CONNECTIONS BETWEEN REPRESENTATION-FINITE AND KÖTHE RINGS

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Abstract. A ring $R$ is called left $k$-cyclic if every left $R$-module is a direct sum of indecomposable modules which are homomorphic image of $R R^k$. In this paper, we give a characterization of left $k$-cyclic rings. As a consequence, we give a characterization of left Köthe rings, which is a generalization of Köthe-Cohen-Kaplansky theorem. We also characterize rings which are Morita equivalent to a basic left $k$-cyclic ring. As a corollary, we show that $R$ is Morita equivalent to a basic left Köthe ring if and only if $R$ is an artinian left multiplicity-free top ring.

1. Introduction

Let $R$ be an associative ring with unit. $R$ is called left pure semisimple if every left $R$-module is a direct sum of finitely generated left $R$-modules. $R$ is said to be representation-finite if it is left artinian and has only finitely many non-isomorphic finitely generated indecomposable left $R$-modules. According to Auslander [2], Ringel and Tachikawa [19] and Fuller and Reiten [10], a ring $R$ is representation-finite if and only if it is left and right pure semisimple. The pure semisimplicity conjecture, which says that left pure semisimple rings are representation-finite, is still open.

Recall that every finitely generated (abelian group) $\mathbb{Z}$-module is a direct sum of cyclic modules. Köthe proved that artinian principal ideal rings have this property [15]. A ring $R$ is called left (resp., right) Köthe if every left (resp., right) $R$-module is a direct sum of cyclic modules. Köthe also posed the problem, known as Köthe’s problem, to classify the left (resp., right) Köthe rings. The Köthe’s problem is still open. Nakayama gave an example of a right Köthe ring $R$ which is not a principal right ideal ring [17, page 289]. Later, Cohen and Kaplansky proved that if a commutative ring $R$ is Köthe, then $R$ is an artinian principal ideal ring [5]. In 1961, Kawada completely solved the Köthe problem for the basic finite dimensional $K$-algebras [12, 13, 14] (see also [18]). Kawada’s papers contain a set of 19 conditions which characterize Kawada algebras, as well as, the list of all possible finitely generated indecomposable modules. Ringel showed that any finite dimensional $K$-algebra of finite representation type is Morita equivalent to a Köthe algebra. By using the multiplicity-free of top and soc of finitely generated indecomposable modules, he also gave a characterization of Kawada algebras [18]. Behboodi et al. proved that if $R$ is a left Köthe ring in which all idempotents are central, then $R$ is an artinian principal right ideal ring [8]. Recently Ghorbani et al. gave a characterization of pure semisimple rings via $\rho$-dimension [11]. They proved that for a ring $R$ and an epic class
\( \rho \) of indecomposable \( R \)-modules, \( R \) is left and right pure semisimple if and only if every (left) right module is a direct sum of modules of \( \rho \)-dimension at most \( n \) for some \( n \), if and only if, \( \text{Mat}_n(R) \) is a left Köthe ring for some positive integer \( n \). In this paper, we will continue the study of left Köthe rings.

A ring \( R \) is called left \( k \)-cyclic if every left \( R \)-module is a direct sum of indecomposable \( k \)-generated modules. In this paper, we first give a characterization of left \( k \)-cyclic rings. We show that \( R \) is a left \( k \)-cyclic ring if and only if \( R \) is a left artinian ring and for each finitely generated indecomposable left \( R \)-module \( M \), \( c_i(top(M)) \leq kp_R(i) \) for each \( 1 \leq i \leq m \), where \( \{e_1, \ldots, e_m\} \) is a basic set of idempotents of \( R \), \( R \cong \bigoplus_{i=1}^{m} (Re_i)^{p_R(i)} \) and \( c_i(top(M)) \) is the number of composition factors of \( top(M) \) which are isomorphic to the \( Re_i/Je_i \) (see Theorem 3.1). As a corollary, we give a characterization of left Köthe rings (see Corollary 3.2). In fact, this corollary is a generalization of the Köthe-Cohen-Kaplansky theorem and Theorem 3.1 of [3]. We also give a characterization of rings which are Morita equivalent to a basic left \( k \)-cyclic ring. We show that a ring \( R \) is Morita equivalent to a basic left Köthe ring if and only if \( R \) is an artinian left multiplicity-free top ring. It is known that any left Köthe ring is representation-finite. Finally, we show that a ring \( R \) is representation-finite if and only if there exists a basic ring \( S \) and a positive integer \( n \) such that \( \text{Mat}_n(S) \) is a left Köthe ring and \( R \) is Morita equivalent to \( \text{Mat}_n(S) \) if and only if \( R \) is a left \( n \)-cyclic ring for some positive integer \( n \) if and only if \( \text{Mat}_m(R) \) is a left Köthe ring for some positive integer \( m \). Our results in this paper generalize and unify some results of [3], [5], [11], [15] and [18].

The paper is organized as follows. In Section 2, we prove some preliminary results that will be needed later in the paper. In Section 3, we give a characterization of left \( k \)-cyclic rings and then we show that all known results about the Köthe’s problem are just corollary of this characterization. Finally in Section 4, we characterize the class of rings which are Morita equivalent to the given left Köthe ring.

1.1. Notation. Throughout this paper, all rings have identity elements and all modules are unital. Let \( R \) be a ring. We denote by \( R\text{-Mod} \) (resp., \( \text{Mod-}R \)) the category of all left (resp., right) \( R \)-modules and by \( J \) the Jacobson radical of \( R \). For a left \( R \)-module \( M \), we denote by \( \text{soc}(M) \), \( \text{top}(M) \), \( \text{rad}(M) \) and \( \ell(M) \) its socle, top, radical and length, respectively. Let \( C \) and \( D \) be two categories. We write \( C \approx D \) in case \( C \) and \( D \) are equivalent. When two rings \( R \) and \( S \) are Morita equivalent (i.e., categories \( \text{R-Mod} \) and \( \text{S-Mod} \) are equivalent) we write \( R \approx S \).

