On spatial Fourier spectrum of rogue wave breathers

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1 INTRODUCTION

The study of rogue waves has been of great interest ever since the discovery of soliton solutions of some nonlinear evolution equations. In particular, the focusing nonlinear Schrödinger (NLS) equation and its family of exact analytical solutions of solitons on a nonvanishing background, also known as breathers, have been proposed as prototypes for rogue wave modeling. The literature is far from exhaustive in disseminating rogue waves study based on the NLS model and its corresponding breather solutions, for which the phenomena are observed and encounter applications in hydrodynamics, optical fibers, and photonic crystals among others.1,2

Consider the focusing NLS equation in (1 + 1)-dimension written in a canonical form3,4:

\[ iq_t + q_{xx} + 2|q|^2q = 0, \quad (x, t) \in \mathbb{R}^2, \quad q(x, t) \in \mathbb{C}. \] (1)

In surface gravity waves, the variables \((x, t)\) represent spatial and temporal quantities, respectively, while in nonlinear optics, \(t\) denotes the transversal pulse propagation in space and \(x\) designates the time variable. Unless otherwise indicated, we adopt the interpretation for water waves for the rest of the article. The complex-valued amplitude \(q(x, t)\) describes an envelope of the corresponding wave packet profile \(\eta\), where usually given by the relationship \(\eta(\hat{x}, \hat{t}) = \text{Re} \{ q(x, t)e^{i(k\hat{x} - \omega\hat{t})} \}\), where \(\text{Re}\) denotes the real part of the expression, \(x = \epsilon(\hat{x} - c_k \hat{t}), \quad t = \epsilon^2 \hat{t}, \quad \epsilon \ll 1, \quad c_k\) is the group velocity, and the wavenumber \(k\) and wave frequency \(\omega\) are related by the linear dispersion relationship for the corresponding medium.

The simplest nontrivial solution is space-independent, \(q(x, t) = q_0(t) = e^{2it}\), known as the continuous or plane-wave solution. The simplest solution with dependence on both variables is \(q(x, t) = q_0(t)\text{sech}(x)\). It is often called the fundamental or bright soliton in nonlinear optics and can be derived by means of the inverse scattering transform (IST) or Darboux transformation.5–8 The so-called dark soliton \(q_0(t)\text{tanh}(x)\) is an exact solution of the defocusing NLS equation, where the product of the dispersive and nonlinear coefficients is negative.9–11

We derive exact analytical expressions for the spatial Fourier spectrum of the soliton family on a constant background. Also known as breathers, these solitons are exact solutions of the nonlinear Schrödinger equation and are considered as prototypes for rogue wave models. Depending on the periodicity in the spatial-temporal domain, the characteristics in the wavenumber-temporal domain may feature either a continuous or discrete spectrum.

KEYWORDS
breathers, nonlinear Schrödinger equation, rogue waves, solitons on a constant background, spatial Fourier spectrum

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In particular, the NLS equation admits the following family of solitons on a constant background, and this is where our interest lies:

\[ q_K(x, t) = q_0(t) \left( \frac{\mu^2 \cos(\rho t) + i \mu \sin(\rho t)}{\rho \cosh(\mu x) - 2 \mu \cos(\rho t) - 1} \right), \quad \rho = \mu \sqrt{4 + \mu^2} \]  

(2)

\[ q_P(x, t) = q_0(t) \left( \frac{4(1 + 4it)}{1 + (4it)^2 + (2x)^2} - 1 \right) \]  

(3)

\[ q_A(x, t) = q_0(t) \left( \frac{\kappa^2 \cosh(\sigma t) + i \kappa \sinh(\sigma t)}{2 \kappa \cosh(\sigma t) - (2x)^2 \cos(\sigma t) - \sigma \cos(\kappa x)} - 1 \right), \quad \sigma = \kappa \sqrt{4 - \kappa^2}. \]  

(4)

In our context, \( q_K, q_P, \) and \( q_A \) denote the Kuznetsov–Ma (KM), Peregrine/rational, and Akhmediev breathers, respectively.\(^{12-18}\) The chosen order is more historical rather than its fame or simplicity, even though the trajectories of breather dynamics are depicted in the complex plane for various parameter values.\(^{19}\) Although the discussion of these breathers in the spatial-temporal domain has reached maturity, this article fills a gap in covering their features in the wavenumber-temporal domain by observing their spectra. Note that this “spectrum” should not be confused with the spectrum from the IST but rather the physical spectrum that can be calculated by switching from the space domain into the wavenumber domain by means of the spatial Fourier transform.

Recently, *Frontiers of Physics* has published a sequence of topical research articles describing the latest progress of NLS breathers to commemorate the 10th anniversary of the observation of the Peregrine soliton in nonlinear media, including optical fibers, water waves, and plasma physics.\(^{20-22}\) In particular, the stability properties in energy spaces of these three important NLS breathers have been reviewed by Alejo et al.\(^{23}\) By characterizing the spectral properties of each breather, they demonstrated that all breathers are unstable in the Lyapunov sense.

We begin with the following well-known definitions on Fourier transforms, Fourier series, and spectral decomposition.\(^{24}\)

**Definition 1** (Spatial Fourier transform). Let \( f(x, t) \) be a square-integrable function on the spatial real line, then it can be represented in a dual spatial-wavenumber \((x, k)\) space by fixing the time variable \( t \) as integral transforms

\[ f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k, t)e^{ikx} dk, \]  

(5)

where \( \hat{f}(k, t) \) is the spatial nonunitary Fourier transform written in the terms of angular wavenumber \( k \) and is defined by

\[ \hat{f}(k, t) = \int_{-\infty}^{\infty} f(x, t)e^{-ikx} dx. \]  

(6)

**Definition 2** (Temporal Fourier transform). Alternatively, for a square-integrable function \( f(x, t) \) on the temporal real line, it can also be expressed at a fixed location \( x \) in space as integral transforms between dual temporal-frequency \((t, \omega)\) domain

\[ f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x, \omega)e^{i\omega t} d\omega, \]  

(7)

where \( \hat{f}(x, \omega) \) is the temporal nonunitary Fourier transform written in the terms of angular frequency \( \omega \) and is defined by

\[ \hat{f}(x, \omega) = \int_{-\infty}^{\infty} f(x, t)e^{-i\omega t} dt. \]  

(8)

In this case, the relationship between (5) and (6) is between space and wavenumber domains, where the spatial and wavenumber variables become the transform variables for a fixed time. On the other hand, for (7) and (8), they relate between the time and frequency domains at a fixed location in space by letting time and frequency as the transform variables.\(^{25}\)

**Definition 3** (Spectral decomposition and complex spatial Fourier series). For a spatially periodic function \( f(x, t) \) with a period \( L \neq 0 \), that is, \( f(x, t) = f(x + L, t) \), for all \( x \in \mathbb{R} \), the spectral decomposition of \( f \) is given as a discrete summation of the harmonic oscillation \( e^{ikx} \):
\[ f(x, t) = \sum_{n=-\infty}^{\infty} \hat{f}_n(t) e^{ik_n x} \]  

(9)

where \( \hat{f}_n(t) = \hat{f}(k_n, t) \) are Fourier coefficients of the complex spatial Fourier series for \( f(x, t) \):

\[ \hat{f}_n(t) = \frac{1}{2L} \int_{-L}^{L} f(x, t) e^{-ik_n x} \, dx. \]  

