Review

Emergent electromagnetism in condensed matter

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Abstract: Electrons in solids constitute quantum many-body systems showing a variety of phenomena. It often happens that the eigen states of the Hamiltonian are classified into subgroups separated by energy gaps. Band structures in solids and spin polarization in Mott insulators are two representative examples. The subspace spanned by these wavefunctions belonging to each of this subgroup can be regarded as a manifold in Hilbert space, and concepts concerning differential geometry become relevant. Connection and curvature are two key quantities, which correspond to the vector potential and field strength of electromagnetism, respectively. Therefore, one can construct an effective electromagnetic field from the structure of the Hilbert space, which is called an "emergent electromagnetic field". In this article, we review the physics related to this emergent electromagnetic field in solids, including the gauge theory of strongly correlated electrons, various Hall effects, multiferroics, topological matter, magnetic texture such as skyrmions, and the shift current in noncentrosymmetric materials.

Keywords: strong correlation, Berry phase, topology, Hall effect, skyrmion

I. Introduction

Quantum mechanics has a highly mathematical formulation based on an abstract Hilbert space consisting of the functions. It often rejects our intuitive picture or understanding, which mainly comes from our experience in the macroscopic world governed by the classical physics. However, one can develop an intuition based on an analogy to the vector space in three dimensions. Especially, the curved surface is an analogue of the subspace with a nontrivial geometrical structure described by the differential geometry. In the language of the gauge theory, the connection corresponds to the vector potential, while the curvature corresponds to the field strength. Therefore, one can construct an effective gauge field for the Hilbert space of the quantum system, which we call the "emergent electromagnetic field".

In sharp contrast to flat space, a curved manifold often has a nontrivial topology. For example, the Gauss-Bonnet theorem relates the number of holes, i.e., genus, to the integral of the Gauss curvature over the closed surface. This is the simplest example of the topological index, which remains unchanged against the continuous deformation of the surface. Topology gives the classification of the manifolds identifying those connected by the continuous deformation, and the topological properties of the manifolds, i.e., subspace in Hilbert space. Therefore, the local geometric structure and global topology are important subjects in quantum physics, especially in condensed-matter physics.

Here we review some of the topics that are related to this concept with concrete predictions concerning the physical properties of solids, including the strong correlation effects in transition metal oxides, the transport properties of magnets, spin transport in semiconductors, topological matter, and photovoltaic effects in solids.

II. Berry phase

The Berry phase is a geometric phase associated with an adiabatic change of the quantum system described by the Hamiltonian, $H(X(t))$, with...
$X(t)$ being a time($t$)-dependent set of parameters, where transitions between the energy eigenstates at each $t$ are forbidden. Therefore, the wavefunction is confined within the Hilbert space spanned by the set of eigenfunctions, $\psi_n(X(t))$, and the situation described in the introduction is realized. The Berry connection is the overlap integral between the two neighboring wavefunctions in this confined space, i.e.,

$$\langle \psi_{nX} | \psi_{nX+\Delta X} \rangle = e^{i\Delta X a_{nX}}$$

with

$$a_{nX}(X) = -i(\psi_{nX} | \partial_{X_X} | \psi_{nX})$$

This connection acts as a vector potential in $X$-space, and it is natural to define the “electromagnetic field” from the vector potential as

$$F_{\mu\nu}(X) = \partial_{X_X} a_{\mu\nu} - \partial_{X_\nu} a_{\mu\nu},$$

which corresponds to the Berry curvature. Physically, the parameter set $X$ can be generalized coordinates, such as the real space position, momentum, and other parameters, and the Berry phase can be regarded as being the generalized Aharonov-Bohm phase.

As an example, let us consider the $2 \times 2$ Hamiltonian given by

$$H(X) = X \cdot \sigma,$$

where $X = (X_1, X_2, X_3)$ are the three parameters, while $\sigma = (\sigma^1, \sigma^2, \sigma^3) = (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices. The eigenvalue problem, $H(X) |\psi(X)\rangle = E(X) |\psi(X)\rangle$, can be easily solved, and the energy eigenvalues are given by $E_b(X) = \pm |X|$. The corresponding Berry curvature $b_\pm(X)$ is given by

$$b_\pm(X) = \pm \frac{X}{2|X|},$$

for the upper (+) and lower (−) energy eigen states, respectively. This Berry curvature is that of the magnetic (anti)monopole with a quantized total flux of $\pm 2\pi$, and has a singularity at $X = 0$, where a degeneracy of the eigenstates occurs. Therefore, the degeneracy plays an important role in the physics related to the Berry phase. This simple example is relevant to the quantization of spin in terms of the path integral formalism; also, the Weyl fermion in solids correspond to this Hamiltonian with $X$ being the crystal momentum, $k$, as described in section V.

III. Gauge theory of strongly correlated electrons

One of the examples where the emergent magnetic field plays an essential role is strongly correlated electronic systems. For example, the transition metal oxides are the source of rich physics in strongly correlated electrons in 3d orbitals.5) The strong Coulomb interaction in 3d orbitals produces the spin moment at each atom, which fluctuates or orders to result in a variety of magnetic properties. Especially of interest is cuprates, where high-temperature superconductivity has been discovered.6) In the parent compounds, such as La$_2$CuO$_4$, a strong on-site Coulomb interaction, $U$, with the $d^9$ configuration induces a spin $S = 1/2$ moment, which forms an antiferromagnetically ordered state. With carrier doping, the antiferromagnetic order is destroyed and high-temperature superconductivity appears. Naively, the magnetism and superconductivity compete with each other, since the conventional cooper pairing is the on-site spin singlet, which is suppressed by $U$. However, unconventional pairing, such as $d$-wave pairing, has an amplitude aside from the on-site, although it is a spin singlet. Actually, it is believed that the cooper pairing of cuprate superconductors is $d_{x^2-y^2}$. Therefore, the interplay between the magnetism and superconductivity became the central issue in the physics of electron correlation.

