Perturbative path-integral of string field and the $A_\infty$ structure of the BV master equation

Toru Masuda, Hiroaki Matsunaga

CEICO, Institute of Physics, the Czech Academy of Sciences,
Na Slovance 2, Prague 8, Czech Republic
Mathematical Institute, Faculty of Mathematics and Physics,
Charles University Prague,
Sokolovska 83, Prague 3, Czech Republic

masudatoru@gmail.com matsunaga@karlin.mff.cuni.cz

Abstract

The perturbative path-integral gives a morphism of the (quantum) $A_\infty$ structure intrinsic to each quantum field theory, which we show explicitly on the basis of the homological perturbation. As is known, in the BV formalism, any effective action also solves the BV master equation, which implies that the path-integral can be understood as a morphism of the BV differential. Since each solution of the BV master equation is in one-to-one correspondence with a (quantum) $A_\infty$ structure, the path-integral preserves this intrinsic $A_\infty$ structure of quantum field theory, where $A_\infty$ reduces to $L_\infty$ whenever multiplications of space-time fields are graded commutative. We apply these ideas to string field theory and (re-)derive some quantities based on the perturbative path-integral, such as effective theories with finite $\alpha'$, reduction of gauge and unphysical degrees, S-matrix and gauge invariant observables.
## Contents

1 Introduction
   1.1 Perturbative path-integral ................................. 1
   1.2 (Quantum) $A_\infty$ reduces to (quantum) $L_\infty$ ................................. 4

2 Path-integral as a morphism of BV
   2.1 $A_\infty$ structure of the BV master equation ......................... 7
   2.2 On the BV differential ....................................... 8
   2.3 Path-integral preserves the BV master equation ......................... 10
   2.4 Homological perturbation performs the path-integral I ..................... 12
   2.5 Homological perturbation performs the path-integral II ..................... 14
   2.6 $A_\infty$ structure of the effective theory ............................. 16
   2.7 The classical limit and cyclic $A_\infty$ ................................. 17

3 Path-integral as a morphism of $A_\infty$
   3.1 Tensor trick ............................................... 18
   3.2 Adjusting $A_\infty$ degree .................................. 20
   3.3 Perturbing $A_\infty$ structure ................................ 21
   3.4 Morphism of the cyclic $A_\infty$ structure .......................... 22
   3.5 Cyclicity of the effective $A_\infty$ structure ................................ 24
   3.6 Morphism of the quantum $A_\infty$ structure .......................... 25
   3.7 Twisted $A_\infty$ and source terms ................................ 27

4 Application to string field theory
   4.1 Effective theories with finite $\alpha'$ ................................ 28
   4.2 Light-cone reduction ........................................ 29
   4.3 $S$-matrix and asymptotic string fields ................................ 30

5 Conclusion and Discussions ........................................ 35

## 1 Introduction

In quantum theory, partition functions or expectation values of observables are central objects. For a Lagrangian field theory, the path-integral provides these objects, though how to integrate is obscure except for free theories. The perturbative path-integral is a standard technique that enables us to treat interacting fields in terms of free theories. In this paper, we show explicitly that the perturbative path-integral can be regarded as a morphism of the (quantum) $A_\infty$ structure intrinsic to each quantum field theory. Such a perspective provides simple explanations of some algebraic properties of the quantities based on the perturbative path-integral, which will be useful for calculating the scattering amplitudes, deriving effective theories, gauge-fixing, studying exact renormalization group flow and so on.

Homotopy algebras, such as quantum $A_\infty$ or $L_\infty$, arise naturally in the context of the ordinary Lagrangian description of quantum field theory. As is known among experts, they describe not only the gauge invariance of Lagrangian but also the Feynman graph expansion. Thus, theoretical
physicists already know some of these structures, albeit implicitly, even for the theory without
gauge degrees.\footnote{String field theories will be typical examples revealing these explicitly.} The Batalin-Vilkovisky (BV) formalism makes these structures visible and provides
the translation between Lagrangian field theories and homotopy algebras \cite{1}. The BV formalism
is one of the most powerful and general frameworks for quantization of gauge theories, which is
based on the homological perturbation \cite{2, 3}. For a given Lagrangian field theory, we can defines
a complex with an appropriate BV differential by solving the BV master equation, which is one
equivalent description of a quantum $A_\infty$ algebra \cite{6, 7}. This $A_\infty$ algebra reduces an $L_\infty$ algebra
whenever multiplications of space-time fields are graded commutative. Since the BV formalism
assigns a homotopy algebra to each quantum field theory, we can extract the intrinsic $A_\infty$ structure
explicitly by casting the BV master action into the homotopy Maurer-Cartan form \cite{38, 39, 40}.

As is well-known, in the BV formalism, any effective action also solves the BV master equation.
This fact implies that the path-integral can be understood as a morphism of the BV differential.
Since any solution of the BV master equation is in one-to-one correspondence with a quantum
$A_\infty$ structure, the path-integral preserves this intrinsic $A_\infty$ structure of quantum field theory.
Although these properties may valid for non-perturbative path-integral, in this paper, we consider
the perturbative path-integral. We first show that the perturbative path-integral can be performed
as a result of the homological perturbation for the intrinsic $A_\infty$ structure and thus it gives a
morphism of this (quantum) $A_\infty$ structure in any BV-quantizable Lagrangian field theory. Then,
we apply these ideas to string field theory and consider some quantities based on the perturbative
path-integral of string fields. As a result of the homological perturbation, we derive effective theory
with finite $\alpha'$ \cite{8}, the Light-cone reduction \cite{9, 10}, and string S-matrix in a simple way. In addition
to these (re-)derivations, we explain that this approach may enable us to use unconventional pieces
of perturbative calculus. We discuss the open string S-matrix based on unconventional propagators
whose 4-point amplitude reproduces the gauge invariant quantity given by \cite{11} directly.

This paper is organized as follows. In section 2, after giving a brief review of the BV formalism
and its relation to the quantum $A_\infty$ structure, we show explicitly that the homological perturbation
indeed performs the perturbative path-integral. This result would be known among experts except
for incidental details. The quantum $A_\infty$ structure of effective theory and the classical limit are
also discussed. In section 3, we translate the results based on the BV formalism into corresponding
results based on the intrinsic quantum $A_\infty$ structure, in which we find out a natural $A_\infty$ degree by
introducing string-field-inspired basis. We show that when the original BV master action includes
source terms, its effective theory must have a twisted $A_\infty$ structure. In section 4, we apply these
results to string field theory. We (re-)derive effective theories with finite $\alpha'$, reduction of gauge
degrees, string amplitudes, and gauge invariant quantities discussed in \cite{11} in a simple way. In
section 5, we conclude with summary and mentioning earlier works. In the rest of this section, we
summarize basic facts of the perturbative path-integral and the relation between $A_\infty$ and $L_\infty$.

1.1 Perturbative path-integral
A classical action $S_{\text{cl}}[\psi_{\text{cl}}] = S_{\text{cl free}}[\psi_{\text{cl}}] + S_{\text{cl int}}[\psi_{\text{cl}}]$ is a functional of classical fields $\psi_{\text{cl}}$. For a given
field theory, we write $\langle \psi_1, \psi_2 \rangle$ for an appropriate inner product of fields $\psi_1, \psi_2$. The action consists
of the kinetic term \( S_{\text{cl free}}[\psi_{cl}] \) and the interacting terms \( S_{\text{cl int}}[\psi_{cl}] \), which we write

\[
S_{\text{cl free}}[\psi_{cl}] \equiv -\frac{1}{2} \langle \psi_{cl}, \mu_1 \psi_{cl} \rangle, \quad S_{\text{cl int}}[\psi_{cl}] \equiv -\sum_{n} \frac{1}{n+1} \langle \psi_{cl}, \mu_n(\psi_{cl}, \ldots, \psi_{cl}) \rangle.
\] (1.1)

In this section, we assume that (1.1) consists of the physical degrees only for simplicity. In a Lagrangian field theory, the expectation value of observables \( \langle \ldots \rangle \) is described by using the path-integral of fields \( \psi_{cl} \) as follows

\[
Z_{J}^{-1} \int \mathcal{D}[\psi_{cl}] \left( \ldots \right) e^{S[\psi_{cl}]+J\psi_{cl}}, \quad Z_{J} = \int \mathcal{D}[\psi_{cl}] e^{S[\psi_{cl}]+J\psi_{cl}}
\] (1.2)

where \( Z_{J} \) denotes the partition function. Although the (non-perturbative) path-integral of interacting fields is a deep question, we can perform it for free theories since free actions are at most quadratic. When the free theory \( S_{\text{cl free}}[\psi_{cl}] \) is solved and the value of \( \sqrt{\det \mu_{J}^{-1}} \) is given, the path-integral integral of free fields is performed as a Gaussian integral and is normalized as

\[
1 = \int \mathcal{D}[\psi_{cl}] e^{S_{\text{cl free}}[\psi_{cl}]}.
\] (1.3)

The perturbative expansion enables us to perform the path-integral of interacting fields formally, which is a standard procedure in a Lagrangian field theory. In terms of the free theory, which should be well solved, we can represent (1.2) as the following expectation values

\[
\langle \ldots e^{S_{\text{cl int}}[\psi_{cl}]} \ldots \rangle_{\text{free, }J} = Z_{J}^{-1} \int \mathcal{D}[\psi_{cl}] \left( \ldots e^{S_{\text{cl int}}[\psi_{cl}]} \ldots \right) e^{S_{\text{cl free}}[\psi_{cl}]+J\psi_{cl}}.
\] (1.4)

The partition function \( Z_{J} \) can be represented as \( Z_{J} = \langle e^{S_{\text{cl int}}[\psi_{cl}]} \rangle_{\text{free, }J} \). We then consider to replace a given functional of \( \psi_{cl} \) by using a formal power series of \( \psi_{cl} \), for which we write \( F[\psi_{cl}] \), and replace the expectation value of a given functional by \( \langle F[\psi_{cl}] \rangle_{\text{free, }J} \) formally. This type of integral reduces to the Gaussian integral (1.3) because of \( F[\psi] e^{J\psi} = F[J\psi] e^{J\psi} \). Hence, whenever the free theory is well solved, we can perform the perturbative path-integral of (1.4) as follows

\[
\langle \ldots F'[\psi_{cl}] \ldots \rangle_{\text{free, }J} \equiv \left( \ldots F'[\partial J] \ldots \right) \left( e^{\frac{1}{2} J \mu_{J}^{-1} J} \right),
\] (1.5)

where \( F'[\psi_{cl}] \) is a formal power series of \( \psi_{cl} \). The expectation value \( \langle \ldots \rangle_{J} \equiv \langle \ldots e^{S_{\text{int}}[\psi_{cl}]} \ldots \rangle_{\text{free, }J} \) is always defined by the perturbative path-integral (1.5) in the rest of this paper. The Feynman graph expansion of \( F[\psi_{cl}] \) is an alternative representation of (1.5) with \( F'[\psi_{cl}] = F[\psi_{cl}] e^{S_{\text{int}}[\psi_{cl}]} \). As is well-known, by adding the source term \( e^{J\psi_{ev}} \), (1.5) can be cast as

\[
\langle \ldots F[\psi_{cl}] \ldots \rangle_{J} \equiv \left( e^{\frac{1}{2} \partial \psi_{ev} \mu_{J}^{-1} \partial \psi_{ev}} \right) \left( \ldots F[\psi_{ev}] \ldots e^{S_{\text{cl int}}[\psi_{ev}]} \right) e^{J\psi_{ev}} \bigg|_{\psi_{ev}=0}.
\] (1.6)

In this paper, the words “perturbative path-integral” always mean (1.5) or (1.6). Note that fields \( \psi_{cl} \) are integrated by using (1.3) in both representations (1.5) and (1.6).

---

\(^3\text{We set } \hbar = 1 \text{ for convenience. If necessary, we write } \hbar \text{ explicitly, such as } \exp(\hbar^{-1}S).\)
1.2 (Quantum) $A_\infty$ reduces to (quantum) $L_\infty$

In this paper, we regard a multiplication of space-time fields $\psi_1$ and $\psi_2$ as the tensor product $\psi_1 \otimes \psi_2$, which may be non-commutative $\psi_1 \cdot \psi_2 \neq (-)^{\psi_1 \psi_2} \psi_2 \cdot \psi_1$ in general. The space spanned by the polynomials of space-time fields is identified with the tensor algebra $T(\mathcal{H})$, where $\mathcal{H}$ denotes the state space of fields. The multiplication of $\psi_1, \ldots, \psi_n$ is given by

$$\psi_1 \cdot \psi_2 \cdots \psi_n \equiv \psi_1 \otimes \cdots \otimes \psi_n.$$  \hfill (1.7)

Note that these multiplications of fields themselves may not appear in the action (1.1) and should be distinguished from vertices or products $\mu_n$ appearing in the action. In the action, multiplications (1.7) always appear with some algebraic structure, such as coupling constant, delta functions of momentum conservation, contractions of indices, space-time differentials or structure constants of Lie algebras. We identify algebraic structures $\mu(\psi, \ldots, \psi)$ appearing in the action with properties of multilinear maps $\mu_n$ acting on the tensor product $\psi_1 \otimes \cdots \otimes \psi_n$, which we write

$$\mu_n(\psi_1, \ldots, \psi_n) \equiv \mu_n(\psi_1 \otimes \cdots \otimes \psi_n).$$ \hfill (1.8)

When multiplications of space-time fields are graded commutative, such as $\psi_1 \cdot \psi_2 = (-)^{\psi_1 \psi_2} \psi_2 \cdot \psi_1$, we replace $T(\mathcal{H})$ by the symmetric tensor algebra $S(\mathcal{H})$, which does not affect $\mu$ itself. Then, multiplications of space-time fields (1.7) can be represented as

$$\psi_1 \cdot \psi_2 \cdots \psi_n = \frac{1}{n!} \psi_1 \wedge \cdots \wedge \psi_n,$$ \hfill (1.9)

where the factor $n!$ comes from the definiton of the symmetrized tensor product

$$\psi_1 \wedge \cdots \wedge \psi_n \equiv \sum_{\sigma \in \mathcal{S}} (-)^{\sigma(\psi)} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}.$$ \hfill (1.10)

Since (1.10) is a natural product of the symmetric tensor algebra $S(\mathcal{H})$, instead of (1.8), natural algebraic structures $\mu^{\text{sym}}(\psi, \ldots, \psi)$ appearing in the action of commutative quantum field theory should be identified with properties of multilinear maps $\mu_n$ acting on the symmetrized tensor product $\psi_1 \wedge \cdots \wedge \psi_n$ as follows

$$\mu_n^{\text{sym}}(\psi_1, \ldots, \psi_n) \equiv \mu_n(\psi_1 \wedge \cdots \wedge \psi_n).$$ \hfill (1.11)

Note that $\mu$ and $\mu^{\text{sym}}$ are distinguished by just inputs states. The commutativity of multiplications of space-time fields is a property of (1.7) or (1.9). Whenever multiplications of space-time fields are graded commutative, $\psi_1 \otimes \cdots \otimes \psi_n \in S(\mathcal{H})$ from the beginning and we find

$$\mu_n(\psi_1, \ldots, \psi_n) = \frac{1}{n!} \mu_n^{\text{sym}}(\psi_1, \ldots, \psi_n).$$ \hfill (1.12)

Under these identifications, our algebraic structures $\mu$ of quantum field theory do not be affected by the graded commutativity of multiplications of space-time fields. As (1.12), our $\mu$ reduces to $\mu^{\text{sym}}$ automatically whenever we consider ordinary quantum field theory. Actually, the relation of $\mu$ and $\mu^{\text{sym}}$ is nothing but that of (quantum) $A_\infty$ and $L_\infty$. The quantum $A_\infty$ structure appearing in this paper reduces to the quantum $L_\infty$ structure if multiplications are commutative.
The grading of $A_\infty$ algebras which appear in this paper is just a label and does not change the physics. As we see later, physical gradings, such as the space-time ghost number or Grassmann parity of fields, do not give the $A_\infty$ degree. Hence, we often set the grading of all $A_\infty$ products to have degree 1 by using appropriate (de-)suspenion maps, which we call a natural $A_\infty$ degree. \footnote{When we consider other gradings, such as $2-n$ for $\mu_n$, the same relations hold as (1.13), (1.14), (1.15) or (1.16) but the sign factors take a little complicated form because of (de-)suspenion maps. See also [12–14].}

An $A_\infty$ structure $\mu = \mu_1 + \mu_2 + \cdots$ is a (co-)derivation acting on $\mathcal{T}(\mathcal{H})$ such that $(\mu)^2 = 0$. For fixed $n \geq 1$, the $A_\infty$ relations $(\mu)^2 = 0$ can be represented as follows

$$\sum_{k+l=n} \sum_{m=0}^{k} (-)^{\epsilon(\psi)} \mu_{k+1}(\underbrace{\psi_1, \ldots, \psi_m}_{m}, \mu_l(\underbrace{\psi_{m+1}, \ldots, \psi_{m+l}}_{m}), \underbrace{\psi_{m+l+1}, \ldots, \psi_n}_{k-m}) = 0, \quad (1.13)$$

where $\epsilon(\psi)$ denotes the sign factor arising from $\mu_l$ passing $\psi_1 \otimes \cdots \otimes \psi_m$. Let $\omega$ be a graded symplectic structure of degree $-1$ and $\{e_{-s}, e_{1+s}\}_{s \geq 0}$ be a set of complete basis such that $\omega(e_{-s}, e_{1+s}) = (-)^s \delta_{s, s'}$. A cyclic $A_\infty$ structure is an $A_\infty$ structure $\mu$ satisfying $\omega(\mu \otimes 1 + 1 \otimes \mu) = 0$, which is the classical limit of a quantum $A_\infty$ structure.

A quantum $A_\infty$ structure $\mu + \hbar \mathfrak{L}$ is a linear map acting on $\mathcal{T}(\mathcal{H})$ such that $(\mu + \hbar \mathfrak{L})^2 = 0$ where $\mu = \sum_n \mu_n[0] + \sum_{n,g} \hbar^g \mu_n[g]$ is a (co-)derivation and $\mathfrak{L}$ is a second order (co-)derivation. For fixed $n \geq 1$ and $g \geq 0$, the quantum $A_\infty$ relations $(\mu + \hbar \mathfrak{L})^2 = 0$ can be represented as

$$\sum_{k+l=n} \sum_{m=0}^{k} (-)^{\epsilon(\psi)} \mu_{k+1,[g_1]}(\underbrace{\psi_1, \ldots, \psi_m}_{m}, \mu_l,[g_2](\underbrace{\psi_{m+1}, \ldots, \psi_{m+l}}_{m}), \underbrace{\psi_{m+l+1}, \ldots, \psi_n}_{k-m})$$

$$+ \sum_{s \in \mathbb{Z}} \sum_{i=0}^{n} \sum_{j=0}^{n-i} (-)^{\epsilon(s, i, j)} \mu_{n+2,[g-1]}(\underbrace{\psi_1, \ldots, \psi_{i-1}, e_{s}, \psi_{i+1}, \ldots, \psi_{j}, e_{1+s}, \psi_{j+1}, \ldots, \psi_n}_{i+j}) = 0, \quad (1.14)$$

where the sign factor $\epsilon(s, i, j)$ arises from $e_{1+s}$ passing $\psi_1 \otimes \cdots \otimes \psi_{i+j}$ and $e_{-s}$ passing $\psi_1 \otimes \cdots \otimes \psi_i$.

