A group commutator involving the last distance matrix and dual distance matrix of a \(Q\)-polynomial distance-regular graph

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Abstract

Let \(\Gamma\) denote the Hamming graph \(H(D, r)\) with \(r \geq 3\). Consider the distance matrices \(\{A_i\}_{i=0}^D\) of \(\Gamma\). Fix a vertex \(x\) of \(\Gamma\), and consider the dual distance matrices \(\{A_i^*\}_{i=0}^D\) of \(\Gamma\) with respect to \(x\). We investigate the group commutator \(A_D^{-1}A_D^*-1A_D^*A_D\). We show that this matrix is diagonalizable. We compute its eigenvalues and their eigenspaces. Let \(T\) denote the subconstituent algebra of \(\Gamma\) with respect to \(x\). We describe the action of \(A_D^{-1}A_D^*-1A_D^*A_D\) on each irreducible \(T\)-module.

1 Introduction

Let \(\Gamma\) denote a \(Q\)-polynomial distance-regular graph with diameter \(D\) and distance matrices \(\{A_i\}_{i=0}^D\). Fix a vertex \(x\) of \(\Gamma\), and let \(\{A_i^*\}_{i=0}^D\) denote the dual distance matrices of \(\Gamma\) with respect to \(x\) (formal definitions will begin in Section 2). It is known that \(A_D\) and \(A_D^*\) are invertible [6, Theorem 6.6]. Motivated by this, we consider the group commutator \(A_D^{-1}A_D^*-1A_D^*A_D\). It is natural to ask, is this matrix diagonalizable, and if so, what are its eigenvalues and their eigenspaces. In this paper we investigate this question. For the sake of concreteness, we assume that \(\Gamma\) is the Hamming graph \(H(D, r)\) with \(r \geq 3\). Our results are summarized as follows. We show that the
product $A_D^{-1}A_D^{*-1}A_D A_D^*$ is diagonalizable with eigenvalues

$$(1 - r)^s \quad - D \leq s \leq D.$$ 

For each eigenvalue $(1 - r)^s$, we describe the corresponding eigenspace in terms of the split decomposition (see [10]). We show that the dimension of this eigenspace is

$$\sum_{0 \leq i, j \leq D \atop i + j \geq D \atop j - i = s} \binom{D}{i} \binom{i}{D - j} (r - 2)^{i + j - D}.$$ 

It is known that the matrices $\{A_i\}_{i=0}^D$ form a basis for the Bose-Mesner algebra $M$ of $\Gamma$. Similarly, the matrices $\{A_i^*\}_{i=0}^D$ form a basis for the dual Bose-Mesner algebra $M^*$ of $\Gamma$ with respect to $x$. Recall that $M$ and $M^*$ generate the subconstituent algebra $T$ of $\Gamma$ with respect to $x$. Let $W$ denote an irreducible $T$-module. We consider the action of $A_D^{-1}A_D^{*-1}A_D A_D^*$ on $W$. We show that this action is diagonalizable with eigenvalues

$$(1 - r)^s \quad -d \leq s \leq d, \quad d - s \text{ is even},$$

where $d$ is the diameter of $W$. For this action, we show that each eigenspace has dimension 1.

This paper is organized as follows. In Section 2, we discuss some basic facts about $Q$-polynomial distance-regular graphs. In Section 3, we give a detailed description of the subconstituent algebra $T$ for the complete graph $K_r$. In Section 4, we focus on the Hamming graph $H(D, r)$; this section contains our main results.

## 2 Preliminaries

Let $\mathbb{C}$ denote the complex number field. Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We observe that $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We call $V$ the standard module. We endow $V$ with the Hermitean inner product $\langle \cdot , \cdot \rangle$ that satisfies $\langle u, v \rangle = u^t \overline{v}$ for all $u, v \in V$. Here $t$ denotes the transpose, and $\overline{\cdot}$ denotes complex conjugation. For each $y \in X$, let $\hat{y}$ denote the element of $V$ with a one in the $y$ coordinate and zero in all other coordinates. We observe that $\{\hat{y} \mid y \in X\}$
is an orthonormal basis for $V$. Let $1$ denote the all 1’s vector in $V$. Observe that $1 = \sum_{y \in X} \hat{y}$.

Throughout the paper, $\Gamma$ denotes a finite, undirected, connected graph with vertex set $X$ and path length distance function $\partial$. Define $D = \max\{\partial(x,y) | x, y \in X\}$. We call $D$ the diameter of $\Gamma$. Let $k$ denote a non-negative integer. Then $\Gamma$ is said to be regular with valency $k$ whenever every vertex of $\Gamma$ is adjacent to exactly $k$ distinct vertices of $\Gamma$.