(10)

**Definition 4** (Complex temporal Fourier series). For a temporally periodic function \( f(x, t) \) with a period \( T \neq 0 \), that is, \( f(x, t) = f(x, t + T), t > 0 \), the spectral decomposition of \( f \) is given as a discrete summation of the harmonic oscillation \( e^{i\omega_n x} \):

\[ f(x, t) = \sum_{n=-\infty}^{\infty} \hat{f}_{\omega_n}(x, \omega_n) e^{i\omega_n t} \]  

(11)

where \( \hat{f}_{\omega_n}(x) = \hat{f}(x, \omega_n) \) are Fourier coefficients of the complex temporal Fourier series for \( f(x, t) \):

\[ \hat{f}_{\omega_n}(x) = \frac{1}{2T} \int_{-T}^{T} f(x, t) e^{-i\omega_n t} \, dt. \]  

(12)

Throughout the rest of this article, unless otherwise mentioned, we focus our discussion on the spatial Fourier transform and (complex) Fourier series for the family of breather soliton solutions of the NLS Equation (1), that is, (5), (6), (9), and (10). Before moving on to the next section, we end this introduction by stating the spatial Fourier transform of a unitary constant function as the following lemma.

**Lemma 1.** The Fourier transform of the constant function \( f(x) = 1 \) is given by the Dirac delta function \( \delta(k) \), generally in the sense of distributions rather than measures, that is,

\[ \hat{f}(k) = \int_{-\infty}^{\infty} 1 e^{-ikx} \, dx = 2\pi \delta(k). \]

The article is organized as follows. Section 2 covers the spatial Fourier transform for the breather solitons. It covers the three breathers mentioned earlier and provides analytical expressions for the corresponding spectra. We also illustrate the spectra by plotting them. Section 4 concludes our discussion.

## 2 Spatial Fourier Spectrum

In this section, we derive the spatial Fourier spectrum for the breather solitons of the NLS equation. For the KM breather and Peregrine soliton, the spectra are continuous in wavenumber \( k \), while for the Akhmediev breather, the spectrum is discrete due to its spatial periodicity.

### 2.1 KM breather

Before proving the theorem for the spectrum of the KM breather, we need to verify the following lemma.

**Lemma 2.** For \( k \neq 0, \mu > 0 \), and \( b > |c| > 0 \), we have the following definite integral (Formula 3.983.1(b) in Gradshteyn and Ryzhik):

\[ \int_{-\infty}^{\infty} \frac{\cos(kx)}{b \cosh(\mu x) + c} \, dx = 2\pi \frac{\sinh \left( \frac{k}{\mu} \arccos \frac{c}{b} \right)}{\mu \sqrt{b^2 - c^2} \sinh \frac{k}{\mu} \pi}. \]  

(13)

**Proof.** We consider the complex-valued function

\[ f(z) = \frac{e^{ikz}}{b \cosh(\mu z) + c} \]
and the rectangular contour γ_K with corners at ±R and ±R + 2πi/μ, for R > 0. See Figure 1. To find the poles and residues of f, we need to solve \( b \cosh(\mu z) + c = 0, \) \( z \in \mathbb{C} \). This gives

\[
\frac{b}{2} (e^{\mu z} + 1) + ce^{\mu z} = 0
\]

\[
e^{\mu z} = -c \pm \sqrt{c^2 - b^2} \frac{b}{b} = -c \pm i \sqrt{1 - c^2 \frac{b^2}{b^2}} = \cos(\pi \mp \theta) \pm i \sin(\pi \mp \theta)
\]

\[
z = \ln 1 \pm \frac{i}{\mu} (\pi \mp \theta + 2n\pi), \quad n \in \mathbb{Z}
\]

where for \( 0 \leq \theta < \pi/2 \), we take

\[
\cos \theta = \frac{c}{b}, \quad \sin \theta = \sqrt{1 - \frac{c^2}{b^2}}, \quad \tan \theta = \frac{\sqrt{b^2 - c^2}}{c} > 0.
\]

Inside the rectangular contour, the function has two simple poles at the following points:

\[
z_2 = \frac{i}{\mu} (\pi - \theta) \quad \text{and} \quad z_3 = \frac{i}{\mu} (\pi + \theta)
\]

that lie on the imaginary axis. The corresponding residue for each of these poles is given as follows:

\[
\text{res}( f, z_2) = \frac{e^{-\frac{k}{\mu} (\pi - \theta)}}{b\mu \sinh i(\pi - \theta)} = \frac{e^{-\frac{k}{\mu} (\pi - \theta)}}{ib\mu \sin(\pi - \theta)} = \frac{e^{-\frac{k}{\mu} (\pi - \theta)}}{i\mu \sqrt{b^2 - c^2}}
\]

\[
\text{res}( f, z_3) = \frac{e^{-\frac{k}{\mu} (\pi + \theta)}}{b\mu \sinh i(\pi + \theta)} = \frac{e^{-\frac{k}{\mu} (\pi + \theta)}}{ib\mu \sin(\pi + \theta)} = -\frac{e^{-\frac{k}{\mu} (\pi + \theta)}}{i\mu \sqrt{b^2 - c^2}}.
\]

Using the residue theorem, we know that

\[
\int_{\gamma_K} f(z) \, dz = 2\pi i \sum_{n=2}^{3} \text{res}( f, z_n) = \frac{2\pi}{\mu \sqrt{b^2 - c^2}} \left[ e^{-\frac{k}{\mu} (\pi - \theta)} - e^{-\frac{k}{\mu} (\pi + \theta)} \right]
\]

\[
= \frac{2\pi e^{-\frac{k}{\mu} \pi}}{\mu \sqrt{b^2 - c^2}} \left[ e^{\frac{k}{\mu} \theta} - e^{-\frac{k}{\mu} \theta} \right] = \frac{2\pi \cdot 2e^{-\frac{k}{\mu} \pi}}{\mu \sqrt{b^2 - c^2}} \sinh \left( \frac{k}{\mu} \theta \right),
\]
On the other hand, expressing the integral on the left-hand side of (14) as a combination of line integrals yields

\[
\int_{\gamma_k} f(z)\,dz = \int_{-R}^{R} \frac{e^{ikx}\,dx}{b\cosh(\mu x) + c} + \int_{0}^{2\pi} \frac{e^{ik(R+iy)}\,id\,y}{b\cosh(\mu + iy) + c}
\]

\[
- \int_{-R}^{R} \frac{e^{ik(x+2\pi/i\mu)}\,dx}{b\cosh(\mu x) + c} - \int_{0}^{2\pi} \frac{e^{ik(-R+iy)}\,id\,y}{b\cosh(-R + iy) + c}
\]

\[
= (1 - e^{-2\frac{k}{\mu}}) \int_{-R}^{R} \frac{e^{ikx}}{b\cosh(\mu x) + c} \,dx + i \int_{0}^{2\pi} \frac{e^{ikRe^{-iy}}\,d\,y}{b\cosh(\mu R + iy) + c} - i \int_{0}^{2\pi} \frac{e^{-ikRe^{-iy}}\,d\,y}{b\cosh(\mu R - iy) + c}
\]

