Large enhancement of conductivity in Weyl semimetals with tilted cones: pseudo-relativity and linear response

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We study the conductivity of two-dimensional graphene-type materials with tilted cones as well as their three-dimensional Weyl counterparts and show that a covariant quantum Boltzmann equation is capable of providing an accurate description of these materials’ transport properties. The validity of the covariant Boltzmann approach is corroborated by calculations within the Kubo formula. We find a strong anisotropy in the conductivities parallel and perpendicular to the tilt direction upon increase of the tilt parameter $\eta$, which can be interpreted as the boost parameter of a Lorentz transformation. While the ratio between the two conductivities is $\sqrt{1-\eta^2}$ in the two-dimensional case where only the conductivity perpendicular to the tilt direction diverges for $\eta \to 1$, both conductivities diverge in three-dimensional Weyl semimetals, where $\eta = 1$ separates a type-I (for $\eta < 1$) from a type-II Weyl semimetal (for $\eta > 1$).

I. INTRODUCTION

Recently, condensed matter systems having Dirac-like linear energy dispersions proved to be a fertile ground to rediscover fundamental particles of nature. The Weyl semimetal (WSM) is a newly discovered phase of matter which hosts yet another emergent excitation of quantum field theory: a long-sought fundamental particle of nature first proposed in 1929 by Herman Weyl as a massless solution to the Dirac equation. Crystals featuring WSM phases, contrary to other types of Dirac matter, have unique topological features such as: the zero-energy excitations in the semimetallic bulk are associated with the chiral Weyl Fermions, having definite handedness near two distinct nodes whereas these nodes are connected only at the boundary of the crystal via a peculiar half loop surface states known as the Fermi arcs. These materials exhibit intriguing and distinct phenomena when exposed to electromagnetic fields, such as chiral anomaly, negative magnetoresistance, and the chiral magnetic effect.

Dirac materials, possessing anisotropic and tilted energy cones, where due to the strain the Fermi surface gains eccentricity and deviates from a standard circular shape, have been reported to exist in the two dimensional organic conductors $\alpha$-(BEDT-TTF)$_2$I$_3$ subjected to pressure and uniaxial strain. In these materials the Dirac crossing occurs away from the high symmetry points of the Brillouin zone and their associated spectrum is modeled with a modified Weyl Hamiltonian. The presence of the tilt largely impacts the magneto-electronic and optical properties in these two-dimensional systems with tilted cones. Recently, another type of Weyl cone with no relativistic analogue, due to the violation of Lorentz invariance, in transition-metal dichalcogenides hosting Weyl fermions has been reported. As compared to the abovementioned (moderately) tilted cones in so-called type-I WSMs, the cones are now “overtilted” such that the isoeenergy lines are no longer closed ellipses but open hyperbolas. The search for this new type of WSM, coined type II, is ongoing and some candidate materials have been predicted theoretically.

Similar to two-dimensional systems, the tilt of the conical spectrum improves the transport qualities of three-dimensional WSMs by increasing the mobility and conductivity of the carriers. These features indicates their better electronic and spintronic functionality compared to other types of Dirac matter. For instance, the reported extremely large and non-saturating magnetoresistance, up to 450,000% in low field and $13 \times 10^6\%$ in high fields, suggests a suitable magnetic memory and spintronics applications. Other peculiar properties of WSMs with tilted cone are the large conductivity of about $(10^6 \ \Omega^{-1}\text{cm}^{-1})$, and enlarged chiral anomaly whose origins remain yet unclear and need further theoretical investigation. As evidence to the significance of the Dirac cone tilting, the transport calculations show that the anomalous Hall and thermal Hall conductivity, Berry curvature and density of states increase with tilt and peak around the critical tilt value, while optical absorption shows no upper bound. Furthermore, the critical angle between the tilt and the magnetic field sets a threshold for the collapse of the Landau level formation and magnetic breakdown.

A very distinct interpretation of tilted Dirac and Weyl systems comes from a relativistic perspective where the tilt is identified as the rapidity of a specific Lorentz transformation. In the presence of external fields exploiting the Lorentz covariance, the tilt parameter becomes an essential variable in identifying a boosted frame where the fields transform trivially and this in turn facilitates the study of the problem at hand. Besides this rich attribute, there is another motivation that appeals the use of relativistic argumentation in describing the physics of the titled Weyl cone. The direct diagonalization of the Dirac equation in the presence of external fields is a cumbersome task, while by utilizing a rel-
ativistic picture and redefining the fields, this problem can be tackled easily.\textsuperscript{16,18,38,44} This suggests the use of covariant formalism that allows for a better understanding of the physics of the tilted Weyl cone phase.

In the present paper, we address the electric transport properties of two- and three-dimensional Dirac systems and WSMs as a function of the tilt parameter. We show that a relativistic viewpoint in the form of a covariant Boltzmann equation allows for a quantitatively accurate description of the magnetoconductivity in the diffusive transport regime. The tilt in the electronic energy dispersion in the general Weyl Hamiltonian, characterized by the tilt velocity $v_0$, is equivalent to drift velocity $v_{\text{drift}} = E/B$ under a suitable Lorentz transformation.\textsuperscript{16,18} After obtaining expressions in the relativistically simplified picture where only the magnetic field is present, an inverse Lorentz boost recovers the results in the original frame. Additionally, we demonstrate and elucidate how the DC conductivity increases in moderately tilted type-I WSM in terms of the tilting degree. This large enhancement, which is about 7 to 10 times larger than in standard WSMs without tilt, is the focal point of recent experimental debates concerning electronic transport in WSMs.\textsuperscript{28,29,45} In order to demonstrate the validity of our approach in terms of the covariant Boltzmann equation, we perform conductivity calculations using the Kubo formula and find that apart from quantum corrections at energies close to the band-contact points due to interband coupling which are neglected in the Boltzmann equation; both approaches show a high quantitative agreement.

The plan of the paper is as follows. In Sec. \textsuperscript{II} we discuss the basic construction of the manifestly covariant Boltzmann equation to study transport in the electron’s co-moving frame of reference which is an inertial frame moving with velocity equal to the electron’s drift velocity relative to the laboratory frame of reference. In Sec. \textsuperscript{III} we compute the conductivity of a two-dimensional anisotropic and tilted Weyl system using the covariant Boltzmann equation as well as Kubo formula and compare the two results. In Sec. \textsuperscript{IV} we repeat the same calculation for a three-dimensional system of type I WSM having tilt in $k_x$-direction and compute the longitudinal and perpendicular (to the direction of the tilt) bulk conductivities using covariant Boltzmann formula as well as Kubo formula and then compare them. Furthermore, we provide a qualitative understanding of our findings for the conductivities by computing the tilt-induced renormalization of Fermi velocity, the density of states and Einstein’s diffusion relation.

