ON THE ROTATION CLASS OF KNOTTED LEGENDRIAN TORI IN $\mathbb{R}^5$

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Abstract. In this paper we show how to combinatorially compute the rotation class of a large family of embedded Legendrian tori in $\mathbb{R}^5$ with the standard contact form. In particular, we give a formula to compute the Maslov index for any loop on the torus and compute the Maslov number of the Legendrian torus. These formulas are a necessary component in computing contact homology. Our methods use a new way to represent knotted Legendrian tori called Lagrangian hypercube diagrams.

1. Introduction

Compared to Legendrian knots in $\mathbb{R}^3$, little is known about knotted Legendrian submanifolds $L^n$ embedded in $\mathbb{R}^{2n+1}$. One reason is that in higher dimensions there are no standard representations of embedded Legendrian submanifolds that enable one to study with the same facility as front projections or Lagrangian projections of Legendrian knots in $\mathbb{R}^3$. For example, one may easily compute the classical invariants of Thurston-Bennequin and rotation numbers by looking at the front projection of a knot in $\mathbb{R}^3$. Moreover, the classical invariants are quite effective at distinguishing many knots up to Legendrian isotopy: torus knots, for example have been shown to be classified by their classical invariants (cf. [10]).

While the Thurston-Bennequin number may be generalized to higher dimensions, it is not always as useful as it is for knots in dimension 3. In the case we study in this paper, knotted Legendrian tori $L \subset \mathbb{R}^5$, the Thurston-Bennequin invariant is well defined (cf. [26]), but uninteresting since it is always equal to zero. In fact, the Thurston-Bennequin number in $\mathbb{R}^{2n+1}$ equals $(-1)^{n+1} \frac{1}{2} \chi(L)$ when $n$ is even. Furthermore, while topological knot type provides an additional invariant for Legendrian knots in $\mathbb{R}^3$, all knotted Legendrian surfaces in $\mathbb{R}^5$ are topologically equivalent provided they are of the same genus.

The rotation class is harder to generalize to higher dimensions. Unlike the Thurston-Bennequin number, which may be defined in terms of a linking number, the rotation number requires the computation of the homotopy class of a map from $L$ to the space of Lagrangians of $\mathbb{R}^4$ with symplectic structure induced by the contact form on $\mathbb{R}^5$. Since writing down this map is non-trivial this invariant is more difficult to compute in higher dimensions.

Lagrangian hypercube diagrams, defined in Section 4, overcome the difficulties involved in studying knotted Legendrian tori in $\mathbb{R}^5$ by providing a way to construct explicit embeddings of Legendrian tori (cf. Section 5) from Lagrangian grid diagrams (cf. Section 3). Using the explicit map defined by a Lagrangian hypercube diagram we demonstrate that the rotation class (cf. Section 2) may be calculated combinatorially as follows:

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Theorem 1. Suppose $H \Gamma = (C, \{W, X, Y, Z\}, G_{zx}, G_{wy})$ is a Lagrangian hypercube diagram with Lagrangian grid diagram projections $G_{zx}$ and $G_{wy}$ in $\mathbb{R}^2$. The diagram $H \Gamma$ determines a Lagrangian torus in $\mathbb{R}^4$ that lifts to an embedded Legendrian torus $L \subset \mathbb{R}^5$. The immersed curves determined by $G_{zx}$ and $G_{wy}$ lift to loops, $\tilde{\gamma}_{zx}$ and $\tilde{\gamma}_{wy}$, in $L$ such that $H_1(L) \cong \langle \tilde{\gamma}_{zx}, \tilde{\gamma}_{wy} \rangle$ (cf. Theorem 6.1). In this setup, the rotation class of $L$, $r(L) \in H_1(L)$, satisfies:

$$r(L) = (w(G_{zx}), w(G_{wy})),$$

where $w(G_{zx})$ is the winding number of the immersed curve determined by $G_{zx}$. In particular, each winding number can be computed combinatorially from the Lagrangian grid diagram projection:

$$w(G) = \frac{1}{4} \left( \#(\text{counterclockwise oriented corners of } G) - \#(\text{clockwise oriented corners of } G) \right).$$

Example 1.1. Let $H \Gamma$ be the Lagrangian hypercube diagram constructed from the Lagrangian grid diagrams shown in Figure 1 (by applying Theorem 8.4). The Lagrangian hypercube determines an immersed Lagrangian torus $T$ (Theorem 5.1). The lift of the Lagrangian torus $T$ is a knotted, embedded Legendrian torus $L$ (Theorem 6.1). By Theorem 1, the rotation class of the Legendrian torus $L$ is $r(L) = (1, 0)$.

Figure 1: Unknots with rotation number 1 and 0 respectively.

Recall that the Maslov index, as defined in [24] and [9], may be viewed as a map $\mu : H_1(L) \to \mathbb{Z}$.

Corollary 2. If $A \in H_1(L)$ such that $A = a[\tilde{\gamma}_{zx}] + b[\tilde{\gamma}_{wy}]$, then the Maslov index is

$$\mu(A) = 2aw(G_{zx}) + 2bw(G_{wy}).$$

The Maslov number of the torus $L$ is the smallest positive number that is the Maslov index of some nontrivial loop (cf. [9]). Thus Corollary 2 enables us to compute the Maslov number of $L$ as follows:

Corollary 3. The Maslov number of $L$ is the non-negative number $2\gcd(w(G_{zx}), w(G_{wy}))$.

In [9], Ekholm, Etnyre, and Sullivan compute the classical invariants for Legendrian tori obtained by front-spinning. In particular, they show that the rotation class of the surface so obtained is determined by the rotation number of the front projection used in the construction. Thus, their construction leads to tori with rotation class of the form $(0, r)$. Not only are we able to construct Legendrian tori in which both factors of the torus are knotted, we also show that Legendrian tori constructed from hypercube diagrams realize every possible pair of integers under the isomorphism defined by $H \Gamma$. In particular, we get examples where the rotation class is $(0, r)$ in the following theorem by taking one of the knots to be a trivial knot with rotation number zero:
Theorem 4. Let \((m, k) \in \mathbb{Z}^2\), and \(K_1, K_2\) be any two topological knots in \(\mathbb{R}^3\). There exists a hypercube diagram, \(H\Gamma = (C, \{W, X, Y, Z\}, G_{xx}, G_{wy})\), such that \(G_{xx}\) and \(G_{wy}\) are Lagrangian grid diagrams whose lifts are Legendrian knots in \(\mathbb{R}^3\) with the same topological knot types as \(K_1\) and \(K_2\), and whose rotation numbers are \(m\) and \(k\), respectively. Moreover, \(H\Gamma\) specifies an immersed Lagrangian torus that lifts to a smoothly embedded Legendrian torus \(L \subset \mathbb{R}^5\) that satisfies \(r(L) = (m, k)\).

Theorem 4 is a statement about the existence of Lagrangian hypercube diagrams that represent smoothly embedded Legendrian tori. The methods used in the proof to find Lagrangian hypercube diagrams lead in general to excessively large diagrams. In practice, however, Lagrangian hypercube diagrams are easy to build by hand. Knot theory benefited greatly because of the development of nice representations for the knots: braids, knot projections, grid diagrams, etc. Theorem 1 and 4 together can be viewed as our attempt to create similar useful representations of Legendrian tori in \(\mathbb{R}^5\). While it is still unclear whether or not all Legendrian tori arise in this way, computers can easily generate and compute many examples (see Theorem 8.4).

In fact, in proving Theorem 4, we also created a computationally useful representation of Legendrian knots in \(\mathbb{R}^3\). The following theorem provides a grid diagram representation similar to Theorem 4 above (cf. Proposition 2.2 in [20]).

**Theorem 5.** Given any topological knot \(K \subset \mathbb{R}^3\) and \(m \in \mathbb{Z}\), there exists a Lagrangian grid diagram whose lift in \(\mathbb{R}^3\) is isotopic to a smoothly embedded Legendrian knot with the same topological type as \(K\) and rotation number \(m\).

This paper stands alone in providing one of the easiest ways to algorithmically compute classical Legendrian invariants using grid and hypercube diagrams for a large class of knotted Legendrian submanifolds in \(\mathbb{R}^{2n+1}\) for \(n \geq 1\) (cf. [9]). We see the potential for much more: this paper contains key elements in the computing the gradings and dimensions of the moduli spaces used in computing the differential in contact homology. Our future work will be on how to use the representations and the calculations in this paper to compute the contact homology algorithmically directly from Lagrangian hypercube diagrams.

