Sharp bounds on the compactness of relativistic charged spheres

Håkan Andrénsson
Mathematical Sciences, University of Gothenburg
Mathematical Sciences, Chalmers University of Technology, S-41296 Göteborg, Sweden

Abstract. The problem of finding an upper bound on the mass $M$ of a charged spherically symmetric static object with area radius $R$ and charge $Q < M$ is addressed. This problem has resulted in a number of papers in recent years but neither a transparent nor a general inequality similar to the case without charge, i.e., $M \leq 4R/9$, has been found. Here, we discuss the surprisingly transparent inequality

$$\sqrt{M} \leq \sqrt{\frac{R}{3}} + \sqrt{\frac{R}{9} + \frac{Q^2}{3R}}.$$  

The inequality holds for any solution which satisfies $p + 2p_T \leq \rho$, where $p \geq 0$ and $p_T$ are the radial- and tangential pressures respectively and $\rho \geq 0$ is the energy density. Moreover, the inequality is sharp and in particular, sharpness holds for infinitely thin shell solutions.

1. Introduction

Black holes for which the charge or angular momentum parameter equals the mass are called extremal black holes. They are very central in black hole thermodynamics due to their vanishing surface gravity and they represent the absolute zero state of black hole physics. It is quite generally believed that extremal black holes are disallowed by nature but a proof is missing. One possibility to obtain an extremal black hole is to produce one from the collapse of an already extremal object. Previous mainly numerical studies [5, 8] have concluded that when $Q < M$ collapse always takes place at a critical stability radius $R_c$ outside the outer horizon, and as $Q$ approaches $M$, this value approaches the horizon. This is similar to the non-charged case where the Buchdahl inequality implies that collapse will take place when $R < 9M/4$, i.e., $R_c = 9M/4$, cf. [7]. In the charged case the critical value is expected to be smaller due to the Coulomb repulsion, and in particular as $Q \to M$ the stability radius does approach the outer horizon. For more information on the relation of this topic to black hole thermodynamics and extremal black holes we refer to [5], [9] and [12] and the references therein.

The problem of finding a similar bound as the classical Buchdahl bound for charged objects have resulted in several papers; some of these are analytical, cf. [12], [15], [9], [10], [13] and [16], whereas others are numerical or use a mix of numerical and analytical arguments, cf. [5], [8], and [11] to mention some of them. We refer the reader to the sources for the details of these studies but in none of them a transparent bound has been obtained (except in very special cases), on the contrary they have been quite involved and implicit. Moreover, most of these studies rely on the assumptions made by Buchdal, i.e., the energy density is assumed to be non-increasing.
and the pressure to be isotropic. In [1] the inequality
\[ \sqrt{M} \leq \frac{\sqrt{R}}{3} + \sqrt{\frac{R}{9} + \frac{Q^2}{3R}}, \]  

is shown to hold whenever \( Q < R \) and \( p + 2p_T \leq \rho \), where \( p \geq 0 \) and \( p_T \) are the radial- and tangential pressures respectively and \( \rho \geq 0 \) is the energy density. Moreover, the inequality is shown to be sharp.

In the non-charged case a general proof of the Buchdahl inequality \( 2m/r \leq 8/9 \), in the case when \( p + 2p_T \leq \rho \), was first given in [2]. A completely different proof was then given by Stalker and Karageorgis [14] where also several other situations were considered, e.g. the isotropic case where \( p = p_T \). The advantage of the method in [14] (which is related to the method by Bondi [6] which however is non-rigorous) compared to the method in [2] is that it is shorter and that it is more flexible in the sense that other assumptions than \( p + 2p_T \leq \rho \) can be treated. On the other hand the result in [14] is weaker than the result in [2] in the sense that the latter method implies that the steady state that saturates the inequality is unique, it is an infinitely thin shell.

The method in [14] also shows sharpness but only in the sense that there are steady states with \( 2m/r \) arbitrary close to \( 8/9 \), leaving open the possibility that different kinds of steady states might share this feature. Moreover, since the assumption \( p + 2p_T \leq \rho \) is satisfied by solutions of the Einstein-Vlasov system it is natural to ask if there exist regular static solutions to the coupled system which can have \( 2m/r \) arbitrary close to \( 8/9 \). This question is given an affirmative answer in the non-charged case, cf. [3], where in particular it is shown that arbitrary thin shells which are regular solutions of the spherically symmetric Einstein-Vlasov system do exist. On the contrary, the matter quantities and the corresponding spacetimes constructed in [14] for showing sharpness cannot be realized by regular solutions of the Einstein-Vlasov system. The construction in [14] gives that a solution which nearly saturates the inequality \( 2m/r \leq 8/9 \) satisfies \( p + 2p_T = \rho \), and in addition \( p_T \) and \( \rho \) are discontinuous. Neither of these two properties can be realized by regular solutions of the (massive) Einstein-Vlasov system. In the charged case there are no rigorous studies of solutions of the Einstein-Maxwell-Vlasov system but in [4] numerical solutions are constructed to this system which have the property that they satisfy the inequality (1) and that the difference of the left- and right hand sides in the inequality can be as small as one wishes.

The proof of the inequality (1) given in [1] is an adaption of the method in [14] to the charged case and we refer to [1] for the mathematical formulation and for the details of the proof. It should be stressed that the inequality (1) also holds anywhere inside the static object, cf. [1]. In [1] a proof is also given that infinitely thin shell solutions saturate the inequality.

References
[1] Andréasson H 2008 Sharp bounds on the critical stability radius for relativistic charged spheres Preprint arXiv:0804.1882
[2] Andréasson H 2008 J. Diff. Equations 245 2243
[3] Andréasson H 2007 Commun. Math. Phys. 274 409
[4] Andréasson H and Eklund M 2009 A numerica investigation of the steady states of the spherically symmetric Einstein-Vlasov-Maxwell (in preparation)
[5] Anninos P and Rothman T 2001 Phys. Rev. D, 62 024003
[6] Bondi H 1964 Proc. R. Soc. A 282 303
[7] Buchdahl H A 1959 Phys. Rev. 116, 1027
[8] de Felice F, Siming L and Yunqiang Y 1999 Class. Quantum Grav. 16, 2669
[9] Farrugia C J and Hajicek P 1979 Commun. Math. Phys. 68, 291
[10] Fayos F, Senovilla J M M, and Torres R 2003 Class. Quantum Grav. 20 2579
[11] Ghezzi C R 2005 Phys. Rev. D 72, 104017
[12] Giuliani A and Rothman T 2007 Gen. Rel. Gravitation DOI 10.1007/s10714-007-0539-7
[13] Harko T and Mak M K 2000 J. Math. Phys. 41 4752
[14] Karageorgis P and Stalker J 2008 Class. Quantum Grav. 25 195021
[15] Mak M K, Dobson P N and Harko T 2001 Europhys. Lett. 55, 310
[16] Yunqiang Y and Siming L 2000 Commun. Theor. Phys. 33, 571