ACYCLIC ORIENTATION POLYNOMIALS AND THE SINK THEOREM FOR CHROMATIC SYMMETRIC FUNCTIONS

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Abstract. We define the acyclic orientation polynomial of a graph to be the generating function for the sinks of its acyclic orientations. Stanley proved that the number of acyclic orientations is equal to the chromatic polynomial evaluated at $-1$ up to sign. Motivated by this result, we develop “acyclic orientation” analogues for theorems concerning the chromatic polynomial by Birkhoff, Whitney, and Greene-Zaslavsky. As the main application, we provide a new proof for Stanley’s sink theorem for chromatic symmetric functions $X_G$, which gives a relation between the number of acyclic orientations with a fixed number of sinks and the coefficients in the expansion of $X_G$ with respect to elementary symmetric functions.

1. Introduction

The purpose of this paper is to introduce acyclic orientation polynomials, present their several expressions, and give a new proof for Stanley’s sink theorem from these expressions. Throughout this paper, let $G = (V, E)$ be a simple graph with $|V| = d$ vertices. The variable associated to a vertex $v \in V$ will be denoted by the same notation $v$.

Our object of study is an acyclic orientation of the graph $G$, an assignment of a direction to each edge so that the orientation induces no directed cycles. Denote by $\mathcal{A}(G)$ the collection of acyclic orientations of $G$. The number of acyclic orientations of $G$ is a Tutte-Grothendieck invariant, i.e., this number obeys a deletion-contraction recursion. As a refinement of the quantity, we introduce the acyclic orientation polynomial (Definition 2.1). For $\sigma \in \mathcal{A}(G)$, let $\text{Sink}(G, \sigma)$ be the set of sinks of $\sigma$, and define the acyclic orientation polynomial $A_G(V)$ to be

$$A_G(V) = \sum_{\sigma \in \mathcal{A}(G)} \prod_{v \in \text{Sink}(G, \sigma)} v.$$ 

Specializing $v = t$ for all $v \in V$ in our polynomial $A_G(V)$, one obtains the polynomial $a_G(t)$ whose coefficient of $t^j$ counts the number of acyclic orientations with $j$ sinks [12, 8]. We prove that $A_G(V)$ satisfies the deletion-contraction recurrence (Theorem 2.3) with a change of variables:

$$v_e = u_1 + u_2 - u_1 u_2, \text{ or } (1 - v_e) = (1 - u_1)(1 - u_2),$$

where $v_e$ is the vertex of the graph $G/e$ obtained from $G$ by contracting an edge $e = u_1 u_2 \in E$.

Using this deletion-contraction recurrence, we shall give several expressions for $A_G(V)$. Stanley [11] showed that the number of acyclic orientations of $G$ is equal to $(-1)^d \chi_G(-1)$, where $\chi_G(n)$ is the chromatic polynomial of $G$. This result motivates us to develop “acyclic
orientation” analogues for theorems concerning $\chi_G(n)$. Let us recall four famous expressions for the chromatic polynomial $\chi_G(n)$:

\begin{align*}
\chi_G(n) &= \sum_{S \subseteq E} (-1)^{|S|} n^{d-|S|} \text{ [The subgraph expansion]} \\
&= \sum_{S \in B_G} (-1)^{|S|} n^{d-|S|} \quad [15 \text{ Whitney’s Theorem}] \\
&= \sum_{\pi \in L_G} \mu_G(\hat{0}, \pi) n^{|\pi|} \quad [2 \text{ Birkhoff’s Theorem}] \\
&= \sum_{\sigma \in \mathcal{A}(G)} (-1)^{d-|\pi(\sigma)|} n^{|\pi(\sigma)|} \quad [7 \text{ Corollary 7.4}],
\end{align*}

where $S \subseteq E$ is a spanning subgraph of $G$, $B_G$ is the broken circuit complex, $L_G$ is the bond lattice, and $\pi : \mathcal{A}(G) \to L_G$ is the map defined in [7, Section 7].

As analogues for (2), (3), (4), and (5), our theorems (Theorems 3.2, 3.5, 3.8, and 3.13) provide four expressions for $A_G(V)$. To a connected graph $C$, we associate a variable

$$v_C = 1 - \prod_{v \in V(C)} (1 - v),$$

which generalizes (1). Denote by $\mathcal{C}(S)$ the set of connected components of a subgraph $S$ and let $s(S)$ be the corank of $S$. In Theorems 3.2 and 3.5, the terms corresponding to each subgraph $S$ in (2) and (3) are replaced by $(-1)^{s(S)} \prod_{G \in \mathcal{C}(S)} v_G$, respectively. Similarly, in Theorems 3.8 and 3.13, the terms in the summations in (4) and (5) are replaced by the variables defined for vertex partitions $\pi \in L_G$, respectively.

Various known results for acyclic orientations are represented as coefficients in our expressions for $A_G(V)$. The linear terms (Corollaries 3.6 and 3.9) give [7, Theorem 7.3], which says that the number of acyclic orientations with the unique sink at a fixed $v \in V$ equals the Möbius invariant. Using [7, Theorem 7.3] and Weisner’s theorem, we present an alternative proof for Theorem 3.8. Comparing Theorems 3.8 and 3.13 yields [7, Theorem 7.4] representing the cardinalities of images under the map $\pi$ in terms of Möbius functions. Note that this theorem gives an expression for the number of acyclic orientations whose sinks are in $U \subseteq V$, which is directly derived from our expression for $A_G(V)$.

The main application of our expressions for $A_G(V)$ is to give a new proof of the sink theorem [12, Theorem 3.3] for the chromatic symmetric function $X_G$. The theorem asserts

$$\text{sink}(G, j) = \sum_{\lambda \vdash d, l(\lambda) = j} c_\lambda,$$

where $\text{sink}(G, j)$ is the number of acyclic orientations of $G$ with $j$ sinks, the numbers $c_\lambda$ are defined by the expansion $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$ in terms of elementary symmetric functions $e_\lambda$, and $l(\lambda)$ is the length of a partition $\lambda$. The original proof relies on the theory of quasi-symmetric functions and $P$-partitions, which inspired Stanley [13, 14] to ask a simple and conceptional proof for the theorem.
This paper is organized as follows. Section 2 introduces acyclic orientation polynomials \( A_G(V) \) and proves their deletion-contraction recurrence. Section 3 presents various expressions for \( A_G(V) \). Section 4 provides a new proof of Stanley’s sink theorem for chromatic symmetric functions. Section 5 presents how \( A_G(V) \) and \( a_G(t) \) work on distinguishing graphs.

