A direct numerical reconstruction algorithm for the 3D Calderón problem

Fabrice Delbary\textsuperscript{1}, Per Christian Hansen\textsuperscript{1}, Kim Knudsen\textsuperscript{2}
\textsuperscript{1}Department of Informatics and Mathematical Modelling, Technical University of Denmark, 2800 Kgs. Lyngby, Denmark
\textsuperscript{2}Department Mathematics, Technical University of Denmark, 2800 Kgs. Lyngby, Denmark
E-mail: fdel@imm.dtu.dk, pch@imm.dtu.dk, k.knudsen@mat.dtu.dk

Abstract. In three dimensions Calderón’s problem was addressed and solved in theory in the 1980s in a series of papers, but only recently the numerical implementation of the algorithm was initiated. The main ingredients in the solution of the problem are complex geometrical optics solutions to the conductivity equation and a (non-physical) scattering transform. The resulting reconstruction algorithm is in principle direct and addresses the full non-linear problem immediately. In this paper we will outline the theoretical reconstruction method and describe how the method can be implemented numerically. We will give three different implementations, and compare their performance on a numerical phantom.

1. Introduction
The Calderón problem (or the so-called inverse conductivity problem) was formulated in Calderón’s seminal paper [4]. The problem concerns the unique determination and reconstruction of an electric conductivity distribution in a bounded domain from knowledge of the Dirichlet-to-Neumann (or Voltage-to-Current) map on the boundary of the domain. Among the numerous applications of Calderón’s problem we single out the emerging technology for medical imaging called Electrical Impedance Tomography [3, 8], where the Dirichlet-to-Neumann map can be obtained (in principle) from electrostatic boundary measurements.

In this paper we consider the particular domain $\Omega = B(0, 1) \subset \mathbb{R}^3$, the unit ball in $\mathbb{R}^3$, and denote by $\sigma \in C^\infty(\Omega)$ the conductivity distribution in $\Omega$. Suppose $\sigma \geq c > 0$, where $c$ is a constant, and suppose further that $\sigma = 1$ near $\partial \Omega$. Imposing a voltage potential $f \in H^{1/2}(\partial \Omega)$ on $\partial \Omega$ induces a voltage potential $u \in H^1(\Omega)$ given as the unique solution to

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega,$$
$$u = f \text{ on } \partial \Omega. \quad (1)$$

The resulting current flux through the boundary is given by $g = \nu \cdot \nabla u|_{\partial \Omega} = \partial_n u|_{\partial \Omega}$, where $\nu(x) = x$ is the outward unit normal to $\partial \Omega$ at $x$. This defines the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto g.$$

The injectivity of $\sigma \mapsto \Lambda_\sigma$, i.e. the uniqueness question for the inverse problems, was settled affirmatively in [15] using the so-called complex geometrical optics solutions to the governing equation, and the theoretical framework for a reconstruction algorithm was developed in [11, 12]. The algorithm
essentially divides the non-linear problem into three steps involving the solution of linear equations: first a boundary integral equation for the boundary trace of the complex geometrical optics solutions is solved and a scattering transform is computed, then the inverse Fourier transform is applied, and finally a boundary value problem is solved for the conductivity. The first numerical reconstructions following this approach were obtained in [1, 2, 9], but in these papers only the zeroth order approximation of the solution to the boundary integral equation was employed.

The reconstruction algorithm we consider here solves the full non-linear problem directly. In contrast to the reconstruction methods based on the formulation of the inverse problem as an optimization problem solved by iterative schemes, we do not face the potential problem of convergence to local minima. We emphasize, that even though our implementation focus on the sphere geometry and take several assumptions on the conductivity, the algorithm is valid in general domains and general $C^2$ conductivities.

In this paper we will outline the theoretical reconstruction method and discuss three different approximations and implementations. The first reconstruction method is based on the solution of the boundary integral equation, however we suggest to replace the Faddeev Green’s function by the usual fundamental solution for the Laplacian in $\mathbb{R}^3$. The second reconstruction method is based on the zero order approximation in the boundary integral equation, while the third reconstruction is a linearization method tantamount to Calderón’s linear reconstruction method.

