QUANTUM INVARIANTS OF DEFORMED FOURIER MATRICES

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ABSTRACT. We study the deformed tensor products of complex Hadamard matrices, $L_{ia,jb} = Q_{ib}H_{ij}K_{ab}$. One problem is that of reconstructing the quantum group $G_L \subset S_{NM}^+$ out of the quantum groups $G_H \subset S_N^+, G_K \subset S_M^+$ and of the parameter matrix $Q \in M_{N \times M}(\mathbb{T})$, and we obtain here several results, including complete results in the Fourier matrix case, $H = F_N, K = F_M$, when the parameter matrix $Q$ is generic.

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INTRODUCTION

A complex Hadamard matrix is a matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, $|H_{ij}| = 1$, and whose rows are pairwise orthogonal. The basic example is the Fourier matrix, $F_N = (\omega^{ij})$ with $\omega = e^{2\pi i/N}$:

$$F_N = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \ldots & \omega^{N-1} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \omega^{N-1} & \omega^{2(N-1)} & \ldots & \omega^{(N-1)^2}
\end{pmatrix}$$

Popa discovered in [35] that these matrices parametrize the orthogonal MASA in the simplest von Neumann algebra, $M_N(\mathbb{C})$. While very basic, this observation is extremely
heavy, conceptually speaking. In brief, it shows that the “core” of the von Neumann algebra theory is not exactly functional analysis, as it was believed for a long time, but rather linear algebra, combinatorics, Fourier analysis, perhaps some algebraic and differential geometry, plus of course, definitely, some low-dimensional topology.

As of now, this is not very surprising, and fits with the modern trend in mathematical physics, or mathematics in general, where operators on Hilbert spaces $T \in B(H)$ are slowly replaced by more concrete objects, such as matrices $M \in M_N(\mathbb{C})$.

As a last historical remark, the other main subfields of operator algebras have followed as well this general trend, and have now strong connections with particle physics, number theory, random matrices, ergodic theory, logic, and quantum information.

Let us go back now to an arbitrary complex Hadamard matrix $H \in M_N(\mathbb{C})$, as defined above, and try to explain what the problem is. There are two approaches here:

1. Subfactors. The idea here is that the pair of algebras $\Delta \perp H \Delta H^*$ produces via basic construction a subfactor $M \subset N$. The planar algebra $P = (P_k)$ of this subfactor has a combinatorial description in terms of $H$, and the problem is to compute its Poincaré series, $f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$. See [26], [28], [36].

2. Quantum groups. The idea here is that associated to $H$ is a certain quantum permutation group $G \subset S_N^+$, and the problem is to compute the spectral measure $\mu \in \mathcal{P}(\mathbb{R}_+)$ of the main character $\chi : G \to \mathbb{C}$. This is the same problem as above, because $f$ is the Stieltjes transform of $\mu$. See [4], [8], [10].

It goes without saying that the above two approaches are equivalent, but with a few slight differences, however. Whether the problem belongs to subfactors, or to quantum groups, or perhaps to something more complicated, is an open question. See [5].

In this paper we discuss the question for a well-known class of complex Hadamard matrices, constructed by Dita in [19]. Given two complex Hadamard matrices $H \in M_N(\mathbb{T})$ and $K \in M_M(\mathbb{T})$, the Dita deformation of the tensor product $(H \otimes K)_{ia,jb} = H_{ij} K_{ab}$, with matrix of parameters $Q \in M_{N\times M}(\mathbb{T})$, is by definition $H \otimes_Q K = (Q_{ib} H_{ij} K_{ab})_{ia,jb}$.

In matrix form, by using the lexicographic order on the double indices:

$$H \otimes_Q K = \begin{pmatrix}
Q_{11} H_{11} K_{11} & \cdots & Q_{1M} H_{11} K_{1M} \\
\vdots & & \vdots \\
Q_{11} H_{11} K_{M1} & \cdots & Q_{1M} H_{11} K_{MM}
\end{pmatrix} \begin{pmatrix}
Q_{11} H_{1N} K_{11} & \cdots & Q_{1M} H_{1N} K_{1M} \\
\vdots & & \vdots \\
Q_{11} H_{1N} K_{M1} & \cdots & Q_{1M} H_{1N} K_{MM}
\end{pmatrix} \begin{pmatrix}
Q_{N1} H_{N1} K_{11} & \cdots & Q_{NM} H_{N1} K_{1M} \\
\vdots & & \vdots \\
Q_{N1} H_{N1} K_{M1} & \cdots & Q_{NM} H_{N1} K_{MM}
\end{pmatrix} \begin{pmatrix}
Q_{N1} H_{NN} K_{11} & \cdots & Q_{NM} H_{NN} K_{1M} \\
\vdots & & \vdots \\
Q_{N1} H_{NN} K_{M1} & \cdots & Q_{NM} H_{NN} K_{MM}
\end{pmatrix}
$$

The computation of the quantum invariants of these matrices is a well-known problem, see [33]. The main result here is due to Burstein, who proved in [17] that in the Fourier case, $H = F_N, K = F_M$, the associated subfactors are of Bisch-Haagerup type [15].
In this paper we study the quantum groups associated to these matrices. The main problem here is that of reconstructing the quantum group $G_L \subset S^+_{N,M}$ out of the quantum groups $G_H \subset S^+_N, G_K \subset S^+_M$ and of the parameter matrix $Q \in M_{N \times M}(\mathbb{T})$. We will obtain here several results on this subject, our first result being as follows:

**Theorem A.** The quantum group associated to $L = H \otimes Q K$ satisfies

$$G_L \subset J_K \ast G_H$$

where $J_K \subset S^+_M$ is the quantum automorphism group of the magic matrix of $K$.

The quantum group $J_K$, related to Jones’ second commutant formula in\cite{26}, is known to contain $G_K$, and to be equal to it for instance when $K = F_M$.

As for $\ast$, this is a free wreath product, a notion introduced in\cite{12}.

The origins of the above result go back to our previous work\cite{8}, where the use of free wreath products in dealing with Dit\c{t}ă deformations was suggested, with some wrong statements (Theorem 6.8 there), however. Wrong as well is the $N = M = 2$ computation in\cite{10} (Theorem 6.1 there), and repairing these errors will be of course one of the goals of this paper. Our second result, repairing the error in\cite{10}, is as follows:

**Theorem B.** The quantum group associated to $L = F_2 \otimes Q F_2$ is given by

$$G_L = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ast \mathbb{Z}_4 & \text{if } n = 1 \text{ and } m = 1, 2 : : m = 4 \\ D_{2n}^{-1} \ast DC_{n}^{-1} & \text{if } 1 < n < \infty \text{ and } m \notin 4\mathbb{N} : : m \in 4\mathbb{N} \\ \mathbb{Z}_2 \ast \mathbb{Z}_2 & \text{if } n = \infty \end{cases}$$

where $q = ad/bc$, with $Q = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, and where $m = \text{ord}(q), n = \text{ord}(q^4)$.

In this statement $D_{2n}^{-1}, DC_{n}^{-1}$ are twists of the dihedral and dicyclic groups, constructed by Nikshych in\cite{34}. The proof uses Theorem A, and methods from\cite{6}.

Our third, and main result, is as follows:

**Theorem C.** The quantum group associated to $L = F_N \otimes Q F_M$ satisfies

$$G_L \subset (\mathbb{Z}^{(N-1)(M-1)} \rtimes \mathbb{Z}_M) \rtimes \mathbb{Z}_N$$

and in the case where $Q \in M_{N \times M}(\mathbb{T})$ is generic, this inclusion is an equality.

The proof uses Theorem A, and a number of further factorizations.

We should mention that Theorems B-C are of course inspired as well from Burstein’s work in\cite{17}. There is probably an abstract way of deducing them from Burstein’s results, and vice versa, but technically speaking, this would require quite a lot of work, because the quantum group interpretation of the Bisch-Haagerup subfactors is not known yet, unless in some special cases, as those discussed by Izumi and Kosaki in\cite{23}.

Finally, we will discuss some consequences of Theorems B-C, regarding the spectral measure $\mu = \text{law}(\chi)$. First, in the context of Theorem B, the principal graph is $D_{2n+2}$,
as already noted by Burstein in [17]. According now to [9], the associated measure is:

$$\mu = \frac{1}{2}(\delta_0 + \Phi_4 \varepsilon_4)$$

Here $\varepsilon_k$ is the uniform measure on the $k$-roots of unity, and $\Phi(z) = 4Re(z)^2$.

