Abstract. We use the Heun type solutions given in [8] and a new generated that equation for the radial Teukolsky equation for Kerr-Newman-de Sitter geometry to calculate reflection coefficient for waves coming from the de Sitter horizon and reflected at the outer horizon of the black hole.

1 Introduction

It is well known that a scalar particle in the background of any D type metric gives Heun type solutions [1]. A recent paper also showed the separability of conformally coupled scalar field equation in general (off-shell) Kerr-NUT-Ads spacetimes in all dimensions [2]. Work in this direction was done in the past. To mention few of them, one may cite Carter [3], who showed that the scalar wave equation is separable in the Kerr-Newman-de Sitter geometries. Teukolsky [4] generalized this for spinors, electromagnetic fields, gravitational fields and gravitinos for the Kerr-Newman and Kerr-Newman-de Sitter class of geometries. In this respect, one can also cite [5,6,7]. In the two papers [8,9], Suzuki et al. obtained the exact solutions of the Teukolsky equations in terms of Heun type functions [10,11,12,13,14]. Suzuki et al. expanded the Heun solutions in terms of infinite series of hypergeometric functions, and used this infinite series solution in their work. Quoting from their second paper [9], "they chose the solution which satisfied the incoming boundary conditions at the outer horizon of the black hole and examined the asymptotic behavior at the de Sitter horizon. They evaluated the absorption rate of the Kerr-de Sitter and the Kerr-Newman-de Sitter black holes by using the analytic solution. They constructed the conserved current by evaluating the Wronskian, and obtained an expression of the absorption rate. From this, they showed explicitly that super-radiance occurs for the boson, similarly to the Kerr geometry case [15]. Then, they derived an analytic expressions of the incident, the reflection and the transmission amplitudes. They also derived the conserved current from which they derived the absorption rate. They also studied the asymptotic limits of their solutions."

A peculiar characteristic of the Teukolsky equations is that, after one obtains the wave equations for the radial and angular variables, one finds that they are in very similar forms. In fact, sometimes they can be made exactly the same, by choosing the correct transformation, as in the Kerr-de Sitter case [16,17], in the limit when the mass of the black hole goes to zero.

Heun functions were not very popular even at the end of the twentieth century. In the last decade, in different indices, one finds that the number of papers on Heun functions [10,11,12,13,14] more than doubled. In different citation index collections, like WOS or Scopus, one can find many papers which give their results in terms of these functions. One can also find connection formulae for Heun functions expanded around different points [18,19]. Furthermore, new mathematical software, like Maple, includes Heun solutions. In many examples, once one knows that the solution will be of the Heun form and the transformation of coordinates to obtain that solution, I believe, it is easier to solve the equation by hand, since, Maple often gives page long solutions. These are much shorter if the calculation is done by hand.

Here we used the wave equations given in [8] for the radial case, obtained the Heun solutions around different points for the Kerr-Newman-de Sitter metric and used the exact solutions in terms of Heun functions [8,9]. By using the connection formulae given in [18], we calculate the reflection coefficients for waves coming from the de Sitter horizon in a closed form.

In the next section, we summarize the relevant information given in [8]. In section 3, we solve the wave equation using expansions both for around the outer horizon and around the inner de Sitter horizon exactly in terms of General Heun functions. Before doing this, we use a transformation to bring infinity to unity, and the outer horizon to zero. Then, we use these solutions and the connection formulae to right the reflection coefficient at the outer horizon. We end by our conclusions.
2 The radial Teukolsky equation for Kerr-Newman-de Sitter geometry

Our reference [7] gives the radial Teukolsky equation as

\[
\Delta^{-1} \frac{d}{dr} \Delta s + \frac{1}{\Delta} \left( (1 + \alpha) \omega (K - \frac{Qr}{s} - \frac{Q}{1+\alpha}) \right)^2 \\
- is(1 + \alpha) (K - \frac{Qr}{s} - \frac{Q}{1+\alpha}) \frac{d}{dr} R + \left( \frac{4is(1 + \alpha)\omega r}{a^2} \right) R = 0, \tag{1}
\]