2. Preliminaries

A ring \( R \) is called semiperfect if \( R/J \) is a semisimple ring and idempotents lift modulo \( J \). We recall that a set \( \{e_1, \ldots, e_m\} \) of idempotents of a semiperfect ring \( R \) is called basic in case they are pairwise orthogonal, \( Re_i \cong Re_j \) for each \( i \neq j \) and for each indecomposable projective left \( R \)-module \( P \), there exists \( i \) such that \( P \cong Re_i \). Clearly, the cardinal of any two basic sets of idempotents of a semiperfect ring \( R \) are equal. An idempotent \( e \) of a semiperfect ring \( R \) is called basic idempotent in case \( e \) is the sum \( e = e_1 + \cdots + e_m \) of a basic set \( \{e_1, \ldots, e_m\} \) of idempotents of \( R \). A semiperfect ring \( R \) is called basic if \( 1_R \) is a basic idempotent (see [1]).
Let \( \{e_1, \cdots, e_m\} \) be a basic set of idempotents of a semiperfect ring \( R, M \) be a left \( R \)-module of finite length and \( 1 \leq i \leq m \), we denote by \( c_i(M) \), the number of composition factors of \( M \) which are isomorphic to the \( Re_i/Je_i \). Recall that in \cite{22}, \( c_i(\text{top}(M)) \) is denoted by \( \text{Gen}(M, Re_i/Je_i) \).

The following proposition is a generalization of \cite{22} Lemma 1.8. \( \text{\textbf{Proposition 2.1.}} \) Let \( R \) be a semiperfect ring, \( M \) be a finitely generated left \( R \)-module and \( k \in \mathbb{N} \). Then the following conditions are equivalent.

1. \( M \) is a \( k \)-generated left \( R \)-module.
2. \( c_i(\text{top}(M)) \leq kp_R(i) \) for each \( 1 \leq i \leq m \), where \( \{e_1, \cdots, e_m\} \) is a basic set of idempotents of \( R \) and \( R \cong \bigoplus_{i=1}^{m} (Re_i)^{pr(i)} \).

**Proof.** Assume that \( R \) is a semiperfect ring and \( M \) is a finitely generated left \( R \)-module. Then \( R \cong \bigoplus_{i=1}^{m} (Re_i)^{pr(i)} \), where \( m \in \mathbb{N} \), each \( p_R(i) \in \mathbb{N} \) and \( \{e_1, \cdots, e_m\} \) is a basic set of idempotents of \( R \) and so by \cite{1} Corollary 15.18, Proposition 27.10, \( \text{top}(M) \cong \left( Re_1/Je_1 \right)^{s_1} \oplus \cdots \oplus \left( Re_m/Je_m \right)^{s_m} \). Thus there exists a projective cover \( P(M) \rightarrow M \) of \( M \), where \( P(M) = \left( Re_1 \right)^{s_1} \oplus \cdots \oplus \left( Re_m \right)^{s_m} \). Now assume that \( M \) is a \( k \)-generated left \( R \)-module. Then there exists an epimorphism \( f : R^{(k)} \rightarrow M \). Therefore there exists a morphism \( g : R^{(k)} \rightarrow P(M) \) such that \( pg = f \). Since \( \text{Ker}(g) \) is a superfluous submodule of \( P(M) \), \( g \) is an epimorphism and hence \( P(M) \) is a direct summand of \( R^{(k)} \). It follows that \( c_i(\text{top}(M)) \leq kp_R(i) \) for each \( 1 \leq i \leq m \). Now assume that for each \( 1 \leq i \leq m \), \( c_i(\text{top}(M)) \leq kp_R(i) \). Then \( s_i \leq kp_R(i) \) and so \( P(M) \) is a direct summand of \( R^{(k)} \). Therefore \( M \) is a \( k \)-generated left \( R \)-module. \( \square \)

One of the considerable properties of Morita theory which we use it in this paper is the fact that submodule lattices are preserved by equivalences. Let \( M \) be a left \( R \)-module and \( K \) be a submodule of \( M \). We denote by \( i_{K \leq M} : K \rightarrow M \) the inclusion monomorphism.

**Lemma 2.2.** Let \( R \) and \( S \) be Morita equivalent rings via an equivalence \( F : R-\text{Mod} \rightarrow S-\text{Mod} \) and \( M \) be a left \( R \)-module. If \( \text{top}(M) \) is a semisimple left \( R \)-module, then \( F(\text{top}(M)) \cong \text{top}(F(M)) \) as left \( S \)-modules.

**Proof.** Assume that \( \mathcal{L}_M \) (resp., \( \mathcal{L}_F(M) \)) is the lattice of submodules of \( M \) (resp., \( F(M) \)). We define the map \( \Lambda_M : \mathcal{L}_M \rightarrow \mathcal{L}_F(M) \) with \( \Lambda_M(K) = \text{Im}(F(i_{K \leq M})) \) for each \( K \in \mathcal{L}_M \). Then by \cite{1} Proposition 21.7, \( \Lambda_M \) is a lattice isomorphism. Let \( \text{top}(M) \) be a semisimple left \( R \)-module. We have an exact sequence

\[
0 \rightarrow \text{rad}(M) \xrightarrow{i_{\text{rad}(M) \leq M}} M \rightarrow \text{top}(M) \rightarrow 0.
\]

