Relativistic electron beam driven longitudinal wake-wave breaking in a cold plasma

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Abstract

Space-time evolution of relativistic electron beam driven wake-field in a cold, homogeneous plasma, is studied using 1D-fluid simulation techniques. It is observed that the wake wave gradually evolves and eventually breaks, exhibiting sharp spikes in the density profile and sawtooth like features in the electric field profile [1]. It is shown here that the excited wakefield is a longitudinal Akhiezer-Polovin mode [2] and its steepening (breaking) can be understood in terms of phase mixing of this mode, which arises because of relativistic mass variation effects. Further the phase mixing time (breaking time) is studied as a function of beam density and beam velocity and is found to follow the well known scaling presented in ref. [3].
I. INTRODUCTION

Plasma based acceleration schemes have shown promising results in recent years [4–7]. Plasmas form an attractive medium for future generation of accelerators, because they can support electric fields of the order of several hundred GV/m, which is many orders of magnitude higher than that produced by conventional RF based accelerators [8]. These extreme fields are generated by relativistically intense longitudinal plasma waves, which are excited when an ultra-intense laser pulse or an ultra-relativistic beam pulse propagates through the plasma [9–17]. Based on the mechanism of excitation of the plasma wave (wake wave), plasma based acceleration schemes are categorized into two types, Laser Wakefield Acceleration (LWFA) and Plasma Wake-field Acceleration (PWFA). In Laser Wakefield Acceleration (LWFA) scheme, an ultrashort, intense laser pulse is employed to drive a relativistically intense plasma wave. Charged particles either externally injected or trapped from the background plasma ride on this excited plasma wake wave and get accelerated to high energies. This acceleration process has been confirmed in a number of experiments by accelerating electrons to GeV energies [18–20]. In Plasma Wakefield Acceleration (PWFA) scheme, an intense, near light-speed electron beam is used instead of a laser pulse to excite plasma wave which has a phase velocity equal to the velocity of the beam. A late coming bunch of charged particles rides on this wave and gets accelerated to high energies. As “plasma afterburners” this scheme is most suitable to boost the energy of the existing linacs. In 2007, Blumenfeld et al. [21] have accelerated electrons from the tail of a driver bunch having energy 42 GeV up-to a maximum energy of 85 GeV, in a meter long plasma at SLAC (Stanford Linear Accelerator Center). But the accelerated electrons had a very broad energy spectrum. Recently in 2014, Litos et al. [22] have also demonstrated the success of PWFA scheme achieving a much lower spread of accelerated beam energy (hardly 2 percent) by injecting a discrete trailing bunch.

The structure of the wakefield excited by an ultra-relativistic electron beam pulse propagating through a plasma, has been studied extensively by Rosenzweig et al. [23], Amatuni et al. [24] and Ruth et al. [25]. Due to intrinsic interest in their non-linear properties numerous investigations have been carried out, both numerically and analytically, in this area. In an earlier study, Rosenzweig et al. [23] gave an analytical expression, in 1-D, for
the wake electric field excited by an ultra-relativistic electron beam having density \( n_b \) less than or equal to half the plasma density \( n_0 \). In our recent work \[1\], we reported a detailed analytical and numerical study of relativistic electron beam driven wakefield, where we analytically extended Rosenzweig’s work to arbitrary beam densities and numerically verified our analytical results using 1-D fluid simulation. Our simulation result exhibited a good match with the analytical results for several plasma periods. However, it was observed that at late times in the simulation, the perturbed density in all cases show spiky features, which is accompanied by sawtooth like structures in the electric field profile. This particular behavior was absent in our analytically derived profile of perturb density and electric field (see ref. \[1\]). The spiky features in the perturb density profile and sawtooth like structures in the electric field profile are well known signatures of wave-breaking \[26, 27\]. In our previous report, we had stated that the excited wake wave is a longitudinal Akhiezer-Polovin wave \[2\] which breaks when perturbed longitudinally \[28\]. In our case, perturbation is produced by numerical noise.

