Expected utility operators and coinsurance problem

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Abstract
The expected utility operators introduced in a previous paper offer a framework for a general risk aversion theory, in which risk is modeled by a fuzzy number \(A\). In this paper, we formulate a coinsurance problem in the possibilistic setting defined by an expected utility operator \(T\). Some properties of the optimal saving \(T\)-coinsurance rate are proved, and an approximate calculation formula of this is established with respect to the Arrow–Pratt index of the utility function of the policyholder, as well as the expected value and the variance of a fuzzy number \(A\). Various formulas of the optimal \(T\)-coinsurance rate are deduced for a few expected utility operators in case of a triangular fuzzy number and of some HARA- and CRRA-type utility functions.

Keywords Expected utility operators · Coinsurance

1 Introduction

In most cases, economic and financial activities are accompanied by risk, which generates pecuniary losses for the agents. A risk-averse agent will try to diminish losses caused by risk by closing an insurance contract. By Eeckhoudt et al. (2005), in the component of an insurance contract enter a premium \(P\) paid by the agent (policyholder) to an insurer and a real function \(I(\cdot)\) which specifies the part of the loss that is recovered: if the loss has the size \(x\), then the insurer will pay the agent the amount \(I(x)\).

Usually, the function \(I(\cdot)\) is defined by setting a coinsurance rate \(\beta\); if \(x\) is the loss, then the policyholder will receive the amount \(I(x) = \beta x\). The agent will choose that \(\beta\) maximizing the expected utility of her final wealth. So an optimization problem occurs, called the coinsurance problem.

A probabilistic model of risk assumes that this is mathematically represented by a random variable. The coinsurance problem from Eeckhoudt et al. (2005) and Mossin (1968) is such a probabilistic model, in which the loss is a random variable, and the agent is described by a utility function. The construction of the possibilistic model for coinsurance problem is done inside von Neumann–Morgenstern EU theory, whose main concept is the expected utility associated with a random variable and a utility function.

On the other hand, the probabilistic models do not reflect in a suitable manner all uncertain situations in which risk appears (e.g., whenever the information on the economical and financial phenomena is extracted from an insufficiently large data set). If the decisions are based on the judgments of a group of experts, then one can use Zadeh’s possibility theory Zadeh (1978). Possibility theory offers a different modeling compared with probability theory: “while probability theory offers a quantitative model for randomness and indecisiveness, possibility theory offers a qualitative model of incomplete knowledge” (Dubois et al. 2004, p. 227).

The notions of possibility measure and necessity measure are fundamental in possibility theory, and the risk is represented by possibilistic distributions (see Carlsson and Fullér 2011; Dubois and Prade 1988; Georgescu 2012a). The fuzzy numbers are the most important class of possibilistic distributions, with operations and algebraic properties very similar with those of real numbers (Carlsson and Fullér 2011; Dubois and Prade 1988). Moreover, the possibilistic indicators of fuzzy numbers (possibilistic expected value, possibilistic variance and covariance, possibilistic moments, etc.) allow us to obtain synthetical knowledge on the risk phenomena. An advantage to work with fuzzy numbers is
that they have good computational properties. Due to all these remarks, fuzzy numbers are capable to model large classes of decision-making problems (see Appadoo and Thavaneswaran 2010; Carlsson and Fullér 2011; Collan et al. 2017; Dubois and Prade 1988; Georgescu and Fono 2019; Majlender 2004; Thavaneswaran et al. 2009). For example, in Collan et al. (2017) it is illustrated how triangular fuzzy numbers are very suitable in modeling the insurance of giga investment projects.

In the paper Georgescu (2013), there is studied a possibilistic-type coinsurance problem: the agent is represented by a utility function, but the loss caused by risk is a fuzzy number.

The coinsurance problem from Georgescu (2013) is formulated by maximizing a possibilistic expected utility, defined in Georgescu (2011) and Georgescu (2012) in the framework of a possibilistic treatment of risk aversion. On the other hand, in Georgescu (2011) there is a second notion of possibilistic expected utility, and the expected utility operators from Georgescu (2012) allow the definition of a general notion of possibilistic expected utility which generalizes the two mentioned above. By this, with each expected utility operator \( T \) one associates a possibilistic expected utility theory (called \( T \)-possibilistic \( EU \)-theory), in which various topics on risk theory can be discussed.

In interpretation, an expected \( T \)-utility operator fixes a construction method of a decision problem and possibilistic analysis on the basis of imprecise information described by a fuzzy number \( A \).

The aim of this paper is to study the coinsurance problem in the framework offered by possibilistic expected utility operators. The formulation of an abstract coinsurance problem can be done inside an arbitrary \( T \)-possibilistic \( EU \)-theory, but the proofs of the optimal coinsurance rate and its approximate computation assume \( T \) to fulfill a supplementary property. For this purpose, the \( D \)-operators have been chosen, a class of expected utility operators introduced in Georgescu and Fono (2019) by a preservation condition of derivability of the utility function with respect to a parameter.

Section 2 presents two notions of possibilistic expected utility from Georgescu (2009) and Georgescu (2011), as well as some operators associated with fuzzy numbers (possibilistic expected utility, possibilistic variance).

In Sect. 3, the definition of expected utility operators is recalled from Georgescu (2012a), Georgescu (2012b) and the \( D \)-operators are introduced.

To formulate the coinsurance problem in the context of a \( T \)-possibilistic \( EU \)-theory, in Sect. 4 a few basic notions are defined: the coinsurance contract, the \( T \)-premium for insurance indemnity, the \( T \)-coinsurance rate, etc. Assuming that \( T \) is a \( D \)-operator, the \( T \)-coinsurance rate can be computed as a solution of a first-order condition.

The results on the optimal \( T \)-coinsurance \( \beta^* \) are contained in Sect. 5. A first result is a possibilistic version of a Mossin theorem (Mossin 1968 or Eeckhoudt et al. 2005, p. 51). The main result is an approximate calculation formula of \( \beta^* \), with respect to the expected value \( E_f (A) \) of the fuzzy number \( A \) representing the risk, the \( T \)-variance \( \text{Var}_T (A) \) of \( A \), and the Arrow–Pratt index of the agent’s utility function. It is also demonstrated a formula that approximates the maximal total expected utility (obtained by the choice of the \( T \)-coinsurance rate \( \beta^* \)).

Another result of the section asserts that if an agent \( u_1 \) is more risk averse than an agent \( u_2 \), then the optimal \( T \)-coinsurance rate is higher for \( u_1 \) than for \( u_2 \).

In Sect. 6, forms of the approximation formula of the optimal \( T \)-coinsurance rate in the particular case of the expected utility operators \( T_1 \), \( T_2 \) from Georgescu (2009), Georgescu (2011) and a risk represented by a triangular fuzzy number are obtained. These formulas are applied for HARA- and CRRA-type utility functions.

In the concluding remarks section, a few open issues are commented and the result from Appendix presents a necessary condition for the positivity of the optimal \( T \)-coinsurance rate.

### 2 Indicators of fuzzy numbers

Fuzzy numbers are generalizations of real numbers, able to express imprecise information. Using Zadeh’s extension principle (Zadeh 1978), the operations with real numbers can be extended to fuzzy numbers, in such a way that most of the algebraic properties are preserved (Dubois and Prade 1980, 1988). At the same time, fuzzy numbers can be thought of as possibilistic distributions (Dubois and Prade 1988; Carlsson and Fullér 2011). By parallelism with probabilistic distributions, but in a completely different way, with each fuzzy number possibilistic indicators can be associated: expected value, variance, covariance, higher-order moments, etc. (Carlsson and Fullér 2001; Carlsson et al. 2005; Dubois and Prade 1987; Fullér and Majlender 2004; Georgescu 2009; Zhang and Whang 2007). All these make the fuzzy numbers an effective tool in the possibilistic treatment of some topics on risk theory (Carlsson and Fullér 2011; Collan et al. 2017; Georgescu 2012a; Thavaneswaran et al. 2009; Zhang and Whang 2007).

