The module structure of the Solomon-Tits algebra of the symmetric group

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Abstract

Let \((W, S)\) be a finite Coxeter system. Tits defined an associative product on the set \(\Sigma\) of simplices of the associated Coxeter complex. The corresponding semigroup algebra is the Solomon-Tits algebra of \(W\). It contains the Solomon algebra of \(W\) as the algebra of invariants with respect to the natural action of \(W\) on \(\Sigma\). For the symmetric group \(S_n\), there is a 1-1 correspondence between \(\Sigma\) and the set of all set compositions (or ordered set partitions) of \(\{1, \ldots, n\}\). The product on \(\Sigma\) has a simple combinatorial description in terms of set compositions.

We study in detail the representation theory of the Solomon-Tits algebra of \(S_n\) over an arbitrary field, and show how our results relate to the corresponding results on the Solomon algebra of \(S_n\). This includes the construction of irreducible and principal indecomposable modules, a description of the Cartan invariants, of the Ext-quiver, and of the descending Loewy series. Our approach builds on a (twisted) Hopf algebra structure on the direct sum of all Solomon-Tits algebras.
1 Introduction

The present work is part of the programme to lift the theory of descent algebras to the enriched setting of twisted descent algebras as recently introduced in [PS]. From a different point of view, this is the attempt to pursue the study of symmetric and quasi-symmetric functions in non-commuting variables initiated in [Wol36] and developed a great deal further in [BRRZ, BZ, RS].

The Solomon-Tits algebra $T_n$ of the symmetric group $S_n$ occurs as a homogeneous component of the free twisted descent algebra. We give here a complete description of the module structure of $T_n$ over an arbitrary field. This includes: the construction of primitive idempotents; a decomposition into principal indecomposable modules; a description of the Cartan matrix, of the Ext-quiver and, in fact, of the entire descending Loewy series of $T_n$.

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The Solomon algebra $D_n$ of $S_n$, discovered by Solomon in the more general context of finite Coxeter groups, is a subalgebra of the integral group algebra $\mathbb{Z}S_n$ of $S_n$ and was originally designed as a noncommutative superstructure of the character ring of the underlying group [Sol76]. A huge body of research papers on the subject, accumulated during the past fifteen years, provides surprising links to many different fields in geometry, combinatorics, algebra and topology ([Bau01, BD92, Bro00, ES, Gar89, GKL+95, JR01, KLT97, MR95, Ren98], to name but a few; see [PS, Sch04b] for a more exhaustive list of references).

The Solomon-Tits algebra $T_n$ of $S_n$ arises from a semigroup structure on the set of simplices of the Coxeter complex $\Sigma_n$ associated with $S_n$ and was first considered by Tits in an appendix to Solomon’s original paper [Sol76]. The simplices in $\Sigma_n$ are in 1-1 correspondence with set compositions of $[n]:=$
\{1, \ldots, n\}$, that is, with $l$-tuples $(P_1, \ldots, P_l)$ of mutually disjoint non-empty subsets $P_i$ of $[n]$ such that $P_1 \cup \cdots \cup P_l = [n]$. The associative product on the simplices in $\Sigma_n$, defined by Tits in geometrical terms, corresponds to the product $\land_n$ on the set $\Pi_n$ of all set compositions of $[n]$, given by

$$(P_1, \ldots, P_l) \land_n (Q_1, \ldots, Q_k) = (P_1 \cap Q_1, P_1 \cap Q_2, \ldots, P_1 \cap Q_k, \ldots, P_l \cap Q_1, P_l \cap Q_2, \ldots, P_l \cap Q_k)$$

for all $(P_1, \ldots, P_l), (Q_1, \ldots, Q_k) \in \Pi_n$ (see [Bro04, Section 2]). Here the superscript $\#$ indicates that empty sets are deleted. It is readily seen that $P \land_n P = P$ for all $P \in \Pi_n$ and that $([n])$ is a two-sided identity in $(\Pi_n, \land_n)$. This amounts to saying that $(\Pi_n, \land_n)$ is an idempotent semigroup with identity and $T_n = \mathbb{Z}\Pi_n$ is the integral semigroup algebra of this semigroup.

The natural action of $S_n$ on subsets of $[n]$ extends to an action on $\Pi_n$, defined by

$$(P_1, \ldots, P_l) \pi = (P_1 \pi, \ldots, P_l \pi),$$

for all $(P_1, \ldots, P_l) \in \Pi_n$ and $\pi \in S_n$. This action respects the product $\land_n$ on $\Pi_n$. Thus the fixed space $B_n$ of $S_n$ in $T_n$ is a subalgebra of $T_n$. Bidigare showed that $B_n$ is naturally isomorphic to $D_n$, thereby clarifying Tits’ original line of reasoning [Bid97]; see Section 3.

We believe that the often challenging algebraic combinatorics related to the Solomon algebra can be put to order when $D_n$ is viewed as the ring of invariants in $T_n$. The notion of twisted descent algebra was introduced in [PS] as a formal framework for this programme.

Let $\mathbb{N}$ denote the set of positive integers. In accordance with the functorial setup of [PS], all the semigroups $(\Pi_A, \land_A)$ are considered simultaneously, where $\Pi_A$ consists of the set compositions of an arbitrary finite subset $A$ of $\mathbb{N}$ (with the product $\land_A$ defined accordingly). The direct sum $T = \bigoplus_A \mathbb{Z}\Pi_A$ carries a second, graded product $\vee$ and a coproduct $\Delta$, in addition to the
internal product $\wedge_A$ on each homogeneous component, and there are fundamental rules linking all three structures. The vector space $\mathcal{T}$ with this triple algebraic structure is the free twisted descent algebra.

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In all of what follows, $F$ is an arbitrary field. We will study the module structure of the Solomon-Tits algebra $(F\Pi_n, \wedge_n)$ over $F$. Note that any bijection $[n] \to A$ gives rise to an isomorphism of algebras $F\Pi_n \to F\Pi_A$. In the body of this paper, with an eye to later applications, results will often be stated for arbitrary finite subsets $A$ of $\mathbb{N}$, rather than for $A = [n]$ only.

The representation theory of any finite idempotent semigroup is governed by its support structure. The support structure of $(\Pi_n, \wedge_n)$ arises from the refinement relation $\preceq$ on $\Pi_n$ defined by $Q \preceq P$ if each component of $Q$ is contained in a component of $P$. As a general consequence, the irreducible $F\Pi_n$-modules $M_P$ are one-dimensional and naturally labelled by (unordered) set partitions $P = \{P_1, \ldots, P_l\}$ of $[n]$ or, more conveniently for our purposes, by those set compositions $P$ contained in

$$\Pi_n^\preceq = \{ (P_1, \ldots, P_l) \in \Pi_n \mid \min P_1 < \min P_2 < \cdots < \min P_l \}.$$ 

This is briefly recovered in Section 2. The reader is also referred to the appendices of [Bro04] as an excellent reference on the subject.

In Section 3 we demonstrate the impact on the Solomon algebra of these basic observations and, as an illustration of the general concept, prove in a few lines results of Solomon (in characteristic zero) and Atkinson and Willigenburg (in positive characteristic) on the radical and the irreducible representations of $\mathcal{D}_n$.

The free twisted descent algebra $\mathcal{T}$ with its additional product $\vee$ and its coproduct $\Delta$ is then described in Section 4. Employing the coproduct $\Delta$,
allows us to construct a “generic” primitive idempotent $e_A$ in $(F\Pi_A, \wedge_A)$ for each finite subset $A$ of $\mathbb{N}$. As a by-product, we obtain a description of the primitive elements in $(\mathcal{T}, \Delta)$ which might be interesting in its own right.

Each of the products $e_Q = e_{Q_1} \lor \cdots \lor e_{Q_k}$, where $Q = (Q_1, \ldots, Q_k) \in \Pi_n$, is a primitive idempotent in $F\Pi_n$, and these elements form a linear basis of $F\Pi_n$. Properly grouping together basis elements, yields a decomposition of $F\Pi_n$ into indecomposable left ideals $\Lambda_Q$, where $Q \in \Pi_n^<$. This is shown in Section 5.

Our key result is Theorem 6.2, a multiplication rule for the basis $\{e_Q \mid Q \in \Pi_n\}$. As an immediate consequence, there is a description of the Cartan invariant $C_{PQ}$ of $F\Pi_n$ (which describes the multiplicity of $M_P$ in a composition series of $\Lambda_Q$) as a product of factorials. It also follows that $C_{PQ} = 0$ unless $Q \preceq P$.

Among other results, in Section 7 we show that the occurrence of $M_P$ in a composition series of $\Lambda_Q$ is restricted to a single Loewy layer of $\Lambda_Q$ (if it occurs at all, that is, if $Q \preceq P$). As a consequence, the Ext-quiver of $F\Pi_n$ coincides with the Hasse diagram of the support lattice of $F\Pi_n$. This is shown in Section 8.

The results on the module structure of $F\Pi_n$ are independent of the underlying field $F$. We demonstrate in Section 9 how they relate to the corresponding results of Garsia and Reutenauer [GR89] and Blessenohl and Laue [BL96, BL02] on the Solomon algebra $\mathcal{D}_n$ (identified with the subalgebra $\mathcal{B}_n$ of $\mathcal{T}_n$) over a field of characteristic zero. Roughly speaking, all the results mentioned above remain true when the Solomon-Tits algebra is replaced by the Solomon algebra and compositions $q = (q_1, \ldots, q_k)$ of $n$ are considered instead of set compositions $Q = (Q_1, \ldots, Q_k)$ of $[n]$. Most strikingly, the $k$th radical of $\mathcal{D}_n$ over $F$ is the invariant space of the $k$th radical of $\mathcal{T}_n$ over $F$, for all $k \geq 0$. Furthermore, via symmetrisation, the primitive idempotents in $\mathcal{T}_n$ become
Lie idempotents in $\mathcal{D}_n$.

Finally, we show that the structure of $\mathcal{T}_n$ as a module for $\mathcal{D}_n$ is intimately linked to the Garsia-Reutenauer characterisation of $\mathcal{D}_n$ as the common stabiliser of a family of vector spaces associated with the Poincaré-Birkhoff-Witt Theorem [GR89, Theorem 4.5]. More precisely, the indecomposable $\mathcal{T}_n$-modules $\Lambda_Q$, when viewed as modules for $\mathcal{D}_n$, turn out to be isomorphic to these spaces.

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The representation theory of the modular Solomon algebra (over a field of positive characteristic $p$) is not yet understood. However, the little we know has recently proved to be of tremendous help in the study of modular Lie representations of arbitrary groups [BS, ES]. It would therefore be desirable to obtain more information on the modular Solomon algebra. From the “twisted” point of view, the difficulties in the modular case arise from the multiplicities which occur when elements of $\mathcal{T}_n$ are symmetrised.

The Solomon algebra $\mathcal{D}_W$ of an arbitrary finite Coxeter group $W$ is the algebra of $W$-invariants of the Solomon-Tits algebra associated with the Coxeter complex of $W$. The approach presented here for type $A$ may apply to other Coxeter groups as well, to a certain degree. It could thereby help understand the representation theory of $\mathcal{D}_W$ for arbitrary $W$, knowledge of which is very restricted at the moment.

More generally, there is the question to which extent the results on the Solomon-Tits algebra of $S_n$ can be lifted to an arbitrary finite idempotent semigroup. We may, for instance, expect the Ext-quiver to coincide with the Hasse diagram of the support lattice for a much larger class of such semigroups.
2 Support structure and Jacobson radical

Let $S$ be a finite idempotent semigroup with identity. Due to Brown [Bro04], the representation theory of $S$ is governed by its support structure. We display basic definitions and results in this section.

Recall that a partially ordered set $(L, \subseteq)$ is a lattice if any two elements $x, y$ in $L$ have a least upper bound $x \lor y$ and a greatest lower bound $x \land y$ in $(L, \subseteq)$. Assigning to each pair of elements $(x, y)$ in $L \times L$ their greatest lower bound $x \land y$ defines the structure of an abelian idempotent semigroup on $L$.

Associated with the semigroup $S$, there is a support lattice $(L, \subseteq)$ together with a surjective support map $s : S \to L$, such that $(xy)s = xs \land ys$, and $xs \subseteq ys$ if and only if $x = xyx$, for all $x, y \in S$. (We use the opposite order of that considered in [Bro04] since this seems more natural for the Solomon-Tits algebra.)

Support lattice and support map of $S$ are unique up to isomorphisms of ordered sets, thanks to the second condition above. The first condition says that $s$ is an epimorphism of semigroups.

Recall that, if $\mathcal{A}$ is a finite-dimensional associative $F$-algebra, then the Jacobson radical $\text{rad} \mathcal{A}$ of $\mathcal{A}$ is the smallest ideal $R$ of $\mathcal{A}$ such that $\mathcal{A}/R$ is semisimple.

**Theorem 2.1 (Brown, 2004).** The support map $s$ extends linearly to an epimorphism of semigroup algebras $s : FS \to FL$ such that $\text{rad} FS = \ker s$.

Furthermore, $FS/\text{rad} FS \cong FL$ is abelian and split semisimple, that is, isomorphic to a direct sum of copies of the ground field $F$.

Hence the irreducible representations of $FS$ are all linear, and in 1-1 correspondence to the elements of the support lattice $L$.

