Multiple periodic solutions of Lagrangian systems of relativistic oscillators

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Abstract. - Let $B_L$ the open ball in $\mathbb{R}^n$ centered at 0, of radius $L$, and let $\phi$ be a homeomorphism from $B_L$ onto $\mathbb{R}^n$ such that $\phi(0) = 0$ and $\phi = \nabla \Phi$, where the function $\Phi : \overline{B_L} \to [-\infty, 0]$ is continuous and strictly convex in $\overline{B_L}$, and of class $C^1$ in $B_L$. Moreover, let $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a function which is measurable in $[0, T]$, of class $C^1$ in $\mathbb{R}^n$ and such that $\nabla_x F$ satisfies the $L^1$-Carathéodory conditions. Set

$$K = \{ u \in \text{Lip}([0, T], \mathbb{R}^n) : |u'(t)| \leq L \text{ for a.e. } t \in [0, T], u(0) = u(T) \} ,$$

and define the functional $I : K \to \mathbb{R}$ by

$$I(u) = \int_0^T (\Phi(u'(t)) + F(t, u(t)))dt$$

for all $u \in K$. In [1], Brezis and Mawhin proved that any global minimum of $I$ in $K$ is a solution of the problem

$$\begin{cases}
(\phi(u'))' = \nabla_x F(t, u) & \text{in } [0, T] \\
u(0) = u(T), \quad u'(0) = u'(T) .
\end{cases}$$

(P$_{\phi, F}$)

In the present paper, we provide a set of conditions under which the functional $I$ has at least two global minima in $K$. This seems to be the first result of this kind. The main tool of our proof is the well-posedness result obtained in [3].

1. - Introduction

As the reader can notice, the title of the present paper is, intentionally, almost identical to the one of [1].

Actually, it is our aim to show how to obtain the multiplicity of periodic solutions for the systems mentioned in the title making a joint use of the theory developed by Brezis and Mawhin in [1] with that we developed in [3].

To be more precise, we now fix some notations that we will keep throughout the paper and recall the main result of [1].

$L, T$ are two fixed positive numbers. For each $r > 0$, we set $B_r = \{ x \in \mathbb{R}^n : |x| < r \}$ ($|\cdot|$ being the Euclidean norm on $\mathbb{R}^n$) and $\overline{B_r}$ is the closure of $B_r$.

We denote by $A$ the family of all homeomorphisms $\phi$ from $B_L$ onto $\mathbb{R}^n$ such that $\phi(0) = 0$ and $\phi = \nabla \Phi$, where the function $\Phi : \overline{B_L} \to [-\infty, 0]$ is continuous and strictly convex in $\overline{B_L}$, and of class $C^1$ in $B_L$. Notice that 0 is the unique global minimum of $\Phi$ in $\overline{B_L}$.

We denote by $B$ the family of all functions $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ which are measurable in $[0, T]$, of class $C^1$ in $\mathbb{R}^n$ and such that $\nabla_x F$ is measurable in $[0, T]$ and, for each $r > 0$, one has $\sup_{x \in \overline{B_r}} |\nabla_x F(\cdot, x)| \in L^1([0, T])$.

Given $\phi \in A$ and $F \in B$, we consider the problem

$$\begin{cases}
(\phi(u'))' = \nabla_x F(t, u) & \text{in } [0, T] \\
u(0) = u(T), \quad u'(0) = u'(T) .
\end{cases}$$

(P$_{\phi, F}$)
A solution of this problem is any function $u : [0, T] \to \mathbb{R}^n$ of class $C^1$, with $u'(t) \in B_L$, $u(0) = u(T)$, such that the composite function $\phi \circ u'$ is absolutely continuous in $[0, T]$ and one has $(\phi \circ u')(t) = \nabla_x F(t, u(t))$ for a.e. $t \in [0, T]$.

Now, we set
$$K = \{ u \in \text{Lip}([0, T], \mathbb{R}^n) : |u'(t)| \leq L \text{ for a.e. } t \in [0, T], u(0) = u(T) \},$$

$\text{Lip}([0, T], \mathbb{R}^n)$ being the space of all Lipschitzian functions from $[0, T]$ into $\mathbb{R}^n$.