Then \( 0 \rightarrow F(\text{rad}(M)) \xrightarrow{F(i_{\text{rad}(M) \leq M})} F(M) \rightarrow F(\text{top}(M)) \rightarrow 0 \) is an exact sequence. Thus \( F(M)/\Lambda_M(\text{rad}(M)) \cong F(\text{top}(M)) \). Since \( F(M)/\Lambda_M(\text{rad}(M)) \) is a semisimple left \( S \)-module, \( \text{rad}(F(M)) \subseteq \Lambda_M(\text{rad}(M)) \). On the other hand, by \cite{1} Proposition 21.7 and the fact that a submodule \( K \) of \( M \) is maximal if and only if \( M/K \) is a simple module, we have \( \Lambda_M(\text{rad}(M)) \subseteq \text{rad}(F(M)) \). Therefore \( F(\text{top}(M)) \cong \text{top}(F(M)) \) as left \( S \)-modules. \( \square \)

**Lemma 2.3.** Let \( R \) be a semiperfect ring and \( M \) be a finite length left \( R \)-module. Then \( \ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M) \right) = c_i(M) \) for each \( 1 \leq i \leq m \), where \( \{e_1, \cdots, e_m\} \) is a basic set of idempotents of \( R \).
Proof. Assume that $R$ is a semiperfect ring. Then $R \cong \bigoplus_{i=1}^{m} (Re_i)^{pR(i)}$, where $m \in \mathbb{N}$, each $p_R(i) \in \mathbb{N}$ and \{e_1, \ldots, e_m\} is a basic set of idempotents of $R$. Let $1 \leq i \leq m$. We show that Hom$_R(Re_i, Re_i/Je_i)$ is a simple left End$_R(Re_i)$-module. Let $0 \neq \alpha, \beta \in$ Hom$_R(Re_i, Re_i/Je_i)$. Then there exists $f \in$ End$_R(Re_i)$ such that $\alpha f = \beta$. Therefore Hom$_R(Re_i, Re_i/Je_i)$ is a simple left End$_R(Re_i)$-module. Now we show that for each $j \neq i$, Hom$_R(Re_i, Re_j/Je_j) = 0$. Let $i \neq j$ and $0 \neq \gamma \in$ Hom$_R(Re_i, Re_j/Je_j)$. Then $\gamma$ is an epimorphism and so Ker$(\gamma)$ is a maximal submodule of $Re_j$. Therefore by [10 Proposition 27.10], $i = j$ which is a contradiction. Let $M$ be a finite length left $R$-module. Then by the above argument and [23 Proposition 32.4], Hom$(Re_i, M)$ is a finite length left End$_R(Re_i)$-module. Now by induction on length of $M$ we show that $\ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M) \right) = c_i(M)$ for each $1 \leq i \leq m$. Assume that $M$ is a simple left $R$-module. Then by [10 Proposition 27.10], there exists $1 \leq j \leq m$ such that $M \cong Re_j/Je_j$. Thus Hom$_R(Re_i, M) \cong$ Hom$_R(Re_i, Re_j/Je_j)$ and so $\ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M) \right) = \ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, Re_j/Je_j) \right)$. Hence by the above argument, $\ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M) \right) = c_i(M)$. Now assume that $M = t > 1$ and $0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M$ is a composition series for $M$. Then $0 = M_{t-1}/M_{t-1} \subset M_{t-2}/M_{t-1} \subset \cdots \subset M_1/M_{t-1} \subset M/M_{t-1}$ is a composition series for $M/M_{t-1}$. Therefore by the induction, $\ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M/M_{t-1}) \right) = c_i(M/M_{t-1})$. We consider the exact sequence $0 \rightarrow M_{t-1} \rightarrow M \rightarrow M/M_{t-1} ightarrow 0$. Then $0 \rightarrow \text{Hom}_R(Re_i, M_{t-1}) \rightarrow \text{Hom}_R(Re_i, M) \rightarrow \text{Hom}_R(Re_i, M/M_{t-1}) \rightarrow 0$ is an exact sequence. So

$$\ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M) \right) = \ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M_{t-1}) \right) + c_i(M/M_{t-1}).$$

If $M_{t-1} \cong Re_i/Je_i$, then $c_i(M/M_{t-1}) = c_i(M)$ and Hom$_R(Re_i, M_{t-1}) = 0$. It follows that $\ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M) \right) = c_i(M)$. Now assume that $M_{t-1} \cong Re_i/Je_i$. Then by the above argument, Hom$_R(Re_i, M_{t-1})$ is a simple left End$_R(Re_i)$-module. Consequently, $\ell \left( \text{End}_R(Re_i) \text{Hom}_R(Re_i, M) \right) = c_i(M)$. □

Let \{M_1, \ldots, M_r\} be the complete set of non-isomorphic finitely generated indecomposable left $R$-modules and \{e_1, \ldots, e_r\} be a basic set of idempotents of $R$. For each $1 \leq l \leq r$, put $q_R(l) = \max \{c_i(\text{top}(M_j)) \mid 1 \leq j \leq t\}$.

**Proposition 2.4.** Let $R$ be a representation-finite ring which is Morita equivalent to a ring $S$. Then $r$ is the cardinal number of the basic set of idempotents of $S$ and $q_R(l) = q_S(l)$ for each $1 \leq l \leq r$, where $r$ is the cardinal number of the basic set of idempotents of $R$.

