AN ANALYSIS OF ISING TYPE MODELS ON CAYLEY TREE BY A CONTOUR ARGUMENT

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Abstract

In the paper the Ising model with competing $J_1$ and $J_2$ interactions with spin values $\pm 1$, on a Cayley tree of order 2 (with 3 neighbors) is considered. We study the structure of the ground states and verify the Peierls condition for the model. Our second result gives description of Gibbs measures for ferromagnetic Ising model with $J_1 < 0$ and $J_2 = 0$, using a contour argument which we also develop in the paper. By the argument we also study Gibbs measures for a natural generalization of the Ising model. We discuss some open problems and state several conjectures.

Keywords: Cayley tree, configuration, Ising model, competing interactions, ground state, contour, Gibbs measure.

1 Introduction

The Ising model, with two values of spin $\pm 1$ was considered in [Pr],[Za] and became actively researched in the 1990’s and afterwards (see for example [BG], [BRZ], [BRSSZ]).

In the paper we consider an Ising model on a Cayley tree with competing interactions and some a natural generalization of the model. The goal of the paper is to study of ground states and Gibbs measures of the model. The method of our investigation is a contour method on the Cayley tree, which we will develop here.

Contour methods have been used in the mathematical physics community for many years. In the simplest application, one first rewrites the model under consideration in terms of contour representing the boundaries between regions where the spin variable in question is constant, and then uses a so-called Peierls argument to show that large contours are rare, thus proving that the leading configurations consist of large oceans of the one spin value, with only small islands of minority spins ([M],[S]). The techniques of this method is globally known as Pirogov-Sinai theory or contour arguments. This technique was pioneered by Peierls [P] in his study of the Ising model, later formalized more precisely by Griffiths and Dobrushin [GD]. The original argument benefited from the particular symmetries of the Ising model. The adaptation of the method to the treatment of non-symmetric models is not trivial, and was developed by Pirogov and Sinai [M], [S], [rozikovu@yandex.ru]
Later, a particularly enlightening alternative version of the argument was put forward by Zahradnik [Z].

Note, that Pirogov-Sinai theory on Cayley tree is not developed. The method used for the description of Gibbs measures on Cayley tree is the method of Markov random field theory and recurrent equations of this theory (See for example [BG], [GR],[GR1],[GR2], [MR], [NR], [R1],[R2], [RS]). But, if we consider non-symmetric models on Cayley tree, then the description of Gibbs measures by the method becomes a difficult problem: in this situation, a nonlinear operator $W$ that maps $\mathbb{R}^r$ (for some $r \geq 1$) into itself appears and the problem is then to describe the fixed points of this operator. Also implementing this method it is very difficult to prove extremity of a disordered Gibbs measure. This problem is not easy even for symmetric models on Cayley tree, which have been studied in [BRZ] for Ising model and in [GR1] for Potts model on Cayley tree.

Note, that extremal Gibbs measures are important, since they describe the possible macro states of physical system. The Gibbs measures of models on $\mathbb{Z}^d$ described using Pirogov-Sinai theory are automatically extremal. So it is crucial to develop Pirogov-Sinai theory on Cayley tree.

2 Definitions and preliminary results

The Cayley tree $\Gamma^k$ (See [Ba]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V,L,i)$, where $V$ is the set of vertexes of $\Gamma^k$, $L$ is the set of edges of $\Gamma^k$ and $i$ is the incidence function associating each edge $l \in L$ with its endpoints $x,y \in V$. If $i(l) = \{x,y\}$, then $x$ and $y$ are called nearest neighboring vertexes, and we write $l = \langle x,y \rangle$. The distance $d(x,y), x,y \in V$ on the Cayley tree is defined by the formula

$$d(x,y) = \min\{d|\exists x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \text{ such that the pairs } \langle x_0, x_1 \rangle, ..., \langle x_{d-1}, x_d \rangle \text{ are nearest neighboring vertexes}\}.$$ 

For the fixed $x^0 \in V$ we set

$$W_n = \{x \in V \mid d(x,x^0) = n\},$$

$$V_n = \cup_{m=1}^n W_m = \{x \in V \mid d(x,x^0) \leq n\},$$

$$L_n = \{l = \langle x,y \rangle \in L \mid x,y \in V_n\}.$$ 

Denote $|x| = d(x,x^0), x \in V$.

A collection of the pairs $\langle x, x_1 \rangle, ..., \langle x_{d-1}, y \rangle$ is called a path from $x$ to $y$ and we write $\pi(x,y)$ . We write $x < y$ if the path from $x^0$ to $y$ goes through $x$.

It is known that there exists a one-to-one correspondence between the set $V$ of vertexes of the Cayley tree of order $k \geq 1$ and the group $G_k$ of the free products of $k+1$ cyclic groups of the second order with generators $a_1, a_2, ..., a_{k+1}$. 

$\text{[PS1],[PS2],[Z1].}$
Let us define a group structure on the group $\Gamma_k$ as follows. Vertices which correspond to the "words" $g, h \in G_k$ are called nearest neighbors and are connected by an edge if either $g = ha_i$ or $h = ga_j$ for some $i$ or $j$. The graph thus defined is a Cayley tree of order $k$.

Consider a left (resp. right) transformation shift on $G_k$ defined as: for $g_0 \in G_k$ we put

$$T_{g_0}h = g_0h \quad \text{(resp. } T_{g_0}h = hg_0, \text{)} \quad \forall h \in G_k.$$ 

It is easy to see that the set of all left (resp. right) shifts on $G_k$ is isomorphic to the group $G_k$.

### 2.1 Configuration space and the model

We consider models where the spin takes values in the set $\Phi = \{-1, 1\}$. A configuration $\sigma$ on $V$ is then defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$. Assume on $\Omega$ the group of spatial shifts acts. We define a periodic configuration as a configuration $\sigma(x)$ which is invariant under a subgroup of shifts $G^*_k \subset G_k$ of finite index. For a given periodic configuration the index of the subgroup is called the period of the configuration. A configuration that is invariant with respect to all shifts is called translational-invariant.

The Hamiltonian of the Ising model with competing interactions has the form

$$H(\sigma) = J_1 \sum_{<x,y>} \sigma(x)\sigma(y) + J_2 \sum_{x,y \in V: d(x,y)=2} \sigma(x)\sigma(y)$$

where $J_1, J_2 \in R$ are coupling constants and $\sigma \in \Omega$.