Clearly, one has
$$\sup_{[0, T]} |u| \leq LT + \inf_{[0, T]} |u|$$

for all $u \in K$. To see this, take $t_0 \in [0, T]$ such that $|u(t_0)| = \inf_{[0, T]} |u|$ and observe that, for each $t \in [0, T]$, one has
$$|u(t) - u(t_0)| = \left| \int_{t_0}^{t} u'(\tau) d\tau \right| \leq LT.$$

Next, consider the functional $I : K \to \mathbb{R}$ defined by
$$I(u) = \int_{0}^{T} (\Phi(u'(t)) + F(t, u(t))) dt$$

for all $u \in K$.

The basic result of the theory developed in [1] is as follows:

**THEOREM 1.1** ([1], Theorem 5.2). - *Any global minimum of $I$ in $K$ is a solution of problem $(P_{\phi, F})$.*

Well, the aim of the present paper is to provide a set of conditions under which the functional $I$ has at least two global minima in $K$.

As far as we know, this is the first result of this kind, and so we cannot do any proper comparison with previous ones.

Notice that some multiplicity results for problem $(P_{\phi, F})$ are already available in the literature. In this connection, we refer to the numerous references contained in the very recent survey by Mawhin [2] and in [4]. But, as we repeat, in those papers the multiple solutions of problem $(P_{\phi, F})$ are not shown to be global minima of the functional $I$ in $K$.

As we said at the beginning, our main tool is provided by the main result obtained in [3] which is recalled in the next section.

Finally, Section 3 contains the statement of our multiplicity result, its proof and various related remarks.

2. - A well-posedness theorem

In this section, we summarize the theory developed in [3].

So, let $X$ be a Hausdorff topological space, $J, \Psi$ two real-valued functions defined in $X$, and $a, b$ two numbers in $[-\infty, +\infty]$, with $a < b$.

If $a \in \mathbb{R}$ (resp. $b \in \mathbb{R}$), we denote by $M_a$ (resp. $M_b$) the set of all global minima of the function $J + a\Psi$ (resp. $J + b\Psi$), while if $a = -\infty$ (resp. $b = +\infty$), $M_a$ (resp. $M_b$) stands for the empty set. We adopt the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$.

We also set
$$\alpha = \max \left\{ \inf_{X} \Psi, \sup_{M_a} \Psi \right\},$$
$$\beta = \min \left\{ \sup_{X} \Psi, \inf_{M_b} \Psi \right\}.$$

One proves that $\alpha \leq \beta$.  

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As usual, given a function \( f : X \to \mathbb{R} \) and a set \( C \subseteq X \), we say that the problem of minimizing \( f \) over \( C \) is well-posed if the following two conditions hold:
- the restriction of \( f \) to \( C \) has a unique global minimum, say \( \hat{x} \);
- every sequence \( \{x_n\} \) in \( C \) such that \( \lim_{n \to \infty} f(x_n) = \inf_C f \), converges to \( \hat{x} \).

A set of the type \( \{x \in X : f(x) \leq r\} \) is said to be a sub-level set of \( f \).

The main result of [3] is as follows:

**THEOREM 2.1 ([3], Theorem 1).** - Assume that \( \alpha < \beta \) and that, for each \( \lambda \in [a, b] \), the function \( J + \lambda \Psi 
\) has sequentially compact sub-level sets and admits a unique global minimum in \( X \).

Then, for each \( r \in [\alpha, \beta] \), the problem of minimizing \( J \) over \( \Psi^{-1}(r) \) is well-posed.

Moreover, if we denote by \( \hat{x}_r \) the unique global minimum of \( J_{\Psi^{-1}(r)} \) (\( r \in [\alpha, \beta] \)), the functions \( r \to \hat{x}_r \) and \( r \to J(\hat{x}_r) \) are continuous in \( [\alpha, \beta] \), and, for some \( \hat{\lambda}_r \in [a, b] \), \( \hat{\lambda}_r \) is the global minimum in \( X \) of the function \( J + \hat{\lambda}_r \Psi \).

