ALGEBRAIC SOLUTIONS OF DIFFERENTIAL EQUATIONS OVER $\mathbb{P}^1 - \{0, 1, \infty\}$

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Abstract: The Grothendieck–Katz $p$-curvature conjecture predicts that an arithmetic differential equation whose reduction modulo $p$ has vanishing $p$-curvatures for almost all $p$, has finite monodromy. It is known that it suffices to prove the conjecture for differential equations on $\mathbb{P}^1 - \{0, 1, \infty\}$. We prove a variant of this conjecture for $\mathbb{P}^1 - \{0, 1, \infty\}$, which asserts that if the equation satisfies a certain convergence condition for all $p$, then its monodromy is trivial. For those $p$ for which the $p$-curvature makes sense, its vanishing implies our condition. We deduce from this a description of the differential Galois group of the equation in terms of $p$-curvatures and certain local monodromy groups.

1. Introduction

The Grothendieck–Katz $p$-curvature conjecture was originally raised as a question on linear homogeneous systems of first-order differential equations (see Conjecture (I) in [Kat72, Introduction] for more details)

$$\frac{dy}{dx} = A(x)y.$$ 

Here $A(x)$ is a square matrix of rational functions of $x$ with coefficients in some number field $K$ and $y$ is a vector-valued function. For all but finitely many primes $p$ of $K$, it makes sense to reduce this system modulo $p$ and to define an invariant, the $p$-curvature, in terms of the resulting system. According to the conjecture, if almost all (that is, all but finitely many) $p$-curvatures vanish, then the original system admits a full set of solutions in algebraic functions.

The conjecture generalizes to a smooth variety $X$ equipped with a vector bundle with an integrable connection $(M, \nabla)$ defined over some number field $K$. It is known that the general version of the conjecture reduces to the case when $X = \mathbb{P}^1_K - \{0, 1, \infty\}$. (See [Bos01], [Kat82], Thm. 10.5, and [And04], 7.1.4).

In this paper, we prove a variant of the conjecture for $X = \mathbb{P}^1_K - \{0, 1, \infty\}$ where the condition for almost all $p$ is replaced by a condition for all $p$. A slightly informal formulation of our main theorem is the following.

Theorem. (Theorem 2.2.1) Let $(M, \nabla)$ a vector bundle with a connection over $X = \mathbb{P}^1_K - \{0, 1, \infty\}$. If the $p$-curvature of $(M, \nabla)$ vanishes for all $p$, then $(M, \nabla)$ admits a full set of rational solutions, that is, $M^{\nabla=0}$ generates $M$ as an $\mathcal{O}_X$-module.

Let us explain the meaning of the condition of vanishing $p$-curvature at all primes $p$: at primes where $p$-curvature is either not defined or non-vanishing, we impose a condition on the $p$-adic radius of convergence of the parallel sections of $(M, \nabla)$. When $(M, \nabla)$ has an integral model at a prime $p$ so that one can make sense of
its reduction mod $p$, this convergence condition is implied by the vanishing of the $p$-curvature.

Katz has shown in [Kat82, Thm. 10.2] that if the $p$-curvature conjecture holds, then for any vector bundle with an integrable connection $(M, \nabla)$ on a smooth variety $X$ over $K$ as above, the Lie algebra $\mathfrak{g}_{\text{gal}}$ of the differential Galois group $G_{\text{gal}}$ of $(M, \nabla)$ is in some sense generated by the $p$-curvatures. Namely, let $K(X)$ be the function field of $X$. The $p$-curvature conjecture implies that $\mathfrak{g}_{\text{gal}}$ is the smallest algebraic Lie subalgebra of $\mathfrak{g}_n(K(X))$ such that for almost all $p$ the reduction of $\mathfrak{g}_{\text{gal}}$ mod $p$ contains the $p$-curvature.

We use Theorem 2.2.1 to prove a result analogous to Katz’s theorem when $X = \mathbb{P}^1_K - \{0, 1, \infty\}$. Of course, this result (Theorem 2.2.5) involves a condition at every prime $p$, but as a compensation we describe $G_{\text{gal}}$ and not only its Lie algebra.

The main tools used to prove Theorem 2.2.1 are the algebraicity results of André [And04, Thm. 5.4.3] and Bost–Chambert-Loir [BCL09, Thm. 6.1, Thm. 7.8]. These results generalize the classical Borel–Dwork criterion for the rationality of a formal power series. This type of results requires estimating the radius of convergence of solutions for $(M, \nabla)$ at each place of $K$. These techniques have been used previously by André [And04, Sec. 6] and Bost [Bos01, 2.4.2] to study the Grothendieck–Katz conjecture in the case when the algebraic monodromy group of $(M, \nabla)$ is solvable.

The paper is organized as follows. In section 2, we formulate our main result, and in particular the condition which substitutes for the vanishing of the $p$-curvature when it does not make sense to reduce $(M, \nabla)$ mod $p$. We then use the main result to deduce a description of the differential Galois group following Katz.

In section 3, we use the criterion in [And04] to prove that a vector bundle with a connection $(M, \nabla)$, as in the theorem, is locally trivial for the étale topology of $X$. To do this, we apply André’s criterion to the formal horizontal sections of $(M, \nabla)$ centered at a specific point $x_0$. We obtain a lower bound for André’s analogue of their radii of convergence at archimedean places, using the uniformization of $\mathbb{P}^1_\mathbb{C} - \{0, 1, \infty\}$ by the unit disc, which arises from its interpretation as the moduli space of elliptic curves with level 2 structure. The chosen point $x_0$ corresponds to the elliptic curve with smallest stable Faltings’ height and we use the Chowla–Selberg formula to deduce the lower bound.

In section 4, we apply the rationality criterion in [BCL09] to prove the main theorem. We give a lower bound for the local capacity of $\Omega$, the image in $\mathbb{P}^1_\mathbb{C}$ of a standard fundamental domain for $\Gamma(2)$ under the uniformization mentioned above. Together with the algebraicity of our formal solution proved in section 3, this allows us to apply the criterion in [BCL09], and deduce that the solutions of $(M, \nabla)$ are rational.

Section 5 is devoted to an interpretation of our computations in section 3 in terms of the stable Faltings height, obtained by relating our estimate for archimedean places to the Arakelov degree of the restriction of the tangent bundle to some point.

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2. Statement of the main results

Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. Let $X \subset \mathbb{P}^1_{\mathcal{O}_K} - \{0, 1, \infty\}$ and $M$ a vector bundle with a connection $\nabla: M \rightarrow \Omega^1_{X_K} \otimes M$ over $X_K$. For a finite place $v$ of $K$ lying over a prime $p$, let $K_v$ be the completion of $K$ with respect to $v$ and denote by $\mathcal{O}_v$ and $k_v$ the ring of integers and residue field of $K_v$. For $\Sigma$ a finite set of finite rational primes, we set $\mathcal{O}_{K, \Sigma} = \mathcal{O}_K[1/p]_{p \in \Sigma} \subset K$.

2.1. The $p$-curvature and $p$-adic differential Galois groups.

2.1.1. For $\Sigma$, as above, sufficiently large, $(M, \nabla)$ extends to a vector bundle with connection (again denoted $(M, \nabla)$) over $X_{\mathcal{O}_{K, \Sigma}}$. In particular, if $p \notin \Sigma$ we can consider the pull back of $(M, \nabla)$ to $X \otimes \mathbb{Z}/p\mathbb{Z}$. If $D$ is a derivation on $X \otimes \mathbb{Z}/p\mathbb{Z}$, so is $D^p$. Let $\nabla(D)$ be the map $(D \otimes \text{id}) \circ \nabla$. Then on $X \otimes \mathbb{Z}/p\mathbb{Z}$, the $p$-curvature is given by (see 
Kat82, Sec. VII) for details) \[ \psi_p(D) := \nabla(D^p) - \nabla(D)^p \in \text{End}_{\mathcal{O}_{X \otimes \mathbb{Z}/p\mathbb{Z}}}(M \otimes \mathbb{Z}/p\mathbb{Z}). \]

In particular, $\psi_p\left(\frac{d}{dx}\right) = -\left(\nabla\left(\frac{d}{dx}\right)^p\right)$. Since $\psi_p(D)$ is $p$-linear in $D$, for $X = \mathbb{P}^1_{\mathcal{O}_K} - \{0, 1, \infty\}$, the equation $\psi_p \equiv 0$ is equivalent to $-\left(\nabla\left(\frac{d}{dx}\right)^p\right) \equiv 0$.

In general, the $\psi_p$ depends on the choice of extension of $(M, \nabla)$ over $X_{\mathcal{O}_{K, \Sigma}}$. However, any two such extensions are isomorphic over $X_{\mathcal{O}_{K, \Sigma}}$, for some sufficiently large $\Sigma'$.