Combining Brown’s theorem with an observation of Bauer [Bau01] 5.5 Hilfs-
Corollary 2.2. For any subalgebra \( B \) of \( FS \), \( \text{rad} \ B = B \cap \ker s \). In particular, \( B/\text{rad} \ B \cong Bs \) is split semisimple.

The short proof follows for the reader’s convenience.

Proof. Let \( R = \text{rad} FS \), then \( R \cap B \) is a nilpotent ideal of \( B \) and therefore contained in \( \text{rad} B \). Conversely, the only nilpotent element of \( B/R \cap B \cong (B + R)/R \subseteq FS/R \) is 0, since \( FS/R \) is split semisimple by Theorem 2.1.

This implies \( \text{rad} B \subseteq R \) and \( \ker s|_B = B \cap \ker s = B \cap R = \text{rad} B \) as asserted. Furthermore, \( FL \) is split semisimple, hence so is the subalgebra \( Bs \) of \( FL \).

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Let \( n \in \mathbb{N} \). The support structure of the semigroup \( (\Pi_n, \wedge_n) \) can be described as follows (see [Bro04, Example 3.2]). Let \( Q = (Q_1, \ldots, Q_k) \) and \( P = (P_1, \ldots, P_l) \) be set compositions of \([n]\). We write

\[ Q \preceq P \]

if each component \( Q_i \) of \( Q \) is contained in a component \( P_j \) of \( P \) or, equivalently, if \( Q = Q \wedge_n P = Q \wedge_n P \wedge_n Q \). The relation \( \preceq \) on \( \Pi_n \) is reflexive and transitive, and will be of crucial importance in the sequel. We have \( Q \preceq P \) and \( P \preceq Q \) if and only if \( Q \) may be obtained by rearranging the components of \( P \). In this case, write

\[ Q \approx P. \]

The set of equivalence classes, \( L \), in \( \Pi_n \) with respect to \( \approx \) is a support lattice of \( (\Pi_n, \wedge_n) \), with order inherited from \( (\Pi_n, \preceq) \). The greatest lower bound of two equivalence classes \([P]_\approx \) and \([Q]_\approx \) in \( L \) is \([P]_\approx \wedge [Q]_\approx = [P \wedge_n Q]_\approx \), for
all $P, Q \in \Pi_n$. The support map $s$ sends $P$ to $[P]_\infty$. Using more illustrative terms, $L$ may be identified with the set of (unordered) set partitions of $[n]$ and $s$ with the map that forgets the ordering of set compositions: $(P_1, \ldots, P_l)s = \{P_1, \ldots, P_l\}$. Applying Theorem 2.1 we get:

**Corollary 2.3.** Let $n \in \mathbb{N}$, then $\text{rad} \, F\Pi_n$ is linearly generated by the elements

$$Q - Q' \quad (Q, Q' \in \Pi_n, Q \approx Q').$$

Furthermore, $F\Pi_n/\text{rad} \, F\Pi_n$ is split semisimple with dimension equal to the number of (unordered) set partitions of $[n]$.

### 3 The Solomon algebra

Let $n \in \mathbb{N}$. The Solomon algebra $D_n$ occurs naturally as a subalgebra of the Solomon-Tits algebra $Z\Pi_n$. Thus we can apply Corollary 2.2 and deduce basic structural information on $D_n$ (over the field $F$). The results are not new, but their proofs demonstrate the advantage in viewing the Solomon algebra as the ring of $S_n$-invariants of the Solomon-Tits algebra at this stage already. A more detailed analysis of $D_n$ follows in Section 3.

We recall the necessary definitions. A finite sequence $q = (q_1, \ldots, q_k)$ of positive integers with sum $n$ is a *composition* of $n$, denoted by $q \models n$. We write $S_q$ for the usual embedding of the direct product $S_{q_1} \times \cdots \times S_{q_k}$ in $S_n$.

The length of a permutation $\pi$ in $S_n$ is the number of inversions of $\pi$. Each right coset of $S_q$ in $S_n$ contains a unique permutation of minimal length. Define $\Xi^q$ to be the sum in the integral group ring $\mathbb{Z}S_n$ of all these minimal coset representatives of $S_q$ in $S_n$. Due to Solomon [Sol76, Theorem 1], the $\mathbb{Z}$-linear span of the elements $\Xi^q (q \models n)$ is a subring of $\mathbb{Z}S_n$ of rank $2^{n-1}$. This is the *Solomon algebra* $D_n$ of $S_n$. 

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For any $Q = (Q_1, \ldots, Q_k) \in \Pi_n$, the type of $Q$ is $\text{type}(Q) := (#Q_1, \ldots, #Q_k)$. Hence $\text{type}(Q) \models n$. Two set compositions $P, Q \in \Pi_n$ belong to the same $S_n$-orbit if and only if $\text{type}(P) = \text{type}(Q)$. As a consequence, the ring of $S_n$-invariants $\mathcal{B}_n$ in $\mathbb{Z}\Pi_n$ has $\mathbb{Z}$-basis consisting of the elements

$$X^q = \sum_{\text{type}(Q) = q} Q \quad (q \models n).$$

Here is Bidigare’s remarkable observation ([Bid97], see also [Bro00, Section 9.6]).

**Theorem 3.1 (Bidigare, 1997).** The linear map, defined by $X^q \mapsto \Xi^q$ for all $q \models n$, is an isomorphism of algebras from $\mathcal{B}_n$ onto $\mathcal{D}_n$.

Now consider the Solomon algebra $\mathcal{D}_{n,F} = F \otimes_{\mathbb{Z}} \mathcal{D}_n$ over $F$. The results of the previous section yield a description of the radical of $\mathcal{D}_{n,F}$ and its quotient in a straightforward way.

Some additional definitions are needed. A composition $r$ of $n$ is a partition of $n$ if $r$ is weakly decreasing. In this case, we write $r \vdash n$. Furthermore, a partition $r$ is $p$-regular (with respect to a positive integer $p$) if no component of $r$ occurs more than $p-1$ times in $r$. Finally, we write $q \approx \tilde{q}$ if $q$ is obtained by rearranging the components of $\tilde{q}$, for all $q, \tilde{q} \models n$.

**Corollary 3.2.** The quotient $\mathcal{D}_{n,F}/\text{rad} \mathcal{D}_{n,F}$ is split semi-simple. Furthermore, if $F$ has characteristic zero, then

$$\text{rad} \mathcal{D}_{n,F} = \langle \Xi^q - \Xi^{\tilde{q}} \mid q, \tilde{q} \models n, q \approx \tilde{q} \rangle_F,$$

and the dimension of $\mathcal{D}_{n,F}/\text{rad} \mathcal{D}_{n,F}$ is equal to the number of partitions of $n$. If $F$ has prime characteristic $p$, then

$$\text{rad} \mathcal{D}_{n,F} = \langle \Xi^q - \Xi^{\tilde{q}} \mid q, \tilde{q} \models n, q \approx \tilde{q} \rangle_F \oplus \langle \Xi^r \mid r \vdash n, r \text{ not } p\text{-regular} \rangle_F,$$

and the dimension of $\mathcal{D}_{n,F}/\text{rad} \mathcal{D}_{n,F}$ is equal to the number of $p$-regular partitions of $n$.  

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This is due to Solomon [Sol76] if $F$ has characteristic zero, and due to Atkin-son and Willigenburg [AvW97] if $F$ has prime characteristic.

**Proof.** Theorem 3.1 allows us to consider $B_{n,F} = F \otimes \mathbb{Z} B_n$ instead of $D_{n,F}$. We will drop the index $F$ in what follows. Let $\tilde{\Pi}_n$ denote the set of (unordered) set partitions of $[n]$, and let $s : F\Pi_n \to F\tilde{\Pi}_n$ be the linear extension of the support map that forgets the ordering. Then by Corollary 2.2, we have $\text{rad } B_n = B_n \cap \ker s$, and $B_n/\text{rad } B_n \cong B_n s$ is split semisimple.

For each $q | n$, set $c_q := m_1! \cdots m_n!$, where $m_i$ denotes the multiplicity of the entry $i$ in $q$ for all $i \in [n]$. For each $r = (r_1, \ldots, r_l) \vdash n$, let $x^r$ denote the sum in $F\tilde{\Pi}_n$ of all set partitions $\tilde{Q} = \{\tilde{Q}_1, \ldots, \tilde{Q}_l\}$ of $[n]$ whose elements have cardinalities $r_1, \ldots, r_l$. Then, if $q \vdash n$ and $r \vdash n$ with $q \approx r$,

$$X^q s = \sum_{\text{type}(Q) = q} QS = c_q x^r = c_r x^r.$$

As a consequence, $B_n s$ is linearly generated by the elements $x^r$ where $r \vdash n$ such that $\text{char } F$ does not divide $c_r$. Furthermore, $(X^q - X^{\tilde{q}}) s = 0$ whenever $q \approx \tilde{q}$, and even $X^q s = 0$ whenever $\text{char } F$ divides $c_q$.

Comparing dimensions, completes the proof. \qed

### 4 The free twisted descent algebra

Let $\text{Fin}$ denote the set of all finite subsets of $\mathbb{N}$. If $A \in \text{Fin}$ has order $n$, then the set $\Pi_A$ of all set compositions of $A$ is an idempotent semigroup with identity $(A)$, with respect to the product defined in the introduction for $A = [n]$. This product is denoted by $\land_A$. It is clear that the semigroups $(\Pi_n, \land_n)$ and $(\Pi_A, \land_A)$ are isomorphic.

Aiming at structural properties of the Solomon-Tits algebra $F\Pi_n$, it is advantageous to consider several of the semigroups $\Pi_A$ simultaneously and to
employ inductive techniques which arise from the structure of the free twisted descent algebra.

Let \( \Pi = \bigcup_{A \in \text{Fin}} \Pi_A \). The products \( \wedge_A \) extend to an internal, or *intersection product* \( \wedge \) on the direct sum

\[
F\Pi = \bigoplus_{A \in \text{Fin}} F\Pi_A ,
\]

by orthogonality:

\[
P \wedge Q := \begin{cases} 
P \wedge_A Q & \text{if } \bigcup P = A = \bigcup Q, \\
0 & \text{otherwise}, \end{cases}
\]

for all \( P, Q \in \Pi \). A second, external, or *concatenation product* \( \vee \) on \( F\Pi \) is defined by

\[
P \vee Q := \begin{cases} 
(P_1, \ldots, P_l, Q_1, \ldots, Q_k) & \text{if } \bigcup P \cap \bigcup Q = \emptyset, \\
0 & \text{otherwise}, \end{cases}
\]

for all \( P = (P_1, \ldots, P_l), Q = (Q_1, \ldots, Q_k) \in \Pi \), and linearity. If \( A = \emptyset \), then \( \Pi_A \) consists of the unique set composition of \( A \): the empty tuple \( \emptyset \), which acts as a two-sided identity in \( (F\Pi, \vee) \).

If \( A \in \text{Fin} \) and \( P \in \Pi_A \), set \( P|_X := (P_1 \cap X, \ldots, P_l \cap X)^\# \in \Pi_X \) for all \( X \subseteq A \). We define a coproduct on \( F\Pi \) by

\[
\Delta(P) = \sum_{X \subseteq A} P|_X \otimes P|_{A \setminus X}
\]

for all \( A \in \text{Fin}, P \in \Pi_A \), and linearity. This coproduct is cocommutative.

The algebra \((F\Pi, \vee)\) is a free associative algebra in the category of \( \text{Fin} \)-graded vector spaces, with one generator in each degree, and a free twisted descent algebra (for details, see [PS]). Background from the theory of twisted algebras, however, is not needed here.
We recall three simple, but crucial set-theoretical observations [PS, Theorem 17, Corollary 18], linking the two products and the coproduct on $F \Pi$. Their proofs are sketched for the reader’s convenience. We denote by $\wedge_2$ (respectively, by $\vee_2$) the (componentwise) product on $F \Pi \otimes F \Pi$ induced by $\wedge$ (respectively, by $\vee$).

Proposition 4.1. $(F \Pi, \vee, \Delta)$ is a Fin-graded bialgebra, that is: if $A, B \in \text{Fin}$ are disjoint, then $F \Pi_A \vee F \Pi_B \subseteq F \Pi_{A \cup B}$ and

\[
\Delta(f \vee g) = \Delta(f) \vee_2 \Delta(g)
\]

for all $f \in F \Pi_A$, $g \in F \Pi_B$.

For the proof, it suffices to consider $P \in \Pi_A$ and $Q \in \Pi_B$, by linearity. In this case, set $C := A \cup B$, then

\[
\Delta(P \vee Q) = \sum_{X \subseteq C} (P \vee Q)|_X \otimes (P \vee Q)|_{C \setminus X}
\]

\[
= \sum_{U \subseteq A} \sum_{V \subseteq B} (P|_U \vee Q|_V) \otimes (P|_{A \setminus U} \vee Q|_{B \setminus V})
\]

\[
= \Delta(P) \vee_2 \Delta(Q).
\]

Proposition 4.2. $(F \Pi, \wedge, \Delta)$ is a bialgebra.

The proof is equally simple. \qed

Let $m_\vee : F \Pi \otimes F \Pi \to F \Pi$ be the linearisation of the product $\vee$ on $F \Pi$.

Proposition 4.3. $(f \vee g) \wedge h = m_\vee \left((f \otimes g) \wedge_2 \Delta(h)\right)$, for all $f, g, h \in F \Pi$.