### 3. - The main result

Here is our main result:

**THEOREM 3.1.** - Let \( \phi \in \mathcal{A} \), \( F \in \mathcal{B} \), \( G \in C^1(\mathbb{R}^n) \), \( \psi \in L^1([0, T]) \setminus \{0\} \), with \( \psi \geq 0 \). Moreover, let \( \gamma : [0, +\infty] \to \mathbb{R} \) be a convex strictly increasing function such that \( \lim_{s \to +\infty} \frac{\gamma(s)}{s} = +\infty \). Assume that the following assumptions are satisfied:

(i) for a.e. \( t \in [0, T] \) and for every \( x \in \mathbb{R}^n \), one has
\[
\gamma(|x|) \leq F(t, x) ;
\]

(ii) \( \liminf_{|x| \to +\infty} \frac{G(x)}{|x|^2} > -\infty ; \)

(iii) the function \( G \) has no global minima in \( \mathbb{R}^n ; \)

(iv) there exist two points \( x_1, x_2 \in \mathbb{R}^n \) such that
\[
\inf_{x \in \mathbb{R}^n} \int_0^T F(t, x) dt < \max \left\{ \int_0^T F(t, x_1) dt, \int_0^T F(t, x_2) dt \right\}
\]
and
\[
G(x_1) = G(x_2) = \inf_{B_c} G
\]
where
\[
c = LT + \gamma^{-1} \left( \frac{1}{T} \max \left\{ \int_0^T F(t, x_1) dt, \int_0^T F(t, x_2) dt \right\} \right) .
\]

Then, there exist \( \hat{\lambda} > 0 \) such that the problem
\[
\begin{cases}
(\phi(u'))' = \nabla_x (F(t, u) + \hat{\lambda} \psi(t)G(u)) & \text{in } [0, T] \\
u(0) = u(T), \ u'(0) = u'(T)
\end{cases}
\]
has at least two solutions which are global minima in \( K \) of the functional
\[
u \to \int_0^T (\Phi(u'(t)) + F(t, u(t)) + \hat{\lambda} \psi(t)G(u(t))) dt .
\]

**PROOF.** Let \( C^0([0, T], \mathbb{R}^n) \) be the space of all continuous functions from \( [0, T] \) into \( \mathbb{R}^n \), with the norm \( \sup_{[0, T]} |u| \). We are going to apply Theorem 2.1 taking: \( a = 0, \ b = +\infty, \ X = K \) regarded as a subset of \( C^0([0, T], \mathbb{R}^n) \) with the relative topology and
\[
J(u) = \int_0^T \psi(t)G(u(t)) dt ,
\]
\[ \Psi(u) = \int_0^T (\Phi(u'(t)) + F(t, u(t)))dt \]

for all \( u \in K \). Fix \( \lambda > 0 \). By \((i_2)\), for a suitable constant \( \delta > 0 \), we have

\[-\delta(|x| + 1) \leq G(x) \]

for all \( x \in \mathbb{R}^n \). For each \( u \in K \), in view of \((i_1)\), \((1.1)\) and of the convexity of \( \gamma \), using Jensen inequality, we get

\[
\begin{align*}
\int_0^T \psi(t)G(u(t))dt + \lambda \int_0^T (\Phi(u'(t)) + F(t, u(t)))dt & \geq -\delta \int_0^T \psi(t)|u(t)|dt + \lambda \int_0^T \gamma(|u(t)|)dt - \delta \int_0^T \psi(t)dt + \lambda \Phi(0)T \\
& \geq -\delta \int_0^T \psi(t)dt \sup_{[0,T]} |u| + \lambda T \gamma \left( \frac{1}{T} \int_0^T |u(t)|dt \right) - \delta \int_0^T \psi(t)dt + \lambda \Phi(0)T \\
& \geq -\delta \int_0^T \psi(t)dt \sup_{[0,T]} |u| + \lambda T \gamma \left( \inf_{[0,T]} |u| \right) - \delta \int_0^T \psi(t)dt + \lambda \Phi(0)T \\
& = -\delta \int_0^T \psi(t)dt \sup_{[0,T]} |u| + \lambda T \gamma \left( \sup_{[0,T]} |u| - LT \right) - \delta \int_0^T \psi(t)dt + \lambda \Phi(0)T .
\end{align*}
\]  \((3.1)\)