2.1.2. Let $L$ be a finite extension of $K$ and $w$ a place of $L$ over $v$. We view $L$ as a subfield of $\mathbb{C}_p$ via $w$. Fix an $x_0 \in X(L_w)$. Given a positive real number $r$, we denote by $D(x_0, r)$ the open rigid analytic disc of radius $r$, with center $x_0$. Thus

$D(x_0, r) = \{ x \in X(\mathbb{C}_p) \text{ such that } |x - x_0|_p < r \}$,

where $| \cdot |_p$ is normalized so that $|p|_p = p^{-1}$.

Let $M^\vee$ be the dual vector bundle of $M$. It is naturally endowed with the connection such that for any local sections $m, l$ of $M$ and $M^\vee$ respectively,

$\langle d(l, m), m \rangle = \langle \nabla_{M^\vee}(l), m \rangle + \langle l, \nabla_M(m) \rangle$.

Definition 2.1.3. If $(V, \nabla)$ is a vector bundle with connection over some scheme or rigid space, we denote by $(V, \nabla)^\otimes$, or simply $(V)^\otimes$, if there is no risk of confusion regarding the connection $\nabla$, the category of $\nabla$-stable subquotients of all the tensor products $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ for $m, n \geq 0$. This is a Tannakian category.

Definition 2.1.4. Let $F_w$ be the field of fractions of the ring of all rigid analytic functions on $D(x_0, r)$ and $\eta_w: \text{Spec}(F_w) \rightarrow X$ the natural map. Consider the fiber functor

$\eta_w: (M|_{D(x_0, r)})^\otimes \rightarrow \text{Vec}_{F_w}: V \mapsto V_{\eta_w}$.

The $p$-adic differential Galois group $G_w(x_0, r)$ is defined to be the automorphism group $\text{Aut}^p \eta_w$ of $\eta_w$.\footnote{We could have defined the $p$-curvatures by considering derivations on $X_{k_v}$ for $v$ a place of $K$. For primes which are unramified in $K$, the two definitions are essentially equivalent, and the present definition will allow us to formulate the inequalities which arise below in a more uniform manner.}
For \( v|p \) a finite place of \( K \), we will say that \((M, \nabla)\) has good reduction at \( v \) if \((M, \nabla)\) extends to a vector bundle with connection on \( X_{O_v} \). The following lemma gives the basic relation between the \( p \)-curvature and the \( p \)-adic differential Galois group.

**Lemma 2.1.5.** Let \( x_0 \in X(O_{L_v}) \) and suppose that \((M, \nabla)\) has good reduction at \( v \). If the \( p \)-curvature vanishes, then the local differential Galois group \( G_w(x_0, p^{\frac{1}{w(p-1)}}) \) is trivial.

**Proof.** To show that \( G_w(x_0, p^{\frac{1}{w(p-1)}}) \) is trivial, we have to show that the restriction of \( M \) to \( D(x_0, p^{\frac{1}{w(p-1)}}) \) admits a full set of solutions. It is well known that this is the case when \( \psi_w \equiv 0 \), but for the convenience of the reader we sketch the argument. See [Bos01, section 3.4.2, prop. 3.9] for related arguments.

Assume there is an extension of \((M, \nabla)\) to a vector bundle with connection on \( X_{O_v} \). If \( m_0 \) is any section of \( M \), then a formal section in the kernel of \( \nabla \) is given by

\[
m = \sum_{i=0}^{\infty} \nabla \left( \frac{d}{dx} \right)^i (m_0) \frac{(x-x_0)^i}{i!} (-1)^i.
\]

Since \( \psi_w \equiv 0 \) (recall that this means the \( p \)-curvature vanishes on \( X_{O_v} \otimes \mathbb{Z}/p\mathbb{Z} \)), we have \( \nabla \left( \frac{d}{dx} \right)^i (m_0) \subset pM \). Hence \( \nabla \left( \frac{d}{dx} \right)^i (m_0) \subset p^{\left\lceil \frac{i}{d} \right\rceil} M \), and one sees easily that the series defining \( m \) converges on \( D(x_0, p^{\frac{1}{w(p-1)}}) \).

**Remarks 2.1.6.**

1. Unlike the notion of \( p \)-curvature, the definition of \( G_w(x_0, r) \) does not require \((M, \nabla)\) to have good reduction. It depends only on the \( O_v \)-model of \( X \) (which we of course always take to be \( P^1_{O_v} - \{0,1,\infty\} \)), which is used to define \( D(x_0, r) \), but not on how \((M, \nabla)\) is extended.

2. If \((M, \nabla)\) has good reduction with respect to \( X_{O_v} \), and it admits a Frobenius structure with respect to some Frobenius lifting on \( X_{O_v} \), then \( G_w(x_0, 1) \) is trivial whenever \( x_0 \in X(O_v) \). See for example [Ked10] 17.2.2, 17.2.3].

From now on we set \( x_0 = \frac{1+\sqrt{3}i}{2} \), which corresponds to the elliptic curve with smallest stable Faltings height. In section 5 we will give a theoretical explanation of why this choice gives the best possible estimates. We set \( G_w = G_w \left( \frac{1+\sqrt{3}i}{2}, p^{\frac{1}{w(p-1)}} \right) \), and we take \( L \) to be a number field containing \( K(\sqrt{3}i) \).

By Lemma 2.1.5, the local differential Galois group \( G_w \) is trivial when the vector bundle with connection \((M, \nabla)\) has good reduction over \( v \), and \( \psi_w \equiv 0 \). This motivates the following definition:

**Definition 2.1.7.** We say that the \( p \)-curvatures of \((M, \nabla)\) vanish for all \( p \) if

1. \( \psi_w \equiv 0 \) for all but finitely many \( p \),
2. \( G_w = \{1\} \) for all primes \( w \) of \( L \).

By what we have just seen, for all but finitely many \( p \), the condition (1) makes sense, and implies (2). Thus (2) is only an extra condition at finitely many primes. As above, the definition does not depend on the extension of \((M, \nabla)\) to \( X_{O_{K,E}} \) or the choice of primes \( \Sigma \).
2.2. The main theorem and a Tannakian consequence.

**Theorem 2.2.1.** Let \((M, \nabla)\) be a vector bundle with a connection over \(X_K = \mathbb{P}^1_K - \{0, 1, \infty\}\), and suppose that the \(p\)-curvatures of \((M, \nabla)\) vanish for all \(p\). Then \((M, \nabla)\) admits a full set of rational solutions.

The proof of this theorem is the subject of sections 3, 4.

**Remarks 2.2.2.** We follow the idea of the proof of Theorem 10.2 in [Kat82]. See also [And04, 1.3, 1.4]).

**Corollary 2.2.3.** If \((M, \nabla)\) is defined over \(X\) and the \(p\)-curvature vanishes for all primes, then \((M, \nabla)\) admits a full set of rational solutions.

2.2.4. As in [Kat82], we can use our main theorem to give a description of the differential Galois group of any vector bundle with a connection over \(X_K\).

Let \(K(X)\) be the function field of \(X_K\). Let \(\omega\) be the fibre functor on \(\langle M \rangle^\oplus\) given by restriction to the generic point of \(X_K\). Write \(G_{\text{gal}} = \text{Aut}^\circ \omega \subset GL(M_{K(X)})\) for the corresponding differential Galois group (see [Kat82, Ch. IV] and [And04, 1.3, 1.4]).

Let \(G\) be the smallest closed subgroup of \(GL(M_{K(X)})\) such that:

1. For almost all \(p\), the reduction of \(\text{Lie } G\) mod \(p\) contains \(\psi_p\).
2. \(G \otimes F_w\) contains \(G_w\) for all \(w\), where, as above, \(F_w\) is the field of fractions of the ring of rigid analytic functions on \(D(x_0, p^{-\frac{1}{W(r-1)}})\).

Let \(\mathfrak{g}\) be the smallest Lie subalgebra of \(GL(M_{K(X)})\) such that for almost all \(p\), the reduction of \(\mathfrak{g}\) mod \(p\) contains \(\psi_p\). As proved in [Kat82, Prop. 9.3], \(\mathfrak{g}\) is contained in \(\text{Lie } G_{\text{gal}}\). Moreover, \(G_w\) is contained in \(G_{\text{gal}} \otimes F_w\) by definition. Hence \(G\) is a subgroup of \(G_{\text{gal}}\). We will see from the proof of the following theorem that (in the presence of the condition (1)), to define \(G\) we only need to impose the condition (2) at finitely many primes.

**Theorem 2.2.5.** Let \((M, \nabla)\) be a vector bundle with a connection defined over \(X_K = \mathbb{P}^1_K - \{0, 1, \infty\}\). Then \(G = G_{\text{gal}}\).

