Trace formulas for Schrödinger operators with complex potentials on a half line

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Abstract
We consider Schrödinger operators with complex-valued decaying potentials on the half line. Such operator has essential spectrum on the half line plus eigenvalues (counted with algebraic multiplicity) in the complex plane without the positive half line. We determine trace formula: sum of imaginary part of these eigenvalues plus some singular measure plus some integral from the Jost function. Moreover, we estimate sum of imaginary part of eigenvalues and singular measure in terms of the norm of potentials. In addition, we get bounds on the total number of eigenvalues, when the potential is compactly supported.

Keywords Complex potentials · Trace formulas

Mathematics Subject Classification 47E05 · 34L20 · 34L40 · 81Q10

1 Introduction and main results

1.1 Introduction

We consider a Schrödinger operator \( Hf = -f'' + qf \), \( f(0) = 0 \) on \( L^2(\mathbb{R}_+) \). We assume that the potential \( q \) is complex and satisfies:

\[
\int_0^\infty (1 + x)|q(x)|dx < \infty. \tag{1.1}
\]

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It is well known that the operator $H$ has essential spectrum $[0, \infty)$ plus $N \in [0, \infty]$ eigenvalues (counted with multiplicity) in the cut domain $\mathbb{C}\setminus[0, \infty)$. We denote the eigenvalues (counted with multiplicity) in the cut domain $\mathbb{C}\setminus[0, \infty)$ by $E_j, j = 1, \ldots, N$. Note that the multiplicity of each eigenvalue equals 1, but we call the multiplicity of the eigenvalue its algebraic multiplicity. Instead of the energy $E \in \mathbb{C}\setminus\mathbb{R}_+$, we will use the momentum defined by $k = \sqrt{E} \in \mathbb{C}_+$, where $\mathbb{C}_\pm = \{ \pm \text{Im} z > 0 \}$. We call $k_j = \sqrt{E_j} \in \mathbb{C}_+$ also the eigenvalues of the operator $H$. Of course, $E$ is really the energy, but since $k$ is the natural parameter, we will abuse terminology. We assume that $k_1, \ldots, k_N \in \mathbb{C}_+$ are labeled by

$$\text{Im} k_1 \geq \text{Im} k_2 \geq \text{Im} k_3 \geq \cdots \geq \text{Im} k_n \geq \cdots \tag{1.2}$$

Recall that, in general, a trace formula is an identity connecting the integral of the potential and various sums of eigenvalues and integrals of coefficients of S-matrix of the operator Schrödinger. We shortly describe results about trace formulas:

- In 1960, Buslaev and Faddeev [2] obtained the classical results about trace formulas for Schrödinger operators with real decaying potentials on half line.
- There are a lot of results about one-dimensional case, see [12] and references therein.
- The multidimensional case was studied in by Buslaev [1]. Trace formulas for Stark operators and magnetic Schrödinger operators were discussed in by Korotyaev and Pushnitsky [21,22].
- The trace formulas for Schrödinger operators with real periodic potentials were obtained in Kargaev [11] and Korotyaev [17]. They were used to obtain two-sided estimates of potential in terms of gap lengths (or the action variables for KdV) in [18] via the conformal mapping theory for the quasimomentum.
- Trace formulas for Schrödinger operators with complex potentials were considered on the lattice $\mathbb{Z}^d$ (see [15,20,24]) and on $\mathbb{R}^3$ [16].

Our main goal is to determine trace formulas for Schrödinger operators with complex potentials on the half line. Our trace formula is the identity (1.23), where the left-hand side is the integral of the real part of potential, the sum of $\text{Im} k_j$ and the integral of the singular measure and the right-hand side is the integral from $\log|\psi(k + i0)\psi(-k + i0)|$ on the positive half line, where $\psi$ is the Jost function. Here, we have the new term, the singular measure, which is absent for real potentials. However, in (1.27) we estimate the singular measure and the sum of $\text{Im} k_j$ in terms of the potential. In our consideration we apply the technique from [16] about trace formulas for Schrödinger operators with complex-valued potentials on $\mathbb{R}^3$, where the Hardy spaces in the upper half-space were used. Note that in the case of lattice [15,20] the Hardy spaces on the unit disk were used.

In contrast to the trace formula for complex potentials, there are many results on estimates of eigenvalues in terms of potentials, see recent articles [5,6] and references therein. In our one-dimensional case, there exist many results about bounds on sums of powers of eigenvalues of Schrödinger operators with complex-valued potentials in terms of $L^p$-norms of the potentials see [3,10,23,26,27] and references therein.
1.2 The Jost function and the Hardy spaces

We recall the well-known facts about the operator $H$, see, e.g., [4, 25]. Introduce the Jost solutions $f_+(x, k)$ of the equation

$$-f''_+ + qf_+ = k^2 f_+, \quad x \geq 0, \quad k \in \mathbb{C}_+ \setminus \{0\},$$

(1.3)

with the conditions $f_+(x, k) = e^{i x k} + o(1)$ as $x \to \infty$ and $k \in \mathbb{R} \setminus \{0\}$. We define the Jost function $\psi(k) = f_+(0, k)$. The Jost function $\psi$ is analytic in $\mathbb{C}_+$, is continuous up to the real line and satisfies

$$\psi(k) = 1 - \frac{Q_0 + o(1)}{i k} \quad \text{as} \quad |k| \to \infty,$$

(1.4)

uniformly in $\arg k \in [0, \pi]$, where $Q_0 = \frac{1}{2} \int_0^\infty q(t) dt$. The function $\psi$ has $N \in [0, \infty]$ zeros in $\mathbb{C}_+$ given by $k_j = \sqrt{E_j} \in \mathbb{C}_+$, counted with algebraic multiplicity of $E_j$. Let $\sigma_d = \{k_1, \ldots, k_N \in \mathbb{C}_+\}$. Due to (1.4), the set $\sigma_d$ is bounded and satisfies

$$\sigma_d \subset \{k \in \mathbb{C}_+ : \psi(k) = 0\} \subset \{k \in \mathbb{C}_+ : |k| \leq r_c\}, \quad r_c = \|q\|,$$

(1.5)

where $\|q\|_\alpha = \int_0^\infty x^\alpha |q(x)| dx, \quad \alpha \geq 0, \quad \|q\| = \|q\|_0$.

Let a function $F(k), k = u + i v \in \mathbb{C}_+$ be analytic on $\mathbb{C}_+$. For $0 < p \leq \infty$, we say that $F$ belongs the Hardy space $H_p = H_p(\mathbb{C}_+)$ if $F$ satisfies $\|F\|_{H_p} < \infty$, where $\|F\|_{H_p}$ is given by

$$\|F\|_{H_p} = \begin{cases} \sup_{v > 0} \frac{1}{2\pi} \left( \int_{\mathbb{R}} |F(u + i v)|^p du \right)^{\frac{1}{p}} & \text{if} \quad 0 < p < \infty, \\ \sup_{k \in \mathbb{C}_+} |F(k)| & \text{if} \quad p = \infty. \end{cases}$$

Note that the definition of the Hardy space $H_p$ involves all $v = \text{Im} k > 0$.

We remark that the Jost function $\psi \in H_\infty$, since $\psi$ is uniformly bounded in $\mathbb{C}_+$. Due to (1.5), all zeros of $\psi$ are uniformly bounded, and then, we can define the Blaschke product $B$ by

$$B(k) = \prod_{j=1}^N \left( \frac{k - k_j}{k - \bar{k}_j} \right), \quad k \in \mathbb{C}_+. \quad (1.6)$$

We describe the basic properties of the Blaschke product $B$ as an analytic function in $\mathbb{C}_+$, see, e.g., [27].