In fact, we are particularly interested in studying the contact homology of embedded Legendrian tori in \(\mathbb{R}^5\) (or \(S^5\)) because of their relationship to Special Lagrangian Cones used to study the String Theory Model in physics. Briefly, according to this model, our universe is a product of the standard Minkowsky space \(\mathbb{R}^4\) with a Calabi-Yau 3-fold \(X\). Based upon physical grounds, the SYZ-conjecture of Strominger, Yau, and Zaslov (cf. [25]) expects that this Calabi-Yau 3-fold can be given a fibration by Special Lagrangian 3-tori with possibly some singular fibers. To make this idea rigorous one needs control over the singularities, which are not understood well. One method used to study these singularities (cf. Haskins [12] and Joyce [13]) is to model them locally as special Lagrangian cones \(C \subset \mathbb{C}^3\). A special Lagrangian cone can be characterized by its associated link \(L = C \cap S^5\) (the link of the singularity), which turns out to be a minimal Legendrian surface. When the link type of \(L\) is a sphere, then \(C\) must be a special Lagrangian plane. The interesting tractable case appears to be when the link type is an embedded torus. Several authors (cf. Castro-Urbano [6], Haskins [12], Joyce [13]) have shown that there exist infinite families of nontrivial special Lagrangian cones arising from minimal embedded Legendrian tori. Some work is already being done by Aganagic, Ekholm, Ng, and Vafa [1] to understand the connection between contact homology and Lagrangian fillings. We see this paper as possibly laying groundwork for developing combinatorial tools to understand Lagrangian cones and special Lagrangian cones through the lens of contact homology.
In Section 2 we present a definition for the rotation class in dimension 5 and prove that it is characterized by a pair of integers. Section 3 discusses Lagrangian grid diagrams, and proves Theorem 5, enabling us to define Lagrangian hypercube diagrams in Section 4. In Section 5 we prove that a Lagrangian hypercube diagram represents an immersed Lagrangian torus in dimension 4. This torus is shown in Section 6 to lift to a Legendrian torus in $\mathbb{R}^5$ with the standard contact structure. We then prove Theorem 1 (Section 7) and close with a proof of Theorem 4 and further examples (Section 8).

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2. Rotation class for embedded Legendrian tori in $\mathbb{R}^5$

In Section 3 of [9], the classical Legendrian invariants of Thurston-Bennequin number and rotation number in dimension 3 are generalized to dimension 2. We recall the definition of rotation class for $\mathbb{R}^5$ here. Let $\mathbb{R}^5$ be parametrized using $wxyzt$-coordinates. The contact 1-form, $\alpha = dt - ydw - xdz$, represents the standard contact structure on $\mathbb{R}^5$. The contact hyperplanes are given by:

$$\xi = \ker(\alpha) = \{\partial_x, \partial_y, \partial_w + y\partial_t, \partial_z + x\partial_t\}.$$ 

Let $f: L \to (\mathbb{R}^5, \xi)$ be a Legendrian immersion. The image of $df_x: T_xL \to T_{f(x)}\mathbb{R}^5$ is a Lagrangian subspace of the contact hyperplane $\xi_{f(x)}$. Choose the complex structure $J: \xi_{(w,x,y,z,t)} \to \xi_{(w,x,y,z,t)}$ such that $J(\partial_w + y\partial_t) = \partial_y$, $J(\partial_y) = -(\partial_w + y\partial_t)$, $J(\partial_z + x\partial_t) = \partial_x$, and $J(\partial_x) = -(\partial_z + x\partial_t)$. Using this complex structure, the complexification $df_C: TL \otimes \mathbb{C} \to \xi$ is a fiberwise bundle isomorphism. With this setup, the homotopy class of $(f, df_C)$ is called the rotation class.

When $f$ is an embedding given by inclusion, we denote the rotation class by $r(L)$. Note that the Lagrangian projection $\pi_t: \mathbb{R}^5 \to \mathbb{C}^2$ gives a complex isomorphism between $(\xi, J)$ and the trivial bundle with fiber $\mathbb{C}^2$. Composing $df_C$ with $\pi_t$ we get a trivialization $TL \otimes \mathbb{C} \to \mathbb{C}^2$, which we identify with $df_C$. Furthermore, we choose Hermitian metrics on $TL \otimes \mathbb{C}$ and $\mathbb{C}^2$ so that $df_C$ is unitary. Thus $f$ gives rise to an element of $U(TL \otimes \mathbb{C}, \mathbb{C}^2)$. The group of continuous maps $C(L, U(2))$ acts freely and transitively on $U(TL \otimes \mathbb{C}, \mathbb{C}^2)$ and hence $\pi_0(U(TL \otimes \mathbb{C}, \mathbb{C}^2))$ is in one to one correspondence with $[L, U(2)]$. From this point forward, we will consider $r(L)$ as an element $[L, U(2)]$.

In general, if $L$ is a genus $g$ Legendrian surface in $\mathbb{R}^5$, then the rotation class is an element of $[\Sigma_g, U(2)]$. When $g = 0$, $[S^2, U(2)] \cong \pi_2(U(2))$, and hence, the rotation class is always trivial, and uninteresting (for spheres, neither classical invariant yields any useful information). However, when $g \geq 1$, the rotation class can be nontrivial. In fact,

**Theorem 2.1.** The rotation class for an embedded Legendrian torus $L \subset \mathbb{R}^5$ can be thought of as an element in $\mathbb{Z} \times \mathbb{Z}$ via the isomorphism $[L, U(2)] \cong \pi_1(U(2)) \times \pi_1(U(2))$.

**Proof.** Given a map of the standard torus, $i: T^2 \to \mathbb{R}^5$, let $a = i(1 \times S^1)$ and $b = i(S^1 \times 1)$. For $\pi_1(U(2))$, choose basepoint $1 \in U(2)$. Define $H: [L, U(2)] \to \pi_1(U(2)) \times \pi_1(U(2))$ to be the map $f \mapsto (f|_a, f|_b)$. $H$ is surjective since $H(fg)(p,q) = (fg|_a(p), fg|_b(q)) = (f(p), g(q))$ for any pair $f, g \in \pi_1(U(2))$. The $\ker(H)$ is the the set of homotopy classes of maps $f : L \to U(2)$ such that the $f|_{a \cup b}$ is nullhomotopic. Since $U(2)$ is aspherical, any map such that $f|_{a \cup b}$ is nullhomotopic must itself be nullhomotopic. Hence, the kernel is trivial and $H$ is an isomorphism. \[ \square \]
The existence of the isomorphism in Theorem 2.1 depends upon the choice of loops on the torus used to define the map. In particular, a generic embedding $i : T^2 \to \mathbb{R}^5$ does not have a preferred basis for homology (one can precompose with any element of $SL(2, \mathbb{Z})$ for example). However, Lagrangian hypercube diagrams do provide preferred, albeit not canonical, choices for these loops—specifically, $\tilde{\gamma}_{zx}$ and $\tilde{\gamma}_{wy}$ in Theorem 6.1. It is these choices together with Theorem 2.1 that allows us to associate ordered pairs of integers to the rotation class and the Maslov index in the theorems above. The calculations are important to our future work in computing contact homology of knotted Legendrian tori algorithmically. While all of our calculations in computing the contact homology from a Lagrangian hypercube diagram will depend upon these choices, the contact homology will not.

Before moving on to the definition of a Lagrangian hypercube diagram, we begin with a discussion of Lagrangian grid diagrams.

3. Lagrangian Grid Diagrams

Let $\mathbb{R}^3$ be given $wyt$-coordinates. The contact 1-form $\bar{\pi} = dt - ydw$ represents the standard contact structure on $\mathbb{R}^3$. The contact planes are given by:

$$\xi = \ker(\bar{\pi}) = \{\partial_y, \partial_w + y\partial_t\}.$$ 

A Legendrian knot in $(\mathbb{R}^3, \xi)$ is an embedding $L : S^1 \to \mathbb{R}^3$ whose tangent vectors always lie in the contact planes determined by $\xi$. Let $\theta \mapsto (w(\theta), y(\theta), t(\theta))$ be a parametrization of $L$. There are two standard projections used to study Legendrian knots, the front projection:

$$\Pi_L := \Pi \circ L : S^1 \to \mathbb{R}^2 \text{ such that } \theta \mapsto (w(\theta), t(\theta)),$$

and the Lagrangian projection:

$$\pi_L := \pi \circ L : S^1 \to \mathbb{R}^2 \text{ such that } \theta \mapsto (w(\theta), y(\theta)).$$

There is a natural way to convert a grid diagram into a front projection of a Legendrian knot, and they have frequently been used in this context (cf. [22], [21], [18]). While not all grid diagrams may be viewed as the Lagrangian projection of a Legendrian knot, under certain conditions, they may. In what follows, we discuss conditions under which a grid diagram can be viewed as the Lagrangian projection of a Legendrian knot.

In general, a given knot diagram will not represent the Lagrangian projection of a Legendrian knot. However, an immersion, $\gamma : S^1 \to \mathbb{R}^2$ such that $\theta \mapsto (w(\theta), y(\theta))$, will correspond to the Lagrangian projection of a Legendrian knot in $(\mathbb{R}^3, \xi)$ if the following conditions hold:

\begin{align}
(3.1) \quad & \int_0^{2\pi} y(\theta)w'(\theta)d\theta = 0, \\
(3.2) \quad & \int_{\theta_0}^{\theta_1} y(\theta)w'(\theta)d\theta \neq 0 \text{ whenever } \gamma(\theta_0) = \gamma(\theta_1) \text{ and } 0 < \theta_1 - \theta_0 < 2\pi.
\end{align}

Note that in the integrals above (and in the integrals below) we have parametrized $S^1$ by $[0, 2\pi]$ such that $\gamma(0) = \gamma(2\pi)$, and that $[\theta_0, \theta_1] \subset [0, 2\pi]$.

