VELOCITY ADDITION AND A CLOSED TIME CYCLE IN LORENTZ-NONINVARIANT THEORIES

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In theories whose Lorentz invariance is violated by the presence of an external tensor of any rank, we show that a signal velocity, understood as the group velocity of a wave, is added to the velocity of the reference frame according to the standard relativistic rule for adding velocities. In the case where we have a superluminal signal, this observation allows creating a closed time cycle and thus coming to a conclusion about a causality violation even in the absence of relativistic invariance. We also reveal an optical anisotropy of a moving medium that is isotropic at rest.

Keywords: Lorentz-invariance violation, causality, superluminal propagation velocity, moving medium, closed time cycle

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1. Introduction

About the covariant description of Lorentz-noninvariant theories. Much attention in recent years has been paid to models without relativistic invariance. Investigations in this direction were initiated in [1]. In this case, the goal was to establish the possibility of small deviations from the Standard Model, manifesting a violation of the special theory of relativity. The object of investigation dealt with in those works is a special rank-four tensor peculiar to the vacuum and not formed by the current fields. A special case of a Lorentz-noninvariant theory beyond the Standard Model is space–time noncommutative theories (see [2], [3]). On the other hand, in the framework of the Standard Model, we often consider theories where the invariance under Lorentz boosts and also under spatial rotations is violated by the presence of external fixed tensors. These can be due to the presence of a medium, external fields, a nontrivial metric, or combinations of these factors. In the presence of a medium, the role of an external tensor (of the first rank) is played by the four-velocity vector of the medium. In other cases, these are the tensors of external fields or the tensors characterizing the external metrics and also the noncommutativity 4×4 tensor when a noncommutative theory is considered.

In all the mentioned cases, the theory can be given a Lorentz-covariant form: we can define the action as a Lorentz-scalar combination of fields and external tensors. The generating functionals of the Green’s functions and the vertex functions have the same property (see, e.g., [4]). The Green’s functions
and one-particle-irreducible vertices, which are second- and higher-rank polarization tensors, obtained by respectively differentiating those functionals over the vector currents and fields, are formed as matrices constructed using external tensors in addition to the particle momenta or coordinates.

Such a program was realized in quantum electrodynamics with external electric and magnetic fields in [5] (also see [6]). Apparently, the first example of a covariant representation of the polarization tensor in a (moving) medium using its four-velocity can be found in [7]. The case of a medium in a magnetic field was considered in [6], [8]. A microscopic foundation for this way of introducing the four-velocity was given in [9] based on the temperature Green’s function formalism; we further comment on this in Sec. 2.3 below. A covariant interpretation of the noncommutativity matrix as a second-rank tensor was used for the same purpose in [10], where the third-rank polarization tensor in external electric and magnetic fields responsible for the nonlinear photon-splitting processes was also similarly constructed. A covariant representation for the third-rank polarization tensor in a medium without an external field was obtained in [11].

The considered covariant approach is in fact a realization of the “extended relativity principle,” which is that an inertial Lorentz reference frame A with an external tensor in it is equivalent to a reference frame B that moves with respect to A with a constant velocity \( V \) if the external tensor in the frame B is obtained from the external tensor in A by the Lorentz transformation corresponding to the velocity \( V \). The gauge-invariant energy–momentum tensor in this theory is not symmetric, and its antisymmetric part is responsible for the nonconservation of the generators of the angular momentum and Lorentz boosts [12].

