The Dimension and Minimum Distance of Two Classes of Primitive BCH Codes

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Abstract

Reed-Solomon codes, a type of BCH codes, are widely employed in communication systems, storage devices and consumer electronics. This fact demonstrates the importance of BCH codes – a family of cyclic codes – in practice. In theory, BCH codes are among the best cyclic codes in terms of their error-correcting capability. A subclass of BCH codes are the narrow-sense primitive BCH codes. However, the dimension and minimum distance of these codes are not known in general. The objective of this paper is to determine the dimension and minimum distances of two classes of narrow-sense primitive BCH codes with design distances \( \delta = (q-1)q^m - 1 - q^{\left\lfloor (m-1)/2 \right\rfloor} \) and \( \delta = (q-1)q^m - 1 - q^{\left\lfloor (m+1)/2 \right\rfloor} \). The weight distributions of some of these BCH codes are also reported. As will be seen, the two classes of BCH codes are sometimes optimal and sometimes among the best linear codes known.

Index Terms

BCH codes, cyclic codes, linear codes, secret sharing, weight distribution, weight enumerator.

I. INTRODUCTION

Throughout this paper, let \( q \) be a power of a prime \( p \). An \([n, k, d]\) linear code \( C \) over \( \text{GF}(q) \) is a \( k \)-dimensional subspace of \( \text{GF}(q)^n \) with minimum Hamming distance \( d \). Let \( A_i \) denote the number of codewords with Hamming weight \( i \) in a linear code \( C \) of length \( n \). The weight enumerator of \( C \) is defined by

\[
1 + A_1 z + A_2 z^2 + \cdots + A_n z^n.
\]

The weight distribution of \( C \) is the sequence \((1, A_1, \ldots, A_n)\).

An \([n, k]\) linear code \( C \) over \( \text{GF}(q) \) is called cyclic if \((c_0, c_1, \cdots, c_{n-1}) \in C \) implies \((c_{n-1}, c_0, c_1, \cdots, c_{n-2}) \in C \). We can identify a vector \((c_0, c_1, \cdots, c_{n-1}) \in \text{GF}(q)^n \) with

\[
c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} \in \text{GF}(q)[x]/(x^n - 1).
\]

In this way, a code \( C \) of length \( n \) over \( \text{GF}(q) \) corresponds to a subset of the quotient ring \( \text{GF}(q)[x]/(x^n - 1) \). A linear code \( C \) is cyclic if and only if the corresponding subset in \( \text{GF}(q)[x]/(x^n - 1) \) is an ideal of the ring \( \text{GF}(q)[x]/(x^n - 1) \).

It is well-known that every ideal of \( \text{GF}(q)[x]/(x^n - 1) \) is principal. Let \( C = \langle g(x) \rangle \) be a cyclic code, where \( g(x) \) is monic and has the smallest degree among all the generators of \( C \). Then \( g(x) \) is unique and called the generator polynomial, and \( h(x) = (x^n - 1)/g(x) \) is referred to as the check polynomial of \( C \).

From now on, let \( m > 1 \) be a positive integer, and let \( n = q^m - 1 \). Let \( \alpha \) be a generator of \( \text{GF}(q^m)^* \), which is the multiplicative group of \( \text{GF}(q^m) \). For any \( i \) with \( 0 \leq i \leq q^m - 2 \), let \( m_i(x) \) denote the minimal polynomial of \( \alpha^i \) over \( \text{GF}(q) \). For any \( 2 \leq \delta < n \), define

\[
g_{(q,m,\delta)}(x) = \text{lcm}(m_1(x), m_2(x), \cdots, m_{\delta - 1}(x)),
\]
where lcm denotes the least common multiple of these minimal polynomials. We also define
\[ \tilde{g}_{(q,m,\delta)}(x) = (x-1)g_{(q,m,\delta)}(x). \]

Let \( C_{(q,m,\delta)} \) and \( \tilde{C}_{(q,m,\delta)} \) denote the cyclic code of length \( n \) with generator polynomial \( g_{(q,m,\delta)}(x) \) and \( \tilde{g}_{(q,m,\delta)}(x) \), respectively. This set \( C_{(q,m,\delta)} \) is called a narrow-sense primitive BCH code with design distance \( \delta \), and \( \tilde{C}_{(q,m,\delta)} \) is called a primitive BCH code with design distance \( \delta \).

By definition, the code \( \tilde{C}_{(q,m,\delta)} \) is a subcode of \( C_{(q,m,\delta)} \) and
\[ \dim(\tilde{C}_{(q,m,\delta)}) = \dim(C_{(q,m,\delta)}) - 1. \]

Clearly, \( C_{(q,m,\delta)} \) and \( C_{(q,m,\delta')}, \delta' \) are identical for two different \( \delta \) and \( \delta' \), as long as \( \delta \) and \( \delta' \) are from the same \( q \)-cyclotomic coset modulo \( n \). The largest design distance of \( C_{(q,m,\delta)} \) is called the Bose distance of the code and denoted by \( d_B \).

The cyclic codes \( C_{(q,m,\delta)} \) are treated in almost every book on coding theory. However, the following questions about the code \( C_{(q,m,\delta)} \) are still open in general.
1) What is the dimension of \( C_{(q,m,\delta)} \)?
2) What is the Bose distance (i.e., the maximum design distance) of \( C_{(q,m,\delta)} \)?
3) What is the minimum distance \( d \) (i.e., the minimum nonzero weight) of \( C_{(q,m,\delta)} \)?

The dimension of \( C_{(q,m,\delta)} \) is known when \( \delta \) is small, and is open in general. There are lower bounds on the dimension of \( C_{(q,m,\delta)} \), which are very bad in many cases. The minimum distance \( d \) of \( C_{(q,m,\delta)} \) is known only in a few cases. Only when \( \delta \) is very small or when \( C_{(q,m,\delta)} \) is the Reed-Solomon code, both the dimension and minimum distance of \( C_{(q,m,\delta)} \) are known. Hence, we have very limited knowledge of the narrow-sense primitive BCH codes, not to mention BCH codes in general. Thus, BCH codes are far from being well understood and studied. For known results on BCH codes, the reader is referred to \([7, 9, 15, 16]\).

In the 1990’s, there were a few papers on the primitive BCH codes \([3, 4, 7, 13, 29, 30]\). However, in the last sixteen years, little progress on the study of these codes has been made. As pointed out by Charpin in \([9]\), it is a well-known hard problem to determine the minimum distance of primitive BCH codes.

The objective of this paper is to determine the dimensions and minimum distances of the codes \( C_{(q,m,\delta_2)} \) and \( \tilde{C}_{(q,m,\delta_3)} \) with design distances \( \delta_2 = (q-1)q^{m-1} - 1 - q^{\lfloor (m-1)/2 \rfloor} \) and \( \delta_3 = (q-1)q^{m-1} - 1 - q^{\lfloor (m+1)/2 \rfloor} \). The weight distributions of some of these BCH codes are also reported.

An \([n,k,d]\) linear code is said to be optimal if its parameters meet a bound on linear codes, and almost optimal if the parameters \([n,k+1,d]\) or \([n,k,d+1]\) meet a bound on linear codes. To investigate the optimality of the codes studied in this paper, we compare them with the tables of the best linear codes known maintained by Markus Grassl at \( \text{http://www.codetables.de} \) which is called the Database later in this paper. Sometimes we will employ the tables of best cyclic codes in \([14]\) to benchmark some BCH codes dealt with in this paper.

II. SOME AUXILIARY RESULTS

The \( q \)-cyclotomic coset modulo \( n \) containing \( i \) is defined by
\[ C_i = \{ iq^j \mod n : 0 \leq j < \ell_i \}, \]
where \( \ell_i \) is the smallest positive integer such that \( q^{\ell_i}i \equiv i \pmod{n} \), and is called the size of \( C_i \). The smallest integer in \( C_i \) is called the coset leader of \( C_i \).

It is easily seen that the minimal polynomial \( m_i(x) \) of \( \alpha^i \) over \( \text{GF}(q) \) is given by
\[ m_i(x) = \prod_{j \in C_i} (x - \alpha^j). \]
According to the BCH bound, the minimum distance \( d \) of the code \( C_{(q,m,\delta)} \) satisfies
\[
d \geq d_B \geq \delta,
\]
where \( d_B \) denotes the Bose distance of the code \( C_{(q,m,\delta)} \). In some cases the difference \( d - \delta \) is very small or zero. In many cases the difference \( d - \delta \) is very large and in such cases the design distance does not give much information on the minimum distance \( d \), but the Bose distance \( d_B \) may be very close to the minimum distance. In fact, we have the following conjecture [9].

