CONVERGENCE RATES OF SPECTRAL DISTRIBUTION OF LARGE DIMENSIONAL QUATERNION SAMPLE COVARIANCE MATRIX

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Abstract. In this paper, we study the convergence rates of empirical spectral distribution of large dimensional quaternion sample covariance matrix. Assume that the entries of $X_n (p \times n)$ are independent quaternion random variables with mean zero, variance 1 and uniformly bounded sixth moments. Denote $S_n = \frac{1}{n} X_n X_n^*$. Using Bai inequality, we prove that the expected empirical spectral distribution (ESD) converges to the limiting Marčenko-Pastur distribution with the ratio of the dimension to sample size $y_p = p/n$ at a rate of $O \left( \frac{n^{-1/2} a_n^{-3/4}}{n} \right)$ when $a_n > n^{-2/5}$ or $O \left( \frac{n^{-1/5}}{n} \right)$ when $a_n \leq n^{-2/5}$, where $a_n = (1 - \sqrt{y_p})^2$ is the lower bound for the M-P law. Moreover, the rates for both the convergence in probability and the almost sure convergence are also established. The weak convergence rate of the ESD is $O \left( \frac{n^{-2/5} a_n^{-2/5}}{n} \right)$ when $a_n > n^{-2/5}$ or $O \left( \frac{n^{-1/5}}{n} \right)$ when $a_n \leq n^{-2/5}$. The strong convergence rate of the ESD is $O \left( \frac{n^{-2/5} + \eta a_n^{-2/5}}{n} \right)$ when $a_n > n^{-2/5}$ or $O \left( \frac{n^{-1/5}}{n} \right)$ when $a_n \leq n^{-2/5}$ for any $\eta > 0$.

Keywords: Empirical Spectral Distribution; Marčenko-Pastur Law; Weak Convergence Rate; Strong Convergence Rate; Quaternion Sample Covariance Matrix.

1. Introduction

Let $A$ be a $p \times p$ Hermitian matrix and denote its eigenvalues by $s_j, j = 1, 2, \cdots, p$. The empirical spectral distribution (ESD) of $A$ is defined by

$$F^A(x) = \frac{1}{p} \sum_{j=1}^{p} I(s_j \leq x),$$

where $I(D)$ is the indicator function of an event $D$. Huge data sets with large dimension and large sample size lead to failure of the applications of the classical limit theorems. In recent decades, the theory of random matrices

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(RMT) has been actively developed which enables us to find the solutions to this issue. The sample covariance matrix is one of the most important random matrices in RMT, which can be traced back to Wishart (1928) [19]. In [16], Marčenko and Pastur proved that ESD of large dimensional complex sample covariance matrices tends to the M-P law $F_y(x)$ with the density function

$$f_y(x) = \begin{cases} \frac{1}{2\pi y \sigma^2} \sqrt{(b-x)(x-a)}, & a \leq x \leq b, \\ 0, & \text{otherwise}, \end{cases}$$

where $a = \sigma^2 (1 - \sqrt{y})^2$, $b = \sigma^2 (1 + \sqrt{y})^2$, $\sigma^2$ is the scale parameter, and the constant $y$ is the limiting ratio of dimension $p$ to sample size $n$. If $y > 1$, $F_y(x)$ has a point mass $1 - 1/y$ at the origin. After the limiting spectral distribution (LSD) of the sample covariance matrices is found, two important problems arise. The first is the bound on extreme eigenvalues; the second is the convergence rate of the ESD with respect to sample size. Yin, Bai and Krishnaiah (1988) [22] proved that the largest eigenvalue of the large dimensional real sample covariance matrix tends to $\sigma^2 (1 + \sqrt{y})$, a.s.. Bai and Yin (1993) [9] established the conclusion that the smallest eigenvalue of the large dimensional real sample covariance matrix strongly converges to $\sigma^2 (1 - \sqrt{y})$.

For convergence rate, since Bai [6] established a Berry-Essen type inequality, much work has been done (see [7, 3, 4, 8, 11, 12], among others). Here the readers are referred to three books [2, 5, 17] for more details.

As the wide applications of quaternions and quaternion matrices in quantum physics, robot technology and artificial satellite attitude control, etc., it is necessary to study the quaternion sample covariance matrix. In [15], it was proved that the ESD of large dimensional quaternion sample covariance matrix tends to the M-P law. From [14], we have known the limits of extreme eigenvalues of quaternion sample covariance matrix. Convergence rates of the ESD of the quaternion sample covariance matrix are considered in this paper.

In what follows, we introduce some notations about quaternions. The quaternion base can be represented by four $2 \times 2$ matrices as

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$ denotes the imaginary unit. Thus, a quaternion can be written by a $2 \times 2$ complex matrix as

$$x = a \cdot e + b \cdot i + c \cdot j + d \cdot k = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} \triangleq \begin{pmatrix} \lambda & \omega \\ -\bar{\omega} & \bar{\lambda} \end{pmatrix}$$

where the coefficients $a, b, c, d$ are real. The conjugate of $x$ is defined as

$$\bar{x} = a \cdot e - b \cdot i - c \cdot j - d \cdot k = \begin{pmatrix} a - bi & -c - di \\ c - di & a + bi \end{pmatrix} = \begin{pmatrix} \bar{\lambda} & -\bar{\omega} \\ \bar{\omega} & \bar{\lambda} \end{pmatrix}$$
and its norm as
\[ \|x\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{|\lambda|^2 + |\omega|^2}. \]
More details can be found in [11, 10, 23, 13, 17, 24, 18]. It is worth mentioning that any \( n \times n \) quaternion matrix \( Y \) can be represented as a \( 2n \times 2n \) complex matrix \( \psi(Y) \). Consequently, we can deal with quaternion matrices as complex matrices.

The following two tools play a key role in establishing the convergence rates of the ESD. The first is Bai inequality:

**Lemma 1.1.** (Bai inequality in [6]) Let \( F \) be a distribution function and \( G \) be a function of bounded variation satisfying \( \int |F(x) - G(x)|\,dx < \infty \). Denote their Stieltjes transforms by \( f(z) \) and \( g(z) \), respectively, where \( z = u + iv \in \mathbb{C}^+ \). Then we have

\[
\|F - G\| \overset{\text{def}}{=} \sup_x |F(x) - G(x)| \\
\leq \frac{1}{\pi (1 - \kappa)(2\gamma - 1)} \left[ \int_{-A}^{A} |f(z) - g(z)| \,du \\
+ 2\pi v^{-1} \int_{|x| > B} |F(x) - G(x)| \,dx \\
+ v^{-1} \sup_x \int_{|s| \leq 2v a} |G(x + s) - G(x)| \,ds \right],
\]

(1.1)

where \( a, \gamma, A \) and \( B \) are positive constants such that \( A > B \),

\[ \gamma = \frac{1}{\pi} \int_{|u| < a} \frac{1}{u^2 + 1} \,du > \frac{1}{2}, \quad \text{and} \quad \kappa = \frac{4B}{\pi(A - B)(2\gamma - 1)} < 1. \]

The other is the form of the inverse of some matrices related to quaternions:

**Lemma 1.2** (see [15] or [21]). For all \( n \geq 1 \), if a complex matrix \( \Omega_n \) is invertible and of Type-III, then \( \Omega_n^{-1} \) is a Type-I matrix.

In Lemma 1.2, the Type-III and Type-I are defined as follows:

**Definition 1.3.** A matrix is called Type-I matrix if it has the following structure:

\[
\begin{pmatrix}
  t_1 & 0 & a_{12} & b_{12} & \cdots & a_{1n} & b_{1n} \\
  0 & t_1 & c_{12} & d_{12} & \cdots & c_{1n} & d_{1n} \\
  d_{12} & -b_{12} & t_2 & 0 & \cdots & a_{2n} & b_{2n} \\
  -c_{12} & a_{12} & 0 & t_2 & \cdots & c_{2n} & d_{2n} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  d_{1n} & -b_{1n} & d_{2n} & -b_{2n} & \cdots & t_n & 0 \\
  -c_{1n} & a_{1n} & -c_{2n} & a_{2n} & \cdots & 0 & t_n
\end{pmatrix}.
\]

Here all the entries are complex.
Definition 1.4. A matrix is called Type-III matrix if it has the following structure:
\[
\begin{pmatrix}
t_1 & 0 & a_{12} & b_{12} & \cdots & a_{1n} & b_{1n} \\
0 & t_1 & -\bar{b}_{12} & a_{12} & \cdots & -\bar{b}_{1n} & \bar{a}_{1n} \\
\bar{a}_{12} & -\bar{b}_{12} & t_2 & 0 & \cdots & \bar{a}_{2n} & b_{2n} \\
\bar{b}_{12} & a_{12} & 0 & t_2 & \cdots & \bar{b}_{2n} & \bar{a}_{2n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\bar{a}_{1n} & -\bar{b}_{1n} & \bar{a}_{2n} & -\bar{b}_{2n} & \cdots & t_n & 0 \\
\bar{b}_{1n} & a_{1n} & \bar{b}_{2n} & a_{2n} & \cdots & 0 & t_n \\
\end{pmatrix}
\]
Here all the variables are complex numbers.

