1. Introduction

In the recent decade, there has been a lot of interest in the study of physics on a noncommutative space–time due to the fact that space–time may exhibit its noncommutativity at the scale of quantum gravity. Especially, string theory, which is considered as the most promising candidate for a theory of quantum gravity. especially, string theory, which is considered as the most promising candidate for a theory of quantum gravity, gives rise to space–time noncommutativity [1]. Apart from the string theory motivation, it is interesting to investigate the space–time noncommutativity in a more familiar set-up, like quantum mechanics. Especially, since the result [2], combining Heisenberg’s time noncommutativity in a more familiar set-up, like quantum string theory motivation, it is interesting to investigate the space–time noncommutativity [1].

A part from the as the most promising candidate for a theory of quantum gravity. Especially, string theory, which is considered as the most promising candidate for a theory of quantum gravity, gives rise to space–time noncommutativity [1]. Apart from the string theory motivation, it is interesting to investigate the space–time noncommutativity in a more familiar set-up, like quantum mechanics. Especially, since the result [2], combining Heisenberg’s uncertainty principle with Einstein’s theory of classical gravity, is quantum mechanical in spirit, the purely quantum mechanical treatment of a noncommutative space–time becomes interesting. In [2] one considers a gedanken experiment at very high energy where the high density of the energy–momentum tensor would result in the formation of black holes through the Einstein equations. In this case it would no longer be possible to measure lengths up to arbitrary precision, but space–time would become noncommutative in a similar way as phase-space becomes noncommutative in quantum mechanics.

Various approaches to quantum mechanics on noncommutative space–time have been proposed in [3–6]. Its space coordinate operator $X_i$ is characterized by the relation

$$[\hat{X}_i, \hat{X}_j] = i\theta_{ij},$$

where $i = 1, 2, 3$ stands for the three space coordinates and the constant $\theta_{ij}$ is the noncommutativity parameter. Here we have taken the time direction to be commutative $[\hat{X}_0, \hat{X}_i] = 0$, due to the problems with unitarity [7] and causality [8] for a noncommuting time direction. We represent the noncommutativity of space coordinates through the Weyl–Moyal correspondence, in which to each function of operators $f(\hat{X})$ corresponds a Weyl symbol $f(x)$, defined on the commutative counterpart of the space. This amounts to replacing the usual commutative product of functions of operators $f(\hat{X})g(\hat{X})$ by the Moyal star-product of Weyl symbols, $f(x) \star g(x)$, where

$$f(x) \star g(x) = f(x) \exp\left[\frac{i}{\theta} \oint \frac{d\theta}{2\pi} g(X)\right],$$

and $x$ are the commutative space coordinates. The canonical quantization condition between the quantum mechanical coordinate $\hat{X}_i$ and momentum $\hat{P}_i$ is the same as in ordinary quantum mechanics,

$$[\hat{X}_i, \hat{P}_j] = i\hbar\delta_{ij}, \quad [\hat{P}_i, \hat{P}_j] = 0,$$

but with the additional relations

$$[\hat{X}_i, \hat{X}_j] = i\theta_{ij}, \quad [\hat{X}_i, \theta_{jk}] = [\hat{P}_i, \theta_{jk}] = 0.$$
Aharanov–Bohm (AB) effect [9–11], the hydrogen atom spectrum and the Lamb shift [5,12], the Hall effect [13], the Aharanov–Casher effect [14] and so on.

Since all the observables in quantum mechanics should be gauge invariant quantities, it is important to examine the gauge invariance of physical quantities in NCQM. For instance, the gauge invariance (or covariance) of the phase factor of a wave function is directly related to many of the physical observables, such as, the Aharanov–Bohm effect, the Aharanov–Casher effect and the Berry phase.

In this Letter, we show that the naive path integral formulation of NCQM and an approach where one shifts the coordinates of NCQM [11] lead neither to a gauge invariant nor to a gauge covariant Aharanov–Bohm phase factor.1 Instead, we propose a gauge covariant formulation of the AB phase which is consistent with the noncommutative Schrödinger equation.

