section 4 after some preliminaries in Sections 2, 3. The argument uses the same basic approach as previous proofs of the Ionescu–Wainger theorem in the literature (particularly [14, Theorem 2.1]), which we summarize as follows.

(i) One begins by exploiting “Type II superorthogonality” (following the terminology recently introduced by Pierce [19]) of the terms \(T_m \mathcal{S}_A f\) (arising from “denominator orthogonality”) of the rational set \(\mathcal{S}_{\leq k}\) to estimate the \(\ell^{2r}(\mathbb{Z}^d; H)\) norm of \(\sum_{A \in \mathcal{S}_{\leq k}} \epsilon_A T_m \mathcal{S}_A f\) by the \(\ell^{2r}(\mathbb{Z}^d; H^{[\leq k]})\) of the square function \((T_m \mathcal{S}_A f)_{A \in \mathcal{S}_{\leq k}}\) that takes values in \(H^{[\leq k]} := H^{(\mathcal{S}_{\leq k})}\), in the spirit of reverse square function estimates of Khintchine type.
(ii) Using a “nonconcentration estimate”, one estimates this square function norm by an expression summing over various “sunflowers” in $S$.

(iii) By exploiting “numerator orthogonality” of the rational set $\Sigma \leq k$, one estimates this resulting sum over sunflowers by a more tractable square function involving the functions $T_{m;\alpha+\Sigma \subseteq A_0} f$ that are summed over cosets of a fixed finite subgroup $\Sigma \subseteq A_0 = \mathbb{T}^d[A_0]$ of $\mathbb{T}^d$.

(iv) At this point the symbol $m$ can be disposed of using the Marcinkiewicz–Zygmund theorem, and then the resulting quantity can be estimated using a square function estimate of Rubio de Francia type [21].

Our main innovations are to eliminate logarithmic losses in (i) using the probabilistic decoupling trick (cf. [17]), and to obtain efficient bounds in (ii) by using recent progress [20] on the sunflower conjecture of Erdős and Rado [4].

We also interpret these results through the lens of adelic harmonic analysis, following the slogan

Major arc analysis on $\mathbb{Z}^d \approx$ Low frequency analysis on $A_\mathbb{Z}^d$

recently advocated (in the one-dimensional setting $d = 1$) in [8]. As reviewed in Appendix A, we have an inclusion homomorphism

$$\iota: \mathbb{Z}^d \to A_\mathbb{Z}^d$$

and a addition homomorphism

$$\pi: \mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d \to \mathbb{T}^d$$

that is Fourier adjoint to $\iota$, with $\pi$ being given explicitly by

$$\pi(\theta, \alpha) := \alpha + \theta$$

for $\theta \in \mathbb{R}^d$ and $\alpha \in (\mathbb{Q}/\mathbb{Z})^d$.

There is a sampling map $S: S(A_\mathbb{Z}^d) \to S(\mathbb{Z}^d)$, where $S(\mathbb{G})$ denotes the Schwartz-Bruhat space on $\mathbb{G}$ (as defined in Appendix A), defined by

$$Sf := f \circ \iota.$$

As in [8], we say that a compact subset $\Omega$ of adelic frequency space $\mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d$ is non-aliasing if the projection map $\pi$ is injective on $\Omega$. In [8, (4.6)] it was shown that the sampling map extends to a unitary transformation

$$S: L^2(A_\mathbb{Z}^d, \Omega) \to \ell^2(\mathbb{Z}^d)^{\pi(\Omega)}$$

for any non-aliasing compact $\Omega \subset \mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d$ (the argument was presented for $d = 1$, but extends to arbitrary dimension). In particular
one has an interpolation map

\[ S_\Omega^{-1} : \ell^2(\mathbb{Z}^d)_{\pi(\Omega)} \to L^2(A_{\mathbb{Z}^d}^d, \Omega) \]

that inverts \( S \); we extend these operators to vector-valued functions taking values in a finite-dimensional vector space in the obvious fashion. For instance, if \( \Omega \) is a set of the form \([-\varepsilon, \varepsilon]^d \times \Sigma\) for some \( \varepsilon > 0 \) and some finite \( \Sigma \subset (\mathbb{Q}/\mathbb{Z})^d \), then \( \Omega \) is non-aliasing if the elements of \( \Sigma \) are separated from each other by more than \( 2\varepsilon \) (in the \( \ell^\infty \) metric), and then \( \pi(\Omega) = \Sigma + [-\varepsilon, \varepsilon]^d \) and every element \( f \) of \( \ell^2(\mathbb{Z}^d)_{\pi(\Omega)} \) then has a unique Fourier representation of the form

\[ f(n) = \sum_{\alpha \in \Sigma} \int_{[-\varepsilon, \varepsilon]^d} e(-n \cdot (\alpha + \theta)) \mathcal{F}_{\mathbb{Z}^d} f(\alpha + \theta) \, d\theta \]

for \( n \in \mathbb{Z}^d \), and the interpolated function \( S_\Omega^{-1} f \in L^2(A_{\mathbb{Z}^d}^d, \Omega) \) is then given by the formula

\[ S_\Omega^{-1} f(x, y) = \sum_{\alpha \in \Sigma} \int_{[-\varepsilon, \varepsilon]^d} e(-x \cdot \theta - y \cdot \alpha) \mathcal{F}_{\mathbb{Z}^d} f(\alpha + \theta) \, d\theta \]

and then it is clear that \( f = SS_\Omega^{-1} f \).

From unitarity we have

\[ \|S_\Omega^{-1} f\|_{L^2(A_{\mathbb{Z}^d}^d)} = \|f\|_{\ell^2(\mathbb{Z}^d)} \]

whenever \( f \in \ell^2(\mathbb{Z}^d)_{\pi(\Omega)} \). In many cases we can extend this \( L^2 \) isometry to an \( L^p \) equivalence. For instance, we have

**Proposition 1.5** (Quantitative Shannon sampling theorem). Let \( 1 \leq p \leq \infty \), and \( V \) be a finite-dimensional normed vector space. If \( \Omega \) is the (non-aliasing) set \( \Omega := [-c, c]^d \times \mathbb{T}^d(\mathbb{Q}) \) for some positive integer \( Q \) and \( 0 < c < \frac{1}{2} \) then \( S_\Omega \) extends to a bounded invertible linear operator from \( \ell^p(\mathbb{Z}^d; V)_{\pi(\Omega)} \) to \( L^p(\mathbb{Z}^d; V)_{\Omega} \) with

\[ \|S_\Omega^{-1} f\|_{L^p(A_{\mathbb{Z}^d}^d; V)} = \exp(O_c(d)) \|f\|_{\ell^p(\mathbb{Z}^d; V)} \]

for all \( f \in \ell^p(\mathbb{Z}^d; V)_{\pi(\Omega)} \), or equivalently

\[ \|S F\|_{L^p(A_{\mathbb{Z}^d}^d; V)} = \exp(O_c(d)) \|F\|_{L^p(A_{\mathbb{Z}^d}^d; V)} \]

for all \( F \in L^p(A_{\mathbb{Z}^d}^d; V)_{\Omega} \).

**Proof.** See [8, Theorem 4.18], after generalizing from \( d = 1 \) to general \( d \) and carefully tracking the dependence on constants. The result also extends to \( 0 < p < 1 \) (after allowing the implied constants to depend on \( p \) as well as \( c \)), but we will only need the \( p \geq 1 \) case here. \( \square \)
Proposition 1.5 can be used to partially explain the sampling principle, Proposition 1.2. First observe that if \([-\varepsilon, \varepsilon]^d \times \Sigma\) is a non-aliasing set then we have the identity

\[ T_{m;\Sigma}Sf = ST_{m\otimes1;\Sigma}f \]  \hspace{1cm} (1.5)

for any \(m \in C_c^\infty(\mathbb{R}^d)\) supported on \([-\varepsilon, \varepsilon]^d \times \Sigma\) and any \(f \in S(\mathbb{A}^d)\), where the tensor product \(\otimes\) is defined in Section 1.2; see [8, Lemma 4.12] (extended to general dimension \(d\) in the obvious fashion). Now suppose that \(Q \geq 1\) is an integer and \(m \in C_c^\infty(\mathbb{R}^d)\) is supported in \([-cQ, cQ]^d\) for some \(0 < c < \frac{1}{2}\). Then we may use (1.5) and basic Fourier-analytic manipulations to factorize

\[ T_{m;\Sigma}T_{\hat{d}[Q]}f = T_{m;\Sigma}SS^{-1}_\Omega T_{\varphi;\Sigma}[Q]f = ST_{m;\Sigma}SS^{-1}_\Omega T_{\varphi;\Sigma}[Q]f \]

where \(\varphi \in C_c^\infty(\mathbb{R}^d)\) is a function of the form

\[ \varphi(\xi_1, \ldots, \xi_d) := \prod_{j=1}^d \varphi_0(Q\xi_j), \]

\(\varphi_0 \in C_c^\infty(\mathbb{R})\) is supported on \([-c', c']\) for some \(c < c' < \frac{1}{2}\) that equals 1 on \([-c, c]\), and \(\Omega := [-c', c']^d \times \mathbb{Z}^d\). From Proposition 1.2 applied to \(\varphi\) we have

\[ \|T_{\varphi;\Sigma}[Q]\|_{B(\mathcal{L}^p(\mathbb{Z}^d;H))} \leq O_{c,c'}(1)^d \]

while from working on each fibre \(\mathbb{R}^d \times \{y\}\) of \(\mathbb{A}_Z^d = \mathbb{R}^d \times \mathbb{Z}^d\) separately and using the Marcinkiewicz–Zygmund theorem (Theorem 1.8), we have

\[ \|T_{m;\Sigma}\|_{B(\mathcal{L}^p(\mathbb{A}_Z^d;H))} = \|T_{m}\|_{B(\mathcal{L}^p(\mathbb{R}^d;H))} = \|T_{m}\|_{B(\mathcal{L}^p(\mathbb{R}^d))} \]

and hence from Proposition 1.5 we have

\[ \|T_{m;\Sigma}\|_{B(\mathcal{L}^p(\mathbb{Z}^d;H))} \leq O_{c,c'}(1)^d\|T_{m}\|_{B(\mathcal{L}^p(\mathbb{R}^d))} \]

which recovers a slightly weaker version of Proposition 1.2 in fact with a bit more effort (applying a smooth partition of unity to \(m\) followed by the triangle inequality) one can in fact recover the full strength of Proposition 1.2. Admittedly, there is some circularity here since Proposition 1.2 was used in the proof, but only for the bump function \(\varphi\) and not for arbitrary multipliers \(m\).

It turns out that there is a similar phenomenon for major arcs:

**Theorem 1.6** (Major arc sampling). Let \((d, k, S, \varepsilon)\) be a major arc parameter set, which is \((r, c)\) good for some \(r \geq 1\) and \(0 < c < 1\). Set \(\Omega := [-\varepsilon, \varepsilon]^d \times \Sigma_{\leq k}\). Then for any finite-dimensional Hilbert space \(H\)
and \((2r)' \leq p \leq 2r\), the interpolation operator \(S^{-1}_\Omega\) extends to a bounded invertible linear operator from \(\ell^p(\mathbb{Z}^d; H)^{\pi(\Omega)}\) to \(L^p(\mathbb{Z}^d; H)^{\Omega}\), with
\[
\|S^{-1}_\Omega f\|_{L^p(\mathbb{A}_d^2; H)} = \exp\left( O_c(d) + O(k \log(r \log(k))) \right) \|f\|_{\ell^p(\mathbb{Z}^d; H)}
\]
for all \(f \in \ell^p(\mathbb{Z}^d; H)^{\pi(\Omega)}\), or equivalently
\[
\|SF\|_{\ell^p(\mathbb{Z}^d; H)} = \exp\left( O_c(d) + O(k \log(r \log(k))) \right) \|F\|_{L^p(\mathbb{A}_d^2; H)}
\]
for all \(F \in L^p(\mathbb{A}_d^2; H)^{\Omega}\).

We prove this theorem in Section 4 by reusing the machinery used to establish Theorem 1.3. As a consequence of this sampling theorem, we can obtain a more general “adelic” version of the Ionescu–Wainger multiplier theorem, in which one transfers a multiplier on \(\mathbb{A}_d^2 = \mathbb{R}^d \times \hat{\mathbb{Z}}^d\) rather than on \(\mathbb{R}^d\) to the lattice \(\mathbb{Z}^d\), or equivalently one allows the use of a different multiplier on each major arc. More precisely, given \(m \in L^\infty(\mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d)\) and a finite set \(\Sigma\), define the multiplier \(T_{m;\Sigma}: \ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)\) by the formula
\[
T_{m;\Sigma} := \sum_{\alpha \in \Sigma} T_{m,(.,\alpha);\alpha}
\]
or equivalently
\[
T_{m;\Sigma} f(n) = \sum_{\alpha \in \Sigma} \int_{\mathbb{R}^d} m(\theta, \alpha) e(-n \cdot (\alpha + \theta)) \mathcal{F}_{\mathbb{Z}^d} f(\alpha + \theta) \, d\theta.
\]
Note that the previous definition (1.1) corresponds to the special case in which the adelic symbol \(m(\theta, \alpha)\) does not depend on the arithmetic component \(\alpha\).

