A unified approach to Fierz identities

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Abstract. We summarize a unified and computationally efficient treatment of Fierz identities for form-valued pinor bilinears in various dimensions and signatures, using concepts and techniques borrowed from a certain approach to spinors known as “geometric algebra”. Our formulation displays the real, complex and quaternionic structures in a conceptually clear manner, which is moreover amenable to implementation in various symbolic computation systems.

Keywords: supergravity, supersymmetry, differential geometry
PACS: 04.65.+e, 11.30.Pb, 02.40.-k

1. INTRODUCTION

Computations involving Fierz identities in curved backgrounds for various dimensions and signatures are a cumbersome ingredient of supergravity and string theory and their applications. This problem can be alleviated by using geometric algebra techniques, which afford a unified treatment of Fierz identities for form-valued pinor bilinears. Our study [1, 2] connects previous work ([3, 4, 5, 6, 7]) with techniques and ideas belonging to the theory of Kähler-Atiyah bundles and modules over such, otherwise known as “geometric algebra”.

Notations and conventions. Let \((M,g)\) denote any smooth, connected and oriented pseudo-Riemannian manifold of dimension \(d = p + q\), where \(p\) and \(q\) are the numbers of positive and negative eigenvalues of the metric tensor \(g\). We further assume that \(M\) is paracompact, so that we have partitions of unity subordinate to any open cover. The space of \(\mathbb{R}\)-valued smooth inhomogeneous and globally-defined differential forms on \(M\) is a \(\mathbb{Z}\)-graded \(\Omega^\infty(M,\mathbb{R})\)-module denoted \(\Omega(M) \overset{\text{def}}{=} \Gamma(M,\wedge^* TM)\), with fixed rank components \(\Omega^k(M) = \Gamma(M,\wedge^k TM)\) for \(k = 0,\ldots,d\). The (real) volume form of \((M,g)\) is denoted by \(v = \text{vol}_M \in \Omega^d(M)\) and satisfies the following properties:

\[
v \odot v = (-1)^{q+\left\lfloor \frac{d}{2} \right\rfloor} 1_M = \begin{cases} (-1)^{p-q} 1_M, & \text{if } d \text{ is even} \\ (-1)^{q-p} 1_M, & \text{if } d \text{ is odd} \end{cases}, \quad v \odot \omega = \pi^{d-1}(\omega) \odot v, \quad \forall \omega \in \Omega(M),
\]

with respect to the geometric product \(\odot\) (see [2]). Hence \(v\) is central in the Kähler-Atiyah algebra \((\Omega(M),\odot)\) when \(d\) is odd and twisted central (i.e., \(v \odot \omega = \pi(\omega) \odot v\), where \(\pi\) is the grading or main automorphism) when \(d\) is even. In Table 1, at the intersection of each row and column of the first sub-table, we indicate the values of \(p - q \mod 8\) for which the volume form \(v\) has the corresponding properties. Some general aspects of the geometric algebra formalism which we use here can also be found in [8] and are discussed in detail in [1, 2].

A real pinor bundle on \((M,g)\) is defined as an \(\mathbb{R}\)-vector bundle endowed with a morphism of bundles of algebras \(\gamma : (\wedge^* TM,\odot) \rightarrow (\text{End}(S),\odot)\) which turns \(S\) into a bundle of modules over the the Kähler-Atiyah bundle of \((M,g)\), which is the exterior bundle \(\wedge^* TM\) endowed with the geometric product \(\odot\). A real pin bundle is a pinor bundle for which \(S\) is a bundle of simple modules over the Kähler-Atiyah bundle. A real spin(or) bundle is a bundle \(S\) of (simple) modules over the even rank sub-bundle \(\wedge^\text{ev} T^* M\) of the Kähler-Atiyah bundle.

Spin projectors and spin bundles. Giving a direct sum bundle decomposition \(S = S_+ \oplus S_-\) amounts to giving a product structure on \(S\), a nontrivial globally-defined bundle endomorphism \(\mathcal{R} \in \Gamma(M,\text{End}(S))\) \(\backslash\{-\text{id}_S,\text{id}_S\}\) satisfying:

\[
\mathcal{R}^2 = \text{id}_S.
\]