**Charpin’s Conjecture:** The minimum distance \( d \leq d_B + 4 \) for the narrow-sense primitive BCH codes.

In view of this conjecture, it is very valuable to determine the Bose distance for the narrow-sense primitive BCH codes.

Given a design distance \( \delta \), it is a difficult problem to determine the Bose distance \( d_B \), not to mention the minimum distance \( d \). However, we have \( d_B = \delta \) if \( \delta \) is a coset leader. Therefore, it is imperative to choose the design distance to be a coset leader. In the next two sections, we will choose the design distances of several classes of BCH codes in this way.

### III. The Parameters of the BCH Codes \( C_{(q,m,\delta_2)} \) and \( \tilde{C}_{(q,m,\delta_2)} \), Where

\[
\delta_2 = (q - 1)q^{m-1} - 1 - q^{\lfloor (m-1)/2 \rfloor}
\]

We need to prove a few lemmas before stating and proving the main results of this section. The following two lemmas are fundamental, and were proved in [15].

**Lemma 1.** The largest \( q \)-cyclotomic coset leader modulo \( n = q^m - 1 \) is \( \delta_1 = (q - 1)q^{m-1} - 1 \). Furthermore, \( |C_{\delta_1}| = m \).

**Theorem 2.** The code \( \tilde{C}_{(q,m,\delta_1)} \) has parameters \([n, m, \delta_1 + 1] \), and meets the Griesmer bound. The code \( C_{(q,m,\delta_1)} \) has parameters \([n, m+1, \delta_1] \), and meets the Griesmer bound.

We will need the following lemma shortly.

**Lemma 3.** The second largest \( q \)-cyclotomic coset leader modulo \( n \) is \( \delta_2 = (q - 1)q^{m-1} - 1 - q^{\lfloor (m-1)/2 \rfloor} \). Furthermore,

\[
|C_{\delta_2}| = \begin{cases} m & \text{if } m \text{ is odd}, \\ \frac{m}{2} & \text{if } m \text{ is even}. \end{cases}
\]

**Proof:** The proof is divided into the following two cases according to the parity of \( m \).

**Case I, i.e., \( m \) is odd:** In this case, we have
\[
\delta_2 = (q - 1)q^{m-1} - 1 - q^{\lfloor (m-1)/2 \rfloor} = n - \left( q^{\lfloor (m-1)/2 \rfloor + 1} \right) q^{\lfloor (m-1)/2 \rfloor}.
\]

It is easily seen that
\[
q\delta_2 \mod n = n - (q^{\lfloor (m+1)/2 \rfloor + 1}).
\]

One can then verify that
\[
C_{\delta_2} = \left\{ n - \left( q^{\lfloor (m+1)/2 \rfloor + 1} \right) q^i : i = 0, 1, \ldots, \frac{m-3}{2} \right\} \cup \left\{ n - \left( q^{\lfloor (m-1)/2 \rfloor + 1} \right) q^i : i = 0, 1, \ldots, \frac{m-1}{2} \right\}.
\]

Therefore, \( \delta_2 \) is the smallest integer in \( C_{\delta_2} \) and is thus the coset leader. Clearly, we have \( |C_{\delta_2}| = m \). Let \( t = (m - 3)/2, 1 \leq i \leq q^{\lfloor (m-1)/2 \rfloor - 1} - 1 \) and let \( J_i = (q - 1)q^{n-1} - 1 - i \). Notice that
\[
q^{\lfloor (m-1)/2 \rfloor - 1} = (q - 1)q^t + (q - 1)q^{t-1} + \ldots + (q - 1)q + 1.
\]
The \( q \)-adic expansion of \( i \) must be of the form

\[
i = i_t q^t + i_{t-1} q^{t-1} + \ldots + i_1 q + i_0,
\]

where each \( i_j \) satisfies \( 0 \leq i_j \leq q - 1 \), but at least one of the \( i_j \)’s is nonzero. It then follows that the \( q \)-adic expansion of \( J_i \) is given by

\[
J_i = (q - 2)q^{m-1} + (q - 1)q^{m-2} + (q - 1)q^{m-3} + \ldots + (q - 1)q^{t+1} + \\
(q - 1 - i_t)q^t + (q - 1 - i_{t-1})q^{t-1} + \ldots + (q - 1 - i_1)q + q - 1 - i_0.
\]

**Subcase I.1, i.e., \( q = 2 \):** In this subcase, we have

\[
J_i = 2^{m-2} + 2^{m-3} + \ldots + 2^t + (1 - i_t)2^t + (1 - i_{t-1})2^{t-1} + \ldots + (1 - i_1)2 + 1 - i_0.
\]

If \( i_0 = 1 \), then \( J_i/2 \) and \( J_i \) are in the same 2-cyclotomic coset modulo \( n \). Hence, \( J_i \) cannot be a coset leader.

We now assume that \( i_0 = 0 \). Since \( i \neq 0 \), one of the \( i_\ell \)’s must be nonzero. Let \( \ell \) denote the largest one such that \( i_\ell = 1 \). One can then verify that

\[
J_i 2^{m-1-\ell} \mod n < J_i.
\]

Whence, \( J_i \) cannot be a coset leader.

**Subcase I.2, i.e., \( q > 2 \):** If \( i_\ell > 1 \) for some \( \ell \) with \( 0 \leq \ell \leq t \), then \( J_i q^{m-1-\ell} \mod n < J_i \). In this case, \( J_i \) cannot be a coset leader.

We now assume that all \( i_\ell \in \{0, 1\} \). Since \( i \geq 1 \), at least one of the \( i_\ell \)’s must be 1. Let \( \ell \) denote the largest one such that \( i_\ell = 1 \). One can then verify that

\[
J_i q^{m-1-\ell} \mod n < J_i.
\]

Whence, \( J_i \) cannot be a coset leader.

Summarizing all the conclusions above, we conclude that \( \delta_2 \) is the second largest coset leader for the case that \( m \) is odd.

**Case II, i.e., \( m \) is even:** In this case, we have

\[
\delta_2 = (q - 1)q^{m-1} - 1 - q^{(m-2)/2} = n - \left(q^{m/2} + 1\right)q^{(m-2)/2}.
\]

It is easily seen that

\[
C_{\delta_2} = \left\{ n - \left(q^{m/2} + 1\right)q^i : i = 0, 1, \ldots, \frac{m-2}{2} \right\}.
\]

Therefore, \( \delta_2 \) is the smallest integer in \( C_{\delta_2} \) and is the coset leader. Obviously, \( |C_{\delta_2}| = m/2 \).

When \( m = 2 \), \( \delta_2 = \delta_1 + 1 \), where \( \delta_1 \) was defined in Lemma \[1\] There does not exist any coset leader between \( \delta_1 \) and \( \delta_2 \). Therefore, we now assume that \( m \geq 4 \).

Let \( t = (m-4)/2 \), \( 1 \leq i \leq q^{(m-2)/2} - 1 \) and let \( J_i = (q - 1)q^{m-1} - 1 - i \). Notice that

\[
q^{(m-2)/2} - 1 = (q - 1)q^t + (q - 1)q^{t-1} + \ldots + (q - 1)q + q - 1.
\]

The \( q \)-adic expansion of \( i \) must be of the form

\[
i = i_t q^t + i_{t-1} q^{t-1} + \ldots + i_1 q + i_0,
\]

where each \( i_j \) satisfies \( 0 \leq i_j \leq q - 1 \), but at least one of the \( i_j \)’s is nonzero. It then follows that the \( q \)-adic expansion of \( J_i \) is given by

\[
J_i = (q - 2)q^{m-1} + (q - 1)q^{m-2} + (q - 1)q^{m-3} + \ldots + (q - 1)q^{t+1} + \\
(q - 1 - i_t)q^t + (q - 1 - i_{t-1})q^{t-1} + \ldots + (q - 1 - i_1)q + q - 1 - i_0.
\]
**Subcase II.1, i.e., q = 2:** In this subcase, we have

$$J_i = 2^{m-2} + 2^{m-3} + \ldots + 2^{i+1} + (1-i)2^i + (1-i_{i-1})2^{i-1} + \ldots + (1-i_1)2 + 1 - i_0.$$  

If \(i_0 = 1\), then \(J_i/2 < J_i\). But \(J_i/2\) and \(J_i\) are in the same 2-cyclotomic coset modulo \(n\). Hence, \(J_i\) cannot be a coset leader.

We now assume that \(i_0 = 0\). Since \(i \neq 0\), one of the \(i_\ell\)'s must be nonzero. Let \(\ell\) denote the largest one such that \(i_\ell = 1\). One can then verify that

$$J_i 2^{m-1-\ell} \mod n < J_i.$$  

Hence, \(J_i\) cannot be a coset leader.

