NEVILLE’S PRIMITIVE ELLIPTIC FUNCTIONS: THE CASE \( g_3 = 0 \)

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Abstract. The vanishing of the invariant \( g_3 \) attached to a lattice \( \Lambda \) singles out a midpoint lattice and yields a square-root of the associated Weierstrass function \( \wp_\Lambda \).

Neville’s Jacobian Elliptic Functions [1] is a peerless classic in its field. It is therefore with some reticence that we draw attention to a minor oversight in its presentation of the primitive functions as meromorphic square-roots of the shifted Weierstrass function \( \wp \).

The oversight occurs on page 50 of [1]: there it is stated that ‘The zeros of \( \wp_z \) are simple, and the branches of \( (\wp_z)^\frac{1}{2} \) can not be separated’. This is not quite correct: if we denote by \( \Lambda \) the period/pole lattice of \( \wp \) then the zeros of \( \wp \) are simple except in case the invariant \( g_3(\Lambda) \) is zero. We note that the statement asserting simplicity of the zeros of \( \wp \) is not made in the prequel [2].

In a little more detail, let \( \Lambda \subset \mathbb{C} \) be any lattice; the associated Weierstrass function \( \wp = \wp_\Lambda \) is then defined by

\[
\wp(z) = z^{-2} + \sum_{0 \neq \lambda \in \Lambda} \{(z - \lambda)^{-2} - \lambda^{-2}\}
\]

and has \( \Lambda \) as both its period lattice and its pole lattice. The zeros of the derivative \( \wp' \) are precisely those \( z \notin \Lambda \) such that \( 2z \in \Lambda \) and they make up three congruent lattices. If \( \Lambda = \{2n_1\omega_1 + 2n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\} \) and \( \omega_1 + \omega_2 + \omega_3 = 0 \) then these three midpoint lattices are \( \omega_1 + \Lambda, \omega_2 + \Lambda, \omega_3 + \Lambda; \) the values of \( \wp \) at each point of these lattices are denoted by \( e_1, e_2, e_3 \) respectively. Among their many properties, these distinct midpoint constants satisfy

\[
e_1 + e_2 + e_3 = 0
\]

and

\[
e_1e_2e_3 = g_3/4
\]

where the invariant \( g_3 = g_3(\Lambda) \) is defined by

\[
g_3 = 140 \sum_{0 \neq \lambda \in \Lambda} \lambda^{-6}.
\]

It follows at once that if \( g_3 = 0 \) then precisely one of the midpoint constants vanishes: say \( 0 = e_p = \wp(\omega_p); \) as \( \wp'(\omega_p) = 0 \) also, \( \omega_p \) is a double zero of the second-order elliptic function \( \wp \).

As its poles are also double, the Weierstrass function \( \wp \) itself has meromorphic square-roots (by the Weierstrass product theorem, for instance). It also follows that if \( g_3 \neq 0 \) then none of the midpoint constants vanishes, so that \( \wp \) has simple zeros and no meromorphic square-roots.

