GENERIC MEAN CURVATURE FLOWS WITH CYLINDRICAL SINGULARITIES

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Abstract. This paper examines the dynamics of mean curvature flow as it approaches a cylindrical singularity. We reveal the mechanism for the isolatedness of cylindrical singularities in terms of the normal form of the asymptotic expansion of the rescaled mean curvature flow and formulate the notion of nondegeneracy of a cylindrical singularity.

Our findings show that, generically, such a non-degenerate singularity is robust and isolated. In the presence of more complicated singularity sets such as in the example of a marriage ring, we demonstrate how to make the first-time singularity set a singleton by an arbitrarily small initial perturbation. These discoveries have implications for the level set flows’ generic low regularity and the type-I nature of generic rotational MCFs. Additionally, the paper introduces the concept of a “firecracker” type singularity and its significance in studying the isolatedness of cylindrical singularities. The research combines techniques and concepts from geometric flows, dynamical systems, and semilinear heat equations.

1. Introduction

The paper focuses on investigating generic mean curvature flows with cylindrical singularities. Mean curvature flow (MCF) is a fundamental geometric flow that has attracted considerable attention in diverse fields, including geometry, partial differential equations, and applied mathematics.

MCF is defined as a family of hypersurfaces \( \{M_t\}_{t \in I} \) evolving in \( \mathbb{R}^{n+1} \) according to the equation \( \partial_t x = \vec{H}(x) \), where \( \vec{H} \) is the mean curvature vector. Singularities must appear in MCFs that originate from smooth embedded closed hypersurfaces. Therefore, understanding singularities is crucial to understanding MCFs. The analysis of singularities involves a blow-up procedure. Huisken [Hui90] introduced the concept of rescaled mean curvature flow (RMCF) to study these singularities. The RMCF is a family of hypersurfaces \( \{M_t\}_{t \in [0, \infty)} \) satisfying the equation the equation \( \partial_t x = \vec{H} + \frac{x}{2} \). When the first singularity of the mean curvature flow \( \{M_t\}_{\tau \in [-1, 0)} \) occurs at the spacetime point \((0, 0) \in \mathbb{R}^{n+1} \times \mathbb{R}\), the corresponding RMCF \( \{M_t\}_{t \in [0, \infty)} \) that captures this singularity is defined as follows:

(1.1) \( M_t = e^{1/2}M_{e^{-t}}, \quad t \in [0, \infty). \)

The singularity model, known as a shrinker, is obtained as a (subsequential) limit of the RMCF, and it satisfies the equation \( \vec{H} + \frac{x}{2} = 0 \).

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The RMCF can be viewed as a dynamical system, a perspective introduced by Colding-Minicozzi in a series of papers [CM12, CM15, CM19, CM21]. To describe this system, let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface, and define its Gaussian area as follows:

$$\mathcal{F}(\Sigma) := (4\pi)^{-n/2} \int_{\Sigma} e^{-\frac{|x|^2}{4}} d\mathcal{H}^n(x),$$

where $\mathcal{H}^n$ represents the $n$-dimensional Hausdorff measure. In this context, a RMCF $M_t$ corresponds to the negative gradient flow of the Gaussian area, and a shrinker is a fixed point of this flow.

The investigation of generic mean curvature flows (MCFs) naturally arose due to the intricate dynamics exhibited by MCFs. Specifically, two major observations motivated this study:

1. The abundance of shrinkers: The vast number of shrinkers makes it seemingly impossible to provide a complete classification of them.
2. Complexity of singular sets: When focusing on generic shrinkers, as classified by Colding-Minicozzi [CM12], the associated singular sets can exhibit highly complicated behavior [CM19]. In particular, there exist MCFs with continuous families of singularities, as illustrated by the marriage ring example below.

Therefore, a natural question arises: Are shrinkers and singular sets more tractable when we study MCFs with generic initial data? In this context, the following two conjectures, found in the problem list of Ilmanen [Ilm03], bear significance. Conjecture 1.2 was also noted by Colding-Minicozzi-Pedersen in [CMP15, Conjecture 7.1].

**Conjecture 1.1** (Huïskens conjecture). \(\text{MCF in } \mathbb{R}^3 \text{ with generic initial data has only spherical and cylindrical singularities.}\)

**Conjecture 1.2** (Generic isolatedness conjecture). \(\text{MCF with generic initial data only has isolated spacetime singularities.}\)

Colding and Minicozzi initiated the program of investigating the generic dynamics of mean curvature flows (MCFs) through a series of papers [CM12, CM19, CM21]. In their work [CM12], Colding and Minicozzi focused on the variational stability of shrinkers. They introduced a quantity known as entropy and demonstrated that only spheres and cylinders are entropy stable shrinkers. Moreover, they proposed a method to avoid unstable singularities by perturbing the initial conditions close to the singular time in subsequent papers [CM19, CM21]. Additionally, in [SX21b, SX21a] and [CCMS20], strategies were developed to avoid unstable singularities modeled on compact shrinkers and conical ones through generic perturbations of initial conditions, via different ideas and techniques. Recently, Chodosh-Choid-Schulze [CCS23] also developed a strategy to avoid unstable singularities modeled on asymptotically cylindrical shrinkers.

It is worth noting that in [Wan16], Lu Wang proved that compact and asymptotically conical shrinkers, as well as asymptotically cylindrical shrinkers, are all possible types of
shrinkers in $\mathbb{R}^3$. Furthermore, Ilmanen conjectured that asymptotically cylindrical shrinkers must coincide with the standard cylinder.

To approach Conjecture 1.2, we assume the generalized Huisken conjecture in $\mathbb{R}^{n+1}$, and this allows us to focus on proving the generic isolatedness of spherical and cylindrical singularities. A singularity is called spherical or cylindrical if it is respectively modeled by a sphere $S^n(\sqrt{2n})$ or a generalized cylinder $\Sigma^k := S^{n-k}(\rho) \times \mathbb{R}^k$, $k = 1, \ldots, n$, where $\rho = \sqrt{2(n-k)}$ denotes the radius of the sphere. It is known that spherical singularities are necessarily isolated. In this paper, our primary focus is on studying generic MCF with cylindrical singularities as a step towards making progress on Conjecture 1.2. We establish and prove Conjecture 1.2 in two distinct settings:

- Locally, we show that each cylindrical singularity can be made isolated within a parabolic ball through a generic perturbation of the initial condition (Theorem 1.6, 1.9 and 1.10).
- Globally, we investigate the first-time singular set of a generic mean convex MCF. We demonstrate that it either contains finitely many isolated singularities or includes a singularity known as the firecracker (Definition 1.14, Theorem 1.15). Additionally, for MCFs with a first-time singular set in the form of a smooth curve, such as in the case of the marriage ring example, we can find an arbitrarily small initial perturbation that results in a perturbed MCF having a single first-time singularity (Theorem 1.12).

In addition to addressing Conjecture 1.2, we have also made progress in areas unrelated to the conjecture but of independent interest.

- A new and significantly simplified proof of $C^1$ and $H^1$ normal form expansion of RMCF near a generalized cylinder (Theorem 1.3).
- The level set flow starting from a generic mean convex hypersurface has low regularity (Theorem 1.17).
- Concerning rotational graphs, we have shown that the MCF generically exhibits first-time singularities that are of type I (Theorem 1.18).
- The first-time singularity of MCF starting from a generic surface in $\mathbb{R}^3$ either has finitely many isolated singularities, or some pathological behavior shows up (Theorem 1.19).

In this paper, our focus is solely on the MCF of closed embedded hypersurfaces. However, it is worth noting that several of our results have potential applications in other scenarios, including noncompact MCF or immersed MCF. Our basic assumption is that

\[(\star) \ \{M_\tau\}_{\tau \in [-1,0)} \text{ is an MCF with a cylindrical singularity modeled on } \Sigma^k \text{ at the spacetime point } (0,0), \text{ and } \{M_t\}_{t \in [0,\infty)} \text{ is its associated RMCF that converges to } \Sigma^k \text{ in the } C^{\infty}_{\text{loc}} \text{-sense.} \]

Our first result is the asymptotic of the RMCF in the weighted Sobolev norm ($H^1$ norm).

### 1.1. Generic isolatedness for RMCFs

The starting point of our research is the following crucial result, which presents a normal form for the RMCF that is asymptotic to the cylinder. We employ coordinates denoted as $z = (\theta, y) \in S^{n-k}(\rho) \times \mathbb{R}^k \subset \mathbb{R}^{n-k+1} \times \mathbb{R}^k$. Here, $\{y_i\}_{i=1}^k$
represents the coordinates on $\mathbb{R}^k$, and $\{\theta_j\}_{j=1}^{n-k+1}$ denotes the restriction of the coordinate functions of $\mathbb{R}^{n-k+1}$ to $\mathbb{S}^{n-k}(\theta)$.

**Theorem 1.3** ($H^1$ normal form theorem). Assume $(\star)$. Then there exist $K > 0$ and $T > 0$ such that for $t > T$, the RMCF is a graph over $\Sigma^k \cap B_{K\sqrt{t}}$, and the graphical function $u(\cdot, t) : \Sigma^k \cap B_{K\sqrt{t}} \to \mathbb{R}$ of $M_t \cap B_{K\sqrt{t}}$ has the following $H^1$ normal form in the (weighted) up to a rotation in $\mathbb{R}^k$

\begin{equation}
(1.2) \quad u(z, t) = \sum_{i \in I} \frac{\theta}{4t}(y_i^2 - 2) + O(1/t^2),
\end{equation}
as $t \to \infty$, where $I \subset \{1, 2, \ldots, k\}$.

We also prove the following $C^1$ normal form theorem.

**Theorem 1.4** ($C^1$ normal form theorem). Denote

\begin{equation}
(1.3) \quad f(z, t) = \theta \sqrt{1 + \sum_{i \in I} \frac{(y_i^2 - 2)}{2t}} - \theta
\end{equation}

and assume $(\star)$. Then there exist $K > 0$ and $T > 0$ such that for $t > T$, the RMCF is a graph over $\Sigma^k \cap B_{K\sqrt{t}}$, and the graphical function $u(\cdot, t) : \Sigma^k \cap B_{K\sqrt{t}} \to \mathbb{R}$ has the following $C^1$ normal form in $B_{K\sqrt{t}}$ up to a rotation in $\mathbb{R}^k$

\begin{equation}
(1.4) \quad u(z, t) = f(z, t) + o(1),
\end{equation}
as $t \to \infty$, where $I \subset \{1, 2, \ldots, k\}$.

Indeed, the normal form presented in this paper is not a new theorem. A similar normal form was previously obtained by Gang Zhou in a series of works [Gan21, Gan22] for the case of $n = 4$ and $k = 3$, and for a larger graphical radius. However, the novelty in our work lies in the significantly simpler proof of the normal form. Our approach leverages results from the study of semi-linear heat equations, particularly Velázquez’s work [Vel92, Vel93], in combination with ideas from dynamical systems. This combination of techniques allows for a more straightforward and efficient derivation of the normal form for the RMCF near the cylindrical singularities.

It is worth mentioning that in the investigation of ancient flows, Angenent-Daskalopoulos-Sesum [ADS19] made the novel discovery that the asymptotic form can provide insights into the geometry of ancient flows. Furthermore, in recent research on the classification of ancient solutions of mean curvature flow [ADS19, ADS20, BC19, BC21, CHH22, CHHW22, DZ22], the asymptotic expansion of the rescaled mean curvature flow (RMCF) over the cylinders at $-\infty$ time plays a crucial role. However, it is important to note that the backward expansion differs from the forward expansion presented in this work, yielding distinct geometric information.

From the normal form that we derived, a natural definition arises.

**Definition 1.5.** A cylindrical singularity as in Theorem 1.3 (equivalently as in Theorem 1.4) is nondegenerate if $I = \{1, 2, \ldots, k\}$, and it is degenerate otherwise.
The concept of nondegeneracy for neckpinching singularities was initially introduced and investigated by Angenent-Velázquez [AV97] in the context of rotationally symmetric mean curvature flows. In their work, they studied the cylindrical singularity that is modeled by $\Sigma^1$, and they also provided examples of mean curvature flows with degenerate singularities.

In this paper, we study several geometric and dynamic properties of nondegenerate singularities. The first significant result from the normal form is the following theorem concerning the isolatedness of nondegenerate cylindrical singularities.

**Theorem 1.6** (Isolatedness theorem). Assume $(\star)$ and the singularity is nondegenerate, then the singularity is isolated. In other words, there exists $\delta_1 > 0$ such that $(0,0)$ is the only singularity of $M_\tau$ in the spacetime neighborhood $B_{\delta_1} \times (-\delta_1^{1/2},0]$.

Another relevant property is the mean convex neighborhood. The mean convex neighborhood property is related to the nonfattening property of level set flow, see [HW20]. It was conjectured that any cylindrical singularity has a mean convex neighborhood. This conjecture was proved by Choi-Haslhofer-Heshkovitz [CHH22] for dimension $n = 2$, by Choi-Haslhofer-Heshkovitz-White [CHHW22] for dimension $\geq 3$ with 2-convexity assumption, and by Gang Zhou [Gan21] for the singularity model $S^1 \times \mathbb{R}^3$ with a nondegeneracy assumption. It is noteworthy that in our case as well, the nondegeneracy assumption is required.

**Theorem 1.7** (Mean convex neighbourhood of a nondegenerate singularity). Assume $(\star)$ and that the singularity is nondegenerate. Then there exist constants $\delta_2 > 0$ and $\tau_0 > 0$, such that for any $\tau \in (-\tau_0,0)$, $M_\tau \cap B_{\delta_2}(0)$ has positive mean curvature and is diffeomorphic to $\Sigma^k$.

Another crucial property of nondegenerate singularities is the type-I curvature condition. A singularity $(y,T)$ of a mean curvature flow (MCF) is classified as type-I if the curvature blows up with a speed of at most $O(1/\sqrt{T-\tau})$. This means that there exist constants $r > 0$ and $C > 0$ such that for any $t < T$, in $M_t \cap B_r(y)$, we have $|A|(x,t) \leq \frac{C}{\sqrt{T-\tau}}$, where $A$ represents the second fundamental form. On the other hand, if the curvature blows up faster than this rate, the singularity is called type-II. There are examples of type-II singularities; for instance, in [AAG95], the authors constructed a surface called a “peanut,” and the MCF starting from such a peanut develops type-II singularities known as “degenerate neckpinching.” In addition, Angenent-Velázquez [AV97] constructed a large family of type-II singularities in their work.

**Theorem 1.8** (Type-I curvature condition of a nondegenerate singularity). Assume $(\star)$ and that the singularity is nondegenerate. Then there exist constants $\delta_2 > 0$ and $\tau_0 > 0$, such that for any $\tau \in (-\tau_0,0)$, $M_\tau \cap B_{\delta_2}(0)$ has curvature bound $|A| \leq C(-\tau)^{-1/2}$ for some constant $C$.

Next, we show that for degenerate singularities, we can always find an arbitrarily small perturbation to make them nondegenerate (Theorem 1.9). Moreover, nondegenerate singularities are robust, as outlined in Theorem 1.10.
Theorem 1.9 (Denseness of nondegenerate singularities). Assume \((\star)\) and that the singularity is degenerate. Then for any \(\varepsilon > 0\) and \(\delta > 0\), there exists \(f : \mathcal{M}_{-1} \to \mathbb{R}\) with \(\|f\|_{C^2} \leq \varepsilon\), the perturbed MCF \(\{\tilde{M}_\tau\}\) starting from \(\tilde{M}_{-1} := \{x + f(x)n : x \in \mathcal{M}_{-1}\}\) has a singularity in a \(\delta\)-neighbourhood of \((0, 0)\), that is modeled by \(\Sigma^k\), and is nondegenerate.

The stability of nondegenerate cylindrical singularities is a highly subtle matter. In [CM12], Colding-Minicozzi introduced a quantity called entropy defined as

\[
\lambda(\Sigma) = \sup_{x_0 \in \mathbb{R}^{n+1}, t_0 \in (0, \infty)} \int_\Sigma e^{-\frac{|x-x_0|^2}{4t_0}} \, d\mathcal{H}^n(x).
\]

The entropies of spheres and cylinders were calculated by Stone and follow a particular order:

\[
\lambda(\Sigma^{n-1}) > \lambda(\Sigma^{n-2}) > \cdots > \lambda(\Sigma^1) > \lambda(\mathbb{S}^n(\sqrt{2}n)) > 1 = \lambda(\mathbb{R}^{n+1}).
\]

The entropy is invariant under translations and dilations. Notably, for a shrinker \(\Sigma\), it was proven that \(\lambda(\Sigma) = F(\Sigma)\). Additionally, Colding-Minicozzi showed in [CM12, Theorem 0.12] that all generalized cylinders are entropy stable.

However, in general, a degenerate cylindrical singularity is not stable. For instance, the mean curvature flow starting from the “peanut” surface constructed in [AAG95] has a degenerate cylindrical singularity that can be perturbed into either a spherical or a cylindrical singularity depending on the chosen perturbations. A perturbation of an MCF with a degenerate first time cylindrical singularity may have the first time singularity with strictly less entropy value, no matter how small the perturbation is. Nonetheless, our theorem demonstrates that perturbations can be constructed to make a degenerate cylindrical singularity nondegenerate without reducing its entropy.

Indeed, nondegenerate singularities exhibit a remarkable stability property under perturbations.

Theorem 1.10 (Stability of nondegenerate singularities). Assume \((\star)\) and the singularity is nondegenerate. Then there is an \(\varepsilon_0 > 0\) such that for all \(u : \mathcal{M}_{-1} \to \mathbb{R}\) with \(\|u\|_{C^2} \leq \varepsilon_0\), the perturbed MCF with initial condition \(\tilde{M}_{-1} = \text{Graph}\{x + u(x)n(x) | x \in \mathcal{M}_{-1}\}\) admits a unique nondegenerate singularity in an \(C\varepsilon_0\)-spacetime neighborhood of \((0, 0)\) modeled by \(\Sigma^k\).

The local stability of cylindrical singularities has been investigated by Schulze-Sesum in [SS20], specifically focusing on the neckpinching case where \(k = 1\). Schulze-Sesum’s study employs geometric measure theory to define nondegeneracy, which differs from the notion presented in our work. Despite the different definitions, there are certain relationships between them. Our Theorem 1.10, similar to Theorem 1.3 of [SS20], implies that a singularity that is nondegenerate in our sense is also nondegenerate in the sense of [SS20]. However, it is important to note that their notion of nondegeneracy and ours are distinct. For instance, the degenerate neckpinching case constructed in [AV97] with the leading term \(H_{2m}\) (where integer \(m \geq 2\)) and a positive coefficient is considered degenerate in our sense, but not degenerate in the sense of [SS20].
1.2. **Pinch a thin ring.** The last two theorems are specifically stated for the RMCF, which means they have a local nature when translated into statements about the MCF. Now, our attention turns to the global problem of perturbing an MCF with a complex singularity set to achieve isolatedness of the first-time singularity.

The “marriage ring” is a classical example of a mean curvature flow (MCF) with a complicated singularity set. This example involves considering a $\text{SO}(n-1)$-invariant hypersurface diffeomorphic to $S^{n-1} \times S^1$ in $\mathbb{R}^{n+1}$, where the $S^1$ factor has an extremely small radius. As a consequence of the symmetry in the configuration, the MCF starting from such a hypersurface undergoes a collapse, leading to a singular set $S^{n-1} \times \{\ast\}$. In the specific case when $n = 2$, the singularity set takes the form of a smooth $S^1$.

Conjecturally, for any closed $C^1$-curve $\gamma \subset \mathbb{R}^3$, there exists an MCF $\{M_\tau\}_{\tau \in [0,\tau_0)}$ such that $M_0$ is the boundary of a tubular neighbourhood of $\gamma$, and $M_\tau$ collapse to a singular set $\gamma$. The authors heard the following conjecture from Colding-Minicozzi.

**Conjecture 1.11** (Colding-Minicozzi). Any $C^1$ curve $\gamma \subset \mathbb{R}^3$ can be the singular set of a (mean convex) MCF.

The $C^1$ regularity of the singular set is required by [CM16a]. We refer to such an MCF as a **thin ring**. White [Whi02, Section 5] conjectured that MCF in $\mathbb{R}^3$ can only have the singular set consisting of isolated singularities and curves.

We show that when the singular set is a smooth closed curve, even a small initial perturbation can lead to an isolated first-time singularity. This result holds true for various scenarios, including the example of a marriage ring. Heuristically, we can create an isolated singularity by slightly squeezing the marriage ring.

**Theorem 1.12.** Suppose $\{M_\tau\}_{\tau \in [-1,0)}$ be a smooth embedded MCF in $\mathbb{R}^{n+1}$ and the singular set at time 0 is a smooth closed curve $\gamma$, and the tangent flows are multiplicity 1 cylinders. Then for any $p \in \gamma$ and any $\varepsilon_0 > 0$ there exists a function $u_0 \in C^{2,\alpha}(M_{-1})$ with $\|u_0\|_{C^{2,\alpha}} < \varepsilon_0$ such that the perturbed MCF $\{\tilde{M}_\tau\}$ starting from $\tilde{M}_{-1} := \{x + u_0(x) \mathbf{n}(x) : x \in M_{-1}\}$ has a single first-time singularity in an $\varepsilon_0$-neighborhood of $p$.

1.3. **Generic mean convex MCFs.** Hypersurfaces in $\mathbb{R}^{n+1}$ are called mean convex (resp. strictly mean convex) if the mean curvature $H \geq 0$ (resp. $H > 0$) at every point. The study of MCF originating from mean convex initial conditions has been extensively studied in the literature such as [HS99, Whi00, Whi03, SW09, Whi15, HK17], among others. It is well-known that mean convexity is preserved under the MCF, and the only possible singularities are either spherical or cylindrical (c.f. [HS99] for first-time singularities and [Whi00, Whi03, Whi15] for later time singularities). Despite this, the singular set for mean convex MCFs can still exhibit a highly complex structure, as illustrated by the marriage ring example discussed in the previous subsection.

In [CM16a], Colding-Minicozzi investigated the partial regularity of the singular set of mean convex MCFs. In this paper, we study the singular set for generic mean convex MCFs. To this end, we consider the space $\mathcal{G}$ comprising mean convex hypersurfaces and introduce the $C^r$-topology with $r > 2$ (refer to [Whi87] for a comprehensive discussion on this space).
**Theorem 1.13.** There is an open dense subset of $G$ such that for any MCF starting from a hypersurface in this set, the singular set consists of at least one isolated singularity that is either spherical or non-degenerate cylindrical.

The main challenge that prevents us from showing that all first-time singularities, not just one, are isolated lies in the lack of uniformity in the size $\delta_1$ of the isolating neighborhood for all singularities. For instance, consider the marriage ring example, where we can introduce perturbations that preserve a discrete subgroup of $S^1$ with arbitrarily large cardinality $N$, resulting in an MCF with exactly $N$ isolated singularities. However, the size of the isolating neighborhood for each singularity is less than $1/N$. Consequently, it becomes difficult to establish a uniform bound on the size of the isolating neighborhoods for all singularities.

Motivated by this issue, we introduce a property called (isolated-or-firecracker):

**Definition 1.14.** A mean convex hypersurface $M$ is said to have property (isolated-or-firecracker), if the first-time singular set of MCF starting from $M$ satisfies one of the following conditions:

1. It consists of finitely many nondegenerate singularities.
2. It contains a degenerate singularity (referred to as a “firecracker”) that is the limit of a sequence of nondegenerate spacetime singularities.

**Theorem 1.15.** The property (isolated-or-firecracker) holds for generic initial data of closed mean convex hypersurfaces.

Indeed, the generic initial data refers to a residual set within the space of mean convex hypersurfaces. The presence of the pathological firecracker singularity in property (isolated-or-firecracker) (2) is highly counter-intuitive, motivating us to propose the following conjecture:

**Conjecture 1.16.** The firecracker singularity exists in MCF, but does not exist generically.

Recently Simon [Sim23] constructed examples of minimal hypersurfaces with any given prescribed compact singular set. It is conceivable that a parabolic version of Simon’s construction would yield a firecracker type singularity for MCF.

