On the Douglas–Rachford and Peaceman–Rachford algorithms in the presence of uniform monotonicity and the absence of minimizers

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May 21, 2024

Abstract

The Douglas–Rachford and Peaceman–Rachford algorithms have been successfully employed to solve convex optimization problems, or more generally find zeros of monotone inclusions. Recently, the behaviour of these methods in the inconsistent case, i.e., in the absence of solutions, has triggered significant consideration. It has been shown that under mild assumptions the shadow sequence of the Douglas–Rachford algorithm converges weakly to a generalized solution when the underlying operators are subdifferentials of proper lower semicontinuous convex functions. However, no convergence behaviour has been proved in the case of Peaceman–Rachford algorithm. In this paper, we prove the strong convergence of the shadow sequences associated with the Douglas–Rachford algorithm and Peaceman–Rachford algorithm in the possibly inconsistent case when one of the operators is uniformly monotone and $3^*$ monotone but not necessarily a subdifferential. Several examples illustrate and strengthen our conclusion. We carry out numerical experiments using example instances of optimization problems.

2010 Mathematics Subject Classification: Primary 49M27, 65K05, 65K10, 90C25; Secondary 47H14, 49M29.

Keywords: convex optimization problem, Douglas–Rachford splitting, inconsistent constrained optimization, least squares solution, normal problem, Peaceman–Rachford splitting, projection operator, proximal mapping.

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1 Introduction

Throughout, we assume that

\[ X \text{ is a real Hilbert space}, \]

with inner product \( \langle \cdot, \cdot \rangle : X \times X \to \mathbb{R} \) and induced norm \(|\cdot|\). Let \( A : X \rightrightarrows X \). The graph of \( A \) is \( \operatorname{gra} A = \{(x, x^*) \in X \times X \mid x^* \in Ax\} \). Recall \( A \) is monotone if \( \{(x, x^*), (y, y^*)\} \subseteq \operatorname{gra} A \) implies that \( \langle x - y, x^* - y^* \rangle \geq 0 \), and \( A \) is maximally monotone if \( A \) is monotone and \( \operatorname{gra} A \) has no proper extension (in terms of set inclusion) that preserves the monotonicity of \( A \). The resolvent of \( A \) is \( J_A = (\text{Id} + A)^{-1} \) and the reflected resolvent of \( A \) is \( R_A = 2J_A - \text{Id} \), where \( \text{Id} : X \to X \).

Throughout this paper, we assume that \( A : X \rightrightarrows X \) and \( B : X \rightrightarrows X \) are maximally monotone. (2)

Consider the monotone inclusion problem:

Find \( x \in X \) such that \( x \in \text{zer}(A + B) := \{x \in X \mid 0 \in Ax + Bx\}. \) (P)

Problem (P) is of significant interest in optimization. Indeed, thanks to Rockafellar’s fundamental result [35, Theorem A] we know that the subdifferential operator \( \partial f \) of a proper lower semicontinuous convex function \( f : X \to ]-\infty, \infty[ \) is maximally monotone. Set \( (A, B) = (\partial f, \partial g) \), where \( f \) and \( g \) are proper lower semicontinuous convex functions on \( X \). Under appropriate constraint qualifications problem (P) amounts to finding a minimizer of \( f + g \), equivalently; a zero of \( A + B \), provided that one exists. For detailed discussions on problem (P) and the connection to optimization problems, we refer the reader to [7, 17, 19, 21, 26, 36, 38, 39, 43, 44] and the references therein.

The Douglas–Rachford algorithm [29, Algorithm 2] is a successful optimization technique to find a zero of \( A + B \) (assuming that one exists) provided that we have access to the resolvents \( J_A \) and \( J_B \). Under the additional assumption that \( A \) is uniformly monotone (see Definition 2.7(i)) the Peaceman–Rachford algorithm [29, Algorithm 1] can also be used to solve (P). Let \( \lambda \in [0, 1] \) and set

\[ T = T_\lambda = (1 - \lambda) \text{Id} + \lambda R_BR_A. \] (3)

Both algorithms operate by iterating the so-called splitting operator \( T_\lambda \), where\(^1\) \( \lambda = \frac{1}{2} \) in the case of classical Douglas–Rachford and \( \lambda = 1 \) in the case of Peaceman–Rachford. Static\(^2\) connection to the set of zeros of \( A + B \) is via the identity\(^3\) (see, e.g., [7, Proposition 26.11(iii)(a)])

\[ \text{zer}(A + B) = J_A(\text{Fix} R_BR_A) = J_A(\text{Fix} T_\lambda). \] (4)

\(^1\)In passing, we point out that for \( \lambda \in [0, 1] \) \( T_\lambda \) is a relaxation of the Douglas–Rachford operator, and the corresponding governing and shadow sequences enjoy the same convergence behaviour of the Douglas–Rachford algorithm, see [41, page 240] and also [7, Theorem 26.11].

\(^2\)Throughout this paper, we adopt the terminology that static refers to results or properties that involves nonsequential behaviour, whereas dynamic refers to results or properties that involve algorithmic behaviour.

\(^3\)Let \( T : X \to X \). The set of fixed points of \( T \) is \( \text{Fix} T := \{x \in X \mid x = Tx\} \).
Let \( x \in X \). The algorithms produce two sequences: the governing sequence \((T^n x)_{n \in \mathbb{N}}\) and the shadow sequence \((J_A T^n x)_{n \in \mathbb{N}}\). The dynamic behaviour of the governing sequence is beautifully ruled and clearly determined by fundamental results from fixed point theory. Indeed, in the case of Douglas–Rachford, because \( T \) is firmly nonexpansive exactly one of the following happens: (i) \( \text{zer}(A + B) \neq \emptyset \); equivalently, \( \text{Fix} \ T \neq \emptyset \). In this case \( T^n x \rightharpoonup x \in \text{Fix} \ T \) (see [29, Proposition 2]) and \( J_A T^n x \rightharpoonup J_A x \in \text{zer}(A + B) \) (see [40]), or (ii) \( \text{zer}(A + B) = \emptyset \); equivalently, \( \text{Fix} \ T = \emptyset \). In case (ii) we have that \( \| T^n x \| \to +\infty \) (see [1, Corollary 2.2]). However, and in a pleasant surprise, the shadow sequence in this case appears to have a will of its own! Indeed, when \( (A, B) = (\partial f, \partial g) \) weak convergence of the shadow sequence has been proved in various scenarios under the mild assumption that the normal problem (see (29) below) has a solution. The normal problem is obtained by perturbing the sum \( A + B \) in a specific manner using the so-called minimal displacement vector (see (28) below). For a comprehensive reference of these results we refer the reader to [10], [11], [14], [15] and the references therein. Related results for detection of infeasibility appear in, e.g., [5], [6], [30] and [37].

Nonetheless, no convergence results are known for the case of two maximally monotone operators that are not necessarily subdifferential operators. Furthermore, unlike Douglas–Rachford algorithm, the behaviour of Peaceman–Rachford algorithm in the inconsistent case remains a terra incognita, even in optimization settings.

The goal of this paper is to provide a comprehensive exploration of the behaviour of the Douglas–Rachford and Peaceman–Rachford algorithms for two maximally monotone operators when one of the underlying operators is \( 3^* \) monotone (see Definition 2.7(ii)) and uniformly monotone under additional mild assumptions. Our analysis tackles this behavior in the possibly inconsistent situation, i.e., when \( \text{zer}(A + B) = \emptyset \), a situation that is hard-to-analyze but common-to-encounter.

Our main contributions are summarized as follows:

\begin{enumerate}
  \item In Lemma 3.6 we provide useful identities and consequent inequalities which serve as key ingredients in the convergence analysis of the algorithms.
  \item Our first main result appears in Theorem 4.2 where we prove the strong convergence of the Douglas–Rachford algorithm when one of the operators is uniformly monotone with a supercoercive\(^4\) modulus (this is always true when the operator is a subdifferential of a uniformly convex function). The convergence relies on the mild assumption that a normal solution exists.
  \item Our second main result appears in Theorem 4.3 which provides an analogous conclusion to the result in (C2) for the Peaceman–Rachford algorithm. Up to the authors’ knowledge, this is the first proof of the (strong) convergence of the shadow sequence of the Peaceman–Rachford algorithm in the inconsistent case.
  \item The main optimization results are presented in Section 5; namely, in Theorem 5.3 and Theorem 5.4. In Proposition 5.5 we provide a situation where a unique normal solution exists. Computationally useful estimates of the gap vector (between the two disjoint domains) are provided in Fact 5.2.
  \item The above results on the Douglas–Rachford and Peaceman–Rachford algorithms for (pos-
\end{enumerate}

\(^4\)Recall that \( f \) is supercoercive when \( f(\cdot)/\| \cdot \| \) is coercive.
sibly inconsistent) optimization problems are numerically illustrated by means of example problems in $d$ variables, for $d = 2$ (for which visualization is possible) and for $d \in \{70, 100, 1000\}$. These example problems involve an affine set and a box as the disjoint domains of the two operators, respectively. Numerical experiments are carried out to investigate the “best” values of the algorithmic parameters, including the relaxation parameter $\lambda$, for various instances of the problems. Comparisons are also made with (the celebrated) Dykstra’s projection algorithm [18].

Organization and notation

This paper is organized as follows: Section 2 contains a collection of auxiliary results and examples of resolvents that are not necessarily proximal mappings. In Section 3 we provide key results concerning the generalized fixed point set and connection to the normal (generalized) solutions. Our main results appear in Section 4 and the optimization counterpart of these results appear in Section 5. Numerical experiments are presented in Section 6. Finally, Section 7 provides concluding remarks and future directions of research.

Our notation is standard and follows largely, e.g., [34] and [7].

2 Auxiliary results

2.1 Nonexpansive mappings and their minimal displacement vector

Definition 2.1. Let $C$ be a nonempty closed convex subset of $X$ and fix $x \in X$. Then $p \in C$ is the projection of $x$ onto $C$ if $\|x - p\| = \inf_{y \in C} \|x - y\| =: d_C(x)$. We denote $p$ as $P_C(x)$.

Lemma 2.2. Let $\alpha \in \mathbb{R}$ and let $C$ be a nonempty closed convex subset of $X$. Then $P_{\alpha C}(0) = \alpha P_C(0)$.

Proof. Indeed, let $p \in C$. Then, by [7, Theorem 3.16], we have that $p = P_C(0) \iff (\forall c \in C) \langle 0 - p, c - p \rangle \leq 0 \iff (\forall c \in C) \langle 0 - \alpha p, ac - \alpha p \rangle \leq 0 \iff \alpha p = P_{\alpha C}(0)$. ■

Definition 2.3. Let $T : X \to X$ and let $(x, y) \in X \times X$ and let $\lambda \in [0, 1]$. Recall that $T$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ and $T$ is $\lambda$-averaged if $\|Tx - Ty\|^2 + \lambda \|(Id - T)x - (Id - T)y\|^2 \leq \|x - y\|^2$.

The next result guarantees the convexity of the range of the displacement mapping associated with nonexpansive mappings.

