REGULARITY OF INFINITESIMAL CR AUTOMORPHISMS

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Abstract. We study the regularity of infinitesimal CR automorphisms of abstract CR structures which possess a certain microlocal extension and show that there are smooth multipliers, completely determined by the CR structure, such that if $X$ is such an infinitesimal CR automorphism, then $\lambda X$ is smooth for all multipliers $\lambda$. As an application, we study the regularity of infinitesimal automorphisms of certain infinite type hypersurfaces in $\mathbb{C}^n$.

1. Introduction and statement of results

Regularity of CR diffeomorphisms has been intensely studied in the real-analytic setting. We recall here the paper of Baouendi, Jacobowitz, and Treves [3] stating that every smooth CR diffeomorphism of an essentially finite real-analytic submanifold $M$ of $\mathbb{C}^N$ extending to a wedge with edge $M$ is actually real-analytic. The smoothness assumptions on the map can be relaxed considerably, and only a certain finite smoothness will suffice in order to guarantee real analyticity of the map.

In the setting where the regularity of the underlying manifold is reduced from real-analytic to smooth, much less is known. Early results concentrated on the setting of strictly pseudoconvex hypersurfaces, following Fefferman’s celebrated mapping theorem [10], as in the paper of Nirenberg-Webster-Yang [20], but have been based on methods which do not carry over to more degenerate situations. The few regularity results we know are regularity theorems for finitely nondegenerate smooth submanifolds of $\mathbb{C}^N$ (see the paper of the second author [18]), and more recently, the work of Berhanu and Ming on the regularity of embeddings [4]; rougher regularity results are also implicit in the construction of a complete system as in the work of Ebenfelt [9].

However, all of these results only apply to integrable smooth CR structures, that is, CR structures which can be realized as smooth submanifolds of some $\mathbb{C}^N$; the recent work of Berhanu and Ming actually does away with the requirement that the source manifold is integrable, but the target manifold still is required to be integrable; their work has actually inspired the research presented here.

In the current paper, we tackle the purely abstract setting. This requires us to part with all techniques relying on the use of CR functions, as our abstract CR structures will in general not have any solutions. However, as they might still possess symmetries, the question of the regularity properties of these symmetries is actually interesting. Our approach to the problem is inspired by the approach of Berhanu and Xiao [4], to which this paper owes a lot.

Before we can state our main theorem, we need some definitions. For definitions and details regarding the notion of abstract CR manifolds and infinitesimal CR automorphisms, see section 2. In what follows, we consider an abstract CR manifold $(M, \mathcal{V})$ with CR bundle $\mathcal{V} \subset \mathbb{C}TM$. We write $\dim_{\mathbb{R}} M = 2n + d$, where $\dim_{\mathbb{C}} \mathcal{V}_p = n$ for $p \in M$, and set $N = n + d$.

**Definition 1.** Let $(M, \mathcal{V})$ be an abstract CR manifold, and $X$ an infinitesimal CR diffeomorphism (with distributional coefficients, see section 2) of $M$. We say that $X$ extends microlocally to a wedge with edge $M$ if there exists a set $\Gamma \subset T^0M$ such that for each $p \in M$, the fiber $\Gamma_p \subset T^0_pM \setminus \{0\}$ is a closed, convex cone, and

$$WF(\omega(X)) \subset \Gamma^0$$

for every holomorphic form $\omega \in \Gamma(M, T'M)$.

Our first result is that there exists an ideal $\mathcal{S} \subset \mathcal{E}(M)$ of smooth functions (determined by the CR structure alone) such that every infinitesimal CR automorphism $X$ of $M$ which extends microlocally to a wedge with edge $M$ has the property that $\lambda X$ is smooth on $M$ for every $\lambda \in \mathcal{S}$.
The ideal \( S \) is constructed in the following manner. Starting with the space \( E_0 = \Gamma(M, T^0M) \) we define an increasing sequence of submodules \( E_k \subset \Gamma(M, T^kM) \) by

\[
E_k = \text{span}_{E(M)} \{ K \mapsto \omega([L, K]) : L \in \Gamma(M, \mathcal{V}), \omega \in E_{k-1} \}, \quad k \geq 1,
\]

\( E = \bigcup_k E_k \).

We then define \( S = \bigwedge N E \) and have the following:

**Theorem 1.** Let \((M, \mathcal{V})\) be an abstract, smooth CR structure, and \(X\) an infinitesimal CR diffeomorphism of \(M\) with distributional coefficients which extends microlocally to a wedge with edge \(M\). Then, for any \(\omega \in E\), the evaluation \(\omega(X)\) is smooth, and for any \(\lambda \in S\), the vector field \(\lambda X\) is smooth.

In analogy to the integrable case, we will say that \(M\) is finitely nondegenerate if \(S = E(M)\). Therefore, we have the following

**Corollary 1.** Let \((M, \mathcal{V})\) be an abstract, smooth, finitely nondegenerate CR structure, and \(X\) a locally integrable infinitesimal CR diffeomorphism of \(M\) with distributional coefficients which extends microlocally to a wedge with edge \(M\). Then \(X\) is smooth.

However, the condition that \(M\) is actually finitely nondegenerate is far too restrictive. We shall say that \((M, \mathcal{V})\) is CR-regular if for every \(p \in M\) there exists a \(\lambda \in S\) with the property that near \(p\), the zero set of \(\lambda\) is a real hypersurface in \(M\), and such that \(\lambda\) does not vanish to infinite order at \(p\).

**Theorem 2.** Let \((M, \mathcal{V})\) be an abstract CR structure, \(p \in M\), and assume that \(M\) is CR regular near \(p\). Then any locally integrable infinitesimal CR diffeomorphism of \(M\) which extends microlocally to a wedge with edge \(M\) is smooth.

Without boundedness conditions on \(X\), this theorem is actually in some sense optimal (even in the real-analytic case), as examples show (see section 7). The preceding theorem also implies a result in the embedded setting for so-called “weakly nondegenerate” hypersurfaces. Weakly nondegenerate hypersurfaces are defined by the requirement that there exist coordinates \((z, w) \in \mathbb{C}^n \times \mathbb{C}\) and a \(k \in \mathbb{N}\) such that \(p = 0\) in these coordinates and that near \(p = 0, M\) is given by an equation of the form

\[
\text{Im} w = (\text{Re} w)^m \varphi(z, \bar{z}, \text{Re} w),
\]

where

\[
\frac{\partial^{|\alpha|} \varphi}{\partial z^\alpha}(0, 0, 0) = \frac{\partial^{|\alpha|} \varphi}{\partial \bar{z}^\alpha}(0, 0, 0) = 0, \quad |\alpha| \leq k,
\]

and

\[
\text{span}_{\mathbb{C}} \{ \varphi_{zz^\alpha}(0, 0, 0) : |\alpha| \leq k \} = \mathbb{C}^n.
\]

If \(k_0\) is the smallest \(k\) for which the preceding condition holds, we say that \(M\) is weakly \(k_0\)-nondegenerate at \(p\).

**Corollary 2.** Let \(M \subset \mathbb{C}^N\) be a smooth hypersurface, \(p \in M\), and assume that \(M\) is weakly \(k\)-nondegenerate at \(p\). Then any locally integrable infinitesimal CR diffeomorphism of \(M\) which extends microlocally to a wedge with edge \(M\) near \(p\) is smooth near \(p\).

The paper is structured as follows: In section 2 we gather the necessary preliminaries concerning infinitesimal CR automorphisms of abstract CR structures. In the following section 3 we collect and prove the results of microlocal analysis which we will need. Section 4 states a (rather simple) division theorem for smooth functions. The following section 5 and section 6 give the proofs of the main results. An example illustrating the role of the multipliers is presented in section 7.

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2. Preliminaries

In this section, we gather basic definitions and properties. More details and proofs of well known results which we do not prove here can be found in e.g. [2].

An abstract CR manifold is a smooth real manifold $M$ together with a formally integrable smooth subbundle $\mathcal{V} \subset \mathbb{C}T^*M$ which satisfies $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. $\mathcal{V}$ is called the CR bundle of $M$ and sections of $\mathcal{V}$ are called CR vector fields. Throughout this paper, we do not assume $M$ to be integrable, i.e. there might be no (or just a few) solutions of the structure or CR functions (functions annihilated by all CR vector fields). $\dim_{\mathbb{C}} \mathcal{V} = n$ is referred to as the CR dimension of $M$ and we will write $\dim_{\mathbb{R}} M = 2n + d$.

A map $H: M \supset U \rightarrow M$ of class $C^1$ is said to be CR (on $U$) if $dH \mathcal{V}_p \subset \mathcal{V}_{H(p)}$ for all $p \in U$. A vector field $X: M \supset U \rightarrow TM$ is an infinitesimal CR automorphism if its local flow $H_\tau$, defined for $\tau \in \mathbb{R}$ small, has the property that for some $\varepsilon > 0$, $H_\tau$ is a CR map if $|\tau| < \varepsilon$.

We will refer to the bundle $T'M := \mathcal{V}^\perp \subset \mathbb{C}T^*M$ as the holomorphic cotangent bundle of $M$, sections of $T'M$ are called holomorphic forms. Its real subbundle $T^0M \subset T'M$, consisting of all real dual vectors that annihilate $\mathcal{V} + \mathcal{V}$, is the characteristic bundle. Sections of $T^0M$ are called characteristic forms.

Let $\mathcal{Y} \in \Gamma(M, (T'M)^*)$. Recall that this is just the dual bundle to the space of holomorphic forms; in analogy to the notion of a holomorphic form, we will refer to such a $\mathcal{Y}$ as a holomorphic vector field (even though it is not holomorphic in the usual sense). Note that $(T'M)^* \subset \mathbb{C}TM/\mathcal{V}$. Every vector field $X \in \Gamma(M, TM)$ gives rise to a holomorphic vector field by restricting $X$ to $T'M$. The following Lemma provides a converse.

**Lemma 1.** Let $\mathcal{Y} \in \Gamma(M, (T'M)^*)$. Then there exists a unique real vector field $X \in \Gamma(M, TM)$ such that $\mathcal{Y}$ is induced by $X$ if and only if
\[
\mathcal{Y}(\tau) = \overline{\mathcal{Y}(\tau)}
\]
for all characteristic forms $\tau \in \Gamma(M, T^0M)$.

