Extremal inscribed and circumscribed complex ellipsoids

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Abstract
We prove that if a convex set in $\mathbb{C}^n$ contains two inscribed complex ellipsoids of maximal volume, then one is a translate of the other. On the other hand, the circumscribed complex ellipsoid of minimal volume is unique. As application we prove the complex analogue of Brunn’s classical characterization of ellipsoids.

Keywords Ellipsoids · Extremal volumes · Complex affine spaces

Mathematics Subject Classification 52A40

1 Introduction
Let $A$ be a non-flat, compact subset of euclidean space $\mathbb{R}^n$. Denote by $\hat{A}$ the convex closure of $A$. An ellipsoid $E$ is called circumscribed if $A \subset E$, and it is inscribed if $E \subset \hat{A}$. We say that $E$ is a minimal circumscribed ellipsoid (MiCE) if it has the minimal volume among all circumscribed ellipsoids. On the other hand, $E$ is a maximal inscribed ellipsoid (MaIE) if $E$ has maximal volume among all inscribed ellipsoids.

The existence of these ellipsoids follows from standard arguments. There is a sphere big enough to contain $A$ and a non-zero sphere contained in $\hat{A}$. All ellipsoids are easily parametrized by a matrix and a vector; among them, we can consider only those that contain the small sphere and are contained in the big sphere. This set is compact in the
parameter space. Moreover, the volume function is continuous so that the existence of a MiCE and a MaIE follows.

The outstanding fact about MiCE and MaIE is that they are unique. The first proofs of this in its full generality seem to appear independently in Danzer et al. (1957) and Zaguskin (1958). MiCE and MaIE are known as Löwner–John Ellipsoids and have many applications in several areas of mathematics (see, e.g., Heil and Martini (1993), Petty (1983), Sect. 2.12 of Martini et al. (2019) and Henk (2012) with the references therein).

This paper deals with similar questions in the space $\mathbb{C}^n$, where ellipsoids are the unit balls of norms defined by inner products. The only previous result seems to be in the paper of Gromov (1967), where he proved the uniqueness of the circumscribed ellipsoid of minimal volume among those centered at the origin, when $A$ is the unit ball of a Banach space over $\mathbb{C}$ (Lemma 1 of Gromov (1967)).

We prove in Sect. 3 that if a convex set in $\mathbb{C}^n$ (convexity is inherited from $\mathbb{R}^{2n}$) contains two complex MaIE, then one is a translate of the other. In Sect. 4 we prove that the complex MiCE is unique. In Sect. 5 we give some applications. The most remarkable of which is the analogue of Brunn’s Theorem for complex ellipsoids. In Sect. 6 we state two straightforward computations as lemmas.

Whenever we find it possible, we use a unifying approach for the real and complex cases, and try to highlight their differences.

2 Preliminaries

Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. The vectors, also called points, of $\mathbb{K}^n$ will be denoted in boldface. Affine subspaces of dimension 1 of $\mathbb{K}^n$ are called lines, and affine subspaces of codimension 1 of $\mathbb{K}^n$ are called hyperplanes. Bare in mind that, unless the contrary is explicitly stated, all objects are general; that is, for example, a line is not a real or a complex line, it is a line in $\mathbb{K}^n$.

For $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$, let $\overline{x} = (\bar{x}_1, \ldots, \bar{x}_n)$, where the bar above denotes the complex conjugate. Also, we denote by $x \odot y$ the Hadamard product of $x$ and $y$; that is, the coordinatewise product (see for example Horn and Johnson (1991) Chapter 5).

A scalar product in $\mathbb{K}^n$ is a sesquilinear, Hermitian, positive-definite functional $\mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ denoted by $\langle x \cdot y \rangle$ for any $x$ and $y$ in $\mathbb{K}^n$. For each scalar product there is a matrix $M$ (Hermitian, positive-definite) such that $\langle x \cdot y \rangle = x^T M y$. In the case that $M$ is the identity matrix, the scalar product is the usual Hermite’s product in $\mathbb{K}^n$. It is well known that the eigenvalues of $M$ are real positive numbers and so is its determinant. It is also known that $M$ is diagonalizable by a unitary transformation.