### 2.2 Gibbs measure

We consider a standard $\sigma$-algebra $\mathcal{F}$ of subsets of $\Omega$ generated by cylinder subsets, all probability measures are considered on $(\Omega, \mathcal{F})$. A probability measure $\mu$ is called a Gibbs measure (with Hamiltonian $H$) if it satisfies the DLR equation: $\forall n = 1, 2, \ldots$ and $\sigma_n \in \Phi^{V_n}$:

$$\mu\left(\{\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\}\right) = \int_{\Omega} \mu(d\omega)\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n)$$

where $\nu^{V_n}_{\omega|_{W_{n+1}}}$ is the conditional probability

$$\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) = Z^{-1}(\omega|_{W_{n+1}}) \exp(-\beta H(\sigma_n|\omega|_{W_{n+1}})).$$

where $\beta > 0$. Here $\sigma_n|_{V_n}$ and $\omega|_{W_{n+1}}$ denote the restriction of $\sigma, \omega \in \Omega$ to $V_n$ and $W_{n+1}$ respectively. Next, $\sigma_n$ is a configuration in $V_n$ and $H(\sigma_n|\omega|_{W_{n+1}})$ is defined as the sum $H(\sigma_n) + U(\sigma_n, \omega|_{W_{n+1}})$ where

$$H(\sigma_n) = J_1 \sum_{<x,y>: x,y \in V_n} \sigma(x)\sigma(y) + J_2 \sum_{d(x,y)=2: x,y \in V_n} \sigma(x)\sigma(y)$$
Finally, $Z(\omega|_{W_{n+1}})$ stands for the partition function in $V_n$ with the boundary condition $\omega|_{W_{n+1}}$:

$$Z(\omega|_{W_{n+1}}) = \sum_{\tilde{\sigma} \in \tilde{\mathcal{V}}^n} \exp(-\beta H(\tilde{\sigma})|\omega|_{W_{n+1}}).$$

It is known (see [S]) that for any sequence $\omega^{(n)} \in \Omega$, any limiting point of the measures $\tilde{\nu}^{V_n}_{\omega^{(n)}}|_{W_{n+1}}$ is a Gibbs measure. Here $\tilde{\nu}^{V_n}_{\omega^{(n)}}|_{W_{n+1}}$ is a measure on $\Omega$ such that $\forall n' > n$:

$$\tilde{\nu}^{V_n}_{\omega^{(n)}}|_{W_{n+1}} (\{\sigma \in \Omega : \sigma|_{V_{n'}} = \sigma_{n'}\}) = \left\{ \begin{array}{ll} \nu^{V_n}_{\omega^{(n)}}|_{W_{n+1}} (\sigma_{n'}|_{V_n}), & \text{if } \sigma_{n'}|_{V_n \setminus V_n} = \omega^{(n)}|_{V_n \setminus V_n} \\ 0, & \text{otherwise.} \end{array} \right.$$  

## 3 Ground states

In the sequel for the simplicity we will consider Cayley tree of order two i.e $k=2$.

The ground states for models on the cubic lattice $Z^d$ it was studied in [GD], [K], [HS], [PS1], [PS2].

For a pair of configurations $\sigma$ and $\varphi$ that coincide almost everywhere, i.e. everywhere except for a finite number of positions, we consider a relative Hamiltonian $H(\sigma, \varphi)$, the difference between the energies of the configurations $\sigma, \varphi$ of the form

$$H(\sigma, \varphi) = J_1 \sum_{\langle x, y \rangle} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)) + J_2 \sum_{x, y \in V: d(x, y) = 2} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)),$$  

where $J = (J_1, J_2) \in \mathbb{R}^2$ is an arbitrary fixed parameter.

Let $M$ be the set of unit balls $V_1$ with vertexes in $V$. We call the restriction of the configuration $\sigma$ to the ball $b \in M$ the bounded configuration $\sigma_b$. We shall say that two boundary configurations $\sigma_b$ and $\sigma'_b$ belong to the same class if one of them can be obtained from the other by replacing all values of the configuration $\sigma'_b$ or $\sigma_b$ by the opposite values and up to any motion in $G_2$.

It is easy to show that the number of such classes is 4. Let $b = \{e, a_1, a_2, a_3\}$ be the ball with the center $e$, where $e$ is the identity of $G_2$ and $a_i, i = 1, 2, 3$ are the generators of the group. Then $\sigma^e_b = \{-1, -1, -1, -1\}$, $\sigma^2_b = \{-1, -1, -1, 1\}$, $\sigma^3_b = \{1, -1, 1, -1\}$, $\sigma^4_b = \{1, 1, -1, -1\}$, are representatives of each 4 classes.

By $C_i, i = 1, 2, 3, 4$ we denote the $i-th$ class of configurations.

Define the energy of ball $b$ for configuration $\sigma$ by

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\langle x, y \rangle, x, y \in b} \sigma(x)\sigma(y) + J_2 \sum_{x, y \in b: d(x, y) = 2} \sigma(x)\sigma(y),$$  

where $J = (J_1, J_2) \in \mathbb{R}^2$. 
The function $U(\sigma_b)$ on every class $\mathcal{C}_i$ takes one and the same value, i.e., if $\sigma_b \in \mathcal{C}_i$ and $\sigma_{b'} \in \mathcal{C}_i$ then $U(\sigma_b) = U(\sigma_{b'})$, and therefore we denote by $U_i$ the value of $U(\sigma_b)$ on class $i = 1, 2, 3, 4$. It is easy to see that

$$U_1 = \frac{3}{2} - J_1 + 3J_2, \quad U_2 = \frac{1}{2} - J_1 - J_2, \quad U_3 = -\frac{3}{2} - J_1 + 3J_2, \quad U_4 = -\frac{1}{2} - J_1 - J_2. \quad (5)$$

**Lemma 3.1.** The relative Hamiltonian (3) has the form

$$H(\sigma, \varphi) = \sum_{b \in M} (U(\sigma_b) - U(\varphi_b)). \quad (6)$$

**Proof.** Note that for any two points $x$ and $y$ such that $<x, y>$ there are exactly 2 unit balls for which $x$ and $y$ are vertexes. Also, for any two points $u$ and $v$ such that $d(u,v) = 2$ there exist a unique ball for which $u$ and $v$ are vertexes. This completes the proof.

**Lemma 3.2.** For any class $\mathcal{C}_i$ and for any bounded configuration $\sigma_b \in \mathcal{C}_i$ there exists a periodic configuration $\varphi$ with period non exceeding 2 such that $\varphi_b \in \mathcal{C}_i$ for any $b' \in M$ and $\varphi_b = \sigma_b$.

**Proof.** Consider 4 separate cases.

*Case $\mathcal{C}_1$.** In this case configuration $\varphi$ coincides with translational-invariant configuration $\varphi^+ = \{\varphi(x) \equiv +1\}$ or $\varphi^- = \{\varphi(x) \equiv -1\}$. Thus the period of $\varphi$ is 1.

*Case $\mathcal{C}_2$.** We continue the bounded configuration $\sigma_b \in \mathcal{C}_2$ to whole lattice $\Gamma^2$ by means of shifts through $g \in \mathcal{H}_1 = \{x \in G_2 : \text{number of the } a_1 \text{ in } x \text{ is even}\}$. Note (see [GR2]) that $\mathcal{H}_1$ is normal subgroup of $G_2$ of index 2. So we obtain a periodic configuration with period 2 (=index of the subgroup) which we denote by $\varphi$; then $\varphi_b = \sigma_b$. It is easy to see that all restrictions $\varphi_b, b \in M$ configuration $\varphi$ belong to $\mathcal{C}_2$. Consequently, $U(\varphi_b) = U(\varphi_{b'})$ for any $b, b' \in M$.