2. The reconstruction algorithms
For the solution $u$ to (1) define $v = \sigma^{1/2} u$. The $v$ satisfies

\begin{equation}
(-\Delta + q)v = 0 \text{ in } \Omega,
\end{equation}

\begin{equation}
v = f \text{ on } \partial\Omega.
\end{equation}

with $q = \Delta \sigma^{1/2}/\sigma^{1/2}$. Extend $q$ by $q = 0$ outside $\Omega$ and consider

\begin{equation}
(-\Delta + q)v = 0 \text{ in } \mathbb{R}^3.
\end{equation}

The reconstruction algorithm is now based on the scattering transform of $q$ defined by

\begin{equation}
t(\xi, \zeta) = \int_{\Omega} e^{-ix\cdot(\xi+\zeta)} \psi(x, \zeta) q(x) dx,
\end{equation}

where $\psi(x, \zeta)$ denotes the complex geometrical optics solutions to (3) satisfying the asymptotic condition $e^{-ix\cdot\zeta}\psi(x, \zeta) \approx 1$ for large $x$. The parameter $\zeta \in \mathbb{C}^3$ is assumed to satisfy $\zeta^2 = (\xi + \zeta)^2 = 0$. In a sense $t$ can be understood as a non-linear Fourier transform of $q$. Integration by parts yields

\begin{equation}
t(\xi, \zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}(\Lambda_\sigma - \Lambda_1)\psi(x, \zeta) dx.
\end{equation}

In order to formulate an equation for $\psi|_{\partial\Omega}$ we define the single-layer potential

\begin{equation}
(S_\zeta \phi)(x) = \int_{\partial\Omega} G_\zeta(x-y)\phi(y, \zeta) ds(y),
\end{equation}

with the Faddeev Green’s function

\begin{equation}
G_\zeta(x) = \frac{e^{ix\cdot\zeta}}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\xi \cdot \zeta} d\xi.
\end{equation}

The reconstruction algorithm consists of three steps:

$\Lambda_\sigma \overset{(i)}{\rightarrow} t(\xi, \zeta) \overset{(ii)}{\rightarrow} q(x) \overset{(iii)}{\rightarrow} \sigma(x).$
(i) $\psi|_{\partial \Omega}$ can be computed from boundary measurements by solving
\[ \psi + S_\zeta (\Lambda_\sigma - \Lambda_1) \psi = e^{ix \cdot \zeta}, \quad x \in \partial \Omega \] (6)
and $t$ can be computed by (5).

(ii) $q$ can be computed from $t$ using
\[ \lim_{\xi \to \infty} t(\xi, \zeta) = \hat{q}(\xi). \]

(iii) $\sigma$ can be computed from $q$ by solving
\[ (\Delta + q)\sigma^{1/2} = 0 \text{ in } \Omega, \quad \sigma^{1/2}|_{\partial \Omega} = 1. \] (7)
It can be shown that ill-posedness of the problem is present only in step (i). Steps (ii)-(iii) give rise to well-posed problems.

2.1. The $t_0$ approximation
To avoid working with the exponentially growing Faddeev Green’s function in the boundary integral equation (6) we suggest to consider
\[ \psi_0 + S_0 (\Lambda_\sigma - \Lambda_1) \psi_0 = e^{ix \cdot \zeta}, \quad x \in \partial \Omega, \] (8)
where the kernel of $S_0$ is the usual fundamental solution for the Laplacian in $\mathbb{R}^3$, $G_0(x) = 1/(4\pi|x|)$. It can be shown [6] that this equation is uniquely solvable for any $\zeta$. We then compute $t_0$ by inserting $\psi_0$ for $\psi$ in (5), that is
\[ t_0(\xi, \zeta) = \int_{\partial \Omega} e^{-ix(\xi+\zeta)} (\Lambda_\sigma - \Lambda_1) \psi_0(x, \zeta) dx, \] (9)
and carry out steps (ii) and (iii) of the algorithm as before. We will call the obtained reconstruction $\sigma_0$. For the 2D problem a similar approximation was used in [7, 10].