In the framework of the discussed approach, we here obtain the transformation law for the group velocity of an electromagnetic wave packet from one inertial frame to another and show that it coincides exactly with the standard relativistic rule for adding the two velocities, the group velocity of the packet and the relative velocity of the two inertial reference frames. We consider the group velocity because it is the propagation velocity of a wave packet [13]. This fact plays an important role in the discussion of causality issues in Sec. 3, although some authors often refer to the “phase” velocity, which is incorrect in this case. The only reservation usually made about the group velocity is that it can exceed the speed of light near a resonance, causing the so-called anomalous dispersion. But this reservation is easily dismissed [14] by considering the real part of the complex group velocity as a function of the real momentum instead of the real group velocity as a function of the real part of the complex momentum. The velocity of the wave front, regarded as the signal velocity according to Brillouin [15], always propagates with the speed of light in the vacuum because it is determined by the infinite-frequency limit \( k_0 = \infty \) and the effect of factors violating the Lorentz invariance, such as an external field or a medium, disappear in this limit. Consequently, this velocity is not important in discussing causality issues.

We present the derivation of the transformation law for the group velocity in Sec. 2.2, preceded by a description of the covariant approach in Sec. 2.1 and followed by a consideration of the interesting special case of light propagation in a moving medium, discussed as an illustration in Sec. 2.3. We show that the dielectric tensor of an isotropic medium becomes anisotropic in a moving frame and the principal axis is determined by the directions of the wave vector and the velocity \( V \) of the medium. In Sec. 3, among other causality issues, we discuss the construction of a closed time cycle in a Lorentz-noninvariant theory.

2. Lorentz transformation of the group velocity

2.1. Polarization operator and dispersion equations. The second pair of Maxwell equations linearized over a homogeneous background for the four-vector potential \( A^\rho(k) \) of a free electromagnetic wave with the momentum \( k_\mu \) can be written as

\[
(k^2 g_{\mu\rho} - k_\mu k_\rho)A^\rho(k) - \Pi_{\mu\rho}(k)A^\rho(k) = 0,
\]

(1)
where $\Pi_{\mu\nu}(k)$ is the polarization operator defined in the configuration space as

$$
\Pi_{\mu\tau}(x, y) = \frac{\delta^2 \Gamma}{\delta A_\mu(x) \delta A_\tau(y)} \bigg|_{F=\bar{F}={\text{const}}},
$$

where the effective action $\Gamma$ is the generating functional of one-particle-irreducible vertices, of which the polarization operator is one [4]. We here allow the effective action $\Gamma$ to depend not only on external tensors but also on space–time derivatives of any order of the electromagnetic field tensor $F_{\mu\nu}$, but we assume that all of them vanish after the variational differentiations in (2) are performed. The Greek indices range the Minkowski space.

We present the polarization operator in the diagonal form

$$
\Pi_{\mu\tau}(k,p) = \delta(k-p)\Pi_{\mu\tau}(k), \quad \Pi_{\mu\tau}(k) = \sum_{a=1}^{3} \kappa^{(a)}_{\mu} \hat{\lambda}_{\mu}^{(a)}(\bar{\rho}(a))^2,
$$

where $\hat{\lambda}_{\mu}^{(a)}$ are its eigenvectors, namely,

$$
\Pi_{\mu\tau} \hat{\lambda}_{\mu}^{(a)} = \kappa_{\mu}^{(a)} \hat{\lambda}_{\mu}^{(a)}, \quad a = 1, \ldots, 4.
$$

The effective action is a Lorentz scalar formed using all the external tensors and fields, and it does not depend on coordinates explicitly. Hence, polarization operator (2) is a tensor in Minkowski space, and its eigenvalues $\kappa_{\mu}$ are scalars. The appearance of the delta function conserving the energy–momentum in (3) is a consequence of the assumption that as factors violating the Lorentz invariance, we consider only those that are gauge-invariant and independent of space and time and hence do not violate the translation invariance.

The fourth eigenvector is trivial, $\hat{\lambda}_{\mu}^{(4)} = k_{\mu}$, and the fourth eigenvalue therefore vanishes, $\kappa_{4} = 0$, because the polarization operator is four-transverse: $\Pi_{\mu\tau} k_{\tau} = 0$. All the eigenvectors are mutually orthogonal, $\hat{\lambda}_{\mu}^{(a)} \hat{\lambda}_{\mu}^{(b)} \sim \delta_{ab}$, and this means that the first three are four-transverse: $\hat{\lambda}_{\mu}^{(a)} k_{\mu} = 0$. The connection between the polarization operator $\Pi_{\mu\tau}$ and the rank-four tensor studied and attempted to be measured with the approach in [1] was discussed in [12].

