A New Monotone Quantity along the Inverse Mean Curvature Flow in $\mathbb{R}^n$

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Abstract

We find a new monotone increasing quantity along smooth solutions to the inverse mean curvature flow in $\mathbb{R}^n$. As an application, we derive a sharp geometric inequality for mean convex, star-shaped hypersurfaces which relates the volume enclosed by a hypersurface to a weighted total mean curvature of the hypersurface.

1 Statement of the Result

Monotone quantities along hypersurfaces evolving under the inverse mean flow have many applications in geometry and relativity. In [3], Huisken andIlmanen applied the monotone increasing property of Hawking mass to give a proof of the Riemannian Penrose Inequality. In a recent paper [1], Brendle, Hung and Wang discovered a monotone decreasing quantity along the inverse mean curvature flow in Anti-Desitter-Schwarzschild manifolds and used it to establish a Minkowski-type inequality for star-shaped hypersurfaces.

In this note, we provide a new monotone increasing quantity along smooth solutions to the inverse mean curvature flow in $\mathbb{R}^n$:

**Theorem 1.** Let $\Sigma$ be a smooth, closed, embedded hypersurface with positive mean curvature in $\mathbb{R}^n$. Let $I$ be an open interval and $X : \Sigma \times I \to \mathbb{R}^n$ be a smooth map satisfying

$$\frac{\partial X}{\partial t} = \frac{1}{H'},$$

(1.1)
where $H$ is the mean curvature of the surface $\Sigma_t = X(\Sigma, t)$ and $\nu$ is the outward unit normal vector to $\Sigma_t$. Let $\Omega_t$ be the bounded region enclosed by $\Sigma_t$ and $r = r(x)$ be the distance from $x$ to a fixed point $O$. Then the function

$$Q(t) = e^{-\frac{n-1}{n-2}t} \left[ n\text{Vol}(\Omega_t) - \frac{1}{n-1} \int_{\Sigma_t} r^2 H d\mu \right]$$

(1.2)

is monotone increasing and $Q(t)$ is a constant function if and only if $\Sigma_t$ is a round sphere for each $t$. Here $\text{Vol}(\Omega)$ denotes the volume of a bounded region $\Omega$ and $d\mu$ denotes the volume form on a hypersurface.

As an application, we derive a sharp inequality for star-shaped hypersurfaces in $\mathbb{R}^n$ which relates the volume enclosed by a hypersurface to an $r^2$-weighted total mean curvature of the hypersurface.

**Theorem 2.** Let $\Sigma$ be a smooth, star-shaped, closed hypersurface embedded in $\mathbb{R}^n$ with positive mean curvature. Then

$$n\text{Vol}(\Omega) \leq \frac{1}{n-1} \int_{\Sigma} r^2 H d\mu$$

(1.3)

where $\text{Vol}(\Omega)$ is the volume of the region $\Omega$ enclosed by $\Sigma$, $r$ is the distance to a fixed point $O$ and $H$ is the mean curvature of $\Sigma$. Furthermore, equality in (1.3) holds if and only if $\Sigma$ is a sphere centered at $O$.

We give some remarks about Theorem 1 and Theorem 2. The discovery of the monotonicity of $Q(t)$ in Theorem 1 is motivated by the recent work of Brendle, Hung and Wang in [1, Section 5]. To prove Theorem 1 we also need a result due to Ros [7] which was proved using Reilly’s formula [6]. Having known $Q(t)$ is monotone increasing, to prove Theorem 2 it may be attempting to ask whether $\lim_{t \to \infty} Q(t) = 0$. We do not know if this is true because both $\text{Vol}(\Omega_t)$ and $\int_{\Sigma_t} r^2 H d\mu$ grow like $e^{\frac{n-1}{n-2}t}$ when $\{\Sigma_t\}$ are spheres while there is only a factor of $e^{-\frac{n-1}{n-2}t}$ in (1.2). Instead, we take an alternate approach by first proving Theorem 2 for a convex hypersurface $\Sigma_t$. The proof in that case again makes use of Reilly’s formula. When $\Sigma$ is merely assumed to be mean convex and star-shaped, we prove Theorem 2 by reducing it to the convex case using solutions to the inverse mean curvature flow provided by the works of Gerhardt [2] and Urbas [8]. If a stronger result of Huisken and Ilmanen in [4] is applied, Theorem 2 indeed can be shown to hold for star-shaped surfaces with nonnegative mean curvature. We will discuss this case in the end.
2 Proof of the Theorems

