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A statistical mechanical interpretation of algorithmic information theory: Total statistical mechanical interpretation based on physical argument

Kohtaro Tadaki
Research and Development Initiative, Chuo University, 1–13–27 Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
E-mail: tadaki@kc.chuo-u.ac.jp

Abstract. The statistical mechanical interpretation of algorithmic information theory (AIT, for short) was introduced and developed by our former works [K. Tadaki, Local Proceedings of CiE 2008, pp. 425–434, 2008] and [K. Tadaki, Proceedings of LFCS’09, Springer’s LNCS, vol. 5407, pp. 422–440, 2009], where we introduced the notion of thermodynamic quantities, such as partition function $Z(T)$, free energy $F(T)$, energy $E(T)$, statistical mechanical entropy $S(T)$, and specific heat $C(T)$, into AIT. We then discovered that, in the interpretation, the temperature $T$ equals to the partial randomness of the values of all these thermodynamic quantities, where the notion of partial randomness is a stronger representation of the compression rate by means of program-size complexity. Furthermore, we showed that this situation holds for the temperature $T$ itself, which is one of the most typical thermodynamic quantities. Namely, we showed that, for each of the thermodynamic quantities $Z(T)$, $F(T)$, $E(T)$, and $S(T)$ above, the computability of its value at temperature $T$ gives a sufficient condition for $T \in (0, 1)$ to satisfy the condition that the partial randomness of $T$ equals to $T$. In this paper, based on a physical argument on the same level of mathematical strictness as normal statistical mechanics in physics, we develop a total statistical mechanical interpretation of AIT which actualizes a perfect correspondence to normal statistical mechanics. We do this by identifying a microcanonical ensemble in the framework of AIT. As a result, we clarify the statistical mechanical meaning of the thermodynamic quantities of AIT.

1. Introduction
Algorithmic information theory (AIT, for short) is a framework for applying information-theoretic and probabilistic ideas to recursive function theory. One of the primary concepts of AIT is the program-size complexity (or Kolmogorov complexity) $H(s)$ of a finite binary string $s$, which is defined as the length of the shortest binary input for a universal decoding algorithm $U$, called an optimal computer, to output $s$. By the definition, $H(s)$ is thought to represent the amount of randomness contained in a finite binary string $s$, which cannot be captured in a computational manner. In particular, the notion of program-size complexity plays a crucial role in characterizing the randomness of an infinite binary string, or equivalently, a real. In [4] Chaitin introduced the halting probability $\Omega$ as an example of random real. His $\Omega$ is defined as the probability that the optimal computer $U$ halts, and plays a central role in the metamathematical development of AIT. The first $n$ bits of the base-two expansion of $\Omega$ solve the halting problem of $U$ for inputs of length at most $n$. By this property, the base-two expansion of $\Omega$ is shown to be a random infinite binary string.

In the works [10, 11] we introduced the notion of partial randomness for a real as a stronger representation of the compression rate of a real by means of program-size complexity. At the same time, we generalized Chaitin’s halting probability $\Omega$ to $\Omega(D)$ by

$$\Omega(D) = \sum_{p \in \text{dom } U} 2^{-\frac{|p|}{H(p)}}, \tag{1}$$

where $p$ is a program in $U$ and $H(p)$ is the program-size complexity of $p$.
so that the partial randomness of $\Omega(D)$ can be controlled by a real $D$ with $0 < D \leq 1$. Here, $\text{dom} \ U$ denotes the set of all programs for $U$ (i.e., the set of all halting inputs for $U$). As $D$ becomes larger, the partial randomness of $\Omega(D)$ increases. When $D = 1$, $\Omega(D)$ becomes a random real, i.e., $\Omega(1) = \Omega$.

In 2006 Calude and Stay [3] pointed out a formal correspondence between $\Omega(D)$ and a partition function in statistical mechanics. In statistical mechanics, the partition function $Z_{\text{sm}}(T)$ at temperature $T$ is defined by

$$Z_{\text{sm}}(T) = \sum_{x \in X} e^{-\frac{E_x}{k_B} T},$$

where $X$ is a complete set of energy eigenstates of a statistical mechanical system and $E_x$ is the energy of an energy eigenstate $x$. The positive constant $k_B$ is called the Boltzmann Constant. Calude and Stay [3] pointed out, in essence, that the partition function $Z_{\text{sm}}(T)$ has the same form as $\Omega(D)$ by performing the following replacements in $Z_{\text{sm}}(T)$:

**Replacements 1.1.**

(i) Replace the complete set $X$ of energy eigenstates $x$ by the set $\text{dom} \ U$ of all programs $p$ for $U$.

(ii) Replace the energy $E_x$ of an energy eigenstate $x$ by the length $|p|$ of a program $p$.