The scalar eigenvalues $\kappa_{a}$ can depend on all basic scalars in the theory, including $k^2$ and those that can be formed by tensors of any rank characteristic of the medium or the vacuum after they are contracted with the photon four-momentum $k_{\mu}$, for instance, $k^2 F^2 k$, $k \theta F k$, etc. Here, $\bar{F}_{\mu\nu}$ is the external field strength tensor, and $\theta_{\mu\nu}$ is the noncommutativity tensor. The set of these external tensors can also contain the medium four-velocity vector, if matter is present. The arguments of $\kappa$ also include momentum-independent scalars, such as the external field invariants $\tilde{G} = -1/4 \bar{F}^2$ and $\mathcal{G} = -1/4 \bar{F} \bar{F}$, where $\bar{F}$ is the dual electromagnetic tensor, but these are inessential for our consideration.

The photon dispersion laws for each of three polarization modes can be found from the equations

$$
k^2 = \kappa_{a}(k), \quad a = 1, 2, 3,
$$

which are the solvability conditions for Eq. (1). Relations (4) determine the frequency $k_{0}$ as a function of the wave-vector components $k_{i}$. The eigenvectors $\hat{\lambda}_{\mu}^{(a)}$ serve as four-vector potentials for free eigenwaves. There are three eigenmodes in all, not two, because possible massive vector particles are also included in propagation equation (1). For instance, such can be the electron–positron states (mutually free or bound into the positronium atom) with which the photon unites into a mixed polariton state [16]. Another example of the third polarization degree of freedom, characteristic of massive vector particles, is supplied by the known longitudinal modes in a medium. We comment further on this in Sec. 2.3.
The dielectric permittivity tensor $\varepsilon_{nj}$ of the anisotropic “medium,” to which the vacuum with the broken Lorentz invariance is equivalent regardless of the presence or absence of a real medium, is related to the polarization tensor components as

$$
\varepsilon_{nj} = \delta_{nj} + \frac{\Pi_{nj}}{k_0^2}, \quad n, j = 1, 2, 3.
$$

(5)

The Lorentz-noninvariant vacuum manifests itself as an equivalent anisotropic “medium” because tensor (5) differs from $\delta_{nj}$. If a homogeneous medium, isotropic in its rest frame, is present alone in a Lorentz-invariant vacuum (i.e., where there are no external tensors except the medium four-velocity), then it becomes anisotropic in a moving frame. We study this phenomenon in Sec. 2.3.

The refractive indices (in every eigenmode) are defined on solutions of dispersion equations (4) as

$$
n_a(k_0, k) = \frac{|k|}{k_0} = \left(1 + \frac{x_a(k_0, k)}{k_0^2}\right)^{1/2}.
$$

(6)

In contrast to the eigenvalues $x_a$, these are not Lorentz scalars.

2.2. Derivation of the addition rule for velocities. Let $I_s, s = 1, 2, \ldots$, denote the momentum-dependent Lorentz invariants including $k^2 = -k_0^2 + k_i^2$. The number of them depends on the problem. The derivative $\partial I_s/\partial k_\mu = P^{(s)\mu}$ is a Lorentz vector. The group velocity of a photon is the three-vector defined as the frequency differentiated with respect to the wave vector,

$$
v^\text{gr}_i = \frac{\partial k_0}{\partial k_i}, \quad i = 1, 2, 3,
$$

(7)

and calculated on a solution of dispersion equation (4). We hereafter omit the indexing of the photon modes $a$, understanding that our manipulation relates to each of the three separately. Assuming summation over all invariants $I_s$, we obtain

$$
v^\text{gr}_i = -\left(\frac{\partial(k^2 - x)}{\partial k_i}\right)\left(\frac{\partial(k^2 - x)}{\partial k_0}\right)^{-1} = \frac{2k_i - X_i P^{(s)}_i}{2k_0 - X_s P^{(s)}_0},
$$

(8)

where the quantities $X_s = \partial x/\partial I_s$ are Lorentz invariant.