2. Main theorem

In this section, we establish the main theorems about convergence rates of the ESD of the quaternion sample covariance matrix. They can be stated as follows.

Theorem 2.1. Suppose that \( X_n = (x_{jk}^{(n)})_{p \times n} \) is a quaternion random matrix whose entries are independent. Furthermore, assume that
\[
E x_{jk}^{(n)} = 0, E \left\| x_{jk}^{(n)} \right\|^2 = 1, \sup_n \sup_{jk} E \left\| x_{jk}^{(n)} \right\|^6 \leq M.
\]
Then, denoting the ESD of \( S_n = \frac{1}{n} X_n X_n^* \) as \( F_{S_n} \), we have
\[
\left\| EF_{S_n} - F_{y_p} \right\| = \begin{cases} O \left( n^{-1/2} a_n^{-3/4} \right), & \text{if } a_n > n^{-2/5}, \\ O \left( n^{-1/5} \right), & \text{otherwise}, \end{cases}
\]
where \( y_p = p/n \) and \( a_n = (1-\sqrt{y_p})^2 \).

Remark 2.2. For brevity, we shall drop the superscript \((n)\) from the variables and denote \( \left\| EF_{S_n} - F_{y_p} \right\| \) by \( \Delta \).

Remark 2.3. Note that
\[
\left\| EF_{S_n} - F_y \right\| \geq \left\| F_Y - F_{y_p} \right\| - \left\| EF_{S_n} - F_{y_p} \right\|.
\]
Consequently, the convergence rate of \( \left\| EF_{S_n} - F_Y \right\| \) relies on that of \( |y_p - y| \). Therefore, it is impossible to establish the convergence rate of \( \left\| EF_{S_n} - F_Y \right\| \), unless we know the rate of \( |y_p - y| \). Thus, we have to consider the convergence rate of \( \left\| EF_{S_n} - F_{y_p} \right\| \).

Remark 2.4. To prove Theorem 2.1, it suffices to show that (2.1) is true when \( y_p \leq 1 \).
In fact, for $y_p > 1$, write $W_p = \frac{1}{p}X_n^*X_n$ and denote by $G_n(x)$ the ESD of $W_p$. It is known that $X_n^*X_n$ and $\bar{X}_n^*\bar{X}_n$ have the same nonzero eigenvalues. By calculation, one gets

\[ F_{S_n}(x) = y_p^{-1}G_n(y_p^{-1}x) + (1 - y_p^{-1})I(x \geq 0) \]

which implies that

\[ \|F_{S_n} - F_{y_p}\| = y_p^{-1}\|G_n - F_{1/y_p}\| . \]

Therefore, the convergence rate for $y_p > 1$ can turn into that for $1/y_p < 1$.

**Theorem 2.5.** Under the assumptions in Theorem 2.1, we have

\[ \|F_p - F_{y_p}\| = \begin{cases} O_p\left(n^{-1/5}\right), & \text{if } a_n < n^{-2/5}, \\ O_p\left(n^{-2/5}a_n^{-2/5}\right), & \text{if } n^{-2/5} \leq a_n < 1. \end{cases} \]

**Theorem 2.6.** Under the assumptions in Theorem 2.1, we have

\[ \|F_p - F_{y_p}\| = \begin{cases} O_{a.s.}\left(n^{-1/5}\right), & \text{if } a_n < n^{-2/5}, \\ O_{a.s.}\left(n^{-2/5+\eta a_n^{-2/5}}\right), & \text{if } n^{-2/5} \leq a_n < 1. \end{cases} \]

### 3. Preliminaries

Before proving the Theorem 2.1, we first truncate the entries of the matrix and renormalize them in order to obtain the bound of $\|x_{jk}\|$, without changing the convergence rate of $F_{S_n}$. The results are listed in Subsection 3.4.

#### 3.1. Truncation

We truncate the variables $x_{jk}$ at $n^{1/4}$. Denote the truncated entries and matrix by $\tilde{x}_{jk} = x_{jk}I(\|x_{jk}\| < n^{1/4})$ and $\tilde{X}_n = (\tilde{x}_{jk})$, respectively. Furthermore, let $\tilde{F}_{S_n}$ denote the ESD of the quaternion sample covariance matrix $\frac{1}{n}\tilde{X}_n\tilde{X}_n^*$. Then, by rank inequality Lemma 8.1, we have

\[ \|F_{S_n} - \tilde{F}_{S_n}\| \leq \frac{1}{p} \sum_{jk} I(\|x_{jk}\| \geq n^{1/4}) . \] (3.1)

Note that

\[
\begin{align*}
\mathbb{E}\left(n^{1/2}p^{-1}\sum_{jk} I(\|x_{jk}\| \geq n^{1/4})\right) & \leq n^{1/2}p^{-1}\sum_{jk} \mathbb{P}(\|x_{jk}\| \geq n^{1/4}) \\
& \leq n^{-1}p^{-1}\sum_{jk} \mathbb{E}\|x_{jk}\|^6 I(\|x_{jk}\| \geq n^{1/4}) \\
& \leq M
\end{align*}
\]
and
\[
\text{Var} \left( n^{1/2} p^{-1} \sum_{j,k} I \left( \| x_{jk} \| \geq n^{1/4} \right) \right) \leq np^{-2} \sum_{j,k} P \left( \| x_{jk} \| \geq n^{1/4} \right)
\]
\[
\leq n^{-1/2} p^{-2} \sum_{j,k} E \| x_{jk} \|^6 I \left( \| x_{jk} \| \geq n^{1/4} \right)
\]
\[= Mn^{-1/2}.\]

By Bernstein’s inequality (see Lemma 8.2), for all small \( \varepsilon > 0 \) and large \( p, n \), it follows that
\[
P \left( \left| \sum_{j,k} n^{1/2} p^{-1} I \left( \| x_{jk} \| \geq n^{1/4} \right) \right| \geq M + \varepsilon \right)
\]
\[
\leq P \left( \left| \sum_{j,k} n^{1/2} p^{-1} \left[ I \left( \| x_{jk} \| \geq n^{1/4} \right) - P \left( \| x_{jk} \| \geq n^{1/4} \right) \right] \right| \geq \varepsilon \right)
\]
\[
\leq 2 \exp \left\{ -\frac{\varepsilon^2}{2(Mn^{-1/2} + n^{1/2}p^{-1}\varepsilon)} \right\}
\]
\[\triangleq 2 \exp \{-cn^{1/2}\} \quad (c > 0)\]

which is summable. Applying Borel-Cantelli lemma, we have
\[
n^{1/2} p^{-1} \sum_{j,k} I \left( \| x_{jk} \| \geq n^{1/4} \right) \leq M + \varepsilon \text{ a.s.} \quad (3.2)
\]

Together with (3.1), (3.2) and Lemma 8.3, one has
\[
L \left( F_{S_n}, \tilde{F}_{S_n} \right) = O_{a.s.} \left( n^{-1/2} \right).
\]

3.2. Centralization. Write \( \tilde{x}_{jk} = x_{jk} - E x_{jk} \) and \( \tilde{X}_n = (\tilde{x}_{jk}) \). Denote by \( \tilde{F}_{S_n} \) the ESD of the quaternion sample covariance matrix \( \frac{1}{n} \tilde{X}_n \tilde{X}_n^* \). Using Lemma 8.4 we get
\[
(3.3) \quad L \left( \tilde{F}_{S_n}, \tilde{F}_{S_n} \right) \leq 2 \left\| \frac{1}{\sqrt{n}} \tilde{X}_n \right\|_2 \left\| \frac{1}{\sqrt{n}} E \tilde{X}_n \right\|_2 + \left\| \frac{1}{\sqrt{n}} E \tilde{X}_n \right\|_2^2.
\]

