A Prediction–Correction ADMM for Multistage Stochastic Variational Inequalities

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Abstract
The multistage stochastic variational inequality is reformulated into a variational inequality with separable structure through introducing a new variable. The prediction–correction ADMM which was originally proposed in He et al. (J Comput Math 24:693–710, 2006) for solving deterministic variational inequalities in finite-dimensional spaces is adapted to solve the multistage stochastic variational inequality. Weak convergence of the sequence generated by that algorithm is proved under the conditions of monotonicity and Lipschitz continuity. When the sample space is a finite set, the corresponding multistage stochastic variational inequality is actually defined on a finite-dimensional Hilbert space and the strong convergence of the algorithm naturally holds true. Some numerical examples are given to show the efficiency of the algorithm.

Keywords Multistage stochastic variational inequality · Monotonicity · Alternating direction method of multiplier · Nonanticipativity · Weak convergence

Mathematics Subject Classification 65K10 · 65K15 · 90C25 · 90C15

1 Introduction
Variational inequalities (VIs), as the first-order necessary conditions of convex programming problems, equilibrium problems and optimal control problems, provide us
with a powerful tool to solve various types of application problems. The deterministic variational inequalities have been extensively studied from the aspects of both the theories and algorithms in the past decades, for more details, see [10] and [11].

In many practical problems, especially in finance, economy and management, the decision makers have to face the uncertainty brought by some stochastic factors. When the impact of the stochastic factors cannot be ignored, the deterministic variational inequalities may not be the suitable/precise models for those problems. In order to model the problems with uncertainty, in recent years, various types of stochastic variational inequalities (SVIs) are proposed.

The first class of stochastic variational inequalities which has attracted the attention of scholars is the one-stage stochastic variational inequalities. Based on whether the stochastic information is known or not before the decision-making, three kinds of one-stage stochastic variational inequalities are proposed, i.e., the Wait-and-See model, the Expected-Value model [13] and the Expected-Residual-Minimization model [5].

In a Wait-and-See model, the stochastic information is assumed to be known when one makes a decision, and the solution to such stochastic variational inequality is a response function of scenarios or a function of a certain random variable. Solving the Wait-and-See stochastic variational inequality is equivalent to solving individually a collection of variational inequalities with stochastic parameters. The Expected-Value stochastic variational inequality is a deterministic variational inequality with the map represented by the expected value of some map with stochastic parameters. In the Expected-Value model, the decision should be made before the stochastic information is observed and hence the decision set is independent of the stochastic factors. The basic idea of Expected-Residual-Minimization model comes from finding a common (deterministic) solution to a collection of variational inequalities with stochastic parameters. In general, such common solution might not exist. The Expected-Residual-Minimization model focuses on finding a (deterministic) solution through minimizing the expectation of a residual function for parameterized variational inequalities. Particularly, when the optimal value of that minimization problem is zero, any minimizer is almost surely a common solution of the corresponding parameterized variational inequalities. Both the Expected-Value model and the Expected-Residual-Minimization model are called the Here-and-Now model.

The one-stage model does not take into account the increasing levels of observed information in the process of decision-making. In practice, there exist a large number of problems in which the decisions can be made step by step. Clearly, in such situation, the observed information until the current step may affect the decision of next step. In other words, the decision in each step should be a response to the historical observation data. In order to model the dynamical decision problems that the decision makers can use the historical observation data at each step when he/she makes a decision, the two-stage and multistage stochastic variational inequalities are proposed.

Under the assumption that the uncertainty is described by a random vector and the random vector can be observed completely in the second stage, the two-stage stochastic variational inequality is to find a pair of solutions to a coupled stochastic variational inequality system: a Here-and-Now solution to an Expected-Value model in stage one and a solution to the Wait-and-See model in stage two. We refer the readers to [6] and [29] for the definitions and examples of the two-stage stochastic
variational inequalities. In [26], the authors first introduced the notion of multistage stochastic variational inequality with nonanticipativity constraints. It is shown that, both the one-stage Expected-Value type stochastic variational inequality and the two-stage stochastic variational inequality are contained in the framework of the multistage model of [26].

There are two main approaches to solve the two-stage and multistage stochastic variational inequalities. The first one is the sample average approximation (SAA) method. The SAA method was first proposed for solving stochastic programming problems (see, e.g., [27]), one-stage stochastic variational inequalities of Expected-Value model (see, e.g., [13]) and the one-stage stochastic variational inequalities of Expected-Residual-Minimization model (see, e.g., [5, 9]). Then it was extended to solve the two-stage stochastic variational inequalities and two-stage stochastic generalized equations, see [6, 7]. The second one is the progressive hedging algorithm (PHA). The PHA was first proposed in [25] for solving the stochastic programming problem with nonanticipativity constraint and then adapted by Rockafellar and Sun in [22] to solve the two-stage and multistage stochastic variational inequalities in the discrete cases. Recently, the PHA was developed to solve various types of two-stage and multistage stochastic variational inequalities and stochastic games, see, for instance, [23, 31, 34]. When the probability space under consideration is not discrete, [8] proposed a discrete approximation method for two-stage stochastic linear complementarity problems.

In this paper, we shall adapt the definition of the multistage stochastic variational inequalities given in [26]. A slight difference is that, rather than the discrete cases, we shall give the definition of multistage stochastic variational inequalities in a general probability space. One main motivation of extending the concept of multistage stochastic variational inequalities into the general probability space is for its particular use in stochastic optimal control problems. We refer the readers to Example 2.2 for more details. We shall see that the multistage stochastic variational inequality is indeed a variational inequality defined on the Hilbert space of square-integrable random vectors. Theoretically, it can be solved by the projection-type algorithms for deterministic variational inequalities. The main difference between the deterministic variational inequality and the multistage stochastic variational inequality is that, in the stochastic case, the calculation of the projection onto a subset of random vectors is much more complicated. Especially, to find the projection of a random vector onto the nonanticipativity subspace one needs to compute a collection of conditional expectations, which is quite different from the usual metric projection onto a nonempty closed convex set (see Remark 3.1 for more details).

One of the key idea of PHA is treating the metric projection onto a nonempty closed convex subset and the projection onto the nonanticipativity subspace separately. It was shown in [30] and [19] that the PHA is equivalent to the alternating direction method of multipliers (ADMM). The ADMM, as an extension of inexact augmented Lagrangian method (ALM), was first proposed by Glowinski and Marrocco in [12]. In the past few decades, the ADMM and its various extensions have been deeply studied by many scholars for solving mathematical programming problems and variational inequality problems under the deterministic framework. For related work, we refer the reader to the review articles [3] and [14].
Note that the PHA (or the ADMM) for multistage stochastic variational inequalities is an implicit iterative algorithm. In each iteration one needs to solve a collection of variational inequalities with stochastic parameters. In this paper, we shall adapt the prediction–correction ADMM, which originally proposed in [15] for solving deterministic finite-dimensional variational inequalities with separable structures, to solve the multistage stochastic variational inequalities. Different from the PHA, the prediction–correction ADMM is an explicit iterative algorithm, in which the calculation of each iteration becomes relatively easier than that of PHA. The weak convergence of the prediction–correction ADMM for multistage stochastic variational inequalities is proved in the general probability space under the conditions of monotonicity and Lipschitz continuity. When the sample space is a finite set, as discussed in [26], the multistage stochastic variational inequalities is actually defined on a finite-dimensional Hilbert space and the strong convergence of the algorithm naturally holds true. Two numerical examples are given in such case to show the efficiency of the algorithm.

The rest of this paper is organized as follows. Some basic notions and results in probability, set-valued and variational analysis and multistage stochastic variational inequalities are given in Sect. 2. In Sect. 3, the prediction–correction ADMM for multistage stochastic variational inequalities and its weak convergence are studied. The discrete cases and numerical examples are discussed in Sect. 4. Some concluding remarks are given in Sect. 5. A proof of a technical result is given in the Appendix.

2 Preliminaries

In this section, we recall some basic notions and preliminary results in probability and set-valued and variational analysis. Then we give the definition of the multistage stochastic variational inequalities in the general probability space.

2.1 Concepts and Results in Probability

First, we recall some basic concepts and results in probability. We refer the readers to [28] for more details.

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. Here \(\Omega\) is the sample space, \(\mathcal{F}\) is a \(\sigma\)-field defined on \(\Omega\), and \(P\) is a probability measure defined on \((\Omega, \mathcal{F})\). Any element of \(\Omega\), denoted by \(\omega\), is called a sample point. As usual, when the context is clear, we omit the \(\omega \in \Omega\) argument in the defined maps/functions. Denote by \(\emptyset\) the collection of all \(P\)-null sets. We say that a property holds almost surely (a.s.) if there is a set \(A \in \emptyset\) such that the property holds for every \(\omega \in \Omega \setminus A\). Denote by \(\mathcal{F}_0\) the trivial \(\sigma\)-field which only contains \(\emptyset\) and \(\Omega\). Let \(\mathcal{A}\) be a family of subsets of \(\Omega\). The smallest \(\sigma\)-field containing \(\mathcal{A}\), denoted by \(\sigma(\mathcal{A})\), is called the \(\sigma\)-field generated by \(\mathcal{A}\). For a metric space \(Y\), denote by \(\mathcal{B}(Y)\) the Borel \(\sigma\)-field of \(Y\). Let \(\mathbb{R}^n\) be an \(n\)-dimensional Euclidean space with Borel \(\sigma\)-field \(\mathcal{B}(\mathbb{R}^n)\). A map \(\xi : \Omega \to \mathbb{R}^n\) is called an \(\mathcal{F}\)-measurable random vector if

\[
\xi^{-1}(A) := \{\omega \in \Omega \mid \xi(\omega) \in A\} \in \mathcal{F}, \quad \forall A \in \mathcal{B}(\mathbb{R}^n).
\]
Denote by \( L^0(\Omega, \mathcal{F}, R^n) \) the set of all \( \mathcal{F} \)-measurable random vectors. When a random vector \( \xi \in L^0(\Omega, \mathcal{F}, R^n) \) is integrable with respect to the probability measure \( P \), we call the integral \( E \xi = \int_{\Omega} \xi(\omega) P(d\omega) \) the expectation of \( \xi \). Let \( \langle \cdot, \cdot \rangle \) and \( |\cdot| \) be, respectively, the inner product and norm in \( R^n \). Denote by \( L^2(\Omega, \mathcal{F}, R^n) \) the Hilbert space of all the square-integrable random vectors taking values in \( R^n \), i.e.,

\[
L^2(\Omega, \mathcal{F}, R^n) = \left\{ \xi \in L^0(\Omega, \mathcal{F}, R^n) \mid E|\xi|^2 < +\infty \right\}. \tag{2.1}
\]

For any \( \xi, \eta \in L^2(\Omega, \mathcal{F}, R^n) \), the norm of \( \xi \) is defined by

\[
\|\xi\|_{L^2} := \left[ E|\xi|^2 \right]^{\frac{1}{2}} = \left[ \int_{\Omega} |\xi(\omega)|^2 P(d\omega) \right]^{\frac{1}{2}},
\]

and the inner product of \( \xi \) and \( \eta \) is defined by

\[
\langle \xi, \eta \rangle_{L^2} := E\langle \xi, \eta \rangle = \int_{\Omega} \langle \xi(\omega), \eta(\omega) \rangle P(d\omega).
\]

Let \( \{x^k\}_{k=1}^{\infty} \) be a sequence in \( L^2(\Omega, \mathcal{F}, R^n) \). We write \( x^k \rightharpoonup x \) to indicate that the sequence \( \{x^k\}_{k=1}^{\infty} \) converges weakly to \( x \) and \( x^k \to x \) to indicate that the sequence \( \{x^k\}_{k=1}^{\infty} \) converges strongly to \( x \).

**Definition 2.1** [28, Definition 1, P. 213] Let \( \xi \in L^0(\Omega, \mathcal{F}, R^n) \), \( E|\xi| < +\infty \) and \( \mathcal{G} \) be a sub-\( \sigma \)-field of \( \mathcal{F} \). The conditional expectation of \( \xi \) with respect to the \( \sigma \)-field \( \mathcal{G} \), denoted by \( E[\xi | \mathcal{G}] \), is a random vector such that

(i) \( E[\xi | \mathcal{G}] : \Omega \to R^n \) is \( \mathcal{G} \)-measurable;
(ii) \( \int_A \xi(\omega) P(d\omega) = \int_A E[\xi | \mathcal{G}](\omega) P(d\omega), \ \forall \ A \in \mathcal{G} \).

The conditional expectation has the following basic properties.

**Lemma 2.1** [28, P.215] Suppose that \( \xi, \eta \in L^0(\Omega, \mathcal{F}, R^n) \), \( E|\xi| < +\infty \), \( E|\eta| < +\infty \), \( E \left| \langle \xi, \eta \rangle \right| < +\infty \), \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) and \( \mathcal{G} \) is a sub-\( \sigma \)-field of \( \mathcal{F} \). Then, the following assertions hold true:

(i) If \( v \) is a constant vector and \( \xi = v \) a.s., then \( E[\xi | \mathcal{G}] = v \) a.s.;
(ii) If \( \xi \) is \( \mathcal{G} \)-measurable, then \( E[\xi | \mathcal{G}] = \xi \) a.s.;
(iii) \( E[\xi | \mathcal{F}_0] = E[\xi] \) a.s.;
(iv) \( E[E[\xi | \mathcal{G}]] = E[\xi] \);
(v) If \( \xi \) is \( \mathcal{G} \)-measurable, then \( E[\langle \xi, \eta \rangle | \mathcal{G}] = \langle \xi, E[\eta | \mathcal{G}] \rangle \) a.s.

**Remark 2.1** Let \( \mathcal{G} \) be a sub-\( \sigma \)-field included in \( \mathcal{F} \). Define

\[
L^2(\Omega, \mathcal{G}, R^n) = \left\{ \xi \in L^2(\Omega, \mathcal{F}, R^n) \mid \xi \text{ is } \mathcal{G} \text{-measurable} \right\},
\]
and let $\xi \in L^2(\Omega, \mathcal{F}, R^n)$. Clearly, $L^2(\Omega, \mathcal{G}, R^n)$ is a closed linear subspace of $L^2(\Omega, \mathcal{F}, R^n)$ and, by Lemma 2.1,
\[
E\langle \xi - E[\xi | \mathcal{G}], v \rangle = E\langle \xi, v \rangle - E\{E[\langle \xi, v \rangle | \mathcal{G}]\} = 0, \quad \forall \, v \in L^2(\Omega, \mathcal{G}, R^n).
\]
Then, for any $\eta \in L^2(\Omega, \mathcal{G}, R^n)$, by $\eta - E[\xi | \mathcal{G}] \in L^2(\Omega, \mathcal{G}, R^n)$ we have
\[
E\vert \xi - E[\xi | \mathcal{G}] \vert^2 = E\vert \xi - \eta \vert^2 + E\vert \eta - E[\xi | \mathcal{G}] \vert^2 + 2E\langle \xi - \eta, \eta - E[\xi | \mathcal{G}] \rangle
\leq E\vert \xi - \eta \vert^2.
\]
Therefore, $E[\xi | \mathcal{G}]$ is the metric projection of $\xi$ onto the subspace $L^2(\Omega, \mathcal{G}, R^n)$ in the sense of the $L^2$ norm.

