Prime ends for domains in metric spaces

Tomasz Adamowicz

Department of Mathematics, Linköpings universitet,
SE-581 83 Linköping, Sweden; tadamowi@gmail.com

Anders Björn

Department of Mathematics, Linköpings universitet,
SE-581 83 Linköping, Sweden; anders.bjorn@liu.se

Jana Björn

Department of Mathematics, Linköpings universitet,
SE-581 83 Linköping, Sweden; jana.bjorn@liu.se

Nageswari Shanmugalingam

Department of Mathematical Sciences, University of Cincinnati,
P.O. Box 210025, Cincinnati, OH 45221-0025, U.S.A.; shanmun@uc.edu

April 27, 2012

Abstract. In this paper we propose a new definition of prime ends for domains in metric spaces under rather general assumptions. We compare our prime ends to those of Carathéodory and Näkki. Modulus ends and prime ends, defined by means of the $p$-modulus of curve families, are also discussed and related to the prime ends. We provide characterizations of singleton prime ends and relate them to the notion of accessibility of boundary points, and introduce a topology on the prime end boundary. We also study relations between the prime end boundary and the Mazurkiewicz boundary. Generalizing the notion of John domains, we introduce almost John domains, and we investigate prime ends in the settings of John domains, almost John domains and domains which are finitely connected at the boundary.

Key words and phrases: Accessibility, almost John domain, capacity, doubling measure, end, finitely connected at the boundary, John domain, locally connected, Mazurkiewicz distance, metric space, $p$-modulus, Poincaré inequality, prime end, uniform domain.

Mathematics Subject Classification (2010): Primary: 30D40; Secondary: 30L99, 31B15, 31B25, 31C45, 31E05, 35J66, 54F15, 54F35.

Contents

1. Introduction 2
2. Preliminaries 4
3. Carathéodory ends and prime ends 7
4. Ends and prime ends 8
5. Examples and comparison with Carathéodory’s definition 10
6. Modulus ends and modulus prime ends 12
1. Introduction

The classical Dirichlet boundary value problem associated with a differential operator \( L \) consists in finding a function \( u \) which satisfies the equation \( Lu = 0 \) on a domain \( \Omega \) and the boundary condition \( u = f \) on \( \partial \Omega \) for given boundary data \( f: \partial \Omega \rightarrow \mathbb{R} \).

This problem has been studied extensively for various elliptic differential operators, including the Laplacian \( \Delta \) and its nonlinear counterpart the \( p \)-Laplacian \( \Delta_p \). Perhaps the most general method for solving the Dirichlet problem for these equations is the Perron method introduced independently by Perron \([65]\) and Remak \([66]\), and further refined in the linear case by Wiener and Brelot (and therefore often called the PWB method in the linear case). For the nonlinear case see Heinonen–Kilpeläinen–Martio \([34]\) and the notes therein, and Björn–Björn–Shanmugalingam \([13]\).

For linear operators such as \( \Delta \), this drawback has been earlier addressed on \( \mathbb{R}^n \) and Riemannian manifolds using the Martin boundary, see Martin \([54]\), Ancona \([4]\), \([5]\) and Anderson–Schoen \([7]\). The minimal Martin kernel functions, which compose the Martin boundary of the domain, are analogs of Poisson kernels for more irregular domains and provide us with integral representations for the solutions of the corresponding Dirichlet problem. In the slit disk one can see that there are two distinct minimal Martin kernels corresponding to each point in the slit (except for the tip). Although, as shown by e.g. Holopainen–Shanmugalingam–Tyson \([41]\) and Lewis–Nyström \([51]\), a Martin boundary can be defined even for nonlinear operators such as the \( p \)-Laplacian and its generalizations to metric spaces, we cannot hope to use the Martin boundary as a kernel for solving the corresponding Dirichlet problem in the nonlinear case.

The Dirichlet problem, as posed above with \( f \) defined on the topological boundary \( \partial \Omega \), is in many cases unnecessarily restrictive. For example, in the slit disk (see Example 5.2) one boundary value is prescribed for each point in the slit, even though it would be more natural to have two boundary values at those points (except for the tip), obtained by approaching the slit from either side. On the other hand, in some domains with complicated boundary there may be nontrivial parts of the boundary which are essentially invisible for the solutions and therefore should be treated accordingly in the Dirichlet problem.

For linear operators such as \( \Delta \), this drawback has been earlier addressed on \( \mathbb{R}^n \) and Riemannian manifolds using the Martin boundary, see Martin \([54]\), Ancona \([4]\), \([5]\) and Anderson–Schoen \([7]\). The minimal Martin kernel functions, which compose the Martin boundary of the domain, are analogs of Poisson kernels for more irregular domains and provide us with integral representations for the solutions of the corresponding Dirichlet problem. In the slit disk one can see that there are two distinct minimal Martin kernels corresponding to each point in the slit (except for the tip). Although, as shown by e.g. Holopainen–Shanmugalingam–Tyson \([41]\) and Lewis–Nyström \([51]\), a Martin boundary can be defined even for nonlinear operators such as the \( p \)-Laplacian and its generalizations to metric spaces, we cannot hope to use the Martin boundary as a kernel for solving the corresponding Dirichlet problem in the nonlinear case.

The goal of this paper is to instead develop an alternative notion of boundary, called the prime end boundary, which can give rise to a more comprehensive potential theory suitable for the Perron method and taking the above geometrical concern into account. Prime ends were introduced by Carathéodory \([20]\) in 1913 for simply connected planar domains. His approach is suitable also for finitely connected planar domains, but to be able to treat more general domains satisfactorily we propose a different definition. Even though we work in metric spaces, our results are new also for simply connected planar domains, and our prime ends are different from Carathéodory’s also in this case.

Roughly speaking, a prime end corresponds to a part of the topological boundary which can be reached from the domain in a connected way. In very special domains, such as uniform domains and domains which are locally connected at the boundary, the prime end boundary coincides with the topological boundary, but in more
general domains it need not even satisfy the T2 separation condition. In between, there is a large class of domains for which the prime end boundary behaves well and provides more flexibility in the Dirichlet problem than the topological boundary.

We introduce a natural topology on the prime end boundary, with the aim at solving the Dirichlet problem with respect to the prime end boundary. Such a Dirichlet problem, and the related potential theory, are studied in a companion paper to this one, Björn–Björn–Shanmugalingam [15]. Even the standard Dirichlet problem with respect to the topological boundary benefits from the study of prime ends, see A. Björn [9].

As already mentioned, the notion of prime ends was first proposed by Carathéodory for simply connected planar domains. It was later used successfully by e.g. Beurling [8], Ohtsuka [64], Ahlfors [2], Näkki [60] and Minda–Näkki [58] to study problems related to boundary regularity of conformal and quasiconformal mappings in Euclidean domains. Others who have formulated versions of prime ends include Kaufmann [48], Mazurkiewicz [56], Epstein [26] and Karmazin [46]. More recently, prime ends have been used by e.g. Ancona [6] and Rempe [67] in various settings to study problems related to potential theory and dynamical systems. These studies are set in Euclidean domains or domains in a topological manifold (as in [56]), and generally require that the domain be a simply connected planar domain, or that it is locally connected at the boundary. The literature on prime ends is quite substantial, and we cannot hope to provide an exhaustive list of references here; we recommend the interested reader to also consider papers cited in the above references.

The exposition of this paper is as follows. Section 2 is devoted to the preliminary notions and definitions needed in the paper. Our aim is to develop prime ends in metric spaces under rather general assumptions. We use the standard assumptions that the metric space is complete and equipped with a doubling measure supporting a Poincaré inequality, but often these assumptions can be substantially weakened; this is pointed out at the end of Section 2. Examples of spaces satisfying the standard assumptions mentioned above include various manifolds, Heisenberg and Carnot groups, Carnot–Carathéodory spaces, certain fractals and some closed subsets of \( \mathbb{R}^n \), see Appendix A in Björn–Björn [10].

In Section 3 we give a brief description of Carathéodory’s notion of prime ends, the prototype for most subsequent notions of prime ends, including ours. The discussion in Section 3 provides a framework for our definition of ends and prime ends introduced in Section 4. We illustrate our notion of ends and prime ends through various examples in Section 5, where we also compare our ends and prime ends with Carathéodory’s and discuss some advantages of our approach.

To tie in the nonlinear potential theory, in Section 6 we further refine the notions of ends and prime ends by introducing \( \text{Mod}_p \)-ends and \( \text{Mod}_p \)-prime ends, which through the \( p \)-modulus of curve families take into account the geometry of the domain. This part is somewhat motivated by the works of Ahlfors [2], Beurling [8] and Näkki [60]. We also relate our prime ends to \( p \)-parabolicity and point out some geometrical differences arising from the conformal and nonconformal \( p \)-modulus.

In [22], Collingwood and Lohwater provided classifications of Carathéodory prime ends. For us, the class of singleton prime ends is the most interesting because of its connection to (path)accessibility and finite connectedness at the boundary. Hence in Section 7 we study ends with singleton impressions, provide geometric conditions for an end to be a prime end, and present a simple way of constructing prime ends at accessible boundary points. For these results our modification of Carathéodory’s definition is essential.

The goal of Section 8 is to construct a topology on the collection of (prime) ends, or rather on the (prime) end closure of the domain. Here we also characterize the convergence of a sequence of ends to an end in terms of the Mazurkiewicz distance
Section 9 is devoted to studying the topology on the prime end boundary. We show that the set of singleton prime ends is homeomorphic to the Mazurkiewicz boundary obtained by completing the domain with respect to its Mazurkiewicz distance, and this homeomorphism extends in a natural way to the domain itself.

Section 10 focuses on studying the prime end boundary of domains which are finitely connected at the boundary. We show that in such domains the prime end closure of the domain is metrizable, and that all prime ends have singleton impressions and can be obtained as connected components of small neighborhoods of boundary points.

The final section of this paper, Section 11, focuses on a more special class of domains, namely John domains. For a more extensive theory, we introduce almost John domains, which e.g. makes it possible to include some cusp domains in our discussion. Here we show that such domains are finitely connected at the boundary, have only singleton prime ends, and that each boundary point is accessible and corresponds to the impression of at least one prime end. For certain values of $p$ we also show that prime ends in almost John domains are exactly the $\text{Mod}_p$-ends, which reflects the use of extremal length in the construction of Carathéodory prime ends due to Schlesinger [68]. For the more special class of uniform domains some additional results are obtained.

Auxiliary results related to modulus and capacitary estimates used in this paper are collected in an appendix. Some of them are also of independent interest. Examples are given throughout the paper to illustrate various geometric situations and properties of prime ends. In fact, all the examples given in this paper are Euclidean domains, and indicate the possibilities which can occur even in the Euclidean setting when the domains are not as nice as the ones considered in the works of Ahlfors, Carathéodory and Näkki. The theorems, on the other hand, are formulated under least possible assumptions to emphasize the generality of our theory.

Acknowledgement. This research began in 2008 when T. A. was a postdoc at the University of Cincinnati. Part of the research was done during the visit of A. B. and J. B to the University of Cincinnati in 2010, and during the visit of N. S. to Linköpings universitet in 2011. The authors wish to thank these institutions for their kind hospitality. We also wish to thank David Herron, David Minda and Raimo Näkki for helpful discussions related to this research.

A. B. and J. B. were supported by the Swedish Science Research Council. A. B. was also a Fulbright scholar during his visit to the University of Cincinnati, supported by the Swedish Fulbright Commission, while J. B. was a Visiting Taft Fellow during her visit to the University of Cincinnati, supported by the Charles Phelps Taft Research Center at the University of Cincinnati. N. S. was supported by the Taft Research Center and by the Simons Foundation grant #200474.

A. B. and J. B. belong to the European Science Foundation Networking Programme Harmonic and Complex Analysis and Applications and to the NordForsk network Analysis and Applications.

2. Preliminaries

Let $(X,d,\mu)$ be a metric measure space equipped with a metric $d$ and a measure $\mu$ (and containing more than one point). We will assume that $\mu$ is a Borel measure such that $0 < \mu(B) < \infty$ for all balls $B$ in $X$.

We also let $1 \leq p < \infty$ be fixed. We shall sometimes impose additional assumptions on $p$. Throughout the paper, $\Omega \subset X$ will be a bounded domain in $X$, i.e. a bounded nonempty connected open subset of $X$ that is not $X$ itself.
A wide class of metric measure spaces of current interest, including weighted and unweighted Euclidean spaces, Riemannian manifolds, Heisenberg groups, and other Carnot–Carathéodory spaces, all have locally doubling measures that support a Poincaré inequality locally. Since we are interested in unifying potential theory on all these spaces, we will assume these properties for the metric spaces considered in this paper. Because the domain under consideration is assumed to be bounded, there is no loss of generality in assuming the doubling condition and the Poincaré inequality as global properties, with only simple modifications needed to go from spaces with globally held properties to spaces with locally held properties.

A measure \( \mu \) is said to be **doubling** if there is a constant \( C_\mu > 0 \) such that for all balls \( B = B(x,r) = \{ y \in X : d(x,y) < r \} \),

\[
\mu(2B) \leq C_\mu \mu(B),
\]

where \( \lambda B(x,r) = B(x,\lambda r) \). If \( \mu \) is doubling, then \( X \) is complete if and only if it is proper (i.e. every closed bounded set is compact), see Proposition 3.1 in Björn–Björn [10].

A consequence of the doubling condition is the following lower mass bound. There exist \( C, Q > 0 \) such that for all \( x \in X, 0 < r \leq R \) and \( y \in B(x,R) \),

\[
\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq \frac{1}{C} \left( \frac{r}{R} \right)^Q.
\]

Indeed, \( Q = \log_2 C_\mu \) and \( C = C_\mu^2 \) will do, see Lemma 3.3 in Björn–Björn [10], but there may be a better, i.e. smaller, exponent \( Q \). Note also that (2.1) implies that \( \mu \) is doubling, i.e. \( \mu \) is doubling if and only if there is an exponent \( Q \) such that (2.1) holds.

If \( X \) is also connected then there exist constants \( C > 0 \) and \( q > 0 \) such that for all \( x \in X, 0 < r \leq R \) and \( y \in B(x,R) \),

\[
\frac{\mu(B(y,r))}{\mu(B(x,R))} \leq C \left( \frac{r}{R} \right)^q.
\]

Note that we always have \( 0 < q \leq Q \) and that any \( 0 < q' < q \) and \( Q' > Q \) will do as well.

We say that \( X \) is **Ahlfors \( Q_0 \)-regular** if there is a constant \( C \) such that

\[
\frac{1}{C} r^{Q_0} \leq \mu(B(x,r)) \leq C r^{Q_0}
\]

for all balls \( B(x,r) \subset X \) with \( r < 2 \text{ diam} \ X \). In this case, the best choices for \( q \) and \( Q \) in (2.1) and (2.2) are to let \( q = Q = Q_0 \). We emphasize that in this paper we do not restrict ourselves to Ahlfors regular metric spaces.

Garofalo–Marola [27] introduced the pointwise dimension \( q_0(x) \) (called \( Q(x) \) in [27]) as the supremum of all \( q > 0 \) such that

\[
\frac{\mu(B(x,r))}{\mu(B(x,R))} \leq C_\eta \left( \frac{r}{R} \right)^q.
\]

for some \( C_\eta > 0 \) and all \( 0 < r \leq R \leq \text{ diam} \ X \). Since the analysis considered in this paper is local, we do not need the above inequality for all \( R > 0 \).

**Definition 2.1.** Given \( x \in X \) we consider the pointwise dimension set \( Q(x) \) of all possible \( q > 0 \) for which there are constants \( C_\eta > 0 \) and \( R_q > 0 \) such that (2.3) holds for all \( 0 < r \leq R \leq R_q \).

Observe that \( Q(x) \) is a bounded interval. Indeed, \( Q(x) = (0,q_0] \) or \( Q(x) = (0,q_0) \) for some nonnegative number \( q_0 \), as in the following examples.
Example 2.2. Let $X = \mathbb{R}^n$ with the measure $d\mu = w \, dx$, where

$$w(x) = \max\{1, \log(1/|x|)\}.$$ 

Then for all $0 < r \leq 1/e$ we have

$$\mu(B(0, r)) = \alpha_n r^n \left(\frac{1}{n} + \log \frac{1}{r}\right),$$

where $\alpha_n$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. It follows that for $x = 0$, (2.3) holds for all $q < n$ but not for $q = n$, i.e. $Q(0) = (0, n)$. On the other hand, $Q(x) = (0, n)$ for $x \neq 0$.

Example 2.3. Let $X = \mathbb{R}^n$ be equipped with the doubling measure $d\mu(x) = |x|^\alpha \, dx$ for some fixed $\alpha > -n$. Then $\mu(B(0, r))$ is comparable to $r^{n+\alpha}$, while for $x \neq 0$, $\mu(B(x, r))$ is comparable to $r^n$ with comparison constant depending on $|x|$. Thus $Q(0) = (0, n + \alpha)$ whereas $Q(x) = (0, n)$ if $x \neq 0$. It follows that (2.1) and (2.2) hold with $q = \min\{n, n + \alpha\}$ and $Q = \max\{n, n + \alpha\}$. Note that for $\alpha$ close to $-n$ we have $q$ close to 0.

Note that $q \leq q_0 \leq Q$, where $q$ and $Q$ are as in (2.1) and (2.2). If the measure $\mu$ is Ahlfors $Q_0$-regular, then $q_0 = Q_0$ and $Q(x) = (0, Q_0]$ for all $x$. The pointwise dimension $Q(x)$ will appear in some of our results in connection with the capacity and the modulus of curve families.

Definition 2.4. A nonnegative Borel function $g$ on $X$ is an upper gradient of an extended real-valued function $u : X \rightarrow [-\infty, \infty]$ if for all nonconstant rectifiable curves $\gamma : [0, l_\gamma] \rightarrow X$, parameterized by arc length $ds$, we have

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_{\gamma} g \, ds$$

whenever both $u(\gamma(0))$ and $u(\gamma(l_\gamma))$ are finite, and $\int_{\gamma} g \, ds = \infty$ otherwise. If $g$ is a nonnegative measurable function (not necessarily Borel) on $X$ and if (2.4) holds for $p$-a.e. nonconstant rectifiable curve, then $g$ is a $p$-weak upper gradient of $u$.

By saying that a property holds for $p$-a.e. rectifiable curve, we mean that it fails only for a curve family $\Gamma$ with zero $p$-modulus, i.e that there is a Borel function $\rho \in L^p(X)$ such that $\int_\gamma \rho \, ds = \infty$ for all $\gamma \in \Gamma$. This is consistent with the definition of $p$-modulus in (6.1) below. Since the $p$-weak upper gradient $g$ can be modified on a set of measure zero to obtain a Borel function, it can be shown that $\int_{\gamma} g \, ds$ is defined (with a value in $[0, \infty]$) for $p$-a.e. rectifiable curve.

Here and throughout the paper we require curves to be nonconstant, unless otherwise stated explicitly.

Upper gradients were introduced by Heinonen and Koskela [35], [36] (where they were called very weak gradients), whereas $p$-weak upper gradients were first defined in Koskela–MacManus [49]. In [49] it was also shown that if $g \in L^p(X)$ is a $p$-weak upper gradient of $u$, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of $u$ such that $g_j \rightarrow g$ in $L^p(X)$. If $u$ has an upper gradient in $L^p(X)$, then it has a minimal $p$-weak upper gradient $g_u \in L^p(X)$ in the sense that for every $p$-weak upper gradient $g \in L^p(X)$ of $u$, $g_u \leq g$ a.e., see Corollary 3.7 in Shanmugalingam [70] and Theorem 7.16 in Hajłasz [30].

Next we define a version of Sobolev spaces on the metric space $X$ due to Shanmugalingam [69].
Definition 2.5. Whenever $u : X \to [-\infty, \infty]$ is a measurable function, let

$$
\|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},
$$

where the infimum is taken over all upper gradients $g$ of $u$. The Newtonian space on $X$ is

$$
N^{1,p}(X) = \{ u : \|u\|_{N^{1,p}(X)} < \infty \}/\sim,
$$

where two functions $u$ and $v$ are said to be equivalent, denoted $u \sim v$, if $\|u - v\|_{N^{1,p}(X)} = 0$.

We say that $X$ supports a $p$-Poincaré inequality if there exist constants $C_{PI} > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$ and all $u \in N^{1,p}(X)$,

$$
\frac{\int_B |u - u_B| \, d\mu}{\|u_B\|_{N^{1,p}(X)}} \leq C_{PI} (\text{diam } B) \left( \frac{\int_B g_{u_B}^p \, d\mu}{\int_{AB} g_{\mu_{AB}}^p \, d\mu} \right)^{1/p},
$$

(2.5)

where $u_B := \frac{1}{|B|} \int_B u \, d\mu := \mu(B)^{-1} \int_B u \, d\mu$. Such Poincaré inequalities are sometimes called weak since we allow for $\lambda > 1$.

From now on we assume that the space $X$ is complete and supports a $p$-Poincaré inequality, and that the measure $\mu$ is doubling.

A consequence of the above standing assumptions is that $X$ is $L$-quasiconvex, i.e. for every $x, y \in X$ there is a rectifiable curve with length at most $Ld(x, y)$ connecting $x$ and $y$, where $L$ only depends on the doubling constant and the constants in the $p$-Poincaré inequality. This result is due to Semmes, see Theorem 17.1 in Cheeger [21], Proposition 4.4 in Hajłasz–Koskela [32] or Theorem 4.32 in Björn–Björn [10] for a proof. Theorem 4.32 in [10] contains an explicit estimate for $L$.

A direct consequence of the quasiconvexity is that $X$ is locally connected. Many of the results in this paper hold under the weaker assumption that $X$ is a locally connected proper metric space. This is true for all the results in Sections 4–10, except for the results concerning $\text{Mod}_{p-}(\text{prime})$-ends. The results in Section 11, apart from the $\text{Mod}_{p-}$-results, hold under the assumptions that $X$ is a quasiconvex proper metric space and $\mu$ is doubling.

Remark 2.6. Recall that $X$ is locally (path)connected if every neighborhood of a point $x \in X$ contains a (path)connected neighborhood. If $X$ is a locally connected proper metric space, then $X$ is locally pathconnected by the Mazurkiewicz–Moore–Menger theorem, see Theorem 1, p. 254, in Kuratowski [50]. In particular every component of an open set is open and pathconnected, see Theorem 2, p. 253, in [50].

3. Carathéodory ends and prime ends

In this section we give a brief overview of Carathéodory’s definitions of ends and prime ends from his 1913 paper [20] for simply connected planar domains.

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. A cross-cut of $\Omega$ is a Jordan arc in $\Omega$ with endpoints on the boundary of $\Omega$. A sequence $\{c_k\}_{k=1}^\infty$ of cross-cuts is called a chain if for every $k$, (1) $\overline{c_k} \cap \overline{c_{k+1}} = \emptyset$, and (2) every cross-cut $c_k$ separates $\Omega$ into exactly two subdomains, one containing $c_{k-1}$ and the other containing $c_{k+1}$; let $D_k$ be the latter subdomain. The impression of the chain is $\bigcap_{k=1}^\infty D_k$, which is a nonempty connected compact set.