**Theorem 1.7** (Ionescu–Wainger multiplier theorem, adelic form). Let \((d, k, S, \varepsilon)\) be a major arc parameter set, and let \(H\) be a finite-dimensional Hilbert space. Let \(m \in S(\mathbb{A}_d^2)\) be supported on \([-\varepsilon, \varepsilon]^d \times (\mathbb{Q}/\mathbb{Z})^d\). Then if \((d, k, S, \varepsilon)\) is \((r, c)\)-good for some \(r \geq 1\) and \(0 < c < 1\), one has
\[
\|T_{m;\Sigma,_{\leq k}}\|_{B(\ell^p(\mathbb{A}_d^2; H))} \leq O_c(1)^d O(r \log k)^{O(k)} \|T_m\|_{B(\ell^p(\mathbb{A}_d^2))}
\]
for any \((2r)' \leq p \leq 2r\).

In principle this theorem converts the analysis of linear Fourier multipliers on major arcs to that of linear Fourier multipliers on the adelic space \(\mathbb{A}_d^2\), which in principle is a simpler setting due to the product structure on \(\mathbb{A}_d^2 = \mathbb{R}^d \times \hat{\mathbb{Z}}^d\). We remark that the method of proof also extends to bilinear or multilinear Fourier multipliers (as long as all exponents \(p\) involved lie strictly between 1 and \(\infty\), but we do not
have applications in mind for this extension\footnote{For instance, the bilinear estimates considered in [8] typically involve the end-point space $\ell^1$ (or even $\ell^p$ for some $p < 1$), and also take values in variational norm spaces rather than Hilbert spaces, so would not be able to be directly treated by a bilinear variant of Theorem 1.7.} and so we leave it to the interested reader.

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1.2. **Notation.** Random variables will be denoted in boldface, and deterministic quantities in non-boldface. We use $\mathbb{N} = \{0, 1, \ldots \}$ to denote the natural numbers, and $\mathbb{Z}_+ = \{1, 2, \ldots \}$ to denote the positive integers.

We use $X = O(Y)$ to denote an estimate of the form $|X| \leq CY$ for some constant $C$. We write $X \sim Y$ if $X = O(Y)$ and $Y = O(X)$. If one needs the constant $C$ to depend on parameters, we indicate this by subscripts, for instance $X \leq O_c(Y)$ denotes the bound $X \leq C_c Y$ for some $C_c$ depending only on $c$.

If $(X, \mu)$ is a measure space, $V = (V, \|\|)$ is a finite-dimensional normed vector space, and $1 \leq p \leq \infty$, we define $L^p(X; V)$ to denote the space of measurable functions $f : X \to V$ whose norm $\|f\|_{L^p(X; V)} := (\int_X \|f(x)\|^p \, d\mu(x))^{1/p}$ is finite, up to almost everywhere equivalence, with the usual modifications at $p = \infty$. We write $L^p(X)$ for $L^p(X; \mathbb{C})$, and when $\mu$ is counting measure we write $\ell^p$ for $L^p$. For any $1 \leq p \leq \infty$, we define the dual exponent $1 \leq p' \leq \infty$ by $1/p + 1/p' = 1$.

All Hilbert spaces will be over the complex numbers. Given a bounded linear operator $T : V \to W$ between (quasi-)normed vector spaces $V, W$, we use $\|T\|_{B(V \to W)}$ to denote its operator norm; if $V = W$, we abbreviate $B(V \to V)$ as $B(V)$. We recall

**Theorem 1.8** (Marcinkiewicz–Zygmund theorem). \cite{10} Let $X, Y$ be measure spaces, let $0 < p < \infty$, and let $T : L^p(X) \to L^p(Y)$ be a linear operator. Then for any finite-dimensional Hilbert space $H$, one has

$$\|T\|_{B(L^p(X; H) \to L^p(Y; H))} \leq \|T\|_{B(L^p(X) \to L^p(Y))}.$$
Proof. We normalize \( \|T\|_{B(L^p(X) \to L^p(Y))} = 1 \). Taking orthonormal bases, it suffices to show that
\[
\int_Y \left( \left| \sum_{i=1}^n T f_i \right|^2 \right)^{p/2} \leq \int_X \left( \left| \sum_{i=1}^n f_i \right|^2 \right)^{p/2}
\]
for any \( f_1, \ldots, f_n \in L^p(X) \). If we let \( g_1, \ldots, g_n \) be independent complex gaussian variables of mean zero and variance 1, we have from hypothesis that
\[
\int_Y \left| \sum_{i=1}^n g_i T f_i \right|^p \leq \int_X \left| \sum_{i=1}^n g_i f_i \right|^p.
\]
Taking expectations of both sides and noting that the sum of independent gaussians is again a gaussian, we conclude that
\[
C_p \int_Y \left( \left| \sum_{i=1}^n T f_i \right|^2 \right)^{p/2} \leq C_p \int_X \left( \left| \sum_{i=1}^n f_i \right|^2 \right)^{p/2}
\]
where \( C_p := \mathbb{E}|g|^p \) with \( g \) a complex gaussian of mean zero and variance 1. Since \( 0 < C_p < \infty \), the claim follows. \( \square \)

If \( E \) is a finite set, we use \( |E| \) to denote its cardinality. If \( E, F \) are subsets of an additive group \( G = (\mathbb{G}, +) \) (such as the torus \( \mathbb{T}^d \)), we write \( E + F := \{ \xi + \eta : \xi \in E, \eta \in F \} \) for their sumset. If \( \xi \in \mathbb{G} \), we write \( \xi + E = E + \xi = E + \{ \xi \} \) for the translate of \( \xi \) by \( E \). These sumset notions also extend in the obvious fashion to the setting in which one of the summands lies in \( \mathbb{G} \) and the other lies in a quotient \( \mathbb{G}/\mathbb{H} \) (for instance, if one lies in \( \mathbb{R}^d \) and the other in \( \mathbb{T}^d \)). We use \( 1_E \) to denote the indicator function of a set \( E \), and \( 1_S \) the indicator of a statement \( S \), thus for instance \( 1_E(x) = 1_{x \in E} \) is equal to 1 when \( x \in E \), and equal to 0 otherwise.

We will need the following combinatorial concepts:

**Definition 1.9 (Nonces and sunflowers).** Let \( A_1, \ldots, A_n \) be a collection of sets.

(i) A *nonce* of the collection \( A_1, \ldots, A_n \) is an element \( s \) that belongs to exactly one of the \( A_i \). A collection is *nonce-free* if there does not exist a nonce.

(ii) The collection \( A_1, \ldots, A_n \) is a *sunflower* if there is a set \( A_0 \) contained in \( A_1, \ldots, A_n \) (the *core* of the sunflower) such that the *petals* \( A_1 \setminus A_0, \ldots, A_n \setminus A_0 \) are all disjoint.

Thus for instance the sets \( \{1, 2\}, \{1, 3\}, \{2, 4\} \) contain 3 and 4 as nonces, whereas \( \{1, 2\}, \{1, 3\}, \{2, 3\} \) are nonce-free, while \( \{1, 2\}, \{1, 3\}, \{1, 4\} \)
is a sunflower with core \{1\} and petals \{2\}, \{3\}, \{4\}. The property of having a nonce is also referred to as the *uniqueness property* in [7], [14], [19].

If \( f : X \to \mathbb{C} \) and \( g : Y \to \mathbb{C} \) are functions, we define the tensor product \( f \otimes g : X \times Y \to C \) by the formula
\[
(f \otimes g)(x, y) := f(x)g(y).
\]

If \( H \) is a finite-dimensional Hilbert space and \( S \) is a finite set, we use \( H^S \) for the space of tuples \((u_s)_{s \in S}\) with \( u_s \in H \) with inner product
\[
\langle (u_s)_{s \in S}, (v_s)_{s \in S} \rangle = \sum_{s \in S} \langle u_s, v_s \rangle.
\]

For any natural number \( k \), we use \( H \otimes^k \) to denote the \( k \)-fold tensor product of \( H \) with itself, spanned by vectors \( u_1 \otimes \cdots \otimes u_k, u_1, \ldots, u_k \in H \) with
\[
\langle u_1 \otimes \cdots \otimes u_k, v_1 \otimes \cdots \otimes v_k \rangle = \prod_{i=1}^k \langle u_i, v_i \rangle.
\]

2. **Superorthogonality**

The (upper) Khintchine inequality asserts that
\[
\left( \mathbb{E} \left[ \sum_{i=1}^n |\epsilon_i z_i|^p \right] \right)^{1/p} \leq O \left( (p^{1/2}) \left( \sum_{i=1}^n |z_i|^2 \right)^{1/2} \right)
\]
for any \( 1 \leq p \leq \infty \) and complex numbers \( z_1, \ldots, z_n \), where \( \epsilon_1, \ldots, \epsilon_n \) are independent random signs in \{-1, +1\} of mean zero. In the case where \( p = 2r \) is an even integer, this inequality can be proven by direct combinatorial expansion of the left-hand side. As laid out recently in [19], this latter argument can be abstracted to more general “Type II superorthogonal systems”. We give the relevant definitions (as well as an extension to hypersystems) as follows.

**Definition 2.1** (Type II superorthogonality). Let \( S \) be a finite set, let \( X = (X, \mu) \) be a measure space, let \( r \) be a positive integer, and let \( H \) be a Hilbert space.

(i) A collection \((f_s)_{s \in S}\) of functions \( f_s \in L^{2r}(X; H) \) indexed by \( S \) is said to be a *Type II 2r-superorthogonal system* if one has
\[
\int_X \prod_{j=1}^r \langle f_{s_j}, f_{s_{r+j}} \rangle_H \ d\mu = 0
\]
whenever \( s_1, \ldots, s_{2r} \in S \) is such that the singleton sets \( \{s_1\}, \ldots, \{s_{2r}\} \)
contain a nonce (as defined in Definition 1.9).

(ii) A collection \( (f_A)_{A \in \mathcal{A}} \) of functions \( f_A \in L^2(X;H) \)
indexed by some family \( \mathcal{A} \) of subsets of a set \( S \) is said to be a Type II
\( 2r \)-superorthogonal hypersystem if one has

\[
\int_X \prod_{j=1}^r \langle f_{A_j}, f_{A_{r+j}} \rangle_H \, d\mu = 0 \quad (2.2)
\]

whenever \( A_1, \ldots, A_{2r} \in \mathcal{A} \) is such that the sets \( A_1, \ldots, A_{2r} \)
contain a nonce.

Note that any \( 2r \)-superorthogonal system \( (f_s)_{s \in S} \) can also be viewed
as a \( 2r \)-superorthogonal hypersystem \( (f_A)_{A \in \binom{S}{r}} \) by identifying each element \( s \in S \) with the associated singleton \( \{s\} \in \binom{S}{r} \).
The nomenclature “Type II” is due to Pierce [19]; there is also a stronger notion of Type I superorthogonality and a weaker notion of Type III superorthogonality discussed in that paper, but we will not need these notions here.

Several examples of superorthogonal systems are given in [19]. Our primary concern will come from functions supported on major arcs, but we can give another representative example of a superorthogonal hypersystem here:

**Example 2.2 (Polynomials of random variables).** Let \( k, R \) be positive integers. Let \( (X_s)_{s \in S} \) be \( R \)-wise independent random variables (thus \( X_{s_1}, \ldots, X_{s_r} \) are jointly independent for any \( r \leq R \) and distinct \( s_1, \ldots, s_r \in S \)), and for each \( A \in \binom{S}{r} \) let \( f_A \) be an complex random variable of the form \( f_A = \sum_{J_0} \prod_{s \in \bar{A}} f_{A,s,j}(X_s) \), where \( J_A \) is a finite set, \( c_{A,j} \) are complex coefficients, and each \( f_{A,s,j}(X_s) \) is a function of \( X_s \) of mean zero; thus for instance \( f_\emptyset \) is a constant. Then for any \( 1 \leq r \leq R/2k \), \( (f_A)_{A \in \binom{S}{r}} \) is a \( 2r \)-superorthogonal hypersystem over the ambient sample space of the random variables. Indeed, if \( A_1, \ldots, A_{2r} \) contains a nonce \( s \), then the expression in (2.2) expands into a sum of finitely many terms, each of which consists of the expectation of a product of an expression of the form \( f_{A,s,j}(X_s) \), times at most \( 2r - 1 \) \( r \leq R - 1 \) other expressions depending on other random variables than \( X_s \), and each of these terms vanishes by the \( 2R \)-wise independent nature of the \( X_s \). If the \( X_s \) are scalar random variables, then any polynomial of degree at most \( k \) in the \( X_s \) can be expressed in the form \( \sum_{A \in \binom{S}{r}} f_A \) for some hypersystem \( (f_A)_{A \in \binom{S}{r}} \) as above by removing the expectation from every monomial \( X_s^a \) appearing in the polynomial and regrouping terms.
We now give the Khintchine inequalities for superorthogonal systems and hypersystems.

