Dynamics Reflect Gapless Edge Modes for Topological Superconductor

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The dynamical feature for a $p$-wave superconductor model in different parameter regions in terms of the appearance of topologically gapless edge modes in reel geometry is investigated. First, the parameter region with gapless edge modes versus a parameter and quasi-momentum is shown. The parameter diagram can be reflected by the expectations of Pauli matrices in global manners. In another view, the dynamical feature of the excitation behave differently in the parameter regions with topological gapless edge modes and not. And the cusps of dynamical return rate vanish as the parameter pass the topological phase boundary slowly enough. It is found that the dynamics in the parameter region with gapless edge modes behaves differently to that without edge modes and related mostly to the eigenenergy gap between the pre-and post-quench eigenstates. The cusps of the dynamical return rate behave robustly against the noise occurs during evolution in the lattice until localization behavior dominates. This work benefits detecting topological edge modes by dynamical manners.

1. Introduction

Different matter in one phase belong to the same equivalence class according to certain criteria. Distinct from Landau’s theory about describing phase of matter based on order parameter, topological invariants act as new criteria in distinguishing phases of matter, like insulators, semimetals, and superconductors by a novel mechanism.[1–3] The topological $p$-wave superconductors which support gapless edge modes, are crucial for their implementation as ingredients for quantum-computing devices,[4–8] and have attracted wide attention.[9–11] Such superconductors can be realized by depositing magnetic atoms on a conventional $s$-wave superconductor substrate.[12,13] The topological phase can be characterized by topological invariants.[14,15] We would focus on a topological superconductor model in this work.

Topological invariants of loop constructed from time-evolution operator can represent the dynamical topology and opens up a novel avenue to classify quench dynamics.[19] Dynamically topological order is found closely related to the singularity of the Bogoliubov angle, but further investigation is needed for the quenches in the view of topology.[20] Topology-changing quenches have been investigated followed by dynamical phase transitions for various topological models.[21] The dynamical feature of localization–delocalization transition for a 1-dimensional (1D) incommensurate lattice described by the Aubry–André model has been investigated by quenching.[22] This inspires us checking the localization behavior during dynamics in this work.

In this work, we consider a topological superconductor model of reel geometry originated from Kitaev’s model.[5] First, the parameter region for the appearance of gapless edge modes is shown. The appearance of edge modes is also reflected by the expectations of Pauli matrices in global manners. Second, the dynamical properties in terms of gapless edge modes are checked including dynamical return rate, dynamics of excitations, and localization during evolution.

Following in Section 2, the $p$-wave superconductor model which we focus on is introduced. In Section 3, the parameter diagram in terms of the appearance of gapless edge modes and the relation with expectation values of Pauli matrices are studied. In Section 4, the dynamical features for this model compared to the appearance of topological edge modes are checked. Finally, we conclude in Section 5.

2. The Topological Superconductor

We consider a 2D topological $p$-wave superconductor model at the mean-field level described by the lattice Hamiltonian in real
Considering the equations \(-i\dot{a}_{k_x n_y} = [H, a_{k_x n_y}]\) and the anti-commutation relations between fermion operators, and taking \(\alpha_n\) and \(\beta_n\) as the amplitudes corresponding to \(a_{k_x n_y}\) and \(a_{k_x n_y}^\dagger\), respectively, we can obtain the time-dependent states by solving the differential equations:

\[
\begin{align*}
-\dot{\alpha}_{k_x n_y} &= (\mu + 2\cos k_x)\alpha_{k_x n_y} + \alpha_{k_x n_y+1} + \alpha_{k_x n_y-1} + 2\lambda_x (i\sin k_x)\beta_{k_x n_y} + \lambda_x (\beta_{k_x n_y-1} - \beta_{k_x n_y+1}) + 2\lambda_y (i\sin k_x)\alpha_{k_x n_y} - \lambda_y (\alpha_{k_x n_y-1} - \alpha_{k_x n_y+1}) \\
-\dot{\beta}_{k_x n_y} &= (\mu - \cos k_x)\beta_{k_x n_y} + \beta_{k_x n_y+1} - \beta_{k_x n_y-1} + 2\lambda_x (i\sin k_x)\alpha_{k_x n_y} - \lambda_x (\alpha_{k_x n_y-1} - \alpha_{k_x n_y+1})
\end{align*}
\]

for \(n_y = 1, 2, \ldots, N_y\). We would focus on the parameter diagram about the appearance of gapless edge modes, and the dynamical features in different parameter regions for this model.

