Comments on the Stress-Energy Tensor Operator in Curved Spacetime.

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To my wife Francesca and my daughter Bianca.

Abstract: The technique based on a *-algebra of Wick products of field operators in curved spacetime, in the local covariant version proposed by Hollands and Wald, is straightforwardly generalized in order to define the stress-energy tensor operator in curved globally hyperbolic spacetimes. In particular, the locality and covariance requirement is generalized to Wick products of differentiated quantum fields. Within the proposed formalism, there is room to accomplish all of physical requirements provided that known problems concerning the conservation of the stress-energy tensor are assumed to be related to the interface between quantum and classical formalism. The proposed stress-energy tensor operator turns out to be conserved and reduces to the classical form if field operators are replaced by classical fields satisfying the equation of motion. The definition is based on the existence of convenient counterterms given by certain local Wick products of differentiated fields. These terms are independent from the arbitrary length scale (and any quantum state) and they classically vanish on solutions of Klein-Gordon equation. Considering the averaged stress-energy tensor with respect to Hadamard quantum states, the presented definition turns out to be equivalent to an improved point-splitting renormalization procedure which makes use of the nonambiguous part of the Hadamard parametrix only that is determined by the local geometry and the parameters which appear in the Klein-Gordon operator. In particular, no extra added-by-hand term $g_{ab}Q$ and no arbitrary smooth part of the Hadamard parametrix (generated by some arbitrary smooth term $w_0$) are involved. The averaged stress-energy tensor obtained by the point-splitting procedure also coincides with that found by employing the local $\zeta$-function approach whenever that technique can be implemented.

1 Introduction.

In [1, 2, 3] the issue is addressed concerning the definition of Wick products of field operators (and time-ordered products of field operators) in curved spacetime and remarkable results are found (see Section 3). The general goal is the definition of the perturbative $S$-matrix formalism and
corresponding renormalization techniques for self-interacting quantum fields in curved spacetime. The definition proposed by Hollands and Wald in [3] also assumes some locality and covariance requirements which (together with other properties) almost completely determine local Wick products. Some of the results on Wick polynomials algebra presented in [3] are straightforward generalizations of Minkowski-spacetime results obtained by Dütsch and Fredenhagen in [4]. A more general approach based on locality and covariance is presented in [5]. Using the machinery introduced in [3], a stress-energy tensor operator could be defined, not only its formal averaged value (also see [1] and comments in [2], where another definition of stress-energy operator was proposed in terms of a different definition of Wick products). However, the authors of [3] remark that such a stress-energy operator would not satisfy the conservation requirement.

This paper is devoted to show that, actually, a natural (in the sense that it coincides with the classical definitions whenever operators are replaced by classical fields) definition of a well-behaved stress-energy tensor operator may be given using nothing but local Wick products defined by Hollands and Wald, provided one considers the pointed-out problem as due to the interface between classical and quantum formalism.

The way we follow is related to the attempt to overcome some known drawbacks which arise when one tries to define a natural point-splitting renormalization procedure for the stress-energy tensor averaged with respect to some quantum state. Let us illustrate these well-known drawbacks [6, 7].

Consider a scalar real field $\phi$ propagating in a globally hyperbolic spacetime $(M, g)$. Assume that the field equations for $\phi$ are linear and induced by some Klein-Gordon operator $P = -\Delta + \xi R + m^2$ and let $\omega$ be a quntum state of the quantized field $\hat{\phi}$. A widely studied issue is the definition of techniques which compute averaged (with respect to $\omega$) products of pairs of field operators $\hat{\phi}(z)$ evaluated at the same event $z$. In practice, one is interested in formal objects like $\langle \hat{\phi}(z)\hat{\phi}(z) \rangle_\omega$. The point-splitting procedure consists of replacing classical terms $\phi(z)\phi(z)$ by some argument-coincidence limit of a integral kernel representing a suitable quantum two-point function of $\omega$. A natural choice involves the Hadamard two-point function $G^{(1)}_\omega(x, y)$, which is regular away from light-related arguments for Hadamard states (see 2.2). The cure for ultraviolet divergences which arise performing the argument-coincidence limit consists of subtracting the "singular part" of $G^{(1)}_\omega(x, y)$, (see 2.2), before taking the coincidence limit $(x, y) \to (z, z)$. This is quite a well-posed procedure if the state is Hadamard since, in that case, the singular part of the two-point functions is known by definition and is almost completely determined by the geometry and the K-G operator. The use of such an approach for objects involving derivatives of the fields, as the stress-energy tensor, turns out to be more problematic. The naïve point-splitting procedure consists of the following limit