**Proof.** We follow the idea of the proof of Theorem 10.2 in [Kat82]. See also [And04, Prop. 3.2.2].

By a theorem of Chevalley, there exists \(W\) in \(\langle M \rangle^\oplus\) and a line \(L' \subset W_{K(X)}\) such that \(G\) is the intersection of \(G_{\text{gal}}\) with the stabilizer of \(L'\). Let \(W'\) be the smallest \(\nabla\)-stable submodule of \(W_{K(X)}\) containing \(L'\). Then \(W'\) has a \(K(X)\)-basis of the form \(\{l, \nabla l, \ldots, \nabla^{r-1} l\}\) where \(l \in L', r = rW'\), and we have written \(\nabla^l\) for \(\nabla(\frac{d}{dx})^l(l)\). Replacing \(W'\) by \(W' \cap W\), we may assume that \(W_{K(X)} = W'\). Then \(L = L' \cap W\) is a line bundle in \(W\).

As above, let \(\mathfrak{g}\) be the smallest algebraic Lie subalgebra of \(GL(M_{K(X)})\) such that for almost all \(p\) the reduction of \(\mathfrak{g}\) mod \(p\) contains \(\psi_p\). Let \(\Sigma\) be a finite set of primes of \(\mathbb{Q}\) such that \((M, \nabla)\) extends to a vector bundle \(\mathcal{M}\) with connection \(\nabla: \mathcal{M} \to \mathcal{M} \otimes \Omega^1_{X_{\Sigma}, \mathcal{E}}\) over \(X_{\Sigma, \mathcal{E}}\), and \(\mathfrak{g}\) mod \(p\) contains \(\psi_p\) for \(p \notin \Sigma\). We also assume that \(\Sigma\) contains all primes \(p \leq r\).

Let \(U \subset X_{\Sigma, \mathcal{E}}\) be a non-empty open subset such that \(l \in L|_U\), \(L\) and \(W\) extend to vector bundles with connection \(\mathcal{L}\) and \(\mathcal{W}\) respectively, in \(\langle M|_U \rangle^\oplus\), and...
\{l, \nabla l, \cdots, \nabla^{r-1}l\} \text{ forms a basis of } W. \text{ Let } \mathcal{N} := \operatorname{Sym}^r W \otimes (\det W^\vee)^r \text{ with the induced connection. The argument in } \text{[Kat82]} \text{ implies that for } p \notin \Sigma, \text{ the } p\text{-curvature of } (\mathcal{N}, \nabla) \text{ vanishes. Let } N := N_{X_K/V}. \text{ We will use the condition (2) in the definition of } G \text{ to show that } G_w \text{ acts trivially on } N_{\eta_w}. \text{ We already know this for } p \notin \Sigma, \text{ by Lemma 2.1.5. Thus we will only need to use (2) for } p \in \Sigma. \text{ Assuming this for a moment, we can apply Theorem 2.2.1 to } (N, \nabla) \text{ and conclude that it has trivial global monodromy. Hence } G_{\text{gal}} \text{ acts as a scalar on } W. \text{ In particular, } G_{\text{gal}} \text{ stabilizes } L \text{ so, by the definition of } L, \text{ } G_{\text{gal}} = G.

Let } D := D(x_0, p^{-\frac{1}{m-n}}). \text{ Recall that the category } \langle M|_{D(x_0, r)} \rangle \otimes F_w \text{ is obtained from } \langle M|_{D(x_0, r)} \rangle \text{ by taking the same collection of objects and tensoring the morphisms by } F_w. \text{ By the definition of } L, \text{ the group } G_w \text{ acts as a character } \chi \text{ on } L_{\eta_w}. \text{ The morphism } L_{\eta_w} \rightarrow W_{\eta_w} \text{ is a map between } G_w \text{-representations. By the equivalence of categories between } \langle M|_{D(x_0, r)} \rangle \otimes F_w \text{ and the category of linear representations of } G_w \text{ over } F_w, \text{ this morphism is a finite } F_w \text{-linear combination of maps } L_{|D} \rightarrow W_D \text{ in } \langle M|_{D(x_0, r)} \rangle \otimes F_w. \text{ In other words, there are a finite number of } \nabla\text{-stable line bundles } W_i \subset W, \text{ with } G_w \text{ acting on } W_i_{\eta_w} \text{ as } \chi \text{ such that } L_{|D} \subset \sum W_i. \text{ In particular, } \{l, \nabla l, \cdots, \nabla^{r-1}l\}_{|D} \text{ is } \nabla\text{-stable, } \nabla^n l \subset \sum W_i \text{ and } G_w \text{ acts as } \chi \text{ on } \nabla^n l_{|D}. \text{ As } W_{\eta_w} \text{ is generated by } \{l, \nabla l, \cdots, \nabla^{r-1}l\}_{|D}, \text{ the group } G_w \text{ acts as } \chi \text{ on } W_{\eta_w}. \text{ Hence } G_w \text{ acts trivially on } N_{\eta_w}. \quad \Box

Using the same idea as in the last paragraph of the proof above, we have the following lemma which is of independent interest.

**Lemma 2.2.6.** Let } H_w \subset G_{\text{gal}} \text{ be the smallest closed subgroup such that } G_w \subset H_w \otimes_{K(X)} F_w. \text{ Then } H_w \text{ is normal in } G_{\text{gal}}.

**Proof.** We need the following fact (see [And92, Lem. 1]): Assume that } G \text{ is a algebraic group over some field } E. \text{ Let } H \subset G \text{ be a closed subgroup and } V \text{ an } E\text{-linear faithful algebraic representation of } G. \text{ Then } H \text{ is a normal subgroup of } G \text{ if for every tensor space } V^{m,n} := V^\otimes m \otimes (V^\vee)^\otimes n \text{, for every character } \chi \text{ of } H \text{ over } E, G \text{ stabilizes } (V^{m,n})^\chi, \text{ the subspace of } V^{m,n} \text{ where } H \text{ acts as } \chi. \text{ If } G \text{ is connected, then these two conditions are equivalent.}

We apply this result to } H_w \subset G_{\text{gal}} \text{ and } V = M_K(X). \text{ Let } L \subset V^{m,n} \text{ be a line, and } W \subset V^{m,n} \text{ the smallest } \nabla\text{-stable subspace containing } L. \text{ It suffices to show that, if } H_w \text{ acts via } \chi \text{ on } L, \text{ then } H_w \text{ acts via } \chi \text{ on } W. \text{ This shows that } (V^{m,n})^\chi \text{ is } \nabla\text{-stable, and hence that } G_{\text{gal}} \text{ stabilizes } (V^{m,n})^\chi.

As in the proof of the theorem above, } G_w \text{ acts on } W \text{ via } \chi. \text{ Hence } H_w \text{ is contained in the subgroup of } G_{\text{gal}} \text{ which acts on } W \text{ via } \chi. \quad \Box

3. **Algebraicity: an application of André’s theorem**

The main goal of this section is to prove a weaker version of Theorem 2.2.1. Namely, that if } (M, \nabla) \text{ is a vector bundle with a connection over } X_K = \mathbb{P}_K^1 - \{0, 1, \infty\} \text{ all of whose } p\text{-curvatures vanish, then } (M, \nabla) \text{ admits a full set of algebraic solutions.}

3.1. **André’s algebraicity criterion.**

3.1.1. As the coordinate ring of } X_K \text{ a principal ideal domain, } M \text{ is free. Hence we may view } \nabla \text{ as a system of first-order homogeneous differential equations. Thus
M \cong \mathcal{O}_{X, k}^n$ and $\nabla (\frac{d}{dx})y = \frac{dy}{dx} - A(x)y$, where $y$ is a section of $M$, $x$ is the coordinate of $X$, and $A(x)$ is an $m \times m$ matrix with entries in $\mathcal{O}_X = K[x^\pm, (x-1)^\pm]$.

As above, we set $x_0 = \frac{1}{2}(1 + \sqrt{3})$. If $y_0 \in L^m$, there exists $y \in L[[x - x_0]]^m$ such that $y(x_0) = y_0$ and $\nabla(y) = 0$. Our goal is to show that if the $p$-curvatures of $(M, \nabla)$ vanishes for all $p$, then $y$ is algebraic.

3.1.2. Now let $y \in K[[x]]$, and let $\nu$ be a place of $K$. If $\nu$ is finite, we denote by $\nu$ the characteristic of the residue field. Let $| \cdot |_\nu$ be the $\nu$-adic norm normalized so that $|p|_\nu = p^{-\frac{[K, Q]}{[\mathbb{Q}, \mathbb{Q}]}}$ if $\nu$ is finite, and $|x|_\nu = |x|_\infty^{\frac{[K, \mathbb{Q}]}{[\mathbb{Q}, \mathbb{Q}]}}$ for $x \in K$, if $\nu$ is archimedean, where $|x|_\infty$ denotes the Euclidean norm on $K$. When there is no confusion, we will also write $| \cdot |$ for $| \cdot |_\infty$. For a positive real number $R$, we denote by $D_\nu(0, R)$ the rigid analytic $\nu$-disc of $\nu$-adic radius $R$. That is $D_\nu(0, R)$ is defined by the inequality $|z|_\nu < R$.

We first state the definition of $\nu$-adic uniformization and the associated radius $R_\nu$ defined in André’s paper ([And04] Definition 5.4.1).

