Local structures of homogeneous Hall MHD turbulence

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Abstract. Local structures of decaying homogeneous and isotropic Hall MHD turbulence are studied by means of direct numerical simulations. Regions of strong vorticity and strong current density in Hall MHD turbulence are compared to those of single-fluid MHD turbulence. An analysis by the use of a low-pass filter reveals that the introduction of the Hall term can modify not only small-scale structures of the current density but also structures of the vorticity field, especially at the scales smaller than the ion skin depth.

1. Introduction

Numerical simulations of single-fluid magnetohydrodynamic (MHD) equations are often carried out to study macroscopic motions of hot plasmas. In fusion studies, MHD simulation studies are conducted frequently for the purpose of studying instability of a force-balance state and nonlinear saturations of the unstable modes in a magnetically confined device. However, an increase of the numerical resolution of MHD simulations brings about overlaps of the scales resolved in MHD simulations and the scales neglected in the MHD approximation. For example, full-three-dimension MHD simulations in a helical-type magnetically confined device such as the Large Helical Device resolve the evolutions of moderate wave number ballooning modes, the wave length of which can be comparable to the ion skin depth (Miura & Nakajima, 2010). Plasma motions in small scales are turbulent (micro-turbulence), being dominated by many effects such as the $E \times B$ and diamagnetic drifts which are discarded in the MHD approximation. Thus hot plasma turbulence may not necessarily be described by single-fluid MHD equations correctly, and should be studied either by the Vlasov-Boltzmann (gyrokinetic) simulations or extended MHD simulations. The former approach is not very easy because we have to resolve the five- or six-dimensional phase space, even though this approach can give more precise descriptions of plasma motions than the MHD and extended MHD models. On the other hand, extended MHD models can offer reasonable physical results within a reasonable computational cost even though the models sometimes have intrinsic theoretical difficulties to close the system of equations.

In this paper we study small-scale plasma motions described by the Hall MHD model, which is among simplest extended-MHD equations. While the Hall MHD model includes only the Hall (ion skin depth) term, it can release the magnetic field from the restriction of the frozen-in condition and enrich structures of turbulence. Simulations of Hall MHD turbulence have been
carried out with various motivations (Mininni et al., 2007; Galtier & Buchlin, 2007; Hori & Miura, 2008; Miura & Hori, 2009; Miura, 2010). Among some Hall term effects, we are concerned about its influence on local structures of the vorticity and the current density. While it is well known that the most dominant local structures in isotropic MHD turbulence is vortex sheets and current sheets, these structures can be modified because the Hall term enables a magnetic reconnection at a thin current sheet. It has been found in our earlier research by the use of the low-pass filter that current sheets are torn into pieces but still somewhat appear sheet-like, while the vortex structures are tubular at the scales of the ion skin depth (Miura & Hori, 2009). In this paper, we study local structures of decaying homogeneous and isotropic Hall MHD turbulence more closely by means of direct numerical simulations (DNSes).

This paper is organized as follows. In §2, outlines of our DNSes are shown. Our earlier results are also reviewed there. In §3, modification of the local structures depending on spatial scales are studied. Concluding remarks are shown in §4.

2. Outlines of DNSes and earlier results

In this section we see outlines of our DNSes, a few aspects of which have been reported earlier (Miura & Hori, 2009; Miura, 2010), in order to understand a basic nature of the turbulent field we study in the next section. DNSes of decaying, homogeneous and isotropic turbulence have been carried out for incompressible Hall MHD equations

\[
\frac{\partial u_i}{\partial t} = 0,
\]

\[
\frac{\partial u_i}{\partial t} = \frac{\partial (u_i u_j)}{\partial x_j} - \frac{\partial p}{\partial x_i} - \epsilon_{ijk} J_j B_k + \frac{1}{Re_p} \frac{\partial^2 u_i}{\partial x_j \partial x_j},
\]

