NOTES ON THE SZEGŐ MINIMUM PROBLEM.
I. MEASURES WITH DEEP ZEROES

BY

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ABSTRACT
The classical Szegő polynomial approximation theorem states that the polynomials are dense in the space $L^2(\rho)$, where $\rho$ is a measure on the unit circle, if and only if the logarithmic integral of the measure $\rho$ diverges. In this note we give a quantitative version of Szegő’s theorem in the special case when the divergence of the logarithmic integral is caused by deep zeroes of the measure $\rho$ on a sufficiently rare subset of the circle.

1. Introduction
Denote by $\mathcal{P}$ the linear space of algebraic polynomials, and by $\mathcal{P}_n$ its subspace of polynomials of degree $n$. Given a finite positive measure $\rho$ on the unit circle $\mathbb{T}$, put

$$e_n(\rho) = \min_{q_0, \ldots, q_{n-1}} \sqrt{\int_{\mathbb{T}} |q_0 + q_1 t + \cdots + q_{n-1} t^{n-1} + t^n|^2 \, d\rho(t)} = \text{dist}_{L^2(\rho)}(t^n, \mathcal{P}_{n-1}).$$

Then

$$\lim_{n \to \infty} e_n(\rho) = \exp\left(\frac{1}{2} \int_{\mathbb{T}} \log \rho' \, dm\right),$$

where $m$ is the Lebesgue measure on $\mathbb{T}$ normalized by condition $m(\mathbb{T}) = 1$, and $\rho' = d\rho/dm$ is the Radon–Nikodym derivative. This is a classical result, first, proven by Szegő for absolutely continuous measures $\rho$, and then, independently, by Verblunsky and Kolmogorov in the general case [4, Section 3.1] and [9, Chapters 1 and 2]. Noting that for $j \geq 0$, $e_{n+j}(\rho)$ coincides with the distance in $L^2(\rho)$ from $t^{-j}$ to the linear span of $\{t^k : -j + 1 \leq k \leq n\}$, and recalling that the trigonometric polynomials are dense in $L^2(\rho)$, one sees that the density of algebraic polynomials $\mathcal{P}$ in $L^2(\rho)$ is equivalent to the condition

$$\lim_{n \to \infty} e_n(\rho) = 0,$$

and therefore, to the divergence of the logarithmic integral

$$\int_{\mathbb{T}} \log \rho' \, dm = -\infty.$$

In these notes we will be occupied by the following question:

Question 1: Suppose $\rho$ is a measure on $\mathbb{T}$ with divergent logarithmic integral. Estimate the rate of decay of the sequence $e_n(\rho)$. 
Our interest in this question came from the linear prediction for stationary processes. If \( \xi : \mathbb{Z} \rightarrow \mathbb{C} \) is a stationary random sequence with spectral measure \( \rho \), then, according to Kolmogorov and Wiener, \( e_n(\rho) \) is the error of the best mean-quadratic linear prediction of \( \xi(n) \) by \( \xi(0), \ldots, \xi(n-1) \), i.e.,

\[
e^2_n(\rho) = \min_{q_0, \ldots, q_{n-1}} \mathbb{E} \left[ \| \xi(n) - \sum_{0 \leq j \leq n-1} q_j \xi(j) \|^2 \right].
\]

In the case when the logarithmic integral converges, \( e_n(\rho) \) has a positive limit \( e_\infty(\rho) \), and dependence of the rate of convergence on the smoothness of the density of \( \rho \) is well-understood [3, 5]. In the case of a divergent logarithmic integral the situation is quite different and not much is known. If the closed support of \( \rho \) is not the whole circle, then it is not difficult to show that \( e_n(\rho) \) tends to zero at least exponentially. In the other direction, a version of the classical result of Erdős and Turán says that if \( \rho' > 0 \) m-a.e. on \( \mathbb{T} \), then the measure \( \rho \) is regular, i.e., \( e_n(\rho)^{1/n} \rightarrow 1 \). Later, stronger criteria for regularity of \( \rho \) were found by Widom, Ullman, and Stahl and Totik; see [10, Chapter 4].

In these notes we show that in several special but interesting situations it is not difficult to estimate decay of the sequence \( e_n(\rho) \) using only simple classical tools. Here, we consider the case when the divergence of the logarithmic integral is caused by deep zeroes of the measure \( \rho \) on a sufficiently rare subset of \( \mathbb{T} \). The results presented in this note extend Theorems 8 and 9 from [1].

Our main idea is that in the case when the measure \( d\rho = \Phi \, dm \) has divergent logarithmic integral (i.e., \( \int_\mathbb{T} \log \Phi \, dm = -\infty \)), the value \( |\log e_n(\rho)| \) can be controlled by the integral

\[
\int_\mathbb{T} \min \{ \log \left( \frac{1}{\Phi} \right), A \} \, dm
\]

of the cut-off of \( \log \Phi^{-1} \) on an appropriate large level \( A \) depending on \( n \). This can be viewed as a quantitative version of the regularization of the weight \( \Phi \) by \( \Phi_\varepsilon = \Phi + \varepsilon \) with \( 0 < \varepsilon \ll 1 \) used by Szegő in the proof of his theorem. We succeeded to make this work only under additional regularity assumptions on \( \Phi \).