**Definition 3.1.3.**

1. For $R \in \mathbb{R}^+$, a $\nu$-adic uniformization of $y$ by $D_\nu(0, R)$ is a pair of meromorphic $\nu$-adic functions $g(z)$, $h(z)$ on $D_\nu(0, R)$ such that $h(0) = 0$, $h'(0) = 1$ and $y(h(z))$ is the germ at 0 of the meromorphic function $g(z)$.
2. Let $R_\nu$ be the supremum of the set of positive real $R$ for which a $\nu$-adic uniformization of $y$ by $D_\nu(0, R)$ exists. We call $R_\nu$ the $\nu$-adic radius (of uniformizability).

3.1.4. In order to state the algebraicity criterion, we need to introduce two constants $\tau(y)$, $\rho(y)$, which play similar roles as the global-boundedness condition in the Borel–Dwork rationality criterion. Let $y = \sum_{n=0}^\infty a_n x^n$. We define

$$\tau(y) = \inf \limsup_i \sum_{n, p \geq 1} \frac{1}{n} \sup_{j \leq n} \log^+ |a_j|_\nu,$$

$$\rho(y) = \sum_{\nu} \limsup_i \sup_{n, j \leq n} \frac{1}{n} \log^+ |a_j|_\nu,$$

where $\log^+$ is the positive part of log, that is $\log^+(a) = \log(a)$ if $a > 1$ and is zero otherwise. The following is a slight reformulation of André’s criterion.

**Theorem 3.1.5.** ([And04] Theorem 5.4.3) Let $y \in K[[x]]$ such that $\tau(y) = 0$ and $\rho(y) < \infty$. Let $R_\nu$ be the $\nu$-adic radius of $y$. If $\prod_\nu R_\nu > 1$, then $y$ is algebraic over $K$.

In general the $\nu$-adic radius $R_\nu$ may be infinity or zero. We refer the reader to André’s paper for a precise definition of the infinite product in such situations. In our applications of this theorem, $R_\nu$ will always be non-zero. We remark that we could have also used Thm. 6.1 and Prop. 5.15 of [BCL09] in place of André’s Theorem.

Suppose that $y$ is a (component of a) formal solution of $(M, \nabla)$ as above. By [And04], Corollary 5.4.5, if the $p$-curvatures of $(M, \nabla)$ vanish for all places over a set of rational primes of density one then $\tau(y) = 0$ and $\rho(y) < \infty$. Hence, in order to prove that $y$ is the germ of an algebraic function, we only need to prove that $\prod_\nu R_\nu > 1$. 


3.2. Estimate of the radii at archimedean places. We begin with the following simple lemma.

Lemma 3.2.1. Suppose that \( \phi: D(0, 1) \to \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\} \) is a holomorphic map such that \( \phi(0) = x_0 \). Then for any archimedean place \( w \) of the number field \( L \) where the connection and the initial conditions \( x_0, y_0 \) are defined, \( R_w \geq |\phi'(0)|_w \).

Proof. Let \( z \) be the complex coordinate on \( D(0, 1) \). Consider the formal power series \( \phi^*y \). The vector valued power series \( g = \phi^*y \) is a formal solution of the differential equations \( \frac{dg}{dz} = (\phi'(z))^{-1}A(\phi(z))g \) which is associated to the vector bundle with connection \( (\phi^*M, \phi^*V) \). Since \( D(0, 1) \) is simply connected, \( g \) arises from a vector valued holomorphic function on \( D(0, 1) \) which we again denote by \( g \).

Let \( t = \phi'(0)z \), and set \( R = |\phi'(0)|_\infty \). Then we may identify \( D(0, 1) \) with the \( t \)-disc \( D(0, R) = D_w(0, |\phi'(0)|_w) \) and the map \( \phi \) with a map
\[
\hat{\phi}: D(0, R) \to \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}
\]
which satisfies \( \hat{\phi}'(0) = 1 \). By the definition of \( R_w \), we have \( R_w \geq |\phi'(0)|_w \). \( \square \)

3.2.2. Before giving a lower bound for the radius associated to archimedean places, we recall the definition of \( \theta \)-functions and their classical relation with the uniformization of \( \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\} \). Following the notation of \cite{Igu62} and \cite{Igu64}, let
\[
\theta_{00}(t) = \sum_{n \in \mathbb{Z}} \exp(\pi in^2t), \quad \theta_{01}(t) = \sum_{n \in \mathbb{Z}} \exp(\pi i(n^2t+n)), \quad \theta_{10}(t) = \sum_{n \in \mathbb{Z}} \exp(\pi in(n+\frac{1}{2})^2t)
\]
These series converge pointwise to holomorphic functions on \( \mathcal{H} \), which we denote by the same symbols.

Lemma 3.2.3. \cite[p. 243]{Igu64} These holomorphic functions \( \theta_{00}^4, \theta_{01}^4, \theta_{10}^4 \) are modular forms of weight 2 and level \( \Gamma(2) \). Moreover, there is an isomorphism from the ring of modular forms of level \( \Gamma(2) \) to \( \mathbb{C}[X,Y,Z]/(X-Y-Z) \) given by sending \( \theta_{00}^4, \theta_{01}^4, \theta_{10}^4 \) to \( X, Y \) and \( Z \) respectively.

We need the following basic facts mentioned in \cite[p. 180]{Igu62} and \cite[p. 244]{Igu64} in this section and section 5.

Lemma 3.2.4.

1. Let \( \eta \) be the Dedekind eta function defined by \( \eta = q^{1/24}\prod(1-q^n) \), where \( q = e^{2\pi it} \). We have \( 2^8\eta^2 = (\theta_{00}\theta_{01}\theta_{10})^8 \). In particular, the holomorphic functions \( \theta_{00}, \theta_{01}, \theta_{10} \) are everywhere nonzero on the upper half plane.

2. The derivative \( \lambda(t_0) = \pi i \theta_{00}(t_0)^4 \theta_{10}(t_0)^4 \).

3. The holomorphic function \( \frac{1}{2}(\theta_{00}^4 + \theta_{01}^4 + \theta_{10}^4) \) is the weight 4 Eisenstein form of level \( \text{SL}_2(\mathbb{Z}) \) with constant term 1 in its Fourier expansion; the holomorphic function \( \frac{1}{2}(\theta_{00}^4 + \theta_{01}^4)(\theta_{00}^4 + \theta_{10}^4)(\theta_{00}^4 - \theta_{10}^4) \) is the weight 6 Eisenstein form of level \( \text{SL}_2(\mathbb{Z}) \) with constant term 1 in its Fourier expansion.

3.2.5. Let \( \lambda = \frac{\theta_{00}(t)}{\theta_{10}(t)} \). \( \mathcal{H} \to \mathbb{P}^1(\mathbb{C}) \) and \( t_0 = \frac{1}{2}(-1 + \sqrt{3}i) \). Then \( \lambda: \mathcal{H} \to \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\} \) is a covering map with \( \Gamma(2) \) as the deck transformation group (\cite[II, §7]{Ch}), and it is the isomorphic to the elliptic curve \( \mathbb{C} / (\mathbb{Z} + t\mathbb{Z}) \) (see loc. cit.).

Lemma 3.2.6. The map \( \lambda \) sends \( t_0 \) to \( x_0 \).
Proof. Since the automorphism group of the lattice \(\mathbb{Z} + t_0 \mathbb{Z}\), hence that of the elliptic curve \(\mathbb{C}/(\mathbb{Z} + t_0 \mathbb{Z})\) is of order 6, the automorphism group of the elliptic curve \(v^2 = u(u - 1)(u - \lambda(t_0))\) must also be of order 6. In particular, \(\lambda\) must send \(t_0\) to either \(\frac{1}{2}(1 + \sqrt{3}i)\) or \(\frac{1}{2}(1 - \sqrt{3}i)\) (the roots of 0 = \(j(t_0) = 2^{8(\lambda(t_0)^2 - \lambda(t_0) + 1)^3}\)). Moreover, from the definition of \(\theta\), we can easily see that \(\lambda(t_0)\) has positive imaginary part. □

**Proposition 3.2.7.** Let \(y\) be a component of the formal solution of the differential equations. Then \(R_{t_0}^{[\frac{1}{2}(\sqrt{3}i)]} \geq \frac{3\Gamma(1/3)^3}{2^{10/3} \pi^3} = 5.632 \cdots\).

Proof. Consider the map \(\lambda \circ \alpha : D(0,1) \rightarrow X,\) where \(\alpha : D(0,1) \rightarrow \mathcal{H}\) is a holomorphic isomorphism such that \(\alpha(0) = t_0\), that is, \(\alpha : z \mapsto -\frac{1}{2} + \frac{\sqrt{3}i}{2} \frac{z-i}{z+i}\). We would like to apply Lemma 3.2.1 to the map \(\lambda \circ \alpha\), which maps \(0 \in D(0,1)\) to \(x_0\) since \(\lambda(t_0) = \lambda \left(\frac{1}{2}(-1 + \sqrt{3}i)\right) = x_0\) by Lemma 3.2.6.

