A Dirac Particle in a Complex Potential

Khaled Saaidi\textsuperscript{1}

Department of Science, University of Kurdistan, Pasdaran Ave., Sanandaj, Iran
Institute for Studies in Theoretical Physics and Mathematics, P.O.Box, 19395-5531, Tehran, Iran

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Abstract

It has been observed that a quantum mechanical theory need not to be Hermitian to have a real spectrum. In this paper we obtain the eigenvalues of a Dirac charged particle in a complex static and spherically symmetric potential. Furthermore, we study the Complex Morse and complex Coulomb potentials.

\textsuperscript{1}E-mail-1: KSaaidi@ipm.ir
E-mail-2 : ksaaidi@uok.ac.ir
1 Introduction

The first interest study in the non-Hermitian quantum theory date back to an old paper by Caliceti et al [1]. In [1] the imaginary cubic oscillator problem in the context of perturbation theory has been studied. The energy spectrum of that model is real and discrete. It shows that one may construct many new Hamiltonians that have real spectrum, although, their Hamiltonians are not Hermitian. The key idea of the new formalism (non-Hermitian quantum theory) lies in the empirical observation that the existence of the real spectrum need not necessarily be attributed to the Hermiticity of the Hamiltonian. In non-Hermitian Hamiltonian, the current Hermiticity assumption $H^\dagger = H$ is replaced by the PT-symmetry condition as $H^\dagger = \hat{P}HT\hat{P}T$ [2, 17], where $\hat{P}$ denotes the parity operator; $(\hat{P}\psi)(x) = \psi(-x)$ and $\hat{T}$ the time reversal operator; $(\hat{T}\psi)(x) = \psi^*(x)$ or generally by Pseudo-Hermitian condition as $H^\dagger = \tau^{-1}H\tau$, where $\tau$ is an invertible Hermitian operator[17]. Such non-Hermitian formalism, for the context of Schrödinger Hamiltonian, has been studied for many different subjects with several techniques [1-20]. Also, some explicit studies of the Hermitian and non-Hermitian Hamiltonians have performed in the context of Dirac Hamiltonian. For example, the solution of ordinary (Hermitian) Dirac equation for Coulomb potential including its relativistic bound state spectrum and wave function was investigated in [22,23]. Also by adding off-diagonal real linear radial term to the ordinary Dirac operator, the relativistic Dirac equation with oscillator potential has been introduced [24, 25] moreover the energy spectrum of corresponding eigenfunctions have been obtained. The ordinary (Hermitian) Dirac equation for a charged particle in static electromagnetic field, is studied for Morse potential [26].

In this paper, we consider the non-Hermitian Dirac Hamiltonian for complex Morse and complex Coulomb potential. We consider a charged particle in static and spherically symmetric four component complex potential. By applying a unitary transformation to Dirac equation, we obtain the second order Schrödinger like equation, therefore comparison with well-known non-relativistic problems is transparent. Using, correspondence between parameters of the two problems (the Schrödinger equation and the Schrödinger like equation which is obtained after applying the unitary transformation on Dirac equation for a potential ) we can obtain the bound states spectrum and wave function.

The structure of this article is as follows. In sec.2, we study the non-Hermitian version of Dirac equation for a charged particle with static and spherically symmetric potential, then by applying a unitary transformation we obtain the proper gauge fixing condition and Schrödinger like differential equation. In sec.3, we discuss the Dirac equation for complex Morse potential then we obtain the real energy spectrum and corresponding eigenfunctions. In sec.4, we consider the Dirac equation for a complex Coulomb potential and we obtain the real eigenvalue of it.