**Subcase II.2, i.e., q > 2:** If \(i_\ell > 1\) for some \(\ell\) with \(0 \leq \ell \leq t\), then \(J_i q^{m-1-\ell} \mod n < J_i\). In this case, \(J_i\) cannot be a coset leader.

We now assume that all \(i_\ell \in \{0, 1\}\). Since \(i \geq 1\), at least one of the \(i_\ell\)'s must be 1. Let \(\ell\) denote the largest one such that \(i_\ell = 1\). One can then verify that

$$J_i q^{m-1-\ell} \mod n < J_i.$$  

Hence, \(J_i\) cannot be a coset leader.

Summarizing all the conclusions above, we deduce that \(\bar{d}_2\) is the second largest coset leader for the case that \(m\) is even.

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**TABLE I**

Weight distribution of \(\tilde{C}_{(2,m,\bar{d}_1)}\) for odd \(m\).

| Weight \(w\) | No. of codewords \(A_w\) |
|-------------|-------------------------|
| \(2^m - 2^{(m-1)/2}\) | 1 |
| \(2^{m-1} - 2^{(m-1)/2}\) | \((2^m - 1)(2^{m-1} + 1)2^{(m-3)/2}\) |
| \(2^{m-1} + 2^{(m-1)/2}\) | \((2^m - 1)(2^{m-1} - 1)2^{(m-3)/2}\) |

**TABLE II**

Weight distribution of \(\tilde{C}_{(2,m,\bar{d}_1)}\) for even \(m\).

| Weight \(w\) | No. of codewords \(A_w\) |
|-------------|-------------------------|
| \(2^{m-1} - 2^{(m-2)/2}\) | 1 |
| \((2^{m-1} - 1)(2^{m-2} - 2^{(m-2)/2})\) | \((2^{m-3} - 1)(2^{m-1} - 2^{(m-2)/2})\) |

**TABLE III**

Weight distribution of \(\tilde{C}_{(q,m,\bar{d}_2)}\) for odd \(m\).

| Weight \(w\) | No. of codewords \(A_w\) |
|-------------|-------------------------|
| \(q-1)q^{m-1} - q^{i(m-1)/2}\) | 1 |
| \((q-1)(q^{m-1} - q^{i-1})(q^{m-1} - 2^{(m-1)/2})/2\) |
| \((q-1)(q^{m-1} - q^{i-1})/2\) |

**Theorem 4.** The code \(\tilde{C}_{(q,m,\bar{d}_2)}\) has parameters \([n, \bar{k}, \bar{d}]\), where \(\bar{d} \geq \bar{d}_2 + 1\) and

\[
\bar{k} = \begin{cases} 
2m & \text{for odd } m, \\
\frac{3m}{2} & \text{for even } m.
\end{cases}
\]
When \( q = 2 \) and \( m \) is odd, \( d = \delta_2 + 1 \) and the weight distribution of the code is given in Table II. When \( q = 2 \) and \( m \) is even, \( d = \delta_2 + 1 \) and the weight distribution of the code is given in Table III.

When \( q \) is an odd prime, \( d = \delta_2 + 1 \) and \( \tilde{C}(q, m, \delta_2) \) is a three-weight code with the weight distribution of Table III for odd \( m \) and Table IV for even \( m \).

**Proof:** The conclusions on the dimension \( k \) follow from Lemmas 1 and 3. By the BCH bound, the minimum distance \( d \geq \delta_2 + 1 \).

When \( q = 2, d = \delta_2 + 1 \) and the weight distribution of \( \tilde{C}(q, m, \delta_2) \) was determined in [17], [18], [20].

We now treat the weight distribution of \( \tilde{C}(q, m, \delta_2) \) for the case that \( q \) is an odd prime. When \( q \) is an odd prime and \( m \) is odd, the weight distribution of \( \tilde{C}(q, m, \delta_2) \) was settled in [27]. When \( q \) is an odd prime and \( m \) is even, the weight distribution of \( \tilde{C}(q, m, \delta_2) \) is not documented in the literature, although the work in [22] is related to and may be extended to settle the weight distribution of the code in this case. Hence, we will provide a proof of the weight distribution of \( \tilde{C}(q, m, \delta_2) \) for both cases. As will be seen below, our proof here can characterize all codewords of \( \tilde{C}(q, m, \delta_2) \) with minimum weight.

From now on in the proof, we assume that \( q \) is odd prime. Let \( \eta' \) and \( \eta \) denote the quadratic characters of \( GF(q^m) \) and \( GF(q) \), respectively. Let \( \chi_1' \) and \( \chi_1 \) denote the canonical additive characters of \( GF(q^m) \) and \( GF(q) \), respectively. We will need the following results regarding Gauss sums [21, Section 5.2]:

\[
G(\eta, \chi_1) = \sum_{y \in GF(q)^*} \eta(y)\chi_1(y) = \begin{cases} \sqrt{q} & \text{if } q \equiv 1 \pmod{4} \\ \sqrt[2]{q} & \text{if } q \equiv 3 \pmod{4} \end{cases}
\]

where \( \iota = \sqrt{-1} \) and

\[
G(\eta, \chi_a) = \eta(a)G(\eta, \chi_1)
\]

for all \( a \in GF(q)^* \), where \( \chi_a(x) = \chi_1(ax) \) for all \( x \in GF(q) \).

It follows from the definition of \( \tilde{C}(q, m, \delta_2) \) and Lemmas 1 and 3 that the check polynomial of this code is \( m_{\delta_1(x)}m_{\delta_2(x)} \). Notice that \( \delta_1 = n - q^{m-1} \) and

\[
\delta_2 = (q - 1)q^{m-1} - 1 - q^{(m-1)/2} = n - (q^{m-1} + q^{(m-1)/2}).
\]

From Delsarte’s Theorem [12], we then deduce that \( \tilde{C}(q, m, \delta_2) \) is equivalent to the following code (up to coordinate permutation)

\[
\tilde{C}_{\delta_2} = \left\{ \left( \Tr(ax^{1+q^{(m-1)/2}}+bx) \right)_{x \in GF(q^m)} : a \in GF(q^m), b \in GF(q^m) \right\},
\]

herein and hereafter \( \Tr \) denotes the trace function from \( GF(q^m) \) to \( GF(q) \). In the definition of the code \( \tilde{C}_{\delta_2} \), we do not specify the order in which the elements of \( GF(q^m) \) are arranged when the codewords are defined, due to the fact that the codes resulted from different orderings of the elements of \( GF(q^m)^* \) are equivalent, and thus have the same weight distribution.

Define \( h = [(m-1)/2] + 1 \) and let

\[
f(x) = \Tr(ax^{1+q^h}+bx)
\]
where \( a \in \text{GF}(q^m) \) and \( b \in \text{GF}(q^m) \).

We now consider the Hamming weight of the codeword

\[
e_{(a,b)} = (f(x))_{x \in \text{GF}(q^m)}
\]

where \( a \in \text{GF}(q^m) \) and \( b \in \text{GF}(q^m) \). It is straightforward to deduce that

\[
\text{wt}(e_{(a,b)}) = (q - 1)q^{m-1} - \frac{1}{q} \sum_{z \in \text{GF}(q)^*} \sum_{x \in \text{GF}(q^m)} \chi'_1(zf(x))
\]

\[
= (q - 1)q^{m-1} - \frac{1}{q} \sum_{z \in \text{GF}(q)^*} \sum_{x \in \text{GF}(q^m)} \chi'_1(zax^{1+q^h} + zbx).
\]

We treat the weight distribution of \( \tilde{e}_{\delta_2} \) according to the parity of \( m \) as follows.

**Case 1: \( q \) is an odd prime and \( m \geq 3 \) is odd**

In this case, we have the following basic facts that will be employed later:

- **F1:** \( h = (m+1)/2 \).
- **F2:** \( \gcd(h,m) = 1 \).
- **F3:** \( \eta' = \eta \) for all \( z \in \text{GF}(q)^* \) (due to the fact that \( (q^m - 1)/(q - 1) \) is odd).
- **F4:** \( \chi'_1(x) = \chi_1(\text{Tr}(x)) \) for all \( x \in \text{GF}(q^m) \).
- **F5:** \( F(x) := a^h x^q + ax = a^h x^{q} + ax \) is a permutation polynomial on \( \text{GF}(q^m) \) for each \( a \in \text{GF}(q^m)^* \), as \( x^{1+q^h} \) is a planar monomial over \( \text{GF}(q^m) \).

