Analysis and Application of Salop Model under Network Effect

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Abstract. Based on Salop's ring city model, this paper applies the mathematical model to solve the problem of benefit game of Bertrand competition under the environment of network effect. The equilibrium solutions for both players are given in this paper, and the equilibrium solution of Salop's ring city model depends on the effect of network externalities. The magnitude of network externalities determines the changes in welfare.

Keywords: Salop model, network effect, uneven distribution, theory of games.

1. Introduction
Salop model plays an important role in game theory. Economides once pointed out used duopoly model to solve game problems involving horizontal merger[1]. The conclusions of Mason and Wang are similar to the conclusion that of Economides[2-3]. However, from an intuitive perspective, this conclusion does not match reality. To explain this “paradox”, many scholars have modified the research scope and conditions of the Salop circle model.

At present, the research on benefit game theory includes motivation of participants related to their efficiency [4], the impact of product variety changes on the enterprise merger[5], consideration of the effect based on the internal incentive mechanism of the parties to the game[6], application problems with the internal incentive mechanism and effect in M-type enterprises[7], the Salop model-based discussion on the motives and the welfare of the actors under network externalities, and empirical studies on the frequency of the game model[8].

It is worth noting that the above studies do not consider the geographic distribution of the inhabitants or assume that people are evenly distributed with a probability density of 1, which is inconsistent with the actual situation.

In real life, the geographic distribution of residents is uneven, and generally, the pedestrian volume in the downtown areas is much larger than that in the suburbs.

In this context, multiple enterprises that produce homogeneous products are used as study subjects, and the differences in the commodity are the manufacturer location difference and the network effect. Based on the Salop circular city model, this paper studies the effect on the competition behaviour of game players in a network effect environment where people are unevenly distributed. On this basis, equilibrium solutions of the model are discussed, and the corresponding conclusions are obtained.

2. Model Establishment
Suppose an oligarchic competition industry composed of n players is evenly distributed on the ring city with unit perimeter, and Bertrand competition is implemented in this industry (participator
\[ i_{n+1} \text{ is located at } x=0 \). The probability density distribution function that inhabitants obey is 
\[ f(x) = \begin{cases} 1 + \cos 2\pi nx, & 0 \leq x \leq 1 \\ 0, & \text{other} \end{cases} \]
and the total number is normalized to 1.

Each game player \( i \) provides homogeneous products \( n_i, i = 1,2,\ldots,n \), except for the differences in the location of the actor and network externalities.

To simplify the analysis, this paper assumes that the marginal cost of the product is zero. Because the distribution density of inhabitants is symmetric, only the selection behaviour of users between participants \( i \) and \( i+1 \) needs to be considered.

If the distance between the user and contestant \( i \) is \( x_i \), the transportation cost function is quadratic, and the unit transportation cost is \( \tau \). In this case, the transportation cost of users at position \( x_i \) is \( \tau x_i^2 \) from \( i \) and is \( \tau \left( \frac{1}{n} - x_i \right)^2 \) from contender \( i+1 \).

Due to the network effect, competitors’ obtained utility from \( n_i \) is divided into two parts (i.e., the return derived from \( n_i \) itself \( R - p_i - \tau x_i^2 \) and the influence from network externalities \( uE(q_i) \)). \( p_i \) is the price of \( n_i \), \( q_i \) is the quantity demanded of \( n_i \), \( u \) represents the network effect coefficient of \( n_i \), and \( E(q_i) \) represents the value of expectation of \( n_i \).