Carathéodory then defined the concept of division of a chain by another chain and says that two chains are equivalent if they divide each other. This leads to an
equivalence relation for which the equivalence classes are called ends. The impression is independent of the representative chain of an equivalence class.

The ends are naturally partially ordered by division, and a prime end is an end divisible only by itself, or in other terms, is minimal with respect to the partial ordering. The impression of a prime end is always a subset of \( \partial \Omega \), see Theorem 9.2 in Collingwood–Lohwater [22].

Later it was realised that if one imposes an extremal length condition on the chains then the corresponding ends are automatically minimal, and they are therefore called prime ends. When this approach is used there are no ends (other than prime ends) and no need for weeding out ends which are not prime ends. This approach leads to the same class of prime ends as in Carathéodory’s approach. According to our investigations, the first use of extremal length in connection with prime ends is due to Schlesinger [68].

One of the main motivations for Carathéodory was the (nowadays) well-known correspondence between points on the unit circle and prime ends of the image \( \Phi(D) \) of a conformal mapping \( \Phi : D \rightarrow \mathbb{C} \), where \( D \) is the unit disk in the complex plane. This is one reason why prime ends are important tools in various situations, and the theory works very well for simply (and finitely) connected planar domains. For infinitely connected domains, and for general domains in higher dimensions, the theory does not work quite so well, cf. Kaufmann [48], Mazurkiewicz [56] and Nakkı [60]. However, when restricted to certain domains it has proved useful also in higher dimensions, see Nakkı [59], Ohtsuka [64] and Karmazin [46], [47].

We want to study prime ends in quite general metric spaces, with a view towards a theory that lends itself to the study of Dirichlet problems. We therefore give two approaches. In the first one we start by defining ends and then say that the prime ends are the ends which are minimal (with respect to the partial order). In the other approach we require the ends initially to satisfy a \( p \)-modulus condition, and to distinguish these ends from the earlier ones we call them \( \text{Mod}_p \)-ends. Here we have a choice of \( p \geq 1 \), leading us to different notions. Since extremal length is connected with the 2-modulus in \( \mathbb{R}^2 \), the \( p \)-modulus condition seems to be a natural generalization to consider. In our generality \( \text{Mod}_p \)-ends need not be minimal, and we therefore also introduce \( \text{Mod}_p \)-prime ends.

4. Ends and prime ends

We are now ready to give our definition of ends and prime ends.

**Definition 4.1.** A bounded connected set \( E \subsetneq \Omega \) is an acceptable set if \( E \cap \partial \Omega \) is nonempty.

Since an acceptable set \( E \) is bounded and connected, we know that \( \overline{E} \) is compact and connected. Moreover, \( E \) is infinite, as otherwise we would have \( \overline{E} = E \subset \Omega \). Therefore, \( \overline{E} \) is a continuum. Recall that a continuum is a connected compact set containing more than one point.

**Definition 4.2.** A sequence \( \{E_k\}_{k=1}^{\infty} \) of acceptable sets is a chain if

(a) \( E_{k+1} \subset E_k \) for all \( k = 1, 2, \ldots \);

(b) \( \text{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) > 0 \) for all \( k = 1, 2, \ldots \);

(c) The impression \( \bigcap_{k=1}^{\infty} E_k \subset \partial \Omega \).

**Remark 4.3.** As \( \{E_k\}_{k=1}^{\infty} \) is a decreasing sequence of continua, the impression is either a point or a continuum. Moreover, (a) and (b) above imply that \( E_{k+1} \subset \text{int} E_k \). In particular, \( \text{int} E_k \neq \emptyset \).
Definition 4.4. We say that a chain \( \{E_k\}_{k=1}^\infty \) \textit{divides} the chain \( \{F_k\}_{k=1}^\infty \) if for each \( k \) there exists \( l \) such that \( E_l \subset F_k \). (Here we implicitly assume that \( k \) and \( l \) are positive integers. We make similar implicit assumptions throughout the paper to enhance readability.) Two chains are \textit{equivalent} if they divide each other. A collection of all mutually equivalent chains is called an \textit{end} and denoted \( [E_k] \), where \( \{E_k\}_{k=1}^\infty \) is any of the chains in the equivalence class. The \textit{impression} of \( [E_k] \), denoted \( I[E_k] \), is defined as the impression of any representative chain.

The collection of all ends is called the \textit{end boundary} and is denoted \( \partial E \Omega \).

Note that the impression of an end is independent of the choice of representative chain. Indeed, if \( \{E_k\}_{k=1}^\infty \) divides \( \{F_k\}_{k=1}^\infty \) then \( I[E_k] \subset I[F_k] \), and the opposite inclusion holds similarly if the two chains are equivalent. Hence the above definition of impression of an end makes sense.

Note also that if a chain \( \{F_k\}_{k=1}^\infty \) divides \( \{E_k\}_{k=1}^\infty \), then it divides every chain equivalent to \( \{E_k\}_{k=1}^\infty \). Furthermore, if \( \{F_k\}_{k=1}^\infty \) divides \( \{E_k\}_{k=1}^\infty \), then every chain equivalent to \( \{F_k\}_{k=1}^\infty \) also divides \( \{E_k\}_{k=1}^\infty \). Therefore, the relation of division extends in a natural way from chains to ends, giving a partial order on ends.

Remark 4.5. Let \( \{E_k\}_{k=1}^\infty \) be a chain. By Remark 4.3, \( E_{k+1} \subset \text{int} \ E_k \). Unfortunately, \( \text{int} \ E_k \) is not necessarily connected. However, let \( G_k \) be the component of \( \text{int} \ E_k \) containing \( E_{k+1} \). Since \( X \) is locally connected, we see that \( G_k \) is open, and hence is an acceptable set. As \( \partial G_k \subset \partial \text{int} \ E_k \subset \partial E_k \), we get that \( \{G_k\}_{k=1}^\infty \) is a chain. Since \( E_{k+1} \subset G_k \) for each \( k \), we see that \( \{G_k\}_{k=1}^\infty \) is divisible by \( \{E_k\}_{k=1}^\infty \). On the other hand, \( \{G_k\}_{k=1}^\infty \) clearly divides \( \{E_k\}_{k=1}^\infty \), and thus they are equivalent and \( \{G_k\} = [E_k] \). As a consequence, we could have required that acceptable sets are open to obtain an equivalent definition of ends. This observation is used in the proofs of Propositions 7.10, 8.5 and Theorem 11.9. On the other hand, with our definition we have a bit more freedom when constructing examples.

Let us next show that there is a certain redundancy in the collection of ends.

Example 4.6. Let \( \Omega = (0,1)^2 \) be the unit square in the plane and let \( E_k = (0,1) \times (0,1/k) \) and \( F_k = \Omega \cap B((\frac{1}{2},0),1/2k) \), \( k = 1,2,\ldots \). Then the chain \( \{E_k\}_{k=1}^\infty \) is divisible by the chain \( \{F_k\}_{k=1}^\infty \), but \( \{F_k\}_{k=1}^\infty \) is not divisible by \( \{E_k\}_{k=1}^\infty \).

Such a redundancy might not cause a problem in some applications (see e.g. Miklyukov [57], where the analogues of acceptable sets are not even required to be connected), but since one of our aims is to use the notion of ends to construct a more general boundary of a domain, such a redundancy creates a difficulty in using the collection of all ends as a boundary. To overcome this type of redundancy, we consider the minimal ends in the following sense.

Definition 4.7. An end \( [E_k] \) is a \textit{prime end} if it is not divisible by any other end. The collection of all prime ends is called the \textit{prime end boundary} and is denoted \( \partial_p \Omega \).

The following is a natural problem about existence of prime ends. We have only been able to solve it in a special case, see Proposition 10.7. The difficulty in the more general setting is that we do not know if a totally ordered uncountable collection of ends (ordered by division) has a maximal end, and hence we are unable to use Zorn’s lemma. (Zorn’s lemma was actually first stated by Kuratowski, see the discussion on p. 30 in Bourbaki [18].)

Open problem 4.8. Is it true that every end is divisible by some prime end?
5.1 Examples and comparison with Carathéodory’s definition

We shall see later that in certain domains, every boundary point corresponds to at least one prime end. However, the following example illustrates that in some situations one may need to also consider ends which are not prime ends.

Example 5.1. (The topologist’s comb, see Figure 1.) Let $\Omega$ be the unit square $(0,1)^2 \subset \mathbb{R}^2$ with the segments $S_k = [\frac{1}{2}, 1] \times \{2^{-k}\}$ removed for each $k$. Let $x_0 = (\frac{1}{2}, 0)$ and let $I = \{0, 1\} \times \{0\}$ be the set of inaccessible points, see Definition 7.6. Then the sets

$$E_k = \left(\left\{\frac{1}{2} - 2^{-k}, 1\right\} \times \{2^{-k}\}\right) \cap \Omega, \quad k = 1, 2, \ldots,$$

define an end with the impression $I \cup \{x_0\}$. However, this is not a prime end, as it is divisible by the chain $\left\{B(x_0, 2^{-k}) \cap \Omega\right\}_{k=1}^{\infty}$, which defines a prime end with impression $\{x_0\}$. Note that there is no prime end with impression containing a point from $I$. We point out here that by Proposition 7.5 the prime ends of this domain are also $\text{Mod}_p$-prime ends for $1 \leq p \leq 2$, in the sense of Definition 6.1.

Note that with Carathéodory’s prime ends the situation is different. In this case $\{x_0\}$ is not the impression of any Carathéodory prime end, but instead $I \cup \{x_0\}$ is the impression of a Carathéodory prime end.

The following example is a major motivation for us.

Example 5.2. Let $\Omega$ be the slit disk $B((0,0), 1) \setminus \{(-1,0) \times \{0\}\} \subset \mathbb{R}^2$. Then for each $x \in [-1,0] \times \{0\}$ there are two prime ends with the impression $\{x\}$ (one from
the upper half-plane and one from the lower half-plane). For \( x \in \partial \Omega \setminus \{(-1, 0) \times \{0\}\} \) there is exactly one prime end with the impression \( \{x\} \). These are the only prime ends associated with \( \Omega \). In this example the Carathéodory prime ends are the same as our prime ends.

**Proposition 5.3.** Let \( \Omega \) be a bounded simply connected domain in the plane. If \( \{c_k\}_{k=1}^\infty \) is a Carathéodory end with impression in the boundary, in particular if it is a Carathéodory prime end, then \( [D_k] \) is an end in our sense, where \( D_k \) are defined as in Section 3.

This follows directly from the definitions and the fact that the impression of a Carathéodory prime end is always a subset of the boundary, see Section 3. (Of course, if \( \{c_k\}_{k=1}^\infty \) is a Carathéodory end with impression containing some point in \( \Omega \), then \( [D_k] \) is not an end in our sense.)

Observe that not every Carathéodory prime end is a prime end in our sense, see Example 5.1. This is due to the fact that we have more ends in some cases, which again depends on the fact that we only require that an acceptable set \( E \) is connected, not that its relative boundary \( \Omega \cap \partial E \) is connected.

That we do not recover Carathéodory’s prime ends in the simply connected planar case is a drawback in our theory, and in some situations our theory is not as useful as Carathéodory’s. On the other hand, it is well known that Carathéodory’s theory is limited to simply and finitely connected planar domains. We will see in Section 7 that there is a close connection between our prime ends and accessibility of boundary points, a connection lost with Carathéodory’s definition as shown by Example 5.1. \( x_0 \) is an accessible point but there is no Carathéodory prime end with impression equal to \( \{x_0\} \). This connection is crucial for our results in Sections 9-11.

We now give one more example showing that Carathéodory’s prime ends and our prime ends need not be the same in general.

**Example 5.4.** (Double equilateral comb, see Figure 1.) Let \( \Omega \subset \mathbb{R}^2 \) be the domain obtained from the unit square \((0, 1)^2\) by removing the collection of segments \((0, \frac{1}{2^n}) \times \{2^n\} \) and \((\frac{1}{2^n}, 1) \times \{3 \cdot 2^{n-2}\} \) for \( n = 1, 2, \ldots \). From the point of view of Carathéodory’s theory introduced in the beginning of Section 3 we take the chain of cross-cuts

\[
c_k = (0, \frac{1}{2}) \times \{3 \cdot 2^{-k-2}\} \quad \text{for } k = 1, 2, \ldots,
\]

which gives a Carathéodory prime end with impression \([0, 1] \times \{0\}\).

To obtain a prime end in our sense we define the acceptable sets

\[
E_k = \Omega \cap \left(\left(\frac{1}{2} - 2^{-k-2}, \frac{3}{2} + 2^{-k-2}\right) \times (0, 2^{-k})\right) \quad \text{for } k = 1, 2, \ldots.
\]

Then \( [E_k] \) is a prime end with impression \( I[E_k] = [\frac{1}{2}, \frac{3}{2}] \times \{0\} \). Thus the Carathéodory prime end above is not a prime end in our sense, as it is divisible by \([E_k]\).

We point out here that \([E_k]\) is also a Modp-prime end for all \( p \geq 1 \), in the sense of Definition 6.1.

**Remark 5.5.** Our definition of prime ends differs from earlier definitions. The boundaries of our acceptable sets correspond to Carathéodory’s cross-cuts, and our acceptable sets correspond to the components \( D_k \) in Carathéodory’s definition. The following are the main differences between our definition and Carathéodory’s definition:

(a) Carathéodory’s cross-cuts are connected, while the boundaries of acceptable sets need not be.

(b) Cross-cuts break the domain into exactly two components, whereas the boundaries of acceptable sets break the underlying domain into at least two components.
There are several reasons for these differences. First, the topology of a metric space is more complicated than that of $\mathbb{R}^n$. (The reader could think of $\mathbb{R}^n$ with a number of holes removed as a particular example of a metric space under consideration.) But even in the simply connected planar case it would have been more restrictive had we required the boundaries of acceptable sets to be connected, as in Carathéodory’s definition, see Example 5.1. In more complicated geometries the difference is even larger, as e.g. not even boundaries of balls need to be connected.

Our modification of the definition of ends and prime ends is essential for many of the results in this paper. In particular, in Section 7 we obtain a close connection between singleton prime ends and accessibility, a connection which fails for Carathéodory ends, as is again demonstrated by Example 5.1.

In Remark 6.4 we discuss Näkki’s definition of prime ends on $\mathbb{R}^n$ from [60]. Näkki, following Carathéodory, requires cross-sets to be connected, and so his definition has the same drawback as Carathéodory’s in connection with the results in this paper, the main difference being again (a) and (b).

Among the many definitions of prime ends given by Karmazin [46], [47] is probably the one closest to ours. Karmazin however studies only prime ends on $\mathbb{R}^n$ and with different applications of the theory than ours.

6. Modulus ends and modulus prime ends

The notion of ends and prime ends discussed in the previous sections does not take into account the potential theory associated with the domain. Using the following notion of $p$-modulus, in this section we give a subclass of ends and prime ends associated with the potential theory. Here, $1 \leq p < \infty$ is fixed.

Let $\Gamma$ be a family of (nonconstant) rectifiable curves in $X$. The $p$-modulus of the family $\Gamma$ is

$$\text{Mod}_p(\Gamma) := \inf_{\rho} \int_X \rho^p \, d\mu,$$

(6.1)

where the infimum is taken over all nonnegative Borel functions $\rho$ on $X$ such that $\int_X \rho \, ds \geq 1$ for every $\gamma \in \Gamma$ and $ds$ denotes the arc length measure. (As usual $\inf \emptyset := \infty$.) It is straightforward to verify that $\text{Mod}_p$ is an outer measure on the collection of all rectifiable curves on $X$. In particular, if $\Gamma_1 \subset \Gamma_2$ are two families of (nonconstant) rectifiable curves in $X$ such that $\Gamma_1 \subset \Gamma_2$, then $\text{Mod}_p(\Gamma_1) \leq \text{Mod}_p(\Gamma_2)$. This monotonicity property will be useful in this paper. For more on $p$-modulus we refer the interested reader to Heinonen [33] and Väisälä [71].

The $n$-modulus in $\mathbb{R}^n$ can be used to define and investigate extremal length as well as (quasi)conformal and quasiregular maps. Further applications of the $p$-modulus include its relation to capacities and Loewner spaces. See Väisälä [71], Heinonen–Koskela [36], Kallunki–Shanmugalingam [44], Vuorinen [73], and Lemma A.1.

For nonempty sets $U \subset X$, $E \subset U$ and $F \subset U$, we let $\Gamma(E,F,U)$ denote the family of all (nonconstant) rectifiable curves $\gamma : [0,l_\gamma] \to U$ such that $\gamma(0) \in E$ and $\gamma(l_\gamma) \in F$. As in [71], the modulus of the curve family $\Gamma(E,F,U)$ is

$$\text{Mod}_p(E,F,U) := \text{Mod}_p(\Gamma(E,F,U)).$$

Definition 6.1. A chain $\{E_k\}_{k=1}^\infty$ (see Definition 4.2) is a $\text{Mod}_p$-chain if

$$\lim_{k \to \infty} \text{Mod}_p(E_k, K, \Omega) = 0$$

(6.2)

for every compact set $K \subset \Omega$. An end $[E_k]$ is a $\text{Mod}_p$-end if there is a $\text{Mod}_p$-chain representing it. A $\text{Mod}_p$-end $[E_k]$ is a $\text{Mod}_p$-prime end if the only $\text{Mod}_p$-end dividing it is $[E_k]$ itself.
For $p > 1$ it is equivalent to assume that (6.2) holds for some compact $K \subset \Omega$ with positive capacity, see Lemma A.11. We do not know if this is true for $p = 1$.

**Lemma 6.2.**
(a) A chain dividing a Mod$_p$-chain is also a Mod$_p$-chain.
(b) Any chain representing a Mod$_p$-end is a Mod$_p$-chain.
(c) An end dividing a Mod$_p$-end is also a Mod$_p$-end.
(d) A Mod$_p$-end is a prime end if and only if it is a Mod$_p$-prime end.

**Proof.**
(a) Let $\{F_k\}_{k=1}^\infty$ be a chain dividing the Mod$_p$-chain $\{E_k\}_{k=1}^\infty$. Then for each $k$ there exists $n_k$ such that $F_{n_k} \subset E_k$. The monotonicity of Mod$_p$ then implies that for every compact $K \subset \Omega$,

$$\lim_{k \to \infty} \text{Mod}_p(F_k, K, \Omega) = \lim_{k \to \infty} \text{Mod}_p(F_{n_k}, K, \Omega) \leq \lim_{k \to \infty} \text{Mod}_p(E_k, K, \Omega) = 0,$$

and hence $\{F_k\}_{k=1}^\infty$ is a Mod$_p$-chain.

(b) Let $\{E_k\}_{k=1}^\infty$ be a chain representing a Mod$_p$-end $[E_k]$. By definition there is a Mod$_p$-chain $\{F_k\}_{k=1}^\infty$ representing the end $[E_k]$. As $\{E_k\}_{k=1}^\infty$ and $\{F_k\}_{k=1}^\infty$ are equivalent chains, it follows from (a) that $\{E_k\}_{k=1}^\infty$ is also a Mod$_p$-chain.

(c) This also follows from (a).

(d) Let $[E_k]$ be a Mod$_p$-end. If $[E_k]$ is also a prime end, then there is no other end dividing it, let alone any other Mod$_p$-end dividing it. Thus $[E_k]$ must be a Mod$_p$-prime end. Conversely, if $[E_k]$ is a Mod$_p$-prime end and $[F_k]$ is an end dividing $[E_k]$, then (c) shows that $[F_k]$ is a Mod$_p$-end. Hence $[F_k] = [E_k]$, and thus $[E_k]$ is a prime end.

**Example 6.3.** Let $\Omega$ be the unit ball in $\mathbb{R}^n$, $n \geq 3$, with a radius removed. Then for every boundary point $x \in \partial \Omega$ there is a prime end $[F_k^x]$ with $\{x\}$ as its impression, see Corollary 7.8.

Let $I$ be a closed subsegment of the removed radius and let

$$E_k = \{x \in \Omega : \text{dist}(x, I) < 2^{-k}\}.$$

Then $[E_k]$ is an end with $I$ as its impression. This is not a prime end as it is divisible by $[F_k^x]$ for every $x \in I$. If $p \leq n - 1$, then Mod$_p(E_k, K, \Omega) \to 0$ as $k \to \infty$, and thus $[E_k]$ is a Mod$_p$-end but not a Mod$_p$-prime end.

Under some conditions, see e.g. Section 11, all Mod$_p$-ends are Mod$_p$-prime ends, and in this case one does not need to do the further subdivision.

Recall that if $\{E_k\}_{k=1}^\infty$ is a chain, then $\{\overline{E_k}\}_{k=1}^\infty$ is a decreasing sequence of continua, and so the impression is either a point or a continuum. Lemmas A.10 and A.11 imply that if a sequence $\{E_k\}_{k=1}^\infty$ of open acceptable sets satisfies the conditions (a) and (b) of Definition 4.2 and $\lim_{k \to \infty} \text{Mod}_p(E_k, K, \Omega) = 0$ for some compact set $K \subset \Omega$ with positive $p$-capacity, $p > Q - 1$, then $\{E_k\}_{k=1}^\infty$ satisfies all the conditions of Definition 4.2, and is therefore a chain. Thus, in view of Remark 4.5, Mod$_p$-ends could be equivalently defined using only (a) and (b) in Definition 4.2 together with (6.2), when $p > Q - 1$.

The notion of Mod$_p$-prime end is similar to the concept of $p$-parabolic prime ends discussed in Miklyukov [57] and Karmazin [45]. The name $p$-parabolicity has been used in the literature to denote spaces where there is not enough room out at infinity in the sense that the collection of all curves that start from a fixed ball and leave every compact subset of the space has $p$-modulus zero. A prime end is a Mod$_p$-prime end if there is insufficient room close to the impression of the prime end. In this sense one could think of Mod$_p$-prime ends as $p$-parabolic ends of the domain. See [23], [24], [28], [29], [38], [39], [40], [43], [52], [53] and [63] for some applications of parabolic ends.
The condition \( \lim_{k \to \infty} \text{Mod}_p(E_k, K, \Omega) = 0 \) depends heavily on \( p \). For example, if \( p > Q \), where \( Q \) is from (2.1), then the collection of all curves in \( X \) passing through \( x \) has positive \( p \)-modulus. For Ahlfors \( Q \)-regular \( X \) this follows from Theorem 4.3 in Adamowicz-Shanmugalingam [1], and the proof therein holds also in our case. Hence in general there are no \( \text{Mod}_p \)-chains with \( x \) in their impressions. However, it can happen that for some \( x \in \partial \Omega \), and every compact \( K \subset \Omega \) we have

\[
\lim_{r \to 0} \text{Mod}_p(B(x, r) \cap \Omega, K, \Omega) = 0, \tag{6.3}
\]

even if \( p > Q \). This is the case e.g. if \( \Omega \subset \mathbb{R}^n \) (unweighted) has an outward polynomial cusp at \( x \) of degree \( m \) and \( p \leq m + n - 1 \), see Durand-Cartagena–Shanmugalingam–Williams [25, Example 2.2].