Indeed, since $(\mathbb{C}TM)^* = \mathcal{V}^\perp + \mathcal{V}^\perp$ and $\mathbb{C}T^0M = (\mathcal{V} \oplus \mathcal{V})^\perp$, we can decompose any form $\omega = \alpha + \beta$ with $\alpha, \beta$ holomorphic forms in a nonunique manner. Thus $\mathcal{Y}$ gives rise to a real vector field $X$ via
\[
X(\omega) = \frac{1}{2} (\alpha(\mathcal{Y}) + \beta(\mathcal{Y}))
\]
which is well defined provided that $\mathcal{Y}(\tau) = \overline{\mathcal{Y}(\tau)}$ for all $\tau \in \Gamma(M, \mathbb{C}T^0M)$ or equivalently, that $\mathcal{Y}(\tau) = \overline{\mathcal{Y}(\tau)}$ for all $\tau \in \Gamma(M, T^0M)$, both of which are equivalent to the definition of $X$ above being independent of the decomposition $\omega = \alpha + \beta$. We shall not distinguish between $X$ as a real vector field and as an element of $\Gamma(M, (T'M)^*)$.

Using the well known identity, see e.g. [13],
\[
\mathcal{L}_L \omega(K) = d\omega(L, K) + K \omega(L) = L \omega(K) - \omega([L, K]),
\]
valid for arbitrary complex forms $\omega$ and complex vector fields $L, K$, we see that the Lie derivative
\[
\mathcal{L}_L \alpha(K) = d\alpha(L, K)
\]
of a holomorphic form $\alpha$ with respect to a CR vector field $L$ is again a holomorphic form. We say that $\mathcal{Y} \in \Gamma(M, (T'M)^*)$ is CR if
\[
L \alpha(\mathcal{Y}) = d\alpha(L, \mathcal{Y})
\]
for every CR vector field $L$ and every holomorphic form $\alpha$. In particular, if $X$ is a real vector field, then $X$ is CR if and only if $\alpha([L, X]) = 0$ for every CR vector field $L$ and every holomorphic form $\alpha$.

**Proposition 1.** If $X$ is an infinitesimal CR automorphism of $M$, then $X \in \Gamma(M, (T'M)^*)$ is CR.

**Proof.** Let $H_\tau = \text{Fl}_\tau^X$ denote the flow of $X$. By definition, $H_\tau$ satisfies the following differential equation:
\[
\frac{dH_\tau}{d\tau}(p) = X \circ H_\tau(p).
\]
We note that $H_0 = \text{Id}_M$ is trivially a CR map, but by assumption we know that if $\tau$ is small then

$$\omega((H_\tau), L) = 0$$

for any CR vector field $L$ and any holomorphic form $\omega$, i.e., $\omega(L) = 0$.

We begin with the following general claim: For any triple $(Y, B, \alpha)$, where

$$Y = \sum_{j=1}^{m} Y_j \frac{\partial}{\partial x_j} \quad Y_j \in \mathbb{R}$$

$$B = \sum_{j=1}^{m} B_j \frac{\partial}{\partial x_j}$$

$$\alpha = \sum_{j=1}^{m} \alpha^j dx^j$$

are defined near 0 and $\alpha(B) = 0$, we have, if $K_\tau = \text{Fl}_Y^\tau$,

$$\left. \frac{d}{d\tau} (K_\tau^* \alpha(B)) \right|_{\tau=0} = \alpha([B, Y])$$

near the origin. For the convenience of the reader, we shall include the computation below.

Recalling the fact

$$K_\tau^* \alpha(B)(p) = \alpha((K_\tau)_* B)(K_\tau(p)) = \sum_{j=1}^{m} \sum_{k=1}^{m} (\alpha^k \circ K_\tau)(p) B_j(p) \frac{\partial K^k}{\partial x_j}(p)$$

we can compute

$$\frac{d}{d\tau} (K_\tau^* \alpha(B))(p) = \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{d}{d\tau} \left( (\alpha^k \circ K_\tau)(p) \frac{\partial K^k}{\partial x_j}(p) B_j(p) \right)$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{\ell=1}^{m} \left( \frac{\partial \alpha^k}{\partial y_\ell} \circ K_\tau \right)(p) (Y_\ell \circ K_\tau)(p) \frac{\partial K^k}{\partial x_j}(p) B_j(p)$$

$$+ \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{\ell=1}^{m} (\alpha^k \circ K_\tau)(p) \left( \frac{\partial Y_\ell}{\partial y_\ell} \circ K_\tau \right)(p) \frac{\partial K^k}{\partial x_j}(p) B_j(p).$$

This leads immediately to

$$\left. \frac{d}{d\tau} (K_\tau^* \alpha(B)) \right|_{\tau=0} = \sum_{k=1}^{m} \sum_{\ell=1}^{m} \left( \frac{\partial \alpha^k}{\partial x_\ell} Y_\ell B_k + \alpha^k \frac{\partial Y_\ell}{\partial x_\ell} B_k \right)$$

$$= \sum_{k=1}^{m} \sum_{\ell=1}^{m} \left( -\alpha^k Y_\ell \frac{\partial B_k}{\partial x_\ell} + \alpha^k \frac{\partial Y_\ell}{\partial x_\ell} B_k \right)$$

$$= \alpha([B, Y]).$$

Now we set $Y = X$, $B = L$ and $\alpha = \omega$ as above. Then we have

$$0 = \left. \frac{d}{d\tau} (H_\tau^* \omega(L)) \right|_{\tau=0} = \omega([L, X])$$

and hence $X$ is CR.

We can now define what an infinitesimal CR diffeomorphism with distributional coefficients is.

We say that a function $f : M \to \mathbb{C}$ is locally integrable if for any parametrization $\varphi : U \to M$, $f \circ \varphi$ is locally integrable on $M$. If $\text{vol}(M)$ is the (complex) density bundle of $M$ we define

$$\mathcal{D}(M, \text{vol}(M)) := \left\{ \psi \in \Gamma(M, \text{vol}(M)) : \text{supp } \psi \subset M \right\}$$
the space of compactly supported sections of $\text{vol}(M)$ equipped with the usual topology. Its strong dual $D'(M)$ is the space of distributions on $M$, c.f. e.g. [6] or [12]. A function $f : M \rightarrow \mathbb{C}$ is locally integrable if and only if

$$\int_M |f\tau| < \infty$$

for all $\tau \in \mathcal{D}(M, \text{vol}(M))$. Therefore any locally integrable function $f$ can be viewed as a distribution on $M$ in the usual way.

Furthermore we set

$$\mathcal{D}(M, T'M \otimes \text{vol}(M)) = \{\omega \in \Gamma(M, T'M \otimes \text{vol}(M)) : \text{supp} \omega \subset M\}$$

with the usual topology. The strong dual $\mathcal{D}'(M, (T'M)^{*}) := (\mathcal{D}(M, T'M \otimes \text{vol}(M)))'$ is the space of distributions (or generalized sections) on $M$ with values in $(T'M)^{*}$. If $U \subset M$ is an open set where $\omega^{1}, \ldots, \omega^{N} \in \Gamma(U, T'M)$ form a basis, and $\omega_{j} = (\omega^{j})^{*} \in \Gamma(U, (T'M)^{*})$ is the dual basis then an element $\mathfrak{g} \in \mathcal{D}'(M, (T'M)^{*})$, when restricted to $\mathcal{D}(M, (T'M \otimes \text{vol}(M))|_{U})$, is of the form

$$\mathfrak{g}|_{U} = \sum_{j=1}^{N} c_{j}\omega_{j},$$

where $c_{j}$ is a distribution on $U$ for $j = 1, \ldots, N$. (We also assumed that w.l.o.g. $U$ is small enough such that $\text{vol}(M)|_{U} \cong U \times \mathbb{C}$.) We shall say that $\mathfrak{g} \in \mathcal{D}'(M, (T'M)^{*})$ is locally integrable if for any representation of the form (1) we have that $c_{j}$ are locally integrable functions on $U$.

We denote the usual duality bracket for $\mathfrak{g} \in \mathcal{D}'(M, (T'M)^{*})$ and $\omega \in \mathcal{D}(M, T'\otimes \text{vol}(M))$ by $\langle \mathfrak{g}, \omega \rangle \in \mathbb{C}$. However, we can also consider a different bracket, i.e.

$$\{ \ldots \} : \mathcal{D}'(M, (T'M)^{*}) \times \Gamma(M, T'M) \longrightarrow \mathcal{D}'(M),$$

which is defined locally as follows: On $U \subset M$ open as above we have the local representation (1) for $\mathfrak{g}$ and we can write $\omega|_{U} = \sum_{j} f_{j}\omega^{j}$ with $f_{j} \in \mathcal{E}(U)$. We define

$$\{ \mathfrak{g}, \omega \}|_{U} := f_{j}c_{j} \in \mathcal{D}'(U).$$

We may write $\mathfrak{g}(\omega) = \omega(\mathfrak{g}) = \{ \mathfrak{g}, \omega \}$.

**Definition 2.** Let $\mathfrak{g} \in \mathcal{D}'(M, (T'M)^{*})$. We say that $\mathfrak{g}$ is an infinitesimal CR diffeomorphism with distributional coefficients if

$$\mathfrak{g}(\tau) = \overline{\mathfrak{g}(\tau)}$$

for all $\tau \in \Gamma(M, T^{0}M)$ and if $\text{Lo}(\mathfrak{g}) = (\mathcal{L}_{\text{Lo}}\alpha)(\mathfrak{g})$ for every $L \in \Gamma(M, \mathcal{V})$ and every $\alpha \in \Gamma(M, T'M)$.

As already mentioned in the introduction, analogously to the integrable case, we consider the increasing sequence of $\mathcal{E}(M, \mathbb{C})$ modules of forms

$$E_{k} = \langle \mathcal{L}_{K_{1}} \cdots \mathcal{L}_{K_{q}} \theta : j \leq k, \ K_{q} \in \Gamma(M, \mathcal{V}), \ \theta \in \Gamma(M, T^{0}M) \rangle.$$

We note that $E_{0} = \Gamma(M, T^{0}M)$, and $E_{j} \subset \Gamma(M, T'M)$ for all $j$, and set $E = \bigcup_{j} E_{j}$.