Proof. Assume that $R$ is a representation-finite ring. Then there exists a basic set of idempotents of $R$. Let $R$ be Morita equivalent to a ring $S$ via an equivalence $F : R-\text{Mod} \rightarrow S-\text{Mod}$ and \{e_1, \ldots, e_r\} be a basic set of idempotents of $R$. Then there exists a basic set of idempotents of $S$ and we show that $r$ is the cardinal number of the basic set of idempotents of $S$. It is enough to show that $r$ is the cardinal number of the complete set of non-isomorphic finitely generated indecomposable projective left $S$-modules. We know that \{F(Re_1), \ldots, F(Re_r)\} is a set of non-isomorphic finitely generated indecomposable projective left $S$-modules. Let $G : S-\text{Mod} \rightarrow R-\text{Mod}$ be the inverse equivalence of $F$ and $Q$ be a finitely generated indecomposable projective left $S$-module. Then $G(Q)$ is a finitely generated indecomposable projective left $R$-module and so $G(Q) \cong Re_j$ for some $1 \leq j \leq r$. Thus
\{F(Re_i), \ldots, F(Re_t)\}\) is the complete set of non-isomorphic finitely generated indecomposable projective left \(S\)-modules. Therefore \(r\) is the cardinal number of the basic set of idempotents of \(S\). Now we show that \(q_R(l) = q_S(l)\) for each \(1 \leq l \leq r\). Assume that \(\{M_1, \ldots, M_t\}\) is the complete set of non-isomorphic finitely generated indecomposable left \(R\)-modules. Then \(\{F(M_1), \ldots, F(M_t)\}\) is the complete set of non-isomorphic finitely generated indecomposable left \(S\)-modules. Let \(1 \leq l \leq r\) and \(1 \leq j \leq t\). Since \(\text{Hom}_R(Re_l, \text{top}(M_j)) \cong \text{Hom}_S(F(Re_l), F(\text{top}(M_j)))\) and \(\text{End}_R(Re_l) \cong \text{End}_S(F(Re_l))\), \(\ell(\text{End}_R(Re_l)\text{Hom}_R(Re_l, \text{top}(M_j))) = \ell(\text{End}_S(F(Re_l))\text{Hom}_S(F(Re_l), F(\text{top}(M_j))))\). Therefore by Lemma 2.2 \(\text{Hom}_S(F(Re_l), F(\text{top}(M_j))) \cong \text{Hom}_S(F(Re_l), \text{top}(F(M_j)))\). Since by Lemma 2.3 \(q_R(l) = q_S(l)\).

\[\square\]

3. A characterization of left Köthe rings

We now give a characterization of left \(k\)-cyclic rings.

**Theorem 3.1.** Let \(R\) be a ring and \(k \in \mathbb{N}\). Then the following conditions are equivalent.

1. \(R\) is a left \(k\)-cyclic ring.
2. \(R\) is a left artinian ring and for each finitely generated indecomposable left \(R\)-module \(M\), \(c_i(\text{top}(M)) \leq kp_R(i)\) for each \(1 \leq i \leq m\), where \(\{e_1, \ldots, e_m\}\) is a basic set of idempotents of \(R\) and \(R \cong \bigoplus_{i=1}^m (Re_i)^{pr(i)}\).
3. \(R\) is a representation-finite ring and \(q_R(i) \leq kp_R(i)\) for each \(1 \leq i \leq m\), where \(\{e_1, \ldots, e_m\}\) is a basic set of idempotents of \(R\) and \(R \cong \bigoplus_{i=1}^m (Re_i)^{pr(i)}\).

**Proof.** (1) \(\Rightarrow\) (2). Assume that \(R\) is a left \(k\)-cyclic ring. Then by [23, Proposition 53.6], \(R\) is a left artinian ring and so \(R \cong \bigoplus_{i=1}^m (Re_i)^{pr(i)}\), where \(m \in \mathbb{N}\), each \(p_R(i) \in \mathbb{N}\) and \(\{e_1, \ldots, e_m\}\) is a basic set of idempotents of \(R\). Let \(M\) be a finitely generated indecomposable left \(R\)-module and \(1 \leq i \leq m\). Then \(M\) is \(k\)-generated and so by Proposition 2.1 \(c_i(\text{top}(M)) \leq kp_R(i)\).

(2) \(\Rightarrow\) (3). It follows from [23, Proposition 54.3].

(3) \(\Rightarrow\) (1). Assume that \(R\) is a representation-finite ring and \(q_R(i) \leq kp_R(i)\) for each \(1 \leq i \leq m\), where \(\{e_1, \ldots, e_m\}\) is a basic set of idempotents of \(R\) and \(R \cong \bigoplus_{i=1}^m (Re_i)^{pr(i)}\). Let \(\{N_1, \ldots, N_s\}\) be the complete set of non-isomorphic finitely generated indecomposable left \(R\)-modules and \(1 \leq i \leq m\). Then \(c_i(\text{top}(N_j)) \leq kp_R(i)\) for each \(1 \leq j \leq s\) and so by Proposition 2.1 \(N_j\) is a \(k\)-generated left \(R\)-module. Let \(M\) be a left \(R\)-module. Since \(R\) is representation-finite, by [23, Propositions 53.6 and 54.3], \(M\) is a direct sum of finitely generated indecomposable left \(R\)-modules. Thus \(R\) is a left \(k\)-cyclic ring.

\[\square\]

**Corollary 3.2.** The following conditions are equivalent for a ring \(R\).

1. \(R\) is a left Köthe ring.
2. \(R\) is a left artinian ring and for each finitely generated indecomposable left \(R\)-module \(M\), \(c_i(\text{top}(M)) \leq p_R(i)\) for each \(1 \leq i \leq m\), where \(\{e_1, \ldots, e_m\}\) is a basic set of idempotents of \(R\) and \(R \cong \bigoplus_{i=1}^m (Re_i)^{pr(i)}\).

3. \(R\) is a representation-finite ring and \(q_R(i) \leq p_R(i)\) for each \(1 \leq i \leq m\), where \(\{e_1, \ldots, e_m\}\) is a basic set of idempotents of \(R\) and \(R \cong \bigoplus_{i=1}^m (Re_i)^{pr(i)}\).
A finitely generated indecomposable left $R$-module $M$ is called *multiplicity-free top* if composition factors of $\text{top}(M)$ are pairwise non-isomorphic. Also, a ring $R$ is called *left multiplicity-free top* if every finitely generated indecomposable left $R$-module is multiplicity-free top. A ring $R$ is called *multiplicity-free top* if it is a left and right multiplicity-free top ring (see [18]).