The organization of this Letter is as follows. In Section 2, we discuss the path integral formulation of NCQM following the result of [9,10] especially focusing on the gauge covariance of the formulation. We shall stress the difference between the commutative and noncommutative cases and point out how gauge covariance is broken in the noncommutative case. Section 3 is devoted to another approach to NCQM where one shifts the coordinates to satisfy the usual commutation relations of ordinary quantum mechanics. This approach also breaks gauge invariance but preserves some exotic kind of gauge invariance. In Section 4, we propose a gauge covariant AB phase factor which is represented by the path-ordered exponential and is consistent with the Schrödinger equation. Section 5 contains summary and discussion.

2. Path integral formulation of NCQM

In this section, we present the path integral formulation of NCQM following the derivation of [9,10]. We consider a particle with mass $m$ and charge $e$, under the noncommutative $U(1)$ gauge group, in a magnetic field. The corresponding gauge potential is $A_i(x)$. The noncommutative Hamiltonian is given by

$$H = \frac{1}{2m} \left( p_i + \frac{e}{\hbar c} A_i(x) \right)^2,$$

where $p_i = -i\nabla_i$. The $U(1)$ gauge field strength is defined by

$$F_{ij} = \partial_i A_j - \partial_j A_i + \frac{e}{\hbar c} [A_i, A_j].$$

The transition amplitude from the initial state $\Psi_i$ to the final state $\Psi_f$, $(\Psi_f, e^{-i\frac{Ht}{\hbar}} \Psi_i)$, is invariant under the following noncommutative gauge transformations,

$$\Psi(x) \rightarrow U(x) \Psi(x),$$

$$A_i(x) \rightarrow A_i(x) \pm U^{-1}(x) - \frac{i\hbar}{e} U(x) \partial_i U^{-1}(x),$$

$$p_i \rightarrow p_i \pm U^{-1} \partial_i U(x) \mp i\hbar U^{-1}(x).$$

Here $\Psi(x)$ is the wave function and $U(x)$ is defined by $U(x) = e^{iu(x)}$, with a real function $u(x)$. The $U(1)$ element $U(x)$ satisfies $U^{-1} \ast U = U \ast U^{-1} = 1$. The Hamiltonian transforms covariantly under the gauge transformation,

$$H(x) \rightarrow U(x) \ast H(x) \ast U^{-1}(x),$$

while in the commutative case, $H$ is invariant under the $U(1)$ gauge transformation.

The propagator $K_t(x, y)$ is represented by the bi-local kernel [9,10]

$$K_t(x, y) = \langle x|e^{-i\frac{Ht}{\hbar}}|y \rangle = \int \frac{d^3q}{(2\pi \hbar)^3} \left( e^{-i\frac{q}{\hbar}x} \ast U^{-1}(x) \ast U(x) \ast e^{i\frac{q}{\hbar}y} \right).$$

Note that the action of $H(x)$ on $e^{i\frac{q}{\hbar}y}$ is via the star-product defined in (2). This propagator is bi-locally gauge covariant provided the noncommutative Schrödinger equation.

The path integral of the AB phase in the path-integral formulation of NCQM. The propagator can be represented by the products of short-time propagators in the infinite time evolution by separating the time interval into $N$-pieces and taking $N \rightarrow \infty$.

$$K_t(x, y) = \lim_{N \rightarrow \infty} \int d^3x_{N-1} \cdots d^3x_1 K_e(x, x_{N-1}) \cdots \times K_e(x_2, x_1).$$

Here $\epsilon \equiv t/N$ and we have used the identity $e^{-i\frac{q(x)}{\hbar}x} \ast e^{-i\frac{q(y)}{\hbar}y} = e^{-i\frac{q}{\hbar}(x+y)/2}$. The reason why gauge covariance is lost in [9,10] is that the quantum mechanical Hamiltonian corresponding to (6) should be treated in the Weyl-ordered form as if we use the midpoint prescription in the path-integral formulation. This is in turn a consequence of the fact that the Hamiltonian contains a mixing term between $\hat{P}_i$ and $\hat{X}_i$. This means that the short-time propagator has to be evaluated in the midpoint of $x$ and $y$, namely, $H = H(\bar{x})$ where $\bar{x} = (x + y)/2$. In this case, the propagator is not bi-locally gauge covariant anymore,

$$K_t(x, y) \rightarrow K_t^0(x, y)$$

$$= \int \frac{d^3q}{(2\pi \hbar)^3} \left( U(\bar{x}) \ast e^{-i\frac{q}{\hbar}x} \ast U^{-1}(\bar{x}) \ast U(x) \ast e^{i\frac{q}{\hbar}y} \right) \times \left( e^{-i\frac{q}{\hbar}y} \ast U^{-1}(y) \right) \neq U(x) \ast K_t(x, y) \ast U^{-1}(y).$$