**Remark 6.4.** Based on the notion of \( n \)-modulus, Näkki [60] introduced another variant of prime ends in \( \mathbb{R}^n \): A connected subset \( A \) of a domain \( \Omega \subset \mathbb{R}^n \) is called a *cross-set* if (1) it is relatively closed in \( \Omega \), (2) \( \overline{A} \cap \partial \Omega \neq \emptyset \), and (3) \( \Omega \setminus A \) consists of two components whose boundaries intersect \( \partial \Omega \). A collection \( \{A_k\}_{k=1}^\infty \) of cross-sets is called a *Näkki chain* if \( A_k \) separates \( A_{k-1} \) and \( A_{k+1} \) (within \( \Omega \)) for all \( k \). A Näkki chain is a *Näkki prime chain* if (a') \( \text{Mod}_n(A_{k+1}, A_k, \Omega) < \infty \), and (b') for any continuum \( K \subset \Omega \) we have that \( \lim_{k \to \infty} \text{Mod}_n(A_k, K, \Omega) = 0 \). The equivalence classes with respect to division of Näkki prime chains define Näkki prime ends.

If \( A_k \) is a cross-set, then the component of \( \Omega \setminus A_k \) containing \( A_{k+1} \) is an acceptable set in our sense. Denote this component by \( E_k \).

In the domains \( \Omega \subset \mathbb{R}^n \) considered by Näkki (so-called quasiconformally collared domains), condition (a') is equivalent to \( \text{dist}(A_{k+1}, A_k) > 0 \), by Lemma 2.3 in [60]. Similarly, for such domains with \( p = n \), condition (b) of Definition 4.2 is equivalent to

\[
\text{Mod}_p(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k, \Omega) < \infty.
\]

The same holds if \( p = Q \) and \( \Omega \) is Ahlfors \( Q \)-regular and Loewner, by (3.9) in Heinonen–Koskela [36]. (For definition and discussion of Loewner spaces see [36] and Heinonen [33].) However, \( \Omega \) is in general not Ahlfors \( Q \)-regular and Loewner, even if \( X \) happens to be.

In the nonconformal case \( p \neq Q \), Example 2.7 in Adamowicz–Shanmugalingam [1] and Example 6.5 below show that the corresponding equivalence can fail even in nice domains, and in more general metric spaces there is usually no value of \( p \) for which it is true. We therefore explicitly require that chains \( \{E_k\}_{k=1}^\infty \) satisfy

\[
\delta_k := \text{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) > 0.
\]

This modification automatically implies that \( \text{Mod}_p(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k, \Omega) < \infty \), since the function \( p = \chi_\Omega/\delta_k \) is admissible in the definition of \( \text{Mod}_p(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k, \Omega) \).

**Example 6.5.** Let \( \Omega = B((0, 0), 2) \subset \mathbb{R}^2 \), \( E = [-1, 0] \times \{0\} \) and \( F = [0, 1] \times \{0\} \). If \( 1 \leq p < 2 \), then \( \text{Mod}_p(E, F, \Omega) < \infty \) even though \( \text{dist}(E, F) = 0 \), as we shall next see. In fact \( E \cap F \neq \emptyset \).

Let \( \Gamma_0 \) be the family of (nonconstant) rectifiable curves in \( \Omega \) passing through the origin. Since singletons have zero \( p \)-capacity in \( \mathbb{R}^2 \), we have \( \text{Mod}_p(\Gamma_0) = 0 \). We shall therefore in this example only consider curves which do not pass through the origin. Let \( \gamma : [0, l_\gamma] \to \Omega \) be such a rectifiable curve connecting \( E \) to \( F \) in \( \Omega \). Joining \( \gamma \) with its reflection in the real axis makes a closed curve \( \tilde{\gamma} \) in \( \Omega \) around the origin. The residue theorem now yields that

\[
\int_{\tilde{\gamma}} \frac{dz}{z} = 2\pi i n,
\]
when $\tilde{\gamma}$ is positively oriented and $n \geq 1$ is an integer. Using symmetry we obtain that

$$n\pi = \frac{1}{2} \left| \int_{\tilde{\gamma}} \frac{dz}{z} \right| = \left| \int_{\gamma} \frac{dz}{z} \right| \leq \int_{\gamma} \frac{|dz|}{|z|} = \int_{\gamma} \frac{ds}{|\gamma(s)|},$$

where $ds$ denotes the arc length measure. It follows that the function $\rho(z) = 1/|z|$ is admissible in the definition of $\text{Mod}_p(E, F, \Omega)$ and hence

$$\text{Mod}_p(E, F, \Omega) \leq \int_{\Omega} \rho^p \, dx \, dy = 2\pi^{1-p} \int_{0}^{2} r^{1-p} \, dr = \frac{2^{3-p} \pi^{1-p}}{2 - p} < \infty.$$

## 7. Singleton ends and accessibility

It is useful to have criteria identifying ends which are prime ends and $\text{Mod}_p$-prime ends. Ends are naturally divided into two classes, those with singleton impressions and those with larger (continuum) impressions. The former are simpler to handle, and the main focus in the later sections will be on singleton ends. A **singleton end** is an end with a singleton impression.

The classification of ends is a classical topic considered initially by Carathéodory [20]. See Sections 9.7 and 9.8 in Collingwood–Lohwater [22] for an extensive classification of prime ends for simply connected planar domains. For us it is enough to distinguish between singleton ends and non-singleton ends.

**Proposition 7.1.** If an end has a singleton impression, then it is a prime end.

Note, however, that there are prime ends with non-singleton impressions, see Example 5.4. Proposition 7.1 follows directly from the following two lemmas.

**Lemma 7.2.** Let $[E_k]$ be an end. Then

$$\text{diam} I[E_k] = \lim_{k \to \infty} \text{diam} E_k.$$

In particular, $[E_k]$ is a singleton end if and only if $\text{diam} E_k \to 0$ as $k \to \infty$.

**Proof.** Since $E_{k+1} \subset E_k$, it is clear that the limit on the right-hand side exists. As $I[E_k] = \bigcap_{k=1}^{\infty} E_k$ and $\text{diam} E_k = \text{diam} E_k$, one inequality is obvious.

For the converse inequality, let $\varepsilon > 0$ be arbitrary and choose $x_k, y_k \in E_k$ so that $d(x_k, y_k) \geq (1 - \varepsilon) \text{diam} E_k$ for $k = 1, 2, \ldots$. By compactness, both $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ have converging subsequences $x_{k_j} \to x_0 \in I[E_k]$ and $y_{k_j} \to y_0 \in I[E_k]$. It follows that

$$\text{diam} I[E_k] \geq d(x_0, y_0) = \lim_{j \to \infty} d(x_{k_j}, y_{k_j}) \geq (1 - \varepsilon) \lim_{j \to \infty} \text{diam} E_{k_j} = (1 - \varepsilon) \lim_{k \to \infty} \text{diam} E_k.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof.

**Lemma 7.3.** Let $[E_k]$ and $[F_k]$ be two ends such that $[E_k]$ divides $[F_k]$ and assume that $\lim_{k \to \infty} \text{diam} F_k = 0$. Then $[E_k] = [F_k]$.

The following observation will be used in the proof of Lemma 7.3 and also later in the paper.

**Remark 7.4.** If a connected set $F \subset \Omega$ intersects both $A$ and $\Omega \setminus A$, then $F \cap (\Omega \cap \partial A)$ is nonempty.

A direct consequence is that if $E_k, E_{k+1}$ and $F$ are connected subsets of $\Omega$ with $E_{k+1} \subset E_k$, $E_{k+1} \cap F \neq \emptyset$ and $F \setminus E_k \neq \emptyset$, then $F$ meets both $\Omega \cap \partial E_{k+1}$ and $\Omega \cap \partial E_k$, which implies in turn that $\text{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) \leq \text{diam} F$.
Proof of Lemma 7.3. Assume that \([E_k] \neq [F_k]\), i.e. that \([F_k]\) does not divide \([E_k]\).
Then it follows that there exists \(I\) such that for each \(n\) we can find \(m_n \geq n\) with \(F_{m_n} \setminus E_I \neq \emptyset\). By the nested property of the chain \(\{F_k\}_{k=1}^\infty\) we get that \(F_k \setminus E_I \neq \emptyset\) for all \(k\). From this we infer that for all \(k\) there is a point \(y_k \in F_k \setminus E_I\).

As \([E_k]\) divides \([F_k]\), for every \(k\) there exists \(j_k \geq 1\) such that \(E_{j_k} \subset F_k\). Let \(x_k \in E_{j_k}\) be arbitrary. Then \(x_k \in F_k \cap E_{j_k+1}\) and \(y_k \in F_k \setminus E_I\). As \(F_k\) is connected, Remark 7.4 implies that
\[
\text{dist}(\Omega \cap \partial E_{j_k+1}, \Omega \cap \partial E_I) \leq \text{diam} F_k \to 0 \quad \text{as} \quad k \to \infty,
\]
contradicting the fact that \([E_k]_{k=1}^\infty\) is a chain.

For \(\text{Mod}_p\)-prime ends we have the following result.

**Proposition 7.5.** If \([E_k]\) is a singleton end with impression \(I[E_k] = \{x\}\) and \(1 \leq p \in Q(x) \neq (0,1]\), then \([E_k]\) is a \(\text{Mod}_p\)-prime end.

**Proof.** By Lemma 7.2, \(\text{diam} E_k \to 0\) as \(k \to \infty\), and thus \([E_k]\) is a \(\text{Mod}_p\)-end by Lemma A.4. Moreover, Proposition 7.1 shows that \([E_k]\) is a prime end, and hence it is a \(\text{Mod}_p\)-prime end by Lemma 6.2(d).

**Definition 7.6.** We say that a point \(x \in \partial \Omega\) is an accessible boundary point if there is a (possibly nonrectifiable) curve \(\gamma : [0,1] \to X\) such that \(\gamma(1) = x = \gamma((0,1)) \subset \Omega\). Moreover, if \([E_k]\) is an end and there is a curve \(\gamma\) as above such that for every \(k\) there is \(0 < t_k < 1\) with \(\gamma((t_k,1)) \subset E_k\), then \(x \in \partial \Omega\) is accessible through \([E_k]\).

Note that \(x \in \partial \Omega\) can be accessible through \([E_k]\) only if \(x \in I[E_k]\).

In the following lemma we use curves to construct prime ends at accessible points. A similar construction has been used by Karmazin [46].

**Lemma 7.7.** Let \(\gamma : [0,1] \to X\) be a curve such that \(\gamma((0,1)) \subset \Omega\) and \(\gamma(1) = x \in \partial \Omega\). Let also \(\{r_k\}_{k=1}^\infty\) be a strictly decreasing sequence converging to zero as \(k \to \infty\). Then there exist a sequence \(\{\delta_k\}_{k=1}^\infty\) of positive numbers smaller than 1 and a prime end \([F_k]\) such that \(I[F_k] = \{x\}\), \(\gamma((\delta_k,1)) \subset F_k\) and \(F_k\) is a component of \(\Omega \cap B(x,r_k)\) for all \(k = 1,2,\ldots\). In particular, \(x\) is accessible through \([F_k]\). If \(1 \leq p \in Q(x) \neq (0,1]\), then this prime end is also a \(\text{Mod}_p\)-prime end.

**Proof.** Note first that by the continuity of \(\gamma\), for each \(k = 1,2,\ldots\), there exists \(0 < \delta_k < 1\) such that
\[
\gamma((\delta_k,1)) \subset \Omega \cap B(x,r_k).
\]
Let \(F_k\) be the component of \(\Omega \cap B(x,r_k)\) containing \(\gamma(\delta_k)\). It follows directly that \(\gamma((\delta_k,1)) \subset F_k\) and hence \(x \in F_k\), showing that \(F_k\) is an acceptable set. Also, by construction, \(F_{k+1} \subset F_k\) for each \(k = 1,2,\ldots\).

Since \(\Omega \cap \partial F_k \subset \partial B(x,r_k)\), it follows that for all \(k = 1,2,\ldots\),
\[
\text{dist}(\Omega \cap \partial F_{k+1}, \Omega \cap \partial F_k) \geq r_k - r_{k+1} > 0.
\]

Also, as \(F_k \subset B(x,r_k)\), we have that \(I[F_k] = \{x\}\).

Finally, Proposition 7.1 implies that \([F_k]\) is a prime end. Moreover, if \(1 \leq p \in Q(x) \neq (0,1]\), then it is also a \(\text{Mod}_p\)-prime end by Proposition 7.5.

**Corollary 7.8.** Let \(x \in \partial \Omega\) be an accessible boundary point. Then there is a prime end \([F_k]\) with \(I[F_k] = \{x\}\). If \(1 \leq p \in Q(x) \neq (0,1]\) then this prime end is a \(\text{Mod}_p\)-prime end.

**Proposition 7.9.** Let \([E_k]\) be an end and \(x \in I[E_k]\) be accessible through \([E_k]\).
Then the following are equivalent:

\[
\begin{align*}
\text{(i) } & \text{\([E_k]\) is a \(\text{Mod}_p\)-prime end;} \\
\text{(ii) } & \text{\([E_k]\) is a prime end;} \\
\text{(iii) } & \text{\([E_k]\) is accessible through \([E_k]\)}.
\end{align*}
\]
(a) \([E_k]\) is a prime end;
(b) \([E_k]\) is a singleton end.

If \(1 \leq p \in Q(x) \neq (0, 1)\) then the following statement is also equivalent to the statements above:
(c) \([E_k]\) is a \(\text{Mod}_p\)-prime end.

The assumption of accessibility is essential in Proposition 7.9. That (a) \(\Rightarrow\) (b) fails without this assumption follows from Example 5.4.

Proof. (b) \(\Rightarrow\) (a) This follows from Proposition 7.1.

(a) \(\Rightarrow\) (b) As \(x\) is accessible through \([E_k]\), there exists a curve \(\gamma : [0, 1] \rightarrow X\) and an increasing sequence of positive numbers \(t_k \rightarrow 1\) as \(k \rightarrow \infty\), such that \(\gamma(1) = x\) and for \(k = 1, 2, \ldots\), \(\gamma([t_k, 1)) \subset E_k\). Lemma 7.7 with e.g. \(r_k = 2^{-k}\) provides us with a prime end \([F_k]\) such that \(I[F_k] = \{x\}\) and \(\gamma([\delta_k, 1)) \subset F_k\) for some \(0 < \delta_k < 1, k = 1, 2, \ldots\).

We shall show that \([F_k]\) divides \([E_k]\). If not, then there exists \(k\) such that for every \(l \geq k + 1\) there is a point \(x_l \in F_l \setminus E_k\). Since \(t_j \rightarrow 1\) as \(j \rightarrow \infty\), for every \(l \geq k + 1\) we can find \(j_l \geq l + 1\) such that \(t_{j_l} \geq \delta_l\) and hence \(y_l := \gamma(t_{j_l}) \in E_{j_l} \subset E_{k+1}\) and \(y_l \in F_l\). As \(x_l \notin E_k\) and \(y_l \in E_{k+1}\), Remark 7.4 yields

\[
\text{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) \leq \text{diam } F_l \rightarrow 0 \quad \text{as } l \rightarrow \infty,
\]

which contradicts the definition of chains. Hence, \([F_k]\) divides \([E_k]\), and as \([E_k]\) is a prime end, it follows that \([E_k] = [F_k]\), and in particular \(I[E_k] = \{x\}\).

Let us finally assume that \(1 \leq p \in Q(x) \neq (0, 1)\).

(b) \(\Rightarrow\) (c) This follows from Proposition 7.5.

(c) \(\Rightarrow\) (a) This follows from Lemma 6.2(d). \(\square\)

**Proposition 7.10.** If \([E_k]\) is an end and \(I[E_k] = \{x\}\), then \(x\) is accessible through \([E_k]\).

Proof. By Remark 4.5, we can assume that each \(E_k\) is open. As \(X\) is locally connected and \(E_k\) is connected, it follows that \(E_k\) is pathconnected, see Remark 2.6. Choose \(x_k \in E_k \setminus E_{k+1}\) for \(k = 1, 2, \ldots\). Since both \(x_k\) and \(x_{k+1}\) belong to the pathconnected set \(E_k\), there exists a curve \(\gamma_k : [1 - 1/k, 1 - 1/(k+1)] \rightarrow E_k\) connecting \(x_k\) to \(x_{k+1}\). Let \(\gamma\) be the union of all these curves. More precisely, let \(\gamma : [0, 1] \rightarrow X\) be given by \(\gamma(t) = \gamma_k(t)\) if \(t \in [1 - 1/k, 1 - 1/(k+1)]\), \(k = 1, 2, \ldots\), and \(\gamma(1) = x\). Because \(\text{diam } E_k \rightarrow 0\), we know that \(\gamma\) is continuous at 1, and hence \(x\) is accessible along \(\gamma\) through \([E_k]\). \(\square\)

The following two corollaries summarize some of the results in this section.

**Corollary 7.11.** A prime end \([E_k]\) is a singleton end if and only if its impression \(I[E_k]\) contains a point which is accessible through \([E_k]\).

Proof. This follows directly from Propositions 7.9 and 7.10. \(\square\)

**Corollary 7.12.** Let \(x \in \partial \Omega\). Then the following are equivalent:

(a) \(x\) is accessible;
(b) there is an end \([E_k]\) with \(I[E_k] = \{x\}\);
(c) there is a prime end \([E_k]\) with \(I[E_k] = \{x\}\).

If \(1 \leq p \in Q(x) \neq (0, 1)\), then also the following statements are equivalent to the statements above:

(d) there is a \(\text{Mod}_p\)-end \([E_k]\) with \(I[E_k] = \{x\}\);
(e) there is a \(\text{Mod}_p\)-prime end \([E_k]\) with \(I[E_k] = \{x\}\).
Proof. (a) ⇒ (e) This follows from Corollary 7.8.
(e) ⇒ (b) This is trivial.
(b) ⇒ (a) This follows from Proposition 7.10.
Let us finally assume that
1 ≤ p ∈ Q(x) ≠ (0, 1].
(b) ⇒ (e) This follows from Proposition 7.5
(e) ⇒ (d) ⇒ (b) These implications are trivial.

8. The topology on ends and prime ends

We would like to find homeomorphisms between the prime end boundary \( \partial P \Omega \) and other boundaries. To do so we need a topology on \( \partial P \Omega \), and in fact on the prime end closure \( \Omega^P := \Omega \cup \partial P \Omega \). We will introduce a topology on the larger set \( \Omega \cup \partial P \Omega \), where \( \partial P \Omega \) is the end boundary. It then naturally induces a topology on \( \Omega^P \) and also on the boundaries connected with \( \text{Mod}_P \)-prime ends.

Definition 8.1. We say that a sequence of points \( \{x_n\}_{n=1}^{\infty} \) in \( \Omega \) converges to the end \([E_k]\), and write \( x_n \to [E_k] \) as \( n \to \infty \), if for all \( k \) there exists \( n_k \) such that \( x_n \in E_k \) whenever \( n \geq n_k \).

If \( x_n \to [E_k] \) as \( n \to \infty \), and \([E_k]\) divides \([F_k]\), then \( x_n \) also converges to \([F_k]\).
Thus the limit of a sequence need not be unique, and we therefore avoid writing \( \lim_{n \to \infty} x_n \). It is less obvious that this problem remains even if we restrict our attention to prime ends, see Example 8.9 below.

Definition 8.2. A sequence of ends \( \{[E_k^n]\}_{n=1}^{\infty} \) converges to the end \([E_k^\infty]\) if for every \( k \) there is \( n_k \) such that for each \( n \geq n_k \) there exists \( l_{n,k} \) with \( E_{i_{n,k}} \subset E_k^\infty \).

Note that the integers \( n_k \) and \( l_{n,k} \) in Definitions 8.1 and 8.2 depend on the representative chain of the corresponding ends. However, both notions of convergence are independent of the choice of representative chain.

Definition 8.3. Convergence of points and ends defines a topology on \( \Omega \cup \partial P \Omega \).
In this topology, a collection \( C \subset \Omega \cup \partial P \Omega \) of points and ends is closed if whenever (a point or an end) \( y \in \Omega \cup \partial P \Omega \) is a limit of a sequence in \( C \), then \( y \in C \).

In this topology, a sequence \( \{x_n\}_{n=1}^{\infty} \) of points in \( \Omega \) converges to a point \( y \in \Omega \) as given by the metric topology, and no sequence of ends converges to a point in \( \Omega \).

Proposition 8.4. The topology defined above is indeed a topology on \( \Omega \cup \partial P \Omega \).

Proof. (1) The empty set and \( \Omega \cup \partial P \Omega \) are clearly closed.
(2) Let \( C_1 \) and \( C_2 \) be closed subsets of \( \Omega \cup \partial P \Omega \). Assume that \( \{y_n\}_{n=1}^{\infty} \) is a sequence in \( C_1 \cup C_2 \) such that \( y_n \to y_\infty \). Then there is a subsequence \( \{y_{n_k}\}_{k=1}^{\infty} \) lying entirely either in \( C_1 \) or else in \( C_2 \). Since a subsequence of a convergent sequence converges to the same limit, it follows that \( y_\infty \in C_1 \) or \( y_\infty \in C_2 \). Hence \( y_\infty \in C_1 \cup C_2 \). By induction, for any positive integer \( N \) we have that \( \bigcup_{n=1}^{\infty} C_n \) is closed whenever \( C_1, \ldots, C_N \) are closed.
(3) Now, let \( \{C_i\}_{i \in I} \) be a collection of closed subsets of \( \Omega \cup \partial P \Omega \). Consider a sequence \( \{y_n\}_{n=1}^{\infty} \subset \bigcap_{i \in I} C_i \). If \( y_n \to y_\infty \) as \( n \to \infty \), then \( y_\infty \in C_i \) for all \( i \in I \), since the \( C_i \) are closed. Therefore, \( y_\infty \in \bigcap_{i \in I} C_i \) and the intersection is closed.

In the rest of this section we discuss the topology on \( \Omega \cup \partial P \Omega \) and the induced topology on the prime end closure \( \Omega^P \) to gain a better understanding for them. Given an open set \( G \subset \Omega \), let \( G^E \) be the union of \( G \) and all the ends \([E_k]\) such that \( E_k \subset G \) for some \( k \). The letter \( E \) in the superscript stands for “ends”.

Proposition 8.5. The collection of sets

\[ C := \{G, G^E : G \subset \Omega \text{ is open}\} \]

forms a basis for our topology.

Proof. We first prove that \( G^E \) is open in our topology if \( G \subset \Omega \) is open. For this, we show that \( F = (\Omega \cup \partial \Omega) \setminus G^E \) is closed in the sense of Definition 8.3. Since \( F \cap \Omega = \emptyset \) if \( G^E \) is closed in \( \Omega \), if a sequence \( \{x_n\}_{n=1}^{\infty} \subset F \cap \Omega \) converges to \( x_\infty \in \Omega \), then \( x_\infty \in F \). Next, if \( \{x_n\}_{n=1}^{\infty} \subset F \cap \Omega \) converges to an end \([E_k]\), then for each \( k \) we can find \( n \) such that \( x_n \in E_k \). In particular, \( E_k \cap F = \emptyset \), or equivalently \( E_k \not\subset G \). As this holds for all \( k \), we see that \([E_k] \not\subset G^E\), i.e. \([E_k] \in F\).