*Case $\mathcal{C}_3$.** In this case let us consider $\mathcal{H}_2 = \{x \in G_2 : |x| \text{ is even}\}$. $\mathcal{H}_2$ is also normal subgroup of $G_2$ of index 2. The proof is then similar to proof of the case $\mathcal{C}_2$.

*Case $\mathcal{C}_4$.** The proof is similar to proof of the case $\mathcal{C}_2$ here one can take $\mathcal{H}_{12} = \{x \in G_2 : \omega_1(x) + \omega_2(x) \text{ is even}\}$, where $\omega_i(x)$ is the number of $a_i$ in $x \in G_2$. Note (see [GR2]) that $\mathcal{H}_{12}$ is also normal subgroup of $G_2$ of index 2. The lemma is proved.

**Definition 3.3.** A periodic configuration $a$ is called a ground state for the relative Hamiltonian $H$ if

$$U(a_b) = \min \{U_1, U_2, U_3, U_4\}, \quad \text{for any } b \in M. \quad (7)$$

**Remark 1.** A ground state can be defined differently as a periodic configuration such that for any configuration $\sigma$ that coincides with $a$ almost everywhere $H(a, \sigma) \leq 0$. 

5
It is easy to see that from definition 3.3 follows second definition i.e. \( H(a, \sigma) \leq 0 \). In [PS1],[PS2] it was proved that these two definitions are equivalent for Hamiltonians on \( \mathbb{Z}^d \). But there is a problem to prove of the equivalentness for Hamiltonians on Cayley tree: normally the ratio of the number of boundary sites to the number of interior sites of a lattices becomes small in the thermodynamic limit of a large system. For the Cayley tree it does not, since both numbers grow exponentially like \( k^n \).

Correspondingly, we make a

**Conjecture 1.** The conditions (7) and \( H(a, \sigma) \leq 0 \) are equivalent.

4 A separation of the set of parameter \( J \in R^2 \).

We set

\[
U_i(J) = U(\sigma_b, J), \text{ if } \sigma_b \in C_i, i = 1, 2, 3, 4.
\]

The quantity \( U_i(J) \) is a linear function of the parameter \( J \in R^2 \). For every fixed \( m = 1, 2, 3, 4 \) we denote by \( A_m \) the set of points \( J \) such that

\[
U_m(J) = \min\{U_1(J), U_2(J), U_3(J), U_4(J)\}. \quad (8)
\]

It is easy to check that

\[
A_1 = \{J \in R^2 : J_1 \leq 0; \ J_1 + 4J_2 \leq 0\};
\]
\[
A_2 = \{J \in R^2 : J_1 \leq 0; \ J_1 + 4J_2 \geq 0\};
\]
\[
A_3 = \{J \in R^2 : J_1 \geq 0; \ J_1 - 4J_2 \leq 0\};
\]
\[
A_4 = \{J \in R^2 : J_1 \geq 0; \ J_1 - 4J_2 \geq 0\};
\]

and \( R^2 = \bigcup_{i=1}^4 A_i \).

5 Description of ground states

For every point \( J \in R^2 \) we divide the sets \( A_i, i = 1, 2, 3, 4 \) into two classes of sets \( A_{m_1}, ..., A_{m_r} \) and \( A_{m_{r+1}}, ..., A_{m_4} \) such that

\[
J \in \cap_{q=1}^r A_{m_q} \text{ and } J \notin \bigcup_{q=r+1}^4 A_{m_q}, \quad (9)
\]

where \( 1 \leq r \leq 4; m_q \in \{1, 2, 3, 4\} \) and \( m_q \neq m_p \) if \( p \neq q; p, q \in \{1, 2, 3, 4\} \).

**Lemma 5.1.** A periodic configuration \( a \) is a ground state for the relative Hamiltonian \( H \) if and only if for \( b \in M \) it is true that \( a_b \in C(m_1,...,m_r) \), where \( C(m_1,...,m_r) = \bigcup_{q=1}^r C_{m_q} \).
Proof. The condition (9) corresponds exactly to
\[ U_{m_1} = \ldots = U_{m_r} < \min\{U_{m_q}, q = r + 1, \ldots, 4\}. \]
What we require follows from definition 3.3.

Suppose two unit balls \( b \) and \( b' \) are neighbors, i.e., have a common edge. We shall then say that the two bounded configurations \( \sigma_b \) and \( \sigma_{b'} \) are compatible if they coincide on the common edge of the balls \( b \) and \( b' \).

Lemma 5.2. Suppose condition (9) is satisfied and that for any \( \sigma_b \in C(m_1, \ldots, m_r) \) and for any ball \( b' \) that neighbors \( b \) there exists exactly one bounded configuration \( \sigma_{b'} \) that belongs to \( C(m_1, \ldots, m_r) \) and is compatible with \( \sigma_b \); then the period of the ground state \( a \) with \( a_b \in C(m_1, \ldots, m_r) \) for any \( b \in M \) does not exceed 2.

Proof. We fix some \( b_0 \in M \). By lemma 3.2 there exists a periodic configuration \( \sigma \) with period not exceed 2 such that \( \sigma_{b_0} = a_{b_0} \) and \( \sigma_b \in C(m_1, \ldots, m_r) \) for any \( b \in M \). We show that \( a = \sigma \). If this is not so, there exist two neighboring balls \( b \) and \( b' \) such that \( a_b = \sigma_b \) and \( a_{b'} \neq \sigma_{b'} \), i.e. for the bounded configuration \( a_b \in C(m_1, \ldots, m_r) \) and for the ball \( b' \) that is the neighbor of \( b \) there exist two different bounded configurations \( \sigma_{b'} \) and \( a_{b'} \) belonging to \( C(m_1, \ldots, m_r) \). We have obtained an assertion that contradicts the condition of lemma 5.2. Therefore, the period of \( a \) does not exceed 2.

Lemma 5.3. Suppose condition (9) is satisfied and that for some \( \sigma_b \in C(m_1, \ldots, m_r) \) and for some ball \( b' \) that is a neighbor of \( b \) there exists two bounded configurations \( \sigma_{b'} \) and \( \sigma_{b'}^2 \) that belongs to \( C(m_1, \ldots, m_r) \) and compatible with \( \sigma_b \); then the relative Hamiltonian \( H \) has infinitely many ground states.