We now translate Conditions 3.1 and 3.2 in the context of grid diagrams. Let $\hat{G}$ be a $wy$-oriented grid diagram. For a complete definition of oriented grid diagrams, see Section 2.1 of [2] (further details may be found in [8], [15], [16], [21], [23]). We recall some of the details here. A grid diagram
consists of an $n \times n$ grid together with a set of markings that, when connected by edges, represent a knot diagram. Typically one assigns the vertical segments at any crossing to be the over-crossing, and in the case of a $wy$-oriented grid diagram, the $y$-parallel segments in $\hat{G}$ would be considered vertical. However, in the following definition we will ignore such crossing conditions, and think of $\hat{G}$ as an immersed $S^1$.

**Definition 3.1.** An immersed grid diagram is an oriented grid diagram $G$ with no crossing data specified.

An immersed grid diagram $G$ may be thought of as a mapping $\gamma : S^1 \to \mathbb{R}^2$ such that $\theta \mapsto (w(\theta), y(\theta))$. Since $w'(\theta)$ is 0 along any segment in $G$ parallel to the $y$-axis, and $y(\theta)$ is constant along any segment parallel to the $w$-axis, Condition 3.1 translates into

$$\int_{0}^{2\pi} y(\theta)w'(\theta)d\theta = \sum_{i=1}^{n} \sigma(a_i) \cdot y_i \cdot \text{length}(a_i) = 0,$$

where $\{a_i\}$ is the collection of segments of $G$ parallel to the $w$-axis, $y_i$ is the $y$-coordinate of $a_i$, and $\sigma(a_i)$ is +1 if $a_i$ is oriented left to right and −1 otherwise. Given a crossing in $G$ with $0 < \theta_1 - \theta_0 < 2\pi$ and $\gamma(\theta_0) = \gamma(\theta_1)$, Condition 3.2 becomes:

$$\int_{\theta_0}^{\theta_1} y(\theta)w'(\theta)d\theta = \sum_{i=1}^{m} \sigma(a'_i) \cdot y_i \cdot \text{length}(a'_i) \neq 0,$$

where $\{a'_i\}$ is the set of $w$-parallel segments in the loop given by $\gamma|_{[\theta_0, \theta_1]}$. Condition 3.1 guarantees that choosing the other loop given by $\gamma([\theta_1, 2\pi] \cup [0, \theta_0])$ will give the negative of the integral above. Therefore any immersed grid diagram $G$ satisfying Conditions 3.1 and 3.2 lifts to a piecewise linear Legendrian knot in $(\mathbb{R}^3, \xi)$ as follows: choose some $t_0 \in \mathbb{R}$ and define the $t$-coordinate at $\gamma(0)$ to be $t_0$. Then the lift $\theta \mapsto (w(\theta), y(\theta), t(\theta))$ is determined by

$$t(\theta) = t_0 + \int_{0}^{\theta} y(u)w'(u)du.$$  

(3.3)

Condition 3.1 guarantees that in defining the $t$-coordinate this way, the lift will be a closed loop. Condition 3.2 guarantees that the vertical and horizontal segments at a crossing will have different $t$-coordinates.

**Definition 3.2.** A Lagrangian grid diagram is an immersed grid diagram $G$ satisfying Conditions 3.1 and 3.2.

**Example 3.3.** In Figure 2, observe that $\int_{G} ydw = \frac{1}{2} + \frac{5}{2} - 2(\frac{3}{2}) = 0$, and there is a path, $\gamma$, connecting the crossing to itself such that $\int_{\gamma} ydw = \frac{5}{2} - \frac{3}{2} = 1$. Hence, the unknot shown in Figure 2 is a Lagrangian grid diagram. Set the $t$-coordinate of the $w$-mark in column 1 to 0 and define the lift as in Equation 3.3. The front projection corresponding to the lift of $G$ is shown in Figure 2.

The Legendrian knots produced using the above method will be piecewise linear, not smooth. However, we can produce smoothly embedded knots as follows. Choose $0 < \epsilon << 1$. Delete an $\epsilon$ neighborhood of each vertex of $G$ and replace it with a smooth curve (cf. Figure 3). Such a smoothing may be accomplished so as to guarantee that the diagram is smooth at the boundary of the $\epsilon$ neighborhood as well. For example, the image of the map

$$E(t) = (w - \epsilon sin(t), y - \epsilon cos(t)),$$
where $t \in [0, \pi/2]$ allows one to replace the corner shown in Figure 3 with a smooth arc, but the resulting rounded corner will only be $C^1$ at the boundary of the $\epsilon$ neighborhood. Note that the smoothing may be done so that the resulting curve is symmetric about the line of slope $\pm 1$ through the vertex of the bend. Furthermore, given a choice of a smoothing at a corner such that the area enclosed by the smooth curve and the original bend is $A$, one may obtain a different smoothing so that the area enclosed is $rA$ where $r \in \mathbb{R}$ such that $0 < r \leq 1$.

Proposition 3.4. Let $\gamma : S^1 \to \mathbb{R}^2$ be the piecewise linear immersion determined by the Lagrangian grid diagram, $G$. There exists a $\delta > 0$ such that for any $0 < \epsilon \leq \delta$ there is a choice of smoothing curves based upon $\epsilon$ such that the immersion determined by the smoothed grid, $\gamma_\epsilon : S^1 \to \mathbb{R}^2$ satisfies the following:

- the lift of $\gamma_\epsilon$ is $C^0$-close to the lift of $\gamma$, and
- for any two $\epsilon, \epsilon' < \delta$ the Legendrian knots $L, L'$ are Legendrian isotopic.

Proof. Set $\delta = \frac{1}{8n}$, where $n$ is the size of $G$, and let $\epsilon \leq \delta$. Recall that a grid diagram has exactly two corners in each row (cf. Section 2.1 of [2]). Let $B$ be the set of corners of $G$. Enumerate the corners, $b_{i,j} \in B$, so that corner $b_{i,1}$ is the corner on the left hand side of row $i$ and $b_{i,2}$ is the corner on the right hand side of row $i$. Let $A_{i,j}$ be the absolute value of the area of the region enclosed by the smoothed arc and the original corner, $b_{i,j} \in B$. Construct each smoothing so that $|A_{i,j}| \leq \epsilon$. Denote by $r_i$ the horizontal edge in row $i$. Then we have the following:
\[ \int_G y dw = \sum_{i=1}^n \sigma(r_i) \cdot (i \cdot \text{length}(r_i) + \tau_1(i)A_{i,1} + \tau_2(i)A_{i,2}) = \sum_{i=1}^n \sigma(r_i) \cdot (\tau_1(i)A_{i,1} + \tau_2(i)A_{i,2}) \]

where \( \sigma(r_i) \) is +1 if the edge is directed left to right and −1 otherwise, \( \tau_j(i) \) is +1 if the smoothing lies above the horizontal edge, and −1 otherwise.

Since not all of \( \sigma(r_i) \cdot \tau_1(i) \) will evaluate to +1 (respectively, all −1), we may choose the smoothings so that

\[ \sum_{i=1}^n \sigma(r_i) \cdot (\tau_1(i)A_{i,1} + \tau_2(i)A_{i,2}) = 0. \]

Since the value of the integral in Condition 3.2 is an integer for the piecewise linear calculation and the smoothing changes the calculation by an amount less than \( \frac{1}{4} \), the smoothed diagram has the same crossing data as the original Lagrangian grid diagram. The second condition of the Lemma is clear. \( \square \)

**Corollary 3.5.** Let \( \gamma_\epsilon \) be parametrized by \( \theta \mapsto (w_\epsilon(\theta), y_\epsilon(\theta)) \). For any \( 0 \leq \theta_0 < \theta_1 \leq 2\pi \),

\[ \left| \int_{\theta_0}^{\theta_1} y(\theta) w'(\theta) d\theta - \int_{\theta_0}^{\theta_1} y_\epsilon(\theta) w'_\epsilon(\theta) d\theta \right| < \frac{1}{4}. \]

Proposition 3.4 and Corollary 3.5 show that a Lagrangian grid diagram corresponds to a smoothly embedded Legendrian knot that does not depend on the choice of epsilon used in the smoothing. Hence we may refer to the Legendrian knot corresponding to a Lagrangian grid diagram.

Given a Lagrangian projection of a Legendrian knot \( L \), one may compute the rotation number as follows. Use the vector field \( \frac{\partial}{\partial y} \) to trivialize \( \xi|_L \). The rotation number of \( L \), the 3-dimensional version of rotation class, is the winding number of the tangent vector to \( L \) with respect to this trivialization:

\[ r(L) = w(\pi_L). \]

For a Lagrangian grid \( G \) that corresponds to \( L \), this can be computed by a signed count of the corners of \( G \). Let \( B \) be the collection of corners in \( G \). As done in [23] and [22], corners may be named by the marking and the type, as shown in Figure 4). For example, a \( W : NE \) corner consists of a northeast pointing corner with a \( W \) marking. For a corner \( b_{i,j} \in B \), let \( \eta(b_{i,j}) \) be a function that assigns a value of +1 to any corner of type \( W : NE, Y : NW, W : SW, \) and \( Y : SE \) (i.e. a counterclockwise oriented corner), and a value of −1 to any corner of type \( W : NW, Y : NE, W : SE, \) and \( Y : SW \) (i.e. a clockwise oriented corner). We observe that:

**Lemma 3.6.** Given a Lagrangian grid diagram \( G \) with Legendrian lift \( L \), the rotation number of \( L \) satisfies:

\[ r(L) = \frac{1}{4} \sum_{b_{i,j} \in B} \eta(b_{i,j}). \]

It can easily be shown that the sum on the right hand side of Equation 3.4 can be expressed as the sum of terms of the form \( \eta(b_{i,j})A_{i,j} \). Hence, if the sum of the \( \eta(b_{i,j})'s \) is 0, then the \( A_{i,j} \) terms may all be taken to have the same value, as in Example 3.7 below. If not, the \( A_{i,j} \)'s must be carefully chosen as in Example 3.8.
Example 3.7. The rotation number of the unknot in Figure 2 is easily computed from its projection since $G$ has 3 corners that are assigned a value of $+1$ and 3 that are assigned a value of $-1$. Hence, we may choose to smooth all corners in the same way to obtain the Lagrangian projection of a smoothly embedded Legendrian knot in $(\mathbb{R}^3, \xi)$ with rotation number 0.