Using Sweedler’s notation, this reads $(f \vee g) \wedge h = \sum (f \wedge h^{(1)}) \vee (g \wedge h^{(2)})$.

For the proof, it suffices again to consider $P, Q, R \in \Pi$, by linearity. Let $A = \bigcup P$, $B = \bigcup Q$ and $C = \bigcup R$, then

\[
m_\vee \left((P \otimes Q) \wedge_2 \Delta(R)\right) = \sum_{X \subseteq C} (P \wedge R|_X) \vee (Q \wedge R|_{C \setminus X}).
\]
This term does not vanish if and only if $A \subseteq C$ and $B = C \setminus A$, that is, if $C$ is the disjoint union of $A$ and $B$. The same is true for the term $(P \lor Q) \land R$.

And in this case,

$$m_\lor \left( (P \otimes Q) \land_2 \Delta(R) \right)$$

$$= (P \land R|_A) \lor (Q \land R|_B)$$

$$= (P_1 \cap R_1, \ldots, P_1 \cap R_m, \ldots, P_l \cap R_1, \ldots, P_l \cap R_m)^\#$$

$$\lor (Q_1 \cap R_1, \ldots, Q_1 \cap R_m, \ldots, Q_k \cap R_1, \ldots, Q_k \cap R_m)^\#$$

$$= (P \lor Q) \land R,$$

where $P = (P_1, \ldots, P_l)$, $Q = (Q_1, \ldots, Q_k)$ and $R = (R_1, \ldots, R_m)$.

Proposition 4.3 often allows us to transfer calculations from $(F \Pi_A, \land_A)$ to $(F \Pi, \lor, \Delta)$ and $(F \Pi_B, \land_B)$ for some (proper) subsets $B$ of $A$. This inductive idea will be frequently used in what follows.

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Let $A \in \text{Fin}$. An element $e$ of $F \Pi_A$ is an idempotent if $e^2 = e \land e = e$ and $e \neq 0$. Such an idempotent is primitive if the left ideal $F \Pi_A \land e$ of $F \Pi_A$ is indecomposable as an $F \Pi_A$-module. (Equivalently, whenever $f, g \in F \Pi_A$ such that $e = f + g$, $f^2 = f$, $g^2 = g$ and $f \land g = 0 = g \land f$, then $f = 0$ or $g = 0$.)

An element $E \in F \Pi$ is $\Delta$-primitive if $\Delta(E) = E \otimes \emptyset + \emptyset \otimes E$.

In concluding this section, we illustrate the inductive method by exploring a relation between the $\Delta$-primitive elements of $F \Pi_A$ and certain primitive idempotents in $F \Pi_A$.  

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Corollary 4.4. Let $A \in \text{Fin}$ and suppose that $E = \sum_{P \in \Pi_A} k_P P \in F\Pi_A$ is $\Delta$-primitive, then $P \wedge E = 0$ for all $P \in \Pi_A \setminus \{(A)\}$.

In particular, $E \wedge E = k_{(A)} E$.

Proof. If $P \in \Pi_A \setminus \{(A)\}$, say, $P = R \vee S$ with $R, S \in \Pi \setminus \{\emptyset\}$, then

$$P \wedge E = (R \vee S) \wedge E = m_\vee((R \otimes S) \wedge_2 (E \otimes \emptyset + \emptyset \otimes E)) = 0,$$

by Proposition 4.3. It follows that $E \wedge E = k_{(A)}(A) = k_{(A)} E$. \hfill \Box

Note that, if $k_{(A)} \neq 0$, then $e = 1/k_{(A)} E$ is in fact a primitive idempotent in $F\Pi_A$. For, if $f, g \in F\Pi_A$ such that $e = f + g$, $f^2 = f$, $g^2 = g$ and $f \wedge g = 0 = g \wedge f$, then $e \wedge f = f$ and $e \wedge g = g$. Hence $f$ and $g$ are $\Delta$-primitive as well, by Proposition 4.2. The preceding corollary implies $c_{(A)} g = f \wedge g = 0$ and $c_{(A)} f = f \wedge f = f$, where $c_{(A)}$ denotes the coefficient of $(A)$ in $f$. Thus $f = 0$ or $g = 0$.

If $P = (P_1, \ldots, P_l) \in \Pi$, then $\ell(P) := l$ is the length of $P$. We set

$$\Pi_A^* := \{(P_1, \ldots, P_l) \in \Pi_A \mid \min A \in P_l\}$$

for all $A \in \text{Fin}$.

Lemma 4.5. Let $A \in \text{Fin}$, then the element

$$e_A = \sum_{P \in \Pi_A^*} (-1)^{\ell(P)-1} P$$

is $\Delta$-primitive. In particular, $e_A$ is a primitive idempotent in $F\Pi_A$.

Proof. Let $R, S \in \Pi \setminus \{\emptyset\}$ such that $R \vee S \in \Pi_A$. We need to show that the coefficient $c_{R,S}$ of $R \otimes S$ in $\Delta(e_A)$ is zero. Let $X = \bigcup R$. It suffices to consider the case where $a^* := \min A \in X$, since $\Delta$ is cocommutative. We have

$$c_{R,S} = \sum_{P \in X} (-1)^{\ell(P)-1},$$

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where $X = \{ P \in \Pi' \mid P|_X = R, P|_{A\setminus X} = S \}$. Let $P = (P_1, \ldots, P_l) \in X$, then there exists an index $i \in [l]$ such that $P_i \setminus X \neq \emptyset$, since $X \neq A$. Choose $i$ minimal with this property. If also $P_i \cap X \neq \emptyset$, then set $P' := (P_1, \ldots, P_{i-1}, P_i \cap X, P_i \setminus X, P_{i+1}, \ldots, P_l) \in X$.

If $P_i \cap X = \emptyset$, then $a^* \in P_1 \cap X$ implies that $i > 1$, and we define $P' := (P_1, \ldots, P_{i-2}, P_{i-1} \cup P_i, P_{i+1}, \ldots, P_l) \in X$.

Then $(P')' = P$ and $(-1)^{\ell(P')} = -(-1)^{\ell(P)}$ for all $P \in X$, that is, $P \mapsto P'$ defines is a sign-reversing pairing of the summands of $c_{R, S}$. This implies that $c_{R, S} = 0$, hence that $e_A$ is $\Delta$-primitive. The additional claim follows from Corollary 4.4 and its subsequent remark.

The idempotent $e_A$ is displayed in Table 1 for some particular choices of $A$. Note that several curly brackets and commas have been omitted; for instance, $(12, 3)$ means $([1, 2], \{3\})$.

Table 1: The idempotent $e_A$ for $A = \{1\}, \{1, 2\}, \{1, 2, 3\}$ and $\{2, 5, 6\}$.

The Lie product $\circ$ associated with the product $\vee$ on $F\Pi$ is defined by $f \circ g = f \vee g - g \vee f$ for all $f, g \in F\Pi$. The set $\text{Prim } F\Pi$ of all $\Delta$-primitive elements in $F\Pi$ is a Lie subalgebra of $(F\Pi, \circ)$. As a by-product of the above considerations, there is the following description of $\text{Prim } F\Pi$. 

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Corollary 4.6. The $\Delta$-primitive Lie algebra of $F\Pi$ is

$$\text{Prim } F\Pi = \bigoplus_{A \in \text{Fin}} e_A \wedge F\Pi_A.$$ 

Its $A$-graded component $e_A \wedge F\Pi_A$ is equal to the linear span of all idempotents $e \in F\Pi_A$ such that $e \wedge F\Pi_A = e_A \wedge F\Pi_A$, and has dimension $2\#\Pi_{n-1}$.

Proof. Let $A \in \text{Fin}$, and suppose $e \in F\Pi_A$ is $\Delta$-primitive. Then $e = (A) \wedge e = e_A \wedge e \in e_A \wedge F\Pi_A$, by Corollary 4.4. Conversely, $e_A \wedge f$ is $\Delta$-primitive for any $f \in F\Pi_A$, by Proposition 4.2.

The set of idempotents $e$ in $F\Pi_A$ with $e \wedge F\Pi_A = e_A \wedge F\Pi_A$ is

$$e_A + e_A \wedge F\Pi_A \wedge ((A) - e_A),$$

as a general fact. The linear span of this set is equal to $e_A \wedge F\Pi_A$, as claimed, since $P \wedge e_A = 0$ for all $P \in \Pi_A$ with $\ell(P) > 1$.

The dimension formula will be obtained in Remark 6.5 (1).

5 Primitive idempotents and principal indecomposable modules

We are now in a position to study the module structure of the Solomon-Tits algebra in more detail. In this section, for any finite subset $A$ of $\mathbb{N}$, we construct a linear basis of $F\Pi_A$ which consists of primitive idempotents. Properly grouping together basis elements, we then obtain a decomposition of $F\Pi_A$ into indecomposable left ideals.

In what follows, $E_A$ is a $\Delta$-primitive idempotent in $F\Pi_A$, that is, we have $\Delta(E_A) = E_A \otimes \emptyset + \emptyset \otimes E_A$ and $E_A \wedge E_A = E_A$, for all $A \in \text{Fin}$. (One possible choice for $E_A$ is the element $e_A$, defined in Lemma 4.3.) We shall make
other choices for $E_A$ in Section 9.) Note that $E_A \in (A) + \langle \Pi_A \setminus \{(A)\} \rangle_F$ by Corollary 4.4. We set

$$E_Q := E_{Q_1} \lor \cdots \lor E_{Q_k}$$

for all $Q = (Q_1, \ldots, Q_k) \in \Pi$. Then

$$E_Q \in (A) + \langle R \in \Pi_A \mid R \preceq Q, R \not\approx Q \rangle_F \quad (5.1)$$

for all $A \in \text{Fin}$, $Q \in \Pi_A$. Here $R \preceq Q$ means that each component of $R$ is contained in a component of $Q$, and $R \approx Q$ means that $R$ is a rearrangement of $Q$ (as in the case $A = [n]$). From (5.1), using a triangularity argument, we get:

**Proposition 5.1.** \( \{ E_Q \mid Q \in \Pi_A \} \) is a linear basis of $F\Pi_A$, for all $A \in \text{Fin}$.  

The basis \( \{ e_Q \mid Q \in \Pi_3 \} \) of $F\Pi_3$ (arising from the idempotents $e_A$ defined in Lemma 4.5), is displayed in Table 2. Some technical preparations are needed

| Basis Elements | Description |
|---------------|-------------|
| $e_{(1,2,3)}$ | $(1, 2, 3)$ |
| $e_{(2,1,3)}$ | $(2, 1, 3)$ |
| $e_{(3,1,2)}$ | $(3, 1, 2)$ |
| $e_{(12,3)}$ | $(12, 3) - (1, 2, 3)$ |
| $e_{(1,23)}$ | $(1, 23) - (1, 2, 3)$ |
| $e_{(13,2)}$ | $(13, 2) - (1, 3, 2)$ |
| $e_{(123)}$ | $(123) - (12, 3) - (1, 23) + (13, 2) + (1, 2, 3) + (1, 3, 2)$ |

Table 2: A basis of $F\Pi_3$ consisting of primitive idempotents.

to describe the left-regular representation of $F\Pi_A$ in terms of this basis. Let $Q = (Q_1, \ldots, Q_k) \in \Pi_A$. We set

$$Q_I := (Q_{i_1}, \ldots, Q_{i_m})$$
for any subset $I = \{i_1, \ldots, i_m\}$ of $[k]$ with $i_1 < \cdots < i_m$. If $P = (P_1, \ldots, P_l) \in \Pi_A$ such that $Q \preceq P$, then there exists a unique set composition $I = (I_1, \ldots, I_l)$ of the index set $[k]$ of $Q$ such that $Q_{I_j} \in \Pi_{P_j}$ for all $j \in [l]$. Note that, in this case, $P \land Q = Q_{I_1} \lor \cdots \lor Q_{I_l}$. For example, if $A = \{3, 5, 6, 7, 8\}$, $Q = (8, 67, 3, 5)$ and $P = (3, 67, 58)$, then $Q \preceq P$, $I = (23, 14)$ and $P \land Q = (67, 3, 8, 5)$.

**Lemma 5.2.** Let $A \in \text{Fin}$ and $P = (P_1, \ldots, P_l), Q = (Q_1, \ldots, Q_k) \in \Pi_A$. Suppose $f^{(j)} \in F \Pi_{P_j}$ for all $j \in [l]$, then

$$
(f^{(1)} \lor \cdots \lor f^{(l)}) \land E_Q = \begin{cases} 
(f^{(1)} \land E_{Q_{I_1}}) \lor \cdots \lor (f^{(l)} \land E_{Q_{I_l}}) & \text{if } Q \preceq P, \\
0 & \text{otherwise,}
\end{cases}
$$

where, in case $Q \preceq P$, $(I_1, \ldots, I_l)$ is the unique set composition in $\Pi_k$ such that $Q_{I_j} \in \Pi_{P_j}$ for all $j \in [l]$.