In this section, we will present after (Georgescu 2009, 2011, 2012a) two notions of expected utility associated with a triple consisting of a utility function (representing an agent), a fuzzy number (representing the risk), and a weighting function. Also, we will recall the definition of expected value and two variances associated with a fuzzy number (Carlsson and Fullér 2001, 2011; Georgescu 2012a; Zhang and Whang 2007).
We fix a mathematical framework consisting of three entities:

- a weighting function \( f : [0, 1] \rightarrow \mathbb{R} \) (\( f \) is a non-negative and increasing function that satisfies \( \int_0^1 f(\gamma) d\gamma = 1 \));
- a continuous utility function \( u : \mathbb{R} \rightarrow \mathbb{R} \);
- a fuzzy number \( A \) whose level sets have the form \([a_1(\gamma), a_2(\gamma)]\) for all \( \gamma \in [0, 1] \).

Then, the support of a fuzzy number \( A \) will be \( supp(A) = \{ x \in \mathbb{R} | A(x) > 0 \} = (a_1(0), a_2(0)) \).

In formulating the general definition, one will assume that the utility function is continuous, but the analysis of such themes of risk theory (e.g., coinsurance problem) requires imposing some supplementary hypotheses (e.g., \( u \) to be of class \( C^2 \)).

Following Georgescu (2012a) Section 4.2, we will define the following two notions of possibilistic expected utility:

\[
E_1(f, u(A)) = \frac{1}{2} \int_0^1 [u(a_1(\gamma)) + u(a_2(\gamma))] f(\gamma) d\gamma
\]

\[
E_2(f, u(A)) = \frac{1}{2} \int_0^1 \left[ a_2(\gamma) - a_1(\gamma) \int_{a_1(\gamma)}^{a_2(\gamma)} u(x) dx \right] f(\gamma) d\gamma
\]  

(2.1)

(2.2)

Setting in (2.1) or (2.2) \( u = 1_R \) (the identity function of \( R \)), one obtains the possibilistic expected value (Dubois and Prade 1987; Carlsson and Fullér 2001; Fullér and Majlender 2004; Majlender 2004):

\[
E_f(A) = E_1(f, 1_R(A)) = E_2(f, 1_R(A))
\]

\[
= \frac{1}{2} \int_0^1 [a_1(\gamma) + a_2(\gamma)] f(\gamma) d\gamma
\]  

(2.3)

If \( supp(A) \subseteq \mathbb{R}_+ \), then from (2.3) it follows that \( E_f(A) \geq 0 \).

For \( u(x) = (x - E_f(A))^2 \), two different notions of possibilistic variance follow Carlsson and Fullér (2001), Fullér and Majlender (2004), Georgescu (2009) and Zhang and Whang (2007):

\[
Var_1(f, A) = \frac{1}{2} \int_0^1 [(a_1(\gamma) - E_f(A))^2 + (a_2(\gamma) - E_f(A))^2] f(\gamma) d\gamma
\]  

(2.4)

\[
Var_2(f, A) = \int_0^1 \left[ \frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - E_f(A))^2 \right] f(\gamma) d\gamma
\]  

(2.5)

3 Expected utility operators and D-operators

In this section, we will recall the definitions of the expected utility operators and the \( D \)-operators, introduced in Georgescu (2011), respectively Georgescu and Fono (2015), as an abstraction of the two possibilistic expected utilities \( E_1(f, u(A)) \) and \( E_2(f, u(A)) \) from the previous section.

Let \( \mathcal{F} \) be the set of fuzzy numbers, \( C(\mathbb{R}) \) the set of real continuous functions (mapped from \( \mathbb{R} \) to \( \mathbb{R} \)), and \( \mathcal{U} \) a subset of \( C(\mathbb{R}) \) satisfying the following properties:

- \( (U_1) \mathcal{U} \) contains constant functions and first- and second-degree polynomial functions;
- \( (U_2) \mathcal{U} \) is closed under linear combinations: if \( a, b \in \mathbb{R} \) and \( g, h \in \mathcal{U} \), then \( ag + bh \in \mathcal{U} \).

For each \( a \in \mathbb{R} \), we denote \( \bar{a} : \mathbb{R} \rightarrow \mathbb{R} \) the constant function \( \bar{a}(x) = a \), for \( x \in \mathbb{R} \). \( 1_R \) will be the identity function of \( \mathbb{R} \). Then, \( a, 1_R \) belong to \( \mathcal{U} \). In particular, we can consider \( \mathcal{U} = C(\mathbb{R}) \).

We fix a weighting function \( f : [0, 1] \rightarrow \mathbb{R} \) and a family \( \mathcal{U} \) with the properties \((U_1)\) and \((U_2)\).

Definition 3.1 Georgescu (2012a), Georgescu (2012b) A \( (f\)-weighted) expected utility operator is a function \( T : \mathcal{F} \times \mathcal{U} \rightarrow \mathbb{R} \) such that for any \( a, b \in \mathbb{R} \), \( g, h \in \mathcal{U} \) and \( A \in \mathcal{F} \) the following conditions are fulfilled:

- (a) \( T(A, 1_R) = E_f(A) \);
- (b) \( T(A, \bar{a}) = a \);
- (c) \( T(A, ag + bh) = aT(A, g) + bT(A, h) \);
- (d) \( g \leq h \) implies \( T(A, g) \leq T(A, h) \).

Example 3.2 Georgescu (2009), Georgescu (2012a) We consider the function \( T_1 : \mathcal{F} \times C(\mathbb{R}) \rightarrow \mathbb{R} \) defined as follows: for any fuzzy number \( A \) and for any \( g \in C(\mathbb{R}) \):

\[
T_1(A, g) = E_1(f, g(A))
\]  

(3.1)

Then, \( T_1 \) is an expected utility operator.

Example 3.3 Georgescu (2009), Georgescu (2012a) We consider the function \( T_2 : \mathcal{F} \times C(\mathbb{R}) \rightarrow \mathbb{R} \) defined as follows: for any fuzzy number \( A \) and for any \( g \in C(\mathbb{R}) \):

\[
T_2(A, g) = E_2(f, g(A))
\]  

(3.2)

Then, \( T_2 \) is an expected utility operator.

An expected utility operator \( T \) is strictly increasing if for any \( A \in \mathcal{F} \) and \( g, h \in \mathcal{U} \), \( g < h \) implies \( T(A, g) < T(A, h) \). One can prove that the expected utility operators \( T_1, T_2 \) are strictly increasing.

If \( f \) is a strictly increasing operator, then for any \( A \in \mathcal{F} \) and \( h \in \mathcal{U} \), \( h > 0 \) implies \( T(A, h) > 0 \).
One notices that conditions (a)-(d) of Definition 3.1 have been abstracted from the properties of $T_1$ and $T_2$. Therefore, the real number $T(A, g)$ will be called generalized possibilistic expected utility (shortly, $T$-expected utility) and it will represent the starting point of a possibilistic $EU$-theory associated with $T$. Sometimes, instead of $T(A, g)$ we will use the notation $T(A, g(x))$.