We would like to stress that the propagator is bi-locally gauge covariant in the commutative case, namely,

$$K_t(x, y) \rightarrow K_t^c(x, y)$$

$$= U(x) K_c(x, y) U^{-1}(y) \quad \text{(commutative case)}.$$

If one goes ahead with the midpoint prescription in the noncommutative case, one arrives at a phase shift $\delta \phi$ for an electron wave function after moving around the path $C$ in the noncommutative space given by

$^2$ There is another problem with the midpoint prescription in NCQM. There is an ambiguity in how to define the star-product between $e^{-i\frac{q}{\hbar}x}$ and $e^{i\frac{q}{\hbar}y}$ in the kernel. Here we have simply assumed that it is given by $\ast$. It could also be given by $\ast$, but this does not change the outcome. The propagator is still not bi-locally gauge covariant.

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1 The shift of coordinates of NCQM has previously been used in [3,5,15].
\[ \delta \phi = \frac{e}{\hbar c} \int dx_i A_i + \frac{e m}{4 \hbar c} \theta \]
\[ \cdot \int dx_i \left[ (\vec{\nabla} \times \vec{\nabla} A_i) - \frac{e m}{\hbar c} (\vec{A} \times \vec{\nabla} A_i) \right] + O(\theta^2). \] (15)

Here the component of \( \vec{\theta} \) is defined by \( \theta_{ij} = \epsilon_{ijk} \theta_{jk} \). This is the result obtained in the path-integral formulation in the midpoint prescription [9,10]. The same result has been obtained by the perturbative analysis of the Schrödinger equation [16].

We can explicitly check that this result is neither gauge invariant nor covariant under the \( O(\theta) \) gauge transformations
\[ \delta A_i^{(0)} = -\partial_i \lambda, \]
\[ \delta A_i^{(1)} = \frac{e}{\hbar c} \theta_{ij} \partial_k A_i \partial_j \lambda. \] (16)

Here \( A_i^{(n)} \) is an \( n \)th order expansion of \( A_i \) in the noncommutativity parameter \( \theta_{ij} \). As we mentioned, this gauge non-covariance originates from the Weyl ordering of the quantum mechanical Hamiltonian and hence, from the midpoint prescription in the path-integral. In the next section, we will use another approach to derive the AB phase in NCQM. From here on, for simplicity, we shall use \( \hbar = c = m = e = 1 \).

3. The phase factor in terms of a shift of coordinates

It is known that the noncommutativity of space in quantum mechanics can be interpreted as ordinary quantum mechanics with deformed Hamiltonian. This deformation can be performed via a shift of coordinates [3,5,15].

Consider quantum mechanics on a noncommutative space, with the commutation relation among coordinate and momentum operators as
\[ [\hat{x}_i, \hat{x}_j] = i \theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i \delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \] (17)

Following the procedure adopted in [3,5], the shifted coordinate and momentum
\[ \hat{x}_i = \hat{x}_i + \frac{1}{2} \theta_{ij} \hat{p}_j, \]
\[ \hat{p}_i = \hat{p}_i, \] (18)

satisfy
\[ [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i \delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \] (20)

Thus NCQM now reduces to ordinary quantum mechanics but with deformed Hamiltonian \( H(\hat{X}, \hat{P}) \rightarrow H(\hat{X}, \hat{P}) \). The gauge potential in the Hamiltonian can be expanded as
\[ A_i(\hat{x}) = A_i(\hat{x}) - \frac{1}{2} \theta_{ij} \partial_k A_i(\hat{x}) + O(\theta^2). \] (21)

