Seidel elements and mirror transformations

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Abstract The goal of this article is to give a precise relation between the mirror symmetry transformation of Givental and the Seidel elements for a smooth projective toric variety $X$ with $-K_X$ nef. We show that the Seidel elements entirely determine the mirror transformation and mirror coordinates.

Keywords Seidel elements · Mirror transformations · Batyrev relations · Fano toric variety · Nef toric variety

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1 Introduction

Let $X$ be a smooth projective toric variety. The variety $X$ can be explicitly written as the symplectic reduction of the Hermitian space $\mathbb{C}^m$ by a Hamiltonian action of a torus $(S^1)^r$, where $r$ is the Picard number of $X$. Let $D_1, \ldots, D_m$ denote the
classes in $H^2(X)$ Poincaré dual to the toric divisors. Let $t_i$ denote the coordinates in $H^2(X)$ with respect to an integral, nef basis $p_1, \ldots, p_r$, and let $q_i = \exp(t_i)$ be the exponential coordinates. Recall that the mirror theorem of Givental [9] states that if $c_1(X) = -K_X = D_1 + \cdots + D_m$ is semipositive (nef), then the cohomology valued function (of the B-model)

$$I_X(y, z) = e^{\sum_{i=1}^r p_i \log y_i/z} \sum_{d \in \text{NE}(X)_{\mathbb{Z}}} \prod_{i=1}^m \left( \prod_{k=-\infty}^0 \frac{(D_i + kz)}{(D_i + D_j)} \right) y_1^{d_1} \cdots y_r^{d_r}$$

determines the $J$-function $J_X(q, z)$ (of the the A-model or Gromov–Witten theory)

$$J_X(q, z) = e^{\sum_{i=1}^r p_i \log q_i/z} \left( 1 + \sum_{\alpha} \sum_{d \in \text{NE}(X)_{\mathbb{Z}} \setminus \{0\}} \left( \frac{\phi_\alpha}{z(z-\psi)} \right)^{D_1} \phi_\alpha q_1^{d_1} \cdots q_r^{d_r} \right)$$

via a change of coordinates $\log q_i = \log y_i + g_i(y)$, $i = 1, \ldots, r$, in such a way that $I_X(y, z) = J_X(q, z)$. Here, the variables $y_1, \ldots, y_r$ of the B-model are called mirror coordinates, and this change of variables is called mirror transformation (or mirror map). This relation can be used to show that the small quantum cohomology ring $QH(X)$ differs from the original presentation suggested by Batyrev [2] only by this change of coordinates. We refer to Givental [9] and the text by Cox and Katz [4] for further details on this discussion.

Let $(Y, \omega)$ denote a symplectic manifold. For a loop $\lambda$ in the group of Hamiltonian symplectomorphisms on $Y$, Seidel [19], assuming $Y$ monotone, constructed an invertible element $S(\lambda)$ in quantum cohomology counting sections of the associated clutched Hamiltonian fibration $E_\lambda \to \mathbb{P}^1$ with fibre $Y$. The Seidel element $S(\lambda)$ defines an element in $\text{Aut}(QH(Y))$ via quantum multiplication, and the association $\lambda \mapsto S(\lambda)$ a representation of $\pi_1(\text{Ham}(Y))$ on $QH(Y)$. This construction was later extended for all symplectic manifolds, see for instance McDuff and Tolman [17] where Seidel’s construction was used to study the underlying symplectic topology in toric manifolds.

In the case where the loop $\lambda$ is a (relatively simple) circle action, the asymptotic form of $S(\lambda)$ can be written explicitly in terms of geometric and Morse theoretic information [17, Theorem 1.10]. Regarding $X$ as a Hamiltonian space, they consider the Seidel element $S_j$ associated with an action $\lambda_j$ that fixes the toric divisor $D_j$ (see Sect. 3.2). It is proved that $\tilde{S}_j$ is a series of the form $\tilde{S}_j = D_j + O(q)$ if $-K_X$ is nef, and $\tilde{S}_j = D_j$ in the Fano case ($-K_X$ is ample). Moreover, it is shown that these elements satisfy the following Batyrev’s relations:

$$\prod_{j: \langle D_j, d \rangle > 0} \tilde{S}_j^{\langle D_j, d \rangle} = q^d \prod_{j: \langle D_j, d \rangle < 0} \tilde{S}_j^{-\langle D_j, d \rangle} \quad \text{in } QH(X) \quad (2)$$

1 Here, $\tilde{S}_j$ is a variant of the Seidel element $S_j = S(\lambda_j)$ given by $S_j = q_0 \tilde{S}_j$, where $q_0$ is the variable corresponding to the maximal section class of the associated bundle $E_j$. 