In this paper, we have extended our earlier work and present a detailed study of the breaking of wake wave and its dependence on the electron beam density and velocity. We have followed the space-time evolution of the electron beam driven wake wave in a cold plasma using 1-D fluid simulation. We have carried out the simulation for a long enough time for the wake wave to break and exhibit spiky features. In section II, we present the equations governing the evolution of wake field. To clearly study the breaking of wake wave, we have neglected the beam evolution in the self consistent field of the wake wave. This is valid in the limit \( \gamma_b \gg 1 \), where \( \gamma_b \) is the Lorentz factor associated with the beam velocity \( v_b \). Section III contains a brief discussion of the simulation techniques and our simulation results. It is observed that the wake wave evolves in time and breaks after several plasma periods. In section IV, the physical mechanism underlying wake wave breaking is discussed and the numerical results are compared with the well known scaling of Akhiezer-Polovin wave breaking time with phase velocity of the wake wave \( \beta_{ph} \) and maximum fluid velocity \( u_m \) \[3\].
II. GOVERNING EQUATIONS

The basic equations governing the space and time evolution of ultra-relativistic electron beam driven wakefield in a cold plasma are the relativistic fluid-Maxwell equations for the plasma electrons. As stated in the introduction, we work in the limit of $\gamma_b \gg 1$, where the beam evolution equations may be neglected. Also ion dynamics is neglected, as ions do not respond in these time scales. Ions are only assumed to provide a stationary neutralizing background. We consider beam to be moving along $z$-direction in an infinite, homogeneous cold plasma. Neglecting the variation of plasma parameters in the transverse (transverse to the beam propagation) direction, the basic governing equations are the continuity and momentum equation for plasma electrons and Poisson’s equation, which in normalized form in 1-D are given as

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial z} = 0 \quad (1)$$

$$\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial z} = -E \quad (2)$$

$$\frac{\partial E}{\partial z} = (1 - n - n_b) \quad (3)$$

where $p = \gamma v$ is the $z$-component of momentum of plasma electrons having $z$-component of velocity $v$ and $\gamma = (1 - v^2)^{-1/2}$ is the relativistic factor for plasma electrons. $E$ and $n_0$ are the $z$-component of the self-consistent electric field and equilibrium plasma density respectively. Here we have used the normalization factors as, $t \rightarrow \omega_{pe} t$, $z \rightarrow \frac{\omega_{pe} z}{c}$, $E \rightarrow \frac{eE}{m_e c \omega_{pe}}$, $v \rightarrow \frac{v}{c}$, $p \rightarrow \frac{p}{m_e c}$, $n \rightarrow \frac{n}{n_0}$, $n_b \rightarrow \frac{n_b}{n_0}$, $\omega_{pe}$ being the non-relativistic plasma frequency and $n_0$ is the equilibrium plasma density. The above equations (equation (1-3)) are the main key equations required to study 1-D electrostatic relativistic electron beam driven wakefield excitation in a cold plasma.

III. FLUID SIMULATION OF RELATIVISTIC ELECTRON BEAM DRIVEN WAKEFIELD

In this section, we briefly discuss the numerical techniques used to study the relativistic electron beam driven wakefield excitation in a cold plasma and present our simulation results. We have developed a 1D fluid code using a set of subroutines (LCPFCT) which is based
on flux-corrected transport scheme \[29\], to study the space and time evolution of an ultra-relativistic electron beam driven wakefield in a cold plasma. The principle of this scheme is based on the generalization of two step Lax-Wendroff method \[30\]. We have simulated equations (1), (2) and (3) with non-periodic boundary conditions. Beam is considered to be rigid. We have initiated the simulation using the profiles of electric field, density and velocity from the analytical work of Rosenzweig et al. \[23\], although our results are independent of the initial choice of profiles. Here the driver beam is allowed to propagate inside the plasma starting from one end of the simulation window and the wake field is evolved according to equations (1), (2) and (3) \[1\]. The simulation results are shown in figures (1-6) for different values of beam density \(n_b\) and beam velocity \(v_b\). Fig. (1) and (2) respectively show the perturbed electron density and the wake electric field for \(n_b = 0.3\) and \(v_b = 0.99\). Same quantities are shown in figures (3) and (4) for a different beam density \(n_b = 0.4\), keeping the beam velocity fixed (at \(v_b = 0.99\)). Finally figures (5) and (6) respectively show the perturbed electron density and wake electric field for \(n_b = 0.4\) and a different beam velocity \(v_b = 0.8\). In all the figures, numerical results are shown in magenta and the analytical results (derived in ref. \[1\] and reproduced in the next section, for completeness) are shown in blue.