**Example 2.1** [28, P.78] Assume that the $\sigma$-field $\mathcal{G}$ is generated by disjoint subsets $A_1, A_2, \ldots, A_m$ ($m \in \mathbb{N}$) with $A_i \in \mathcal{F}$, $P(A_i) > 0$, $i = 1, 2, \ldots, m$ and $\Omega = \bigcup_{i=1}^{m} A_i$. Then, for any $\xi \in L^2(\Omega, \mathcal{F}, R^n)$,
\[
E[\xi | \mathcal{G}] = \sum_{i=1}^{m} \frac{E[\xi \chi_{A_i}]}{P(A_i)} \chi_{A_i},
\]
where $\chi_{A_i}$ is the characteristic function of $A_i$, i.e.,
\[
\chi_{A_i}(\omega) = \begin{cases} 1, & \omega \in A_i, \\ 0, & \omega \notin A_i, \end{cases} \quad \forall \, i = 1, 2, \ldots, m.
\]
Let $B_1, B_2, \ldots, B_\ell$ ($\ell \in \mathbb{N}$) be disjoint subsets, $u_1, u_2, \ldots, u_\ell \in R^n$ and define $\xi = \sum_{j=1}^{\ell} u_j \chi_{B_j}$. Then we have
\[
E[\xi | \mathcal{G}] = \sum_{i=1}^{m} \sum_{j=1}^{\ell} u_j P(B_j | A_i) \chi_{A_i},
\]
where $P(B_j | A_i) = P(A_i \cap B_j) / P(A_i)$ is the probability of $B_j$ under the condition that $A_i$ has occurred.

### 2.2 Set-Valued and Variational Analysis

In this subsection, we introduce some elemental results in set-valued and variational analysis. We refer the readers to [1] for more details.

Let $(X, \mathcal{X}, \mu)$ be a complete $\sigma$-finite measure space, $Y$ be a complete separable metric space. A set-valued map $\Phi : X \rightrightarrows Y$ is characterized by its graph $Gph(\Phi)$, a
subset of the product space $X \times Y$ defined by
\[
Gph(\Phi) := \{(x, y) \in X \times Y \mid y \in \Phi(x)\}.
\]
For any $x \in X$, the $\Phi(x)$, which is a subset of $Y$, is called the value of $\Phi$ at $x$. The domain of $\Phi$ is the subset of elements $x \in X$ such that $\Phi(x)$ is nonempty, i.e., $\text{Dom}(\Phi) := \{x \in X \mid \Phi(x) \neq \emptyset\}$. The image of $\Phi$ is defined by $\text{Im}(\Phi) := \bigcup_{x \in X} \Phi(x)$. Suppose that $\Phi : X \rightrightarrows Y$ is a set-valued map with closed values. $\Phi$ is called $\mathcal{S}$-measurable if
\[
\Phi^{-1}(A) := \{x \in X \mid \Phi(x) \cap A \neq \emptyset\} \in \mathcal{S}, \quad \forall A \in \mathcal{B}(Y).
\]
A single-valued map $\varphi : X \to Y$ is called a measurable selection of $\Phi$ if $\varphi$ is $\mathcal{S}$-measurable and $\varphi(x) \in \Phi(x)$ for $\mu$-a.e. $x \in X$.

**Lemma 2.2** [1, Theorem 8.1.3, page 308] Let $(X, \mathcal{S}, \mu)$ be a complete $\sigma$-finite measure space, $Y$ a complete separable metric space, $\Phi : X \rightrightarrows Y$ a measurable set-valued map with nonempty closed values. Then there exists a measurable selection of $\Phi$.

Next, we introduce the concepts of monotonicity for a set-valued map. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|_\mathcal{H}$ and inner product $\langle \cdot, \cdot \rangle_\mathcal{H}$.

**Definition 2.2** [1, Definition 3.5.1,3.5.4] A set-valued map $\Phi : \mathcal{H} \rightrightarrows \mathcal{H}$ is monotone if its graph is monotone in the sense that
\[
\langle u - v, x - y \rangle_\mathcal{H} \geq 0, \quad \forall (x, u), (y, v) \in Gph(\Phi).
\]
A set-valued map $\Phi$ is maximal monotone if

(i) $\Phi$ is monotone;

(ii) there is no other monotone set-valued map whose graph strictly contains the graph of $\Phi$.

**Lemma 2.3** Let $\Phi$ be maximal monotone. Then its graph is weakly-strongly closed in the sense that if $x_n$ converges weakly to $x$ and if $u_n \in \Phi(x_n)$, $u_n$ converges strongly to $u$, then $u \in \Phi(x)$.

**Proof** The proof is similar to [1, Proposition 3.5.6], so we omit it. \hfill $\Box$

Let $\mathcal{S}$ be a nonempty closed convex subset of $\mathcal{H}$. The normal cone $N_\mathcal{S}(x)$ of $\mathcal{S}$ on $x$ is defined by
\[
N_\mathcal{S}(x) = \left\{ v \in \mathcal{H} \mid \langle v, y - x \rangle_\mathcal{H} \leq 0, \quad \forall y \in \mathcal{S} \right\}.
\]
The metric projection of $x \in \mathcal{H}$ onto $\mathcal{S}$ with norm $\|\cdot\|_\mathcal{H}$ is defined by
\[
\Pi_{\mathcal{S}}(x) := \left\{ \bar{x} \in \mathcal{S} \mid \|x - \bar{x}\|_\mathcal{H} = \inf_{y \in \mathcal{S}} \|x - y\|_\mathcal{H} \right\}.
\] (2.2)

By the fundamental theory in convex analysis, $\Pi_{\mathcal{S}}(x)$ is a singleton and satisfies
(i) \((x - \Pi_S(x), y - \Pi_S(x))_H \leq 0, \quad \forall y \in S;\)

(ii) \(\|\Pi_S(x) - \Pi_S(y)\|_H \leq \|x - y\|_H, \quad \forall x, y \in H.\)

**Definition 2.3** Let \(H\) be a Hilbert space, \(S \subseteq H\) be a nonempty closed convex set, and \(F: H \rightarrow H\) be a given map. The variational inequality problem VI\((F, S)\) is to find an \(x^* \in S\) such that

\[
(F(x^*), y - x^*)_H \geq 0, \quad \forall y \in S.
\]

Clearly, \(x^* \in S\) is a solution to VI\((F, S)\) if and only if \(0 \in F(x^*) + N_S(x^*)\).

**Lemma 2.4** [21, Theorem 3] Let \(H\) be a Hilbert space, \(S \subseteq H\) be a nonempty closed convex set, and, \(F: H \rightarrow H\) be a monotone hemi-continuous map (i.e., continuous from line segments in \(H\) to the weak topology in \(H\)), then the set-valued map \(x \rightsquigarrow F(x) + N_S(x)\) is maximal monotone.

### 2.3 Multistage Stochastic Variational Inequalities

In this subsection, we introduce the definition of the multistage stochastic variational inequalities. The concept of multistage stochastic variational inequalities was first introduced in [26] in the cases that the uncertainty was represented by some discrete random vectors. We shall see that the definition given in [26] also fits for a much general case.

Let \(N \in \mathbb{N}, m_1, m_2, \ldots, m_{N-1} \in \mathbb{N}, \xi_j \in \mathcal{L}^0(\Omega, \mathcal{F}, R^{m_j}), j = 1, 2, \ldots, N - 1, \mathcal{O}\) be the collection of all the \(P\)-null sets. For \(i \in \mathbb{N}, 1 \leq i \leq N - 1\), define

\[
\mathcal{F}_i := \sigma\{\sigma(\xi_1, \xi_2, \ldots, \xi_i) \cup \mathcal{O}\}.
\]  

(2.3)

Here, \(\sigma(\xi_1, \xi_2, \ldots, \xi_i)\) is the \(\sigma\)-field generated by \(\xi_1, \xi_2, \ldots, \xi_i\) and we call \(\mathcal{F}_i\) the augmented \(\sigma\)-field generated by \(\xi_1, \xi_2, \ldots, \xi_i\). Let \(\mathcal{F}_0 := \{\emptyset, \Omega\}\). It is clear that \(\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_{N-1} \subseteq \mathcal{F}\). Let \(n_0, n_1, \ldots, n_{N-1} \in \mathbb{N}, n_0 + n_1 + \cdots + n_{N-1} = n\).

Define

\[
\mathcal{N} := \left\{x = (x_0, x_1, \ldots, x_{N-1}) \in \mathcal{L}^2(\Omega, \mathcal{F}, R^n) \quad \mid \quad x_i \in \mathcal{L}^0(\Omega, \mathcal{F}_i, R^{n_i}), i = 0, 1, \ldots, N - 1 \right\}.
\]  

(2.4)

i.e., \(\mathcal{N}\) is the set of all \(x = (x_0, x_1, \ldots, x_{N-1}) \in \mathcal{L}^2(\Omega, \mathcal{F}, R^n)\) such that for any \(i = 0, 1, \ldots, N - 1\), \(x_i\) is an \(\mathcal{F}_i\)-measurable \(R^{n_i}\)-valued random vector. Clearly, \(\mathcal{N}\) is a closed linear subspace of \(\mathcal{L}^2(\Omega, \mathcal{F}, R^n)\). We call \(\mathcal{N}\) the nonanticipativity subspace.

**Remark 2.2** By the definition of \(\mathcal{F}_i\), \((\Omega, \mathcal{F}_i, P)\) is a complete probability space for any \(i = 1, \ldots, N - 1\). In addition, by Doob–Dynkin’s Lemma, \(x_0\) is a deterministic vector in \(R^{n_0}\), and, for \(i = 1, 2, \ldots, N - 1\), there are Borel measurable maps \(\varphi_i\) such that \(x_i = \varphi_i(\xi_1, \xi_2, \ldots, \xi_i)\) a.s.
Also, we consider the orthogonal complementary subspace of $N$ defined by

$$M = N^\perp = \left\{ y = (y_0, y_1, \ldots, y_{N-1}) \in L^2(\Omega, \mathcal{F}, R^n) \mid E \langle x, y \rangle = 0, \forall x \in N \right\}. \quad (2.5)$$

By the properties of conditional expectation, for any $x \in N$ and any $y \in L^2(\Omega, \mathcal{F}, R^n)$,

$$E \langle x, y \rangle = \sum_{i=0}^{N-1} E \langle x_i, y_i \rangle = \sum_{i=0}^{N-1} E \left[ E \langle x_i, y_i \mid \mathcal{F}_i \rangle \right].$$

Then, we have

$$M = \left\{ y = (y_0, y_1, \ldots, y_{N-1}) \in L^2(\Omega, \mathcal{F}, R^n) \mid E[y_i \mid \mathcal{F}_i] = 0 \text{ a.s.}, \forall i = 0, 1, \ldots, N-1 \right\}. \quad (2.6)$$

Let $C_i : \Omega \rightsquigarrow R^{n_i}$ be given $\mathcal{F}_i$-measurable set-valued map with nonempty closed convex values, $i = 0, 1, \ldots, N-1$. Consider the set-valued map $C : \Omega \rightsquigarrow R^n$ such that

$$C(\omega) = C_0(\omega) \times C_1(\omega) \times \cdots \times C_{N-1}(\omega) \subset R^{n_0} \times R^{n_1} \times \cdots \times R^{n_{N-1}}, \text{ a.s. } \omega \in \Omega.$$

Using set-valued map $C$, we define a subset $C$ of $L^2(\Omega, \mathcal{F}, R^n)$ by

$$C := \left\{ x = (x_0, x_1, \ldots, x_{N-1}) \in L^2(\Omega, \mathcal{F}, R^n) \mid x_i(\omega) \in C_i(\omega) \text{ a.s. } \omega \in \Omega, i = 0, \ldots, N-1 \right\}, \quad (2.6)$$

i.e., each element of $C$ is an $L^2$-integrable selection of set-valued map $C$. By the definition of $C$ and Lemma 2.2, the set $C$ is a nonempty closed convex set of $L^2(\Omega, \mathcal{F}, R^n)$.

The multistage stochastic variational inequality considered in this paper is defined as follows.

**Definition 2.4** Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $F$ be a given map from $L^2(\Omega, \mathcal{F}, R^n)$ to $L^2(\Omega, \mathcal{F}, R^n)$, $N$ and $C$ be defined by (2.4) and (2.6), respectively, the multistage stochastic variational inequality MSVI($F, C \cap N$) is: To find $x^* \in C \cap N$ such that

$$-F(x^*) \in N_{C \cap N}(x^*), \quad (2.7)$$

where $N_{C \cap N}(x^*)$ is the normal cone of $C \cap N$ on $x^*$ in $L^2(\Omega, \mathcal{F}, R^n)$.

**Remark 2.3** The set $C(\omega)$ represents the set of all available decisions for each $\omega \in \Omega$. The random vectors $\xi_1, \xi_2, \ldots, \xi_{N-1}$ can be regarded as the observed information of the decision maker. In Definition 2.4, the constraint $x \in C$ means that any admissible strategy $x$ should be valued in the decision set $C(\omega)$ for a.s. $\omega \in \Omega$, while the
nonanticipativity constraint \( x \in \mathcal{N} \) means that the admissible strategy \( x \) should be a function (or a reaction) of the observed information up to now, but not rely on the observed information in the future.

The multistage stochastic variational inequality MSVI \( (F, C \cap \mathcal{N}) \) (2.7) is closely related to the following multistage stochastic variational inequality in extensive form.

**Definition 2.5** Let \((\Omega, F, P)\) be a complete probability space, \( F \) be a given map from \( L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \) to \( L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \), \( \mathcal{N} \) and \( \mathcal{C} \) be defined by (2.4) and (2.6), respectively. The multistage stochastic variational inequality in extensive form is: To find \( x^* \in \mathcal{C} \cap \mathcal{N} \) and \( v^* \in \mathcal{M} \) such that

\[
-F(x^*) - v^* \in N_{\mathcal{C}}(x^*). \tag{2.8}
\]

**Lemma 2.5** Let \( F : L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \), \( C \) be the nonempty closed convex set defined by (2.6). Then, for a random vector \( x^* \in \mathcal{C} \), the condition

\[
E\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{C} \tag{2.9}
\]

is equivalent to the pointwise (for sample point) type condition

\[
\langle F(x^*)(\omega), x - x^*(\omega) \rangle \geq 0, \quad \forall x \in C(\omega), \text{ a.s. } \omega \in \Omega. \tag{2.10}
\]

Lemma 2.5 has been proved in [26] in the discrete cases. For the convenience of readers, we provide a proof of Lemma 2.5 in the general probability space in the appendix.

By the definition of the normal cone of a closed convex set, the multistage stochastic variational inequality in extensive form (2.8) can be rewritten as

\[
E\langle F(x^*) + v^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{C}. \tag{2.11}
\]

By Lemma 2.5, it is equivalent to finding \( x^* \in \mathcal{C} \cap \mathcal{N} \) and \( v^* \in \mathcal{M} \) such that for a.s. \( \omega \in \Omega \),

\[
\langle F(x^*)(\omega) + v^*(\omega), x - x^*(\omega) \rangle \geq 0, \quad \forall x \in C(\omega). \tag{2.12}
\]

Since \( N_{\mathcal{C}}(x^*) + N_{\mathcal{N}}(x^*) \subset N_{C \cap \mathcal{N}}(x^*) \), any solution to the multistage stochastic variational inequality in extensive form (2.8) is a solution to MSVI \( (F, C \cap \mathcal{N}) \) (2.7). When the sum rule

\[
N_{C \cap \mathcal{N}}(x) = N_{\mathcal{C}}(x) + N_{\mathcal{N}}(x), \quad \forall x \in C \cap \mathcal{N} \tag{2.13}
\]

is satisfied, (2.7) is equivalent to (2.8).

In what follows, we give a sufficient condition under which the sum rule (2.13) holds true.