The essence of the strong correlation can be taken into account by excluding electron double occupancy at each orbital, which restricts the Hilbert space. This constraint can be expressed by the gauge field.7–9) Actually, the 2D Heisenberg model can be mapped to a lattice gauge theory in the strong coupling limit, where the Hilbert space is confined in the single occupancy at each atomic site.10) When the carriers are doped, there appear the charge degrees of freedom, and the constraint becomes an inequality. When the hole is doped, there are three possible states at each orbital, i.e., (i) vacancy, (ii) a single electron with spin up, and (iii) a single electron with spin down. This means that the number of electrons at each site is less than or equal to one. Therefore, we need some theoretical tool to transform the inequality to the equality, which is called the slave particle method. In the slave boson method, the electron creation operator, $c^\dagger_{\sigma i}$, is expressed as ref. 6

$$c^\dagger_{\sigma i} = f^\dagger_{\sigma i} b_i,$$

where $i$ specifies the atom/site, while $\sigma = \uparrow, \downarrow$ the spin; $f^\dagger_{\sigma i}$ is the fermion creation operator, while $b_i$ is the boson annihilation operator. They satisfy the constraint

$$b_i^\dagger b_i + f_{\sigma i}^\dagger f_{\sigma i} + f_{\sigma i}^\dagger f_{\sigma i} = 1,$$
each term of which represents one of the three states (i), (ii), and (iii), respectively. Accordingly, \(b_i^\dagger, b_i\) represent the vacancy and is called holon operators, while \(f_i^\dagger, f_i\) to the single electron occupancy called spinon operators, respectively. The constraint, Eq. [7], is regarded as being the local conservation of “charge”, i.e., corresponding to the local gauge transformation as

\[
f_{i\sigma} \rightarrow e^{i\varphi_i} f_{i\sigma},
\]

\[
b_i \rightarrow e^{i\varphi_i} b_i.
\]

In order to make the theory gauge invariant, one needs to introduce the gauge field, which appears naturally once one goes beyond the mean field theory to take into account the Gaussian fluctuation.\(^7\)-\(^9\)

More explicitly, \(a_0\) is the Lagrange multiplier to impose the constraint Eq. [7], while \(a\) comes from the phase of the hopping order parameter \(\chi_{ij} = \sum_{\sigma} \langle f_{i\sigma} f_{j\sigma} \rangle\). The Lagrangian for the holon-spinon-gauge field coupled system is

\[
L = \int df_i^\dagger \left[ \frac{(-i \hbar \nabla - \mathbf{a})^2}{2m_f} - a_0 - \mu_f \right] f_i + b \left[ \frac{(-i \hbar \nabla - \mathbf{a} + e \mathbf{A})^2}{2m_b} - a_0 - \mu_b \right] b,
\]

where there is no Lagrangian for the gauge field to start with. This is because integration over the gauge field, \(a_0\), gives the constraint Eq. [7]. However, \(a\) is interacting with the fermions and bosons, which leads to the effective Lagrangian, i.e., it obtains its dynamics.\(^8\)\(^9\) Namely, its propagator is given by

\[
D_{00}(q, \omega) = N(0),
\]

\[
D_{ab}(q, \omega) = \left( \delta_{ab} - \frac{q_i q_j}{q^2} \right) \frac{1}{\sigma_j |\omega| + \chi q^2},
\]

where \(\sigma_j\) is the sum of the conductivities of fermions and bosons, while \(\chi\) is that of the diamagnetic susceptibilities.

The physical meaning of this gauge field is now discussed. Let us introduce the Stratonovich-Hubbard transformation of the Hubbard interaction in the path-integral formalism as follows. First let us rewrite the Hubbard interaction as

\[
U n_{i\uparrow} n_{i\downarrow} = -\frac{U}{2} (n_{i\uparrow} - n_{i\downarrow})^2 + \frac{U}{2} (n_{i\uparrow} + n_{i\downarrow})
\]

\[
= -2U (S_i^z)^2 + \frac{U}{2} (n_{i\uparrow} + n_{i\downarrow})
\]

\[
= -\frac{2U}{3} (\mathbf{S}_i)^2 + \frac{U}{2} (n_{i\uparrow} + n_{i\downarrow}),
\]

Fig. 1. Scalar spin chirality, i.e., the solid angle subtended by the spins, which acts as an emergent magnetic field for the electron coupled to spins. The effective flux \(\Phi\) penetrating the triangle is given by the product \(\mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k)\).