**Proof.** For $l = 1$, there is nothing to prove (since $Q \preceq (A)$). Let $l > 1$, then Proposition 4.1 implies that

$$
\Delta(E_Q) = \Delta(E_{Q_{I_1}}) \lor_2 \cdots \lor_2 \Delta(E_{Q_{I_l}}) = \sum_{I \subseteq [k]} E_{Q_{I_1}} \otimes E_{Q_{[k] \setminus I}}.
$$

Thus, setting $\tilde{f} := f^{(2)} \lor \cdots \lor f^{(l)}$ and applying Proposition 4.3, we get

$$
(f^{(1)} \lor \tilde{f}) \land E_Q = \sum_{I \subseteq [k]} (f^{(1)} \land E_{Q_{I_1}}) \lor (\tilde{f} \land E_{Q_{[k] \setminus I}}).
$$

This term vanishes unless there exists a subset $I_1$ of $[k]$ such that $Q_{I_1} \in \Pi_{P_1}$, in which case $(f^{(1)} \lor \tilde{f}) \land E_Q = (f^{(1)} \land E_{Q_{I_1}}) \lor (\tilde{f} \land E_{Q_{[k] \setminus I_1}})$. The proof may be completed by a simple inductive step. 

Special cases of the preceding lemma are:

$P \land E_Q = \begin{cases} 
E_{P \land Q} & \text{if } Q \preceq P, \\
0 & \text{otherwise,}
\end{cases}$

for all $P, Q \in \Pi_A$ \hspace{1cm} (5.2)
(with \( f^{(i)} = (P_i) \) for all \( i \in [l] \)), and

\[
E_P \wedge E_Q = E_P \quad \text{whenever} \quad P \approx Q
\]

(5.3)

(with \( f^{(i)} = E_{P_i} \) for all \( i \in [l] \)). Considering \( P = Q \), we obtain:

**Corollary 5.3.** \( E_Q \wedge E_Q = E_Q \) for all \( Q \in \Pi \).

As we shall see below, each \( E_Q \) is in fact a *primitive* idempotent. Note that (5.1) and (5.2) imply furthermore that

\[
E_P \wedge E_Q = 0 \quad \text{unless} \quad Q \preceq P.
\]

(5.4)

\[
\ast \quad \ast \quad \ast
\]

In what follows, all modules are left modules. If \( \mathcal{A} \) is a finite-dimensional associative algebra with identity and \( M \) is an \( \mathcal{A} \)-module, then the \( \mathcal{A} \)-*radical* \( \text{rad}_\mathcal{A} M \) of \( M \) is the intersection of all maximal \( \mathcal{A} \)-submodules of \( M \). In particular, \( \text{rad} \mathcal{A} = \text{rad}_\mathcal{A} \mathcal{A} \) is the Jacobson radical of \( \mathcal{A} \), and \( \text{rad}_\mathcal{A} M = (\text{rad} \mathcal{A}) M \).

We observe that the set

\[
\Pi^<_\mathcal{A} := \{ (Q_1, \ldots, Q_k) \in \Pi_\mathcal{A} \mid \min Q_1 < \cdots < \min Q_k \}
\]

is a transversal for the equivalence classes in \( \Pi_\mathcal{A} \) with respect to \( \approx \), for each \( \mathcal{A} \in \text{Fin} \). (Hence the irreducible \( F\Pi_\mathcal{A} \)-modules are in 1-1 correspondence to the elements of \( \Pi^<_\mathcal{A} \), by Corollary 2.3.)

**Theorem 5.4.** For each \( Q \in \Pi_\mathcal{A} \),

\[
\Lambda_Q := F\Pi_\mathcal{A} \wedge E_Q = \langle E_{Q'} \mid Q' \in \Pi_\mathcal{A}, \ Q' \approx Q \rangle_F
\]

is an indecomposable \( F\Pi_\mathcal{A} \)-left module with radical

\[
\text{rad}_{F\Pi_\mathcal{A}} \Lambda_Q = \langle E_Q - E_{Q'} \mid Q' \in \Pi_\mathcal{A}, \ Q' \approx Q \rangle_F
\]
of codimension 1. In particular,
\[ F\Pi_A = \bigoplus_{Q \in \Pi_A^<} \Lambda_Q \]
is a decomposition into indecomposable submodules, and
\[ \text{rad } F\Pi_A = \langle E_Q - E_{Q'} \mid Q \in \Pi_A^<, Q' \in \Pi_A, Q' \approx Q \rangle_F. \]

**Proof.** The elements \( E_Q', Q' \approx Q \), constitute a linear basis of \( \Lambda_Q = F\Pi_A \wedge E_Q \), by (5.2), (5.3) and Proposition 5.1. In particular, \( \Lambda_Q = \Lambda_{Q'} \) whenever \( Q \approx Q' \). Hence \( F\Pi_A = \bigoplus_{Q \in \Pi_A^<} \Lambda_Q \) is a decomposition into left ideals by Proposition 5.1.

Since the dimension of \( F\Pi_A/\text{rad } F\Pi_A \) is equal to \( \#\Pi_A^< \), by Corollary 2.3 it follows that \( \Lambda_Q/\text{rad } F\Pi_A \Lambda_Q \) is one-dimensional for all \( Q \). This implies that \( \Lambda_Q \) is indecomposable and also the description of \( \text{rad } F\Pi_A \Lambda_Q \).

As an immediate consequence, each of the idempotents \( E_Q \) is primitive in \( F\Pi_A \). Besides, \( \dim \Lambda_Q = \ell(Q)! \) for all \( Q \in \Pi_A \).

**Corollary 5.5.** The one-dimensional spaces \( M_Q = \Lambda_Q/\text{rad } F\Pi_A \Lambda_Q, Q \in \Pi_A^< \), form a complete set of mutually non-isomorphic irreducible \( F\Pi_A \)-modules.

**Remark 5.6.** Suppose \( \tilde{E}_A \) is another \( \Delta \)-primitive idempotent in \( F\Pi_A \), for all \( A \in \text{Fin} \), and \( \tilde{\Lambda}_Q = F\Pi_A \wedge \tilde{E}_Q \) denotes the corresponding indecomposable \( F\Pi_A \)-module for all \( Q \in \Pi_A \). Then \( \Lambda_Q \cong \tilde{\Lambda}_Q \) as \( F\Pi_A \)-modules.

Indeed, from Corollary 1.2 we get that \( E_A \wedge \tilde{E}_A = \tilde{E}_A \) for all \( A \in \text{Fin} \). Hence \( E_{Q'} \wedge \tilde{E}_Q = \tilde{E}_{Q'} \) whenever \( Q' \approx Q \) by Lemma 5.2. It follows that \( f \mapsto f \wedge \tilde{E}_Q \) defines an isomorphism from \( \Lambda_Q \) onto \( \tilde{\Lambda}_Q \).
6 Cartan invariants

Taking $E_A = e_A$ for all $A \in \text{Fin}$, we obtain the linear basis $\{ e_Q \mid Q \in \Pi_A \}$ of $F\Pi_A$ from Lemma 4.3 and Proposition 5.1. It is well adapted to the module structure of $F\Pi_A$, as will be further stressed in the sections that follow.

The key result is Theorem 6.2, a multiplication rule for the basis $\{ e_Q \mid Q \in \Pi_A \}$, which allows us to determine the Cartan matrix of $F\Pi_A$ at once.

The Lie product \( \circ \) associated with the product \( \lor \) on $F\Pi$ has been considered at the end of Section 4 already. It occurs in a natural way when two basis elements $e_P$ and $e_Q$ are multiplied together. For example, if $P = (A)$ and $Q = (X, Y) \in \Pi_A$, then

\[
e_A \land e_{(X, Y)} = \begin{cases} e_X \circ e_Y & \text{if } \min A \in Y, \\ 0 & \text{otherwise.} \end{cases}
\]

Indeed, if $\min A \in X$, then

\[
e_A \land e_{(X, Y)} = \sum_{P \in \Pi_A^*} (-1)^{\ell(P)-1} P \land e_{(X, Y)} = (A) \land e_{(X, Y)} - (X, Y) \land e_{(X, Y)} = 0,
\]

by (5.2). Similarly, if $\min A \in Y$, then

\[
e_A \land e_{(X, Y)} = (A) \land e_{(X, Y)} - (Y, X) \land e_{(X, Y)} = e_{(X, Y)} - e_{(Y, X)}.
\]

A more systematical approach follows. The (right-normed) Dynkin mapping $F\Pi \to F\Pi, f \mapsto f^\circ$ is defined recursively by $(A)^\circ = (A)$ for all $A \in \text{Fin}$,

\[(Q_1, \ldots, Q_k)^\circ := Q_1 \circ ((Q_2, \ldots, Q_k)^\circ),\]

for all $(Q_1, \ldots, Q_k) \in \Pi$ with $k > 1$, and linearity. For $A \in \text{Fin}$, set

\[\Pi_A^\dagger := \{ (Q_1, \ldots, Q_k) \in \Pi_A \mid \min A \in Q_k \} .\]

We will need the following folklore result on right-normed multilinear Lie monomials (see [Sch04a, Proposition 2.3] for the left-normed case).
Proposition 6.1. Let $A \in \text{Fin}$ and $Q = (Q_1, \ldots, Q_k) \in \Pi_A^\dagger$, then

$$Q^o \in Q + \langle R \in \Pi_A \mid R \approx Q, R \notin \Pi_A^\dagger \rangle_F .$$

(6.2)

In particular, $\{ \check{Q}^o \mid \check{Q} \in \Pi_A^\dagger, Q \approx \check{Q} \}$ is a linear basis of $\langle R^o \mid R \approx Q \rangle_F$ of order $(k - 1)!$.

Proof. The first claim follows by a simple induction on $k$ and implies linear independency of the set considered in the second claim. The rest follows by comparing dimensions.

For all $Q = (Q_1, \ldots, Q_k), P = (P_1, \ldots, P_l) \in \Pi_A$, we write

$$Q \preceq^\dagger P$$

if there exists a set composition $(I_1, \ldots, I_l) \in \Pi_{[k]}$ such that $Q_{I_j} \in \Pi_{P_j}^\dagger$ for all $j \in [l]$. In this case, $(I_1, \ldots, I_l)$ is unique, and certainly $Q \preceq P$. For example, $(4, 5, 13, 2) \preceq (123, 45)$, but not $(4, 5, 13, 2) \preceq^\dagger (123, 45)$, since $I_1 = \{3, 4\}$ and $Q_{I_1} = (13, 2) \notin \Pi_{\{1,2,3\}}^\dagger$.

Theorem 6.2. Let $P = (P_1, \ldots, P_l), Q = (Q_1, \ldots, Q_k) \in \Pi_A$, then

$$e_P \wedge e_Q = \begin{cases} e_{Q_{I_1}} \lor \cdots \lor e_{Q_{I_l}} & \text{if } Q \preceq^\dagger P, \\ 0 & \text{otherwise} \end{cases}$$

where, in case $Q \preceq^\dagger P$, the set composition $(I_1, \ldots, I_l)$ of $[k]$ is so chosen that $Q_{I_j} \in \Pi_{P_j}^\dagger$ for all $j \in [l]$.

In particular, $e_A \wedge e_Q = e_{Q^o}$ if $Q \in \Pi_A^\dagger$, and $e_A \wedge e_Q = 0$ otherwise.

Here the map $e : Q \mapsto e_Q$ has been extended linearly to $F \Pi$, so that by definition

$$e_{\Sigma f_R R} = \sum f_R e_R$$

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for all $\sum f_R R \in E\Pi$. For example, we have $e_{(123,45)} \land e_{(4,5,13,2)} = 0$, since $(4,5,13,2) \preceq^t (123,45)$ does not hold, while

$$e_{(234,1)} \land e_{(34,1,2)} = e_{(34,2')} \lor e_{(1)} = e_{(34,2')-(2,34)} \lor e_{(1)} = e_{(34,2,1)} - e_{(2,34,1)}.$$  

**Proof of Theorem 6.2.** The goal is to prove the assertion in the special case where $P = (A)$, mentioned in the second part, since the general case then follows from Lemma 5.2, applied to $f^{(i)} = e_{P_i}$ for all $i \in [l]$.

This will be done in four steps. Choose $i \in [k]$ such that $a^* := \min A \in Q_i$.

**Step 1.** Suppose $P \in \Pi_A^*$ such that $Q \preceq P$ and set $Q' := P \land Q$. Then $a^* \in P_1$ and $Q' \cong Q$. Furthermore, from $Q' = (Q'_1,\ldots,Q'_k) = (P_1 \cap Q_1,\ldots,P_1 \cap Q_l,\ldots)$ it follows that $a^* \in Q'_j$ for some $j \leq i$; and we have $i = j$ if and only if $\bigcup_{m \leq i} Q_m \subseteq P_1$.

**Step 2.** Suppose that $i < k$, and let $Q' \cong Q$ such that $a^* \in Q'_i$. Put $X := \{ P \in \Pi_A^* | Q' = P \land Q \}$. Then $\bigcup_{m \leq i} Q_m \subseteq P_1$ for each $P \in X$, by Step 1. Hence the set $X$ decomposes into subsets $X_\equiv$ and $X_\not\equiv$, where $P = (P_1,\ldots,P_l) \in X$ belongs to $X_\equiv$ or $X_\not\equiv$ according as $\bigcup_{m \leq i} Q_m = P_1$ or not.

