Effects of curvature on dynamics

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Abstract

In this article we discuss the effect of curvature on dynamics when a physical system moves adiabatically in a curved space. These effects give a way to measure the curvature of the space intrinsically without referring to higher dimensional space. Two interesting examples, the Foucault Pendulum and the perihelion shift of planetary orbits, are presented in a simple geometric way. A paper model is presented to see the perihelion shift.
I. Introduction

How do we know whether the space in which we live is curved or flat? When we imagine a curved line or a curved surface, like the surface of a sphere, the inherent picture is the line or the sphere kept in a flat space of higher dimension. We can then describe the curvature of these objects in terms of a flat euclidian co-ordinate system. Imagine two dimensional creatures on the surface of the earth with no access to height and depth. How do they know they are on the surface of a sphere and not on a piece of infinite flat land. Can an ant know whether the wire it is crawling on is straight or curved?

The above situations are hypothetical. But how do we answer if the same question is asked about the three dimensional space we live in.

There is an intrinsic notion of curvature which can be measured without referring to the external higher dimensional coordinates\(^1\). This is given by the Gauss-Bonet theorem\(^2\) which describes curvature of a surface through geometrical properties only within the surface. This is like the curl of a vector field. Though we can express it in terms of a coordinate system we can define it in terms of the Stoke’s Theorem without referring to any coordinate system. The Stoke’s theorem uses only the value of a vector field along a closed curve to measure how much the vector field curls around a region enclosed by the curve. There are a number of phenomenon in physics where we can see the effect of a property over a region, within a given boundary, on an integral done only over the boundary. When a quantum system is moved adiabatically along a closed curve \(C\) the eigenstates evolve and return to the same state. It however picks up a phase dependent on the curve \(C\). This is purely a geometrical effect independent of the rate at which the curve \(C\) is traversed. This phase is called the geometric phase or the Berry’s phase\(^3\). Though these phases are ignorable for the complete state of the system they can be observed through interference effects as in Aharanov-Bohm effect\(^4\). A similar effect in optics using polarization states of light was shown by S. Panchratnam\(^5\). A comprehensive analysis of such phenomena can be found in\(^6\).

In a flat space vectors at different locations can be compared since we have a global coordinate system. We say two unit vectors at different locations are same if they are parallel. If we transport one vector to the location of the other they will coincide. This action of moving a vector from one location to another is called parallel transport of a vector. On a curved space when we parallel transport a vector from one location to another
the final vector is not unique. It depends upon the path along which we do the parallel transport. When the path is a closed loop the vector finally may not coincide with itself. The change in the orientation of the vector depends upon the curvature of surface enclosed by the loop.

Laws of Mechanics govern relationships between dynamical vectors. As the experimental set up moves in space or a particle moves along its trajectory, the dynamical vectors will undergo parallel transport. We analyze the effect of curvature on these vectors and how can such effects indicate the curvature of space intrinsically.

In the next section we present the description of the process of parallel transport of a vector in a given space. Then we present two interesting effects of curvature on dynamics. In section 3 we discuss how the curvature of the latitude along which a Foucault Pendulum travels with the earth, affects its plane of oscillation. In section 4 we discuss in a very simplified and approximate way how the curvature of space due to gravity causes the perihelion shifts of the elliptical orbits of planets around the sun. In section 5 we present the conclusion.

II. PARALLEL TRANSPORT

The fifth postulate of Euclid states that “Given a straight line and a point outside it, one and only one line can be drawn parallel to it”. This postulate gives a very well defined notion of parallel vectors at different locations in space. Given a vector $\vec{v}$ at one location, vectors at other locations are parallel to it if they are contained by the unique line parallel to $\vec{v}$. So at every location we can only have one vector parallel to $\vec{v}$. Here we are only interested in the direction of the vectors and hence we will only consider unit vectors. Other vectors are obtained by just multiplying this unit vectors by real numbers. On a curved surface the fifth postulate of Euclid don’t hold. We don’t have a well defined notion of parallel lines on a curved surface. If parallel lines are defined as lines on the surface that don’t intersect then from a point outside a line we can either have several lines parallel to it or no lines parallel to it. The examples of the two types are a hyperboloid surface and an ellipsoidal surface respectively.