We say that $\Gamma$ is distance-regular whenever for all integers $h, i, j$ ($0 \leq h, i, j \leq D$) and for all $x, y \in X$ with $\partial(x,y) = h$, the number $p^h_{ij} := |\{z \in X | \partial(x,z) = i \text{ and } \partial(y,z) = j\}|$ is independent of $x$ and $y$. The constants $p^h_{ij}$ ($0 \leq h, i, j \leq D$) are called the intersection numbers of $\Gamma$. For the rest of this paper, assume that $\Gamma$ is distance-regular with $D \geq 1$. Observe that $\Gamma$ is regular with valency $k = p^0_{11}$. For each integer $i$ ($0 \leq i \leq D$), let $A_i$ denote the matrix in $Mat_X(\mathbb{C})$ with $(x,y)$-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \quad (x, y \in X).$$

We call $A_i$ the $i$th distance matrix of $\Gamma$. For convenience, define $A_i = 0$ if $i < 0$ or $i > D$. We abbreviate $A = A_1$, and call this the adjacency matrix of $\Gamma$. Observe

$$A_0 = I,$$
$$\sum_{i=0}^D A_i = J \quad (J \text{ is all 1's matrix}),$$
$$A_i^t = A_i \quad (0 \leq i \leq D),$$
$$A_iA_j = \sum_{h=0}^D p^h_{ij}A_h \quad (0 \leq i, j \leq D).$$

Let $M$ denote the subalgebra of $Mat_X(\mathbb{C})$ generated by $A$. We call $M$ the Bose-Mesner algebra of $\Gamma$. The matrices $A_0, A_1, \ldots, A_D$ form a basis for $M$. By [2, p. 45], $M$ has a second basis $E_0, E_1, \ldots, E_D$ such that

$$E_0 = |X|^{-1}J,$$
$$\sum_{i=0}^D E_i = I,$$
$$E_i^t = E_i \quad (0 \leq i \leq D),$$
$$E_iE_j = \delta_{ij}E_i \quad (0 \leq i, j \leq D).$$
We call $E_0, E_1, \ldots, E_D$ the primitive idempotents of $\Gamma$.

Let $\circ$ denote the entry-wise product in $\text{Mat}_X(\mathbb{C})$. Observe

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \leq i, j \leq D).$$

Consequently, $M$ is closed under $\circ$. Therefore, there exist complex scalars $q_{ij}^h$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D).$$

By [1, p. 170], the scalar $q_{ij}^h$ is real and nonnegative for $0 \leq h, i, j \leq D$. The $q_{ij}^h$ are called the Krein parameters of $\Gamma$. The graph $\Gamma$ is said to be $Q$-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ whenever the following hold for $0 \leq h, i, j \leq D$:

(i) $q_{ij}^h = 0$ if one of $h, i, j$ is greater than the sum of the other two;

(ii) $q_{ij}^h \neq 0$ if one of $h, i, j$ equals the sum of the other two.

For the rest of this paper, assume that $\Gamma$ is $Q$-polynomial with respect to $E_0, E_1, \ldots, E_D$.

For the rest of this section, fix $x \in X$. For each integer $i$ ($0 \leq i \leq D$), let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y, y)$-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X).$$

We call $E_i^*$ the $i$th dual idempotent of $\Gamma$ with respect to $x$. For convenience, define $E_i^* = 0$ if $i < 0$ or $i > D$. Observe

$$\sum_{i=0}^{D} E_i^* = I,$$

$$E_i^* = E_i^* \quad (0 \leq i \leq D),$$

$$E_i^* = E_i^* \quad (0 \leq i \leq D),$$

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad (0 \leq i, j \leq D).$$

The matrices $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{C})$, called the dual Bose-Mesner algebra of $\Gamma$ with respect to $x$ [8, p. 378].

We now give another basis for $M^*$. For each integer $i$ ($0 \leq i \leq D$), let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y, y)$-entry
\((A_i^*)_{yy} = |X|(E_i)_{xy} \quad (y \in X)\).

We call \(A_i^*\) the \(i\)th dual distance matrix of \(\Gamma\) with respect to \(x\) \cite[p. 379]{8}. We abbreviate \(A^* = A_1^*\), and call this the dual adjacency matrix of \(\Gamma\) with respect to \(x\).

The matrices \(A_0^*, A_1^*, \ldots, A_D^*\) form a basis for \(M^*\) such that
\[
\begin{align*}
A_0^* &= I, \\
\sum_{i=0}^{D} A_i^* &= |X|E_0^*,
\end{align*}
\]
\[
\begin{align*}
\overline{A_i^*} &= A_i^* \quad (0 \leq i \leq D), \\
A_i^{*t} &= A_i^* \quad (0 \leq i \leq D), \\
A_i^*A_j^* &= \sum_{h=0}^{D} q_{ij}^h A_h^* \quad (0 \leq i, j \leq D).
\end{align*}
\]

Observe that the algebra \(M^*\) is generated by \(A^*\) \cite[Lemma 3.11]{8}.

\textbf{Definition 1.} \cite[Definition 3.3]{8} Let \(T = T(x)\) denote the subalgebra of \(\text{Mat}_X(\mathbb{C})\) generated by \(M\) and \(M^*\). We call \(T\) the subconstituent algebra (or Terwilliger algebra) of \(\Gamma\) with respect to \(x\).

