Entanglement entropy of integer quantum Hall states in polygonal domains

Iván D Rodríguez$^1$ and Germán Sierra$^2$

$^1$ National University of Maynooth, Dublin, Ireland
$^2$ Instituto de Física Teórica, UAM-CSIC, Madrid, Spain
E-mail: rodriguez@fi.infn.it and german.sierra@uam.es

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Abstract. The entanglement entropy of the integer quantum Hall states satisfies the area law for smooth domains with a vanishing topological term. In this paper we consider polygonal domains for which the area law acquires a constant term that only depends on the angles of the vertices and we give a general expression for it. We study also the dependence of the entanglement spectrum on the geometry and give it a simple physical interpretation.

Keywords: fractional QHE (theory), entanglement in extended quantum systems (theory)

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1. Introduction

The area law satisfied by the entanglement entropy of the low energy states of quantum many-body systems in condensed matter and field theory has become one of the most fundamental tools for studying the physical properties of these complex systems [1]–[3]. To define this entropy one considers a low energy state $\psi$, usually the ground state, and computes the reduced density matrix $\rho_A$ by tracing out the degrees of freedom outside a domain $A$. The entanglement entropy $S_A(\psi)$ associated with the state $\psi$ and the domain $A$ is defined as the von Neumann entropy of the reduced density matrix $\rho_A$, i.e.

$$ S_A(\psi) = - \text{tr} \rho_A \log \rho_A. $$

The area law states that $S_A(\psi)$ is proportional to the size of the boundary of $A$. In 3D this size is the area separating $A$ from its environment, which gives the law its name. For lower dimensions one should rather use the terms perimeter law for 2D and zeroth law for 1D, but those names are not customary.

Three issues are important regarding the area law: violations, fluctuations and subleading corrections. They all provide a great amount of information about the system. In conformally invariant 1D models, the area law shows a log violation proportional to the central charge of the corresponding CFT and the open/closed topology of the system [4]–[7]. Fluctuations around the log law for the Renyi entropy in Luttinger liquids allow a determination of the Luttinger parameter [8], and subleading terms contain information about the scaling dimensions of the operators [9].

For 2D, much less is known about corrections to the area law. In systems with topological order, the correction is a constant term $\gamma$, called the topological entanglement entropy [10, 11], and whose value is given by the logarithm of the total quantum dimension of the anyonic excitations. The area law of the fractional quantum Hall (FQH) states has been the target of several recent studies [12]–[16], carried out in order to confirm its validity and to compute the value of $\gamma$ predicted in [10, 11]. Reference [16] uses Chern–Simons theory, finding the predicted value of $\gamma$; however the linear behaviour of $S_A$ is not captured, due to the purely topological nature of this theory. There are numerical studies using the Laughlin wavefunction [12]–[14] and exact diagonalization [14, 15], for filling fractions $\nu = 1/3, 1/5$ and the $\nu = 5/2$ Pfaffian state. The approaches of [12]–[15]...
use the orbital basis for the Landau levels. The close relationship of this basis to the spatial partitioning of the blocks leads to an area law of the form

$$ S_A = c \ast P - \gamma + O(1/P), $$

where $P$ is the perimeter of the domain $A$. It is easy to see that for spherical and disc geometries, $P \propto \sqrt{I_A}$, with $I_A$ the number of Landau orbitals in the block $A$, while for torus and cylindrical geometries $P$ is independent of $I_A$ \cite{14, 17}, as will become clear later.

The numerical values of $\gamma$ computed in the spherical geometry \cite{12} and the torus geometry \cite{14, 15} agree, within some precision, with their theoretical values, despite the fact that the systems analysed are not very large.

For non-topological models there are also some results. For those with a Fermi surface, the area law exhibits a log violation \cite{18}–\cite{23} reminiscent of that of the 1D conformal systems, which suggests that higher dimensional bosonization methods may give an appropriate description \cite{24}. Corrections to the area law in non-smooth domains, in $2+1$ dimensions, have been obtained for relativistic free bosons and fermions in \cite{25, 26}, in an interacting CFT using the AdS/CFT conjecture in \cite{27} and in two-dimensional quantum systems in a topological phase in \cite{28}. The 2D quantum critical models of Fradkin and Moore \cite{29} with critical exponent $z = 2$ also have universal subleading contributions (see also \cite{30}–\cite{32}).

