Coupled Painlevé VI system with $E_6^{(1)}$-symmetry

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Abstract
We present a new system of ordinary differential equations with affine Weyl group symmetry of type $E_6^{(1)}$. This system is expressed as a Hamiltonian system of sixth order with a coupled Painlevé VI Hamiltonian.

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Introduction

The Painlevé equations $P_J$ ($J = I, \ldots, VI$) are ordinary differential equations of second order. It is known that these $P_J$ admit the following affine Weyl group symmetries [O1]:

\[
\begin{array}{c|c|c|c|c|c|c}
J & P_{II} & P_{III} & P_{IV} & P_{V} & P_{VI} \\
\hline
I & A_1^{(1)} & A_1^{(1)} & A_1^{(1)} & A_1^{(1)} & D_4^{(1)} \\

\end{array}
\]

Several extensions of the Painlevé equations have been studied from the viewpoint of affine Weyl group symmetry. The Noumi–Yamada system is a generalization of $P_{II}$, $P_{IV}$ and $P_{V}$ for $A_1^{(1)}$-symmetry [NY1]. The coupled Painlevé VI system with $D_{2n+2}^{(1)}$-symmetry is also studied [S]. In this paper, we present a new system of ordinary differential equations with $E_6^{(1)}$-symmetry. Our system can be expressed as a Hamiltonian system of sixth order with a coupled Painlevé VI Hamiltonian.

In order to obtain this system, we consider a similarity reduction of a Drinfeld–Sokolov hierarchy of type $E_6^{(1)}$. The Drinfeld–Sokolov hierarchies are extensions of the KdV (or mKdV) hierarchy [DS]. They are characterized by graded Heisenberg subalgebras of affine Lie algebras. They also imply several Painlevé systems by similarity reduction as follows [AS, FS1, FS2, KIK, KK1, KK2]:
As is seen above, the coupled Painlevé VI system is derived from the $D_{2n+2}^{(1)}$-hierarchy associated with the graded Heisenberg subalgebra of type $(1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)$. We apply a similar method to the case of $E_6^{(1)}$ by choosing the graded Heisenberg subalgebra of type $(1, 1, 0, 1, 0, 1, 0)$; see figures 1 and 2. The hierarchy defined thus implies our new system by similarity reduction.

This paper is organized as follows. In section 1, we present an explicit formula of a coupled Painlevé VI system with $E_6^{(1)}$-symmetry. In section 2, we recall the affine Lie algebra $g(E_6^{(1)})$ and its graded Heisenberg subalgebra of type $(1, 1, 0, 1, 0, 1, 0)$. In section 3, we formulate a similarity reduction of a Drinfeld–Sokolov hierarchy of type $E_6^{(1)}$. In section 4, we derive the coupled Painlevé VI system from the similarity reduction.

1. Main result

The Painlevé equation $P_{VI}$ can be expressed as the following Hamiltonian system [IKSY, O2]:

$$
\begin{align*}
    s(s-1) \frac{dq}{ds} &= \frac{\partial H_{VI}}{\partial p}, \\
    s(s-1) \frac{dp}{ds} &= -\frac{\partial H_{VI}}{\partial q},
\end{align*}
$$

| Lie algebra | Gradation | Painlevé system |
|-------------|-----------|-----------------|
| $A_1^{(1)}$ | $(1, 1)$  | $P_{II}$ $P_{IV}$ |
|             | $(1, 0)$  |                 |
| $A_2^{(1)}$ | $(1, 1, 1)$ | $P_{II}$ $P_{IV}$ |
|             | $(2, 1, 1)$ |                |
|             | $(1, 0, 0)$ |                |
| $A_3^{(1)}$ | $(1, 1, 1, 1)$ | $P_{II}$ |
| $A_n^{(1)} (n \geq 4)$ | $(1, \ldots, 1)$ | Noumi–Yamada system |
| $D_4^{(1)}$ | $(1, 1, 0, 1, 1)$ | $P_{VI}$ |
| $D_{2n+2}^{(1)} (n \geq 2)$ | $(1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)$ | Coupled $P_{VI}$ |
with the Hamiltonian $H_{VI} = H_{VI}(p, q, s; \beta_0, \beta_1, \beta_2, \beta_3)$ defined by

$$H_{VI} = q(q-1)(q-s)p^2 - (\beta_1 - 1)q(q-1) + \beta_3 q(q-s) + \beta_4(q-1)(q-s)p + \beta_2(\beta_0 + \beta_2)q,$$