**Theorem 2.1** Let \((\Omega, \mathcal{F}, P)\) be a complete probability space, \( \mathcal{N} \) and \( \mathcal{C} \) be defined by (2.4) and (2.6), respectively. Suppose that for \( i = 0, 1, \ldots, N - 1 \), \( C_i \) are \( \mathcal{F}_i \)-measurable. Then, the sum rule (2.13) holds true.
Proof Let \( x \in \mathcal{C} \cap \mathcal{N} \) and define
\[
\mathcal{L}(x) := \left\{ \xi = (\xi_0, \xi_1, \ldots, \xi_{N-1}) \in \mathcal{N} \mid \xi_i(\omega) \in N_{C_i(\omega)}(x_i(\omega)) \text{ a.s. } \omega \in \Omega, \right. \\
\forall i = 0, 1, \ldots, N - 1 \}.
\] (2.14)

We claim that
\[
N_{\mathcal{C} \cap \mathcal{N}}(x) = \mathcal{M} + \mathcal{L}(x).
\]

For any \( \eta \in \mathcal{M}, \xi \in \mathcal{L}(x) \) and \( y \in \mathcal{C} \cap \mathcal{N} \),
\[
\mathbb{E} \langle \eta + \xi, y - x \rangle = \mathbb{E} \langle \xi, y - x \rangle = \sum_{i=0}^{N-1} \mathbb{E} \langle \xi_i, y_i - x_i \rangle \leq 0.
\]

It implies \( \mathcal{M} + \mathcal{L}(x) \subseteq N_{\mathcal{C} \cap \mathcal{N}}(x) \).

On the other hand, for any \( \xi \in N_{\mathcal{C} \cap \mathcal{N}}(x) \) we have
\[
\xi = \Pi_{\mathcal{N}}(\xi) + \Pi_{\mathcal{M}}(\xi),
\]
where \( \Pi_{\mathcal{N}}(\xi) \) and \( \Pi_{\mathcal{M}}(\xi) \) are, respectively, the projections of \( \xi \) onto \( \mathcal{N} \) and \( \mathcal{M} \). Clearly \( \Pi_{\mathcal{M}}(\xi) \in \mathcal{M}, \Pi_{\mathcal{N}}(\xi) \in \mathcal{N} \). By \( \xi \in N_{\mathcal{C} \cap \mathcal{N}}(x) \), we have
\[
0 \geq \mathbb{E} \langle \xi, y - x \rangle = \mathbb{E} \langle \Pi_{\mathcal{N}}(\xi) + \Pi_{\mathcal{M}}(\xi), y - x \rangle
\] \[
= \mathbb{E} \langle \Pi_{\mathcal{N}}(\xi), y - x \rangle, \forall y \in \mathcal{C} \cap \mathcal{N}.
\]

For any fixed \( i = 0, 1, \ldots, N - 1 \) and any \( y_i \in L^2(\Omega, \mathcal{F}_i, \mathbb{R}^n) \) such that \( y_i(\omega) \in C_i(\omega) \) a.s. \( \omega \in \Omega \), we define \( \tilde{y} = (x_0, x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_{N-1}) \). Then, \( \tilde{y} \in \mathcal{C} \cap \mathcal{N} \) and
\[
0 \geq \mathbb{E} \langle \Pi_{\mathcal{N}}(\xi), \tilde{y} - x \rangle = \mathbb{E} \langle (\Pi_{\mathcal{N}}(\xi))_i, y_i - x_i \rangle.
\] (2.15)

Here, \((\Pi_{\mathcal{N}}(\xi))_i\) is the \( i \)th component of \( \Pi_{\mathcal{N}}(\xi) \). Clearly, \((\Pi_{\mathcal{N}}(\xi))_0 \in N_{C_0}(\xi_0)\). For any \( i = 1, 2, \ldots, N - 1 \), by the definition of \( \mathcal{F}_i, (\Omega, \mathcal{F}_i, P) \) is a complete probability space. Then, by (2.15), the \( \mathcal{F}_i \)-measurability of \( C_i \) and a similar proof of Lemma 2.5, we conclude that
\[
(\Pi_{\mathcal{N}}(\xi))_i(\omega) \in N_{C_i(\omega)}(x_i(\omega)) \text{ a.s. } \omega \in \Omega.
\]

Therefore, \( \Pi_{\mathcal{N}}(\xi) \in \mathcal{L}(x) \). Consequently, \( N_{\mathcal{C} \cap \mathcal{N}}(x) \subseteq \mathcal{M} + \mathcal{L}(x) \). This proves \( N_{\mathcal{C} \cap \mathcal{N}}(x) = \mathcal{M} + \mathcal{L}(x) \).

Since
\[
N_{\mathcal{C}}(x) + N_{\mathcal{N}}(x) = N_{\mathcal{C}}(x) + \mathcal{M} \subset N_{\mathcal{C} \cap \mathcal{N}}(x),
\]
to prove (2.13), we only need to prove that \( \mathcal{L}(x) \subset N_{\mathcal{C}}(x) \). Using a similar proof of Lemma 2.5 again, we deduce that
\[
N_{\mathcal{C}}(x) = \left\{ \xi \in L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \mid \xi_i(\omega) \in N_{C_i(\omega)}(x_i(\omega)) \text{ a.s. } \omega \in \Omega, \right. \\
\]
Then, the conclusion follows immediately from the definition of $\mathcal{L}(x)$. \hfill \square

**Remark 2.4** Let $\mathcal{H}$ be a Hilbert space, $S_1, S_2 \subset \mathcal{H}$ be nonempty closed convex subsets of $\mathcal{H}$. When $\mathcal{H}$ is a finite-dimensional space, by [20, Corollary 23.8.1],

\begin{equation}
N_{S_1 \cap S_2}(x) = N_{S_1}(x) + N_{S_2}(x)
\end{equation}

holds true if $ri(S_1) \cap ri(S_2) \neq \emptyset$, where $ri(S_1)$ and $ri(S_2)$ are the relative interiors of $S_1$ and $S_2$, respectively. However, in the general infinite-dimensional cases, the condition $ri(S_1) \cap ri(S_2) \neq \emptyset$ is not enough to ensure the sum rule (2.17). A counterexample can be found in [2]. In the infinite-dimensional cases, the sum rule (2.17) holds true if

$$\text{cone}(S_1 - S_2)$$

is a closed linear subspace of $\mathcal{H}$. (2.18)

Here, cone$(S_1 - S_2)$ is the cone generated by $S_1 - S_2$. (2.18) is called the Attouch-Brezis qualification condition and a proof the sum rule (2.17) under condition (2.18) can be found in [18].

Note that $\mathcal{N}$ is a closed linear subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n)$. When the sample space $\Omega$ is a finite set, $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n)$ is isomorphic to a finite-dimensional Euclidean space (see [26] or Section 4 of this paper for more details). In this case, the equality (2.13) holds true if $ri(C) \cap \mathcal{N} \neq \emptyset$. It is proved in [26, Theorem 2.3] that when the sample space $\Omega$ is a finite set, $ri(C) \cap \mathcal{N} \neq \emptyset$ if there is $\hat{x} \in \mathcal{N}$ such that $\hat{x}(\omega) \in ri(C(\omega))$ for all $\omega \in \Omega$. In the general cases, $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n)$ is an infinite-dimensional Hilbert space. In order that the sum rule (2.13) holds true, it seems a natural way to assume the Attouch-Brezis qualification condition for $C$ and $\mathcal{N}$. However, in the stochastic cases, the Attouch-Brezis qualification condition may fail even when $C(\omega)$ is a polyhedral for any $\omega \in \Omega$.

Let us consider a simple example. Let $\Omega = [0, 1]$, $\mathcal{F}$ be the $\sigma$-field of Lebesgue measurable sets and $P$ be the Lebesgue measure on [0, 1]. Clearly, $(\Omega, \mathcal{F}, P)$ is a complete probability space. We consider the special case of one-stage with $C(\omega) \equiv [0, +\infty)$. Then, $C = \{x \in \mathcal{L}^2(0, 1) \mid x(\omega) \geq 0, \text{ a.e. } \omega \in [0, 1]\}$ and $\mathcal{N} = \{x \in \mathcal{L}^2(0, 1) \mid x(\omega) = a \text{ a.e. } \omega \in [0, 1], a \in \mathbb{R}\}$. By Theorem 2.1, $N_{C \cap \mathcal{N}}(x) = N_C(x) + N_{\mathcal{N}}(x)$ for any $x \in C \cap \mathcal{N}$. For any $x \in \mathcal{L}^\infty(0, 1)$,

$$x(\omega) = x(\omega) + \|x\|_{L^\infty} - \|x\|_{L^\infty}, \text{ a.e. } \omega \in [0, 1],$$

where $\|x\|_{L^\infty}$ is the $L^\infty$ norm of $x$. Letting $\tilde{x}(\omega) = x(\omega) + \|x\|_{L^\infty}$, a.e. $\omega \in [0, 1]$, we have $\tilde{x} \in C$ and

$$x = \tilde{x} - \|x\|_{L^\infty} \in C - \mathcal{N}.$$

It follows that $\mathcal{L}^\infty(0, 1) \subset \text{cone}(C - \mathcal{N})$. Therefore, if cone$(C - \mathcal{N})$ is a closed linear subspace of $\mathcal{L}^2(0, 1)$, it must be $\mathcal{L}^2(0, 1)$. However, for any $x \in \mathcal{L}^2(0, 1) \setminus \mathcal{L}^\infty(0, 1)$ such that $x(\omega) < 0$ a.e. $\omega \in (0, 1)$, $x \notin \text{cone}(C - \mathcal{N})$. Therefore, the Attouch-Brezis qualification condition does not hold true in this example.
As illustrated in [26], one of the motivations for studying MSVI \((F, C \cap N)\) \((2.7)\) is to solve the multistage stochastic convex optimization problem: To find \(x^* \in C \cap N\) such that

\[
f(x^*) = \min_{x \in C \cap N} f(x),
\]

where \(f(x) = E g(x(\omega), \omega)\) and \(g: R^n \times \Omega \rightarrow R\) is a given function which is continuously differentiable and convex with respect to the first variable and \(\mathcal{F}\)-measurable with respect to the second variable. If \(E |g(0, \omega)| < +\infty\) and there is an \(\mathcal{F}\)-measurable nonnegative random variable \(\eta\) with \(E \eta^2(\omega) < +\infty\) such that

\[
|g(u, \omega) - g(v, \omega)| \leq \eta(\omega)|u - v| \quad \text{a.s. } \omega \in \Omega, \quad \forall u, v \in R^n,
\]

then, the function \(f: L^2(\Omega, \mathcal{F}, R^n) \rightarrow R\) is a well-defined differentiable convex function on \(L^2(\Omega, \mathcal{F}, R^n)\) with its Gâteaux derivative

\[
Df(x) = \nabla_x g(x(\omega), \omega) \quad \text{a.s. } \omega \in \Omega, \quad \forall x \in L^2(\Omega, \mathcal{F}, R^n).
\]

Indeed, for any \(x \in L^2(\Omega, \mathcal{F}, R^n)\),

\[
|f(x)| = |E g(x(\omega), \omega)| \leq E |g(x(\omega), \omega) - g(0, \omega)| + E |g(0, \omega)| \\
\leq E \eta(\omega)|x(\omega)| + E |g(0, \omega)| < +\infty.
\]

This proves that \(f\) is well-defined on \(L^2(\Omega, \mathcal{F}, R^n)\). For any \(d \in L^2(\Omega, \mathcal{F}, R^n)\), by \((2.19)\) we have

\[
\left| \frac{g(x(\omega) + td(\omega), \omega) - g(x(\omega), \omega)}{t} \right| \leq \eta(\omega)|d(\omega)| \quad \text{a.s. } \omega \in \Omega
\]

and \(E \eta(\omega)|d(\omega)| < +\infty\). By Lebesgue Dominated Convergence Theorem, we have

\[
\lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t} = \lim_{t \to 0^+} E \left[ \frac{g(x(\omega) + td(\omega), \omega) - g(x(\omega), \omega)}{t} \right] \\
= E \langle \nabla_x g(x(\omega), \omega), d(\omega) \rangle.
\]

Then, by Riesz Representation Theorem, we have \((2.20)\).

By the basic theory of convex optimization, solving the above multistage stochastic convex optimization problem is equivalent to solving the MSVI \((F, C \cap N)\) \((2.7)\) with \(F(x) = Df(x)\).

To end this section, we provide an application of MSVI \((F, C \cap N)\) \((2.7)\) in stochastic optimal control problem. We refer the reader to [33] for some basic notions in stochastic control theory.

**Example 2.2** Let \((\Omega, \mathcal{F}, \mathbb{F}, P)\) be a complete filtered probability space with the filtration \(\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq 1}\), on which a one-dimensional standard Wiener process \(W(\cdot)\) is
defined such that $\mathcal{F}$ is the natural filtration generated by $W(\cdot)$ (augmented by all the $P$-null sets), i.e., for any $t \in [0, 1]$,

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^W \cup \mathcal{O})$$

with $\mathcal{F}_t^W := \sigma(W(s) : s \in [0, t])$ and $\mathcal{O}$ being the collection of all $P$-null sets.

Let $A, \tilde{A} : [0, 1] \to \mathbb{R}^{n \times n}$ and $B, D : [0, 1] \to \mathbb{R}^{n \times m}$ be given bounded Borel measurable matrix-valued maps, $G \in \mathbb{R}^{n \times n}$ be a given positive semi-definite matrix and $U \subset \mathbb{R}^m$ be a given closed convex set. Consider the controlled linear stochastic differential equation

$$dx(t) = \left( A(t)x(t) + B(t)u(t) \right) dt + \left( \tilde{A}(t)x(t) + D(t)u(t) \right) dW(t), \quad t \in [0, 1]$$

with $x(0) = x_0$

$$J(u) = \frac{1}{2} \mathbb{E} \langle G(x(1) - \eta), x(1) - \eta \rangle. \quad (2.22)$$

Here $u \in \mathcal{U}$ is called the control,

$$\mathcal{U} := \left\{ u \in L^2_{\mathbb{F}}(0, 1; \mathbb{R}^m) \left| u(t) \in U, \text{ a.e. } t \in [0, 1], \text{ a.s.} \right. \right\},$$

$L^2_{\mathbb{F}}(0, 1; \mathbb{R}^m)$ is the space of $\mathbb{R}^m$-valued $\mathcal{F}([0, 1]) \otimes \mathcal{F}$-measurable stochastic processes $\varphi$ such that for any $t \in [0, 1]$, $\varphi(t)$ is $\mathcal{F}_t$-measurable and $\left[ \mathbb{E} \int_0^1 |\varphi(t)|^2 dt \right]^{\frac{1}{2}} < \infty$. $x$ is the state valued in $\mathbb{R}^n$ with initial datum $x_0 \in \mathbb{R}^n$ and control $u$, and, $\eta : \Omega \to \mathbb{R}^n$ is an $\mathcal{F}_1$-measurable random vector. $x$ is a solution to (2.21) if

$$x(t) = x_0 + \int_0^t \left( A(s)x(s) + B(s)u(s) \right) ds + \int_0^t \left( \tilde{A}(s)x(s) + D(s)u(s) \right) dW(s) \text{ a.s., } \forall t \in [0, 1].$$

Here, $\int_0^t \varphi(s) dW(s)$ is the Itô’s stochastic integral of stochastic process $\varphi$ on $[0, t]$ with respect to Wiener process $W$. In order that the Itô’s stochastic integral is well-defined, we usually assume $\varphi$ is a square-integrable stochastic process and for any $t \in [0, 1]$, $\varphi(t)$ is $\mathcal{F}_t$-measurable. By the standard theory of stochastic analysis (see for instance [33, Chapter 1]), for any $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}$, (2.21) admits unique solution. We consider the following stochastic optimal control problem with control constraints: To find $u^* \in \mathcal{U}$ such that

$$J(u^*) = \min_{u \in \mathcal{U}} J(u). \quad (2.23)$$

The continuous-time stochastic optimal control problem (2.23) has important applications in mathematical finance (see, for instance [17]) and can be regarded as a stochastic convex programming problem defined on the Hilbert space of square-integrable stochastic processes. Its discretized approximation problem is a multistage
stochastic convex programming problem with nonanticipativity constraint. To see this, let \( N \) be a given large enough natural number, \( \Delta = 1/N \), \( \Delta W_i = W((i + 1)/N) - W(i/N) \), \( i = 0, 1, \ldots, N - 1 \) and consider the Euler approximation of the stochastic differential equation (2.21):

\[
x_{i+1} = x_i + [A_i x_i + B_i u_i] \Delta + [\tilde{A}_i x_i + D_i u_i] \Delta W_i, \quad i = 0, \ldots, N - 1,
\]

\( x_0 \in \mathbb{R}^n \).