As mentioned in the introduction and also in ref. \[1\], it is observed that in all cases the simulated wakefield profile gradually deviates from analytical profile, with time and eventually breaks after several plasma periods. The signature of breaking of the wake wave is seen as density spikes in the wake wave. The electric field also exhibits into a sawtooth structure close to breaking time. It is observed that for a fixed beam velocity, higher the beam density shorter is the wake wave breaking time; whereas for a fixed beam density, higher the beam velocity longer is the wake wave breaking time. Fig. (7) and (8) show the variation of breaking time with the beam velocity (phase velocity of the wake wave) for fixed beam densities \(n_b = 0.3\) and 0.4 respectively and fig. (9) shows the variation of wake wave breaking time with the max. fluid velocity \(u_m\) ("\(u_m\)" is related to the beam density, as discussed in the next section). In these figures (7, 8 and 9), the points are obtained from simulation and continuous lines present our understanding of the breaking mechanism of wake wave in terms of breaking of longitudinal Akhiezer-Polovin wave. This is presented in the next section.
IV. ANALYSIS OF WAKE WAVE BREAKING

In this section, we present a detailed discussion on the physical mechanism of wake wave breaking and the dependence of wake wave breaking time on the electron beam velocity and density. For the sake of completeness we first present the analytical expression for wakefield profile behind the beam. In terms of wave frame variable \( \tau = (t - \frac{z}{c \beta_{ph}}) \) and in the limit \( \beta_{ph} \to 1 \), the wake wave profile behind the beam may be written in parametric form as

\[
E = \left( \frac{\gamma_m^2 - 1}{\gamma_m + \sqrt{\gamma_m^2 - 1} \cos(2\phi)} \right)^{1/2} \sin(2\phi) \tag{4}
\]

\[
\tau = \tau_f + 2\sqrt{\gamma_m + \sqrt{\gamma_m^2 - 1}} \left[ E(\phi_f, m) - E(\phi, m) \right] \tag{5}
\]

where \( \tau_f = \frac{\omega_{pe} l_b}{c} \), \( l_b \) being the length of the beam and \( \gamma_m \) and \( \phi_f \) are constants; \( \gamma_m = (1 - u_m^2)^{-1/2} \) is the Lorentz factor associated with the maximum fluid velocity \( \text{“}u_m\text{”} \) behind the beam and \( \phi_f \) is the value of the parameter \( \phi \) at \( \tau = \tau_f \). \( E(\phi, m) \), \( E(\phi_f, m) \) are incomplete elliptic integral of second kind with \( m^2 = 2(\sqrt{\gamma_m^2 - 1})/(\gamma_m + \sqrt{\gamma_m^2 - 1}) \). The perturbed density is given by \( n_1 = \frac{1}{2} x^2 \) where \( x \) is defined as \( x = (1 - \beta/\sqrt{1 - \beta^2}) \), \( \beta \) being the fluid velocity; \( x \) is related to the parameter \( \phi \) as \( x = \gamma_m + (\sqrt{\gamma_m^2 - 1}) \cos(2\phi) \). Thus all the wake field variables, perturbed density, electric field, and fluid velocity are represented in terms of the parameter \( \phi \). The constants \( \gamma_m \) and \( x_f \) (value of \( x \) corresponding to \( \phi_f \)) are evaluated using the beam density \( \alpha (= n_b) \) and length \( l_b \) (or \( \tau_f \)) as

\[
\gamma_m = (1 - \alpha) + \alpha x_f \tag{6}
\]

where \( x_f \) is related to \( \tau_f \) through the implicit relations

\[
x_f = \frac{1 - 2\alpha \sin^2 \psi_f}{1 - 2\alpha} \tag{7}
\]

and

\[
\tau_f = 2(1 - 2\alpha)^{-1}[E(k) - E(\psi_f, k_1)] \tag{8}
\]