The toy example is the absolutely continuous measure \( d\rho = e^{-H} \, dm \), where \( H(e^{2\pi i \theta}) = h(\theta) \), \( h : \mathbb{R} \rightarrow [0, +\infty] \) is a 1-periodic even function, continuous and decreasing on \( (0, \frac{1}{2}] \), and such that \( \int_0^\infty h(\theta) \, d\theta = +\infty \). Then, under mild
assumptions on $h$, we obtain

$$|\log e_n(\rho)| \simeq \int_T h_A \, dm,$$

where $h_A = \max(h, A)$, and $A = A(n)$ is a solution to the equation $nh^{-1}(a) = a$, $h^{-1}$ is the inverse to the restriction of $h$ on $(0, \frac{1}{2}]$. Throughout the paper we use the following notation: for positive $A$ and $B$, $A \lesssim B$ means that there is a positive numerical constant $C$ such that $A \leq CB$, while $A \gtrsim B$ means that $B \lesssim A$, and $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$.

In the forthcoming second note, we will consider the opposite case when the bulk of the measure $\rho$ is concentrated on a rare subset of $\mathbb{T}$.

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2. Preliminaries

Here and elsewhere, $H : \mathbb{T} \to [0, +\infty]$ is a measurable function with

$$\int_T H \, dm = +\infty.$$

By $\lambda_H(a) = m\{H > a\}$ we denote the distribution function of $H$. For $A \geq 1$, we put $H_A(t) = \min(H(t), A)$. To estimate from below and above $\log e_n$, we will use the integrals

$$\int_T H_A \, dm = \int_0^A \lambda_H(a) \, da = A\lambda_H(A) + \int_{\{H \leq A\}} H \, dm$$

with some $A = A(n)$.

We record several simple observations, which we frequently use throughout the paper.

2.1. First, we note that under mild regularity assumptions one of the two terms on the RHS can be discarded. If $\lambda_H(a)$ satisfies

$$\limsup_{a \to \infty} \frac{\lambda_H(a)}{\lambda_H(2a)} < 2$$

(i.e., decays not faster than $a^{-p}$ with some $p < 1$), then

$$\int_T H_A \, dm \simeq A\lambda_H(A).$$
2.2. On the other hand, if the function \( a \mapsto a \lambda_H^2(a) \) does not increase (i.e., \( \lambda_H(a) \) decays as \( 1/\sqrt{a} \), or faster), then

\[
\int_{\mathbb{T}} H_A \, dm \simeq \int_{\{H \leq A\}} H \, dm,
\]

provided that \( \lambda_H(A) \) is separated from 1 (i.e., \( A \) is sufficiently large). To see this, denote by \( H^*: [0, 1] \to [0, +\infty] \) the decreasing rearrangement of \( H \), that is, the function inverse to \( \lambda_H \). Then, the function \( s \mapsto s^2 H^*(s) \) does not decrease. Letting \( \alpha = \lambda_H(A) \) (i.e., \( A = H^*(\alpha) \)), we obtain

\[
A \lambda_H(A) = \alpha H^*(\alpha) \lesssim \alpha^2 H^*(\alpha) \int_\alpha^1 \frac{ds}{s^2} = \int_\alpha^1 H^*(s) \, ds = \int_{\{H \leq A\}} H \, dm.
\]

2.3. Furthermore, if \( A/\lambda_H(A) \simeq B/\lambda_H(B) \), then

\[
\int_{\mathbb{T}} H_A \, dm \simeq \int_{\mathbb{T}} H_B \, dm.
\]

Assume, for instance, that \( A \leq B \) and that \( B/\lambda_H(B) \leq C \cdot A/\lambda_H(A) \). Since \( \lambda_H \) does not increase, \( B \leq C \cdot A \). Then,

\[
\int_{\{H \leq B\}} H \, dm \leq \int_{\{H \leq A\}} H \, dm + \int_{\{A < H \leq C \cdot A\}} H \, dm \leq \int_{\{H \leq A\}} H \, dm + C \cdot A \lambda_H(A) \leq C \int_{\mathbb{T}} H_A \, dm,
\]

and similarly,

\[
B \lambda_H(B) = \frac{B}{\lambda_H(B)} \cdot \lambda_H^2(B) \leq C \cdot \frac{A}{\lambda_H(A)} \cdot \lambda_H^2(A) = C \cdot A \lambda_H(A).
\]

Thus,

\[
\int_{\mathbb{T}} H_B \, dm \leq C \int_{\mathbb{T}} H_A \, dm.
\]

Since \( B \geq A \), the opposite estimate is obvious.
2.4. Our last remark concerns regularity of $H$. Since we will be interested only in rather crude lower and upper bounds for $e_n(\rho)$ under conditions $d\rho \geq e^{-H} \, dm$ (for lower bounds) or $\int_{\mathbb{T}} e^H \, d\rho < \infty$ (for upper bounds), our estimates will not distinguish between the sequences $e_n(\rho)$ and $C e_n(\rho)$, and we can always replace the function $H$ by any function $\tilde{H}$ with $|\tilde{H} - H| \leq 1$ without affecting our estimates. Keeping this in mind, we always assume that, for any positive $C$, the equation $C \lambda_H(A) = A$ has a unique solution.

In Section 5 (Theorem 9) we will be using the same tacit assumption for the equation $C \lambda^*_H(A) = A$, where $\lambda^*_H(a)$ is the length of the longest open interval within the set $\{H > a\}$.

In the same way, we can always assume that $e_n(\rho) < 1/2$.

3. The lower bound for $e_n$ via the Remez-type inequality

**Theorem 2:** Suppose $d\rho \geq e^{-H} \, dm$. Then

$$|\log e_n(\rho)| \lesssim \int_0^A \lambda_H(a) \, da,$$

where $A = A(n)$ solves the equation $n \lambda_H(A) = A$.