**Case 1.1:** Let \( a \neq 0 \) and \( b \neq 0 \). Recall that \( F(x) = a^h x^q + ax \) is a permutation polynomial over \( \text{GF}(q^m) \) for any \( a \in \text{GF}(q^m)^* \). Let \( x_0 \) be the unique solution of \( F(x) = a^h x^q + ax = -b^{q^h} \) for any \( a \in \text{GF}(q^m)^* \) and \( b \in \text{GF}(q^m) \). Put

\[
u = \text{Tr}
\left(ax_0^{1+q^h}\right).
\]

In this subcase, it follows from Theorem 1 in [11] that

\[
\sum_{x \in \text{GF}(q^m)} \chi'_1(axf(x)) = \begin{cases} q^{m/2} \eta'(-a) \chi'_1(zax_0^{1+q^h}), & \text{if } q \equiv 1 \pmod{4} \\ t^{3m}q^{m/2} \eta'(-a) \chi'_1(zax_0^{1+q^h}), & \text{if } q \equiv 3 \pmod{4} \end{cases}
\]

\[
= \begin{cases} q^{m/2} \eta'(-a) \eta(z) \chi_1(zu), & \text{if } q \equiv 1 \pmod{4} \\ t^{3m} \eta'(-a) q^{m/2} \eta(z) \chi_1(zu), & \text{if } q \equiv 3 \pmod{4} \end{cases}
\]

where \( \bar{t} \) denotes the complex conjugate of the complex number \( \ell \).

When \( u = 0 \), we have

\[
\sum_{z \in \text{GF}(q)^*} \sum_{x \in \text{GF}(q^m)} \chi'_1(axf(x)) = \begin{cases} \sum_{z \in \text{GF}(q)^*} q^{m/2} \eta'(-a) \eta(z), & \text{if } q \equiv 1 \pmod{4} \\ \sum_{z \in \text{GF}(q)^*} t^{3m} \eta'(-a) q^{m/2} \eta(z), & \text{if } q \equiv 3 \pmod{4} \end{cases}
= 0.
\]

Consequently, \( \text{wt}(e_{(a,b)}) = (q - 1)q^{m-1} \).

When \( u \neq 0 \), we have

\[
\sum_{z \in \text{GF}(q)^*} \sum_{x \in \text{GF}(q^m)} \chi'_1(axf(x)) = \begin{cases} q^{m/2} \eta'(-a) \mathcal{G}(\eta, \chi_u), & \text{if } q \equiv 1 \pmod{4} \\ t^{3m} \eta'(-a) q^{m/2} \mathcal{G}(\eta, \chi_u), & \text{if } q \equiv 3 \pmod{4} \end{cases}
\]

\[
= \begin{cases} q^{(m+1)/2} \eta'(-a) \eta(u), & \text{if } q \equiv 1 \pmod{4} \\ q^{(m+1)/2} t^{3m+1} \eta'(-a) \eta(u), & \text{if } q \equiv 3 \pmod{4} \end{cases}
\]

Consequently, \( \text{wt}(e_{(a,b)}) = (q - 1)q^{m-1} \pm q^{(m-1)/2} \).
Case 1.2: Let $a \neq 0$ and $b = 0$: In this case, it follows from Theorem 2.3 in [11] that
\[
\sum_{x \in \text{GF}(q^m)} \chi'_1(af(x)) = \sum_{x \in \text{GF}(q^m)} \chi'_1(ax^{1+q^h})
\]
\[
= \begin{cases} 
q^{m/2}\eta'(za), & \text{if } q \equiv 1 \pmod{4} \\
q^m q^{m/2}\eta'(za), & \text{if } q \equiv 3 \pmod{4}
\end{cases}
\]
As a result, we obtain
\[
\sum_{z \in \text{GF}(q)^*} \sum_{x \in \text{GF}(q^m)} \chi'_1(af(x)) = 0.
\]
Hence, $\text{wt}(c_{(a,b)}) = (q-1)q^{m-1}$.

Case 1.3: Let $a = 0$ and $b \neq 0$: In this case, $f(x) = \text{Tr}(bx)$. Obviously, $\text{wt}(c_{(0,b)}) = (q-1)q^{m-1}$.

Summarizing the conclusions of Cases 1.1, 1.2, and 1.3, we see that the code $\tilde{C}_{\delta_2}$ has the following three nonzero weights:
\[
w_1 = (q-1)q^{m-1} - q^{(m-1)/2}, \\
w_2 = (q-1)q^{m-1}, \\
w_3 = (q-1)q^{m-1} + q^{(m-1)/2}.
\]
Let $A_{w_i}$ be the total number of codewords with Hamming weight $w_i$ in $\tilde{C}_{\delta_2}$. It is straightforward to see that the minimum distance of the dual of $\tilde{C}_{\delta_2}$ is at least 3. Then the first three Pless power moments yield the following set of equations [19] p. 259):
\[
\begin{aligned}
A_{w_1} + A_{w_2} + A_{w_3} &= q^{2m} - 1, \\
w_1 A_{w_1} + w_2 A_{w_2} + w_3 A_{w_3} &= q^{2m-1}(q-1)(q^{m}-1), \\
w_1^2 A_{w_1} + w_2^2 A_{w_2} + w_3^2 A_{w_3} &= q^{2m-2}(q-1)(q^{m+1} - q^m - q + 2).
\end{aligned}
\]

Solving this system of equations gives the $A_{w_i}$’s in Table [III]

Case 2: $q$ is an odd prime and $m \geq 4$ is even

In this case, we have the following basic facts that will be used subsequently:

H1: $h = [(m-1)/2] + 1 = m/2$.

H2: $\gcd(h,m) = h = m/2$.

H3: $\eta'(z) = 1$ for all $z \in \text{GF}(q)^*$ (due to the fact that $(q^m-1)/(q-1)$ is even).

H4: $\chi'_1(x) = \chi_1(\text{Tr}(x))$ for all $x \in \text{GF}(q^m)$.

H5: The equation $x^{q^h} + y = 0$ has $q^h$ solutions $y$ in $\text{GF}(q^m)$ (it follows from Lemma 2.2 in [11]).

H6: $F(x) := a^h x^{2h} + ax = (a^h + a)x$ is a permutation polynomial on $\text{GF}(q^m)$ for $q^m - q^{m/2} - 1$ nonzero $a \in \text{GF}(q^m)$, and is not a permutation polynomial on $\text{GF}(q^m)$ for $q^{m/2} - 1$ nonzero elements $a \in \text{GF}(q^m)$.

We now consider the Hamming weight $\text{wt}(c_{(a,b)})$ of the codeword $c_{(a,b)}$ case by case for Case 2.
Case 2.1: Let \( a \neq 0, \ a^{q^h} + a \neq 0 \) and \( b \neq 0 \): In this subcase, let \( x_0 = -b^{q^h}/(a^{q^h} + a) \). Then we have
\[
a^{1+q^h}a_{x_0} = \frac{ab^{1+q^h}}{(a^{q^h} + a)^2}.
\]
Put
\[
u = \text{Tr} \left( a^{1+q^h}a_{x_0} \right) = \text{Tr} \left( \frac{ab^{1+q^h}}{(a^{q^h} + a)^2} \right).
\]
It then follows from Theorem 1 in [11] that
\[
\sum_{x \in \GF(q^m)} \chi'_1(af(x)) = -q^{m/2} \chi'_1(a\varpi^{1+q^h}a) = -q^{m/2} \chi'_1(zu).
\]
We obtain then
\[
\sum_{z \in \GF(q^m)} \sum_{x \in \GF(q^m)} \chi'_1(af(x)) = \begin{cases} -(q - 1)q^{m/2}, & \text{if } \nu = 0, \\ q^{m/2}, & \text{if } \nu \neq 0. \end{cases}
\]
Consequently,
\[
\text{wt}(c_{(a,b)}) = \begin{cases} (q - 1) \left( q^{m-1} + q^{m-2}/2 \right), & \text{if } \nu = 0, \\ (q - 1)q^{m-1} - q^{m-2}/2, & \text{if } \nu \neq 0. \end{cases}
\]

Case 2.2: Let \( a \neq 0, \ a^{q^h} + a = 0 \) and \( b \neq 0 \): In this subcase \( F(x) = -b^{q^h} \) has no solution \( x \in \GF(q^m) \).

It then follows from Theorem 2 in [11] that
\[
\sum_{x \in \GF(q^m)} \chi'_1(af(x)) = 0.
\]
As a result, we have
\[
\sum_{z \in \GF(q^m)} \sum_{x \in \GF(q^m)} \chi'_1(af(x)) = 0.
\]
It then follows from [5] that \( \text{wt}(c_{(a,b)}) = (q - 1)q^{m-1} \).