Note that $i < k$ implies that $\ell(P) > 1$ for all $P \in X_\equiv$. The mapping

$$X_\equiv \to X_\not\equiv, \quad P \mapsto \tilde{P} := (P_1 \cup P_2, P_3,\ldots,P_l)$$

is a bijection with inverse given by

$$X_\not\equiv \to X_\equiv, \quad \tilde{P} \mapsto P := (\bigcup_{m \leq i} Q_m, \tilde{P}_1 \setminus \bigcup_{m \leq i} Q_m, \tilde{P}_2,\ldots,\tilde{P}_l),$$

and $\ell(\tilde{P}) = \ell(P) - 1$ for all $P \in X_\equiv$.

**Step 3.** We are now in a position to prove $e_A \land e_Q = 0$ if $i < k$, by induction on $i$. For,

$$e_A \land e_Q = e_A \land (e_A \land e_Q), \text{ by Lemma 4.6}$$
\[
\begin{align*}
&= e_A \land \sum_{Q' \approx Q} \sum_{P \in \Pi^*_A} (-1)^{\ell(P)-1} e_{Q'}, \text{ by } (5.2) \\
&= e_A \land \sum_{j=1}^{i} \sum_{Q' \approx Q} \left( \sum_{P \in \Pi^*_A} (-1)^{\ell(P)-1} \right) e_{Q'}, \text{ by Step 1} \\
&= \sum_{j=1}^{i-1} \sum_{Q' \approx Q} \left( \sum_{P \in \Pi^*_A} (-1)^{\ell(P)-1} \right) e_A \land e_{Q'}, \text{ by Step 2.}
\end{align*}
\]

Thus, if \(i = 1\), then \(e_A \land e_Q = 0\) follows directly, while for \(i > 1\), we may conclude by induction.

**Step 4.** Assume now that \(i = k\). We will show that \(e_{Q^o} = e_A \land e_Q\) by induction on \(k = \ell(Q)\).

For \(k = 1\), this is Lemma 15. Let \(k > 1\), and set \(X = Q_1\), \(\tilde{Q} = (Q_2, \ldots, Q_k)\) and \(Y = \bigcup \tilde{Q}\), then

\[
\begin{align*}
e_{Q^o} &= e_X \lor e_{\tilde{Q}}^o - e_{\tilde{Q}}^o \lor e_X \\
&= e_X \lor (e_Y \land e_{\tilde{Q}}) - (e_Y \land e_{\tilde{Q}}) \lor e_X, \text{ by induction} \\
&= (e_{(X,Y)} - e_{(Y,X)}) \land e_Q, \text{ by Lemma } b.2 \\
&= e_A \land e_{(X,Y)} \land e_Q, \text{ by } (6.1) \\
&= e_A \land (e_X \lor (e_Y \land e_{\tilde{Q}})), \text{ by Lemma } b.2 \\
&= e_A \land (e_X \lor e_{\tilde{Q}}^o), \text{ by induction} \\
&= e_A \land e_Q, \text{ by } (6.2) \text{ and Step 3.}
\end{align*}
\]

The proof is complete. \(\square\)

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Corollary 6.3. The elements $e_T$, $T \in \Pi_A^<$, form a complete set of mutually orthogonal primitive idempotents in $F\Pi_A$, and $\sum_{T \in \Pi_A^<} e_T = (A)$.

Their linear span is a complement of $\text{rad} F\Pi_A$ in $F\Pi_A$.

Proof. We have $T \preceq S$ if and only if $S = T$, for all $S, T \in \Pi_A^<$. Hence the claim follows from Theorem 6.2, Corollary 5.3 and Theorem 5.4. \hfill $\Box$

* * *

Let $C_A = (C_{P,Q})$ denote the Cartan matrix of $F\Pi_A$, that is, $C_{P,Q}$ equals the multiplicity of $M_P$ in a composition series of $\Lambda_Q$, for all $P, Q \in \Pi_A^<$. Equivalently,

$$C_{P,Q} = \dim e_P \land \Lambda_Q = \dim e_P \land F\Pi_A \land e_Q,$$

since $M_P$ has dimension one. We already now that

$$C_{P,Q} \neq 0 \text{ implies } Q \preceq P,$$

by (5.4) and Theorem 5.4. In this case, Proposition 6.1 and Theorem 6.2 yield the following linear basis of the Peirce component $e_P \land F\Pi_A \land e_Q$.

Theorem 6.4. Let $P = (P_1, \ldots, P_l), Q = (Q_1, \ldots, Q_k) \in \Pi_A^<$ such that $Q \preceq P$, then the elements

$$e_{Q(1)} \lor \cdots \lor e_{Q(l)}$$

with $Q(i) \in \Pi_{P_i}^1$ for all $i \in [l]$ and $Q(1) \lor \cdots \lor Q(l) \approx Q$, form a linear basis of $e_P \land F\Pi_A \land e_Q$. In particular,

$$C_{P,Q} = (m_1 - 1)! \cdots (m_l - 1)!,$$

where $m_j = \# \{ i \in [k] \mid Q_i \subseteq P_j \}$ for all $j \in [l]$. 27
The Cartan matrix of $F\Pi_3$ is displayed in Table 3.

**Remarks 6.5.** (1) For each $Q \in \Pi_A^<$, the sum of the $Q$-column of $C_A$ is $\ell(Q)!$, the dimension of $\Lambda_Q$. The sum of the $P$-row of $C_A$ is equal to the dimension of the right ideal $e_P \wedge F\Pi_A$ of $F\Pi_A$, for all $P \in \Pi_A^<$. This right ideal is indecomposable as an $F\Pi_A$-right module. If $\text{type}(P) = (p_1, \ldots, p_l)$, then there is the explicit formula

$$
\dim e_P \wedge F\Pi_A = 2^l |\Pi_{p_1-1}| \cdots |\Pi_{p_l-1}|.
$$

(6.3)

To see this, let $n = |A|$ and consider first the case where $P = (A)$. We have

$$
\dim e_A \wedge F\Pi_A = \sum_{Q \in \Pi_A^<} C_{(A),Q} = \sum_{Q \in \Pi_A^<} (\ell(Q) - 1)! = \#\Pi_A^* = \#\Pi_n^*.
$$

To any $R = (R_1, \ldots, R_l) \in \Pi_{\{2, \ldots, n\}}$ can be associated two set compositions $(\{1\}, R_1, \ldots, R_l)$ and $(R_1 \cup \{1\}, R_2, \ldots, R_l)$ in $\Pi_n^*$. The identity $\#\Pi_n^* = 2\#\Pi_{n-1}$ readily follows, completing the proof of (6.3) in this case. For example, we have $\dim e_{(123)}F\Pi_3 = 2\#\Pi_2 = 6$. For arbitrary $P \in \Pi_A^<$, (6.3) now follows from the factorisation

$$
\dim e_P \wedge F\Pi_A = \sum_{Q \in \Pi_A^<} C_{P,Q} = \sum_{Q \in \Pi_A^<} C_{P,Q} = \prod_{i=1}^{l} \sum_{Q \in \Pi_A^<} C_{P_i,Q}.
$$
The Cartan matrices $C_A$, $A \in \text{Fin}$, have a very simple fractal structure. Namely, if $Q = (Q_1, \ldots, Q_k) \in \Pi_A$ is of length $k$, then the non-zero part of the $Q$-column of $C_A$ coincides with the (rightmost) $(1, 2, \ldots, k)$-column of $C_{[k]}$ when $Q_i$ is “identified” with $i$ for all $i \in [k]$. More formally, let $P = (P_1, \ldots, P_l) \in \Pi_A$ such that $Q \preceq P$, then

$$C_{P,Q} = C_{I, (1, \ldots, k)},$$

where $I = (I_1, \ldots, I_l)$ is the set composition in $\Pi_k$ such that $Q_{I_j} \in \Pi_{P_j}$ for all $j \in [l]$. (Indeed, both values are equal to $(|I_1| - 1)! \cdots (|I_l| - 1)!$, by Theorem 6.4.)

There is the following interesting structural explanation for the identity (6.4). Let $K$ denote the annihilator of $\Lambda_Q$ in $\mathbb{F}\Pi_A$, then

$$\mathbb{F}\Pi_A = K \oplus \mathbb{F}\Pi_Q$$

by (5.2), where $\Pi_Q$ is the sub-semigroup of $\Pi_A$ consisting of all set partitions $P \in \Pi_A$ such that $Q \preceq P$. There is an isomorphism of algebras $\iota : \mathbb{F}\Pi_k \to \mathbb{F}\Pi_Q$ mapping

$$(I_1, \ldots, I_l) \mapsto \left( \bigcup_{i \in I_1} Q_i, \bigcup_{i \in I_2} Q_i, \ldots, \bigcup_{i \in I_l} Q_i \right)$$

for all $(I_1, \ldots, I_l) \in \Pi_k$. We have $e_I \iota \equiv e_{I_\iota}$ modulo $K$ for all $I \in \Pi_k$, since $\min Q_1 < \cdots < \min Q_k$.

Using the isomorphism $\iota$, $\Lambda_Q$ may be viewed as a module for $\mathbb{F}\Pi_k$. More precisely, the mapping $e_{(i_1, \ldots, i_k)} \mapsto e_{(Q_{i_1}, \ldots, Q_{i_k})}$ defines an isomorphism of $\mathbb{F}\Pi_k$-modules from $\Lambda_1 \cdots \Lambda_k$ onto $\Lambda_Q$. Thus, if $P \in \Pi_Q$ and $I \in \Pi_k$ such that $I \iota = P$, then

$$C_{P,Q} = \dim e_{I_\iota} \land \Lambda_Q = \dim e_I \iota \land \Lambda_Q = \dim e_I \land \Lambda_1 \cdots \Lambda_k = C_{I, (1, \ldots, k)},$$

recovering (6.4).
7 Descending Loewy series

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Suppose $\mathcal{A}$ is a finite-dimensional associative algebra over $F$ with identity. If $M$ is an $\mathcal{A}$-module, then the descending Loewy series $(\text{rad}_\mathcal{A}^k M)_{k \in \mathbb{N}_0}$ of $M$ is defined recursively by $\text{rad}_\mathcal{A}^0 M = M$ and $\text{rad}_\mathcal{A}^{k+1} M := \text{rad}_\mathcal{A} (\text{rad}_\mathcal{A}^k M)$ for all $k \in \mathbb{N}_0$. The $k$th Loewy layer of $M$ is $\text{rad}_\mathcal{A}^k M / \text{rad}_\mathcal{A}^{k+1} M$. It is the largest semi-simple quotient of $\text{rad}_\mathcal{A}^k M$ as an $\mathcal{A}$-module.

Throughout this section, $\mathcal{A} ∈ \mathsf{Fin}$. We will describe the descending Loewy series of $F \Pi \mathcal{A}$. For convenience, we set $\mathcal{A} := F \Pi \mathcal{A}$ and write $fg$ instead of $f \land g$ for $f, g \in \mathcal{A}$. Furthermore, for $P, Q \in \Pi \mathcal{A}$, we write $P \rightarrow Q$ if $Q \preceq P$ and $\ell(Q) - \ell(P) = 1$.

Lemma 7.1. Let $Q = (Q_1, \ldots, Q_k) \in \Pi_\leq \mathcal{A}$ and $\overline{Q} = (Q_k, \ldots, Q_1)$, then

$$\text{rad}_\mathcal{A} \Lambda_Q = \sum_{P \rightarrow Q} \Lambda_P e_{\overline{Q}}.$$ 

Proof. If $k = 1$, then both sides are equal to zero. Suppose $k > 1$. The right hand side is a nilpotent left ideal of $\mathcal{A}$, by (5.4), and contained in $\Lambda_Q$, thus actually contained in $\text{rad}_\mathcal{A} \Lambda_Q$.

Let $Q' = (Q'_1, \ldots, Q'_k) \approx Q$ and $i \in [k - 1]$. Set $P := (Q'_1, \ldots, Q'_{i-1}, Q'_i \cup Q'_{i+1}, Q'_{i+2}, \ldots, Q'_k)$, then $P \rightarrow Q$. In fact, $Q \in \Pi_\leq \mathcal{A}$ implies that $\overline{Q} \preceq^\flat P$. Hence there exists a sign $\epsilon \in \{+1, -1\}$ such that

$$(*) \quad \epsilon \left(e_{Q'_1, \ldots, Q'_i, Q'_{i+1}, \ldots, Q'_k} - e_{Q'_1, \ldots, Q'_{i+1}, Q'_i, \ldots, Q'_k}\right) = e_P e_{\overline{Q}} \in \sum_{P \rightarrow Q} \Lambda_P e_{\overline{Q}},$$

by Theorem 6.2. The elements of the form $(*)$ linearly generate $\text{rad}_\mathcal{A} \Lambda_Q$, by Theorem 5.4. This completes the proof. 

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As a consequence of Lemma 7.1, we have
\[ \text{rad } A = \sum_{Q \in \Pi_A^<} \text{rad}_A \Lambda_Q = \langle e_S e_T | S, T \in \Pi_A, S \to T \rangle_F. \tag{7.1} \]

Using the triangularity property (5.4), it is a routine matter now to obtain a description of \( \text{rad}_A^{(k)} \Lambda_Q \) (and thus of the \( k \)th Loewy layer of \( \Lambda_Q \)), by induction.

