Brownian flights over a circle

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Our goal is to find the stationary distribution $P(\rho, \phi)$ of this process, where by $\rho, \phi$ we denote the polar coordinates on the plane. Clearly, $P(\rho, \phi) = P(\rho)$, and the rotation-invariant drift field $v$ is specified by its restriction to one ray.

The time-dependent distribution $P(\rho, \phi, t)$ of the Brownian motion in the rotation-invariant drift field $v$ satisfies
the nonstationary Fokker-Planck (FP) equation, which in polar coordinates can be written as follows:

\[
\frac{\partial P}{\partial t} = D \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial P}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 P}{\partial \phi^2} \right] - \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v \rho P) - \frac{1}{\rho} \frac{\partial}{\partial \phi} (v \phi P) \tag{1}
\]

where \( v \rho, v \phi \) are polar coordinates of the vector \( \mathbf{v}(\phi, \rho) \) and \( D \) is the diffusion coefficient.

Suppose the drift field \( \mathbf{v} \) is such that the corresponding diffusion \( P(\rho, \phi, t) \) stays away from the disc of radius \( R \), is ergodic with a stationary distribution \( P(\rho, \phi) = P(\rho) \) and the prescribed mean linear velocity \( V \). Denote by \( \Omega(R, V) \) the set of all corresponding drift fields \( \mathbf{v} \). According to the Schilder theorem, the large deviation rate function for the Brownian motion is the square of the velocity, see [1, 6]. Hence the drift field \( \mathbf{v} \in \Omega(R, V) \) we are looking for, is the one which minimizes the action \( S \) (the "entropy" of ensemble of Brownian trajectories)

\[
S(\mathbf{v}) = \int_{R}^{\infty} v^2(\rho, \phi) P(\rho, \phi) \rho d\rho d\phi \tag{2}
\]

over \( \Omega(R, V) \).

First we get the closed expression for the action \( S \) in terms of \( P(\rho) \) only, expressing \( v \rho \) and \( v \phi \) as functionals of \( P(\rho) \) by solving separately two auxiliary problems. Then we study the limiting behavior of the stationary distribution \( P(\rho, \phi) \) as \( R \) tends to the infinity, and we see the emergence of the Airy distribution in that limit.

### A. Determination of radial velocity, \( v_\rho(\rho) \)

Our first remark is that if \( \mathbf{v} \) is rotation-invariant and \( P \) is a stationary rotation-invariant solution of the equation [5], then the radial velocity \( v_\rho(\rho) \) is defined by \( P \). Indeed, the distribution \( P \) and velocity \( v_\phi \) do not depend on \( t \) and \( \phi \), so the Fokker-Planck (FP) equation [5] boils down to

\[
D \frac{\partial}{\partial \rho} \left( \rho \frac{\partial P(\rho)}{\partial \rho} \right) - \frac{\partial}{\partial \rho} (\rho v \rho P(\rho)) = 0 \tag{3}
\]

Integrating the FP equation [3], we get

\[
D \frac{\partial P(\rho)}{\partial \rho} = v_\rho P(\rho) + \frac{C_0}{\rho}, \tag{4}
\]
where $C_0$ is the integration constant. Recall that $P$ is a stationary rotation-invariant distribution for which the difference $D\frac{\partial P(\rho)}{\partial \rho} - v_\rho P(\rho)$ vanishes, since it equals to the mass transfer through the circle of radius $\rho$ which is zero in the stationary case. Thus, $C_0 = 0$, and for the radial velocity one gets the expression

$$v_\rho(\rho) = \frac{D}{P(\rho)} \frac{\partial P(\rho)}{\partial \rho}$$

(5)

### B. Determination of angular velocity, $v_\phi(\rho)$

In order to find the minimizer of the functional $S(v)$ of (2), we proceed as follows. First, for any distribution $P(\rho)$ we will look for the rotation-invariant velocity field $v_P$ which minimizes the functional

$$S_P(v) = \int_{R}^{\infty} \int_{0}^{2\pi} v_\rho^2(\rho) P(\rho) \rho d\rho d\phi$$

(6)

and then we look for the distribution $P$ which will minimize the functional $S_P(v)$. We consider the functional $S_P(*)$ on the space of velocities with the given average tangential velocity, $V$, that is

$$\int_{R}^{\infty} \int_{0}^{2\pi} \frac{v_\phi(\rho)}{\rho} P(\rho) \rho d\rho d\phi = \frac{V}{R}$$

(7)

Collecting (6) and (7) and introducing the Lagrange multiplier, $\lambda$, we can rewrite the auxiliary minimization problem in the following form:

$$S'_P = 2\pi \int_{R}^{\infty} \left( v_\phi^2(\rho) \rho + \lambda v_\phi(\rho) \right) P(\rho) d\rho \rightarrow \min$$