Similarly, if \( \{[F_n]\}_{n=1}^{\infty} \subset F \cap \partial \Omega \) converges to an end \([E_k^\infty]\), then for every \( k \) there are \( n \) and \( l \) such that \([F_n] \subset [E_k]\). Since \([F_n]\not\subset G^E\) we must have that \([F_n]\not\subset G\), and in particular \([E_k^\infty]\not\subset G\), showing that \([E_k^\infty]\in F\).

Thus all the sets in \( C \) are open. Since \( C \) contains all the open subsets of \( \Omega \) it is enough to show that if \( H \subset \Omega \cup \partial \Omega \) is open in our topology and \([E_k]\in H\), then there is an open set \( G \subset \Omega \) such that \([E_k] \in G^E \subset H\). By Remark 4.5 we may assume that the sets \( E_k \) are open. We shall show that \( E_k^E \subset H \) for some \( k \), i.e. \( G = E_k \) will do.

Assume that this is false. Then by passing to a subsequence if necessary, either
(a) there are points \( x_n \in ([E_n^E] \cap \Omega) \setminus H = E_n \setminus H \) for all \( n \); or
(b) there are ends \([F_n] \in ([E_n^E] \cap \partial \Omega) \setminus H \) for all \( n \).

In case (a), \( x_n \to [E_k] \) as \( n \to \infty \), by definition. As \([E_k] \in H\), this contradicts the fact that \( H \) is open. Also in case (b) we see that \([F_n] \to [E_k] \in H\) as \( n \to \infty \), by definition, contradicting the openness of \( H \) again.

We thus conclude that indeed \( E_k^E \subset H \) for some \( k \), and thus \( C \) is a basis for our topology. \( \square \)

One may ask if the collection

\[ \{G_1 \cup G_2^E : G_1, G_2 \subset \Omega \text{ are open}\} \quad (8.1) \]

may contain all open sets in our topology. Example 8.6 below shows that this is not true.

When restricting to prime ends it directly follows from Proposition 8.5 that the collection

\[ \{G, G^P : G \subset \Omega \text{ is open}\}, \]

where \( G^P = G^E \cap \Pi^P \), forms a basis for our topology on \( \Pi^P \). This time it follows from Example 8.7 that not all open sets can be written in the form \( G_1 \cup G_2^E \).

Example 8.6. Let \( \Omega = (0, 3)^2 \subset \mathbb{R}^2 \), \( I = [0, 3] \times \{0\} \) and \( E_k = (0, 3) \times (0, 1/k) \). Then \([E_k]\) is an end with impression \( I\). Furthermore, let \( G_1 = (0, 2)^2 \) and \( G_2 = (1, 3) \times (0, 2) \). If

\[ G_1^E \cup G_2^E \subset G_1^E \cup G_4 \quad (8.2) \]

for some open sets \( G_3, G_4 \subset \Omega \), then \([E_k] \in G_1^E \). But \([E_k] \not\in G_1^E \cup G_2^E \). Thus we cannot have equality in (8.2) and hence the collection (8.1) does not contain all open sets.

Example 8.7. Let \( \tilde{\Omega} \subset \mathbb{R}^2 \) be the double equilateral comb from Example 5.4, and let \( \Omega = \tilde{\Omega} \times (0, 1) \). Note that \( \Omega \subset \mathbb{R}^3 \) is simply connected and homeomorphic to a ball. For \( a, b \in [0, 1] \), let \( I^{a,b} \) be the closed line segment with end points \((a, 0, a)\) and \((b, 0, b)\). Let further

\[ E_k^{a,b} = \{x \in \Omega : \text{dist}(x, I^{a,b}) < 1/k\} \].
Then \([E_k^{a,b}]\) is a prime end with impression \(I^{a,b}\). Let next
\[
G_1 = \tilde{\Omega} \times (0, \frac{2}{3}) \quad \text{and} \quad G_2 = \tilde{\Omega} \times (\frac{1}{3}, 1).
\]
If
\[
G_1^P \cup G_2^P \subset G_3^P \cup G_4 \tag{8.3}
\]
for some open sets \(G_3, G_4 \subset \Omega\), then \([E_k^{0,1}] \in G_3^P\). But \([E_k^{0,1}] \notin G_1^P \cup G_2^P\). Thus we cannot have equality in (8.3) and hence the collection
\[
\{G_1 \cup G_2^P : G_1, G_2 \subset \Omega \text{ are open}\}
\]
does not contain all open sets in \(\Omega^P\).

Observe also that \(\{[E_k^{a,1-a}] : 0 \leq a \leq 1\}\) is an uncountable collection of prime ends such that no pair of them can be separated as in the T2 separation condition below, since the sequence \(\{(\frac{1}{2n+1}, \frac{1}{2})\}_{n=1}^{\infty}\) converges to all of them.

Recall that a topological space \(Y\) satisfies the T1 separation condition if any two distinct points can be separated, i.e. if each point lies in an open set which does not contain the other point. (An equivalent way to formulate the T1 separation condition is to require that every singleton set is closed.) If the two open sets can be chosen to be disjoint, then \(Y\) satisfies the T2 separation condition. A topological space is Hausdorff if it satisfies the T2 separation condition.

If \([E_k]\) and \([F_k]\) are two distinct ends such that \([E_k]\) divides \([F_k]\), then any neighborhood of \([F_k]\) contains \([E_k]\), and thus the topology on \(\Omega \cup \partial E\Omega\) does not satisfy the T1 separation condition. If we however restrict ourselves to prime ends, i.e. to \(\Omega^P\), then the T1 separation condition is satisfied.

**Proposition 8.8.** The topology on \(\Omega^P\) satisfies the T1 separation condition.

**Proof.** If \(x \in \Omega\), then \(\{x\}\) is closed in our topology. Thus to verify the T1 separation condition it suffices to show that a singleton set \(\{E_k\}\) is closed for any prime end \([E_k]\). We thus need to consider the sequence \(\{[E_k^{n}]\}_{n=1}^{\infty}\), with \([E_k^{n}] = [E_k]\) for all \(n\). Assume that \(\{[E_k^{n}]\}_{n=1}^{\infty}\) converges to a prime end \([E_k^{\infty}]\). As the sequence is constant it is not hard to see that \([E_k^{\infty}]\) must divide \([E_k^{\infty}]\). Since \([E_k^{\infty}]\) is a prime end, we thus must have \([E_k] = [E_k^{\infty}]\). Hence the set \(\{[E_k]\}\) is closed. \(\square\)

The topology obtained on \(\Omega^P\) does not need to satisfy the T2 separation condition, and can thus be nonmetrizable, as shown by Example 8.7 and the following example. In Corollary 10.9 we will show that this topology is metrizable if \(\Omega\) is finitely connected at the boundary.

**Example 8.9.** (See Figure 2.) Let \(\Omega \subset \mathbb{R}^2\) be obtained from the rectangle \((-1,1) \times (0,1)\) by removing the segments
\[
(-1,-2^{-k}) \times \{2^{-k}\}, \quad [2^{-k},1) \times \{2^{-k}\} \quad \text{and} \quad [-1+2^{-k},1-2^{-k}] \times \{3 \cdot 2^{-k-1}\},
\]
\(k = 1,2,\ldots\) Then the sets
\[
E_k = \Omega \cap ((-1,2^{-k}) \times (0,2^{-k})) \quad \text{and} \quad F_k = \Omega \cap ((-2^{-k},1) \times (0,2^{-k}))
\]
define two prime ends with impressions
\[
I[E_k] = [-1,0] \times \{0\} \quad \text{and} \quad I[F_k] = [0,1] \times \{0\}.
\]
These prime ends are clearly different but the sequence \(\{(0,2^{-n})\}_{n=1}^{\infty}\) converges to both of them. It follows that any neighborhood of any of these two prime ends
contains all but a finite number of points from this sequence. Hence these two prime ends do not have disjoint neighborhoods, or in other terms the T2 separation condition fails. It is easy to modify Ω so that the impressions of the two prime ends have a common interval and not just a common point.

The domain Ω above is not simply connected. To get a simply connected domain consider

\[ \Omega' = (\Omega \times (0, 1)) \cup ((-1, 1) \times (0, 1) \times (1, 2)) \]

or Example 8.7.

Definition 8.2 implies that if \([E_n^k] \to [E_\infty^k]\) as \(n \to \infty\), then there are sequences \(\{x_i^n\}_{i=1}^\infty, \ n = 1, 2, \ldots, \) and \(\{x_i^\infty\}_{i=1}^\infty\) in Ω, which converge to \([E_n^k]\) and \([E_\infty^k]\) respectively as \(i \to \infty\), and satisfy

\[ \lim_{n \to \infty} \limsup_{i \to \infty} d(x_i^n, x_i^\infty) = 0. \] (8.4)

However, even with the additional assumption that the diameters of \([E_n^k]\) and \([E_\infty^k]\) converge to 0, this sequential criterion does not imply convergence of prime ends. Consider e.g. the slit disk (Example 5.2) and let \(x_i^n\) converge to a point on the slit from one side and \(x_i^\infty\) from the other side. Instead, one can use the Mazurkiewicz distance associated with the connectedness properties of the domain.

**Definition 8.10.** We define the **Mazurkiewicz distance** \(d_M\) on Ω by

\[ d_M(x, y) = \inf \text{diam } E, \]

where the infimum is over all connected sets \(E \subset \Omega\) containing \(x, y \in \Omega\).

Clearly, \(d_M\) is a metric on Ω. When \(x, y \in \Omega\), we have \(d_M(x, y) \geq d(x, y)\).

**Lemma 8.11.** Let \(\{[E_n^k]\}_{n=1}^\infty, \ n = 1, 2, \ldots, \) and \([E_\infty^k]\) be ends. Then the following are equivalent:

(a) The sequence of ends \(\{[E_n^k]\}_{n=1}^\infty\) converges to the end \([E_\infty^k]\) and \([E_\infty^k]\) is a singleton end.

(b) Whenever \(\{x_i^n\}_{i=1}^\infty, \ n = 1, 2, \ldots, \) and \(\{x_i^\infty\}_{i=1}^\infty\) are sequences in Ω, which converge to \([E_n^k]\) and \([E_\infty^k]\), respectively, as \(i \to \infty\), we must have

\[ \lim_{n \to \infty} \limsup_{i \to \infty} d_M(x_i^n, x_i^\infty) = 0. \] (8.4)
Recall that by Lemma 7.2 an end \([E_k]\) has a singleton impression if and only if \(\lim_{k \to \infty} \diam E_k = 0\).

**Proof.** (a) \(\Rightarrow\) (b) For all \(k\) there exists \(n_k\) such that for each \(n \geq n_k\) we can find \(l_{n,k}\) such that \(E_{l_{n,k}}^n \subset E_k^\infty\). Let \(\{x^n_i\}_{i=1}^\infty\) and \(\{x^\infty_i\}_{i=1}^\infty\) converge to \([E^n_k]\), \(n = 1, 2, \ldots\), and \([E^\infty_k]\), respectively. Fix \(k, n\) and \(l_{n,k}\) as above for a moment. Then there exists \(m_{n,k}\) such that for all \(i \geq m_{n,k}\) we have \(x^n_i \in E_{l_{n,k}}^n \subset E_k^\infty\). Similarly, there is \(m_k\) such that for \(i \geq m_k\) we have that \(x^\infty_i \in E_k^\infty\). Hence for \(i \geq \max\{m_{n,k}, m_k\}\) we have that
\[
\diam(x^n_i, x^\infty_i) \leq \diam E_k^\infty.
\]
Since \(\diam E_k^\infty \to 0\) we conclude that
\[
0 \leq \limsup_{n \to \infty} \limsup_{i \to \infty} \diam(x^n_i, x^\infty_i) \leq \limsup_{n \to \infty} \diam E_k^\infty \to 0.
\]

(b) \(\Rightarrow\) (a) Assume first that \(\{[E^n_k]\}_{n=1}^\infty\) does not converge to the end \([E^\infty_k]\). Then there exists \(k_0\) such that for all \(n\) there is \(m_{n} \geq n\) with the property that \(E_{m_{n}}^n \setminus E_k^\infty \neq \emptyset\) for all \(i = 1, 2, \ldots\). For each \(n = 1, 2, \ldots\) we define a sequence \(\{x_{k_0}^n\}_{i=1}^\infty\) by choosing \(x_{k_0}^n \in E_{m_{n}}^n \setminus E_k^\infty\) if this set is nonempty and \(x_{k_0}^n \in E_k^n\) otherwise. By construction, \(x_{k_0}^n \to [E_k^n]\), as \(i \to \infty\). Let also \(x_i^\infty \in E_k^\infty\) be arbitrary, \(i = 1, 2, \ldots\). Then \(x_i^{m_{n}} \notin E_k^\infty\) and \(x_i^\infty \in E_k^{k_0+1}\) whenever \(i > k_0\). Remark 7.4 yields
\[
\diam(x_{k_0}^{m_{n}}, x_i^\infty) \geq \limsup_{n \to \infty} \limsup_{i \to \infty} \diam(x_{k_0}^{m_{n}}, x_i^\infty) > 0.
\]
Letting \(i \to \infty\) and then \(n \to \infty\) implies that
\[
\limsup_{n \to \infty} \limsup_{i \to \infty} \diam(x_{k_0}^{m_{n}}, x_i^\infty) > 0,
\]
i.e. (8.4) fails. Thus \([E^n_k]\) \(\not\to [E^\infty_k]\).

To see that \([E^\infty_k]\) is a singleton end, choose \(x^\infty_k \in E_k^\infty\) so that \(d(x^\infty_k, y^\infty_k) \geq \frac{1}{2} \diam E_k^\infty\), \(k = 1, 2, \ldots\). Let \(z_k \in E_k^n\), \(n, k = 1, 2, \ldots\), be arbitrary. Observe that \(z_k \to [E_k^n]\), \(y^\infty_k \to [E_k^\infty]\) and \(z_k \to [E_k^n]\) for \(n = 1, 2, \ldots\), as \(j \to \infty\). We then have by the triangle inequality that
\[
\diam E_k^\infty \leq 2d(x_k^{\infty}, y_k^{\infty}) \leq 2\diam(x_k^{\infty}, y_k^n) \leq 2(\diam(x_k^{\infty}, z_k^n) + \diam(z_k^n, y_k^{\infty})).
\]
Letting \(k \to \infty\) and then \(n \to \infty\) together with (8.4) (used twice) completes the proof. \(\square\)

### 9. Prime ends and the Mazurkiewicz boundary

We now focus on describing embeddings and homeomorphisms between the prime end boundary and two other boundaries, the topological boundary and the Mazurkiewicz boundary. Our investigations are motivated by the fact that such mappings allow us to discuss the correspondence between prime ends and their impressions, with a view towards boundary value problems.

In Björn–Björn–Shanmugalingam [15] the Dirichlet problem for \(p\)-harmonic functions, with boundary data defined on the Mazurkiewicz boundary, is studied in domains which are finitely connected at the boundary (see Definition 10.1 for the notion of finite connectedness at the boundary). By Theorem 10.8 this is equivalent to studying the Dirichlet problem with respect to the prime end boundary for such domains. We refer to [15] for further details on the Dirichlet problem, but this is another important motivation for this and the next section.

We saw in Section 7 that accessibility of a boundary point determines whether there is a prime end with a singleton impression at this point. Furthermore,
Lemma 8.11 tells us that there is a strong link between the Mazurkiewicz distance on $\Omega$ and the topology of $\partial P \Omega$. Motivated by these, we consider the boundary of $\Omega$ with respect to the Mazurkiewicz distance in this section. Recall that the Mazurkiewicz distance was introduced in Definition 8.10.

**Remark 9.1.** Because $X$ is locally connected, $d_M$ and $d$ define the same topology on $\Omega$.

The completion of the metric space $(\Omega, d_M)$ is denoted $\overline{\Omega}^M$, and $d_M$ extends in the standard way to $\overline{\Omega}^M$: For $d_M$-Cauchy sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq \Omega$ we define the equivalence relation

$$\{x_n\}_{n=1}^{\infty} \sim \{y_n\}_{n=1}^{\infty} \quad \text{if} \quad \lim_{n \to \infty} d_M(x_n, y_n) = 0.$$ 

Note that every Cauchy sequence is trivially equivalent to any of its subsequences.

The collection of all equivalence classes of $d_M$-Cauchy sequences can be formally considered to be $\overline{\Omega}^M$, but we will identify equivalence classes of $d_M$-Cauchy sequences having a limit in $\Omega$ with that limit point. By considering equivalence classes of $d_M$-Cauchy sequences without limits in $\Omega$ we define the boundary of $\Omega$ with respect to $d_M$ as $\partial_M \Omega = \overline{\Omega}^M \setminus \Omega$. Since $X$ is proper, we know that $\Omega$ is locally compact with respect to $d_M$, and it follows that $\Omega$ is an open subset of $\overline{\Omega}^M$. We extend the original metric $d_M$ on $\Omega$ to $\overline{\Omega}^M$ by setting

$$d_M(x^*, y^*) = \lim_{n \to \infty} d_M(x_n, y_n),$$

if $x^* = \{x_n\}_{n=1}^{\infty} \subseteq \overline{\Omega}^M$ and $y^* = \{y_n\}_{n=1}^{\infty} \subseteq \overline{\Omega}^M$. This is well defined and an extension of $d_M$.

By the construction of $\overline{\Omega}^M$ and Remark 9.1, every point in $\Omega$ can be identified with exactly one equivalence class of $d_M$-Cauchy sequences in $\Omega$. This is, of course, not true on the boundary of $\Omega$ in general, as illustrated by the following example.

**Example 9.2.** Consider the planar slit disk in Example 5.2. To every point $x \in [-1, 0) \times \{0\}$ there correspond exactly two points in the Mazurkiewicz boundary given by sequences approaching $x$ from the upper and lower half-planes, respectively. For instance the sequence $x_n = (x, (-1)^n/n)$ for $n = 2, 3, \ldots$ converges to $x$ in the Euclidean metric but is not a $d_M$-Cauchy sequence as

$$d_M(x_n, x_{n+1}) \geq 2|x| \quad \text{for } n = 2, 3, \ldots.$$

In the next example we show that a point in the topological boundary need not correspond to a limit of any $d_M$-Cauchy sequence in the Mazurkiewicz boundary.

**Example 9.3.** (The topologist’s comb) Let $\Omega$ be the topologist’s comb as in Example 5.1. Then no point in the bottom segment $I = \left(\frac{1}{2}, 1\right) \times \{0\}$ corresponds to an element of $\partial_M \Omega$. Namely, any sequence of points in $\Omega$ converging to a point in $I$ fails the Cauchy condition with respect to the Mazurkiewicz distance.

**Lemma 9.4.** There is a continuous map $\Psi : \overline{\Omega}^M \to \overline{\Omega}$ such that $\Psi|_{\Omega}$ is the identity map and $\Psi|_{\partial_M \Omega} : \partial_M \Omega \to \partial \Omega$.

This mapping need not be injective nor surjective in general, as demonstrated by the slit disk and the topologist’s comb in Examples 9.2 and 9.3, respectively.

**Proof.** Let $\{x_n\}_{n=1}^{\infty}$ be a $d_M$-Cauchy sequence in $\Omega$ representing a point in $\overline{\Omega}^M$. Since $d(x_i, x_j) \leq d_M(x_i, x_j)$, it follows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the given metric $d$ as well, and so by the completeness of $X$, we can set

$$\Psi(\{x_n\}_{n=1}^{\infty}) = \lim_{n \to \infty} x_n \in \overline{\Omega}.$$
The map \( \Psi \) is well defined, since every sequence representing the same point in \( \Omega^M \) converges to the same limit in the given metric \( d \).

To prove the continuity of \( \Psi \), consider \( \{x_n\}_{n=1}^\infty \), \( \{y_n\}_{n=1}^\infty \in \Omega^M \) and let \( x = \Psi(\{x_n\}_{n=1}^\infty) \) and \( y = \Psi(\{y_n\}_{n=1}^\infty) \). Then by definition we have that
\[
d_M(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) = \lim_{n \to \infty} d_M(x_n, y_n) \geq \lim_{n \to \infty} d(x_n, y_n) = d(x, y).
\]

Therefore \( d(\Psi(\{x_n\}_{n=1}^\infty), \Psi(\{y_n\}_{n=1}^\infty)) \leq d_M(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) \), that is, \( \Psi \) is 1-Lipschitz continuous.

Next, we show that under rather general assumptions, the prime end boundary and the Mazurkiewicz boundary coincide.

**Theorem 9.5.** Assume that every prime end in \( \Omega \) has a singleton impression. Then there is a homeomorphism \( \Phi : \partial_p \Omega \to \partial_M \Omega \).

This is a special case of the following result. Recall from Proposition 7.1 that every singleton end is a prime end.

**Theorem 9.6.** Let \( \partial SP \Omega \) be the set of all singleton ends. Then there is a homeomorphism \( \Phi : \Omega \cup \partial SP \Omega \to \Omega^M \) such that \( \Phi|_{\Omega} \) is the identity map and \( \Phi|_{\partial SP \Omega} : \partial SP \Omega \to \partial_M \Omega \).

Recall that by Lemma 7.2 an end \( [E_k] \) has a singleton impression if and only if \( \lim_{k \to \infty} \text{diam } E_k = 0 \). We will use this fact (implicitly) several times in the proof below.

**Proof.**

**Step 1. Definition of \( \Phi \).** Let \( [E_k] \in \partial SP \Omega \). For each \( k \) choose \( x_k \in E_k \). Then for \( l \geq k \) we have that \( x_k, x_l \in E_k \) and as \( E_k \) is connected, this implies that
\[
d_M(x_k, x_l) \leq \text{diam } E_k \to 0, \quad \text{as } k \to \infty.
\]
Thus, \( \{x_k\}_{k=1}^\infty \) is a \( d_M \)-Cauchy sequence and corresponds to a point \( y \in \Omega^M \). If \( y \) belonged to \( \Omega \), then we would have \( y \in \bigcap_{k=1}^\infty E_k \cap \Omega = I[E_k] \cap \Omega = \emptyset \), which is a contradiction. Thus \( y \in \partial_M \Omega \), and we define
\[
\Phi([E_k]) = y.
\]

For \( x \in \Omega \) we, of course, set \( \Phi(x) = x \).

**Step 2. \( \Phi \) is well defined.** Assume that \( \{E_k\}_{k=1}^\infty \) and \( \{E'_k\}_{k=1}^\infty \) are equivalent chains, and let \( x_k \in E_k \) and \( x'_k \in E'_k \), \( k = 1, 2, \ldots \). Then for every \( k \), there exists \( l_k \geq k \) such that \( E_{l_k} \subset E'_k \). Hence for all \( l \geq l_k \), we have that \( x_l \in E_l \subset E_{l_k} \subset E'_k \). Thus
\[
\lim_{l \to \infty} d_M(x_l, x'_l) \leq \text{diam } E'_k \to 0 \quad \text{as } k \to \infty,
\]
showing that \( \{x_k\}_{k=1}^\infty \) and \( \{x'_k\}_{k=1}^\infty \) are equivalent as \( d_M \)-Cauchy sequences. Hence \( \Phi \) is well-defined.

