M/D/1 Queues with LIFO and SIRO Policies

STEVEN FINCH

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Abstract. While symbolics for the equilibrium M/D/1-LIFO waiting time density are completely known, corresponding numerics for M/D/1-SIRO are derived from recursions due to Burke (1959). Implementing an inverse Laplace transform-based approach for the latter remains unworkable.

We examined in [1] a first-in-first-out M/D/1 queue alongside unlimited waiting space, where the input process is Poisson with rate $\lambda$ and the service times are constant with value $1/\mu = a$. The FIFO policy of sorting clients, which is plainly fair and has minimal waiting time variance [2], is also known as FCFS: first-come-first-serve.

Keeping all other features the same, we wonder about the effect of replacing FIFO by LIFO: last-in-first-out. This policy, which seems patently unfair and has maximal waiting time variance [3], is also known as LCFS: last-come-first-serve.

Intermediate to FIFO and LIFO is SIRO: serve-in-random-order. This policy is variously known as ROS: random-order-of-service and RSS: random-selection-for-service. Analysis of SIRO is more difficult than that of LIFO, as will be seen.

1. LIFO

Let $W_{\text{que}}$ denote the waiting time in the queue (prior to service). Under equilibrium (steady-state) conditions and traffic intensity (load) $\rho = \lambda/\mu < 1$, the probability density function $f(x)$ of $W_{\text{que}}$ has Laplace transform [4, 5, 6, 7]

$$F(s) = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\infty} \exp(-sx)f(x)dx = 1 - \rho + \frac{\lambda - \lambda \Theta(s)}{s + \lambda - \lambda \Theta(s)} = F_{\text{alt}}(s) + 1 - \rho$$

where

$$\theta(x) = \sum_{k=1}^{\infty} e^{-k \rho} \frac{(k \rho)^{k-1}}{k!} \delta(x - a k), \quad \Theta(s) = -\frac{1}{\rho} \omega(-\rho \exp(-\rho - a s)), \quad \delta(x) \text{ is the Dirac delta and } \omega(s) \text{ is the principal branch of the Lambert omega:}$$

$$\omega(s)e^{\omega(s)} = s, \quad -1 \leq \omega(x) \in \mathbb{R} \quad \forall \ x \geq -1/e, \quad \exists \text{ branch cut for } x < -1/e.$$
From
\[(1 - \rho)s + \lambda [1 - \Theta(s)](1 - \rho + 1) = s F(s) + \lambda F(s) [1 - \Theta(s)]\]
we have
\[(1 - \rho)s + \lambda (2 - \rho - F(s)) [1 - \Theta(s)] = s F(s),\]
i.e.,
\[F(s) = 1 - \rho + \lambda (2 - \rho - F(s)) \left[\frac{1}{s} - \frac{\Theta(s)}{s}\right]\]
hence
\[f(x) = (1 - \rho)\delta(x) + \kappa + \lambda \int_0^x \left((2 - \rho)\delta(t) - f(t)\right) \left[1 - \int_0^t \theta(u)du\right] dt\]
\[= (1 - \rho)\delta(x) + \lambda + \lambda(2 - \rho) \left[1 - \int_0^x \theta(u)du\right] - \lambda \int_0^x f(t) \left[1 - \int_0^x \theta(u)du\right] dt.\]

The indicated condition \(\kappa = \lambda\) is true by the initial value theorem [8]:
\[\lim_{\varepsilon \to 0^+} f(\varepsilon) = \lim_{s \to 1^-} s F_{alt}(s).\]

Differentiating, we obtain
\[f'(x) = \lambda(2 - \rho) \left[0 - \theta(x)\right] - \lambda f(x) \left[1 - 0\right] - \lambda \int_0^x f(t) \left[0 - \theta(x - t)\right] dt\]
\[= -\lambda(2 - \rho)\theta(x) - \lambda f(x) + \lambda \int_0^x f(t)\theta(x - t)dt\]
\[= -\lambda(2 - \rho)\theta(x) - \lambda f(x) + \lambda \int_0^x f(t) \sum_{k=1}^\infty \frac{e^{-k\rho}(k\rho)^{k-1}}{k!} \delta(x - t - ak)dt\]
\[= -\lambda(2 - \rho)\theta(x) - \lambda f(x) + \lambda \sum_{k=1}^\infty \frac{e^{-k\rho}(k\rho)^{k-1}}{k!} \int_0^x f(t)\delta(x - t - ak)dt\]
\[= -\lambda(2 - \rho)\theta(x) - \lambda f(x) + \lambda \sum_{k=1}^\infty \frac{e^{-k\rho}(k\rho)^{k-1}}{k!} f(x - ak).\]

For \(0 < x < a\),
\[f'(x) = -\lambda f(x), \quad f(0^+) = \lambda\]
implies
\[ f(x) = \lambda e^{-\lambda x}. \]