$$\langle \hat{T}_{\mu\nu}(z) \rangle_\omega = \lim_{(x,y) \to (z,z)} D_{\mu\nu}(x, y)[G^{(1)}_\omega(x, y) - (Z_n(x, y) + W(x, y))]$$

where $Z_n$ is the expansion of the singular part of $G^{(1)}_\omega$ in powers of the squared geodesic distance $s(x, y)$ of $x$ and $y$ truncated at some sufficiently large order $n$, and $D_{\mu\nu}(x, y)$ is a non-local

\footnote{The use of the Hadamard function rather than the (Wightman) two-point functions is a matter of taste, since the final result does not depend on such a choice as a consequence of the bosonic commutation relations.}
differential operator obtained by point-splitting the form of the stress-energy tensor \( \text{(see 3 in 2.1)} \). \( W \) is an added smooth function of \( x, y \).

That procedure turns out to be plagued by several drawbacks whenever \( D = \text{dim}(M) \) is even \((D = 4 \text{ in particular})\). Essentially, (a) the produced averaged stress-energy tensor turns out not to be conserved (in contrast with Wald’s axioms on stress-energy tensor renormalization \( \text{[7]} \)) and (b) it does not take the conformal anomaly into account \( \text{[7]} \) which also arises employing different renormalization approaches \( \text{[3]} \). (c) The choice of the term \( W \) turns out to be quite messy. Indeed, a formal expansion of \( W \) is known in terms of powers of the squared geodesic distance \( \text{[8]} \), but it is completely determined only if the first term \( W_0(x, y) \) of the expansion is given. However, it seems that there is no completely determined natural choice for \( W_0 \) (see discussion and references in \( \text{[8, 7, 9]} \)). It is not possible to drop the term \( W \) if \( D \) is even. Indeed, an arbitrary length scale \( \lambda \) is necessary in the definition of \( Z_n \) and changes of \( \lambda \) give rise to an added term \( W \). As a minor difficulty we notice that (d) important results concerning the issue of the conservation of the obtained stress-energy tensor \( \text{[8]} \) required both the analyticity of the manifold and the metric in order to get convergent expansions for the singular part of \( G^{(1)}_{\omega} \).

The traditional cure for (a) and (b) consists of by hand improving the prescription as

\[
\langle \hat{T}_{\mu\nu}(z) \rangle_\omega = \lim_{(x,y)\to(z,z)} D_{\mu\nu}(x,y)[G^{(1)}_{\omega}(x,y) - (Z_n(x,y) + W(x,y))] + g_{\mu\nu}(z)Q(z),
\]

where \( Q \) is a suitable scalar function of \( z \) determined by imposing the conservation of the final tensor field. A posteriori, \( Q \) seems to be determined by the geometry and \( P \) only.

Coming back to the stress-energy operator, one expects that any conceivable definition should produce results in agreement with the point-splitting renormalization procedure, whenever one takes the averaged value of that operator with respect to any Hadamard state \( \omega \). However, the appearance of the term \( Q \) above could not allow a definition in terms of local Wick products of field operators only.

In Section 2 we prove that it is possible to ”clean up” the point-splitting procedure. In fact, we suggest an improved procedure which, preserving all of the relevant physical results, is not affected by the drawbacks pointed out above. In particular, it does not need added-by-hand terms as \( Q \), employing only mathematical objects completely determined by the local geometry and the operator \( P \). The ambiguously determined term \( W \) (not only the first term \( W_0 \) of its expansion) is dropped, barring the part depending on \( \lambda \) as stressed above. Finally, no analyticity assumptions are made. Our prescription can be said “minimal” in the sense that it uses the local geometry and \( P \) only. The only remaining ambiguity is a length scale \( \lambda \). We also show that the presented prescription produces the same renormalized stress-energy tensor obtained by other definitions based on the Euclidean functional integral approach.