Case 2.3: Let \( a \neq 0 \) and \( b = 0 \): In this case, it follows from Theorem 2.4 in [11] that
\[
\sum_{x \in \GF(q^m)} \chi'_1(af(x)) = \begin{cases} -q^{m/2}, & \text{if } a^{q^h} + a \neq 0, \\ q^m, & \text{if } a^{q^h} + a = 0. \end{cases}
\]
Hence,
\[
\sum_{z \in \GF(q^m)} \sum_{x \in \GF(q^m)} \chi'_1(af(x)) = \begin{cases} -(q - 1)q^{m/2}, & \text{if } a^{q^h} + a \neq 0, \\ (q - 1)q^m, & \text{if } a^{q^h} + a = 0. \end{cases}
\]
We then deduce that
\[
\text{wt}(c_{(a,b)}) = \begin{cases} (q - 1) \left( q^{m-1} + q^{m/2} \right), & \text{if } a^{q^h} + a \neq 0, \\ 0, & \text{if } a^{q^h} + a = 0. \end{cases}
\]
Case 2.4: Let \( a = 0 \) and \( b \neq 0 \): In this subcase, we have \( f(x) = \text{Tr}(bx) \). Obviously, \( \text{wt}(c_{(a,b)}) = (q - 1)q^{m-1} \).

Summarizing the conclusions of Cases 2.1, 2.2, 2.3, and 2.4, we conclude that \( \tilde{C}_{\delta_2} \) has the following three nonzero weights:

\[
\begin{align*}
  w_1 &= (q - 1)q^{m-1} - q^{(m-2)/2}, \\
  w_2 &= (q - 1)q^{m-1}, \\
  w_3 &= (q - 1)\left( q^{m-1} + q^{(m-2)/2} \right).
\end{align*}
\]

Let \( A_{w_i} \) be the total number of codewords with Hamming weight \( w_i \) in \( \tilde{C}_{\delta_2} \). It is straightforward to see that the minimum distance of the dual of \( \tilde{C}_{\delta_2} \) is at least 3. Then the first three Pless power moments yield the following system of equations:

\[
\begin{align*}
  A_{w_1} + A_{w_2} + A_{w_3} &= q^{3m/2} - 1, \\
  w_1 A_{w_1} + w_2 A_{w_2} + w_3 A_{w_3} &= q^{(3m-2)/2}(q - 1)(q^m - 1), \\
  w_1^2 A_{w_1} + w_2^2 A_{w_2} + w_3^2 A_{w_3} &= q^{(3m-4)/2}(q - 1)(q^m - 1)(q^{m+1} - q^m - q + 2).
\end{align*}
\]  

Solving this system of equations gives the \( A_{w_i} \)'s in Table \(V\). This completes the proof of this theorem.

We remark that the proof of Theorem 4 given above may be modified to one for the same conclusions that hold for the case that \( q \) is an odd prime power. To this end, the results in [11] should be generalized.

On the other hand, the proof of Theorem 4 actually characterizes all the codewords in \( \tilde{C}_{\delta_2} \) with the minimum Hamming weight. This characterisation is given in the next theorem and may be employed to derive \( t \)-designs with the code \( \tilde{C}_{\delta_2} \) (see [2] for information about \( t \)-designs associated with linear codes).

**Theorem 5.** Let \( q \) be an odd prime, and let \( m \geq 4 \). When \( m \) is odd, all the codewords of \( \tilde{C}_{\delta_2} \) with minimum weight \( \delta = \delta_2 + 1 \) are those \( c_{(a,b)} \) such that

- \( ab \neq 0 \) and \( \eta'(-a)\eta\left( \text{Tr}(ax_0^{1+q^{(m+1)/2}}) \right) = 1 \) if \( q \equiv 1 \pmod{4} \), where \( x_0 \) is the unique solution of \( aq^{(m+1)/2}x^q + ax + bx^{q^{(m+1)/2}} = 0 \); and
- \( ab \neq 0 \) and \( t^{3m+1}\eta'(-a)\eta\left( \text{Tr}(ax_0^{1+q^{(m+1)/2}}) \right) = 1 \) if \( q \equiv 3 \pmod{4} \), where \( x_0 \) is the unique solution of \( a^q x^q + ax + bx^{q^{(m+1)/2}} = 0 \).

When \( m \) is even, all the codewords of \( \tilde{C}_{\delta_2} \) with minimum weight \( \delta = \delta_2 + 1 \) are those \( c_{(a,b)} \) such that \( a^q + a \neq 0 \), \( b \neq 0 \) and

\[
\text{Tr}\left( \frac{ab^{1+q^{m/2}}}{(a^q + a)^2} \right) \neq 0.
\]

**Table V**

**Examples of \( \tilde{C}_{(q,m,\delta_1)} \) of Theorem 4**

| \( n \) | \( k \) | \( d = \delta_2 + 1 \) | \( m \) | \( q \) | Optimality |
|--------|------|----------------------|------|-----|-----------|
| 15     | 6    | 6                    | 4    | 1   | Optimal   |
| 31     | 10   | 12                   | 5    | 2   | Optimal   |
| 63     | 9    | 28                   | 6    | 2   | Optimal   |
| 127    | 14   | 56                   | 7    | 2   | Optimal   |
| 256    | 12   | 120                  | 8    | 2   | Best known|
| 26     | 6    | 15                   | 3    | 3   | Optimal   |
| 80     | 6    | 51                   | 4    | 3   | Optimal   |
| 242    | 10   | 153                  | 5    | 3   | Best known|
Examples of the code $\tilde{C}_{(q,m,\delta_2)}$ are summarized in Table VII. They either are optimal, or have the same parameters as the best linear codes in the Database.

The following theorem is proved in [13], [20].

**Theorem 6.** The minimum distance $\tilde{d}^\perp$ of the dual of $\tilde{C}_{(2,m,\delta_2)}$ is equal to 5 when $m \geq 5$ is odd, and 3 when $m \geq 4$ is even.

**Theorem 7.** The code $C_{(q,m,\delta_2)}$ has parameters $(n,k,d)$, where $d \geq \delta_2$ and

$$k = \begin{cases} 
2m + 1 & \text{for odd } m, \\
\frac{3m}{2} + 1 & \text{for even } m.
\end{cases}$$

(8)

Furthermore, $d = \delta_2$ if $q$ is a prime.

**Proof:** The conclusions on the dimension $k$ follow from Lemmas 1 and 3. By the BCH bound, the minimum distance $d \geq \delta_2$.

It follows from the definition of $C_{(q,m,\delta_2)}$ and Lemmas 1 and 3 that the check polynomial of this code is $(x - 1)m_{\delta_1(x)}m_{\delta_2(x)}$. Notice that $\delta_1 = n - q^{m-1}$ and

$$\delta_2 = (q - 1)q^{m-1} - 1 - q^{\frac{(m-1)}{2}} = n - (q^{m-1} + q^{\frac{(m-1)}{2}}).$$

From Delsarte’s Theorem we then deduce that $C_{(q,m,\delta_2)}$ is equivalent to the following code

$$C_{\delta_2} = \left\{ \left( \text{Tr}(a x + b x^{\frac{(m-1)}{2}} + c) \right)_{x \in \mathbb{F}(q^m)} : a \in \mathbb{F}(q^m), b \in \mathbb{F}(q^m), c \in \mathbb{F}(q) \right\}. \quad (9)$$

To prove that $d = \delta_2$ for the case that $q$ is a prime, one can refine the proof of Theorem 4 with the quadratic expression of (9) to obtain the weight distribution of the code. We leave the details to interested readers.

---

**TABLE VI**

Examples of $C_{(q,m,\delta_1)}$ of Theorem 7

| $n$ | $k$ | $d = \delta_2$ | $m$ | $q$ | Optimality |
|-----|-----|----------------|-----|-----|------------|
| 15  | 7   | 5              | 4   | 2   | Yes        |
| 31  | 11  | 11             | 5   | 2   | Yes        |
| 63  | 10  | 27             | 6   | 2   | Best known |
| 127 | 15  | 55             | 7   | 2   | Best known |
| 255 | 13  | 119            | 8   | 2   | Best known |
| 26  | 7   | 14             | 3   | 3   | Optimal    |
| 80  | 7   | 50             | 4   | 3   | Optimal    |
| 242 | 11  | 152            | 5   | 3   | Best known |

Examples of the code $C_{(q,m,\delta_2)}$ are summarized in Table VII. They are sometimes optimal, and sometimes have the same parameters as the best linear codes in the Database. When $(q,m) = (2,6)$, the cyclic code $C_{(q,m,\delta_2)}$ has parameters [63,10,27], which are the best possible parameters according to [14] p. 258. When $(q,m) = (3,3)$, the cyclic code $C_{(q,m,\delta_2)}$ has parameters [26,7,14], which are the best possible parameters according to [14] p. 300.

