On the (0, 4) Conformal Field Theory of the Throat

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Abstract

In SO(32) heterotic string theory, the space–time at the core of $N$ coincident NS–fivebranes is an infinite throat, $\mathbb{R} \times S^3$. As shown by Witten, the throat signals a singularity in the usual heterotic string conformal field theory and a non–perturbative $USp(2N)$ gauge group appears, due to the $N$ small instantons at the fivebranes’ core. Nevertheless, we look for some trace of the non–perturbative physics in a description of the heterotic string infinitely far down the throat. Our guide is a D1–brane probing $N$ D5–branes in type I, which yields a 1+1 dimensional (0, 4) supersymmetric model with ADHM data in its couplings, as shown by Douglas. The neighbourhood of the classical boundary of the hypermultiplet moduli space of the theory flows to an exact conformal field theory description of the throat theory. Ironically, the remnant of the non–perturbative symmetry is indeed found in the conformal field theory, lurking in the structure of the partition function, and encoded in a family of deformations of the theory along flat directions. The deformations have an explicit description using the flow from the type I theory, and have a hyperKähler structure. Similar results hold true for the analogous (4, 4) supersymmetric situation in the type IIB theory, as is evident in the work of Diaconescu and Seiberg.

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1. Fivebranes and Instantons

In the type I string theory, D5–branes are forced\(^1\) to travel in pairs by the orientifold projection \(\Omega\). This dynamical unit carries an \(SU(2)\) gauge symmetry on their 5+1 dimensional world–volume\(^2\). Consider a coincidence of \(N\) of these pairs. The gauge symmetry on the world volume is then \(USp(2N)\).

In the dual \(SO(32)\) heterotic string theory, the \(N\) brane pairs become \(N\) Neveu–Schwarz (NS) fivebranes with zero size instantons at their core, and the metric, dilaton and \(H\)–field are\(^3\):

\[
\begin{align*}
&ds^2_H = \left(-dt^2 + \sum_{i=1}^{5} dy_i^2\right) + \left(e^{2\Phi_0} + \frac{N\alpha'}{x^2}\right) \left(dx^2 + x^2 d\Omega^2_3\right) \\
&e^\Phi = \left(e^{2\Phi_0} + \frac{N\alpha'}{x^2}\right)^{\frac{1}{2}}, \quad H = -\epsilon_{\mu\nu\lambda}^\kappa \partial_\kappa \Phi.
\end{align*}
\]

(1.1)

where \(t\) is time, the \(y_i\)'s are coordinates on the world–volume, \(d\Omega^2_3\) is the metric on a round unit three sphere, and \(x\) is the radial variable in the transverse space.

Near the core of the configuration (\(i.e.,\) near \(x=0\) —or equivalently, for large \(N\)— the metric may be written in terms of a new radial coordinate\(^4\) \(\sigma = \sqrt{N\alpha'} \log(x/\sqrt{N\alpha'})\):

\[
\begin{align*}
&ds^2_H = -dt^2 + \sum_{i=1}^{5} dy_i^2 + (d\sigma^2 + N\alpha' d\Omega^2_3) \\
&\Phi = -\frac{\sigma}{\sqrt{N\alpha'}} + \text{constant}, \quad H = -N\alpha' \epsilon_3,
\end{align*}
\]

(1.2)

which is \(\mathbb{R}^7 \times S^3\), where \(\epsilon_3\) is the \(S^3\) volume form.

The solution is smooth everywhere, (in contrast to the type I situation where the three–sphere vanishes at finite proper distance). The asymptotic flat spacetime has decoupled from the solution and is infinitely far away in these coordinates.

The appearance of the infinite throat \(\mathbb{R} \times S^3\) in the transverse geometry, with a negative linear dilaton\(^5\) signaling the runaway growth of the string coupling far down it, has been of interest for some time. In the current language\(^2\), the appearance of such a throat, which occurs only when the instantons are of zero size, signals a singularity in the perturbative description of the heterotic string. Non–perturbative effects become important in this regime, and they have the distinction that they persist no matter how small one can make the heterotic string coupling far away from the branes: the strong coupling effects merely occur farther down the throat, the mouth of which is at a point in \(\mathbb{R}^4\).

