Aspects of the decoherence in high spin environments: Breakdown of the mean-field approximation

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Abstract

The study of the decoherence of qubits in spin systems is almost restricted to environments whose constituents are spin-$\frac{1}{2}$ particles. In this paper we consider environments that are composed of particles of higher spin, and we investigate the consequences on the dynamics of a qubit coupled to such baths via Heisenberg $XY$ and Ising interactions. It is shown that while the short time decay in both cases gets faster as the magnitude of the spin increases, the asymptotic behavior exhibits an improvement of the suppression of the decoherence when the coupling is through Heisenberg $XY$ interactions. In the case of a transverse Ising model, we find that the mean field approximation breaks down for high values of the spin.

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I. INTRODUCTION

Modeling the environments to which quantum systems are coupled is a central topic in the study of the decoherence and the entanglement phenomena \[1\]. Indeed, it has become widely accepted that the decoherence as well as the degradation of the entanglement resource (e.g., entanglement sudden death), are solely a consequence of the interaction of the physical systems with their surroundings \[2–4\]. This concept constitutes the main idea behind the topic of open quantum systems \[5\], which has attracted a great deal of interest during the past decades.

The internal structure of the environment is rather complicated in most cases. As a matter of fact, the large number of its constituents makes it very difficult to deal, in an exact analytical manner, with the evolution of the system of interest. This explains the reason for which one has to resort, very often, to simplified models that capture the essential features of the environment. Needless to say that these features depend on the physical nature of the degrees of freedom characterizing its constituents. Therefore, it is no surprise that most of the investigations have dealt with developing various techniques that enable the elimination of the spin or the bosonic degrees of freedom of the environment \[6–17\], which allows one to focus on the evolution of the central system. The mathematical tools needed for such calculations vary depending on whether the environment is of spin or bosonic nature. Nevertheless, the main idea behind these techniques is the same and rests in the partial trace operation, which means that the reduced density matrix corresponding to the central system may be obtained by taking the trace over the environmental degrees of freedom.

With the rapid progresses in the field of spintronics, it became evident that spin systems should be among the first options that have to be exploited for the implementation of new quantum technologies \[18, 19\]. The advantages brought by such systems reside in the fact that they can be prepared and fabricated in a scalable form that facilitates the implementation of quantum algorithms \[20–25\]. Therefore, it is of practical importance to consider environments that are of spin nature. In most studies, it is usually assumed that the constituents are electrons or, more generally, spin-\(\frac{1}{2}\) particles.

Recently, a great interest has been given to the so-called single-molecule magnets \[26, 27\]. Common examples of such systems include \([\text{Mn}_3\text{O(O}_2\text{CEt})_3\text{mpko}_3\text{]}(\text{ClO}_4)\) (also known as Mn\(_3\)), and \([\text{Fe}_8\text{O}_2\text{(OH)}_2\text{tacn}]_6^{8+}\). They are characterized by large magnetic moments, and
a slow magnetic relaxation. In addition, they exhibit configurations with large values of the spin. For example, the ground state of the single-molecule magnet \( \text{Mn}_3 \) has a spin \( S = 6 \); that corresponding to the ground state of \([\text{Fe}_8\text{O}_2(\text{OH})_12(\text{tacn})_6]^{8+}\) is \( S = 10 \). Thus, it is tempting to consider the scenario in which systems of particles of higher spins constitute the environment to which a central qubit is coupled.

In this paper the emphasis is on a particular case of spin environments, for which the whole information about the intra-bath interactions as well as the coupling to the central system are encoded in the total spin operators. We have already addressed this problem earlier for the case of spin-\( \frac{1}{2} \) particles (see for example [15]). Here we shall generalize the investigation to environments formed by particles of arbitrary value of the spin. This will enable us to compare the obtained results, and to draw some conclusions about the advantages and the inconveniences of using this kind of environments.

The manuscript is organized as follows. Section II is devoted to the study of an environment whose intra-bath interactions as well as its coupling to a central two-level system (a qubit) is of Heisenberg XY type; we derive an explicit expression for the degeneracy of the total angular momentum, and we deduce its probability distribution when the size of the spin bath is sufficiently large. This is followed by the analysis of the asymptotic behavior of the state of the central system. Furthermore, we use the Holstein-Primakoff transformation to establish the connection between the studied spin model and a bosonic Jaynes-Cummings model. Section III deals with a transverse Ising model; there the mean field approximation is used to linearize the problem, and we identify the critical point of the system. Then we study analytically the decoherence of the qubit, and we compare the obtained results with the exact solution in the case of a vanishing transverse field. The paper is ended with a brief conclusion.

II. CASE OF HEISENBERG XY INTERACTIONS

A. Model

The first model we shall investigate describes the coupling of a central spin-\( \frac{1}{2} \) particle (a qubit) to a set of \( N \) spin-\( S \) particles through Heisenberg XY interactions. The central particle is subject to the effect of an applied magnetic field, the strength of which is denoted
by $2\mu$. The full Hamiltonian of the system is then given by

$$H = H_S + H_{SB} + H_B,$$  \hspace{1cm} (1)

where

$$H_S = \mu \sigma_z$$  \hspace{1cm} (2)

is the Hamiltonian of the free central spin,

$$H_{SB} = \frac{\alpha}{\sqrt{N}} \left[ \sum_{i=1}^{N} \sigma_x S^i_x + \sum_{i=1}^{N} \sigma_y S^i_y \right]$$  \hspace{1cm} (3)

denotes the Hamiltonian describing the coupling of the spin to its environment, and

$$H_B = \frac{g}{N} \sum_{i \neq j}^{N} \left( S^i_x S^j_x + S^i_y S^j_y \right)$$  \hspace{1cm} (4)

is the Hamiltonian of the spin bath. In the above, $\sigma_x$, $\sigma_y$ and $\sigma_z$ denote the usual Pauli matrices, whereas $S^i_{x,y,z}$ are the components of the spin operator of the $i$'th spin in the environment. The constants $\alpha$ and $g$ are the coupling strengths, and are assumed positive.

It is worth noting that the case for which the environment is composed of particles with spin $-\frac{1}{2}$ has been thoroughly investigated in Ref. 15. There, the rescaling of the coupling constants $\alpha$ and $g$ by the factors $\sqrt{N}$ and $N$, respectively, enabled us to discuss the case when $N$ is infinite. From a statistical point of view, the above rescaling ensures the well behavior of the free energy of the spin bath. In this work, we shall follow a slightly different approach to deal with the case of a large number of environmental spins, namely, we shall derive a probability distribution for the total spin angular momentum, that can be used for large but finite number of spins within the environment.