**Corollary 3:** Suppose $H$ belongs to the weak $L^1(m)$-space, i.e., $\lambda_H(a) \lesssim 1/a$ for $a \geq 1$. Then $e_n(\rho)$ does not decay to zero faster than a negative power of $n$.

Similarly, if $H$ belongs to the weak $L^p(m)$-space with $0 < p < 1$, that is $\lambda_H(a) \lesssim a^{-p}$, then

$$|\log e_n(\rho)| \lesssim n^{\frac{1-p}{1+p}}.$$

If $\lambda_H(a) \lesssim (\log a)^{-1}$ for $a \geq e$ (in particular, if $\log H$ is integrable), then

$$|\log e_n(\rho)| \lesssim \frac{n}{\log^2 n},$$

and so on, until we arrive at the classical Erdős-Turán theorem, which states that

$$|\log e_n(\rho)| = o(n), \quad n \to \infty,$$

provided that $H < +\infty$ a.e. on $\mathbb{T}$ (that is, $\lambda_H(a) \to 0$ as $a \to \infty$).
Proof of Theorem 2. Let \( P \) be an extremal algebraic polynomial of degree \( n \) such that \( P(0) = 1 \) and
\[
\int_T |P|^2 \, d\rho = e_n(\rho)^2.
\]
Then
\[
0 \leq \int_T \log |P|^2 \, dm = \int_{\{H \leq A\}} \log |P|^2 \, dm + \int_{\{H > A\}} \log(|P|^2 e^{-H}) \, dm + \int_{\{H \leq A\}} H \, dm.
\]
Estimating the first and the second integrals on the RHS we let \( n \) be so large that
\[
\int_{\{H \leq A\}} H \, dm \geq 1
\]
and
\[
(1) \quad e^A > \rho(T).
\]
By Jensen’s inequality,
\[
\int_{\{H \leq A\}} \log(|P|^2 e^{-H}) \, dm \leq \frac{1}{e} + \log \left( \int_{\{H \leq A\}} |P|^2 e^{-H} \, dm \right)
\]
\[
\leq \frac{1}{e} + \log \left( \int_T |P|^2 \, d\rho \right) = \frac{1}{e} + 2 \log e_n(\rho),
\]
and similarly,
\[
\int_{\{H > A\}} \log |P|^2 \, dm = \lambda_H(A) \left( \frac{1}{\lambda_H(A)} \int_{\{H > A\}} \log |P|^2 \, dm \right)
\]
\[
\leq \lambda_H(A) \log \left( \frac{1}{\lambda_H(A)} \int_{\{H > A\}} |P|^2 \, dm \right)
\]
\[
\leq \frac{1}{e} + \lambda_H(A) \log \left( \int_T |P|^2 \, dm \right).
\]
Next, applying the $L^2$-version of the classical Remez inequality (which follows, for instance, from a more general Nazarov’s result [8]), we obtain

$$
\int_T |P|^2 \, dm \leq e^{Cn\lambda_H(A)} \int_{\{H \leq A\}} |P|^2 \, dm \\
\leq e^{Cn\lambda_H(A)+A} \int_{\{H \leq A\}} |P|^2 e^{-H} \, dm \\
\leq e^{Cn\lambda_H(A)+A} \int_T |P|^2 \, d\rho \\
\leq e^{Cn\lambda_H(A)+A} \rho(T) \quad \text{(by extremality of } P, \int_T |P|^2 \, d\rho \leq \rho(T)) \\
\leq e^{CA} \quad \text{(by (1) and the equality } n\lambda_H(A) = A),
$$

whence

$$
\lambda_H(A) \log\left(\int_T |P|^2 \, dm\right) \leq CA \lambda_H(A).
$$

Therefore

$$
0 \leq \frac{2}{e} + CA \lambda_H(A) + 2 \log e_n(\rho) + \int_{\{H \leq A\}} H \, dm,
$$

and finally

$$
|\log e_n(\rho)| \lesssim A \lambda_H(A) + \int_{\{H \leq A\}} H \, dm = \int_T H_A \, dm,
$$

proving Theorem 2. \qed

4. The upper bound for $e_n$ via Taylor polynomials of an outer function

We give two upper bounds for $e_n(\rho)$. Both of them are based on the construction of monic polynomials of large degree with a good estimate for the $L^2(\rho)$-norm. The first bound uses Taylor polynomials of an outer function $F$ such that $1/F$ mimics the behaviour of $\rho$. It is better adjusted to the case when the distribution function $\lambda_H(a)$ decays relatively fast as $a \to \infty$. The second bound uses classical Chebyshev’s polynomials and starts working only when $\lambda_H(a)$ decays at infinity slower than $1/a$. 
Let \( \varphi: [0, \frac{1}{2}] \to (0, +\infty) \) be a continuous decreasing function, \( \varphi(0) = \infty, \varphi(\frac{1}{2}) \leq \inf_T H \). Given \( \tau \in [-\frac{1}{2}, \frac{1}{2}] \), denote by \( \theta_\tau \) the solution to the equation \( \varphi(\theta_\tau) = H(e^{2\pi i \tau}) \). We call the function \( H \) subordinated to \( \varphi \) if, for any \( \tau \),

\[
H(e^{2\pi i \theta}) \leq \varphi(\theta + \theta_\tau - \tau), \quad \tau - \theta_\tau < \theta \leq \tau,
\]

\[
H(e^{2\pi i \theta}) \leq \varphi(\tau + \theta_\tau - \theta), \quad \tau \leq \theta < \tau + \theta_\tau.
\]

Note that an equivalent way to express the \( \varphi \)-subordination is to say that the function \( (\varphi^{-1} \circ H)(e^{2\pi i \theta}) \) is a non-negative Lipschitz function on \( [-\frac{1}{2}, \frac{1}{2}] \) with the Lipschitz constant at most one.