where \(\mathbf{S}_i = \frac{1}{2} \sum_{\alpha\beta} \epsilon_{\alpha\beta}^i \mathbf{c}_{\alpha\beta} \mathbf{c}_{\alpha\beta} \) is the spin operator of the electron at site \(i\). Here, we used the identity \(n_{i\sigma}^2 = n_{i\sigma}\), since the eigenvalue of \(n_{i\sigma}\) is 0 or 1. Next, we introduce the spin fluctuation field, \(\varphi_i\), and using the identity

\[
(\varphi_i)^2 + 2\varphi_i \cdot \mathbf{S}_i = (\varphi_i - \mathbf{S}_i)^2 - (\mathbf{S}_i)^2,
\]

one can express the on-site Hubbard repulsive interaction by coupling of the electron spin to the fluctuating spin field, \(\varphi_i\), with the Hund’s coupling of the order of the Coulomb interaction, \(U\). This picture is similar to the double-exchange model, where the electrons are classified into two, i.e., the localized electrons in some orbitals, such as \(t_{2g}\) d-orbitals, and the itinerant ones, such as \(e_g\) electrons. The former acts as the localized spins coupled to the latter with Hund’s coupling. With this picture in mind, suppose there are three localized spins (\(\mathbf{S}_1, \mathbf{S}_2\) and \(\mathbf{S}_3\)) that are coupled to the conduction electron, as shown in Fig. 1. The conduction electron spin strongly coupled to this spin structure is enforced to be parallel to the localized spin at each site due to the Hund’s coupling.

Therefore, the effective transfer integral, \(t_{ij}\), between the two sites \(i\) and \(j\) is given by

\[
t_{ij} = t(\chi_i |\chi_j|),
\]

where \(|\chi_i\rangle\) and \(|\chi_j\rangle\) are the spinor spin wavefunction at sites \(i\) and \(j\), respectively. This quantity is a complex number, whose phase corresponding to the spatial components \(\alpha\) of the gauge field, which is determined by the configuration of the localized spins. When the conduction electron is moving around the triangle 123, it obtains a phase factor that is 1/2 of the solid angle subtended by the three localized spins, which is called scalar spin chirality.\(^9\)

Therefore, the gauge flux, \(b = \partial_\alpha a_\alpha - \partial_\alpha a_\alpha\), is the emergent magnetic field, whose propagator is given by

\[
\langle b(q, \omega) b(-q, -\omega) \rangle = \frac{q^2}{\sigma \omega^2 + \chi q^2}
\]

from Eq. [12], which
has a singular structure in the low-energy/long-wavelength limit, and causes the anomalous behavior of the system.

The picture obtained above is that the system is described by the spinful fermions and spinless bosons, which are coupled through the strongly fluctuating gauge field, i.e., the emergent electromagnetic field. By integrating over this gauge field, one can obtain the physical response functions from those of fermions and bosons. For example, the resistivity, $\rho$, of the system is given by the sum of those of fermions, $\rho_F$, and bosons, $\rho_B$. Since the boson density is that of the doped holes, $x$, while that of fermions is $1 - x$. Therefore, $\rho_B$ is much larger than $\rho_F$. The low-energy bosons are scattered effectively by the gauge field, which results in $\rho_B \propto T$, where $T$ is the temperature. Other physical observables are obtained by combining the contributions from bosons and fermions, which are different for different quantities. This resolved the dichotomy between the two pictures, i.e., the small number $x$ of hole carriers and the large Fermi surface with Luttinger volume, $1 - x$.

IV. Emergent electromagnetic field due to spin textures

A. Anomalous Hall effect due to spin chirality. The emergent electromagnetic field discussed in the previous section is fluctuating both quantum mechanically and thermally. The strong coupling nature makes it very difficult to treat the fluctuation on a solid basis. One way is to introduce a fictitious parameter, such as $N$ (number of fermion or boson species), and expand with respect to $1/N$, assuming a large $N$. In this limit, the perturbative treatment of the gauge field fluctuation is justified, while the applicability to the realistic case of $N = 2$ is not well-founded. Nonperturbative effects, such as confinement, remains an important issue to pursue, which is related to fractionalization of the electrons. On the other hand, there are several situations where the emergent magnetic field is static without any fluctuation. In this case, the problem is reduced to that of a single-particle, and one can solve the problem exactly.

The non-collinear spin structures offer an ideal arena to study this possibility. Especially, the non-coplanar spin structure with the solid angle subtended by the spins produces the static emergent magnetic field, leading to the Hall effect. This possibility was tested in the non-coplanar spin structure in pyrochlore ferromagnet NdMo$_2$O$_7$, where the strong single-spin anisotropy enforces the directions of the rare-earth (Nd) moments to point outward from or inward to the center of the tetrahedron, which are coupled to the conduction electrons of Mo atoms. Therefore, the conduction electrons are subject to the emergent magnetic field, and show the anomalous Hall effect. However, note that the periodicity of the electronic state does not change due to the magnetic ordering, and hence the band structure is well defined with the original first-Brillouin zone in this material. Therefore, it is more appropriate to consider the Berry phase of the Bloch wavefunctions in momentum space rather than the emergent magnetic field in real space.

B. Skyrmion. The concept of the emergent electromagnetic field in real space applies well to the spin texture, called skyrmion, as schematically shown in Fig. 2. It is realized in a noncentrosymmetric ferromagnet, as observed experimentally by neutron-scattering experiments and Lorentz electron transmission microscopy. Theoretically, skyrmions are described by the Hamiltonian

$$H_S = \int d^3x \left[ \frac{J}{2a} (\nabla \mathbf{n})^2 + \frac{D}{a^2} \mathbf{n} \cdot [\nabla \times \mathbf{n}] - \frac{\mu}{a^3} \mathbf{H} \cdot \mathbf{n} \right].$$

[16]

where $a$ is the lattice constant, $J$ is the ferromagnetic exchange coupling, and $D$ is the Dzyaloshinskii-Moriya (DM) spin-orbit interaction. The DM interaction introduces a twist of the spins, and stabilizes the skyrmion crystal (SkX) configuration, $\mathbf{n}(\mathbf{x})$, in some interval of the magnetic field, $\mathbf{H} = H_z$. Here, the size $\lambda$ of the skyrmion is determined by the

