GROUP ACTIONS ON RIEMANN-ROCH SPACE

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ABSTRACT. Let $G$ be a group acting on a compact Riemann surface $\mathcal{X}$ and $D$ be a $G$-invariant divisor on $\mathcal{X}$. The action of $G$ on $\mathcal{X}$ induces a linear representation $L_G(D)$ of $G$ on the Riemann-Roch space associated to $D$.

In this paper we give some results on the decomposition of $L_G(D)$ as sum of complex irreducible representations of $G$, for $D$ an effective non-special $G$-invariant divisor. In particular, we give explicit formulae for the multiplicity of each complex irreducible factor in $L_G(D)$. We work out some examples on well known families of curves.

1. INTRODUCTION

Let $\mathcal{X}$ be a compact Riemann surface, and let $G$ be a group of automorphisms of $\mathcal{X}$. If $D$ is a divisor on $\mathcal{X}$ which is stable under the action of $G$, then $G$ acts on the Riemann-Roch space $L(D)$ associated to $D$.

The problem of determining the decomposition of the induced linear representation $L_G(D)$ of $G$ on $L(D)$ as sum of irreducible representations of $G$ was originally considered by A. Hurwitz [6] in the case $D$ a canonical divisor and $G$ a cyclic group. C. Chevalley and A. Weil [4], extended this result to any finite group and $D$ a canonical divisor.

Since then, many authors have worked on this problem for certain types of divisors. See for instance Borne [1], Ellingsrud and Lønsted [5], Kani [8], Köch, [9], and Nakajima [10]. In the case where $D$ is a non-special divisor, an equivariant Riemann-Roch formula was given for the character of $L_G(D)$, see for instance [1]. Also in this case, Joyner and Ksir [7] gave explicit formula for the multiplicity of each rational irreducible factor when $L_G(D)$ is a rational representation of $G$.

In this paper we extend the Joyner-Ksir’s results to the general case: that is, when $L_G(D)$ is a complex representation of $G$. We give explicit formulae for the multiplicity of each complex irreducible factor in the decomposition of $L_G(D)$ as a sum of complex irreducible representations of $G$. To illustrate this decomposition we give some examples of group actions on Riemann-Roch spaces for divisors on well known families of curves.

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In order to formulate the results, we use the following notation: Let $t_\omega$ satisfying the Riemann-Hurwitz formula

\begin{equation}
X
\end{equation}
on a compact Riemann surface $a$. Rational and Analytical Representations.

The subgroup $G$ of $P$ analytic representation $G$, linear representations of $G$.

\begin{equation}
\omega
\end{equation}

Theorem 2.1.

The group $G$ acts on a compact Riemann surface $X$ of genus $g$ with branching data $(\gamma; m_1, \cdots, m_r)$ if and only if $G$ has a generating vector of type $(\gamma; m_1, \cdots, m_r)$ satisfying the Riemann-Hurwitz formula \((2.1)\).

In order to formulate the results, we use the following notation: Let $G$ be a group acting on a compact Riemann surface $X$ with branching data $(\gamma; m_1, \cdots, m_r)$ and generating vector $(a_1, \cdots, a_\gamma, b_1, \cdots, b_\gamma, c_1, \cdots, c_r)$. For each $P \in X$ let $G_P = \langle c_P \rangle$ be the stabilizer of $P$ in $G$, of order $m_P \geq 1$.

The subgroup $G_P$ acts on the cotangent space $X(P)$ at $P$ by a $\mathbb{C}$-character $\omega_P$. This is the ramification character of $X$ at $P$. Since $G_P$ is cyclic we have that $\omega_P$ is a primitive $m_Pth$-root of the unity. Particularly, for $P_j$ a branch point we will write $G_{P_j} = G_j$ with order $m_{P_j} = m_j$ and $\omega_P = \omega_j$ the corresponding ramification character.

For $V$ a complex irreducible representation of $G$, let $N_{P}^{V}$ be the number of times that $\omega_P$ is an eigenvalue of $V(c_P)$.