In turn, since \( \lim_{s \to +\infty} \frac{\gamma(s) - LT}{s} = +\infty \), we infer from \((3.1)\) that, for every \( \rho \in \mathbb{R} \), there is \( M > 0 \) such that

\[
\left\{ u \in K : \int_0^T \psi(t)G(u(t))dt + \lambda \int_0^T (\Phi(u'(t)) + F(t, u(t)))dt \leq \rho \right\} \subseteq \left\{ u \in K : \sup_{[0,T]} |u| \leq M \right\} . \quad (3.2)
\]

Now, observe that \( K \) is a closed subset of \( C^0([0,T], \mathbb{R}^n) \). On the other hand, from Lemma 4.1 of \([1]\), it follows that the functional \( J + \lambda \Psi \) is lower semicontinuous in \( K \). Summarizing: the functions belonging to any sub-level set of \( J + \lambda \Psi \) are equi-continuous (since they are in \( K \)) and equi-bounded in view of \((3.2)\). Hence, by the Ascoli-Arzelà theorem, any sub-level set of \( J + \lambda \Psi \) is relatively sequentially compact in \( C^0([0,T], \mathbb{R}^n) \). But, for the remarks above, the same set is closed in \( C^0([0,T], \mathbb{R}^n) \), and so it is sequentially compact in \( K \). Next, observe that the functional \( J \) ha no global minima in \( K \). Since the constant functions lie in \( K \), it is clear that

\[
\inf_K J = \inf_{\mathbb{R}^n} G \int_0^T \psi(t)dt .
\]

Hence, if \( G \) is unbounded below, so \( J \) is too. Now, suppose that \( G \) is bounded below. Arguing by contradiction, assume that \( \hat{u} \in K \) is a global minimum of \( J \). Then, we would have

\[
\int_0^T \psi(t) \left( G(\hat{u}(t)) - \inf_{\mathbb{R}^n} G \right) dt = 0 ,
\]

and so, since the integrand is non-negative, it would follow

\[
\psi(t) \left( G(\hat{u}(t)) - \inf_{\mathbb{R}^n} G \right) = 0
\]

for a.e. \( t \in [0,T] \). Therefore, since \( \psi \neq 0 \), for some \( t \in [0,T] \), we would have \( G(\hat{u}(t)) = \inf_{\mathbb{R}^n} G \), against \((i_3)\). Notice that the absence of global minima for \( J \) implies that

\[
\beta = \sup_K \Psi .
\]

Moreover, since \( \lim_{s \to +\infty} \gamma(s) = +\infty \), from \((i_1)\) it follows that

\[
\sup_K \Psi = +\infty .
\]
Furthermore, since \( b = +\infty \), we have
\[
\alpha = \inf_{K} \Psi .
\]

Clearly
\[
\inf_{K} \Psi \leq \inf_{x \in \mathbb{R}^n} \int_{0}^{T} F(t,x)dt + \Phi(0)T .
\]

Now, put
\[
r = \max \left\{ \int_{0}^{T} F(t,x_1)dt, \int_{0}^{T} F(t,x_2)dt \right\} + \Phi(0)T .
\]

By the above remarks and by the inequality in (i4), we have
\[
\alpha < r < \beta .
\]

Fix \( u \in \Psi^{-1}([-\infty, r]) \). By (i1) and Jensen inequality again, we have
\[
r \geq \int_{0}^{T} (\Phi(u'(t)) + F(t,u(t)))dt \geq \int_{0}^{T} \gamma(|u(t)|)dt + \Phi(0)T \geq T \gamma \left( \frac{1}{T} \int_{0}^{T} |u(t)|dt \right) + \Phi(0)T ,
\]
and so
\[
\gamma \left( \frac{1}{T} \int_{0}^{T} |u(t)|dt \right) \leq \frac{1}{T} \max \left\{ \int_{0}^{T} F(t,x_1)dt, \int_{0}^{T} F(t,x_2)dt \right\} .
\]