Fig. 2. Schematic view of a skyrmion structure. It is a swirling vortex-like spin texture with the spin at the core pointing downward. The spins point in all the directions, which wrap the unit sphere. This is described by the topological index, $Q$, defined in Eq. [19] in the text.
ratio of the DM interaction, $D$, and the ferromagnetic exchange interaction, $J$. Namely, $\lambda \equiv a(J/D)$ with $a$ being the lattice constant, which is on the order of $20–100$ nm, and is typically larger than the mean free path of the conduction electrons. There are two consequences of this relatively large size: (i) the emergent magnetic field in real space is more relevant to this case, since its spatial variation is small within the mean free path and the lattice constant. (ii) The large number of spins are involved in a skyrmion even in the 2D case, \textit{i.e.}, atomic layer, which justifies the continuum approximation.

Therefore, the motion of the skyrmion can be regarded as being classical. The real-space emergent magnetic field due to the spin texture is given by

$$h = \nabla \times a = \frac{\hbar c}{2e} (n \cdot \partial_t n \times \partial_y n) \hat{z}, \quad \text{[17]}$$

while the electric field is

$$e_i = -\frac{1}{c} \frac{\partial a_i}{\partial t} = \frac{\hbar c}{2e} (n \cdot \partial_t n \times \partial_i n). \quad \text{[18]}$$

These emergent electromagnetic fields act on the conduction electrons similarly to the Maxwell electromagnetic field, and drives the electron motion. The integral of the emergent magnetic field over the space defines the topological index, called the skyrmion number, $Q$, defined as

$$Q = \frac{1}{4\pi} \int_{\text{uc}} d^2 x n \cdot (\partial_y n \times \partial_y n) = \pm 1. \quad \text{[19]}$$

Namely, the total emergent flux, $\Phi$, of the skyrmion is given by $\Phi = \frac{\pi}{\sqrt{2}} Q$. This emergent magnetic field produces the Hall effect of the conduction electrons called the topological Hall effect (THE).\textsuperscript{15} Furthermore, the motion of the skyrmion results in the time-dependence of the flux, leading to the emergent electromagnetic induction and the emergent electric field, which is observed as a reduction of THE due to the motion of skyrmions. This effect was predicted theoretically\textsuperscript{20} and observed experimentally.\textsuperscript{21}

The skyrmion number, $Q$, cannot be changed by a continuous deformation of the spin configuration, and hence protects the skyrmion topologically. Furthermore, $Q$ enters into the effective action for the center-of-mass motion of a skyrmion as $Q(X_{2Y} - Y_{2X})$, which indicates the canonical conjugate relation between the two components $X$ and $Y$ of the position. This is analogous to the charged particle under the magnetic field, and the velocity becomes perpendicular to the force. This Gyro-dynamics results in the unique motion of the skyrmion, especially under the current. For example, the threshold current density for the spin-transfer torque effect is orders of magnitude smaller than that of the domain wall, because skyrmions avoid the impurity potential due to the Gyro-dynamics.\textsuperscript{15}

C. Multiferroics. The concept of the emergent electromagnetic field can be generalized to the non-Abelian case. The most natural extension is the SU(2) gauge field corresponding to the spin degrees of freedom. Starting from the Dirac equation, one obtains the effective Lagrangian for the positive energy space by expanding with respect to $1/(mc^2)$, as\textsuperscript{22,23}

$$L = i \psi \bar{D}_0 \psi + \psi \left( \frac{D^2}{2m} + \frac{q^2}{4} A^2 \cdot A^2 \right) \psi. \quad \text{[20]}$$

Here, $\psi$ is now the 2-component spinor corresponding spin up and spin down, and the gauge covariant derivatives are $D_0 = \partial_0 + i e A_0 + i q A^0 \sigma^2$, and $D_i = \partial_i - i e A_i - i q A^i \sigma^2$ ($i = 1, 2, 3$) with $q$ being a quantity proportional to the Bohr magneton.\textsuperscript{22,23} The SU(2) emergent electromagnetic field, given by $A^0 = B_0$, $A^i = \epsilon_{iad} E_i$, which is coupled to the 4-component spin current, $j^i_0 = \psi \sigma^i \psi$, $j^i = \frac{1}{2m} (\psi \sigma^i D_i \psi - D_i \psi \sigma^i \psi)$. This relation leads to an interesting relation between the spin current and the electric polarization, $P$. Namely, $P$ is given by the derivative of the Lagrangian with respect to the electric field, $E$, hence, one obtains the relation

$$P_i \propto \epsilon_{iad} j^i_0, \quad \text{[21]}$$

which means that the spin current produces the ferroelectric moment.\textsuperscript{24,25}

Therefore, once the spin ordering produces the spin current, the electric polarization emerges, which offers a mechanism of ferroelectricity of spin origin, \textit{i.e.}, multiferroics.\textsuperscript{26,27} This mechanism explains well the first discovery of the magnetic control of ferroelectric polarization in manganite.\textsuperscript{28}

The commutators among the spin components,

$$[S^i, S^j] = i \hbar \epsilon_{ijk} S^k, \quad \text{[22]}$$

are translated into

$$[S^i, S^\pm] = \pm i \hbar S^\pm, \quad \text{[23]}$$

with $S^\pm = S^x \pm i S^y$. The “phase” $\theta$ of the $xy$-components of the spin is defined by $S^\pm \approx e^{i \theta}$, and Eq. 23 results in

$$[S^i, \theta] = i \hbar. \quad \text{[24]}$$
This is reasonable, since $S^z$ is the generator for the rotation of the $xy$-components of the spin. In analogy to the relation between the particle number, $n$, and the phase, $\varphi$, of the bosonic field operator, one can see that the spatial gradient of the phase, $\nabla \varphi$, leads to the “superfluid” spin current with the spin polarization along the $z$-axis. (Note that the spin current is a tensor quantity having indices for the real space and spin space.) Generalizing this consideration to the generic direction of the spin current, Eq. [21] leads to the electric polarization, $P$, as

$$P = \eta e_i \times (S_i \times S_j),$$  \[25\]

where $\eta$ is a constant that depends on the spin-orbit interaction.\textsuperscript{24,25} This simple formula, Eq. [25], became a guiding principle to search for the multiferroic materials.\textsuperscript{27} (See Fig. 3 for the schematic illustration of Eq. [25].)

V. Emergent electromagnetic field in momentum space

A. Anomalous Hall effect. The Berry phase of the Bloch wavefunctions in solids is an important issue related to many hot topics in the current condensed-matter physics. The Bloch wavefunction is written as

$$\psi_{nk\sigma}(r, s) = e^{i(k \cdot r)} u_{nk}(r) \chi_{\sigma}(s),$$  \[26\]

with $u_{nk}(r)$ being a periodic function of coordinate $r$ with respect to the translation of the lattice vectors, $\chi_{\sigma}(s)$ is the spin wavefunction for $\sigma = \uparrow, \downarrow$ with the spin coordinate $s = \pm 1/2$. $n$ is the band index, and $k$ is the crystal momentum. The energy dispersions, $\varepsilon_n(k)$’s, with different $n$’s are usually separated by the energy gaps at each $k$, except for the band crossing point. Therefore, this is again the typical situation, where the Berry connection plays an important role, which is given by

$$a_{nk}(k) = -i[u_{nk} \partial_{k} |u_{nk} \rangle]$$  \[27\]

with $a = x, y, z$, and the corresponding Berry curvature is

$$b_{n}(k) = \nabla_{k} \times a_{n}(k),$$  \[28\]

analogously to the magnetic field.

This Berry phase modifies the equation of motion for a wave-packet made of the Bloch states.\textsuperscript{29,30} Due to the canonical conjugate relation, the position operator, $x$, is usually defined as $x_\mu = \dot{x}_\mu / \partial_{k}$. When the wave-packet is made from the subspace characterized by the Berry connection, it should be modified into the gauge covariant form as

$$x_\mu = i \partial / \partial k_\mu + a_{nk}(k).$$  \[29\]

Therefore, $a_{nk}(k)$ has the meaning of “intracell coordinate”, i.e., the shift of the center of mass of the electron wavepacket within the unit cell, although it is a gauge-dependent quantity.\textsuperscript{31} This also leads to the commutator $[x, y] = i b_\mu(k)$, i.e., the real space coordinates do not commute with each other. With this non-commutative nature, the equations of motion for the wavepacket

$$\frac{dx_\mu}{dt} = -i[x_\mu, H],$$
$$\frac{d\pi_\mu}{dt} = -i[\pi_\mu, H],$$  \[30\]

are modified.\textsuperscript{29,30} Let the Hamiltonian, $H$, be given by $H = \varepsilon_n(k + eA) + V(x)$, where $\varepsilon_n(k)$ is the energy dispersion of band $n$, and $V(x)$ is the potential; $A$ is the vector potential for the Maxwell electromagnetic field. Putting this into Eq. [30], we obtain

$$v_\mu = \frac{dx_\mu}{dt} = \frac{\partial \varepsilon_n(k)}{\partial k_\mu} |_{k = A} - i[x_\mu, \psi] \frac{\partial V(x)}{\partial x_\nu},$$
$$= \frac{\partial \varepsilon_n(k)}{\partial k_\mu} + \varepsilon_{\mu \lambda} b_\lambda(k) \frac{\partial V(x)}{\partial x_\nu},$$
$$= \frac{\partial \varepsilon_n(k)}{\partial k_\mu} + (b \times F)_\mu,$$
$$\frac{dk_\mu}{dt} = -\varepsilon_{\mu \lambda} ev_\lambda B_\lambda(k) - \frac{\partial V(x)}{\partial x_\nu} = F_\mu,$$  \[31\]

where $F = -ev \times B - \nabla_x V(x) = -ev \times B - eE$ is the force acting on the electron. Equation [31] clearly demonstrates the duality between the real-space and the momentum space. As can be seen from

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The figure shows a schematic illustration of the spin-current mechanism of the electric polarization. The polarization, $P$, is induced, as given by $P = \eta e_i \times (S_i \times S_j)$ (Eq. [25]), with $e_i$ being the unit vector connecting the atoms $i$ and $j$. The figure highlights the transfer integral, spin current, and the relationship between the electric polarization and the spin current.
Eq. [31], the Berry curvature produces a velocity transverse to the external force, which is called the anomalous velocity. This naturally leads to the Hall effect, such as the anomalous Hall effect discussed below.\(^{32}\)

Note that there is one sharp difference between the Maxwell magnetic field, \(B\), and Berry curvature, \(b\). Namely, \(B\) is divergence-free, \(\text{i.e.}, \nabla_z \cdot B(x) = 0\), corresponding to the absence of a magnetic monopole, while \(\nabla_k \cdot b(k)\) can have a magnetic monopole at a band crossing point, as discussed in Eq. [5], with \(X\) being replaced by \(k\).