The author assumes that each participant have rational expectations (i.e., \( E(q_i) = q_i \)); therefore, for residents who are located between \( i \) and \( i+1 \) before the cooperation between \( i \) and \( i+1 \) at a distance \( x_i \) away from \( i \), the utility of \( n_i \) and \( n_{i+1} \) for the people is
\[
U_{x_i}^i = R - p_i - \tau x_i^2 + uq_i \\
U_{x_i}^{i+1} = R - p_{i+1} - \tau \left( \frac{1}{n} - x_i \right)^2 + uq_{i+1} \quad i = 1,2,\ldots,n
\]

This paper only considers the cooperation between two actors. If \( i \) and \( i-1 \) work together, the incompatibility between \( n_i \) and \( n_{i-1} \) is eliminated due to it. Therefore, for the inhabitants located between participants \( i \) and \( i+1 \) after the consociation and at a distance \( x_i \) away from \( i \), the obtained network externality of \( n_{i-1} \) is \( u(q_{i-1} + q_i) \), and that of \( n_{i+1} \) is \( uq_{i+1} \). Hence, after the cooperation between \( i \) and \( i+1 \), the utility for the people at \( x_i \) for \( n_i \) (\( n_{i-1} \)) and \( n_{i+1} \) is
\[
U_{x_i}^{i} = R - p_i - \tau x_i^2 + u(q_{i-1} + q_i) \\
U_{x_i}^{i+1} = R - p_{i+1} - \tau \left( \frac{1}{n} - x_i \right)^2 + uq_{i+1}
\]

3. Equilibrium Solution of the Model

The probability density of the population distribution mentioned in this paper is \( f(x) = \begin{cases} 1 + \cos 2\pi nx, & 0 \leq x \leq 1 \\ 0, & \text{other} \end{cases} \), therefore, each competitor is symmetrical to its population. The fluctuation of the population distribution along the circular city has \( n \) cycles, and the circular city has exactly \( n \) participants to compete, which is in line with the actual situation.

Since the model adopts a circular structure, \( p_{n+1} = p_1 \) and \( p_n = p_0 \) can be set. Before the consociation, the symmetry can indicate that only the demand function of any \( n_i, i = 1,2,\ldots,n \) is required.

If inhabitants, who are located between \( i \) and \( i+1 \) at a distance \( x_i \) away from \( i \), have no difference in the effectiveness \( (U_{x_i}^{i} = U_{x_i}^{i+1}) \) between \( n_i \) and \( n_{i+1} \), the demand function of \( n_i \) is
After the consociation of $i$ and $i-1$, if the utility for each $n_i$ are indifferent ($U_i^{i'} = U^{i+1}_{i+1}$) between $n_i(n_{i-1})$ and $n_{i+1}$ when residents are located between $i$ and $i+1$ at a distance $x_i$ away from $i$, then the demand function of $n_i(n_{i-1})$ is

$$q_i = 2x_i, i = 1, 2, ..., n$$

(3)

The demand function for each $n_i$, all the participants who did not participate in the collaboration, is

$$q_{-i} = \frac{1}{n-2} (1 - \frac{1}{n} - 2x_i)$$

(5)

The following conclusions can be obtained by comparing the demand of $i$ before and after the merger:

**Proposition 1:** If the distributions of $n$ participators and inhabitants meet the conditions of the model described in this paper before the consociation, then due to symmetry, the demand of each $n_i$ is $q_i = \frac{1}{n}, i = 1, 2, ..., n$, and the indifference point between $n_i$ and $n_{i+1}$ for the utility is

$$x_i = (p_{i+1} - p_i + \frac{\tau}{n^2} - \frac{u}{n}) \frac{n}{2\tau - 2nu}$$

(6)

After the cooperation between $i$ and $i-1$, the demand function of $n_i$ and $n_{i-1}$ is

$$q_i = \frac{1}{n} + (p_{i+1} - p_i + \frac{\tau}{n^2}) \frac{n}{\tau - nu},$$

the demand function of each uninvolved is $q_{-i} = \frac{1}{n-2} \left[1 - \frac{1}{n} - (p_{i+1} - p_i + \frac{\tau}{n^2}) \frac{n}{\tau - nu}\right]$, and the indifference point between $n_i(n_{i-1})$ and $n_{i+1}$ for utility is

$$x_i = (p_{i+1} - p_i + \frac{\tau}{n^2}) \frac{n}{2\tau - 2nu}$$

(7)

Please see the appendix for the proof.

This conclusion shows that a horizontal collaboration expands the user scale and eliminates the incompatibility between $n_i$ and $n_{i-1}$ of the two involved players. Therefore, after the cooperation, the demand for all $n_i$ is asymmetric.