We associate to the increasing chain $E_{k}$ the increasing sequence of ideals $\mathcal{S}_{k} \subset \mathcal{E}(M, \mathbb{C})$, where

$$\mathcal{S}_{k} = \bigwedge_{N} E_{k} = \left( \text{det} \begin{pmatrix} V^{1}(\mathfrak{g}_{1}) & \cdots & V^{1}(\mathfrak{g}_{N}) \\ \vdots & \ddots & \vdots \\ V^{N}(\mathfrak{g}_{1}) & \cdots & V^{N}(\mathfrak{g}_{N}) \end{pmatrix} : V^{j} \in E_{k}, \ \mathfrak{g}_{j} \in \Gamma(M, (T'M)^{*}) \right).$$

Every $\mathcal{S}_{k}$ is an ideal; locally, one can find smaller sets of generators: Let $U \subset M$ be open, and assume that $L_{1}, \ldots, L_{n}$ is a local basis for $\Gamma(U, \mathcal{V})$, that $\theta^{1}, \ldots, \theta^{d}$ is a local basis for $\Gamma(U, T^{0}M)$, and that $\omega^{1}, \ldots, \omega^{N}$ is a local basis of $T'M$. We write $\mathcal{L}_{j} = \mathcal{L}_{L_{j}}$ for $j = 1, \ldots, n$ and $\mathcal{L}^{\alpha} = \mathcal{L}_{L_{1}^{\alpha_{1}}} \cdots \mathcal{L}_{L_{n}^{\alpha_{n}}}$ for any multi-index $\alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in \mathbb{N}^{n}$. We note that, since $\mathcal{V}$ is formally integrable, the $\mathcal{L}^{\alpha}$, where $|\alpha| = k$, generate all $k$-th order homogeneous differential operators in the $\mathcal{L}_{j}$, and we thus have

$$E_{k}|_{U} = \langle \mathcal{L}^{\alpha}\theta^{j} : 1 \leq j \leq d, \ |\alpha| \leq k \rangle.$$
We can expand
\[ \mathcal{L}^\alpha \theta^j = \sum_{\ell=1}^N A_\ell^{\alpha, j} \omega^\ell \]
and for any choice \( \alpha = (\alpha^1, \ldots, \alpha^N) \) of multiindices \( \alpha^1, \ldots, \alpha^N \in \mathbb{N}^n \) and \( r = (r_1, \ldots, r_N) \in \{1, \ldots, d\}^N \) we define the functions
\[ D(\alpha, r)(q) = \det \begin{pmatrix} A_1^{\alpha^1, r_1} & \cdots & A_N^{\alpha^N, r_1} \\ \vdots & \ddots & \vdots \\ A_1^{\alpha^1, r_N} & \cdots & A_N^{\alpha^N, r_N} \end{pmatrix}. \]

With this notation, we have
\[ S^k|_U = (D(\alpha, r) : |\alpha^j| \leq k); \]
we shall denote the stalk of \( S^k \) at \( p \) by \( S^k_p \).

3. Microlocal Analysis for Vector-Valued Distributions

We gather in this section the necessary preliminary results about the wavefront set of sections of bundles satisfying a system of PDEs. 1971 Hörmander [14, 15] introduced the notion of wavefront set. One of the first consequences of its definition is the microlocal elliptic regularity theorem, i.e. for any distribution \( u \) and (pseudo-)differential operator \( P \) we have
\[ \text{WF } u \subseteq \text{WF } Pu \cup \text{Char } P. \]

For a CR distribution \( v \) on a CR manifold \((M, \nu)\) the fact above amounts to saying that \( \text{WF } v \subseteq T^\alpha M \). In order to prove the analogous fact for a CR section \( \mathfrak{H} \) of \((T^*M)^*\), we need that the microlocal elliptic regularity theorem holds also for vector-valued distributions and \( P \) being a square matrix of differential operators.

Indeed, a simple adaption of the arguments that establish [4] in the scalar case also provides a proof in the multidimensional situation. However, despite relation [4] for scalar operators being a classical result in microlocal analysis that is treated in numerous books e.g. [16, 19, 11, 21] and the analogous statement for vector-valued distributions implicitly mentioned in the literature, see e.g. [7], we were not able to find a definite source for the vector-valued case with precisely the statements proven we need. Hence for the convenience of the reader who are not acquainted with microlocal analysis and pseudodifferential operators we try here to give a rather self-contained proof of [4] for vector-valued distributions and matrix differential operators. We mainly follow the exposition of [17], see also [11, 19, 21].

Let \( \Omega \subseteq \mathbb{R}^n \) always be an open set. A set \( \Gamma \subseteq \mathbb{R}^n \) is a cone if \( \lambda \cdot x \in \Gamma \) for all \( x \in \Gamma \) and \( \lambda > 0 \). We say that a subset \( V \subseteq T^*\Omega \setminus \{0\} = \Omega \times (\mathbb{R}^n \setminus \{0\}) \) is conic, if for all \((x, \xi) \in V \) and real numbers \( \lambda > 0 \) we have \((x, \lambda \xi) \in V \). Sometimes we call also a conic set \( V \subseteq T^*\Omega \setminus \{0\} \) a cone. A conic neighbourhood of a point \( \xi_0 \) is an open cone \( \Gamma \) containing \( \xi_0 \). Similarly we call an open conic set \( V \subseteq T^*\Omega \setminus \{0\} \) a neighbourhood of the point \((x_0, \xi_0)\), if \((x_0, \xi_0) \in V \).

The space of smooth functions on \( \Omega \) with values in \( \mathbb{C}^\nu \) will be denoted by \( \mathcal{E}(\Omega, \mathbb{C}^\nu) \). If \( \nu = 1 \) we also write simply \( \mathcal{E}(\Omega) \). As usual the space of test functions \( \mathcal{D}(\Omega, \mathbb{C}^\nu) \) consists of all functions \( f \in \mathcal{E}(\Omega, \mathbb{C}^\nu) \) with compact support. The space of vector-valued distributions on \( \Omega \) is denoted by \( \mathcal{D}'(\Omega, \mathbb{C}^\nu) \equiv (\mathcal{D}'(\Omega))^\nu \), whereas the space of distributions of compact support is written as \( \mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega) \). We recall that the Fourier transform
\[ \hat{v}(\xi) = \mathfrak{F}(v)(\xi) = \int e^{-ix\xi} v(x) \, dx \]
of \( v \in \mathcal{E}'(\Omega) \) is an analytic function. In general, the integral means the duality bracket for distributions. It is well known that a distribution \( v \in \mathcal{E}' \) is smooth iff its Fourier transform \( \hat{v}(\xi) \) is rapidly decreasing for \( |\xi| \to \infty \). Now this observation leads to definition of the wavefront set of a distribution \( u \in \mathcal{D}'(\Omega) \).

**Definition 3.** Let \((x_0, \xi_0) \in T^*\Omega \setminus \{0\} \). The point \((x_0, \xi_0) \) is not in WF \( u \) iff there is \( \varphi \in \mathcal{D}(\Omega) \) with \( \varphi \equiv 1 \) near \( x_0 \) and a conic neighbourhood \( \Gamma \) of \( \xi_0 \) such that
\[ \sup_{\xi \in \Gamma} |\xi^N \hat{\varphi}(\xi)| < \infty \quad \forall N. \]
We note that if \((x_0, \xi_0) \notin \text{WF} u\) then \((x_0, -\xi_0) \notin \text{WF} \bar{u}\), here the conjugate \(\bar{u}\) of \(u \in \mathcal{D}'\) is defined by \((\bar{u}, \varphi) = (u, \bar{\varphi})\).

The wavefront set of \(u = (u_1, \ldots, u_n) \in \mathcal{D}'(\Omega, \mathbb{C}^n)\) is then defined as \(\text{WF} u = \bigcup_{j=1}^n \text{WF} u_j\). Obviously we have the following characterization of \(\text{WF} u\): A point \((x_0, \xi_0) \in T^*\Omega \setminus \{0\}\) is not in \(\text{WF} u\) iff there is a test function \(\varphi \in \mathcal{D}(\Omega)\) with \(\varphi \equiv 1\) near \(x_0\) such that \(\tilde{\varphi} u_j\) on a common conic neighbourhood \(\Gamma\) of \(\xi_0\). It is easy to see that \(\text{WF} u\) is a conic set.

Before we can finally begin with the proof of (4), we have to introduce matrix-valued pseudodifferential operators. We may assume that the theory of scalar-valued pseudodifferential operators on open sets is known to the reader, a detailed introduction can be found in [10,14,11].

A pseudodifferential operator \(P\) of order \(m\) is an operator \(P : \mathcal{D}(\Omega, \mathbb{C}^n) \to \mathcal{E}(\Omega, \mathbb{C}^n)\) of the form

\[
P = \begin{pmatrix}
p^{11} & \cdots & p^{1\mu} \\
\vdots & \ddots & \vdots \\
p^{\nu1} & \cdots & p^{\nu\mu}
\end{pmatrix}
\]

where \(p^{jk} \in \Psi^m_{\mu\lambda}(\Omega)\) are scalar pseudodifferential operators of order \(m\). The symbol of \(P\) is the matrix

\[
p(x, \xi) = \begin{pmatrix}
p^{11}(x, \xi) & \cdots & p^{1\mu}(x, \xi) \\
\vdots & \ddots & \vdots \\
p^{\nu1}(x, \xi) & \cdots & p^{\nu\mu}(x, \xi)
\end{pmatrix}
\]

whose entries \(p^{jk} \in S^m(\Omega \times \mathbb{R}^n)\) are the symbols of the operators \(p^{jk}\). The class of pseudodifferential operators of order \(m\) between vector-valued functions is denoted by \(\Psi^m(\Omega, \mathbb{C}^n)\). As in the case of scalar operators the elements of the set \(\Psi^{-\infty}(\Omega) = \bigcap \Psi^m(\Omega)\) are called smoothing operators; if \(Q \in \Psi^{-\infty}(\Omega)\) then \(Q(\mathcal{E}'(\Omega, \mathbb{C}^n)) \subseteq \mathcal{E}(\Omega, \mathbb{C}^n)\). As with scalar operators we can associate to each pseudodifferential operator \(P\) a properly supported pseudodifferential operator \(\tilde{P}\), i.e.

\[
\tilde{P} : \mathcal{E}'(\Omega, \mathbb{C}^n) \to \mathcal{E}'(\Omega, \mathbb{C}^n)
\]

and

\[
\tilde{P} : \mathcal{D}'(\Omega, \mathbb{C}^n) \to \mathcal{D}'(\Omega, \mathbb{C}^n)
\]

respectively, with \(P - \tilde{P} \in \Psi^{-\infty}\) since we can repeat the procedure in the scalar case (see e.g. [19]) in each entry separately to obtain the desired operator. Similarly we can construct to each sequence \(a_j \in S^{m-j}\) a symbol \(a \in S^m\) such that \(a = \sum_{j<N} a_j \in S^{m-N}\) by also repeating the proof from the scalar case, cf. [19]. We will use the notation \(a \sim \sum a_j\).