A product structure \(\mathcal{R}\) is called a spin endomorphism if it also satisfies the condition:

\[
[\mathcal{R}, \gamma(\omega)]_{\odot} = 0, \quad \forall \omega \in \Omega^\text{ev}(M).
\]
A spin endomorphism exists only when \( p - q \equiv 0, 4, 6, 7 \). When \( S \) is a pin bundle, the restriction \( \gamma^\text{def.} = \gamma|_{\wedge^2 T^* M} : (\wedge^2 T^* M, \omega) \to (\text{End}(S), \omega) \) is fiberwise reducible iff. \( S \) admits a spin endomorphism \( \mathcal{R} \), in which case we define the spin projectors determined by \( \mathcal{R} \) to be the globally-defined endomorphisms \( \mathcal{P}_+^\text{def.} = \frac{1}{2}(\text{id}_S \pm \mathcal{R}) \), which are complementary idempotents in \( \Gamma(M, \text{End}(S)) \). Thus the eigen-subbundles \( S^\pm = \mathcal{P}_\pm(S) \) corresponding to the eigenvalues \( \pm 1 \) of \( \mathcal{R} \) are complementary in \( S \), i.e. \( S = S^+ \oplus S^- \), and \( \mathcal{R} \) determines a nontrivial direct sum decomposition \( \gamma^\text{def.} = \gamma^+ \oplus \gamma^- \).

The effective domain of definition of \( \gamma \). Let \( \wedge^2 T^* M \) denote the bundle of twisted (anti-)selfdual forms [2]. Its space \( \Omega^\pm(M) = \Gamma(M, \wedge^2 T^* M) \) of smooth global sections is the \( \mathbb{C}^m(M, \mathbb{R}) \)-module consisting of those forms \( \omega \in \Omega(M) \) which satisfy the condition \( \omega \circ \nu = \pm \omega \). Defining:

\[
\wedge^2 T^* M \quad \text{def.} = \begin{cases} 
\wedge^2 T^* M, & \text{if } \gamma \text{ is fiberwise injective (simple case)}, \\
\wedge^2 T^* M, & \text{if } \gamma \text{ is not fiberwise injective (non-simple case)},
\end{cases}
\]

\[
\wedge^- T^* M \quad \text{def.} = \begin{cases} 
0, & \text{if } \gamma \text{ is fiberwise injective (simple case)}, \\
\wedge^- T^* M, & \text{if } \gamma \text{ is not fiberwise injective (non-simple case)},
\end{cases}
\]

one finds that \( \gamma \) restricts to zero on \( \wedge^- T^* M \) and to a monomorphism of vector bundles on \( \wedge^2 T^* M \).

2. SCHUR ALGEBRAS AND REPRESENTATION TYPES

Definition. Let \( S \) be a pin bundle of \( (M, g) \) and \( \nu \) be any point of \( M \). The Schur algebra of \( \gamma \) is the unital subalgebra \( \Sigma_{\gamma, \nu} \) of \( \text{End}(S, \nu) \) defined through:

\[
\Sigma_{\gamma, \nu} \quad \text{def.} = \{ T_\nu \in \text{End}(S, \nu) \mid [T_\nu, \gamma_\nu(\omega)]_{-\nu} = 0, \forall \omega \in \wedge^2 T^* M \}.
\]

The subset \( \Sigma_{\gamma} = \{ (x, T_x) \mid x \in M, T_x \in \Sigma_{\gamma, \nu} \} = \bigcup_{x \in M} \Sigma_{\gamma, \nu} \) is a sub-bundle of unital algebras of the bundle of algebras \( \text{End}(S, \nu) \), which we shall call the Schur bundle of \( \gamma \). The isomorphism type of the fiber \( \Sigma_{\gamma, \nu} \) is independent of \( x \) and is denoted by \( \Sigma \), being called the Schur algebra of \( \gamma \). Real pin bundles \( S \) are of three types: normal, almost complex or quaternionic, depending on whether their Schur algebra \( \Sigma \) is isomorphic with \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \). We summarize some of their properties in Table 2. Since \( \gamma \) is fiberwise irreducible in the cases of interest below, its Schur algebra \( \Sigma \) depends only on \( p - q \) (mod 8). In Tables 1, we indicate in parentheses the corresponding Schur algebras. Note that the real Clifford algebra \( Cl(p, q) \) is non-simple iff. \( p - q \equiv 1, 5 \), which we indicate in tables through the blue shading.