**Theorem 2.3** (Superorthogonal Khintchine inequality). Let \( k, r \in \mathbb{Z}_+ \), let \( X = (X, \mu) \) be a measure space, \( S \) a finite set, and \( H \) a finite-dimensional Hilbert space.

(i) (Khintchine for superorthogonal systems) If \((f_s)_{s \in S}\) is a Type II \( 2r \)-superorthogonal system in \( L^2(X; H) \) indexed by \( S \), then
\[
\left\| \sum_{s \in S} f_s \right\|_{L^{2r}(X; H)} \leq O(r)^{1/2} \left\| (f_s)_{s \in S} \right\|_{L^{2r}(X; H)} .
\]

(ii) (Khintchine for superorthogonal hypersystems) If \((f_A)_{A \in (S \leq k)}\) is a Type II \( 2r \)-superorthogonal hypersystem in \( L^2(X; H) \) then
\[
\left\| \sum_{A \in (S \leq k)} f_A \right\|_{L^{2r}(X; H)} \leq O(r)^{k/2} \left\| (f_A)_{A \in (S \leq k)} \right\|_{L^{2r}(X; H)} .
\]

where we adopt the notation
\[
H^{[\leq k]} := H^{(S \leq k)} .
\]

Part (i) is standard (see e.g., [19, §3.1]). Part (ii) (without any losses of \( \log |S| \) factors) appears to new; with logarithmic losses one can obtain a result of this type from [13, Lemma 5.1], an iteration of (i), and the triangle inequality.

**Proof.** We begin with (i). We may index \( S = \{1, \ldots, n\} \). The desired estimate may be rewritten as
\[
\int_X \left\| \sum_{s=1}^n f_s \right\|_{H}^{2r} \, d\mu \leq O(r)^{r} \int_X \left( \sum_{s=1}^n \left\| f_s \right\|_{H}^{2r} \right) \, d\mu .
\]

From the binomial theorem, the Cauchy-Schwarz inequality, and the triangle inequality, for any \( u, v \in H \) we have
\[
\left\| u + v \right\|_{H}^{2r} = \left\| u \right\|_{H}^{2r} + 2r \Re \langle v, u \rangle \left\| u \right\|_{H}^{2r-2} + O \left( \sum_{j=2}^{2r} \binom{2r}{j} \left\| v \right\|_{H}^{j} \left\| u \right\|_{H}^{2r-j} \right) .
\]

Observe for any odd \( 2j + 1 \) between 1 and \( 2r \) that
\[
\binom{2r}{2j+1} \sim \binom{2r}{2j}^{1/2} \binom{2r}{2j+2}^{1/2}
\]
(since \( k! \sim ((k-1)! (k+1)!)^{1/2} \) for any \( k \in \mathbb{Z}_+ \), and hence by Young’s inequality we may restrict the \( j \) summation here to even integers, thus
\[
\|u + v\|_H^{2r} = \|u\|_H^{2r} + 2r \operatorname{Re} \langle v, u \rangle \|u\|_H^{2r-2} + O \left( \sum_{j=1}^{r} \binom{2r}{2j} \|v\|_H^{2j} \|u\|_H^{2r-2j} \right)
\]
and in particular
\[
\|u + v\|_H^{2r} \leq 2r \operatorname{Re} \langle v, u \rangle \|u\|_H^{2r-2} + \sum_{j=0}^{r} C^{1+j \geq 1} \left( \binom{2r}{2j} \|v\|_H^{2j} \|u\|_H^{2r-2j} \right)
\]
for some absolute constant \( C > 1 \). As a special case, for any \( v_1, \ldots, v_n \in H \), one has
\[
\left\| \sum_{s=1}^{n} v_s \right\|_H^{2r} \leq \operatorname{Re} Z + \sum_{j=0}^{r} C^{1+j \geq 1} \left( \binom{2r}{2j} \|v_1\|_H^{2j} \sum_{s=2}^{n} v_s \right)^{2r-2j}
\]
where \( Z \) is a linear combination of expressions of the form \( \prod_{j=1}^{r} \langle v_{s_j}, v_{s_{r+j}} \rangle \) where \( \{s_1\}, \ldots, \{s_{2r}\} \) contains a nonce. Iterating this identity \( n \) times, we conclude that
\[
\left\| \sum_{s=1}^{n} v_s \right\|_H^{2r} \leq \operatorname{Re} Z’ + \sum_{j=0}^{r} C^{1+j_1 \geq 1, \ldots, j_n \geq 1} \left( \binom{2r}{2j_1, \ldots, 2j_n} \|v_1\|_H^{2j_1} \cdots \|v_n\|_H^{2j_n} \right)
\]
where \( Z’ \) is also a linear combination of expressions of the form \( \prod_{j=1}^{r} \langle v_{s_j}, v_{s_{r+j}} \rangle \) with \( \{s_1\}, \ldots, \{s_{2r}\} \) containing a nonce, and \( \sum^* \) denotes a sum over tuples \( (j_1, \ldots. j_n) \in \mathbb{N}^n \) with \( j_1 + \cdots + j_n = r \). Applying this with \( v_i := f_i(x) \), integrating in \( X \), and using the Type II \( 2r \)-superorthogonality hypothesis (2.1) as well as the bound \( C^{1+j_1 \geq 1, \ldots, j_n \geq 1} \leq C^r \), we conclude that
\[
\int_X \left\| \sum_{s=1}^{n} f_s \right\|_H^{2r} d\mu \leq \sum^* \binom{2r}{2j_1, \ldots, 2j_n} \int_X \|f_1\|_H^{2j_1} \cdots \|f_n\|_H^{2j_n} d\mu.
\]
On the other hand, we have
\[
\int_X \left( \sum_{s=1}^{n} \|f_s\|_H^2 \right)^r = \sum^* \binom{r}{j_1, \ldots, j_n} \int_X \|f_1\|_H^{2j_1} \cdots \|f_n\|_H^{2j_n} d\mu.
\]
To finish the proof it will suffice to establish the inequality
\[
\binom{2r}{2j_1, 2j_2} \leq (2r)^r \binom{r}{j_1, \ldots, j_n}
\]
for any \( j_1, \ldots, j_n \geq 0 \) summing to \( r \). But this follows from the combinatorial observation that given a partition of \( \{1, \ldots, 2r\} \) into \( n \) classes of cardinality \( 2j_1, 2j_2, \ldots, 2j_n \) respectively, one can remove \( j_1 \) elements from the first class, then \( j_2 \) elements from the second class, and so forth until one is left with a partition of \( r \) elements of \( \{1, \ldots, 2r\} \) into \( n \) classes of cardinality \( j_1, \ldots, j_n \). There are at most \((2r)^r\) ways to remove these
elements in the order indicated, and \( \binom{r}{j_1, \ldots, j_n} \) ways to partition the remaining elements, with the original partition being recoverable from this data. This gives (i).

Now we prove (ii). By the triangle inequality it suffices to establish this result with \( \binom{S}{< k} \) replaced by \( \binom{S}{k} \). The case \( k = 0 \) is trivial, so suppose \( k \geq 1 \). We apply the probabilistic decoupling method (cf., [17]), which can be viewed as a substitute for the random partitioning lemma in [13] Lemma 5.1] that avoids logarithmic losses. We form a random partition \( S = S_1 \uplus \cdots \uplus S_k \) by setting \( S_i := \{ s \in S : i_s = i \} \), where \( i_s, s \in S \) are independent random variables drawn uniformly at random from \( \{1, \ldots, k\} \). Observe that if \( A \subset \binom{S}{k} \), then \( A \) takes the form \( \{s_1, \ldots, s_k\} \) with \( s_i \in S_i \) for \( i = 1, \ldots, k \) with probability precisely \( \frac{k!}{k^k} \) (this is the probability that the tuple \((i_s)_{s \in A}\) forms a permutation of \( \{1, \ldots, k\}\)). Thus we have

\[
\sum_{A \subset \binom{S}{k}} f_A = \frac{k!}{k^k} \mathbb{E} \sum_{s_1 \in S_1, \ldots, s_k \in S_k} f_{\{s_1, \ldots, s_k\}}
\]

and hence by the triangle inequality

\[
\left\| \sum_{A \subset \binom{S}{k}} f_A \right\|_{L^2(X; \mathcal{H})} \leq \frac{k!}{k^k} \mathbb{E} \left\| \sum_{s_1 \in S_1, \ldots, s_k \in S_k} f_{\{s_1, \ldots, s_k\}} \right\|_{L^2(X; \mathcal{H})}.
\]

Using the Taylor expansion

\[
e^k = \frac{k^0}{0!} + \frac{k^1}{1!} + \cdots + \frac{k^k}{k!} + \cdots \geq \frac{k^k}{k!}
\]

it will thus suffice to establish the deterministic inequality

\[
\left\| \sum_{s_1 \in S_1, \ldots, s_k \in S_k} f_{\{s_1, \ldots, s_k\}} \right\|_{L^2(X; \mathcal{H})} \leq O(r)^{k/2} \left\| (f_{\{s_1, \ldots, s_k\}})_{s_1 \in S_1, \ldots, s_k \in S_k} \right\|_{L^2(X; \mathcal{H}^{S_1 \times \cdots \times S_k})}
\]

whenever \( S = S_1 \uplus \cdots \uplus S_k \) is a partition of \( S \). By induction, it suffices to establish the bound

\[
\left\| \left( \sum_{s_1 \in S_1, \ldots, s_k \in S_k} f_{\{s_1, \ldots, s_k\}} \right)_{s_1 \in S_1, \ldots, s_{i-1} \in S_{i-1}} \right\|_{L^2(X; \mathcal{H}^{S_1 \times \cdots \times S_{i-1}})} \leq O(r)^{1/2} \left\| \left( \sum_{s_{i+1} \in S_{i+1}, \ldots, s_k \in S_k} f_{\{s_1, \ldots, s_k\}} \right)_{s_1 \in S_1, \ldots, s_i \in S_i} \right\|_{L^2(X; \mathcal{H}^{S_1 \times \cdots \times S_i})}
\]
for all $1 \leq i \leq k$. But this follows by applying part (i) to the Hilbert space $H^{S_1 \times \cdots \times S_{i-1}}$ and the functions

$$f_{s_i} := \left( \sum_{s_{i+1} \in S_{i+1}, \ldots, s_k \in S_k} f_{\{s_1, \ldots, s_k\}} \right)_{s_1 \in S_1, \ldots, s_{i-1} \in S_{i-1}}$$

for $s_i \in S_i$, which one can verify to form a $2r$-superorthogonal system in $L^{2r}(X; H^{S_1 \times \cdots \times S_{i-1}})$.

As a sample application of Theorem 2.3, we can specialize to the situation in Example 2.2 to conclude

**Corollary 2.4** (Hoeffding-type inequality). Let the notation and hypotheses be as in Example 2.2. If we have the bound

$$\sum_{A \in (\mathcal{S} \leq k) : A \neq \emptyset} |f_A|^2 \leq \sigma^2$$

almost surely for some $\sigma > 0$, then one has

$$\mathbb{P}\left( \left| \sum_{A \in (\mathcal{S} \leq k)} f_A - f_\emptyset \right| \geq \lambda \sigma \right) \leq O(1)^k \left( \exp(-ck\lambda^2/k) + \exp(-cR) \right)$$

for all $\lambda > 0$ and some absolute constant $c > 0$.

See [16], [22] for some previous Hoeffding-type inequalities for sums of $R$-wise independent random variables.

**Proof.** We may normalize $f_\emptyset = 0$. By shrinking $\lambda$ if necessary we can also assume that $\lambda \leq (R/k)^{k/2}$ (otherwise the first term on the right-hand side is dominated by the second). We can also assume that $\lambda \geq C^k$ and $R \geq Ck$ for a large constant $C$, as the bound is trivial otherwise. Let $1 \leq r \leq R/2k$ be an integer to be chosen later. By Markov’s inequality one has

$$\mathbb{P}\left( \left| \sum_{A \in (\mathcal{S} \leq k)} f_A \right| \geq \lambda \sigma \right) \leq \lambda^{-2r} \sigma^{-2r} \mathbb{E} \left( \sum_{A \in (\mathcal{S} \leq k)} |f_A|^2 \right)^{2r}.$$ 

Applying Theorem 2.3(ii) to the $2r$-superorthogonal hypersystem $(f_A)_{A \in (\mathcal{S} \leq k)}$, we obtain

$$\mathbb{E} \left( \sum_{A \in (\mathcal{S} \leq k)} f_A \right)^{2r} \leq O(r)^{kr} \mathbb{E} \left( \sum_{A \in (\mathcal{S} \leq k)} |f_A|^2 \right)^r.$$
Combining this with the preceding inequality and (2.3), we conclude that
\[ \mathbb{P} \left( \left| \sum_{A \in \binom{S}{\leq k}} f_A \right| \geq \lambda \sigma \right) \leq \left( O(r) / \lambda^{2/k} \right)^{kr} . \]

If we set \( r := \lfloor c \lambda^{2/k} \rfloor \) for a sufficiently small absolute constant \( c > 0 \), we obtain the claim. \( \square \)

We now discuss the sharpness of the estimates in Corollary 2.4. The first example below shows that the first term on the right-hand side of (2.4) is reasonably sharp; the second example shows the second term in (2.4) only has a small amount of room for improvement.