We observe that \(T\) is generated by \(A\) and \(A^*\). Moreover \(T\) has finite dimension. By \cite[Lemma 3.4]{8}, the algebra \(T\) is semisimple.

\textbf{Definition 2.} An element \(C \in T\) is called central whenever \(CB = BC\) for all \(B \in T\). Define \(Z(T) = \{C \in T | C\) is central\}\). We call \(Z(T)\) the center of \(T\).

\textbf{Definition 3.} By a \(T\)-module we mean a subspace \(W \subseteq V\) such that \(BW \subseteq W\) for all \(B \in T\).

Let \(W\) denote a \(T\)-module. Then \(W\) is said to be irreducible whenever \(W\) is nonzero, and \(W\) contains no \(T\)-modules other than 0 and \(W\). By \cite[Corollary 6.2]{5} any \(T\)-module is an orthogonal direct sum of irreducible \(T\)-modules. In particular the standard module \(V\) is an orthogonal direct sum of irreducible \(T\)-modules. By \cite[Lemma 3.3]{3}, any two non-isomorphic irreducible \(T\)-modules are orthogonal.

Let \(W\) denote an irreducible \(T\)-module. By the endpoint of \(W\) we mean \(\min \{i | 0 \leq i \leq D, E_i^*W \neq 0\}\). By the diameter of \(W\) we mean \(|\{i | 0 \leq i \leq D, E_i^*W \neq 0\}\} - 1\). By the dual endpoint of \(W\) we mean \(\min \{i | 0 \leq i \leq D, E_iW \neq 0\}\). By the dual diameter
of $W$ we mean $|\{i|0 \leq i \leq D, E_i W \neq 0\}| - 1$. The diameter of $W$ is equal to the dual diameter of $W$ \cite{7, Corollary 3.3}. The $T$-module $W$ is thin whenever $\dim(E_i^* W) \leq 1$ for $0 \leq i \leq D$. In this case $\dim(E_i W) \leq 1$ for $0 \leq i \leq D$ \cite{8, Lemma 3.9}. By the displacement of $W$ we mean the integer $\rho + \tau + d - D$, where $\rho, \tau, d$ denote the endpoint, dual endpoint, and diameter of $W$, respectively. By \cite{4, Theorem 8.3, Theorem 8.4} there is a unique irreducible $T$-module $W$ with $E_0^* W \neq 0$ and $E_0 W \neq 0$, called the primary $T$-module. The primary $T$-module has basis $A_0 \hat{x}, \ldots, A_D \hat{x}$ \cite{8, Lemma 3.6}. The primary $T$-module is thin \cite{4, Theorem 8.4}.

By \cite{6, Theorem 6.6} $A_D$ and $A_D^*$ are invertible. Motivated by this, we consider the group commutator $A_D^{-1} A_D^* A_D A_D^*$. For the sake of concreteness, we focus on the Hamming graph $H(D, r)$ with $r \geq 3$. Before we discuss $H(D, r)$, it is useful to discuss the complete graph $K_r$. We do this in the next section.

3 The complete graph $K_r$

From now on, fix an integer $r \geq 3$ and assume that $|X| = r$.

**Definition 4.** The complete graph $K_r$ has vertex set $X$, and any two distinct vertices are adjacent.

For the rest of this section, assume that $\Gamma$ is the complete graph $K_r$. We have $E_0 = r^{-1}J$, $E_1 = I - E_0 = I - r^{-1}J$ and $A = J - I = (r - 1)E_0 - E_1$. For the rest of this section, fix $x \in X$ and let $T = T(x)$ be the corresponding subconstituent algebra. We have $A^* = (r - 1)E_0^* - E_1^*$.

We have a comment about writing matrices in $\text{Mat}_X(\mathbb{C})$. When we list the elements of $X$ we list $x$ first. We have

$$E_0^* = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad E_1^* = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (1)$$

**Lemma 5.** For the graph $K_r$, the matrices $E_0$ and $E_0^*$ generate $T$.

**Proof.** The algebra $T$ is generated by $A$ and $A^*$. Also $A = rE_0 - I$ and $A^* = rE_0^* - I$. \hfill $\Box$
Lemma 6. For the graph $K_r$, the following hold:

(i) \( rE_0E_0^*E_0 = E_0. \)

(ii) \( rE_0^*E_0E_0^* = E_0^*. \)

Proof. By matrix multiplication using (1) and $E_0 = r^{-1}J$.  \( \square \)

Note that for the complete graph $K_r$, the primary $T$-module has dimension 2. Up to isomorphism, there exists a unique non-primary irreducible $T$-module, and this has dimension 1. In particular, every irreducible $T$-module is thin.

Lemma 7. For the graph $K_r$, the algebra $T$ is isomorphic to $Mat_2(\mathbb{C}) \oplus \mathbb{C}$.  

