LADDER SYSTEM UNIFORMIZATION ON TREES I & II

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Abstract. Given a tree $T$ of height $\omega_1$, we say that a ladder system colouring $(f_\alpha)_{\alpha \in \text{lim} \omega_1}$ has a $T$-uniformization if there is a function $\varphi$ defined on a subtree $S$ of $T$ so that for any $s \in S_\alpha$ of limit height and almost all $\xi \in \text{dom} f_\alpha$, $\varphi(s \upharpoonright \xi) = f_\alpha(\xi)$. In sharp contrast to the classical theory of uniformizations on $\omega_1$, J. Moore proved that CH is consistent with the statement that any ladder system colouring has a $T$-uniformization (for any Aronszajn tree $T$). Our goal is to present a fine analysis of ladder system uniformization on trees pointing out the analogies and differences between the classical and this new theory. We show that if $S$ is a Suslin tree then CH implies that there is a ladder system colouring without $S$-uniformization, but $MA(S)$ implies that any ladder system colouring has even an $\omega_1$-uniformization. Furthermore, it is consistent that for any Aronszajn tree $T$ and ladder system $C$ there is a colouring of $C$ without a $T$-uniformization; however, and quite surprisingly, $\diamondsuit^+$ implies that for any ladder system $C$ there is an Aronszajn tree $T$ so that any monochromatic colouring of $C$ has a $T$-uniformization. We also prove positive uniformization results in ZFC for some well-studied trees of size continuum.

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1. Introduction

A ladder system on $\omega_1$ is a sequence $C = (C_\alpha)_{\alpha \in \text{lim} \omega_1}$ indexed by the set of countable limit ordinals, so that $C_\alpha$ is a cofinal subset of $\alpha$ in order type $\omega$. Given a colouring of the ladder system $C$, that is, a sequence of maps $f = (f_\alpha)_{\alpha \in \text{lim} \omega_1}$ so that $\text{dom} f_\alpha = C_\alpha$, one might ask if there is a single global function $\varphi$ defined on $\omega_1$ that agrees with all the local colourings $f_\alpha$ almost everywhere on $C_\alpha$; we refer to such a global map $\varphi$ as an $\omega_1$-uniformization. A priori, nothing prevents the existence of $\varphi$ since two local maps $f_\alpha, f_\beta$ are only defined on finitely many common points.

Date: June 12, 2018.
2010 Mathematics Subject Classification. 03E05, 03E35, 03E50.
Key words and phrases. ladder system, uniformization, Suslin tree, special tree, Aronszajn tree, diamond, colouring.
It turns out, such $\omega_1$-uniformizations may or may not exist and this topic has been extensively studied due to various connections to algebra, in particular to the Whitehead problem and its relatives [9, 10, 22, 23], to topology [3, 5, 24, 34], and to fundamental questions in set theory [14, 19], in particular, to the study of forcing axioms that are compatible with the CH [1, 26].

Our current interest lies in understanding a relatively new version of the uniformization property introduced by Justin Moore, a notion that played a key role in understanding uncountable minimal linear orders [19]. If $T$ is a tree of height $\omega_1$, we say that $S \subseteq T$ is a subtree if $S$ is downward closed and pruned in $T$, i.e., if any $s \in S$ has extensions with arbitrary large height below $\text{ht}(T) = \omega_1$. Given some $s \in S$ and $\xi < \text{ht}(s)$, we let $s \upharpoonright \xi$ be the unique predecessor of $s$ in $S$ of height $\xi$.

Now, the main definition is the following.

**Definition 1.1.** [19] Suppose that $T$ is a tree of height $\omega_1$, $C$ is a ladder system and $f$ a colouring of $C$. A $T$-uniformization of $f$ is a map $\varphi$ on a subtree $\text{dom} \varphi = S \subseteq T$ so that for all $s \in S$ of limit height $\alpha < \omega_1$ and almost all $\xi \in C_\alpha$,

$$\varphi(s \upharpoonright \xi) = f_\alpha(\xi).$$

In the above situation, we say that $\varphi$ uniformizes $f$ (on $S$). In Figure 1 we aimed to emphasize that $\varphi$ is only required to agree with the local colouring $f_\alpha$ along those branches that are bounded in $S$.

Prior to Moore’s and our work, a similar theme of ‘uniformization on trees’ was investigated by Zoran Spasojevic [29], in connection to tree topologies, but mostly for a restricted class of colourings that correspond to topological separation axioms.

The goal of our paper is to study and understand the similarities and more importantly, the crucial differences between $T$-uniformizations and $\omega_1$-uniformizations.

First, note that any subtree of $\omega_1$ (in our convention of a subtree) must be $\omega_1$ itself, and so $\text{dom} \varphi = \omega_1$ for any $\omega_1$-uniformization $\varphi$. We will say that a $T$-uniformization $\varphi$ of some ladder system colouring is **full** if $\text{dom} \varphi = T$. It is easily seen that if a ladder system colouring has an $\omega_1$-uniformization then it has a full $T$-uniformization for any tree $T$ of height $\omega_1$ (and the global colouring on $T$ can be chosen so that it only depends on
the levels). In turn, positive results from the classical theory of uniformizations extend to positive results on $T$-uniformizations.

However, there is great flexibility in picking very different domains for $T$-uniformizations of different colourings $f$. Furthermore, given a $T$-uniformization $\varphi$ and some level $\alpha$ of the tree, the number of $\xi \in C_\alpha$ so that $\varphi(s \upharpoonright \xi) \neq f_\alpha(\xi)$ generally depends on the choice of $s \in S_\alpha$. Thus, it looks (and actually is) harder to negate the existence of $T$-uniformizations.

Given a ladder system $C$ on $\omega_1$, an $n$-colouring of $C$ is a sequence of maps $f_\alpha : C_\alpha \rightarrow n$. We say that $f$ is monochromatic if each $f_\alpha$ is constant. Now, we will write

- $\text{Unif}_n(T, C)$ if any $n$-colouring of $C$ has a $T$-uniformization, and
- $\text{cUnif}_n(T, C)$ if all monochromatic $n$-colouring of $C$ has a $T$-uniformization.

Let us briefly review some fundamental results about ladder system uniformizations: the study was initiated by Saharon Shelah in the 1970s, as he isolated the uniformization property as the combinatorial essence of the Whitehead problem (after solving the problem in [25]). First, the two extreme cases are summarized below.

(I) $\text{MA}_{\omega_1}$ implies $\text{Unif}_\omega(\omega_1, C)$ for any ladder system $C$ [26], and

(II) $2^{\aleph_0} < 2^{\aleph_1}$ implies $\neg \text{cUnif}_2(\omega_1, C)$ for any ladder system $C$ [7].

We should mention that for any ladder system colouring, there is a (proper) poset which introduces an $\omega_1$-uniformization but which adds no new reals. However, the latter result implies that these posets cannot be iterated without adding reals, and hence the uniformization property is one of the fundamental barriers to the consistency of forcing axioms compatible with the CH, at least for many reasonably large classes of posets.

At this point, the following theorem of Moore on $T$-uniformizations might come as a surprise:

(III) it is consistent with CH that $\text{Unif}_{\omega_1}(T, C)$ holds for any Aronszajn tree $T$ and ladder system $C$ [19, Theorem 1.9].

As earlier said, Moore’s main motivation was to study uncountable linear orders, and he proved that in any model as above, the only minimal uncountable linear orders are $\omega_1$ and its reverse $-\omega_1$ [19, Lemma 3.3]. Answering a question of James Baumgartner, this result was later extended by the present author to show that it is consistent with CH that there is a Suslin tree $R$ and $\text{Unif}_\omega(T, C)$ holds for all $C$ and any Aronszajn tree $T$ that embeds no derived subtree of $R$ [27, Theorem 2.1 and Corollary 2.3].

Initially, our motivation was to understand if Moore’s model can contain any Suslin trees, or if our latter theorem is optimal and CH does not allow uniformization on Suslin trees. Upon answering this question, we extended some classical theorems to uniformizations on trees, but more importantly, found several unexpected positive uniformization results.

Non-uniformization results. In Section 2, we show how variations of the diamond principle imply that there are ladder system colourings without $T$-uniformizations; as expected, stronger diamonds imply that we can take care of more trees and find even monochromatic 2-colourings without $T$-uniformizations. We show that $2^{\aleph_0} < 2^{\aleph_1}$ implies $\neg \text{cUnif}_2(T, C)$ for all Suslin trees $T$, and so Moore’s model from point (III) contains no Suslin trees, and our [27] Theorem 2.1 and Corollary 2.3] are optimal.

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1We will use this definition for $n = 2$ and $n = \omega$.

2i.e., a tree of height $\omega_1$ with countable levels but no uncountable chains

3A Suslin tree is an Aronszajn tree with no uncountable antichains.
Moreover, we show that the parametrized weak diamond \( \Phi(\text{non}(M)) \) implies \( \neg \text{Unif}_2(T, C) \) for any \( \aleph_1 \)-tree \( T \). To complete the picture, we present a model of the GCH (and in fact, of \( \diamondsuit \)) in which for any \( \aleph_1 \)-tree \( T \) and any ladder system \( C \), \( \neg \text{Unif}_2(T, C) \) holds.

**Uniformization results.** Given the classical results and the work we outlined above on non-uniformization, it is somewhat unexpected that positive result on uniformization can be proved in ZFC or using strong diamond principles.

In Section 3 we prove one of our main results: if \( Q \) is the tree of all well ordered sets of rational numbers with a maximum then any ladder system colouring has a \( Q \)-uniformization; moreover, this fact will be witnessed by a single master colouring regardless of the choice of ladder system and colouring. One might say that this is not so surprising as the tree \( Q \) is far from Aronszajn: although it has no uncountable chains, the levels are of size continuum and furthermore, \( Q \) satisfies rather strong closure properties. We present another variation to produce uniformizations defined on non-special trees, still in ZFC.

However, we also show that \( \diamondsuit^+ \) implies that for any ladder system \( C \), there is an Aronszajn tree \( T \) so that \( \text{cUnif}_2(T, C) \) holds. The tree \( T \) can either be made special or alternatively, we can ensure that any colouring of \( C \) has a \( T \)-uniformization defined on a Suslin-subtree of \( T \). We present these constructions in Section 4 and reflect on the sizes of antichains in such trees in Section 5.

Finally, in Section 6 we return to Suslin trees: we show that a natural ccc forcing that introduces an \( \omega_1 \)-uniformization for a given ladder system colouring actually preserves all Suslin trees. In turn, it is consistent that there are Suslin trees and \( \text{Unif}_2(\omega_1, C) \) holds.

Our paper ends with a list of open problems in Section 7.

**Notations.** We use standard notations from set theory consistent with classical textbooks [13]. In forcing arguments, stronger conditions are smaller.

For us, a tree \((T, <_T)\) is a partially ordered set so that \( t^+ = \{s \in T : s <_T t\} \) is well ordered for any \( t \in T \); we will usually omit the subscript from \( <_T \) if it leads to no confusion. Let \( \text{ht}(t) \) denote the order type of \( t^+ \), and let \( T_\alpha \) denote the \( \alpha \)th level of \( T \) i.e., all elements \( t \in T \) with \( \text{ht}(t) = \alpha \) (and let \( T_{\leq \beta} = \cup_{\alpha < \beta} T_\alpha \)). Given \( t \in T_\beta \) and \( \alpha < \beta \), we let \( t \upharpoonright \alpha \) denote the unique element of \( T_\alpha \) below \( t \). Since all the subtrees \( S \subset T \) we work with are downward closed, the levels \( S_\alpha \) are just \( S \cap T_\alpha \).

An \( \aleph_1 \)-tree is a tree of height \( \omega_1 \) with countable levels. An Aronszajn tree is an \( \aleph_1 \)-tree with no uncountable chains, and a Suslin tree is an \( \aleph_1 \)-tree without uncountable chains or uncountable antichains.

We will use the following standard fact multiple times.

**Fact 1.2.** Suppose that \( S \) is an \( \aleph_1 \)-tree and \( M \) is a countable elementary submodel of some \( H(\Theta) \) with \( S \in M \). Let \( Y \subset S \) be an element of \( M \) and \( t \in Y \setminus M \). Then

1. if \( S \) has no uncountable antichains then \( Y \cap M \cap t^+ \neq \emptyset \), and
2. if \( S \) has no uncountable chains then \( Y \cap M \setminus t^\downarrow \neq \emptyset \).

**Proof.** (1) Indeed, take a maximal antichain \( A \subset Y \) so that \( A \in M \), and note that \( A \subset M \) as \( A \) is countable. Then \( \varepsilon = \sup \text{ht}[A] + 1 \in M \) and \( M \models t \upharpoonright \varepsilon \) is compatible with some element of \( Y \). So there must be some \( s \in A \) so that \( s \) is compatible with \( t \upharpoonright \varepsilon \) and so \( s \leq t \) \( \varepsilon \leq t \).

(2) If we choose \( A \subset Y \) to be a maximal chain in \( A \) then again, \( A \) is countable and we must have some element of \( A \) off the branch \( t^+ \). \(\square\)
Acknowledgments. Let us thank S.-D. Friedman and M. Levine for valuable discussions and helpful comments on preliminary drafts of this paper. The author was supported in part by the FWF Grant I1921 and OTKA 113047.

2. Non-uniformization results from (weak) diamonds

In this section, we prove results showing that various versions of the diamond principle negate the uniformization property on trees. As expected, stronger diamonds will imply that the uniformization property fails for more trees and even monochromatic colourings. Lastly, we will get to a theorem showing that consistently, for any $\aleph_1$-tree $T$ and ladder system $C$, $\mathsf{Unif}_2(T, C)$ fails.