Given a compact Riemannian manifold \((\Omega, g)\) with boundary \(\Sigma\), we recall that Reilly’s formula [6] asserts

\[
\int_{\Omega} |\nabla^2 u|^2 + \langle \nabla(\Delta u), \nabla u \rangle + \text{Ric}(\nabla u, \nabla u) \, dv = \int_{\Sigma} (\Delta u) \frac{\partial u}{\partial \nu} - \|\nabla \Sigma u, \nabla \Sigma u\| - 2(\Delta \Sigma u) \frac{\partial u}{\partial \nu} - H \left( \frac{\partial u}{\partial \nu} \right)^2 d\mu. \tag{2.1}
\]

Here \(u\) is a smooth function on \(\Omega\); \(\nabla^2, \Delta\) and \(\nabla\) denote the Hessian, the Laplacian and the gradient on \(\Omega\); \(\Delta \Sigma\) and \(\nabla \Sigma\) denote the Laplacian and the gradient on \(\Sigma\); \(\nu\) is the unit outward normal vector to \(\Sigma\); \(\|II\|\) and \(H\) are the second fundamental form and the mean curvature of \(\Sigma\) with respect to \(\nu\); and \(\text{Ric}\) is the Ricci curvature of \(g\).

To prove Theorem 1, we need a result of Ros [7], which was proved by choosing \(\Delta u = 1\) on \(\Omega\) and \(u = 0\) at \(\Sigma\) in the above Reilly’s formula.

**Theorem 3** (Ros [7]). Let \((\Omega, g)\) be an \(n\)-dimensional compact Riemannian manifold with nonnegative Ricci curvature with boundary \(\Sigma\). Suppose \(\Sigma\) has positive mean curvature \(H\), then

\[
n \text{Vol}(\Omega) \leq (n - 1) \int_{\Sigma} \frac{1}{H} d\mu. \tag{2.2}
\]

and equality holds if and only if \((\Omega, g)\) is isometric to a round ball in \(\mathbb{R}^n\).

**Proof of Theorem 3.** We use ‘ to denote differentiation w.r.t \(t\). Some basic formulas along the inverse mean curvature flow (1.1) in \(\mathbb{R}^n\) are

\[
H' = -\Delta_{\Sigma_t} \left( \frac{1}{H} \right) - \frac{\|II\|^2}{H}, \quad d\mu' = d\mu, \quad \text{Vol}(\Omega_t)' = \int_{\Sigma_t} \frac{1}{H} d\mu. \tag{2.3}
\]

Let \(u = r^2\), then \(u\) satisfies

\[
\nabla^2 u = 2g \quad \text{and} \quad \Delta u = 2n, \tag{2.4}
\]