2.1. Rational and Analytical Representations. The action of $G$ on $X$ induces two linear representations of $G$, the rational representation $\rho_r : G \to GL(H^1(X, \mathbb{Z}) \otimes \mathbb{Q})$ and the analytic representation $\rho_a : G \to GL(H^{1,0}(X, \mathbb{C}))$. Both are related by

\begin{equation}
\rho_r \otimes \mathbb{C} \cong \rho_a \oplus \rho_a^*
\end{equation}
where $\rho_a^*$ is the complex conjugate of $\rho_a$.

The multiplicity of each complex irreducible factor in the decomposition of $\rho_a$ as sum of complex irreducible representations of $G$ was given by Chevalley-Weil in [4], as follows.

**Theorem 2.2.** Let $V$ be a non-trivial complex irreducible representation of $G$. Then the multiplicity $a(V)$ of $V$ in the decomposition of $\rho_a$ as sum of complex irreducible representations of $G$ is given by

$$a(V) = \dim_{\mathbb{C}}(V)(\gamma - 1) + \sum_{j=1}^r m_j^{-1} \sum_{k=1}^{N_{j_k}^V} \left\langle -\frac{k}{m_j} \right\rangle$$

where $\langle q \rangle = q - \lfloor q \rfloor$ is the fractional part of the rational number $q$.

Furthermore, for the trivial representation $V_0$ we have $a(V_0) = \gamma$.

For the rational representation $\rho_r$ similar formulae was given by Broughton [3] as follows.

**Theorem 2.3.** Let $V$ be a non-trivial complex irreducible representation of $G$. Then the multiplicity $r(V)$ of $V$ in the decomposition of $\rho_r \otimes \mathbb{C}$ as sum of complex irreducible representations is given by

$$r(V) = 2(\gamma - 1) \dim_{\mathbb{C}}(V) + \sum_{j=1}^r (\dim_{\mathbb{C}}(V) - \dim_{\mathbb{Q}}(V^{G_j}))$$

where $V^{G_j}$ denotes the fixed subspace of $V$ under the action of $G_j$.

With these results we obtain the following

**Corollary 2.4.** If $V$ is a non-trivial absolutely irreducible representation of $G$, then

$$a(V) = a^*(V) = (\gamma - 1) \dim_{\mathbb{Q}}(V) + \frac{1}{2} \left( \sum_{j=1}^r \dim_{\mathbb{Q}}(V) - \dim_{\mathbb{Q}}(V^{G_j}) \right)$$

where $a^*(V)$ is the multiplicity of $V$ in the decomposition of $\rho_a^*$.

2.2. The Ramification Module. The following definition was introduced in [8] (also see [1], [7] and [10]). The ramification module for the cover $\Pi : \mathcal{X} \rightarrow \mathcal{X}_G$ with branching data $(\gamma; m_1, \ldots, m_r)$ and generating vector $(a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_r)$ is defined by

$$\Gamma_G = \sum_{j=1}^r \text{Ind}_{G_j}^{G} \left( \sum_{\alpha=1}^{m_j-1} \alpha \omega_j^\alpha \right).$$

The following result was proved by Kani [8] and Nakajima [10].
**Theorem 2.5.** Let $G$ be a group acting on $X$ and $\Gamma_G$ the associated ramification module. Then there is a unique $G$-module $\tilde{\Gamma}_G$ such that

$$\Gamma_G = \tilde{\Gamma}_G^{[G]}.$$  

Considering $\tilde{\Gamma}_G^*$ the dual $G$-module of $\tilde{\Gamma}_G$, an interesting relationship between the analytical representation (character) and the representation (character) of $G$ on $\tilde{\Gamma}_G^*$ is given by the following result [8, Theorem 2 and Corollary.]

**Theorem 2.6.** Let $G$ be a group acting on $X$. If $\chi_a$ is the character of the analytical representation and $\chi_{\tilde{\Gamma}_G}^*$ is the character of the representation of $G$ on $\tilde{\Gamma}_G^*$, then

$$\chi_a = \chi_0 + (\gamma - 1)\chi_{\text{reg}} + \chi_{\tilde{\Gamma}_G}^*$$

where $\chi_{\text{reg}}$ is the regular character and $\chi_0$ is the trivial character of $G$.

