A turbulent ring and dynamo in a precessing sphere

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Abstract. A new ring structure of high activity, both in vorticity and magnetic flux density, is observed in MHD turbulence in a precessing sphere of which the spin and precession axes are orthogonal. This ring is fixed to the precession frame being localized near a great circle whose normal is inclined slightly from the spin axis. Both the velocity and magnetic fields are activated near the cross sections of the ring with the equatorial plane, and their fluctuations make a prograde motion.

1. Introduction

The origin and the structure, as well as the chaotic reversal, of geomagnetic field have long attracted peoples’ attention. It is generally believed that the dynamo action of the electrically conducting fluid (composed mainly of the molten iron) in the outer core of the Earth may be responsible for the geomagnetic field and that the buoyancy forces and precession may be the possible sources for driving the fluid motion. So far most theoretical studies and numerical simulations have been devoted to the former, while very few for the latter (e.g. Tilgner 2005, Wu & Roberts 2009). In this paper we focus our attention on the precession driven dynamo in a spherical container and perform a DNS analysis to investigate the generation mechanism and the structure of magnetic field.

2. Governing equations

We consider the MHD dynamo driven by incompressive flows of an electrically conducting fluid in a precessing sphere with the spin and precession angular velocities being constant in time and the two axes being orthogonal. The evolution equations in the sphere for the fluid velocity \( u(r,t) \) and the magnetic flux density \( b(r,t) \) may be written in the precession frame \((x,y,z)\) which is rotating with the precession angular velocity \( \Omega_p = \Omega_p \hat{x} \) as

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u \times (\nabla \times u) - 2\Gamma \hat{x} \times u - \nabla P + (\nabla \times b) \times b + \frac{1}{Re} \nabla^2 u, \\
\frac{\partial b}{\partial t} &= \nabla \times (u \times b) + \frac{1}{Re_m} \nabla^2 b, \quad \nabla \cdot u = 0, \quad \nabla \cdot b = 0,
\end{align*}
\]

where \( P \) is the modified pressure including the centrifugal force, \( \Gamma = \Omega_p / \Omega_s \) the Poincaré number, \( Re = a^2 \Omega_s / \nu \) the Reynolds number, \( Re_m = a^2 \Omega_s \mu \sigma \) the Magnetic Reynolds number,

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\[ \Omega_s = \Omega_s \hat{x} \] the spin angular velocity, \( a \) the sphere radius, \( \nu \) the kinematic viscosity, \( \mu \) the magnetic permeability and \( \sigma \) the electrical conductivity of fluid. The length has been normalized by \( a \), the time by \( 1/\Omega_s \), and the magnetic flux density by \( \sqrt{\mu \sigma \Omega_s} \). The outside of the sphere is vacuum, where the magnetic flux density \( b^{(0)} \) obeys \( \nabla \cdot b^{(0)} = 0 \) and \( \nabla \times b^{(0)} = 0 \). These equations are supplemented by

\[ u = \hat{z} \times r \quad b = b^{(0)} \quad \text{(on } r = 1), \tag{2} \]

which are the boundary conditions derived from the assumptions that the flow is non-slip on the boundary and that the magnetic permeability of the fluid is equal to that of vacuum. We also assume that \( b^{(0)} \) is zero at infinity.

### 3. Critical magnetic Reynolds number

We solve the above set of equations numerically by the spectral method (Kida & Nakayama 2008). The velocity and magnetic fields are expressed by the toroidal and poloidal scalar functions, \( U, W, B \) and \( J \) as

\[ u = \nabla \times (\nabla \times rU) + \nabla \times rW \quad \text{and} \quad b = \nabla \times (\nabla \times rB) + \nabla \times rJ. \]

These scalar functions are expanded by the Zernike polynomials and spherical harmonics; for example,

\[ U(r, t) = \sum_{m=-M}^{M} \sum_{l=|m|}^{L} \sum_{n+l \text{ even}}^{N} U_{nlm}(t) \Phi_{nlm}(r) Y_{lm}(\phi, \theta), \tag{3} \]