**TABLE 1.** Properties of \( \nu \) according to \( p - q \) (mod 8) and fiberwise character of real pin representations \( \gamma \).

| \( \nu \) is central | \( \nu \) is not central | \( \nu \) injective | \( \nu \) non-injective |
|---------------------|------------------------|-------------------|---------------------|
| 0(\mathbb{R}), 2(\mathbb{R}) | 1(\mathbb{R}) | 3(\mathbb{C}), 7(\mathbb{C}) | 4(\mathbb{H}), 6(\mathbb{H}) |
| injective | non-injective |

**TABLE 2.** Summary of pin bundle types. \( N = \text{rk}_R S \) is the real rank of \( S \) while \( \Delta = \text{rk}_R S \) is the Schur rank of \( S \). The non-simple cases are indicated through the blue shading of the corresponding table cells. The red color indicates those cases for which a spin endomorphism can be defined.

| \( \mathbb{S} \) | \( p - q \) (mod 8) | \( \wedge^2 T^* M \approx Cl(p, q) \) | \( \Delta \) | \( N \) | Number of choices for \( \gamma \) | \( \gamma^\sim(\wedge^2 T^* M) \) | Fibrewise injectivity of \( \gamma \) |
|----------------|-----------------|-----------------|------|------|-----------------|----------------|------------------|
| \( \mathbb{R} \) | 0, 2 | Mat(\Delta, \mathbb{R}) | 2(\mathbb{Z}) = 2(\mathbb{Z}) | 2(\mathbb{Z}) | 1 | Mat(\Delta, \mathbb{R}) | injective |
| \( \mathbb{H} \) | 4, 6 | Mat(\Delta, \mathbb{H}) | 2(\mathbb{Z}) = 2(\mathbb{Z}) | 2(\mathbb{Z}) | 1 | Mat(\Delta, \mathbb{H}) | injective |
| \( \mathbb{C} \) | 3, 7 | Mat(\Delta, \mathbb{C}) | 2(\mathbb{Z}) = 2(\mathbb{Z}) | 2(\mathbb{Z}) | 1 | Mat(\Delta, \mathbb{C}) | injective |
| \( \mathbb{H} \) | 5 | Mat(\Delta, \mathbb{H}) | 2(\mathbb{Z}) = 2(\mathbb{Z}) | 2(\mathbb{Z}) | 1 | Mat(\Delta, \mathbb{H}) | injective |
| \( \mathbb{R} \) | 1 | Mat(\Delta, \mathbb{R}) | 2(\mathbb{Z}) = 2(\mathbb{Z}) | 2(\mathbb{Z}) | 1 | Mat(\Delta, \mathbb{R}) | injective |

Well-known facts from the representation theory of Clifford algebras imply the following:

1. \( \gamma \) is fiberwise injective iff. \( Cl(p, q) \) is simple as an associative \( \mathbb{R} \)-algebra, i.e. iff. \( p - q \not\equiv 1, 5 \) (called simple case).
2. When \( \gamma \) is fiberwise non-injective (i.e. when \( p - q \not\equiv 1, 5 \), the so-called non-simple case), we have \( \gamma(\nu) = \epsilon_\nu \text{id}_S \), where the sign factor \( \epsilon_\nu \in \{-1, 1\} \) is called the signature of \( \gamma \). The two choices for \( \epsilon_\nu \) lead to two inequivalent choices for \( \gamma \). The fiberwise injectivity and surjectivity of \( \gamma \) are summarized in the second sub-table of Table 1.
3. GEOMETRIC FIERZ IDENTITIES FOR REAL PINORS

Admissible bilinear pairings. A non-degenerate bilinear pairing \( \mathcal{B} \) on \( S \) is called admissible [6, 7] if:

1. \( \mathcal{B} \) is either symmetric or skew-symmetric, i.e. \( \mathcal{B}(\xi, \xi') = \sigma_{\mathcal{B}} \mathcal{B}(\xi', \xi) \), \( \forall \xi, \xi' \in \Gamma(M, S) \) with symmetry factor \( \sigma_{\mathcal{B}} \in \{-1, +1\} \);

2. For any \( \omega \in \Omega(M) \), we have:

\[
\gamma(\omega)' = \gamma(\tau_{\mathcal{B}}(\omega)) \iff \mathcal{B}(\gamma(\omega)\xi, \xi') = \mathcal{B}(\xi, \gamma(\tau_{\mathcal{B}}(\omega))\xi') , \forall \xi, \xi' \in \Gamma(M, S) ,
\]

where \( \tau_{\mathcal{B}} \overset{\text{def}}{=} \tau \circ \pi_{\mathcal{B}}^{-1} = \begin{cases} \tau, & \text{if } \epsilon_{\mathcal{B}} = +1 \\ \tau \circ \pi, & \text{if } \epsilon_{\mathcal{B}} = -1 \end{cases} \) with \( \tau(\omega(\xi)) = (-1)^{\frac{q-1}{2}}, \forall \xi \in \Omega^k(M) \) is the \( \mathcal{B} \)-modified reversion and the sign factor \( \epsilon_{\mathcal{B}} \in \{-1, 1\} \) is called the type of \( \mathcal{B} \);