Consequently, the noncommutative Hamiltonian \( H(\hat{X}, \hat{P}) = \frac{1}{2} \hat{P}_i + A_i(\hat{x}) \hat{X}_i \) is interpreted as the deformed Hamiltonian
\[ \hat{H}(\hat{X}, \hat{P}) = \frac{1}{2} \left( \hat{P}_i - A_i(\hat{x}) - \frac{1}{2} \partial_k \theta_{ij} \partial_i A_i(\hat{x}) \right)^2 + O(\theta^2). \] (22)

in ordinary quantum mechanics. The Hamiltonian (22) is no longer star-gauge covariant as a consequence of shifting the coordinates. This is because the potential \( A_i(\hat{x}) \) is given in the noncommutative space and it transforms as
\[ A_i(\hat{x}) \rightarrow A_i(\hat{x}) = U(\hat{x}) A_i(\hat{x}) U^{-1}(\hat{x}) \]

\[ - i U(\hat{x}) \partial_k U^{-1}(\hat{x}). \] (23)

However, the potential \( A_i(\hat{x}) \) is not given in this type of noncommutative space, but the ordinary quantum mechanical one, and consequently does not transform similarly to (23). Therefore, the star gauge covariance of the Hamiltonian is lost in (22).

The Schrödinger equation corresponding to (22) is
\[ i \frac{\partial \psi(\hat{x})}{\partial t} = \frac{1}{2} \left( \hat{P}_i - A_i(\hat{x}) - \frac{1}{2} \partial_k \theta_{ij} \partial_i A_i(\hat{x}) \right)^2 \psi(\hat{x}). \] (24)

The solution to this equation is obtained from the commutative solution through the shift of coordinates
\[ \psi(x) = \psi(x) \exp \left[ i \int \frac{\hat{x}}{c} \left( A_i(\hat{x}) + \frac{1}{2} \partial_k \theta_{ij} \partial_j A_i(\hat{x}) \right) \right]. \] (25)

where \( \psi \) is the solution of the equation with vanishing gauge potential and \( p_i \) is now the eigenvalue of \( \hat{p}_i \) as \( \hat{p}_i \) only acts on \( \psi \) in (24) because of the antisymmetry of \( \theta_{ij} \). It was shown in [11] that the phase shift in this solution is equivalent to the path integral result obtained in [9,10], i.e., Eq. (15) in the previous section and thus is neither gauge invariant nor covariant.

A comment is in order about the gauge invariance of this approach. In view of the shifted coordinate, the Hamiltonian and any physical observables are manifestly invariant under the coordinate shifted gauge transformation but not under the ordinary star gauge transformation. Here the coordinate shifted gauge transformation is defined by the commutative \( U(1) \) gauge transformation evaluated in the shifted coordinate \( x_i - \frac{1}{2} \theta_{ij} p_j \).

4. The gauge covariant phase factor: The Wilson loop

In this section we propose a gauge covariant phase factor which can be obtained with the help of the Wilson loop operator. Let us first consider the AB phase in commutative quantum mechanics. The Schrödinger equation in the presence of a time independent vector potential is
\[ i \frac{\partial \psi_{\text{Comm}}}{\partial t} = \frac{1}{2} \left( p_i + A_i(x) \right)^2 \psi_{\text{Comm}}. \] (26)

This equation is solved by

\[ \psi_{\text{Comm}}(x,t) = \psi(x,t) \exp \left[ -i \int \frac{\hat{x}}{c} \left( A_i(\hat{x}) \right) \right]. \] (27)

Here \( \psi(x,t) \) is the solution of the Schrödinger equation in the absence of the vector potential. The integral is performed along a path \( C \) which ends in the point \( x \).

The phase factor \( \exp[-i \int \frac{\hat{x}}{c} \partial \psi_{\text{Comm}}(A_i(\hat{x}))] \) in (27) is clearly gauge invariant under the \( U(1) \) gauge transformation \( \delta A_i = -\theta_{ij} A_j \). The AB phase in the commutative case is evaluated as the gauge invariant magnetic field \( \hat{B} \) through Stokes theorem \( \int_{C} d\hat{\epsilon} \cdot \hat{A} = \int_{S} d\hat{\epsilon} \cdot \hat{B} \) where the boundary of \( S \) is the closed path \( C \). Consequently the observable is gauge invariant (see, e.g., [17,18]).