**Corollary 3.3.** Let $R$ be a basic ring. Then $R$ is a left Köthe ring if and only if $R$ is an artinian left multiplicity-free top ring.

**Proof.** It follows from Corollary 3.2 and [23] Proposition 54.3. \qed

In the following, we show that Corollary 3.2 is a generalization of the Köthe-Cohen-Kaplansky theorem and Theorem 3.1 of [3]. In fact, all known results related to the characterization of left Köthe rings obtain from Corollary 3.2.

A left $R$-module $M$ is called *local* if it has a unique maximal submodule which contains any proper submodule of $M$. A ring $R$ is called of *left local type* if every finitely generated indecomposable left $R$-module is local (see [21]). An idempotent $e \in R$ is called *left* (resp., *right*) *semicentral* if $Re = eRe$ (resp., $eR = eRe$) (see [4]).

**Proposition 3.4.** Let $R$ be a semiperfect ring that all primitive idempotents of $R$ are left semicentral. Then the following conditions are equivalent.

1. $R$ is a left multiplicity-free top ring.
2. $R$ is of left local type.

**Proof.** (1) $\Rightarrow$ (2). Since $R$ is a semiperfect ring, $R = \bigoplus_{i=1}^{n} Re_i$ with $e_1 + \cdots + e_n = 1_R$, where $e_1, \cdots, e_n$ are orthogonal primitive idempotents of $R$ and each $e_i Re_i$ is a local ring. Set $R_i = e_i Re_i$ for each $1 \leq i \leq n$. Since all primitive idempotents of $R$ are left semicentral, $R = \bigoplus_{i=1}^{n} R_i$ is a basic ring. Therefore by [11 Proposition 27.10], $\{Re_1/Je_1, \cdots, Re_n/Je_n\}$ is the complete set of non-isomorphic simple left $R$-modules. Let $M$ be a finitely generated indecomposable left $R$-module. Since $R$ is left multiplicity-free top, $\text{top}(M) = S_1 \oplus \cdots \oplus S_t$, where $t \leq n$ and $S_j \cong Re_j/Je_j$ for each $j$. On the other hand, since $R$ is a finite direct sum of the rings $R_i$, there exists $1 \leq i \leq n$ such that $M$ is a left $R_i$-module. Let $1 \leq l \leq t$. Then $R_j S_l = 0$ for each $j \neq i$. It follows that $l = i$ and so $\text{top}(M) = S_i$. Therefore $R$ is of left local type.

(2) $\Rightarrow$ (1) is clear. \qed

Recall that a left $R$-module $M$ is called *uniserial* if its submodules are linearly ordered by inclusion. Also, a ring $R$ is called *left* (resp., *right*) *uniserial* if it is uniserial as a left (resp., right) $R$-module (see [1]).

**Theorem 3.5.** Let $R$ be a left artinian ring that all primitive idempotents of $R$ are left semicentral. If $R$ is a left multiplicity-free top ring, then $R$ is an artinian principal right ideal ring.

**Proof.** Assume that $R$ is a left multiplicity-free top ring. Then by Proposition 3.4, $R$ is of left local type. Since $R$ is left artinian, $R = \bigoplus_{i=1}^{n} Re_i$ with $e_1 + \cdots + e_n = 1_R$, where $e_1, \cdots, e_n$ are orthogonal primitive idempotents of $R$ and each $e_i Re_i$ is a local ring. Set $R_i = e_i Re_i$ for each $1 \leq i \leq n$. Since all primitive idempotents of $R$ are left
In this section, we will prove that for any semicentral ring \( R = \bigoplus_{i=1}^{n} R_i \), where each \( R_i \) is a local ring, it follows that each \( R_i \) is of left local type. Consequently, by [20, Theorem 2.4], each \( R_i \) is a right uniserial ring. On the other hand, since each \( R_i \) is a left artinian ring of left local type, there is a finite upper bound for the lengths of finitely generated indecomposable modules in \( R_i \)-Mod. Thus by [23, Proposition 54.3], each \( R_i \) is an artinian ring. Therefore each \( R_i \) is an artinian right uniserial ring. Consequently, by [23, Proposition 56.3], \( R \) is an artinian principal right ideal ring. □

As consequences of Corollary 3.2, Theorem 3.5 and [15], we have the following results.

**Corollary 3.6.** Let \( R \) be a left artinian ring that all primitive idempotents of \( R \) are left semicentral. Then \( R \) is a multiplicity-free top ring if and only if \( R \) is an artinian principal right ideal ring.

**Corollary 3.7.** ([15, 5, Köthe-Cohen-Kaplansky Theorem]) A commutative ring \( R \) is a Köthe ring if and only if \( R \) is an artinian principal ideal ring.

**Corollary 3.8.** ([3, Theorem 3.1]) Let \( R \) be a ring in which all idempotents are central. If \( R \) is a left Köthe ring, then \( R \) is an artinian principal right ideal ring.

The following example shows that there exists an artinian left multiplicity-free top local ring \( R \) which is not principal left ideal ring.

**Example 3.9.** Let \( F \) be a field which is isomorphic to the its proper subfield \( \overline{F} \) such that \( \dim(\overline{F}) = 2 \) (for example let \( F = \mathbb{Z}_2(y) \), where \( \mathbb{Z}_2(y) \) is the quotient field of polynomial ring \( \mathbb{Z}_2[y] \) and let \( \overline{F} = \mathbb{Z}_2(y^2) \)). Let \( \alpha \) be the isomorphism from \( F \) to \( \overline{F} \) and \( F[x; \alpha] \) be a skew polynomial ring with a usual polynomial addition and multiplication given by \( \lambda x = x\alpha(\lambda) \) for each \( \lambda \in F \). Set \( R := F[x; \alpha]/<x^2> \). Then \( R \) is a local ring. Let \( \mathcal{M} \) be the maximal ideal of \( R \) and \( \{1, a\} \) be a basis for the vector space \( F \) over \( \overline{F} \). Therefore \( \mathcal{M} = xR = Rx \oplus Rxa \) and \( \mathcal{M}^2 = 0 \). Set \( Q = R/\mathcal{M} \). Consequently, \( \dim(Q\mathcal{M}) = 2 \) and \( \dim(\mathcal{M}Q) = 1 \), where \( \mathcal{M}(\mathcal{M}Q) = 1 \). Hence by [6, Proposition 3], \( R \) is an artinian left multiplicity-free top ring but it is not principal left ideal ring.