In Section 3 we show that the improved procedure straightforwardly suggests a natural form of the stress-energy tensor operator written in terms of local Wick products of operators which generalize those found in \( \text{\[3, 4\]} \). This operator is conserved, reduces to the usual classical form, whenever operators are replaced by classical fields satisfying the field equation, and agrees with the point-splitting result if one takes the averaged value with respect to any Hadamard state. To define the stress-energy tensor operator as an element of a suitable *-algebra of formal operators
smeared by functions of $\mathcal{D}(M)$, we need to further develop the formalism introduced in [3]. This is done in the third section, where we generalize the notion of local Wick products given in [3] to differentiated local Wick products proving some technical propositions.

Concerning notations and conventions, throughout the paper a spacetime, $(M, g)$, is a connected $D$-dimensional smooth (Hausdorff, second-countable) manifold with $D \geq 2$ and equipped with smooth Lorentzian metric $g$ (we adopt the signature $-, +, \cdots, +$). $\Delta$ denotes the Laplace-Beltrami-D’Alembert operator on $M$, locally given by $\nabla_\mu \nabla^\mu$, $\nabla$ being the Levi-Civita covariant derivative associated with the metric $g$. A spacetime is supposed to be oriented, time oriented and in particular globally hyperbolic (see the Appendix.A and [10]), also if those requirements are not explicitly stated. Throughout $\mu_g$ denotes the natural positive measure induced by the metric on $M$ and given by $\sqrt{-g(x)} \, dx^1 \wedge \cdots \wedge dx^n$ in each coordinate patch. The divergence, $\nabla \cdot T$, of a tensor field $T$ is defined by $(\nabla \cdot T)^{\alpha\cdots\beta} = \nabla_\mu T^{\mu \alpha\cdots\beta}$ in each coordinate patch. Finally, throughout the paper, “smooth” means $C^\infty$.

2 Cleaning up the Point-Splitting Procedure.

2.1. Classical framework. Consider a smooth real scalar classical field $\varphi$ propagating in a smooth $D$-dimensional globally hyperbolic spacetime $(M, g)$. $P\varphi = 0$ is the equation of motion of the field the Klein-Gordon operator $P$ being

$$
P \overset{\text{def}}{=} -\Delta + \xi R(x) + V(x) = -\Delta + m^2 + \xi R(x) + V'(x),$$

where $\xi \in \mathbb{R}$ is a constant, $R$ is the scalar curvature, $m^2 \geq 0$ is the mass of the field and $V' : M \to \mathbb{R}$ is any smooth function. The symmetric stress-energy tensor, obtained by variational derivative with respect to the metric of the action [10], reads

$$
T_{\alpha\beta}(x) = \nabla_\alpha \varphi(x) \nabla_\beta \varphi(x) - \frac{1}{2} g_{\alpha\beta}(x) \left( \nabla_\gamma \varphi(x) \nabla_\gamma \varphi(x) + \varphi^2(x) V(x) \right)
+ \xi \left[ \left( R_{\alpha\beta}(x) - \frac{1}{2} g_{\alpha\beta}(x) R(x) \right) + g_{\alpha\beta}(x) \Delta - \nabla_\alpha \nabla_\beta \right] \varphi^2(x). \quad (3)
$$

Concerning the “conservation relation” of $T_{\alpha\beta}(x)$, if $P\varphi = 0$, a direct computation leads to

$$
\nabla^\alpha T_{\alpha\beta}(x) = -\frac{1}{2} \varphi^2(x) \nabla_\beta V'(x). \quad (4)
$$

It is clear that the right-hand side vanishes provided $V' \equiv 0$ and (4) reduces to the proper conservation relation. The trace of the stress-energy tensor can easily be computed in terms of $\varphi^2(x)$ only. In fact, for $P\varphi = 0$, one finds

$$
g_{\alpha\beta}(x) T^{\alpha\beta}(x) = \left[ \frac{\xi D - \xi}{4 \xi D - 1} \Delta - V(x) \right] \varphi^2(x). \quad (5)
$$

For $\xi = 1/6$, in (four dimensional) Minkowski spacetime and on solutions of the field equations, this tensor coincides with the so called “new improved” stress-energy tensor [11].
where \( \xi_D = (D - 2)/[4(D - 1)] \) defines the \textit{conformal coupling}: For \( \xi = \xi_D \), if \( V \equiv 0 \) and \( m = 0 \), the action of the field \( \varphi \) turns out to be invariant under local conformal transformations \( (g(x) \rightarrow \lambda(x)g(x), \varphi(x) \rightarrow \lambda(x)^{1/2-D/4}\varphi(x)) \) and the trace of \( T_{\alpha\beta}(x) \) vanishes on field solutions by \((\ref{6})\).