An $L_\infty$ structure $\mu^\text{sym} = \mu^\text{sym}_1 + \mu^\text{sym}_2 + \cdots$ is a (co-)derivation acting on $\mathcal{S}(\mathcal{H})$ such that $(\mu^\text{sym})^2 = 0$. For fixed $n \geq 1$, the $L_\infty$ relations $(\mu^\text{sym})^2 = 0$ can be represented as follows

$$\sum_{k+l=n} \sum_{\sigma \in S_{l,k}} (-)^{\sigma(\psi)} \mu^\text{sym}_{k+1}(\mu^\text{sym}_{l}(\psi_{\sigma(1)}, \ldots, \psi_{\sigma(l)}), \psi_{\sigma(l+1)}, \ldots, \psi_{\sigma(n)}) = 0, \quad (1.15)$$

where $\sigma(\psi)$ denotes the sign factor arising from the $(l, k)$-unshuffle of $\psi_{\sigma(1)} \wedge \cdots \wedge \psi_{\sigma(n)}$. A cyclic $L_\infty$ structure is an $L_\infty$ structure $\mu^\text{sym}$ satisfying $\omega(\mu^\text{sym} \otimes 1 + 1 \otimes \mu^\text{sym}) = 0$, which is the classical limit of a quantum $L_\infty$ structure.

A quantum $L_\infty$ structure $\mu^\text{sym} + \hbar \mathfrak{L}$ is a linear map acting on $\mathcal{S}(\mathcal{H})$ such that $(\mu^\text{sym} + \hbar \mathfrak{L})^2 = 0$ where $\mu^\text{sym} = \sum_n \mu^\text{sym}_n[0] + \sum_{n,g} \hbar^g \mu^\text{sym}_n[g]$ is a (co-)derivation and $\mathfrak{L}$ is a second order (co-)derivation. For fixed $n > 0$ and $g \geq 0$, the quantum $L_\infty$ relations $(\mu^\text{sym} + \hbar \mathfrak{L})^2 = 0$ can be represented as

$$\sum_{k+l=n} \sum_{\sigma \in S_{l,k}} (-)^{\sigma(\psi)} \mu^\text{sym}_{k+1,[g_1]}(\mu^\text{sym}_{l,[g_2]}(\psi_{\sigma(1)}, \ldots, \psi_{\sigma(l)}), \psi_{\sigma(l+1)}, \ldots, \psi_{\sigma(n)})$$

$$+ \frac{1}{2} \sum_{s \in \mathbb{Z}} \mu^\text{sym}_{n+2,[g-1]}(e_{-s}, e_{1+s}, \psi_1, \ldots, \psi_n) = 0. \quad (1.16)$$
As long as we consider an $A_\infty$ structure $\mu$ that can be represented by the form of (1.8), the (quantum) $A_\infty$ structure $\mu$ of commutative quantum field theory reduces the (quantum) $L_\infty$ structure $\mu^{\text{sym}}$ automatically just as (1.12). We end this section by giving two examples.

### 4 point amplitude

The amplitudes of Lagrangian field theory have a quantum $A_\infty$ structure. Let us consider the cubic action, which is (1.7) with $\mu_{i>2} = 0$. It can be a non-commutative field theory. We write $\mu^{-1}_1$ for a propagator of this theory. The 4 point amplitude $A_4$ is given by

$$A_4 \sim \langle \psi_0, \mu_2(\mu^{-1}_1\mu_2(\psi_1, \psi_2), \psi_3) \rangle + \langle \psi_0, \mu_2(\psi_1, \mu^{-1}_1\mu_2(\psi_2, \psi_3)) \rangle. \quad (1.17)$$

It consists of the $S$-channel and $T$-channel. When multiplications of space-time fields are commutative, $\mu$ reduces to $\mu^{\text{sym}}$ as (1.12). Then, the expression (1.17) reduces to

$$A_4 \sim \langle \psi_0, \mu_2^{\text{sym}}(\mu^{-1}_1\mu_2^{\text{sym}}(\psi_1, \psi_2), \psi_3) \rangle + \langle \psi_0, \mu_2^{\text{sym}}(\mu^{-1}_1\mu_2^{\text{sym}}(\psi_2, \psi_3), \psi_1) \rangle$$

$$+ \langle \psi_0, \mu_2^{\text{sym}}(\mu^{-1}_1\mu_2^{\text{sym}}(\psi_3, \psi_1), \psi_2) \rangle. \quad (1.18)$$

It consists of the $S$-channel, the $T$-channel and the $U$-channel. As is known, this is a 4 point amplitude of commutative Lagrangian field theory.

### Yang-Mills theory

Let us consider the $A_\infty$ structure of the ordinary Yang-Mills action $S[A] = -\frac{1}{2} \int \langle F, \star F \rangle$, which is a commutative Lagrangian field theory. Yang-Mills fields $A$ are Lie-algebra-value 1-forms. The first $A_\infty$ structure is given by the kinetic operator

$$\mu_1(A_1) = d \star d A_1, \quad (1.19)$$

where $d$ denotes the exterior differential and $\star$ denotes the Hodge dual operation. By casting the Yang-Mills action as the form of (1.3), vertices provides higher $A_\infty$ products

$$\mu_2(A_1, A_2) = d \star (A_1 \wedge A_2) - (\star d A_1) \wedge A_2 + A_1 \wedge (\star d A_2), \quad (1.20)$$

$$\mu_3(A_1, A_2, A_3) = A_1 \wedge (\star (A_2 \wedge A_3)) - (\star (A_1 \wedge A_2)) \wedge A_3, \quad (1.21)$$

where $\wedge$ denotes exterior products of forms. Note that this $\wedge$ is different from the symmetrized tensor product of (1.10) and should be replaced by the non-commutative star product for a non-commutative Yang-Mills theory. As a symmetrization of exterior products, we can consider the graded commutator of exterior products, $[A_1, A_2]_{\wedge} \equiv A_1 \wedge A_2 - (-)^{A_1 A_2} A_2 \wedge A_1$. We find

$$\mu^{\text{sym}}_2(A_1, A_2) = d \star [A_1, A_2]_{\wedge} - [\star d A_1, A_2]_{\wedge} + [A_1, \star d A_2]_{\wedge}, \quad (1.22)$$

$$\mu^{\text{sym}}_3(A_1, A_2, A_3) = [A_1, \star [A_2, A_3]_{\wedge}]_{\wedge} + [A_2, \star [A_3, A_1]_{\wedge}]_{\wedge} + [A_3, \star [A_1, A_2]_{\wedge}]_{\wedge}. \quad (1.23)$$

These are the $L_\infty$ structure of the Yang-Mills theory. These $A_\infty$ and $L_\infty$ structures are related by (1.12) each other. As is known, this is just a piece of the full $A_\infty$ (or $L_\infty$) structure of the BV master action for the Yang-Mills theory.
2 Path-integral as a morphism of BV

In this section, on the basis of the BV formalism, we show explicitly a statement that the homological perturbation performs the perturbative path-integral and discuss several properties effective theories have as a consequence of it. This statement would be known among experts and thus a review except for incidental details. We first explain that solving the BV master equation is equivalent to extracting the quantum $A_\infty$ structure intrinsic to each Lagrangian field theory. Next, we give a brief review of basic facts of the BV formalism which are related to properties of the path-integral. Then, we show the statement and properties of the effective $A_\infty$ structure. Note that quantum field theories without gauge degrees can be treated within the BV formalism. Although it trivially solves the BV master equation, it provides non-trivial results, which we explain.

The BV formalism is one of the most general and systematic prescription to quantize gauge theories, which enable us to treat open or redundant gauge algebras. As is known, some gauge-fixing is necessary to perform the path-integral in a given gauge theory, to which we can apply the BV formalism even if ordinary methods such as fixing-by-hand, deriving the Dirac bracket, brute-force computations and the BRST procedure do not work. To carry out (1.2), the action must provides a regular Hessian. Also, symmetries proportional to the equations of motion are redundant and must be taken into account. We thus introduce antifields $\psi^*_{\text{cl}}$, ghost fields $c$, antifields for ghosts $c^*$ and pairs of higher fields-antifields as much as needed,

$$S_{\text{cl}}[\psi_{\text{cl}}] \rightarrow S[\psi] = S_{\text{cl}}[\psi_{\text{cl}}] + \psi^*_{\text{cl}}(S, \psi) + c^* (S, c) + \cdots.$$ (2.1)

We write $\psi$ for the sum of all fields and antifields. This extended action $S[\psi]$ is called a BV master action, which can provide a regular Hessian. The BV master action $S[\psi]$ is a solution of the BV master equation

$$\hbar \Delta e^{S[\psi]} = \left[ \hbar \Delta S[\psi] + \frac{1}{2} \left( S[\psi], S[\psi] \right) \right] e^{S[\psi]} = 0.$$ (2.2)

The BV master equation guarantees that the theory is independent of gauge-fixing conditions and has no gauge anomaly arising from the measure factor of the path-integral. A gauge-fixing is carried out by choosing appropriate gauge-fixing fermions, which determines a Lagrangian submanifold.

We write $\psi_g$ and $\psi^*_g$ for fields and antifields having space-time ghost number $g$ and $-g - 1$ respectively: $\psi_0 \equiv \psi_{\text{cl}}, \psi^*_0 \equiv \psi^*_{\text{cl}}, \psi_1 \equiv c$ and $\psi^*_1 \equiv c^*$ for example. The BV Laplacian $\Delta$ is a second-order odd derivative, which is defined by

$$\Delta \equiv \sum_g (-)^g \frac{\partial}{\partial \psi_g} \frac{\partial}{\partial \psi^*_g} = \frac{\partial}{\partial \psi_{\text{cl}}} \frac{\partial}{\partial \psi^*_{\text{cl}}} - \frac{\partial}{\partial c} \frac{\partial}{\partial c^*} + \cdots.$$ (2.3)

It is a fundamental object in the BV formalism and has geometrical meaning. The BV bracket is defined by $(-)^F (F, G) \equiv \Delta (FG) - \langle \Delta F \rangle G - \langle - \rangle F \Delta G$, where $F$ and $G$ are any functionals of fields and antifields. The BV bracket can be cast as

$$(F, G) = \sum_g \left[ \frac{\partial_r F}{\partial \psi_g} \frac{\partial}{\partial \psi^*_g} - \frac{\partial_r F}{\partial \psi^*_g} \frac{\partial}{\partial \psi_g} \right].$$ (2.4)

Note that $\partial_r$ denotes the right derivative and it satisfies $\frac{\partial}{\partial \psi_g} F = (-)^{\phi(F+1)} \frac{\partial_r}{\partial \psi_g} F$. 7
2.1 $A_\infty$ structure of the BV master equation

Suppose that for a given Lagrangian field theory, its BV master action $S[\psi]$ was obtained by solving the BV master equation. When the theory consists of physical degrees only, the BV master action is the classical action itself. We start with a given BV master action $S$.

We first consider the simplest case. Suppose that a solution $S$ of the classical master equation $(S, S) = 0$ also solves the quantum master equation $\hbar \Delta S + \frac{1}{2}(S, S) = 0$ without any modification. Then, the derivative $(S, \psi_g^*)$ induces the cyclic $A_\infty$ structure $\mu$ as follows

\[
(S, \psi_g^*) = (-)^g \frac{\partial S}{\partial \psi_g} = -\sum_{n=1}^{\infty} \mu_n(\psi, ..., \psi) \bigg|_{-g},
\]

where $\mu_n(\psi, ..., \psi) \big|_{-g}$ denotes the restriction onto the space-time ghost number $-g$ sector. The BV master action $S[\psi]$ has neutral ghost number and the BV derivation $(S, \psi_g^*)$ has ghost number one, although $\psi = \sum \psi_g + \sum \psi_g^*$ has indefinite ghost number. The $A_\infty$ relation can be read from

\[
0 = (S, (S, \phi_g^*)) = \sum_n \sum_{k+l=n} \sum_{m=0}^k (\epsilon^{(\psi)}(\psi, ..., \psi, \mu(\psi, ..., \psi), \psi, ..., \psi) \bigg|_{m} 1^{-g}),
\]

where $\epsilon(\psi)$ denotes the sum of $\psi$’s ghost numbers appearing between $\mu_{k+1}$ and $\mu_l$. In terms of these $A_\infty$ products, the BV master action $S[\psi]$ can be always cast into the following form of homotopy Maurer-Cartan action,

\[
S[\psi] = -\frac{1}{2} \left\langle \psi, \mu_1 \psi \right\rangle - \sum_{n=2}^{\infty} \left\langle \psi, \mu_n(\psi, ..., \psi) \right\rangle.
\]

Note that the space-time ghost number is not a natural grading of the $A_\infty$ structure $\mu$. Since $\mu$ consists of kinetic operators and interacting vertices, $\mu$ has neutral ghost number.

Next, we consider a generic case. Suppose that a solution $S_{[0]}$ of the classical master equation $(S_{[0]}, S_{[0]}) = 0$ does not solve the quantum master equation $\hbar \Delta S_{[0]} \neq 0$. Then, we need to construct correcting terms $\hbar S_{[1]} + \hbar^2 S_{[2]} + \cdots$ such that $S \equiv S_{[0]} + \hbar S_{[1]} + \hbar^2 S_{[2]} + \cdots$ satisfies the quantum master equation $\hbar \Delta S + \frac{1}{2}(S, S) = 0$. In this case, the quantum BV master action $S$ induces the quantum $A_\infty$ structure $\mu_{n,[l]}$ as follows

\[
(S, \phi_g^*) = \frac{\partial S_{[0]}}{\partial \psi_g} + \sum_{l>0} \hbar^l \frac{\partial S_{[l]}}{\partial \psi_g} = \sum_n \left[ \mu_{n,[0]}(\psi, ..., \psi) + \sum_l \hbar^l \mu_{n,[l]}(\psi, ..., \psi) \bigg|_{-g} \right].
\]

The quantum BV master action $S$ provides a natural nilpotent operation $\Delta_S$ defined by

\[
\hbar \Delta_S \equiv \hbar \Delta + (S, ).
\]

The quantum $A_\infty$ relation is encoded in (2.9) as follows

\[
(\hbar \Delta_S)^2 \phi_g^* = \sum_{n,l} \left[ \hbar \sum_{s \leq 2} \sum_{i=0}^{n-i} \sum_{j=0}^{n-i} (\epsilon^{(s,i,j)}(\psi, ..., \psi, e_{-s}, \psi, ..., \psi, e_{1+s}, \psi, ..., \psi) \bigg|_{i} ) \mu_{n+2, [l-1]}(\psi, ..., \psi, \psi, ..., \psi, \psi, ..., \psi) \right.
\]

\[
+ \sum_{n_1+n_2=n \geq 0} \sum_{l_1+l_2=l} \sum_{m=0}^{n_1} (\epsilon^{(\psi)}(\psi, ..., \psi, \mu_{n_1+1,[l_1]}(\psi, ..., \psi, \mu_{n_2,[l_2]}(\psi, ..., \psi, \psi, ..., \psi) \bigg|_{m} ) ) \left. \right|_{1-g},
\]

The quantum $A_\infty$ structure is then defined by

\[
(\hbar \Delta_S)^{n+1} \phi_g^* = \sum_{n,l} \left[ \hbar \sum_{s \leq 2} \sum_{i=0}^{n-i} \sum_{j=0}^{n-i} (\epsilon^{(s,i,j)}(\psi, ..., \psi, e_{-s}, \psi, ..., \psi, e_{1+s}, \psi, ..., \psi) \bigg|_{i} ) \mu_{n+2, [l-1]}(\psi, ..., \psi, \psi, ..., \psi, \psi, ..., \psi) \right.
\]

\[
+ \sum_{n_1+n_2=n \geq 0} \sum_{l_1+l_2=l} \sum_{m=0}^{n_1} (\epsilon^{(\psi)}(\psi, ..., \psi, \mu_{n_1+1,[l_1]}(\psi, ..., \psi, \mu_{n_2,[l_2]}(\psi, ..., \psi, \psi, ..., \psi) \bigg|_{m} ) \left. \right|_{1-g}. \]
where the sign factor $\epsilon(s,i,j)$ arises from $e_{1+s}$ passing $\psi^{\otimes (i+j)}$ and $e_{-s}$ passing $\psi^{\otimes i}$. For $s \geq 0$, these $e_{-s}$ and $e_{1+s}$ are defined by $e_{-s} \equiv \frac{\partial}{\partial \psi^s}$ and $e_{1+s} \equiv (-)^s \partial_{\psi^s}$ respectively. They enable us to get the following useful representation

$$\Delta \mu_{n,[l]}(\ldots) = \sum_{s \in \mathbb{Z}} (-)^s \epsilon(s) \mu_{n,[l]}(\ldots, e_{-s}, \ldots, e_{1+s}, \ldots). \quad (2.11)$$

Note that the condition $(\hbar \Delta S)^2 = 0$ is equivalent to the BV master equation (2.2). Hence, a solution of the BV master equation assigns a quantum $A_{\infty}$ structure to each Lagrangian field theory. In terms of these quantum $A_{\infty}$ products, the quantum BV master action $S[\psi]$ can be cast into the form of homotopy Maurer-Cartan action

$$S[\psi] = -S_{[0]}[\psi] - \sum_{n,l} \hbar l \mu_{n,[l]}(\psi, \mu_{n,[l]}(\psi, \ldots, \psi)),$$

where the classical master action $S_{[0]}[\psi]$ takes the same form as (2.7) with $\mu_{n,[0]} \equiv \mu_n$.

### 2.2 On the BV differential

In the classical theory, a solution of the equations of motion determines physical states up to gauge degrees. These information are encoded into the classical BV differential

$$Q_S \equiv (S, \quad) \quad (2.13)$$

acting on the space $\mathcal{F}(\mathcal{H})$ spanned by functionals of fields and antifields $\psi \in \mathcal{H}$. Note that $\mathcal{F}(\mathcal{H})$ includes the space of polynomials of space-time-field multiplications, which is identified with the tensor algebra $\mathcal{T}(\mathcal{H})$ in this paper. For a given master action (2.12), the equation of motion for the field $\psi_g$ can be represented by using the BV differential and its antifield $\psi^*_g$ as follows

$$0 = (-)^g \frac{\partial S}{\partial \psi^*_g} = \sum_n \mu_n(\phi, \ldots, \phi)\big|_{-g} = Q_S \psi^*_g. \quad (2.14)$$

It implies that the on-shell states are $Q_S$-closed. Likewise, the gauge transformation of the master action implies that its gauge degrees are $Q_S$-exact. The space $\mathcal{H}_{\text{phys}}$ of the physical states are described by the $Q_S$-cohomology. The observables $F$ are functionals of physical states, which we write $F \in \mathcal{F}(\mathcal{H}_{\text{phys}})$. Solving the classical theory is equivalent to finding the cohomology of complex with the classical BV differential (2.13),

$$\left( \mathcal{F}(\mathcal{H}), Q_S \right) \xrightarrow{p} \left( \mathcal{F}(\mathcal{H}_{\text{phys}}), 0 \right),$$

where $p$ denotes a restriction to on-shell and $i$ denotes an embedding to off-shell.