with five regular singularities at \( r_+^{\pm}, r_-^{\pm}, r_+^{d}, r_-^{d} \) and infinity. Here \( r_+ \) is the outer horizon, \( r_- \) is the inner horizon, \( r_+^{d} \) is the outer de Sitter horizon, \( r_-^{d} \) is the inner de Sitter horizon. Here \( \Delta \) is the cosmological constant, \( M \) its mass, \( \alpha, \omega, Q \) its angular momentum, \( Q \) its charge, \( K = \omega (r^2 + a^2) \), \( \alpha = \frac{a \omega}{r^2} \) and

\[
\Delta = (r^2 + a^2) (1 - \frac{\alpha}{a}) - Mr + Q^2 \\
= - \frac{\alpha}{a^2} (r - r_+)(r - r_-)(r - r_+^{d})(r - r_-^{d}). \tag{2}
\]

By using the new variable

\[
z = \left( \frac{r_+^{d} - r_-^{d}}{r_+^{d} - r_-} \right) (r - r_-) \tag{3}
\]

one gets a new wave equation. Again quoting [8], "the new equation has regular singularities at 0, 1, \( z_r \), \( z_{inf} \) and at infinity." Note that now for \( r \) equal to \( r_+^{d} \), \( z \) goes to infinity. Furthermore,

\[
z_r = \left( \frac{r_+^{d} - r_-^{d}}{r_+^{d} - r_-} \right) (r - r_-) \tag{4}
\]

and

\[
z_{inf} = \left( \frac{r_+^{d} - r_-^{d}}{r_+^{d} - r_-} \right), \tag{5}
\]

which are both take negative values, outside our realm of interest.

In [9], the authors use a different independent variable \( x = 1 - z \), which maps the inner horizon \( r_- \) to one, the outer horizon to zero, inner de Sitter horizon to infinity, outer de Sitter horizon to

\[
x_r = \left( \frac{r_+^{d} - r_-^{d}}{r_+^{d} - r_-} \right) (r - r_-) \tag{6}
\]

and infinity to

\[
x_{inf} = \left( \frac{r_+^{d} - r_-^{d}}{r_+^{d} - r_-} \right). \tag{7}
\]

We choose to use the variables in the first paper of of our reference [5].

As shown in [8], one can factor out the singularity at \( z = z_{inf} \) using the transformations

\[
R(z) = z^B_1(z - 1)^B_2(z - z_{inf})^{B_3} g(z). \tag{8}
\]

We think this is very remarkable, since here, by a single s-homotopic transformation, i.e. by multiplying the dependent variable by a power, one gets rid of both linear and quadratic powers of \( \frac{1}{z - z_{inf}} \) multiplying the dependent variable, as well as the same term multiplying the derivative of the dependent variable. Normally, in differential equations, such a transformation gets rid of only one term, usually the inverse quadratic term in the original equation. Only the special form of the used metric enables this important result.

Here

\[
B_1 = \frac{1}{2} (-s \pm \frac{2(1 + \alpha)a^2(\omega r^2 + a^2) - am - \frac{Qr}{1+\alpha}}{\alpha(r_+^{d} - r_-)(r_+^{d} - r_-)(r_+^{d} - r_-^{d}) - is}), \tag{9}
\]

\[
B_2 = \frac{1}{2} (-s \pm \frac{2(1 + \alpha)a^2(\omega r^2 + a^2) - am - \frac{Qr}{1+\alpha}}{\alpha(r_+^{d} - r_-)(r_+^{d} - r_-)(r_+^{d} - r_-^{d}) - is}), \tag{10}
\]

\[
B_3 = \frac{1}{2} (-s \pm \frac{2(1 + \alpha)a^2(\omega r^2 + a^2) - am - \frac{Qr}{1+\alpha}}{\alpha(r_+^{d} - r_-)(r_+^{d} - r_-)(r_+^{d} - r_-^{d}) - is}). \tag{11}
\]