Applying \( \gamma^{-1} \), we get
\[
\frac{1}{T} \int_{0}^{T} |u(t)|dt \leq \gamma^{-1} \left( \frac{1}{T} \max \left\{ \int_{0}^{T} F(t,x_1)dt, \int_{0}^{T} F(t,x_2)dt \right\} \right)
\]
and hence
\[
\inf_{[0,T]} |u| \leq \gamma^{-1} \left( \frac{1}{T} \max \left\{ \int_{0}^{T} F(t,x_1)dt, \int_{0}^{T} F(t,x_2)dt \right\} \right) .
\]

In view of (1.1), we then infer that
\[
\sup_{[0,T]} |u| \leq LT + \gamma^{-1} \left( \frac{1}{T} \max \left\{ \int_{0}^{T} F(t,x_1)dt, \int_{0}^{T} F(t,x_2)dt \right\} \right) . \tag{3.3}
\]

In turn, in view of (i4), (3.3) implies that
\[
J(x_1) = J(x_2) \leq J(u) .
\]

Since \( x_1, x_2 \in \Psi^{-1}([-\infty, r]) \), we then conclude that \( x_1, x_2 \) are two distinct global minima of \( J|_{\Psi^{-1}([-\infty, r])} \).

Now, arguing by contradiction, assume that, for every \( \lambda > 0 \), the functional \( J + \lambda \Psi \) has a unique global minimum in \( K \). Then, by Theorem 2.1 (recall that \( J + \lambda \Psi \) has sequentially compact sub-level sets), there would exist \( \hat{\lambda}_r > 0 \) and \( \hat{u}_r \in \Psi^{-1}(r) \) such that \( \hat{u}_r \) is the unique global minimum of \( J + \hat{\lambda}_r \Psi \) in \( K \). Then, for \( i = 1, 2 \), we would have
\[
\inf_{u \in K} (J(u) + \hat{\lambda}_r \Psi(u)) \leq J(x_i) + \hat{\lambda}_r \Psi(x_i) \leq J(\hat{u}_r) + \hat{\lambda}_r \Psi(\hat{u}_r) = \inf_{u \in K} (J(u) + \hat{\lambda}_r \Psi(u)) .
\]

That is to say, \( x_1 \) and \( x_2 \) would be two distinct global minima in \( K \) of the functional \( J + \hat{\lambda}_r \Psi \), a contradiction. So, there exists some \( \lambda > 0 \) such that the functional \( J + \lambda \Psi \) has at least two global minima in \( K \). To conclude the proof, take \( \hat{\lambda} = \frac{1}{\lambda} \) and apply Theorem 1.1. \( \triangle \)
In the sequel, the following further result from [1] will be useful:

**PROPOSITION 3.1 ([1], Proposition 3.2). -** Let $\phi \in \mathcal{A}$, $p > 1$ and $\mu > 0$. Then, for every $\omega \in L^1([0,T],\mathbb{R}^n)$, the problem

$$\begin{cases}
(\phi(u'))' = \mu|u|^{p-2}u + \omega(t) & \text{in } [0,T] \\
u(0) = u(T), \ u'(0) = u'(T)
\end{cases}$$

has a unique solution.

The next three examples (where $\phi \in \mathcal{A}$) show that, in Theorem 3.1, none of $(i_2) - (i_4)$ can be removed at all.

**EXAMPLE 3.1. -** Take: $F(x) = |x|^2$, $G(x) = \langle z, x \rangle$, with $z \in \mathbb{R}^n \setminus \{0\}$, $\psi = 1$, $\gamma(s) = \frac{s^2}{2}$. Clearly, $(i_1) - (i_3)$ are satisfied, but, for every $\lambda \in \mathbb{R}$, the problem

$$\begin{cases}
(\phi(u'))' = u + \lambda z & \text{in } [0,T] \\
u(0) = u(T), \ u'(0) = u'(T)
\end{cases}$$

has a unique solution by Proposition 3.1.