3. If \( p-q \equiv \delta 0, 4, 6, 7 \) (thus \( S = S^+ \oplus S^- \) where \( S^\pm \subset S \) are real spin bundles), then \( S^+ \) and \( S^- \) are either \( \mathcal{B} \)-orthogonal to each other or \( \mathcal{B} \)-isotropic. The isotropy of \( \mathcal{B} \) is the sign factor \( t_{\mathcal{B}} \in \{-1, 1\} \) defined through:

\[
t_{\mathcal{B}} \overset{\text{def}}{=} \begin{cases} +1, & \text{if } \mathcal{B}(S^+, S^-) = 0 \\ -1, & \text{if } \mathcal{B}(S^+, S^-) = 0 \end{cases} .
\]

When \( p-q \neq 0, 4, 6, 7 \), the isotropy \( t_{\mathcal{B}} \) is undefined.

The number and properties of independent admissible bilinear pairings (studied in detail in [6, 7]) depend on \( p \) and \( q \).

Local expressions. Let \( e^m \) be a pseudo-orthonormal local coframe of \((M, g)\) defined above an open subset \( U \subset M \). Then property (1) amounts to \((\gamma^m)' = e_{\mathcal{B}} \gamma^m \iff \mathcal{B}(\gamma^m \xi, \xi') = e_{\mathcal{B}} \mathcal{B}(\xi, \gamma^m \xi') , \forall m = 1 \ldots d \), which in turn implies:

\[
(\gamma^A)' = e_{\mathcal{B}}^{[A]} (\gamma^A) \equiv A \equiv A = (m_1, \ldots, m_k), \text{ with } 1 \leq m_1 < \ldots < m_k \leq d ,
\]

(3)

\[
(\gamma^A)^{-1} = e_{\mathcal{B}}^{[A]} \mathcal{B}(\gamma^A)^{-1} \xi, \xi' = e_{\mathcal{B}}^{[A]} \mathcal{B}(\xi, \gamma^A \xi') , \forall \xi, \xi' \in \Gamma(M, S) .
\]

(4)

If \((e_i)_{i=1 \ldots N} \) is an arbitrary local frame of \( S \) defined above \( U \) (with dual local frame \( (e^i)_{i=1 \ldots N} \) of \( S^* \)), then:

\[
T|_{U \xi} = \sum_{j=1}^N T_{ij}e^j = \sum_{j=1}^N e^j(T_{i}e_j) , \forall T \in \Gamma(M, \text{End}(S)) , \text{ where } T_{ij} \overset{\text{def}}{=} e'(T|_{U \xi}) \in \mathcal{C}^\infty(U, \mathbb{R}) .
\]

Preparations. Given an admissible fiberwise bilinear pairing \( \mathcal{B} \) on \( S \), we define endomorphisms \( E_{\xi, \xi'} \) of \( S \) through \( E_{\xi, \xi'}(\xi'') \overset{\text{def}}{=} \mathcal{B}(\xi', \xi'')\xi \) for any \( \xi, \xi' \in \Gamma(M, S) \) (see [2]). Then the following identities are satisfied:

\[
E_{\xi, \xi'} \circ E_{\xi', \xi''} = \mathcal{B}(\xi, \xi') E_{\xi', \xi''} , \forall \xi, \xi', \xi'' \in \Gamma(M, S) ,
\]

(6)

\[
\text{tr}(T \circ E_{\xi, \xi'}) = \mathcal{B}(T, \xi') , \forall \xi, \xi' \in \Gamma(M, S) .
\]

(7)

3.1. Normal case

This occurs when \( S \approx \mathbb{R} \), i.e. for \( p-q \equiv \delta 0, 1, 2 \), in which case \( N = \Delta = 2|\xi| \). It is characterized by two admissible bilinear pairings \( \mathcal{B}_0, \mathcal{B}_1 \), which one can take to be related through \( \mathcal{B}_1 = \mathcal{B}_0 \circ (\text{id} \otimes \gamma(\nu)) \) and whose properties are given in [1, 6, 7]. These two pairings are independent when \( p-q \equiv \delta 0, 2 \) (the simple normal cases) and proportional to each other when \( p-q \equiv \delta 1 \) (the non-simple normal case). We summarize some properties of the subcases of the normal case in Table 3. Here and below, we use the abbreviations M=Majorana, MW=Majorana-Weyl, SM=symplectic Majorana, SMW=symplectic Majorana-Weyl, DM=double Majorana for the (sometimes conflicting) physics terminology. The green shading indicates those cases for which a spin endomorphism can be defined. We have \( \gamma(\nu) = \gamma^{p+1} = \gamma^1 \circ \ldots \circ \gamma^d \) in any local positively-oriented pseudo-orthonormal coframe of \((M, g)\).