(15)

since \(\cosh(\mu x + 2\pi i) = \cosh(\mu x) \cos(2\pi) + i \sinh(\mu x) \sin(2\pi) = \cosh(\mu x)\). Furthermore, \(|\cosh(\mu \pm iy)|^2 = \cosh^2(\mu R) - \sinh^2(\mu R) = \cosh^2(\mu R) - 1 \geq \cosh^2(\mu R), |b\cosh(\mu \pm iy)| \geq b\cosh(\mu R),\) and for sufficiently large \(R, |b\cosh(\mu \pm iy) + c| \geq b\cosh(\mu R) - c\). Hence,

\[
\left| i \int_{0}^{2\pi} \frac{e^{ikR\cosh(\mu y) + c}}{b\cosh(\mu R + iy) + c} \,d\,y \right| \leq \int_{0}^{2\pi} \frac{e^{ikR\cosh(\mu y) + c}}{b\cosh(\mu R + iy) + c} \,d\,y \leq \int_{0}^{2\pi} \frac{e^{-ky}}{b\cosh(\mu R - c)} \,d\,y
\]

\[
\leq \frac{1 - e^{-2k\mu}}{k(b\cosh(\mu R - c))}
\]

which tends to 0 as \(R \to \infty\). Similarly, for sufficiently large \(R\)

\[
\left| i \int_{0}^{2\pi} \frac{e^{-ikR\cosh(\mu y) + c}}{b\cosh(\mu R - iy) + c} \,d\,y \right| \leq \int_{0}^{2\pi} \frac{e^{-ikR\cosh(\mu y) + c}}{b\cosh(\mu R - iy) + c} \,d\,y \leq \int_{0}^{2\pi} \frac{e^{-ky}}{b\cosh(\mu R - c)} \,d\,y
\]

\[
\leq \frac{1 - e^{-2k\mu}}{k(b\cosh(\mu R - c))}
\]

which again tends to 0 as \(R \to \infty\). Hence, by letting \(R \to \infty\), we observe from (14) and (15) that

\[
(1 - e^{-2\frac{k}{\mu}}) \int_{-\infty}^{\infty} \frac{e^{ikx}}{b\cosh(\mu x) + c} \,dx = \frac{2\pi \cdot (2e^{-\frac{1}{\mu}})}{\mu \sqrt{b^2 - c^2}} \sinh \left( \frac{k}{\mu} \right).
\]

(16)

By taking the real part of the integral on the left-hand side of (16) and pulling out the factor \(2e^{-\frac{1}{\mu}}\) from the product in front of it, we obtain

\[
\int_{-\infty}^{\infty} \cos(kx)\,dx = \frac{2\pi \sinh \left( \frac{k}{\mu} \right)}{\mu \sqrt{b^2 - c^2}} \sinh \left( \frac{k}{\mu} \right) = \frac{2\pi \arccos \left( \frac{c}{b} \right)}{\mu \sqrt{b^2 - c^2}} \sinh \left( \frac{k}{\mu} \right).
\]

This completes the proof.

\[
\hat{q}_k(t) = \begin{cases} 2\pi e^{2it} \left\{ \frac{\mu^3 \cos(\rho t) + i\mu \sin(\rho t)}{\mu^3 \sqrt{\mu^2 + \sin^2(\rho t)}} \sin \left( \frac{2\mu \arccos \left(-\frac{c}{b}\right)}{\mu \sin(\rho t)} \right) \right\} - \delta(k), & \text{for } k \neq 0 \\ 2e^{2it} \left\{ \frac{\mu^3 \cos(\rho t) + i\mu \sin(\rho t)}{\mu^3 \sqrt{\mu^2 + \sin^2(\rho t)}} \arccos \left(-\frac{2\mu}{\rho} \cos(\rho t)\right) \right\} - \pi \delta(0), & \text{for } k = 0. \end{cases}
\]

\[
\textbf{Theorem 1.} \text{ The spatial Fourier spectrum of the KM breather is given by}
\]

(17)
Proof. Using Definition 1 and the fact that \( q_K(x, t) \) is an even function with respect to \( x \), the spatial Fourier transform of the KM breather \( q_K \) can be written as follows:

\[
\hat{q}_K(k, t) = \int_{-\infty}^{\infty} e^{2i\mu} \left( \frac{\mu^3 \cos(\rho t) + i \mu \rho \sin(\rho t)}{\rho \cosh(\mu x) - 2 \mu \cos(\rho t)} \right) e^{-ikx} dx - \int_{-\infty}^{\infty} e^{-ikx} dx
\]

\[
= e^{2i\mu} \left[ \frac{\mu^3 \cos(\rho t) + i \mu \rho \sin(\rho t)}{\rho \cosh(\mu x) - 2 \mu \cos(\rho t)} \right] \int_{-\infty}^{\infty} \left( \frac{\cos(kx)}{\rho \cosh(\mu x) - 2 \mu \cos(\rho t)} \right) dx - \int_{-\infty}^{\infty} e^{-ikx} dx.
\]

Using Lemma 1, taking \( b = \rho \) and \( c = -2\mu \cos(\rho t) \) in Lemma 2, and because \( \sqrt{b^2 - c^2} = \mu \sqrt{\mu^2 + \sin^2(\rho t)} \) for \( \mu > 0 \), we obtain

\[
\hat{q}_K(k, t) = 2\pi e^{2i\mu} \left\{ \frac{\mu^3 \cos(\rho t) + i \mu \rho \sin(\rho t)}{\mu^2 \sqrt{\mu^2 + \sin^2(\rho t)}} \right\} \frac{\sin \left( \frac{k}{\mu} \arccos \left( \frac{-2\mu \cos(\rho t)}{\rho} \right) \right)}{\sinh \left( \frac{k}{\mu} \pi \right)} - \delta(k), \quad \text{for } k \neq 0.
\]

Taking the limit of the expressions in Lemma 2 as \( k \to 0 \) yields

\[
\int_{-\infty}^{\infty} \frac{dx}{\rho \cosh(\mu x) - 2 \mu \cos(\rho t)} = \frac{2}{\mu^2 \sqrt{\mu^2 + \sin^2(\rho t)}} \arccos \left( \frac{-2\mu}{\rho} \cos(\rho t) \right).
\]

Hence,

\[
\hat{q}_K(0, t) = 2e^{2i\mu} \left\{ \frac{\mu^3 \cos(\rho t) + i \mu \rho \sin(\rho t)}{\mu^2 \sqrt{\mu^2 + \sin^2(\rho t)}} \right\} \arccos \left( \frac{-2\mu}{\rho} \cos(\rho t) \right) - \pi\delta(0).
\]

This completes the proof. \( \square \)

Figure 2 displays the spatial Fourier spectrum of the KM breather as a function of time for several values of wavenumber \( k = k_0 \). All panels correspond to the parameter value \( \mu = 1 \). The real part, imaginary part, and modulus of the spectrum are depicted as dashed blue, dotted red, and solid black curves, respectively. We observe that all spectrum components are periodic with respect to time. For small \( k \), the spectrum modulus reaches local maximum values of larger than six while the local minimum values are also positive and larger than one. As the values of \( k \) increase, both local maximum and minimum values of the spectrum decrease, where the latter could vanish for particular temporal values. Furthermore, the spectrum modulus feature a local minimum that is sandwiched between local maxima for \( k \leq 1 \) in the neighborhood when \( t = 0 \). This feature starts to disappear as the wavenumber increases and eventually, only local maxima are visible.