The composition of two properly supported pseudodifferential operators \(A, B\) of order \(m_1\) and \(m_2\) respectively is the operator \(C\) given by the matrix with entries

\[
C^{j\ell} = \sum_{k=1}^\nu A^{jk} B^{k\ell}.
\]

For the symbol \(c\) of \(C\) we write \(a^*b\). We have that the symbol \(a^{jk}b^{k\ell}\) of \(A^{jk} \circ B^{k\ell}\) must satisfy the following expansion (c.f. [19])

\[
a^{jk}b^{k\ell} \sim \sum_\alpha \frac{\partial^\alpha a^{jk}(x, \xi)D^\alpha_x b^{k\ell}(x, \xi)}{\alpha!},
\]

hence

\[
c^{j\ell} \sim \sum_{k=1}^\nu \sum_\alpha \frac{\partial^\alpha a^{jk}(x, \xi)D^\alpha_x b^{k\ell}(x, \xi)}{\alpha!}.
\]

We see that the analogous formula has to be valid in the matrix case

\[
a^*b \sim \sum_\alpha \frac{\partial^\alpha a(x, \xi)D^\alpha_x b(x, \xi)}{\alpha!}.
\]

As in the case of scalar operators (see e.g. [11]) we say that a properly supported operator \(P \in \Psi^m_{ps}(\Omega, \mathbb{C}^n)\) is a classical pseudodifferential operator if there are smooth functions \(p_m\) on \(T^*\Omega \setminus \{0\}\) that are homogenous
of degree $m - j$ in the second variable and $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi \equiv 1$ near the origin such that the symbol $p$ of $P$ satisfies the following asymptotic expansion

$$\psi(x, \xi) \sim \sum_j (1 - \psi(\xi)) p_{m-j}(x, \xi).$$

In slight abuse of notation we write also

$$p \sim \sum p_{m-j}$$

and we will sometimes refer to the (formal) series $\sum p_{m-j}$ as the symbol of $P$. The term of highest order $p_m$ in the series is called the principal symbol of $P$. The class of classical operators of order $m$ will be denoted by $\Psi^m_\text{cl}(\Omega, \mathbb{C}^r)$ and the term of highest order in the asymptotic expansion is called the principal symbol of the operators. If $A, B$ are two classical pseudodifferential operators of order $m_1$ and $m_2$, resp., then we have that $C = A \circ B$ is a classical operator of order $m_1 + m_2$ and, if $c \sim \sum c_{m-\ell}$ and $m = m_1 + m_2$,

$$c_{m-\ell} = \sum_{j+k+|\alpha| = \ell} \frac{1}{\alpha!} \partial^\alpha_x a_{m-j}(x, \xi) D^\alpha_x b_{m-k}(x, \xi).$$

We see that for the principal symbols, i.e. $\ell = 0$, the above equation is just $c_m = a_m b_m$.

We close this very short introduction with two definitions, that are completely analogous to the definitions for scalar operators, see [19].

**Definition 4.** The essential support $\text{essupp} A \subseteq T^*\Omega \setminus \{0\}$ of $A \in \Psi^m_\text{ps}(\Omega, \mathbb{C}^r)$ is defined by saying that a point $(x_0, \xi_0)$ is not in $\text{essupp} A$ if there is a conic neighbourhood of $(x_0, \xi_0)$ such that

$$\sup_{(x, \xi) \in \Gamma} \left| (\partial^\alpha_\xi \partial^\beta_x a^{jk}) |(x, \xi)|^N \right| < \infty$$

for all $\alpha, \beta \in \mathbb{N}^n$, $N \in \mathbb{N}$ and $j, k = 1, \ldots, \nu$.

If $A, B \in \Psi^\infty_\text{ps}(\Omega, \mathbb{C}^r)$ then $\text{essupp} AB \subseteq \text{essupp} A \cap \text{essupp} B$.

**Definition 5.** An operator $A \in \Psi^m_\text{ps}(\Omega, \mathbb{C}^r)$ is elliptic or non-characteristic at $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$ if the principal symbol $a_m$ of $A$ is invertible at $(x_0, \xi_0)$. We set

$$\text{Char} A := \{(x, \xi) \in T^*\Omega \setminus \{0\} \mid a_m(x, \xi) \text{ is not invertible}\}$$

Now we are able to start with the proof of [14] for vector-valued distributions.

**Theorem 3.** Let $P \in \Psi^m_\text{cl}(\Omega, \mathbb{C}^r)$ be elliptic at $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. Then there are operators $Q \in \Psi^{-m}_\text{cl}(\Omega, \mathbb{C}^r)$ and $R, S \in \Psi^0_\text{cl}(\Omega, \mathbb{C}^r)$ such that

$$Q P = \text{Id} + R$$

$$(x_0, \xi_0) \notin \text{essupp} R$$

$$P Q = \text{Id} + S$$

$$(x_0, \xi_0) \notin \text{essupp} S$$

**Proof.** We set $\tilde{q}_m = (p_m)^{-1}$ in some conic neighbourhood $\Gamma$ of $(x_0, \xi_0)$ where $\det p_m \neq 0$. Recursively we define on $\Gamma$

$$\tilde{q}_{m-N}(x, \xi) = -(p_m(x, \xi))^{-1} \sum_{j+k+|\alpha| = N} \frac{1}{\alpha!} \partial^\alpha_\xi \tilde{q}_{m-j}(x, \xi) D^\alpha_x b_{m-k}(x, \xi)$$

Using a suitable cut-off function $\psi \in \mathcal{E}(T^*\Omega \setminus \{0\})$, i.e. $\text{supp } \psi \subseteq \Gamma$, $\psi \equiv 1$ near $(x_0, \xi_0)$ and $\psi$ is homogeneous of degree 0 in the second variable, we can extend the functions $\tilde{q}_{m-k}$ to the whole space $T^*\Omega \setminus \{0\}$ by putting $q_{m-k} = \psi \tilde{q}_{m-k}$. Let $Q$ be the classical pseudodifferential operator associated to the symbol $\sum q_{m-k}$. Then clearly $Q P \in \Psi^0_\text{cl}(\Omega, \mathbb{C}^r)$ and therefore $R := \text{Id} - Q P \in \Psi^0_\text{cl}(\Omega, \mathbb{C}^r)$. If $\sum r_{m-j}$ is the symbol of $R$ then it follows that $r_{m-j} \equiv 0$ in some conic neighbourhood of $(x_0, \xi_0)$ by construction. Hence $(x_0, \xi_0) \notin \text{essupp} R$.

Analogously we can construct $Q_1 \in \Psi^{-m}_\text{cl}(\Omega, \mathbb{C}^r)$ such that $Q P Q_1 = \text{Id} + S_1$ with $S_1 \in \Psi^0_\text{cl}(\Omega, \mathbb{C}^r)$ and $(x_0, \xi_0) \notin \text{essupp} S_1$. Following an argument in [19] we conclude that

$$Q = Q(PQ_1 - S_1) = (\text{Id} + R)Q_1 - QS_1 = Q_1 + RQ_1 - QS_1$$

and

$$P Q = P Q_1 + PRQ_1 - PQS_1 = \text{Id} + S_1 + PRQ_1 - PQS_1 = \text{Id} + S$$

where $S = S_1 + PRQ_1 - PQS_1 \in \Psi^0_\text{cl}(\Omega, \mathbb{C}^r)$. Clearly $(x_0, \xi_0) \notin \text{essupp} S$. \qed
Proposition 2. Let $u \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$. Then we have

$$\text{WF } u = \bigcap_{P \in \Psi^\nu(\Omega, \mathbb{C}^\nu)} \text{Char } P.$$

Proof. If $(x_0, \xi_0) \notin \text{WF } u$ then there is a test function $\varphi \in \mathcal{D}(\Omega)$ such that $\hat{\varphi u}_k$ is rapidly decreasing for each $k = 1, \ldots, \nu$ on an open cone containing $\xi_0$. The multiplication with $\varphi$ is a differential operator, that we will denote by $\Phi$ in the scalar case. If $v \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$ the operator $\Phi_v$ acting on $\mathcal{D}'(\Omega, \mathbb{C}^\nu)$ given by $(\Phi_v v)_j = \varphi v_j$ (i.e. $\Phi_v = \Phi \cdot \text{Id}$) is also a differential operator of order 0.

Let $\psi \in \mathcal{E}(S^{n-1})$ such that $\psi \equiv 1$ near $\xi_0/|\xi_0|$ and supp $\psi \subset \subset \Gamma \cap S^{n-1}$ and set

$$a(x, \xi) = \varphi(x) \psi \left( \frac{\xi_0}{|\xi_0|} \right)$$

and $A \in \Psi^0_{cl}(\Omega)$ with

$$Aw(x) = \frac{1}{(2\pi)^n} \int e^{i \xi \cdot a(x, \xi)} \hat{\varphi}(\xi) \, d\xi$$

for $w \in \mathcal{D}(\Omega)$.