In the context of Theorem C now, our result is as follows:

**Theorem D.** For $L = F_N \otimes Q F_M$ with $Q \in M_{N \times M}(\mathbb{T})$ generic we have

$$N \int \left(\frac{\chi}{N}\right)^k = 1 + \left(\frac{k}{2}\right)(M - 1)N^{-1} + O(N^{-2})$$

for any $k \geq 1$, in the $N \to \infty$ limit.

We do not know how to explicitely compute $\mu$, nor how $\mu$ should specialize at non-generic values of $Q$. There are probably many interesting questions here.

The paper is organized as follows: 1 is a preliminary section, in 2-3-4-5 we state and prove Theorems A-B-C-D, and 6 contains some final remarks.

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1. **Quantum Permutations**

A complex Hadamard matrix is a matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, $|H_{ij}| = 1$, and whose rows are pairwise orthogonal.

The basic example is the Fourier matrix, $F_N = (\omega^{ij})$ with $\omega = e^{2\pi i/N}$.

It is also known that the tensor product of two Hadamard matrices, $(H \otimes K)_{ia,jb} = H_{ij}K_{ab}$, is Hadamard. As a basic example here, the Fourier matrix of a finite abelian group $F_G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_K}$ is the Hadamard matrix $F_G = F_{N_1} \otimes \ldots \otimes F_{N_K}$.

We have as well the following key construction, due to Diţă [19]:

**Proposition 1.1.** If $H \in M_N(\mathbb{C})$ and $K \in M_M(\mathbb{C})$ are Hadamard, then so is

$$H \otimes Q K = (Q_{ia}H_{ij}K_{ab})_{ia,jb}$$

for any choice of a rectangular matrix of parameters $Q \in M_{N \times M}(\mathbb{T})$.

**Proof.** Indeed, the scalar products between the rows are given by:

$$< R_{ia}, R_{kc} > = \sum_{jb} Q_{ia}H_{ij}K_{ab} = N\delta_{ik} \sum_b Q_{ib}K_{ab} = N M \delta_{ik} \delta_{ac}$$

Thus $H \otimes Q K$ is indeed complex Hadamard, as claimed.
As a first example, by deforming the Klein group matrix \( F_{2,2} = F_2 \otimes F_2 \) with the matrix of parameters \( Q = (1 \ 1 \ q, 1 \ q) \), we obtain the following Hadamard matrix:

\[
F^q_{2,2} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & q & -1 & -q \\
1 & -q & -1 & q
\end{pmatrix}
\]

Observe that Haagerup’s result in [21] tells us that any complex Hadamard matrix at \( N \leq 5 \) appears as a Dita deformation of a Fourier matrix. See [40].

Let us explain now what the quantum invariants are. We use:

**Definition 1.2.** A unitary \( u \in M_N(A) \) over a \( C^* \)-algebra is called “magic” if all its entries are projections, summing up to 1 on each row and column.

The basic example here is provided by the coordinate functions on \( S_N \subset O_N \) coming from the permutation matrices, which are given by:

\[
u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}
\]

Now let \( C(S_N^+) \) be the universal \( C^* \)-algebra generated by the entries of a \( N \times N \) magic matrix, with comultiplication, counit and antipode defined as follows:

\[
\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}
\]

Wang proved in [42] that this algebra satisfies Woronowicz’s axioms in [43]. Thus \( S_N^+ \) is a compact quantum group, called quantum permutation group.

If \( H \in M_N(\mathbb{C}) \) is Hadamard, with rows denoted \( H_1, \ldots, H_N \in \mathbb{T}^N \), then:

\[
\left\langle \frac{H_i}{H_j}, \frac{H_i}{H_k} \right\rangle = \sum_r \frac{H_{ir}}{H_{jr}} \cdot \frac{H_{kr}}{H_{jr}} = \sum_r \frac{H_{kr}}{H_{jr}} = \langle H_k, H_j \rangle = \delta_{jk}
\]

Thus the matrix of vectors \( \xi_{ij} = H_i/H_j \) has the property that each of its rows is an orthogonal basis of \( \mathbb{C}^N \). A similar computation works for columns, and so the matrix of rank one projections \( P_{ij} = Proj(H_i/H_j) \) is magic in the sense of Definition 1.2.

Note that, since \( Proj(\xi) = \frac{1}{||\xi||^2} (\xi \overline{\xi})_{ij} \), the projections \( P_{ij} \) are given by:

\[
P_{ij} = \frac{1}{N} \left( \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \right)_{kl}
\]

Now with the above observation in hand, we can formulate:
Definition 1.3. Associated to a complex Hadamard matrix $H \in M_N(\mathbb{C})$ is the minimal subgroup $G \subset S^+_N$ producing a factorization of type

$$C(S^+_N) \xrightarrow{\pi_H} M_N(\mathbb{C}) \xrightarrow{\pi_{\mathcal{G}}} C(G),$$

where $\pi_H(u_{ij}) = P_{ij}$, with $P_{ij} = \text{Proj}(H_i/H_j)$, where $H_1, \ldots, H_N$ are the rows of $H$.

Here the existence and uniqueness of $C(G)$ follow from [7].

As a basic example, for $H = F_N$ we obtain $G = \mathbb{Z}_N$. It is also known that for a tensor product $H = H' \otimes H''$ we have $G = G' \times G''$, so for a generalized Fourier matrix $F_G = F_{N_1} \otimes \ldots \otimes F_{N_k}$ we obtain the group $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$ itself. See [8], [10].

Let $u = (u_{ij})$ be the fundamental representation of $G$, and denote by $\mu \in \mathcal{P}(\mathbb{R}^+)$ the law of its character, $\chi = \sum_i u_{ii}$. Observe that the moments of $\mu$ are integers:

$$\int_G \text{Tr}(u)^k = \int_G \text{Tr}(u^\otimes k) = \dim(\text{Fix}(u^\otimes k))$$

Observe also that $0 \leq \chi \leq N$, and so $\text{supp}(\mu) \subset [0, N]$.

The above moments, or rather the spaces on the right, can be computed by using:

Theorem 1.4. We have an equality of complex vector spaces

$$\text{Hom}(u^\otimes k, u^\otimes l) = \text{Hom}(P^\otimes k, P^\otimes l)$$

where for $X \in M_N(A)$ we set $X^\otimes k = (X_{i_1j_1} \ldots X_{i_kj_k})_{i_1 \ldots i_k, j_1 \ldots j_k}$.

Proof. This follows from Woronowicz’s Tannakian duality results in [44], because the spaces on the right provide a model for the Hopf image. See [7].

Recall from [26] that the planar algebra associated to $H$ is given by $P_k = \text{Fix}(P^\otimes k)$. By using Theorem 1.4 we obtain $P_k = \text{Fix}(u^\otimes k)$, and so the Stieltjes transform of $\mu$ is the Poincaré series of the planar algebra, $f(z) = \sum_{k=0}^\infty \dim(P_k)z^k$. See [1], [8].

As an illustration, by using the fact that $P$ is magic, we obtain:

$$\text{Fix}(u) = \text{Fix}(P) = \left\{ \xi \bigg| \sum_j \xi_j P_{ij} = \xi_i 1, \forall i \right\} = \mathbb{C}$$

At $k = 2$ now, we have the following result, due to Jones [26]:

Proposition 1.5. We have $\text{End}(u) = C(\Gamma^o_H)$, where $\Gamma^o_H$ is the set of connected components of the graph $\Gamma_H$ defined as follows:

1. The vertices are the pairs of indices $(i, j)$.
2. The edges $(i, j) - (k, l)$ are where $<\xi_{ik}, \xi_{jl}> \neq 0$. 
Proof. We must prove that $End(u)$ is abelian, and that its dimension equals the number of connected components of $\Gamma_H$. By using the fact that $\{\xi_{jl}\}_j$ is an orthogonal basis, and then the fact that $\{\xi_{ik}\}_k$ is an orthogonal basis as well, we obtain:

$$T \in End(u) \iff T \in End(P)$$
$$\iff \sum_r T_{ir} P_{rl} = \sum_r P_{ir} T_{rl}$$
$$\iff T_{ij} \xi_{jl} = \sum_r T_{rl} < \xi_{jl}, \xi_{ir}> \xi_{ir}$$
$$\iff T_{ij} < \xi_{jl}, \xi_{ik}> = T_{kl} < \xi_{jl}, \xi_{ik}>$$
$$\iff (T_{ij} - T_{kl}) < \xi_{jl}, \xi_{ik}> = 0$$

Thus $T \in End(u)$ is equivalent to $T_{ij} = T_{kl}$, for each edge $(i, j) - (k, l)$, so the entries of $T$ must indeed be constant over the connected components of $\Gamma_H$. \(\square\)

As an example, consider the Fourier matrix $H = F_N$. Here we have $H_i = \rho^i$, with $\rho = (1, \omega, \omega^2, \ldots, \omega^{N-1})$, so the edges of $\Gamma_H$ are those of type $(i, j) - (i + r, j + r)$. Thus $\Gamma_H$ is a disjoint union of $N$ copies of the complete $N$-graph, and $\Gamma_H^0 = \{1, \ldots, N\}$.