where \(g\) is the Euclidean metric. Now

\[
\left( \int_{\Sigma_t} uH d\mu \right)' = \int_{\Sigma_t} (u'H + uH' + uH) d\mu. \tag{2.5}
\]
Let \( \langle \cdot, \cdot \rangle \) be the Euclidean inner product. By (2.3), (2.4) and the divergence theorem, we have
\[
\int_{\Sigma_t} u' H d\mu = \int_{\Sigma_t} \langle \nabla u, \frac{1}{H} \nu \rangle H d\mu = \int_{\Omega_t} \Delta u dV = 2n \text{Vol}(\Omega_t). \tag{2.6}
\]
By (2.4), we also have
\[
\Delta_{\Sigma_t} u = \Delta u - H \frac{\partial u}{\partial \nu} - \nabla^2 u(\nu, \nu) = 2(n-1) - H \frac{\partial u}{\partial \nu},
\]
which together with (2.3) - (2.4) implies
\[
\int_{\Sigma_t} u H' d\mu = \int_{\Sigma_t} \left( -\frac{\Delta_{\Sigma_t} u}{H} - \frac{u\|\mathbb{I}\|^2}{H} \right) d\mu = \int_{\Sigma_t} \left( -\frac{2(n-1)}{H} + \frac{\partial u}{\partial \nu} - \frac{u\|\mathbb{I}\|^2}{H} \right) d\mu = -\int_{\Sigma_t} \frac{2(n-1)}{H} d\mu + 2n \text{Vol}(\Omega_t) - \int_{\Sigma_t} \frac{u\|\mathbb{I}\|^2}{H} d\mu. \tag{2.7}
\]
Substituting (2.6) and (2.7) into (2.5) yields
\[
\left( \int_{\Sigma_t} u H d\mu \right)' = 4n \text{Vol}(\Omega_t) + \int_{\Sigma_t} \left[ -\frac{2(n-1)}{H} - \frac{u\|\mathbb{I}\|^2}{H} + uH \right] d\mu \\
\leq 4n \text{Vol}(\Omega_t) + \int_{\Sigma_t} \left[ -\frac{2(n-1)}{H} - \frac{uH}{n-1} + uH \right] d\mu \\
= 4n \text{Vol}(\Omega_t) + \int_{\Sigma_t} \left[ -\frac{2(n-1)}{H} + \frac{n-2}{n-1} uH \right] d\mu = 4n \text{Vol}(\Omega_t) + \frac{n-2}{n-1} \int_{\Sigma_t} uH d\mu \\
\leq 4n \text{Vol}(\Omega_t) - 2n \text{Vol}(\Omega_t) + \frac{n-2}{n-1} \int_{\Sigma_t} uH d\mu = 2n \text{Vol}(\Omega_t) + \frac{n-2}{n-1} \int_{\Sigma_t} uH d\mu \tag{2.8}
\]
where we have used \( \|\mathbb{I}\|^2 \geq \frac{1}{n-1} H^2 \) in line 2 and Theorem 3 in line 4. On the other hand, by Theorem 3 again, we have
\[
\text{Vol}(\Omega_t)' = \int_{\Sigma_t} \frac{1}{H} d\mu \geq \frac{n}{n-1} \text{Vol}(\Omega_t). \tag{2.9}
\]
It follows from (2.8) and (2.9) that
\[
\left[ n(n - 1)\text{Vol}(\Omega_t) - \int_{\Sigma_t} uHd\mu \right] \geq \frac{n - 2}{n - 1} \left[ n(n - 1)\text{Vol}(\Omega_t) - \int_{\Sigma_t} uHd\mu \right]
\]
or equivalently
\[
e^{-\frac{n - 2}{n - 1}t} \left( n\text{Vol}(\Omega_t) - \frac{1}{n - 1} \int_{\Sigma_t} r^2Hd\mu \right) \geq 0. \tag{2.10}
\]
We conclude that \( Q(t) \) is monotone increasing, moreover \( Q(t) \) is a constant function if and only if equalities in (2.8) and (2.9) hold. By Theorem 3, we know these equalities hold if and only if \( \Sigma_t \) is a round sphere for all \( t \). This completes the proof of Theorem 1. \( \Box \)

Next, we prove Theorem 2 in the case that \( \Sigma \) is a convex hypersurface.