As a simple application of the above result we obtain a generalization of [7, Proposition 5 and Corollary 6].

**Corollary 2.7.** Let $V$ be a complex irreducible representation of $G$. Then

$$\langle \chi_{\tilde{\Gamma}_G}, \chi_V \rangle = \begin{cases}  
\langle \chi_{\tilde{\Gamma}_G}, \chi_0 \rangle = 0 & \text{if } V = V_0 \text{ is the trivial representation:}  
\langle \chi_{\tilde{\Gamma}_G}, \chi_V \rangle = a^*(V) + (1 - \gamma) \dim_{\mathbb{C}}(V) & \text{if } V \text{ is a non-trivial representation.}
\end{cases}$$

where $\chi_V$ is the character of $V$ and $\langle , \rangle$ is the usual inner product of characters.

**Proof.** According to Theorem 2.6 observe that if $\chi_a^*$ is the dual character of $\chi_a$ then

$$\chi_a^* = \chi_0 + (\gamma - 1)\chi_{\text{reg}} + \chi_{\tilde{\Gamma}_G}.$$  

It follows that

1. the multiplicity of the trivial representation $V_0$ of $G$ in $\tilde{\Gamma}_G$ is

$$\langle \chi_{\tilde{\Gamma}_G}, \chi_0 \rangle = \gamma + (1 - \gamma) - 1 = 0;$$

2. the multiplicity of any non-trivial complex irreducible representation $V$ of $G$ in $\tilde{\Gamma}_G$ is

$$\langle \chi_{\tilde{\Gamma}_G}, \chi_V \rangle = a^*(V) + (1 - \gamma) \dim_{\mathbb{C}}(V).$$

$\square$
3. Decomposition of $L_G(D)$

Let $X$ be a compact Riemann surface of genus $g$ and $D$ a divisor on $X$. We recall that the Riemann-Roch space associated to $D$ is defined by

$$L(D) = \{ f \in \mathbb{C}^*(X) \mid \operatorname{div}(f) \geq -D \} \cup \{0\}$$

and the dimension of $L(D)$ is given by the Riemann-Roch Theorem

$$\dim_{\mathbb{C}}(L(D)) = \deg(D) - g + 1 + \dim_{\mathbb{C}}(\Omega(D))$$

where $\Omega(D) = \{ \omega / \omega \text{ is an abelian differential with } \operatorname{div}(\omega) \geq D \} \cup \{0\}$.

A divisor $D$ is called non-special if $\dim_{\mathbb{C}}(\Omega(D)) = 0$, or, equivalently, if $\dim_{\mathbb{C}}(L(K-D)) = 0$ for some canonical divisor $K$ on $X$.

**Remark 3.1.** As was mentioned earlier, if $D$ is a divisor on $X$ which is stable under the action of $G$, then $G$ acts on the Riemann-Roch space $L(D)$ associated to $D$ by the linear representation $L_G(D)$.

For each $P$ in $X$, consider the (basic) $G$-invariant divisor given by

$$D_b(P) = \frac{1}{m_P} \sum_{g \in G} g(P) \quad \text{where } m_P = |G_P|.$$

Then the set of the basic divisors generates the group $\operatorname{Div}(X)^G$ of the $G$-invariant divisors on $X$.

We recall the definition of the equivariant degree, as can be seen for example in [7].

**Definition 3.2.** The equivariant degree is a map from $\operatorname{Div}(X)^G$ to the Grothendieck group $R_k(G) = \mathbb{Z}[G^*_k]$,

$$\deg_{eq} : \operatorname{Div}(X)^G \rightarrow R_k(G)$$

defined by the following conditions:

1. $\deg_{eq}$ is additive on the $G$-invariant divisors of disjoint support;
2. If $D = r_PD_b(P)$, then

$$\deg_{eq}(D) = \begin{cases} 
\operatorname{Ind}_{G_P}^G \left( \sum_{k=1}^{r_P} \omega_{P}^{-k} \right) & \text{if } r_P > 0; \\
-\operatorname{Ind}_{G_P}^G \left( \sum_{k=0}^{-(r_P+1)} \omega_{P}^{k} \right) & \text{if } r_P < 0; \\
0 & \text{if } r_P = 0.
\end{cases}$$
where $\omega_P$ is the ramification character of $X$ at $P$.