We define $B := A \cdot \text{Id} \in \Psi^0_{cl}(\Omega, \mathbb{C}^\nu)$ and $Q := B \circ \Phi_v$. Now $Q$ acts on $u$ by

$$(Qu)_k = \frac{1}{(2\pi)^n} \int e^{i \xi \cdot a(x, \xi)} \hat{\varphi u}(\xi) \, d\xi = \frac{1}{(2\pi)^n} \varphi(x) \int e^{i \xi \cdot \psi} \left( \frac{\xi}{|\xi|} \right) \hat{\varphi u}(\xi) \, d\xi$$

where the integral is for the moment only seen as $\mathbb{F}^{-1}(a(x, \cdot) \hat{\varphi u}(\cdot))$. Since by assumption

$$G_k(\xi) = \psi \left( \frac{\xi}{|\xi|} \right) \hat{\varphi u}(\xi) \in S(\mathbb{R}^n)$$

we have $(Qu)_k \in \mathcal{D}(\Omega) \subseteq \mathcal{E}(\Omega)$. By construction $(x_0, \xi_0) \in \text{Char } Q$.

Now let $P \in \Psi^0_{cl}(\Omega, \mathbb{C}^\nu)$ with $(x_0, \xi_0) \notin \text{Char } P$ and $Pu \in \mathcal{E}(\Omega, \mathbb{C}^\nu)$. According to Theorem 3 there is an operator $R \in \Psi^0_{cl}(\Omega, \mathbb{C}^\nu)$ with $(x_0, \xi_0) \notin \text{essupp } R$ and

$$u + Ru \in \mathcal{E}(\Omega, \mathbb{C}^\nu).$$

If we choose $\varphi$ and $\psi$ similarly to above and set $\theta(x, \xi) = (\psi(\xi) \varphi(x)) I$ we can assume that $\text{supp } \theta \cap \text{essupp } R = \emptyset$. Therefore $\Theta R \in \Psi^{-\infty}$ and $\Theta u \in \mathcal{E}(\Omega, \mathbb{C}^\nu)$. Actually we have as above

$$(\Theta u)_k = \frac{1}{(2\pi)^n} \int e^{i \xi \cdot \psi(\xi)} \hat{\varphi u}(\xi) \, d\xi$$

By Lemma A.1.2 in [21] we have that $(\Theta u)_k \in S(\mathbb{R}^n)$ and hence $\psi \hat{\varphi u} \in S(\mathbb{R}^n)$. It follows promptly that $\hat{\varphi u}$ has to decrease rapidly in a conic neighbourhood of $\xi_0$. \hfill \Box

Proposition 3. Let $u \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$ and $P \in \Psi^m_{cl}(\Omega, \mathbb{C}^\nu)$. Then

$$\text{WF } Pu \subseteq \text{WF } u \cap \text{essupp } P.$$

Proof. Let $(x_0, \xi_0) \notin \text{WF } u$. By Proposition 2 there is a classical pseudodifferential operator $Q_1$ with $(x_0, \xi_0) \notin \text{Char } Q_1$ such that $Q_1 u \in \mathcal{E}(\Omega, \mathbb{C}^\nu)$. Theorem 3 in turn provides an operator $Q_2$ such that $Q = Q_2 Q_1 = \text{Id} + R$ where $R \in \Psi^0_{cl}(\Omega, \mathbb{C}^\nu)$ and $(x_0, \xi_0) \notin \text{essupp } R$. Apparently $Qu \in \mathcal{E}$, hence $PQ \in \mathcal{E}$. On the other hand

$$QPu = PQu + [Q, P] u$$

and

$$[Q, P] = (\text{Id} + R)P - P(\text{Id} + R) = RP - PR = [R, P]$$

Thus $(x_0, \xi_0) \notin \text{essupp } [Q, P]$. As in the proof of Proposition 2 we construct a classical pseudodifferential operator $S$ that is elliptic at the point $(x_0, \xi_0)$ and satisfies $\text{essupp } S \cap \text{essupp } R = \emptyset$. It follows that $\text{essupp } S \cap \text{essupp } [Q, P] = \emptyset$ and therefore $S[Q, P] u \in \mathcal{E}$ and $SQPu \in \mathcal{E}$. The operator $SQ$ is non-characteristic at $(x_0, \xi_0)$. Hence $(x_0, \xi_0) \notin \text{WF } Pu$.

Now let $(x_0, \xi_0) \notin \text{essupp } P$. Again we construct an operator $S$ satisfying $(x_0, \xi_0) \notin \text{Char } S$ and $\text{essupp } S \cap \text{essupp } P = \emptyset$. Thus $SP \in \Psi^{-\infty}$ and $SPu \in \mathcal{E}$. It follows $(x_0, \xi_0) \notin \text{WF } (Pu)$. \hfill \Box
Theorem 4. If \( P \in \Psi_{cl}^m(\Omega, \mathbb{C}^\nu) \) then we have for all \( u \in \mathcal{D}'(\Omega, \mathbb{C}^\nu) \) that
\[
\text{WF } u \subseteq \text{WF } (Pu) \cup \text{Char } P.
\]

Proof. If \( (x_0, \xi_0) \notin \text{WF } (Pu) \cup \text{Char } P \) then \( P \) is elliptic at the point \( (x_0, \xi_0) \). By Theorem 3 there are an operator \( Q \in \Psi_{cl}^{-m} \) elliptic at \( (x_0, \xi_0) \) and an operator \( R \in \Psi_{cl}^0 \) with \( (x_0, \xi_0) \notin \text{esssupp } R \) such that \( QP = \text{Id} + R \). We have \( (x_0, \xi_0) \notin \text{WF } (QPu) \) by Proposition 3. On the other hand \( QPu = u + Ru \) and \( (x_0, \xi_0) \notin \text{WF } (Ru) \) again by Proposition 3. Hence \( (x_0, \xi_0) \notin \text{WF } u. \)
\[\Box
\]

4. A Division Theorem

The aim of this section is to study the following question: Suppose that \( \lambda \) is a smooth function and \( u \), say, a locally integrable function such that \( f = \lambda \cdot u \) is smooth. Can we conclude that \( u \) itself has to be smooth? Obviously this is a local problem and the only points of interest are the zeros of \( \lambda \), since \( u = f/\lambda \) must be smooth whenever \( \lambda \neq 0 \).

On the other hand the example \( \lambda(x) = e^{-1/x^2} \) and \( u = |x| \) shows that \( u \) may have singularities at the points where \( \lambda \) is flat, and furthermore the example \( \lambda(x,y) = x^2 + y^2 \) shows that the structure of the zero set of \( \lambda \) is of importance.

We are going to only give a simple sufficient condition on \( \lambda \) adapted to the applications which we have in mind. It remains to study the situation near zeros of finite order of \( \lambda \). We begin with the one-dimensional case.

**Lemma 2.** Let \( \lambda \) be a smooth function near \( 0 \in \mathbb{R} \) such that there is some \( k \in \mathbb{N} \) with \( \lambda^{(j)}(0) = 0 \) for \( 0 \leq j < k \) and \( \lambda^{(k)}(0) \neq 0 \). Furthermore let \( u \) be locally integrable near \( 0 \) such that \( f := \lambda u \) is smooth near \( 0 \). Then \( u \) is smooth near \( 0 \), too.

**Proof.** First, we note that the zero of \( \lambda \) at \( 0 \) is isolated. By the Fundamental Theorem of Calculus we obtain easily the existence of a smooth function \( \tilde{\lambda} \) with \( \tilde{\lambda}(0) \neq 0 \) such that
\[
\lambda(x) = x^k \tilde{\lambda}(x)
\]
near the origin.

In order to proceed we need a similar decomposition for \( f \). But, since we do not know the values of the derivatives of \( f \) at the origin a-priori, the Fundamental Theorem of Calculus only says that there is a smooth function \( f_1 \) such that \( f = xf_1 \). If \( k > 1 \) then in a punctured neighbourhood of \( 0 \) we have
\[
u(x) = \frac{x^{1-k} f_1(x)}{\tilde{\lambda}(x)}
\]
and if \( f_1(0) \neq 0 \) then \( u(x) \sim x^{1-k} \) for \( x \to 0 \). This is a contradiction to \( u \) being locally integrable. Therefore \( f_1(0) = 0 \) and there is a smooth function \( f_2 \) near the origin such that \( f(x) = x^2 f_2(x) \).

Iterating this argument if necessary we obtain that there is a smooth function \( f_k \) near \( 0 \) such that \( f(x) = x^k f_k(x) \). Hence we obtain in some punctured neighbourhood of \( 0 \) the following representation of \( u \)
\[
u(x) = \frac{f_k(x)}{\tilde{\lambda}(x)},
\]
where the right-hand side of this equation can be extended smoothly to the origin. \[\Box
\]

One cannot expect that the analogous result to Lemma 2 holds in several variables (c.f. [5]). However, one can adapt the proof of Lemma 2 to show a partial result for smooth functions whose zero set satisfies additionally certain geometric conditions.

**Proposition 4.** Let \( p_0 \in \mathbb{R}^n \) and \( \lambda \) a smooth function defined near \( p_0 \). Suppose that \( \lambda^{-1}(0) \) is a real hypersurface in \( \mathbb{R}^n \) near \( p_0 \) in \( \lambda^{-1}(0) \) and that there are \( v \in \mathbb{R}^n \) and \( k \in \mathbb{N} \) such that \( \partial_{x}^{j} \lambda(p) = 0 \), for \( j < k \) and \( p \in \lambda^{-1}(0) \) close by \( p_0 \), and \( \partial_{x}^{k} \lambda(p) \neq 0 \).