We call the unbounded continuous decreasing function \( \varphi \) on \( (0, \frac{1}{2}] \) regular if it satisfies at least one of the following two conditions:

\( \text{(Reg1)} \quad \varphi(\frac{\theta}{2}) \lesssim \varphi(\theta) \) and \( \varphi(\theta) \gtrsim \frac{1}{\theta} \).

\( \text{(Reg2)} \quad \varphi(\theta/2) \lesssim \varphi(\theta) \) and \( \varphi(\theta) \gtrsim \frac{1}{\theta} \).

**Theorem 4:** Suppose that

\[
\int_T e^H d\rho < \infty,
\]

with \( H \) subordinated to a regular function \( \varphi \). Then

\[
\left| \log e_n(\rho) \right| \gtrsim \int_0^A \lambda_H(a) \, da,
\]

where \( A \) solves the equation \( n\varphi^{-1}(A) = A \) when \( \varphi \) satisfies condition (Reg1), and \( A = \sqrt{n} \) when \( \varphi \) satisfies condition (Reg2).

**Corollary 5:** In the assumptions of Theorem 4, suppose that \( \log \varphi(\theta) \gtrsim \log \frac{1}{\theta} \).

If

\[
\liminf_{a \to \infty} a\lambda_H(a) > 0,
\]

then \( e_n(\rho) \) decays to zero at least as a negative power of \( n \).

Furthermore, \( e_n(\rho) \) decays to zero faster than any negative power of \( n \), provided that

\[
\lim_{a \to \infty} a\lambda_H(a) = \infty.
\]

Note that we need to impose the additional condition \( \log \varphi(\theta) \gtrsim \log \frac{1}{\theta} \) in this Corollary only in the case when \( \varphi \) satisfies the first regularity condition (Reg1).
4.1. TAYLOR POLYNOMIALS. Denote by $P_r$ the Poisson kernel for the unit disk evaluated at the point $r \in [0, 1)$.

**Lemma 6:** Let $H$ be a weight such that

$$H_A * P_{1-\delta} \lesssim H + 1 \quad \text{everywhere on } \mathbb{T},$$

with $\log \delta^{-1} \lesssim A$. Suppose that

$$\int_\mathbb{T} e^H \, d\rho < \infty.$$

Then there exists a positive constant $C$ such that, for $n \geq CA/\delta$, we have

$$|\log e_n(\rho)| \geq \int_\mathbb{T} H_A \, dm.$$

**Proof of Lemma 6.** Let $M$ be a positive constant such that

$$H_A * P_{1-\delta} \leq M(H + 1),$$

and let $F_A$ be an outer function in $\mathbb{D}$ with the boundary values $|F_A|^2 = e^{H_A/M}$, i.e.,

$$\log |F_A(rt)| = \frac{1}{2M} (H_A * P_r)(t).$$

We expand $F_A((1-\delta)z)$ into the Taylor series

$$F_A((1-\delta)z) = \sum_{k \geq 0} f_k z^k,$$

and consider the Taylor polynomials

$$P_A(z) = \sum_{k=0}^n f_k z^k.$$

Then

$$e_n(\rho)^2 \leq |P_A(0)|^{-2} \int_\mathbb{T} |P_A|^2 \, d\rho.$$

First, we note that

$$|P_A(0)| = |F_A(0)| = \exp \left( \int_\mathbb{T} \log |F_A| \, dm \right) = \exp \left( \frac{1}{2M} \int_\mathbb{T} H_A \, dm \right),$$

i.e.,

$$e_n(\rho)^2 \leq \int_\mathbb{T} |P_A|^2 \, d\rho \cdot \exp \left( - \frac{1}{M} \int_\mathbb{T} H_A \, dm \right).$$

Next,

$$\int_\mathbb{T} |F_A((1-\delta)t)|^2 \, d\rho(t) = \int_\mathbb{T} \exp \left( \frac{1}{M} H_A * P_{1-\delta} \right) \, d\rho \leq \int_\mathbb{T} \exp(H + 1) \, d\rho \lesssim 1,$$
so it remains to estimate the remainder
\[ |F_A((1 - \delta)z) - P_A(z)| \leq \sum_{k > n} |f_k|. \]
By Cauchy’s estimates,
\[ |f_k| \leq (1 - \delta)^k \max_{\mathbb{T}} |F_A| \leq (1 - \delta)^k e^{A/(2M)}, \]
whence
\[ \sum_{k > n} |f_k| \lesssim \delta^{-1} e^{A/(2M)} - n \delta \lesssim 1, \]
provided that \( A/\delta \lesssim n \) (here, we use that \( \log \delta^{-1} \lesssim A \)). Thus
\[ \int_{\mathbb{T}} |P_A|^2 d\rho \lesssim 1, \]
which proves the lemma.

4.2. Estimates of the Poisson integral. Put \( p_{\delta}(\theta) = \varphi(1 - 2\pi \theta) \) and recall that \( p_{\delta}(\theta) \lesssim \min(\delta^{-1}, \delta \theta^{-2}) \).