Now we compute the multiplicity of any complex irreducible representation of $G$ in the decomposition of $\deg_{eq}(D)$, for $D$ a positive multiple of a basic divisor.

**Proposition 3.3.** Consider $D = r_P D_b(P)$ with $r_P > 0$ and put $r_P = l_P + s_P m_P$, where $0 \leq l_P < m_P$. If $V$ is a complex irreducible representation of $G$, then the multiplicity $d(V)$ of $V$ in the decomposition of $\deg_{eq}(D)$ as sum of complex irreducible representations of $G$ is given by

$$d(V) = s_p \dim_{\mathbb{C}}(V) + \epsilon_P \left( \dim_{\mathbb{C}}(V) - \sum_{k=0}^{m_P-(l_P+1)} N_{P_k}^V \right)$$

where $\epsilon_P = 0$ if $l_P = 0$ and $\epsilon_P = 1$ if $l_P \neq 0$.

**Proof.** Since $\omega_P$ is a primitive $m_P$th-root of the unity, we have

$$\rho_{reg} = \text{Ind}_{\{1\}}^G(\chi_0) = \text{Ind}_{G_P}^G(\text{Ind}_{\{1\}}^{G_P} \chi_0) = \text{Ind}_{G_P}^G \left( \sum_{k=1}^{m_P} w_P^k \right) = \text{Ind}_{G_P}^G \left( \sum_{k=1}^{m_P} w_P^{-k} \right).$$

With this we obtain

$$\text{Ind}_{G_P}^G \left( \sum_{k=1}^{r_P} \omega_P^{-k} \right) = \text{Ind}_{G_P}^G \left( \sum_{k=1}^{l_P+s_P m_P} \omega_P^{-k} \right) = \begin{cases} s_p \rho_{reg} + \text{Ind}_{G_P}^G \left( \sum_{k=1}^{l_P} \omega_P^{-k} \right) & \text{if } l_P \neq 0 \\ s_p \rho_{reg} & \text{if } l_P = 0 \end{cases}$$

Furthermore, if $l_P \neq 0$ we have

$$\text{Ind}_{G_P}^G \left( \sum_{k=1}^{l_P} \omega_P^{-k} \right) + \text{Ind}_{G_P}^G \left( \sum_{k=0}^{m_P-(l_P+1)} \omega_P^k \right) = \rho_{reg}$$

and in this way

$$\left\langle \text{Ind}_{G_P}^G \left( \sum_{k=1}^{l_P} \omega_P^{-k} \right), V \right\rangle = s_p \dim_{\mathbb{C}}(V) - \sum_{k=0}^{m_P-(l_P+1)} N_{P_k}^V.$$

Hence

$$d(V) = \left\langle \text{Ind}_{G_P}^G \left( \sum_{k=1}^{r_P} \omega_P^{-k} \right), V \right\rangle = s_p \dim_{\mathbb{C}}(V) + \epsilon_P \left( \dim_{\mathbb{C}}(V) - \sum_{k=0}^{m_P-(l_P+1)} N_{P_k}^V \right)$$

where $\epsilon_P = 0$ if $l_P = 0$ and $\epsilon_P = 1$ if $l_P \neq 0$. □

The following result can be seen in [7, Lemma 4] (also see [1]).
Lemma 3.4. Let $D$ be a $G$-invariant non-special divisor on $X$ and $\chi_L$ the character of the representation $L_G(G)$ of $G$. Then

$$\chi_L = (1 - \gamma)\chi_{\text{reg}} + \deg_{\text{eq}}(D) - \chi_{\tilde{\Gamma}_G}$$

Now we are able to prove our main result.