**IV. THE PARAMETERS OF THE BCH CODES $C_{(q,m,\delta_3)}$ AND $\tilde{C}_{(q,m,\delta_3)}$, WHERE**

$$\delta_3 = (q - 1)q^{m-1} - 1 - q^{\frac{(m+1)}{2}}$$

Before determining the parameters of the codes $C_{(q,m,\delta_3)}$ and $\tilde{C}_{(q,m,\delta_3)}$, we have to prove the following lemma.

**Lemma 8.** Let $m \geq 4$. Then the third largest $q$-cyclotomic coset leader modulo $n$ is $\delta_3 = (q - 1)q^{m-1} - 1 - q^{\frac{(m+1)}{2}}$. In addition, $|\delta_3| = m$.

**Proof:** The proof is divided into the following two cases according to the parity of $m$. 

---
Case I, i.e., \( m \) is odd: In this case, we have
\[
\delta_3 = (q - 1)q^{m-1} - 1 - q^{(m+1)/2} = n - \left( q^{(m-3)/2} + 1 \right) q^{(m+1)/2}.
\]
It can be verified that
\[
C_{\delta_3} = \left\{ n - \left( q^{(m+3)/2} + 1 \right) q^i : i = 0, 1, \ldots, \frac{m-5}{2} \right\} \cup \left\{ n - \left( q^{(m-3)/2} + 1 \right) q^i : i = 0, 1, \ldots, \frac{m+1}{2} \right\}.
\]

Therefore, \( \delta_3 \) is the smallest integer in \( C_{\delta_3} \) and is thus the coset leader. Clearly, we have \( |C_{\delta_3}| = m \).

Let \( t = (m - 1)/2 \). By definition,
\[
\delta_2 = (q - 1)q^{m-1} - 1 - q^{(m-1)/2} = (q - 2)q^{m-1} + (q - 1)q^{m-2} + (q - 1)q^{m-3} + \ldots + (q - 1)q^t + 1
\]
\[
= (q - 2)q^t + (q - 1)q^{t-1} + (q - 1)q^{t-2} + \ldots + (q - 1)q + (q - 1).
\]

Observe that \( \delta_2 - \delta_3 = (q - 1)q^t \). We need to prove that \( J_i := \delta_2 - i \) is not a coset leader for all \( i \) with \( 1 \leq i \leq (q - 1)q^t - 1 \).

Notice that
\[
(q - 1)q^t - 1 = (q - 2)q^t + (q - 1)q^{t-1} + (q - 1)q^{t-2} + \ldots + (q - 1)q + 1.
\]

The \( q \)-adic expansion of \( i \) must be of the form
\[
i = i_t q^t + i_{t-1} q^{t-1} + \ldots + i_0 q + i_0,
\]
where \( i_\ell \) satisfies \( 0 \leq i_\ell \leq q - 1 \) for all \( 0 \leq \ell \leq t - 1 \) and \( 0 \leq i_t \leq q - 2 \), but at least one of the \( i_\ell \)'s is nonzero. It then follows that the \( q \)-adic expansion of \( J_i \) is given by
\[
J_i = (q - 2)q^{m-1} + (q - 1)q^{m-2} + (q - 1)q^{m-3} + \ldots + (q - 1)q^t + 1
\]
\[
= (q - 2 - i_t)q^t + (q - 1 - i_{t-1})q^{t-1} + (q - 1 - i_{t-2})q^{t-2} + \ldots + (q - 1 - i_1)q + q - 1 - i_0.
\]

Subcase I.1, i.e., \( q = 2 \): In this subcase, we have \( i_t = 0 \) and
\[
J_i = 2^{m-2} + 2^{m-3} + \ldots + 2^t + 1 - i_{t-1} 2^{t-1} + \ldots + (1 - i_1) 2 + 1 - i_0.
\]

If \( i_0 = 0 \), then \( J_i/2 < J_i \). But \( J_i/2 \) and \( J_i \) are in the same 2-cyclotomic coset modulo \( n \). Hence, \( J_i \) cannot be a coset leader.

We now assume that \( i_0 \neq 0 \). One of the \( i_\ell \)'s must be nonzero. Let \( \ell \) denote the largest one such that \( i_\ell = 1 \). One can then verify that
\[
J_i 2^{m-1-\ell} \mod n < J_i.
\]

Whence, \( J_i \) cannot be a coset leader.

Subcase I.2, i.e., \( q > 2 \): If \( i_t \geq 1 \), then \( J_i q^{m-1-t} \mod n < J_i \). In this case, \( J_i \) cannot be a coset leader.

If \( i_\ell \geq 2 \) for some \( \ell \) with \( 0 \leq \ell \leq t - 1 \), then \( J_i q^{m-1-\ell} \mod n < J_i \). In this case, \( J_i \) cannot be a coset leader.

We now assume that all \( i_\ell \in \{0, 1\} \) for all \( 0 \leq \ell \leq t - 1 \) and \( i_t = 0 \). Since \( i \geq 1 \), at least one of the \( i_\ell \)'s must be 1. Let \( \ell \) denote the largest one such that \( i_\ell = 1 \). One can then verify that
\[
J_i q^{m-1-\ell} \mod n < J_i.
\]

Whence, \( J_i \) cannot be a coset leader.

Summarizing all the conclusions above, we deduce that \( \delta_3 \) is the third largest coset leader for the case that \( m \) is odd.
\[ \delta_3 = (q - 1)q^{m-1} - 1 - q^{(m+2)/2} = n - \left( q^{(m-4)/2} + 1 \right) q^{(m+2)/2}. \]

It is easily seen that
\[
C_{\delta_3} = \left\{ n - \left( q^{(m-4)/2} + 1 \right) q^i : i = 0, 1, \ldots, \frac{m+2}{2} \right\} \cup \left\{ n - \left( q^{(m+4)/2} + 1 \right) q^i : i = 0, 1, \ldots, \frac{m-6}{2} \right\}.
\]

Therefore, \( \delta_3 \) is the smallest integer in \( C_{\delta_3} \) and is the coset leader. Obviously, \( |C_{\delta_3}| = m \).

Similarly as in the case that \( m \) is odd, one can prove that \( \delta_3 \) is the third largest coset leader for the case that \( m \) is even. Details are omitted here.

### Table VII

**The weight distribution of \( \tilde{C}_{(2,m,\delta_3)} \) for odd \( m \).**

| Weight \( w \) | No. of codewords \( A_w \) |
|----------------|-----------------------------|
| 0 | 1 |
| \( 2^{m-1} - 2^{(m+1)/2} \) | \( 2^{m-1} \cdot 2^{m-5} / 2 \cdot (2^{(m-3)/2} + 1) \cdot (2^{m-1} - 1)/3 \) |
| \( 2^{m-1} - 2^{(m-1)/2} \) | \( 2^{m-1} \cdot 2^{m-3} / 2 \cdot (2^{(m-1)/2} + 1) \cdot 5 \cdot 2^{m-1} + 4)/3 \) |
| \( 2^{m-1} \) | \( 2^{m-1} \cdot 9 \cdot 2^{m-4} + 3 \cdot 2^{m-3} + 1 \) |
| \( 2^{m-1} + 2^{(m-1)/2} \) | \( 2^{m-1} \cdot 2^{m-3} / 2 \cdot (2^{m-1}/2 - 1) \cdot 5 \cdot 2^{m-1} + 4)/3 \) |
| \( 2^{m-1} + 2^{(m+1)/2} \) | \( 2^{m-1} \cdot 2^{m-5} / 2 \cdot (2^{(m-3)/2} - 1) \cdot (2^{m-1} - 1)/3 \) |

### Table VIII

**The weight distribution of \( \tilde{C}_{(2,m,\delta_3)} \) for even \( m \).**

| Weight \( w \) | No. of codewords \( A_w \) |
|----------------|-----------------------------|
| 0 | 1 |
| \( 2^{m-1} - 2^m / 2 \) | \( 2^{m-1} \cdot 2^{m-3} + 2^{(m-4)/2} / 2^{m-1} + 2^m - 1) / 3 \) |
| \( 2^{m-1} - 2^{(m-2)/2} \) | \( 2^{m-1} \cdot 2^{m-1} + 2^{(m-2)/2} / 2^{m} + 2^{(m+2)/2} + 4)/3 \) |
| \( 2^{m-1} \) | \( 2^{m-1} \cdot 2^{m-4} + 2^{3m-4} / 2 - 2^{m-2} + 2^m + 1 \) |
| \( 2^{m-1} + 2^{(m-2)/2} \) | \( 2^{m-1} \cdot 2^{m-1} + 2^{(m-2)/2} / 2^{m} + 2^{(m+2)/2} + 4)/3 \) |
| \( 2^{m-1} + 2^m / 2 \) | \( 2^{m-1} \cdot 2^{m-3} - 2^{(m-4)/2} / 2^{m} + 2^{(m+1)/2} - 1) / 3 \) |