The following example shows that the converse of Theorem 3.5 and Corollary 3.8 are not true in general.

**Example 3.10.** Let \( H \) be a division ring which is isomorphic to the its proper subdivision ring \( \overline{H} \) such that \( \dim(\overline{H}) = 3 \) (see [24, Theorem]). Let \( \alpha \) be the isomorphism from \( H \) to \( \overline{H} \) and \( H[x; \alpha] \) be a skew polynomial ring. Set \( R := H[x; \alpha]/<x^2> \). Then \( R \) is a local ring. Let \( \mathcal{M} \) be the maximal ideal of \( R \) and \( \{1, a, b\} \) be a basis for the vector space \( H \) over \( \overline{H} \). Then \( \mathcal{M} = xR = Rx \oplus Rxa \oplus Rxb \) and \( \mathcal{M}^2 = 0 \). Thus by [9, Theorem 9], \( R \) is a right uniserial ring and \( \dim(Q\mathcal{M}) = 3 \) and \( \dim(\mathcal{M}Q) = 1 \), where \( Q = R/\mathcal{M} \). It follows that \( t(RR) = 2 \). Let \( u, v, w \) be the linearly independent elements of \( Q\mathcal{M} \). The similar argument as in the proof of [7, Lemma 3.1] shows that \( T = (R \oplus R \oplus R)/D \), where \( D = \{u\lambda, v\lambda, w\lambda \mid \lambda \in R\} \) is an indecomposable right \( R \)-module with \( t(\text{soc}(T_R)) = 2 \). Therefore by [21, Theorem B], \( R \) is not of left local type. Hence by Proposition 3.4, \( R \) is not a left multiplicity-free top ring but \( R \) is an artinian principal right ideal ring.
4. A characterization of representation-finite rings

Let $\mathcal{U}$ be a class of left $R$-modules and $M$ be a left $R$-module. Then $\text{Tr}(\mathcal{U}, M) = \sum \{ \text{Im}(h) \mid h \in \text{Hom}_R(U, M), U \in \mathcal{U} \}$ is called trace of $\mathcal{U}$ in $M$. An idempotent $e$ of $R$ is called full idempotent if $ReR = R$. We recall that for a full idempotent $e \in R$, $\text{Tr}(Re, R) = ReR = R$ and so $Re$ is a generator in $R\text{-Mod}$ (see [23, Exercise 13.10(1)]).

**Theorem 4.1.** Let $S$ be a basic ring and $k \in \mathbb{N}$. Then the following conditions are equivalent.

1. $S$ is a left $k$-cyclic ring.
2. Any ring Morita equivalent to $S$ is left $k$-cyclic.
3. Any ring $R$ which is Morita equivalent to $S$ is artinian and for each indecomposable left $R$-module $M$, $c_i(\text{top}(M)) \leq k$ for each $1 \leq i \leq m$, where $\{e_1, \ldots, e_m\}$ is a basic set of idempotents of $R$.
4. For each full idempotent $e \in S$, $eSe$ is a left $k$-cyclic ring.
5. There exists a full idempotent $e \in S$ such that $eSe$ is a left $k$-cyclic ring.

**Proof.** Let $S$ be a basic ring. Then $S = \bigoplus_{j=1}^t Sf_j$, where $t \in \mathbb{N}$ and $\{f_1, \ldots, f_t\}$ is a basic set of idempotents of $S$.

(1) $\Rightarrow$ (2). Assume that $S$ is a left $k$-cyclic ring. Then by [23, Proposition 53.6], $S$ is left artinian and so there is a finite upper bound for the lengths of finitely generated indecomposable modules in $S\text{-Mod}$. Thus by [23, Proposition 54.3], $S$ is a representation-finite ring. Let $R$ be Morita equivalent to $S$. Then $R$ is a representation-finite ring and so there is a finite upper bound for the lengths of finitely generated indecomposable modules in $R\text{-Mod}$. Thus by [23, Proposition 54.3], $R$ is a representation-finite ring. Consequently, by [23, Proposition 54.3], $R$ is an artinian ring. It follows that $R \cong \bigoplus_{i=1}^m (Re_i)^{p_R(i)}$, where $m \in \mathbb{N}$, each $p_R(i) \in \mathbb{N}$ and $\{e_1, \ldots, e_m\}$ is a basic set of idempotents of $R$. So by Theorem 2.4 $t = m$ and $q_R(i) = q_S(i)$ for each $1 \leq i \leq t$. Hence by Theorem 3.1 $q_R(i) \leq k$ for each $1 \leq i \leq t$. Let $M$ be an indecomposable left $R$-module. Then $M$ is a $k$-generated module. Therefore $c_i(\text{top}(M)) \leq k$ for each $1 \leq i \leq t$.

(2) $\Rightarrow$ (1). It follows from Theorem 3.1.

(2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (4). Assume that $S$ is a left $k$-cyclic ring and $e$ is a full idempotent of $S$. Then $Se$ is a generator. Hence by [1 Corollary 22.4], $S \cong eSe$. Therefore by (2), $eSe$ is a left $k$-cyclic ring.