Here, \( A_i = A(i/N) \), \( B_i = B(i/N) \), \( \tilde{A}_i = \tilde{A}(i/N) \), \( D_i = D(i/N) \), \( \mathcal{U}^N := (u_0, u_1, \ldots, u_{N-1}) : \Omega \to (\mathbb{R}^m)^N \) is the discretized control and \( \mathcal{X}^N := (x_0, x_1, \ldots, x_N) : \Omega \to (\mathbb{R}^n)^{N+1} \) is the corresponding discretized state. Denote \( \Psi_i = I + A_i \Delta + \tilde{A}_i \Delta W_i, \Lambda_i = B_i \Delta + D_i \Delta W_i, \) \( i = 0, 1, \ldots, N - 1 \). We have

\[
x_N = \left[ \prod_{i=0}^{N-1} \Psi_i \right] x_0 + \sum_{i=0}^{N-1} \left[ \prod_{j=i+1}^{N-1} \Psi_j \right] \Lambda_i u_i. \quad (2.24)
\]

Then, the corresponding discretized cost function is represented by

\[
j^N(\mathcal{U}^N) = \frac{1}{2} \mathbb{E} \left\{ G \left[ \left[ \prod_{i=0}^{N-1} \Psi_i \right] x_0 + \sum_{i=0}^{N-1} \left[ \prod_{j=i+1}^{N-1} \Psi_j \right] \Lambda_i u_i - \eta \right] , \left[ \prod_{i=0}^{N-1} \Psi_i \right] x_0 + \sum_{i=0}^{N-1} \left[ \prod_{j=i+1}^{N-1} \Psi_j \right] \Lambda_i u_i - \eta \right\}.
\]

(2.25)

By the definition of standard Wiener process, \( \Delta W_i : \Omega \to \mathbb{R}, k = 0, 1, \ldots, N - 1 \) is a sequence of mutually independent Gaussian random variables with expectation 0 and variance \( 1/N \). Let \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \),

\[
\mathcal{F}_i = \sigma\left\{ \sigma(\Delta W_0, \Delta W_1, \ldots, \Delta W_{i-1}) \cup \emptyset \right\}, \quad i = 1, 2, \ldots, N - 1
\]

and define

\[
\mathcal{N} := \left\{ \mathcal{U}^N = (u_0, u_1, \ldots, u_{N-1}) \in \mathcal{L}^2(\Omega, \mathcal{F}, (\mathbb{R}^m)^N) \mid u_i \in \mathcal{L}^0(\Omega, \mathcal{F}_i, \mathbb{R}^m), \right. \\
i = 0, 1, \ldots, N - 1 \}
\]

In addition, we define

\[
\mathcal{C} := \left\{ \mathcal{U}^N = (u_0, u_1, \ldots, u_{N-1}) \in \mathcal{L}^2(\Omega, \mathcal{F}, (\mathbb{R}^m)^N) \mid u_i(\omega) \in U \text{ a.s. } \omega \in \Omega, \\
i = 0, 1, \ldots, N - 1 \right\}
\]
The discretized approximation problem for the continuous-time stochastic optimal control problem (2.23) is: To find $(\mathcal{U}^N)^* \in \mathcal{C} \cap \mathcal{N}$ such that

$$J^N((\mathcal{U}^N)^*) = \min_{\mathcal{U}^N \in \mathcal{C} \cap \mathcal{N}} J^N(\mathcal{U}^N).$$  \tag{2.26}$$

Clearly, (2.26) is a multistage stochastic convex programming problem with nonanticipativity constraint.

Denote

$$\zeta = \left[ \prod_{i=0}^{N-1} \Psi_i \right] x_0, \quad Z = \left( \left[ \prod_{j=1}^{N-1} \Psi_j \right] \Lambda_0, \left[ \prod_{j=2}^{N-1} \Psi_j \right] \Lambda_1, \ldots, \Lambda_{N-1} \right)$$

and let $b = Z^\top G (\zeta - \eta)$ and $M = Z^\top G Z$. The derivative of $J^N$ with respect to $\mathcal{U}^N$ is

$$DJ^N(\mathcal{U}^N) = M \mathcal{U}^N + b$$  \tag{2.27}$$

and the first-order necessary condition for the stochastic convex programming problem (2.26) is

$$-DJ^N((\mathcal{U}^N)^*) \in N_{\mathcal{C} \cap \mathcal{N}}((\mathcal{U}^N)^*),$$

which is a multistage stochastic variational inequality with nonanticipativity constraint.

3 Algorithm and Convergence Analysis

In this section, we shall propose the prediction–correction ADMM for solving MSVI($F, \mathcal{C} \cap \mathcal{N}$) (2.7) and prove that the sequence generated by that algorithm converges weakly in $L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$ to a solution to MSVI($F, \mathcal{C} \cap \mathcal{N}$) (2.7).

We assume that the map $F : L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \to L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$ satisfies the following conditions.

(A1) The map $F$ is monotone.

(A2) The map $F$ is Lipschitz continuous with constant $L_F > 0$, i.e.,

$$\|F(x) - F(y)\|_{L^2} \leq L_F \|x - y\|_{L^2}, \quad \forall x, y \in L^2(\Omega, \mathcal{F}, \mathbb{R}^n).$$

The following is a simple example in which the above assumptions are satisfied.

**Example 3.1** Let $f(x) = \mathbb{E} g(x(\omega), \omega)$ and $g : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a given function which is continuously differentiable and convex with respect to the first variable. Assume $\mathbb{E} |g(0, \omega)| < +\infty$ and there is an $\mathcal{F}$-measurable nonnegative random variable $\eta$ with $\mathbb{E} \eta^2(\omega) < +\infty$ such that (2.19) holds true. Moreover, assume that there is a constant $L > 0$ such that

$$|\nabla_x g(u, \omega) - \nabla_x g(v, \omega)| \leq L|u - v| \text{ a.s. } \omega \in \Omega, \quad \forall u, v \in \mathbb{R}^n.$$  \tag{3.1}$$

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The $F(\cdot) = Df(\cdot)$ is monotone and Lipschitz continuous on $L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$. Actually, by the convexity of $g$ with respect to the first variable, we have, for any $x, y \in L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$,

$$\langle \nabla_x g(x(\omega), \omega) - \nabla_x g(y(\omega), \omega), x(\omega) - y(\omega) \rangle \geq 0 \text{ a.s. } \omega \in \Omega$$

which implies that

$$E( F(x) - F(y), x - y) = E(\langle \nabla_x g(x(\omega), \omega) - \nabla_x g(y(\omega), \omega), x(\omega) - y(\omega) \rangle \geq 0.$$  

In addition, by (3.1), for any $x, y \in L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$,

$$\|F(x) - F(y)\|_{L^2} = \left[ E|\nabla_x g(x(\omega), \omega) - \nabla_x g(y(\omega), \omega)|^2 \right]^{1/2} \leq L \left[ E|x - y|^2 \right]^{1/2} = L \|x - y\|_{L^2}.$$  

### 3.1 Prediction–Correction ADMM

In this subsection, we will introduce the prediction–correction ADMM for MSVI $(F, C \cap N)$ (2.7). In order to illustrate the key idea of this algorithm clearly, let us go back for a moment to the multistage stochastic convex optimization problem: To find $x^* \in C \cap N$ such that

$$f(x^*) = \min_{x \in C \cap N} f(x) \quad (3.2)$$

with $f(x) = E [g(x(\omega), \omega)]$ and $g: \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$. Clearly, problem (3.2) is equivalent to the following optimization problem with a separable objective function and a linear equality constraint:

$$\min \left\{ f(x) + I_N(y) \mid x \in C, y \in L^2(\Omega, \mathcal{F}, \mathbb{R}^n), x - y = 0 \right\}, \quad (3.3)$$

where $I_N(\cdot)$ is the indicator function of the nonanticipativity subspace $N$ defined by

$$I_N(y) := \begin{cases} 0, & \text{if } y \in N; \\ +\infty, & \text{if } y \notin N. \end{cases}$$

Denote $\left[ L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \right]^3 = L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \times L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \times L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$. Let $\beta > 0$ and define the augmented Lagrange function for problem (3.3) by

$$L_\beta(x, y, \lambda) = f(x) + I_N(y) - E(\langle \lambda, x - y \rangle) + \frac{\beta}{2} E|x - y|^2, \quad (3.4)$$

for any $(x, y, \lambda)^T \in \left[ L^2(\Omega, \mathcal{F}, \mathbb{R}^n) \right]^3$. 

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Given a triplet \((x^k, y^k, \lambda^k)^\top \in C \times \mathcal{N} \times \mathcal{L}^2(\Omega, \mathcal{F}, R^n)\), the classical ADMM generates a new iterate \((x^{k+1}, y^{k+1}, \lambda^{k+1})^\top\) via the following procedure

\[
x^{k+1} = \arg\min_{x \in C} \left\{ f(x) - \mathbb{E} \langle \lambda^k, x - y^k \rangle + \frac{\beta}{2} \mathbb{E} |x - y^k|^2 \right\},
\]

\[
y^{k+1} = \arg\min_{y \in \mathcal{L}^2(\Omega, \mathcal{F}, R^n)} \left\{ I_\mathcal{N}(y) - \mathbb{E} \langle \lambda^k, x^{k+1} - y \rangle + \frac{\beta}{2} \mathbb{E} |x^{k+1} - y|^2 \right\}, \tag{3.5}
\]

\[
\lambda^{k+1} = \lambda^k - \beta(x^{k+1} - y^{k+1}). \tag{3.6}
\]

By the first-order necessary condition in optimization, we have \((x^{k+1}, y^{k+1}, \lambda^{k+1})^\top\) satisfies

\[
- Df(x^{k+1}) + \lambda^k - \beta(x^{k+1} - y^k) \in N_C(x^{k+1}),
\]

\[
- \lambda^k + \beta(x^{k+1} - y^{k+1}) \in N_\mathcal{N}(y^{k+1}),
\]

\[
\lambda^{k+1} = \lambda^k - \beta(x^{k+1} - y^{k+1}). \tag{3.6}
\]

By \(Df(x^{k+1}) = \nabla_x g(x^{k+1}(\omega), \omega)\), a.s. \(\omega \in \Omega\) and Lemma 2.5, the first variational inclusion in (3.6) is equivalent to the following (sample point) pointwise version

\[- \nabla_x g(x^{k+1}(\omega), \omega) + \lambda^k(\omega) - \beta(x^{k+1}(\omega) - y^k(\omega)) \in N_{C(\omega)}(x^{k+1}(\omega))\text{ a.s. } \omega \in \Omega.\]

Letting \(\hat{u}^k := x^{k+1}, u^{k+1} := y^{k+1}, v^k := -\lambda^k\), the second variational inclusion in (3.6) can be rewritten equivalently as \(u^{k+1} = \Pi_\mathcal{N}(\hat{u}^k + \frac{1}{\beta}v^k)\). In addition, if we choose \(v^0 := -\lambda^0 \in \mathcal{M}\), then we can obtain by induction that \(v^k \in \mathcal{M}\),

\[
u^{k+1} = v^k + \beta(\hat{u}^k - \Pi_\mathcal{N}(\hat{u}^k)) = v^k + \beta \Pi_\mathcal{M}(\hat{u}^k),\]

and \(v^{k+1} = v^k + \beta(\hat{u}^k - \Pi_\mathcal{M}(\hat{u}^k)) = v^k + \beta \Pi_\mathcal{M}(\hat{u}^k)\). Now, taking \(F(\cdot) = Df(\cdot)\), (3.6) becomes

\[
- F(\hat{u}^k(\omega)) - v^k(\omega) - \beta(\hat{u}^k(\omega) - u^k(\omega)) \in N_{C(\omega)}(\hat{u}^k(\omega))\text{ a.s. } \omega \in \Omega,
\]

\[
u^{k+1} = v^k + \beta \Pi_\mathcal{M}(\hat{u}^k), \tag{3.7}
\]

which coincides with the progressive hedging algorithm in [22].

Note that in (3.6) (resp. (3.7)), to obtain \(\lambda^{k+1}\) (resp. \(\hat{u}^k\)) one has to solve a collection of parameterized (with parameter \(\omega \in \Omega\)) finite-dimensional variational inequalities. When the structure of those variational inequalities is complicated (for example, high nonlinearity of \(F(\cdot)\) or \(Df(\cdot)\)), solving those finite-dimensional variational inequalities will become time-consuming.

Taking \(F(x) = Df(x)\), by the properties of metric projection, (3.6) can be rewritten as
\[ x^{k+1} = \Pi_C \left[ x^k - \frac{1}{\beta} \left( F(x^{k+1}) - \lambda^k + \beta(x^k - y^k) \right) \right], \]
\[ y^{k+1} = \Pi_N \left[ y^k - \frac{1}{\beta} \left( \lambda^k - \beta(x^{k+1} - y^k) \right) \right]; \]
\[ \lambda^{k+1} = \lambda^k - \beta(x^{k+1} - y^{k+1}). \]  
(3.8)

To obtain an explicit iterative algorithm, one natural way is to replace \( x^{k+1} \) by \( x^k \) directly on the right side of the first equation in (3.8). However, as pointed out in [15] for deterministic structured variational inequalities, such modification might lead to divergence and some proper correction is necessary to ensure the convergence. Here we shall adapt the prediction–correction ADMM which was proposed in [15] to solve the multistage stochastic variational inequality MSVI \( F, C \cap N \) (2.7). Before giving the algorithm, we introduce some useful notations which will be used in the sequel.

Let \( \beta, r > 0 \) and define \( G \) from \( \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n) \) to \( \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n) \) by

\[ G(\theta) = \begin{pmatrix} \beta r x \\ \beta y \\ \frac{1}{\beta} \lambda \end{pmatrix}, \quad \forall \theta = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} \in \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n). \]  
(3.9)

Clearly, \( G \) is a symmetry operator and \( G^{-1} \) exists with

\[ G^{-1}(\vartheta) = \begin{pmatrix} \frac{1}{\beta r} u \\ \frac{1}{\beta} v \\ \beta w \end{pmatrix}, \quad \forall \vartheta = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n). \]

Let \( \theta^k = (x^k, y^k, \lambda^k)^T, \tilde{\theta}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)^T \in \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n) \) \((k \in \mathbb{N})\), define

\[ \xi_x^k := F(x^k) - F(\tilde{x}^k) + \beta(x^k - \tilde{x}^k), \]  
(3.10)

and denote

\[ \zeta^k := \begin{pmatrix} \xi_x^k \\ 0 \\ 0 \end{pmatrix} \]

and

\[ d(\theta^k, \tilde{\theta}^k, \zeta^k) := \theta^k - \tilde{\theta}^k - G^{-1} \zeta^k. \]  
(3.11)

The prediction–correction ADMM for MSVI \( F, C \cap N \) (2.7) is defined as follows.

**Remark 3.1** (i) In Step 1.1 of Algorithm 3.1, by the properties of metric projection onto a nonempty closed convex set, the projection

\[ \tilde{x}^k = \Pi_C \left[ x^k - \frac{1}{\beta r} \left( F(x^k) - \lambda^k + \beta(x^k - y^k) \right) \right]. \]
Algorithm 3.1 Prediction–correction ADMM

Step 0. Set $\beta, r > 0$, $r > \frac{L F}{\beta} + 1$, $\alpha \in (0, 1)$, $\theta^0 = (x^0, y^0, \lambda^0)^\top \in C \times N \times L^2(\Omega, \mathcal{F}, R^n)$, $k = 0$.