The aim of this paper is to calculate the entanglement entropy of the IQHE ground state in arbitrary polygonal domains. We obtain that the entanglement entropy is given by the area (or perimeter) law plus a contribution due to the vertices, $\gamma$, that only depends on the angle of the vertices and the density of the fluid of electrons.

In the following we shall concentrate on the IQHE state with filling fraction $\nu = 1$ defined on a cylindrical geometry, but the calculation of $\gamma$ for higher filling fractions and different geometries are easily generalizable.

Let us consider the Landau model for a particle in a cylinder of size $L_x \times L_y$. The one-particle wavefunction in the lowest Landau level (LLL), in the gauge $A = B(0, x)$, is (in units with the magnetic length $\ell$ equal to 1)

$$ \phi_{k_y}(x, y) = \frac{1}{\pi^{1/4} L_y^{1/2}} e^{i k_y y} e^{-(x - k_y)^2/2}. $$

For the cylinder, the identification of the wavefunction along the $y$ direction implies

$$ k_y = \frac{2\pi n}{L_y}, \quad -\frac{n_0}{2} + 1 \leq n \leq \frac{n_0}{2}. $$

The number of LLLs, $n_0$, is obtained by imposing that the particle lives in the strip $|x| \leq L_x/2$, which yields $n_0 = (L_x L_y/2\pi)$. This value also gives the total number of quantum fluxes through the box. The electron operator can be written as

$$ \psi(x, y) = \sum_{k_y} \phi_{k_y}(x, y) c_{k_y} + \text{higher LLs}, $$

where $c_{k_y}$ is the fermionic annihilation operator of the LLL labelled by $k_y$. The extra term in (4) involves the remaining Landau levels, which are empty for filling fraction $\nu = 1$.

The ground state for $\nu = 1$ is given by

$$ |\Phi_0\rangle = \Pi_{k_y} c_{k_y}^\dagger |0\rangle, $$

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We compute the entropy of entanglement between the domain $D = D_A \cup D_B$ (including two vertices with angle $\alpha$) and its complement, $D_C$, on the cylinder (we assume periodic boundary conditions in the $y$ direction). Varying the parameter $l_x = L \cot(\alpha/2)/2$ we can obtain the correction to the area law, $\gamma$, for $\alpha \in [0, \pi]$.

where $|0\rangle$ is the Fock vacuum. The two-point fermion correlator in this state is

$$C_{r,r'} = \langle \Phi_0 | \psi^\dagger(x,y) \psi(x',y') | \Phi_0 \rangle.$$  

Using (4) and (5) one finds

$$C_{r,r'} = \sum_{k_y} \phi_{k_y}^\ast(x,y) \phi_{k_y}(x',y').$$  

We want to compute the entanglement entropy, $S_D$, of the state $\Phi_0$, in a polygonal domain $D$ embedded in a cylinder of radius $L_y$ such as that shown in figure 1. This entropy is given by the formula $S_D = - \text{tr} \rho_D \log \rho_D$ where $\rho_D = \text{tr}_{D_c} |\Phi_0\rangle \langle \Phi_0|$, and $D_c$ is the complement of $D$ in the cylinder. The computation of $S_D$ is done in two steps [5,33]. First one restricts the correlation matrix $C_{r,r'}$, to the domain $D$, i.e.

$$\tilde{C}_{r,r'} = C_{r,r'}, \quad r, r' \in D.$$  

Next, one diagonalizes $\tilde{C}_{r,r'}$, i.e.

$$\int_D d^2r' \tilde{C}_{r,r'} g(r') = \lambda g(r).$$  

The entropy $S_D$ is obtained by means of

$$S_D = \sum_m H(\lambda_m),$$  

where $H(x) = -x \log x - (1 - x) \log(1 - x)$. We have introduced a parameter $l_x$ that allows us to vary the angle $\alpha$ (see figure 1). If $l_x = 0$, the domain $D(l_x = 0) = D_0$ becomes a half-cylinder. The eigenvalues of the correlator (8) for this case were given in [17]:

$$\lambda_n = \int_{D_0} \phi_n^\ast(x,y) \phi_n(x,y) dx dy = \frac{1}{2} \left( 1 - \text{Erf} \left( \frac{2n\pi}{L_y} \right) \right), \quad n = -\infty, \ldots, \infty.$$  