**Proposition 1.1** Let a potential $q$ satisfy (1.1). Then, $\psi \in H_\infty$ is continuous up to the real line and satisfies

$$|\psi(k)| \leq e^{\omega}, \quad \forall \ \omega \in \left\{ \|q\|_1, \ \frac{\|q\|}{|k|} \right\}, \quad (1.7)$$
and if \( \|q\|_1 < \omega_n \), where \( \omega_n e^{\omega_n} = 1 \), then the operator \( H \) does not have eigenvalues.

(i) The zeros \( (k_j)^N \) of \( \psi \) in the upper half plane \( \mathbb{C}_+ \) (counted with multiplicity) labeled by (1.2) satisfy

\[
\sum_{j=1}^{N} \text{Im} k_j < \infty. \tag{1.8}
\]

The Blaschke product \( B(k) \) given by (1.6) converges absolutely and uniformly in every bounded disk in \( \mathbb{C}_+ \setminus \sigma_d \), and the function \( B \in \mathcal{H}_\infty \) with \( \|B\|_{\mathcal{H}_\infty} \leq 1 \).

(ii) It has an analytic continuation from \( \mathbb{C}_+ \) into the domain \( \{|k| > r_c\} \), where \( r_c = \|q\| \) and has the following Taylor series

\[
\log B(z) = -i \frac{B_0}{k} - i \frac{B_1}{2k^2} - i \frac{B_2}{3k^3} - \cdots, \quad \text{as} \quad |k| > r_c, \tag{1.9}
\]

\[
B_0 = 2 \sum_{j=1}^{N} \text{Im} k_j, \quad B_n = 2 \sum_{j=1}^{N} \text{Im} k_j^{n+1}, \quad n \geq 1,
\]

where each sum \( B_n, n \geq 1 \) is absolutely convergence and satisfies

\[
|B_n| \leq 2 \sum_{j=1}^{N} |\text{Im} k_j^{n+1}| \leq \pi(n + 1)r_c^n B_0. \tag{1.10}
\]

Remark (1) Estimate (1.8) is well known, see, e.g., [27]. This fact follows from two points: The function \( \psi \in \mathcal{H}_\infty \) and the zeros of \( \psi \) are uniformly bounded.

(2) The function \( B \) has complicated properties in the disk \( \{|k| < r_c\}, r_c = \|q\| \) and very good properties in the domain \( \{|k| > r_c\} \).

(3) We use asymptotics of \( B \) at large \( |k| \) to determine the trace formulas in Theorem 1.4.

Example Consider the potential \( q(x) = Atx^2 - 1, x \in (0, 1), t > 0 \) and \( q(x) = 0 \) for \( x > 1 \), where \( A \in \mathbb{C} \). We have \( \|q\| = \frac{|A|}{t} \) and \( \|q\|_1 = \frac{|A|}{1+t^2} \). If \( t \) is small, then the complex potential \( q \) is rather big and due to (1.5) all eigenvalues belong to the half disk with the radius \( \|q\| \), but if \( \|q\|_1 < \omega_n \), where \( \omega_n e^{\omega_n} = 1 \), then by Proposition 1.1, the operator \( H \) does not have eigenvalues.

1.3 Trace formulas and estimates

We describe the Jost function \( \psi \in \mathcal{H}_\infty \) in terms of a canonical factorization.

Theorem 1.2 Let a potential \( q \) satisfy (1.1). Then, \( \psi \) has a canonical factorization in \( \mathbb{C}_+ \) given by

\[
\psi = \psi_{\text{in}} \psi_{\text{out}}, \tag{1.11}
\]

where
• \( \psi_{\text{in}} \) is the inner factor of \( \psi \) having the form
\[
\psi_{\text{in}}(k) = B(k)e^{-iK(k)}, \quad K(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\nu(t)}{k-t}, \quad \forall \, k \in \mathbb{C}_+.
\]

• \( B \) is the Blaschke product for \( \text{Im} \, k > 0 \) given by (1.6) and \( d\nu(t) \geq 0 \) is some singular compactly supported measure on \( \mathbb{R} \), which satisfies
\[
\nu(\mathbb{R}) = \int_{\mathbb{R}} d\nu(t) < \infty,
\]
\[
\text{supp} \, \nu \subset \{ z \in \mathbb{R} : \psi(z) = 0 \} \subset [-r_c, r_c].
\]

• The function \( K(\cdot) \) has an analytic continuation from \( \mathbb{C}_+ \) into the cut domain \( \mathbb{C} \setminus [-r_c, r_c] \) and has the following Taylor series in the domain \( \{|k| > r_c\} \):
\[
K(k) = \sum_{j=0}^{\infty} \frac{K_j}{k^{j+1}}, \quad K_j = \frac{1}{\pi} \int_{\mathbb{R}} t^j d\nu(t).
\]

• \( \psi_{\text{out}} \) is the outer factor given by
\[
\psi_{\text{out}}(k) = e^{iM(k)}, \quad M(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log |\psi(t)|}{k-t} dt, \quad k \in \mathbb{C}_+,
\]
where the function \( \log |\psi(t+i0)| \) belongs to \( L^1_{\text{loc}}(\mathbb{R}) \).

**Remark** (1) Due to (1.4), the integral \( M(k) \) in (1.15) converges absolutely.

(2) We have \( |\psi_{\text{in}}| \leq 1 \), since \( |B| \leq 1 \), \( \text{Im} \, K \leq 0 \) and \( \text{Im} \, \frac{1}{k-t} \leq 0 \) for all \( k \in \mathbb{C}_+ \).

(3) These results are crucial to determine trace formulas in Theorem 1.4. The canonical factorization is a first trace formula. It is a generating function; these results will be used in the proof of trace formulas in Theorem 1.4.

Let \( R(k) = (H - k^2)^{-1} \) and \( R_0(k) = (H_0 - k^2)^{-1} \), where \( H_0 \) is the operator \( H \) with \( q = 0 \). The differentiation of a canonical factorization produces a trace formula for \( \text{Tr}(R(k) - R_0(k)) \).

**Corollary 1.3** Let a potential \( q \) satisfy (1.1). Then, the trace formula
\[
-2k \text{ Tr}
\left(R(k) - R_0(k)\right) = \sum_{j=1}^{m} \frac{2i \text{ Im} \, k_j}{(k-k_j)(k-k_j^*)} + \frac{i}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{(t-k)^2},
\]
holds true for any \( k \in \mathbb{C}_+ \setminus \sigma_d \), where the measure \( d\mu(t) = \log |\psi(t)| dt - d\nu(t) \) and the series converges uniformly in every bounded disk in \( \mathbb{C}_+ \setminus \sigma_d \).