Theorem 3.5. Let $G$ be a group acting on $X$ and $D = \sum_{P \in X} r_P D_b(P)$ be an effective non-special divisor on $X$.

For each $P \in X$ write $r_P = l_P + s_P m_P$ with $0 \leq l_P < m_P$. If $V$ is a non-trivial complex irreducible representation of $G$, then the multiplicity $m(V)$ of $V$ in the decomposition of $L_G(D)$ as sum of irreducible complex representations of $G$ is given by

$$m(V) = \sum_{P \in X} s_P \dim_C(V) + \sum_{P \in X} \epsilon_P \left( \dim_C(V) - \sum_{k=0}^{m_P - (l_P + 1)} N_P^V_k \right) - a^*(V)$$

where $\epsilon_P = 0$ if $l_P = 0$ and $\epsilon_P = 1$ if $l_P \neq 0$.

Furthermore, for the trivial representation $V_0$ we have

$$m(V_0) = 1 - \gamma + \sum_{P \in X} s_P.$$

Proof. According to Lemma 3.4 we have

$$\chi_L = (1 - \gamma)\chi_{\text{reg}} + \deg_{\text{eq}}(D) - \chi_{\tilde{\Gamma}_G}.$$

Hence

$$m(V) = \langle \chi_L, \chi_V \rangle = (1 - \gamma) \dim_C(V) + \langle \deg_{\text{eq}}(D), \chi_V \rangle - \langle \chi_{\tilde{\Gamma}_G}, \chi_V \rangle.$$
and the result follows.

**Remark 3.6.** Let \( D = \pi^*(D_0) \) be a divisor on \( X \) which is a pullback of an effective divisor \( D_0 = \sum_{Q \in X_G} \alpha_Q Q \) on \( X_G \).

Then
\[
D = \pi^*(D_0) = \sum_{Q \in X_G} \alpha_Q \sum_{P \in \pi^{-1}(Q)} m_P P = \sum_{Q \in X_G} \alpha_Q \sum_{g \in G} g(P) = \sum_{Q \in X_G} \alpha_Q m_P D_b(P),
\]
fixing \( P \in \pi^{-1}(Q) \). Hence, with the notation of the Theorem 3.5, we have \( l_P = 0 \) and \( s_P = \alpha_Q \), for all \( Q \in X_G \).

Our last result of this section is a generalization of [7, Theorems 1 and 2].

**Corollary 3.7.** Let \( D = \pi^*(D_0) \) be a non-special divisor on \( X \) which is a pullback of an effective divisor \( D_0 \) on \( X_G \). If \( V \) is a non-trivial complex irreducible representation of \( G \), then the multiplicity \( m(V) \) of \( V \) in the decomposition of \( L_G(D) \) as sum of irreducible complex representations of \( G \) is given by
\[
m(V) = \deg(D_0) \dim_{\mathbb{C}}(V) - a^*(V).
\]
Furthermore, for the trivial representation \( V_0 \) we have \( m(V_0) = \deg(D_0) + 1 - \gamma \).
In particular, if \( V \) is a non-trivial absolutely irreducible representation of \( G \), then
\[
m(V) = \dim_{\mathbb{Q}}(V)(\deg(D_0) + 1 - \gamma) - \frac{1}{2} \left( \sum_{j=1}^{r} \dim_{\mathbb{Q}}(V) - \dim_{\mathbb{Q}}(V^{G_j}) \right).
\]