(iii) Set the Boltzmann Constant $k_B$ to $1/\ln 2$, where the $\ln$ denotes the natural logarithm.

Inspired by this suggestion of Calude and Stay, starting from the works [13, 14] we have developed so far a statistical mechanical interpretation of AIT, where $\Omega(D)$ appears as a partition function.

Generally speaking, in order to give a statistical mechanical interpretation to a framework which looks unrelated to statistical mechanics at first glance, it is important to identify a microcanonical ensemble in the framework. Once we can do so, we can easily develop an equilibrium statistical mechanics on the framework according to the theoretical development of normal equilibrium statistical mechanics. Here, the microcanonical ensemble is a certain sort of uniform probability distribution. In fact, in the work [12] we developed a statistical mechanical interpretation of the noiseless source coding scheme in information theory by identifying a microcanonical ensemble in the scheme. Then, based on this identification, in [12] the notions in statistical mechanics such as statistical mechanical entropy, temperature, and thermal equilibrium are translated into the context of noiseless source coding.

Thus, in order to develop a total statistical mechanical interpretation of AIT, it is appropriate to identify a microcanonical ensemble in the framework of AIT. Note, however, that AIT is not a physical theory but a purely mathematical theory. Therefore, in order to obtain significant results for the development of AIT itself, we have to develop a statistical mechanical interpretation of AIT in a mathematically rigorous manner, unlike in normal statistical mechanics in physics where arguments are not necessarily mathematically rigorous. A fully rigorous mathematical treatment of statistical mechanics is already developed (see Ruelle [9]). At present, however, it would not as yet seem to be an easy task to merge AIT with this mathematical treatment in a satisfactory manner.

In our former works [13, 14], for mathematical strictness we developed a statistical mechanical interpretation of AIT in a different way from the idealism above. In [13] we introduced the notion of thermodynamic quantities at temperature $T$, such as partition function $Z(T)$, free energy $F(T)$, energy $E(T)$, statistical mechanical entropy $S(T)$, and specific heat $C(T)$, into AIT by performing Replacements 1.1 for the corresponding thermodynamic quantities at temperature $T$ in statistical mechanics. These thermodynamic quantities in AIT are real functions of a real

1 In [10, 11], $\Omega(D)$ is denoted by $\Omega^D$. 
argument $T > 0$. We prove that if the temperature $T$ is a computable real with $0 < T < 1$ then, for each of these thermodynamic quantities, the partial randomness (and therefore the compression rate) of its value equals to $T$. Thus, the temperature $T$ plays a role as the partial randomness of the values of the thermodynamic quantities in our statistical mechanical interpretation of AIT. Furthermore, we showed that this situation holds for the temperature $T$ itself, which is one of the most typical thermodynamic quantities. Namely, we showed in [13, 14] that for each of the thermodynamic quantities $Z(T), F(T), E(T),$ and $S(T)$, the computability of its value at temperature $T$ gives a sufficient condition for $T \in (0, 1)$ to be a fixed point on partial randomness (see Theorems 3.4, 3.5, 3.6, and 3.7 below).

In this paper we show that, if we do not stick to the mathematical strictness of an argument, we can certainly develop a total statistical mechanical interpretation of AIT which attains a perfect correspondence to normal statistical mechanics. In the total interpretation, we identify a microcanonical ensemble in AIT in a similar manner to [12], based on the probability measure which gives Chaitin’s $\Omega$ the meaning of the halting probability actually. This identification enables us to clarify the statistical mechanical meaning of the thermodynamic quantities of AIT, which are originally introduced by [13] in a rigorous manner.

The paper is organized as follows. We begin in Section 2 with some preliminaries to AIT and partial randomness. In Section 3, we review the major rigorous results on the statistical mechanical interpretation of AIT, obtained so far in our former works [13, 14]. In Section 4, based on a physical argument on the same level of mathematical strictness as normal statistical mechanics, we develop a total statistical mechanical interpretation of AIT. We conclude this paper with a mention to the future direction of this work in Section 5.

2. Preliminaries

We start with some notation about numbers and strings which will be used in this paper. $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Q}$ is the set of rationals, and $\mathbb{R}$ is the set of reals. $\{0, 1\}^*$ is the set of finite binary strings, where $\lambda$ denotes the empty string. For any $s \in \{0, 1\}^*$, $|s|$ is the length of $s$. A subset $S$ of $\{0, 1\}^*$ is called prefix-free if no string in $S$ is a prefix of another string in $S$. $\{0, 1\}^\infty$ is the set of infinite binary strings, where an infinite binary string is infinite to the right but finite to the left. For any function $f$, the domain of definition of $f$ is denoted by dom $f$. Normally, $o(n)$ denotes any function $f: \mathbb{N}^+ \rightarrow \mathbb{R}$ such that $\lim_{n \rightarrow \infty} f(n)/n = 0$.