By elementary calculation, one obtains
\[
(3.4) \quad \left\| \frac{1}{\sqrt{n}} E \tilde{X}_n \right\|_2 \leq \sqrt{n} \max_{j,k} E \| x_{jk} \| I \left( \| x_{jk} \| \geq n^{1/4} \right)
\]
\[
\leq n^{-3/4} E \| x_{jk} \|^6 = O \left( n^{-3/4} \right).
\]
By Remark 2.3 in [14], we know that
\[
\limsup \left\| \frac{1}{\sqrt{n}} \tilde{X}_n \right\|_2 \leq 1 + \sqrt{y}, \text{a.s.}
\]
which implies that
\[
\limsup \left\| \frac{1}{\sqrt{n}} \tilde{X}_n \right\|_2 \leq \limsup \left\| \frac{1}{\sqrt{n}} \hat{X}_n \right\|_2 + \frac{1}{\sqrt{n}} E \tilde{X}_n \right\|_2 \\
\leq 1 + \sqrt{y}, \text{a.s.}
\]
Together with (3.3), (3.4) and (3.5), we can show that
\[
L \left( \tilde{F}^{S_n} , \hat{F}^{S_n} \right) = O \text{ a.s.} \left( n^{-3/4} \right).
\]

3.3. Rescaling. Write \( \bar{x}_{jk} = \sigma_{jk}^{-1} \tilde{x}_{jk} \) and \( \bar{X}_n = (\bar{x}_{jk}) \) where \( \sigma_{jk}^2 = E \| \tilde{x}_{jk} \|^2 \).
Moreover, let \( \tilde{F}^{S_n} \) denote the ESD of the quaternion sample covariance matrix \( \frac{1}{n} \tilde{X}_n \bar{X}_n \). By Lemma 8.4, one has
\[
L \left( \tilde{F}^{S_n} , \hat{F}^{S_n} \right) \leq 2 \left\| \frac{1}{\sqrt{n}} \hat{X}_n \right\|_2 \left\| \frac{1}{\sqrt{n}} \left( \hat{X}_n - \bar{X}_n \right) \right\|_2 + \left\| \frac{1}{\sqrt{n}} \left( \tilde{X}_n - \bar{X}_n \right) \right\|_2^2.
\]
By element calculation, we get
\[
\left\| \frac{1}{\sqrt{n}} \left( \tilde{X}_n - \bar{X}_n \right) \right\|_2^2 \leq 2 \frac{1}{n} \sum_{jk} \| \tilde{x}_{jk} \|^2 (\sigma_{jk}^{-1} - 1)^2 \\
\leq 2 p \max_{jk} |\sigma_{jk}^{-1} - 1|^2 \frac{1}{np} \sum_{jk} E \| \tilde{x}_{jk} \|^2, \text{a.s.}
\]
\[
= 2 p \max_{jk} |\sigma_{jk} - 1|^2, \text{a.s.}
\]
\[
= O \text{ a.s.} \left( n^{-1} \right)
\]
where the second inequality follows from that
\[
E \left( \frac{1}{np} \sum_{jk} (\| \tilde{x}_{jk} \|^2 - E \| \tilde{x}_{jk} \|^2) \right)^2 \leq \frac{1}{n^2 p^2} \sum_{jk} E \| \tilde{x}_{jk} \|^4 \leq C n^{-2}
\]
and the last equality follows from that
\[
|\sigma_{jk} - 1| \leq 1 - \sigma_{jk}^2 = E \| x_{jk} \|^2 I (\| x_{jk} \| \geq n^{1/4}) \leq n^{-1} E \| x_{jk} \|^6 = O \left( n^{-1} \right).
\]
From (3.6) and (3.7), we can show that
\[
L \left( \tilde{F}^{S_n} , \hat{F}^{S_n} \right) = O \text{ a.s.} \left( n^{-1/2} \right).
\]
3.4. **Conclusion.** Combining the three subsections above, Lemma 8.5 and Remark 8.6, we get

\[ \| F^{S_n} - F_{yp} \|_2 \leq C \max \left\{ \| F^{S_n} - F_{yp} \|_2, \frac{1}{\sqrt{n\alpha} + \sqrt{n}} \right\}. \]

For brevity, we still use \( x_{jk} \) to denote the variables after truncation and renormalization. Thus, to complete the proof of Theorem 2.1, we can further assume that

1. \( E x_{jk} = 0, \ E \| x_{jk} \|^2 = 1, \)
2. \( \| x_{jk} \| < n^{1/4}, \)
3. \( \sup_{jk} E \| x_{jk} \|^6 \leq M. \)

4. **Proof of Theorem 2.1**

The Stieltjes transform of M-P law \( F_{yp}(x) \) is given by

\[ s(z) = \int_{-\infty}^{+\infty} \frac{1}{x-z} dF_{yp}(x) = \frac{1 - y_p - z + \sqrt{(z - 1 - y_p)^2 - 4y_p}}{2y_pz} \]

where \( z = u + vi \in \mathbb{C}^+. \) And the Stieltjes transform of \( F^{S_n}(x) \) is

\[ s_p(z) = \int_{-\infty}^{+\infty} \frac{1}{x-z} dF^{S_n}(x) = \frac{1}{2p} \tr \left( S_n - z I_{2p} \right)^{-1}. \]

Applying Lemma 8.7, one has

\[ s_p(z) = \frac{1}{2p} \sum_{k=1}^{p} \tr \left( \frac{1}{n} \phi'_k \bar{\phi}_k - z I_2 - \frac{1}{n^2} \phi'_k X_{nk}^* \left( \frac{1}{n} X_{nk} X_{nk}^* - z I_{2p-2} \right)^{-1} X_{nk} \bar{\phi}_k \right)^{-1} \]

where \( X_{nk} \) is the matrix resulting from deleting the \( k \)-th quaternion row of \( X_n \), and \( \phi'_k \) is the quaternion vector of order \( 1 \times n \) obtained from the \( k \)-th quaternion row of \( X_n \). Set

\[ \varepsilon_k = \frac{1}{n} \phi'_k \bar{\phi}_k - z I_2 - \frac{1}{n^2} \phi'_k X_{nk}^* \left( \frac{1}{n} X_{nk} X_{nk}^* - z I_{2p-2} \right)^{-1} X_{nk} \bar{\phi}_k \]

\[ - (1 - z - y_p - y_p z E s_p(z)) I_2. \]

We can show that

\[ E s_p(z) = \frac{1}{1 - z - y_p - y_p z E s_p(z)} + \delta_n \]

where

\[ \delta_n = - \frac{1}{2p (1 - z - y_p - y_p z E s_p(z))} \]
(4.1) \[
\sum_{k=1}^{p} \text{Etr} \left\{ \varepsilon_k ((1 - z - y_p - y_p z E_s p(z)) I_2 + \varepsilon_k)^{-1} \right\}.
\]

From [15], we have known that the root of the equation above is
\[
E_s p(z) = \frac{1 - z - y_p + y_p z \delta_n + \sqrt{(1 - z - y_p - y_p z \delta_n)^2 - 4y_p z}}{2y_p z}.
\]

To begin with, we estimate the first integral in (1.1). Since
\[
|E_s p(z) - s(z)| \leq \left| \frac{1}{2} \right| \left| \frac{2(z + y_p - 1) + y_p z \delta_n}{\sqrt{(z + y_p - 1)^2 - 4y_p z + \sqrt{(z + y_p - 1 + y_p z \delta_n)^2 - 4y_p z}}} \right|,
\]
we need to find a bound for \(|\delta_n|\). For brevity, we shall use the following notation:
\[
v_{y_p} = \sqrt{a_n + \sqrt{\bar{v}}} = 1 - \sqrt{y_p + \sqrt{\bar{v}}}
\]
\[
S_{nk} = \frac{1}{n} X_{nk} X_{nk}^*
\]
\[
b_n = b_n(z) = \frac{1}{z + y_p - 1 + y_p z E_s p(z)}
\]
\[
\xi_k = \xi_k(z) = \left( (z + y_p - 1 + y_p z E_s p(z)) I_2 - \varepsilon_k \right)^{-1}
\]

Using Lemma [12] we get the form of \((S_n - z I_{2p})^{-1}\) as
\[
\left( \begin{array}{cccc}
t_1 & 0 & a_{12} & b_{12} & \cdots \\
0 & t_1 & c_{12} & d_{12} & \cdots \\
d_{12} & -b_{12} & t_2 & 0 & \cdots \\
-c_{12} & a_{12} & 0 & t_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \right).
\]