### 1.4. Generic level set flows.

Level set flow (LSF) is a weak formulation of MCF, where the evolution of a hypersurface $M_\tau$ is represented by the level sets of a function $f(x, \tau)$ satisfying the equation

$$\partial_\tau f = |\nabla f| \text{div} \left( \frac{\nabla f}{|\nabla f|} \right).$$

The level set flow PDE was first proposed by Osher-Sethian [OS88] in the context of numerical analysis. Later, Chen-Giga-Goto [CGG91] and Evans-Spruck [ES91] proved the existence of a viscosity solution to the level set flow PDE. If $f(x, \tau)$ is a solution to the level set flow PDE and $|\nabla f(x, \tau)| \neq 0$, then $M_\tau := \{ x : f(x, \tau) = 0 \}$ is a smooth MCF when $\{M_\tau\}$ is a family of codimension 1 hypersurfaces. Consequently, one can define a weak solution to MCF as the level sets of a level set flow function. When $M_0 := \{ x : f(x, 0) = 0 \}$ is
mean convex, the MCF evolves monotonically. This allows one to introduce an arrival time function $g(x) := f(x,t) - t$, where $g$ satisfies the equation $-1 = |\nabla g| \cdot \text{div} \left( \frac{\nabla g}{|\nabla g|} \right)$.

The regularity of the LSF is a natural question in the context of PDE. Huisken [Hui93] proved that the arrival time function of a convex hypersurface has at least $C^2$ regularity, and later Sesum [Ses08] constructed examples with at most $C^3$ regularity. In our previous work [SX22], we demonstrated that for a generic convex hypersurface, the arrival time function does not have $C^3$ regularity.

When the initial hypersurface is mean convex but not convex, the regularity becomes more subtle due to the emergence of cylindrical singularities. Colding-Minicozzi [CM16b, CM18] studied the relationship between the singular set’s structure of mean curvature flow and the regularity of the arrival time function. Specifically, they showed that if the singular time is not unique, the arrival time function is $C^{1,1}$ but not $C^2$. In this paper, we prove a generic regularity result for the arrival time function.

**Theorem 1.17.** The space of mean convex closed hypersurfaces (denoted by $\mathcal{V}$) in $\mathbb{R}^{n+1}$ with $C^r$-topology $r \geq 2$ (see [SX22] for details), has the following structure: there is an open and dense subset $U = U_1 \cup U_2$ of $\mathcal{V}$, such that

- Any arrival time function of a hypersurface in $U_1$ has at most $C^2$-regularity. Moreover, the level set flow has only an isolated spherical singularity;
- Any arrival time function of a hypersurface in $U_2$ has at most $C^{1,1}$-regularity. Moreover, the level set flow has at least one isolated cylindrical singularity.

The subspace $U_1$ is studied in [SX22]. In this paper we only focus on the space $U_2$. The explicit definition of $U_2$ is as follows:

$U_2$ is the set of mean convex closed hypersurfaces such that the corresponding LSFs have at least two distinct singular times.

It would also be interesting to know the relation between $U_1$ and $U_2$. The peanut example by Altschuler-Angenent-Giga [AAG95] shows that there exist hypersurfaces that belong to both closures $\overline{U}_1$ and $\overline{U}_2$, showing that the $\overline{U}_1$ and $\overline{U}_2$ are not disjoint.

1.5. **Generic MCF of rotational graphs and type-I condition.** In this section, we explore the application of the main results to rotational graphs. The MCF of rotational graphs was studied in [AAG95], where the authors utilized a Liouville-Sturm theorem to prove that all singularities of the MCF of rotational graphs are isolated. Consequently, firecracker singularities cannot appear.

Let $x \in \mathbb{R}$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ represent the coordinates in $\mathbb{R}^{n+1}$, where $r = |y| = \sqrt{y_1^2 + \cdots + y_n^2}$ is the radius function in $\mathbb{R}^n$. A hypersurface $\Gamma_0$ is called a rotational graph if it is obtained by rotating the graph of a function $r = u(x) > 0$ for $a < x < b$ around the $x$-axis. The MCF starting from such a rotational graph is denoted by $\Gamma(t)$, and the MCF is in the sense of the level set flow.

It is a folklore conjecture that generic MCF has only type-I singularities, see [GIKW21]. In [GIKW21], Garfinkle-Isenberg-Knopf-Wu presented a numerical example to show that in
the non-compact case, type-II blow-up may not be avoidable at infinity. However, in the context of closed MCFs, our analysis of non-degenerate singularities suggests that type-I singularities should be the generic behavior for cylindrical singularities.

**Theorem 1.18.** Starting from a generic rotational graphical closed hypersurface, the first-time singularities of the MCF are either spherical or nondegenerate cylindrical. In both cases, the singularities are type-I.

To make the genericity in Theorem 1.18 precise, we recall that the space of rotational graphs is a Banach manifold, where the local atlas is given by the SO($n$)-invariant $C^r$ functions, which form a closed linear subspace of the Banach space $C^r$. Let us call the space of rotational graphs $R$. The statement of genericity in Theorem 1.18 means that we have an open and dense subset of $R$ for $r \geq 2$ with the desired properties.

1.6. **Generic MCF.** For generic MCFs in $\mathbb{R}^3$ without mean convexity assumption, combined with [CCMS20, SX21b, SX21a], we have the following singularity structure theorem:

**Theorem 1.19.** The first time singular set of an MCF starting from a generic set of surfaces in $\mathbb{R}^3$ contains at least one of the following:

1. A spherical singularity;
2. Isolated cylindrical singularities;
3. Firecrackers;
4. Singularities modeled by shrinkers with cylindrical ends;
5. Singularities modeled by higher multiplicity shrinkers.

The expectation is that for generic mean convex MCF, only (1) and (2) types of singularities occur, and thus Conjecture 1.2 holds, at least for the first-time singular set. However, so far, the tools developed have not been able to rule out the possibility of (3), or (5) type singularities. The nonexistence of (4) and (5) for an arbitrary MCF is conjectured by Ilmanen. Recently, Chodosh-Choi-Schulze [CCS23] showed that (4) type singularities will not occur in generic MCF. It also appears that (5) type singularities do not occur in generic MCF, at least in situations where the model shrinker is closed, as suggested in [Sun23]. Recently Bamler-Kleiner [BK23] proved the Multiplicity One Conjecture by Ilmanen, showing that (5) does not appear for MCF in $\mathbb{R}^3$.

It should be noted that for $n \geq 3$, the MCF in $\mathbb{R}^{n+1}$ can have more complicated singularity types beyond (1) and (2).

1.7. **Outline of the proof.** The methodology used in this work is substantially more challenging and distinct from the one-sided perturbation approach presented in [CM12, CCMS20, SX21b, SX21a]. To illustrate this point, let us consider an example. Suppose $M$ is an $m \times m$ matrix with distinct eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_m$. For a generic vector $v$, we have $\lim_{n \to \infty} \lambda_1^n (M^n v)$ converges to $v_1$, the eigenvector corresponding to $\lambda_1$. However, to obtain $v_2$, the eigenvector corresponding to $\lambda_2$, we need to consider $\lim_{n \to \infty} \lambda_2^n (M^n v)$, where $v$ is
generic in \( \mathbb{R}^n / (\mathbb{R}v_1) \). Therefore, obtaining the eigenvector corresponding to an eigenvalue other than the first eigenvalue from a generic vector is much harder.

In the context of generic MCF, we analyze the linearized operator \( L_{\Sigma} = \Delta - \frac{1}{2}x \cdot \nabla + (|A|^2 + \frac{1}{2}) \) restricted to the shrinker. When dealing with nonspherical and noncylindrical shrinkers, our approach in \([SX21b, SX21a]\) involves introducing an initial positive perturbation to increase the difference between the perturbed RMCF and the unperturbed one towards the direction of the first eigenfunction, analogous to finding \( v_1 \) in the previous paragraph.

However, when dealing with spherical and cylindrical shrinkers, the first several eigenfunctions correspond to translations and dilations, which do not provide new geometric evolution. Instead, new geometric information is hidden in some higher Fourier modes. Hence, the situation is analogous to finding \( v_2 \) in the previous paragraph, and we need to consider the higher eigenfunctions modulo the conformal linear transformations\(^1\). In \([SX22]\), when dealing with spherical singularities, we deciphered the information of regularity of the LSF lying in the eigen direction of the first negative eigenvalue. We then introduced a centering map to effectively modulo the translations and dilation and ensure that an initial perturbation eventually grows towards the eigen direction of the first negative eigenvalue.

However, dealing with cylindrical singularities presents much greater complexity. In this case, there exist functions \( h_2(y_i) = c_2(y_i^2 - 2) \) that lie in the kernel of \( L_{\Sigma} \) and play a crucial role in determining the isolatedness of the singularity. Our objective is to guide our initial perturbation towards \( h_2 \) since it represents the essential Fourier mode. One might initially consider that eliminating the translations and dilations would be sufficient. However, there are also eigenfunctions \( \{y_i\}_{i=1}^k \) associated with a positive eigenvalue \( \frac{1}{2} \) that cannot be eliminated by a conformal linear transformation in general. This observation is a key point in the proof of \( F \)-instability of cylinders as shown in \([CM12]\).

To address this difficulty, we need to understand the geometric significance of the eigenfunctions \( y_i \). The key insight is that these eigenfunctions correspond to translations along the axis direction and can be eliminated by translation if there is a nontrivial \( h_2 \)-component in the normal form (see Remark 2.2). Thus, we encounter a situation where the elimination of \( y_i \) and the revelation of \( h_2 \) are intertwined. In the proof, we introduce a dynamic procedure that involves infinitely many steps of conformal linear transformations along the RMCF to eliminate all the exponentially growing modes progressively.

The second difficulty we encounter is a recurring challenge that arises in the analysis of RMCF with noncompact shrinkers. Specifically, the manifold at each moment \( t \) cannot be globally represented as a graph over the shrinker \( \Sigma_k \), necessitating the introduction of a cutoff. Consequently, a crucial aspect is to obtain a reliable estimate of the solution to the linearized equation along the RMCF, particularly in close proximity to the boundary of the graphical scale where the cutoff is implemented.

To address this difficulty, we proceed in two steps. In Step 1, we initially derive a normal form with a graphical radius of \( O(\sqrt{\log t}) \) using solely dynamical arguments. Then, in Step

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\(^1\)By conformal linear transformations, we mean the group of tranformations generated by rigid transformations in \( \mathbb{R}^{n+1} \) (including translations and rotations) and dilations.
2, we extend the graphical scale from $O(\sqrt{\log t})$ to $O(\sqrt{t})$. To accomplish this extension, we make use of the Ornstein-Uhlenbeck regularization estimate, which was established by Velázquez in [Vel92] and referred to as the "linear regularization effect" in the paper. This estimate allows us to obtain a favorable $C^2$ bound for the RMCF over the region $\Sigma^k \cap B_{K\sqrt{t}}$ for some $K > 0$ as $t \to \infty$. The success of this estimate is attributed to the drift term in the Ornstein-Uhlenbeck operator. As we will observe in the proof, the $O(\sqrt{t})$ graphical radius is vital in providing significant geometric insights for the MCF.

When the manifold is close to the limiting cylinder, we can represent it within the graphical radius as the graph of a function $u$ that evolves according to the equation $\partial_t u = L_{\Sigma} u + Q(u)$, where $Q$ is the explicit nonlinearity defined in (A.2). Our work is influenced by studies on singularities of semi-linear heat equations defined on $\mathbb{R}^k$, which have been investigated by many authors such as [Vel92, Vel93, HV92b, Zaa02], where they study the blow-up of the semi-linear equation $\partial_t u = \Delta u + |u|^{p-1}u$. The blowup of the equation near a type-I singularity involves a similar linear term $L_{\mathbb{R}^k} u$. The power $p$ plays a crucial role as higher powers of $p$ make the system supercritical, leading to more complex singularity models. In our case, the nonlinearity $Q$ is indeed more intricate, as it involves not only high powers but also higher-order derivatives. Here, we utilize the pseudolocality property of MCF to control the derivatives and manage the complexity arising from the nonlinearity $Q$.

Organization of the paper. The paper is organized as follows. In Section 2, we give a preliminary on the eigenfunctions of the $L$-operator. Of particular importance for later proofs is our interpretation of the eigenfunction $y_i$. In Section 3, we show how to extend the graphical scale from $O(\sqrt{\log t})$ to $O(\sqrt{t})$ for the rescaled mean curvature flows with cylindrical singularities. In Section 4, we study the local dynamics of the RMCF near the cylindrical shrinker and prove that the linear part of the equation approximates very well the local dynamics. In Section 5, we prove normal form Theorem 1.3 and 1.4. In Section 6, we prove that nondegenerate singularities are isolated (Theorem 1.6). In Section 7 and 8, we prove the genericity and stability of nondegenerate singularities (Theorem 1.9 and 1.10). In Section 10, we give the proof of the global theorem 1.12. All other theorems mentioned above will be proved in Section 9 as an application of Theorem 1.9 and 1.10. Finally, we have two appendices. In Appendix A we derive the nonlinear term $Q$ in the equation $\partial_t u = Lu + Q$ that is the RMCF equation for the graphical function on $\Sigma^k$. In Appendix B, we present the important Ornstein-Uhlenbeck regularization estimate of Velázquez [Vel92] and Kammerer-Zaag [FKZ00] adapted to our setting (Proof of Proposition 3.8).

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2. Eigenvalues and Eigenfunctions of the $L$-operator

In this section we summarize some previously known results on cylindrical singularities. We will be working in the weighted Sobolev space. Given two functions $f, g$ defined on a hypersurface $\Sigma$, we define an inner product $\langle f, g \rangle_{L^2} = \int_{\Sigma} f(x)g(x)e^{-\frac{|x|^2}{2}}dH^n(x)$. Then we define the weighted $L^2$-norm by $\|f\|_{L^2} = \langle f, f \rangle_{L^2}^{\frac{1}{2}}$, and the weighted $L^2(\Sigma)$ space consists of function $f$ with $\|f\|_{L^2} < +\infty$. Similarly we define weighted higher Sobolev space $H^k$ with $H^k(\Sigma) := \left\{ f : \sum^k_{i=0} \|\nabla^i f\|_{L^2}^2 < +\infty \right\}$. Throughout the paper, for simplicity of notations, we use $\| \cdot \|$ to denote the $H^1$-norm if not otherwise mentioned.

Recall that the linearized operator on a shrinker is defined as $L := \Delta - \frac{1}{2}(x, \nabla \cdot) + (|A|^2 + 1/2)$. In the special case that the shrinker is $\Sigma^k$, we have

\begin{equation}
L_{\Sigma}u = \Delta_{S^{n-k}(\varrho)}u + L_{\mathbb{R}^k}u, \quad L_{\mathbb{R}^k}u = \Delta_{\mathbb{R}^k}u - \sum^k_{i=1} \frac{1}{2}y_i \partial_{y_i}u + u.
\end{equation}

Throughout this paper, the eigenvalue $\lambda$ of an operator $L$ is the number such that there exists a function $f$ such that $Lf = \lambda f$. The following fact was proved in [SWZ20, Section 5.2]: Suppose the eigenvalues of $\Delta$ on $S^{n-k}(\varrho)$ are given by $\mu_0 \geq \mu_1 \geq \mu_2 \geq \cdots$ with corresponding eigenfunctions $\phi_0, \phi_1, \phi_2, \cdots$, and suppose the eigenvalues of $L_{\mathbb{R}^k}$ on $\mathbb{R}^k$ is given by $\nu_0 \geq \nu_1 \geq \nu_2 \geq \cdots$ with corresponding eigenfunctions $\psi_0, \psi_1, \psi_2, \cdots$, then the eigenvalues of $L_{\Sigma}$ are given by $\{\mu_i + \nu_j - 1\}^{\infty}_{i,j=0}$ with corresponding eigenfunctions $\{\phi_i \psi_j\}^{\infty}_{i,j=0}$.

The spectrum of $\Delta_{S^{n-k}(\varrho)}$ is $0, -1/2, -\frac{n-k+1}{n-k}, \cdots$, and the eigenfunctions are known to be the restriction of homogeneous harmonic polynomials. The first several eigenfunctions are listed as follows: constant functions for eigenvalue $0$; $\theta_i$, the restriction of linear functions in $\mathbb{R}^{n-k+1}$ to $S^{n-k}(\varrho)$, for eigenvalue $-1/2$; and $\theta_i^2 - \theta_j^2, \cdots$ for eigenvalue $-\frac{n-k+1}{n-k}$. The spectrum of $L_{\mathbb{R}^k}$ on $\mathbb{R}^k$ is given by half integers $1 - \frac{m}{2}$, $m = 0, 1, 2, \ldots$, and the eigenfunctions for eigenvalue $1 - \frac{m}{2}$ are given by $h_{m_1}(y_1) \cdots h_{m_k}(y_k)$ with $m_1 + \cdots + m_k = m$, where $h_{m_1}(y_1) = c_{m_1}h_{m_1}(y_1/2)$, and $h_{m_1}$ are standard Hermite polynomials with $c_{m_1} = 2^{-m_1/2}(4\pi)^{-1/4}(m_1!)^{-1/2}$ are normalizing factors such that $\|h_{m_1}\|_{L^2(\mathbb{R})} = 1$. In particular, we have $\tilde{h}_0(x) = 1$, $\tilde{h}_1(x) = x$, $\tilde{h}_2(x) = 4x^2 - 2$ and $c_0 = (4\pi)^{-1/4}$, $c_1 = 4^{-1/2}\pi^{-1/4}$, $c_2 = 2^{-2}\pi^{-1/4}$. We shall use the following fact (c.f. Appendix B of [HV92a]).

\textbf{Lemma 2.1.} Let $A_{m,n,\ell} = \int_{\mathbb{R}} h_m(x)h_n(x)e^{-\frac{|x|^2}{2}}dx$. Then $A_{m,n,\ell} = 0$ unless we have $m + n + \ell$ is even and $n \leq m + \ell$, $m \leq n + \ell$, $\ell \leq m + n$, in which case we have

$$A_{m,n,\ell} = (4\pi)^{-1/4}(m!n!\ell!)^{1/2} \left( \left( \frac{m + n - \ell}{2} \right)! \left( \frac{n + \ell - m}{2} \right)! \left( \frac{m + \ell - n}{2} \right)! \right)^{-1}.$$ 

In particular, we have $A_{2,2,2} = 2\pi^{-1/4} = 8c_2$.

Combining the spectra together, we obtain the first three eigenvalues and their corresponding eigenfunctions of $L_{\Sigma^k}$, see Table 1. These eigenfunctions have geometric meanings:
• Constant 1 is the mean curvature on the generalized cylinder, representing infinitesimal (spacetime) dilation.

• $\theta_i$ and $y_j$ are both infinitesimal translations. Specifically, $\theta_i$’s represent the translations in the directions of the spherical components, and $y_j$’s represent the translations in the axis directions (see Remark 2.2).

• $\theta_i y_j$ represents the infinitesimal rotation.

• $h_2(y_i) = c_2(y_i^2 - 2)$ is known to be the non-integrable Jacobi field. It represents non-degenerate neckpinching in the related direction on the axis.

• $y_i, y_j$’s show up if we rotate $h_2(y_i)$’s in the $\mathbb{R}^k$ space. e.g. $(\frac{y_i y_j}{\sqrt{2}})^2 - 2 = \frac{y_i^2 - 2}{2} + \frac{y_j^2 - 2}{2} + \sqrt{2} y_i y_j$.

| eigenvalues of $L_{\Sigma^k}$ | corresponding eigenfunctions |
|-------------------------------|-----------------------------|
| 1                             | $\theta_i y_j$, $i = 1, 2, \ldots, n - k + 1$, $j = 1, 2, \ldots, k$ |
| $1/2$                         | $\theta_i y_j$, $h_2(y_j) = c_2(y_j^2 - 2)$, $y_j y_{j_2}$ |
| $0$                           | $\max\{-1/(n-k), -1/2\} \ldots$ |

Table 1. Eigenvalues and eigenfunctions of $L_{\Sigma^k}$.

Note that while $h_{m_i}(y)$ is a normalized eigenfunction in $\| \cdot \|_{L^2(\mathbb{R})}$, $h_{m_i}(y_i)$ is not a normalized eigenfunction in $\| \cdot \|_{L^2(\Sigma^k)}$. Thus, we define $H_{m_1, m_2, \ldots, m_k}$ be a multiple of $h_{m_1} h_{m_2} \cdots h_{m_k}$, such that $H_{m_1, m_2, \ldots, m_k}$ is a normalized eigenfunction in $\| \cdot \|_{L^2(\Sigma^k)}$. In particular, $H_2(y_i) = c_0^{k-1} G(S^{n-k})^{-1/2} h_2(y_i)$, $H_{1,1}(y_i, y_j) = c_0^{k-2} G(S^{n-k})^{-1/2} h_1(y_i) h_1(y_j)$. Here

$$G(S^{n-k}) = \int_{S^{n-k}(\mathbb{R})} e^{-\frac{|x|^2}{2}} dx = \int_{S^{n-k}(\mathbb{R})} e^{-\frac{|x|^2}{2}} \omega_{n-k},$$

where $\omega_{n-k}$ is the area of the $(n - k)$-dimensional unit sphere.

Among all the eigenfunctions, we would like to draw readers’ attention to $y_i$ corresponding to the eigenvalue $1/2$. Perturbing the cylinder in the $y_i$ direction can decrease the value of the $F$-functional strictly, which cannot be compensated by translations or dilations. This is the key idea in the proof of $F$-instability of cylinders in [CM12, Section 11]. However, one of the key observations in the present paper is that in the presence of a nontrivial $h_2(y_i) = c_2(y_i^2 - 2)$ component, a perturbation in the $y_i$-direction can be killed. Let us elaborate on this point in the following remark.

**Remark 2.2** (Geometric meaning of the eigenfunctions $y_i$). The eigenfunctions $y_j$’s represent the infinitesimal translations in the cylinder axis directions. These translations may not be seen in many cases. For example, if the hypersurface is exactly the generalized cylinder, it is translation invariant in the axis direction, then one can not see the infinitesimal translation represented by $y_i$’s.
However, if the RMCF converges to $\Sigma^k$ in a “pinching” manner, then $y_i$ shows up as an infinitesimal translation. For simplicity, let us suppose $M_t$ is an MCF with a spacetime singularity at $(0,0)$, that is modeled by a generalized cylinder $\Sigma^1 = S^{n-1} \times \mathbb{R}$, where $\mathbb{R}$ has coordinate $y$. We assume the singularity is non-degenerate and in later section, we will prove that the dominant term in the RMCF is $h_2 = c_2(y^2 - 2)$, namely if we write the RMCF $M_t$ as a graph of function $u(\cdot,t)$ over $\Sigma^1$, in some large ball, then $u(\cdot,t) = \frac{t}{2}(y^2 - 2) + o(1/t)$ for some constant $a$. As a consequence, if we take a look at the spacetime blowing up of the MCF at a fixed nearby spacetime point $(\bar{z},0)$ where $\bar{z} = (0,z) \in \mathbb{R}^n \times \mathbb{R}$. Then the RMCF for $(\bar{z},0)$ can be express as a graph of

$$v(\cdot,t) = \frac{a}{t}((e^{t/2}\bar{z} + y)^2 - 2)) + o(1/t) = \frac{ae^t}{t}\bar{z}^2 + 2\frac{a}{t}e^{t/2}\bar{z}y + \frac{a}{t}(y^2 - 2) + o(1/t).$$

Because $\bar{z}$ is a constant, $\frac{ae^t}{t}\bar{z}^2$ represent a spacetime dilation term. Then this implies that $y$ represents the translation in the axis direction, after modulo some dilations.

3. Extension of the graphical scale

The starting point of our work is the rough estimate of order $O(\sqrt{\log t})$ of the size of balls by [CM15], within which the RMCF can be written as the graph of a function, as well as a rough estimate of the decay of $L^2$ norm of the function. To extend the graphical scale from $O(\sqrt{\log t})$ to $O(\sqrt{t})$ as in Theorem 1.3. We shall invoke the Ornstein-Uhlenbeck regularization property of the drifted Laplacian of [FKZ00, Vel92, Vel93] to study the linear equation $\partial_t u = Lu$.

3.1. Coarse graphical scale and $L^2$ estimate. Assuming $(\star)$, by Colding-Minicozzi’s Lojasiewicz inequality [CM15], we know that the limiting shrinker $\Sigma^k$ is unique. In [CM15], Colding-Minicozzi introduced the notion of cylindrical scale.