**Proposition 1.** Let \( \Sigma \) be a smooth, closed, convex hypersurface embedded in \( \mathbb{R}^n \). Then
\[
n\text{Vol}(\Omega) \leq \frac{1}{n - 1} \int_{\Sigma} r^2Hd\mu \tag{2.11}
\]
where \( \text{Vol}(\Omega) \) is the volume of the region \( \Omega \) enclosed by \( \Sigma \), \( r \) is the distance to a fixed point \( O \) and \( H \) is the mean curvature of \( \Sigma \). Moreover, equality in (2.11) holds if and only if \( \Sigma \) is a sphere centered at \( O \).

**Remark 4.** Proposition 1 generalizes an inequality of the first author in [5, Theorem 3.2 (1)].

**Proof.** Apply Reilly’s formula (2.1) to the Euclidean region \( \Omega \) and choose \( u = r^2 \), we have
\[
4n(n - 1)\text{Vol}(\Omega) = \int_{\Sigma} II(\nabla^{\Sigma}u, \nabla^{\Sigma}u) + 2(\Delta_{\Sigma}u)\frac{\partial u}{\partial \nu} + H \left( \frac{\partial u}{\partial \nu} \right)^2 d\mu
\]
where
\[
\Delta_{\Sigma}u = \Delta u - H \frac{\partial u}{\partial \nu} - \nabla^2 u(\nu, \nu) = 2(n - 1) - H \frac{\partial u}{\partial \nu}.
\]
Therefore,
\[
\int_{\Sigma} H \left( \frac{\partial u}{\partial \nu} \right)^2 d\mu = \int_{\Sigma} II(\nabla^{\Sigma}u, \nabla^{\Sigma}u)d\mu + 4n(n - 1)\text{Vol}(\Omega). \tag{2.12}
\]
Since $\Sigma$ is convex, $\mathbb{II}(\cdot,\cdot)$ is positive definite. Hence, (2.12) implies

$$n(n-1)\text{Vol}(\Omega) \leq \frac{1}{4} \int_{\Sigma} H\langle \nabla(r^2), \nu \rangle^2 d\mu \leq \int_{\Sigma} Hr^2 d\mu. \quad (2.13)$$

When $n(n-1)\text{Vol}(\Omega) = \int_{\Sigma} Hr^2 d\mu$, we must have $\mathbb{II}(\nabla \Sigma u, \nabla \Sigma u) = 0$, hence $\nabla \Sigma u = 0$. This implies that $u = r^2$ is a constant on $\Sigma$, which shows that $\Sigma$ is a sphere centered at $O$. \hfill \square

To deform a star-shaped hypersurface to a convex hypersurface through the inverse mean curvature flow, we make use of a special case of a general result of Gerhardt [2] and Urbas [8].

**Theorem 5** (Gerhardt [2] and Urbas [8]). Let $\Sigma$ be a smooth, closed hypersurface in $\mathbb{R}^n$ with positive mean curvature, given by a smooth embedding $X_0 : S^{n-1} \to \mathbb{R}^n$. Suppose $\Sigma$ is star-shaped with respect to a point $P$. Then the initial value problem

$$\begin{cases}
\frac{\partial X}{\partial t} = \frac{1}{H} \nu \\
X(\cdot,0) = X_0(\cdot)
\end{cases} \quad (2.14)$$

has a unique smooth solution $X : S^{n-1} \times [0, \infty) \to \mathbb{R}^n$, where $\nu$ is the unit outer normal vector to $\Sigma_t = X(S^{n-1}, t)$ and $H$ is the mean curvature of $\Sigma_t$. Moreover, $\Sigma_t$ is star-shaped with respect to $P$ and the rescaled hypersurface $\tilde{\Sigma}_t$, parametrized by $\tilde{X}(\cdot, t) = e^{-\frac{t}{n-1}} X(\cdot, t)$, converges to a sphere centered at $P$ in the $C^\infty$ topology as $t \to \infty$.

Now we can complete the proof of Theorem [2].