Proof. By the comment below Lemma 6 and since $T$ is semisimple. \( \square \)

Lemma 8. For the graph $K_r$, the following hold:

(i) \( \dim(T) = 5. \)

(ii) \( I, E_0, E_0^*, E_0^*E_0^*, E_0^*E_0 \) form a basis for $T$.

Proof. (i) By Lemma 7

(ii) By Lemma 5, Lemma 6 and (i) above. \( \square \)

Definition 9. Define $e_0 \in Mat_X(\mathbb{C})$ to be the projection onto the primary $T$-module. Define $e_1 \in Mat_X(\mathbb{C})$ to be the projection onto the span of the non-primary irreducible $T$-modules.

Observe that

\[ V = e_0V + e_1V \quad (\text{orthogonal direct sum}). \]  \( (2) \)

Lemma 10. For the graph $K_r$, the following hold:

(i) \( E_0V = \mathbb{C}1 \) and \( E_0^*V = \mathbb{C}\hat{x}. \)

(ii) the vectors $\hat{x}, 1$ form a basis for $e_0V$.

(iii) \( e_0V = E_0V + E_0^*V \) (direct sum).
Proof. (i) By matrix multiplication.
(ii) Observe that $\hat{x}, 1$ are linearly independent and contained in the primary $T$-module. Since the primary $T$-module has dimension 2, the result follows.
(iii) By (i) and (ii).

Lemma 11. For the graph $K_r$, with respect to the basis $\hat{x}, 1$ the matrices representing $A, A^*$ are

$$A : \begin{bmatrix} -1 & 0 \\ 1 & r-1 \end{bmatrix}, \quad A^* : \begin{bmatrix} r-1 & r \\ 0 & -1 \end{bmatrix}.$$

Proof. By matrix multiplication using $A = J - I$ and $A^* = (r-1)E_0^* - E_1^*$.

Lemma 12. For the graph $K_r$, the following hold:

(i) $\dim(e_1 V) = r - 2$.

(ii) $e_1 V = \text{span}\{\hat{y} - \hat{z} \mid y, z \in X \text{ and } x, y, z \text{ are mutually distinct}\}$.

Proof. (i) View $e_1 V = (e_0 V)^\perp$. By Lemma 10, $\dim(e_1 V) = r - 2$.
(ii) Define $S = \text{span}\{\hat{y} - \hat{z} \mid y, z \in X \text{ and } x, y, z \text{ are mutually distinct}\}$. We will show that $S = e_1 V$. First we show that $S \subseteq e_1 V$. Let $y, z \in X$ such that $x, y, z$ are mutually distinct. Then $\langle \hat{y} - \hat{z}, \hat{x} \rangle = 0$ and $\langle \hat{y} - \hat{z}, 1 \rangle = 0$. The vectors $\hat{x}$ and $1$ span $e_0 V$ by Lemma 10, so $\langle \hat{y} - \hat{z}, e_0 V \rangle = 0$. Now $\hat{y} - \hat{z} \in e_1 V$ by (2). So $S \subseteq e_1 V$. To show equality, we show that $S$ has dimension $r - 2$. To do this, we display $r - 2$ linearly independent vectors in $S$. Fix $y \in X$ with $y \neq x$. There exist $r - 2$ vertices $z \in X$ such that $x, y, z$ are mutually distinct. Observe that $\{\hat{y} - \hat{z} \mid z \in X \text{ and } y, z \text{ are mutually distinct}\}$ are linearly independent vectors in $S$. Therefore $\dim(S) = r - 2$. By these comments, $S = e_1 V$.

Lemma 13. For the graph $K_r$, the following hold:

(i) $e_0^2 = e_0$.

(ii) $e_1^2 = e_1$.

(iii) $e_0 e_1 = e_1 e_0 = 0$.

(vi) $e_0 + e_1 = I$.

Proof. By construction of $e_0$ and $e_1$. 
Lemma 14. For the graph $K_r$, the following hold:

(i) $e_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{r-1} & \frac{1}{r-1} & \cdots & \frac{1}{r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{r-1} & \frac{1}{r-1} & \cdots & \frac{1}{r-1} \end{bmatrix}$.

(ii) $e_0^t = e_0$.

(iii) $e_0^t = e_0$.

(iv) $e_0 = \frac{n}{n-1}(E_0 + E_0^* - E_0E_0^* - E_0^*E_0)$.

(v) $e_0, e_1 \in T$.

Proof. (i)–(iv) By construction.
(v) By (iv) and Lemma 8, $e_0 \in T$. Since $e_0 + e_1 = I$, $e_1 \in T$.

Lemma 15. For the graph $K_r$, the following hold:

(i) $e_0E_0^* = E_0^*e_0 = E_0^*$.

(ii) $e_0E_0 = E_0e_0 = E_0$.

Proof. (i) By Lemma 14(i) and (1).
(ii) Similar to (i).