In a flat space performing parallel transport of a vector is straightforward. We have a global cartesian coordinate system and we just transport the vector along any path keeping
its direction fixed with respect to this cartesian coordinate system. On a curved surface we don’t have a global cartesian coordinate system since we can’t have a family of parallel lines. We only have local tangent space and we can have coordinate system and vectors only in this tangent space. As we move along the curve the tangent plane changes. How do we compare vectors in two different tangent space? The tangent space represents the vector space at a point on the curved space. So, as we move from one point to another, there should be no relative motion between the curved surface and the tangent space. Such a movement of the tangent plane over the curved surface is called rolling without slipping. This is like the perfect rolling of a wheel on a surface, where the part of the wheel near the point of contact is always at rest with respect to the surface. The vector being parallel transported must retain its direction with respect to the coordinate system on this rolling plane. This process defines the parallel transport of a vector in a curved space.

The direction of motion at each point on the path is along the tangent to the path. This is also a vector on the rolling tangent plane. As we move, this vector may change its orientation with respect to the coordinate system on the tangent plane. If it doesn’t then the path is called a geodesic on the curved surface. A geodesic is equivalent to a straight line on a flat surface. In a flat space frames that move in a straight line with uniform speed are inertial frames. Frames that move along a curve that is not straight in flat space are non inertial. Likewise, in a curved space frames that move along a geodesic with uniform speed are inertial frames while frames that move along any other path on the curved space are non inertial.

Euclidean or flat spaces are characterized by a unique or global tangent space all over. So whichever path we travel from point X to point Y, the final tangent space at Y is the same, and it is the same as the one we had at the point X. If the path is a loop starting and ending at X, the tangent spaces at the beginning and the end of the journey will coincide. So every vector parallel transported along a loop will coincide with itself in a flat space. In a curved space the tangent vector spaces we obtain after rolling from point X to point Y depend upon the path taken. This is shown in Figure[1]a and b) where we roll the tangent plane, over a spherical surface, along two paths. Path 1 is along a great circle from X to Y while path 2 is via a third point Z. The paths from X to Z and Z to Y are great circles which intersect at right angle at the point Z. We can see that the final orientation of the tangent plane at Y is different in the two cases. Let us combine the two paths to form a
loop, starting from $X$, via $Z$, to $Y$ along path 2, and coming back from $Y$ to $X$ along path 1. See Figure 1(c) We can see that the orientation of the tangent plane at the end of the journey changes from the one we started with.

![Figure 1](image)

FIG. 1: $X$, $Y$, $Z$ are three points on the sphere connected by geodesics. (a) and (b) shows the tangent plane being parallel transported along two different paths. The final orientations of the tangent plane in the two cases are different. In (c) the plane is parallel transported along a closed loop. The final orientation of the tangent plane is rotated by $90^\circ$ with respect to the initial one.

The change of orientation of the tangent plane is the same as the change in orientation of any vector parallel transported along the curve on the surface since the parallel transported vector always maintains its direction constant with respect to the co-rolling tangent planes. So after being parallel transported along a closed curve a vector coincides with itself on a flat space while it changes orientation with respect to itself in a curved space. This change of orientation is related to the curvature over the area of the surface enclosed by the loop through the Gauss-Bonnet Theorem. If the vector rotates by an angle $\alpha$ as it goes around the curve $C$ then

$$2\pi - \alpha = \oint_S \frac{da}{\hat{R}^2}$$

(1)