**Step 3. \( \Phi \) is surjective.** Let \( \{x_n\}_{n=1}^\infty \) be a \( d_M \)-Cauchy sequence in \( \Omega \), corresponding to a point in \( \partial_M \Omega \). We can assume that for all \( j, k \geq n \),
\[
d(x_j, x_k) \leq d_M(x_j, x_k) < 2^{-n-1}. \quad (9.1)
\]
It follows that \( \{x_n\}_{n=1}^\infty \) is a \( d \)-Cauchy sequence and converges to some \( x \in \partial \Omega \), and moreover,
\[
d(x_k, x) \leq 2^{-k-1}. \quad (9.2)
\]
For each $k = 1, 2, \ldots$, let $E_k$ be the component of $\Omega \cap B(x, 2^{-k})$ containing $x_k$. Then for all $j \geq k$, (9.1) implies that there exists a connected set $F_j \subset \Omega$ such that $x_j, x_k \in F_j$ and $\text{diam} F_j < 2^{-k-1}$. From (9.2) it follows that $F_j \subset \Omega \cap B(x, 2^{-k})$. As $E_k$ is a component of $\Omega \cap B(x, 2^{-k})$ and $x_k \in E_k$, we obtain that the connected set $F_j \subset E_k$ and thus $x_j \in E_k$ for all $j \geq k$. Letting $j \to \infty$ shows that $x \in E_k$ for $k = 1, 2, \ldots$.

This also shows that $x_{k+1} \in E_k$ and as $E_{k+1}$ is connected, we obtain that $E_{k+1} \subset E_k$ for all $k$. Since $\Omega \cap \partial E_k \subset \partial B(x, 2^{-k})$, we see that

$$\text{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) \geq 2^{-k-1} > 0.$$ 

By construction we know that $\text{diam} E_k \to 0$, and hence $\{E_k\}_{k=1}^\infty$ is a chain with impression $\{x\}$. By Proposition 7.1, $[E_k]$ is a prime end. Moreover, $\Phi([E_k]) = \{x_n\}_{n=1}^\infty$. Thus $\Phi$ is surjective. (That $\Phi|_{\Omega}$ is bijective is clear.)

**Step 4.** $\Phi$ is injective. Let $[E_k]$ and $[F_k]$ be two distinct singleton prime ends. So $\{E_k\}_{k=1}^\infty$ does not divide $\{F_k\}_{k=1}^\infty$. Hence, there exists $k$ such that for each $l$ we can find a point $y_l \in F_l \setminus E_k$. We need to show that $\{y_l\}_{l=1}^\infty$ is not equivalent to any sequence representing $\Phi([E_k])$. Let $x_l \in E_l$ for each $l$. Since $x_l \in E_{k+1}$ and $y_l \notin E_k$ for $l > k$, Remark 7.4 yields that every connected set $A$ containing both $x_l$ and $y_l$ satisfies

$$\text{diam} A \geq \text{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k).$$

Hence, for each $l \geq k + 1$ we have that

$$d_M(x_l, y_l) \geq \text{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) > 0.$$ 

Thus the two sequences $\{x_l\}_{l=1}^\infty$ and $\{y_l\}_{l=1}^\infty$ are not equivalent, and $\Phi$ is injective.

**Step 5.** $\Phi$ is continuous. We need to show that preimages of closed sets are closed. Since the topologies on $\overline{\Omega}^M$ and $\Omega \cup \partial_{SP}$ are given by converging sequences, it suffices to consider sequential continuity. As $\Phi|_{\Omega}$ is continuous, it is enough to show that the image of every sequence with a limit in $\partial_{SP}\Omega$ has the correct limit. There are two such types of sequences we need to consider.

Assume first that the sequence of singleton prime ends $\{[E_k^n]\}_{n=1}^\infty$ converges to a singleton prime end $[E_k^\infty]$. Let $\Phi([E_k^n]) = \{x_k^n\}_{n=1}^\infty$ and $\Phi([E_k^\infty]) = \{x_k^\infty\}_{n=1}^\infty$, where $x_k^n \in E_k^n$, $n = 1, 2, \ldots$, and $x_k^\infty \in E_k^\infty$ are provided by Steps 1 and 2. Then it is clear that $\{x_k^n\}_{n=1}^\infty$ converges to $[E_k]$ for each $n$, and that $\{x_k^\infty\}_{n=1}^\infty$ converges to $[E_k^\infty]$. By Lemma 8.11, it follows that $\lim_{n \to \infty} \limsup_{k \to \infty} d_M(x_k^n, x_k^\infty) = 0$, i.e. $\lim_{n \to \infty} \limsup_{k \to \infty} d_M(\Phi([E_k^n]), \Phi([E_k^\infty])) = 0$. This shows that $\{\Phi([E_k^n])\}_{n=1}^\infty$ converges in $d_M$ to $\Phi([E_k^\infty])$ as $n \to \infty$.

Assume next that $\Omega \ni y_n \to [E_k] \in \partial_{SP}\Omega$ as $n \to \infty$. Thus for each $k$ we can find $n_k$ such that $y_n \in E_k$ whenever $n \geq n_k$. As $E_k$ is connected we see that

$$d_M(y_n, y_n) \leq \text{diam} E_k \text{ for } l, n \geq n_k.$$ 

Since $\text{diam} E_k \to 0$ as $k \to \infty$, this shows that $\{y_n\}_{n=1}^\infty$ is a $d_M$-Cauchy sequence. As $y_{n_k} \in E_k$, the sequence $\{y_{n_k}\}_{k=1}^\infty$ represents $\Phi([E_k])$ and is equivalent to $\{x_n\}_{n=1}^\infty$, which is the limit of the sequence $\{y_n\}_{n=1}^\infty$ in $\overline{\Omega}^M$.

Thus we conclude that $\Phi$ is continuous.

**Step 6.** $\Phi^{-1}$ is continuous. As in Step 5 there are two types of sequences we need to consider.

Assume first that the sequence of singleton prime ends $\{[E_k^n]\}_{n=1}^\infty$ does not converge to the singleton prime end $[E_k^\infty]$. Then by Lemma 8.11, there are sequences $\{x_k^n\}_{k=1}^\infty$ converging to $[E_k^n]$ for each $n$, and $\{x_k^\infty\}_{k=1}^\infty$ converging to $[E_k^\infty]$, such that

$$\limsup_{n \to \infty} \limsup_{k \to \infty} d_M(x_k^n, x_k^\infty) > 0.$$
Because the ends \( [E^n_k] \) and \([E^\infty_k]\) are singleton ends, it follows that \( \Phi([E^n_k]) \) is represented by \( \{x_k\}_{k=1}^\infty \) and \( \Phi([E^\infty_k]) \) is represented by \( \{x_k\}_{k=1}^\infty \). Therefore, it is not true that \( \lim_{n \to \infty} d_M(\Phi([E^n_k]), \Phi([E^\infty_k])) = 0 \).

Assume next that \( \{x_n\}_{n=1}^\infty \) is a sequence of points in \( \Omega \) which does not converge to the singleton prime end \([E^\infty_k]\). Then we can find \( k \) and an increasing sequence \( n_i \to \infty \) for which \( x_{n_i} \notin E^\infty_k \). Let \( y_l \in E^\infty_l \) for \( l = 1, 2, \ldots \), i.e. \( \Phi([E^n_k]) = \{y_l\}_{l=1}^\infty \).

As the sequence \( \{x_{n_i}\}_{i=1}^\infty \) lies entirely in \( \Omega \setminus E^\infty_k \), Remark 7.4 shows that
\[
d_M(y_l, x_{n_i}) \geq \text{dist}(\Omega \cap \partial E^\infty_{k+1}, \Omega \cap \partial E^\infty_k) \quad \text{for } l > k \text{ and all } i.
\]
Thus
\[
d_M(\Phi([E^n_k]), x_{n_i}) \geq \text{dist}(\Omega \cap \partial E^\infty_{k+1}, \Omega \cap \partial E^\infty_k) > 0,
\]
which shows that \( \{x_{n_i}\}_{i=1}^\infty \) cannot converge to \( \Phi([E^n_k]) \), and hence neither can \( \{x_{n_i}\}_{i=1}^\infty \) converge to \( \Phi([E^\infty_k]) \).

This shows that \( \Phi^{-1} \) is continuous and so \( \Phi \) is a homeomorphism. \( \square \)

10. Domains finitely connected at the boundary

In general not all prime ends have singleton impressions, as demonstrated by Example 5.4. In this section and the next section we explore conditions under which prime ends have this property (cf. Section 7).

Here we present a topological condition, finite connectedness at the boundary, which guarantees that all prime ends have singleton impressions. Finite connectedness at the boundary is equivalent to the compactness of \( \Omega \cup \partial \Omega \), where \( \partial \Omega \) is the set of all singleton prime ends, see Theorem 10.10 and the comments after it.

**Definition 10.1.** We say that \( \Omega \) is finitely connected at a point \( x_0 \in \partial \Omega \) if for every \( r > 0 \) there is an open set \( G \) (open in \( X \)) such that \( x_0 \in G \subset B(x_0, r) \) and \( G \cap \Omega \) has only finitely many components. If \( \Omega \) is finitely connected at every boundary point, then it is called finitely connected at the boundary.

This terminology follows Näkki [59], who seems to have first used it in print. (Näkki [61] has informed us that he learned about it from Väisälä, who however first seems to have used it in print in [71].) Beware that the notion of finitely connected domains is a completely different notion.

We now introduce some further notation. Fix \( x_0 \in \partial \Omega \) (we do not assume that \( \Omega \) is finitely connected here). For each \( r > 0 \) let \( \{G_j(r)\}_{j=1}^{N(r)} \) be the family of components of \( B(x_0, r) \cap \Omega \) which have \( x_0 \) in their boundary, i.e. \( x_0 \in \partial G_j(r) \). Here \( N(r) \) is either a nonnegative integer or \( \infty \). Let
\[
H(r) = (B(x_0, r) \cap \Omega) \setminus \bigcup_{j=1}^{N(r)} G_j(r)
\]
be the union of the remaining components (if any). (The sets \( G_j(r) \) and \( H(r) \) of course depend on \( x_0 \).)

**Example 10.2.** (See Figure 3.) Let \( \Omega \) be obtained from the unit square \((0,1)^2\) in \( \mathbb{R}^2 \) by removing the segments
\[
S_k = \left\{(x,y) : 0 < x \leq \frac{1}{2k^2} \text{ and } y = kx\right\}, \quad k = 1, 2, \ldots.
\]
Then \( \Omega \) is finitely connected at the boundary but there are infinitely many prime ends with the origin as their impression. For each “wedge” there is one such singleton prime end. The sequence consisting of these prime ends converges to the
singleton prime end defined by the acceptable sets
\[ E_k = \left\{ (x, y) : 0 < x < \frac{1}{2k^2} \text{ and } kx < y < \frac{1}{2k} \right\}, \quad k = 1, 2, \ldots \]

Example 10.3. (See Figure 3.) Let
\[ \Omega = \left( (-1, 1) \times (0, 1) \right) \setminus \left\{ \left\{ 0, \frac{1}{2}, \frac{3}{8}, \ldots \right\} \times (0, 1) \right\} \]
(see the above figure). Then for the point \( x_0 = (0, 0) \) we have that \( N(r) = 1 \) for all \( 0 < r < 1 \) but the domain is not finitely connected at \( x_0 \).

The following characterization of finite connectedness is useful, see Björn–Björn–Shanmugalingam [16] for a proof.

**Proposition 10.4.** The set \( \Omega \) is finitely connected at \( x_0 \) if and only if for each \( r > 0 \) we have \( N(r) < \infty \) and \( x_0 \notin H(r) \).

We next present two auxiliary results which will primarily be applied to ends, but we state them here for more general sets since it will be useful later in the proofs of Theorem 10.8 and Proposition 10.13.

**Lemma 10.5.** Assume that \( \Omega \) is finitely connected at \( x_0 \in \partial \Omega \). Let \( A_k \subseteq \Omega \) be such that \( A_{k+1} \subseteq A_k \), \( x_0 \in \overline{A_k} \) and \( \text{dist}(x_0, \Omega \cap \partial A_k) > 0 \) for each \( k = 1, 2, \ldots \). Furthermore, let \( 0 < r_k < \text{dist}(x_0, \Omega \cap \partial A_k) \) be a sequence decreasing to zero. Then for each \( k = 1, 2, \ldots \) there is a component \( G_{j_k}(r_k) \) of \( B(x_0, r_k) \cap \Omega \) intersecting \( A_l \) for each \( l = 1, 2, \ldots \), and such that \( x_0 \in G_{j_k}(r_k) \) and \( G_{j_k}(r_k) \subset A_k \).

**Proof.** Consider the components \( G_1(r_k), \ldots, G_N(r_k) \) of \( B(x_0, r_k) \cap \Omega \) which have \( x_0 \) in their boundary. Let \( H(r_k) = (\Omega \cap B(x_0, r_k)) \setminus \bigcup_{j=1}^{N(r_k)} G_j(r_k) \). As \( \Omega \) is finitely connected at \( x_0 \), Proposition 10.4 shows that \( x_0 \notin \overline{H(r_k)} \), so for each \( l = 1, 2, \ldots \), at least one of \( G_1(r_k), \ldots, G_N(r_k) \) has a nonempty intersection with \( A_l \). Since there are only finitely many components \( G_j(r_k), \ j = 1, 2, \ldots, N(r_k) \), at least one of them intersects infinitely many (and thus all) \( A_l \). Call this component \( G_{j_k}(r_k) \). As it is connected and \( r_k < \text{dist}(x_0, \Omega \cap \partial A_k) \), we must have \( G_{j_k}(r_k) \subset A_k \). \( \square \)
Lemma 10.6. Assume that \( \Omega \) is finitely connected at \( x_0 \in \partial \Omega \). Let \( A_k \subseteq \Omega \) and \( r_k > 0 \) be as in the statement of Lemma 10.5.

Then there exists a prime end \( \{F_k\} \) such that \( I[F_k] = \{x_0\} \), \( F_k = G_{j_k}(r_k) \) for some \( 1 \leq j_k \leq N(r_k) \) and \( F_k \subset A_k \), \( k = 1, 2, \ldots \).

Proof. Consider the rooted tree whose vertices are \( G_j(r_k) \), \( j = 1, 2, \ldots, N(r_k) \), \( k = 1, 2, \ldots \), and where two vertices are connected by an edge if and only if they are \( G_j(r_k) \) and \( G_i(r_{k+1}) \) for some \( i, j \) and \( k \). Consider the collection \( P \) of all descending paths in the tree starting from the root (including finite ones). We introduce a metric \( t \) on \( P \) by letting \( t(p, q) = 2^{-n} \), where \( n \) is the level where the paths \( p \) and \( q \) branch (or end), i.e. \( n \) is the largest integer such that \( p \) and \( q \) have a common vertex \( G_j(r_n) \). Since \( \Omega \) is finitely connected at \( x_0 \), for each \( l = 1, 2, \ldots \), there are only finitely many vertices in the first \( l \) levels of the tree. It follows that \( P \) is totally bounded in the metric \( t \).

For each \( k = 1, 2, \ldots \), we consider the subcollection \( P_k \) consisting of all paths \( p \in P \) for which there exists a component \( G_j(r_k) \subset A_k \) such that \( p \) passes through the vertex \( G_j(r_k) \). Lemma 10.5 guarantees that each \( P_k \) is nonempty. Let \( p \in P_{k+1} \) and let \( G_j(r_{k+1}) \) be a vertex \( p \) passes through. Let \( G_j(r_k) \) be the component of \( \partial \Omega \cap B(x_0, r_k) \) containing \( G_j(r_{k+1}) \). Since \( A_{k+1} \subset A_k \), we see that \( G_j(r_k) \cap A_k \) is nonempty and as \( G_j(r_k) \) is connected and \( r_k \leq \text{dist}(x_0, \partial \Omega) \), we conclude that \( G_j(r_k) \subset A_k \). Hence \( P_{k+1} \subset P_k \) for \( k = 1, 2, \ldots \).

We now verify that each \( P_k \) is complete. Indeed, if \( \{p_n\}_{n=1}^{\infty} \subset P_k \) is a Cauchy sequence in the metric \( t \), then for every \( l = 1, 2, \ldots \), there exists \( n_l \) such that the paths \( p_{n_l} \) and \( p_{n_m} \) have the first \( l \) vertices in common, whenever \( n, m \geq n_l \). This makes it possible to construct a path \( p \in P_k \) which for every \( l = 1, 2, \ldots \), has the first \( l \) vertices in common with all \( p_{n_l}, n \geq n_l \), i.e. \( p_n \to p \) in the metric \( t \).

As \( P \) is totally bounded, it follows that all \( P_k, k = 1, 2, \ldots \), are compact. Hence \( \{F_k\}_{k=1}^{\infty} \) is a decreasing sequence of nonempty compact sets, and thus there exists an infinite path \( q \in \bigcap_{k=1}^{\infty} P_k \). The vertices through which it passes define the end \( [F_k] \) such that \( F_k = G_{j_k}(r_k) \subset A_k \), \( k = 1, 2, \ldots \). Since \( \text{diam} F_k \leq 2r_k \to 0 \) as \( k \to \infty \), this end is a prime end by Proposition 7.1 (and Lemma 7.2).

Proposition 10.7. Assume that \( \Omega \) is finitely connected at \( x_0 \in \partial \Omega \). If \( \{E_k\} \) is an end with \( x_0 \in I[E_k] \), then there is a prime end \( \{F_k\} \) dividing \( \{E_k\} \) such that \( I[F_k] = \{x_0\} \). Moreover, \( E_k = \{x_0\} \) is a prime end, then \( I[E_k] = \{F_k\} \) and \( I[F_k] = \{x_0\} \).

Proof. As \( \text{dist}(\partial \Omega \cap \partial E_k) > 0 \), at least one of \( \text{dist}(x_0, \partial \Omega \cap \partial E_k) \) and \( \text{dist}(x_0, \partial \Omega) \) must be positive. We can therefore choose a subsequence of \( \{E_k\}_{k=1}^{\infty} \) to obtain an equivalent chain, also denoted \( \{E_k\}_{k=1}^{\infty} \), where all those distances are positive. Hence, we can inductively construct a sequence \( \{r_k\}_{k=1}^{\infty} \) decreasing to zero, such that \( 0 < r_k \leq \text{dist}(x_0, \partial \Omega) \). Lemma 10.6, applied with \( A_k = E_k \), provides a prime end \( [F_k] \) with the desired properties. If moreover \( [E_k] \) is a prime end, then we must have \( [E_k] = [F_k] \) and thus \( I[E_k] = \{x_0\} \).

Theorem 10.8. Assume that \( \Omega \) is finitely connected at the boundary. Then all prime ends have singleton impressions, and every \( x \in \partial \Omega \) is the impression of a prime end and is accessible.

Furthermore, if \( 1 \leq p \in Q(x) \neq (0,1] \) for each \( x \in \partial \Omega \), then \( \partial_p \Omega \) is also the \( \text{Mod}_p \)-prime end boundary.

Proof. That all prime ends have singleton impressions follows from Proposition 10.7. If \( x \in \partial \Omega \), then applying Lemma 10.6 with \( A_k = \Omega \setminus \{y\} \) for some \( y \in \Omega \) yields a prime end \( \{F_k\} \) with \( \{x\} \) as its impression. Proposition 7.10 shows that \( x \) is accessible. Finally, Proposition 7.5 shows that all prime ends are also \( \text{Mod}_p \)-prime ends if \( p \) is as in the statement of the theorem.
The next few results relate prime ends to the Mazurkiewicz boundary. The conclusions about metrizability and compactness will be important for future studies on Dirichlet problems with respect to prime end boundaries.

The following result follows directly from Theorems 9.6 and 10.8.

Corollary 10.9. Assume that $\Omega$ is finitely connected at the boundary. Then there is a homeomorphism $\Phi : \bar{\Omega}^P \to \bar{\Omega}^M$ such that $\Phi|_{\Omega}$ is the identity map. Moreover, the prime end closure $\bar{\Omega}^P$ is metrizable with the metric $m_P(x, y) := d_M(\Phi(x), \Phi(y))$.

The topology on $\bar{\Omega}^P$ given by this metric is equivalent to the topology given by the sequential convergence discussed in Section 8.

Theorem 10.10. The following are equivalent:

(a) $\Omega$ is finitely connected at the boundary;
(b) $\bar{\Omega}^P$ is compact and all prime ends have singleton impressions;
(c) $\Omega \cup \partial_{SP} \Omega$ is compact;
(d) $\bar{\Omega}^M$ is compact.

Proof. (a) $\Leftrightarrow$ (d) This is shown in Björn–Björn–Shanmugalingam [16].
(c) $\Leftrightarrow$ (d) This follows directly from Theorem 9.6.
(a) $\Rightarrow$ (b) By Theorem 10.8, all prime ends have singleton impressions. Hence $\bar{\Omega}^P = \Omega \cup \partial_{SP} \Omega$, which is compact by the already shown implication (a) $\Rightarrow$ (c).
(b) $\Rightarrow$ (c) Since all prime ends have singleton impressions, we have that $\Omega \cup \partial_{SP} \Omega = \bar{\Omega}^P$, which is compact by assumption.

The fact that all prime ends have singleton impressions is on its own not sufficient for $\Omega$ to be finitely connected at the boundary, see e.g. the topologist’s comb in Example 9.3 whose prime end closure $\bar{\Omega}^P$ is not compact. On the other hand, the double comb below has a compact prime end closure but is not finitely connected at the boundary and has a nonsingleton prime end. (Note that the double equilateral comb in Example 5.4 does not have a compact prime end closure.)

Example 10.11. (Double comb, see Figure 4.) Let $\Omega \subset \mathbb{R}^2$ be the domain obtained from the unit square $(0, 1)^2$ by removing the collection of segments $\{0, 1 - 2^{-n}\} \times \{2^{-n}\}$ and $\{2^{-n}, 1\} \times \{3 \cdot 2^{-n-2}\}$ for $n = 1, 2, \ldots$. Then $\Omega$ has a prime end with impression $[0, 1] \times \{0\}$. Note also that this prime end is a Mod$_p$-prime end for all $p \geq 1$. 

Figure 4. Example 10.11.
We end this section by providing more details on prime ends at certain boundary points. Note that if $\Omega$ is finitely connected at a boundary point then $N(r) \geq 1$ at that point and $r \mapsto N(r)$ is nonincreasing, see Björn–Björn–Shanmugalingam [16].

**Definition 10.12.** Assume that $\Omega$ is finitely connected at $x_0 \in \partial \Omega$ and let

$$N = \lim_{r \to 0^+} N(r).$$

Then $\Omega$ is $N$-connected at $x_0$ if $N < \infty$, and locally connected at $x_0$ if $N = 1$.

If $\Omega$ is locally connected at every boundary point, then $\Omega$ is said to be locally connected at the boundary.