Example 3.8. Starting at the corner given by the lower left $W$ marking and following the orientation of the knot, the values of $\eta(b_{i,j})$ are as follows: $+1$, $+1$, $-1$, $+1$, $-1$, $+1$, $+1$, $+1$. We can use these values to smooth the corners. For each $+1$ corner choose a smoothing with area $A$, for some $A$ sufficiently small (as in the proof of Proposition 3.4). For each $-1$ corner, choose a smoothing of area $\frac{7}{3}A$. The sum of the signed areas will be 0, and the smoothing can be lifted to a smoothly embedded Legendrian unknot in $(\mathbb{R}^3, \xi)$. Using the values of $\eta$ above, the rotation number is 1.

The Legendrian lift of the smoothed Lagrangian grid diagram is unique up to Legendrian isotopy (Proposition 3.4). By Corollary 3.5, we can do integer calculations directly from the Lagrangian grid diagram instead of the smoothing.

The following theorem, Theorem 3.9, allows us to translate the Lagrangian grid conditions (3.1 and 3.2) into simple area calculations. Before stating the theorem, observe that a grid diagram of size $n$ contains $n - 1$ bands, $[i - \frac{1}{2}, i + \frac{1}{2}] \times [0, n]$, for $i = 1, ..., n - 1$. The union of the bands contains all of the horizontal edges of the grid diagram. Furthermore, in each band there is an even number of edges that intersect the interior of the band, half of which are oriented left to right, and half of which are oriented right to left. Hence, one may pair edges with opposite orientations. Each pair of edges, together with the vertical edges joining them, forms an oriented rectangle of width 1. The orientation of the rectangle is induced by the orientation of the horizontal edges.

Theorem 3.9. Given an immersed grid diagram with parametrization $\gamma : S^1 \to \mathbb{R}^2$ such that $\theta \mapsto (w(\theta), y(\theta))$. For each band of the grid diagram, pair each horizontal edge with one of opposite
orientation to obtain a set of oriented rectangles for the band. Let $\mathcal{R}$ be the set of all such rectangles for the entire grid. Then

$$\int_{0}^{2\pi} y(\theta)w'(\theta)d\theta = \sum_{R \in \mathcal{R}} \sigma(R) \text{Area}(R),$$

where $\sigma(R)$ is 1 if $R$ is oriented clockwise, and $-1$ if $R$ is oriented counterclockwise.

**Proof.** Each $R \in \mathcal{R}$ corresponds to two horizontal edges of the grid diagram, $e$ and $f$, with $y$-coordinates $y_e$ and $y_f$, respectively, such that $y_e > y_f$. Since the length of $e$ and $f$ is 1, the contribution of this pair of edges to the integral in Equation 3.3 (with $\theta = 2\pi$) is given by

$$\sigma(R)(y_e \cdot \text{length}(e) - y_f \cdot \text{length}(f)) = \sigma(R)(y_e - y_f) \cdot 1 = \sigma(R) \text{Area}(R).$$

Taking the sum, we obtain the result. $\square$

The previous theorem immediately provides an easy way to recognize when a given grid diagram will lift (using Equation 3.3) to a closed loop:

**Corollary 3.10.** Given an immersed grid diagram with parametrization $\gamma : S^1 \to \mathbb{R}^2$ such that $\theta \mapsto (w(\theta), y(\theta))$. The loop $\gamma$ lifts to a closed, immersed Legendrian loop in $(\mathbb{R}^3, \xi)$ if and only if

$$\sum_{R \in \mathcal{R}} \sigma(R) \text{Area}(R) = 0.$$

**Example 3.11.** For the grid diagram in Figure 6, we see by computing the signed areas shown that the integral in Equation 3.3 evaluate to $-7$. Hence, it is not a Lagrangian grid diagram.

![Figure 6: Decomposition of grid into rectangles.](image)

In practice, the area calculation described in the previous example may be carried out by choosing convenient polygonal regions. For example, Figure 6 has 3 easily identifiable polygonal regions with area 2, $-1$, and $-8$. The signed area of these polygonal regions will correspond to the integrals defined in Conditions 3.1 and 3.2. For convenience, in the proofs that follow, we will use this signed area calculation to quickly compute the integrals defined in Conditions 3.1 and 3.2.

**Scholium 3.12.** Given an immersed grid diagram with parametrization $\gamma : S^1 \to \mathbb{R}^2$ such that $\theta \mapsto (w(\theta), y(\theta))$, and a loop $\gamma|_{[\theta_0, \theta_1]}$ that goes from a crossing in $G$ to itself such that $0 < \theta_1 - \theta_0 < 2\pi$ and $\gamma(\theta_0) = \gamma(\theta_1)$. For each band of the grid diagram such that the interior of the band intersects $\gamma|_{[\theta_0, \theta_1]}$, pair each horizontal edge of the intersection with one of opposite orientation to obtain a set of rectangles for the band. Let $\hat{\mathcal{R}}$ be the set of all such rectangles for the entire grid. Then

$$\int_{\theta_0}^{\theta_1} y(\theta)w'(\theta)d\theta = \sum_{R \in \hat{\mathcal{R}}} \sigma(R) \text{Area}(R),$$
where $\sigma(R)$ is 1 if $R$ is oriented clockwise, and $-1$ if $R$ is oriented counterclockwise.

**Proof.** The loop $\gamma|_{[\theta_0, \theta_1]}$ has enough of the structure of a grid diagram to allow the proof of Theorem 3.9 to proceed as before. \qed

The integral in Scholium 3.12 computes the difference in the $t$-coordinates at any crossing and hence will be used in many calculations that follow. Let $\mathcal{C}$ be the set of crossings for a grid diagram. For any crossing $c \in \mathcal{C}$ and loop $\gamma|_{[\theta_0, \theta_1]}$ from $c$ to itself, we define $\Delta t(c)$ to be this difference:

$$\Delta t(c) := \sum_{R \in \hat{R}} \sigma(R) \text{Area}(R).$$

The absolute value of $\Delta t(c)$ is the length of the Reeb chord associated to the crossing, $c$, and in some contexts is called the *action*.

Given an immersed grid diagram that lifts to a closed loop (that is, it satisfies Condition 3.1), Scholium 3.12 provides an easy way to determine whether or not the lift is embedded. In particular, we obtain:

**Corollary 3.13.** Let $G$ be an immersed grid diagram satisfying Condition 3.1 with parametrization $\gamma : S^1 \to \mathbb{R}^2$ such that $\theta \mapsto (w(\theta), y(\theta))$, and let $\mathcal{C}$ be the set of crossings in $G$. Then $\Delta t(c) \neq 0$ for all crossings $c \in \mathcal{C}$ if and only if $\gamma$ lifts to an embedded Legendrian loop in $(\mathbb{R}^3, \xi)$.

![Figure 7: An unknot that lifts to a closed, but not embedded loop.](image)

**Example 3.14.** The diagram shown in Figure 7 lifts to a closed loop in $\mathbb{R}^3$, but the center crossing does not separate in the lift. Notice that the dark shading indicates rectangles with a signed area of $-1$ and the light shading indicates rectangles with a signed area of 1. Using Corollary 3.10 we see that the diagram lifts to a closed loop. However, a path beginning and ending at the center crossing will have a total change in the $t$-coordinate of 0. Hence, by Corollary 3.13, the grid does not satisfy Condition 3.2.

**Example 3.15.** The diagram shown in Figure 2 has already been shown to lift to an embedded loop. One can see an obvious set of 2 rectangles as described in the proof of Theorem 3.9, one with signed area 1, and one with signed area $-1$. Using Corollaries 3.10 and 3.13, we can quickly verify the fact that the diagram lifts to an embedded loop.

Before proceeding with the proof of Theorem 5, we introduce some definitions and lemmas that we will use only for the proofs in this paper.