Note that \( \lim_{x \to 0^+} f(a k + \varepsilon) = 0 \) for each \( k \geq 1 \) because, if a client arrives at the same moment the server becomes available, the client is taken immediately (by LIFO) and there is no waiting. For \( a < x < 2a \),
\[ f'(x) = -\lambda f(x) + \lambda e^{-\rho} \cdot \lambda e^{-\lambda (x-a)} \]
\[ = -\lambda f(x) + \lambda^2 e^{-\lambda x} \]
coupled with \( f(a^+) = 0 \) implies
\[ f(x) = \lambda^2 (x-a)e^{-\lambda x}. \]

For \( 2a < x < 3a \),
\[ f'(x) = -\lambda f(x) + \lambda e^{-\rho} \cdot \lambda^2 (x-2a)e^{-\lambda (x-a)} + \lambda e^{-2\rho} \frac{2 \rho}{2!} \cdot \lambda e^{-\lambda (x-2a)} \]
\[ = -\lambda f(x) + \lambda^3 (x-a)e^{-\lambda x} \]
coupled with \( f(2a^+) = 0 \) implies
\[ f(x) = \frac{1}{2} \lambda^3 x(x-2a)e^{-\lambda x}. \]

For \( 3a < x < 4a \),
\[ f'(x) = -\lambda f(x) + \lambda e^{-\rho} \cdot \frac{1}{2} \lambda^3 (x-a)(x-3a)e^{-\lambda (x-a)} \]
\[ + \lambda e^{-2\rho} \frac{2 \rho}{2!} \cdot \lambda^2 (x-3a)e^{-\lambda (x-2a)} + \lambda e^{-3\rho} \frac{(3 \rho)^2}{3!} \cdot \lambda e^{-\lambda (x-3a)} \]
\[ = -\lambda f(x) + \frac{1}{2} \lambda^4 x(x-2a)e^{-\lambda x} \]
coupled with \( f(3a^+) = 0 \) implies
\[ f(x) = \frac{1}{6} \lambda^4 x^2(x-3a)e^{-\lambda x}. \]

More generally, for \( k a < x < (k+1)a \), we obtain
\[ f(x) = \frac{1}{k!} \lambda^{k+1} x^{k-1}(x-k a)e^{-\lambda x} \]
and thus the waiting time density for LIFO is completely understood. Prabhu [9] and Peters [10] evidently hold priority in discovering this formula, the latter correcting an
error in \[11\]. The density for M/D/1-FIFO is likewise completely understood \[12\] – no surprises occur here – although it is M/U/1-FIFO densities which are only \textit{effectively} known \[1\]. Stitching the fragments together gives the LIFO density function pictured in Figure 1, for parameter values \(\lambda = 2\) and \(\mu = 3\); hence \(\rho = 2/3\) and \(a = 1/3\). It is interesting to compare this plot with Figure 3 of \[1\], the FIFO density for identical parameter values.

Let \(\xi = a^2\) and \(\eta = a^3\). Moments of \(W_{\text{que}}\) for LIFO are \[13, 14\]
\[
\text{mean} = -F'(0) = \frac{\lambda \xi}{2(1 - \rho)}, \quad \text{variance} = F''(0) - F'(0)^2 = \frac{\lambda \eta}{3(1 - \rho)^2} + \frac{\lambda^2 \xi^2(1 + \rho)}{4(1 - \rho)^3}
\]
giving \(\frac{1}{3}\) and \(\frac{7}{9}\) respectively. The mean of \(W_{\text{que}}\) for FIFO is the same as that for LIFO; the corresponding variance is smaller:
\[
\frac{\lambda \eta}{3(1 - \rho)} + \frac{\lambda^2 \xi^2}{4(1 - \rho)^2}
\]
giving \(\frac{5}{27}\). (\(\xi\) and \(\eta\) are second and third service time moments, used for consistency with earlier work.)

We clarify that, while \(\theta(x)\) is the notation for service time density in \[1\], \(\theta(x)\) here is the notation for busy period length density. The probability \{a busy period \(X\) is of length exactly \(ka\)\} is equal to the probability that \{exactly \(k\) clients of a queue, having precisely 1 starting client and traffic intensity \(\rho\), are served before the queue first vanishes\}. This is known as the Borel distribution \[15, 16, 17\], which satisfies \[18, 19, 20\]
\[
\mathbb{E}\left(\exp\left(-s\frac{X}{a}\right)\right) = -\frac{1}{\rho} \omega(-\rho \exp(-\rho - s))
\]
for M/D/1; thus the formula for \(\Theta(s) = \mathbb{E}(\exp(-sX))\) follows.

Study of busy period lengths is possible for M/U/1 – the density involves modified Bessel functions of the first kind \[21\] – it is far more complicated than the density for M/D/1.