Let $\alpha$ be an arbitrary real. We denote by $\alpha|_n \in \{0, 1\}^*$ the first $n$ bits of the base-two expansion of $\alpha - |\alpha|$ with infinitely many zeros, where $|\alpha|$ is the greatest integer less than or equal to $\alpha$. For example, in the case of $\alpha = 5/8$, $\alpha|_6 = 101000$. A real $\alpha$ is called computable if there exists a total recursive function $f: \mathbb{N}^+ \rightarrow \mathbb{Q}$ such that $|\alpha - f(n)| < 1/n$ for all $n \in \mathbb{N}^+$.

2.1. Algorithmic information theory

In the following we concisely review some definitions and results of AIT [4, 5]. A computer is a partial recursive function $C: \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that dom $C$ is a nonempty prefix-free set. For each computer $C$ and each $s \in \{0, 1\}^*$, $H_C(s)$ is defined by $H_C(s) = \min \{|p| \mid p \in \{0, 1\}^* \land C(p) = s\}$ (may be $\infty$). A computer $U$ is said to be optimal if for each computer $C$ there exists $d \in \mathbb{N}$ such that, for every $s \in \{0, 1\}^*$, $H_U(s) \leq H_C(s) + d$. It is easy to see that there exists an optimal computer. We choose a particular optimal computer $U$ as the standard one for use, and define $H(s)$ as $H_U(s)$, which is referred to as the program-size complexity of $s$ or the Kolmogorov complexity of $s$. A finite binary string in dom $U$ is called a program for $U$.

For each $s \in \{0, 1\}^*$, $P(s)$ is defined as $\sum_{p \in \text{dom } U} 2^{-|p|}$. Chaitin’s halting probability $\Omega$ is defined by $\Omega = \sum_{p \in \text{dom } U} 2^{-|p|}$. For any $\alpha \in \mathbb{R}$, we say that $\alpha$ is weakly Chaitin random if there exists $c \in \mathbb{N}$ such that $n - c \leq H(\alpha|_n)$ for all $n \in \mathbb{N}^+$ [4, 5]. Then [4] showed that $\Omega$ is weakly
Chaitin random. For any $\alpha \in \mathbb{R}$, we say that $\alpha$ is \textit{Chaitin random} if $\lim_{n \to \infty} H(\alpha|_n) - n = \infty$ \cite{4, 5}. Obviously, for every $\alpha \in \mathbb{R}$, if $\alpha$ is Chaitin random, then $\alpha$ is weakly Chaitin random. We can show that the converse also hold. Thus, for every $\alpha \in \mathbb{R}$, $\alpha$ is weakly Chaitin random if and only if $\alpha$ is Chaitin random (see Chaitin \cite{5} for the proof and historical detail). Thus $\Omega$ is Chaitin random.

2.2. Partial randomness

In the works \cite{10, 11}, we generalized the notion of the randomness of a real so that \textit{the degree of the randomness}, which is often referred to as \textit{the partial randomness} recently \cite{2, 8, 3}, can be characterized by a real $T$ with $0 \leq T \leq 1$ as follows.

\textbf{Definition 2.1} (weak Chaitin $T$-randomness). Let $T \geq 0$. For any $\alpha \in \mathbb{R}$, we say that $\alpha$ is \textit{weakly Chaitin $T$-random} if there exists $c \in \mathbb{N}$ such that $Tn - c \leq H(\alpha|_n)$ for all $n \in \mathbb{N}^+$.

\textbf{Definition 2.2} ($T$-compressibility). Let $T \geq 0$. For any $\alpha \in \mathbb{R}$, we say that $\alpha$ is $T$-compressible if $H(\alpha|_n) \leq Tn + o(n)$, which is equivalent to $\limsup_{n \to \infty} H(\alpha|_n)/n \leq T$.

In the case of $T = 1$, the weak Chaitin $T$-randomness results in the weak Chaitin randomness. For every $T \in [0, 1]$ and every $\alpha \in \mathbb{R}$, if $\alpha$ is weakly Chaitin $T$-random and $T$-compressible, then

$$\lim_{n \to \infty} \frac{H(\alpha|_n)}{n} = T. \quad (3)$$

The left-hand side of (3) is referred to as the \textit{compression rate} of a real $\alpha$ in general. Note, however, that (3) does not necessarily imply that $\alpha$ is weakly Chaitin $T$-random. Thus, the notion of partial randomness is a stronger representation of the notion of compression rate.

