On White Noise Solutions of mSQG Equations on $\mathbb{R}^2$

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Siyu Liang $^{1 \ 2 \ 3}$

Abstract

In this paper, we show existence of white noise solutions for weak formulations of modified Surface Quasi-Geostrophic (mSQG) equations. Based on previous results ([10]) on white noise solutions for mSQG equations on the torus $\mathbb{T}^2$, we show a similar result for the whole space $\mathbb{R}^2$ by letting the volume of the torus go to infinity and applying compactness methods (Skorokhod’s theorem).

Key words: white noise solutions, weak formulation, mSQG equations, Skorokhod’s theorem

1 Introduction

In this paper, we study the stationary solutions of the following modified Surface Quasi-Geostrophic equations (mSQG equations) on the torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$ and the whole space $\mathbb{R}^2$

\[
\begin{aligned}
\partial_t \omega + u \cdot \nabla \omega &= 0, \\
u &= \nabla^\perp (-\Delta)^{-1/(1+\epsilon)/2} \omega,
\end{aligned}
\]

(mSQG)

where $0 < \epsilon < 1$, $\nabla^\perp = (-\partial_2, \partial_1)$, and $(-\Delta)^{-1/(1+\epsilon)/2}$ is the fractional Laplacian operator, the definition of which is given in Section 2.2.

When $\epsilon = 1$, the above equation becomes the Euler equation, and for $\epsilon = 0$ it is called the Surface Quasi-Geostrophic (SQG) equation.

The SQG equations are approximations to the shallow water equations with a small Rossby number (which goes to 0 in the limit), where a small Rossby number means the system is mainly determined by the Coriolis force which is caused by earth rotation. It is also called "(nearly) in geostrophic balance".

The SQG equation is obtained from the 3D Quasi-Geostrophic equation by assuming the potential vorticity to be identically 0. The SQG equation ($\epsilon = 0$) is introduced in [7],

1Department of Mathematics, University of Bielefeld, D-33615 Bielefeld, Germany, sliang@math.uni-bielefeld.de
2Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
3School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
where a striking mathematical and physical analogy is developed between the structure and formation of singular solutions of SQG equations and the potential formation of finite-time singular solutions for the 3D Euler equations. For a more physical background of Quasi-Geostrophic equations and the formulation of SQG equations we refer to [13, 18, 8, 25] and [29].

The classical incompressible Euler equations are well-known and have been studied extensively in the literature, see for example, [17], [6] and [19]. It is constructed in [10] a white noise solution of Euler equations by the following point-vortex system:

$$\omega^N_t = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \delta_{X^n_t}$$

and for every $N \in \mathbb{N}$, the finite dimensional dynamics

$$\frac{dX^i_t}{dt} = \sum_{j=1}^{N} \frac{1}{\sqrt{N}} \xi_j K(X^i_t - X^j_t) \quad i = 1, ..., N$$

in $(\mathbb{T}^2)^N$ with initial condition $(X^1_0, ..., X^N_0) \in (\mathbb{T}^2)^N \setminus \Delta_N$, $\Delta_N := \{(x_1, x_2, ..., x_N) \in (\mathbb{T}^2)^N; x_i = x_j \text{ for some } i \neq j, i, j = 1, 2, ..., N\}$, where $K$ is the Biot-Savart kernel on $\mathbb{T}^2$ (we set $K(0) = 0$), and the intensities $\xi_1, ..., \xi_N$ are (random) numbers of any sign.

Exploiting the similarity to the Euler equations, many classical results have also been obtained for SQG and mSQG equations. For example, global existence of weak solutions to SQG equations is known in the spaces $L^p(\mathbb{R}^2)$, for $p \in (4/3, \infty)$ (see [26, 22]). In [3], non-uniqueness of weak solutions is proved in a certain class by using the methods of convex integration. mSQG equations, which are equations between SQG equations and Euler equations, have also been studied by many papers, such as [14, 15, 5, 12]. In a recent work [4], nontrivial global (classical) solutions of the mSQG equations have been constructed.

Similarly to Euler equations, there are also some results via point-vortex model to approximate mSQG equations, such as [10], [20], [21], [11] (for more general models), [12], and [27]. In [10], the point-vortex approximation is used to show the existence of white noise solutions of the weak formulation of mSQG equations on the torus (see Definition 3.1 for the definition of white noise solutions).

However, for the case of $\mathbb{R}^2$, there is no result of existence of white noise solutions as far as we know. In this paper, we will generalize the result of the existence of white noise solutions of mSQG equations to $\mathbb{R}^2$. But we will prove it in a different way. Since there have been previous results of the existence of white noise solutions on $\mathbb{T}^2$ ([10]), we do not use vortex systems to approximate solutions. Instead, since the existence of white noise solutions holds on the torus of any volume, we will let the volume of torus go to infinity and apply the compactness methods.

The reason that we consider the mSQG equations on $\mathbb{R}^2$ is that the kernel corresponding to $\nabla^\perp(-\Delta)^{-1(1+\epsilon)/2}$ is dominated by $C\frac{1}{|x|^{3-\epsilon}}$. Therefore, the kernel corresponding to mSQG equations ($0 < \epsilon < 1$) has a better behaviour at infinity compared to 2D Biot-Savart kernel ($\epsilon = 1$). When $\epsilon = 0$, the behaviour of the kernel at infinity is even better. However, its behaviour at the origin is bad. Therefore, in the case of SQG equations ($\epsilon = 0$), it is difficult to obtain even the existence of white noise solutions on the torus $\mathbb{T}^2$. 

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Main results of this paper: we prove the existence of white noise solutions to the weak formulations of (mSQG) on $\mathbb{R}^2$ (Theorem 3.7).

In Section 2, we first introduce some function and distribution spaces. Then we show the properties and the relations of the kernel of (mSQG) on $\mathbb{T}^2$ and $\mathbb{R}^2$. Finally we introduce the definition of space white noise. In Section 3, we define the nonlinear term by approximating sequences. In Section 4, we prove our main result.

2 Preliminaries

2.1 Function and distribution spaces

In this section we introduce some function and distribution spaces.

2.1.1 Function and distribution spaces on $\mathbb{R}^2$

Denote by $S(\mathbb{R}^2)$ the Schwartz space and $S'(\mathbb{R}^2)$ its dual space. Denote by $C^\infty_c(\mathbb{R}^2)$ the space of smooth functions on $\mathbb{R}^2$ with compact support. Denote by $C^k_c(\mathbb{R}^2)$ the space of compact supported functions on $\mathbb{R}^2$ which have $k$th continuous derivatives.

On $\mathbb{R}^2$, we recall the classical (non-homogeneous) Sobolev spaces:

$$H^s(\mathbb{R}^2) := \left\{ u \in S'(\mathbb{R}^2); \| u \|_{H^s(\mathbb{R}^2)} := \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

(1)

where $s \in \mathbb{R}$, and

$$\hat{u}(\xi) = \mathcal{F}u(\xi) := \int_{\mathbb{R}^2} u(x)e^{-ix\cdot\xi}dx,$$

denotes the Fourier transform of $u$ on $\mathbb{R}^2$. One knows that $H^s(\mathbb{R}^2)$ is a Hilbert space with $H^{-s}(\mathbb{R}^2)$ as its dual space.

For $s \in \mathbb{R}$, we define the spaces of vector fields $H^s(\mathbb{R}^2; \mathbb{R}^2)$ to be the sets of the vector-valued functions with both components in $H^s(\mathbb{R}^2)$. For simplicity, from now on, we will use the same notations of vector fields and function spaces when there is no confusion.

We introduce the following weighted Sobolev norms and spaces.

**Definition 2.1** (Weighted Sobolev norms and spaces). Let $\rho \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\rho(x) \geq 0$. Define the weighted Sobolev norms $\| \cdot \|_{H^s(\mathbb{R}^d, \rho)}$ by

$$\| \cdot \|_{H^s(\mathbb{R}^d, \rho)} := \| \rho \cdot \|_{H^s(\mathbb{R}^d)}.$$

Define the weighted Sobolev spaces $H^s(\mathbb{R}^d, \rho)$ as the subspace of $S'(\mathbb{R}^d)$ such that $\| \cdot \|_{H^s(\mathbb{R}^d, \rho)}$ finite.

Since we always consider the 2D case, from now on for simplicity we use the notation $H^s(\rho)$ instead of $H^s(\mathbb{R}^2, \rho)$ when no confusion occurs. Moreover, we define the space $H^{-1}(\rho)$ as the space $\bigcap_{\epsilon > 0} H^{-1-\epsilon}(\rho)$ with the following Frechet metric $d$:

$$d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1 + \| u - v \|_{H^{-1-\epsilon}(\rho)}}{1 + \| u - v \|_{H^{-1-\epsilon}(\rho)}}.$$
Then, convergence in $H^{-1-\epsilon}(\rho)$ is equivalent to convergence in $H^{-1-\epsilon}(\rho)$ for each $\epsilon > 0$. Let

$$\rho_\sigma(x) := \frac{1}{\langle x \rangle^\sigma}$$

and

$$\rho_{\sigma'}(x) := \frac{1}{\langle x \rangle^{\sigma'}},$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

The following lemma is proved in [28, Theorem 6.31]:

**Lemma 2.2.** For $0 < \sigma' < \sigma$, and $s' > s$, the distributional space $H^{s'}(\rho_{\sigma'})$ is compactly embedded in $H^{s}(\rho_{\sigma})$.

### 2.1.2 Function and distribution spaces on $\mathbb{T}^2$

Denote by $C^\infty(\mathbb{T}^2)$ the space of smooth functions on $\mathbb{T}^2$. Noting that $\{\frac{1}{2\pi}e^{ik\cdot x}\}_{k \in \mathbb{Z}^2}$ is the orthonormal basis of $L^2(\mathbb{T}^2; \mathbb{C})$, for $u \in L^2(\mathbb{T}^2)$, we consider the Fourier expansion of $u$:

$$u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k \frac{1}{2\pi} e^{ik\cdot x} \quad \text{with} \quad \hat{u}_k = \overline{u}_{-k},$$

where $\hat{u}_k := \frac{1}{2\pi} \int_{\mathbb{T}^2} u(x) e^{-ik\cdot x} dx$ denotes the $k$th Fourier coefficient of $u$ on $\mathbb{T}^2$. It follows from Fourier-Plancherel equality that the above series is convergent in $L^2(\mathbb{T}^2)$. Define the Sobolev norm for $s \in \mathbb{R}$:

$$\|u\|_{H^s(\mathbb{T}^2)}^2 := \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^s |\hat{u}_k|^2. \quad (2)$$

We define the Sobolev spaces $H^s(\mathbb{T}^2)$ as the completion of $C^\infty(\mathbb{T}^2)$ with respect to the norm $\| \cdot \|_{H^s(\mathbb{T}^2)}$. For $s \in \mathbb{R}$, we define the space of vector fields $H^s(\mathbb{T}^2; \mathbb{R}^2)$ to consist of the vectors with both components in $H^s(\mathbb{T}^2)$.