**Example 2.5.** Let \( n, k \) be positive integers, and let \( X_{i,j} \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, n \) be independent Bernoulli random variables taking values in \( \{-1, +1\} \) of mean zero. Then the random variable \( \prod_{i=1}^{k} \sum_{j=1}^{n} X_{i,j} \) can be expanded in the form \( \sum_{A} f_A \) where \( f_A = X_{1,j_1} \ldots X_{k,j_k} \) when \( A \) is of the form \( \{(1,j_1), \ldots, (k,j_k)\} \) and \( f_A = 0 \) otherwise. One then easily verifies that (2.3) holds with \( \sigma = n^{k/2} \), and that
\[ \mathbb{P} \left( \left| \sum_{A \in \binom{S}{\leq k}} f_A \right| \geq \lambda \sigma \right) = 2^{-nk} \]
when \( \lambda = n^{k/2} \) and \( S = \{1, \ldots, k\} \times \{1, \ldots, n\} \). Here one can take \( R \) to be arbitrary. This shows that the first-term on the right-hand side of (2.4) cannot be improved except possibly for the \( O(1)^k \) factor or in the explicit value of \( c \). Modifications of this example can also be used to illustrate the sharpness of Theorems 2.3; we leave the details to the interested reader.

**Example 2.6.** Let \( R \) be a natural number, let \( p \) be a prime greater than \( R \), and let \( P \) be a random polynomial of degree at most \( R - 1 \) with coefficients in \( \mathbb{Z}/p\mathbb{Z} \), drawn uniformly among all such polynomials. Then the random variables \( P(i) \) for \( i = 1, \ldots, p \) are \( R \)-wise independent, since the Lagrange interpolation formula shows that for any distinct \( i_1, \ldots, i_R \), the map from polynomials \( P \) to evaluations \( (P(i_1), \ldots, P(i_R)) \) is a bijection. We then have
\[ \mathbb{P} \left( \left| \sum_{i=1}^{p} (1_{P(i)=0} - 1/p) \right| = p - 1 \right) = p^{-R} \]
Comparing this with (2.3) with \( \sigma^2 = p, k = 1 \), and \( \lambda = p^{-1} / \sqrt{p} \), and taking \( p \) comparable to \( CR \log R \), we see that the \( \exp(-cR) \) term in (2.4) cannot be improved to more than \( \exp(-C R \log R) \) for some constant \( C \).
One can construct similar examples for higher values of $k$ by considering the random variable $\prod_{j=1}^{k} \sum_{i=1}^{p} (1_{P_j(i)=0} - 1/p)$ where $P_1, \ldots, P_k$ are independent copies of $P$; we leave the details to the interested reader.

3. Sunflower bound

If $k, r \in \mathbb{Z}_+$, let $\text{Sun}(k, r)$ denote the smallest natural number with the property that any family of $\text{Sun}(k, r)$ distinct sets of cardinality at most $k$ contains $r$ distinct elements $A_1, \ldots, A_r$ that form a sunflower (as defined in Definition 1.9). The celebrated Erdős-Rado theorem asserts that $\text{Sun}(k, r)$ is finite; in fact Erdős and Rado gave the bounds

$$(r - 1)^k \leq \text{Sun}(k, r) \leq (r - 1)^k k! + 1.$$ 

The sunflower conjecture asserts in fact that the upper bound can be improved to $\text{Sun}(k, r) \leq O(1)$. This remains open at present; the best bound known currently (in the regime where $k, r$ are both large) is

$$\text{Sun}(k, r) \leq O(r \log(kr))^k$$

for all $k, r \in \mathbb{Z}_+$, due to Rao (building upon a recent breakthrough of Alweiss, Lovett, Wu, and Zhang [1]).

We can give a probabilistic version of the Erdős-Rado theorem:

**Lemma 3.1** (Probabilistic Erdős-Rado theorem). Let $k, r \in \mathbb{Z}_+$, let $S$ be a finite set, and let $A$ be a random subset of $S$ of cardinality $k$ (i.e., a random element of $\binom{S}{k}$). Let $A_1, \ldots, A_r$ be $r$ independent copies of $A$. Then with probability at least $(4\text{Sun}(k, r))^{-r}$, $A_1, \ldots, A_r$ form a sunflower.

**Proof.** If there is a set $A \in \binom{S}{k}$ with $\mathbb{P}(A = A) \geq (4\text{Sun}(k, r))^{-1}$, then with probability at least $(4\text{Sun}(k, r))^{-r}$ we have $A_1 = \cdots = A_r = A$. Since $A, \ldots, A$ is a sunflower, this gives the claim.

Now suppose that $\mathbb{P}(A = A) < (4\text{Sun}(k, r))^{-1}$ for all $A \in \binom{S}{k}$. We form $2\text{Sun}(k, r)$ independent samples $A_1, \ldots, A_{2\text{Sun}(k, r)}$ of $A$. Consider the event $E$ that these samples only consist of at most $\text{Sun}(k, r)$ distinct sets. On this event, there are at most $\sum_{m \leq \text{Sun}(k, r)} \binom{2\text{Sun}(k, r)}{m}$ ways to a maximal collection $A_{i_1}, \ldots, A_{i_m}$ of distinct samples for some $m \leq \text{Sun}(k, r)$; if we fix these indices $i_1, \ldots, i_m$, we see from hypothesis that each of the other samples $A_i$ has a probability at most $1/4$ of matching one of these distinct samples. From the union bound, we conclude that

$$\mathbb{P}(E) \leq 2^{2\text{Sun}(k, r)}(1/4)^{\text{Sun}(k, r)} \leq 1/2.$$
If we now condition to the complement of $E$, the samples $A_1, \ldots, A_{2\text{Sun}(k,r)}$ necessarily contain a sunflower, hence by symmetry and then undoing the conditioning we see that $A_1, \ldots, A_r$ is a sunflower with probability at least
\[
\frac{1}{2} \left( \frac{2\text{Sun}(k,r)}{r} \right)^{-1} \geq 2^{-1}(2\text{Sun}(k,r))^{-r},
\]
giving the claim.

In the converse direction, we can find a collection $A_1, \ldots, A_{\text{Sun}(k,r)-1}$ of distinct sets of cardinality $k$, such that no distinct $r$ elements in this collection form a sunflower. If $A_1, \ldots, A_r$ are drawn uniformly from this collection, then the probability that they form a sunflower is then precisely $(\text{Sun}(k,r)-1)^{1-r}$ (the probability that all the $A_i$ coincide). Thus the bound of $(4\text{Sun}(k,r)-r)$ in the above lemma cannot be dramatically improved.

**Lemma 3.1** lets us control square functions:

**Corollary 3.2** (Sunflower bound on square function). Let $S$ be a finite set, let $k \in \mathbb{Z}_+$, let $X$ be a measure space, and let $H$ be a finite-dimensional Hilbert space. Let $(f_A)_{A \in (S)_k}$ be a finite collection of functions $f_A \in L^2_r(X;H)$. Then
\[
\left\| (f_A)_{A \in (S)_k} \right\|_{L^{2r}(X;H^{[k]})}^{2r} \leq (4\text{Sun}(k,r))^r \sum_{A_0 \in \binom{S}{\leq k}} \sum_{i=1}^r \left\| f_{A_0 \cup A_i} \right\|_H^2
\]
and conversely
\[
\sum_{A_0 \in \binom{S}{\leq k}} \sum_{i=1}^r \left\| f_{A_0 \cup A_i} \right\|_H^2 \leq \left\| (f_A)_{A \in (S)_k} \right\|_{L^{2r}(X;H^{[k]})}^{2r},
\]
where $\sum^*\sum^*$ denotes the sum over tuples $(A_1, \ldots, A_r)$ of sets $A_1, \ldots, A_r \in \binom{S \setminus A_0}{k-|A_0|}$ that are pairwise disjoint (or equivalently, that $A_0 \cup A_1, \ldots, A_0 \cup A_r$ form a sunflower), and $H^{[k]} := H^{(S)_k}$.

See [7, Lemma 2.3] for a version of this result in the $k = 1$ case;

**Proof.** We begin with the first inequality. Expanding out both sides, it suffices to establish the pointwise estimate
\[
\left( \sum_{A \in (S)_k} \left\| f_A(x) \right\|_H^r \right)^r \leq (4\text{Sun}(k,r))^r \sum_{A_0 \in \binom{S}{\leq k}} \sum_{i=1}^r \left\| f_{A_0 \cup A_i}(x) \right\|_H^2
\]

(3.2)
for all $x$.

Fix $x$. We may normalise the left-hand side of (3.2) to equal 1. We can then view the sequence $(\|f_A(x)\|_{L^2})_{A \in (S)}$ as the probability density function for a random subset $A$ of $S$ of cardinality $k$, and the inequality then can be written as

$$1 \leq (4\text{Sun}(k,r))^r \mathbb{P}(A_1, \ldots, A_r \text{ form a sunflower}).$$

The claim now follows from Lemma 3.1. The second inequality similarly follows from the trivial bound

$$\mathbb{P}(A_1, \ldots, A_r \text{ form a sunflower}) \leq 1.$$

\[\square\]

4. Proof of main theorems

Let $(d, k, S, \varepsilon)$ be a major arc parameter set. We now explore the additive structure of the major arcs associated to this set. We first observe from the Chinese remainder theorem that

$$\Sigma_{A_1} + \Sigma_{A_2} = \Sigma_{A_1 \cup A_2} \quad (4.1)$$

whenever $A_1, A_2 \subseteq S$ are disjoint. For $A_0 \in \left( \frac{S}{\leq k} \right)$, we also define the set

$$\Sigma_{(A_0)} := \bigcup_{A \in \left( \frac{S \setminus A_0}{\leq k - |A_0|} \right)} \Sigma_A.$$

From (4.1) we then have the inclusion

$$\Sigma_{A_0} + \Sigma_{(A_0)} \subseteq \Sigma_{\leq k}. \quad (4.2)$$

Let $H$ be a finite dimensional Hilbert space. Define a major arc system adapted to $(d, k, S, \varepsilon)$ taking values in $H$ to be a collection $(f_\alpha)_{\alpha \in \Sigma_{\leq k}}$ of functions $f_\alpha \in \ell^2(\mathbb{Z}^d; H)$ with Fourier support in $\alpha + [-\varepsilon, \varepsilon]^d$ for each $\alpha \in \Sigma_{\leq k}$, thus

$$f_\alpha \in \ell^2(\mathbb{Z}^d; H)^{\alpha + [-\varepsilon, \varepsilon]^d}.$$

For any $\Sigma \subseteq \Sigma_{\leq k}$, we define

$$f_\Sigma := \sum_{\alpha \in \Sigma} f_\alpha.$$

Lemma 4.1 (Orthogonality properties). Let $(f_\alpha)_{\alpha \in \Sigma_{\leq k}}$ be a major arc system adapted to a major arc parameter set $(d, k, S, \varepsilon)$ taking values in a Hilbert space $H$. Suppose that the parameter set $(d, k, S, \varepsilon)$ is $(r, c)$-good for some $r \in \mathbb{Z}_+$ and $0 < c < 1$.
(i) The major arcs $\alpha + [-\epsilon, \epsilon]^d$, $\alpha \in \Sigma_{\leq k}$ are disjoint. (Indeed, the $\alpha \in \Sigma_{\leq k}$ are at least $2\epsilon/c$-separated in the $\ell^\infty$ metric.)

(ii) (Denominator orthogonality) The hypersystem $(f_{\Sigma A})_{A \in (S_{\leq k})}$ is Type II $2r$-superorthogonal.

(iii) (Numerator orthogonality) If $A_1, \ldots, A_r \in (S_{\leq k})$ form a sunflower with core $A_0$ and petals $A_1 \setminus A_0, \ldots, A_r \setminus A_0$, then the functions
\[
\prod_{i=1}^{r} f_{\Sigma A_{\alpha} + \alpha_i} \in \ell^2(\mathbb{Z}^d; H^\otimes r)
\]
for $\alpha_1 \in \Sigma_{A_1 \setminus A_0}, \ldots, \alpha_r \in \Sigma_{A_r \setminus A_0}$ are pairwise orthogonal in the Hilbert space $L^2(\mathbb{T}^d; H^\otimes r)$, where we use the product notation $r \prod_{i=1}^{r} f_i(x) := f_1(x) \otimes \cdots \otimes f_r(x)$.

Proof. Note from Definition 1.3 that the coefficients of every element of $\Sigma_{\leq k}$ is a rational number with denominator at most $q^k_{\text{max}}$. In particular, if $\alpha, \alpha'$ are two distinct elements of $\Sigma_{\leq k}$, then $\alpha, \alpha'$ differ in $\ell^\infty$ metric by at least $1/q^k_{\text{max}}$. The claim (i) now follows (with room to spare) from (1.4).