**Definition 3.1** (Graphical radius; called cylindrical scale in [CM15]). Let $\varepsilon_0, \ell$ and $C_\ell$ be some fixed constants. Given a hypersurface $\Sigma$, the cylindrical scale (throughout the paper, we call it the graphical radius) $r(\Sigma)$ is the largest radius such that $\Sigma \cap B_{r(\Sigma)}$ is the graph a function $u : \Sigma^k \cap B_{r(\Sigma)} \to \mathbb{R}$ with $\|u\|_{C^{2,\alpha}} \leq \varepsilon_0$ and the part of hypersurface $\Sigma^k \cap B_{r(\Sigma)}$ has curvature bound $|\nabla^\ell A| \leq C_\ell$.

We need some result from [CM15] to serve as initial input for our proofs. The result is summarized in the following proposition.

**Proposition 3.2.** Assume $(\star)$ and let $r(t)$ be the graphical radius of the manifold $M_t$ in the RMCF. Then we have the estimate $r(t) \geq (\alpha \log t)^{1/2}$ for some small $\alpha > 0$ and all $t > t_0$ for some large $t_0$. Moreover, we have the estimate for the graphical function $u : \Sigma^k \cap B_{r(\Sigma)} \to \mathbb{R}$

$$\|u\|_{L^2(B_{r(t)})} \leq Ct^{-\frac{\ell}{4} + c}$$

for some constant $C$ and sufficiently small $c > 0$. 


Proof. In Theorem 6.1 of [CM15] we choose $1/\tau = 3 - \varepsilon$, then we get the estimate $F(\Sigma_{t-1}) - F(\Sigma_{t+1}) \leq t^{-\frac{3}{2}} = t^{-(3-\varepsilon)}$ by Lemma 6.9 of [CM15] (We remark that the statement of Theorem 6.1 of [CM15] gives only some $\tau \in (1/3, 1)$. However, the proof there is constructive. It is not hard to see that $\tau$ can be chosen close to $1/3$). Defining $R(t)$ (shrinker scale in [CM15]) through $e^{-\frac{R(t)^2}{4}} = F(\Sigma_{t-1}) - F(\Sigma_{t+1})$, we get $R(t) \geq \sqrt{2} \sqrt{\frac{1}{\tau} \log t}$. By Theorem 5.3 of [CM15] on the graphical scale $r(t) \geq (1 + \mu) R(t)$ for some $\mu > 0$, we get the estimate

$$r(t) \geq \frac{\sqrt{2}}{\sqrt{\tau}} (1 + \mu) \sqrt{\log t} = \sqrt{2(3-\varepsilon)}(1 + \mu) \sqrt{\log t}.$$  

To estimate $\|u\|_{L^2}$, we invoke $(\ast_2)$ of [CM15, Page 259] (see the proof of Theorem 0.26 in Section 4 therein) to yield

$$\|u(t)\|_{L^2(B_r(t))} \leq C r(t) \rho^\varepsilon (\|\phi\|_{L^2(B_r(t))} + e^{-(1-\varepsilon)\frac{r(t)^2}{\rho^2}}),$$

for some uniform constants $C$ and $\rho$, and arbitrarily small $\varepsilon$, where $\phi$ is a function with the estimate (see equation (6.6) of [CM15])

$$\|\phi\|_{L^2(B_r(t))} \leq C (F(\Sigma_{t-1}) - F(\Sigma_{t+1})) \leq t^{-1/\tau}.$$ 

Combining the two estimates, we get the estimate of $\|u\|_{L^2}$ as stated. \hfill \Box

We remark that the graphical radius estimate and the time $t_0$ here are uniform in entropy for all cylindrical singularities (see footnote 5 of [CM15]).

3.2. Equations of motion. To study the asymptotics of the RMCF approaching the limiting cylinder, we need to study the evolution of graphical function. In Proposition A.1 in Appendix A, we write the graph of the RMCF over a shrinker given by

$$\partial_t u = Lu + Q(J^2 u)$$

where $u(\cdot, t) : \Sigma^k \cap B_{r(t)} \to \mathbb{R}$ whose graph is the RMCF $M_t$ restricted to the ball $B_{r(t)}$, and $Q$ is at least quadratic in $u$ given in (A.2) and $J^2 u := (u, Du, D^2 u)$ means the 2-jet of $u$. We shall study the evolution of $u$ under the differential equation.

Since $M_t$ cannot be written as a global graph over $\Sigma^k$, we introduce a smooth cutoff function $\chi(t) : \Sigma^k \to \mathbb{R}$ that is 1 over $B_{r(t)-1}$ and vanishes outside $B_{r(t)}$, so that $|\nabla^\ell \chi| \leq C$ for any $\ell > 0$. We define $A_t = \Sigma^k \cap (B_{r(t)+1} \setminus B_{r(t)})$. Then we derive the equation for $\chi u$:

$$\partial_t (\chi u) = L(\chi u) + \chi Q(J^2 u) + (u(\Delta \chi + \partial_t \chi) + 2\langle \nabla \chi, \nabla u \rangle + \frac{1}{2} \langle x, \nabla \chi \rangle u),$$

where the last term on the RHS is supported on the annulus $A_{r(t)}$. For simplicity, we write

(3.1) \hfill $$\partial_t (\chi u) = L(\chi u) + B(J^2 u).$$

The natural function space to study these equations is $H^1(\Sigma^k)$, which admits a natural direct sum decomposition into eigenspaces of the $L$-operator. Lying in the kernel of $L$, there are functions generating rotations, which are not important in our analysis of isolatedness of
singularities, thus we choose to modulo them. Let \( G(t) \) be a curve in the group \( \text{SO}(n+1) \) of rotations, and \( \overline{v} \) is the function of the difference of the graph after the rotation by \( G(t) \). We determine rotation so that the projection of \( \overline{v} \) to the space of infinitesimal rotations \( \mathfrak{so}(n+1) \) is zero. By implicit function theorem, one can choose a curve of matrices \( G(t) \in \text{SO}(n+1) \), so that the evolution of \( v \) satisfies the equation

\[
\partial_t \overline{v} = L \overline{v} + \mathcal{B}(J^2 \overline{v}) + \chi \langle G(t)x, n \rangle,
\]

where the RHS is zero when projected to the eigenmodes corresponding to rotations, and \( \mathcal{B}(J^2 \overline{v}) = \mathcal{B}(J^2 \overline{v}) + O(|G(t)|v) \), see [BC19, Lemma 2.4, 2.5]. We have the following lemma.

**Lemma 3.3.** Let \( A := \chi \langle G(t)x, n \rangle \). There is a constant \( C \) such that \( \|A\|_{L^2} \leq C \|\overline{B}\|_{L^2} \).

**Proof of Lemma 3.3.** The argument is similar to [BC19, Page 39]. From the definition of \( A \), we have \( \int (\mathcal{B}(J^2 u) - A(t)) A(t) e^{-\frac{|x|^2}{4}} dx = 0 \) and by Plancheral and Cauchy-Schwarz inequality,

\[
\|A\|_{L^2} \leq C \|\mathcal{B}(J^2 u)\|_{L^2}.
\]

Therefore, we can absorb \( A := \chi \langle G(t)x, n \rangle \) into \( \overline{B} \), and \( \overline{B} \) has similar bound as \( B \) by Lemma 3.3. In the following we remove all the overline notations, for notational simplicity.

Let \( E = H^1(\Sigma^k)/\mathfrak{so}(n+1) \) be the function space under consideration, and we introduce the direct sum decomposition \( E = E^+ \oplus E^0 \oplus E^- \) where \( E^+ \) (respectively \( E^0 \) and \( E^- \)) is spanned by eigenfunctions of \( L \) with positive (respectively 0 and negative) eigenvalues. We denote by \( \Pi^+ \), \( \Pi^- \), \( \Pi_0 \) the natural projection to the spaces \( E^+ \), \( E^- \), \( E^0 \) respectively.

### 3.3. Pseudolocality extension

Starting from this section, we shall extend the graphical scale from \( O(\sqrt{\log t}) \) to \( O(\sqrt{t}) \). One of the main ingredients of the extension is the pseudolocality theorem, which says that if we have two hypersurfaces that are close to each other in a large domain, then under the MCF or RMCF they will still be close to each other for a short but definite amount of time. This allows us to extend the graphical scale of the RMCF to exponentially growing radius for a short time. However, the derivative estimates of the graph deteriorate as time goes on. To address this problem, we use the regularization property of the Ornstein-Uhlenbeck operator to show that within the graphical radius of order \( O(\sqrt{t}) \), various derivative estimates of the graph actually improve as time goes on.

The following pseudolocality argument was first studied by Ecker-Huisken in [EH91]. Later Ilmanen-Neves-Schulze [INS19, Section 9] gave a simple proof using White’s regularity theorem [Whi05]. There they study the MCF of graphs over a hyperplane, and we study the MCF of graphs over a cylinder.

**Theorem 3.4.** For any \( \varepsilon \), there exist \( R_0 > 0 \) and \( \delta_0 > 0 \) with the following significance: suppose \( M_t \) is an RMCF and \( M_{t_0} \) is a graph of a function \( u(\cdot, t_0) \) over \( \Sigma^k \cap B_R \) for some \( R > R_0 \) and \( \|u(\cdot, t_0)\|_{C^1(B_R)} \leq \varepsilon \). Then
(1) when \( t \in [t_0, t_0 + \delta_0] \), \( M_t \) is a graph of function \( u(\cdot, t) \) over \( \Sigma^k \) in the ball of radius \( e^{(t-t_0)/2}R \) with \( \| \nabla^k u(\cdot, t) \|_{L^\infty} \leq C e^{-(k-1)(t-t_0)/2} \varepsilon, k = 0, 1; \)

(2) and when \( t \in [t_0 + \delta_0/2, t_0 + \delta_0] \), we have \( \| \nabla^k u(\cdot, t) \|_{L^\infty} \leq C_k \varepsilon, k \in \mathbb{N} \).

In the theorem, the factor \( e^{-(k-1)(t-t_0)/2} \) is from the exponential expansion property of the RMCF. In particular, when \( k = 0 \), the \( L^\infty \) norm of \( u \) increases very fast. We need to use the equation of the RMCF \( \partial_t u = Lu + Q \) to show that the growth rate is actually not that fast.

Theorem 3.4 follows from the following theorem for MCF considering the scaling between MCF and RMCF.

**Theorem 3.5.** For any \( \tau_0 \in (-1, 0) \) and \( \varepsilon_0 > 0 \), there exists \( \eta_0 > 0 \) such that for any \( \eta \in (0, \eta_0) \), if \( M_{-1} \cap B_{\eta^{-1}} \) is the graph of a function \( v \) over \( \Sigma^k \) with \( \| v \|_{C^1} \leq \eta \), then

1. \( M_{-1} \cap B_{\eta^{-1}}^+ \) is smooth MCF for \( \tau \in (-1, \tau_0) \);
2. \( M_{-1} \cap B_{\eta^{-1}}^- \) has a singularity before time \( -\tau_0 \);
3. \( M_{-1} \cap B_{\eta^{-1}}^- \) can be written as a graph of function \( u(\cdot, \tau) \) over \( \sqrt{-\tau} \Sigma^k \) for \( \tau \in (-1, \tau_0) \),
   with \( \| u(\cdot, \tau) \|_{C^1} \leq \varepsilon_0 \).

**Proof.** We first prove item (1) by contradiction. Suppose not, then we can find a sequence of hypersurfaces \( M^\varepsilon_{-1} \), such that each one of them is a graph of function \( v_\varepsilon \) inside the ball of radius \( k \), with \( \| v_\varepsilon \|_{C^1} \leq k^{-1} \), but each one of them has singularity before time \( \tau_0 \). Then we can use the compactness of Brakke flow (see [Ilm94, Section 7]) to pass to a limiting weak MCF \( M^\infty_{-1} \). By Ecker-Huisken’s curvature estimate (see [EH91]). This is actually the key ingredient to the pseudolocality of MCF, and this is also why we need to shrink the radius from \( \eta^{-1} \) to \( \frac{\eta^{-1}}{2} \), \( M^\infty_{-1} \) is a smooth MCF when \( \tau \) is close to \(-1 \). Also, the sequence \( M^\varepsilon_{-1} \) has varifold limit being the standard cylinder \( \Sigma^k \). Thus, the limit flow \( M^\infty_{-1} \) is exactly the shrinking soliton, with only one singularity at time 0. Then by the upper semi-continuity of Gaussian density, we know that the singularity of \( M^\varepsilon_{-1} \) must have singular time converging to 0, which is a contradiction. The proof of items (2) and (3) is basically verbatim and we omit it here. \( \square \)

### 3.4. Extending the graphical radius.

In this section, we extend the graphical radius from \( O(\sqrt{\log t}) \) to \( O(\sqrt{t}) \).

#### 3.4.1. The main result.

**Proposition 3.6.** Suppose \( M_t \) is an RMCF converging to \( \Sigma^k \) in the \( C^\infty_{loc} \)-sense. Then for any \( \varepsilon_0 > 0 \), there exist \( T_0 > 0 \) and \( K > 0 \), such that when \( t > T_0 \), \( M_t \) can be written as a graph of function \( u(\cdot, t) \) over \( \Sigma^k \cap B_K \sqrt{\tau} \), with \( \| u(\cdot, t) \|_{C^2( B_K \sqrt{\tau})} \leq \varepsilon_0 \).

The proof of Proposition 3.6 relies on the following technical lemma.

**Lemma 3.7.** For any \( \varepsilon_0 > 0 \), there exist \( t_0 > 0 \), \( \delta_0 > 0 \) and \( R_0 > 0 \) and \( K > 0 \) with the following significance: suppose \( M_T \) is a graph over \( \Sigma^k \cap B_R \) as a function of \( u(\cdot, T) \) with
\[ u(\cdot, T) \|_{C^2(B_{R_0})} \leq \varepsilon_0, \text{ where } T > t_0 \text{ and } R > R_0. \] Then when \( t \in [T + \delta_0/2, T + \delta_0] \), we have
\[ u(\cdot, t) \|_{C^2} \leq \varepsilon_0 \text{ on } \Sigma^k \cap B_r(t), \text{ where } r(t) = K\sqrt{(t-T)+(R/K)^2}. \]

We will prove Lemma 3.7 in the following subsection. Now we first prove Proposition 3.6 using Lemma 3.7.

**Proof of Proposition 3.6.** Let us fix \( t_0, \delta_0 \) and \( R_0 \) as in Lemma 3.7. We suppose \( T \) is sufficiently large so that \( \|u(\cdot, T)\|_{C^2(B_{R_0})} \leq \varepsilon_0 \). Now we apply Lemma 3.7, which shows that for
\[ t \in [T + \delta_0/2, T + \delta_0], \quad \|u(\cdot, t)\|_{C^2(B_{r(t)})} \leq \varepsilon_0. \]

Next, we can apply Lemma 3.7 by replacing \( T \) by \( T + \delta_0/2 \), and replacing \( R_0 \) by \( r(T+\delta_0/2) \). Then we obtain
\[ \|u(\cdot, t)\|_{C^2(B_{r(t)})} \leq \varepsilon_0 \text{ for } t \in [T + \delta_0, T + (3/2)\delta_0], \]
where
\[ \bar{r}(t) = K\sqrt{(t-(T+\delta_0/2))} + (r(T+\delta_0/2)/K)^2 = r(t). \]
Therefore, we have extended the estimate \( \|u(\cdot, t)\|_{C^2(B_{r(t)})} \leq \varepsilon_0 \) to the interval \( t \in [T + \delta_0/2, T + (3/2)\delta_0] \).

Iterating the above process shows that \( \|u(\cdot, t)\|_{C^2(B_{r(t)})} \leq \varepsilon_0 \) for \( t > T + \delta_0/2 \). Finally, we have \( r(t) \sim \sqrt{t} \) when \( t \) is sufficiently large. Thus when \( t \) is sufficiently large, we obtain
\[ \|u(\cdot, t)\|_{C^2(B_{K\sqrt{t}})} \leq \varepsilon_0. \]

\[ \square \]

3.4.2. The Ornstein-Uhlenbeck regularization and proof of Lemma 3.7. In the following, we prove Lemma 3.7. From now on we choose sufficiently small \( \varepsilon_0 \) and \( \tau_0 \) so that Theorem 3.4 holds and \( \varepsilon_0 e^{\tau_0/2} \leq 1/2 \), and Proposition A.1 applies for the graph function with \( C^2 \)-norm less than \( 3\varepsilon_0 \). Recall that we have the equation \( \partial_t (\chi u) = L(\chi u) + B(J^2 u) \) as in Section 3.2. To get the \( C^2 \) bound of \( u \), we note that \( B \) is quadratic small in the \( C^2 \) norm of \( u \), thus it remains to give a good \( C^2 \) control on the linear part. For the linear equation, we have the following estimates.

**Proposition 3.8** (Proposition 2.13 of [FKZ00]). Let \( v: \Sigma^k \to \mathbb{R} \) be a solution to the equation
\[ \partial_t v = Lv + P(\cdot, t)v \]
with initial condition \( \|v(\cdot, t_0)\|_{L^2} < \infty \), where \( |P| \) is uniformly bounded by \( C_0/t \) as \( t \to \infty \). Then there exist \( K_0 > 0 \) and \( \delta_1 > 0 \) such that we have the estimates
\[ \|v(\cdot, s')\|_{C^0(B_{K_0\sqrt{t'}})} \leq C(K_0, \delta_1)(s')^2 t_0^{-2+C_0}\|v(\cdot, t_0)\|_{L^2}, \]
whenever we have
\[ s' = K_0 + s, \quad e^{(s-t_0)/2} \leq K_0\sqrt{s}, \quad s \geq t_0 + \delta_1, \quad \text{and } s \geq \frac{1}{15}K_0. \]

Moreover, if for some \( \ell \in \mathbb{Z}_+, \alpha \in (0, 1), \quad \|P(\cdot, t)\|_{C^{\ell, \alpha}} \leq C_0/t, \) then
\[ \|v(\cdot, s')\|_{C^{\ell, 2\alpha}(B_{K_0\sqrt{t'}})} \leq C(K_0, \ell, \delta_1)(s')^2 t_0^{-2+C_0}\|v(\cdot, t_0)\|_{L^2}. \]
We postpone the proof to Appendix B because the proof is essentially the same as [FKZ00]. Roughly speaking, this proposition shows that the solution to the variational equation has nice $C^2$ control on the scale up to $O(\sqrt{t})$ provided we have $L^2$ control of the initial value.

**Lemma 3.9.** For any $\varepsilon_0 > 0$ small, let $t_0$ and $R_0$ be such that we have the bound $\|u(\cdot, T)\|_{C^2(B_R)} \leq \varepsilon_0$ for the graphical function $u(\cdot, T)$ of $M_T$ over $\Sigma^k \cap B_R$ where $T > t_0$ and $R > R_0$. Let $v$ be the solution to the Cauchy problem \[
\begin{aligned}
\partial_t v &= L v, \\
v(\cdot, T) &= \chi u(\cdot, T).
\end{aligned}
\]
Then when $T$ is sufficiently large, for $t \in [T + 2\delta_1, T + 4\delta_1]$ and $K^2 T \geq R$, where the constants $K_0, \delta_1$ are as in Proposition 3.8, we have $\|v(\cdot, t)\|_{C^2(B_r(t))} \leq C t^{-\frac{3}{4} + \varepsilon}$ where $r(t) = K_0 \sqrt{(t - T) + (R/K_0)^2}$.

**Proof.** We only need to carefully choose parameters in Proposition 3.8. In Proposition 3.8, let $K_0 \leq \delta_1$, $t_0 = T$, $s' = t$. With these choices of parameters, one can check that the conditions (3.4) hold whenever $T$ is sufficiently large. Then Proposition 3.8 shows that $\|v(\cdot, t)\|_{C^2(B_r(t))} \leq C \|v(\cdot, T)\|_{L^2}$, where $C$ is a fixed constant once we fix $K_0$ and $\delta_1$.

Notice that we have proved $\|u\|_{L^2(B_{r(t)})} \leq C t^{-\frac{3}{4}} \varepsilon_0$ in Proposition 3.2. Next, since we have $r(t) \leq e^{(t-t_0)/2} R$ whenever $K_0$ is fixed and $R$ is sufficiently large, Theorem 3.4 shows $\|\chi u\|_{C^2(B_r(t))} \leq 3\varepsilon_0$ for $t \in [T + 2\delta_0, T + 4\delta_0]$. Thus we have for $t \in [T + 2\delta_0, T + 4\delta_0]$ that $\|(\chi u)(\cdot, t)\|_{L^2(B_r(t))} \leq \frac{C}{t^{\frac{3}{4}} \varepsilon_0}$. This completes the proof by applying Proposition 3.8.

We are now ready to give the proof of Lemma 3.7. The idea is that we will use Lemma 3.9 to control the growth of the linearized solution on a larger scale, then use the linearized solution to control the real solution of RMCF on a larger scale.

**Proof of Lemma 3.7.** Let $K = K_0$ be as in Proposition 3.8 and $T, R$ be as in the statement of Lemma 3.9, and let $r(t) := K \sqrt{(t - T) + (R/K)^2}$. We only need to bound the $C^1$-norm of $u$ in $B_r(t)$, since then Theorem 3.4 shows that $\|\nabla^2 u\|_{L^\infty}$ is nicely bounded in the ball of radius $r(t)$.

Let $v$ be as in Lemma 3.9, and let $w = v - \chi u$. Then $w$ satisfies the equation $\partial_t w = Lw + B(J^2 u)$ with the initial condition $w(\cdot, T) = 0$. By Duhamel principle, Theorem 3.4, and the quadratic decay of $B$, we have $\|w\|_{C^0(B_{r(t)})} \leq C \delta_0 \varepsilon_0^2$ when $t \in [T, T + \delta_0]$. Here we point out that in $B(J^2 u)$, most of the terms related to $J^2 u$ are uniformly bounded, except for a term $\frac{1}{2} \langle x, \nabla \chi \rangle u$. However, this term is supported outside $B_{r(t)}$ because $\nabla \chi$ is supported outside $B_{r(t)}$, so it does not affect the estimate.

Use Lemma 3.9, when $t \in [T + \delta_0/2, T + \delta_0]$ with $\delta_0 = 4\delta_1$, we have $\|u\|_{C^0(B_{r(t)})} \leq C \delta_0 \varepsilon_0^2 + C t^{-\frac{3}{4}} < \varepsilon_0^2$. Finally, using the interpolation inequality (see Lemma B.1 in [CM15]) with $\|\nabla^{(n+4)} u\|_{L^\infty(B_{r(t)})} \leq C_{n+4}$
and \( \| \nabla^{(n+6)} u \|_{L^\infty(B_{r(t)})} \leq C_{n+6} \) (see Theorem 3.4), we have
\[
\| \nabla u \|_{L^\infty(B_{r(t)})} \leq C \| u \|_{L^\infty(B_{r(t)})}^{1/2 + 1/(2n+4)} \| \nabla^{(n+2)} u \|_{L^\infty(B_{r(t)})}^{1/2} \| \nabla^{(n+6)} u \|_{L^\infty(B_{r(t)})} \leq C \varepsilon_0^{1+1/3}, \quad \text{and}
\]
\[
\| \nabla^2 u \|_{L^\infty(B_{r(t)})} \leq C \| u \|_{L^\infty(B_{r(t)})}^{1/2 + 1/(2n+6)} \| \nabla^{(n+6)} u \|_{L^\infty(B_{r(t)})} \leq C \varepsilon_0^{1+1/3}\]
on a slightly smaller ball as well. When \( \varepsilon_0 \) is sufficiently small, we have \( \| u \|_{C^2(B_{r(t)})} \leq \varepsilon_0 \). Therefore we extend the estimate from \( t = T \) through \( t \in [T, T + \delta_0] \), and we can repeat the above analysis starting from \( T + \delta_0 \). Thus we obtain Lemma 3.7 for all \( t \geq T \) by iterating the above discussion.

4. The invariant cones

In this section, we prove that the dynamic of the RMCF is controlled by the linear equation \( \partial_t u = L \Sigma^2 u \). The idea is similar to that used in [SX21b, SX21a]. The main technical difficulty is that the cutoff \( \chi \) creates a boundary term when we perform an integration by parts. Due to the \( K \sqrt{t} \) estimate of the graphical scale and the fast decay of Gaussian weight, the boundary term is actually of order \( e^{-K^2 t \over t} \), which is much smaller than the \( o(1/t) \) remainder in the normal form theorems.