Before the collaboration, each $i$ simultaneously adopt prices competition strategy to maximize their profits. The profit function of each $i$ can be obtained using symmetry and Equation (3):

$$\pi_i = \max_{p_i} \left\{p_i (p_{i+1} - p_i + \frac{\tau}{n^2} - \frac{u}{n}) \frac{n}{\tau - nu}\right\}$$

Based on the first-order condition, the equilibrium of $i$ ($i = 1, 2, ..., n$) before the merger are

$$p_i^* = \frac{\tau}{n^2} - \frac{u}{n}$$

$$\pi_i^* = \frac{\tau}{n^3} - \frac{u}{n^2}, i = 1, 2, ..., n$$

(8)

Therefore, in a market with network externalities, the network effect coefficient is positively correlated with the utility of $n_i$. Therefore, under competition, each $i$ may adopt a strategy to expand the scale of sales. Equation (5) shows that the strengthening of the network effect might lead to intensified competition, lowered equilibrium solution, and reduced earnings.
After the consociation, the merged and uninvolved sides start competing for incomes. It is worth noting that the remaining $n-2$ uninvolved satisfy symmetry. Based on Equation (4), the equilibrium solutions are

$$p_{i}^{**} = \frac{(\tau - nu)(n+1) + \tau}{3n^2}$$
$$p_{-i}^{**} = \frac{(\tau - nu)(2n-1) - \tau}{3n^2}$$
$$\pi_{i}^{**} = \frac{[(\tau - nu)(n+1) + \tau]^2}{9n^3(\tau - nu)}$$
$$\pi_{-i}^{**} = \frac{1}{n-2} \frac{[(\tau - nu)(2n-1) - \tau]^2}{9n^3(\tau - nu)} \quad (9)$$

The comparison of the interests of the involved side before and after the cooperation shows that when

$$u \in \left(0, \frac{\tau}{n}\right) \cup \left(\frac{\tau}{n(n+1+\sqrt{18})}, \frac{\tau}{n(n+1-\sqrt{18})}\right)$$

all uninvolved $i$ could be expelled from the game, and the cooperator $i$ could become a monopolist. In a monopolistic market, for consumers located between $i$ and $i+1$ at a distance $\frac{1}{n}$ away from $i$, utility of $n_i$ is the lowest due to symmetry. In this case, the equilibrium solution of $i$ are

$$p_{i}^{M} = \frac{R - \tau}{n^2} + u$$
$$\pi_{i}^{M} = \frac{R - \tau}{n^2} + u$$

When $p_{-i}^{**} > 0$, $u \in \left(0, \frac{(2n-2)\tau}{(2n-1)n}\right)$, participants who did not participate in the collaboration can remain in the market. Under this circumstance, the equilibrium solution of them before and after the merger is $\Delta p_{-i} = p_{-i}^{**} - p_{-i}^{*}$:

$$\Delta p_{-i} \geq 0 \quad 0 < u \leq \frac{(2n-5)\tau}{(2n-4)n}$$
$$\Delta p_{-i} < 0 \quad \frac{(2n-5)\tau}{(2n-4)n} < u \leq \frac{(2n-2)\tau}{(2n-1)n}$$

This calculation also shows that when they remain in the market to participate in the competition, the change in the equilibrium solution of the firm that implements a cooperative strategy is $\Delta p_{i} = p_{i}^{**} - p_{i}^{*} > 0$. 
We categorize the participants who practiced the cooperative strategy as category I, and the rest as category II, the plays choose A as the strategy for raising prices and B as the strategy for raising prices. The above analysis shows that the category I might increase their prices to maximize profit.