### 3. The Parameter Diagram

In this work, we take periodic boundary condition along \(x\) direction and open boundary condition along \(y\) direction. Then \(k_x\) acts as a good quantum number and a parameter in the Hamiltonian. The topological phase can be reflected by integrals [27-29]. For example, the integral in the Brillouin zone (FBZ): 

\[
\sigma_{xy} = \frac{1}{4\pi} \oint_{FBZ} \tilde{h} \cdot (\tilde{h} \times \tilde{\partial}_y) dk_x dk_y,
\]

where \(\tilde{h}\) is the normalized vector of \((h_x, h_y, h_z)\) in the Equation \((2)\). After some algebra, one can find

\[
\sigma_{xy} = \frac{1}{4\pi} \oint_{FBZ} \tilde{n} \cdot (\tilde{n} \times \tilde{\partial}_y) dk_x dk_y,
\]

where \(\tilde{n}\) is the normalization for \((\langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle)\) under the ground state. [27,29] These calculations can be examined by using the ground state in the eigen-equations for the Hamiltonian \((2)\) as below

\[
H(\pm) = \pm \Omega |\pm\rangle
\]

where \(\Omega = \sqrt{\hbar_x^2 + \hbar_y^2 + \hbar_z^2}\), and

\[
|\pm\rangle = \frac{\pm 1}{\sqrt{2\Omega(|\pm\rangle \mp \hbar_z)}}, |\hbar_x \mp \hbar_z\rangle
\]

After calculating \(\sigma_{xy}\), it can be verified that the topological non-trivial parameter interval is \(\mu \in (-4, 4)\).

In topological classification of matter, bulk-boundary correspondence is a remarkable characteristic. The appearance of topological gapless edge modes is a signature for topological phase. As we focus on reel-shape lattice, we pay special attention on the parameter region with the appearance of the gapless edge modes versus \(\mu\) and \(k_x\) as shown in Figure 1a. We found that the gapless edge modes locate between the parameter boundaries described by the relations:

\[
\mu = -2 \cos k_x \pm 2
\]

The region for the appearance of the topological gapless edge modes is coincident with Figure 1a shown by a cartoon in the Supporting Information. [30]

To check the details of the topological edge modes, the eigenenergy spectra in quasi-momentum space (with Fourier...
Topological edge modes have remarkable localization characteristic compared to trivial modes. The localization for an state can be revealed by inverse participation ratio, IPR=$\frac{1}{|\langle \psi_j | m_{k_{x}} | \psi_j \rangle|^2}$, here $\psi_{m,k_{x}}$ denotes the projection of the $m$th eigenstate on the $j$th site for certain $k_{x}$. The localization for the eigenstates identified by $m$ versus $k_{x}$ are shown in Figure 1d. It can be seen that the region for the appearance of gapless edge modes is consistent with the parameter diagram in Figure 1a,c.

Topological phase of matter is usually indicated by the invariants like Zak phase, Chern number, and so on. In this work, the expectations of Pauli matrices under ground state, namely, $\langle \sigma_{x} \rangle$, $\langle \sigma_{y} \rangle$, and $\langle \sigma_{z} \rangle$ can also be applied to distinguish different parameter regions in terms of gapless edge modes. For example, the vector of $\langle \sigma_{x} \rangle$, $\langle \sigma_{y} \rangle$, $\langle \sigma_{z} \rangle$ plays such a role for this model shown in Figure 1e. A reference loop is parameterized as $(\cos k_{x}, \sin k_{y})$. The vectors wind this loop a full round when there is topological gapless edge modes ($k_{x} = \pi/4$ as an example) whereas not when there is no gapless edge modes ($k_{x} = 3\pi/4$ as an example), obtained for $k_{y}$ being ergodic ($-\pi, \pi$). This agrees with Figure 1a,c. Besides, the trajectories of $\langle \sigma_{x} \rangle - \langle \sigma_{y} \rangle - \langle \sigma_{z} \rangle$ form different loops in terms of the appearance of gapless edge modes on the Bloch sphere as shown in Figure 1f. These results suggest that the appearance of topological gapless edge modes can be revealed by expectation values of operators in global manifors. This benefits checking the topological phase of matter in experiments.