Proof. In accordance with lemma 3.2, there exist periodic configurations \( \sigma^1 \) and \( \sigma^2 \) with period not exceeding 2 whose restriction to ball \( b' \) are bounded configurations \( \sigma_{b'}^1 \) and \( \sigma_{b'}^2 \) respectively. We construct the following configuration \( a^t \) :

\[
a^t(x) = \begin{cases} 
\sigma^1(x) & \text{if } |x| \in [2tp - 2t; 2tp) \\
\sigma^2(x) & \text{if } |x| \in [2tp; 2tp + 2t)
\end{cases}
\]

where \( t, p \in \{1, 2, \ldots\} \). For any \( b \in M \) it is true that \( a_b^t \in C(m_1, \ldots, m_r) \), and therefore \( a^t \) is a ground state. Since \( t \in \{1, 2, \ldots\} \) is arbitrary, the number of ground states is infinite. The lemma is proved.

Summaries, we have

Theorem 5.4. For any \( J \in R^2 \setminus \{J \in R^2 : J_1 = 0\} \cup \{J \in R^2 : J_1 = \pm 4J_2, J_2 \geq 0\} \) the period of a ground state for the relative Hamiltonian \( H \) does not exceed 2. On \( \{J \in R^2 : J_1 = 0\} \cup \{J \in R^2 : J_1 = \pm 4J_2, J_2 \geq 0\} \) there are infinitely many ground states.
6 The Peierls condition

By theorem 5.4, it is obvious that there exist not more than \(2^4 = 16\) ground states with period not exceeding 2. For every point \(x \in V\) we denote by \(V_2(x)\) the ball

\[V_2(x) = \{y \in V : d(x, y) \leq 2\}\]

We denote the restriction of configuration \(\sigma\) to \(V_2(x)\) by pr\((\sigma, V_2(x))\).

**Definition 6.1.** Let \(\sigma^1, ..., \sigma^q\) be the complete set of all ground states of the relative Hamiltonian \(H\) and suppose their period does not exceed 2. The ball \(V_2(x)\) is said to be an improper ball of the configuration \(\sigma\) if pr\((\sigma, V_2(x)) \neq pr(\sigma^j, V_2(x))\) for any \(j = 1, \ldots, q\). The union of the improper balls of the configuration \(\sigma\) is called the boundary of the configuration and denoted by \(\partial(\sigma)\).

**Definition 6.2.** The relative Hamiltonian \(H\) with ground states \(\sigma^1, ..., \sigma^q\) satisfies the Peierls condition if for any \(j = 1, ..., q\) and any configuration \(\sigma\) coinciding almost everywhere with \(\sigma^j\),

\[H(\sigma, \sigma^j) \geq \lambda|\partial(\sigma)|,\]

where \(\lambda\) is a positive constant that does not depend on \(\sigma\), and \(|\partial(\sigma)|\) is the number of unit balls in \(\partial(\sigma)\).

**Theorem 6.3.** Suppose the sets \(A_m\) are split in such a way that condition (9) holds. If any bounded configuration \(\sigma_b \in \mathcal{C}(m_1, ..., m_r)\) and for any \(b'\) that is a neighbor of \(b\) there exist exactly one bounded configuration \(\sigma^j_{b'} \in \mathcal{C}(m_1, ..., m_r)\) compatible with \(\sigma_b\) then the Peierls condition is satisfied.

**Proof.** It follows from lemma 5.2 that the period of a ground state not exceed 2, so that there are not more than \(2^4\) ground states. We prove the fulfillment of the Peierls condition. Suppose \(\sigma\) coincides almost everywhere with the ground state \(\sigma^j\) and \(b \in \partial(\sigma)\). Among the four vertexes of the ball \(b\) there is a vertex \(x \in b\) such that \(V_2(x) \subset \partial(\sigma)\). Indeed, one can take the center of \(b\) for \(x\). In the ball \(V_2(x)\) there exists a unit ball \(b'\) such that \(U(\sigma_{b'}) - U(\sigma^j_{b'}) \geq \epsilon\), where \(\epsilon = \min\{U_{m_p}, p = r + 1, ..., 4\} - U_{m_1}\). Since \(\sigma^j\) is a ground state, \(U(\sigma_b) \geq U(\sigma^j_b)\) for any \(b \in M\). Thus for any \(b \in \partial(\sigma)\),

\[\sum_{b' \in \hat{V}(b)} (U(\sigma_{b'}) - U(\sigma^j_{b'})) \geq \epsilon,\]

where \(\hat{V}(b) = \cup_{x \in b} V_2(x)\). Since \(|\hat{V}(b)| = 21\),

\[H(\sigma, \sigma^j) = \sum_{b \in M} (U(\sigma_b) - U(\sigma^j_b)) = \sum_{b \in \partial(\sigma)} (U(\sigma_b) - U(\sigma^j_b)) \geq \frac{\epsilon}{21} |\partial(\sigma)|.\]

Therefore, the Peierls condition is satisfied for \(\lambda = \frac{1}{21}\). The theorem is proved.
Conjecture 2. The models is considered here satisfies the Peierls condition iff the number of ground states is finite (cf. [HS], [Pe]).

Remark 2. If one want to prove the conjecture 2 by well known arguments (see for example [HS], [K]) then appears the problem mentioned in Remark 1. We hope there is an other argument to prove it.

7 The existence of two Gibbs measures

In this section we consider the model (1) with $J_2 = 0$ and shall prove that there are at least two Gibbs measure for the model. Note that the result it was proved [Pr] using theory of Markov random fields and recurrent equations of this theory. Here we shall use our “contour method” on Cayley tree. The existence of several Gibbs measures (for some values of parameters $J_1, J_2$) is a mathematical expression of the well-known physical phenomenon—the coexistence of several aggregate states (or phases) of the matter.

We recall that the Ising model ((1) with $J_2 = 0$) is a lattice spin system described by a configuration of “spins” inside $\Lambda \subset V$, $\sigma = \{\sigma(x) \in \{-1, 1\}, \ x \in \Lambda\}$, in which only neighboring spins interact. The energy $H_\Lambda(\sigma|\varphi)$ of the configuration $\sigma$ in the presence of boundary configuration $\varphi = \{\varphi(x), x \in V \setminus \Lambda\}$ is expressed by the formula

$$H_\Lambda(\sigma|\varphi) = J_1 \sum_{<x,y>: \ x,y \in \Lambda} \sigma(x)\sigma(y) + J_1 \sum_{x \in \Lambda, y \in V \setminus \Lambda} \sigma(x)\varphi(y).$$  (10)

We consider the case $J_1 < 0$—the so-called ferromagnetic Ising model. For simplicity (without lose of the generality) we put $J_1 = -1$. By theorem 5.4 and theorem 6.3 the Hamiltonian (10) satisfies the Peierls condition with two ground states $\varphi_+ \equiv 1$ and $\varphi_- \equiv -1$. The particular interest for the Ising model is due primarily to the fact that for the Ising model with $k = 2$ the thermodynamic functions can be calculated explicitly.