4.1. The cone theorem. We introduce a double cone construction that is used to suppress the \( E^+ \)-components and manifest the \( E^0 \)-component. Let \( \kappa \in (0, 1) \) be a small number. We introduce two cones \( \mathcal{K}_{\geq 0} \) and \( \mathcal{K}_0 \) as follows
\[
\mathcal{K}_{\geq 0}(\kappa) := \{ u = (u_+, u_0, u_-) \in E^+ \oplus E^0 \oplus E^- \mid \| u_+ + u_0 \| \geq \kappa \| u \| \}
\]
is a \( \kappa \)-cone around \( E^+ \oplus E^0 \) and
\[
\mathcal{K}_0(\kappa) := \{ u = (u_+, u_0, u_-) \in E^+ \oplus E^0 \oplus E^- \mid \| u_0 \| \geq \kappa \| u \| \}
\]
is a \( \kappa \)-cone around \( E^0 \). Both are narrower when \( \kappa \) is closer to 1.

We shall also need to compare the difference between the perturbed RMCF and the unperturbed one under evolution. For this purpose, we introduce the following setting. Let \( u_1, u_2 \) be two graphical RMCF over \( \Sigma^k \), and \( \| u_i(\cdot, t) \|_{C^{2,\alpha}} \leq \varepsilon_0 \) inside \( \Sigma^k \cap B_{K \sqrt{t}} \). Then we write \( v = \chi(u_1 - u_2) \), where \( \chi \) is a smooth cutoff function that is 0 outside the ball \( B_{K \sqrt{t}} \) and is 1 inside the ball \( B_{K \sqrt{t} - 1} \), and we can calculate
\[
(4.1) \quad \partial_t v = Lv + \delta \mathcal{B}.
\]
where \( \delta \mathcal{B} = \delta \mathcal{B}(J^2 u_1, J^2 u_2) = \mathcal{B}(J^2 u_1) - \mathcal{B}(J^2 u_2) \). In \( \delta \mathcal{B} \), we replace \( u_i \) by \( \chi u_i \) creating an error supported on \( A_{K \sqrt{t}} \) denoted by \( \mathcal{E}(J^2 u_1, J^2 u_2) \). Thus we can write
\[
(4.2) \quad \delta \mathcal{B} = P v + \mathcal{E}(J^2 u_1, J^2 u_2), \quad P = \int_0^1 D_u \mathcal{B}(sJ^2(\chi u_1) + (1 - s)J^2(\chi u_2)) ds.
\]
**Proposition 4.1.** There exists a constant $C > 0$, such that for all $\varepsilon_0$ sufficiently small the following holds. Let $v(t) = \chi(t)(u_1(t) - u_2(t)) : \Sigma^k \cap B_{K^\sqrt{t}} \to \mathbb{R}$ be as above a solution to

\[ (4.1) \]

satisfying $\varepsilon_0 \|v(n)\| \geq \|v(t)\|_{C^2(\mathbb{R}_+)} e^{-\frac{k^2}{8}t}$ for all $t \in [n, n + 1]$ for some $n$ sufficiently large. Then we have

\[ \|v(n + 1) - e^Lv(n)\| \leq C\varepsilon_0\|v(n)\|. \]

The assumption $\varepsilon_0 \|v(n)\| \geq \|v(t)\|_{C^2(\mathbb{R}_+)} e^{-\frac{k^2}{8}t}$ in Proposition 4.1 is verified by the following lemma.

**Lemma 4.2.** Suppose there is a sequence $t_n \to \infty$ such that $\|v(t_n)\| \geq e^{-\frac{k^2}{8}t_n}$. Then there exists $N$ sufficiently large such that we have $\|v(t)\| \geq e^{-\frac{k^2}{8}t}$ for all $t > t_N$.

The following cone theorem follows immediately from Proposition 4.1 and this lemma (details can be found in the proof of Theorem 3.1 of [SX22]).

**Theorem 4.3** (Cone theorem). Let $M_t$ be the RMCF and $u : \Sigma^k \cap B_{K^\sqrt{t}} \to \mathbb{R}$ be the graphical function of $M_t \cap B_{K^\sqrt{t}}$ with $\|u(t)\| \to 0$ as $t \to \infty$. Then for all $\kappa \in (0, 1)$, there exists $n$ such that

1. if $u(n) \in K_{\geq 0}(\kappa)$, then $u(n') \in K_{\geq 0}(\kappa)$ for all $n' \geq n$;
2. if $u(n) \in K_0(\kappa)$, then $u(n') \in K_0(\kappa)$ for all $n' \geq n$.

Moreover, if $u(n)$ does not enter $K_{\geq 0}(\kappa)$ for all $n$ large, then we have $\|u(n)\| \leq e^{-\frac{k^2}{8}n}$ for all $n$ large.

### 4.2. Proof of Proposition 4.1.

In the following, we work on the proof of Proposition 4.1.

**Proof of Proposition 4.1.** Let $w$ be a solution to the linearized RMCF equation, i.e. $\partial_t w = Lw$ with the initial condition $w(n) = \chi(n)v(n)$ for some $n$ large. We next estimate the evolution of $v - w$. We first compute the time derivative of $\|v - w\|^2_{L^2}$, which is

\[ \partial_t \int (v - w)^2 e^{-\frac{|x|^2}{4}} = 2 \int (v - w)(L(v - w) + \delta B)e^{-\frac{|x|^2}{4}} \]

\[ = -2 \int |\nabla(v - w)|^2 e^{-\frac{|x|^2}{4}} + 2 \int (v - w)^2 e^{-\frac{|x|^2}{4}} + 2 \int (v - w)\delta Be^{-\frac{|x|^2}{4}}. \]

Similarly, we compute the time derivative of $\|\nabla(v - w)\|^2_{L^2}$:

\[ \partial_t \int |\nabla(v - w)|^2 e^{-\frac{|x|^2}{4}} = 2 \int \nabla(v - w) \cdot \nabla(L(v - w) + \delta B)e^{-\frac{|x|^2}{4}} \]

\[ = -2 \int |\mathcal{L}(v - w)|^2 e^{-\frac{|x|^2}{4}} + 2 \int |\nabla(v - w)|^2 e^{-\frac{|x|^2}{4}} + 2 \int (\mathcal{L}(v - w))\delta Be^{-\frac{|x|^2}{4}}. \]

We bound $2 \int (\mathcal{L}(v - w))\delta Be^{-\frac{|x|^2}{4}}$ by $\frac{1}{10}\|\mathcal{L}(v - w)\|^2_{L^2} + 10\|\delta B\|^2_{L^2}$. Since we have $\delta B = P_v + \mathcal{E}$, we get $\|\delta B\|^2_{L^2} \leq 2\|P_v\|^2_{L^2} + 2\|\mathcal{E}\|^2_{L^2}$. Notice that $\delta B$ is defined in equation (4.1) independent
of \( w \), and \( \delta B = P v + \mathcal{E} \), where \( \mathcal{E} \) is only supported on \( \mathbb{A}_{K\sqrt{T}} \). If \( \| v \|_{C^2(B_{K\sqrt{T}})} \leq \varepsilon_0 \) by the definition of graphical radius, it is straightforward that

\[
\| \delta B \|_{L^2}^2 \leq \varepsilon_0^2 \| J^2 v \|_{L^2}^2 + e^{\frac{K^2}{4}} \| v \|_{C^2(B_{K\sqrt{T}})}^2 \leq \varepsilon_0^2 \| J^2 v \|_{L^2}^2 + \| v \|_{C^2(B_{K\sqrt{T}})}^2 e^{\frac{K^2}{4}},
\]

where we used Proposition 3.2.

The term \( \| J^2 v \|_{L^2}^2 \) is bounded by a multiple of \( \| \mathcal{L} v \|_{L^2}^2 + \| \nabla v \|_{L^2}^2 \leq \| v \|_{H^2}^2 \) using Bochner formula (Details can be found in [CM16a, Section 5] and [SX22, Proposition 4.6]). Moreover, by assumption, we have \( \| v \|_{C^2(B_{\kappa(t)})} e^{\frac{K^2}{4}} \leq \varepsilon_0^2 \| v(n) \|_{H^2}^2 \). In conclusion, we have

\[
\partial_t \| v - w \| \leq C \| v - w \| + \varepsilon_0^2 \| v(t) \|_{H^2}^2 + \varepsilon_0^2 \| v(n) \|_{H^2}^2 + \varepsilon_0^2 \| v(n) \|^2.
\]

with initial condition \( \| v(n) - w(n) \| = 0 \). Noting that \( v(n+1) - w(n+1) = v(n+1) - e^L v(n) \), we thus get by Gronwall inequality that

\[
\| v(n+1) - e^L v(n) \|^2 \leq e^C \varepsilon_0^2 (\int_0^1 \| v(t+n) \|^2_{H^2} dt + \| v(n) \|^2_{H^2}).
\]

It remains to estimate \( \int_0^1 \| v(t+n) \|^2_{H^2} dt \). Repeating the above calculations with \( w = 0 \) and the same estimate (4.4). Substituting (4.4) into (4.3) with \( w = 0 \) and integrating over time \( 1 \), we get

\[
\int_0^1 \| v(t+n) \|^2_{H^2} dt \leq C \| v(n) \|^2.
\]

Substituting the last estimate to the above estimate of \( \| v(n+1) - e^L v(n) \|^2 \), we get the estimate in the statement.

\[
\square
\]

4.3. **Control the boundary term.** In this section, we finally prove the technical Lemma 4.2.

**Proof of Lemma 4.2.** We follow the proof of Proposition 4.1 until equation (4.5), which does not need the assumption of \( e^{\frac{K^2}{4}} \| v(t) \|_{C^2(B_{\kappa(t)})} \leq \varepsilon_0 \| v(t_n) \| \). Let \( x = \| v_-(t) \|^2, y = \| v_0(t) \|^2, z = \| v_+(t) \|^2 \), then we get\[
\begin{cases}
\dot{x} &\leq -c x + O(\varepsilon_0 e^{-\frac{K^2}{4} t})
\\
\dot{y} &\leq \varepsilon (x + y + z) + O(\varepsilon_0 e^{-\frac{K^2}{4} t}), \text{ where } -c = \max\{-1/(n-k), -1/2\} < 0.
\\
\dot{z} &\geq \varepsilon_0 e^{-\frac{K^2}{4} t},
\end{cases}
\]

If we have

1. for all time \( |x(t)| \geq \xi (|y(t)| + |z(t)|) \) for some constant \( \xi \) (i.e. \( v(t) \notin \mathcal{K}_{\xi}(\kappa) \) for all \( t \), where \( \kappa := (1 + \delta)^{-1/2} \), and
2. \( |x(t)| + |y(t)| + |z(t)| \rightarrow 0 \),

then we get \( |x| + |y| + |z| \leq C \varepsilon_0 e^{-\frac{K^2}{4} t} \) by integrating the differential inequalities for \( \dot{x} \).
Next, suppose there is a sequence of times \( t_n \to \infty \) such that \( e^{-\frac{K^2}{8} t_n} \leq \|v(t_n)\| \). By the above reasoning, \( |x(t)| \geq \xi(|y(t)| + |z(t)|) \) cannot hold for all time, i.e. there is a sequence \( s_n \) such that \( v(s_n) \in \mathcal{K}_{\geq 0}(\kappa) \). Furthermore, without loss of generality, we choose some large \( n \) with \( s_n = t_n \) such that we have

\[ (*) \quad \text{both } e^{-\frac{K^2}{8} t_n} \leq \|v(t_n)\| \text{ and } v(t_n) \in \mathcal{K}_{\geq 0}(\kappa) \text{ where } \kappa \text{ is close to } 1, \text{ so that the cone } \mathcal{K}_{\geq 0}(\kappa) \text{ is very narrow.} \]

This can be achieved. Indeed, let \([t_n, t'_n]\) be the sequence of maximal intervals on which we have \( e^{-\frac{K^2}{8} t} \leq \|v(t)\| \) and suppose that on the interval \([t'_n, t_{n+1}]\), we have \( v(t) \notin \mathcal{K}_{\geq 0}(\kappa) \), in addition to \( e^{-\frac{K^2}{8} t} \geq \|v(t)\| \). We can then repeat the ODE argument in the first paragraph get \( \|v(t)\| < e^{-\frac{K^2}{8} t} \) on the interval \( t \in [t'_n, t_{n+1}] \) and obtain a contradiction to the definition of \( t_{n+1} \). Thus, to get growth of \( \|v(t)\| \) such that at time \( t_{n+1} \) we have \( e^{-\frac{K^2}{8} t_{n+1}} \leq \|v(t_{n+1})\| \), we must have \( v(t_{n+1}) \in \mathcal{K}_{\geq 0}(\kappa) \).

With \((*)\), we repeat the proof of Proposition 4.1 for a very short time interval \([t_n, t_n + \Delta]\), where \( \Delta \) is chosen such that \( \|v(t)\|_{C^2(\mathbb{R}^n)} e^{-\frac{K^2}{8} t} \leq \varepsilon_0 \|v(t_n)\| \) for \( t \in [t_n, t_n + \Delta] \) by continuity. Equation (4.5) remains to be true on the \( \Delta \)-interval. Thus we get \( \|v(t_n + \Delta) - e^{\Delta L} v(t_n)\| \leq \varepsilon_0 \|v(t_n)\| \) as Proposition 4.1, which gives \( \|v(t_n + \Delta)\| \geq (1 - \varepsilon_0) \|v(t_n)\| \). Choosing \( t_n \) large, we can make \( \varepsilon_0 \) as small as we wish, and for small \( \varepsilon_0 < \frac{K^2}{8} \), we get that \((*)\) remains to hold at time \( t_n + \Delta \). Thus we can iterate the procedure starting from time \( t_n + \Delta \). This completes the proof.

\[ \square \]

5. The normal forms

In this section, we prove the normal form theorems 1.3 and 1.4 based on the following observation. Indeed, Fourier modes in \( E^- \) decay exponentially and those in \( E^+ \) grow exponentially. The convergence of the RMCF to the cylinder implies that exponential growth is impossible, which implies that the \( E^+ \) component should be small. Thus \( E^0 \) component dominates the graphical function. The normal form then follows by projecting the RMCF equation to the \( E^0 \) component and analyzing the resulting ODE.

5.1. The \( H^1 \)-normal form. We next give the proof of the \( H^1 \)-normal form. The proof consists of the following steps

1. We first apply the cone theorem to show that the projection to \( E^+ \oplus E^- \) is much smaller than that of \( E^0 \). This gives us an ODE for the \( E^0 \) component (Lemma 5.2).
2. Next we show that the ODE is diagonalizable, so that the \( y_i y_j \) modes can be eliminated by rotations, and we are only left with the \( h_2(y_i) \) modes with coefficients \( \frac{K}{t^2} \) (see Proposition 5.1 below).
3. Finally, we show that components in \( E^+ \) and \( E^- \) are of order \( O(1/t^2) \).

We do not only show the dominated term is spanned by \( y_i^2 - 2 \), and we calculated the explicit coefficients as well.
Proposition 5.1. The coefficients of $a^2 - 2$ in Theorem 1.3 are explicitly given by $a_i(t) = \frac{b}{t} + o(1)$, where $b$ is either 0 or $\varrho/4$.

Throughout this subsection, we set $v(t) = \chi(t)u(t)$, where $\chi$ is again the cut-off function used in Section 4, supported on the ball of radius $K\sqrt{t}$. The idea is that after modulo the rotations, the remaining neutral modes are $h_2(y_i)$'s and $h_1(y_i)h_1(y_j)$'s. Recall that we have defined the normalized (in $\| \cdot \|_{L^2(\Sigma^{2\delta})}$) eigenfunctions $H_2(y_i) = c_0^{1/2}g(S^{n-k})^{-1/2}h_2(y_i)$, $H_{1,1}(y_i, y_j) = c_0^{-2}g(S^{n-k})^{-1/2}h_1(y_i)h_1(y_j)$. Here $g(S^{n-k}) = \int_{S^{n-k}(\varrho)} e^{-\frac{|x|^2}{4\varrho}} dx = \varrho^{n-k}e^{-\frac{\varrho^2}{4}}\omega_{n-k}$, where $\omega_{n-k}$ is the area of the $(n-k)$-dimensional unit sphere.

Define

$$m_{ii}(t) := \langle v, H_2(y_i) \rangle = c_0^{1/2}g(S^{n-k})^{-1/2}v, h_2(y_i),$$

and

$$m_{ij}(t) = m_{ij}(t) := \langle v, H_{1,1}(y_i, y_j) \rangle = c_0^{-2}g(S^{n-k})^{-1/2}v, h_1(y_i)h_1(y_j).$$

Then

$$v = \Pi_0v + \Pi_{\neq 0}v = \sum_i m_{ii}H_2(y_i) + \sum_{i<j} m_{ij}H_{1,1}(y_i, y_j) + \Pi_{\neq 0}v$$

$$= \sum_i m_{ii}c_0^{1/2}g(S^{n-k})^{-1/2}h_2(y_i)$$

$$+ \sum_{i<j} m_{ij}c_0^{-2}g(S^{n-k})^{-1/2}h_1(y_i)h_1(y_j) + \Pi_{\neq 0}v$$

We also define $\bar{m}_{ii}(t) = a\langle v, h_2(y_i) \rangle$ with $a = \sqrt{2}c_0$ and $\bar{m}_{ij}(t) = \langle v, h_1(y_i)h_1(y_j) \rangle$. The reason that we put $a = c_0$ in $m_{ii}$ is to make sure that later when we compute the derivatives of $\bar{m}_{ii}$ and $\bar{m}_{ij}$, we will get uniform expressions. We remark that our choice of $a$ matches the choice of coefficients in [Vel92, (3.3)]. We keep $a$ in the following computation and the choice of $a$ will become clear.

Let $M(t)$ be an $n \times n$ matrix, whose entries are given by $\bar{m}_{ii}$ and $\bar{m}_{ij}$. $M(t)$ is a symmetric matrix.

Lemma 5.2. Suppose $\|\Pi_{\neq 0}v(t)\| \leq \varepsilon(t)\|\Pi_0v(t)\|$, where $\varepsilon(t) \to 0$ as $t \to \infty$. Then $M(t)$ satisfies the following ODE:

$$M'(t) = -\gamma M^2(t) + O(\varepsilon_1(t)|M|^2) + O(e^{-\frac{\varrho^2}{4}}, t),$$

where $\varepsilon_1(t) \to 0$ as $t \to \infty$, and $\gamma = \frac{1}{2}c_0^{2(k-1)}g(S^{n-k})^{-1}$ is a fixed constant.

Proof. From Proposition A.1, we write the reminder $Q(J^2u)$ as

$$Q(J^2u) = -\frac{1}{2\varrho}(u^2 + 4u\Delta u + 2|\nabla u|^2) + C(J^2u),$$
where $C(J^2u) = O(\|u\|_{C^{2,\alpha}}(\|u\|^2 + |\nabla u|^2 + |\nabla^2 u|))$. Then we take the time derivative of $\bar{m}_{ii}$ and $\bar{m}_{ij}$ to get

\begin{align}
\bar{m}_{ii}'(t) &= a \int (Lv + \chi Q(v)) h_2(y_i) e^{-\frac{\|y\|^2}{4}} + O \left( e^{-\frac{K^2}{4} t} \right) \\
&= a \int \left( -\frac{1}{2q}(v^2 + 4v \Delta_g v + 2|\nabla_g v|^2) + C(J^2 v) \right) h_2(y_i) e^{-\frac{\|y\|^2}{4}} + O \left( e^{-\frac{K^2}{4} t} \right), \\
\bar{m}_{ij}'(t) &= \int \left( -\frac{1}{2q}(v^2 + 4v \Delta_g v + 2|\nabla_g v|^2) + C(J^2 v) \right) h_1(y_i) h_1(y_j) e^{-\frac{\|y\|^2}{4}} + O \left( e^{-\frac{K^2}{4} t} \right),
\end{align}

where the $O \left( e^{-\frac{K^2}{4} t} \right)$ error is created by the cutoff $\chi(t)$ since the graphical scale is $O(K \sqrt{t})$.

We next show that the RHS of (5.2) is dominated by terms involving $v^2$. On the RHS of (5.2), we substitute

$$v = \sum_i a^{-1} \bar{m}_{ii} c_0^{(2)} G(S^{n-k})^{-1} h_2(y_i) + \sum_{i<j} \bar{m}_{ij} c_0^{(2)} G(S^{n-k})^{-1} h_1(y_i) h_1(y_j) + \Pi_{\neq 0} v.$$

First, terms involving $4v \Delta_g v + 2|\nabla_g v|^2$ depend only on $\Pi_{\neq 0} v$ since $\Pi_0 v$ has no $\theta$-dependence. Applying the cone theorem 4.3 item (2), we get $\|\Pi_{\neq 0} v\| \leq \varepsilon_1(t) \|v\|$, where $\varepsilon_1(t) \to 0$ as $t \to \infty$. Thus we get that on the RHS of (5.2), the integral of terms involving $4v \Delta_g v + 2|\nabla_g v|^2$ is estimated as $C \|\Pi_{\neq 0} v\|^2 \leq C \varepsilon_1(t)^2 \|v\|^2$.

Next, terms involving $C(J^2 v)$ after integration are bounded by $\varepsilon_1(t) \|v\| + e^{-K^2 t}$ using Proposition A.1(3).

Thus we have proved that the RHS of (5.2) is dominated by terms involving $v^2$. Moreover, using the cone theorem 4.3 item (2) again, on the RHS of (5.2), we replace $v^2$ by $(\Pi_0 v)^2$ creating another error bounded by $\varepsilon_1(t) \|v\|^2$. For $\Pi_0 v$, its Fourier expansion has only finitely many terms, then we can write it into the Fourier expansion and calculate $\bar{m}_{ii}'$ and $\bar{m}_{ij}'$ explicitly using Lemma 2.1. We have

\begin{align}
\bar{m}_{ii}' &= -\gamma \sum_{\ell=1}^k \bar{m}_{ii}^2 + O(\varepsilon_1(t) \|M\|^2) + O(e^{-\frac{K^2}{4} t}) \\
\bar{m}_{ij}' &= -\gamma \sum_{\ell=1}^k \bar{m}_{ii} \bar{m}_{\ell j} + O(\varepsilon_1(t) \|M\|^2) + O(e^{-\frac{K^2}{4} t}).
\end{align}

Using Lemma 2.1, we can calculate that

$$\gamma = \frac{1}{2q} A_{22,2,2}(4\pi)^{k-1} c_0^{(k-1)} G(S^{n-k})^{-1} a^{-1} = \frac{1}{2q} A_{21,1,1}(4\pi)^{k-2} c_0^{(k-2)} G(S^{n-k})^{-1} a = \frac{1}{q} c_0^{2(k-1)} G(S^{n-k})^{-1}.$$

We can also see why we need to choose $a = \sqrt{2} c_0$ from the calculation. $\square$
Proof of Proposition 5.1. The proof is almost the same as [Vel93, Section 3] with the only difference being that here (5.1) contains a term $O(e^{-\frac{\lambda^2}{t}})$, which decays so fast that it does not affect the argument in [Vel93]. Therefore we obtain the desired formula as in [Vel93, Lemma 3.4].

For the readers’ convenience, let us sketch the proof here. We diagonalize $M(t)$ using matrices in $SO(k)$. Let $\lambda_1(t), \ldots, \lambda_k(t)$ be the eigenvalues of the matrix $M(t)$. Then by classical perturbation theory for linear operator (see proof of [Vel93, Lemma 3.1]) $\lambda'_i = -\gamma \lambda_i^2 + O(\varepsilon_i(t) \sum_{i=1}^k \lambda_i^2) + O(e^{-\frac{\lambda_i^2}{4t}})$. Thus, the eigenvalues satisfy a Riccati equation system. Note that the majorant part of the ODE is $\lambda' = -\gamma \lambda^2$, and the solution converging to 0 as $t \to \infty$ is either $\frac{1}{\gamma t}$, or constant 0. Then one can show that each $\lambda_i$ has either of the asymptotics. Thus we get that in the expansion of $v$, the coefficient of $(y_i^2 - 2) = c_i^{-1} h_2(y_i)$ is given by $\frac{1}{\gamma t} a^{-1} c_0^{2(k-1)} g(S^{n-k})^{-1} c_2^{-1} = \frac{\varphi}{4t}$.