We recall the well-known results about the asymptotics of the Jost function. Introduce the Sobolev space \( W_m \) given by
\[
W_m = \left\{ q \in L^1(\mathbb{R}_+) : xq \in L^1(\mathbb{R}_+), q^{(j)} \in L^1(\mathbb{R}_+), \, j = 1, \ldots, m \right\}, \quad m \geq 0.
\]
If \( q \in W_{m+1}, m \geq 0 \), then the function \( \psi(\cdot) \) is analytic in \( \mathbb{C}_+ \) is continuous up to the real line and satisfies

\[
i \log \psi(k) = -\frac{Q_0}{k} - \frac{Q_1}{k^2} - \frac{Q_2}{k^3} + \cdots - \frac{Q_m + o(1)}{k^{m+1}},
\]

as \( |k| \to \infty \) uniformly in \( \arg k \in [0, \pi] \), see [2], where

\[
Q_0 = \frac{1}{2} \int_0^\infty q(x)dx, \quad Q_1 = - \frac{i}{2} q(0), \quad Q_2 = \frac{1}{2^3} \left(q'(0) + \int_0^\infty q^2(x)dx\right).
\]

In Theorem 1.4, we show that if \( q \in W_{m+1} \) then the function \( M(k), k \in \mathbb{C}_+ \) defined by (1.15) satisfies

\[
M(k) = \frac{\mathcal{J}_0 + iI_0}{t} + \frac{\mathcal{J}_1 + iI_1}{t^2} + \cdots + \frac{\mathcal{J}_m + iI_m + o(1)}{t^{m+1}},
\]

as \( \Im k \to \infty \), where the real constants \( I_j \) and \( \mathcal{J}_j \in \mathbb{R} \) are given by

\[
I_j = \Im Q_j, \quad \mathcal{J}_j = \text{v.p.} \frac{1}{\pi} \int_{\mathbb{R}} h_{j-1}(t)dt, \quad h_j(t) = t^{j+1}(h(t) + P_j(t)),
\]

\[
h(t) = \log |\psi(t)| = h_{-1}(t), \quad P_j(t) = \frac{I_0}{t} + \frac{I_1}{t^2} + \cdots + \frac{I_j}{t^{j+1}},
\]

\( j = 0, 1, 2, \ldots, m \). In particular, we have

\[
\mathcal{J}_0 = \text{v.p.} \frac{1}{\pi} \int_{\mathbb{R}} h(t)dt, \quad \mathcal{J}_1 = \text{v.p.} \frac{1}{\pi} \int_{\mathbb{R}} (th(t) + I_0)dt.
\]

If \( m = 1 \), then \( \mathcal{J}_0 \) in (1.22) converges since (1.18) gives \( h(t) = \Re \frac{iQ_0}{t} + \frac{O(1)}{t^2} \) as \( t \to \pm \infty \).

**Theorem 1.4** (Trace formulas) Let a potential \( q \) satisfy (1.1). Then,

\[
B_0 + \frac{v(\mathbb{R})}{\pi} + \frac{1}{2} \int_0^\infty \Re q(x)dx = \mathcal{J}_0,
\]

where \( \mathcal{J}_0 \) converges. Let, in addition, a potential \( q \in W_{m+1} \) for some integer \( m \geq 1 \). Then, the function \( M \) satisfies (1.20) and the following identities hold true

\[
\frac{B_j}{j+1} + K_j + \Re Q_j = \mathcal{J}_j, \quad j = 1, \ldots, m,
\]
in particular

\[ \frac{B_1}{2} + K_1 + \frac{\text{Im } q(0)}{4} = J_1, \]
\[ \frac{B_2}{3} + K_2 + \text{Re } Q_2 = J_2. \]  
(1.25)

**Remark** (1) Recall that \( B_0 \geq 0 \) and \( K_0 = \frac{\nu(\mathbb{R})}{\pi} \geq 0 \). Thus, in order to estimate \( B_0 + \frac{\nu(\mathbb{R})}{\pi} \geq 0 \) in terms of the potential \( q \) we need to estimate the integral \( J_0 \) in terms of the potential \( q \).

Introduce the function \( \phi(r) = \frac{2}{\pi} \int_0^r \frac{\sin t}{t} \, dt, \ r \geq 0 \) and the integral \( \mathcal{I} \) by

\[ \mathcal{I} = \frac{1}{2} \int_0^\infty \text{Re } q(x) \phi(2x \|q\|) \, dx. \]  
(1.26)

Note that \( \phi(r) \in [0, 1) \) for all \( r \geq 0 \) and \( |\mathcal{I}| \leq \frac{1}{2} \int_0^\infty |\text{Re } q(x)| \, dx \).

**Theorem 1.5** (Estimates) Let a potential \( q \) satisfy (1.1). Then, the following hold true

\[ B_0 + \frac{\nu(\mathbb{R})}{\pi} + \mathcal{I} \leq 2 \pi \left( \|q\|_1 + \|q\|(C_0 + \log^+ \|q\|) \right), \]  
(1.27)

where \( C_0 = e^2 + e^{\frac{4+1}{4}} \) and \( \log^+ z = \begin{cases} \log^+ z & z > 1 \\ 0 & 0 \leq z \leq 1 \end{cases} \).

**Remark** (1) This estimate of \( \nu(\mathbb{R}) \) plus sum of \( \text{Im } k_j \) in terms of the potential is new.

(2) If \( \text{Re } q = 0 \), then we have \( \mathcal{I} = 0 \). If \( \pm \text{Re } q \geq 0 \), then we have \( \pm \mathcal{I} \geq 0 \).

(3) In [5], there is a following estimate \( \sum_j \text{dist}(E_j, \mathbb{R}^+) \leq C \|q\|^2 \) for some constant \( C \).

(4) In [27], there is a following estimate \( B_0 \leq C_\alpha \left( \|q\|_\alpha \|q\|^\alpha + \|q\| \right) \) for any \( \alpha \in (0, 1) \) and some constant \( C_\alpha \) depending on \( \alpha \) only.

Consider estimates for complex compactly supported potentials. In this case, the Jost function \( \psi(k) = f_+(0, k) \) is entire and due to (1.5) it has a finite number of zeros in \( \mathbb{C}^+ \).

**Theorem 1.6** Let \( q \in L^1(\mathbb{R}^+) \) and let \( \text{supp } q \subset [0, \gamma] \) for some \( \gamma > 0 \). Then, the number of zeros \( \mathcal{N}_+(\psi) \) of \( \psi \) (counted with multiplicity) in \( \mathbb{C}^+ \) satisfies

\[ \mathcal{N}_+(\psi) \leq C_1 + C_2 \gamma \|q\|, \]  
(1.28)

where the constants \( C_1 < 11 \), and \( C_2 < 2 \) are given in Lemma 5.1.

Note that the estimate of \( \mathcal{N}_+ \) was determined in Frank et al. [7], when \( q \) decays exponentially at infinity.
In our paper, we use classical results from complex analysis about the Hardy space in the upper half plane. In particular, we use a so-called canonical factorization of analytic functions from Hardy spaced via its inner and outer factors, see Sect. 4. This gives us a new class of trace formula for the spectrum of Schrödinger operators with complex-valued potentials on the half line.

We shortly describe the plan of the paper. In Sect. 2, we present the main properties of the Jost function. In Sect. 3, we prove main theorems. Section 4 is a collection of needed facts about Hardy spaces. In Sect. 5, we discuss the case of compactly supported potentials.

2 Fundamental solutions

2.1 Fundamental solutions

It is well known that the Jost solution \( f_+(x, k) \) of Eq. (1.3) satisfies the integral equation

\[
 f_+(x, k) = e^{ikx} + \int_x^\infty \frac{\sin k(t - x)}{k} q(t) f_+(t, k) dt, \quad (x, k) \in [0, \infty) \times \mathbb{C}^+.
\]

In order to study properties of the Jost function \( \psi(k) = f_+(0, k) \), we define the function \( y(x, k) = e^{-ikx} f_+(x, k) \), which satisfies the integral equation

\[
 y(x, k) = 1 + \int_x^\infty G(t - x, k) q(t) y(t, k) dt, \quad G(t, k) = \frac{\sin kt}{k} e^{ikt},
\]

\( \forall (x, k) \in [0, \infty) \times \mathbb{C}^+ \). The standard iterations give \( y(x, k) \) and the Jost function \( \psi(k) = y(0, k) \):

\[
 y(x, k) = 1 + \sum_{n \geq 1} y_n(x, k), \quad y_n(x, k) = \int_x^\infty G(t - x, k) q(t) y_{n-1}(t, k) dt, \quad y_0 = 1,
\]

\[
 \psi(k) = 1 + \sum_{n \geq 1} \psi_n(k), \quad \psi_n(k) = y_n(0, k),
\]

where \( \psi_1(k) = \int_0^\infty \frac{\sin kt}{k} e^{ikt} q(t) dt = -\frac{q_0}{2ik} + \hat{q}(k) \frac{2ik}{2ik} \),

\( q_0 = \int_0^\infty q(t) dt \) and \( \hat{q}(k) = \int_0^\infty e^{ikt} q(t) dt \) is the Fourier transform. The identity (2.2) gives

\[
 \psi(k) = y(0, k) = 1 + \int_0^\infty \frac{\sin kt}{k} q(t) f_+(t, k) dt.
\]
We recall well-known properties of the functions introduced above (see, e.g., [4]).