(8)

Solution of the equation $\delta S'_P = 0$ gives

$$v_\phi(\rho) = -\frac{\lambda}{\rho}$$

(9)

Substituting (9) back into (7), we get the expression for the Lagrange multiplier $\lambda$:

$$\lambda = \frac{V}{R} \left( 2\pi \int_{R}^{\infty} \frac{P(\rho)}{\rho} d\rho \right)^{-1}$$

(10)

Thus, we find the angular velocity, $v_\phi \equiv (v_P)_\phi$, as a functional of $P(\rho)$:

$$v_\phi = \frac{V}{2\pi R \rho} \left( \int_{R}^{\infty} \frac{P(\rho)}{\rho} d\rho \right)^{-1}$$

(11)

### III. MINIMIZATION OF THE ACTION

#### A. General formalism

To find the minimizer $P(\rho)$ of the action $S$, defined in (2), we substitute (5) and (11) into (2), getting the following expression for $S\{P(\rho)\}$:

$$S\{P(\rho)\} = \int_{R}^{\infty} \left[ v_\rho^2(\rho) + v_\phi(\rho)^2 \right] P(\rho) \rho d\rho d\phi = 2\pi \int_{R}^{\infty} \frac{D^2}{P(\rho)} \left( \frac{\partial P(\rho)}{\partial \rho} \right)^2 \rho d\rho +$$

$$2\pi \frac{V^2}{4\pi^2 R^2} \int_{R}^{\infty} \frac{P(\rho)}{\rho} \left( \int_{R}^{\infty} \frac{P(\rho)}{\rho} d\rho \right)^{-2} d\rho = 2\pi D^2 \int_{R}^{\infty} \frac{1}{P(\rho)} \left( \frac{\partial P(\rho)}{\partial \rho} \right)^2 \rho d\rho + \frac{V^2}{2\pi R^2} \left( \int_{R}^{\infty} \frac{P(\rho)}{\rho} d\rho \right)^{-1}$$

(12)
We have the normalization condition
\[ \int_{-\infty}^{\infty} \int_{0}^{2\pi} P(\rho) \rho d\rho d\phi = 1, \] (13)
which enter in the action \( S\{P(\rho)\} \) with the Lagrange multiplier \( \gamma \).

Let us make the substitution
\[ P(\rho) = Q^2(\rho) \] (14)
and plug it into (12)–(13). We obtain the functional
\[ S'\{Q(\rho)\} = S\{Q^2(\rho)\} + 2\pi \gamma \int_{\rho}^{\infty} Q^2(\rho) \rho d\rho. \] (15)
Collecting all terms together, we can explicitly write (15) in the following form:
\[ S'\{Q(\rho)\} = 8\pi D^2 \int_{\rho}^{\infty} \dot{Q}^2(\rho) \rho d\rho + \frac{V^2}{2\pi R^2} \left( \int_{\rho}^{\infty} \frac{Q^2(\rho)}{\rho} d\rho \right)^{-1} + 2\pi \gamma \int_{\rho}^{\infty} Q^2(\rho) \rho d\rho. \] (16)

In Eq.(16) we have used the notation \( \dot{Q} = \frac{\partial Q}{\partial \rho} \). It should be pointed out that the action (16) has the atypical second term, which, however, still allows us to proceed with the standard Euler-Lagrange minimization of the action (16). Equation \( \delta S'\{Q(\rho)\} = 0 \) leads to the following ordinary differential equation
\[ \frac{d}{d\rho} \left( \rho \frac{dQ(\rho)}{d\rho} \right) + \left( \frac{V^2}{16\pi^2 D^2 C^2 R^2 \rho^2} - \frac{\gamma \rho}{4D^2} \right) Q(\rho) = 0 \] (17)
where
\[ C = \int_{\rho}^{\infty} \frac{P(\rho)}{\rho} d\rho \] (18)
Developing the first term in (17), we arrive at the boundary problem
\[ \begin{cases} \frac{d^2 Q(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{dQ(\rho)}{d\rho} + \frac{1}{4D^2} \left( \frac{V^2}{4\pi^2 C^2 R^2 \rho^2} - \gamma \right) Q(\rho) = 0 \\ Q(\rho = R) = 0 \\ Q(\rho \to \infty) \to 0 \end{cases} \] (19)

The constants \( C \) and \( \gamma \) should be determined self-consistently. We find \( Q(\rho | C, \gamma) \) by solving the boundary problem (19), then we plug \( P = Q^2(\rho | C, \gamma) \) defined in (14) into expression for \( C \) getting
\[ C = \int_{\rho}^{\infty} \frac{Q^2(\rho | C, \gamma)}{\rho} d\rho \] (20)
and \( \gamma \) we determine from the normalization \( \int_{\rho}^{\infty} \rho d\rho \int_{0}^{2\pi} d\phi P(\rho) = 1: \)
\[ 2\pi \int_{\rho}^{\infty} Q^2(\rho | C, \gamma) \rho d\rho = 1 \] (21)