2 Preliminaries

The Hamiltonian of a Dirac particle for a complex electromagnetic field is $(c = \hbar = 1)$

$$H = \hat{\alpha}.(\hat{p} - e\hat{A}) + \hat{\beta}m + eV,$$  \hspace{1cm} (1)

where the Dirac matrices $\hat{\alpha}, \hat{\beta}$ have their usual meaning, and setting $A_0$ equal to $V$. In (1), $\hat{A}$ and $V$ are the vector and scalar complex field respectively, where $A^\dagger \neq \hat{A}$ and $V^\dagger \neq V$. Then the Dirac Hamiltonian (1) is not Hermitian. It is well known that the local gauge symmetry in
quantum electrodynamic implies an invariance under the transformation as:

\[
(V, \hat{A}) \rightarrow (V', \hat{A}') = (V + \frac{\partial \Lambda}{\partial t}, \hat{A} + \nabla \Lambda).
\] (2)

Here \(\Lambda(t, \vec{r})\) is a complex scalar field. Suppose that the charge distribution is static with spherical symmetry, so the gauge invariance implies that \(V' = V\) and \(\hat{A}' = \hat{r}A(r)\), where \(\hat{r}\) is the radial unit vector[26]. One can denoted the correspondence wave function of (1) as:

\[
\Psi = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}.
\] (3)

In this case one can obtain

\[
(m + eV - E_r)\Phi = i \left[ \hat{\sigma} \cdot \nabla - e(\hat{\sigma} \cdot \dot{r})A(r) \right] \chi,
\]

\[
(eV - m - E_r)\chi = i \left[ \hat{\sigma} \cdot \nabla + e(\hat{\sigma} \cdot \dot{r})A(r) \right] \Phi.
\] (4)

Here \(\hat{\sigma}\)'s are the three Pauli spin matrices, \(E_r\) is the relativistic energy eigenvalue, then we replaced \(ie\hat{\sigma} \cdot \hat{A}(-ie\hat{\sigma} \cdot A)\) in first(second) equation of (4) instead of \(e\hat{\sigma} \cdot \hat{A}\), respectively. Note that, because of the spherical symmetry of the complex field, the angular-momentum operator \(\hat{J}\) and the parity operator, \(\hat{P}\), commute with the Hamiltonian and the two spinors \(\Phi\) and \(\chi\) have opposite parity also. So the correspondence wave functions are denoted by

\[
\Phi = ig(r)\Omega_{\kappa,\mu}(\theta, \varphi),
\]

\[
\chi = f(r)\sigma_r\Omega_{-\kappa,\mu}(\theta, \varphi).
\] (5)

It is seen that

\[
(\hat{\sigma} \cdot \nabla)ig(r)\Omega_{\kappa,\mu}(\theta, \varphi) = i\sigma_r\Omega_{\kappa,\mu}(\partial_r + \frac{1}{r} + \frac{\kappa}{r})g(r),
\] (6)

\[
(\hat{\sigma} \cdot \nabla)(f(r)\sigma_r\Omega_{-\kappa,\mu}(\theta, \varphi)) = \sigma_r\Omega_{-\kappa,\mu}(\partial_r + \frac{1}{r} - \frac{\kappa}{r})f(r),
\] (7)

where \(\kappa\) is the spin orbit coupling operator which defined as:

\[
\kappa = \hat{\sigma} \cdot \hat{L} + \hbar I.
\] (8)

and we have used from

\[
\kappa\Omega_{\pm\kappa,\mu}(\theta, \varphi) = \pm \kappa \hbar \Omega_{\pm\kappa,\mu}(\theta, \varphi),
\] (9)

in which

\[
\kappa = \begin{cases} 
-(l + 1) = -(j + \frac{1}{2}) & \text{for } j = l + \frac{1}{2} \\
 l = (j + \frac{1}{2}) & \text{for } j = l - \frac{1}{2}
\end{cases}
\] (10)

Therefore by defining \(u_1 = g(r)/r\), \(u_2 = f(r)/r\), we obtain the following two component radial Dirac equation [28]

\[
(m + eV - E_r)u_1(r) = (\partial_r - \frac{k}{r} - eA(r))u_2(r),
\]

\[
(eV - m - E_r)u_2(r) = -(\partial_r + \frac{k}{r} + eA(r))u_1(r).
\] (11)