Now we prove (ii). From inspecting the Fourier transform, it suffices to show that
\[
\sum_{j=1}^{r} (\alpha_j + \theta_j) - \sum_{j=r+1}^{2r} (\alpha_j + \theta_j) \neq 0
\]
in $\mathbb{T}^d$ whenever $\alpha_j \in \Sigma_{A_j}$ and $\theta_j \in [-\epsilon, \epsilon]^d$ for $j = 1, \ldots, 2r$. As the $A_1, \ldots, A_{2r}$ contain a nonce, there exists an $A \subseteq S$ which contains all but exactly one of the $A_1, \ldots, A_{2r}$. This implies that $Q_A(\alpha_1 + \cdots + \alpha_r - \alpha_{r+1} - \cdots - \alpha_{2r})$ has precisely one non-zero term, and hence the point $\alpha_1 + \cdots + \alpha_r - \alpha_{r+1} - \cdots - \alpha_{2r} \in \mathbb{T}^d$ is non-zero. Observe that the coordinates of this point consist of rational numbers of denominator at most $1/q^k_{\text{max}}$. The claim now follows from (1.4) and the triangle inequality.

Now we prove (iii). Inspecting the Fourier transform, it suffices to show that
\[
\sum_{j=1}^{r} (\alpha_{0,j} + \alpha_j + \theta_j) - \sum_{j=r+1}^{2r} (\alpha_{0,j} + \alpha_j + \theta_j) \neq 0
\]
in $\mathbb{T}^d$ whenever $\alpha_{0,j} \in \Sigma_{A_0}$ and $\theta_j \in [-\epsilon, \epsilon]^d$ for $j = 1, \ldots, 2r$, and
\[
(\alpha_1, \ldots, \alpha_r), (\alpha_{r+1}, \ldots, \alpha_{2r}) \in \Sigma_{A_1 \setminus A_0} \times \cdots \times \Sigma_{A_r \setminus A_0}
\]
are distinct. Multiplying by \( Q_{A_0} \) to cancel the \( \alpha_{0,j} \) factors, it suffices to show that
\[
\sum_{j=1}^{r} Q_{A_0}(\alpha_j + \theta_j) - \sum_{j=r+1}^{2r} Q_{A_0}(\alpha_j + \theta_j) \neq 0.
\]
Note that \( Q_{A_0} \leq q_{\max}^{\lfloor |A_0| \rfloor} \). On the other hand, from the sunflower hypothesis and the Chinese remainder theorem we see that the point \( Q_{A_0}(\alpha_1 + \cdots + \alpha_r - \alpha_{r+1} - \cdots - \alpha_{2r}) \) is non-zero. The coordinates of this point consist of rational numbers of denominator at most \( \frac{1}{q_{\max}^{2r(1-|A_0|)}} \), and the claim now follows from (1.4) and the triangle inequality. □

We can exploit these orthogonality properties to obtain a description of the \( \ell^{2r} \) norm of a sum \( f_{\Sigma \leq k} \) associated to a major arc system, as well as a companion result that will be useful in the sequel.

**Theorem 4.2** (Applying orthogonality). Let \((d, k, S, \varepsilon)\) be a major arc parameter set which is \((r, c)\)-good for some \( r \in \mathbb{Z}_+ \) and \( 0 < c < 1 \). Let \( H \) be a finite-dimensional Hilbert space.

(i) *(Description of \( \ell^{2r} \) norm)* If \((f_\alpha)_{\alpha \in \Sigma \leq k} \) is a major arc system adapted to \((d, k, S, \varepsilon)\), then we have
\[
O(1)^d \| f_{\Sigma \leq k} \|_{\ell^{2r}(\mathbb{Z}^d; H)} \leq \left( \sum_{A_0 \in \{S \leq k\}} \left( \left\| (f_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \right\|_{\ell^{2r}(\mathbb{Z}^d; H)} \right)^{2r} \right)^{1/2r} \leq O_c(1)^d O(1)^k \| f_{\Sigma \leq k} \|_{\ell^{2r}(\mathbb{Z}^d; H)}.
\]

(ii) *(Rubio de Francia type estimate)* Let \( \varphi_0 \in C^\infty_c(\mathbb{R}) \) be a bump function supported on \([-1, 1]\), and let \( \varphi \in C^\infty_c(\mathbb{R}^d) \) be the symbol
\[
\varphi(\xi_1, \ldots, \xi_d) = \prod_{j=1}^{d} \varphi_0 \left( \frac{\xi_j}{\varepsilon} \right).
\]
Then for any \( 2 \leq p \leq \infty \) and \( f \in \ell^p(\mathbb{Z}^d; H) \), one has the inequality
\[
\left( \sum_{A_0 \in \{S \leq k\}} \left( \left\| (T_{\varphi, \alpha + \Sigma A_0} f)_{\alpha \in \Sigma(A_0)} \right\|_{\ell^p(\mathbb{Z}^d; H)} \right)^p \right)^{1/p} \leq O_{\varphi_0}(1)^d O(1)^k \| f \|_{\ell^p(\mathbb{Z}^d; H)}.
\]
Proof. We begin with (ii), as this will be used in the proof of (i). By interpolation it suffices to establish the claims for $p = 2, \infty$. For $p = 2$ the claim follows from Lemma 4.1(i) and Plancherel’s theorem, noting that each $T_{\varphi, \alpha} f$ has Fourier transform supported in $\Sigma_{\alpha_0} + \alpha + [-\varepsilon, \varepsilon]^d$, and each $\beta \in \Sigma_{\alpha_0}$ has at most $2^k$ representations of the form $\beta = \alpha + \alpha_0$ with $A_0 \in \binom{S}{\leq k}$, $\alpha_0 \in \Sigma_{\alpha_0}$, $\alpha \in \Sigma_{(\alpha_0)}$. For $p = \infty$, it suffices by translation invariance to show that

$$
\left\| (T_{\varphi, \alpha} f(0))_{\alpha \in \Sigma_{(A_0)}} \right\|_{L^\infty(A_0)} \leq O_{\varphi_0}(1)^d O(1)^k \|f\|_{L^\infty(\mathbb{Z}^d; H)}
$$

for any $f \in L^\infty(\mathbb{T}^d; H)$ and $A_0 \in \binom{S}{<k}$. By the inclusion-exclusion formula, and conceding a factor of $2^k$, it suffices to show that

$$
\left\| (T_{\varphi, \alpha} f(0))_{\alpha \in \Sigma_{(A_0)}} \right\|_{L^\infty(A_0)} \leq O_{\varphi_0}(1)^d \|f\|_{L^\infty(\mathbb{Z}^d; H)}
$$

for any $A'_0 \subseteq A_0$. By duality, this bound is equivalent to the assertion that

$$
\left\| \sum_{\alpha \in \Sigma_{(A_0)}} c_\alpha T^*_{\varphi, \alpha + \Sigma_{\leq A_0} \delta_0} \right\|_{L^1(\mathbb{Z}^d; H)} \leq O_{\varphi_0}(1)^d \|c_\alpha\|_{L^\infty(\mathbb{Z}^d; H)}
$$

for any sequence $c_\alpha \in H$, $\alpha \in \Sigma_{(A_0)}$, where $\delta_0$ is the Kronecker delta function. Observe that the integrand on the left-hand side is actually supported on $(Q_{A_0} Z)^d$. By Cauchy-Schwarz (and (1.4)), it then suffices to show that

$$
\left\| \sum_{\alpha \in \Sigma_{(A_0)}} c_\alpha w T^*_{\varphi, \alpha + \Sigma_{\leq A_0} \delta_0} \right\|_{L^1(\mathbb{Z}^d; H)} \leq O_{\varphi_0}(1)^d \varepsilon^{-d/2} Q_{A_0}^{d/2} \|c_\alpha\|_{L^\infty(\mathbb{Z}^d; H)}
$$

where $w$ is the weight function

$$
w(n_1, \ldots, n_d) := \prod_{j=1}^d (1 + \varepsilon^2 n_j^2).
$$

From Lemma 4.1(i) we see that the functions $w T^*_{\varphi, \alpha + \Sigma_{\leq A_0}} \delta_0$ have disjoint Fourier supports and are thus pairwise orthogonal. Each such function can be split into $Q_{A_0}$ orthogonal components $w T^*_{\varphi, \alpha + \alpha_0} \delta_0$ with $\alpha_0 \in \Sigma_{\leq A_0}$. By the Pythagorean theorem, it thus suffices to establish the bound

$$
\|w T^*_{\varphi, \alpha + \alpha_0} \delta_0\|_{L^2(\mathbb{Z}^d)} \leq O_{\varphi_0}(1)^d \varepsilon^{d/2}
$$

for each $\alpha \in \Sigma_{(A_0)}$, $\alpha_0 \in \Sigma_{\leq A_0}$. The magnitude of the expression inside the norm of the left-hand side does not actually depend on $\alpha + \alpha_0$, so we may assume that $\alpha + \alpha_0 = 0$. The left-hand side then factors as
a tensor product and it now suffices to establish the claim for \( d = 1 \), that is to say to show that

\[
\sum_{n \in \mathbb{Z}} (1 + \varepsilon^2 n^2) \varepsilon^2 |\hat{\varphi}_0(\varepsilon n)|^2 \leq O_{\varphi_0}(\varepsilon)
\]

which follows from the rapid decrease of \( \hat{\varphi} \) (and noting from (1.4) that \( \varepsilon \leq 1 \)). This completes the proof of (ii).

Now we prove (i). By Lemma 4.1(ii) and Theorem 2.3(ii), we have

\[
\|f_{\Sigma_k}\|_{\ell^2(\mathbb{Z}^d; H)} \leq O(r)^{k/2} \left\| f_{\Sigma_A} \right\|_{\ell^2(\mathbb{Z}^d; H \preceq k)}.
\]

If we then apply Corollary 3.2 with the sunflower bound (3.1), and apply the triangle inequality to sum in \( k \), we conclude that

\[
\|f_{\Sigma_k}\|_{\ell^2(\mathbb{Z}^d; H)} \leq O(r \log^{1/2}(kr))^{k} \left( \sum_{A_0 \in (s \leq k)} X_{A_0} \right)^{1/2r}
\]

(4.5)

where

\[
X_{A_0} := \sum_{\alpha_1 \in \Sigma_{A_1}, \ldots, \alpha_r \in \Sigma_{A_r}} \prod_{i=1}^r \left\| f_{A_0 \cup A_i} \right\|_{\ell^2(\mathbb{Z}^d; H \preceq r)}^2
\]

and \( \sum_{\alpha_1 \in \Sigma_{A_1}, \ldots, \alpha_r \in \Sigma_{A_r}} \) denotes a sum over tuples \((A_1, \ldots, A_r)\) of disjoint sets \( A_1, \ldots, A_r \in (s \leq k - |A_0|)\). For \( A_0, A_1, \ldots, A_r \) as above, we can split

\[
\prod_{i=1}^r f_{A_0 \cup A_i} = \sum_{\alpha_1 \in A_1, \ldots, \alpha_r \in A_r, A_i} \prod_{i=1}^r f_{\alpha_i + A_0}.
\]

From Lemma 4.1(iii) and the Pythagorean theorem, we may thus write

\[
X_{A_0} = \sum_{\alpha_1 \in \Sigma_{A_1}, \ldots, \alpha_r \in \Sigma_{A_r}} \prod_{i=1}^r \left\| f_{\alpha_i + A_0} \right\|_{\ell^2(\mathbb{Z}^d; H \preceq r)}^2.
\]

We drop the hypothesis of disjointness in the \( \sum_{\alpha_1 \in \Sigma_{A_1}, \ldots, \alpha_r \in \Sigma_{A_r}} \) sum to obtain the upper bound

\[
X_{A_0} \leq \sum_{A_1, \ldots, A_r \in (s \leq k - |A_0|)} \left( \sum_{\alpha_1 \in \Sigma_{A_1}, \ldots, \alpha_r \in \Sigma_{A_r}} \left\| \prod_{i=1}^r f_{\alpha_i + A_0} \right\|_{\ell^2(\mathbb{Z}^d; H \preceq r)}^2 \right)
\]

which by the Fubini–Tonelli theorem can be rearranged as

\[
X_{A_0} \leq \left\| (f_{\alpha + A_0})_{\alpha \in \Sigma(A_0)} \right\|_{\ell^2(\mathbb{Z}^d; H_{\Sigma(A_0)})}^{2r}.
\]

This gives the first inequality in (4.3).
Now we establish the second inequality in (1.3). Let $c' := \frac{1+c}{2}$, so that $c < c' < 1$. Let $\varphi \in C^\infty_c(\mathbb{R}^d)$ be a multiplier of the form

$$\varphi(\xi_1, \ldots, \xi_d) := \prod_{j=1}^d \varphi_0(\xi_j/\varepsilon),$$

where $\varphi_0 \in C^\infty_c(\mathbb{R})$ is a fixed real even bump function (depending only on $c$) supported on $[-c'/c, c'/c]$ that equals 1 on $[-1, 1]$. From Lemma 4.1(i) (with $c$ replaced by $c'$, and $\varepsilon$ replaced by $\frac{\varepsilon}{c}$) we have

$$f_{\alpha + \Sigma A_0} := T_{\varphi; \alpha + \Sigma A_0} f_{\xi \leq k}$$

and the claim now follows from (ii) (setting $p = 2r$).

