THE SERIES THAT RAMANUJAN MISUNDERSTOOD

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Abstract. We give a new appraisal of the function $\Delta(x)$ and its zeroes in the equation $f(x) = g(x) + \Delta(x)$ where $f(x) = \sum_{n \in \mathbb{Z}} 2^n x^{2n}$ and $g(x) = 1/((\log 2)(\log(1/x)))$.

1. Introduction

Consider the bilateral infinite series that converges in the unit disc $|x| < 1$,

\begin{equation}
(1.1) \quad f(x) = \sum_{n \in \mathbb{Z}} 2^n x^{2n} = \ldots + 1024x^{1024} + 512x^{512} + 256x^{256} + 128x^{128} + 64x^{64} + 32x^{32} + 16x^{16} + 8x^8 + 4x^4 + 2x^2 + x + \frac{x^{1/2}}{2} + \frac{x^{1/4}}{4} + \frac{x^{1/8}}{8} + \frac{x^{1/16}}{16} + \ldots
\end{equation}

Next also for $|x| < 1$, consider the function,

\begin{equation}
(1.2) \quad g(x) = \frac{1}{\log 2 \log(1/x)}
\end{equation}

Ramanujan, in his theory of prime numbers in his pre-Cambridge days, seemed to believe that for all real $0 < x < 1$, $f(x) = g(x)$. In Hardy’s famous book on Ramanujan, we can form a view that Ramanujan was familiar with the Euler-McLaurin summation formula from the Carr Synopsis book he referred to constantly, and that this formula omitted the oscillating term $\Delta(x)$. As a result, Ramanujan inferred many things about the distribution of prime numbers as if there were no analytic theory introduced by Riemann in his landmark paper of 1859 which put the now-named Riemann Hypothesis, and gave the first proof of the Riemann zeta functional equation. Using contour integration and the residue theorem, the reality is that

\begin{equation}
(1.3) \quad f(x) = g(x) + \Delta(x),
\end{equation}

where $\Delta(x)$ oscillates around zero, and the amplitude of the oscillations only are noticeable around the third or fourth decimal place. Indeed, both $f(x)$ and $g(x)$ satisfy the functional equation $2f(x^2) = f(x)$, and $f(x) = g(x)$ approximately to 3 or 4 decimal places. The $\Delta(x)$ oscillations become more wriggly as $x$ approaches 1 near its limiting boundary value of convergence. G H Hardy was able to explain

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to Ramanujan that $\Delta(x)$ is an oscillating periodic function of $\log(\log(1/x))$. The correct formula corresponding to (1.3) is for $|x| < 1$,

$$
\sum 2^k x^{2k} = \frac{1}{\log 2 \log(1/x)} \left\{ 1 - \sum' \Gamma \left( 1 + \frac{2ki\pi}{\log 2} \right) \left( \log \left( \frac{1}{x} \right) \right)^{-2ki\pi/\log 2} \right\},
$$

with the sum $\sum$ over all integers $k$, and the sum $\sum'$ over all nonzero integers $k$.

The problem is to locate the zeroes of $\Delta(x)$, and so find where (1.3) above becomes $f(x) = g(x)$.

### 2. Approximation with self-similar oscillating function

#### 2.1. Function $\Delta_0(x)$

At $x$ close to 1, $\log(1/x) \approx (1 - x)$, and

$$
\Delta(x) = \frac{1}{\log 2 \log(1/x)} \sum' \Gamma \left( 1 + \frac{2ki\pi}{\log 2} \right) (\log(1/x))^{-2ki\pi/\log 2} \approx
$$

$$
\Delta_0(x) = \frac{1}{\log 2} \sum' \Gamma \left( 1 + \frac{2ki\pi}{\log 2} \right) (1 - x)^{-1 - 2ki\pi/\log 2}
$$

$\Delta_0(x)$ is a self-similar function, such that $\Delta_0((x + 1)/2) = 2\Delta_0(x)$. As $x$ approaches 1, period of oscillations of $\Delta_0(x)$ exponentially decreases, and its amplitude exponentially increases. Fig. (1) shows the plot of $\Delta_0(x)$. Due to such periodicity of $\Delta_0(x)$, it is enough to study this function at any interval $[x, (x + 1)/2]$ for complete knowledge of the function. The function $\Delta_0(x)$ is dominated by the largest ($k = \pm 1$) terms of the sum (2.2), and these two terms add to a function of the form

$$
\frac{b}{1 - x} \cos \left( \log(1 - x) \frac{2\pi}{\log 2} + \phi \right).
$$

It is sinusoidal, which get squeezed horizontally as $x$ approaches 1, and get stretched vertically.

We can study and write more about the function $\Delta_0(x)$ if needed. In the paper by Campbell [3] he refers to an ingenious approach to finding zeroes of a similar oscillating function examined in a study by Mahler [2], which may be applicable for the functions in our current paper.
Of particular interest to us are zeroes of $\Delta_0(x)$. The first zero of $\Delta_0(x)$ is $x_0 \approx 0.23628629$. All consecutive zeroes are given by

$$x_n = 1 - \frac{1 - x_0}{2^{n/2}}.$$  

2.2. Approximation of $\Delta(x)$ by $\Delta_0(x)$. How well is $\Delta(x)$ approximated by $\Delta_0(x)$? Fig. (2) shows the plots of both $\Delta_0(x)$ and $\Delta(x)$. At smaller $x$, $\Delta_0(x)$ and $\Delta(x)$ differ considerably, but as $x$ approaches 1, $\Delta_0(x)$ and $\Delta(x)$ converge.