Figure 3 illustrates the spatial Fourier spectrum of the KM breather as a function of the wavenumber for several values of time \( t = t_0 \). We also take the breather parameter \( \mu = 1 \) in all cases. The real and imaginary parts of the spectrum are given by dashed blue and dotted red curves, respectively. For the former, the presence of the Dirac delta function is noticeable when \( k = 0 \), where the curves go to infinity negatively. The spectrum reduces to a real-valued function when \( t = 0 \) and its corresponding periodic time-scale values. Both spectrum components vanish as \( k \to \pm \infty \). While the imaginary part always reaches a local maximum at \( k = 0 \) for \( t \neq 0 \), the real part does not always behave like that. As shown at the bottom right panel, the real part may reach a local minimum in a limiting sense when \( k \to 0 \) at \( t_0 = \pi/(4\rho) \).

2.2 | Peregrine soliton

Before deriving the spectrum expression for the Peregrine soliton, we state and verify the following lemma.

**Lemma 3.** For \( a > 0 \) and \( k \in \mathbb{R} \), we have the following integral:

\[
\int_{-\infty}^{\infty} \frac{\cos(kx)}{a + (2x)^2} \, dx = \frac{\pi}{2} \frac{e^{-\frac{a}{2}}}{\sqrt{a}}. \tag{18}
\]
FIGURE 2  The spatial Fourier spectrum evolution for the Kuznetsov–Ma (KM) breather $\hat{q}_K(k_0, t)$ as a function of time for $\mu = 1, k_0 = 0.1$ (top left), $k_0 = 0.5$ (top right), $k_0 = 1$ (bottom left), and $k_0 = 2$ (bottom right). The real part (dashed blue), the imaginary part (dotted red), and the modulus (solid black) are depicted separately [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 3  The spatial Fourier spectrum of the Kuznetsov–Ma (KM) breather $\hat{q}_K(k, t_0)$ as a function of the wavenumber $k$ for $\mu = 1$ when $t_0 = 0$ (top left), $t_0 = \pi/(8\rho)$ (top right), $t_0 = \pi/(6\rho)$ (bottom left), and $t_0 = \pi/(4\rho)$ (bottom right). The real and imaginary parts are given by dashed blue and dotted red plots, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

**Proof.** We show the proof using the semicircular contour theorem from complex analysis, for example, Theorem 9.1 in Howie\textsuperscript{27} (on pages 154–155). Let

$$f(z) = \frac{e^{i|\lambda|z}}{a + (2z)^2}$$

be a complex-valued function and meromorphic in the upper half-plane. We observe that $f$ has no poles on the real axis, and it has only two poles at $z = i\sqrt{a}/2$ and $z = -i\sqrt{a}/2$. Only the first of these is in the upper half-plane. We
calculate that
\[
\text{res}\left( f, i\frac{\sqrt{a}}{2} \right) = \frac{e^{-\frac{a}{4}} \sqrt{a}}{4i\sqrt{a}}.
\]

In the upper half-plane, for all sufficiently large \(|z|\), we have
\[
|zf(z)| = \left| \frac{ze^{i|k|z}}{a + (2z)^2} \right| = \left| \frac{ze^{i|k|x} e^{-|k|y}}{4z^2 + a} \right| \leq \frac{|z|}{4|z|^2 - a}.
\]

We have adopted the relationship \(\text{Re}(z) = x\) and \(\text{Im}(z) = y\) and the fact that \(e^{i|k|x} = 1\) and \(|e^{-|k|y}| \leq 1\) for \(y \geq 0\). It follows that \(|zf(z)|\) tends uniformly to 0 as \(|z| \to \infty\) in the upper half-plane. Let \(\gamma_P : [-R, R] \cup \{z : |z| = R, \text{Im} z \geq 0\}\), traversed in the positive orientation with \(R > \sqrt{a}/2\) [Colour figure can be viewed at wileyonlinelibrary.com], then
\[
(PV) \int_{-\infty}^{\infty} f(x) \, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx = \int_{\gamma_P} f(z) \, dz - \lim_{R \to \infty} \int_{0}^{\pi} f(Re^{i\theta})iRe^{i\theta} \, d\theta,
\]
where \(PV\) stands for the Cauchy principal value of the improper integral. Indeed, from the condition of \(|zf(z)|\) uniformly in the upper half-plane as \(|z| \to \infty\), it implies that for all \(\epsilon > 0\), there exists \(S > 0\) such that \(|zf(z)| < \epsilon\) for all \(z\) in the upper half-plane such that \(|z| > S\). Consequently, for all \(R > S\),
\[
\left| \int_{0}^{\pi} f(Re^{i\theta})iRe^{i\theta} \, d\theta \right| \leq \int_{0}^{\pi} \left| \frac{e^{i|k|Re^{i\theta}}iRe^{i\theta}}{a + 4R^2e^{2i\theta}} \right| \, d\theta \leq \frac{\pi R}{4R^2 - a} < \pi \epsilon,
\]
and thus,
\[
\int_{0}^{\pi} f(Re^{i\theta})iRe^{i\theta} \, d\theta \to 0 \quad \text{as } R \to \infty.
\]

Since there exists a constant \(\widehat{S} > 0\) such that \(|f(x)| \leq \widehat{S}/x^2\) for large \(|x|\), then \(\int_{-\infty}^{\infty} f(x) \, dx\) exists and equals to its Cauchy principal value. Hence, we can dispense with the principal value and by applying the residue theorem, we deduce that
\[
\int_{-\infty}^{\infty} \frac{e^{i|k|x}}{a + (2x)^2} \, dx = \int_{-\infty}^{\infty} \frac{\cos(kx) + i \sin(|k|x)}{a + (2x)^2} \, dx = 2\pi i \text{ res}\left( f, i\frac{\sqrt{a}}{2} \right) = \frac{\pi e^{-\frac{a}{4}} \sqrt{a}}{2\sqrt{a}}.
\]
Equating the real parts gives the desired result, and the proof is completed. \(\square\)

**Theorem 2.** The spatial Fourier spectrum of the Peregrine soliton \(q_P(x, t)\) is given by
\[
\hat{q}_P(k, t) = 2\pi e^{2it} \left( \frac{1 + 4it}{\sqrt{1 + 4t^2}} e^{-\frac{1}{2} \sqrt{1 + (4t)^2}} \delta(k) - \delta(k) \right).
\]
Proof. Using Definition (1) and the fact that \( q_P(x, t) \) is an even function with respect to \( x \), we can write the spatial Fourier transform \( \hat{q}_P(k, t) \) as follows:

\[
\hat{q}_P(k, t) = \int_{-\infty}^{\infty} e^{2it} \left( \frac{4(1+4it)}{1+(4t)^2+(2x)^2} \right) e^{-ikx} dx - \int_{-\infty}^{\infty} 1 e^{-ikx} dx
\]

\[
= 4e^{2it}(1+4it) \int_{-\infty}^{\infty} \frac{\cos(kx)}{1+(4t)^2+(2x)^2} dx - \int_{-\infty}^{\infty} e^{-ikx} dx.
\]

Using Lemmas 1 and 3, we obtain the desired result:

\[
\hat{q}_P(k, t) = 4e^{2it}(1+4it) \left( \frac{\pi}{2\sqrt{1+(4t)^2}} e^{-\frac{1}{2} \sqrt{1+(4t)^2}} \right) - 2\pi \delta(k)
\]

\[
= 2\pi e^{2it} \left( \frac{1+4it}{\sqrt{1+(4t)^2}} e^{-\frac{1}{2} \sqrt{1+(4t)^2}} - \delta(k) \right).
\]

This completes the proof of the theorem. \( \square \)

Figure 5 shows the spatial Fourier spectrum of the Peregrine breather as a function of time for various values of wavenumber \( k = k_0 \). The real part, imaginary part, and modulus of the spectrum are depicted in dashed blue, dotted red, and solid black curves, respectively. All spectrum components tend to vanish as \( t \to \pm\infty \). Both the real part and modulus of the spectrum reach local maxima at \( t = 0 \), while the imaginary part takes negative and positive values for \( t < 0 \) and \( t > 0 \), respectively. It reaches local minimum and maximum for some values \( t \) not too far away from \( t = 0 \). The value of the local maxima for the spectrum modulus decreases as the value of the wavenumber increases. For small wavenumber, the local maximum reaches the value of six as \( k \to 0 \).