**Definition 3.16.** An *almost Lagrangian grid diagram* is an immersed grid diagram such that:

- the top right corner has a marking,
- there is a parametrization $\gamma : S^1 \to \mathbb{R}^2$ in which $\gamma(0)$ and $\gamma(2\pi)$ are that marked point.
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\[ \int_{\theta_0}^{\theta_1} y(\theta)w'(\theta)d\theta \neq 0 \text{ whenever } 0 < \theta_1 - \theta_0 < 2\pi \text{ and } \gamma(\theta_0) = \gamma(\theta_1). \]

As before, choose some \( t_0 \in \mathbb{R} \). The lift, \( \theta \mapsto (w(\theta), y(\theta), t(\theta)) \), is determined by setting the \( t \)-coordinate at \( \gamma(0) \) to be \( t_0 \) and,

\[ t(\theta) = t_0 + \int_{0}^{\theta} y(u)w'(u)du. \]

Thus the last condition of Definition 3.16 guarantees that an almost Lagrangian grid diagram gives rise to an embedded Legendrian arc. Since the endpoints of this arc project to the top right corner marking and differ only in their \( t \)-coordinates, an almost Lagrangian grid diagram still gives rise to a knot in \( \mathbb{R}^3 \) by attaching the endpoints by a segment parallel to the \( t \)-axis.

**Lemma 3.17.** An almost Lagrangian grid diagram can always be modified (using configurations listed in Table 1) to get a Lagrangian grid diagram whose lift has the same topological knot type and winding number as the knot given by the almost Lagrangian grid diagram.

**Proof.** An almost Lagrangian grid diagram lifts to a Legendrian arc whose endpoints have \( t \)-coordinates that differ by some \( k \in \mathbb{Z} \). Attach one of the configurations shown in Table 1. Each time such a configuration is attached, the resulting grid will again be an almost Lagrangian grid diagram, but the distance between the end points of the new Legendrian arc will be reduced by 1 or 2. Continue reducing this distance until the arc closes up to give a Lagrangian grid diagram. Note that adding these configurations do not change the knot type or the winding number. \( \square \)

| Table 1. Configurations used to convert an almost Lagrangian grid diagram into a Lagrangian grid diagram. The change in the distance between the endpoints follows from Theorem 3.9. |
|-------------------------------------------------------------|
|               -2               | -1               | 1               | 2               |
| ![Diagram](image) |

**Lemma 3.18.** Let \( k \in \mathbb{Z} \). Any almost Lagrangian grid diagram can be modified to obtain a Lagrangian grid diagram with rotation number \( k \).

**Proof.** Let \( k \in \mathbb{Z} \). We can easily modify the rotation number of an almost Lagrangian grid diagram by attaching one of the configurations shown in Figure 8. We then apply Lemma 3.17 to obtain a Lagrangian grid diagram whose lift has the same topological knot type as the original almost Lagrangian grid diagram. \( \square \)

We now proceed with the proof of Theorem 5, stated in the introduction.
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Figure 8: Configuration to change the winding number of an immersed grid diagram.

Proof of Theorem 5. We use Lenhard Ng’s arguments, found in Section 2.1 of [20], as a guide to construct Lagrangian grid diagrams. Recall from [22] that a grid diagram (in the usual sense) may be thought of as a front projection of a Legendrian knot. Given such a front projection, we may resolve the front to obtain the Lagrangian projection of a knot isotopic to the one determined by the front. This Lagrangian projection will have the same crossing data as the original grid, and, as a diagram, is isotopic to the original grid after adding loops at each southeast corner.

We follow a similar procedure, but modify it so that we obtain a Lagrangian grid diagram. Given a grid diagram (in the usual sense), stabilize at each southeast corner as shown in Figure 9. The resulting horizontal edge of length 1 may be moved, using commutation moves (cf. Section 3.1 of [3]), to the bottom of the grid to obtain a simple front (cf. Definition 2.7 of [20]).

By applying another stabilization in the right-most column, and then commutation moves, we may ensure that this grid has a marking in the top right corner. Then add a loop at each southeast corner, as is done in constructing the front resolution (cf. Definition 2.1 of [20]). By possibly inserting some number of empty rows and columns, we may adjust the enclosed areas so that we obtain a diagram whose lift represents the same knot in $\mathbb{R}^3$ as the grid diagram we started with. This diagram will, in general, not be a grid diagram, since it contains empty rows and columns. At the top right corner, attach a configuration as shown in Figure 10 to fill in any empty rows and columns, and thus obtain an almost Lagrangian grid diagram. Then, by applying Lemma 3.18, we may obtain a Lagrangian grid diagram representing the same topological knot type as the original grid diagram, and having any rotation number $k$. $\square$

4. LAGRANGIAN HYPERCUBE DIAGRAMS IN DIMENSION 4

In this section we define Lagrangian hypercube diagrams—the structure we use to construct Lagrangian tori in $\mathbb{R}^4$ (Section 5) that lift to Legendrian tori in $\mathbb{R}^5$ (Section 6).

Lagrangian hypercube diagrams are closely related to grid, cube, and hypercube diagrams. To construct a grid, cube, or hypercube diagram, one places markings in a 2, 3, or 4 dimensional
Cartesian grid, while ensuring that certain marking conditions and crossing conditions hold (cf. Section 2 and 3 in [2], and Section 2 in [3]). In each case, the markings determine a link (cf. Figure 11). For a hypercube diagram, there is an algorithm for constructing a Lagrangian torus associated to the hypercube diagram, such as the one shown in the last picture in Figure 11 (cf. Theorem 5.1 in [2]).

The tori constructed from hypercube diagrams in [2] are Lagrangian, but they do not generally lift to Legendrian tori in \( \mathbb{R}^5 \), since Conditions 3.1 and 3.2 are not required to be satisfied. The definition of Lagrangian hypercube diagram presented here is a modification of the construction in [2], specifically designed so that the Lagrangian tori that we build in Section 5 will lift to Legendrian tori in \( \mathbb{R}^5 \) (which we do in Section 6). Before stating the definition of a Lagrangian hypercube diagram, we present some preliminary definitions.

Let \( n \) be a positive integer and let \( C = [0, n] \times [0, n] \times [0, n] \times [0, n] \subset \mathbb{R}^4 \) be thought of as a 4-dimensional Cartesian grid, i.e., a grid with integer valued vertices with axes \( w, x, y, \) and \( z \). Orient \( \mathbb{R}^4 \) with the orientation \( w \wedge x \wedge y \wedge z \).

Recall that a grid diagram’s marking conditions require reference to rows and columns (cf. Section 2.1 of [2])). We next define what will be the 4-dimensional analogs of the rows and columns used to state the marking conditions of a Lagrangian hypercube diagram.

A **flat** is any right rectangular 4-dimensional polytope with integer valued vertices in \( C \) such that there are two orthogonal edges at a vertex of length \( n \) and the remaining two orthogonal edges are of length 1. (Each flat is congruent to the product of a unit square and an \( n \times n \) square.) Moreover, the flat will be named by the two edges of length \( n \). Although a flat is a 4-dimensional object, the name references the fact that a flat is a 2-dimensional array of unit hypercubes. For example,
an \(xy\)-flat is a flat that has a face that is an \(n \times n\) square that is parallel to the \(xy\)-plane. In a hypercube of size \(n = 3\), one example of a \(xy\)-flat would be the subset \([0, 1] \times [0, 3] \times [0, 3] \times [2, 3]\) (shown in Figure 12).

A stack is a set of \(n\) flats that form a right rectangular 4-dimensional polytope with integer vertices in \(C\) in which there are three orthogonal edges of length \(n\) at a vertex, and the remaining edge has length 1. (Each stack is the product of a cube with edges of length \(n\) and a unit interval.) A stack is named by the three edges of length \(n\). An example of a \(wxz\)-stack in a hypercube of size 3 is the subset \([0, 3] \times [0, 3] \times [2, 3] \times [0, 3]\) (shown at the top of Figure 12). Further examples of flats and stacks may be found in Figure 12.

A marking is a labeled point in \(\mathbb{R}^4\) with half-integer coordinates in \(C\). Unit hypercubes of the 4-dimensional Cartesian grid will either be blank, or marked with a \(W\), \(X\), \(Y\), or \(Z\) such that the following marking conditions hold:

- each stack has exactly one \(W\), one \(X\), one \(Y\), and one \(Z\) marking;
- each stack has exactly two flats containing exactly 3 markings in each;
- for each flat containing exactly 3 markings, the markings in that flat form a right angle such that each ray is parallel to a coordinate axis;
- for each flat containing exactly 3 markings, the marking that is the vertex of the right angle is \(W\) if and only if the flat is a \(zw\)-flat, \(X\) if and only if the flat is a \(wx\)-flat, \(Y\) if and only if the flat is a \(xy\)-flat, and \(Z\) if and only if the flat is a \(yz\)-flat.

Figure 12: A schematic for displaying a Lagrangian hypercube diagram. The outer \(w\) and \(y\) coordinates indicate the “level” of each \(zx\)-flat. The inner \(z\) and \(x\) coordinates start at \((0, 0)\) for each of the nine \(zx\)-flats. With these conventions understood, it is then easy to display \(xy\)-flats, \(xyz\)-stacks, \(wxz\)-stacks, \(wxy\)-stacks, etc. The second picture is a schematic of a Lagrangian hypercube diagram.
The 4th condition rules out the possibility of either \(wy\)-flats or a \(zx\)-flats with three markings (see Figure 12). As with oriented grid diagrams and cube diagrams, we obtain an oriented link from the markings by connecting each \(W\) marking to an \(X\) marking by a segment parallel to the \(w\)-axis, each \(X\) marking to a \(Y\) marking by a segment parallel to the \(x\)-axis, and so on.