If \( u \) is a locally integrable function near \( p_0 \) with the property that \( f = \lambda \cdot u \) is smooth, then \( u \) has to be smooth near \( p_0 \).
Proof. We can choose coordinates \((x_1, \ldots, x_n) = (x', x_n)\) in a neighbourhood \(V\) of \(p_0\) such that \(p_0 = 0\) in these coordinates, \(\lambda^{-1}(0) \cap V = \{(x', x_n) \in V \mid x_n = 0\}\) and for \((x', 0)\) we have
\[
\frac{\partial^j \lambda}{\partial x_n^j}(x', 0) = 0, \quad j < k,
\]
and
\[
\frac{\partial^k \lambda}{\partial x_n^k}(x', 0) \neq 0.
\]
As in the proof of Lemma 2 we conclude, if we shrink \(V\), that there is a smooth function \(\tilde{\lambda}\) on \(V\) with \(\tilde{\lambda}(x) \neq 0\) for \(x \in V\) such that \(\lambda(x', x_n) = x_n^k \tilde{\lambda}(x', x_n)\). There is also a smooth function \(f_1\) on \(V\) such that \(f(x', x_n) = x_n f_1(x', x_n)\). We want to show as in the one-dimensional case that \(f_1(x', 0) = 0\) for \((x', 0) \in V\) if \(k > 1\): Suppose that there exists some \(y, 0\) such that \(x_n f_1(x', 0) = 0\) and also \(\tilde{\lambda}(x) \neq 0\) for \(x \in W\). W.l.o.g. the open set \(W\) is of the form \(W = W' \times I \subset \mathbb{R}^{n-1} \times \mathbb{R}\) and
\[
F(x_n) := \int_{W'} \left| \frac{f_1}{\tilde{\lambda}}(x', x_n) \right| dx' > 0
\]
for \(x_n \in I\). We conclude that
\[
\int_{W} |u(x)| dx = \int_{I} |x_n|^{1-k} F(x_n) dx_n = \infty
\]
and hence \(u\) is not locally integrable near \((y, 0)\) which contradicts our assumption.

Therefore we obtain by iteration a smooth function \(\tilde{f}\) defined near the origin in \(\mathbb{R}^n\) such that \(f(x', x_n) = x_n^k \tilde{f}(x', x_n)\). Hence \(u = \tilde{f}/\tilde{\lambda}\) is also smooth near the origin. 

5. Proof of Theorem 1

Let \(X\) be an infinitesimal CR diffeomorphism as in the statement of the theorem. The assertion of the theorem can be checked locally, so we restrict ourselves to an open set \(U \subset M\) on which we are given a basis \(L_1, \ldots, L_n\) of CR vector fields, a basis \(\omega^1, \ldots, \omega^N\) of holomorphic forms, and a generating set \(\theta^1, \ldots, \theta^d\) of characteristic forms. We also assume that \(X\) extends microlocally to a wedge with edge \(M\).

Since \(L_{k}\) maps holomorphic forms to holomorphic forms, we can write
\[
d\omega^j(L_k, \cdot) = \sum_{\ell=1}^{N} B^j_{k, \ell} \omega^\ell(\cdot)
\]
for functions \(B^j_{k, \ell}\) which are smooth on \(U\). By Lemma 1 we can regard \(X\) as a holomorphic vector field and write
\[
X = \sum_{j=1}^{N} X_j (\omega^j)^*.
\]
The assumption that \(X\) is an infinitesimal CR diffeomorphism implies that
\[
L_k X_j = L_k(\omega^j(X)) = d\omega^j(L_k, X) = \sum_{\ell=1}^{N} B^j_{k, \ell} X_\ell
\]
and
\[
\theta(X) = \overline{\theta(X)}
\]
for any characteristic form \(\theta\).

The proof of Theorem 1 follows now from the next statement.
Proposition 5. Let \((M, V)\) be an abstract, smooth CR structure and let \(X\) be a distributional CR holomorphic vector field; that is a distributional section of \((T^*M)^*\) that satisfies (15). Assume further that \(X\) fulfills (15) for any characteristic form and extends microlocally to a wedge with edge \(M\). Then for any \(\lambda \in \Gamma(M, S^k)\) the section \(\lambda X\) is smooth.

Proof. Similarly to above we can write locally on \(U \subset M\)

\[
X = \sum_{j=1}^{N} X_j(\omega^j)^*
\]

where now \(X_j \in \mathcal{D}'(U)\).

If we consider the vector

\[
\tilde{X} = (X_1, \ldots, X_1, \ldots, X_N, \ldots, X_N),
\]

then (14) implies that \(\tilde{X}\) satisfies the equation \(P\tilde{X} = K\tilde{X}\), where

\[
P = \begin{pmatrix}
L_1 & 0 & \ldots & 0 \\
0 & L_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & L_n
\end{pmatrix}.
\]

and

\[
K = \begin{pmatrix}
B_{1,1}^1 & 0 & \ldots & B_{1,2}^1 & 0 & \ldots & \ldots & B_{1,N}^1 & 0 \\
B_{1,1}^2 & 0 & \ldots & B_{1,2}^2 & 0 & \ldots & \ldots & B_{1,N}^2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
B_{n,1}^1 & 0 & \ldots & B_{n,2}^1 & 0 & \ldots & \ldots & B_{n,N}^1 & 0 \\
B_{n,1}^2 & 0 & \ldots & B_{n,2}^2 & 0 & \ldots & \ldots & B_{n,N}^2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
B_{n,1}^N & 0 & \ldots & B_{n,2}^N & 0 & \ldots & \ldots & B_{n,N}^N & 0 \\
B_{n,1}^N & 0 & \ldots & B_{n,2}^N & 0 & \ldots & \ldots & B_{n,N}^N & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
B_{n,1}^N & 0 & \ldots & B_{n,2}^N & 0 & \ldots & \ldots & B_{n,N}^N & 0 \\
B_{n,1}^N & 0 & \ldots & B_{n,2}^N & 0 & \ldots & \ldots & B_{n,N}^N & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

We can thus apply Theorem 4 in order to see that the components \(X_j\) of \(X\) have their wavefront sets contained in the characteristic directions of the \(L_k\), which means, they are restricted to \(T^0M\). By assumption, we know that there exists a closed convex cone \(\Gamma^0 \subset T^0M \setminus \{0\}\) such that \(WF(X_j) \subset \Gamma^0\) for every \(j = 1, \ldots, N\). If we denote by \(W^+ = (\Gamma^0)^c \subset T^0M\) then we have \(WF(X_j) \cap W^+ = \emptyset\) for all \(j = 1, \ldots, N\). For simplicity, we shall say that \(X_j\) extends above; similarly, with \((-\Gamma^0)^c = W^-\), we have that \(WF(X_j) \cap W^- = \emptyset\), and say that \(\tilde{X}_j\) extends below. Obviously the same is true for any derivative of the \(X_j\) or \(\tilde{X}_j\), respectively.

By (15) we know that \(\theta(X) = \bar{\theta}(\tilde{X})\) for every characteristic form \(\theta\). Recall from (2) that we write

\[
L^n = \sum_{\ell=1}^{N} A^n_{\ell}^j \omega^\ell,
\]

hence, in coordinates, the equation \(\theta(X) = \bar{\theta}(\tilde{X})\) becomes

\[
\sum_{\ell=1}^{N} A^n_{\ell}^j X^\ell \sum_{\ell=1}^{N} A^n_{\ell}^j \tilde{X}^\ell,
\]

(16)
and the left hand side of that equation extends below, while the right hand side extends above. Choose any \( N \)-tuple \( \alpha = (\alpha^1, \ldots, \alpha^N) \in \mathbb{N}^N \), with \( |\alpha| \leq k \) for \( j = 1, \ldots, N \), and \( r = (r_1, \ldots, r_N) \in \{1, \ldots, d\}^N \). Applying \( L^\alpha \) to \( \theta(X) = \theta \) (i.e. to (16)) gives

\[
\sum_{\ell=1}^N A^\alpha_{\beta,\ell} X_\ell = \sum_{\ell=1,\ldots,N} C_{\beta,\ell} L^\beta \bar{X}_\ell,
\]

where \( C_{\beta,\ell} \) are smooth functions on \( U \). We thus have, for the above choice of \( \alpha \) and \( r \), the following system of equations:

\[
\begin{pmatrix}
A^{\alpha^1,r_1} & \cdots & A^{\alpha^N,r_1} \\
\vdots & \ddots & \vdots \\
A^{\alpha^1,r_N} & \cdots & A^{\alpha^N,r_N}
\end{pmatrix}
\begin{pmatrix}
X_1 \\
\vdots \\
X_N
\end{pmatrix}
= \begin{pmatrix}
\sum C_{\beta,\ell} L^\beta \bar{X}_\ell \\
\vdots \\
\sum C_{\beta,\ell} L^\beta \bar{X}_\ell
\end{pmatrix}.
\]

We multiply this system by the classical adjoint of the matrix

\[
\begin{pmatrix}
A^{\alpha^1,r_1} & \cdots & A^{\alpha^N,r_1} \\
\vdots & \ddots & \vdots \\
A^{\alpha^1,r_N} & \cdots & A^{\alpha^N,r_N}
\end{pmatrix}
\]

and obtain for each \( j = 1, \ldots, N \) that

\[
D(\alpha, r)X_j = \sum_{|\beta| \leq k} D_{\beta,\ell} L^\beta \bar{X}_j,
\]

where the \( D_{\beta,\ell} \) are smooth functions on \( U \); the right hand side of this equation therefore extends below.

Hence, for any \( \alpha \) and \( r \) with \( |\alpha| \leq k \), we have that \( D(\alpha, r)X_j \) extends above and below; in particular, we have that \( \text{WF}(\bar{X}_j) = \emptyset \) so that we can conclude that \( D(\alpha, r)X_j \) is actually smooth. Since any \( \lambda \) in the statement of Proposition 5 can, over \( U \), be written as a smooth linear combination of \( D(\alpha, r) \) with \( |\alpha| \leq k \), the proof is finished. \( \square \)

6. Proof of the further statements in Section 1

In this section we give the proofs of Theorem 2 and Corollary 2. The statement of Theorem 2 follows immediately from Theorem 1 and Proposition 4. By assumption there is a multiplier \( \lambda \in \mathcal{S} \) near \( p \) whose zero set is a real hypersurface near \( p \) and \( \lambda \) is not flat at \( p \). Theorem 1 implies that \( \lambda X \) is smooth near \( p \) and since \( X \) is assumed to be locally integrable we can apply Proposition 4 to conclude that \( X \) also has to be smooth near \( p \).

In order to prove Corollary 2 we now have to show that a weakly \( k \)-nondegenerate real hypersurface \( M \) is CR-regular, i.e. on \( M \) there is a multiplier \( \lambda \) that can be written in suitable local coordinates as

\[
\lambda(z, \bar{z}, s) = s^\ell \psi(z, \bar{z}, s)
\]

with \( \psi \) being a smooth function that does not vanish for \( s = 0 \) and \( \ell \in \mathbb{N} \).