The Gibbs measure on the space $\Omega_\Lambda = \{-1, 1\}^\Lambda$ with boundary condition $\varphi$ is defined in the usual way.

$$\mu_{\Lambda,\beta}(\sigma/\varphi) \equiv \mu_{\Lambda,\beta}^{\varphi}(\sigma) = Z^{-1}(\Lambda, \beta, \varphi) \exp(-\beta H_\Lambda(\sigma|\varphi)),$$

where $Z(\Lambda, \beta, \varphi)$ is the normalizing factor (statistical sum). The main goal of the section is to prove the following

Theorem 7.1. For all sufficiently large $\beta$ there are at least two Gibbs measure for the two-dimensional (i.e. $k = 2$) ferromagnetic Ising model on Cayley tree.

Proof. Let us consider a sequence of balls on $\Gamma^2$

$$V_1 \subset V_2 \subset ... \subset V_n \subset ..., \ \cup V_n = V,$$

and two sequences of boundary conditions outside these balls:

$$\varphi_{n,+} \equiv 1, n = 1, 2, ..., \ \varphi_{n,-} \equiv -1, n = 1, 2, ...$$
Each of two sequences of measures \( \{ \mu_{V_n, \beta}^+, n = 1, 2, \ldots \} \) and \( \{ \mu_{V_n, \beta}^-, n = 1, 2, \ldots \} \) contains a convergent subsequence (See [S, Theorem 1.2]). We denote the corresponding limits by \( \mu_+^\beta, \mu_-^\beta \) for the first and second sequence respectively. Our purpose is to show for a sufficiently large \( \beta \) these measures are different. Now we describe a boundary of configuration which is more simple than it was defined at the section 6.

Consider \( V_n \), let \( V_n' \subseteq V_n \), \( V_n' = \{ t \in V_n : \sigma(t) = -1 \} \). For any \( A \subseteq V \) denote

\[
\partial A = \{ x \in V \setminus A : \exists y \in A, < x, y > \}. 
\]

Consider the graph \( G^n = (V_n', L'_n) \), where

\[
L'_n = \{ l = < x, y > \in L : x, y \in V_n' \}. 
\]

It is clear, that for a fixed \( n \) the graph \( G^n \) contains a finite number of connected subgraphs \( G_j^n \) i.e.

\[
G^n = \{ G_1^n, \ldots, G_m^n \}, \quad G_i^n = (V'_{n,i}, L_{n,i}^{(i)}).
\]

**Definition 7.2.** The boundary \( \partial V'_n \) is called the boundary of the configuration \( \sigma(V_n) = \{ \sigma(x), x \in V_n \} \) and denoted by \( \partial(\sigma(V_n)) \). The set \( \partial V'_{n,i}, i = 1, \ldots, m \) is called the contour of the boundary \( \partial(\sigma(V_n)) \). The set \( V'_{n,i} \) is called the interior of the contour \( \partial V'_{n,i} \) and the set \( V'_n \) is called the interior of the boundary \( \partial(\sigma(V_n)) \).

Note that any collection of contours uniquely determines configuration \( \sigma \) inside \( V_n \) (for a fixed constant configuration \( \varphi \) outside \( V_n \)). Indeed, going from a point \( x \) of boundary of \( \Lambda \) to \( x^0 \in V \) (where \( x^0 \) is a point, which corresponds to \( e \in G_2 \)) through the unique path \( \pi(x, x^0) \) we put +1 until of the first point of \( \partial(\sigma(\Lambda)) \) on \( \pi(x, x^0) \), crossing the point we put −1 until the second point of \( \partial(\sigma(\Lambda)) \) on \( \pi(x, x^0) \) and crossing this point we put +1 and so on.

**Lemma 7.3.** Let \( \gamma \) be a fixed contour and \( p_+(\gamma) = \mu_+^\beta \{ \sigma : \gamma \subset \partial(\sigma(V_n)) \} \). Then

\[
p_+(\gamma) \leq \exp\{-2|\beta||\gamma|\},
\]

where \( |\gamma| \) stands for a number of elements of the set \( \gamma \).

**Proof.** Denote

\[
|\partial(\sigma(V_n))| = \sum_{\gamma : \gamma \subset \partial(\sigma(V_n))} |\gamma|.
\]

For any \( \sigma \), which coincides with \( \varphi_+ \equiv 1 \) outside of \( V_n \) we have

\[
H_{V_n}(\sigma) = H(\sigma(V_n)) + H(\sigma(V_n)|\varphi_+(V \setminus V_n)) = 1 - |V_{n+1}| + 2|\partial(\sigma(V_n))|.
\]

Indeed,

\[
H_{V_n}(\sigma) = - \sum_{<x,y>, \{x,y\} \cap V_n \neq \emptyset} \sigma(x)\sigma(y) = -\Sigma^+ - \Sigma^-.
\]
where $\Sigma^+(\Sigma^-)$ is the part of sum taken for such $\{x, y\}$, that $\sigma(x) = \sigma(y),$ $(\sigma(x) = -\sigma(y))$

It is easy to see that $-\Sigma^- = |\partial(\sigma(V_n))|$. The total number of $\{x, y\}$ such that $\{x, y\} \cap V_n \neq \emptyset$ is equal to $|V_n| + |\partial V_n| = 1 = |V_{n+1}| - 1$.

Consequently,

$$\Sigma^+ = |V_{n+1}| - 1 - |\Sigma^-| = |V_{n+1}| - 1 + \Sigma^-.$$ 

Thus,

$$H_{V_n}(\sigma) = 1 - |V_{n+1}| - 2\Sigma^- = 1 - |V_{n+1}| + 2|\partial(\sigma(V_n))|.$$

By definition we have (where $\Lambda = V_n$)

$$p_+(\gamma) = \frac{\sum_{\sigma(\Lambda) \subset \partial(\sigma(\Lambda))} \exp\{-\beta H_\Lambda(\sigma)\}}{\sum_{\sigma(\Lambda)} \exp\{-\beta H_\Lambda(\sigma)\}} =$$

$$= \frac{\sum_{\sigma(\Lambda) \subset \partial(\sigma(\Lambda))} \exp\{\beta|\Lambda| + \beta|\partial\Lambda| - 2\beta|\partial(\sigma(\Lambda))| - \beta\}}{\sum_{\sigma(\Lambda)} \exp\{\beta|\Lambda| + \beta|\partial\Lambda| - 2\beta|\partial(\sigma(\Lambda))| - \beta\}} =$$

$$= \frac{\sum_{\sigma(\Lambda) \subset \partial(\sigma(\Lambda))} \exp\{-2\beta|\partial(\sigma(\Lambda))|\}}{\sum_{\sigma(\Lambda)} \exp\{-2\beta|\partial(\sigma(\Lambda))|\}}.$$

Denote

$$F_\gamma = \{\sigma(\Lambda) : \gamma \subset \partial(\sigma(\Lambda))\},$$

$$F^-_\gamma = \{\sigma(\Lambda) : \gamma \cap \partial(\sigma(\Lambda)) = \emptyset\}.$$

Define the map $\chi_\gamma; F_\gamma \to F^-_\gamma$ as following: for $\sigma(\Lambda) \in F_\gamma$ we destroy the contour $\gamma$ changing the values $\sigma(x)$ inside of $\gamma$ to $+1$. The constructed configuration is $\chi_\gamma(\sigma(\Lambda)) \in F^-_\gamma$.