Drawing a hypercube diagram as in Figure 11 becomes difficult for all but the most trivial examples. A hypercube schematic (cf. Figure 12, and Figure 13 below) overcomes this difficulty by conveniently displaying the markings of a Lagrangian hypercube diagram.

Let \(\pi_{xz}, \pi_{wy} : \mathbb{R}^4 \to \mathbb{R}^2\) be the natural projections, projecting out the \(x, z\) and \(w, y\) directions respectively. The projection \(\pi_{xz}(C)\) produces an \(n \times n\) square in the \(wy\)-plane. If we project the \(W\) and \(Y\) markings of the hypercube to this square as well, the markings satisfy the conditions for an immersed grid diagram, which we denote \(G_{wy} := (\pi_{xz}(C), \pi_{xz}(W), \pi_{xz}(y))\), where \(W\) and \(y\) are the sets of \(W\) and \(Y\) markings respectively. Similarly, we define \(G_{zx} := (\pi_{wy}(C), \pi_{wy}(Z), \pi_{wy}(X))\), where \(Z\) and \(X\) are the sets of \(Z\) and \(X\) markings respectively.

In a grid diagram, one typically requires a crossing condition, namely that the vertical segment crosses over the horizontal segment. For a Lagrangian hypercube diagram, the crossing conditions are determined as follows. We require that the two immersed grid diagrams, \(G_{zx}\) and \(G_{wy}\), are Lagrangian grid diagrams (that is, they satisfy Conditions 3.1 and 3.2). By Proposition 3.4, a Lagrangian grid diagram lifts to a smoothly embedded Legendrian knot. Hence the crossing conditions of the grid are determined by this lift. We require one additional product lift condition that the pair \(G_{zx}\) and \(G_{wy}\) must satisfy (recall that \(\Delta t(c)\) in the definition is the length of the Reeb chord associated to \(c\)).

**Definition 4.1.** For two Lagrangian grid diagrams, \(G_{wy}\) and \(G_{zx}\), let \(\mathcal{C} = \{c_i\}\) be the crossings in \(G_{zx}\) and \(\mathcal{C}' = \{c'_j\}\) be the crossings in \(G_{wy}\). The pair of grid diagrams is said to satisfy the product lift condition if \(|\Delta t(c_i)| \neq |\Delta t(c'_j)|\) for all \(i, j\).

The two Lagrangian grid diagrams, \(G_{wy}\) and \(G_{zx}\), lift to embedded Legendrian knots in \(\mathbb{R}^3\). In Sections 5 and 6 we show how to construct a Lagrangian torus in \(\mathbb{R}^4\) that is essentially a product of the grid diagrams (cf. Figure 13) that lifts to a “product” of the two knots in \(\mathbb{R}^5\). Peter Lambert-Cole uses this idea to define Legendrian products in greater generality in [14]. While any two Lagrangian grid diagrams give rise to Legendrian knots in \(\mathbb{R}^3\) whose product is an immersed Legendrian torus in \(\mathbb{R}^5\), our goal is to construct embedded Legendrian tori. Definition 4.1 provides the necessary restriction on the pair, \(G_{wy}\) and \(G_{zx}\), that guarantees the torus will be embedded.

**Definition 4.2.** A Lagrangian hypercube diagram, denoted \(HT = (C, \{W, X, Y, Z\}, G_{zx}, G_{wy})\), is a set of markings \(\{W, X, Y, Z\}\) in \(C\) that (1) satisfy the marking conditions, (2) \(G_{wy}\) and \(G_{zx}\) are Lagrangian grid diagrams, and (3) \(G_{wy}\) and \(G_{zx}\) satisfy the product lift condition.

**5. Building a Torus from a Lagrangian Hypercube Diagram**

The Lagrangian grid diagrams, \(G_{zx}\) and \(G_{wy}\), associated to a Lagrangian hypercube diagram may be displayed on a hypercube schematic directly. To display \(G_{wy}\) in the schematic, note that each \(zx\)-flat containing a \(W\) and \(Z\) marking will project, via \(\pi_{xz}\), to a cell of \(G_{wy}\) containing a \(W\) marking, and each \(zx\)-flat containing an \(X\) and \(Y\) marking will project to a cell of \(G_{wy}\) containing a \(Y\) marking. In Figure 13, the blue shading indicates the diagram associated to \(G_{wy}\). To see \(G_{zx}\) in the schematic, note that each pair of markings in a \(zx\)-flat on the schematic corresponds to an
edge of the Lagrangian grid diagram $G_{zx}$. Placing all of these segments on a single $n \times n$ grid will produce a copy of $G_{zx}$.

Figure 13: Lagrangian hypercube diagram with unknotted $G_{zx}$ and $G_{wy}$ and rotation class $(1,0)$.

The immersed torus specified by the Lagrangian hypercube diagram is the product of $G_{zx}$ and $G_{wy}$, determined as follows: place a copy of the immersed grid $G_{zx}$ at each $zx$-flat on the schematic that contains a pair of markings (shown in red on Figure 13). Doing so produces a schematic with two copies of $G_{zx}$ with the same $y$-coordinates and two with the same $w$-coordinates. For each pair of copies sharing the same $w$-coordinates, we may translate one parallel to the $w$-axis toward the other. Doing so traces out an immersed tube connecting these two copies of $G_{zx}$. Similarly, we may translate parallel to the $y$-axis to produce an immersed tube connecting two copies of $G_{zx}$ with the same $y$-coordinates. Since we are connecting copies of $G_{zx}$ in flats corresponding to the markings of $G_{wy}$, the tube will close to produce an immersed torus. Thus we obtain:

**Theorem 5.1.** A Lagrangian hypercube diagram determines an immersed Lagrangian torus $i : T^2 \to \mathbb{R}^4$. Furthermore, the map determines a preferred set of loops, $\gamma_{zx} = S^1 \times 1$ and $\gamma_{wy} = 1 \times S^1$, that map to curves projecting to the Lagrangian grid diagrams $G_{zx}$ and $G_{wy}$.

Since the torus is formed by the translation of $x$- and $z$-parallel segments to the $w$- and $y$-axes, we see that only $wx$-, $wz$-, $yz$-, and $xy$-rectangles are used in the construction of the torus. Since $wy$- and $zx$-rectangles are never used in the construction of the torus, it is Lagrangian with respect to the symplectic form $dw \wedge dy + dz \wedge dx$. Furthermore, just as in the case of Lagrangian grid diagrams, we obtained a smooth embedding by carefully smoothing corners. We may obtain a smooth embedding of the torus in $\mathbb{R}^5$ by first smoothing $G_{zx}$ and $G_{wy}$ as in Lemma 3.4.
The immersed torus has only two types of singularities: double point circles and intersections of double point circles. Each crossing of $G_{zx}$ generates a double point circle as shown by the yellow dots in Figure 13. Similarly each crossing of $G_{wy}$ generates a double point circle, which is visible in the schematic as the $zx$-flat where a $w$-parallel tube passes through a $y$-parallel tube. In Figure 13 this is shown by the yellow diagram. The green dot in Figure 13 corresponds to an intersection of two double point circles.

6. Lifting the Hypercube to $\mathbb{R}^5$

Let $i : T \to \mathbb{R}^4$ be the immersed torus obtained from a Lagrangian hypercube diagram as given by Theorem 5.1. Note that, $\alpha|_{wxyz}$-hyperplane $= \omega = dw \wedge dy + dz \wedge dx$ is a symplectic form on $\mathbb{R}^4$-hyperplanes in $\mathbb{R}^5$. We will show that $HT$ represents the Lagrangian projection of a Legendrian surface in $\mathbb{R}^5$ with respect to the standard contact structure $\xi$.

In order to lift $i(T)$, we begin by choosing some point $p \in i(T)$ to have $t$ coordinate equal to some $t_0 \in \mathbb{R}$. If we attempt to lift $i(T)$ to a Legendrian surface with respect to $\alpha$ we should choose to define the $t$-coordinate of $p'$ to be:

$$t = t_0 + \int \gamma ydw + \int \gamma xdz,$$

where $\gamma$ is a path from $p$ to $p'$. This integral will be independent of path precisely when the 1-form $i^*(ydw + xdz)$ is 0 on $H_1(T)$. Recall that $H_1(T)$ is generated by $\gamma_{zx}$ and $\gamma_{wy}$.

In order check for path-independence of the integral in Equation 6.1, we evaluate the following:

$$i^*(ydw + xdz)[i^*(\gamma_{zx})] = \int_{\gamma_{zx}} i^*(ydw + xdz) = \int_{\gamma_{zx}} ydw + \int_{\gamma_{zx}} xdz.$$

$$i^*(ydw + xdz)[i^*(\gamma_{wy})] = \int_{\gamma_{wy}} i^*(ydw + xdz) = \int_{\gamma_{wy}} ydw + \int_{\gamma_{wy}} xdz.$$

It is clear that $\int_{\gamma_{zx}} ydw$ and $\int_{\gamma_{wy}} xdz$ both evaluate to 0. Since $G_{zx}$ and $G_{wy}$ are Lagrangian grid diagrams, the remaining integrals both evaluate to 0 and we get a well-defined lift to a Legendrian torus in $\mathbb{R}^5$ using Equation 6.1. Furthermore, the product lift condition guarantees that the lift will be embedded. Let $L$ be the lift of $i(T)$ obtained from Equation 6.1. Define $\pi_t : \mathbb{R}^5 \to \mathbb{R}^4$ to be the projection $(w, x, y, z, t) \mapsto (w, x, y, z)$. Then $\pi_t(L) = i(T)$, i.e. the torus determined by $HT$ is the Lagrangian projection of the Legendrian torus $L$. Thus we obtain the following:

**Theorem 6.1.** The torus determined by a Lagrangian hypercube diagram $HT$ lifts to an embedded Legendrian torus $L \subset (\mathbb{R}^5, \xi)$. Furthermore, the generators $\gamma_{zx}$ and $\gamma_{wy}$ lift to curves $\tilde{\gamma}_{zx}$ and $\tilde{\gamma}_{wy}$ that generate $H_1(L)$.