An ellipsoid (centered at the origin) is a set

$$\left\{ x \in \mathbb{K}^n \mid x^T M \overline{x} \leq 1 \right\},$$

where $M$ is Hermitian and positive-definite. If $M$ is the identity matrix, this ellipsoid is the unit ball $\mathcal{B}(\mathbb{K}^n)$ and its boundary is the unit sphere $\mathcal{S}(\mathbb{K}^n)$.

The set of unitary transformations is the subgroup of $\mathcal{G}L(\mathbb{K}^n)$ that preserves the unit sphere. The modulus of a scalar $\lambda \in \mathbb{K}$ will be denoted by $|\lambda|$. The set $\mathcal{G}(\mathbb{K}^1)$ is
the multiplicative group of modulus 1 scalars in $\mathbb{K}$. We will denote by $\|\cdot\|$ the usual norm in $\mathbb{K}^n$, that is, the norm defined by Hermite’s product.

Let $B \overset{\text{def}}{=} \sqrt{M^T}$. We have $M^T = BB$ and hence, $M = B^T B^T = B^T B$. From this, we obtain

$$x^T M^T = x^T B^T B^T = (Bx)^T (Bx) = \|Bx\|^2.$$  

Therefore, the ellipsoids in $\mathbb{K}^n$ can also be written in the form

$$\left\{ B^{-1} u \mid u \in \mathcal{B} \left( \mathbb{K}^n \right) \right\}.$$  

If we use a unitary transformation to bring $M$ to the diagonal form, then this can be rewritten as

$$\mathcal{E}_1(\lambda) \overset{\text{def}}{=} \left\{ \lambda \odot u \mid u \in \mathcal{B} \left( \mathbb{K}^n \right) \right\},$$  

where $\lambda$ is a vector in $\mathbb{R}^n_+$. Since $M^{\frac{1}{2}} = B$, the two forms 1 and 2 are related by the fact that $\lambda$ is the diagonal of $M^{-\frac{1}{2}}$. The map $x \mapsto \lambda \odot x = M^{-\frac{1}{2}} x$ is an invertible linear map in $GL (\mathbb{K}^n)$ which maps the unit sphere $\mathcal{B} (\mathbb{K}^n)$ into the ellipsoid $\mathcal{E}_1(\lambda)$. Therefore

$$\text{Vol} \mathcal{E}_1(\lambda) = \det M^{-\frac{1}{2}} \text{Vol} \mathcal{B} (\mathbb{K}^n).$$  

The results that follow do not depend on the translation-invariant measure chosen to define the volume. We just need the validity of equation 3.

Let $\det \lambda$ denote the product of coordinates of $\lambda$. Of course, we have $\det \lambda = \det A^{-\frac{1}{2}}$ and

$$\text{Vol} \mathcal{E}_1(\lambda) = \det \lambda \text{Vol} \mathcal{B} (\mathbb{K}^n).$$  

Translates of ellipsoids centered at the origin are also called ellipsoids, and translations do not change volume.

### 3 Maximal inscribed ellipsoids

**Theorem 1** Let $A$ be a non-flat compact in $\mathbb{K}^n$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two MaIE contained in $\hat{A}$. Then, there is a vector $c \in \mathbb{K}^n$ such that $\mathcal{E}_2 = \mathcal{E}_1 + c$. If $\mathbb{K} = \mathbb{R}$ then $\mathcal{E}_1 = \mathcal{E}_2$.

**Proof** Using a suitable affine transformation we can assume that $\mathcal{E}_1$ is the unit ball $\mathcal{E}_1(1)$. Suppose that the center of $\mathcal{E}_2$ is the point $c \in \mathbb{K}^n$. We can use a unitary transformation to diagonalize the matrix of $\mathcal{E}_2 - c$. And therefore, $\mathcal{E}_2 = \mathcal{E}_1(\lambda) + c$ for some $\lambda \in \mathbb{R}^n_+$.