where $\beta_i (i = 0, \ldots, 4)$ are complex parameters satisfying

$$\beta_0 + \beta_1 + 2\beta_2 + \beta_3 + \beta_4 = 1.$$

We define a coupled Hamiltonian $H$ by

$$H = H_{VI}(p_1, q_1, s; \alpha_1, 1 - \alpha_1 - 2\alpha_2 - 2\alpha_3, \alpha_1, \alpha_3) + H_{VI}(p_2, q_2, s; \alpha_3, 1 - 2\alpha_3 - 2\alpha_4, \alpha_5, \alpha_3) + H_{VI}(p_3, q_3, s; \alpha_3, 1 - \alpha_0 - 2\alpha_3 - 2\alpha_6, \alpha_0, \alpha_3) + \sum_{1 \leq i < j \leq 3} \{ (q_i - 1)p_i + \alpha_2 \} \{ (q_j - 1)p_j + \alpha_2 \} \{ (q_i q_j + s) \},$$

(1.1)

where $\alpha_i (i = 0, \ldots, 6)$ are complex parameters satisfying

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = 1.$$

Note that these parameters correspond to the simple roots of type $E_6^{(1)}$. We consider a Hamiltonian system with the Hamiltonian (1.1),

$$s(s-1) \frac{dq_i}{ds} = \{ H, q_i \}, \quad s(s-1) \frac{dp_i}{ds} = \{ H, p_i \} \quad (i = 1, 2, 3),$$

(1.2)

where $\{ \cdot, \cdot \}$ stands for the Poisson bracket defined by

$$\{ p_i, q_j \} = \delta_{i,j}, \quad \{ p_i, p_j \} = \{ q_i, q_j \} = 0 \quad (i, j = 1, 2, 3).$$

The affine Weyl group $W(E_6^{(1)})$ is generated by the transformations $r_i (i = 0, \ldots, 6)$ acting on the simple roots as

$$r_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i \quad (i, j = 0, \ldots, 6),$$

where $A = (a_{ij})_{i, j=0}^6$ is the generalized Cartan matrix of type $E_6^{(1)}$ defined by

$$A = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 2
\end{bmatrix}.$$

Let $\pi_i (i = 1, 2)$ be Dynkin diagram automorphisms acting on the simple roots as

$$\pi_i(\alpha_j) = \alpha_{\sigma_i(j)} \quad (i = 1, 2; j = 0, \ldots, 6),$$

where $\sigma_i (i = 1, 2)$ are permutations defined by

$$\sigma_1 = (01)(26), \quad \sigma_2 = (05)(46).$$

We consider an extension of $W(E_6^{(1)})$

$$\tilde{W} = \langle r_0, r_1, r_2, r_3, r_4, r_5, r_6, \pi_1, \pi_2 \rangle.$$
with the fundamental relations
\[ r_i^2 = 1 \quad (i = 0, \ldots, 6), \]
\[ (r_i r_j)^{2-a_{ij}} = 0 \quad (i, j = 0, \ldots, 6; i \neq j), \]
\[ \pi_i^2 = 1 \quad (i = 1, 2), \]
\[ (\pi_i \pi_j)^3 = 1, \]
\[ \pi_i r_j = r_{\sigma(i,j)} \pi_j \quad (i = 1, 2; j = 0, \ldots, 6). \]

The action of the group \( \tilde{W} \) can be lifted to canonical transformations of the Hamiltonian system (1.2). Denoting by
\[ \varphi_0 = q_0 - 1, \quad \varphi_1 = q_1 - 1, \quad \varphi_2 = p_1, \quad \varphi_3 = q_1 q_2 q_3 - s, \]
\[ \varphi_4 = p_2, \quad \varphi_5 = q_2 - 1, \quad \varphi_6 = p_3, \]
we obtain

**Theorem 1.1.** The system (1.2) with (1.1) is invariant under the action of birational canonical transformations \( r_i (i = 0, \ldots, 6) \) and \( \pi_i (i = 1, 2) \) defined by
\[ r_i (\alpha_j) = \alpha_j - a_{ij} \alpha_i, \quad r_i (\varphi_j) = \varphi_j + \frac{\alpha_i}{\varphi_i} \{ \varphi_i, \varphi_j \} \quad (i, j = 0, \ldots, 6), \]
and
\[ \pi_i (\alpha_j) = \alpha_{\sigma(i,j)}, \quad \pi_i (\varphi_j) = \varphi_{\sigma(i,j)} \quad (i = 1, 2; j = 0, \ldots, 6). \]

**2. Affine Lie algebra**

Following the notation of [Kac], we recall the affine Lie algebra \( \mathfrak{g} = \mathfrak{g}(E_6^{(1)}) \) and its graded Heisenberg subalgebra of type \( (1, 1, 0, 1, 0, 1, 0) \).