\[
\frac{\partial B_i}{\partial t} = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left[ \epsilon_{k\ell m} (u_m - \epsilon J_m) B_n - \frac{1}{Re_{\eta}} J_k \right]
\]

where \( u_i, B_i \) and \( J_i := \epsilon_{ijk} \partial B_k / \partial x_j \) are the \( i \)-th components of the velocity vector, the magnetic field vector and the current density vector, respectively. The symbols \( \epsilon_{ijk}, \epsilon, Re_p \) and \( Re_{\eta} \) denotes the Levi-Civita symbol, the Hall parameter, the Reynolds number associated with the viscosity and the Reynolds number associated with the resistivity, respectively. The Hall MHD equations are solved by the pseudo-spectral method and the Runge-Kutta-Gill scheme under the triply periodic boundary condition over a \((2\pi)^3\) cube. The aliasing errors in the pseudo-spectral computations are removed by the 2/3-rule. We set parameters as \((Re_p, Re_{\eta}, \epsilon) = (500, 500, 0.05)\), and the number of grid points \( N^3 = 512^3 \) (thus the maximum wavenumber \( k_{\text{max}} = 170 \)). The initial condition of the velocity and the magnetic field vectors is given by the energy spectra proportional to \( k^2 \exp(-k^2/2k_0^2) \), where \( k = \sqrt{k_1^2 + k_2^2 + k_3^2} \) is the magnitude of the wavenumber vector \( (k_1, k_2, k_3) \), and random phases. We make use of a DNS result of the single-fluid MHD turbulence with parameters \((Re_p, Re_{\eta}, \epsilon) = (500, 500, 0)\), the same \( N \) and the same initial condition, so that we can compare the DNS results of the Hall MHD turbulence to that of the single-fluid MHD turbulence.

In Fig.2(a), the time evolution of the kinetic energy \( E_K = \langle u_i u_i / 2 \rangle \) and the magnetic energy \( E_M = \langle B_i B_i / 2 \rangle \) are shown, where the symbol \( \langle \cdot \rangle \) denotes the volume average. Figure 2(a) shows that \( E_K \) and \( E_M \) are exchanged each other. We have shown earlier through low-pass filtered simulations that the energy is considered to be exchanged at relatively low wavenumber region in Hall MHD turbulence while it is at relatively large wave number region in single-fluid MHD turbulence (Miura, 2010). In Fig.2(b), the kinetic energy spectrum \( E_K(k) := \langle u_i u_i \rangle_k \) and the magnetic energy spectrum \( E_M(k) := \langle B_i B_i \rangle_k \) are shown at \( t = 1.0 \). Here the symbol \( \langle \cdot \rangle_k \) represents the shell average in the wavenumber space, and the symbol\( \sim \)
denotes a Fourier coefficient. As have been shown by some numerical studies (Mininni et al., 2007; Miura, 2010), the introduction of the Hall term affects the magnetic energy transfer, especially at the high wavenumber range. In Fig.2(b) the two spectra \( E_K(k) \) and \( E_M(k) \) are shown. The energy spectra might be scaled as \( E_k(k) \propto k^{-5/3} \) and \( E_M(k) \propto k^{-5/3} \) or \( \propto k^{-7/3} \), as has been discussed in some numerical studies (Mininni et al., 2007; Galtier & Buchlin, 2007; Hori & Miura, 2008). However, the Reynolds numbers of our DNSes are not sufficiently large so that the scaling law of the energy spectra is not very clear.

![Figure 1](image1.png)

Figure 1. (a) Time evolutions of the kinetic energy \( E_K = \langle u_i u_i / 2 \rangle \) and the magnetic energy \( E_M = \langle B_i B_i / 2 \rangle \). (b) The kinetic energy spectrum and the magnetic energy spectrum at \( t = 1 \).

![Figure 2](image2.png)

Figure 2. Isosurfaces of the enstrophy density and the current density operated in (a) Hall MHD and (b) single-fluid MHD turbulence at \( t = 1 \). The blue and the white isosurfaces are for the enstrophy density and the current density, respectively. The thresholds of the isosurfaces are six times of the standard derivation of the quantities above their mean values.