Step 1: Prediction

Step 1.1. Compute
\[
\tilde{x}^k = \Pi_C\{x^k - \frac{1}{\beta r} [F(x^k) - \lambda^k + \beta(x^k - y^k)]\}.
\] (3.12)

Step 1.2. Compute
\[
\tilde{y}^k = \Pi_N\{y^k - \frac{1}{\beta} [\lambda^k - \beta(\tilde{x}^k - y^k)]\}.
\] (3.13)

Step 1.3. Update $\tilde{\lambda}^k$ via
\[
\tilde{\lambda}^k = \lambda^k - \beta(\tilde{x}^k - \tilde{y}^k).
\] (3.14)

Step 2: Correction

Set
\[
\theta^{k+1} = \theta^k - \alpha d(\theta^k, \hat{\theta}^k, \xi^k),
\] (3.15)

let $k := k + 1$ and return to Step 1.

can be rewritten as
\[
E\langle x^k - \frac{1}{\beta r} [F(x^k) - \lambda^k + \beta(x^k - y^k)] - \tilde{x}^k, z - \tilde{x}^k \rangle \leq 0, \quad \forall z \in C.
\] (3.16)

By Lemma 2.5, the inequality (3.16) is equivalent to the following pointwise form
\[
\langle x^k(\omega) - \frac{1}{\beta r} [F(x^k)(\omega) - \lambda^k(\omega) + \beta(x^k(\omega) - y^k(\omega))] - \tilde{x}^k(\omega), z - \tilde{x}^k(\omega) \rangle \leq 0, \quad \forall z \in C(\omega), \text{ a.s. } \omega \in \Omega.
\]

Therefore, to compute $\tilde{x}^k$ we can compute pointwisely the projections in finite-dimensional space
\[
\tilde{x}^k(\omega) = \Pi_{C(\omega)}\{x^k(\omega) - \frac{1}{\beta r} [F(x^k)(\omega) - \lambda^k(\omega) + \beta(x^k(\omega) - y^k(\omega))]\}
\]
for almost every $\omega \in \Omega$. This approach is feasible when the sample space $\Omega$ is a finite set. In the general cases, solving $\tilde{x}^k$ is equivalent to solving the following convex stochastic optimization problem without the nonanticipativity constraint:
\[
\min_{z \in C} E \left| z - x^k + \frac{1}{\beta r} [F(x^k) - \lambda^k + \beta(x^k - y^k)] \right|^2.
\]

(ii) In Step 1.2 of Algorithm 3.1, we need to compute the projection onto the closed linear subspace $N$, which is quite different from the projection in Step 1.1.
illustrated in the above, in Step 1.1., the projection can be transformed into a collection of metric projections (with parameter \( \omega \in \Omega_1 \)) in finite-dimensional space. On the other hand, to calculate the projection onto the closed linear subspace \( N \) one needs to compute a collection of conditional expectations. Indeed, by Remark 2.1, \( \tilde{y}^k = (\tilde{y}^k_0, \tilde{y}^k_1, \ldots, \tilde{y}^k_{N-1}) \) can be represented by

\[
\tilde{y}^k_i = \mathbb{E} \left\{ y^k_i - \frac{1}{\beta} \left[ \lambda^k_i - \beta (\tilde{x}^k_i - y^k_i) \right] \mid F_i \right\}, \quad i = 0, 1, \ldots, N - 1.
\]

If \( \xi_1, \xi_2, \ldots, \xi_{N-1} \) are discrete random vectors, then, by Example 2.1, the explicit formulas of those conditional expectations can be obtained. In the general cases, the Monte Carlo method has to be used to compute those conditional expectations. That will be much more complicated and time-consuming.

**Remark 3.2** In [15], two correction approaches are given. Note that in the second correction approach given by [15] one needs to compute one more metric projection. As we have seen in Remark 3.1, to compute the metric projection in the stochastic cases will take up much time. Therefore, we adapt here only the first approach of [15].

### 3.2 Convergence

In this subsection, we shall prove the weak convergence of the sequence generated by Algorithm 3.1.

Let \( K := \mathcal{C} \times \mathcal{N} \times \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n) \) and define map \( T \) from \([\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n)]^3\) to \([\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{R}^n)]^3\) by

\[
T(\theta) := \begin{pmatrix}
F(x) - \lambda \\
\lambda \\
x - y
\end{pmatrix}, \quad \forall \theta = \begin{pmatrix}
x \\
y \\
\lambda
\end{pmatrix} \in K.
\] (3.17)

Obviously, under condition (A1)–(A2), \( T \) is monotone and Lipschitz continuous. The following result gives the relationship among VI\((T, K)\), MSVI\((F, \mathcal{C} \cap \mathcal{N})\) (2.7) and the multistage stochastic variational inequality in extensive form (2.8).

**Lemma 3.1** The following assertions hold true.

(i) \( \theta^* = (x^*, y^*, \lambda^*)^\top \) is a solution to variational inequality VI\((T, K)\) if and only if \( x^* = y^* \) and \((x^*, -\lambda^*)\) is a solution to the multistage stochastic variational inequality in extensive form (2.8).

(ii) If \( \theta^* = (x^*, y^*, \lambda^*)^\top \) is a solution to variational inequality VI\((T, K)\), then \( x^* = y^* \), and \( x^* \) is a solution to MSVI\((F, \mathcal{C} \cap \mathcal{N})\) (2.7).

(iii) If the sum rule \( N_{\mathcal{C} \cap \mathcal{N}}(x^*) = N_{\mathcal{C}}(x^*) + N_{\mathcal{N}}(x^*) \) holds true and \( x^* \) is a solution to MSVI\((F, \mathcal{C} \cap \mathcal{N})\) (2.7), then there is \( \lambda^* \in \mathcal{M} \) such that \( \theta^* := (x^*, x^*, \lambda^*)^\top \) is a solution to VI\((T, K)\).
\textbf{Proof} By the definition of $T$ and $K$, $\theta^*$ is a solution to variational inequality $VI(T, K)$ if and only if $x^* = y^*$, $y^* \in N$, $-\lambda^* \in M$, and

$$-F(x^*) + \lambda^* \in N_C(x^*).$$

This proves the first assertion. Then, by [26, Theorem 3.2], the second and the third assertions hold true. \hfill \Box

From Lemma 3.1 we see that a solution to MSVI($F, C \cap N$) (2.7) can be found by solving the variational inequality $VI(T, K)$ if variational inequality $VI(T, K)$ or equivalently, the multistage stochastic variational inequality in extensive form (2.8) has a solution. In what follows, we always assume that the solution set of variational inequality $VI(T, K)$ is nonempty.

Let $G$ be defined by (3.9). Define the $G$-norm and $G$-inner product on $[L^2(\Omega, \mathcal{F}, R^n)]^3$, respectively, by

$$\|\theta\|_{L^2,G}^2 = \beta r \|x\|_{L^2}^2 + \beta \|y\|_{L^2}^2 + \frac{1}{\beta} \|\lambda\|_{L^2}^2,$$  \hspace{1cm} (3.18)

and

$$\langle \theta, \vartheta \rangle_{L^2,G} = \langle \theta, G\vartheta \rangle_{L^2} = \beta r \langle x, u \rangle_{L^2} + \beta \langle y, v \rangle_{L^2} + \frac{1}{\beta} \langle \lambda, w \rangle_{L^2},$$  \hspace{1cm} (3.19)

for any

$$\theta = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \quad \vartheta = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in [L^2(\Omega, \mathcal{F}, R^n)]^3.$$  

In addition, using the $G$-norm, we define the $G$-metric projection operator on to $K$ by

$$\Pi_{K,G}(\theta) = \{ \tilde{\theta} \in K \mid \| \tilde{\theta} - \theta \|_{L^2,G} = \min_{\vartheta \in K} \| \vartheta - \theta \|_{L^2,G} \}. \hspace{1cm} (3.20)$$

Clearly, the $G$-metric projection has the following properties:

(i) $\langle \theta - \Pi_{K,G}(\theta), \vartheta - \Pi_{K,G}(\theta) \rangle_{L^2,G} \leq 0$, $\forall \vartheta \in K$.

(ii) $\| \Pi_{K,G}(\theta) - \Pi_{K,G}(\vartheta) \|_{L^2,G} \leq \| \theta - \vartheta \|_{L^2,G}, \forall \theta, \vartheta \in [L^2(\Omega, \mathcal{F}, R^n)]^3$.

Now, we are in a position to prove the convergence of Algorithm 3.1. Let $k \in \mathbb{N}$, $\theta^k = (x^k, y^k, \lambda^k) \in C \times N \times L^2(\Omega, \mathcal{F}, R^n)$ be the random vector obtained in the $k$th iteration, and $\tilde{\theta}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in C \times N \times L^2(\Omega, \mathcal{F}, R^n)$ be the vector generated by Step 1 of Algorithm 3.1.

Define

$$\varphi(\theta^k, \tilde{\theta}^k, \zeta^k) = \langle \lambda^k - \tilde{\lambda}^k, \tilde{y}^k - y^k \rangle_{L^2} + \langle \theta^k - \tilde{\theta}^k, Gd(\theta^k, \tilde{\theta}^k, \zeta^k) \rangle_{L^2},$$  \hspace{1cm} (3.21)

where $d(\theta^k, \tilde{\theta}^k, \zeta^k)$ is defined by (3.11). We have the following estimate for $\varphi(\theta^k, \tilde{\theta}^k, \zeta^k)$.
Proposition 3.1 Assume (A2). Choose \( r > 1 + \frac{L_F}{\beta} \), where \( L_F \) is the Lipschitz constant of \( F \). Then

\[
\varphi(\theta^k, \tilde{\theta}^k, \zeta^k) \geq \frac{1}{2} \|d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2,G}^2. \tag{3.22}
\]

Proof By the definition of \( d(\theta^k, \tilde{\theta}^k, \zeta^k) \),

\[
\varphi(\theta^k, \tilde{\theta}^k, \zeta^k) = \langle \lambda^k - \tilde{\lambda}^k, \tilde{y}^k - y^k \rangle_{L^2} + \langle x^k - \tilde{x}^k, r\beta(x^k - \tilde{x}^k) - \zeta^k \rangle_{L^2}
+ \langle y^k - \tilde{y}^k, \beta(y^k - \tilde{y}^k) \rangle_{L^2} + \langle \lambda^k - \tilde{\lambda}^k, \frac{1}{\beta} (\lambda^k - \tilde{\lambda}^k) \rangle_{L^2}. \tag{3.23}
\]

By \( \lambda^k - \tilde{\lambda}^k = \beta(\tilde{x}^k - \tilde{y}^k) \), we have

\[
\langle \lambda^k - \tilde{\lambda}^k, \tilde{y}^k - y^k \rangle_{L^2} + \frac{1}{2} \langle \lambda^k - \tilde{\lambda}^k, \frac{1}{\beta} (\lambda^k - \tilde{\lambda}^k) \rangle_{L^2} + \frac{1}{2} \langle y^k - \tilde{y}^k, \beta(y^k - \tilde{y}^k) \rangle_{L^2}
= \frac{\beta}{2} \|y^k - \tilde{y}^k - \frac{1}{\beta} (\lambda^k - \tilde{\lambda}^k)\|_{L^2}^2
= \frac{\beta}{2} \|y^k - \tilde{y}^k - \tilde{x}^k + \tilde{y}^k\|_{L^2}^2
= \frac{\beta}{2} \|y^k - \tilde{x}^k\|_{L^2}^2. \tag{3.24}
\]

Let \( \zeta^k \) be defined by (3.10). Then, by the Lipschitz continuity of \( F \),

\[
\|\zeta^k\|_{L^2} = \|\beta(x^k - \tilde{x}^k) + F(x^k) - F(\tilde{x}^k)\|_{L^2} \leq (L_F + \beta)\|x^k - \tilde{x}^k\|_{L^2}
\leq r\beta\|x^k - \tilde{x}^k\|_{L^2}.
\]

It implies that

\[
\frac{1}{2} \langle \lambda^k - \tilde{\lambda}^k, \frac{1}{\beta} (\lambda^k - \tilde{\lambda}^k) \rangle_{L^2} + \frac{1}{2} \langle y^k - \tilde{y}^k, \beta(y^k - \tilde{y}^k) \rangle_{L^2}
+ \langle x^k - \tilde{x}^k, r\beta(x^k - \tilde{x}^k) - \zeta^k \rangle_{L^2}
\geq \frac{1}{2} \langle \lambda^k - \tilde{\lambda}^k, \frac{1}{\beta} (\lambda^k - \tilde{\lambda}^k) \rangle_{L^2} + \frac{1}{2} \langle y^k - \tilde{y}^k, \beta(y^k - \tilde{y}^k) \rangle_{L^2}
+ \frac{1}{2} \langle x^k - \tilde{x}^k, r\beta(x^k - \tilde{x}^k) \rangle_{L^2} - \langle x^k - \tilde{x}^k, \zeta^k \rangle_{L^2} + \frac{1}{2r\beta} \langle \zeta^k, \zeta^k \rangle_{L^2}
= \frac{1}{2\beta} \|\lambda^k - \tilde{\lambda}^k\|_{L^2}^2 + \frac{\beta}{2} \|y^k - \tilde{y}^k\|_{L^2}^2
+ \frac{1}{2} \left\langle x^k - \tilde{x}^k - \frac{1}{r\beta} \zeta^k, r\beta \left(x^k - \tilde{x}^k - \frac{1}{r\beta} \zeta^k \right) \right\rangle_{L^2}
= \frac{1}{2} \|d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2,G}^2. \tag{3.25}
\]
Combining (3.23), (3.24) with (3.25), we obtain that
\[ \varphi(\theta^k, \tilde{\theta}^k, \zeta^k) \geq \frac{B}{2} \|y^k - \tilde{x}^k\|_{L^2}^2 + \frac{1}{2} d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2, G}^2 \geq \frac{1}{2} d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2, G}^2. \]

Denote by Sol VI(T, K) the solution set of VI(T, K). The following theorem shows the contractive property of the sequence \{\theta^k\} generated by Algorithm 3.1.