where \( \Phi_{nlm}(r) \) is the Zernike polynomial and \( Y_{lm}(\phi, \theta) \) is the spherical harmonics, \((M, L, N)\) is the number of truncation modes, and \((r, \theta, \phi)\) is the spherical polar coordinate with \( \theta \) being the polar angle from the \( z \)-axis and \( \phi \) the azimuthal angle from the \( x \)-axis. The evolution equations for the expansion coefficients \( U_{nlm} \) and the corresponding ones for \( W, B \) and \( J \) are derived and integrated numerically by the Adams-Bashforth and Crank-Nicolson schemes. First, we generate the statistically stationary turbulence starting from an arbitrary initial condition with \((M, L, N) = (85, 85, 170)\). We set \( Re = 10000 \) and \( \Gamma = 0.1 \) for which the flow can be turbulent (Goto et al. 2007). Then we seed a magnetic field of small amplitude and integrate the full set of equations (1) for various values of \( Re_m \). To examine whether the dynamo action is effective or not, we monitor the temporal evolution of the magnetic energy \( E_{M_s} \) inside the sphere which is defined by (5a) below. As shown in figure 1, the magnetic energy decays exponentially in time for \( Re_m < 6000 \) but it is maintained for \( Re_m > 8000 \).

Now that we have realized dynamo action by the present numerical simulation, we shall examine the dynamo mechanism by analysing the velocity and magnetic fields for \( Re_m = 10000 \). For better resolution we raise the number of truncation modes to \((M, L, N) = (127, 127, 254)\).
4. Energy Dynamics

In the present system the energy is supplied from outside to the fluid motion as the kinetic energy by precessing motion of the sphere, and is consumed as thermal energy through the viscous dissipation and Joule heating. During this process the magnetic field can get some energy from the fluid motion through their couplings (the third term in the first equation of (1) and the first term in the second). If this happens, it is said that dynamo action is effective. Here we examine how the energy is partitioned between various kinds of modes.

The analysis of energy balance would be made simplest if the dynamics of fluid motion is described in the coordinate system (called the body frame) moving with the sphere because then the work due to the viscous stress and the Lorentz force on the sphere boundary vanishes. The kinetic energy of fluid motion is written as

\[ E_K = \int_{V_i} \frac{1}{2} |\mathbf{u}|^2 dV, \]  

where \( V_i \) stands for the volume of the sphere. Since the magnetic field is distributed both inside and outside the sphere, we divide the magnetic energy into the contribution \( E_{Mi} \) from the inside and \( E_{Mo} \) from the outside as

\[ E_{Mi} = \int_{V_i} \frac{1}{2} |\mathbf{b}|^2 dV, \]  

\[ E_{Mo} = \int_{V_o} \frac{1}{2} |\mathbf{b}|^2 dV, \]

where \( V_o \) stands for the outside of the sphere. Then the total magnetic energy is

\[ E_M = E_{Mi} + E_{Mo}. \]

In figure 2 we plot the temporal evolution of kinetic and magnetic energies in a statistically steady period (100 \( \leq t/2\pi \leq 110 \)). It is seen that the magnetic energy is much less than the kinetic energy.

In order to examine the energy dynamics in the entire system we consider the energy balance equations. By multiplying \( \mathbf{u} \) to the evolution equation of the velocity field written in the body frame and integrating over the whole sphere, we obtain

\[ \frac{dE_K}{dt} = -\int_{V_i} \mathbf{u} \cdot (\nabla \times \mathbf{b}) dV + \Gamma \int_{V_i} \hat{\mathbf{y}}_b \cdot (r \times \mathbf{u}) dV - \frac{1}{Re} \int_{V_i} |\nabla \times \mathbf{u}|^2 dV, \]

where \( \mathbf{y}_b \) is the Lorentz force, \( \Gamma \) is the Poincaré force, and \( Re \) is the Reynolds number. The terms in the equation are associated with different sources of energy.
where \( \hat{y}_b = \hat{x}\sin t + \hat{y}\cos t \). Similarly, by multiplying \( b \) to the evolution equation for the magnetic flux density and integrating over the whole sphere, we obtain

\[
\frac{d\mathcal{E}_{M_i}}{dt} = \int_{V_i} \mathbf{u} \cdot \left[ \mathbf{b} \times (\nabla \times \mathbf{b}) \right] dV - \frac{1}{\Re_m} \int_{V_i} \left| \nabla \times \mathbf{b} \right|^2 dV + \frac{1}{\Re_m} \int_S [\mathbf{b} \times (\nabla \times \mathbf{b})] \cdot dS, \tag{7}
\]

where \( S \) stands for the sphere surface. Furthermore, we can show that the time-derivative of (5b) is written as

\[
\frac{d\mathcal{E}_{M_i}}{dt} = -\frac{1}{\Re_m} \int_S [\mathbf{b} \times (\nabla \times \mathbf{b})] \cdot dS. \tag{8}
\]

The first (Lorentz) term on the right-hand side of (6) for the kinetic energy represents the exchange with the magnetic energy in the sphere, the second (Poincaré) term the power supply by precession, and the third (viscous) term the viscous dissipation. The first (Lorentz) term on the right-hand side of (7) for the magnetic energy represents the exchange with the kinetic energy, the second (Joule) term the dissipation by Joule heating, and the third (Poynting) term the exchange with the magnetic energy outside the sphere.

