Iwasawa decomposition of the Lie supergroup

\[ SL(n, m, \mathbb{C}) \]

F. Pellegrini

Institut de mathématiques de Luminy,
163, Avenue de Luminy, 13288 Marseille, France

Abstract

We show that the superanalogue of the Iwasawa decomposition exists for supergroup \( SL(n, m, \mathbb{C}) \). The first component of the decomposition is the compact real form \( SU(n, m) \), which was constructed following the idea of our article [3]. The second component is the super version of the \( AN \) group, that we define in this article.

Keywords: Lie supergroup; real form; Iwasawa decomposition.

1 Introduction

Iwasawa decomposition of a simple complex Lie group \( D \) has many useful applications in group theory. For instance, it gives rise to Poisson-Lie structure on the compact real form \( G \) of \( D \), such that the dual Poisson-Lie group is identified with the subgroup \( AN \) of \( D = G^C \). In fact, \( D \) is said to be the Lu-Weinstein Drinfeld double of \( G \) and \( AN \). One of our motivation for superizing the Iwasawa decomposition is to define super Lu-Weinstein Drinfeld double. The reader may be surprised that the superization of the Iwasawa decomposition has not yet been considered in the literature. The reason is simple: it was reported in [5] that real forms of complex simple Lie superalgebra are never compact. As the compacity is an important ingredient of the standard Iwasawa decomposition, this result seemed to imply that there is no super-version of the decomposition. Recently, we have shown in [3] that

\footnote{e-mail: pelleg@iml.univ-mrs.fr}
the definition of the real form given in [5] for the supercase was too restrictive. In fact, we argued that there is a notion of graded real form which is more flexible than [5] one. In particular, we show that working with this new concept, each supergroup of the series $OSp(2r, s, \mathbb{C})$ has precisely one compact graded real form. In this article we extend the results of [3] in two directions. First, we show the existence of compact graded real form also for $SL(n, m, \mathbb{C})$ supergroup. Secondly, we argue that our new concept is very natural and powerful since it does allow to construct the super-analogue of the Iwasawa decomposition.

In section 2, we give the definition of the notion of normal and graded real form. Next, we define the supergroup $SL(n, m, \mathbb{C})$. Finally, we construct the compact graded real form $SU(n, m)$ of the supergroup $SL(n, m, \mathbb{C})$ following the ideas of our paper [3].

In section 3, we define the three ingredients used in the Iwasawa decomposition i.e. the superalgebra of complex ”functions” on respectively the real supergroup $SL(n, m, \mathbb{C})$, $SU(n, m)$ and $s(AN)$. This last supergroup is the superization of the real group $AN$. Next, we show the existence of the Iwasawa decomposition $SL(n, m, \mathbb{C}) = SU(n, m)s(AN)$. Our method uses the superization of the Gram-Schmidt orthonormalisation of a family of vectors.

## 2 Compact graded real form of $SL(n, m, \mathbb{C})$

We first give the definitions of complex matrix Lie supergroup and of its real form. We illustrate these notions on the supergroup $SL(n, m, \mathbb{C})$.

**Definition 2.1** A complex matrix Lie supergroup $\mathcal{H}$ is a complex superbialgebra generated by finite set of odd and even generators subject to polynomials relations. Those relations are supposed to generate a superideal of superbialgebra such that the quotient can be given the structure of a Hopf superalgebra (i.e. the antipode can be defined).

Now we turn to the definition of normal and graded real form:

**Definition 2.2** A normal real form of a complex Lie supergroup $\mathcal{H}$ is a pair $(\mathcal{H}, \sigma)$ where $\sigma$ is an even map from $\mathcal{H}$ to $\mathcal{H}$ such that:

$$(\sigma \otimes \sigma)\Delta(x) = \Delta(\sigma(x)), \quad (1)$$
\[ \epsilon(\sigma(x)) = \overline{\epsilon(x)}, \]  
\[ \sigma(\lambda x + \mu y) = \overline{\lambda} \sigma(x) + \overline{\mu} \sigma(y), \]  
\[ \sigma(xy) = \sigma(x)\sigma(y), \]  
\[ S \circ \sigma \circ S \circ \sigma(x) = x, \]  
\[ \sigma(\sigma(x)) = x, \]  
\[ \]  
with \( x, y \in H \) and \( \lambda, \mu \in \mathbb{C} \). If the two last properties are replaced by the following:

\[ S \circ \sigma \circ S \circ \sigma(x) = (-1)^{|x|}x, \]
\[ \sigma(\sigma(x)) = (-1)^{|x|}x, \]

then we have a graded real form (cf. [3]).