2.2. \textit{Hadamard quantum states and Hadamard parametrix}. From now on \( A(M, g) \) denotes the abstract \(*\)-algebra with unit \( 1 \) generated by \( 1 \) and the abstract field operators \( \varphi(f) \) smeared by the functions of \( \mathcal{D}(M) := C^\infty_0(M, \mathbb{C}) \). The abstract field operators enjoy the following properties where \( f, h \in \mathcal{D}(M) \) and \( E \) is the advanced-minus-retarded fundamental solution of \( P \) which exists in globally hyperbolic spacetimes \([12]\).

(a) \textbf{Linearity:} \( f \mapsto \varphi(f) \) is linear,
(b) \textbf{Field equation:} \( \varphi(Pf) = 0 \),
(c) \textbf{CCR:} \( [\varphi(f), \varphi(h)] = E(f \otimes h)1 \), \( E \) being the \textit{advanced-minus-retarded} bi-solution \([12]\),
(d) \textbf{Hermiticity:} \( \varphi(\overline{f}) = \varphi(f)^* \).

An algebraic quantum state \( \omega : A(M, g) \rightarrow \mathbb{C} \) on \( A(M, g) \) is a linear functional which is normalized \( (\omega(1) = 1) \) and positive \( (\omega(a^*a) \geq 0 \) for every \( a \in A(M, g)) \). The GNS theorem \([13]\) states that there is a triple \( (\mathcal{H}_\omega, \Pi_\omega, \Omega_\omega) \) associated with \( \omega \). \( \mathcal{H}_\omega \) is a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle_\omega \). \( \Pi_\omega \) is a \(*\)-algebra representation of \( A(M, g) \) which takes values in a \(*\)-algebra of unbounded operators defined on the dense invariant linear subspace \( \mathcal{D}_\omega \subset \mathcal{H}_\omega \). The distinguished vector \( \Omega_\omega \in \mathcal{H}_\omega \) satisfies both \( \Pi_\omega(A(M, g))\Omega_\omega = \mathcal{D}_\omega \) and \( \omega(a) = \langle \Omega_\omega, \Pi_\omega(a)\Omega_\omega \rangle_\omega \) for every \( a \in A(M, g) \). Different GNS triple associated to the same state are unitarily equivalent. From now on, \( \hat{\varphi}_\omega(f) \) denotes the closeable field operator \( \Pi_\omega(\varphi) \) and \( A_\omega(M, g) \) denotes the \(*\)-algebra \( \Pi_\omega(A(M, g)) \). Wherever it does not produce misunderstandings we write \( \hat{\varphi} \) instead of \( \hat{\varphi}_\omega \) and \( \langle \cdot, \cdot \rangle_\omega \) instead of \( \langle \cdot, \cdot \rangle_\omega \).

The \textit{Hadamard} two-point function of \( \omega \) is defined by
\[
G^{(1)}_\omega \overset{\text{def}}{=} \text{Re } G^{(+)}_\omega ,
\]
\( G^{(+)}_\omega \) being the \textit{two-point} function of \( \omega \), i.e., the linear map on \( \mathcal{D}(M) \times \mathcal{D}(M) \)
\[
G^{(+)}_\omega : f \otimes g \rightarrow \omega(\varphi(f)\varphi(g)) = \langle \Omega_\omega, \hat{\varphi}(f)\hat{\varphi}(g)\Omega_\omega \rangle .
\]

We also assume that \( \omega \) is \textit{globally Hadamard} \([12, 4, 14]\), i.e., it satisfies the

\textbf{Hadamard requirement:} \( G^{(+)}_\omega \in \mathcal{D}'(M \times M) \) and takes the singularity structure of \( \text{(global)} \)
\textit{Hadamard form} in a causal normal neighborhood \( N \) of a Cauchy surface \( \Sigma \) of \( M \).