That is to say, \(\varepsilon_k\) is a scalar matrix. Denote by \(\alpha_k\) the first column of \(\phi_k\) and by \(\beta_k\) the second column of \(\phi_k\), then, \(\varepsilon_k = \theta_k I_2\) where \(\theta_k = \frac{1}{n} \alpha_k^t \alpha_k - z - \frac{1}{n^2} \alpha_k^t X_{nk}^* (S_{nk} - z I_{2p-2})^{-1} X_{nk} \alpha_k - (1 - z - y_p - y_p z E_s p(z)) \). Rewrite
\[
\xi_k = -\frac{1}{n} \alpha_k^t \bar{\alpha}_k - z - \frac{1}{n^2} \alpha_k^t X_{nk}^* (S_{nk} - z I_{2p-2})^{-1} X_{nk} \bar{\alpha}_k I_2.
\]

Noting that
\[
\Im \left( \frac{1}{n} \alpha_k^t \bar{\alpha}_k - z - \frac{1}{n^2} \alpha_k^t X_{nk}^* (S_{nk} - z I_{2p-2})^{-1} X_{nk} \bar{\alpha}_k \right)
\]
We are now in a position to estimate (4.2)

\[ \left( 1 + \frac{1}{n^2} \alpha_k^* X_{nk}^* (S_{nk} - zI_{2p-2})^{-1} (S_{nk} - zI_{2p-2})^{-1} X_{nk} \alpha_k \right) < -v, \]

one gets \[ \frac{1}{n} \alpha_k^* \alpha_k - z - \frac{1}{n} \alpha_k^* X_{nk}^* (S_{nk} - zI_{2p-2})^{-1} X_{nk} \alpha_k \leq v^{-1}. \]

We are now in a position to estimate \(|\delta_n|\). By (4.2) and the fact \(\xi_k = b_n I_2 + b_n\xi_k \sigma_k\), one has

\[
|\delta_n| \leq \frac{1}{2p} \sum_{k=1}^{p} \left( b_n^2 |E\epsilon_k| + b_n^3 |E|tr\epsilon_k^2| + b_n^4 |E|tr\epsilon_k^3| + b_n^4 |v^{-1}E|tr\epsilon_k^4| \right)
\]

\[
= \frac{1}{2p} \sum_{k=1}^{p} \left( b_n^2 |E\epsilon_k| + \frac{b_n^3}{2} |E|tr\epsilon_k^2| + \frac{b_n^4}{4} |E|tr\epsilon_k^3| + \frac{b_n^4}{8} v^{-1}E|tr\epsilon_k^4| \right).
\]

Next, we shall complete the estimation of \(|\delta_n|\) by the following four steps under the conditions \(v > n^{-1/2}\) and \(|b_n| \leq 2/\sqrt{y_p}|\).

**Step 1: the estimator of \(|E\epsilon_k||\).**

By Lemma 3.6 in [15], we have

\[ |E\epsilon_k| \leq \frac{C}{nv}. \]

**Step 2: the estimator of \(||E\epsilon_k||^2|\).**

Let \(E(\cdot)\) denote the conditional expectation given \(\{x_j, j = 1, \ldots, n; j \neq k\}\), then we get

\[ E|\epsilon_k|^2 \leq 3 \left[ E|\epsilon_k - E\epsilon_k|^2 + E|E\epsilon_k - E\epsilon_k|^2 + |E\epsilon_k|^2 \right]. \]

By Lemma 8.8 and Lemma 7.1, it follows that

\[
E|\epsilon_k - E\epsilon_k|^2 \leq 2E \left| tr \left( \frac{1}{n} \phi_k^* \phi_k - I_2 \right)^2 + \frac{2}{n^4} E \left| tr \phi_k^* X_{nk}^* (S_{nk} - zI_{2p-2})^{-1} X_{nk} \phi_k \right. \\
- \left. tr X_{nk}^* (S_{nk} - zI_{2p-2})^{-1} X_{nk} \right|^2 \right| \leq \frac{8}{n} + \frac{C}{n^4} Etr \left( X_{nk}^* (S_{nk} - zI_{2p-2})^{-1} X_{nk} X_{nk}^* (S_{nk} - zI_{2p-2})^{-1} X_{nk} \right) \leq C \left[ \frac{1}{n} + \frac{|u|^2}{n^2} Etr \left( (S_{nk} - uI_{2p-2})^2 + v^2 I_{2p-2} \right)^{-1} \right].
\]
\[
(4.4) \quad \leq C \left[ \frac{1}{n} + \frac{|u|^2}{nv^2} \left( \Delta + v/v_y \right) \right].
\]

Applying Lemma 7.2, we obtain
\[
E \left| \tilde{\text{tr}} \epsilon_k - \text{tr} \epsilon_k \right|^2
\leq \frac{|z|^2}{n^2} \left[ E \left| \text{tr} \left( S_{nk} - zI_{2p-2} \right)^{-1} - \text{tr} (S_{nk} - zI_{2p})^{-1} \right|^2 + \frac{1}{v^2} \right]
\leq C |z|^2 \left[ E \left| s_p (z) - \text{Es}_p (z) \right|^2 + \frac{1}{n^2 v^2} \right]
\leq C |z|^2 \left[ n^{-2} v^{-4} \left( \Delta + v/v_y \right) + \frac{1}{n^2 v^2} \right]
(4.5)
\]

Together with Step 1, (4.3), (4.4), and (4.5), we get
\[
E \left| \text{tr} \epsilon_k \right|^2 \leq C \left( \frac{1}{n} + \frac{|z|^2}{nv^2} \left( \Delta + v/v_y \right) \right).
\]

**Step 3: the estimator of** \(E \left| \text{tr} \epsilon_k \right|^4\).

Similar to (4.3), the estimand can be written as
\[
(4.6) \quad E \left| \text{tr} \epsilon_k \right|^4 \leq 27 \left[ E \left| \text{tr} \epsilon_k - \tilde{\epsilon}_k \right|^4 + E \left| \tilde{\epsilon}_k - \text{tr} \epsilon_k \right|^4 + |\text{tr} \epsilon_k|^4 \right].
\]

First of all, we estimate the first term of righthand side of (4.6). Using Lemma 8.8 and Lemma 7.1, it follows that
\[
E \left| \text{tr} \epsilon_k - \tilde{\epsilon}_k \right|^4
\leq 8E \left| \text{tr} \left( \frac{1}{n} \phi_k \bar{\phi}_k - I_2 \right) \right|^4 + \frac{8}{n^8} E \left| \text{tr} \phi_k^* X_{nk} (S_{nk} - zI_{2p-2})^{-1} X_{nk} \phi_k \right|^4
\leq \frac{128}{n^2} + C \left\{ v_8 \text{tr} \left( (S_{nk} - zI_{2p-2})^{-1} S_{nk} (S_{nk} - \bar{z}I_{2p-2})^{-1} S_{nk} \right)^2 \right\}
\]

\[ \leq C \frac{n^2}{n^4} + C \frac{n}{n^4} \left\{ \varphi_n \left( n + \frac{|z|^4}{v^2} \text{tr} \left( (S_{nk} - u I_{2p-2})^2 + v^2 I_{2p-2} \right)^{-1} \right) \right. \\
+ \left. \left[ n + |z|^2 \text{tr} \left( (S_{nk} - u I_{2p-2})^2 + v^2 I_{2p-2} \right)^{-1} \right]^2 \right\} \]
\]
\[ \leq C \left[ \frac{1}{n^2} + \frac{|z|^2}{n^2 v^4} (\Delta + v/v_y)^2 \right] \]

where \( \varphi_n \leq M n^{1/2} \). Employing Lemma 7.2, one has
\[ E \left| \text{tr} \varepsilon_k - \text{Etr} \varepsilon_k \right|^4 \]
\[ = \left| \frac{|z|^4}{n^4} E \left| \text{tr} \left( S_{nk} - z I_{2p-2} \right)^{-1} - \text{Etr} \left( S_{nk} - z I_{2p-2} \right)^{-1} \right| \right|^4 \]
\[ \leq C \frac{|z|^4}{n^4} \left[ E \left| \text{tr} \left( S_n - z I_{2p} \right)^{-1} - \text{Etr} \left( S_n - z I_{2p} \right)^{-1} \right|^4 + \frac{1}{v^4} \right] \]
\[ = C |z|^4 \left[ E \left| s_p (z) - E s_p (z) \right|^4 + \frac{1}{n^4 v^4} \right] \]
\[ \leq C |z|^4 \left[ n^{-4} v^{-8} (\Delta + v/v_y)^2 + \frac{1}{n^4 v^4} \right] \]
\[ \leq C |z|^4 \frac{n^{-2} v^4 (\Delta + v/v_y)^2}{n^2 v^4} \]