Note that \(|x_0| = |1 - x_0| = 1\), so we have \(|\theta_{00}(t_0)| = |\theta_{01}(t_0)| = |\theta_{10}(t_0)|\). By Lemma 3.2.4 we have

\[|\lambda'(t_0)| = |\pi i (\theta_{00}(t_0)\theta_{10}(t_0))/\theta_{01}(t_0)|^4 = |\pi |\theta_{00}(t_0)|^4 = |2^{8/3} \eta^3(t_0)|^{1/6}.\]

We now apply the Chowla–Selberg formula (see [SC67]) to \(Q(\sqrt{3}i)\):

\[|\eta(t_0)|^4 \Im(t_0) = \frac{1}{4 \pi \sqrt{3}} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3.\]

Then we have

\[|\lambda'(t_0)| = |2^{8/3} \eta^3(t_0)|^{1/6} = \frac{\pi^{24/3}}{4 \pi \sqrt{3} \Im(t_0)} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3.\]

We get

\[|\lambda'(t_0)| \cdot |\alpha'(0)| = \frac{\pi^{24/3}}{4 \pi \sqrt{3} \Im(t_0)} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3 \cdot 3 \Im(t_0) = \frac{3 \Gamma(1/3)^6}{2^{20/3} \pi^{18}}\]

by the fact \(\Gamma(1/3) \Gamma(2/3) = \frac{2 \pi}{\sqrt{3}}\). □

**3.3. Algebraicity of the formal solutions.**

**Proposition 3.3.1.** Let \((M, \nabla)\) be a vector bundle with a connection over \(\mathbb{P}^1_K - \{0,1,\infty\}\), and assume that the \(p\)-curvatures of \((M, \nabla)\) vanish for all \(p\). Then \((M, \nabla)\) is locally trivial with respect to the étale topology of \(\mathbb{P}^1_K - \{0,1,\infty\}\).

Proof. Consider \(y \in L[[x-x_0]]\). By Proposition 3.2.7 we have

\[\prod_{w|\infty} R_w \geq 5.632 \cdots.\]

If \(w|p\) is a finite place of \(L\), then since \(G_w\) is trivial, \((M, \nabla)\) has a full set of solutions over \(D(x_0, |p|^{-1/p^{1-p}})\). In particular, \(y\) is analytic on \(D(x_0, |p|^{-1/p^{1-p}})\). Hence

\[\prod_{w|p} R_w \geq \prod_{w|p} \left|p^{-1/p^{1-p}}\right| = p^{-\frac{1}{p^{1-p}}}.\]

and

\[\log\left(\prod_w R_w\right) \geq \log 5.6325 \cdots - \sum_p \frac{\log p}{p(p-1)} > 0.967 \cdots.\]
Applying Theorem 3.1.5 we have that \( y \) is algebraic. Hence \((M, \nabla)\) is étale locally trivial.

\[ \square \]

**Remarks 3.3.2.** It is possible to define \( G_w \) using different radii such that the proof of the above proposition continues to hold. For example, we may work with \( G'_w := G_w(x_0, \frac{1}{3}) \) for all \( w \mid 2 \) and \( G'_w = G_w(x_0, 1) \) for other \( w \). We can define \( G' \) in the same way as \( G \) in section 2.2.4 but replacing \( G_w \) by \( G'_w \). In this situation, we have \( \log(\prod_w R_w) \geq \log 5.6325 \cdots \log 4 > 0.342 \cdots \). Applying the same argument as in Theorem 2.2.5 we have \( \text{Lie} G' = \text{Lie} G_{\text{gal}} \).

In this case, we give a example showing that the condition (2) in section 2.2.4 which asserts that all \( p \)-adic differential Galois groups are trivial does not suffice to guarantee the equality of Lie algebras, in the absence of condition (1). (The condition (1) is used to guarantee the assumption that \( \tau(y) = 0, \rho(y) < \infty \) in Theorem 3.1.5.)

We consider the Gauss–Manin connection on \( H^1_{\text{dR}} \) of the Legendre family of elliptic curves. This family has \( G_w = \{1\} \) for \( w \)'s over \( p > 2 \) because of the Frobenius structure since it has good reduction at these primes (see Remark 2.1.6 and \( G_w(x_0, \frac{1}{4}) = \{1\} \) for \( w \) lying over 2 by a direct computation: as in section 3.2 below, we see that the matrix giving the connection lies in \( \frac{1}{2} \text{End}(M_{\mathcal{O}_K}) \otimes \Omega^1_{X_{\mathcal{O}_K}} \) and a formal horizontal section of a general differential equation of this form will have convergence radius \( \frac{1}{4} \). Hence the smallest group containing all \( p \)-adic differential Galois groups is trivial while \( \text{Lie} G_{\text{gal}} = \mathfrak{sl}_2 \). However, in this special case, \( G' \) is the smallest group containing almost all \( \psi_p \) and we recover [Kat82, thm. 11.2].

### 4. Rationality: An Application of a Theorem of Bost and Chambert-Loir

In this section, we will first review the rationality criterion due to Bost and Chambert-Loir for an algebraic formal function using capacity norms. Then we will use the moduli interpretation of \( X \) to compute the capacity norm and verify that in our situation this theorem is applicable.

#### 4.1. Review of the rationality criterion

We will review the definition of adèlic tube adapted to a given point, the definition of capacity norms for the special case we need, and the rationality criterion in [BCL09].

**Definition 4.1.1.** ([BCL09] Definition 5.16) Let \( Y \) be a smooth projective curve over \( K \), and let \( (x_0) \) be the divisor corresponding to a given point \( x_0 \in Y(L) \) for some number field \( L \supseteq K \). For each finite place \( w \) of \( L \), let \( \Omega_w \) be a rigid analytic open subset of \( Y_{\text{rig}} \) containing \( x_0 \). For each archimedean place \( w \), we choose one embedding \( \sigma : L \to \mathbb{C} \) corresponding to \( w \) and we let \( \Omega_w \) be an analytic open set of \( Y_{\sigma}(\mathbb{C}) \) containing \( x_0 \). The collection \( (\Omega_w) \) is an adèlic tube adapted to \( (x_0) \) if the following conditions are satisfied:

1. For an archimedean place, the complement of \( \Omega_w \) is non-polar (e.g. a finite collection of closed domains and line segments); if \( w \) is real, we further assume that \( \Omega_w \) is stable under complex conjugation.
2. For a finite place, the complement of \( \Omega_w \) is a non-empty affinoid subset;
3. For almost all finite places, \( \Omega_w \) is the tube of the specialization of \( x_0 \) in the special fiber of \( Y \). That is, \( \Omega_w \) is the open unit disc with center at \( x_0 \).

We call \( (\Omega_w) \) a weak adèlic tube if we drop the condition that \( \Omega_w \) is stable under complex conjugation when \( w \) is real.
4.1.2. Now let $Y = \mathbb{P}^1_{\mathcal{O}_K}$. The weak adèlic tube that we will use can be described as follows:

1. For an archimedean place, $\Omega_w$ will be an open simply connected domain inside $\mathbb{P}^1_\mathbb{C} - \{0, 1, \infty\}$.
2. For a finite place, $\Omega_w$ will be chosen to be an open disc of form $D(x_0, \rho_w)$.
3. For almost all finite places, $\rho_w = 1$.

4.1.3. For $\Omega_w$ as above, Bost and Chambert-Loir have defined the local capacity norms $|| \cdot ||_{\text{cap}}^w$ (see [BCL09, Chapter 5]). These are norms on the line bundle $T_{x_0}X$ over $\text{Spec}(\mathcal{O}_L)$. The Arakelov degree of $T_{x_0}X$ with respect to these norms plays the same role as $\log(\prod R_w^i)$ in section 3. This degree can be computed as a local sum after choosing a section of this bundle. We will use the section $\frac{d}{dx}$, in which case one has the following simple description of local capacity norms:

1. For an archimedean place, let $\phi: D(0, R) \to \Omega_w$ be a holomorphic isomorphism that maps 0 to $x_0$, then $|| \frac{d}{dx} ||_{\text{cap}}^w = |R\phi'(0)|^{-1}w^i$ (see [Bos99, Example 3.4]).
2. For a finite place, $|| \frac{d}{dx} ||_{\text{cap}}^w = \rho_w^{-1}$ (see [BCL09, Example 3.12]).

Now, we can state the rationality criterion:

**Theorem 4.1.4.** ([BCL09, Theorem 7.8]) Let $(\Omega_w)$ be an adèlic tube adapted to $(x_0)$. Suppose $y$ is a formal power series over $X$ centered at $x_0$ satisfying the following conditions:

1. For all $w$, $y$ extends to an analytic meromorphic function on $\Omega_w$;
2. The formal power series $y$ is algebraic over the function field $K(X)$.
3. The Arakelov degree of $T_{x_0}X$ defined as $\sum_w - \log(|| \frac{d}{dx} ||_{\text{cap}}^w)$ is positive.