\section{Covariant Boltzmann Equation}

Tilted Dirac cones in a magnetic field, in both 2D and 3D materials, can be elegantly described within a covariant formulation,\textsuperscript{18,40,44} in which the tilt parameter is associated with an effective electric field.

In order to see this, consider the minimal Dirac Hamiltonian, in which we omit the valley degree of freedom and set $\hbar = 1$ (for more details see Sec. \textsuperscript{III}), in arbitrary dimensions representing the tilted Weyl cone in $x$-direction and in the presence of homogeneous electromagnetic fields in the Landau gauge $\mathbf{A} = -By \mathbf{x}$.

$$H = v_F (\mathbf{k} - e \mathbf{A}) \cdot \sigma + v_0 k_x \sigma_0 - e(E-v_0 B) y.$$  \hspace{1cm} (1)

Here $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices and $\mathbf{k}$ is the 3D momentum vector and the (isotropic) Fermi velocity $v_F$ can be thought of as an effective speed of light within the covariant description. The electromagnetic field is accounted for by the so-called minimal coupling $\mathbf{k} \rightarrow \mathbf{k} - e \mathbf{A}$ and $H \rightarrow H + e \phi$, where $\phi$ and $\mathbf{A}$ are the scalar and vector potentials, respectively, and $e$ is the electric charge of the relativistic particle. The potentials yield the electric and magnetic fields, with $\mathbf{E} = -\nabla \phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$, respectively, which we consider to be spatially homogeneous in the following. Note that the second term in the Hamiltonian only shifts the energy spectrum of the Landau levels by a constant as $k_z$ is a good quantum number, and this term can thus be absorbed in the dispersion.\textsuperscript{16,18,38,44} Furthermore, the Hamiltonian \textsuperscript{(1)} can be mapped under an appropriate Lorentz transformation in the direction of the tilt, where the rapidity gives $\eta = \tan \theta = E/v_F B$, to a relatively simple Hamiltonian, namely,$\textsuperscript{16,18,42,44}$

$$H' = e^\frac{2}{\hbar} \sigma_y \left( H - v_0 k_x \sigma_0 \right) e^{-\frac{2}{\hbar} \sigma_y} = v_F \left( \mathbf{k} - e \mathbf{A}' \right) \cdot \sigma,$$  \hspace{1cm} (2)

where $\mathbf{A}'$ and $\mathbf{k}'$ are the corresponding vector potential and momentum vector in the co-moving frame.\textsuperscript{18,42} One notes that for $v_0 = v_{\text{drift}}$, the electric field vanishes in the new frame of reference and the spectrum gives $\varepsilon(E,B) - v_0 k_x = \varepsilon'(B')$. Hence the Lorentz boost provides a useful tool in studying the dynamics of carriers in tilted systems under external fields. In the framework of the Liouville equation, the quantum mechanical phase-space distribution function of the carriers satisfies the equation $i\partial_t f = \{H,f\}_{P,B}$, where $f$ indicates Wigner’s distribution function and $\{X,Y\}_{P,B} = \partial_{k_x} X \partial_{\sigma_y} Y - \partial_{\sigma_y} X \partial_{k_x} Y$ stands for the Poisson’s bracket.\textsuperscript{46,47} The additional term to the energy in general changes the Liouville equation, however, for the spatially homogeneous distribution function which we consider here its effect vanishes as the Poisson bracket $\{v_0 k_x \sigma_0, f\}_{P,B}$ becomes itself zero. This in turn proves that Hamiltonian \textsuperscript{(1)} and its boosted equivalent \textsuperscript{(2)} yield the same collisionless Boltzmann equation. Ignoring the effect of the additional term in the collision kernel, we utilize this similarity and compute the physical quantity of interest in an appropriate inertial frame by implementing the coordinate-independent Boltzmann transport equation and then restore to the original frame of reference via the Lorentz transformation law.\textsuperscript{16,47} In order to justify these assumptions and the correctness of the results obtained by the covariant approach, we will
make use of the established Kubo formula of linear response for the conductivity and compare the outcomes of both the methods.

This first section is therefore devoted to the covariant formulation of Boltzmann’s transport equation. The distribution function is a Lorentz scalar since it relates to the number of particles, \( dN = f(x^\mu, k^\nu) dx^\mu dk_\mu \), through the phase space volume\(^{46,48,51} \) Lorentz covariance, as a basic structural property of the Dirac equation, can be implemented in investigating the statistical kinetics of the carriers in Weyl systems. Additionally, we throughout our calculations ignore the spin degree of freedom, as the spin component affected by the Lorentz boost undergoes a precession due to the Wigner rotation\(^{22} \) whereas our main concern is the study of electronic transport of a tilted system. The ratio \( \eta = v_{\text{drift}}/v_F < 1 \), there exists a Lorentz transformation to a frame of reference that allows us to get rid of the electric field \((\text{magnetic regime})\), while the electric regime is associated with a drift velocity that is larger than the speed of light \((\eta > 1)\), in which case one can get rid of the magnetic field by an appropriate Lorentz boost. In the remainder of this paper, we concentrate on the magnetic regime, which happens to be relevant for the covariant description of type-I Weyl semimetals with moderately tilted cones.

The covariant Boltzmann equation with electromagnetic fields in manifest covariant form is\(^{46,48,50} \)

\[
k^\mu \partial_\mu f + e F^{\mu\nu} k_\nu \frac{\partial f}{\partial k^\mu} = -\frac{k^\mu}{v_F^2} \frac{\delta f}{\tau},
\]

where \( \delta f = f - f^0 \), is the deviation from the equilibrium distribution. The collision kernel is approximated with a suitable relaxation time ansatz where \( u_\mu \) is the 4-velocity for carrier flow\(^{46,50} \) which in electron’s local rest frame takes the form \( u_\mu = (v_F, 0, 0, 0) \). Using relativistic notations, the 4-momentum is \( k^\mu = (\varepsilon/v_F, k) \) where \( \varepsilon \) and \( v_F \) are the energy and the Fermi velocity of the carriers, respectively. Note that for the massless carriers with Dirac dispersion we have \( k = \varepsilon v_F^2 \), in terms of the 4-velocity \( v_\mu = (v_F, v) \), and \( \tau \) is the carriers scattering time as the time interval between two successive collisions. Writing the electromagnetic tensor \( F^{\mu\nu} \) explicitly in terms of the fields \( F^{00} = -\partial v^0 = E_0/v_F \), and \( F^{ij} = -\varepsilon_{ij} B_\ell \), the equations of motion read

\[
k^\mu \partial_\mu = \frac{\varepsilon}{v_F} (\partial t + v \cdot \nabla k),
\]

\[
e^{\mu\nu} k_\nu \frac{\partial}{\partial k^\mu} = \frac{\varepsilon}{v_F^2} \eta \left( E \cdot \nabla k + v \times B \cdot \nabla k \right).
\]