The affine Lie algebra \( \mathfrak{g} \) is generated by the Chevalley generators \( e_i, f_i, \alpha^\vee_i (i = 0, \ldots, 6) \) and the scaling element \( d \) with the fundamental relations
\[ (a d e_i)^{1-a_{ij}} (e_j) = 0, \quad (a d f_i)^{1-a_{ij}} (f_j) = 0 \quad (i \neq j), \]
\[ [\alpha^\vee_i, \alpha^\vee_j] = 0, \quad [\alpha^\vee_i, e_j] = a_{ij} e_j, \quad [\alpha^\vee_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{i,j} \alpha^\vee_j, \]
\[ [d, \alpha^\vee_j] = 0, \quad [d, e_i] = \delta_{i,0} e_0, \quad [d, f_i] = -\delta_{i,0} f_0. \]

for \( i, j = 0, \ldots, 6 \). We denote the Cartan subalgebra of \( \mathfrak{g} \) by
\[ \mathfrak{h} = \bigoplus_{j=0}^6 \mathbb{C} \alpha^\vee_j \oplus \mathbb{C} d. \]

The canonical central element of \( \mathfrak{g} \) is given by
\[ K = \alpha^\vee_0 + \alpha^\vee_1 + 2 \alpha^\vee_2 + 3 \alpha^\vee_3 + 2 \alpha^\vee_4 + \alpha^\vee_5 + 2 \alpha^\vee_6. \]

The normalized invariant form \((\ , \ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}\) is determined by the conditions
\[ (\alpha^\vee_i | \alpha^\vee_j) = a_{ij}, \quad (e_i | f_j) = \delta_{i,j}, \quad (\alpha^\vee_i | e_j) = (\alpha^\vee_i | f_j) = 0, \]
\[ (d | d) = 0, \quad (d | \alpha^\vee_j) = \delta_{0,j}, \quad (d | e_j) = (d | f_j) = 0, \]
for \( i, j = 0, \ldots, 6 \).

Consider the gradation \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \) of type \( (1, 1, 0, 1, 0, 1, 0) \) by setting
\[ \text{deg } \mathfrak{h} = \text{deg } e_i = \text{deg } f_i = 0 \quad (i = 2, 4, 6), \]
\[ \text{deg } e_i = 1, \quad \text{deg } f_i = -1 \quad (i = 0, 1, 3, 5). \]
With an element $\vartheta \in \mathfrak{h}$ such that
\[
(\vartheta | \alpha^i) = \begin{cases} 0 & (i = 2, 4, 6), \\ 1 & (i = 0, 1, 3, 5), \end{cases}
\]
this gradation is defined by
\[
g_k = \{ x \in g | [\vartheta, x] = kx \} \quad (k \in \mathbb{Z}).
\]
Note that $\vartheta$ is given explicitly by
\[
\vartheta = 6d_1 + 4d_2 + 7d_3 + 10d_4 + 10d_5 + 10d_6 + 4d_7 + 5d_8.
\]
We denote by
\[
\mathfrak{g}_{<0} = \bigoplus_{k < 0} \mathfrak{g}_k, \quad \mathfrak{g}_{\geq 0} = \bigoplus_{k \geq 0} \mathfrak{g}_k.
\]
Such gradation implies the Heisenberg subalgebra of $\mathfrak{g}$
\[
\mathfrak{s} = \{ x \in g | [x, \Lambda_1] = \mathbb{C}K \},
\]
with an element of $\mathfrak{g}_1$
\[
\Lambda_1 = e_1 + 2e_3 + e_5 + e_{21} + e_{60} + e_{23} + e_{234} + e_{236} + e_{436} + 2e_{634},
\]
where
\[
e_{i_1i_2,...,i_j} = \text{ade}_{i_1}\text{ade}_{i_2}, \ldots, \text{ade}_{i_j}(e_j).
\]
Note that $\mathfrak{s}$ admits the gradation of type $(1, 1, 0, 1, 0, 1, 0)$, namely
\[
\mathfrak{s} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, \quad \mathfrak{g}_k \subset \mathfrak{g}_k.
\]
We also remark that the positive part of $\mathfrak{s}$ has a graded base $\{\Lambda_k\}_{k=1}^{\infty}$ satisfying
\[
[\Lambda_k, \Lambda_l] = 0, \quad [\vartheta, \Lambda_k] = n_k \Lambda_k \quad (k, l = 1, 2, \ldots),
\]
where $n_k$ stands for the degree of element $\Lambda_k$ defined by
\[
n_{6k+1} = 6k + 1, \quad n_{6k+2} = 6k + 1, \quad n_{6k+3} = 6k + 2,
\]
\[
n_{6k+4} = 6k + 4, \quad n_{6k+5} = 6k + 5, \quad n_{6k+6} = 6k + 5.
\]
We formulate the Drinfeld–Sokolov hierarchy of type $E_6^{(1)}$ associated with the Heisenberg subalgebra $\mathfrak{s}$ by using these $\Lambda_k$ in the following section.