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In obtaining this formula we assumed that $L_x$ is effectively infinite. Computing now the entanglement entropy (10) using (11), one obtains $S_{D_0} = 0.203\,291* L_y$. Observe that $L_y$ coincides with the perimeter of $D_0$, but this is a general result that holds for any smooth domain $D_{\text{smooth}}$, i.e.

$$S_{D_{\text{smooth}}} = c* P,$$

where $P$ is the perimeter of $D_{\text{smooth}}$ and the constant

$$c = 0.203\,291$$

is independent of the geometry of the system but varies with the number of fully occupied Landau levels [17].

2. Correction to the area law

The aim of this section is to show that the area law for a polygonal domain $D$ is given by

$$S_D = S_{D_{\text{smooth}}} + \gamma,$$

where $S_{D_{\text{smooth}}}$ is the entanglement entropy of a smooth domain $D_{\text{smooth}}$ with the same perimeter as $D$, given by equation (12), and $\gamma$ is a constant term due to the corners of the domain.

To prove equation (14) we start by diagonalizing the correlator (8) in a generic domain $D$. Using (2) and (7), equation (9) can be written as

$$\sum_n \phi^*_n(x, y) \int_D \phi_n(x', y') g(x', y') = \sum_n \phi^*_n(x, y) A_n = \lambda g(x, y),$$

where we have defined

$$A_n = \int_D \int_D \phi_n(x', y') g(x', y').$$

The vanishing eigenvalues $\lambda$ of equation (9) do not contribute to $S_D$, so one can focus on the non-vanishing ones. Using equation (15), the corresponding eigenfunctions $g(x, y)$ can be written as

$$g(x, y) = \frac{1}{\lambda} \sum_m \phi^*_m(x, y) A_m.$$

Finally, substituting (17) in (16) one obtains the following equation for the eigenvalues:

$$\sum_m (F_D)_{nm} A_m = \lambda A_n,$$

where $F_D$ is a matrix with elements

$$(F_D)_{nm} = \int_D \phi_n(x, y) \phi^*_m(x, y).$$

Observe that the continuous eigenvalue equation (9) has been converted into a discrete one (18), where the eigenvalues of the matrix $F_D$ are those of the correlator (8). This fact will allow us to apply numerical methods to find the eigenvalues $\lambda$. 

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Indeed, let us first show the validity of equation (14) for some simple domains such as those depicted in figure 2. The domains $D_n$, with $n = 4, 6, 8, 10$, are four different polygons with $n$ right angles. For each of these domains we first compute the matrix $F_D$ (see (19)), and we diagonalize it for different values of $P$. The corresponding entropies $S_{D_n}$ can be fitted to the following formulae:

(a) $S_{D_4} = -0.341\,315 + 0.203\,291 \times P_{D_4}$,
(b) $S_{D_6} = -0.512\,909 + 0.203\,295 \times P_{D_6}$,
(c) $S_{D_8} = -0.683\,902 + 0.203\,291 \times P_{D_8}$,
(d) $S_{D_{10}} = -0.854\,12 + 0.203\,289 \times P_{D_{10}}$.

Notice that the constant multiplying the perimeters $P_{D_n}$ agrees, to great accuracy, with the value $c$ in (12) for smooth domains. From these examples one can extract the value of $\gamma$ for right angles:

$$\gamma\left(\frac{\pi}{2}\right) = a\left(\frac{\pi}{2}\right) \times \text{(number of } \pi/2 \text{ vertices)},$$

with $a(\pi/2) = -0.0855$. On the basis of (14), (20) we can propose a general expression for the entanglement entropy of an arbitrary domain $D$ with $n$ vertices parametrized by angles $\alpha_i$ ($i = 1, \ldots, n$):

$$S_D = S_{D_{\text{smooth}}} + \gamma, \quad \text{with } \gamma(\{\alpha_i\}) = \sum_i a(\alpha_i) \times n_i, \quad \alpha_i \in [0, 2\pi],$$

with $n_i$ the number of vertices with angle $\alpha_i$ in the boundary of $D$. Each of these angles varies between 0 and $2\pi$, but as we show below, the parameter $\gamma$ is invariant under the
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Figure 3. Plot of $a(\alpha)$ versus $\alpha$ for $\alpha \in [0, \pi]$: the dotted line corresponds to the values of $a(\alpha)$ obtained by diagonalization of the $F_D$ matrix (18) and the continuous line shows fit (23). There is a perfect matching between the two curves.