On the other hand, the Schrödinger equation in NCQM is
\[ i \frac{\partial \psi(x,t)}{\partial t} = \frac{1}{2} \left( \hat{P}_i + A_i(x) \right)^2 \psi(x,t), \] (28)

where all \( x \)-dependent terms are evaluated by the star-product with respect to \( x \). We recall that a gauge invariant quantity in a non-Abelian gauge theory is the Wilson loop. Wilson loops have been previously used in the context of noncommutative gauge field theories for constructing observable quantities, as well as new representations of the noncommutative gauge groups, formed by the no-go theorem of noncommutative gauge theories (see, e.g., [19–21] and references therein). They are defined by the gauge trace of the path-ordered exponential. Inspired by this, we consider the Ansatz for the solution to (28) as
\[ \Psi(x, x_0, t) = \text{Pexp}_{x_0} \left[ -i \int_0^t ds_0 \frac{d\xi(s)}{ds} A_i(x_0 + \xi(s)) \right] \]

(29)

Here the symbol \( \text{P} \) stands for path ordering. The parameter \( 0 \leq s \leq 1 \) parameterizes the path \( C \) with endpoints \( x_0 + \xi(0) = x_0 \) and \( x_0 + \xi(1) = x_0 + t = x \), where \( \xi(0) = 0 \) and \( \xi(1) = t \). \( \Psi(x, x_0, t) \) is the solution of the free Schrödinger equation

\[ -\nabla^2 \psi(x, x_0, t) = \frac{i}{\hbar} \frac{\partial \psi(x, x_0, t)}{\partial t}. \]

(30)

In the case of the AB experiment, \( x_0 \) represents the location of the source of electrons and \( x \) represents the point at which the intensity of the beam is evaluated. The free solution \( \psi(x, x_0, t) \) can also be viewed as a wavefunction at the point \((x_0, t_0)\) from which it is taken to \((x, t)\) by the free propagator, \( \mathcal{K}_{\text{free}}(x, t; x_0, t_0)\).

The definition of the path-ordered exponential is

\[ \mathcal{U}(x, x_0, C) \equiv \text{Pexp}_{x_0} \left[ -i \int_0^1 ds \frac{d\xi(s)}{ds} A_i(x_0 + \xi(s)) \right] = 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^1 ds_1 \int_0^{s_2} ds_2 \ldots \int_0^{s_n} \frac{d\xi(s_1)}{ds_1} \frac{d\xi(s_2)}{ds_2} \ldots \frac{d\xi(s_n)}{ds_n} \times \prod_{i=1}^n A_i(x_0 + \xi(s_1)) \ast x_0 \ldots x_0 A_i(x_0 + \xi(s_n)) \] \]

(31)

This is nothing but a Wilson line in noncommutative gauge theory [19] and under NC gauge transformations it transforms as:

\[ \mathcal{U}(x, x_0, C) \rightarrow U(x) \ast x \mathcal{U}(x, x_0, C) \ast x_0 U^{-1}(x_0). \]

(32)

It can be shown (see Appendix A) that this path-ordered exponential satisfies the equation

\[ \tilde{V}_i \mathcal{U}(x, x_0, C) = -i \tilde{A}(x) \ast x \mathcal{U}(x, x_0, C). \]

(33)

Let us check the Ansatz (29), starting with the r.h.s. of the NC Schrödinger equation (28), which reads:

\[ H \ast x \Psi = \frac{1}{2} \left[ -\nabla^2 \Psi - 2i\tilde{A} \ast x \Psi - i(\tilde{\psi} \cdot \tilde{A}) \ast x \Psi \right] \]

\[ + \tilde{A} \ast x \tilde{A} \ast x \Psi. \]

(34)

For the evaluation of (34) we shall need:

\[ \tilde{\psi} \Psi = -i\tilde{A} \ast x e_p \ast x_0 \Psi + e_p \ast x_0 \tilde{\psi} \Psi. \]

(35)

\[ -\nabla^2 \Psi = -i(\tilde{\psi} \cdot \tilde{A}) \ast x e_p \ast x_0 \Psi + \tilde{\psi} \AST x \ast \tilde{A} \ast x e_p \ast x_0 \Psi \]

\[ -i\tilde{A} \ast x e_p \ast x_0 \tilde{\psi} \Psi - i\tilde{A} \ast x \ast \tilde{A} \ast x \tilde{\psi} \Psi - i\tilde{A} \ast x e_p \ast x_0 \tilde{\psi} \Psi + e_p \ast x_0 \tilde{\psi} \Psi. \]

(36)

where \( \Psi = e_p \ast x_0 \Psi \) and \( e_p \) stands for \( \text{Pexp}_{x_0}[-i \int_0^1 ds \frac{d\xi(s)}{ds} A_i(x_0 + \xi(s))] \).