Luckily, we have a concise perturbative description of these heterotic non–perturbative effects in the dual type I string theory\(^2\): A gauge symmetry \(USp(2N)\) (from 5–5 strings) appears on the world–volume of the dual D5–branes, to accompany the \(SO(32)\) from
the D9–branes (from 9–9 strings), accompanied by hypermultiplets in the \((2N, 32)\), (5–9 strings) and the antisymmetric tensor of \(USp(2N)\) (5–5 strings).

One might wonder, however, if there is still some way of capturing some sign of this non–perturbative physics in a heterotic string description. Necessarily, this description will have to be of the physics down the throat, in order to be consistent with the previous two paragraphs. As the region deep inside the throat is far away from the outside, it is conceivable that this may be possible, with the expectation that the transition between the description inside the throat and the description outside must lie outside heterotic perturbation theory.

With this reasoning in mind, one is led to be optimistic that there is a \textit{conformal field theory} description of the heterotic string down the throat, and that it might somehow contain the \(USp(2N)\) structure. There is a limit to the optimism, of course. There is simply no way that we can expect to find a conformal field theory description containing \(USp(2N)\) as a \textit{gauge symmetry}, as the central charge of an affine algebra based on that group rapidly exceeds 26 as \(N\) increases. Therefore, the \(USp(2N)\) must appear in a more subtle way.

2. \textit{Probing in Type I}

Let us return to the type I picture for a while, in order to gather clues. There, the D1–brane represents an infinitely heavy heterotic string soliton, which ultimately becomes the light fundamental heterotic string in the strong coupling limit. We will consider one D1–brane. (Formulae relevant to a collection of \(2M\) D1–brane probes may be found in ref. [6].)

We locate the D5–branes at a point in \((x^6, x^7, x^8, x^9)\), and the D1–brane’s world–volume will lie in \((x^0, x^1)\). This arrangement breaks the Lorentz group up as follows:

\[
SO(1, 9) \supset SO(1, 1)^{01} \times SO(4)^{2345} \times SO(4)^{6789},
\]

(2.1)

where the superscripts denote the sub–spacetimes in which the surviving factors act. Following refs. [7,8], we may label the worldsheet fields according to how they transform under the covering group (which acts as a global symmetry of our final 1+1 dimensional model):

\[
G = [SU(2)' \times SU(2)']_{2345} \times [SU(2)_R \times SU(2)_L]_{6789},
\]

(2.2)

with doublet indices \((A', \tilde{A}', A, Y)\), respectively.

Accordingly, the 16 supercharges of the type I string decompose under (2.1) first (due to the D–strings) as \(16=8_++8_-\) where ± subscripts denote a chirality with respect to \(SO(1, 1)\), and furthermore (due to the D5–branes) each \(8\) decomposes into a pair of \(4\)’s of the \(SO(4)\)’s. The orientation projection \(\Omega\) will later pick out one of these spinors to carry
the supersymmetry on the D1–brane world–volume, giving a 1+1 dimensional system with
(0, 4) supersymmetry.

The spectrum of massless fields in the model produces a family of fields on the world–
volume, with coordinates \((x^0, x^1)\). The supersymmetry algebra is of the form[7]:

\[
\{Q^{AA'}, Q^{BB'}\} = \epsilon^{AB} \epsilon^{A'B'} P_-, \tag{2.3}
\]

where \(P_- = -i\partial/\partial\sigma^-, \) where \(\sigma^\pm = (x^0 \pm x^1)/2\). Here, \(\epsilon^{AB}\) and \(\epsilon^{A'B'}\) are the antisymmetric
tensors of \(SU(2)\) and \(SU(2)\)\(^\prime\), respectively, \(A, B, A'\) and \(B'\) being doublet indices.

The various fields which arise in the model from the 1–1, 1–5 and 1–9 sector may be
obtained using the standard D–brane calculus, reviewed in refs.[9,10]. For the specific case
here, the fields are already worked out in ref.[8], where it was noticed that the model is
isomorphic to the theory constructed in ref.[7].