Let us mention that from [19], where $T$-uniformizations were introduced and studied first, the only non-uniformization result that can be extracted immediately is [19, Lemma 3.3]: under $2^{\aleph_0} < 2^{\aleph_1}$, any Aronszajn tree $T$ that is club-embeddable into all its subtrees and any ladder system $C$, there is a monochromatic 2-colouring of $C$ without a $T$-uniformization. Such trees can be constructed by forcing as done by Abraham ans Shelah [3], or by using $\diamondsuit^+$ to construct a minimal lexicographically ordered Aronszajn tree [5] [28] and then invoke [19, Lemma 2.9].

We start by recalling some definitions and in particular the basis of the result of Devlin and Shelah from point (II).

Theorem 2.1. [7] $2^{\aleph_0} < 2^{\aleph_1}$ if and only if for any $F : 2^{<\omega_1} \to 2$ there is a $g \in 2^{\omega_1}$ so that for any $f \in 2^{\omega_1}$ the set $\{\alpha \in \omega_1 : g(\alpha) \neq F(f \upharpoonright \alpha)\}$ is stationary.

The latter statement if often referred to as the weak diamond, denoted by $\Phi_{\omega_1}^2$ usually, which was discovered in search for the minimal assumptions that imply the failure of uniformization on $\omega_1$. Applications of this principle usually involve coding a countable structure into a countable sequence of 0s and 1s to define $F$. In fact, we will use the following form of $\Phi_{\omega_1}^2$, where $H(\Theta)$ denotes the collection of sets of hereditary cardinality $< \Theta$.

Theorem 2.2. [19] Theorem 3.2] $2^{\aleph_0} < 2^{\aleph_1}$ implies that for any $F : H(\aleph_1) \to 2$ there is a $g : \omega_1 \to 2$ so that for any $U \in H(\aleph_2)$ there is a countable elementary submodel $M \prec H(\aleph_2)$ containing $U$ so that $g(\omega_1 \cap M) \neq F(U^M)$.

The function $g$ from the theorem will be referred to as an oracle for $F$.

By Moore’s result from point (III), $\Phi_{\omega_1}^2$ by itself is not strong enough to show the existence of colourings without a $T$-uniformization for an arbitrary Aronszajn tree $T$. However, $\Phi_{\omega_1}^2$ does have consequences for Suslin trees.

Theorem 2.3. $\Phi_{\omega_1}^2$ implies $\neg \mathsf{cUnif}_2(T, C)$ for any $\aleph_1$-tree $T$ without uncountable antichains and any ladder system $C$.

This result generalizes Devlin and Shelah’s theorem from point (I), but applies to Suslin trees as well. So, we see that Moore’s model cannot contain Suslin trees, and the current author’s result from [27] is optimal. Let us also mention that $\Phi_{\omega_1}^2$ easily implies that for any

\[4\text{e}., \text{ for any subtree } S \subseteq T \text{ there is a club } D \subseteq \omega_1 \text{ and an order-preserving injection } \bigcup_{d \in D} T_d \to \bigcup_{u \in U^M} S_u.\]

\[\text{is the image of } U \text{ under the transitive collapse of } M. \text{ Typically, this will coincide with } U \cap M. \text{ Furthermore, we can always assume that } M \text{ contains whatever countably many parameters we require form } H(\aleph_2), \text{ not just } U.\]
ladder system $C$ and Aronszajn tree $T$ there is a monochromatic 2-colouring of $C$ without a full $T$-uniformization.

**Proof.** Fix an $\aleph_1$-tree $T$ without uncountable antichains and ladder system $C$; we can assume that $T \subseteq \omega^{<\omega_1}$ and so $T_{<\alpha} \in H(\aleph_1)$ for all $\alpha < \omega_1$.

We need to find a monochromatic 2-colouring of $C$ with no $T$-uniformization, and to do so, we first define a function $F : H(\aleph_1) \to 2$. In fact, we will only specify the values of $F$ on maps $\varphi : S \to 2$ in $H(\aleph_1)$ where $S$ is a subtree of $T_{<\alpha}$ for some limit $\alpha < \omega_1$. On all other elements of $H(\aleph_1)$, we let $F$ be constant 1.

Two cases are distinguished.

**Case 1.** Suppose that there is an $s \in S$ and $i < 2$, so that for any $t \in T_0$ above $s$ such that $t^+ \subseteq S$, the set
$$\{ \xi \in C_\alpha : \varphi(t \upharpoonright \xi) = i \}$$
is cofinal in $\alpha$. Pick such an $s, i$ canonically (using some well order of $H(\aleph_1)$), and define $F(\varphi) = i$.

**Case 2.** If there is no such pair $s, i$ then we let $F(\varphi) = 1$.

Now, let $g \in 2^{<\omega_1}$ be the oracle for $F$ given by $\Phi^g_\omega$. We claim that there is no $T$-uniformization for the monochromatic coloring $f$ that is defined by taking $f_\alpha$ to be constant $g(\alpha)$.

Otherwise, let $S \subseteq T$ be a subtree and suppose $\varphi : S \to 2$ is a $T$-uniformization of $f$. Now, we reach a contradiction: find a countable elementary submodel $M \prec H(\aleph_2)$ containing $T, \varphi$ and $g$ so that $g(\alpha) \neq F(\varphi \cap M)$. Let $\alpha = M \cap \omega_1$.

**Claim 2.4.** We used the second case definition for $F(\varphi \cap M)$.

**Proof.** Otherwise, we used the first case definition of $F$ with some node $s \in S_{<\alpha}$ and $i < 2$. But then for any $t \in T_0$ above $s$ so that $t^+ \subseteq S$, $\varphi(t \upharpoonright \xi) = i$ for infinitely many $\xi \in C_\alpha$.

However, $g(\alpha) = 1 - i$ and so for every $t \in S_\alpha$, $\varphi(t \upharpoonright \xi) = 1 - i$ holds for almost all $\xi \in C_\alpha$.

In turn, there is no element of $S \cap T_0$ above $s$, which contradicts that $S$ was pruned.

In particular, we deduce that $F(\varphi \cap M) = 1$ and so $g(\alpha) = 0$. We know that Case 1 failed with any $s \in S_{<\alpha}$ and $i = 0$, so we can find $t_0 \in T_0$ so that $t_0^+ \subseteq S_{<\alpha}$ and $\varphi(t_0 \upharpoonright \xi) = 1$ for almost all $\xi \in C_\alpha$. Consider the set
$$X = \{ t \in T : \text{ht}(t) = \beta, g(\beta) = 0, t^+ \subseteq S \text{ and } \varphi(t \upharpoonright \xi) = 1 \text{ for almost all } \xi \in C_\beta \}.$$

Note that $X \subseteq M$ and $t_0 \in X$ by Fact [1,2] we can find some $t \in t_0^+ \cap X$. However, any $t \in t_0^+$ is also in $S$, so $\varphi(t \upharpoonright \xi) = g(\beta) = 0$ for almost all $\xi \in C_\beta$ (where $\beta = \text{ht}(t)$). This contradicts that $t \in X$ (i.e., that $\varphi(t \upharpoonright \xi) = 1$ for almost all $\xi \in C_\beta$), which in turn finishes the proof of the theorem.

In Section [3] we show that even Unif$_\omega(\omega_1, C)$ for all $C$ is consistent with the existence of Suslin trees (but necessarily, CH will fail in such a model). However, we are not sure if it suffices to assume in the above result that $T$ has no stationary antichains.

Our next goal is to find an assumption that gives ladder system colourings which have no $T$-uniformization for any $\aleph_1$-tree $T$ that is not necessarily Suslin. To prove such a result, we employ the technique of parametrized weak diamonds [21, 11]. Let $M$ stand for the ideal of meager sets in $2^\omega$. 
Definition 2.5. Let $\diamondsuit^{\omega_1}(\text{non}(M))$ denote the following statement: if $X$ is an $\omega_1$ set of ordinals and $F : \bigcup_{\alpha<\omega_1} \alpha^\alpha \to M$ so that $F \upharpoonright \alpha^\alpha \in L(\mathbb{R})[X]$ for all $\alpha < \omega_1$, then there is a $g : \omega_1 \to 2^\omega$ so that for all $f : \omega_1 \to \omega_1$ the set
\[ \{ \alpha < \omega_1 : f \upharpoonright \alpha \in \alpha^\alpha \text{ and } g(\alpha) \notin F(f \upharpoonright \alpha) \} \]
is stationary.

Recall that $L(\mathbb{R})$ is the class of sets constructible from $\mathbb{R}$ (in the sense of Gödel) and $L(\mathbb{R})[X]$ is the minimal model extending $L(\mathbb{R})$ which contains $X$. See [12] Chapter 13 for more details on constructibility. So, compared to $\Phi^2_\omega$, we changed the range of $F$ from 2 to $M$, required $F$ to be nicely definable and the oracle $g$ will stationarily often avoid the 'bad' set given by $F$.

$\diamondsuit^{\omega_1}(\text{non}(M))$ implies the existence of Suslin trees [29] Theorem 3.1 and follows from the classical diamond principle $\diamondsuit^1$. However, $\diamondsuit^{\omega_1}(\text{non}(M))$ has the great advantage that it can hold in various models with $2^\omega = 2^{\omega_1}$ (see [13] for more details).

Theorem 2.6. $\diamondsuit^{\omega_1}(\text{non}(M))$ implies $\neg \text{Unif}_{f}(T, C)$ for any $\aleph_1$-tree $T$ and ladder system $C$.

Claim 2.7. $F(\varphi)$ is a meager subset of $2^\omega$.

Proof. It suffice to note that the set
\[ \bigcap_{k \geq n} \{ \psi \in 2^\omega : \varphi(t \upharpoonright \alpha_k) = \psi(k) \} \]
is nowhere dense for any choice of $t \in T_\alpha$ and $n < \omega$. \qed

Now, suppose $g : \omega_1 \to 2^\omega$ is the oracle for $F$ witnessing $\diamondsuit^{\omega_1}(\text{non}(M))$, and define the colouring $f_\alpha : C_\alpha \to 2$ by $f_\alpha(\alpha_k) = g(\alpha)(k)$ for $k < \omega$. We claim that this (not necessarily monochromatic) 2-colouring $f = (f_\alpha)_{\alpha<\omega_1}$ has no $T$-uniformization.

Otherwise, suppose that $\varphi : S \to 2$ is a $T$-uniformization. Now, we can find some limit $\alpha < \omega_1$ so that $g(\alpha) \notin F(\varphi \upharpoonright S_{<\alpha})$. This means that for any $t \in S_{\omega_1} \subseteq T_{\alpha}$,
\[ \varphi(t \upharpoonright \alpha_k) \neq g(\alpha)(k) = f_\alpha(\alpha_k) \]

\[ \text{Recall that } \diamondsuit \text{ says that there is a sequence } W = (W_\alpha)_{\alpha<\omega_1} \text{ so that for any } X \subseteq \omega_1, \text{ the set } \{ \alpha < \omega_1 : X \cap \alpha = W_\alpha \} \text{ is stationary.} \]
for infinitely many $k$, a contradiction.

Next, one would like to find a monochromatic 2-colouring without a $T$-uniformization, much like in the case of Suslin trees. We will use $\diamondsuit$ to do this but if the interested reader looks closely at the preceding proofs for an optimal assumption then he or she will be lead to a fairly technical common strengthening of $\Phi^2_{\omega_1}$ and $\diamondsuit^{\omega_1}(\text{non}(\mathcal{M}))$ that suffices for the same argument to work; however, we chose to present the simpler $\diamondsuit$-based argument here as we see no further application or real advantage of the other, more technical approach at this point.

**Theorem 2.8.** $\diamondsuit$ implies that for any $\aleph_1$-tree $T$, there is a ladder system $C$ so that $\neg c\text{-Unif}_2(T, C)$.

Note that, given a tree $T$, we are choosing both the ladder system and the monochromatic 2-colouring without $T$-uniformization; we will see in Section 3 that this is best possible from diamond-type assumptions.

We use the following equivalent form of $\diamondsuit$ (see [20]): for any $F : H(\aleph_1) \to 2^\omega$ there is a $g : \omega_1 \to 2^\omega$ so that for any $U \in H(\aleph_2)$ there is a countable elementary $M \prec H(\aleph_2)$ containing $U$ so that $F(U^M) = g(M \cap \omega_1)$.

**Proof.** We fix an $\aleph_1$-tree $T$, and we define $F : H(\aleph_1) \to 2^\omega$ as follows. First, take bijections $e_\alpha : \omega \to \alpha$ for all $\alpha < \omega_1$. Suppose that $\varphi : S \to 2$ for some pruned $S \subseteq T_{<\alpha}$, and we distinguish two cases.

**Case 1.** Suppose that there is an $s \in S$ and $i < 2$, so that for any $t \in T_\alpha$ above $s$ such that $t^i \subset S$, the set

$$I_t = \{ \xi < \alpha : \varphi(t \upharpoonright \xi) = i \}$$

is cofinal in $\alpha$. Pick such an $s, i$ canonically (using some well order of $H(\aleph_1)$), and define $F(\varphi)(0) = 1 - i$.

Now, let $B_s = \{ t \in T_\alpha : t^i \subset S, s \leq t \}$. We choose the values $F(\varphi)(k)$ for $k \geq 1$ so that

1. $E_\varphi = \{ e_\alpha(k) : k \geq 1, F(\varphi)(k) = 1 \}$ is a cofinal, type $\omega$ subset of $\alpha$, and
2. for any $t \in B_s$, there are infinitely many $\xi \in E$ such that $\varphi(t \upharpoonright \xi) = i$.