2. SIRO

The probability density function \(f(x)\) of \(W_{\text{que}}\) has Laplace transform \[22, 23, 24, 25\]
\[
F(s) = 1 + \frac{\lambda(1 - \rho)}{s} \int_{\Theta(s)}^{1} \Phi(s, z) \Psi(s, z) dz = F_{\text{alt}}(s) + 1 - \rho
\]
where
\[
\Phi(s, z) = \frac{1 - z - as/(1 - \rho)}{z - e^{-a s/(1 - \rho)}} - \frac{1 - z}{z - e^{-\rho(1 - z)}},
\]
\[ \Psi(s, z) = \exp \left( -\int_{z}^{\infty} \frac{dy}{y - e^{-as - \rho(1-y)}} \right). \]

The integral underlying \( \Psi(s, z) \) is intractable; our symbolic approach for FIFO & LIFO seems inapplicable for SIRO.

We therefore turn to a numeric approach developed by Burke [26], which is based on certain simplifying properties of M/D/1 that do not easily generalize. Since \( \mu = 1 \) is assumed in [26], we take \( \lambda = 2/3 \) and rescale at the end. Two recursions are key:

\[ P_n = P_{n-1} e^{\lambda} - (P_0 + P_1) \frac{\lambda^{n-1}}{(n-1)!} - \sum_{j=2}^{n-1} P_j \frac{\lambda^{n-j}}{(n-j)!}; \]

\[ P_0 = 1 - \lambda, \quad P_1 = P_0 e^{\lambda} - P_0 \]

where the empty sum convention holds for \( n = 2 \), and

\[ Q_i(n, m) = \begin{cases} 
(1 - \frac{1}{n}) e^{-\lambda} \sum_{k=0}^{m} \frac{\lambda^k}{k!} Q_{i-1}(n + k - 1, m) & \text{if } n > 0, \\
0 & \text{otherwise}; 
\end{cases} \]

\[ Q_i(1, m) = \delta_i, \quad Q_0(n, m) = \begin{cases} 
\frac{1}{n} & \text{if } n > 0, \\
0 & \text{otherwise} 
\end{cases} \]

where \( \delta_i \) is the Kronecker delta and \( m \) is suitably large. These lead to the probability that waiting time is \( \leq t \):

\[ H(t, m) = (1 - \lambda) + \lambda \sum_{n=1}^{\infty} P_{n-1} \cdot \left( (t - \lfloor t \rfloor) Q_{\lfloor t \rfloor}(n, m) + \sum_{i=0}^{\lfloor t \rfloor - 1} Q_i(n, m) \right) \]

in the limit as \( m \to \infty \), where again the empty sum convention holds for \( 0 \leq t < 1 \).

Clearly the preceding expression simplifies to

\[ (1 - \lambda) + \lambda t \sum_{n=1}^{\infty} \frac{P_{n-1}}{n} \]

for all \( 0 \leq t \leq 1 \) and any \( m \geq 0 \). The dependence of \( H(t, m) \) on \( m \) becomes visible for \( t > 1 \). To find the waiting time density value, let \( t' = t + \frac{1}{4} \) (an arbitrary choice) and assume \( 0 < t < t' < 1 \). Then

\[ 4 (H(t', m) - H(t, m)) = \lambda \sum_{n=1}^{\infty} \frac{P_{n-1}}{n} \]
and, rescaling,
\[
12 \left( H(t', m) - H(t, m) \right) = 2 \sum_{n=1}^{\infty} \frac{P_{n-1}}{n} = 1.176773\ldots
\]

This corresponds to the height of the leftmost rectangle in Figure 2, surmounting the interval \([0, \frac{1}{3}]\).

For \(1 < t < t' < 2\), the rescaled density value is
\[
\lim_{m \to \infty} 12 \left( H(t', m) - H(t, m) \right) = 2 \sum_{n=2}^{\infty} \frac{P_{n-1}}{n} \left( 1 - \frac{1}{n} \right) e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{1}{n + k - 1}
\]
\[
= 0.392099\ldots
\]
which corresponds to the height of the rectangle surmounting the interval \([\frac{1}{3}, \frac{2}{3}]\).

Unlike earlier, the requirement that \(m\) is large becomes essential. For \(2 < t < t' < 3\), the formula for \(Q_2(n, m)\) involving \(\{Q_0(n, m), Q_1(n, m)\}\) is more complicated than that for \(Q_1(n, m)\). It is better to implement the recursion than to write the formula.