**Theorem 3.1** Assume (A1)–(A2) and let \( r > \frac{L_F}{\beta} + 1, \alpha \in (0, 1) \). Then, for any \( \theta^* \in \text{Sol VI}(T, K) \),
\[ \|\theta^{k+1} - \theta^*\|_{L^2, G}^2 \leq \|\theta^k - \theta^*\|_{L^2, G}^2 - \alpha(1 - \alpha)\|d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2, G}^2. \]  

**Proof** By the definition of \( \theta^{k+1} \), we have
\[
\|\theta^{k+1} - \theta^*\|_{L^2, G}^2 = \|\theta^k - \alpha d(\theta^k, \tilde{\theta}^k, \zeta^k) - \theta^*\|_{L^2, G}^2 = \|\theta^k - \theta^*\|_{L^2, G}^2 + \alpha^2 \|d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2, G}^2 - 2\alpha \langle \theta^k - \theta^*, GD(\theta^k, \tilde{\theta}^k, \zeta^k) \rangle_{L^2} = \|\theta^k - \theta^*\|_{L^2, G}^2 + \alpha^2 \|d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2, G}^2 - 2\alpha \langle \theta^k - \tilde{\theta}^k, GD(\theta^k, \tilde{\theta}^k, \zeta^k) \rangle_{L^2} - 2\alpha \langle \lambda^k - \tilde{\lambda}^k, \tilde{\zeta}^k - y^k \rangle_{L^2} + 2\alpha \langle \lambda^k - \tilde{\lambda}^k, \tilde{\zeta}^k - y^k \rangle_{L^2} = \|\theta^k - \theta^*\|_{L^2, G}^2 + \alpha^2 \|d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2, G}^2 - 2\alpha \varphi(\theta^k, \tilde{\theta}^k, \zeta^k) + 2\alpha \langle \lambda^k - \tilde{\lambda}^k, \tilde{\zeta}^k - y^k \rangle_{L^2}.
\]

We claim that
\[ \langle \theta^* - \tilde{\theta}^k, GD(\theta^k, \tilde{\theta}^k, \zeta^k) \rangle_{L^2} + \langle \lambda^k - \tilde{\lambda}^k, \tilde{\zeta}^k - y^k \rangle_{L^2} \leq 0. \]  

Indeed, since \( \theta^* \in \text{Sol VI}(T, K) \), we have \( x^* \in C, y^* \in N, x^* = y^*, \lambda^* \in M, \) and
\[ \langle F(x^*) - \lambda^*, x - x^* \rangle_{L^2} \geq 0, \forall x \in C. \]

Note that \( \lambda^k - \tilde{\lambda}^k = \beta(\tilde{x}^k - \tilde{y}^k) \). By (3.10) and the definition of \( \tilde{x}^k \),
\[ \tilde{x}^k = \Pi_C \left\{ x^k - \frac{1}{\beta r} [F(x^k) - \lambda^k + \beta(x^k - y^k)] \right\} = \Pi_C \left\{ x^k - \frac{1}{\beta r} [F(\tilde{x}^k) - \tilde{\lambda}^k - \beta(y^k - \tilde{y}^k) + \tilde{\zeta}^k] \right\}.
\]

Then, we have
\[ \langle x^* - \tilde{x}^k, F(\tilde{x}^k) - \tilde{\lambda}^k - \beta(y^k - \tilde{y}^k) + \tilde{\zeta}^k - \beta r(x^k - \tilde{x}^k) \rangle_{L^2} \geq 0. \]
Combining (3.29) with (3.30), we obtain from the monotonicity of $F$ that

$$
\langle x^* - \tilde{x}^k, \lambda^* - \tilde{\lambda}^k - \beta (y^k - \tilde{y}^k) + \xi_x^k - \beta r (x^k - \tilde{x}^k) \rangle_{L^2} \geq 0.
$$

(3.31)

Similarly, by

$$
\tilde{y}^k = \Pi_N \{ y^k - \frac{1}{\beta} (\tilde{\lambda}^k - \beta (\tilde{y}^k - y^k)) \},
$$

we have $\tilde{y}^k \in N$ and

$$
\langle y^* - \tilde{y}^k, \tilde{\lambda}^k - \beta (\tilde{y}^k - y^k) \rangle_{L^2} = \langle x^* - \tilde{x}^k, \lambda^* - \beta r (x^k - \tilde{x}^k) \rangle_{L^2} \geq 0.
$$

(3.32)

Since $\lambda^* \in M, y^*, \tilde{y}^k \in N, \langle \lambda^* - \tilde{\lambda}^k, y^* - \tilde{y}^k \rangle_{L^2} = 0$. Then, by (3.32),

$$
\langle y^* - \tilde{y}^k, \tilde{\lambda}^k - \lambda^* \rangle_{L^2} \geq 0.
$$

It implies that

$$
\langle y^* - \tilde{y}^k, \beta (y^k - \tilde{y}^k) + (\tilde{\lambda}^k - \lambda^*) \rangle_{L^2} \geq \langle y^* - \tilde{x}^k, \beta (y^k - \tilde{y}^k) \rangle_{L^2}.
$$

(3.33)

In addition, by $\tilde{\lambda}^k = \lambda^k - \beta (\tilde{x}^k - \tilde{y}^k)$ and $x^* - y^* = 0$,

$$
\langle \lambda^* - \tilde{\lambda}^k, \frac{1}{\beta} (\tilde{\lambda}^k - \lambda^*) \rangle_{L^2} = \langle \lambda^* - \tilde{x}^k, y^k - \tilde{y}^k \rangle_{L^2} = \langle \lambda^* - \tilde{x}^k, y^k - y^* - (\tilde{x}^k - x^*) \rangle_{L^2}.
$$

(3.34)

Combining (3.31), (3.33) with (3.34), we obtain

$$
\langle \theta^* - \tilde{\theta}^k, Gd(\theta^k, \tilde{\theta}^k, \xi^k) \rangle_{L^2} = \langle x^* - \tilde{x}^k, r \beta (x^k - \tilde{x}^k) - \xi_x^k \rangle_{L^2} + \langle y^* - \tilde{y}^k, \beta (y^k - \tilde{y}^k) \rangle_{L^2}
$$

$$
+ \langle \lambda^* - \tilde{\lambda}^k, \frac{1}{\beta} (\lambda^k - \tilde{\lambda}^k) \rangle_{L^2}, \langle \theta^* - \tilde{\theta}^k, Gd(\theta^k, \tilde{\theta}^k, \xi^k) \rangle_{L^2} = \langle x^* - \tilde{x}^k, \lambda^* - \tilde{\lambda}^k - \beta (y^k - \tilde{y}^k) \rangle_{L^2} + \langle y^* - \tilde{y}^k, \tilde{\lambda}^k - \lambda^* - \beta (\tilde{y}^k - y^k) \rangle_{L^2}
$$

$$
+ \langle \lambda^* - \tilde{\lambda}^k, (\tilde{x}^k - x^*) - (\tilde{y}^k - y^*) \rangle_{L^2}
$$

$$
= \langle x^* - \tilde{x}^k - y^* + \tilde{y}^k, \beta (\tilde{y}^k - y^k) \rangle_{L^2}
$$

$$
= \langle \lambda^* - \tilde{\lambda}^k, \tilde{y}^k - y^k \rangle_{L^2}.
$$

(3.35)

This proves (3.28).

Finally, by (3.27), (3.28) and Proposition 3.1, we have

$$
\|\theta^{k+1} - \theta^*\|_{L^2,G}^2 \leq \|\theta^k - \theta^*\|_{L^2,G}^2 + \alpha^2 d(\theta^k, \tilde{\theta}^k, \xi^k)_{L^2,G}^2 - 2\alpha \varphi(\theta^k, \tilde{\theta}^k, \xi^k)_{L^2,G}^2.
$$
\[ \leq \| \theta^k - \theta^* \|_{L^2, G}^2 + \alpha^2 \| d(\theta^k, \tilde{\theta}^k, \zeta^k) \|_{L^2, G}^2 - \alpha \| d(\theta^k, \tilde{\theta}^k, \zeta^k) \|_{L^2, G}^2 \]

\[ = \| \theta^k - \theta^* \|_{L^2, G}^2 - \alpha (1 - \alpha) \| d(\theta^k, \tilde{\theta}^k, \zeta^k) \|_{L^2, G}^2. \]

This completes the proof of Theorem 3.1 \[ \Box \]

**Theorem 3.2** Assume (A1)–(A2). Let \( r > \frac{L_F}{\beta} + 1 \) and \( \alpha \in (0, 1) \). Then, the sequence \( \{ \theta^k \}_{k=0}^\infty \) generated by Algorithm 3.1 converges weakly to some \( \theta^* \in \text{Sol VI}(T, K) \).

**Proof** The proof is divided into four steps.

Step 1: In this step, we shall prove that there is a constant \( K > 0 \) s.t.

\[ \| e_G(\tilde{\theta}^k, T, K) \|_{L^2, G} \leq K \| \theta^k - \tilde{\theta}^k \|_{L^2, G}, \] (3.36)

where

\[ e_G(\tilde{\theta}^k, T, K) = \tilde{\theta}^k - \Pi_{K, G}(\tilde{\theta}^k - G^{-1} T(\tilde{\theta}^k)). \] (3.37)

Define \( \Lambda : L^2(\Omega, \mathcal{F}, R^n) \to [L^2(\Omega, \mathcal{F}, R^n)]^3 \) by

\[ \Lambda(x) = \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix}, \quad \forall \ x \in L^2(\Omega, \mathcal{F}, R^n). \] (3.38)

Then, Step 1 of Algorithm 3.1 can be rewritten as

\[ \tilde{\theta}^k = \Pi_{K, G}\{ \theta^k - G^{-1}[T(\tilde{\theta}^k) - \beta \Lambda(y^k - \tilde{y}^k) + \zeta^k] \}. \] (3.39)

It follows from the Lipschitz continuity of \( G \)-metric projection that

\[ \| e_G(\tilde{\theta}^k, T, K) \|_{L^2, G} \]

\[ = \| \Pi_{K, G}\{ \theta^k - G^{-1}[T(\tilde{\theta}^k) - \beta \Lambda(y^k - \tilde{y}^k) + \zeta^k] \} - \Pi_{K, G}(\tilde{\theta}^k - G^{-1} T(\tilde{\theta}^k)) \|_{L^2, G} \]

\[ \leq \| \theta^k - \tilde{\theta}^k - G^{-1}[\beta \Lambda(y^k - \tilde{y}^k) + \zeta^k] \|_{L^2, G} \]

\[ \leq \| \theta^k - \tilde{\theta}^k \|_{L^2, G} + \| G^{-1} \beta \Lambda(y^k - \tilde{y}^k) \|_{L^2, G} + \| G^{-1} \zeta^k \|_{L^2, G}. \] (3.40)

By the definition of \( \Lambda \),

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\[ \| G^{-1} \beta A (\tilde{y}^k - y^k) \|_{L^2, G} = \left\| G^{-1} \begin{pmatrix} \beta (\tilde{y}^k - y^k) \\ - \beta (\tilde{y}^k - y^k) \\ 0 \end{pmatrix} \right\|_{L^2, G} \]
\[ = \sqrt{\frac{1 + r}{r} \left[ \beta \| \tilde{y}^k - y^k \|^2_{L^2} \right]} \]
\[ \leq \sqrt{\frac{1 + r}{r} \| \tilde{\theta}^k - \theta^k \|^2_{L^2}}. \]

It follows from (3.40) that
\[ \| e_G(\tilde{\theta}^k, T, K) \|_{L^2, G} \]
\[ \leq \| \theta^k - \tilde{\theta}^k \|_{L^2, G} + \sqrt{\frac{1 + r}{r} \| \tilde{\theta}^k - \theta^k \|^2_{L^2, G}} + \| G^{-1} \xi^k \|_{L^2, G}. \] (3.41)

In addition, by the definition of \( \zeta^k \),
\[ \| G^{-1} \xi^k \|^2_{L^2, G} = \frac{1}{r \beta} \| \xi^k \|^2_{L^2} \]
\[ = \frac{1}{r \beta} \| F(x^k) - F(\tilde{x}^k) + \beta (x^k - \tilde{x}^k) \|^2_{L^2} \]
\[ \leq \frac{1}{r \beta} (L_F + \beta)^2 \| x^k - \tilde{x}^k \|^2_{L^2} \]
\[ \leq \left( \frac{L_F + \beta}{r \beta} \right)^2 \| \theta^k - \tilde{\theta}^k \|^2_{L^2, G}. \] (3.42)

Combining (3.41) with (3.42), we obtain that
\[ \| e_G(\tilde{\theta}^k, T, K) \|_{L^2, G} \leq K \| \theta^k - \tilde{\theta}^k \|_{L^2, G}, \]
where
\[ K = 1 + \sqrt{\frac{1 + r}{r} + \frac{L_F + \beta}{r \beta}}. \]

Step 2: In this step, we shall prove that, for any \( \theta^* \in Sol\ VI(T, K) \),
\[ \| \theta^{k+1} - \theta^* \|^2_{L^2, G} \leq \| \theta^k - \theta^* \|^2_{L^2, G} - (\alpha - \alpha^2) (1 - \frac{L_F + \beta}{r \beta})^2 \| \theta^k - \tilde{\theta}^k \|^2_{L^2, G}. \] (3.43)

By Theorem 3.1,
\[ \| \theta^{k+1} - \theta^* \|^2_{L^2, G} \leq \| \theta^k - \theta^* \|^2_{L^2, G} - (\alpha - \alpha^2) \| d(\theta^k, \tilde{\theta}^k, \xi^k) \|^2_{L^2, G}. \] (3.44)
Since \( r > \frac{L_F}{r} + 1, \frac{L_F + \beta}{r \beta} \in (0, 1) \). Then, by (3.42), we have

\[
\|d(\theta^k, \tilde{\theta}^k, \zeta^k)\|_{L^2, G} \leq \|\theta^k - \tilde{\theta}^k - G^{-1} \zeta^k\|_{L^2, G} \leq \|\theta^k - \tilde{\theta}^k\|_{L^2, G} - 2\langle \theta^k - \tilde{\theta}^k, G^{-1} \zeta^k \rangle_{L^2, G} + \|G^{-1} \zeta^k\|_{L^2, G} \geq \|\theta^k - \tilde{\theta}^k\|_{L^2, G} - 2\|\tilde{\theta}^k\|_{L^2, G}\|G^{-1} \zeta^k\|_{L^2, G} + \|G^{-1} \zeta^k\|_{L^2, G} \geq (\|\theta^k - \tilde{\theta}^k\|_{L^2, G} - \|G^{-1} \zeta^k\|_{L^2, G})^2 \geq \left(1 - \frac{L_F + \beta}{r \beta}\right)^2 \|\theta^k - \tilde{\theta}^k\|_{L^2, G}.
\] (3.45)

Substituting (3.45) into (3.44), we obtain (3.43).

Step 3: In this step, we prove that any weak cluster point of \( \{\theta^k\}_{k=0}^{\infty} \) is a solution of \( VI(T, K) \).

By (3.43), \( \{\theta^k\}_{k=0}^{\infty} \) is bounded. Moreover,

\[
(\alpha - \alpha^2) \left(1 - \frac{L_F + \beta}{r \beta}\right)^2 \sum_{k=0}^{m} \|\theta^k - \tilde{\theta}^k\|_{L^2, G} \leq \|\theta^0 - \theta^*\|_{L^2, G} - \|\theta^{m+1} - \theta^*\|_{L^2, G} \leq \|\theta^0 - \theta^*\|_{L^2, G}, \quad \forall \ m \in \mathbb{N}.
\] (3.46)

Letting \( m \to \infty \), we obtain that

\[
(\alpha - \alpha^2) \left(1 - \frac{L_F + \beta}{r \beta}\right)^2 \sum_{k=0}^{\infty} \|\theta^k - \tilde{\theta}^k\|_{L^2, G} \leq \|\theta^0 - \theta^*\|_{L^2, G}. \tag{3.46}
\]

Therefore,

\[
\lim_{k \to \infty} \|\theta^k - \tilde{\theta}^k\|_{L^2, G} = 0. \tag{3.47}
\]

Define

\[
\tilde{\theta}^k = \Pi_{K^*_G}(\tilde{\theta}^k - G^{-1} T(\tilde{\theta}^k)).
\]

Clearly,

\[
\tilde{\theta}^k = \Pi_{K^*_G}\{\tilde{\theta}^k - G^{-1}[T(\tilde{\theta}^k) + G(\tilde{\theta}^k - \tilde{\theta}^k)]\} = \Pi_K\{\tilde{\theta}^k - [T(\tilde{\theta}^k) + G(\tilde{\theta}^k - \tilde{\theta}^k)]\}.
\]

Letting \( N_K(\tilde{\theta}^k) \) be the normal cone to \( K \) at \( \tilde{\theta}^k \in K \), it follows that

\[
-T(\tilde{\theta}^k) - G(\tilde{\theta}^k - \tilde{\theta}^k) \in N_K(\tilde{\theta}^k). \tag{3.48}
\]

Adding \( T(\tilde{\theta}^k) \) to both sides of (3.48), we have that

\[
T(\tilde{\theta}^k) - T(\tilde{\theta}^k) - G(\tilde{\theta}^k - \tilde{\theta}^k) \in N_K(\tilde{\theta}^k) + T(\tilde{\theta}^k), \tag{3.49}
\]
i.e.,
\[(\tilde{\theta}^k, T(\tilde{\theta}^k) - T(\tilde{\theta}^k) - G(\tilde{\theta}^k - \tilde{\theta}^k)) \in Gph(N_K(\cdot) + T(\cdot)).\] (3.50)

By (3.36) and (3.47),
\[
\lim_{k \to \infty} \|\tilde{\theta}^k - \tilde{\theta}^k\|_{L^2,G} = \lim_{k \to \infty} \|\epsilon_G(\tilde{\theta}^k, T, K)\|_{L^2,G} \leq \lim_{k \to \infty} K \|\theta^k - \tilde{\theta}^k\|_{L^2,G} = 0.
\] (3.51)

By condition (A2), T is Lipschitz continuous. Then we have
\[
\lim_{k \to \infty} \|T(\tilde{\theta}^k) - T(\tilde{\theta}^k) - G(\tilde{\theta}^k - \tilde{\theta}^k)\|_{L^2,G} = 0.
\]

Since \(\{\theta^k\}_{k=0}^\infty\) is bounded, it possesses a weakly convergent subsequence \(\{\theta^j_k\}_{j=0}^\infty\).
Assume \(\theta^j_k \to \theta^\infty\). By (3.47) and (3.51), we have \(\tilde{\theta}^j_k \to \theta^\infty\) and \(\tilde{\theta}^j_k \to \theta^\infty\). Under condition (A1)–(A2), T is Lipschitz continuous and monotone. By Lemma 2.4, \(N_K(\cdot) + T(\cdot)\) is maximal monotone. Then, by (3.49) and Lemma 2.3, we have
\[
0 \in T(\theta^\infty) + N_K(\theta^\infty),
\] (3.52)
i.e., \(\theta^\infty \in Sol VI(T, K)\).