Then $y$ is rational.

**Corollary 4.1.5.** The theorem still holds if we only assume that $(\Omega_w)$ is a weak adèlic tube.

**Proof.** The idea is implicitly contained in the discussion in [Bos99, section 4.4]. We only need to prove that $y$ is rational over $X_{L'}$, where $L'/L$ is a finite extension which we may assume does not have any real places. Let $w$ be a place of $L$ and $w'$ a place of $L'$ over $w$.

For $w$ archimedean, choose the embedding $\sigma': L' \to \mathbb{C}$ corresponding to $w'$ which extends the chosen embedding $\sigma: L \to \mathbb{C}$ corresponding to $w$. We have a natural identification $Y_{\sigma'}(\mathbb{C}) = Y_{\sigma}(\mathbb{C})$, and we take $\Omega_{w'} := \Omega_w$. If $w$ is a finite place, we set $\Omega_{w'} = \Omega_w \otimes_{\mathcal{O}_w} L_{w'}$.

Since $L'$ does not have any real places, the weak adèlic tube $(\Omega_{w'})$ is an adèlic tube. The first two conditions in Theorem 4.1.4 still hold and the Arakelov degree of $T_{x_0}X$ with respect to $(\Omega_{w'})$ is the same as that of $T_{x_0}X$ with respect to $(\Omega_w)$. We can apply Theorem 4.1.4 to $y$ over $X_{L'}$ and conclude that $y$ is rational. □

4.2. Proof of the main theorem. Let $y$ be the algebraic formal function which is one component of the formal horizontal section $y$ of $(M, \nabla)$ over $X_K$.

**Lemma 4.2.1.** Let $y$ be as above. Then this formal power series centered at $x_0$ has convergence radius equal to 1 for almost all finite places.

**Proof.** Since the covering induced by $y$ is finite étale over $X_L$, by Proposition 3.3.1 it is étale over $X_{\mathcal{O}_w}$ at $x_0$ for almost all places. For such places, we have $\rho_w = 1$ by lifting criterion for étale maps. □
4.2.2. We now define an adélic tube \((\Omega_w)\) adapted to \(x_0\). For an archimedean place \(w\), we choose the embedding \(\sigma: L \to \mathbb{C}\) corresponding to \(w\) such that \(\sigma(x_0) = (1 + \sqrt{3}i)/2\). Let \(\hat{\Omega}\) be the open region in the upper half plane cut out by the following six edges (see the attached figure): \(\Re(t) = -\frac{3}{2}, |t + 2| = 1, |t + \frac{2}{3}| = \frac{1}{3}, |t - \frac{1}{3}| = 1, \) and \(\Re(t) = \frac{1}{2}\). This is a fundamental domain of the arithmetic group \(\Gamma(2) \subset \text{SL}_2(\mathbb{Z})\).

We define \(\Omega_w\) to be \(\lambda(\hat{\Omega})\).

For \(w\) finite, we choose \(\Omega_w\) to be \(D(x_0, 1)\) if \(y\) is étale over \(X_{O_w}\) at \(x_0\); otherwise, we choose \(\Omega_w\) to be \(D(x_0, \frac{p-1}{p}(p-1))\).

The collection \((\Omega_w)\) is a weak adélic tube and \(y\) extends to an analytic (in particular meromorphic) function on each \(\Omega_w\) by Lemma 4.2.1, Lemma 3.2.1, and Lemma 2.1.5.

**Lemma 4.2.3.** The Arakelov degree of \(T_{x_0}X\) with respect to the adélic tube \((\Omega_w)\) defined above is positive.

**Proof.** We want to give a lower bound of \((||\frac{d}{dx}||_{\omega}^{\text{cap}})^{-1}\), the capacity of \(\Omega_w\). Let \(a = -\frac{3}{2} + \sqrt{3}i\). On the line \(\Re(t) = -\frac{3}{2}\), the point \(a\) is the closest point to \(t_0 = \frac{1}{2}(-1 + \sqrt{3}i)\) with respect to Poincaré metric. The stabilizer of \(t_0\) in \(\text{SL}_2(\mathbb{Z})\) has order 3, and permutes the geodesics \(\Re(t) = -\frac{3}{2}, |t + \frac{2}{3}| = \frac{1}{3}, |t - 1| = 1, \) and this action preserves the Poincaré metric. Using this, together with the fact that the distance to \(t_0\) is invariant under \(z \mapsto -1 - \bar{z}\), one sees that the distance from any point on the boundary of \(\hat{\Omega}\) to \(t_0\) is at least that from \(a\) to \(t_0\). Since \(\alpha: D(0, 1) \to \mathcal{H}\) (defined in the proof of Prop. 3.2.7) preserves the Poincaré metrics, \(\alpha^{-1}(\hat{\Omega})\) contains a disc with respect to the Poincaré radius equal to the distance from \(t_0\) to \(a\).

In \(D(0, 1)\), a disc with respect to Poincaré metric is also a disc in the Euclidean sense. Hence \(\alpha^{-1}(\hat{\Omega})\) contains a disc of Euclidean radius

\[
|\alpha^{-1}(a)| = ||(a - t_0)/(a - \bar{t}_0)|| = 0.45685\cdots.
\]

Since \(\lambda\) maps the fundamental domain \(\hat{\Omega}\) isomorphically onto \(\Omega_w\), by 4.1.3 the local capacity \((||\frac{d}{dx}||_{\omega}^{\text{cap}})^{-1}\) is at least \(||(a - t_0)/(a - \bar{t}_0)|| \cdot |\lambda'(\frac{1}{2}(-1 + \sqrt{3}i))|\).
By \(\text{4.1.3}\) we have \(-\log(\| \frac{d}{dx} ||_{w}^{\text{cap}}) \geq -\frac{\log p}{p(p-1)}\) when \(w | p\). Recall in Proposition \(\text{3.2.7}\) we have \(|\lambda'(\frac{1}{2}(-1 + \sqrt{3}i))| = 5.632 \cdots\), hence the Arakelov degree of \(T_{x_0}X\) is
\[
\sum_{w} -\log(\| \frac{d}{dx} ||_{w}^{\text{cap}}) > \log(5.6325 \cdots \times 0.45685 \cdots ) - \sum_{p} \frac{\log p}{p(p-1)} > 0.184 \cdots .
\]
\[\square\]

Now we are ready to prove Theorem \(\text{2.2.1}\):

Proof. Applying Proposition \(\text{3.3.1}\) we have a full set of algebraic solutions \(y\). Choosing the weak adélic tube as in \(\text{4.2.2}\) and applying Corollary \(\text{4.1.5}\) (the assumptions are verified by \(\text{4.2.2}\) and Lemma \(\text{4.2.3}\), we have that these algebraic solutions are actually rational.

This shows that \((M, \nabla)\) has a full set of rational solutions over \(X_L\). Since formation of \(\ker(\nabla)\) commutes with the finite extension of scalars \(\otimes_K L\), this implies that \((M, \nabla)\) has a full set of rational solutions over \(X_K\).

\[\square\]

5. Interpretation using the Faltings height

In this section, we view \(X_{\mathbb{Z}[\frac{1}{2}]}\) as the moduli space of elliptic curves with level 2 structure. Let \(\lambda_0 \in X(\overline{\mathbb{Q}})\) and \(E\) the corresponding elliptic curve. Using the Kodaira–Spencer map, we will relate the Faltings height of \(E\) with our lower bound for the product of radii of uniformizability (see section 3) at archimedean places of the formal solutions in \(\hat{O}_{X_K, \lambda_0}\). We will focus mainly on the case when \(\lambda_0 \in X(\overline{\mathbb{Z}})\) and sketch how to generalize to \(\lambda_0 \in X(\overline{\mathbb{Q}})\) at the end of this section. In this section, unlike the previous sections, we will use \(\lambda\) as the coordinate of \(X\).

5.1. Hermitian line bundles and their Arakelov degrees.

5.1.1. Let \(K\) be a number field, and \(\mathcal{O}_K\) its ring of integers. Recall that an Hermitian line bundle \((L, || \cdot ||_\sigma)\) over \(\text{Spec}(\mathcal{O}_K)\) is a line bundle \(L\) over \(\text{Spec}(\mathcal{O}_K)\), together with an Hermitian metric \(|| \cdot ||_\sigma\) on \(L \otimes_K \mathbb{C}\) for each archimedean place \(\sigma: K \to \mathbb{C}\).

Given an Hermitian line bundle \((L, || \cdot ||_\sigma)\), its (normalized) Arakelov degree is defined as:
\[
\hat{\text{deg}}(L) := \frac{1}{[K : \mathbb{Q}]} \left( \log(\#(L/s\mathcal{O}_K)) - \sum_{\sigma: K \to \mathbb{C}} \log ||s||_\sigma \right),
\]
where \(s\) is any section.