The fields are as follows: In the 1–1 sector we get a family of four–component scalars
\(b^{A'}\tilde{A'}\) which parametrize the motion of the D1–brane inside the D5–branes. There is also
a hypermultiplet field \(b^{AY}(x^0, x^1)\), parametrizing the motion of the D1–brane transverse
to the D5–branes.

The superpartners consist of the four component fermion \(\psi^{A'}\tilde{A'}\), the right–moving super–
partner of the four component scalar field \(b^{A'}\tilde{A'}\), and \(\psi^{AY}\) is the right–moving superpart–
ner of \(b^{AY}\). The supersymmetry transformations are:

\[
\delta b^{A'}\tilde{A'} = i\epsilon_{AB} \eta^A_{\pm} \psi^{B}\tilde{A'} \quad \text{and} \quad \delta b^{AY} = i\epsilon_{A'B'} \eta^{A'A'} \psi^{B'Y}. \tag{2.4}
\]

From the 1–9 sector, we have left–moving fermions \(\lambda_+\) in the 32, coming from the fact that
the 1–9 strings end on the D9–branes which carry a gauge symmetry \(SO(32)\). These are the
current algebra fermions of the heterotic string[11]. We denote a component by \(\chi^M_+\),
where \(M\) is a D9–brane index, whenever we need to explicitly show the index structure
under the global symmetry of the D9–brane gauge group.

The 1–5 string excitations give a supermultiplet in the 2N of the D5–branes’ \(USp(2N)\)
gauge group, with components \((\phi^{A'm}, \chi_{-m})\):

\[
\delta \phi^{A'm} = i\epsilon_{AB} \eta^A_{\pm} \chi^{Bm}. \tag{2.5}
\]

Here, \(m\) is a D5–brane group theory index. (Note that because \(\Omega\) forces the D5–branes
to be paired, \(m\) is even, and so \(\phi\) naturally has components in multiples of four, like a
hypermultiplet. Actually, it is a “twisted” hypermultiplet.) Also, (with components \(\chi^Y_{-m}\)),
there are left moving fermions \(\chi^Y_+\) transforming in the 2N.

The 9–5 and 5–5 hypermultiplets of the 5+1 dimensional theory appear as couplings in
the 1+1 dimensional theory. From the 5–5 sector there is the antisymmetric of \(USp(2N)\)
denoted $X_{mn}^{AY}$. Meanwhile, the 9–5 sector produces a $(32, \2N)$, denoted $h_{M}^{Am}$, with $m$ and $M$ showing off its choices in D5– and D9–brane group theory.

We can now display the supersymmetry transformation relating them to the left moving fields:

$$\delta \lambda^{M}_{+} = \eta^{AA'}^{+} C_{AA'}^{M}$$
$$\delta \chi^{m}_{+} = \eta^{AA'}^{+} C_{AA'}^{Ym},$$

where

$$C_{AA'}^{M} = h_{A}^{M} m_{A}^{φ_{A'm}},$$
$$C_{AA'}^{Ym} = φ_{A'}^{m} (X_{A'n}^{m} - b_{A}^{A'} Y_{m}).$$

These precise transformations allow us to write the non–trivial part of the $(0, 4)$ supersymmetric 1+1 dimensional Lagrangian containing the Yukawa couplings and the potential:

$$L_{tot} = L_{kinetic} - \frac{i}{4} \int d^{2}σ \left[ \lambda^{M}_{+} \left( \epsilon^{BD} \frac{∂C^{M}_{BB'}}{∂b_{D'}Y} \psi_{-}^{B'}Y + \epsilon^{B'D'} \frac{∂C^{M}_{BB'}}{∂φ_{B'D'}m} χ_{-}^{Bm} \right) + \chi^{m}_{+} \left( \epsilon^{BD} \frac{∂C^{Ym}_{BB'}}{∂b_{D'}Y} \psi_{-}^{B'}Y + \epsilon^{B'D'} \frac{∂C^{Ym}_{BB'}}{∂φ_{B'D'}m} χ_{-}^{Bm} \right) \right. \left. + \frac{1}{2} \epsilon^{AB} \epsilon^{A'B'} \left( C_{AA'}^{M} C_{BB'}^{M} + C_{AA'}^{Ym} C_{BB'}^{Ym} \right) \right].$$