Let us start from the local relation [3]:

\[
\sum_{A=\text{ordered}} (\gamma_{A}^{-1})_{jk}(\gamma_{A})_{lm} = \frac{2d}{N} \delta_{jm} \delta_{lk} ,
\]
TABLE 3. Summary of subcases of the normal case.

| $p - q$ mod 8 | Cl($p, q$) | $\gamma(\mathcal{F})$ is injective | $\mathcal{F}$ (real spinors) | name of pinors | $\gamma(V)$ | $\nu \circ V$ is central |
|--------------|----------|-------------------------------|-------------------------------|----------------|---------------|-------------------|
| 0           | simple   | Yes                            | N/A                           | $\gamma(V)$ (MW) | M             | $\gamma(V)$ +1    | No                |
| 1           | non-simple | No                             | $\pm 1$                       | N/A             | M             | $\pm 1$          | +1                | Yes               |
| 2           | simple   | Yes                            | N/A                           | N/A             | M             | $\gamma(V)$ -1    | No                |

where $A$ runs over strictly-ordered multi-indices with components from the set $\{1, \ldots, d\}$. Multiplying by $T_{kj}$ (see (5)) and summing over $j,k$ gives the completeness relation for the normal case:

$$T = \sum_{A=ordered}^{N} \frac{1}{2A} \text{tr}(\gamma_A^{-1} \circ T) \gamma_A, \quad \forall T \in \Gamma(M,End(S)).$$

Setting $T = E_{\xi, \xi'}$ in relation (8) gives the following expansion upon using (4) and (7):

$$E_{\xi, \xi'} = \frac{N}{2A} \sum_{A=ordered}^{\text{tr}(\gamma_A^{-1} \circ E_{\xi, \xi'} \gamma_A)} = \frac{N}{2A} \sum_{A=ordered}^{\mathcal{B}_0(\xi, \gamma \xi') \mathcal{E}_A}(\xi, \gamma \xi') \gamma_A.$$

Relation (9) implies that the inhomogeneous forms $\tilde{E}_{\xi, \xi'} = (\gamma|_{\Omega^1(M)})^{-1}(E_{\xi, \xi'}) \in \Omega^1(M)$ have the following expansion in terms of the basic admissible pairing $\mathcal{B}_0$:

$$\tilde{E}_{\xi, \xi'} = \frac{N}{2A} \sum_{A=ordered}^{\mathcal{E}_A(\mathcal{B}_0(\xi, \gamma \xi'))(\xi, \gamma \xi')} \gamma_A \mathcal{E}_A(\xi, \gamma \xi') \mathcal{B}_0(\xi, \gamma \xi'), \quad \forall \xi, \xi' \in \Gamma(M,S).$$

where we used $\mathcal{E}_A(\xi) \mathcal{E}_A(\gamma \xi) = \gamma^{-1}(\mathcal{A})$. Relation (6) implies the geometric Fierz identities for the normal case:

$$\tilde{E}_{\xi_1, \xi_2} = \mathcal{B}_0(\xi_1, \xi_2) \mathcal{E}_{\xi_1, \xi_2}, \quad \forall \xi_1, \xi_2, \xi_3, \xi_4 \in \Gamma(M,S).$$