After all these transformations are made, we end up with a wave equation for \( g(z) \) which reads

\[
\left( \frac{d^2}{dz^2} + \frac{2B_1 + s + 1}{z - 1} + \frac{2B_2 + s + 1}{z - 1} + \frac{2B_3 + s + 1}{z - z_r} \right) g(z) = 0, \tag{12}
\]

\[
\left( \sigma_z + \sigma + v \right) g(z) = 0, \tag{13}
\]

where

\[
\sigma_z = \frac{2B_1 + s + 1}{z - 1} + \frac{2B_2 + s + 1}{z - 1} + \frac{2B_3 + s + 1}{z - z_r} \tag{14}
\]

and

\[
A = \frac{2a^4(1 + \alpha)^2(r_+^{d} - r_-^{d})^2(r_+^{d} - r_-)^2(r_+^{d} - r_-^{d})}{\omega (r_+^{d} - r_-)}, \tag{15}
\]

\[
B = -a^2(1 + \alpha)^2(r_+^{d} - r_-^{d})(r_+^{d} - r_-)(r_+^{d} - r_-^{d}) \tag{16}
\]

\[
+ 2\omega(\omega a - m)r_+^{d}(r_+^{d} - r_-^{d}) - a^2(\omega a - m)^2(2r_+^{d} - r_-^{d}). \tag{17}
\]
where

\[ H = \frac{a}{a^2}(-27M^4 + 36(1 - \alpha)M^2(a^2 + Q^2) - 8(1 - \alpha)^2(a^2 + Q^2)^2 - \frac{16a^2}{a^4}(a^2 + Q^2)^3) \]  

At the end, we find that aside from the terms multiplying it, the solution of Eq. (8) is

\[ g(z) = H_G(z, -v; \sigma_+, \sigma_-; 2B_1 + s + 1, 2B_2 + s + 1; z) \]  
in the standard given in [20].

3 The reflection coefficients

Here we use the information we obtained from [8] and first try to calculate the possible scattering for waves coming from the inner de Sitter horizon at the outer horizon. We first calculate the two solutions at $r_r^d$, then use the connection formula given in [18] to write the solution at the outer horizon in terms of the wave coming from and reflected to the inner de Sitter horizon. Unfortunately, this formula [18] works only between two finite points. That is why we first use a transformation of the independent variable $z$ in Eq. (5),

\[ t = \frac{1 - z}{z - z_r} \]  

which will be zero at the outer horizon, and $-1$ at the inner de Sitter horizon. This transformation yields the following equation for the dependent variable.

\[
\begin{align*}
\frac{d^2 R}{dt^2} + \left(1 - \frac{(\sigma_+ + \sigma_-)}{t + 1}\right) + \frac{2B_2 + s + 1}{t} + \frac{2B_1 + s + 1}{t + \frac{1}{z_r}} \frac{dR}{dt} \\
+ \left(\frac{(\sigma_+ \sigma_-)}{(t + 1)^2} - \frac{M}{t(t + 1)(z_r + 1)}\right)R = 0. 
\end{align*}
\]  

This equation is not of the Heun form. To put it to the Heun form, we make a s-homotopic transformation

\[ R = (t + 1)^{\kappa} S_1(t) \]  

and find the first solution as $\kappa = \sigma_+$, and the second as $\kappa = \sigma_-$. We choose the first solution, which yields the differential equation

\[
\begin{align*}
\frac{d^2 S_1}{dt^2} + \left(1 + \frac{\sigma_+ - \sigma_-}{t + 1}\right) &+ \frac{2B_2 + s + 1}{t} + \frac{2B_1 + s + 1}{t + \frac{1}{z_r}} \frac{dS_1}{dt} \\
+ \left(\frac{\sigma_+ (\sigma_+ - s - B_3)}{(t + 1)(t + \frac{1}{z_r})} - \frac{v - \sigma_+ (s - B_2 + 1)}{t(t + 1)(z_r + 1)}\right)S_1 &- 0. 
\end{align*}
\]  

This equation is not in the standard Heun form. Recalling that here $t$ is between $-1$ to zero, we define $u$ as the absolute value of $t$. Then the solution is

\[ S_1 = H_G(\frac{1}{z_r}, -v + \sigma_+ (s - B_2 + 1); \sigma_+, \sigma_+ - s - B_3; 2B_2 + s + 1, 1 + \sigma_+ - \sigma_-; u). \]  

In the standard form [20], the parameters given as $H_G(a, q; \alpha, \beta; \gamma, \delta; z)$ for the differential equation

\[
\begin{align*}
\frac{d^2 H_G(x)}{dx^2} + \left(\frac{\delta}{x - 1} + \frac{\epsilon}{x - a} + \frac{\gamma}{x}\right) \frac{dH_G(x)}{dx} \\
+ \left(\frac{\alpha \beta x - Q}{x(x - 1)(x - a)}\right)H_G(x) &- 0. 
\end{align*}
\]  