Now we can prove Theorem 1.4. Let the notation and hypotheses be as in that theorem. We normalize $\|T_m\|_{B(L^2)_{\ell^2}} = 1$ and $\|f\|_{\ell^2(\mathbb{Z}^d; H)} = 1$ (we can also assume by limiting arguments that $f \in \ell^2(\mathbb{Z}^d; H)$ to avoid technicalities), and our task is to show that

$$\left\| \sum_{A \in \mathcal{S}} \varepsilon_A T_{m; \Sigma A} f \right\|_{\ell^2(\mathbb{Z}^d; H)} \leq O_c(1)^d O(r \log^{1/2}(kr))^k.$$  

Applying Theorem 4.2(i) to the hypersystem $(\varepsilon_A T_{m; \Sigma A} f)_{A \in \mathcal{S}}$, it suffices to show that

$$\left( \sum_{A_0 \in \mathcal{S}_0} \left\| (T_{m; \alpha + \Sigma A_0} f)_{\alpha \in \Sigma(A_0)} \right\|_{\ell^2(\mathbb{Z}^d; H)}^{2r} \right)^{1/2r} \leq O_c(1)^{d+k}. \quad (4.7)$$

With $\varphi$ as in (4.4), we may use Lemma 4.1(i) to factor

$$T_{m; \alpha + \Sigma A_0} f = T_{m; \alpha + \Sigma A_0} T_{\varphi; \alpha + \Sigma A_0} f.$$  

Next, from the Magyar–Stein–Wainger sampling principle (Proposition 1.2) we have

$$\|T_{m; \Sigma A_0} f\|_{\ell^2(\mathbb{Z}^d; H)} \leq O(1)^d \|f\|_{\ell^2(\mathbb{Z}^d; H)}$$

for any $F \in \ell^2(\mathbb{Z}^d; H)$, hence by the Marcinkiewicz–Zygmund theorem (Theorem 1.8) one has

$$\left\| (T_{m; \Sigma A_0} F)_{\alpha \in \Sigma(A_0)} \right\|_{\ell^2(\mathbb{Z}^d; H)} \leq O(1)^d \left\| F_{\alpha \in \Sigma(A_0)} \right\|_{\ell^2(\mathbb{Z}^d; H)}$$

for any $F_{\alpha} \in \ell^2(\mathbb{Z}^d; H)$, which by the modulation symmetries of the Fourier transform imply that

$$\left\| (T_{m; \alpha + \Sigma A_0} F_{\alpha})_{\alpha \in \Sigma(A_0)} \right\|_{\ell^2(\mathbb{Z}^d; H)} \leq O(1)^d \left\| F_{\alpha \in \Sigma(A_0)} \right\|_{\ell^2(\mathbb{Z}^d; H)}.$$
Putting all this together, we reduce to showing that
\[
\left( \sum_{A_0 \in \mathcal{S}_k} \left\| \left( T_{\nu;\alpha + \Sigma A_0} f \right)_{\alpha \in \Sigma(A_0)} \right\|_{L^{2r}(\mathbb{Z}^d; H^{\Sigma(A_0)})} \right)^{1/2r} \leq O_c(1)^d O(1). \]

But this follows from Theorem 4.2(ii). This concludes the proof of Theorem 4.4.

Now we observe an arithmetic analogue of Theorem 4.2, in which the spatial scale parameter \( \varepsilon \) becomes irrelevant:

**Theorem 4.3** (Applying orthogonality, arithmetic limit). Let \((d, k, S, \varepsilon)\) be a major arc parameter set. Let \( H \) be a finite-dimensional Hilbert space. For each \( \alpha \in \Sigma_{\leq k} \), let \( f_{\alpha} \in \mathcal{S}(\mathbb{A}_d^2) \) have Fourier support in \( \mathbb{R}^d \times \{ \alpha \} \), and define \( f_{\Sigma} := \sum_{\alpha \in \Sigma} f_{\alpha} \) as before. Then for every positive integer \( r \), we have
\[
O(r \log^{1/2} (kr))^{-k} \| f_{\Sigma_{\leq k}} \|_{L^{2r}(\mathbb{A}_d^2; H)} \leq \left( \sum_{A_0 \in \mathcal{S}_k} \left\| \left( f_{\alpha + \Sigma A_0} \right)_{\alpha \in \Sigma(A_0)} \right\|_{L^{2r}(\mathbb{A}_d^2; H^{\Sigma(A_0)})} \right)^{1/2r} \leq O(1)^d O(1)^k \| f_{\Sigma_{\leq k}} \|_{L^{2r}(\mathbb{A}_d^2; H)}.
\]

(4.8)

Also, we have
\[
\| T_{\Sigma_{\leq k}} \|_{B(L^{2r}(\mathbb{Z}^d; H))} \leq O(1)^d O(r \log^{1/2} (kr))^{k}. \quad (4.9)
\]

**Proof.** One can establish (4.8) by direct repetition of the proof of Theorem 4.2, but we shall instead deduce this theorem as a limiting case of Theorem 4.2 (basically by sending \( \varepsilon \) to zero). By splitting \( \mathbb{A}_d^2 \) into fibres \( \{ x \} \times \mathbb{Z}^d \) for \( x \in \mathbb{R}^d \) and using the Fubini–Tonelli theorem, it suffices to establish the analogous claim for \( \mathbb{Z}^d \), that is to say to establish the bound
\[
O(r \log^{1/2} (kr))^{-k} \| f_{\Sigma_{\leq k}} \|_{L^{2r}(\mathbb{Z}^d; H)} \leq \left( \sum_{A_0 \in \mathcal{S}_k} \left\| \left( f_{\alpha + \Sigma A_0} \right)_{\alpha \in \Sigma(A_0)} \right\|_{L^{2r}(\mathbb{Z}^d; H^{\Sigma(A_0)})} \right)^{1/2r} \leq O(1)^d O(1)^k \| f_{\Sigma_{\leq k}} \|_{L^{2r}(\mathbb{Z}^d; H)}
\]

(4.10)

where for each \( \alpha \in \Sigma_{\leq k} \), \( f_{\alpha} \in \mathcal{S}(\mathbb{Z}^d; H) \) has Fourier support in \( \{ \alpha \} \), that is to say \( f_{\alpha}(y) = c_{\alpha} e(-y \cdot \alpha) \) for some \( c_{\alpha} \in H \). Let \( 0 < \varepsilon < 1 \) be a
small parameter, let \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) be a Schwartz function whose Fourier transform is supported in \([-1, 1]^d\) with normalization \( \| \varphi \|_{L^2(\mathbb{R}^d)} = 1 \), and let \( f_{\alpha, \varepsilon} \in \mathcal{S}(\mathbb{Z}^d; H) \) be the functions

\[
f_{\alpha, \varepsilon}(n) := f_{\alpha}(\hat{i}(n))\varphi(\varepsilon n) = c_{\alpha} e(-n \cdot \alpha)\varphi(\varepsilon n)
\]

where \( \hat{i}: \mathbb{Z}^d \to \mathbb{Z}^d \) is the canonical embedding. Then \( (f_{\alpha, \varepsilon})_{\alpha \in \Sigma_{\leq k}} \) is a major arc system adapted to \((d, k, S, \varepsilon)\). For \( \varepsilon \) small enough, this set of parameters is \((r, 1/2)\)-good, and so we see from Theorem 4.2 that

\[
O(r \log^{1/2}(kr))^{-k} \| f_{\Sigma_{\leq k}, \varepsilon} \|_{\ell^r(\mathbb{Z}^d; H)}^{1/2r} \leq \left( \sum_{A_0 \in \left( \frac{S}{k} \right)} \left\| (f_{\alpha + \Sigma A_0, \varepsilon})_{\alpha \in \Sigma(A_0)} \right\|_{L^2(\mathbb{Z}^d; H^{\varepsilon(A_0)})}^{2r} \right)^{1/2r} \leq O(1)^{d+k} \| f_{\Sigma_{\leq k}, \varepsilon} \|_{\ell^r(\mathbb{Z}^d; H)}
\]

(4.11)

where for any \( \Sigma \subseteq \Sigma_{\leq k} \) we denote

\[
f_{\Sigma, \varepsilon}(n) := \sum_{\alpha \in \Sigma} f_{\alpha, \varepsilon}(n) = f_{\Sigma}(\hat{i}(n))\varphi(\varepsilon n).
\]

The functions \( f_{\Sigma} \circ \hat{i} \) are all periodic with period \( Q_S \). By Riemann integrability one then has

\[
\varepsilon^{1/2r} \| f_{\Sigma_{\leq k}, \varepsilon} \|_{\ell^r(\mathbb{Z}^d; H)} \to \| f_{\Sigma_{\leq k}} \|_{\ell^r(\mathbb{Z}^d; H)}
\]

and similarly

\[
\varepsilon^{1/2r} \left\| (f_{\alpha + \Sigma A_0, \varepsilon})_{\alpha \in \Sigma(A_0)} \right\|_{L^2(\mathbb{Z}^d; H^{\varepsilon(A_0)})} \to \left\| (f_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \right\|_{L^2(\mathbb{Z}^d; H^{\varepsilon(A_0)})}
\]

as \( \varepsilon' \to 0 \) for any \( A_0 \). Multiplying (4.11) by \( \varepsilon^{1/2r} \) and taking the limit \( \varepsilon \to 0 \), we obtain the claim (4.8).

We now prove (4.9). Let \( F \in L^2(\hat{\mathbb{Z}^d}; H) \), then we have \( T_{1_{\Sigma \leq k}} F = F_{\Sigma_{\leq k}} \), where \( F_{\alpha}(y) := e(-y \cdot \alpha)F_{\mathbb{Z}^d}(\alpha) \) and \( F_{\Sigma} := \sum_{\alpha \in \Sigma} F_{\alpha} \) for any \( \Sigma \subseteq \Sigma_{\leq k} \). By (4.8) we then have

\[
\| T_{1_{\Sigma \leq k}} F \|_{L^2(\hat{\mathbb{Z}^d}; H)} \leq O(r \log^{1/2}(kr))^{k} \left( \sum_{A_0 \in \left( \frac{S}{k} \right)} \left\| (F_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \right\|_{L^2(\mathbb{Z}^d; H^{\varepsilon(A_0)})}^{2r} \right)^{1/2r}
\]

so it will suffice to establish the bound

\[
\left( \sum_{A_0 \in \left( \frac{S}{k} \right)} \left\| (F_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \right\|_{L^p(\mathbb{Z}^d; H^{\varepsilon(A_0)})}^{p} \right)^{1/p} \leq O(1)^d \| F \|_{L^p(\hat{\mathbb{Z}^d}; H)}
\]
for all $2 \leq p \leq \infty$. By interpolation it suffices to establish this for $p = 2$ and $p = \infty$. The claim $p = 2$ is immediate from Bessel’s inequality. For $p = \infty$ it suffices by translation invariance to show that
\[
\left( \sum_{\alpha \in \Sigma(A_0)} \| F_{\alpha + \Sigma A_0}(0) \|_H^2 \right)^{1/2} \leq O(1)^d \| F \|_{L^\infty(\mathbb{Z}, H)}
\]
which by duality is equivalent to the assertion that
\[
\int_{\mathbb{Z}^d} \left\| \sum_{\alpha \in \Sigma(A_0)} c_\alpha \sum_{\alpha_0 \in \Sigma A_0} e(-y \cdot (\alpha + \alpha_0)) \right\|_H^2 \, d\mu_{\mathbb{Z}^d}(y) \leq O(1)^d \left( \sum_{\alpha \in \Sigma(A_0)} \| c_\alpha \|_H^2 \right)^{1/2}
\]
for any $c_\alpha \in H$ for $\alpha \in \Sigma(A_0)$.

Observe that the integrand vanishes unless the projection of $y$ to $(\mathbb{Z}/Q A_0 \mathbb{Z})^d$ vanishes, thus the integrand is supported on a set of measure $Q_{A_0}^d$. By Cauchy-Schwarz, it thus suffices to show that
\[
\int_{\mathbb{Z}^d} \left\| \sum_{\alpha \in \Sigma(A_0)} c_\alpha \sum_{\alpha_0 \in \Sigma A_0} e(-y \cdot (\alpha + \alpha_0)) \right\|_H^2 \, d\mu_{\mathbb{Z}^d}(y) \leq O(1)^d Q_{A_0}^d \sum_{\alpha \in \Sigma(A_0)} \| c_\alpha \|_H^2.
\]
But this is immediate from Plancherel’s theorem since $|\Sigma A_0| = Q_{A_0}^d$. \qed

Now we can prove Theorem 1.6. To abbreviate the notation we write $X \lesssim Y$ for $X \leq \exp(O_c(d) + O(k \log(r \log k))) Y$ and $X \approx Y$ for $X \lesssim Y \lesssim X$.