This was derived in ref.[7] as the most general $(0, 4)$ supersymmetric Lagrangian with these types of multiplets, providing that the $C$ satisfy the condition:

$$C_{AA'}^{M} C_{BB'}^{M} + C_{AA'}^{Ym} C_{BB'}^{Ym} + C_{BA'}^{M} C_{AB'}^{M} + C_{BA'}^{Ym} C_{AB'}^{Ym} = 0,$$

which they do[8]. $L_{kinetic}$ contains the usual kinetic terms for all of the fields, and the required terms which complete them into gauge invariant terms.

This theory is very interesting. The structure of (part of) the moduli space of vacua $b_{AY}$ was shown in ref.[7] (see also ref.[8] for the full translation into D1–brane probe language) to be equivalent to specifying the data shown by Atiyah, Hitchin, Drinfeld and Manin[12] required for the full specification of instantons. This proved that D5–branes were indeed instantons of the D9–brane gauge theory, and that under duality, they were indeed related to the NS fivebranes which are instantons of the heterotic gauge group.

As noted in ref.[7], the potential is of the form

$$V = (h^{2} + (X - b)^{2})φ^{2}.$$
condition (2.9), the $X$ control the positions and orientation of the $N$ instantons while $h$ determines their size.

Our interest here is in the small instanton limit with the $2N$ D5–branes coincident, so we initially set the values of $X$ and $h$ to zero. Now the potential is of the form $b^2 \phi^2$, and another branch of moduli space, parameterized by $b=0, \phi \neq 0$, seems to opens up. This classical analysis seems to tell us that the branches touch each other at $\phi = b = 0$, but an analysis of the global symmetries of the two branches suggests that the origins of the $\phi$ branches are infinitely far away: The $\text{spin}(4)$ — the covering group of the $SO(4)_{2345}$ — acts on the bosons $\phi^{A^m}$ of the $\phi$ branch, but only on the fermions $\psi^{A^Y}$ of the $b^{AY}$ branch. Meanwhile, the $SU(2)_R$ acts on the bosons $b^{AY}$ of the ‘$b$ branch’ but only on the fermions $\chi^{A^m}_\pm$ of the ‘$\phi$ branch’. The two branches of moduli space have different symmetries and therefore should not touch each other, and the quantum theory should reveal this.

In two dimensions, the phrase “moduli space of vacua” is used with the understanding that wavefunctions can (and do) actually spread all over the moduli space. Therefore, for consistency with the previous paragraph, there must be an infinite wormhole separating the two

We can proceed further by finding the metric on the $b$ branch. General considerations give at one loop (it is not enough to do a tree level calculation on the hypermultiplet branch with only $(0,4)$ supersymmetry) gives:

$$ds^2 = \left( C + \frac{N\alpha'}{x^2} \right) (dx^2 + x^2 d\Omega_3^2), \quad H = -N\alpha' \epsilon_3,$$

with the obvious coordinates on the four–space parameterized by $b^{AY}$. The constant $C$ is the result of the tree level calculation, giving the hyperKähler space $\mathbb{R}^4$, and the $N/x^2$ results from letting the $N$ 1–5 hypermultiplet fields run around a loop. Strictly speaking, we have allowed the $X$ fields to be merely near zero to generate the appropriate cubic couplings implied by this result. This is equivalent to letting the D5–branes have the freedom to separate slightly. Furthermore, in the expression above, we have worked with $N(2N-1)$ equal masses $m_i$ for the 1–5 fields $\phi_i$, corresponding to $N$ coincident D5–branes. In the case where we allow such terms to be different (i.e., use up the full family of values for the $X^{AY}_{mn}$), the prefactor in the metric becomes $C + \sum_{i=1}^{N} |x - m_i|^{-2}$, a fact which will be important later.

The constant $C$ in the metric is actually proportional to $1/g_{YM}^2 \sim 1/g_1^2 \sim g_{5}^2 = e^{2\Phi_0}$.