**Lemma 2.1** Let \( q \in L^1(\mathbb{R}_+) \) and let \( x \in [0, \infty) \). Then, the functions \( f_+(x, \cdot), f'_+(x, \cdot) \) are analytic in \( \mathbb{C}_+ \) and continuous up to the real line without the point 0.

Let, in addition, \( \|q\|_1 = \int_0^\infty x|q(x)|\,dx < \infty \). Then, these functions are continuous up to the real line. Moreover, the functions \( y \) and \( \psi \) satisfy

\[
|y_n(x, k)| \leq \frac{\omega^n}{n!}, \quad \omega = \min \left\{ \|q\|_1, \frac{\|q\|}{|k|} \right\}, \quad n \geq 1, \tag{2.5}
\]

and

\[
|y(x, k)| \leq e^{\omega}, \\
|y(x, k)| \leq \omega e^{\omega}, \\
|y(x, k)| \leq \omega^2 e^{\omega}, \quad n \geq 1, \tag{2.6}
\]

and

\[
|\psi(k)| \leq e^{\omega}, \quad |\psi(k)| \leq \omega e^{\omega}, \\
|\psi(k)| \leq \frac{\omega^2}{2} e^{\omega}. \tag{2.7}
\]

In particular, if \( \|q\|_1 < \omega_n \), where \( \omega_n e^{\omega_n} = 1 \), then the operator \( H \) has no eigenvalues.

**Proof** Let \( q \in L^1(\mathbb{R}_+) \) and let \( \omega = \frac{\|q\|}{|k|} \). Substituting the estimate \( |G(t, k)| \leq \frac{1}{|k|} \) for all \( t \geq 0, \ k \in \mathbb{C}_+ \setminus \{0\} \) into the identity

\[
y_n(x, k) = \int_{x=t_0}^{x=t_n} \left( \prod_{1 \leq j \leq n} G(t_j - t_{j-1}, k)q(t_j) \right) dt_1 dt_2 \cdots dt_n, \tag{2.8}
\]

we obtain

\[
|y_n(x, k)| \leq \frac{1}{|k|^n} \int_{x=t_0}^{x=t_n} \left( \prod_{1 \leq j \leq n} |q(t_j)| \right) dt_1 dt_2 \cdots dt_n
\]

\[
= \frac{1}{|k|^n} \int_{x=t_0}^{x=t_n} |q(t_1)q(t_2) \cdots q(t_n)| dt_1 dt_2 \cdots dt_n
\]

\[
= \frac{\|q\|^n}{n! |k|^n} = \frac{\omega^n}{n!}. \tag{2.9}
\]

This shows that for each \( x \geq 0 \) the series (2.3) converges uniformly on bounded subset of \( \mathbb{C}_+ \setminus \{|k| > \varepsilon\} \) for any \( \varepsilon > 0 \). Each term of this series is an analytic function in \( \mathbb{C}_+ \). Hence, the sum is an analytic function in \( \mathbb{C}_+ \). Summing the majorants we obtain estimates (2.6)–(2.7) for \( \omega = \|q\|/|k| \). Thus, the functions \( f_+(x, \cdot), f'_+(x, \cdot), x \in \mathbb{R} \) are analytic in \( \mathbb{C}_+ \) and continuous up to the real line without the point 0.
Consider the second case: let \( \omega = \|q\|_1 = \int_0^\infty x|q(x)|\,dx < \infty \). The function \( G(t, k) \) satisfies \( |G(t, k)| \leq t \) for all \( k \in \mathbb{C}_+, t > 0 \). Then, using above arguments we obtain

\[
|y_n(x, k)| \leq \int_{x=t_0}^{t_n} \left( \prod_{1 \leq j \leq n} |(t_j - t_j^{-1})q(t_j)| \right) dt_1 \, dt_2 \cdots dt_n
\]

\[
\leq \int_{x=t_0}^{t_n} \left( \prod_{1 \leq j \leq n} |t_j q(t_j)| \right) dt_1 \, dt_2 \cdots dt_n = \frac{\|q\|^n_1}{n!} = \omega^n. \tag{2.10}
\]

This shows that for each \( x \geq 0 \) the series (2.3) converges uniformly on bounded subset of \( \mathbb{C}_+ \). Each term of this series is an analytic function in \( \mathbb{C}_+ \). Hence the sum is an analytic function in \( \mathbb{C}_+ \). Summing the majorants, we obtain estimates (2.6)–(2.7) for \( \omega = \|q\|_1 \). Thus, the functions \( f(x, \cdot), f'(x, \cdot), x \geq 0 \) are analytic in \( \mathbb{C}_+ \) and continuous up to the real line.

We have \( |f(k) - 1| \leq ae^a < 1, a = \|q\|_1 < \omega^* \) for any \( k \in \mathbb{C}_+ \), which yields \( |f(k)| \geq 1 - ae^a > 0 \). \( \square \)

3 Proof of main theorems

3.1 Hardy spaces and Jost functions

In order to study zeros of the Jost function \( \psi \) in the upper half plane, we need to study the Blaschke product \( B \), defined by (1.6). We describe the basic properties of the Blaschke product \( B \) as an analytic function in \( \mathbb{C}_+ \).

**Proof of Proposition 1.1** Lemma 2.1 yields that the Jost function \( \psi \) is analytic in \( \mathbb{C}_+ \), is continuous up to the boundary and satisfies (1.7). Moreover, asymptotics (2.6) implies that all zeros of \( \psi \) are uniformly bounded. Note that (see p 53 in [8]), in general, in the upper half plane the condition (1.8) is replaced by

\[
\sum \frac{\text{Im} \, k_j}{1 + |k_j|^2} < \infty, \tag{3.1}
\]

and the Blaschke product with zeros \( k_j \) has the form

\[
B(k) = \frac{(k - i)^m}{(k + i)^m} \prod_{k_j \neq 0}^N \frac{1 + k_j^2}{1 + k_j^2} \frac{k - k_j}{k - k_j}, \quad k \in \mathbb{C}_+. \tag{3.2}
\]

If all moduli \( |k_n| \) are uniformly bounded, estimate (3.1) becomes \( \sum \text{Im} \, k_j < \infty \) and the convergence factors in (3.2) are not needed, since \( \prod_{k_j \neq 0}^N \frac{k - k_j}{k - k_j} \) already converges.

Results about zeros for the case \( \|q\|_1 < \omega^* \) follow from Lemma 2.1.

The statement (i) is a standard fact for functions from \( \mathcal{H}_\infty \), see Sect. VI in [14].

The statement (ii) follows from Lemma 4.1. \( \square \)
We describe the Jost function $\psi(k), k \in \mathbb{C}_+$ in terms of a canonical factorization.