The following notion will play a key role in what follows:

**Definition 1.6.** Associated to $H \in M_N(\mathbb{C})$ is the quantum group $J_H \subset S_N^+$ given by

$$C(J_H) = C(S_N^+)/ < T \in End(v), \forall T \in End(P) >$$

with the usual convention $End(P) = \{T \in M_N(\mathbb{C})|(T \otimes 1)P = P(T \otimes 1)\}$.

It is possible to think of $J_H$ as being the quantum automorphism group of the colored oriented graph $\Gamma_P$ having as vertices the numbers $1, \ldots, N$, and with all the oriented edges $i \to j$ drawn, and “colored” by the matrices $P_{ij} \in M_N(\mathbb{C})$. See [4].

We denote by $u, v$ the fundamental representations of $G_H, J_H$.

**Proposition 1.7.** The quantum group $J_H$ has the following properties:

(1) We have an inclusion $G_H \subset J_H$.

(2) For $H = F_N$ we have $G_H = J_H = \mathbb{Z}_N$.

(3) $End(u) = End(v) = End(P) = C(\Gamma_H^0)$.

Proof. We know from Theorem 1.4 that $End(u) = End(P)$, and this gives:

$$C(J_H) = C(S_N^+)/ < T \in End(v), \forall T \in End(u) >$$

Now by using Woronowicz’s Tannakian duality in [14] this gives both the inclusion $G_H \subset J_H$ in (1), and the equality $End(u) = End(v)$ in (3). Observe that this actually finishes the proof of (3), because the last equality follows from Proposition 1.5.

Regarding now (2), one way of viewing this is by saying that $\Gamma_P$ is circulant, with $i \to j$ colored by $P_{ij} = \text{Proj}(\rho^{i-j})$, and so $J_H = \mathbb{Z}_N$ as claimed. See [4]. \(\square\)
For more results on $J_H$, or rather on its planar algebra, see Jones [26]. Also, for more on quantum groups of type $J_H$, and their associated planar algebras, see [4], [14].

Summarizing, we have several approaches to the computation of $\mu$. In what follows we will use the quantum group approach, where the method is as follows:

**Fact 1.8.** The quantum invariants of $H \in M_N(\mathbb{C})$ can be computed as follows:

1. Compute the associated quantum permutation group $G \subset S_N^+$.
2. Compute then the spectral measure $\mu$ of the main character $\chi : G \to \mathbb{C}$.

The power of this method essentially comes from the fact that it splits the problem into two parts, which are roughly of equivalent difficulty. For a number of general methods in dealing with these two problems, see [1], [7], [18] and [4], [20], [37].

Note also that choosing as “output” a real probability measure $\mu$, and more precisely the spectral measure of a sum of projections, $\chi = \sum_i u_{ii}$, is certainly a good thing.

In addition to this, at least for certain special classes of complex Hadamard matrices, some powerful subfactor methods are available as well. See [17], [24], [31].

We should mention, however, that most of the known examples of Hadamard matrices resist so far both quantum group and subfactor techniques. This is unfortunately the case for all the Hadamard matrices at $N = 6$ found in [11], [16], [21], [29], [39], [41].

Finally, quite puzzling here is the case of circulant Hadamard matrices [16], [22], whose obvious symmetry is not understood yet, at the quantum algebra level. More precisely, the following question is open: “can $G_H$ see the fact that $H$ is circulant?”.

2. Free wreath products

Let us go back now to the Ditâ deformations in Proposition 1.1. In order to compute the associated quantum groups, we use the following notion, from [12]:

**Definition 2.1.** Let $C(S^+_M) \to A$ and $C(S^+_N) \to B$ be Hopf algebra quotients, with fundamental corepresentations denoted $u, v$. We let

$$A \ast_w B = A^{\ast N} \ast B/ < [u^{(i)}_{ab}, v_{ij}] = 0 >$$

with the Hopf algebra structure making $w_{ia, jb} = u^{(i)}_{ab} v_{ij}$ a corepresentation.

The fact that we have indeed a Hopf algebra follows from the fact that $w$ is magic. In terms of quantum groups, if $A = C(G)$, $B = C(H)$, we write $A \ast_w B = C(G \wr_* H)$:

$$C(G) \ast_w C(H) = C(G \wr_* H)$$

The $\wr_*$ operation is then the free analogue of $\wr$, the usual wreath product. See [12]. With these notations, we have the following result:
Lemma 2.2. The representation associated to \( L = H \otimes_Q K \) factorizes as

\[
C(S^+_{NM}) \xrightarrow{\pi_L} M_{NM}(\mathbb{C}) \xrightarrow{\gamma} C(S^+_M \rtimes G_H)
\]

the formulae for the map on the right being \( U_{ab}^{(i)} = \sum_j P_{ia,jb}, V_{ij} = \sum_a P_{ia,jb} \).

Proof. The rows of \( L = H \otimes_Q K \) are the vectors \( L_{ia} = (Q_{ic}H_{ik}K_{ac})_{kc} \), and so the corresponding magic basis, formed by the quotients of these vectors, is given by:

\[
\xi_{ia,jb} = \frac{L_{ia}}{L_{jb}} = \left( \frac{Q_{ic}H_{ik}K_{ac}}{Q_{ic}H_{ik}K_{ac}} \right)_{kc}
\]

Now recall the general projection formula \( P_\xi = \frac{1}{||\xi||^2} (\xi_\xi)_{ij} \). In double index notation we have \( P_\xi = \frac{1}{||\xi||^2} (\xi_\xi)_{kc,ld} \), so the projections \( P_{ia,jb} = P_{\xi_{ia,jb}} \) are given by:

\[
P_{ia,jb} = \frac{1}{NM} \left( \frac{Q_{ic}H_{ik}K_{ac}}{Q_{ic}H_{ik}K_{ac}} \right)_{kc,ld}
\]

According to Definition 2.1, we have to prove that: (1) the elements \( V_{ij} = \sum_a P_{ia,jb} \) do not depend on \( b \), and generate a copy of \( C(G_H) \), (2) for any \( i \), the elements \( U_{ab}^{(i)} = \sum_j P_{ia,jb} \) form a magic matrix, and (3) we have \( U_{ab}^{(i)}V_{ij} = V_{ij}U_{ab}^{(i)} = P_{ia,jb} \).