Let us first prove that the ellipsoid $\mathcal{E}_3 = \mathcal{E}_1 \left( \frac{1}{2} (\lambda + 1) \right) + \frac{1}{2} c$ is contained in the convex closure of $\mathcal{E}_1 \cup \mathcal{E}_2$, that is,

$$\mathcal{E}_3 = \mathcal{E}_1 \left( \frac{\lambda + 1}{2} \right) + \frac{c}{2} \subset \mathcal{E}_1(1) \cup (\mathcal{E}_1(\lambda) + c).$$
For any $x \in \mathcal{E}_3$ there is $u \in \mathcal{B}(\mathbb{R}^n) = \mathcal{E}_1(1) = \mathcal{E}_1$, such that

$$x = \frac{\lambda + 1}{2} \odot u + \frac{c}{2}.$$ 

The point $y = \lambda \odot u + c$ is in $\mathcal{E}_2$ and

$$\frac{y + u}{2} = \frac{\lambda \odot u + c + u}{2} = x.$$

This means that $x$ is the middle point of the segment joining $y$ and $u$, and proves that

$$\mathcal{E}_3 \subseteq \mathcal{E}_1 \cup \mathcal{E}_2,$$

as we wished.

Since $\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \hat{A}$, we have $\mathcal{E}_3 \subseteq \mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \hat{A}$.

We know that $\text{Vol}(\mathcal{E}_1) = \text{Vol}(\mathcal{E}_2) = \det \lambda \text{Vol}(\mathcal{E}_1)$. Therefore, $\det \lambda = 1$. On the other hand, $\text{Vol}(\mathcal{E}_3) = \det(\frac{1}{2}(\lambda + 1)) \text{Vol}(\mathcal{E}_1)$. If $\lambda \neq 1$, Lemma 1 on page 1 implies that $\det(\frac{1}{2}(\lambda + 1)) > 1$, and hence $\text{Vol}(\mathcal{E}_3) > \text{Vol}(\mathcal{E}_1)$; but this contradicts that $\mathcal{E}_1$ is a MalE. So, we have $\lambda = 1$ and therefore, $\mathcal{E}_2 = \mathcal{E}_1 + c$, which proves the first part of the theorem.

For the second part, we have $\kappa = \mathbb{R}$ and that $\mathcal{E}_1$ and $\mathcal{E}_2$ are two unit balls whose centers are at distance $2\alpha$. Let $e_1 = (1, 0, ..., 0)$ be the first basis vector. Using a suitable unitary affine transformation, we may assume that $\mathcal{E}_1$ has its center at $\alpha e_1$ and $\mathcal{E}_2$ has its center at $-\alpha e_1$. Now, we shall prove that $\mathcal{E}_3 = \mathcal{E}_1(\alpha e_1 + 1)$ is contained in the convex closure of $\mathcal{E}_1 \cup \mathcal{E}_2$, that is,

$$\mathcal{E}_1(\alpha e_1 + 1) \subseteq (\mathcal{E}_1(1) + \alpha e_1) \cup (\mathcal{E}_1(1) - \alpha e_1).$$

Given $x \in \mathcal{E}_1(\alpha e_1 + 1)$, there exists $u = (u_1, ..., u_n) \in \mathcal{B}(\mathbb{R}^n)$ such that $x = (\alpha e_1 + 1) \odot u$.

Let $y = u + \alpha e_1 \in \mathcal{E}_1$ and $z = u - \alpha e_1 \in \mathcal{E}_2$. To prove that $x$ is in the segment joining the points $y$ and $z$, we have to find $t \in [0, 1] \subset \mathbb{R}$ such that $x = ty + (1 - t)z$. For all coordinates but the first, any value of $t$ is a solution of the corresponding equation. In the first coordinate, we have the equation

$$(\alpha + 1) u_1 = t(u_1 + \alpha) + (1 - t)(u_1 - \alpha).$$

For $\alpha \neq 0$, the unique solution is $t = \frac{1}{2}(u_1 + 1)$. Moreover, $t$ is in the interval $[0, 1]$, because the coordinates of vectors in the unit ball are real numbers in the interval $[-1, 1]$. This proves that $\mathcal{E}_3 \subseteq \mathcal{E}_1 \cup \mathcal{E}_2$ and therefore, that $\mathcal{E}_3 \subseteq \hat{A}$.