Lemma 2.4. Let $\lambda \geq 0$ and let $N : X \to X$ be nonexpansive. Then $\text{ran} (\lambda (Id - N))$ is convex.

Proof. This is a direct consequence of [7, Example 20.29 and Corollary 21.14]. ■
Lemma 2.5. Let $T: X \to X$ and let $w \in X$. Then $\text{Fix}(T \cdot + w) = -w + \text{Fix}(w + T)$.

Proof. Indeed, let $x \in X$. Then $x \in \text{Fix}(T \cdot + w) \iff x = T(x + w) \iff -w + x + w = T(x + w) \iff x + w \in \text{Fix}(w + T) \iff x \in -w + \text{Fix}(w + T)$.

Lemma 2.6. Let $T: X \to X$ be nonexpansive, let $v = P_{\text{ran}(\text{Id} - T)}0$ and suppose that $x \in \text{Fix}(v + T)$. Let $n \in \mathbb{N}$. Then $T^n x = x - nv$.

Proof. We proceed by induction on $n \in \mathbb{N}$. The base case at $n = 0$ is clear. Now suppose for some $n \in \mathbb{N}$ we have $T^n x = x - nv$. The definition of $v$ and the nonexpansiveness of $T$ imply $\|v\| \leq \|T^n x - T^{n+1} x\| \leq \|T x\| = \|v\|$. The first inequality follows from the fact that $v$ is the element in $\text{ran}(\text{Id} - T)$ with minimum norm, and the fact that $(\text{Id} - T)T^n x \in \text{ran}(\text{Id} - T) \subseteq \text{ran}(\text{Id} - T)$. The last equality follows from the definition of $x$. Therefore, $\|T^n x - T^{n+1} x\| = \|v\|$. By the uniqueness of $v$ we conclude that $T^n x - T^{n+1} x = v$. Now use the inductive hypothesis to complete the proof.

Let $\lambda \in [0,1]$ and suppose that $T: X \to X$ is $\lambda$-averaged. Let $x \in X$. Recall that (see, e.g., [20, Proposition 1.2])

$$T^n x - T^{n+1} x \to P_{\text{ran}(\text{Id} - T)}0.$$ (5)

### 2.2 Further notions of monotonicity

Definition 2.7. Let $C: X \rightrightarrows X$ be monotone. Then

(i) $C$ is uniformly monotone with a modulus $\phi: \mathbb{R}_+ \to [0, +\infty]$ if $\phi$ is increasing, vanishes only at 0, and

$$\{(x, x^*), (y, y^*)\} \subseteq \text{gra} \ C \Rightarrow \langle x - y, x^* - y^* \rangle \geq \phi(\|x - y\|).$$ (6)

(ii) $C$ is strongly monotone with a constant $\beta > 0$ if $C - \beta \text{Id}$ is monotone, i.e.,

$$\{(x, x^*), (y, y^*)\} \subseteq \text{gra} \ C \Rightarrow \langle x - y, x^* - y^* \rangle \geq \beta \|x - y\|^2.$$ (7)

(iii) $C$ is $3^*$ monotone if $(\forall (y, z^*) \in \text{dom} C \times \text{ran} C)$

$$\inf_{(x, x^*) \in \text{gra} C} \langle x - y, x^* - z^* \rangle > -\infty.$$ (8)

(iv) $C$ is cyclically monotone if, for every $n \geq 2$, for every $(x_1, \ldots, x_{n+1}) \in X^{n+1}$ and every $(x_1^*, \ldots, x_n^*) \in X^n$

$$\begin{align*}
\{(x_1, x_1^*), (x_2, x_2^*), \ldots, (x_n, x_n^*) \in \text{gra} \ C \}
\quad \implies \quad \sum_{i=1}^n \langle x_{i+1} - x_i, x_i^* \rangle \leq 0.
\end{align*}$$ (9)
The importance of uniform (respectively strong) monotonicity is, in fact, motivated by its close connection to the notions of uniform (respectively strong) convexity as we see below.

**Fact 2.8.** Let $f : X \to ]-\infty, +\infty].$

(i) Suppose that $f$ is uniformly convex with modulus $\phi$. Then $\partial f$ is uniformly monotone with modulus $2\phi$.

(ii) Suppose that $f$ is $\beta$-strongly convex for some $\beta > 0$. Then $\partial f$ is $\beta$-strongly monotone.

**Proof.** (i): See [42, Theorem 3.5.10] and also [7, Example 22.4(iii)].

(ii): See [7, Example 22.4(iv)]. □

### 2.3 Examples of resolvents that are not necessarily proximal mappings

We now present a collection of resolvents of maximally monotone operators that are not necessarily subdifferentials. These results are interesting on their own, since the computation of the resolvent is not straightforward in general.

**Proposition 2.9.** Let $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}$. Suppose that $S : X \to X$ is continuous, linear, and single-valued such that $S$ and $-S$ are monotone and $S^2 = -\gamma \text{Id}$ where $\gamma \geq 0$. Suppose that $A = \alpha \text{Id} + \beta S$. Then

$$J_A = \frac{1}{(1+\alpha)^2 + \beta^2 \gamma} ((1 + \alpha) \text{Id} - \beta S),$$

and

$$R_A = \frac{1 - \alpha^2 - \beta^2 \gamma}{(1+\alpha)^2 + \beta^2 \gamma} \text{Id} - \frac{2\beta}{(1+\alpha)^2 + \beta^2 \gamma} S. \quad (11)$$

**Proof.** Indeed, set $T = \frac{1}{(1+\alpha)^2 + \beta^2 \gamma} ((1 + \alpha) \text{Id} - \beta S)$. Then

$$(\text{Id} + A)T = \frac{1}{(1+\alpha)^2 + \beta^2 \gamma} ((1 + \alpha) \text{Id} + \beta S)((1 + \alpha) \text{Id} - \beta S)$$

$$= \frac{1}{(1+\alpha)^2 + \beta^2 \gamma} ((1 + \alpha)^2 \text{Id} + (1 + \alpha)\beta S - \beta^2 S^2 - (1 + \alpha)\beta S)$$

$$= \text{Id}. \quad (12c)$$

This proves (10). The formula in (11) is a direct consequence of (10). □

Let $L : X \to X$ be continuous and linear. Recall that the adjoint of $L$ is the unique linear operator $L^* : X \to X$ that satisfies $(\forall (x, y) \in X \times X) \langle x, L^* y \rangle = \langle Lx, y \rangle$.

In the following, we use $B(z; \rho)$ to denote the closed ball in $X$ with centre $z$ and radius $\rho$. Namely,

$$B(z; \rho) := \{ y \in X : \| z - y \| \leq \rho \}.$$ 

Given a closed set $C \subseteq X$, the normal cone operator associated with $C$ is denoted by $N_C$ and defined as

$$N_C(x) := \left\{ \begin{array}{ll}
{\{ w \in X : \langle w, y - x \rangle \leq 0, \forall y \in C \}}, & \text{if } x \in C, \\
\emptyset, & \text{if } x \notin C.
\end{array} \right.$$
Proposition 2.10. Let \((\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}\). Suppose that \(S \colon X \to X\) is continuous, linear, and single-valued such that \(S\) is monotone, \(S^* = -S\) and \(S^2 = -\gamma \text{Id}\) where \(\gamma \geq 0\). Suppose that \(L = \alpha \text{Id} + \beta S\) and set \(A = L + N_{B(0;1)}\). Then \(A\) is maximally monotone and

\[
J_A x = \begin{cases} 
\frac{1}{(1+\alpha)^2 + \beta^2 \gamma}((1+\alpha)\text{Id} - \beta S)x, & \|x\|^2 \leq (1+\alpha)^2 + \beta^2 \gamma; \\
\frac{1}{\|x\|^2} \left(\sqrt{\|x\|^2 - \beta^2 \gamma \text{Id} - \beta S}\right)x, & \text{otherwise.} 
\end{cases}
\tag{13}
\]

Proof. The maximal monotonicity of \(A\) follows from, e.g., [7, Corollary 25.5(i)]. We now turn to \((13)\). Indeed, let \(u \in X\) and observe that \(u = J_A x \iff x \in (1+\alpha)u + \beta Su + N_{B(0;1)} u \iff (\exists r \geq 0)
\]

\[
x = (1+\alpha)u + \beta Su + ru,
\tag{14}
\]

where we used [7, Example 6.39]. We proceed by cases. \textbf{Case 1:} \(\|u\| < 1\). In this case \(r = 0\) and \((14)\) yields \(x = (1+\alpha)u + \beta Su\). By \((14)\) with \(r = 0\) we deduce that \(\|x\|^2 = ((1+\alpha)^2 + \gamma \beta^2)\|u\|^2\) and also that \(u = J_A x\). Now combine with Proposition 2.9 applied with \(A\) replaced by \(L\). \textbf{Case 2:} \(\|u\| = 1\). It follows from \((14)\) that \(r = \langle x, u \rangle - (1+\alpha)\). Therefore, \((14)\) becomes \(x = \langle x, u \rangle u + \beta Su\). Hence \(\|x\|^2 = \|\langle x, u \rangle u + \beta Su\|^2 = \langle x, u \rangle^2 + \beta^2 \gamma\). We therefore learn that \(\langle x, u \rangle^2 = \|x\|^2 - \beta^2 \gamma\). Because \(J_A\) is (maximally) monotone by, e.g., [7, Corollary 23.11(i)] and \(\{(0,0), (x,u)\} \subseteq \text{gra} J_A\) we learn that \(\langle x, u \rangle \geq 0\). Therefore, we conclude that \(\langle x, u \rangle = \sqrt{\|x\|^2 - \beta^2 \gamma}\). Altogether we rewrite \((14)\) as

\[
x = \sqrt{\|x\|^2 - \beta^2 \gamma u + \beta Su}.
\tag{15}
\]

It is straightforward to verify that \(u = \frac{1}{\|x\|^2}(\sqrt{\|x\|^2 - \beta^2 \gamma} \text{Id} - \beta S)x\) satisfies \((15)\). The proof is complete. \hfill \blacksquare

Example 2.11. Suppose that \(X = \mathbb{R}^2\), let \(\theta \in \left[0, \frac{\pi}{2}\right]\), set

\[
\mathcal{R} = \mathcal{R}_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},
\tag{16}
\]

and set \(A = \mathcal{R} + N_{B(0;1)}\). Then \(A\) is maximally monotone and strongly monotone and

\[
J_A x = \begin{cases} 
\frac{1}{2(1+\cos(\theta))}((1+\cos(\theta))\text{Id} + \mathcal{R}_\theta)x, & \|x\|^2 \leq 2(1 + \cos \theta); \\
\frac{1}{\|x\|^2} \left(\sqrt{\|x\|^2 - \sin^2(\theta) - \cos(\theta)}\right)\text{Id} + \mathcal{R}_\theta x, & \text{otherwise.}
\end{cases}
\tag{17}
\]

