Investigation of the solutions of the Cauchy problem and boundary-value problems for the ordinary differential equations with continuously changing order of the derivative

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Abstract. As is known, the problems for the differential equations with continuously changing order of the derivatives are not considered completely. In this paper we consider the initial and boundary value problems for this type of linear ordinary differential equations with constant coefficients and obtain analytic representations of solutions of these problems.

It should be noted that this area is one of the less studied fields of modern mathematics and there are not effective methods for the study of problems for such differential equations, just as we study the problem for partial differential equations, with both with additive and a multiplicative derivatives.

The method used here is based on an invariant for the above mentioned derivative, i.e. on the functions that do not change for the derivative of any real positive order.

Keywords. Cauchy problem, boundary value problem, ordinary differential equations, constant coefficients, changing order of the derivative, initial conditions, manifolds.

AMS Subject Classification:

1 Introduction

In the additive analysis both in discrete [1], [2] and continuous cases [3], [4], various problems are well studied [5], [6]. Not so extensive, but still investigated as discrete [7], [8] and continuous problems for the differential equations with multiplicative derivatives [9], [10]. For the partial differential equations, the number of initial conditions under consideration coincides with the order of the derivative of the equation with respect to time [11]. Concerning the boundary conditions for partial differential equations, the number of non-local boundary conditions (if the boundary of the domain is divided into two parts) also coincides with the order of the derivative with respect to the space variables [12].

If we consider the ordinary differential equations of fractional order, then the number of conditions not related to the order of the derivative in this equation, but to the step of change of the order of the derivative [13].

We here consider as the Cauchy problem and boundary value problems for the linear ordinary differential equations with constant coefficients, where the order of the derivative is changed continuously. In solution of this problem we use mainly invariant function technique.

Euler was the first to construct an invariant with respect to usual additive derivative function $e^x$. Then he constructed the function $e^{ax}$ by the help of which
ordinary linear homogeneous differential equations with constant coefficients he corresponded some algebraic equation (characteristic equation), i.e. he indeed algebrized the differential equation [14].

Then, with the help of Mittag-Leffler function the invariant function for fractional derivative was constructed [15], [16].

In this paper, using the Volterra function [17] the invariant is constructed for the derivative, the order of which is changing continuously.

2 Equation and its general solution.

Consider the equation

\[
\int_0^{\sqrt{2}} D^\alpha y(x) d\alpha = 0, \quad x > a > 0,
\]

where \(\sqrt{2}\) is an order of equation, the order of which changes continuously from zero to \(\sqrt{2}\).

Following the Volterra function [17], consider the function that by \(\lambda = 1\) is an invariant for any real order

\[
h_{+0}(x, \lambda) = \int_{-1}^{\infty} \frac{x^v}{v!} \lambda^v d\nu,
\]

where \(v!\) by \(v > -1\) is defined by the Euler’s gamma function \(\Gamma[15]\) and is assumed that \(v! = \infty\) if \(v \leq -1\).

Note that the function (2) by \(\lambda = 1\) is a Volterra function if the integral takes the beginning not from -1 and from zero. Now it is easy to see that for any \(0 < \alpha \in \mathbb{R}\),

\[
D^\alpha h_{+0}(x, \lambda) = \lambda^\alpha h_{+0}(x, \lambda).
\]

Thus the solution of the equation (1) we seek as

\[
y(x) = h_{+0}(x, \lambda).
\]

Substituting (3) into (1) we get

\[
\int_0^{\sqrt{2}} \lambda^\alpha h_{+0}(x, \lambda) d\alpha = 0,
\]

or

\[
\int_0^{\sqrt{2}} \lambda^\alpha d\alpha = 0.
\]

Note that the equation (4) is a generalization of the polynomial when the order of the variable changes continuously.

It easy to see that calculation of the integral for the characteristic equation (5) leads us to

\[
\frac{\lambda^{\sqrt{2}} - 1}{\ln \lambda} = 0,
\]
Thus considering \((8)\) from \((5)\) for the general solution of the equation \((1)\) we get
\[
y(x) = \sum_{k \in \mathbb{Z}, k \neq 0} c_k h_{+0}(x, \lambda_k),
\]
where \(c_k\) are arbitrary constants and
\[
h_{+0}(x, \lambda_k) = \int_{-1}^{\infty} x^v \frac{d^v}{v!} h_k(x) dv, k \in \mathbb{Z}, k \neq 0,
\]
are partial solutions of the equation \((1)\).