Another important remark is that the symmetries give the following constraint. The time-reversal symmetry, \(T\), gives the relation \(b_{\nu}(k) = -b_{\nu}(-k)\), while the spatial inversion symmetry, \(I\), gives \(b_{\nu}(k) = b_{\nu}(-k)\). Therefore, when both \(T\) and \(I\) symmetries are there, and the Berry curvature, \(b_{\nu}(k)\), vanishes. Also, in the noncentrosymmetric system, where \(T\)-symmetry is absent, \(b_{\nu}(k)\) can be nonzero, although the contributions from \(k\) and \(-k\) cancel with each other when \(T\) is intact.

We now discuss the Hall effect driven by the Berry curvature (emergent magnetic field) in momentum space.\(^{32}\) From the above consideration, the Hall conductivity \(\sigma_H\) is given by the integral of the Berry curvature over the occupied states, \(\text{i.e.},\)

\[
\sigma_H = \frac{e^2}{h} \sum_n \int \frac{dk}{(2\pi)^d} f(\varepsilon_n(k)) b_{nz}(k), \tag{32}
\]

where \(f(\varepsilon)\) is the Fermi distribution function. In the band insulator, the \(k\)-integral is over the first Brillouin zone for the occupied band, \(n\). Especially in 2D, the Stokes theorem leads to

\[
\sigma_H = \frac{e^2}{h} \sum_{\text{occupied}} \frac{\int d\mathbf{k}}{(2\pi)^2} \mathbf{a}_n(k) = \frac{e^2}{h} \sum_{\text{occupied}} N_c(n), \tag{33}
\]

where \(N_c(n) = \frac{\int d\mathbf{k}}{(2\pi)^2} \mathbf{a}_n(k)\) is the integer due to the single-valuedness of the wavefunction, called the Chern number. This is the so-called TKNN formula for the integer quantum Hall effect.\(^{33}\)

Equation [32] represents the intrinsic Hall effect due to the geometrical nature of the Bloch wavefunctions, which is also applicable to a metallic system. This offers a modern interpretation of the old theory by Karplus-Luttinger for the anomalous Hall effect (AHE) in metallic ferromagnets.\(^{34}\) However, there has been a long-term controversy about the origin of the AHE. After a paper by Karplus-Luttinger, the effects of the impurity scatterings are proposed to be essential for the AHE, and extrinsic mechanisms, \(\text{i.e.},\) the skew scattering\(^{35}\) and side jump,\(^{36}\) were proposed. A unified theoretical treatment of this issue has recently been developed, and now the respective role of intrinsic and extrinsic mechanisms are now clarified.\(^{32}\) Beyond the perturbative treatment of the spin-orbit interaction, it is now recognized that the band crossings near the Fermi energy play an essential role, and act as the monopoles of the Berry curvature. Therefore, intrinsic Hall conductivity has a topological meaning and is robust against impurity scatterings. Therefore, as a function of the diagonal conductivity, there are three regions of the AHE, \(\text{i.e.}:\) (i) the strongly disordered case, where the intrinsic contribution is reduced and approximately the relation \(\sigma_H \approx \sigma_{xx}^0\) holds, (ii) the intermediate case, where the intrinsic contribution is dominant and \(\sigma_H\) is almost constant, and (iii) the clean case, where the skew scattering contribution is dominant.

Now first-principles band structure calculations can predict the intrinsic anomalous Hall conductivity with reasonable accuracy to be compared with experiments, and there are several materials where the intrinsic AHE is established. Band structure calculations have revealed many band crossings near the Fermi energy, and their resonant contribution to the Hall conductivities, which is a common feature of ferromagnetic metals.

Also, one can ask if the AHE can be quantized in 2D without an external magnetic field. This issue is related to Haldane’s work on the quantized Hall effect without the Landau levels.\(^{37}\) He introduced the nearest neighbor and next-nearest neighbor hopping integrals, the latter of which is complex. We have shown that the tight-binding model for a ferromagnet with the spin-orbit interaction leads to a similar model showing the non-zero Chern numbers, and proposed the quantized AHE (QAHE).\(^{38,39}\)

Recently, QAHE has been predicted and observed in the surface state of a magnetic topological insulator.\(^{40,41}\)

**B. Spin Hall effect.** A direction to generalize the idea of intrinsic AHE is to consider the spin current instead of the charge current. This corresponds to the generalization of the Berry phase to a non-Abelian group, \(\text{i.e.},\) the SU(2) gauge field in spin space. Note that this non-Abelian gauge field can be finite, even with both \(T\) and \(I\) symmetries. In the presence of the SU(2) Berry curvature in momentum space, the generalized equation of motion for the wavepacket of the particle with spin is given by ref. 42
where $n$ is the band index, and $i, j$ are 1, 2, 3. $z$ is the two-component spinor wavefunction for the spin, i.e., $z = (z_1, z_2)$, and $A^a$ is the $2 \times 2$ SU(2) vector potential; $F_{ij}^a = \partial_i A^a_j - \partial_j A^a_i + [A^a_i, A^a_j]$ is the corresponding field strength. This non-Abelian gauge leads to the flow of the spin, i.e., spin current, in the form of the spin Hall effect (SHE), represented by the equation for the spin current, $j^a_j$, as

$$ j^a_j = \sigma_a \epsilon_{ijk} E_k, \quad [37] $$

where $\epsilon_{ijk}$ is the totally antisymmetric tensor, and $\sigma_a$ is the spin Hall conductivity. Note that this form is the “dual” to Eq. [21], and hence it is expected that the relativistic spin-orbit interaction is relevant to the SHE.\(^{42-44}\)

