Quantum mechanics in curved space-time II

C.C. Barros Jr.

Instituto de Física, Universidade de São Paulo,
C.P. 66318, 05315-970, São Paulo, SP, Brazil

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This paper is a sequence of the work presented in [1], where, the principles of the general relativity have been used to formulate quantum wave equations taking into account the effect of the electromagnetic and strong interactions in the space-time metric of quantum systems. Now, the role of the energy-momentum tensor in this theory is studied, and it is consistent with the formulation of the general quantum mechanics shown in [1]. With this procedure, a dynamical cut-off is generated and the constant $A$ of the field equation is calculated.

I. INTRODUCTION

A major question in the formulation of physics is if the concepts used to study large scale systems, as galaxies and even in the cosmology, may be applied to very small systems, such as atoms and elementary particles. Should the basic principles of physics depend on the size of the object?

In practical terms, one aspect of this question is if the microscopic world interactions, the weak, electromagnetic and the strong, may affect significantly the space-time metric and if this proposition may have any observable effect. In this description, the gravitational forces may be neglected, and it is a good approximation, due to the small masses of the considered particles. In [1], these ideas have been formulated considering a particle in a region with a potential, that affects the metric, and the wave equations for spin-0 and spin-1/2 particles, based on the general relativity principles, have been proposed, generating very interesting results.

The simplest systems where this theory could be tested are the one electron atoms, and the calculation of the deuterium spectrum has shown a clear numerical improvement when compared with the usual Dirac spectrum [2], [3] (also proposed by Sommerfeld [4]), with a percentual deviation from the experimental results approximately five times smaller, near one additional digit of precision.

An interesting fact that appeared from this theory, is the existence of black holes inside these quantum systems, with sizes that are not negligible. This propriety, that is related with the existence of a trapping surface at $r_0$, as it was defined by Penrose [6], in [5] have been successfully used in order to describe quark confinement. Solving the quantum wave equations [5], quark confinement has been obtained, without the need of introducing confining potentials, as it is currently done [7]-[10]. The confinement obtained in this way is a strong confinement, as the quark wave functions are not continuos at $r_0$.

In order to compliment this theory, proposed in [1] and [5], the energy-momentum tensor must be included in this formulation, and this study is the main purpose of this paper. We look for consistence with the previous results in an approach based on the energy-momentum tensor $T^{\mu\nu}$.

II. FIELD EQUATIONS AND METRIC

In this section, the field equations for particles subjected to electromagnetic and strong interactions, neglecting the effect of the gravitational field, will be obtained. For this purpose we will make a brief review of the results of [1], and then, relate them with a formulation based on the energy-momentum tensor.

As a first step, a system with spherical symmetry will be considered, but the basic ideas can be generalized to systems with arbitrary metrics. If the spherical symmetry is considered, the space-time may be described by a Schwarzschild-like metric [11], [12],

$$ ds^2 = \xi \, dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) - \xi^{-1} dr^2 , $$

(1)

where $\xi(r)$ is determined by the interaction potential $V(r)$, and is a function only of $r$, for a time independent
interaction. So, the metric tensor $g_{\mu\nu}$ is diagonal

$$g_{\mu\nu} = \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & -\xi^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}.$$  \hfill (2)

The energy relation for this system is \[1\]

$$E = \sqrt{p^2 c^2 + m_0^2 c^4},$$  \hfill (3)

therefore,

$$E(\vec{\beta} = 0) = E_0 \xi^{1/2} = E_0 + V,$$

where $\vec{\beta}$ is the particle velocity. This relation means that in the rest frame of the particle, the energy is simply due to the sum of its rest mass $E_0$ with the potential, and then,

$$\xi^{1/2} = 1 + \frac{V}{mc^2}.$$  \hfill (4)

Applying these ideas in the study of one electron atoms, one must consider a Coulomb interaction

$$V(r) = -\frac{\alpha Z}{r},$$  \hfill (5)

and $\xi$ is given by

$$\xi = 1 - \frac{2\alpha Z}{mc^2r} + \frac{\alpha^2 Z^2}{m^2 c^4 r^2}.$$  \hfill (6)

These expressions determine the horizon of events at $r_0$, that appears from the metric singularity $\xi(r_0)=0$, and using the values of \[13\], one finds

$$r_0 = \frac{\alpha Z}{mc^2} = 2.818 \text{ Z fm},$$  \hfill (7)

that is not a negligible value at the atomic scale.