**Theorem 7.2.** Let \( Q \in \Pi_A^< \) and \( k \in \mathbb{N}_0 \), then
\[ \text{rad}_A^{(k)} \Lambda_Q = \sum_{Q \leq R \in \Pi_A, \ell(Q) - \ell(R) \geq k} e_R \Lambda_Q \oplus \bigoplus_{Q \leq T \in \Pi_A^<, \ell(Q) - \ell(T) = k} e_T \Lambda_Q. \]

**Proof.** For \( k = 0 \), this follows from Theorem 5.4. Let \( k > 0 \) and choose \( Q' \approx Q \). Suppose \( R \in \Pi_A \) such that \( Q \preceq R \) and \( \ell(Q) - \ell(R) \geq k \). Then \( e_R e_Q' \in \text{rad}_A \Lambda_Q \), in fact, \( e_R e_Q' \in e_R \text{rad}_A \Lambda_Q \). Hence
\[ e_R e_Q' \in \langle (e_R e_P)(e_{PEQ}) | R \geq P \to Q \rangle_F, \]
by Lemma 7.1 and (5.4)
\[ \subseteq (\text{rad}_A^{(k-1)})(\text{rad}_A \Lambda_Q), \]
by induction
\[ = \text{rad}_A^{(k)} \Lambda_Q. \]

Conversely,
\[ \text{rad}_A^{(k)} \Lambda_Q = (\text{rad}_A \text{rad}_A^{(k-1)} \Lambda_Q) \]
\[ \subseteq \sum_{S \to T \geq R \geq Q, \ell(Q) - \ell(R) \geq k-1} (e_S e_T)(e_R \Lambda_Q) \subseteq \sum_{Q \leq S \in \Pi_A, \ell(Q) - \ell(S) \geq k} e_S \Lambda_Q, \]
by (5.4), (7.1) and induction. This proves the first equality and implies the inclusion from right to left in the second equality.

To prove the remaining part of the second equality, choose \( R \in \Pi_A \) such that \( Q \preceq R \) and \( \ell(Q) - \ell(R) \geq k \), then by Corollary 6.3 and (5.4),
\[ e_R \Lambda_Q = \left( \sum_{T \in \Pi_A^<} e_T \right) e_R \Lambda_Q = \sum_{R \leq T \in \Pi_A^<} e_T e_R \Lambda_Q \subseteq e_T \Lambda_Q + \sum_{Q \leq T \in \Pi_A^<, \ell(Q) - \ell(T) \geq k+1} e_T \Lambda_Q, \]
by (5.4), (7.1) and induction. This proves the first equality and implies the inclusion from right to left in the second equality.
where $T'$ is the set composition in $\Pi_A^<$ with $T' \approx R$. Finally, the sum

$$\text{rad}_A^{(k+1)} \Lambda_Q + \sum_{\substack{Q \leq T \in \Pi_A^< \\ell(Q) - \ell(T) = k}} e_T \Lambda_Q$$

is direct, by (5.4), Corollary 6.3 and the description of $\text{rad}_A^{(k+1)} \Lambda_Q$ given by the first equality.

From the second description of $\text{rad}_A^{(k)} \Lambda_Q$ above, it follows that

$$e_T \text{rad}_A^{(k)} \Lambda_Q \not\equiv 0 \pmod{\text{rad}_A^{(k+1)} \Lambda_Q}$$

if and only if $Q \preceq T$ and $\ell(Q) - \ell(T) = k$, for all $Q, T \in \Pi_A^<$. This gives:

**Corollary 7.3.** Let $Q, T \in \Pi_A^<$, then $M_T$ occurs as a direct summand in the $k$th Loewy layer of $\Lambda_Q$ if and only if $Q \preceq T$ and $\ell(Q) - \ell(T) = k$.

Thus the occurrence of $M_T$ in $\Lambda_Q$ is restricted to a single Loewy layer (if it occurs at all, that is, if $Q \preceq T$), with multiplicity $C_{T,Q}$. For example, the Loewy structure of $\Lambda_{(1,2,3)}$ may be obtained by inspection of the last column of Table 3. It is displayed in Figure 1.

$$\begin{align*}
M_{(1,2,3)} \\
\Lambda_{(1,2,3)} &= M_{(12,3)} M_{(13,2)} M_{(1,23)} \\
&\quad M_{(123)} M_{(123)}
\end{align*}$$

**Figure 1:** Loewy structure of $\Lambda_{(1,2,3)}$

As another consequence, we can determine the nilindex of $\text{rad} F \Pi_A$, that is, the smallest $k$ such that $\text{rad}^{(k)} F \Pi_A = 0$. 32
Corollary 7.4. The nilindex of $\text{rad} \, F\Pi_A$ is equal to $|A|$. More precisely, if $A = \{a_1, \ldots, a_n\}$ is of order $n$ and $a_1 < \ldots < a_n$, then

$$\text{rad}^{(n-1)} F\Pi_A$$

$$= \ e_A \wedge F\Pi_A \wedge e_{(a_1,a_2,\ldots,a_n)} = \langle e_{(x_1,\ldots,x_n)} \mid (x_1,\ldots,x_n) \in \Pi_A, x_n = a_1 \rangle_F$$

has dimension $(n-1)!$, while $\text{rad}^{(n)} F\Pi_A = 0$.

Proof. If $P, Q \in \Pi_A^{<}$, then $Q \preceq P$ implies $0 \leq \ell(Q) - \ell(P) \leq n - 1$, with equality on the right if and only if $Q = (a_1,a_2,\ldots,a_n)$ and $P = (A)$. Hence the claims follow from Theorems 7.2 and 6.4. □

Let $\Pi^\dagger = \bigcup_{A \in \text{Fin}} \Pi^\dagger_A$. To conclude this section, we construct a linear basis of $F\Pi_A$ which is well adapted to the descending Loewy series of $F\Pi_A$:

Corollary 7.5. The elements $e_{Q(1)^o} \vee \cdots \vee e_{Q(m)^o}$, where

(a) $m \in \mathbb{N}_0$;

(b) $Q(i) \in \Pi^\dagger$ for all $i \in [m]$;

(c) $Q = Q(1)^o \vee \cdots \vee Q(m)^o \in \Pi_A$;

(d) $\left( \bigcup Q(1), \ldots, \bigcup Q(l) \right) \in \Pi_A^{<}$;

form a linear basis of $F\Pi_A$. If $k \in \mathbb{N}_0$, then those amongst these basis elements for which

(e) $\ell(Q) = m + k$,

span a linear complement of $\text{rad}^{(k+1)} F\Pi_A$ in $\text{rad}^{(k)} F\Pi_A$. 

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Proof. For $k \in \mathbb{N}_0$, we have

$$\text{rad}^{(k)} F \Pi_A = \bigoplus_{Q \in \Pi^*_A} \text{rad}^{(k)} F \Pi_A \Lambda_Q = \bigoplus_{Q \in \Pi^*_A} \bigoplus_{Q \leq T \in \Pi^*_A} \bigoplus_{\ell(Q) - \ell(T) = k} e_T \wedge \Lambda_Q$$

by Theorem 5.4 and the second equality in Theorem 7.2. Applying Theorem 6.4 to each of the spaces $e_T \wedge \Lambda_Q$ on the right, shows that those elements satisfying all five conditions (a)–(e) are linearly independent and span a linear complement of $\text{rad}^{(k+1)} F \Pi_A$ in $\text{rad}^{(k)} F \Pi_A$, as asserted.

Remark 7.6. Corollary 7.5 has a transparent explanation in terms of a Poincaré-Birkhoff-Witt theorem for $(F \Pi, \vee, \Delta)$, viewed as a free $\mathbb{F}$-graded bialgebra, with one generator in each degree.

Indeed, the primitive Lie algebra of $F \Pi$ is generated by the elements $e_{Q^0}$, $Q \in \Pi^1$, by Corollary 4.6, Theorem 6.2, and Proposition 6.1. (These are the basis elements for $m = 1$ above.) Corollary 7.5 says that a basis of $F \Pi$ is obtained by taking (non-vanishing) increasing concatenation products of these basis elements of $\text{Prim} F \Pi$, with respect to a certain order on the basis. Products of basis elements $b_1 \in F \Pi_A \cap \text{Prim} F \Pi$, $b_2 \in F \Pi_B \cap \text{Prim} F \Pi$ are zero unless $A$ and $B$ are disjoint. It is therefore enough to consider this case and to define $b_1 < b_2$ if $\min A < \min B$. This imposes condition (d) above.

From this point of view, the parameter $m$ given in (a) is the number of Lie factors in the PBW basis element, condition (c) picks out basis elements of total degree $A$, and condition (b) ensures that basis elements of $\text{Prim} F \Pi$ occur as factors only.

It seems quite remarkable that there is such a simple connection between the PBW structure of the external algebra $(F \Pi, \vee)$ and the module structure of the internal algebra $(F \Pi, \wedge)$. The same phenomenon for the Solomon algebra was discovered by Blessenohl and Laue [BL02, Theorem 2.1]. It is not hard to derive from the above the following analogue of their result:
For each $k \in \mathbb{N}_0$, we have $\text{rad}^{(k)}(F\Pi, \wedge) = \gamma^{(k)}(F\Pi, \vee)$, where $\gamma^{(k)}(F\Pi, \vee)$ is the $k$th component of the descending central series of $(F\Pi, \vee)$.

(For details on the descending central series, see [BL02, Section 2].)

8 Ext-quiver

Let $A$ be a finite subset of $\mathbb{N}$, and recall that we write $P \to Q$ if $P \preceq Q$ and $\ell(Q) - \ell(P) = 1$, that is, if $P$ is obtained by assembling two components of $Q$ (and keeping the remaining ones, in some order). Bearing in mind the considerations at the end of Section 2, we see that $(\Pi^<_A, \to)$ is isomorphic to the Hasse diagram of the support lattice of $(\Pi_A, \wedge_A)$. As a consequence, the Ext-quiver of $F\Pi_A$ (which encodes the occurrence of an irreducible module $M_P$ in the first Loewy layer of a principal indecomposable module $\Lambda_Q$, see [Ben98, 4.1.6]) has the following simple description.

**Theorem 8.1.** The Ext-quiver of $F\Pi_A$ is given by the Hasse diagram of the support lattice of $(\Pi_A, \wedge_A)$, that is, it is isomorphic to $(\Pi^<_A, \to)$.

**Proof.** For all $P, Q \in \Pi^<_A$, the irreducible module $M_P$ occurs in the first Loewy layer of $\Lambda_Q$ if and only if $P \to Q$, by Corollary 7.3, with multiplicity $C_{P,Q} = 1$, by Theorem 6.4. \qed

The Ext-quiver of $F\Pi_3$ is displayed in Figure 2.

Theorem 8.1 suggests that there might be a link between the Ext-quiver of $FS$ and the support lattice of $S$ for a larger class of idempotent semigroups $S$; for example, for those semigroups associated to arbitrary Coxeter complexes. In fact, Theorem 8.1 remains true when dihedral groups (instead of symmetric groups) are considered.
9 The module structure of the Solomon algebra

Let \( n \in \mathbb{N} \). The Solomon algebra \( D_n \) is isomorphic to the ring of \( S_n \)-invariants, \( B_n \), of the Solomon-Tits algebra \( Z\Pi_n \), as explained in Section 3. In this section, we use this isomorphism to relate the module structure of \( F\Pi_n \) to the module structure of \( D_{n,F} \). Here we assume that \( F \) is a field of characteristic zero. (The modular case is not yet understood.) We will drop the index \( F \) in our notations and write, for instance, \( D_n \) instead of \( D_{n,F} \) in what follows.

9.1 Lie idempotents

We start with a brief analysis of \( \Delta \)-primitive idempotents in the Solomon algebra.

For each finite subset \( A \) of \( \mathbb{N} \), let \( S_A \) denote the symmetric group on \( A \). If \( A \) has order \( n \), then \( S_A \) is isomorphic to \( S_n \), and everything we have said in Section 3 remains true when \( A \) instead of \( [n] \) is considered. In particular, the algebra of \( S_A \)-invariants \( B_A \) in \( F\Pi_A \) is a subalgebra of \( F\Pi_A \) and isomorphic
to the Solomon algebra $\mathcal{D}_A$ of $S_A$. The linear map

$$ f \mapsto \mathcal{J} = \sum_{\pi \in S_A} f^\pi $$

is (up to the factor $\frac{1}{n!}$) a projection from $F\Pi_A$ onto $\mathcal{B}_A$, since $F$ has characteristic zero.

If $\pi \in S_A$ and $X \subseteq A$, then the natural action of $\pi$ on the subsets of $A$ gives rise to a linear map $F\Pi_X \to F\Pi_{X\pi}$, defined by (1.1), which we also denote by $\pi$. It is easy to see that

$$ \Delta(f^\pi) = \Delta(f)^{\pi \otimes \pi} \quad (9.1) $$

for all $f \in F\Pi_A$. For, it suffices to consider $f = P \in \Pi_A$, by linearity. We have $(P^\pi)|_X = (P|_{X^{\pi^{-1}}})^\pi$ for all $X \subseteq A$, hence

$$ \Delta(P^\pi) = \sum_{X \subseteq A} \left( P|_{X^{\pi^{-1}}} \otimes P|_{A \setminus X^{\pi^{-1}}} \right)^{\pi \otimes \pi} = \Delta(P)^{\pi \otimes \pi}. $$

Combining (9.1) with Corollary 4.4, we obtain the following result.

**Proposition 9.1.** Suppose $A \in \text{Fin}$ has order $n$. If $E$ is a $\Delta$-primitive idempotent in $F\Pi_A$, then $\frac{1}{n!}E$ is a $\Delta$-primitive idempotent in $\mathcal{B}_A$.