Equation [37] has been explicitly derived for the p-type GaAs, where the 4-fold degeneracy of the band occurs at the $\Gamma$-point. It acts as the Yang-monopole, i.e., a generalization of the U(1) monopole to the SU(2) case, and hence the gauge field strength, $F_{ij}$ is given by

$$ F_{ij} \propto \pm \epsilon_{ijk} \frac{k_l}{k^2}. \quad [38] $$

Here, the $\pm$ sign depends on the spin parallel or antiparallel to the momentum $k$. This spin-dependent Berry curvature leads to the SHE described by Eq. [37].\(^{42}\)

Since the emergent electromagnetic field is not directly related to the real electromagnetic charge, one can expect that it is also relevant to the neutral particles. For example, photon is the constrained system, i.e., the polarization of the electric and magnetic fields are perpendicular to the wavevector, and hence the Berry phase appears. Therefore, one can derive the equation of motion for the wavepacket of the photon, similar to Eq. [36]. This leads to the Hall effect of light, where the shifts of the reflected and transmitted light beams occur transverse to the incident direction with the opposite directions for right and left-circular light.\(^{45}\) Another example is the Hall effect of magnons, where, e.g., the relativistic DM interaction gives the phase factor for the propagation of the magnon, leading to the Berry curvature in momentum space in ferromagnets with multiple atoms in the unit cell.\(^{46}\) The thermal Hall effect in pyrochlore insulating ferromagnet Lu$_2$V$_2$O$_7$ has been analyzed from this viewpoint showing the excellent agreement between the theory and experiment.\(^{47}\)

### C. Ferroelectricity and shift current

It is now well known that the Berry phase is relevant to the electronic contribution to the electric polarization.\(^{48,49}\) The idea is to integrate the polarization current over the displacement of the atoms from the symmetric positions. This polarization current is driven by the Berry phase. More explicitly, consider the two-dimensional space of the momentum, $k$, and the displacement, $D$. Adiabatically increasing $D(t)$ from 0 to the physically realized value $D(T) = D_0$, the polarization current is given by the integral over the first Brillouin zone as

$$ J_z(D) = \int_{-\pi}^{\pi} \frac{dk_z}{2\pi} F_{k_z,D}(k_z, D) \frac{dD}{dt}. \quad [39] $$

Note that this relation is analogous to the Hall effect replacing $k_y$ by $D$. By using the Stokes theorem and with the gauge choice that the connection is zero at $D = 0$,

$$ P(D_0) = \int_0^T dt J_z(D(t)) = \int_{-\pi}^{\pi} \frac{dk_z}{2\pi} A_{k_z}(k_z, D_0). \quad [40] $$

Namely, the polarization is given by the integral of the intracell coordinate over the first Brillouin zone. Note that the Berry phase (curvature) is zero for the $I$ and $T$ symmetric systems, and the polarization is finite only when the $I$ symmetry is broken. Namely, the $I$-symmetry breaking is encoded by the Berry phase of the Bloch wavefunctions.

Recently, an extension of the ferroelectricity to the nonequilibrium situation has been studied extensively in the field of nonlinear optics. As mentioned in the Introduction, the concept of the Berry phase and the emergent electromagnetic field are useful concepts when the wavefunctions are confined in the manifold in Hilbert space. However, generalization of the geometric phase without the adiabatic limit is possible; see e.g., the paper by Aharonov and Anandan.\(^{50}\) Especially, for the interband transition by high-energy photons, the electrons jump from one manifold to the other. Even in this high-energy process, the concept of the Berry phase is useful. Intuitively, the change in the intracell coordinate associated with the interband transition creates a current, which is called the shift current.\(^{51}\) This shift current can be a dc current by steady
photoexcitation, although it is closely related to the polarization current. Namely, the integral of the intracell coordinate over the occupied band is the polarization, and the interband transition partly changes this polarization to result in the current. The reason why it can be the dc current is that the relaxation is “neutral”, i.e., the relaxation has no preferred direction while the excitation is asymmetric between right and left.

This phenomenon can be formulated by the Floquet formalism combined with the Keldysh Green function method. Consider the 2 × 2 Hamiltonian,\(^{52}\)
\[
h = \begin{pmatrix} 
\epsilon_1^0 & iF\psi_{12}^0 \\
-iF\psi_{21}^0 & \epsilon_2^0 
\end{pmatrix} \equiv \epsilon + d \cdot \sigma,
\]
where \(F = E/\Omega\) is the electric field of light divided by its frequency. The current operator is given by
\[
nev = -e \frac{\partial h}{\partial k} \equiv b_0 + b \cdot \sigma.
\]
The expectation value of the current reads
\[
J = \int dk \frac{\pi E^2}{2\Omega^2} \frac{\Gamma}{\sqrt{\Gamma^2 + \Gamma^2}} \delta(d_2) |\psi_{12}^0|^2 
\times \left[ \frac{d}{dk} \text{Im}(\log \psi_{21}^0) + a_{22} - a_{11} \right],
\]
where \(a_{11}\) and \(a_{22}\) are the Berry connection of the conduction and valence bands, respectively, and \(\frac{d}{dk} \text{Im}(\log \psi_{21}^0) + a_{22} - a_{11}\) is the gauge-invariant shift of the intracell coordinate. \(I\) represents the relaxation of the electrons, and the factor \(\frac{\Gamma}{\sqrt{\Gamma^2 + \Gamma^2}}\) describes the competition between the neutral relaxation and the induced emission, which brings back the shift and reduces the shift current.