2. Acyclic orientation polynomials and their recurrence

Let \( G \) be a graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). In this paper, let \( |V| = d \) and assume that \( G \) is simple, i.e., \( G \) has no loops or multiple edges. For an edge \( u_1u_2 \in E \), a direction \( \overrightarrow{u_1u_2} \) (resp. \( \overrightarrow{u_2u_1} \)) means that the oriented edge is toward \( u_2 \) (resp. \( u_1 \)). An orientation \( \sigma \) is an assignment of a direction \( \overrightarrow{u_1u_2} \) or \( \overrightarrow{u_2u_1} \) to each edge \( u_1u_2 \in E \). An orientation \( \sigma \) is said to be acyclic if \( \sigma \) has no directed cycles. Let \( \mathcal{A}(G) \) be the set of acyclic orientations of \( G \). For \( \sigma \in \mathcal{A}(G) \), a sink of \( \sigma \) is a vertex \( v \) such that the direction of each edge incident to \( v \) is toward to \( v \). Let \( \text{Sink}(G, \sigma) \) be the set of sinks of an orientation \( \sigma \) and \( \text{sink}(G, \sigma) = |\text{Sink}(G, \sigma)| \).

We associate a variable to each vertex \( v \in V \), and use the same notation \( v \) for this variable. Then \( V \) also denotes the set of the variables corresponding to vertices. Assume that all the variables commute with each other. To each acyclic orientation \( \sigma \in \mathcal{A}(G) \), we assign the following monomial

\[
\prod_{v \in \text{Sink}(G, \sigma)} v.
\]

We introduce the definition of the main object in this paper.

Definition 2.1. For a graph \( G = (V, E) \), define the acyclic orientation polynomial \( A_G(V) \) of \( G \) to be the generating function for sinks of acyclic orientations of \( G \), i.e.,

\[
A_G(V) = \sum_{\sigma \in \mathcal{A}(G)} \prod_{v \in \text{Sink}(G, \sigma)} v.
\]

Let \( a_G(t) \) be the polynomial obtained from \( A_G(V) \) by setting \( v = t \) for each \( v \in V \), i.e.,

\[
a_G(t) = \sum_{\sigma \in \mathcal{A}(G)} t^{\text{sink}(G, \sigma)}.
\]

Take a non-empty subset \( U \) of \( V \). Let \( \mathcal{A}(G, U) \) be the set of acyclic orientations \( \sigma \) of \( G \) with \( \text{Sink}(G, \sigma) = U \). Then the coefficient of \( \prod_{v \in U} v \) is equal to \( |\mathcal{A}(G, U)| \), which will be denoted by \( a(G, U) \) shortly. When \( U \) consists of a single vertex \( u \), we write \( a(G, u) \) instead of \( a(G, \{u\}) \).

Example 2.2. Let us consider two graphs \( G_1 = (\{v_1, v_2, v_3\}, \{v_1v_2, v_2v_3, v_3v_1\}) \) and \( G_2 = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_1, v_1v_4\}) \). Their acyclic orientation polynomials are

\[
A_{G_1}(\{v_1, v_2, v_3\}) = 2(v_1 + v_2 + v_3), \quad \text{and} \quad A_{G_2}(\{v_1, v_2, v_3, v_4\}) = 2(v_1 + v_2 + v_3 + v_4 + v_2v_4 + v_3v_4).
\]

In Figure 1, we list all the acyclic orientations of \( G_2 \) with corresponding monomials.

We show that the acyclic orientation polynomial \( A_G(V) \) satisfies the deletion-contraction recurrence with a change of variables. Our theorem generalizes the fact that the number of acyclic orientations satisfies the deletion-contraction recurrence. Take an edge \( e = u_1u_2 \in E(G) \). The deletion \( G \setminus e \) is the graph obtained from \( G \) by deleting \( e \). The contraction \( G/e \)
Theorem 2.3. The acyclic orientation polynomial $A_G(V)$ satisfies the deletion-contraction recurrence:

\begin{equation}
A_G(V) = A_{G/e}(V) + A_{G/e}(V/e).
\end{equation}

Proof. Let $e = u_1u_2$ be an edge of $G$. By the relation (6), we can regard $A_{G/e}(V/e)$ as a polynomial in variables $V$. Fix a non-empty subset $U$ of $V$ and let $a$, $a_D$ and $a_C$ be the coefficients of $\prod_{v \in U} v$ in $A_G(V)$, $A_{G/e}(V)$ and $A_{G/e}(V/e)$, respectively. It is clear that $a = a(G, U)$ and $a_D = a(G \setminus e, U)$. From the relation $v_e = u_1 + u_2 - u_1u_2$, we see that

\[
a_C = \begin{cases} 
-|\mathcal{A}(G/e, U \setminus \{u_1, u_2\} \cup \{v_e\})|, & \text{if } u_1, u_2 \in U, \\
|\mathcal{A}(G/e, U \setminus \{u_1\} \cup \{v_e\})|, & \text{if } U \cap \{u_1, u_2\} = \{u_i\}, \\
|\mathcal{A}(G/e, U)|, & \text{if } u_1, u_2 \notin U.
\end{cases}
\]

To prove equation (7), it suffices to show that

\[a = a_D + a_C.\]

Suppose that $U$ contains two endpoints of an edge in $G \setminus e$. Then we have $a = a_D = a_C = 0$ since sinks of an acyclic orientation are not adjacent. From now on, we may assume that $U$ has no vertices that are adjacent in $G \setminus e$. We need to consider the following three cases.

Case 1: $u_1, u_2 \in U$. In this case, $u_1$ and $u_2$ are adjacent in $G$, and hence we have $a = 0$. Also merging $u_1$ and $u_2$ gives a bijection between $\mathcal{A}(G \setminus e, U)$ and $\mathcal{A}(G/e, U \setminus \{u_1, u_2\} \cup \{v_e\})$, which shows $a_D + a_C = 0$.

Case 2: $U \cap \{u_1, u_2\} = \{u_i\}$. Without loss of generality we assume that $u_1 \in U$ and $u_2 \notin U$. Deleting $e$ induces a bijection between $\mathcal{A}(G, U)$ and $\mathcal{A}(G \setminus e, U \cup \{v_e\}) \cup \mathcal{A}(G \setminus e, U \setminus \{u_2\})$. Moreover, since both $u_1$ and $u_2$ do not have outgoing edges in acyclic orientations in $\mathcal{A}(G \setminus e, U \cup \{u_2\})$, we can merge $u_1$ and $u_2$ in $G \setminus e$ to get a one-to-one correspondence between $\mathcal{A}(G \setminus e, U \cup \{u_2\})$ and $\mathcal{A}(G/e, U \setminus \{u_1\} \cup \{v_e\})$. Therefore, we have $a = a_D + |\mathcal{A}(G \setminus e, U \cup \{u_2\})| = a_D + a_C$.