For convenience we separate the electric field into two parts \( E = E_0 + \delta E \) where only the second term \( \delta E \ll E_0 \) corresponds to a small bias in the chemical potential between the leads, while the first one \( E_0 \) represents a strong electric field that eventually corresponds to the tilt parameter of the cones, as we discuss in detail in the next section. The drift velocity associated with the latter field, \( v_{\text{drift}} = E_0 \times B / B^2 \), determines the appropriate Lorentz boost defined as (for instance in the \( x \)-direction of space)

\[
\Delta_\mu = \begin{pmatrix} \gamma & -\gamma \eta & 0 & 0 \\ -\gamma \eta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

(6)

to the frame of reference where \( E_0 \) vanishes as long as we remain in the magnetic regime, while the small bias field is transformed to \( E' = \delta E' = \gamma \delta E_\perp + \delta E_\parallel \) as well as the magnetic field \( B'_x = \gamma^{-1} B_z \), \((B'_{x,y} = 0)\), in terms of the Lorentz factor

\[
\gamma^{-1} = \sqrt{1 - \eta^2}, \quad \eta = v_{\text{drift}}/v_F.
\]

(7)

We thus obtain solution of the covariant Boltzmann equation for a stationary and homogeneous system, the nonequilibrium distribution, as\(^{53} \)

\[
\delta f = \frac{-\varepsilon \tau}{1 + \omega^2 \tau^2} \left( \delta E' + \tau \omega' \frac{B'}{B^2} \times \delta E' \right) \cdot \nabla_k \varepsilon' \left( \frac{f^{(0)}}{\varepsilon'} \right),
\]

(8)

where, in the co-moving frame, the cyclotron frequency of Dirac fermions is given by

\[
\omega' = \frac{v_F^2}{\varepsilon} B'.
\]

(9)

The nonequilibrium current due to an infinitesimal bias is defined through the formula

\[
J = -e \text{Tr}(\nu \, \delta f),
\]

(10)

where the trace (Tr) represents the summation over the momentum and other degrees of freedom. While computing the current in electron’s rest frame, we use the distribution as calculated in Eq. (8) and then perform the trace by taking into the account that the components of the velocity of the particle in the co-moving frame should transform according to the relativistic velocity addition formula, in directions parallel and perpendicular to the boost, as

\[
v' = \frac{v_\parallel + v_{\text{drift}}}{1 + \frac{v_\parallel v_{\text{drift}}}{v_F^2} v_\perp}, \quad v_\perp = \sqrt{1 - \frac{v_\parallel^2}{v_F^2}},
\]

(11)

One ultimately obtains the conductivity in the lab frame via the transformation law, namely,

\[
\delta E_\parallel' = \delta E_\parallel, \quad \delta E_\perp' = \gamma \delta E_\perp,
\]

(12)

and the formula \( \sigma_{\mu\nu} = \delta j_{\mu}/\delta E_\nu \) for the infinitesimal bias fields\(^{24} \) Using these relations, we will restore the diagonal conductivities parallel and perpendicular to the boost direction and compare them with the result obtained from the Kubo formula and semiclassical Boltzmann calculations.
III. 2D ANISOTROPIC TILTED WEYL HAMILTONIAN

The emergence of Dirac physics in condensed matter makes it possible to probe relativistic effects in materials even if in the latter, the electronic properties are generically described by more general Hamiltonians taking into the account the anisotropy and the tilt of the energy dispersion. Restricted to two dimensions, the general form of a Dirac Hamiltonian with two bands intersecting linearly in a conical shape reads \[ H_{\text{Weyl}} = v_0' k_\mu \sigma_0 + v_\nu' k_\nu \sigma_\mu. \] (13)

where the indices \( \mu \) and \( \nu \) run over the spatial dimensions \( x, y \). Here and in the remainder of this paper, we use a system of units with \( \hbar = 1 \), and \( \sigma^0 \) is the 2 \times 2 identity matrix. In addition to the anisotropic Fermi velocity described by the parameter \( v^\nu \), this Hamiltonian is specified by two parameters, \( v_0' \) and \( v_\nu' \), that characterize the tilt of the Dirac cones. This type of Hamiltonian describes the dispersion of a strained graphene sheet and organic conductors of the \( \alpha-(\text{BEDT-TTF})_2\text{I}_3 \) family. \[ \text{The additional term } v_0' k_x \sigma^0 \text{ can be understood on the basis of tight-binding models for graphene-like systems. Indeed, second-nearest-neighbor hopping induces the tilt of the Dirac cones if the latter are dragged away from the } K \text{ and } K' \text{ points of the first Brillouin zone (e.g. by strain) and breaks the particle-hole symmetry of the system.} \]

This sufficiently general form of the Hamiltonian (13) can be simplified by rescaling and rotating in momentum and pseudospin spaces. \[ e^{i \frac{\xi}{v_F} \sigma_z} H_{\text{Weyl}}(R^{-1} \mathbf{k}) e^{-i \frac{\xi}{v_F} \sigma_z} \longrightarrow H_{\text{Weyl}}(\mathbf{k}), \] (14)

These transformations remove the anisotropy and bring the tilt into the \( x \)-direction such that one obtains the minimal Hamiltonian

\[ H_{\text{Weyl}} = v_F k \cdot \sigma + v_0 k_x \sigma^0, \] (15)

where, for simplicity, we set the new Fermi and tilt velocities as \( v_F = v_x = v_y \) and \( v_0 = v_0' = \eta \) \( v_F \), respectively. The roto-scaling transformation \( R \) is

\[ R(\xi_{\text{tilt}}) = \begin{pmatrix} \cos \xi_{\text{tilt}} & \frac{v_y}{v_x} \sin \xi_{\text{tilt}} \\ -\sin \xi_{\text{tilt}} & \frac{v_x}{v_y} \cos \xi_{\text{tilt}} \end{pmatrix}, \] (16)

where \( \xi_{\text{tilt}} = \cos^{-1} \left( \eta^{-1} \frac{v_y}{v_x} \right) \), and \( 0 < \eta < 1 \) is the tilt parameter defined as

\[ \eta = \sqrt{\left( \frac{v_0'}{v_x} \right)^2 + \left( \frac{v_0'}{v_y} \right)^2}. \] (17)

Note that the roto-scaling transformation in real space is a pure \( \text{SO}(2) \) rotation that maps the original coordinates into a modified coordinates in order to remove the anisotropy in \( x - y \) plane, namely, \( (x, y)^\dagger \rightarrow (\cos \xi_{\text{tilt}} + i \sigma_y \sin \xi_{\text{tilt}})(x, \frac{v_y}{v_x} y)^\dagger \), and \( t \) stands for the matrix transpose. \[ 14,18 \]

Under external fields in the Landau gauge with vector potential \( A = -By \hat{x} \), this Hamiltonian reads

\[ H_{\text{Weyl}} - v_0 k_x \sigma_0 = v_F (k - eA) \cdot \sigma - e (E - v_0 B) y \sigma^0. \] (18)

As we stated in Sec. III in our gauge choice, \( k_x \) stays a good quantum number such that the term \( v_0 k_x \) on the left hand side is diagonal in pseudospin and momentum space – it only shifts the spectrum in energy and can thus be absorbed into the Hamiltonian. One notices that the Hamiltonian on the right hand side in Eq. (18) has the same covariant form as that of massless particles in a magnetic field \( B \hat{z} \) and an effective electric field \( E_{\text{eff}} = E - v_0 B \) in the \( y \)-direction. Notice that if we choose \( E_0 = v_0 B \), this effective electric field vanishes, and we obtain the Hamiltonian of graphene in the presence of a magnetic field. This evidently indicates that the tilt velocity plays the role of the drift velocity, \( v_0 = v_{\text{drift}} \).