Particularizing $g$, from $T(A, g)$ various possibilistic indicators are obtained. By axiom (a), for $g = 1_R$, the possibilistic expected value $E_f(A)$ follows. For $g(x) = (x - E_f(A))^2$, we have the notion of $T$-covariance:

$$Var_T(A) = T(A, (x - E_f(A))^2) \quad (3.3)$$

Using condition (d) of Definition 3.1, it follows immediately that $Var_T(A) \geq 0$.

**Remark 3.4** As we have seen previously, the two operators $T_1$ and $T_2$ introduce the possibilistic variances $Var_{T_1}(A) = Var_1(f, A)$, respectively, $Var_{T_2}(A) = Var_2(f, A)$.

The two possibilistic variances $Var_{T_1}(A), Var_{T_2}(A)$ have been used in the application of some models in possibilistic risk theory (Appadoo and Thavaneswaran 2010; Carlsson and Fullér 2011; Collan et al. 2017; Georgescu 2013; Majlender 2004; Thavaneswaran et al. 2009).

In case of probabilistic risk, the risk aversion of an agent is described by the Arrow–Pratt index (Arrow 1965, 1970; Pratt 1964): for a utility function $u$ of class $C^2$, the Arrow–Pratt index is defined by

$$r_u(w) = -\frac{u''(w)}{u'(w)} \text{ for } w \in \mathbb{R} \quad (3.4)$$

If $u_1$, $u_2$ are the utility functions of the two agents, then the Arrow–Pratt theorem Arrow (1965), Arrow (1970) and Pratt (1964) asserts that “the agent $u_1$ is more risk averse than the agent $u_2$” iff $r_{u_1}(w) \geq r_{u_2}(w)$ for any $w \in \mathbb{R}$.

Papers Georgescu (2009), Georgescu (2011) contain two distinct possibilistic treatments of risk aversion when the risk is a fuzzy number. These possibilistic theories of risk aversion are based on the possibilistic utilities $T_1(A, u), T_2(A, u)$. In particular, in both cases a Pratt-type theorem is proved. A surprising result is obtained: the possibilistic risk aversion is characterized in terms of the Arrow–Pratt index. In a certain sense, we could say that “the possibilistic risk aversion (in the sense of papers Georgescu 2009, 2011) is equivalent to the possibilistic risk aversion” (Arrow 1965, 1970; Pratt 1964).

The expected utility operators allow a generalization of possibilistic risk aversion theories from Georgescu (2009) and Georgescu (2011).

In this general framework, it is defined what it means that “an agent is more risk averse than another agent” and it is proved a Pratt-type theorem which characterizes this property. The main tool used in proving this result is the approximation formula from the following proposition.

**Proposition 3.5** Georgescu (2011, 2012a) Let $T$ be an expected utility operator, $A$ a fuzzy number, and $u$ a utility function of class $C^2$. Then,

$$T(A, u) \approx u(E_f(A)) + \frac{1}{2}u''(E_f(A))Var_T(A) \quad (3.5)$$

Proposition 3.5 will be used in this paper to prove the approximation formula from Theorem 5.9.

In paper Georgescu and Fono (2019), the $D$-operators have been introduced to study a possibilistic portfolio choice problem. We will present next the definition of the $D$-operators.

For a utility function $g(x, \lambda)$, in which $\lambda$ is a parameter, we consider the following properties:

(i) $g(x, \lambda)$ is continuous with respect to the argument $x$ and derivable with respect to the argument $\lambda$;

(ii) for any $\lambda \in \mathbb{R}$, the function $\frac{\partial g(x, \lambda)}{\partial x} : \mathbb{R} \rightarrow \mathbb{R}$ is derivable.

**Definition 3.6** Georgescu and Fono (2019) An expected utility operator $T : \mathcal{F} \times \mathcal{C}(\mathbb{R}) \rightarrow \mathbb{R}$ is a $D$-operator if for any fuzzy number $A$ and for any function $g(x, \lambda)$ with (i) and (ii), the following properties are fulfilled:

(D1) The function $\lambda \mapsto T(A, g(\cdot, \lambda))$ is derivable (with respect to $\lambda$);

(D2) $T(A, \frac{\partial g(\cdot, \lambda)}{\partial x}) = \frac{d}{d\lambda} T(A, g(\cdot, \lambda))$.

By Proposition 1 from Georgescu and Fono (2019), $T_1$ and $T_2$ are $D$-operators.

**Remark 3.7** Conditions (D1) and (D2) make it possible the use of first order conditions in solving some optimization problems in which the objective function is a $T$-expected utility. In paper Georgescu and Fono (2019), the $D$-operators offer the framework to find some approximate solutions of a possibilistic portfolio problem. In the next sections, the properties (D1) and (D2) will be intensely used to determine the optimal coinsurance rate in a coinsurance problem formulated in the context of expected utility operators.

**Proposition 3.8** Let $T, S$ be two expected utility operators and $c \in \mathbb{R}$.

(a) $U = cT + (1 - c)S$ is an expected utility operator;

(b) If $S, T$ are $D$-operators, then $U$ is a $D$-operator.

**Proof** (a) By Georgescu (2012a), Proposition 5.2.5.
(b) Condition \((D_1)\) is immediate. We will verify \((D_2)\). Since \(S, T\) fulfill condition \((D_2)\), we will have the following equalities:

\[
\frac{d}{d\lambda} U(A, g(\cdot, \lambda)) = c \frac{d}{d\lambda} T(A, g(\cdot, \lambda)) + (1-c) \frac{d}{d\lambda} D(A, g(\cdot, \lambda))
\]

\[
S(A, g(\cdot, \lambda)) = c T \left( A, \frac{\partial g(\cdot, \lambda)}{\partial \lambda} \right) + (1-c) 
\]

for any fuzzy number \(A\) and for any utility function \(g(x, \lambda)\) which fulfils hypotheses (i) and (ii).

\(\square\)

4 The coinsurance problem

In this section, we will deal with the coinsurance problem in the context of expected utility operators. First, we will introduce a few entities by which we will define this coinsurance problem; then, we will restrict the universe of discussion to \(D\)-operators in order to start the study of optimal coinsurance rate.

Consider an agent with a utility function \(u\) of class \(C^2\) such that \(u' > 0\) and \(u'' < 0\). Assume that the agent possesses an initial wealth subject to risk. To retrieve a part of the loss caused by this risk, the agent will close an insurance contract. By Eeckhoudt et al. (2005), p. 46, an insurance contract has two components:

- a premium \(P\) to be paid by the policyholder;
- an indemnity schedule \(I(x)\) indicates the amount to be paid by the insurer for a loss \(x\).

We will think of \(I(x)\) as a utility function, and the premium \(P\) will be defined with respect to the mathematical modeling of risk. In case of a probabilistic model, the loss will be a random variable \(X \geq 0\), and \(P\) will be defined by means of (probabilistic) expected utility \(EI(X)\) (see Eeckhoudt et al. 2005, p. 49).

The possibilistic form of the coinsurance problem from Georgescu (2013) has as hypothesis the fact that the risk is a fuzzy number \(A\) with the property that \(\text{supp}(A) \subset \mathbb{R}_+\) and \(\text{supp}(A)\) does not reduce to a single point. In particular, this hypothesis assures that \(E_f(A) > 0\).

To extend this to a \(EU\)-theory associated with an expected utility operator \(T\), we will fix \(T\) and a weighting function \(f : [0, 1] \to \mathbb{R}\).

The \(T\)-premium for insurance indemnity is defined by

\[
P = (1 + \lambda)T(A, u)
\]

where \(\lambda \geq 0\) is a loading factor.

**Remark 4.1** (i) The expression \((4.1)\) of \(P\) is inspired from the form of the premium for insurance indemnity from the probabilistic model (Eeckhoudt et al. 2005, p. 42).