We now imagine that an inertial Lorentz frame $W$, to which the preceding equations relate, moves with respect to the initial frame $W'$ (marked with a prime) with the velocity $\mathbf{V}$. We intend to demonstrate that the group velocity $\mathbf{v}^\text{gr}$ in the frame $W'$ is related to the group velocity in the frame $W$ by the standard relativistic rule $\mathbf{v}' = \mathbf{v} \oplus \mathbf{V}$ for adding a signal velocity $\mathbf{v}$ to the velocity of the reference frame, with the group velocity taken for $\mathbf{v}$. Namely, we show that

$$
v^\text{gr}_\parallel = v^\text{gr}_\parallel \oplus \mathbf{V} \equiv \frac{V + v^\text{gr}_\parallel}{1 + V v^\text{gr}_\parallel},
$$

(9)

$$
v^\text{gr}_\perp = v^\text{gr}_\perp \oplus \mathbf{V} \equiv \frac{v^\text{gr}_\perp \sqrt{1 - V^2}}{1 + V v^\text{gr}_\parallel},
$$

(10)

where the subscripts $\parallel$ or $\perp$ indicate the directions parallel and orthogonal to $\mathbf{V}$.

We assume that the physical processes ensuring the signal propagation with the velocity $\mathbf{v}$ in the rest frame $W$ and the processes responsible for its propagation with the velocity $\mathbf{v} \oplus \mathbf{V}$ in the moving frame $W'$ are described by the same equations but only covariantly transformed in passing from the frame $W$ to
the frame $W'$. In other words, we accept that the physical carrier of the signal is an electromagnetic wave process governed by invariant equations (1) and that the group velocity of a propagating packet plays the role of the signal velocity. To effectively take the mentioned Lorentz transformation of the electromagnetic signal into account, we essentially rely on the fact established in Sec. 2.1 that the right-hand side $\varkappa$ of dispersion equation (4) is a Lorentz scalar.

Using equality (8) in expression (9), we obtain the relation

$$v'^{gr} \oplus v = \left( V + \frac{2k_0 - X_sP_0^{(s)}}{2k_0 - X_sP_0^{(s)}} \right) \left( 1 + V \frac{2k_0 - X_sP_0^{(s)}}{2k_0 - X_sP_0^{(s)}} \right)^{-1} =$$

$$= \frac{V(2k_0 - X_sP_0^{(s)}) + 2k_0 - X_sP_0^{(s)}}{2k_0 - X_sP_0^{(s)} + V(2k_0 - X_sP_0^{(s)})}.$$ (11)

Because the Lorentz transformations for the vectors $\partial I_s/\partial k_\mu = P^{(s)\mu}$ (and the analogous transformations for the momenta $k'^\mu$) in passing from one frame to the other have the form

$$P'_\| = \frac{P_\| + VP_0}{\sqrt{1 - V^2}}, \quad P'_0 = \frac{P_0 + VP_\|}{\sqrt{1 - V^2}},$$ (12)

it follows from (11) that

$$v'^{gr} \oplus V = \frac{2k'_\| - X_sP_0^{(s)}}{2k'_0 - X_sP_0^{(s)}}.$$ (13)

which is just the group velocity $v'^{gr}$ given by (8) (more precisely, its parallel projection) calculated in the frame $W'$ with the Lorentz invariance of the $X_s$ taken into account. We have thus proved relation (9).