**Proof of Theorem 1.2** Lemma 2.1 gives that the Jost function $\psi \in \mathcal{H}_\infty$, $\psi$ is continuous in $\mathbb{C}_+$ up to the boundary and satisfies (2.7). Thus, from Theorem 4.3 we obtain all results in Theorem 1.2. □

We prove the first result about the trace formulas.

**Proof of Corollary 1.3** Differentiating (1.11) and using Theorem 1.2, we obtain

$$\frac{\psi'(k)}{\psi(k)} = \frac{B'(k)}{B(k)} = \frac{i}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{(k-t)^2},$$

$$B'(k) = \sum \frac{2i \text{Im} k_j}{(k-k_j)(k-\bar{k}_j)}, \quad \forall k \in \mathbb{C}_+,$$ (3.3)

where $d\mu(t) = h(t)dt - dv(t)$. Define $Y_0(k) = |q|^2 R_0(k)q_1, k \in \mathbb{C}_+$, where $q_1 = |q|^2 e^{i \text{arg} V}$. Recall that $Y_0(k)$ is a trace class operator and the Jost function satisfies the following identity $\psi = \det(I + Y_0)$ in $\mathbb{C}_+$. The derivative of the determinant $\psi = \det(I + Y_0(k))$ satisfies

$$\frac{\psi'(k)}{\psi(k)} = -2k \text{Tr} \left( R(k) - R_0(k) \right), \quad \forall k \in \mathbb{C}_+,$$ (3.4)

see [9]. Combining (3.3) and (3.4), we obtain (1.16). Note that the series converges uniformly in every bounded disk in $\mathbb{C}_+ \setminus \sigma_d$, since $\sum \text{Im} k_j < \infty$. □

**Proof of Theorem 1.4** Let a potential $q$ satisfy (1.1). From Lemma 2.1 and from asymptotics (2.3), we deduce that the Jost function $\psi(k) = 1 - \frac{1}{2\pi i} (q_0 - \tilde{q}(k)) + \frac{O(1)}{k^2}$ as $k \to \pm \infty$. Thus, from Theorem 4.4 we obtain (1.23).

Let $q \in W_{m+1}, m \geq 1$. Then, due to Lemma 2.1, asymptotics (1.18) and Theorem 4.4, we obtain all results in Theorem 1.4 for the case $m \geq 1$. □

**Proof of Theorem 1.5** We estimate $\mathcal{J}_0 = \frac{1}{\pi} \int_0^\infty \xi(t)dt$, where $\xi(t) = \log |\psi(t)\psi(-t)|, t > 0$ for the case $r_c = \|q\| > 1$, the proof for the case $r_c < 1$ is similar. We rewrite $\mathcal{J}_0$ in the following form

$$\mathcal{J}_0 = \mathcal{J}_{01} + \mathcal{J}_{02}, \quad \mathcal{J}_{01} = \frac{1}{\pi} \int_0^{r_c} \xi(t)dt, \quad \mathcal{J}_{02} = \frac{1}{\pi} \int_{r_c}^\infty \xi(t)dt.$$ (3.5)

Consider $\mathcal{J}_{01}$. Estimate (2.7) with $\omega = \|q\|_1$ for the interval $(0, 1)$ and with $\omega = \|q\|/|k|$ for $(1, r_c)$ gives

$$\mathcal{J}_{01} = \frac{1}{\pi} \int_0^1 \xi(t)dt + \frac{1}{\pi} \int_1^{r_c} \xi(t)dt \leq \frac{2}{\pi} \int_0^1 \|q\|_1 dt + \frac{2\|q\|}{\pi} \int_1^{r_c} \frac{dt}{t} = \frac{2\|q\|}{\pi} + 2\|q\|/\log r_c.$$ (3.6)
Consider \( J_{02} \). Recall that \( \psi_1 \) is given by (2.3). Let \( \tilde{\psi} = \psi - 1 - \psi_1 \). Thus, we have \( \psi = 1 + g \), where \( g := \psi_1 + \tilde{\psi} \). Estimate (2.5) gives

\[
|\psi(k)| \leq e^{\omega}, \quad |\psi(k) - 1| \leq \omega e^{\omega}, \quad |\tilde{\psi}(k)| \leq \frac{\omega^2}{2} e^{\omega},
\]

(3.7)

where

\[
\omega = \frac{\|q\|}{|k|}, \quad \psi_1(k) = \int_0^\infty \frac{\sin kx}{k} q(x) dx.
\]

(3.8)

Let \( F^-(k) = F(-k) \). This and the identities \( g + g^- = (\psi_1 + \psi^-_1) + (\tilde{\psi} + \tilde{\psi}^-) = f_0 + (\tilde{\psi} + \tilde{\psi}^-) \) yield

\[
f = \psi \psi^- = (1 + g)(1 + g^-) = 1 + g + g^- + gg^- = 1 + f_0 + f_1,
\]

\[
f_0(k) := \psi_1(k) + \psi_1(-k) = \frac{1}{k} \int_0^\infty q(x) \sin(2kx) dx,
\]

(3.9)

\[
f_1 := \tilde{\psi} + \tilde{\psi}^- + gg^-.
\]

Then, estimate (3.7) implies

\[
|\tilde{\psi}(k) + \tilde{\psi}(-k)| \leq \omega^2 e^{2\omega}, \quad |g(k)g(-k)| \leq \omega^2 e^{2\omega},
\]

(3.10)

which yields

\[
f = 1 + f_0 + f_1, \quad |f_0| \leq \omega, \quad |f_1(k)| \leq 2\omega^2 e^{2\omega}.
\]

(3.11)

This yields

\[
f \bar{f} = (1 + f_0 + f_1)(1 + \bar{f}_0 + \bar{f}_1) = 1 + 2 \text{Re}(f_0 + f_1) + |f_0 + f_1|^2
\]

(3.12)

and then

\[
\xi = \frac{1}{2} \log |f|^2 \leq \frac{1}{2}(2 \text{Re} f_0 + F), \quad F = 2 \text{Re} f_1 + |f_0 + f_1|^2.
\]

(3.13)

Thus, we obtain

\[
J_{02} = \frac{1}{2\pi} \int_{r_c}^{\infty} \log |f|^2(k) dk \leq \frac{1}{2\pi} \int_{r_c}^{\infty} (2 \text{Re} f_0 + F) dk.
\]

(3.14)

Here, due to (3.9) the first integral has the form

\[
\frac{1}{\pi} \int_{r_c}^{\infty} \text{Re} f_0 dk = \frac{1}{2} \int_0^\infty \text{Re} q(x) \varphi(x) dx = \frac{1}{2} \int_0^\infty \text{Re} q(x)(1 - \phi(r)) dx, \quad r = 2x r_c.
\]

(3.15)
where \( \phi(r) = \frac{2}{\pi} \int_{0}^{r} \frac{\sin t}{t} dt \) and \( \phi(r) \in [0, 1] \), since the function \( \varphi(x) = \frac{2}{\pi} \int_{r_c}^{\infty} \frac{\sin kx}{k} dk \) satisfies:

\[
\varphi(x) = \frac{2}{\pi} \int_{r_c}^{\infty} \frac{\sin 2kx}{k} dk = \frac{2}{\pi} \int_{r}^{\infty} \frac{\sin t}{t} dt = \varphi(0) - \phi(r) = 1 - \phi(r). \tag{3.16}
\]