**Case 2.** If Case 1 fails, we let $F(f)$ be the constant 1 function.

Now, let $g : \omega_1 \to 2^\omega$ be the oracle for $F$ that witnesses $\diamondsuit$. We define the ladder system $C$: if $\{ e_\alpha(k) : k \geq 1, g(\alpha)(k) = 1 \}$ has type $\omega$ and is cofinal in $\alpha$ then we let this set be $C_\alpha$. Otherwise, $C_\alpha$ is any type $\omega$, cofinal subset of $\alpha$. To colour $C$, we simply let $f_\alpha : C_\alpha \to 2$ be constant $g(\alpha)(0)$.

We claim that $f$ has no $T$-uniformization. Otherwise, let $\varphi : S \to 2$ be a $T$-uniformization of $f$. Now, find a countable elementary submodel $M \prec H(\aleph_2)$ containing $T, \varphi$ and $g(\alpha) = F(\varphi \cap M)$. Let $g(0) = 1 - i = F(\varphi \cap M)(0)$ and $\alpha = \omega_1 \cap M$.

**Claim 2.9.** We used the second case definition for $F(\varphi \cap M)$.

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7Let $\diamondsuit^{\omega_1}(2 \times \text{non}(\mathcal{M}))$ stand for the following: if $X$ is an $\omega_1$ set of ordinals and $F : \bigcup_{\alpha < \omega_1} \alpha^\omega \to 2 \times \mathcal{M}$ so that $F \upharpoonright \alpha^\omega \in L(\mathbb{R})[X]$ for all $\alpha < \omega_1$, then there is a $g : \omega_1 \to 2 \times 2^\omega$ so that for all $f : \omega_1 \to \omega_1$ the set $\{ \xi < \omega_1 : f \upharpoonright \alpha \in \alpha^\omega \text{ and } F(f \upharpoonright \alpha) = (i, c) \} \text{ implies } g(\alpha) = (1 - i, c)$ where $c \notin C$ is stationary.

$\diamondsuit^{\omega_1}(2 \times \text{non}(\mathcal{M}))$ follows from $\diamondsuit$ and most likely also consistent with the negation of CH.
Proof. Otherwise, we used the first case definition of $F$ with some node $s \in S_{<\alpha}$ and $i < 2$. But then for any $t \in T_\alpha$ above $s$ so that $t^i \subset S$, $\varphi(t \upharpoonright \xi) = i$ for infinitely many $\xi \in E_{\varphi \cap M} = C_\alpha$. However, $g(\alpha) = 1 - i$ and so for every $t \in S_\alpha$, $\varphi(t \upharpoonright \xi) = 1 - i$ holds for almost all $\xi \in C_\alpha$. In turn, there is no element of $S \cap T_\alpha$ above $s$, which contradicts that $S$ was pruned.

In particular, we deduce that $F(\varphi \cap M)(0) = 1$ and so $g(\alpha) = 0$. In turn, by elementarity, there are arbitrary large $\alpha' < \alpha$ so that $g(\alpha') = 0$ as well. Hence, for any $t \in T_\alpha$ such that $t^i \subset S$ and any $\varepsilon < \alpha$, there is $\xi \in \alpha \setminus \varepsilon$ so that $\varphi(t \upharpoonright \xi) = 0$. But this implies that Case 1 does hold for $\alpha$ with $i = 0$ and any choice of $s$. This contradiction (from the existence of a $T$-uniformization for $f$) finishes the proof.

Finally, we end the section with a relatively simple result showing that consistently all trees and all ladder systems fail the uniformization property in the strongest sense.

**Theorem 2.10.** After adding $\aleph_2$ Cohen subsets to $\omega_1$ for any $\aleph_1$-tree $T$ and ladder system $\mathcal{C}$, $\neg \text{Unif}_2(T, \mathcal{C})$.

Proof. Any $\aleph_1$-tree $T$ and ladder system $\mathcal{C}$ of the final extension appears at some intermediate stage of the iteration, so we will assume $T, \mathcal{C}$ are in the ground model $V$ and show that if $g \in 2^{|\omega_1|}$ is Cohen generic over $V$ then the colouring $f$ defined by $f_\alpha$ is constant $g(\alpha)$ has no $T$-uniformization in $V[g]$. Since the whole forcing is $\sigma$-closed, the rest of the iteration cannot add a uniformization for this colouring in later stages.

Now, suppose that $f$ does have a uniformization and we reach a contradiction. Find a Cohen condition $p$ (i.e., a countable partial function from $\omega_1$ to 2) and name $\check{\varphi}$, so that $p \Vdash \check{\varphi} : \check{\delta} \to 2$ is a $T$-uniformization of $f$. Take a continuous, increasing sequence of countable elementary submodels $(M_n)_{n \leq \omega}$ of $H(\aleph_2)$ so that $T, \check{g}, \check{\varphi}, p \in M_0$. Let $\check{\delta}_n = M_n \cap \omega_1$ for $n \leq \omega$, and list $T_{\check{\delta}_n}$ as $\{t_n : n \in \omega\}$. We will define $q_0 \geq q_1 \geq q_2 \ldots$ conditions below $p$ so that $q_n \in M_{n+1}$, $q_n$ decides $\check{\varphi} \upharpoonright \check{\delta}_n$, and $q_{n+1} \Vdash t_n \upharpoonright \check{\delta}_{n+1} \notin \check{S}$ and so $t_n \notin \check{S}$ by the downward closure of $\check{S}$. In turn, any lower bound $q_\omega$ to this sequence will force that $t_n \notin \check{S}$ for all $n < \omega$ and so $q_\omega \Vdash \check{S} \cap T_{\check{\delta}_\omega} = \emptyset$, a contradiction.

Let us describe the general step in the construction, getting $q_{n+1}$ from $q_n$ (where $q_{-1} = p$). Given $q_n \in M_{n+1}$, we first find an extension $r \in M_{n+2}$ so that $r \in M_{n+1}$ and $r$ is $M_{n+1}$-generic; this can be done since the forcing is $\sigma$-closed. Now, $r$ decides $\check{\varphi} \upharpoonright \check{\delta}_{n+1}$ and decides whether $t_n \upharpoonright \check{\delta}_{n+1} \in \check{S}$ or not. In the former case, there is an $i < 2$ so that $r$ forces that $\check{\varphi}(t_n \upharpoonright \xi) = i$ for infinitely many $\xi \in C_{\delta_{n+1}}$. Since $r \in M_{n+1}$, we can find an extension $q_{n+1} \Vdash r$ in $M_{n+2}$ that forces $\check{g}(\delta_{n+1}) = 1 - i$. In turn, $q_{n+1}$ forces that $t_n \upharpoonright \delta_{n+1} \notin \check{S}$ and so $t_n \notin \check{S}$ as well.

3. Uniformization results in ZFC

The goal of this section is to present positive uniformization results that are provable in ZFC. The results from the previous section show that we must use wide $\aleph_1$-trees (i.e., trees with uncountable levels) and indeed, we will mostly work with the tree $Q$ of all well ordered subsets $t \subset Q$ of the rational numbers such that $t$ has a maximum. The set $Q$, ordered by

That is, iterating posets of the form $\{p : \text{dom } p \in \omega_1, \text{ran } p = 2\}$ with countable support in $\omega_2$ stages.
end-extension, is a special tree i.e., the union of countably many antichains (witnessed by the map \( t \mapsto \max(t) \in Q \)), and was used by Duro Kurepa to construct the first (special) Aronszajn tree in ZFC \cite{13}.

The tree \( Q \) naturally sits in a larger set \( Q^* = \{ t \subset Q : t \text{ is well ordered and bounded in } Q \} \). Now, there is still an order preserving map from \( Q^* \) to the real numbers \( \mathbb{R} \), namely \( t \mapsto \sup_\mathbb{R}(t) \), so there are no uncountable chains in \( Q^* \); however \( Q^* \) is non-special \cite{31}.

We start by stating our results and then proceed with the proofs.

**Theorem 3.1.** There is \( h : Q \to \omega \) so that for any ladder system \( \omega \)-colouring \( f \), there is a (necessarily) special Aronszajn tree \( T \subset Q \) so that \( h \upharpoonright T \) uniformizes \( f \).

**Theorem 3.2.** There is \( h^* : Q^* \to \omega \) so that for any ladder system \( \omega \)-colouring \( f \), there is a non-special tree \( T \subset Q^* \) so that \( h^* \upharpoonright T \) uniformizes of \( f \).

In other words, for both trees \( Q \) and \( Q^* \) there is a single master colouring which witnesses that any ladder system colouring has a \( Q \)-and \( Q^* \)-uniformization.

**Corollary 3.3.** \( \text{Unif}_Q(Q,C) \) and \( \text{Unif}_{Q^*}(Q^*,C) \) holds for any ladder system \( C \).

Let us mention that, unlike \( Q \) or \( Q^* \), no \( R_1 \)-tree can have a single colouring that uniformizes all ladder system colourings at once.

**Proposition 3.4.** Suppose that \( T \) is an \( R_1 \)-tree and \( h : T \to \omega \). Then for any ladder system \( C \), there is a monochromatic 2-colouring of \( C \) so that \( h \upharpoonright S \) is not a \( T \)-uniformization of \( f \) if \( S \subset T \) is pruned.

The argument is a simple, forcing-free variant of the proof of Theorem \ref{2.10}.

**Proof.** Take any increasing sequence of limit ordinals \( (\alpha_n)_{n<\omega} \) with supremum \( \alpha_\omega \). List \( T_\alpha \) as \( (t_n)_{n<\omega} \) and define \( f_{\alpha_n} \) to be constant 0 if and only if there are infinitely many \( \xi \in C_{\alpha_n} \) so that \( h(t_n \upharpoonright \xi) = 1 \). One can immediately check that if \( h \upharpoonright S \) uniformizes \( f \) for some downward closed subset \( S \subset T \) then \( t_n \upharpoonright \alpha_n \notin S \) for all \( n < \omega \) so \( S \cap T_{\alpha_n} = \emptyset \).

Let us present the proofs of the theorems now, starting by some preliminaries.

Throughout the rest of this section we only work with subtrees of \( Q \) and \( Q^* \). In this context, we say that a tree \( T \) (possibly of countable height) is strongly pruned if for any \( \varepsilon < \delta < \text{ht}(T) \) and \( s \in T_\varepsilon \) there are infinitely many \( t \in T_\delta \) extending \( s \) with \( |\sup_\mathbb{R}(t) - \sup_\mathbb{R}(s)| \) arbitrary small. In case of \( T \subset Q \), we could have used \( \max_Q \) instead of the supremum. For example, any initial segment \( Q_{<\alpha} \) of \( Q \) is strongly pruned for \( \alpha < \omega_1 \).

We say that a colouring \( h : T \to \omega \) is rich if for any \( \varepsilon < \delta < \text{ht}(T) \), \( s \in T_\varepsilon \) and \( n < \omega \) there are infinitely many \( t \in T_\delta \) extending \( s \) with \( h(t) = n \) and \( |\sup_\mathbb{R}(t) - \sup_\mathbb{R}(s)| \) arbitrary small. In turn, rich maps must be defined on strongly pruned trees.

The following observation should make it clear how this definition will be useful.

**Observation 3.5.** Suppose \( T \) is a subtree of \( Q \) (or \( Q^* \)) of countable limit height \( \alpha < \omega_1 \) and \( h : T \to \omega \) is rich. If \( f_\alpha : C_\alpha \to \omega \) for some \( C_\alpha \) that is cofinal, type \( \omega \) in \( \alpha \), then there is a \( t \in Q \) (or \( t \in Q^* \), respectively) so that \( t^i \subset T \) and \( h(t \upharpoonright \xi) = f_\alpha(\xi) \) for all \( \xi \in C_\alpha \).

**Proof.** Indeed, simply define \( t_0 < t_1 < \ldots \) in \( T \) inductively so that for all \( k < \omega \),

1. \( \text{ht}(t_k) = \xi_k \) where \( \xi_k \) is the \( k \)th element of \( C_\alpha \),
2. \( h(t_k) = f_\alpha(\xi_k) \),
3. \( |\sup_\mathbb{R}(t_k) - \sup_\mathbb{R}(t_{k+1})| \leq \frac{1}{2^k} \).
Then \( t = \cup_{k<\omega} t_k \in \mathcal{Q} \) is as desired.

In fact, when proving the theorem, we need a slightly stronger notion: a map \( h : T \to \omega \) is flush if it is rich and for any \( \varepsilon < \text{ht}(T) \), cofinal branch \( b \subset T_{<\varepsilon} \) which has an upper bound in \( T_{\varepsilon} \) and \( n < \omega \), there are infinitely many \( t \in T_{\varepsilon} \) above \( b \) so that \( h(t) = n \) and \( |\sup_{\mathcal{Q}}(\cup b) - \max_{\mathcal{Q}}(t)| \) is arbitrary small.\(^9\)

We will show that there are flush maps on both \( \mathcal{Q} \) and \( \mathcal{Q}^* \), and that any flush map witnesses Theorem 3.1 and Theorem 3.2.

**Lemma 3.6.** There are flush maps on \( \mathcal{Q} \) and \( \mathcal{Q}^* \).

**Proof.** We construct flush maps \( h_\alpha : \mathcal{Q}_{<\alpha} \to \omega \) for \( \alpha < \omega_1 \) by induction, so that \( h_\beta \) extends \( h_\alpha \) for \( \alpha < \beta \). The final union \( h = \bigcup_{\alpha<\omega} h_\alpha \) is the desired map on \( \mathcal{Q} \) then.