(1) We have indeed the following computation, where \( (P_{ij}) \) is the magic for \( H \):

\[
V_{ij} = \frac{1}{NM} \left( \sum_a \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \right)_{kc,ld}
\]

\[
= \frac{1}{N} \left( \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \delta_{cd} \right)_{kc,ld}
\]

\[
= \frac{1}{N} \left( \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \delta_{cd} \right)_{kc,ld}
\]

\[
= ((P_{ij})_{kld} \delta_{cd})_{kc,ld}
\]

\[
= P_{ij} \otimes 1
\]

(2) Since \( P = (P_{ia,jb}) \) is magic, the elements \( U_{ab}^{(i)} = \sum_j P_{ia,jb} \) are self-adjoint, and we have \( \sum_b U_{ab}^{(i)} = \sum_j P_{ia,jb} = 1 \). It remains to prove that the elements \( U_{ab}^{(i)} \) are idempotents,
and that they satisfy \( \sum_a U^{(i)}_{ab} = 1 \). In order to do so, observe first that we have:

\[
U^{(i)}_{ab} = \frac{1}{NM} \left( \sum_j Q_{ic}Q_{jd} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \right)_{kc,ld}
\]

The condition \( \sum_a U^{(i)}_{ab} = 1 \) can be checked as follows:

\[
\sum_a U^{(i)}_{ab} = \frac{1}{NM} \left( \sum_j Q_{ic}Q_{jd} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \right)_{kc,ld}
\]

\[
= \frac{1}{N} \left( \sum_j \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \delta_{cd} \right)_{kc,ld}
\]

\[
= \frac{1}{N} \left( \sum_j \delta_{cd} \right)_{kc,ld}
\]

\[
= 1
\]

Let us check now that that each \( U^{(i)}_{ab} \) is idempotent. We have:

\[
((U^{(i)}_{ab})^2)_{kc,ld} = \sum_{me} (U^{(i)}_{ab})_{kc,me} (U^{(i)}_{ab})_{me,ld}
\]

\[
= \frac{1}{N^2 M^2} \sum_{mejn} Q_{ic}Q_{je}Q_{nd}Q_{ie} \cdot \frac{H_{ik}H_{jm}}{H_{il}H_{jn}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{be}}
\]

\[
= \frac{1}{N^2 M^2} \sum_{mejn} Q_{ic}Q_{je}Q_{nd}Q_{ne} \cdot \frac{H_{ik}H_{jm}H_{nl}}{H_{il}H_{jm}H_{nm}} \cdot \frac{K_{ac}K_{bd}}{K_{bc}K_{ad}}
\]

\[
= \frac{1}{NM} \sum_{ejn} Q_{ic}Q_{je}Q_{jd}Q_{ne} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jn}} \cdot \frac{K_{ac}K_{bd}}{K_{bc}K_{ad}}
\]

\[
= \frac{1}{NM} \sum_{ejn} Q_{ic}Q_{je}Q_{jd}Q_{ne} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jn}} \cdot \delta_{jn} \cdot \frac{K_{ac}K_{bd}}{K_{bc}K_{ad}}
\]

\[
= \frac{1}{NM} \sum_{ejn} Q_{ic}Q_{je}Q_{jd}Q_{ne} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jn}} \cdot \frac{K_{ac}K_{bd}}{K_{bc}K_{ad}}
\]

\[
= \frac{1}{NM} \sum_{ejn} Q_{ic}Q_{je}Q_{jd}Q_{ne} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jn}} \cdot K_{ac}K_{bd}
\]

\[
= \frac{1}{NM} \sum_{ejn} Q_{ic}Q_{je}Q_{jd}Q_{ne} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jn}} \cdot K_{bc}K_{ad}
\]

\[
= (U^{(i)}_{ab})_{kc,ld}
\]
(3) First, we have the following computation:

\[ (U_{ab}^{(i)}V_{ij})_{kc,ld} = \sum_{me} (U_{ab}^{(i)})_{kc,me} (V_{ij})_{me,ld} \]

\[ = \frac{1}{N} \sum_{me} (U_{ab}^{(i)})_{kc,me} \cdot \frac{H_{im}H_{jl}}{H_{il}H_{jm}} \cdot \delta_{ed} \]

\[ = \frac{1}{N} \sum_{m} (U_{ab}^{(i)})_{kc,md} \cdot \frac{H_{im}H_{jl}}{H_{il}H_{jm}} \]

\[ = \frac{1}{N^2M} \sum_{mn} Q_{ic}Q_{nd} \cdot \frac{H_{ik}H_{nm}H_{jl}}{H_{il}H_{jm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{N^2M} \sum_{mn} Q_{ic}Q_{nd} \cdot \frac{H_{ik}H_{jm}}{H_{il}H_{jm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{n} Q_{ic}Q_{jd} \cdot \frac{H_{ik}H_{jl}}{H_{jk}H_{il}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{nc} Q_{id}Q_{nc} \cdot \frac{H_{ik}H_{jm}}{H_{il}H_{jm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{N^2M} \sum_{mn} Q_{ic}Q_{nd} \cdot \frac{H_{ik}H_{jm}}{H_{il}H_{jm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{n} Q_{id}Q_{nc} \cdot \frac{H_{ik}H_{jm}}{H_{il}H_{jm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{m} Q_{ic}Q_{jd} \cdot \frac{H_{ik}H_{jl}}{H_{jk}H_{il}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{nc} Q_{id}Q_{nc} \cdot \frac{H_{ik}H_{jm}}{H_{il}H_{jm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{m} Q_{ic}Q_{jd} \cdot \frac{H_{ik}H_{jl}}{H_{jk}H_{il}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{nc} Q_{id}Q_{nc} \cdot \frac{H_{ik}H_{jm}}{H_{il}H_{jm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{m} Q_{ic}Q_{jd} \cdot \frac{H_{ik}H_{jl}}{H_{jk}H_{il}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{nc} Q_{id}Q_{nc} \cdot \frac{H_{ik}H_{jm}}{H_{il}H_{jm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

We have as well the following computation:

\[ (V_{ij}U_{ab}^{(i)})_{kc,ld} = \sum_{me} (V_{ij})_{kc,me} (U_{ab}^{(i)})_{me,ld} \]

\[ = \frac{1}{N} \sum_{me} (V_{ij})_{kc,me} \cdot (U_{ab}^{(i)})_{me,ld} \]

\[ = \frac{1}{N} \sum_{m} (V_{ij})_{kc,md} \cdot (U_{ab}^{(i)})_{me,ld} \]

\[ = \frac{1}{N^2M} \sum_{mn} Q_{ic}Q_{nd} \cdot H_{ik}H_{jm} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{n} Q_{ic}Q_{jd} \cdot H_{ik}H_{jl} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{nc} Q_{id}Q_{nc} \cdot H_{ik}H_{jm} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{m} Q_{ic}Q_{jd} \cdot H_{ik}H_{jl} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = \frac{1}{NM} \sum_{nc} Q_{id}Q_{nc} \cdot H_{ik}H_{jm} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \]

\[ = (P_{sa,jb})_{kc,ld} \]
Thus we have indeed $U_{ab}^{(i)} V_{ij} = V_{ij} U_{ab}^{(i)} = P_{ia,jb}$, and we are done. □

We are ready now to state and prove our main result in this section. This is a finer factorization of $\pi_L$, that we believe to be optimal, or at least not far from that:

**Theorem 2.3.** The representation associated to $L = H \otimes Q K$ factorizes as

$$C(S^+_N) \xrightarrow{\pi_L} M_{NM}(\mathbb{C})$$

$$\xrightarrow{C(J_K \cdot G_H)} C(J_K \cdot G_H)$$

when $J_K \subset S^+_N$ is the quantum automorphism group of the magic matrix of $K$.

**Proof.** We use the factorization found in Lemma 2.2. In order to prove the result, we must construct a factorization of the map found there, as follows:

$$C(S^+_M \cdot G_H) \xrightarrow{\pi_L} M_{NM}(\mathbb{C})$$

$$\xrightarrow{C(J_K \cdot G_H)} C(J_K \cdot G_H)$$

For this purpose, fix $i \in \{1, \ldots, N\}$, consider the matrix $W_{ab} = U_{ab}^{(i)}$, and let $C(G)$ with $G \subset S^+_M$ be the Hopf image of the representation $u_{ab} \rightarrow W_{ab}$. It is enough to show that we have $G \subset J_K$, and by using twice Theorem 1.4, this is the same as proving:

$$End(P) \subset End(W)$$

Here, and in what follows, $P$ is the magic matrix for $K$, given by:

$$(P_{ab})_{cd} = \frac{1}{M} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}}$$

According to the formulae in the proof of Lemma 2.2, we have:

$$W_{ab} = \frac{1}{NM} \left( \sum_j \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \right)_{kc,ld}$$