Combining the two inequalities above with Step 1, (4.6) can be estimated by
\[ E \left| \text{tr} \varepsilon_k \right|^4 \leq C \left[ \frac{1}{n^2} + \frac{|z|^4}{n^2 v^4} (\Delta + v/v_y)^2 \right] \]

**Step 4: the estimator of \( E \left| \text{tr} \varepsilon_k \right|^3 \).**

By Cauchy’s inequality, Step 2 and Step 3, we can easily acquire
\[ E \left| \text{tr} \varepsilon_k \right|^3 \leq \left( E \left| \text{tr} \varepsilon_k \right|^2 \right)^{3/2} \left( E \left| \text{tr} \varepsilon_k \right|^4 \right)^{1/2} \]
\[ \leq \left[ \frac{1}{n^3/2} + \frac{|z|^3}{n^3/2 v^3} (\Delta + v/v_y)^{3/2} \right] \]

Assume that \( |b_n| \leq 2/\sqrt{3p |z|} \). Then, the four steps above yield
\[ (4.7) \quad |\delta_n| \leq C_0 n^{-1} v^{-3} (\Delta + v/v_y)^2 \]

By Lemma 8.15, if \( |\delta_n| < v/ \left[ v_y 10 (A + 1)^2 \right] \), then \( \Delta \leq C v/v_y \). Therefore, our next goal is to find a possible value of the set \( \{ v : |\delta_n| \leq v/ \left[ v_y 10 (A + 1)^2 \right] \} \)
with $vv_{yp} \sim Cn^{-1/2}$. Define

$$\mathcal{F} = \left\{ vv_{yp}^{1/2} : vv_{yp}^{1/2} > M_0 n^{-1/2}, |\delta_n| \leq v/ \left[v_{yp}, 10 (A + 1)^2\right]\right\},$$

where $M_0 = \sqrt{C_0 (C_2 + 1)^2 / (10(A + 1)^2)}$ and $C_2$ is the constant given in Lemma 8.15. It is not difficult to verify $\mathcal{F} \neq \emptyset$. In fact, by (4.2), (4.3) and

$$|b_n| \leq \frac{1}{\mathcal{S}(z + y_p - 1 + y_p z E_{\bar{y}}(z))} \leq v^{-1},$$

we have

$$|\delta_n| \leq \frac{1}{2pv^2} \sum_{k=1}^{p} \left[|E tr e_k| + v^{-1} \left|E_{tr} e_k^2\right|\right] \leq \frac{C_1}{n v^5}.$$

Choosing $v_0 = \sqrt{10 C_1 (1 + A)^2 / n}$, the inequality above turns out to be

$$|\delta_n| \leq v_0 / \left[10 (A + 1)^2\right].$$

This indicates $v_0 \in \mathcal{F}$, for all large $n$.

We assert that the infimum of $\mathcal{F}$ is $M_0 n^{-1/2}$, which is denoted by $v_1 vv_{yp}^{1/2} (v_1)$. If it is not the case, then by the continuity of various functions involved, there must exist $z_2 = u_2 + v_2 i$ with $u_2 \in [-A, A], v_2 vv_{yp}^{1/2} (v_2) \in \mathcal{F}$ and such that $|\delta_n (z_2)| = v_2 / \left[v_{yp} (v_2) 10 (A + 1)^2\right]$ and $|\delta_n (z_1)| \leq v_2 / \left[v_{yp} (v_2) 10 (A + 1)^2\right]$ for any $z_1 = u + iv_2, u \in [-A, A]$. Then, by Lemma 8.14 $|b_n (z_1)| \leq \frac{2}{\sqrt{v_p (z_1)}}$, the inequality (4.7) holds. By Lemma 8.15, we get

$$\Delta \leq C_2 v_2 / v_{yp} (v_2).$$

Combining the equality above with $v_2 vv_{yp}^{1/2} (v_2) > M_0 n^{-1/2}$, it follows that

$$|\delta_n (z_2)| < v_2 / \left[v_{yp} (v_2) 10 (A + 1)^2\right].$$

This leads to a contradiction with $|\delta_n (z_2)| = v_2 / \left[v_{yp} (v_2) 10 (A + 1)^2\right]$. Hence, $v_1 vv_{yp}^{1/2} (v_1) = M_0 n^{-1/2}$ and $|\delta_n| \leq v_1 / \left[v_{yp} (v_1) 10 (A + 1)^2\right]$. By Lemma 8.14 and Lemma 8.15, we get $\Delta \leq C v_1 / v_{yp} (v_1)$.

We shall complete the proof of Theorem 2.1 in the following two cases.

**Case 1:** When $a_n > v_1$, we get

$$v_1 \sqrt{a_n} = v_1 vv_{yp}^{1/2} (v_1) = M_0 n^{-1/2} \Rightarrow v_1 \leq M_0 n^{-1/2} / \sqrt{a_n}$$

and

$$\Delta \leq C v_1 / \sqrt{a_n} \leq C n^{-1/2} a_n^{-3/4}.$$
Case 2: When \( a_n \leq v_1 \), one acquires
\[
v_1^{5/4} \leq v_1 v_{yp}^{1/2} (v_1) = M_0 n^{-1/2} \Rightarrow v_1 \leq M_0 n^{-2/5}
\]
and
\[
\Delta \leq C \sqrt{v_1} \leq C n^{-1/5}.
\]
Note that \( a_n = v_1 \) is equivalent to \( v_1 = \left( \frac{M_0}{2} \right)^{4/5} n^{-2/5} \). Thus, the above two cases is the same as stated in Theorem 2.1. This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.5

Applying Lemma 1.1, Lemma 7.3, and Lemma 8.16, one has
\[
\begin{align*}
    E \| F_p - F_{yp} \| & \leq \left( \int_{-A}^{A} E \left( |s_p(z) - s_{yp}(z)| \right) \right) + 2\pi v^{-1} \int_{|x| > B} |1 - EF_p(x)| \, dx \\
    & \quad + v^{-1} \sup_{x} \int_{|t| < 2av} |F_{yp}(x + t) - F_{yp}(x)| \, dt \\
    & \leq C \left( \int_{-A}^{A} E \left( |s_p(z) - Es_p(z)| \right) \right) + \int_{-A}^{A} \left( |Es_p(z) - s_{yp}(z)| \right) \, du \\
    & \quad + o \left( n^{-2} + v/v_{yp} \right).
\end{align*}
\]
By Lemma 7.2 we have
\[
E \left( |s_p(z) - Es_p(z)| \right) \leq \left[ E \left( |s_p(z) - Es_p(z)|^2 \right) \right]^{1/2} \leq C n^{-1} v^{-2} (\Delta + v/v_{yp})^{1/2}.
\]
We will estimate \( E \| F_p - F_{yp} \| \) according to the following two cases.

Case 1: When \( a_n < n^{-2/5} \), choosing \( v = M_1 n^{-2/5} \) and due to \( \Delta = O \left( n^{-1/5} \right) \), it follows that
\[
E \left( |s_p(z) - Es_p(z)| \right) \leq C n^{-3/10}.
\]
From the proof of Theorem 2.1 we have known that
\[
\int_{-A}^{A} \left( |Es_p(z) - s_{yp}(z)| \right) \, du = O \left( n^{-1/5} \right).
\]
Consequently, we obtain
\[
E \| F_p - F_{yp} \| = O \left( n^{-1/5} \right).
\]
Case 2: When \( a_n > n^{-2/5} \), selecting \( v = M_2 n^{-2/5} a_n^{1/10} \) and owing to \( \Delta = \left( n^{-1/2} a_n^{-3/4} \right) \), one gets

\[
E \left( |s_p(z) - Es_p(z)| \right) \leq C n^{-2/5} a_n^{-2/5}.
\]

From the proof of Theorem 2.1, it has been proved that

\[
\int_{-A}^{A} \left( |Es_p(z) - s_{yp}(z)| \right) du = O \left( n^{-1/2} a_n^{-3/4} \right).
\]

Hence, we have

\[
E \left\| F_p - F_{yp} \right\| = O \left( n^{-2/5} a_n^{-2/5} \right).
\]

The proof of Theorem 2.5 is complete.