**Proof of Theorem [2]** By Theorem 5, there exists a smooth solution \{\Sigma_t\} to the inverse mean curvature flow with initial condition $\Sigma$. Moreover, the rescaled hypersurface $\tilde{\Sigma}_t = \{e^{-\frac{t}{n-1}} x \mid x \in \Sigma_t\}$ converges exponentially fast in the $C^\infty$ topology to a sphere. In particular, $\tilde{\Sigma}_t$ and hence $\Sigma_t$, must be convex for large $t$.

Let $T$ be a time when $\Sigma_T$ becomes convex. By Proposition [11], we have

$$n\text{Vol}(\Omega_T) \leq \frac{1}{n-1} \int_{\Sigma_T} r^2 H d\mu.$$

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i.e. $Q(T) \leq 0$. By Theorem 1 we know $Q(t)$ is monotone increasing, hence

$$Q(0) \leq Q(T) \leq 0$$

which proves (1.3).

If the equality in (1.3) holds, then $Q(0) = 0$. It follows from the monotonicity of $Q(t)$ and the fact $Q(t) \leq 0$ for large $t$ that

$$Q(t) = 0, \quad \forall \ t.$$ 

By Theorem 1 this implies that $\Sigma_t$ is a sphere for each $t$. By Proposition 1 $\Sigma_t$ is a sphere centered at $O$ for large $t$. Therefore, we conclude that the initial hypersurface $\Sigma$ is a sphere centered at $O$. 

\[ \square \]

## 3 The case of nonnegative mean curvature

Suppose $\Sigma$ is a star-shaped hypersurface with nonnegative mean curvature in $\mathbb{R}^n$. By approximating $\Sigma$ with star-shaped hypersurfaces with positive mean curvature, it is not hard to see that the inequality (1.3) still holds for $\Sigma$. (For instance, such an approximation can be provided by the short time solution to the mean curvature flow with initial condition $\Sigma$.)

To see that the rigidity part of (1.3) also holds for such a $\Sigma$, we resort to a result of Huisken and Ilmanen in [4, Theorem 2.5]:

**Theorem 6** (Huisken and Ilmanen [4]). Let $X_0 : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n$ be an embedding such that $\Sigma = X_0(\mathbb{S}^{n-1})$ is a $C^1$, star-shaped hypersurface with measurable, bounded, nonnegative weak mean curvature. Then

$$\frac{\partial X}{\partial t} = \frac{1}{H} \nu$$

has a smooth solution $X : \mathbb{S}^{n-1} \times (0, \infty) \rightarrow \mathbb{R}^n$ such that as $t \rightarrow 0^+$, the hypersurface $\Sigma_t = X(\mathbb{S}^{n-1}, t)$ converges to $\Sigma$ uniformly in $C^0$.

**Remark 7.** In the above theorem, if the initial surface $\Sigma$ is assumed to be smooth, the same proof in [4] together with the upper estimate of $H$ for smooth solutions (c.f. [3, (1.4)]) shows that as $t \rightarrow 0^+$, $\Sigma_t$ converges to $\Sigma$ in $W^{2,p}$ norm for any $1 < p < \infty$. On the other hand, by Theorem 5, $\Sigma_t$ converges to a sphere in the $C^\infty$ topology after rescaling, as $t \rightarrow \infty$. In particular, $\Sigma_t$ is convex for large enough $t > 0$. 

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It follows from Theorem 1, Proposition 1, Theorem 6 and Remark 7 that

**Theorem 8.** Let $\Sigma$ be a smooth, star-shaped, closed hypersurface embedded in $\mathbb{R}^n$ with nonnegative mean curvature. Then

$$n\text{Vol}(\Omega) \leq \frac{1}{n-1} \int_{\Sigma} r^2 H \, d\mu$$

(3.2)

where $\text{Vol}(\Omega)$ is the volume of the region $\Omega$ enclosed by $\Sigma$, $r$ is the distance to a fixed point $O$ and $H$ is the mean curvature of $\Sigma$. Furthermore, equality in (1.3) holds if and only if $\Sigma$ is a sphere centered at $O$.

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