For $T \in M_N(\mathbb{C})$, we therefore have the following formulae:

$$(WT)_{ab} = \sum_f W_{af} T_{fb} = \frac{1}{NM} \left( \sum_j \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \sum_f \frac{K_{ac}K_{fd}}{K_{ad}K_{fc}} \cdot T_{fb} \right)_{kc,ld}$$

$$(TW)_{ab} = \sum_f T_{af} W_{fb} = \frac{1}{NM} \left( \sum_j \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \sum_f T_{af} \cdot \frac{K_{fc}K_{bd}}{K_{fd}K_{bc}} \right)_{kc,ld}$$
Now observe that the last sums on the right are respectively:

\[ \sum_f \frac{K_{ac}K_{fd}}{K_{ad}K_{fc}} \cdot T_{fb} = \sum_f (P_{af})_{cd} T_{fb} = \left( \sum_f P_{af} T_{fb} \right)_{cd} = ((PT)_{ab})_{cd} \]

\[ \sum_f T_{af} \cdot \frac{K_{fc}K_{bd}}{K_{fd}K_{bc}} = \sum_f T_{af} (P_{fb})_{cd} = \left( \sum_f T_{af} P_{fb} \right)_{cd} = ((TP)_{ab})_{cd} \]

Thus we have \( T \in \text{End}(P) \implies T \in \text{End}(W) \), and we are done. \( \square \)

As a main consequence, when \( G_K = J_K \), as is for instance the case for the Fourier matrices \( K = F_M \), we have the following factorization:

\[ C(S_{NM}^+) \xrightarrow{\pi_L} M_{NM}(\mathbb{C}) \]
\[ C(G_K \wr \ast G_H) \]

In general, the representation in Lemma 2.2 does not seem to further factorize, unless in some special cases. The point is that \( T \in \text{End}(P^\otimes k) \implies T \in \text{End}(W^\otimes k) \), needed for further factorizing \( \pi_L \), would basically need the assumption \( Q_{id} = Q_d \) for certain numbers \( Q_1, \ldots, Q_M \), which means that we are in a usual tensor product situation.

Summarizing, finding the exact assumptions on \( H, K, Q \) which ensure a factorization through \( C(G_K \wr \ast G_H) \) remains an open problem, that we would like to raise here.

### 3. Fourier matrices

In this section and in the next two ones we study the Ditţă deformations of the Fourier matrices, \( L = F_N \otimes_Q F_M \). As explained in the introduction, there is some overlapping here with the results of Burstein in [17], waiting to be better understood, in the general context of Bisch-Haagerup subfactors [15].

Our starting point is Theorem 2.3 in the case \( H = F_N, K = F_M \), namely:

**Proposition 3.1.** The representation associated to \( L = F_N \otimes_Q F_M \) factorizes as

\[ C(S_{NM}^+) \xrightarrow{\pi_L} M_{NM}(\mathbb{C}) \]
\[ C(\mathbb{Z}_M \wr \ast \mathbb{Z}_N) \]

the map on the right being given by \( u_{ab}^{(i)} \to \sum_j P_{ia,jb}, v_{ij} \to \sum_a P_{ia,jb} \).
Proof. This follows of course from Theorem 2.3, but in the present case there are actually many simplifications. Indeed, with \( \omega = e^{2\pi i/N} \) and \( \theta = e^{2\pi i/M} \), we have:

\[
(P_{ia,jb})_{kc,ld} = \frac{1}{NM} \cdot \frac{Q_{ic} Q_{jd}}{Q_{id} Q_{jc}} \cdot \omega^{(i-j)(k-l)} \theta^{(a-b)(c-d)}
\]

With this formula in hand, the computations in the proof of Lemma 2.2 apply, with several simplifications, and give the factorization there. Regarding now the second factorization, from Theorem 2.3, we can use here the following standard fact:

\[
C(\mathbb{Z}_M) = C(S^+_M)/ <v_{ij} = v_{kl}, \forall i-j = k-l(M)>
\]

Indeed, the above formula shows that \( P_{ia,jb} \) depends only on \( a-b(M) \), and so by summing over \( j \) we obtain \( U_{ab}^{(i)} = U_{cd}^{(i)} \) if \( a-b = c-d(M) \), which gives the result. \( \square \)

Now our task is to analyse the representation \( \pi_Q \), and our first concern is to better understand the algebra \( C(\mathbb{Z}_M \rtimes \mathbb{Z}_N) \). For this purpose, recall the following notion:

**Definition 3.2.** If \( H \sim \Gamma \) is a finite group acting by automorphisms on a discrete group, the corresponding crossed coproduct Hopf algebra is

\[
C^*(\Gamma) \rtimes C(H) = C^*(\Gamma) \otimes C(H)
\]

with comultiplication \( \Delta(r \otimes \delta_k) = \sum_{h \in H} (r \otimes \delta_h) \otimes (h^{-1} \cdot r \otimes \delta_{h^{-1}k}) \), for \( r \in \Gamma, k \in H \).

Observe that \( C(H) \) is a subcoalgebra, and that \( C^*(\Gamma) \) is not a subcoalgebra. See [30].

The quantum group corresponding to \( C^*(\Gamma) \rtimes C(H) \) is denoted \( \hat{\Gamma} \rtimes H \).

Our motivating example is coming from the symmetric group \( S_N \), which acts by permutations of the copies on any free product group power \( G^*N \). By restriction we have an action of \( \mathbb{Z}_N \sim G^*N \), and we have \( C(G \rtimes \mathbb{Z}_N) \simeq C^*(G^*N) \rtimes C(\mathbb{Z}_N) \).

In the particular case \( G = \mathbb{Z}_M \), the precise result is as follows:

**Proposition 3.3.** We have an isomorphism of Hopf algebras

\[
C(\mathbb{Z}_M \rtimes \mathbb{Z}_N) \simeq C^*(\mathbb{Z}^*_M) \rtimes C(\mathbb{Z}_N)
\]

given by \( v_{ij} \to 1 \otimes v_{ij}, u_{ab}^{(i)} \to \frac{1}{M} \sum_c \theta^{(b-a)c} g_c^i \otimes 1 \), where \( g_0, \ldots, g_{N-1} \) are the standard generators of \( \mathbb{Z}^*_M \).

Proof. Our first claim is that we have isomorphisms of algebras, as follows:

\[
C(\mathbb{Z}_M \rtimes \mathbb{Z}_N) = C(\mathbb{Z}_M)^*N \otimes C(\mathbb{Z}_N) \simeq C^*(\mathbb{Z}^*_M) \otimes C^*(\mathbb{Z}_N)
\]

Indeed, recall that \( C(\mathbb{Z}_M \rtimes \mathbb{Z}_N) \) is the quotient of \( C(\mathbb{Z}_M)^*N \ast C(\mathbb{Z}_N) \) by the relations \( u_{ab}^{(i)} v_{ij} = v_{ij} u_{ab}^{(i)} \). Since \( C(\mathbb{Z}_N) \) is the quotient of \( C(S^+_N) \) by \( v_{ij} = v_{kl} \) if \( i-j = k-l(N) \), in this situation, as an algebra, we just have the tensor product \( C(\mathbb{Z}_M)^*N \otimes C(\mathbb{Z}_N) \).

Regarding now the map on the right, this comes from the Fourier transform, the isomorphism being realized by using the elements \( h = \sum_k \omega^k v_{k0} \) and \( g_i = \sum_a \theta^a u_{ab}^{(i)} \).
Proposition 3.4. With $\pi$ the representation that the subfactor associated to $F$ is trivial, the quantum group associated to $F$ is given by $Q_L = \prod_{i=1}^{N} \epsilon_{i}$. These elements generate $C(ZM \otimes ZN)$, and are particularly convenient for understanding the representation $\pi_Q$. First, we can decompose $\pi_Q$ by changing the basis of $\mathbb{C}^{NM}$.

**Proof.** Indeed, in terms of the basis in the statement, we have:

$$P_{ia,jb}(\varepsilon_{kc}) = \frac{1}{M} \delta_{i,j,k} \sum_{d} \frac{Q_{id}Q_{jc}}{Q_{ic}Q_{jd}} \cdot \theta^{(b-a)(c-d)} \varepsilon_{(i-j)d}$$

The announced formulae are direct verifications. The last claim is immediate. 

As a first illustration, let us discuss the case $N = M = 2$. Burstein has shown in [17] that the subfactor associated to $F_{2,2}^q$ has principal graph $\tilde{D}_{2n+2}$, where $n = ord(q^4)$. Thus, according to [6], [13], and more precisely to the ADE tables in [6], the associated quantum group must be one of the twists $D_{2n}^1$, $DC_{n}^{-1}$ constructed by Nikshych in [34].