A. Conductivity from the Boltzmann equation

The above reasoning allows us to find the Lorentz boost to the appropriate frame of reference, where we can easily calculate the transport coefficients. Notice also that the separation between a strong electric field \( E_0 \) and a bias field \( \delta E \), which was somewhat artificial in the previous section, is now much more natural – while \( \delta E \) describes a true electric field used to drive a current through the system, \( E_0 \) simply represents the tilt of the Dirac cones and not a physical electric field. Now assuming that the electric field is oriented in the \( y \)-direction as in Eq. (18), the bias and magnetic fields under the Lorentz boost in the \( x \)-direction transform as

\[ E'_x = \delta E_x, \quad E'_y = \gamma \delta E_y, \quad B' \simeq \gamma^{-1} B, \] (19)

where we have neglected a small correction \( \eta \delta E_x / c \ll B \) to the magnetic field in the last expression. Additionally, the components of the velocity of the electron in the lab frame, using Eq. (15), relate to those in the co-moving frame via

\[ v_x = v_F \frac{\cos \phi' + \eta}{1 + \eta \cos \phi'}, \quad v_y = v_F \frac{\sin \phi'}{\gamma (1 + \eta \cos \phi')} \] (20)

In lights of these relations, due to the relativistic aberration of angles under a boost, the polar angle transforms as

\[ d\phi = \frac{d\phi'}{\gamma (1 + \eta \cos \phi')}, \] (21)

which resembles the relativistic Doppler factor in relativistic beaming. This alternatively indicates the Jacobian of the transformation from the lab into the co-moving frame of reference. Using the transformed form of the velocities \( v_\mu \) in Eq. (20), the Lorentz invariant distribution function \( f_s \) in the co-moving frame and that
∇_{k′} ε′ = v_F (\cos \phi′, \sin \phi′), one can easily compute the 
magnetococonductivity tensor in the co-moving frame,

\begin{align}
J_x' = -\frac{e^2}{2\pi} \int \frac{d\varepsilon'}{\gamma} \left( \frac{\partial f^{(0)}}{\partial \varepsilon'} \right) \\
\int_0^{2\pi} \frac{d\phi'}{2\pi} (\cos \phi' + \eta) \cos \phi' / (1 + \eta \cos \phi')^2 \delta E_x',
\end{align}

(22)

\begin{align}
J_y' = -\frac{e^2}{2\pi} \int \frac{d\varepsilon'}{\gamma} \left( \frac{\partial f^{(0)}}{\partial \varepsilon'} \right) \\
\int_0^{2\pi} \frac{d\phi'}{2\pi} (\sin \phi')^2 / (1 + \eta \cos \phi')^2 \delta E_y'.
\end{align}

(23)

In restoring the nonequilibrium current in the lab frame 
we only need to consider the transformation of the bias 
fields as given in \( [19] \). Now noting the transformation 
of the energy as \( \varepsilon = \gamma^{-1} \varepsilon' \) and that at zero temperature the 
Dirac distribution gives \( \left( \partial f^{(0)} / \partial \varepsilon' \right) = -\gamma^{-1} \delta(\varepsilon-\varepsilon_F) \), 
the current and consequently the conductivity perpendicular 
to the boost direction (which is identified with the tilt direction) in the lab frame reads

\begin{align}
\sigma_{\text{perp}} = \sigma_{yy} = \sigma_0 \left( \frac{1 - \eta}{\eta^2} \right) \varepsilon \tau,
\end{align}

(24)

where \( \sigma_0 = e^2/h \) is a fundamental constant such that 
h/e^2 = 25.8 kΩ. Similarly, we find for the conductivity 
in the direction of the boost (tilt direction)

\begin{align}
\sigma_{\text{tilt}} = \sigma_{xx} = \sigma_0 \left( \frac{1 - \eta^2}{\eta^2} \right) \varepsilon \tau,
\end{align}

(25)

and one thus realizes that the ratio

\begin{align}
\frac{\sigma_{\text{perp}}}{\sigma_{\text{tilt}}} = \gamma = (1 - \eta^2)^{-1/2}
\end{align}

(26)

is a direct measure of the relativistic factor and thus of 
the tilt strength \( \eta \).

Notice that the above expressions for the conductivity 
calculated in the framework of the covariant Boltzmann 
equation coincide, in the limit \( \gamma = 1 \), with those obtained 
previously within semiclassical non-relativistic Boltzmann calculations.\(^6\) We emphasize that both the relativistic 
and the non-relativistic Boltzmann approaches do not 
account for quantum corrections in the close vicinity 
of the band contact point that are responsible, e.g., for 
the minimal conductivity of graphene.\(^6\) These are better 
taken into account within the Kubo formalism, which is 
discussed in the following subsection.

B. Kubo formula

To verify the validity of the covariant approach, we 
compute the conductivity using the Kubo formula and 
check the agreement of both methods. Noting that \( v_0 = \eta \ v_F \), the unitary transformation \( U = \exp(-i \pi \sigma \cdot \hat{n} / 4) \),

\begin{align}
\varepsilon_\alpha = v_F \ k \ (\eta \cos \phi + \alpha),
\end{align}

(27)

where \( \phi \) is the momentum vector polar angle and \( \alpha = \pm 1 \) is 
the band index. The corresponding Fermi surface is 
an ellipse. As in the sections above, we consider a system 
with a moderate tilt \( \eta < 1 \), such that we remain in 
the magnetic regime, where we can get rid of the tilt by 
the appropriate Lorentz boost. In the limit of zero 
temperature \( (T = 0) \) for noninteracting systems, the Kubo-Streda formula for the diagonal conductivity gives\(^6\)

\begin{align}
\Pi_{\mu\mu} = i \frac{e^2}{\pi} \sum_k d\varepsilon \left( -\frac{\partial f^{(0)}}{\partial \varepsilon} \right) \Pi_{\mu\mu},
\end{align}