We have

$$\frac{\text{Vol}(\mathcal{E}_3)}{\text{Vol}(\mathcal{E}_1)} = \det(\alpha e_1 + 1) = \alpha + 1.$$ 

If $\mathcal{E}_1 \neq \mathcal{E}_2$, then $\alpha > 0$ and therefore, $\text{Vol}(\mathcal{E}_3) > \text{Vol}(\mathcal{E}_1)$. This contradicts the maximality of $\mathcal{E}_1$ and proves that $\mathcal{E}_1 = \mathcal{E}_2$. \hfill \square

**Remark 1** The proof of the second part of the theorem does not work for complex ellipsoids, because we are using that $u_1$ is a real number in an essential manner.
Moreover, the second part of this theorem is not true for complex ellipsoids. In $\mathbb{C}^1$, complex ellipsoids are disks. A rectangle of sides 2 and 4 contains many unit disks which are of maximal volume. It is easy to generalize this counterexample to all dimensions.

### 3.1 Symmetry

A subset of $\mathbb{K}^n$ is *symmetric* if it is preserved by the multiplicative group of modulus 1 scalars in $\mathbb{K}$; the origin is its *center*. A translate of a symmetric set is also called symmetric. It is easy to see that an affine image of a symmetric set is also symmetric. All ellipsoids are symmetric as they are affine images of the unit ball.

In $\mathbb{C}^n$, additionally to the complex ellipsoids, there are real ellipsoids which are inherited from $\mathbb{R}^{2n}$. Observe that all complex ellipsoids are real but not the other way around. The difference between real and complex ellipsoids is clearly exposed by the following result from Bracho and Montejano (2021).

**Theorem 2** Any symmetric ellipsoid in $\mathbb{C}^n$ is complex.

Now, we shall see that symmetry guarantees uniqueness of the MaIE.

**Theorem 3** If $A$ is a non-flat, compact symmetric set in $\mathbb{C}^n$, then its complex MaIE is unique.

**Proof** We can assume that the center of $A$ is the origin. Let $E$ be the unique real MaIE in $A$ (Theorem 1). Let $\zeta$ be a scalar of modulus 1. Since $\zeta E \subset \zeta A = A$ and $\text{Vol}(\zeta E) = \text{Vol}(E)$ then, the uniqueness of $E$ implies that $E$ is symmetric. Using Theorem 2, we get that $E$ is complex. There is not another complex MaIE because any complex ellipsoid is real. $\square$

### 4 Minimal circumscribed ellipsoids

**Theorem 4** Let $A$ be a non-flat, compact set in $\mathbb{K}^n$. Let $E_1$ and $E_2$ be two MiCE containing $A$. Then, $E_1 = E_2$.

**Proof** Using an affine transformation, we can make $E_2$ the unit ball. Then, using a unitary transformation, we can diagonalize the matrix of $E_1$. Finally, we translate so that the center of $E_1$ is the origin and the center of $E_2$ is some vector $c \in \mathbb{K}^n$. Then,

$$E_1 = \{ x \in \mathbb{K}^n \mid \sum \lambda_i |x_i|^2 \leq 1 \} = E_1(\beta)$$

$$E_2 = \{ x \in \mathbb{K}^n \mid \sum |x_i - c_i|^2 \leq 1 \} = E_1(1) + c,$$

where $\beta = \lambda^{-\frac{1}{2}}$. Therefore,

$$A \subset E_1 \cap E_2 \subset E_3 \overset{\text{def}}{=} \left\{ x \in \mathbb{K}^n \mid \sum \lambda_i |x_i|^2 + |x_i - c_i|^2 \leq 2 \right\}.$$
Using Lemma 2, the inequality is transformed into
\[
\sum (\lambda_i + 1) \left| \left( x_i - \frac{c_i}{\lambda_i + 1} \right) \right|^2 \leq 2 - \sum \left| \frac{\lambda_i}{\lambda_i + 1} \right| c_i^2 \leq 2. \tag{4}
\]