For the category I, when the effect of network externalities is small, i.e., \( 0 < u \leq \frac{(2n-5)\tau}{(2n-4)n} \), they might increase their prices to sell products. When the network externalities are strengthened, i.e., \( \frac{(2n-5)\tau}{(2n-4)n} < u \leq \frac{(2n-2)\tau}{(2n-1)n} \), they might play strategy B. They might be expelled from the game, when the effect of network externalities is strong, and the network coefficient increases to

\[
U \in \left( \frac{(2n-2)\tau}{(2n-1)n}, \frac{\tau}{n} \right) \cup \left( \frac{\tau}{n}, \frac{\tau}{n(n+1+\sqrt{18})}, \frac{\tau}{n(n+1-\sqrt{18})} \right).
\]

With the enhancement of network externalities, the utility of \( n_i \) increases, the demand for \( n_i \) is the key for the contestant. The above analysis shows that the gains from the expansion of the network effect cannot compensate for the loss, so simply adopting strategy B should be unwise.

4. The Analysis of the Solution

With the enhancement of network externalities, the participant in category I must continue to choose strategy B, and when \( P_i < c_i \) they all are forced to withdraw from the game. Therefore, changes in \( W \) can be discussed based on whether the actor in category I are expelled. \( W \) is the sum of profit and surplus \( CS \). From Equation (5),

\[
W^* = \sum_{i=1}^{n} \pi_i^* + CS = n\pi_i^* + \int_{2n}^{2n+1} U_i^i \, dx = R + \frac{u}{n} - \frac{\tau}{12n^2}
\]  

When the contestants in category I have not been expelled \( (u \in \left( 0, \frac{(2n-2)\tau}{(2n-1)n} \right) ) \), the change of \( W \) is

\[
\Delta W = W^{**} - W^* = \pi_i^{**} + (n-2)\pi_i^{**} + 2 \int_{2n}^{2n+1} U_i^j \, dx + \int_{2n}^{2n+1} U_i^{ji} \, dx - W^* \\
\geq \frac{7n^2 - 12n + 8}{9n^2} u + \frac{-11n^2 + n + 10}{36n^2} \tau
\]  

Thus, we have:

\[
\Delta W < 0 \, u \in \left( 0, \frac{11n^2 - n - 10}{4(7n^2 - 12n + 8)} \frac{\tau}{\tau} \right)
\]

\[
\Delta W \geq 0 \, u \in \left[ \frac{11n^2 - n - 10}{4(7n^2 - 12n + 8)} \frac{\tau}{\tau}, \frac{2n - 2}{2n - 1} \frac{\tau}{\tau} \right]
\]

When the contenders in category II are expelled one by one and all the competitors becomes one, the change of \( W \) is

\[
\Delta W^* = \pi^* + 2 \int_{2n}^{2n+1} U_i^i \, dx - W^* \\
= \frac{n - 1}{n} u - \frac{(n-2)(n-1)}{12n^2} \tau
\]

\[
\Delta W^* < 0 \, u \in \left( -2n - 2 \frac{\tau}{(2n-1)n}, \frac{n - 2}{12n} \frac{\tau}{\tau} \right)
\]
The above analysis shows that the change in $W$ depends on the strength of network externalities. It is worth noting that if $n = 3$ in Equations (10), (11), and (12), the calculation results are exactly consistent with $W$ changes among the three competitors described in Ref.[7]. Therefore, the change in $W$ due to the collaboration of game players under the uneven inhabitants distribution and network effect analysed in this paper is just the generalization of the situation described in Ref.[7]. This paper assumes that residents follow a nonuniform distribution, indicating that $W$ is not affected by the population distribution.

5. Conclusion
This paper studies the impact of a cooperative strategy between $i$ in the context of network externalities and the uneven population distribution. On this basis, the motives of the cooperative strategy and changes in $W$ before and after the tactic are discussed. The conclusions show that under the environment of network externalities, the participants generally have motives for the cooperative strategy, that can place the competitors who did not adopt the cooperator tactic at a competitive disadvantage.

With the gradual increase in network externalities of $n$, the adverse impact on the participant in category $I$ also increases. No matter how network externalities change, the game players in category $I$ can obtain the higher return.

When network externalities increase, the contenders in category $I$ can be reasonable with choosing strategy B, but when the effect of network externalities become stronger, the advantage is more pronounced.

The players in category $I$ keep adopting strategy B, so they might eventually be expelled from the game. The analysis of $W$ reveals that it changes in line with the strength of network externalities.