Figure 6 features the spatial Fourier spectrum of the Peregrine breather as a function of the wavenumber for several values of time \( t = t_0 \). The real and imaginary parts of the spectrum are plotted as dashed blue and dotted red curves, respectively. Similar to the previous case, the appearance of the Dirac delta function is visible at \( k = 0 \) as the curve extends infinitely negative. The imaginary part vanishes when \( t_0 = 0 \), and thus, the spectrum reduces to a real-valued function. For \( t > 0 \), both real and imaginary parts tend to vanish as \( k \to \pm\infty \). They also reach local maxima at \( k = 0 \), and depending
on the values of $k$, the local maxima for the real part can be larger, equal, or smaller than the ones for the imaginary part. The largest of these quantities take the value of slightly larger than six at $k = 0$.

2.3 Akhmediev breather

Lemma 4. For $0 < |a| < b$, the following trigonometric integral holds; compare Formula 3.613.1 in Gradshteyn and Ryzhik:

$$
\int_0^{2\pi} \frac{\cos(n \xi)}{b - a \cos \xi} d\xi = \frac{2\pi}{\sqrt{b^2 - a^2}} \left( \frac{b - \sqrt{b^2 - a^2}}{a} \right)^n, \quad \text{for } n \in \mathbb{Z}.
$$

(20)

Proof. By performing the integration in the complex plane, we can write the integral as the real part of another integral expressed on the right-hand side of the following equation:

$$
\int_0^{2\pi} \frac{\cos(n \xi)}{b - a \cos \xi} d\xi = \text{Re} \left( \int_0^{2\pi} \frac{e^{in\xi}}{b - a \cos \xi} d\xi \right).
$$

Let $C : z = e^{i\xi}$, $0 \leq \xi \leq 2\pi$ be the unit circle in the complex plane, then $e^{in\xi} = z^n$, $\cos \xi = \frac{1}{2}(z + z^{-1})$, and $d\xi = -idz/z$. See Figure 7. Substituting these into the integral (20), we can express it as

$$
\int_0^{2\pi} \frac{\cos(n \xi)}{b - a \cos \xi} d\xi = 2i \oint_C \frac{z^n}{az^2 - 2bz + a} dz = 2i \oint_C \frac{z^n}{a(z - z_1)(z - z_2)} dz
$$

where

$$
z_1 = \frac{b + \sqrt{b^2 - a^2}}{a} \quad \text{and} \quad z_2 = \frac{b - \sqrt{b^2 - a^2}}{a}
$$

are the roots of the quadratic equation $az^2 - 2bz + a = 0$. The integrand has simple poles at $z = z_1$ and $z = z_2$. Observe that

$$
|z_1| = \frac{b + \sqrt{b^2 - a^2}}{a} > \frac{b}{a} > 1
$$
and thus, \( z_1 \) does not lie within the contour of \( C \). Furthermore, since \( z_1z_2 = a/a = 1 \), we deduce that \( |z_2| < 1 \). The residue at \( z = z_2 \) is given by

\[
\frac{z_2^n}{a(z_2 - z_1)} = \frac{1}{-2\sqrt{b^2 - a^2}} \left( \frac{b + \sqrt{b^2 - a^2}}{a} \right)^n.
\]

Using Cauchy's residue theorem, we obtain the desired trigonometric integral:

\[
\int_0^{2\pi} \cos(n\xi) \frac{d\xi}{b - a \cos \xi} = \frac{(2\pi i)(2l)}{-2\sqrt{b^2 - a^2}} \left( \frac{b + \sqrt{b^2 - a^2}}{a} \right)^n = \frac{2\pi}{\sqrt{b^2 - a^2}} \left( \frac{b - \sqrt{b^2 - a^2}}{a} \right)^n.
\]

This completes the proof. \( \square \)

**Corollary 1.** For \( 0 \leq |a| < 1 \), we have the following integral:

\[
\int_0^{2\pi} \frac{d\xi}{1 - a \cos \xi} = \frac{2\pi}{\sqrt{1 - a^2}}.
\]

**Proof.** By taking \( b = 1 \) and \( n = 0 \) in Lemma 4, that is, Equation (19), we obtain the result immediately. \( \square \)

In what follows, we state and prove the spatial Fourier spectrum for the Akhmediev breather. A similar expression of this spectrum is available in Akhmediev and Ankiewicz, but the authors did not provide any proof. In the meantime, a derivation proof using trigonometric integrals and series has been attempted by the author in an appendix of his PhD thesis.

**Theorem 3.** The spatial Fourier amplitude spectrum for the Akhmediev breather \( q_A(x, t) \) is given by

\[
\hat{q}_A^{(n)}(k, t) = \hat{q}_A^{(n)}(\kappa, t) \begin{cases} 
\left( \epsilon \left( \frac{k^3 \cosh(\kappa t) + i\kappa \sinh(\kappa t)}{\sqrt{4\kappa^2 \cosh^2(\kappa t)} - a^2} \right) - 1 \right) \cdot 
\left( \epsilon \left( \frac{2\kappa^2 \cosh(\kappa t) - \sqrt{4\kappa^2 \cosh^2(\kappa t)} - a^2}{\kappa} \right) \right) \quad & \text{for } n = 0 \\
\frac{1}{2\pi} \left( \frac{k^3 \cosh(\kappa t) + i\kappa \sinh(\kappa t)}{\sqrt{4\kappa^2 \cosh^2(\kappa t)} - a^2} \right) \quad & \text{for } n \in \mathbb{N}.
\end{cases}
\]

**Proof.** The Akhmediev breather is periodic in space with the period \( L = 2\pi/\kappa \). Furthermore, we consider the wavenumber values \( k_n \) at a discrete multiple of the modulation wavenumber \( \kappa \), that is, \( k_n = n\kappa \), \( n \in \mathbb{Z} \). Hence,

\[
\hat{q}_A^{(n)}(\kappa, t) = \frac{\kappa}{4\pi} \int_{-2\pi/\kappa}^{2\pi/\kappa} q_A(x, t)e^{-inx} \, dx.
\]
Employing the variable substitution \( \xi = \kappa x \), and using the fact that \( q_A \) is an even function with respect to the spatial variable \( x \), it reduces to

\[
\hat{q}_A^{(n)}(\kappa, t) = \frac{1}{4\pi} \int_{-2\pi}^{2\pi} e^{2it} \left( \frac{\kappa^3 \cosh(\sigma t) + i\kappa \sigma \sinh(\sigma t)}{2\kappa \cosh(\sigma t) - \sigma \cos \xi} - 1 \right) \cos(n\xi) d\xi = \frac{1}{2\pi} \int_{0}^{2\pi} e^{2it} \left( \frac{\kappa^3 \cosh(\sigma t) + i\kappa \sigma \sinh(\sigma t)}{2\kappa \cosh(\sigma t) - \sigma \cos \xi} - 1 \right) \cos(n\xi) d\xi.
\]