Proof. Set \(S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and observe that \(S\) is maximally monotone, \(S^T = -S\), \(S^2 = -\text{Id}\), and \(\mathcal{R} = \cos(\theta) \text{Id} + \sin(\theta) S\). It is straightforward to verify that, likewise \(\mathcal{R}\), \(A\) is \(\cos(\theta)\)-strongly monotone. The conclusion now follows from applying Proposition 2.10 with \((\alpha, \beta, \gamma)\) replaced by \((\cos(\theta), \sin(\theta), 1)\). \hfill \blacksquare

Example 2.12. Suppose that \(X = \mathbb{R}^2\), let \((u, u^+) \in \mathbb{R}^2 \times \mathbb{R}^2\) such that \(\|u\| = \|u^+\| = 1\), \(\langle u, u^+ \rangle = 0\), and set \(K = \{x \in X \mid \langle x, u \rangle \leq 0\}\). Let \(L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{R}^{2 \times 2}\) be monotone\(^5\) and set \(A = L + N_K\). Set

\[^5\text{Recall that } L\text{ is monotone if and only if } \alpha \geq 0, \delta \geq 0 \text{ and } 4\alpha \delta \geq (\beta + \gamma)^2\text{ (see [9, Lemma 6.1]).}\]
\( \kappa = 1 + \langle u^+, Lu^+ \rangle \geq 1 \) by the monotonicity of \( L \). Let \( x \in \mathbb{R}^2 \). Then we have

\[
J_L = \frac{1}{(1+\alpha)(1+\gamma)-\beta^2} \begin{pmatrix} 1+\delta & -\beta \\ -\gamma & 1+\alpha \end{pmatrix},
\]

and

\[
J_Ax = \begin{cases} J_Lx, & \langle J_Lx, u \rangle < 0; \\ \frac{1}{\kappa} \langle x, u^+ \rangle u^+, & \text{otherwise}. \end{cases}
\]

**Proof.** The formula in (18) is clear. We verify (19). Let \( (x, z) \in X \times X \). Then \( z = J_Ax \Leftrightarrow x \in z + Lz + N_Kz \). Observe that this implies that \( z \in K \) and therefore we consider two cases.

**Case 1:** \( \langle z, u \rangle < 0 \). This implies that \( N_Kz = \{0\} \) and, therefore, \( x = (\text{Id} + L)z \). Equivalently, \( z = J_Lx \).

**Case 2:** \( \langle z, u \rangle = 0 \). Observe that in this case \( N_Kz = \mathbb{R}^+u \). That is, \( (\exists r \geq 0) \) such that \( x = z + Lz + ru \). Therefore, \( r = \langle x - Lz, u \rangle \). We claim that \( z = \frac{1}{\kappa} \langle x, u^+ \rangle u^+ \) solves the equation

\[
x = z + Lz + \langle x - Lz, u \rangle u.
\]

Indeed, substituting for \( z = \frac{1}{\kappa} \langle x, u^+ \rangle u^+ \) in the right hand side of (20) yields

\[
z + Lz + \langle x - Lz, u \rangle u = z + \langle Lz, u \rangle u + \langle Lz, u^+ \rangle u^+ + \langle x, u \rangle u - \langle Lz, u \rangle u
\]

\[
= z + \langle Lz, u^+ \rangle u^+ + \langle x, u \rangle u
\]

\[
= \frac{1}{\kappa} \langle x, u^+ \rangle u^+ + \frac{1}{\kappa} \langle Lu^+, u^+ \rangle \langle x, u^+ \rangle u^+ + \langle x, u \rangle u
\]

\[
= \frac{1}{\kappa} \langle 1 + \langle Lu^+, u^+ \rangle \rangle \langle x, u^+ \rangle u^+ + \langle x, u \rangle u
\]

\[
= \langle x, u^+ \rangle u^+ + \langle x, u \rangle u = x.
\]

Hence, \( z = \frac{1}{\kappa} \langle x, u^+ \rangle u^+ \) solves (20) as claimed. The proof is complete. \( \square \)

**Example 2.13.** Suppose that \( X = \mathbb{R}^2 \), let \( \theta \in ]0, \pi/2[ \), set

\[
\mathcal{R} = \mathcal{R}_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},
\]

and let \( U \) be nonempty closed convex subset in \( \mathbb{R}^2 \). Set

\[
A = \mathcal{R} + N_U.
\]

Then the following hold:

(i) \( A \) is maximally monotone and strongly monotone.

(ii) \( A \) is \( 3^\ast \) monotone.

(iii) \( A \) is not cyclically monotone. Hence, \( A \) is not a subdifferential operator.
Proof. Set \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and observe that \( R = \cos(\theta) \text{Id} + \sin(\theta)S \).

(i): Clearly \( R \) is maximally monotone and \( \cos \theta \)-strongly monotone and \( N_U \) is maximally monotone. The maximal monotonicity of \( A \) follows from, e.g., [7, Corollary 25.5(i)]. The strong monotonicity of \( A \) is an immediate consequence of the strong monotonicity of \( R \).

(ii): Combine (i) and [7, Example 25.15(iv)].

(iii): It follows from [2, Example 4.6] that \( A \) is not cyclically monotone. Therefore by [35, Theorem B] \( A \) is not a subdifferential operator. □

Example 2.14. Suppose that \( X = \mathbb{R}^2 \), let

\[
R = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},
\]

and let \( K = \mathbb{R}_- \times \mathbb{R} \).

Then the following hold:

(i) \( A \) is maximally monotone and strongly monotone.
(ii) \( A \) is \( 3^* \)-monotone.
(iii) \( A \) is not cyclically monotone. Hence, \( A \) is not a subdifferential operator.
(iv) We have

\[
J_A : (\xi_1,\xi_2) \mapsto \begin{cases} \frac{1}{2} (\text{Id} - R)(\xi_1,\xi_2), & \xi_1 + \sqrt{3}\xi_2 < 0; \\ \frac{1}{2} (0,\xi_2), & \text{otherwise}. \end{cases}
\]

Proof. (i)–(iii): Apply Example 2.13(i)–(iii) with \((\theta, U)\) replaced by \((\frac{\pi}{3}, K)\). (iv): Write \( K = \{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid \langle (\xi_1,\xi_2), (1,0) \rangle \leq 0 \} \). Now apply Example 2.12 with \((u,u^+)\) replaced by \(((1,0),(0,1))\) and \((\alpha,\beta,\gamma,\delta)\) replaced by \((\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2})\). □

3 Generalized solutions and generalized fixed points

Throughout the remainder of this paper, we set

\[
T = T_{(A,B)} = \frac{1}{2} (\text{Id} + R_B R_A).
\]

The well-defined \textit{minimal displacement vector} associated with \( T \) (see Lemma 2.4) is

\[
v = \Pi_{\text{ran}(\text{Id} - T)}(0),
\]
and the generalized solution set (this is also known as the set of normal solutions (see [12])) is

\[
Z = \text{zer}(-v + A + B(\cdot - v)) = \{ x \in X \mid 0 \in -v + Ax + B(x - v) \}. \tag{29}
\]

We recall that (see [32, Theorem 2.5])

\[\text{ran} \ (\text{dom} \ A - \text{dom} \ B) \cap (\text{ran} \ A + \text{ran} \ B). \tag{30}\]

The following remark gives situations under which assumption (30) holds. We recall that case (i) below automatically holds when \((A, B) = (\partial f, \partial g)\) for \(f, g\) convex, proper and lsc functions.

**Remark 3.1.** Assumption (30) holds if one of the following holds (see [13, Theorem 5.2]): (i) \(A\) and \(B\) are \(3^∗\) monotone. (ii) \((\exists C \in \{A, B\}) \text{ dom } C = X\) and \(C\) is \(3^∗\) monotone. (iii) \((\exists C \in \{A, B\}) \text{ ran } C = X\) and \(C\) is \(3^∗\) monotone.

**Lemma 3.2.** Let \(Z\) be as in (29). We have \(Z = J_A(\text{Fix}(v + T))\).

**Proof.** Indeed, it follows from [12, Proposition 2.24] that \(T_{(-v + A, B(\cdot - v))} = T(\cdot + v)\) Combining with (4), applied with \((A, B)\) replaced by \((-v + A, B(\cdot - v))\), and [7, Proposition 23.17(ii)], we learn that \(Z = J_{-v + A}(\text{Fix} T(\cdot + v)) = J_A(v + \text{Fix} T(\cdot + v))\). The claim now follows from combining the last equality with Lemma 2.5 applied with \(w\) replaced by \(v\). \[\Box\]

**Lemma 3.3.** Let \(\lambda \in [0, 1]\), set

\[T_\lambda = (1 - \lambda) \text{ Id} + \lambda R_B R_A, \tag{31}\]

and set \(v_\lambda = \text{ran } (\text{dom } T, T_\lambda)(0)\). Then \(v_\lambda\) is well defined. Moreover the following hold:

(i) \(\text{Id} - T_\lambda = \lambda(\text{Id} - R_B R_A) = 2\lambda(J_A - J_B R_A)\).

(ii) \(v_\lambda = 2\lambda v\).

(iii) \(\text{Fix}(v_\lambda + T_\lambda) = \text{Fix}(v + T)\).

**Proof.** The claim that \(v_\lambda\) is well defined follows from applying Lemma 2.4 with \(N\) replaced by \(R_B R_A\). (i): This fact clearly follows from (31) and the definitions. (ii): Recalling (27), observe that \(T = T_{1/2}\) and hence \(\text{ran } (\text{dom } T, T) = \text{ran } (\text{dom } (1 - \lambda) \text{ Id} - \lambda R_B R_A) = \text{ran } (-\lambda(\text{Id} - R_B R_A)) = \text{ran } (2\lambda(\text{Id} - \lambda R_B R_A)) = 2\lambda\text{ran } (\text{Id} - T)\). Now combine with Lemma 2.2 applied with \((\alpha, C)\) replaced by \((2\lambda, \text{ran } (\text{dom } T))\). (iii): Indeed, let \(x \in X\). Then \(x \in \text{Fix}(v_\lambda + T_\lambda) \iff x = v_\lambda + (1 - \lambda) x + \lambda R_B R_A x \iff \lambda x = v_\lambda + \lambda R_B R_A x \iff x = \frac{v_\lambda}{\lambda} + \frac{1}{\lambda}(x + R_B R_A x)\). Now combine with (ii). \[\Box\]

**Proposition 3.4.** Let \(\lambda \in [0, 1]\) and set \(T_\lambda = (1 - \lambda) \text{ Id} + \lambda R_B R_A\). Let \(x \in X\) and let \(n \in \mathbb{N}\). Suppose that \(y \in \text{Fix}(v + T)\). Then the following hold:

(i) \(T_\lambda^n y = y - 2\lambda n v \in \text{Fix}(v + T)\).

(ii) Suppose that \(Z = \{ \bar{x} \}\). Then \(J_A T_\lambda^n y = J_A(y - 2\lambda n v) = \bar{x}\).

Suppose that \(Z = \{ \bar{x} \}\) and \(\lambda \in [0, 1]\). Then we additionally have:
Proof. (i): Combine Lemma 3.3(ii) and Lemma 2.6 with \( T \) replaced by \( T_\lambda \). (ii): The first identity is a direct consequence of (i). Now on the one hand, we have \( Z = J_A(\text{Fix}(\nu + T)) = \{ \overline{x} \} \). On the other hand, by (i) we have \( J_A(y - 2\lambda n\nu) \in J_A(\text{Fix}(\nu + T)) \). Altogether, the conclusion follows.