4. Dynamical Features

Since the appearance of gapless edge modes is a remarkable signature for topological phase of matter and the energy gap influence the transition probability between corresponding eigenstates, and then the dynamics of the excitations. One may conjecture that the presence of gapless edge modes may be reflected by the dynamics of excitations. Thus we check the dynamical features versus the appearance of gapless edge modes, including quenching by turning the parameter $\mu$ suddenly and the evolution of the excitation in different parameter regions in the following.

4.1. Quench

In conventional thermodynamics and statistical physics, non-analyticities of free energy density at critical temperatures indicate phase transition in thermodynamic limit. Similarly, dynamical phase transition describes the analog behavior occurs in quantum systems when time evolution acts as temperature variation with non-analytic behavior during evolution. Quenching is a candidate to reflect the dynamical phase transition by instantaneously changing the parameter(s) of a Hamiltonian starting from a ground state. The Loschmidt overlap defined as

$$G(t) = |\langle \psi_{\mu,0} | e^{-itH} | \psi_{\mu,0} \rangle|$$

Figure 1. a) The parameter diagram drew according to Equation (7), indicates the regions for the appearance of gapless edge modes for the real-shape topological superconductor with periodic and open boundary conditions in $x$ and $y$ directions, respectively. The blue region denotes that with gapless edge modes and the other regions denote those without gapless edge modes. b) Dispersion versus $k_{x}$ and $k_{y}$ with periodic boundary conditions in both $x$ and $y$ directions where $N_{x} = 32$. c) Branches of the dispersion relation versus $k_{x}$ when $\mu = -2$, which is the main research situation in this work. The insets (c1) and (c2) show the distributions $\langle \psi_{m}|\psi_{m} \rangle$ for the edge modes $\langle |\psi_{m} \rangle$ (half of the eigenstate is shown due to redundancy of the Hilbert space). d) IPR for the eigenstates (numbered by $m$) versus $k_{x}$ when $\mu = -2$. e) The vector of $\langle \sigma_{x} \rangle$, $\langle \sigma_{y} \rangle$, $\langle \sigma_{z} \rangle$ for two examples in different parameter regions ($k_{x} = \pi/4$ and $k_{y} = 3\pi/4$ when $\mu = -2$ for a full period of $k_{y}$, f) A range of trajectories for $\langle \sigma_{x} \rangle$, $\langle \sigma_{y} \rangle$, $\langle \sigma_{z} \rangle$ when the trajectories are drew of $k_{y}$ for three regions of $k_{y} \in [-x, x]$ with different dominant colors (slightly gradient color for different $k_{x}$).
is a quantity applied in quench. The initial state $|\psi_{\mu,0}\rangle$ is the ground state with parameter $\mu$ in the Hamiltonian (3). This quantity measures the overlap between the time evolved state $e^{-i\mu t}|\psi_{\mu,0}\rangle$ and the initial state $|\psi_{\mu,0}\rangle$. Following a sudden change of the parameter $\mu$ in the post-quench Hamiltonian, namely, $H(\mu)$ quench $H(\mu')$ in this work.

