Interpreting Lambda Calculus in Domain-Valued Random Variables

Robert Furber∗  Radu Mardare†  Prakash Panangaden‡  Dana Scott§

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Abstract

We develop Boolean-valued domain theory and show how the lambda-calculus can be interpreted in using domain-valued random variables. We focus on the reflexive domain construction rather than the language and its semantics. The notion of equality has to be interpreted in the Boolean algebra and when we say that an equation is valid in the model we mean that its interpretation is the top element of the Boolean algebra.

1 Introduction

There has been burgeoning interest in probabilistic programming languages in the last ten years and workshops on the topic now figure regularly as satellites of both programming languages conferences and machine learning conferences. The main motivation is building compositional models of probabilistic processes and performing inference on them; see, for example, the thesis of Dan Roy [25] or the papers on Church [16] and Anglican [33]. For machine learning applications the notion of conditioning is fundamental and the striking results of [1, 2] show that this is a subtle issue.

The combination of probability and higher type programming has been both technically challenging and important for the development of semantics for such languages. There are a variety of approaches based on probabilistic coherence spaces [9] or cones [10], quasi-Borel spaces [31, 17, 32] and Boolean-valued models [3] which was based on Dana Scott’s vision [28]. Stochastic lambda-calculi have appeared [6, 7, 8] with important contributions to the understanding of probability theory at higher type. The work on quasi-Borel spaces gives a cartesian closed category that can serve as the foundation for a typed higher-order probabilistic programming language [17].

∗University of Edinburgh
†University of Strathclyde
‡McGill University
§Carnegie Mellon University
In the work described in [3] a Boolean-valued domain theory was developed. The idea is to use one of the standard set-theoretic models of the $\lambda$-calculus but interpreted in a suitable Boolean-valued universe of sets. However, the basic domain theoretic definitions of directed set, supremum, reflexive dcpo and continuity were interpreted as usual. The theory presented there is rather complicated and had artificial restrictions on the way randomness was handled.

In the present paper a completely Boolean-valued point of view is adopted and even basic concepts, like equality and order, are all interpreted in a Boolean-valued logic. This leads to a much simpler theory but still in line with the vision of [28].

We show that untyped $\lambda$-calculus can be interpreted in domain-valued random variables. In order to employ the theory of Boolean-valued sets, for most of the article we actually use the Boolean-valued power set rather than random variables, and then show that there is an isomorphism between the two at the end. We focus on the reflexive domain construction rather than the language and its semantics; it is the completely Boolean-valued reconstruction of domain theory that we feel is the interesting contribution of the present paper. The notion of equality has to be interpreted in the Boolean algebra and when we say that an equation is valid in the model we mean that its interpretation is the top element of the domain.

What makes the theory of Boolean-valued sets necessary is that domain-valued random variables do not form a continuous dcpo. However, when “continuous dcpo” is given its interpretation in Boolean-valued sets, they are.

The version of Boolean-valued sets we have used is the original version in terms of a cumulative hierarchy. If the reader prefers, they may rephrase the arguments results in terms of topos theory in a Boolean topos, and we outline the connection of the two in Section 3, though the only part of this theory that we use is the ability to define functions.

As an example of how to apply Boolean-valued domain theory, we give a $\lambda$-calculus proof of the Kleene-Post theorem [20, 2.2 Corollary 1] that there are incomparable many-one degrees, similar to Spector’s [30, Theorem 2] proof using measure theory.

There are a number of directions for future work. First, one can use this construction to give semantics to a $\lambda$-calculus extended with probabilistic choice as was done in [3]. In this model it will be interesting to see which equations involving the interplay of choice and the standard $\lambda$-calculus constructions are valid. One could then relate it to an operational semantics as, for example, in [8] but one would need a Boolean-valued notion of operational semantics. More interestingly one could define conditioning as a primitive and explore its semantics.

A second line of research is the notion of approximation. A notion of “approximate equality” has been developed recently [22]; the connection to the notion of equality used in the present paper is unclear but there is a similarity in that in both cases equality may only hold partially. Whether this would prove to be a useful synthesis of the ideas remains to be seen.

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1 We formulated it in terms of many-one degrees for simplicity. It could also be redone with Turing degrees instead.
2 Background on Boolean-Valued Set Theory

In this section we describe how statements and proofs in ordinary set theory can be re-interpreted in a Boolean-valued sense. This interpretation is originally due to Dana Scott [26] and Robert Solovay; textbook expositions can be found in [19, Chapter 14] and [5]. The reader who wishes to ignore set theory entirely and rephrase results using topos theory (taking responsibility for redoing the proofs) should skip to Section 3.

Throughout this section we let $A$ be an arbitrary complete Boolean algebra. The idea is that $A$ will play the role that two-element Boolean algebra $2$ plays in ordinary logic. We can build up the class of $A$-valued sets, $\mathcal{V}_A$, by considering an $A$-valued set $X$ to be a partial function that assigns to each element $x$ of its domain the amount, valued in $A$, that $x$ is an element of $X$. More accurately, we build up $\mathcal{V}_A$ by the following iterative construction, which generalizes von Neumann’s construction:

\[
\begin{align*}
V_A^0 &= \emptyset \\
V_A^{\alpha+1} &= \{ f : V_A^\alpha \to A \} \\
V_A^\alpha &= \bigcup_{\beta < \alpha} V_A^\beta \quad \text{for } \alpha \text{ a limit ordinal.}
\end{align*}
\]

Then either informally, or using proper classes, $V_A = \bigcup_{\alpha \in \text{Ord}} V_A^\alpha$.

We use the term $A$-valued set for the elements of $V_A$, but we also will use the shorter term $A$-set, and this helps to avoid certain opportunities for confusion arising from the term “valued”.

We can then interpret $\in$, $\subseteq$ and $=$ by the following mutually recursive formulas\(^2\):

\[
\begin{align*}
\| x \in X \| &= \bigvee_{t \in \text{dom} X} \| x = t \| \land X(t) \\
\| X \subseteq Y \| &= \bigwedge_{t \in \text{dom} X} X(t) \Rightarrow \| t \in Y \| \\
\| x = y \| &= \| x \subseteq y \| \land \| y \subseteq x \|.
\end{align*}
\]

In the above, and all that follows, we use an operator precedence convention for $\lor$ and $\land$ that agrees with operator precedence for $\exists$ and $\forall$, so there is an implicit bracket around everything to the right of such a join or meet.

Recall that the first-order language of set theory, which we write as $\mathcal{L}_{\text{Set}}$, is the usual first-order language for a signature with equality and one two-place relation symbol, namely $\in$. We write $\mathcal{L}_{\text{Set}}(V^A)$ for this language extended with constants from $V^A$.

In any Boolean algebra, we can interpret the connectives of propositional logic, and therefore any propositional formula. Using the completeness of $A$, we can interpret quantifiers by interpreting the universal quantifier as a meet and

\(^2\)We have used $\| - \|$ instead of semantic brackets because we require the semantic brackets later on for interpreting $\lambda$-calculus.
the existential quantifier as a join. All together, this gives us an interpretation of $\mathcal{L}_{\text{Set}}(V^A)$ in $V^A$.

**Meta-Theorem 2.1** ($V^A$ as a model).

(i) If $\phi \in \mathcal{L}_{\text{Set}}(V^A)$ is a theorem of ZFC set theory, then $\|\phi\| = 1$ in $V^A$.

(ii) The inference rules of first-order logic can be applied to theorems of ZFC set theory and statements about elements of $V^A$ in $\mathcal{L}_{\text{Set}}(V^A)$.

(iii) If $\|\exists x. \Phi(x, X_1, \ldots, X_n)\| = a$, where $X_1, \ldots, X_n \in V^A$ (and we allow $n = 0$), then there exists $X_0 \in V^A$ such that $\|\Phi(X_0, X_1, \ldots, X_n)\| = a$.

**Proof.** For (i) and (ii), see [5, Theorem 1.17]; (iii) is a particular instance of what we shall later call the “completeness” of $V^A$—see [19, Lemma 14.19] or [5, Lemma 1.27].

It is helpful to make certain constructions of $A$-sets explicit. If $x, y \in V^A$, we can define the singleton $\{x\}^A$, unordered pair $\{x, y\}^A$ and ordered pair $(x, y)^A$ as follows:

- $\text{dom}(\{x\}^A) = \{x\}$
- $\{x\}^A(x) = 1$
- $\text{dom}(\{x, y\}^A) = \{\{x\}^A, \{x, y\}^A\}$
- $(x, y)^A(\{x\}^A) = 1$
- $(x, y)^A(\{x, y\}^A) = 1$.

For $X, Y \in V^A$ we define $X \times_A Y$ as follows:

- $(X \times_A Y) : \{(x, y)^A \mid x \in \text{dom}(X), y \in \text{dom}(Y)\} \rightarrow A$
- $(X \times_A Y)(x, y)^A = \|x \in X\| \land \|y \in Y\|$.  

For each $A$-set $X$, the $A$-valued power set $\mathcal{P}^A(X)$ is defined by

- $\text{dom}(\mathcal{P}^A(X)) = \{u \in V^A \mid \text{dom}(u) = \text{dom}(X) \text{ and } \forall t \in \text{dom}(u). u(t) \leq X(t)\}$
- $\mathcal{P}^A(X)(u) = 1$

For all $S, X \in V^A$ we have:

- $\|S \in \mathcal{P}^A(X)\| = \|S \subseteq X\|$,  

which is how the power set axiom is proved for $V^A$.

If $\Phi$ is a set-theoretic formula we can define the $A$-set

- $\{x \in X \mid \Phi(x)\}^A : \text{dom}(X) \rightarrow A$
- $\{x \in X \mid \Phi(x)\}^A(t) = X(t) \land \|\Phi(t)\|$,  


and this is how the axiom of separation is proved for $V^A$.

If we now consider von Neumann’s universe $V$ of classic set theory constructed inductively on ordinals $\alpha \in \text{Ord}$, then for each set $S \in V_\alpha$, there is a corresponding element $\tilde{S} \in V_\alpha^A$ defined recursively. The domain of $\tilde{S}$ is \{x \mid x \in S\}, and it is defined by:

$$\tilde{\emptyset} = \emptyset$$
$$\tilde{S}(\tilde{x}) = 1 \quad \text{for all } x \in S.$$  

To describe how a statement in set theory about an ordinary set in $V$ translates to a statement about $\tilde{X}$ in $V^A$, we need the notion of a $\Delta_0$ statement. Bounded quantifiers are those of the form $\forall x \in X \Phi(x)$ and $\exists x \in X \Psi(x)$, i.e. in the language of set theory $\forall x . x \in X \Rightarrow \Phi(x)$ and $\exists x . x \in X \land \Psi(x)$. A $\Delta_0$ formula $\Phi \in \Delta_{\text{set}}(V^A)$ is one containing only bounded quantifiers. We have $\Delta_0$ invariance, see [19, Lemma 14.21] or [3, Theorem 1.23 (v)].

**Meta-Theorem 2.2** ($\Delta_0$ invariance). Let $\Phi(x_1, \ldots, x_n)$ be a $\Delta_0$-formula, with free variables $x_1, \ldots, x_n$. Let $X_1, \ldots, X_n \in V$. Then

$$\Phi(X_1, \ldots, X_n) \iff \|\Phi(\tilde{X}_1, \ldots, \tilde{X}_n)\| = 1$$

We point out some useful consequences of Meta-Theorem 2.2. The statements “$S$ is a singleton whose only element is $x$”, “$S$ is an unordered pair of $x$ and $y$” and “$S$ is an ordered pair, first element $x$, second element $y$” are all $\Delta_0$ statements, and so the following all hold:

$$\tilde{\{x\}} = \{\tilde{x}\}^A \quad \tilde{\{x, y\}} = \{\tilde{x}, \tilde{y}\}^A \quad \tilde{(x, y)} = (\tilde{x}, \tilde{y})^A.$$

Another important consequence of Meta-Theorem 2.2 is that $\tilde{\omega}$ is the smallest inductively defined $A$-set in $V^A$, so that $\tilde{\omega}$ is the $\omega$ of $V^A$. Given any set $X$, we say a subset $S \subseteq X$ is finite if there exists $n \in \omega$ and $f : n \to X$ such that $S = \text{im}(f)$. We write $\mathcal{P}_{\text{fin}}(X)$ for the set of all finite subsets of $X$. If we interpret this with $A$-sets, for each $X \in V^A$ we can define $\mathcal{P}^A_{\text{fin}}(X)$ to be the set of finite $A$-subsets of $X$, using the axiom of separation. The following proposition is a consequence of the distributive law - see [21, 3.1.11].

**Proposition 2.3.** For all sets $X$:

$$\|\mathcal{P}^A_{\text{fin}}(\tilde{X}) = \mathcal{P}^A_{\text{fin}}(\tilde{X})\| = 1$$

We can create categories $\text{Rel}_A$ and $\text{Set}_A$ from $V^A$ as follows. Each has $V^A$ as its class of objects. For each $X, Y \in V^A$, the set of binary relations can be defined as $\text{dom}(\mathcal{P}^A(X \times_A Y))$, and the set of functions $\{X \stackrel{A}{\to} Y\} \in V^A$ can be defined using products and the axiom of separation. We define:

$$\text{Rel}_A(X, Y) = \text{dom}(\mathcal{P}^A(X \times_A Y))/ \sim$$
$$\text{Set}_A(X, Y) = \{f \in \text{dom}(\{X \stackrel{A}{\to} Y\}) \mid \|f \in \{X \stackrel{A}{\to} Y\}\| = 1\}/ \sim,$$

Note that Bell uses the alternative terminology “restricted” for $\Delta_0$.

In the sense used to formulate the axiom of infinity in ZF.
where $\sim$ is the equivalence relation defined by $f \sim g$ iff $\|f = g\| = 1$. Identity elements are defined as identity relations, and composition by composition of relations, which is well-defined with respect to $\sim$.

3 Boolean-Valued Setoids

We now discuss the kind of “Boolean-valued sets” that are essentially $A$-valued models of the theory of $\models$. These can be considered an alternative description of either sheaves or separated presheaves on $A$, and are usually discussed in the more general case where $A$ is a Heyting algebra, such as in [12, 15 §11.9] and [23, Definition 5.1]. Although we will use only one definition of $A$-valued sets, the distinction between $A$-valued relations that satisfy the definition of a function internally and functions that do so gives us two distinct categories.

In order to distinguish the $A$-valued sets we will be discussing from elements of $\mathrm{V}^A$, we call them $A$-valued setoids, or simply $A$-setoids, because setoid is a long-established term for a set equipped with an equivalence relation. An $A$-setoid is a pair $(X, \|=\|_X)$ where $\|=\|_X : X \times X \to A$. The two axioms of $\|=\|_X$ are symmetry and transitivity, i.e.

\[
\forall x, y \in X. \|x = y\|_X = \|y = x\|_X
\]

so $A$-valued setoids are “$A$-valued partial equivalence relations”.

Since reflexivity is not assumed, the statement $\|x = x\|_X$ does not represent a tautology, but its intended interpretation is, in some sense, the degree to which $x \in X$, or the place where $x \in X$. So we define the notation $\varepsilon_X(x) = \|x = x\|_X$.

We say $(X, \|=\|_X)$ is total if $\|x = x\|_X = 1$ for all $x \in X$, i.e. if reflexivity holds; it is strict if $\|x = y\|_X = 1$ implies $x = y$.

**Definition 3.1.** Let $(X, \|=\|_X)$ be an $A$-setoid. We say it is **completed**\(^5\) if either, hence both, of the following equivalent properties holds of it:

(i) For any family of elements $(a_i)_{i \in I}$ in $A$ and corresponding family $(x_i)_{i \in I}$ in $X$ such that for all $i, j \in I$, $a_i \land a_j \leq \|x_i = x_j\|_X$, there exists $x \in X$ such that for all $i \in I$, $a_i \leq \|x_i = x\|_X$.

(ii) For any pairwise disjoint family of elements $(a_i)_{i \in I}$ in $A$ and corresponding family $(x_i)_{i \in I}$ in $X$ such that for all $i \in I$, $a_i \leq \|x_i = x_i\|_X$, there exists $x \in X$ such that for all $i \in I$, $a_i \leq \|x_i = x\|_X$.

\(^5\)This definition is slightly different from that used in [12, 4.10.4.11] and elsewhere, which would not suit us because under that definition $\mathrm{P}^A(X)$ is not complete, when considered as an $A$-setoid in the sense we will define in Definition 3.13.

**Definition 3.2.** If $(X, \|=\|_X)$ is an $A$-setoid and $(Y, \|=\|_Y)$ is a set and a function $\|=\|_Y : Y \times Y \to A$, and $f : X \to Y$ is a bijection such that for all $x_1, x_2 \in X$:

$$\|f(x_1) = f(x_2)\|_Y = \|x_1 = x_2\|_X,$$

then we say $f$ is a **strict isomorphism**, and it follows that $(Y, \|=\|_Y)$ is an $A$-setoid, and is total, strict or complete iff $(X, \|=\|_X)$ is.
Definition 3.3. The product \((X, \|\cdot\|_X) \times (Y, \|\cdot\|_Y)\) is defined to have
\[
\|(x, y)\|_{X \times Y} = \|x\|_X \land \|y\|_Y.
\]

Definition 3.4. A predicate on \((X, \|\cdot\|_X)\) is a function \(S : X \to A\) such that:

(i) For all \(x_1, x_2 \in X\), \(\|x_1 = x_2\|_X \leq S(x_1) \Leftrightarrow S(x_2)\).
(ii) For all \(x \in X\), \(S(x) \leq \varepsilon_X(x)\).

A binary relation \(X \to Y\) is simply a predicate on \(X \times Y\).

We can now define the first two categories of \(A\)-setoids, the category of \(A\)-setoids with relations \(\text{SetoidRel}_A\), and the category of \(A\)-setoids with “relational functions”, \(\text{SetoidR}_A\).

Definition 3.5. The categories \(\text{SetoidRel}_A\) and \(\text{SetoidR}_A\) have \(A\)-setoids as objects. The hom set \(\text{SetoidRel}_A(X, Y)\) is the set of binary relations \(X \to Y\). In this case there is no need to take a quotient by any equivalence relation.

The identity relation \(\text{id}_X : X \to X\) is defined by:
\[
\text{id}_X(x_1, x_2) = \|x_1 = x_2\|_X,
\]
composition of relations \(R : X \to Y\) and \(S : Y \to Z\) is defined by
\[
(S \circ R)(x, z) = \bigvee_{y \in Y} S(y, z) \land R(x, y).
\]

The category \(\text{SetoidR}_A\) has the same objects, identity maps, definition of composition and equivalence relation on morphisms, but the a morphism \(f : X \to Y\) must satisfy the additional conditions:

(iii) For all \(x \in X\), \(y_1, y_2 \in Y\), \(f(x, y_1) \land f(x, y_2) \leq \|y_1 = y_2\|_Y\).
(iv) For all \(x \in X\), \(\varepsilon_X(x) \leq \bigvee_{y \in Y} f(x, y)\).

We define the category of \(A\)-setoids and “functional functions”, \(\text{SetoidF}_A\), as follows.

Definition 3.6. The category \(\text{SetoidF}_A\) has \(A\)-setoids as objects. The morphisms are defined starting with an \(A\)-setoid:
\[
X \to Y = \{ f : X \to Y \mid \forall x_1, x_2 \in X, \|x_1 = x_2\|_X \leq \|f(x_1) = f(x_2)\|_Y \}
\]
\[
\|f_1 = f_2\|_{X \to Y} = \bigwedge_{x \in X} \varepsilon_X(x) \Rightarrow \|f_1(x) = f_2(x)\|_Y.
\]

Then the hom set \(\text{SetoidF}_A(X, Y)\) is the quotient set of this\(^6\) i.e. \(X \to Y\) modulo the equivalence relation:
\[
f_1 \sim f_2 \text{ iff } \forall x \in X, \varepsilon_X(x) \leq \|f_1(x) = f_2(x)\|_Y.
\]
Identity maps and composition are defined as for functions.

\(^6\)Monro calls these admissible functions in [23, Definition 5.2].
The category \( \text{SetoidF}_A \) is called \( \text{Mod}_0(A) \) in \[23\] Definition 5.2, where it is proved to be a quasi-topos. The full subcategory on non-empty complete \( A \)-setoids is a topos \[23\] Proposition 5.6 (ii) equivalent to the category of sheaves on the canonical topology on \( A \).

By definition, \( \text{SetoidR}_A \) is a subcategory of \( \text{SetoidRel}_A \). We now show that mapping a functional function to its graph defines a faithful functor from \( \text{SetoidF}_A \rightarrow \text{SetoidR}_A \).

**Definition 3.7.** The following defines a faithful functor \( \gamma : \text{SetoidF}_A \rightarrow \text{SetoidR}_A \). On objects, \( \gamma(X, \|=\|_X) = (X, \|=\|_X) \). For \( f \in \text{SetoidF}_A(X, Y) \), we define
\[
\gamma(f)(x, y) = \varepsilon^X(x) \land \|f(x) = y\|_Y.
\]

We can formulate a notion of poset that does not require us to define \( |=\|_X \) but can instead be defined in terms of it. It turns out to be useful in the general theory, as well as for dealing with posets that are \( A \)-setoids or \( A \)-sets.

**Definition 3.8.** An \( A \)-poset is a pair \( (X, \leq \leq^X) \), where \( X \) is a set and \( \leq \leq^X : X \times X \rightarrow A \) is such that for all \( x_1, x_2, x_3 \in X \):

(i) \( \|x_1 \leq x_2\|_X \land \|x_2 \leq x_3\|_X \leq \|x_1 \leq x_3\|_X \).

(ii) \( \|x_1 \leq x_2\|_X \leq \|x_1 \leq x_1\|_X \land \|x_2 \leq x_2\|_X \).

We then define \( \|=\|_X \) by
\[
\|x_1 = x_2\|_X = \|x_1 \leq x_2\|_X \land \|x_2 \leq x_1\|_X.
\]

Then \( (X, \|=\|_X) \) is an \( A \)-setoid, and \( \leq \leq^X \in \text{SetoidRel}_A(X, X) \) is a reflexive, antisymmetric, transitive relation when these statements are interpreted in the \( A \)-valued sense. If \( (X, \|=\|_X) \) is an \( A \)-setoid and \( \leq \leq^X \in \text{SetoidRel}_A(X, X) \) is a relation that is reflexive, antisymmetric and transitive, in the \( A \)-valued sense, then \((X, \leq \leq^X)\) is an \( A \)-poset in the above sense, and \( |=\|_X \) satisfies (3.9). □

Since \( |=\|_X \) is defined in terms of \( \leq \leq^X \), the following fact is convenient for proving that a function on underlying sets \( f : X \rightarrow Y \) not only defines an element of \( \text{SetoidF}_A(X, Y) \) but also a monotone function in the \( A \)-valued sense.

**Lemma 3.10.** Let \((X, \leq \leq^X), (Y, \leq \leq^Y)\) be \( A \)-posets, and suppose that \( f : X \rightarrow Y \) is \( A \)-monotone in the sense that for all \( x_1, x_2 \in X \)
\[
\|x_1 \leq x_2\|_X \leq \|f(x_1) \leq f(x_2)\|_Y.
\]

(This is monotonicity using the interpretation of logic in Section 2.) Then \( f \in X \rightarrow Y \). Every \( f \in \text{SetoidF}_A(X, Y) \) that is monotone in the \( A \)-valued interpretation of the term is of this form. □

The following definition and theorem are based on \[24\] Proposition 3.3.
Definition 3.11. Let \((X, \|=|_X), (Y, \|=|_Y)\) be \(A\)-setoids such that \(Y\) is complete, and let \(f \in \text{SetoidR}_A(X, Y)\). For each \(x \in X\), there exists \(\mathcal{F}(f)(x) \in Y\) such that
\[
f(x, y) \leq \|\mathcal{F}(f)(x) = y\|_Y. \tag{3.12}
\]
Any choice of \(\mathcal{F}(f)(x)\) for all \(x \in X\) defines an element \(\mathcal{F}(f) \in \text{SetoidF}_A(X, Y)\) such that \(\gamma(\mathcal{F}(f)) = f\), and in fact this establishes a \(\text{SetoidF}_A\) isomorphism
\[
\text{SetoidR}_A(X, Y) \cong \text{SetoidF}_A(X, Y).
\]

Every \(A\)-set defines an \(A\)-setoid as follows.

Definition 3.13. Let \(X \in V^A\). Define \(\|=|_X : \text{dom}(X) \times \text{dom}(X) \to A\) by
\[
\|x = y\|_X = \|x \in X\| \land \|y \in X\| \land \|x = y\|
\]
(3.14)
Then \((\text{dom}(X), \|=|_X)\) is an \(A\)-setoid, which we write \(\text{Oid}(X)\) when we need to be unambiguous.

Definition 3.15. Let \(X \in V^A\). For each \(S \in V^A\) such that \(\|S \subseteq X\| = 1\), then \(e(S)\), defined as follows, is a predicate on \(\text{Oid}(X)\):
\[
e(S)(x) = \|x \in S\|
\]
where \(x \in \text{Oid}(X)\). In particular, this holds if \(S \in \text{dom}(P^A(X))\).

Definition 3.16. The following defines \(\text{Oid} : \text{Rel}_A \to \text{SetoidRel}_A\) and \(\text{Oid} : \text{Set}_A \to \text{SetoidR}_A\) as functors, where \(f : X \to Y\) in either \(\text{Rel}_A\) or \(\text{Set}_A\):
\[
\text{Oid}(f)(x, y) = \|(x, y)^A \in f\|
\]
These functors are full, faithful and essentially surjective, and therefore define equivalences \(\text{Rel}_A \simeq \text{SetoidRel}_A\) and \(\text{Set}_A \simeq \text{SetoidR}_A\).

Theorem 3.17. For each \(X \in V^A\), \(\text{Oid}(P^A(X))\) is complete. Moreover, if \(\Phi\) is a formula of \(\mathcal{L}_\text{Set}(V^A)\) and \(\exists S \in P^A(X) . \Phi(S, \ldots)\| = a \in A\), then there exists \(S \in \text{Oid}(P^A(X))\) such that \(\|\Phi(S, \ldots)\| = a\).

Corollary 3.18. For each \(X, Y \in V^A\), \(\text{Oid}(P^A(X \times_A Y))\) and \(\text{Oid}(\{X \overset{A}{\to} Y\})\) are complete, and if we prove a relation or a function exists in the Boolean-valued sense, satisfying some formula \(\Phi \in \mathcal{L}_\text{Set}(V^A)\), then there exists a corresponding element of \(\text{Oid}(P^A(X \times_A Y))\) or \(\text{Oid}(\{X \overset{A}{\to} Y\})\) satisfying \(\Phi\).

4 Models of Untyped \(\lambda\)-calculus in \(A\)-sets

The purpose of this section is to show that we can model untyped \(\lambda\)-calculus in \(A\)-valued sets. We use the Engeler model \([11]\) as the main example. We will repeatedly use Meta-Theorem \([27]\) to prove statements in \(V^A\) by proving them in set theory first. The reader who would prefer to use a topos formulation might prefer to use the categorical formulation of the Engeler model from \([18]\).
4.1 Background on Domain Theory

A dcpo \((D, \leq)\) is a poset that is directed complete, i.e. such that every directed set has a supremum. Every complete lattice is a dcpo. The least element of a dcpo \((D, \leq)\), if it exists, is called the bottom element and is written \(\bot\). In a dcpo \(D\), where \(d, e \in D\), we say that \(d\) is way below \(e\), written \(d \ll e\), if for each directed set \(\{e_i\}_{i \in I}\) such that \(e \leq \bigvee_{i \in I} e_i\), there exists some \(j \in I\) such that \(d \leq e_j\). The relation \(\ll\) is transitive and antisymmetric, but in general is neither reflexive nor irreflexive. We write each directed set \((D, \leq)\) is an dcpo isomorphism, d that the directed set has a supremum. Every complete lattice is a dcpo. The least element of a dcpo \((D, \leq)\) say it is a countably-based domain according to Definition 4.1. A dcpo is called a complete lattice. A basis or base for a domain \(D\) is a subset \(B \subseteq D\) such that for all \(d \in D\), \(\downarrow d \cap B\) is directed, and \(d = \bigvee (\downarrow d \cap B)\). It follows that if \(D\) is continuous, then \(D\) is a base for \(D\). If \(D\) has a countable base, we say it is a countably-based domain. See [4] for more related concepts.

If \(D\) and \(E\) are dcpos, a function \(f : D \to E\) is said to be Scott continuous if it is monotone and preserves directed suprema. We write \([D \to E]\) for the set of Scott-continuous maps \(D \to E\). This is a dcpo when given the pointwise ordering, and directed suprema are calculated pointwise. When we refer to this as a dcpo, we always use this structure. The Scott topology on a dcpo \(D\) is the topology whose open sets are the sets \(U \subseteq D\) such that \(U\) is an up set and for all directed sets \((d_i)_{i \in I}\) such that \(\bigvee_{i \in I} d_i \in U\) there exists \(i \in I\) such that \(d_i \in U\). If \(D\) and \(E\) are dcpos, a function \(f : D \to E\) is Scott continuous if it is a continuous map from \(D\) to \(E\) equipped with their respective Scott topologies.

**Definition 4.1.** A reflexive dcpo \([D, \text{fun}, \text{lam}]\) is a triple \((D, \text{fun}, \text{lam})\) where \(D\) is a dcpo with bottom, and \(\text{fun} : D \to [D \to D]\) and \(\text{lam} : [D \to D] \to D\) are Scott-continuous maps making \([D \to D]\) a retract of \(D\), i.e. \(\text{fun} \circ \text{lam} = \text{id}_{[D \to D]}\). A reflexive dcpo is extensional if \(\text{fun}\), or equivalently \(\text{lam}\) is an isomorphism, i.e. iff additionally \(\text{lam} \circ \text{fun} = \text{id}_D\).

It is often convenient to formulate this notion using the uncurried form of \(\text{fun}\), which is to say, we can equivalently define a reflexive dcpo to be a triple \((D, \cdot, \text{lam})\) where \(D\) is a dcpo with bottom, and \(\cdot : D \times D \to D\) and \(\text{lam} : [D \to D] \to D\) are Scott-continuous maps such that for each \(f \in [D \to D]\) and \(d \in D\) we have \(\text{lam}(f) \cdot d = f(d)\).

To finish this subsection, we describe how the language of untyped \(\lambda\)-calculus can be interpreted in a reflexive dcpo. Given a reflexive dcpo \((D, \cdot, \text{lam})\), and a set of variables \(\text{Var}\), a valuation \(\rho : \text{Var} \to D\). We also choose a set of constants \(\mathcal{R} \subseteq D\), which is allowed to be any subset, including \(\emptyset\) and \(D\) itself. We then define the language of \(\lambda\)-calculus \(\Lambda(D, \text{Var}, \mathcal{R})\) as follows.

**Definition 4.2.** The language \(\Lambda(D, \text{Var}, \mathcal{R})\) is defined inductively by the rules

\[
\begin{align*}
&x \in \text{Var} \quad d \in \mathcal{R} \\
&\frac{}{x \in \Lambda(D, \text{Var}, \mathcal{R})} & d \in \Lambda(D, \text{Var}, \mathcal{R}) \\
&x \in \text{Var} & M \in \Lambda(D, \text{Var}, \mathcal{R}) \\
&\frac{}{\lambda x.M \in \Lambda(D, \text{Var}, \mathcal{R})}
\end{align*}
\]

\(^*\)This is what is meant by a cpo in [3, Definition 1.2.1 (ii)].
\[ M \in \Lambda(D, \text{Var}, \mathcal{R}) \quad N \in \Lambda(D, \text{Var}, \mathcal{R}) \]

\[ MN \in \Lambda(D, \text{Var}, \mathcal{R}) \]

We write \( \Lambda(\text{Var}) \) for \( \Lambda(D, \text{Var}, \emptyset) \), because this does not depend on \( D \). The elements of \( \Lambda(\text{Var}) \) are called pure \( \lambda \)-terms. We write \( \text{fv}(M) \) for the set of free variables of \( M \) defined as usual.

We make the following observation about how the above definition is formulated in set theory:

**Example 4.3.** There exist \( \Delta_0 \) formulas \( \Lambda(D, \text{Var}, \mathcal{R}) \)-ind and \( \Lambda(\text{Var}) \)-ind, such that the sets \( \Lambda(D, \text{Var}, \mathcal{R}) \) and \( \Lambda(\text{Var}) \) are respectively definable as the smallest sets satisfying their corresponding formulas.

**Proof.** To define \( \Lambda(\text{Var}) \)-ind and \( \Lambda(D, \text{Var}, \mathcal{R}) \)-ind, we need to choose some way of representing the syntax of \( \lambda \)-calculus. For example, we could take the ordinals 0 to mean a variable, 1 a \( \lambda \)-abstraction, 2 an application and 3 a constant, so that a variable appears as \((0, x)\) for \( x \in \text{Var} \), a \( \lambda \)-abstraction as \((1, (x, M))\) where \( x \in \text{Var} \) and \( M \) is a \( \lambda \)-term, an application as \((2, (M, N))\) where \( M, N \) are \( \lambda \)-terms, and a constant as \((4, d)\) where \( d \in \mathcal{R} \).

Then we can define:

\[
\Lambda(\text{Var})\text{-ind}(X) = (\forall x \in \text{Var}. \exists p \in X.p = (0, x))
\]

\[
\& (\forall x \in \text{Var}. \forall M \in X. \exists p \in X.p = (1, (x, M)))
\]

\[
\& (\forall M \in X. \forall N \in X. \exists p \in X.p = (2, (M, N))).
\]

Strictly speaking this is not quite a \( \Delta_0 \) formula, but all that needs to be done is to show that the statements \( p = (0, x) \), \( p = (1, (x, M)) \) and \( p = (2, (M, N)) \) are expressible as \( \Delta_0 \) formulas (with the usual set-theoretic representations of ordered pairs), which is left as an exercise to the reader. Then we can also define

\[
\Lambda(D, \text{Var}, \mathcal{R})\text{-ind}(X) = \Lambda(\text{Var})\text{-ind}(X) \land (\forall d \in \mathcal{R}. \exists p \in X.p = (3, d))
\]

It is then clear that \( \Lambda(\text{Var}) \) and \( \Lambda(D, \text{Var}, \mathcal{R}) \) are the least (with respect to \( \subseteq \) ) sets satisfying their respective formulas.

**Proposition 4.4.** Let \( \text{Var} \) be a set. For any complete Boolean algebra \( A \), by Meta-Theorem 2.1 (i) and (iii) there exists \( \Lambda(\text{Var})^A \) that satisfies the definition of \( \Lambda(\text{Var}) \) in the Boolean-valued sense, i.e. \( \| \Lambda(\text{Var})\text{-ind}(\Lambda(\text{Var})^A) \| = 1 \) and for all \( X \in V^A \) such that \( \| \Lambda(\text{Var})\text{-ind}(X) \| = 1 \), we have \( \| \Lambda(\text{Var})^A \subseteq X \| = 1 \).

Then \( \| \Lambda(\text{Var}) = \Lambda(\text{Var})^A \| = 1 \).

**Proof.** We know that \( \Lambda(\text{Var})\text{-ind} \) holds of \( \Lambda(\text{Var}) \), so by Meta-Theorem 2.2 \( \| \Lambda(\text{Var})\text{-ind}(\Lambda(\text{Var})) \| = 1 \). By the defining property of \( \Lambda(\text{Var})^A \), we have \( \| \Lambda(\text{Var})^A \subseteq \Lambda(\text{Var}) \| = 1 \), and we only need to show the opposite inclusion.

---

9 This is needed to make the proof of Proposition 4.3 and later results that deal with viewing the syntax of \( \lambda \)-calculus from inside \( V^A \) more comprehensible.

10 Not Boolean-valued, just the usual kind of set.
We do this by showing that if $X \in V^A$ such that $\|\Lambda(\text{Var})\text{-}\text{ind}(X)\| = 1$, then $\|\Lambda(\text{Var}) \subseteq X\| = 1$. By expanding the definitions, $\|\Lambda(\text{Var}) \subseteq X\| = 1$ is equivalent to $\forall M \in \Lambda(\text{Var}). \|\widetilde{M} \in X\| = 1$, so we show that this follows from $\|\Lambda(\text{Var})\text{-}\text{ind}(X)\| = 1$ by induction on the structure of elements of $\Lambda(\text{Var})$.

- **Base case for a variable:**
  
  If $(0, x) \in \Lambda(\text{Var})$ where $x \in \text{Var}$, we start with the observation that $(0, x) = (0^A, \check{x})^A$. Since $\|\Lambda(\text{Var})\text{-}\text{ind}(X)\| = 1$ we have $\|(0^A, \check{x})^A \in X\| = 1$, and therefore $\|(0, x) \in X\| = 1$, as required.

- **Inductive step for a $\lambda$-abstraction:**
  
  If $(1, (x, M)) \in \Lambda(\text{Var})$ such that $x \in \text{Var}$ and $M \in \Lambda(\text{Var})$ and $\|(\check{x}, M)^A \in X\| = 1$, then as in the previous cases, we start with $(1, (x, M)) = (1^A, (\check{x}, M)^A)^A$. As $\|\Lambda(\text{Var})\text{-}\text{ind}(X)\| = 1$ implies $\|(1^A, (\check{x}, M)^A)^A \in X\| = 1$, and therefore $\|(1, (x, M)) \in X\| = 1$.

- **Inductive step for an application:**
  
  If $(2, (M, N)) \in \Lambda(\text{Var})$ such that $M, N \in \Lambda(\text{Var})$ and $\|\widetilde{M} \in X\| = \|\widetilde{N} \in X\| = 1$, then as in the previous cases, we start with $(2, (M, N)) = (2^A, (M, N)^A)^A$. As $\|\Lambda(\text{Var})\text{-}\text{ind}(X)\| = 1$ we have $\|(2^A, (M, N)^A)^A \in X\| = 1$, so $\|(2, (M, N)) \in X\| = 1$.

This completes the induction.

The equational theory of $\lambda$-calculus, called $\lambda$, is described following [4]:

**Definition 4.5.** Sentences of $\lambda$ are of the form $M = N$, where $M, N \in \Lambda(\text{Var})$. The theory $\lambda$ is generated by the following rules, under modus ponens and $\alpha$-conversion.

(i) $(\lambda x. M) N = M[x := N]$, where $M[x := N]$ is capture-avoiding substitution of $N$ for $x$.
(ii) $M = M$.
(iii) $M = N \Rightarrow N = M$.
(iv) $M = N, N = L \Rightarrow M = L$.
(v) $M = N \Rightarrow MZ = NZ$.
(vi) $M = N \Rightarrow ZM =ZN$.
(vii) $M = N \Rightarrow \lambda x. M = \lambda x. N$.

We write $\lambda \vdash M = N$ to say that the equation $M = N$ is derivable in the theory $\lambda$. 

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Example 4.6. Here is a brief description of the set-theoretic formulation of \( \lambda \). We can simply consider it to be a subset of \( \Lambda(\text{Var}) \times \Lambda(\text{Var}) \), and interpret \( \lambda \vdash M = N \) as \( (M, N) \in \lambda \). We can define a formula \( \lambda \text{-ind}(S) \), which we shall no longer require to be \( \Delta_0 \), that encodes the rules of forming \( \lambda \) from Definition 4.5, and then \( \lambda \) can be characterized as the least set (with respect to \( \subseteq \)) that such that \( \lambda \text{-ind}(\lambda) \). This allows us to define \( \lambda^A \) in \( V^A \).

We can then prove that \( \| \lambda \subseteq \lambda^A \| = 1 \) by induction on the structure of an element of \( \lambda \), as was done to prove \( \| \lambda(\text{Var}) \subseteq \lambda(\text{Var})^A \| = 1 \) in Proposition 4.4.

Definition 4.7. Following [4, Definition 5.4.2], we define the interpretation of \( \lambda \)-terms from \( M \in \Lambda(D, \text{Var}, \mathcal{R}) \) in \( D \), using a valuation \( \rho \) such that \( \text{fv}(M) \subseteq \text{dom}(\rho) \), by

\[
\begin{align*}
[x]_\rho &= \rho(x) & & x \in \text{Var} \\
[K]_\rho &= k & & k \in \mathcal{R} \\
[MN]_\rho &= [M]_\rho \cdot [N]_\rho \\
[\lambda x.M]_\rho &= \text{lam}(\lambda d.[M]_{\rho(x:=d)}),
\end{align*}
\]

where \( \lambda \) is the “meta-lambda”, and \( \rho(x := d) \) is the partial function where \( \text{dom}(\rho(x := d)) = \text{dom}(\rho) \cup \{x\} \) and

\[
\rho(x := d)(y) = \begin{cases} 
\rho(y) & \text{if } y \in \text{dom}(\rho) \setminus \{x\} \\
d & \text{if } y = x
\end{cases}
\]

This is proved to define a model of \( \lambda \)-calculus in [4, Theorem 5.4.4], which is to say, that for all \( M, N \in \Lambda(D, \text{Var}), \) if \( \lambda \vdash M = N \), then for all valuations \( \rho \), \( [M]_\rho = [N]_\rho \). For closed terms \( M \in \Lambda(D, \text{Var}, \mathcal{R}) \), we write \( [M] \) for \( [M]_\rho \).

We now give the Boolean-valued version.

Theorem 4.8. Let \( D \in V^A \) be strict, total and complete as an \( A \)-setoid, and let there be \( \text{lam} \) and \( \cdot \) in \( V^A \) such that \( \| (D, \text{lam}, \cdot) \| = 1 \) is a reflexive dcpo \( = 1 \). Take \( \text{Var} \) to be a set of variables and \( \mathcal{R} \in V^A \) such that \( \| \mathcal{R} \subseteq D \| = 1 \) an \( A \)-set of constants. Then for each \( \rho \) that defines a partial function from \( \text{Var} \) to \( D \), there are functions \( \llbracket . \rrbracket^A_\rho \in \text{SetoidF}_A(\Lambda(D, \text{Var}, \mathcal{R}), D) \) and \( \llbracket . \rrbracket^A_\rho \in \text{SetoidR}_A(\Lambda(\text{Var}), D) \) satisfying Definition 4.7 in the \( A \)-valued sense.

Furthermore, for all \( M, N \in \Lambda(\text{Var}) \), if \( \lambda \vdash M = N \), then for all \( \rho \) we have \( [M]_\rho = [N]_\rho \).

Proof. It follows from Meta-Theorem 4.1 applied to Definition 4.7 that there exist some elements of \( V^A \) representing functions \( \llbracket . \rrbracket^A_\rho \) with the required properties holding with Boolean value 1. So by Corollary 5.15 we can take \( \llbracket . \rrbracket^A_\rho \in \text{Set}_A(\Lambda(D, \text{Var}, \mathcal{R}), D) \) and also \( \llbracket . \rrbracket^A_\rho \in \text{Set}_A(\Lambda(\text{Var}), D) \) for pure terms (using Proposition 4.4). So these define maps in \( \text{SetoidF}_A(\Lambda(D, \text{Var}, \mathcal{R}), D) \) and \( \text{SetoidR}_A(\Lambda(\text{Var}), D) \) by Definition 5.10. Therefore, by completeness of \( D \), we can apply Definition 5.11 to get maps in \( \text{SetoidF}_A(\Lambda(D, \text{Var}, \mathcal{R}), D) \) and \( \text{SetoidF}_A(\Lambda(\text{Var}), D) \).
From this, we get that if $\lambda \vdash M = N$, then $\| (M, N) \|_\lambda = 1$, and therefore for all $\rho$, $\| [M]_\rho = [N]_\rho \|_\rho = 1$. Since $D$ is strict, this implies that actually $[M]_\rho = [N]_\rho$ in the usual sense as well.

4.2 The Engeler Model

We start with a basic result about the power set.

**Proposition 4.9.** For any set $X$, the power set $\mathcal{P}(X)$, when ordered using the subset relation $\subseteq$, is a dcpo (in fact, a complete lattice). If $S, T \in \mathcal{P}(X)$, $S \ll T$ iff $S$ is finite and $S \subseteq T$. The set of finite sets $\mathcal{P}_{\text{fin}}(X)$ is a base, and if $X$ is countable, it is a countable base.

We can then apply this in $V^A$ as follows.

**Proposition 4.10.** For any $X \in V^A$, the $A$-set $\mathcal{P}^A(X)$, ordered with the subset relation, is a continuous lattice with base $\mathcal{P}^A_{\text{fin}}(X)$, interpreted in the $A$-valued sense. As an $A$-setoid, $\mathcal{P}^A(X)$ is complete and total. If $X = \bar{Y}$, then $\mathcal{P}_{\text{fin}}(\bar{Y})$ is a base, and $\mathcal{P}^A(\bar{Y})$ is strict.

**Proof.** We get the first part by applying Meta-Theorem 2.1 to Proposition 4.9. By Theorem 3.17, $\mathcal{P}^A(X)$ is complete, and it is clear from its definition that it is total. Now, if $X = \bar{Y}$, we use the fact that $\| \mathcal{P}_{\text{fin}}(\bar{Y}) = \mathcal{P}_{\text{fin}}^A(\bar{Y}) \| = 1$ (Proposition 2.3), so $\mathcal{P}_{\text{fin}}(\bar{Y})$ is also a base for $\mathcal{P}^A(\bar{Y})$ with Boolean value 1.

Finally, to show that $\mathcal{P}^A(\bar{Y})$ is strict, let $S, T \in \mathcal{P}^A(\bar{Y})$ such that $\| S = T \|_{\mathcal{P}^A(\bar{Y})} = 1$, which is equivalent to $\| S = T \| = 1$. First observe that $\text{dom}(S) = \text{dom}(T)$, and then that for all $\bar{y} \in \text{dom}(\bar{Y})$, we have $\| \bar{y} \in S \| = S(\bar{y})$ and likewise for $T$. Since $\| S = T \| = 1$, for all $y \in Y$ we have

$$S(\bar{y}) = \| \bar{y} \in S \| = \| \bar{y} \in T \| = T(\bar{y}),$$

so $S = T$.

As in [11] and [4, Definition 5.4.5] we can define the Engeler model, an adaptation of the Scott-Plotkin graph model that uses set-theoretic operations instead of Gödel numbering. We define it as follows.

Let $E_0 = \{ \emptyset \}$. We define $E_{n+1} = (\mathcal{P}_{\text{fin}}(E_n) \times E_n) \cup E_n$ and $E = \bigcup_{n=0}^\infty E_n$. Then we have a map $(-, -) : \mathcal{P}_{\text{fin}}(E) \times E \to E$ defined by pairing, since for each $S \in \mathcal{P}_{\text{fin}}(E)$ we have $S \subseteq E_n$ for some $n \in \mathbb{N}$. For reasons that will become clear later, we require that the existence of the Engeler model in the following manner.

**Proposition 4.11.** Let $E$ be a set equipped with an injective mapping $(-, -) : \mathcal{P}_{\text{fin}}(E) \times E \to E$. Then if we define $\text{lam} : [\mathcal{P}(E) \to \mathcal{P}(E)] \to \mathcal{P}(E)$ and $\cdots$ by

$$\text{lam}(f) = \{(K, q) \mid q \in f(K)\} = \{x \in E \mid \exists K \in \mathcal{P}_{\text{fin}}(E), q \in f(K), x = (K, q)\}$$

$$F : X = \{q \in E \mid \exists K \in \mathcal{P}_{\text{fin}}(E).K \subseteq X \text{ and } (K, q) \in F\},$$

What we really need of $E$ is for it to be countable, non-empty, and an algebra of the functor $\mathcal{P}_{\text{fin}} \times \text{Id}$, such that the structure map $\mathcal{P}_{\text{fin}}(E) \times E \to E$ is injective, which implies $E$ must be infinite. We cannot use the initial algebra of $\mathcal{P}_{\text{fin}} \times \text{Id}$, because it is $\emptyset$. 

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where \( f \in [\mathcal{P}(E) \to \mathcal{P}(E)] \), and \( F, X \in \mathcal{P}(E) \). Then \((\mathcal{P}(E), \cdot, \text{lam})\) is a reflexive dcpo.

We omit the proof as it is essentially proved in the references already given.

We can now build the Engeler model in \( VA \), as follows.

**Theorem 4.12.** Let \( E \) be the set defined before Proposition 4.11. \((\mathcal{P}(\mathcal{E}), \|, \cdot, \text{lam})\) is an \( A \)-valued reflexive dcpo, where \( \cdot \) and \( \text{lam} \) are defined by the \( A \)-valued interpretation of the definitions given in Proposition 4.11.

**Proof.** By Meta-Theorem 2.2, \((\cdot, \cdot) \) defines an injective function \( \mathcal{P}(\mathcal{E}) \times \mathcal{E} \to \mathcal{E} \). Since \( \|\mathcal{P}(\mathcal{E})\| = 1 \) by Prop. 2.3, it also defines an injective function \( \mathcal{P}(\mathcal{E}) \times \mathcal{E} \to \mathcal{E} \). We apply Meta-Theorem 2.1 (i) to Prop. 4.11 to conclude that \( \mathcal{P}(\mathcal{E}) \) is a reflexive dcpo. 

By combining Theorem 4.12 with Proposition 4.10, we can apply Theorem 4.8 to obtain a function \([\cdot]^{\mathcal{E}}\) that interprets \( \lambda \)-terms in a manner that respects provable equations between them.

The following lemma shows that if we only use pure \( \lambda \)-terms we do not get anything new by using \( \mathcal{P}(\mathcal{E}) \) instead of \( \mathcal{P}(E) \) as a model of \( \lambda \)-calculus.

**Lemma 4.13.** Let \( M \in \Lambda(\text{Var}) \) and \( \rho \) a valuation such that \( \text{fv}(M) \subseteq \text{dom}(\rho) \), and consider \([\cdot]^{\mathcal{E}}\) and \([\cdot]^{\mathcal{E}}\) as defined for \( \mathcal{P}(\mathcal{E}) \) and \( \mathcal{P}(E) \) respectively. Then

\[
\|\llbracket \hat{M} \rrbracket^{\mathcal{E}}\| = \|\llbracket M \rrbracket^{\mathcal{E}}\| = 1.
\]

In particular, since \( \emptyset = \emptyset \), if \( M \) is closed we have

\[
\|\llbracket M \rrbracket^{\mathcal{E}}\| = \|\llbracket M \rrbracket^{\mathcal{E}}\| = 1.
\]

**Proof.** We prove it by induction on the structure of \( M \). The inductive hypothesis is for the statement to hold for all valuations \( \rho \), we do not do a separate induction proof for each valuation.

- **Base case \( M = x \) with \( x \in \text{Var} \):**
  
  By definition, \([x]^{\mathcal{E}}_{\hat{\rho}} = \hat{\rho}(\hat{x})\), by which we really mean \(\|(\hat{x}, [x]^{\mathcal{E}}_{\hat{\rho}})\| = 1\).
  
  We also have \(\|(x, [x]_{\rho})\| = 1\) and \((\hat{x}, [x]_{\rho})^{\mathcal{E}} = (x, [x]_{\rho})\), so since \(\hat{\rho}\) is a partial function internally in \( VA \), we can conclude that \(\|\llbracket x \rrbracket^{\mathcal{E}}_{\hat{\rho}} = \|\llbracket x \rrbracket^{\mathcal{E}}_{\rho}\| = 1\).

- **Inductive step for an application:**
  
  We first prove the key step, which is that if \( S, T \subseteq E \), then

  \[
  \|\hat{S} \cdot \hat{T} = \|\llbracket S \cdot T \rrbracket^{\mathcal{E}} = 1.
  \] (4.14)
For all \( q \in E \), we have
\[
\|q \in \tilde{S} \cdot \tilde{T}\| = \|\exists K \in \mathcal{P}_{\text{fin}}(E), K \subseteq \tilde{T} \text{ and } (K, \tilde{q}) \in \tilde{S}\|
\]
\[
= \|\exists K \in \mathcal{P}_{\text{fin}}(E), K \subseteq \tilde{T} \text{ and } (K, \tilde{q})^A \in \tilde{S}\| \quad \text{Prop. 2.3}
\]
\[
= \bigvee_{K \in \mathcal{P}_{\text{fin}}(E)} \| K \subseteq \tilde{T}\| \land \| (K, \tilde{q}) \in \tilde{S}\|
\]
\[
= \bigvee_{K \in \mathcal{P}_{\text{fin}}(E)} \| K \subseteq \tilde{T}\| \land \| (K, q) \in \tilde{S}\|
\]
\[
= [\exists K \in \mathcal{P}_{\text{fin}}(E), K \subseteq T \text{ and } (K, q) \in \tilde{S}]_A
\]
\[
= [q \in S \cdot T]_A = \|\tilde{q} \in \tilde{S} \cdot \tilde{T}\|,
\]
where we have used the \( A \)-valued Iverson bracket \([\_]_A\) above.

Then the proof of this step proceeds as follows, where for convenience we write at each step simply \( A = B \) instead of \( \|A = B\| = 1\):
\[
[MN]_\rho^A = [\hat{M} \hat{N}]_\rho^A
\]
\[
= [\hat{M}]_\rho^A \cdot [\hat{N}]_\rho^A
\]
\[
= [\hat{M}]_\rho^A \cdot [\hat{N}]_\rho^A \quad \text{by inductive hypothesis}
\]
\[
= [M]_\rho^A \cdot [N]_\rho^A
\]
\[
= [MN]_\rho^A.
\]

- Inductive step for a \( \lambda \)-abstraction:

For this step, we first point out that when interpreting \([\lambda x. M])_\rho\) in the Engeler model, we can simplify the expression
\[\text{\textbf{lam}}(@d. [M])_\rho(x:=d) = \{(K, q) \in E \mid q \in [M]_\rho(x:=K)\}\]

So for \( p \in E \) we have:
\[
\|\tilde{p} \in [\lambda x. M]_\rho^A\|
\]
\[
= \|\exists K \in \mathcal{P}_{\text{fin}}(E), q \in \mathcal{E}. \tilde{p} = (K, q) \text{ and } q \in [M]_\rho^A(x:=K)\|
\]
\[
= \|\exists K \in \mathcal{P}_{\text{fin}}(E), \tilde{q} \in \mathcal{E}. \tilde{p} = (K, \tilde{q}) \text{ and } \tilde{q} \in [M]_\rho^A(x:=K)\| \quad \text{Prop. 2.3}
\]
\[
= \|\exists K \in \mathcal{P}_{\text{fin}}(E), \tilde{q} \in \mathcal{E}. \tilde{p} = (K, \tilde{q}) \text{ and } \tilde{q} \in [M]_\rho^A(x:=K)\| \quad \text{by ind. hyp.}
\]
\[
= [p \in [\lambda x. M]_\rho^A] = \|\tilde{p} \in [\lambda x. M]_\rho^A\|.
\]

\[\square\]

### 4.3 Injective Spaces and Oracles

We can encode the Booleans \( \{\bot, \top\} \), and natural numbers \( \mathbb{N} \) in \( \lambda \)-calculus in several ways, such as with the Church numerals. The details of the encoding do not matter, only the following.
Definition 4.15. Let $(D, \text{fun}, \text{lam})$ be a reflexive dcpo, $\bot, \top$ be closed $\lambda$-terms, and $(c_n)_{n \in \mathbb{N}}$ be a family of closed $\lambda$-terms. Then $(D, \text{fun}, \text{lam}, \bot, \top, (c_n)_{n \in \mathbb{N}})$ is a reflexive dcpo with numerals if

(i) $\llbracket \bot \rrbracket$ and $\llbracket \top \rrbracket$ are distinct elements of $D$.

(ii) There exists a closed $\lambda$-term $\text{id}$ such that for all $\lambda$-terms $M, N$, $\lambda \vdash \text{id}TN = M$ and $\lambda \vdash \text{id}\bot M = N$.

(iii) There exist closed $\lambda$-terms $\text{succ}$ and $\text{pred}$ representing the successor and predecessor operations on $(c_n)_{n \in \mathbb{N}}$, i.e. for all $n \in \mathbb{N}$, $\lambda \vdash \text{succ}c_n = c_{n+1}$ and $\lambda \vdash \text{pred}c_{n+1} = c_n$ and $\lambda \vdash \text{pred}c_0 = c_0$.

(iv) There exists a closed $\lambda$-term $0^\uparrow$ such that $\lambda \vdash 0^\uparrow c_0 = \top$ and for all $n > 0$, $\lambda \vdash 0^\uparrow c_n = \bot$.

Proposition 4.16. The Church Booleans and Church numerals make the Engeler model into a reflexive continuous lattice with numerals.

The proof is a standard working out of basic facts about Church numerals and Booleans.

Corollary 4.17. The Engeler model in $V^A$, as described in Theorem 4.12, is a reflexive continuous lattice with numerals, when given (of) the Church Booleans and Church numerals.

Proof. By Lemma 4.13, $\llbracket \bot \rrbracket^A = \llbracket \top \rrbracket = 1$, and likewise for $\top$ and the Church numerals.

Then part (i) of Definition 4.15 is a $\Delta_0$ statement, so it follows by applying Meta-Theorem 4.16 to Proposition 4.16. Parts (ii) - (iv) then follow by Example 4.6.

In the following, for a set $A \subseteq \mathbb{N}$, we write $\chi_A$ for the function $A \to 2$ such that

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases},$$

while if $\{\bot, \top\}$ are part of the structure of a reflexive dcpo with numerals $D$, we write $\chi_A^D$ for the function $\mathbb{N} \to \{\llbracket \bot \rrbracket, \llbracket \top \rrbracket\}$ such that:

$$\chi_A^D(x) = \begin{cases} \llbracket \top \rrbracket & \text{if } x \in A \\ \llbracket \bot \rrbracket & \text{if } x \notin A \end{cases}.$$
(i) For every computable function \( f : \mathbb{N}^m \to \mathbb{N} \), there exists a closed \( \lambda \)-term \( M \) such that for all \((n_1, \ldots, n_m) \in \mathbb{N}^m\), \( \lambda \vdash M c_{n_1} \cdots c_{n_m} = c f(n_1, \ldots, n_m) \), and therefore \( \llbracket M c_{n_1} \cdots c_{n_m} \rrbracket = \llbracket c f(n_1, \ldots, n_m) \rrbracket \).

(ii) \( \llbracket \bot \rrbracket \) and \( \llbracket c_n \rrbracket \) are discrete in the Scott topology of \( D \), and also are distinct elements, i.e. \( \llbracket c_m \rrbracket = \llbracket c_n \rrbracket \) implies \( m = n \).

If \( D \) is additionally a continuous lattice, then:

(iii) For every set \( A \subseteq \mathbb{N} \), there is a \( d_g \in D \) such that for all \( n \in \mathbb{N} \), \( d_g : \llbracket c_n \rrbracket = \chi A^D(n) \).

**Proof.** (i) By parts (iii) and (iv) of Definition 6.4.3 and Proposition 6.4.3, \( (c_n)_{n \in \mathbb{N}} \) is an adequate numeral system by Definition 6.4.1 and 6.4.2, which means that for each computable function there exists a closed \( \lambda \)-term \( M \) such that \( \lambda \vdash M c_n \). As \( (D, \llbracket \cdot \rrbracket) \) is a \( \lambda \)-model by Theorem 5.4.4, this implies \( \llbracket M c_n \rrbracket = \llbracket c f(n) \rrbracket \).

(ii) As Scott topologies are \( T_0 \), part (i) of Definition 6.4.3 implies that there is a Scott-open set \( U \subseteq D \) containing one of \( \{ \llbracket \top \rrbracket, \llbracket \bot \rrbracket \} \) but not the other. We start under the assumption that \( \llbracket \top \rrbracket \in U \) and \( \llbracket \bot \rrbracket \notin U \). This implies that \( \llbracket \top \rrbracket \) is an open subset of the subspace \( \{ \llbracket \top \rrbracket, \llbracket \bot \rrbracket \} \).

The map \( \text{fun}(\llbracket \cdot \rrbracket) : D \to D \) is Scott continuous, and \( \text{fun}(\llbracket \bot \rrbracket)(\llbracket \bot \rrbracket) = \llbracket \top \rrbracket \) and vice-versa. So \( \text{fun}(\llbracket \bot \rrbracket)^{-1}(U) \) is an open set containing \( \llbracket \bot \rrbracket \), but not \( \llbracket \top \rrbracket \), proving that \( \{ \llbracket \top \rrbracket, \llbracket \bot \rrbracket \} \) is an open subset of the subspace \( \{ \llbracket \top \rrbracket, \llbracket \bot \rrbracket \} \). This proves that \( \{ \llbracket \top \rrbracket, \llbracket \bot \rrbracket \} \) is a discrete subspace of \( D \), and the proof starting with \( \llbracket \bot \rrbracket \in U \) is similar.

To prove the discreteness and distinctness of \( \{ \llbracket c_n \rrbracket \}_{n \in \mathbb{N}} \), we will need the fact that there exist closed \( \lambda \)-terms \( m^n \) such that \( \lambda \vdash m^n c_n = \top \) if \( m = n \) and \( \lambda \vdash m^n c_n = \bot \) otherwise. It is not difficult to prove directly that we can take \( m^n = \lambda m. \text{if}(0^m) \cdot (0^m \cdot \text{pred}(m)) \), and \( m^n = \lambda m. n - 1'(\text{pred}(m)) \). It follows that \( \text{fun}(m^n)(\llbracket c_m \rrbracket) = \llbracket \top \rrbracket \) if \( n = m \) and \( \llbracket \bot \rrbracket \) otherwise.

Now, if \( m \neq n \) are elements of \( \mathbb{N} \), we have \( \lambda \vdash m^n c_m = \top \), and \( \lambda \vdash m^n c_n = \bot \), so \( \text{fun}(m^n)(\llbracket c_m \rrbracket) = \llbracket \top \rrbracket \neq \llbracket \bot \rrbracket = \text{fun}(m^n)(\llbracket c_n \rrbracket) \). As \( \text{fun}(m^n) \) is a function, it follows that \( \llbracket c_m \rrbracket \neq \llbracket c_n \rrbracket \).

To prove the discreteness of \( \{ \llbracket c_n \rrbracket \}_{n \in \mathbb{N}} \), we show that for all \( m \in \mathbb{N} \), the singleton \( \{ \llbracket c_m \rrbracket \} \) is relatively open in \( \{ \llbracket c_n \rrbracket \}_{n \in \mathbb{N}} \). Let \( U \subseteq D \) be a Scott-open set such that \( \{ \top \} \in U \), but \( \{ \bot \} \notin U \). Then \( V = \text{fun}(m^n)^{-1}(U) \) is a Scott-open set such that \( \llbracket m \rrbracket \in V \) but \( \llbracket n \rrbracket \notin V \) for all \( n \in \mathbb{N} \) such that \( n \neq m \).

(iii) Given \( A \subseteq \mathbb{N} \), define \( g : \{ \llbracket c_n \rrbracket \}_{n \in \mathbb{N}} \to D \) by \( g(\llbracket c_n \rrbracket) = \chi A^D(n) \). By the discreteness of \( \{ \llbracket c_n \rrbracket \}_{n \in \mathbb{N}} \), proved in the previous part, this is continuous. Since \( D \) is a continuous lattice, and therefore injective by Theorem 2.12, \( g \) extends to a Scott-continuous map \( \overline{g} : D \to D \), and since \( (D, \text{lam}, \text{fun}) \) is a reflexive dcpo we can define \( d_g = \text{lam} \overline{g} \), and then for all \( n \in \mathbb{N} \):

\[
d_g : \llbracket c_n \rrbracket = \overline{g}(\llbracket c_n \rrbracket) = \chi A^D(n).
\]

\(^{12}\)Strictly speaking, we must use \( \lambda x. \text{if}(0^x) \cdot \text{TF} \), because Barendregt requires \( T \) and \( F \) to be the Church Booleans, whereas we allow \( \bot \) and \( \top \) to be any representation of Booleans.
We can formulate the theory of many-one degrees [29, Definition 4.8 (i), (iii)] in any continuous lattice with numerals.

**Proposition 4.19.** Let $D$ be a reflexive continuous lattice with numerals. The following are equivalent for Proposition 4.19.

(i) $S_1 \preceq_m S_2$, i.e. there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $S_1 = f^{-1}(S_2)$.

(ii) $\chi_{S_1} \preceq_m \chi_{S_2}$, i.e. there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi_{S_1} = \chi_{S_2} \circ f$.

(iii) There exists a closed $\lambda$-term $M \in \Lambda(D, \text{Var})$ such that for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\lambda \vdash Mc_n = c_m$, and there exist $d_{S_1}, d_{S_2} \in D$ such that for all $n \in \mathbb{N}$, $[d_{S_1}c_n] = \chi_{S_1}^D(n)$, $[d_{S_2}c_n] = \chi_{S_2}^D(n)$ and $[d_{S_2}(Mc_n)] = [d_{S_1}c_n]$. 

(iv) There exists a closed $\lambda$-term $M \in \Lambda(D, \text{Var})$ such that for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\lambda \vdash Mc_n = c_m$, and for all $d_{S_1}, d_{S_2} \in D$ such that for all $n \in \mathbb{N}$, $[d_{S_1}c_n] = \chi_{S_1}^D(n)$ and $[d_{S_2}c_n] = \chi_{S_2}^D(n)$, we have that for all $n \in \mathbb{N}$, $[d_{S_2}(Mc_n)] = [d_{S_1}c_n]$.

**Proof.**

• (i) ⇔ (ii):
Let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary total function. For any $n \in \mathbb{N}$, we have $(\chi_{S_2} \circ f)(n) = 1$ iff $f(n) \in S_2$ iff $n \in f^{-1}(S_2)$ iff $\chi_{f^{-1}(S_2)}(n) = 1$. So if (i) holds, and $f$ is a total computable function such that $f^{-1}(S_2) = S_1$, then we have (ii). Likewise, if (ii) holds, and $f$ is a total computable function such that $\chi_{S_1} = \chi_{S_2} \circ f$, then $\chi_{S_1} = \chi_{f^{-1}(S_2)}$ so (i) holds.

• (ii) ⇒ (iii):
Suppose that there exists a total computable $f : \mathbb{N} \to \mathbb{N}$ such that $\chi_{S_1} = \chi_{S_2} \circ f$. By Lemma [13], (i), there is a closed $\lambda$-term $M$ such that $\lambda \vdash Mc_n = c_{f(n)}$. By (iii) of the same lemma, there are $d_{S_1}, d_{S_2} \in D$ such that for all $n \in \mathbb{N}$, $d_{S_1} \cdot [c_n] = [\bot]$ if $\chi_{S_1}(n) = 0$ and $d_{S_1} \cdot [c_n] = [\top]$ if $\chi_{S_1}(n) = 1$, and likewise for $d_{S_2}$. Then for all $n \in \mathbb{N}$ we have $[d_{S_1}c_n] = \chi_{S_1}^D(n)$ and:

$$[d_{S_2}(Mc_n)] = [d_{S_2}c_{f(n)}] = d_{S_2} \cdot [c_{f(n)}] = \chi_{S_2}^D(f(n)) = \chi_{S_2}^D(n).$$

• (iii) ⇒ (iv):
Suppose that (iii) holds, and let $d_{S_1}^*, d_{S_2}^* \in D$ such that for all $n \in \mathbb{N}$, $[d_{S_1}^*c_n] = \chi_{S_1}^D(n)$ and $[d_{S_2}^*c_n] = \chi_{S_2}^D(n)$. Then we have $[d_{S_1}^*c_n] = [d_{S_1}c_n]$ and likewise for $S_2$. Let us write $f(n)$ for the $m \in \mathbb{N}$ such that $[Mc_n] = [c_n]$. Then we have:

$$[d_{S_2}^*(Mc_n)] = d_{S_2} \cdot [c_{f(n)}] = [d_{S_2}(Mc_n)] = [d_{S_2}c_n] = [d_{S_2}^*c_n],$$

using (iii) for the second-to-last step, and proving (iv).
• (iv) ⇒ (ii):

Suppose that (iv) holds. The mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $\lambda \mapsto M_{c_n} = c_{f(n)}$ is computable, and total by the assumption of (iv). By Lemma 4.18 (iii), there exist $d_{S_1}, d_{S_2} \in D$ such that for all $n \in \mathbb{N}$, $[d_{S_1}c_n] = \chi_{S_1}^D(n)$ and $[d_{S_2}c_n] = \chi_{S_2}^D(n)$, so by (iv):

$$\chi_{S_1}^D(n) = [d_{S_1}c_n] = [d_{S_2}(M_{c_n})] = [d_{S_2}(f(n))] = \chi_{S_2}^D(f(n)).$$

From this it follows that $\chi_{S_2} \circ f = \chi_{S_1}$, and we have proved (ii).

\[\square\]

Using random variables, we will be able in Theorem 5.9 to prove the existence of incomparable many-one degrees.

## 5 Random Variables

A *negligibility space* $(X, \Sigma, N)$ is a measurable space $(X, \Sigma)$ equipped with a $\sigma$-ideal $N$ such that $\Sigma/N$ is a complete Boolean algebra, not just $\sigma$-complete. We introduce the notation $A(X)$ for $\Sigma/N$, as it is the complete Boolean algebra associated to $X$ or simply the algebra associated to $X$.

For any measure space $(X, \Sigma, \mu)$, we can define the null ideal $N(\mu)$ to be the $\sigma$-ideal of sets $N \in \Sigma$ with $\mu(N) = 0$. Then $\Sigma/N(\mu)$ is a $\sigma$-complete Boolean algebra. If $\mu$ is a probability measure, then $\Sigma/N(\mu)$ is in fact a complete Boolean algebra [13, 322B, 322C], so $(X, \Sigma, N(\mu))$ is a negligibility space. The complete Boolean algebra $A(X, \Sigma, N(\mu))$ is called the measure algebra of $(X, \Sigma, N(\mu))$.

Let $Y$ be a countable set. We first define, for each $y \in Y$, $B_y = \{S \subseteq Y \mid y \in S\}$, which form a subbasis of open sets for the positive topology (which equals the Scott topology) on $\mathcal{P}(Y)$. All notions of measurability for $\mathcal{P}(Y)$ will be with respect to the Borel $\sigma$-algebra defined by this topology, and since this is a second countable topology, $(B_y)_{y \in Y}$ is also a countable generating set for this $\sigma$-algebra.

**Definition 5.1.** Let $X = (X, \Sigma_X)$ be a measurable space and $Y$ a countable set. We define $L^0(X; \mathcal{P}(Y))$ to be the set of measurable functions $X \to \mathcal{P}(Y)$, using the Borel $\sigma$-algebra of the Scott topology of $\mathcal{P}(Y)$. We define a $\Sigma_X$-valued order as follows:

$$\|a \leq b\|_{L^0} = \{x \in X \mid a(x) \subseteq b(x)\}.$$

The corresponding notion of equality is

$$\|a = b\|_{L^0} = \{x \in X \mid a(x) = b(x)\}.$$  

If $X = (X, \Sigma_X, N_X)$ is a negligibility space, define $L^0(X; \mathcal{P}(Y))$ to be the set of measurable functions $L^0(X; \mathcal{P}(Y))$ modulo the relation $a \sim b$ if $X \setminus \|a = b\|_{L^0} \in N_X$, which is the usual “almost everywhere” equivalence. Then $\|\cdot\|_{L^0}$ and $\|\cdot\|_{L^0}$, defined by composing the corresponding notions for $L^0(X; \mathcal{P}(Y))$ with $[\cdot] : \Sigma \to A(X)$, are well-defined and make $L^0(X; \mathcal{P}(Y))$ into an $A(X)$-poset.
Proof. We prove that \( \|a \leq b\|_{\mathcal{L}^0} \) is measurable as follows. First, define \( \gamma \subseteq \mathcal{P}(Y) \times \mathcal{P}(Y) \) to be the graph of the \( \subseteq \) relation, i.e. \( \gamma = \{ (S, T) \in \mathcal{P}(Y) \times \mathcal{P}(Y) \mid S \subseteq T \} \).

\[ \|a \leq b\|_{\mathcal{L}^0} = (a, b)^{-1}(\gamma), \] so since \( (a, b) : X \to \mathcal{P}(Y) \times \mathcal{P}(Y) \) is measurable, we only need to show that \( \gamma \) is measurable with respect to the product measurable space structure. We have

\[
\gamma = \{ (S, T) \in \mathcal{P}(Y) \times \mathcal{P}(Y) \mid \forall y \in Y, y \in S \Rightarrow y \in T \}
= \bigcap_{y \in Y} \{ (S, T) \in \mathcal{P}(Y) \times \mathcal{P}(Y) \mid y \notin S \text{ or } y \in T \}
= \bigcap_{y \in Y} \{ (S, T) \in \mathcal{P}(Y) \times \mathcal{P}(Y) \mid y \notin S \} \cup \{ (S, T) \in \mathcal{P}(Y) \times \mathcal{P}(Y) \mid y \in T \}
= \bigcap_{y \in Y} (\mathcal{P}(Y) \setminus B_y) \times \mathcal{P}(Y)) \cup (\mathcal{P}(Y) \times B_y).
\]

Measurability then follows from the countability of \( Y \).

For the well-definedness of \( \|\leq\|_{\mathcal{L}^0} \), it is clear that if \( a \sim a' \) and \( b \sim b' \), then \( \|a \leq b\|_{\mathcal{L}^0} \) and \( \|a' \leq b'\|_{\mathcal{L}^0} \) can only differ where \( a \) differs from \( a' \) or \( b \) differs from \( b' \), and the set of all such points is in \( \mathcal{N}_X \). It is easy to deduce the transitivity (in the sense of Definition \( \ref{def:transitivity} \) (i)) from transitivity of \( \subseteq \), and part (ii) of the definition of an \( A(X) \)-poset follows because \( \|a \leq a\|_{\mathcal{L}^0} = 1 \) for all \( a \in L^0(X; \mathcal{P}(Y)) \).

We define \( G_X : L^0(X; \mathcal{P}(Y)) \to \mathcal{P}^{A(X)}(\hat{Y}) \) as follows, with \( a \in L^0(X; \mathcal{P}(Y)) \) and \( y \in Y \):

\[
G_X([a])(y) = [a^{-1}(B_y)] = \{ x \in X \mid y \in a(x) \} \quad (\text{5.2})
\]

The first way of giving the definition shows clearly that \( a^{-1}(B_y) \in \Sigma \) (by the measurability of \( a \)).

**Proposition 5.3.** For any negligibility space \((X, \Sigma, \mathcal{N})\) and any countable set \( Y \), \( G_X \) is a strict isomorphism of \( A(X) \)-posets \( L^0(X; \mathcal{P}(Y)) \to \mathcal{P}^{A(X)}(\hat{Y}) \).

Proof. We show that for all \( a, b \in L^0(X; \mathcal{P}(Y)) \), \( ||G_X([a])||_{\mathcal{P}^{A(X)}(\hat{Y})} = ||a||_{\mathcal{L}^0} \leq ||b||_{\mathcal{L}^0} \). This shows, in one go, that \( G_X \) is an \( A(X) \)-monotone function and also injective (using the fact that \( L^0(X; \mathcal{P}(Y)) \) is total, i.e. \( \varepsilon_{L^0(X; \mathcal{P}(Y))}(a) = [X] \) for all \( a \in L^0(X; \mathcal{P}(Y)) \)).
We start by expanding the definitions:

\[
\|G_X([a])\| \leq G_X([b]) \|_{\mathcal{P}(X)}(Y)
\]
\[
= \|G_X([a])\| \leq G_X([b])
\]
\[
= \bigwedge_{y \in Y} \|t \in G_X([a])\| \Rightarrow \|t \in G_X([b])\|
\]
\[
= \bigwedge_{y \in Y} G_X([a])(\hat{y}) = G_X([b])(\hat{y})
\]
\[
= \bigwedge_{y \in Y} \{x \in X \mid y \in a(x)\} \Rightarrow \{x \in X \mid y \in b(x)\}
\]
\[
= \bigcap_{y \in Y} \{x \in X \mid y \in a(x) \text{ implies } y \in b(x)\}
\]
\[
= \{x \in X \mid \forall y \in Y, y \in a(x) \text{ implies } y \in b(x)\}
\]
\[
= \{x \in X \mid a(x) \subseteq b(x)\} = \|[a] \leq [b]\|_{L^0}.
\]

Since \(L^0(X; \mathcal{P}(Y))\) is a strict \(A(X)\)-setoid, and \(\mathcal{P}^{A(X)}(Y)\) is total, implying that \(G_X\) is an injective function. So to prove that \(G_X\) is a strict isomorphism, we only need to show that \(G_X\) is surjective.

Let \(b \in \mathcal{P}^{A(X)}(Y)\). For each \(y \in Y\), pick \(S_y \in \Sigma\) such that \([S_y] = b(\hat{y})\). Define a function \(a : X \rightarrow \mathcal{P}(Y)\) by \(a(x) = \{y \in Y \mid x \in S_y\}\). For each \(y \in Y\), we have

\[
a^{-1}(B_y) = \{x \in X \mid a(x) \in B_y\} = \{x \in X \mid y \in a(x)\} = \{x \in X \mid x \in S_y\}
\]

\[
= S_y \in \Sigma.
\]

Since \((B_y)_{y \in Y}\) generates the Borel \(\sigma\)-algebra of \(\mathcal{P}(Y)\), this implies that \(a\) is measurable. For all \(y \in Y\), we then have

\[
G_X([a])(\hat{y}) = [a^{-1}(B_y)] = [S_y] = b(\hat{y}),
\]

and therefore \(G_X([a]) = b\), as required to prove \(G_X\) surjective.

Since \(\mathcal{P}(E)\) is a reflexive dcpo and has a binary operation \(\cdot\) for application, for any measurable space \(X\) we can extend this pointwise to \(L^0(X; \mathcal{P}(E))\) by defining for \(a, b \in L^0(X; \mathcal{P}(E))\) and \(x \in X\):

\[
(a \cdot b)(x) = a(x) \cdot b(x).
\]

For a negligibility space \(X\) we can then extend this to \(L^0(X; \mathcal{P}(E))\) by defining

\[
[a] \cdot [b] = [a \cdot b].
\]

**Proposition 5.4.** Let \((X, \Sigma, \mathcal{N})\) be a negligibility space. Then the above definition of \(\cdot : L^0(X; \mathcal{P}(E)) \times L^0(X; \mathcal{P}(E)) \rightarrow L^0(X; \mathcal{P}(E))\) is well-defined, and \(G_X : L^0(X; \mathcal{P}(E)) \rightarrow \mathcal{P}^{A(X)}(E)\) is an “application homomorphism”, i.e. for all \([a], [b] \in L^0(X; \mathcal{P}(E))\),

\[
G_X([a] \cdot [b]) = G_X([a]) \cdot G_X([b]).
\]
Proof. In fact, we will prove the first part by deducing it from the second part. So let \( a, b \in L^0(X; \mathcal{P}(E)) \). Then for all \( q \in E \) we have

\[
(G_X([a]) \cdot G_X([b]))(\bar{q}) = \|\exists K \in \mathcal{P}_{fin}(E). K \subseteq G_X([b]) \text{ and } (K, \bar{q})^{A(X)} \in G_X([a])\|
\]

\[
= \|\exists K \in \mathcal{P}_{fin}(E). K \subseteq G_X([b]) \text{ and } (K, \bar{q})^{A(X)} \in G_X([a])\|
\]

\[
= \Bigg\| \bigcap_{K \in \mathcal{P}_{fin}(E)} (\exists K \subseteq G_X([b]) \land (K, \bar{q})^{A(X)} \in G_X([a])) \Bigg\|
\]

so all together we have proved that \( G_X([a]) \cdot G_X([b]) = G_X([a \cdot b]) \). In passing, we have proved that for all \( q \in E \), \( (a \cdot b)^{-1}(B_q) \) is measurable, and therefore that \( a \cdot b \in L^0(X; \mathcal{P}(E)) \).

If \( a', b' \in L^0(X; \mathcal{P}(E)) \) such that \( [a'] = [a] \) and \( [b'] = [b] \), then by Proposition 5.3 and the fact that \( \mathcal{P}^{A(X)}(E) \) is a strict \( A(X) \)-setoid:

\[
G_X([a \cdot b]) = G_X([a]) \cdot G_X([b]) = G_X([a']) \cdot G_X([b']) = G_X([a' \cdot b']),
\]

and therefore \( [a \cdot b] = [a' \cdot b'] \), so the definition \( [a \cdot b] = [a \cdot b] \) genuinely defines a function \( L^0(X; \mathcal{P}(E)) \times L^0(X; \mathcal{P}(E)) \to L^0(X; \mathcal{P}(E)) \). Putting this together with what we proved above gives us \( G_X([a]) \cdot G_X([b]) = G_X([a \cdot b]) \). \( \square \)

We introduce the following notation. If \( S \subseteq E \), define \( K_S \in L^0(X; \mathcal{P}(E)) \) to be the function taking the constant value \( S \). This relates to \( - \) in the following way.

**Lemma 5.5.** Let \( Y \) be an arbitrary countable set. For all \( S \subseteq Y \) we have

\[
\|G_X([K_S])\| = \|S\| = 1
\]

**Proof.** It suffices to show that for all \( y \in Y \), \( \|\bar{y} \in G_X([K_S])\| = \|\bar{y} \in \hat{S}\| \). We have

\[
\|y \in G_X([K_S])\| = G_X([K_S])(\bar{y}) = \{x \in X \mid y \in K_S(x)\} = \{x \in X \mid y \in S\}
\]

\[
= \|\bar{y} \in \hat{S}\|.
\]

\( \square \)
Recall that a subbase of clopens for the product topology of \(2^\omega\) is given by
the sets \((C_{n,b})_{n \in \omega, b \in 2}\), where
\[
C_{n,b} = \{f \in 2^\omega \mid f(n) = b\}.
\]
These sets also generate the Borel \(\sigma\)-algebra of \(2^\omega\).

In the following, we will also have to consider the (isomorphic) product space
\(2^\omega \times 2^\omega\), but for which we will need to describe the subsets in more detail. So
for \(i \in \{1, 2\}\), \(n \in \omega\) and \(b \in 2\) we write
\[
D_{i,n,b} = \{(f_1, f_2) \in 2^\omega \times 2^\omega \mid f_i(n) = b\}
\]
Then for all \(i \in \{1, 2\}\), \(n \in \omega\) and \(b \in 2\) we have
\[
D_{i,n,b} = \pi_i^{-1}(C_{n,b}).
\]
Let \(X = 2^\omega \times 2^\omega\) equipped with its Borel \(\sigma\)-algebra as a measurable space.
We define \(\mu_X\) to be the usual independent fair coin measure, which is to say,
it is the unique measure such that for all finitely-supported partial functions
\(f_1, f_2 : \omega \to 2\)
\[
\mu\left(\bigcap_{n \in \text{dom}(f_1)} D_{1,n,f_1(n)} \cap \bigcap_{n \in \text{dom}(f_2)} D_{2,n,f_2(n)}\right) = 2^{-\left(|\text{dom}(f_1)| + |\text{dom}(f_2)|\right)}.
\]
We take the null ideal of this measure as the negligible sets of \(X\). Then \(\tau_1, \tau_2\)
are random variables in \(L^0(X; 2^\omega)\). They define \(S_1, S_2 \in \mathcal{P}^{A(X)}(\tilde{\omega})\) as follows,
taking \(i \in \{1, 2\}\):
\[
S_i(\hat{n}) = [\tau_i^{-1}(C_{n,1})] = [D_{i,n,1}].
\]
We remind the reader at this point that \(\|\hat{n}\| = S_i(\hat{n})\) because of the way
Boolean-valued equality behaves for elements of \(\tilde{\omega}\).

**Lemma 5.7.** For all \(f : \omega \to \omega\), neither \(S_1\) nor \(S_2\) is the preimage of a finite
\(A(X)\)-subset of \(\tilde{\omega}\), i.e. \(\exists K \in \mathcal{P}^{A(X)}(\tilde{\omega}) \forall n < \omega.n \in S_i \Leftrightarrow f(n) \in K = 0\), and
likewise for \(S_2\).

**Proof.** We only give the proof for \(S_1\), as the proof for \(S_2\) is essentially the same.
Using Proposition 2.3 we have
\[
\|\exists K \in \mathcal{P}^{A(X)}(\tilde{\omega}) \forall n < \omega.n \in S_1 \Leftrightarrow f(n) \in K\| = \|
\exists K \in \mathcal{P}^{A(X)}(\tilde{\omega}) \forall n < \omega.n \in S_1 \Leftrightarrow f(n) \in K\|
\]
\[
= \bigvee_{K \in \mathcal{P}^{A(X)}(\tilde{\omega})} \bigwedge_{n < \omega} \|\hat{n} \in S_1\| \Leftrightarrow \|f(n) \in K\|
\]
\[
= \bigvee_{K \in \mathcal{P}^{A(X)}(\tilde{\omega})} \bigwedge_{n < \omega} \|D_{1,n,1}\| \Leftrightarrow \|f(n) \in K\|_{A(X)}.
\]
This being equal to 0 is equivalent to the statement that for all \(K \in \mathcal{P}^{A(X)}(\tilde{\omega})\):
\[
\bigwedge_{n < \omega} \|D_{1,n,1}\| \Leftrightarrow \|f(n) \in K\|_{A(X)} = 0.
\]

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Then

\[ \bigwedge_{n \in \omega} [D_{1,n}] \equiv [f(n) \in K]_{A} = \bigwedge_{n \in \omega} [D_{1,n}[f(n) \in K_2] = \left[ \bigcap_{n \in \omega} D_{1,n}[f(n) \in K_2] \right] \]

Since it is an infinite intersection of distinct sets for distinct indices, the measure of the intersection above \( \leq 2^n \) for all \( n \), and so it has measure zero, so the above element has Boolean value 0, as required.

**Proposition 5.8.** Then for all functions \( f : \omega \to \omega \), we have \( \| S_1 = \hat{f}^{-1}(S_2) \| = 0 \) and

\( \| S_2 = \hat{f}^{-1}(S_1) \| = 0 \), i.e. these sets cannot be mapped to each other by any classical function.

**Proof.** We only give the proof that \( \| S_1 = \hat{f}^{-1}(S_2) \| = 0 \), as the proof of the other way is analogous. We start by ruling out some possibilities of what \( f \) is like. Assume for a contradiction that \( \| S_1 = \hat{f}^{-1}(S_2) \| > 0 \). If the image of \( f \) is finite, then since \( \hat{f}^{-1}(S) = \hat{f}^{-1}(S \cap \text{im}(f)) \) classically, by Meta-Theorem 2.1 we have \( \| S_1 = \hat{f}^{-1}(S_2) \| = \| S_1 = \hat{f}^{-1}(S_2 \cap \text{im}(f)) \| \) and that \( \| S_2 \cap \text{im}(f) \| \in \mathcal{P}_{\omega}(\omega) \| = 1 \). But this contradicts Lemma 5.7 so either \( \| S_1 = \hat{f}^{-1}(S_2) \| = 0 \) after all, or the image of \( f \) is infinite, so we reduce to the latter case.

If the image of \( f \) is infinite, then by the well-orderedness of \( \omega \) there exists an infinite set \( I \subseteq \omega \) such that \( f|_I \) is injective, and we fix such a set until the end of the proof.

We now convert \( \| S_1 = \hat{f}^{-1}(S_2) \| \) into Boolean-valued set theory.

\[ \| S_1 = \hat{f}^{-1}(S_2) \| = \| \forall n \in \omega, n \in S_1 \implies \hat{f}(n) \in S_2 \| = \bigwedge_{n \in \omega} \| \hat{\bar{n}} \in S_1 \implies \| \hat{\bar{f}}(\bar{n}) \in S_2 \| \]

We can then analyse what \( \| \hat{\bar{f}}(\bar{n}) \in S_2 \| \) means as a formula in set theory:

\[ \| \hat{\bar{f}}(\bar{n}) \in S_2 \| = \| \forall m \in \bar{\omega}, (\bar{n}, m) \in \hat{f}^{-1} \implies m \in S_2 \| = \bigwedge_{m \in \omega} \| (\bar{n}, m) \in \hat{f}^{-1} \| \implies \| m \in S_2 \| = \| \hat{\bar{f}}(\bar{n}) \in S_2 \| . \]

So

\[ \| S_1 = \hat{f}^{-1}(S_2) \| = \bigwedge_{n \in \omega} \| \bar{n} \in S_1 \| \implies \| \hat{\bar{f}}(\bar{n}) \in S_2 \| = \bigwedge_{n \in \omega} S_1(\bar{n}) \implies S_2(\hat{\bar{f}}(\bar{n})) \]

\[ \leq \bigwedge_{n \in I} S_1(\bar{n}) \implies S_2(\hat{\bar{f}}(\bar{n})) \]

\[ = \bigwedge_{n \in I} [D_{1,n}] \equiv [D_{2,f(n)}] \]

\[ = \left[ \bigcap_{n \in I} (D_{1,n} \cap D_{2,f(n)}) \cup (D_{1,n} \cap D_{2,f(n)}) \right] . \]
We will show that this is equal to 0 by showing that the set inside the brackets has measure zero. Recall the finite distributive law, that if we have a finite set $K$ and sequences of sets $(S_{n,0})_{n \in K}$ and $(S_{n,1})_{n \in K}$ then
\[
\bigcap_{n \in K} S_{n,0} \cup S_{n,1} = \bigcup_{g : K \to \{0,1\}} \bigcap_{n \in K} S_{n,g(n)}.
\]
This is easily proved by induction on $|K|$. Note also that if for each $n \in K$, the sets $S_{n,0}$ and $S_{n,1}$ are disjoint, then the sets $\bigcap_{n \in K} S_{n,g(n)}$ are pairwise disjoint as $g$ ranges over $2^K$.

So for all $K \in \mathcal{P}_{\text{fin}}(I)$, we have
\[
\mu_X \left( \bigcap_{n \in K} \left( D_{1,n,1} \cap D_{2,f(n),1} \right) \cup \left( D_{1,n,0} \cap D_{2,f(n),0} \right) \right)
\]
\[
= \mu_X \left( \bigcup_{g : K \to \{0,1\}} \bigcap_{n \in K} D_{1,n,g(n)} \cap D_{2,f(n),g(n)} \right)
\]
\[
= 2^{|K|} \mu_X \left( \bigcap_{n \in K} D_{1,n,g(n)} \cap D_{2,f(n),g(n)} \right) = 2^{|K|} 2^{-2|K|} = 2^{-|K|},
\]
where in the second-to-last step we have used the fact that $f|_I$ is injective (and therefore $f|_K$ is too). Since we can exhaust $I$ by a countable increasing sequence of finite sets, the result follows.

\begin{theorem}
There exist sets $T_1, T_2 \subseteq \mathbb{N}$ that are many-one incomparable, i.e. we neither have $T_1 \leq_m T_2$ nor $T_2 \leq_m T_1$.
\end{theorem}

\begin{proof}
We start by doing Boolean-valued reasoning about $S_1$ and $S_2$, as defined in \text{[11]}, taking $A = A(2^{\omega \times \omega})$, and use the fact that the Engeler model $\mathcal{P}^A(E)$ is (A-valuedly) a reflexive continuous lattice with numerals (Corollary \text{[17]} and the characterization of many-one reductions from Proposition \text{[19]}.

First, by the set theory meta-theorem applied to Lemma \text{[18]} (iii), there exist $d_1, d_2 \in \mathcal{P}^A(E)$ such that for all $n \in \omega$ and $i \in \{1,2\}$ we have $\|d_i \cdot c_n = \top\| = \|\bar{n} \in S_i\|$ and $\|d_i \cdot \bar{c}_n = \bot\| = -\|\bar{n} \in S_i\|$.

Suppose $M \in A(\text{Var})$ such that for all $n \in \omega$, there exists $m \in \omega$ such that $\lambda \vdash Mc_n = c_m$. In particular, there exists some function $f : \omega \to \omega$ such that for all $m \in \omega$, $\lambda \vdash Mc_n = c_{f(n)}$.

Now we consider $[d_2(Mc_n)]^A$ and $[d_1\bar{c}_n]^A$. In the following, for ease of notation, we write an equality where the fact that an $A$-valued equality is 1 is meant:
\[
[d_2(Mc_n)]^A = d_2 \cdot [Mc_n]^A = d_2 \cdot [\bar{c}_{f(n)}]^A
\]
by Theorem \text{[18]}.

Therefore
\[
\|d_2(Mc_n)]^A = \|\top\|^A = \|d_2 \cdot \bar{c}_{f(n)}]^A = \|\top\|^A = \|f(n) \in S_2\|,
\]
and the corresponding negative statement for $\bot$. Likewise
\[
\|d_1\bar{c}_n]^A = \|\top\|^A = \|\bar{n} \in S_1\|,
\]

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Proposition 4.19, that $S$ can then define $1$ has measure so that $T$ can conclude that $1$ has measure $\| \forall$.

It is proved in Proposition 5.8 that $i \in \{ \}$ used above, this means that the set of $\bot$ for all $\Lambda(\text{Var})$, we have

$$0 = \bigcap_{n \in \omega} \| [a_2] \cdot [K_{[Mc_n]}] = [a_1] \cdot [K_{[c_n]}] \|_{L^0}$$

so the set inside the square brackets has measure zero.

Since (as long as Var is countable) $\Lambda(\text{Var})$ is countable, the set of $M \in \Lambda(\text{Var})$ that map numerals to numerals is also countable. Therefore the set of $x \in X$ such that for all $M \in \Lambda(\text{Var})$ mapping numerals to numerals we have $a_2(x) \cdot [Mc_n] \neq a_1(x) \cdot [c_n]$ has measure 1.

For all $i \in \{ 1, 2 \}$, we have, by the definition of $d_i$, for all $n \in \omega$ the statement $\| d_i \cdot [c_n] = [\top] \| \vee \| d_i \cdot [c_n] = [\bot] \| = 1$. By a similar argument to that used above, this means that the set of $x \in X$ such that $a_i(x) \cdot [c_n] \in \{ [\top], [\bot] \}$ has measure 1, and by the countability of $\omega$ the set where this is true for both $i \in \{ 1, 2 \}$ and all $n \in \omega$ is also of measure 1.

Therefore the intersection of the sets defined in the previous two paragraphs has measure 1, and so is non-empty and so there exists a point $x$ in it. We can then define $T_i = \{ n \in \omega \mid a_i(x) \cdot [c_n] = [\top] \}$, and we have proved, by Proposition 4.19 that $T_1 \not\subseteq_m T_2$.

To get the final result, we re-run the argument swapping the roles of $S_1$ and $S_2$, and take $x$ to be in the intersection of the corresponding sets of measure 1, so that $T_2 \not\subseteq_m T_1$ either.

We finish by showing, by means of a counterexample, that we could not have done this by using the “fact” that $L^0(X; \mathcal{P}(E))$ is a continuous dcpo, because this is not true.

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Proposition 5.10. Let $X = (X, \Sigma, \mathcal{N})$ be a negligibility space such that $A(X)$ is not atomic, and let $Y$ be a non-empty countable set. Then $L^0(X; \mathcal{P}(Y))$ is not a continuous dcpo.

Proof. First, let $p = \forall \{a \in A(X) \mid a \text{ an atom}\}$. As $A(X)$ is not atomic, $\neg p \neq 0$, so there is a set $S \in \Sigma$ such that $S \notin \mathcal{N}$ and $[S] = \neg p$. By definition there are no atoms below $[S]$. Define

$$b(x) = \begin{cases} Y & \text{if } x \in S \\ \emptyset & \text{otherwise,} \end{cases}$$

where $x \in X$. This defines a measurable function $X \to \mathcal{P}(Y)$. As $S \notin \mathcal{N}$, $[b] \neq [K_0]$. We show that $L^0(X; \mathcal{P}(Y))$ is not continuous by showing that $[b]$ is not the supremum of elements way below it, which will follow from the fact that the only element of $L^0(X; \mathcal{P}(Y))$ that is way below $[b]$ is $[K_0]$.

If the only element below $[b]$ is $[K_0]$, then we are finished, so we reduce to the case that there is at least one $[a] \in L^0(X; \mathcal{P}(Y))$ such that $[K_0] < [a] < [b]$. Define $T = a^{-1}(\mathcal{P}(Y) \setminus \emptyset)$, which is in $\Sigma$ because $a$ is measurable and $\mathcal{P}(Y) \setminus \emptyset = \bigcup_{y \in Y} B_y$ and is therefore a Borel set. We have $T \setminus S \in \mathcal{N}$, so we redefine $T$ to be $T \cap S$ if necessary to make $T \subseteq S$.

Since there are no atoms below $[S]$, there are none below $[T]$ either, so there exists a non-zero element $a_1 \in A(X)$ such that $a_1 \leq [T]$ and $a_1$ has no atoms below it. We can repeat this argument to form a strictly descending sequence $(a_i)_{i \in \mathbb{N}}$ where for all $i \in \mathbb{N}$, $a_i$ has no atoms below it and $a_i \leq [T]$. We can define $b_i = a_i \setminus \bigcap_{i=1}^{\infty} a_i$, to get such a strictly decreasing sequence $(b_i)_{i \in \mathbb{N}}$, now with $\bigcap_{i=1}^{\infty} b_i = 0$. Since $A(X) = \Sigma/\mathcal{N}$, we can find a sequence $(V_i)_{i \in \mathbb{N}}$ of elements of $\Sigma$ such that for all $i \in \mathbb{N}$, $[V_i] = b_i$, and by adjusting negligible sets it can be arranged that $(V_i)_{i \in \mathbb{N}}$ is also strictly descending. Defining $T_i = T \setminus U_i$, then $(T_i)_{i \in \mathbb{N}}$ is strictly increasing with respect to $\subseteq$, maps to a strictly increasing sequence under $\subseteq$, and $T \setminus \bigcup_{i=1}^{\infty} T_i \in \mathcal{N}$. Define, for each $i \in \mathbb{N}$, $c_i : X \to \mathcal{P}(Y)$ as follows:

$$c_i(x) = \begin{cases} Y & \text{if } x \in T_i \cup S \setminus T \\ \emptyset & \text{if } x \in T \setminus T_i \cup X \setminus S, \end{cases}$$

where $x \in X$. Each $c_i$ is measurable for the same reasons that $b$ is, and $(\{c_i\})_{i \in \mathbb{N}}$ is a strictly increasing sequence in $L^0(X; \mathcal{P}(Y))$. Since for all $i \in \mathbb{N}$, $c_i(x) = \emptyset$ for all $x \in T \setminus T_i$ and $T \setminus T_i \notin \mathcal{N}$, while $a(x) \neq \emptyset$ for all $x \in T \setminus T_i$, we have $[a] \nleq [c_i]$. But $\bigvee_{i=1}^{\infty} c_i = \bigvee_{i=1}^{\infty} [c_i] = [b]$ because $T \setminus \bigcup_{i=1}^{\infty} T_i \in \mathcal{N}$. Therefore $a$ is not way below $b$. \hfill \qedsymbol

6 Conclusions

We have developed domain theory in a Boolean-valued universe of sets. Using the measure algebra as the Boolean algebra we obtained a domain of random variables which can be seen to be a reflexive domain in the internal language of the Boolean-valued set theory. We have focused on the pure $\lambda$-calculus here but it should be straightforward to extend this to a $\lambda$-calculus with probabilistic choice as an explicit primitive.

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