To fully describe the state of the qubit, we need to determine the evolution in time of its density matrix $\rho$. Usually, one assumes that the qubit is initially uncorrelated with its environment, and that the latter is in thermal equilibrium at temperature $T$. Since the evolution of the total system is unitary, it is sufficient to eliminate the environmental degrees of freedom from the evolution equation by tracing out the states of the bath. In our case it is more convenient to work in the basis composed of the common eigenvectors of the operators $J^2$ and $J_z$, where

$$\vec{J} = \sum_{i=1}^{N} \vec{S}^i.$$  \hspace{1cm} (5)
These vectors are denoted by $|j, m\rangle$ such that (we assume $\hbar = 1$)

$$J^2|j, m\rangle = j(j + 1)|j, m\rangle, \quad J_z|j, m\rangle = m|j, m\rangle. \quad (6)$$

The evolution in time of the density matrix of the system we are interested in is thus given by [note that $\beta = 1/(k_B T)$, $k_B$ being Boltzmann’s constant]

$$\rho(t) = \frac{1}{Z} \sum_{j,m} \nu(j, N; S) \langle j, m | \exp(-iHt) \rho_S(0) \otimes \exp(-\beta H_B) \exp(iHt) | j, m \rangle. \quad (7)$$

The quantity $\nu(j, N; S)$ stands for the degeneracy corresponding to the value $j$ of the total spin angular momentum. Knowing the degeneracy, one can decompose the total spin space of the environment as follows:

$$\mathbb{C}^{2S+1 \otimes N} = \bigoplus_{j}^{SN} \nu(j, N; S) \mathbb{C}^{2j+1}, \quad (8)$$

where $\mathbb{C}$ is the field of complex numbers, and the sum over $j$ runs from $0 \ (\frac{1}{2})$ to $NS$, when $NS$ is even (odd). Hence our next task resides in the determination of the degeneracy $\nu$, which we fulfill in the next subsection.

### B. The degeneracy and the distribution of the quantum number $j$

In order to find the expression of the degeneracy $\nu$, we introduce the spaces:

$$F_m = \{\mathcal{V}_{jm} \in \mathbb{C}^{2S+1 \otimes N}, \ J_z \mathcal{V}_{jm} = m \mathcal{V}_{jm}\}, \quad (9)$$

and

$$E_{j,m} = \{\mathcal{V}_{jm} \in \mathbb{C}^{2S+1 \otimes N}, \ J_z \mathcal{V}_{jm} = m \mathcal{V}_{jm}, \ J^2 \mathcal{V}_{jm} = j(j + 1) \mathcal{V}_{jm}\}. \quad (10)$$

It can easily be verified that the dimension of the space $F_m$ is given by

$$\dim F_m = \sum_{L_S}^{N} \frac{N!}{2^S \prod_{k=0}^{L_S}(L_S+k)!} \delta \left( \sum_{\rho=0}^{2S} L_{-S+\rho}, N \right) \delta \left( \sum_{\rho=0}^{2S} (S-\rho) L_{-S+\rho}, m \right). \quad (11)$$
In the above equation, the quantities $L_i$ (with $i = -S, -S + 1, -S + 2, \cdots S - 1, S$) are integer numbers taking on values from 0 to $N$. The expression (11) can be further simplified to the form

$$
\dim F_m = \sum_{L_{-S+2}, L_{-S+3}, \cdots, L_S}^{N} \frac{N!}{\prod_{k=2}^{2S} (L_{-S+k})!} \times \left[ \left( SN + m - \sum_{\rho=2}^{2S} \rho L_{-s+\rho} \right)! \left( (1 - S) N - m + \sum_{\rho=2}^{2S} (\rho - 1) L_{-S+\rho} \right)! \right]^{-1}.
$$

Now since

$$
F_m = \bigoplus_{j=m}^{NS} E_{j,m},
$$

which means that

$$
\dim E_{j,m} = \dim F_j - \dim F_{j+1} = \nu(j, N; S),
$$

we obtain after some algebra

$$
\nu(j, N; S) = \sum_{L_{-S+2}, L_{-S+3}, \cdots, L_S=0}^{N} A \frac{N!}{\prod_{k=2}^{2S} (L_{-S+k})!} \left[ \left( 2S - 1 \right) N + 2j + 1 - \sum_{\rho=2}^{2S} (2\rho - 1) L_{-s+\rho} \right] \left[ SN + j + 1 - \sum_{\rho=2}^{2S} \rho L_{-S+\rho} \right]^{-1}.
$$

Notice that in the particular case where $S = 1/2$, the degeneracy simplifies to

$$
\left( \frac{N}{N/2-j} \right).
$$

In this way we can assign to the quantum number $j$ a probability distribution, which we designate by $P(j)$, as follows:

$$
P(j) = \frac{2j + 1}{(2S + 1)^N} \nu(j, N; S).
$$

This distribution corresponds to a tracial state of a randomly distributed set of $N$ independent spin-$S$ particles. It allows, under convergence conditions, for the calculation of the expectation value of any quantity that depends on the total angular momentum number $j$; the latter has to be dealt with as a continued real random variable when $N$ is sufficiently large.
Before we proceed further, let us notice that equation (15) implies that whatever the values of \( N \) and \( S \) are, we have:

\[
\nu(\text{NS}, N; S) = 1, \quad \nu(\text{NS} - 1, N; S) = N. \quad (17)
\]

Furthermore, since

\[
\mathbb{C}^{(2S+1)\otimes N} \otimes \mathbb{C}^{2S+1} = \bigoplus_{j=0}^{NS} \nu(j, N; S) \mathbb{C}^{(2j+1)} \otimes \mathbb{C}^{2S+1} \]

\[
= \bigoplus_{j} \nu(j, N; S) \bigoplus_{j'=|j-S|} \mathbb{C}^{(2j'+1)}, \quad (18)
\]

it follows that

\[
\nu(j, N + 1; S) = \sum_{j'=|j-S|} j' + S \nu(j', N; S). \quad (19)
\]

Actually, the above equality is a special case of the property [16]

\[
\nu(J, N_1 + N_2; S) = \frac{1}{2J + 1} \sum_{j_1, j_2, m_1, m_2, M} \nu(j_1, N_1; S) \nu(j_2, N_2; S)
\]

\[
\times \langle j_1 m_1 j_2 m_2 | JM \rangle^2. \quad (20)
\]

When \( N \) becomes very large, the probability distribution \( P(j) \) approaches a Gaussian distribution, the form of which may be inferred from the expression of \( \nu \) together with the fact that

\[
\frac{1}{(2S+1)^N} \text{tr} \left( \sum_{k,j=1}^{N} \tilde{S}^k \tilde{S}^j \right) = \frac{3}{(2S+1)^N} \text{tr} \left( \sum_{k,j=1}^{N} S_z^k S_z^j \right)
\]

\[
= NS(S + 1). \quad (21)
\]

Taking into account equations (15) and (21), we find that the explicit expression of the distribution \( P(j) \) takes the form

\[
P(j) = \frac{6j^2}{NS(S + 1)} \sqrt{\frac{3}{2\pi NS(S + 1)}} \exp \left( - \frac{3j^2}{2S(S + 1)N} \right). \quad (22)
\]

Hence given a function \( f \) of the random variable \( j \), its mean value can be evaluated as:

\[
\langle f(j) \rangle = \int_0^\infty P(j)f(j) dj. \quad (23)
\]
In particular, we find that the mean value of \( j \) reads
\[ \langle j \rangle = 2 \sqrt{\frac{2}{3\pi}} \sqrt{N S(S + 1)}, \] (24)
which leads to
\[ (\Delta j)^2 = \langle j^2 \rangle - \langle j \rangle^2 = \left(1 - \frac{8}{3\pi}\right) NS(S + 1). \] (25)
This shows that the width of the distribution is proportional to \( \sqrt{N} \), which explains, once more, the reason for which the coupling constants \( g \) and \( \alpha \) have been rescaled by the respective powers of the size of the environment. As we have mentioned above the rescaling ensures that the Helmholtz free energy of the system is extensive. This is necessary in order to study what is referred to as the thermodynamic limit (more precisely, the limit \( N \to \infty \)). For example, the order of magnitude of \( N \) is \( 10^6 \) in a quantum dot. On the other hand, for the known molecules, the order of magnitude of the spin is \( S \sim 10 \). Note, nevertheless, that even for the case where \( S \sim 3 \), one is actually dealing with a large spin, for which the quasiclassical approximation may be employed \[29\]. In the latter approximation, the environmental spin is dealt with as a classical vector. It is also possible to use spin coherent states for large \( S \) to express the Hamiltonian classically, and to find the quantum corrections to the latter in the form of a series of powers of \( 1/S \) \[30\]. The present work deals with the environmental spins quantum mechanically.

Another point worth mentioning, relative to the large \( S \) limit, is the connection with a bosonic environment. This link can be established using the Holstein-Primakoff transformation which maps the spin operators \( S^k_\pm \) to bosonic ones as follows: \[31\]
\[ S^k_- = \sqrt{2S} \sqrt{1 - \frac{a_k^\dagger a_k}{2S}} a_k, \quad S^k_+ = \sqrt{2S} a_k^\dagger \sqrt{1 - \frac{a_k^\dagger a_k}{2S}}, \] (26)
with \([a_k, a_{k'}^\dagger] = \delta_{kk'}\). When \( S >> 1 \), we may write as a first approximation
\[ S^k_- \approx \sqrt{2S} a_k, \quad S^k_+ \approx \sqrt{2S} a_k^\dagger. \] (27)
Let us introduce the operators
\[ B = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} a_k, \] (28)
\[ B^\dagger = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} a_k^\dagger. \] (29)
It is easily verified that these operators satisfy \([B, B^\dagger] = 1\), that is they are also bosonic operators. Then, to a good approximation, we can write in the limit \(N \to \infty\)

\[
H_{SB} = 2\alpha\sqrt{2S}(\sigma_- B^\dagger + \sigma_+ B),
\]

\[
H_B = 2gSB^\dagger B.
\]

This shows that under the above assumptions, the model is equivalent to a Jaynes-Cummings model \[32\]. We see also that the coupling constants of the new Hamiltonian are proportional to the magnitude of the spin, which is analogous to the results obtained in the context of the spin wave theory \[33\]. (See Appendix B for more details about the time evolution of this bosonic model.)

C. Time evolution

The time evolution of the central qubit can be studied in the large \(N\) limit using the time evolution operator that has been derived in Ref \[15\]. Using that form, the elements of the reduced density matrix may be calculated by virtue of equation (7). The trace operation over the environmental degrees of freedom is carried out using the probability distribution of \(j\) provided that \(N\) is sufficiently large. This considerably facilitates the calculation, since the direct use of the degeneracy \(\nu\) requires the evaluation of sums of terms that grow rapidly with \(N\) and \(j\). Furthermore, the analytical form of \(P\) makes it possible to find in close analytical form the expression of the asymptotic reduced density matrix (see below).

The only difficulty we have to face here rests in the fact that the quantum number \(m\) should be dealt with as a random variable that is dependent on \(j\). To overcome this difficulty, we adopt, as a first step, the approximation in which \(m^2\) is replaced by \(\theta j^2\) where \(\theta\) is a yet-to-be-determined parameter. Then, for sufficiently large values of \(N\) we find, for instance, that

\[
\rho_{12}(t) = \rho_{21}(t) = \frac{\rho_{12}(0)}{Z} \int_0^\infty P(j)e^{-\frac{\mu^2\sqrt{\nu}j^2}{N}} \left\{ \cos^2(t\sqrt{\mu^2 + (1 - \theta)\alpha^2 j^2/N}) \right. \\
- \frac{\mu^2}{\mu^2 + \alpha^2(1 - \theta)j^2/N} \sin^2(t\sqrt{\mu^2 + (1 - \theta)\alpha^2 j^2/N}) \\
+ \frac{i\mu}{\sqrt{\mu^2 + \alpha^2(1 - \theta)j^2/N}} \sin(2t\sqrt{\mu^2 + (1 - \theta)\alpha^2 j^2/N}) \right\} dj,
\]

\[32\]
where

$$Z = \left[ 1 + 2S(S+1)g\beta(1-\theta)/3 \right]^{-3/2}.$$  \ \ \ \ \ \ (33)

Similarly, the diagonal elements of the reduced density matrix are calculated as:

$$\rho_{11}(t) = 1 - \rho_{22}(t) = \frac{\rho_{11}(0)}{Z} \int_0^\infty P(j) e^{-\frac{g\beta(1-\theta)^2}{N}} \left\{ \cos^2(t\sqrt{\mu^2 + (1-\theta)\alpha^2j^2/N}) + \frac{\alpha^2}{\mu^2 + \alpha^2(1-\theta)j^2/N} \sin^2(t\sqrt{\mu^2 + (1-\theta)\alpha^2j^2/N}) \right\} dj + \frac{\rho_{22}(0)}{Z} \int_0^\infty P(j) e^{-\frac{g\beta(1-\theta)^2}{N}} \times \left\{ \frac{\alpha^2}{\mu^2 + \alpha^2(1-\theta)j^2/N} \sin^2(t\sqrt{\mu^2 + (1-\theta)\alpha^2j^2/N}) \right\} dj.$$  \ \ \ \ \ \ (34)

Having determined the analytical forms of the elements of \(\rho\), it is now possible to deduce their asymptotic values by making use of the Riemann-Lebesgue lemma. This yields

$$\psi = \lim_{t \to \infty} \frac{\rho_{12}(t)}{\rho_{12}(0)} = \frac{1}{2} - \frac{\left(\mu/\alpha\right)^2}{\left(\mu/\alpha\right)^3} \left[ \beta g + \frac{3}{2(1-\theta)S(S+1)} \right]$$
$$- \sqrt{\pi} \left(\frac{\mu}{\alpha}\right)^3 \left[ \beta g + \frac{3}{2(1-\theta)S(S+1)} \right]^{3/2}$$
$$\times \exp \left[ (\mu/\alpha)^2 \left( \beta g + \frac{3}{2(1-\theta)S(S+1)} \right) \right]$$
$$\times \text{erfc} \left[ \frac{\mu}{\sqrt{\alpha \left( \beta g + \frac{3}{2(1-\theta)S(S+1)} \right)}} \right],$$  \ \ \ \ \ \ (35)

where \(\text{erfc}(x)\) designates the complementary error function, and we have used the symbol \(\psi\) for later convenience and ease of notation. One can see that the right-hand side of the latter equation is independent of the number of environmental spins, which is a direct consequence of the rescaling of the coupling strengths \(g\) and \(\alpha\).

Comparing the latter result with that obtained for \(S = 1/2\), we come to the conclusion that the parameter \(\theta\) should vanish, i.e \(\theta \equiv 0\). In order to explain this result, it suffices to notice that the interaction between the bath’s spins is of Heisenberg \(XY\) type whose form includes only the \(x\) and \(y\) components of the spin operators. The spin coupling makes it more probable for the total spin vector to lie within the \(x-y\) plane. Hence it is plausible to neglect \(m^2\) compared to \(j^2\) which, obviously, contains the contribution of the three components of the total spin operator \(\vec{J}\). We only need to put \(\theta = 0\) into equation (35) in order to determine
the long-time behavior of the off-diagonal element $\rho_{12}$; that corresponding to the diagonal element $\rho_{11}$ reads

$$\lim_{t \to \infty} \rho_{11}(t) = \rho_{11}(0)\left\{ \frac{1}{2} + \left(\frac{\mu}{\alpha}\right)^2 \left[ \beta g + \frac{3}{2S(S+1)} \right] + \sqrt{\pi} \left(\frac{\mu}{\alpha}\right)^3 \left[ \beta g + \frac{3}{2S(S+1)} \right]^{3/2} \exp \left[ (\mu/\alpha)^2 \left( \beta g + \frac{3}{2S(S+1)} \right) \right] \times \text{erfc} \left[ \frac{\mu}{\alpha} \sqrt{\left( \beta g + \frac{3}{2S(S+1)} \right)} \right] \right\} + \rho_{22}(0) \left\{ \sqrt{\pi} (\mu/\alpha)^3 \left[ \beta g + \frac{3}{2S(S+1)} \right]^{3/2} \times \exp \left[ (\mu/\alpha)^2 \left( \beta g + \frac{3}{2S(S+1)} \right) \right] \text{erfc} \left[ \frac{\mu}{\alpha} \sqrt{\left( \beta g + \frac{3}{2S(S+1)} \right)} \right] \right\}. \quad (36)$$

**FIG. 1**: (Color online) The asymptotic value of the coherence $\psi$ for different values of $S$ and $T$. The remaining parameters are $g = 1$, $\mu = \alpha$.

We have displayed in Figures 1 and 2 the variation of the asymptotic value of the off-diagonal element $\rho_{12}$ as a function of $S$, $\beta$ and $\mu/\alpha$ for $g = 1$. It is clear that the above matrix element assumes larger values as $S$ increases, indicating that the partial suppression of the decoherence may be improved in environments whose constituents are of high spin. This effect becomes more apparent as the temperature increases.

From a statistical point of view, the above results imply that the amount of information accessible at long times to the central system, which initially has been leaked to the spin bath, is greater when the magnitude of $S$ increases. Indeed, initially the qubit looses rapidly its coherence due to the coupling to the environment; afterwards, as things randomize, the qubit has the tendency to adhere the state that minimizes the loss of coherence. The number
of accessible bath’s state vectors is equal to \((2S + 1)^N\). Clearly this number increases as \(S\) increases, allowing more options for the central system to select among the above states, those which display the minimal decoherence due to the coupling to the bath. This process cannot go on indefinitely, since a saturation does emerge as the magnitude of the spin \(S\) increases; indeed, when \(S >> 1\), we find that

\[
\rho_{12}(\infty)/\rho_{12}(0) = \frac{1}{2} - \left(\frac{\mu}{\alpha}\right)^2 \beta g - \sqrt{\pi} \left(\frac{\mu}{\alpha}\right)^3 (\beta g)^{3/2} \times \exp[(\mu/\alpha)^2 \beta g] \text{erfc} \left[\frac{\mu}{\alpha} \sqrt{\beta g}\right].
\]  

(37)

FIG. 2: (Color Online) The asymptotic value of the coherence \(\psi\) for different values of \(S\) and \(\mu/\alpha\). The other parameters are \(g = 1, \beta = 0.1\).

At short times, the decay of the reduced density matrix is purely Gaussian, which is typical for the non-Markovian dynamics. For example, the evolution of the off-diagonal element can be approximated at short times by

\[
|\rho_{12}(t)| = |\rho_{12}(0)| e^{-t^2/\tau_d^2}
\]

(38)

where the decoherence time \(\tau_D\) can be determined via the second-order master equation, describing the evolution of the open system (see, e.g., [16]). Explicitly, we find that

\[
\tau_D = \frac{1}{\alpha} \sqrt{\beta g + \frac{3}{2S(S + 1)}}.
\]

(39)

From the latter expression, we deduce that the larger \(S\), the shorter \(\tau_D\), meaning that the decoherence is faster when \(S\) is large. The minimum value of \(\tau_D\) is given by

\[
\tau_D^{\text{min}} = \frac{1}{\alpha} \sqrt{\beta g}.
\]

(40)
Hence, as far as the discussion is concerned with the magnitude of $S$, there exits an apparent competition between the speed of the loss of coherence of the central system at short times and its asymptotic state. To be more specific, we note that, while the decoherence time constant takes its smallest values for high spins, the partial recovery of quantum interferences gets better, and \textit{vice versa}. This means that if one is interested in the short time behavior, it is much better to use a spin bath that is composed of spin-$\frac{1}{2}$ particles, which yields the largest values of $\tau_D$. On the contrary, if the application requires the optimal state at long times, then it is more convenient to use a spin environment with $S$ sufficiently large.

III. CASE OF TRANSVERSE ISING MODEL

A. Model

The second model we shall investigate is a generalization of that studied by Lucamarini \textit{et al} \[17\]. Here we would like to describe the interaction of a qubit with a ferromagnetic symmetry-broken spin environment that is composed of $N$ spin-$S$ particles; the latter are coupled to each other via Ising type interactions and are subject to a transverse magnetic field along the $x$ direction. The Hamiltonian of the total system is given by

$$H = H_0 + H_{SB} + H_B$$

with

$$H_0 = \mu S_z^0, \quad (42)$$

$$H_{SB} = -\frac{J_0}{\sqrt{N}} S_z^0 \sum_{i=1}^N S_i^z, \quad (43)$$

$$H_B = -w \sum_{i=1}^N S_i^x - \frac{J}{N} \sum_{i,j}^N S_i^z S_j^z. \quad (44)$$

Notice that the coupling constant of the qubit to the environment has been denoted by $J_0$, whereas the strength of the long range intra-bath interactions has been designated by $J$. Moreover, $\mu$ and $w$ are the strengths of the applied magnetic fields.

The bath’s Hamiltonian can be linearized using the mean field approximation; afterwards, the problem may be fully studied analytically as we shall see below. Indeed, the mean field
approximation yields the following form of the Hamiltonian

$$H_{B}^{mf} = -w \sum_{i=1}^{N} S_x^i - 2Jm \sum_{i=1}^{N} S_z^i + NJm^2,$$  \hspace{1cm} (45)

where $m$ is the order parameter of the phase transition, the value of which can be fixed by the self-consistency condition that arises from minimizing the free energy $F = -1/(N\beta) \ln Z_N$.

In the above, the partition function corresponding to the mean-field approximation Hamiltonian $H_{B}^{mf}$ is given by

$$Z_N = \text{tr} e^{-\beta H_{B}^{mf}}.$$  \hspace{1cm} (46)

The value of $m$ runs from $S$ to 0, provided that the temperature varies within the interval $0 \leq T \leq T_c$, where $T_c$ is a critical temperature that will be derived shortly.

It can be shown by a suitable rotation within the $x$-$z$ plane that for $S$ even (the case $S$ odd yields the same results):

$$Z_N = e^{-\beta m^2 JN} \prod_{k=1}^{N} \left[ 1 + \text{tr} \bigoplus_{\ell=1}^{S} \{ \cosh(\beta \ell \tan(\phi)) I_2 + \frac{\sinh(\ell \tan(\phi))}{\beta \tan(\phi)} \sigma_z \} \right],$$  \hspace{1cm} (47)

where $I_2$ refers to the two-dimensional unit matrix, and $\phi$ is given by

$$\tan(\phi) = \sqrt{w^2 + 4J^2 m^2} := \Theta.$$  \hspace{1cm} (48)

It follows that

$$Z_N = e^{-\beta m^2 JN} \left[ 1 + 2 \sum_{\ell=1}^{S} \cosh(\ell \beta \Theta) \right]^N.$$  \hspace{1cm} (49)

To evaluate the above expression, it suffices to use the exponential form of the $\cosh$ function; then one gets two sums involving geometric series, that can easily be calculated to yield

$$Z_N = e^{-\beta m^2 JN} \left[ 1 + 2 \cosh \left( \frac{S+1}{2} \beta \Theta \right) \left( \frac{\sinh(S \beta \Theta/2)}{\sinh(\beta \Theta/2)} \right) \right]^N.$$  \hspace{1cm} (50)

By minimizing the free energy, we obtain the following self-consistency equation

$$\frac{\Theta}{J} = \frac{S \sinh(\beta(S+1)\Theta) - (S+1) \sinh(\beta S \Theta)}{\sinh(\beta \Theta/2) \sinh(\beta(2S+1)\Theta/2)}.$$  \hspace{1cm} (51)

As a consequence, the critical point at which the phase transition occurs is given by

$$T_c = \frac{2JS(S+1)}{3k_B}.$$  \hspace{1cm} (52)
Remarkably, this result, which is based on the mean field approximation, agrees with the probability distribution \( P(j) \), shown in equation (22). Indeed, to ensure the convergence of the expectation value of the bath’s function \( \exp\{J\beta j^2/N\} \), we should have \( J\beta/N < 3/(2NS(S + 1)) \), which yields exactly the same critical temperature as (52). A similar expression of the critical temperature has been derived in Ref. 34, a close one is reported in Ref. 29 where the mean field theory and semiclassical arguments are used. This dependence on the spin is obvious since the size of the spin space increases, leading to an enlargement of the ordered phase, and hence an increase of the critical temperature.

The ordered phase corresponds to the temperature interval \( 0 \leq T \leq T_c \); in this phase, the supplementary condition

\[
\frac{w}{J} < \frac{S \sinh(\beta(S + 1)w) - (S + 1) \sinh(\beta Sw)}{\sinh(\beta w/2) \sinh(\beta(2S + 1)w/2)}
\]

is satisfied.

B. Coherence evolution

Without loss of generality, we suppose that \( S = 1 \); this particular case captures the main features of the dynamics, and is relatively simple to study analytically. Larger values of \( S \) show the same behavior, but are a bit more complicated since the mathematical formulas are more cumbersome (see Appendix A for \( S = \frac{3}{2} \) and \( S = 2 \)). The physical conclusions drawn from all these cases are the same.

If the spin bath is in thermal equilibrium at temperature \( T \), then under the mean-field approximation, the evolution in time of the reduced density matrix of the central qubit is described by

\[
\rho(t) = \text{tr}_B \sum_{\ell,n} \frac{\rho_{\ell n}(0)}{Z_N} \exp \left\{ -it \left( \mu S_z^0 - \frac{J_0}{\sqrt{N}} \sum_{i=1}^{N} S_i^z - w \sum_{i=1}^{N} S_i^x - 2Jm \sum_{i=1}^{N} S_i^z + NJm^2 \right) \right\} \\
\times |\ell\rangle \langle n| \exp \left\{ -\beta \left( -w \sum_{i=1}^{N} S_i^i - 2Jm \sum_{i=1}^{N} S_i^z + NJm^2 \right) \right\} \\
\times \exp \left\{ it \left( \mu S_z^0 - \frac{J_0}{\sqrt{N}} \sum_{i=1}^{N} S_i^z - w \sum_{i=1}^{N} S_i^x - 2Jm \sum_{i=1}^{N} S_i^z + NJm^2 \right) \right\},
\]

(54)
where $\ell, n \equiv \pm 1$, and $S_0^z|\ell\rangle = \ell/2|\ell\rangle$. In particular:

$$
\rho_{12}(t) \equiv \rho_{+-}(t) = \rho_{+-}(0)g(t),
$$

(55)

where the function $g$ describes the decoherence of the qubit and is defined by

$$
g(t) = \frac{1}{[1 + 2 \cosh(\beta\Theta)]^N} \prod_{k=1}^{N} \text{tr} \exp \left\{ it \left[ \left( \frac{J_0}{2\sqrt{N}} + 2Jm \right) S_z^k + wS_x^k \right] \right\} 
\times \exp \left\{ \beta(wS_x^k + 2JmS_z^k) \right\} \exp \left\{ it \left[ \left( \frac{J_0}{2\sqrt{N}} - 2Jm \right) S_z^k - wS_x^k \right] \right\}.
$$

(56)

We see that the magnetic field $\mu$ does not affect the off-diagonal elements of the reduced density function; this is the reason for which we neglect it in the subsequent discussion.

The function $g(t)$ can be calculated as follows. First let us introduce the operator

$$
G = \alpha S_z + \gamma S_x,
$$

(57)

where $\alpha, \gamma$ are complex numbers, and

$$
S_x = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}, \quad S_z = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}.
$$

(58)

Then, it can be shown by induction that for $k \neq 0$,

$$
G^{2k} = (\alpha^2 + \gamma^2)^{k-1}G^2, \quad G^{2k+1} = (\alpha^2 + \gamma^2)^kG.
$$

(59)

Therefore,

$$
e^{-\kappa G} = I_3 + \left( \frac{\cosh(\kappa \sqrt{\alpha^2 + \gamma^2}) - 1}{\alpha^2 + \gamma^2} \right) G^2 - \left( \frac{\sinh(\kappa \sqrt{\alpha^2 + \gamma^2})}{\sqrt{\alpha^2 + \gamma^2}} \right) G,
$$

(60)

where $I_3$ denotes the three-dimensional unit matrix and $\kappa \in \mathbb{C}$.

Taking into account the formula (60) we can show that equation (56) reduces, after neglecting the $O(1/N)$ terms, to

$$
g(t) = \frac{1}{[1 + 2 \cosh(\beta\Theta)]^N} \left\{ \left( 2 \cosh(\beta\Theta/2) \right)^2 \left[ \cos \left( \frac{JJ_0mt}{\Theta \sqrt{N}} \right) + i \frac{\Theta}{J} \sin \left( \frac{JJ_0mt}{\Theta \sqrt{N}} \right) \right]^2 - 1 \right\}^N.
$$

(61)
Expanding the cosine and the sine functions in Taylor series, we find that the modulus of $g$ is given by

$$|g(t)|^2 = \left[ \frac{\left(2 \cosh(\frac{\beta \Theta}{2})\right)^2 - 1}{1 + 2 \cosh(\beta \Theta)} \right]^{2N} \left\{ 1 + \frac{8m^2 J_0^2 J^2 t^2 \cosh(\frac{\beta \Theta}{2})^2}{[1 + 2 \cosh(\beta \Theta)]^2 \Theta^2 N} \right\} \times \left[ 3 + 2 \cosh(\beta \Theta) \right] \frac{\Theta^2}{J^2} - 1 - 2 \cosh(\theta) + O(1/N^2) \right\}^N. \quad (62)$$

We notice that the quantity within the square braces in the latter equation is identically equal to one. Hence, by taking the limit $N \to \infty$, it turns out that

$$\lim_{N \to \infty} |g(t)|^2 = \exp \left\{ -\frac{8m^2 J_0^2 J^2 t^2 \cosh(\frac{\beta \Theta}{2})^2}{[1 + 2 \cosh(\beta \Theta)]^2} \times \left[ 3 + 2 \cosh(\beta \Theta) \right] \frac{\Theta^2}{J^2} - 2 \cosh(\beta \Theta) - 3 \right\}. \quad (63)$$