**Remark 2.1.** The isomorphism classes of the Heisenberg subalgebras are in one-to-one correspondence with the conjugacy classes of the finite Weyl group $[KP]$. In the notation of $[C]$, the Heisenberg subalgebra $\mathfrak{s}$ introduced above corresponds to the regular primitive conjugacy class $E_6(a_2)$ of the Weyl group $W(E_6)$; see $[DF]$.

### 3. Drinfeld–Sokolov hierarchy

In this section, we formulate a similarity reduction of a Drinfeld–Sokolov hierarchy of type $E_6^{(1)}$ associated with the Heisenberg subalgebra $\mathfrak{s}$.

In the following, we use the notation of infinite dimensional groups
\[
G_{<0} = \exp(\hat{\mathfrak{g}}_{<0}), \quad G_{\geq 0} = \exp(\hat{\mathfrak{g}}_{\geq 0}),
\]
where $\hat{\mathfrak{g}}_{<0}$ and $\hat{\mathfrak{g}}_{\geq 0}$ are the completions of $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{\geq 0}$, respectively.
Let $X(0) \in G_{>0}G_{\geq 0}$. Introducing the time variables $t_k (k = 1, 2, \ldots)$, we consider a $G_{>0}G_{\geq 0}$-valued function

$$X = X(t_1, t_2, \ldots) = \exp \left( \sum_{k=1}^{\infty} t_k \Lambda_k \right) X(0).$$

Then we have a system of partial differential equations

$$X \partial_k X^{-1} = \partial_k - \Lambda_k \quad (k = 1, 2, \ldots), \quad (3.1)$$

where $\partial_k = \partial / \partial t_k$, defined through the adjoint action of $G_{>0}G_{\geq 0}$ on $\hat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geq 0}$. Via the decomposition

$$X = W^{-1}Z, \quad W \in G_{<0}, \quad Z \in G_{\geq 0},$$

the system (3.1) implies a system of partial differential equations

$$\partial_k - B_k = W(\partial_k - \Lambda_k)W^{-1} \quad (k = 1, 2, \ldots), \quad (3.2)$$

where $B_k$ stands for the $\mathfrak{g}_{\geq 0}$-component of $WA_kW^{-1} \in \hat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geq 0}$. The Zakharov–Shabat equations,

$$[\partial_k - B_k, \partial_l - B_l] = 0 \quad (k, l = 1, 2, \ldots), \quad (3.3)$$

follow from the system (3.2).

Under the system (3.2), we consider the operator

$$\mathcal{M} = W \exp \left( \sum_{k=1}^{\infty} t_k \Lambda_k \right) \vartheta \exp \left( - \sum_{k=1}^{\infty} t_k \Lambda_k \right) W^{-1}.$$

Then the operator $\mathcal{M}$ satisfies

$$[\partial_k - B_k, \mathcal{M}] = 0 \quad (k = 1, 2, \ldots). \quad (3.4)$$

Note that

$$\mathcal{M} = W \vartheta W^{-1} - \sum_{k=1}^{\infty} n_k t_k W \Lambda_k W^{-1}. \quad (3.5)$$

Now we require that the similarity condition $\mathcal{M} \in \mathfrak{g}_{>0}$ be satisfied. Then we have

$$\mathcal{M} = \vartheta - \sum_{k=1}^{\infty} n_k t_k B_k.$$

We also assume that $t_k = 0$ for $k \geq 3$. Then systems (3.3) and (3.4) are equivalent to

$$[\partial_1 - B_1, \partial_2 - B_2] = 0,$n_k t_k W \Lambda_k W^{-1}.