By assumption we have that there are coordinates \((z, w) \in \mathbb{C}^n \times \mathbb{C} \) such that \( M \) is given locally by

\[
\text{Im} \, w = (\text{Re} \, w)^m \varphi(z, \bar{z}, \text{Re} \, w)
\]

where \( m \in \mathbb{N} \) and \( \varphi \) is a smooth real-valued function defined near \( 0 \) with the property that \( \varphi_{z^0}(0) = \varphi_{\bar{z}^0}(0) = 0 \) for \( |\alpha| \leq k \) and

\[
\text{span}_{\mathbb{C}} \{ \varphi_{z^0}(0, 0, 0): 0 < |\alpha| \leq k \} = \mathbb{C}^n.
\]

A local basis of the CR vector fields on \( M \) is given by

\[
L_j = \frac{\partial}{\partial \bar{z}_j} - i s^m \frac{\varphi_{\bar{z}_j}}{1 + i(s^m \varphi)_s} \frac{\partial}{\partial s}, \quad 1 \leq j \leq n.
\]
The characteristic bundle is spanned near the origin by
\[
\theta = -ds - i \frac{s^n}{1 + i(s^m \varphi)_s} \sum_{j=1}^n \varphi_{\bar{z}_j} dz_j + i \frac{s^n}{1 - i(s^m \varphi)_s} \sum_{j=1}^n \varphi_{z_j} d\bar{z}_j
\]
and \( \theta \) together with the forms \( \omega^j = dz_j \) constitute a local basis of \( T' M \).

For simplicity we set
\[
b^j = -i \frac{s^n}{1 + i(s^m \varphi)_s} \varphi_{\bar{z}_j}
\]
for \( 1 \leq j \leq n \). If we consider a general holomorphic form
\[
\eta = \sigma \theta + \sum_{j=1}^n \rho_j \omega^j = -\sigma ds + \sum_{j=1}^n \sigma b^j dz_j + \sum_{j=1}^n (\sigma b_j + \rho^j) d\bar{z}_j
\]
with \( \sigma, \rho^j \) being smooth functions on \( M \), then we obtain
\[
d\eta(L_{\ell}, \cdot) = ((\sigma b^j)_s + \sigma_{\bar{z}_j}) \theta + \sum_{j=1}^n (L_{\ell} \rho^j + \sigma (L_{\ell} b^j - \bar{L}_j b^j)) \omega^j.
\]

For a multi-index \( \alpha \in \mathbb{N}_0^n \) with length \( |\alpha| = r \) we define the following sequence of multi-indices
\[
\begin{align*}
\alpha(1) &= e_1 \\
\alpha(2) &= 2e_1 \\
&\vdots \\
\alpha(\alpha_1) &= \alpha_1 e_1 \\
\alpha(\alpha_1 + 1) &= \alpha_1 e_1 + e_2 \\
&\vdots \\
\alpha(\alpha_1 + \alpha_2) &= \alpha_1 e_1 + \alpha_2 e_2 \\
&\vdots \\
\alpha(|\alpha|) &= \alpha.
\end{align*}
\]
Furthermore let \( L_j^\alpha = -\partial_{\bar{z}_j} - b^j \partial_s - b_s^j \) be the formal adjoint of \( L_j \) (c.f. [16]) and \( b^{j^*} := b^j, j = 1, \ldots, n \).

Iterative application of (17) leads to
\[
\mathcal{L}^\alpha \theta = ((-L^\ell)^{\alpha_1}) \theta + \sum_{\ell=1}^n \sum_{\nu=1}^{|\alpha|} L^{\alpha(\nu)}((-L^\ell)^{\alpha-\alpha(\nu)}1)(L^{\alpha(\nu)-\alpha(\nu-1)}b^\ell - \bar{L}_\ell b^{\alpha(\nu)-\alpha(\nu-1)}) \omega^\ell
\]
\[
= A^\alpha_0 \theta + \sum_{\ell=1}^n A^\alpha_\omega \omega^\ell.
\]

We claim that \( A^\alpha_0 = s^m B^\alpha_0 \) where \( B^\alpha_0 \) are some smooth functions and \( B^\alpha_0(0) = 2i \varphi_{\bar{z}_sz_z}(0) \) for \( |\alpha| \leq k \).

First, we observe that for \( 1 \leq j, \ell \leq n \)
\[
L_j b^\ell - \bar{L}_\ell b^j = s^m \left( \frac{i \varphi_{\bar{z}_jz_z}(1 + i(s^m \varphi)_s + \varphi_{z_z}(s^m \varphi_{z_z})_s)}{(1 + i(s^m \varphi)_s)^2} + \varphi_{\bar{z}_j}(s^m \varphi_{z_z})_s(1 + i(s^m \varphi)_s - i s^m \varphi_{z_z}(s^m \varphi)_s)ight)
\]
\[
+ \frac{i \varphi_{\bar{z}_jz_z}(1 + i(s^m \varphi)_s + \varphi_{z_z}(s^m \varphi_{z_z})_s)}{(1 + i(s^m \varphi)_s)^2}
\]
\[
= s^m \lambda^j_\ell
\]
and \( \lambda^j_\ell(0) = 2i \varphi_{\bar{z}_jz_z}(0) \) by the assumptions on \( \varphi \). Furthermore we remark that
\[
(-L^\ell)^{\beta} = (-L^\ell)^{\beta(|\beta|-1)} b^\ell
\]
where \( r \in \{1, \ldots, n\} \) is the greatest integer such that \( \beta_r \neq 0 \).

If we also recall the two simple facts for smooth functions \( f, g \): \( (s^q f)_s = s^{q-1} f + s^q f_s \) for \( q \geq 2 \) whereas \( (s g)_s = g + s g_s \) we see the following: If \( m \geq 2 \) we have

\[
A^\alpha_\ell(z, \bar{z}, s) = s^{m} \frac{2i \varphi_{z \bar{z}}(z, \bar{z}, s)}{1 + (s^m \varphi(z, \bar{z}, s))^2} + s^{2m-1} R^\alpha_\ell(z, \bar{z}, s) = s^m B^\alpha_\ell(z, \bar{z}, s).
\]

On the other hand we obtain for \( m = 1 \) the following representation

\[
A^\alpha_\ell(z, \bar{z}, s) = s \frac{2i \varphi_{z \bar{z}}(z, \bar{z}, s)}{(\varphi(z, \bar{z}, s) + s \varphi_s(z, \bar{z}, s))^2} + s S^\alpha_\ell(z, \bar{z}, s) + s^2 T^\alpha_\ell(z, \bar{z}, s) = s B^\alpha_\ell(z, \bar{z}, s),
\]

where \( S^\alpha_\ell \) is a sum of products of rational functions with respect to \( \varphi \) and its derivatives. Each of these summands contains at least one factor of the form \( \varphi_{z \beta} \) or \( \varphi_{\bar{z} \beta} \) with \( |\beta| \leq |\alpha| \leq k \) and therefore \( S^\alpha_\ell(0) = 0 \). The claim follows.

By assumption there are multi-indices \( \alpha^1, \ldots, \alpha^n \neq 0 \) of length \( \leq k \) such that

\[
\{ \varphi_{z \beta} (0), \ldots, \varphi_{z \bar{z} \beta} (0) \}
\]

is a basis for \( \mathbb{C}^n \). Now we choose \( \alpha = (0, \alpha^1, \ldots, \alpha^n) \) and calculate according to \( [3] \) the multiplier \( D(\alpha) = D(\alpha, 1) \) (note that \( d = 1 \)):

\[
D(\alpha) = \det \begin{pmatrix}
1 & 0 & \cdots & 0 \\
A^{a_1}_\ell & A^{a_1}_1 & \cdots & A^{a_1}_n \\
\vdots & \vdots & \ddots & \vdots \\
A^{a_n}_\ell & A^{a_n}_1 & \cdots & A^{a_n}_n \\
\end{pmatrix} = s^{n-m} \det \begin{pmatrix}
1 & 0 & \cdots & 0 \\
A^{a_1}_\ell & B^{a_1}_1 & \cdots & B^{a_1}_n \\
\vdots & \vdots & \ddots & \vdots \\
A^{a_n}_\ell & B^{a_n}_1 & \cdots & B^{a_n}_n \\
\end{pmatrix} = s^{n-m} Q(\alpha)
\]

where

\[
Q(\alpha) = \det \begin{pmatrix}
1 & 0 & \cdots & 0 \\
A^{a_1}_\ell & B^{a_1}_1 & \cdots & B^{a_1}_n \\
\vdots & \vdots & \ddots & \vdots \\
A^{a_n}_\ell & B^{a_n}_1 & \cdots & B^{a_n}_n \\
\end{pmatrix} = \det \begin{pmatrix}
B^{a_1}_1 & \cdots & B^{a_1}_n \\
\vdots & \ddots & \vdots \\
B^{a_n}_1 & \cdots & B^{a_n}_n \\
\end{pmatrix},
\]

hence

\[
Q(\alpha)(0) = (2i)^n \det \begin{pmatrix}
\varphi_{z a_1} (0) \\
\vdots \\
\varphi_{z a_n} (0)
\end{pmatrix} \neq 0.
\]

The proof of Corollary 2 is complete.