(ii) If \(T\) is the operator \(T_1\) from Example 3.2, then we obtain the notion of possibilistic premium for insurance indemnity from Georgescu (2013).

We will assume that \(I(x) = \beta x\) for all \(x\). Following the terminology from Eeckhoudt et al. (2005), \(\beta\) will be called coinsurance rate, and \(1 - \beta\) will be called retention rate. The coinsurance rate \(\beta\) represents the fraction from the size of the loss the insured gets following an insurance contract.

Similar with Eeckhoudt et al. (2005), p. 49 or Georgescu (2013), we will make the hypothesis that the policyholder chooses \textit{apriori} a coinsurance rate \(\beta\). Then, the corresponding \(T\)-premium for insurance indemnity \(P(\beta)\) will have the form:

\[
P(\beta) = (1 + \lambda)T(A, \beta x) = \beta (1 + \lambda)T(A, x).
\]

By the axiom (a) from Definition 3.1, \(P(\beta)\) will be written:

\[
P(\beta) = \beta P_0
\]

(4.3)

If \(\beta\) is the coinsurance rate and \(x\) is the size of the loss, then the agent remains ultimately with the following amount:

\[
g(x, \beta) = w_0 - P(\beta) - x + \beta x = w_0 - \beta P_0 - (1 - \beta)x
\]

(4.4)

Consider the function which gives the utility of amount \(g(x, \beta)\):

\[
h(x, \beta) = u(g(x, \beta)) = u(w_0 - \beta P_0 - (1 - \beta)x)
\]

(4.5)

Then,

\[
H(\beta) = T(A, h(x, \beta))
\]

(4.6)

is the total \(T\)-utility associated with a possibilistic risk \(A\), an initial wealth \(w_0\) and an insurance contract with a coinsurance rate \(\beta\).
Since the agent wants to maximize this total utility, he will choose \( \beta \) as the solution of the coinsurance problem:

\[
\max_{\beta} H(\beta) \tag{4.7}
\]

To be able to study the existence and the computation of the solution of problem (4.7), we will assume from now on that \( T \) is a \( D \)-operator.

Taking into account the properties \((D_1)\) and \((D_2)\) of Definition 3.1, from (4.9) it follows

\[
H'(\beta) = T(A, u'(g(x, \beta)))(x - P_0)
\]

Analogously, we obtain the second derivative:

\[
H''(\beta) = T(A, u''(g(x, \beta)))(x - P_0)^2 \tag{4.9}
\]

By hypothesis, \( u''(g(x, \beta)) < 0 \). Applying the axiom (d) of Definition 3.1, from (4.9) it follows \( H''(\beta) \leq 0 \); thus, \( H \) is a concave function. Moreover, if the expected utility operator \( T \) is strictly increasing, then \( H \) is strictly concave.

We can consider then the solution \( \beta^* \) of the optimization problem 4.7: \( H''(\beta^*) = \max H(\beta) \). The determination of the optimal coinsurance rate \( \beta^* \) and the total utility function \( H(\beta^*) \) is one of the agent’s important problems. When it exists, the optimal coinsurance rate \( \beta^* \) verifies the first-order condition \( H'(\beta) = 0 \). Taking into account (4.8), the first-order condition \( H'(\beta) = 0 \) will be written:

\[
T(A, (x - P_0)u'(g(x, \beta^*))) = 0 \tag{4.10}
\]

Let us consider the case of \( D \)-operators \( T_1 \) and \( T_2 \). By (3.1) and (3.2), the first-order condition (4.10) gets the following form:

- for the \( D \)-operator \( T_1 \):

\[
E_1(f, (A - P_0)u'(g(A, \beta^*_{T_1}))) = 0 \tag{4.11}
\]

The coinsurance problem formulated in \( EU \)-theory associated with the operator \( T_1 \) has been studied in Georgescu (2013). In particular, using the first-order condition (4.11), in Georgescu (2013) an approximate calculation formula of the optimal coinsurance has been proved.

5 The properties and the computation of the optimal coinsurance rate

Let \( f : [a, b] \to \mathbb{R} \) be a weighting function, \( T \) a \( D \)-operator, \( u : \mathbb{R} \to \mathbb{R} \) a utility function, and \( A \) a fuzzy number. As in the previous section, we will make the following assumptions on \( u \) and \( A \):

- \( u \) is of class \( C^2 \), \( u' > 0 \) and \( u'' < 0 \);
- \( supp(A) \subseteq \mathbb{R}_+ \) and \( supp(A) \) is not a point set.

According to the second hypothesis, one gets \( E_f(A) > 0 \). Let \( \beta^* = \beta^*_T \) be the solution of the insurance problem (4.7). We will keep all notations from Section 4.

**Proposition 5.1**

(i) If \( \lambda = 0 \), then \( \beta^* = 1 \);

(ii) If \( \lambda > 0 \), then \( \beta^* < 1 \).

**Proof**

(i) Setting \( \beta = 1 \) in (4.4), we have \( g(x, 1) = w_0 - P_0 \). Then, by Definition 3.1 it follows

\[
T(A, (x - P_0)u'(g(x, 1))) = T(A, (x - P_0)u'(w_0 - P_0)) = u'(w_0 - P_0)T(A, x - P_0) = u'(w_0 - P_0)(E_f(A) - P_0) = -\lambda E_f(A)u'(w_0 - P_0)
\]

If \( \lambda = 0 \), then \( \beta^* = 1 \) verifies the first-order condition (4.10).

(ii) Since \( supp(A) \subseteq \mathbb{R}_+ \) and \( supp(A) \) is not a point set, it follows

\[
E_f(A) = \frac{1}{2} \int_0^1 [a_1(\gamma) + a_2(\gamma)]f(\gamma)d\gamma > 0
\]

From \( \lambda > 0 \), \( u' > 0 \) and the proof of (i), we deduce

\[
H'(1) = T(A, (x - P_0)u'(g(x, 1))) = -\lambda E_f(A)u'(w_0 - P_0) < 0
\]

Suppose by absurdum that \( \beta^* \geq 1 \). Since \( H \) is concave, its derivative \( H' \) is decreasing; thus, \( H'(\beta) \leq H'(1) < 0 \). This contradicts the first-order condition \( H'(\beta^*) = 0 \); thus, \( \beta^* < 1 \).

By Proposition 5.1, the inequality \( \beta^* \leq 1 \) holds. In the formulation (4.7) of the coinsurance problem, no restriction
has been made on $\beta$ (as in Eeckhoudt et al. 2005, Section 3.2, for the probabilistic coinsurance rate). Therefore, the solution $\beta^*$ of (4.7) may not satisfy the inequality $0 < \beta^* \leq 1$. In Appendix, we will establish a necessary condition for $0 < \beta^*$. 

**Remark 5.2** Proposition 5.1 is a result analogous to Mossin theorem (Mossin 1968 or Eeckhoudt et al. 2005, Proposition 3.1). Then, when $T$ is the operator $T_1$, one obtains Proposition 4 from Georgescu (2013).

An exact solution for the maximization problem (4.7) is difficult to find. Therefore, it is more convenient to find approximate solutions of equation (4.10).