The l.h.s. of the NC Schrödinger equation (28) is

\[ i \frac{\partial}{\partial t} \Psi \ast x_0 i \frac{\partial}{\partial t} \Psi \]

\[ = -\frac{1}{2} \left[ -\nabla^2 \Psi - i(\tilde{\psi} \cdot \tilde{A}) \ast x e_p \ast x_0 \Psi + \tilde{A} \ast x \tilde{A} \ast x e_p \ast x_0 \Psi \right] \]

\[ + 2i \tilde{A} \ast x e_p \ast x_0 \tilde{\psi} \Psi \]

\[ = \frac{1}{2} \left[ -\nabla^2 \Psi - i(\tilde{\psi} \cdot \tilde{A}) \ast x \Psi + \tilde{A} \ast x \tilde{A} \ast x \Psi \right]. \]

(37)

This is exactly \( H \ast x \Psi \) as in (34). Thus the Ansatz (29) satisfies

\[ i \frac{\partial}{\partial t} \Psi = H \ast x \Psi. \]

(38)

The path-ordered exponential (31) is hard to evaluate explicitly but it can be done for an infinitesimal closed path \( C_1 \) in the \( 1-2 \) plane depicted in Fig. 1. We can show that

\[ \mathcal{U}(x, x_0, C_1) \equiv \mathcal{U}(x, x_0 + \epsilon e_2) \ast x_0 \mathcal{U}(x + \epsilon e_1, x + \epsilon e_1 + \epsilon e_2) \]

\[ \ast x_0 \mathcal{U}(x + \epsilon e_1, x) \]

\[ = \text{exp}[ -i \epsilon^2 (\tilde{A}_1 \Psi(x) - \tilde{A}_2 \Psi(x)) + \epsilon^2 (\tilde{A}_1(x), \tilde{A}_2(x))] \]

\[ + C(3)^2 \]

\[ = \text{exp}[ -i \epsilon^2 \tilde{F}_{12}] + C(3)^2. \]

(39)

where \( \epsilon \ll 1 \) is the infinitesimal parameter and \( e_1, e_2 \) are unit vectors along the directions 1 and 2. The star-product is evaluated at \( x \) and the field strength is defined by (7). The result is manifestly gauge covariant. A generalization of this result to \( U_4(N) \) is possible by replacing \( \tilde{A}_i \) by \( A^T_i T^a \), where \( T^a \) are the generators of \( U(N) \).

The NCAB phase factor for a path \( a \) from \( x_0 \) to \( x \) is given by

\[ e^{i \phi_{BNC}(x, x_0, a)} = \text{Pexp}_{x_0} \left[ -i \int_0^1 ds \frac{d\xi(s)}{ds} A_i(x_0 + \xi(s)) \right]. \]

(40)

where the path \( a \) is parametrized appropriately in the line integral. In view of the gauge transformation (32), it transforms as

\[ e^{i \phi_{BNC}(x, x_0, a)} \rightarrow U(x) \ast x \epsilon^{\frac{i}{2} \phi_{BNC}(x, x_0, a)} \ast x_0 U^{-1}(x_0), \]

(41)

under a gauge transformation.

The path-ordered phase factor appearing here is quite similar to the non-Abelian counterpart of the AB phase [22]. This would be related to the topological features of the phase factor which will be studied elsewhere [23].

One important consistency check for the Ansatz (29) is its gauge covariance. The wave function \( \Psi(x, x_0, t) \) has to transform in the fundamental representation of \( U_4(1) \), and its Hermitian conjugate, correspondingly, in the antifundamental representation,

\[ \Psi(x, x_0, t) \rightarrow U(x) \ast x \Psi(x, x_0, t), \]

\[ \Psi^+(x, x_0, t) \rightarrow \Psi^+(x, x_0, t) \ast x U^{-1}(x_0), \]

(42)

in order to insure the gauge covariance of the NC Schrödinger equation. One can show that the gauge transformation (32) of the path-ordered exponential is compatible with this gauge covariance requirement. Indeed, since \( \Psi(x, x_0, t) \) is solution of the NC Schrödinger equation, the path-ordered exponential (31) is invariant under gauge transformations.