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We also assume that $t_k = 0$ for $k \geq 3$. Then systems (3.3) and (3.4) are equivalent to

$$[\partial_1 - B_1, \partial_2 - B_2] = 0,$n_k t_k W \Lambda_k W^{-1}.$$
In this section, we derive the Hamiltonian system (1)

\[ \Lambda_1 = e_1 + 2e_3 + e_5 + e_{21} + e_{60} + e_{23} + e_{43} + e_{234} + e_{236} + e_{436} + 2e_{6234}, \]

\[ \Lambda_2 = 2e_0 - 2e_3 - 2e_5 - 2e_{21} - 2e_{45} + 2e_{23} + 2e_{43} - 7e_{63} - 4e_{234} + 5e_{236} - 4e_{436} - 2e_{6234}. \]

In the following, we use the notation of a \( \mathfrak{g}_{>0} \)-valued 1-form \( B = B_1 dt_1 + B_2 dt_2 \) with respect to the coordinates \( t = (t_1, t_2) \). Then the similarity reduction (3.5) is expressed as

\[ d_t M = [B, M], \quad d_t B = B \wedge B, \]

where \( d_t \) stands for an exterior differentiation with respect to \( t \). Denoting by

\[ M_1 = -t_1 \Lambda_1 - t_2 \Lambda_2, \quad B_1 = \Lambda_1 dt_1 + \Lambda_2 dt_2, \]

we can express the operators \( M \) and \( B \) in the form

\[ M = \theta + \sum_{i=2,4,6} \xi_i e_i + \sum_{i=2,4,6} \psi_i f_i + M_1, \]

\[ B = u + \sum_{i=2,4,6} x_i e_i + \sum_{i=2,4,6} y_i f_i + B_1, \]

where

\[ \theta = \vartheta + \sum_{i=0}^6 \theta_i \alpha^i, \quad u = \sum_{i=0}^6 u_i \alpha^i. \]

The system (3.7) is expressed in terms of these variables as follows:

\[ d_t \theta_i = x_i \psi_i - y_i \xi_i, \quad d_t \theta_j = 0, \]

\[ d_t \xi_i = \left( u \alpha^i \right) \xi_i - x_i \left( \theta \alpha^i \right), \]

\[ d_t \psi_i = -\left( u \alpha^i \right) \psi_i + y_i \left( \theta \alpha^i \right) \]

and

\[ d_t u_i = x_i \wedge y_i + y_i \wedge x_i, \quad d_t u_j = 0, \]

\[ d_t x_i = \left( u \alpha^i \right) \wedge x_i, \quad d_t y_i = -\left( u \alpha^i \right) \wedge y_i, \]

for \( i = 2, 4, 6 \) and \( j = 0, 1, 3, 5 \).

In this section, we proposed three representations (3.5), (3.6) and (3.7) of the similarity reduction. In the following, we use the system (3.7) in order to derive the system (1.2).

**4. Derivation of coupled \( P_{\nu 1} \)**

In this section, we derive the Hamiltonian system (1.2) from the similarity reduction (3.7). Let \( n_+ \) be the subalgebra of \( \mathfrak{g} \) generated by \( e_i (i = 0, \ldots, 6) \) and \( b_+ = \mathfrak{h} \oplus n_+ \) the Borel subalgebra of \( \mathfrak{g} \). We introduce below a gauge transformation for the system (3.7)

\[ \mathcal{M}^+ = \exp(\text{ad}(\Gamma)) M, \quad d_t B^+ = \exp(\text{ad}(\Gamma))(d_t B), \]

with \( \Gamma \in \mathfrak{g}_{>0} \) such that \( \mathcal{M}^+ \) and \( B^+ \) should take values in \( b_+ \).