Here \( E(\psi_f, k_1) \) and \( E(k_1) \) are respectively the incomplete and complete elliptic integrals of second kind with \( k_1^2 = 2\alpha \). Equations (6), (7) and (8) are derived using wakefield equations inside the beam and using the continuity conditions at the end of the beam (for complete details see ref. [1]). Equation (7) and (8) are valid for \( \alpha < 1/2 \), which is the range of
beam density within which we have limited our present set of simulation. The frequency of the wake wave behind the beam is given by $\omega_{\text{wake}} = \pi \left(2\sqrt{(\gamma_m + \sqrt{\gamma_m^2 - 1})E(m)}\right)^{-1}$.

Using equations (6-8) for a given beam density $\alpha$ and beam length $l_b$, the wakefield profiles behind the beam (perturbed electron density and electric field) are plotted along with the simulation results in Fig. (1)-(4). In figure (5) and (6), the simulation results are compared with the numerical solution of the wakefield differential equations, $\beta_{ph} \neq 1$ (see ref. [1]). As mentioned earlier, simulation results match well with the analytical expressions for several plasma periods, but at late times a marked deviation between the two is observed. Sharp spikes in perturbed density is accompanied with sawtooth profile in wake electric field. These features are well known signature of wake wave breaking. In order to understand this phenomenon, we first identify the wake wave with a longitudinal Akhiezer-Polovin mode.

It is well known that the stationary wave frame solution of the relativistic fluid-Maxwell equations in 1-D, for a cold homogeneous plasma with infinitely massive ions (equations (11)-(3) without the beam terms in the Poisson’s equation) is a longitudinal Akhiezer-Polovin mode which is parameterized in terms of maximum fluid velocity “$u_m$”. and phase velocity “$\beta_{ph}$” [2, 31]. Thus the wakefield behind the beam, which is a solution of equations (11)-(3) with $\alpha = 0$, is nothing but a longitudinal Akhiezer-Polovin mode, where the parameter $u_m$ is related to the beam density $\alpha$ and the length of the beam $l_b$ through the equations (7-8). Also using the identity $E(\frac{2\sqrt{k}}{1+k}) = \frac{1}{1+k} (2E(k) - k^2 K(k))$ [32], with $k^2 = \frac{\gamma_m - 1}{\gamma_m + 1}$ and $k'^2 = 1 - k^2$, the expression for wake becomes identical with the expression for Akhiezer-Polovin wave frequency (equation (11) of ref. [28] and equation (5) of ref. [3]). To further emphasize the equivalence between the wake wave excited by an ultra-relativistic electron beam with beam density $\alpha = n_b$ and length $\tau_f$ (or $l_b$), and a longitudinal Akhiezer-Polovin mode with parameter “$u_m$” and “$\beta_{ph}$”, we first estimate “$u_m$” for the parameters of fig. (1) using equations (6-8). Using this value of $u_m$ and $\beta_{ph} = \beta_b \rightarrow 1$, and following the method outlined in refs. [3, 28], we plot the appropriate Akhiezer-Polovin mode along with the wakefield behind the beam. This is shown in fig. (10), which clearly establishes that the two are identical.