Analogously, substituting equality (8) in expression (10), we obtain

$$v'^{gr} \oplus V = \left( \frac{2k_\| - X_sP_0^{(s)}}{2k_0 - X_sP_0^{(s)}} \sqrt{1 - V^2} \right) \left( 1 + V \frac{2k_0 - X_sP_0^{(s)}}{2k_0 - X_sP_0^{(s)}} \right)^{-1} =$$

$$= \frac{(2k_\| - X_sP_0^{(s)})\sqrt{1 - V^2}}{2k_0 - X_sP_0^{(s)} + V(2k_0 - X_sP_0^{(s)})} = \frac{2k'_\| - X_sP_0^{(s)}}{2k'_0 - X_sP_0^{(s)}}.$$ (14)

We again use the second equation in (12) for the transformation of $P_0^{(s)}$ and the analogous transformation for $k_0$ and also the fact that the vector components perpendicular to the relative velocity $V$ of the two reference frames do not change under the Lorentz boost along $V$. Comparing (8) and (14), we see that (14) is just the perpendicular group velocity $v'^{gr}$ as calculated by the observer in the reference frame $W'$. The relativistic rule for adding velocities of perpendicular component (10) has thus also been proved.

The considered derivation of the relativistic law for adding velocities can by no means be applied to the phase velocity $v_{ph} = k_0/|k|$ in correspondence with a well-known fact [17]: the phase velocity cannot serve as a signal velocity and is not even a three-vector.

**2.3. Moving isotropic medium.** An important example is an isotropic medium, whose presence violates the Lorentz invariance. The electromagnetic field equations can be given a Lorentz-covariant form by introducing the four-velocity vector $u_\mu$, $u^2 = -1$, $u_0 = (1 - V^2)^{-1/2}$, $u = V(1 - V^2)^{-1/2}$, where $V$ is the velocity three-vector of the moving medium. The scalars discussed in Sec. 2.2 are then $I_1 = k^2$ and $I_2 = (uk)^2$. 817
Introducing this vector allows a covariant treatment of the linear [7] and nonlinear [11] Maxwell equations of an initially isotropic medium also after it is set in motion with a constant velocity as a whole and moreover placed in an external field [8]. A microscopic theory justifying such an approach in the temperature Green’s function method in relativistic statistics developed in [7] can be found in [9]. This theory is based on writing the density matrix in the form \( \rho = e^{-\beta(\mu P)} \), where \( \beta \) is the Lorentz-invariant inverse temperature parameter and \( P \) is the four-momentum. The thermodynamical potential, playing the role of the Lagrangian density in plasma, is then written as a Lorentz scalar using the four-velocity vector. We recall that the standard density matrix expression, to which it reduces in the plasma rest frame, has the form \( \rho = e^{-\beta H} \), where \( \mu = (1, 0, 0, 0) \) and \( H = P^0 \) is the Hamiltonian.

The “extended relativity principle” mentioned in the introduction now states that any two inertial frames in a medium are equivalent after the Lorentz transformation is performed on the velocity of the medium in passing from one frame to the other. In more sensory terms, this means that an observer in a frame in motion relative to a medium, air for example, certainly feels the motion because the wind can be felt. In this respect, the usual relativity principle is violated together with the Lorentz invariance. But after the motion of the medium with respect to the observer is excluded by its Lorentz transformation, the situation for the observer returns to equivalence to that viewed by an observer at rest with respect to the medium.

The most general covariant tensor decomposition [7] for the polarization operator of an isotropic homogeneous medium can be rewritten in the diagonal form [11], [18]

\[
\Pi_{\mu\nu}(k) = \kappa \sum_{b=1,2} \frac{c^{(b)}_{\mu} c^{(b)}_{\nu}}{(c^{(b)})^2} + \kappa_3 \frac{a_{\mu} a_{\nu}}{a^2},
\]

where \( a_{\mu} = u_{\mu} k^2 - k_{\mu} (uk) \), \( a^2 = k^2(k^2 - (uk)^2) \), \( (au) = 0 \).