(iii): It follows from Lemma 3.3(i) that \( J_AT_\lambda^n x - J_BR_AT_\lambda^n x = \frac{1}{2\lambda}(\text{Id} - T_\lambda)T_\lambda^n x \). Now combine with (5) applied with \( T \) replaced by \( T_\lambda \) in view of Lemma 3.3(ii).

(iv): Combine (ii) and (iii) applied with \( x \) replaced by \( y \). \( \square \)

We now recall the following key result by Minty which is of central importance in our proofs.

Fact 3.5 (Minty’s Theorem). Let \( C : X \rightrightarrows X \) be monotone. Then

\[
\text{gra } C = \{ (J_\nu x, J_{\nu^{-1}} x) \mid x \in \text{ran } (\text{Id} + C) \}. \tag{32}
\]

Moreover,

\[
C \text{ is maximally monotone } \iff \text{ran } (\text{Id} + C) = X. \tag{33}
\]

Proof. See [31]. \( \square \)

Lemma 3.6. Let \( \lambda \in [0,1] \) and set

\[
T_\lambda = (1 - \lambda) \text{Id} + \lambda R_BR_A. \tag{34}
\]

Let \( (x, y) \in X \times X \). Then

\[
\lambda \|x - y\|^2 - \lambda \|T_\lambda x - T_\lambda y\|^2 = (1 - \lambda)\|\text{Id} - T_\lambda\|x - (\text{Id} - T_\lambda) y\|^2
\]

\[
= 4\lambda^2 \langle J_Ax - J_Ay, J_{A^{-1}}x - J_{A^{-1}}y \rangle + 4\lambda^2 \langle J_BR_Ax - J_BR_Ay, J_{B^{-1}}R_Ax - J_{B^{-1}}RAy \rangle. \tag{35}
\]

Consequently we have:

(i) \( \|x - y\|^2 - \|T_\lambda x - T_\lambda y\|^2 \geq 4\lambda \langle J_Ax - J_Ay, J_{A^{-1}}x - J_{A^{-1}}y \rangle \).

(ii) \( \|x - y\|^2 - \|T_\lambda x - T_\lambda y\|^2 \geq 4\lambda \langle J_BR_Ax - J_BR_Ay, J_{B^{-1}}R_Ax - J_{B^{-1}}RAy \rangle \).

Proof. Indeed, observe that

\[
T_\lambda = (1 - 2\lambda) \text{Id} + \lambda (\text{Id} + R_BR_A) \tag{36}
\]

and

\[
\text{Id} - T_\lambda = \lambda (\text{Id} - R_BR_A). \tag{37}
\]

In view of (36) and (37) we have

\[
\lambda \|x - y\|^2 - \lambda \|T_\lambda x - T_\lambda y\|^2 = (1 - \lambda)\|\text{Id} - T_\lambda\|x - (\text{Id} - T_\lambda) y\|^2 \tag{38a}
\]

\[
= \langle \lambda((x - y) + (T_\lambda x - T_\lambda y)), (x - y) - (T_\lambda x - T_\lambda y) \rangle - (1 - \lambda)\|\text{Id} - T_\lambda\|x - (\text{Id} - T_\lambda) y\|^2 \tag{38b}
\]

\[
= \langle \lambda((x - y) + (T_\lambda x - T_\lambda y)) - (1 - \lambda)((x - y) - (T_\lambda x - T_\lambda y)), (x - y) - (T_\lambda x - T_\lambda y) \rangle \tag{38c}
\]

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and the conclusion follows using the monotonicity of $A$. Upholding the notation of Section 3 we recall that Lemma 3.2 implies that

$$
\langle T_\lambda x - T_\lambda y, (x - y) - (T_\lambda x - T_\lambda y) \rangle
$$

(38d)

$$
= \lambda^2 \langle (x - y) + (R_B R_A x - R_B R_A y), (x - y) - (R_B R_A x - R_B R_A y) \rangle
$$

(38e)

$$
= \lambda^2 \left( \|x - y\|^2 - \|R_B R_A x - R_B R_A y\|^2 \right)
$$

(38f)

$$
= \lambda^2 \left( \|x - y\|^2 - \|R_A x - R_A y\|^2 + \|R_A x - R_A y\|^2 - \|R_B R_A x - R_B R_A y\|^2 \right)
$$

(38g)

$$
= \lambda^2 \left( \langle (\text{Id} + R_A) x - (\text{Id} + R_A) y, (\text{Id} - R_A) x - (\text{Id} - R_A) y \rangle

+ \lambda^2 \left( \langle (\text{Id} + R_A) R_A x - (\text{Id} + R_A) R_A y, (\text{Id} - R_A) R_A x - (\text{Id} - R_A) R_A y \rangle

\right) \right)
$$

(38h)

$$
= 4 \lambda^2 \langle JAx - JAy, J_{A^{-1}} x - J_{A^{-1}} y \rangle + 4 \lambda^2 \langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y \rangle.
$$

(38i)

This proves (35).

(i) & (ii): Observe that the monotonicity of $A$ (respectively $B$) and the Minty parametrization (see Fact 3.5) of gra $A$ (respectively gra $B$) imply that $\langle JAx - JAy, J_{A^{-1}} x - J_{A^{-1}} y \rangle \geq 0$ (respectively $\langle J_B R_A x - J_B R_A y, J_{B^{-1}} R_A x - J_{B^{-1}} R_A y \rangle \geq 0$). Now combine with (35). □

**Corollary 3.7.** Let $\lambda \in [0,1]$ and set

$$
T_\lambda = (1 - \lambda) \text{Id} + \lambda R_B R_A.
$$

(39)

Let $(x, y) \in X \times X$. Then the following hold:

(i) $\langle J_A T^n_\lambda x - J_A T^n_\lambda y, J_{A^{-1}} T^n_\lambda x - J_{A^{-1}} T^n_\lambda y \rangle \to 0$.

(ii) $\langle J_B R_A T^n_\lambda x - J_B R_A T^n_\lambda y, J_{B^{-1}} R_A T^n_\lambda x - J_{B^{-1}} R_A T^n_\lambda y \rangle \to 0$.

**Proof.** (i): It follows from Lemma 3.6(i) that ($\forall n \in \mathbb{N}$)

$$
\|T^n_\lambda x - T^n_\lambda y\|^2 - \|T^{n+1}_\lambda x - T^{n+1}_\lambda y\|^2 \geq 4 \lambda \langle J_A T^n_\lambda x - J_A T^n_\lambda y, J_{A^{-1}} T^n_\lambda x - J_{A^{-1}} T^n_\lambda y \rangle.
$$

(40)

Telescoping yields

$$
\sum_{n=0}^{\infty} \langle J_A T^n_\lambda x - J_A T^n_\lambda y, J_{A^{-1}} T^n_\lambda x - J_{A^{-1}} T^n_\lambda y \rangle < +\infty,
$$

(41)

and the conclusion follows using the monotonicity of $A$ in view of Minty’s parametrization Fact 3.5. (ii): Proceed similar to the proof of (i) using Lemma 3.6(ii) and the monotonicity of $B$ in view of Fact 3.5. □

### 4 Dynamic consequences

Upholding the notation of Section 3 we recall that Lemma 3.2 implies that

$$
Z \neq \emptyset \iff \text{Fix}(\nu + T) \neq \emptyset \iff \nu \in \text{ran (Id} - T).
$$

(42)

**Lemma 4.1.** Suppose that $A: X \rightharpoonup X$ is uniformly monotone with modulus $\phi$. Let $(x, y) \in X \times X$. Then

$$
\langle JAx - JAy, J_{A^{-1}} x - J_{A^{-1}} y \rangle \geq \phi(\|JAx - JAy\|).
$$

(43)
Proof. This is a direct consequence of Minty’s theorem Fact 3.5. □

Let \( C : X \rightarrow X \) be uniformly monotone with modulus \( \phi \) and suppose that \( \text{zer} \ C \neq \emptyset \). Then \( C \) is strictly monotone and it follows from, e.g., [7, Proposition 23.35] that

\[
\text{zer} \ C \text{ is a singleton.} \tag{44}
\]

Moreover, it follows from [7, Example 25.15(iii)] that

\[
\phi \text{ is supercoercive } \iff \ C \text{ is } 3^* \text{ monotone.} \tag{45}
\]

We are now ready for the main results of this section.

**Theorem 4.2 (convergence of Douglas–Rachford algorithm).** Let \( \lambda \in [0,1] \) and set

\[
T_\lambda = (1 - \lambda) \text{Id} + \lambda R_B R_A. \tag{46}
\]

Suppose that \( Z \neq \emptyset \) and that \( (\exists C \in \{ A, B \}) \) such that \( C \) is uniformly monotone with a supercoercive modulus. Then \( (\exists x \in X) \) such that \( Z = \{ x \} \). Let \( x \in X \). Then the following hold:

(i) \( J_A T_\lambda^n x \rightarrow \overline{x} \).

(ii) \( J_B R_A T_\lambda^n x \rightarrow \overline{x} - v \).

Proof. To lighten the notation, throughout the proof we set \( T = T_\lambda \). By assumption it is straightforward to verify that \( (\exists \overline{C} \in \{ -v + A, B(\cdot + v) \}) \) such that \( \overline{C} \) is uniformly monotone and \( 3^* \) monotone (see (45)). Hence, \( -v + A + B(\cdot + v) \) is uniformly monotone and therefore \( Z \) is a singleton by (44) applied with \( C \) replaced by \( -v + A + B(\cdot + v) \). Now combine with (42) to learn that \( \text{Fix}(v + T) \neq \emptyset \). (i)&(ii): Indeed, let \( y \in \text{Fix}(v + T) \) and observe that Proposition 3.4(ii) implies that \((\forall n \in \mathbb{N}) J_A T^n y = \overline{x} \). First suppose that \( C = A \). Combining Corollary 3.7(i) and Lemma 4.1 applied with \((x,y)\) replaced by \((T^n x, T^n y)\) we learn that \( \phi(\| J_A T^n x - \overline{x} \|) = \phi(\| J_A T^n x - J_A T^n y \|) \rightarrow 0 \). Hence \( J_A T^n x \rightarrow \overline{x} \). Now combine with Proposition 3.4(iii) to prove (ii). For the case \( C = B \), proceed similar to above but use Proposition 3.4(iv) and Corollary 3.7(ii) instead. □

**Theorem 4.3 (convergence of Peaceman–Rachford algorithm).** Set

\[
\overline{T} = R_B R_A. \tag{47}
\]

Suppose that \( Z \neq \emptyset \) and that \( A \) is uniformly monotone with supercoercive modulus. Then \( (\exists x \in X) \) such that \( Z = \{ x \} \). Let \( x \in X \). Then \( J_A \overline{T}^n x \rightarrow \overline{x} \).