The precise result at $N = M = 2$ is in fact as follows:

**Theorem 3.5.** The quantum group associated to $L = F_2 \otimes_Q F_2$, with $Q = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, is

$$G_L = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 : \mathbb{Z}_4 & \text{if } n = 1 \text{ and } m = 1, 2 : m = 4 \\ D_{2n}^1 : DC_{n}^{-1} & \text{if } 1 < n < \infty \text{ and } m \not\in 4\mathbb{N} : m \in 4\mathbb{N} \\ \mathbb{Z}_2 \otimes \mathbb{Z}_2 & \text{if } n = \infty \end{cases}$$

where $q = ad/bc$, with $Q = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, and where $m = ord(q), n = ord(q^4)$.

**Proof.** Our first claim is that we have $F_2 \otimes_Q F_2 \simeq F_{2,2}^q$, with $q = ad/bc$, and so we can assume $Q = (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})$. Indeed, this follows by dephasing our matrix:

$$\left( \begin{array}{ccc} a & b & a & b \\ a & -b & a & -b \\ c & d & -c & -d \\ c & -d & -c & d \end{array} \right) \otimes \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \frac{a}{b} & \frac{d}{b} & -\frac{c}{a} & -\frac{d}{a} \\ -\frac{a}{b} & -\frac{d}{b} & \frac{c}{a} & \frac{d}{a} \end{array} \right) \otimes \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & \frac{ad}{bc} & -1 & -\frac{ad}{bc} \\ 1 & -\frac{ad}{bc} & -1 & \frac{ad}{bc} \end{array} \right) \simeq \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & \frac{ad}{bc} & -1 & -\frac{ad}{bc} \\ 1 & -\frac{ad}{bc} & -1 & \frac{ad}{bc} \end{array} \right)$$
As a first remark, the \( n = 1 \) case is clear, because at \( q = \pm 1 \) we have the generalized Fourier matrix \( F_{2,2} = F_2 \otimes F_2 \), which produces the Klein group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), and at \( q = \pm i \) we have the Fourier matrix \( F_4 \), which produces the group \( \mathbb{Z}_4 \). See \[8\], \[10\].

Now recall that \( C(\mathbb{Z}_2 \wr \mathbb{Z}_2) \) is generated by elements \( g_0, g_1, h \), satisfying:

\[
g_0^2 = 1 = g_1^2 = h^2, \quad hg_0 = g_0 h, \quad hg_1 = g_1 h
\]

In terms of these generators, the representation \( \pi_Q^{(1)} \) is given by:

\[
h \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_0 \rightarrow \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}, \quad g_1 \rightarrow \begin{pmatrix} 0 & q^{-1} \\ q & 0 \end{pmatrix}
\]

Now put \( z = g_0 g_1, t = g_0 \) and \( h = a \). This enables us to identify \( C(\mathbb{Z}_2 \wr \mathbb{Z}_2) \) with the Hopf algebra \( A_h(2) \) in Section 7 in \[6\], whose quotients are classified there. More precisely, the non-trivial quotients are:

1. \( A(k, 1) \): quotient by the relation \( z^k = 1 \).
2. \( A(k, -1) \): quotient by the relation \( z^k = a \).
3. \( C^*(D_\infty) \): quotient by the relation \( a = 1 \), and its quotients.

The third case cannot hold by the previous computation of \( \pi_Q^{(1)}(h) \). Now with this concrete description in hand we can directly describe the minimal factorization of \( \pi_L \), and this together with Proposition 7.2 in \[6\] finishes the proof.

We refer to Burstein \[17\] for a number of computations in index \( NM = 6 \), that for the moment we cannot fully recover, by using the techniques developed here.

4. The generic case

In this section we go back to the case where \( N, M \in \mathbb{N} \) are arbitrary. In order to further factorize the representation \( \pi_Q \) from Proposition 3.1, we use:

**Definition 4.1.** We let \( \Gamma_{N,M} \) be the group generated by elements \( g_0, \ldots, g_{N-1} \) with the relations \( [g_1 \ldots g_{i_M}, g_{j_1} \ldots g_{j_M}] = 1 \) and \( g_0^M = \ldots = g_{N-1}^M = 1 \).

Let us study now the kernel of the restrictions of \( \pi_Q, \pi_Q^{(1)} \) to \( \mathbb{Z}_N^* \). We see that the images of \( g_1 \ldots g_{i_M} \) are diagonal matrices, so the commutators \( [g_1 \ldots g_{i_M}, g_{j_1} \ldots g_{j_M}] \) belong to this kernel. This observation gives the following result, which refines Proposition 3.1:

**Proposition 4.2.** We have a factorization

\[
C(S_{NM}^+) \xrightarrow{\pi_L} M_{NM}(\mathbb{C}) \xrightarrow{\pi_R} C^*(\Gamma_{N,M}) \rtimes C(\mathbb{Z}_N)
\]

where \( \mathbb{Z}_N \) acts on \( \Gamma_{N,M} \) by cyclic permutations of the generators \( g_0, \ldots, g_{N-1} \).
Lemma 4.3. We have the following results:

Proof. Assume that we have a representation \( \rho : C^*(\Gamma) \times C(H) \rightarrow A \), let \( U \) be an \( H \)-stable normal subgroup of \( \Gamma \), so that \( H \) acts on \( \Gamma/U \) and that we can form the crossed coproduct \( C^*(\Gamma/U) \times C(H) \), and assume that \( \rho \) is trivial on \( U \). Then \( \rho \) factorizes as:

\[
C^*(\Gamma) \times C(H) \xrightarrow{\rho} \xrightarrow{\rho} A
\]

With \( \Gamma = \mathbb{Z}_M^N \), \( U \) = the subgroup generated by the above commutators, and \( \Gamma/U = \Gamma_{M,N} \), we can factorize the representation in Proposition 3.1, as claimed. \( \square \)

We call \( p_1, \ldots, p_m \in \mathbb{T} \) root independent if for any \( r_1, \ldots, r_m \in \mathbb{Z} \):

\[
p_1^{r_1} \cdots p_m^{r_m} = 1 \implies r_1 = \ldots = r_m = 0
\]

Also, a matrix \( Q \in M_{N,M}(\mathbb{T}) \), taken to be dephased \( (Q_{0c} = Q_{d0} = 1) \), will be called “generic” if the elements \( Q_{ic} \), \( 1 \leq i \leq N - 1 \), \( 1 \leq c \leq M - 1 \) are root independent.

We denote by \( \epsilon_0, \ldots, \epsilon_{M-1} \) the standard basis of \( \mathbb{C}^M \).

Lemma 4.3. We have the following results:

1. \( \Gamma_{N,M} \simeq \mathbb{Z}^{(N-1)(M-1)} \times \mathbb{Z}_M \) via \( g_0 \rightarrow (1, t) \) and \( g_i \rightarrow (a_{i0}, t) \) for \( 1 \leq i \leq N - 1 \), where \( a_{ic} \) and \( t \) are the standard generators of \( \mathbb{Z}^{(N-1)(M-1)} \) and \( \mathbb{Z}_M \).

2. For \( \theta \in M_{NM}(\mathbb{T}) \) with \( \prod c \theta_{ic} = \prod i \theta_{ic} = 1 \), and with \( \theta_{ic} \), \( 1 \leq i \leq N - 1 \), \( 0 \leq c \leq M - 2 \) root independent, we have an embedding \( \Gamma_{N,M} \subset U_M \), \( g_i(\epsilon_c) = \theta_{ic} \epsilon_{c-1} \).

3. The embedding in (2) is projectively faithful, in the sense that its composition with the canonical quotient map \( U_M \rightarrow PU_M \) is faithful.

4. If \( Q \in M_{NM}(\mathbb{T}) \) is generic we have an projectively faithful embedding \( \Gamma_{N,M} \subset U_M \) given by \( g_i(\epsilon_c) = \theta_{ic} \epsilon_{c-1} \), where \( \theta_{ic} = \frac{Q_{ic} Q_{i-1,c-1}}{Q_{ic} Q_{i-1,c-1}} \).