**Remark 2.1** The map \( \sigma \) is the generalisation to the supergroup framework of the concept of star structure. The latter is well-known in the Hopf algebra literature, where the real form of a complex Hopf algebra is by definition the star structure. Thus, we have adapted to the supergroup context the notion of real form, as it is defined in the Hopf algebra setting.

Now, we turn to the definition of \( SL(n, m, \mathbb{C}) \). First, we recall the definition of the complex superbialgebra of formal power series \( \mathbb{C}[x_{ij}], \ i, j = 1, \ldots, n+m \). The coproduct and counit are defined on the generators by:

\[ \Delta(x_{ij}) = 1 \otimes x_{ij} + \sum_{k=1}^{n+m} x_{ij} \otimes 1 + x_{ik} \otimes x_{kj}, \]
\[ \epsilon(x_{ij}) = 0. \]

Moreover, it is also enlightening to evaluate the coproduct of the elements \( y_{ij} = \delta_{ij} + x_{ij} \). We have:

\[ \Delta(y_{ij}) = \sum_{k=1}^{n+m} y_{ik} \otimes y_{kj}. \]  
\[ \]  
These maps are defined on all elements of \( \mathbb{C}[x_{ij}] \) by the morphism property of \( \Delta, \epsilon \). The gradation of the generators \( x_{ij} \) is \( |x_{ij}| = |i| + |j| \) where \( |i| = 0, |j| = 1 \) for respectively \( i = 1 \ldots n, j = n + 1 \ldots n + m \). We have the following standard Grassmann rules \( x_{ij}x_{mn} = (-1)^{(|i|+|j|)(|m|+|n|)}x_{mn}x_{ij} \) for the product in \( \mathbb{C}[x_{ij}] \).
Definition 2.3 The complex Lie supergroup $SL(n,m,\mathbb{C})$ or better $Hol(SL(n,m,\mathbb{C}))$ is the quotient of the superbialgebra $\mathbb{C}[x_{ij}]$ by the ideal generated by the polynomial $\text{sdet}(\delta + x) - 1 = 0$. The antipode is defined on the quotient by the following superalgebra-antimorphism i.e $S(x_{ij}x_{mn}) = (-1)^{(|i|+|j|)(|m|+|n|)}S(x_{mn})S(x_{ij})$:

$$S(x_{ij}) = -\delta_{ij} + (\delta + x)_{ij}^{-1},$$

with $i, j = 1 \ldots n + m$.

Here $(\delta + x)^{-1}$ means the inverse of the supermatrices which have for elements at the row $i$ and the column $j$: $(\delta_{ij} + x_{ij})$. The definition of the inverse of a supermatrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ reads (see [1]):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix},$$

when $A, D$ are even invertible matrices and nothing is assumed for the odd matrices $B, C$.

Remark 2.2 We shall use the notation $SL(n,m,\mathbb{C})$ and $Hol(SL(n,m,\mathbb{C}))$ interchangeably. In particular, we shall adopt the latter notation when we want to stress that in the super setting we deal with the holomorphic "functions" on the supergroup. We frequently write "functions" in inverted commas, the reason is that, in fact, we are working with formal series on the Lie supergroup. But, we think that the reader will understand this abuse of notations in the sense that a lot of ideas of this article are more natural in thinking about it as if we are working on functions on some space.

Now we follow our paper [3] and we equip $SL(n,m,\mathbb{C})$ with a graded real form as follows:

Theorem 2.1 The even antilinear superalgebra-morphism:

$$\sigma(x_{ij}) = (-1)^{(|i|+|j|)|j|}S(x_{ji})$$

The superdeterminant of a supermatrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is given by $\text{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\text{det}(A - BD^{-1}C)}{\text{det}(D)}$ where $\text{det}$ is the usual determinant of matrices (see [1]).
introduce the structure of graded real form on $SL(n,m,\mathbb{C})$ in the sense of Definition 2.2.

**Proof:**

In fact, it is enough to prove the properties (1) to (6) just for the generators $x_{ij}$ because of the property of superalgebra-morphism of $\sigma$. Thus, the property (4) is clearly fulfilled. The antilinearity (property (3)) stems also from the definition of $\sigma$.