After a change of variables to $\sigma = \sqrt{N\alpha'} \log(x/\sqrt{N\alpha'})$ (i.e., concentrating attention near the origin of the $b$ branch) we see that we develop the anticipated infinite throat geometry. We have recovered the fact that the origin of the $\phi$ branch is infinitely far away. In the

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1 See ref.[14]. See also refs.[15,16] for analysis of the $b$–branch and the $\phi$–branch, respectively, using anomaly arguments.
stringy strong coupling limit, this model flows to an infra–red fixed point and the model
is appropriate to the world–sheet theory of the heterotic string.

(It is worth noting here that this result also applies to the more general (0,4) models
derived in ref.[6], in the context of type I D1–branes on an ALE space: The results of
ref.[17] showed that even in the absence of D5–branes, the type I model of ref.[4] still
contains instantons hidden in the ALE spaces. This was confirmed directly with the probe
models of ref.[6], as it was shown that there are additional fields in the model coming from
1–9 strings ‘trapped’ at the fixed points and therefore mimicking the behaviour of 1–5
strings. The resulting fields are therefore analogues of the \( \phi \) fields discussed here. Even
in the absence of D5–branes therefore, the resulting metric on the moduli space would see
the throat–type core, but in that situation connected to an external spacetime with a \( \mathbb{Z}_2 \)
action at infinity.)

3. The Conformal Field Theory of the Throat

In the stringy strong coupling limit, this model flows to an infra–red fixed point.

The form of the throat geometry, with the linear dilaton is very suggestive of an exact
conformal field theory description based on affine \( SU(2)_N \) (a WZW model\[18\]) times a
Feigen–Fuchs\[19\] theory with screening charge \( (N\alpha')^{-1/2} \). This has been discussed many
times before, particularly for the (4,4) supersymmetric case in the context of type II string
theory\[4,20\].

We would like to try to understand more details about this theory, as it is the conformal
field theory description of the throat which we hoped to be led to earlier in the discussion.

3.1 Basic Content

In the case in hand, we need to find a description in terms of a (0,4) supersymmetric
conformal field theory. We have two clues from the heterotic side:

\( (i) \) The right moving sector must be supersymmetric, with \( c_R=15 \), and so we can expect
that the theory on the right will match the type II case.

\( (ii) \) The full \( SO(32) \) gauge symmetry must be present, as in the small instanton limit,
it is restored. Therefore, the left moving sector must contain an affine \( SO(32)_1 \), which
has \( c=16 \). (This is consistent with the fact that there is a left–moving \( SO(32) \) current
algebra carried by the \( \lambda_+ \) before the flow: It survives the limit.) The remaining part
of the left–moving sector must supply a central charge 10 for a total of \( c_L=26 \).

Let us remind ourselves of how the right–moving supersymmetric sector works. The six
world–volume directions each supply a free boson and majorana fermion, giving a central
charge \( 6 \times \frac{3}{2} = 9 \). The \( SU(2)_N \) theory is supersymmetried\[21\] by adding three majorana
fermions with values in the Lie algebra. They can be made into free fermions by a chiral
rotation. This is at the cost of shifting the level by \( N \mapsto N - 2 \). The central charge of the theory is therefore 9 from the world-volume, \( 3 - \frac{6}{N} + \frac{3}{2} \) from the supersymmetric affine \( SU(2)_N \) theory, together with \( 1 + \frac{6}{N} + \frac{1}{2} \) from the Fiegen–Fuchs theory, where we have added a fermion for supersymmetry. This gives \( c_R = 15 \).

For the left-moving sector, we take the \( c=16 \) affine \( SO(32)_1 \) theory and the \( c=6 \) contribution from the flat non-supersymmetric world-volume directions. The remaining central charge of 4 comes from taking only the bosonic part of the previous paragraph: \( 3 - \frac{6}{N} + 1 + \frac{6}{N} = 4 \), giving \( c_L = 26 \), as required.

It is important to note that the final central charges are independent of \( N \) (of course) in a way which requires a balance between the dilaton (Fiegen–Fuchs) sector, the WZW model, and the fermions.

So it appears that we can construct the required heterotic \((0, 4)\) conformal field theory as the theory on the hypermultiplet branch, representing the infinite throat physics.