Proof. (1,2) Consider the morphism \( \Gamma_{N,M} \rightarrow \mathbb{Z}_M = < t > \) given by \( g_i \rightarrow t \), whose kernel \( T \) is generated by the elements of type \( g_{i1} \ldots g_{im} \). We get an exact sequence:

\[
1 \rightarrow T \rightarrow \Gamma_{N,M} \rightarrow \mathbb{Z}_M \rightarrow 1
\]

This sequence splits by \( t \rightarrow g_0 \), so we have \( \Gamma_{N,M} \simeq T \times \mathbb{Z}_M \). Now by the definition of \( \Gamma_{N,M} \), \( T \) is an abelian group. We now determine some generators. We put:

\[
a_{ic} = g_0^{c-1} g_i g_0^{-c}, \ 0 \leq i \leq N - 1, \ 0 \leq c \leq M - 1
\]

We claim that \( T \) is generated by the following elements:

\[
a_{ic} = g_0^{c-1} g_i g_0^{-c}, \ 1 \leq i \leq N - 1, \ 0 \leq c \leq M - 2
\]

Indeed, it is clear that these elements belong to \( T \). We have \( a_{0c} = 1 \) so we do not need to consider those elements. Let \( T_0 \) be the subgroup of \( \Gamma_{N,M} \) generated by the elements:

\[
a_{ic} = g_0^{c-1} g_i g_0^{-c}, \ 1 \leq i \leq N - 1, \ 0 \leq c \leq M - 1
\]
It is not difficult to check, using the commutativity relations in $T$, that:
\[ g_i a_j g_i^{-1} = a_{j,c+1}, \quad g_i^{-1} a_j g_i = a_{j,c-1} \]

It follows that $T_0$ is a normal subgroup of $\Gamma_{N,M}$. The elements $g_0^{-1} g_i = a_0$ belong to $T_0$, and thus it follows that $[\Gamma_{N,M} : T_0] \leq M$. But then we have:
\[ M = [\Gamma_{N,M} : T] \leq [\Gamma_{N,M} : T_0] \leq M \]

Thus $T_0 = T$. Finally the identity $a_0 a_1 a_2 \ldots a_{i,M-1} = 1$ shows that $T = T_0$ is indeed generated by the elements $a_{ic} = g_0^{-1} g_i g_0^{-c}, \quad 1 \leq i \leq N - 1, \quad 0 \leq c \leq M - 2$.

We must prove that $T$ is freely generated by the elements $a_{ic}$, and that $\pi_\theta : \Gamma_{N,M} \to U_M$ given by $\pi_\theta(g_i)(\epsilon_c) = \theta_{ic} \epsilon_c^{-1}$ is faithful. We have ker $\pi_\theta \subset T$, since the elements of $T$ are the only ones having their image by $\pi_\theta$ formed by diagonal matrices. Now since $T$ is generated by the elements $a_{ic}, 1 \leq i \leq N - 1, 0 \leq c \leq M - 2$, to prove our two assertions, it is enough to show that for $R_{ic} \in \mathbb{Z}, \quad 1 \leq i \leq N - 1, \quad 0 \leq c \leq M - 2$, we have:
\[ \pi_\theta \left( \prod_{i=1}^{N-1} \prod_{c=0}^{M-2} a_{ic}^{R_{ic}} \right)(\epsilon_0) = 1 \implies R_{ic} = 0, \quad \forall i, c \]

We have $\pi_\theta(a_{ic})(\epsilon_0) = \theta_{ic} \theta_{0c}^{-1} \epsilon_0$. Hence:
\[
\pi_\theta \left( \prod_{i=1}^{N-1} \prod_{c=0}^{M-2} a_{ic}^{R_{ic}} \right)(\epsilon_0) = \left( \prod_{i=1}^{N-1} \prod_{c=0}^{M-2} \theta_{ic}^{R_{ic}} \theta_{0c}^{-R_{ic}} \right) \epsilon_0 = \left( \prod_{i=1}^{N-1} \prod_{c=0}^{M-2} \theta_{ic}^{R_{ic}} \theta_{0c}^{-\sum_{i=1}^{N-1} R_{ic}} \right) \epsilon_0 = \left( \prod_{j=1}^{N-1} \prod_{c=0}^{M-2} \theta_{jc}^{R_{jc}} \theta_{0c}^{-\sum_{i=1}^{N-1} R_{ic}} \right) \epsilon_0 = \left( \prod_{j=1}^{N-1} \prod_{c=0}^{M-2} \theta_{jc}^{R_{jc} + \sum_{i=1}^{N-1} R_{ic}} \right) \epsilon_0
\]

We conclude by root independence.

(3) We have to prove that $\pi_\theta(g) \in \mathbb{C}1 \implies g = 1$. As before an element $g$ with $\pi_\theta(g) \in \mathbb{C}1$ belongs to $T$. We have $\pi_\theta(a_{ic})(\epsilon_1) = \theta_{i,c+1} \theta_{0,c+1}^{-1} \epsilon_0$. Put $\lambda_c = \sum_{i=1}^{N-1} R_{ic}$. A direct computation shows that:
\[
\pi_\theta \left( \prod_{i=1}^{N-1} \prod_{c=0}^{M-2} a_{ic}^{R_{ic}} \right)(\epsilon_1) = \left( \prod_{i=1}^{N-1} \theta_{i0}^{-R_{i,M-2} - \lambda_{M-2}} \right) \left( \prod_{i=1}^{N-1} \prod_{c=1}^{M-2} \theta_{ic}^{R_{i,c-1} + \lambda_{c-1} - R_{i,M-2} - \lambda_{M-2}} \right) \epsilon_1
\]

By comparing with the last formula in the proof of (2), and by using root independence, we conclude that $\pi_\theta$ is indeed projectively faithful.

(4) This follows from (2,3). \qed
We will need as well the following lemma:

**Lemma 4.4.** Let \( \pi : C^\ast(\Gamma) \times C(H) \to L \) be a surjective Hopf algebra map, such that \( \pi|_{C(H)} \) is injective, and such that for \( r \in \Gamma \) and \( f \in C(H) \), we have:
\[
\pi(r \otimes 1) = \pi(1 \otimes f) \implies r = 1
\]
Then \( \pi \) is an isomorphism.

**Proof.** We use here various tools from [2], [32]. Put \( A = C^\ast(\Gamma) \rtimes C(H) \). We start with the following Hopf algebra exact sequence, where \( i(f) = 1 \otimes f \) and \( p = \varepsilon \otimes 1 \):
\[
C \to C(H) \xrightarrow{i} A \xrightarrow{p} C^\ast(\Gamma) \to C
\]
Since \( \pi \circ i \) is injective, and Hopf subalgebra \( \pi \circ i(C(H)) \) is central in \( L \), we can form the quotient Hopf algebra \( L = L/(\pi \circ i(C(H)))^+ L \), and we get another exact sequence:
\[
C \to C(H) \xrightarrow{\pi \circ i} L \xrightarrow{q} L \to C
\]
Note that this sequence is indeed exact, e.g. by centrality (see [2], [38]). So we get the following commutative diagram with exact rows, with the Hopf algebra map on the right surjective:
\[
\begin{array}{cccccccc}
C & \longrightarrow & C(H) & \xrightarrow{i} & A & \xrightarrow{p} & C^\ast(\Gamma) & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C & \longrightarrow & C(H) & \xrightarrow{\pi \circ i} & L & \xrightarrow{q} & L & \longrightarrow & C
\end{array}
\]
Since a quotient of a group algebra is still a group algebra, we get a commutative diagram with exact rows as follows:
\[
\begin{array}{cccccccc}
C & \longrightarrow & C(H) & \xrightarrow{i} & A & \xrightarrow{p} & C^\ast(\Gamma) & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C & \longrightarrow & C(H) & \xrightarrow{\pi \circ i} & L & \xrightarrow{q'} & C^\ast(\Gamma) & \longrightarrow & C
\end{array}
\]
Here the vertical Hopf algebra map on the right is induced by a surjective group morphism \( u : \Gamma \to \overline{\Gamma}, g \mapsto \overline{g} \). By the five lemma (see e.g. [32], or [3]) we just have to show that \( u \) is injective.

So, let \( g \in \Gamma \) be such that \( u(g) = 1 \). We thus have \( q' \pi(g \otimes 1) = up(g \otimes 1) = u(g) = \overline{g} = 1 \).