(28)

where the polarization tensor reads

\begin{align}
\Pi_{\mu\mu} = \text{Tr} (v_\mu \text{Im} G^R v_\mu \text{Im} G^R),
\end{align}

(29)

\( \Pi_{\mu\mu} = \text{Tr} (v_\mu \text{Im} G^R v_\mu \text{Im} G^R) \),

(29)

\( \text{Im} G^R = (G^R - G^A)/2i \) and the Heisenberg equation gives 
the velocity operator as \( v_\mu = i [\hat{x}_\mu, H_{\text{Weyl}}] \) 
where \( \hat{x}_\mu \) is the position operator. In the self-consistent Born approximation, 
the first contribution to the advanced (A) and 
retarded (R) Green’s function of graphene in the energy 
eigenbasis reads

\begin{align}
G^{R/A}(\varepsilon) = \sum_{\alpha=\pm} \left( 1 + \alpha \frac{\varepsilon}{\varepsilon - v \kappa (\eta \cos \phi + \alpha) \pm i \Gamma} \right),
\end{align}

(30)

\begin{align}
\text{Im} G^R = \frac{1}{\Gamma} \left( A_+ \quad 0 \right),
\end{align}

(31)

where the decay term is proportional to the scattering 
(relaxation) time \( \tau \) via \( \Gamma^{-1} = 2 \tau \). Using the above definition, 
the spectral function is a diagonal matrix in energy 
basis and may be written as

\begin{align}
\mathfrak{A}_+ = \frac{1}{|z - y(\eta \cos \phi \pm 1)|^2 + 1},
\end{align}

(32)

where the dimensionless variables \( z = \varepsilon / \Gamma \) 
and \( y = v_F k / \Gamma \). Now to compute conductivity, we first 
express the velocity matrix in the helicity basis as

\begin{align}
v_\mu = e v_F \left( \begin{array}{c}
v_\mu^\alpha \\
v_\mu^{\alpha'}
\end{array} \right),
\end{align}

(33)

where the band velocities are given by

\begin{align}
v_\mu^\alpha = (\eta + \alpha \cos \phi, \sin \phi),
\end{align}

(34)

while the off-diagonal velocities read

\begin{align}
v_\mu^{\alpha'} = (i \sin \phi e^{-i \phi}, -i \cos \phi e^{-i \phi}).
\end{align}

(35)
We then obtain the polarization as

$$
\Pi_{\mu\nu} = \sum_{\alpha} |\mathbf{A}_\alpha v^\alpha|^2 + \sum_{\alpha\neq\alpha'} |\mathbf{A}_\alpha v^\alpha A_{\alpha'} v'^{\alpha'}|,
$$

$$
= \mathbf{A}_x^2 (v_x^+)^2 + \mathbf{A}_x^2 (v_x^-)^2 + 2 \mathbf{A}_y (v_y^-)^2,
$$

where the second term in the first lines shows the quantum coherent mixing of the bands whereas the first term gives the intraband contribution and we remind that \(\alpha\) and \(\alpha'\) are the band indices. Now writing the momentum summation in terms of the momentum and angular integrals as \(\sum_k \to (2\pi)^{-2} \int_0^\infty dy \int_0^{2\pi} d\phi\) and performing the integrals we finally obtain

$$
\sigma_{yy} = \frac{\sigma_0}{\pi} \left[ \frac{\gamma - 1}{\eta^2} (1 + z \tan^{-1} z) + \frac{2z (\tan^{-1} z - z) (1 - z/\sqrt{z^2 + \eta^2}) + \eta^2}{2\eta^2} \right],
$$

$$
\sigma_{xx} = \frac{\sigma_0}{\pi} \left[ \frac{\gamma - 1}{\eta^2 \gamma} (1 + z \tan^{-1} z) + \frac{\eta^2 + 2(z - \tan^{-1} z)(z - \sqrt{z^2 + \eta^2})}{2\eta^2} \right],
$$

for the diagonal conductivities in the \(x-\) and \(y-\)directions.

Equations (47) and (48) are the main result of this section and merit a detailed discussion. First, let us investigate the zero-tilt case, \(\eta \to 0\), in which we obtain

$$
\sigma_{xx} = \sigma_{yy} = \sigma_0 \left[ \frac{1}{2\pi} (1 + z \tan^{-1} z) + \frac{\tan^{-1} z}{2\pi z} \right].
$$

In the high-energy (diffusive) limit \(z \gg 1\), we then retrieve the standard result, in agreement with the semiclassical Boltzmann approach, namely

$$
\sigma_{xx} = \sigma_{yy} = \sigma_0 \varepsilon \tau / 2,
$$

which coincides with Eqs. (24) and (25) in the static limit \(\omega = 0\). We emphasize that in the opposite limit of \(z \ll 1\), i.e. at energies close to the Dirac point, band mixing yields quantum corrections that are beyond the reach of the semiclassical Boltzmann approach, within the first Born and relaxation time approximation, and therefore a full quantum treatment is needed. Indeed, the Kubo and Boltzmann approaches yield different results then, as we will show below.

Before comparing both approaches, let us discuss in detail some aspects of the conductivities calculated from the Kubo formula. For the case of non-zero tilt in the diffusive limit, \(z = \varepsilon / \Gamma = 2\varepsilon \tau \gg 1\), the conductivities (37) and (38) can be rewritten as

$$
\sigma_{\text{tilt}}(= \sigma_{xx}) = \sigma_0 \left( \frac{\gamma - 1}{\eta^2 \gamma} \frac{z}{2} + \frac{1}{4z} \right),
$$

$$
\sigma_{\text{perp}}(= \sigma_{yy}) = \sigma_0 \left( \frac{\gamma - 1}{\eta^2 \gamma} \frac{z}{2} + \frac{1}{4z} \right),
$$

where the last term takes into account the first (quantum) correction in \(1/z\). The conductivities (41) and (42) are plotted in Fig. 1 as a function of the tilt parameter \(\eta\) for different values of \(z\). First notice that the result coincides with that in Eq. (40) for zero tilt, i.e. in the limit \(\eta \to 0\), where we retrieve also \(\sigma_{xx} = \sigma_{yy}\). Figure 2 shows a comparison between the conductivities (41) and (42) obtained from the Kubo approach (dashed lines) and those from the covariant Boltzmann formula, Eqs. (25) and (24). The only difference resides in the offset of \(\delta \sigma = \sigma_0 / 4z = \sigma_0 / 8\varepsilon \tau\), which is due to the quantum corrections that are neglected in the latter approach. This offset becomes less relevant at larger values of the tilt where the conductivities are enhanced. This enhancement can globally be understood from the density of states that increases with the tilt parameter. However, this argument in terms of the density of states does not explain the strong anisotropy in the conductivities. While the conductivity \(\sigma_{\text{tilt}}\) along the tilt direction remains finite and saturates at a value of

$$
\sigma_{\text{tilt}}(\eta \to 1) = \sigma_0 \left( \varepsilon \tau + \frac{1}{8\varepsilon \tau} \right),
$$

i.e. the Boltzmann contribution is doubled with respect to the \(\eta \to 0\) limit (40), and the conductivity in the perpendicular direction diverges (43) as

$$
\sigma_{\text{perp}}(\eta \to 1) \sim \sigma_0 \varepsilon \tau / \sqrt{1 - \eta^2}.
$$