Step 4: In this step, we prove that \(\{\theta^k\}\) has only one weak cluster point.
Assume \(\{\theta^k_i\}_{i=0}^\infty \subset \{\theta^k_i\}_{k=0}^\infty\), \(\{\theta^k_j\}_{j=0}^\infty \subset \{\theta^k_i\}_{k=0}^\infty\), \(\theta^k_i \to \theta^\infty_1\) as \(i \to \infty\), and \(\theta^k_j \to \theta^\infty_2\) as \(j \to \infty\). By step 3, \(\theta^\infty_1, \theta^\infty_2 \in Sol VI(T, K)\). Then, replacing \(\theta^*\) in (3.43) by \(\theta^\infty_1\) and \(\theta^\infty_2\), respectively, we have \(\|\theta^k - \theta^\infty_1\|_{L^2,G}^2\) and \(\|\theta^k - \theta^\infty_2\|_{L^2,G}^2\) are decreasing and bounded. Assume \(\|\theta^k - \theta^\infty_1\|_{L^2,G}^2 \to l_1\) and \(\|\theta^k - \theta^\infty_2\|_{L^2,G}^2 \to l_2\). Then,
\[
\begin{align*}
\|\theta^k - \theta^\infty_1\|_{L^2,G}^2 - \|\theta^k - \theta^\infty_2\|_{L^2,G}^2 &= \|\theta^k - \theta^\infty_2 + \theta^\infty_2 - \theta^\infty_1\|_{L^2,G}^2 - \|\theta^k - \theta^\infty_2\|_{L^2,G}^2 \\
&= \|\theta^\infty_2 - \theta^\infty_1\|_{L^2,G}^2 + 2(\theta^k - \theta^\infty_2, \theta^\infty_2 - \theta^\infty_1)_{L^2,G}.
\end{align*}
\] (3.53)

By \(\theta^{k_i} \to \theta^\infty_1\),
\[
\begin{align*}
l_1 - l_2 &= \lim_{i \to \infty} \left[\|\theta^{k_i} - \theta^\infty_1\|_{L^2,G}^2 - \|\theta^{k_i} - \theta^\infty_2\|_{L^2,G}^2\right] \\
&= \lim_{i \to \infty} \left[\|\theta^\infty_2 - \theta^\infty_1\|_{L^2,G}^2 + 2(\theta^{k_i} - \theta^\infty_2, \theta^\infty_2 - \theta^\infty_1)_{L^2,G}\right] \\
&= \|\theta^\infty_2 - \theta^\infty_1\|_{L^2,G}^2 - 2\|\theta^\infty_2 - \theta^\infty_1\|_{L^2,G} \\
&= -\|\theta^\infty_2 - \theta^\infty_1\|_{L^2,G}^2.
\end{align*}
\] (3.54)

On the other hand, by \(\theta^{k_j} \to \theta^\infty_2\),
\[
\begin{align*}
l_1 - l_2 &= \lim_{j \to \infty} \left[\|\theta^{k_j} - \theta^\infty_1\|_{L^2,G}^2 - \|\theta^{k_j} - \theta^\infty_2\|_{L^2,G}^2\right]
\end{align*}
\]
\[ \lim_{j \to \infty} \left[ \| \theta_2^\infty - \theta_1^\infty \|^2_{L^2,G} + 2(\theta^{kj} - \theta_2^\infty, \theta_2^\infty - \theta_1^\infty)_{L^2,G} \right] = \| \theta_2^\infty - \theta_1^\infty \|^2_{L^2,G}. \]

(3.55)

By (3.54) and (3.55), \( \| \theta_2^\infty - \theta_1^\infty \|^2_{L^2,G} = 0 \). This proves the uniqueness of the weak cluster.

This completes the proof of Theorem 3.2.

\[ \square \]

**Theorem 3.3** Under the conditions in Theorem 3.2, the sequence \( \{ x^k \}_{k=0}^\infty \) converges weakly to a solution \( x^* \) to MSVI \(( F, C \cap N ) \) (2.7).

**Proof** It follows directly from Theorem 3.2 and Lemma 3.1.

\[ \square \]

**4 The Discrete Cases and Numerical Examples**

In this section, we shall consider the special case that the sample space is a finite set. On such discrete sample space, all the random vectors are discrete type random vectors, and in such case, as discussed in [26], the multistage stochastic variational inequality is actually defined on a finite-dimensional Hilbert space.

Let \( m \in \mathbb{N} \). Consider the sample space

\[ \hat{\Omega} := \{ \hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_m \}. \]

Define \( \hat{\mathcal{F}} = 2^{\hat{\Omega}} \), i.e., \( \hat{\mathcal{F}} \) is the \( \sigma \)-field composed by all the subsets of \( \hat{\Omega} \). Then, any map \( \hat{\Omega} \to \mathbb{R}^n \) are \( \hat{\mathcal{F}} \)-measurable. Given \( \{ p_i \}_{i=1}^m, p_i > 0 \) for any \( i = 1, 2, \ldots, m \) and \( \sum_{i=1}^m p_i = 1 \). Let \( \hat{P} : \hat{\mathcal{F}} \to [0, 1] \) be defined by \( \hat{P}(\emptyset) = 0, \hat{P}(\{ \hat{\omega}_i \}) = p_i, i = 1, 2, \ldots, m \) and

\[ \hat{P}(A) = \sum_{\hat{\omega}_i \in A} p_i, \quad \forall \ A \in \hat{\mathcal{F}}. \]

Then, \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \) is a (discrete) probability space.

Let \( \hat{\xi} : \hat{\Omega} \to \mathbb{R}^n \) be a random vector belonging to \( L^2(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{R}^n) \). The \( \hat{\xi} \) can be treated as a finite-dimensional vector in the product Euclidean space \( (\mathbb{R}^n)^m \) defined by \( \hat{\xi} = (\hat{\xi}_1^\top, \hat{\xi}_2^\top, \ldots, \hat{\xi}_m^\top)^\top \) with \( \hat{\xi}_i = \hat{\xi}(\hat{\omega}_i), i = 1, 2, \ldots, m \). The norm of \( \xi \) is

\[ \| \hat{\xi} \|_{L^2} = \left( \sum_{i=1}^m |\hat{\xi}_i|^2 p_i \right)^{\frac{1}{2}} = \left[ (\hat{\xi}_1^\top, \hat{\xi}_2^\top, \ldots, \hat{\xi}_m^\top)^\top \mathcal{P}(\hat{\xi}_1^\top, \hat{\xi}_2^\top, \ldots, \hat{\xi}_m^\top)^\top \right]^\frac{1}{2}, \]

(4.1)
where
\[ \mathcal{P} := \begin{bmatrix} p_1 I & \cdots & p_m I \end{bmatrix} \]

is a positive definite matrix in \( R^{mn \times mn} \). Here \( I \in R^{n \times n} \) is the \( n \times n \)-identity matrix. In addition, for \( \hat{\xi}, \hat{\eta} \in L^2(\hat{\Omega}, \hat{\mathcal{F}}, R^n) \), the inner product of \( \hat{\xi} \) and \( \hat{\eta} \) is
\[ \left\langle \hat{\xi}, \hat{\eta} \right\rangle_{L^2} = (\hat{\xi}_1^\top, \hat{\xi}_2^\top, \ldots, \hat{\xi}_m^\top) \mathcal{P}(\hat{\eta}_1^\top, \hat{\eta}_2^\top, \ldots, \hat{\eta}_m^\top)^\top. \]

Therefore, \( L^2(\hat{\Omega}, \hat{\mathcal{F}}, R^n) \) is isomorphic to \( (R^n)^m \) (with norm \( \|x\|_{\mathcal{P}} = x^\top \mathcal{P} x \) and inner product \( \langle x, y \rangle_{\mathcal{P}} = x^\top \mathcal{P} y \) for any \( x, y \in (R^n)^m \)). Similarly, in the discrete case, the space \( \left[ L^2(\hat{\Omega}, \hat{\mathcal{F}}, R^n) \right]^3 \) with norm \( \| \cdot \|_{L^2, \mathcal{G}} \) and inner product \( \langle \cdot, \cdot \rangle_{L^2, \mathcal{G}} \) (See (3.18) and (3.19) for their definitions) is also isomorphic to a finite-dimensional Hilbert space.

Since any bounded sequence in finite-dimensional space is sequential compact, when the sample space is a finite set, the convergence of Algorithm 3.1 follows directly from Theorem 3.2 and 3.3.

**Theorem 4.1** Suppose the conditions in Theorem 3.2 hold true. Then, in the case that the sample space is a finite set, the sequence \( \{\theta^k\} \) generated by Algorithm 3.1 converges to some \( \theta^* \in \text{Sol V1}(T, \mathcal{K}) \). Consequently, the sequence \( \{x^k\}_{k=0}^\infty \) converges to a solution \( x^* \) to MSVI(\( F, \mathcal{C} \cap \mathcal{N} \)) (2.7).

In the following we give some test examples.

**Example 4.1** Let \( m \in \mathbb{N} \), \( \hat{\Omega} := \{\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_m\} \). Generate randomly a discrete probability distribution \( \hat{P}((\hat{\omega}_i)) = p_i, i = 1, 2, \ldots, m \) and numbers \( \hat{\xi}_i \in R, i = 1, 2, \ldots, m \). Let random variable \( \hat{\xi} : \hat{\Omega} \rightarrow R \) be defined by \( \hat{\xi}(\hat{\omega}_i) = \hat{\xi}_i, \), \( i = 1, 2, \ldots, m \) and assume
\[ \hat{P}(\hat{\xi} = \hat{\xi}_i) = \hat{P}((\hat{\omega}_i)) = p_i, i = 1, 2, \ldots, m. \]

Let \( \hat{\mathcal{F}} = 2^{\hat{\Omega}}, (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \) and \( L^2(\hat{\Omega}, \hat{\mathcal{F}}, R^n) \) be defined as above. Let \( n_0, n_1 \in \mathbb{N} \), \( n_0 + n_1 = n \). For any \( x \in L^2(\hat{\Omega}, \hat{\mathcal{F}}, R^n) \), let \( x(\hat{\omega}_i) = (x_0(\hat{\omega}_i), x_1(\hat{\omega}_i)) \), where \( x_0(\hat{\omega}_i) \in R^{n_0}, x_1(\hat{\omega}_i) \in R^{n_1}, i = 1, 2, \ldots, m \). Let \( \hat{\mathcal{F}}_0 = \{\emptyset, \hat{\Omega}, \hat{\mathcal{F}}_1 = \sigma(\hat{\xi}) \). From the definition of \( \hat{\xi}, \hat{\mathcal{F}}_1 = 2^{\hat{\Omega}} \) and
\[ \mathcal{N} = \left\{ x : \hat{\Omega} \rightarrow R^n \mid x(\hat{\omega}_i) = (x_0, x_1(\hat{\xi}(\hat{\omega}_i))), i = 1, 2, \ldots, m \right\}, \]
i.e., the first \( n_0 \) components of \( x \in \mathcal{N} \) is deterministic and equals to some \( x_0 \in R^{n_1} \), the last \( n_1 \) components of \( x \) is a function of \( \hat{\xi} \). In addition,
\[ \mathcal{M} = \mathcal{N}^\perp = \left\{ y : \hat{\Omega} \rightarrow R^n \mid y(\hat{\omega}_i) = (y_0(\hat{\omega}_i), 0) \right\}. \]
i = 1, 2, \ldots, m, \mathbb{E}y_0 = \sum_{i=1}^{m} y_0(\hat{\omega}_i) p_i = 0$, i.e., the first $n_0$ components of $y \in \mathcal{M}$ is a random vector taking values in $\mathbb{R}^{n_0}$ with expectation 0, the last $n_1$ components of $y$ is identically 0.

Let $C(\hat{\omega}_i) \equiv [-1, 1]^n$. Then,

$$C = \{ x : \hat{\Omega} \rightarrow \mathbb{R}^n \mid x(\hat{\omega}_i) \in [-1, 1]^n, i = 1, 2, \ldots, m \}.$$ 

Generate randomly nonzero positive semi-definite matrices $M_i \in \mathbb{R}^{n \times n}$ and vectors $b_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$, and assume $M(\hat{\xi}_i) = M_i$ and $b(\hat{\xi}_i) = b_i$. Define $\hat{F} : \mathcal{L}^2(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{R}^n) \rightarrow \mathcal{L}^2(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{R}^n)$ by

$$\hat{F}(x) = M(\hat{\xi})x + b(\hat{\xi}), \quad \forall x \in \mathcal{L}^2(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{R}^n).$$

It is easy to check that $\hat{F}$ is monotone and Lipschitz continuous.

Now we solve the MSVI($\hat{F}, C \cap \mathcal{N}$) numerically by Algorithm 3.1. Choose arbitrarily

$$\theta^0 = (x^0, y^0, \lambda^0) \in C \times \mathcal{N} \times \mathcal{L}^2(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{R}^n).$$

Let $\{ \theta^k \}_{k=0}^{\infty} = \{(x^k, y^k, \lambda^k)\}_{k=0}^{\infty}$ be the sequence generated by Algorithm 3.1. By Lemma 3.1, $x^k$ is a solution to MSVI($\hat{F}, C \cap \mathcal{N}$) if and only if $x^k = y^k, \lambda^k \in \mathcal{M}$ and

$$-\hat{F}(x^k)(\hat{\omega}_i) + \lambda^k(\hat{\omega}_i) \in N_{C(\hat{\omega}_i)}(x^k(\hat{\omega}_i)), \quad i = 1, 2, \ldots, m.$$ 

That leads us to define the stopping criterion

$$Err(x^k) := \max_i |x^k(\hat{\omega}_i) - \Pi_C(\hat{\omega}_i)(x^k(\hat{\omega}_i) - \hat{F}(x^k)(\hat{\omega}_i) + \lambda^k(\hat{\omega}_i))|$$

$$+ \sum_{i=1}^{m} |x^k(\hat{\omega}_i) - y^k(\hat{\omega}_i)|^2 p_i < \varepsilon$$

for some sufficiently small $\varepsilon > 0$.