6. Proof of Theorem 2.6

By (5.1) and the proof of Theorem 2.5, it suffices to show that

\[
\int_{-A}^{A} E \left( |s_p(z) - s_{yp}(z)| \right) du = \begin{cases} 
O_{a.s.} \left( n^{-1/5} \right), & \text{if } a_n < n^{-2/5}, \\
O_{a.s.} \left( n^{-2/5+\eta} a_n^{-2/5} \right), & \text{if } n^{-2/5} \leq a_n < 1.
\end{cases}
\]

By Lemma 7.2, we have

\[
E \left( |s_p(z) - Es_p(z)|^{2l} \right) \leq C n^{-2l} v^{-4l} \left( \Delta + v/v_{yp} \right)^l.
\]

We proceed to complete the proof by two cases.

Case 1: When \( a_n < n^{-2/5} \), choosing \( v = M_1 n^{-2/5} \), we obtain

\[
n^{2l/5} E \left( |s_p(z) - Es_p(z)|^{2l} \right) \\
\leq C n^{2l/5} n^{-2l} v^{-4l} \left( \Delta + v/v_{yp} \right)^l \\
\leq C n^{-l/5}.
\]

Then the result follows by choosing \( l > 5 \).

Case 2: When \( a_n > n^{-2/5} \), selecting \( v = M_2 n^{-2/5} a_n^{1/10} \), one has

\[
n^{2l(2/5-\eta)} n^{4l/5} E \left( |s_p(z) - Es_p(z)|^{2l} \right) \\
\leq C n^{2l(2/5-\eta)} n^{-2l} v^{-4l} \left( \Delta + v/v_{yp} \right)^l \\
\leq C n^{-2l\eta}.
\]

Then the result follows by choosing \( l > 1/(2\eta) \).

This completes the proof of Theorem 2.6.
7. Some Auxiliary Lemmas

In this section, we establish three lemmas which are of importance in proving the main theorems.

Lemma 7.1. For \( z = u + iv \) with \( v > 0 \), one gets

\[
\text{Etr}
\left(
(S_{nk} - uI_{2p-2})^2 + v^2I_{2p-2}
\right)^{-1}
\leq \frac{C}{v^2} \left( \Delta + v/v_{yp} \right).
\]

Proof. By Lemma 8.12 and Lemma 8.13, it follows that

\[
\text{Etr}
\left(
(S_{nk} - uI_{2p-2})^2 + v^2I_{2p-2}
\right)^{-1}
= \frac{1}{v} \text{E} \text{E}^\prime \text{tr}
\left(
S_{nk} - zI_{2p-2}
\right)^{-1}
\leq \frac{1}{v} \text{E} \text{E}^\prime \text{tr}
\left(
S_{n} - zI_{2p}
\right)^{-1} + \frac{1}{v^2}
\leq \frac{C}{v} \left( |Es_p(z) - s_{yp}(z)| + |s_{yp}(z)| \right) + \frac{1}{v^2}
\leq \frac{C}{v} \left( \Delta + v + 1/(\sqrt{v}v_{yp}) \right) + \frac{1}{v^2}
\leq \frac{C}{v^2} \left( \Delta + v/v_{yp} \right).
\]

\[\square\]

Lemma 7.2. If \( |z| < A, \ v > n^{-1/2}, \ |b_n| \leq 2/\sqrt{y_p|z|}, \) and \( l \geq 1 \), then

\[
\text{E} |s_p(z) - Es_p(z)|^{2l} \leq \frac{C}{n^{2l}v^{4l}y_p^{2l}} \left( \Delta + v/v_{yp} \right)^l
\]

where \( A \) is defined in Lemma 1.1.

Proof. Write \( \text{E}_k(.) \) as the conditional expectation given \( \{x_{lj}; l \leq k, j \leq n\} \). Then

\[s_p(z) - Es_p(z) = \frac{1}{p} \sum_{k=1}^{p} \left[ \text{E}_k \text{tr} \left( S_n - zI_p \right)^{-1} - \text{E}_{k-1} \text{tr} \left( S_n - zI_p \right)^{-1} \right]
\]

\[= \frac{1}{p} \sum_{k=1}^{p} \gamma_k,
\]

where

\[\gamma_k = (E_k - E_{k-1}) \left[ \text{tr} \left( S_n - zI_{2p} \right)^{-1} - \text{tr} \left( S_{nk} - zI_{2p-2} \right)^{-1} \right]
\]

\[= - (E_k - E_{k-1}) \text{tr} \left[ \left( \frac{1}{n^2} \phi_k^* X_{nk}^* (S_{nk} - zI_{2p-2})^{-2} X_{nk} \phi_k + I_2 \right) \xi_k \right]
\]

\[\Delta (E_k - E_{k-1}) \sigma_k.
\]
By Lemma 8.9 we have $|\sigma_k| \leq 2/v$. Applying Lemma 8.11 one has

$$(7.1) \quad E |s_p (z) - E s_p (z)|^2 \leq C n^{-2l} \left\{ E \left( \sum_{k=1}^{p} E_{k-1} |\gamma_k|^2 \right) + \sum_{k=1}^{p} E |\gamma_k|^2 \right\}.$$ 

Note that $\varepsilon_k = \theta_k I_2$ and $\xi_k = b_n I_2 + b_n \xi_k \varepsilon_k$, then, $\gamma_k$ can be written as

$$\gamma_k = - b_n n^{-2} \left( E_k \text{tr} \phi_k^* X_n \left( S_n - z I_{2p-2} \right)^{-2} X_n \phi_k \right)$$

$$(7.2) \quad - E_{k-1} \text{tr} X_n^* \left( S_n - z I_{2p-2} \right)^{-2} X_n \right) + (E_k - E_{k-1}) b_n \theta_k \sigma_k.$$ 

Using Lemma 8.8 and the condition $|b_n| \leq 2/\sqrt{v}$, we have

$$n^{-4} |b_n|^2 E_{k-1} \left[ \left| \text{tr} \phi_k^* X_n \left( S_n - z I_{2p-2} \right)^{-2} X_n \phi_k \right| \right]^2$$

$$\leq \frac{C}{n^2} |b_n|^2 E_{k-1} \left( (S_n - z I_{2p-2})^{-2} S_n (S_n - z I_{2p-2})^{-2} S_n \right)$$

$$\leq \frac{C}{n^2 v^2 |z|} E_{k-1} \left( (S_n - z I_{2p-2})^{-1} S_n (S_n - z I_{2p-2})^{-1} S_n \right)$$

$$\leq \frac{C}{n^2 v^2 |z|} E_{k-1} \left[ n + |z|^2 \text{tr} \left( (S_n - z I_{2p-2})^{-1} (S_n - z I_{2p-2})^{-1} \right) \right]$$

$$\leq \frac{C}{n^2 v^2 |z|} E_{k-1} \left[ n + |z|^2 + \frac{n |z|^2}{v} \text{tr} s_p (z) \right]$$

$$(7.3) \quad \leq \frac{C}{nv^3} E_{k-1} \left( 1 + \text{tr} s_p (z) \right).$$

Recall that

$$\text{tr} \varepsilon_k = \text{tr} \left( \frac{1}{n} \phi_k^* \phi_k - I_2 + y_p I_2 + y_p z E s_p (z) I_2 - \frac{1}{n^2} \phi_k^* X_n^* (S_n - z I_{2p-2})^{-1} X_n \phi_k \right)$$

$$= \text{tr} \left( \frac{1}{n} \phi_k^* \phi_k - I_2 \right) + \frac{2}{n} + \frac{z}{n} \left( \text{tr} (S_n - z I_{2p})^{-1} - \text{tr} (S_n - z I_{2p})^{-1} \right)$$

$$- \frac{1}{n^2} \left( \text{tr} \phi_k^* X_n^* (S_n - z I_{2p-2})^{-1} X_n \phi_k - \text{tr} X_n^* (S_n - z I_{2p-2})^{-1} X_n \right),$$
then we obtain

\[ E_{k-1} (|E_k - E_{k-1}| b_n \theta_k \sigma_k)^2 \]

\[ \leq \frac{1}{y_p |z| \nu^2} E_{k-1} |\text{tr} \varepsilon_k|^2 \]

\[ \leq \frac{C}{y_p |z| \nu^2} \left[ E \left| \text{tr} \left( \frac{1}{n} \phi_k^* \phi_k - I_2 \right) \right|^2 + \frac{|z|^2}{n^2} E_{k-1} \left| \text{tr} (S_n - z I_{2p})^{-1} - \text{Etr} (S_n - z I_{2p})^{-1} \right|^2 \right. \]

\[ + \left. \frac{|z|^2}{n^2} E_{k-1} \left| \text{tr} (S_n - z I_{2p})^{-1} - \text{tr} (S_{nk} - z I_{2p-2})^{-1} \right|^2 + \frac{1}{n^2} \right] \]