The calculation of $a(\alpha)$ can be done considering the domain of figure 1. For different values of $\alpha \in [0, \pi]$ (or $l_x$ in figure 1), we diagonalize the $F_D$ matrix, calculate $S_D$ and finally using (21), we obtain the value of $a(\alpha)$ by means of the equation

$$a(\alpha) = \frac{1}{2} (S_D - 0.20329 \star P_D),$$

with $P_D$ the perimeter of $D$. The factor $\frac{1}{2}$ in (22) arises from the fact that the domain of figure 1 contains two vertices with angle $\alpha$. The numerical determination of $a(\alpha)$ is given in figure 3. This curve can be fitted with the following expansion in the variable $2l_x/L_y = |\cot(\alpha/2)|$ (see figure 1):

$$a(\alpha) = 0.00723706|\cot(\alpha/2)| - 0.155709|\cot(\alpha/2)|^2 + 0.0876488|\cot(\alpha/2)|^3 - 0.0306014|\cot(\alpha/2)|^4 + 0.00698332|\cot(\alpha/2)|^5 - 0.00105786|\cot(\alpha/2)|^6 + 0.000105322|\cot(\alpha/2)|^7 - 6.616 \cdot 10^{-6}|\cot(\alpha/2)|^8 + 2.375 \cdot 10^{-7}|\cot(\alpha/2)|^9 - 3.7112 \cdot 10^{-9}|\cot(\alpha/2)|^{10}. \quad (23)$$

Note that (23) satisfies the relation $a(\alpha) = a(2\pi - \alpha)$ that together with (21) implies that $S_D = S_{D_c}$ ($D_c$ is the complement of $D$) as explained above. Hence, we can restrict ourselves to the interval $\alpha \in [0, \pi]$ where $\cot(\alpha/2) \geq 0$, so we can drop the absolute values in (23).

In order to assess the accuracy of the fit (23) we can compare the entanglement entropies obtained by diagonalization of the $F_D$ matrix (18), for the complicated domains of figure 4, with the theoretical formula (21) using the extrapolation (23). In the case of diagonalization of the $F_D$ matrix we obtain

$$S_a = 38.04348652 \quad S_b = 50.02330873 \quad \text{and} \quad S_c = 48.92750022 \quad (24)$$

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and using (21) and (23) we arrive at

\[
\begin{align*}
S_a &= 0.20329 \times P + 4 \times a(\pi/2) + 6 \times a(\arctan(9/15)) = 38.047, \\
S_b &= 0.20329 \times P + 2 \times a(\pi/2) + 4 \times a(\arctan(9/15)) + 5 \times a(2 \times \arctan(9/15)) = 50.0253 \quad \text{and} \\
S_c &= 0.20329 \times P + 2 \times a(\pi/2) + 2 \times a(\arctan(9/15)) + 2 \times a(\arctan(7/15) + \arctan(9/15)) + 5 \times a(2 \times \arctan(7/15)) = 48.927.
\end{align*}
\]
The comparison of (24) and (25) shows clearly the validity of (21) and that of the extrapolation (23). The previous results have been obtained for a cylindrical geometry; however it can be shown that $a(\alpha)$ is the same function for the sphere and the plane and thus $\gamma$ is also independent of the geometry.

3. Analytical calculation of $a(\alpha)$

In this section we shall give an analytical justification of equation (21). A crucial point in our analysis is that the eigenvalue $\lambda_n(l_x, L_y)$ of the $F_D$ matrix (18) corresponds to a deformation of the eigenvalue (11) for the case of the smooth domain $D_0$ ($l_x = 0$ in figure 1). This result can be shown by diagonalization of $F_D$ for many different values of $l_x$ and $L_y$. This deformation is parametrized by a function $\kappa(l_x, L_y)$ and it is given by

$$\lambda_n(l_x, L_y) = \frac{1}{2} \left( 1 - \text{Erf} \left[ \frac{2n\pi}{L_y} (1 - \kappa(l_x, L_y)) \right] \right).$$  

(26)

In figure 5 we show, for the particular case $l_x = 21$ and $L_y = 30$ ($\alpha = 71.075^\circ$ in figure 5), the perfect matching between the eigenvalues obtained by diagonalization of the $F_D$ matrix and those given by (26).

To calculate the function $\kappa(l_x, L_y)$ we can use equations (10), (21), (26) and the change of variables $x = (2\pi n/L_y)(1 - \kappa(l_x, L_y))$, obtaining

$$S_D = cP_D + 2a(\alpha) = \int_{-\infty}^{\infty} H[\lambda_n(l_x, L_y)] \, dn = \frac{L_y c}{1 - \kappa(l_x, L_y)},$$  

(27)

with $P_D = 2\sqrt{(L_y/2)^2 + l_x^2}$ the perimeter of $D$ and $c$ given in (13). From equation (27), the deformation $\kappa(l_x, L_y)$ is expressed as

$$\kappa(l_x, L_y) = 1 - \frac{L_y c}{cP_D + 2a(\alpha)},$$  

(28)

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with $a(\alpha)$ given by the extrapolation (23) or by the theoretical value (33) to be obtained below.

Equation (26) is rather interesting since it shows that the whole spectrum of the matrix $F_D$ is given in terms, basically, of the function $\kappa(l_x, L_y)$. Hence, we expect that overall quantities like $\text{Tr } F_D^p = \sum_n \lambda_n^p(l_x, L_y)$ will also depend on the latter function. If, for some $p$, we are able to compute this trace, then we will know $\kappa(l_x, L_y)$ as well $a(\alpha)$ by means of equation (28). The simplest choice is $p = 1$, but in this case the trace of $F_D$ does not depend on $\kappa(l_x, L_y)$. This follows from the relations

$$\text{Tr } F_D = \sum_n \lambda_n(l_x, L_y) = \int_D C(x, y) \, dx \, dy \sim \rho_0 A,$$

where $C(x, y)$ is the correlator (8), $\rho_0$ is the density of electrons and $A \gg 1$ is the area of the domain $D$.

The next choice is $p = 2$ for which $\text{Tr } F_D^2$ does depend on $\kappa(l_x, L_y)$. Indeed, using (26) and taking the limit $n_0 \gg 1$ we obtain

$$\text{Tr } F_D^2 = \sum_{n=-n_0}^{n_0} \lambda_n^2(l_x, L_y) = n_0 + \frac{L_y}{2\sqrt{2} \pi^{3/2} (-1 + \kappa(l_x, L_y))}.$$  \hspace{1cm} (30)

Then from (30) and (27), the entropy $S_D$ and $\text{Tr } F_D^2$ are related by

$$S_D = -2\sqrt{2} \pi^{3/2} c(\text{Tr } F_D^2 - n_0).$$  \hspace{1cm} (31)

Therefore if we found an alternative formula for $\text{Tr } F_D^2$, we would obtain a theoretical formula for $S_D$ and justify in this way equation (21). This is done in the appendix for the domain $D$ of figure 1 with the result

$$\text{Tr } F_D^2 = -\frac{P_D}{2\sqrt{2} \pi^{3/2}} + n_0 + \frac{4 + (\pi - \alpha)(-1 + 3 \cos(\alpha)) \csc(\alpha)}{4\pi^2}$$  \hspace{1cm} (32)

where $P_D$ is the perimeter of the domain $D$. Finally, from (31) and (32) and restoring the magnetic length $\ell$ we obtain

$$S_D = \frac{c}{\ell} P_D + \gamma(\alpha) \quad \text{with}$$

$$\gamma(\alpha) = 2a(\alpha) = -\frac{c}{\sqrt{2\pi}} (4 + (\pi - \alpha)(-1 + 3 \cos(\alpha)) \times \csc(\alpha)).  \hspace{1cm} (33)$$

Observe that $a(\alpha)$ is independent of $\ell$ and only depends on the density of the fluid of electrons, as can be shown by including higher Landau levels in the previous analysis. Figure 6 shows the overlap between the extrapolation $a(\alpha)$ in (23) obtained by diagonalization of the $F_D$ matrix and the theoretical expression (33). The small difference between the curves is due to the approximation used in the calculation of $\text{Tr } F_D^2$ (see equation (A.8) in the appendix).

In conclusion, in this section we have proved the validity of our proposal of equation (21) for the entanglement entropy in a non-smooth domain.
Figure 6. Plot of $a(\alpha)$ versus $\alpha$. The continuous curve corresponds to the extrapolation of $a(\alpha)$, equation (23), obtained by exact diagonalization of the $F_D$ matrix. The dashed curve corresponds to the theoretical approximation for $a(\alpha)$ in equation (33).

4. The entanglement spectrum

The reduced density matrix $D$ of a domain contains of course much more information than just the entropy $S_D$. In the context of the QHE this information is related to the physical degrees of freedom of the edge excitations [34]–[37]. This is a rather surprising conjecture since, after all, the edge of $D$ is an arbitrary curve within the whole system. The entanglement Hamiltonian, $H_D$, defined as $\rho_D = e^{-H_D}$, is then expected to be intimately related to the Hamiltonian describing the excitations of a real edge. In the QHE the edge excitations are described by a chiral CFT, which also describes the excitations of the bulk. For the IQHE this result follows from the entanglement spectrum that is given by the eigenvalues of the operator [38]

$$Q_D = 1/2 - C_D,$$

with $C_D$ the two-point correlator defined in (8). In the smooth domain $D_0$ (obtained considering $l_x = 0$ in figure 1), the eigenvalues of $C_{D_0}$ are given by equation (11), so the entanglement spectrum will be

$$q_n = \frac{1}{2} \text{Erf} \left( \frac{2\pi n}{L_y} \right), \quad n = -\infty, \ldots, \infty,$$

with $q_n$ the eigenvalues of $Q_{D_0}$. In the limit $L_y \gg 1$, this spectrum becomes $q_n = 2\sqrt{\pi} n / L_y$, which is that of a free chiral boson moving on a circle of length $L_y$ with a velocity $v = 1/\sqrt{\pi}$. If we now deform the domain as in figure 2, the entanglement spectrum in the boundary of $D$ can be obtained from equation (26):

$$q'_n = \frac{1}{2} \text{Erf} \left( \frac{2\pi n (1 - \kappa(l_x, L_y))}{L_y} \right), \quad n = -\infty, \ldots, \infty.$$

In the limit $L_y \gg 1$, equation (28) implies $P_D = L_y / (1 - \kappa(l_x, L_y))$, and the spectrum (36) becomes $q'_n = 2\sqrt{\pi} n / P_D$, which is that of a free chiral boson moving on a circle of length $P_D$ with a velocity $v = 1/\sqrt{\pi}$. That circle has the same length as the boundary of $P_D$. doi:10.1088/1742-5468/2010/12/P12033
The latter boundary has two singular points on it, i.e. two vertices with angles $\alpha$ and $2\pi - \alpha$, but they have subleading effects on both the entropy $S_D$ and the entanglement spectrum. This phenomenon is analogous to what happens for real systems where the electrons surround the hills and valleys of the potential travelling all the way through the sample.

In summary, we have computed in this paper the subleading term for the area law for IQHE states on polygonal domains, which is given by the expression

$$\gamma = \sum_i a(\alpha_i) \cdot n_i$$

(37)

where $n_i$ is the number of vertices in the domain with angle $\alpha_i$ and $a(\alpha_i)$ is a function that only depends on the angles of the vertices and the density of the fluid. For the IQHE with filling fraction $\nu = 1$, we have found numerical and analytical expressions of the function $a(\alpha)$, given by equations (23) and (33), which agree rather well. The fact that the correction (37) is a constant has its origin in the gapped character of the IQHE. For gapless 2 + 1 systems one may expect a size dependent subleading term. Indeed, this has been confirmed for relativistic free bosons and fermions in [25, 26] and in an interacting CFT using the AdS/CFT conjecture in [27]. Also, the critical models with dynamical exponent $z = 2$ of Fradkin and Moore [29] exhibit a logarithmic subleading correction of the form

$$\beta = f(\alpha) \cdot \log(L)$$

(38)

with $L$ the perimeter of the domain and $\alpha$ the angle of the vertex. The function $f(\alpha)$ has a similar behaviour to $a(\alpha)$, in the sense that both curves satisfy $f(\alpha), a(\alpha) \leq 0$ and $(\partial f(\alpha)/\partial \alpha), (\partial a(\alpha)/\partial \alpha) \geq 0$. These properties are a consequence of the strong subadditivity relation satisfied by the entanglement entropy [27].

Finally, in section 4 we have obtained the entanglement spectrum of non-smooth domains which corresponds to a chiral free boson moving on the boundary, in agreement with the conjecture of [34].

As a final remark we want to mention that it is natural to expect corrections similar to (37) in other 2D gapped systems and this will be studied in a future work.

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Appendix

In this appendix we calculate the trace (30) analytically. Let us first write it as

$$\text{Tr } F_{D_A \cup D_B}^2 = \text{Tr } F_{D_A}^2 + \text{Tr } F_{D_B}^2 + 2 \text{Tr } (F_{D_A} F_{D_B}),$$

(A.1)
with $D_A$ and $D_B$ as given in figure 1. From the definition of $F_D$ (see equation (18)), the first term in (A.1) reads

$$\text{Tr } F_{D_A}^2 = \int \int_{D_A} d^2r' d^2r \sum_{m,n=-n_0}^{n_0} \phi_m(x', y') \phi_n(x', y') \phi_n(x, y) \phi_m(x, y)$$

$$= \frac{1}{4\pi^2} \int \int_{D_A} e^{-(1/2)(x-x')^2 - (1/2)(y-y')^2}$$

$$= \int_{(l_x/2)}^{(l_x/2)} dx' dx' \left( e^{-(1/2)(x-x')^2 - (1/8)a^2(l_x+x+x')^2} \frac{2\pi^2}{a^2} \right)$$

$$+ \frac{a e^{-(1/2)(x-x')^2}(x'-x) \text{Erf}[a(x-x')/2\sqrt{2}]}{4\sqrt{2}\pi^{3/2}} + \frac{a e^{-(1/2)(x-x')^2}}{4\sqrt{2}\pi^{3/2}} \times (l_x + x + x') \text{Erf} \left[ \frac{a(l_x + x + x')/2\sqrt{2}}{2\sqrt{2}} \right] = A + B + C + D, \quad (A.2)$$

where $a = L_y/l_x$. The integrals $A$ and $B$ in (A.2) can be easily found:

$$A = \frac{\pi + 2 \arctan [1/a - a/4]}{4\pi^2 a}, \quad B = \frac{4}{(4 + a^2)\pi^2} - \frac{\sqrt{2}l_x}{4 + a^2\pi^{3/2}}. \quad (A.3)$$

To compute the integral $C$ we first integrate the variables $y, y'$ and $x$, obtaining that

$$C = \int_{-(l_x/2)}^{(l_x/2)} dx' \left( \frac{a}{2\sqrt{2}\pi^{3/2}} e^{-(1/8)(l_x-2x')^2} \text{Erf} \left[ \frac{a(l_x - 2x')}{4\sqrt{2}} \right] \right)$$

$$- \frac{a^2}{2\pi^{3/2} \sqrt{(2 + a^2)}} \text{Erf} \left[ \frac{\sqrt{4 + a^2(l_x - 2x')}}{4\sqrt{2}} \right] = C_1 + C_2. \quad (A.4)$$