In Fig.2, the isosurfaces of the enstrophy density \( Q := \omega_i \omega_i / 2 \) and the current density \( J := J_i J_i / 2 \) in (a) Hall MHD and (b) single-fluid MHD turbulence are shown, where \( \omega_i := \epsilon_{ijk} \partial u_k / \partial x_j \) is the \( i \)-th component of the vorticity vector. We observe that the isosurfaces of \( Q \) (blue) and \( J \) (white) are sheet-like both in Fig.2(a) and (b). A clear difference between the two figures is that the isosurfaces in the Hall MHD turbulence are torn into pieces of thin and narrow
sheets. However, we have studied that the isosurfaces of $Q$ in Hall MHD turbulence are not simple sheets. By operating the low-pass filter with the cut-off wavenumber $k_c$ ($k_c = 32$ in the reference), we find tubular structures in the vorticity field (Miura & Hori, 2009). This is the point we study more closely in the next section.

3. Tubular vortices and current sheets

The vorticity and the current density field are studied by the use of the low-pass filter with various $k_c$’s. Isosurfaces of the enstrophy density $Q^< := \omega_i^< \omega_i^< / 2$ and the current density $J^< := j_i^< j_i^< / 2$ are shown in Fig.3 over a region of 256$^3$ grid points. The symbol $< \,$ represents that the quantity is low-pass-filtered. The cut-off filter is set (a)$k_c = 128$, (b)$64$ and (c)$32$ in Hall MHD turbulence, and (d)$k_c = 128$, (e)$64$ and (f)$32$ in single-fluid MHD turbulence, respectively. In Fig.3(a), isosurfaces of both $Q^<$ and $J^<$ are shown. Though the isosurfaces are torn into small pieces in comparison to their single-fluid-MHD counterparts in Fig.3(d), the structures are not very different from those observed in Fig.2(a). However, by the use of the low-pass filter with $k_c = 64$, some isosurfaces of $Q^<$ in Hall MHD turbulence in Fig.3(b) become tubular (typical regions are encircled yellow), while the current density $J^<$ in Hall MHD turbulence is less sensitive to the change of $k_c$ in comparison to $Q^<$. By lowering the cut-off wavenumber to $k_c = 32$, the difference between structures of $Q^<$ and those of $J^<$ in Hall MHD turbulence, and between $Q^<$ in Hall MHD and $Q^<$ in MHD turbulence become the clearest. While many of $Q^<$ are tubular in Fig.3(e), only a few of them are tubular in Fig.3(f). On the other hand, many of the current structures are sheet-like both in Fig.3(c) and (e).

The change of the vortical field at the moderate wavenumber might be studied through the kinetic energy transfer function. By manipulating Fourier representation of eq.(2), we can describe the kinetic energy budget in the wavenumber space as

$$ \frac{d}{dt} E_K(k) = T^K_u(k) + T^K_B(k) - \frac{1}{Re_\nu} k^2 E_K(k) \quad (4) $$

where $T^K_B(k)$ and $T^K_B(k)$ come from the advection term in eq.(2) $- u_j (\partial u_i / \partial x_j)$ and the Lorentz term $\epsilon_{ijk} J_j B_k$ in eq.(2), respectively. The magnetic energy transfer is omitted because the change of the vortex structures is concerned here. The kinetic energy transfer functions $T^K_u(k)$ and $T^K_B(k)$ in (a)Hall MHD and (b)single-fluid MHD are plotted in Fig.4(a) and (b), respectively. By comparing Fig.4(a) and (b), we find that the energy transfer by $T^K_u(k)$ is activated in Hall MHD turbulence. Furthermore, the positive energy transfer by the Lorentz term is localized at $20 \leq k \leq 30$ in Fig.4(a), while it is not very localized in Fig.4(b). It may be that the activated energy transfer $T^K_u(k)$ and $T^K_B(k)$ enhance rolling-up of vortex sheets to tubes when the “frozen-in” condition is broke by the Hall term.