3.1.1. Example: One real pinor in nine Euclidean dimensions

In this case ($p = 9, q = 0$) the pin bundle $S$ is an $\mathbb{R}$-vector bundle of rank $N = 2^4 = 16$. Since $d \equiv 8$ and $p - q \equiv 1$, we are in the normal non-simple case and thus $\gamma(V) = e_\nu |d_0$. Choosing the signature $\epsilon_\nu = +1$, we realize the subalgebra $(\Omega^+(M), \circ)$ of twisted self-dual forms through the truncated model $(\Omega^-(M), \bullet_\pm)$, where $\Omega^-(M) = \bigoplus_{k=0}^4 \Omega^k(M)$ and $\bullet_\pm$ is the reduced geometric product discussed in [2]. Details on the truncated models of the Kähler-Atiyah algebra can be found in loc. cit. Since $\gamma(V) = d_0$ we have only one admissible pairing $\mathcal{B}$ on $S$, which has $\sigma_\mathcal{B} = +1$ and $\mathcal{E}_\mathcal{B} = +1$. We can assume that $\mathcal{B}$ is positive-definite and thus is a scalar product on $S$ and we denote the corresponding norm through $|| \cdot ||$. The isotropy $\nu$ is not defined. In the case of one pinor $\xi \in \Gamma(M,S)$ (which we normalize through $|| \xi || = 1$), we are interested in pinor bilinears such as $\tilde{E}(k) = \frac{1}{24} \mathcal{B}(\xi, \gamma a_1 \ldots a_k \xi) \epsilon^{a_1 \ldots a_k} \in \Omega^k(M), \forall a_1, \ldots, a_k \in \Gamma(S)$. Using (3) and the properties of the bilinear pairing $\mathcal{B}$, we can construct (up to twisted Hodge duality on $(M,g)$):

$$\mathcal{B}(\xi, \xi) = 1, \quad V \mathcal{E}(V) \mathcal{E}(\Phi) \mathcal{E}(\phi) = \frac{1}{24} \mathcal{B}(\xi, \gamma a_1 \ldots a_k \xi) \epsilon^{a_1 \ldots a_k} \mathcal{E}(\xi, \gamma a_1 \ldots a_k \xi) \epsilon^{a_1 \ldots a_k}.$$

In this case, the truncated model of the Fierz algebra admits a basis consisting of a single element, constructed from the lower truncation of $\tilde{E}$ — namely $\tilde{E}_c = \frac{N}{2A} \tilde{E}_c = \frac{1}{24} \sum_{k=0}^{\infty} \tilde{E}(k) = \frac{1}{24}(1 + V + \Phi)$. The truncated geometric Fierz identity follows easily from (6), upon using the definition of the reduced product $\bullet_\pm$ in terms of $\circ$ (see [2]):

$$\tilde{E}_c \bullet_\pm \tilde{E}_c = \frac{1}{2} \tilde{E}_c \quad (\iff \tilde{E} \circ \tilde{E} = \tilde{E}) \iff V \bullet_\pm V + \Phi \bullet_\pm \Phi + V \bullet_\pm \Phi + \Phi \bullet_\pm V = 15 + 14V + 14\Phi.$$

Solving the system of equations obtained by separating rank components in (10) gives, upon using the definition of the twisted Hodge star ($\bar{\omega} = \omega \circ V$ for any $\omega \in \Omega^k(M)$), the following system of conditions on the forms $V$ and $\Phi$:

$$||V||^2 = 1, \quad ||\Phi||^2 = 14, \quad t_\nu \Phi = 0, \quad \bar{\omega}(\Phi \land \Phi) = 14V, \quad \bar{\omega}(V \land \Phi) = \Phi, \quad \Phi \land_2 \Phi = -12\Phi.$$
3.2. Almost complex case

This occurs when $\mathbb{S} \cong \mathbb{C}$, which happens for $p - q \equiv 3, 7$. In this case, $d$ is odd and we have $N = 2\Delta = 2^{\frac{d+1}{2}} + 1$. There exist two complex structures on the bundle $\mathcal{S}$ — the two globally-defined endomorphisms $J \in \Gamma(M, \text{End}(\mathcal{S}))$ given by $J = \pm \gamma(\nu)$, which satisfy:

$$J^2 = -\text{id}_\xi, \quad [J, \gamma(\omega)]_{-\circ} = 0, \quad \forall \omega \in \Omega(M).$$

The results of [3] imply that there also exists a globally-defined endomorphism $D \in \Gamma(M, \text{End}(\mathcal{S}))$ which satisfies:

$$D \circ \gamma(\omega) = \gamma(\pi(\omega)) \circ D, \quad \forall \omega \in \Omega(M), \quad [D, J]_{+\circ} = 0,$$

$$D^2 = (-1)^{\frac{p+q+1}{4}} \text{id}_\xi = \begin{cases} -\text{id}_\xi, & \text{if } p - q \equiv 3 \\ +\text{id}_\xi, & \text{if } p - q \equiv 7 \end{cases}.$$

| $p - q \mod 8$ | CI($p, q$) | $\gamma$ | $\epsilon_\gamma$ | $D^2$ | $\mathcal{B}$ (real spinors) | name of pinor | $\gamma(\nu)$ | $\nu \circ \nu$ | $\nu$ is central |
|--------|--------|--------|--------|--------|----------------|-------------|-------------|-------------|----------------|
| 3      | simple | Yes    | N/A    | $-\text{id}_\xi$ | N/A         | $\mathcal{M}$ | $\pm \xi$   | $-1$        | Yes           |
| 7      | simple | Yes    | N/A    | $+\text{id}_\xi$ | $D$ (Majorana) | $DM$       | $-1$        | Yes         |