Recall that we take \( \omega = \frac{r_c}{k} \). We estimate the second integral:

\[
F = 2 \Re f_1 + |f_0 + f_1(k)|^2 \leq 4\omega^2 e^{2\omega} + (\omega + 2\omega^2 e^{2\omega})^2 \leq 8\omega^2 e^{2\omega} + \omega^2 + 4\omega^2 e^{4\omega},
\]

\[
\int_{r_c}^{\infty} \omega^2 e^{\lambda \omega} dt = \int_{r_c}^{\infty} \frac{r_c^2 e^{\lambda \omega}}{t^2} dt = -\frac{r_c}{A} e^{\frac{\lambda r_c}{A}} \bigg|_{r_c}^{\infty} = \frac{r_c}{A} e^A, \quad \forall A \neq 0,
\]

\[
\frac{1}{2\pi} \int_{r_c}^{\infty} F dk \leq \frac{1}{2\pi} \int_{r_c}^{\infty} (8\omega^2 e^{2\omega} + \omega^2 + 4\omega^2 e^{4\omega}) dk = \frac{r_c}{2\pi} (4e^2 + 1 + e^4)
\]

\[
= \frac{2r_c}{\pi} C_0. \tag{3.17}
\]

where \( C_0 = \frac{4e^2 + 1 + e^4}{4} \). Substituting (3.6), (3.14), (3.15) and (3.17) into (1.23), we obtain

\[
B_0 + \frac{\nu(\mathbb{R})}{\pi} + \frac{1}{2} \int_{0}^{\infty} \Re q(x) dx = J_0
\]

\[
\leq \frac{2}{\pi} \left( \|q\|_1 + \|q\| \log r_c \right) + \left( \frac{1}{2} \int_{0}^{\infty} \Re q(x) (1 - \phi(r) dx + \frac{2r_c}{\pi} C_0 \right),
\]

which yields (1.27).

\[\square\]

4 Analytic functions in the upper half plane

We discuss different properties of functions from Hardy spaces. Recall that if \( f \in \mathcal{H}_\infty(\mathbb{C}_+), \) then the Blaschke product \( B \in \mathcal{H}_\infty \) with \( \|B\|_{\mathcal{H}_\infty} \leq 1 \) and

\[
\lim_{v \to +0} B(u + iv) = B(u + i0), \quad |B(u + i0)| = 1 \quad \text{a.e on} \ \mathbb{R}, \quad \tag{4.1}
\]

\[
\lim_{v \to 0} \int_{\mathbb{R}} \log |B(u + iv)| du = 0. \quad \tag{4.2}
\]

We recall the needed results about the Blaschke product (see, e.g., [15]).

Lemma 4.1 Let \( f \in \mathcal{H}_\infty(\mathbb{C}_+) \) and let \( \{z_j\} \) be the zeros of \( f \) in \( \mathbb{C}_+ \), which are uniformly bounded by \( r_0 \). Define \( B_n = 2 \sum_j \Im z_j^{n+1} \) for all \( n \geq 0 \). Then,

\[
|B_n| \leq 2 \sum_j |\Im z_j^{n+1}| \leq \frac{\pi}{2} (n + 1) r_0^n B_0 \quad \forall \ n \geq 1. \tag{4.3}
\]
Moreover, the function \( \log B(z) \) is analytic in \( \{ |z| > r_0 \} \) and has the corresponding Taylor series given by

\[
\log B(z) = -\frac{iB_0}{z} - \frac{iB_1}{2z^2} - \frac{iB_2}{3z^3} - \cdots - \frac{iB_{n-1}}{nz^n} - \cdots. \tag{4.4}
\]

We need some results about functions from Hardy spaces. We begin with asymptotics. Consider the integral

\[
M(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{h(t)}{k-t} \, dt, \quad k \in \mathbb{C}_+,
\]

where \( h \) belongs to a class \( \mathcal{X}_m \) defined by

**Definition** A function \( h \) belongs to the class \( \mathcal{X}_m = \mathcal{X}_m(\mathbb{R}) \) if \( h \in L^1_{\text{real,loc}}(\mathbb{R}) \) and has the form

\[
h(t) = -P_m(t) + \frac{h_m(t)}{t^{m+1}}, \quad P_m(t) = \frac{I_0}{t} + \frac{I_1}{t^2} + \cdots + \frac{I_m}{t^{m+1}},
\]

\[
h_m(t) = o(1) \quad \text{as} \quad t \to \pm \infty
\]

for some real constants \( I_0, I_1, \ldots, I_m \) and integer \( m \geq 0 \) and there exist following integrals

\[
\mathcal{J}_m = \text{v.p.} \frac{1}{\pi} \int_{\mathbb{R}} h_{m-1}(t) \, dt, \quad M_m(k) = \text{v.p.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{h_m(t)}{k-t} \, dt, \quad k \in \mathbb{C}_+.
\]

Note that if \( h \in \mathcal{X}_m \) for some \( m \geq 0 \), then there exist finite integrals (the principal value):

\[
\mathcal{J}_j = \frac{1}{\pi} \lim_{s \to \infty} \int_{-s}^{s} h_{j-1}(t) \, dt, \quad h_j = t^{j+1}(h(t) + P_j(t)), \quad h_{-1} = h,
\]

for all \( j = 0, 1, 2, \ldots, m - 1 \).

**Lemma 4.2** (i) Let \( M(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{h(t)}{k-t} \, dt \) for some \( h \in \mathcal{X}_m, \, m \geq 0 \). Then,

\[
M(k) = \frac{\mathcal{J}_0 + iI_0}{k} + \frac{\mathcal{J}_1 + iI_1}{k^2} + \cdots + \frac{\mathcal{J}_m + iI_m}{k^{m+1}} + \frac{M_m(k)}{k^{m+1}}, \quad \forall \, k \in \mathbb{C}_+,
\]

where the constants \( \mathcal{J}_0, \ldots, \mathcal{J}_m \in \mathbb{R} \) and the function \( M_m \) is given by (4.7)–(4.8).

(ii) Let \( h \) be a real function from \( L^1_{\text{loc}}(\mathbb{R}) \) and satisfy (4.6) for some \( m \geq 0 \) and

\[
h_m = \alpha + \text{Re} \hat{\beta}, \quad \text{where} \quad \alpha \in L^s(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad s \in [1, \infty),
\]

\[
\hat{\beta}(t) = \int_0^\infty e^{2itx} \beta(x) \, dx, \quad \beta \in L^1(\mathbb{R}).
\]

\[\square\] Springer
Then, \( h \in \mathcal{X}_m \) and the function \( M_m \) from (4.7) is analytic in \( \mathbb{C}_+ \) and

\[
M_m(k) = o(1) \quad \text{as} \quad \text{Im} \, k \to \infty.
\]  

(4.11)

**Proof** The statement (i) was proved in Korotyaev [16].

(ii) We consider \( M_m \) in (4.7). It is clear that the function \( \tilde{\alpha}(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\alpha(t)}{k-t} \, dt \) is well defined and is analytic in \( \mathbb{C}_+ \) and \( \tilde{\alpha}(k) = o(1) \) as \( \text{Im} \, k \to \pm \infty \).