One thus obtains the same behavior (26) for the ratio between the conductivities as in the Boltzmann analysis. This difference in the conductivities stems from the anisotropy of the velocities and mobilities in two directions. Experimentally observed results show that, when...
applying strain to the graphene crystal, the group velocity of the carriers along the strain drops by increasing the strain whereas the group velocity along the perpendicular direction to the strain increases. The tilt parameter for energies at $\sigma_{\text{tilt}}/\sigma_0$ and $\sigma_{\perp}/\sigma_0$ of the tilted Graphene in terms of the tilt parameter for energies at $\gamma_z = 1, 5$ and 10. Conductivity Calculated from the Kubo formula (dashed) agree with the covariant Boltzmann approach (solid) in Ref. [25, 27]. While the perpendicular conductivity diverges in critical limit; the conductivity parallel to tilt direction saturates at finite value.

**FIG. 2.** (Color online) Comparison between the conductivities $\sigma_{\text{tilt}}/\sigma_0$ and $\sigma_{\perp}/\sigma_0$ of the tilted Graphene in terms of the tilt parameter for energies at $\gamma_z = 1, 5$ and 10. Conductivity Calculated from the Kubo formula (dashed) agree with the covariant Boltzmann approach (solid) in Ref. [25, 27]. While the perpendicular conductivity diverges in critical limit; the conductivity parallel to tilt direction saturates at finite value.

**IV. TYPE-I WEYL SEMIMETAL**

**A. Boltzmann equation**

In this section, we present the core result of the paper and investigate the magnetoconductivity of type-I Weyl semimetals based on the covariant formalism introduced in Sec. II. Consider a Weyl material in the presence of perpendicular electric and magnetic fields – again, we choose the magnetic field to be oriented in the $z$-direction and a tilt in the $x$-direction, i.e. the associated electric field is oriented in the $y$-direction. The Hamiltonian of the system reads

$$H = v_F \sigma \cdot (k - eA) - eE_0y.$$  \hspace{1cm} (45)

We apply a Lorentz boost in the $x$-direction with drift velocity $v_0 = E_0/B$ to work in a frame where the electric field vanishes. Noting the representation of the Lorentz boost, the bias fields transform along the boost direction as $\delta E_{y(z)} = \gamma \delta E_{y(z)}$, $\delta E_{x} = \delta E_{x}$ and the magnetic field as $B_y = \gamma^{-1} B_z$. Taking the axis of the polar coordinate, $\theta = 0$, along the boost direction simplifies the expressions, and using this parametrization in the co-moving frame we obtain

$$\nabla_{k'} \epsilon' = v_F (\cos \theta', \sin \theta' \cos \phi', \sin \theta' \sin \phi').$$

Next, implementing the relativistic addition formula, the velocities transform accordingly as

$$v_x = v_F \frac{\eta + \cos \theta'}{1 + \eta \cos \theta'},$$  \hspace{1cm} (46)

$$v_y = v_F \frac{\eta \sin \theta'}{1 + \eta \cos \theta'},$$  \hspace{1cm} (47)

$$v_z = v_F \frac{\sin \theta' \sin \phi'}{1 + \eta \cos \theta'}.$$  \hspace{1cm} (48)

Considering the relativistic aberration of the polar angle $\theta$ under the Lorentz boost, (21), then the solid angle and consequently the conic cross section transforms as $d\Omega = \gamma^{-2} d\Omega'/(1 + \eta \cos \theta')^2$ where $d\Omega = d(\cos \theta') d\phi'/ (4\pi)$ and the azimuthal angle as $d\phi = d\phi'$. Thus, the nonequilibrium current in the co-moving frame for each spatial direction will give

$$J_x' = -\frac{\epsilon^2}{2\pi^2} \int \frac{\epsilon'^2 d\epsilon'}{v_F} \frac{\partial f^{(0)}}{\partial \epsilon} \int \frac{d\Omega'}{\gamma^2} \frac{\cos \theta' + \eta \cos \theta'}{(1 + \eta \cos \theta')^3} \delta E_z,$$  \hspace{1cm} (49)

$$J_y' = -\frac{\epsilon^2}{2\pi^2} \int \frac{\epsilon'^2 d\epsilon'}{v_F} \frac{\partial f^{(0)}}{\partial \epsilon} \int \frac{d\Omega'}{\gamma^2} \frac{\sin \theta' \cos \theta'}{(1 + \eta \cos \theta')^3} \delta E_y' ,$$  \hspace{1cm} (50)

$$J_z' = -\frac{\epsilon^2}{2\pi^2} \int \frac{\epsilon'^2 d\epsilon'}{v_F} \frac{\partial f^{(0)}}{\partial \epsilon} \int \frac{d\Omega'}{\gamma^3} \frac{\sin \theta' \sin \phi' \sin \theta'}{(1 + \eta \cos \theta')^3} \delta E_z.$$  \hspace{1cm} (51)
Next, by taking into the account the transformation of the bias fields, we obtain in the lab frame for the conductivities parallel to the boost

\[ \sigma_{\text{tilt}} = \sigma_0 \frac{\varepsilon^2}{\pi v_F} \left( \frac{\tanh^{-1} \eta - \eta}{\eta^3} \right). \] (52)

Similarly for the conductivity in direction perpendicular to the boost we get

\[ \sigma_{\text{perp}} = \sigma_0 \frac{\varepsilon^2}{\pi v_F} \left( \frac{\gamma \eta - \tanh^{-1} \eta}{2\eta^3} \right). \] (53)

In the zero-tilt limit, \( \gamma \to 1 \), we obtain the isotropic conductivity as \( \sigma_{\text{tilt}} = \sigma_{\text{perp}} = \sigma_0 \varepsilon^2 / (3\pi v_F) \), recovering the same result as obtained using semiclassical Boltzmann equation in zero temperature, reported elsewhere in the literature.\(^64-66\)

### B. Kubo formalism

Again, we compare the result obtained from the covariant Boltzmann equation to the conductivity calculated within linear response theory. We write a minimal type-I Weyl semimetal Hamiltonian with a tilt in the direction \( \mathbf{k}_z \) (Fig. 3), parametrized by \( \eta = v_0 / v < 1 \) as

\[ H = v_F \mathbf{k} \cdot \mathbf{\sigma} + \eta v_F k_z. \] (54)