(4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (1). Assume that there exists a full idempotent $1_S \neq e \in S$ such that $eSe$ is a left $k$-cyclic ring. Then $Se$ is a progenerator and by [23, Propositions 53.6 and 54.3], $eSe$ is a representation-finite ring. Hence by [1 Corollary 22.4], $eSe$ is Morita equivalent to $S$ via
an equivalence $Se \otimes_{eSe} - : eSe - \text{Mod} \to S - \text{Mod}$. It follows that $S$ is a representation-finite ring. It is sufficient to show that every finitely generated indecomposable left $S$-module is $k$-generated. Let $Y$ be a finitely generated indecomposable left $S$-module. Then there exists a finitely generated indecomposable left $eSe$-module $X$ such that $Y \cong Se \otimes_{eSe} X$. Since $eSe$ is left $k$-cyclic, there exists an epimorphism $(eSe)^k \to X$. Thus there exists an epimorphism $Se \otimes_{eSe} (eSe)^k \to Se \otimes_{eSe} X$. Since $Se \otimes_{eSe} eSe \cong Se$ as left $S$-module and $e$ is idempotent, there exists an epimorphism $S^k \to Se \otimes_{eSe} X$. Therefore $Y$ is a $k$-generated left $S$-module.

**Corollary 4.2.** The following conditions are equivalent for a basic ring $S$.

1. $S$ is a left Köthe ring.
2. Any ring Morita equivalent to $S$ is left Köthe.
3. Any ring Morita equivalent to $S$ is an artinian left multiplicity-free top.
4. For each full idempotent $e \in S$, $eSe$ is a left Köthe ring.
5. There exists a full idempotent $e \in S$ such that $eSe$ is a left Köthe ring.

The following example shows that there exists a basic ring $S$ and an idempotent $e$ of $S$ such that $eSe$ is left Köthe but $S$ is not a left Köthe ring.

**Example 4.3.** Let $R$ be a simple artinian ring and $A$ be a basic finite dimensional algebra which is not left Köthe and $R$ is not isomorphic to each direct summand of $A$. Then $S = A \oplus R$ is a basic ring which is not left Köthe but $(0,1)S(0,1)$ is a left Köthe ring.

**Corollary 4.4.** Let $R$ be a left Köthe ring. Then there exists a positive integer $k$ such that every ring Morita equivalent to $R$ is left $k$-cyclic.

**Proof.** Assume that $R$ is a left Köthe ring. Then $R$ is left artinian and so $R \cong \bigoplus_{i=1}^m (Re_i)^{pR(i)}$, where $m \in \mathbb{N}$, each $pR(i) \in \mathbb{N}$ and $\{e_1, \ldots, e_m\}$ is a basic set of idempotents of $R$. Set $e = e_1 + \cdots + e_m$, $S = eRe$ and $k = \max\{pR(i) \mid 1 \leq i \leq m\}$. Thus $S$ is a basic ring and by [11, Proposition 27.10], $P = eR$ generate all simple right $R$-modules. So by [11, Proposition 17.9], $P$ is a generator in $\text{Mod-R}$. It follows that there is a right $R$-module $R'$ such that $P^{(k)} \cong R \oplus R'$ and also by [11, Corollaries 22.4 and 22.5], $R \cong S$ via an equivalence $P \otimes_R - : R - \text{Mod} \to S - \text{Mod}$. Let $Y$ be a finitely generated indecomposable left $S$-module. Then there exists a finitely generated indecomposable left $R$-module $X$ such that $Y \cong P \otimes_X X$. Since $R$ is left Köthe, there exists an epimorphism $R \to X$ and so there exists an epimorphism $P \otimes_R R \to Y$. On the other hand, by [23, Proposition 11.10] and [11, Proposition 4.5], we have $S$-isomorphisms

$$SS^{(k)} \cong \text{Hom}_R(P, P)^{(k)} \cong \text{Hom}_R(P^{(k)}, P) \cong \text{Hom}_R(R \oplus R', P) \cong P \oplus \text{Hom}_R(R', P).$$

Consequently, there exists an epimorphism $S^{(k)} \to Y$. Therefore by [23] Propositions 53.6 and 54.3, $S$ is a left $k$-cyclic ring. Thus by Theorem 4.1, every ring Morita equivalent to $R$ is left $k$-cyclic.

**Remark 4.5.** Let $R \cong \bigoplus_{i=1}^m (Re_i)^{pR(i)}$, where $m \in \mathbb{N}$, each $pR(i) \in \mathbb{N}$ and $\{e_1, \ldots, e_m\}$ is a basic set of idempotents of $R$. Let $k$ be a positive integer such that $k \leq pR(i)$ for each
$1 \leq i \leq m$. Assume that any ring which is Morita equivalent to $R$ is left $k$-cyclic. Then by Theorem 4.1, $R$ is artinian and for each indecomposable left $R$-module $M$, $c_i(\text{top}(M)) \leq k$ for each $1 \leq i \leq m$. Therefore by Corollary 3.2, $R$ is a left Köthe ring. In fact, if $k \leq p_R(i)$ for each $1 \leq i \leq m$, then the converse of Corollary 4.4 is true.

The following example shows that the converse of Corollary 4.4 is not true in general.

**Example 4.6.** Let $Q$ be the quiver

\[
\begin{array}{c}
2 \\
\rightarrow \\
\downarrow \\
3 \\
\rightarrow \\
\downarrow \\
1 \\
\rightarrow \\
\downarrow \\
4 \\
\rightarrow \\
\downarrow \\
5
\end{array}
\]

and $A = KQ$ be the path algebra of $Q$ over an algebraically closed field $K$. We identify $A-\text{mod} \approx \text{rep}_K(Q)$. Clearly $A$ is a basic representation-finite $K$-algebra. Let $M$ be the representation

\[
\begin{array}{c}
K \\
[1 \ 0] \\
K \\
[0 \ 1] \\
K \\
[1 \ 1] \\
K
\end{array}
\]

Then $M$ is a finitely generated indecomposable left $A$-module and it is easy to see that $c_1(\text{top}(M)) = 2$. Thus by Corollary 3.2, $A$ is not left Köthe. By using Theorem 3.1, it is easy to see that $A$ is a left 2-cyclic ring. Therefore by Theorem 4.1, every ring Morita equivalent to $A$ is left 2-cyclic.