For a finite place \(v\) over \(p\), the integral structure of \(L\) defines a norm \(|| \cdot ||_v\) on \(L_{K_v}\). More precisely, if \(s_v\) is a generator of \(L_{\mathcal{O}_{K_v}}\) and \(n\) is an integer, we define \(||p^n s_v||_v = p^{-n[K_v: \mathbb{Q}_p]}\). We obtain a norm on \(\mathcal{O}_v\) by viewing it as the trivial line bundle. We will use \(|| \cdot ||_v\) for the norms on different line bundle as no confusion would arise. We may rewrite the Arakelov degree using the \(p\)-adic norms:
\[
\hat{\text{deg}}(L) = \frac{1}{[K : \mathbb{Q}]} \left( -\sum_v \log ||s||_v \right),
\]
where \(v\) runs over all places of \(K\). It is an immediate corollary of the product formula that the right hand side does not depend on the choice of \(s\).
5.1.2. Let $E$ be an elliptic curve over a number field $K$, and denote by $e: \text{Spec } K \to E$ and $f: E \to \text{Spec } K$ the identity and structure map respectively. For each $\sigma: K \to \mathbb{C}$, we endow $e^*\Omega^1_{E/K} = f_*\Omega^1_{E/K}$ with the Hermitian norm given by $||\alpha||_\sigma = \left( \frac{1}{2\pi} \int_{\sigma(E)} |\alpha \wedge \partial \bar{\alpha}| \right)^{1/2}$, where $e_\sigma$ is 1 for real embeddings and 2 otherwise.

This can be used to define the Faltings’ height of $E$, which we recall precisely only in the case when $E$ has good reduction over $\mathcal{O}_K$. Denote by $f: \mathcal{E} \to \text{Spec } \mathcal{O}_K$ the elliptic curve over $\mathcal{O}_K$ with generic fibre $E$, and again write $e$ for the identity section of $\mathcal{E}$. The norms $||\alpha||_\sigma$ make $e^*\Omega^1_{\mathcal{E}/\text{Spec}(\mathcal{O}_K)} = f_*\Omega^1_{E/\text{Spec}(\mathcal{O}_K)}$ into a Hermitian line bundle, and we define the (stable) Faltings height $h_F$ by

$$h_F(E = \overline{\text{deg}}(f_*\Omega^1_{E/\text{Spec}(\mathcal{O}_K)}).$$

Notice that $h_F(E)$ does not depend on the choice of $K$. Here we use Deligne’s definition for convenience [Del85, 1.2]. This differs from Faltings’ original definition (see [Fal86]) by a constant $\log(\pi)$.

In general, the elliptic curve $E$ would have semi-stable reduction everywhere after some field extension. We assume this is the case and $E$ has a Neron model $f: \mathcal{E} \to \text{Spec } \mathcal{O}_K$ which endows $f_*\Omega^1_{\mathcal{E}/\text{Spec}(\mathcal{O}_K)}$ a canonical integral structure. With the same Hermitian norm defined as above, we have a similar definition of Faltings height in the general case. See [Fal86] for details. As in the good reduction case, this definition does not depend on the choice of $K$.

5.1.3. We will assume that $\lambda_0$ and $\lambda_0 - 1$ are both units at each finite place. Given such a $\lambda_0$, consider the elliptic curve $E_{\lambda_0}$ over $\mathbb{Q}(\lambda_0)$ defined by the equation $y^2 = x(x-1)(x-\lambda_0)$. Then $E_{\lambda_0}$ has good reduction at primes not dividing 2, and potentially good reduction everywhere, since its $j$-invariant is an algebraic integer. Let $K$ be a number field such that $(E_{\lambda_0})_K$ has good reduction everywhere. We denote by $E_{\lambda_0}$ the elliptic curve over $\mathcal{O}_K$ with generic fiber $E_{\lambda_0}$.

5.1.4. To express our computation of radii in terms of Arakelov degrees, we endow the $\mathcal{O}_K$-line bundle $T_{\lambda_0}(X_{\mathcal{O}_K})$, the tangent bundle of $X_{\mathcal{O}_K}$ at $\lambda_0$, with the structure of an Hermitian line bundle as follows. For each archimedean place $\sigma: K \to \mathbb{C}$, we have the universal covering $\lambda: \mathcal{H} \to \sigma X$, introduced in [5.2.5]. The $\text{SL}_2(\mathbb{R})$-invariant metric $\frac{dt}{23(t_0)^2}$ on the tangent bundle of $\mathcal{H}$ induces the desired metric on the tangent bundle via push-forward. As in the proof of Proposition [5.2.7], our lower bound on the radius of the formal solution is $\frac{23(t_0)\lambda(t_0)}{||d\lambda||_\sigma} = \frac{||d\lambda||_\sigma}{||\lambda||_\sigma}$, where $t_0$ is a point on $\mathcal{H}$ mapping to $\lambda_0$. It is easy to see the left hand side does not depend on the choice of $t_0$. Under the assumptions in [5.1.3], the tangent vector $\frac{d}{d\lambda}$ is an $\mathcal{O}_K$-basis vector for the tangent bundle $T_{\lambda_0}(X_{\mathcal{O}_K})$, and we have

$$\overline{\text{deg}}(T_{\lambda_0}X) = \frac{1}{[K: \mathbb{Q}]} \left( - \sum_{\sigma: K \to \mathbb{C}} \log \left| \frac{d}{d\lambda} \right|_\sigma \right) \leq \frac{1}{[K: \mathbb{Q}]} \log(\prod \text{R}_\sigma),$$

where the $\text{R}_\sigma$ are the radius of uniformization discussed in section [5.2].

5.2. The Kodaira–Spencer map. Consider the Legendre family of elliptic curves $E \subset \mathbb{P}^2[\mathbb{Z}[\frac{1}{2}]] \times X_{\mathbb{Z}[\frac{1}{2}]}$ over $X_{\mathbb{Z}[\frac{1}{2}]}$ given by $y^2 = x(x-1)(x-\lambda)$. We have the Kodaira–Spencer map [FC90, Ch. III.9, Kat72, 1.1]):

$$(5.2.1) \quad KS: (f_*\Omega^1_{E/X_{\mathbb{Z}[\frac{1}{2}]}})^{\otimes 2} \to \Omega^1_{X_{\mathbb{Z}[\frac{1}{2}]}} \otimes (\alpha \otimes \beta \mapsto \langle \alpha, \nabla \beta \rangle),$$
Lemma 5.2.5. The Kodaira–Spencer map preserves the Hermitian metrics: a related argument and Lemma 3.2.4 (2) for relation between
do this, one studies both sides for the Tate curve. See for example [MB90, 2.2] for
are the same up to a constant and we only need to compare them at the cusps. To
SL are
| |
5.2.4
λ
Lemma 5.2.3. Given λ
preserves the Oν-generators of (f∗Ω1
X 1/2
|λ0) and Ω1
X 1/2
|λ0 when λ0 and
λ0 − 1 are both v-units.
5.2.2. Following Kedlaya’s notes ([Ked Sec. 1.3]), we choose \{dx, zdv\} to be an integral basis of \(H^1_{dR}(E/X)|_{\lambda_0}\) and compute the Gauss–Manin connection:
\[
\nabla \frac{dx}{2y} = \frac{1}{2(1 - \lambda)} \frac{dx}{2y} \otimes d\lambda + \frac{1}{2\lambda(\lambda - 1)} \frac{xdx}{2y} \otimes d\lambda.
\]
The Kodaira–Spencer map then sends \((\frac{dx}{2y})^\otimes 2\) to \(\frac{1}{2\lambda(\lambda - 1)} d\lambda\).
This computation shows:

Lemma 5.2.3. Given \(v\) a finite place not lying over 2, the Kodaira–Spencer map \([5.2.1]\) preserves the \(O_v\)-generators of \((f∗\Omega^1_{E/X_{\eta_{1/2}}})^\otimes 2|_{\lambda_0}\) and \(\Omega^1_{X_{\eta_{1/2}}}|_{\lambda_0}\) when \(\lambda_0\) and
The archimedean places \(\sigma\), we consider \(f∗\Omega^1_{\sigma E/\text{Spec } \mathbb{C}}\) with the metrics \(||\alpha||_\sigma\) defined in section 5.1 and we endow \(\Omega^1_{X_{\eta_{1/2}}}|_{\lambda_0}\) the Hermitian line bundle structure as the dual of the tangent bundle.

To see the Kodaira–Spencer map preserves the Hermitian norms on both sides, one may argue as follows. Notice that the metrics on \((f∗\Omega^1_{\sigma E/\text{Spec } \mathbb{C}})^\otimes 2\) and \(\Omega^1_{X_{\eta_{1/2}}}|_{\lambda_0}\) are \(\text{SL}_2(\mathbb{R})\)-invariant (see for example [ZP09, Remark 3 in Sec. 2.3]). Hence they are the same up to a constant and we only need to compare them at the cusps. To do this, one studies both sides for the Tate curve. See for example [MB90, 2.2] for a related argument and Lemma 3.2.4 (2) for relation between \(\theta\)-functions and \(\Omega^1_X\).