We suppose that \( h_\alpha \) has been constructed already and extend it to \( \mathcal{Q}_\alpha \) that defines \( h_{\alpha+1} \) (in limit steps, we simply take unions). If \( \alpha = \alpha_0 + 1 \) is successor then being flush compared to being rich gives no extra requirements. Note that any element \( s \in \mathcal{Q}_\alpha \) has infinitely many successors \( t \in \mathcal{Q}_\alpha \) so that \( |\max_{\mathcal{Q}}(t) - \max_{\mathcal{Q}}(s)| \) is arbitrary small. In turn, we can define \( h_{\alpha+1} \) on these elements (for each such maximal \( s \) independently) so that all colours appear infinitely often on these successors.

If \( \alpha \) is limit then any cofinal branch \( b \subset \mathcal{Q}_{<\alpha} \) that is bounded on the \( \alpha \)th level of \( \mathcal{Q} \) has infinitely many bounds \( t \) such that \( |\sup_{\mathcal{Q}}(\cup b) - \max_{\mathcal{Q}}(t)| \) is arbitrary small. For different branches we obviously have different extensions, so we can define \( h_{\alpha+1} \) independently on these sets while obeying our requirements on realizing each colour infinitely often. Any such extension will preserve that the colouring is flush.

The same argument with suprema instead of maximums gives a flush map on \( \mathcal{Q}^* \).

We are ready to prove our theorems.

**Proof of Theorem 3.1.** Our goal is to show that any flush map \( h : \mathcal{Q} \to \omega \) will work.

Fix an \( \omega \)-colouring \( f \) of some ladder system \( \mathcal{C} \). The corresponding tree \( T \subset \mathcal{Q} \) will be constructed by defining its initial segments \( T_{<\alpha} \) for \( \alpha < \omega_1 \) by induction on \( \omega_1 \) so that

\begin{enumerate}
  \item \( T_{<\alpha} \) is a countable, downward closed and strongly pruned subtree of \( \mathcal{Q} \) of height \( \alpha \),
  \item \( T_{<\beta} \) is an end extension of \( T_{<\alpha} \) for \( \alpha < \beta \),
  \item \( h \upharpoonright T_{<\alpha} \) is rich, and
  \item \( h \upharpoonright T_{<\alpha} \) uniformizes \( f \upharpoonright \alpha \).
\end{enumerate}

Clearly, if we succeed then \( T = \bigcup_{\alpha<\omega_1} T_{<\alpha} \) is the tree we were looking for.

In limit cases, we simply take unions. Suppose that \( T_{<\alpha} \) is defined and we will find \( T_{<\alpha+1} = T_{<\alpha} \cup T_\alpha \) now.

Given some \( s \in T_{<\alpha} \) and \( j < \omega \), we look at the poset \( \mathcal{P}_{s,j} \) of all finite, non empty chains \( p \subset T_{<\alpha} \) above \( s \) so that for all \( t \in p \),

\begin{enumerate}
  \item if \( \xi = \text{ht}(t) \in C_\alpha \) then \( h(t) = f_\alpha(\xi) \), and
  \item \( |\max_{\mathcal{Q}}(t) - \max_{\mathcal{Q}}(s)| < \frac{1}{4} \).
\end{enumerate}

**Claim 3.7.** For any \( \varepsilon < \alpha \), \( D_\varepsilon = \{ q \in \mathcal{P}_{s,j} : \text{ht}(\cup q) \geq \varepsilon \} \) is dense in \( \mathcal{P}_{s,j} \).

\(^9\)Note that the trees \( \mathcal{Q} \) and \( \mathcal{Q}^* \) do branch at limit levels, so this is not an impossible assumption a priori.

\(^{10}\)That is, the trees grow up: for any \( t \in T_{<\beta} \setminus T_{<\alpha} \), \( t \upharpoonright \alpha \subset T_{<\alpha} \).
Suppose that 

Proof of Theorem 3.2. Given any \( p \in \mathcal{P}_{s,j} \) let \( t_0 \) be the largest element of \( p \) and suppose \( \varepsilon > \text{ht}(t_0) \). We list all elements of \( C_\alpha \cap (\varepsilon + 1) \setminus \text{ht}(t_0) \) as \( \xi_1 < \xi_2 < \cdots < \xi_k \). Let and let \( d > 0 \) be so small such that

\[
|\max_Q(t_0) - \max_Q(s)| + d \cdot (k + 1) < \frac{1}{j}.
\]

Find nodes \( t_1 < t_2 < \cdots < t_k \) in \( T_{<\alpha} \) above \( t_0 \) so that

1. \( h(t_1) = \xi_1 \).
2. \( h(t_i) = f_\alpha(\xi_i) \), and
3. \( |\max_Q(t_i) - \max_Q(t_{i-1})| < d \).

This is possible since \( h \upharpoonright T_{<\alpha} \) was rich. Finally, if \( \varepsilon > \xi_k \) then take any \( t_{k+1} \) above \( t_k \) of height \( \varepsilon \) so that \( |\max_Q(t_{k+1}) - \max_Q(t_k)| < d \). If \( \xi_k = \varepsilon \) then let \( t_{k+1} = t_k \); now \( q = p \cup \{t_i : 1 \leq i \leq k + 1\} \in \mathcal{P}_{s,j} \) and \( q \in D_\varepsilon \) as desired.

Now, take a finite support product \( \mathcal{P} = \Pi_{i<\omega} \mathcal{P}_i \) so that for all \( s, j \) as above, there are infinitely many \( i < \omega \) so that \( \mathcal{P}_i = \mathcal{P}_{s,j} \). It is easy to find a sufficiently generic filter \( G \subset \mathcal{P} \) so that we get pairwise different, cofinal branches \( b_i = \cup G(i) \) in \( T_{<\alpha} \) from the \( i \)th coordinate of \( G \). Note that if \( \mathcal{P}_i = \mathcal{P}_{s,j} \) then \( |\sup_Q b_i - \max_Q s| \leq \frac{1}{j} \).

Enumerate \( \omega \) as \( (k_i)_{i<\omega} \) so that for each \( s, j \) as above and each \( k < \omega \), there are infinitely many \( i < \omega \) so that \( \mathcal{P}_i = \mathcal{P}_{s,j} \) and \( k_i = k \). Since \( h \) is flush on \( Q \), we can find some \( t_i \in Q_\alpha \) above the branch \( b_i \) so that \( h(t_i) = k_i \) and \( |\max_Q(s) - \max_Q(t_i)| < \frac{1}{j} \).

Finally, let

\[
T_\alpha = \{t_i : i < \omega\}.
\]

This defines \( T_{<\alpha+1} = T_{<\alpha} \cup T_\alpha \), and note that our choices ensure that

1. \( T_{<\alpha+1} \) is a countable and strongly pruned end-extension of \( T_{<\alpha} \),
2. \( h \upharpoonright T_{<\alpha+1} \) is rich, and
3. \( h(t \upharpoonright \xi) = f_\alpha(\xi) \) for all \( t \in T_\alpha \) and almost all \( \xi \in C_\alpha \).

This finishes the inductive construction and hence the proof of the theorem.

Finally, we show how to modify the above argument for \( Q^* \) so that we produce uniformizations on non-special trees \( T \) (however, these trees will not be Aronszajn any more).

Proof of Theorem 3.3. Suppose that \( h^* : Q^* \to \omega \) is flush and take any ladder system colouring \( f \).

The corresponding non-special tree \( T \subset Q^* \) will be constructed by defining its initial segments \( T_{<\alpha} \) for \( \alpha < \omega_1 \) by induction on \( \omega_1 \) so that

1. \( T_{<\alpha} \) is a downward closed and strongly pruned subtree of \( Q^* \) of height \( \alpha \),
2. \( T_{<\beta} \) is an end extension of \( T_{<\alpha} \) for \( \alpha < \beta \),
3. \( h \upharpoonright T_{<\alpha} \) is rich, and
4. \( h \upharpoonright T_{<\alpha} \) uniformizes \( f \upharpoonright \alpha \).

In limit steps, we will simply take unions, and in successor steps \( \alpha + 1 \) where \( \alpha \) is also a successor, we only aim to preserve the above properties. This so far is the same as the proof of Theorem 3.1 except we allowed \( T_{<\alpha} \) to have uncountable levels.

\[1\]That is, the trees grow up: for any \( t \in T_{<\beta} \setminus T_{<\alpha} \), \( t \upharpoonright \alpha \subset T_{<\alpha} \).
Now, the difference comes in at stages when $T_{<\alpha}$ is already defined for some limit $\alpha$ and we aim to construct the $\alpha$th level $T_\alpha$. In the end, we need to make sure that $T$ is non-special i.e., for any partition $g : T \to \omega$ there is some $t' < t \in T$ such that $g(t) = g(t')$.

First, we need a definition: an $\alpha$-control tuple is a 4-tuple $(A,s,g,i)$ that satisfies the following:

(i) $A \subset T_{<\alpha}$ is a countable, downward closed and pruned subtree of height $\alpha$, and $s \in A$, 
(ii) $h_{<\alpha} \upharpoonright A$ is rich, $g : A \to \omega$ and $i < 2$, and 
(iii) if $i = 0$ then there are subtrees $A_\ell \subset A$ for $\ell < \omega$ of height $< \alpha$ so that 
\begin{itemize}
  \item $A = \bigcup_{\ell < \omega} A_\ell$ and $s \in A_0$, 
  \item $h \upharpoonright A_\ell$ is rich for all $\ell < \omega$, 
  \item for any $n < \omega$, $g^{-1}(n)$ is either dense in each $A_k$ above $s$ (we call $n$ a large colour of $g$ this case) \footnote{\text{I.e., for any} \ s \leq s' \in A_k \text{ there is} \ s' \leq s'' \in A_k \text{ so that} \ g(s'') = n.}$ or empty above $s$.
\end{itemize}

Enumerate all $\alpha$-control tuples $(A,s,g,i)$ extended by a pair of natural numbers $1 \leq m \in \omega$ and $m' \in \omega$ as $\{(A_\nu,s_\nu,g_\nu,i_\nu,m_\nu,m'_\nu) : \nu < \epsilon\}$, each $\epsilon$ times. If $i_\nu = 0$ then we fix $A_\nu = \bigcup_{\ell < \omega} A_\epsilon$ that witnesses condition (e).

We will define $T_\alpha$ to be $\{t_\nu : \nu < \epsilon\}$ so that for any $\nu < \epsilon$,
\begin{itemize}
  \item $s_\nu < t_\nu$ and $t_\nu \subset A_\nu$, 
  \item $|\sup R s_\nu - \sup R t_\nu| < \frac{1}{m_\nu}$ and $h(t_\nu) = m'_\nu$, 
  \item if $i_\nu = 0$ and $n < \omega$ is a large colour of $g_\nu$ then $g_\nu(t_\nu \upharpoonright \varepsilon) = n$ for some $\varepsilon < \alpha$, and 
  \item if $i_\nu = 1$ and $n < \omega$ then for some $\varepsilon < \alpha$, $g^{-1}_\nu(n)$ is either dense or empty above $t_\nu \upharpoonright \varepsilon$.
\end{itemize}

If we succeed, then conditions (c) and (d) will ensure that $T_{<\alpha+1}$ is still strongly pruned and $h \upharpoonright T_{\alpha+1}$ is rich. The last two conditions will help us prove that the final tree is non-special (although, this might not be clear at this point).

We construct $t_\nu$ for $\nu < \epsilon$ by induction, so suppose we defined $(t_\mu)_{\mu < \nu}$ already. Consider the tuple $(A_\nu,s_\nu,g_\nu,i_\nu,m_\nu,m'_\nu)$, and let us list $C_\alpha \setminus \{\text{ht}(s_\nu) + 1\}$ as $\xi_0 < \xi_1 < \ldots$. We find nodes $(t_\nu)_{\nu \in 2^{<\omega}}$ inside $A_\nu$ with the following properties:

1. $s_\nu < t_\theta$ and $t_\nu < t_\nu \in A_\nu$ for all $\nu < \omega$.
2. $|\sup R s_\nu - \sup R t_\nu| < \frac{1}{m_\nu}$ for any $\nu \in 2^{<\omega}$,
3. $\text{ht}(t_\nu) \geq \xi_n$ for all $\nu < 2^n$ and $n < \omega$,
4. if $\xi \in (\text{ht}(t_\nu) + 1) \cap C_\alpha$ then $h(t_\nu \upharpoonright \xi) = f_\alpha(\xi)$ for any $\nu < 2^{<\omega}$,
5. if $i = 0$ and $n < \omega$ is a large colour of $g_\nu$ then $g_\nu(t_\nu \upharpoontright \varepsilon) = n$ for all $\nu \in 2^{n+1}$, and 
6. if $i = 1$ and $n < \omega$ then for any $\nu \in 2^{n+1}$, $g^{-1}_\nu(n)$ is either dense or empty above $t_\nu$.

Observe that these assumptions ensure that for any $x \in 2^{<\omega}$, the branch $b_x = \{t_{x|n} : n < \omega\}$ is cofinal in $A_\nu$ and has an upper bound in $Q^\nu$. Moreover, $|\sup R s_\nu - \sup R b_x| \leq \frac{1}{2m_\nu}$ and $b_x \neq b_y$ if $x \neq y \in 2^{<\omega}$, so there is an $x = x_{\alpha} \in 2^{<\omega}$ such that the branch $b_\alpha := \{t_{x|n} : n < \omega\}$ is distinct from all the previously defined branches $(t_\mu^i : \mu < \nu)$.