2.2. The $t_{\exp}$ approximation
A crude approximation of $t$ is obtained by substituting the asymptotic value of $\psi \approx e^{ix \cdot \zeta}$ for $\psi$ in (5). This was suggested for the 2D problem in [14]. The resulting approximation of $t$ is denoted by $t_{\exp}$, i.e.
\[ t_{\exp}(\xi, \zeta) = \int_{\partial \Omega} e^{-ix(\xi+\zeta)} (\Lambda_\sigma - \Lambda_1) e^{ix \cdot \zeta} dx. \] (10)
We then propose to implement steps (ii) and (iii) with $t_{\exp}$ for $t$, thus obtaining an approximation of $\sigma$ denoted by $\sigma_{\exp}$. Effectively this approach gives a linearization of step (i) in the algorithm.

2.3. The Calderón inversion
Step (ii) and (iii) in the algorithm can be linearized as well. This gives the reconstruction formula
\[ \sigma_{\exp}(x) = 1 - \frac{1}{2(2\pi)^n} \int_{|\xi|^2} t_{\exp}(\xi, \zeta) e^{ix \cdot \xi} d\xi. \] (11)
3. Details on the numerical implementation

To test the different reconstruction schemes we chose a numerical phantom composed of three objects with constant conductivity different from the conductivity of the background medium (conductivity 1).

- A ball with center in the $(Oxy)$ plane and with a conductivity larger than 1.
- Two prolate ellipsoids with axis in the $(Oxy)$ plane, with slightly different sizes and with a same conductivity smaller than 1.

The phantom is illustrated in figure 1.

![Figure 1. Phantom](image)

In order to be in the framework of the theory, the conductivity distribution must be smooth. Hence we have smoothed out the discontinuities in the phantom. A cross sectional plot of the resulting conductivity distribution and its support can be seen in figure 2.

![Figure 2. Left: profile of the conductivity $\sigma$ in the $(Oxy)$ plane. Right: support of $\sigma$.](image)

The Dirichlet-to-Neumann map $\Lambda_\sigma$ is computed using the numerical phantom and represented in the basis of spherical harmonics $Y_{nm}$ at the boundary of $\Omega$. We fix $f = Y_{nm}$ for $n, m$ up to a fixed order and solve numerically the Schrödinger equation (2) for $v$ using the moment method. Then we expand $\Lambda_\sigma Y_{nm} = \partial_\nu v |_{\partial\Omega}$ in the same basis of spherical harmonics at the boundary and store the coefficients up to the fixed order.

Quadrature points are chosen on the unit sphere, as defined in [16] and [5] for solving boundary integral equations of the second kind involved in acoustic scattering. All computations involved in the reconstruction methods we propose are done using pointwise values of functions at these quadrature points.

To briefly summarize, we consider a positive integer $N$ and the $N^2$ points $x_{pq}$, $1 \leq p \leq N$, $0 \leq q \leq 2N - 1$ on the unit sphere $\partial\Omega$ given by the spherical coordinates

$$x_{pq} = (\sin \theta_p \cos \varphi_q, \sin \theta_p \sin \varphi_q, \cos \theta_p),$$
where $\theta_p = \arccos t_p$ and $\varphi_q = \pi q/N$, the real numbers $t_p$ being the $N$ zeros of the Legendre polynomial $P_N$. These quadrature points allow us to construct an approximation formula for functions $\psi \in C(\partial \Omega)$ by linear combinations of spherical harmonics, using pointwise values of $\psi$ at the quadrature points, more precisely

$$
\psi \simeq \frac{\pi}{N} \sum_{p=1}^{N} \sum_{q=0}^{2N-1} \alpha_p \psi(x_{pq}) \sum_{n=0}^{N-1} \sum_{m=-n}^{n} Y_n^{-m}(x_{pq}) Y_n^m, \tag{12}
$$

where $\alpha_p$ are the weights of the Gauss-Legendre quadrature rule on $[-1,1]$ and $Y_n^m$ are the usual spherical harmonics. For analytic functions $\psi$, the approximation (12) converges exponentially in the maximum norm as $N \to \infty$. From this approximation formula ensues the quadrature formula

$$
\int_{\partial \Omega} \psi \, ds \simeq \frac{\pi}{N} \sum_{p=1}^{N} \sum_{q=0}^{2N-1} \alpha_p \psi(x_{pq}), \tag{13}
$$

and the approximation formula

$$
\int_{\partial \Omega} \frac{\psi(y)}{|x-y|} \, ds(y) \simeq \frac{\pi}{N} \sum_{p=1}^{N} \sum_{q=0}^{2N-1} \alpha_p \psi(x_{pq}) \sum_{n=0}^{N-1} P_n(x_{pq} \cdot x) \text{ for } x \in \partial \Omega. \tag{14}
$$