### 3 Cauchy problem

Now let us consider Cauchy problem for the equation \((1)\) with the following initial conditions
\[
D^\alpha y(x)|_{x=a} = \varphi(\alpha), \alpha \in [0, \sqrt{2}).
\]

To define the constants \(c_k\) included to the general solution of the equation \((9)\) we substitute its general solution into \((11)\). Then we obtain
\[
\sum_{k \in \mathbb{Z}, k \neq 0} c_k e^{\sqrt{2}k\pi\alpha i} h_{+0}(a, \lambda_k) = \varphi(\alpha), \alpha \in [0, \sqrt{2}).
\]

The functions \(e^{\sqrt{2}k\pi\alpha i}\) form orthogonal system by \(\alpha \in [0, \sqrt{2})\) i.e.
\[
\int_0^{\sqrt{2}} e^{\sqrt{2}k\pi\alpha i} e^{\sqrt{2}n\pi\alpha i} d\alpha = \int_0^{\sqrt{2}} e^{\sqrt{2}(k+n)\pi\alpha i} d\alpha = \frac{\sqrt{2}(k+n)\pi n i}{\sqrt{2}k(n+1)\pi i} \bigg|_0^{\sqrt{2}} = 0, \quad k + n \neq 0,
\]
\[
\frac{\sqrt{2}(k+n)\pi n i}{\sqrt{2}k(n+1)\pi i} = \begin{cases} 
0, & k + n \neq 0, \\
\sqrt{2}, & k + n = 0.
\end{cases}
\]

Multiplying \((12)\) by \(e^{-\sqrt{2}n\pi\alpha i}\) and integrating from zero \(\sqrt{2}\) we find
\[
\sum_{k \in \mathbb{Z}, k \neq 0} c_k h_{+0}(a, \lambda_k) \int_0^{\sqrt{2}} e^{\sqrt{2}k\pi\alpha i} e^{-\sqrt{2}n\pi\alpha i} d\alpha = \int_0^{\sqrt{2}} \varphi(\alpha) e^{-\sqrt{2}n\pi\alpha i} d\alpha,
\]
or
\[
\sum_{k \in \mathbb{Z}, k \neq 0} c_k h_{+0}(a, \lambda_k) \frac{e^{2(k-n)\pi i} - 1}{\sqrt{2}(k-n)\pi i} + c_n h_{+0}(a, \lambda_n) \sqrt{2} = \int_0^{\sqrt{2}} \varphi(\alpha) e^{-\sqrt{2}n\pi\alpha i} d\alpha.
\]
Taking into account (13) one may get
\[ c_n = \frac{1}{\sqrt{2h+0(a, \lambda_n)}} \int_{0}^{\frac{\pi}{\sqrt{2}}} \varphi(\alpha)e^{-\sqrt{2n\pi\alpha}i}d\alpha, \quad n \in Z, n \neq 0. \] (14)

Substituting (14) into (9) we define the solution of the Cauchy problem (1), (11) in the form
\[ y(x) = \sum_{k \in Z; k \neq 0} \frac{h+0(x, \lambda_k)}{\sqrt{2h+0(a, \lambda_k)}} \int_{0}^{\frac{\pi}{\sqrt{2}}} \varphi(\alpha)e^{-\sqrt{2k\pi\alpha}i}d\alpha. \] (15)

Thus it was proved

**Theorem.** Let \( \varphi(\alpha) \) be continuous function. Then the problem (1), (11) has a solution that may be presented in the form (15).

4 Boundary problem

Here we consider the equation (1) on \((a, b)\) with boundary condition
\[ a_0 D^\alpha y(x)|_{x=a} + b_0 D^\alpha y(x)|_{x=b} = \varphi(\alpha), \quad \alpha \in [0, \sqrt{2}], \] (16)
where \( \sqrt{2} \) is an order of the differential equation (1), \( a_0, b_0 \) are constants, \( \varphi(\alpha) \) is a given real valued smooth function.

In this case the arbitrary constants \( c_k \) included into the general solution of the equation (9) are determined by substituting (9) into the boundary condition (16). Then we have
\[ a_0 \sum_{k \in Z; k \neq 0} c_k D^\alpha y_k(x)|_{x=a} + b_0 \sum_{k \in Z; k \neq 0} c_k D^\alpha y_k(x)|_{x=b} = \varphi(\alpha), \]
or
\[ \sum_{k \in Z; k \neq 0} c_k \lambda_k^\alpha [a_0 h+0(a, \lambda_k) + b_0 h+0(b, \lambda_k)] = \varphi(\alpha). \] (17)

Following (8) let us consider the functions
\[ \lambda_k^\alpha = e^{\sqrt{2k\pi\alpha}i}, \quad k \in Z, \quad k \neq 0, \alpha \in [0, \sqrt{2}) \] (18)

It is easy to see that by \( k \neq m \)
\[
(\lambda_k^\alpha, \lambda_m^\alpha) = \int_{0}^{\frac{\pi}{\sqrt{2}}} e^{\sqrt{2k\pi\alpha}i} e^{-\sqrt{2m\pi\alpha}i} d\alpha = \int_{0}^{\frac{\pi}{\sqrt{2}}} e^{\sqrt{2(k-m)\pi\alpha}i} d\alpha = \frac{e^{\sqrt{2(k-m)\pi\alpha}i}}{\sqrt{2(k-m)\pi i}} \bigg|_{\alpha=0} =
\]
\[ = \frac{e^{\sqrt{2(k-m)\pi\alpha}i_{\alpha=0}} - 1}{\sqrt{2(k-m)\pi i}} = 0, \]
and 
\[(\lambda_m^a, \lambda_m^a) = \sqrt{2},\]
i.e. the functions (18) are orthogonal.