We develop respectively the expressions $\sigma \otimes \sigma(\Delta(x_{ij}))$ and $\Delta(\sigma(x_{ij}))$ by using the equations (7) and (11). Thereby, we obtain respectively:

$$\sigma \otimes \sigma(\Delta(x_{ij})) = 1 \otimes S(x_{ji})(-1)^{(|i|+|j|)|j|} + (-1)^{(|i|+|j|)|j|} S(x_{ji}) \otimes 1 + (-1)^{(|i|+|k|)|k|+(|j|+|k|)|k|} S(x_{ki}) \otimes S(x_{jk}),$$

and

$$\Delta(\sigma(x_{ij})) = (1)^{(|i|+|j|)|j|}\Delta(S(x_{ji}))$$

= $(-1)^{(|i|+|j|)|j|}\tau(S \otimes S(1 \otimes x_{ji} + x_{ji} \otimes 1 + x_{jk} \otimes x_{ki}))$

= $(-1)^{(|i|+|j|)|j|}1 \otimes S(x_{ji}) + (-1)^{(|i|+|j|)|j|} S(x_{ji}) \otimes 1 + (-1)^{|j|(|i|+|j|)+(|i|+|k|)|k|} S(x_{ki}) \otimes S(x_{jk}).$

Here $\tau$ is the flip with the following property $\tau(f \otimes g) = (-1)^{|f||g|} g \otimes f$. Hence, $\sigma \otimes \sigma \Delta(x_{ij}) = \Delta(\sigma(x_{ij}))$, which corresponds to the property (1).

Now we turn to the property (2):

$$\epsilon(\sigma(x_{ij})) = \epsilon((-1)^{|i|+|j|})|j| S(x_{ji})) = (-1)^{|i|+|j|}|j| \epsilon(x_{ij}) = 0 = \overline{\epsilon(x_{ij})}.$$
Here we have use the property \( S(S(x)) = x, \forall x \in SL(n,m,\mathbb{C}) \).

Finally, it remains to prove \( \sigma(\sigma(x_{ij})) = (-1)^{|i|+|j|}x_{ij} \). We need to evaluate \( \sigma(S(x_{ji})) \). For this we first develop the following identity:

\[
\delta_{ij} = \sigma((\delta_{ik} + x_{ik})(\delta_{kj} + S(x_{kj}))),
\]

thus from (11) and the antilinearity of \( \sigma \) we deduce:

\[
\delta_{ij}(-1)^{|j|} = (\delta_{kj} + \sigma(S(x_{kj})))((\delta_{ik} + S(x_{ki}))(-1)^{|k|}|j|)
\]

Next, we multiply each member of the last equation by the right by \((\delta_{ip} + x_{ip})\) and we obtain:

\[
(\delta_{kj} + \sigma(S(x_{kj})))\delta_{kp}(-1)^{|k|}|j| = (-1)^{|j|}(\delta_{jp} + x_{jp}).
\]

We deduce:

\[
\sigma(S(p_{jj})) = (-1)^{|l|+|j|}x_{jp}.
\]

Thus, from this last identity we have:

\[
\sigma(\sigma(x_{ij})) = \sigma(S(x_{ji}))(-1)^{|i|+|j|}|j| = (-1)^{|i|+|j|}x_{ij} = (1)^{|i|+|j|}x_{ij},
\]

which ends up the proof.

\[\blacksquare\]

**Remark 2.3** If \( m = 0 \) (i.e there are no odd generators), our definition reduces to the standard compact real form \( SU(n,\mathbb{C}) \) of \( SL(n,\mathbb{C}) \). For this reason, we call the pair \((\text{Hol}(SL(n,m,\mathbb{C}),\sigma))\) the compact graded real form of \( SL(n,m,\mathbb{C}) \). With a slight ambiguity of notations, we note the compact graded real form of \( SL(n,m,\mathbb{C}) \) by \( SU(n,m) \).