\textbf{Definition 2.3} (Chaitin $T$-randomness, Tadaki \cite{10, 11}). Let $T \geq 0$. For any $\alpha \in \mathbb{R}$, we say that $\alpha$ is \textit{Chaitin $T$-random} if $\lim_{n \to \infty} H(\alpha|_n) - Tn = \infty$.

In the case of $T = 1$, the Chaitin $T$-randomness results in the Chaitin randomness. Obviously, for every $T \in [0, 1]$ and every $\alpha \in \mathbb{R}$, if $\alpha$ is Chaitin $T$-random, then $\alpha$ is weakly Chaitin $T$-random. However, in 2005 Reimann and Stephan \cite{8} showed that, in the case of $T < 1$, the converse does not necessarily hold.

3. The previous works: Rigorous results

In this section, we summarize the main rigorous results of the statistical mechanical interpretation of AIT, developed by our former works \cite{13, 14}. We first introduce the notion of thermodynamic quantities into AIT in the following manner.

In statistical mechanics, the thermodynamic quantities: free energy $F_{\text{sm}}(T)$, energy $E_{\text{sm}}(T)$, entropy $S_{\text{sm}}(T)$, and specific heat $C_{\text{sm}}(T)$ at temperature $T$ are given as follows.

$$F_{\text{sm}}(T) = -k_B T \ln Z_{\text{sm}}(T), \quad E_{\text{sm}}(T) = \frac{1}{Z_{\text{sm}}(T)} \sum_{x \in X} E_x e^{-\frac{E_x}{k_B T}},$$

$$S_{\text{sm}}(T) = \frac{E_{\text{sm}}(T) - F_{\text{sm}}(T)}{T}, \quad C_{\text{sm}}(T) = \frac{d}{dT} E_{\text{sm}}(T), \quad (4)$$

where $Z_{\text{sm}}(T)$ is the partition function given by (2). We introduce the notion of thermodynamic quantities into AIT by performing Replacements 1.1 for the thermodynamic quantities (2) and (4) in statistical mechanics.\(^1\) For that purpose, we choose a particular enumeration $q_1, q_2, q_3, q_4, \ldots$ of the infinite set $\text{dom} \ U$ first. Then, motivated by the formulae (2) and (4), and taking into account Replacements 1.1, we introduce the notion of thermodynamic quantities into AIT as follows.

\(^1\) For the thermodynamic quantities in statistical mechanics, see e.g. Chapter 16 of \cite{1} and Chapter 2 of \cite{15}.

\(^2\) To be precise, the partition function is not a thermodynamic quantity but a statistical mechanical quantity.
Definition 3.1 (thermodynamic quantities in AIT, [13]). Let $T$ be any real with $T > 0$.

(i) The partition function $Z(T)$ at temperature $T$ is defined as $\lim_{k \to \infty} Z_k(T)$ where $Z_k(T) = \sum_{i=1}^{k} 2^{-\|q_i\|/T}$.

(ii) The free energy $F(T)$ at temperature $T$ is defined as $\lim_{k \to \infty} F_k(T)$ where $F_k(T) = -T \log_2 Z_k(T)$.

(iii) The energy $E(T)$ at temperature $T$ is defined as $\lim_{k \to \infty} E_k(T)$ where $E_k(T) = \sum_{i=1}^{k} |q_i| 2^{-\|q_i\|/T}/Z_k(T)$.

(iv) The statistical mechanical entropy $S(T)$ at temperature $T$ is defined as $\lim_{k \to \infty} S_k(T)$ where $S_k(T) = \{E_k(T) - F_k(T)\}/T$.

(v) The specific heat $C(T)$ at temperature $T$ is defined as $\lim_{k \to \infty} C_k(T)$ where $C_k(T) = E_k'(T)$, the derived function of $E_k(T)$.

Note that $Z(T)$ certainly equals to $\Omega(T)$, which is defined by (1).

Theorem 3.2 (properties of $Z(T)$ and $F(T)$, [10, 11, 13]). Let $T \in \mathbb{R}$.

(i) If $0 < T \leq 1$ and $T$ is computable, then each of $Z(T)$ and $F(T)$ converges and is weakly Chaitin $T$-random and $T$-compressible.

(ii) If $1 < T$, then $Z(T)$ and $F(T)$ diverge to $\infty$ and $-\infty$, respectively.

Theorem 3.3 (properties of $E(T)$ and $S(T)$, [13]). Let $T \in \mathbb{R}$.

(i) If $0 < T < 1$ and $T$ is computable, then each of $E(T)$, $S(T)$, and $C(T)$ converges and is Chaitin $T$-random and $T$-compressible.

(ii) If $1 \leq T$, then both $E(T)$ and $S(T)$ diverge to $\infty$. In the case of $T = 1$, $C(T)$ diverges to $\infty$.