Proof. Observe that, likewise \( A, -v + A \) is uniformly monotone and \( 3^* \) monotone (see (45)). Hence, \( -v + A + B(\cdot + v) \) is uniformly monotone and therefore \( Z \) is a singleton by applying (44) with \( C \) replaced by \( -v + A + B(\cdot + v) \). Now combine with (42) to learn that \( \text{Fix}(v + T) \neq \emptyset \). Let \( y \in \text{Fix}(v + T) \) and observe that Proposition 3.4(ii) implies that \((\forall n \in \mathbb{N}) J_A T^n y = \overline{x} \). Therefore Corollary 3.7(i) implies that

\[
\langle J_A \overline{T}^n x - \overline{x}, J_A^{-1} \overline{T}^n x - J_A^{-1} \overline{T}^n y \rangle \rightarrow 0. \tag{48}
\]
Combining (48) with Lemma 4.1 applied with \( x \) replaced by \( \bar{T}^n x \) yields \( \phi(\|J_A \bar{T}^n x - \bar{r}\|) \to 0 \), and the conclusion follows. ■

Before proceeding to the illustrative example in this section we recall the following fact.

**Fact 4.4.** Suppose that \( (\exists C \in \{A, B\}) \) such that \( C \) is \( 3^* \) monotone and surjective. Then

\[
\text{ran} (\text{Id} - T) = (\text{dom} A - \text{dom} B) \cap (\text{ran} A + \text{ran} B) = \text{dom} A - \text{dom} B.
\]  

(49)

**Proof.** See [32, Theorem 2.5] and also [13, Theorem 5.2]. ■

**Example 4.5.** Suppose that \( X = \mathbb{R}^2 \), let \( \theta \in [0, \frac{\pi}{2}] \), let \( (\beta, \gamma) \in \mathbb{R}^2 \), let \( r > 0 \), set \( b = (0, \beta) \), set \( c = (\gamma, 0) \), set

\[
\mathcal{R} = \mathcal{R}_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},
\]  

(50)

set \( A = \mathcal{R} + N_{B(0,1)} \) and set \( B = b + N_{B(r)} \). Let \( x \in X \), let \( \lambda \in [0, 1] \) and set

\[
T_\lambda = (1 - \lambda) \text{Id} + \lambda R_B R_A = \text{Id} - 2\lambda J_A + 2\lambda J_B(2J_A - \text{Id}).
\]  

(51)

Then

\[
J_A x = \begin{cases} 
\frac{1}{\sqrt{1 + \cos(\theta)}} (\text{Id} + \mathcal{R}_-\theta) x, & \|x\|^2 \leq 2(1 + \cos\theta); \\
\frac{1}{\|x\|^2} \left( \sqrt{\|x\|^2 - \sin^2(\theta) - \cos(\theta)} \right) \text{Id} + \mathcal{R}_-\theta) x, & \text{otherwise},
\end{cases}
\]  

(52)

\[
J_B x = P_{B(r)}(x - b) = \begin{cases} 
x - b, & \text{if } \|x - (b + c)\| \leq r; \\
x - b + \gamma r, & \text{otherwise},
\end{cases}
\]  

(53)

Moreover, we have:

(i) \( A \) is maximally monotone and strongly monotone, hence \( A \) is uniformly monotone.

(ii) \( B \) is maximally monotone.

(iii) \( \text{ran} (\text{Id} - T) = \text{dom} A - \text{dom} B = \text{dom} A - \text{dom} B = B(-c; r + 1) \).

(iv) \( v = (\text{sign}(\gamma), \min\{0, r + 1 - |\gamma|\}, 0) \).

(v) Exactly one of the following holds:

(a) \( |\gamma| < (r + 1) \) and \( A + B \) is maximally monotone, in which case \( v = (0, 0) \in \text{ran} (\text{Id} - T) \) and \( Z \) is a singleton.

(b) \( |\gamma| \geq (r + 1) \) and \( v = (\text{sign}(\gamma)(r + 1) - \gamma, 0) \), in which case

\[
|Z| = \varnothing \iff Z = \{\text{sign}(\gamma)(1, 0)\} \iff \beta = -\text{sign}(\gamma) \sin \theta.
\]

(vi) Suppose that \( \beta = -\text{sign}(\gamma) \sin \theta \). Then \( Z = \{\text{sign}(\gamma)(1, 0)\} \) and the following hold:

(a) \( J_A T_\lambda^n x \to \text{sign}(\gamma)(1, 0) \).

(b) Suppose that \( \lambda \in [0, 1] \). Then \( J_B R_A J_\lambda^n x \to (\gamma - \text{sign}(\gamma)r, 0) \).

**Proof.** The formula in (52) follows from applying Proposition 2.10 with \((a, \beta, \gamma) \) replaced by \((\cos \theta, \sin \theta, 1) \) and \( S \) replaced by \((\frac{1}{2}, 0) \). The first identity in (53) follows from [7, Proposition 23.17(ii)] whereas the second identity follows from , e.g., [7, Proposition 29.10].
Figure 1: A GeoGebra [28] snapshot illustrating Example 4.5(vi) with \( \theta = \pi/4, \gamma = -3.5, r = 3/2 \) and \( \lambda = 1/2 \). The first few iterates of the governing sequence \( (x_n)_{n \in \mathbb{N}} = (T^n x)_{n \in \mathbb{N}} \) (the blue dots) and the shadow sequence \( (y_n)_{n \in \mathbb{N}} = (J A T^n x)_{n \in \mathbb{N}} \) (the red dots) are also depicted.

(i): It is straightforward to verify that \( A \) is \( \cos \theta \)-strongly monotone. The maximal monotonicity of \( A \) follows from, e.g., [7, Corollary 25.5(i)].

(ii): This is clear by observing that \( B = \partial (\iota B(c; r) + \langle b, \cdot \rangle) \).

(iii): On the one hand, we have \( \text{dom } A = B(0; 1) \) and \( \text{dom } B = B(c; r) \). Therefore \( \text{ran } A = \text{ran } B = \mathbb{R}^2 \) by, e.g., [7, Corollary 21.15]. On the other hand, combining (i) and [7, Example 25.15(iv)] we learn that \( A \) is \( 3^* \) monotone. Moreover, \( B \) is \( 3^* \) monotone by, e.g., [7, Example 25.14]. Now, apply Fact 4.4 with \( C \) replaced by either \( A \) or \( B \) to learn that \( \text{ran } (\text{Id} - T) = B(0; 1) - B(c; r) = B(-c; r + 1) \).

(iv): Observe that \( -c = (-\gamma, 0) \). If \( |\gamma| < r + 1 \) then \( (0, 0) \in B(-c; r + 1) \) and hence \( \nu = (0, 0) \). Else, if \( |\gamma| \geq r + 1 \) then \( \nu = (\text{sign}(\gamma)(r + 1) - \gamma, 0) \) and the conclusion follows.

(v)(a): Indeed, in view of (iii) and (iv) we have \( (0, 0) \in \text{int } B(-c; r + 1) = \text{int } (\text{dom } A - \text{dom } B) \). Therefore, \( A + B \) is maximally monotone by, e.g., [7, Corollary 25.5(iii)] and strongly monotone by (i). Hence, \( \text{zer } (A + B) \) is a singleton by, e.g., [7, Corollary 23.37].

(v)(b): The formula for \( \nu \) follows from (iv). Observe that \( Z \subseteq \text{dom } (-\nu + A) \cap \text{dom } B(\cdot - \nu) = B(0; 1) \cap B(c + \nu; r) = \{ \text{sign}(\gamma)(1, 0) \} \). This proves the first equivalence. We now turn to the second equivalence. Indeed, set \( z = \text{sign}(\gamma)(1, 0) \). Then \( 0 \in -\nu + Rz + N B(0; 1) z + b + N B(c; r) z \iff 0 \in -\nu + Rz + b + \mathbb{R} \times \{ 0 \} = Rz + b + \mathbb{R} \times \{ 0 \} \iff \text{sign}(\gamma)(\cos \theta, \sin \theta) + (0, \beta) \in \mathbb{R} \times \{ 0 \} \iff \beta = -\text{sign}(\gamma) \sin \theta \).

(vi)(a): Combine (v)(b), Theorem 4.2(i) and Theorem 4.3. (vi)(b): Combine (v)(b) and Theorem 4.2(ii).
5 Application to optimization problems

In this section we assume that
\[ f \text{ and } g \text{ are proper lower semicontinuous convex functions on } X, \]  
and we set
\[ (A, B) = (\partial f, \partial g). \]  

We use the abbreviations
\[ (R_f, R_g) = (2 \text{Prox}_f - \text{Id}, 2 \text{Prox}_g - \text{Id}). \]  

Hence,
\[ T = T_{(\partial f, \partial g)} = \frac{1}{2} (\text{Id} + R_g R_f) = \text{Id} - \text{Prox}_f + \text{Prox}_g R_f, \]  
and
\[ Z = \{ x \in X \mid 0 \in -v + \partial f(x) + \partial g(x - v) \}. \]

Lemma 5.1. Suppose that \( \exists h \in \{ f, g \} \) such that \( h \) is uniformly convex on \( X \). Then the following hold:

(i) \( \text{ran}(\text{Id} - T) = \text{dom } f - \text{dom } g. \)

(ii) \( v = P_{\text{dom } f - \text{dom } g}^{\partial h}. \)

(iii) \( \partial h \) is uniformly monotone.

Suppose that \( Z \neq \emptyset \). Then there exists \( \pi \in X \) such that the following hold:

(iv) \( -\langle \cdot, v \rangle + f + g(\cdot - v) \) is uniformly convex, lower semicontinuous and proper.

(v) \( Z = \text{argmin}(-\langle \cdot, v \rangle + f + g(\cdot - v)) = \{ \pi \}. \)

Proof. (i): It follows from [42, Theorem 3.5.5(i)] that \( h^* \) is uniformly smooth, hence \( \text{dom } h^* = X \). Now combine this with [32, Theorem 3.3(i)].

(ii): Combine (i) and (28).

(iii): This is [7, Example 22.4(iii)].

(iv): It is straightforward to verify that \( \exists \tilde{h} \in \{ -\langle \cdot, v \rangle + f, g(\cdot - v) \} \) such that \( \tilde{h} \) is uniformly convex and therefore \( -\langle \cdot, v \rangle + f + g(\cdot - v) \) is uniformly convex. The lower semicontinuity of \( -\langle \cdot, v \rangle + f + g(\cdot - v) \) is a direct consequence of the lower semicontinuity of \( f \) and \( g \). Finally observe that \( \emptyset \neq Z \subseteq \text{dom } f \cap \text{dom } g(\cdot - v) \). Hence, \( -\langle \cdot, v \rangle + f + g(\cdot - v) \) is proper.

(v): Indeed, recall that \( Z = \{ x \in X \mid 0 \in -v + \partial f(x) + \partial g(x - v) \}. \) On the one hand, it follows from (iii) that \( -v + \partial f(x) + \partial g(x - v) \) is uniformly monotone. On the other hand, (44) applied with \( C \) replaced by \( -v + \partial f(x) + \partial g(x - v) \) implies that \( Z \) is a singleton. Now observe that \( Z \subseteq \text{argmin}(-\langle \cdot, v \rangle + f + g(\cdot - v)) \), and the latter set is a singleton by combining (iv) and, e.g., [7, Proposition 17.26(iii)].