where $R$ is the radius of curvature at every point of the surface and $S$ is the region over the surface enclosed by the closed curve $C$. The quantity $da/\hat{R}^2$ on the r.h.s of Eq. 1 is the solid angle $d\Omega$ subtended by the area element $da$ on the surface at the local center of curvature. The angle $\alpha$ is the amount by which the tangent vector to the curve rotates on the surface of the manifold as it completes the loop. So it is a measure of the curvature of the curve within the manifold, called the intrinsic curvature of the curve. The curvature at a point on a surface is given as $K = 1/R^2$, called the gaussian curvature. The solid angle $\oint_S Kda$
gives a measure of the total amount by which the surface curves over the area enclosed by the closed curve $C$. In terms of $\mathcal{K}$ Eq.1 becomes

$$2\pi - \alpha = \oint_S \mathcal{K} da$$

(2)

The Gauss-Bonnet theorem given by Eq.2 gives a way to access the curvature of the manifold, with respect to the external higher dimensional euclidian space, through the intrinsic curvature of a curve which can be measured within the manifold.

III. FOUCAULT PENDULUM

A pendulum performs a simple harmonic motion in the gravitational field of the earth. A Foucault Pendulum consist of a heavy spherical bob suspended by a very long thin suspension wire. In the case of small oscillations the bob of the pendulum lies on the tangent plane $P$ to the surface of the earth as shown in Figure 2.

The advantage of a foucault pendulum over an ordinary pendulum is that an ordinary pendulum can oscillate only in a fixed vertical plane determined by its suspension mechanism. A foucault pendulum can be set to oscillate along any vertical plane and hence it is not constrained to change its plane of oscillation along with the suspension mechanism. The rotation will just cause a torsion in the thin suspension wire which can get removed by causing a spin in the bob about the vertical.

FIG. 2: The Foucault Pendulum at a latitude $\theta$ on the earth. The bob of the pendulum traces a small segment AB on the local tangent plane P to the earth. As the pendulum is dragged clockwise along the latitude with the rotating earth, the segment AB is parallel transported along the latitude.
As the bob of the pendulum oscillates, it describes a small line segment AB on the plane \( P \). The velocity vector of the bob is along this line. It is observed that this line segment rotates with respect to the surface of the earth as the earth rotates. The rate at which it rotates depends upon the latitude of the location of the Foucault Pendulum. It is fastest at the pole and slows down as one moves towards the equator. If \( \omega_0 \) is the angular velocity of earth’s rotation and \( \theta \) is the local latitude then the the angular speed of rotation of line AB is given by \( \omega = \omega_0 \sin \theta \). At the poles this speed is same as that of the earth’s rotation while at the equator it is 0. Also, due to the factor \( \sin \theta \), the sense of rotation is opposite in the two hemispheres. After one complete rotation of the earth the segment AB rotates by an angle \( 2\pi \sin \theta \). The local gravity everywhere is normal to the plane \( P \). As the earth rotates the pendulum is carried along a latitude. There is no horizontal force on the bob to change the direction of its velocity vector, and hence, that of the orientation of the line AB. This kind of transport of a vector along a curve is called a parallel transport of the vector along the curve. The rotation of the plane of oscillation of the Foucault pendulum is thus the effect of curvature on the parallel transport of a vector. The latitude of the earth is not a geodesic. Hence the frame of reference, fixed with the surface of the earth, carried along the latitude is not inertial. The rotation of the vector AB in this non inertial frame is ascribed to a pseudo force called the coriolis force. So by observing this rotation of AB a two dimensional creature can only decipher that the surface is rotating with an angular velocity \( \omega_0 \sin \theta \). It can’t decipher that it is on a curved surface. However if the creature starts from a point X, traverses a closed loop, and comes back to the same point, the parallel transported AB should coincide with its original direction in a flat space. The change in orientation of AB from its original direction thus becomes a handle for the two dimensional creature confined to the surface to measure the curvature of the surface. Let us do the parallel transport of the vector AB along the closed curve of the latitude of the earth. As the tangent plane rolls along the latitude the envelope of the tangents is a cone. So it is equivalent to do a transport of this vector on the surface of this tangent cone. A cone can be cut and opened to make it flat as shown in Figure 3. The curve of the latitude is shown as the arc of a circle of radius \( R \cot \theta \). Along this arc the segment AB moves parallely, and gradually the angle it makes with the local latitude changes, as shown in the Figure 3. Note that the arc of the circle making the rim of the cone is not complete. Since the length of
FIG. 3: The parallel transport of AB along a latitude $\theta$ is equivalent to doing it on the tangent cone touching the earth along the latitude. This is shown in the left figure. $A'B'$ is the final orientation after one complete rotation of the earth. The cone is cut along the dotted line and shown on the right as a flat sector of a circle of angle $\alpha$. AB starts from a point X and moves parallel to itself along the latitude $\theta$. It is seen that the angle between AB and the latitude gradually changes and finally when it returns to point X the final orientation of $A'B'$ changes by $\alpha$.