It is clear that

$$\partial(\sigma(\Lambda)) = \partial(\chi_\gamma(\sigma(\Lambda))) \cup \gamma,$$

$$|\partial(\sigma(\Lambda))| = |\partial(\chi_\gamma(\sigma(\Lambda)))| + |\gamma|.$$ 

For a given $\gamma$ the map $\chi_\gamma$ is one-to-one map.

Further, we can write

$$p_+(\gamma) = \frac{\sum_{\sigma(\Lambda) \in F_\gamma} \exp\{-2\beta|\partial(\sigma(\Lambda))|\}}{\sum_{\sigma(\Lambda)} \exp\{-2\beta|\partial(\sigma(\Lambda))|\}} \leq$$

$$= \frac{\sum_{\sigma(\Lambda) \in F_\gamma} \exp\{-2\beta|\partial(\sigma(\Lambda))|\}}{\sum_{\sigma(\Lambda) \in F^-_\gamma} \exp\{-2\beta|\partial(\sigma(\Lambda))|\}} =$$

$$= \frac{\sum_{\sigma(\Lambda) \in F_\gamma} \exp\{-2\beta|\partial(\sigma(\Lambda))|\}}{\sum_{\sigma(\Lambda) \in F^-_\gamma} \exp\{-2\beta|\partial(\chi_\gamma(\sigma(\Lambda)))|\}} = \exp\{-2\beta|\gamma|\}.$$

The lemma is proved.

**Lemma 7.4.** For all sufficiently large $\beta$, there is a constant $C = C(\beta) > 0$, such that

$$\mu^{+}_{\beta}\{\sigma(\Lambda) : |\gamma| > C \ln |\Lambda| \text{ for some } \gamma \subset \partial(\sigma(\Lambda))\} \to 0.$$
as \(|\Lambda| \to \infty\).

**Proof.** Denote by \(N_\gamma(r) = |\{\gamma : \ t \in \gamma, |\gamma| = r\}|\) - the number of different contours with \(t \in \gamma\) and \(|\gamma| = r\). Note that \(N_\gamma(r) \leq 12^{2r-1}\).

Suppose \(\beta > \ln 12\), then

\[
\mu_\beta^+ \{\sigma(\Lambda) : \gamma \subset \partial(\sigma(\Lambda)), \ t \in \gamma, |\gamma| = r\} \leq 12^{2r-1} \cdot e^{-2\beta r}.
\]

\[
\mu_\beta^+ \{\sigma(\Lambda) : \gamma \subset \partial(\sigma(\Lambda)), \ t \in \gamma, |\gamma| > C_1 \ln |\Lambda|\} \leq \frac{1}{12} \sum_{r \geq C_1 \ln |\Lambda|} 12^{2r} \cdot e^{-2\beta r} \leq \frac{1}{12} \sum_{r \geq C_1 \ln |\Lambda|} (144 \cdot e^{-2\beta})^r = \frac{(144 \cdot e^{-2\beta})^{C_1 \ln |\Lambda|}}{12(1 - 144e^{-2\beta})} = \frac{|\Lambda|^{C_1(\ln 144 - 2\beta)}}{12(1 - 144e^{-2\beta})},
\]

where \(C_1\) will be defined later.

Thus, we have

\[
\mu_\beta^+ \{\sigma(\Lambda) : \exists \gamma \subset \partial(\sigma(\Lambda)), \ |\gamma| > C_1 \ln |\Lambda|\} \leq \frac{|\Lambda|^{C_1(\ln 144 - 2\beta) + 1}}{12(1 - 144e^{-2\beta})}.
\]

The last expression tends to zero if \(|\Lambda| \to \infty\) and \(C_1 > \frac{1}{2\beta - \ln 144}\). The lemma is proved.

**Lemma 7.5.** If \(e \in \Lambda\). Then uniformly by \(\Lambda\)

\[
\mu_\beta^+ \{\sigma(\Lambda) : \sigma(e) = -1\} \to 0,
\]

as \(\beta \to \infty\).

**Proof.** If \(\sigma(e) = -1\), then \(e\) is point for interior of some contour, we shall write this as \(e \in \text{Int}\gamma\). Assume \(t \in \gamma\) and \(e \in \text{Int}\gamma\) then for any such contour we have \(|\gamma| \geq |t| + 2\).

Consequently,

\[
\mu_\beta^+ \{\sigma(\Lambda) : e \in \text{Int}\gamma, \ t \in \gamma, |\gamma| < C_1 \ln |\Lambda|\} \leq \frac{1}{12} \sum_{r = |t| + 2}^{C_1 \ln |\Lambda|} (144 \cdot e^{-2\beta})^r \leq \frac{(144e^{-2\beta})^{|t|+2}}{12(1 - 144e^{-2\beta})}.
\]

\[
\mu_\beta^+ \{\sigma(e) = -1\} \leq \mu_\beta^+ \{\sigma(\Lambda) : e \in \text{Int}\gamma, \ \gamma \subset \partial(\sigma(\Lambda))\} \leq \frac{1}{12} \sum_{|t|=1}^{C_1 \ln |\Lambda|} \frac{(144 \cdot e^{-2\beta})^{|t|+2}}{1 - 144e^{-2\beta}} + \mu_\beta^+ \{\sigma(\Lambda) : \exists \gamma \subset \partial(\sigma(\Lambda))\}, \text{such that, } |\gamma| \geq C_1 \ln |\Lambda|\} \leq \frac{12^5 e^{-6\beta}}{(1 - 144e^{-2\beta})^2} + \frac{|\Lambda|^{C_1(\ln 144 - 2\beta) + 1}}{12(1 - 144e^{-2\beta})}.
\]

(11)

For \(|\Lambda| \to \infty\) and \(\beta \to \infty\) from (11) we get \(\mu_\beta^+ \{\sigma(e) = -1\} \to 0\). The lemma is proved.
Let us continue the proof of theorem 7.1. By lemma 7.5 we have
\[ \mu^+_\beta \{ \sigma(e) = -1 \} < \frac{1}{2}. \] (12)
Using the similar argument one can prove
\[ \mu^-_\beta \{ \sigma(e) = +1 \} < \frac{1}{2}. \] (13)
By (13) we have
\[ \mu^-_\beta \{ \sigma(e) = -1 \} > \frac{1}{2}. \] (14)
Thus, from (12) and (14) we have \( \mu^+_\beta \neq \mu^-_\beta \). The theorem 7.1 is proved.