Before proving a formula for the approximate calculation of $\beta^*$, let us denote $w = w_0 - P_0$. Then, formula (4.4) becomes:

$$g(x, \beta) = w - (1 - \beta)(x - P_0)$$ \tag{5.1}

**Theorem 5.3** An approximate value of the optimal $T$-coinsurance rate $\beta^*$ is:

$$\beta^* \approx 1 + \frac{u'(w)}{u''(w)} \frac{\lambda E_f(A)}{Var_T(A) + \lambda^2 E_f^2(A)}$$ \tag{5.2}

**Proof** By (5.1), $u'(g(x, \beta)) = u'(w - (1 - \beta)(x - P_0))$. We consider the first-order Taylor approximation of $u'(w - (1 - \beta)(x - P_0))$ around $w$:

$$u'(g(x, \beta)) \approx u'(w) - (1 - \beta)(x - P_0)u''(w)$$

from which it follows

$$(x - P_0)u'(g(x, \beta)) \approx u'(w)(x - P_0) - (1 - \beta)u''(w)(x - P_0)^2$$

Taking into account (4.8) and Definition 3.1, we have

$$H'(\beta) = T(A, (x - P_0)u'(g(x, \beta)) \approx T(A, u'(w)(x - P_0) - (1 - \beta)u''(w)(x - P_0)^2)$$

$$= u'(w)T(A, x - P_0) - (1 - \beta)u''(w)T(A, (x - P_0)^2)$$

We notice that

$$T(A, x - P_0) = T(A, x) - P_0 = E_f(A) - P_0$$

$$T(A, (x - P_0)^2) = T(A, x^2 - 2P_0x + P_0^2)$$

$$= T(A, x^2) - 2P_0T(A, x) + P_0^2$$

$$= T(A, x^2) - 2P_0E_f(A) + P_0^2$$

$$= (T(A, x^2) - E_f^2(A))$$

$$+ (E_f(A) - P_0)^2$$

$$= Var_T(A) + (E_f(A) - P_0)^2$$

Replacing $T(A, x - P_0)$ and $T(A, (x - P_0)^2)$ in the approximate expression of $H'(\beta)$, one obtains:

$$H'(\beta) \approx u'(w)(E_f(A) - P_0) - (1 - \beta)u''(w)$$

$$[Var_T(A) + (E_f(A) - P_0)^2]$$

Then, the first-order condition $H'(\beta^*) = 0$ can be written as

$$u'(w)(E_f(A) - P_0) - (1 - \beta^*)u''(w)[Var_T(A) + (E_f(A) - P_0)^2] \approx 0$$

from where

$$\beta^* \approx 1 - \frac{u'(w)}{u''(w)} \frac{E_f(A) - P_0}{Var_T(A) + (E_f(A) - P_0)^2}$$

Since $P_0 = (1 + \lambda)E_f(A)$, we have $E_f(A) - P_0 = E_f(A) - (1 + \lambda)E_f(A) = -\lambda E_f(A)$. With this, the approximate value of $\beta^*$ gets the form (5.2)

$$\beta^* \approx 1 + \frac{u'(w)}{u''(w)} \frac{\lambda E_f(A)}{Var_T(A) + \lambda^2 E_f^2(A)}$$

\[\square\]

Taking into account the definition of the Arrow–Pratt index from (3.4), one obtains

**Corollary 5.4**

$$\beta^* \approx 1 - \frac{\lambda}{r_u(w) Var_T(A) + \lambda^2 E_f^2(A)}$$ \tag{5.3}

By particularizing the operator $T$, different approximation formulas of the optimal coinsurance rate $\beta$ are obtained from (5.3). If $T$ is the $D$-operator $T_1$ from Example 3.2, then the approximation formula (22) from Georgescu (2013) is obtained.

**Remark 5.5** The approximate value of $\beta^*$ given by (5.3) gives us the way the optimal $T$-insurance depends on the risk aversion of the agent who closes the insurance contract, and it depends on the expected value and the variance of the fuzzy number representing the risk. The following result will give a more precise form of the relation between the risk aversion and the $T$-coinsurance rate: an increase in risk aversion will generate an increase in coinsurance rate.

We consider two agents whose utility functions $u_1$, $u_2$ are of class $C^2$ and verify the conditions $u_1' > 0$, $u_2' > 0$, $u_1'' < 0$ and $u_2'' < 0$. Let $\beta_1^*$, $\beta_2^*$ be the optimal $T$-coinsurance rates associated with the utility functions $u_1$, $u_2$, the weighting function $f$, the $D$-operator $T$, and the fuzzy number $A$.

**Proposition 5.6** If the agent $u_1$ is more risk averse than $u_2$, then $\beta_1^* > \beta_2^*$. 

\[\square\]
Proof We consider the Arrow–Pratt indices of the utility functions \( u_1, u_2 \):

\[
\beta_1 \approx 1 - \frac{\lambda}{\lambda - u_1(w)} E_f(A) = \frac{\lambda}{\lambda - u_1(w)} E_f(A) + \lambda^2 E^2_f(A) \tag{5.4}
\]

\[
\beta_2 \approx 1 - \frac{\lambda}{\lambda - u_2(w)} E_f(A) = \frac{\lambda}{\lambda - u_2(w)} E_f(A) + \lambda^2 E^2_f(A) \tag{5.5}
\]

By hypothesis, \( u_1(w) \geq u_2(w) > 0 \); thus, \( 0 < \frac{\lambda}{\lambda - u_1(w)} < \frac{\lambda}{\lambda - u_2(w)} \). Since \( E_f(A) > 0 \) and \( Var_f(A) \geq 0 \), from (5.4) and (5.5) it follows immediately \( \beta_1^* \geq \beta_2^* \). \( \square \)

Let \( T, S \) be two \( T \)-operators and \( c \in \mathbb{R} \). By Proposition 3.8, \( U = cT + (1 - c)S \) is a \( D \)-operator. We consider the coinsurance problems associated with the \( D \)-operators \( T, S, U \), and in rest, keeping the same data which define the coinsurance problem (4.7).

**Proposition 5.7** Let \( \beta_T^*, \beta_S^* \), and \( \beta_U^* \) be the optimal coinsurance rates corresponding to the \( D \)-operators \( T, S, U \). Then,

\[
\beta_U^* \approx 1 - \frac{1}{c - \beta_T^* + \frac{1 - c}{1 - \beta_S^*}} \tag{5.6}
\]

Proof By (5.3), we have the following approximate values of \( \beta_T^*, \beta_S^* \) and \( \beta_U^* \):

\[
\beta_T^* \approx 1 - \frac{\lambda}{r_a(w)} Var_f(A) + \lambda^2 E^2_f(A)
\]

\[
\beta_S^* \approx 1 - \frac{\lambda}{r_a(w)} Var_f(A) + \lambda^2 E^2_f(A)
\]

\[
\beta_U^* \approx 1 - \frac{\lambda}{r_a(w)} Var_f(A) + \lambda^2 E^2_f(A)
\]

from where it follows:

\[
1 - \beta_T^* \approx \frac{\lambda}{r_a(w)} Var_f(A) + \lambda^2 E^2_f(A)
\]

\[
1 - \beta_S^* \approx \frac{\lambda}{r_a(w)} Var_f(A) + \lambda^2 E^2_f(A)
\]

\[
1 - \beta_U^* \approx \frac{\lambda}{r_a(w)} Var_f(A) + \lambda^2 E^2_f(A)
\]

By Georgescu (2012a), Proposition 5.1.5, we have \( Var_U(A) = cVar_f(A) + (1 - c) Var_f(A) \). Then, by taking into account (5.7)-(5.9) the following equalities hold:

\[
\frac{1 - \beta_T^*}{c - \beta_T^*} + \frac{1 - c}{1 - \beta_S^*} \approx \frac{r_a(w)}{c - \beta_T^* + \frac{1 - c}{1 - \beta_S^*}} Var_f(A) + \lambda^2 E^2_f(A)
\]

From here, it follows

\[
\beta_U^* \approx 1 - \frac{1}{c - \beta_T^* + \frac{1 - c}{1 - \beta_S^*}} \tag{5.10}
\]

Remark 5.8 The previous proposition allows to obtain the optimal coinsurance rates for all convex combinations of two \( D \)-operators. In particular, if we take \( c = \frac{1}{2} \), then \( U = \frac{1}{2}T + \frac{1}{2}S \) and by (5.6), the optimal coinsurance rate of \( U \) will be:

\[
\beta_U^* \approx 1 - \frac{2}{1 - \beta_T^* + \frac{1 - c}{1 - \beta_S^*}} \tag{5.11}
\]

The approximation formula of the optimal insurance \( \beta^* \) from Corollary 5.4 will be used in the following to approximate the total expected utility \( H(\beta^*) = T(A, h(x, \beta^*)) \).