The following fact will be used throughout the examples in this section to calculate \( v \).
**Fact 5.2.** Let \( x_0 \in X \) and \( \lambda \in ]0, 1] \). Update \( x_0 \) via \( x_{n+1} = T_\lambda x_n \) where \( T_\lambda \) is as in (3). The following hold.

(i) \( \frac{x_n}{n} \to -2\lambda v \).

(ii) Suppose \( \lambda \in ]0, 1[ \). Then \( x_n - x_{n+1} \to 2\lambda v \).

*Proof.* See [33] for (i) and [1] or [20] for (ii).  

**Theorem 5.3 (convergence of Douglas–Rachford algorithm).** Let \( \lambda \in ]0, 1[ \) and set

\[
T_\lambda = (1 - \lambda) \text{Id} + \lambda R_g R_f .
\]  

(59)  

Suppose that \( Z \neq \emptyset \) and that \( (\exists h \in \{f, g\}) \) such that \( h \) is uniformly convex on \( X \). Then \( (\exists \overline{x} \in X) \) such that \( Z = \text{argmin}(-\langle \cdot, v \rangle + f + g(\cdot - v)) = \{\overline{x}\} \). Let \( x \in X \). Then the following hold:

(i) \( \text{Prox}_f T_\lambda^n x \to \overline{x} \).

(ii) \( \text{Prox}_g R_f T_\lambda^n x \to \overline{x} - v \).

*Proof.* (i)&(ii): Combine Theorem 4.2(i)&(ii), applied with \((A, B, C)\) replaced by \((\partial f, \partial g, \partial h)\), and Lemma 5.1(iii)&(v).  

**Theorem 5.4 (convergence of Peaceman–Rachford algorithm).** Set

\[
\widetilde{T} = R_g R_f .
\]  

(60)  

Suppose that \( Z \neq \emptyset \) and that \( f \) is uniformly convex on \( X \). Then \( (\exists \overline{x} \in X) \) such that \( Z = \text{argmin}(-\langle \cdot, v \rangle + f + g(\cdot - v)) = \{\overline{x}\} \). Let \( x \in X \). Then \( \text{Prox}_f \widetilde{T}^n x \to \overline{x} \).

*Proof.* Combine Theorem 4.3 applied with \((A, B, C)\) replaced by \((\partial f, \partial g, \partial h)\), and Lemma 5.1(iii)&(v).  

**Proposition 5.5.** Suppose that \( X \) is finite-dimensional, let \( w \in X \) and let \( U \) and \( V \) be nonempty polyhedral subsets\(^6\) of \( X \). Let \( r > 0 \) and set \( f = \frac{r}{2} \| \cdot - w \|^2 + \iota_U \) and set \( g = \iota_V \). Let \( x \in X \). Then \( (\exists \overline{x} \in X) \) such that the following holds:

(i) \( f \) is strongly convex.

(ii) \( \text{ran} (\text{Id} - T) = U - V \).

(iii) \( v \in U - V = \text{dom} f - \text{dom} g \).

(iv) \( Z = \text{argmin}(-\langle \cdot, v \rangle + \frac{r}{2} \| \cdot - w \|^2 + \iota_U + \iota_V (\cdot - v)) = \{\overline{x}\} \).

(v) \( v \in \text{ran} (\text{Id} - T) \).

(vi) \( \text{Prox}_f x = P_U \left( \frac{r}{r+1} x + \frac{1}{r+1} w \right) \).

(vii) \( \text{Prox}_g x = P_V x \).

(viii) Let \( \lambda \in ]0, 1[ \) and set \( T_\lambda = (1 - \lambda) \text{Id} + \lambda R_g R_f \). Then we have:

(a) \( \text{Prox}_f T_\lambda^n x \to \overline{x} \).

(b) \( \text{Suppose that } \lambda \in ]0, 1[ \). Then \( \text{Prox}_g R_f T_\lambda^n x \to \overline{x} - v \).

\(^6\)A subset \( U \) of \( X \) is a polyhedral if it is a finite intersection of closed halfspaces.
Proof. (i): This is a direct consequence of the strong convexity of $\frac{1}{2}\|x - w\|^2$.

(ii)\&(iii): Observe that $\text{dom } f - \text{dom } g = U - V$ is polyhedral by \cite[Corollary 19.3.2]{34} hence closed. Now combine with (i) and Lemma 5.1(i)\&(ii).

(iv): It follows from (iii) that $\text{dom } N_U \cap \text{dom } N_V (\cdot - v) = U \cap (v + V) \neq \emptyset$. Hence $N_U + N_V (\cdot - v) = \partial (\text{id}_U + t_U (\cdot + v))$ by, e.g., \cite[Theorem 16.47(iii)]{7}, and therefore is maximally monotone. Applying Fact 3.5 with $A$ replaced by $\frac{1}{2}(r w + N_U + N_V (\cdot - v))$ yields that $-v + r \text{id} - rw + N_U + N_V (\cdot - v)$ is surjective, hence $\mathbb{Z} \neq \emptyset$. Now combine with Lemma 5.1(v).

(v): Combine (iv) and (42).

(vi)\&(vii): This is clear by, e.g., \cite[Examples 23.3\&23.4]{7}.

(viii)(a): Apply Theorem 5.3(i) (for the case $\lambda \in ]0, 1[$) and Theorem 5.4 (for the case $\lambda = 1$) in view of (i) and (iv).

(viii)(b): Apply Theorem 5.3(ii) in view of (i) and (iv).

\section*{Example 5.6}
Suppose that $X = \mathbb{R}^2$, let $(a_i, \beta_i) \in \mathbb{R}^2$ be such that $a_i \leq \beta_i, i \in \{1, 2\}$, let $U = \mathbb{R} \times \{0\}$, let $w = (w_1, 0) \in U$ and let $V = [a_1, \beta_1] \times [a_2, \beta_2]$. Set $f = \frac{1}{2}\| \cdot - w \|^2 + t_U$ and set $g = t_V$. Let $x = (x_1, x_2) \in \mathbb{R}^2$. Then the following hold:

(i) $f$ is strongly convex.
(ii) $\text{ran } (\text{id} - T) = U - V = \mathbb{R} \times [-\beta_2, -a_2]$. 
(iii) $v \in U - V = \text{dom } f - \text{dom } g$.
(iv) $\mathbb{Z} = \text{argmin} (\frac{1}{2}\| \cdot - w \|^2 + t_U + t_V (\cdot - v)) = \{(P_{[a_1, \beta_1]} w_1, 0)\}$.
(v) $v = (0, \min \{\max \{0, -\beta_2\}, -a_2\}) \in \text{ran } (\text{id} - T)$.
(vi) $\text{Prox}_g x = \frac{r}{r+1} P_U x + \frac{1}{r+1} w$.
(vii) $\text{Prox}_g x = P_V x$, where $(P_V x)_i = \min \{\max \{x_i, a_i\}, \beta_i\}$.
(viii) Let $\lambda \in ]0, 1[$ and set $T_\lambda = (1 - \lambda) \text{id} + \lambda R_g R_f$. Then we have:

(a) $\text{Prox}_f T_\lambda x \rightarrow (P_{[a_1, \beta_1]} w_1, 0)$.
(b) Suppose that $\lambda \in ]0, 1[$. Then $\text{Prox}_g R_f T_\lambda x \rightarrow (P_{[a_1, \beta_1]} w_1, \max \{\min \{0, \beta_2\}, a_2\})$.

\section*{Proof}
(i)\&(iii): This is Proposition 5.5(i)\&(iii) applied with $X$ replaced by $\mathbb{R}^2$ and $r = 1$ by observing that $U$ and $V$ are nonempty polyhedral subsets of $\mathbb{R}^2$.

(iv): Indeed,

\begin{align}
Z &= \{ x \in \mathbb{R}^2 \mid 0 \in -v + x - w + N_U x + N_V (x - v) \} \\
&= \{ x \in \mathbb{R}^2 \mid 0 \in -v + x - w + U^\perp + N_V (x - v) \} \\
&= \{ x \in \mathbb{R}^2 \mid 0 \in x - w + U^\perp + N_V (x - v) \} \\
&\subseteq \text{argmin} (\frac{1}{2}\| x - w \|^2 + t_U + t_V (\cdot - v)),
\end{align}

where (61c) follows from combining (iii) and \cite[Remark 2.8(ii)]{8}. Now on the one hand, Proposition 5.5(iv) implies that $Z$ is a singleton. On the other hand, the strong convexity of $\frac{1}{2}\| x - w \|^2$
Figure 2: A GeoGebra [28] snapshot illustrating Example 5.6 with \( \lambda = \frac{1}{2} \), \((\alpha_1, \beta_1) = (-1, 1)\), \((\alpha_2, \beta_2) = (1, 3)\) and \( w = (1, 0) \). The first few iterates of the governing sequence \( (x_n)_{n \in \mathbb{N}} = (T^n x)_{n \in \mathbb{N}} \) (the red dots) and the shadow sequence \( (y_n)_{n \in \mathbb{N}} = (J_A T^n x)_{n \in \mathbb{N}} \) (the blue dots) are also depicted.

 guarantees that \( \text{argmin}(\frac{1}{2} \| x - w \|^2 + \iota_U + \iota_V(\cdot - v)) \) is a singleton. Altogether, this verifies the first identity in (iv). We now turn to the second identity. Set \( x = (P_{[\alpha_1, \beta_1]} w, 0) \in U \) and observe that by (61c) we have \( \{ x \} = Z \iff w - x \in U^\perp + N_V(x - v) \). We examine three cases.

**CASE 1:** \( w_1 \in [\alpha_1, \beta_1] \). Then \( w - x = (0, 0) \in U^\perp + N_V(x - v) \). **CASE 2:** \( w_1 < \alpha_1 \). In this case \( N_V(x - v) \) is \( \{ \mathbb{R}_- \times \mathbb{R}_+, \mathbb{R}_- \times \mathbb{R}_- \} \), hence \( w - x \in \mathbb{R}_- \times \mathbb{R} = U^\perp + N_V(x - v) \). **CASE 3:** \( w_1 > \alpha_1 \). Proceed similar to **CASE 2.**

(v): This is a direct consequence of (ii).

(vi): Combine Proposition 5.5(vi) and [27, 5.13(i)].

(vii): This follows from, e.g., [16, Lemma 6.26].

(viii)(a)&(viii)(b): Combine (iv), (v) and Proposition 5.5(viii)(a)(viii)(b). ■

**Example 5.7.** Suppose that \( X = \mathbb{R}^d \) and let \( x = (x_1, \ldots, x_d) \in X \). Let \( U = \mathbb{R}_d^+ \) and \( V = \{ x \in X \mid L x = b \} \) where \( L \in \mathbb{R}^{m \times d} \) has full rank, \( b \in \mathbb{R}^m \), \( m \leq d \). Set \( f = \frac{r}{2} \| \cdot - z \|^2 + \iota_U \) where \( r > 0 \), \( z \in X \), \( \gamma := \frac{1}{1 + \gamma} \in ]0, 1[ \) and set \( g = \iota_V \). Then the following hold:

(i) \( f \) is strongly convex.
(ii) \( \text{ran} (\text{Id} - T) = U - V \).
(iii) \( v \in U - V = \text{domain} f - \text{domain} g \).
(iv) \( Z = \text{argmin}(\frac{r}{2} \| \cdot - w \|^2 + \iota_U + \iota_V(\cdot - v)) = \{ x \} \).
(v) $v \in \text{ran} (\text{Id} - T)$.