**Lemma 7:** Let \( \varphi: (0, \frac{1}{2}] \to [0, \infty) \) be an unbounded continuous decreasing function, let \( \tilde{\varphi} \) be its even 1-periodic extension on \( \mathbb{R} \), and \( \tilde{\varphi}_A = \min(\tilde{\varphi}, A) \). Then
\[ \tilde{\varphi}_A * p_{\delta} \lesssim \varphi + 1 \] everywhere on \( \mathbb{R} \),

provided that at least one of the following holds:

(i) \( \) the function \( \theta \mapsto \theta^2 \varphi(\theta) \) does not decrease, and \( \delta \lesssim \varphi^{-1}(A) \);
(ii) \( \varphi(\theta/2) \lesssim \varphi(\theta), \varphi(\theta) \gtrsim 1/\theta, \) and \( \delta \lesssim 1/A. \)

Note that condition (i) is weaker than condition (Reg1) in Theorem 4, i.e., the lemma is a bit stronger than what we will use for the proof of Theorem 4. We need this version of Lemma 7 for the proof of Theorem 14. We also note that condition (i) yields estimate \( \varphi(\theta/2) \lesssim \varphi(\theta) \) from condition (ii).

**Proof of Lemma 7.** We take a sufficiently small \( \tau_0 > 0 \) so that \( \varphi(\tau_0) \geq 1 \), fix \( \tau \in (0, \tau_0] \), and estimate the convolution \( (\tilde{\varphi}_A * p_{\delta})(\tau) \). There is nothing to prove if \( A \leq \varphi(\frac{1}{2}\tau) \) since in this case
\[ (\tilde{\varphi}_A * p_{\delta})(\tau) \leq \max_{[-\frac{\tau}{2}, \frac{\tau}{2}]} \tilde{\varphi}_A = A \leq \varphi(\frac{1}{2}\tau) \lesssim \varphi(\tau) \]
for any \( \delta > 0 \). Hence, in what follows, we assume that \( A \geq \varphi(\frac{1}{2}\tau), \) i.e., \( \tau \geq 2\varphi^{-1}(A) \).
First, we note that for any $\theta \in [0, \frac{1}{2}]$, we have $\bar{\varphi}_A(\tau + \theta) \leq \bar{\varphi}_A(\tau - \theta)$ (to see this, one needs to consider three cases: $0 \leq \theta \leq \tau$, $\tau \leq \theta \leq \frac{1}{2} - \tau$, and $\frac{1}{2} - \tau \leq \theta \leq \frac{1}{2}$). Therefore, 

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\varphi}_A(\tau - \theta)p_\delta(\theta)\,d\theta = \int_0^{\frac{1}{2}} [\bar{\varphi}_A(\tau - \theta) + \bar{\varphi}_A(\tau + \theta)]p_\delta(\theta)\,d\theta \leq 2A \int_{0 \leq \theta \leq \frac{\varphi^{-1}(A)}{\varphi^{-1}(A)}} p_\delta(\theta)\,d\theta + 2 \int_{\theta > \tau - 2, \varphi^{-1}(A), \varphi(\tau - \theta)p_\delta(\theta)\,d\theta = I + II.}$$

Before we start estimating integrals on the RHS, observe that $\delta \lesssim \tau$. In the case (i) it is obvious since $\delta \lesssim \varphi^{-1}(A) \leq \tau/2$, in the case (ii) it is also obvious since then $\delta \lesssim 1/A \leq \varphi^{-1}(A) \leq \tau/2$. Therefore, in the first integral $\theta \geq \tau - \varphi^{-1}(A) \geq \tau/2 \gtrless \delta$. Recalling the standard estimate of the Poisson kernel $p_\delta(\theta) \lesssim \min(\delta^{-1}, \delta\theta^{-2})$, we get

$$I \lesssim A\delta \int_{\varphi^{-1}(A)}^{\tau + \varphi^{-1}(A)} \frac{d\theta}{\theta^2} \lesssim \frac{A\delta\varphi^{-1}(A)}{\tau^2}.$$ 

In both cases (i) and (ii) the RHS is $\lesssim \varphi(\tau)$. Indeed, if (i) holds, then it is bounded by $\varphi^{-1}(A)^2 A/\tau^2 \lesssim \tau^2 \varphi(\tau)/\tau^2 = \varphi(\tau)$. If (ii) holds, then it is bounded by $\varphi^{-1}(A)/\tau^2 \lesssim 1/\tau \lesssim \varphi(\tau)$.

We split the second integral into four parts

$$\int_{0 \leq \theta \leq \frac{\varphi^{-1}(A)}{\varphi^{-1}(A)}} = \int_0^{\min(\delta, \frac{1}{2} \tau)} + \int_{\frac{1}{2} \tau}^{\min(\delta, \frac{1}{2} \tau)} + \int_{\varphi^{-1}(A) \leq \theta - \tau \leq \frac{1}{2} \tau} + \int_{\frac{1}{2} \tau}^{1/2} \varphi(\tau - \theta)p_\delta(\theta)\,d\theta \lesssim \frac{1}{\delta} \min(\delta, \frac{1}{2} \tau) \cdot \varphi(\frac{1}{2} \tau) \lesssim \varphi(\tau),$$

and estimate them one by one. We have

$$\int_0^{\min(\delta, \frac{1}{2} \tau)} \varphi(\tau - \theta)p_\delta(\theta)\,d\theta \leq \frac{1}{\delta} \int_0^{\min(\delta, \frac{1}{2} \tau)} \varphi(\tau - \theta)\,d\theta \leq \frac{1}{\delta} \min(\delta, \frac{1}{2} \tau) \cdot \varphi(\frac{1}{2} \tau) \lesssim \varphi(\tau),$$