3
Note that, $A(r)$ is a gauge field, which has a symmetry such as (2), therefore, it must be fixed. It is seen that fixing this gauge degree of freedom by $\nabla \cdot \mathbf{A} \equiv \frac{\partial A}{\partial r} = 0$ is not a suitable choice. Remark that in this paper, instead of solving Dirac equation we want to solve the 2nd order differential equation, which is obtained by eliminating one component of equation (11). Note that, this result is not Schrödinger like. One can obtain the proper gauge fixing by applying the global unitary transformation on two components $u_1$ and $u_2$ as:

$$
\begin{align*}
  u_1 &= a \phi^u - b \phi^l, \\
  u_2 &= b \phi^u + a \phi^l,
\end{align*}
$$

(12)

where $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$ and $\phi^u, \phi^l$ are the upper and lower component of spinor. This unitary transformation create two results. Firstly, it makes a gauge fixing condition such as:

$$
(2eV + Cm - E_r)\phi^u + \left(\frac{CeV}{S} - Sm - \frac{d}{dr}\right)\phi^l = 0,
$$

(14)

and

$$
\left(\frac{CeV}{S} - Sm + \frac{d}{dr}\right)\phi^u + (-Cm - E_r)\phi^l = 0,
$$

(15)

where $C = a^2 - b^2$. However, by eliminating $\phi^l$ in (14) and (15), one can obtain the Schrödinger like differential equation for radial upper component, $\phi^u$, as:

$$
-\frac{d^2 \phi^u}{dr^2} + V_{eff}\phi^u + (m^2 - E_r^2)\phi^u = 0,
$$

(16)

where

$$
V_{eff} = \left(\frac{CeV}{S}\right)^2 + (2eE_rV - \frac{eC}{S} \frac{dV}{dr}).
$$

(17)

### 3 The complex Morse potential

The complex Morse potential in Schrödinger equation, which holds discrete spectrum is given by [18]

$$
V^{CM}(x) = (B_R + iB_I)^2 e^{-2x} - (B_R + iB_I)(1 + 2D)e^{-x},
$$

(18)

the corresponding Schrödinger equation is as:

$$
-\frac{d^2 \psi}{dx^2} + V^{CM}(x)\psi - E\psi = 0.
$$

(19)

This Schrödinger equation is exactly solvable. It is well known that the energy eigenvalues of the Schrödinger equation for a complex Morse potential (18) is a function of $(D + \frac{1}{2})$, which explains that the spectrum of that is real [19], which is obtained as:

$$
E_n = -(n - D)^2,
$$

(20)
where \( n \) is an integer number between 0 and \( D \). So we assume that, the complex potential \( V(r) \) in (17) is:

\[
V(r) = -(\zeta + i\eta)e^{-r}. \tag{21}
\]

Here \( \zeta, \eta \in \mathbb{R} \). By using equation (17), one finds the effective complex Morse potential, \( V_{CM}^{eff} \), as:

\[
V_{CM}^{eff} = \left( \frac{eC}{S} \right)^2 (\zeta + i\eta)^2 e^{-2r} - \frac{Ce}{S}(\zeta + i\eta)(1 - 2eE_r)e^{-r}. \tag{22}
\]

By comparing equations (16) and (22) with (19) and (18), respectively, we can obtain a correspondence between those parameters as:

\[
\begin{align*}
\frac{Ce}{S} \zeta &= B_R, \\
\frac{Ce}{S} \eta &= B_I, \\
-2eE_r &= 2D, \\
m^2 - E_r^2 &= -E.
\end{align*} \tag{23}
\]

Therefore, from (20) and (23), it is clearly shown that the real spectrum of Dirac particle with complex Morse potential is:

\[
E_{r,n} = \frac{1}{1 + e^2} \left\{ -en + \sqrt{(1 + e^2)m^2 - n^2} \right\} \tag{24}
\]

where \( n = 0, 1, 2, \ldots, n_{\text{max}} \leq m\sqrt{1 + e^2} \). Lastly, we can obtain the eigenfunction of (16), \( \phi^u(r) \), with complex effective potential (22) as:

\[
\phi^u_n(z) = N_n z^{\nu_n} e^{-\frac{z}{2}} L_n^{2\nu_n}(z), \tag{25}
\]

where, \( N_n \) is the normalization constant and

\[
\begin{align*}
z &= 2\left( \frac{Ce}{S} \right)(\zeta + i\eta)e^{-r}, \\
\nu_n &= -eE_{r,n} - n. \tag{26, 27}
\end{align*}
\]