Now, let us turn our attention to a description based on the energy-momentum tensor. If one consider a field generated by the electromagnetic interaction, the energy-momentum tensor is

$$T_{\mu\nu} = \epsilon_0 \left( F_{\mu\alpha} F^\nu_{\alpha} - \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right),$$  \hfill (8)

and it is related to the space-time geometry by the field equations

$$R^\mu_{\nu} - \frac{1}{2} R \delta^\mu_{\nu} = -A T^\mu_{\nu},$$

where $A$ is a constant to be determined.

In an one electron atom, $T^{\mu\nu}$ is determined by an electrostatic field, with the nonvanishing components

$$T^0_0 = \frac{1}{2} \left( \epsilon_0 E^2 + \frac{B^2}{\mu_0} \right),$$  \hfill (9)

and $T^r_r = -T^0_0$, that for a central electrostatic field with charge $Ze$, $T^0_0$ is just

$$T^0_0 = \frac{\epsilon_0 E^2}{2} = \frac{Ze^2 \alpha}{8\pi r^4}.$$  \hfill (10)
Eq. (10) may be used to determine the $\xi$ function observing that in the given metric

$$
e^{-B}\left(r\frac{dB}{dr} - 1\right) + \frac{1}{r^2} = AT_0^0$$

$$
e^{-B}\left(r\frac{dB}{dr} + 1\right) - \frac{1}{r^2} = -AT_0^0,$$

that have the solution

$$
\xi(r) = e^{-B} = 1 - \frac{c^2 AM(r)}{4\pi r}
$$

with

$$M(r) = \int_0^r 4\pi c^2 (r')^2 T_0^0(r') dr' .$$

(13)

If the particle is outside the black hole, that is the case of the electron in an atom, it will be affected just by the part of the field located in the region external to the horizon of events, and the integration must be performed in the interval $r_0 \leq r' \leq r$,

$$M(r) = m_0 + \int_{r_0}^r 4\pi c^2 (r')^2 T_0^0(r') dr' = m_0 + \frac{Z^2 \alpha}{2 c^2} \left(\frac{1}{r_0} - \frac{1}{r}\right),$$

(15)

where $m_0$ is a constant of integration. So,

$$M(r) = m_0 + \frac{mZ}{2} - \frac{Z^2 \alpha}{2 c^2 r},$$

(16)

and

$$\xi = 1 - \frac{c^2 AZ}{4\pi r} \left[m_0 + \frac{mZ}{2}\right] + \frac{AZ^2 \alpha}{8\pi r^2},$$

(19)

that is solution of (13). The constants $A$ and $m_0$ may be obtained now, comparing the expression, that is determined by the energy-momentum tensor and the field equations, with (7), determined by the energy relation (3).

Identifying the terms $r^{-1}$ and $r^{-2}$ we have

$$A = \frac{8\pi \alpha}{m^2 c^4},$$

$$m_0 = \frac{mZ}{2}.$$  

(20)

The field equation is then

$$R^\mu_\nu - \frac{1}{2} R g^\mu_\nu = -\frac{8\pi \alpha}{m^2 c^4} T^\mu_\nu.$$  

(21)

So, in this system, due to the interaction, the electron energy decrease with $r$, until it reaches the value $E(r_0) = 0$ at the horizon of events. At large distances, $E(\infty) = m$.

Considering that the strong interactions may be approximated by a strong coulomb field, with $\alpha \sim 1$, one may use the field equation in order to study strongly interacting systems. But if one wants to describe the strong interactions by the Yang-Mills field, determined by the correct coupling constant $\alpha$, and an Yang-Mills energy-momentum tensor.
III. SUMMARY AND CONCLUSIONS

In this paper the study of quantum systems in curved spaces has been continued. The metric is determined by the interaction of quantum objects, such as electrons and quarks. The effect of the energy-momentum tensor of electromagnetic and strong interactions in the space-time has been considered, and with this procedure the constant of the field equations has been calculated. The results obtained are consistent with the ones of [1].

One can observe the presence of the mass in the constant $A$, what does not happen in the general relativity, fact that is due to the equivalence principle. Observing that $A \propto g/m^2$, one concludes that the effect of the curvature of space-time decreases for large masses and increases for small masses and for large coupling constants. It is interesting to note that a dynamical cut-off is determined by this theory, as it was used in eq. (16), providing correct results.

Another observation that must be made is that initially, spherical symmetry has been supposed, but eq. (21) is general, independent of the symmetry of the system. This equation may also be used with the inclusion of other interactions, as the Yang-Mills one, for example, considering the Yang-Mills field tensor $F_{\mu\nu}^a$ in the construction of the energy-momentum tensor, and quark confinement, from the results of this theory, is expected to occur.

One must remark that with the development of the general quantum mechanics, we are being able to explain many characteristics of the studied physical systems, using the new proprieties that appears from the understanding of the geometry of the space-time.

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