In the quantum theory, the stationary point of (2.14) does not completely determine the physical states. In addition to solve (2.14), we need to replace functionals $F$ of physical states by their expectation values $\langle F \rangle$, which is given by the path-integral

$$F \xrightarrow{P} \langle F \rangle \equiv \int \mathcal{D}[\psi] F e^{S[\psi]}.$$
Solving the BV master equation was necessary to define a regular Hessian for $S[\psi]$. As the case of $F = 1$, the integrand $F e^S$ must be $\Delta$-closed in order to obtain the gauge independent path-integral. Hence, for a given theory $S[\psi]$, its observables $F = F[\psi]$ satisfy

$$\hbar \Delta_S F[\psi] = 0.$$  (2.17)

The equation of motions can be also cast as $\hbar \Delta_S \psi = 0$. Note however that the $\Delta_S$-exact transformation, such as $\delta \psi = \hbar \Delta_S \epsilon$, is not the invariance of the theory defined by the above path-integral. For example, the $\Delta_S$-exact deformation $F \mapsto F + \Delta_S \Lambda$ does not change the expectation value $\langle F \rangle$ because of $\int D[\psi] \Delta(... = 0$. In this sense, the physics of quantizable Lagrangian field theory is described by the $\Delta_S$-cohomology,

$$\begin{align*}
(\mathcal{F}(\mathcal{H}), \hbar \Delta + Q_S) & \xrightarrow{P} (\mathcal{F}(\mathcal{H}_{\text{q-phys}}), 0) \xrightarrow{i_{\text{ev}}} (\mathcal{F}(\mathcal{H}_{\text{phys}}), 0),
\end{align*}$$  (2.18)

where $\mathcal{H}_{\text{q-phys}}$ denotes the space of physical state in the quantum theory. While $p$ and $i$ of (2.15) denote restriction and embedding to on-shell and off-shell respectively, $\text{ev}$ and $i$ of (2.18) denote substituting the expectation values and returning values to variables respectively. The path-integral $P$ should be identified with $P = \text{ev} \circ p$, the composition of $p$ and $\text{ev}$, because it not only gives the expectation value but also condenses field configurations onto the stationary point.

In this paper, we consider the perturbative path-integral (1.6), which is written in terms of the free theory (1.3). For the perturbative path-integral, there would be two options for realizing $\text{ev}$ of (2.18). The first option is to identify $\text{ev} : \mathcal{F}(\mathcal{H}_{\text{phys}}) \rightarrow \mathcal{F}(\mathcal{H}_{\text{q-phys}})$ with imposing

$$\Delta_{\text{free}} F[\psi] = 0$$  (2.19)

for any $F[\psi] \in \mathcal{F}(\mathcal{H}_{\text{phys}})$. The second option is to identify $\text{ev}$ with performing the Gaussian integral (1.3). While $p$ gives $p(\psi) = \psi_{\text{phys}}$ for $\psi = \psi_{\text{phys}} + \psi_{\text{gauge}} + \psi_{\text{unphys}}$, the map $\text{ev}$ evaluates the expectation value of a free physical field $\psi_{\text{phys}} \in \mathcal{H}_{\text{phys}}$, namely,

$$\text{ev} (\psi_{\text{phys}}) = 0,$$  (2.20)

which comes from $\langle \psi_{\text{phys}} \rangle_{\text{free}} = 0$. We may write $\psi_{\text{ev}} = i(\langle \psi_{\text{phys}} \rangle)$ and $\text{ev}(\psi_{\text{phys}}) = \langle \psi_{\text{phys}} \rangle$ with $\text{ev} \circ i = 1$, although both of them are 0 for the perturbative path-integral. As we will see, both of (2.19) and (2.20) provide the perturbative path-integral map $P$. The equivalence of these two options comes from the fact that the Gaussian integral (1.3) can be understood as a result of the homological perturbation for the BV differential of the free theory.

As we see later, we do not need to require (2.19) or (2.20) explicitly when we consider the path-integral to obtain $S$-matrix or to remove gauge and unphysical degrees. The condition (2.19) or (2.20) should be imposed explicitly when we consider the path-integral of fields whose momentum are higher than some cut-off scale or the path-integral of all massive fields for example.

### 2.3 Path-integral preserves the BV master equation

Let us consider a BV master action $S[\psi]$. We split fields $\psi$ into two components $\psi'$ and $\psi''$,

$$\psi = \psi' + \psi''.$$  (2.21)
By performing the path-integral of the fields $\psi''$, we obtain the BV effective action $A[\psi']$ from the original BV action $S[\psi' + \psi'']$. The effective action can be written as follows

$$A[\psi'] \equiv \ln \int D[\psi''] e^{S[\psi' + \psi'']},$$

(2.22)

where $\psi'$ is independent of $\psi''$ but $A[\psi']$ depends on the on-shell of $\psi''$. In general, if there exist interactions between $\psi'$ and $\psi''$, the path-integral of $\psi''$ may impose constraints arising from $\frac{\partial S}{\partial \psi''} = 0$ on the on-shell of remaining fields $\psi'$, which is described by (2.22). It is well-known that the BV effective action $A[\psi']$ also solves the BV master equation

$$\hbar \Delta' A[\psi'] + \frac{1}{2} \left( A[\psi'], A[\psi'] \right)' = 0.$$  

(2.23)

The effective BV Laplacians $\Delta'$ and $\Delta''$ satisfying $\Delta = \Delta' + \Delta''$ are defined by

$$\Delta' \equiv \sum (-)^g \left. \frac{\partial}{\partial \psi'^*_g} \frac{\partial}{\partial \psi^*_g} \right|_{\psi''}, \quad \Delta'' \equiv \sum (-)^g \left. \frac{\partial}{\partial \psi'^*_g} \frac{\partial}{\partial \psi''_g} \right|_{\psi'}.$$  

(2.24)

As $\Delta$ provides the BV master equation (2.24), the effective BV Laplacian $\Delta'$ also provides the effective BV bracket $(-)^A(A, B)' \equiv \Delta'(AB) - (\Delta'A)B - (-)^A A(\Delta'B)$. Because of the effective BV master equation (2.22), the operator $\hbar \Delta'' A \equiv \hbar \Delta' + (A, ')$ satisfies $(\hbar \Delta'' A)^2 = 0$. Hence, the effective action has a quantum $A_{\infty}$ structure $\mu'$ and takes the homotopy Maurer-Cartan form

$$A[\psi'] = -\sum_{n} \frac{1}{n+1} \langle \psi', \mu_{n,1}(\psi', \ldots, \psi') \rangle - \sum_{n,l} \frac{\hbar^l}{n+1} \langle \psi', \mu_{n,1,l}(\psi', \ldots, \psi') \rangle.$$  

(2.25)

This fact implies that the path-integral of fields $\psi''$ gives a morphism $P$ preserving the BV master equation such that

$$P \Delta_S = \Delta'_A.$$

(2.26)

As long as the original action $S[\psi]$ satisfies the BV master equation, these operators $\Delta_S$ and $\Delta'_A$ are nilpotent and the morphism $P$ preserves the cohomology. Because of $\mu(\psi, \ldots, \psi)|_{-g} = \Delta_S(\psi^*_g)$ and $\mu'(\psi', \ldots, \psi')|_{-g} = \Delta'_A(\psi'^*_g)$, as we will explain in section 3, this $P$ induces a morphism $p$ between these $A_{\infty}$ structures $\mu$ and $\mu'$ such that

$$p(\mu_1 + \mu_2 + \cdots) = (\mu'_1 + \mu'_2 + \cdots) p.$$  

(2.27)

In the rest of this section, we show that the path-integral can be understood as a morphism $P$ preserving the cohomology of the BV differentials,

$$\left( \mathcal{F}(\mathcal{H}' \oplus \mathcal{H}''), \Delta_S \right) \xrightarrow{p} \left( \mathcal{F}(\mathcal{H}' \oplus \mathcal{H}'_{\text{phys}}), \Delta'_A \right),$$  

(2.28)

where $\mathcal{H}'$ and $\mathcal{H}''$ denotes the state spaces of $\psi'$ and $\psi''$ respectively, $\mathcal{H}'_{\text{phys}}$ denotes the physical space of $\psi''$ on-shell. On the basis of the homological perturbation, we can construct this morphism $P$ explicitly and show that $P$ gives

$$P(F[\psi]) = \langle F[\psi'' + \psi''] \rangle'' \equiv Z^{-1}_{\psi''} \int D[\psi''] F[\psi] e^{S[\psi' + \psi'']} ,$$  

(2.29)

where $F[\psi]$ is any functional of fields $\psi$ and $Z_{\psi'}$ is defined by

$$Z_{\psi'} \equiv \int D[\psi''] e^{S[\psi' + \psi'']}.$$  

(2.30)
2.4 Homological perturbation performs the path-integral I

In this section, we construct a morphism \( \hat{P} \) performing the path-integral without normalization. The perturbative path-integral \( P \) is constructed by using this \( \hat{P} \) in the next section. We split the action \( S = S_{\text{free}} + S_{\text{int}} \) into the kinetic part \( S_{\text{free}} \) and interacting part \( S_{\text{int}} \). Since the perturbative path-integral is based on the free theory, we construct a map \( \hat{P} \) such that

\[
\hat{P}(e^{S_{\text{int}}[\psi']}) = \langle e^{S_{\text{int}}[\psi']} \rangle_{\text{free}} = \int \mathcal{D}[\psi'] e^{S[\psi']} ,
\]

where we set \( \psi' = 0 \) for simplicity. Clearly, such \( \hat{P} \) satisfies \( \hat{P}(1) = 1 \) as (\ref{2.3}) and describes the perturbative path-integral based on the free field theory.

We assume that the kinetic terms of \( \psi' \) and \( \psi'' \) have no cross term \( S_{\text{free}}[\psi' + \psi''] = S_{\text{free}}[\psi'] + S_{\text{free}}[\psi''] \). We also assume that the free theory of \( \psi'' \) is solved and takes

\[
S_{\text{free}}[\psi''] = \frac{1}{2} \langle \psi'', \mu_1'' \psi'' \rangle = \frac{1}{2} \langle \psi''_0, K_0 \psi''_0 \rangle + \sum_g \langle \psi''_{g-1}, K_g \psi''_g \rangle ,
\]

where \( \psi''_g \) is the \( g \)-th ghost field of \( \psi'' = \sum_g \psi'' + \sum_g \psi''^* \) and \( K_g \) is its kinetic operator. We write \( K_1^{-1} \) for a propagator of the kinetic operator \( K_g \). In order to derive the propagators, we need to add an appropriate gauge-fixing fermion into the action with trivial pairs. Note that we now consider the path-integral over a corresponding Lagrangian submanifold and thus \( \psi''^* \) should be understood as functionals of fields and trivial pairs determined by the gauge-fixing fermion.

Let us consider a projection \( \pi : \mathcal{H}'' \to \mathcal{H}''_{\text{phys}} \) onto the physical space of the \( \psi'' \) fields, in which the free equations of motion \( \mu'' \psi = 0 \) holds. We may represent \( \iota \pi = \iota \) by using a natural embedding \( \iota : \mathcal{H}''_{\text{phys}} \to \mathcal{H}'' \) satisfying \( \mu'' \iota = 0 \). We write \( K_1^{-1} = \sum_g K_g^{-1} \) and \( \mu_1'' = \sum_g K_g \) for brevity. Once \( K_1^{-1} \) is given, we get the (abstract) Hodge decomposition

\[
\mu_1'' K_1^{-1} + K_1^{-1} \mu_1'' = 1 - \iota \pi .
\]

We often impose the conditions \( \pi K_1^{-1} = 0, K_1^{-1} \iota = 0 \) and \( (K_1^{-1})^2 = 0 \), which is always possible without additional assumptions \cite{15}. As we see later, it is related to the \( i\epsilon \)-trick of ordinary Lagrangian field theory. The decomposition (\ref{2.33}) induces a homotopy contracting operator \( k_{\psi''}^{-1} \) for \( Q_{S_{\text{free}}[\psi'']} = (S_{\text{free}}[\psi''], ) \) and provides its BV version

\[
Q_{S_{\text{free}}[\psi'']} k_{\psi''}^{-1} + k_{\psi''}^{-1} Q_{S_{\text{free}}[\psi'']} = 1 - \iota \pi .
\]

Note that \( k_{\psi''}^{-1} \) decreases space-time ghost number 1 since \( Q_{S_{\text{free}}[\psi'']} \) increases 1. In terms of the kinetic operators \( K_g \) and their propagators \( K_g^{-1} \), these BV operations can be represented as

\[
Q_{S_{\text{free}}[\psi'']} = -K_0 \psi'' \frac{\partial}{\partial \psi''} - \sum_{g > 0} K_g \left[ \psi''_{g-1} \frac{\partial}{\partial \psi''} + \psi''_g \frac{\partial}{\partial \psi''_{g-1}} \right] ,
\]

\[
k_{\psi''}^{-1} = -\frac{K_0^{-1}}{n_0} \psi'' \frac{\partial}{\partial \psi''} - \sum_{g > 0} K_g^{-1} \left[ \psi''_g \frac{\partial}{\partial \psi''_{g-1}} + \psi''_{g-1} \frac{\partial}{\partial \psi''_g} \right] ,
\]
where \( n_0 \) and \( n_g \) are determined by the relation (2.34). In the above normalization, the operator \( n_g \) counts the \( \psi_g - \psi_{g-1} \) polynomial degree as \( n_g(\psi_g)^{\otimes m} (\psi_{g-1})^{\otimes 1} = (m+n)(\psi_g)^{\otimes m} (\psi_{g-1})^{\otimes n} \). Likewise, by identifying \( \psi_{g-1} = \psi^{*} \) for \( g \geq 0 \), we find \( n_0(\psi_0)^{\otimes m} (\psi^{*})^{\otimes n} = (m+n)(\psi_0)^{\otimes m} (\psi^{*})^{\otimes n} \) and \( n_g(\psi^{*})^{\otimes m} (\psi_g)^{\otimes n} = (m+n)(\psi^{*})^{\otimes m} (\psi_g)^{\otimes n} \). We thus obtain \( n_g(\psi)^{\otimes m} = n(\psi)^{\otimes n} \).

Now, we have the following homological data of the classical theory of free fields:

\[
k_{\psi}^{-1} \subset \left( \mathcal{F}(\mathcal{H}^\prime \oplus \mathcal{H}^\prime\prime), Q_{\text{free}[\psi]} + Q_{\text{free}[\psi^{*}]} \right) \xrightarrow{\pi} \left( \mathcal{F}(\mathcal{H}^\prime \oplus \mathcal{H}^\prime\prime_{\text{phys}}), Q_{\text{free}[\psi]} \right),
\]

which is called a deformation retract. Note that we must solve the equations of motion of \( \psi^{*} \) to specify \( \pi \) or \( \iota \). In order to define a propagator \( k_{\psi^{*}}^{-1} \), we have to specify the off-shell and carry out its gauge-fixing if \( \psi^{*} \) has any gauge or unphysical degree. Therefore, we must know how to solve the theory to obtain this homological data.

We expect that the perturbative path-integral (2.31) can be found by transferring the relation (2.37) into its quantum version (2.18) without interactions since (2.31) is an expectation value of the free theory. The homological perturbation lemma enables us to perform such a transfer of homological data. Clearly, we can take a perturbation \( \hbar \Delta \) since \( \hbar \Delta_{\text{free}[\psi^{*}]} = \hbar \Delta + Q_{\text{free}[\psi^{*}]} \) is nilpotent. As a result of the homological perturbation, we obtain a new deformation retract

\[
\hat{K}^{-1} \subset \left( \mathcal{F}(\mathcal{H}^\prime \oplus \mathcal{H}^\prime\prime), \hbar \Delta_{\text{free}[\psi^{*}]} \right) \xrightarrow{\hat{P}} \left( \mathcal{F}(\mathcal{H}^\prime \oplus \mathcal{H}^\prime\prime_{\text{phys}}), \hbar \Delta_{\text{free}[\psi]} \right),
\]

where morphisms \( \iota \) and \( \pi \) and a contracting homotopy \( k_{\psi^{*}}^{-1} \) of the initial data (2.37) are replaced by perturbed ones

\[
\hat{\iota} = (1 + k_{\psi^{*}}^{-1} \hbar \Delta)^{-1} \iota, \quad \hat{\pi} = \pi (1 + \hbar \Delta k_{\psi^{*}}^{-1})^{-1}, \quad \hat{K}^{-1} = k_{\psi^{*}}^{-1} (1 + \hbar \Delta k_{\psi^{*}}^{-1})^{-1}.
\]

These operators satisfy the abstract Hodge decomposition with \( \hbar \Delta_{\text{free}[\psi^{*}]} \) as (2.34) on the right side of (2.38). Note that \( \hat{\iota} = \iota \) follows from \( k_{\psi^{*}}^{-1} \Delta^\prime + \Delta^\prime k_{\psi^{*}}^{-1} = 0 \), \( (\Delta^\prime)^2 = (\Delta) \) \( = 0 \), and \( \hbar \Delta (\Delta^\prime) \iota = (\Delta^\prime) \iota = 0 \). On the right side of (2.38), a new differential operator is given by

\[
\hbar \Delta_{\text{free}[\psi^{*}]} = Q_{\text{free}[\psi^{*}]} + \pi \hbar \Delta \hat{\iota} = Q_{\text{free}[\psi^{*}]} + \hbar \pi \Delta \iota.
\]

Note that the differential \( \pi \Delta^\prime \iota \) must vanish on \( \mathcal{F}(\mathcal{H}^\prime\prime_{\text{phys}}) \) to obtain \( \Delta^\prime = \pi \Delta \iota \), which is automatic when we consider the path-integral of off-shell states or gauge-and-unphysical degrees.\(^5\) In order to consider the path-integral of physical fields, we assume (2.19) or (2.20) explicitly. While (2.19) gives \( \pi \hbar \Delta^\prime \iota = 0 \) directly, (2.20) replaces \( \pi \) by the composition \( \mathbf{ev} \circ \pi \) that provides \( \mathbf{ev} \circ \pi(\psi) = 0 \) for all \( \psi \in \mathcal{H} \) and removes physical degrees via (1.3).