On $\mathbb{T}^2$, define Fréchet space $H^{-1-}$ to be the linear space $\bigcap_{n \geq 1} H^{-1-\frac{1}{n}}$ with the distance as follows:

$$\rho_{H^{-1-}}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_{H^{-1-\frac{1}{n}}}}{1 + \|x - y\|_{H^{-1-\frac{1}{n}}}}.$$

**Remark 2.3.**

1. In this paper, by the notation $\sum_{k \in \mathbb{Z}^2}$, we always mean $\lim_{N \to \infty} \sum_{|k_1|, |k_2| \leq N}$, which is particularly important when the series is not absolutely convergent.

2. From now on we may suppress the domain ($\mathbb{R}^2$) or ($\mathbb{T}^2$) in the notation of these function spaces, when no confusion occurs.
2.2 Introduction of weak formulations of mSQG equations

2.2.1 Kernel of (mSQG) on $\mathbb{R}^2$

On the whole space, we know that the operator $(-\Delta)^{-(1+\epsilon)/2}$ and $\nabla^\perp (-\Delta)^{-(1+\epsilon)/2}$ are defined by the Fourier multiplier $|\xi|^{-(1+\epsilon)}$ and $i\xi^\perp |\xi|^{-(1+\epsilon)}$, respectively. Hence if we write them in the forms of the convolution, they are equivalent to the convolution with $\mathcal{F}^{-1}(|\xi|^{-(1+\epsilon)})$ and $\mathcal{F}^{-1}(i\xi^\perp |\xi|^{-(1+\epsilon)})$, respectively.

Define

$$K_\epsilon := \mathcal{F}^{-1}(i\xi^\perp |\xi|^{-(1+\epsilon)}).$$

Recall that on $\mathbb{R}^2$, the Fourier transform and Fourier inverse transform are defined as follows:

$$\hat{f}(\xi) = \mathcal{F} f(\xi) := \int_{\mathbb{R}^2} f(x) e^{-ix\cdot\xi} dx,$$

and

$$\mathcal{F}^{-1} f(\xi) := \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(x) e^{ix\cdot\xi} dx.$$

Thus we know that $K_\epsilon$ is dominated by $C_\epsilon \frac{1}{|x|}$ for some constant $C_\epsilon$. The kernel is singular at the origin.

2.2.2 Kernel of (mSQG) on the torus and the relations to the kernel on $\mathbb{R}^2$

For fixed $M$, denote $\mathbb{T}^2_M = (\mathbb{R}/M\mathbb{Z})^2$ to be the torus of length $M$. Let $f$ be a distribution in some Sobolev space $H^{-N}(\mathbb{T}^2_M)$, for some $N > 0$ with the Fourier expansion

$$f(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \hat{f}_k^M \hat{e}_k^M \quad \text{with} \quad \hat{f}_k^M = \hat{f}_{-k}^M,$$

where $\hat{f}_k^M := \frac{1}{M} \int_{\mathbb{T}^2_M} f(x) e^{-2\pi ik \cdot x/M} dx$ denotes the $k$th Fourier coefficient of $f$ on $\mathbb{T}^2_M$ and $\hat{e}_k^M(x) = \frac{1}{M} e^{2\pi ik \cdot x/M}$. The operator $(-\Delta)^{-(1+\epsilon)/2}$ on the torus $\mathbb{T}^2_M$ is defined as:

$$(-\Delta)^{-(1+\epsilon)/2} f = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{M}{2\pi |k|} \right)^{1+\epsilon} \hat{f}_k^M \hat{e}_k^M.$$

Therefore,

$$\nabla^\perp (-\Delta)^{-(1+\epsilon)/2} f = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{M}{2\pi} \right)^{\epsilon} \frac{i k^\perp}{|k|^{1+\epsilon}} \hat{f}_k^M \hat{e}_k^M.$$

If we write it in the form of convolution,

$$\nabla^\perp (-\Delta)^{-(1+\epsilon)/2} f = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{M}{2\pi} \right)^{\epsilon} \frac{i k^\perp}{|k|^{1+\epsilon}} \frac{1}{M^2} \int_{\mathbb{T}^2_M} f(\xi) e^{-2\pi ik \cdot \xi/M} d\xi e^{2\pi ik \cdot x/M}$$

$$= \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{M}{2\pi} \right)^{\epsilon} \frac{i k^\perp}{|k|^{1+\epsilon}} \frac{1}{M^2} e^{2\pi ik \cdot x/M} * f$$

$$=: K_\epsilon^M * f,$$
where the convolution is defined on the torus $\mathbb{T}_M^2 = [-M/2, M/2]^2$.

Now we want to show that similar to the case of $\mathbb{R}^2$, $|x|^{2-\epsilon} K_\epsilon^M(x)$ can also be bounded by a constant which does not depend on $x$ and $M$.

For $x \in \mathbb{T}_M^2$, 

$$|x|^{2-\epsilon} K_\epsilon^M(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{2\pi} \right)^\epsilon \frac{i k \perp}{|k|^{1+\epsilon}} \left( \frac{|x|}{M} \right)^{2-\epsilon} e^{2\pi ik \cdot x / M}.$$ 

Let $\eta = \frac{x}{M} \in [-\frac{1}{2}, \frac{1}{2}]^2 \setminus \{(0,0)\}$, then 

$$|x|^{2-\epsilon} K_\epsilon^M(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{2\pi} \right)^\epsilon \frac{i k \perp}{|k|^{1+\epsilon}} |\eta|^{2-\epsilon} e^{2\pi ik \cdot \eta}.$$ 

(4)

The next lemma tells us exactly what we want.

**Lemma 2.4.** Define $|l|_\infty = \max\{|l_1|, |l_2|\}$. For any $\eta \in [-\frac{1}{2}, \frac{1}{2}]^2 \setminus \{(0,0)\}$,

1. $\lim_{N \to \infty} \sum_{|l|_\infty \leq N} \mathcal{F}^{-1}(\frac{i \xi \perp}{|\xi|^{1+\epsilon}})(\eta + l)$ exists, which is denoted by $\sum_{l \in \mathbb{Z}^2} \mathcal{F}^{-1}(\frac{i \xi \perp}{|\xi|^{1+\epsilon}})(\eta + l)$, and one has 

$$\left| \sum_{l \in \mathbb{Z}^2} \mathcal{F}^{-1}(\frac{i \xi \perp}{|\xi|^{1+\epsilon}})(\eta + l) \right| \lesssim |\eta|^{-2+\epsilon}. \quad (5)$$

2. $\sum_{l \in Z^2} \mathcal{F}^{-1}(\frac{i \xi \perp}{|\xi|^{1+\epsilon}})(\eta + l)$ is a smooth function of $\eta$.

3. It holds 

$$\sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{2\pi} \right)^\epsilon \frac{i k \perp}{|k|^{1+\epsilon}} e^{2\pi ik \cdot \eta} = \sum_{l \in \mathbb{Z}^2} \mathcal{F}^{-1}(\frac{i \xi \perp}{|\xi|^{1+\epsilon}})(\eta + l). \quad (6)$$

Then by combining 1 and 3, one obtains that (4) is bounded by a constant.

**Proof.** One knows that $\mathcal{F}^{-1}(\frac{i \xi \perp}{|\xi|^{1+\epsilon}}) = C \nabla^\perp(\eta|^{-1+\epsilon}) = C \frac{\eta \perp}{\eta|^{1-\epsilon}}$, where $C$ is some constant which depends only on $\epsilon$ (see, for example, Proposition 1.29 of [1]). Set 

$$\mathbb{Z}^2_+ = \{(x_1, x_2) \in \mathbb{Z}^2; x_1 > 0\} \cup \{(0, x_2); x_2 \in \mathbb{N}^+\},$$ 

$$\mathbb{Z}^2_- = \{x \in \mathbb{Z}^2; -x \in \mathbb{Z}^2_+\}.$$ 

Thus we have $\mathbb{Z}^2 = \mathbb{Z}^2_+ \cup \mathbb{Z}^2_- \cup \{(0,0)\}$. Then we obtain 

$$\sum_{|l|_\infty \leq N} \frac{(\eta + l) \perp}{|\eta + l|^{3-\epsilon}} = \frac{\eta \perp}{|\eta|^{3-\epsilon}} + \sum_{l \in \mathbb{Z}^2_+ \setminus \{(0,0)\}} \left[ \frac{(\eta + l) \perp}{|\eta + l|^{3-\epsilon}} - \frac{(l - \eta) \perp}{|l - \eta|^{3-\epsilon}} \right].$$

Note that when $x \neq 0$, $|\nabla (\frac{x \perp}{|x|^{3-\epsilon}})| \lesssim \frac{1}{|x|^{3-\epsilon}}$. Hence we deduce 

$$\left| \sum_{|l|_\infty \leq N} \frac{(\eta + l) \perp}{|\eta + l|^{3-\epsilon}} \right| \lesssim |\eta|^{-2+\epsilon} + \sum_{l \in \mathbb{Z}^2_+ \setminus \{(0,0)\}} |\eta| \sup_{\xi \in [0, l+\eta]} \frac{1}{|\xi|^{3-\epsilon}}.$$
As a result,
where \([l - \eta, l + \eta] = [l_1 - \eta_1] \times [l_2 - \eta_2]\).
Since \(\eta \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2\), \(l \in \mathbb{Z}^2_+\), \(l - \eta\) is uniformly away from the origin, we obtain \(\sup_{\xi \in [l - \eta, l + \eta]} \frac{1}{|\xi|^{1+\varepsilon}} \leq \frac{1}{|\eta|^{1+\varepsilon}}\). Thus, we find

\[
\sup_{N} \sum_{(l_1, l_2) \in \mathbb{Z}^2_+} \frac{1}{|l|^{1+\varepsilon}} \leq |\eta| \leq |\eta|^{2+\varepsilon}.
\]

As a result, \(\left| \sum_{|l| \leq N} \frac{(\eta+l_1^\perp)}{|\eta+l_1^\perp|^{1+\varepsilon}} \right|\) has a uniform bound \(C|\eta|^{-2+\varepsilon}\), where \(C\) is some constant independent of \(N\). Moreover, by the same argument we obtain that \(\left\{ \sum_{|l| \leq N} \frac{(\eta+l_1^\perp)}{|\eta+l_1^\perp|^{1+\varepsilon}} \right\}_{N \geq 1}\) is a Cauchy sequence, hence the limit \(\lim_{N \to \infty} \sum_{|l| \leq N} \mathcal{F}^{-1}\left( \frac{i\xi^\perp}{|\xi|^{1+\varepsilon}} \right)(\eta + l)\) exists and (5) holds, which finishes the proof of 1.

2 follows from the fact that each derivative of \(\sum_{(l_1, l_2) \in \mathbb{Z}^2_+} \frac{(\eta+l_1^\perp)}{|\eta+l_1^\perp|^{1+\varepsilon}} - \frac{(l-\eta_1^\perp)}{|l-\eta_1^\perp|^{1+\varepsilon}}\) converges uniformly with respect to \(\eta \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2\), which can be easily obtained by the same argument of 1.

For 3, when we view \(\sum_{l \in \mathbb{Z}^2} \mathcal{F}^{-1}\left( \frac{i\xi^\perp}{|\xi|^{1+\varepsilon}} \right)(\eta + l)\) as a function of \(\eta\) on \(T^2_1\), the \(k\)th Fourier coefficient is

\[
\int_{[0,1] \times [0,1]} \sum_{l \in \mathbb{Z}^2} \mathcal{F}^{-1}\left( \frac{i\xi^\perp}{|\xi|^{1+\varepsilon}} \right)(x + l)e^{-2\pi ik \cdot x} dx = \sum_{l \in \mathbb{Z}^2} \int_{[0,1] \times [0,1]} \mathcal{F}^{-1}\left( \frac{i\xi^\perp}{|\xi|^{1+\varepsilon}} \right)(y)e^{-2\pi ik \cdot (y-l)} dy
\]

\[
= \sum_{l \in \mathbb{Z}^2} \int_{[0,1] \times [0,1]} \mathcal{F}^{-1}\left( \frac{i\xi^\perp}{|\xi|^{1+\varepsilon}} \right)(y)e^{-2\pi ik \cdot y} dy
\]

\[
= \int_{\mathbb{R}^2} \mathcal{F}^{-1}\left( \frac{i\xi^\perp}{|\xi|^{1+\varepsilon}} \right)(y)e^{-2\pi ik \cdot y} dy
\]

\[
= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \mathcal{F}^{-1}\left( \frac{i\xi^\perp}{|\xi|^{1+\varepsilon}} \right) \left( \frac{y}{2\pi} \right)e^{-ik \cdot y} dy
\]

\[
= \left( \frac{1}{2\pi} \right)^{\epsilon} i \frac{2\pi i\xi^\perp}{|2\pi\xi|^{1+\varepsilon}} (y)e^{-ik \cdot y} dy
\]

where the last equality is due to \(\mathcal{F}\mathcal{F}^{-1} = id\).
The proof of 3 is finished.

Thus to conclude, combining the case of \(\mathbb{R}^2\), we have proved the following lemma:
Lemma 2.5. Let
\[ K_\epsilon = \mathcal{F}^{-1}(i\xi^{\perp}|\xi|^{-(1+\epsilon)}) , \]
\[ K_\epsilon^M = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{M}{2\pi} \right)^{\epsilon} \frac{ik^{\perp}}{|k|^{1+\epsilon}} \frac{1}{M^2} e^{2\pi ik \cdot x/M} \]
be the kernel corresponding to the operator \( \nabla^{\perp}(-\Delta)^{-(1+\epsilon)/2} \) on \( \mathbb{R}^2 \) and \( \mathbb{T}_M^2 \), respectively. Then there exists a common constant \( C_\epsilon \) which does not depend on \( M \), such that
\[ |K_\epsilon(x)| \leq \frac{C_\epsilon}{|x|^{2-\epsilon}}, \]
and
\[ |K_\epsilon^M(x)| \leq \frac{C_\epsilon}{|x|^{2-\epsilon}}, \]
for any \( x \in \mathbb{R}^2, x \in \mathbb{T}_M^2 \), respectively.

Moreover, note that if we fix some \( x \neq 0 \) and let \( M \) goes to infinity, the sum
\[ \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{M}{2\pi} \right)^{\epsilon} \frac{ik^{\perp}}{|k|^{1+\epsilon}} \frac{1}{M^2} e^{2\pi ik \cdot x/M} = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{i2\pi k^{\perp}}{|k|^{1+\epsilon}} \left( \frac{2\pi}{M} \right)^2 e^{2\pi ik \cdot x/M} \]
converges to the integration
\[ \frac{1}{4\pi^2} \int_{\mathbb{R}^2} i\xi^{\perp}|\xi|^{-(1+\epsilon)} e^{ix \cdot \xi} d\xi, \]
which is exactly the Fourier inverse transform of \( i\xi^{\perp}|\xi|^{-(1+\epsilon)} \).

In other word, we have the following lemma

Lemma 2.6. For any \( x \in \mathbb{R}^2 \setminus \{0\} \), \( K_\epsilon^M(x) \) converges pointwisely to \( K_\epsilon \) as \( M \) goes to infinity.

\[ \square \]

2.3 Weak formulation of mSQG equations

In this section we do some (at least formally) transformation to transform the equation to a weak form. A similar transformation can be found in [9, 10]. We put it here for completeness. Recall the mSQG equation on both \( \mathbb{T}_M^2 \) and \( \mathbb{R}^2 \):
\[ \partial_t \omega + u \cdot \nabla \omega = 0. \]

Let \( \phi \) be a test function, i.e. \( \phi \in C^\infty(\mathbb{T}_M^2) \) in the case of \( \mathbb{T}_M^2 \) and \( \phi \in C^\infty_c(\mathbb{R}^2) \) in the case of \( \mathbb{R}^2 \). Then we obtain
\[ \langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle u(s) \cdot \nabla \omega_s, \phi \rangle ds. \]

(7)
Note that \( u = K_\epsilon \ast \omega \) (on \( \mathbb{R}^2 \)) or \( u = K_\epsilon^M \ast \omega \) (on \( \mathbb{T}_M^2 \)), and both \( K_\epsilon \) and \( K_\epsilon^M \) are anti-symmetric. Therefore, we can transform (7) to

\[
\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_{\phi,\epsilon} \rangle \, ds
\]

on \( \mathbb{R}^2 \), and

\[
\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H^M_{\phi,\epsilon} \rangle \, ds
\]

on \( \mathbb{T}_M^2 \), where

\[
H_{\phi,\epsilon}(x, y) := \frac{1}{2} K_\epsilon(x - y) (\nabla \phi(x) - \nabla \phi(y)),
\]

and

\[
H^M_{\phi,\epsilon}(x, y) := \frac{1}{2} K^M_\epsilon(x - y) (\nabla \phi(x) - \nabla \phi(y)).
\]

### 2.4 Introduction of space white noise

#### 2.4.1 Space white noise on \( \mathbb{T}^2 \)

First, we recall the definition and the construction of the space white noise distribution on the torus \( \mathbb{T}^2 = \mathbb{T}_T^2 = (\mathbb{R}/2\pi\mathbb{Z})^2 \) (see, for example, [9]). The following definition and construction mainly come from [9], which we write here for completeness.

A **space white noise (variable)** \( \omega \) on \( \mathbb{T}^2 \) is a Gaussian distributional valued random variable mapping from some probability space \((\Xi, \mathcal{F}, \mathbb{P})\) to \( C^\infty(\mathbb{T}^2)' \) such that

- For any \( \phi \in C^\infty(\mathbb{T}^2) \), \( \langle \omega, \phi \rangle \) is a real valued Gaussian random variable with zero mean.
- For any \( \phi, \psi \in C^\infty(\mathbb{T}^2) \),
  \[
  \mathbb{E}\langle \omega, \phi \rangle \langle \omega, \psi \rangle = \langle \phi, \psi \rangle_{L^2(\mathbb{T}^2)}.
  \]

We call the distribution of a space white noise on \( C^\infty(\mathbb{T}^2)' \) the **space white noise distribution** (on \( C^\infty(\mathbb{T}^2)' \)). Now we show the existence of the space white noise variable by constructing it.

Define

\[
\omega = \sum_{n \in \mathbb{Z}^2} G_n(\theta) \frac{1}{2\pi} e^{inx},
\]

where \( \theta \in \Xi \), \( G_n = \overline{G_{-n}} \), and \( G_n, n \in \mathbb{Z}^2_+ \cup \{0\} \) are independent random variables with standard (complex) Gaussian distributions. Thus we have for \( m, n \in \mathbb{Z}^2_+ \)

\[
\mathbb{E}[G_n G_m] = \delta_{mn}.
\]

Hence it is easy to verify that \( \omega \) is a space white noise, the details of which can be found in [9].

**Remark 2.7.**
• We know that $\omega \in H^{-1-\epsilon} \mathbb{P}$-a.s. for any $\epsilon > 0$, the proof of which can be found in [9, Section 2.1]. Therefore, the space white noise distribution is supported in $H^{-1-}$.

• By the definition of the space white noise on $\mathbb{T}^2$, any random variable with space white noise distribution on $\mathbb{T}^2$ in some probability space could be expanded by the series $\omega = \sum_{n \in \mathbb{Z}^2} G_n(\theta) \frac{1}{2\pi} e^{inx}$, where $G_n$, $n \in \mathbb{Z}^2 \cup \{0\}$ are independent random variables with standard Gaussian distributions in the same probability space.

• From now on we do not distinguish the notion of a space white noise (variable) and the space white noise distribution when no confusion occurs.

### 2.4.2 Space white noise on $\mathbb{T}^2_M$

Similarly, we define space white noise on $\mathbb{T}^2_M = (\mathbb{R}/M\mathbb{Z})^2$ in the same way. A **space white noise (variable)** $\omega$ on $\mathbb{T}^2_M$ is a Gaussian distributional valued random variable mapping from some probability space $(\Xi, \mathcal{F}, \mathbb{P})$ to $C^\infty(\mathbb{T}^2_M)^\prime$ such that

- For any $\phi \in C^\infty(\mathbb{T}^2_M)$, $\langle \omega, \phi \rangle$ is a real valued Gaussian random variable.
- For any $\phi, \psi \in C^\infty(\mathbb{T}^2_M)$,
  $\mathbb{E}\langle \omega, \phi \rangle \langle \omega, \psi \rangle = \langle \phi, \psi \rangle_{L^2(\mathbb{T}^2_M)}$.

We call the distribution of a space white noise on $C^\infty(\mathbb{T}^2_M)^\prime$ the **space white noise distribution** (on $C^\infty(\mathbb{T}^2_M)^\prime$).

**Fourier transform and Sobolev spaces on $\mathbb{T}^2_M$**

For $u \in C^\infty(\mathbb{T}^2_M)$, we consider the following Fourier expansion of $u$ on the torus:

$$u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k^M e^{inx} \quad \text{with} \quad \hat{u}_k^M = \overline{\hat{u}_{-k}^M},$$

where $\hat{u}_k^M := \frac{1}{M} \int_{\mathbb{T}^2_M} u(x) e^{-2\pi i k \cdot x} dx$ denotes the $k$th Fourier coefficient of $u$ on $\mathbb{T}^2_M$.

Define the Sobolev norm on $\mathbb{T}^2_M$ for $s \in \mathbb{R}$:

$$\|u\|^2_{H^s(\mathbb{T}^2_M)} := \sum_{k \in \mathbb{Z}^2} \left(1 + \frac{2\pi |k|}{M}\right)^s |\hat{u}_k^M|^2.$$

Define the space $H^s(\mathbb{T}^2_M)$ as the completion of $C^\infty(\mathbb{T}^2_M)$ under the norm $\| \cdot \|_{H^s(\mathbb{T}^2_M)}$.

Similar to the case of $\mathbb{T}^2$, a space white noise variable has the following form: for some probability space $(\Xi, \mathcal{F}, \mathbb{P})$, define

$$\omega^M = \sum_{n \in \mathbb{Z}^2} G_n^M(\theta) e_n^M, \quad \theta \in \Xi,$$  (11)
where $e_n^M = \frac{1}{M} e^{2\pi in/M} x^M, G_n^M = \overline{G_{-n}^M}$, and $G_n^M, n \in \mathbb{Z}_+^2 \cup \{0\}$ are independent random variables with standard (complex) Gaussian distributions. Thus we have

$$\mathbb{E}[G_n^M G_m^M] = \delta_{mn}$$

for $m, n \in \mathbb{Z}_+^2$.

### 2.4.3 Space white noise on $\mathbb{R}^2$

When it comes to the cases of the whole space $\mathbb{R}^2$, first we recall the definition of a space white noise $\omega$ on $\mathbb{R}^2$ as a Gaussian distributional valued random variable mapping from some probability space $(\Xi, \mathcal{F}, \mathbb{P})$ to $C_c^\infty(\mathbb{R}^2)'$ such that

- For any $\phi \in C_c^\infty(\mathbb{R}^2)$, $\langle \omega, \phi \rangle$ is a real valued Gaussian random variable.
- For any $\phi, \psi \in C_c^\infty(\mathbb{R}^2)$,
  $$\mathbb{E}\langle \omega, \phi \rangle \langle \omega, \psi \rangle = \langle \phi, \psi \rangle_{L^2(\mathbb{R}^2)}.$$

We will construct a space white noise on $\mathbb{R}^2$ by taking the limit of space white noise on the torus and letting the volume of the torus go to infinity. We extend (11) periodically to a distribution $\bar{\omega}^M$ on $\mathbb{R}^2$ (with the Fourier series we can expand it directly by viewing it as the series on $\mathbb{R}^2$). That is,

$$\bar{\omega}^M = \sum_{n \in \mathbb{Z}^2} C_n^M(\theta)e_n^M \text{ in } \mathbb{R}^2.$$

However, $\bar{\omega}^M$ is not uniformly bounded with respect to $M$ in the sense of $H^{-1-\nu}(\mathbb{R}^2)$ but only uniformly bounded in some weighted Sobolev spaces. We have the following lemma.

**Lemma 2.8.** For any $\nu > 0$ and $\sigma > 2$, the distribution of $\{\bar{\omega}^M\}_{M \geq 1}$ is tight in the weighted Sobolev space $H^{-1-\nu}(\rho_\sigma)$. Hence $\{\bar{\omega}^M\}_{M \geq 1}$ is tight in the metric space $H^{-1-\nu}(\rho_\sigma)$. Moreover, denote by $\mu_M$ the distribution of $\bar{\omega}^M$ in $H^{-1-\nu}(\rho_\sigma)$. Then $\mu_M$ converges weakly to the space white noise distribution on $\mathbb{R}^2$ in $H^{-1-\nu}(\rho_\sigma)$ as $M \to \infty$.

**Proof.** By the definition of the weighted Sobolev norm,

$$\mathbb{E}\|\bar{\omega}^M\|^2_{H^{-1-\nu}(\rho_\sigma)}$$

$$= \mathbb{E}\|\rho_\sigma \bar{\omega}^M\|^2_{H^{-1-\nu}(\mathbb{R}^2)}$$

$$= \mathbb{E}\|\sum_{n \in \mathbb{Z}^2} \rho_\sigma e_n^M G_n^M\|^2_{H^{-1-\nu}(\mathbb{R}^2)}$$

$$= \int_{\mathbb{R}^2} \mathbb{E} \left( \sum_{n \in \mathbb{Z}^2} G_n^M \int_{\mathbb{R}^2} \frac{e^{2\pi in x/M}}{\langle x \rangle^{\sigma}} e^{-i\xi x} dx \right)^2 (1 + |\xi|^2)^{-1-\nu} d\xi$$

$$\leq \frac{1}{M} \int_{\mathbb{R}^2} \sum_{n \in \mathbb{Z}^2} \left( \int_{\mathbb{R}^2} \frac{e^{2\pi in x/M - i \xi x}}{\langle x \rangle^{\sigma}} dx \right)^2 (1 + |\xi|^2)^{-1-\nu} d\xi,$$
where the last inequality is due to the reason that \( G_n^M = \overline{G_{-n}^M} \), and \( G_n^M, n \in \mathbb{Z}_+ \cup \{0\} \) are independent random variables with standard (complex) Gaussian distributions. Note that

\[
\frac{1}{M^2} \sum_{n \in \mathbb{Z}^2} \left( \int_{\mathbb{R}^2} e^{2\pi i n \cdot x/M - \frac{1}{2} |\xi|} \langle x, \xi \rangle^{\eta} \, dx \right)^2 = \frac{1}{M^2} \sum_{n \in \mathbb{Z}^2} [F(\rho_\sigma)(\xi - \frac{2\pi n}{M})]^2 \\
= \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}^2} \left( \frac{2\pi}{M} \right)^2 [F(\rho_\sigma)(\xi - \frac{2\pi n}{M})]^2.
\]

(12)

Since \( \sigma > 2 \), \( \rho_\sigma = \frac{1}{(1+|x|^2)^{\frac{1}{4}}} \in L^1 \), \( F(\rho_\sigma) \) is continuous and bounded. Moreover, since \( \rho_\sigma \) is infinitely smooth with all the derivatives bounded and \( L^1 \)-integrable, \( F(\rho_\sigma)(x) \) decays faster than \( (1 + |x|)^{-N} \) for any \( N > 0 \) when \( x \) goes to infinity. Therefore, (12) is bounded by \( \frac{C}{M^2} \sum_{n \in \mathbb{Z}^2} (1 + |\xi - \frac{2\pi n}{M}|)^{-4} \lesssim M^2 \), which is independent of \( \xi \). And when \( M \) goes to infinity, (12) will converge to

\[
\frac{1}{4\pi^2} \int_{\mathbb{R}^2} [F(\rho_\sigma)(x)]^2 \, dx,
\]

where the rate of convergence is obviously independent of \( \xi \). Hence (12) is uniformly bounded for any \( \xi \in \mathbb{R}^2 \) and \( M \geq 1 \). Therefore,

\[
\mathbb{E}[\|\hat{\omega}^M\|^2_{H^{-1-\nu}(\rho_\sigma)}] \lesssim \int_{\mathbb{R}^2} (1 + |\xi|^2)^{-1-\nu} \, d\xi \\
\lesssim 1.
\]

Note that \( H^{-1-\nu}(\rho_\sigma) \) can be compactly embedded into \( H^{-1-2\nu}(\rho_{\sigma'}) \), where \( \rho_{\sigma'}(x) := \frac{1}{\langle x, \xi \rangle^{\eta}} \) and \( \sigma' > \sigma \). Hence we obtain the tightness of \( \{\mu_M\}_{M \geq 1} \) in \( H^{-1-\nu}(\rho_\sigma) \).

Therefore, by the Prokhorov’s theorem and Skorokhod’s theorem, there exists a subsequence \( M_k \) such that \( \mu_{M_k} \) converges weakly to some limit \( \bar{\mu} \). Moreover, there exists a sequence of random variables \( \hat{\omega}^{M_k} \) on another probability space \( (\Xi', \mathcal{F}', \mathbb{P}') \), which have distributions \( \mu_{M_k} \), such that \( \hat{\omega}^{M_k} \) converges to \( \hat{\omega} \) in \( H^{-1-\nu}(\rho_\sigma) \mathbb{P}' - a.e. \), such that \( \hat{\omega} \) has the distribution \( \bar{\mu} \). We claim that \( \bar{\mu} \) is the space white noise distribution on \( \mathbb{R}^2 \). Indeed, we know that for \( \phi \in C_c^\infty(\mathbb{R}^2) \), there exists some \( k_0 \) such that for \( k \geq k_0 \), \( \phi \) is supported on the ball with the radius smaller than \( \frac{M_k}{2} \), then we can view \( \phi \) as a function \( \hat{\phi}_M \) on the torus \( \mathbb{T}_M^2 \), thus for \( k \geq k_0 \),

\[
\langle \hat{\omega}^{M_k}, \phi \rangle = \langle \hat{\omega}^{M_k}, \phi_{M_k} \rangle
\]

is centred Gaussian, therefore, \( \langle \hat{\omega}, \phi \rangle \) is centred Gaussian. Moreover, similarly, from the argument of the explanation of Definition 4 of [23], if we fix \( \phi, \psi \in C_c^\infty(\mathbb{R}^2) \), when \( k \) is large enough such that \( \phi \) and \( \psi \) are supported in the ball with the radius smaller than \( \frac{M_k}{2} \), we have

\[
\mathbb{E}[\langle \hat{\omega}^{M_k}, \phi \rangle \langle \hat{\omega}^{M_k}, \psi \rangle] = \langle \phi, \psi \rangle_{L^2(\mathbb{R}^2)},
\]

thus we have

\[
\mathbb{E}[\langle \hat{\omega}, \phi \rangle \langle \hat{\omega}, \psi \rangle] = \langle \phi, \psi \rangle_{L^2(\mathbb{R}^2)},
\]

which finishes the proof of our claim.

\( \square \)
3 Main result

3.1 Definition of the nonlinear term and the white noise solutions of mSQG on $\mathbb{R}^2$

After the preparations, we will introduce our main result. First we introduce the definition of the white noise (stationary) solution of the weak formulation form of (mSQG).

Definition 3.1. Fix any $T > 0$. We say that a measurable map $\omega : \Xi \times [0, T] \to C_c^\infty (\mathbb{R}^2)'$ (where $(\Xi, \mathcal{F}, \mathbb{P})$ is some probability space) with trajectories of class $C([0, T] ; (C^2_c)')$ (see Definition 4.4 for the definition of the topological space $(C^2_c)'$) is a white noise (weak) solution of (mSQG), if it satisfies the following:

1. For fixed $t$, $\omega_t$ is a space white noise on $\mathbb{R}^2$.
2. For any $\phi \in C_c^\infty (\mathbb{R}^2)$,

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_{\phi, \epsilon} \rangle \, ds$$

holds $\mathbb{P}$-a.e., where

$$H_{\phi, \epsilon} (x, y) := \frac{1}{2} K_\epsilon (x - y) (\nabla \phi (x) - \nabla \phi (y)),$$

and we will introduce the definition of the term $\langle \omega_s \otimes \omega_s, H_{\phi, \epsilon} \rangle$ later in Theorem 3.4.

Similarly to the paper [9], here we also need to define the nonlinear term by constructing an approximating sequence.

Since for $\sigma > 2$, the space white noise $\tilde{\omega}$ is $\mathbb{P}$-a.e. in the weighted Sobolev space $H^{-\sigma} (\rho_\sigma)$ (see Lemma 2.8), $\tilde{\omega} \otimes \tilde{\omega}$ is in $H^{-\sigma} (\mathbb{R}^4, \rho_\sigma \times \rho_\sigma)$ $\mathbb{P}$-a.e. $\langle \tilde{\omega} \otimes \tilde{\omega}, f \rangle$ is defined when $f \in H^2 (\mathbb{R}^4, \rho_\sigma^{-1} \times \rho_\sigma^{-1})$, where $H^2 (\mathbb{R}^4, \rho_\sigma^{-1} \times \rho_\sigma^{-1})$ is the union of the spaces $H^{2+\nu} (\mathbb{R}^4, \rho_\sigma^{-1} \times \rho_\sigma^{-1})$ for all $\nu > 0$. In particular, it can be defined when $f \in C_c^\infty (\mathbb{R}^2 \times \mathbb{R}^2)$. However, $H_{\phi, \epsilon}$ does not belong to the space $H^{2+} (\mathbb{R}^4, \rho_\sigma^{-1} \times \rho_\sigma^{-1})$. Thus similarly to [9], we need to define the nonlinear term by constructing approximating sequence.

First of all, the following lemma gives for smooth and compactly supported function $\phi$, $H_{\phi, \epsilon} \in L^2 (\mathbb{R}^2 \times \mathbb{R}^2)$.

Lemma 3.2. For $\phi \in C_c^\infty (\mathbb{R}^2)$, $H_{\phi, \epsilon} \in L^2 (\mathbb{R}^2 \times \mathbb{R}^2)$.

Proof. We prove directly by calculation. Since

$$H_{\phi, \epsilon} (x, y) := \frac{1}{2} K_\epsilon (x - y) (\nabla \phi (x) - \nabla \phi (y)),$$
and $K_\epsilon (x - y) \leq \frac{1}{|x - y|^{2-\epsilon}}$, $0 < \epsilon < 1$, assuming that $\phi$ is supported in the ball of radius $R$, we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |H_{\phi, \epsilon}(x, y)|^2 dxdy \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\nabla \phi (x) - \nabla \phi (y)|^2}{|x - y|^{4-2\epsilon}} dxdy$$

$$\leq \int_{|x| \leq 2R} \int_{|y| \leq 2R} \frac{\|D^2 \phi\|_{L^\infty}^2}{|x - y|^{4-2\epsilon}} dxdy + 2 \int_{|x| \leq R} \int_{|y| \geq 2R} \frac{|\nabla \phi (x) - 0|^2}{|x - y|^{4-2\epsilon}} dxdy$$

$$\leq C(R) \|D^2 \phi\|_{L^\infty}^2 + 2\|\nabla \phi\|_{L^\infty}^2 \pi R^2 \int_{|y| \geq 2R} \frac{1}{(|y| - R)^{4-2\epsilon}} dy$$

$$\leq C(R) (\|D^2 \phi\|_{L^\infty}^2 + \|\nabla \phi\|_{L^\infty}^2),$$

where $C(R)$ is a constant which only depends on $R$ and the second inequality is due to the symmetric property of $H_{\phi, \epsilon}(x, y)$. Since we only used the property $K_\epsilon (x - y) \leq \frac{1}{|x - y|^{2-\epsilon}}$, by Lemma 2.5 we immediately have the following corollary:

**Corollary 3.3.** Let $\phi \in C^2_c(\mathbb{R}^2)$ be a function supported in $[-\frac{M_0}{2}, \frac{M_0}{2}]$. Then for any $M > M_0$, $\phi$ can be viewed as a function in $C^2(\mathbb{T}_M^2)$. For any $M > M_0$, we have

$$H_{\phi, \epsilon}^M(x, y) := \frac{1}{2} K_\epsilon^M (x - y) \left( \nabla \phi (x) - \nabla \phi (y) \right) \in L^2(\mathbb{T}_M^2 \times \mathbb{T}_M^2).$$

Moreover, there exists a constant $C_\phi$ which does not depend on $M$, such that

$$\|H_{\phi, \epsilon}^M(x, y)\|_{L^2(\mathbb{T}_M^2 \times \mathbb{T}_M^2)} \leq C_\phi.$$

Similar as Theorem 8 of [9], we will prove the following theorem which gives the approximating sequence.

**Theorem 3.4.** Fix $\phi \in C^2(\mathbb{R}^2 \times \mathbb{R}^2)$. Assume that $f_n \in C^\infty_c(\mathbb{R}^2 \times \mathbb{R}^2)$ are symmetric and approximate $H_{\phi, \epsilon}$ in the following sense:

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f_n - H_{\phi, \epsilon})^2 (x, y) dxdy = 0$$

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} f_n(x, x) dx = 0.$$

Then the sequence of r.v.’s $\langle \hat{\omega} \otimes \hat{\omega}, f_n \rangle$ is a Cauchy sequence in mean square. We denote by

$$\langle \hat{\omega} \otimes \hat{\omega}, H_{\phi, \epsilon} \rangle$$

its limit. Moreover, the limit is the same if $f_n$ is replaced by $\tilde{f}_n$ with the same properties and such that $\lim_{n \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\tilde{f}_n - f_n)^2 (x, y) dxdy = 0$. 


Proof. Without loss of generality, we assume that for each $n$, $f_n$ is supported in $[-\frac{n}{4}, \frac{n}{4}]^4$.

In the following we will use the relation between white noise on the torus and the whole space, the details of which can be found in [23].

Since $f_n$ is supported in $[-\frac{n}{4}, \frac{n}{4}]^4$, it could also be viewed as a smooth function on $T^2_M \times T^2_M$ when $M \geq n$.

By the explanation of Definition 4 in [23], we know
\[
\langle \tilde{\omega} \otimes \bar{\omega}, f_n \rangle = \langle \omega^n \otimes \omega^n, f_n \rangle = \langle \omega^m \otimes \omega^m, f_n \rangle,
\]
for $m \geq n$, where $f_n$ is understood as a function on $\mathbb{R}^2$, $T^2_n$ and $T^2_m$ respectively.

Therefore, now what we need to prove is the following:
\[
\langle \omega^n \otimes \omega^n, f_n \rangle \text{ converges in } L^2(\Xi) \text{ as } n \text{ goes to } +\infty.
\]
To prove the convergence it suffices to prove it is a Cauchy sequence.

Since $\lim_{n \to \infty} \int_{\mathbb{R}^2} f_n(x,x) \, dx = 0$, it is equivalent to show that $\langle \omega^n \otimes \omega^n, f_n \rangle - \int f_n(x,x) \, dx$ is a Cauchy sequence in mean square. We have for $m \geq n$,
\[
\mathbb{E}\left[ \left| \langle \omega^n \otimes \omega^n, f_n \rangle - \int_{\mathbb{R}^2} f_n(x,x) \, dx - \langle \omega^m \otimes \omega^m, f_m \rangle + \int_{\mathbb{R}^2} f_m(x,x) \, dx \right|^2 \right]
= \mathbb{E}\left[ \left| \langle \omega^n \otimes \omega^n, f_n \rangle - \int_{\mathbb{R}^2} f_n(x,x) \, dx - \langle \omega^m \otimes \omega^m, f_m \rangle + \int_{\mathbb{R}^2} f_m(x,x) \, dx \right|^2 \right]
= \mathbb{E}\left[ \left| \langle \omega^m \otimes \omega^m, (f_n - f_m) \rangle - \int_{\mathbb{R}^2} (f_n - f_m)(x,x) \, dx \right|^2 \right],
\]
where the second equality is due to (13).

By (ii) and (iii) of the Corollary 6 in [9], (for completeness we attach the corollary later in Corollary 3.5) we know that (14) equals
\[
2 \int_{T^2_M} \int_{T^2_M} (f_n - f_m)^2 \, dxdy = 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f_n - f_m)^2 \, dxdy,
\]
which implies the Cauchy property of $\langle \tilde{\omega} \otimes \bar{\omega}, f_n \rangle$ in mean square. Hence $\langle \tilde{\omega} \otimes \bar{\omega}, H_{\phi,\epsilon} \rangle$ is well defined.

Moreover, by a similar way we prove that if we replace $f_n$ by $\tilde{f}_n$ with the same properties and such that $\lim_{n \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\tilde{f}_n - f_n)^2 \, dxdy = 0$, $\langle \tilde{\omega} \otimes \bar{\omega}, \tilde{f}_n \rangle$ also converges in mean square to $\langle \tilde{\omega} \otimes \bar{\omega}, H_{\phi,\epsilon} \rangle$.\]

Corollary 3.5. See [9, Corollary 6]  

i) If $\omega^M : \Xi \to C^\infty(T^2_M)^d$ is a white noise and $f \in H^2(\mathbb{T}^2_M \times \mathbb{T}^2_M)$, then for every $p \geq 1$ there is a constant $C_{p,M} > 0$ such that
\[
\mathbb{E}\left[ \left| \langle \omega^M \otimes \omega^M, f \rangle \right|^p \right] \leq C_{p,M} \|f\|^{p}_{L^x}.
\]

ii) We have $\mathbb{E}\left[ \langle \omega^M \otimes \omega^M, f \rangle \right] = \int_{\mathbb{T}^2_M} f(x,x) \, dx$.

iii) If $f$ is symmetric, then
\[
\mathbb{E}\left[ \langle \omega^M \otimes \omega^M, f \rangle - \mathbb{E}\left[ \langle \omega^M \otimes \omega^M, f \rangle \right]^2 \right] = 2 \int_{\mathbb{T}^2_M} \int_{\mathbb{T}^2_M} f(x,y)^2 \, dxdy.
\]
We now give an example of the approximating sequence \( \{f_n\}_{n \geq 1} \).