In statistical physics, the zeros of a partition function correspond to the cusps of free energy density in thermodynamic limit as a function of temperature. Similarly, the Loschmidt overlap in Equation (8) plays the role as the partition function. Since $G(t)$ scales with the size of the system $N$, another quantity, return rate is usually employed in quenching

$$f(t) = -\frac{1}{N}\ln G(t) \quad (9)$$

This quantity behaves non-analytically versus time when the Loschmidt overlap $G(t)$ vanishes. We employ this quantity to check the dynamics in terms of the appearance of gapless edge modes. $f(t)$ is shown versus time $t$ and $k_x$ in Figure 2a when $\mu$ changes suddenly from $-2$ to $-5$. While $\mu = -2$ and $k_x \in (-\pi/2, \pi/2)$ in the Brillouin zone, the gapless edge modes appear, and when $k_x$ belongs to the complementary set, the gapless edge modes disappear, as shown in Figure 1c. While $\mu = -5$, this model is in the topologically trivial phase in the whole Brillouin zone without gapless edge modes. As can be seen in Figure 2a, the behavior of $f(t)$ is different obviously whether or not $\mu$ quenching across the parameter boundary of the regions with and without gapless edge modes.

Besides the sudden change of parameters in quench, it is natural to consider what happens if the parameter $\mu$ continuously crossing such parameter boundaries within finite time. Among various manners of the parameter crossing the boundaries, we consider the case

$$\mu(t) = \begin{cases} 
\mu_0 + \frac{(\mu_n - \mu_0)}{\tau} & t \leq \tau, \\
\mu_n & t > \tau 
\end{cases} \quad (10)$$

$\mu_n = -5$ and $\mu_0 = -2$ in this work. The larger $\tau$ is, the further it from quenching and the closer to adiabatic process. As shown in Figure 2b and the color map between Figure 2b and Figure 2b', with increasing $\tau$, that is reducing the speed of $\mu$ passing the phase boundary, the cusps tends to disappear. And the cusps appear and delay until $\tau$ becomes too large. The slower $\mu(t)$ crossing the parameter boundaries, the latter the cusps appear since it takes time for reaching the boundary. Different response for the quench across the parameter boundary may provide the reference to design control strategies to finish certain research on this model.

### 4.2. Overlap between Eigenstates

We conjecture the overlap between the eigenstates of the post-quench Hamiltonian and the initial state (pre-quench, $|\psi_{\mu,2n-1,0}\rangle$) is mostly related to the appearance of the cusps during the evolution. Here $\nu$ is the number for different eigenstates. Thus we check the projection $|\langle \psi_{\mu,2n-1,0} | \phi_{\mu',\nu}\rangle|^2$, where $|\phi_{\mu',\nu}\rangle$ are the eigenstates of the post-quench Hamiltonian with the parameter $\mu$ having changed to $\mu'$ abruptly. This projection measures the distance between $|\psi_{\mu,2n-1,0}\rangle$ and $|\phi_{\mu',\nu}\rangle$. The larger $\nu$ corresponds to the larger eigenenergy gap between the corresponding eigenenergies of $|\psi_{\mu,2n-1,0}\rangle$ and $|\phi_{\mu',\nu}\rangle$. The results are shown in Figure 3. Comparing Figure 3a with Figure 1a, the projection $|\langle \psi_{\mu,2n-1,0} | \phi_{\mu',\nu}\rangle|^2$ approaches zero when $\mu$ changes from the region with (without) gapless edge modes to that without (with) such modes. By comparing Figure 3a–d, when it quenches to the same
4.3. Dynamical Features in Different Parameter Regions

The dynamical feature of excitations may behave distinctively in different parameter regions in terms of the appearance of gapless edge modes. To examine this and enrich the contents reflecting topological characteristic by dynamics, we employ a quantity \( f'(t) \) with the definition formally like the return rate \( f(t) \). It is a quantity that does not use an eigenstate of the pre-quench Hamiltonian as the initial state in \( f'(t) \), but the excitation with full population on one site as the initial state. And we show the evolution of \( f'(t) \) versus \( k_x \) in Figure 4a and the evolution of two examples of the excitation on the lattice in different parameter regions in Figure 4b,c. As shown in Figure 4b,c, the free dynamics of the excitation behaves distinctively in different parameter regions in terms of edge modes. And a full range of examination is in the Supporting Information.\(^{[33]}\) Such different evolutions results to the distinctive behavior of \( f'(t) \) in Figure 4a. These results hint that the dynamics of the excitation provides a candidate to denote parameter regions with gapless edge modes or not.