**Definition 7.6.** A Gibbs measure \( \mu_\beta \), for Hamiltonian \( \beta H \) is called **small deviation** of a fixed configuration \( \varphi \in \Omega \) if for any \( n > 0 \)
\[ \lim_{\beta \to \infty} \sup_{x \in V} \mu_\beta(\sigma : \sigma(V_n(x)) \neq \varphi(V_n(x))) = 0. \] (15)

**Theorem 7.7.** The Gibbs measures \( \mu^+_\beta \), \( \mu^-_\beta \) of ferromagnetic Ising model on Cayley tree of order 2 are small deviations of the ground states \( \varphi_+ \equiv 1 \) and \( \varphi_- \equiv -1 \) respectively.

**Proof.**
\[ \mu^+_\beta(\sigma : \sigma(V_n(x)) \neq \varphi(V_n(x))) \leq \sum_{A \subseteq V_n(x)} \mu^+_\beta \{ \sigma(A) = -1 \} \leq \sum_{A \subseteq V_n(x)} |A| \mu^+_\beta \{ \sigma(e) = -1 \} = C(n) \mu^+_\beta \{ \sigma(e) = -1 \}, \] (16)
where \( C(n) = \sum_{A \subseteq V_n(x)} |A| \). It is easy to see that \( C(n) \) depends only on \( n \). Using lemma 7.5 from (16) we get (15). The theorem is proved.

From the theorem 7.7. it follows an additional information about the structure of the “typical” configurations for each of the constructed measures. Namely, for the \( \mu^+_\beta \) (\( \mu^-_\beta \)) almost every configuration \( \sigma \) is such that on a connected set whose density on the Cayley tree tends to unity as \( \beta \to \infty \) the configuration \( \sigma \) coincides with \( \varphi_+ \) (\( \varphi_- \)), and all the connected components of the set \( \{ x : \sigma(x) \neq \varphi_+(x) \} \) (\( \{ x : \sigma(x) \neq \varphi_-(x) \} \)) are finite.

### 8 Some generalizations

In this section we consider a generalization of previous results describing the Ising model. We consider a “spin” model with two values of “spin” : \( v_1 \) and \( v_2 \). The state of such a system on the Cayley tree \( \Gamma^2 \) is determined by the configuration \( \omega = \{ \omega(x), x \in V \} \), \( \omega(x) = v_1 \) or \( v_2 \). The energy \( H \) of the configuration \( \omega \) inside a finite set \( \Lambda \subset V \) is
\[ H_\Lambda(\omega_\Lambda|\omega_{V \setminus \Lambda}) = \sum_{x,y \in \Lambda} \lambda(\omega(x),\omega(y)) + \sum_{x \in \Lambda, y \in V \setminus \Lambda} \lambda(\omega(x),\omega(y)), \] (17)
here $\omega_\Lambda = \{\omega(x), \ x \in \Lambda\}$ and $\omega_{V\setminus \Lambda} = \{\omega(x), \ x \in V\setminus \Lambda\}$ are parts of the configuration $\omega$ inside the set and outside it. The interaction $\lambda(v_i, v_j) = \lambda_{ij}, \ i, j = 1, 2$ is given by a matrix $M = (\lambda_{ij})_{i,j=1,2}$ of second order. If we set $v_1 = -1, v_2 = 1$ and $\lambda(v_1, v_2) = -v_1 v_2$ then we get the ferromagnetic Ising model.

Now we consider two outer constant configurations

$$\bar{\omega}_V^{(i)} = v_i, \ i = 1, 2.$$ 

Denote by $H^{(i)}_\Lambda(\omega_\Lambda)$ the energy $H_\Lambda(\omega_\Lambda|\bar{\omega}_V^{(i)})$ corresponding to the configuration $\bar{\omega}_V^{(i)}$ and by $P^{(i)}_{\Lambda, \beta}, \ i = 1, 2$ the corresponding Gibbs measure.

Denote by $\omega^{(i)}_\Lambda$ the configuration $\omega_\Lambda$ extended by $\bar{\omega}_V^{(i)}$ and by $\partial(\omega^{(i)}_\Lambda)$ its boundary as was explained in section 7, so that $\partial(\omega^{(i)}_\Lambda)$ consists of points $x$ of the tree such that $\omega(x) = v_i$ and there is at least one $y \in S_1(x) = \{u \in V : d(x, u) = 1\}$, such that $\omega(x) \neq \omega(y)$.

**Lemma 8.1.** Let $K$ be a connected subgraph of $\Gamma^2$, such that $|K| = n$, then $|\partial K| = n + 2$.

**Proof.** We shall use the induction over $n$. For $n = 1$ and 2 the assertion is trivial. Assume for $n = m$ the lemma is true i.e from $|K| = m$ follows $|\partial K| = m + 2$. We shall prove the assertion for $n = m + 1$ i.e. for $\bar{K} = K \cup \{x\}$. Since $\bar{K}$ is connected graph we have $x \in \partial K$ and there is unique $y \in S_1(x)$ such that $y \in K$. Thus $\partial \bar{K} = (\partial K \setminus \{x\}) \cup (S_1(x) \setminus \{y\})$. Consequently,

$$|\partial \bar{K}| = |\partial K| - 1 + 2 = m + 3.$$ 

The lemma is proved.

**Lemma 8.2.** The energy $H^{(i)}_\Lambda(\omega_\Lambda), \ i = 1, 2$ has the form

$$H^{(1)}_\Lambda(\omega_\Lambda) = (\lambda_{21} + \lambda_{22} - 2\lambda_{11})|\partial(\omega_\Lambda)| + 3m(\lambda_{11} - \lambda_{22}) + \lambda_{11}(|V_{n+1}| - 1);$$

$$H^{(2)}_\Lambda(\omega_\Lambda) = (\lambda_{12} + \lambda_{11} - 2\lambda_{22})|\partial(\omega_\Lambda)| + 3m(\lambda_{22} - \lambda_{11}) + \lambda_{22}(|V_{n+1}| - 1);$$

where $m \equiv m(\omega_\Lambda) = |\{\gamma : \gamma \subset \partial(\omega_\Lambda)\}|$ - the number of different contours of $\partial(\omega_\Lambda)$, i.e. $\partial(\omega_\Lambda) = \{\gamma_1, ..., \gamma_m\}$.