**Remark 6.2.** If we omit the product lift condition from the definition of a Lagrangian hypercube diagram, then the above procedure will still produce an immersed Legendrian torus in $\mathbb{R}^5$.

**Example 6.3.** Figure 13 shows a schematic picture of a Lagrangian hypercube diagram where all grid-projections are unknots as in Example 3.7. By Theorem 6.1, the torus determined by this Lagrangian hypercube diagram lifts to a Legendrian torus in $(\mathbb{R}^5, \xi)$. 
ON THE ROTATION CLASS OF KNOTTED LEGENDRIAN TORI IN $\mathbb{R}^5$

7. Proof of Theorem 1

With the rotation class understood to be an element of $[L, U(2)]$, we see from Theorem 2.1 that the class may be identified with a pair of integers corresponding to the elements of $\pi_1(U(2))$ determined by a meridian and longitude of the torus. Before proving Theorem 1 we identify an explicit generator of $\pi_1(U(2))$. Recall that $U(2)$ parametrizes framed Lagrangians of $(\mathbb{R}^4, \omega)$, where $\omega = \text{d}\alpha|_{xwyz}$-hyperplane = $\text{d}w \wedge \text{d}y + \text{d}z \wedge \text{d}z$. This provides a natural identification of $\mathbb{R}^4 = \{w, x, y, z\}$ with $\mathbb{C}^2 = \{w + iy, z + ix\}$, and allows us to identify the $yx$-, $xy$-, $yz$-, and $zy$-planes with the following matrices:

$$U_{xy} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad U_{yx} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad U_{yz} = \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix}, \quad U_{zy} = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}.$$ 

Note that $U_{xy}, U_{yx}, U_{yz},$ and $U_{zy}$ correspond to unitary Lagrangian frames (cf. Section 2.3 of [19]):

$$U_{xy} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad U_{yx} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad U_{yz} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U_{zy} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

Note that, as maps from $\mathbb{R}^2 \to \mathbb{R}^4$, these frames produce $xy$-, $(-x)y$-, $(-z)y$-, and $zy$-planes respectively. Geometrically, this matches up with the fact that the Lagrangian planes along an $xz$-slice of the hypercube will be given by a positively or negatively oriented $\partial_x$ or $\partial_z$ vector paired with a positively oriented $\partial_y$-vector.

Choose $U_{xy}$ to be the basepoint. We define a loop $\gamma : [0, 1] \to U(2)$ that begins at $U_{xy}$ and rotates through $U_{yz}, U_{yx}, U_{zy}$.

We will define $\gamma$ in 4 pieces. First, define a map $\hat{\gamma} : [0, 1] \to U(2)$ as follows:

$$\hat{\gamma}(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi}{3}it} \end{pmatrix}.$$ 

Then, define $\gamma_1(t) = \hat{\gamma}(t)U_{xy}, \gamma_2(t) = \hat{\gamma}(t)U_{yz}, \gamma_3(t) = \hat{\gamma}(t)U_{yx},$ and $\gamma_4(t) = \hat{\gamma}(t)U_{zy}$. Finally, define $\gamma(t) = \gamma_1 \ast \gamma_2 \ast \gamma_3 \ast \gamma_4$. Thus, $\gamma$ corresponds to a rotation of Lagrangian planes, beginning at an $xy$-plane, and rotating through $yz$-, $yx$-, and $zy$-planes.

Lemma 7.1. The loop $\gamma$ represents a generator of $\pi_1(U(2))$.

Proof. Observe that the determinant, $\text{det} : U(2) \to U(1)$ induces an isomorphism on $\pi_1$ that takes $\gamma$ to a generator of $\pi_1(U(1))$. \[\square\]

As above, we can now identify $wx$-, $yw$-, and $yx$-oriented planes with the following matrices:

$$U_{wx} = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \quad U_{yw} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad U_{zw} = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}.$$ 

Note that while $U_{yz}$ corresponds to a unitary Lagrangian frame giving rise to the Lagrangian plane $\{-\partial_x, \partial_y\}$, $U_{yx}$ gives rise to the Lagrangian plane $\{\partial_x, -\partial_y\}$, and hence, both refer to the same oriented plane. The same argument as that given in the proof of Lemma 7.1 will then show that there is a generator for $\pi_1(U(2))$ given by acting on matrices $U_{xy}, U_{yz}, U_{zw},$ and $U_{wx}$ on the left by:

$$\hat{\gamma}(t) = \begin{pmatrix} e^{\frac{2\pi}{3}it} & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Much of the content of the paper to this point has been building up toward presenting the following proof. Our discussion of Lagrangian grid diagrams in Section 3 enables us to define an immersed
Lagrangian torus corresponding to a Lagrangian hypercube diagram as in Theorem 5.1. Theorem 6.1 shows how to obtain a Legendrian torus from the Lagrangian hypercube diagram. Having determined easy methods for computing the rotation number of the Lagrangian grid diagrams (Lemma 3.6), we are ready to prove Theorem 1.

**Proof of Theorem 1.** Theorem 6.1 guarantees that the lift, \( L \), exists. We must see that the image of \( r(L) \in [L, U(2)] \) under the isomorphism defined in Theorem 2.1 is \( (w(G_{zx}), w(G_{wy})) \). \( G_{zx} \) and \( G_{wy} \) each correspond to one of the two factors of \( T \). Let \( [f_{zx}] \) and \( [f_{wy}] \) be the elements of \( \pi_1(U(2)) \) determined by \( G_{zx} \) and \( G_{wy} \) (since \( G_{zx} \) and \( G_{wy} \) are constant, choice of base point is irrelevant). Then the isomorphism defined in Theorem 2.1 maps \( r(L) \) to \( ([f_{zx}], [f_{wy}]) \). We must show that \( [f_{zx}] = w(G_{zx})[\gamma] \).

Clearly, \( w(G_{zx}) \) computes how many times the tangent vector to the grid \( G_{zx} \) wraps around the loop \( \gamma \). By Lemma 7.1, \( [\gamma] \) generates \( \pi_1(U(2)) \). A similar argument shows that \( [f_{wy}] = w(G_{wy})[\gamma] \). \( \square \)

Corollaries 2 and 3 show how to calculate the Maslov index, \( \mu : H_1(L) \to \mathbb{Z} \), and the Maslov number directly from a Lagrangian hypercube diagram. Recall from Theorem 6.1 that \( \tilde{\gamma}_{zx} \) and \( \tilde{\gamma}_{wy} \) are the lifts of \( \gamma_{zx} \) and \( \gamma_{wy} \), respectively. We are now ready to prove the corollaries.

**Proof of Corollary 2.** Given an embedded loop \( \gamma : S^1 \to L \) representing a primitive class \( A \in H_1(L) \), for any \( p \in S^1 \), \( T_{\gamma(p)}L \) is a Lagrangian plane, \( P_{\gamma(p)} \). Thus we obtain a map \( S^1 \to \text{Lag}(\mathbb{C}^2) \) such that \( p \mapsto P_{\gamma(p)} \). The isomorphism defined in the proof of Theorem 1 is valid here as well, once we identify planes that differ only in orientation, which produces a factor of 2. \( \square \)

**Proof of Corollary 3.** Follows directly from the previous corollary and the fact that the Maslov number is the smallest positive number that is the Maslov index of a non-trivial loop in \( H_1(L) \) and 0 if every non-trivial loop has Maslov index 0 (cf. [9]). \( \square \)

8. **Proof of Theorem 4 and Examples**

Before proceeding with the proof of Theorem 4 we establish a few preliminary results. The construction of Theorem 8.4 can be used to produce a hypercube diagram (in the sense of [2]) given any pair of Lagrangian grid diagrams. However if the product lift condition is not satisfied by the pair of Lagrangian grid diagrams, the resulting Legendrian torus will not be embedded (cf. Remark 6.2). Theorem 8.1, 8.2, and Corollary 8.3 show that for any pair of topological knots, and any rotation numbers, one may find a pair of Lagrangian grid diagrams such that the product lift condition is satisfied and hence construct a Lagrangian hypercube diagram that lifts to an embedded Legendrian torus.

**Theorem 8.1.** Let \( G \) be a Lagrangian grid diagram with a marking in the upper-right corner. Enumerate the crossings of \( G \) by \( \{c_i\} \). For any \( M > 0 \), there is another Lagrangian grid diagram \( G' \), whose lift represents the same topological knot and has the same rotation number as the lift of \( G \), such that \( |\Delta t(c_i)| > M \) for all \( i \).