**Theorem 9.** Let \( m \geq 4 \). The code \( \tilde{C}_{(q,m,\delta_3)} \) has parameters \( [n, \tilde{k}, \tilde{d}] \), where \( \tilde{d} \geq \delta_3 + 1 = (q - 1)q^{m-1} - q^{(m+1)/2} \) and

\[
\tilde{k} = \begin{cases} 
3m/2 & \text{for odd } m, \\
5m & \text{for even } m.
\end{cases}
\]

Furthermore, when \( q = 2 \) and \( m \) is odd, the binary code \( \tilde{C}_{(q,m,\delta_3)} \) has minimum distance \( \tilde{d} = \delta_3 + 1 \) and its weight distribution is given in Table VII. When \( q = 2 \) and \( m \) is even, the binary code \( \tilde{C}_{(q,m,\delta_3)} \) has minimum distance \( \tilde{d} = \delta_3 + 1 \) and its weight distribution is given in Table VII.

When \( q \) is an odd prime and \( m \geq 4 \) is even, the code \( \tilde{C}_{(q,m,\delta_3)} \) has minimum distance \( \tilde{d} = \delta_3 + 1 \) and its weight distribution is given in Table IX.

When \( q \) is an odd prime and \( m \geq 5 \) is odd, the code \( \tilde{C}_{(q,m,\delta_3)} \) has minimum distance \( \tilde{d} = \delta_3 + 1 \) and its weight distribution is given in Table IX.

**Proof:** The conclusions on the dimension \( \tilde{k} \) follow from Lemmas 1, 3, and 8. By the BCH bound, the minimum distance \( \tilde{d} \geq \delta_3 + 1 \).
It follows from the definition of $\tilde{C}(q,m,\delta_3)$ and Lemmas [1] [3] and [8] that the check polynomial of this code is $m_{\delta_1(x)}m_{\delta_2(x)}m_{\delta_3(x)}$. Notice that

$$\delta_3 = (q-1)q^{m-1} - 1 - q^{\lceil (m+1)/2 \rceil} = n - (q^{m-1} + q^{\lceil (m+1)/2 \rceil}).$$

From Delsarte’s Theorem we then deduce that $\tilde{C}(q,m,\delta_3)$ is equivalent to the following code

$$\tilde{C}_{\delta_3} = \left\{ \left( \text{Tr} \left( ax + bx^1 + q^h + cx^1 + q^{h+1} \right) \right)_{x \in GF(q^m)}, \ a \in GF(q^m), \ b \in GF(q^m), \ c \in GF(q^m) \right\},$$

where $h = [(m-1)/2] + 1$.

When $q = 2$, the binary code $\tilde{C}_{\delta_3}$ has minimum distance $\tilde{d} = \delta_3 + 1$ and its weight distribution was settled in [20].

When $q$ is an odd prime and $m \geq 4$ is even, the code $\tilde{C}_{\delta_3}$ has minimum distance $\tilde{d} = \delta_3 + 1$ and its weight distribution in Table IX is a special case of Theorem 2 in [32].

When $q$ is an odd prime and $m \geq 5$ is odd, we have $h = (m+1)/2$ and $h + 1 = (m+3)/2$. It is easy to see that $\text{gcd}(m,h) = 1$ and

$$1 + q^{3h} \equiv 1 + q^{h+1} \pmod{n}.$$

It then follows that

$$\tilde{C}_{\delta_3} = \left\{ \left( \text{Tr} \left( ax + bx^1 + q^h + cx^1 + q^{h+1} \right) \right)_{x \in GF(q^m)}, \ a \in GF(q^m), \ b \in GF(q^m), \ c \in GF(q^m) \right\}.$$

In this case, the weight distribution of $\tilde{C}_{\delta_3}$ is a special case of Theorem 2 in [31].

### TABLE IX

| Weight $w$ | No. of codewords $A_w$ |
|------------|------------------------|
| $(q-1)q^{m-1} - q^{m/2}$ | $(q^m - 1)(q^{3m-6}/2 + q^{m-2}) + 2(q^{m-2}/2 - 1)(q^{m-3} + q^{(m-4)/2})/2(q+1)$ |
| $(q-1)(q^{m-1} - q^{(m-2)/2})$ | $(q^{m-1} - 1)(q^{m-1} - 1)(q^{m-1} - 1)/2$ |
| $(q-1)(q^{m-1} - q^{m-2})/2$ | $(q^{m-1} - 2q^{m} + q)(q^{m-2} - 1)(q^{m-1} + q^{m-2})/2$ |
| $(q-1)(q^{m-1} - q^{m-2})/2$ | $(q^{m-1} - 1)(q^{m-1} - 1)(q^{m-1} - 1)/2(q+1)$ |
| $(q-1)q^{m-1} + q^{(m-2)/2}$ | $(q^{m-1} + q^{m-2})(q^{m-2} - 1)/2(q+1)$ |
| $(q-1)q^{m-1} + q^{m-2}$ | $(q^{m-1} + q^{m-2})(q^{m-2} - 1)/2(q+1)$ |
| $(q-1)(q^{m-1} + q^{m-2})/2$ | $(q^{m-1} + q^{m-2})(q^{m-2} - 1)/2(q+1)$ |

### TABLE X

| Weight $w$ | No. of codewords $A_w$ |
|------------|------------------------|
| $(q-1)q^{m-1} - q^{(m+1)/2}$ | $(q^m - 1)(q^{m-3} + q^{(m-3)/2})(q^{m-1} - 1)/2$ |
| $(q-1)(q^{m-1} - q^{m-2})/2$ | $(q^{m-1} - 1)(q^{m-1} + q^{m-1})(q^{m-2} + q - 1)/2$ |
| $(q-1)(q^{m-1} + q^{m-2})/2$ | $(q^{m-1} - 1)(q^{m-1} + q^{m-1})(q^{m-2} + q - 1)/2$ |
| $(q-1)q^{m-1}$ | $(q^{m-1} - 1)(q^{m-1} + q^{m-1})(q^{m-2} + q - 1)/2$ |
| $(q-1)q^{m-1} + q^{m-1}/2$ | $(q^{m-1} - 1)(q^{m-1} + q^{m-1})(q^{m-2} + q - 1)/2$ |
| $(q-1)q^{m-1} + q^{m-1}$ | $(q^{m-1} - 1)(q^{m-1} + q^{m-1})(q^{m-2} + q - 1)/2$ |
| $(q-1)(q^{m-1} + q^{m-1})/2$ | $(q^{m-1} - 1)(q^{m-1} + q^{m-1})(q^{m-2} + q - 1)/2$ |
| $(q-1)q^{m-1} + q^{m-1}$ | $(q^{m-1} - 1)(q^{m-1} + q^{m-1})(q^{m-2} + q - 1)/2$ |

The following theorem is proved in [20].
Theorem 10. The minimum distance $d^\perp$ of the dual of $\tilde{C}_{(2,m,\delta_3)}$ is equal to 7 when $m \geq 5$ is odd, and 5 when $m \geq 6$ is even.

| $n$ | $k$ | $d = \delta_3 + 1$ | $m$ | $q$ | Optimality |
|-----|-----|---------------------|-----|-----|------------|
| 15  | 10  | 4                   | 4   | 2   | Yes        |
| 31  | 15  | 8                   | 5   | 2   | Yes        |
| 63  | 15  | 24                  | 6   | 2   | Yes        |
| 127 | 21  | 48                  | 7   | 2   | Best known |
| 255 | 20  | 112                 | 8   | 2   | Best known |
| 26  | 10  | 9                   | 3   | 3   | No         |
| 80  | 10  | 45                  | 4   | 3   | Best known |
| 242 | 15  | 135                 | 5   | 3   | No         |

Example 1. Let $(q, m) = (2, 4)$. Then $\delta_3 = 3$, and $\tilde{C}_{(q,m,\delta_3)}$ has parameters $[15,10,4]$ and weight enumerator $1 + 105z^4 + 280z^6 + 435z^8 + 168z^{10} + 35z^{12}$.

Example 2. Let $(q, m) = (2, 5)$. Then $\delta_3 = 7$, and $\tilde{C}_{(q,m,\delta_3)}$ has parameters $[31,15,8]$ and weight enumerator $1 + 465z^8 + 8680z^{12} + 18259z^{16} + 5208z^{20} + 155z^{24}$.