In the case \( \beta = \beta_1 + i \beta_2 \), we use the following identities

\[
\begin{align*}
\Re \hat{\beta}(t) &= \int_0^\infty \Re e^{i2tx} \beta(x) \, dx, \\
\Re e^{i2tx} \beta &= \Re (c+is)(\beta_1 + i \beta_2) = c \beta_1 - s \beta_2, \\
\int_{\mathbb{R}} \frac{\hat{g}(t)}{k-t} \, dt &= -2\pi i \int_0^\infty e^{ikx} g(x) \, dx, \\
\int_{\mathbb{R}} \frac{\hat{g}(t)}{k-t} \, dt &= 0, \quad \forall \, k \in \mathbb{C}_+,
\end{align*}
\]

for \( g \in L^1(\mathbb{R}_+) \) and where \( c = \cos 2tx, s = \sin 2tx \). These identities yield

\[
\frac{1}{\pi} \int_{\mathbb{R}} \frac{\Re \hat{\beta}(t)}{k-t} \, dt = -\frac{i}{2} \hat{\beta}(k), \quad \forall \, k \in \mathbb{C}_+.
\]

Thus, the function \( M_m \) from (4.7) is analytic in \( \mathbb{C}_+ \) and \( M_m(k) = o(1) \) as \( \text{Im} \, k \to \pm \infty \).

The proof of the existence of \( \mathcal{J}_m = \text{v.p.} \frac{1}{\pi} \int_{\mathbb{R}} h_{m-1}(t) \, dt \) in (4.7) is similar, since \( h_{m-1} = \frac{1}{i} (-I_m + h_m) \) and \( h_m = \alpha + \Re \hat{\beta} \) and \( h_{m-1}, h_m \in L^1_{\text{loc}}(\mathbb{R}) \).

We recall the standard facts about the canonical factorization, see, e.g., [8,14] and in the needed form for us from [16].

**Theorem 4.3** Let a function \( f \in \mathcal{H}_p \) for some \( p \geq 1 \) be continuous in \( \overline{\mathbb{C}}_+ \) and \( f(k) = 1 + O(1/k) \) as \( |k| \to \infty \), uniformly in \( \arg k \in [0, \pi] \). Then, \( f(k), k \in \mathbb{C}_+ \) has a canonical factorization in \( \mathbb{C}_+ \) given by

\[
\begin{align*}
f &= f_{\text{in}} f_{\text{out}}, \quad f_{\text{in}}(k) = B(k) e^{-iK(k)}, \quad K(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{dv(t)}{k-t}, \\
f_{\text{out}}(k) &= e^{iM(k)}, \quad M(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log |f(t)|}{k-t} \, dt.
\end{align*}
\]  

(4.12)

\- \( B \) is the Blaschke product for \( \text{Im} \, k > 0 \) given by (1.6) and \( dv(t) \geq 0 \) is some singular compactly supported measure on \( \mathbb{R} \) and for some \( r_c > 0 \) satisfies

\[
\begin{align*}
v(\mathbb{R}) &= \int_{\mathbb{R}} dv(t) < \infty, \\
\text{supp} \, v &\subset [-r_c, r_c], \quad (4.13) \\
\text{supp} \, v &\subset \{ k \in \mathbb{R} : f(k) = 0 \} \subset [-r_c, r_c]. \quad (4.14)
\end{align*}
\]
• The function $K(\cdot)$ has an analytic continuation from $\mathbb{C}_+$ into the domain $\mathbb{C}\setminus[-r_c, r_c]$ and has the following Taylor series

$$K(k) = \sum_{j=0}^{\infty} \frac{K_j}{k^{j+1}}, \quad K_j = \frac{1}{\pi} \int_{\mathbb{R}} t^j \, dv(t). \quad (4.15)$$

• The function $\log |f(t + i0)|$ belongs to $L^1_{loc}(\mathbb{R})$.

**Remark** The integral $M(k)$ in (4.12) converges absolutely since $f(t) = 1 + \frac{O(1)}{t}$ as $t \to \pm \infty$.

In order to describe the Jost function $\psi$ in terms of a canonical factorization, we introduce the corresponding class of functions. Let $f \in \mathcal{H}_p$ for some $0 < p \leq \infty$ and for integer $m \geq 0$ $f$ satisfies

$$-i \log f(k) = \frac{Q_0}{k} + \frac{Q_1}{k^2} + \frac{Q_2}{k^3} + \ldots + \frac{Q_m + o(1)}{k^{m+1}}, \quad (4.16)$$

as $|k| \to \infty$ uniformly in $\arg k \in [0, \pi]$, for some constants $Q_j \in \mathbb{C}, j = 0, 1, 2, \ldots, m$. Then, the function $h = \log |f(t)|, t \in \mathbb{R}$ has the form

$$h(t) = -P_m(t) + \frac{h_m(t)}{t^{m+1}}, \quad P_m(t) = \frac{I_0}{t} + \frac{I_1}{t^2} + \ldots + \frac{I_m}{t^{m+1}}, \quad (4.17)$$

$$I_j = \text{Im } Q_j, \ j = 0, 1, \ldots, m, \quad h_m(t) = o(1) \quad \text{as} \quad t \to \pm \infty.$$

We describe a canonical factorization of functions from $\mathcal{H}_p$ from Korotyaev [16].

**Theorem 4.4** Let a function $f \in \mathcal{H}_p$ satisfy (4.16) for some $m \geq 0, p \geq 1$ and let the function $h(t) = \log |f(t)|, t \in \mathbb{R}$ has the form (4.17), where $h_m$ satisfies (4.10). Then, $f$ has a canonical factorization $f = f_{\text{in}} f_{\text{out}}$ in $\mathbb{C}_+$ given by Theorem 4.3, where the function $M$ has the form

$$M(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{h(t)}{k-t} \, dt = \mathcal{J}_0 + i \frac{I_0}{k} + \mathcal{J}_1 + i \frac{I_1}{k^2} + \ldots + \mathcal{J}_m + i \frac{I_m + M_m(k)}{k^{m+1}}, \quad (4.18)$$

for any $k \in \mathbb{C}_+$, where

$$M_m(k) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{h_m(t)}{k-t} \, dt, \quad \mathcal{J}_j = \text{v. p.} \frac{1}{\pi} \int_{\mathbb{R}} h_{j-1}(t) \, dt,$$

$$h_j(t) = t h_{j-1} - I_j, \quad (4.19)$$

$$M_m(k) = o(1) \quad \text{as} \quad \text{Im } k \to \infty, \quad (4.20)$$

$j = 0, 1, \ldots, m$, and $h_{-1} = h$. Moreover, the following trace formulas hold true:

$$B_j + K_j + \text{Re } Q_j = \mathcal{J}_j, \quad j = 0, 1, \ldots, m. \quad (4.21)$$
5 Schrödinger operators with compactly supported potentials

5.1 Entire functions

An entire function $f(k)$ is said to be of exponential type if there is a constant $\beta$ such that $|f(k)| \leq \text{const} e^{\beta |k|}$ everywhere. The infimum of the set of $\beta$ for which such inequality holds is called the type of $f$.

**Definition** Let $\mathcal{E}_\gamma, \gamma > 0$ denote the class of exponential type functions $f$ satisfying

$$
|f(k) - 1| \leq \omega(k)e^{2\gamma k - \omega(k)} \quad \forall \ k \in \mathbb{C},
$$

$$
k_- = \frac{1}{2}(|\text{Im} \ k| - \text{Im} \ k) \geq 0, \quad \omega(k) = \min\{Q_1, \frac{Q}{|k|}\},
$$

(5.1)

for some positive constants $Q_1$, $Q$ and if each its zero $z$ in $\mathbb{C}_+$ satisfies $|z| \leq Q$.

Note that if $q \in L^1(\mathbb{R}_+)$ and $\text{supp} \ q \subset [0, \gamma]$, then a Jost function $\psi \in \mathcal{E}_\gamma$, see the proof of Theorem 1.6. Define the disk $D_r(u) = \{z : |z - u| < r\}$ for $u \in \mathbb{C}$ and $r > 0$.