We show that the above \( \hat{P} \) obtained as a result of the homological perturbation indeed realizes the perturbative path-integral (2.31). Note that when we impose \( \pi K^{-1} = 0 \) in (2.33), the operator \( k_{\psi^{*}}^{-1} \) commutes with \( \Delta^\prime \) and vanishes on \( \mathcal{H}^\prime\prime_{\text{phys}} \). We thus consider off-shell fields \( \pi \psi^{*} = 0 \), where \( \psi^{*} = \sum_{g \in \mathbb{Z}} \psi_{g}^{*} \) with \( \psi_{g}^{*} \equiv \psi_{g-1}^{*} \) having ghost number \( -g \). We find

\[
\hat{P}(\psi^{*} \otimes 2n) = \pi \left( \hbar \Delta k_{\psi^{*}}^{-1} \right)^{n} (\psi^{*} \otimes 2n) = \pi \frac{1}{n!} \left( \frac{\hbar}{2} \sum_{g} K_{g}^{-1} \frac{\partial}{\partial \psi_{g}^{*}} \frac{\partial}{\partial \psi_{g-1}^{*}} \right)^{n} (\psi^{*} \otimes 2n).
\]

---

\(^5\)For the Hodge decomposition \( \psi = \psi_{p} + \psi_{g} + \psi_{u} \), the BV Laplacian \( \Delta \) takes the form \( \frac{\partial}{\partial \psi_{p}} \frac{\partial}{\partial \psi_{p}} = \frac{\partial}{\partial \psi_{g}} \frac{\partial}{\partial \psi_{g}} + \frac{\partial}{\partial \psi_{u}} \frac{\partial}{\partial \psi_{u}} + \frac{\partial}{\partial \psi_{p}} \frac{\partial}{\partial \psi_{g}} + \frac{\partial}{\partial \psi_{p}} \frac{\partial}{\partial \psi_{u}} \). The physical term \( \pi \frac{\partial}{\partial \psi_{p}} \frac{\partial}{\partial \psi_{p}} \) remains unlike the other terms.
because of \( \pi (\hbar \Delta k_{\psi''}^{-1})^m (\psi'' \otimes 2^n) = 0 \) for \( m \neq n \) and

\[
\pi (\hbar \Delta k_{\psi''}^{-1})^n (\psi'' \otimes 2^n) = \pi (\hbar \Delta k_{\psi''}^{-1})^{n-1} \left( \frac{\hbar}{2n} \sum_g K_g^{-1} \frac{\partial}{\partial \psi''_g} \frac{\partial}{\partial \psi''_{-g}} \right) (\psi'' \otimes 2^n)
\]

\[
= \pi (\hbar \Delta k_{\psi''}^{-1})^{n-2} \frac{1}{n(n-1)} \left( \frac{\hbar}{2} \sum_g K_g^{-1} \frac{\partial}{\partial \psi''_g} \frac{\partial}{\partial \psi''_{-g}} \right)^2 (\psi'' \otimes 2^n) .
\]

(2.42)

It leads the Feynman graph expansion (1.4) thanks to \( \pi K^{-1} = 0 \). The condition \( \pi K^{-1} \) can be understood as the \( i\epsilon \)-trick for a propagator with \( \pi K^{-1} \neq 0 \). When we apply the \( i\epsilon \)-trick to the propagator in order for choosing a contour avoiding the on-shell poles, each operators of the Hodge decomposition (2.34) are \( i\epsilon \)-modified. Then we can impose \( \pi \psi'' = 0 \) on the original physical space \( \mathcal{H}_{\text{phys}}'' \) because the mass shell is \( i\epsilon \)-shifted and take \( \epsilon \to 0 \) after computations. We obtain

\[
\hat{P} (e^{S_{\text{int}}[\psi'+\psi'']} ) = \pi \sum_{n=0}^{\infty} (\hbar \Delta k_{\psi''}^{-1})^n (e^{S_{\text{int}}[\psi'+\psi'']} )
\]

\[
\equiv \pi \exp \left[ \frac{\hbar}{2} \sum_g K_g^{-1} \frac{\partial}{\partial \psi''_g} \frac{\partial}{\partial \psi''_{-g}} \right] (e^{S_{\text{int}}[\psi'+\psi'']} ) ,
\]

(2.43)

which gives a functional of on-shell states \( \pi \psi'' \in \mathcal{H}_{\text{phys}}'' \). If we introduce a source term \( e^{J\psi''} \) as (1.6), we can represent (2.43) by the formula

\[
\hat{P} (e^{S_{\text{int}}[\psi'+\psi'']+J\psi'']} ) = \exp \left[ \frac{\hbar}{2} \sum_g K_g^{-1} \frac{\partial}{\partial \psi''_g} \frac{\partial}{\partial \psi''_{-g}} \right] (e^{S_{\text{int}}[\psi'+\psi'']+J\psi'']} ) \bigg|_{\psi''=\pi \psi''} .
\]

(2.44)

These (2.43) and (2.44) are nothing but the Feynman graph expansion (1.6) in the perturbative quantum field theory. Note that it includes the term \( \frac{\hbar}{2} K_0^{-1} (\partial_{\psi'})^2 \) which consists of classical fields and their propagators, and thus it provides non-zero value after removing all antifields (and also ghosts) from (2.44). Hence, quantum field theory without gauge degree can be treat in this BV framework and provides non-trivial results after the homological perturbation, although its BV master action is the same as the classical action and the BV master equation looks trivial.

Note that although the condition (2.19) or (2.20) is not necessary to derive (2.43) or (2.44), we find that the formula (2.44) takes the completely same form as (1.6) under (2.20). Thus, (2.20) may provide more conventional calculations, rather than (2.19). Actually, when we calculate S-matrix with (2.20) providing \( (\text{ev} \circ \pi) \psi'' = 0 \) we need to use \( e^{\psi_{\text{as}} + \mu_1 \psi_{\text{phys}}} \) as usual. When we calculate S-matrix with (2.19), \( \pi \psi'' \) of (2.44) should be understood as \( \psi_{\text{as}} \) as we see in section 4.

2.5 Homological perturbation performs the path-integral II

In this section, we construct a morphism \( P \) performing the perturbative path-integral such that

\[
P(...) \equiv Z_{\psi'}^{-1} \hat{P}((...) e^{S[\psi'+\psi'']-S_{\text{free}}[\psi'']} ) \mid_{\text{ev}^{\text{as}}} \int \mathcal{D}[\psi'] (... ) e^{S[\psi'+\psi']} .
\]

(2.45)
We expect that it can be found by transferring the homological data of (2.37) into its fully quantum version including interactions. Again, the homological perturbation enables us to perform such a transfer. We can take $\hbar \Delta S_{\text{int}}[\psi] = \hbar \Delta + Q_{\text{int}}[\psi]$ as a perturbation since $\hbar \Delta S[\psi] = \hbar \Delta + Q_S[\psi]$ is nilpotent. After the perturbation, we obtain a new homological data as follows

$$K^{-1} \circ \left( \mathcal{F}(\mathcal{H}' \oplus \mathcal{H}''), \hbar \Delta S_{\psi'} + \psi'' \right) \xrightarrow{P} \left( \mathcal{F}(\mathcal{H}' \oplus \mathcal{H}'_{\text{phys}}), \hbar \Delta'_{A[\psi']} \right).$$

(2.46)

The perturbation lemma tell us how to construct morphisms $I$ and $P$ explicitly,

$$I = \left(1 + \kappa^{-1}_{\psi'} \hbar \Delta S_{\text{int}[\psi]}\right)^{-1}, \quad P = \pi \left(1 + \hbar \Delta_{\text{int}[\psi]} \kappa^{-1}_{\psi''}\right)^{-1}.$$

(2.47)

Likewise, a contracting homotopy for $\hbar \Delta S_{\psi}$ and the induced differential $\hbar \Delta'_{A[\psi']}$ are given by

$$K^{-1} = \kappa^{-1}_{\psi''} \left(1 + \hbar \Delta S_{\text{int}[\psi]} \kappa^{-1}_{\psi''}\right)^{-1} \quad \hbar \Delta'_{A[\psi']} = P \hbar \Delta S_{\psi} \iota = \pi \hbar \Delta S_{\psi} I.$$

(2.48)

Although we can prove the statement (2.45) by tedious but direct calculation using these, we follows a pedagogical approach given by [16]. See also [17] and [18].

As is known, the homological perturbation transfers a given deformation retract to a new deformation retract. It therefore enables us to obtain the new Hodge decomposition

$$(1 - I P) F = \left[\left(\hbar \Delta_{S[\psi]}\right) K^{-1} + K^{-1} (\hbar \Delta_{S[\psi]}^2)\right] F$$

(2.49)

for any $F[\psi] \in \mathcal{H}' \oplus \mathcal{H}''$. Note that since $[\kappa^{-1}_{\psi''}, \Delta] = \sum_g n_g^{-1} K_{g}^{-1} \partial_{\psi''_{\psi}} \partial_{\psi''}$ acts on the off-shell states satisfying $K \psi'' \neq 0$, it does not act on $\iota(...)$. Because of $\hat{P} I (...) = \hat{P} \iota (...) \pi \kappa^{-1}_{\psi''} = 0$, we find the following property of $I$ and $\hat{P}$,

$$\hat{P} \left( I (...) e^{S_{\text{int}[\psi'] + \psi''}} \right) = \pi \left[\left((1 + \hbar \Delta''_{\psi''} \kappa^{-1}_{\psi''}) e^{S_{\text{int}[\psi'] + \psi''}}\right) \iota (...)\right]$$

$$= \hat{P} \left(e^{S_{\text{int}[\psi'] + \psi''}}\right) \pi \iota (...) .$$

(2.50)

In other words, since $\pi \iota = 1$ on $\mathcal{H}''_{\text{phys}}$, we proved that $I(...) \pi \iota(...) \mathcal{H}''_{\text{phys}}$ passes the $\psi''$ integral as follows

$$Z_{\psi'}^{-1} \int D[\psi''] I (PF) e^{S[\psi' + \psi'']} = Z_{\psi'}^{-1} \int D[\psi''] \iota (PF) e^{S[\psi' + \psi'']} = PF .$$

(2.51)

The abstract Hodge decomposition (2.49) elucidates that our morphism $P$, a result of homological perturbation, indeed performs the perturbative path-integral as follows

$$P(F) = Z_{\psi'}^{-1} \int D[\psi''] F e^{S[\psi' + \psi'']} - Z_{\psi'}^{-1} (\text{extra}).$$

(2.52)

We show that the extra term vanishes

$$(\text{extra}) \equiv \hbar \hat{P} \left(\left(\Delta_{S[\psi]} K^{-1} F + K^{-1} \Delta_{S[\psi]} F\right) e^{S[\psi' + \psi''] - S_{\text{free}[\psi'']}}\right) = 0 .$$

(2.53)
The second term is trivially zero when we use $k^{−1}_\psi$ satisfying the subsidiary condition $(k^{−1}_\psi)^2 = 0$ and $\pi k^{−1}_\psi = 0$, which can be always imposed by dressing the old $k^{−1}_\psi$ without any additional condition $[13]$. Note that $(k^{−1}_\psi)^2 = 0$ gives \( \hat{P} K^{-1} = \pi K^{-1} \). Thus, $\pi k^{−1}_\psi = 0$ provides

\[
\hat{P} \left( K^{-1}(...) e^{S_{\text{int}}[\psi'] + \psi''} \right) = \pi \left[ k^{−1}_\psi (1 + \hbar \Delta s_{\text{int}}[\psi] k^{−1}_\psi)^{-1} (...) e^{S_{\text{int}}[\psi'] + \psi''} \right] = 0 .
\]

This fact implies that after the path-integral, as expected, the $K^{-1}$-exact quantities vanish

\[
\int \mathcal{D}[\psi''] K^{-1}(...) e^{S[\psi'] + \psi''} = 0 .
\]

Actually, the first term vanishes for similar reasons. The morphism \( \hat{P} \) satisfies \( \hat{P} \Delta_{s_{\text{free}}[\psi']} = \Delta' \hat{P} \) because of its defining properties \( \hat{P} \Delta_{s_{\text{free}}[\psi]} = \Delta' \hat{P} \) and \( \hat{P} e^{S_{\text{free}}[\psi']} = e^{S_{\text{free}}[\psi']} \hat{P} \). We find

\[
\hat{P} \left( [\Delta_{s[\psi]}(...)] e^{S[\psi'] - s_{\text{free}}[\psi'']} \right) = \hat{P} \left( \Delta_{s_{\text{free}}[\psi']} [(...)] e^{S[\psi] - s_{\text{free}}[\psi'']} \right) = \Delta' \hat{P} \left( [(...)] e^{S[\psi] - s_{\text{free}}[\psi'']} \right) .
\]

It implies that the $\psi''$ integrals maps the $\Delta_{s[\psi]}$-exacts into $\Delta'$-exact quantities,

\[
\int \mathcal{D}[\psi''] \Delta_{s[\psi]}(...) e^{S[\psi'] + \psi''} = \Delta' \left[ \int \mathcal{D}[\psi''] (...) e^{S[\psi'] + \psi''} \right] .
\]

After applying this property, the integrand of the first term becomes $K^{-1}$-exact and gives zero. Hence, the statement (2.45) is proved. Note also that because of $Z_{\psi'} P(1) = \hat{P}(1) = e^{A[\psi']}$, the relation (2.57) is nothing but the condition of morphism

\[
P \Delta_{s[\psi]} = Z^{-1}_{\psi'} \Delta' Z_{\psi'} P = \Delta'_{A[\psi']} P .
\]

2.6 $A_\infty$ structure of the effective theory

In the rest of this section, we explain several properties that effective theories have as a result of the homological perturbation. We consider the (quantum) $A_\infty$ structure of the effective theory,

\[
\mu'(\psi', \ldots, \psi') = \mu'_1(\psi') + \mu'_2(\psi', \ldots, \psi') ,
\]

which is given by $\mu'(\psi') \equiv \hbar \Delta'_{A[\psi']} \psi'$ for $\psi' = \sum_g [\psi'_g + \psi'^*_g]$. The $A_\infty$ structure of the effective theory can be obtained by calculating the perturbed BV differential $\hbar \Delta'_{A[\psi']}$. Since $k^{−1}_\psi$ commutes with $\Delta'$, we find that it takes

\[
\hbar \Delta'_{A[\psi']} = Q_{s_{\text{free}}[\psi]} + \pi \sum_n (\hbar \Delta''_{s_{\text{int}}[\psi'] + \psi''} k^{−1}_\psi)^n \hbar \Delta'_{s_{\text{int}}[\psi'] + \psi''} .
\]

Note that the commutator of the full perturbation $\hbar \Delta_{s_{\text{int}}[\psi]}$ and the propagator $k^{−1}_\psi$,\[
[\hbar \Delta_{s_{\text{int}}[\psi]}, k^{−1}_\psi] = \hbar \sum_g \frac{K_g}{n_g} \frac{-\partial}{\partial \psi'_g} \frac{\partial}{\partial \psi''_g} + \sum_g \frac{K_g}{n_g} \mu_{\text{int}}(\psi, \ldots, \psi'_g) \frac{\partial}{\partial \psi''_g} ,
\]

(2.60)
naturally includes the loop operator $L_{\psi''\psi''}$ and the tree grafting operator $T_{\psi''}$ defined by
\[
\hbar L_{\psi''\psi''} \equiv \hbar \sum_g \frac{K^{-1}_g}{n_g} \frac{\partial}{\partial \bar{\psi}''_g} \frac{\partial}{\partial \psi''_g}, \quad T_{\psi''} \equiv \sum_g \frac{K^{-1}_g}{n_g} \mu_{\text{int}}(\psi'', \ldots, \psi'' + \psi'') \frac{\partial}{\partial \bar{\psi}''_g}. \tag{2.62}
\]

These provide basic manipulations of the $\psi''$ Feynman graphs as follows
\[
L_{\psi''\psi''} \mu_{n+2}(\psi, \ldots, \psi) \big|_{\psi''=0} = \frac{1}{2} \sum_{s \in \mathbb{Z}} \sum_{i,j} \mu_{n+2} \left( \psi_s', \ldots, \psi_s', K_s^{-1}e_s, \psi_s', \ldots, e_s, \psi_s', \ldots, \psi_s' \right), \tag{2.63}
\]
\[
T_{\psi''} \mu_{n+1}(\psi, \ldots, \psi) \big|_{\psi''=0} = \sum_k \mu_{n+1} \left( \psi'_k, \ldots, \psi'_k, \mu_{\text{int}}(\psi'_k, \ldots, \psi'_k), \psi'_n, \ldots, \psi'_n \right). \tag{2.64}
\]

Note that since $n_g \psi'' \otimes m = m \psi'' \otimes m$, each graph has appropriate coefficient, such as
\[
T_{\psi''} T_{\psi''} \mu_{\text{int}} \big|_{\psi''=0} = \sum \mu_{\text{int}} \left( \ldots, K^{-1} \mu_{\text{int}} \ldots, K^{-1} \mu_{\text{int}} \ldots \right) + 2 \sum \frac{1}{2} \mu_{\text{int}} \ldots, K^{-1} \mu_{\text{int}} \ldots, K^{-1} \mu_{\text{int}} \ldots. \tag{2.65}
\]

We write $\pi(\psi' + \psi'') = \varphi$ and $\iota(\varphi) = \psi'$, although $\varphi = \psi'$ for our perturbative path-integral. By acting $\hbar \Delta'_{A[\varphi]}$ on $\varphi$, we obtain the quantum $A_\infty$ structure of the effective theory as follows
\[
\mu'(\varphi, \ldots, \varphi) = \mu_1(\varphi) + \sum_{n=0}^{\infty} \left[ \hbar L_{\psi'\psi'} + T_{\psi''} \right]^{n} \mu_{\text{int}}(\varphi + \psi'_e, \ldots, \varphi + \psi'_e) \bigg|_{\psi'_e=0}. \tag{2.66}
\]

Note that the effective vertices $\mu'_{\text{int}} = \mu'_2 + \mu'_3 + \cdots$ have the $\hbar$ dependent parts,
\[
\mu'_n(\varphi, \ldots, \varphi) = \mu'_{n[0]}(\varphi, \ldots, \varphi) + \hbar \mu'_{n[1]}(\varphi, \ldots, \varphi) + \hbar^2 \mu'_{n[2]}(\varphi, \ldots, \varphi) + \cdots \tag{2.67}
\]

We consider $\varphi(t)$ such that $\varphi(0) = 0$ and $\varphi(1) = \varphi$ for $t \in \mathbb{R}$. As a functional of $\varphi$, by using $\varphi(t)$, the effective action (2.25) can be cast as
\[
A[\varphi] = - \int_0^1 dt \left\langle \partial_t \varphi(t), \mu' \left( \varphi(t), \ldots, \varphi(t) \right) \right\rangle. \tag{2.68}
\]

### 2.7 The classical limit and cyclic $A_\infty$

The classical part of the effective theory has a cyclic $A_\infty$ structure. The effective $A_\infty$ structure (2.66) has the non-trivial classical limit $\mu'_{\text{tree}} = \lim_{\hbar \to 0} \mu'$, which is obtained by setting $\hbar \to 0$ in (2.67) as follows,
\[
\mu'_{\text{tree}}(\varphi, \ldots, \varphi) = \sum_{n=0}^{\infty} \left[ \sum_g \frac{K^{-1}_g}{n_g} \mu_{\text{int}}(\varphi + \psi''_e, \ldots, \varphi + \psi''_e) \right]^{n} \mu(\varphi + \psi''_e) \bigg|_{\psi''_e=0}. \tag{2.69}
\]
We write $A_{\text{tree}}[\varphi]$ for the classical part of the effective action $A[\varphi]$, which consists of tree graphs only. By construction of (2.66) and $\mu'_{\text{tree}}(\varphi) = Q_{A_{\text{tree}}[\varphi]} \varphi$, we find

$$Q_{A_{\text{tree}}[\varphi]} = \pi Q S[\varphi] I_{\text{tree}} = P_{\text{tree}} Q S[\varphi] t,$$

where $I_{\text{tree}}$ and $P_{\text{tree}}$ are the classical limits of $I$ and $P$ respectively,

$$I_{\text{tree}} = \left(1 + k_{\psi'}^{-1} Q S_{\text{int}}[\varphi]\right)^{-1} t, \quad P_{\text{tree}} = \pi \left(1 + Q S_{\text{int}}[\varphi] k_{\psi'}^{-1}\right)^{-1}.$$

We can obtain these classical limits as a result of the perturbation $Q S_{\text{int}}[\varphi]$ to (2.37),

$$K_{\text{tree}}^{-1} \left(\mathcal{F}(\mathcal{H}' + \mathcal{H}''), Q S[\varphi]\right) \xrightarrow{P_{\text{tree}}}{_{I_{\text{tree}}} \left(\mathcal{F}(\mathcal{H}' + \mathcal{H}_{\text{phys}}''), Q A[\varphi]\right).}$$

This fact implies that a morphism $P_{\text{tree}}$ performs the classical part of the perturbative path-integral, or the Feynman graph expansion grafting only trees, as follows

$$P_{\text{tree}}(...) = (Z_{\varphi}^{\text{tree}})^{-1} \lim_{h \to 0} \int D[\psi''(\ldots)] e^{S[\psi' + \psi'']}.$$

We assumed that the perturbative partition function $Z_{\varphi}$ splits into the tree and loop parts,

$$Z_{\varphi} = Z_{\varphi}^{\text{tree}}, Z_{\varphi}^{\text{loop}}, \quad Z_{\varphi}^{\text{tree}} \equiv e^{A_{\text{tree}}[\varphi]}.$$

Thus, if we interested in the tree part only, the classical perturbation (2.72) is enough. Actually, by using these $I_{\text{tree}}, P_{\text{tree}}$, a first few terms of (2.69) are also calculated as follows

$$\mu'_{\text{tree},1}(\varphi) = \mu_1(\varphi), \quad \mu'_{\text{tree},2}(\varphi, \varphi) = \mu_2(\varphi, \varphi),$$
$$\mu'_{\text{tree},3}(\varphi, \varphi, \varphi) = \mu_3(\varphi, \varphi, \varphi) + \mu_2(K^{-1} \mu_2(\varphi, \varphi), \varphi) + \mu_2(\varphi, K^{-1} \mu_2(\varphi, \varphi)),$$
$$\mu'_{\text{tree},4}(\varphi, \varphi, \varphi) = \mu_4(\varphi, \varphi, \varphi) + \sum \mu_3(\varphi, \varphi, K^{-1} \mu_2(\varphi, \varphi)) + \mu_2(K^{-1} \mu_2(\varphi, \varphi), K^{-1} \mu_2(\varphi, \varphi))$$
$$+ \sum \mu_2(\varphi, K^{-1} \mu_3(\varphi, \varphi, \varphi)) + \sum \mu_2(\varphi, K^{-1} \sum \mu_2(\varphi, K^{-1} \mu_2(\varphi, \varphi))),$$

where $\Sigma$ denotes the cyclic sum. Note that as we mentioned in (2.63), the propagator $k_{\psi''}^{-1}$ adjusts the coefficients and the restricting $\psi''_{\text{ev}} = 0$ picks up the correct ones in the calculations.