Corollary 16. For the graph $K_r$, the elements $e_0, e_1$ form a basis for $Z(T)$.

Proof. Combining Lemma 8 and Lemma 15 we have $e_0 \in Z(T)$. Since $e_0 + e_1 = I$, $e_1 \in Z(T)$. By construction, $e_0, e_1$ are linearly independent. By Lemma 7 the dimension of $Z(T)$ is 2. The result follows.

Lemma 17. For the graph $K_r$, the subspace $e_1V$ is orthogonal to each of $E_0V$ and $E_0^*V$.

Proof. By (2) and Lemma 10(iii).

Lemma 18. For the graph $K_r$,

$$V = E_0^*V + E_0V + e_1V \quad (\text{direct sum}).$$
Proof. By (2) and Lemma 10(iii).

Lemma 19. For the graph $K_r$,

$$e_1V = E_1V \cap E_1^*V.$$  

Proof. We have the orthogonal direct sums $V = E_0V + E_1V$ and $V = E_0^*V + E_1^*V$. Using these and Lemmas 17, 18,

$$e_1V = (E_0V + E_0^*V)^\perp \cap (E_0^*V)^\perp = E_1V \cap E_1^*V.$$  

Lemma 20. For the graph $K_r$, each of $A, A^*$ acts on $e_1V$ as $-I$.

Proof. Since $A$ acts on $E_1V$ as $-I$ and $A^*$ acts on $E_1^*V$ as $-I$, each of $A, A^*$ acts on $E_1V \cap E_1^*V$ as $-I$. By Lemma 19, the result follows.

Lemma 21. For the graph $K_r$, the matrix $A^{-1}A^*-1AA^*$ is diagonalizable. Its eigenspaces are $E_0^*V, E_0V, e_1V$. The corresponding eigenvalues are

$$1 - r, \quad (1 - r)^{-1}, \quad 1.$$  

Proof. We first show that $\hat{x}$ (resp. $1$) is an eigenvector for $A^{-1}A^*-1AA^*$ with eigenvalue $1 - r$ (resp. $(1 - r)^{-1}$). Let $B$ (resp. $B^*$) denote the matrix representing $A$ (resp. $A^*$) with respect to the basis $\hat{x}, 1$ as in Lemma 11. By matrix multiplication,

$$B^{-1}B^*-1BB^* = \begin{bmatrix} 1 - r & 0 \\ 0 & \frac{1}{1-r} \end{bmatrix}. \quad \text{The results follow in view of Lemma 10(i) and Lemmas 18, 20}$$

4 The Hamming graph $H(D, r)$

In this section we give our main results, which are about the Hamming graph $H(D, r)$.

Definition 22. For an integer $D \geq 1$, the Hamming graph $H(D, r)$ has vertex set the Cartesian product of $D$ copies of $X$, with two vertices adjacent whenever they differ in precisely one coordinate.
By [2, p. 27] the Hamming graph $H(D, r)$ is distance-regular with diameter $D$. By [2, p. 261] $H(D, r)$ is $Q$-polynomial. Observe that for $D = 1$, $H(D, r)$ is the complete graph $K_r$.

In order to distinguish between $H(D, r)$ and $K_r$, we use the following notations. When we talk about $H(D, r)$, we write everything in bold. For example, for $H(D, r)$ the vertex set is denoted by $X$ and the standard module is denoted by $V$. On the other hand, when we discuss $K_r$, we retain the notation that we set up earlier. Observe that $X = X \times X \times \cdots \times X$ (Cartesian product of $D$ terms). Therefore we have a tensor product $V = V^{\otimes D}$. The algebras $M^*$ and $T$ are with respect to the vertex $(x, x, \ldots, x)$ where $x$ is from below Definition 4.

We now state our first main result.

**Theorem 23.** For the graph $H(D, r)$ the matrix $A^{-1}_D A^*_D A^{-1}_D A^*_D$ is diagonalizable, with eigenvalues

$$(1 - r)^s - D \leq s \leq D.$$ 

We will prove Theorem 23 after Theorem 32.

We now consider for each eigenvalue of $H(D, r)$, what is the corresponding eigenspace and its dimension? To do this, it is convenient to bring in the split decomposition of $V$ (see [10]). We now recall this decomposition. Using $V = V^{\otimes D}$ and (3),

$$V = (E_0^* V + E_0 V + e_1 V)^{\otimes D}. \quad (4)$$

Expanding (4) we obtain

$$V = \sum V_1 \otimes V_2 \otimes \cdots \otimes V_D \quad \text{(direct sum)}, \quad (5)$$

where the sum is over all sequences $V_1, V_2, \ldots, V_D$ of elements taken from $E_0^* V, E_0 V, e_1 V$.

For each summand in (5) define

$$\alpha = |\{j|1 \leq j \leq D, V_j = E_0^* V\}|, \quad (6)$$

$$\beta = |\{j|1 \leq j \leq D, V_j = E_0 V\}|, \quad (7)$$

$$\eta = |\{j|1 \leq j \leq D, V_j = e_1 V\}|. \quad (8)$$

We call $\eta$ the displacement.