Fig. 1. Closed path in 1–2 plane.
Schroedinger equation (28) with the initial condition \( \Psi(x_0, t_0) = \Psi(x_0, t_0) \), it follows that, according to (42), the initial condition will transform under gauge transformations as

\[
\Psi(x_0, t_0) \rightarrow U(x_0) \Psi(x_0, t_0).
\]

(43)

On the other hand, the formal general solution of (28) can be written using the total propagator \( K(x, t; x_0, t_0) \):

\[
\Psi(x, t) = K(x, t; x_0, t_0) \Psi(x_0, t_0).
\]

(44)

The total propagator factorizes into the free propagator and the gauge-field-dependent phase factor, such that the solution can be written as:

\[
\Psi(x, t) = \text{Pexp}_{x_0} \left[ -i \int_0^t ds \frac{ds}{d\xi} A_i(x_0 + \xi(s)) \right] \Psi(x_0, t_0).
\]

(45)

By comparing (29) with (45), it is clear that

\[
\Psi(x, t) = K(x, t; x_0, t_0) \Psi(x_0, t_0).
\]

(46)

and, in view of the fact that the free propagator does not transform under gauge transformations, while the initial solution \( \Psi(x_0, t_0) \) transforms as (43), the solution \( \Psi(x, t) \) of the free Schroedinger equation will have the peculiar gauge transformation:

\[
\Psi(x, t) \rightarrow U(x_0) \Psi(x, t).
\]

(47)

We should point out that \( \Psi(x, t) \) is not actually a genuine solution of a free Schroedinger equation, but an artifact of the factorization of the total propagator as in (45). In other words, from the dynamical point of view \( \Psi(x, t) \) satisfies the free Schroedinger equation, while inheriting at the same time the gauge transformation property (43) of the initial solution of (28).

The gauge transformations (32) and (47) provide the consistency check for the gauge covariance of \( \Psi(x, t) \) defined by the Ansatz (29). As a result, the noncommutative Schroedinger equation (28) is covariant under a noncommutative gauge transformation. This guarantees that the observable probability density \( P(x, t) \), for the AB-effect of two waves differing by a phase depending on the paths \( a \) or \( b \),

\[
P(x, t) = \left( \Psi^\dagger(x, t) \Psi(x_0, t) e^{-i\phi_N(x, x_0, a)} + \Psi^\dagger(x, t) \Psi(x_0, t) e^{-i\phi_N(x, x_0, b)} \right) \Psi(x_0, t),
\]

(48)

is gauge invariant.

5. Summary and discussion

In this Letter, we have studied the gauge covariance of the wave function phase factor in the framework of NCQM. Due to the fact that the phase factor in a wave function is frequently related to a physical observable, it is important to investigate the gauge invariance and covariance of it in NCQM. The AB phase factor is probably the most familiar observable phase factor in quantum mechanics. The naive path-integral formulation of NCQM violates the star gauge covariance of the AB phase. The origin of this violation comes from the Weyl ordered quantum mechanical Hamiltonian and midpoint prescription in the short-time propagator. This is quite different from the commutative case where the Hamiltonian itself is \( U(1) \) gauge invariant and hence the propagator is bi-locally gauge covariant.

The same result is obtained by shifting the coordinates of NCQM, whence the \( U(1) \) gauge invariance/covariance is broken. However, some exotic gauge invariance, the “shifted gauge invariance” (see end of Section 3) is preserved although the physical meaning of this type of gauge invariance is not clear.

We have found a gauge covariant AB phase factor which is defined by the path-ordered exponential. This resembles the well-known Wilson loop in non-Abelian gauge theory. We have shown that the path-ordered exponential is consistent with the noncommutative Schroedinger equation. We would like to stress that our result is quite similar to the non-Abelian AB phase proposed in [22]. This is very natural because the \( U(1) \) gauge symmetry is essentially non-Abelian, which can be seen from Eq. (7).