The temporal evolutions of these terms are plotted in figure 3. As seen in figure 3(a) for the kinetic energy, the energy is supplied to \( \mathcal{E}_K \) through the Poincaré term (\( \cdots \cdots \)), the most of which is dissipated by viscosity (\(-\cdots\)) and the rest (2 \(\sim\) 3\%) is transferred to \( \mathcal{E}_{M_i} \) through the Lorenz term (\( \cdots \cdots \)). Figure 3(b) for the magnetic energy, on the other hand, shows that \( \mathcal{E}_{M_i} \) receives energy from \( \mathcal{E}_K \) through the Lorenz term (\( \cdots \cdots \)), the most of which dissipates by Joule heating (\( \cdots \cdots \)) and the rest is shared with \( \mathcal{E}_{M_o} \).

![Figure 3. Energy transfer. (a) Power by Lorentz force \( \times 10 \) (\( \cdots \cdots \)), power by Poincaré force (\( \cdots \cdots \)), and viscous dissipation (\(-\cdots\)) for the kinetic energy. (b) Supply by Lorentz force (\( \cdots \cdots \)), Joule heating (\( \cdots \cdots \)), and Poynting flux \( \times 10^4 \) (\(-\cdots\)) for the magnetic energy.](image)

5. Dynamical Behaviour

Now we examine the structures of the velocity and magnetic fields and their dynamical interactions.
5.1. High-speed stream and turbulent ring

Since the Reynolds number \((Re = 10000)\) is relatively large, a thin boundary layer of thickness about 0.05 develops along the sphere surface. Outside the boundary layer we observe a band of high-speed stream in which the ortho-radial component of velocity, about 0.5 in magnitude, is dominant. In figure 4 we plot this component of velocity on three spheres of radii (a) 0.85, (b) 0.9, (c) 0.95. The spin and precession axes are shown respectively by solid and dotted straight lines and the equator by a circle surrounding the sphere. A closed band of high speed stream, in which the swirling is in the same direction as that of the spin of the sphere, is clear in each figure. The streamlines in this band are nearly circular, and the normal of the circles varies with the radius \(r\) in a simple way; as \(r\) decreases from 0.95 to 0.7, the polar angle \(\theta\) changes linearly with \(r\) from 33° to 42°, whereas the azimuthal angle \(\varphi\) stays nearly constant around 50°.

The activity of the velocity field is represented more clearly by the vorticity magnitude, a snapshot of which, at \(t/2\pi = 110\), is drawn with iso-surface in figure 5(a). It is impressive that the high-vorticity region (called a turbulent ring) is concentrated around a great circle whose normal is close to that of the high-speed stream band near the sphere boundary.

5.2. Turbulence generation

An animation of the turbulent ring shows that many blobs of iso-surfaces move along the ring in the prograde direction (anti-clockwise in the perspective of figure 5) and that they move more violently near the cross-sections of the ring with the equatorial plane, suggesting that the turbulence is mostly generated there.

More quantitative pictures of the temporal evolution of the turbulent ring may be seen in the slices of vorticity field on spheres of various radii. Figure 6 is an example of such slices, where four successive snapshots of the vorticity field on the sphere of \(r = 0.95\) are shown on the Mercator coordinates \((\theta', \varphi')\) in the 'ring frame' that is rotated from the precession frame so that the turbulent ring should be on the equator \(\theta' = \pi/2\) at \(r = 0.8\). The time elapses from (a) to (d). The vorticity magnitude is represented by gray scale with darker gray for higher values, and the in-plane component by arrows. We see that several high-vorticity hairpins are generated at two places, around \((\theta', \varphi') = (0, \pi/2)\) and \((0, 3\pi/2)\), which are the cross-sections of the turbulent ring with the equatorial plane.

![Figure 4. High-speed-stream band. On the spheres of radii (a) 0.85, (b) 0.9 and (c) 0.95 drawn with arrows are the velocity vectors of the mean field averaged over \(100 \leq t/2\pi \leq 110\).](image)
Figure 5. Turbulent ring and enhanced magnetic field. The high-activity regions of (a) vorticity and (b) magnetic flux density are shown by isosurfaces. Their distributions on the equatorial plane are drawn with gray scales. $t/2\pi = 110$.