### 3 Iwasawa decomposition of \( SL(n,m,\mathbb{C}) \)

Here, we turn to the definition of the main actors which occur in the Iwasawa decomposition of \( SL(n,m,\mathbb{C}) \). We first recall the Iwasawa decomposition (see [2]) in the non-supercase. When \((m = 0)\), the Iwasawa decomposition is the statement that the group \( SL(n,\mathbb{C}) \) viewed as a real group can be decomposed as \( SL(n,\mathbb{C}) = GAN \)\(^2\) where \( G = SU(n,\mathbb{C}) \) is the compact real form

\[\text{More precisely, it exists two unique maps } g,b \text{ from } SL(n,\mathbb{C}) \text{ in (respectively) } SU(n,\mathbb{C}),AN \text{ such that for all } d \in SL(n,\mathbb{C}) \text{ we have } d = g(d)b(d).\]
of $SL(n, \mathbb{C})$ and $AN$ is the real subgroup of $SL(n, \mathbb{C})$ of upper triangular matrices with real positive elements on the diagonal and determinant equal to one. In the supercase, we need to work with dual objects i.e. “functions” on the supergroup. So we need: 1) the Hopf superalgebra $Fun(SL(n, m, \mathbb{C}))$ of ”functions” on $SL(n, m, \mathbb{C})$ viewed as a real supergroup, 2) the Hopf superalgebra $Fun(SU(n, m))$ of ”functions” on $SU(n, m)$ the compact graded real form of $SL(n, m, \mathbb{C})$ and 3) the Hopf superalgebra $Fun(s(AN))$ of ”functions” on $s(AN)$ which is the superization of the previous Lie group $AN$. We discuss the three ingredients separately.

1) For finding $Fun(SL(n, m, \mathbb{C}))$ we borrow some inspiration from the [4] paper where the (non-super) q-analogue of the Iwasawa decomposition was considered. Thus the space of complex ”functions” on the real supergroup $SL(n, m, \mathbb{C})$ is $\text{Hol}(SL(n, m, \mathbb{C})) \otimes \mathbb{C} \overline{\text{Hol}}(SL(n, m, \mathbb{C}))$. $\overline{\text{Hol}}(SL(n, m, \mathbb{C}))$ is a ”copy” of $\text{Hol}(SL(n, m, \mathbb{C}))$ where the generators $x_{ij}$ are named $\overline{x}_{ij}$. The first copy of the tensor product corresponds to ”holomorphic functions” on $SL(n, m, \mathbb{C})$ while the second copy to ”antiholomorphic functions” on $SL(n, m, \mathbb{C})$. Note that we use the notation $\text{Hol}(SL(n, m, \mathbb{C}))$ for $SL(n, m, \mathbb{C})$ viewed as the complex supergroup and the notation $Fun(SL(n, m, \mathbb{C}))$ for $SL(n, m, \mathbb{C})$ viewed as the real supergroup. So, we have the following definition:

**Definition 3.1** The space of complex ”functions” $Fun(SL(n, m, \mathbb{C}))$ on the real supergroup $SL(n, m, \mathbb{C})$ is the Hopf superalgebra:

$$Fun(SL(n, m, \mathbb{C})) = \text{Hol}(SL(n, m, \mathbb{C})) \otimes \mathbb{C} \overline{\text{Hol}}(SL(n, m, \mathbb{C})).$$

(12)

2) The construction of the Hopf superalgebra of ”functions” on the compact graded real form of the supergroup $OSp(2r, s, \mathbb{C})$ was performed in detail in [3]. Here we adapt this construction to the compact graded real form of $SL(n, m, \mathbb{C})$, thud the Hopf superalgebra $Fun(SU(n, m))$ is the quotient: $\text{Hol}(SL(n, m, \mathbb{C})) \otimes \overline{\text{Hol}}(SL(n, m, \mathbb{C}))/I$. Here $I$ is the superideal generated by the polynomial equations $\sigma(x_{ij}) - x_{ij} = 0$ ($\sigma$ is the map of the theorem 2.1). The fact that $I$ is a Hopf superideal is a consequence of the relations (1-6) and the properties of the antipode for $Fun(SL(n, m, \mathbb{C}))$. Thus we have the definition:

**Definition 3.2** The space of complex ”functions” on the compact graded real form $SU(n, m)$ of $SL(n, m, \mathbb{C})$ is the following Hopf superalgebra:

$$Fun(SU(n, m)) = Fun(SL(n, m, \mathbb{C}))/I.$$

(13)
Here I is the Hopf superideal generated by the polynomial equations:

\[ \sigma(x_{ij}) - x_{ij}^\dagger = 0, \]  

with \( \sigma(x_{ij}) = (-1)^{|i|+|j|} |j| S(x_{ji}). \)

3) Finally, the definition of \( \text{Fun}(s(AN)) \) reads:

**Definition 3.3** The superalgebra \( \text{Fun}(s(AN)) \) of complex "functions" on the real supergroup \( s(AN) \) is the following Hopf superalgebra:

\[ \text{Fun}(s(AN)) = \text{Fun}(SL(n,m,\mathbb{C}))/J. \]

Here, \( J \) is the Hopf superideal generated by the following polynomials relations:

\[ x_{ij} = x_{ij}^\dagger = 0 \text{ for } i > j, \]

\[ x_{ii} - x_{ii}^\dagger = 0. \]

**Remark 3.1** For \( m = 0 \), the Hopf superalgebra \( \text{Fun}(s(AN)) \) reduces to the Hopf algebra \( \text{Fun}(AN) \) of "functions" on the Lie group \( AN \). If \( m \neq 0 \), the fact that \( \text{Fun}(s(AN)) \) is a good definition follows from the fact that the super Iwasawa decomposition can be formulated with it.

**Remark 3.2** As \( \text{Fun}(s(AN)) \) and \( \text{Fun}(SU(n,m)) \) are both quotients of \( \text{Fun}(SL(n,m,\mathbb{C})) \), we have the canonical projections \( i \) and \( j \) which are the superalgebra-morphisms:

\[ i : \text{Fun}(SL(n,m,\mathbb{C})) \rightarrow \text{Fun}(SU(n,m)), \]  

\[ j : \text{Fun}(SL(n,m,\mathbb{C})) \rightarrow \text{Fun}(s(AN)). \]

Both \( i \) and \( j \) map an element \( f \in \text{Fun}(SL(n,m,\mathbb{C})) \) into its respective cosets.

Now we turn to the main theorem of this article, the Iwasawa decomposition of the real supergroup \( \text{Fun}(SL(n,m,\mathbb{C})) \):
Theorem 3.1 (Iwasawa decomposition)
There exist a unique pair \((\phi, \psi)\) of superalgebra-morphisms:

\[
\phi : \text{Fun}(SU(n,m)) \longrightarrow \text{Fun}(SL(n,m,\mathbb{C})) \\
\psi : \text{Fun}(s(AN)) \longrightarrow \text{Fun}(SL(n,m,\mathbb{C}))
\]

such that:

\[
\phi(i(f(1))).\psi(j(f(2))) = f, \quad \forall f \in SL(n,m,\mathbb{C}).
\] (17)

Here \(\Delta(f) = f(1) \otimes f(2)\) and \(\cdot\) is the standard commutative multiplication in the superalgebra \(\text{Fun}(SL(n,m,\mathbb{C}))\).

Remark 3.3 This theorem in the non-super case \((m=0)\) is the dualisation of the Iwasawa decomposition of the Lie group \(SL(n,\mathbb{C})\) i.e. it gives the Iwasawa decomposition on the space of complex "functions" on \(SL(n,\mathbb{C})\).

When \(m = 0\), note that the maps \(\phi, \psi\) are, respectively, the pullbacks of the maps \(g,b\) defined in the footnote 2 for the Lie group \(SL(n,\mathbb{C})\).

Before giving the proof of the theorem, we have to introduce some notations. We said that a column supervector \(X\):

\[
X = \begin{pmatrix}
X_1 \\
\vdots \\
X_n \\
\chi_1 \\
\vdots \\
\chi_m
\end{pmatrix}
\] (20)

is even (odd) when \(X_i\) are even (odd) and \(\chi_m\) are odd (even). We define the supertranspose of the supervector \(X\) by:

\[
X^{st} = \begin{pmatrix}
(-1)^{|X|}X_1 & \cdots & (-1)^{|X|}X_n & \chi_1 & \cdots & \chi_m
\end{pmatrix}.
\] (21)

We use also the notation:

\[
X^\dagger = \begin{pmatrix}
X_1^\dagger \\
\vdots \\
X_n^\dagger \\
\chi_1^\dagger \\
\vdots \\
\chi_m^\dagger
\end{pmatrix}
\] (22)
We give the definition of the scalar product of two supervectors $X, Y$:

$$(X, Y) = X^\dagger Y.$$  \hspace{1cm} (23)

This scalar product have the following properties:

$$(X, Y)^\dagger = (-1)^{|X|+|Y|} (Y, X),$$  \hspace{1cm} (24)

$$(X\lambda, Y) = (-1)^{|X|+1} \lambda^\dagger (X, Y),$$  \hspace{1cm} (25)

where $\lambda$ are possible odd or even polynoms of the generators $x_{ij}, x^\dagger_{ij}$. Moreover, the norm of a supervector $X$ is noted and defined by $||X|| = \sqrt{\langle X, X \rangle}$.