We have one more extremely vital thing to specify, however. We need to establish that we can indeed construct a modular invariant partition function for the theory, and verify that it is the correct one implied by the flow from the type I theory. Is this possible?

Let us focus on the non-trivial \( SU(2)_N \) part, for a moment. If this were the type II theory, we would have the same supersymmetric structures on the left and the right, and the obvious “A-type” diagonal combination of the characters would work perfectly, as well as other non-diagonal “D-” and “E-type” combinations. Overall, we are of course, forced to have a non-diagonal combination in this heterotic case.

One set of modular invariants can be deduced as follows: Let us start with with a \((4, 4)\) model with A, D, or E modular invariants chosen for the level \( N - 2 \) WZW sector. Let us gather the eight majorana fermions on each side into an affine \( SO(8)_1 \) current algebra. We can use the diagonal modular invariant combination of those, for that sector. Similarly for the bosonic Fiegen–Fuchs theory.

Now we can construct a heterotic modular invariant as follows: On the left, take out the \( SO(8)_1 \) and replace it with an \( SO(32)_1 \). There is a modular invariant combination of these two theories, after a trivial exchange of the sign of the spinor and vector representations. We now have \( c_R = 15, c_L = 26 \).

### 3.2 Constraints from Duality

Note that this is in contrast to what has been done in interesting other work on the symmetric throat conformal field theory in the literature, to make heterotic backgrounds. There, the full supersymmetric \( c_L = c_R = 6 \) theory was used as a building block to construct \((0, 4)\) models with some of the heterotic gauge symmetry broken, effectively embedding the spin connection into the gauge group in the standard way.
These are not the conformal field theories implied by the flow of the linear sigma model from type I to heterotic. In particular, we know that the conformal field theory has a family of deformations with a particular structure: This is encoded in the freedom to generate mass–difference terms $m_i$ for the $N$ 1–5 fields $\phi^A_m$, corresponding to different expectation values of the $X_{mn}^{AB}$ hypermultiplets, allowing the D5–branes to move apart. The flow from the linear sigma model implies that the final conformal field theory must contain these deformations.

That the freedom to give vacuum expectation values to the $X_{AB}^{mn}$ fields (generating mass differences $m_i$ for the $\phi^A_m$’s) in the gauge theory corresponds to a family of deformations of our conformal field theory is straightforward to see. For example, adding a single mass difference $m_i$ will produce the geometry:

$$ds^2 = \left( C + \frac{1}{x-m} + \frac{N-1}{x} \right) (dx^2 + x^2 d\Omega_3^2).$$

(3.1)

So we have two throats after the flow of the model to the fixed point, one at $x=0$ and the other at $x=m$. The first will contain the affine $SU(2)$ at level 1, while the other has level $N-1$. Had we not generated $m_1$, we would have had a single throat containing affine $SU(2)$ and level $N$. However, recalling the discussion near the beginning of this section, all of these conformal field theories have $(c_R=15, c_L=26)$ due to the interplay of supersymmetry and the linear dilaton. In other words, from the point of view of (say) the first conformal field theory $(N)$, we have added an operator which produced a genuine deformation, as it preserved the overall central charge. The resulting conformal field theories $(N-1$ and 1) have now flown infinitely far apart.

This structure is all inherited from the fact that we can move branes apart in the dual theory, a freedom parametrized by the $USp(2N)$ antisymmetric tensor quartet $X_{mn}^{AY}$, the mass differences of the 1–5 fields $\phi^A_m$. Furthermore, as this costs no energy (the D5–branes are BPS states) we see that our family of conformal field theory deformations define true flat directions in this space of $(0,4)$ conformal field theories. (Note also that this family of deformations has a hyperKähler structure, inherited from the parent instanton moduli space.)

This realization of a family of deformations of the $(0,4)$ conformal field theory is amusing and satisfyingly explicit.