In [18], the authors, in their equation (A.15), for a finite interval, give the necessary formulea, to write the $H_G(z)$, solution expanded in terms of $z$, in terms of two solutions of the equation, expanded around 1 $- z$. This is a formula to write

\[ y_1(z) = C_1 y_3(1 - z) + C_2(1 - z)^{\kappa} y_4(1 - z). \]  

This formula, for general Heun functions, reads

\[ H_G(a, Q; \alpha, \beta; \gamma, \delta; z) = C_1 H_G(1 - a, -Q - \alpha \beta; \alpha, \beta; 1 + \alpha + \beta - \gamma - \delta, \delta; 1 - z) + C_2(1 - z)^{\gamma + \delta - \alpha - \beta} \\
H_G(1 - a, Q; \gamma + \delta - \alpha - \beta, \alpha, \delta + \gamma - \beta; 1 + \gamma + \delta - \alpha - \beta, \delta; 1 - z). \]  

Here

\[ C_1 = H_G(a, q; \alpha, \beta; \gamma, \delta; 1), \]  

\[ C_2 = H_G(a, -Q - a\gamma; \gamma + \delta - \alpha - \beta; \gamma + \delta - \alpha - \beta, \gamma, \delta; 1). \]
In our example, we have

\[ y_1 = H_G(1 - \frac{1}{z_r}, Q_3; \sigma_+, \sigma_+ - B_3 - s; 1 + \sigma_+ - \sigma_-, 2B_2 + s + 1; u), \]

\[ Q_1 = -\frac{v - \sigma_+(2B_2 + s + 1)}{z_r}, \]

\[ y_3 = H_G(1 - \frac{1}{z_r}, Q_5; \sigma_+, \sigma_+ - B_3; 1 + \sigma_+ - \sigma_-, 2B_2 + s + 1; 1 - u), \]

\[ Q_3 = \frac{v - \sigma_+(2B_2 + s + 1)}{z_r} + \sigma_+(\sigma_+ - s - B_3), \]

\[ C_1 = H_G(1 - \frac{1}{z_r}, v - \frac{\sigma_+(2B_2 + s + 1)}{z_r}; \sigma_+, \sigma_+ - 2B_3 - s; 2B_1 + s + 1, 2B_2 + s + 1; 1). \]

In terms of the parameters giving in the original equation, one writes

\[ \kappa = -\sigma_+ - \sigma_- = -i(2(1 + \alpha)a^2(\omega(r^2 + a^2) - am - \frac{Q\omega}{1+\alpha})\alpha(r^+_+ - r_+)(r^-_+ - r^-_+) - is), \]

\[ y_4 = H_G(1 - \frac{1}{z_r}, Q_3; \sigma_-, \sigma_- - 2B_3 - s; 1 + \sigma_- - \sigma_+, 2B_2 + s + 1; 1 - u), \]

\[ Q_4 = \frac{v - \sigma_-(2B_2 + s + 1)}{z_r} + \sigma_-(\sigma_+ - s - B_3), \]

\[ C_2 = H_G(1 - \frac{1}{z_r}, v - \frac{\sigma_-(2B_2 + s + 1)}{z_r}; \sigma_-, \sigma_- - 2B_3 - s; 2B_1 + s + 1, 2B_2 + s + 1; 1). \]

Then the wave goes as, up to a decaying power,

\[ e^{i\pi n(1-u)}y_3 + Re^{-i\pi n(1-u)}y_4. \]

The reflection coefficient R is given by

\[ R = \left| \frac{M}{N} \right|^2 \]

Here M and N are given by

\[ M = H_G(1 - \frac{1}{z_r}, v - \frac{\sigma_-(2B_2 + s + 1)}{z_r}; \sigma_-, \sigma_- - 2B_3 - s; 2B_1 + s + 1, 2B_2 + s + 1; 1), \]

\[ N = H_G(1 - \frac{1}{z_r}, v - \frac{\sigma_+(2B_2 + s + 1)}{z_r}; \sigma_+, \sigma_+ - 2B_3 - s; 2B_1 + s + 1, 2B_2 + s + 1; 1). \]