We define a $(m + n) \times (m + n)$ supermatrix $P$ by specifying either its entries or its column supervectors. In the first case, we note $p_{ij}$ the entries of the supermatrix at the $i$th row and $j$th column. In the second case, $P$ reads:

$$P = (P_1, \ldots, P_{n+m}),$$  \hspace{1cm} (26)

where

$$P_i = \begin{pmatrix}
  p_{1i} \\
  \vdots \\
  p_{ni} \\
  p_{n+1 i} \\
  \vdots \\
  p_{m+n i}
\end{pmatrix}.$$  \hspace{1cm} (27)

Finally, we end this sequence of notations with two definitions:

**Definition 3.4** We say that a supermatrix $P$ with a unit superdeterminant is a $SU(n,m)$-supermatrix if its diagonal elements are normalized formal series and $P$ fulfills $^\dagger P_{\text{det}} P = 1$.

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3 A normalized formal serie is a formal serie where monomial of degree zero is equal to one.

4 The supertranspose of a supermatrix $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is $N^{\dagger} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}$, where $A^t, B^t, C^t, D^t$ is the usual transposition of the matrice $A, B, C, D$. Moreover, the entries of $N^\dagger$ at the $i$th row and $j$th column are $n^\dagger_{ij}$. 

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10
Definition 3.5 We say also that a supermatrix $Q$ with a unit superdeterminant is a $s(AN)$-supermatrix if its diagonal elements are real\footnote{For the conjugation $\dagger$, an element is real when it is equal to its conjugate.} normalized formal series and $Q$ is an upper triangular supermatrix.

Proof of the theorem 3.1:

The proof of the theorem gets organized in two parts. In the first part, we explicitly describe the two superalgebra-morphisms $\phi$ and $\psi$. Next, we show that they fullfill the property (17). In the second part, in turn, we show the unicity of these maps.

Let $\tilde{\phi}, \tilde{\psi}$ be two superalgebra-automorphisms of $Fun(SL(n, m, \mathbb{C}))$. Consider supermatrices $M, \Phi, \Psi$ with elements $\delta_{ij} + x_{ij}, \phi(\delta_{ij} + x_{ij})$ and $\psi(\delta_{ij} + x_{ij})$, respectively. We know from definition 3.1 that $sdet(M) = 1$. Denote $M_l, \Phi_l, \Psi_l$ the columns of $M, \Phi, \Psi$, respectively. Set

$$\Phi_l = \frac{V_l}{||V_l||}, \quad (28)$$

where the supervectors $V_l$ are defined recursively by:

$$V_1 = M_1, \quad V_l = M_l - \sum_{k=1}^{l-1} V_k \frac{(V_k, M_l)}{(V_k, V_k)}, \quad l = 2, \ldots, n + m. \quad (29)$$

It is easy to see that (28) imply that $\Phi$ is a $SU(n, m)$-supermatrix (cf. Definition 3.4), hence $\Phi$ is invertible. Then, we also set:

$$\Psi = \Phi^{-1}M. \quad (30)$$

We deduce from (30) that $\Psi$ is a $s(AN)$-supermatrix (cf. Definition 3.5).

The fact that $\Phi$ is a $SU(n, m)$-supermatrix imply that $\tilde{\phi}(I) = 0$ ($I$ was defined in definition 3.2). Hence, $\tilde{\phi}$ gives rise to a superalgebra-morphism $\phi$ from $Fun(SU(n, m))$ to $Fun(SL(n, m, \mathbb{C}))$ by:

$$\phi(i(\delta_{ij} + x_{ij})) = \Phi_{ij}, \quad \phi(i(\delta_{ij} + x_{ij}^\dagger)) = \Phi_{ij}^\dagger. \quad (31)$$
The fact that $\Psi$ is a $s(AN)$-supermatrix imply that $\tilde{\psi}(J) = 0$ ($J$ was defined in definition 3.3). Hence, $\tilde{\psi}$ gives rise to a superalgebra-morphism $\psi$ from $\text{Fun}(s(AN))$ to $\text{Fun}(\text{SL}(n, m, \mathbb{C}))$ by:

$$\psi(i(\delta_{ij} + x_{ij})) = \Psi_{ij}, \quad \psi(i(\delta_{ij} + x^\dagger_{ij})) = \Psi^\dagger_{ij}. \quad (32)$$

Furthermore, the fact that $\phi, \psi, i, j$ are superalgebra-morphisms makes sufficient to prove (17) just for the generators $x_{ij}$ and $x^\dagger_{ij}$. Now from eq. (30) we deduce $M = \Phi \Psi$. Hence, we have also $M^\dagger = \Phi^\dagger \Psi^\dagger$. These two last equalities give directly the validity of (17) for the morphisms $(\phi, \psi)$ defined by (31), (32). Thus the existence of $(\phi, \psi)$ is proved.