One can show that for the case \( n < -eE_{r,n} \), the \( \phi^u_n(\pm\infty) = 0 \), and the orthogonality condition is satisfied [19]. Furthermore, from (15), one can obtain the lower spinor component as:

\[
\phi^l_n(z) = \frac{1}{mC + E_{r,n}} (-Sm - \frac{1}{2}z - z \frac{d}{dz}) \phi^u_n(z), \tag{28}
\]

\[
= -\frac{N_n}{mC + E_{r,n}} z^{\nu_n} e^{-\frac{z}{2}} \left[ (Sm + \nu_n + n)L_n^{2\nu_n}(z) + (2\nu_n + n) L_{(n-1)}^{2\nu_n}(z) \right], \tag{29}
\]

where (15), (25), (26), (27) and recursion properties of the Laguerre polynomials have been used.
3.1 The complex Coulomb potential

We assume that the complex Coulomb potential is:

\[ eV = \frac{iZ\alpha}{r}, \tag{30} \]

where \( \alpha \) is fine structure constant and \( Z \) is atomic number. This equation represents the interaction of a point charge, \(-e\), and an imaginary point charge \( iZe \). It is obvious that one has the freedom to use any unitary transformation. So for two coupled first order differential equation (11), with complex Coulomb potential (30), is obtained the gauge fixing condition and Schrödinger like differential equation, by applying the another global unitary transformation, instead of (12). In this case we apply the global unitary transformation such \( e^{\frac{2i}{\sigma} \sigma \sigma} \), which is rewritten as:

\[
U = \begin{pmatrix}
a & ib \\
ib & a
\end{pmatrix}, \tag{31}
\]

where \( a, b \in \mathbb{R} \) and \( a^2 + b^2 = 1 \). By applying (31) to \( \phi^u \) and \( \phi^l \) and institute it in (11), we have

\[
(m - E_r C)\phi^u + \left[ \frac{i(S^2 - C^2)}{S}eV - iSE_r - \partial_r \right] \phi^l = 0,
\]
\[
\left[ \frac{i(S^2 - C^2)}{S}eV - iSE_r + \partial_r \right] \phi^u - (m + E_r C)\phi^l = 0, \tag{32}
\]

where, \( S = 2ab, C = a^2 - b^2 \) and we have used from a gauge fixing condition as:

\[ eV = \frac{iS}{C}(eA + \frac{k}{r}). \tag{33} \]

However, we eliminate the \( \phi^l \) component in (32), and obtain the radial differential equation for \( \phi^u \) as:

\[-\frac{d^2}{dr^2} \phi^u(r) + V_{\text{CC}}^\prime(r)\phi^u - (E_r^2 - m^2)\phi^u(r) = 0, \tag{34} \]

where

\[ V_{\text{CC}}^\prime(r) = \frac{\gamma(\gamma + 1)}{r^2} + \frac{2i\alpha Z(S^2 - C^2)E_r}{r}, \tag{35} \]

in which

\[
\gamma = \frac{(S^2 - C^2)\alpha Z}{S},
\]
\[
= \sqrt{\kappa^2 + \alpha^2 Z^2}, \tag{36} \]
\[
= \sqrt{(j + \frac{1}{2})^2 + \alpha^2 Z^2},
\]