**Theorem 5.9** The total expected utility \( H(\beta^*) \) corresponding to the optimal coinsurance rate \( \beta^* \) can be approximated by

\[
H(\beta^*) \approx u \left( w + \frac{1}{r_a(w)} Var_f(A) + \lambda^2 E^2_f(A) \right)
\]

\[
+ \frac{\lambda^2}{2} E^2_f(A) Var_f(A)
\]

\[
\approx \left( w + \frac{1}{r_a(w)} Var_f(A) + \lambda^2 E^2_f(A) \right)^2 u''
\]

\[
\approx H(\beta^*) \approx u(\beta, \beta^* + \lambda^2 E^2_f(A))
\]

Proof We consider the unidimensional function

\[
v(x) = h(x, \beta^*) = u(w - (1 - \beta^*) \lambda(x - P_0)) \tag{5.11}
\]

One will notice that \( H(\beta^*) = T(A, v) \). \( v \) is a utility function of class \( C^2 \); thus, we can apply the approximation formula (3.5):

\[
H(\beta^*) \approx v(E_f(A)) + \frac{v''(E_f(A))}{2} Var_f(A) \tag{5.12}
\]
Since $E_f(A) - P_0 = -\lambda E_f(A)$, it follows the approximation

$$w - (1 - \beta^*) (E_f(A) - P_0) = w + \lambda (1 - \beta^*) E_f(A)$$  

(5.13)

By Corollary 5.4, one has

$$1 - \beta^* \approx -\frac{\lambda}{r_u(w)} \frac{E_f(A)}{Var_f(A) + \lambda^2 E_f^2(A)}$$  

(5.14)

Deriving twice in (5.11), it follows

$$v''(x) = (1 - \beta^*)^2 u''(w - (1 - \beta^*) (x - P_0))$$  

(5.15)

From (5.11), (5.15) and (5.14), we can deduce

$$v(E_f(A)) \approx u(w + \frac{1}{r_u(w)} \frac{\lambda^2 E_f^2(A)}{Var_f(A) + \lambda^2 E_f^2(A)})$$  

(6.1)

$$v''(E_f(A)) \approx \frac{\lambda^2}{r_u(w)} \frac{E_f^2(A)}{(Var_f(A) + \lambda^2 E_f^2(A))^2} u''\left(w + \frac{1}{r_u(w)} \frac{\lambda^2 E_f^2(A)}{Var_f(A) + \lambda^2 E_f^2(A)}\right)$$  

(6.2)

Replacing $v(E_f(A))$ and $v''(E_f(A))$ with their approximate values from (5.14) and (5.17), the approximation formula from the statement of the theorem follows. □

6 Particular cases and examples

In this section, we will study the optimal $T$-coinsurance rate for some particular $D$-operators, making the following assumptions on the weighting function $f$ and the fuzzy number $A$:

- $f(t) = 2t$, for any $t \in [0, 1]$;
- $A$ is the triangular fuzzy number $(a, \alpha, \beta)$:

$$A(t) = \begin{cases} 
  1 - \frac{a - t}{\alpha} & a - \alpha \leq t \leq a \\
  1 - \frac{t - a}{\beta} & a \leq t \leq a + \beta \\
  0 & \text{otherwise}
\end{cases}$$

As to the $D$-operator $T$, we will consider the following particular cases:

- (a) $T$ is the $D$-operator $T_1$. By Georgescu (2012a), Examples 3.3.10 and 3.4.10, we have

$$E_f(A) = a + \frac{\beta - \alpha}{6}$$  

(6.3)

$$Var_{T_1}(A) = \frac{\alpha^2 + \beta^2 + \alpha \beta}{18}$$  

(6.4)

Replacing $E_f(A)$ and $Var_{T_1}(A)$ in (5.3), the optimal $T_1$-coinsurance rate $\beta^*_1 = \beta^*_1$ gets the form:

$$\beta^*_1 \approx 1 - \frac{\lambda}{r_u(w)} \frac{a + \frac{\beta - \alpha}{6}}{\alpha^2 + \beta^2 + \alpha \beta + \frac{\lambda^2}{18} (a + \frac{\beta - \alpha}{6})^2}$$  

(6.5)

- (b) $T$ is the $D$-operator $T_2$. By Georgescu (2012a), Example 3.4.10 we have

$$Var_{T_2}(A) = \frac{\alpha^2 + \beta^2}{36}$$  

(6.6)

Replacing $E_f(A)$ and $Var_{T_2}(A)$ in (5.3), the optimal $T_2$-coinsurance rate $\beta^*_2$ becomes:

$$\beta^*_2 \approx 1 - \frac{\lambda}{r_u(w)} \frac{a + \frac{\beta - \alpha}{6}}{\alpha^2 + \beta^2 + \alpha \beta + \frac{\lambda^2}{36} (a + \frac{\beta - \alpha}{6})^2}$$  

(6.7)

- (c) We consider the $D$-operator $U = \frac{1}{2} T_1 + \frac{1}{2} T_2$ (by Proposition 3.8). For the computation of the optimal $T_2$-coinsurance rate $\beta^*_u$, we will recall the formula from Remark 5.8. Using (6.3) and (6.5), one obtains:

$$1 - \frac{\beta^*_1}{1 - \beta^*_1} + \frac{1}{1 - \beta^*_2} \approx \frac{r_u(w)}{\lambda} \frac{a^2 + \beta^2 + \alpha \beta}{18} + \frac{\alpha^2 + \beta^2}{36} + \frac{2\lambda^2 (a + \frac{\beta - \alpha}{6})^2}{a + \frac{\beta - \alpha}{6}}$$

(6.8)

By Remark 5.8, one gets

$$\beta^*_u = 1 - \frac{2\lambda}{r_u(w)} \frac{a + \frac{\beta - \alpha}{6}}{\alpha^2 + \beta^2 + \alpha \beta + \frac{\lambda^2}{36} (a + \frac{\beta - \alpha}{6})^2}$$  

(6.9)

We ask the problem of comparing the two coinsurance rates $\beta^*_1$, $\beta^*_2$ from (6.3) and (6.5). First, we notice that if $\lambda = 0$, then by Proposition 5.1 (i), we have $\beta^*_1 = \beta^*_2 = 1$. 

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Proposition 6.1 If $\lambda > 0$, then there is the following dependence relation between $\beta_1^*$ and $\beta_2^*$:

$$\frac{1}{1 - \beta_1^*} - \frac{1}{1 - \beta_2^*} \approx \frac{(\alpha + \beta)^2}{36\lambda E_f(A)} r_u(w)$$

(6.7)

Proof Formulas (6.3) and (6.5) can be written as

$$\frac{\alpha^2 + \beta^2 + a\beta}{18} + \lambda^2 E_f^2(A) \approx \frac{\lambda E_f(A)}{r_u(w)(1 - \beta_1^*)}$$

$$\frac{\alpha^2 + \beta^2}{36} + \lambda^2 E_f^2(A) \approx \frac{\lambda E_f(A)}{r_u(w)(1 - \beta_2^*)}$$

(By Proposition 5.1 (ii), $1 - \beta_1^* > 0$ and $1 - \beta_2^* > 0$).