and

$$\int_{\min(\delta, \frac{1}{2} \tau)}^{\frac{1}{2} \tau} \varphi(\tau - \theta)p_\delta(\theta)\,d\theta \lesssim \int_{\min(\delta, \frac{1}{2} \tau)}^{\frac{1}{2} \tau} \frac{\varphi(\tau - \theta)}{\theta^2}\,d\theta \lesssim \delta \cdot \varphi(\frac{1}{2} \tau) \int_{\min(\delta, \frac{1}{2} \tau)}^{\infty} \frac{d\theta}{\theta^2} \lesssim \varphi(\tau).$$
Next,
\[
\int_{\varphi^{-1}(A) \leq |\theta - \tau| \leq \frac{1}{2}\tau} \varphi(\tau - \theta)p_\delta(\theta) \, d\theta \lesssim \delta \int_{\varphi^{-1}(A) \leq |\theta - \tau| \leq \frac{1}{2}\tau} \frac{\varphi(\tau - \theta)}{\theta^2} \, d\theta \\
\lesssim \frac{\delta}{\tau^2} \int_{\varphi^{-1}(A)} \varphi(\xi) \, d\xi.
\]
In the case (i), the integral on the RHS equals
\[
\frac{\delta}{\tau^2} \int_{\varphi^{-1}(A)} \frac{\xi^2 \varphi(\xi)}{\xi^2} \, d\xi \leq \frac{\delta}{\tau^2} \frac{(\tau/2)^2 \varphi(\tau/2)}{\varphi^{-1}(A)} \lesssim \frac{\varphi(\tau)}{\tau},
\]
while in the case (ii), it does not exceed
\[
\frac{\delta}{\tau^2} \cdot A\tau/2 \lesssim \frac{1}{\tau} \frac{\tau \varphi(\tau) \gtrless 1}{\gtrless A} \lesssim \varphi(\tau).
\]
At last,
\[
\int_{\frac{1}{2}\tau}^{\frac{1}{2}\tau} \varphi(\tau - \theta)p_\delta(\theta) \, d\theta \lesssim \delta \int_{\frac{1}{2}\tau}^{\frac{1}{2}\tau} \frac{\varphi(\theta - \tau)}{\theta^2} \, d\theta \lesssim \delta \varphi(\tau/2) \cdot \frac{1}{\tau} \frac{\delta \lesssim \tau}{\gtrsim} \varphi(\tau),
\]
completing the proof of the lemma. 

4.2.1. The Poisson integral of $H_A$.

**Lemma 8:** Let $H$ be subordinated to a regular function $\varphi$, and $H_A = \min(H, A)$. Then
\[
H_A \ast \mathbb{P}_{1-\delta} \lesssim H + 1 \quad \text{everywhere on } \mathbb{T},
\]
provided that $\delta \lesssim \varphi^{-1}(A)$ when $\varphi$ satisfies condition (Reg1), and $\delta \lesssim A^{-1}$ when $\varphi$ satisfies condition (Reg2).

Clearly, Lemma 6 and Lemma 8 combined together yield Theorem 4.

**Proof of Lemma 8.** We write $H(e^{2\pi i \theta}) = h(\theta)$, fix the point $\tau \in [-\frac{1}{2}, \frac{1}{2}]$ with $h(\tau) < \infty$ at which we will estimate the convolution $(h_A \ast p_\delta)(\tau)$, and choose $\theta_\tau$ so that $\varphi(\theta_\tau) = h(\tau)$. Similarly to the proof of the previous lemma, we assume that $A \geq \varphi(\frac{1}{2}\theta_\tau)$, i.e., that $\varphi^{-1}(A) \leq \frac{1}{2}\theta_\tau$; otherwise,
\[
(h_A \ast p_\delta)(\tau) \leq A \leq \varphi(\frac{1}{2}\theta_\tau) \lesssim \varphi(\theta_\tau) = h(\tau),
\]
and we are done. Now
\[
(h_A \ast p_\delta)(\tau) = \left( \int_{|\theta - \tau| \gtrsim \theta_\tau - \varphi^{-1}(A)} + \int_{|\theta - \tau| \lesssim \theta_\tau - \varphi^{-1}(A)} \right) h_A(\theta)p_\delta(\tau - \theta) \, d\theta = I + II.
\]
To estimate the first integral, we note that, since \( \theta - \varphi^{-1}(A) \geq \theta / 2 \), we have

\[
I \leq A \int_{|\theta - \tau| \geq \theta / 2} p_\delta(\tau - \theta) \, d\theta \leq 2A \int_{\theta / 2}^{1/2} p_\delta(\theta) \, d\theta
\]

\[
\lesssim A \delta \int_{\theta / 2}^{\infty} \frac{d\theta}{\theta^2} \lesssim \frac{A \delta}{\theta}.
\]

In the first case, the RHS is

\[
\delta \lesssim \varphi^{-1}(A) \frac{A \varphi^{-1}(A)}{\theta} \lesssim \left( \frac{1}{2} \theta \cdot \varphi \left( \frac{1}{2} \theta \right) \right) \lesssim \varphi(\theta).
\]

In the second case,

\[
A \delta / \theta \lesssim 1 / \theta \lesssim \varphi(\theta).
\]

Therefore, in both cases, the first integral is \( \lesssim \varphi(\theta) = h(\tau) \).