The above two theorems show that if $T$ is a computable real with $0 < T < 1$ then the partial randomness (and therefore the compression rate) of the values of all the thermodynamic quantities in Definition 3.1 equals to the temperature $T$. These theorems also show that the values of all the thermodynamic quantities diverge when the temperature $T$ exceeds 1. This phenomenon might be regarded as some sort of phase transition in statistical mechanics.

In statistical mechanics or thermodynamics, among all thermodynamic quantities one of the most typical thermodynamic quantities is temperature itself. Inspired by this fact in physics and by the above observation that the temperature $T$ equals to the partial randomness of the values of the thermodynamic quantities in the statistical mechanical interpretation of AIT, the following question arises naturally: Can the partial randomness of the temperature $T$ equal to the temperature $T$ itself in the statistical mechanical interpretation of AIT? This question is rather self-referential. However, we can answer it affirmatively in the following form.

Theorem 3.4 (fixed point theorem by $Z(T)$, [13]). For every $T \in (0, 1)$, if $Z(T)$ is computable, then $T$ is weakly Chaitin $T$-random and $T$-compressible.

Theorem 3.5 (fixed point theorem by $F(T)$, [14]). For every $T \in (0, 1)$, if $F(T)$ is computable then $T$ is weakly Chaitin $T$-random and $T$-compressible.

Theorem 3.6 (fixed point theorem by $E(T)$, [14]). For every $T \in (0, 1)$, if $E(T)$ is computable then $T$ is Chaitin $T$-random and $T$-compressible.

Theorem 3.7 (fixed point theorem by $S(T)$, [14]). For every $T \in (0, 1)$, if $S(T)$ is computable then $T$ is Chaitin $T$-random and $T$-compressible.

4 It is still open whether $C(T)$ diverges or not in the case of $T > 1.$
Each of the above four theorems is just a fixed point theorem on partial randomness, where the computability of the value of thermodynamic quantity of AIT at temperature $T$ gives a sufficient condition for a real $T \in (0, 1)$ to be a fixed point on partial randomness, i.e., to satisfy the property that the partial randomness of $T$ equals to $T$ itself, and therefore to satisfy that
\[
\lim_{n \to \infty} \frac{H(T|n)}{n} = T. \tag{5}
\]
This means that the compression rate of $T$ equals to $T$ itself. Intuitively, we might interpret the meaning of (5) as follows: Consider imaginarily a file of infinite size whose content is

“The compression rate of this file is 0.100111001 . . . .”

When this file is compressed, the compression rate of this file actually equals to 0.100111001 . . . , as the content of this file says. This situation is self-referential and forms a fixed point.

4. Total statistical mechanical interpretation of AIT: Physical argument

In what follows, based on a physical argument we develop a total statistical mechanical interpretation of AIT which attains a perfect correspondence to normal statistical mechanics. In consequence, we justify the interpretation of $\Omega(D)$ as a partition function and clarify the statistical mechanical meaning of the thermodynamic quantities of AIT in Definition 3.1. In the work [12], we developed a statistical mechanical interpretation of the noiseless source coding scheme based on an absolutely optimal instantaneous code by identifying a microcanonical ensemble in the scheme. In a similar manner to [12] we develop a total statistical mechanical interpretation of AIT in what follows. This can be possible because the set dom $U$ is prefix-free and therefore the action of the optimal computer $U$ can be regarded as an instantaneous code which is extended over an infinite set. Note that, in what follows, we do not stick to the mathematical strictness of the argument and we make an argument on the same level of mathematical strictness as statistical mechanics in physics. We start with some reviews of statistical mechanics.

In statistical mechanics we consider a quantum system $S_{\text{total}}$ which consists in a large number of identical quantum subsystems. Let $N$ be the number of such subsystems. For example, $N \sim 10^{22}$ for 1 cm$^3$ of a gas at room temperature. We assume here that each quantum subsystem can be distinguishable from others. Thus, we deal with quantum particles which obey Maxwell-Boltzmann statistics and not Bose-Einstein statistics or Fermi-Dirac statistics. Under this assumption, we can identify the $i$th quantum subsystem $S_i$ for each $i = 1, \ldots, N$. In quantum mechanics, each quantum system is described by a quantum state completely. In statistical mechanics, among all quantum states, energy eigenstates are of particular importance. Any energy eigenstate of each subsystem $S_i$ can be specified by a number $n = 1, 2, 3, \ldots$, called a quantum number, where the subsystem in the energy eigenstate specified by $n$ has the energy $E_n$. Then, any energy eigenstate of the system $S_{\text{total}}$ can be specified by an $N$-tuple $(n_1, n_2, \ldots, n_N)$ of quantum numbers. If the state of the system $S_{\text{total}}$ is the energy eigenstate specified by $(n_1, n_2, \ldots, n_N)$, then the state of each subsystem $S_i$ is the energy eigenstate specified by $n_i$ and the system $S_{\text{total}}$ has the energy $E_{n_1} + E_{n_2} + \cdots + E_{n_N}$. Then, the fundamental postulate of statistical mechanics, called the principle of equal probability, is stated as follows.