Then going back to (17) we get
\[
\sum_{k \in \mathbb{Z}, k \neq 0} c_k [a_0 h_{+0}(a, \lambda_k) + b_0 h_{+0}(b, \lambda_k)] \int_0^{\sqrt{2}} \lambda_k \alpha \lambda_k^\alpha d\alpha = \int_0^{\sqrt{2}} \varphi(\alpha) \lambda_m^\alpha d\alpha,
\]
or
\[
c_m \sqrt{2} [a_0 h_{+0}(a, \lambda_m) + b_0 h_{+0}(b, \lambda_m)] = \int_0^{\sqrt{2}} \varphi(\alpha) \lambda_m^\alpha d\alpha. \tag{19}\]

If
\[
a_0 h_{+0}(a, \lambda_m) + b_0 h_{+0}(b, \lambda_m) \neq 0, m \in \mathbb{Z}, m \neq 0, \tag{20}\]
Then from (19) we obtain
\[
c_m = \frac{\int_0^{\sqrt{2}} \varphi(\alpha) \lambda_m^\alpha d\alpha}{\sqrt{2} [a_0 h_{+0}(a, \lambda_m) + b_0 h_{+0}(b, \lambda_m)]}, m \in \mathbb{Z}, m \neq 0. \tag{21}\]

Putting computed in (21) constants \(c_m\) into (9) we obtain the solution of the problem (1), (16) in the form
\[
y(x) = \sum_{k \in \mathbb{Z}, k \neq 0} \frac{\int_0^{\sqrt{2}} \varphi(\alpha) \lambda_k^\alpha d\alpha}{\sqrt{2} [a_0 h_{+0}(a, \lambda_k) + b_0 h_{+0}(b, \lambda_k)]} h_{+0}(x, \lambda_k), \quad x \in [a, b]. \tag{22}\]

Finally for the considered boundary problem (1), (16) the following theorem is proved.

**Theorem.** If \(0 < \alpha < b, a_0, b_0\) are given numbers, \(\varphi(\alpha)\) is continuous real function and the relation function (20) is valid, then solution of the boundary problem (1), (16) exists and may be presented in the form of (??).

**Note.** The case when the coefficients \(a_0, b_0\) of the boundary condition (16) are the functions of the variable \(\alpha\) is an open problem.

Список литературы

[1] Gelfand A.O. Calculus of Finite Differences, Moscow, Nauka, 1967, 375 p.

[2] Izadi F.A., Aliev N. A., Bagirov G., Discrete Calculus by Analogy, Book December, 3, 2009, 154 p.

[3] Trikomi F. Differential Equations, IL, Moscow, 1962, 349 ð.

[4] Общая теория граничных задач, Сборник научных трудов, Киев, Наукова Думка, 1983, 328 стр.
[5] Aliev N. A., Bagirov G., Izadi F.A., Discrete-additive Analysis, Tarbiat Moallem University Publishers, Tabriz, Iran, 1993, 220 p.

[6] Aliev N.A. Decomposition of the matrix functions over the solutions of the spectral problem, Differential Equations, Vol.2, No.6, 1966, pp.847-852.

[7] Hassani O. L., Aliev N. Analytic approach to solve specific linear and nonlinear differential equations, International Mathematical Forum, Journal for Theory and Applications, Vol.33-36(2008), No.3, pp.1623-1631.

[8] Jahansahi M., Ahmadkhanlu A., Aliev N., Fatemi M. Discrete additive and multiplicative differentiation and integration and their invariant functions, Journal of Contemporary Applied Mathematics, Vol.1, No.1, 2011, pp.28-35.

[9] Bashirov A.E., Kurpinar E.M., Ilzyapici A. Multiplikative calculus and its applications, J. Math. Anal., 2008, pp.36-48.

[10] Luc Florack, Hans van Assen. Multiplicative calculus in biomedical image analysis, J. Math. Emaging, 2011, 12 pages.

[11] Kurant R. Partial Differential Equations, Moscow, Nauka, 1964, 830 p.

[12] Bahrami F., Aliev N., Hosseini S.M. A method for the reduction of four dimensional mixed problems with general boundary conditions to a system of second kind Fredholm integral equations, Italian Journal of Pure and Applied Mathematics, No.17, 2005, pp.91-104

[13] Aliev N., Pashavand A.A., Boundary value problem for a fractional order ordinary linear differential equation with a constant coefficient, Proceedings of IAM, Vol.4, No.1, 2015, pp.3-7.

[14] Arnold V.I. Ordinary Differential Equations, Moscow, Nauka, 1971, 239p.

[15] Mittaq-Leffler G., Acta Math., Vol.29, 1905, pp.81-101.

[16] Samko S.G., Kilbas A.A., Marichev O.I., Integrals and Derivatives of Fractional Order and Some Their Applications, Minsk, Nauka i Tekhnika, 1987, 687 p.

[17] Volterra V. Theory of Functionals, Integral and Integro-differential Equations, Moscow, Nauka, 1982, 304 p.

[18] Andre Ango. Mathematics for Electro and Radio Engineers, Moscow, Nauka, 1967, 778 p.