Figure 1. Acyclic orientations of $G_2$ with corresponding monomials below.
Case 3: $u_1, u_2 \notin U$. Let $\mathcal{A}_{\sim e}$ be the set of acyclic orientations of $G \setminus e$ which do not contain a directed path from $u_1$ to $u_2$ nor vice-versa. Define

\[
\begin{align*}
\mathcal{A}_1 &= \{ o \in \mathcal{A}(G, U) \mid o \setminus e \notin \mathcal{A}(G \setminus e, U) \}, \\
\mathcal{A}_2 &= \{ o \in \mathcal{A}(G, U) \mid o \setminus e \in \mathcal{A}(G \setminus e, U), o \setminus e \notin \mathcal{A}_{\sim e} \}, \\
\mathcal{A}_3 &= \{ o \in \mathcal{A}(G, U) \mid o \setminus e \in \mathcal{A}(G \setminus e, U), o \setminus e \in \mathcal{A}_{\sim e} \}, \\
\mathcal{A}_{3,+} &= \{ o \in \mathcal{A}_3 \mid \overrightarrow{u_1 u_2} \in o \}, \quad \text{and} \quad \mathcal{A}_{3,-} = \{ o \in \mathcal{A}_3 \mid \overrightarrow{u_2 u_1} \in o \}.
\end{align*}
\]

Then

\[
\mathcal{A}(G, U) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3, \quad \text{and} \quad \mathcal{A}_3 = \mathcal{A}_{3,+} \cup \mathcal{A}_{3,-}.
\]

It is enough to show that $a_D = \left| \mathcal{A}_2 \cup \mathcal{A}_{3,-} \right|$ and $a_C = \left| \mathcal{A}_1 \cup \mathcal{A}_{3,+} \right|$, which will follow from the fact that the following two maps are bijections:

\[
\begin{align*}
\mathcal{A}_2 \cup \mathcal{A}_{3,-} &\longrightarrow \mathcal{A}(G \setminus e, U) \quad \text{and} \quad \mathcal{A}_1 \cup \mathcal{A}_{3,+} \longrightarrow \mathcal{A}(G/e, U) \\
\text{if } o \text{ contains a directed path from } u_1 \text{ to } u_2, &\quad \text{then } o \longmapsto o \setminus e, \\
\text{if } o \text{ contains a directed path from } u_2 \text{ to } u_1, &\quad \text{then } o \longmapsto o/e.
\end{align*}
\]

To verify this fact, we exhibit their inverses below.

To obtain the former identity, define a map from $\mathcal{A}(G \setminus e, U)$ to $\mathcal{A}_2 \cup \mathcal{A}_{3,-}$ as follows: for $o \in \mathcal{A}(G \setminus e, U)$,

\[
\begin{align*}
o \mapsto \begin{cases}
on \cup \overrightarrow{u_1 u_2} \in \mathcal{A}_2, & \text{if } o \text{ contains a directed path from } u_1 \text{ to } u_2, \\
on \cup \overrightarrow{u_2 u_1} \in \mathcal{A}_2, & \text{if } o \text{ contains a directed path from } u_2 \text{ to } u_1, \\
\end{cases}
\end{align*}
\]

Note that by acyclicity of $o$, the first and second cases are mutually exclusive. One can check that this map is the inverse of deleting $e$ from $o$ in $\mathcal{A}_2 \cup \mathcal{A}_{3,-}$.

Similarly, let us construct the inverse of the map $o \mapsto o/e$ from $\mathcal{A}_1 \cup \mathcal{A}_{3,+}$ to $\mathcal{A}(G/e, U)$. For $o \in \mathcal{A}(G/e, U)$, let $o'$ be the acyclic orientation of $G \setminus e$ naturally obtained from $o$ by splitting $v_e$ into $u_1$ and $u_2$. Since $v_e$ is not a sink in $o$, it follows that $o'$ cannot have both $u_1$ and $u_2$ as its sinks at the same time. Furthermore, there are no paths from $u_1$ to $u_2$ nor vice-versa in $o'$ because of acyclicity of $o$. Define a map from $\mathcal{A}(G/e, U)$ to $\mathcal{A}_1 \cup \mathcal{A}_{3,+}$ as follows:

\[
\begin{align*}
o \mapsto \begin{cases}
o' \cup \overrightarrow{u_1 u_2} \in \mathcal{A}_1, & \text{if } u_1 \text{ is a sink of } o', \\
o' \cup \overrightarrow{u_2 u_1} \in \mathcal{A}_1, & \text{if } u_2 \text{ is a sink of } o', \\
o' \cup \overrightarrow{u_1 u_2} \in \mathcal{A}_{3,+}, & \text{otherwise.}
\end{cases}
\end{align*}
\]

This map is the inverse of contracting $e$ from $o$ in $\mathcal{A}_1 \cup \mathcal{A}_{3,+}$.

Example 2.4. Let $H$ be the graph whose vertex set is $V(H) = \{v_1, v_2, v_3, v_4\}$ and edge set is $E(H) = \{v_1 v_2, v_2 v_3, v_3 v_1, v_1 v_4, v_3 v_4\}$. The graphs $H, H \setminus v_3 v_4$, and $H/v_3 v_4$ are shown in Figure 2. Using the deletion-contraction recurrence together with the computations in Example 2.2, we obtain

\[
\begin{align*}
A_H(V) &= A_{H_v v_3 v_4}(V) + A_{H/v_3 v_4}(\{v_1, v_2, v_3, v_4 = v_3 + v_4 - v_3 v_4\}) \\
&= 2(v_1 + v_2 + v_3 + v_4 + v_2 v_4 + v_3 v_4) + 2(v_1 + v_2 + v_3 + v_4 - v_3 v_4) \\
&= 4(v_1 + v_2 + v_3 + v_4) + 2v_2 v_4,
\end{align*}
\]

and hence we have $a_H(t) = 16t + 2t^2$. 5
3. Four expressions for acyclic orientation polynomials

3.1. Subgraph expansions. We will expand our acyclic orientation polynomial $A_G(V)$ with respect to spanning subgraphs of $G$. Let $S$ be a subset of the edge set $E(G)$. The set $S$ will be identified with the spanning subgraph of $G$ whose edge set is $S$. Denote by $|S|$ the number of edges of $S$. Let $S(G)$ be the collection of spanning subgraphs of $G$. For an edge $e = u_1u_2 \in E(G)$, define

$$S(G)^e = \{ S \in S(G) \mid e \notin E(S) \}, \text{ and } S(G)_e = S(G) \setminus S(G)^e.$$ 

Note that a subgraph in $S(G)_e$ has a connected component which contains both $u_1$ and $u_2$.

**Proposition 3.1.** For an edge $e \in E(G)$, deleting $e$ yields $S(G)^e = S(G \setminus e)$ and contracting $e$ gives a bijection between $S(G)_e$ and $S(G/e)$.

Let $C(S)$ be the set of connected components of a subgraph $S$. For each connected component $C \in C(S)$, define the variable $v_C$ to be

$$v_C = 1 - \prod_{v \in V(C)} (1 - v).$$

Note that $v_C = v_e$, where $C$ is a graph with two vertices and one edge $e$. Let $s(S)$ be the corank of $S$ defined as $s(S) = |S| - d + |C(S)|$.

**Theorem 3.2.** For a graph $G = (V, E)$, its acyclic orientation polynomial $A_G(V)$ equals

$$A_G(V) = \sum_{S \subseteq E} (-1)^{s(S)} \prod_{C \in C(S)} v_C.$$

Hence,

$$a_G(t) = \sum_{S \subseteq E} (-1)^{s(S)} \prod_{C \in C(S)} (1 - (1 - t)^{|V(C)|}).$$

**Proof.** We prove the theorem by induction on the number of edges, the base case that $G$ has no edges being trivial. Take an edge $e = u_1u_2 \in E(G)$. Thanks to the deletion-contraction recurrence (7) for $A_G(V)$, it is enough to show that

$$A_{G \setminus e}(V) = \sum_{S \in S(G)^e} (-1)^{s(S)} \prod_{C \in C(S)} v_C \quad \text{and} \quad A_{G/e}(V/e) = \sum_{S \in S(G)_e} (-1)^{s(S)} \prod_{C \in C(S)} v_C.$$

The first equation follows immediately from Proposition 3.1 and the induction hypothesis. To verify the second equation, let $S \in S(G)_e$ be a subgraph of $G$ containing $e$. Since

![Figure 2. Graphs $H, H \setminus v_3v_4$, and $H/v_3v_4$.](image)
contracting $e$ leaves connected components not containing $e$ unchanged, we only consider the connected component $C_e$ of $S$ having $e$. Let $S' \in S(G/e)$ be the subgraph of $G/e$ obtained from $S$ by contracting $e$ and $C'_e$ the connected component containing $v_e$. Using the relation $1 - v_e = (1 - u_1)(1 - u_2)$ and the fact that $V(C'_e) \setminus \{v_e\} \cup \{u_1, u_2\} = V(C_e)$, we obtain $v_{C_e} = v_{C'_e}$. Since contracting $e$ preserves the corank of $S$, we deduce

$$( -1 )^{s(S)} \prod_{C \in \mathcal{C}(S)} v_C = ( -1 )^{s(S')} \prod_{C' \in \mathcal{C}(S')} v_{C'},$$

and the proof follows from Proposition 3.1 and the induction hypothesis. \hfill \Box

**Example 3.3.** Let $P$ be the path graph of length 2. See Figure 3. We can compute $\prod_{C \in \mathcal{C}(S)} v_C$ for each spanning subgraph $S$. For instance, let $S = \{v_1v_2\}$ be the second subgraph in Figure 3. $S$ has the two connected components whose vertex sets are $\{v_1, v_2\}$ and $\{v_3\}$. Then the corresponding term is

$$\prod_{C \in \mathcal{C}(S)} v_C = (1 - (1 - v_1)(1 - v_2)) v_3.$$

Note that the corank of each subgraph of $P$ is equal to 0. Using the previous theorem, we have

$$A_P(V) = (1 - (1 - v_1)(1 - v_2)(1 - v_3)) + (1 - (1 - v_1)(1 - v_2)) v_3$$

$$+ (1 - (1 - v_2)(1 - v_3)v_1 + v_1v_2v_3$$

$$= v_1 + v_2 + v_3 + v_1v_3.$$

![Figure 3](image)

**Figure 3.** A list of all spanning subgraphs of $P$ and their corresponding terms.

3.2. **Broken circuit complexes.** We will give an expression for $A_G(V)$ in terms of the broken circuit complex $B_G$ as an analogue of Whitney’s theorem [13]. Let the edge set $E(G)$ be linearly ordered. A broken circuit is a cycle with its smallest edge removed. The **broken circuit complex** $B_G$ is the collection of all spanning subgraphs $S$ which do not contain a broken circuit. For an edge $e \in E(G)$, define

$$(B_G)^e = \{ S \in B_G \mid e \notin E(S) \}, \text{ and } (B_G)_e = B_G \setminus (B_G)^e.$$

Recall the following proposition appeared in [3] and [6, Lemma 3.7].

**Proposition 3.4.** If $e \in E(G)$ is the largest edge in $E(G)$, then deleting $e$ yields $(B_G)^e = B_{G \setminus e}$ and contracting $e$ gives a bijection between $(B_G)_e$ and $B_{G/e}$.

**Proof.** A proof follows from the choice of $e$. \hfill \Box
Theorem 3.5. For a graph \( G = (V, E) \), the acyclic orientation polynomial \( A_G(V) \) equals

\[
A_G(V) = \sum_{S \in B_G} \prod_{C \in \mathcal{C}(S)} v_C.
\]

Proof. Replacing Proposition 3.4 by 3.1 in the proof of Theorem 3.2 proves this theorem. \( \square \)

The linear terms of equation (9) give the following corollary (Theorem 7.3).

Corollary 3.6. Let \( G \) be a connected graph whose vertex set is \( V \) with \( |V| = d \). For any vertex \( v \in V(G) \), the number of acyclic orientations of \( G \) with the unique sink \( v \) is equal to the number of broken circuits with \( d - 1 \) edges.

Proof. For \( S \in B_G \), the graph \( S \) is connected if and only if \( |S| = d - 1 \). Since \( v_B \) has no constant terms for a connected component \( C \in \mathcal{C}(S) \), the degree of each term of \( \prod_{C \in \mathcal{C}(S)} v_C \) is greater than 1 if \( |S| < d - 1 \). Thus, by equation (9), the linear terms of \( A_G(V) \) are the same as those of

\[
\sum_{S \in B_G, |S| = d - 1} v_S = \sum_{S \in B_G, |S| = d - 1} \left( 1 - \prod_{v \in V} (1 - v) \right).
\]

Therefore, the coefficient of \( v \in V \) in \( A_G(V) \) is equal to \( |\{S \in B_G \mid |S| = d - 1\}| \).

3.3. Bond lattices. We express \( A_G(V) \) in terms of the bond lattice \( L_G \) as an analogue of Birkhoff’s Theorem [2]. For a partition \( \pi \) of a set, an element \( B \in \pi \) is called a block. A bond is a vertex partition each of whose blocks induces a connected graph. The set of bonds of \( G \) forms the lattice \( L_G \) partially ordered by refinement, called the bond lattice of \( G \).