This shift current can be the mechanism of the photovoltaic effect in perovskite materials.\(^{53}\) A first-principles band structure calculation can predict the shift current as a function of the energy of the incident light. For example, a detailed comparison has been done for the photovoltaic effect in BaTiO\(_3\), showing a good agreement between theory and experiment.\(^{54}\) This shift current could be relevant to the high-efficiency solar-cell action. Theoretically, the shift current does not require the free carriers analogously to the fact that the polarization current can exist in insulators. Actually, it is proposed theoretically that the shift current remains even when the photon energy is below the particle-hole continuum, i.e., at the exciton absorption peak.\(^{55}\) The shift current can be regarded as being one of the nonreciprocal responses in noncentrosymmetric systems, and a unified understanding of them is an important future problem.\(^{56}\)

VI. Topological states of electrons in solids

The emergent electromagnetic field corresponds to the curvature defined locally at each point of the manifold. It is the most remarkable result in differential geometry that one can derive the topological invariant by the integral of the local curvature over the manifold. The classic example of this fact is the Gauss-Bonnet theorem, where the integral of the Gauss curvature over the closed surface in three-dimensions gives the number of “holes” called genus.\(^1\) This wisdom can be applied to the electronics states in solids. The Chen number discussed for the quantum Hall effect is an example of this application.\(^{33}\) The gap protects this topological index, i.e., it remains unchanged as long as the gap does not close. In other words, the gap must close to have any change in the topological index. This leads to the “bulk-edge correspondence”. Namely, the vacuum outside of the solid is a “trivial” state, and the gap must close at the surface when the electronic states inside the solid has a nontrivial topological index. Therefore, the gapless states should appear at the surface or edge of the system, which is protected by the topology of the bulk states.

Looking into a more explicit example, it is a natural question as to whether the analogue of the quantum Hall effect exists also for the SHE. The simplest case is that the spin-up and spin-down electrons are decoupled, showing the opposite sign of the quantized Hall conductance, i.e., quantized spin Hall effect. In general, however, the spin is not conserved in the presence of the spin-orbit interaction and the spin components are mixed. Therefore, it is a highly nontrivial issue as to how to define the quantum SHE in this general case. Actually, it has been revealed by a seminal paper by Kane and Mele\(^{57}\) that one can define the \(Z_2\) topological index, which is not related to the spin Hall effect, but is related to the edge channels.\(^{58},^{59}\) This \(Z_2\) index is closely related to the time-reversal symmetry, \(\mathcal{T}\), and Kramer’s theorem coming from the fact that \(\mathcal{T}^2 = -1\) for the half-odd integer spin, \(S\). The helical edge channel remains gapless as long as the perturbations, such as the impurity scattering respect the time-reversal symmetry, \(\mathcal{T}\). Namely, the crossing of the dispersion of the helical edge channel is the Kramer’s doublet at the time-reversal symmetric momentum.
This two-dimensional quantum SHE was the first discovery of topological insulators (TIs), which are now fully classified by K-theory in any dimensions and symmetry classes.\textsuperscript{58,59} The three-dimensional TI has no analogue to the quantum Hall system. It is characterized by the surface gapless Weyl fermion with the spin and momentum locked, as observed by the spin-resolved ARPES experiment.\textsuperscript{58} This corresponds to half of the Dirac fermions, \textit{i.e.}, an example of the electron fractionalization. This momentum-spin locking corresponds to the strong coupling limit of the spin-orbit interaction, and is hence expected to be very useful in spintronics applications. Especially, when magnetization normal to the surface is introduced, there appears a gap of the surface Weyl fermion. This is a system showing the parity anomaly in two-dimensions, which shows the Hall conductance, $e^2/(2h)$. When the magnetization direction is the same for both the top and bottom surfaces, the Hall conductance of the total system is $e^2/h$ with the chiral edge channel circulating along the side surface. This is an ideal laboratory to realize the quantized anomalous Hall effect predicted theoretically. Actually, it was discovered experimentally in 2013 in magnetically doped TI Cr:Bi$_2$Te$_3$, as discussed above.\textsuperscript{41} It has been also proposed that the magnetic surface of TI behaves as a two-dimensional multiferroics.\textsuperscript{60}

**VII. Conclusions**

We have reviewed physical phenomena related to emergent electromagnetism in solids. They are all related to the geometrical properties of the wavefunctions on the manifold in Hilbert space. The gauge field naturally and ubiquitously appears when the system is constrained, and the nontrivial geometry is possible accordingly. This gauge field is called an “emergent electromagnetic field” analogously to the Maxwell electromagnetic field.

The emergent electromagnetic fields are classified into those in real space and momentum space, but they can be unified into the phase space, \textit{i.e.}, 6-dimensions including both the real and momentum spaces. Furthermore, adding the parameters characterizing the Hamiltonian, one can consider the even higher dimensional spaces providing much richer structures of theories. Actually, in cold atom systems, the “synthetic dimensions” is designed to realize \textit{e.g.} the four-dimensional quantum Hall state.

Another future direction is to consider the time-dependence of the emergent electromagnetic field more. Especially, the dynamics of the emergent electromagnetic field in momentum space has not been yet explored very much thus far. There remain many interesting and important problems related to emergent electromagnetic field to be studied in the future, which will provide a more coherent and unified view of the electronic systems in solids.

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Profile

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