Once we obtain the asymptotic expansion for the eigenvalues of the matrix $M(t)$, let $W(t)$ denote the eigenspace associated to the eigenvalues $\lambda_i$’s with $\lambda_i \approx \frac{1}{\gamma t}$, and let $\Pi_{W(t)}$ denote the orthogonal projection to $W(t)$. Then the classical eigen theory (see [Vel93, Lemma 3.6]) shows that $E := \lim_{t \to \infty} \Pi_{W(t)}$ exists, and $E$ is the orthogonal projection on an $\ell$-dimensional space. Then up to a rotation, $E$ is a diagonal matrix with first $\ell$ entries 1. Thus we have the desired expansion.

Proof of Theorem 1.3 on $H^1$-normal form. The cone theorem gives an $H^1$ normal form as in Theorem 1.3 but with error $o(1/t)$ in $H^1$. Indeed, by assumption, we have $u(t) \to 0$ in the $C^\infty_{loc}$-sense as $t \to \infty$, thus $\|u(t)\| \to 0$ as $t \to \infty$. By item (2) of the Cone Theorem 4.3, we see that

1. either $u(t)$ remains in the cone $K_0(\kappa)$ for all time sufficiently large,
2. or $u(t)$ does not enter the cone $K_0(\kappa)$ for all time sufficiently large.

In the former case, since $\kappa$ can be chosen to be arbitrarily small, we see that the $u_0$ part dominates. Thus, we have verified the assumption of Lemma 5.2 and Proposition 5.1, which gives the $H^1$ normal form as in Theorem 1.3 but with $o(1/t)$ error in $H^1$.

So to complete the proof of the theorem, we need to improve the $o(1/t)$ estimate of the error to $O(1/t^2)$ estimate. We then substitute the normal form into the equations of motion (3.1). Let $\sigma := \theta \sum_{i \in I} \frac{h_2(y_i)}{4c_2 t}$, which solves

$$\langle \partial_t \sigma, h_2(y_i) \rangle_{L^2} = -\frac{1}{2\theta} \langle \sigma^2, h_2(y_i) \rangle_{L^2}. \tag{5.4}$$

Applying Proposition A.1, we get the equation for $w = u - \sigma = o(1/t)$ as follows,

$$\partial_t w = Lw + Q(J^2(\sigma + w)) + \frac{1}{t} \sigma.$$
When projected to each \( h_2(y_i) \) modes in \( E^0 \), noting that \( Q(J^2(\sigma)) \) cancels with \( \frac{1}{t} \sigma \) by equation (5.4), we get the equation

\[
(5.5) \quad w_i = -\frac{2}{t} w_i + O(w_i^2) + O(1/t^3)
\]

for the coefficients of each Fourier modes \( h_2(y_i) \), where the term \(-\frac{2}{t} w_i\) is obtained as

\[
-\frac{1}{2\varrho} (2\sigma w, h_2(y_i))_{L^2} = -\frac{1}{2\varrho} \cdot 2 \cdot A_{2,2,2} \cdot \frac{\varrho}{4c_2} w_i = -\frac{2}{t} w_i
\]

coming from the mixed term \(-\frac{1}{2\varrho} \cdot 2w\sigma\) in \( Q(J^2(w+\sigma)) \), and the \( O(1/t^3) \) term appears as an application of item (3) of Proposition A.1. By assumption, we have \( w_i = o(1/t) \) thus \( O(w_i^2) \) is of higher order compared with \( \frac{2}{t} w_i \). The solution to the leading order equation \( \dot{w}_i = -\frac{2}{t} w_i \) is \( O(1/t^2) \). Thus we complete the proof.

\[\square\]

5.2. The \( C^1 \) normal form. In this section, we give the proof of Theorem 1.4.

The main difficulty is as follows. In the equation of motion for \( u \), we have \( \partial_t u = Lu + Q(J^2u) \) (see Lemma A.1). However, in the radius \( O(\sqrt{t}) \), \( h_2/t \) is not small (when \( |y| \approx \varepsilon_0\sqrt{t}, h_2/t \approx \varepsilon_0^2 \)). Therefore, in the \( C^0 \) norm, the nonlinear term \( Q(J^2u) \) is \( O(\varepsilon_0^2) \), which leads to cumulative errors to the estimate of \( u \) as \( t \to \infty \). The way to address the difficulty is to invoke a formalism of [AV97], also used in [Gan21, Gan22], followed by the Ornstein-Uhlenbeck regularization estimate in [Ve82].

Proof of Theorem 1.4 for \( C^1 \) normal form. Using Proposition 3.6, we have the equation of \( u \) in (A.5). Now we rewrite (A.5) as \( \partial_t u = \mathcal{L} u + \tilde{m} \), where we introduced the nonlinear operator

\[
\mathcal{L} f = \frac{\varrho^2}{(\varrho + f)^2} \Delta_\theta f + \Delta_{\mathbb{R}^k} f - \frac{1}{2} y_i \partial_i f + \frac{\varrho + f}{2} - \frac{\varrho^2}{2(\varrho + f)}
\]

and \( \tilde{m} = O(|\nabla u|^4 + |\nabla u|^2 |\nabla^2 u| + |\nabla_\theta u|^2) \).

We next study the solution to the equation \( \partial_t f = \mathcal{L} f \).

Lemma 5.3. Suppose \( 1 \leq \ell \leq k \). Let

\[
f(\theta, y, t) = \varrho \sqrt{1 + \frac{\sum_{i=1}^{\ell} (y_i^2 - 2)}{2t}} = \varrho \left[ \sqrt{1 + \frac{\sum_{i=1}^{\ell} (y_i^2 - 2)}{2t}} + 1 \right].
\]

Then we have

\[
(5.6) \quad \partial_t f - \mathcal{L} f = -\frac{(n - k)^2 (2\sum_{i=1}^{\ell} y_i^2)}{t^2(\varrho^2 + \frac{\varrho^2}{2t} \sum_{i=1}^{\ell} (y_i^2 - 2))^{3/2}}.
\]
Moreover in the $H^1$-norm, we have

$$f - \frac{\theta}{4t} \sum_{i=1}^\ell (y_i^2 - 2) = O(t^{-2}).$$ \hfill (5.7)

**Proof of Lemma 5.3.** To find an approximation solution of $\partial_t f = \mathcal{L} f$, we use the idea of [AV97] to apply the change of variable $y = \sqrt{t} \xi$, then if $\partial_t f = \mathcal{L} f$ and $g(\theta, \xi, t) = f(\theta, \sqrt{t} \xi, t) + \varrho$, we get

$$\partial_t g = \frac{\Delta \xi g}{t} - \frac{1}{2} \xi \cdot \nabla \xi g + \frac{\varrho^2 - \varrho^4}{2g} + \frac{1}{2\sqrt{t}} \xi \cdot \nabla \xi g,$$ \hfill (5.8)

and when $t$ is sufficiently large, the dominant part of the right-hand side is $-\frac{1}{2} \xi \cdot \nabla \xi g + \frac{\varrho^2 - \varrho^4}{2g}$. So an approximation solution is $g(\theta, \xi, t) = \varrho \sqrt{1 + \left(\sum_{i=1}^\ell (y_i^2 - 2) \right)}$, which yields an approximation solution $\tilde{g}(\theta, y, t) = \varrho \sqrt{1 + \left(\sum_{i=1}^\ell (y_i^2 - 2) \right)} + \varrho$. Notice that we expect the dominant mode to be $\frac{\varrho}{2}(\sum_{i=1}^\ell (y_i^2 - 2))$. So we slightly modify the function to get an approximated solution $f$ as in the statement.

We can compute $\partial_y f = \frac{\varrho^2 y_j \delta_{ij}}{2t \sqrt{\varrho^2 + \frac{\varrho^4}{2t} \sum_{i=1}^\ell (y_i^2 - 2)}}$ and

$$\partial^2_{y_j y_m} f = \frac{\varrho^4 \delta_{jm} \delta_{ij} \delta_{ml}}{4t \sqrt{\varrho^2 + \frac{\varrho^4}{2t} \sum_{i=1}^\ell (y_i^2 - 2)}} - \frac{\varrho^4 y_j y_m \delta_{ij} \delta_{ml}}{(2t)^2 (\varrho^2 + \frac{\varrho^4}{2t} \sum_{i=1}^\ell (y_i^2 - 2))^{3/2}}.$$ This gives (5.6). Next, (5.7) follows from the following direct computation

$$f - \frac{\theta}{4t} \sum_{i=1}^\ell (y_i^2 - 2) = \frac{\theta}{8t^2} \left(\sum_{i=1}^\ell (y_i^2 - 2)\right)^2 \left(1 + \frac{1}{2t} \sum_{i=1}^\ell (y_i^2 - 2) + 1\right)^{-2}. \hfill \square$$

We can check that $w := u - f$ satisfies the following equation

$$\partial_t w = \mathcal{L} u - \mathcal{L} f + \tilde{m} - \frac{\varrho^4 (\sum_{i=1}^\ell y_i^2)}{(2t)^2 (\varrho^2 + \frac{\varrho^4}{2t} \sum_{i=1}^\ell h_2(y_i))^{3/2}}.$$ Using the mean value theorem we obtain the equation to be

$$\partial_t w = \mathcal{L} w + \mathcal{M}(w) - \frac{\varrho^4 (\sum_{i=1}^\ell y_i^2)}{(2t)^2 (\varrho^2 + \frac{\varrho^4}{2t} \sum_{i=1}^\ell h_2(y_i))^{3/2}}, \hfill (5.10)$$

where $\mathcal{M}(w)$ is the nonlinear term consists of the terms of $w \Delta w$, $w^2$, $|\nabla w|^2$, and higher power terms of $w$ and $|\nabla w|$, as well as $t^{-1}|y||w|$, $t^{-1}|y||\nabla w|$, $t^{-1}|y||\nabla w|$; it also consists of terms like $t^{-2}|y|^2$ and higher powers, coming from the derivatives of $f$. 

Using Kato’s inequality $\Delta g \cdot \text{sgn}(g) \leq \Delta |g|$ and bounding $D^2w$ by a constant (see Proposition 3.6), we get

$$\partial_t |w| - L|w| \leq C(|w|^2 + |\nabla w|^2 + \frac{|y|^2}{t^2} + \frac{1}{t^2}).$$

Similarly, we can obtain the equations for $|\nabla \rho w|$ and $|\nabla_y w|$ (we need to bound $D^3w$ by a constant, for which purpose, we invoke the “Moreover” part of Proposition 3.8 noting that $P = 0$ in the current setting, which improves Proposition 3.6 to $C^\ell$ bound, $\ell > 2$). Then, letting $Z = (|w| + |\nabla w|) \chi_{\varepsilon_0\sqrt{t}}$, we get

$$\partial_t Z - LZ - \frac{C}{t} Z \leq C(Z^2 + \frac{|y|^2}{t^2} + \frac{1}{t^2} + \chi_1(y, t)),$$

where $\chi_1(y, t)$ is 1 outside $\varepsilon_0\sqrt{t}$, and 0 elsewhere. Now we are already in the setting of Velázquez [Vel92] (2.14).

**Lemma 5.4** (Proposition 2.3 in [Vel92]). Let $Z \geq 0$ satisfy equation (5.11) with $\|Z\|_{L^2} = o(t^{-1})$. Then we have $\max_{|y| \leq \varepsilon_0\sqrt{t}} Z(y, t) = o(1)$, as $t \to \infty$.

For the proof of Lemma 5.4, we refer readers to [Vel92, equation (2.14)] (proof of Proposition 2.3 in [Vel92]) and thereafter. The idea of the proof is also similar to the proof of Proposition 3.8, which is given in Appendix B.

In our case, we have $\|Z\|_{L^2} \leq C\|w\| = o(t^{-1})$ because of (5.7) and the $H^1$ normal form Theorem 1.3. Thus, by the last lemma, we get $\|w(\theta, y, t)\|_{L^\infty(|y| \leq \varepsilon_0\sqrt{t})} = o(1)$ as $t \to \infty$. This shows that $u(z, t)$ converges to $f(z, t) = \varrho \sqrt{1 + \frac{1}{2t} \sum_{i=1}^\ell (y_i^2 - 2)} - \varrho$ in $C^1$-norm for $y$ in the ball of radius $\varepsilon_0\sqrt{t}$. This completes the proof of Theorem 1.4 for the $C^1$-normal form. \qed

6. Isolatedness of Nondegenerate Singularities

In this section, we shall apply our $C^1$-normal form to obtain geometric consequences on the MCF. The first consequence is the isolatedness theorem 1.6 for nondegenerate singularities. In addition to the normal form, another key ingredient in the proof is a pseudolocality theorem developed particularly for MCFs with cylindrical singularities. We also prove the mean convex neighborhood theorem for nondegenerate singularities.

6.1. **Proof of the isolatedness theorem.** We first present some necessary ingredients for the proof of Theorem 1.6. The first ingredient is Theorem 1.4, where we get a $C^1$ normal form in the ball of radius $O(\sqrt{t})$.

Another ingredient is a pseudolocality lemma. The intuition behind the lemma is the following: Theorem 1.4 implies that the graph of the RMCF in $B_{K\sqrt{t}}$ is given by the nondegenerate normal form (in $C^1$ norm) $u(y, \theta, t) \approx \frac{\varrho}{4t} \sum_{i=1}^\ell (y_i^2 - 2)$ to the leading order, where the constant $C$ is close to $\varrho/4$ when $|y|$ is small, and $u$ is close to a fixed constant when $|y| \approx K\sqrt{t}$ (see (5.9)). Then for any $\xi, \eta \in \mathbb{R}^\ell$, when $T$ is sufficiently large, $u(\eta - \sqrt{T}\xi, \theta, T) \approx C|T^{-1/2}\eta - \xi|^2 > 0$. In particular, this implies that near $\sqrt{T}\xi$, the
RMCF $M_T$ is almost a cylinder with a larger radius than $\varrho$. Roughly speaking, this lemma shows that if a MCF is almost a cylinder in a large range, but the radius is strictly larger than the radius of the self-shrinking cylinder, then it does not shrink to a singularity before the time that the self-shrinking cylinder shrinks to singularities.

**Lemma 6.1** (Pseudolocality of MCF graph over cylinders). For any $\varepsilon_0 > 0$, and a vector $V \in \mathbb{R}^k$ with $0 < |V| < \varepsilon_0$, there exist $T_0 > 0$, $R_1 > 0$ and $\varepsilon_1 > 0$ with the following significance. Suppose $T > T_0$ and $M_T$ is a MCF, at time $\tau = 0$ is a graph of function $u$ over the cylinder $\Sigma^k$ inside a ball of radius $\varepsilon_0 \sqrt{T}$, with $\|u(\theta, y) - C|T^{-1/2}y - V|^2\|_{C^1} < \varepsilon_1$, where $C > 0$ is some constant. Then for $\tau \in [0, 1]$, $M_T \cap B_{R_1}$ is a smooth MCF.

The proof of Lemma 6.1 is a straightforward application of pseudolocality.

**Proof of Lemma 6.1.** We use the pseudolocality property of MCF. We observe that the graph of the function $C|T^{-1/2}y - V|^2$ is a paraboloid over the cylinder when $|y| < \varepsilon_0 \sqrt{T}$. Moreover, for any fixed $R > 0$, inside the ball of radius $R$, we have $C|T^{-1/2}y - V|^2 \rightarrow C|V|^2$ as $T \rightarrow \infty$. So the graph of the function $C|T^{-1/2}y - V|^2$ converges to the cylinder $\mathbb{S}^{n-k}(\varrho + C|V|^2) \times \mathbb{R}^k$ smoothly compactly as $T \rightarrow \infty$. By the pseudolocality of MCF (See Theorem 3.4), as $T \rightarrow \infty$, the MCF $M_T^\tau$ starting from the graph of function $C|T^{-1/2}y - V|^2$ converges to the shrinking cylinder MCF with initial radius $\varrho + C|V|^2$, which is smooth for time $\tau \in [0, 1]$. This shows that when $T$ is sufficiently large, $M_T^\tau$ is smoothly close to the shrinking cylinder MCF with initial radius $\varrho + C|V|^2$ with $\tau \in [0, 1]$, inside the ball of radius $2R_1$. Finally, when $\varepsilon_1$ is sufficiently small, a similar argument shows the lemma.

With this pseudolocality lemma, we can prove Theorem 1.6.

**Proof of Theorem 1.6.** Theorem 1.4 shows that the RMCF is a graph over $\Sigma^k$ in a ball of radius $\varepsilon_0 \sqrt{T}$ when $t$ is sufficiently large. Let us cover the RMCF $M_t \cap (B_{\varepsilon_0 \sqrt{T}} \setminus B_{2-\varepsilon_0 \sqrt{T}})$ with several balls $B_{R_1}(p_i)$ of radius $R_1 := 2^{-1} \varepsilon_0 \sqrt{T}$ centered at $p_i \in B_{\varepsilon_0 \sqrt{T}} \setminus B_{2-\varepsilon_0 \sqrt{T}}$, $i = 1, 2, \ldots, m$, and Theorem 1.4 implies that we can apply Lemma 6.1 to get that the MCF starting from $M_t$ is smooth in $B_{R_1}(p_i)$ for time 1. Rescaling back, we see that $M_T$ is smooth in $B_{e^{-t/2}R_3}(e^{-t/2}p_i)$ for $\tau \in [-e^{-t}, 0]$. Because $t$ can be chosen arbitrarily large, $M_T \cap (B_3(0) \setminus \{0\})$ is smooth for $\tau \in [-1, 0]$ for some $\delta > 0$ (for example, we can choose it to be $e^{-T/2} \varepsilon_0 \sqrt{T}$ for some fixed sufficiently large $T$).

Moreover, the region $\varepsilon_0 \sqrt{T} \leq |y| \leq \varepsilon_0 \sqrt{T}$, $t \in [T, \infty)$ for the RMCF scale, corresponds to the region $\varepsilon_0 (\log(-\tau))^{-1/2} \leq |y| \leq \varepsilon_0 (\log(-\tau))^{-1/2}$, $\tau \in [-e^{-T}, 0)$, which shrinks to the origin $0$ as $\tau \rightarrow 0$. Thus the singularity is isolated in the backward $\delta$-spacetime neighborhood.

6.2. Mean convex neighborhood and Type-I. In this section, we give the proof of the mean convex neighborhood Theorem 1.7 and the Type-I curvature condition Theorem 1.8. In fact, Lemma 6.1 shows that when $T$ is sufficiently large and $\varepsilon_0$ is sufficiently small, the MCF is still a graph over the cylinder. Then the RMCF is roughly the graph of the function $\sim \varepsilon_0 |V|^2 e^{t/2}$. This gives a description of the neighborhood of the non-degenerate singularity.
Proof of Theorem 1.7. We only need to prove that for the corresponding RMCF \( \{M_t\} \), when \( t \) is sufficiently large, \( M_t \cap B_{\delta_2 e^{t/2}} \) is mean convex.

First, we show that when \( t \) is sufficiently large, \( M_t \cap B_{\delta_2 \sqrt{t}} \) is mean convex, where \( \delta_2 \) is chosen slightly smaller than \( K \) in Proposition 3.6. In previous section, we have proved that \( M_t \cap B_{\delta_2 \sqrt{t}} \) can be written as a graph of the function \( u(\cdot, t) \) over \( \Sigma^k \), with \( u(\theta, y, t) \) converges to \( \sqrt{1 + \sum_{i=1}^{k} h_2(y_i)} \) in \( C^1 \)-norm for \( y \) in the ball of radius \( \delta_2 \sqrt{t} \). In particular, this shows that \( u(\theta, y, t) \) converges to \( \|u\|_{C^2(\delta_2 \sqrt{t})} \leq \varepsilon_0 \).

The mean curvature of the graph of \( u(\theta, y, t) = \frac{\sqrt{1 + \sum_{i=1}^{k} h_2(y_i)}}{1 + \sqrt{1 + (2t)^{-1} |y|^2}} \) is explicitly given by (A.3), and it is bounded from below by \( \frac{\rho}{2} - C\varepsilon_0 \), where \( \rho/2 \) is the mean curvature of \( \Sigma^k \), and \( \varepsilon_0 \) is the \( C^2 \)-bound of \( u \). Thus, if initially, we fixed \( \varepsilon_0 \) to be a sufficiently small number, we see the mean curvature is positive.

Finally, from the scale \( \delta_2 \sqrt{t} \) to \( \delta_2 e^{t/2} \), we use the same argument as the proof of Theorem 1.6. When \( t \) is sufficiently large, we can decompose \( M_t \cap B_{\delta_2 \sqrt{t}} \) into the union of several pieces, each piece is very close to cylinders with various radii. Under the evolution of MCF, each piece remains close to some cylinder. In particular, when \( t \) is sufficiently large, each piece is mean convex. This concludes the proof.

\[ \square \]

Proof of Theorem 1.8. We only need to prove that for the corresponding RMCF \( \{M_t\} \), when \( t \) is sufficiently large, \( M_t \cap B_{\delta_2 e^{t/2}} \) has bounded \( |A| \). This is true just as the proof of Theorem 1.7.

\[ \square \]

7. Stability of nondegenerate singularities

In this section, we study the perturbed RMCF and give the proof of Theorem 1.9 and 1.10.

7.1. The key perturbation lemma. Let us consider a perturbation of \( M_0 \) given by \( \widetilde{M}_0^n := \{ x + \eta v_0(x) n(x) \mid x \in M_0 \} \). Taking \( \eta \) sufficiently small, the RMCF \( \widetilde{M}_t^n \) starting from \( \widetilde{M}_0^n \) remains \( O(\eta) \)-close to the unperturbed RMCF \( M_t \) for \( t \in [0, T] \) (Proposition 3.3 of [SX21b]).

Evolving the RMCF \( \widetilde{M}_t^n \) and writing it as the graph of a function over \( \Sigma^k \cap B_r(T) \), it is most likely that the Fourier modes with positive eigenvalues of \( L \) grow exponentially, which may quickly dominate the \( h_2 \)-component that we added by the perturbation at time \( T \) by applying the last lemma. This is related to an essential drawback of RMCF: it can only detect a single spacetime point. If the RMCF is zoomed in at a point that is not a singularity, it will not converge to a shrinker. In our case, a perturbation can shift the spacetime position of the singularity. Thus we need to keep track of the spacetime position of the singularity, to make sure that we are zooming in at the correct point.
In terms of the local dynamics of the RMCF near the shrinker, positive Fourier modes correspond to conformal linear transformations and have exponential growth under the dynamics, and we can choose appropriate conformal linear transformations to eliminate them. In particular, by Remark 2.2, the presence of an $h_2(y_i)$ term can be used to kill the $y_i$-modes by a translation in the axis direction of the cylinder.

We use $\mathcal{T}$ to denote the group of transformations generated by translations, rotations and dilations. It is natural to equip $\mathcal{T}$ with vector norm because $\mathcal{T}$ is a subgroup of the direct sum of the rigid transformation group and the dilation group of the Euclidean space. We will only consider the connected component of the identity element $\text{id}$.

We first establish a proposition saying that if the graphical function $u$ of a hypersurface in a sufficiently large ball has large enough $E^0$ component, then most of the Fourier modes in the $E^+$ component can be eliminated by conformal linear transformations under certain assumptions.

**Proposition 7.1.** For any small $\xi > 0$, there exist $\varepsilon_0 > 0$ and $R_0$ sufficiently large, such that for any $R > R_0$ the following holds for the graphical function $u : \Sigma^k \cap B_{R+10} \to \mathbb{R}$ satisfying

1. $\|u\|_{C^2} \leq \varepsilon_0$ and $\|\chi_R^2 u\| \geq C e^{-R^2/4}\|u\|_{C^2};$
2. $\|\Pi_0 \chi_R^2 u\| = \frac{1}{|I|} \sum_{i \in I} h_2(y_i) + \text{h.o.t.}$, where $I \subset \{1, 2, \ldots, k\}$ and the h.o.t. has $H^1$ norm less than $\frac{1}{10}\xi$;
3. $\|\Pi_0 \chi_R^2 u\| \geq (1 - 10\xi)\|\chi_R^2 u\|;$
4. $\|\Pi_0 \chi_R^2 u\| \geq (1 - \frac{1}{2}\xi)\|\chi_R^2 u - \Pi_1 \chi_R^2 u\|$ where $\Pi_1$ is spanned by $1, y_i, \theta_j, j = 1, 2, \ldots, n - k + 1$ and $i \in I$.