Parameter region, \( |\langle \psi_{\nu=-2,0}\mid \phi_{k_x,0} \rangle|^2 \) takes largest values when \( \nu = 0 \). Since larger \( \nu \) denotes the eigenstate with an eigenvalue further from the one corresponding to \( |\psi_{\nu=0}\rangle \). \( |\langle \psi_{\nu=-2,0}\mid \phi_{k_x,0} \rangle|^2 \) with larger \( \nu \) shows weaker correlation to whether it quenches to different parameter regions. Thus, it means the eigenstate with the nearest eigenenergy in the post-quench Hamiltonian is correlated prominently to the cusps in the dynamics.

4.4. Localization Resulting from Noise

Noise and disorders are usually inevitable in reality. The robust property against disorders of topological edge modes makes them be potential ingredients for quantum computation. As mentioned previously, the appearance of the gapless edge modes is related to the appearance of cusps in \( f(t) \). We explore the behavior of \( f(t) \) in the presence of the time-dependent noise in \( \mu \) after quenching from the region with gapless edge modes to that without gapless edge modes. The noise is introduced as \( \mu(t) = \mu + \delta \mu(t) \), where \( \delta \mu(t) \) denotes the fluctuation during evolution with values randomly distributed in \([0, \delta]\). Different from the noise in thermodynamics and statistical physics which denotes the thermal vibration of the particles and leads to thermalization, this noise results from the random vibration of the lattice. In Figure 5a, we show the dynamics of averaged \( f(t) \) versus \( \delta \) and time. It can be seen that cusps occurs until the noise makes \( \mu(t) \) cross the parameter boundary between regions with and without gapless edge modes as shown in Figure 5a.

To check the influence of the noise on the dynamics in detail, three examples of the dynamics quenching from the gapless edge modes are shown in Figure 5b1–b3. The dynamics is depressed, and localization occurs as the excitation is mainly bounded near the edge with \( \delta \) increasing. Meanwhile, the cusps in \( f(t) \) vanish meanwhile the dynamical pattern changes to those in the region without gapless edge modes, like that in Figure 4c.

Disorder in real space leads to localization of quantum states due to Anderson localization mechanism.\(^{[14]}\) This provides the

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**Figure 4.** a) \( f'(t) \) versus time \( t \) and \( k_x \) starting from the initial state with all population on the middle of the lattice of \( N_y = 32 \). The other parameters in the Hamiltonian are similar to those in Figure 2. b,c) The projections of the excitation on the lattice during evolution when \( k_x = 0 \) (with gapless edge modes) and \( -2 \) (without gapless edge modes), respectively. And a full examination of \( k_x \) in the Brillouin zone is in the Supporting Information.\(^{[33]}\) b1) and c1) show the corresponding \( f'(t) \) for (b) and (c), respectively.

**Figure 5.** a) \( f(t) \) averaged over 100 samples versus the amplitude \( \delta \) of the noise and time \( t \) when \( N_y=24 \) and \( k_x=\pi/4 \). The other parameters are same to those in Figure 2. b1–b3) The dynamics of the excitations when the noise amplitude \( \delta = 0, 2, \) and 5, respectively.
clue to get insight into the mechanism of the above localization in future works. However, if the noise becomes too large, it is beyond the scope of this mean-field model in this work. The above result also hints that the topological phase can be reflected by dynamics.

5. Conclusion

The parameter diagram and the dynamical features in terms of the appearance of topological gapless edge modes have been explored for a topological superconductor model with periodic and open boundary conditions in two orthogonal directions. The appearance of the gapless edge modes can be reflected by the expectations of Pauli matrices in two global manners. The dynamical excitation of the return rate occurs until the parameter passes the parameter boundary very slowly. The dynamical pattern of the excitation in the region with gapless edge modes is different to that in the region without gapless edge modes. The cusps behave robustly until the noise of the lattice leads to topological phase transition and localization. This work indicates one can detect topological phase in dynamical manners.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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