\[ + \frac{1}{n^3} E_{k-1} \left| \text{tr} \phi_k^* X_{nk}^* (S_{nk} - z I_{2p-2})^{-1} X_{nk}^* (S_{nk} - z I_{2p-2})^{-1} X_{nk} \right|^2 \]

\[ \leq \frac{C}{y_p |z| \nu^2} \left( \frac{1}{n} + \frac{y_p^2 |z|^2 E_{k-1} |s_p(z) - \text{Es}_p(z)|^2}{n^2 \nu^2} \right) \]

\[ + \frac{1}{n^3} E_{k-1} \text{tr} \left( X_{nk}^* (S_{nk} - z I_{2p-2})^{-1} X_{nk} X_{nk}^* (S_{nk} - z I_{2p-2})^{-1} X_{nk} \right) \]

\[ \leq \frac{C}{y_p |z| \nu^2} \left( \frac{|z|^2}{n} + \frac{|u|^2}{n^2 \nu} E_{k-1} \text{tr} (S_{nk} - z I_{2p-2})^{-1} + \frac{y_p^2 |z|^2 E_{k-1} |s_p(z) - \text{Es}_p(z)|^2}{n^2 \nu^2} \right) \]

\[ \leq \frac{C}{y_p |z| \nu^2} \left( \frac{|z|^2}{n} + \frac{y_p |u|^2}{n^2 \nu} E_{k-1} \text{Es}_p(z) + \frac{y_p^2 |z|^2 E_{k-1} |s_p(z) - \text{Es}_p(z)|^2}{n^2 \nu^2} \right) \]

(7.4)

\[ \leq \frac{C}{y_p |z| \nu^2} E_{k-1} (1 + \text{Es}_p(z)) \]

Combining (7.2), (7.3), and (7.4), for large \( n \), the first term on the right-hand side of (7.1) is dominated by

\[ C n^{-2l} E \left( \sum_{k=1}^{p} E_{k-1} |\gamma_k|^2 \right)^l \leq \frac{C}{n^2 \nu^2 \nu^4} E (1 + \text{Es}_p(z))^l + \frac{C}{n^2 \nu^2 \nu^4} E |s_p(z) - \text{Es}_p(z)|^{2l}. \]

Similarly, by (7.2), one gets

\[ n^{-4l} |b_n^{2l}| E \left( \text{tr} \phi_k^* X_{nk}^* (S_{nk} - z I_{2p-2})^{-2} X_{nk} \phi_k \right. \]

\[ \left. - \text{tr} (X_{nk}^* (S_{nk} - z I_{2p-2})^{-2} X_{nk}) \right)^{2l} \]

\[ \leq \frac{C}{n^{2l} |b_n|^{2l}} E \left\{ \text{tr} \left( (S_{nk} - z I_{2p-2})^{-2} S_{nk} (S_{nk} - z I_{2p-2})^{-2} S_{nk} \right)^l \right\} \]

\[ + \left[ \text{tr} \left( (S_{nk} - z I_{2p-2})^{-2} S_{nk} (S_{nk} - z I_{2p-2})^{-2} S_{nk} \right)^l \right] \]
where the third inequality follows from the fact that $\varphi_{4l} \leq C n^{l-1}$. And

$$\mathbb{E} \left| (E_k - E_{k-1}) b_n \theta_k \sigma_k \right|^{2l}$$

$$\leq \frac{C}{y_p |z|^{2l}} \mathbb{E} \left| \text{tr} \phi_k \bar{\phi}_k - I \right|^{2l} + \frac{|z|^{2l}}{n^{2l}} \mathbb{E} \left| \text{tr} (S_n - z I_p) - E \right|^{2l}$$

$$\leq \frac{C}{y_p |z|^{2l}} \left\{ \frac{1}{n^l} + \frac{y_p^2 |z|^{2l} \mathbb{E} |s_p(z) - E s_p(z)|^{2l}}{n^{2l} v^{2l}} + \frac{|z|^{2l}}{n^{2l} v^{2l}} \right\}$$

$$\leq \frac{C}{y_p |z|^{2l}} \left\{ \frac{|z|^{2l}}{n^l} + \frac{1}{n^{2l}} \mathbb{E} \left[ \varphi_4 \left( (S_n - z I_{2p-2})^{-1} S_n (S_n - z I_{2p-2})^{-1} S_n \right)^t \right] \right\}$$
\[
+ y_p^l |z|^{2l} E |s_p(z) - E s_p(z)|^{2l}
\]
\[
\leq \frac{C}{n^l y_p^l |z|^{2l}} \left( |z|^{2l} + v^{-2l+1} |z|^{2l} E \Im s_p(z) + v^{-l} |z|^{2l} E (\Im s_p(z))' + \frac{1}{nv^{2l}} \right)
\]
\[
+ \frac{C v}{v^{2l}} E |s_p(z) - E s_p(z)|^{2l}
\]
(7.6)
\[
\leq \frac{C}{n^l y_p^{3l} y_p^l} E \left( 1 + v^{-l+1} \Im s_p(z) + \frac{1}{nv^l} \right) + \frac{C}{v^{2l}} E |s_p(z) - E s_p(z)|^{2l}.
\]
Therefore, together with the two inequalities above, the second term on the right hand side of (7.1) is bounded by
\[
C n^{-2l} \sum_{k=1}^{p} E |\gamma_k|^{2l} \leq \frac{C}{n^{3l-1} v^{3l} y_p^l} E \left( 1 + v^{-l+1} \Im s_p(z) + \frac{1}{nv^l} \right)
\]
\[
+ \frac{C}{n^{2l-1} v^{2l}} E |s_p(z) - E s_p(z)|^{2l}.
\]
Consequently, we obtain
\[
E |s_p(z) - E s_p(z)|^{2l} \leq \frac{C}{n^{2l} v^{2l} y_p^l} \left[ 1 + E (1 + \Im s_p(z))' + n^{-l+1} v^{-l+1} E \Im s_p(z) \right]
\]
\[
+ \frac{C}{n^{2l-1} v^{2l}} E |s_p(z) - E s_p(z)|^{2l}
\]
(7.7)
\[
\leq \frac{C}{n^{2l} v^{2l} y_p^l} \left[ 1 + E (\Im s_p(z))' + n^{-l+1} v^{-l+1} E \Im s_p(z) \right].
\]
Now, we shall complete the proof of the lemma by using induction on \(l\) and the inequality (7.7).

**step 1:** When \(l = 1\), by Lemma 8.12 and Lemma 8.13, one has
\[
E |s_p(z) - E s_p(z)|^2 \leq \frac{C}{n^{2v^2} y_p^l} [1 + E (\Im s_p(z))]
\]
\[
\leq \frac{C}{n^{2v^2} y_p^l} [1 + |E s_p(z) - s_{yp}(z)| + |s_{yp}(z)|]
\]
\[
\leq \frac{C}{n^{2v^2} y_p^l} [1 + \Delta/v + 1/ (\sqrt{y_p v_{yp}})]
\]
\[
\leq \frac{C}{n^{2v^2} y_p^l} (\Delta + v/ v_{yp}.
\]

**step 2:** In the final step, we need the case \(l \in (\frac{1}{2}, 1)\). Therefore, we shall extend the lemma to \(l \in (\frac{1}{2}, 1)\). By Lemma 8.10 and the first step, it
follows that
\[
E|s_p(z) - E_{sp}(z)|^{2l} \leq \frac{C}{n^{2l}} E\left(\sum_{k=1}^{p} |\gamma_k|^2\right)^l \\
\leq \frac{C}{n^{2l}} \left(\sum_{k=1}^{p} E|\gamma_k|^2\right)^l \\
\leq C (E|s_p(z) - E_{sp}(z)|^2)^l \\
\leq \frac{C}{n^{2l} \delta^4 l^{2l}} (\Delta + v/v_{yp})^l.
\]

**step 3:** Suppose that, for \( l \in (2^{t-1}, 2^t) \), \( t = 0, 1, \ldots, k - 1 \), the lemma is true. Then consider the case \( l \in (2^k, 2^{k+1}) \). By (7.7), we have
\[
E|s_p(z) - E_{sp}(z)|^{2l} \leq \frac{C}{n^{2l} \delta^4 l^{2l}} \left[1 + E|s_p(z) - E_{sp}(z)|^l + |E_{sp}(z)|^l \right] \\
\leq \frac{C}{n^{2l} \delta^4 l^{2l}} \left[1 + \left(\frac{\Delta + v/v_{yp}}{\sqrt{n} l^{2l} y_p^l}\right)^{l/2} + \left|\frac{\Delta}{v} + 1/\left(\sqrt{y_p} v_{yp}\right)\right|^l \right] \\
\leq \frac{C}{n^{2l} \delta^4 l^{2l}} \left(\Delta + v/v_{yp}\right)^l.
\]
Then, the proof of the lemma is complete. \(\square\)

**Lemma 7.3.** Under the conditions of Theorem 2.1 and the additional assumption \( \|x_{jk}\| \leq n^{-1/4} \), for any fixed \( t > 0 \), we have
\[
\int_B^\infty |EF_p(x) - F_{yp}(x)| \, dx = o\left(n^{-t}\right),
\]
where \( B = b_n + 1 = \sqrt{y_p} + 2 \).