It is noted here that the isosurfaces of $Q^<$ and $J^<$ are located closely to each other in Fig.3(a), (b) and (d)-(f), but not in Fig.3(c). To the contrary, in Fig.3(c), isosurfaces of $Q^<$ and $J^<$ with $k_c = 32$ are located separately. Our current understanding is as follows. In MHD turbulence, large-scale plasma motions can behave as if the magnetic field is frozen into the flow field. Magnetic reconnection events are inhibited until the current sheets become so thin that the resistivity can change the topology of the magnetic field associated with the current sheets. Consequently, only a few current sheets experiences magnetic reconnection, and many of the current sheets remain stably, keeping the vortex sheets in the neighborhood of them. However, in Hall MHD turbulence, the Hall term with $\epsilon = 0.05$ influences the magnetic induction at the scale $k \simeq 20$. Because of the Hall term, magnetic reconnection is allowed and frozen-in condition is broken at the scales $k \simeq 20$. Then vortex structures and current structures can evolve separately, and vortex sheets can roll up to tubes with a help of activated energy transfer in Fig.4.
Figure 3. Isosurfaces of the enstrophy density and the current density operated by the low-pass filter. The two quantities with $k_c = 128$, 64 and 32 in the Hall MHD turbulence are shown in (a), (b), (c). Their counterparts in the single-fluid MHD turbulence are shown in (d), (e), (f), respectively.
The separation of isosurfaces of $Q^<$ and $J^<$ in Fig.3(c) can be expressed that the small-scale contributions to the isosurfaces in Hall MHD turbulence make the isosurfaces of $Q^<$ and $J^<$ locate closely to each other, as if the former reattach to the latter (or vice versa). That is, the isosurfaces of $Q^<$ and $J^<$ are located separately in Fig.3(c) for $|k_1|, |k_2|, |k_3| \leq k_c = 32$ but they are located closely to each other in Fig.3(a) and (b), when Fourier coefficients of $|k_1|, |k_2|, |k_3| > 32$ are taken into account. Recall that $E_K(k) < E_M(k)$ at a large wavenumber range, as we see in Fig.2. It means $|\tilde{\omega}(k)|^2 << |\tilde{J}_i(k)|^2$ at the high wavenumber range. Then, we can expect in the real space $|uj(\partial u_i/\partial x_j)| << |\epsilon_{ijk}J_jB_k|$ at regions where high wavenumber coefficients contribute. By neglecting all the terms in the right hand side but the Lorentz force in eq.(2) and the Hall term in eq.(3), the velocity and the magnetic field are dominated by the Lorentz force and its rotation, respectively. (The dissipation terms are omitted for simplicity but we have to keep in mind that they play important roles in the high wavenumber range.) We postulate that the reattachment of high $Q^<$ regions to high $J^<$ regions are cause because the velocity and the magnetic fields are both dominated by the single term $\epsilon_{ijk}J_jB_k$.

4. Concluding Remarks

Local structures of decaying homogeneous and isotropic Hall MHD turbulence are studied by the use of the low-pass filter. The introduction of the Hall term brings about not only tearing of current sheets into pieces, but also formation of tubular structures in the moderate scales. The qualitative change of spatial structures might be related to the change of the energy transfer associated with the Lorentz force. Our observation also suggests that the high wavenumber region of both the vorticity and the current density field are dominated by the Lorentz force and its rotation, not by the fluid inertia, leading to the reattachment of the strong vorticity region to the strong current density region. The numerical work was performed on the NEC SX-8 and on the HITACHI SR16000 of the National Institute for Fusion Science (NIFS), and supported by the NIFS Collaborative Research Program (NIFS10KTA8001, NIFS09KNXN154,NIFS10KTB003,NIFS11KNS0016) and Grant-in-Aid for Scientific Research, KAKENHI (22540509,23340182,23540583).

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