Let
\[
\mathcal{E}_4 \overset{\text{def}}{=} \left\{ x \in \mathbb{K}^n \mid \sum (\lambda_i + 1) \left| \left( x_i - \frac{c_i}{\lambda_i + 1} \right) \right|^2 \leq 2 \right\},
\]
which is an ellipsoid. From inequality 4, we obtain that
\[
A \subset \mathcal{E}_1 \cap \mathcal{E}_2 \subset \mathcal{E}_3 \subset \mathcal{E}_4.
\]
Let \( z \in \mathbb{K}^n \) be the vector with coordinates \( c_i / (\lambda_i + 1) \). We have
\[
\mathcal{E}_4 = z + \left\{ x \in \mathbb{K}^n \mid \sum (\lambda_i + 1) |x_i|^2 \leq 2 \right\} = \mathcal{E}_4 \left( \left( \frac{\lambda + 1}{2} \right)^{-\frac{1}{2}} \right) + z.
\]
Since \( \det \beta = 1 \), we obtain \( \det \lambda = 1 \). If \( \lambda \neq 1 \), the hypothesis of Lemma 1 are satisfied, then
\[
\Delta \overset{\text{def}}{=} \det \left( \frac{\lambda + 1}{2} \right) > 1.
\]
Therefore \( \text{Vol} \mathcal{E}_4 = \Delta^{-\frac{1}{2}} \text{Vol} \mathcal{E}_2 < \text{Vol} \mathcal{E}_2 \), which contradicts the minimality of \( \mathcal{E}_2 \).

So, we conclude that \( \lambda = 1 \), \( \mathcal{E}_1 \) is the unit ball and \( \mathcal{E}_2 = \mathcal{E}_1 + c \). In this case, the inequality 4 transforms to the following
\[
\sum \left| \left( x_i - \frac{c_i}{2} \right) \right|^2 \leq 1 - \sum \left| \frac{c_i}{2} \right|^2.
\]
Therefore, \( \mathcal{E}_3 \) is a ball with center in \( c/2 \), and radius
\[
r \overset{\text{def}}{=} \sqrt{1 - \sum \left| \frac{c_i}{2} \right|^2}.
\]
If \( c \) is not the origin then \( r < 1 \) and \( \mathcal{E}_3 \) has volume strictly less than that of the unit ball \( \mathcal{E}_1 \). Therefore, \( c = 0 \) and \( \mathcal{E}_2 = \mathcal{E}_1 \).

\[\square\]

5 Applications

5.1 Centered ellipsoids

There are other extremal ellipsoids when the extremum is searched for ellipsoids that have a fixed center (say the origin). In this setting, all extremal ellipsoids are unique: maximal inscribed or minimal circumscribed, real or complex.
To see this, one can modify the proofs of the previous theorems (which is not difficult), or use the symmetrization of sets. If \( A \) is a set in \( \mathbb{R}^n \), its symmetrization around the origin is
\[
S(A) = \bigcup_{|\zeta| = 1} \zeta A.
\]
It is not hard to prove that if \( A \) is non-flat and compact then \( S(A) \) also is non-flat and compact.

Applying previous theorems, we obtain unique extremal ellipsoids circumscribed or inscribed in \( S(A) \). These extremal ellipsoids contain the origin and it is not difficult to see that they are the unique extremal ellipsoids centered at the origin.

### 5.2 Polarity

As one is introduced in the subject of Löwner–John Ellipsoids, there is a strong feeling of certain symmetry in the concepts and proofs. This subsection partially explains that feeling.

Let \( A \) be a non-flat, compact, convex subset of \( \mathbb{R}^n \) containing the origin in its interior. The polar of \( A \) is
\[
A^* = \{ x \in \mathbb{R}^n \mid x \cdot a \leq 1 \quad \forall a \leq 1 \}.
\]
Polarity reverses the inclusion relation. It is a fact that \( A^{**} = A \) (see for example Barvinok (2002) Chapter IV). If \( B \) is a symmetric, positive-definite matrix then
\[
\{ Bu \mid u \in \mathcal{B}(\mathbb{R}^n) \}^* = \left\{ B^{-1} u \mid u \in \mathcal{B}(\mathbb{R}^n) \right\},
\]
that is, the polars of ellipsoids centered at the origin are ellipsoids centered at the origin. Moreover, if \( \mathcal{E} \) is an ellipsoid centered at the origin then
\[
Vol(\mathcal{E}) Vol(\mathcal{E}^*) = 1. \tag{5}
\]

Denote by \( \mathcal{MI}(A) \) the ellipsoid of minimal volume among all centered at the origin ellipsoids containing \( A \). Denote by \( \mathcal{MA}(A) \) the ellipsoid of maximal volume among all centered at the origin ellipsoids contained in \( A \).