It is well known that the amplitude of a Akhiezer-Polovin mode is limited by the wave
breaking limit which is given by $E_{WB} = \sqrt{2(\gamma_{ph} - 1)}$, where $\gamma_{ph} = (1 - \beta_{ph}^2)^{-1}$ is the Lorentz factor associated with the phase velocity of the wave. For $\beta_{ph} \to 1$, wave breaking limit $E_{WB} \to \infty$, and the mode in-principle should never break. But this is contrary to what is observed in our simulations; the wake wave breaks at a much lower amplitude. Recently it has been shown that an Akhiezer-Polovin mode can break at an amplitude well below its wave breaking limit via a process called phase mixing, when it is subjected to an arbitrary small longitudinal perturbation [28]. It has been shown in ref. [3] that addition of an arbitrary small longitudinal perturbation to a longitudinal Akhiezer-Polovin mode results in the frequency of the mode acquiring a spatial dependence due to relativistic mass variation effects. In the present case, perturbation arises due to numerical noise. Because of the spatial dependence in frequency, different “pieces” of the wave slowly go out phase with each other as time progresses. The process of phase mixing is clearly visible in the electric field profile, where the phase difference between simulated wake field and analytically obtained wake field slowly increases with time. Phase mixing eventually leads to breaking of the wake wave. The phenomenon of phase mixing leading to wave breaking of a relativistically intense longitudinal wave have been studied extensively by several author in different contexts [26, 27, 33, 34]. It is shown in ref. [3], that the time in which wake breaks scales with the phase velocity $\beta_{ph}$ and maximum fluid velocity $u_m$ as $\tau_{mix} \sim \frac{2\pi \beta_{ph}}{3 \delta} \left(\frac{1}{u_m^2} - \frac{1}{4}\right)$, where “$\delta$” is the amplitude of the perturbation. We have verified this scaling in our simulations by first keeping “$u_m$” fixed (i.e. $\alpha = n_b$ and $l_b$ fixed) and varying $\beta_b = \beta_{ph}$, and then keeping $\beta_b = \beta_{ph}$ fixed and varying $u_m$ (i.e. by varying $\alpha$). Note the continuous lines in fig. (7), (8) and (9); the scaling of phase mixing time $\tau_{mix}$ with $\beta_{ph}$ and $u_m^2$ compares well with out simulation results.

V. SUMMARY

We have studied space-time evolution of relativistic electron beam driven wake wave in a cold homogeneous plasma using 1-D fluid simulation. It is found that at times, which depend of on the electron beam density and velocity, the wake wave breaks via a phenomenon called phase mixing. The wake wave is further identified with a longitudinal Akhiezer-Polovin mode and its breaking time scales with phase velocity ($\beta_{ph}$) and maximum fluid velocity
\((u_m)\) according to a relation as suggested in ref. [3].

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FIG. 1. Numerical and analytical normalized perturbed electron density ($n_1$) profile at different times for the normalized beam density ($n_b$)=0.3 and beam velocity ($v_b$) =0.99
FIG. 2. Numerical and analytical normalized electric field ($E$) profile at different times for the normalized beam density ($n_b$)=0.3 and beam velocity ($v_b$) =0.99
FIG. 3. Numerical and analytical normalized perturbed electron density ($n_1$) profile at different times for the normalized beam density ($n_b$)=0.4 and beam velocity $v_b = 0.99$. 
FIG. 4. Numerical and analytical normalized electric field ($E$) profile at different time for the normalized beam density ($n_b$)=0.4 and beam velocity $v_b = 0.99$
FIG. 5. Numerical normalized perturbed electron density ($n_1$) profile at different times for the normalized beam density ($n_b$)=0.4 and beam velocity $v_b = 0.8$.
FIG. 6. Numerical normalized electric field ($E$) profile at different times for the normalized beam density ($n_b$) = 0.4 and beam velocity $v_b = 0.8$. 

\[
\omega_{pe t=0} \quad \omega_{pe t=40} \quad \omega_{pe t=65}
\]
FIG. 7. Plot for numerically obtained (circles) and fitted (solid) scaling of Phase mixing time ($\tau_{\text{mix}}$) with the phase velocity ($\beta_{\text{ph}}$) for the normalized beam density ($n_b$)=0.3
FIG. 8. Plot for numerically obtained (circles) and fitted (solid) scaling of Phase mixing time ($\tau_{\text{mix}}$) with the phase velocity ($\beta_{\text{ph}}$) for the normalized beam density ($n_b$)=0.4
FIG. 9. Plot for analytical (solid) and numerical (circles) scaling of Phase mixing time ($\tau_{mix}$) as a function of maximum fluid velocity ($u_m$).
FIG. 10. Plot of perturbed density profile ($n_1$) of wake wave (magenta) and Akhiezer-plovin mode (blue circles)