**EXAMPLE 3.2. -** Take: $F(x) = \frac{|x|^2}{2}$, $G = 0$, $\gamma(s) = \frac{s^2}{2}$. Clearly, $(i_1)$, $(i_2)$ and $(i_4)$ are satisfied, but, by Proposition 3.1, 0 is the unique solution of the problem

$$\begin{cases}
(\phi(u'))' = u & \text{in } [0,T] \\
u(0) = u(T), \ u'(0) = u'(T)
\end{cases}$$

**EXAMPLE 3.3. -** Take: $F(x) = |x|^2$, $G(x) = \begin{cases} 0 & \text{if } |x| \leq LT + 1 \\
-(|x| - LT - 1)^3 & \text{if } |x| > LT + 1 \end{cases}$, $\psi = 1$, $\gamma(s) = s^2$. Clearly, $(i_1)$, $(i_3)$ and $(i_4)$ are satisfied. In particular, $(i_4)$ is satisfied by any pair of distinct points $x_1, x_2 \in \mathbb{R}^n$ such that $|x_1| = |x_2| \leq 1$. However, for each $\lambda > 0$, the functional $u \to \int_0^T (\Phi(u'(t)) + |u(t)|^2 + \lambda G(u(t)))dt$ is unbounded below in $K$.

We conclude with a joint consequence of Theorem 3.1 and Proposition 3.1.

**THEOREM 3.2. -** Let $\phi \in \mathcal{A}$, $p > 1$, $G \in C^1(\mathbb{R}^n)$, $\psi \in L^1([0,T]) \setminus \{0\}$, with $\psi \geq 0$. Assume that $G$ satisfies assumptions $(i_2)$, $(i_3)$ of Theorem 3.1 and the following:

$(j_1)$ there is $\rho > LT$ such that $G$ is constant in $B_\rho$.

Then, there exists $\tilde{\lambda} > 0$ such that the problem

$$\begin{cases}
(\phi(u'))' = |u|^{p-2}u + \tilde{\lambda}\psi(t)\nabla G(u) & \text{in } [0,T] \\
u(0) = u(T), \ u'(0) = u'(T)
\end{cases}$$

has at least one solution which is a global minimum in $K$ of the functional

$$u \to \int_0^T \left( \Phi(u'(t)) + \frac{|u(t)|^p}{p} + \tilde{\lambda}\psi(t)G(u(t)) \right) dt$$

and whose range is contained in $\mathbb{R}^n \setminus \overline{B_{\rho-LT}}$.  

PROOF. Apply Theorem 3.1 with $F(t, x) = \frac{|x|^p}{p}$ and $\gamma(s) = \frac{s}{p}$. Concerning $(i_4)$, notice that it is satisfied by any pair of distinct points $x_1, x_2 \in \mathbb{R}^n$, such that $|x_1| = |x_2| \leq \rho - LT$. This comes from $(j_1)$ after observing that $\gamma^{-1} \left( \frac{1}{T} \int_0^T F(t, x_1) dt \right) = |x_1|$. So, by Theorem 3.1, there exists $\tilde{\lambda} > 0$ such that the problem

\[
\begin{cases}
(\phi(u'))' = |u|^{p-2} u + \tilde{\lambda} \psi(t) \nabla G(u) & \text{in } [0, T] \\
u(0) = u(T) , \ u'(0) = u'(T)
\end{cases}
\]

has at least one non-zero solution which is a global minimum in $K$ of the functional

\[
u \rightarrow \int_0^T \left( \Phi(u'(t)) + \frac{|u(t)|^p}{p} + \tilde{\lambda} \psi(t) G(u(t)) \right) dt .
\]

Denote by $w$ such a solution. To complete the proof, we have to show that $\inf_{[0, T]} |w| > \rho - LT$. Arguing by contradiction, assume that $\inf_{[0, T]} |w| \leq \rho - LT$. Then, by (1.1), we would have

\[
\sup_{[0, T]} |w| \leq \rho .
\]

By $(j_1)$, this would imply that $w$ is a solution of the problem

\[
\begin{cases}
(\phi(u'))' = |u|^{p-2} u & \text{in } [0, T] \\
u(0) = u(T) , \ u'(0) = u'(T)
\end{cases}
\]

and hence $w = 0$ by Proposition 3.1, which is a contradiction. The proof is complete. \(\triangle\)
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