There are four independent choices $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ for the non-degenerate admissible pairing, which we can take to be related through $\mathcal{B}_3 = \mathcal{B}_0 \circ (\text{id}_\delta \otimes J)$, $\mathcal{B}_2 = \mathcal{B}_0 \circ (\text{id}_\delta \otimes D)$, $\mathcal{B}_1 = \mathcal{B}_0 \circ [\text{id}_\delta \otimes (D \circ J)]$. Using again the completeness relations of [3], a more involved, but similar, derivation to that given in the normal case, we obtain (see [1]) local expansions of the inhomogeneous differential forms which ‘dequantize’ $E_{\xi, \xi'}$ expressed using the basic admissible pairing $\mathcal{B}_0$. A crucial difference from the normal case is that the bundle morphism $\gamma$ is not surjective, having image equal to $\text{End}_\mathbb{C}(\mathcal{S}) = \text{End}_{\mathbb{C}_p}(\mathcal{S})$, where the complex structure on $\mathcal{S}$ is defined by $J$. To take this into account, we use the fact (see [1]) that there exists a unique decomposition $E_{\xi, \xi'} = E_{\xi, \xi'}^{(0)} + D \circ E_{\xi, \xi'}^{(1)}$, with $E_{\xi, \xi'}^{(0)}, E_{\xi, \xi'}^{(1)} \in \Gamma(M, \text{End}_\mathbb{C}(\mathcal{S}))$, $\forall \xi, \xi' \in \Gamma(M, \mathcal{S})$. Since $\gamma$ is injective, this allows us to define $E_{\xi, \xi'}^{(0)} \equiv \gamma^{-1}(E(0)) \in \Omega(M)$ and $E_{\xi, \xi'}^{(1)} \equiv \gamma^{-1}(E(1)) \in \Omega(M)$, which have the expansions [1]:

$$E_{\xi, \xi'}^{(0)}(\xi_1, \xi_2) = \frac{\Delta}{2d} \sum_{A=\text{ordered}} (\gamma^A) \mathcal{B}_0(\xi, \gamma^A \xi') e_A,$$

$$E_{\xi, \xi'}^{(1)}(\xi_1, \xi_2) = \frac{\Delta}{2d} \sum_{A=\text{ordered}} (\gamma^A) \mathcal{B}_0(\xi, D \circ \gamma^A \xi') e_A.$$

One also finds that (6) implies the geometric Fierz identities for the almost complex case:

$$E_{\xi_1, \xi_2}^{(0)} \circ E_{\xi_3, \xi_4}^{(0)} + (\gamma^A) \mathcal{B}_0(\xi_3, \xi_2) E_{\xi_1, \xi_4}^{(0)},$$

$$\pi(E_{\xi_1, \xi_2}^{(1)} \circ E_{\xi_3, \xi_4}^{(1)} + E_{\xi_1, \xi_2}^{(1)} \circ E_{\xi_3, \xi_4}^{(1)}) = \mathcal{B}_0(\xi_3, \xi_2) E_{\xi_1, \xi_4}^{(1)}.$$

3.3. Quaternionic case

This occurs for $\mathbb{S} \cong \mathbb{H}$, which happens for $p - q \equiv 8, 4, 6$. Then $N = 4\Delta = 2^{\frac{d+1}{2}} + 1$. The Schur algebra is isomorphic with the $\mathbb{R}$-algebra $\mathbb{H}$ of quaternions, while the Schur bundle is locally (over small enough open subsets $U \subset M$) generated by four linearly-independent elements $J_\alpha \in \Gamma(U, \text{End}(\mathcal{S})) (\alpha = 0 \ldots 3)$ which we can take to correspond to the quaternion units. Hence $J_0 = \text{id}_\delta$ while $J_1, J_2, J_3$ satisfy:

$$J_1 \circ J_0 = -\delta_{1j} J_0 + \epsilon_{jkl} J_k, \quad \forall i, j, k = 1 \ldots 3 \implies [J_1, J_0]_{+\circ} = 0, \quad J_1^2 = -\text{id}_\delta,$$

$$[J_1, \gamma(\omega)]_{-\circ} = 0, \quad \forall \omega \in \Omega(U).$$
where \( \varepsilon_{ijk} \) is the Levi-Civita symbol. We thus have \( \Gamma(U, \text{End}_{\mathbb{H}}(S)) \equiv \{ T \in \Gamma(U, \text{End}(S)) | [T, J_i]_{-0} = 0, \forall i = 1 \ldots 3 \} \).