The latitude is $2\pi R \cos \theta$ the angle it subtends at the center is

$$\alpha = \frac{2\pi R \cos \theta}{R \cot \theta} = 2\pi \sin \theta$$

It is clear from the Figure that after completing one full rotation from point X to the point X, i.e after one day, AB rotates by an amount $\alpha$ with respect to the local latitude. According to the Gauss-Bonnet Theorem this change of orientation of AB suggests a curvature of the surface of the earth enclosed by the latitude. The total curvature is given by a solid angle measuring $2\pi - \alpha = 2\pi (1 - \sin \theta)$. For a spherical earth this is precisely equal to the solid angle subtended at its center by the area enclosed by the latitude $\theta$.

IV. PERIHELION SHIFT

Planets move around the sun in an elliptic orbit with the sun at the focus. The nearest point to the sun in the orbit of a planet is called its perihelion. If the sun is at the origin of the coordinate system, then the gravitational field in the region around the sun is given as $k \hat{r}/r^2$. Bound motion in this gravitational field can be shown to be elliptic orbits with the sun at the focus. The energy $E$ and the angular momentum $\mathbf{L}$ are the constants of motion.
in this field. These constants determine the shape and size of the orbits, i.e the length of the semi-major axis and the eccentricity of the ellipse. Since $\mathbf{L}$ is a constant, the orbit of the planet stays in a fixed plane which is perpendicular to $\mathbf{L}$. The orientation of the orbit on this plane also remains constant. This orientation also depends upon the initial position and velocity of the planet, just like the energy and the angular momentum. This orientation is defined by a vector on the plane of the orbit, called the Laplace Runge Lenz vector\footnote{10} $\mathbf{A}$, given by

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\frac{k\mathbf{r}}{r}$$

$\mathbf{A}$ is a constant of motion in the Kepler problem. It can be shown to be directed along the perihelion of the orbit. So the planetary orbits have a fixed orientation in the plane.

This however fails for mercury, the nearest planet to the sun. It was known since much before Newton’s laws of motion and the theory of gravity was used to understand planetary orbits. The perihelion of mercury is not fixed in space, contradicting the constancy of $\mathbf{A}$. Effects due to gravitational pull of venus and the deviation of the shape of the sun from sphere couldn’t explain this discrepancy. This shift however happens at a very very slow rate. It is about 43 seconds of an arc in a century. This shift cannot be understood from Newtonian theory of gravity.