**Theorem 5** \( \mathcal{MA}(A)^* = \mathcal{MI}(A^*) \).

**Proof** Denote \( E = \mathcal{MA}(A) \) and \( F = \mathcal{MI}(A^*) \). We have \( A^* \subset E^* \) and by the minimality of \( F \) we have \( VolF \leq VolE^* \). By equation 5 \( VolF VolE \leq 1 \). On the other hand, we have \( F^* \subset A \) and by the maximality of \( E \) we have \( VolE \geq VolF^* \). By equation 5, \( VolF VolE \geq 1 \). Therefore, \( VolF VolE = 1 \).

The ellipsoid \( E^* \) contains \( A^* \) and has the same volume as \( F \). By the uniqueness of \( \mathcal{MI}(A^*) \) we have \( E^* = F \). \( \square \)

**Corollary 1** The two theorems about the uniqueness of extremal real ellipsoids centered at the origin are polar to each other.
5.3 A characterization of ellipsoids

**Theorem 6** Let $A$ be a non-flat, compact, convex subset of $\mathbb{K}^n$. Then, $A$ is an ellipsoid if given any two non-interior points of $A$, there is an affine isomorphism which maps one of them to the other and preserves $A$.

**Proof** Let $f$ be an affine isomorphism that preserves $A$. Since $A$ is non-flat and convex, it has non-zero volume, therefore $f$ preserves volumes. Let $E$ be the unique ellipsoid of minimum volume containing $A$. The ellipsoid $f(E)$ contains $f(A) = A$ and has the same volume than $E$. Since $E$ is unique, $f(E) = E$. Since any affinity is continuous,

$f(\partial E) = \partial (E)$. Denote by $\partial A$ the boundary of $A$. There exists $p \in \partial E \cap \partial A$, because $E$ is minimal. For any point $q \in \partial A$, there is, by hypothesis, an affinity $f$ which preserves $A$ and such that $f(p) = q$. Since $f(\partial E) = \partial E$, then $q \in \partial E$. This implies that $\partial A \subseteq \partial E$. But both are homeomorphic to spheres of the same dimension, so $\partial A = \partial E$. Therefore, $A = \partial A = \partial E = E$. □

5.4 Brunn’s Theorem

Let $A$ be a non-flat, compact subset of $\mathbb{K}^n$. We will call $A$ a *puck* if any non empty intersection with a line is convex and symmetric, i.e., if $\mathbb{K} = \mathbb{R}$ then it is a line segment; if $\mathbb{K} = \mathbb{C}$ then it is a disk. When $\mathbb{K} = \mathbb{R}$, pucks are just convex bodies.

**Theorem 7** A puck in $\mathbb{K}^n$ is an ellipsoid if and only if for any line $\ell$, the centers of all intersections of $A$ with lines parallel to $\ell$, lie in a hyperplane.

**Proof** For the only if part, observe that the property holds for balls and that it is preserved by affine transformations.

We shall prove the if part using Theorem 6. For this, let $x$ and $y$ be two different non-interior points of $A$ and let $\ell$ be the unique affine line that contains $x$ and $y$. The line $\ell$ defines a hyperplane $H$ as stated in the hypothesis. Let $o$ be the point $\ell \cap H$. Let us make a translation $\tau_o$ which sends $o$ to the origin. Denote $A' = A - o$, $x' = x - o$, $y' = y - o$, etc.

Since $x'$ and $y'$ belong to the same linear subspace $\ell'$ which is of dimension 1, there exists a scalar $\lambda \in \mathbb{K}$ such that $y' = \lambda x'$. The center of $\ell \cap A$ is $o$, therefore the center of $\ell' \cap A'$ is the origin. Moreover since $x'$ and $y'$ are non-interior points of $A'$ they must lie in the boundary of $\ell' \cap A'$ and therefore, $\|x'\| = \|y'\|$, because $A$ is a puck. Hence, we obtain $|\lambda| = \frac{\|y'\|}{\|x'\|}^{-1} = 1$.