The least element \( \hat{0} \) of \( L_G \) is the bond each of whose block has only one vertex, and the greatest element \( \hat{1} \) of \( L_G \) is the bond each of whose block is the vertex set of a connected component of \( G \). Let \( \mu_G(\cdot, \cdot) \) be the Möbius function of \( L_G \). The Möbius invariant of \( G \) is defined as \( \mu(G) = |\mu_G(\hat{0}, \hat{1})| \). Note that \( |\mu_G(\hat{0}, \pi)| = (-1)^{d-|\pi|} |\mu_G(\hat{0}, \pi) | \) for \( \pi \in L_G \).

Proposition 3.7. Let \( e = u_1u_2 \in E(G) \) and \( \pi \in L_G \). Denote by \( \hat{0}_e \) the bond whose blocks are singletons except the one block \( \{u_1, u_2\} \). Then \( \mu_G(\hat{0}, \pi) = \mu_G, e(\hat{0}, \pi) - \mu_G(\hat{0}, \pi) \) if \( \pi \in L_G \setminus \hat{0}, e \), and \( \mu_G(\hat{0}, \pi) = -\mu_G(\hat{0}, \pi) \) otherwise.

Proof. This proposition can be proved by the deletion-contraction recurrence for \( \mu(G) \). \( \square \)

Let us take a bond \( \pi \in L_G \). Let \( G/\pi \) be the graph obtained from \( G \) by contracting each block \( B \in \pi \) to the vertex \( v_B \), and denote by \( V/\pi \) the vertex set of \( G/\pi \). For \( B \in \pi \), define the variable \( v_B \) associated with the vertex \( v_B \) to be

\[
v_B = 1 - \prod_{B \in \pi} (1 - v).
\]

Theorem 3.8. For a graph \( G = (V, E) \), its acyclic orientation polynomial \( A_G(V) \) equals

\[
A_G(V) = \sum_{\pi \in L_G} (-1)^{d-|\pi|} \mu_G(\hat{0}, \pi) \prod_{B \in \pi} v_B.
\]

By the Möbius inversion formula, equation (10) is equivalent to

\[
\sum_{\pi \in L_G} (-1)^{d-|\pi|} A_{G/\pi}(V/\pi) = \prod_{v \in V} v.
\]
Proof. In the proof of Theorem 3.2 using Proposition 3.1 in place of 3.7 proves the theorem.

For a non-empty subset $U$ of $V$, define

$$\mathcal{R}(U) = \{ \pi \in L_G \mid B \cap U \neq \emptyset \text{ for every } B \in \pi \},$$

and then extracting the coefficient of $\prod_{v \in U} v$ from equation (10) yields

$$a(G, U) = \sum_{\pi \in \mathcal{R}(U)} (-1)^{d-|\pi|} \mu_G(\hat{0}, \pi).$$

As in Corollary 3.6 Theorem 3.8 gives the following corollary.

**Corollary 3.9 (\cite{7} Theorem 7.3).** Let $G$ be a connected graph with the vertex set $V$. For any vertex $v \in V$, we have $|a(G, v)| = \mu(G)$.

\cite{7} Theorem 7.3 was proved via the theory of hyperplane arrangements, and its three more proofs were also presented in \cite{5}. In the remaining of the section, we shall alternatively give a non-inductive proof for Theorem 3.8 using \cite{7} Theorem 7.3 and Weisner’s theorem. For a proof of Weisner’s theorem, see \cite{14} Corollary 3.9.3.

**Theorem 3.10** (Weisner’s theorem). Let $L$ be a finite lattice with at least two elements, $\mu_L(\cdot, \cdot)$ its Möbius function, $0_L$ its least element, and $\hat{1}_L$ its greatest element. For $a \in L$ with $a \neq \hat{1}_L$,

$$\sum_{t : t \wedge a = \hat{0}_L} \mu_L(t, \hat{1}_L) = 0,$$

where $a \wedge b$ is the largest element $p$ satisfying $p \leq a$ and $p \leq b$.

Let $G^U$ be the graph whose vertex set is $V \cup \{u_0\}$ and edge set is $E \cup \{u_0v \mid v \in U\}$. For $\sigma \in \mathcal{A}(G^U, \{u_0\})$, deleting $u_0$ from $\sigma$ yields an acyclic orientation whose sinks are contained in $U$. This procedure is bijective, which gives the following identity:

$$a(G^U, u_0) = a(G, \subseteq U),$$

where $a(G, \subseteq U)$ denotes the number of acyclic orientations of $G$ whose sinks belong to $U$.

The following proposition is the key ingredient for proving equation (11).

**Proposition 3.11.** For a non-empty proper subset $U$ of $V$, we have

$$\sum_{\pi \in L_G} (-1)^{d-|\pi|} a(G/\pi, \subseteq U/\pi) = 0,$$

where $U/\pi$ is the subset of $V/\pi$ corresponding to $U$, i.e., $U/\pi = \{v_B \mid B \in \pi, B \cap U \neq \emptyset\}$.

**Proof.** Define the map $\iota : L_G \rightarrow L_{G^U}$ by $\iota(\pi) = \pi \cup \{\{u_0\}\}$ for each $\pi \in L_G$. The lattice $L_G$ is embedded in $L_{G^U}$ via this map. The least element and greatest element of $L_{G^U}$ will be denoted by $\hat{0}_{G^U}$ and $\hat{1}_{G^U}$, respectively.

The quantity $a(G/\pi, \subseteq U/\pi)$ in equation (11) is equal to

$$a(G/\pi, \subseteq U/\pi) = a(G^U/\iota(\pi), u_0) = \mu_{G^U/\iota(\pi)}(G^U/\iota(\pi)) = (-1)^{|\pi|} \mu_{G^U}(\iota(\pi), \hat{1}_{G^U}),$$

where the first equality is verified by equation (13) and $(G/\pi)^{U/\pi} \simeq G^U/\iota(\pi)$, the second by \cite{7} Theorem 7.3, and the third by the fact the bond lattice of $G^U/\iota(\pi)$ is isomorphic to the interval $[\iota(\pi), \hat{1}_{G^U}]$ in $L_{G^U}$ and $|\iota(\pi)| = |\pi| + 1$. 

It remains to show that
\[
(-1)^d \sum_{\pi \in L_G} \mu_G(\iota(\pi), \hat{1}_Gv) = 0.
\]
Denote by \(\tilde{\pi}_0\) the bond in \(L_{Gv}\) consisting of \(U \cup \{u_0\}\) and \(d - |U|\) singletons. Let \(\mathcal{M} = \{\iota(\pi) \wedge \tilde{\pi}_0 \in L_{Gv} \mid \pi \in L_G\}\). Then \(L_G\) is partitioned as \(L_G = \cup_{\tilde{\pi} \in \mathcal{M}} \{\pi \in L_G \mid \iota(\pi) \wedge \tilde{\pi}_0 = \tilde{\pi}\}\), which shows that the left-hand side of equation (15) is
\[
(\cdot \cdot \cdot ) = \sum_{\tilde{\pi} \in \mathcal{M}} \sum_{\pi \in L_G} \mu_G(\iota(\pi), \hat{1}_Gv).
\]
Fix \(\tilde{\pi} \in \mathcal{M}\). Note that for an element \(\pi' \in L_{Gv}\) with \(\pi' \wedge \tilde{\pi}_0 = \tilde{\pi}\), there exists a unique \(\pi \in L_G\) with \(\pi' = \iota(\pi)\). Since \(L_{Gv}/\tilde{\pi}\) is isomorphic to the interval \([\tilde{\pi}, \hat{1}_G]\) in \(L_{Gv}\), applying Theorem 3.10 to the lattice \(L_{Gv}/\tilde{\pi}\) shows that the summation in (16) associated with \(\tilde{\pi}\) is equal to 0, which finishes the proof.

For a non-empty proper subset \(U \subset V\), let us evaluate the left-hand side of equation (11) for \(v = 1\) if \(v \in U\), and \(v = 0\) otherwise. From the definition of \(v_B\), we see \(v_B = 1\) if and only if \(v = 1\) for some \(v \in B\). It follows that \(v_B = 1\) if \(B \cap U \neq \emptyset\) for \(B \in \pi\). By the definition of \(U/\pi\), our evaluation is equal to the left-hand side of equation (14), which is 0 by Proposition 3.11.

By the definition of \(v_B\), the left-hand side of equation (11) is a square-free polynomial in variables \(V\) without constant term. Hence, we write
\[
\sum_{\pi \in L_G} (-1)^{d - |\pi|} A_{G/\pi}(V/\pi) = \sum_{U \subseteq V} c_U \prod_{v \in U} v.
\]
The values \(c_U\) must be 0 for all \(U \subseteq V\). If not, there exists a minimal \(U \subseteq V\) such that \(c_U \neq 0\). By the evaluation in the previous paragraph, the right-hand side is \(\sum_{W \subseteq U} c_W = 0\), and by minimality, \(c_W = 0\) if \(W \subseteq U\). Hence, we have \(c_U = 0\), which is a contradiction.

We claim that \(c_V = 1\), which will complete the proof. Since \(v_B\) is a polynomial of degree \(|B|\), the degree of the polynomial \(A_{G/\pi}(V/\pi)\) is less than \(|V|\) if \(G/\pi\) has an edge. The polynomial \(A_{G/\pi}(V/\pi)\) where \(\pi\) is the greatest element \(\hat{\pi}\) of \(L_G\) is the only term which can contribute to \(c_V\), and the coefficient of \(\prod_{v \in V} v\) in the polynomial is \((-1)^{d - |\pi|}\), which proves the claim.

3.4. The map \(A(G)\) to \(L_G\). We present an expression of \(A_G(V)\) in terms of the map \(\pi : A(G) \to L_G\) introduced in Section 7. Let us fix an ordering of \(V\), and take \(\sigma \in A(G)\).

For \(j \geq 1\), suppose that \(B_1, B_2, \ldots, B_{j-1} \subseteq V\) are defined. Denote by \(s_j\) the smallest element in \(V \setminus \bigcup_{i=1}^{j-1} B_i\). Let \(B_j\) be the collection of vertices in \(V \setminus \bigcup_{i=1}^{j-1} B_i\) reachable to \(s_j\) by a directed path in \(\sigma\). Define \(\pi(\sigma) = \{B_1, \cdots, B_q\}\), where \(q\) is the largest integer with \(B_q \neq \emptyset\). Clearly, \(\pi(\sigma) \in L_G\). The blocks in \(\pi(\sigma)\) are called the sink-components [1].

Recall the bijections which were used to prove the deletion-contraction recursion of the number of acyclic orientations [11]. Take an edge \(e = u_1u_2 \in E(G)\). Let \(A_{e\sigma}\) be the set of acyclic orientations of \(G \setminus e\) which do not contain a directed path from \(u_1\) to \(u_2\) nor vice-versa. Define
\[
A(G)_e = \{ \sigma \cup \overrightarrow{u_2u_1} \mid \sigma \in A_{e\sigma} \}
\] and \(A(G)^e = A(G) \setminus A(G)_e\).
Deleting $e$ from an acyclic orientation in $\mathcal{A}(G)e$ gives a bijection between $\mathcal{A}(G)e$ and $\mathcal{A}(G\setminus e)$, while contracting $e$ yields a bijection between $\mathcal{A}(G)e$ and $\mathcal{A}(G/e)$.

**Proposition 3.12.** Let $e = u_1u_2$ with $u_1 < u_2$ be the smallest edge of $E(G)$ in the lexicographic order. Then deleting $e$ from an orientation in $\mathcal{A}(G)e$ preserves its image, and contracting $e$ from an orientation in $\mathcal{A}(G)e$ leads to merging $u_1$ and $u_2$ in the image.

**Proof.** The choice of $u_1u_2$ proves the proposition. □

The following theorem is an analogue of [7, Corollary 7.4] for the chromatic polynomial.

**Theorem 3.13.** For a graph $G = (V, E)$, its acyclic orientation polynomial $A_G(V)$ equals

$$A_G(V) = \sum_{o \in \mathcal{A}(G)e} \prod_{B \in \pi(o)} v_B.$$

**Proof.** A proof follows from the proof of Theorem 3.2 by replacing Proposition 3.1 by 3.12. □

Comparing Theorems 3.8 and 3.13 gives a proof for [7, Theorem 7.4] whose original proof exploits Corollary 3.9 ([7, Theorem 7.3]).

**Theorem 3.14** ([7, Theorem 7.4]). For a bond $\pi \in L_G$, the cardinality of the image $\{o \in \mathcal{A}(G) \mid \pi = \pi(o)\}$ is equal to $|\mu_G(\hat{0}, \pi)| = (-1)^{d-|\pi|}\mu(\hat{0}, \pi)$.

**Proof.** The set $\{\prod_{B \in \pi} v_B \mid \pi \in L_G\}$ is linearly independent over $\mathbb{Z}$. This fact can be proved using induction and its proof will be omitted. Then equating coefficients of equation (10) in Theorem 3.8 and equation (17) in Theorem 3.13 proves the theorem. □

If each element in $U \subseteq V$ is smaller than any elements in $V \setminus U$, then for $o \in \mathcal{A}(G)$, its sinks are in $U$ if and only if $\pi(o)$ lies in $R(U)$, which together with [7, Theorem 7.4] gives

$$a(G, \subseteq U) = \sum_{\pi \in R(U)} (-1)^{d-|\pi|}\mu_G(\hat{0}, \pi).$$

Note that the following identity

$$\sum_{U \subseteq V, \pi \in R(U)} (-1)^{|U|} = \prod_{B \in \pi} \left( \sum_{B' \subseteq B, B' \neq \emptyset} (-1)^{|B'|} \right) = (-1)^{|\pi|}$$

leads to the fact that equation (18) is equivalent to equation (12). Hence, we remark that Theorem 3.8 could be alternatively proved using [7, Theorem 7.4].

We close this section with a generating function for $a(G, \subseteq U)$ given by

$$\sum_{U \subseteq V} \left( a(G, \subseteq U) \prod_{v \in U} v \right) = \sum_{\pi \in L_G} (-1)^{d-|\pi|}\mu_G(\hat{0}, \pi) \prod_{B \in \pi} \left( \prod_{v \in B} (1 + v) - 1 \right).$$

4. **Chromatic symmetric functions and Stanley’s sink theorem**

In this section, we present a new proof for [12, Theorem 3.3], which expresses the number of acyclic orientations with a fixed number of sinks as the sum of the coefficients of elementary symmetric functions $e_\lambda$ with a fixed length in the expansion of the chromatic symmetric function. Our proof does not require the theory of quasi-symmetric functions and $P$-partitions, which was used in the original proof of [12, Theorem 3.3].
We begin with the definition of the chromatic symmetric function $X_G$ for a graph $G = (V, E)$ with $|V| = d$. Let $x_1, x_2, \ldots$ be commuting indeterminates. A proper coloring $\kappa$ of $G$ is a function $\kappa : V \rightarrow \{1, 2, 3, \ldots\}$ such that $\kappa(v) \neq \kappa(v')$ whenever $v, v' \in V$ are adjacent.

Definition 4.1 ([12, Definition 2.1]). The chromatic symmetric function $X_G$ is defined as

$$X_G = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)},$$

where the sum is over all proper colorings $\kappa$ of $G$.

We will expand the symmetric function $X_G$ in terms of power sum symmetric functions and elementary symmetric functions. For $n \geq 1$, the $n$-th power sum symmetric function $p_n$ and elementary symmetric function $e_n$ are

$$p_n = \sum_{i \geq 1} x_i^n \quad \text{and} \quad e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n},$$

with $p_0 = e_0 = 1$. For a sequence $\alpha = (\alpha_1, \ldots, \alpha_j)$ of positive integers, define

$$p_\alpha = \prod_{i=1}^j p_{\alpha_i} = p_{\alpha_1} \cdots p_{\alpha_j}, \quad \text{and} \quad e_\alpha = \prod_{i=1}^j e_{\alpha_i} = e_{\alpha_1} \cdots e_{\alpha_j}.$$  

The length $l(\alpha)$ of $\alpha$ is defined as the number of parts of $\alpha$. A sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_j)$ of positive integers with $\alpha_1 + \cdots + \alpha_j = n$ is a composition of $n$ and the set of such sequences is denoted by Comp($n$). For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j > 0)$ with $\lambda_1 + \cdots + \lambda_j = n$, the sequence $\lambda$ is called a partition of $n$, denoted by $\lambda \vdash n$. Note that $\{p_\lambda \mid \lambda \vdash n\}$ and $\{e_\lambda \mid \lambda \vdash n\}$ form bases for the space of all homogeneous symmetric functions of degree $n$.

Let us collect expansions of $X_G$ with respect to power sum symmetric functions $p_\lambda$:

\begin{align*}
(20) \quad X_G &= \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)} \quad [12 \, \text{Theorem 2.5}] \\
(21) &= \sum_{S \in B_G} (-1)^{|S|} p_{\lambda(S)} \quad [12 \, \text{Theorem 2.9}] \\
(22) &= \sum_{\pi \in L_G} \mu(\hat{0}, \pi) p_{\text{type}(\pi)} \quad [12 \, \text{Theorem 2.6}] \\
(23) &= \sum_{\sigma \in A(G)} (-1)^{d-|\sigma(\sigma)|} p_{\text{type}(\pi(\sigma))} \quad [1 \, \text{Proposition 5.2}],
\end{align*}

where $\lambda(S)$ and type$(\pi)$ are the non-increasing sequences of the sizes of connected components of a spanning subgraph $S$, and elements of a vertex partition $\pi$, respectively.

The following proposition shows that in the expansion of $p_n$ with respect to $e_\lambda$, the sum of coefficients corresponding to partitions of a fixed length is a binomial coefficient up to sign. The proof below uses the generating functions. For its determinantal expression, see e.g. [9 Exercise I.2.8].

Proposition 4.2. Let $p_n = \sum_{\lambda \vdash n} b_\lambda e_\lambda$ be the expansion of $p_n$ in terms of $e_\lambda$. Then

\[ (-1)^{n-1} \sum_{\lambda \vdash n} b_\lambda = (-1)^{j-1}{n \choose j}. \]


Proof. The generating function for \((-1)^{n-1}p_n\) is

\[
\sum_{n \geq 1} p_n (-z)^{n-1} = \sum_{i \geq 1} \frac{x_i}{1 + x_i z} = \frac{d}{dz} \log \prod_{i \geq 1} (1 + x_i z) = \frac{d}{dz} \log E(z) = \frac{E'(z)}{E(z)}
\]

\[
= \left( \sum_{n \geq 1} n e_n z^{n-1} \right) \left( \sum_{j \geq 1} \left( - \sum_{n \geq 1} e_n z^n \right)^{j-1} \right),
\]

where \(E(z) = \sum_{n \geq 0} e_n z^n\). Equating coefficients of \(z^{n-1}\) on both sides yields

\[
(-1)^{n-1}p_n = \sum_{\alpha \in \text{Comp}(n)} (-1)^{l(\alpha)-1} \alpha_1 e_\alpha.
\]

The proposition then follows from the following identities

\[
\sum_{\alpha \in \text{Comp}(n)} \alpha_1 = \sum_{\alpha \in \text{Comp}(n)} \left| \{(a, \alpha_1 + 1 - a, \alpha_2, \alpha_3, \ldots) \mid 1 \leq a \leq \alpha_1 \} \right| = \sum_{\alpha \in \text{Comp}(n+1)} 1 = \binom{n}{1}.
\]

\[
\square
\]

Example 4.3. For the graph \(H\) in Example 2.4, the expansion of \(X_H\) with respect to \(p_\lambda\) is given as follows:

\[
X_H = -4p_4 + 6p_{(3,1)} + 2p_{(2,2)} - 5p_{(2,1,1)} + p_{(1,1,1,1)}.
\]

From relations shown in the proof of Proposition 4.2

\[
p_1 = e_1, \quad p_2 = e_1^2 - 2e_2, \quad p_3 = e_1^3 - 3e_2e_1 + 3e_3, \quad p_4 = e_1^4 - 4e_2e_1^2 + 4e_3e_1 + 2e_2^2 - 4e_4,
\]

we derive \(X_H = 16e_4 + 2e_{(3,1)}\).

Let \(\Lambda\) be the \(\mathbb{Q}\)-algebra of symmetric functions with \(\mathbb{Q}\)-coefficients. Since \(\{e_n\}_{n \geq 1}\) is algebraically independent and generates \(\Lambda\) as a \(\mathbb{Q}\)-algebra, assigning a value to each \(e_n\) determines an algebra homomorphism on \(\Lambda\). Let \(\mathbb{Q}[t]\) be the ring of polynomials in an indeterminate \(t\) with \(\mathbb{Q}\)-coefficients. Define an algebra homomorphism \(\phi : \Lambda \to \mathbb{Q}[t]\) by \(\phi(e_n) = t\) for each \(n \geq 1\). Then

\[
\phi(e_\lambda) = t^{l(\lambda)}
\]

For this homomorphism \(\phi\), the image \(\phi(p_n)\) is computed as follows.

Lemma 4.4. The image of \(p_n\) under \(\phi\) is equal to

\[
\phi(p_n) = (-1)^{n-1}(1 - (1 - t)^n).
\]

Proof. By Proposition 4.2, for \(p_n = \sum_{\lambda \vdash n} b_\lambda e_\lambda\),

\[
\phi(p_n) = \sum_{j=1}^{n} \left( \sum_{\lambda \vdash n, \quad l(\lambda) = j} b_\lambda \phi(e_\lambda) \right) = (-1)^{n-1} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} t^j = (-1)^{n-1}(1 - (1 - t)^n).
\]

\[
\square
\]
We are ready to prove Stanley’s sink theorem [12, Theorem 3.3]. The proof below uses theorems (Theorem 3.2 and equation (20)) involving subgraph expansions. Instead, one can employ Theorems (Theorem 3.5 and equation (21)) concerning broken circuit complexes, theorems (Theorem 3.8 and equation (22)) concerning bond lattices, or theorems (Theorem 3.13 and equation (23)) concerning the map from $A(G)$ to $L_G$.

**Theorem 4.5** ([12, Theorem 3.3]). Let $X_G = \sum_{\lambda \vdash d} c_{\lambda} e_{\lambda}$ be the expansion of the chromatic symmetric function $X_G$ in terms of elementary symmetric functions $e_{\lambda}$ and let sink($G, j$) be the number of acyclic orientations of $G$ with $j$ sinks. Then

\begin{equation}
\text{sink}(G, j) = \sum_{\lambda \vdash d \atop i(\lambda)=j} c_{\lambda}.
\end{equation}

**Proof.** The statement that equation (26) holds for every $j \geq 1$ is equivalent to

\begin{equation}
a_G(t) = \sum_{\lambda \vdash d} c_{\lambda} t^{l(\lambda)}.
\end{equation}

To prove this equation, we apply $\phi$ to the two expansions of $X_G$ in terms of $e_{\lambda}$ and $p_{\lambda}$. By equation (24),

\[ \phi(X_G) = \phi(\sum_{\lambda \vdash d} c_{\lambda} e_{\lambda}) = \sum_{\lambda \vdash d} c_{\lambda} \phi(e_{\lambda}) = \sum_{\lambda \vdash d} c_{\lambda} t^{l(\lambda)}. \]

On the other hand, we obtain

\[ \phi(X_G) = \sum_{S \subseteq E} (-1)^{s(S)} p_{\lambda(S)} \]
\[ = \sum_{S \subseteq E} (-1)^{s(S)} \prod_{C \in \mathcal{C}(S)} (-1)^{|V(C)| - 1} \phi(p_{|V(C)|}) \]
\[ = \sum_{S \subseteq E} (-1)^{s(S)} \prod_{C \in \mathcal{C}(S)} \left(1 - (1 - t)^{|V(C)|}\right) \]
\[ = a_G(t), \]

where the first equality uses equation (20), the second follows from the definition of the corank $s(S) = |S| - d + |\mathcal{C}(S)|$, the third is verified by Lemma 4.4 and the last by equation (8) in Theorem 3.2. We conclude that equation (27) holds, completing the proof. \hfill \Box

**Example 4.6.** Let $H$ be the graph in Example 2.4. Since $X_H = 16e_4 + 2e_{(3,1)}$, the previous theorem gives sink($H, 1$) = 16 and sink($H, 2$) = 2, or $a_H(t) = 16t + 2t^2$. This coincides with the computation in Section 2.

5. **On distinguishing graphs by acyclic orientation polynomials**

We end this paper with a discussion on how well graphs are distinguished by the polynomials $A_G(V)$ and $a_G(t)$ in relation to $X_G$. Stanley [12] asked the question of whether the chromatic symmetric polynomial $X_G$ distinguishes non-isomorphic trees. The answer to the question is still unknown, but several graph polynomials [6, 10] associated with $X_G$ have been introduced in the endeavor to resolve the question. For example, the noncommutative version $Y_G$ of $X_G$ is a complete isomorphism invariant, i.e., $Y_G$ distinguishes among all simple graphs $G$. 

The (univariate) polynomial \( a_G(t) \) is a \textit{weaker} invariant than \( X_G \) by Stanley’s sink theorem [12, Theorem 3.3]. Trees with 10 or fewer vertices are distinguished by \( a_G(t) \). But there exist non-isomorphic trees \( T_1, T_2 \) having the identical \( a_G(t) \) with 11 vertices. These two trees shown in Figure 4 were introduced in [4] as the smallest instances which are not distinguished by the subtree data.

![Figure 4. Two non-isomorphic trees with the same \( a_G(t) \).](image)

The (multivariate) acyclic orientation polynomial \( A_G(V) \) is a \textit{complete} isomorphism invariant. To see this, we first assume that \( G \) is connected. For two vertices \( u_1, u_2 \in V \), the coefficient of \( u_1u_2 \) in \( A_G(V) \) is zero if and only if \( u_1 \) and \( u_2 \) are adjacent. Hence, the polynomial \( A_G(V) \) can recover the original graph \( G \). For any graph \( G \), the polynomial \( A_G(V) \) has a factorization into those of its connected components, and therefore using the previous argument applied to each component shows that \( G \) can be determined by \( A_G(V) \).

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