For \( g \in \Gamma \), put:
\[
g A = \{ a \in A \mid p(a_1) \otimes a_2 = g \otimes a \}
\]
\[
\overline{g} L = \{ l \in L \mid q'(l_1) \otimes l_2 = \overline{g} \otimes l \}
\]
The commutativity of the right square ensures that \( \pi(g A) \subset \overline{g} L \). Then with the previous \( g \), we have \( \pi(g \otimes 1) \in \overline{g} L = \pi i(C(H)) \) (exactness of the sequence), so \( \pi(g \otimes 1) = \pi(1 \otimes f) \) for some \( f \in C(H) \). We conclude by our assumption that \( g = 1 \). \( \Box \)
Theorem 4.5. The minimal factorization for $L = F_N \otimes_Q F_M$ with $Q$ generic is

$$C(S_{NM}^+) \xrightarrow{\pi_L} C^*(\Gamma_{N,M}) \rtimes C(\mathbb{Z}_N) \rightarrow M_{NM}(\mathbb{C})$$

where $\Gamma_{N,M} \simeq \mathbb{Z}^{(N-1)(M-1)} \rtimes \mathbb{Z}_M$ is the discrete group constructed above.

Proof. We want to apply Lemma 4.4 to the morphism $C[\Gamma_{N,M}] \rtimes C(\mathbb{Z}_N) \rightarrow L$ arising from the factorization in Proposition 4.2, where $L$ denotes the Hopf image of $\pi_L$.

The first observation is that the injectivity assumption on $C(\mathbb{Z}_M)$ holds by Proposition 3.4, and that the restriction of $\pi_L$ to each $W(i)$ in Proposition 3.4 consists of scalar matrices. Also, the restriction to $\Gamma_{N,M}$ of the representation on the right, restricted to $W(1)$, is the representation in Lemma 4.3 (4), which is projectively faithful under our assumption. Therefore we can apply indeed Lemma 4.4, and we are done. \qed

5. Spectral measures

In this section we study the spectral measure $\mu$ of the quantum groups found in the previous sections. Let us begin with a full computation at $M = N = 2$:

Proposition 5.1. For $L = F_2 \otimes_Q F_2$ with $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $q = ad/bc$, $n = ord(q^4)$,

$$\mu = \frac{1}{2}(\delta_0 + \Phi \epsilon_{4n})$$

where $\epsilon_k$ is the uniform measure on the $k$-roots of unity, and $\Phi(z) = 4Re(z)^2$.

Proof. Recall that $\widetilde{D}_n$ with $n \geq 4$ is the following graph, having $n + 1$ vertices:

$$\widetilde{D}_n = \bullet \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \circ \rightarrow \circ$$

We know from Theorem 3.5 and from the ADE tables in [6] that the principal graph is $\widetilde{D}_{2n+2}$, a fact already noted by Burstein in [17]. On the other hand, according to [9], for $\widetilde{D}_{m+2}$ with $m \in \{2, 3, \ldots, \infty\}$ we have $\mu = \frac{1}{2}(\delta_0 + \Phi \epsilon_{2m})$, and this gives the result. \qed

In the generic case now, we have the following result:

Theorem 5.2. For $L = F_N \otimes_Q F_M$ with $Q \in M_{N \times M}(\mathbb{T})$ generic we have

$$N \int \left( \frac{k}{N} \right)^k = 1 + \binom{k}{2} (M-1)N^{-1} + O(N^{-2})$$

for any $k \geq 1$, in the $N \rightarrow \infty$ limit.
Proof. We use the equality $C(G_M) = C^*(\Gamma_{N,M}) \rtimes C(\mathbb{Z}_N)$ in Theorem 4.5. According to the various results in sections 2-4 above, the precise factorization is:

$$
C(S_{N,M}^+) \to C(\mathbb{Z}_M \rtimes \mathbb{Z}_N) \to C^*(\Gamma_{N,M}) \rtimes C(\mathbb{Z}_N)
$$

where

$$
u_{ia,jb} \to u_{ab}^{(i)} v_{ij} \to \frac{1}{M} \sum_c \theta^{(b-a)c} g_i^c \otimes v_{ij}
$$

By using the fact that $v_{ii} = \delta_1$ is the Dirac mass at $1 \in \mathbb{Z}_N$, for any $i$, we therefore obtain the following formula for the main character:

$$
\chi = \frac{1}{M} \sum_{iac} g_i^c \otimes v_{ii} = \sum_{ic} g_i^c \otimes v_{ii} = \left( \sum_{ic} g_i^c \right) \otimes \delta_1
$$

It is well-known that the Haar functional on a crossed coproduct $C^*(\Gamma) \rtimes C(H)$ is the tensor product of the Haar functionals of $C^*(\Gamma), C(H)$. Thus if we denote by $\times$ the multiplicative convolution, we have:

$$
\text{law}(\chi) = \text{law} \left( \sum_{ic} g_i^c \right) \times \left( \frac{N-1}{N} \delta_0 + \frac{1}{N} \delta_1 \right)
$$

Thus, we must estimate the law of the following element:

$$
\sum_{ic} g_i^c = N + (g_0 + \ldots + g_{N-1}) + (g_0^2 + \ldots + g_{N-1}^2) + \ldots + (g_{M-1}^M + \ldots + g_{N-1}^M)
$$

The point now is that, when raising this element to the power $k$ and integrating, the main contribution will come from the $N$ factors, and will equal $N^k$. As for the second order term, according to the definition of $\Gamma_{N,M}$, this can only come from $k-2$ copies of the $N$ factor, and a product of type $g_i^c g_{M-c} = 1$, with $i \in \{1, \ldots, M-1\}$. Thus:

$$
\int \left( \sum_{ic} g_i^c \right)^k = N^k + \binom{k}{2} (M-1) N^{k-1} + O(N^{k-2})
$$

Now by rescaling and doing the above $\times$ operation, this gives the result. \qed

In general, the above method can be probably used to fully compute $\mu$, but we don’t have results here. As an example, let us discuss the case $M = 2$.

At $N = 2$ we have $\sum_{ic} g_i^c = 2 + g_0 + g_1$, whose moments $1, 2, 6, 20, 70, 252, \ldots$ are the numbers $(2k)$. Thus we have $\int \chi^k = \frac{1}{2} (2k)_k$, and since these latter numbers count the $2k$-loops on $\widetilde{D}_\infty$, we recover the formula in Proposition 5.1, at $n = \infty$. 


In general now, the moments of $\sum_{ic} g_i^c = N + g_0 + \ldots + g_{N-1}$ are:

\[
\begin{align*}
M_0 &= 1 \\
M_1 &= N \\
M_2 &= N(N + 1) \\
M_3 &= N^2(N + 3) \\
M_4 &= N(N^3 + 6N^2 + 2N - 1) \\
M_5 &= N^2(N^3 + 10N^2 + 10N - 5)
\end{align*}
\]

We do not know what the associated measure is. The first problem appears at $M = 3$, where the moment sequence starts with $1, 3, 12, 54, 258, 1278$.

6. Conclusion

We have seen in this paper that the quantum invariants of Hadamard matrices of type $L_{ia,jb} = Q_{ia} H_{ij} K_{ab}$ can be computed by using quantum permutations.

Our results have a non-trivial overlapping with the subfactor work on the subject, and notably with the results of Burstein in [17]. It is of course our hope that:

1. The precise relation between the quantum group and subfactor results will be at some point clarified, in the general context of Bisch-Haagerup subfactors [15].

2. Much more importantly, that the various subfactor and quantum group methods can one day lead to results in index 6, where the difficult problems are.

We should perhaps comment a bit more on (2). As explained in the introduction, the year 1983 has seen two important events in operator algebras, namely the paper [25] by Jones, laying the foundations of subfactor theory, and the paper [35] by Popa, making the link between orthogonal MASA and complex Hadamard matrices.

As of now, 30 years after, the main problems arising from [25], [35] remain basically the same. The main problem is that of classifying the finite index subfactors of the Murray-von Neumann hyperfinite factor $R$, the main subproblems being:

1. Classify the small index subfactors. This question has been in recent years subject to some spectacular developments. For a brief account of the results, and of the heavy load of work involved in this project, see [27].

2. Compute the Hadamard subfactors. Here there are few advances so far. One problem is that the quantum permutations, a theory partly designed for attacking the problem, doesn’t in fact fully apply to the problem. See [4].

Regarding now the index 6 subfactors coming from Hadamard matrices, this is of course a subject which is of interest in connection with both (1,2). The problems here look terribly complicated, and we believe that only a joint combination of all available techniques (planar algebras, quantum permutations) could produce some results here.
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