B. Analysis of the stationary asymptotic distribution

We are interested in the asymptotic behavior of the stationary measure \( P(\rho) \) in the vicinity of large circle of radius \( R \), i.e. when \( \rho = R + r \) and \( 0 < r \ll R \). First, it is convenient to make in (19) the substitution \( Q(\rho) = U(\rho) \rho^{-1/2} \). In terms of the function \( U(\rho) \), equation (19) reads
\[ \frac{d^2 U(\rho)}{d\rho^2} - \frac{1}{4} \left( \frac{\gamma}{D^2} - \frac{\Omega^2}{\rho^2} \right) U(\rho) = 0 \] (22)
where $\Omega^2 = 1 + \frac{V^2}{4\pi^2 C^2 D^2 R^2}$. Substituting $\rho = R + r$ in (22) and expanding it near $R$ up to linear terms in $r$, we get for $0 \leq r \ll R$:

$$
\frac{d^2U(r)}{dr^2} - \frac{1}{4} \left( \frac{\gamma}{D^2} - \frac{\Omega^2}{R^2} \right) U(r) = 0
$$

$$
U(r = 0) = 0
$$

$$
U(r \to \infty) \to 0
$$

(23)

Making use of the linear transform $r = u + vx$, we rewrite the boundary problem (23) as follows

$$
\frac{d^2U(x)}{dx^2} - xU(x) = 0
$$

$$
U \left( x = -\frac{u}{v} \right) = 0
$$

$$
U(x \to \infty) \to 0
$$

(24)

where

$$
u = \frac{R}{2} \left( 1 - \frac{\gamma R^2}{D^2 \Omega^2} \right); \quad v = \frac{2^{1/3} R}{\Omega^{2/3}} (25)$$

The general solution of (24)–(25) is

$$
U(x) = C_1 \text{Ai} \left( x + a_1 + \frac{u}{v} \right)
$$

(26)

where $\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{t^3}{3} + tx \right) dt$. Hence,

$$
U(r) = C_1 \text{Ai} \left( \frac{r - u}{v} + a_1 + \frac{u}{v} \right) \equiv C_1 \text{Ai} \left( \frac{r}{v} + a_1 \right)
$$

(27)

where $C_1 > 0$ is the constant to be determined via (20) and $a_1 \approx -2.33811$ is the first (smallest in absolute value) zero of the Airy function, i.e. the solution of the equation $\text{Ai}(a_1) = 0$. Correspondingly

$$
U(r) = C_1 \text{Ai} \left( \left( \frac{\Omega^2}{2} \right)^{1/3} \frac{r}{R} + a_1 \right) \equiv C_1 \text{Ai} \left( \frac{\mu^{1/3} r}{R} + a_1 \right)
$$

(28)

where we have defined

$$
\mu = \frac{\Omega^2}{2} = \frac{1}{2} \left( 1 + \left( \frac{V}{2\pi C D R} \right)^2 \right)
$$

(29)

Using (28) and taking into account that $Q(r) = U(r)/\sqrt{R + r}$ (where $0 \leq r \ll R$), we get the explicit expression for the requested distribution function, $Q(r)$:

$$
Q(r) = \frac{C_1}{\sqrt{R + r}} \text{Ai} \left( \frac{\mu^{1/3} r}{R} + a_1 \right)
$$

(30)

The constants $C_1, C$ are fixed by two auxiliary conditions (20) and (21), namely:

$$
C = C_1^2 \int_0^\infty \frac{dr}{(R + r)^2} \text{Ai}^2 \left( \frac{\mu^{1/3} r}{R} + a_1 \right)
$$

(31a)

$$
1 = 2\pi C_1^2 \int_0^\infty dr \text{Ai}^2 \left( \frac{\mu^{1/3} r}{R} + a_1 \right)
$$

(31b)
Evaluating the integral in (31b), we determine the constant $C_1$:

$$C_1^2 = \frac{\mu^{1/3}}{2\pi \left(\text{Ai}'(a_1)\right)^2 R}; \quad \text{Ai}'(a_1) = \frac{d\text{Ai}(x)}{dx} \bigg|_{x=a_1} \quad (32)$$

The constant $C$ we find perturbatively expanding the denominator in (31a) in the power series in $R \ll 1$. In the zero’s order approximation we have

$$\frac{1}{(R+r)^2} \approx \frac{1}{R^2}, \quad (33)$$

which allows us to evaluate $C$ in (31a) in the zero’s leading term in $0 \geq \frac{r}{R} \ll 1$:

$$C \approx \frac{1}{2\pi R^2} \quad (34)$$

The first-order correction to $C$ is derived in Appendix.