By construction

$$\alpha + \beta + \eta = D. \quad (9)$$
**Definition 24.** For $0 \leq \eta \leq D$ let $\mathbb{V}_\eta$ denote the sum of the terms in (5) that have displacement $\eta$.

**Lemma 25.** For $0 \leq \eta \leq D$,

$$
\mathbb{V} = \sum_{\eta=0}^{D} \mathbb{V}_\eta \quad \text{(orthogonal direct sum).}
$$

(10)

**Proof.** By (5) along with Lemma 17 and the construction. \hfill \Box

**Definition 26.** For $-1 \leq i, j \leq D$ define

$$
\mathbb{V}_{ij} = (\mathbb{E}_0^* \mathbb{V} + \mathbb{E}_1^* \mathbb{V} + \cdots + \mathbb{E}_i^* \mathbb{V}) \cap (\mathbb{E}_0 \mathbb{V} + \mathbb{E}_1 \mathbb{V} + \cdots + \mathbb{E}_j \mathbb{V}).
$$

Observe that $\mathbb{V}_{ij} = 0$ if $i = -1$ or $j = -1$. Also for $0 \leq i, j \leq D$, $\mathbb{V}_{i-1,j} \subseteq \mathbb{V}_{ij}$ and $\mathbb{V}_{i,j-1} \subseteq \mathbb{V}_{ij}$. So $\mathbb{V}_{i-1,j} + \mathbb{V}_{i,j-1} \subseteq \mathbb{V}_{ij}$.

**Definition 27.** For $0 \leq i, j \leq D$ define

$$
\tilde{\mathbb{V}}_{ij} = \text{orthogonal complement of } \mathbb{V}_{i-1,j} + \mathbb{V}_{i,j-1} \text{ in } \mathbb{V}_{ij}.
$$

By [10, Corollary 5.8],

$$
\mathbb{V} = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{\mathbb{V}}_{ij} \quad \text{(direct sum).}
$$

(11)

By Lemma 17 and Definitions 26, 27, we have

$$
\tilde{\mathbb{V}}_{ij} = \mathbb{V}_{ij} \cap \mathbb{V}_\eta \quad \text{where } \eta = i + j - D.
$$

(12)

**Lemma 28.** For $0 \leq i, j \leq D$, $\tilde{\mathbb{V}}_{ij}$ is the sum of the terms in (5) such that

$$
\alpha = D - i, \quad \beta = D - j, \quad \eta = i + j - D.
$$

(13)

**Proof.** Let $\mathbb{V}'_{ij}$ denote the sum of the terms in (5) that satisfy (13). We show that $\mathbb{V}'_{ij} = \tilde{\mathbb{V}}_{ij}$. We first show that $\mathbb{V}'_{ij} \subseteq \tilde{\mathbb{V}}_{ij}$. Let $u$ denote a summand in (5) that satisfies (13). We show that $u \subseteq \tilde{\mathbb{V}}_{ij}$. Without loss of generality, $u = (E_0^*V)^{\otimes \alpha} \otimes (e_1V)^{\otimes \eta} \otimes (E_0V)^{\otimes \beta}$. Observe that

$$
u \subseteq (E_0^*V)^{\otimes \alpha} \otimes V^{\otimes i} \quad \text{since } \beta + \eta = i
$$

$$
u = (E_0^*V)^{\otimes \alpha} \otimes (E_0^*V + E_1^*V)^{\otimes i}$$

$$
u \subseteq E_0^*V + E_1^*V + \cdots + E_i^*V.$$
Similarly,
\[
    u \subseteq V^\otimes j \otimes (E_0V)^\otimes \beta
\]
\[
    = (E_0V + E_1V)^\otimes j \otimes (E_0V)^\otimes \beta
\]
\[
    \subseteq E_0V + E_1V + \cdots + E_jV.
\]

By the above comments and Definition 26, \( u \subseteq \mathbb{V}_{ij} \). By construction \( u \subseteq \mathbb{V}_\eta \). Thus \( u \subseteq \mathbb{V}_{ij} \cap \mathbb{V}_\eta = \tilde{\mathbb{V}}_{ij} \). We have shown

\[
    \mathbb{V}_{ij} \subseteq \tilde{\mathbb{V}}_{ij} \quad 0 \leq i, j \leq D. \tag{14}
\]

By (5) and the construction,

\[
    \mathbb{V} = \sum_{i=0}^{D} \sum_{j=0}^{D} \mathbb{V}_{ij}' \quad \text{(direct sum)}. \tag{15}
\]

Combining (11), (14) and (15), we obtain \( \mathbb{V}_{ij}' = \tilde{\mathbb{V}}_{ij} \) for \( 0 \leq i, j \leq D \). The result follows.