(Incidentally, there is another reason why the above choices cannot be relevant to our problem: We could have replaced the left $SO(8)_1$ with level 1 affine $E_8 \times E_8$ and arrived at the same solutions. This is of course reminding us of the other heterotic string. However, we know from multiple duality considerations that the behaviour of small $E_8 \times E_8$ instantons is very different [2,25], and not obtainable in this way. The fact that the $E_8 \times E_8$ current algebra can be inserted in such a trivial way is further evidence that these modular invariants proposed in the previous paragraph are not the solutions relevant to this discussion, and should be regarded as artifacts.)
In summary, the linear sigma model implies the existence of a family of deformations (but not symmetries, of course) of the $(0,4)$ conformal field theory of the throat, with a $USp(2N)$ structure. This data is encoded in the conformal field theory in terms of certain restrictions on how the various primary fields in the theory are constructed out of the left and right–moving sectors. In other words, the $USp(2N)$ is encoded in the required modular invariant mass matrix $M_{ij}$ appearing in the partition function:

$$Z = \text{Tr}(q^{L_0} \frac{C_R}{2\pi} q^{\bar{L}_0} \frac{C_R}{2\pi}) = \sum_{ij} M_{ij} \chi^i(q) \bar{\chi}^j(\bar{q}).$$

(3.2)

We do not propose to embark on a search for an explicit form for this $USp(2N)$ modular invariant partition function here, but merely stress that its existence is ensured by duality. Of course, it is more standard to discuss modular invariants’ isomorphic structure to the Dynkin diagrams of simply laced Lie algebras. Here, we are invoking an isomorphism to the non–simply laced algebra $C_N$. This is not unreasonable. Recall that the isomorphism is between the possible indices which can appear in the sum (3.2) and the eigenvalues of the adjacency matrix of the Dynkin diagram, given essentially by integers called the “exponents”. The exponents of $C_N$ are simply the odd integers $1,3,\ldots,2N-1$, while for $A_N$ it is the integers up to $1,2,\ldots,N$ and for $D_N$ it is $1,3,\ldots,2N-3$, and $N-1$. We expect that the exponents of $C_N$ will appear in the sum instead of those of $D_N$ by a projection which removes characters corresponding to the extra exponent $N-1$ and replaces it with $2N-1$. So in this way only the exponents of $C_N$ will appear in the sum, and so it is not unthinkable to find $C_N$ characterizing a modular invariant.

4. Closing Remarks

We have found what we were looking for.

• There is a heterotic description of the physics of the infinite throat in terms of a $(0,4)$ conformal field theory, designed as a heterotic variant of the exact conformal field theory presented in the literature for the $(4,4)$ type II case. The conformal field theory is IR limit of the linear sigma model description of the apparent boundary of the $b^{AY}$ hypermultiplet branch of moduli space of the theory on the world–volume of a D1–brane probing the D5–branes of the type I theory.

• There is a remnant of the non–perturbative (from the point of view of the usual heterotic variables) $USp(2N)$ symmetry living in this new description of the infinite throat. It

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3 It would be interesting to find it explicitly, though. It may be possible to obtain a description as a “heterotic coset model” [26].

4 In terms of Dynkin diagrams, this is tantamount to projecting the diagram of $D_N$ so that the two dovetailed roots at one end becomes the single long root at the end of the $C_N$ diagram.
encodes the modular invariant combination of affine characters which build the conformal field theory and is explicitly realized as an important family of deformations of the theory along truly flat directions. It did not appear as a gauge symmetry, as anticipated from general considerations of consistency.

This picture is consistent with what happens in the type IIB case. There, we may also engage in a discussion involving D1–branes and D5–branes, or fundamental strings and NS–fivebranes in the dual type IIB theory. In that case, the theory is (4, 4) supersymmetric, and the situation was studied by Diaconescu and Seiberg [28]. The conformal field theory (arising this time from the Coulomb branch: vector multiplet moduli space) is the usual symmetric $SU(2)_N$ WZW+ Feigen–Fuchs. The diagonal modular invariant $A_{N-1}$ encodes the fact that there is an $SU(N)$ gauge symmetry on the D5–branes not seen directly by the conformal field theory of the fundamental strings, but by the D1–strings. In the case where there is an orientifold O5–plane present (as distinct from an O9–plane), there is an $SO(2N)$ gauge symmetry on the D5–branes, and the resulting conformal field theory has the $D_{N-2}$ modular invariant.

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