(vi) $\text{Prox}_f x = P_U(\gamma x + (1 - \gamma)z)$, where $(P_U x)_i = \min \{x_i, 0\}$.

(vii) $\text{Prox}_g x = P_V x = x - L^T (L L^T)^{-1} (L x - b)$.

(viii) Let $\lambda \in [0, 1]$ and set $T_\lambda = (1 - \lambda) \text{Id} + \lambda R_g R_f$. Then we have:

(a) $\text{Prox}_f T_\lambda^n x \to \overline{x}$.

(b) Suppose that $\lambda \in ]0, 1[$. Then $\text{Prox}_g R_f T_\lambda^n x \to \overline{x} - v$.

Proof. (i)–(iii): This is Proposition 5.5(i)–(iii) applied with $X$ replaced by $\mathbb{R}^d$ by observing that $U$ and $V$ are nonempty polyhedral subsets of $\mathbb{R}^d$.

(iv)–(v): This is Proposition 5.5(iv)–(v).

(vi): This follows from [16, Lemma 6.26] and [7, Prop. 24.8(i)].

(vii) This follows from [16, Lemma 6.26].

(viii)(a)–(viii)(b) This is Proposition 5.5(viii)(a)–(viii)(b).

Example 5.8. Suppose that $X = \mathbb{R}^d$ and let $x = (x_1, \ldots, x_d) \in X$. Let $U = \{x \in X \mid \alpha_i \leq x_i \leq \beta_i\}$ such that $0 \leq \alpha_i \leq \beta_i$ for $i = 1, \ldots, d$ and let $V = \{x \in X \mid L x = b\}$ where $L \in \mathbb{R}^{m \times d}$ has full rank, $b \in \mathbb{R}^m$, $m \leq d$. Set $f = \frac{r}{2} \| \cdot - z \|^2 + \iota_U$ where $r > 0$, $\gamma := \frac{1}{1 + \tau} \in ]0, 1[$, $z \in X$ and set $g = \iota_V$. Then the following hold:

(i) $f$ is strongly convex.

(ii) $\overline{\text{ran}} \ (\text{Id} - T) = U - V$.

(iii) $v \in U - V = \text{dom} f - \text{dom} g$.

(iv) $Z = \text{argmin} (\frac{r}{2} \| \cdot - w \|^2 + \iota_U + \iota_V (\cdot - v)) = \{ \overline{x} \}$.

(v) $v \in \text{ran} (\text{Id} - T)$.

(vi) $\text{Prox}_f x = P_U(\gamma x + (1 - \gamma)z)$, where $(P_U x)_i = \max \{x_i, \alpha_i\}, \beta_i\}$.

(vii) $\text{Prox}_g x = P_V x = x - L^T (L L^T)^{-1} (L x - b)$.

(viii) Let $\lambda \in [0, 1]$ and set $T_\lambda = (1 - \lambda) \text{Id} + \lambda R_g R_f$. Then we have:

(a) $\text{Prox}_f T_\lambda^n x \to \overline{x}$.

(b) Suppose that $\lambda \in ]0, 1[$. Then $\text{Prox}_g R_f T_\lambda^n x \to \overline{x} - v$.

Proof. (i)–(iii): This is Proposition 5.5(i)–(iii) applied with $X$ replaced by $\mathbb{R}^d$ by observing that $U$ and $V$ are nonempty polyhedral subsets of $\mathbb{R}^d$.

(iv)–(v): This is Proposition 5.5(iv)–(v).

(vi): This follows from [16, Lemma 6.26] and [7, Prop. 24.8(i)].

(vii): This follows from [16, Lemma 6.26].

(viii)(a)–(viii)(b) This is Proposition 5.5(viii)(a)–(viii)(b).
6 Numerical experiments

This section contains numerical experiments with the inconsistent problems in Example 5.7–Example 5.8. We introduce Dykstra’s algorithm to compare performance with the Douglas–Rachford algorithm and Peaceman–Rachford algorithm. Following this we give experiments using different values of the algorithmic parameters $\gamma$ and $\lambda$ to determine which choices provide optimal performance, i.e. (i) the smallest number of iterations, (ii) an accurate approximate solution $\bar{x}$, and (iii) an accurate approximation of the gap vector $v$.

Let $A, B$ be closed convex subsets of $X$ and $z \in X$. Dykstra’s projection algorithm (see [18]) operates as follows: Set $a_0 := z$, $p_0 := 0$ and $q_0 := 0$. Given $a_n, p_n, q_n$, where $n \geq 0$, update

$$
\begin{align*}
b_n &:= P_B(a_n + q_n), \quad q_{n+1} := a_n + q_n - b_n, \\
a_{n+1} &:= P_A(b_n + p_n), \quad p_{n+1} := b_n + p_n - a_{n+1}.
\end{align*}
$$

It is known that both $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ converge strongly (i.e., converge in norm), to $P_{A \cap B}(z)$.

From [3, Corollary 3.4] we have that the sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ satisfy

$$
b_n - a_n, b_n - a_{n+1} \to v. \quad (62)
$$

Recall that the set $V$ in Example 5.7–Example 5.8 is given as $V = \{ x \in X \mid Lx = b \}$. In these numerical experiments the $m \times d$ matrix $L$ and $m \times 1$ vector $b$ were generated using uniformly distributed random numbers between $-50$ and $50$. The numerical experiments for Example 5.7–Example 5.8 with the same dimensions $m, d$ use the same matrix $L$ and vector $b$. To ensure that $U \cap V = \emptyset$ in Example 5.7 we used Farkas’ Lemma to choose the sign of the columns of $L$ in such a way that we can guarantee that $x \in V$ would not intersect with $U = \mathbb{R}_+^d$. In Example 5.8 we choose the box $U$ as a subset of $\mathbb{R}_+^d$ to ensure the intersection remains empty. We can write Example 5.7–Example 5.8 as

$$
\begin{align*}
\text{minimize} \quad & \phi(x) := \frac{r}{2} \|x - z\|^2 \\
\text{subject to} \quad & x \in U \cap V.
\end{align*}
$$

6.1 Computational algorithms

Below are the algorithms as implemented in the experiments. Recall that for Peaceman–Rachford $\lambda$ defined in Step 1 below is fixed as 1.

Algorithm 1. (Douglas–Rachford Relaxation and Peaceman–Rachford)

Step 1 (Initialization) Choose a parameter $r > 0$, $\gamma := 1/(r + 1)$, $\lambda \in [0, 1]$ and the initial iterate $x_0$ arbitrarily. Choose a small parameter $\varepsilon > 0$, and set $n = 0$.

Step 2 (Projection onto $U$) Set $x^- := x_n$. Compute $\bar{x} = P_U(\gamma x^- + (1 - \gamma)z)$ using Example 5.7(vi) or Example 5.8(vi).
Step 3 (Projection onto $V$) Set $x^{-} := 2\bar{x} - x$. Compute $\hat{x} = P_{V}(x^{-})$ using Example 5.7(vii) or Example 5.8(vii).

Step 4 (Update) Set $x_{n+1} := x_{n} + 2\lambda(\hat{x} - \bar{x})$.

Step 5 (Stopping criterion) If $\|\bar{x} - \bar{x}_{n}\|_{\infty} \leq \epsilon$, then return $\bar{x}$ and $\nu$ using Fact 5.2(i) and (ii), and stop. Otherwise, set $\bar{x}_{n+1} := \bar{x}$, $n := n + 1$, and go to Step 2.

Algorithm 2. (Dykstra)

Step 1 (Initialization) Choose the initial iterates $x_{0} = 0$, $p_{0} = 0$ and $q_{0} = 0$. Choose the initial iterate $\bar{x}_{0}$ arbitrarily. Choose a small parameter $\epsilon > 0$, and set $n = 0$.

Step 2 (Projection onto $U$) Set $x^{-} := x_{n} + q_{n}$. Compute $\bar{x} = P_{U}(x^{-})$ using Example 5.7(vi) or Example 5.8(vi).

Step 3 (Projection onto $V$) Set $x^{-} := \bar{x} + p_{n}$. Compute $\hat{x} = P_{V}(x^{-})$ using Example 5.7(vii) or Example 5.8(vii).

Step 4 (Update) Set $x_{n+1} := \bar{x}$, $q_{n+1} := x_{n} + q_{n} - \bar{x}$ and $p_{n+1} := \bar{x} + p_{n} - \bar{x}$.

Step 5 (Stopping criterion) If $\|\bar{x} - \bar{x}_{n}\|_{\infty} \leq \epsilon$, then return $\bar{x}$ and $\nu$ using Fact 5.2(i) and (ii), and stop. Otherwise, set $\bar{x}_{n+1} := \bar{x}$, $n := n + 1$, and go to Step 2.

6.2 Computations

Table 1–Table 2 show the error in the objective $\phi$, the gap vector $\nu$ and the solution $\bar{x}$. The “exact” solution used in these calculations was found using Dykstra’s algorithm with a tolerance $\epsilon = 10^{-12}$ (except for Example 5.8 when $(m, d) = (50, 1000)$ with $\epsilon = 3 \times 10^{-12}$ since Dykstra was unable to terminate with $\epsilon = 10^{-12}$). For $d > 1000$ we have not been able to consistently reach our desired accuracy. Further investigation is needed to understand the cause of this issue.

In Figure 3, we show plots of the number of iterations taken by the DR relaxation with $\lambda \in \{0.25, 0.5, 0.75, 0.9, 1\}$, against the parameter $\gamma \in ]0,1]$. We obtained these plots for 500 values of $\gamma \in ]0,1]$. For the sake of brevity we only show four of these plots but we have generated these for every problem instance found in Table 1–Table 2. These plots have informed our choice of a value of $\gamma$ for each problem instance that provides optimal performance of the algorithms.

In Figure 3 for Example 5.8 with $(m, d) = (500, 1000)$, $\lambda \in \{0.25, 0.75\}$ and $(m, d) = (50, 1000)$, we observe that for values of $\gamma$ close to 1, the methods terminate in less than 10 iterations though we have not arrived at the solution. In these examples we thus chose the next best values of $\gamma$. This also occurred for Example 5.8 with $(m, d) = (10, 100)$ so we again chose the next best value of $\gamma$. Though we have not included the figure for Example 5.8 with $(m, d) = (65, 70)$ this example showed $n = 2$ for all $\gamma \in [0.6, 1]$ and every value of $\lambda$, thus in Table 2 we set $\gamma = 0.6$ but any $\gamma \in [0.6, 1]$ could have been chosen.
Figure 3: Parameter curves with $m = 50$, $d = 1000$ (top) and $m = 500$, $d = 1000$ (bottom).