## 3 Path-integral as a morphism of $A_{\infty}$

All results obtained in section 2 can be written in terms of the (quantum) $A_{\infty}$ algebras and morphisms of $A_{\infty}$ directly. In this section, we explicitly construct a morphism $p$ between two $A_{\infty}$ structures $\mu$ and $\mu'$ such that

$$p(\mu_1 + \mu_2 + \cdots) = (\mu'_1 + \mu'_2 + \cdots) p,$$

where $\mu = \mu_1 + \mu_2 + \cdots$ and $\mu' = \mu'_1 + \mu'_2 + \cdots$ are (higher order) differentials acting on tensor algebras. The perturbative path-integral map $P$ induces such $p$, which we explain.
3.1 Tensor trick

In order to extract $A_\infty$ products from (2.34), we consider the state space $\mathcal{H}$ instead of $\mathcal{F}(\mathcal{H})$, on which $Q_{\text{free}[\psi]}\psi = \mu_1(\psi)$ and $\kappa^{-1}_\psi \psi = K^{-1}(\psi)$ hold. For brevity, we write (2.37) as

$$\kappa^{-1}_{\psi'} \odot \left( \mathcal{H}, \mu_1 \right) \xrightarrow{\pi} \left( \mathcal{H}', \mu'_1 \right).$$

(3.2)

By applying the tensor trick to each component of (3.2), we can obtain corresponding deformation retract of tensor algebras. The identity map $\mathbb{I}$ of $\mathcal{H}$ and morphisms $\pi$ and $\iota$ can be extended to the identity $1 = 1_{\mathcal{T}(\mathcal{H})}$ of $\mathcal{T}(\mathcal{H})$ and morphisms $\pi$ and $\iota$ of tensor algebras by defining

$$1|_{\mathcal{H}^\otimes n} = (\mathbb{I})^\otimes n, \quad \pi|_{\mathcal{H}^\otimes n} \equiv (\pi)^\otimes n, \quad \iota|_{\mathcal{H}^\otimes n} \equiv (\iota)^\otimes n.$$ 

(3.3)

These are morphisms of tensor algebra preserving the cohomology

$$\pi (1 \otimes 1) = \pi \otimes \pi, \quad \iota (1 \otimes 1) = \iota \otimes \iota,$$

(3.4)

where $\otimes$ is the product $\otimes : \mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ of the tensor algebra. The tensor algebra $\mathcal{T}(\mathcal{H})$ can be regarded as a coalgebra. Note that these $\pi$ and $\iota$ are also coalgebra morphisms

$$\Delta \pi = (\pi \otimes \pi) \Delta, \quad \Delta \iota = (\iota \otimes \iota) \Delta,$$

(3.5)

where $\Delta$ denotes the coproduct $\Delta : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{H})$ of coalgebra. The $k$-linear map $\mu_k : \mathcal{H}^\otimes k \rightarrow \mathcal{H}$ can be extended to a linear map $\mu_k$ acting on the tensor algebra, which becomes a derivation of $\mathcal{T}(\mathcal{H})$, and the contracting homotopy $\kappa^{-1}_\psi$ between $\mathbb{I}$ and $\iota \pi$ becomes a homotopy $\kappa^{-1}$ between two morphisms $1$ and $\iota \pi$ of $\mathcal{T}(\mathcal{H})$ by defining

$$\mu_k|_{\mathcal{H}^\otimes n} \equiv \sum_l \mathbb{I}^\otimes n-l \otimes \mu_k \otimes \mathbb{I}^\otimes l-k, \quad \kappa^{-1}|_{\mathcal{H}^\otimes n} \equiv \sum_l \mathbb{I}^\otimes n-l-1 \otimes \kappa^{-1}_\psi \otimes (\iota \pi)^\otimes l.$$ 

(3.6)

While $\mu_k$ is a derivation of the tensor algebra, $\kappa^{-1}$ is a $(1, \iota \pi)$-derivation as follows

$$\mu_k (1 \otimes 1) = \mu_k \otimes 1 + 1 \otimes \mu_k, \quad \kappa^{-1} (1 \otimes 1) = \kappa^{-1} \otimes \iota \pi + 1 \otimes \kappa^{-1}.$$ 

(3.7)

Note that derivation $\mu_k$ is also a coderivation and $\kappa^{-1}$ is also a $(1, \iota \pi)$-coderivation

$$\Delta \mu_k (1 \otimes 1) = (\mu_k \otimes 1 + 1 \otimes \mu_k) \Delta, \quad \Delta \kappa^{-1} = (\kappa^{-1} \otimes \iota \pi + 1 \otimes \kappa^{-1}) \Delta,$$

(3.8)

and thus $\kappa^{-1}$ satisfies the characteristic property with the coproduct $\Delta$ as follows

$$(\kappa^{-1} \otimes 1 - 1 \otimes \kappa^{-1}) \Delta \kappa^{-1} = (\kappa^{-1} \otimes \kappa^{-1}) \Delta.$$ 

(3.9)

We obtain the abstract Hodge decomposition on $\mathcal{T}(\mathcal{H})$

$$1 - \iota \pi = \mu_1 \kappa^{-1} + \kappa^{-1} \mu_1,$$

(3.10)

and thus we can consider a deformation retract of tensor algebras, induced from (3.2),

$$\kappa^{-1} \odot \left( \mathcal{T}(\mathcal{H}), \mu_1 \right) \xrightarrow{\pi} \left( \mathcal{T}(\mathcal{H}'), \mu'_1 \right),$$

(3.11)

which can be also regarded as a deformation retract of coalgebras. The similar construction can be applied to (2.38) or (2.72). Note that $\pi$ and $\iota$ are $A_\infty$ morphisms such that

$$\pi \mu_1 = \mu'_1 \pi, \quad \mu_1 \iota = \iota \mu'_1.$$ 

(3.12)
3.2 Adjusting $A_\infty$ degree

Since the $g$-th ghost fields $\psi_g$ have space-time ghost number $g$ and their antifields $\psi_g^*$ has $-1-g$ respectively, the sum of them $\psi = \sum_g \psi_g + \psi_g^*$ includes odd and even fields. One way to remedy a natural $A_\infty$ degree is to introduce string-field-inspired basis $\{\hat{e}_{-g}, \hat{e}_{-g}^*\}_g$ for given fields-antifields $\{\psi_g, \psi_g^*\}_g$. These basis have Grassman parity, which we call basis ghost number. The sum of the space-time and basis ghost numbers gives the $A_\infty$ degree. We assign a base $\hat{e}_{-g}$ having Grassmann parity $-g$ for each field $\psi_g$ having space-time ghost number $g$ such that the total states such as $\Psi_g \equiv \hat{e}_{-g} \otimes \psi_g$ and $\Psi_g^* \equiv \hat{e}_{-g}^* \otimes \psi_g^*$, become $A_\infty$ degree zero as follows

$$\Psi \equiv \sum \hat{e}_{-g} \otimes \psi_g + \sum \hat{e}_{-g}^* \otimes \psi_g^*, \quad (3.13)$$

It determines the basis ghost number of $\hat{e}_{-g}^*$ to be $1+g$. For $g \geq 0$, we may write

$$\psi_{1-g} \equiv \psi_g^*, \quad \psi_{1-g}^* \equiv \psi_g, \quad (\psi_g^*)^* \equiv \psi_g, \quad (3.14)$$

$$\hat{e}_{1+g} \equiv (-)^g \hat{e}_{-g}^*, \quad \hat{e}_{1+g}^* \equiv (-)^g \hat{e}_{-g}, \quad (e_{-g}^*)^* \equiv e_{-g}, \quad (3.15)$$

$$\Psi_{1-g} \equiv (-)^g \Psi_g^*, \quad \Psi_{1-g}^* \equiv (-)^g \Psi_g, \quad (\Psi_g^*)^* \equiv \Psi_g. \quad (3.16)$$

Let $\hat{H} \equiv E \otimes H$ be the state space of $(3.13)$, where $E$ denotes the space of basis. We can define an $A_\infty$ map $\mu_n : \hat{H}^\otimes n \to \hat{H}$ of degree 1 by

$$\mu_n(\Psi_{g_1} \otimes \cdots \otimes \Psi_{g_n}) \big|_{-g} \equiv -\hbar \Delta S[\psi] \Psi_{1-g} \equiv (-)^{g+1} \hat{e}_{1+g} \otimes \mu_n(\psi_{g_1}, \ldots, \psi_{g_n}) \big|_{-g}. \quad (3.17)$$

Note that $\mu$ has no ghost number, $e_{1+g}$ has basis ghost number $1+g$, space-time fields $\psi_{g_1} \otimes \cdots \otimes \psi_{g_n}$ carry space-time ghost number $-g = \sum_{k=1}^n g_k$, and hence the $A_\infty$ structure $\mu$ on $T(\hat{H})$ indeed has degree 1. The sign factor of $\mu$’s $A_\infty$ relations is simpler that $(2.10)$.

As we see, the base $\hat{e}_{-g}^*$ is determined for each pair of given input-fields $\psi_{g_1}, \ldots, \psi_{g_n}$ such that it compatible with the BV master action $(2.12)$.

We introduce an inner product $\langle \hat{e}_{-g}, \hat{e}_{-g}^* \rangle_E$ and a symplectic form $\omega(\hat{e}_{-g}, \hat{e}_{-g}^* \rangle_E$ as follows

$$\langle \hat{e}_{-g}, \hat{e}_{-g}^* \rangle_E = (-)^g \delta_{g,g'}, \quad \omega(\hat{e}_{-g}, \hat{e}_{1-g}) \equiv (-)^g \langle \hat{e}_{-g}, \hat{e}_{1+g} \rangle_E = (-)^g \delta_{g,g'}, \quad (3.18)$$

and define a symplectic form $\omega : \hat{H}^\otimes 2 \to \mathbb{C}$ satisfying $\omega (A, B) = -(-)^{AB} \omega (B, A)$ as follows

$$\omega(\hat{e}_{p_1} \otimes \psi_{g_1}, \hat{e}_{p_2} \otimes \Psi_{g_2}) \equiv -(-)^{p_2 g_1} \omega(\hat{e}_{p_1}, \hat{e}_{p_2}) \langle \psi_{g_1}, \psi_{g_2} \rangle. \quad (3.19)$$

The $A_\infty$ structure $\mu$ on $T(\hat{H})$ is determined such that the homotopy Maurer-Cartan action

$$\hat{S}[\Psi] = -\sum_n \frac{1}{n+1} \omega(\Psi, \mu_n(\Psi \otimes \cdots \otimes \Psi)) \quad (3.20)$$

---

7 We omit the label $a$ distinguishing species of fields $(\psi_a)_g$ having the same space-time ghost number for brevity. The total state $(3.13)$ should be understood as $\hat{e}_{-g} \otimes \psi_g \equiv \sum_a (\hat{e}_a)_{-g} \otimes (\psi_a)_g$.

8 A different pair $a = (\psi_{a_1})_{g_1} \otimes \cdots \otimes (\psi_{a_n})_{g_n}$ gives different base $(e_a)^*_{g_1+\ldots+g_n}$, which can be read from a given BV master action explicitly.
equals to the usual BV master action \( (2.12) \), \( \hat{S}[\Psi] = S[\psi] \). The homotopy Maurer-Cartan action \( \hat{S}[\Psi] \) consists of fields \( \Psi \) of degree 0, the (quantum) \( A_\infty \) structure of degree 1, and the graded symplectic form of degree \(-1\), which has the simplest \( A_\infty \) degrees. The \( A_\infty \) relations of \( \mu \), which have simple sign factors, reduce to the original (quantum) \( A_\infty \) relations induced from the BV master action, which may include complicated sign factors, by extracting the basis.

Note that in this degree-adjusted notation, as the derivation \( \mu_1 \) acting on \( T(\hat{H}) \) is defined by \( (3.17) \), its contracting homotopy \( \kappa^{-1} \) are given by

\[
\kappa^{-1} (\Psi_g) \equiv -k_1^{-1} \Psi_g = (-)^{g+1} \hat{e}_{1+g} \otimes K^{-1} (\psi_g) .
\]

Likewise, morphisms \( \pi \) and \( \iota \) of \( (3.11) \) are extended in a natural way

\[
\pi(\Psi_g) \equiv \hat{e}_{-g} \otimes \pi(\psi_g) , \quad \iota(\Psi_g) \equiv \hat{e}_{-g} \otimes \iota(\psi_g) .
\]

These degree-adjusted operators satisfy the abstract Hodge decomposition

\[
1 - \iota \pi = \mu_1 \kappa^{-1} + \kappa^{-1} \mu_1 ,
\]

where 1 denotes the unit of the tensor algebra \( T(\hat{H}) \). Therefore, instead of \( (3.11) \), we can consider a deformation retract of the degree-adjusted \( A_\infty \) algebra

\[
\kappa^{-1} \big( (T(\hat{H}), \mu_1) \big) \xrightarrow{\pi} \big( T(\hat{H}'), \mu_1' \big) ,
\]

which has the same algebraic or coalgebraic properties as \( (3.11) \).

### 3.3 Perturbing \( A_\infty \) structure

We consider the degree-adjusted \( A_\infty \) structure \( (3.17) \) and the perturbation of \( (3.24) \). We would like to extract information of the perturbation of \( A_\infty \) structure from \( (2.72) \) or \( (2.46) \). Let us consider a derivation \( Q_{\int \Psi} \) acting on the tensor algebra \( T(\hat{H}) \), via the tensor trick. We find

\[
Q_{\int \Psi} \Psi \otimes n = \sum_{k=1}^{n} \Psi \otimes k-1 \otimes \sum_{m \geq 2} \mu_m (\Psi \otimes m) \otimes \Psi \otimes n-k = \sum_{m \geq 2} \mu_m \Psi \otimes n+m-1 .
\]

These derivations give the same results on the tensor algebra \( T(\hat{H}) \), which becomes explicit if we take the sum of \( \Psi \otimes n \). We thus consider the group-like element of the tensor algebra \( T(\hat{H}) \),

\[
\frac{1}{1 - \Psi} \equiv 1 + \Psi + \Psi \otimes \Psi + \Psi \otimes \Psi \otimes \Psi + \cdots + \Psi \otimes n + \cdots .
\]

By using this element \( (1 - \Psi)^{-1} \in T(\hat{H}) \), we find the equality of (co-)derivations

\[
\left[ Q_{\int \Psi} - \mu_{\text{int}} \right] \frac{1}{1 - \Psi} = 0 .
\]

Hence, as long as we restrict the space \( \mathcal{F}(\hat{H}) \) onto the vector space \( T(\hat{H}) \), we obtain the same results as the previous section even if we replace \( Q_{\int \Psi} \) by \( \mu_{\text{int}} \) in the homological perturbation.
We can extend the BV Laplacian $\Delta$ to a linear map acting on the tensor algebra $T(\mathcal{H})$, a second order derivation of the tensor algebra $T(\mathcal{H})$, which provides

$$
\hbar \Delta \Psi^\otimes n = \hbar \sum_{k,l} \sum_{s \in \mathbb{Z}} \Psi^\otimes k-1 \otimes \hat{e}_{-s} \otimes \Psi^\otimes n-k-l \otimes \hat{e}_{1+s} \otimes \Psi^{l-1} = \hbar \mathcal{L} \Psi^\otimes n-2.
$$

(3.28)

The higher order coderivation $\mathcal{L}$ is defined as follows. For a given base $\hat{e}_{-s} (= I \otimes \hat{e}_{-s}) \in \mathcal{H}$, we consider a derivation $\hat{e}_{-s}$ acting on $T(\mathcal{H})$ by defining $\hat{e}_{-s}|_{\mathcal{H} \otimes n} : \mathcal{H}^n \rightarrow \mathcal{H}^{n+1}$ as follows

$$
\hat{e}_{-s}|_{\mathcal{H} \otimes n} = \sum_l I \otimes l \otimes \hat{e}_{-s} \otimes I \otimes n-l,
$$

(3.29)

which is also a coderivation $\hat{\Delta} \hat{e}_{-s} = (\hat{e}_{-s} \otimes 1 + 1 \otimes \hat{e}_{-s}) \hat{\Delta}$. Then, a higher order coderivation $\mathcal{L}$ is defined by

$$
\mathcal{L}|_{\mathcal{H} \otimes n} = \sum_{l,m} \sum_{s \in \mathbb{Z}} I \otimes l \otimes \hat{e}_{-s} \otimes I \otimes m \otimes \hat{e}_{1+s} \otimes I \otimes n-l-m.
$$

(3.30)

This $\mathcal{L}$ does not satisfy (3.8) as $\hat{\Delta} \mathcal{L} = (\mathcal{L} \otimes 1 + \sum_s (-) \hat{e}_{-s} \otimes \hat{e}_{1+s} + 1 \otimes \mathcal{L}) \hat{\Delta}$. Instead, it satisfies the relation of order 2 coderivation as follows

$$
(\hat{\Delta} \otimes 1) \hat{\Delta} \mathcal{L} - (\hat{\Delta} \mathcal{L} \otimes 1 + 1 \otimes \hat{\Delta} \mathcal{L}) \hat{\Delta} + (\mathcal{L} \otimes 1 \otimes 1 + 1 \otimes \mathcal{L} \otimes 1 + 1 \otimes 1 \otimes \mathcal{L}) (\hat{\Delta} \otimes 1) \hat{\Delta} = 0.
$$

(3.31)

The equivalence of $\hbar \Delta$ and $\mathcal{L}$ becomes manifest if we take the sum of $\Psi^\otimes n$,

$$
\left[ \hbar \Delta - \hbar \mathcal{L} \right] \frac{1}{1 - \Psi} = 0.
$$

(3.32)

Hence, we can obtain the perturbed (quantum) $A_\infty$ structure directly by replacing $\hbar \Delta_{S_{\text{int}}}$ by $\hbar \mathcal{L} + \mu_{\text{int}}$ in the homological perturbation.