Here we give another argument:

Lemma 5.2.5. The Kodaira–Spencer map preserves the Hermitian metrics:
\[
||\frac{(dx)^{\otimes 2}}{2y}||_\sigma = \left|\frac{d\lambda}{2\lambda_0(\lambda_0 - 1)}\right||\sigma.
\]

Proof. Let \(dz\) be an invariant holomorphic differential of \(\mathbb{C}/(\mathbb{Z} \oplus t_0\mathbb{Z})\), where \(\lambda(t_0) = \lambda_0\). By the theory of the Weierstrass-\(\wp\) function, we have a map from the complex torus to the elliptic curve
\[
\frac{v^2}{w^2} = 4v^3 - g_2(t_0)v - g_3(t_0)
\]
such that \(dz\) maps to \(\frac{dv}{w}\). Here \(g_2\) is the weight 4 modular form of level \(\text{SL}_2(\mathbb{Z})\) with \(4\pi^3\) as the constant term in its Fourier series and \(g_3\) is the weight 6 modular form with \(8\pi^3\) as the constant term. Using Lemma 3.2.4 (3), we see that the right hand side has three roots: \(\frac{\pi^2}{3}(\theta_{00}^4(t_0) + \theta_{01}^4(t_0)), -\frac{\pi^2}{3}(\theta_{00}^4(t_0) + \theta_{10}^4(t_0)), \frac{\pi^2}{3}(\theta_{10}^4(t_0) - \theta_{01}^4(t_0))\). Hence this curve is isomorphic to \(y^2 = x(x-1)(x-\lambda)\) via the map
\[
x = \frac{u}{\frac{4}{3} \pi^2(\theta_{00}^4(t_0) + \theta_{01}^4(t_0))}, y = \frac{u}{2(-\pi^2 \theta_{01}^4(t_0))^3/2},
\]
and we have
\[
\frac{dx}{2y} = \frac{\pi i \theta_{01}^2(t_0)}{u} \frac{dv}{u} = \pi i \theta_{01}^2(t_0) dz.
\]
Hence
\[
||\frac{(dx)^{\otimes 2}}{2y}||_\sigma = |\pi^2 \theta_{01}^4(t_0) \cdot \left( \frac{1}{2\pi} \int_{E(\mathbb{C})} |dz \wedge d\bar{z}| \right)^{\sigma} = |\pi \theta_{01}^4(t_0) \Im(t_0)|^{\sigma}.
\]
On the other hand, using Lemma 3.2.4 (2), we have
\[
\| \frac{d\lambda}{2\lambda_0(\lambda_0 - 1)^{1/2}} \|_{\sigma} = \left| \frac{2\delta(t_0)\lambda(t_0)}{2\lambda_0(\lambda_0 - 1)^{1/2}} \right| = \left| \frac{\delta(t_0)\pi \theta_{10}(t_0)\theta_{01}(t_0)}{\theta_{01}(t_0)\lambda_0(\lambda_0 - 1)^{1/2}} \right| = |\pi \theta_{01}(t_0)\delta(t_0)|.
\]

\[
\]

Proposition 5.2.6. If \( \lambda_0 \) and \( \lambda_0 - 1 \) are both units at every finite places, we have \( \tilde{\deg}(T_{\lambda_0}X) = -2h_F(E_{\lambda_0}) + \frac{\log 2}{3} \).

Proof. By lemma 5.2.3 and lemma 5.2.5 we have
\[
(5.2.7)
\]
\[
\tilde{\deg}(T_{\lambda_0}X) = \deg(\Omega_{X^{\mathcal{O}_K}}^{1})|_{\lambda_0} = \left[ \frac{1}{K:Q} \left( -\sum_{v} \log \left( \| \frac{d\lambda}{2\lambda_0(\lambda_0 - 1)^{1/2}} \|_{v} \right) \right) \right]
\]
\[
= \left[ \frac{1}{K:Q} \left( -\sum_{v,\infty} \log \left( \| \frac{d\lambda}{2\lambda_0(\lambda_0 - 1)^{1/2}} \|_{v} \right) \right) \right] - \sum_{v \not\text{ finite}} \log \| 1/2 \|_{v}
\]
\[
= 2h_F(E_{\lambda_0}) + \left[ \frac{1}{K:Q} \sum_{v \mid 2} \log \| (\frac{dx}{2y})^{\otimes 2} \|_{v} \right] - \log 2.
\]

Now we study \( \| (\frac{dx}{2y})^{\otimes 2} \|_{v} \) given \( v \mid 2 \). The sum \( \frac{1}{K:Q} \sum_{v \mid 2} \log \| (\frac{dx}{2y})^{\otimes 2} \|_{v} \) does not change after extending \( K \), hence we may assume that \( \mathcal{E}_{\lambda_0} \) over \( \mathcal{O}_v \) has the Deuring normal form \( u^2 + auw + u = u^3 \) (see [Sil09] Appendix A Prop. 1.3 and the proof of Prop. 1.4 shows in the good reduction case, \( \alpha \) is a \( \mathfrak{v} \)-integer). An invariant differential generating \( f_{\mathcal{E}_{\lambda_0}/\mathop{\text{Spec}} \mathcal{O}_K[4]} \) is \( \frac{du}{u^3 + auw + 1} \).

Because both \( \frac{du}{u^3 + auw + 1} \) and \( \frac{dx}{2y} \) are invariant differentials, we have \( \| (\frac{dx}{2y})^{\otimes 2} \|_{v} = \| \Delta_1/\Delta_2 \|_{v}^{1/2} \| (\frac{dw}{u^3 + auw + 1}) \|_{v} \), where \( \Delta_1 \) and \( \Delta_2 \) are the discriminant of the Deuring normal form and that of the Legendre form respectively. Since \( E \) has good reduction, \( \| \Delta_1 \|_{v} = 1 \) (see the proof of loc. cit.). Hence \( \| (\frac{dx}{2y})^{\otimes 2} \|_{v} = \| (\frac{du}{u^3 + auw + 1}) \|_{v} \cdot \| 1/16 \|_{v}^{1/12} = \| 2 \|_{v}^{-1/3} \).

Hence \( \tilde{\deg}(T_{\lambda_0}X) = -2h_F(E_{\lambda_0}) - \frac{2}{3} \log 2 + \log 2 = -2h_F(E_{\lambda_0}) + \frac{\log 2}{3}. \)

\[
\]

5.2.8. As pointed out by Deligne ([Del85, 1.5]), the point \( \frac{1 + \sqrt{3}}{2} \) corresponds to the elliptic curve with smallest height. Hence, our choice \( \frac{1 + \sqrt{3}}{2} \) gives the largest \( \tilde{\deg}(T_{\lambda_0}X) \) among those \( \lambda_0 \) such that \( \lambda_0 \) and \( \lambda_0 - 1 \) are units at every prime.

5.3. The general case. For the general case when \( \lambda_0 \in X(\overline{\mathbb{Q}}) \), using a similar argument as in section 5.2 we have
(5.3.1) \[
\frac{1}{[K:Q]} \left( - \sum_{\sigma:K \to \mathbb{C}} \log \left\| \frac{d}{d\lambda} \right\|_\sigma \right) \leq -2h_F(E_{\lambda_0}) + \frac{\log 2}{3} + \frac{1}{[K:Q]} \left( \sum_{v \text{ finite}} \log^+ ||\lambda_0||_v + \log(||N_{\lambda_0}(\lambda_0-1)||) \right)
\]

and equality holds if and only if \(\lambda_0 \in X(\mathbb{Z}_2)\). As discussed in 5.1.4, the left hand side is the sum of the logarithms of our estimates of the radii of uniformizability at archimedean places.

To apply Theorem 3.1.5, we also need to modify the estimate of the radii at finite places in Lemma 2.1.5. A possible estimate for \(R_v\) is \(p^{-1}p^{-1} \cdot \min\{||\lambda_0||_v, ||\lambda_0 - 1||_v, 1\}\). The later factor comes from the fact we cannot rule out the possibility that one has local monodromy at 0, 1, \(\infty\).

Compared to the case when \(\lambda_0 \in X(\mathbb{Z})\), our estimate for the sum of the logarithms of the archimedean radii increases by at most \(\frac{1}{[K:Q]} \left( \sum_{v \text{ finite}} \log^+ ||\lambda_0||_v + \log(||N_{\lambda_0}(\lambda_0-1)||) \right)\), while the estimate for the sum of logarithms of the radii at finite places becomes smaller by \(\sum_v \max\{\log^+ ||\lambda_0||_v, \log^+ ||(\lambda_0-1)^{-1}||_v\} \). An explicit computation shows that the later is larger than the former. Hence the estimate for the product of the radii does not become larger than the case when \(\lambda_0 \in X(\mathbb{Z})\).

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