The vector \( c^{(1)}_{\mu} \) is defined as an arbitrary four-vector normal to the hyperplane spanned by the vectors \( k_{\mu} \) and \( a_{\mu} \). The vector \( c^{(2)}_{\mu} \equiv \varepsilon_{\mu\rho\lambda\epsilon} c^{(1)}_{\rho} a_{\lambda} k_{\epsilon} \) is also normal to that hyperplane and also to \( c^{(1)}_{\mu} \). The four vectors \( k_{\mu}, a_{\mu}, \) and \( c^{(1,2)}_{\mu} \) are eigenvectors of the polarization operator:

\[
\Pi_{\mu\nu} c^{(1,2)}_{\mu} c^{(1,2)}_{\nu} = \kappa_1 c^{(1,2)}_{\mu} c^{(1,2)}_{\mu}, \quad \kappa_1 = \kappa_2 = \kappa, \quad \Pi_{\mu\nu} a_{\mu} = \kappa_3 a_{\mu}, \quad \Pi_{\mu\nu} k_{\mu} = 0.
\]

Only three of them are involved in decomposition (15) because one eigenvalue is zero as a result of the transversality \( \Pi_{\mu\nu} k_{\nu} = 0 \).

The three basis vectors \( a_{\mu} \) and \( c^{(1,2)}_{\mu} \) are four-vector potentials of the electromagnetic eigenwaves. The orientations of the corresponding fields—electric \( e_i \sim k_0 a_i - k_i a_0 \) (or \( e_i \sim k_0 c^{(1,2)}_{i} - k_i c^{(1,2)}_{0} \)) and magnetic \( h_i = \varepsilon_{ijk} k_j a_k \) (or \( h_i = \varepsilon_{ijk} c^{(1,2)}_{j} \))—calculated based on these vector potentials were described in detail in [11]. In the Lorentz frame in which the medium is at rest, modes 1 and 2 are electromagnetic waves transversely polarized in the plane orthogonal to the wave vector \( k \), while mode 3 is a purely electric wave (the magnetic field is equal to zero) with the electric field longitudinally polarized along \( k \), \( e \sim k(k_0^2 - k^2) \).

This wave can be realized under the condition that the dispersion equation has a massive solution with \( k_0^2 - k^2 \neq 0 \). We note all this only to illustrate how these well-known facts appear in the considered formalism.

The degeneracy \( \kappa_1 = \kappa_2 \) in (16) reflects the symmetry of the problem under rotations around the direction of the velocity \( \mathbf{V} \). Dispersion equations (4) become

\[
k^2 = \kappa_{1,2}(k^2, (uk)^2), \quad k^2 = \kappa_3(k^2, (uk)^2),
\]

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where we explicitly indicate the dependence of the eigenvalues on the two Lorentz scalars $k^2$ and $(uk)^2$.

In accordance with representation (15), the spatial part $\Pi_{nj}$ of the polarization tensor in addition to the unit tensor $\delta_{nj}$ is formed by the wave vector $k$ and the velocity $V$ of the moving medium:

$$\Pi_{nj} = A\delta_{nj} + Bk_nk_j + C(k_nV_j + V_nk_j) + DV_nV_j,$$

where the coefficients $A$, $B$, $C$, and $D$ are certain rotational scalars, functions of $k^2$, $V$, $(V \cdot k)$, and $k_0$. Therefore, the vector $d$ orthogonal to the plane spanned by these two vectors, $(d \cdot k) = (d \cdot V) = 0$, is a universal eigenvector of three-dimensional dielectric permittivity tensor (5):

$$\varepsilon_{nj}d_j = \frac{A}{k_0}d_i.$$

The orientations of the other two eigenvectors in this plane depend on the individual orientations of the vectors $k$ and $V$. This situation is typical for crystals of a monoclinic system [17].