### 3.4 Morphism of the cyclic $A_\infty$ structure

Recall that the perturbed BV differential $\hbar \Delta'_A$ provides the perturbed $A_\infty$ structure of (2.66), which is a result of the homological perturbation (2.38). We can derive the perturbed (quantum) $A_\infty$ structure $\mu'$ directly by applying homological perturbation to this coalgebraic homological data (3.24). We first consider the classical part. We can take the derivation

$$
\mu_{\text{int}} = \mu_2 + \mu_3 + \mu_4 + \cdots
$$

(3.33)

acting on $T(\mathcal{H})$ as a perturbation for (3.24) because of the $A_\infty$ relations of $\mu = \mu_1 + \mu_{\text{int}}$. Note also that this $\mu_{\text{int}}$ is also a coderivation $\Delta \mu_{\text{int}} = (\mu_{\text{int}} \otimes 1 + 1 \otimes \mu_{\text{int}}) \hat{\Delta}$ and the coderivation $\mu = \mu_1 + \mu_{\text{int}}$ is nilpotent $(\mu)^2 = 0$. We obtain the deformation retract of tensor algebras or coalgebras

$$
k^{-1} \left( T(\mathcal{H}), \mu_1 + \mu_{\text{int}} \right) \xrightarrow{p_{\text{i}}} \left( T(\mathcal{H}'), \mu'_1 + \mu'_{\text{int}} \right),
$$

(3.34)
where $p$ and $i$ are morphisms preserving its cohomology and $k^{-1}$ is a contracting homotopy. The perturbed data also satisfy the abstract Hodge decomposition

$$1 - ip = \mu k^{-1} + k^{-1} \mu.$$  \hfill (3.35)

Note that the morphisms of tensor algebra $p$ and $i$ and the contracting homotopy $k^{-1}$ of tensor algebra are given by solutions of

$$p = \pi - p \mu_{\text{int}} k^{-1}, \quad i = \iota - k^{-1} \mu_{\text{int}} i, \quad k^{-1} = k^{-1} - k^{-1} \mu_{\text{int}} k^{-1},$$  \hfill (3.36)

where $\pi$ and $\iota$ are morphisms satisfying (3.4) and (3.3), $\mu_{\text{int}}$ is a (co-)derivation satisfying (3.7) and (3.8), and $\kappa$ is a contracting homotopy of the tensor algebra satisfying (3.7) and (3.8). By using this morphism $p$ or $i$, the effective $A_\infty$ structure $\mu' = \mu_1' + \mu_1'$ is given by

$$\mu_{\text{int}}' = p \mu_{\text{int}} \iota = \pi \mu_{\text{int}} i.$$  \hfill (3.37)

where the second equality follows form (3.36) quickly

$$p \mu_{\text{int}} (i + k^{-1} \mu_{\text{int}} i) = (p + p \mu_{\text{int}} k^{-1}) \mu_{\text{int}} i.$$  \hfill (3.38)

As a result of the perturbation, the $A_\infty$ relations are automatic

$$\left( \mu_1' + \mu_{\text{int}}' \right)^2 = 0,$$  \hfill (3.39)

which come from $(\mu_1 + \mu_{\text{int}})^2 = 0$ and the defining properties (3.36),

$$(\mu_1')^2 = p \mu_{\text{int}} (i \pi) \mu_{\text{int}} i = p (\mu_{\text{int}})^2 i + \left[ -p \mu_{\text{int}} k \right] \mu_1 \mu_{\text{int}} i + p \mu_{\text{int}} \mu_1 \left( -k \mu_{\text{int}} i \right),$$

$$= p \left[ (\mu_{\text{int}})^2 + \mu_{\text{int}} \mu_1 + \mu_1 \mu_{\text{int}} \right] i - \mu_1' \left( \pi \mu_{\text{int}} i \right) - \left( p \mu_{\text{int}} i \right) \mu_1'. \hfill (3.40)$$

Likewise, the morphisms $p$ and $i$ become $A_\infty$ morphisms such that

$$p \mu = \mu' p, \quad i \mu' = \mu i,$$  \hfill (3.41)

as long as the assumptions of the perturbation $\mu_{\text{int}} k^{-1} \neq -1$ and $k^{-1} \mu_{\text{int}} \neq -1$ are provided. Apparently, when $\sum_n (-\mu_{\text{int}} k^{-1})^n$ converges, $1 + \mu_{\text{int}} k^{-1}$ is invertible and (3.41) follows from

$$p \mu - \mu' p = (\pi - p \mu_{\text{int}} k^{-1}) \mu_1 + p \mu_{\text{int}} (k^{-1} \mu_1 + \iota \pi + \mu_1 k^{-1}) - \mu' p$$

$$= (\mu_1' + p \mu_{\text{int}} \iota) \pi + p (\mu_{\text{int}} \mu_1) k^{-1} - \mu' p$$

$$= (\mu_1' + \mu_{\text{int}}') (\pi - p) - p (\mu_1 + \mu_{\text{int}}) \mu_{\text{int}} k^{-1}$$

$$= (\mu' p - p \mu) \mu_{\text{int}} k^{-1}. \hfill (3.42)$$

Likewise, we find $\mu - i \mu' = -k^{-1} \mu_{\text{int}} (\mu - i \mu')$ and (3.41) follows from it.

By solving (3.36), we obtain the following expression of morphisms $p$ and $i$,

$$p = \pi \frac{1}{1 + k^{-1} \mu_{\text{int}}}, \quad i = \frac{1}{1 + k^{-1} \mu_{\text{int}}}. \hfill (3.43)$$
Note that these expressions enable us to obtain (3.41) directly from the second line of (3.42), $p(\mu_1 + \mu_{\text{int}}) = (\mu'_1 + \mu'_{\text{int}}) \pi - p(\mu_1 + \mu_{\text{int}}) \mu_{\text{int}} \kappa^{-1}$. By substituting (3.43) into (3.37), the effective cyclic $A_\infty$ structure can be cast as follows

$$
\mu' = \mu'_1 + \pi \mu_{\text{int}} \frac{1}{1 + \kappa^{-1} \mu_{\text{int}} t}.
$$

(3.44)

In this tree-level case, because of the coalgebraic properties (3.8) of $\mu_{\text{int}}$, tensor algebra morphisms $p$ and $i$ are also coalgebra morphisms and the derivation $\mu'$ is also a coderivation

$$
\hat{\Delta} p = (p \otimes p) \hat{\Delta}, \quad \hat{\Delta} i = (i \otimes i) \hat{\Delta}, \quad \hat{\Delta} \mu' = (\mu' \otimes 1 + 1 \otimes \mu') \hat{\Delta}.
$$

(3.45)

The third property quickly follows from the first or second property. As long as $\mu_{\text{int}}$ is a well-defined perturbation such that $\mu_{\text{int}} \kappa^{-1} \neq -1$ or $\kappa^{-1} \mu_{\text{int}} \neq -1$, the first property follows from

$$
(p \otimes p) \hat{\Delta} (1 + \mu_{\text{int}} \kappa^{-1}) = (\pi \otimes \pi) \hat{\Delta} = \hat{\Delta} \pi = \hat{\Delta} p (1 + \mu_{\text{int}} \kappa^{-1}),
$$

(3.46)

where the first equality follows from direct computation

$$
(p \otimes p) \hat{\Delta} (\mu_{\text{int}} \kappa^{-1}) = (p \mu_{\text{int}} \otimes p + p \otimes p \mu_{\text{int}}) \hat{\Delta} (\kappa^{-1})
$$

$$
= \left[ p \mu_{\text{int}} \otimes \pi + \pi \otimes p \mu_{\text{int}} - (p \mu_{\text{int}} \otimes p \mu_{\text{int}} \kappa^{-1} + p \mu_{\text{int}} \kappa^{-1} \otimes p \mu_{\text{int}}) \right] \hat{\Delta} (\kappa^{-1})
$$

$$
= (p \mu_{\text{int}} \kappa^{-1} \otimes \pi + \pi \otimes p \mu_{\text{int}} \kappa^{-1}) \hat{\Delta} (\kappa^{-1}) = (\pi \otimes \pi) \hat{\Delta}.
$$

(3.47)

Likewise, the second property of (3.45) holds. Again, (3.43) can solve (3.45) easily.

### 3.5 Cyclicity of the effective $A_\infty$ structure

Before adding the quantum part, we consider the cyclicity. Note that the cyclic property of the (perturbed) $A_\infty$ structure is manifest from the beginning, as long as it is induced from the (effective) BV master action. However, we would like to show that the homological perturbation itself preserves the cyclic $A_\infty$ structure whenever a contracting homotopy satisfies the compatibility condition (3.52). In quantum field theory, it is nothing but a Hermitian property of the propagators. In string field theory, it is the BPZ property.

We may write $\langle \omega | A \otimes B \equiv \omega(A, B)$ for the symplectic structure on $\hat{\mathcal{H}}$ for simplicity. The cyclic property of the $A_\infty$ structure $\mu = \mu_1 + \mu_{\text{int}}$ can be cast as follows

$$
\langle \omega | \mu_n \otimes \mathbb{I} = -\langle \omega | \mathbb{I} \otimes \mu_n.
$$

(3.48)

The perturbed $A_\infty$ structure $\mu'$ provides an effective homotopy Maurer-Cartan action

$$
\hat{A}[\Psi] = -\sum_n \frac{1}{n+1} \omega' (\Psi', \mu'_n (\Psi', ..., \Psi')),
$$

(3.49)
where $\Psi' \in \hat{H'} \equiv E \otimes H'$ denotes effective fields. When $\mu$ is given by (3.17), it equals to (2.25). The symplectic structure $\omega'$ on $\hat{H'}$ is defined by using the inner product on $\hat{H'}$,

$$\langle A', B' \rangle' = \langle \iota A', \iota B' \rangle \quad A', B' \in \hat{H'},$$

and the symplectic form $\hat{\omega}$ on $E$, just as (3.18). We may write $\langle \hat{\omega}'|A' \otimes B' = \hat{\omega}'(A', B')$ for this symplectic structure on $\hat{H}$ for simplicity, which enables us to have

$$\langle \hat{\omega}'|\iota \Pi = \langle \hat{\omega}|\Pi \otimes \iota.$$  

(3.51)

When we take Hermit propagators $K^{-1}$, we find $\omega'(\kappa^{-1}A, B) = (-)^A\omega(A, \kappa^{-1}B)$ quickly. This compatibility of $\kappa^{-1}$ and $\omega$ can be cast as follows

$$\langle \omega'|\iota \Pi = \langle \omega|\Pi \otimes \iota.$$  

(3.52)

As we see, this property (3.52) ensures the cyclicity of the perturbed $A_\infty$ structure.

When we have (3.48) and (3.52), the abstract Hodge decomposition (3.23) implies

$$\langle \omega|\Pi i \otimes \iota = \langle \omega|\Pi \otimes \iota.$$  

(3.53)

Because of (3.12), it provides the cyclic property of $\mu'_1 = \pi \mu_1 \iota$ quickly

$$\langle \omega'|\mu'_1 \otimes \iota' = -\langle \omega'|\iota' \otimes \mu'_1.$$  

(3.54)

where $\langle \omega'| = \langle \omega|\iota \otimes \iota$ is the symplectic form on $\hat{H'}$ and $\Pi' \equiv \pi \iota$ denotes the unit of $\hat{H'}$. Likewise, (3.48) and (3.52) guarantees the cyclic property of $\mu'_{\text{int}} = \pi \mu_{\text{int}} \iota$ as follows

$$\langle \omega|\mu_{\text{int}} i \otimes (i + \kappa^{-1} \mu_{\text{int}}) = -\langle \omega|(i + \kappa^{-1} \mu_{\text{int}}) \otimes \mu_{\text{int}} i.$$  

(3.55)

Hence, as long as we take $\kappa^{-1}$ satisfying (3.52), the cyclic property of $\mu'$ is manifest

$$\langle \omega'|\mu'_1 + \mu'_{\text{int}} \otimes \iota' = -\langle \omega'|\iota' \otimes (\mu'_1 + \mu'_{\text{int}}).$$  

(3.56)

Note also that in the context of the quantum $A_\infty$ algebra, the cyclic property is already included in the quantum $A_\infty$ relations from the beginning. If the path-integral or homological perturbation can be understood as a morphism of the quantum $A_\infty$ structure, the above computations arise as a natural consequence of it.

### 3.6 Morphism of the quantum $A_\infty$ structure

Finally, we include the quantum part. Suppose that a solution $S$ of the classical master equation also solves the quantum one $\Delta S = 0$. Then, the cyclic $A_\infty$ structure $\mu = \mu_1 + \mu_{\text{int}}$ induced from $S$ satisfies the quantum $A_\infty$ relation

$$(\mu + h \mathcal{L})^2 = 0,$$  

(3.57)
which is the coalgebraic representation of (2.10). Since $\mu + \hbar \mathcal{L}$ is a nilpotent linear map acting on the vector space $\mathcal{T}(\mathcal{H})$, we can take the following perturbation for (3.24),

$$\mu_{\text{int}} + \hbar \mathcal{L}.$$  \hfill (3.58)

As a result of the homological perturbation, we obtain the deformation retract describing the perturbative path-integral of quantum field theory

$$K^{-1} \bigcirc (\mathcal{T}(\hat{\mathcal{H}}), \mu + \hbar \mathcal{L}) \xrightarrow{\mathcal{P}} (\mathcal{T}(\hat{\mathcal{H}}'), \mu'_{1} + \mu'_{\text{eff}}).$$  \hfill (3.59)

Morphisms $\mathcal{P}$ and $\mathcal{L}$ and a contracting homotopy $K^{-1}$ satisfy the abstract Hodge decomposition

$$1 - \mathcal{L} \mathcal{P} = (\mu + \hbar \mathcal{L}) K^{-1} + K^{-1} (\mu + \hbar \mathcal{L}) .$$  \hfill (3.60)

These morphisms $\mathcal{P}$ and $\mathcal{L}$ are given by solutions of the recursive relations

$$\mathcal{P} = \pi - \mathcal{P} (\hbar \mathcal{L} + \mu_{\text{int}})^{-1}, \quad \mathcal{L} = \iota - \kappa^{-1} (\hbar \mathcal{L} + \mu_{\text{int}}) \iota .$$  \hfill (3.61)

Note however that these $\mathcal{P}$ and $\mathcal{L}$ are not coalgebra morphisms and do not satisfy (3.8) because $\mathcal{L}$ is a higher coderivative and does not satisfy (3.8). The morphism $\mathcal{P}$ or $\mathcal{L}$ enables us to obtain the effective quantum $A_{\infty}$ structure $\mu' = \mu'_{1} + \mu'_{\text{eff}}$ with

$$\mu'_{\text{eff}} = \pi (\hbar \mathcal{L} + \mu_{\text{int}}) \iota = \mathcal{P} (\hbar \mathcal{L} + \mu_{\text{int}}) \iota .$$  \hfill (3.62)

Note also that derivation $\mu'$ is not a coderivation and does not satisfy (3.8), which may be regarded as a higher order coderivation in $IBA_{\infty}$ or $IBL_{\infty}$, since $\mathcal{L}$ is a second order. These maps $\mathcal{P}$ and $\mathcal{L}$ are just morphisms of the vector space $\mathcal{T}(\mathcal{H})$ such that

$$\mathcal{P} (\mu + \hbar \mathcal{L}) = (\mu'_{1} + \mu'_{\text{int}}) \mathcal{P}, \quad (\mu + \hbar \mathcal{L}) \mathcal{L} = 1 (\mu'_{1} + \mu'_{\text{int}}) ,$$  \hfill (3.63)

which we call a morphism of the (quantum) $A_{\infty}$ structure. These relations (3.63) are proven by the same way as (3.41), which follows from the homological perturbation lemma.

When a given solution $S_{[0]}$ of the classical master equation $(S_{[0]}, S_{[0]}) = 0$ does not solve the quantum master action, $\Delta S_{[0]} \neq 0$, it necessitates quantum corrections $\hbar S_{[1]} + \hbar^{2} S_{[2]} + \cdots$ such that $S = S_{[0]} + \hbar S_{[1]} + \hbar^{2} S_{[2]} + \cdots$ satisfies $\hbar \Delta S + \frac{1}{2} (S, S) = 0$. Then, the cyclic $A_{\infty}$ structure $\mu_{[0]}$ induced from $S_{[0]}$ does not satisfy the quantum $A_{\infty}$ relation, $(\mu_{[0]} + \hbar \mathcal{L})^{2} \neq 0$. Each correction $S_{[i]}$ gives components of quantum $A_{\infty}$ maps $\mu_{n,[i]} : \mathcal{H}^{\otimes m} \to \mathcal{H}$, which is extended to coderivation of $\mathcal{T}(\hat{\mathcal{H}})$ by defining $\mu_{n,[i]}|_{\hat{\mathcal{H}}^{\otimes m}} : \hat{\mathcal{H}}^{m} \to \hat{\mathcal{H}}^{\otimes m-n+1}$ for $m \geq n$ otherwise zero as (3.6). For a given $S_{[i]}$, we obtain the coderivation

$$\mu_{[i]} = \mu_{0,[i]} + \mu_{1,[i]} + \mu_{2,[i]} + \cdots ,$$  \hfill (3.64)

which are necessary for the quantum $A_{\infty}$ relations $(\mu_{[0]} + \sum_{l} \hbar^{l} \mu_{[l]} + \hbar \mathcal{L})^{2} = 0$. Hence, in this case, the above $\mu_{\text{int}}$ of (3.58) must be replaced by

$$\mu_{\text{int}} = \mu_{\text{int},[0]} + \sum_{l \geq 1} \hbar^{l} \mu_{[l]} .$$  \hfill (3.65)
This replacement enables us to obtain the appropriate perturbed data \((3.58)\) in the same way. We can express the solutions of the defining equations \((3.61)\) as follows

\[
P = \pi \frac{1}{1 + (\hbar \mathcal{L} + \mu_{\text{int}}) \kappa^{-1}}, \quad l = \frac{1}{1 + \kappa^{-1}(\hbar \mathcal{L} + \mu_{\text{int}})} \iota. \tag{3.66}
\]

The perturbed quantum \(A_\infty\) structure can be written as

\[
\mu' = \mu_1' + \pi (\hbar \mathcal{L} + \mu_{\text{int}}) \frac{1}{1 + \kappa^{-1}(\hbar \mathcal{L} + \mu_{\text{int}})} \iota, \tag{3.67}
\]

which takes the same form as \((2.66)\). Its homotopy Maurer-Cartan action is given by \((3.49)\) with replacing \(\mu'\) by \((3.67)\), which equals to \((2.25)\) or \((2.68)\) derived in section 2.

### 3.7 Twisted \(A_\infty\) and source terms

The BV master action including the source terms gives an effective theory with a twisted \(A_\infty\) structure. For a given BV master action, we can add source terms \(V\)

\[
S_V[\Psi] \equiv S[\Psi] - \omega(\Psi, V) \tag{3.68}
\]

and suppose that this \(S_V[\Psi]\) and its free part also satisfy the BV master equation \(\hbar \Delta e^{S_V[\Psi]} = 0\). Then, source terms \(V\) must satisfy the following properties with the \(A_\infty\) structure of \(S[\Psi]\),

\[
\mu_1(V) = 0, \quad \sum_n \sum_{k=0}^n \mu_{n+1}(\underbrace{\Psi, \ldots, \Psi}_{k}, V, \Psi, \ldots, \Psi) = 0, \tag{3.69}
\]

which we call gauge invariant source terms. Then, the source terms \(\mu_0 \equiv V\) become the zeroth product of a twisted \(A_\infty\) structure \(\mu + V\). Note that \(\mu\) itself is the \(A_\infty\) structure and thus this \(\mu + V\) is stronger than a generic twisted \(A_\infty\) structure. If we add \(V\) to our perturbation, we find that the effective \(A_\infty\) structure is twisted by \(\kappa^{-1}V\) as follows

\[
\mu'_{V'} = e^{-\kappa^{-1}V} \mu' e^{\kappa^{-1}V} + V. \tag{3.70}
\]

It becomes a twisted (quantum) \(A_\infty\) structure, whose zeroth element is

\[
\mu'_{V,0} = V + \sum_{k=0}^{\infty} \mu'_k(\underbrace{(\kappa^{-1}V)^{\otimes k}}_{k}). \tag{3.71}
\]

Note that \(\mu'\) itself is the \(A_\infty\) structure and the twisted \(n\)-product is given by

\[
\mu'_{V,n}(\Psi^{\otimes n}) = \sum_{k=0}^{\infty} \sum_{k_0 + \ldots + k_n = k} \mu'_k(\underbrace{(\kappa^{-1}V)^{\otimes k_0}, \Psi, (\kappa^{-1}V)^{\otimes k_1}, \ldots, \Psi, (\kappa^{-1}V)^{\otimes k_n}}_{n}). \tag{3.72}
\]

\(^9\)It is an \(A_\infty\) structure including the zeroth product \(\mu'_0\), which is also called a weak (or curved) \(A_\infty\) structure.
4 Application to string field theory

In string field theories, fortunately, we have classical or quantum BV master actions, except for a few cases. Hence, we can apply the previous results and perform the perturbative path-integral of string fields on the basis of the homological perturbation. Let us consider the BV master action of string field theory

\[ S[\Psi] = -\frac{1}{2} \omega(\Psi, \mu_1(\Psi)) \sum_n \frac{1}{n+1} \omega(\Psi, \mu_n(\Psi, \ldots, \Psi)), \quad (4.1) \]

where \( \mu_1 = Q \) is the BRST operator of strings and the (quantum) \( A_\infty \) or \( L_\infty \) structure \( \mu \) is given by \( \sum_n \mu_n(\Psi^\otimes) = -\hbar \Delta S[\Psi] \Psi \). The state (3.13) is now a string field \( \Psi \) where \( \psi \) denotes space-time fields and adjusting basis \( \hat{e}_g \) are just the (suspended) complete basis of the considering conformal field theory. By solving the free theory, we can obtain the Hodge decomposition

\[ 1 - i\pi = \mu_1 \kappa^{-1} + \kappa^{-1}_1 \mu_1, \quad (4.2) \]

which is the starting point of performing the perturbative path-integral.