**Lemma 5.1**

(i) Let $f \in \mathcal{E}_\gamma$ and let $Q_1 < \omega$, where $\omega_\gamma e^{\omega} = 1$. Then $f(k) \neq 0$ for any $k \in \mathbb{C}_+$.

(ii) Let $f \in \mathcal{E}_\gamma$, and let $\omega(r) \leq \frac{1}{2}$ for some $r > 0$. If $k \in \mathbb{C}_+, |k| \leq r$, then

$$
|f(k)| \geq 1 - \frac{\sqrt{e}}{2}.
$$

(5.2)

Moreover, the number of zeros $N(\rho)$ of $f$ (counted with multiplicity) in disk $D_\rho(it)$ with the center $it = i2Q$ and the radius $\rho = \sqrt{\alpha}Q$ for any $\alpha \in (5, 8)$ satisfies

$$
N(\rho) \leq C_1 + C_2 \gamma Q,
$$

(5.3)

where the constants $C_1 = \frac{1}{C_*} \left(\frac{\sqrt{2}+2}{4} + \log \frac{2}{2-\sqrt{2}}\right)$ and $C_2 = \frac{4-\pi}{\pi C_*}$ and $C_* = \frac{1}{2} \log \frac{8}{\alpha} > 0$.

In particular, if we take $\alpha - 5$ small enough, then $C_1 \sim 10$ and $C_1 \sim 1$, and the domain $\{k \in \mathbb{C}_+, |k| \leq Q\} \subset D_\rho(it)$.

**Proof**

(i) We have $|f(k) - 1| \leq Q_1 e^{Q_1} < 1$ for any $k \in \mathbb{C}_+$, which yields $|f(k)| \geq 1 - Q_1 e^{Q_1} > 0$.

(ii) From (5.1) and $\omega(r) \leq \frac{1}{2}$, we obtain $|f(k) - 1| \leq \omega(r) e^{\omega(r)} \leq \frac{1}{2}e^{\frac{1}{2}}$, which yields (5.2).

Recall the Jensen formula (see p. 2 in [13]) for an entire function $F$ any $r > 0$:

$$
\log |F(0)| + \int_0^r \frac{N_\phi(F)}{s} \, ds = \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\phi})| \, d\phi,
$$

(5.4)
where \( N_s(F) \) is the number of zeros of \( F \) in the disk \( \mathbb{D}_s(0) \). We take the function \( F(z) = f(it + z) \) and the disk \( \mathbb{D}_r(it) \), \( it = i2Q \) with the radius \( r = \sqrt{8} Q \). Thus, (5.4) implies

\[
\log |f(it)| + \int_0^r \frac{N(s)}{s} ds = S, \quad S = \frac{1}{2\pi} \int_0^{2\pi} \log |f(k\phi)| d\phi, \quad k\phi = it - ire^i\phi, \tag{5.5}
\]

where \( N(s) = N_s(f(it + \cdot)) \). From (5.1), we obtain at \( k\phi = it - ire^i\phi \) and \( a = \frac{\pi}{4} : \)

\[
S = \frac{1}{2\pi} \int_0^{2\pi} \log |f(k\phi)| d\phi = S_0 + S_1, \quad S_0 = \frac{1}{2\pi} \int_0^a \log |f(k\phi)| d\phi, \quad S_1 = \frac{1}{2\pi} \int_a^a \log |f(k\phi)| d\phi \leq \frac{1}{2\pi} \int_a^a \log (1 + \omega e^{i\omega}) d\phi
\]

\[
= \frac{3}{4} \log (1 + \omega e^{i\omega}), \tag{5.6}
\]

where \( \omega = \omega(2Q) \leq \frac{1}{2} \). On the circle \( |it - iz| = r \), using (5.1) we get

\[
\min_{0 \leq \phi \leq a} |it - ire^i\phi| = (\sqrt{8} - 2) Q,
\]

\[
\omega_1 = \max_{0 \leq \phi \leq a} \omega(it - ire^i\phi) = \frac{1}{\sqrt{8} - 2} = \frac{\sqrt{2} + 1}{2}, \quad \log (1 + \omega_1 e^{i\omega_1}) \leq 2\omega_1. \tag{5.7}
\]

Define the integral \( S_00 = \frac{2\gamma}{\pi} \int_0^a k_- d\phi \), where we have \( k_- = r \sin \phi - t = r(\sin \phi - \sin a) \) for \( \phi \in (-a, a) \) and (5.1), (5.7) give

\[
S_0 = \frac{1}{2\pi} \int_0^a \log |f(k\phi)| d\phi \leq \frac{1}{\pi} \int_0^a \log (1 + \omega_1 e^{2\gamma k_- + i\omega_1}) d\phi
\]

\[
\leq S_00 + \frac{1}{\pi} \int_0^a \log (1 + \omega_1 e^{i\omega_1}) d\phi = S_00 + \frac{\omega_1}{2} \leq S_00 + \frac{\log (1 + \omega_1 e^{i\omega_1})}{4}, \tag{5.8}
\]

and

\[
S_00 = \frac{2\gamma}{\pi} \int_0^a k_- d\phi = \frac{2\gamma}{\pi} \int_0^a r(\cos \phi - \cos a) d\phi = \frac{2\gamma r}{\pi} (\sin a - a \cos a)
\]

\[
= \gamma Q \frac{(4 - \pi)}{\pi}. \tag{5.9}
\]

Collecting (5.6)–(5.9), we obtain

\[
S \leq \frac{1}{4} + \frac{\sqrt{2} + 1}{4} + \gamma Q \frac{(4 - \pi)}{\pi} = \frac{\sqrt{2} + 2}{4} + \gamma Q \frac{(4 - \pi)}{\pi}. \tag{5.10}
\]
Thus, if \( \rho = \sqrt{\alpha} Q \) for any \( \alpha \in (5, 8) \), then we get \( \mathbb{D}_\rho(it) \subset \mathbb{D}_r(it) \), which yields

\[
\int_{\rho}^{r} \frac{N(t)dt}{t} \geq N(\rho) \int_{\rho}^{r} \frac{dr}{t} = N(\rho) \log \frac{r}{\rho} = C_* N(\rho), \quad C_* = \frac{1}{2} \log \frac{8}{\alpha} > 0.
\]

(5.11)

Then, substituting (5.2), (5.10), (5.11) into (5.5) we obtain

\[
\log \frac{2 - \sqrt{e}}{2} + C_* N(\rho) \leq \log |f(it)| + C_* N(\rho) \leq \frac{\sqrt{2} + 2}{4} + \gamma Q \frac{(4 - \pi)}{\pi},
\]

which yields (5.3). If \( \alpha - 5 \) is small enough, then \( C_1 \sim 10 \) and \( C_2 \sim 1 \), and the domain \( \{ \Im k \geq 0, |k| \leq Q \} \subset \mathbb{D}_\rho(it) \).

\( \square \)

5.2 Proof of Theorem 1.6

Consider the Schrödinger operator \( H \), when the potential \( q \in L^1(\mathbb{R}^+) \) is complex and \( \text{supp} q \subset [0, \gamma] \) for some \( \gamma > 0 \). We recall known results: The Jost function \( \psi \) is entire and satisfies \( |\psi(k) - 1| \leq \omega(k) e^{2\gamma k + \omega(k)} \) for all \( k \in \mathbb{C} \), where \( \omega(k) = \min\{\|q\|_1, \frac{\|q\|_1}{|k|}\} \) (see, e.g., [19]). Thus, due to this fact and (1.5) we obtain that the function \( \psi \in \mathcal{E} \) with \( Q = \|q\| \) and \( Q_1 = \|q\|_1 \). Then, Lemma 5.1 gives estimate (1.28).

\( \square \)

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