We first consider a gauge transformation

\[ \mathcal{M}^+ = \exp(\text{ad}(\Gamma_1)) M, \quad d_t B^+ = \exp(\text{ad}(\Gamma_1))(d_t B), \]

with \( \Gamma_1 \in \mathfrak{g}_{>0} \cap b_+ \) such that

\[ \exp(\text{ad}(\Gamma_1))(M_1) = \sum_{i=0,1,3,5} c_i e_i + e_{21} + e_{45} + e_{60} + e_{23} + e_{43} + c_6 e_{63} + e_{234}. \]
Note that \( c_0, c_1, c_3, c_5 \) and \( c_{63} \) are algebraic functions in \( t_1 \) and \( t_2 \). Then we have
\[
d_t \mathcal{M}^* = [B^*, \mathcal{M}^*], \quad d_t B^* = B^* \wedge B^*.
\] (4.1)

With the notation
\[
\mathcal{M}_t^* = \exp(\text{ad}(\Gamma_1))(\mathcal{M}_1), \quad B_t^* = \exp(\text{ad}(\Gamma_1))(B_1),
\]
the operators \( \mathcal{M}^* \) and \( B^* \) are expressed in the form
\[
\mathcal{M}^* = \theta^* + \sum_{i=2,4,6} \xi_i^* e_i + \sum_{i=2,4,6} \psi_i^* f_i + \mathcal{M}_1^*, \quad
B^* = \mu^* + \sum_{i=2,4,6} x_i^* e_i + \sum_{i=2,4,6} y_i^* f_i + B_1^*,
\]
where
\[
\theta^* = \theta + \sum_{i=0}^6 \theta_i^* a_i^*, \quad \mu^* = \sum_{i=0}^6 \mu_i^* a_i^*.
\]

We next consider a gauge transformation
\[
\mathcal{M}^* = \exp(\text{ad}(\Gamma_2))(\mathcal{M}^*), \quad d_t B^* = \exp(\text{ad}(\Gamma_2))(d_t - B^*),
\]
with \( \Gamma_2 = \sum_{i=2,4,6} \lambda_i f_i \) such that \( \mathcal{M}^*, B^* \in \mathfrak{b}^+ \), namely
\[
\xi_i^* \lambda_i^2 - (\theta^*|a_i^*) \lambda_i - \psi_i^* = 0 \quad (i = 2, 4, 6)
\] (4.2)

and
\[
d_t \lambda_i = x_i^* \lambda_i^2 - (\mu^*|a_i^*) \lambda_i - y_i^* \quad (i = 2, 4, 6).
\] (4.3)

Here we have

**Lemma 4.1.** Under the system (4.1), equation (4.3) follows from equation (4.2).

**Proof.** The system (4.1) can be expressed as
\[
d_t \theta_i^* = x_i^* \psi_i^* - y_i^* \xi_i^*, \quad d_t \theta_i^* = 0, \quad
\]
\[
d_t \xi_i^* = (\mu^*|a_i^*) \xi_i^* - x_i^* (\theta^*|a_i^*),
\]
\[
d_t \psi_i^* = - (\mu^*|a_i^*) \psi_i^* + y_i^* (\theta^*|a_i^*),
\] (4.4)

for \( i = 2, 4, 6 \) and \( j = 0, 1, 3, 5 \). By using (4.4) and \( (d_t \theta^*|a_i^*) = 2d_t \theta_i^* \), we obtain
\[
d_t (\xi_i^* \lambda_i^2 - (\theta^*|a_i^*) \lambda_i - \psi_i^*) = \left\{ 2 \xi_i^* \lambda_i - (\theta^*|a_i^*) \right\} \left\{ d_t \lambda_i - x_i^* \lambda_i^2 + (\mu^*|a_i^*) \lambda_i + y_i^* \right\}
\]
\[
(i = 2, 4, 6).
\]

It follows that equation (4.2) implies (4.3) or
\[
\lambda_i = \frac{(\theta^*|a_i^*)}{2 \xi_i^*} \quad (i = 2, 4, 6).
\] (4.5)

Hence, it is enough to verify that equation (4.3) follows from (4.5). Together with (4.4), equation (4.5) implies
\[
d_t \lambda_i = \frac{(d_t \theta^*|a_i^*) \xi_i^* - (\theta^*|a_i^*) d_t \xi_i^*)}{2 (\xi_i^*)^2}
\]
\[
= x_i^* \lambda_i^2 - (\mu^*|a_i^*) \lambda_i - y_i^* + \frac{x_i^* \left\{ 4 \xi_i^* \psi_i^* + (\theta^*|a_i^*) \right\}}{4 (\xi_i^*)^2}.
\] (4.6)
On the other hand, we obtain
\[ 4\xi_i^* \theta_i^* + (\theta^* [\alpha_i^*])^2 = 0 \] (4.7)
by substituting (4.5) into (4.2). Combining (4.6) and (4.7), we obtain equation (4.3).