Collecting (30), (32) and (34), we get the final expression for the distribution function $P(\rho)$ where $\rho = R + r$:

$$P(\rho) = \frac{\mu^{1/3}}{2\pi \left(\text{Ai}'(a_1)\right)^2 R \rho} \text{Ai}^2 \left(\frac{\mu^{1/3}(\rho - R)}{R} + a_1\right) \quad (\rho \geq R), \quad (35)$$

where $\mu$ is obtained by substituting (34) into (29):

$$\mu = \frac{1}{2} \left(1 + \frac{V^2 R^2}{2D^2}\right) \bigg|_{R \gg 1} \approx \frac{V^2 R^2}{2D^2}. \quad (36)$$

IV. CONCLUSION

We have shown in the paper that the stationary radial distribution, $P(\rho)$ of the random walk which winds with the tangential velocity $V$ around the impenetrable disc of radius $R$ for $R \gg 1$ converges to the Airy distribution given by the expression

$$P(\rho) = \frac{V^{2/3}}{2^{4/3}\pi \left(\text{Ai}'(a_1)\right)^2 D^{2/3} R^{1/3} \rho} \text{Ai}^2 \left(\frac{V^2 \left(\frac{\rho - R}{R^{1/3}} + a_1\right)}{2D^2}\right) \quad (\rho \geq R), \quad (37)$$

where $a_1 \approx -2.33811$ is the first zero of the Airy function. Typical trajectories are localized in the circular strip $[R, R + \delta R^{1/3}]$ where $\delta$ is the constant which depends on the parameters $D$ and $V$ and is independent on $R$.

It should be emphasized, that the presence of the impenetrable disc which restricts Brownian flights is crucial for the asymptotic behavior (37) and for the localization of trajectories within the strip of size $R^{1/3}$. It seems instructive to compare two models which look pretty similar. The first model represents the "stretched" random walk of $N = cR$ steps evading a disk of radius $R$ and in main respects repeats the problem considered in our paper (see also [5].) (definitely, in the lattice version on the square lattice $c > 4$). The second model discussed in [3], studies the deviations from the typical "Wulf shape" of strongly inflated random loop of $N$ steps enclosing the algebraic area $A = cN^2$. At $N \gg 1$ the fluctuations in the first model scale as $N^{1/3}$, while remain Gaussian with the typical width $N^{1/2}$ in the second model. In both models the trajectories are pushed to very improbable tiny regions of the phase space, however the presence of a large deviation regime seems not to be a sufficient condition to affect the statistics and the convexity of the solid boundary on which trajectories recline, is very important.

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Appendix A

Let us note that if we keep the next order correction in (33), i.e. take $\frac{1}{(\mu + r)} \approx \frac{1}{r} \left(1 - \frac{2}{\mu} \right)$, the selfconsistency equation (31a) becomes as follows (compare to (34))

$$C \approx C^2 \pi R^2 \int_0^\infty dr \, \text{Ai}^2 \left(\frac{\mu^{1/3} r}{R} + a_1\right) - 2C^2 \int_0^\infty r dr \, \text{Ai}^2 \left(\frac{\mu^{1/3} r}{R} + a_1\right) = \frac{1}{2\pi R^2} \left(1 - \frac{4|a_1|}{3\pi \mu^{1/3}}\right)$$ (A1)

Substituting in (A1) the general expression for $\mu$ from (29), we get the following equation for the constant $C$

$$2\pi CR^2 \approx 1 - \frac{2^{7/3}}{3} \frac{|a_1|}{V^2} \left(1 + \frac{V^2}{4\pi^2 C^2 D^2 R^2}\right)^{1/3}$$ (A2)

By definition (20) the constant $C$ should be positive. Seeking the solution of (A2) in the perturbative form, we may write $C$ as follows at $R \gg 1$:

$$C \approx \frac{1}{2\pi R^2} + \frac{\kappa_1}{R^3} + \frac{\kappa_2}{R^4}$$ (A3)

where $\kappa_1$ and $\kappa_2$ are yet unknown coefficients which should be determined by solving (A2) in the limit $R \gg 1$. Substituting the ansatz (A3) into (A2) we immediately find that $\kappa_1 = 0$ and $\kappa_2$ is defined by the expression

$$\kappa_2 = -\frac{2^{4/3}}{3\pi} \frac{|a_1|D^2}{V^2}$$ (A4)

Thus, in the first-order expansion in $r/R \ll 1$ the constant $C$, which enters in the integral (20), reads

$$C \approx \frac{1}{2\pi R^2} - \frac{2^{4/3} |a_1|D^2}{3\pi V^2 R^4}$$ (A5)

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