**The Principle of Equal Probability:** If the energy of the system $S_{\text{total}}$ is known to have a constant value in the range between $E$ and $E + \delta E$, where $\delta E$ is the indeterminacy in measurement of the energy of the system $S_{\text{total}}$, then the system $S_{\text{total}}$ is equally likely to be in any energy eigenstate specified by $(n_1, n_2, \ldots, n_N)$ such that $E \leq E_{n_1} + E_{n_2} + \cdots + E_{n_N} \leq E + \delta E$.

Let $\Theta(E, N)$ be the total number of energy eigenstates of $S_{\text{total}}$ specified by $(n_1, n_2, \ldots, n_N)$ such that $E \leq E_{n_1} + E_{n_2} + \cdots + E_{n_N} \leq E + \delta E$. The above postulate states that any
energy eigenstate of $\mathcal{S}_{\text{total}}$ whose energy lies between $E$ and $E + \delta E$ occurs with the probability $1/\Theta(E, N)$. This uniform distribution of energy eigenstates whose energy lie between $E$ and $E + \delta E$ is called a microcanonical ensemble. In statistical mechanics, the entropy $S(E, N)$ of the system $\mathcal{S}_{\text{total}}$ is then defined by $S(E, N) = k_B \ln \Theta(E, N)$. The average energy $\varepsilon$ per one subsystem is given by $E/N$. In a normal case where $\varepsilon$ has a finite value, the entropy $S(E, N)$ is proportional to $N$. On the other hand, the indeterminacy $\delta E$ of the energy contributes to $S(E, N)$ through the term $k_B \ln \delta E$, which can be ignored compared to $N$ unless $\delta E$ is too small. Thus the magnitude of the indeterminacy $\delta E$ of the energy does not matter to the value of the entropy $S(E, N)$ unless it is too small. The temperature $T(E, N)$ of the system $\mathcal{S}_{\text{total}}$ is defined by

$$\frac{1}{T(E, N)} = \frac{\partial S}{\partial E}(E, N).$$

Thus the temperature is a function of $E$ and $N$.

Now we give a total statistical mechanical interpretation to AIT. As considered in [4], think of the optimal computer $U$ as a decoding equipment at the receiving end of a noiseless binary communication channel. Regard its programs (i.e., finite binary strings in $\text{dom } U$) as codewords and regard the result of the computation by $U$, which is a finite binary string, as a decoded “symbol.” Since $\text{dom } U$ is a prefix-free set, such codewords form what is called an “instantaneous code,” so that successive symbols sent through the channel in the form of concatenation of codewords can be separated.

For establishing the total statistical mechanical interpretation of AIT, we assume that the infinite binary string sent through the channel is generated by infinitely repeated tosses of a fair coin. Under this assumption, the infinite binary string is referred to as the channel infinite string hereafter. For each $r \in \{0, 1\}^*$, let $Q(r)$ be the probability that the channel infinite string has the prefix $r$. It follows that $Q(r) = 2^{-|r|}$. Thus, the channel infinite string is the random variable drawn according to Lebesgue measure on $\{0, 1\}^\infty$. Note that the success probability for $U$ to decode one symbol at the receiving end of the channel infinite string equals to Chaitin’s halting probability $\Omega$ since $\Omega = \sum_{p \in \text{dom } U} Q(p)$. Since this success probability can be regarded as a probability for $U$ to halt at the receiving end of the channel infinite string, we will establish the total statistical mechanical interpretation of AIT, based on a probability measure which gives Chaitin’s $\Omega$ the meaning of halting probability actually. In addition, note that the probability to get a finite binary string $s$ as the first decoded symbol by $U$ at the receiving end of the channel infinite string equals to $P(s)$ since $P(s) = \sum_{U(p) = s} Q(p)$.