Thanks to lemma 4.1, the gauge parameters \( \lambda_i (i = 2, 4, 6) \) are determined by equation (4.2). Hence we obtain the system on \( b_* \)
\[ d_t M^* = [B^*, M^*], \quad d_t B^* = B^* \land B^*, \] (4.8)
with dependent variables \( \lambda_i \) and \( \mu_i = \xi_i^* (i = 2, 4, 6) \). The operator \( M^+ \) is described as
\[ M^+ = 1 + \sum_{i=2,4,6} \mu_i e_i + (c_0 + \lambda_6) e_0 + (c_1 + \lambda_2) e_1 + (c_3 + \lambda_4 + c_63 \lambda_6 - \lambda_2 \lambda_4) e_3 \]
\[ + (c_5 + \lambda_4) e_5 + e_{21} + e_{45} + e_{60} + (1 - \lambda_4) e_{23} + (1 - \lambda_2) e_{43} + c_63 e_{63} + e_{234}, \]
where \( \kappa \in \mathfrak{h} \). Note that \( d_t \kappa = 0 \).
Let \( s_1 \) and \( s_2 \) be independent variables defined by
\[ s_1 = \frac{c_{63}(1 + c_3 - c_0 c_{63})}{6}, \quad s_2 = \frac{c_{63}(1 + c_1)(1 + c_3)}{6}. \]
We now regard the system (4.8) as a system of ordinary differential equations
\[ s(s - 1) \frac{d}{ds} - B, M^* \right] = 0, \] (4.9)
with respect to the independent variable \( s = s_1 \) by setting \( s_2 = 1 \). The operator \( B \) is expressed in the form
\[ B = \sum_{i=0}^{6} u_i \alpha_i^* + \sum_{i=0}^{6} x_i e_i + x_{21} e_{21} + x_{45} e_{45} + x_{23} e_{23} + x_{43} e_{43} \]
\[ + x_{63} e_{63} + x_{234} e_{234} + x_{236} e_{236} + x_{436} e_{436} + x_{634} e_{634}. \]
Each coefficient of \( B \) is a polynomial in \( \lambda_i \) and \( \mu_i \); we do not give the explicit formula.
Let \( q_i, p_i (i = 1, 2, 3) \) be dependent variables defined by
\[ q_1 = \frac{1 - \lambda_2}{1 + c_1}, \quad q_2 = \frac{1 - \lambda_4}{1 + c_5}, \quad q_3 = \frac{1 + c_3 - c_0 c_{63}}{1 + c_3 + c_{63} \lambda_6}, \]
\[ p_1 = \frac{(1 + c_1) \mu_2}{6}, \quad p_2 = -\frac{(1 + c_5) \mu_4}{6}, \]
\[ p_3 = \frac{(1 + c_3 + c_{63} \lambda_6) \mu_6 + c_{63}(\kappa [\alpha_i^*])}{6 c_{63}(1 + c_3 - c_0 c_{63})}. \] (4.10)
We also set
\[ \alpha_i = \frac{k [\alpha_i^*]}{6} (i = 0, \ldots, 6). \]
Then we obtain

**Theorem 4.2.** The system (4.9) is equivalent to the system (1.2) with (1.1).

**Remark 4.3.** The system (1.2) with (1.1) can be regarded as the compatibility condition of a
Lax pair
\[ M^+ w = 0, \quad s(s - 1) \frac{dw}{ds} = B w, \] (4.11)
where \( w = \exp(\Gamma)W \exp\left(\sum_{k=1}^{\infty} t_k \Lambda_k\right) \). On the other hand, the affine Lie algebra \( \mathfrak{g}(E_n) \) is realized as a central extension of the loop algebra \( \mathfrak{g}(E_6[z, z^{-1}]) \) with a derivation \( zd/dz \). In this framework, the system (4.11) can be identified with a Lax pair

\[
\frac{dz}{dw} = Mw, \quad s(s-1)\frac{dw}{ds} = Bw,
\]

where \( M = (6d - M^+)/6 \).