**Proof.** Scale \( G \) by \( k \in \mathbb{Z} \) (each segment of the diagram of length \( \ell \) becomes a segment of length \( k\ell \)). This produces a diagram satisfying the Lagrangian conditions (Conditions 3.1 and 3.2), but, of course, it will not be a grid diagram, due to empty rows and columns. However, the area of each rectangle (as in Corollary 3.10) will be multiplied by \( k^2 \). Therefore, \( |\Delta t(c_i)| \) may be made
arbitrarily large for all $i$. We must then show that the empty rows and columns may be filled in, while preserving the Lagrangian grid conditions.

By following the techniques of Theorem 5, we may assume that the upper-right corner of $G$ (prior to scaling) has a horizontal and vertical edge of length 1 or 2. Begin by inserting one additional row and column at the upper-right corner. The additional area created by this will be either $2k + 1$, $3k + 1$, or $4k + 1$ depending on the initial lengths of the horizontal and vertical edges of the upper-right corner. Then attach the configuration shown in Figure 14. The unshaded regions will be equal in area, but with opposite sign due to the symmetry between empty rows and columns after scaling the initial grid. The dark-grey regions will also be equal in magnitude but with opposite sign. The light-grey region at the top right may be extended so that it is of area $2k + 1$, $3k + 1$, or $4k + 1$ (an even or odd area may be achieved by placing an additional box as shown by the dotted lines at the upper-right corner of Figure 14).

Finally, observe that for all of the original crossings, $\Delta t$ has been scaled up by a factor of $k^2$. However, this procedure creates 4 additional crossings: $d_1$, $d_2$, $d_3$, and $d_4$. By choosing $k$ sufficiently large, and possibly making our initial grid diagram larger, we may ensure that $\min|\Delta t(d_i)| \geq jk + 1$ for $j = 2, 3, 4$.

![Figure 14: Configurations used to fill in empty rows and columns ($j = 2, 3, 4$).](image)

We showed in the previous theorem that the minimum value of $|\Delta t(c_i)|$ may be made arbitrarily large for a Lagrangian grid diagram, the following theorem shows that we may make Lagrangian grid diagrams arbitrarily large, while not allowing $\Delta t(c_i)$ to become large.

**Theorem 8.2.** Given a Lagrangian grid diagram $G$ of size $n$, there exists $m > n$ such that one may modify $G$ to obtain a Lagrangian grid diagram, $G'$ of size $n'$ for any $n' > m$, whose lift has the same topological type and rotation number as the lift of $G$. Moreover, if $\Delta$ is the maximum over $|\Delta t(c_i)|$ for crossings $\{c_i\}$ in $G$ and $\Delta'$ is defined similarly for $G'$, then $\Delta' \leq \Delta + |a| + 1$, where $|a|$ is the length of the right-most vertical edge in $G$. 
Proof. We may assume that $G$ has a marking in the upper-right corner. Let $k \in \mathbb{Z}$. At the top right corner of the grid, we stabilize and attach a configuration of size $2k$ as shown in Figure 15. Since we began with a Lagrangian grid diagram, if we choose the top right corner as the base-point, each new crossing will have $t$-coordinates that differ by either $a \pm 1$ or $a$, and at the new top right corner, the $t$-coordinates in the lift will differ by $a \pm 1$. We then apply Lemma 3.17 to obtain a Lagrangian grid diagram. By carefully choosing which configurations we use in applying Lemma 3.17, we may ensure that the Lagrangian grid diagram we obtain has even or odd size. The statement about the bound on $\Delta'$ is clear from the construction. \hfill \Box

Corollary 8.3. Suppose $G_1$ and $G_2$ are two Lagrangian grid diagrams of size $m$ and $n$, respectively. There exist Lagrangian grid diagrams, $G'_1$ and $G'_2$, both of the same size, whose lifts represent the same two topological knots with the same rotation numbers as the lifts of $G_1$ and $G_2$, respectively. Furthermore, if $\{c_i\}$ is the set of crossings in $G'_1$ and $\{d_j\}$ is the set of crossings in $G'_2$, then $|\Delta t(c_i)| < |\Delta t(d_j)|$ for all $i, j$.

Proof. Using the techniques in the proof of Theorem 5, we may assume that $G_1$ and $G_2$ each have markings in the upper right corner. Apply Theorem 8.1 to $G_1$, choosing $M$ sufficiently large to guarantee that $M > \max\{\Delta t(d_j)\} + |a| + 1$ where $a$ is the signed area as shown in Figure 15. This guarantees that $\min\{\Delta t(c_i)\} > \max\{\Delta t(d_j)\} + |a| + 1$. Finally, apply Theorem 8.2 to $G_2$ so that both grids are the same size. \hfill \Box

Theorem 8.4. Let $G_1$ and $G_2$ be two Lagrangian grid diagrams of the same size. If $G_1$ and $G_2$ satisfy the product lift condition, then there exists a Lagrangian hypercube diagram, $H\Gamma = (C, \{W, X, Y, Z\}, G_{zx}, G_{wy})$, such that $G_{zx} = G_1$ and $G_{wy} = G_2$.

Proof. For convenience, relabel the markings of $G_1$ by $Z$ and $X$, following the orientation of the grid: $Z_0, X_0, Z_1, X_1, \ldots$ etc. Do the same for $G_2$, but relabel using $W$ and $Y$ markings. Denote the coordinates of $W_i$ by $(w_{w,i}, y_{w,i})$, $Y_i$ by $(w_{y,i}, y_{y,i})$, etc. Place $\tilde{Z}_i$ in the hypercube at position $(w_{w,i}, x_{z,i}, y_{w,i}, z_{z,i})$, $\tilde{W}_i$ at position $(w_{w,i}, x_{z,i}, y_{w,i}, z_{z,i})$, $\tilde{X}_i$ at position $(w_{y,i}, x_{y,i}, y_{y,i}, z_{z,i})$, and $\tilde{Y}_i$ at position $(w_{y,i}, x_{y,i+1}, y_{y,i}, z_{z,i+1})$, where $i$ is taken modulo $n$. Let $W$ be the collection of $\tilde{W}$ markings, and similar for $Y$, $Z$ and $X$. It is easy to check that the markings of $H\Gamma = (C, \{W, X, Y, Z\}, G_{zx}, G_{wy})$ satisfy the marking conditions, and hence $H\Gamma$ is a Lagrangian hypercube diagram with $G_{zx} = G_1$ and $G_{wy} = G_2$. \hfill \Box

Having developed the results on Lagrangian grid diagrams in Section 3, and having shown in Theorems 8.4, 8.2, and Corollary 8.3 we now have the necessary framework to complete the proof of Theorem 4 below.
Proof of Theorem 4. Given \((m, k) \in \mathbb{Z}^2\), and two knot types \(K_1\) and \(K_2\). Theorem 5 allows one to construct Lagrangian grid diagrams \(G_1\) and \(G_2\) representing \(K_1\) and \(K_2\) with rotation numbers \(m\) and \(k\) respectively. Corollary 8.3 allows one to find Lagrangian grid diagrams, \(G'_1\) and \(G'_2\), of the same size representing the same topological knots and having the same rotation numbers as \(G_1\) and \(G_2\). Applying Theorem 8.4 enables us to construct a Lagrangian hypercube diagram such that \(G_{zx} = G'_1\) and \(G_{wy} = G'_2\).

Example 8.5. One may construct a Lagrangian grid diagram for the unknot with arbitrary rotation number by following the construction shown in Figure 16. To realize rotation number \(r > 0\), construct the diagram as in Figure 16. The resulting diagram will have size \(2r + 3\), and the distance between the \(t\)-coordinates at each crossing is \(r + 1\). Let \(G_{zx}\) be such a grid diagram. Let \(G_{wy}\) be the Lagrangian grid diagram for the unknot of size \(2r + 3\) given by the construction shown in Figure 17. Note that the distance between the \(t\)-coordinates at the crossing will be \((r + 1)(r + 2)/2\) for the diagram of size \(2r + 3\). Then applying Theorem 8.4, Theorem 6.1 and Theorem 1 we obtain a Lagrangian hypercube diagram with rotation class \((r, 0)\). Figure 13 shows the construction for \(r = 1\).

Finally, we note that the embedded Legendrian tori given by Lagrangian hypercube diagrams may have interesting grid projections in which both diagrams, \(G_{zx}\) and \(G_{wy}\), represent non-trivial Legendrian knots in \(\mathbb{R}^3\). This suggests that the Lagrangian hypercube diagram construction is different from front spinning.

Example 8.6. Figure 18 shows a Lagrangian hypercube diagram with \(G_{zx}\) representing a trefoil, and \(G_{wy}\) representing a \((5, 2)\) torus knot. One may check that \(G_{wy}\) has rotation number 0, \(G_{zx}\) has rotation number 1, and hence, the Lagrangian hypercube diagram has rotation class \((1, 0)\).

Question 6. Are there Legendrian tori obtained from Lagrangian hypercube diagrams that cannot be obtained by front spinning a Legendrian knot?

The answer to this question is probably yes, and warrants further study.
Figure 18: Hypercube diagram with $G_{wy}$ representing a (5,2) torus knot, and $G_{zx}$ representing a trefoil.

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