By subtraction, from the two previous relations it follows:

$$\frac{(\alpha + \beta)^2}{36} \approx \frac{\lambda E_f(A) + 1}{r_u(w)(1 - \beta_1^*)} - \frac{1}{1 - \beta_1^*}$$

which implies (6.7). \qed

Corollary 6.2 If $\lambda > 0$, then $\beta_1^* > \beta_2^*$.

Proof Since $u'(w) > 0$, $u''(w) < 0$ implies $r_u(w) = -\frac{u''(w)}{u'(w)} > 0$. We have $\lambda > 0$ and $E_f(A) > 0$; therefore, the right-hand side of (6.7) is positive. Further, using (6.7) one obtains the inequality $\frac{1}{1 - \beta_1^*} > \frac{1}{1 - \beta_2^*}$, from where it follows $\beta_1^* > \beta_2^*$. \qed

Formulas (6.3), (6.5) and (6.6) may get different forms with respect to the utility function $u$.

Example 6.3 Assume that the utility function $u$ is CARA-type (Gollier 2004, Section 3.6)

$$u(w) = \zeta(\eta + \frac{w}{\gamma})^{1-\gamma}, \text{ for } \eta + \frac{w}{\gamma} > 0$$

(6.8)

By Gollier (2004), Section 3.6, $r_u(w) = (\eta + \frac{w}{\gamma})^{-1}$; thus, formulas (6.3), (6.5) and (6.6) will get the form:

$$\beta_1^* \approx 1 - \lambda \left(\eta + \frac{w}{\gamma}\right) \frac{a + \frac{\beta - \alpha}{6}}{18 + \frac{\beta^2 + a\beta}{6} + \lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.9)

$$\beta_2^* \approx 1 - \lambda \left(\eta + \frac{w}{\gamma}\right) \frac{a + \frac{\beta - \alpha}{6}}{36 + \frac{\beta^2}{36} + \lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.10)

$$\beta_U^* \approx 1 - 2\lambda \left(\eta + \frac{w}{\gamma}\right) \frac{a + \frac{\beta - \alpha}{6}}{18 + \frac{(\alpha + \beta)^2 + 2(\alpha^2 + \beta^2)}{36} + 2\lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.11)

If $A$ is a symmetric triangular fuzzy number $(\alpha, \alpha)$, then setting $\beta = \alpha$ in (6.9)-(6.11) we find the following forms of the three optimal coinsurance rates:

$$\beta_1^* \approx 1 - \lambda \left(\eta + \frac{w}{\gamma}\right) \frac{6a}{\lambda^2 + 6\lambda^2 a^2}$$

(6.12)

$$\beta_2^* \approx 1 - \lambda \left(\eta + \frac{w}{\gamma}\right) \frac{18a}{\lambda^2 + 18\lambda^2 a^2}$$

(6.13)

$$\beta_U^* \approx 1 - \lambda \left(\eta + \frac{w}{\gamma}\right) \frac{9a}{2a^2 + 18\lambda^2 a^2}$$

(6.14)

Example 6.4 Assume that the utility function $u$ is CRRA-type:

$$u(w) = \left\{ \begin{array}{ll}
\frac{w^{1-\gamma}}{1-\gamma} & \gamma > 1 \\
\ln(w) & \gamma = 1
\end{array} \right.$$ 

(6.15)

Then, by Eeckhoudt et al. (2005), p. 21, $r_u(w) = \frac{\zeta}{w}$ for $\gamma > 1$ and $r_u(w) = \frac{1}{w}$ for $\gamma = 1$. Then, by (6.3), (6.5) and (6.6) the following formulas for the optimal coinsurance rates $\beta_1^*, \beta_2^*, \beta_U^*$ follow:

- for $\gamma > 1$:

$$\beta_1^* \approx \frac{1 - \zeta w}{a + \frac{\beta - \alpha}{6} + \lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.16)

$$\beta_2^* \approx \frac{1 - \zeta w}{a + \frac{\beta - \alpha}{6} + \lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.17)

$$\beta_U^* \approx \frac{1 - 2\lambda w}{\frac{a + \beta - \alpha}{6} + \lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.18)

- for $\gamma = 1$:

$$\beta_1^* \approx \frac{1 - \zeta w}{\frac{a^2 + (\alpha + \beta)^2 + a\beta}{18} + \lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.19)

$$\beta_2^* \approx \frac{1 - \zeta w}{\frac{a^2 + (\alpha + \beta)^2}{36} + \lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.20)

$$\beta_U^* \approx \frac{1 - 2\lambda w}{\frac{(a + \beta)^2 + 2a^2 + 2(\alpha + \beta)^2}{36} + 2\lambda^2 \left(a + \frac{\beta - \alpha}{6}\right)^2}$$

(6.21)

Example 6.5 We consider the $T$-coinsurance problem with the following initial data:

- the weighting function is $f(t) = 2t, t \in [0, 1]$;
- $A$ is the triangular fuzzy number $A = (6, 2, 3)$;
- $u$ is the utility function of CRRA-type $u(w) = \ln(w)$; thus, $r_u(w) = 1$;

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the initial wealth is \( w_0 = 40 \) and the loading factor is \( \lambda = \frac{1}{2} \).

Formulas (6.1)-(6.3) give the following indicators of the fuzzy number \( A \):

\[
E_f(A) = \frac{37}{6}, \quad Var_{T_1}(A) = \frac{19}{18}, \quad Var_{T_2}(A) = \frac{13}{36}
\]

By (4.1), \( P_0 = (1+\lambda)E_f(A) = \frac{37}{4} \); thus, \( w = w_0 - P_0 = \frac{132}{4} \).

Applying formulas (6.3), (6.5) or (6.7), the two optimal coinsurance rates have the approximate values:

\[
\beta_1^* \approx 1 - \lambda w = \frac{E_f(A)}{Var_{T_1}(A) + \lambda^2 E_f^2(A)} = -10.71
\]

\[
\beta_2^* \approx 1 - \lambda w = \frac{E_f(A)}{Var_{T_2}(A) + \lambda^2 E_f^2(A)} = -11.5
\]

The example above emphasized two \( T \)-coinsurance problems in which the \( T \)-coinsurance rates have been strictly negative. In Appendix, we will find a necessary condition for the \( T \)-coinsurance rate \( \beta_1^* \) to be strictly positive. We do not know a necessary and sufficient condition for \( \beta_1^* > 0 \). For a particular case of the utility function, the following property will give us a sufficient condition for \( \beta_1^* > 0 \).

**Proposition 6.6** Assume that the utility function \( u \) is defined by: \( u(x) = -e^{-x} \), for \( x \in \mathbb{R} \). If \( \lambda > \frac{1}{E_f(A)} \), then \( \beta_1^* > 0 \).

**Proof** One notices immediately that \( E_f(A) > 0 \); thus, \( \lambda > 0 \). Also, \( r_u(x) = -\frac{u'(x)}{u(x)} = 1 \) for any \( x \in \mathbb{R} \); thus, according to (5.3), the optimal \( T \)-coinsurance rate \( \beta^* = \beta_1^* \) can be approximated as:

\[
\beta^* \approx 1 - \frac{\lambda E_f(A)}{Var_{T}(A) + \lambda^2 E_f^2(A)}
\]

(6.22)

By hypothesis, \( \lambda > \frac{1}{E_f(A)} \), we will have

\[
E_f^2(A)\lambda^2 - \lambda E_f(A) + Var_{T}(A) = (E_f(A)\lambda - 1)^2 + \lambda E_f(A) - 1 + Var_{T}(A) > 0.
\]

Dividing both members of the previous inequality by \( Var_{T}(A) + \lambda^2 E_f^2(A) > 0 \) and taking into account (6.22), it follows \( \beta^* > 0 \). \( \Box \)

**Example 6.7** We consider the following hypotheses:

- the weighting function is \( f(t) = 2t \), \( t \in [0,1] \);
- the risk is represented by the triangular fuzzy number \( A = (2,4,1) \);
- the utility function is \( u(x) = -e^{-x} \), for \( x \in \mathbb{R} \);
- the loading factor is \( \lambda > 0 \).