It is known that the class of left Köthe rings is a proper subclass of the class of representation-finite rings. In the following, we show that the class of representation-finite rings and the class of rings which are Morita equivalent to the left Köthe rings are coincide.

**Proposition 4.7.** The following conditions are equivalent for a ring $R$.

1. $R$ is a representation-finite ring.
2. There exists a basic ring $S$ and a positive integer $n$ such that $\text{Mat}_n(S)$ is a left Köthe ring and $R \approx \text{Mat}_n(S)$.

*Proof. (1) $\Rightarrow$ (2). Assume that $R$ is a representation-finite ring. Then there exists a basic ring $S$ such that $R \approx S$. Let $S = \bigoplus_{j=1}^r Sf_j$, where $r \in \mathbb{N}$ and $\{f_1, \ldots, f_r\}$ is a basic set of idempotents of $S$. Set $d = q_S(1) + \cdots + q_S(r)$ and $T = \text{Mat}_d(S)$. Then by [1, Corollary 22.6], $R \approx T$ and so $T$ is a representation-finite ring. It follows that $T \cong \bigoplus_{k=1}^s (Th_k)^{pr(k)}$, where $s \in \mathbb{N}$, each $pr(k) \in \mathbb{N}$ and $\{h_1, \ldots, h_s\}$ is a basic set of idempotents of $T$. Since $S$ is basic, $pr(j) = d$ for each $1 \leq j \leq s$. On the other hand, by Proposition 2.4, $r = s$. 

Therefore, $R \approx \text{Mat}_s(S)$, which is a left Köthe ring.
and $q_S(j) = q_T(j)$ for each $1 \leq j \leq r$. Consequently, $q_T(j) = q_S(j) \leq d = p_T(j)$ for each $1 \leq j \leq r$. Therefore by Corollary 3.2, $T$ is a left Köthe ring.

(2) ⇒ (1). It follows from [23, Propositions 53.6 and 54.3]. □

**Remark 4.8.** Let $R$ be a representation-finite ring which is not left Köthe. Then there exists a basic ring $S$ and $n \in \mathbb{N}$ such that $\text{Mat}_n(S)$ is a left Köthe ring and $R \approx \text{Mat}_n(S)$. In fact left Köthe property is not a Morita invariant property.

**Proposition 4.9.** Let $R$ be a ring and $n \in \mathbb{N}$. Then the following conditions are equivalent.

1. $\text{Mat}_n(R)$ is a left $k$-cyclic ring.
2. $R$ is a left $kn$-cyclic ring.
3. For each $m \geq n$, $\text{Mat}_m(R)$ is a left $k$-cyclic ring.

**Proof.** (1) ⇒ (2). Assume that $\text{Mat}_n(R)$ is a left $k$-cyclic ring. Then by [23, Propositions 53.6 and 54.3], $\text{Mat}_n(R)$ is a representation-finite ring. Since $R \approx \text{Mat}_n(R)$, $R$ is a representation-finite ring. By [23, Proposition 54.3], it is sufficient to show that every non-cyclic finitely generated indecomposable left $R$-module is $kn$-generated. Let $M$ be a non-cyclic finitely generated indecomposable left $R$-module and $F : R\text{-Mod} \to \text{Mat}_n(R)\text{-Mod}$ be an equivalence. Then $F(M)$ is a finitely generated indecomposable left $S$-module. Consequently, $F(M)$ is a $k$-generated left $S$-module. Therefore by [16, Example 17.23], $M$ is a $kn$-generated left $R$-module.

(2) ⇒ (3). Assume that $R$ is a left $kn$-cyclic ring. Then by [23, Propositions 53.6 and 54.3], $R$ is a representation-finite ring. Let $m$ be a positive integer such that $m \geq n$. Set $T = \text{Mat}_m(R)$. Since $R \approx T$, $T$ is a representation-finite ring. By [23, Theorem 54.3], it is sufficient to show that every finitely generated indecomposable left $T$-module is $k$-generated. Let $X$ be a finitely generated indecomposable left $T$-module and $G : T\text{-Mod} \to R\text{-Mod}$ be an equivalence. Then $G(X)$ is a finitely generated indecomposable left $R$-module. It follows that $G(X)$ is a $kn$-generated module. Thus there exists an epimorphism $\alpha : R^{kn} \to G(X)$. Consequently, by [16, Example 17.23], there exists an epimorphism $S^k \to X$. Therefore $X$ is a $k$-generated left $S$-module.

(3) ⇒ (1) is clear. □

**Corollary 4.10.** The following conditions are equivalent for a ring $R$.

1. $R$ is a representation-finite ring.
2. There exists a basic ring $S$ and a positive integer $n$ such that $\text{Mat}_n(S)$ is a left Köthe ring and $R \approx \text{Mat}_n(S)$.
3. There exists a positive integer $n$ such that $R$ is a left $n$-cyclic ring.
4. There exists a positive integer $n$ such that $\text{Mat}_n(R)$ is a left Köthe ring.
5. There exists a positive integer $n$ such that for each $m \geq n$, $\text{Mat}_m(R)$ is a left Köthe ring.

**Proof.** (1) ⇔ (2). It follows from Proposition 4.7
(2) ⇒ (3). It follows from Corollary 4.4
(3) ⇔ (4) ⇔ (5). It follows from Proposition 4.9
(4) ⇒ (1) is clear. □
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