Then there exists an element $T \in \mathcal{T}$ such that $\|T\| \leq C\xi\|\chi_R^2 u\|$ and $T(M)$ is a graph of function $v$ over $\Sigma^k \cap B_{R+9}$ satisfying $\|\Pi_0 \chi_R v\| \geq (1 - \xi)\|\chi_R^2 v\|$.

**Proof.** The Fourier modes $\theta_j, j = 1, 2, \ldots, n - k + 1$ correspond to infinitesimal translations in the $S^{n-k}$ direction, constant 1 corresponds to infinitesimal dilation and $y_i$ to infinitesimal translations in the $y_i$ direction. We take Fourier expansion of $u$ and denote the Fourier coefficients of these Fourier modes by $\alpha \in \mathbb{R}^{n-k+1}, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{|I|}$ respectively. We choose a conformal linear transformation $T$ composed by rigid transformations and dilations generated by these Fourier modes with norms $|\alpha_j|, |\beta|, |\gamma|$ respectively.

Item (3) implies that the ratio $\|\Pi_0 \chi_R^2 u\|/\|\chi_R^2 u\|$ is of order 1 and the ratio $\|\Pi_1 \chi_R^2 u\|/\|\chi_R^2 u\|$ is of order $\xi$. Then we have $\|T\| \leq C\xi\varepsilon_0$. By Remark 2.2, the Fourier modes $y_i$ can be eliminated in the presence of $h_2(y_i)$ by a translation in the $y_i$ direction. For small enough $\xi$, the error between the conformal linear transformation and its linearization generated by $\Pi_1$ Fourier modes is estimated by $O(\xi^2)$. We refer the readers to [SX22, Appendix A] for the detailed computations of the linearizations of the translations and dilations.

The transformation $T$ eliminates the Fourier modes of $u$ in $\Pi_1$ but introduces an additional error of nonlinearity of order $O(\xi^2)$ as well as an error of order $e^{-R^2/2}\xi\|u\|_{C^2}$ due to the change of $\chi_R^{2}(\cdot)$ to $\chi_R^{2}(\cdot + \gamma_i y_i)$. For small $\xi$, these errors are bounded by $\frac{1}{2}\xi\|\chi_R^2 v\|$. Thus, we obtain the estimate in the conclusion part of the proposition from item (4).
7.2. Stability of nondegenerate singularities. In this section we prove Theorem 1.10. The idea is to apply the last proposition at every time $n$ to eliminate possible exponentially growing modes.

Proof of Theorem 1.10. Suppose $\{M_t\}_{t \in [0, \infty)}$ is the RMCF corresponding to $M_\tau$, $\tau \in [-1, 0)$, zoomed in at $(0, 0)$. Let $\eta u_0 : M_{-1} \to \mathbb{R}$ be an initial perturbation for any $u$ with $\|u\|_{C^2} \leq 1$ and $\eta$ sufficiently small. We consider the RMCF $\tilde{M}_t$ starting from $\tilde{M}_0 := \{x + \eta u_0(x)n(x) : x \in \tilde{M}_0\}$.

The proof consists of the following two steps.

1. The cone preservation property and the centering map.
2. The neutral dynamics on $E^0$.

We first show how to use conformal linear transformations to eliminate Fourier modes with positive eigenvalues in order to make sure that the cone preservation property holds for all time $n > T$. By the normal form Theorem 1.3 and the cone Theorem 4.3, for any $\varepsilon > 0$ small, there exists $T_0 > 0$ such that for all $t > T_0$, the unperturbed RMCF $M_t$ can be written as a graph over $\Sigma^k \cap B_{K\sqrt{t}}$ of a function $u : \Sigma^k \cap B_{K\sqrt{t}} \to \mathbb{R}$ satisfying $\chi u \in \mathcal{K}_0(1 - \frac{1}{2}\xi)$. For sufficiently small initial perturbation, the perturbed RMCF $\tilde{M}_t$ is written as the graph of a function $w(t) : \Sigma^k \cap B_{K\sqrt{t}} \to \mathbb{R}$ that is sufficiently close to $u$, thus we have $\chi w \in \mathcal{K}_0(1 - \xi)$ for some fixed $T > T_0$. Thus we get both $u(T)$ and $w(T)$ are in the cone $\mathcal{K}_0(\kappa)$ for $1 - \xi = \kappa$. To apply the cone theorem 4.3 to $w$, we have to make sure that $w_+ \to 0$ in the $C^\infty_{\text{loc}}$ sense as $t \to \infty$. However, this is in general not true. To fix this issue, we apply the centering map along with the flow whenever the orbit $w(n)$ at time $n$ leaves the cone $\mathcal{K}_0(\kappa)$. For each $n > T_0$, we introduce a $\tilde{T}_n$ in the group of conformal linear transformations as follows.

1. If $\chi w(n) \in \mathcal{K}_0(\kappa)$, then $\tilde{T}_n = 0$.
2. If $\chi w(n) \in \mathcal{K}_0(\kappa)$ but $\chi w(n + 1) \notin \mathcal{K}_0(\kappa)$, then we apply Proposition 7.1 to choose a conformal linear transformation $\tilde{T}_n$ such that re-centered MCF $\tilde{T}_nM_t$ has corresponding RMCF $\tilde{M}_t$ at time $n+1$ written as a graph $\tilde{w}(n+1) : \Sigma^k \to \mathbb{R}$ with $\chi \tilde{w}(n + 1) \in \mathcal{K}_0(\kappa)$.

Since $|w_0(n)|$ decays to zero like $1/t$, using the correspondence between MCF and RMCF, we get the estimate $|\tilde{T}_n| \leq \frac{e^{-t/2}}{t}$. This implies that the composition of the sequence of centering maps converges. When $\eta \to 0$, the centering map converges to identity. Here we consider only $\{\tilde{T}_n\}$ generated by Fourier modes with positive eigenvalues. We do not consider rotations since we have chosen to modulo rotations in Section 3.2. Indeed, by Lemma 3.3 and the normal form (Theorem 1.3), the size of elements in $SO(n + 1)$ that we use to modulo rotations decays like $O(1/t^2)$. We shall perform a step of centering map at each time $n$. Thus we always have $\tilde{w}(n) \in \mathcal{K}_0(\kappa)$ and we get for large $t$, the neutral modes in $E^0$ dominate.
We next analyze the neutral dynamics in the finite dimensional space $E^0$. We give a basis $(h_2(y_1), \ldots, h_2(y_k))$ of $E^0$. For the unperturbed flow $M_t$, by the normal form (Theorem 1.3), we get $\Pi_0u(t) = \sum_{i=1}^{k} \frac{\partial}{\partial y_i} h_2(y_i) + O(1/t^2)$ up to a rotation in $\mathbb{R}^k$, thus we get that $\Pi_0u(t)$ almost points to the diagonal direction $(1, \ldots, 1)$ in $E^0$. To show that the perturbed RMCF $\tilde{M}_t$ has also a nondegenerate singularity, it is enough to show that $\Pi_0w(t)$ also approaches the diagonal direction $(1, \ldots, 1)$ in $E^0$. In other words, it is enough to show that the solutions to the equations $\dot{a}_i = -\gamma a_i^2 + o(1/t^2)$ is robust under small perturbation of initial conditions ($\gamma$ is a positive constant appeared in Lemma 5.2). This is similar to the proof of Proposition 5.1 and we omit it here.

Iterating the procedure, we get that $\mathbf{M}_t := \prod_{i=1}^{n} \mathbf{T}_i \mathbf{M}_t, t \in [-1, 0)$ gives an MCF with corresponding RMCF $\tilde{M}_t, t \in [0, \infty)$ converges to $\Sigma^k$ in the $C^\infty_{loc}$-sense with leading terms in the normal form $\sum_{i=1}^{k} \frac{1}{t} h_2(y_i)$. Thus the perturbed MCF $\mathbf{M}_t$ admits a nondegenerate singularity at the spacetime point $(0, 0)$ modeled on $\Sigma^k$.

8. Denseness of nondegenerate singularities

In this section we prove Theorem 1.9. As we have explained in the introduction, the proof differs quite a lot from the perturbation argument as in [SX21b, SX21a, SX22], due to the presence of the $y_i$-eigenmodes. The difficulty is overcome by combining the observation in Remark 2.2 as well as an argument applying a centering map for infinitely many steps along the flow. Moreover, the neutral eigenspace is in general not a one-dimensional space, but essentially $k$ dimensions, spanned by $h_2(y_i)$ for $i = 1, 2, \ldots, k$. Thus, it is possible that the original RMCF is degenerate in some directions of $y_i$, and nondegenerate in some other directions $y_j$.

To prove the denseness of nondegenerate singularities, we need to understand how to perturb a degenerate singularity to obtain a nondegenerate one. Let us first recall Lemma 5.2 of [HV92b] and its $H^1$-version proved in [SX22].

**Lemma 8.1.** Let $S(\tau, s)$ be the fundamental solution to the variational equation $\partial_\tau v = L_{M_\tau} v$ such that the function $S(\tau, s)v(s)$ solve the equation with initial condition $v(s) : M_s \to \mathbb{R}$. Then the operator $S(\tau, 0) : L^2(M_0) \to H^p(M_t), p \in \mathbb{N}$, has dense image.

We refer the readers to [SX22, Appendix B] for a proof of Lemma 8.1 for $p = 1$, and the $p > 1$ case is similar.

Let us assume that $M_t$ is the RMCF, corresponding to an MCF $\mathbf{M}_{\tau}, \tau \in [-1, 0)$. Moreover, we assume the leading term of $M_t$ is $\sum_{i \in \mathcal{I}} \frac{\partial}{\partial y_i} (y_i^2 - 2)$, where $\mathcal{I}$ is a proper subset of $\{1, 2, \ldots, k\}$. To proceed, without loss of generality and for simplicity, we assume that $k = 2$ and the normal form of $M_t$ has leading term $\frac{\partial}{\partial y_1} (y_1^2 - 2)$. In other words, the $y_1$-direction is degenerate and the $y_2$-direction is nondegenerate. Since the case of all the directions being degenerated is simpler than this mixed type problem, so we only discuss the more difficult mixed case, i.e. some directions are degenerate, others are nondegenerate here.
8.1. Constructing the initial perturbation. Throughout the proof, $\chi = \chi(t)$ is a smooth cut-off function that is constant 1 on $\Sigma^k \cap B_{K,\sqrt{T-1}}$ and 0 outside $\Sigma^k \cap B_{K,\sqrt{T}}$.

Let $\varepsilon_0$ be a small number as in Proposition 7.1 and suppose $T$ is a large time such that $M_T \cap B_{K,\sqrt{T}}$ can be written as a graph of $u : \Sigma^k \cap B_{K,\sqrt{T}} \to \mathbb{R}$ with $\|u\|_{C^2} \leq \varepsilon_0$. We first apply Lemma 8.1 to find some initial condition $v_0$ of the variational equation $\partial_t v = L_M v$, for some fixed small $\xi > 0$ such that $v(T)$ satisfies $\|\Pi_{h_2(y_i)} \chi(T)\| \geq (1 - \frac{1}{3})\|\chi(T)\|$, where $\bar{v}$ is the pullback of $v$ from $M_T \cap B_{K,\sqrt{T}}$ to $\Sigma^k \cap B_{K,\sqrt{T}}$.

Now we use the variational equation to approximate the RMCF. Suppose $\bar{M}_t^n$ is the RMCF starting from $\{x + \eta_0 n(x) \mid x \in M_t\}$. Then $\bar{M}_t^n$ can be written as a graph of function $\eta \bar{v}(T)$ over $M_T$, and $\|\eta \bar{v}(T) - \eta v(T)\|_{C^2} \leq C\eta^{1+\sigma_0}$ for some $\sigma_0 > 0$ (Proposition 3.3 of [SX21b]). Next, we know that the function $\bar{v}$ satisfies (4.1) and (4.2) when pulled back to $\Sigma^k \cap B_{K,\sqrt{T}}$, whenever $t > T$ and $\|\eta \bar{v}\|_{C^2}$ is sufficiently small.

In the following, we shall show that there is a converging sequence of conformal linear transformations applied to the MCF $M^n_T$ corresponding to the RMCF $\bar{M}_t^n$ such that the limiting MCF has a nondegenerate cylindrical singularity at the spacetime point $(0,0)$.

Because both $M_T$ and $\bar{M}_T$ are close to $\Sigma^k \cap B_{K,\sqrt{T}}$, from our construction of the perturbation, $\bar{M}_T$ can be written as the graph of a function $\bar{u}$ over $\Sigma^k \cap B_{K,\sqrt{T}}$ with $\|\Pi_{h_2(y_i)} \chi w\| \geq (1 - \xi)\|\chi w\|$ for $w = \bar{u} - u$ (details for the difference of two graphs that are close to each other can be found in [SX22, Appendix C] and [SX21b, Appendix C]). In the following we take $1 - \xi = \kappa$ as in Proposition 7.1. We shall run the two equations (c.f. (4.1) and (3.1) respectively)

$$\partial_t (\chi w) = L(\chi w) + \delta B(\chi w) \quad \text{and} \quad \partial_t (\chi u) = L(\chi u) + B(\chi u)$$

simultaneously.

We have (4.1), (4.2),

$$\partial_t (\chi w) = L(\chi w) + P(\chi w) + \mathcal{E}, \quad P = \int_0^1 D_u B(s\bar{u} + (1 - s)u) ds = \int_0^1 D_u B(u + sw) ds.$$

Since $w$ is the perturbation, in the following we shall also assume $\|\chi w\|_{C^2} \leq \|\chi u\|_{C^2}$, otherwise, $\bar{u}$ has already grown large enough such that $h_2(y_i)$ is very large for $i = 1, 2$. Note from Proposition A.1, we have that $B = -\frac{1}{\theta} \left( \frac{u^2}{2} + 2u \Delta \theta u + |\nabla \theta u|^2 \right) + C(J^2 u)$. Thus we get by the normal form Theorem 1.3

$$\partial_t (\chi w) = L(\chi w) - \frac{1}{\theta}(u + \frac{1}{2}(\chi w))(\chi w) + O(t^{-2})(\chi w) + \mathcal{E}.$$

Using Proposition 3.8 we bound the $L^2$ norm of the boundary term $\mathcal{E}$ by $\|\chi w\| t^C e^{-K^2 t}$.

8.2. The dynamics of the neutral modes. In this section, we study the $\chi w$-equation in (8.1) projected to the $E^0$ subspace, under the assumption that the cone condition $\|\Pi_0 \chi w\| \geq (1 - \xi)\|\chi w\|$ holds. We shall verify the cone condition in the next subsection.
Suppose
\[ \Pi_0\chi w = a_1(t)h_2(y_1) + a_2(t)h_2(y_2) + a_{12}(t)h_1(y_1)h_1(y_2), \]
then we have the following lemma.

**Lemma 8.2.** Suppose that the difference function \( w \) satisfies the cone condition \( \|\Pi_0\chi w\| \geq (1 - \xi)\|\chi w\| \) for all time \( t > T \). Then, when projected to the \( E^0 \)-components, we get the following system of ODEs (see proof of Proposition 5.1 for the quadratic term in \( B \) and Item (3) of Proposition A.1 for higher order terms)
\[
\begin{align*}
\dot{a}_1 &= -\frac{\sqrt{e_\alpha}}{\theta}(a_1)^2 - \frac{e_\alpha^2}{\sqrt{e_\alpha}}(a_2)^2 + O((|a_1| + |a_2| + |a_{12}|)/t^2) \\
\dot{a}_2 &= -\left(\frac{2}{\eta}a_2 + \frac{\sqrt{e_\alpha}}{\theta}a_2^2\right) + O((|a_1| + |a_2| + |a_{12}|)/t^2) \\
\dot{a}_{12} &= O(\frac{1}{t^2}) - \frac{1}{\sqrt{e_\alpha}}(a_1 + a_2)a_{12} + O((|a_1| + |a_2| + |a_{12}|)/t^2).
\end{align*}
\]

(8.2)

Suppose we have \( |a_2(T)| + |a_{12}(T)| \leq \varepsilon_0|a_1(T)| \) for some \( \varepsilon_0 > 1/T^{1/2} \) and \( \eta \ll 1/T \), then we have the following estimate for \( t \in [0, O(\eta^{-1})] \)
\[
a_1(t + T) = \frac{a_1(T)}{1 + a_1(T)\frac{\sqrt{e_\alpha}t}{\theta} + O(\frac{a_1(T)}{(T + t)^2})},
\]
\[
|a_2(t + T)| + |a_{12}(t + T)| \leq 2\varepsilon_0|a_1(t + T)|.
\]

**Remark 8.3.** Here we only give estimate of \( a_1 \) up to time of order \( O(\eta^{-1}) \), at which time, the difference function \( w \) is of order \( 1/t \), comparable to the graphical function \( u \) for the unperturbed flow.

**Proof.** In order to derive the equations of \( \dot{a}_1, \dot{a}_2 \) and \( \dot{a}_{12} \), we follow the setting in Proposition 5.1, but replacing \( v \) there by \( \chi w \). Then \( a_i = a^{-1}m_{ii}c_0^{2(k-1)}G(S^{n-1})^{-1} \) and \( a_{ij} = m_{ij}c_0^{2(k-2)}G(S^{n-1})^{-1} \). Then the calculations are similar to Lemma 5.2, except that there is an extra \( -\frac{1}{\theta}w\chi w \) term in the equation of \( \partial_t(\chi w) \). Notice that the leading order term of \( u \) is \( \frac{2}{\eta}c_2^{-1}h_2(y_2) \), which is orthonormal to \( h_2(y_1) \) and \( h_1(y_1)h_2(y_1) \), so this term will contribute an extra term only to \( \dot{a}_2 \), which is
\[
e_0^{2(k-1)}G(S^{n-k})^{-1}\langle -\frac{1}{\theta}w(a_2h_2(y_2)), h_2(y_2) \rangle = -\frac{a_2}{4t\alpha_2}A_{2,2,2} = \frac{2}{t}a_2.
\]

Here we notice that the integration in the components other than \( y_2 \) direction will cancel with \( e_0^{2(k-1)}G(S^{n-k})^{-1} \). As the error term \( \mathcal{E} \) is bounded by \( \|\chi w\|^tC e^{-K^2t} \) in \( L^2 \), which is absorbed into the \( \mathcal{O} \)-terms in (8.2).

We next show how to obtain the estimate of the solutions. We shall assume that \( 2\varepsilon_0|a_1(t)| > |a_2(t)| + |a_{12}(t)| \) for all \( t \in [T, T'] \) and solve the above ODE system over the time interval \([T, T']\) for some \( T' \). This assumption reduces the first ODE into the form \( \dot{a}_1 = (O(\frac{1}{t^2}) - \frac{\sqrt{e_\alpha}}{\theta}a_1)a_1 \) to the leading order. Note that the equation \( \dot{a}_1 = \frac{1}{t^2}a_1 \) has solution of the form \( a_1(t) = Ce^{-\frac{t}{T}} \), i.e. the solution stabilizes, thus as \( t \) is large, the behavior of the solution is dominated...
by $\dot{a}_1 = -\sqrt{\frac{a_0}{\epsilon}} a_1^2$ and the solution eventually stabilizes to $\frac{a_1(T)}{1+a_1(T)^2}$ to the leading order. Thus, on the time interval $t \in [T, T']$, we have $a_1 = O(\eta)$ and $a_2 = O(\varepsilon_0 \eta)$. For the second equation, with the assumptions we have $\dot{a}_2 = -\left(\frac{2}{\epsilon}a_2 + O(\eta \varepsilon_0) a_2\right) + O(|a_1|^2 \eta^2)$. Integrating the equation over $[T, T']$ using the fact that $O(\eta \varepsilon_0) \ll \frac{2}{\epsilon}$ for $t \in [T, O(\eta^{-1})]$, we get the oscillation of $a_2$ is much less than $|a_1(T)| / T \ll \varepsilon_0 |a_1(T)|$. We estimate the $\dot{a}_{12}$-equation in a similar way. Thus we get the estimate $|a_2(t)| + |a_{12}(t)| \leq 2\varepsilon_0 |a_1(t)|$ on the time interval $[T, T']$. The estimate shows that there is no obstruction to extend the time $T'$ to $O(\eta^{-1})$. This completes the proof.

\[ \square \]

8.3. The perturbation argument: using conformal linear transformations to eliminate all positive eigenmodes. In this section, we show how to use conformal linear transformations to guarantee the cone condition $\|\Pi_0 \chi w\| \geq (1 - \xi)\|\chi w\|$ assumed in the last lemma.

Suppose $\tilde{u}$ undergoes a translation of size $d$ in the direction of $y_1$, then we have

$$
\tilde{u}(t, \theta, y_1 + d, y_2) = u(t, \theta, y_1, y_2) - \tilde{u}(t, \theta, y_1, y_2) + (\tilde{u}(t, \theta, y_1, y_2) - u(t, \theta, y_1, y_2))
$$

\[ (8.3) \]

We have

$$
\Pi_{y_1} \frac{\partial \tilde{u}}{\partial y_1} = c_1 \int y_1 \frac{\partial}{\partial y_1} \tilde{u} e^{-\frac{|y|^2}{4}} = \frac{c_1}{2c_2} \int \frac{\partial}{\partial y_1} h_2(y_1) \frac{\partial}{\partial y_1} \tilde{u} e^{-\frac{|y|^2}{4}} = -\frac{c_1}{2c_2} \int \mathcal{L}_{y_1} h_2(y_1) \tilde{u} e^{-\frac{|y|^2}{4}}
$$

$$
= \frac{c_1}{c_2} \int h_2(y_1) \tilde{u} e^{-\frac{|y|^2}{4}} = 2\Pi_{y_1} h_2(y_1) \tilde{u} = 2\Pi_{y_1} h_2(y_1) u + 2\Pi_{y_1} h_2(y_1) w.
$$

Lemma 8.4. There exists a sequence of conformal linear transformations $\mathcal{T}_n$ at each time $t_n = T + 10n$ with norm $\|\mathcal{T}_n\| \leq \eta$, such that after transforming $\tilde{M}_n$ by $\mathcal{T}_n$, we have

1. the cone condition satisfied (letting $\xi_n = e^{-(5-10\varepsilon_0)n} \xi$)

$$
\|\Pi_{h_2(y_1)} \chi w(t_n)\| \geq (1 - \xi_n)\|\chi w(t_n)\|
$$

(8.5)

at each moment $t_n$.

2. $\|\mathcal{T}_n \circ \cdots \circ \mathcal{T}_0\| \leq 10\eta$ and $\|\chi u(t_n)\| \to 0$ as $n \to \infty$.

Proof. Suppose the condition (8.5) is satisfied at time $t_{n-1} = T + 10(n-1)$, but violated at time $t_n = T + 10n$. By Proposition 4.1, we have the estimate

- $\|\Pi_+ \chi w(t_n)\| \leq e^{10(1+\varepsilon_0)} \|\Pi_+ \chi w(t_{n-1})\|$, 
- $e^{-10(1-2-\varepsilon_0)} \xi_{n-1} \|\chi w(t_{n-1})\| \leq \|\Pi_+ \chi w(t_{n-1})\| \leq \xi_{n-1} \|\chi w(t_{n-1})\|$. 

The former is easy, and we have the latter in order for (8.5) to be satisfied at time $t_{n-1}$ and violated for time $t_n$, because $\Pi_+ \chi w(t_{n-1})$ gets expanded by a rate at least $e^{10(1/2-\varepsilon_0)}$. 

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We next show how to apply a conformal linear transformation to make the condition (8.5) satisfied at time \( t_n \). In order to apply Proposition 7.1 to eliminate growing positive eigenmodes using conformal linear transformations, it remains to verify the assumption (3) and (4) therein, which are used to eliminate \( y_1 \)- and \( y_2 \)-modes, since other positive modes are straightforward to eliminate. For that purpose, we shall need the nondegenerate \( \Pi_{h_2(y_1)} \bar{u} \) and \( \Pi_{h_2(y_2)} \bar{u} \) (see (8.3)).