Lemma 29. For \( 0 \leq \eta \leq D \),

\[
    \mathbb{V}_\eta = \sum_{0 \leq i, j \leq D} \tilde{\mathbb{V}}_{ij}. \tag{16}
\]

Proof. By Lemma 28, the sum (16) is equal to the sum of the terms in (5) that have displacement \( \eta \). By Definition 24, that sum is equal to \( \mathbb{V}_\eta \).

Lemma 30. For the graph \( H(D, r) \), the following hold for \( 0 \leq i, j \leq D \).

(i) Assume that \( i + j < D \). Then \( \tilde{\mathbb{V}}_{ij} = 0 \).

(ii) Assume that \( i + j \geq D \). Then \( \dim(\tilde{\mathbb{V}}_{ij}) = \binom{D}{i}(D-i)(r-2)^{i+j-D} \).

Proof. We refer to the description of \( \tilde{\mathbb{V}}_{ij} \) from Lemma 28.

(i) Suppose that \( \tilde{\mathbb{V}}_{ij} \neq 0 \). Then there exists at least one term in (5) that satisfies (13). For this term, the displacement \( \eta \) is non-negative by (3), so by (13) we have \( i + j \geq D \), a contradiction. Therefore \( \tilde{\mathbb{V}}_{ij} = 0 \).

(ii) Consider a summand in (5) that contributes to \( \tilde{\mathbb{V}}_{ij} \). This summand is described below (5) and in Lemma 28. In this summand, the 1-dimensional subspace \( E_0^*V \) appears with multiplicity \( \alpha \), the 1-dimensional subspace \( E_0V \) appears with multiplicity
\[ \beta, \text{ and the } (r - 2)\text{-dimensional subspace } e_1 V \text{ appears with multiplicity } \eta. \] Therefore this summand has dimension \((r - 2)\eta\). Also, the number of summands with any given \(\alpha, \beta, \eta\) is \(\binom{D}{\alpha} \binom{D - \alpha}{\beta} \binom{D - \alpha - \beta}{\eta}\). Hence

\[
\dim(\tilde{V}_{ij}) = \binom{D}{\alpha} \binom{D - \alpha}{\beta} \binom{D - \alpha - \beta}{\eta} (r - 2)\eta
\]

Lemma 31. For the graph \(H(D, r)\) and \(0 \leq i, j \leq D\), the matrix \(A_D^{-1} A_D^{* - 1} A_D A_D^{*}\) acts on \(\tilde{V}_{ij}\) as \((1 - r)^{i-j} I\).

Proof. We view \(A_D = A^{\otimes D}\) and \(A_D^{*} = (A^{*})^{\otimes D}\). Thus \(A_D^{-1} A_D^{* - 1} A_D A_D^{*} = (A^{-1} A^{*-1} A A^{*})^{\otimes D}\).

We refer to the description to \(\tilde{V}_{ij}\) from Lemma 28. On each summand in (5) that satisfies (13), \(A_D^{-1} A_D^{* - 1} A_D A_D^{*}\) acts as \((1 - r)^{\alpha - \beta}\) in view of Lemma 21. By (13) we have \(\alpha - \beta = j - i\). The result follows.

Theorem 32. For the graph \(H(D, r)\) and for \(-D \leq s \leq D\), the subspace

\[
\sum_{0 \leq i, j \leq D \atop i+j \geq D \atop j-i=s} \tilde{V}_{ij}
\]

is an eigenspace for the matrix \(A_D^{-1} A_D^{* - 1} A_D A_D^{*}\). The dimension of this eigenspace is

\[
\sum_{0 \leq i, j \leq D \atop i+j \geq D \atop j-i=s} \binom{D}{\alpha} \binom{i}{D - j} (r - 2)^{i+j-D}.
\]

The corresponding eigenvalue is \((1 - r)^s\).

Proof. Applying Lemma 31 and using (11), the sum (17) is an eigenspace for \(A_D^{-1} A_D^{* - 1} A_D A_D^{*}\) with eigenvalue \((1 - r)^s\). Its dimension is obtained using Lemma 30.

Proof of Theorem 23. Combine (11), Lemma 30(i) and Theorem 32.

Recall the subconstituent algebra \(T\) for \(H(D, r)\). We now discuss the irreducible \(T\)-modules. By [9, p. 195], every irreducible \(T\)-module is thin.

Lemma 33. [9, p. 202] For the graph \(H(D, r)\), let \(W\) denote an irreducible \(T\)-module. Then the endpoint of \(W\) is equal to the dual endpoint of \(W\).
In the next lemma we refer to Definition 24.

**Lemma 34.** For $0 \leq \eta \leq D$, the subspace $\mathcal{V}_\eta$ is spanned by the irreducible $T$-modules with displacement $\eta$.