7. An Example

In this section we are going to present an example to show that the local integrability condition in Theorem 2 and Corollary 2, respectively, is essential for the conclusions in these statements to hold. More precisely, we construct two different infinitesimal diffeomorphisms with distributional coefficients on a real hypersurface in \( \mathbb{C}^2 \) such that the two diffeomorphisms are not locally integrable. We also construct a multiplier such that the products of this multiplier with each diffeomorphism coincide and are smooth. We further note that the coefficients of both diffeomorphisms are closely related to the non-extendable CR distribution for nonminimal CR submanifolds given by Baouendi and Rothschild [4].
We begin with the calculation of the multiplier in a more general setting in order to simplify the computation. We will later on restrict ourselves to real hypersurfaces in \( \mathbb{C}^2 \). Let \((M, \mathcal{V})\) be a 3-dimensional abstract CR structure of hypersurface type that is generated in some coordinates by the vector field

\[
L = \frac{\partial}{\partial \bar{z}} + s^m b(z, \bar{z}) \frac{\partial}{\partial z}.
\]

The characteristic bundle \( T^0 M \) is spanned by

\[
\theta = -ds + s^m b(z, \bar{z}) dz + s^m b(z, \bar{z}) d\bar{z}
\]

and thus the forms \( \omega = dz \) and \( \theta \) form a basis of \( T^1 M \). We obtain

\[
d\theta(L, \cdot) = -2is^m \text{Im} \left( \frac{\partial b}{\partial z} \right)(z, \bar{z}) \omega + ms^{m-1} b(z, \bar{z}) \theta.
\]

We calculate the simplest nontrivial multiplier: for \( \alpha^1 = 0 \), \( \alpha^2 = 1 \) and \( r = (1,1) \) (note that \( N = 2 \) and \( d = 1 \)) we have

\[
D(\Omega, r) = \det \begin{pmatrix} 1 & 0 \\ ms^{m-1} b(z, \bar{z}) & -2is^m \text{Im} \left( \frac{\partial b}{\partial z} \right)(z, \bar{z}) \end{pmatrix} = -2is^m \text{Im} \left( \frac{\partial b}{\partial z} \right)(z, \bar{z}).
\]

Now let \( m = 1 \), \( b = -i \frac{\psi}{1+\bar{w}} \) for some smooth real-valued function \( \psi \) defined in an open neighbourhood \( V \) of \( 0 \in \mathbb{C} \), i.e. \( M \) is an embedded real hypersurface in \( \mathbb{C}^2 \) given near the origin by the defining function

\[
\rho(z, \bar{z}, w, \bar{w}) = \text{Im} w - \text{Re} w \cdot \psi(z, \bar{z}).
\]

Then the multiplier \( D(\Omega, r) \) from above is of the form

\[
D(\Omega, r) = 2is \left( \frac{\psi z \bar{z}}{|\Psi|^2} - 2 \frac{\psi z \bar{z} \psi \bar{z}}{|\Psi|^4} \right) = 2is G(z, \bar{z}),
\]

where we have set \( \Psi := 1 + i\psi \). Note also that \( \omega_1 = \omega = dz \) and \( \omega_2 = dw = \Psi ds + is\psi dz + is\psi d\bar{z} \) is an alternative basis for \( T^1 M \) in this case.

Since \( M \) is a real hypersurface in \( \mathbb{C}^2 \) we have the following decomposition of an open neighbourhood \( \Omega \) of \( 0 \in \mathbb{C}^2 \)

\[
\Omega = U_+ \cup M \cup U_-
\]

with \( U_+ = \{(z, w) \in \Omega: \rho(z, \bar{z}, \bar{z}, w) > 0\} \) and \( U_- = \{(z, w) \in \Omega: \rho(z, \bar{z}, \bar{z}, w) < 0\} \) being open subsets of \( \Omega \). We shall also assume that \( \Omega \cap (\mathbb{C} \times \{0\}) = V \times \{0\} \).

If we consider the holomorphic function

\[
F: (z, w) \mapsto \frac{1}{w}
\]

on \( \mathbb{C} \times \mathbb{C} \setminus \{0\} \) then we see that \( F \) is of slow growth for \( w \to 0 \) on both \( U_+ \) and \( U_- \). We write \( u_+ = b_+ F \) for the boundary value of \( F|_{U_+} \) and \( u_- = b_- F \) for the boundary value of \( F|_{U_-} \), respectively. Note that by the Plemelj-Sokhotski jump relations (see, e.g., [8]) we have

\[
u_0 = u_+ - u_- = \frac{2\pi i}{\Psi} (1 \otimes \delta).
\]

Note also that \( u_0 \) is essentially (up to the factor \( -2\pi i \)) the non-extendable CR distribution from [1], c.f. also [2], for the hypersurface \( M \).

We claim that \( \text{WF} u_+ = \mathbb{R}_+ \theta|_{V \times \{0\}} \) and \( \text{WF} u_- = \mathbb{R}_- \theta|_{V \times \{0\}} \), respectively: Note that \( u_+ \) and \( u_- \) are smooth outside \( V \times \{0\} \subset M \) and that \( \text{WF} u_0 = (\mathbb{R} \setminus \{0\}) \theta|_{V \times \{0\}} \). Furthermore we know that \( \text{WF} u_+ \) and \( \text{WF} u_- \) must each be contained in \( (\mathbb{R} \setminus \{0\}) \theta \) since both are CR distributions. However, since \( u_+ \) extends holomorphically to \( U_+ \) it follows that \( \text{WF} u_+ \cap \mathbb{R}_+ \theta = \emptyset \) (see e.g. [18]) and by symmetry we have also \( \text{WF} u_- \cap \mathbb{R}_- \theta = \emptyset \). Now let \( p = (z, 0) \in V \times \{0\} \) and suppose that, e.g., \( \mathbb{R}_+ \theta \cap \text{WF} u_+ = \emptyset \). Then we would have that \( \mathbb{R}_+ \theta_p \cap \text{WF} u_0 = \emptyset \) which is obviously a contradiction to above.
We consider the following vector fields with distributional coefficients
\[ X_+ = u_+ \frac{\partial}{\partial z} \bigg|_M + \bar{u}_+ \frac{\partial}{\partial \bar{z}} \bigg|_M \]
and
\[ X_- = u_- \frac{\partial}{\partial z} \bigg|_M + \bar{u}_- \frac{\partial}{\partial \bar{z}} \bigg|_M . \]
We claim that both vector fields constitute infinitesimal CR diffeomorphisms on \( M \) if
\[ \frac{\partial \psi}{\partial x} = \psi \frac{\partial \psi}{\partial y} \]
where \( z = x + iy \). We show this for \( X_+ \), the argument for \( X_- \) is completely analogous of course. First we see that \( X_+ \) is real since
\[ X_+ = \text{Re} u_+ \frac{\partial}{\partial x} \bigg|_M + \text{Im} u_+ \frac{\partial}{\partial y} \bigg|_M. \]
Furthermore note that the regular distributions \((\nu > 0)\)

\[ u_\nu = \frac{1}{s\Psi + i\nu} \]
on \( M \) converge to \( u_+ \) in \( D' \) for \( \nu \to 0 \). We have
\[ X_+ \rho = -s\psi_x \text{Re} u_+ - s\psi_y \text{Im} u_+ = \lim_{\nu \to 0} \left(-s\psi_x \text{Re} u_\nu - s\psi_y \text{Im} u_\nu\right) = \lim_{\nu \to 0} \left(-s^2\psi_x - s\psi_y + s\nu \right) = \lim_{\nu \to 0} \text{Re} u_\nu^2 = 0 \]
with convergence in \( D' \). Hence \( X_+ \in D'(M, TM) \). We conclude further
\[ L(\omega_1(X_+)) = Lu_+ = 0, \]
\[ L(\omega_2(X_+)) = 0 \]
and since \( d\omega_j = 0, (j = 1, 2) \)
\[ d\omega_1(L, X_+) = 0, \]
\[ d\omega_2(L, X_+) = 0. \]

Since \( \omega_1(X_+) = \omega_1(X_-) = u_+, \omega_2(X_+) = \omega_2(X_-) = 0 \) and \( \omega_1(X_-) = u_- \) all the assumptions of [Theorem 1](#) are satisfied for both \( X_+ \) and \( X_- \).

Indeed
\[ D(\alpha, r)u_+ = D(\alpha, r)u_- = 2i \frac{G(z, \bar{z})}{\Psi(z, \bar{z})} \in \mathcal{E}(M) \]
hence \( D(\alpha, r)X_+ = D(\alpha, r)X_- \in \mathcal{E} \). Note also that \( D(\alpha, r)u_0 = 0 \).

**References**

[1] M. S. Baouendi and Linda Preiss Rothschild. Cauchy-Riemann functions on manifolds of higher codimension in complex space. *Invent. Math.*, 101(1):45–56, 1990.

[2] M Salah Baouendi, Peter Ebenfelt, and Linda Rothschild. *Real submanifolds in complex space and their mappings*, volume 47 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1999.

[3] M Salah Baouendi, H Jacobowitz, and François Trèves. On the analyticity of CR mappings. *Annals of Mathematics. Second Series*, 122(2):365–400, 1985.

[4] S. Berhanu and Ming Xiao. On the $C^\infty$ version of the reflection principle for mappings between CR manifolds. *Amer. J. Math.*, 137(5):1365–1400, 2015.

[5] Jan Boman. Differentiability of a function and of its compositions with functions of one variable. *Math. Scand.*, 20:249–268, 1967.
[6] Jacques Chazarain and Alain Piriou. *Introduction to the theory of linear partial differential equations*, volume 14 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York, 1982. Translated from the French.

[7] Nils Dencker. On the propagation of polarization sets for systems of real principal type. *J. Funct. Anal.*, 46(3):351–372, 1982.

[8] J. J. Duistermaat and J. A. C. Kolk. *Distributions*. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2010. Theory and applications, Translated from the Dutch by J. P. van Braam Houckgeest.

[9] Peter Ebenfelt. Finite jet determination of holomorphic mappings at the boundary. *The Asian Journal of Mathematics*, 5(4):637–662, 2001.

[10] Charles Fefferman. The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *Inventiones Mathematicae*, 26:1–65, 1974.

[11] Alain Grigis and Johannes Sjöstrand. *Microlocal analysis for differential operators*, volume 196 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994. An introduction.

[12] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.

[13] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[14] Lars Hörmander. Fourier integral operators. I. *Acta Math.*, 127(1-2):79–183, 1971.

[15] Lars Hörmander. Linear differential operators. In *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Tome 1, pages 121–133. Gauthier-Villars, Paris, 1971.

[16] Lars Hörmander. *The analysis of linear partial differential operators. I*. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition.

[17] M. S. Joshi. Lectures on Pseudo-differential Operators. *ArXiv Mathematics e-prints*, June 1999.

[18] Bernhard Lamel. A $C^\infty$-regularity theorem for nondegenerate CR mappings. *Monatsh. Math.*, 142(4):315–326, 2004.

[19] Nicolas Lerner. *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, volume 3 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010.

[20] L. Nirenberg, S. Webster, and P. Yang. Local boundary regularity of holomorphic mappings. *Comm. Pure Appl. Math.*, 33(3):305–338, 1980.

[21] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson.