We give two classes of spherically symmetric exact solutions of the couple gravitational and electromagnetic fields with charged source in the tetrad theory of gravitation. The first solution depends on an arbitrary function $H(R, t)$. The second solution depends on a constant parameter $\eta$. These solutions reproduce the same metric, i.e., the Reissner–Nordström metric. If the arbitrary function which characterizes the first solution and the arbitrary constant of the second solution are set to be zero, then the two exact solutions will coincide with each other. We then calculate the energy content associated with these analytic solutions using the superpotential method. In particular, we examine whether these solutions meet the condition which Møller required for a consistent energy-momentum complex: Namely, we check whether the total four-momentum of an isolated system behaves as a four-vector under Lorentz transformations. It is then found that the arbitrary function should decrease faster than $1/\sqrt{R}$ for $R \to \infty$. It is also shown that the second exact solution meets the Møller's condition.
1. Introduction

At present, teleparallel theory seems to be popular again, and there is a trend of analyzing the basic solutions of general relativity with teleparallel theory and comparing the results. It is considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory [1]∼[7] or metric-affine gravity [8] as well as a possible physical relevant geometry by itself-teleparallel description of gravity [9, 10]. Teleparallel approach is used for positive-gravitational-energy proof [11]. A relation between spinor Lagrangian and teleparallel theory is established [12]. It has been shown that the teleparallel equivalent of general relativity (TEGR) is not consistent in presence of minimally coupled spinning matter [13]. Demonstration of the consistency of the coupling of the Dirac fields to the TEGR has been done [14]. However, it has been shown that this demonstration is not correct [15, 16].

The tetrad theory of gravitation based on the geometry of absolute parallelism [17]∼[24] can be considered as the closest alternative to general relativity, and it has a number of attractive features both from the geometrical and physical viewpoints. Absolute parallelism is naturally formulated by gauging spacetime translations and underlain by the Weitzenböck spacetime, which is characterized by the metric condition and by the vanishing of the curvature tensor. Translations are closely related to the group of general coordinate transformations which underlies general relativity. Therefore, the energy-momentum tensor represents the matter source in the field equation for the gravitational field just like in general relativity.

The tetrad formulation of gravitation was considered by Møller in connection with attempts to define the energy of gravitational field [25]∼[27]. For a satisfactory description of the total energy of an isolated system it is necessary that the energy density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well-known that there exists no covariant, nontrivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum are related to the geometrical description of the gravitational field rather than are an intrinsic drawback of the theory [28, 29].

Møller proposed [26] the three conditions which any energy-momentum complex must satisfy:
(1) It must be an affine tensor density which satisfies the conservation law.
(2) For an isolated system the four-momentum is constant in time and transform as a 4-vector under linear coordinate transformations.
(3) The superpotential transforms as a tensor density of rank 3 under the group of the spacetime transformations.
Then he showed [27] that such an energy-momentum complex can be constructed in the tetrad theory of gravitation.

It is the aim of the present work to find spherically symmetric solutions in the tetrad theory of gravitation for the coupled gravitational and electromagnetic fields. We obtain two classes of exact analytic solutions, and then calculate the energy of these solutions using the superpotential given by Møller [27] and Mikhail et.al. [30]. We shall then confirm that these solutions meet the Møller’s conditions when the asymptotic conditions are imposed appropriately.
The general form of the tetrad field, $b_\mu^i$, having spherical symmetry was given by Robertson [31]. In the quasi-orthogonal coordinate system it can be written as:

$$
\begin{align*}
    b_0^0 &= A, \\
    b_\alpha^0 &= C x^\alpha, \\
    b_0^\alpha &= D x^\alpha \\
    b_\alpha^\alpha &= \delta_\alpha^\alpha B + F x^\alpha x^\alpha + \epsilon_{\alpha\beta\gamma} S x^\beta,
\end{align*}
$$

(1)

where $A$, $C$, $D$, $B$, $F$, and $S$ are functions of $t$. It can be shown that the functions $D$ and $F$ can be eliminated by coordinate transformations [19, 32], i.e., by making use of freedom to redefine $t$ and $r$, leaving the tetrad field (1) having four unknown functions in the quasi-orthogonal coordinates. Thus the tetrad field (1) without the functions $D$ and $F$ will be used in the following sections for the calculations of the field equations of gravity and electromagnetism but in the spherical polar coordinate.

In §2 we derive the field equations for the coupled gravitational and electromagnetic fields. In §3 we first apply the tetrad field (1) without the $S$-term to the derived field equations. We then give derivation for the general solution without the $S$-term, and express the exact solution in terms of an arbitrary function denoted by $H(R, t)$. A relation between this solution and a previous one [33] is also established in §3. We also study the general, spherically symmetric solution with a non-vanishing $S$-term in §3. In §4 we calculate the energy content of these two exact analytic solutions. Following Møller [26], we require that the total four-momentum of an isolated system be transformed as a four-vector under global, linear coordinate transformations. Using Lorentz transformations we show that the arbitrary function $H(R, t)$ should decrease faster than $1/\sqrt{R}$ for $R \to \infty$. We also examine the asymptotic behavior of the solution with the non-vanishing $S$-term and we find that its associated energy is consistent with the Møller’s condition. The final section is devoted to discussion and conclusion.

2. The tetrad theory of gravitation and electromagnetism

In the Weitzenböck spacetime the fundamental field variables describing gravity are a quadruplet of parallel vector fields [19] $b_\mu^i$, which we call the tetrad field in this paper, characterized by

$$
D_\nu b_\mu^i = \partial_\nu b_\mu^i + \Gamma^\mu_{\lambda\nu} b_\lambda^i = 0,
$$

(2)

where $\Gamma^\mu_{\lambda\nu}$ define the nonsymmetric affine connection coefficients. The metric tensor $g_{\mu\nu}$ is given by $g_{\mu\nu} = \eta_{ij} b_\mu^i b_\nu^j$ with the Minkowski metric $\eta_{ij} = \text{diag}(-1, +1, +1, +1)$. Equation (2) leads to the metric condition and the identically vanishing curvature tensor.

The gravitational Lagrangian $L_G$ is an invariant constructed from $g_{\mu\nu}$ and the contorsion tensor $\gamma_{\mu\nu\rho}$ given by

$$
\gamma_{\mu\nu\rho} = b_\mu^i b_\nu^j b_\rho^k.
$$

(3)

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general gravitational Lagrangian density invariant under parity operation is given by the

*In this paper Latin indices ($i, j, ...$) represent the vector number, and Greek indices ($\mu, \nu, ...$) represent the vector components. All indices run from 0 to 3. The spatial part of Latin indices are denoted by ($a, b, ...$), while that of Greek indices by ($\alpha, \beta, ...$).
form \[18, 19\]
\[ \mathcal{L}_G = \sqrt{-g} L_G = \sqrt{-g} (\alpha_1 \Phi^\mu \Phi_\mu + \alpha_2 \gamma^{\mu\nu\rho} \gamma_{\mu\nu\rho} + \alpha_3 \gamma^{\mu\nu\rho} \gamma_{\rho\mu\nu}) \]  
(4)

with \( g = \text{det}(g_{\mu\nu}) \) and \( \Phi_\mu \) being the basic vector field defined by \( \Phi_\mu = \gamma^\rho_{\mu\rho} \). Here \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) are constants determined such that the theory coincides with general relativity in the weak fields \[18, 27\]:

\[ \alpha_1 = -\frac{1}{\kappa}, \quad \alpha_2 = \frac{\lambda}{\kappa}, \quad \alpha_3 = \frac{1}{\kappa}(1 - \lambda), \]  
(5)

where \( \kappa \) is the Einstein constant and \( \lambda \) is a free dimensionless parameter\(^*\). The vanishing of this dimensionless parameter will reproduce the teleparallel equivalent theory of general relativity.

The electromagnetic Lagrangian density \( L_{e.m.} \) is \[22\]
\[ L_{e.m.} = -\frac{1}{4} g^{\rho\sigma} g^{\mu\nu} F_{\mu\nu} F_{\rho\sigma}, \]  
(6)

with \( F_{\mu\nu} \) being given by\(^\dagger\) \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

The gravitational and electromagnetic field equations for the system described by \( L_G + L_{e.m.} \) are the following:

\[ G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \]  
(7)

\[ K_{\mu\nu} = 0, \]  
(8)

\[ \partial_\nu \left( \sqrt{-g} F^{\mu\nu} \right) = 0 \]  
(9)

with \( G_{\mu\nu} \) being the Einstein tensor of general relativity. Here \( H_{\mu\nu} \) and \( K_{\mu\nu} \) are defined by

\[ H_{\mu\nu} = \lambda \left[ \gamma_{\rho\sigma\mu} \gamma^{\rho\sigma\nu} + \gamma_{\rho\sigma\mu} \gamma^{\rho\sigma\nu} + \gamma_{\rho\sigma\nu} \gamma^{\rho\sigma\mu} + g_{\mu\nu} \left( \gamma_{\rho\sigma\lambda} \gamma^{\rho\sigma\lambda} - \frac{1}{2} \gamma_{\rho\sigma\lambda} \gamma^{\rho\sigma\lambda} \right) \right], \]  
(10)

and

\[ K_{\mu\nu} = \lambda \left[ \Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_\rho \left( \gamma^{\rho}_{\mu\nu} - \gamma^{\rho}_{\nu\mu} \right) + \gamma_{\mu\nu}^{\rho} \right], \]  
(11)

and they are symmetric and antisymmetric tensors, respectively. The energy-momentum tensor \( T^{\mu\nu} \) is given by

\[ T^{\mu\nu} = -g_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \]  
(12)

It can be shown \[19\] that in spherically symmetric case the antisymmetric part of the field equation (8) implies that the axial-vector part of the torsion tensor, \( a_\mu = (1/3) \varepsilon_{\mu\rho\sigma} \gamma^{\rho\sigma} \), should be vanishing. Then the \( H_{\mu\nu} \) of (10) vanishes, and the field equations (7)–(9) reduce to the coupled Einstein-Maxwell equation in teleparallel equivalent of general relativity. The equation (7) then determines the tetrad field only up to local Lorentz transformations

\[ b^k_\mu \rightarrow \Lambda(x)^k_\ell b^\ell_\mu, \]

which retain the condition \( a_\mu = 0 \). Hereafter we shall refer to this property of the field equations as \textit{restricted local Lorentz invariance}.

\(^*\)Throughout this paper we use the relativistic units, \( c = G = 1 \) and \( \kappa = 8\pi \).

\(^\dagger\)Heaviside-Lorentz rationalized units will be used throughout this paper.
3. Family of Reissner-Nordström solutions

In this section we are going to study two cases of the tetrad field (1).

Case I: The vanishing S-term.

For the tetrad field (1) without the $S$-term the axial-vector part of the torsion tensor, $a_\mu$, is identically vanishing, and the remaining field equations possess the restricted local Lorentz invariance. Thus, the general solution for the tetrad field (1) without the $S$-term can be obtained from the diagonal tetrad field for the Reissner-Nordström metric by a local Lorentz transformation which keeps spherical symmetry [32]

\[
(\Lambda_{kl}) = \begin{pmatrix}
-L & H \sin \theta \cos \phi & H \sin \theta \sin \phi & H \cos \theta \\
-H \sin \theta \cos \phi & 1 + (L - 1) \sin^2 \theta \cos^2 \phi & (L - 1) \sin^2 \theta \sin \phi \cos \phi & (L - 1) \sin \theta \cos \theta \cos \phi \\
-H \sin \theta \sin \phi & (L - 1) \sin^2 \theta \sin \phi \cos \phi & 1 + (L - 1) \sin^2 \theta \sin^2 \phi & (L - 1) \sin \theta \cos \theta \sin \phi \\
-H \cos \theta & (L - 1) \sin \theta \cos \theta \cos \phi & (L - 1) \sin \theta \cos \theta \sin \phi & 1 + (L - 1) \cos^2 \theta
\end{pmatrix},
\]

where $H$ is an arbitrary function of $t$ and $R$, and

\[L = \sqrt{H^2 + 1}.
\]

Namely, we see that

\[b_k^\mu = \eta^{kl} \Lambda_{ik} b_l^{(0)\mu}
\]

is the most general, spherically symmetric solution without the $S$-term. Here $b_l^{(0)\mu}$ is the diagonal tetrad field which is given in the spherical polar coordinates by [34]

\[
(b_l^{(0)\mu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & X \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
0 & X \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \sin \theta \\
0 & X \cos \theta & -\sin \theta / R & 0
\end{pmatrix},\]

where $X$ and $R$ are defined by

\[X = \left[1 - \frac{2m}{R} + \frac{q^2}{R^2}\right]^{1/2}, \quad R = r / B.
\]
The explicit form of the $b_i^\mu$ is then given by

$$
(b_i^\mu) = \left( \begin{array}{cccc}
\frac{L}{X} & HX & 0 & 0 \\
\frac{H \sin \theta \cos \phi}{X} & \frac{LX \sin \theta \cos \phi}{R} & \frac{\cos \theta \cos \phi}{R \sin \theta} & -\frac{\sin \phi}{R \sin \theta} \\
\frac{H \sin \theta \sin \phi}{X} & \frac{LX \sin \theta \sin \phi}{R} & \frac{\cos \theta \sin \phi}{R} & \frac{\cos \phi}{R \sin \theta} \\
\frac{H \cos \theta}{X} & \frac{LX \cos \theta}{R} & -\frac{\sin \theta}{R} & 0
\end{array} \right).
$$

(17)

If we apply the tetrad field (17) to the field equations (7)∼(9) then, the vector potential $A_\mu$, the antisymmetric electromagnetic tensor $F_{\mu\nu}$ and $T_{\mu\nu}$ take the form

$$
A_t(R) = -\frac{q}{2\sqrt{\pi}R}, \quad F_{Rt} = -\frac{q}{2\sqrt{\pi}R^2}, \quad T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = \frac{q^2}{8\pi R^4},
$$

(18)

The metric associated with the tetrad field (17) is by definition given by the Reissner-Nordström solution.

Now let us compare the solution (17) with that given before: Nashed [33] obtained a solution with an arbitrary function $\mathcal{B}$ for the tetrad (1) with three unknown function in the spherical polar coordinate. The tetrad field of that solution can be obtained from (17) if the function $H$ is chosen as

$$
H = \left( \frac{R^2 \mathcal{B}^2 - 2R \mathcal{B}' + \frac{2m}{R} - \frac{q^2}{R^2}}{X} \right)^{1/2}.
$$

(19)

**Case II: The non-vanishing $S$-term.**

Let us next look for spherically symmetric solutions of the form (1) with non-vanishing $S$-term by using the result that the antisymmetric part of the field equation (8) requires the axial-vector part of the torsion tensor, $a_\mu$, to be vanishing for spherically symmetric case [19]. For this purpose we start with the tetrad field (1) with the six unknown functions of $t$ and $r$. In order to study the condition that the $a_\mu$ vanishes it is convenient to start from the general expression for the covariant components of the tetrad field $b_\mu^\alpha$,

$$
b_0^0 = -\ddot{A}, \quad b_0^\alpha = \ddot{C}x^\alpha, \quad b_\alpha^0 = -\ddot{D}x^\alpha, \\
b_\alpha^\alpha = \delta_{\alpha\beta}\ddot{B} + \ddot{F}x^\alpha x^\alpha + \epsilon_{\alpha\beta\gamma}\ddot{S}x^\gamma,
$$

(20)

where the six unknown functions, $\ddot{A}$, $\ddot{C}$, $\ddot{D}$, $\ddot{B}$, $\ddot{F}$, $\ddot{S}$, are connected with the six unknown functions of (1). We can assume without loss of generality that the two functions, $\ddot{D}$ and $\ddot{F}$, are vanishing by making use of the freedom to redefine $t$ and $r$ [19, 32]. We transform the tetrad field (20) to
the spherical polar coordinates \((t, r, \theta, \phi)\):

\[
(b_{\mu}) = \begin{pmatrix}
\bar{A} & 0 & 0 & 0 \\
r\hat{C}\sin\theta\cos\phi & B\sin\theta\cos\phi & r\hat{B}\cos\theta\cos\phi + r^2\hat{S}\sin\phi & -r\hat{B}\sin\theta\sin\phi + r^2\hat{S}\sin\theta\cos\theta\cos\phi \\
r\hat{C}\sin\theta\sin\phi & B\sin\theta\sin\phi & r\hat{B}\cos\theta\sin\phi - r^2\hat{S}\cos\phi & r\hat{B}\sin\theta\cos\phi + r^2\hat{S}\sin\theta\cos\theta\sin\phi \\
r\hat{C}\cos\theta & \hat{B}\cos\theta & -r\hat{B}\sin\theta & -r^2\hat{S}\sin^2\theta
\end{pmatrix}.
\]

The condition that the axial-vector part \(a_{\mu}\) vanishes is then expressed by [32]

\[
0 = \sqrt{(-g)}a^\mu = \begin{cases} 
3\hat{B}\hat{S} + r(\hat{B}\hat{S} - \hat{B}'\hat{S}), & \mu = 0, \\
2\hat{C}\hat{S} + (\hat{S}\hat{B} - \hat{S}\hat{B}), & \mu = 1
\end{cases}
\]

with \(\hat{S}' = d\hat{S}/dr\) and \(\hat{S} = d\hat{S}/dt\). This condition can be solved to give

\[
\hat{C} = 0, \quad \hat{S} = \frac{\eta}{r^3}\hat{B},
\]

where \(\eta\) is a constant with dimension of \((\text{length})^2\). The tetrad field (21) then gives the following expression for the line element:

\[
ds^2 = -\bar{A}^2dt^2 + \bar{B}^2dr^2 + r^2\bar{B}^2\left(1 + \frac{\eta^2}{r^4}\right)d^2\Omega.
\]

The symmetric part of the field equations now coincides with the Einstein equation. The metric tensor must be the Reissner-Nordström solution when the Schwarzschild radial coordinate \(R\) is used. Therefore we choose the new radial coordinate

\[
R = r\hat{B}\sqrt{1 + \frac{\eta^2}{r^4}},
\]

and require that the line-element written in the coordinate \((t, R, \theta, \phi)\) coincides with the Reissner-Nordström metric. Then we have

\[
\bar{A}(r) = X(R), \quad \frac{dR}{dr} = \bar{B}(r)X(R)
\]

where \(X(R)\) is defined by (16) with the constants \(m\) and \(q\) being interpreted as the total mass and the total charge, respectively, of the central body. Eliminating \(\bar{B}\) from (25) and the second equation of (26), we obtain a differential equation for \(R(r)\), which can easily be solved to give

\[
r^2 = |\eta|\sinh Y(R)
\]

with the function \(Y(R)\) being defined by

\[
Y(R) = 2\int \frac{dR}{RX} = \ln \left[\frac{(R - m + \sqrt{R^2 - 2mR + q^2})^2}{2|\eta|}\right],
\]

where \(X(R)\) is defined by (16) with the constants \(m\) and \(q\) being interpreted as the total mass and the total charge, respectively, of the central body. Eliminating \(\bar{B}\) from (25) and the second equation of (26), we obtain a differential equation for \(R(r)\), which can easily be solved to give
where the additive integration constant is fixed in the last equation by requiring the asymptotic condition \(r/R \to 1\) as \(R \to \infty\). Using (27) in (25) gives

\[r \dot{B}(r) = R \tanh Y(R),\]  

which together with (23) and (25) leads to

\[r^2 \dot{S} = \frac{\eta}{r^2} (r \dot{B}) = \frac{\eta R}{|\eta| \cosh Y(R)},\]  

(30)

Now it is straightforward to obtain the covariant components of the tetrad field, \(b^\mu_\nu\), with the non-vanishing \(S\)-term for the Reissner-Nordström solution in the coordinate system \((t, R, \theta, \phi)\):

The non-vanishing components are given by

\[b^0_0 = X \]
\[b^1_1 = \frac{\sin \theta \cos \phi}{X} \]
\[b^1_2 = R \left( \tanh Y \cos \theta \cos \phi + \frac{\eta \sin \phi}{|\eta| \cosh Y} \right) \]
\[b^1_3 = R \left( -\tanh Y \sin \phi + \frac{\eta \cos \theta \cos \phi}{|\eta| \cosh Y} \right) \sin \theta \]
\[b^2_1 = \frac{\sin \theta \sin \phi}{X} \]
\[b^2_2 = R \left( \tanh Y \cos \theta \sin \phi - \frac{\eta \cos \phi}{|\eta| \cosh Y} \right) \]
\[b^2_3 = R \left( \tanh Y \cos \phi + \frac{\eta \cos \theta \sin \phi}{|\eta| \cosh Y} \right) \sin \theta \]
\[b^3_1 = \frac{\cos \theta}{X} \]
\[b^3_2 = -R \tanh Y \sin \theta \]
\[b^3_3 = -R \left( \frac{\eta \sin^2 \theta}{|\eta| \cosh Y} \right).\]  

(31)

Or equivalently in the quasi-orthogonal coordinate system, in which the spatial coordinates are given by \((x^\alpha) = (R \sin \theta \cos \phi, R \sin \theta \sin \phi, R \cos \theta)\), the space-space components \(b^a_\alpha\) are expressed in a more compact form:

\[b^a_\alpha = \tanh Y \delta_{a\alpha} + \left( \frac{1}{X} - \tanh Y \right) \frac{x^a x^\alpha}{R^2} + \left( \frac{\eta}{|\eta| \cosh Y} \right) \epsilon_{a\alpha \beta} x^\beta \frac{1}{R}.\]  

(32)

It is of interest to note that solution (31) is reduced to solution (29) obtained before [34] when \(q = 0\) and \(m\) is replaced by \(m(1 - e^{-R^3/r^3})\).

Finally we notice that if the constant \(\eta\) is set equal to zero the tetrad field (31) reduces to the matrix inverse of the solution (17) with \(H = 0\).
4. The energy associated with each solution

The superpotential is given by [27, 30]

\[ U_\mu^\nu\lambda = \frac{(-g)^{1/2}}{2\kappa} P_\chi_{\rho \sigma}^{\tau \nu \lambda} [\Phi^\rho g^{\sigma \chi} g_{\mu \tau} - \lambda g_{\tau \mu} \gamma^{\lambda \rho \sigma} - (1 - 2\lambda) g_{\tau \mu} \gamma^{\sigma \rho \chi}], \tag{33} \]

where \( P_\chi_{\rho \sigma}^{\tau \nu \lambda} \) is

\[ P_\chi_{\rho \sigma}^{\tau \nu \lambda} \overset{\text{def.}}{=} \delta^\tau_\chi g_{\rho \sigma}^{\nu \lambda} + \delta^\nu_\rho g_{\sigma \chi}^{\nu \lambda} - \delta^\lambda_\sigma g_{\chi \rho}^{\nu \lambda} \tag{34} \]

with \( g_{\rho \sigma}^{\nu \lambda} \) being a tensor defined by

\[ g_{\rho \sigma}^{\nu \lambda} \overset{\text{def.}}{=} \delta^\nu_\rho \delta^\lambda_\sigma - \delta^\lambda_\rho \delta^\nu_\sigma. \tag{35} \]

The energy contained in the sphere with radius \( R \) is expressed by the surface integral [36]

\[ E(R) = \int_{r=R} U_0^{0\alpha} n_\alpha d^2S, \tag{36} \]

where \( n_\alpha \) is the unit 3-vector normal to the surface element \( d^2S \).

Let us first discuss the solution given by (17). Calculating the necessary components of the superpotential in the quasi-orthogonal coordinates \((t, x^\alpha)\),

\[ U_0^{0\alpha} = \frac{2X x^\alpha}{\kappa R^2} (L - X), \tag{37} \]

and substituting it into (36), we obtain

\[ E(R) = XR(L - X), \tag{38} \]

which depends on the arbitrary function \( H \). Since this arbitrary function originates from the restricted local Lorentz invariance of the field equations (7) and (9), the result (38) shows that the energy content of a sphere with constant \( R \) is not invariant under restricted local Lorentz transformations.

Next let us turn to the solution (31). Calculating the necessary components of the superpotential,

\[ U_0^{0\alpha} = \frac{2X x^\alpha}{\kappa R^2} (\tanh Y - X), \tag{39} \]

and substituting it into (36), we have

\[ E(R) = XR(\tanh Y - X). \tag{40} \]

For large \( R \) this is rewritten as

\[ E(R) \approx m - \frac{q^2 + m^2}{2R}, \tag{41} \]
where only those terms up to order \( O(1/R) \) are retained. In this approximation the total energy is independent of the constant \( \eta \). Finally we notice that the result (41) agrees with that given before [33, 37].

We now turn to study whether the obtained solutions (17) and (31) satisfy the Møller’s three conditions (1)~(3) recapitulated in the Introduction. Since the two conditions (1) and (3) are satisfied in the tetrad theory of gravitation [27], we shall focus our attention on the condition (2).

We start with the solution (17). The asymptotic form of the tetrad field \( b_i^\mu \) is expressed up to \( O(1/R^2) \) in the quasi-orthogonal spatial coordinates \((x^\alpha) = (R \sin \theta \cos \phi, R \sin \theta \sin \phi, R \cos \theta)\) by

\[
\begin{align*}
b_0^0 &= 1 + \frac{H^2}{2} + \frac{m}{R} \left( 1 + \frac{H^2}{2} \right) - \frac{q^2}{2R^2} + \frac{3m^2}{2R^2}, \\
b_0^\alpha &= \left[ H - \frac{mH}{R} \right] n^\alpha, \\
b_a^0 &= \left[ H + \frac{mH}{R} \right] n^a, \\
b_a^\alpha &= \delta_a^\alpha + \left[ \frac{H^2}{2} - \frac{m}{R} \left( 1 + \frac{H^2}{2} \right) + \frac{q^2}{2R^2} - \frac{m^2}{2R^2} \right] n^a n^\alpha.
\end{align*}
\]

We calculate the energy separately according to the asymptotic behavior of the arbitrary function \( H(R) \).

**Case I:** \( H(R, t) \sim f(t)/\sqrt{R^{1-\epsilon}} \), where \( 0 < \epsilon < 1 \).

The calculation of energy for such asymptotic behavior shows that it is divergent as \( R \to \infty \), so we exclude this case from our consideration.

**Case II:** \( H(R, t) \sim f(t)/\sqrt{R^{1+\epsilon}} \), where \( 0 < \epsilon \).

The calculation of energy for such an asymptotic behavior of \( H(R, t) \) gives

\[
E(R) = m - \frac{q^2 + m^2}{2R},
\]

up to order \( O(1/R) \) in agreement with the result (41) for the solution (31), and the Møller’s condition (2) is satisfied.

**Case III:** \( H(R, t) \sim f(t)/\sqrt{R} \).

The non-vanishing components of the superpotential (33) are given asymptotically by

\[
\begin{align*}
U_0^{0\alpha} &= \frac{2n^\alpha}{\kappa R^2} \left[ m - \frac{q^2}{2R} + \frac{f^2(t)}{2} - \frac{m f^2(t)}{2R} - \frac{m^2}{2R} \right], \\
U_\gamma^{0\beta} &= \frac{1}{\kappa R^2} \left[ \left( \frac{f^3(t)}{4R} + \frac{m f(t)}{\sqrt{R}} \right) \delta_\gamma^\beta - \left( \frac{f^3(t)}{4\sqrt{R}} - \frac{m f(t)}{\sqrt{R}} \right) n^\gamma n^\beta \right] = -U_\gamma^{0\beta}, \\
U_\gamma^{\beta\alpha} &= \frac{1}{\kappa R^2} \left[ \frac{f^2(t)}{2} - \frac{f^4(t)}{8R} - \frac{mf^2(t)}{2R} + \frac{q^2}{2R} + \frac{m^2}{2R} \right] \left( n^\alpha \delta_\gamma^\beta - n^\beta \delta_\gamma^\alpha \right).
\end{align*}
\]

The energy-momentum complex \( \tau_\mu^\nu \) is given by

\[
\tau_\mu^\nu = U_\mu^{\nu\lambda},
\]

(45)
and automatically satisfies the conservation law, $\tau_{\mu}^{\nu}, \nu = 0$. The nonvanishing components of $\tau_{\mu}^{\nu}$ are expressed by

$$
\tau_{0}^{0} = \frac{1}{\kappa R^3} \left[ \frac{q^2}{R} + \frac{f^4(t)}{4R} + \frac{mf^2(t)}{R} + \frac{m^2}{R} \right],
$$

$$
\tau_{\alpha}^{0} = \frac{n^\alpha}{\kappa R^3} \left[ \frac{f^3(t)}{2\sqrt{R}} + \frac{3mf(t)}{\sqrt{R}} \right],
$$

$$
\tau_{\alpha}^{\beta} = \frac{1}{\kappa R^3} \left[ \frac{3f^2(t)}{2} n^\alpha n^\beta - \left\{ \frac{f^2(t)}{2} - \frac{f^4(t)}{4R} - \frac{mf^2(t)}{2R} + \frac{q^2}{R} - \frac{m^2}{2R} \right\} \delta_{\alpha}^{\beta} \right],
$$

where we have neglected higher order terms of $1/R^4$.

Using (44) in (36) and keeping up to $O(1/R)$, we find that the energy $E(R)$ is given by

$$
E(R) = m - \frac{q^2}{2R} + \frac{f^2(t)}{2} - \frac{f^4(t)}{8R} - \frac{mf^2(t)}{2R} - \frac{m^2}{2R},
$$

(47)

where the first two terms represent the standard value of the energy but there are extra terms which contribute to the total energy.

Now let us examine if condition (2) is satisfied or not in the case III. For this purpose we consider the Lorentz transformation

$$
\bar{x}^0 = \gamma (x^0 + vx^1), \quad \bar{x}^1 = \gamma (x^1 + vx^0), \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3,
$$

(48)

where the coordinates $\bar{x}^\mu$ represent the rest frame of an observer moving with speed $v$ to the negative direction of the $x^1$-axis, and $\gamma$ is given by $\gamma = \frac{1}{\sqrt{1 - v^2}}$. Here the speed of light is taken to be unity. The energy-momentum in a volume element $d^3\bar{x}$ on the hyperplane, $\bar{x}^0 = \text{const.}$, is given by [36]

$$
\bar{\tau}_{\mu}^{\nu} d^3\bar{x} = \frac{\partial x^\rho}{\partial \bar{x}^\mu} \frac{\partial \bar{x}^\nu}{\partial x^\sigma} \bar{\sigma}^{\bar{\tau}} d^3x/\gamma.
$$

(49)

Using equations (48) and (49), it is easy to calculate the components $\bar{\tau}_{\mu}^{0}$ as follows:

$$
\bar{\tau}_{\mu}^{0} d^3\bar{x} = \frac{\partial x^\rho}{\partial \bar{x}^\mu} \left( \tau_{\rho}^{0} + vt_1^1 \right) d^3x.
$$

(50)

Integration of (50) over the three dimensional hyperplane with $\bar{x}^0 = \text{constant}$ gives

$$
\int_{\bar{x}^0 = \text{constant}} \tau_{\mu}^{0} d^3\bar{x} = \frac{\partial x^\rho}{\partial \bar{x}^\mu} \left( \int_{\bar{x}^0 = \text{constant}} \left[ \tau_{\rho}^{0} + vt_1^1 \right] d^3x \right).
$$

(51)

Using (44) and (45) allows us to calculate the integral on the right-hand side of (51); for the second term we have

$$
\int \tau_{\rho}^1 d^3x = \frac{f^2}{6} \delta_1^1.
$$

(52)

Thus, we obtain

$$
\bar{P}_{\mu} = \frac{\partial x^\rho}{\partial \bar{x}^\mu} \left\{ P_{\rho} + \frac{vf^2}{6} \delta_1^1 \right\},
$$

(53)
or for the four components,

\[ \bar{P}_\mu = \gamma \left\{ -\left( E + \frac{v^2 f^2}{6} \right), v \left( E - \frac{f^2}{6} \right), 0, 0 \right\}, \text{ where } E = \lim_{R \to \infty} E(R) = m + \frac{f^2}{2}, \tag{54} \]

by virtue of (47). Equation (54) shows that the four-momentum is not transformed as a 4-vector under Lorentz transformations, and the Møller's condition (2) is not satisfied in the case III! Therefore, this case of spherically symmetric solutions, in which the components \( b_a^0 \) behave as \( 1/\sqrt{R} \), is not physically acceptable although it gives Reissner-Nordström metric.

As for the solution with the non-vanishing \( S \)-term, the tetrad field is given by (31) in the quasi-orthogonal coordinate system, and for large \( R \) it tends to the asymptotic form like \( b_i^\mu = \delta_i^\mu + O(1/R) \), and therefore the Møller's condition (2) is satisfied.

5. Main results and discussion

In this paper we have studied the coupled equations of the gravitational and electromagnetic fields in the tetrad theory of gravitation, applying the most general spherically symmetric tetrad field of the form (1) to the field equations. Exact analytic solutions are obtained by studying two cases: The case Without the \( S \)-term and the case with \( S \)-term. In both cases we use the previously derived result [19] that the antisymmetric part of the coupled field equations requires the axial-vector part of the torsion tensor, \( a_{\mu} \), to vanish.

We obtained two exact solutions in which the field equations reduce to those of the Einstein-Maxwell theory in the teleparallel equivalent of general relativity. The metric is then that of the Reissner-Nordström solution. For the tetrad field of the form (1) without the \( S \)-term, the condition \( a_{\mu} = 0 \) is automatically satisfied, and the most general solution can be obtained from the diagonal tetrad field for the Reissner-Nordström metric by applying those local Lorentz transformations which retain the form (1) without the \( S \)-term. Since the general expression for those local Lorentz transformations involves an arbitrary function denoted by \( H(R, t) \), the obtained solution (17) for the tetrad field also involves this arbitrary function and reduces to the previous solution [33] when the function \( H \) is chosen appropriately (19).

For the tetrad field of the form (1) with the non-vanishing \( S \)-term, the solution (31) is derived by requiring the two conditions: The one is \( a_{\mu} = 0 \), and the other is that the metric should coincide with the Reissner-Nordström metric. The solution involves a constant parameter \( \eta \). If this constant is set equal to zero, the tetrad field (31) reduces to the matrix inverse of the solution (17) with \( H = 0 \).

We have used the superpotential method [27, 30] to calculate the energy of the isolated system described by the obtained solutions, and studied the asymptotic conditions imposed by the Møller’s condition (2).

Concerning the solution (17), the energy \( E(R) \), which is contained within the sphere of radius \( R \), is given by (38) and depends on the arbitrary function. In other words, the energy contained in a finite sphere does depend on the tetrad field we use: This can be considered as a manifestation of the pseudotensor character of the gravitational energy-momentum complex.
As for the asymptotic behavior of the function $H$, we conclude that it must decrease faster than $1/\sqrt{R}$ for large $R$. In this case the energy $E(R)$ takes the well-known form (43) for large $R$, and the four-momentum is transformed as a 4-vector. Thus all the Møller’s condition are satisfied. We reach this conclusion of the asymptotic behavior of the function $H$ in the following manner. If the arbitrary function $H$ decreases more slowly than $1/\sqrt{R}$, the $E(R)$ will be divergent for $R \to \infty$. If the arbitrary function $H$ behaves like $1/\sqrt{R}$ for large $R$, the associated energy does not agree with the well-known one, and furthermore, as we have shown, the four-momentum is not transformed as a 4-vector (54), violating the Møller’s condition (2).

Next we have calculated the energy associated with solution (31) with the non-vanishing $S$-term. We obtain expression (40) for $E(R)$, which depends on the parameter $\eta$. It follows from (40) that if $R \to 0$ then $E(R) \to \infty$, and that if $R \to \infty$ then $E(R) \to m$. It is also shown that the four-momentum behaves like a 4-vector, indicating that this solution meets all the Møller’s condition. Thus we have obtained two exact solutions physically different from each other as we have seen from the discussion of the energy. They are identical only when the arbitrary function $H$ and the arbitrary constant $\eta$ are set to be zero.

A summary of the main results is given in the table below. The solutions of spherically symmetric Reissner-Nordström black hole are classified into two groups. The solution without the $S$-term has an arbitrary function and the solution with the $S$-term has a constant parameter $\eta$.

| Field equation | Energy $E(R)$ |
|----------------|----------------|
| **Tetrad field** | **Satisfied identically** | $XR \left( L - X \right)$ |
| without the $S$-term | | Reissner-Nordström solution |
| **Tetrad field** | **satisfied when $a_\mu = 0$** | $XR \left( \tanh Y - X \right)$ |
| with $S$-term | | Reissner-Nordström solution |
Table II: Asymptotic behavior of the arbitrary function

| Arbitrary Function | Energy $E(R)$ | Physically acceptable |
|--------------------|---------------|-----------------------|
| $H \sim 1/\sqrt{R^{1-\varepsilon}}$ | Divergent | No |
| $H \sim 1/\sqrt{R^{1+\varepsilon}}$ | $E(R) = m - \frac{q^2 + m^2}{2R}$ | Yes |
| $H \sim 1/\sqrt{R}$ | $E(R) = m - \frac{q^2 + m^2}{2R} + \frac{f^2(t)}{2} - \frac{f^4(t)}{8R} - \frac{mf^2(t)}{2R}$ | No |

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