At first sight, the above formula reveals that the evolution exhibits a Gaussian behavior, as expected. Nevertheless, a careful investigation shows that in order to ensure the decay of the modulus of $g$, the following condition has to be met

$$\frac{\Theta^2}{J^2} < \frac{1 + 2 \cosh(\beta \Theta)}{3 + 2 \cosh(\beta \Theta)}, \quad (64)$$

otherwise the operator $\rho$ cannot be considered as a density matrix. This indicates that the mean-field approximation breaks down if the above condition is violated. For the numerical calculations, we have to take into account the condition (64) together with the self-consistency equation (and the corresponding inequality for $w$)

$$\frac{\Theta}{J} = \frac{4\sinh(\beta \Theta)}{1 + 2 \cosh(\beta \Theta)}. \quad (65)$$

Some comments are in order here. First of all, the latter equation is used to calculate numerically the order parameter $m$ for any particular values of the model parameters, which is necessary for the subsequent discussion. This means that the values of $m$ depend on the values of the other parameters. As the temperature increases, $m$ decreases, and the Gaussian decay becomes faster, and vice versa. Second, the time parameter will be given in units of the coupling constant $J_0$ which is very convenient as clearly noticed by inspecting the derivations presented above; to interpret the obtained results, it simply suffices to note
that as $J_0$ increases, the Gaussian decay becomes faster and *vice versa*. Third, as will be shown below, the condition \((64)\) determines only the ranges of the model parameters that yield a finite variation of the density matrix; it does not guarantee that the results of the mean-field approximation reproduce the exact dynamics.

![Graph showing $|g(t)|$ as a function of the scaled time $J_0 t$ for a transverse Ising bath of $N = 10000$ spin-1 particles with $T = 2.52$ (solid line), and $T = 2.54$ (dot-dashed line). The other parameters are $w = 1$, $J = 2$. These values yield the order parameters: $m = 0.280$ for $T = 2.52$, and $m = 0.245$ for $T = 2.54$. Note that we set $k_B = 1$.](image)

FIG. 3: (Color online) $|g(t)|$ as a function of the scaled time $J_0 t$ for a transverse Ising bath of $N = 10000$ spin-1 particles with $T = 2.52$ (solid line), and $T = 2.54$ (dot-dashed line). The other parameters are $w = 1$, $J = 2$. These values yield the order parameters: $m = 0.280$ for $T = 2.52$, and $m = 0.245$ for $T = 2.54$. Note that we set $k_B = 1$.

In figure 3 we display the variation of $g(t)$ as a function of time for two different values of the temperature. It is clearly noticed that the decay is Gaussian and gets faster as the temperature approaches $T_c$. This behavior is identical to that corresponding to a spin bath that is composed of spin-$\frac{1}{2}$ particles. Nonetheless, it should be stressed that the direct comparison of the decay in both cases is not possible, since the critical temperatures are not the same. Indeed, even when the baths have the same size, and the same coupling constants, the temperature ranges corresponding to the ordered phase when $S = 1$ and $S = \frac{1}{2}$ do not match. Moreover, it is evident that the order parameters are different.

Despite the breakdown of the mean-field approximation, one can draw some important conclusions from the case of a vanishing transverse magnetic field, that is, when $w = 0$. This instance can be studied exactly without resorting to any kind of approximations. Here the function $g(t)$ is given by
FIG. 4: (Color online) $|g(t)|$ as a function of the scaled time $J_0 t$ when $w = 0$ for $S = \frac{1}{2}$ and $S = 1$. Here $N = 10$ and $J = T$. Here we take $k_B = 1$.

FIG. 5: (Color online) The same as Fig. 4 but for $S = \frac{3}{2}$ and $S = 2$.

\[
g(t) = \frac{\sum_j^N \nu(j, N; S) \sum_{\ell=-j}^j \exp\left\{i J_0 \ell t / \sqrt{N} + \beta J \ell^2 / N\right\}}{\sum_j^N \nu(j, N; S) \sum_{\ell=-j}^j \exp\left\{\beta J \ell^2 / N\right\}},
\]  

which can be evaluated numerically for arbitrary values of the coupling constants. An example of the variation in time of the modulus of the function $g(t)$ is depicted in figures 4 and 5 for several values of the spin $S$ when the size of the spin bath is $N = 10$; it can be seen that the loss of coherence gets faster as $S$ increases. The same result is found to be valid for any higher value of the spin. This is identical to the short-time behavior associated with the dynamics of the qubit when it is coupled to the environment through Heisenberg $XY$ interactions. The other difference that can be inferred from the above figure is the presence of a oscillatory periodic collapse and revival of the quantum coherence for finite
size of the environment. These oscillations are mainly due to the competition between the two terms within the exponential function in the numerator of the expression of $g(t)$. For a given $T$, they become more important as the size of the bath increases as shown in Figure 6. The latter behavior is a mere manifestation of the coupling between the qubit and the environmental constituents which is proportional to the magnitude of the spin. We notice also that the aforementioned oscillations get more and more suppressed as the temperature increases. This suppression is, however, quicker for small $S$ and is slower for larger $S$ which explains why they persist when $S > 1/2$.

![Figure 6](image_url)  

**FIG. 6:** (Color online) $|g(t)|^2$ as a function of the scaled time $J_0 t$ when $w = 0$ for some values of the spin $S$ with $N = 100$ and $J = T$ (we set $k_B = 1$).

When the temperature and the size of the spin bath become sufficiently large, the oscillations completely disappear, and the Gaussian decay becomes dominant. Indeed, for example, at high temperature we have

$$g(t) \approx \left[ \cos \left( \frac{J_0 t}{2\sqrt{N}} \right) \right]^N,$$

when $S = \frac{1}{2}$, whereas for $S = 1$,

$$g(t) \approx 3^{-N} \left[ 1 + 2 \cos \left( \frac{J_0 t}{\sqrt{N}} \right) \right]^N.$$

It follows that when $N$ becomes sufficiently large, the decay is described by the laws:

$$g(t) = \begin{cases} 
e^{-\frac{J^2 t^2}{8}} & \text{for } S = \frac{1}{2}, \\ e^{-\frac{J^2 t^2}{8}} & \text{for } S = 1, \end{cases}$$

which confirms the above observations.
FIG. 7: (Color online) Comparison between the exact (solid line) and the mean-field (dashed line) solutions for $|g(t)|$ when $w = 0$ for a spin bath of spin-1 particles. The parameters are $N = 100$, $J = 3$ and $T = 3.8$; the corresponding critical temperature is $T_c = 4$, and the order parameter is $m = 0.358$.

For completeness, we have compared the results of the mean-field approximation with the exact one when $w = 0$, see Fig. 7 for an example; it is clearly seen that although the mean-field solution produces a finite variation of the coherence, it fails to fairly reproduce quantitatively the exact dynamics even when condition (64) is satisfied. To explain the reason for which the mean-field approximation works for $S = \frac{1}{2}$ and fails for the other values, we note that the only possible values of $S_z$ (which determines the order parameter $m$) in the former case are $\pm \frac{1}{2}$, that is they have the same magnitude. For larger values of $S$, this kind of uniqueness is lost, since $|S_z| = S, S - 1 \cdots 0(1/2)$, meaning that several values of the spin contribute asymmetrically with different statistical weights to the value of the function $g(t)$. Let us stress, in the end, that the mean-field theory, as any other approximation, may or may not reproduce the actual underlying physics of the studied system. It remains, however, a useful tool to study, at least qualitatively, the dynamics in many cases.