We conclude this subsection with the statement that an at-rest isotropic medium behaves as an anisotropic medium when it moves. Its dielectric tensor has three different eigenvalues (all depending on the medium velocity) and three optical axes determined by the direction of that velocity. The refractive indices also depend on the reference frame. There are three eigenwaves with different dispersion laws. Transparent isotropic bodies moving with a relativistic velocity are birefringent.

According to relations (9) and (10), in a moving medium, the velocity of every eigenwave, regarded as its group velocity, is obtained as the relativistic sum of its rest-frame group velocity and the medium velocity. A similar conclusion about a medium that is already anisotropic at rest cannot be drawn based on the results in Sec. 2.2, because it is unclear if the three-dimensional tensors responsible for this anisotropy can be given a relativistically covariant extension using the vector $u_\mu$.

3. Concluding remarks

We have demonstrated that when an electromagnetic wave packet is the physical carrier of information, its group velocity obeys standard relativistic law (9), (10) for adding velocities independently of whether the vacuum is Lorentz invariant or the relativistic invariance is violated by the presence of external vectors or tensors. We thus showed that the light propagation process governed by electromagnetic field equations depending on those tensors guarantees that the group velocity in the rest frame and in the moving frame are related according to the relativistic law for transforming the signal velocity. This indicates that in the vacuum with a violated Lorentz invariance just as in the Lorentz-invariant case [13], the group velocity has the important property of a signal velocity. In particular, if the group velocity and the frame velocity do not exceed $c = 1$ (the speed of light in the Lorentz-invariant vacuum), then the resulting group velocity in the moving frame remains less than unity. Therefore, causality normally survives a violation of relativistic invariance.

It sometimes happens that a superluminal signal appears when dynamical calculations are performed in theories with violated Lorentz invariance. Some examples are light propagation in external metrics [19] and in noncommutative electrodynamics [10], [20] and also in QED with an extremely strong external field [21] and in other systems (although the phase velocity was used as a criterion for superluminosity in [20], the authors’ conclusion turns out to be correct because it relates equally well to the group velocity). There is a temptation to dismiss the seriousness of such results, saying that so long as relativistic invariance has been discarded, there in no longer any reason for concern about causality, following the principle expressed in the Russian proverb “Sniavshi golovu, po volosam ne plachut!” or its German rationalized equivalent “Ist der Kopf abgeschlagen, wird niemand nach dem Hute fragen?” (This means approximately that the one
who is beheaded should not mourn over his hair or hat.) Such a position is based on the statement [22] that in accordance with the "extended relativity principle" as described in Sec. 2.3, a closed time cycle cannot be arranged in a Lorentz noninvariant theory even if a superluminal signal is available. Intuitively, this statement seems paradoxical because it implies that, for instance, faster-than-light propagation of an acoustic signal in an infinitely extended perfectly rigid body, whose presence certainly singles out its reference frame among other frames and hence violates the relativistic invariance, might be compatible with causality.

Nevertheless, we take the opposite standpoint expressed in [23]. Our results allow asserting that when a superluminal signal is carried by an electromagnetic wave packet in a Lorentz-noninvariant theory, the causality principle is violated in the same way as in a Lorentz-invariant theory. The point is that a closed time cycle (a "time machine") can be constructed in a thought experiment despite the absence of relativistic invariance.