**Proof.** Using lemma 8.1 and well known fact that if $K$ is a connected graph then number of edges of $K$ equal $|K| - 1$ we have

$$H^{(1)}_\Lambda(\omega_\Lambda) = \sum_{i=1}^{m} \left( \sum_{<x, y>: x, y \in \text{Int} \gamma_i} \lambda(\omega(x), \omega(y)) + \sum_{<x, y>: x \in \text{Int} \gamma_i, y \in \gamma_i} \lambda(\omega(x), \omega(y)) \right) + \sum_{<x, y>: x, y \in V_{n+1} \cup \bigcup_{i=1}^{m} \text{Int} \gamma_i} \lambda(\omega(x), \omega(y)) =$$
\[
\sum_{i=1}^{m} \left( \lambda_{22}(|\text{Int} \gamma_i| - 1) + \lambda_{21} |\gamma_i| \right) + \lambda_{11} \left( |V_{n+1}| - 1 - \sum_{i=1}^{m} (|\text{Int} \gamma_i| + |\gamma_i| - 1) \right) = \\
\lambda_{22} \sum_{i=1}^{m} (|\gamma_i| - 3) + \lambda_{21} \sum_{i=1}^{m} |\gamma_i| + \lambda_{11} \left( |V_{n+1}| - 1 - 2 \sum_{i=1}^{m} |\gamma_i| + 3m \right) = \\
\lambda_{22} (|\partial (\omega_A)| - 3m) + \lambda_{21} |\partial (\omega_A)| + \lambda_{11} (|V_{n+1}| - 1 - 2|\partial (\omega_A)| + 3m) = \\
(\lambda_{21} + \lambda_{22} - 2\lambda_{11}) |\partial (\omega_A)| + 3m (\lambda_{11} - \lambda_{22}) + \lambda_{11} (|V_{n+1}| - 1); \\
\]

The proof of second part of the lemma is similar. The lemma is proved.

**Lemma 8.3.** Let \( \gamma \) be a fixed contour and \( p^{(i)}_\beta (\gamma) = P^{(i)}_\beta \{ \omega : \gamma \subset \partial (\omega_A) \} \). Then
\[
p^{(i)}_\beta (\gamma) \leq \exp \{ -\beta (\lambda_{21} + \lambda_{22} - 2\lambda_{11}) |\gamma| - 3\beta (\lambda_{11} - \lambda_{22}) \},
\]
and a similar inequality holds for \( p^{(2)}_\beta (\gamma) \).

**Proof.** One can use lemma 8.2 and the simple fact that \( m(\chi_{\gamma} (\omega_\gamma)) = m(\omega_\gamma) - 1 \), where \( \chi_{\gamma} \) defined in proof of lemma 7.3. Then the proof of the lemma 8.3 is similar to proof of the lemma 7.3.

Using lemma 8.3 one can prove analogies of lemma 7.4 and 7.5 then also

**Theorem 8.4.** For all sufficiently large \( \beta \) there are at least two Gibbs measure for the two dimensional model (17) on the Cayley tree of order 2.

**Theorem 8.5.** The Gibbs measure \( P^{(i)}_\beta \) of the model (17) on Cayley tree of order 2 is small deviation of the configuration \( \omega^{(i)} \equiv v_i, i = 1, 2 \).

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**References**
1. [Ba] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, (Academic Press, London/New York, 1982).
2.[BRZ] P.M. Bleher, J. Ruiz, V.A. Zagrebnov. On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice, *Jour. Statist. Phys.* 79 : 473-482 (1995).
3.[BG] P.M. Bleher and N.N. Ganikhodjaev, On pure phases of the Ising model on the Bethe lattice, *Theor. Probab. Appl.* 35:216-227 (1990).
4.[BRSSZ] P.M. Bleher, J. Ruiz, R.H.Schommann, S.Shlosman and V.A. Zagrebnov, Rigidity of the critical phases on a Cayley tree, *Moscow Math. Journ.* 3: 345-362 (2001).
5. [GD] V.M. Gertsik, R.L.Dobrushin, Gibbs states in lattice models with two-step interaction, *Func. Anal. Appl.* 8: 201-211 (1974).
6. [GR] N.N. Ganikhodjaev, U.A. Rozikov, Group representation of the Cayley forest and some applications, *Izvestiya: Math.* 67: 17-27 (2003).

7. [GR1] N.N. Ganikhodjaev and U.A. Rozikov, On disordered phase in the ferromagnetic Potts model on the Bethe lattice, *Osaka Journ. Math.* 37: 373-383 (2000).

8. [GR2] N.N. Ganikhodjaev and U.A. Rozikov, A description of periodic extremal Gibbs measures of some lattice models on the Cayley tree, *Theor. Math. Phys.* 111: 480-486 (1997).

9. [HS] W. Holsztynski, J. Slawny, Peierls condition and the number of ground states, *Commun. Math. Phys.* 61: 177-190 (1978).

10. [K] I.A. Kashapov, Structure of ground states in three-dimensional Ising model with tree-step interaction, *Theor. Math. Phys.* 33: 912-918 (1977).

11. [M] R.A. Minlos, *Introduction to mathematical statistical physics* (University lecture series, ISSN 1047-3998; v.19, 2000)

12. [MR] F.M. Mukhamedov, U.A. Rozikov, On Gibbs measures of models with competing ternary and binary interactions and corresponding von Neumann algebras. *Jour. of Stat.Phys.* 114: 825-848 (2004).

13. [NR] Kh.A. Nazarov, U.A. Rozikov, Periodic Gibbs measures for the Ising model with competing interactions, *Theor. Math. Phys.* 135: 881-888 (2003)

14. [Pe] E.A. Pecherski, The Peierls Condition is not always satisfied, *Select. Math. Sov.* 3: 87-92 (1983/84)

15. [P] R. Peierls, On Ising model of ferromagnetism. *Proc. Cambridge Phil. Soc.* 32: 477-481 (1936)

16. [Pr] C. Preston, *Gibbs states on countable sets* (Cambridge University Press, London 1974).

17. [PS1] S.A. Pirogov, Ya.G. Sinai, Phase diagrams of classical lattice systems, I. *Theor. Math. Phys.* 25: 1185-1192 (1975)

18. [PS2] S.A. Pirogov, Ya.G. Sinai, Phase diagrams of classical lattice systems, II. *Theor. Math. Phys.* 26: 39-49 (1976)

19. [R1] U.A. Rozikov, A description of limit Gibbs measures for $\lambda$-models on the Bethe lattice, *Siberian Math. J.* 39: 373-380 (1998).

20. [R2] U.A. Rozikov, Representation of trees and its application *Math. Notes.* 72: 516-527 (2002).

21. [R3] U.A. Rozikov, Partition structures of the group representation of the Cayley tree into cosets by finite-index normal subgroups and their applications to the description of periodic Gibbs distributions, *Theor. Math. Phys.* 112: 929-933 (1997).

22. [RS] U.A. Rozikov, Yu.M. Suhov, A hard-core model on a Cayley tree: an example of a loss network, *Queueing Syst.* 46: 197-212 (2004)

23. [S] Ya.G. Sinai, *Theory of phase transitions: Rigorous Results* (Pergamon, Oxford, 1982).

24. [Z] M. Zahradnik, An alternate version of Pirogov-Sinai theory. *Comm. Math.*
25. [Z1] M. Zahradník, A short course on the Pirogov-Sinai theory, *Rendiconti Math. Serie VII* 18: 411-486 (1998)

26. [Za] S. Zachary, Countable state space Markov random fields and Markov chains on trees, *Ann. Prob.* 11: 894-903 (1983).