is the relativistic angular momentum. Since the wave function has to be normalized, we have chosen the positive sign for \( \gamma \). Furthermore, it is obviously seen that these solutions, in fact,
exist for all values of $Z^2$. The non Hermitian and PT-symmetric radial Schrödinger-Coulomb differential equation is \[10\]
\[-\frac{d^2}{dr^2} + \frac{l(l + 1)}{r^2} + \frac{iA}{r} - E\psi(r) = 0, \tag{37}\]
and its non-relativistic energy spectrum is:
\[E_n = \frac{A^2}{4n^2}, \tag{38}\]
\[n = n_r + \gamma + 1, \tag{38}\]
\[n_r = 0, 1, 2, \ldots\]
So comparing equations (34) and (35) with (37), gives the following correspondences between the parameters of two problems for the regular solution of $\phi_u$ as:
\[\gamma = l, \tag{39}\]
\[2\alpha Z(S^2 - C^2)E_r = A, \tag{39}\]
\[(E_r^2 - m^2) = E. \tag{39}\]
Using (38) and (39) we obtain the relativistic energy spectrum for complex Coulomb potential as:
\[E_{r,n} = m \left[1 - \frac{\alpha^2 Z^2(S^2 - C^2)^2}{(n_r + \gamma + 1)^2}\right]^{-\frac{1}{2}}, \tag{40}\]
where $n_r = n - j + \frac{1}{2}$ is the radial quantum number, $n$ the principle quantum number, $\gamma = j + \frac{1}{2}$ is the angular momentum quantum number, and consequently,
\[E_{r,n} = m \left[1 - \frac{Z^2\alpha^2(S^2 - C^2)^2}{[n - j - \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 + Z^2\alpha^2}]^2}\right]^{-\frac{1}{2}}, \tag{41}\]
\[n = 1, 2, 3, \ldots \tag{42}\]
\[j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \tag{42}\]
and the upper component of the radial spinor eigenfunction is:
\[\phi_u^n(r) = N_n e^{1+\gamma} e^{-\frac{r}{2}} L^{2\gamma+1}_n(r). \tag{43}\]
Here $r = x - i\theta$, where $x$ is real($x \in [0, \infty)$), $\theta > 0$ and $N_n$ is normalization constant \[10\]. It shows that the integration path has been shifted down from the positive. From (35), it is understood that the potential is zero in two cases. The first one is $Z = 0$ and $S \neq C$, and the second one is $Z \neq 0, S = C$. It is easily seen that for vanishing potential the energy eigenvalue is $m$. Note, for $a = \cos(\frac{\pi}{4})$ and $b = \sin(\frac{\pi}{4})$, $S = 2ab = \sin(\frac{\pi}{4}) = a^2 - b^2 = \cos(\frac{\pi}{4})$. Hence for $\gamma > k^2$, in general, the real part of the wave function shows an oscillatory behavior and for the states that $n = \kappa$, the energy spectrum is imaginary \[28\].

**For the ordinary Coulomb potential** $\gamma = \sqrt{k^2 - Z^2}\alpha^2$ and, therefore, one can conclude from $\gamma$ that for states with $\kappa^2 = 1$ only solution up to $Z \sim 137$ can be constructed. For $(Z\alpha^2) > k^2$, in general, the real part of the wave function shows an oscillatory behavior and for the states that $n = \kappa$, the energy spectrum is imaginary \[28\].
the case which the unitary transformation is $e^{\frac{i\pi}{8} \sigma_x}$ the energy eigenvalue is $m$. Therefore, we assume that the unitary transformation is not identity and also $S \neq C$, then the vanishing of potential is due to $Z = 0$. Then, from (40), it is clearly found that a continuous increase of the coupling strength $Z\alpha(S^2 - C^2)$ from zero, electron states can be pushed up to the positive energy continuum. For $Z\alpha(S^2 - C^2) \ll 1$ the energy formula can be expanded as:

$$E_{r,n} - m \approx \frac{Z^2 \alpha^2 (S^2 - C^2)^2}{2} \left[ \frac{1}{n^2} + \frac{Z^2 \alpha^2}{2n^3} \left( \frac{1}{m} + \frac{3(S^2 - C^2)^2}{4n} \right) \right].$$

(44)