**Proof.** In [14], it has proved that, for any \( \xi > 0 \) and \( m = [\log n] \),
\[
E(\lambda_{max}(S_n))^m \leq (b + \xi)^m.
\]
Note that
\[
1 - F_p(x) \leq I(\lambda_{max}(S_n) \geq x), \text{ for } x \geq 0.
\]
Then, it follows that
\[
\int_B^{\infty} \left| F_p(x) - F_{y_p}(x) \right| \, dx \leq \int_B^{\infty} P(\lambda_{\max}(S_n) \geq x) \\
\leq \int_B^{\infty} \left( \frac{b + \xi}{x} \right)^m \, dx = O \left( \left( \frac{b + \xi}{B} \right)^{m-1} \right)
\]
which completes the proof of this lemma. \(\square\)

8. APPENDIX

In this section, to be self-contained, we shall present some existing results which will be used in the proof of the main theorems.

**Lemma 8.1** (Theorem A.44 in [5]). Let \(A\) and \(B\) be two \(m \times k\) complex matrices. Then,
\[
\|F^{AA^*} - F^{BB^*}\| \leq \frac{1}{m} \text{rank} (A - B)
\]
where \(\|g\| = \sup_x |g(x)|\).

**Lemma 8.2** (Bernstein’s inequality). If \(Y_1, \cdots, Y_k\) are independent random variables with mean zeros and uniformly bounded by \(K\), then, for any \(\varepsilon > 0\),
\[
P \left( \left| \sum_j Y_j \right| \geq \varepsilon \right) \leq 2 \exp \left\{ -\varepsilon^2 / \left[ 2 \left( B_n^2 + K\varepsilon \right) \right] \right\}
\]
where \(B_n^2 = \text{Var} \left( \sum_j Y_j \right)\).

**Lemma 8.3** (Theorem A.45 in [5]). Let \(A\) and \(B\) be two \(m \times m\) Hermitian matrices. Then,
\[
L \left( F^A, F^B \right) \leq \|A - B\|_2
\]
where \(L\) is the Levy distance between two two-dimensional distribution functions \(F\) and \(G\) defined by
\[
L \left( F, G \right) = \inf \{ \varepsilon : F(\xi - \varepsilon, \eta - \varepsilon) - \varepsilon \leq G(\xi, \eta) \leq F(\xi + \varepsilon, \eta + \varepsilon) + \varepsilon \}.
\]

**Lemma 8.4** (Theorem A.47 in [5]). Let \(A\) and \(B\) be two \(m \times k\) complex matrices. Then,
\[
L \left( F^{AA^*}, F^{BB^*} \right) \leq 2 \|A\|_2 \|A - B\|_2 + \|A - B\|_2^2.
\]
Lemma 8.5 (Lemma B.19 in [5]). Let \( F_1, F_2 \) be distribution functions and let \( G \) satisfy \( \sup_x |G(x + \theta) - G(x)| \leq g(\theta) \), for all \( \theta \), where \( g \) is an increasing and continuous function such that \( g(0) = 0 \). Then
\[
\|F_1 - G\|_2 \leq 3 \max \{ \|F_2 - G\|_2, L(F_1, F_2), g(L(F_1, F_2)) \}.
\]

Remark 8.6 (Lemma 8.14 in [5]). For the M-P law with index \( y \leq 1 \), the function \( g \) can be taken as \( g(v) = 2v/ (\sqrt{a} + \sqrt{v}) \).

Lemma 8.7 (Inversion formula for block matrix). Suppose that the matrix \( \Sigma \) is nonsingular and has the partition as given by \( \left( \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array} \right) \). If \( \Sigma_{11} \) is also singular, then, the inverse of \( \Sigma \) has the from
\[
\Sigma^{-1} = \left( \begin{array}{cc} \Sigma_{11}^{-1} + \Sigma_{11}^{-1} \Sigma_{12} \Sigma_{21}^{-1} \Sigma_{11}^{-1} & -\Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \\ -\Sigma_{21} \Sigma_{12} \Sigma_{22}^{-1} & \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{12} \Sigma_{22}^{-1} \end{array} \right)
\]
where \( \Sigma_{22}^{-1} = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} \).

Lemma 8.8 (Lemma 2.18 in [20]). Let \( A = (a_{jk})_{j,k=1}^{2n} \) be a \( 2n \times 2n \) non-random matrix and \( X = (x'_1, \ldots, x'_n)' \) be a random quaternion vector of independent entries. Assume that \( \mathbb{E} x_j = 0 \), \( \mathbb{E} \|x_j\|^2 = 1 \), and \( \mathbb{E} \|x_j\|^4 \leq \varphi_I \). Then, for any \( m \geq 1 \), we have
\[
\mathbb{E} |\text{tr} X^* AX - \text{tr} A|^m \leq C_m \left( (\varphi_4 \text{tr}(AA^*))^{m/2} + \varphi_{2m} \text{tr}(AA^*)^{m/2} \right),
\]
where \( C_m \) is a constant depending on \( m \) only.

Lemma 8.9 (see (A.1.12) in [5]). Let \( z = u + iv \), \( v > 0 \), and let \( A \) be an \( n \times n \) Hermitian matrix, \( A_k \) be the \( k \)-th major sub-matrix of \( A \) of order \( (n - 1) \), to be the matrix resulting from the \( k \)-th row and column from \( A \). Then
\[
|\text{tr}(A - zI_n)^{-1} - \text{tr}(A_k - zI_{n-1})^{-1}| \leq \frac{1}{v}.
\]

Lemma 8.10 (Lemma 2.12 in [5]). Let \( \{\tau_k\} \) be a complex martingale difference sequence with respect to the increasing \( \sigma \)-fields \( \mathcal{F}_k \). Then, for \( p > 1 \), \( \mathbb{E} |\sum \tau_k|^p \leq K_p \mathbb{E}(\sum |\tau_k|^2)^{p/2} \).

Lemma 8.11 (Rosenthal’s inequality). Let \( X_i \) are independent with zero means, then we have, for some constant \( C_k \):
\[
\mathbb{E} \left| \sum X_i \right|^{2k} \leq C_k \left( \sum \mathbb{E} |X_i|^{2k} + \left( \sum \mathbb{E} |X_i|^2 \right)^k \right).
\]

Lemma 8.12 (Lemma 8.17 in [5]). For the Stieltjes transform of the M-P law, we have
\[
|s_{y_p}(z)| \leq \frac{\sqrt{2}}{\sqrt{y_p t y_p}}.
\]
Lemma 8.13 (Lemma B.22 in [5]). Let $G$ be a function of bounded variation. Let $g(z)$ denote its Stieltjes transform. When $z = u + iv$, with $v > 0$, we have
\[
\sup_u |g(z)| \leq \pi v^{-1} \|G\|.
\]

Lemma 8.14 (Lemma 8.17 in [5]). For all $z \in \mathbb{C}^+$, when $|\delta| \leq v/ \left[ v_{yp} 10 (A + 1)^2 \right]$, we have
\[
|b_n| \leq \frac{2}{\sqrt{y_p} \cdot |z|}.
\]

Lemma 8.15 (Lemma 8.21 in [5]). If $|\delta_n| < v/ \left[ v_{yp} 10 (A + 1)^2 \right]$ for all $|z| < A$, then there is a constant $C$ such that
\[
\Delta \leq Cv/v_{yp}
\]
where $A$ is defined in Lemma 7.1 for the M-P law with index $y \leq 1$.

Lemma 8.16. For $v > n^{-1/2}$, we have
\[
\sup_x \int_{|u| < v} \left| F_{yp} (x + u) - F_{yp} (x) \right| du \leq \frac{11\sqrt{2(1+y)}}{3\pi y} v^2/v_{yp},
\]
where $F_{yp}$ is the M-P law with index $y \leq 1$.

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