**Proposition 10.13.** Assume that $\Omega$ is finitely connected at $x_0 \in \partial \Omega$. Then there are exactly $N$ distinct prime ends with impression $\{x_0\}$, where $N$ is as in Definition 10.12. Furthermore, there is no other prime end with $x_0$ in its impression.

**Proof.** Assume first that $N$ is finite. Then there exists $r_0$ such that $N(r) = N$ for all $0 < r \leq r_0$. For each $j = 1, \ldots, N$ and $0 < r < r_0$, consider the components $G_j(r)$ of $B(x_0, r) \cap \Omega$ which have $x_0$ in their boundaries. We label them in such a way that $G_j(r) \subset G_j(r_0)$. It can be directly checked that for each $j = 1, \ldots, N$, the choice of $E_j = G_j(r_0/k), k = 1, \ldots$ gives us an end $[E_j]$ with impression $\{x_0\}$. Clearly, these ends are distinct since they belong to different components of $B(x_0, r_0) \cap \Omega$.

By Proposition 7.1, they are prime ends.

To see that these are the only such prime ends, let $[E_k]$ be a prime end with $x_0 \in I[E_k]$. By Lemma 10.6, applied with $A_k = E_k$, there are positive numbers $r_k$ decreasing to 0 and a singleton prime end $[F_k]$ dividing $[E_k]$ such that $F_k = G_{jk}(r_k)$ for some $1 \leq j_k \leq N(r_k) = N$. As $G_{jk}(r_k) \subset G_{jk}(r_0)$ we must have $j_k = j_0$, i.e. $[F_k] = [E_{j_0}]$. Since $[E_k]$ is a prime end it follows that $[E_k] = [E_k^0]$, showing that there are no more prime ends.

If $N$ is infinite, let $n$ be arbitrary and find $\rho_n$ such that $N(\rho_n) \geq n$. For each $j = 1, 2, \ldots, N(\rho_n)$ apply Lemma 10.6 to the sets $A_k := G_{jk}(\rho_n), k = 1, 2, \ldots$ to obtain $N(\rho_n) \geq n$ distinct prime ends with $x_0$ as their impression. Letting $n \to \infty$ shows that there are infinitely many such distinct prime ends. By Proposition 10.7 there are no other prime ends containing $x_0$ in their impressions.

**Corollary 10.14.** If $\Omega$ is locally connected at the boundary and $[E_k]$ is a prime end in $\Omega$, then $I[E_k] = \{x\}$ for some $x \in \partial \Omega$ and there exist radii $r_k^i > 0$, such that

$$B(x, r_k^i) \cap \Omega \subset E_k, \quad k = 1, 2, \ldots.$$ 

Furthermore, for each $x \in \partial \Omega$, the sets $F_k = G_{1/k}, k = 1, 2, \ldots$, define the only prime end $[F_k]$ with $x$ in its impression.

Moreover, the mapping $\Upsilon : [E_k] \to I[E_k]$ extended by identity in $\Omega$ is a homeomorphism between the prime end closure $\overline{I[E_k]}$ and the topological closure $\overline{\partial \Omega}$.

**Proof.** The pairing between prime ends and boundary points follows from Proposition 10.13, which also shows that it is a bijection.

The continuity of $\Upsilon$ is straightforward since $[E_k^n] \to [E_k^\infty]$ as $n \to \infty$, then for each $l$ there is $n_l$ such that $I[E_k^n] \subset E_k^\infty$ whenever $n \geq n_l$. As diam $E_k^\infty \to 0$, this implies that $I[E_k^n] \to I[E_k^\infty]$.

To see that $\Upsilon^{-1}$ is continuous, assume that $I[E_k^n] \to I[E_k^\infty] = \{x\}$ in the given metric. We can assume that $B(x, r_k^i) \cap \Omega \subset E_k^\infty$ for all $k$. For each $l$ there exists $n_l$ such that $d(I[E_k^n], x) < r_l^i$ whenever $n \geq n_l$. Since diam $E_k^n \to 0$ we get $E_k^n \subset B(x, r_k^i) \cap \Omega \subset E_k^\infty$ for sufficiently large $k$, i.e. $[E_k^n] \to [E_k^\infty]$. 

\[\square\]
11. (Almost) John and uniform domains

We saw in Theorem 10.8 that under some conditions all prime ends are Mod-$p$-ends. The aim of this section is to look at the converse, i.e. when are Mod-$p$-ends automatically prime ends. We will obtain this converse (for $p > Q - 1$) for uniform and John domains. In order to also include outward cusps (which are not John domains) we introduce almost John domains, which to our best knowledge have not appeared earlier in the literature.

In this section $δ_{Ω}(x)$ stands for the distance of the point $x ∈ Ω$ to $X \setminus Ω$ with respect to the given metric $d$.

**Definition 11.1.** A domain $Ω ⊂ X$ is a John domain if there is a constant $C_{Ω} ≥ 1$, called a John constant, and a point $x_{0} ∈ Ω$, called a John center, such that for every $x ∈ Ω$ there exists a rectifiable John curve $γ : [0, l_{γ}] → Ω$ parameterized by arc length, such that $x = γ(0)$, $x_{0} = γ(l_{γ})$ and

$$t ≤ C_{Ω}δ_{Ω}(γ(t)) \quad \text{for} \quad 0 ≤ t ≤ l_{γ}. \quad (11.1)$$

A domain $Ω ⊂ X$ is a uniform domain if there is a constant $C_{Ω} ≥ 1$, called a uniform constant, such that whenever $x, y ∈ Ω$ there is a rectifiable curve $γ : [0, l_{γ}] → Ω$, parameterized by arc length, connecting $x$ to $y$ and satisfying the following two conditions:

$$l_{γ} ≤ C_{Ω}d(x, y),$$

and

$$\min\{t, l_{γ} - t\} ≤ C_{Ω}δ_{Ω}(γ(t)) \quad \text{for} \quad 0 ≤ t ≤ l_{γ}.$$ 

A slit disk or a bounded domain satisfying the interior cone condition are John domains, while for instance outward cusps, such as

$$Ω = \{(x, y) ∈ \mathbb{R}^2 : 0 < y < x^3 < 1\}, \quad (11.2)$$

fail condition (11.1). Among examples of uniform domains we mention quasidisks, bounded Lipschitz domains and domains with fractal boundary such as the von Koch snowflake. See Buckley–Stanoyevitch [19], Heinonen [33], Martio–Sarvas [55], Näkki–Väisälä [62] and Väisälä [72] for more information on John and uniform domains.

Observe that uniform domains are necessarily John domains and that they are locally connected at the boundary, see Proposition 11.2 below. Note however that there are plenty of John domains which are locally connected at the boundary, but not uniform, e.g. inward cusps in $\mathbb{R}^2$ such as

$$B((0, 0), 1) \setminus \{(x, y) : 0 ≤ y ≤ x^3 < 1\}.$$

**Proposition 11.2.** If $Ω$ is a uniform domain, then it is locally connected at the boundary.

**Proof.** Let $x_{0} ∈ \partial Ω$, $r > 0$ and $x, y ∈ B(x_{0}, r/4C_{Ω}) \cap Ω$, where $C_{Ω} ≥ 1$ is a uniform constant of $Ω$. Let $G$ be the component of $B(x_{0}, r) \cap Ω$ containing $x$. Then $x$ and $y$ can be connected by a curve $γ : [0, l_{γ}] → Ω$ with length

$$l_{γ} ≤ C_{Ω}d(x, y) ≤ C_{Ω} \frac{r}{2C_{Ω}} = \frac{r}{2}$$

from which it follows that $γ ⊂ B(x_{0}, r)$ and hence $y ∈ G$. Thus, $G \cup B(x_{0}, r/4C_{Ω})$ is a neighborhood of $x_{0}$ whose intersection with $Ω$ is connected. □
In this section we will show that under some assumptions all Mod$_p$-ends in John domains are prime ends. Let us however first focus on connections with the results in the previous section.

**Theorem 11.3.** Let $\Omega$ be a John domain. Then there is a constant $N$ depending only on the doubling constant $C_\mu$, the John constant $C_\Omega$ and the quasiconvexity constant $L$, such that $\Omega$ is at most $N$-connected at every boundary point.

Recall that quasiconvexity was discussed at the end of Section 2.

This result can also be deduced from Lemma 4.3 in Aikawa–Shanmugalingam [3] by an argument similar to the one at the beginning of the proof below. Our proof is more direct and has been inspired by the proof of Theorem 2.18 in Nakiçi–Väisälä [62] for domains in $\mathbb{R}^n$. It follows from the proof below that $N$ can be chosen as the integer part of $C_\mu^2 (3LC_\Omega)^{\log_2 C_\mu}$.

**Proof.** Let $x_0$ be a John center (with John constant $C_\Omega$) and $x \in \partial \Omega$. Let $B = B(x, r)$ be a ball such that $x_0 \notin 3B$. It is enough to prove that the ball $B$ intersects at most $N$ components of $3B \cap \Omega$. The union of these components together with $B$ then makes the open neighborhood $G$ of $x$ as in Definition 10.1 (for the radius $3r$). Let $G_1, \ldots, G_k$ be some components of $3B \cap \Omega$ which intersect $B$, and let $x_j \in G_j \cap B$ for $j = 1, \ldots, k$. Since $\Omega$ is a John domain, there exist John curves $\gamma_j$ joining $x_j$ to $x_0$. As $x_0 \notin 3B$, we see that $\gamma_j \cap G_j \cap \partial 2B \neq \emptyset$. Choose $y_j = \gamma_j(t_j) \in \gamma_j \cap G_j \cap \partial 2B$. Since $x_j$ is contained in $B$ we have $d(x_j, y_j) > r$, and by the John condition, $\delta_\Omega(y_j) > r/C_\Omega$. Let $B_j = B(y_j, r/LC_\Omega)$. Since $C_\Omega \geq 1$, it follows that $LB_j \subset 3B$. If $B_i \cap B_j$ is nonempty for some $i \neq j$, then there are a point $z \in B_i \cap B_j$ and two curves $\beta_i$ and $\beta_j$ connecting $z$ to $y_i$ and $y_j$ respectively, with lengths at most

$$\max\{Ld(z, y_i), Ld(z, y_j)\} < \frac{r}{C_\Omega}.$$ 

From this it follows that $\beta_i$ and $\beta_j$ are both contained in $LB_i \cup LB_j \subset 3B \cap \Omega$. Because both $\beta_i$ and $\beta_j$ have $z \in B_i \cap B_j$ in common, $B_i$ should be contained in the same component of $3B \cap \Omega$ as $y_j$, which is not possible since $y_i \in B_i \subset G_i$. Hence the balls $B_j$, $j = 1, \ldots, k$, are pairwise disjoint. Thus by (2.1),

$$\mu(3B) \geq \sum_{j=1}^k \mu(B_j) \geq \frac{k}{C_\mu^2 (3LC_\Omega)^{\log_2 C_\mu}} \mu(3B).$$

Hence $k \leq C_\mu^2 (3LC_\Omega)^{\log_2 C_\mu}$. 

By Theorem 4.32 in Björn–Björn [10] we have an explicit estimate $L \leq 192C_\mu^4 C_\Omega$. Hence the control over $N$ can be given solely in terms of $C_\mu$, $C_\Omega$, and the constant $C_\Omega$ associated with the Poincaré inequality, but not on the dilation constant $\lambda$ in the Poincaré inequality.

The above theorem makes it possible to employ the results from the previous section. However, these conclusions and other results in this section hold for somewhat more general domains as well. We therefore introduce the following notion. Recall first that the $s$-dimensional Hausdorff content $H^s_\infty(E)$ of a set $E \subset X$ is the number

$$H^s_\infty(E) := \inf \left\{ \sum_{j=1}^\infty r_j^s : E \subset \bigcup_{j=1}^\infty B(x_j, r_j) \right\}.$$ 

**Definition 11.4.** A domain $\Omega \subset X$ is an almost John domain if for each $r > 0$ there exists a closed set $F \subset \overline{\Omega}$ such that $H^1_\infty(F) < r$ and $\Omega \setminus F$ is a John domain.
Observe that the John constant and John center of Ω \ F are allowed to depend on r. Typical examples of almost John domains which are not John domains are outward cusps such as (11.2), and the domain in Example 10.2. In both cases we can take F = B((0, 0), r) \ ∂Ω.

**Theorem 11.5.** If Ω is an almost John domain, then it is finitely connected at the boundary.

The converse is false as the domain
\[
Ω := \{ (x, y, z) \in \mathbb{R}^3 : 0 < y < x^3 < 1 \text{ and } 0 < z < 1 \}
\]
shows. Note that Ω is locally connected at the boundary.

To prove Theorem 11.5 we will use the following lemma.

**Lemma 11.6.** Let A = \{ x ∈ ∂Ω : Ω is not finitely connected at x \}. Then either A = ∅, i.e. Ω is finitely connected at the boundary, or \( H_∞^1(A) > 0 \).

This is a special case of Lemma 2.1 in Herron–Koskela [37]. They prove their result in \( \mathbb{R}^n \), but the proof is valid for the metric spaces under consideration here. For the reader's convenience we include a proof of our weaker result since our proof is simpler and more self-contained than the one in [37].

**Proof.** Assume that A ≠ ∅ and let \( x_0 \in A \). By Proposition 10.4, there is 0 < r < diam Ω such that either \( N(r) = \infty \) or \( x_0 \in \overline{H(r)} \). In either case there is a sequence \( \{ U_j \}_{j=1}^∞ \) of distinct components of \( \Omega \cap B(x_0, r) \) such that dist(\( U_j, x_0 \)) → 0 as \( j \to \infty \). Since Ω is connected, we must have \( \partial U_j \cap (∂Ω \cap ∂B(x_0, r)) \neq ∅ \).

Let 0 < \( r' < \frac{1}{2}r \). Then for \( j \) large enough, we can find \( x_j \in U_j \) such that \( d(x_0, x_j) = r' \). As \( X \) is complete and hence proper, there is a convergent subsequence \( \{ x_{j_k} \}_{k=1}^∞ \) with limit \( x' \in X \). Since the \( U_j \) are distinct we see that \( x' \in ∂Ω \). It also follows that \( d(x_0, x') = r' \).

Now let 0 < \( r'' < \frac{1}{2}r \). For each sufficiently large \( k \) there is a component \( V_k \) of \( B(x', r'') \cap ∂Ω \) such that \( x_{j_k} \in V_k \subset U_{j_k} \) and
\[
\text{dist}(V_k, x') < \text{dist}(U_{j_k}, x') + 1/k \to 0 \quad \text{as } k \to \infty.
\]
The components \( V_k \) must be distinct. Therefore, either \( x' \) is in the boundary of infinitely many of the sets \( V_k \), or else \( x' \in \overline{H(r'')} \). Again using Proposition 10.4, we see that Ω cannot be finitely connected at \( x' \).

We have thus shown that for every 0 < \( r' < \frac{1}{2}r \) there is a point \( x' \in A \) such that \( d(x_0, x') = r' \). It follows that \( H_∞^1(A) \geq \frac{1}{2}r > 0 \).

**Proof of Theorem 11.5.** Assume that Ω is not finitely connected. Then by Lemma 11.6 we have \( H_∞^1(A) > 0 \), where
\[
A = \{ x ∈ ∂Ω : Ω \text{ is not finitely connected at } x \}.
\]
Let \( F \subset ∂Ω \) be a closed set such that \( H_∞^1(F) < H_∞^1(A) \). Then there is some \( x \in A \setminus F \). As \( F \) is closed, dist(\( x, F \)) > 0. Since finite connectedness at a boundary point is a local property it follows that Ω \ F cannot be finitely connected at x. Hence Ω \ F is not a John domain, by Theorem 11.3. Thus Ω cannot be an almost John domain.

We can now collect the consequences of the results in the previous section.

**Corollary 11.7.** Let Ω be an almost John domain. Then the following are true:

(a) Every end is divisible by some prime end.
(b) Every prime end has a singleton impression.
(c) Every \( x \in \partial \Omega \) is accessible and there is at least one prime end with impression \( \{x\} \).

(d) There is a homeomorphism \( \Phi : \overline{\Omega}^P \to \overline{\Omega}^M \) such that \( \Phi|_{\partial \Omega} \) is the identity map.

(e) If
\[
1 \leq p \in \mathcal{Q}(x) \neq (0,1) \quad \text{for all} \ x \in \partial \Omega,
\]
then \( \partial_p \Omega \) is also the Mod\(_p\)-prime end boundary.

(f) The prime end closure \( \overline{\Omega}^P \) is metrizable and compact.

Proof. By Theorem 11.5, \( \Omega \) is finitely connected at the boundary.

(a) and (b) This follows from Proposition 10.7.

(e) This follows from Theorem 10.8.

(f) This follows from Corollary 10.9 and Theorem 10.10.

We also have the following consequence of a combination of Theorem 11.3, Propositions 11.2 and 10.13 and Corollary 10.14.

**Corollary 11.8.** If \( \Omega \) is a John domain, then there is a positive integer \( N \), depending only on the doubling constant, the John constant and the quasiconvexity constant, such that for every \( x \in \partial \Omega \) there is at least one, and at most \( N \), prime end with impression \( \{x\} \).

If \( \Omega \) is a uniform domain, then for every \( x \in \partial \Omega \) there is exactly one prime end with impression \( \{x\} \). Moreover, there is a homeomorphism \( \Upsilon : \overline{\Omega}^P \to \overline{\Omega} \) such that \( \Upsilon|_{\partial \Omega} \) is the identity map.

We are now ready to formulate and prove the main result of this section.

**Theorem 11.9.** If \( \Omega \) is an almost John domain and \( p > Q - 1 \), then every Mod\(_p\)-end is a prime end with singleton impression.

Note that the conclusion of Theorem 11.9 fails if \( |E_k| \) is merely an end or if \( p \leq Q - 1 \), see Examples 4.6 and 6.3. If (11.3) holds, then the existence of Mod\(_p\)-ends at every \( x \in \partial \Omega \) follows from (c) and (e) in Corollary 11.7.

To prove Theorem 11.9 we need the following lemma about chains of balls in John domains. This lemma is a variant of a chain condition first formulated by Boman, see Boman [17], Hajłasz–Koskela [32], and the references therein. In this paper we use the following chain condition.

**Definition 11.10.** We say that a set \( E \subset \Omega \) is chain-connected to \( B(x_0, \rho_0) \in \Omega \) if there exists \( M > 0 \) such that every \( x \in E \) can be connected to the ball \( B(x_0, \rho_0) = B(x_0, \rho_0) \) by a chain of balls
\[
\{B_{i,j} : i = 0,1, \ldots \text{ and } j = 0,1, \ldots, m_i\}
\]
with the following properties:

(a) For all balls \( B \) in the chain, we have \( 3\lambda B \subset \Omega \).

(b) For all \( i \) and \( j \), the ball \( B_{i,j} \) has radius \( \rho_i = 2^{-i} \rho_0 \) and center \( x_{i,j} \) such that \( d(x_{i,j}, x) \leq M \rho_i \).

(c) For all \( i \), we have \( m_i \leq M \).

(d) For large \( i \), we have \( m_i = 0 \) and the balls \( B_{i,0} \) are centered at \( x \).

(e) The balls \( B_{i,j} \) are ordered lexicographically, i.e. \( B_{i,j} \) comes before \( B_{i',j'} \) if and only if \( i < i' \) or \( i = i' \) and \( j < j' \). If \( B_{i,j} \) and \( B_{i',j'} \) are two neighbors with respect to this ordering, then \( B_{i,j} \cap B_{i',j'} \) is nonempty.

**Lemma 11.11.** Let \( \Omega \) be a John domain with a John center \( x_0 \) and a John constant \( C_\Omega \). Let \( \rho_0 \leq \delta_{B_0}(x_0)/4A \) and \( A = C_\Omega \delta_B(x_0)/\rho_0 \geq 4C_\Omega \lambda \). Then \( \Omega \) is chain-connected to \( B_{0,0} := B(x_0, \rho_0) \) in the sense of Definition 11.10 with \( M = 2A \).
Proof. For $x \in \Omega$, let $\gamma : [0, l_x] \to \Omega$ be a John curve, parameterized by arc length, connecting $x = \gamma(0)$ to $x_0 = \gamma(l_x)$. Choose the smallest possible $i_x \in \mathbb{N}$ such that $4\lambda C_\Omega \rho_{i_x} \leq \frac{1}{2} \delta_\Omega(x)$. Recall that $\rho_{i_x} = 2^{-i_x} \rho_0$.

The first ball $B_{0,0} = B(x_0, \rho_0)$ in the chain clearly satisfies $4\lambda B_{0,0} \subset \Omega$. Also, by (11.1),

$$d(x_0, x) \leq l_x \leq C_\Omega \delta_\Omega(x_0) = A\rho_0.$$  

Suppose that the ball $B_{i,j}$ has already been constructed and that it satisfies (a). Let

$$c = \inf \{ t \in [0, l_x] : \gamma(t) \in B_{i,j} \}.$$  

Assume first that $i < i_x$. If $c \geq 4\lambda C_\Omega \rho_i$, then let $B_{i,j+1} = B(x_{i,j+1}, \rho_i)$ with $x_{i,j+1} = \gamma(c)$ be the successor of $B_{i,j}$. Note that by construction and by (11.1),

$$4\lambda \rho_i \leq \frac{c}{C_\Omega} \leq \delta_\Omega(x_{i,j+1}),$$  

i.e. $4\lambda B_{i,j+1} \subset \Omega$.

If $c < 4\lambda C_\Omega \rho_i$, then let $m_i = j$ and let $B_{i+1,0} = B(x_{i+1,0}, \rho_{i+1})$ with $x_{i+1,0} = \gamma(c)$ be the successor of $B_{i,j}$. Note that (11.4) implies

$$\delta_\Omega(x_{i+1,0}) \geq \delta_\Omega(x_{i,m_i}) - \rho_i \geq 4\lambda \rho_i - \rho_i \geq 4\lambda \rho_{i+1}$$  

and hence $4\lambda B_{i+1,0} \subset \Omega$.

For $i = i_x$ and $c > 0$, let $B_{i_x,j+1} = B(x_{i_x,j+1}, \rho_{i_x})$ with $x_{i_x,j+1} = \gamma(c)$ be the successor of $B_{i_x,j}$. Note that if $c \geq 4\lambda C_\Omega \rho_{i_x}$, then (11.4) implies that $4\lambda B_{i_x,j+1} \subset \Omega$.

On the other hand, if $0 < c < 4\lambda C_\Omega \rho_{i_x}$, then the same conclusion follows from the fact that

$$d(x_{i_x,j+1}, x) \leq c < 4\lambda C_\Omega \rho_{i_x} \leq \frac{1}{2} \delta_\Omega(x)$$  

and hence

$$\delta_\Omega(x_{i_x,j+1}) \geq \delta_\Omega(x) - d(x_{i_x,j+1}, x) \geq \frac{1}{2} \delta_\Omega(x) \geq 4\lambda \rho_{i_x}.$$  

If $i = i_x$ and $c = 0$ or if $i > i_x$, then let $B_{i+1,0} = B(x, \rho_{i+1})$ be the successor of $B_{i,j}$. Then clearly

$$4\lambda \rho_i \leq 4\lambda \rho_{i+1} \leq \frac{1}{2} \delta_\Omega(x)$$  

and thus $4\lambda B_{i+1,0} \subset \Omega$.