Suppose that in (8.3), the term \( w(t_n) \) has a nontrivial \( y_1 \)-mode, which is of order \( O(\eta \xi_n) \) by the first bullet point above, and we shall perform a translation to eliminate it. By (8.4), this needs \( \Pi_{h_2(y_1)} \bar{u}(t_n) \) to be bounded away from zero by a number of order \( O(\eta \xi_n) \) (see Remark 2.2). At each moment \( t_n \), we shall introduce a translation \( d_n \) and a modification of \( \eta_n \) of the scalar factor of the initial perturbation as follows. If the cone condition in (8.5) is satisfied, we shall choose \( d_n = 0 \) and \( \eta_n = \eta_{n-1} \), i.e. we do not apply the translation. Otherwise, if the cone condition in item (1) is violated, we then consider the term \( \Pi_{h_2(y_1)} \bar{u}(t_n) =: \bar{a}_2(t_n) \).

If \( |\bar{a}_2(t_n)| > \frac{1}{2} e^{-(5-10\epsilon_0)n}\eta \), we set \( \eta_n = \eta_{n-1} \) and \( d_n = -\frac{1}{\bar{a}_2(t_n)} \Pi_y w(t_n) \). Otherwise, if \( |\bar{a}_2(t_n)| \leq \frac{1}{2} e^{-(5-10\epsilon_0)n}\eta \), we then modify \( \eta_n = \eta_{n-1} \pm e^{-(5-10\epsilon_0)n}\eta \), where we choose \( \pm \) to be the same as the sign of \( \Pi_{h_2(y_1)} \bar{u} \) in (8.4), and choose the size of translation \( d_n \) by the same formula as the last case. Note that the normal form Theorem 1.3 implies that \( \Pi_{h_2(y_1)} u = O(t^{-2}) \).

On the other hand, the last lemma implies that \( \Pi_{h_2(y_1)} w \) dominates \( \Pi_{h_2(y_1)} u \) for \( t > O(\eta^{-1/2}) \), then we can apply Proposition 7.1 directly to guarantee (8.5). Thus, in the following, we only focus on the period of time \( t \in [T, O(\eta^{-1/2})] \). Since we have the estimate \( \Pi_{h_2(y_1)} w \sim \frac{\eta_{n-1}}{1+\eta_{n-1}} \), on the time interval \( t \in [T, O(\eta^{-1/2})] \), the \( e^{-(5-10\epsilon_0)n}\eta \)-perturbation added to \( \eta_{n-1} \) becomes a perturbation \( (1+\eta(1)) e^{-(5-10\epsilon_0)n}\eta \) added to \( \Pi_{h_2(y_1)} w \).

As \( e^{-5} \simeq 0.00675 \), we see that the sequence of numbers \( \eta_n \) converges to a nonzero limit and the modification of \( \eta_n \) does not affect all earlier steps. In this way, applying the proposition 7.1, we eliminate the \( y_1 \)-mode.

We next consider the \( y_2 \)-mode. This is easier than eliminating the \( y_1 \)-mode, since by a similar calculation as (8.3) and (8.4) with \( y_1 \) replaced by \( y_2 \), we find that \( \Pi_{h_2(y_2)} u \) dominates \( \Pi_{h_2(y_2)} w \) for all time. Thus, we can apply Proposition 7.1 without refining \( \eta \). This completes the proof of the lemma.

\[
\mathcal{T}_n - \text{id} \leq C(\|\chi w(n)\| + \|\chi u(n)\|) \leq C\epsilon_0
\]

by Proposition 7.1. We denote by \( T_n \) the conformal linear transformation of the initial condition of the MCF corresponding to \( T_n \). Using the correspondence between MCF and RMCF, we have \( \|T_n - \text{id}\| \leq e^{-\frac{\pi}{2}}\|T_n - \text{id}\| \). This means the composition \( \cdots \circ T_{T+2} \circ T_{T+1} \circ T_T \) converges to some limit that is \( O(\epsilon_0) \)-close to identity. Here \( \{T_n\} \) consists only of conformal linear transformations generated by Fourier modes with positive eigenvalues. For rotations, i.e. conformal linear transformations with zero eigenvalues of \( L \), note that we have chosen to
modulo rotations in Section 3. By Lemma 3.3, the size of rotation at time \( t \) is estimated as \( O(1/t^2) \) since \( B \) is quadratic in \( u \) and \( u \) decays like \( O(1/t) \), thus the total amount of rotations that we need to perform integrates to a number of order \( \varepsilon_0 \). Thus, we complete the proof of the denseness theorem 1.9.

9. Applications

In this section, we give some applications of our main result to level set flows, rotational graphs, etc.

9.1. Regularity of level set flow. In this section we prove Theorem 1.17. We first prove the following lemma:

**Lemma 9.1.** If the \( \{M_\tau\} \) is a MCF with a nondegenerate singularity that is modeled by \( \Sigma^k \) with \( k \geq 1 \), then \( \{M_\tau\} \) has at least two distinct singular times.

**Proof.** Without loss of generality, assume \((0, 0)\) is the nondegenerate singularity of \( \{M_\tau\} \). We show that \( M_\tau \) is nonvanishing after time 0. Recall that in Proof of Theorem 1.6, we prove that there exists \( \delta > 0 \) such that \( M_\tau \cap (B_\delta(0) \setminus \{0\}) \) is smooth for \( \tau \in [-1, 0] \) for some \( \delta > 0 \). This shows that \( M_\tau \) is nonvanishing after time 0, and hence \( M_\tau \) has at least two distinct singular times. \( \square \)

**Proof of Theorem 1.17.** Let us define \( U_1 \) to be the set of mean convex hypersurfaces that has a single spherical singularity, and the spherical singularity is the most generic one in the sense of [SX22]. Recall we define \( U_2 \) to be the set consisting of mean convex closed hypersurfaces such that the level set flows starting from these hypersurfaces have a nondegenerate cylindrical singularity.

The openness of \( U_1 \) is proved in [SX22], and the openness of \( U_2 \) is proved by the stability of nondegenerate cylindrical singularity Theorem 1.10. Hence \( U = U_1 \cup U_2 \) is open. Next we prove denseness. Given a mean convex hypersurface \( \Sigma \), the first time singularity is either spherical or cylindrical. If the first-time singularity is spherical, from [SX22], \( \Sigma \in \overline{U_1} \). If the first-time singularity is cylindrical, the genericity Theorem 1.9 shows that \( \Sigma \in \overline{U_2} \). Then we proved the denseness of \( U \).

Finally, for hypersurfaces in \( U_1 \), the level set flow starting from them has \( C^2 \) but not \( C^3 \) regularity argued as [SX22]; for hypersurfaces in \( U_2 \), the level set flow starting from them has at least two distinct singular time by Lemma 9.1, hence by [CM18] the level set flow has \( C^{1,1} \) but not \( C^2 \) regularity. Then the proof is concluded. \( \square \)

9.2. Generic isolated first time singularities.

**Proof of Theorem 1.15.** For any closed mean convex hypersurface \( M \) and any \( \varepsilon > 0 \), we show that in an \( \varepsilon \)-\( C^2 \) neighbourhood of \( M \), there is a hypersurface \( \tilde{M} \) has property (isolated-or-firecracker). We apply the perturbations to \( M \) in Theorem 1.9 with initial perturbation \( C^2 \)-norm less than \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \), to perturb a degenerate singularity to a nondegenerate
one. If any one of the perturbations such that the first-time singularities of the perturbed flow are all nondegenerate, then we are done.

Otherwise, we choose \( \varepsilon_i < 2^{-i}\varepsilon \), and we also choose the perturbation which does not change the nondegeneracy of any existing nondegenerate singularities. Then we will get a sequence of perturbations, which generate a sequence of nondegenerate singularities, and each one of them is isolated. By the upper-semi-continuity of Gaussian density, a subsequence of these nondegenerate singularities converges to a limit singularity, which is by definition a firecracker. \( \square \)

9.3. **Generic rotational symmetric MCF has first-time type-I singularity.** In this section, we prove a generic type I singularity theorem for rotational graphs. The starting point of our proof is a result of Altschuler-Angenent-Giga [AAG95] studying the evolution of \( \Gamma(t) \) with \( \Gamma(0) = \Gamma_0 \) being a smooth compact hypersurface that is a rotational graph of a function \( r = u(x) \) defined on a closed connected subset on the \( x \)-axis.

**Theorem 9.2** (Theorem 1.1, Theorem 1.2 in [AAG95]). The evolution of \( \Gamma(t) \) has the following properties:

1. **Outside** \( x \)-axis, \( \Gamma(t) \) is always smooth.
2. There is a finite sequence of times \( 0 = t_0 < t_1 < \cdots < t_l \) such that the hypersurface \( \Gamma(t) \) is smooth and compact, including the \( x \)-axis, when \( t_{j-1} < t < t_j, j = 1, 2, \ldots, l \). The solution is empty for \( t > t_l \).
3. Let \( \Gamma_*(t) \) be a connected component of \( \Gamma(t) \) that becomes singular at time \( t_j \). Then either \( \Gamma_*(t) \) shrinks to a point on the \( x \)-axis, or one or more necks of \( \Gamma_*(t) \) get pinched.
4. The number of singularities is no more than the number of positive local maxima and minima of the function \( r = u(x) \) which defines the initial hypersurfaces.

Here item (4) is crucial for our application. In the last section we have seen that generically the first singular set either consists of nondegenerate singularities, or there exists a firecracker: a singularity that is the limit of nondegenerate singularities. Item (4) here rules out the second possibility. To apply item (4), we need the initial rotational graph function to have finitely many local maxima and minima. The following lemma shows that it is easy to find such rotational graphs. Recall that \( \mathcal{R} \) is the space of rotational graphs endowed with \( C^r \) norm with \( r \geq 2 \).

**Lemma 9.3.** There is an open and dense subset of \( \mathcal{R} \) such that the function generating a smooth closed rotational graph has finitely many critical points.

**Proof.** Since openness is easy, we focus on denseness in the following. Suppose \( \Gamma \) is a smooth closed rotational graph with function \( u(x) \), that is defined on \([a, b]\). We need to show that there exists an arbitrarily close rotational graph that has finitely many critical points. If \( u \) itself has finitely many critical points, then we are done. Now we consider that \( u \) has infinitely many critical points. Because \( \Gamma \) is smooth and closed, we have \( \lim_{x \to a^+} u'(x) = \infty \), \( \lim_{x \to b^-} u'(x) = -\infty \). Thus we may assume the critical points of \( u \) are all lying between \([a + \delta, b - \delta]\) for some \( \delta > 0 \).
Now we consider a smooth cut-off function \( \varphi \), that is supported on \([a + \delta/2, b - \delta/2]\), strictly monotone increasing on \([a + \delta/2, (a+b)/2]\) and strictly monotone decreasing on \([(a+b)/2, b - \delta/2]\). Then by the standard argument as the existence of Morse function, \( u + s\varphi \) has isolated (hence finitely many) critical points for a.e. \( s \in \mathbb{R} \). This implies the denseness. □

Now we give the proof of Theorem 1.18.

**Proof of Theorem 1.18.** Let us start with openness, which is the easier part. Suppose \( \Gamma_0 \) is a smooth closed rotational graphical hypersurface, with MCF \( \Gamma(t) \). If the first singularity of \( \Gamma_0 \) is spherical, then Huisken’s theorem shows that for any nearby smooth closed rotational graphical hypersurface \( \tilde{\Gamma}_0 \), the MCF \( \tilde{\Gamma}(t) \) also has the first singularity to be spherical.

If the first-time singularities of \( \Gamma_0 \) are nondegenerate cylindrical, our Theorem 1.10 shows that after a small perturbation, they are still nondegenerate cylindrical.

Next we prove the denseness. Suppose \( \Gamma_0 \) is a smooth closed rotational graphical hypersurface, with MCF \( \Gamma(t) \). If the first singularity of \( \Gamma_0 \) is spherical, then we are done. So we consider the first singularities of \( \Gamma_0 \) are cylindrical. By Lemma 9.3, we assume that the graph generating \( \Gamma_0 \) has finitely many critical points.

Now we apply the perturbation argument (denseness part) in Section 7, but only restricted to the subspace of \( H^1 \) that are SO(\( n \))-invariant. Here we use the fact that the rotational symmetry is preserved by MCF, and the linearized equation is also rotational symmetric. Moreover, the eigenfunction \( h_2 \) is also rotational symmetric. So the arguments in Section 7 still apply here.

We perturb \( \Gamma_0 \) with respect to each one of its degenerate singularities. Because the number of local maxima and local minima of a function does not change under small perturbation of the function, and each time when we apply the perturbation, we have at least one more nondegenerate cylindrical singularity, so after finitely many times of perturbations, we get a rotational graph \( \tilde{\Gamma}_0 \), such that the first-time singularities of the MCF \( \tilde{\Gamma}(t) \) are nondegenerate cylindrical.

By Huisken’s theorem [Hui90], a spherical singularity must be type-I. By Theorem 1.7, a nondegenerate cylindrical singularity must be type-I. Then we conclude the proof. □

10. Pinch a thin ring

In this section we prove the global Theorem 1.12 on pinching a thin ring.

**Proof of Theorem 1.12.** Without loss of generality, we assume \( \gamma \) is a curve passing through the origin, and we will pinch the MCF at the origin. Let us summarize some known facts about MCF whose singular set is a smooth curve. By the stratification of the singular set ([CHN13]), the tangent flow at any singular point splits along a line, hence must be a cylinder in \( \mathbb{R}^3 \). By Colding-Minicozzi’s analysis of cylinder tangent flow, [CM16a], as the time approaches the singular time, the flow becomes a “tube” around the singular set.

The proof will be divided into the following steps.
Step 1: construction of the perturbation at a large time $T$.
We consider the RMCF $M_{p,t} = e^{t/2}(M_t - e^{-t} - p)$ blown up at each point $p \in \gamma$.

We first apply the concept of graphical scale (Definition 3.1) to find a large time $T$ that is uniform for all $p$, such that the at time $T$, $M_{p,T}$ can be written as a graph over $\Sigma^1$ of a function $u_p : \Sigma^1 \cap B_r(T) \to \mathbb{R}$ with $\|u_p\|_{C^2} \leq \varepsilon_0$. The uniformity of $T$ is given by [CM15] and [CM16a, Corollary 3.2].

We next construct a function $f : M_{-e^{-t}} \to \mathbb{R}$ defined as follows:
- Inside $B_{e^{-T/2}(r(T))}$, let $f$ be the transplantation of $e^{-T/2}(y^2 - 2)$ over $M_{-1} \cap B_r(T)$.
- Outside $B_{e^{-T/2}(r(T) + 1)}$, let $f$ be the constant $e^{-T/2}(r(T))^2 - 2$.
- In the annulus region $B_{e^{-T/2}(r(T) + 1)} \setminus B_{e^{-T/2}(r(T))}$, let $f$ be the smooth interpolation.

Step 2: construction of the perturbation of the initial condition.
Let $\varepsilon_1 > 0$ be a sufficiently small number much smaller than $\varepsilon_0$. We next apply Lemma 8.1 of Herrero-Velázquez to the linearized MCF equation: $\partial_\tau u = \Delta_M u$ on the time interval $\tau \in [-1, -e^{-T}]$ to find a function $u(-1)$ such that $\|u(-e^{-T}) - f\|_{C^2} \leq \varepsilon_1$.

With $u(-1)$, for any small $\eta > 0$ we introduce the perturbed MCF $(\widetilde{M}_\eta)$ with initial condition $\widetilde{M}_{-1} = \text{Graph}\{x + \eta u(-1, x)u(x) \mid x \in M_{-1}\}$.

We next prove that for sufficiently small $\eta > 0$, the perturbed MCF $\widetilde{M}_{\eta}$ verifies the statement of the theorem.

Step 3: updating the set of blowup points iteratively.
Starting from time $T$, we choose a sequence of times $t_n$ with $t_0 = T$ such that $\Delta_n := t_{n+1} - t_n \to \Delta := 2 \log 2$ as $n \to \infty$ and $e^{t_n + 1/2}/r(t_{n+1}) = 2e^{t_n/2}/r(t_n)$.

At time $-e^{-T}$ we choose the set blowup points $\mathcal{P}_T = \{p_i^T\} \subset \gamma$ such that nearby points $p_i^T$ and $p_{i+1}^T$ have distance $\frac{1}{2}e^{-T/2}r(T)$, thus, the RMCFs $M_{p_i^T,T}$ and $M_{p_{i+1}^T,T}$ at time $T$ overlap when we restrict each to the graphical radius $r(T)$.

Going from $t_n$ to $t_{n+1}$, we update the set $\mathcal{P}_{t_n} = \{p_j^{t_n}\}_{j \in \mathcal{I}_n}$ to $\mathcal{P}_{t_{n+1}} = \{p_j^{t_{n+1}}\}_{j \in \mathcal{I}_{n+1}}$ in such a way that

1. $\mathcal{P}_{t_n} \subset \mathcal{P}_{t_{n+1}}$ and $|\mathcal{I}_n| < |\mathcal{I}_{n+1}|$;
2. For any two neighboring points $p_i^{t_n}, p_{i+1}^{t_n}$ in $\mathcal{P}_{t_n}$, we add a new point $p_j^{t_{n+1}}$ with equal distance to the two points, so we have $2|\mathcal{I}_n| = |\mathcal{I}_{n+1}|$.

Thus by the definition of $t_n$, in the scale of RMCF, neighboring points $p_i^{t_n}, p_{i+1}^{t_n}$ in $\mathcal{P}_{t_n}$ are blown up to have distance $\frac{1}{2}r(t_n)$ that holds for all $n$. In particular, the union of an $e^{-t_n/2}r(t_n) = O(e^{-t_n/2}\sqrt{\log t_n})$ neighborhood around each point $p_i^{t_n}$ covers the whole curve $\gamma$.

We always have $0 \in \mathcal{P}_{t_n}$ for all $n$. We denote the two points in $\mathcal{P}_{t_n}$ neighboring 0 by $p_{i_1}^{t_n}$ and $p_{i_2}^{t_n}$ and denote by $p_0^{t_n}$ the point 0.

Step 4: Proof of the theorem.
For each $p \in \mathcal{P}_{t_n}$, we study the difference equation (4.1) between the perturbed RMCF $\widetilde{M}_{p,t} := e^{t/2}(\widetilde{M}_t - e^{-t} - p)$ and the unperturbed one $M_{p,t}$ and denote by $u_p : \Sigma^1 \cap B_r(t_n) \to \mathbb{R}$ the difference of the graphical functions of the two RMCF written over the shrinker.
At the time $t = T$, if $p \in \mathcal{P}_T \setminus \{0\}$, by the choice of $f$ above, we have that $\langle f_p, 1 \rangle_{L^2} \geq c\|f_p\|$ for some constant $c > 0$ independent of $\varepsilon_0, \varepsilon_1$, where $f_p := e^{T/2}f(x-p)\chi_{r(T)}$ and $\chi_{r(T)}$ is a smooth cutoff function that vanishes outside the ball $B_{r(T)}$ and equals to 1 within the ball $B_{r(T)-1}$. Thus we get $f_p \in K_1(\kappa)$ for some $\kappa \in (0, 1)$ independent of $\varepsilon_0, \varepsilon_1$, where we define the cone

$$K_1(\kappa) := \{u \in E \mid \langle u, 1 \rangle_{L^2} \geq \kappa\|u\|\}.$$ 

Since 1 is the eigenfunction of the $L$-operator with eigenvalue 1, the function $\|e^{Lt}f_p\|$ grows exponentially, so the conclusion of Proposition 4.1 holds. Thus analogous to Theorem 4.3, we have

$$w_p(t_n)\chi_{r(t_n)} \in K_1(\kappa)
\tag{10.1}$$

for all $t_n > T$ and $\|w_p(t_n)\| \geq C\|w_p(T)\|e^{\varepsilon_n-t}$. 

Next, for the point $p = 0$, the choice of $f$ gives $f_p \in K_0(\kappa')$ for some $\kappa' < 1$ very close to 1. Then Proposition 4.1 and Theorem 4.3 imply that $w_p(t_n)$ will remain in the cone $K_0(\kappa')$ for all $t_n$, with a possible application of centering map at each step $t_n$ as we did in Step 2 of the proof of Theorem 1.9. Since $\kappa'$ is close to 1, the norm of centering map that we perform at each step is so small that the centering map does not affect the cone condition $w_p(t_n)\chi_{r(t_n)} \in K_1(\kappa)$ for $p \in \mathcal{P}_T \setminus \{0\}$ as we considered in the last step.

With this construction, for the point $p \in \mathcal{P}_{t_N} \setminus \{0\}$ that was first introduced at time $t_N$, the graphical function $w_p$ has exponential growth due to the condition (10.1). By choosing $\eta$ sufficiently small, we see that it takes about time of order $\log \eta^{-1}$ for $w_p$ to grow to size of order 1, and for the unperturbed flow $M_{p,t}$, its graphical function $\|u_p\|_{C^1}$ decays no slower than linear by the normal form (Theorem 1.4), so we get $\|u_p\|_{C^1} \leq \varepsilon_0/\log \eta^{-1} \ll 1$. When $\min w_p(\cdot, t_n) > 0$ is of order 1 for some $n > N$, then the avoidance principle implies that MCF starting from $\tilde{M}_{p,t_n}$ does not develop singularity within time 1, which means that the singular time of the point $p$ in the perturbed MCF $\tilde{M}_p$ is postponed to a positive time. On the other hand, since there is no exponential growth for the point $p = 0$, we get an isolated singularity of the perturbed MCF $\tilde{M}_p$ that is in an $\varepsilon_0$-neighborhood of the spacetime point $(0, 0)$. 

\[\square\]

**Appendix A. Estimates of the nonlinear term**

In this appendix, we derive the equation of motion for the graphical function $u$ of a manifold evolving under RMCF approaching a cylinder $\Sigma^k = S^{n-k}(\varrho) \times \mathbb{R}^k$ where $\varrho = \sqrt{2(n-k)}$ is the radius of the sphere. We use coordinates $z = (\theta, y) \in \Sigma^k$ where $\theta \in \mathbb{R}^{n-k+1}$ denotes point on $S^{n-k}(\varrho)$ and $y$ denotes point on $\mathbb{R}^k$. Note $|\theta| = \varrho$. The following computations are locally around a point $z_0 = (\theta_0, y_0)$, and we choose local orthonormal frame $\{\theta_0\}$ and $\{y_i\}$. Greek letters correspond to the spherical part and $i, j, k$'s correspond to the $\mathbb{R}^k$ part. We will use $\partial_\alpha$ and $\partial_i$ to simplify $\partial_{\theta_\alpha}$ and $\partial_{y_i}$ respectively.
Proposition A.1. Let $M_t$ be an RMCF converging to a cylinder $\Sigma^k$ in the $C^\infty_{loc}$ sense as $t \to \infty$. Writing $M_t$ as a normal graph of a function $u$ over $\Sigma^k$ within the graphical radius, then we have

1. we have $\partial_t u = Lu + Q(J^2 u)$, $J^2 u := (u, \nabla u, \nabla^2 u)$, where we have

\[
|Q(J^2 u)| \leq C(|\nabla u|^4 + |\nabla^2 u| \text{Hess}_u | + |\nabla u|^2 + u^2 + |u| \text{Hess}_u |
\]

if $|u|_{C^0} \leq \varepsilon_0$ for some $\varepsilon_0$ small.

2. The leading terms in $Q$ is given explicitly as

\[
Q(J^2 u) = -(2\varrho)^{-1} \left( u^2 + 4u\Delta u + 2|\nabla u|^2 \right) + C(J^2 u),
\]

where $C(J^2 u)$ consists of terms cubic and higher power in $J^2 u$.