In other words, for \( n = 0, 1, 2, \ldots \), the distributions \( G^{(+)}_\omega - \chi Z^{(+)}_n \in \mathcal{D}'(M \times M) \), can be represented by functions of \( C^n(N \times N) \). \( Z^{(+)}_n \) is the \textit{Hadamard parametrix} truncated at the order \( n \) and defined on test functions supported in \( C_z \times C_z \) for every \( z \in M \), \( C_z \) being a convex normal neighborhood of \( z \) (see the Appendix A and \([12, 14]\) for the definition of \( \chi \) and \( N \)). Since we are interested in the local behavior of the distributions we ignore the smoothing function \( \chi \) in the following because \( \chi(x, y) = 1 \) if \( x \) is sufficiently close to \( y \). The \textit{propagation} of the global

\(^3\)The involution being the adjoint conjugation on \( \mathcal{H}_\omega \) followed by the restriction to \( \mathcal{D}_\omega \).
Hadamard structure in the whole spacetime [15] (see also [12, 6, 18]) entails the independence of the definition of Hadamard state from $\Sigma$, $N$ and $\chi$. It also implies that $G^{(1)}_\omega$ (as well as $G^{(+)}_\omega$) is a smooth function, $(x, y) \mapsto G^{(1)}_\omega(x, y)$ away from the subset of $M \times M$ made of the pairs of points $x, y$ such that either $x = y$ or they are light-like related. If $C_z$ is a convex normal neighborhood of $z$, using Hadamard condition and the content of the Appendix A, one proves that $\text{Re}(Z^{(+)})$ is represented by a smooth kernel $Z(x, y)$ if $s(x, y) \neq 0$ and the map $(x, y) \mapsto G^{(1)}_\omega(x, y) - Z_n(x, y)$ can be continuously extended into a function of $C^n(C_z \times C_z)$. For $s(x, y) \neq 0$,

$$Z_n(x, y) = \beta_D^{(1)} \frac{U(x, y)}{s^{D/2-1}(x, y)} + \beta_D^{(2)} V^{(n)}(x, y) \ln \frac{|s(x, y)|}{\lambda^2} \quad \text{if } D \text{ is even},$$

$$Z_n(x, y) = \beta_D^{(1)} \theta(s(x, y)) \frac{T^{(n)}(x, y)}{s^{D/2-1}(x, y)} \quad \text{if } D \text{ is odd}.$$  

$\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ otherwise. The smooth real-valued functions $U, V^{(n)}, T^{(n)}$ are defined by recursive (generally divergent) expansions in powers of the (signed) squared geodesical distance $s(x, y)$ and are completely determined by the metric and the operator $P$. $\beta_D^{(i)}$ are numerical coefficients. $\lambda > 0$ is an arbitrarily fixed length scale. Details are supplied in the Appendix A.

### 2.3. Classical ambiguities and their relevance on quantum ground

Let us consider the point-splitting procedure introduced in 1.1 by (1). The differential operator $D_{\mu\nu}(x, y)$ (written in (10) below putting $\eta = 0$ therein) is obtained by point-splitting the classical expression for the stress-energy tensor (3) [6, 7]. The crucial point is that the classical stress-energy tensor may be replaced by a classically equivalent object which, at the quantum level, breaks such an equivalence. In particular, classically, we may re-define

$$T_{\mu\nu}(\eta)(z) \overset{\text{def}}{=} T_{\mu\nu}(z) + \eta g_{\mu\nu}(z) \varphi(z) P\varphi(z),$$

where $\eta \in \mathbb{R}$ is an arbitrarily fixed pure number and $T_{\mu\nu}(z)$ is given by (3). It is obvious that $T_{\mu\nu}(\eta)(z) = T_{\mu\nu}(z)$ whenever $\varphi$ satisfies the field equation $P\varphi = 0$. Therefore, there is no difference between the two tensors classically speaking and no ambiguity actually takes place through that way. On quantum ground things dramatically change since $\langle \hat{\varphi}(x) P\hat{\varphi}(x) \rangle_\omega \neq 0$, provided the left-hand side is defined via point-splitting procedure (see also [3] where the same remark appears in terms of local Wick polynomials). Therefore the harmless classical ambiguity becomes a true quantum ambiguity. Actually, we argue that, without affecting the classical stress-energy tensor, the found ambiguity can be used to clean up the point-splitting procedure. By this way, the general principle ”relevant quantum objects must reduce to corresponding well-known classical objects in the formal classical limit, i.e., when quantum observables are replaced by classical observables”, is preserved.