The balls $\{B_{i,j} : i = 0, 1, \ldots$ and $j = 0, 1, \ldots, m_i\}$ cover $\gamma$ in the direction from $x_0$ to $x$ and neighboring balls always have nonempty intersection. Thus (e) is satisfied. Also (a) is satisfied by construction and the comments above.

As for the other properties, note first that if $i > i_x$, then there is only one ball with radius $\rho_i$ and that ball is centered at $x$. This proves (d), so it remains to prove (b) and (c).

For $i = 0$ and all $j \leq m_0$ we have that

$$0 \leq d(x_{0,j}, x) < l_x - j \rho_0 \leq C_\Omega \delta_\Omega(x_0) - j \rho_0 = (A - j) \rho_0,$$

showing that $m_0 \leq A$ and $d(x_{0,j}, x) \leq A\rho_0$.

Similarly, for $0 < i \leq i_x$ we have by construction that

$$0 \leq d(x_{i,j}, x) < 4\lambda C_\Omega \rho_{i-1} - j \rho_i = (8\lambda C_\Omega - j) \rho_i$$  

and hence $j < 8\lambda C_\Omega \leq 2A$. This also shows that $d(x_{i,j}, x) < 2A\rho_i$.

For $i > i_x$, (b) and (c) are obvious.\[\square\]
Corollary 11.12. Let $\Omega$ be a John domain with a John center $x_0$ and a John constant $C_\Omega$. Let $\rho_0 \leq \delta_B(x_0)/4\lambda$ and $B = B(x_0, \rho_0)$. If $p > Q - 1$, then there exists a constant $C > 0$ depending only on $C_\Omega$, $B$, $p$, the doubling constant and the constants in the $p$-Poincaré inequality, such that for all open $E \subset \Omega \setminus B$,

$$\mathcal{H}_\infty^1(E) \leq C \text{Mod}_p(E, B, \Omega).$$

Proof. This follows directly from Lemmas 11.11 and A.6.

Proof of Theorem 11.9. Let $0 < r < \text{diam} \Omega$ and $F$ be the set associated with $r$ as in Definition 11.4. Given an end $[E_k]$, set $E_k' = E_k \setminus F$ and $\Omega' = \Omega \setminus F$. By Remark 4.5 we may assume that the sets $E_k$ are open, and hence so are $E_k'$. Let $x_0$ be a John center of $\Omega'$ and $B = B(x_0, \rho) \Subset \Omega'$. If $k$ is large enough, then $E_k \cap B = \emptyset$. We consider only such $k$ in the rest of the proof.

Every curve connecting $B$ to $E_k$ in $\Omega'$ connects $B$ to $E_k' \supset E_k$ in $\Omega$ and hence

$$\text{Mod}_p(E_k', B, \Omega') \leq \text{Mod}_p(E_k, B, \Omega).$$

As $\Omega'$ is a John domain, this together with Corollary 11.12 implies that

$$\mathcal{H}_\infty^1(E_k') \leq C \text{Mod}_p(E_k', B, \Omega') \leq C \text{Mod}_p(E_k, B, \Omega),$$

where $C$ depends on $r$ but not on $E_k$. Since $E_k$ is connected, it follows that

$$\text{diam} E_k \leq \mathcal{H}_\infty^1(E_k) \leq \mathcal{H}_\infty^1(F) + \mathcal{H}_\infty^1(E_k') \leq r + C \text{Mod}_p(E_k, B, \Omega).$$

As $[E_k]$ is a Mod$_p$-end, we know that $\lim_{k \to \infty} \text{Mod}_p(E_k, B, \Omega) = 0$. Hence,

$$\limsup_{k \to \infty} \text{diam} E_k \leq r.$$ 

Letting $r \to 0$ shows that $\lim_{k \to \infty} \text{diam} E_k = 0$, and an application of Proposition 7.1 (and Lemma 7.2) completes the proof.

Proposition 11.13. Let $\Omega$ be an almost John domain. Then the following are equivalent:

(a) $[E_k]$ is a singleton end;
(b) $[E_k]$ is a prime end.

If moreover, $p > Q - 1$ and $1 \leq p \in Q(x) \neq (0, 1]$ for all $x \in \partial \Omega$, then the following statements are also equivalent to the statements above:

(c) $[E_k]$ is a Mod$_p$-end;
(d) $[E_k]$ is a Mod$_p$-prime end.

In particular, the Mod$_p$-end boundary coincides with the prime end boundary $\partial p\Omega$.

Proof. (a) $\Rightarrow$ (b) This follows from Proposition 7.1.

(b) $\Rightarrow$ (a) This follows from Corollary 11.7(b).

Assume finally that $p$ is as in the statement of the proposition.

(a) $\Rightarrow$ (d) This follows from Proposition 7.5

(d) $\Rightarrow$ (c) This is trivial.

(c) $\Rightarrow$ (a) This follows from Theorem 11.9.

Appendix. Modulus and capacity estimates

In this appendix, we will provide several estimates for the modulus and capacity needed in our study of prime ends. Recall that $p \geq 1$ and that $\lambda \geq 1$ denotes the dilation constant in the $p$-Poincaré inequality.

The following lemma will be important for our estimates.
Lemma A.1. For any choice of disjoint sets $E, F \subset \Omega$ we have
\[
\operatorname{Mod}_p(E, F, \Omega) = \operatorname{cap}_p(E, F, \Omega),
\] (A.1)
where $\operatorname{cap}_p(E, F, \Omega)$ is the $p$-capacity of the condenser $(E, F, \Omega)$ defined by
\[
\operatorname{cap}_p(E, F, \Omega) := \inf_u \int_{\Omega} g_u^p \, d\mu,
\] (A.2)
with the infimum taken over all $u \in N^{1,p}(\Omega)$ satisfying $0 \leq u \leq 1$ on $\Omega$, $u = 1$ on $E$, and $u = 0$ on $F$.

Note that both $\operatorname{Mod}_p$ and $\operatorname{cap}_p$ are symmetric with respect to the first two arguments.

For compact $E$ and $F$, equality (A.1) was obtained by Kallunki–Shanmugalingam [44], Theorem 1.1, with a more involved proof, whereas Heinonen–Koskela [36], Proposition 2.17, obtained this result using a different definition of the capacity. At that time it was not known if the two definitions give the same capacity, and it was the measurability result from Theorem 1.11 in Järvenpää–Järvenpää–Rogovin–Shanmugalingam [42] that made it possible to prove the lemma in its present form. To do so we need the following localization of Theorem 1.11 in [42].

Lemma A.2. Assume that $p \in L^p_{\text{loc}}(\Omega)$ is an upper gradient in $\Omega$ of $u : \Omega \to \mathbb{R}$. Then $u$ is measurable in $\Omega$.

Proof. For $k > 0$, let $u_k = \min\{k, \max\{-k, u\})$ be the truncation of $u$ at levels $\pm k$. Let $\Omega' \Subset \Omega$ be an arbitrary open set and find a nonnegative Lipschitz function $\eta$ with $\operatorname{supp} \eta \subset \Omega$ such that $\eta = 1$ on $\Omega'$ and $0 \leq \eta \leq 1$ on $X$. Let $g \in L^p(\Omega)$ be an upper gradient of $\eta$ in $X$.

We shall show that the function $\rho \chi_{\operatorname{supp} \eta} + kg$ is an upper gradient of $u_k \eta$ in $X$. To do so, let $\gamma : [0, l_\gamma] \to X$ be a nonconstant rectifiable curve in $X$. If $\gamma \subset X \setminus \operatorname{supp} \eta$, then $\left| (u_k \eta)(\gamma(0)) - (u_k \eta)(\gamma(l_\gamma)) \right| = 0 \leq \int_\gamma (\rho \chi_{\operatorname{supp} \eta} + kg) \, ds$. Otherwise, by splitting $\gamma$ into two parts and possibly reversing the orientation, we can assume that $x := \gamma(0) \in \operatorname{supp} \eta$, and that either $\gamma \subset \operatorname{supp} \eta$ or that $\gamma(l_\gamma) \notin \operatorname{supp} \eta$. In the latter case, let $c = \inf\{t : \gamma(t) \notin \operatorname{supp} \eta\}$, so that $\gamma([0, c]) \subset \operatorname{supp} \eta$ and $\eta(\gamma(c)) = 0$. In the former case let $c = l_\gamma$. In both cases we have, with $\gamma' = \gamma|_{[0, c]}$, $z = \gamma(c)$ and $y = \gamma(l_\gamma)$, that
\[
\left| (u_k \eta)(x) - (u_k \eta)(y) \right| = \left| (u_k \eta)(x) - (u_k \eta)(z) \right| \leq |u_k(x)||\eta(x) - \eta(z)| + |\eta(z)||u_k(x) - u_k(z)| \leq k \int_{\gamma'} g \, ds + \int_{\gamma'} \rho \, ds \leq \int_{\gamma} (\rho \chi_{\operatorname{supp} \eta} + kg) \, ds.
\]
As $\gamma$ was arbitrary, $\rho \chi_{\operatorname{supp} \eta} + kg \in L^p(\Omega)$ is an upper gradient of $u_k \eta$ in $X$. Theorem 1.11 in [42] implies that $u_k \eta$ is measurable in $X$. Letting $k \to \infty$ implies that $u = \lim_{k \to \infty} u_k \eta$ is measurable in $\Omega'$. Since $\Omega' \Subset \Omega$ was arbitrary, the result follows.

Proof of Lemma A.1. To see the validity of (A.1), note that by (2.4) and the fact that for the minimal $p$-weak upper gradient $g_v$ of an (everywhere defined) function $v \in N^{1,p}(\Omega)$ there are upper gradients $g_1$ of $v$ such that $g_1 \to g_v$ in $L^p(\Omega)$ (see Koskela–MacManus [49]), we have $\operatorname{cap}_p(E, F, \Omega) \geq \operatorname{Mod}_p(E, F, \Omega)$. On the other
hand, if $\rho \in L^p(\Omega)$ is an admissible function used for computing $\text{Mod}_p(E, F, \Omega)$, then we define a function $u$ on $\Omega$ by

$$u(x) = \min \left\{ 1, \inf_{\gamma \in \mathcal{E}} \int_{\gamma \cap x} \rho \, ds \right\},$$

where the infimum is taken over all rectifiable curves connecting $E$ to $x$ in $\Omega$. Observe that $u = 0$ on $E$, $u = 1$ on $F$ and $\rho$ is an upper gradient of $u$, by Lemma 3.1 in Björn–Björn–Shanmugalingam [14] (or Lemma 5.25 in Björn–Björn [10]). By Lemma A.2, the function $u$ is measurable in $\Omega$, and since $|u| \leq 1$ and $\Omega$ is bounded, it follows that $u \in N^{1,p}(\Omega)$ and

$$\text{cap}_p(E, F, \Omega) \leq \int_{\Omega} g_u^p \, d\mu \leq \int_{\Omega} \rho^p \, d\mu.$$ 

Hence, by taking infimum over all such $\rho$ we conclude that

$$\text{cap}_p(E, F, \Omega) \leq \text{Mod}_p(E, F, \Omega).$$

\textbf{Lemma A.3.} Let $E, F \subset \Omega$ be disjoint and with nonempty interiors. Then

$$\text{Mod}_p(E, F, \Omega) = \text{cap}_p(E, F, \Omega) > 0.$$ 

\textbf{Proof.} The equality follows from Lemma A.1. It is thus enough to show that

$$\int_{\Omega} g_u^p \, d\mu \geq c > 0$$

for every $u \in N^{1,p}(\Omega)$ such that $u = 1$ on $E$ and $u = 0$ on $F$. Note that if there are no such functions $u$, then the theorem holds trivially since then $\text{cap}_p(E, F, \Omega) = \infty > 0$.

Let $x$ and $y$ be points in the interiors of $E$ and $F$, respectively. Since $X$ is quasiconvex, Lemma 4.38 in Björn–Björn [10] implies that $\Omega$ is rectifiable connected and we can thus find a rectifiable curve $\gamma : [0, l_0] \to \Omega$ connecting $x$ to $y$. Let $0 < r < \text{dist}(\gamma, X \setminus \Omega)/3\lambda$ be such that both $B(x, r) \subset E$ and $B(y, r) \subset F$. Cover $\gamma$ by balls $B_j = B(x_j, r)$, $j = 0, 1, \ldots, n$, such that $B_0 = B(x, r)$, $B_n = B(y, r)$ and $B_j \cap B_{j+1}$ is nonempty for $j = 0, 1, \ldots, n-1$. Then $B_{j+1} \subset 3B_j$ and $B_j \subset 3B_{j+1}$ for $j = 1, \ldots, n-1$.

Let $u \in N^{1,p}(\Omega)$ be such that $u = 1$ on $E$ and $u = 0$ on $F$. Then, since $\mu$ is doubling,

$$|u_{B_j} - u_{B_{j+1}}| \leq |u_{B_j} - u_{3B_j}| + |u_{B_{j+1}} - u_{3B_j}| \leq C \int_{3B_j} |u - u_{3B_j}| \, d\mu.$$ 

The $p$-Poincaré inequality then yields that

$$1 = |u_{B_0} - u_{B_n}| \leq \sum_{j=0}^{n-1} |u_{B_j} - u_{B_{j+1}}| \leq C \sum_{j=0}^{n-1} \int_{3B_j} |u - u_{3B_j}| \, d\mu$$

$$\leq C \sum_{j=0}^{n-1} r \left( \int_{3\lambda B_j} g_u^p \, d\mu \right)^{1/p} \leq \tilde{C} \left( \int_{\Omega} g_u^p \, d\mu \right)^{1/p},$$

where $\tilde{C}$ is independent of $u$ (but depends on $r$ and the sequence of balls $\{B_j\}_{j=0}^n$). Taking infimum over all admissible functions $u$ yields the desired result. \hfill \square

The following estimate is crucial for showing that singleton ends are $\text{Mod}_p$-ends in Proposition 7.5. Recall that $Q(x)$ was defined in Definition 2.1.
Lemma A.4. Let \( x \in \overline{\Omega} \). If \( 1 \leq p \in Q(x) \neq (0, 1] \), then for every compact \( K \subset \Omega \setminus \{x\} \),

\[
\lim_{r \to 0} \text{Mod}_p(B(x, r) \cap \Omega, K, \Omega) = 0.
\]  

(A.3)

The following example shows that we cannot allow \( p = 1 \) and \( Q(x) = (0, 1] \) in Lemma A.4.

Example A.5. Let \( X = \mathbb{R} \) (unweighted), \( \Omega = (-1, 1) \), \( K = \{0\} \), \( x = 1 \) and \( 0 < r < 1 \). Then every function \( u \) admissible for \( \text{cap}_p(B(x, r) \cap \Omega, K, \Omega) \) satisfies \( u(0) = 0 \) and \( u(1 - r) = 1 \). Hence

\[
\int_{-1}^{1} |u'(t)| \, dt \geq u(1 - r) - u(0) = 1,
\]

resulting in \( \text{cap}_p(B(x, r) \cap \Omega, K, \Omega) \geq 1 \). In view of Lemma A.1, we see that (A.3) fails.

Proof of Lemma A.4. By Lemma A.1 it suffices to show that

\[
\lim_{r \to 0} \text{cap}_p(B(x, r) \cap \Omega, K, \Omega) = 0.
\]

(A.4)

Assume first that \( 1 < p < q_0 \), where \( q_0 \) is the right end point of \( Q(x) \). Choose \( \varepsilon > 0 \) such that \( q := p + \varepsilon \in Q(x) \). Theorem 3.3 in Garofalo–Marola [27] together with the upper mass bound estimate (2.3) implies that for all sufficiently small \( R > 0 \) there exists \( C(R) > 0 \) such that for all \( 0 < r < R \) we have

\[
\text{cap}_p(B(x, r), X \setminus B(x, R), X) \leq C(R)r^{-p}\mu(B(x, r))
\]

\[
\leq C(R)r^{-p}C_q\mu(B(x, R))\left(\frac{r}{R}\right)^q
\]

\[
= C(R)C_qR^{-q}\mu(B(x, R)) \to 0 \quad \text{as} \quad r \to 0. 
\]

(A.5)

If instead \( Q(x) = (0, q_0] \neq (0, 1] \) and \( p = q_0 \), then \( p > 1 \) and the estimate from Theorem 3.3 in [27] becomes

\[
\text{cap}_p(B(x, r), X \setminus B(x, R), X) \leq C(R)\left(\log\frac{R}{r}\right)^{1-p} \to 0 \quad \text{as} \quad r \to 0.
\]

(A.6)

Now, let \( R > 0 \) be sufficiently small and such that \( B(x, R) \subset X \setminus K \). Since every \( u \) admissible in the definition of \( \text{cap}_p(B(x, r), X \setminus B(x, R), X) \) is also admissible for \( \text{cap}_p(B(x, r) \cap \Omega, K, \Omega) \), we conclude from (A.5) and (A.7) that (A.4) holds for all \( 1 < p \in Q(x) \).

Finally, if \( p = 1 \in Q(x) \neq (0, 1] \), then we have by above that (A.3) holds for some \( q > 1 \). As \( \Omega \) is bounded, the Hölder inequality implies that for every \( \rho \) admissible in the definition of \( \text{Mod}_q(B(x, r) \cap \Omega, K, \Omega) \) (and thus also for \( \text{Mod}_1(B(x, r) \cap \Omega, K, \Omega) \)) we have

\[
\int_{\Omega} \rho \, d\mu \leq \mu(\Omega)^{1-1/q}\left(\int_{\Omega} \rho^q \, d\mu\right)^{1/q}.
\]

Taking infimum over all such \( u \) shows that

\[
\text{Mod}_1(B(x, r) \cap \Omega, K, \Omega) \leq \mu(\Omega)^{1-1/q}\text{Mod}_q(B(x, r) \cap \Omega, K, \Omega)^{1/q}
\]

and (A.3) holds also for \( p = 1 \) in this case. \( \square \)

Next, we shall relate the modulus to the Hausdorff content.
Lemma A.6. Let $E \subset \Omega$ be open and $B(x_0, r) \subseteq \Omega \setminus E$. Assume that there exist $M > 0$ and $0 < \rho_0 \leq r$ such that $E$ can be chain-connected to the ball $B_{0,0} = B(x_0, \rho_0)$ as in Definition 11.10. Let $s > 0$ and $p > Q - s$. Then there exists a constant $C$ depending only on $M$, $p$, $s$, $Q$, $r$, the doubling constant $C_\mu$ and on the constants in the Poincaré inequality such that

$$\mathcal{H}^s_\infty(E) \leq C \text{cap}_p(E, B(x_0, r), \Omega) = C \text{Mod}_p(E, B(x_0, r), \Omega).$$

The proof of this lemma is based on a technique introduced in Hajlasz–Koskela [31].

Proof. In view of Lemma A.1 it suffices to estimate $\mathcal{H}^s_\infty(E)$ using $\text{cap}_p(E, B(x_0, r), \Omega)$. Let $u \in N^{1,p}(\Omega)$ be such that $u = 0$ on $B(x_0, r)$ and $u = 1$ on $E$. Consider $x \in E$ and let $C_x = \{B_{i,j} : i = 0, 1, \ldots$ and $j = 0, 1, \ldots, m_i\}$ be the corresponding chain. For each ball $B$ in the chain let $B^*$ be its immediate successor. Then $B \cap B^*$ is nonempty and $\frac{1}{2} r(B) \leq r(B^*) \leq r(B)$, where $r(B)$ is the radius of $B$. Thus $B^* \subset 3B$ and $B \subset 5B^*$. Note also that properties (b), (c) and (e) imply that for all $i = 0, 1, \ldots$ and $j = 0, 1, \ldots, m_i$, we have

$$d(x_{i,j}, x_0) \leq (M + 1) \sum_{k=0}^i 2\rho_k < 4(M + 1)\rho_0. \quad (A.8)$$

Since $u = 1$ in the open set $E$ containing $x$, a telescopic argument together with assumption (d) implies that

$$1 = \lim_{i \to \infty} |u_{B_{i,0}} - u_{B_{0,0}}| \leq \sum_{B \in C_x} |u_B - u_{B^*}| \leq \sum_{B \in C_x} (|u_B - u_{3B}| + |u_{B^*} - u_{3B}|). \quad (A.9)$$

The doubling property and the $p$-Poincaré inequality yield

$$|u_{B^*} - u_{3B}| \leq C \int_{3B} |u - u_{3B}| \, d\mu \leq C r(B) \left( \int_{3B} g_\mu^p \, d\mu \right)^{1/p}. \quad (A.9)$$

The difference $|u_B - u_{3B}|$ is estimated similarly and inserting both estimates into (A.9) implies that

$$1 \leq C \sum_{B \in C_x} \frac{r(B)}{\mu(3\lambda B)^{1/p}} \left( \int_{3\lambda B} g_\mu^p \, d\mu \right)^{1/p}. \quad (A.9)$$

For each $B \in C_x$, (A.8) together with (2.1) gives

$$\mu(3\lambda B) \geq \mu(B) \geq C \left( \frac{r(B)}{4(M + 1)\rho_0} \right)^Q \mu(4(M + 1)B_{0,0}) \geq C \left( \frac{r(B)}{4(M + 1)\rho_0} \right)^Q \mu(B_{0,0}).$$

The last estimate then becomes

$$1 \leq \frac{C\rho_0^{Q/p}}{\mu(B_{0,0})^{1/p}} \sum_{B \in C_x} r(B)^{1 - Q/p} \left( \int_{3\lambda B} g_\mu^p \, d\mu \right)^{1/p},$$

where $C$ depends only on $M$, $p$, $Q$, the doubling constant $C_\mu$, and on the constants in the Poincaré inequality, but not on $u$.

Since $p > Q - s$, we have $p - Q + s > 0$ and hence

$$1 = C \sum_{i=1}^{\infty} 2^{-i(p-Q+s)/p} \geq C \frac{1}{M} \sum_{B \in C_x} \left( \frac{r(B)}{\rho_0} \right)^{(p-Q+s)/p},$$
where $C$ depends only on $p$, $Q$ and $s$. Comparing the last two estimates we see that there exists a ball $B_x \in \mathcal{C}_z$ such that
\[
\left( \frac{r(B_x)}{r} \right)^{(p-Q+s)/p} \leq \left( \frac{r(B_x)}{\rho_0} \right)^{(p-Q+s)/p} \leq \frac{C r^{Q/p}}{\mu(B_{0,0})} r(B_x)^{1-Q/p} \left( \int_{3\lambda B_z} g_h^p \, d\mu \right)^{1/p},
\]
where $C$ depends only on $M$, $p$, $Q$, $s$, $C_\mu$ and the constants in the Poincaré inequality, but not on $u$ or $x$.