The reader may wish to understand better the origin of the formulas (28), (29). In fact, it is a superanalogue of the Gram-Schimdt procedure. We start with the family of column supervectors $M_1, \ldots, M_{n+m}$. In $(n + m)$-steps we construct a family $V_1, \ldots, V_{n+m}$ of orthogonal supervectors. More precisely, in the $k$-th step of the recursion we modify the $k$-th column in such a way that it becomes orthogonal to the $k - 1$ previous columns. Finally, we obtain (30) by the normalisation of the orthogonal family $V_1, \ldots, V_{n+m}$.

Now, we turn to the unicity of the maps $\tilde{\phi}, \tilde{\psi}$. We assume that there exist two distincts pairs of superalgebra-automorphisms of $\text{Fun} (\text{SL}(n, m, \mathbb{C}))$ $(\tilde{\phi}_k, \tilde{\psi}_k), \ k = 1, 2$ verifying $\tilde{\phi}_k(I) = 0, \tilde{\psi}_k(J) = 0$ and fulfulling (17). They give rise to two pairs of supermatrices $(\Phi_k, \Psi_k)$ defined for $k = 1, 2$ as follows:

$$\tilde{\phi}_k(\delta_{ij} + x_{ij}) = (\Phi_k)_{ij}, \quad \tilde{\phi}_k(\delta_{ij} + x^\dagger_{ij}) = (\Phi_k)^\dagger_{ij}, \quad (33)$$

$$\tilde{\psi}_k(\delta_{ij} + x_{ij}) = (\Psi_k)_{ij}, \quad \tilde{\psi}_k(\delta_{ij} + x^\dagger_{ij}) = (\Psi_k)^\dagger_{ij}. \quad (34)$$

Firstly, from the fact that $\tilde{\phi}_k(I) = 0$ (resp. $\tilde{\psi}_k(J) = 0$) we deduce that the supermatrix $\Phi_k$ (resp. $\Psi_k$) are $SU(n, m)$-supermatrix (resp. $s(AN)$-supermatrix). Secondly, we have the following equality between supermatrices:

$$\Phi_1 \Psi_1 = \Phi_2 \Psi_2. \quad (35)$$

because $\tilde{\phi}_k, \tilde{\psi}_k$ fullfill the relation (17). So we obtain:

$$\Phi_2^{-1} \Phi_1 = \Psi_2 \Psi_1^{-1}. \quad (36)$$

Finally, we remark that $\Phi_2^{-1} \Phi_1$ is a $SU(n, m)$-supermatrix as a matricial product of $SU(n, m)$-supermatrices, whereas $\Psi_2 \Psi_1^{-1}$ is a $s(AN)$-supermatrix.
as a matricial product of $s(AN)$-supermatrices. The unique supermatrix which is both a $SU(n,m)$-supermatrix and $s(AN)$-supermatrix is the unit supermatrix. Hence, we deduce that:

$$\Phi_1 = \Phi_2, \quad \Psi_1 = \Psi_2.$$  \hspace{1cm} (37)

The unicity is therefore proved. ■

4 References

[1] F.A.Berezin, *Introduction to superanalysis*, edited by A.A.Kirillov, MPAM, Reidel Publishing Company, Holland (1984).

[2] Anthony.W. Knapp, *Representation theory of semisimple groups an overview based on examples*, Princeton University Press, Princeton mathematical series (1986).

[3] F.Pellegrini, *Grassmann real form of $OSp(2r, s, \mathbb{C})$*, math.RA/0311240.

[4] P. Podleś and S.L. Woronowicz, *Quantum Deformation of Lorentz Group*. Commun. Math. Phys. **130**, 381-431 (1990).

[5] V.V.Serganova, *Classification of real simple Lie superalgebras and symmetric superspaces*, Functional Analysis **17 n3** (July-September 1983) 46-54.