For \( n = 0 \), applying Lemma 4 or Corollary 1, we obtain

\[
\hat{q}_A^{(0)}(\kappa, t) = e^{2it} \left( \frac{\kappa^3 \cosh(\sigma t) + i\kappa \sigma \sinh(\sigma t)}{\sqrt{4\kappa^2\cosh^2(\sigma t) - \sigma^2}} - 1 \right).
\]

For \( n \neq 0 \), taking \( a = \sigma \) and \( b = 2\kappa \cosh(\sigma t) \), we obtain

\[
\hat{q}_A^{(n)}(\kappa, t) = e^{2it} \left( \frac{\kappa^3 \cosh(\sigma t) + i\kappa \sigma \sinh(\sigma t)}{\sqrt{4\kappa^2\cosh^2(\sigma t) - \sigma^2}} - 1 \right) \left( \frac{2\kappa \cosh(\sigma t) - \sqrt{4\kappa^{-2}\cosh^2(\sigma t) - \sigma^2}}{\sigma} \right)^n.
\]

This completes the proof. \( \square \)

Figure 8 illustrates the spatial Fourier spectrum of the Akhmediev breather for \( n = 0 \) as a function of time \( t \) for \( \kappa_0 = 0.5 \) (top left), \( \kappa_0 = 1 \) (top right), \( \kappa_0 = \sqrt{2} \) (bottom left), and \( \kappa_0 = \sqrt{3} \) (bottom right). The real and imaginary parts are depicted by the dashed blue and solid red curves, respectively, whereas the solid black curve represents the modulus of the amplitude spectrum evolution. For all values of \( \kappa \), the spectrum modulus reaches a minimum value during \( t = 0 \), and for an increasing value of \( \kappa \), these minimum values of the modulus are decreasing until they vanish at \( \kappa_0 = 1 \) and then increase again toward one as \( \kappa_0 \to 2 \). For \( t \to \pm \infty \), the amplitude modulus also goes to one.

Figure 9 displays the moduli of amplitude spectra for several values of \( n \) and \( \kappa \). The solid black curves correspond to the moduli for the main/central wavenumber when \( n = 0 \); the dashed blue curves correspond to the first pair of sidebands \( (n = 1) \), the dotted red curves correspond to the second pair of sidebands \( (n = 2) \), and the dash-dotted green curves correspond to the third pair of sidebands \( (n = 3) \). Although the moduli for the central wavenumber reach
The modulus of spatial Fourier amplitude spectrum of the Akhmediev breather $|\hat{q}_A^{(n)}(\kappa_0, t)|$ as a function of time $t$ for $\kappa_0 = 0.5$ (top left), $\kappa_0 = 1$ (top right), $\kappa_0 = \sqrt{2}$ (bottom left), and $\kappa_0 = \sqrt{3}$ (bottom right) for $n = 0$ (solid black), $n = 1$ (dashed blue), $n = 2$ (dotted red), and $n = 3$ (dash-dotted green) [Colour figure can be viewed at wileyonlinelibrary.com]

The modulus of spatial Fourier amplitude spectrum of the Akhmediev breather $|\hat{q}_A^{(n)}(\kappa, t)|$ when $t = 0$ as a function of the wavenumber $k$ for $\kappa = 0.5$ (top left), $\kappa = 1$ (top right), $\kappa = \sqrt{2}$ (bottom left), and $\kappa = \sqrt{3}$ (bottom right) for $n = 0$ (black), $n = 1$ (blue), $n = 2$ (red), $n = 3$ (green), $n = 4$ (cyan), and $n = 5$ (magenta) [Colour figure can be viewed at wileyonlinelibrary.com]

a local minimum when $t = 0$, all the sidebands moduli reach a local maximum when $t = 0$. The further the sideband pairs from the main wavenumber, the smaller they attain maximum values. For $t \to \pm \infty$, all sideband moduli tend to zero while the central sideband goes to one.

Figure 10 presents the moduli of the spatial Fourier amplitude spectrum of the Akhmediev breather when $t = 0$ as a function of wavenumber for several values of $\kappa$. Each panel shows only up to five pairs of sideband wavenumber as indicated by different colors: the first pair is blue, the second one is red, the third one is green, and the fourth and fifth pairs are cyan and magenta, respectively. Despite the moduli of the sidebands are getting shorter as they progress toward higher-order pairs, the modulus that corresponds to the main wavenumber does not always occupy the highest position.
 Depending on the sideband wavenumber \( \kappa \), it might be taller than the first pairs of sidebands (for example for \( \kappa_0 = \frac{1}{2} \) and \( \kappa_0 = \sqrt{3} \)) or shorter than the first pairs of sidebands (for example for \( \kappa_0 = 1 \) and \( \kappa_0 = \sqrt{2} \)).

Different from the previous two cases where the spectra are continuous in \( k \) (the KM breather and Peregrine soliton), the spectrum for this breather is discrete due to its periodic wave profile. While the initial modulated waves (when \( t < 0 \) but not \( t \to -\infty \)) exhibit the spectrum of the main/central wavenumber accompanied by a pair of sidebands, more pairs of sidebands were generated as the waves evolves in time. As time increases (when \( t > 0 \)), higher-order sidebands in the spectrum disappear and the spectrum configuration returns to the state of the initial condition when \( t < 0 \), where the central wavenumber is accompanied with a pair of sidebands. This behavior suggests that the wave energy is distributed from the main wavenumber to the sideband wavenumbers and is recollected back during the evolution. Interestingly, for a particular value of \( \kappa \), that is, \( \kappa_0 = 1 \), all energy from the central wavenumber is transferred completely to its sidebands, whereas for other values of \( \kappa \), only partial energy is distributed from the central wavenumber to its sidebands.

3 | DISCUSSION

In this section, we discuss some similarities and differences between the three rogue wave solutions’ spectra. First, we compare the KM breather’s and Peregrine soliton’s spectra. Second, we consider the comparison between the spectra of the Peregrine soliton and Akhmediev breather. Finally, the contrast between the KM breather’s and Akhmediev breather’s spectra can be deduced accordingly, which flows naturally from the previous two descriptions. We also comment briefly on other exact solutions of the NLS equation and explain the relationship between the spectrum and generating mechanism of the rogue waves.

Observing them in the time domain \( t \), the spectrum of the KM breather is time-periodic, whereas the spectrum of the Peregrine’s soliton exhibits exponentially decaying behavior as \( t \to \pm \infty \). The period of the former depends on its parameter \( \mu \). The larger the value of \( \mu \), the smaller its period would be. As \( \mu \to 0 \), its period is getting large and the KM breather’s spectrum approaches the Peregrine soliton’s spectrum. By comparing the two spectra as a function of the wavenumber \( k \), we observe that they share several similarities, albeit there also exist some differences. In addition to the appearance of the Dirac delta function in both spectra, both profiles decay to zero as \( k \to \pm \infty \). Furthermore, depending on the values of \( t \), both the real and imaginary parts of the spectra oscillate between their maximum and minimum values. One notable difference occurs at \( k = 0 \) when the KM breather’s spectrum is smooth and is thus differentiable there, whereas the Peregrine soliton’s spectrum has a cusp and fails to be differentiable.