4.1 Effective theories with finite \( \alpha' \)

Each effective theories based on the perturbative path-integral,

\[ A[\Psi'] = -\frac{1}{2} \omega'(\Psi', \mu'_1(\Psi')) \sum_n \frac{1}{n+1} \omega'(\Psi', \mu'_n(\Psi', \ldots, \Psi')), \quad (4.3) \]

always have the (quantum) \( A_\infty \) or \( L_\infty \) structure trivially, as a result of the homological perturbation, as long as the original action \( S[\Psi] \) solves the BV master equation. In addition, when the original action includes source terms \( \omega(\Psi, \psi) \), the effective action (4.3) has a weak (quantum) \( A_\infty \) structure \( \mu'_V = e^{-\kappa^{-1} V} \mu e^{\kappa^{-1} V} \) as shown in the previous section.

We can integrate all massive space-time fields \( \Psi_{M \neq 0} \) out from the string field \( \Psi = \Psi_{M=0} + \Psi_{M \neq 0} \) and get an effective field theory that consists of massless (plus auxiliary) fields \( \Psi_{M=0} \) by using the abstract Hodge decomposition

\[ 1 - i\pi_M = \mu_1 \kappa^{-1}_M + \kappa^{-1}_M \mu_1, \quad (4.4) \]

where \( \kappa^{-1}_M \) denotes propagators of massive fields and \( (i\pi)_M = 0 \) denotes a projector onto the massless fields \( \Psi_{M=0} = (i\pi)_M = 0 \Psi \). We can construct these \( \psi_{M=0} \), \( i\pi_{M=0} \) and \( \kappa^{-1}_M \) explicitly by solving the free theories, which gives the effective action (4.3) with \( \Psi' = \pi_{M=0}(\Psi) = \pi_{M=0}(\Psi_{M=0}) \in \hat{H}' \).

Likewise, we can integrate space-time fields \( \Psi_{p>\Lambda} \) having higher momentum \( p > \Lambda \) out from the string field \( \Psi = \Psi_{p \leq \Lambda} + \Psi_{p > \Lambda} \) and construct a Wilsonian effective action with the cut-off scale \( \Lambda \) perturbatively. It is obtained by using the Hodge decomposition

\[ 1 - (i\pi)_{p \leq \Lambda} = \mu_1 \kappa^{-1}_{p>\Lambda} + \kappa^{-1}_{p>\Lambda} \mu_1, \quad (4.5) \]

where \( (i\pi)_{p \leq \Lambda} \) denotes the restriction onto the lower momentum fields \( (i\pi)_{p \leq \Lambda} \Psi = \Psi_{p \leq \Lambda} \) and \( \kappa^{-1}_{p>\Lambda} \) denotes propagators of the higher momentum fields. It provides (4.3) with \( \Psi' = \pi_{p \leq \Lambda}(\Psi) \in \hat{H}' \). In the same manner, for any decomposition (4.2), we can obtain corresponding effective action.
We just assumed the existence of such well-defined projectors and pursued its algebraic aspects in this paper. We would like to emphasize that the physically important information is in how to construct these projectors and propagators concretely. To give the abstract Hodge decomposition is equivalent to solving the theory. We thus started from the free theory and considered perturbations.

### 4.2 Light-cone reduction

While light-cone theory consists of physical degrees, covariant theory has the gauge invariance. In string field theory, explicit Lorentz covariance is given in return for adding the gauge and unphysical degrees. We can remove these extra degrees by using the path-integral and obtain a light-cone string field theories for each covariant string field theories [9, 10].

We write \( Q \) for the BRST operator of the world-sheet theory of strings and \( \omega \) for the BPZ inner product of its conformal field theory. We consider a covariant string field theory,

\[
S[\Psi] = -\frac{1}{2} \omega(\Psi, Q \Psi) - \frac{1}{3} \omega(\Psi, m_2(\Psi, \Psi)) - \cdots. \tag{4.6}
\]

It has an \( A_\infty \) (or \( L_\infty \)) structure \( m \) with \( m_1 = Q \) as long as it satisfies the BV master equation. There is a similarity transformation \( U \) connecting the BRST operator \( Q \) and the kinetic operator \( L_{lc}^0 \) in light-cone gauge plus the differential \( d \) acting on the gauge and unphysical degrees [20], which diagonalise physical and extra degrees as follows

\[
Q = U^{-1} (c_0 L_{lc}^0 + d) U. \tag{4.7}
\]

Note that \((c_0 L_{lc}^0)^2 = 0\) holds in addition to \((c_0 L_{lc}^0 + d)^2 = 0\) and these are defined on the critical dimention. The similarity transformation \( U \) becomes an isomorphism

\[
\mu U \equiv m_n (U^{-1} \otimes \cdots \otimes U^{-1}) \tag{4.8}
\]

It gives a linear transformation of the conformal basis and thus provides a linear string-field redefinition \( \Phi \equiv U \Psi \). We obtain the diagonalised action with the \( A_\infty \) structure \( \mu \) as follows

\[
S[\Phi] = -\frac{1}{2} \omega(\Phi, (c_0 L_{lc}^0 + d) \Phi) - \sum_{n=2}^{\infty} \frac{1}{n+1} \omega(\Phi, \mu_n(\Phi, \cdots, \Phi)) \tag{4.9}
\]

The extra degrees become the BRST quartets and thus \( d \) has no cohomology unless there is no quartet excitations. As is known, the integration of the BRST quartets is volume 1 since bosonic and fermionic integrations exactly cancel each other. We can start with the BRST quartets,

\[
\kappa^{-1} \bigotimes (\mathcal{H}, d) \xrightarrow{\pi} (\mathcal{H}_{lc}, 0), \tag{4.10}
\]

\[\text{For details, see [120]. For bosonic open strings, these are given by}
\]

\[
d \equiv -p^+ \sum_{n \neq 0} a^-_n c_{-n}, \quad U \equiv \exp \left[ c_0 N_{a^+ b^+ c^+} \right] \exp \left[ \frac{1}{p^+} \sum_{n \neq 0} \frac{1}{n} \hat{L}_n a^+_{-n} \right],
\]

where \( N_{a^+ b^+ c^+} \) counts \( a^+_{n}, b_{n}, c_{n} \) for \( n \neq 0 \) and \( \hat{L}_n \) is a \( \lambda = 1 \) Virasoro generator commuting with \( a^+_{n} \).
where $\kappa^{-1}$ denotes the propagator for $d$ and $\mathcal{H}_{lc}$ is the state space of string fields in the light-cone gauge. We take $\pi : \mathcal{H} \rightarrow \mathcal{H}_{lc}$ and $\iota : \mathcal{H}_{lc} \rightarrow \mathcal{H}$ as natural projection and embedding. We can take $c_0 L^{lc}_0$ as a perturbation to \cite{10} and get

$$\kappa^{-1} \circ (\mathcal{H}, c_0 L^{lc}_0 + d) \xrightarrow{\pi} (\mathcal{H}_{lc}, c_0 L^{lc}_0).$$

(4.11)

It describes the no-ghost theorem of covariant strings \cite{21}. We can take a further perturbation $\mu_{int}$ for \cite{11} because of the $A_\infty$ structure $(\mu_1 + \mu_{int})^2 = 0$ and obtain

$$\kappa^{-1} \circ (T(\mathcal{H}), \mu_1 + \mu_{int}) \xrightarrow{P} (T(\mathcal{H}_{lc}), \nu^{lc}_{1} + \nu^{lc}_{int}).$$

(4.12)

While the left side has the $A_\infty$ structure $\mu$ of the covariant string field theory \cite{4.9}, the right side provides the transferred $A_\infty$ structure $\nu^{lc}$ of the light-cone string field theory. By using the light-cone string field $\varphi \in \mathcal{H}_{lc}$ and the light-cone vertices $\nu^{lc}_{int} \equiv \pi \mu_{int} 1$, we obtain the light-cone string field theory $S_{lc}[\varphi]$ extracted from the covariant theory \cite{4.9},

$$S_{lc}[\varphi] = -\frac{1}{2} \omega(\varphi, c_0 L^{lc}_0 \varphi) - \sum_{n=2}^{\infty} \frac{1}{n+1} \omega(\varphi, \nu^{lc}_{n}(\varphi, \ldots, \varphi)),$$

(4.13)

where we used loose notation $\varphi = \iota(\varphi)$ and $c_0 L^{lc}_0 = \pi(c_0 L^{lc}_0) \iota = \mu_1$ for simplicity. Note that the vertices $\nu^{lc}_{int}$ consists of the original vertices $\mu_{int}$ (with projections and embeddings) and effective vertices $\mu_{eff}$ including propagators $\kappa^{-1}$ as follows

$$\nu^{lc}_{int} = \pi \mu_{int} \iota + \pi \left[ \sum_{n=1}^{\infty} (-)^n \mu_{int} \kappa^{-1} \mu_{int}^n + \sum_g h^g (g\text{-loop}) \right] \iota.$$

In this sense, the light-cone reduction (4.12) can be cast as the form which consists of the light-cone kinetic term, the original vertices, and effective vertices. Hence, the action (4.13) has higher interacting terms and takes the different form from the original covariant theory (4.9) unless all of the effective vertices $\mu_{eff}(\varphi, \ldots, \varphi)$ exactly equal to zero. See \cite{11} for further discussions.

### 4.3 S-matrix and asymptotic string fields

When a given (quantum) $A_\infty$ structure $\mu = \mu_1 + \mu_2 + \cdots$ has no linear part $\mu_1$, it is called minimal. The $S$-matrix is realized as a minimal model, which can be obtained by using the homological perturbation. The uniqueness of the minimal $A_\infty$ structure is ensured by the minimal model theorem in mathematics. In terms of physics, it implies that the on-shell amplitudes are independent of given gauge-fixing condition or propagator and thus are unique.

In addition, our homological techniques suggest that we may use unconventional gauge-fixing conditions and propagators in the usual Feynman graph calculations.

---

\footnote{For the Fock vacuum $|\Omega\rangle \equiv |c\rangle \otimes |a^\pm, b, c\rangle$, we define $\pi : |\Omega\rangle \mapsto |c\rangle$ and $\iota : |c\rangle \mapsto |\Omega\rangle$. For excitations on these vacua, we define $\pi \circ (p^\mu, a_n, c_0; a_n^\pm, e_n, b_n) = (p^\mu, a_n, c_0) \circ \pi$ and $\iota \circ (p^\mu, a_n, c_0) = (p^\mu, a_n, c_0; 0, 0, 0) \circ \iota$ for $n \neq 0$.}
The S-matrix is a set of multi-linear forms \( \{ A_n \}_{n \geq 3} \) defined on the tensor algebra \( T(H_{as}) \) of the state space \( H_{as} \), whose inputs are asymptotic free string fields \( \Psi_{as} \in H_{as} \). We consider the action of asymptotic string fields,

\[
S_{as}[\Psi_{as}] = -\frac{1}{2} \langle \Psi_{as}, Q \Psi_{as} \rangle .
\] (4.14)

The asymptotic string field \( \Psi_{as} \in H_{as} \) has the linear gauge invariance \( \delta \Psi_{as} = Q \lambda_{as} \) and the physical states condense on the cohomology of \( Q \) acting on \( H_{as} \). We assume that the cohomology \( H_{as \ phys} \) of the asymptotic theory is isomorphic to that of the free theory, \( H_{phys} \equiv I(H_{as \ phys}) \).

We first solve the free theory and derive a propagator \( \kappa^{-1} \), which gives the abstract Hodge decomposition (4.2). Then, by defining morphisms \( \iota_{as} \equiv I \) and \( \pi_{as} \equiv I^{-1} \) that satisfy \( \pi_{as} \kappa^{-1} = \iota_{as} \kappa^{-1} = 0 \), we can consider

\[
\kappa^{-1} \circ \left( T(H), \mu_1 \right) \xrightarrow{\iota_{as}} \left( T(H_{as \ phys}), 0 \right) \xrightarrow{I} \left( T(H_{phys}), 0 \right) .
\] (4.15)

Note that \( \iota_{as} \equiv I \) \( (I^{-1}) \) \( \pi_{as} \) \( Q \) vanish and it gives the same decomposition \( 1 - \iota_{as} \pi_{as} = \mu_1 \kappa^{-1} + \kappa^{-1} \mu_1 \). The minimal model is obtained by taking interacting terms \( \mu_{int} \) as the perturbation to (4.15). The (quantum) \( A_\infty \) structure of the \( S \)-matrix is given by the right side of

\[
K_{as}^{-1} \circ \left( T(H), \mu_1 + \mu_{int} \right) \xrightarrow{\pi_{as}} \left( T(H_{as \ phys}), \mu'_{int} \right) .
\] (4.16)

This is a minimal model because \( \mu'_{int} \equiv Q \) vanishes and it has no gauge degree. The morphism \( \pi_{as} \) determines a nonlinear field relation between interacting and asymptotic theories on-shell. The \((n+1)\)-point amplitude is given by the \( \mu'_n \) part of the homotopy Maurer-Cartan action

\[
A[\Psi'] = -\sum_{n} \frac{1}{n+1} \omega' \left( \Psi_{as}', \mu'_n(\Psi_{as}', ..., \Psi_{as}') \right) .
\] (4.17)

It defines multi-linear maps acting on the on-shell asymptotic string fields. As we showed, it is the same as the Feynman graph expansion and thus gives the amplitudes correctly. In addition, as long as it is minimal, the \( A_\infty \) relation \( (\mu'_{int})^2 = 0 \) implies the BRST identities

\[
\omega' \left( Q \Psi_0', \mu'_n(\Psi_1', ..., \Psi_n') \right) + \sum_{k=1}^{n} \omega' \left( \Psi_0', \mu'_n(\Psi_1', ..., Q \Psi_k', ..., \Psi_n') \right) = 0 ,
\] (4.18)

which corresponds to the Stokes theorem. Hence, even if we replace \( H_{as \ phys} \) by \( \text{Ker}[Q] \), the amplitudes (4.17) reproduce the same values because of the BRST identities (4.18).

**Open string field theory**

We obtained a generic result (4.17) which is valid whenever we consider ordinary perturbative calculations, in which propagators of \( S \)-matrix and gauge-fixing conditions should be written in terms of the free theory. In the rest, we consider somewhat unconventional situations where each pieces of \( S \)-matrix may be given by using information of interacting terms.
Let us consider Witten’s open string field theory, which satisfies the classical BV master equation. We can obtain the tree amplitudes on the basis of the classical limit of the homological perturbation. Since it is a cubic theory, the $A_\infty$ structure has no higher product $\mu_n = 0$ for $n > 2$. The interacting vertex $\mu_{\text{int}} = \mu_2$ is given by the star product

$$\mu_2(A, B) \equiv (-)^A A \ast B.$$  

We first consider the Siegel gauge and the linear $b$-gauge, which give a standard perturbative calculus and valid results. Next, we consider unconventional gauges, the dressed $B_{-0}$ gauge and $A_T$ gauge, whose validities look obscure but may be justified by using homological approach.

**Siegel gauge**

In the Siegel gauge $b_0 \Psi = 0$, the propagator $\kappa_{\text{Siegel}}^{-1} \equiv b_0 L_0^{-1}$ has poles on the kernel of $L_0$. We can represent the projector onto the physical states as $\iota_\pi \equiv e^{-\infty \pi}$. Note that the Schwinger representation of the inverse of $L_0$ naturally includes $e^{-\infty \pi}$ as a boundary term.

$$b_0 L_0^{-1} \equiv b_0 \int_0^\infty e^{-t \pi} dt = \frac{b_0}{L_0} (1 - e^{-\infty \pi}).$$  

Since $\mu_1 \equiv Q$ is the BRST operator of open strings, we obtain the decomposition

$$1 - e^{-\infty \pi} = Q (b_0 L_0^{-1}) + (b_0 L_0^{-1}) Q.$$  

As is known, the Siegel gauge is the standard gauge used in perturbative calculations and it provides a conventional propagator.

**Linear $b$-gauge**

Let us consider a linear combination of the oscillators $b_n$, which we write $B(g)$, that can be encoded in a vector field $v(z) = \sum_{n \in \mathbb{Z}} v_n z^{n+1}$. The linear $b$-gauge is given by

$$B(g) \Psi = 0 \quad \text{with} \quad B(g) \equiv \sum_{n \in \mathbb{Z}} v_n b_n = \oint \frac{dz}{2\pi i} v(z) b(z),$$  

where $g$ denotes the label of the space-time ghost number. Note that the BPZ properties $B(-g) = B^*_g$ and $B_{(g-1)}^* = B_{(g)}$ must be satisfied for the consistency. For each $B(g)$ or $B^*_g$, we define a linear combination of the Virasoro generators $L(g) \equiv Q B(g) + B(g) Q$, which appears in propagators.

In general, the linear $b$-gauge may not be invariant under the BPZ conjugation $B(g) \neq B^*_g$ and then we cannot impose the same gauge-fixing condition for all space-time ghost numbers, such as $B_{(g-1)} = B^*_g \neq B_{(g)}$. We write $\Psi = \sum \Psi_g, B \equiv \sum_g B(g)$ and $L_0 \equiv \sum L(g)$ for simplicity. The double Schwinger representation of the propagator

$$\kappa_{\text{double}}^{-1} \equiv (B^* L_0^{-1}) Q (B L_0^{-1}) = \frac{B^*}{L_0} Q \frac{B}{L_0} (1 - e^{-\infty L_0}) (1 - e^{-\infty L_0^*})$$  

\[\text{If this open string field theory gives a well-defined quantum theory, it solves the quantum BV master equation without any modification. Then, we can extend these results to the loop amplitudes since it guarantees that the theory gives amplitudes independent of the gauge-fixing condition.}\]
provides the decomposition (4.2) with $1 - (i\pi)_{\text{double}} \equiv (1 - e^{-\infty L_0})(1 - e^{-\infty L_0^*})[1 + Q(\mathcal{B}_0 - \mathcal{B}_0^*)]$. It gives correct on-shell amplitudes unless the vector field $v(z)$ is singular. Calculations of homological perturbation suggest us an interesting but unconventional propagator. Since the interactions of open string fields are given by the star product, (4.26) gives a gauge-fixing condition $\kappa_{\text{mean}}^{-1} \equiv \frac{1}{2}((B(L_0)^{-1} + B^*(L_0^*)^{-1})(1 - e^{-\infty L_0})(1 - e^{-\infty L_0^*})$ (4.24)

with the gauge-fixing condition $(B + B^*)\Psi = 0$, which gives the decomposition (4.2) with $1 - (i\pi)_{\text{mean}} \equiv (1 - e^{-\infty L_0})(1 - e^{-\infty L_0^*})$. Both of (4.23) and (4.24) reduces to the ordinary propagator $(B + B^*)(L_0 + L_0^*)^{-1}$ with the gauge-fixing condition $(B + B^*)\Psi = 0$ when $B_{(g)}^* = B_{(g)}$ holds.