IV. CONCLUSION

Throughout this paper, we have investigated several features of the decoherence of a qubit, due to its coupling to a spin environment whose constituents are spin-$S$ particles. To achieve this aim, we have considered two particular types of the interaction between the
central system and its surrounding, namely, Heisenberg $XY$ and Ising interactions. In the first case, the form of the adopted Hamiltonian led to the derivation of the degeneracy and the probability distribution of the total angular momentum, obtained by summing the spin vectors of $N$ independent spin-$S$ particles. When the size of the spin bath is sufficiently large, the distribution turns out to be Gaussian, and we have determined its explicit analytical form. Because of the rescaling of the coupling constants of the model, the obtained results are well-behaved when $N$ is large. The long-time behavior has been studied analytically thanks to the probability distribution $P(j)$. The investigation shows that the asymptotic value of the coherence increases with the magnitude of $S$, indicating that the recovery of the quantum interferencies is much better in environments for which $S$ is large. On the contrary, the short time decay is found to be inversely proportional to the spin.

In the case of a transverse Ising spin bath, we used the mean-field approximation to determine the ordered phase, and to identify the corresponding critical temperature; the obtained results agree with those found by making use of the probability distribution of the total spin angular momentum. We have shown by analytical calculations that the mean-field approximation fails to provide us with acceptable physical results regarding the dynamics of the qubit. Indeed, we found that if the problem parameters do not satisfy certain additional conditions, then the off-diagonal elements of the reduced density matrix diverge as the time increases which is, clearly, unphysical. When the transverse magnetic field vanishes, it is shown that a collapse and a revival of the quantum coherence occurs if the size of the bath is not too large. As the size of the spin environment as well as the temperature increase, these oscillations vanish, and the decay is purely Gaussian. It is worthwhile mentioning at the end that the value of the critical temperature depends quadratically on $S$. This is quite important from a practical point of view, since working at low temperatures requires cooling down the whole system; it is obvious that the greater $S$, the larger $T_c$ is, meaning that the system may be exploited for eventual applications at relatively higher temperatures.

The results reported in this study provide more insights into the effect of the spin environments on the dynamics of the quantum bits, which would contribute to the understanding of the decoherence process, and to the quest for reliable experimental techniques that enable one to minimize this undesirable phenomena.
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Appendix A: Mean-field expression of \( g(t) \) for \( S = \frac{3}{2} \), and \( S = 2 \)

In this appendix we display the explicit form of the function \( g \) defined in the main text by means of equation (56) when \( S = \frac{3}{2} \), and \( S = 2 \).

1. \( S = \frac{3}{2} \)

In this case,

\[
g(t) = \frac{\cosh(\beta \Theta / 2)^N}{\left( \cosh(3\beta \Theta / 2) + \cosh(\beta \Theta / 2) \right)^N} \left[ \cos \left( \frac{JJ_0 t \Theta}{\Theta \sqrt{N}} \right) + i \frac{\Theta}{J} \sin \left( \frac{JJ_0 t \Theta}{\Theta \sqrt{N}} \right) \right]^N
\]

\[
\times \left\{ \left( 2 \cosh(\beta \Theta / 2) \right)^2 \cos \left( \frac{JJ_0 t \Theta}{\Theta \sqrt{N}} \right) + i \frac{\Theta}{J} \sin \left( \frac{JJ_0 t \Theta}{\Theta \sqrt{N}} \right) \right\}^2 - 2 \right\}^N. \tag{A1}
\]

From the above expression, we find that the modulus of \( g \) is given by

\[
|g(t)|^2 = \left\{ 1 + \frac{m^2 J_0^2 J^2 t^2}{2 \left[ \cosh(\beta \Theta) \right]^2 \Theta^2 N} \left[ 11 + 12 \cosh(\beta \Theta) + 3 \cosh(2\beta \Theta) \right] \frac{\Theta^2}{J^2}
\]

\[
- \left[ 3 + 4 \cosh(\beta \Theta) + 3 \cosh(2\beta \Theta) \right] + O(1/N^2) \right\}^N. \tag{A2}
\]

It follows that

\[
\lim_{N \to \infty} |g(t)|^2 = \exp \left\{ - \frac{m^2 J_0^2 J^2 t^2}{2 \left[ \cosh(\beta \Theta) \right]^2} \left[ 3 + 4 \cosh(\beta \Theta) + 3 \cosh(2\beta \Theta) \right] \frac{J^2}{\Theta^2}
\]

\[
- \left[ 11 + 12 \cosh(\beta \Theta) + 3 \cosh(2\beta \Theta) \right] \right\}. \tag{A3}
\]

To ensure the Gaussian decay we should have

\[
\frac{\Theta^2}{J^2} < \frac{3 + 4 \cosh(\beta \Theta) + 3 \cosh(2\beta \Theta)}{11 + 12 \cosh(\beta \Theta) + 3 \cosh(2\beta \Theta)}. \tag{A4}
\]
2. \( S = 2 \)

Here the function \( g \) reads

\[
g(t) = \left\{ -2 + \left(2 \cosh(\beta \Theta/2)\right)^4 \left[ \cos\left(\frac{JJ_0 t}{\Theta \sqrt{N}}\right) + i \frac{\Theta}{J} \sin\left(\frac{JJ_0 t}{\Theta \sqrt{N}}\right) \right]^4 - 3 \left( -1 + \left(2 \cosh(\beta \Theta/2)\right)^2 \left[ \cos\left(\frac{JJ_0 t}{\Theta \sqrt{N}}\right) + i \frac{\Theta}{J} \sin\left(\frac{JJ_0 t}{\Theta \sqrt{N}}\right) \right]^2 \right) \right\}^N \times \frac{1}{\left(1 + 2 \cosh(\beta \Theta) + 2 \cosh(2 \beta \Theta)\right)^N}.
\]

This yields

\[
|g(t)|^2 = \left\{ 1 + \frac{8m^2 J^2 J_0^2 t^2 (\cosh(\beta \Theta/2))^2}{[1 + 2 \cosh(\beta \Theta) + 2 \cosh(2 \beta \Theta)]^2 \Theta^2 N} \times \left( [31 + 42 \cosh(\beta \Theta) + 18 \cosh(2 \beta \Theta) + 4 \cosh(3 \beta \Theta)] \frac{\Theta^2}{J^2} - [5 + 10 \cosh(\beta \Theta) + 6 \cosh(2 \beta \Theta) + 4 \cosh(3 \beta \Theta)] \right) \right\}^N + O(1/N^2).
\]

Therefore:

\[
\lim_{N \to \infty} |g(t)|^2 = \exp\left\{ -\frac{8m^2 J^2 J_0^2 t^2 (\cosh(\beta \Theta/2))^2}{[1 + 2 \cosh(\beta \Theta) + 2 \cosh(2 \beta \Theta)]^2} \times \left( [31 + 42 \cosh(\beta \Theta) + 18 \cosh(2 \beta \Theta) + 4 \cosh(3 \beta \Theta)] \frac{J^2}{\Theta^2} - [5 + 10 \cosh(\beta \Theta) + 6 \cosh(2 \beta \Theta) + 4 \cosh(3 \beta \Theta)] \right) \right\},
\]

meaning that

\[
\frac{\Theta^2}{J^2} < \frac{5 + 10 \cosh(\beta \Theta) + 6 \cosh(2 \beta \Theta) + 4 \cosh(3 \beta \Theta)}{31 + 42 \cosh(\beta \Theta) + 18 \cosh(2 \beta \Theta) + 4 \cosh(3 \beta \Theta)}.
\]