Now pick an upper bound $t_\nu \in Q^\nu_{\alpha}$ for $b_\alpha$ so that $h(t_\nu) = m'_\nu$ and $|\sup R b_\alpha - \sup R t_\nu| < \frac{1}{2m_\nu}$. This is possible since $h$ was flush. Conditions (c) and (d) are satisfied by our choice of $t_\nu$, and conditions (c) and (d) will hold by our assumptions on the restrictions $t_{x|n}$.

Let us explain the construction of the binary system $(t_\nu)_{\nu \in 2^{<\omega}}$. First, choose $t_\theta$ with $\text{ht}(t_\theta) = \xi_0$ above $s_\nu$ so that $h_{<\alpha}(t_\theta) = f_\alpha(\xi_0)$.

Suppose that $(t_\nu)_{\nu \in 2^{<\omega}}$ has been defined, and fix some $\nu \in 2^n$ and $j < 2$; our goal is to find the right $t_\nu$ for $v = u^{-j} e \in 2^{n+1}$.\footnote{\text{I.e., for any} \ s \leq s' \in A_k \text{ there is} \ s' \leq s'' \in A_k \text{ so that} \ g(s'') = n.}
Claim 3.8. There is a \( \bar{t} \in A_\nu \) above \( t_u \) of height at least \( \xi_{n+1} \), so that

1. if \( \xi \in (\text{ht}(t) + 1) \cap C_\alpha \) then \( h_{<\alpha}(\bar{t} \upharpoonright \xi) = f_\alpha(\xi) \), and
2. if \( i = 0 \) and \( \bar{t} \in A_\nu \setminus A_{\nu+1} \) then \( \text{ht}(t) > \max(C_\alpha \cap \text{ht}(A_\nu)) \).

Proof. We just look at the elements of \( C_\alpha \) between \( \text{ht}(t_u) \) and \( \xi_{n+1} \), let these be \( \xi_k \leq \xi_{k+1} < \cdots < \xi_{n+1} \). If additionally, \( i = 0 \) then we look at the minimal \( \ell \) so that \( \xi_{n+1} < \text{ht}(A_\nu) \) and \( t_u \in A_\nu \). We extend our previous list \( (\xi_k)_{k \leq n+1} \) by the elements of \( C_\alpha \cap \text{ht}(A_\nu) \), and so we have a list \( \xi_k \leq \cdots < \xi_m \).

Now, in a finite induction, we define \( t_u < t_k < t_{k+1} < \cdots < t_m = \bar{t} \) in \( A_\nu \) (or just in \( A_\nu \) if \( i = 1 \)) so that \( \text{ht}(t_l) = \xi_l \) and \( h_{<\alpha}(t_l) = f_\alpha(\xi_l) \). This is possible since \( h_{<\alpha} \) was rich on \( A_\nu \).

Now, given \( \bar{t} \) as above, if \( i = 0 \), \( \bar{t} \in A_\nu \) and \( n \) is large for \( g_\nu \) then we can find \( t_v \) above \( t_u \) in \( A_\nu \) so that \( g_\nu(t_v) = n \). If \( i = 1 \) then we can find \( t_v \) above \( t \) in \( A_\nu \) so that either \( g_\nu^{-1}(n) \) is empty or dense above \( t_v \).

This defines \( t_v \), and hence finishes the construction of the binary tree.

At this point, we finished the construction of the level \( T_n \) and so the whole tree \( T = \bigcup_{\alpha < \omega} T_\alpha \) is defined. Why is \( T \) non special? Suppose that \( g : T \rightarrow \omega \), and we need to find \( t' < t \) so that \( g(t') = g(t) \). Take a continuous \( \omega + 1 \)-sequence of countable elementary submodels \((M_\delta)_{\delta \leq \omega} \) of some large enough \( H(\Theta) \) with \( g,T,h \in M_0 \) such that \( \alpha = M_\omega \cap \omega_1 \). Let \( A = M_\omega \cap T \) and note that \( h \upharpoonright A \) is rich.

Claim 3.9. There is some \( s \in A \cap M_0 \) so that \((A,s,g \upharpoonright A,0)\) is an \( \alpha \)-control tuple.

Proof. \((A,\emptyset,g \upharpoonright A,1)\) is certainly an \( \alpha \)-control triple so it gets enumerated at some step \( \mu < \omega \). The node \( t_\mu \) that we introduced (above \( \emptyset \)) satisfies that for any colour \( n < \omega \), \( g^{-1}(n) \) is either dense or empty above \( t_\mu \) by \( 6 \). So this property reflects to \( M_0 \), and we can find an \( s \in A \cap M_0 \) so that for any colour \( n < \omega \), \( g^{-1}(n) \) is either dense or empty above \( s \). So, \((A,s,g \upharpoonright A,1)\) is a control tuple witnessed by the sequence \( A_\mu = A \cap M_\mu \).

In turn, at step \( \alpha \), we enumerated \((A,s,g \upharpoonright A,0)\) as \((A_\nu,s_\nu,g_\nu,i_\nu)\) for some \( \nu < \omega \). First, we claim that \( n = g(t_\nu) \) is a large colour of \( g_\nu \). Indeed, \( t_\nu \) witnesses that \( g_\nu^{-1}(n) \) cannot be empty above any restriction of \( t_\nu \). However, in this case there is some \( \varepsilon < \alpha \) so that \( g_\nu(t_\nu \upharpoonright \varepsilon) = n \) by condition \( 7 \). In other words, \( g(t_\nu) = g(t_\nu \upharpoonright \varepsilon) \) as desired.

4. Uniformization results from strong diamonds

Recall that \( \diamond^+ \) asserts the existence of a sequence \( W = (W_\delta)_{\delta < \omega_1} \) so that \( |W_\delta| \leq \omega \) and for any \( X \subseteq \omega_1 \), there is a closed unbounded \( C \subseteq \omega_1 \) so that the set

\[
\{ \delta < \omega_1 : X \cap \delta, C \cap \delta \in W_\delta \}
\]

contains a club. It is well known that the model of Theorem 2.10 satisfies \( \diamond \), but not the stronger \( \diamond^+ \). We will see now that this is no coincidence: \( \diamond^+ \) actually implies that there are Aronszajn trees and ladder systems with the monochromatic uniformization property, which is in surprising contrast with the classical theory of \( \omega_1 \)-uniformizations.

Let us say that a \( T \)-uniformization \( \varphi \) of some colouring is special/non-special/Suslin if \( \text{dom}(\varphi) \) is a special/non-special/Suslin subtree of \( T \). Our main results read as follows.
Theorem 4.2. \( \Diamond^+ \) implies that for any ladder system \( C \), there is a special Aronszajn tree \( T \subset \mathcal{Q} \) so that \( c\text{Unif}_{\omega}(T, C) \).

Theorem 4.2. \( \Diamond^+ \) implies that for any ladder system \( C \), there is an Aronszajn tree \( T \subset 2^{<\omega_1} \) so that any monochromatic \( \omega \)-colouring of \( C \) has a Suslin \( T \)-uniformization.

In the latter result, the tree \( T \) is necessarily non-special but we don’t know if it can be made almost Suslin i.e., to contain no stationary antichains. Equivalently, whether \( c\text{Unif}_{\omega}(T, C) \) implies the existence of stationary antichains in \( T \).

Combined with Theorem 2.6 we also get the following result.

Corollary 4.3. \( \Diamond^+ \) implies that for any ladder system \( C \), there is an Aronszajn tree \( T \) so that \( c\text{Unif}_{\omega}(T, C) \) holds but \( \text{Unif}_{2}(T, C) \) fails.

We are not aware of an analogue of this result in the classical context of \( \omega_1 \)-uniformizations. On a related note, it is proved in [23] that consistently, \( \text{Unif}_{2}(\omega_1, C) \) holds but \( \text{Unif}_{\omega}(\omega_1, C) \) fails for some \( C \).

The idea to prove the above theorems is the following: in the end, no matter how we constructed the tree \( T \), the \( \Diamond^+ \)-sequence \( (W_\delta)_{\delta \in \omega_1} \) will allow us to guess initial segments of any ladder system colouring \( f \) club often. So, at some stage \( \delta \) of constructing the tree (when we constructed \( T_{<\delta} \) already), we look at elements of \( W_\delta \) that look like a \( T_{<\delta} \)-uniformization of some partial ladder system colouring \( f \upharpoonright \delta \) that is also in \( W_\delta \); collect these partial uniformizations into a set \( H_\delta \), which must be countable. We aim to extend \( T_{<\delta} \) by the level \( T_\delta \) so that for any element \( h \) of \( H_\delta \) and any possible colour \( n \) for the ladder \( C_\delta \) there are a lot of \( t \in T_\delta \) so that \( t \uparrow \in dom h \) and \( h(t \uparrow \xi) = n \) for almost all \( \xi \in C_\delta \). This will allow us to extend any map \( h \in H_\delta \) to the level \( \delta \) no matter what is the constant value \( n \) of \( f_\delta \).

We will use rich maps, strongly pruned trees and some arguments that resemble the work in Section 3 so we stick to using the letter \( h \) for the uniformizing functions. We state one extra preliminary lemma before proving the theorems.

**Lemma 4.4.** Let \( \delta_0 < \delta_1 < \omega_1 \) and suppose that \( T \) is a downward closed, strongly pruned subtree of \( \mathcal{Q} \) of height \( \delta_1 \), and \( f \) is a ladder system \( \omega \)-colouring. If \( h_0 \) is a rich \( T_{<\delta_1+1} \)-uniformization of \( f \upharpoonright \delta_0 + 1 \) then there is a rich \( h_1 \) extending \( h_0 \) so that \( h_1 \) is a \( T_{<\delta_1} \)-uniformization of \( f \upharpoonright \delta_1 \).

**Proof.** Look at the poset \( \mathcal{P} \) of all maps \( p \) so that

1. \( \text{dom } p = \bigcup\{t^i : t \in D^p \} \) where \( D^p \subseteq T_{<\delta_1} \) is finite,
2. \( p \cup h_0 \) is still a function, and
3. \( p \) uniformizes \( f \) for any \( t \in \text{dom } p \) of limit height \( \alpha \) and for almost all \( \xi \in \text{dom } f_\alpha \), \( p(t \uparrow \xi) = f_\alpha(\xi) \).

It is easily checked that for any \( p \in \mathcal{P} \) and \( s \in T_{<\delta_1} \), extending some element of \( \text{dom } p \), there is some \( q \in \mathcal{P} \) so that \( q \leq p \) and \( s \in \text{dom } q \). Using this, one can find a filter \( G \subseteq \mathcal{P} \) so that \( h_1 = \bigcup G \cup h_0 \) is defined on a strongly pruned subtree of \( T_{<\delta_1} \). Furthermore, by requiring \( G \) to meet more dense sets, one can ensure that \( h_1 \) is rich as well.

Finally, given a countable set \( W \), we will say that some set \( x \) is \( W \)-definable if \( x \in M \) whenever \( M \) is a countable elementary submodel of \( (H((2^{\aleph_0})^+), \in, \prec) \) with \( W \in M \), where \( \prec \) is some fixed well-order on \( H((2^{\aleph_0})^+) \).

Let us prove the theorems now.
Proof of Theorem \[\vdash\]

Let \( \mathcal{W} = (W_\delta)_{\delta \leq \omega_1} \) denote the \( \Diamond^+ \) sequence, and let \( W_{< \delta} = (W_\alpha)_{\alpha < \delta} \) for \( \delta < \omega_1 \).

The tree \( T \subseteq Q \) will be constructed by defining its initial segments \( T_{< \alpha} \) for \( \alpha < \omega_1 \) by induction on \( \omega_1 \), along with countable sets \( \mathcal{H}_\alpha \) and so called sealing functions \( \mathfrak{s}t_\alpha : \mathcal{H}_\alpha \times \omega \rightarrow [T_\alpha]^\omega \). In the end, these functions will tell us how to continue a partial \( T \)-uniformization.

We assume that these objects satisfy the following properties, which we preserve through the induction:

(a) \( T_{< \alpha} \subseteq Q \) is a countable, downward closed strongly pruned and uniquely \( W_{< \alpha+1} \)-definable tree,
(b) \( T_{< \beta} \) is an end extension of \( T_{< \alpha} \) for \( \alpha < \beta \),
and for any limit \( \alpha < \omega_1 \),
(c) \( \mathcal{H}_\alpha \) is the set of all \( W_{< \alpha+1} \)-definable rich maps \( h : S \rightarrow \omega \) so that \( S \cap T_{< \alpha} \subseteq Q \)-pruned,
(d) \( \mathfrak{s}t_\alpha \) is uniquely \( W_{< \alpha+1} \)-definable, and
(e) for all \( h \in \mathcal{H}_\alpha \),
   (i) \( \text{dom}(h) \cup \mathfrak{s}t_\alpha(h, n) \) is a downward closed, strongly pruned subtree of \( T_{< \alpha} \), and
   (ii) \( h(t \upharpoonright \xi) = n \) for all \( t \in \mathfrak{s}t_\alpha(h, n) \) and almost all \( \xi \in C_\alpha \).

These assumptions ensure that for any \( h \in \mathcal{H}_\alpha \), there is some rich extension \( h' \) of \( h \) that is defined on \( \text{dom}(h) \cup \mathfrak{s}t_\alpha(h, n) \). Moreover, if \( h \) was a uniformization for some colouring \( \mathfrak{f} \upharpoonright \alpha \upharpoonright \mathcal{C} \) \( \alpha \), then \( h' \) is a uniformization of \( \mathfrak{f} \upharpoonright \alpha + 1 \) whenever \( f_\alpha \) is constant \( n \).