Another application will be the reflection of waves coming from the outer horizon at the inner horizon. In S, it is stated that, the transformation they used brought the outer horizon to \( z = 0 \), and the outer horizon to \( z = 1 \). To get the reflection term, we need one solution around \( z = 0 \), and two solutions in terms of the the transformed independent variable at \( z = 1 \). For a solution around \( z \), we use the differential equation, our Eq. (14), given in S. The total solution is given in our Eq. (10). S. The factors that multiply the Heun function are common to all three solutions. The Heun part of the solution is given by

\[ Y_1 = H_G(z_r, -v; \sigma_+, \sigma_-; 2B_1 + s + 1, 2B_2 + s + 1; z). \]

We then, translate the variable to \( 1 - z \) with the new Heun solution

\[ Y_3 = H_G(1 - z_r, v + \sigma_+, \sigma_-; 2B_2 + s + 1, 2B_1 + s + 1; 1 - z). \]

We find the second solution by multiplying \( Y_1 \) by \( (1 - z)^\psi \) and looking for the proper value of \( \psi \) to get a Heun type solution. We find \( \psi \) equal to \( -(2B_2 + s) \) to give us

\[ Y_3 = (1 - z)^{-(2B_2 + s)}H_G(1 - z_r, Q_3; \sigma_-, (2B_2 + s), \sigma_+ - (2B_2 + s); 1 - (2B_2 + s), 1 + 2B_1 + s + 1 - z) \]

\[ Q_4 = v + \sigma_+ \sigma_- + (2B_2 + s)((2B_1 + s)(1 - z_r) + 2B_3 + s + 1). \]

In our Eq. (12) giving \( B_2 \), we choose the plus sign. Then \( \psi \) is an imaginary quantity, aside from \(-s\), giving a decaying solution, as it approaches the inner horizon. After multiplying both sides of the equation

\[ Y_1 = D_3Y_3 + (1 - z)^{-(2B_2 + s)}D_4Y_4, \]

by \((1 - z)^{(B_2 + \psi)}\), we obtain an equation of the form

\[ (1 - z)^{(B_2 + \psi)}Y_1 = (1 - z)^{(B_2 + \psi)}D_3Y_3 + (1 - z)^{-(B_2 + \psi)}D_4Y_4. \]

Up to an overall constant, this equation may be written as an incoming and outgoing waves, as a function of \( 1 - z \) for \( 1 > z > 0 \), equal to the wave evaluated at \( z \) around zero.

The reflection constant \( R \) is given as

\[ \frac{P}{H_G(z_r, -v; \sigma_+; \sigma_--; 2B_1 + s + 1, 2B_3 + s + 1; 1)} \]

\[ P = H_G(z_r, Q_5; \sigma_- - (2B_1 + s), \sigma_+ - (2B_1 + s); 2B_1 + s + 1, 2B_3 + s + 1; 1), \]

\[ Q_5 = -v - z_r(2B_1 + s + 1)(2B_1 + s). \]
Note that we have two formal results, our equations (45,46,49) and (54,55). They are formal solutions, since we do not know if our Heun solution is convergent at $u$ and $z$ equal to unity. We can use the analysis as done by Leaver [21], and conclude that to have a convergent Heun functions in equations (45,46,47) if $z_r > 1$, and if $z_r < 1$ in equations (54,55). Since these two results are incompatible, we choose the first case. Our analysis is correct only for reflection for waves coming from the de Sitter horizon and scatter at the outer horizon.

4 Conclusion

We used the wave equation, obtained, reduced to a manageable form by the authors in [3], and solved for different values of the independent variable in terms of general Heun functions, for the regions between the de Sitter and the outer horizons, and between the outer and inner horizons. They used infinite series expansions of the Heun function. We used the Heun functions directly. We tried to calculate the reflection coefficients for waves for this two regions formally. Unfortunately, we could not get convergent solutions at two regular singular points for both of these regions, by putting constraints on the parameters in the wave equations [21]. We chose to use the constraint, $z_r > 1$. Note that our equations (45,46,47) are consistent equations, and our equations (54,55) may be inconsistent. We may, therefore, dismiss the latter case. The similar problem may be studied in similar metrics [22].

Acknowledgement

We thank Prof. Tolga Birkandan for collaboration in the early part of this work, and for given me our references [6] and [8]. I thank Prof. Reyhan Kaya for technical assistance. This work is morally supported by Science Academy, Istanbul.

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