**Proof.** Let \( D_0 = \sum_{Q \in X_G} \alpha_Q Q \). According to Remark 3.6, we have
\[
D = \pi^*(D_0) = \sum_{Q \in X_G} \alpha_Q \sum_{P \in \pi^{-1}(Q)} m_P P = \sum_{Q \in X_G} \alpha_Q \sum_{g \in G} g(P) = \sum_{Q \in X_G} \alpha_Q m_P D_b(P),
\]
fixing \( P \in \pi^{-1}(Q) \).
Now applying Theorem 3.5 with \( l_P = 0 \) and \( s_P = \alpha_Q \), we have
\[
m(V) = \sum_{P \in X} s_P \dim_{\mathbb{C}}(V) - a^*(V) = \deg(D_0) \dim_{\mathbb{C}}(V) - a^*(V)
\]
and \( m(V_0) = \deg(D_0) + 1 - \gamma \).
Finally, according to Corollary 2.4 for \( V \) a non-trivial absolutely irreducible representation
\[
a^*(V) = (\gamma - 1) \dim_{\mathbb{Q}}(V) + \frac{1}{2} \left( \sum_{j=1}^{r} \dim_{\mathbb{Q}}(V) - \dim_{\mathbb{Q}}(V^{G_j}) \right).
\]
Then in this case
\[
m(V) = \dim_{\mathbb{Q}}(V)(\deg(D_0) + 1 - \gamma) - \frac{1}{2} \left( \sum_{j=1}^{r} \dim_{\mathbb{Q}}(V) - \dim_{\mathbb{Q}}(V^G) \right).
\]

\[\square\]

4. Examples

In this section, to apply our results we give some examples of group actions on Riemann-Roch spaces for divisors on well known families of curves.

We first recall a well known fact:

Remark 4.1. Let \( D \) be a divisor on \( X \). If \( \deg(D) > 2(g - 1) \), then \( D \) is non-special.

Example 4.2. See [7, Example 4]. Let \( X \) be the Klein quartic of genus \( g = 3 \)
\[
\{[X : Y : Z] \in \mathbb{CP}^2 / X^3Y + Y^3Z + Z^3X = 0\}.
\]

Consider the automorphisms of \( X \) given by
\[
\tau[X : Y : Z] = [\eta X : \eta^4 Y : \eta^2 Z] \quad \text{and} \quad \sigma[X : Y : Z] = [Y : Z : X]
\]
where \( \eta \) is a primitive seventh root of the unity. The group \( G = \langle \tau, \sigma \rangle \cong \langle \tau \rangle \rtimes \langle \sigma \rangle \) has order 21 and has characters table given by

| degree | \( \tau \) | \( \sigma \) |
|--------|----------|----------|
| \( \chi_0 \) | 1 | 1 | 1 |
| \( \chi_1 \) | 1 | 1 | \( \zeta \) |
| \( \chi_2 \) | 1 | 1 | \( \zeta^2 \) |
| \( \chi_3 \) | 3 | \( \eta + \eta^2 + \eta^4 \) | 0 |
| \( \chi_4 \) | 3 | \( \eta^3 + \eta^6 + \eta^8 \) | 0 |

with \( \zeta \) a primitive cube root of the unity. The complex irreducible representations associated to \( \chi_3 \) and \( \chi_4 \) are respectively
\[
V_3(\tau) = \begin{pmatrix}
\eta & 0 & 0 \\
0 & \eta^2 & 0 \\
0 & 0 & \eta^4
\end{pmatrix}; \quad V_3(\sigma) = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \equiv \begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta & 0 \\
0 & 0 & \zeta^2
\end{pmatrix}
\]
\[
V_4(\tau) = \begin{pmatrix}
\eta^3 & 0 & 0 \\
0 & \eta^5 & 0 \\
0 & 0 & \eta^8
\end{pmatrix}; \quad V_4(\sigma) = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \equiv \begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta & 0 \\
0 & 0 & \zeta^2
\end{pmatrix}.
\]

The point \( P = [1 : \zeta : \zeta^3] \) is fixed by \( H_P = \langle \sigma \rangle \).

Consider the non-special divisor \( D = D_b(P) = \frac{1}{3} \sum_{g \in G} g(P) \) of degree 7.