**Dressed $B_0^-$ gauge**

Let $z$ be now a coordinate of the silver frame. We set $B_0^- = B_0 + B_0^*$ for $B_0$ defined by $v(z) = z$ of (4.22). Although the $B_0^-$ gauge would be understood as a special case of the linear $b$-gauge defined in the silver frame, it may have more unconventional or non-perturbative aspects. We can regard it as a gauge-fixing condition based on the star product multiplications. In the silver frame, the conformal stress tensor $T(z)$ naturally defines a state

$$K \equiv \int_{-\infty}^{\infty} dz T(z) \mid id \rangle,$$

where $\mid id \rangle$ denotes the identity state of the star product. By using any functions $F = F(K)$ and $G = G(K)$ of the string field $K$, where multiplications are given by the star product $\ast$, we can consider the operator $B_{F,G}$ defined by

$$B_{F,G} \Phi \equiv \frac{1}{2} F(K) \ast B_0^- \left( F(K)^{-1} \ast \Phi \ast G(K) \right) \ast G(K).$$

(4.26)

Since the interactions of open string fields are given by the star product, (4.26) gives a gauge-fixing condition $B_{F,G} \Phi = 0$ written by using information of interacting terms and would be unconventional in a perturbation from the free theory. While the linear $b$-gauge is written in terms of the free theory or the world-sheet theory, the dressed $B_0^-$ gauge needs the star product defining the interacting term and deviates from the free theory. In this sense, it seems that we cannot use (4.26) within an ordinary perturbation from the free theory. It however gives a Hodge decomposition of operators acting on the identity state, which implies that we can apply the homological perturbation. As long as the gauge-fixing condition $B_{F,G} \Phi = 0$ is valid, which is just an assumption unfortunately, it gives (4.17) correctly. For any state $\Phi \in \mathcal{H}$, the identity state $\mid id \rangle$ satisfies

$$\mid id \rangle \ast \Phi = \Phi = \Phi \ast \mid id \rangle.$$  

(4.27)

Recall that we can represent a given state $\Psi$ as a set of operators $\mathcal{O}_\Psi$ acting on the conformal vacuum $\mid 0 \rangle$. Likewise, we may represent $\Psi$ as a set of operators $\bar{\Psi}$ acting on the identity state $\mid id \rangle$,

$$\mathcal{O}_\Psi \mid 0 \rangle = \Psi = (\bar{\Psi})_L \mid id \rangle = (\bar{\Psi})_R \mid id \rangle,$$

(4.28)

where $(\bar{\Psi})_L \Phi = \Psi \ast \Phi$ and $(\bar{\Psi})_R \Phi = (-)^\Phi \Phi \ast \Psi$ for any state $\Phi \in \mathcal{H}$. The propagator (4.26) gives a decomposition on the identity state and in this sense reproduces (4.17).

---

13 For singular $v(z)$, such as a silver frame, we can obtain correct on-shell tree amplitudes. However, for loops, suggests a gauge dependent result.

14 In principle, more unconventional propagator $\frac{1}{2}(B(L_0)^{-1} + B^*(L_0^*)^{-1})$ may be allowed since $(i\pi)$ does not have to be a projector to apply the homological perturbation, which gives $(i\pi) = \frac{1}{2}(e^{-\infty L_0} + e^{-\infty L_0^*})$. 

33
Tachyon-vacuum homotopy operator

In open string field theory, in addition to the perturbative vacuum, the tachyon vacuum is well studied \[24, 28\]. As is known, the tachyon vacuum has empty cohomology, which leads an interesting Hodge decomposition. We show that it provides tree S-matrices based on unconventional propagators whose 4-point amplitude naturally gives the gauge invariant quantity given by \[11\].

Let us consider the tachyon vacuum solution \(\Psi_T\) of Witten’s open string field theory

\[
Q \Psi_T + \Psi_T \ast \Psi_T = 0 .
\] (4.29)

We write \(Q_T\) for the BRST operator around the tachyon vacuum \(\Psi_T\). In terms of (4.28), it can be written as \(Q_T = Q + (\hat{\Psi}_T)_L - (\hat{\Psi}_T)_R\). It is known that the tachyon vacuum has no cohomology and there exist a state \(A_T\) satisfying

\[
Q_T A_T = | id \rangle
\] (4.30)

and \(A_T \ast A_T = 0\). We assume \(A_T \in \mathcal{H}\) and call it a homotopy contracting state \[24\]. Let us consider an operator \(\hat{1}\) defined by \(\hat{1} \Phi \equiv | id \rangle \ast \Phi = \Phi \ast | id \rangle\) and a state \(W \in \mathcal{H}\) defined by

\[
W \equiv \Psi_T \ast A_T + A_T \ast \Psi_T .
\] (4.31)

Since \(Q | id \rangle = 0\) and also \(Q_T | id \rangle = 0\), the relation (4.30) can be cast as

\[
\hat{1} - \hat{W} = Q \hat{A}_T + \hat{A}_T Q ,
\] (4.32)

where \(2 \hat{W} \equiv (\hat{W})_L + (\hat{W})_R\) and \(2 \hat{A}_T \equiv (\hat{A})_L + (\hat{A})_R\). The operator \(\hat{W}\) commutes with \(Q\) and \(\hat{A}_T\) because of \(QW = 0\) and \(W \ast A_T = A_T \ast W\) respectively. We find that the operator

\[
\hat{\kappa}^{-1} \equiv \hat{A}_T (\hat{1} - \hat{W})^{-1}
\] (4.33)

solves the Hodge decomposition (4.12) on \(| id \rangle\). The expression (4.33) should be understood as (4.21) and determine \(\hat{\kappa}\) naturally. If the subspace \(\mathcal{H}_{(1-W)^{-1}} \equiv \hat{\kappa} \mathcal{H}\) equals to the \(Q\)-cohomology \(\mathcal{H}_{\text{phys}}\), we will obtain the on-shell amplitudes (4.17) correctly. Unfortunately, as the case of (4.20), the condition \(\mathcal{H}_{(1-W)^{-1}} = \mathcal{H}_{\text{phys}}\) is just an assumption. However, we can check that (4.33) indeed gives a correct 4-point amplitude. For any state \(\Phi \in \mathcal{H}\), we find

\[
\hat{\kappa}^{-1} \hat{W} \Phi = - (A_T - \hat{\kappa}^{-1}) \Phi .
\] (4.34)

It resembles (4.20) and can be understood as separating the main contribution from the boundary contribution. By using the cyclic property, the 4-point amplitude (4.17) reduces to

\[
\mathcal{A}_4(\Psi', ..., \Psi') = -\frac{1}{2} \langle (\hat{\kappa}^{-1} - \hat{A}_T) \hat{\Psi}' (\hat{W} \hat{\Psi}')^3 \rangle_{\text{sliver}},
\]

where \(\langle ... \rangle_{\text{sliver}}\) denotes the correlation function of the conformal field theory on the sliver frame. As shown by [11], the expression (4.35) reproduces the Veneziano amplitude if we identify \(\hat{\Psi}'\) and \(\hat{\kappa}^{-1}\) of (4.33) with \(\mathcal{O}_i\) and \(A_0\) of [11], which supports the validity of (4.33) as a propagator.

Clearly, the propagator (4.33) needs the star product multiplications as the dressed \(B_0^-\) gauge. In order for calculations without using the star product explicitly, instead of (4.33), we may have to use the operator \(\mathcal{O}_{\kappa^{-1}}\ of \mathcal{O}_{\kappa^{-1}} | 0 \rangle = | \kappa^{-1} \rangle = \hat{\kappa}^{-1} | id \rangle\) and the decomposition

\[
1 - \hat{\kappa} = Q \mathcal{O}_{\kappa^{-1}} + \mathcal{O}_{\kappa^{-1}} Q ,
\] (4.36)

which should be defined by some combination of oscillators appearing in the world-sheet theory concretely. Note that the gauge-fixing condition should be understood as \(\mathcal{O}_{\kappa^{-1}} \Psi = 0\).
5 Conclusion and Discussions

We showed that the perturbative path-integral can be performed as a morphism of the (quantum) $A_{\infty}$ structure intrinsic to each quantum field theory, which is a result of the homological perturbation. As we checked explicitly, the homological perturbation for $A_{\infty}$ is an alternative representation of the perturbative path-integral. Therefore, when the original BV master action includes source terms, its effective theory must have a twisted (quantum) $A_{\infty}$ structure. As long as physicists believe that the path-integral condenses configurations of integrated fields onto the on-shell physical ones, our results seem to be a quite natural (or trivial) because the BV formalism itself is based on the homological perturbation and determines the physical states from it.

As we discussed, Homological approaches may enable us to use unconventional propagators for calculating $S$-matrix, which may provide further applications. As we explained, the BV master equation and the intrinsic $A_{\infty}$ structure play central roles in perturbative quantum field theory. Once we solve the BV master equation, we can quickly obtain each quantities given by the perturbative path-integral, such as effective actions or scattering amplitudes. Thus, it would be important tasks to try to derive BV master actions for some superstring field theories [30–33].

We would like to emphasize that such algebraic approaches to Lagrangian field theory have been exploited since long-time before and not new ideas. However, the link between homotopy algebras and the BV formalism have developed recently and minimal models of quantum homotopy algebras are now available [16,34]. We thus believe that it would be worth studying these approaches more explicitly and physicist-friendly in terms of higher algebraic literature. We end this paper by mentioning related earlier works. The earliest and outstanding work would be [1], which introduced quantum $L_{\infty}$ algebras and established the link to the BV master equation. The geometry and meaning of the classical BV formalism were given by various authors in the early days, for example, see [4,5]. Recently, a nice review was given by [35]. Application of minimal models of homotopy algebras to field theory was given by [6], which pointed out that minimal models give S-matrices. Also, [17] is suggestive. Quantum minimal models is given by [16,34] recently. Derivations of S-matrix based on the homological perturbation were given by many authors. For example, see [3,30,38] for the tree level and see [16,18,34] for the loop. The work [10] discussed effective theory and renormalization group by using the $A_{\infty}$ structure. The work [11] presented that the BV formalism is very useful to discuss quantities based on the path-integral, such as renormalization group flow. Also, the works [12,13] derived Wick’s theorem and Feynman rules for finite-dimensional integrals by using BV differentials. The link between solutions of BV master equation and homotopy algebras originates from their operadic relations, which were studied by [3,4,14].

Acknowledgments

H.M. would like to thank Martin Markl and Jan Plumann for helpful discussions of homological perturbation. Also, the authors would like to thank Ted Erler, Carlo Maccaferri and Yuji Okawa for discussions of SFT at the GGI workshop “String Theory from the World-Sheet Perspective”, at the YITP workshop “Strings and Fields 2019”, or at Italian or Czech restaurants.

This work has been supported by GACR Grant EXPRO 19-28268X. The work of T.M. has been supported by the GACR grant 20-25775X. The work of H.M. was supported by Praemium Academiae and RVO: 67985840 within 2019, in which main part of this work was done.

\[\text{Also, homotopy algebras would be more accessible to mathematicians, rather than the BV formalism.}\]
References

[1] B. Zwiebach, “Closed string field theory: Quantum action and the B-V master equation,” Nucl. Phys. B 390, 33 (1993) [arXiv:hep-th/9206084].

[2] I. A. Batalin and G. A. Vilkovisky, “Gauge Algebra and Quantization,” Phys. Lett. 102B (1981) 27. I. A. Batalin and G. A. Vilkovisky, “Quantization of Gauge Theories with Linearly Dependent Generators,” Phys. Rev. D 28 (1983) 2567 Erratum: [Phys. Rev. D 30 (1984) 508].

[3] M. Henneaux and C. Teitelboim, “Quantization of gauge systems,” (Princeton University Press), 1992.

[4] A. S. Schwarz, “Geometry of Batalin-Vilkovisky quantization,” Commun. Math. Phys. 155 (1993) 249 [hep-th/9205088].

[5] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich, “The Geometry of the master equation and topological quantum field theory,” Int. J. Mod. Phys. A 12 (1997) 1405 [hep-th/9502010].

[6] S. Barannikov, “Solving the noncommutative Batalin-Vilkovisky equation,” Lett. Math. Phys. 103 (2013) 605 [arXiv:1004.2253 [math.QA]]. Modular Operads and Batalin-Vilkovisky Geometry, International Mathematics Research Notices, Vol. 2007, rnm075. [arXiv:1710.08442 [math.QA]]

[7] M. Doubek, B. Jurco and K. Munster, “Modular operads and the quantum open-closed homotopy algebra,” JHEP 1512 (2015) 158 [arXiv:1308.3223 [math.AT]].

[8] A. Sen, “Wilsonian Effective Action of Superstring Theory,” JHEP 1701 (2017) 108 [arXiv:1609.00459 [hep-th]].

[9] H. Matsunaga, “Light-cone reduction of Wittens open string field theory,” JHEP 1904 (2019) 143 [arXiv:1901.08555 [hep-th]].

[10] T. Erler, H. Matsunaga, to appear.

[11] T. Masuda and H. Matsunaga, “Deriving on-shell open string field amplitudes without using Feynman rules,” arXiv:1908.09784 [hep-th].

[12] M. Markl, “Loop homotopy algebras in closed string field theory,” Commun. Math. Phys. 221 (2001) 367 [hep-th/9711045].

[13] M. Herbst, “Quantum A-infinity structures for open-closed topological strings,” hepth/0602018.

[14] K. Munster and I. Sachs, “Quantum Open-Closed Homotopy Algebra and String Field Theory,” Commun. Math. Phys. 321 (2013) 769 [arXiv:1109.4101 [hep-th]].

[15] M. Crainic, “On the perturbation lemma, and deformations,” arXiv:math/0403266 [math.AT].

[16] M. Doubek, B. Jurco and J. Pulmann, “Quantum $L_{\infty}$ Algebras and the Homological Perturbation Lemma,” arXiv:1712.02696 [math-ph].

[17] C. Albert, “Batalin-Vilkovisky Gauge-Fixing via Homological Perturbation Theory,” https://math.unice.fr/~patras/CargeseConference/ACQFT09_CarloALBERT.pdf

[18] J. Pulmann, “S-matrix and homological perturbation lemma,” Master Thesis, 2016, Mathematical Institute of Charles University.

[19] K. Munster and I. Sachs, “Homotopy Classification of Bosonic String Field Theory,” Commun. Math. Phys. 330 (2014) 1227 [arXiv:1208.5626 [hep-th]].

[20] Y. Aisaka and Y. Kazama, “Relating Green-Schwarz and extended pure spinor formalisms by similarity transformation,” JHEP 0404 (2004) 070 [hep-th/0404141].

[21] M. Kato and K. Ogawa, “Covariant Quantization of String Based on BRS Invariance,” Nucl. Phys. B 212 (1983) 443.

[22] E. Witten, “Noncommutative Geometry and String Field Theory,” Nucl. Phys. B 268, 253 (1986).
[23] A. Sen, “String Field Theory as World-sheet UV Regulator,” JHEP 1910 (2019) 119 [arXiv:1902.00263 [hep-th]].
[24] M. Kiermaier, A. Sen and B. Zwiebach, “Linear b-Gauges for Open String Fields,” JHEP 0803 (2008) 050 [arXiv:0712.0627 [hep-th]].
[25] M. Kiermaier and B. Zwiebach, “One-Loop Riemann Surfaces in Schnabl Gauge,” JHEP 0807 (2008) 063 [arXiv:0805.3701 [hep-th]].
[26] T. Erler and M. Schnabl, “A Simple Analytic Solution for Tachyon Condensation,” JHEP 0910 (2009) 066 [arXiv:0906.0979 [hep-th]].
[27] A. Sen, “Universality of the tachyon potential,” JHEP 9912 (1999) 027 [hep-th/9911116].
[28] M. Schnabl, “Analytic solution for tachyon condensation in open string field theory,” Adv. Theor. Math. Phys. 10 (2006) no.4, 433 [hep-th/0511286].
[29] I. Ellwood, B. Feng, Y. H. He and N. Moeller, “The Identity string field and the tachyon vacuum,” JHEP 0107 (2001) 016 [hep-th/0105024].
[30] N. Berkovits, “Constrained BV Description of String Field Theory,” JHEP 1203 (2012) 012 [arXiv:1201.1769 [hep-th]].
[31] H. Matsunaga, “Notes on the Wess-Zumino-Witten-like structure: $L_\infty$ triplet and NS-NS superstring field theory,” JHEP 1705 (2017) 095 [arXiv:1612.08827 [hep-th]].
[32] T. Erler, “Superstring Field Theory and the Wess-Zumino-Witten Action,” JHEP 1710 (2017) 057 [arXiv:1706.02629 [hep-th]].
[33] H. Matsunaga and M. Nomura, “On the BV formalism of open superstring field theory in the large Hilbert space,” JHEP 1805 (2018) 020 [arXiv:1802.04171 [hep-th]].
[34] C. Braun and J. Maunder, “Minimal models of quantum homotopy Lie algebras via the BV-formalism,” J. Math. Phys. 59 (2018) no.6, 063512 [arXiv:1703.00082 [math.QA]].
[35] B. Jurco, L. Raspollini, C. Samann and M. Wolf, “$L_\infty$-Algebras of Classical Field Theories and the Batalin-Vilkovisky Formalism,” Fortsch. Phys. 67 (2019) no.7, 1900025 [arXiv:1809.09899 [hep-th]].
[36] H. Kajiura, “Noncommutative homotopy algebras associated with open strings,” Rev. Math. Phys. 19 (2007) 1. [math/0306332 [math-qa]]. H. Kajiura, “Homotopy algebra morphism and geometry of classical string field theory,” Nucl. Phys. B 630 (2002) 361. [hep-th/0112228].
[37] S. Konopka, “The S-Matrix of superstring field theory,” JHEP 1511 (2015) 187 [arXiv:1507.08250 [hep-th]].
[38] T. Macrelli, C. Samann and M. Wolf, “Scattering amplitude recursion relations in Batalin-Vilkovisky-quantizable theories,” Phys. Rev. D 100 (2019) no.4, 045017 [arXiv:1903.05713 [hep-th]].
[39] B. Jurco, T. Macrelli, C. Samann and M. Wolf, “Loop Amplitudes and Quantum Homotopy Algebras,” arXiv:1912.06695 [hep-th].
[40] T. Nakatsu, “Classical open string field theory: A(infinity) algebra, renormalization group and boundary states,” Nucl. Phys. B 642 (2002) 13 [hep-th/0105272].
[41] K. J. Costello, “Renormalisation and the Batalin-Vilkovisky formalism,” arXiv:0706.1533 [math.QA].
[42] O. Gwilliam and T. Johnson-Freyd, “How to derive Feynman diagrams for finite-dimensional integrals directly from the BV formalism,” Topology and quantum theory in interaction, 175-185, Contemp. Math., 718, Amer. Math. Soc., Providence, RI, 2018 [arXiv:1202.1554 [math-ph]].
[43] T. Johnson-Freyd, “Homological perturbation theory for nonperturbative integrals,” Lett. Math. Phys. 105 (2015) no.11, 1605 [arXiv:1206.5319 [math-ph]].
[44] B. Jurco, Presentation at the Solvay workshop on “Higher Spin Gauge Theories, Topological Field Theory and Deformation Quantization”, Brussels, February 17-21, 2020.