Neville shifts \( \wp \) by the midpoint constants and considers the three functions \( \wp - e_p \) as \( p \) runs over \( \{1, 2, 3\} \). By design, each of these second-order elliptic functions has double zeros on the corresponding midpoint lattice \( \omega_p + \Lambda \) and so has two meromorphic square-roots; Neville (though with an ingenious change of notation, which we recommend) defines the primitive function \( J_p \) to be the meromorphic square-root of \( \wp - e_p \) that satisfies \( zJ_p(z) \to 1 \) as \( z \to 0 \). Of course, our observation calls for no correction to any of this: it is simply the case that if \( g_3 = 0 \) then one of the midpoint constants is actually zero and need not be subtracted; the corresponding primitive function is then naturally preferred.
To take a particularly straightforward example, let \( \omega_1 = 1 \) and \( \omega_2 = i \) so that \( \omega_3 = -1 - i \) and 
\[
\Lambda = \{ 2m + 2ni : m, n \in \mathbb{Z} \}
\]
is the lattice of (even) Gaussian integers; the union \( \frac{1}{2} \Lambda \) of \( \Lambda \) and its three midpoint lattices is the full lattice of Gaussian integers. As multiplication by \( i \) leaves \( \Lambda \) invariant,
\[
\sum_{0 \neq \lambda \in \Lambda} \lambda^{-6} = \sum_{0 \neq \lambda \in \Lambda} (i\lambda)^{-6} = i^{-6} \sum_{0 \neq \lambda \in \Lambda} \lambda^{-6} = - \sum_{0 \neq \lambda \in \Lambda} \lambda^{-6}
\]
whence
\[
g_3(\Lambda) = 0
\]
and a similar calculation reveals that
\[
p(iz) = -p(z).
\]
It follows that \( e_3 = p(\omega_3) = 0 \): indeed, \( p(\omega_2) = p(i) = -p(1) = -p(\omega_1) \) so that \( e_1 + e_2 = 0 \) while \( e_1 + e_2 + e_3 = 0 \) in any case; of course, a direct computation is also possible. For this lattice, the Weierstrass function \( \wp \) has global meromorphic square-roots, namely \( J_3 \) and \( -J_3 \). It may be checked that the identity \( p(iz) = -p(z) \) implies that \( i J_3(iz) = J_3(z) \); it may also be checked that the same symmetry interchanges the other primitive elliptic functions \( J_1 \) and \( J_2 \) in the sense \( J_2(z) = i J_1(iz) \) and \( J_1(z) = i J_2(iz) \).

To summarize the general situation: if the invariant \( g_3(\Lambda) \) vanishes, then one of the midpoint constants vanishes, naturally singling out the corresponding midpoint lattice along with the corresponding primitive elliptic function, which is a meromorphic square-root of the Weierstrass function \( \wp_\Lambda \) itself; if \( g_3(\Lambda) \) does not vanish, then \( \wp_\Lambda \) lacks meromorphic square-roots. To put a part of this another way, the invariant \( g_3(\Lambda) \) is the obstruction to the existence of a global meromorphic square-root of \( \wp_\Lambda \).

Incidentally, an obstruction-theoretic significance also attaches to the invariant
\[
g_2 = g_2(\Lambda) = 60 \sum_{0 \neq \lambda \in \Lambda} \lambda^{-4}
\]
which satisfies
\[
e_2e_3 + e_3e_1 + e_1e_2 = -g_2/4.
\]
Let \( \zeta_4 \) be the fourth-order Eisenstein function defined by
\[
\zeta_4(z) = \sum_{\lambda \in \Lambda} (z - \lambda)^{-4}
\]
so that
\[
\zeta_4 = \frac{1}{6} \wp'' = \wp^2 - \frac{1}{12} g_2.
\]
Evidently, if \( g_2 \) is zero then \( \zeta_4 \) has the functions \( \pm \wp \) as meromorphic square-roots. Assume instead that \( g_2 \) is nonzero and write \( c^2 = \frac{1}{12} g_2 \); if \( \zeta_4 = \wp^2 - c^2 = (\wp - c)(\wp + c) \) is a square then its zeros must be double, so that \( c \) and \( -c \) are midpoint constants; as the three midpoint constants have zero sum, they are \( \pm c \) and 0, whence
\[
-\frac{1}{4} g_2 = e_2e_3 + e_3e_1 + e_1e_2 = c^2 = -\frac{1}{12} g_2
\]
and therefore \( g_2 \) is zero, contrary to assumption. In short, \( \zeta_4 \) admits global meromorphic square-roots precisely when the invariant \( g_2 \) vanishes.

REFERENCES

[1] E.H. Neville, Jacobian Elliptic Functions, Oxford University Press (1944).
[2] E.H. Neville, Elliptic Functions: A Primer, Pergamon Press (1971).

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