First, the less interesting cases: in limit steps, we simply take unions (which preserves being strongly pruned and the definability requirements). If \( \alpha \) is successor and \( T_{< \alpha} \) is defined then we canonically extend \( T_{< \alpha} \) by a level \( T_\alpha \) so that \( T_{< \alpha+1} \) remains strongly pruned.

Now, we describe the construction of \( T_{< \alpha+1} = T_{< \alpha} \cup T_\alpha \) for a limit \( \alpha \) when \( T_{< \alpha} \) is defined already. To this end, fix some \( h \in \mathcal{H}_\alpha \), \( s \in \text{dom} h \), and \( n, j \in \omega \). Consider the poset \( \mathcal{P} = \mathcal{P}_{s, h, n, j} \) of all finite, non-empty chains \( p \subseteq \text{dom} h \) above \( s \) so that for all \( t \in p \),

1. \( \text{ht}(t) \in C_\alpha \) implies \( h(t) = n \), and
2. \( |\max_Q(t) - \max_Q(s)| \leq \frac{1}{2} \).

It is easy to check, using that \( h \) is rich, that for any \( \varepsilon < \omega \), \( D_\varepsilon = \{ t \in \mathcal{P} : \text{ht}_Q(\cup p) \geq \varepsilon \} \) is dense in \( \mathcal{P}_{s, h, n, j} \). Now, take a finite support product \( \mathcal{P} = \prod_{\varepsilon \in \omega} \mathcal{P}_\varepsilon \) so that for all \( s, h, n, j \) as above, there are infinitely many \( i < \omega \) so that \( \mathcal{P}_i = \mathcal{P}_{s, h, n, j} \). It is easy to find a sufficiently generic filter \( G \subseteq \mathcal{P} \) so that we get pairwise different, cofinal branches \( b_i = \cup G(i) \in T_{< \alpha} \) from the \( i \)th coordinate of \( G \). Note that if \( \mathcal{P}_i = \mathcal{P}_{s, h, n, j} \) then \( |\sup_Q b_i - \max_Q s| \leq \frac{1}{2} \), so we can form \( t_\alpha^* = b_i \cup \{ \max_Q(s) + \frac{1}{2} \} \in Q \) and let

\[ T_\alpha = \{ t_\alpha^* : i < \omega \}. \]

The sealing function \( \mathfrak{s}t_\alpha \) is defined as follows: given \( h \in \mathcal{H}_\alpha \) and \( n < \omega \), let
\[ \mathfrak{s}t_\alpha(h, n) = \{ t_\alpha^* : i < \omega, \mathcal{P}_i = \mathcal{P}_{s, h, n, j} \text{ for some } j < \omega, s \in \text{dom} h \}. \]

Since the poset \( \mathcal{P} \) is uniquely definable from \( \mathcal{H}_\alpha \), we can make a canonical choice of \( G \) (using the well order \( < \)), and so \( T_\alpha \) and \( \mathfrak{s}t_\alpha \) are uniquely definable from \( W_{< \alpha+1} \) as well.

This finishes the construction of our (special) strongly pruned Aronszajn tree \( T = \bigcup_{\alpha < \omega_1} T_{< \alpha} \).

Suppose now that \( \mathfrak{f} \) is a monochromatic \( \omega \)-colouring of \( \mathcal{C} \), and we would like to find a \( T \)-uniformization. There is a club \( D \subseteq \omega_1 \) so that \( \delta \in D \) implies that \( \mathcal{C} \upharpoonright \delta, D \cap \delta, \mathfrak{f} \upharpoonright \]

\[13\]That is, the trees grow up: for any \( t \in T_{< \beta} \setminus T_{< \alpha} \), \( t \upharpoonright \alpha < T_{< \alpha} \).
\(\delta, W_{<\delta} \in W_\delta\). Let \(\{\alpha_\xi : \xi < \omega_1\}\) be the increasing enumeration of \(D\), and let \(n_\xi\) denote the constant value of \(f_{\alpha_\xi}\).

We will define a \(\subseteq\)-increasing sequence of functions \((h_\xi)_{\xi<\omega_1}\) so that

1. \(h_\xi \in H_{\alpha_\xi}\), i.e., \(h_\xi\) is a rich \(W_{<\alpha_\xi+1}\)-definable map on a strongly pruned subtree of \(T_{<\alpha_\xi}\), and
2. \(h_\xi\) uniformizes \(f \upharpoonright \alpha_\xi + 1\).

If we succeed, then \(h = \cup_{\xi<\omega_1} h_\xi\) is the desired \(T\)-uniformization of \(f\).

Proof of Theorem 4.2. We still use \(W\) from the parameters \(C\) uniformizes \(f\) in the way), and these parameters are all in \(W\) from the parameters \(C\). Hence \(h_\xi\) is \(W_{<\alpha_\xi+1}\)-definable as desired i.e., \(h_\xi \in H_{\alpha_\xi}\).

This finishes the induction and hence the construction of the \(T\)-uniformization of \(h\).

To summarize, we constructed an Aronszajn tree \(T \subseteq Q\) that satisfies \(cUnif_\omega(T, C)\).

Our next goal is to prove the second theorem about Suslin uniformizations, which will be very similar. We will construct \(T\) as a subtree of \(2^{<\omega_1}\) instead of \(Q\), and make sure that the sealing functions preserve maximal antichains that are guessed by the diamond sequence.

We need a small definition before the proof: suppose that \(S\) is a countable tree of height \(\alpha\) and \(X \subseteq S\) is a maximal antichain. We say that \(X\) is \(\alpha\)-reflecting if the set \(\{\xi \in \alpha \cap \lim(\omega_1) : X \cap S_{<\xi}\text{ is a maximal antichain in } S_{<\xi}\}\) is cofinal in \(\alpha\).

Proof of Theorem 4.2. We still use \(W\) to denote the \(\diamondsuit^+\)-sequence. \(T\) will be built by constructing its initial segments \(T_{<\alpha} \subseteq 2^{<\omega_1}\) along with \(H_\alpha\) and \(s_{\alpha}\), with the following properties:

(a) \(T_{<\alpha} \subseteq 2^{<\omega_1}\) is a countable, downward closed, normal tree and uniquely definable from \(W_{<\alpha+1}\),
(b) \(T_{<\beta}\) is an end extension of \(T_{<\alpha}\) for \(\alpha < \beta\), and for any limit \(\alpha < \omega_1\),
(c) \(H_\alpha\) is the set of all \(W_{<\alpha+1}\)-definable rich maps \(h : S \to \omega\) so that \(S \subseteq T_{<\alpha}\) is downward closed and normal,
(d) \(s_{\alpha}\) is uniquely definable from \(W_{<\alpha+1}\), and
(e) for all \(h \in H_\alpha\),
   (i) \(\text{dom}(h) \cup s_{\alpha}(h, n)\) is a downward closed subtree of \(T_{<\alpha}\),
   (ii) \(h(t \upharpoonright \xi) = n\) for all \(t \in s_{\alpha}(h, n)\) and almost all \(\xi \in C_{\alpha}\), \(^{14}\)

\(^{14}\)I.e., any node has incomparable extensions.
(iii) if $X \in W_\alpha$ is $\alpha$-reflecting maximal antichain in $dom\ h$ then any $t \in sI_\alpha(h,n)$ is above some element of $X$.

As before, the interesting case in the construction is when $T_{<\alpha}$ is constructed already for a limit $\alpha$ and we aim to define $T_\alpha$. For any $h \in H_\alpha$, $s \in dom\ h$ and $n \in \omega$, we now consider the poset $\mathcal{P}_{s,h,n}$ of all finite, non empty chains $p \subset dom\ h$ above $s$ so that for all $t \in p^i$ above $s$, $ht(t) \in C_\alpha$ implies $h(t) = n$.

We take a finite support product $\mathcal{P} = \prod \mathcal{P}_i$ so that for all $s,h,n$ as above, there are infinitely many $i < \omega$ so that $\mathcal{P}_i = \mathcal{P}_{s,h,n}$. In order to find the right generic branches, we need a new density lemma.

**Claim 4.5.** Suppose that $X \subset dom\ h$ is $\alpha$-reflecting maximal antichain. Then $D_X = \{ p \in \mathcal{P}_{s,h,n} : p^i \cap X \neq \emptyset \}$ is dense in $\mathcal{P}_{s,h,n}$.

*Proof.* Fix some $\alpha$-reflecting maximal antichain $X \subset S = dom\ h$, and suppose that $p_0 \in \mathcal{P}_{s,h,n}$ is arbitrary. Let $t_0 = \max_S(p_0)$ and find some limit $\varepsilon < \alpha$ above $t_0$ so that $X \cap S_{<\varepsilon}$ is a maximal antichain in $S_{<\varepsilon}$. List the elements of $C_\alpha$ that are between $t_0$ and $\varepsilon$ as $\alpha_1 < \alpha_2 < \cdots < \alpha_{k-1}$. Using that $h$ is rich, we can find $t_0 < t_1 < t_2 < \cdots < t_{k-1}$ so that $t_i \in S_{\alpha_i}$ and $h(t_i) = n$ for $1 \leq i < k$. In turn, $p_1 = p_0 \cup \{ t_i : i < k \} \in \mathcal{P}_{s,h,n}$ is an extension of $p_0$.

Now, either $X \cap t_{k-1}^i$ already or we can find some $t \in S_{<\varepsilon}$ above $t_{k-1}$ so that $t^i \cap X \neq \emptyset$. Note that no new elements of $C_\alpha$ appear between $t_{k-1}$ and $t$, so we can define $p_2 = p_1 \cup \{ t \}$ which is an extension $p_0$ in $D_X$. \qed

Now, it is easy to find a sufficiently generic filter $G \subset \mathcal{P}$ so that we get pairwise different branches $b_i = \cup G(i)$ in $T_{<\alpha}$ from the $i$th coordinate of $G$ and $b_i^0 \cap X \neq \emptyset$ for any $i < \omega$ and $\alpha$-reflecting maximal antichain $X \subset dom\ h$ so that $X \in W_\alpha$ (where $P_i = \mathcal{P}_{s,h,n}$).

We define $T_\alpha$ by adding unique upper bounds $t_i^\ast$ for each branch $b_i$, just as in the proof of Theorem 3.4. The sealing function $sI_\alpha$ is defined as follows: given $h \in H_\alpha$ and $n < \omega$, let

$$sI_\alpha(h,n) = \{ t_i^\ast : i < \omega, \mathcal{P}_i = \mathcal{P}_{s,h,n}, s \in dom\ h \}.$$ 

Since the poset $\mathcal{P}$ is uniquely definable from $H_\alpha$, we can make a canonical choice of $G$, and so $T_\alpha$ and $sI_\alpha$ is $W_{<\alpha+1}$-definable as well.

This finishes the construction of an Aronszajn tree $T = \bigcup_{\alpha < \omega_1} T_{<\alpha} \subset 2^{\omega_1}$.

Suppose that we are given some monochromatic $\omega$-colouring $f$ of $C$. The diamond sequence and sealing functions provide a unique way to construct a sequence of maps $(h_\xi)_{\xi < \omega_1}$ along a club $D = \{ \alpha_\xi : \xi < \omega_1 \} \subset \omega_1$ so that

1. $h_\xi \in H_{\alpha_\xi}$,
2. $dom\ h_{\xi+1} \cap T_{\alpha_\xi} = sI_{\alpha_\xi}(h_\xi, n_\xi)$ where $n_\xi$ is the constant value of $f_{\alpha_\xi}$, and
3. $h_\xi$ uniformizes $f \upharpoonright \alpha_\xi + 1$.

So $h = \bigcup_{\xi < \omega_1} h_\xi$ is a $T$-uniformization of $f$. We need to show that $dom\ h = S$ is Suslin, so suppose that $X \subset S$ is a maximal antichain. The set $\hat{D}$ of those $\alpha < \omega_1$ so that $X \cap S_{<\alpha} \in W_\alpha$ and $X \cap S_{<\alpha}$ is a reflecting maximal antichain in $S_{<\alpha}$ contains a club. So, we can find some $\xi < \omega_1$ so that $\alpha_\xi \in \hat{D}$. In turn, by the properties of our sealing function, any $t \in sI_{\alpha_\xi}(h_\xi, n_\xi) = S_{\alpha_\xi}$ extends some element of $X \cap S_{<\alpha_\xi}$. So we must have $X \subset S_{<\alpha_\xi}$ and hence $X$ is countable.

This finishes the proof of the theorem.
5. Uniformization and Large Antichains

First, let us explain why large uncountable antichains appear even in the non-special tree $T$ of Theorem 4.2. Fix a ladder system $C$, and consider the colourings $f^\nu$ for $\nu \in \lim \omega_1$ so that $f^\nu_\alpha$ is constant 1 on $C_\alpha$ if $\nu < \alpha < \omega_1$, otherwise $f^\nu_\alpha$ is constant 0. So each $f^\nu$ is a monochromatic 2-colouring of $C$.

We can arrange the $\diamondsuit^\alpha$-sequence $(W_\alpha)_{\alpha < \omega_1}$ so that $\lim (\omega_1) \cap \alpha, f^\nu | \alpha \in W_\alpha$ for $\nu < \alpha$. So, as in the proof of Theorem 4.2, we can build all the uniformizations $(\bigcup \nu)_{\mu < \nu} C$ of all colourings all have uniformizations agreed up to that level as well. However, the solution alone that the above colouring was continuous: if two colourings agreed up to some level, their corresponding uniformizations agreed up to that level as well. However, the fact alone that the above defined colourings all have $T$-uniformizations does not necessarily mean that $T$ has large antichains.