To estimate the second integral, we note that, by the subordination to \( \varphi \), it is bounded by \( 2(\tilde{\varphi}_A * p_\delta)(\theta) \), which, by the previous lemma, is \( \lesssim \varphi(\theta) + 1 = h(\tau) + 1 \).

5. The upper bound for \( e_n \) via Chebyshev polynomials

Here we assume that the function \( H \) is lower semicontinuous, i.e., the sets \( \{ H > a \} \) are open, and denote by \( \lambda^*_H(a) \) the length of the longest open interval within \( \{ H > a \} \).

**Theorem 9:** Suppose that

\[
\int \mathbb{T} e^H \, d\rho < \infty.
\]

Then

\[
| \log e_n(\rho) | \gtrsim A \lambda^*_H(A),
\]

where \( A = A(n) \) solves the equation \( n \lambda^*_H(A) = A \).

The following Corollary combines Theorem 9 with Theorem 2 (and takes into account Observations 2.1 and 2.3).
Corollary 10: Let $d\rho = e^{-H} \, dm$. Suppose that the set $\{H > a\}$ contains an interval with the length comparable to the total length of $\{H > a\}$ (i.e., $\lambda_H^* \gtrsim \lambda_H$), and that the function $\lambda_H$ satisfies
\[
\limsup_{a \to \infty} \frac{\lambda_H(a)}{\lambda_H(2a)} < 2.
\]
Then
\[
|\log e_n(\rho)| \simeq A \lambda_H(A),
\]
where $A = A(n)$ is a solution to the equation $n \lambda_H(a) = a$.

Proof of Theorem 9. We will use the following classical lemma (cf., for instance, [7]):

Lemma 11: For any $\alpha \in (0, \frac{\pi}{2}]$ and for any $n \in 2\mathbb{N}$, there exists a monic polynomial $T_{n,\alpha}$ of degree $n$ such that
\[
\max\{|T_{n,\alpha}(e^{i\theta})|: |\theta| \geq \alpha\} = 2 \cos^n(\alpha/2).
\]

For the reader’s convenience, we recall its proof.

Proof. Put
\[
T_{2m,\alpha}(e^{i\theta}) = 2 \cos^{2m}(\alpha/2)e^{im\theta} \cos \left(2m \arccos \left(\frac{\cos(\theta/2)}{\cos(\alpha/2)}\right)\right).
\]
We only need to show that this is a monic polynomial of degree $n = 2m$. Recall that $\cos(2m \arccos x) = 2^{2m-1}Q_m(x^2)$, where $Q_m$ is a monic polynomial of degree $m$. Then
\[
T_{2m,\alpha}(e^{i\theta}) = 2^{2m} \cos^{2m}(\alpha/2)e^{im\theta} Q_m \left(\frac{\cos^2 \theta/2}{\cos^2 \alpha/2}\right) = 2^{2m} \cos^{2m}(\alpha/2)e^{im\theta} Q_m \left(\frac{e^{i\theta} + e^{-i\theta} + 2}{4 \cos^2 \alpha/2}\right)
\]
and it is easily seen that the RHS is a monic polynomial of degree $2m$, proving the lemma.

Now, we turn to the proof of Theorem 9. Without loss of generality, we assume that $n$ is an even number. Let $J \subset \mathbb{T}$ be the longest arc in the set $\{H > a\}$. We assume that $J = \{t = e^{i\varphi}: |\varphi| \leq \alpha\}$, $\alpha = \pi \lambda_H^*(a)$. Let $T = T_{n,\alpha}$.
be a monic polynomial of degree $n$ as in Lemma 11. Then, by a straightforward computation (or by the classical Remez inequality)

$$\max_J |T| = \max_{|\varphi| \leq \alpha} |T(e^{i\varphi})| \lesssim e^{Cn\alpha} \left( \cos \frac{\alpha}{2} \right)^n.$$ 

Noting that $\cos^n \frac{\alpha}{2} \leq e^{-cn\alpha^2}$, we get

$$\int_T |T|^2 \, d\rho \leq \max_J (|T|^2 e^{-H}) \int_T e^H \, d\rho \lesssim e^{-a} \max_J |T|^2 + \max_{T \setminus J} |T|^2 \lesssim (e^{Cn\lambda^*_H(a) - a} + 1) e^{-cn\lambda^*_H(a)^2}.$$ 

Letting $A_C$ be the unique solution to the equation $Cn\lambda^*_H(A_C) = A_C$, we obtain

$$|\log e_n(\rho)| \gtrsim n\lambda^*_H(A_C)^2 \simeq A_C\lambda^*_H(A_C) \simeq A\lambda^*_H(A),$$

completing the proof of Theorem 9. 

6. Examples

To illustrate our results, we consider the function

$$H = h \circ d_K,$$

where $h: (0, \frac{1}{2}] \to (0, +\infty)$ is a $C^1$-smooth decreasing function, $h(0) = +\infty$, and $d_K(t) = \text{dist}(t, K)$, where $K \subset \mathbb{T}$ is a compact set of zero length (recall that we identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$).