3. For three functions $v, u_1, u_2 : \Sigma \cap B_{R} \to \mathbb{R}$ and the cutoff function $\chi$ that is 1 on $B_{R-1}$ and 0 outside $B_{R}$, we have (denoting $w = u_1 - u_2$)

\[
\int_{\Sigma \cap B_{R}} |v(C(J^2(\chi u_1)) - C(J^2(\chi u_2)))| e^{-\frac{|x|^2}{4}} \leq C \max\{\|u_1\|_{C^2}, \|u_2\|_{C^2}\}^2 (\|w\| \cdot \|v\| + e^{-\frac{|x|^2}{4}}).
\]

Proof. Note that the unit normal vector $\mathbf{n} = \frac{\varrho}{\sqrt{\rho}}$. Later we will use the following fact:

\[
\partial_i \theta = \partial_i \partial_j \theta = 0, \quad \partial_\alpha \theta = \theta_\alpha, \quad \partial_\alpha \partial_\beta \theta = -\delta_{\alpha\beta} \theta.
\]

We consider a graph $\Sigma^k$ locally given by $\{\widetilde{F}(z) = (z) + u(z)\mathbf{n} = (z) + u(z)\frac{\varrho}{\rho}\}$, which induces a frame on $\Sigma^k$ given by $\tilde{\partial}_\alpha = (1 + \frac{u}{\rho})\partial_\alpha + \frac{\alpha_u}{\rho} \cdot \theta$, $\tilde{\partial}_i = \partial_i + \frac{\partial_u}{\rho} \cdot \theta$. The induced metric is given by

\[
\tilde{g}_{\alpha\beta} = (1 + \frac{u}{\rho})^2 g_{\alpha\beta} + \partial_\alpha u \partial_\beta u, \quad \tilde{g}_{ij} = g_{ij} + \partial_i u \partial_j u, \quad \tilde{g}_{\alpha i} = \partial_\alpha u \partial_i u.
\]

The inverse matrix is given by

\[
\tilde{g}^{\alpha\beta} = (1 + \frac{u}{\rho})^{-2} g^{\alpha\beta} - (1 + \frac{u}{\rho})^{-4} \partial_\alpha u \partial_\beta u + m_{\alpha\beta},
\]

\[
\tilde{g}^{ij} = g^{ij} - \partial_i u \partial_j u + m_{ij}, \quad \tilde{g}^{\alpha i} = -\partial_\alpha u \partial_i u + m_{\alpha i}.
\]

We find a unit normal vector field given by

\[
\tilde{\mathbf{n}} = \frac{\varrho - (1 + \frac{u}{\rho})^{-1} \partial_\alpha u \theta_\alpha - \partial_i u \partial_i}{\sqrt{1 + \sum_\alpha (1 + \frac{u}{\rho})^{-2} (\partial_\alpha u)^2 + \sum_\alpha (\partial_i u)^2}} =: \frac{\varrho - (1 + \frac{u}{\rho})^{-1} \partial_\alpha u \theta_\alpha - \partial_i u \partial_i}{S}.
\]

Now we calculate the 2nd fundamental form. We have

\[
\partial_\alpha \partial_\beta F = -(1 + \frac{u}{\rho}) \frac{\delta_{\alpha\beta}}{\rho^2} \theta + \frac{\partial_\beta u}{\rho} \theta_\alpha + \frac{\partial_\alpha u}{\rho} \theta_\beta + \frac{\partial_{\alpha\beta} u}{\rho} \theta,
\]

\[
\partial_i \partial_\alpha F = \frac{\partial_i u}{\rho} \theta_\alpha, \quad \partial_i \partial_j F = \frac{\partial_{ij} u}{\rho} \theta.
\]
Taking inner product with $\tilde{n}$, we get the 2nd fundamental forms:
\[
\tilde{A}_{\alpha\beta} = S^{-1} \left( -(1 + \frac{u}{\rho}) \delta_{\alpha\beta} - 2(1 + \frac{u}{\rho})^{-1} \partial_{\alpha} u \frac{\partial_{\beta} u \rho}{\rho} + \partial_{\alpha\beta} u \right),
\]
\[
\tilde{A}_{i\alpha} = S^{-1} \left( \partial_{i\alpha} u - (1 + \frac{u}{\rho})^{-1} \partial_{\alpha} u \frac{\partial_{i} u \rho}{\rho} \right), \quad \tilde{A}_{ij} = S^{-1} (\partial_{ij} u).
\]

In conclusion, we have (note that the convention in MCF is $H = -\text{tr} A$)
\[
-H = S^{-1} \left( (1 + \frac{u}{\rho})^{-2} \Delta_{\theta} u + \Delta_{\theta} u - \partial_{ij} u \partial_{i} u \partial_{j} u - \partial_{i\alpha} u \partial_{\alpha} u - \partial_{i} u \partial_{\alpha} u \right)
\]
\[
-(1 + \frac{u}{\rho})^{-4} \partial_{\alpha} u \partial_{\beta} u \partial_{\alpha\beta} u - (1 + \frac{u}{\rho})^{-1} \frac{k}{\rho} - (1 + \frac{u}{\rho})^{-3} \frac{\nabla_{\theta} u^2}{\rho} + m \right)
\]

where $m$ is a remainder like $\nabla u$ of order at least 4. More precisely, we have
\[
|m| \leq C|\nabla u|^4.
\]

Next we consider the term $\frac{1}{2} \langle \tilde{z}, \tilde{n} \rangle$. On $\Sigma^k$ near $z_0 = (\theta_0, y_0)$, we have
\[
\langle \tilde{z}, \tilde{n} \rangle = S^{-1} \left( (z) + \frac{\theta}{\rho}, \frac{\theta}{\rho} - (1 + \frac{u}{\rho})^{-1} \partial_{\alpha} u \partial_{\alpha} u - \partial_{i} u \partial_{i} u \right) = S^{-1} (\rho + u - y_i \partial_i u)
\]

Therefore, we can obtain the equation of $u$ from the RMCF equation (modulo diffeomorphism) $(\partial_{\tilde{t}} \tilde{z})^{\perp} = -(\tilde{H} - \frac{\langle \tilde{z}, \tilde{n} \rangle}{2}) \tilde{n}$. Take an inner product of the equation with $\tilde{n}$, we get
\[
S^{-1} \partial_{\tilde{t}} u = S^{-1} \left( (1 + \frac{u}{\rho})^{-2} \Delta_{\theta} u + \Delta_{\theta} u - \partial_{ij} u \partial_{i} u \partial_{j} u - \partial_{i\alpha} u \partial_{\alpha} u - (1 + \frac{u}{\rho})^{-4} \partial_{\alpha} u \partial_{\beta} u \partial_{\alpha\beta} u - (1 + \frac{u}{\rho})^{-1} \frac{k}{\rho} - (1 + \frac{u}{\rho})^{-3} \frac{\nabla_{\theta} u^2}{\rho} + m \right) + S^{-1} \frac{1}{2} (\rho + u - y_i \partial_i u).
\]

Thus we obtain the equation of $u$ as follows:
\[
\partial_{\tilde{t}} u = \left( (1 + \frac{u}{\rho})^{-2} \Delta_{\theta} u + \Delta_{\theta} u - \partial_{ij} u \partial_{i} u \partial_{j} u - \partial_{i\alpha} u \partial_{\alpha} u - (1 + \frac{u}{\rho})^{-1} \frac{n - k}{\rho} \right)
\]
\[
-(1 + \frac{u}{\rho})^{-4} \partial_{\alpha} u \partial_{\beta} u \partial_{\alpha\beta} u - (1 + \frac{u}{\rho})^{-3} \frac{\nabla_{\theta} u^2}{\rho} + m \right) + \frac{1}{2} (\rho + u - y_i \partial_i u).
\]

When $|u|_{C^0} \leq \varepsilon_0$, we can further rewrite the equation as $\partial_{\tilde{t}} u = Lu + Q(J^2 u)$, where $Q(J^2 u) = m_1 + m_2 + m_3 + m_4$, with $m_4$ consisting of all terms like $\partial u \partial u \partial^2 u$ and
\[
m_1 = m, \quad m_2 = -\frac{1}{2\rho} \frac{u^2}{1 + \frac{u}{\rho}} - (1 + \frac{u}{\rho})^{-3} \frac{\nabla_{\theta} u^2}{\rho}, \quad m_3 = -\frac{2u}{\rho} + \frac{u^2}{(1 + \frac{u}{\rho})^2} \Delta_{\theta} u.
\]

From (A.5), we get (A.2) immediately. Indeed, the leading terms come from $m_2$ and $m_3$ respectively. The terms $m_1$ (see (A.4)) and $m_4$ contribute only to $C(J^2 u)$. Estimate (A.1) follows also immediately from (A.5). In particular, when $\|u\|_{C^2} \leq \varepsilon_0$, we have the estimate
\[ |Q| \leq C(|\nabla u|^2 + \varepsilon_0 u). \] For two different functions \( u_1 \) and \( u_2 \), the fundamental theorem of calculus shows that
\[ |Q(J^2 u_1) - Q(J^2 u_2)| \leq C \varepsilon_0 (|u_1 - u_2| + |\nabla u_1 - \nabla u_2| + |\text{Hess} u_1 - \text{Hess} u_2|). \]
We next consider item (3). To get the terms \( ||u|| ||v|| \) on the RHS, we need to perform a step of integration by parts for terms of the form \( v \text{Hess}_u \), which also gives us the boundary term \( e^{-R^2/4} \). Taking terms \( \partial_t u \partial_j u \partial_i u \) in \( m_4 \) as an example, we have
\[ \partial_t u \partial_j u \partial_i u = (\nabla u)^T \nabla^2 u \nabla u = \frac{1}{2} \nabla \cdot (|\nabla u|^2 \nabla u) - \frac{1}{2} \Delta u \cdot |\nabla u|^2. \]
Consider the first term on the RHS
\[ \nabla \cdot (|\nabla (\chi u_1)|^2 \nabla (\chi u_1)) - \nabla \cdot (|\nabla (\chi u_2)|^2 \nabla (\chi u_2)) \]
\[ = \nabla \cdot ((\nabla (\chi w) \cdot \nabla (\chi u_1 + \chi u_2)) \nabla (\chi u_1)) + \nabla \cdot (|\nabla (\chi u_2)|^2 \nabla (\chi w))). \]
When multiplied by \( v \) and taking integration by parts, we see that it can be bounded by the RHS of (3). All other terms can be treated similarly and are easier, so we get (3).

**Appendix B. Extension to scale \( \sqrt{t} \) in the linear equation**

In this section, we give the proof of Proposition 3.8 following the ideas of [Vel92] and [FKZ00, Proposition 2.13] for readers’ convenience. The result gives an important characterization of the solution to the linear PDE \( \partial_t v = Lv + P v \) where \( P \) has reasonable growth or decay, namely, the \( C^2 \)-norm of the solution can be controlled over a region of radius \( O(\sqrt{t}) \). The main novelty here is that we need to take extra care of the spherical component \( S^{n-k} \).

**Step 1.** We first define \( Z(\cdot, t) = v(\cdot, t) t^{-C} \) to get rid of the perturbation \( P(t) \). Then we have \( \frac{\partial Z}{\partial t} \leq LZ \). We next define the following norm
\[
N_r(\Psi) = \sup_{|\xi| \leq r} \left( \int_{\Sigma^k} \Psi(\theta, y)^2 e^{-\frac{|y|^2}{4\tau}} d\theta dy \right)^{1/2}.
\]
Following [Vel92], we use \( S(\tau) \) to denote the semigroup generated by \( L_{\Sigma^k} \) over \( \Sigma^k \), and we introduce the kernel of the semigroup generated to the operator \( L_{\Sigma^k} \) on \( \Sigma^k \)
\[
S(\tau, (\theta, y), (\eta, z)) = G(\tau, \theta, \eta) \frac{e^\tau}{(4\pi(1-e^{-\tau}))^{k/2}} \exp \left( -\frac{|y e^{-\tau/2} - z|^2}{4(1-e^{-\tau})} \right)
\]
where \( G \) is the heat kernel for the semigroup associated to \( \Delta_{S^{n-k}(\omega)} \) on \( S^{n-k}(\omega) \). As a consequence, we have the following generalizations of Velázquez inequality [Vel92].

**Lemma B.1.** For any \( r, \tilde{r} > 0 \) and \( \psi \) such that \( N_{\tilde{r}}(\psi) < +\infty \), we have for some constant \( C \) only depending on \( n, k \),
\[
N_r(S(\tau) \psi) \leq C \frac{e^\tau}{(4\pi(1-e^{-\tau}))^{n/2}} \exp \left( \frac{e^{-\tau}(r - \tilde{r} e^{\tau/2})^2}{4(1-e^{-\tau})} \right) N_{\tilde{r}}(\psi).
\]

The proof is the same as [Vel92]. We provide a proof here for the reader’s convenience.
Proof. For any fixed $\xi \in \mathbb{R}^k$, let us consider

$$I = \int_{\mathbb{R}^k} |S(\tau)\psi(\theta, y)|^2 e^{-\frac{|w-\xi|^2}{4}} d\theta dy = \int_{\mathbb{R}^k} e^{-\frac{|w-\xi|^2}{4}} \left( \int_{\mathbb{S}^k} S(\tau, (\theta, y), (\eta, z))\psi(\eta, z) d\eta dz \right)^2 d\theta dy = \left( \frac{e^\tau}{(4\pi(1-e^{-\tau}))^{k/2}} \right)^2 \int_{\mathbb{R}^k} e^{-\frac{|w-\xi|^2}{4}} \left( \int_{\mathbb{S}^k} G(\tau, \theta, \eta) \exp \left( -\frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})} \right) \psi(\eta, z) d\eta dz \right)^2 d\theta dy.$$

Now let $w \in \mathbb{R}^k$ be any point satisfying $|w| \leq \tilde{r}$. Using Hölder’s inequality, we obtain that

$$\left( \int_{\mathbb{S}^k} G(\tau, \theta, \eta) \exp \left( -\frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})} \right) \psi(\eta, z) d\eta dz \right)^2 \leq \left( \int_{\mathbb{S}^k} e^{-\frac{|w|^2}{4}} \psi(\eta, z)^2 d\eta dz \right) \left( \int_{\mathbb{S}^k} G^2(\tau, \theta, \eta) \exp \left( -2 \frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})} + \frac{|z-w|^2}{4} \right) d\eta dz \right) = \left( \int_{\mathbb{S}^k} e^{-\frac{|w|^2}{4}} \psi(\eta, z)^2 d\eta dz \right) \left( \int_{\mathbb{S}^{n-k}} G^2(\tau, \theta, \eta) d\eta \right) \left( \int_{\mathbb{R}^k} \exp \left( -2 \frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})} + \frac{|z-w|^2}{4} \right) d\eta dz \right).$$

We observe that $\int_{\mathbb{S}^{n-k}} G^2(\tau, \theta, \eta) d\eta = G(2\tau, \theta, \theta)$ by the semigroup property of the heat kernel. We also have the identity

$$-2 \frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})} + \frac{|z-w|^2}{4} = - \frac{1+e^{-\tau}}{4(1-e^{-\tau})} \left| z-w - \frac{2(ye^{-\tau/2} - w)}{1+e^{-\tau}} \right|^2 + \frac{e^{-\tau}|y-we^{\tau/2}|}{2(1+e^{-\tau})}.$$

So we have

$$\int_{\mathbb{R}^k} \exp \left( -2 \frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})} + \frac{|z-w|^2}{4} \right) d\eta dz = \int_{\mathbb{R}^k} \exp \left( - \frac{1+e^{-\tau}}{4(1-e^{-\tau})} \left| z \right|^2 + \frac{e^{-\tau}|y-we^{\tau/2}|}{2(1+e^{-\tau})} \right) d\eta dz.$$

Therefore, we obtain that

$$I \leq N_\tilde{r}(\psi) \left( \int_{\mathbb{S}^k} G(2\tau, \theta, \theta) \exp \left( -\frac{|y-\xi|^2}{4} + \frac{e^{-\tau}|y-we^{\tau/2}|}{2(1+e^{-\tau})} \right) d\theta dy \right) \left( \int_{\mathbb{R}^k} \exp \left( - \frac{1+e^{-\tau}}{4(1-e^{-\tau})} \left| z \right|^2 \right) \right) \left( \frac{e^\tau}{(4\pi(1-e^{-\tau}))^{k/2}} \right)^2.$$

The only part that was not calculated by Velázquez is the integral over the spherical part $\int_{\mathbb{S}^{n-k}} G(2\tau, \theta, \theta) d\theta$. Li-Yau’s estimate of heat kernel suggests that $G(2\tau, \theta, \theta) \leq Ct^{-(n-k)/2}$ when $\tau \leq \varrho$, and $G(2\tau, \theta, \theta) \leq C$ when $\tau > \varrho$. Therefore we obtain that $G(2\tau, \theta, \theta) \leq C(4\pi(1-e^{-\tau})^{-(n-k)/2})$. Thus we obtain the lemma. 

Step 2. For given $t_0$, we estimate $N_{K_0\sqrt{s}}(Z(s))$ for $s > t_0$. Let us suppose $K_0$ is a fixed constant, and $s$ is chosen such that $e^{(s-t_0)/2} \leq K_0\sqrt{s}$. Notice that whenever $K_0 = t_0^{-1/2}$, $s = t_0$, and for any fixed $K_0$, when $t_0$ is sufficiently large, we always have $s > K_0$. 

Denote by \( r(t) = e^{(t-t_0)/2} \). Because \( \frac{\partial Z}{\partial t} \leq LZ \), we have
\[
Z(s) \leq S(s - t_0)Z(t_0) = S(s - t_0 - \delta)S(\delta)Z(t_0),
\]
where we choose \( \delta > 0 \) small to be determined. Then in the \( N_r \)-norm we have
\[
N_r(s)(Z(s)) \leq N_r(s)(S(s - t_0 - \delta)S(\delta)Z(t_0)).
\]
We observe that \( r(t_0+\delta)e^{(s-t_0-\delta)/2} = r(s) \), which suggests that \( (r(s)-r(t_0+\delta)e^{(s-t_0-\delta)/2})_+ = 0 \). Lemma B.1 shows that
\[
N_r(s)(Z(s)) \leq C \left( \frac{e^{s-t_0}}{1 - e^{-(s-t_0-\delta)}} \right)^{n/2} N(r(t_0+\delta))(S(\delta)Z(t_0)).
\]
Finally, we use Lemma B.1 again to obtain
\[
N_r(t_0+\delta)(S(\delta)Z(t_0)) \leq C \left( \frac{e^{t_0}}{1 - e^{-(s-t_0-\delta)}} \right)^{n/2} \|Z(t_0)\|_{L^2}.
\]
Combining the estimates above we get
\[
N_r(s)(Z(s)) \leq C \left( \frac{e^{s-t_0}}{1 - e^{-(s-t_0-\delta)}} \right)^{n/2} \|Z(t_0)\|_{L^2}
\]
Finally, we use \( N_{K_0\sqrt{s}}(Z(s)) \) to bound \( Z(\cdot, s') \) in the domain \( \{|y| \leq K_0\sqrt{s'}\} \) with some \( s < s' \).

Claim: We have the estimate \( Z(\theta, y, s') \leq \tilde{C}(K_0)N_{K_0\sqrt{s'}-K_0}(Z(s' - K_0)) \) for some constant \( \tilde{C}(K_0) \).

Proof of the claim. Just as [FKZ00], we use the kernel of \( L_{\Sigma^k} \) and the upper bound of \( G \) to conclude that
\[
0 \leq Z(\theta, y, s') \leq S(K_0)Z(\theta, y, s' - K_0)
\]
\[
\leq C(K_0) \int_{\Sigma^k} \exp \left( -\frac{|y - K_0|/2 - z|^2}{4(1 - e^{-K_0})} \right) Z(\eta, z, s' - K_0) d\eta dz.
\]
Now for any \( |w| \leq K_0\sqrt{s' - K_0} \), we have
\[
Z(\theta, y, s) \leq C(K_0) \int_{\Sigma^k} \exp \left( -\frac{|y - K_0|/2 - z|^2}{4(1 - e^{-K_0})} + \frac{|z + w|^2}{8} \right) e^{-\frac{|z + w|^2}{8}} Z(\eta, z, s' - K_0) d\eta dz.
\]
Then Cauchy-Schwarz inequality yields that
\[
Z(\theta, y, s') \leq C(K_0) \left( \int |Z(\eta, z, s' - K_0)|^2 e^{-\frac{|z + w|^2}{4}} d\eta dz \right)^{1/2} I(K_0, w, y)^{1/2},
\]
where \( I(K_0, w, y) = \int |Z(\eta, z, s' - K_0)|^2 e^{-\frac{|z + w|^2}{4}} d\eta dz \) is a suitable integral.
where
\[
I(K_0, w, y) = C \int \exp \left(-2 \frac{|ye^{-K_0/2} - z|^2}{4(1 - e^{-K_0})} + \frac{|z + w|^2}{4} \right) dz
\]
\[
= C \left( \frac{4\pi(1 - e^{-K_0})}{1 + e^{K_0}} \right)^{k/2} \exp \left( \frac{|w + ye^{-K_0/2}|^2}{2(1 + e^{-K_0})} \right),
\]
and here \( C \) is simply the area of \( S^{n-k} \). If we take the infimum in \( w \) with \( |w| \leq K_0 \sqrt{s' - K_0} \), and then take the supremum in \( y \) with \( |y| \leq \frac{1}{4} K_0 \sqrt{s'} \), whenever \( s' \geq \frac{16}{15} K_0 \), we get
\[
\sup_{|y| \leq \frac{1}{4} K_0 \sqrt{s}} \inf_{|w| \leq K_0 \sqrt{s} - K_0} |w + ye^{-K_0/2}| = 0.
\]
Thus, we have proved the claim.

Finally, we choose \( s' = s + K_0 \). Using (B.4), we get that when \( |y| \leq \frac{1}{4} K_0 \sqrt{s'} \),
\[
Z(y, \theta, s') \leq \tilde{C}(K_0) C \frac{K_0^2 s}{(1 - (K_0^2 s)^{-1} e^{\delta})^{n/2}} \exp \left( \frac{1}{4(1 - e^{-\delta})} \right) \|Z(t_0)\|_{L^2}.
\]
In conclusion, we have
\[
\|v(\cdot, s')\|_{L^\infty(B_{K_0^2 \theta})} \leq C(K_0, \delta) (s')^{2t_0^2} \|v(\cdot, t_0)\|_{L^2},
\]
whenever \( s' = K_0 + s, e^{(s-t_0)/2} = K_0 \sqrt{s}, s \geq t_0 + \delta \), and \( s \geq \frac{1}{15} K_0 \).

**Step 4.** Finally, we improve the \( L^\infty \) estimate to \( C^{t+2,\alpha} \) estimate. For this purpose, we invoke the parabolic Schauder estimate. First, we suppose the \( C^{0,\alpha} \) bound of \( P \). The derivatives with respect to the \( \theta \)-component can be easily estimated using the heat kernel \( e^{\Delta_{S^{n-k}} \cdot} \), so we focus on the \( y \)-component. Since there is a drift term \( y \cdot \nabla_y \) in the \( L \)-operator (see (2.1)), we perform a rescaling only in the \( \mathbb{R}^k \)-component similar to (1.1). Let \( \tau = -e^{-s+t_0}, x = e^{-s/2} y = \sqrt{-\tau} e^{-t_0/2} y \), and
\[
w(\tau, \theta, x) = -\tau v(t_0 - \log(-\tau), \theta, e^{t_0/2} x/\sqrt{-\tau}) = e^{-s} v(s, \theta, y),
\]
where the \( e^{-s} \) factor is multiplied to get rid of the 1 in the \( L \)-operator. We further set \( \tilde{\tau} = e^{t_0} \tau \in [-1, -e^{-s+t_0}] \), then \( w \) satisfies the heat equation \( \partial_\tau w = \Delta_{\mathbb{R}^k} w + (e^{-t_0} \Delta_{S^{n-k}(\theta)} w + \tilde{P}w) \) on \( S^{n-k}(\theta) \times \mathbb{R}^k \cap B_1 \), where
\[
\tilde{P} = e^{-t_0} P \frac{ds}{d\tau} = e^{-t_0} O(1/s) \frac{1}{\tau} = e^{-t_0} O(1/s) e^{s-t_0} = O(e^{-t_0})
\]
using the assumption \( e^{(s-t_0)/2} = K_0 \sqrt{s} \). Then the parabolic Schauder estimate gives \( \|w(\tau, \cdot, \cdot)\|_{C^{2,\alpha}} \leq C \|w(t, \theta, \cdot)\|_{C^{0,\alpha}} \). Using the relation between \( w \) and \( v \) we get \( \|v(\tau, \theta, \cdot)\|_{C^{2,\alpha}} \leq C \|v(t, \theta, \cdot)\|_{C^{0,\alpha}} \). For \( C^{t,\alpha} \) estimate for a general \( \ell \in \mathbb{Z}_+ \), we just take derivatives of \( v \) and repeat the above discussion. This completes the proof. \( \square \)
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