Summarizing, initial segments of the sought-after density function are
\[
g(x) = \begin{cases} 
  c_1 = 1.176773\ldots & \text{if } 0 < x < \frac{1}{3}, \\
  c_2 = 0.392099\ldots & \text{if } \frac{1}{3} < x < \frac{1}{2}, \\
  c_3 = 0.179926\ldots & \text{if } \frac{1}{2} < x < 1, \\
  c_4 = 0.095228\ldots & \text{if } 1 < x < \frac{2}{3}, \\
  c_5 = 0.054829\ldots & \text{if } \frac{2}{3} < x < \frac{4}{3}, \\
  c_6 = 0.033415\ldots & \text{if } \frac{4}{3} < x < 2, \\
  c_7 = 0.021228\ldots & \text{if } 2 < x < \frac{5}{3}, \\
  c_8 = 0.013923\ldots & \text{if } \frac{5}{3} < x < \frac{7}{3}, \\
  c_9 = 0.009369\ldots & \text{if } \frac{7}{3} < x < 3.
\end{cases}
\]

The Laplace transform of \(g(x)\) is
\[
G(s) = \sum_{j=1}^{\infty} \frac{c_j}{s} \exp \left( -\frac{j-1}{3} s \right) - \exp \left( -\frac{j}{3} s \right)
\]
and verification that \(G(s) = F_{alt}(s)\) can be done experimentally (although not yet theoretically).

We wonder about possible generalizations of this approach: works that cite \([26]\) include \([27, 28, 29]\). It is known (by other techniques) that the mean of \(W_{\text{que}}\) for SIRO is the same as that for FIFO and LIFO; the corresponding variance is between the two extremes \([30, 31, 32]\):
\[
\frac{2 \lambda \eta}{3(1-\rho)(2-\rho)} + \frac{\lambda^2 \xi^2(2+\rho)}{4(1-\rho)^2(2-\rho)}
\]
giving \(\frac{1}{3}\).
Figure 1: Waiting time density plot $y = f(x)$ for Deterministic last-in-first-out service.
Figure 2: Waiting time density plot $y = g(x)$ for Deterministic [$\frac{1}{3}$] serve-in-random-order policy.
3. Addendum

Let us briefly summarize corresponding results for M/M/1. For LIFO, the density $f(x)$ for $W_{q}$ is

$$f(x) = (1 - \rho)\delta(x) + \sqrt{\frac{\mu}{\lambda}} \exp \left[ - (\lambda + \mu)x \right] \frac{I_1 \left( 2\sqrt{\lambda\mu}x \right)}{x}$$

where $I_1(\cdot)$ is the modified Bessel function of first order. Since $\xi = 2/\mu^2$ and $\eta = 6/\mu^3$, the mean and variance are $\frac{2}{3}$ and $\frac{32}{9}$ respectively for parameter values $\lambda = 2$ and $\mu = 3$. Note that $\frac{32}{9} > \frac{8}{9}$, which is the analogous variance for FIFO.

For SIRO, the density $f(x)$ for $W_{q}$ is likewise of the form $(1 - \rho)\delta(x) + \varphi(x)$ where

$$\varphi(x) = 2(\mu - \lambda) \int_0^\pi \frac{\exp \left[ (2\psi(\tau) - \tau) \cot(\tau) \right] \exp \left[ -\lambda \left( 1 - \frac{2\sqrt{\rho}}{\sqrt{\rho}} \cos(\tau) + \frac{1}{\rho} \right) x \right]}{\exp[\pi \cot(\tau)] + 1} \left( 1 - \frac{2\sqrt{\rho}}{\sqrt{\rho}} \cos(\tau) + \frac{1}{\rho} \right) \sin(\tau) d\tau$$

and

$$\psi(\tau) = \begin{cases} \arctan \left( \frac{\sin(\tau)}{\cos(\tau) - \sqrt{\rho}} \right) & \text{if } 0 \leq \tau \leq \arccos \left( \sqrt{\rho} \right), \\ \pi + \arctan \left( \frac{\sin(\tau)}{\cos(\tau) - \sqrt{\rho}} \right) & \text{if } \arccos \left( \sqrt{\rho} \right) < \tau \leq \pi. \end{cases}$$

The mean is the same as that for LIFO; the variance is $\frac{14}{9}$ for $\lambda = 2$ and $\mu = 3$, intermediate to $\frac{8}{9}$ and $\frac{32}{9}$. Plotting the (monotone decreasing) densities for FIFO, LIFO and SIRO together, it becomes evident that

- both short & long waiting times occur more often under SIRO than under FIFO
- both short & long waiting times occur more often under LIFO than under SIRO

consistent with intuition.

We mention an unanswered question from [38]: for M/M/c-FIFO, what is the maximum wait time density associated with the queue over a specified time interval $[0, n]$? This open problem can be extended to LIFO and SIRO as well. See also [39] for treatment (analogous to M/D/1 here) of the D/M/1 queue.

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Steven Finch
MIT Sloan School of Management
Cambridge, MA, USA
*steven_finch@harvard.edu*