Figure 6. Generation of turbulence activity. The vorticity field on the sphere surface of radius $r = 0.95$ is plotted on the ring frame at (a) $t/2\pi = 102.4$, (b) 102.45, (c) 102.5, (d) 102.55. The in-plane component of vorticity vector is drawn by arrows and the magnitude of vorticity is represented by gray scale with darker gray for larger values.

5.3. Intensification of magnetic field
As seen in section 4, the magnetic energy is transferred from the kinetic energy through the Lorentz term (the first term in the right-hand side of (7)) which can be rewritten as

$$[\text{Lorentz term}] = \int_{V_i} b^2 \, \mathbf{b} \cdot (\mathbf{b} \cdot \nabla) u \, dV.$$  
(9)
Note that $\mathbf{b} \cdot (\mathbf{b} \cdot \nabla) \mathbf{u}$ represents the stretching or shrinking of lines of magnetic force according as the sign is positive or negative. This integral is proportional to the weighted (by $b^2$) mean of it. Since this term is actually positive (figure 3), the stretching of lines of magnetic force contributes to the energy supply to the magnetic field.

We may get a glimpse of such stretching events in figure 7, where the regions of high vorticity and high magnetic flux density are simultaneously drawn with iso-surface at four successive instants of time. The plotted region is $0.7 < r < 1$, $0.72 < \theta' < 2.42$ and $\pi/2 < \varphi' < 3\pi/2$ in the ring frame to focus on a part of the turbulent ring. Let us take a look at a large blob of high vorticity (light gray) extended vertically in the middle of figure 7(a) which is wound horizontally by several thin blobs of high magnetic flux density (dark gray) as well as of high vorticity itself.

**Figure 7.** Intensification of magnetic flux density. Iso-surfaces of magnitude of vorticity and the magnetic flux density are drawn with light and dark gray, respectively. The plotted region is $0.7 < r < 1$, $0.72 < \theta' < 2.42$ and $\pi/2 < \varphi' < 3\pi/2$. The size of the solid grids is 0.34 for $\theta'$ (from lower-right to upper-left) and $\pi/5$ for $\varphi'$ (from lower-left to upper-right). The cross-point at the lower-center is $(\theta', \varphi') = (1.06, 7\pi/10)$. The perspective is from the inside of the sphere. $t/2\pi = 100.15, 100.2, 100.25, 100.3$
These thin blobs are well-known stretched structures discovered around a vortex tube (Melander & Hussain 1993).

5.4. Structure of magnetic field
The intensified magnetic field is represented in figure 5(b) by iso-surfaces of the magnitude of magnetic flux density in the full sphere as well as their distributions on the equatorial plane.

![Diagram of magnetic field](image)

**Figure 8.** Magnetic field outside and on the sphere. (a) The magnitude of magnetic flux density on the sphere is shown by gray scale with darker regions for higher values, and lines of magnetic force are drawn. (b) The contours of the radial component of magnetic flux density are drawn on the sphere. The increment of the contour level is 0.002 and the shaded regions indicate positive values. The solid curve indicates the turbulent ring (located at the equator of the ring coordinate). $t/2\pi = 100$. 

The perspective is the same as figure 5(a) for the vorticity field. It is seen that high activity regions of both the magnetic and vorticity fields are close to each other, suggesting their strong interactions.

In figure 8(a) we plot the lines of magnetic force outside the sphere as well as the distribution of the magnitude of magnetic flux density on the surface with darker regions for higher values. The equator is drawn with a dotted curve on the sphere. Among many loops, those aligned obliquely in the left side in the northern hemisphere are conspicuous. They sit on a ridge of high magnetic flux density on the sphere. Figure 8(b) shows the spatial distribution on the sphere surface of the radial component of magnetic flux density represented by contours with positive parts being shaded in the Mercator coordinate \((\theta, \varphi)\) of the precession frame. The solid curve indicates the location of the turbulent ring. Strong magnetic field spreads in the shape of belt along the turbulent ring. It takes large values with alternated signs around the cross-sections of the turbulent ring with the equator, i.e. around \(\varphi = \pi/2\) and \(3\pi/2\) on the solid line.

6. Concluding Remarks

We have realized an MHD dynamo driven by precession in a sphere by DNS. A high-speed stream band was observed which swirls in the prograde direction with the swirl axis being inclined from the spin axis. Turbulence is generated on the high-speed stream around the cross-section with the equatorial plane. The turbulent velocity field intensifies the magnetic field by stretching-and-intensification of lines of magnetic force. The theoretical foundation of this scenario of precession dynamo is one of our next targets.

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