Appendix B: Evolution of the bosonic system defined by equations (30) and (31)

The aim of this appendix is to investigate the dynamics of the qubit using the bosonic model

\[
H_S = \mu \sigma_z, \tag{B1}
\]

\[
H_B = 2gSB^\dagger B, \tag{B2}
\]

\[
H_{SB} = 2\alpha \sqrt{2} S(\sigma_- B^\dagger + \sigma_+ B), \tag{B3}
\]
which has been obtained via the Holstein-Primakoff transformation. The time evolution operator corresponding to the above system can be determined using the Schrödinger equation:

\[
    i \frac{dU(t)}{dt} = (H_S + H_{SB} + H_B)U(t).
\]  

(B4)

In the standard basis of \( \mathbb{C} \), we find that the components of the above operator are given by

\[
    U_{11}(t) = e^{-4igt\hat{n} \cdot \frac{1}{2}} \left[ \cos(tM_1) - \frac{i(gS - \mu)}{M_1} \sin(tM_1) \right],
\]

(B5)

\[
    U_{22}(t) = e^{-4igt\hat{n} \cdot \frac{1}{2}} \left[ \cos(tM_2) + \frac{i(gS - \mu)}{M_2} \sin(tM_2) \right],
\]

(B6)

\[
    U_{12}(t) = U_{21}(t) = -4i\sqrt{2S}B^\dagger e^{-i(\mu+2g\hat{n})t/2} \sin(tM_2)/M_2,
\]

(B7)

where

\[
    \hat{n} = B^\dagger B, 
\]

(B8)

\[
    M_1 = \sqrt{(gS - \mu)^2 + 8\alpha^2 S\hat{n}},
\]

(B9)

\[
    M_2 = \sqrt{(gS - \mu)^2 + 8\alpha^2 S(\hat{n} + 1)}.
\]

(B10)

The partition function of the bath, which is assumed at inverse temperature \( \beta \) can easily be calculated as:

\[
    Z = \sum_{n=0}^{\infty} e^{-2Sg\beta n} = \frac{e^{2S\beta}}{e^{2S\beta} - 1}.
\]

(B11)

Hence, assuming a factorized initial state, the evolution in time of the coherence in this approximation is described by

\[
    \rho_{12}(t) = \rho_{12}(0)e^{-i\alpha^2 t(1 - e^{-2g\beta})} \sum_{n=0}^{\infty} \left[ e^{-2g\beta n} \langle n | U_{11}(t) U_{22}^\dagger(t) | n \rangle \right].
\]

(B12)

For convenience, we displayed in figure 8 an example of the dependence on time of the modulus of \( \rho_{12} \); it can be seen that the latter does not assume constant values at long times, but rather oscillates as the time increases. This is due to the fact that all the terms that depended on the spin under the square root in the Holstein-Primakoff transformation have been omitted. We can however assure that at long times, \( \rho_{12} \) assumes larger values as the magnitude of the spin increases. At short time, the decay is almost independent of the value of \( S \).
FIG. 8: (Color online) Evolution in time of the modulus of $\rho_{12}$ given by Eq. (B12) for: $S = 5$ (dotted line), $S = 8$ (dashed line), and $S = 12$ (solid line). The other parameters are $g = 1$, $\beta = 0.01$, $\mu = 3$ and $\alpha = 0.5$. Note that $\rho_{12}(0)$ is normalized to unity for convenience, and the time is given in unit of $\alpha^{-1}$.

[1] N. V. Prokof’ev and P. C. E. Stamp, Rep. Prog. Phys. 63, 669 (2000).
[2] W. H. Zurek, Phys. Today 44, No. 10, 36 (1991).
[3] D. P. DiVincenzo and D. Loss, J. Magn. Magn. Matter. 200, 202 (1999).
[4] W. H. Zurek, Rev. Mod. Phys. 75, 715-775 (2003).
[5] H. P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford, 2002).
[6] A. V. Khaetskii, D. Loss, and L. Glazman, Phys. Rev. Lett. 88, 186802 (2002).
[7] W. A. Coish and D. Loss, Phys. Rev. B 70, 195340 (2004).
[8] W. Zhang, V. V. Dobrovitski, K. A. Al-Hassanieh, E. Dagotto, and B. N. Harmon, Phys. Rev. B 74, 205313 (2006).
[9] L. Amico, R. Fazio, A. Osterloh, and V. Vacr dal, Rev. Mod. Phys. 80, 517-576 (2008).
[10] X. Z. Yuan, H. S Goan, and K. D. Zhu, Phys. Rev. B 75, 045331 (2007).
[11] Z. Huang, G. Sadiek and S. Kais J. Chem. Phys. 124, 144513 (2006).
[12] B. Alkurtass, G. Sadiek, and S Kais, Phys. Rev. A 84, 022314 (2011).
[13] H. P. Breuer, D. Burgarth and F. Petruccione, Phys. Rev. B 70, 045323 (2004).
[14] Y. Hamdouni, M. Fannes, and F. Petruccione, Pys. Rev. B 73, 245323 (2006).
[15] Y. Hamdouni and F. Petruccione, Phys. Rev. B 76, 174306 (2007).
[16] Y. Hamdouni, J. Phys. A: Math. Theo. 40, 11569 (2007); Y. Hamdouni, J. Phys. A: Math. Theo. 42, 315301 (2009); Y. Hamdouni, Phys. Lett. A 373, 1233-1238 (2009); Y. Hamdouni, J. Phys. A: Math. Theo. 45, 425301 (2012).
[17] M. Lucamarini, S. Paganelli, and S. Mancini, Phys. Rev. A 69, 062308 (2004).
[18] D. Loss and D. P. DiVincenzo Phys. Rev. A 57, 120(1998).
[19] G. Burkard, D. Loss, and D. P. DiVincenzo, Phys. Rev. B 59, 2070 (1999).
[20] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[21] P. W. Shor, Phys. Rev. A 52, R2493 (1995).
[22] A. Barenco, D. Deutsch, A. Ekert, and R. Jozsa, Phys. Rev. Lett. 74 4083 (1995).
[23] J. I. Cirac and P. Zoller, Phys. Rev. Lett. 74 4091 (1995).
[24] J. A. Jones, M. Mosca, and R. H. Hansen, Nature 393 344 (1995).
[25] I. L. Chuang, N. Gershenfeld, and M. Kubinec, Phys. Rev. Lett 80 3408 (1998).
[26] L. Bogani and W. Wernsdorfer, Nat. Mater. 7 179 (2008).
[27] M. N. Leuenberger and D. Loss, Nature 410 789 (2001).
[28] W. Von Waldenfels, Séminaire de probabilité (Starsburg), tome(24), p.349-356 (Springer-Verlag, Berlin, 1990).
[29] S. Das Sarma, E. H. Hwang, and A. Kaminski, Phys. Rev. B 67 155201 (2003).
[30] D. A. Garanin, K. Kladko, and P. Fulde, Eur. Phys. J. B 14 293 (2000).
[31] T. Holstein and H. Primakoff, Phys. Rev. 58 1098 (1940).
[32] E. T. Jaynes and F. W. Cummings, Proc. IEEE 51 89 (1963).
[33] N. Majlis, The Quantum Theory of Magnetism (World Scientific, New Jersey, 2007).
[34] C. Kittel, Introduction to Solid State Physics (Wiley, New York, 1953).