2.2.1. Numerical estimates. A few first zeroes of $\Delta(x)$ and $\Delta_0(x)$, and relative error of approximation, given by $|((\text{zero of } \Delta(x)) - (\text{zero of } \Delta_0(x)))/(1 - (\text{zero of } \Delta(x)))|$ is shown in Table (1).

As $x$ approaches 1, the relative error goes to zero. The estimate of relative error can be given comparing Taylor series expansion for $\Delta(x)$ and $\Delta_0(x)$. If $x_z$ is a zero of $\Delta(x)$, and $x_{z0}$ is corresponding zero of $\Delta_0(x)$, then

$$1 - x_{z0} \approx (1 - x_z) + \frac{1}{2}(1 - x_z)^2 + \frac{1}{3}(1 - x_z)^3,$$

or

$$1 - x_z \approx (1 - x_{z0}) - \frac{1}{2}(1 - x_{z0})^2 - \frac{1}{3}(1 - x_{z0})^3.$$  

3. Arbitrary $a$

The results of the previous section are applicable to the equation with an arbitrary $a$ instead of 2. Consider the bilateral infinite series that converges for real $0 < x < 1$,

$$f(x) = \sum_{n \in \mathbb{Z}} a^n x^n$$

$$= ... + a^{10} x^{10} + a^9 x^9 + a^8 x^8 + a^7 x^7 + a^6 x^6 + a^5 x^5 + a^4 x^4$$

$$+ a^3 x^3 + a^2 x^2 + ax + x + \frac{x^{1/a}}{a} + \frac{x^{1/a^2}}{a^2} + \frac{x^{1/a^3}}{a^3} + \frac{x^{1/a^4}}{a^4}$$

$$+ \frac{x^{1/a^5}}{a^5} + \frac{x^{1/a^6}}{a^6} + \frac{x^{1/a^7}}{a^7} + \frac{x^{1/a^8}}{a^8}$$

$$+ \frac{x^{1/a^9}}{a^9} + \frac{x^{1/a^{10}}}{a^{10}} + ...$$
Next consider the function given by real $0 < x < 1$,

$$g(x) = \frac{1}{((\log a) \log(1/x))}.$$  \hfill (3.2)

$$f(x) = g(x) + \Delta(x),$$  \hfill (3.3)

where $\Delta(x)$ oscillates around zero, and the amplitude of the oscillations only are noticeable around the third or fourth decimal place. Both $f(x)$ and $g(x)$ satisfy the functional equation $af(x^2) = f(x)$, and $f(x) = g(x)$ approximately to 3 or 4 decimal places. The $\Delta(x)$ oscillations become more wriggly as $x$ approaches 1 near its limiting boundary value of convergence. $\Delta(x)$ is an oscillating periodic function.
of log(log(1/x)). The correct formula corresponding to (3.3) is for |x| < 1,

\[
\sum a^k x^k = \frac{1}{((\log a)(\log(1/x)))} \left\{ 1 - \sum' \Gamma \left( 1 + \frac{2ki\pi}{\log a} \right) \left( \log \left( \frac{1}{x} \right) \right)^{-2ki\pi/\log a} \right\},
\]

where the sum \( \sum \) is over all integers \( k \), and the sum \( \sum' \) is over all nonzero integers \( k \).

\( \Delta(x) \) may be approximated by

\[
\Delta_0(x) = \frac{1}{\log a} \sum' \Gamma \left( 1 + \frac{2ki\pi}{\log 2} \right) (1 - x)^{-1-2ki\pi/\log a}
\]

As an example, we consider \( a = 3 \). Fig. (3) shows the plots of both \( \Delta_0(x) \) and \( \Delta(x) \). At smaller \( x \), \( \Delta_0(x) \) and \( \Delta(x) \) differ considerably, but as \( x \) approaches 1, \( \Delta_0(x) \) and \( \Delta(x) \) converge.

Zeroes of \( \Delta(x) \) may be approximated by zeroes of \( \Delta_0(x) \). The first zero of \( \Delta_0(x) \) can be found by numerically solving equation \( \Delta_0(x) = 0 \). Approximately, taking only the first terms of the sum,

\[
\Gamma \left( 1 + \frac{2i\pi}{\log a} \right) (1 - x)^{-1-2i\pi/\log a} + \Gamma \left( 1 - \frac{2i\pi}{\log a} \right) (1 - x)^{-1+2i\pi/\log a} = 0
\]

(3.6)

\[
x_0 \approx 1 - e^{-\log a \left( \frac{\pi}{2} + \arg \left( \frac{1}{1 + \frac{2i\pi}{\log a}} \right) \right)}
\]

(3.7)

All consecutive zeroes are given by

\[
x_n = 1 - \frac{1 - x_0}{a^{n/2}}
\]

(3.8)

References

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