We first establish the claim in the case $p = 2r$. By a limiting argument we may assume that $f \in \ell^2(\mathbb{Z}^d; H)^{\eta(\Omega)}$, thus we can write $f = \sum_{\alpha \in \Sigma \subseteq_k} f_\alpha$ where
\[
f_\alpha(n) := \int_{[-\varepsilon, \varepsilon]^d} e(-n \cdot (\alpha + \theta)) \hat{f}_{\mathbb{Z}^d}(\alpha + \theta) \, d\theta.
\]
We then have $S_\Omega^{-1} f = \sum_{\alpha \in \Sigma \subseteq_k} F_\alpha$ where
\[
F_\alpha(x, y) := e(-y \cdot \alpha) \int_{[-\varepsilon, \varepsilon]^d} e(-x \cdot \theta) \hat{f}_{\mathbb{Z}^d}(\alpha + \theta) \, d\theta.
\]
Writing $f_\Sigma := \sum_{\alpha \in \Sigma} f_\alpha$ and $F_{\Sigma} := \sum_{\alpha \in \Sigma} F_\alpha$ for any $\Sigma \subseteq \Sigma \subseteq_k$, we see from Theorem 1.2(i) (and bounding $r \log^{1/2}(kr) = \exp(O(\log(r \log k)))$) that
\[
\| f \|_{\ell^2(\mathbb{Z}^d; H)} \approx \left( \sum_{\alpha \in \Sigma \subseteq_k} \| (f_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \|_{\ell^2(\mathbb{Z}^d, H^{\eta(A_0)})}^2 \right)^{1/2r}.
\]
and similarly from Theorem 4.3 that
\[ \|S^{-1}_\Omega f\|_{L^{2r}(\mathbb{A}_d^2; H)} \approx \left( \sum_{A_0 \in \left( S_{\leq k} \right)} \| (F_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \|_{L^{2r}(\mathbb{A}_d^2; H; \Sigma(A_0))} \right)^{1/2r} \]
so it will suffice to show that
\[ \| (f_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \|_{L^{2r}(Z^d; H; \Sigma(A_0))} \approx \| (F_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \|_{L^{2r}(\mathbb{A}_d^2; H; \Sigma(A_0))} \]
for each \( A_0 \in \left( S_{\leq k} \right) \). From expanding the definitions, we see that
\[ (F_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} \in L^2(\mathbb{A}_d^2; H; \Sigma(A_0)) \]
and
\[ (f_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)} = S(F_{\alpha + \Sigma A_0})_{\alpha \in \Sigma(A_0)}. \]
The claim now follows from Proposition 1.5 and Lemma 4.1(i).

Now we establish Theorem 1.6 for general \((2r)^{'} \leq p \leq 2r\). We begin with the upper bound
\[ \|S^{-1}_\Omega f\|_{L^{2r}(\mathbb{A}_d^2; H)} \lesssim \|f\|_{L^p(Z^d; H)} \]
for \( f \in L^p(Z^d; H)^{n}(\Omega) \). With \( \varphi \) as in (4.10), we can write
\[ S^{-1}_\Omega f = S^{-1}_\Omega T_{\varphi; \Sigma \leq k} f \]
where \( \Omega := [-\frac{r'}{2}, \frac{r'}{2}]^d \times \Sigma_{\leq k} \). Note that the right-hand side is well defined for all \( f \) in \( L^p(Z^d; H) \) (with no restriction on the Fourier support on \( f \)). Thus it will suffice to show that
\[ \|S^{-1}_\Omega T_{\varphi; \Sigma \leq k}\|_{B(\ell^p(Z^d; H) \rightarrow L^p(\mathbb{A}_d^2; H))} \lesssim 1 \quad (4.12) \]
for all \((2r)^{'} \leq p \leq 2r\). Now that the Fourier restriction has been removed, interpolation becomes available, and it suffices to establish this bound for \( p = 2r, (2r)^{'} \). For \( p = 2r \) the claim follows from the \( p = 2r \) case of Theorem 1.6 already established (with \( c \) replaced by \( c^{'} \)), noting from Theorem 4.3 that
\[ \|T_{\varphi; \Sigma \leq k}\|_{B(\ell^p(Z^d; H))} \lesssim 1 \quad (4.13) \]
for \( p = 2r \) (indeed, this estimate holds for all \((2r)^{'} \leq p \leq 2r\) by duality and interpolation).

For \( p = (2r)^{'} \), we apply duality to write the estimate in the equivalent form
\[ \|T_{\varphi; \Sigma \leq k} S\|_{B(L^{2r}(\mathbb{A}_d^2; H) \rightarrow \ell^{2r}(Z^d; H))} \lesssim 1. \quad (4.14) \]
From (4.5) we have
\[ T_{\varphi; \Sigma \leq k} S = ST_{\varphi \otimes 1 \Sigma \leq k}. \]
From Theorem 4.3 one has
\[ \|T_{\varphi \otimes 1 \Sigma \leq k}\|_{B(L^{2r}(\mathbb{A}_d^2; H))} = \|T_{\varphi}\|_{B(L^{2r}(\mathbb{R}^d; H))}\|T_{1 \Sigma \leq k}\|_{B(L^{2r}(\mathbb{R}^d; H))} \lesssim 1 \]
and the claim now follows from the $p = 2r$ case of Theorem 1.6 already established (with $c$ replaced by $c'$).

Now we obtain the lower bound
\[
\|S_{\Omega}^{-1}f\|_{L^p(A^d; H)} \lesssim \|f\|_{L^p(\mathbb{Z}^d; H)}
\]
for $f \in L^p(\mathbb{Z}^d; H)^{\pi(\Omega)}$. This is equivalent to
\[
\|SF\|_{L^p(\mathbb{Z}^d; H)} \lesssim \|F\|_{L^p(A^d; H)}
\]
for $F \in L^p(\mathbb{Z}^d; H)^{\Omega}$. For such $F$ we have $SF = ST_{\varphi \otimes 1_{\Sigma \leq k}} F$, so it suffices to show that
\[
\|ST_{\varphi \otimes 1_{\Sigma \leq k}}\|_{B(L^p(A^d; H) \to L^p(\mathbb{Z}^d; H))} \lesssim 1
\]
for all $(2r)' \leq p \leq 2r$. By interpolation it suffices to establish this bound for $p = 2r, (2r)'$. For $p = 2r$ the claim follows from (4.14). For $p = (2r)'$ we dualize to
\[
\|T_{\varphi \otimes 1_{\Sigma \leq k}}S_{\Omega}^{-1}\|_{B(L^p(\mathbb{Z}^d; H) \to L^{2r}(A^d; H))} \lesssim 1
\]
and the claim now follows from (4.13) and Theorem 1.6. This concludes the proof of Theorem 1.6 for general $p$.

Now we can prove Theorem 1.7. Let the notation and hypotheses be as in that theorem, and as before let $\varphi$ be the function (4.6). Then by (1.5) (and Lemma 4.1(i)) we can factorize
\[
T_{m; \Sigma \leq k} = T_{m; \Sigma \leq k} T_{\varphi; \Sigma \leq k} = ST_{m}S_{\Omega}^{-1}T_{\varphi; \Sigma \leq k}.
\]
The claim now follows from (4.13) and Theorem 1.6.

5. The Ionescu–Wainger major arc construction

We now describe the specific choice of major arcs that essentially appears in the original work [7] of Ionescu and Wainger, as well as in many subsequent works.

Lemma 5.1. Let $0 < \rho < 1$ be a parameter, and set $k := \lfloor \frac{2}{\rho} \rfloor + 1$. Suppose that $N \geq 2^k$. Then there exists a set $S$ of pairwise coprime natural numbers such that for any $d \in \mathbb{Z}_+$, and $\varepsilon > 0$, the major arc parameter set $(d, k, S, \varepsilon)$ obeys the following properties:

(i) One has $T_d[q] \subseteq \Sigma \leq k$ for all natural numbers $1 \leq q \leq N$.
(ii) One has $\Sigma \leq k \subseteq T_d[q]$ for some $Q \leq 3^N$. 


(iii) All elements of $S$ are bounded by $C^k N^{\rho/2}$ for some absolute constant $C > 1$. In particular, $(d, k, S, \varepsilon)$ will be $(r, \frac{1}{2})$-good whenever

$$\varepsilon < \frac{1}{4rC2^kN^{\rho/2}}.$$  

(5.1)

(iv) $\Sigma \leq k$ is the union of finitely many subgroups of $T^d$, each of the form $T^d[\ell_q]$ for some $q \leq O(1)^{k^2N^{\rho/2}}$. In particular, $|\Sigma| \leq O(1)^{dk^2N^{\rho/2}}$.

Typically $\rho$ (and hence $k$) and $r$ will be fixed in applications. For $N$ sufficiently large depending on $\rho, r, d$, the condition (5.1) can be simplified to $\varepsilon \leq \exp(-N^{\rho})$, and the bounds $q \leq O(1)^{k^2N^{\rho/2}}, \ |\Sigma| \leq \exp(N^\rho)$, the main point here is we can cover the Farey sequence $\bigcup_{1 \leq q \leq N} T^d[q]$ by good major arcs whose width $\varepsilon$ can be as large as $\exp(-N^\rho)$.

Proof. We set $S$ equal to

$$S := \{ \prod_{p \leq N^{\rho/2}} p^{\frac{\log N}{\log p}} \} \cup \{ p^{\frac{\log N}{\log p}} : N^{\rho/2} < p \leq N \}$$

where $p$ is always understood to be restricted to the primes. Clearly the elements of $S$ are pairwise coprime. To prove (i), we have to show that every natural number $1 \leq q \leq N$ is a factor of a product of at most $k$ distinct elements from $S$. But by the fundamental theorem of arithmetic we can write $q = p_1^{a_1} \cdots p_m^{a_m}$ for some primes $1 < p_1 < \cdots < p_m \leq N$ and $1 \leq a_i \leq \frac{\log N}{\log p_i}$. At most $\left\lfloor \frac{2}{\rho} \right\rfloor = k - 1$ of these primes can exceed $N^{\rho/2}$. One can then write $q$ as a factor of $\prod_{p \leq N^{\rho}} p^{\frac{\log N}{\log p}}$ times at most $k - 1$ terms of the form $p^{\frac{\log N}{\log p}}$, giving the claim.

The product $Q_S$ of all the elements of $S$ is equal to

$$\prod_{p \leq N} p^{\frac{\log N}{\log p}} = \text{lcm}(1, \ldots, N) \leq 3^N$$

where the latter inequality is established in [5]. Since $\Sigma \subseteq T^d[Q_S]$, this gives (ii).

For (iii), we trivially bound $p^{\frac{\log N}{\log p}}$ by $N$, and note from the prime number theorem that the number of primes less than $N^{\rho/2}$ is $O(N^{\rho/2}/\log N^{\rho/2}) = O(kN^{\rho/2}/\log N)$, giving (iii) as claimed (noting from the hypothesis $N \geq 2^k$ that $N^{\rho/2} \geq 2^{1/2}$ and hence $N \leq N^O(N^{\rho/2}/\log N^{\rho/2}) = O(1)^{kN^{\rho/2}}$).
Finally to prove (iv), note from definition that $\Sigma_{\leq k}$ is the union of $T^d[q]$ where $q$ is the product of at most $k$ elements of $S$, and the claim now follows from (iii).

As a particular corollary of this construction, we can prove a sampling theorem for the classical major arcs.

**Corollary 5.2 (Classical major arc sampling).** Let $d, N \in \mathbb{Z}_+$, $\varepsilon > 0$, and set

$$\Omega := [-\varepsilon, \varepsilon]^d \times \bigcup_{q=1}^N T^d[q].$$

Let $0 < \rho < 1$ and $1 < p < \infty$ be such that

$$\varepsilon < \exp(-C \max(p, p')\rho^{-2}N^{p/2})$$

(5.2)

for a sufficiently large absolute constant $C$. Then for any finite-dimensional Hilbert space $H$, one has

$$\|S^{-1}_\Omega f\|_{L^p(A^d;H)} = \exp(O(d + \rho^{-1}\log\max(p, p')\log\rho^{-1}))\|f\|_{L^p(\mathbb{Z}^d;H)}$$

for all $f \in \ell^p(\mathbb{Z}^d;H)^\pi(\Omega)$.

**Proof.** Set $r$ to be the first natural number such that $(2r)' \leq p \leq 2r$, then $r \sim \max(p, p')$, and let $S$ be the set constructed by Lemma 5.1 then $\Omega \subseteq [-\varepsilon, \varepsilon]^d \times \Sigma_{\leq k}$. If the constant $C$ in (5.2) is large enough, Lemma 5.1(iii) ensures that $(d, k, S, \varepsilon)$ is $(r, \frac{1}{2})$-good, and the claim now follows from Theorem 1.6. □

**Appendix A. Abstract harmonic analysis**

We define the Pontryagin dual pairs $(G, G^*)$ of LCA groups used in this paper.

(i) If $G = \mathbb{R}$ with Lebesgue measure $\mu_\mathbb{R} = dx$, then $G^* = \mathbb{R}^* = \mathbb{R}$ with Lebesgue measure $\mu_{\mathbb{R}^*} = d\xi$ is a Pontryagin dual, with pairing $x \cdot \xi := x\xi$ mod 1.