**Construction of the approximating sequence** \( \{f_n\}_{n \geq 1} \)

We have proved in Lemma 3.2 that for fixed \( \phi \in C_c^2(\mathbb{R}^2) \), \( H_{\phi, \epsilon} \in L^2(\mathbb{R}^2 \times \mathbb{R}^2) \), thus there exists a sequence of function \( g_n \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) which converge to \( H_{\phi, \epsilon} \) in \( L^2(\mathbb{R}^2 \times \mathbb{R}^2) \). We can also assume that \( g_n \) is symmetric (otherwise, we let \( \tilde{g}_n = \frac{1}{2}(g_n(x, y) + g_n(y, x)) \)).

Without loss of generality we assume that for each \( n \), \( g_n \) is supported in \( [-\frac{n}{4}, \frac{n}{4}]^4 \). Let \( f_n = r_ng_n \), where \( r_n \) is defined as follows:

\[
\begin{cases}
  1, & |x - y| \geq \frac{1}{n^5}; \\
  0, & |x - y| \leq \frac{1}{2n^5}; \\
  \in [0, 1] & \text{such that } r_n \text{ smooth, } \frac{1}{2n^5} \leq |x - y| \leq \frac{1}{n^5}.
\end{cases}
\]

Thus \( f_n \) is also smooth and supported in \( [-\frac{n}{4}, \frac{n}{4}]^4 \). Moreover,

\[
\|f_n - H_{\phi, \epsilon}\|_{L^2} \leq \|g_n - H_{\phi, \epsilon}\|_{L^2} + \|(f_n - g_n)1_{|x-y| \leq \frac{1}{n^5}}\|_{L^2} \\
\leq \|g_n - H_{\phi, \epsilon}\|_{L^2} + \|g_n1_{|x-y| \leq \frac{1}{n^5}}\|_{L^2} \\
\leq 2\|g_n - H_{\phi, \epsilon}\|_{L^2} + \|H_{\phi, \epsilon}1_{(x,y) \in [-\frac{n}{4}, \frac{n}{4}]^4, |x-y| \leq \frac{1}{n^5}}\|_{L^2} \\
\rightarrow 0,
\]

where the last line is due to the fact that \( g_n \) converges to \( H_{\phi, \epsilon} \) in \( L^2 \) and the Lebesgue measure of the set \( \{(x, y) \in [-\frac{n}{4}, \frac{n}{4}]^4, |x-y| \leq \frac{1}{n^5}\} \) goes to 0.

Remark 3.6.

1. Obviously, all the \( f_n \) and \( g_n \) that we defined above rely on \( \phi \) and \( \epsilon \), but for simplicity of the notation, we skip them in our notation.

2. Note that the rate of convergence of \( \langle \bar{\omega} \otimes \bar{\omega}, f_n \rangle \to \langle \bar{\omega} \otimes \bar{\omega}, H_{\phi, \epsilon} \rangle \) in \( L^2(\Xi) \) only depends on the rate of the convergence of \( f_n \) to \( H_{\phi, \epsilon} \) in \( L^2(\mathbb{R}^2 \times \mathbb{R}^2) \), but does not depend on \( \bar{\omega} \) as long as it is a space white noise.

3. From our construction, we can require that \( f_n(x, x) = 0 \) for any \( n \).

After we define the nonlinear term, we manage to define the white noise (weak) solution of \( (mSQG) \) on \( \mathbb{R}^2 \).

### 3.2 Main theorem of the paper

Now we introduce the main result of the paper.

**Theorem 3.7.** There exists a white noise stationary (weak) solution of \( (mSQG) \) according to Definition 3.1.

In other words, we prove a similar result of [10] by letting the volume of the torus go to infinity, which is in the next section.
4 Proof of the Theorem 3.7

In this section we prove the main result (Theorem 3.7) of this paper. First we recall the similar result on the torus. Recalling the Theorem 1 of [10] (also Theorem 1.1 of [21] by letting $\theta = 0$), the following theorem was proved

**Theorem 4.1** (Existence). Let $\epsilon \in (0, 1)$. There exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stationary process $\xi : \Omega \to C([0, T]; H^{-1})$ such that, for all $t \in [0, T]$, $\xi_t$ is a white noise on $\mathbb{T}^2$; and for all $\phi \in C^\infty(\mathbb{T}^2)$, $\mathbb{P}$-a.s. for all $t \in [0, T]$, one has

$$\langle \xi_t, \phi \rangle = \langle \xi_0, \phi \rangle + \int_0^t \langle \xi_s \otimes \xi_s, H_{\phi, \epsilon} \rangle \, ds. \quad (15)$$

Note that for $\epsilon = 1$ it is the Euler equation, the result is also proved in [9].

**Remark 4.2.**

1. In both [9] and [10], it is obvious that the zero set depends on $\phi$. Indeed since the non-linear term is defined in the mean square sense, for any $\phi$ we can change the value of $\langle \xi_s \otimes \xi_s, H_{\phi, \epsilon} \rangle$ in any zero set $N_\phi$.

2. In [9] and [10], the proof of both cases on the torus only use the boundedness of second derivatives of test function $\phi$. In other words, the above theorem also holds when $\phi \in C^2(\mathbb{T}^2)$.

3. In the proof of [9] and [10], the proof does not depend on the radius of the torus. Therefore, the same results also work for $\mathbb{T}^2_M$ for any $M > 0$.

Let $\omega_t^M$ be the solution in the above theorem on the torus $\mathbb{T}^2_M$ on some probability space $(\Omega^M, \mathcal{F}^M, \mathbb{P}^M)$. First we fix $\phi \in C_c^2$. Assume that $\phi$ is supported in $[-A, A]^2$. Thus for $M > A$, $\phi$ could also be viewed as a function on the torus $\mathbb{T}^2_M$. By Theorem 4.1 we have for $M > A$,

$$\langle \omega_t^M, \phi \rangle = \langle \omega_0^M, \phi \rangle + \int_0^t \langle \omega_s^M \otimes \omega_s^M, H_{\phi, \epsilon}^M \rangle \, ds,$$

where $H_{\phi, \epsilon}^M$ is defined in Section 2.2.2 and Section 2.3, and the nonlinear term is defined as in [9].

Similar to before, let $\tilde{\omega}_t^M$ be the periodic extension of $\omega_t^M$ on $\mathbb{R}^2$. Thus we have for any $t \geq 0$,

$$\langle \tilde{\omega}_t^M, \phi \rangle = \langle \omega_t^M, \phi_M \rangle,$$

where the left hand is defined on $\mathbb{R}^2$ and the right hand side is defined on the torus $\mathbb{T}^2_M$. Thus

$$\langle \tilde{\omega}_t^M, \phi \rangle = \langle \tilde{\omega}_0^M, \phi \rangle + \int_0^t \langle \tilde{\omega}_s^M \otimes \tilde{\omega}_s^M, H_{\tilde{\phi}, \epsilon}^M \rangle \, ds,$$

(16) where $\langle \omega_t^M, \phi \rangle$ and $\langle \omega_0^M, \phi \rangle$ are duality products on $\mathbb{R}^2$ but $\langle \omega_s^M \otimes \omega_s^M, H_{\tilde{\phi}, \epsilon}^M \rangle$ is the duality product on $\mathbb{T}^2_M$.

Before we prove the tightness we need some more preparations. We begin with a lemma.
Lemma 4.3. The metric space $C^2_c$ with the $C^2$ Hölder norm is separable.

Proof.

Step 1: to prove that $C^2_c$ can be approximated by $S(\mathbb{R}^2)$

We fix a family of smooth functions which converge to the Dirac function, for example,

$$
\gamma_R(x) = \begin{cases} 
C_R \exp\left( \frac{1}{R|x|^2-1} \right), & |x|^2 \leq \frac{1}{R^2}, \\
0, & |x|^2 \geq \frac{1}{R^2},
\end{cases}
$$

where $C_R$ is a constant such that

$$
\int_{\mathbb{R}^2} \gamma_R(x) \, dx = 1.
$$

For any function $f \in C^2_c$ and any index $|\alpha| \leq 2$,

$$
D^\alpha (\gamma_R * f - f) = C_R \int_{\mathbb{R}^2} D^\alpha (f(x - y) - f(x)) \gamma_R(y) \, dy.
$$

Since $f \in C^2_c$, $D^\alpha (f(x - y) - f(x))$ goes to 0 uniformly as $y$ tends to 0, hence $\gamma_R * f$ converges to $f$ in $C^2$. It is obvious that $\gamma_R * f$ is smooth and has compact support. Therefore, $\gamma_R * f \in S(\mathbb{R}^2)$.

Step 2: to find a Countable Dense Subset of $C^2_c$

Since $S(\mathbb{R}^2)$ is separable, let $\{f_i\}_{i=1}^\infty$ be its countable dense subset. Since $C^\infty_c$ is dense in $S(\mathbb{R}^2)$, for each $f_i$, we can find a sequence $f_{ij} \in C^\infty_c$, such that $f_{ij}$ converges to $f_i$ in $S(\mathbb{R}^2)$ (hence in $C^2$) as $j$ goes to infinity. Therefore, from the above arguments we know $\{f_{ij}\}_{i,j \geq 1}$ is a dense subset of $C^2_c$.

Thus we have proved $C^2_c$ is a separable metric space. \qed

Definition 4.4. We define the following function spaces:

1. Define $(C^2_c)'$ to be the space which contains all the continuous linear functional from $C^2_c$ to $\mathbb{R}$ with weak * topology.

2. Define the time Sobolev space $W^{1,2}([0, T]; (C^2_c)')$ to be the space of all $u \in C([0, T]; (C^2_c)')$ such that $u(\phi) \in L^2([0, T]; \mathbb{R})$ and $\partial_t u(\phi) \in L^2([0, T]; \mathbb{R})$ for any $\phi \in C^2_c$. The topology of $W^{1,2}([0, T]; (C^2_c)')$ is defined to be the weakest topology on $W^{1,2}([0, T]; (C^2_c)')$ such that for any $\psi \in L^2([0, T]; C^2_c)$, the maps

$$
u \mapsto \langle u, \psi \rangle
$$

and

$$
u \mapsto \partial_t \langle u, \psi \rangle
$$

are continuous from $W^{1,2}([0, T]; (C^2_c)')$ to $\mathbb{R}$.

Remark 4.5.

1. $C^2_c$ is not complete. Denote by $C^2_0$ its closure with respect to the $C^2$ norm. Then by Banach–Steinhaus theorem the space $(C^2_c)'$ is the same as the space $(C^2_0)'$. It is obvious that $C^2_0$ is also separable with the same countable dense subset of $C^2_c$. 

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2. Since $C_c^2$ is separable, the closed unit ball of $(C_c^2)'$ is compact metric space by Banach-Alaoglu Theorem, hence also separable. Therefore, $(C_c^2)'$ is also separable.

We have the following tightness results.

**Lemma 4.6.** Let $\{D(\omega_i^M)\}_{M=1}^\infty$ be the distribution of $\omega_i^M$ in $W^{1,2}([0, T]; (C_c^2)')$. Then for any $T > 0$, $\{D(\omega_i^M)\}_{M=1}^\infty$ is tight in $W^{1,2}([0, T]; (C_c^2)')$.

**Proof.** By definition of the topology of $(C_c^2)'$, it suffices to prove that for any $\phi \in C_c^2$,

$$E|\langle \omega_i^M, \phi \rangle|^2 \leq C$$

and

$$E|\partial_t \langle \omega_i^M, \phi \rangle|^2 \leq C,$$

where $C$ is a constant which depends on $\phi$ but not $M$ and $t$. (17) is immediately obtained by Lemma 2.8.

To obtain (18), we note from (16) that for any $t > 0$,

$$\partial_t \langle \omega_i^M, \phi \rangle = \langle \omega_s^M \otimes \omega_s^M, H_{\phi, \epsilon}^M \rangle.$$ 

Assume that $\phi$ is supported in $[-\frac{A}{4}, \frac{A}{4}]^2$. Then for $M > A$, $\phi_M = \phi$. Hence we have for $M > A$,

$$E|\langle \omega_s^M \otimes \omega_s^M, H_{\phi, \epsilon}^M \rangle|^2 = E|\langle \omega_s^M \otimes \omega_s^M, H_{\phi, \epsilon}^M \rangle|^2.$$ 

By Corollary 6 of [10], we deduce

$$E|\langle \omega_s^M \otimes \omega_s^M, H_{\phi, \epsilon}^M \rangle|^2 = 2 \int_{T^2_M} \int_{T^2_M} H_{\phi, \epsilon}^M (x, y)^2 dxdy.$$ 

Recall that $H_{\phi, \epsilon}^M (x, y) = \frac{1}{2}K^M_{\epsilon}(|x - y|)(\nabla \phi(x) - \nabla \phi(y))$, where $|K^M_{\epsilon}(x)| \leq C_{\epsilon}$ and $C_{\epsilon}$ is a uniform constant does not depend on $M$. Moreover, by Corollary 3.3, $H_{\phi, \epsilon}^M$ is uniformly bounded with respect to $M$ in the sense of $L^2(T^2_M \times T^2_M)$-norm. Hence $E|\langle \omega_s^M \otimes \omega_s^M, H_{\phi, \epsilon}^M \rangle|^2$ is uniformly bounded. Since there are only finite positive integers which are smaller than $A$, we conclude (18).

Now we apply the Skorokhod Theorem A.1. Note that the space $W^{1,2}([0, T]; (C_c^2)')$ satisfies the requirement of Theorem A.1, since $W^{1,2}([0, T]; (C_c^2)')$ is separated by the countable dense subset of $L^2([0, T]; C_c^2)$.

By the statement of Theorem A.1, one needs to show that the $\sigma$-algebra generated by the countable dense subset of $L^2([0, T]; C_c^2)$ is the Borel $\sigma$-algebra of $W^{1,2}([0, T]; (C_c^2)')$. By Theorem B.4 it suffices to prove that $W^{1,2}([0, T]; (C_c^2)')$ is a standard Borel space. (See Appendix B.2 for the definition of the standard Borel space).

**Lemma 4.7.** $W^{1,2}([0, T]; (C_c^2)')$ is a standard Borel space.
Proof. Let $X_1 := W^{1,2}([0, T]; (C_c^2)')$ and $X_2 := W^{1,2}([0, T]; L^2)$, where $X_2$ consists of all the functions $v$ such that $v \in L^2([0, T] \times \mathbb{R}^2)$ and $\partial_t v \in L^2([0, T] \times \mathbb{R}^2)$ with the norm

$$
\|v\|_{W^{1,2}([0, T]; L^2)} := \|v\|_{L^2([0, T] \times \mathbb{R}^2)} + \|\partial_t v\|_{L^2([0, T] \times \mathbb{R}^2)}.
$$

It is obvious that $X_2$ is a Polish space and it is continuously embedded in $X_1$. We need to prove

$$
\mathcal{B}(X_2) = \mathcal{B}(X_1) \cap X_2.
$$

Obviously $\mathcal{B}(X_1) \cap X_2 \subset \mathcal{B}(X_2)$. It suffices to show that any open set of $X_2$ is in $\mathcal{B}(X_1) \cap X_2$.

Note that $\{B(x_m, r_n)\}_{m,n \geq 1}$ is a countable topology basis of $X_2$, where $\{x_m\}_{m \geq 1}$ is a countable dense subset of $X_2$ and $\{r_n\}_{n \geq 1}$ is the sequence of all the positive rational numbers. Therefore, we only need to prove

$$
B(x_m, r_n) \in \mathcal{B}(X_1) \cap X_2.
$$

Without loss of generality we only prove it for $x_m = 0$.

Note that

$$
B(0, r_n) = \{x \in X_2; \|x\|_{L^2([0, T] \times \mathbb{R}^2)} + \|\partial_t x\|_{L^2([0, T] \times \mathbb{R}^2)} < r_n\}
$$

$$
= \bigcup_{j \geq 1} \bigcap_{k,l \geq 1} \{x \in X_2; |\langle x, \psi_k \rangle| + |\langle \partial_t x, \psi_l \rangle| < r_n - \frac{1}{j}\}
$$

$$
= X_2 \cap \bigcup_{j \geq 1} \bigcap_{k,l \geq 1} \{x \in X_1; |\langle x, \psi_k \rangle| + |\langle \partial_t x, \psi_l \rangle| < r_n - \frac{1}{j}\},
$$

where $\{\psi_k\}_{k \geq 1}$ is set to be a countable dense subset of the unit ball of $L^2([0, T] \times \mathbb{R}^2)$ such that $\{\psi_k\}_{k \geq 1}$ is also a subset of $L^2([0, T]; C_c^2)$. Then

$$
\{x \in X_1; |\langle x, \psi_k \rangle| + |\langle \partial_t x, \psi_l \rangle| < r_n - \frac{1}{j}\}
$$

is an open set of $X_1$. Hence

$$
B(0, r_n) \in \mathcal{B}(X_1) \cap X_2,
$$

which finishes our proof. \qed

Therefore, there exists another probability space, which we still use the notation $(\Omega, \mathcal{F}, \mathbb{P})$ for simplicity, and a sequence of random variables $\tilde{\omega}_t^{M_k}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, such that

- $\tilde{\omega}_t^{M_k}$ has the same distribution to $\tilde{\omega}_t^{M_k}$ in $W^{1,2}([0, T]; (C_c^2)')$; (we also assume that $M_k$ is increasing to infinity and $M_k \geq k$)

- $\tilde{\omega}_t^{M_k}$ converge $\mathbb{P}$-almost surely to some limit $\tilde{\omega}_t$ in $W^{1,2}([0, T]; (C_c^2)')$.

Hence by the same argument of Lemma 2.8, we obtain that for any fixed $t \in [0, T]$, $\tilde{\omega}_t$ is a space white noise distribution on $\mathbb{R}^2$.

By the definition of the solution on the torus, $\tilde{\omega}_t^M$ has the following form

$$
\tilde{\omega}_t^M = \sum_{n \in \mathbb{Z}^2} \tilde{C}_n(t, \theta_M)e_n^M \text{ on } \mathbb{R}^2,
$$
where \( \theta_M \in \Omega^M, \bar{G}_n^M(\cdot, \theta_M) \in W^{1,2}([0, T]; \mathbb{R}) \) and for each \( t, \bar{G}_n^M(t, \cdot), n \in \mathbb{Z}_n^2 \cup \{0\} \) are independent random variables with standard Gaussian distributions on \( (\Omega^M, \mathcal{F}^M, \mathbb{P}^M) \).

Note that for fixed \( M \), if \( a_n(t) \in W^{1,2}([0, T]; \mathbb{R}) \) and
\[
\sum_{n \in \mathbb{Z}^2} a_n(t) e_n^M \in W^{1,2}([0, T]; (C_c^2)') ,
\]
the map
\[
\sum_{n \in \mathbb{Z}^2} a_n(t) e_n^M \mapsto (a_{n_1}(t), a_{n_2}(t), ..., a_{n_k}(t))
\]
is continuous from \( W^{1,2}([0, T]; (C_c^2)') \) to \( (W^{1,2}([0, T]; \mathbb{R}))^k \) for any \( k \) and \( n_1, n_2, ... n_k \in \mathbb{Z}^2 \).

Therefore, \( \tilde{\omega}_t^M \) also has the form
\[
\tilde{\omega}_t^M = \sum_{n \in \mathbb{Z}^2} \bar{G}_n^M(t, \theta) e_n^M \text{ on } \mathbb{R}^2
\]
on \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \bar{G}_n^M(t, \cdot), \bar{G}_n^M(t, \cdot), ..., \bar{G}_n^M(t, \cdot) \) and \( \bar{G}_n^M(t, \cdot), \bar{G}_n^M(t, \cdot), ..., \bar{G}_n^M(t, \cdot) \) have the same joint distributions on \( (W^{1,2}([0, T]; \mathbb{R}))^k \). Define
\[
\hat{\omega}_t^M = \sum_{n \in \mathbb{Z}^2} \bar{G}_n^M(t, \theta) e_n^M \text{ on } \mathbb{T}_M^2,
\]
i.e. \( \hat{\omega}_t^M \) is an extension of \( \hat{\omega}_t^M \) on \( \mathbb{R}^2 \). Moreover, \( \hat{\omega}_t^M \) has the same distribution as \( \omega_t^M \), hence it also satisfies the equation (15).

Thus it satisfies the same equation as (16) for any \( \phi \in C_c^2(\mathbb{R}^2) \) \( \mathbb{P} \)-a.s.:
\[
\langle \hat{\omega}_t^M, \phi \rangle = \langle \hat{\omega}_0^M, \phi \rangle + \int_0^t \langle \hat{\omega}_s^M \otimes \hat{\omega}_s^M, H_{\phi, \eta}^M \rangle \, ds. \tag{20}
\]

Same as usual, \( \phi \) could also be viewed as a function on \( \mathbb{T}_M^2 \), when we fix \( \phi \) and let \( M_k \) large enough. It suffices to prove for any fixed \( \phi \in C_c^2(\mathbb{R}^2) \), we have \( \mathbb{P} \)-a.s.
\[
\lim_{k \to \infty} \int_0^t \langle \hat{\omega}_s^M \otimes \hat{\omega}_s^M, H_{\phi, \eta}^M \rangle \, ds = \int_0^t \langle \bar{\omega}_s \otimes \bar{\omega}_s, H_{\phi, \eta} \rangle \, ds, \tag{21}
\]
where on the left hand side, \( \langle \omega_s \otimes \omega_s, H_{\phi, \eta}^M \rangle \) is the duality product on the torus and on the right hand side, \( \langle \omega_s \otimes \omega_s, H_{\phi, \eta} \rangle \) is the duality product on \( \mathbb{R}^2 \).

**Proof of (21)**

**Step 1**

Fix \( \eta > 0 \). Recall from Theorem 3.4, for a space white noise distribution \( \bar{\omega} \) on \( \mathbb{R}^2 \) in some probability space, we define \( \langle \omega \otimes \bar{\omega}, H_{\phi, \eta} \rangle \) as the mean square limit of \( \langle \omega \otimes \bar{\omega}, f_n \rangle \), where \( f_n \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) are symmetric and approximate \( H_{\phi, \eta} \) \( \mathbb{P} \)-a.s. in the following sense:

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f_n - H_{\phi, \eta})^2 (x, y) \, dx \, dy = 0
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} f_n (x, x) \, dx = 0.
\]
Moreover, without loss of generality we assume that for each \( n \), \( f_n \) is supported in \([-\frac{n}{4}, \frac{n}{4}]^4\). By 3 of Remark 3.6, we can require \( f_n(x, x) = 0 \). And by 2 of Remark 3.6, we know the approximation is uniform with respect to the time \( t \). Hence we know \( \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi, \epsilon} \rangle \) is the \( L^2(\Omega; L^2([0, T])) = L^2([0, T]; L^2(\Omega)) \) limit of \( \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, f_n \rangle \). Thus we can find an \( n_0 \), such that
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f_{n_0} - H_{\phi, \epsilon})^2 (x, y) dxdy < \frac{\eta}{T},
\]
(22)
thus
\[
\mathbb{E} \int_0^T |\langle \tilde{\omega}_s \otimes \tilde{\omega}_s, f_{n_0} \rangle - \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi, \epsilon} \rangle|^2 dt < \eta.
\]

**Step 2**
Fix \( n_0 \), since \( \tilde{\omega}_s^{M_k} \) converge \( \mathbb{P} \)-a.s. to \( \tilde{\omega}_s \) in \( W^{1,2}([0, T]; (C^2_c)' \), (hence in \( C([0, T]; (C^2_c)') \) )
\( \langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, f_{n_0} \rangle \) converges to \( \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, f_{n_0} \rangle \) in \( C([0, T]; \mathbb{R}) \) \( \mathbb{P} \)-almost surely as \( k \) goes to infinity. Moreover, since when \( k \geq n_0 \), \( \langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, f_{n_0} \rangle = \langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, f_{n_0} \rangle \), where \( f_{n_0} \) can be viewed as the product on the torus \( T^2_{M_k} \) when \( k \geq n_0 \), just as we have shown during the proof of the Theorem 3.4. By Corollary 3.5 it is uniformly integrable. Therefore,
\( \langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, f_{n_0} \rangle \) converges to \( \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, f_{n_0} \rangle \) as \( k \to \infty \) in \( L^2(\Omega; L^2([0, T])) \).

**Step 3**
By step 1 and step 2, we know that there exists some \( k_0 \geq n_0 \) such that when \( k \geq k_0 \),
\[
\mathbb{E} \int_0^T |\langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, f_{n_0} \rangle - \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi, \epsilon} \rangle|^2 ds < 2\eta \tag{23}
\]

**Step 4**
Just as we have mentioned in step 2, when \( k \geq k_0 \geq n_0 \), \( \langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, f_{n_0} \rangle \) is the duality product on the torus \( T^2_{M_k} \).
By ii) iii) of Corollary 3.5 and the definition of \( \langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, H_{\phi, \epsilon} \rangle \),
\[
\mathbb{E} \int_0^T |\langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, f_{n_0} \rangle - \langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, H_{\phi, \epsilon} \rangle|^2 ds \leq T \int_{T^2_{M_k}} \int_{T^2_{M_k}} (f_{n_0} - H_{\phi, \epsilon})^2 (x, y) dxdy.
\]
If we view \( H_{\phi, \epsilon}(x, y) \) as measurable functions on \( \mathbb{R}^2 \) which are 0 valued outside \([-\frac{M}{2}, \frac{M}{2}]^4\), we can view \( \int_{T^2_{M_k}} \int_{T^2_{M_k}} (f_{n_0} - H_{\phi, \epsilon})^2 (x, y) dxdy \) as \( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f_{n_0} - H_{\phi, \epsilon})^2 (x, y) dxdy \).
Since for any \( x \), \( K_M^\epsilon(x) \) goes to \( K_\epsilon(x) \) as \( M \) goes to infinity, \( H_{\phi, \epsilon} \) converges pointwisely to \( H_{\phi, \epsilon} \). Moreover, \( \{H_{\phi, \epsilon}^M\}_{M>0} \) are all dominated by the \( L^2(\mathbb{R}^2 \times \mathbb{R}^2) \) integrable function \( \frac{C(\nabla \phi(x) - \nabla \phi(y))}{|x-y|^{2-2\epsilon}} \) for some constant \( C \) not depending on \( M \), thus the convergence of \( H_{\phi, \epsilon}^M \) to \( H_{\phi, \epsilon} \) also holds in \( L^2(\mathbb{R}^2 \times \mathbb{R}^2) \). Thus combining with (22), we can find some \( k_1 \geq k_0 \), such that when \( k \geq k_1 \), \( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f_{n_0} - H_{\phi, \epsilon})^2 (x, y) dxdy < \frac{2\eta}{T} \), hence for any \( k \geq k_1 \),
\[
\mathbb{E} \int_0^T |\langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, f_{n_0} \rangle - \langle \tilde{\omega}_s^{M_k} \otimes \tilde{\omega}_s^{M_k}, H_{\phi, \epsilon} \rangle|^2 ds < 2\eta. \tag{24}
\]
Hence by (23) and (24), we obtain that (21) holds in \( L^2(\Omega) \).
Since for any \( 0 \leq t \leq T \), \( \langle \tilde{\omega}_t^{M_k}, \phi \rangle \) converges to \( \langle \tilde{\omega}_t, \phi \rangle \) \( \mathbb{P} \)-a.s., the convergence of (21) also holds \( \mathbb{P} \)-a.s.
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A Skorokhod’s Representation Theorem

We show the following Jakubowski’s version of the Skorokhod Theorem in the form given by Brzeźniak and Ondreját [2] Theorem A.1 and it was proved by A. Jakubowski in [16].

Theorem A.1. Let \( \mathcal{Y} \) be a topological space such that there exists a sequence \( f_m \) of continuous functions \( f_m : \mathcal{Y} \to \mathbb{R} \) that separates points of \( \mathcal{Y} \). Let us denote by \( \mathcal{S} \) the \( \sigma \)-algebra generated by the maps \( f_m \). Then

(i) every compact subset of \( \mathcal{Y} \) is metrizable;

(ii) if \( (\mu_m) \) is tight sequence of probability measures on \( (\mathcal{Y}, \mathcal{S}) \), then there exists a subsequence \( (m_k) \), a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with \( \mathcal{Y} \)-valued Borel measurable variables \( \xi_k, \xi \) such that \( \mu_{m_k} \) is the law of \( \xi_k \) and \( \xi_k \) converges to \( \xi \) almost surely on \( \Omega \). Moreover, the law of \( \xi \) is a Radon measure.

B Standard Borel Spaces

First we introduce the following definitions of countably generated Borel Space and standard Borel space.

Definition B.1 (Countably generated Borel space, see [24] Chapter V Definition 2.1). A Borel space \( (X, \mathcal{B}) \) is said to be countably generated if there exists a denumerable class \( \mathcal{D} \subset \mathcal{B} \) such that \( \mathcal{D} \) generates \( \mathcal{B} \).

Definition B.2 (Standard borel space, see [24] Chapter V Definition 2.2). A countably generated Borel space \( (X, \mathcal{B}) \) is called standard if there exists a complete separable metric space \( Y \) such that the \( \sigma \)-algebras \( \mathcal{B} \) and \( \mathcal{B}(Y) \) are \( \sigma \)-isomorphic.

Moreover, we will introduce the following theorem, which is Theorem 2.4 of Chapter V of [24].

Theorem B.3. Let \((X, \mathcal{B})\) be standard, \((Y, \mathcal{C})\) countably generated and \( \varphi \) a one-one map of \( X \) into \( Y \) which is measurable. Then \( Y' = \varphi(X) \in \mathcal{C} \) and \( \varphi \) is a Borel isomorphism between the Borel spaces \((X, \mathcal{B})\) and \((Y', \mathcal{C}_{Y'})\).

By Theorem B.3, we know the following theorem holds.

Theorem B.4. Let \((X, \mathcal{B})\) be any standard Borel space. Assume that \( \{f_n\}_{n \geq 1} \) is a sequence of \( \mathcal{B} \)-measurable functions from \( X \) to \( \mathbb{R} \) which separate the points of \( X \). Denote by \( \sigma_0(X) \) the \( \sigma \)-algebra generated by \( \{f_n\}_{n \geq 1} \). Then \( \sigma_0(X) = \mathcal{B} \).

Proof. Consider the identity map \( id \):

\[
(X, \mathcal{B}) \longrightarrow (X, \sigma_0(X)).
\]

Since each \( f \) is measurable, it is obvious that \( id \) is measurable. Hence by Theorem B.3 we know that \( id \) is a Borel isomorphism, which finishes our proof.
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