It is seen that for $Z\alpha(S^2 - C^2) = 0$ the modified binding energy, $E_{r,n}^{mb} = E_{r,n} - m$, is

$$E_{r,n}^{mb} = E_{r,n} - m = 0,$$

(45)

where $(mb) \equiv$ (modified binding). Hence, we see that with increasing $Z$ the modified binding energy increases also. Note that (40) is nearly real for all values of $Z$ and $n$. Namely, for the states with $n = j + \frac{1}{2}$, energy value can be calculated as:

$$E_{r,n} = m \left[ \frac{n^2 + Z^2 \alpha^2}{n^2 + (2CSZ\alpha)^2} \right]^{\frac{1}{2}}.$$

(46)

Therefore, this results show that for all unitary transformations which apply to Dirac equation for gauge fixing condition and Schrödinger like requirement, the energy eigenvalues of the states with $n = (j + \frac{1}{2})$ is real for all values of $Z$ (see the second footnote on the page 7).

References

[1] E. Caliceti, S. Graffi, M. Maioli; Com. Math. Phys. 75, 51 (1980)
[2] C. M. Bender, S. Boettcher, P. N. Meisinger; J. Math. Phys. 40, 2201 (1999)
[3] C. M. Bender, K. A. Milton; Phys. Rev. Lett. D55, R3255 (1997)
[4] Kh. Saaidi; "More on Exact PT-Symmetry Quantum Mechanics" quant-ph/0307068.
[5] D. T. Trinh, E. Delabaere; J. Phys. A: Math. Gen. 33, 8771 (2000)
[6] E. Delabaere, F. Pham; Phys. Lett. A 250, 25 (1998)
[7] D. Dorey, C. Dunning, R. Tateo; J. Phys. A 34, 5679 (2001)
[8] G. A. Mezincescu; J. Phys. A: Math. Gen 33, 4911 (2000)
[9] C. M. Bender, Q. Wang; J. Phys. A: Math. Gen. 34, 3325, (2001)
[10] M. Znojil, G. levai; Phys. Lett. A 271, 327 (2000)
[11] M. Znojil; Phys. Lett. A 259, 220 (1999); J. Phys. A: Math. Gen. 33, 4911 (1999)
[12] C. M. Bender, S. Boettecher, & V. M. Savage; J. Math. Phys 41, 6381 (2000)
[13] C. M. Bender, E. J. Weniger; J. Math. Phys. 42, 2167 (2001)
[14] C. M. Bender, G. V. Dunne, P. N. Meisinger & M. Simsek; Phys. Lett. A 281, 311 (2001)
[15] C. M. Bender, D. C. Brody, H. F. Jones; Phys. Rev. Lett. 89, 270401 (2002)
[16] C. M. Bender, D. C. Brody, H. F. Jones; ”Must a Hamiltonian be Hermitian” hep-th/0303005
[17] A. Mostafazadeh; J. Math. Phys. A 43, 205; 2914 (2002)
[18] B. Bagchi, C. Quesne; Phys. Lett A 273, 285 (2000)
[19] Z. Ahmed; Phys. Lett. A 290, 19 (2001)
[20] A. Mostafazadeh; J. Phys. A 36, 7081 (2003).
[21] S. Weigert; ”Completeness and orthogonality in PT-symmetric quantum systems”: quant-ph/0306040
[22] L. C. Biedenharn; Phys. Rev. 126, 845 (1962)
[23] B. Goodman, SR. Ignjatovic; Am. J. Phys. 65, 214 (1999)
[24] O. L. de Lange; Phys. A: Math & Gen. 24, 667 (1991)
[25] V. M. Villalba; Phys. Rev. A 49, 586 (1994)
[26] A. D. Alhaidari; Phys. Rev. A 65 42109; 19902 (2002)
   Int. J. Mod. Phys A 17, 4551 (2002)
   J. Phys. A: Math & Gen. 35, 6207 (2002)
[27] O. Mustafa; J. Phys. A: Math & Gen. 36, 5067 (2003)
[28] , W.Grainer; Relativistic quantum mechanics; (springer-verlag, 1990)