Theorem 5.1. Suppose that $F$ is a family of $\aleph_1$ many ladder system colourings (the ladder systems may vary with the colouring). Then if $\diamondsuit$ holds, then there is a Suslin tree $T$ so that any $f \in F$ has a full $T$-uniformization.

One can similarly show, without any extra assumptions beyond ZFC, that for any family $F$ of $\aleph_1$ many ladder system colourings, there is a (necessarily special) Aronszajn tree $T \subseteq Q$ so that any $f \in F$ has a full $T$-uniformization.

Proof. Let $F = \{f^\nu : \nu < \omega_1\}$ where $f^\nu = (f^\nu_\alpha)_{\alpha \in \lim (\omega_1)}$ and dom $f^\nu_\alpha = C^\nu_\alpha$. Let $\mathbf{W}$ denote the diamond sequence.

We aim to construct a Suslin tree $T \subseteq 2^{<\omega_1}$ and the uniformizations $h^\nu : T \to \omega$ for $f^\nu$ simultaneously for each $\nu < \omega_1$. So, by induction on $\alpha < \omega_1$, we construct $T_{<\alpha}$ and $(h^\nu_{<\alpha})_{\nu < \alpha}$ so that

1. $T_{<\alpha}$ is a countable, downward closed and normal subtree of $2^{<\omega_1}$,
2. $h^\nu_{<\alpha} : T_{<\alpha} \to \omega$ is a uniformization of $f^\nu | \alpha$,
3. for all $\nu < \alpha < \beta$, $T_{<\beta}$ is an end-extension of $T_{<\alpha}$ and $h^\nu_{<\beta}$ is an end-extension of $h^\nu_{<\alpha}$,
4. if $\alpha$ is limit and $W_\alpha$ is a maximal antichain of $T_{<\alpha}$ then any $t \in T_\alpha$ extends some element of $W_\alpha$,
5. for any $\Delta \subseteq [\alpha]^{<\omega}$, $n : \Delta \to \omega$, $\varepsilon < \alpha$ and $s \in T_{<\varepsilon}$ there are infinitely many $t \in T_\varepsilon$ above $s$ so that $h^\nu(t) = n(\nu)$

for all $\nu \in \Delta$.

The latter condition is a simultaneous version of the richness condition that we introduced before the proof of Theorem 4.2. It is clear that if we succeed in building these objects then $T = \bigcup_{\alpha < \omega_1} T_{<\alpha}$ is a Suslin tree and $h^\nu = \bigcup_{\nu < \alpha < \omega_1} h^\nu_{<\alpha}$ is a full $T$-uniformization of $f^\nu$. 
In limit steps, we can take unions and all the assumptions are preserved. As usual, the non-trivial step in our construction is when we are given $T_{<\alpha}$ for some limit $\alpha$ along with $h_\alpha^\nu : T_{<\alpha} \to \omega$ for $\nu < \alpha$ with the above conditions. We need to define $T_\alpha$, which is the next level of the tree, and extend all the maps $h_\alpha^\nu$ to $T_\alpha$ (which gives $h_\alpha^{\nu+1}$), and define $h_\alpha^{\nu+1}$ all while preserving condition 5.

We first construct a map $h_\alpha^\nu : T_{<\alpha} \to \omega$ that will serve as the restriction of $h_\alpha^{\nu+1}$ to $T_{<\alpha}$.

Claim 5.2. There is a map $h_\alpha^\nu : T_{<\alpha} \to \omega$ that uniformizes $f^\nu \restriction \alpha$ and so that any $\Delta \in [\alpha+1]^{<\omega}$, $\eta : \Delta \to \omega$, $\epsilon < \alpha$ and $s \in T_{<\epsilon}$, there are infinitely many $t \in T_\epsilon$ above $s$ so that

$$h_\alpha^\nu(t) = \eta(t)$$

for all $\nu \in \Delta$.

Proof. It is straightforward to check that the forcing $\mathcal{P}$ defined in the proof of Lemma 4.3 can be used to provide this map $h_\alpha^\nu \restriction T_{<\alpha}$, by choosing the filter $G \subset \mathcal{P}$ to be generic enough over the maps $\{h_\alpha^\nu : \nu < \alpha\}$. \hfill \Box

Our next goal is to define $T_\alpha$: first, list $T_{<\alpha}$ as $\{s_i : i < \omega\}$, each node infinitely often.

We will inductively construct cofinal branches $b_i \subset T_{<\alpha}$ above $s_i$ so that

(a) $b_i \neq b_j$ for all $j < i$,

(b) $b_i$ extends some element of $W_\alpha$ if $W_\alpha$ is a maximal antichain in $T_{<\alpha}$, and

(c) for any $\nu \leq \alpha$, and almost all $\xi \in C_\alpha^\nu$,

$$h_\alpha^\nu(\bigcup b_i \restriction \xi) = f_\alpha^\nu(\xi).$$

If we succeed then we let $t_i^* = \bigcup b_i$ and $T_\alpha = \{t_i^* : i < \omega\}$. The last condition ensures that for any $\nu \leq \alpha$, $h_\alpha^\nu$ uniformizes $f^\nu \restriction \alpha + 1$ as well and not just $f^\nu \restriction \alpha$.

Suppose that $b_j$ is already found for $j < i$. Let $t_0 \in T_{<\alpha}$ extend $s_i$ so that $t_0 \notin \bigcup_{j < i} b_j$ and $t_0$ extends some element of $W_\alpha$ if $W_\alpha$ is a maximal antichain in $T_{<\alpha}$. List $\alpha + 1$ as $\{\nu_n : n < \omega\}$. We now build a sequence $t_0 < t_1 < \cdots < t_n < \cdots$ for $n < \omega$ in $T_{<\alpha}$ so that

(i) $\alpha_n = \text{ht}(t_n)$ is the $n$th element of $C_\alpha^{\nu_n}$,

(ii) if $\xi \in (\alpha_{n+1} + 1) \setminus (\alpha_n + 1)$ then for any $k \leq n$ such that $\xi \in C_\alpha^{\nu_k}$,

$$h_\alpha^{\nu_k}(t_{n+1} \restriction \xi) = f_\alpha^{\nu_k}(\xi).$$

Clearly, the branch $b_i = \bigcup_{n < \omega} t_i^*$ will satisfy our requirements.

Given $t_n$, we describe the construction of $t_{n+1}$. Consider the finite set

$$\bigcup \{C_\alpha^{\nu_k} : k \leq n\} \cap (\alpha_{n+1} + 1) \setminus (\alpha_n + 1),$$

and let $\{\xi_l : 1 \leq l \leq \ell\}$ denote the increasing enumeration. We need to find an increasing sequence in $T_{<\alpha}$

$$t_n = t_{n,0} < t_{n,1} < \cdots < t_{n,l} < \cdots < t_{n,\ell-1} = t_{n+1}$$

so that $\text{ht}(t_{n,l}) = \xi_l$ and if $\xi_l \in C_\alpha^{\nu_k}$ for some $k \leq n$ then $h_\alpha^{\nu_k}(t_{n,l}) = f_\alpha^{\nu_k}(\xi_l)$ for $1 \leq l \leq \ell$. Given $t_n$, we simply apply the simultaneous richness property 5 of the colourings $(h_\alpha^\nu)_{k \leq n}$ to find $t_{n,l+1}$ (this is done by setting $\epsilon = \xi_{l+1}$, $\Delta = \{\nu_k : k \leq n, \xi_{l+1} \in C_\alpha^{\nu_k}\}$ and $n(\nu_k) = f_\alpha^{\nu_k}(\xi_{l+1})$).

Now that we defined these branches and the level $T_\alpha$, our last job is to define the maps $h_\alpha^{\nu+1}$ (extending $h_\alpha^\nu$) on the new level $T_\alpha$ for $\nu < \alpha$. The only requirement to keep in mind

\footnote{The only role of this assumption is to make sure that $\sup_{n \in \omega} \text{ht}(t_n) = \alpha$.}
is the simultaneous richness condition (5) with $\varepsilon = \alpha$. One can do this by a simple induction of length $\omega$ enumerating all these countably many requirements, or by using the next claim.

**Claim 5.3.** Suppose that $M$ is a countable elementary submodel so that $T_{\leq \alpha} \in M$, and let $(c^\nu_{\leq \alpha})$ be mutually $M$-generic Cohen-functions from $T_{\alpha}$ to $\omega$. If we let $h_{\alpha + 1}^\nu(t) = c^\nu(t)$ for $t \in T_{\alpha}$ then $(h_{\alpha + 1}^\nu)_{\nu \leq \alpha}$ satisfies condition (5).

In any case, we leave the proof to the reader.

With this, the $\alpha$th step of the induction is done and hence we finished our construction of the tree $T$ with the full $T$-uniformizations. The fact that $T$ is Suslin follows from condition (4).

6. **Forcing uniformizations and preserving Suslin trees**

In [19], Moore introduced a forcing that uniformizes a given colouring on an Aronszajn tree $T$, which furthermore can be iterated without adding reals. In turn, consistently, CH holds and $\text{Unif}_\omega(T, C)$ for any ladder systems $C$ and Aronszajn tree $T$.

However, we proved in Theorem 2.3 that whenever CH holds, for any Suslin tree $T$ there are monochromatic 2-colourings of ladder systems which do not have $T$-uniformization. In turn, there could be no Suslin trees in Moore’s model. On the other hand, we proved in [27] how one can preserve CH and a single Suslin tree $R$, while forcing the existence of $T$-uniformizations for all $\omega$-colourings and those trees $T$ which do not embed $R$ in a strong sense.

Our goal in this section is to show that if we do not require CH then even $\text{Unif}_\omega(\omega_1, C)$ for all $C$ is consistent with the existence of Suslin trees. We will do this by proving that for any ladder system colouring $f$, there is a natural ccc forcing $P_f$ which introduces an $\omega_1$-uniformization for $f$ and such that $P_f$ preserves all Suslin trees. One can then apply well known iteration theorems to find the desired model; the details follow below.

**Theorem 6.1.** For any ladder system $C$ and $\omega$-colouring $f$ of $C$, there is a ccc forcing $P_f$ of size $\aleph_1$, so that

1. $\Vdash P_f$ there is an $\omega_1$-uniformization of $f$, and
2. if $R$ is a Suslin tree in the ground model then $\Vdash P_f \not\Vdash \dot{R}$ is Suslin.

**Corollary 6.2.** Suppose CH holds. Then there is a proper, cardinality and cofinality preserving forcing $P$ so that in $V^P$, $\text{Unif}_\omega(\omega_1, C)$ holds for any ladder system $C$, and any Suslin tree in the ground model remains Suslin in $V^P$.

**Proof.** An appropriate countable support iteration (in length $\omega_2$) of posets of the form $P_f$ will force that $\text{Unif}_\omega(\omega_1, C)$ holds for all ladder systems $C$. The fact that any ground model Suslin tree remains Suslin follows from Tadatoshi Miyamoto’s preservation theorem [18].

Similar models that combine features of the constructible universe $L$ and consequences of MA or PFA have received considerable attention [1 2 16 34 350]. However, we could not find a model combining the existence of Suslin trees and the uniformization property.

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16 That is, if successor stages of a proper, countable support iteration preserve a given Suslin tree, then so does the limit steps of the iteration.
For example, in [2], a forcing axiom for stable posets is proved consistent: although the model satisfies Unif$_ω$(ω$_1$, C), it has no Suslin trees.

More recently, the centre of attention has been on models of MA(S) (or PFA(S)) i.e., the forcing axiom for ccc partial orders (or proper posets, respectively) that preserve a fixed (coherent) Suslin tree S. We have the following corollary now.

**Corollary 6.3.** Suppose S is a Suslin tree. Then MA(S) implies Unif$_ω$(ω$_1$, C) for all C.

This should be compared with a result of Paul Larson and Todorcevic: if S is a Suslin tree and C is an S-name for a ladder system, then forcing with S will introduce a colouring of C with no ω$_1$-uniformization [15 Theorem 6.2]. It would be interesting to see how much of that argument carries over to T-uniformizations.

Now, let us prove the theorem. There are various ways to introduce a uniformization for a given ladder system colouring by a ccc forcing. These techniques include posets of

1. countable partial functions p from ω$_1$ to ω that are defined on finite unions of the ladders, and so that p agrees with $f_α$ almost everywhere on its domain [26 Chapter II, Theorem 4.3];

2. finite partial maps p from lim(ω$_1$) to ω so that the maps

$$\{f_α \upharpoonright \{C_α(n) : n \geq p(α)\} : α \in \text{dom } p\}$$

are pairwise compatible [34].

In both cases the order on the posets is extension as functions. We will use another natural variant: forcing with finite partial maps defined on what we call C-closed sets, and we code the uniformization property into the ordering.

**Proof of Theorem 6.1.** Given a ladder system C, we say that D ⊆ ω$_1$ is C-closed if for any α ≠ β ∈ D ∩ lim ω$_1$, $C_α \cap C_β ⊆ D$. Let ⟨E⟩ denote the smallest C-closed superset of E.

We show that a finite set E has finite C-closure, and even more:

**Observation 6.4.** Suppose that C is a ladder system.

1. A finite set E have finite C-closure and max(E) = max E.

2. Any initial segment of a C-closed set is C-closed.

3. If D is C-closed and E is finite then

$$(D \cup E) \setminus (\max E + 1) = D \setminus (\max E + 1).$$

**Proof.** First, let E be finite and we show that E has C-closure by induction on ε = max E; we can assume that ε ∈ lim ω$_1$ otherwise $⟨E⟩ = ⟨E \cap ε⟩ \cup \{ε\}$. Let

$$E^* = (E \cap ε) \cup \bigcup_{α ∈ E} C_α \cap C_ε,$$

and note max $E^* < max E$ since E was finite and that $⟨E⟩ = ⟨E^*⟩ \cup \{ε\}$. So (E) must have size $|⟨E^*⟩| + 1$ which is finite by induction.