Equation (13) is used in the different schemes for numerical integrations involved in (9) and (10) and the formulas (12), (14) are used to build the matrix of the discretized operator $I + S_0(\Lambda_\sigma - \Lambda_1)$ for the $t_0$ method. The computation of $1 - \sigma^{\exp}$ from $t^{\exp}(\xi, \zeta)/|\xi|^2$ (Calderón inversion) or the computation of the approximation of $q$ ($t^{\exp}$ and $t_0$ approximations) is simply done by a fast Fourier transform. For the $t^{\exp}$ and $t_0$ approximations, the solution of the Schrödinger equation (7) is done with a finite elements method of order 1.

### 3.1. The $t_0$ approximation

The reconstruction scheme for the $t_0$ approximation requires steps to compute the solution of (8), thus, we proceed as follows

(i) Build the matrix of the operator $I + S_0(\Lambda_\sigma - \Lambda_1)$ using (12), (14).

(ii) Solve the discretized system of (8), using a Tikhonov regularization with Morozov discrepancy principle because of the ill-posedness, to get values of $\psi$ at the quadrature points.

(iii) Compute (9) using (13) for $|\xi| < \xi_M$ where $\xi_M > 0$ and where $|\xi|$ is chosen of minimal norm to avoid numerical instabilities.

(iv) Compute the fast inverse Fourier transform of $t_0(\xi, \zeta)$ to get the approximation of $q_0$ of $q$.

(v) Solve the Schrödinger equation (7) to get $\sigma_0$ using a finite elements method.

### 3.2. The $t^{\exp}$ approximation

The reconstruction scheme for the $t^{\exp}$ method does not require to solve the first boundary integral equation, hence, the steps are the following

(i) Compute (10) using (13) for $|\xi| < \xi_M$ where $\xi_M > 0$ and where $|\xi|$ is chosen of minimal norm to avoid numerical instabilities.

(ii) Compute the fast inverse Fourier transform of $t^{\exp}(\xi, \zeta)$ to get an approximation $q^{\exp}$ of $q$.

(iii) Solve the Schrödinger equation (7) to get $\sigma^{\exp}$ using a finite elements method.
3.3. The Calderón inversion

The Calderón inversion does not require to solve the Schrödinger equation since the approximation of the conductivity is directly computed from $t^{\exp}$, so that the steps are limited to

(i) Compute (10) using (13) for $|\xi| < \xi_M$ where $\xi_M > 0$ and where $|\zeta|$ is chosen of minimal norm to avoid numerical instabilities.

(ii) Compute the fast inverse Fourier transform of $t^{\exp}(\xi, \zeta)/|\xi|^2$ to get $1 - \sigma^{\text{app}}$.

4. Numerical results

We consider reconstructions with $t_0$ and $t^{\exp}$ approximations and with the Calderón inversion, first with $|\xi| \leq 6$ and then with $|\xi| \leq 8$. We take $N = 16$, that is 512 points on the unit sphere. The cross sectional profile of the reconstructions for the cases $|\xi| \leq 6$ and $|\xi| \leq 8$ can be seen in figure 3 and 4 respectively.

![Figure 3](image)

**Figure 3.** Reconstructions with truncation $\xi_M = 6$: left $\sigma_0$, middle $\sigma^{\exp}$, and right $\sigma^{\text{app}}$.

![Figure 4](image)

**Figure 4.** Reconstructions with truncation $\xi_M = 8$: left $\sigma_0$, middle $\sigma^{\exp}$, and right $\sigma^{\text{app}}$.