From observing these figures we can make some comments on the choice of $\gamma$. One can see in all plots from Figure 3 and the values of $\gamma$ used in Table 1–Table 2 that the optimal $\gamma$ has little variation as $\lambda$ changes compared to the variation between different problem instances. We have also generated plots for the same problem instances $(m, d)$ in Table 1–Table 2 but with different (randomly generated) matrices $L, b$ and have observed that for the majority the optimal $\gamma$ remains more or less unchanged for the same problem instance $(m, d)$. Thus it appears that the optimal value of $\gamma$ depends largely on the problem instance $(m, d)$ rather than the value of $\lambda$. Across all the problem instances we observe that the optimal values of $\gamma$ range from 0.5 to 0.9. In order to determine what specific value of $\gamma$ is optimal for a problem instance we need to make further comparisons which are carried out below.

When comparing the error in $\bar{x}$ and $v$ for different values of $\gamma$ we see that in general the values of $\gamma$ that take the smallest number of iterations also produce the smallest errors. This gives further merit to identifying the optimal value of $\gamma$ for each problem. In general, we observe that the more difficult a problem is to solve (i.e., the larger the number of iterations is), the larger the optimal value of $\gamma$ becomes. From Table 1–Table 2 we see in each example the case where $(m, d) = (65, 70)$ takes the smallest number of iterations and has the smallest values of $\gamma$ while $(m, d) = (50, 1000)$ takes the largest number of iterations and has the largest values of $\gamma$. Problem instances where $d$ is large and/or where $m \ll d$ seem to be more difficult for the algorithms to solve.

From Table 1–Table 2 the relaxed DR algorithm with $\lambda = 0.75$ appears to provide the best per-
than those from $\lambda$ significantly smaller than those using Fact 5.2(i). The limit expression in Fact 5.2(i) contains

$\text{Though PR (})v\text{, the relaxed DR algorithm with}$

$v_\infty\text{ and an accurate approximation of}$

$\phi(x) - \phi(x^*)| \quad \|v - v^*\|_{\infty} \quad \|v - v^*\|_{\infty} \quad \|x - x^*\|_{\infty} \quad \lambda \quad \gamma \quad n$

| $m$ | $d$ | $\phi(x) - \phi(x^*)$ | $\|v - v^*\|_{\infty}$ | $\|v - v^*\|_{\infty}$ | $\|x - x^*\|_{\infty}$ | $\lambda$ | $\gamma$ | $n$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 100 | $1.22 \times 10^{-9}$ | $1.73 \times 10^{-4}$ | $2.48 \times 10^{-8}$ | $1.38 \times 10^{-7}$ | 0.25 | 0.75 | 204 |
| | | $6.65 \times 10^{-10}$ | $1.62 \times 10^{-3}$ | $1.06 \times 10^{-8}$ | $4.35 \times 10^{-8}$ | 0.5 | 0.75 | 109 |
| | | $1.03 \times 10^{-10}$ | $1.57 \times 10^{-3}$ | $4.82 \times 10^{-9}$ | $1.27 \times 10^{-8}$ | 0.75 | 0.75 | 75 |
| | | $2.61 \times 10^{-10}$ | $1.55 \times 10^{-3}$ | $2.18 \times 10^{-8}$ | $8.73 \times 10^{-9}$ | 0.9 | 0.75 | 63 |
| | | $5.17 \times 10^{-10}$ | $1.54 \times 10^{-3}$ | $1.01 \times 10^{-8}$ | $1 \times 10^{-8}$ | 1 | 0.75 | 57 |
| | | $2.82 \times 10^{-9}$ | $-1 \times 10^{-8}$ | $1.74 \times 10^{-8}$ | $2.73 \times 10^{-7}$ | Dykstra | 338 |
| 65 | 70 | $8.77 \times 10^{-8}$ | $2.66 \times 10^{-2}$ | $1.54 \times 10^{-7}$ | $2.25 \times 10^{-7}$ | 0.25 | 0.7 | 44 |
| | | $4.94 \times 10^{-9}$ | $2.98 \times 10^{-2}$ | $6.95 \times 10^{-9}$ | $6.53 \times 10^{-9}$ | 0.5 | 0.55 | 25 |
| | | $3.09 \times 10^{-9}$ | $3.64 \times 10^{-2}$ | $2.46 \times 10^{-9}$ | $2.46 \times 10^{-9}$ | 0.75 | 0.5 | 15 |
| | | $8.03 \times 10^{-10}$ | $2.59 \times 10^{-2}$ | $1.94 \times 10^{-9}$ | $1.53 \times 10^{-9}$ | 0.9 | 0.55 | 16 |
| | | $8.96 \times 10^{-10}$ | $1.62 \times 10^{-2}$ | $-1 \times 10^{-9}$ | $1.95 \times 10^{-9}$ | $1 \times 10^{-9}$ | 1 | 0.55 | 23 |
| | | $1.35 \times 10^{-9}$ | $-2.20 \times 10^{-9}$ | $4.15 \times 10^{-9}$ | $Dykstra$ | 22 |
| 50 | 1000 | $2.81 \times 10^{-8}$ | $3.05 \times 10^{-4}$ | $2.15 \times 10^{-8}$ | $4.24 \times 10^{-7}$ | 0.25 | 0.9 | 662 |
| | | $1.25 \times 10^{-8}$ | $2.93 \times 10^{-4}$ | $1.04 \times 10^{-8}$ | $2.07 \times 10^{-7}$ | 0.5 | 0.9 | 345 |
| | | $7.31 \times 10^{-9}$ | $2.85 \times 10^{-4}$ | $6.49 \times 10^{-9}$ | $1.28 \times 10^{-7}$ | 0.75 | 0.9 | 236 |
| | | $5.76 \times 10^{-9}$ | $2.82 \times 10^{-4}$ | $5.30 \times 10^{-9}$ | $1.02 \times 10^{-7}$ | 0.9 | 0.9 | 199 |
| | | $4.94 \times 10^{-9}$ | $2.79 \times 10^{-4}$ | $-1 \times 10^{-9}$ | $8.64 \times 10^{-8}$ | $1 \times 10^{-8}$ | 1 | 0.9 | 181 |
| | | $2.89 \times 10^{-7}$ | $-3.96 \times 10^{-8}$ | $3.28 \times 10^{-6}$ | $Dykstra$ | 3078 |
| 500 | 1000 | $3.74 \times 10^{-8}$ | $2.48 \times 10^{-3}$ | $2.37 \times 10^{-8}$ | $9.42 \times 10^{-8}$ | 0.25 | 0.7 | 122 |
| | | $3.56 \times 10^{-9}$ | $2.22 \times 10^{-3}$ | $9.70 \times 10^{-9}$ | $2.13 \times 10^{-8}$ | 0.5 | 0.7 | 68 |
| | | $3.08 \times 10^{-9}$ | $2.31 \times 10^{-3}$ | $6.73 \times 10^{-9}$ | $1.73 \times 10^{-8}$ | 0.75 | 0.65 | 47 |
| | | $1.25 \times 10^{-9}$ | $2.20 \times 10^{-3}$ | $1.50 \times 10^{-6}$ | $1.16 \times 10^{-8}$ | 0.9 | 0.65 | 41 |
| | | $1.56 \times 10^{-10}$ | $1.89 \times 10^{-3}$ | $-1 \times 10^{-9}$ | $2.70 \times 10^{-9}$ | $1 \times 10^{-9}$ | 1 | 0.65 | 43 |
| | | $3.72 \times 10^{-8}$ | $-1.37 \times 10^{-8}$ | $8.44 \times 10^{-8}$ | $Dykstra$ | 135 |

Table 1: Example 5.7 with $z = 0$, $\varepsilon = 10^{-8}$, $x \in \mathbb{R}_+^d$. For Dykstra, (62) is used to calculate $v$.

formance in general. We see a trend that as $\lambda$ increases the number of iterations taken decreases. Though PR ($\lambda = 1$) needed the smallest number of iterations to reach a solution of the desired tolerance we cannot apply Fact 5.2(ii) to calculate $v$ and instead must rely on Fact 5.2(i). From Table 1—Table 2 we observe that the errors in $v$ when using Fact 5.2(ii) are almost always significantly smaller than those using Fact 5.2(i). The limit expression in Fact 5.2(i) contains $n$ thus to more accurately compute $v$ we must allow the algorithm to run for more iterations which defeats the performance benefit gained by choosing this method. Thus one could conclude that to find a solution quickly and accurately the relaxed DR algorithm is preferred over PR. When $\lambda = 0.9$ Fact 5.2(ii) is applicable but we see from Table 1—Table 2 that the errors in $v$ are often much larger than those from $\lambda \in \{0.25, 0.5, 0.75\}$ or from Dykstra.

To summarize our observations from the numerical experiments:

- When considering our measures of optimal performance, i.e. a small number of iterations and an accurate approximation of $x$ and $v$, the relaxed DR algorithm with $\lambda = 0.75$ seems to
be the best choice.

- When \( d \) is large and/or where \( m \ll d \) an optimal \( \gamma \) is likely to be \( \gamma \in [0.8, 1] \), else \( \gamma \in [0.5, 0.8] \) may be an optimal choice.

### 7 Conclusion and Future Work

We have proved the strong convergence of the shadow sequences of the Douglas–Rachford algorithm and Peaceman–Rachford algorithm in the inconsistent case. We have conducted numerical experiments comparing the performance of the relaxed Douglas–Rachford algorithm and Peaceman–Rachford algorithm on inconsistent finite-dimensional optimization problems of various sizes.
In the future it will be interesting to see the performance of these algorithms on inconsistent infinite-dimensional optimization problems such as the optimal control problems studied in [25]:

\[
\begin{align*}
(MEC) \quad &\min_{u \in L^2([0,1],\mathbb{R}^m)} \int_0^1 \|u(t)\|^2 \, dt \\
\text{s.t.} \quad &\dot{x}(t) = L(t) x(t) + B(t) u(t), \quad x(0) = x_0, \quad x(1) = x_f, \\
&u_i \leq u_i(t) \leq \bar{u}_i, \quad \text{for all } t \in [0,1], \quad i = 1, \ldots, m,
\end{align*}
\]

where \( x \) and \( u \) are the state and control variables, respectively. Problem \((MEC)\) is referred to as the minimum-energy control problem which is a special case of linear–quadratic optimal control problems. This problem is more general and computationally more challenging than the finite-dimensional linear–quadratic problems we have made experiments with in this paper.

We note that the consistent case of Problem \((MEC)\) has been extensively studied in [4,22,24], including its extension to an \(LQ\) control problem subject to both control and state box constraints [23], all for \( \lambda = 1/2 \). One must also note that Theorem 5.3–Theorem 5.4 hold also for the consistent case. So it will be interesting to solve the optimal control problems in [22–24], for various \( \lambda \in ]0,1] \) and make comparisons, in the consistent case.

Acknowledgements

The research of WMM is partially supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant (NSERC DG). The research of BIC was supported by an Australian Government Research Training Program Scholarship. The research of MS is partially supported by WMM’s NSERC DG.

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