Indeed, let a signal be emitted by an emitter at rest in the reference frame designated by a prime in Sec. 2.2. Let it be emitted at the origin \( x' = 0 \) at the instant \( t' = 0 \) and then propagate with the superluminal speed \( v' > 1 \) along the axis 1 to come to a certain point \( (x'_1, t') \), where \( x'_1 = v't' > t' \). Because this point is spacelike, there exists the Lorentz transformation

\[
x_1 = \frac{x'_1 + Vt'}{\sqrt{1 - V^2}}, \quad t = \frac{t' + Vx'_1}{\sqrt{1 - V^2}}
\]

in passing to a frame (without a prime) moving with the negative subluminal speed \( 1 > |V| > 1/v' \), which reverses the sign of the time \( t < 0 \). For us to be able to reflect the signal using a detector at rest in the unprimed frame such that it arrives at the worldpoint \( x = 0, t = 0 \), the reflected signal must have the speed \( |v| = x/t = (x'_1 + Vt')/(t' + Vx'_1) \), which is just our Eq. (9) for the group velocity in the moving frame. (We assume that the speeds in the direct and opposite directions are the same.) To complete the construction of a closed time cycle, it remains to perform the reverse Lorentz transformation to return to the frame equivalent to the initial frame. The worldpoint \( x = 0, t = 0 \) is then mapped to \( x' = 0, t' = 0 \).

Therefore, once the signal transmission is realized via a wave process governed by Lorentz-invariant dispersion law (4) (with external tensors included), the paradoxical influence of a consequence on its cause is achieved despite the lack of Lorentz invariance. Consequently, situations where the group velocity turns out to be greater than unity should be recognized as contradicting the causality principle in a theory with the Lorentz invariance violated by an external tensor, just exactly as it is so recognized in a Lorentz-invariant theory.

But in [10], where the group velocity of an electromagnetic wave exceeding the speed of light in noncommutative electrodynamics was analyzed, it was noted that the excess is extremely small, \( v_{gr} - 1 \ll 1 \). As a result, the velocity of the detector needed to realize the paradox of the influence of the future on the past becomes too large, beyond any existing human experience. In other words, to realize a closed cycle, it is insufficient to have a superluminal signal available; we also need some macroscopic bodies moving with a speed approaching the speed of light, \( 1 > |V| > 1/v' \). Our current knowledge does not extend to such speeds of macroscopic instruments. Therefore, a violation of the causality principle, although it seems logically absurd, cannot be completely ruled out based on the existing experience when the signal speed only insignificantly exceeds the speed of light.

We now extend that analysis to the superluminal electromagnetic wave solutions in black hole metrics, discovered in the one-loop approximation in the low-momentum limit in [19]. In the Schwarzschild metric, the excess speed is

\[
\Delta v \sim \frac{\alpha}{30\pi} \frac{r_{gr}}{r} \left( \frac{\lambda C}{r} \right)^2,
\]
where $\alpha = 1/137$ is the fine structure constant, $R_{gr}$ is the gravitational radius of the black hole, $\lambda_C \sim 2.4 \times 10^{-10}$ cm is the electron Compton wavelength, and $r$ is the distance from the center of the star. When the latter is equal to its gravitational radius, $r = R_{gr}$, we obtain

$$\Delta v \sim \frac{\alpha}{30\pi} \left(\frac{\lambda_C}{R_{gr}}\right)^2.$$

Even for the smallest black holes with a mass of the order of the solar mass, where $R_{gr} \sim 5 \times 10^5$ cm, this excess is $\Delta v \sim 2 \times 10^{-35}$, which is much smaller than the admissible estimates resulting from the noncommutative theory in [10]. This means that a tachyon, if emitted from inside a black hole, can overcome its horizon with a very small excess of its speed over the speed of light, with the excess reducing to zero as the tachyon further recedes from the black hole. In contrast, deep inside the black hole, where $r$ is small, the speed excess can be large, limited only by what the approximations in [19] allow, i.e., $\Delta v \ll 1$.

We can imagine that a tachyon emitted from the depth of the black hole, as it moves toward the horizon, meets matter falling into the black hole with a speed approaching the speed of light. The matter can reflect the tachyon back toward its source to form the necessary “detail” of the “time machine” depicted above. A prospect of developing an acausal electrodynamics of black holes thus arises, based on not only retarded but also advanced interaction.

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