Let $N$ be a large number, say $N \sim 10^{2^{22}}$. We relate AIT to the statistical mechanics reviewed above in the following manner. Among all infinite binary strings, consider infinite binary strings of the form $p_1p_2 \cdots p_N \alpha$ with $p_1, p_2, \ldots, p_N \in \text{dom } U$ and $\alpha \in \{0, 1\}^\infty$. For each $i$, the $i$th slot fed by $p_i$ corresponds to the $i$th quantum subsystem $S_i$. On the other hand, the ordered sequence of the 1st slot, the 2nd slot, $\ldots$, and the $N$th slot corresponds to the quantum system $\mathcal{S}_{\text{total}}$. We relate a codeword $p \in \text{dom } U$ to an energy eigenstate of a quantum subsystem, and relate a codeword length $|p|$ to an energy $E_p$ of the energy eigenstate of the quantum subsystem. Then, a finite binary string $p_1 \cdots p_N$ corresponds to an energy eigenstate of $\mathcal{S}_{\text{total}}$ specified by $(n_1, \ldots, n_N)$. Thus, $|p_1 \cdots p_N| = |p_1| + \cdots + |p_N|$ corresponds to the energy $E_{n_1} + \cdots + E_{n_N}$ of the energy eigenstate of $\mathcal{S}_{\text{total}}$.

We define a subset $C(L, N)$ of $\{0, 1\}^*$ as the set of all finite binary strings of the form $p_1 \cdots p_N$ with $p_i \in \text{dom } U$ whose total length $|p_1 \cdots p_N|$ lie between $L$ and $L + \delta L$. Then, $\Theta(L, N)$ is defined as the total number of elements of the finite set $C(L, N)$. Therefore, $\Theta(L, N)$ is the total number of all concatenations of $N$ codewords whose total length lie between $L$ and $L + \delta L$. It follows that if $p_1 \cdots p_N \in C(L, N)$, then $2^{-(L+\delta L)} \leq Q(p_1 \cdots p_N) \leq 2^{-L}$. Thus, all concatenations $p_1 \cdots p_N \in C(L, N)$ of $N$ codewords occur in a prefix of the channel infinite string with the same probability $2^{-L}$. Note here that we care nothing about the magnitude of
δL, as in the case of statistical mechanics. Thus, the following principle, called the principle of equal conditional probability, holds.

**The Principle of Equal Conditional Probability:** Given that a concatenation of \( N \) codewords of total length \( L \) occurs in a prefix of the channel infinite string, all such concatenations occur with the same probability \( 1/\Theta(L, N) \).

We introduce a microcanonical ensemble into AIT in this manner. Thus, we can develop a certain sort of statistical mechanics on AIT. Note that, in statistical mechanics, the principle of equal probability is just a conjecture which is not yet proved completely in a realistic physical system. On the other hand, in our total statistical mechanical interpretation of AIT, the principle of equal conditional probability is automatically satisfied.

The *statistical mechanical entropy* \( S(L, N) \) is defined by

\[
S(L, N) = \log_2 \Theta(L, N).
\]  

(6)

The *temperature* \( T(L, N) \) is then defined by

\[
\frac{1}{T(L, N)} = \frac{\partial S}{\partial L}(L, N).
\]  

(7)

Thus, the temperature is a function of \( L \) and \( N \).

According to the theoretical development of equilibrium statistical mechanics,\(^5\) we can introduce a *canonical ensemble* into AIT in the following manner. We investigate the probability distribution of the left-most codeword \( p_1 \) of the channel infinite string, given that a concatenation of \( N \) codewords of total length \( L \) occurs in a prefix of the channel infinite string. For each \( p \in \{0, 1\}^* \), let \( R(p) \) be the probability that the left-most codeword of the channel infinite string is \( p \), given that a concatenation of \( N \) codewords of total length \( L \) occurs in a prefix of the channel infinite string. Then, based on the principle of equal conditional probability, we see that \( R(p) = \Theta(L-|p|, N-1) / \Theta(L, N) \). From the general definition (6) of statistical mechanical entropy, we thus have

\[
R(p) = 2^{S(L-|p|, N-1)-S(L, N)}. \tag{8}
\]

Let \( E(L, N) \) be the expected length of the left-most codeword of the channel infinite string, given that a concatenation of \( N \) codewords of total length \( L \) occurs in a prefix of the channel infinite string. Then, the following equality is expected to hold:

\[
S(L, N) = S(E(L, N), 1) + S(L - E(L, N), N - 1). \tag{9}
\]

Here, the term \( S(L, N) \) in the left-hand side denotes the statistical mechanical entropy of the whole concatenation of \( N \) codewords of total length \( L \). On the other hand, the first term \( S(E(L, N), 1) \) in the right-hand side denotes the statistical mechanical entropy of the left-most codeword of the concatenation of \( N \) codewords of total length \( L \) while the second term \( S(L - E(L, N), N - 1) \) in the right-hand side denotes the statistical mechanical entropy of the remaining \( N - 1 \) codewords of the concatenation of \( N \) codewords of total length \( L \). Thus, the equality (9) represents the additivity of the statistical mechanical entropy. We assume here that the equality (9) holds.