Examining the spectra as temporal evolution, both spectra feature continuous functions. For \( t \to \pm \infty \), the spectrum of the Akhmediev breather for the central wavenumber approaches unity, whereas for higher-order sidebands approaches zero. The latter feature shares similarity with the Peregrine soliton’s spectrum. Looking at the wavenumber domain, the differences are stark. The spectrum of the Peregrine soliton, and also the KM breather, is a continuous function, whereas the spectrum of the Akhmediev breather is a discrete function, where the modulus takes nonzero values only at the central wavenumber and higher-order sidebands. In the spatial domain, the Akhmediev breather’s spectrum approaches the Peregrine soliton’s spectrum as \( \kappa \to 0 \), which corresponds to an infinitely large modulation period.

There are seven types of exact solutions of the NLS equation known in the literature. Three solutions do not depend on any parameter except for suitable gauge and scaling transformations, whereas the other four families of solutions depend on some additional parameters. These three solutions are mentioned earlier, that is, the continuous plane-wave, fundamental bright soliton, and the rational Peregrine soliton. Together with the other two rogue wave solutions, that is, Akhmediev and KM breathers, we have covered five of them and discussed at length the spectra of the latter three. The spatial Fourier spectra for the plane-wave and bright soliton solutions are given by Dirac delta and hyperbolic secant functions, respectively.

The other two exact solutions are beyond the scope of this article. They are called the families of double-periodic and stationary periodic solutions, that can be expressed in terms of the Jacobi elliptic functions and depend on a parameter only. The former also belongs to the breather type of solutions and is periodic in both spatial and temporal domains. In its limiting case, it tends to the bright soliton for \( k \to 0 \) and a special case of the Akhmediev breather for \( k \to 1 \), that is, when \( \kappa = \sqrt{2} \) in (4). For the family of the stationary periodic wave solutions, the limiting behavior also approaches the bright soliton in one direction and degenerates into the plane-wave solution in the other direction. See also earlier studies. However, exact analytical expressions for the spatial Fourier spectra for these solutions seem to be absent in the literature and thus might be a potentially open problem.
Although a recent report suggested replacing nonlinear wave packet modulational (sideband) instability with a probabilistic prediction based on random superposition of steep waves in the generating mechanism for oceanic rogue waves, the former is still considered as a reliable indicator for the examination of rogue wave events in both theoretical model scenarios and experimental laboratory settings. In water waves, modulational instability is also known as Benjamin–Feir instability, whereas in nonlinear optics, it is often called Bespalov–Talanov instability. On the one hand, among the three breather wave solutions, only the Peregrine soliton and Akhmediev breather exhibit modulational instability in the limit of sufficiently large temporal variable, both in the positive and negative directions, along with different phases. The modulated wave envelope blends the plane-wave with a small perturbation that bears this phase information. In the spectral domain, the amplitude spectrum comprises the nonzero modulus main wavenumber accompanied by a pair of sidebands, where the energy is concentrated toward the central part. It turns out that the Akhmediev breather is the nonlinear extension of this linear modulational instability.29,36,37

On the other hand, the modulation for the KM breather is never uniformly small, and thus, it does not correspond to the modulational instability in the classical sense. Generating the KM breather wave profile in a hydrodynamic laboratory is a challenging attempt since the initial condition consists of not only the nonvanishing background of the continuous plane-wave but also a soliton-like profile on top of it, not a modulated wave profile like in the other two cases. Thus, the KM breather must already exist in the wave field from the very beginning of wave profile propagation or wave signal evolution.1,29

Furthermore, in particular cases such as shallow-water waves, the dispersion effect is relatively weak and the uniform wave trains are stable. Hence, the wave focusing mechanism plays an essential role in rogue wave formation, which may be recognized either as dispersive focusing or geometrical focusing for unidirectional or (higher-dimension) multidirectional waves, respectively.38,39 When these breather solutions encounter nonlinear superposition, the NLS equation admits exact analytical higher-order breathers.15,40 In the spectral domain for a large temporal variable in the absolute sense, the family of higher-order Akhmediev breathers exhibits the nonzero modulus main wavenumber supplemented with two pairs of sidebands.28,29,41 Recently, experimental results from hydrodynamics and optics demonstrated that the nonlinear modulational instability phenomenon possesses a wider band of unstable frequencies than the prediction from the linear stability analysis.42 Another theoretical perspective adds that the formation and generating mechanism of higher-order rogue waves through progressive fission and fusion of n degenerate breathers enlightened the total number of wave peaks and the decomposition rule in their circular pattern.43 Although this topic is also of great interest and active research sub-areas, a more detailed discussion on this particular issue is beyond the scope of this paper. The next section concludes this article.

4 | CONCLUSION

In this article, we have considered the Fourier spectrum for rogue wave prototypes from the NLS equation. Also known as the soliton waves on a nonvanishing background, all three breathers are related with a complex-valued parameter. While the Peregrine soliton was discovered theoretically after the discovery of the KM breather and before the Akhmediev breather, it serves as the limiting case for both the KM and Akhmediev breathers. We need to perform integrations in the complex plane when deriving the analytical expressions for the spatial Fourier spectrum of the breathers. Although the computations performed in this article are relatively straightforward, it turns out that the derivation is technically challenging nonetheless.

Since both the KM and Peregrine breathers have infinite periodicity in the spatial domain, their spectra are continuous functions in both wavenumber and temporal domains. The spectrum modulus for the former in the temporal domain shows a pattern of periodicity while for the latter, the period is infinity. In the wavenumber domain, the corresponding spectrum moduli for both breathers exhibit infinite periodicity. For the Akhmediev breather, since it is periodic in space, its spatial spectrum remains continuous in the time domain but transforms into a discrete-type in the wavenumber domain. Remarkably, higher-order sidebands were generated as the initial sideband pairs evolve in time and then returned to an initial state.

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CONFLICT OF INTEREST
The author declares no conflict of interest.

DEDICATION
The author would like to dedicate this article to his late father Zakaria Karjanto (Khouw Kim Soey, 許金瑞) who introduced and taught him the alphabet, numbers, and the calendar in his early childhood. Karjanto senior was born in Tasikmalaya, West Java, Japanese-occupied Dutch East Indies on 1 January 1944 (Saturday Pahing) and died in Bandung, West Java, Indonesia, on 18 April 2021 (Sunday Wage).