Denote by

$$K_{+s} = \{ t : d_K(t) < s \}$$

the $s$-neighbourhood of $K$ and by

$$\psi_K(s) = m(K_{+s})$$

its length. Then

$$\lambda_H(a) = m\{ H > a \} = \begin{cases} \psi_K(h^{-1}(a)), & a \geq h\left(\frac{1}{2}\right), \\ 1, & a < h\left(\frac{1}{2}\right), \end{cases}$$

and

$$\int_0^A \lambda_H(a) \, da = \left( \int_0^{h(1/2)} + \int_{h(1/2)}^A \right) \lambda_H(a) \, da = \int_{h^{-1}(A)}^{1/2} \psi_K |h'| + h\left(\frac{1}{2}\right),$$

provided that $A > h\left(\frac{1}{2}\right)$. 
To estimate the function $\psi_K$ it is convenient to use that
\[
\psi_K(s) \simeq sN_K(s) \simeq sP_K(s),
\]
where $N_K(s)$ is the covering number of $K$ and $P_K(s)$ is the packing number of $K$; see, for instance, [2, Chapter 3]. We call the set $K$ $\gamma$-regular if
\[
\psi_K(s) \simeq s^{1-\gamma}.
\]
For instance, the set $e^{2\pi iC}$, where $C$ is the standard ternary Cantor set, is $\gamma$-regular with $\gamma = \frac{\log 2}{\log 3}$, while the set $\{t = \exp(2\pi in^{-\nu}) : n \in \mathbb{N}\} \cup \{1\}$ is $\gamma$-regular with $\gamma = (\nu + 1)^{-1}$.

6.1. TWO COROLLARIES. We get straightforward corollaries to our results taking $h(s) = s^{-p}$.

**Corollary 12:** Let $K \subset \mathbb{T}$ be a $\gamma$-regular compact set with some $\gamma \in [0,1)$. Suppose that $d\rho = \exp(-d_K^{-1})\,dm$. Then
\[
|\log e_n(\rho)| \simeq \log n.
\]

The second corollary pertains to the case when the length of the longest interval in the set $\{d_K < s\}$ is comparable with the length of the whole set $\{d_K < s\}$. Then Corollary 10 applies.

**Corollary 13:** Let $\nu > 0$, $K = \{t = \exp(2\pi in^{-\nu}) : n \in \mathbb{N}\} \cup \{1\}$, and $d\rho = \exp(-d_K^{-p})\,dm$ with $p > \vartheta = \frac{\nu}{\nu+1}$. Then
\[
|\log e_n(\rho)| \simeq n^{\frac{\nu}{\nu+\vartheta}}.
\]

6.2. MEASURES WITH DEEP ZERO AT ONE POINT. The last illustration on our estimates pertains to the simplest case when the measure $\rho$ has a deep zero at one point and is symmetric with respect to this point. In this case, our estimates yield a relatively complete result.

**Theorem 14:** Let $h : (0, \frac{1}{2}] \to [0, +\infty)$ be a continuous decreasing function such that
\[
\int_0^1 h(a)\,da = +\infty.
\]
Suppose that $h$ satisfies at least one of the following two conditions:

(i) $\theta \mapsto \theta^2 h(\theta)$ does not decrease,
and
\[ |\log \theta| = O(h(\theta)), \quad \theta \to 0; \]

(ii)
\[ \limsup_{a \to \infty} \frac{h^{-1}(a)}{h^{-1}(2a)} < 2. \]

Let \( \rho \) be an absolutely continuous measure on \( \mathbb{T} \) with density \( e^{-h(|\theta|)} \). Then
\[ |\log e_n(\rho)| \simeq \int_0^{1/2} h_A(a) \, da \]
where \( A \) solves the equation \( nh^{-1}(A) = A \) and \( h_A = \min(h, A) \).

The lower bound for \( e_n \) (i.e., the upper bound for \( |\log e_n| \)) follows from Theorem 2 and does not need any regularity assumptions on \( h \). Conditions (i) and (ii) are needed for the proof of the upper bound for \( e_n \). In the case (i), it is a consequence of Lemma 6 combined with the first case of Lemma 7. In the case (ii), it follows from Theorem 9. Note that these two cases overlap, e.g., the function \( h(\theta) = \theta^{-p} \) with \( 1 < p \leq 2 \) satisfies both of them.

The following corollary gives an idea about the rate of decay of \( e_n(\rho) \) for several explicitly written functions \( h \).

**Corollary 15:** Let \( \rho \) be an absolutely continuous measure on \( \mathbb{T} \) with density \( e^{-h(|\theta|)} \). Then for \( n \geq 4 \) we have:

(i) If \( h(\theta) \simeq \theta^{-1} \log^{-1}(1/\theta) \), then \( |\log e_n(\rho)| \simeq \log \log n \).

(ii) If \( p > -1 \) and \( h(\theta) \simeq \theta^{-1} \log^p(1/\theta) \), then \( |\log e_n(\rho)| \simeq (\log n)^{p+1} \).

(iii) If \( p > 1 \) and \( h(\theta) \simeq \theta^{-p} \), then \( |\log e_n(\rho)| \simeq n^{(p-1)/(p+1)} \).

(iv) If \( p > 0 \) and \( h(\theta) \simeq \exp(\theta^{-p}) \), then \( |\log e_n(\rho)| \simeq n(\log n)^{-2/p} \).

Our last remark is that, plausibly, the technique based on the potential theory in the external field developed by Mhaskar–Saff, Rakhmanov, Levin–Lubinsky, Totik and others should allow one to obtain more precise estimates of \( e_n \) in the situation considered in Theorem 14. See, for instance, Theorem 1.22 and Examples 3 and 4 in Section 1.6 in [6] which contain similar results for orthogonal polynomials on the real line. On the other hand, likely, this will require much stronger regularity assumptions on the function \( h \) and more technical proofs.
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