The AB phase factor is related to the Dirac monopole quantization and topological properties of the theory and it would be interesting to find the gauge invariant quantization condition corresponding to the noncommutative Dirac monopole, especially due to the results in [24] on noncommutative monopoles, dyons and soliton solutions. It would also be interesting to investigate the star gauge invariant path-integral formulation of NCQM [23].

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Appendix A

In this appendix the relation

\[
\frac{d}{ds} \hat{U}(x, t_0, C) = -i A_i(x) \Psi \hat{U}(x, t_0, C),
\]

(A.1)

where

\[
\hat{U}(x_0, C) = \text{Pexp}_{x_0} \left[ -i \int_0^1 ds \frac{ds}{d\xi} A_i(x_0 + \xi(s)) \right]
\]

\[
\times \left( 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^s ds_1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} ds_n \right)
\]

\[
\times A_i(x_0 + \xi(s_1)) \Psi \cdots A_i(x_0 + \xi(s_n)) \Psi(x_0, t_0).
\]

(A.2)

is proven. The parametrization of the path \( C \) is as follows: \( x = x_0 + \xi(1) = x_0 + \xi(0) = 0 \), so that \( \xi(1) = 1 \) and \( \xi(0) = 0 \). We will begin by considering the path-ordered exponential as a continuous function of the parameter \( s' \) in the form

\[
\hat{U}(x(s'), x_0, C)
\]

\[
= 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^{s_1} \cdots \int_0^{s_{n-1}} \int_0^{s_n} \frac{ds_1 \cdots ds_n}{ds_1 \cdots ds_n} \times A_i(x_0 + \xi(s_1)) \Psi \cdots A_i(x_0 + \xi(s_n)) \Psi(x_0, t_0).
\]

(A.3)

This can be differentiated with respect to \( s' \) using the result
\[ \frac{b}{a} \int f(x) \, dx = f(b). \]

It gives

\[ \partial_x \mathcal{L}(x(s'), x_0, C) = \partial_x \mathcal{L}(x(s'), x_0, C), \]

(A.4)

\[ \partial_x \mathcal{L}(x(s'), x_0, C) = -i \frac{d \xi(s')}{ds'} \mathcal{L}(x(s'), x_0, C). \]

(A.5)

where the dummy indices of summation have been renamed to \( n - 1 = k \) and \( l_i = i \) in going from Eq. (A.6) to Eq. (A.7). In Eq. (A.8), the integration variables have been renamed from Eq. (A.7) by decrementing the value of \( k \) by 1 in order to make the result more transparent. Note that this calculation could be done because the star-products are taken with respect to \( x_0 \) and do not influence the integration.

The newly obtained relation (A.9) can also be written in the form

\[ \frac{d \xi(s')}{ds'} \mathcal{L}(x(s'), x_0, C) = -i \frac{d \xi(s')}{ds'} \mathcal{L}(x(s'), x_0, C). \]

(A.9)

If we then go back to the path-ordered exponential as given by (A.2) and consider it as a function depending on two points, the initial and final point, we notice that we can interpret \( \xi(s') \) as the point \( l_i = \xi(1) \) in the parametrization of (A.2). This leads to the relation

\[ \frac{d}{dl_i} \mathcal{L}(x(s), x_0, C) = -i A_i(x_0 + l_i) \star_{x_0} \mathcal{L}(x, x_0, C), \]

(A.10)

from Eq. (A.10). This relation can be written in the form (A.1) by noting that since \( x_i = x_0 + l_i \) we have relations of the form

\[ \frac{d}{dl_i} \mathcal{L}(x(s), x_0, C) = -i \frac{d(x_0 + l_i)}{dl_i} \mathcal{L}(x(s), x_0, C), \]

(A.11)

\[ \frac{d}{dl_i} \mathcal{L}(x(s), x_0, C) = -i \frac{d(x_0 + l_i)}{dl_i} \mathcal{L}(x(s), x_0, C), \]

(A.12)

so that the product with respect to \( x_0 \) in (A.11) can safely be transformed into a star-product with respect to \( x \) and therefore we finally have

\[ \frac{d}{dl_i} \mathcal{L}(x(s), x_0, C) = -i A_i(x) \star_{x} \mathcal{L}(x, x_0, C), \]

(A.13)

which is exactly (A.1) or (33).

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