(ii) If $G = \mathbb{Z}$ with counting measure $\mu_\mathbb{Z} = d\xi$, then $G^* = \mathbb{T}$ with counting measure $\mu_{\mathbb{T}} = d\xi$ is a Pontryagin dual, with pairing $x \cdot \xi := x\xi$.

(iii) If $G = \mathbb{Z}/Q\mathbb{Z}$ is a cyclic group for some $Q \in \mathbb{Z}_+$ with normalized counting measure $f(x) := \frac{1}{Q}\mathbb{E}_{x \in \mathbb{Z}/Q\mathbb{Z}} f(x)$, then the dual cyclic group $G^* = \mathbb{T}[Q] = \frac{1}{Q}\mathbb{Z}/\mathbb{Z}$ with counting measure $\mu_{\mathbb{T}[Q]}$ is a Pontryagin dual, with pairing $x \cdot \xi := x\xi$. 

□
(iv) If $G = \hat{\mathbb{Z}} := \lim_{\rightarrow} \mathbb{Z}/Q\mathbb{Z}$ is the compact group of profinite integers with Haar probability measure (using the projection maps from $\mathbb{Z}/Q\mathbb{Z}$ to $\mathbb{Z}/q\mathbb{Z}$ whenever $q$ divides $Q$), then the discrete group $G^* = \hat{\mathbb{Z}}^* = \mathbb{Q}/\mathbb{Z}$ of “arithmetic frequencies” with counting measure $\mu_{\mathbb{Q}/\mathbb{Z}}$ is a Pontryagin dual, with pairing $x \cdot (\frac{a}{q} \mod 1) := \frac{x a \mod q}{q}$.

(v) If $G_1, G_2$ are LCA groups with Pontryagin duals $G_1^*, G_2^*$, then the product $G_1 \times G_2$ (with product Haar measure) is an LCA group with Pontryagin dual $G_1^* \times G_2^*$ and pairing $(x_1, x_2) \cdot (\xi_1, \xi_2) := x_1 \cdot \xi_1 + x_2 \cdot \xi_2$. In particular, if $G = A^d = \mathbb{R}^d \times \hat{\mathbb{Z}}^d$ is the $d$th power of the adelic integers $A_\mathbb{Z} := \mathbb{R} \times \hat{\mathbb{Z}}$ (with the product Haar measure $\mu_{A^d} := \mu_\mathbb{R} \times \mu_{\hat{\mathbb{Z}}^d}$), then an adelic frequency space $G^* = (A^d_\mathbb{Z})^* = \mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d$ is a Pontryagin dual (with product measure $\mu_{\mathbb{R} \times \mathbb{Q}/\mathbb{Z}} := \mu_\mathbb{R} \times \mu_{\mathbb{Q}/\mathbb{Z}}$ and the indicated pairing).

More explicitly: an element of $A^d_\mathbb{Z}$ is of the form $(x, y)$, where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $y = (y_1, \ldots, y_d) \in \hat{\mathbb{Z}}^d$, thus $y_d \mod Q$ is an element of $\mathbb{Z}/Q\mathbb{Z}$ for any positive integer $Q$ (with the compatibility conditions $y_d \mod q = (y_d \mod Q) \mod q$ whenever $q$ divides $Q$), and if $(\xi, \eta) = (\xi_1, \ldots, \xi_d, \frac{a_1}{q} \mod 1, \ldots, \frac{a_d}{q} \mod 1)$ is an element of the dual group $\mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d$, then

\[
(x, y) \cdot (\xi, \eta) = x \cdot \xi + y \cdot \eta
= x_1 \xi_1 + \cdots + x_d \xi_d + a_1 y_1 \mod q + \cdots + a_d y_d \mod q.
\]

We have the canonical inclusion $\iota: \mathbb{Z}^d \to A^d_\mathbb{Z}$ defined by

\[
\iota(n) := (n, (n \mod Q)_{Q \in \mathbb{Z}_+})
\]

and the projection map $\pi: \mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d \to \mathbb{T}^d$ defined by

\[
\pi(\theta, \alpha) := \alpha + \theta;
\]

the two maps enjoy the Fourier adjoint relationship

\[
n \cdot \pi(\theta, \alpha) = \iota(n) \cdot (\theta, \alpha)
\]

for all $n \in \mathbb{Z}^d$ and $(\theta, \alpha) \in \mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d$.

We define the following Schwartz-Bruhat spaces $S(G)$ on various LCA groups:

(i) $S(\mathbb{R}^d)$ is the space of Schwartz functions on $\mathbb{R}^d$.

\footnote{The adelic integers $A_\mathbb{Z}$ should not be confused with the larger ring $A_\mathbb{Q} = A_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ of adelic numbers, which we will not use in this paper.}

\footnote{For a definition of Schwartz-Bruhat spaces on arbitrary LCA groups, see \cite{[13]}.}
(ii) \( S(\mathbb{Z}^d) \) is the space of rapidly decreasing functions on \( \mathbb{Z}^d \), and \( S(\mathbb{T}^d) \) is the space of smooth functions on \( \mathbb{T} \).

(iii) \( S(\hat{\mathbb{Z}}^d) \) is the space of locally constant functions \( f \) on \( \hat{\mathbb{Z}}^d \), or equivalently those functions of the form \( f(x) = f_Q(x \mod Q) \) for some \( Q \in \mathbb{Z}_+ \) and \( f_Q : (\mathbb{Z}/Q\mathbb{Z})^d \to \mathbb{C} \). \( S((\mathbb{Q}/\mathbb{Z})^d) \) is the space of finitely supported functions on \( (\mathbb{Q}/\mathbb{Z})^d \).

(iv) \( S(A_\mathbb{Z}^d) \) is the space of functions of the form \( f(x, y) = f_Q(x, y \mod Q) \) for some \( Q \in \mathbb{Z}_+ \) and \( f_Q : \mathbb{R}^d \times (\mathbb{Z}/Q\mathbb{Z})^d \) that is Schwartz in the \( \mathbb{R} \) variable. \( S(\mathbb{R}^d \times (\mathbb{Q}/\mathbb{Z})^d) \) is the space of functions supported on \( \mathbb{R}^d \times \Sigma \) for some finite set \( \Sigma \subset (\mathbb{Q}/\mathbb{Z})^d \) and Schwartz in the \( \mathbb{R}^d \) variable.

**Appendix B. Losses in the sampling principle**

In this appendix we demonstrate some losses in the Maygar–Stein–Wainger sampling principle (Proposition 1.2) that demonstrate that the \( O(1)^d \) factor in (1.2) is necessary, at least when \( p \) is close to 1 or \( \infty \).

**Proposition B.1** (Losses in the sampling principle). Let \( 1 \leq p \leq \infty \) be sufficiently close to 1 or \( \infty \). Then there is a constant \( C > 1 \) such that for any \( d \in \mathbb{Z}_+ \), there exists a multiplier \( m \in C^\infty_c(\mathbb{R}^d) \) supported in \( [-\frac{1}{2}, \frac{1}{2}]^d \) for which

\[
\| T_m; 0 \|_{B(\ell^p(\mathbb{Z}^d))} > C^d \| T_m \|_{B(L^p(\mathbb{R}^d))}.
\]  

(B.1)

**Proof.** By duality we may assume \( p \) sufficiently close to 1.

We first observe that it will suffice to establish the claim for \( d = 1 \), since if we can find a one-dimensional multiplier \( m \in C^\infty_c(\mathbb{R}) \) supported in \( [-\frac{1}{2}, \frac{1}{2}] \) with

\[
\| T_m; 0 \|_{B(\ell^p(\mathbb{Z}))} > C \| T_m \|_{B(L^p(\mathbb{R}))}
\]

then by taking tensor products (and many applications of the Fubini–Tonelli theorem) we see that \( m^\otimes d \in C^\infty_c(\mathbb{R}^d) \) is supported in \( [-\frac{1}{2}, \frac{1}{2}]^d \) with \( T_m^\otimes d \) the \( d \)-fold tensor product of \( T_m \), and \( T_m^\otimes d; 0 \) similarly the \( d \)-fold tensor product of \( T_m; 0 \), and hence (by many further applications of Fubini–Tonelli)

\[
\| T_m^\otimes d; 0 \|_{B(\ell^p(\mathbb{Z}^d))} = \| T_m; 0 \|_{B(\ell^p(\mathbb{Z}^d))}^d \| T_m \|_{B(L^p(\mathbb{R}))}^d = C^d \| T_m^\otimes d \|_{B(L^p(\mathbb{R}^d))}
\]

giving the claim.

It remains to establish the claim for \( d = 1 \). Suppose for contradiction that the claim fails, then there exists a sequence of \( p > 1 \) converging
to 1 such that we have
\[ \|T_{m;0}\|_{B(p(Z))} \leq \|T_m\|_{B(L^p(\mathbb{R}))} \]
for all \( m \in C_c^\infty(\mathbb{R}) \) supported on \([-1/2, 1/2]\). From the Riesz-Thorin theorem, the \( B(L^p(\mathbb{R})) \) norm is continuous in \( p \), thus on taking limits we conclude that
\[ \|T_{m;0}\|_{B(\ell^1(\mathbb{Z}))} \leq \|T_m\|_{B(L^1(\mathbb{R}))}. \]
But the multiplier operator \( T_m \) is convolution with \( \mathcal{F}_{-1}^{-1}R_m \), hence
\[ \|T_m\|_{B(L^1(\mathbb{R}))} = \|\mathcal{F}_{-1}^{-1}R_m\|_{L^1(\mathbb{R})}. \]
Similarly, from the Poisson summation formula \( T_{m;0} \) is convolution with the restriction of \( \mathcal{F}_{-1}^{-1}R_m \) to the integers, thus we have
\[ \|T_{m;0}\|_{B(\ell^1(\mathbb{Z}))} = \|\mathcal{F}_{-1}^{-1}m|_\mathbb{Z}\|_{\ell^1(\mathbb{Z})}, \]
thus
\[ \sum_{n \in \mathbb{Z}} |\mathcal{F}_{-1}^{-1}m(n)| \leq \int_{\mathbb{R}} |\mathcal{F}_{-1}^{-1}m(\xi)| \, d\xi \]
for all \( m \in C_c^\infty(\mathbb{R}) \) supported on \([-1/2, 1/2]\). Since the class of \( m \) is invariant under modulations we conclude that
\[ \sum_{n \in \mathbb{Z}} |\mathcal{F}_{-1}^{-1}m(n + \theta)| \leq \int_{\mathbb{R}} |\mathcal{F}_{-1}^{-1}m(\xi)| \, d\xi \]
for all \( 0 \leq \theta \leq 1 \). However we have
\[ \int_{0}^{1} \sum_{n \in \mathbb{Z}} |\mathcal{F}_{-1}^{-1}m(n + \theta)| \, d\theta = \int_{\mathbb{R}} |\mathcal{F}_{-1}^{-1}m(\xi)| \, d\xi \]
and the integrand on the left-hand side is continuous in \( \theta \), thus we have
\[ \sum_{n \in \mathbb{Z}} |\mathcal{F}_{-1}^{-1}m(n + \theta)| = \int_{\mathbb{R}} |\mathcal{F}_{-1}^{-1}m(\xi)| \, d\xi \]
for all \( m \in C_c^\infty(\mathbb{R}) \) supported on \([-1/2, 1/2]\) and \( 0 \leq \theta \leq 1 \). In particular
\[ \sum_{n \in \mathbb{Z}} |\mathcal{F}_{-1}^{-1}m(n)| = \int_{\mathbb{R}} |\mathcal{F}_{-1}^{-1}m(\xi)| \, d\xi. \quad (B.2) \]
If we formally set \( m = 1_{[-1/2, 1/2]} \), then \( \mathcal{F}_{-1}^{-1}m(\xi) = \frac{\sin(\pi \xi)}{\pi \xi} \), and the left-hand side evaluates to 1 and the right-hand side is logarithmically divergent. Of course, the indicator function \( 1_{[-1/2, 1/2]} \) is not smooth, but by applying a standard mollification to this function one can then easily construct a counterexample to (B.2), giving the claim.  

With some numerical effort one could refine the above argument to obtain an explicit range \( p \in [1, 1 + \varepsilon] \cup [(1 + \varepsilon)', \infty) \) of exponents \( p \) and an explicit constant \( C > 1 \) for which the above proposition holds, but we will not do so here. It seems likely that this proposition in fact
holds for all $p \neq 2$ (presumably with a constant $C$ that converges to 1 as $p \to 2$), but constructing a concrete counterexample may require locating a multiplier $m$ for which both the continuous operator norm $\|T_m\|_{B(L^p(\mathbb{R}))}$ and the discrete operator norm $\|T_{m,0}\|_{B(\ell^p(\mathbb{Z}))}$ are explicitly computable (or at least bounded above and below with extreme precision), and examples of such multipliers are rare. Some variant of the continuous and discrete Hilbert transforms seem like the most promising candidates for this task; see [2] for some recent progress in this direction.

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UCLA Department of Mathematics, Los Angeles, CA 90095-1555.

E-mail address: tao@math.ucla.edu