Using the formulas (6.1)-(6.3), we obtain the following indicators:

\[
E_f(A) = \frac{3}{2}; \quad Var_{T_1}(A) = \frac{7}{6}; \quad Var_{T_2}(A) = \frac{17}{36}
\]

(6.23)

Then, by applying the approximation (6.22) of \( \beta_1^* \) in case of \( D \)-operators \( T_1, T_2 \) we find:

\[
\beta_{T_1}^* \approx 1 - \frac{18\lambda}{14 + 27\lambda^2};
\]

\[
\beta_{T_2}^* \approx 1 - \frac{54\lambda}{17 + 81\lambda^2}.
\]

In this case, the condition of Proposition 6.6 is \( \lambda > \frac{2}{3} \). In particular for \( \lambda = 1 \), we obtain \( \beta_{T_1}^* = \frac{23}{41}, \quad \beta_{T_2}^* = \frac{22}{49} \).

### 7 Concluding remarks

The basic idea of the paper is the study of the coinsurance problem by the expected utility operators from Georgescu (2011) and Georgescu (2012a). The main contributions of the paper are:

- to build a coinsurance model in the framework offered by the possibilistic \( EU \)-theory associated with an expected utility operator;
- the use of \( D \)-operators defined in Georgescu and Fono (2019) to study the properties of the optimal coinsurance and its approximate calculation;
- the application of the general results to the computation of the coinsurance rates in a few remarkable cases and their comparison.

The particular cases and the examples from Sect. 6 emphasize the fact that the approximation formulas of the optimal coinsurance obtained in the paper offer fast calculation procedures.

We report next a few open problems:

(a) We assume that we have a data set representing values of a probabilistic risk (random variables) which appears as a parameter in the context of a probabilistic model (for example, in the coinsurance problem). Based on the existing data, one could determine those indicators by which we know the phenomenon described by the probabilistic model. Thereby, in case of the coinsurance problem, from data one obtains the statistic mean value and variance; then, we can compute the optimal coinsurance (by an approximate calculation formula similar to (5.3)). In Vercher et al. (2007), Vercher et al. present a method by which from a dataset one can build a trapezoidal fuzzy number. By applying Vercher et al.’s method, the probabilistic model of the coinsurance
turns into a possibilistic model, in which risk is described by this trapezoidal fuzzy number. We compute then the expected value and the variance associated with this trapezoidal fuzzy number; then, by formula (5.3) on can obtain the optimal coinsurance associated with the $T$-possibilistic model. An open problem is to find those formulas describing the way the probabilistic model of coinsurance is turned into a possibilistic model (by Vercher et al’s method), which allows a comparison of the two models.

(b) In papers Athayde and Flores (2004), Garlappi and Skoulakis (2011), Niguez et al. (2016), it is studied the effect of absolute risk aversion, prudence, and temperance on the optimal solution for the standard portfolio choice problem (Eeckhoudt et al. 2005, Section 4.1). A similar problem is investigated in Georgescu and Fono (2019) in the context of $E_U$-theory associated with a $D$-operator. It would be interesting to study refinements of Theorems 5.3 and 5.9 such that the optimal $T$-coinsurance rate and the standard expected utility to be expressed according to the indicators of risk aversion, prudence, and temperance as well as the $T$-moments of the possibilistic risk represented by the fuzzy number $A$.

(c) A third problem is the study of a coinsurance problem with two types of risk: besides the investment risk, a background risk might appear. Both the investment risk and the background risk can be probabilistic (random variables) or possibilistic (fuzzy numbers). Besides the purely possibilistic coinsurance model in which both risks are random variables, we have:

- the possibilistic model, in which risks are fuzzy;
- two mixed models, in which a risk is a fuzzy number, and the other is a random variable.

To define such bidimensional coinsurance models, it is necessary for the notions of multidimensional possibilistic expected utility (Georgescu 2012a, Section 6.1) and the mixed expected utility (Georgescu 2012a, Section 7.1) to be generalized for some “multidimensional expected utility operators.”

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Compliance with ethical standards

Conflict of interest Author Irina Georgescu declares that she has no conflict of interest.

Ethical approval All procedures performed in studies involving human participants were in accordance with the ethical standards of the institutional and/or national research committee and with the 1964 Helsinki Declaration and its later amendments or comparable ethical standards.

Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

8 Appendix

In the following, we will prove a necessary condition for the optimal $T$-coinsurance $\beta^* = \beta_T^*$ to be strictly positive. We will keep the notations from Sections 4 and 5.

Lemma 8.1 If $T$ is an expected utility operator, $A$ a fuzzy number and $u,v$ two continuous utility functions, then

$$
T(A, [u(x) - T(A, u(x))] [v(x) - T(A, v(x))])
= T(A, u(x) v(x)) - T(A, u(x)) T(A, v(x))
$$

Proof One uses conditions (b) and (c) from Definition 3.1.

Proposition 8.2 Let $T$ be a strictly increasing expected utility operator. Assume that $\lambda > 0$. Then, from $\beta_T^* > 0$ the following inequality follows:

$$
\lambda < \frac{T(A, (x - E_f(A)) \{u'(w_0 - x) - T(A, u'(w_0 - x))\})}{E_f(A) T(A, u'(w_0 - x))}
$$

Proof We will denote $\beta_T^* = \beta_T^*$. From (4.4), we have $g(x, 0) = w_0 - x$; thus, by (4.8):

$$
H'(0) = T(A, u'(w_0 - x) (x - P_0))
$$

(a)

Since $T$ is strictly increasing, $H(\beta)$ is a strictly concave function; thus, $H'(\beta)$ is a strictly decreasing function. Then, the following implication holds:

$$
\beta^* > 0 \Rightarrow 0 = H'(\beta^*) < H'(0)
$$

(b)

Applying (a) and Lemma 8.1, it follows

$$
H'(0) = T(A, u'(w_0 - x) [x - P_0 - T(A, x - P_0)])
+ T(A, u'(w_0 - x)) T(A, x - P_0)
$$

Noticing that $T(A, x - P_0) = -\lambda E_f(A)$, the previous inequality gets the form

$$
H'(0) = T(A, (x - E_f(A)) \{u'(w_0 - x)
- T(A, u'(w_0 - x))\}) - \lambda E_f(A) T(A, u'(w_0 - x))
$$
Then, the inequality $H'(0) > 0$ is written as:

$$T(A, (x - E_f(A))[u'(w_0 - x) - T(A, u'(w_0 - x)))] > \lambda E_f(A)T(A, u'(w_0 - x)).$$

Since $T$ is strictly increasing and $u'(w_0 - x) > 0$, we have $T(A, u'(w_0 - x)) > 0$. Also $E_f(A) > 0$; thus, the last inequality from above is equivalent with

$$\lambda < \frac{T(A, (x - E_f(A))[u'(w_0 - x) - T(A, u'(w_0 - x))]}{E_f(A)T(A, u'(w_0 - x))}$$

(c)

From (b) and (c), the implication which we had to prove follows.

□

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