In this case, the bundle morphism \( \gamma \) has image equal to \( \text{End}_{\mathbb{H}}(S) \equiv \text{End}_{\mathbb{C}}(S) \), where the quaternionic structure of \( S \) is given locally by \( (J_a)_{a=0 \ldots 3} \). One can show [1] that any operator \( E_{\xi, \eta} \in \Gamma(M, \text{End}(S)) \) has the unique local decomposition \( E_{\xi, \eta} = \sum_{a=0}^{3} J_a \circ E_{\xi, \eta}^{(a)} \), where \( E_{\xi, \eta}^{(a)} = \frac{\Delta}{2^a} \sum_{A=\text{ordered}} \text{tr}(J_a^{-1} \circ E_{\xi, \eta}) \gamma_A \in \Gamma(U, \text{End}_{\mathbb{H}}(S)) \). Therefore, we can define \( E_{\xi, \eta}^{(a)} = \gamma^{-1}(E_{\xi, \eta}^{(a)}) \in \Omega(U) \). We have eight admissible pairings \( \mathcal{B}_\varepsilon^a (\varepsilon = \pm 1, \alpha = 0 \ldots 3) \), which one can take to be given by \( \mathcal{B}_\varepsilon^a = \mathcal{B}_\varepsilon^0 \circ (\text{id}_S \otimes J_k) \), \( \forall k = 1, 2, 3 \), where \( \mathcal{B}_\varepsilon^0 \) are the so-called basic admissible pairings. Only four of \( \mathcal{B}_\varepsilon^a \) are independent in the quaternionic non-simple case, i.e. when \( p - q \equiv 5 \). Fixing a choice for \( \varepsilon \), we find the local expansions [1] in terms of the basic admissible pairing:

\[
E_{\xi, \eta}^{(0)} = \frac{\Delta}{2^a} \sum_{A=\text{ordered}} \varepsilon^{[A]}_{\alpha \beta \gamma} \mathcal{B}_0(\xi, \gamma \eta') \xi_A', \quad \forall \xi, \eta' \in \Gamma(M, S),
\]

\[
E_{\xi, \eta}^{(1)} = \frac{\Delta}{2^a} \sum_{A=\text{ordered}} \varepsilon^{[A]}_{\alpha \beta \gamma} \mathcal{B}_0(\xi, (J_i \circ \gamma_A) \eta') \xi_A', \quad \forall i = 1 \ldots 3
\]

and (6) implies the geometric Fierz identities for the quaternionic case:

\[
E_{\xi_1, \xi_2}^{(0)} \circ E_{\xi_3, \xi_4}^{(0)} - 3 \sum_{i=1}^{3} E_{\xi_1, \xi_2}^{(i)} \circ E_{\xi_3, \xi_4}^{(i)} = \mathcal{B}_0(\xi_3, \xi_2) E_{\xi_1, \xi_4}^{(0)},
\]

\[
E_{\xi_1, \xi_2}^{(i)} \circ E_{\xi_3, \xi_4}^{(i)} + E_{\xi_1, \xi_2}^{(i)} \circ E_{\xi_3, \xi_4}^{(i)} + 3 \sum_{j, k=1}^{3} \varepsilon_{ijk} E_{\xi_1, \xi_2}^{(j)} \circ E_{\xi_3, \xi_4}^{(k)} = \mathcal{B}_0(\xi_3, \xi_2) E_{\xi_1, \xi_4}^{(i)} \quad (i = 1 \ldots 3).
\]

**ACKNOWLEDGMENTS**

This work was supported by the CNCS projects PN-II-RU-TE (contract number 77/2010) and PN-II-ID-PCE (contract numbers 50/2011 and 121/2011). CIL is supported by the Research Center Program of IBS (Institute for Basic Science) in Korea (CA1205-01). IAC acknowledges her FDP *Open Horizons* scholarship, which financed part of her studies.

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