The Solomon algebra is intimately linked to the free Lie algebra, as was discovered by Garsia and Reutenauer [GR89]. The $\Delta$-primitive idempotents in $\mathcal{B}_n$ (or, via Theorem 3.1 in $\mathcal{D}_n$) are easily identified as the Lie idempotents in $\mathcal{B}_n$. To recall their definition, consider the free associative algebra $A(X) = \bigoplus_{n \geq 0} A_n(X)$ over an infinite set $X$. The $n$th homogeneous component, $A_n(X)$, of $A(X)$ is an $FS_n$-module, via Polya action:

$$ \pi x_1 \cdots x_n := x_{1\pi} \cdots x_{n\pi}, $$

for all $\pi \in S_n$, $x_1, \ldots, x_n \in X$. The Lie commutator $a \circ b = ab - ba$ defines the structure of a Lie algebra on $A(X)$. Let $L(X) = \bigoplus_{n \geq 1} L_n(X)$ denote the
Lie subalgebra of $A(X)$ generated by $X$. Then $L(X)$ is a free Lie algebra, freely generated by $X$. The (right-normed) Dynkin operator $\omega_n \in FS_n$ can be defined by $\omega_n(x_1 \cdots x_n) = x_1 \circ (x_2 \circ (x_3 \circ \cdots (x_{n-1} \circ x_n) \cdots))$ for all $x_1, \ldots, x_n \in X$. The Dynkin-Specht-Wever theorem [Dyn47, Spe48, Wev49] says that $\omega_n^2 = n\omega_n$ or, equivalently, that left action of $\frac{1}{n}\omega_n$ yields a projection from $A_n(X)$ onto $L_n(X)$. Any such idempotent $e$ in $FS_n$ is a Lie idempotent. Equivalently, $e \in FS_n$ is a Lie idempotent if and only if $e\omega_n = \omega_n$ and $\omega_ne = ne$.

If $q = (q_1, \ldots, q_k)$ is a composition of $n$, let $\ell(q) = k$ and $q^* := q_1$. It is a simple, yet striking observation that

$$\omega_n = \sum_{q|n}(-1)^{\ell(q)-1}q^*\Xi^q,$$

hence that $\omega_n \in D_n$ (see [BL96, Proposition 1.2] for the left-normed version of this identity). The corresponding element in $B_n$ is

$$\Omega_n = \sum_{q|n}(-1)^{\ell(q)-1}q^*X^q.$$

Accordingly, we shall refer to elements $E$ in $B_n$ such that $\Omega_n \wedge E = nE$ and $E \wedge \Omega_n = \Omega_n$ as Lie idempotents in $B_n$. From the definition of the idempotent $e_{[n]}$ given in Lemma 4.5, it is readily seen that

$$\Omega_n = \frac{1}{(n-1)!} e_{[n]}.$$

Hence the idempotent $e_{[n]}$ we used for the study of the Solomon-Tits algebra is a refinement of the classical Dynkin operator, and Lemma 4.5 is a refinement of the Dynkin-Specht-Wever theorem, by Proposition 9.1. These refined idempotents are defined over $\mathbb{Z}$ (unlike Lie idempotents). This allowed us to study the Solomon-Tits algebra over a field of arbitrary characteristic.

**Proposition 9.2.** The $\Delta$-primitive idempotents in $B_n$ are precisely the Lie idempotents in $B_n$. 

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This is closely related to [KLT97, Theorem 3.1]. However, some care must be taken when linking the coproduct on the direct sum $\bigoplus_n D_n$ considered in [KLT97] to the coproduct $\Delta$ (for more details, see [PS, Lemma 30]).

**Proof.** Let $e$ be a $\Delta$-primitive idempotent in $B_n$. From Corollary 4.4 it follows that $X^q \wedge e = \sum_{\text{type}(Q)=q} Q \wedge e = 0$ whenever $q \mid n$ with $\ell(q) > 1$, hence that $\Omega_n \wedge e = ne$. Similarly, we get $e \wedge \Omega_n = \Omega_n$, since $\frac{1}{n} \Omega_n$ is also a $\Delta$-primitive idempotent in $B_n$ by Proposition 9.1. Hence $e$ is a Lie idempotent in $B_n$.

Conversely, if $e$ is a Lie idempotent in $B_n$, then $e = \frac{1}{n} \Omega_n \wedge e$. Hence $e$ is $\Delta$-primitive, by Propositions 4.2 and 9.1

### 9.2 Principal indecomposable modules

Let $n \in \mathbb{N}$, and let $\Omega_A := \frac{1}{|A|!} e_A$ denote the Dynkin operator in $B_A$ (up to the factor $|A|$), for all $A \in \text{Fin}$. Each $\Omega_A$ is a $\Delta$-primitive idempotent in $B_A$ by Proposition 9.1. The corresponding linear basis $\{ \Omega_Q \mid Q \in \Pi_n \}$ of $F\Pi_n$ consists of primitive idempotents such that, additionally,

$$\Omega_Q^\pi = \Omega_Q$$

for all $Q \in \Pi_n$, $\pi \in S_n$.

Suppose $Q \in \Pi_n$ has type $q = (q_1, \ldots, q_k)$, then the order of the stabiliser of $Q$ in $S_n$ is $s_q = q_1! \cdots q_k!$, and

$$\bar{\Omega}_Q = \sum_{\pi \in S_n} \Omega_Q^\pi = s_q \sum_{\text{type}(Q')=q} \Omega_Q$$

depends on $q$ only. We set $f_q := \bar{\Omega}_Q$. Then the principal left ideal

$$\Lambda_q := F\Pi_n \wedge f_q$$
of $F\Pi_n$ is $S_n$-invariant and contains $\Omega_P$ whenever $\text{type}(P) \approx q$, since
\begin{equation}
\Omega_P \wedge f_q = s_q \sum_{\text{type}(Q') = q} \Omega_P \wedge \Omega_{Q'} = s_q c_q \Omega_P \tag{9.4}
\end{equation}
in this case, by (5.3). Here $c_q$ is the coefficient defined in the proof of Corollary 3.2. Combined with (9.3) and Theorem 5.4 this implies that
\[ \Lambda_q = \bigoplus_{Q \in \Pi_n^< \atop \text{type}(Q) \approx q} \Lambda_Q = \langle \Omega_P \mid \text{type}(P) \approx q \rangle_F, \]
where $\Lambda_Q = F\Pi_n \wedge \Omega_Q$ for all $Q \in \Pi_n^<$. Hence there is the $S_n$-invariant decomposition
\[ F\Pi_n = \bigoplus_{p \vdash n} \Lambda_p \]
of $F\Pi_n$ into left ideals by Theorem 5.4. These two formulae combined with (9.2), (9.4) and Corollary 3.2 allow us to recover the following result of Garsia and Reutenauer [GR89] and Blessenohl and Laue [BL96].

**Theorem 9.3.** Let $n \in \mathbb{N}$, then
\[ B_n = \bigoplus_{p \vdash n} \overline{\Lambda}_p \]
is a decomposition of $B_n$ into indecomposable left ideals. For each $p \vdash n$, $\overline{\Lambda}_p$ has $F$-basis $\{ f_q \mid q \approx p \}$ consisting (up to non-zero scalar factors) of primitive idempotents, and $\text{rad}_{B_n} \overline{\Lambda}_p$ has $F$-basis $\{ f_p - f_q \mid q \approx p, q \neq p \}$.

Furthermore, the modules $M_p = \overline{\Lambda}_p / \text{rad}_{B_n} \overline{\Lambda}_p$, $p \vdash n$, have dimension one and form a complete set of irreducible $B_n$-modules.

For later use, note that if $Q = (Q_1, \ldots, Q_k) \in \Pi_n$ has type $q$ and $S^Q$ is a right transversal of the stabiliser of $Q$ in $S_n$, then
\begin{equation}
\overline{e}_Q = \sum_{\sigma \in S^Q} \left( \overline{e}_{Q_1} \vee \cdots \vee \overline{e}_{Q_k} \right)^\sigma = s_q \sum_{\sigma \in S^Q} \Omega_Q^\sigma = \overline{\Omega}_Q = f_q. \tag{9.5}
\end{equation}
9.3 The Solomon-Tits algebra as a module for the Solomon algebra

Let \( n \in \mathbb{N} \). It is natural to consider \( F\Pi_n \) as a module for the subalgebra \( \mathcal{B}_n \).

We will study the action of \( \mathcal{B}_n \) on the basis \( \{ e_Q \mid Q \in \Pi_n \} \) and the indecomposable modules \( \Lambda_Q = F\Pi_n \wedge e_Q \) of \( F\Pi_n \).

**Proposition 9.4.** Let \( Q, R \in \Pi_n^\leq \), then \( \Lambda_Q \) and \( \Lambda_R \) are isomorphic as \( \mathcal{B}_n \)-modules if and only if \( \text{type}(Q) \approx \text{type}(R) \). Furthermore, \( M_Q \) is an irreducible \( \mathcal{B}_n \)-module isomorphic to \( M_p \), where \( p \vdash n \) such that \( \text{type}(Q) \approx p \).

**Proof.** It is more convenient here to consider the Dynkin operator \( \Omega_A \) instead of \( e_A \) for all \( A \in \text{Fin} \). We may do so by Remark 5.6.

If \( \text{type}(Q) \approx \text{type}(R) \), then there exist \( \tilde{Q} \approx Q \) and \( \pi \in S_n \) such that \( \tilde{Q}^\pi = R \).

The mapping \( \Lambda_Q \to \Lambda_R \) which sends \( f \mapsto f^\pi \) is then an isomorphism of \( \mathcal{B}_n \)-modules, by (9.2). Conversely, if \( \Lambda_Q \) and \( \Lambda_R \) are isomorphic as \( \mathcal{B}_n \)-modules, then \( f_q \wedge \Lambda_Q \neq 0 \) implies \( f_q \wedge \Lambda_R \neq 0 \), hence \( \tilde{R} \preceq Q^\pi \) for some \( \tilde{R} \approx R \), \( \pi \in S_n \), by (9.3) and (5.4). Interchanging the roles of \( Q \) and \( R \), there exist \( \tilde{Q} \approx Q \) and \( \sigma \in S_n \) such that \( \tilde{Q} \preceq R^\sigma \). This implies \( \text{type}(Q) \approx \text{type}(R) \), completing the proof of the first claim.

The second claim also follows from (9.3) and (5.4), since

\[
f_p \wedge \Omega_Q = s_p \sum_{\text{type}(P)=p} \Omega_P \wedge \Omega_Q = s_p \sum_{\text{type}(P)=p} \Omega_P \notin \text{rad}F\Pi_n \Lambda_Q.
\]

Let \( q \mid n \) and \( Q = (Q_1, \ldots, Q_k) \in \Pi_n \) be of type \( q \), then \( \pi \in S_n \) is a block permutation of \( Q \) if \( \pi \) performs a blockwise permutation of the components (blocks) of \( Q \), that is, more precisely, if \( Q^\pi \approx Q \) and \( \pi|_{Q_i} \) is non-decreasing for all \( i \in [k] \). For instance, \( \pi = 216543 \in S_6 \) is a block permutation of \( Q = (13, 5, 4, 26) \), while \( \sigma = 612543 \in S_6 \) is not a block permutation of...
Q (although $Q^\pi = (26,4,5,13) = Q^\sigma$). Let $G_Q$ denote the subgroup of $S_n$ consisting of all block permutations of $Q$ and set $\alpha_Q = 1/|G_Q| \sum_{\pi \in G_Q} \pi$.

**Lemma 9.5.** Let $Q, \tilde{Q}, Q' \in \Pi_n$ such that $Q \approx \tilde{Q} \approx Q'$. Then, for all $\pi \in G_Q$, we have $e_{\tilde{Q}}^\pi = e_{Q'}^\pi$. Furthermore, $e_Q^{\alpha_Q} = e_{Q'}^{\alpha_Q}$ if and only if $\text{type}(\tilde{Q}) = \text{type}(Q')$.

**Proof.** If $A, B \in \text{Fin}$ and $\tau : A \rightarrow B$ is an order-preserving bijection, then $e_A^\tau = e_B$. Hence

$$e_{\tilde{Q}}^\pi = e_{\tilde{Q}_1}^\pi \lor \cdots \lor e_{\tilde{Q}_k}^\pi = e_{Q_1}^\pi \lor \cdots \lor e_{Q_k}^\pi = e_{Q'}^\pi$$

for all $\pi \in G_Q$. Furthermore, $\text{type}(\tilde{Q}) = \text{type}(Q')$ if and only if there exists a block permutation $\pi \in G_Q$ such that $\tilde{Q}^\pi = Q'$. By the above, this is equivalent to $e_Q^{\alpha_Q} = e_{Q'}^{\alpha_Q}$. \qed

As a consequence, the indecomposable module $\Lambda_Q = \langle e_{\tilde{Q}} \mid \tilde{Q} \approx Q \rangle_F$ is invariant under the action of $G_Q$ and therefore a $(\mathcal{B}_n, F G_Q)$-bimodule.