**Proof.** Let $\mathcal{V}_\eta'$ denote the span of the irreducible $T$-modules with displacement $\eta$. We show $\mathcal{V}_\eta = \mathcal{V}_\eta'$. By [10, Theorem 6.2(i)],

$$\mathcal{V}_\eta' = \sum_{0 \leq i,j \leq D \atop i+j=\eta+D} \tilde{\mathcal{V}}_{ij}. \quad (18)$$

By Lemma 29, $\mathcal{V}_\eta = \mathcal{V}_\eta'$.

Note that $A^{-1}_D A^{* -1}_D A_D A^{*}_D \in T$. Next, we describe the action of $A^{-1}_D A^{* -1}_D A_D A^{*}_D$ on any irreducible $T$-module. We introduce some notation.

**Definition 35.** For $-D \leq s \leq D$, let $F_s \in Mat_X(\mathbb{C})$ denote the projection onto the eigenspace for $A^{-1}_D A^{* -1}_D A_D A^{*}_D$ associated with the eigenvalue $(1 - r)^s$.

We refer to Definition 35. By linear algebra, $F_s$ is a polynomial in $A^{-1}_D A^{* -1}_D A_D A^{*}_D$ and is therefore contained in $T$.

Moreover,

$$F_s F_t = \delta_{st} F_s \quad (-D \leq s, t \leq D),$$

$$\sum_{s=-D}^{D} F_s = I,$$

$$A^{-1}_D A^{* -1}_D A_D A^{*}_D = \sum_{s=-D}^{D} (1 - r)^s F_s.$$

Write $\mathcal{V}$ as an orthogonal direct sum of irreducible $T$-modules : $\mathcal{V} = \sum_W W$.

For $-D \leq s \leq D$,

$$F_s \mathcal{V} = \sum_W F_s W.$$ 

In the above sum, each nonzero summand $F_s W$ is the eigenspace for the action of $A^{-1}_D A^{* -1}_D A_D A^{*}_D$ on $W$ associated with the eigenvalue $(1 - r)^s$.

Let $W$ denote an irreducible $T$-module. Then $W$ is the direct sum of the nonzero terms among $\{F_s W\}_{s=-D}^{D}$.

**Theorem 36.** For the graph $H(D, r)$, let $W$ denote an irreducible $T$-module with diameter $d$. Then for $-D \leq s \leq D$ the following are equivalent:
(i) \( F_sW \neq 0; \)

(ii) \(-d \leq s \leq d \) and \( d - s \) is even.

Now assume (i), (ii) hold. Then \( \dim(F_sW) = 1. \)

Proof. (i) \( \Rightarrow \) (ii) Let \( \eta \) be the displacement of \( W \). By Lemma 34, \( W \subseteq \mathbb{V}_{\eta} \). We have

\[
0 = F_sW \subseteq W \subseteq \mathbb{V}_{\eta}.
\]

By the definition of displacement,

\[
\eta = \rho + \tau + d - D, \tag{19}
\]

where \( \rho, \tau \) denote the endpoint and dual endpoint of \( W \), respectively. By Lemma 33,

\[
\rho = \tau. \tag{20}
\]

By Lemma 29 and Theorem 32, there exist \( 0 \leq i, j \leq D \) such that

\[
i + j = \eta + D, \tag{21}
\]

\[
 j - i = s, \tag{22}
\]

and \( W \) is not orthogonal to \( \tilde{\mathbb{V}}_{ij} \). From (19) minus (20) plus (21) minus (22), we get

\[
d - s = 2(i - \rho). \tag{23}
\]

Therefore \( d - s \) is even. From (19) plus (20) plus (21) plus (22), we get

\[
d + s = 2(j - \tau). \tag{24}
\]

By Definitions 26, 27 and the statement below (22), we have \( i \geq \rho \) and \( j \geq \tau \). By this and (23), (24) we obtain \(-d \leq s \leq d.\)

(ii) \( \Rightarrow \) (i) Define the set \( S = \{d, d-2, d-4, \ldots, -d\} \). Observe that \( |S| = d + 1 \). The \( T \)-module \( W \) is thin with diameter \( d \), so it has dimension \( d + 1 \). Note that \( W \) is the direct sum of the nonzero subspaces among \( \{F_sW\}_{s=-D}^{D} \). These nonzero subspaces have dimension 1 by [10, Lemma 3.5] and [11, Lemma 3.8]. By the earlier part of this proof, \( F_sW = 0 \) unless \( s \in S \). It follows that \( F_sW \neq 0 \) for all \( s \in S \).

Next assume that (i), (ii) hold. We mentioned in the proof of (ii) \( \Rightarrow \) (i) that \( \dim(F_sW) = 1. \)

**Corollary 37.** For the graph \( H(D, r) \), let \( W \) denote an irreducible \( T \)-module with diameter \( d \). Then the action of \( A_D^{-1}A_D^{s-1}A_D A_D^s \) on \( W \) is diagonalizable with eigenvalues

\[
(1 - r)^s -d \leq s \leq d, \quad d - s \text{ is even.}
\]

The corresponding eigenspaces all have dimension 1.

Proof. By Theorem 36 and the comment above it. \( \square \)
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