The integral $C_1$ only depends on the ratio $a = L_y/l_x$. Therefore if we vary $l_x$, maintaining the value of $a$ constant (by adjusting $L_y$), the integral $C_1$ remains the same. Taking the limit $l_x \rightarrow \infty$ in $C_1$ and making the change of variables $u = l_x - 2x'$, one obtains an integral between 0 and $\infty$ whose value is

$$C_1 = \frac{a}{2\pi^2} \arctan \left[ \frac{a}{2} \right]. \quad (A.5)$$

The integral $C_2$ is easily computed:

$$C_2 = \frac{a^2}{(4 + a^2)\pi^2} - \frac{a^2 l_x}{2\sqrt{2}(4 + a^2)^{3/2}}. \quad (A.6)$$

Collecting the previous expressions one gets

$$C = \frac{a}{2\pi^2} \arctan \left[ \frac{a}{2} \right] + \frac{a^2}{(4 + a^2)\pi^2} - \frac{a^2 l_x}{2\sqrt{2}(4 + a^2)^{3/2}}. \quad (A.7)$$

The integral $D$ in (A.2):

$$D = \int_{-(l_x/2)}^{(l_x/2)} \frac{a e^{-(1/2)(x-x')^2}(l_x + x + x') \text{Erf}[a(l_x + x + x')/2\sqrt{2}]}{4\sqrt{2}\pi^{3/2}} \quad (A.8)$$

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cannot be solved analytically. However, one can obtain a very good approximation replacing in (A.8) \((l_x + x + x')\text{Erf}[a(l_x + x + x')/2\sqrt{2}]\) by \((l_x + x + x')\), which upon integration yields

\[
D = -\frac{a l_x}{2\sqrt{2\pi}3/2} + \frac{a l_x^2}{4\pi}.
\]

(A.9)

Finally, from (A.3), (A.7) and (A.9) we arrive at

\[
\text{Tr} F_{D_A}^2 = \frac{\pi + 2 \text{arctan}[1/a - a/4]}{4\pi^2 a} + \frac{4}{(4 + a^2)\pi^2} - \frac{\sqrt{2l_x}}{\sqrt{4 + a^2\pi^3/2}}
\]

\[
+ \frac{a}{2\pi^2} \text{arctan}\left[\frac{a}{2}\right] + \frac{a^2}{(4 + a^2)\pi^2}
\]

\[
- \frac{a^2 l_x}{2\sqrt{2(4 + a^2)\pi^{3/2}}} - \frac{a l_x}{2\sqrt{2\pi^3/2}} + \frac{a l_x^2}{4\pi}.
\]

(A.10)

To complete the calculation one needs the quantities \(\text{Tr} F_{D_B}^2\) and \(2\text{Tr}(F_{D_A} F_{D_B})\) in (A.1).

Their values are easy to obtain and they read

\[
\text{Tr} F_{D_B}^2 = \sum_n (F_{D_B})_{nn} (F_{D_B})_{nn} = n_0 - \frac{L}{2\sqrt{2\pi}3/2} - \frac{Ll_x}{4\pi},
\]

(A.11)

with \(n_0\) defined in (3). In (A.11) we have used the fact that in the domain \(D_B\), the matrix \(F\) is diagonal.

Finally, from (A.10) and (A.11) we obtain that

\[
\text{Tr} F_{D_A \cup D_B}^2 = -\frac{P_{D_A \cup D_B}}{2\sqrt{2\pi}3/2} + n_0 + \frac{4}{(4 + a^2)\pi^2} + \frac{a^2}{(4 + a^2)\pi^2}
\]

\[
+ \frac{1}{4a\pi} - \frac{a}{4\pi} + \frac{\text{arctan}\left[1/a - a/4\right]}{2a\pi^2} + \frac{a \text{arctan}\left[a/2\right]}{2\pi^2},
\]

(A.12)

with \(P_{D_A \cup D_B}\) the perimeter of the domain \(D\) in figure 1. Observe from figure 1 that the ratio \(a = (L_y/L_x)\) is related to the angle, \(\alpha\), of the vertex by \(a = 2\tan(\alpha/2)\). Therefore we can write equation (A.12) in terms of the angle \(\alpha\), obtaining that

\[
\text{Tr} F_{D_A \cup D_B}^2 = -\frac{P_{D_A \cup D_B}}{2\sqrt{2\pi}3/2} + n_0 + \frac{4 + (\pi - \alpha)(-1 + 3\cos(\alpha)) \csc(\alpha)}{4\pi^2}.
\]

(A.13)

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