Example 3. Let $(q, m) = (3, 4)$. Then $\delta_3 = 44$, and $\tilde{C}_{(q,m,\delta_3)}$ has parameters $[80,10,45]$ and weight enumerator

$$1 + 3040z^{45} + 9900z^{48} + 10080z^{51} + 16640z^{54} + 14400z^{57} + 3528z^{60} + 1440z^{63} + 20z^{72}.$$ 

Example 4. Let $(q, m) = (3, 5)$. Then $\delta_3 = 134$, and $\tilde{C}_{(q,m,\delta_3)}$ has parameters $[242,15,135]$ and weight enumerator

$$1 + 29040z^{135} + 359370z^{144} + 3855060z^{153} + 6719372z^{162} + 3188592z^{171} + 182952z^{180} + 14520z^{189}.$$ 

The optimality of the code $\tilde{C}_{(q,m,\delta_3)}$ is marked in Table XI where further examples of the code is documented. As shown in this table, the code $\tilde{C}_{(q,m,\delta_3)}$ is sometimes optimal, and sometimes has the same parameters as the best linear code known.

Theorem 11. Let $m \geq 4$. The code $C_{(q,m,\delta_3)}$ has parameters $[n,k,d]$ where $d = \delta_3 = (q-1)q^{m-1} - 1 - q^{\lfloor (m+1)/2 \rfloor}$ and

$$k = \begin{cases} 
3m + 1 & \text{for odd } m, \\
\frac{2m}{2} + 1 & \text{for even } m.
\end{cases} \quad (11)$$

Proof: The conclusions on the dimension $k$ follow from Lemmas 1, 3 and 8. By the BCH bound, the minimum distance $d \geq \delta_3$.

It follows from the definition of $C_{(q,m,\delta_3)}$ and Lemmas 1, 3 and 8 that the check polynomial of this code is $(x-1)m_{\delta_3(x)}m_{\delta_2(x)}m_{\delta_1(x)}$. Notice that

$$\delta_3 = (q-1)q^{m-1} - 1 - q^{\lfloor (m+1)/2 \rfloor} = n - (q^{m-1} + q^{\lfloor (m+1)/2 \rfloor}).$$

From Delsarte’s Theorem we then deduce that $C_{(q,m,\delta_3)}$ is equal to the following code

$$C_{\delta_3} = \left\{ \text{Tr} \left( ax + bx^{1+q^{\lfloor (m-1)/2 \rfloor}+1} + cx^{1+q^{\lfloor (m+1)/2 \rfloor}+1} \right) + e \right\}_{x \in \text{GF}(q^m)^*},$$

where $a \in \text{GF}(q^m)$, $b \in \text{GF}(q^m)$, $c \in \text{GF}(q^m)$, $e \in \text{GF}(q)$.

Similarly, the weights and their frequencies of the codewords in $C_{(q,m,\delta_3)}$ are determined by the affine and quadratic functions

$$\text{Tr} \left( ax + bx^{1+q^{\lfloor (m-1)/2 \rfloor}} + cx^{1+q^{\lfloor (m+1)/2 \rfloor}+1} \right) + e.$$
Theorem 12. Let $q > 2$. The codes $\tilde{C}(q,3,q^3-q^2-q-2)$ and $C(q,3,q^3-q^2-q-2)$ have parameters $[q^3-1,7,\tilde{d} \geq q^3-q^2-q-1]$ and $[q^3-1,8,d \geq q^3-q^2-q-2]$, respectively.

Proof: When $m = 3$, one can similarly prove that the third largest coset leader $\delta_3 = \delta_2 - 1 = q^3 - q^2 - q - 2$ and $|C_{\delta_3}| = 1$. The conclusions on the dimensions of $\tilde{C}(q,3,q^3-q^2-q-2)$ and $C(q,3,q^3-q^2-q-2)$ follow from Lemmas 1 and 3. The conclusions on the minimum distances follow from the BCH bound.

We conjecture that $\tilde{d} = q^3 - q^2 - q - 1$ and $d = q^3 - q^2 - q - 2$ for the two codes in Theorem 12, and invite the reader to settle this conjecture.

The following theorem follows from Theorem 10 as $C_{(2,m,\delta_3)}$ is the even-weight subcode of $\tilde{C}_{(2,m,\delta_3)}$.

Theorem 13. The minimum distance $d^{\perp}$ of the dual of $C_{(2,m,\delta_3)}$ is equal to 8 when $m \geq 5$ is odd, and 6 when $m \geq 6$ is even.

V. SUMMARY AND CONCLUDING REMARKS

The first contribution of this paper is the determination of the second and third largest $q$-cyclotomic coset leaders $\delta_2$ and $\delta_3$, which are documented in Lemmas 5 and 8. The second contribution is the establishment of the dimensions, minimum distances and the weight distributions of the primitive BCH codes $\tilde{C}(q,m,\delta_2)$, $\tilde{C}(q,m,\delta_3)$, and the settlement of the dimensions and minimum distances of $C(q,m,\delta_2)$ and $C(q,m,\delta_3)$ for the case that $q$ is an odd prime. Sometimes a direct proof of the weight distribution formulas was given. Sometimes a bridge between these BCH codes and some cyclic codes with known weight distribution was established and employed to derive the weight distribution of the BCH codes.

The third contribution of this paper is the characterisation of all the codewords in the code $\tilde{C}(q,m,\delta_2)$ with minimum weight documented in Theorem 5. This may be employed to derive $t$-designs with this code. It is noticed that characterising all the codewords with minimum weight in a linear code is a very difficult problem in general.

Nonbinary Kasami codes were introduced and their weight distributions were settled in [32] and a class of cyclic codes were defined in [31]. The fourth contribution of this paper is the proof of the fact that a subclass of the nonbinary Kasami codes and a subclass of cyclic codes treated in [31] are actually equivalent to the BCH codes $\tilde{C}(q,m,\delta_2)$. The code $\tilde{C}_{\delta_2}$ of [41] was known to be equivalent to a cyclic code.

| $n$ | $k$ | $d = \delta_3$ | $m$ | $q$ | Optimality |
|-----|-----|----------------|-----|-----|------------|
| 15  | 11  | 3              | 4   | 2   | Yes        |
| 31  | 16  | 7              | 5   | 2   | No (optimal $d = 8$) |
| 63  | 16  | 23             | 6   | 2   | Best known |
| 127 | 22  | 47             | 7   | 2   | Best known |
| 255 | 21  | 111            | 8   | 2   | Best known |
| 26  | 11  | 8              | 3   | 3   | No (best $d = 9$) |
| 81  | 11  | 44             | 4   | 3   | No (best $d = 45$) |
| 242 | 16  | 134            | 5   | 3   | No (best $d = 135$) |
in the literature. Our contribution regarding this code is to prove that it is equivalent to the BCH code $\tilde{C}(q,m,\delta_2)$.

A number of examples of the codes $\tilde{C}(q,m,\delta_i)$ and $C(q,m,\delta_i)$ for $i \in \{2,3\}$ were worked out and put into several tables. Most of them are either optimal or almost optimal linear codes. It was also known that $\tilde{C}(q,m,\delta_1)$ and $C(q,m,\delta_1)$ are optimal with respect to the Griesmer bound [15]. The list of tables of best cyclic codes documented in [14] shows that BCH codes are among the best cyclic codes except in a few cases. Hence, it would be worthy to further investigate primitive BCH codes.

To the best of our knowledge, the weight distributions of only a few classes of primitive BCH codes are determined in the literature. These codes are the following:

1) The Reed-Solomon codes.
2) The codes documented in this paper and two more subclasses of binary primitive BCH codes dealt with in [20].

It is in general very difficult to determine the dimensions and minimum distances of BCH codes, let alone their weight distributions. For the weight distribution of cosets of some binary primitive BCH codes, the reader is referred to [8], [10] for information.

Finally, we point out an application of some of the codes of this paper in secret sharing. Any linear code over GF$(q)$ can be employed to construct secret sharing schemes [1], [6], [23], [28]. In order to have such secret sharing scheme with interesting access structures, we need a linear code $C$ over GF$(q)$ such that

$$\frac{w_{\text{min}}}{w_{\text{max}}} > \frac{q-1}{q},$$

where $w_{\text{max}}$ and $w_{\text{min}}$ denote the maximum and minimum nonzero weight in $C$, respectively.

The codes $\tilde{C}(q,m,\delta_2)$ and $\tilde{C}(q,m,\delta_1)$ satisfy the inequality in (12) when $m \geq 5$, and can be employed to obtain secret sharing schemes with interesting access structures using the framework documented in [1], [6], [23], [28].

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