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REFERENCES
1. Onorato M, Residori S, Bortolozzo U, Montina A, Arecchi FT. Rogue waves and their generating mechanisms in different physical contexts. Phys Rep. 2013;528(2):47-89.
2. Dudley JM, Dias F, Erkintalo M, Genty G. Instabilities, breathers and rogue waves in optics. Nat Photonics. 2014;8(10):755-764.
3. Sulem C, Sulem P-L. The Nonlinear Schrödinger Equation—Self-Focusing and Wave Collapse. New York, NY, US: Springer-Verlag; 1999.
4. Fibich G. The Nonlinear Schrödinger Equation—Singular Solutions and Optical Collapse. Cham, Switzerland: Springer; 2015.
5. Zakharov V, Shabat A. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Soviet Physics JETP. 1972;34(1):62.
6. Dyatsev KB, Trulsen K. Note on breather type solutions of the NLS as models for freak-waves. Phys Scr. 1999;T82:48-52.
7. Matveev VB, Salle MA. Darboux Transformations and Solitons. Berlin, Germany: Springer; 1991.
8. Trisetyarso A. Application of Darboux transformation to solve multisoliton solution on non-linear Schrödinger equation. arXiv preprint, arXiv:0910.0901 [math-ph]. Retrieved from https://arxiv.org/abs/0910.0901v1; 2009.
9. Okamoto K. Fundamentals of Optical Waveguides. 3rd ed. Cambridge, MA, US: Academic Press; 2021.
10. Agrawal GP. Nonlinear Fiber Optics. 6th ed. Burlington, MA, US: Academic Press; 2019.
11. Millot G, Tchofo-Dinda P. Optical fiber solitons, physical origin and properties. Encyclopedia of Modern Optics. Amsterdam, the Netherlands: Elsevier; 2005:56-65.
12. Kuznetsov EA. Solitons in a parametrically unstable plasma. Dokl Akad Nauk SSSR (Proc USSR Acad Sci). 1977;236:575-577.
13. Ma Y-C. The perturbed plane-wave solutions of the cubic Schrödinger equation. Stud Appl Math. 1979;60(1):43-58.
14. Peregrine DH. Water waves, nonlinear Schrödinger equations and their solutions. The ANZIAM J. 1983;25(1):16-43.
15. Akhmediev N, Eleonskii VM, Kulagin NE. Generation of periodic trains of picosecond pulses in an optical fiber: exact solutions. Soviet Phys JETP. 1985;62(5):894-899.
16. Akhmediev NN, Korneev VI. Modulation instability and periodic solutions of the nonlinear Schrödinger equation. Theor Math Phys. 1986;69(2):1089-1093.
17. Akhmediev NN, Eleonskii VM, Kulagin NE. Exact first-order solutions of the nonlinear Schrödinger equation. Theor Math Phys. 1987;72(2):809-818.
18. Akhmediev N, Ankiewicz A, Taki M. Waves that appear from nowhere and disappear without a trace. Phys Lett A. 2009;373(6):675-678.
19. Karjanto N. Peregrine soliton as a limiting behavior of the Kuznetsov-Ma and Akhmediev breathers. Frontiers in Phys. 2020;9:599767. Also accessible online at https://arxiv.org/abs/2009.00269, preprint arXiv:2009.00269 [nlin.PS]. Last accessed October 10, 2022.
20. Kibler B, Fatome J, Finot C, et al. The Peregrine soliton in nonlinear fibre optics. Nat Phys. 2010;6:790-795.
21. Chabchoub A, Hoffmann NP, Akhmediev N. Rogue wave observation in a water wave tank. Phys Rev Lett. 2011;106:204502.
22. Bailung H, Sharma SK, Nakamura Y. Observation of Peregrine solitons in a multicomponent plasma with negative ions. Phys Rev Lett. 2011;107(255005).
23. Alejo MA, Fanelli L, Muñoz C. Review on the stability of the Peregrine and related breathers. Frontiers in Phys. 2020;8:591995.
24. Pelinovsky D. Spectral analysis. Encyclopedia of Nonlinear Science. New York, US and London, UK: Routledge; 2005:863-864.
25. Bauck J. A note on Fourier transform conventions used in wave analyses. Retrieved from https://engrxiv.org/ijy76/ and doi:10.31224/sofic/ijy76. Last accessed October 10, 2022; 2019.
26. Gradstedt YS, Ryzhik IM. Table of Integrals, Series, and Products. In: D. Zwillinger, ed. Translated from Russian by Scripta Technica, Inc. 7th ed. Waltham, MA, US: Academic Press; 2014. V. Moll (Scientific Editor).
27. Howie JM. Complex Analysis. London, UK, Berlin Heidelberg, Germany: Springer-Verlag; 2003.
28. Akhmediev NN, Ankiewicz A. Solitons: Nonlinear Pulses and Beams. London, UK: Chapman & Hall; 1997.
29. Karjanto N. Mathematical Aspects of Extreme Water Waves. PhD thesis: University of Twente, the Netherlands. Accessible online at https://arxiv.org/abs/2006.00766, preprint arXiv:2006.00766 [nlin.PS]. Last accessed October 10, 2022.
30. Karjanto N. Fourier spectrum and related characteristics of the fundamental bright soliton solution. Mathematics 2022;10(23):4559. Also available in arXiv:2205.01521.
31. Chen J, Pelinovsky DE, White RE. Rogue waves on the double-periodic background in the focusing nonlinear Schrödinger equation. Phys Rev E. 2019;100(5):52219.
32. Conforti M, Musset A, Kudlinski A, Trillo S, Akhmediev N. Doubly periodic solutions of the focusing nonlinear Schrödinger equation: Recurrence, period doubling, and amplification outside the conventional modulation-instability band. Phys Rev A. 2020;101(2):23843.
33. Gemmrich J, Cicon L. Generation mechanism and prediction of an observed extreme rogue wave. Sci Rep. 2022;12(1):1-10.
34. Benjamin TB, Feir JE. The disintegration of wave trains on deep water. Part 1. Theory J Fluid Mech. 1967;27(3):417-430.
35. Bessalov VI, Talanov VI. On the filament structure of light beams in nonlinear liquids. Sov J Exp Theor Phys Lett. 1966;3:307-312.
36. Andonowati, Karjanto N, van Groesen E. Extreme wave phenomena in down-stream running modulated waves. Appl Math Model. 2007;31(7):1425-1443.
37. Karjanto N, van Groesen E. Qualitative comparisons of experimental results on deterministic freak wave generation based on modulational instability. J Hydro Environ Res. 2010;3(4):186-192.
38. Kharif C, Pelinovsky E. Physical mechanisms of the rogue wave phenomenon. Eur J Mech-B/Fluids. 2003;22(6):603-634.
39. Kharif C, Pelinovsky E, Slunyaev A. Rogue Waves in the Ocean. Berlin Heidelberg, Germany: Springer Verlag; 2009.
40. Akhmediev N, Ankiewicz A, Soto-Crespo JM. Rogue waves and rational solutions of the nonlinear Schrödinger equation. Phys Rev E. 2009;80(2):26601.
41. Karjanto N, van Groesen E. Mathematical physics properties of waves on finite background. In: Lang S. P., Bedore S. H., eds. Handbook of Solitons: Research, Technology and Applications, Chapter 14. Hauppauge, New York: Nova Science Publishers; 2009:501-539. Also accessible online at https://arxiv.org/abs/1610.09059, preprint arXiv:1610.09059. Last accessed October 10, 2022.
42. Vanderhaegen G, Naveau C, Szriftgiser P, et al. “Extraordinary” modulation instability in optics and hydrodynamics. Proc Natl Acad Sci. 2021;118(14):e2019348118.
43. He JS, Zhang HR, Wang LH, Porsezian K, Fokas AS. Generating mechanism for higher-order rogue waves. Phys Rev E. 2013;87(5):52914.

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