Repeating this argument for every $x \in E$, we obtain balls $B_x$, such that
\[
r(B_x)^s \leq \frac{C r^{p+s}}{\mu(B_{0,0})} \int_{3\lambda B_z} g_h^p \, d\mu. \tag{A.10}
\]
Note that $x \in 2M B_z$ for all $x \in E$ by (b). Hence, the balls $\{2M B_z\}_{x \in E}$ cover $E$, as do the balls $\{3\lambda B_z\}_{x \in E}$. The 5-covering lemma (Theorem 1.2 in Heinonen [33]) allows us to choose pairwise disjoint balls $3\lambda B_{z_i}, i = 1, 2, \ldots$, so that $E \subset \bigcup_{i=1}^\infty 15\lambda B_{z_i}$. In particular, the balls $3\lambda B_{z_i}, i = 1, 2, \ldots$, are pairwise disjoint. Thus we get from (A.10) that
\[
\mathcal{H}_\infty^s(E) \leq \sum_{i=1}^\infty r(15\lambda B_{z_i})^s = 15^s M^s \lambda^s \sum_{i=1}^\infty r(B_{z_i})^s \leq \frac{C r^{p+s}}{\mu(B_{0,0})} \sum_{i=1}^\infty \int_{3\lambda B_{z_i}} g_h^p \, d\mu \leq \frac{C r^{p+s}}{\mu(B_{0,0})} \int_\Omega g_h^p.
\]
Taking infimum over all admissible functions $u$ completes the proof. \hfill \qed

**Lemma A.7.** Let $E \Subset \Omega$ and $B = B(x_0, r) \Subset \Omega \setminus E$. Then there exists $0 < \rho_0 < r$ such that $E$ can be chain-connected to the ball $B_{0,0} = B(x_0, \rho_0)$ as in Definition 11.10.

**Proof.** Since $\Omega$ is connected, there exists $0 < \varepsilon < r$ such that both $\overline{B}$ and $\overline{E}$ belong to the same component $G$ of
\[
\Omega_\varepsilon := \{ x \in \Omega : \text{dist}(x, X \setminus \Omega) > \varepsilon \},
\]
see Lemma 4.49 in Björn–Björn [10]. Choose $0 < \rho_0 \leq \varepsilon / 6\lambda$ and let $B_i = B(x_i, \rho_0/2), i = 1, \ldots, N$, be a maximal pairwise disjoint collection of balls with centers in $\Omega_\varepsilon$. By the doubling property, there are only finitely many such balls and their number $N$ depends only on $\varepsilon, \rho_0$ and the doubling constant $C_\mu$. The balls $\{2B_i\}_{i=1}^N$ cover $\Omega_\varepsilon$ and $6\lambda B_i \subset \Omega$ for all $i = 1, 2, \ldots, N$.

Let $x \in E$ be arbitrary. By pathconnectedness of the component $G$, there exists a curve $\gamma$ in $G$ from $x_0$ to $x$. We can therefore among the balls $2B_i, i = 1, 2, \ldots, N$, choose a minimal chain of balls covering $\gamma$. Number these balls in the direction from $x_0$ to $x$ and call them $B_{0,j}, j = 1, 2, \ldots, m_0$. Clearly, $m_0 \leq N$ and neighboring balls in the chain have nonempty intersection. Complete the chain by the balls $B_{i,0} = B(x, \rho_i), \rho_i = 2^{-i} \rho_0, i = 1, 2, \ldots$. It remains to verify that the conditions (a)–(e) of Lemma A.6 are satisfied. The only property that needs some justification is that $d(x_{i,j}, x) \leq M \rho_i$ with $M = \max\{N, 2/\varepsilon \rho_0\}$. For $i \geq 1$, this is trivial and for $i = 0$ we have $d(x_{0,j}, x) \leq \text{diam} \Omega_\varepsilon \leq 2/\varepsilon$. The other properties follow by construction. \hfill \qed
Remark A.8. The proof of Lemma A.7 shows that \( M = \max\{N, 2/\varepsilon p_0\} \). It follows that \( M \) (and hence also \( C \) in Lemma A.6) depends on \( \text{dist}(E, X \setminus \Omega) \). The estimate in Lemma A.6 therefore does not apply if we only know that \( E \subset \Omega \). Indeed, in the topologist’s comb in Example 5.1 we have for \( p \leq 2 \), every compact \( K \subset \Omega \),

\[
\text{Mod}_p(E_k, K, \Omega) = \text{Mod}_p(F_k, K, \Omega) \to 0, \quad \text{as } k \to \infty,
\]

by Proposition 7.5, where \( F_k = \left(\frac{1}{2} - 2^{-k}, \frac{1}{2} + 2^{-k}\right) \times (0, 2^{-k}) \cap \Omega \). On the other hand, \( H^1_\infty(E_k) \geq \frac{1}{2} \) for all \( k = 1, 2, \ldots \). See, however, Lemma 11.11.

Corollary A.9. Let \( E \subset \Omega \) be open and \( B \subset \Omega \setminus E \) be a ball. If \( p > Q - 1 \), then there exists \( C > 0 \) depending on \( \text{dist}(E, X \setminus \Omega) \) such that

\[
H^1_\infty(E) \leq C \text{Mod}_p(E, B, \Omega).
\]

Proof. This follows directly from Lemmas A.6 and A.7. \( \square \)

Lemma A.10. Let \( \{E_k\}_{k=1}^\infty \) be a sequence of open acceptable sets satisfying \( \bigcap_{k=1}^\infty \Omega \subset E_k \) for each \( k \). If \( \lim_{k \to \infty} \text{Mod}_p(E_k, B, \Omega) = 0 \) for some ball \( B \subset \Omega \setminus E_1 \) and \( p > Q - 1 \), then \( I := \bigcap_{k=1}^\infty E_k \subset \partial \Omega \).

Proof. Clearly, \( I \) is a compact connected set and \( I \cap \partial \Omega \) is nonempty. Let

\[
\Omega_\delta = \{x \in \Omega : \text{dist}(x, X \setminus \Omega) > \delta\}.
\]

Corollary A.9 applied to \( E_k \cap \Omega_\delta \) shows that \( H^1_\infty(I \cap \Omega_\delta) = 0 \) for each \( \delta > 0 \). It follows that \( H^1(I \cap \Omega_\delta) = 0 \) and hence also \( H^1(I \cap \Omega) = 0 \), where \( H^1 \) denotes the one-dimensional Hausdorff measure.

Assume that \( I \cap \Omega \) is nonempty and find \( \delta > 0 \) such that \( I \cap \Omega_{2\delta} \neq \emptyset \). Fix \( x \in I \cap \Omega_{2\delta} \) and let \( U \) be the component of \( I \cap \Omega \) containing \( x \). Note that \( U \) is connected but not necessarily pathconnected. Let \( 0 < \varepsilon < \delta/2 \), and cover \( U \) by (finitely or countably many) balls \( B_j = B(x_j, r_j) \) so that \( \sum_j r_j < \varepsilon \). Let \( V \) consist of all points \( y \in U \) for which there exists a chain \( \{B_{j_k}\}_{k=1}^{N_j} \) (depending on \( y \)) of balls from this cover such that \( x \in B_{j_1} \), \( y \in B_{N_j} \), and \( B_{j_k} \cap B_{j_{k+1}} \neq \emptyset \) for all \( k \). If \( y \in V \cap B_j \) for some \( j \), then \( U \cap B_j \subset V \) and hence \( V \) is open in \( U \). Similarly, \( U \setminus V \) is open in \( U \) and must therefore be empty, since \( U \) is connected.

Assume next that \( U \cap \partial \Omega_\delta = \emptyset \). Then the connected set \( I \) could be written as a disjoint union of the nonempty relatively open sets \( U \cap \Omega_\delta \) and \( I \setminus (U \cap \Omega_\delta) \), which is a contradiction. Thus, there exists \( z \in U \cap \partial \Omega_\delta \subset V \). Hence there is a chain \( \{B_{j_k}\}_{k=1}^{N_j} \) of balls from the cover of \( U \) above satisfying \( x \in B_{j_1} \), \( z \in B_{N_j} \), and \( B_{j_k} \cap B_{j_{k+1}} \) nonempty, and so

\[
\delta \leq d(z, x) \leq \sum_{k=1}^{N_j} 2r_{j_k} \leq 2 \sum_j r_j < 2 \varepsilon < \delta,
\]

which is not possible. This contradiction shows that \( I \cap \Omega \) must be empty, i.e. \( I \subset \partial \Omega \). \( \square \)

Lemma A.11. Let \( [E] \) be an end and \( p > 1 \). Then \( \lim_{j \to \infty} \text{Mod}_p(E_j, K, \Omega) = 0 \) for every compact \( K \subset \Omega \) if and only if \( \lim_{j \to \infty} \text{Mod}_p(E_j, K_0, \Omega) = 0 \) for some compact \( K_0 \subset \Omega \) with \( C_p(K_0) > 0 \).

Here

\[
C_p(E) := \inf \|u\|_{N^{1,p}(X)}^p,
\]

where the infimum is taken over all everywhere defined functions \( u \in N^{1,p}(X) \) such that \( u \geq 1 \) on \( E \). We say that a property holds quasieverywhere (q.e.) if the set of points for which it fails has \( C_p \) capacity zero.
An alternative proof of this result, using superharmonic functions, is given as Proposition 5.14 in the forthcoming paper Björn–Björn [11] (which needs to be combined with Lemma A.1 to give Lemma A.11).

**Proof.** Assume that \( \lim_{j \to \infty} \text{Mod}_p(E_j, K_0, \Omega) = 0 \) for some compact set \( K_0 \subset \Omega \) with positive capacity, and let \( K \subset \Omega \) be compact.

By Lemma A.1, we can equivalently work with capacities. We can thus find \( u_j \in N^{1,p}(\Omega) \) admissible in the definition of \( \text{cap}_p(E_j, K_0, \Omega) \), such that \( g_{u_j} \to 0 \) in \( L^p(\Omega) \), as \( j \to \infty \). Lemma 3.2 in Björn–Björn–Parviainen [12] or Lemma 6.2 in Björn–Björn [10] provides us with \( u \in N^{1,p}(\Omega) \) and convex combinations \( v_j = \sum_{i=1}^{N_j} a_{j,i} u_i \) and \( g_j = \sum_{i=1}^{N_j} a_{j,i} g_i \) such that \( v_j \to u \) both in \( L^p(\Omega) \) and q.e., and \( g_j \to g \) in \( L^p(\Omega) \), where \( g \) is a \( p \)-weak upper gradient of \( u \). Note that \( v_j = 0 \) on \( K_0 \) and \( v_j = 1 \) on \( E_{N_j}, j = 1, 2, \ldots \). Since \( g_{u_j} \to 0 \) in \( L^p(\Omega) \), we see that \( g = 0 \) a.e. in \( \Omega \). As \( \Omega \) is connected, the Poincaré inequality implies that \( u \) is constant q.e. in \( \Omega \). Since \( u = 0 \) q.e. on the set \( K_0 \) with positive capacity, we must have \( u = 0 \) q.e. in \( \Omega \), and hence \( v_j \to 0 \) in \( N^{1,p}(\Omega) \).

Corollary 3.9 in Shanmugalingam [69] or Corollary 1.72 in [10] implies that \( v_j \to 0 \) locally quasi-uniformly, i.e. for every \( \varepsilon > 0 \) we can find \( A_\varepsilon \subset \Omega \) such that \( C_p(A_\varepsilon) < \varepsilon \) and \( v_j \to 0 \) uniformly in \( K \setminus A_\varepsilon \). It follows that for sufficiently large \( j \), we have \( v_j < \frac{\varepsilon}{2} \) on \( K \setminus A_\varepsilon \), and thus the function \( w_j = \max\{0, 2(v_j - \frac{\varepsilon}{2})\} \) is admissible in the definition of \( \text{cap}_p(E_{N_j}, K \setminus A_\varepsilon, \Omega) \), i.e.

\[
\text{cap}_p(E_{N_j}, K \setminus A_\varepsilon, \Omega) \leq \int_\Omega g_{w_j}^p \, d\mu \leq 2^p \int_\Omega g_{v_j}^p \, d\mu \leq 2^p \int_\Omega g_j^p \, d\mu. \tag{A.11}
\]

Next, if \( v \in N^{1,p}(X) \) is admissible in the definition of \( C_p(A_\varepsilon) \), then for some nonnegative Lipschitz function \( \eta \) with compact support in \( \Omega \) and such that \( \eta = 1 \) on \( K \), the function \( 1 - vg \) is admissible in the definition of \( \text{cap}_p(E_j, K \cap A_\varepsilon, \Omega) \), for sufficiently large \( j \). Hence by the Leibniz rule (Theorem 2.15 in [10]) we obtain

\[
\text{cap}_p(E_j, K \cap A_\varepsilon, \Omega) \leq \int_\Omega g_{1-vg}^p \, d\mu \leq \int_\Omega (vg_\eta + \eta g) \, d\mu \leq C\|v\|_{N^{1,p}(X)}
\]

for sufficiently large \( j \). Taking infimum over all such \( v \) shows that \( \text{cap}_p(E_j, K \cap A_\varepsilon, \Omega) \leq C\varepsilon \) for sufficiently large \( j \). Combining this with (A.11) we obtain that, for sufficiently large \( j \),

\[
\text{cap}_p(E_{N_j}, K, \Omega) \leq 2^p \int_\Omega g_j^p \, d\mu + C\varepsilon \to C\varepsilon, \quad \text{as } j \to \infty.
\]

Letting \( \varepsilon \to 0 \) finishes the proof, since the converse implication is trivial. \( \square \)

**References**

[1] T. Adamowicz and N. Shanmugalingam, Non-conformal Loewner type estimates for modulus of curve families, *Ann. Acad. Sci. Fenn. Math.* **35** (2010), 609–626.

[2] L. V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York–Düsseldorf–Johannesburg, 1973.

[3] H. Aikawa and N. Shanmugalingam, Carleson-type estimates for \( p \)-harmonic functions and the conformal Martin boundary of John domains in metric measure spaces, *Michigan Math. J.* **53** (2005), 165–188.

[4] A. Ancona, Une propriété de la compactification de Martin d’un domaine euclidien, *Ann. Inst. Fourier (Grenoble)* **29**:4 (1979), ix, 71–90.
[5] A. ANCONA, Negatively curved manifolds, elliptic operators, and the Martin boundary, *Ann. of Math.* **125** (1987), 495–536.
[6] A. ANCONA, Sur la théorie du potentiel dans les domaines de John, *Publ. Mat.* **51** (2007), 345–396.
[7] M. T. ANDERSON and R. SCHOEN, Positive harmonic functions on complete manifolds of negative curvature, *Ann. of Math.* **121** (1985), 429–461.
[8] A. BEURLING, A minimum principle for positive harmonic functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **372** (1965), 1–7.
[9] A. BJÖRN, The Dirichlet problem for p-harmonic functions on the comb, *In preparation.*
[10] A. BJÖRN and J. BJÖRN, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics **17**, European Math. Soc., Zurich, 2011.
[11] A. BJÖRN and J. BJÖRN, Obstacle and Dirichlet problems on arbitrary nonopen sets in metric spaces, and fine topology, *In preparation.*
[12] A. BJÖRN, J. BJÖRN and M. PARVIAINEN, Lebesgue points and the fundamental convergence theorem for superharmonic functions on metric spaces, *Rev. Mat. Iberoam.* **26** (2010), 147–174.
[13] A. BJÖRN, J. BJÖRN and N. SHANMUGALINGAM, The Perron method for p-harmonic functions, *J. Differential Equations* **195** (2003), 398–429.
[14] A. BJÖRN, J. BJÖRN and N. SHANMUGALINGAM, Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces, *Houston J. Math.* **34** (2008), 1197–1211.
[15] A. BJÖRN, J. BJÖRN and N. SHANMUGALINGAM, The Dirichlet problem for p-harmonic functions with respect to the Mazurkiewicz boundary, *In preparation.*
[16] A. BJÖRN, J. BJÖRN and N. SHANMUGALINGAM, The Mazurkiewicz distance and sets which are finitely connected at the boundary, *In preparation.*
[17] J. BOMAN, $L^p$ estimates for very strongly elliptic systems, *Report no. 29*, Department of Mathematics, Stockholm University, Stockholm, 1982.
[18] N. BOURBAKI, *Elements of the History of Mathematics*, Springer, Berlin–Heidelberg, 1994.
[19] S. BUCKLEY and A. STANOYEVITCH, Weak slice conditions, product domains, and quasiconformal mappings, *Rev. Mat. Iberoam.* **17** (2001), 607–642.
[20] C. CARATHÉODORY, Über die Begrenzung einfach zusammenhängender Gebiete, *Math. Ann.* **73** (1913), 323–370.
[21] J. CHEEGGER, Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.* **9** (1999), 428–517.
[22] E. F. COLLINGWOOD and A. J. LOHWARD, *The Theory of Cluster Sets*, Cambridge Tracts in Math. and Math. Phys. **56**, Cambridge Univ. Press, Cambridge, 1966.
[23] T. COULHON, I. HOLOPAINEN and L. SALOUFF-COSTE, Harnack inequality and hyperbolicity for subelliptic p-Laplaceans with applications to Picard type theorems, *Geom. Funct. Anal.* **11** (2001), 1139–1191.
[24] J. L. DOOB, Conformally invariant cluster value theory, *Illinois J. Math.* **5** (1961), 521–549.
[25] E. DURAND-CARTAGENA, N. SHANMUGALINGAM and A. WILLIAMS, $p$-Poincaré inequality vs. $\infty$-Poincaré inequality; some counterexamples, to appear in *Math. Z.*
[26] D. B. A. EPSTEIN, Prime ends, *Proc. Lond. Math. Soc.* **42** (1981), 385–414.
[27] N. GAROFALO and N. MAROLA, Sharp capacitary estimates for rings in metric spaces, *Houston J. Math.* **36** (2010), 681–695.
[28] A. Grigor’yan, Escape rate of Brownian motion on Riemannian manifolds, Appl. Anal. 71 (1999), 63–89.

[29] A. Grigor’yan and L. Saloff-Coste, Heat kernel on connected sums of Riemannian manifolds, Math. Res. Lett. 6 (1999), 307–321.

[30] P. Hajłasz, Sobolev spaces on metric-measure spaces, in Heat Kernels and Analysis on Manifolds, Graphs and Metric Spaces (Paris, 2002), Contemp. Math. 338, pp. 173–218, Amer. Math. Soc., Providence, RI, 2003.

[31] P. Hajłasz and P. Koskela, Sobolev meets Poincaré, C. R. Acad. Sci. Paris Ser. I Math. 320 (1995), 1211–1215.

[32] P. Hajłasz and P. Koskela, Sobolev meets Poincaré, Mem. Amer. Math. Soc. 145:688 (2000).

[33] J. Heinonen, Lectures on Analysis on Metric Spaces, Springer, New York, 2001.

[34] J. Heinonen, T. Kilpeläinen and O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, 2nd ed., Dover, Mineola, NY, 2006.

[35] J. Heinonen and P. Koskela, From local to global in quasiconformal structures, Proc. Nat. Acad. Sci. U.S.A. 93 (1996), 554–556.

[36] J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, Acta Math. 181 (1998), 1–61.

[37] D. A. Herron and P. Koskela, Locally uniform domains and quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A I Math. 20 (1995), 187–206.

[38] I. Holopainen, Nonlinear Potential Theory and Quasiregular Mappings on Riemannian Manifolds, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 74 (1990).

[39] I. Holopainen and N. Shanmugalingam, Measurability of equivalence classes and MECp-property in metric spaces, Rev. Mat. Iberoam. 23 (2007), 811–830.

[40] S. Kakutani, A. P. Karmazin, Parabolicity and hyperbolicity conditions for boundary elements of surfaces, Mat. Zametki 70 (2001), 948–951 (Russian). English transl.: Math. Notes 70 (2001), 866–869.

[41] A. P. Karmazin, The set of pre-ends and the ideal boundary of a manifold without boundary, Mat. Zametki 71 (2002), 505–508. English transl.: Math. Notes 71 (2002), 505–508.

[42] A. P. Karmazin, Quasisymmetries, the Theory of Prime Ends and Metric Structures on Domains, Izdat. Surgut, Surgut, 2008 (Russian).
[50] K. Kuratowski, *Topology*, vol. 2, Academic Press, New York–London, 1968.
[51] J. L. Lewis and K. Nyström, Boundary behaviour and the Martin boundary problem for \( p \)-harmonic functions in Lipschitz domains, *Ann. of Math.* **172** (2010), 1907–1948.
[52] P. Li and L.-F. Tam, Green’s functions, harmonic functions, and volume comparison, *J. Differential Geom.* **41** (1995), 277–318.
[53] P. Li and J. Wang, Weighted Poincaré inequality and rigidity of complete manifolds, *Ann. Sc. Éc Norm. Super.* **39** (2006), 921–982.
[54] R. S. Martin, Minimal positive harmonic functions, *Trans. Amer. Math. Soc.* **49** (1941), 137–172.
[55] O. Martio and J. Sarvas, Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4** (1979), 383–401.
[56] S. Mazurkiewicz, Recherches sur la théorie des bouts premiers, *Fund. Math.* **33** (1945), 177–228.
[57] V. M. Miklyukov, Some criteria for parabolicity and hyperbolicity of the boundary sets of surfaces, *Izv. Ross. Akad. Nauk Ser. Mat.* **60** :4 (1996), 111–158 (Russian). English transl.: *Izv. Math.* **60** (1996), 763–809.
[58] D. Minda and R. Nääki, Invariant metrics on Riemann surfaces, *J. Anal. Math.* **39** (1981), 25–44.
[59] R. Nääki, Boundary behavior of quasiconformal mappings in \( n \)-space, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **484** (1970), 1–50.
[60] R. Nääki, Prime ends and quasiconformal mappings, *J. Anal. Math.* **35** (1979), 13–40.
[61] R. Nääki, Private communication, 2010.
[62] R. Nääki and J. Väisälä, John disks, *Expo. Math.* **9** (1991), 3–43.
[63] T. Napier and M. Ramachandran, Filtered ends, proper holomorphic mappings of Kähler manifolds to Riemann surfaces, and Kähler groups, *Geom. Funct. Anal.* **17** (2008), 1621–1654.
[64] M. Ohtsuka, *Dirichlet Problem, Extremal Length and Prime Ends*, Van Nostrand, New York, 1970.
[65] O. Perron, Eine neue Behandlung der ersten Randwertaufgabe für \( \Delta u = 0 \), *Math. Z.* **18** (1923), 42–54.
[66] R. Remak, Über potentialkonvexe Funktionen, *Math. Z.* **20** (1924), 126–130.
[67] L. Rempe, On prime ends and local connectivity, *Bull. Lond. Math. Soc.* **40** (2008), 817–826.
[68] E. Schlesinger, Conformal invariants and prime ends, *Amer. J. Math.* **80** (1958), 83–102.
[69] N. Shanmugalingam, Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, *Rev. Mat. Iberoam.* **16** (2000), 243–279.
[70] N. Shanmugalingam, Harmonic functions on metric spaces, *Illinois J. Math.* **45** (2001), 1021–1050.
[71] J. Väisälä, *Lectures on \( n \)-dimensional Quasiconformal Mappings*, Lecture Notes in Math. **229**, Springer, Berlin–Heidelberg, 1971.
[72] J. Väisälä, Uniform domains, *Tohoku Math. J.* **40** (1988), 101–118.
[73] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Mathematics. **1319**, Springer–Verlag, Berlin, 1988.