In the implementation of Algorithm 3.1, we choose $\alpha = 0.61, \beta = 1.1L_\hat{F}$ with

$$L_\hat{F} = \max\{\sigma_i \mid i = 1, 2, \ldots, m\},$$

where $\sigma_i$ is the largest eigenvalue of $M_i, i = 1, 2, \ldots, m$, and, $r = 1.1 + (L_\hat{F}/\beta)$.

In Table 1, we report the numerical performance for PHA and Algorithm 3.1(Alg.3.1). We report the average number of iterations (Avg-iter) and the average running time in seconds (Avg-time(s)). It can be found from Table 1 that the average running time of Algorithm 3.1 is shorter than that of PHA even though the average number of iterations of Algorithm 3.1 is more than PHA.
Example 4.2 Let \((\Omega, \mathcal{F}, \mathbb{P}, P)\) be a complete filtered probability space with the filtration \(\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq 1}\), on which a one-dimensional standard Wiener process \(W(\cdot)\) is defined such that \(\mathbb{F}\) is the natural filtration generated by \(W(\cdot)\) (augmented by all the \(P\)-null sets).

Let \(n = m = 1, U = [0, 1]\) and \(\eta \in L^2(\Omega, \mathcal{F}_1, \mathbb{R})\). Consider the discrete control system
\[
\begin{align*}
x_{i+1} &= x_i + [x_i - u_i] \Delta + u_i \Delta W_i, \quad i = 0, 1, \ldots, N - 1, \\
x_0 &= 1
\end{align*}
\] (4.2)
and cost function
\[
J^N(\mathcal{U}^N) = \frac{1}{2} \mathbb{E}|x_N - \eta|^2.
\] (4.3)

Here, \(\mathcal{U}^N = (u_0, u_1, \ldots, u_{N-1}) : \Omega \to \mathbb{R}^N, u_i(\omega) \in U, \text{a.s. } \omega \in \Omega, \mathbb{F}^N = (x_0, x_1, \ldots, x_N) : \Omega \to \mathbb{R}^{N+1}, \Delta = 1/N, \Delta W_i = W((i + 1)/N) - W(i/N), \quad i = 0, 1, \ldots, N - 1\) are independent identically distributed Gaussian random variables with mean 0 and variance \(1/N\).

Clearly, the discrete-times stochastic optimal control problem (2.26) in Example 2.2 with control system (4.2) and cost function (4.3) is a discretized approximation of the continuous-time stochastic optimal control problem (2.23) with control system
\[
\begin{align*}
dx(t) &= (x(t) - u(t))dt + u(t) dW(t), \quad t \in [0, 1], \\
x_0 &= 1
\end{align*}
\]
and cost function
\[
J(u) = \frac{1}{2} \mathbb{E}|x(1) - \eta|^2.
\]

It follows from Example 2.2 that the first-order necessary condition for the discrete-times optimal control problem (2.26) is a multistage stochastic variational inequality defined on a general probability space. In order to obtain an approximation of that multistage stochastic variational inequality in a discrete sample space, we use a sequence of random walks to approximate the Wiener process.

Let us consider a sequence \(\{\xi_i\}_{i=1}^N(\ell \in \mathbb{N})\) of independent identically distributed random variables such that for each \(i\), \(P(\xi_i = 1) = P(\xi_i = -1) = 1/2\), and, we

\[\text{Table 1} \quad \text{Numerical results for Example 4.1}\]

| \(m\) = 10, \(n_1 = n_2 = 5\) | \(m\) = 20, \(n_1 = n_2 = 30\) |
|-----------------|-----------------|
| \text{Avg-time (s)} | \text{Avg-iter} | \text{Avg-time (s)} | \text{Avg-iter} |
| PHA | 1.7628 | 28 | 35.0378 | 153 |
| Alg. 3.1 | 0.2028 | 134 | 14.1337 | 2180 |
| PHA | 34.0394 | 363 | 486.331 | 2001 |
| Alg. 3.1 | 0.8424 | 322 | 24.6326 | 4086 |
define

\[ S_0 = 0, \quad S_i = \sum_{k=1}^{i} \xi_k, \quad i = 1, 2, \ldots, N \ell \]

and

\[ Y_0 = 0, \quad Y_{i}^{\ell} = \frac{1}{\sqrt{N \ell}} S_{i \ell}, \quad i = 1, 2, \ldots, N. \]

By [16, Theorem 4.17, Chapter 2], \((Y_{1}^{\ell}, Y_{2}^{\ell}, \ldots, Y_{N}^{\ell})\) converges to \((W(1/N), W(2/N), \ldots, W(1))\) in law as \(\ell \to \infty\). Define

\[
\hat{\Omega} = \prod_{i=0}^{N-1} \{-1, 1\}.
\] (4.4)

and let \(\hat{\mathcal{F}}\) be all the subsets of \(\hat{\Omega}\) and \(\hat{P}\) be the probability measure induced by the binomial distribution. Then, \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\) is a discrete probability space. Denote \(\Delta Y_{i}^{\ell} := (S_{(i+1)\ell} - S_{i\ell})/\sqrt{N \ell}, i = 0, 1, \ldots, N - 1\) and let \(\hat{\mathcal{F}}_0 = \{\emptyset, \Omega\}, \hat{\mathcal{F}}_i = \sigma(\Delta Y_{0}^{\ell}, \Delta Y_{1}^{\ell}, \ldots, \Delta Y_{i-1}^{\ell}), i = 1, 2, \ldots, N - 1\). We define

\[
\mathcal{N} := \left\{ \hat{u}^N = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{N-1}) \in L^2(\hat{\Omega}, \hat{\mathcal{F}}, R^N) \mid \hat{u}_i \in L^0(\hat{\Omega}, \hat{\mathcal{F}}_i, R), i = 0, 1, \ldots, N - 1 \right\}
\]

and

\[
\mathcal{C} := \left\{ \hat{u}^N = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{N-1}) \in L^2(\Omega, \hat{\mathcal{F}}, R^N) \mid \hat{u}_i(\omega) \in U \text{ a.s. } \omega \in \Omega, i = 0, 1, \ldots, N - 1 \right\}.
\]

Replacing \(\Delta W_i\) by \(\Delta Y_{i}^{\ell}, i = 0, 1, \ldots, N - 1\) in (4.2), we obtain the following stochastic approximation difference equation

\[
\hat{x}_{i+1} = \hat{x}_i + [\hat{x}_i - \hat{u}_i] \Delta + \hat{u}_i \Delta Y_{i}^{\ell}, \quad i = 0, 1, \ldots, N - 1, \quad \hat{x}_0 = 1,
\] (4.5)

Similarly to Example 2.2, we denote \(\hat{\Psi}_i = 1 + \Delta, \hat{\Lambda}_i = -\Delta + \Delta Y_{i}^{\ell}, i = 0, 1, \ldots, N - 1\) and define

\[
\hat{Z}_i = \left[ \prod_{j=i+1}^{N-1} \hat{\Psi}_j \right] \hat{\Lambda}_i, \quad i = 0, 1, \ldots, N - 1, \quad \hat{\xi} = \sum_{i=0}^{N-1} \hat{Z}_i.
\]
Then, \( \hat{\eta} = \prod_{i=0}^{N-1} \hat{\Psi} + \hat{\zeta} \) is the final value of the solution to (4.5) with control \( \hat{\Psi}^N \equiv (1, 1, \ldots, 1) \).

Define the random walk approximation of cost function (4.3) by

\[
\hat{J}^N(\hat{\Psi}^N) = \frac{1}{2} \mathbb{E}\left[\hat{x}_N - \hat{\eta}\right]^2 = \frac{1}{2} \mathbb{E}\left[\sum_{i=0}^{N-1} \hat{Z}_i \hat{u}_i - \hat{\zeta}\right]^2.
\]

(4.6)

The discretized approximation stochastic optimal control problem of random walks is: To find \((\hat{\Psi}^N)^* \in C \cap \mathcal{N}\) such that

\[
\hat{J}^N((\hat{\Psi}^N)^*) = \min_{\hat{\Psi}^N \in C \cap \mathcal{N}} \hat{J}^N(\hat{\Psi}^N).
\]

(4.7)

Clearly, \((\hat{\Psi}^N)^* \equiv (1, 1, \ldots, 1)\) is the optimal solution to (4.7).

Letting

\[
M = (\hat{Z}_0, \hat{Z}_1, \ldots, \hat{Z}_{N-1})^\top(\hat{Z}_0, \hat{Z}_1, \ldots, \hat{Z}_{N-1}), \quad b = \hat{\zeta}(\hat{Z}_0, \hat{Z}_1, \ldots, \hat{Z}_{N-1})^\top,
\]

we have

\[
D \hat{J}^N(\hat{\Psi}^N) = M \hat{\Psi}^N - b, \quad \forall \hat{\Psi}^N \in L^2(\hat{\Omega}, \hat{\mathcal{F}}, R^N).
\]

(4.8)

To solve (4.7), we only need to solve the multistage stochastic variational inequality

\[
D \hat{J}^N((\hat{\Psi}^N)^*) \in N_{C \cap \mathcal{N}}((\hat{\Psi}^N)^*).
\]

(4.9)

Note that the sample space \( \hat{\Omega} \) defined by (4.4) has \( 2^{N\ell} \) sample points. In order to obtain a relatively high approximation accuracy for the original discrete-time optimal control problem (or related continuous-time optimal control problem), the number of the independent identically distributed random variables \( \ell \) for approximating the increment of Wiener process (and the number of the partitions \( N \)) should be large enough. Then, \( \hat{\Omega} \) will contain an extremely large number of sample points and there is no hope to obtain the exact solution \( \tilde{y} \) of the projection onto the nonanticipativity subspace \( \mathcal{N} \) (in Step 1 of Algorithm 3.1). The Monte Carlo method is used to calculate \( \tilde{y} \). Consequently, the calculation for \( \tilde{y} \) will be time-consuming and it is difficult to reduce the calculation error. Let \( \kappa (\in \mathbb{N}) \) be the number of sample size. The numerical results of (4.9) are reported in Table 2. The parameters are determined in the same way as that in Example 4.1.

5 Concluding Remark

This paper is devoted to establishing an explicit type splitting algorithm for multistage stochastic variational inequalities based on the prediction–correction ADMM.
for deterministic variational inequalities with separable structures. As we have seen, the main difference between the deterministic variational inequality and the multistage stochastic variational inequality is that, in the stochastic case, there exists an extra nonanticipativity constraint which leads to some new difficulty in proposing a proper algorithm for multistage stochastic variational inequalities. The key idea of both PHA and Algorithm 3.1 is to treat the projections onto the nonempty closed convex set and the nonanticipativity subspace separately by proper splitting method. The main advantage of Algorithm 3.1 is that it is an explicit iterative algorithm so that the calculation in each step of the algorithm becomes much easier.

In order to simplify the discussion and make the main idea much clear, we simplified the original prediction–correction ADMM (for deterministic variational inequalities) in [15]. It should be remarked that the algorithm proposed in this paper can be further developed and generalized. For instance, the prediction–correction ADMM with variable parameters $\beta$ and/or $\alpha$, some accelerated algorithms based on the prediction–correction ADMM. Furthermore, by Lemma 3.1, under proper conditions, solving original multistage stochastic variational inequality MSVI($F, C \cap N$) is equivalent to solving the variational inequality VI($T, K$). That give us an opportunity to solve multistage stochastic variational inequalities with some other algorithms for deterministic variational inequalities in infinite-dimensional spaces. Some new algorithms for multistage stochastic variational inequalities in the general probability space might be proposed in that way. Further more, some algorithms with strong convergence for multistage stochastic variational inequalities in the general probability space are also valuable for further research.

The multistage stochastic variational inequality in this paper is defined on $L^2(\Omega, \mathcal{F}, R^n)$. One of the main motivation to define the multistage stochastic variational inequality on that space is for its applications in stochastic optimal control problems, in which the admissible controls are usually chosen to be the square-integrable stochastic processes (see Example 2.2). In addition, since $L^2(\Omega, \mathcal{F}, R^n)$ is a Hilbert space, some technical difficulties in the algorithm design and the convergence analysis are avoided when we consider the multistage stochastic variational inequality on that space. It should be remarked that, in some original research articles, the two-stage stochastic programming problems are considered on $L^\infty(\Omega, \mathcal{F}, R^n)$, see for instance [24] or the book [27]. Therefore, it is interesting to investigate the multistage stochastic variational inequality on $L^\infty(\Omega, \mathcal{F}, R^n)$. Also, we can study the multistage stochastic variational inequality on $L^p(\Omega, \mathcal{F}, R^n)$ for any $p \in [1, \infty)$. When $p \neq 2$, the space $L^p(\Omega, \mathcal{F}, R^n)$ is not a Hilbert space (even not a reflexive Banach space if $p = 1$ or $p = \infty$), some new phenomena and new difficulties might appear. It shall be investigated elsewhere.
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Appendix A: Proof of Lemma 2.5

In the appendix, we shall give the proof of Lemma 2.5. The main idea of the proof comes from [32, Lemma 4.6].

Proof It is clear that (2.10) implies (2.9). Then we only need to prove that the converse is also true.

Define

$$A = \{ \omega \in \Omega \mid \exists x \in C(\omega) \text{ s.t. } \langle F(x^*)(\omega), x - x^*(\omega) \rangle < 0 \}.$$  

To prove that (2.10) holds, we only need to prove that $P(A) = 0$. Let

$$G = \{ (\omega, x) \in \Omega \times \mathbb{R}^n \mid x \in C(\omega), \langle F(x^*)(\omega), x - x^*(\omega) \rangle < 0 \}.$$  

Then $G$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable. By [4, Theorem III.23], $A$ is $\mathcal{F}$-measurable.

Take $k, r = 1, 2, \cdots$, define

$$A_{k, r} = \left\{ \omega \in \Omega \mid \exists x \in C(\omega) \cap \bar{B}(0, r) \text{ s.t. } \langle F(x^*)(\omega), x - x^*(\omega) \rangle \leq -\frac{1}{k} \right\}$$

and

$$\Phi_{k, r}(\omega) = \left\{ x \in C(\omega) \cap \bar{B}(0, r) \mid \langle F(x^*)(\omega), x - x^*(\omega) \rangle \leq -\frac{1}{k} \right\}.$$  

Here $\bar{B}(0, r)$ is the closed ball in $\mathbb{R}^n$ of center 0 and radius $r$. Similarly, $A_{k, r}$ is $\mathcal{F}$-measurable. In addition, $\Phi_{k, r}$ is an $\mathcal{F}$-measurable set-valued map and

$$A = \bigcup_{r=1}^{\infty} \bigcup_{k=1}^{\infty} A_{k, r}.$$  

To prove $P(A) = 0$, we only need to prove that for any $k, r$, $P(A_{k, r}) = 0$. Assume that there exist $k, r$ such that $P(A_{k, r}) > 0$. By Lemma 2.2, there exists $\eta \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ such that

$$\eta(\omega) \in \Phi_{k, r}(\omega), \quad \text{a.s. } \omega \in A_{k, r}.$$  

Define $\tilde{\eta} = \eta \chi_{A_{k, r}} + x^* \chi_{\Omega \setminus A_{k, r}}$, then $\tilde{\eta} \in \mathcal{C}$, and

$$\langle F(x^*), \tilde{\eta} - x^* \rangle_{\mathcal{L}^2} = \int_{\Omega} \langle F(x^*)(\omega), \tilde{\eta}(\omega) - x^*(\omega) \rangle P(d(\omega)).$$
\[ \int_{A_{k,r}} \langle F(x^*)(\omega), \eta(\omega) - x^*(\omega) \rangle P(d(\omega)) \leq -\frac{1}{k} P(A_{k,r}) < 0, \]

contradicting (2.9). Thus \( P(A) = 0. \) \( \square \)

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