From figure 3 and 4 we note that the shape of the inclusions are determined better with the lower value of $\xi_M$, whereas the contrast is better achieved with the larger value of $\xi_M$. This is perhaps due to the exponential growth of the involved functions with respect to $\xi$. Also, we see that even though the reconstructions are quite similar the reconstructions $\sigma_0$ and $\sigma^{\exp}$ seem to be slightly better than $\sigma^{\text{app}}$. This holds true with both truncations.

5. Conclusions

In this article we have outlined three different reconstruction methods for the Calderón problem and given their numerical implementations. The performance of the methods was illustrated with a numerical phantom consisting of three inclusions. All three methods produce reconstructions that separate and localize the inclusions, however the contrast is not reliable. A comparison of the three methods shows that the mere linearization performs the poorest, whereas the most involved method based on solving
the perturbed boundary integral equation gives the best reconstructions. However, there is still room for improvement, and this may be achieved by solving the correct boundary integral equation as a first step in the algorithm. The example shows that there is a delicate balance between the noise in the problem, the errors the computation, and the truncation of the scattering transform, but more analysis must be done to reveal the connections and understand the regularization effect.

Acknowledgments
The work of Fabrice Delbary was supported by a post doctoral fellowship from the Villum Kann Rasmussen Foundation.

References
[1] J. Bikowski Electrical Impedance Tomography Reconstructions in Two and Three Dimensions; from Calderón to Direct Methods PhD thesis, Colorado State University, Fort Collins, CO, 2008.
[2] J. Bikowski, K. Knudsen, and J. L. Mueller, Direct numerical reconstruction of conductivities in three dimensions using scattering transforms, Inverse Problems 27 (2011), 015002 (19pp).
[3] L. Borcea, Electrical impedance tomography, Inverse Problems 18 (2002), pp. 99–136.
[4] A-P. Calderón, On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), Soc. Brasil. Mat., Rio de Janeiro, 1980, pp. 65–73.
[5] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, vol. 93 of Applied Mathematical Sciences, Springer-Verlag, Berlin, second ed., 1998.
[6] H. Cornean, K. Knudsen, and S. Siltanen, Towards a d-bar reconstruction method for three-dimensional EIT, J. Inverse Ill-Posed Probl. 14 (2006), no. 2, pp. 111–134.
[7] M. DeAngelo and J. L. Mueller D-bar reconstructions of human chest and tank data using an improved approximation to the scattering transform, Physiological Measurement 31 (2010), no. 2, pp. 221–232.
[8] D. Holder, Electrical impedance tomography, Institute of Physics Publishing, Bristol and Philadelphia, 2005.
[9] K. Knudsen, J. L. Mueller, The Born approximation and Calderón’s method for reconstruction of conductivities in 3-D, Proceedings of the Eighth AIMS International Conference on Dynamical Systems, Differential Equations and Applications, Dresden 2010. To appear.
[10] J. Mueller and S. Siltanen, Direct reconstruction of conductivities from boundary measurements, Siam J. Sci. Comp. 24 (2003), no. 4, 1232–1266.
[11] A. I. Nachman, Reconstructions from boundary measurements, Ann. of Math. (2) 128 (1988), no. 3, pp. 531–576.
[12] R. G. Novikov, A multidimensional inverse spectral problem for the equation \( -\Delta \psi + (\psi(x) - Eu(x))\psi = 0 \), Funktsional. Anal. i Prilozhen. 22 (1988), no. 4, pp. 11–22, 96, translation in Funct. Anal. Appl., 22 (1988), no. 4, pp. 263–272.
[13] A. I. Nachman, J. Sylvester, and G. A. Uhlmann, An n-dimensional Borg-Levinson theorem, Comm. Math. Phys. 115 (1988), no. 4, pp. 595–605.
[14] S. Siltanen, J. Mueller and D. Isaacson An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem, Inverse Problems 16 (2000), pp. 681–699 (Erratum: Inverse problems 17, 1561–1563).
[15] J. Sylvester and G. A. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. (2) 125 (1987), no. 1, pp. 153–169.
[16] L. Wienert, Die numerische Approximation von Randintegrooperatoren für die Helmholtzgleichung im \( \mathbb{R}^3 \), Dissertation, Göttingen (1990).