Similar to the case of graphene, the unitary transformation \( U = e^{-i\mathbf{\Phi} \mathbf{\hat{r}}} \), where \( \mathbf{\hat{n}} = \mathbf{\hat{z}} \times \mathbf{k} / \sqrt{k_x^2 + k_y^2} \), brings the Weyl Hamiltonian into the diagonal form. In this basis, the Green’s function through the polar parametrization reads

\[ G^{R/A}(\varepsilon) = \sum_{\alpha = \pm} \frac{1 + \alpha \sigma_z}{\varepsilon - \nu k(\eta \cos \theta + \alpha)} \pm i\Gamma, \] (55)

and the spectral function, in terms of the dimensionless variables, give similar diagonal form as defined in Eq. 31 with

\[ \mathcal{A}_{\pm} = \frac{1}{(z - y(\eta \cos \theta \pm 1))^2 + 1}. \] (56)

Writing the velocity operators in energy basis, as given in Eq. 39, we obtain the band diagonal velocities (using spherical coordinates)

\[ v_\mu^\alpha = (\alpha \sin \theta \cos \phi, \alpha \sin \theta \sin \phi, \eta + \alpha \cos \theta), \] (57)

and the off-diagonal velocities are

\[ v_{\mu\nu}^{\alpha\alpha'} = e^{-i\phi}(\cos \theta \cos \phi + i \sin \phi), \] (58)

\[ v_{\mu\nu}^{\alpha\alpha'} = e^{-i\phi}(\cos \theta \sin \phi - i \cos \phi), \] (59)

\[ v_{\mu\nu}^{\alpha\alpha'} = -e^{-i\phi} \sin \phi. \] (60)

To compute the conductivity using Eq. 28, we write accordingly the conductivity as

\[ \sigma_{\mu\nu} = \frac{e^2}{\pi v_F} \langle \Pi_{\mu\nu} \rangle, \] (61)

where the polarization tensor is defined as before in Eq. 36 with the velocities given in Eqs. 57-60. Moreover, we define the three-dimensional momentum and angular integrals now as \( \langle \cdots \rangle = (2\pi)^{-3} \int_0^{\infty} y^2 dy \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi. \) One thus finds the conductivity

\[ \sigma_{\perp} (= \sigma_{xx} = \sigma_{yy}) = \sigma_0 \frac{\Gamma}{2\pi v_F} (a_1 z^2 + a_2), \] (62)

\[ \sigma_{\text{tilt}} (= \sigma_{zz}) = \sigma_0 \frac{\Gamma}{2\pi v_F} (a_3 z^2 + a_4), \] (63)

where the second term yields the (intrinsic) quantum correction to the conductivity due to the band mixing effects\(^67\) and the first term is the Drude conductivity which coincides with the result of the Boltzmann equation in the diffusive limit \( z \gg 1 \). The other parameters are defined as

\[ a_1 = \frac{\eta - (1 - \eta^2) \tanh^{-1} \eta}{2\eta^2(1 - \eta^2)}, \] (64)

\[ a_2 = \frac{\eta + (1 - \eta^2) \tanh^{-1} \eta}{2\eta(1 - \eta^2)}, \] (65)

\[ a_3 = \frac{\tanh^{-1} \eta - \eta}{\eta^3}, \] (66)

\[ a_4 = \frac{\tanh^{-1} \eta}{\eta}, \] (67)

and we have restored \( \sigma_0 \) by noting that \( h = 2\pi \). In the limit of zero tilt (\( \eta \to 0 \)), we obtain the isotropic conductivity

\[ \sigma_{xx} (= \sigma_{yy} = \sigma_{zz}) = \sigma_0 \frac{\Gamma}{4\pi v_F} (z^2 / 3 + 1), \] (68)
recovering the results obtained in previous works using linear response calculations. In figure 4, we compare the conductivity of a type-I WSM for zero tilt with Fermi velocity $v \approx c/300$, where $c$ is the speed of light, and constant relaxation time $\tau = 10^{-7}$ s using the Boltzmann approach and Kubo formalism. As in the 2D case, we notice a significant increase of the conductivities with the tilt in the case of moderate tilt. The discrepancy between the Kubo and Boltzmann conductivities is inherited from the zero-tilt case, where we have already noticed that the Kubo formula systematically yields a larger conductivity. In the same manner as in the 2D case discussed above, the main sources of this discrepancy are quantum interference effects between the two bands that are correctly taken into account via the Kubo formula whereas they are not treated yet in the first order linear approximation of the Boltzmann equation and thus are not present in its results. In the zero-tilt limit, the (isotropic) conductivity shows a parabolic behavior in energy that can be understood qualitatively from the behavior of the density of states for Weyl semimetal, which is quadratic in energy.

In the nonzero tilt limit, we find again, from our calculations based on the Kubo formula, that the conductivity $\sigma_{\text{perp}}$ in the direction perpendicular to the tilt increases and diverges for $\eta \rightarrow 1$ and that the conductivity $\sigma_{\text{tilt}}$ in the tilt direction is smaller than $\sigma_{\text{perp}}$. However, contrary to the 2D case, $\sigma_{\text{tilt}}$ now also diverges upon increasing tilt (Fig. 5). The parallel conductivity can be linked to the longitudinal magnetoresistivity if we set the magnetic field in $z$-direction. Therefore the finite value of conductivity in the direction parallel to the tilt is a theoretical evidence of the enhanced longitudinal magnetoresistance observed in WSM having tilted Weyl points.

This anisotropy in the conductivity can be understood from the covariant point of view where the tilt velocity is identified with an effective electric field normal to its direction, $v_0 = E_0 \times B / B^2$. Thus in effect, the auxiliary field enhances the conductivity along the field (perpendicular the tilt direction).

To further corroborate our approach in terms of the covariant Boltzmann equation, we further inspect the anisotropy and the directional features in the conductivity in the framework of a semiclassical non-covariant Boltzmann equation. In this case, we take into account the tilt-induced anisotropy directly in the expression for the averaged velocities, without appealing to Lorentz boosts to a frame of reference, where the dispersion becomes effectively isotropic. We can then write the conductivity as

$$\sigma_{\mu \nu} = \frac{e^2 \tau}{2\pi^2 v_F} \left\langle \frac{v_\mu v_\nu}{(1 + \eta \cos \theta)^2} \right\rangle_\Omega,$$  

in terms of the band velocities given in Eq. 57, where $\langle \cdot \cdot \rangle_\Omega$ denotes the averaging over the random solid angles. In performing the angular integrals, the velocity $v_z$ compensates partially for the diverging behavior of the density of states and, as a result, yields a less divergent expression for $\sigma_{zz}$. In contrast, the $v_x$ and $v_y$ velocities produce a more singular result and thus yield a strongly divergent behavior of $\sigma_{xx}$ and $\sigma_{yy}$. The calculated conductivity using the non-covariant Boltzmann equation then gives the expressions

$$\sigma_{xx} = \sigma_{yy} = \sigma_0 \frac{\varepsilon^2 \tau}{\pi v_F} \frac{\eta - (1 - \eta^2) \tanh^{-1} \eta}{2\eta^3 (1 - \eta^2)},$$

$$\sigma_{zz} = \sigma_0 \frac{\varepsilon^2 \tau}{\pi v_F} \frac{\tanh^{-1} \eta - \eta}{\eta^4},$$

which agree with 55 and 52 obtained from the covariant Boltzmann approach. This further confirms the
accuracy of the results obtained from the Kubo formula and covariant Boltzmann approach.