**Proposition 9.6.** Suppose $Q \in \Pi_n$ is of type $q$ and $p \mid n$ such that $q \approx p$, then $\Lambda_Q = \Lambda_Q^{\alpha_Q} \oplus \Lambda_Q^{\text{id}_n - \alpha_Q}$ is a decomposition into $\mathcal{B}_n$-submodules, and $\Lambda_Q^{\alpha_Q}$ is isomorphic to the indecomposable $\mathcal{B}_n$-module $\overline{\lambda}_p$.

Here $\text{id}_n$ denotes the identity in $S_n$.

**Proof.** The first part is obvious. For the proof of the second part, choose a right transversal $G_Q^Q$ of $G_Q$ in $S_n$. Then the map $\varphi : \Lambda_Q^{\alpha_Q} \rightarrow \overline{\lambda}_p$ which sends $g \mapsto \sum_{\pi \in G_Q} g^\pi$ is an isomorphism of $\mathcal{B}_n$-modules, since $e_{\tilde{Q}}^{\alpha_Q} \varphi = 1/|G_Q| f_{\tilde{q}}$ for all $\tilde{Q} \approx Q$ of type $\tilde{q}$, by (9.5). \qed

There is a surprising link here to a fundamental result of Garsia and Reutenauer [GR89, Theorem 4.5] on the Solomon algebra. This result states that
an element \( \gamma \in FS_n \) lies in the Solomon algebra \( D_n \) if and only if, via Polya action,

\[
\gamma l_1 \cdots l_k \in \langle l_{1\pi} \cdots l_{k\pi} \mid \pi \in S_k \rangle_F,
\]

for all \( k \)-tuples of homogeneous Lie elements \( l = (l_1,\ldots,l_k) \) such that \( l_1 \cdots l_k \in A_n(X) \). As a consequence, there is an action of \( D_n \) on the linear span \( U_l \) of the elements \( l_{1\pi} \cdots l_{k\pi} \) \( (\pi \in S_k) \). The modules \( U_l \) are characteristic for the Solomon algebra in the sense that \( D_n \) is the common stabiliser in \( FS_n \) of these linear spaces with respect to Polya action. In fact, it suffices to consider \emph{generic} tuples \( l = (l_1,\ldots,l_k) \) of Lie monomials only for which the elements \( l_{1\pi} \cdots l_{k\pi} \) \( (\pi \in S_k) \) are linearly independent. The \emph{type} of \( l \) is the composition \( q = (q_1,\ldots,q_k) \) of \( n \) such that \( l_i \in L_{q_i}(X) \) for all \( i \in [k] \).

The \( D_n \)-modules \( U_l \) may, of course, be viewed as modules for \( B_n \).

**Theorem 9.7.** Let \( n \in \mathbb{N} \) and \( q \models n \). Suppose that \( l = (l_1,\ldots,l_k) \) is a generic \( k \)-tuple of homogeneous Lie monomials, of type \( q \). Then \( U_l \cong \Lambda_Q \) as a \( B_n \)-module, for any \( Q \in \Pi_n \) of type \( q \).

\textit{Proof.} If \( I = \{i_1,\ldots,i_m\} \subseteq [k] \) such that \( i_1 < \cdots < i_m \), put \( q_I := (q_{i_1},\ldots,q_{i_k}) \) and \( l_I := l_{i_1} \cdots l_{i_k} \). Let \( r = (r_1,\ldots,r_m) \models n \). Then, on the one hand,

\[
\Xi^r l_1 \cdots l_k = \sum_{(I_1,\ldots,I_m) \in \Pi_k} l_{I_1} \cdots l_{I_m},
\]

by [GR89, Theorem 2.1]. If \( Q = (Q_1,\ldots,Q_k) \in \Pi_n \) has type \( q \), then, on the other hand, there is 1-1 correspondence between the set compositions \( I = (I_1,\ldots,I_m) \in \Pi_k \) such that \( q_{I_j} \models r_j \) and the set compositions \( R \in \Pi_n \) of type \( r \) such that \( Q \preceq R \), namely the map

\[
I \mapsto R_I := \bigcup_{i \in I_1} Q_i \cup \bigcup_{i \in I_2} Q_i \cup \cdots \cup \bigcup_{i \in I_m} Q_i.
\]
already considered in Remark 6.5 (2). It follows from (5.2) that

\[ X^r \land e_Q = \sum_{I = (I_1, \ldots, I_m) \in \Pi_k} e_{R_I \land Q} = \sum_{(I_1, \ldots, I_m) \in \Pi_k} e_{Q_{I_1} \lor \cdots \lor Q_{I_m}}. \]

Thus the map \( e_{(Q_1, \ldots, Q_k)} \mapsto l_1 \cdot \cdots \cdot l_k \) defines an isomorphism of \( B_n \)-modules from \( \Lambda_Q \) onto \( U_l \). \( \square \)

Combining Theorem 9.7 with Proposition 9.6, we see that the projective indecomposable \( D_n \)-module \( \Lambda_p \) is isomorphic to a direct summand of \( U_l \) whenever \( l \) is of type \( p \). This result is due to Mielck [Mie96]. Furthermore, the following description of the decomposition numbers of the characteristic \( D_n \)-modules \( U_l \) is now immediate from Theorem 9.7 and Proposition 9.4.

**Corollary 9.8.** Let \( n \in \mathbb{N} \) and \( q \vdash n \), and suppose \( Q \in \Pi_n^< \) has type \( q \). Then, for any generic tuple \( l \) of homogeneous Lie monomials of type \( q \), the multiplicity of \( M_p, p \vdash n \), in a composition series of \( U_l \) is equal to the sum of all Cartan invariants \( C_{P,Q} \) of \( F\Pi_n \), taken over set compositions \( P \in \Pi_n^< \) with \( \text{type}(P) \approx p \).

A better understanding of the \( D_n \)-modules \( U_l \) would be very interesting. This includes as a (crucial) special case the study of \( FS_n \) as \( D_n \)-module (when \( l \) is of type \((1,1,\ldots,1))\).

### 9.4 Cartan invariants

The most efficient way to recover the well-known description of the Cartan invariants of \( D_n \) (see [GR89], [BL96, Corollary 2.1], [KLT97, Section 3.6]) seems to build on Proposition 9.6 as follows.

Let \( c_{r,q} \) denote the multiplicity of \( M_r \) in a composition series of \( \Lambda_Q \), for all \( q, r \vdash n \). Suppose \( Q \in \Pi_n^< \) has type \( \approx q \). Then, for \( \Lambda_Q = F\Pi_n \land e_Q \), we
obtain
\[ cr_q = \dim f_r \land \Lambda_q = \dim (f_r \land \Lambda_q)^{\alpha_q} = \dim (f_r \land X)^{\alpha_q} = \dim X^{\alpha_q}. \]

We will need the following auxiliary result.

**Lemma 9.9.** Let \( r \vdash n, Q \in \Pi_n^{<} \), and set
\[ X := \bigoplus_{Q \leq T \in \Pi_n^{<} \atop \text{type}(T) \approx r} e_T \land \Lambda_Q. \]

Then \( f_r \land \Lambda_Q = f_r \land X \) and \( \dim f_r \land X = \dim X \).

**Proof.** Set \( k := \ell(Q) - \ell(r) \). If \( k < 0 \), then \( f_r \land \Lambda_Q = 0 \) as is readily seen from (5.1), (5.2) and (9.3), while \( X = 0 \) is clear.

Suppose \( k \geq 0 \), then any occurrence of \( M_R \) in \( \Lambda_Q \), for \( R \) of type \( \approx r \), is restricted to the \( k \)th Loewy layer of \( \Lambda_Q \) (if it occurs at all), by Corollary 7.3.

Hence
\[ f_r \land \Lambda_Q / \text{rad}^{(k)}_{\Pi_n} \Lambda_Q = 0 = f_r \land \text{rad}^{(k+1)}_{\Pi_n} \Lambda_Q, \]
by Proposition 9.4 and (9.3). Now Theorem 7.2 implies
\[ f_r \land \Lambda_Q \subseteq f_r \land \bigoplus_{Q \leq T \in \Pi_n^{<} \atop \ell(Q) - \ell(T) = k} e_T \land \Lambda_Q. \]

Furthermore, if \( T \in \Pi_n^{<} \) such that \( \ell(Q) - \ell(T) = k \) and \( \text{type}(T) \not\approx r \), then \( f_r \land e_T = 0 \). It follows that \( f_r \land \Lambda_Q \subseteq f_r \land X \), the other inclusion is clear.

The dimensions of \( f_r \land \Lambda_Q = f_r \land X \) and of \( X \) both describe the multiplicity of \( M_r \) in a \( B_n \)-composition series of \( \Lambda_Q \), by Proposition 9.4. \( \square \)

As a consequence of the preceding result, we have
\[ cr_q = \dim (f_r \land \Lambda_Q)^{\alpha_q} = \dim (f_r \land X)^{\alpha_q} = \dim X^{\alpha_q}. \]
Put $l := \ell(r)$. Then $X^{\alpha_Q}$ has set of linear generators $(e_{Q(1)} \lor \cdots \lor e_{Q(l)})^{\alpha_Q}$ where $Q(i) \in \Pi$ for all $i \in [l]$ such that $Q(1) \lor \cdots \lor Q(l) \approx Q$ and
\[
\left( \bigcup Q(1), \ldots, \bigcup Q(l) \right) \in \Pi_n^<
\] (9.6)
has type $\approx r$. Using a triangularity argument, we can drop condition (9.6) and obtain the set of generators
\[
\sum_{\pi \in S_l} (e_{Q(1)\pi} \lor \cdots \lor e_{Q(l)\pi})^{\alpha_Q}.
\]
From Lemma 9.3 we know that the map which sends $e_Q^{\alpha_Q} \mapsto \text{type}(\tilde{Q})$ is a linear isomorphism from $\Lambda_Q^{\alpha_Q}$ onto the $F$-linear span (in the free associative algebra $A(\mathbb{N})$ over $\mathbb{N}$) of $\{ \tilde{q} \mid \tilde{q} \approx q \}$. It maps $X^{\alpha_Q}$ onto the linear span $T_{r,q}$ of the elements
\[
\sum_{\pi \in S_l} q(1^{\pi}) \circ \cdots \circ q(l^{\pi})
\]
where $q(i) \models r_i$ for all $i \in [l]$ such that $q(1) \lor \cdots \lor q(l) \approx q$. Here $r.s$ is the (concatenation) product of the compositions $r$ and $s$ in $A(\mathbb{N})$ and $q \mapsto q^\circ$ is the associated right-normed Dynkin mapping. Thus $c_{r,q} = \dim T_{r,q}$.

Let $B$ be a set of compositions and $\beta : B \to L(\mathbb{N})$ be an injective mapping such that $B\beta$ is a linear basis of $L(\mathbb{N})$. Consider a total order on $B$ such that $q \leq r$ whenever the sum of the components of $q$ is less than the sum of the components of $r$. The corresponding (symmetrised) Poincaré-Birkhoff-Witt basis of $A(\mathbb{N})$ consists of the elements
\[
\sum_{\pi \in S_m} (q(1^{\pi})\beta) \circ \cdots \circ (q(m^{\pi})\beta)
\] (9.7)
where $q(i) \in B$ for all $i \in [m]$ and $q(i) \geq q(i+1)$ for all $i \in [m-1]$. Those basis elements (9.7) for which $m = l$, $q(i) \models r_i$ for all $i \in [l]$ and $q(1) \lor \cdots \lor q(l) \approx q$, form a linear basis of $T_{r,q}$. Hence $c_{r,q}$ is equal to the number of these basis elements, in accordance with the results in [BL02, GR89]. An explicit formula for $c_{r,q}$ follows easily (see [BL02, Corollary 2.1]).
9.5 Descending Loewy series

As a general observation on idempotent semigroups, we know from Corollary 2.2 that \( \text{rad} \mathcal{B}_n = \mathcal{B}_n \cap \text{rad} F\Pi_n \). Comparison of the results of Section 7 with those on the Loewy structure of \( \mathcal{D}_n \) derived in [BL96] yields the following surprising generalisation.

**Theorem 9.10.** \( \text{rad}^{(k)} \mathcal{B}_n = \mathcal{B}_n \cap \text{rad}^{(k)} F\Pi_n \), for all \( k \in \mathbb{N}_0 \).

**Proof.** For \( k \in \mathbb{N}_0 \), \( \text{rad}^{(k)} F\Pi_n \) has set of linear generators

\[ e_{Q(1)^o} \lor \cdots \lor e_{Q(m)^o} \]

by Corollary 7.5 where \( Q(i) \in \Pi \) for all \( i \in [m] \) such that \( Q(1) \lor \cdots \lor Q(m) \in \Pi_n \) and \( \sum_{i=1}^{m} \ell(Q(i)) \geq m + k \). Hence by (9.5) \( \text{rad}^{(k)} F\Pi_n = \mathcal{B}_n \cap \text{rad}^{(k)} F\Pi_n \) has set of linear generators \( f_{q(1)^o} \ldots q(m)^o \), where \( q(i) \) is a composition for all \( i \in [m] \) such that \( q(1) \ldots q(m) \models n \) and \( \sum_{i=1}^{m} \ell(q(i)) \geq m + k \). (The symbol \( f \) is understood to be linear with respect to subscripts.) These elements linearly generate \( \text{rad}^{(k)} \mathcal{B}_n \), by [BL96] Theorem 2.5. \( \square \)

Again it would be interesting to know whether Theorem 9.10 holds for other Coxeter groups as well.

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