Now suppose that D is C-closed, let γ < ω$_1$ and α < β ∈ D ∩ γ. Then

$$C_α \cap C_β ⊆ D ∩ β ⊆ D ∩ γ,$$

so D ∩ γ is C-closed.

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17See [22] [23] for examples when the uniformization is done by countable approximations which allow certain fusion arguments.
Finally, we prove the last statement by induction on \( \delta = \max D \). Since \( D \cap \delta \) is \( C \)-closed, we can apply induction to see that
\[
( (D \cap \delta) \cup E ) \setminus (\max E + 1) = (D \cap \delta) \setminus (\max E + 1),
\]
and by adding \( \delta \) to both sides we get the desired equality.

Now, given a ladder system \( C \) and \( \omega \)-colouring \( f \), let \( P = P_\xi \) consist of all finite maps \( p: D^\xi \to \omega \) so that \( D^\xi \subset \omega_1 \) is \( C \)-closed. The extension in \( P \) is defined by \( q \leq p \) if \( q \supseteq p \) and
\[
\{ \xi \} \text{ for all } \alpha \in D^\alpha \cap \lim \omega_1 \text{ and all } \xi \in C_\alpha \cap D^\alpha \setminus D^\xi, \ q(\xi) = f_\alpha(\xi) .
\]

First, we prove a few simple facts on \( P \) that show how a generic filter gives an \( \omega_1 \)-uniformization of \( f \).

**Claim 6.5.** Suppose that \( G \subset P \) is generic and let \( \varphi = \cup G \).

1. \( D_\alpha = \{ q \in P : \alpha \in D^q \} \) is dense in \( P \) for all \( \alpha \in \omega_1 \); 
2. for any \( p \in P \) and \( \alpha \in D^p \), \( p \Vdash \varphi(\xi) = f_\alpha(\xi) \) for all \( \xi \in C_\alpha \setminus D^p \); 
3. \( \Vdash \text{dom } \varphi = \omega_1 \) and for all \( \alpha \in \lim \omega_1 \) and almost all \( \xi \in C_\alpha \), \( \Vdash \varphi(\xi) = f_\alpha(\xi) \).

**Proof.**

1. Given \( p \) and \( \alpha \in \omega_1 \setminus D^p \), we form \( D = \langle D^p \cup \{ \alpha \} \rangle \). This is a finite set and we need to show that there is some \( q \supseteq p \) in \( P \) so that \( \text{dom } q = D \) i.e., that the extension property (\( \dagger \)) is satisfied. To this end, note that for any \( \alpha \in D \setminus D^p \), there is at most one limit \( \alpha \in D^p \) so that \( \xi \in C_\alpha \). So, we can define \( q(\xi) = f_\alpha(\xi) \) if such an \( \alpha \) exists, and let \( q(\xi) = 0 \) otherwise. So \( q \in D_\alpha \) is an extension of \( p \).

2. is a simple consequence of (\( \dagger \)), and (\( \ddagger \)) follows from the genericity of \( G \) and the previous statements.

Finally, we prove that \( P \) is \( \text{ccc} \) and preserves Suslin trees simultaneously.\(^{15}\)

**Lemma 6.6.** \( P \) is \( \text{ccc} \) and for any Suslin tree \( R, \Vdash \check{R} \) is Suslin.

**Proof.** We can assume that \( R = (\omega_1, <_R) \). Suppose that \( p \Vdash \check{X} \subset R, |\check{X}| = R_1 \), and \( A = \{ q_\xi : \xi < \omega_1 \} \) are conditions below \( p \). We will find \( \xi < \zeta \) and a common extension \( q^* \) of \( q_\xi \) and \( q_\zeta \) so that \( q^* \Vdash \check{X} \) is not an antichain in \( R \).\(^{16}\)

First, by extending each \( q_\xi \), we can suppose that there are \( t_\xi \in R \) of height at least \( \xi \), so that \( q_\xi \Vdash t_\xi \in \check{X} \).

Now, take a continuous sequence of elementary submodels \( (M_n)_{n \leq \omega} \) of a large enough \( H(\Theta) \) so that \( M_0 \) contains all the relevant parameters (e.g. \( T, C, f, P, R, A, (t_\xi)_{\xi < \omega_1} \)). Fix some \( q_\xi \in A \setminus M_\omega \), and find a large enough \( n < \omega \) so that \( r = q_\xi \cap M_\omega \) is a subset of \( M_n \) (and so \( r \in M_n \) as well). Let
\[
E = \bigcup \{ C_\alpha \cap M_n : \alpha \in \text{dom}(q_\xi \setminus r) \cap \lim \omega_1 \},
\]
and note that \( E \subset M_n \) is finite, so \( E \in M_n \) and \( \varepsilon = \max E \in M_n \) as well.

Let \( D = \langle \text{dom}(q_\xi \cup E) \rangle \) and note that \( D \setminus (\varepsilon + 1) = \text{dom}(q \setminus r) \) by Claim 6.4.\(^{13}\) We can extend \( q_\xi \) to a condition \( \check{q}_\xi : D \to \omega \) in \( P \), and we let \( \check{r} = \check{q}_\xi \cap M_n \). Note that \( \check{r} \in M_n \).

\(^{15}\)Let us mention that the abstract framework in \( \text{[14]} \) for preserving Suslin trees unfortunately does not fit the uniformization posets.

\(^{16}\)The proof that \( R \) has no uncountable chains in the extension is completely analogous.
Claim 6.7. There is \( q \in M_n \cap A \) and an extension \( q_\xi \in M_n \) with \( \text{dom} \ q_\xi = D' \) such that

1. \( q_\xi \cap \bar{q}_\xi = r \),
2. the unique monotone \( \psi: D \to D' \) is an isomorphism between \( q_\xi \) and \( q_\xi \) which fixes \( D \cap D' = \text{dom} \ r \),
3. if \( \alpha' = \psi(\alpha) \) then \( C_\alpha \cap \varepsilon \subseteq C_{\alpha'} \) and \( f_\alpha(\xi) = f_{\alpha'}(\xi) \) for all \( \xi \in C_\alpha \cap \varepsilon \), and
4. \( t_\xi \leq t_\xi \).

Proof. Indeed, \( M_n \) contains the set \( I \subset \omega_1 \) of all \( \zeta \in \omega_1 \) such that \( q_\zeta \) has some extension \( \bar{q}_\zeta \) which extends \( r \) and is isomorphic to \( \bar{q}_\xi \) in the above sense. Now, for any \( \delta \in M_n \cap \omega_1 \), there is a \( \zeta \in I \) so that \( t_\xi | \delta \leq t_\xi \) witnessed by \( q_\zeta \). Since \( R \) was Suslin, there must be some \( \zeta \in I \cap M_n \) so that \( t_\xi \leq t_\xi \) by Fact 1.2.

Now, we can amalgamate \( q_\xi \) and \( q_\zeta \):

Claim 6.8. \( D \cup D' \) is \( C \)-closed, and \( q^* = q_\xi \cup q_\zeta \) is a condition in \( P \) extending both \( q_\xi \) and \( q_\zeta \).

Proof. First, let’s see why \( D \cup D' \) is \( C \)-closed: let \( \alpha < \beta \in D \cup D' \), and note that the interesting cases are when \( \beta \in D \setminus M_n \) and \( \alpha \in D' \setminus D \). We distinguish two cases: first, assume that \( \alpha = \psi(\beta) \). In this case, \( C_\beta \cap M_n \subset C_\alpha \), so any \( \xi \in C_\alpha \cap C_\beta \) is in \( E \) actually.

Second, if \( \alpha \neq \alpha' = \psi(\beta) \) then \( C_\alpha \cap C_{\alpha'} = C_\alpha \cap C_{\alpha'} \) and the latter is a subset of \( D \).

Finally, the definition of \( E \) and the choice of \( \bar{q}_\xi \) made sure that no element of \( \text{dom} q^* \setminus \text{dom} \bar{q}_\xi \) is at critical levels given by \( C_\alpha \) for some \( \alpha \in D \setminus M_n \).

Hence, \( q^* \) is a common extension of both \( q_\xi \) and \( q_\zeta \), forcing that \( X \) is not an antichain.

Finally, the theorem is proved.

Remark. One can prove that even finite support iterations of posets of the form \( P \) preserve Suslin trees, and so we can prove the above consistency result together with arbitrary large values of the continuum.

7. Closing remarks and open problems

Hopefully, our exposition convinced the reader that there is a diverse theory behind Definition 1.1 well worth studying in detail. At the same time, we believe that our proofs demonstrated rather different applications of a wide spectrum of guessing principles, starting from the weakest of weak diamonds \( (2^{\aleph_0} < 2^{\aleph_0}) \), and length continuum diagonalization arguments in ZFC, to one of the strongest assumptions, \( \diamondsuit^+ \).

In this last section, we collect the still unsolved questions that occurred throughout the paper and say a few words on uniformizations for Kurepa trees.

Various problems. First, it is natural to ask how crucial it was in Theorem 2.3 that all antichains are countable.

Problem 7.1. Suppose CH holds. Does \( \text{cUnif}_{\omega}(T, C) \) for some \( C \) imply the existence of stationary antichains in \( T \)?
In Section 6, we cited [15, Theorem 6.2] stating that a Suslin tree always forces $\neg \text{cUnif}_2(\omega_1, C)$. It seems unclear if that argument can be extended to show the existence of non $T$-uniformizable colourings.

**Problem 7.2.** Suppose $S$ is a Suslin tree and $C$ and $T$ are ($S$-names for) a ladder system and Aronszajn tree. Does $S$ force $\neg \text{Unif}_2(T; C)$ or even $\neg \text{cUnif}_2(T; C)$?

Upon reading Theorem 5.1, it is natural to consider the smallest size $\lambda^*$ of a family $F$ of ladder system colourings such that there is no single Aronszajn $T$ such that any $f \in F$ has a full $T$-uniformization. Note that under MA$_{\aleph_1}$ no such family exists since all ladder system colourings have an $\omega_1$-uniformization. However, if $\lambda^*$ exists then it is at least $\aleph_2$ by Theorem 5.1. Moreover, using the ideas of Theorem 2.10 one can produce models where $\aleph_2 \leq \lambda^* < 2^{\aleph_1}$.

**Problem 7.3.** Is it consistent that $\lambda^*$ exists and is bigger than $\aleph_2$?

The next problem is inspired by Corollary 4.3 and concerns the classical $\omega_1$-uniformization theory.

**Problem 7.4.** Is it consistent that $c\text{Unif}_\omega(\omega_1, C)$ holds but $\text{Unif}_2(\omega_1, C)$ fails for some ladder system $C$?

We conjecture that the answer is yes, and one might start by looking at the [24] and [23] where similar questions are studied. In fact, if the ladder system $C = (C_\alpha)_{\alpha \in I}$ is only defined on a stationary, co-stationary set $I \subset \omega_1$ then the conjunction of CH plus $\text{Unif}_2(\omega_1, C)$ and $\neg \text{Unif}_\omega(\omega_1, C)$ are consistent [23].

Finally, it would be interesting to look at ladder systems and trees on $\aleph_2$, and to study uniformizations there. In the classical setting of uniformization on $\omega_2$, this was initiated in [26, Appendix 3]. We do not know whether the uniformization property here also gives consequences on minimal linear orders of size $\aleph_2$, but it is certainly an intriguing future direction for research.

**Kurepa trees.** We did overlook some well studied class of trees: Canadian trees, that is, $\aleph_1$-trees with at least $\aleph_2$ uncountable branches, and Kurepa trees, which are $\aleph_1$-trees with $2^{\aleph_1}$ uncountable branches. If we take a model of CH with a Canadian/Kurepa tree $T$ and apply Corollary 5.2 then in the extension any $\omega$-colouring of a ladder system $C$ will have a full $T$-uniformization and $T$ remains Canadian/Kurepa (since no cardinals are collapsed).

Recall that that if the weak diamond $\Phi^2_n$ holds then there is a colouring $f$ of any ladder system $C$ without an $\omega_1$-uniformization. So, if $\text{Unif}_\omega(T; C)$ holds then any $T$-uniformization $\varphi$ of this particular $f$ must be defined on an Aronszajn subtree of $T$. In particular, $T$ must have Aronszajn subtrees. Of course, we can make $\text{Unif}_\omega(T; C)$ hold for a Kurepa tree artificially by requiring that $T$ has an Aronszajn subtree $T_0$ that satisfies $\text{Unif}_\omega(T_0; C)$.

Now, it is possible to construct Kurepa trees $T$ without Aronszajn subtrees, say from $V = L$ or strong diamond assumptions [6]. So the next question arises.

**Problem 7.5.** Is there, consistently, a Kurepa tree $T$ without Aronszajn subtrees and a ladder system $C$ so that $\text{Unif}_\omega(T; C)$ holds?

So CH must fail in a model witnessing a positive solution to the above problem. A stronger question might ask the following.

**Problem 7.6.** Is there a model of MA$_{\aleph_1}$ with a Kurepa tree that has no Aronszajn subtrees?
A natural approach would be to take a model of Martin’s axiom and \( c = \aleph_2 \) (or even PFA) and introduce the Kurepa-tree by a \( \sigma \)-closed forcing \(^{32}\) hoping that one can preserve \( \text{MA}_{\aleph_1} \).

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\(^{32}\)We would be surprised if this question was not answered already, however we could not find a reference.
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