Based on the Salop circular model, the author discusses the equilibrium solution and the impact of the tactic for collaboration under network externalities. In contrast, the effects of network externalities and unit transportation costs on the competitive equilibrium of manufacturers under different consumer preferences are not considered, then the model considering these conditions will be studied in the future.

Acknowledgements
This work is supported by Quality engineering project of GDEI (Grant No. 2018jxgg22 and Grant No. 202lzxkc04). The author is very grateful to the referee for valuable comments and suggestions.

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Appendix:
Proof of Proposition 1
Proof: Because the density function for the population distribution in this circular city is
\[ f(x) = \begin{cases} 1 + \cos 2n\pi x, & 0 \leq x \leq 1 \\ 0, & \text{other} \end{cases} \]
according to Equations (1) and (3), the indifference point between \( n_i \) and \( n_{i+1} \) before the merge strategy satisfies:
\[
R - p_i - \tau x_i^2 + u \int_{i-1}^{i} \frac{x_i}{n} (1 + \cos 2n\pi x)dx = R - p_{i+1} - \tau \left(\frac{1}{n} - x_i\right)^2 + 2u \left(\frac{1}{n} - x_i\right)
\]
\[
\Rightarrow -p_i - \tau x_i^2 + u \left[ \frac{i}{n} + x_i - \frac{1}{n} - x_{i-1} \right] + u \int_{i-1}^{i} \frac{x_i}{n} \left[ \sin 2n\pi \left(\frac{i}{n} + x_i\right) - \sin 2n\pi \left(\frac{i-1}{n} + x_{i-1}\right) \right]
\]
\[
= -p_{i+1} - \tau \left( x_i^2 - \frac{1}{n} + \frac{1}{n^2} \right) + 2u \left(\frac{1}{n} - x_i\right)
\]
Because each \( i \) is symmetric with respect to its population, the indifference distance between any two \( n_i \) is equal (\( x_i = x_j, i \neq j, i, j = 1, 2, \ldots, n \)).
Using the first-order Taylor series expansion of \( \sin 2n\pi \left(\frac{i}{n} + x_i\right) - \sin 2n\pi \left(\frac{i-1}{n} + x_{i-1}\right) \), the equation can be simplified as follows:
\[
-p_i - \tau x_i^2 + u \left[ \frac{i}{n} + x_i - \frac{1}{n} - x_{i-1} \right] + u \frac{2n\pi}{2n\pi} \left[ \sin 2n\pi \left(\frac{i}{n} + x_i\right) - \sin 2n\pi \left(\frac{i-1}{n} + x_{i-1}\right) \right]
\]
\[
\Rightarrow x_i = (p_{i+1} - p_i + \frac{\tau}{n_2} - \frac{u}{n}) \frac{n}{2\tau - 2un}
\]
From the above equation and Equation (3), we have: \( q_i = \frac{1}{n}, \ n = 1, 2, \ldots, n \).
After the consociation, the indifference point between \( n_i (n_{i-1}) \) and \( n_{i+1} \) satisfies:
\[
R - p_i - \tau x_i^2 + u \int_{i-1}^{i} \frac{x_i}{n} (1 + \cos 2n\pi x)dx = R - p_{i+1} - \tau \left(\frac{1}{n} - x_i\right)^2 + 2u \left(\frac{1}{n} - x_i\right)
\]
Similarly, we know:
\[
x_i' = (p_{i+1} - p_i + \frac{\tau}{n_2}) \frac{n}{2\tau - 2un}
\]
Based on the above equation and Equations (4) and (5), the demand function of \( n_i \) for the category \( l \) choosing the strategy is
\[
q_i = \frac{1}{n} + (p_{i+1} - p_i + \frac{\tau}{n_2}) \frac{n}{\tau - nu}
\]
The function of \( n_i \) for the category \( l \) is
\[
q_{-i} = \frac{1}{n-2} \left[ 1 - \frac{1}{n} - (p_{i+1} - p_i + \frac{\tau}{n_2}) \frac{n}{\tau - nu} \right]
\]