![Conductivity Graph](image)

**FIG. 6.** (Color online) Conductivity of the tilted type-I WSM perpendicular and parallel to the tilt direction computed from the covariant Boltzmann equation (solid) and Kubo formula (dashed). The conductivities are expressed as a function of the tilt parameter for different values of the normalized energy \( z \). The perpendicular conductivity enhances and diverges at critical value \( \eta = 1 \) while the parallel increases but stays finite in the critical limit.

As in the 2D case and the above-mentioned zero-tilt limit for Weyl semimetals, we therefore find a high degree of accuracy between the Kubo-formula approach and the results from the covariant Boltzmann equation over the whole range of tilt parameters \( \eta \). This is summarized in Fig. 6. Again, the main difference stems from the conductivity offset due to interband contributions that are neglected in the Boltzmann approach. Overall, we find that the increase in the conductivities with the tilt parameter is more pronounced at larger energies, i.e. upon increasing \( z \). This is also represented in the form of a color plot in Fig. 7 where we plot the conductivity \( \sigma_{\perp} / \sigma_w \) in the plane spanned by the tilt parameter and the energy.

![Color Density Plot](image)

**FIG. 7.** (Color online) Density plot of the normalized perpendicular conductivity in terms of the tilt degree and chemical energy.

**C. Density of states as a function of the tilt parameter**

The increase of the conductivity with the tilt of the Weyl cones can be understood qualitatively from an analysis of the density of states (DOS), \( g(\varepsilon) \), which enters into the expression of the conductivity in the Einstein relation for a system of \( d \)-dimension

\[
\sigma_E = \frac{e^2 \bar{v}^2 F}{d} g(\varepsilon) \propto g(\varepsilon)
\]

that we use here for a qualitative analysis. The parameter \( \bar{v} \) represents an average velocity since the Einstein relation does not make a difference between the directions contrary to the more appropriate Boltzmann or Kubo for-

![Density of States Graph](image)

**FIG. 8.** Increasing pattern of the normalized Density of states \( g(\varepsilon) / g_0 \) of type-I WSM by the degree of tilting. \( g_0 = \frac{v^2 F}{2 \pi^2 v_F^2} \).
mula). The tilt of the Weyl cones enlarges the Fermi surface and thus increases the DOS. This yields eventually an enhanced conductivity in the type-I WSM. The hike of the DOS can be justified using both Lorentz covariance and Sommerfeld expansion. From the covariance point of view and due to the length contraction in lab frame in the direction of boost, it is easy to see that the particle density scales as $dn = \gamma dn'$. Furthermore, the transformation of energy and hence the DOS in the inverse Lorentz boost back to the lab frame gives $g(\varepsilon) = \gamma^2 g'(\varepsilon')$ for the relation between the DOS in the co-moving and lab frames of reference. Therefore the expression for the DOS of Weyl semimetals in the lab frame reads

$$g(\varepsilon, \eta) = \frac{\gamma^4}{2\pi^2} \varepsilon^2 \frac{\varepsilon^2 - v_F^2}{3},$$

which is plotted in Fig. 9. One notices that the behavior of the DOS reflects indeed, as expected, that of the conductivities. However, we insist that our argument in terms of the DOS is qualitative and not sufficient to explain the anisotropy in the conductivities $\sigma_{\text{tilt}}$ and $\sigma_{\text{perp}}$. Notice that the increase of the DOS as a function of the tilt parameter can be absorbed into a renormalized Fermi velocity (see Fig. 9), with

$$\frac{v_F^*}{v_F} = (1 - \eta^2)^{2/3}. \quad (74)$$

In the limit of $0 < \eta < 1$ this integral yields the same result (73) as that obtained from the covariance point of view.

V. Conclusion

In conclusion, we have studied the influence of the tilt in the dispersion of graphene (or two-dimensional graphene-like systems) and type-I WSM on the magnetococonductivity. The tilt can be described elegantly within a covariant framework of the Dirac equation since it can be viewed as an effective electric field that conspires with the magnetic field. In the case of type-I WSM with moderate tilts, $\eta < 1$, a Lorentz boost into the co-moving frame of reference characterized precisely by the tilt velocity $v_0$ allows one to get rid of the electric field – this is the so-called magnetic regime in electrodynamics. This relativistic effect, accompanied by a Lorentz transformation back to the lab frame, yields an increased conductivity that we have studied here within the covariant form of the Boltzmann transport equation both in two and three spatial dimensions. We have systematically compared the results of the conductivity from the covariant Boltzmann equation to calculations within the Kubo formula of linear response theory. Furthermore, the conductivities perpendicular to the tilt direction, calculated within the Boltzmann and Kubo approach, are enhanced by an extra factor, both in two- and in three-dimensional cases, as compared to the direction parallel to the tilt. This demonstrates that the transport in a tilted system becomes directional where $\sigma_{\text{tilt}} < \sigma_{\text{perp}}$. This smaller conductivity along the field direction ($z$-direction) agrees with the experimental evidences on the extremely large and nonsaturating longitudinal ($z$-direction) magnetoresistance reported on type-I WSM.

Our findings can qualitatively be understood with the help of the DOS, to which the conductivities are roughly proportional, within the simplified picture provided by Einstein’s relation. The tilt enhances the DOS by some power of the relativistic Lorentz factor $\gamma = 1/\sqrt{1 - \eta^2}$, which generally enters the expressions, for the conductivities on the one hand and for the DOS, scattering time and effective Fermi velocity on the other hand. We further observe that the power of the Lorentz factor that enters in the expressions of the (surface) conductivities, depends on the orientation of the tilt with respect to that of the conductivity as well as on the system’s spatial dimension. For the renormalization of the effective Fermi velocity, for example, we find $v_F^* = v_F \gamma^{-\beta}$ with $\beta_{2\text{D}} = 3/4$ and $\beta_{3\text{D}} = 2/3$. Thus measuring the ratio $v_F^*/v_F$ experimentally would allow for an experimental determination of the tilt and its magnitude in a type-I WSM.

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