In the theory of gravity according to General Relativity\footnote{11}, a massive object causes a curvature of the space-time around it. All objects in the vicinity of the massive body moves in this curved space-time. The trajectory is obtained by evaluating the geodesic path in this space-time. This approach have to be followed when we are near very massive astronomical bodies like a neutron star or a black hole. Since the trajectory of planets in a gravitational field is planar we can consider the effect of curvature of space-time on this plane. The plane of the trajectory will now be a curved surface as shown in fig\[11\]

The curvature of space caused by sun is very small. So the orbit of mercury is essentially an ellipse perturbed only slightly due to the curvature of the space. Locally the planet executes an elliptical orbit on the tangent space. As the planet moves, the tangent plane on the curved space changes. Any vector which remains constant on the local tangent space will be parallel transported. Hence the the Laplace Runge Lenz vector will be parallel transported along the path of the orbit which lies on the curved space. As we complete the loop on one revolution we will find that the vector $\mathbf{A}$ rotated with respect to itself. This
FIG. 4: Space gets curved near a massive body. The figure on the left shows a star $S$ and a flat plane around it. The figure on the right shows how the plane gets curved near the star. will cause the perihelion of the elliptical orbit to shift.

![Figure 4](image)

FIG. 5: Elliptic orbits on conical surface. The position of the sun is at the apex of the cone. In (a) opening angle of the cone is $\sin^{-1}(59/60)$ while in (b) it is $\sin^{-1}(5/6)$. Smaller opening angle indicates high curvature and hence a large perihelion shift.

![Figure 5](image)

We show this with MATLAB plots in Figure 5. Instead of the kind of curved surface as shown in Figure 4, we work with a cone. This approximation is fine for orbits of most planets with low eccentricity. The difference between the nearest and farthest distances in the orbit from the sun is not much. And so we are essentially on the surface of a cone. On a flat plane a planet executes a closed elliptic orbit with the sun at one of the focus. As the plane is curled up into a cone, with the sun at the apex, the ellipse on the plane is no longer closed. The perihelion of the ellipse gradually shifts. The rate at which it shifts depends upon the opening angle of the cone. When the opening angle is small the curvature due to sun is high. In Figure 5 we show perihelion shift on two different conical surface. One with
an opening angle of $\sin^{-1}\left(\frac{59}{60}\right)$ has small perihelion shift while the one with a smaller opening angle of $\sin^{-1}\left(\frac{5}{6}\right)$ shows higher perihelion shift.

FIG. 6: A simple paper model to understand the perihelion shift. After the cone is made trace the ellipse. Observe, how the ellipses continuously traverse from one to the other.

A simple paper model can be used to demonstrate the perihelion shift of elliptic orbits. This is shown in Figure 6. We draw 6 similar elliptic orbits around S as focus with the semi-major axes of successive ellipses rotated by $\pi/3$. Now we will show how the orbit smoothly goes from one orbit to the other when this flat surface is made into a cone. For this cut the paper along the major axis of one of the ellipse from the edge of the paper to the point S. Now slide one of the cut edge over the other till one half of the cut orbit exactly coincides with the other half of the next orbit. In doing this the plane paper curls up into a cone. The position of the Sun is at the apex of the cone. Now trace the orbits. The orbits will smoothly traverse from one to the other. This will give exactly the effect of perihelion shift of the orbit. Of course the actual case of mercury has very little shift in one revolution. This is because the curvature is very very small. To show such small effects in our simple model we have to draw a large number of ellipses with their major axes rotated by very small amount and then make a cone with a very large opening angle. It will be rather difficult to see the effect. So we are just demonstrating a rather drastic effect by a large curvature.
V. CONCLUSION

Curvature of a surface can be generally described in a higher dimensional euclidian space. But in the absence of any higher dimensional quantity or the existence of any higher dimensions, curvature has to be described and measured strictly within the surface. Gauss-Bonet theorem gives a way to do this. A dynamical vector that remains constant as the system moves through space acts naturally as a vector which is parallel transported along a closed path. The change in the orientation of the vector with respect to its original orientation after traversing a closed path is a measure of the total curvature of a surface enclosed by the path. The foucault pendulum and the perihelion shift of the orbit of a planet are two such examples. While foucault pendulum indicates the curvature of the surface of the earth embedded in higher three dimensional space, the perihelion shift indicates the curvature of the space itself which can’t be described in any higher dimensional manifold. Very simple geometric demonstration of these effects are presented.

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