By expanding \( S(L-|p|, N-1) \) around the “equilibrium point” \( L - E(L, N) \), we have

\[
S(L-|p|, N-1) = S(L - E(L, N) + E(L, N) - |p|, N-1) \\
= S(L - E(L, N), N - 1) + \frac{\partial S}{\partial L}(L - E(L, N), N - 1)(E(L, N) - |p|). \tag{10}
\]

\(^5\) We follow the argument of Section 16-1 of Callen [1] in particular.
Here, we ignore the higher order terms than the first order. Since $N \gg 1$ and $L \gg E(L, N)$, using the definition (7) of temperature we have

$$
\frac{\partial S}{\partial L} (L - E(L, N), N - 1) = \frac{\partial S}{\partial L} (L, N) = \frac{1}{T(L, N)}.
$$

(11)

Hence, by (10) and (11), we have

$$
S(L - |p|, N - 1) = S(L - E(L, N), N - 1) + \frac{1}{T(L, N)} (E(L, N) - |p|).
$$

(12)

Thus, using (8), (9), and (12), we obtain $R(p) = 2^{(E(L, N) - T(L, N)S(E(L, N), 1))} 2^{-\frac{|p|}{T(L, N)}}$. Then, according to statistical mechanics or thermodynamics we define the free energy $F(L, N)$ of the left-most codeword of the concatenation of $N$ codewords of total length $L$ by

$$
F(L, N) = E(L, N) - T(L, N) S(E(L, N), 1).
$$

(13)

It follows that

$$
R(p) = 2^{F(L, N)/T(L, N)} 2^{-\frac{|p|}{T(L, N)}}.
$$

(14)

Using $\sum_{p \in \text{dom} \ U} R(p) = 1$ we see that, for any $p \in \text{dom} \ U$,

$$
R(p) = \frac{1}{Z(T(L, N))} 2^{-\frac{|p|}{T(L, N)}},
$$

(15)

where $Z(T)$ is defined by

$$
Z(T) = \sum_{p \in \text{dom} \ U} 2^{-\frac{|p|}{T}} \quad (T > 0).
$$

(16)

$Z(T)$ is called the partition function (of the left-most codeword of the channel infinite string). Thus, in our total statistical mechanical interpretation of AIT, the partition function $Z(T)$ has exactly the same form as $\Omega(D)$, which is defined by (1). The distribution in the form of $R(p)$ is called a canonical ensemble in statistical mechanics.

Then, using (14) and (15), $F(L, N)$ is calculated as

$$
F(L, N) = F(T(L, N)),
$$

(17)

where $F(T)$ is defined by

$$
F(T) = -T \log_2 Z(T) \quad (T > 0).
$$

(18)

On the other hand, from the definition of $R(p)$, $E(N, L)$ is calculated as $E(N, L) = \sum_{p \in \text{dom} \ U} |p| R(p)$. Thus, we have

$$
E(L, N) = E(T(L, N)),
$$

(19)

where $E(T)$ is defined by

$$
E(T) = \frac{1}{Z(T)} \sum_{p \in \text{dom} \ U} |p| 2^{-\frac{|p|}{T}} \quad (T > 0).
$$

(20)

Then, using (13), (17), and (19), the statistical mechanical entropy $S(E(L, N), 1)$ of the left-most codeword of the concatenation of $N$ codewords of total length $L$ is calculated as $S(E(L, N), 1) = S(T(L, N))$, where $S(T)$ is defined by

$$
S(T) = \{E(T) - F(T)\} / T \quad (T > 0).
$$

(21)

Note that the statistical mechanical entropy $S(E(L, N), 1)$ coincides with the Shannon entropy $-\sum_{p \in \text{dom} \ U} R(p) \log_2 R(p)$ of the distribution $R(p)$. 


Finally, the specific heat $C(T)$ of the left-most codeword of the channel infinite string is defined by

$$C(T) = E'(T) \quad (T > 0),$$

(22)

where $E'(T)$ is the derived function of $E(T)$.

Thus, a total statistical mechanical interpretation of AIT can be established, based on a physical argument. We can check that the formulas in this argument: the partition function (16), the free energy (18), the expected length of the left-most codeword (20), the statistical mechanical entropy (21), and the specific heat (22) correspond to the thermodynamic quantities of AIT in Definition 3.1. Thus, the statistical mechanical meaning of the notion of thermodynamic quantities of AIT in Definition 3.1 is clarified by this argument.

5. Concluding remarks

In this paper, we have developed a total statistical mechanical interpretation of AIT which actualizes a perfect correspondence to normal statistical mechanics. However, the argument used in the development is on the same level of mathematical strictness as normal statistical mechanics in physics. Thus, we try to make the argument a rigorous form in a future study. This effort might stimulate a further unexpected development of the mathematical research of AIT.

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