The linear subspaces $H'$ and $\ell'$ are complementary. Therefore, for any vector $z \in \mathbb{K}^n$ there exist unique vectors $z_{\ell'} \in \ell'$ and $z_{H'} \in H'$ such that $z = z_{\ell'} + z_{H'}$. Define $\phi(z) = \lambda z_{\ell'} + z_{H'}$, which is a linear isomorphism of the whole space. Observe that $\phi(x') = \lambda x' = y'$.

The spaces $\ell'$ and $H'$ are invariant subspaces of $\phi$; in fact, $\phi$ is the direct sum of the identity in $H'$ and multiplication by $\lambda$ in $\ell'$. Any parallel line $L$ to $\ell'$ is equal to $\ell' + p$ where $p = L \cap H'$. If $z \in L$ then $\phi(z) = \lambda z_{\ell'} + p$. Therefore, $\phi$ leaves $L$ invariant and it acts inside $L$ like multiplication by $\lambda$. Since $L \cap A'$ is symmetric, $|\lambda| = 1$ and...
the center of \( L \cap A' \) lies in \( H' \) by hypothesis, then \( \phi ( L \cap A') = L \cap A' \). We have that \( A' \) is the disjoint union of all the \( L \cap A' \) with \( L \) parallel to \( \ell' \). Since \( \phi \) preserves each “slice” then \( \phi \) preserves \( A' \).

Now, consider the affine isomorphism

\[
f = \tau_0^{-1} \circ \phi \circ \tau_0 : \mathbb{K}^n \to \mathbb{K}^n .
\]

We have \( f (A) = \tau_0^{-1} ( \phi (A') ) = A \) and \( f (x) = \tau_0^{-1} ( \phi (x') ) = \tau_0^{-1} (y') = y \). So, the hypothesis of Theorem 6 is fulfilled and therefore, \( A \) is an ellipsoid. \( \square \)

Remark 2 If \( \mathbb{K} = \mathbb{R} \), the affine isomorphism \( f \) from the previous proof is called a skew reflection. When \( \mathbb{K} = \mathbb{C} \), it could be called skew rotation.

6 Auxiliary facts

This section contains two lemmas needed in the paper whose proofs are straightforward computations. Recall that \( \mathbf{1} \) is the vector which has all coordinates equal to 1.

Lemma 1 If \( \lambda \in \mathbb{R}^n_+ \) is such that \( \det \lambda = 1 \) and \( \lambda \neq \mathbf{1} \), then

\[
\det \left( \frac{\lambda + \mathbf{1}}{2} \right) > 1.
\]

Proof We have \( (\lambda_i - 1)^2 \geq 0 \) and therefore \( (\lambda_i + 1)^2 \geq 4\lambda_i \). With equality only when \( \lambda_i = 1 \). Taking the product, we get \( \det (\lambda + \mathbf{1})^2 > 4^n \det \lambda \) which is the same as \( \det (\lambda + \mathbf{1}) > 2^n \sqrt{\det \lambda} = 2^n \). This proves the lemma. \( \square \)

Lemma 2 Let \( c, x \) be complex numbers and \( \lambda \) be a real number. Then,

\[
\lambda |x|^2 + |x - c|^2 = (\lambda + 1) \left| x - \frac{c}{\lambda + 1} \right|^2 + \left( \frac{\lambda}{\lambda + 1} \right) |c|^2 .
\]

Proof We have

\[
\lambda |x|^2 + |x - c|^2 = \lambda |x|^2 + (x - c)(x - c) = (\lambda + 1) |x|^2 - (x\overline{c} + c\overline{x}) + |c|^2 ,
\]

and completing the square, we obtain

\[
= (\lambda + 1) \left| x - \frac{c}{\lambda + 1} \right|^2 + \left( \frac{\lambda}{\lambda + 1} \right) |c|^2 .
\]

\( \square \)
The first lemma is used in the proofs of Theorems 1 and 4. The second lemma is used in the proof of Theorem 4.

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