Then \( r_P = 1, l_P = 1 \) and \( s_P = 0 \). Also
\[
\dim_{\mathbb{C}}(\mathcal{L}(D)) = \deg(D) - g + 1 = 5.
\]
Since $m_P = 3$ and $l_P = 1$, we have $m_P - (l_P + 1) = 1$. In this way for each $V_j$ we obtain
\[
\sum_{k=0}^{1} N_{P,k}^{V_j} = 1 ; \sum_{k=0}^{1} N_{P,k}^{V_2} = 0 ; \sum_{k=0}^{1} N_{P,k}^{V_3} = 2 ; \sum_{k=0}^{1} N_{P,k}^{V_4} = 2
\]
The analytic representation of $G$ associated to the action on $X$ is $\rho_a = V_3$. With this $a^*(V_1) = 0$, $a^*(V_2) = 0$, $a^*(V_3) = 0$ and $a^*(V_4) = 1$.

Applying Theorem 3.5 we have
\[
m(V_0) = 1, \ m(V_1) = 0, \ m(V_2) = 1, \ m(V_3) = 1 \text{ and } m(V_4) = 0.
\]
Finally we conclude
\[
L_G(D) \cong V_0 \oplus V_2 \oplus V_3.
\]

**Example 4.3.** Let $p \geq 5$ be a prime number.
Consider the Fermat curve
\[
X = \{[X : Y : Z] \in \mathbb{CP}^2 / X^p + Y^p + Z^p = 0\}
\]
of genus $g = \frac{(p-1)(p-2)}{2}$ and the automorphism of $X$ defined by $\sigma[X : Y : Z] = [\omega X : Y : Z]$ where $\omega$ is a primitive $p^{th}$-root of the unity.

For $G = \langle \sigma \rangle$ the branching data is $(0; p, p, \ldots, p)$ and a generating vector is $(\sigma, \sigma, \ldots, \sigma)$. The non-trivial representations $\{V_1, V_2, \ldots, V_{p-1}\}$ of $G$ are defined by $\sigma \rightarrow w^i$ with $1 \leq i \leq p-1$.

Let $\eta$ be a primitive $2p^{th}$-root of the unity and $P = [0 : \eta : 1] \in X$. Then $P$ is a fixed point by $G$. Consider $D = (p(p-3) + 1)D_b(P)$. Then
\[
\dim_{\mathbb{C}}(L(D)) = \deg(D) - g = 1 = \frac{(p-1)(p-2)}{2}.
\]
Is it not difficult to prove that the set
\[
\beta = \left\{ F_{ab} = \frac{X^a Y^b}{(Y - \eta Z)^{a+b}} / 0 \leq a \leq p-3, \ 0 \leq b \leq p-3, \ a + b \leq p-3 \right\}
\]
is a basis of $L(D)$. For $F_{ab} \in \beta$ the action of $G$ is given by $\sigma(F_{ab}) = \frac{\omega^a X^a Y^b}{(Y - \omega Z)^{a+b}}$.

Hence
\[
L_G(D) \cong (p-2)V_0 \oplus (p-3)V_1 \oplus \cdots \oplus 2V_{p-4} \oplus V_{p-3}.
\]

Now applying Theorem 2.2 the analytic representation of $G$ is
\[
\rho_a \cong (p-2)V_1 \oplus (p-3)V_2 \oplus \cdots \oplus 2V_{p-3} \oplus V_{p-2}.
\]
In this way
\[ \rho_a^* \cong V_2 \oplus 2V_3 \oplus \cdots \oplus (p - 3)V_{p-2} \oplus (p - 2)V_{p-1} \]

We will apply Theorem 3.5 with \( s_P = p - 3 \) and \( l_P = 1 \). We have
\[ m(V_0) = 1 + s_P = p - 2 \]
\[ m(V_j) = -a^*(V_j) + s_P + \dim_{\mathbb{C}}(V_j) - \sum_{k=0}^{p-2} N_{P_k}^{V_j} \]

Finally
\[ m(V_j) = p - 2 - a^*(V_j) - \sum_{k=0}^{p-2} N_{P_k}^{V_j} = \begin{cases} 
    p - 2 - j & \text{for } 1 \leq j \leq p - 3 \\
    0 & \text{for } p - 2 \leq j \leq p - 1
\end{cases} \]

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