Lastly, we note a derivation of the affine Weyl group symmetry for the system (1.2). We define a Poisson structure for the \( \mathfrak{b}_+ \)-valued operator \( M^+ \) by

\[
\{\mu_i, \lambda_j\} = 6\delta_{i,j}, \quad \{\mu_i, \mu_j\} = \{\lambda_i, \lambda_j\} = 0 \quad (i, j = 2, 4, 6).
\]

It is equivalent to

\[
\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, 2, 3),
\]

via the transformation (4.10). Hence \( p_i, q_j (i = 1, 2, 3) \) give a canonical coordinate system associated with the Poisson structure for \( M^+ \).

Thanks to [NY2], we then obtain birational canonical transformations \( r_i (i = 0, \ldots, 6) \) given in theorem 1.1. They are derived from the transformations

\[
r_i(X) = X \exp(-e_i) \exp(f_i) \exp(-e_i) \quad (i = 0, \ldots, 6),
\]

where \( X = \exp\left(\sum_{k=1}^{\infty} t_k \Lambda_k\right)X(0) \).

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References

[AS] Ablowitz M J and Segur H 1977 Exact linearization of a Painlevé transcendent Phys. Rev. Lett. 38 1103–6
[C] Carter R 1972 Conjugacy classes in the Weyl group Compos. Math. 25 1–59
[DF] Delduc F and Fehér L 1995 Regular conjugacy classes in the Weyl group and integral hierarchies J. Phys. A. Math. Gen. 28 5843–82
[DS] Drinfel’d V G and Sokolov V V 1985 Lie algebras and equations of Korteweg–de Vries type J. Phys. Math. 30 1975–2036
[FS1] Fuji K and Suzuki T 2006 The sixth Painlevé equation arising from \( D(1)_{4} \) hierarchy J. Phys. A: Math. Gen. 39 12073–82
[FS2] Fuji K and Suzuki T 2008 Higher order Painlevé system of type \( D(3)_{1} \) arising from integrable hierarchy Int. Math. Res. Not. 1 1–21
[IKSY] Iwasaki K, Kimura H, Shimomura S and Yoshida M 1991 From Gauss to Painlevé—a modern theory of special functions Aspects of Mathematics vol E16 (Braunschweig: Vieweg)
[Kac] Kac V G 1990 Infinite Dimensional Lie Algebras (Cambridge: Cambridge University Press)
[KIK] Kikuchi T, Ieda T and Kakei S 2003 Similiarity reduction of the modified Yajima–Oikawa equation J. Phys. A. Math. Gen. 36 11465–80
[KK1] Kakei S and Kikuchi T 2004 Affine Lie group approach to a derivative nonlinear Schrödinger equation and its similarity reduction Int. Math. Res. Not. 78 4181–209
[KK2] Kakei S and Kikuchi T 2007 The sixth Painlevé equation as similarity reduction of \( \hat{\mathfrak{gl}}_3 \) hierarchy Lett. Math. Phys. 79 221–34
[KP] Kac V G and Peterson D 1985 112 Constructions of the basic representation of the Roop Group of \( E_8 \) Symp. on Anomalies, Geometry and Topology ed W A Baedeen and A R White (Singapore: World Scientific) pp 276–98
[NY1] Noumi M and Yamada Y 1998 Higher order Painlevé equations of type \( A(1)_{1} \) Funkcial. Ekvac. 41 483–503
[NY2] Noumi M and Yamada Y 2001 Birational Weyl group action arising from a nilpotent Poisson algebra Physics and Combinatorics 1999 Proc. Nagoya 1999 Int. Workshop ed A N Kirillov, A Tsuchiya and H Umemura (Singapore: World Scientific) pp 287–319

[O1] Okamoto K 1987 Studies on the Painlevé equations: I Ann. Math. Pura Appl. 146 337–81
  Okamoto K 1987 Studies on the Painlevé equations: II Japan. J. Math. 13 47–76
  Okamoto K 1986 Studies on the Painlevé equations: III Math. Ann. 275 221–56
  Okamoto K 1987 Studies on the Painlevé equations: IV Funkcial. Ekvac. 30 305–32

[O2] Okamoto K 1999 The Hamiltonians associated with the Painlevé equations The Painlevé Property: One Century Later (CRM Series in Mathematical Physics) ed R Conte (Berlin: Springer)

[S] Sasano Y 2006 Higher-order Painlevé equations of type $D_{1}^{(1)}$ RIMS Koukyuroku 1473 143–63