The operator used in the point-splitting procedure corresponding to $T_{\mu\nu}(\eta)$ is obtained by means
of a point-separation and symmetrization of the right-hand side of (3) and (9). It reads

\[ D^{(\eta)}_{(z)\alpha\beta}(x, y) \equiv \frac{1}{2} \left( \delta^{\alpha'}_{\alpha}(z, x)\delta^{\beta'}_{\beta}(z, y)\nabla_{(z)}\alpha\nabla_{(y)}\beta' + \delta^{\alpha'}_{\alpha}(z, y)\delta^{\beta'}_{\beta}(z, x)\nabla_{(y)}\alpha'\nabla_{(x)}\beta' \right) \]

\[ - \frac{1}{2} g_{\alpha\beta}(z) \left( g^{\gamma\gamma'}(z)\delta(z, x)\mu\delta(z, y)\nu\nabla_{(x)}\mu\nabla_{(y)}\nu + V(z) \right) \]

\[ + \xi \left[ \left( R_{\alpha\beta}(z) - \frac{1}{2} g_{\alpha\beta}(z)R(z) \right) + \frac{g_{\alpha\beta}(z)}{2} (\Delta x + \Delta y) \right] \]

\[ - \frac{1}{2} \left( \delta^{\alpha'}_{\alpha}(z, x)\delta^{\beta'}_{\beta}(z, x)\nabla_{(x)}\alpha\nabla_{(x)}\beta' + \delta^{\alpha'}_{\alpha}(z, y)\delta^{\beta'}_{\beta}(z, y)\nabla_{(y)}\alpha'\nabla_{(y)}\beta' \right) \]

\[ + \eta \frac{g_{\alpha\beta}(z)}{2} (P_x + P_y) \],

(10)

\[ \delta(v, u) \] is the operator of the geodesic transport from \( T_u M \) to \( T_v M \). We aim to show that there is a choice for \( \eta, \eta_D \), depending on the dimension of the spacetime manifold \( D \) only, such that

\[ \langle \tilde{\Gamma}_{\mu\nu}^{(UD)}(z) \rangle_{\omega} \equiv \lim_{(x, y) \to (z, z)} D^{(\eta D)}_{(z)\mu\nu}(x, y)[G^{(1)}_{\omega}(x, y) - Z_n(x, y)] \],

(11)

is physically well behaved. To this end a preliminary lemma is necessary.

2.4. A crucial lemma. The following lemma plays a central rôle in the proof of Theorem 2.1 concerning the properties of the new point-splitting prescription. The coefficients of the expansion of \( U \) in (9), \( U_k(z, z) \), which appear below and (a|b) are defined as in the Appendix A.

Lemma 2.1. In a smooth \( D \)-dimensional \( (D \geq 2) \) spacetime \((M, g)\) equipped with the differential operator \( P \) in (9), the associated Hadamard parametrix (8), (8) satisfies the following identities, where the limits hold uniformly.

(a) If \( n \geq 1 \)

\[ \lim_{(x, y) \to (z, z)} P_x Z_n(x, y) = \lim_{(x, y) \to (z, z)} P_y Z_n(x, y) = \delta_D c_D U_{D/2}(z, z) . \]

(12)

Above \( \delta_D = 0 \) if \( D \) is odd and \( \delta_D = 1 \) if \( D \) is even and

\[ c_D \equiv (-1)^{D/2+1} \frac{(2D^2 + 1)(D + 2)}{2^{D-1} \pi^{D/2} \Gamma(D/2)} , \]

(13)

(b) If \( D \) is even and \( n \geq 1 \) or \( D \) is odd and \( n > 1 \),

\[ \lim_{(x, y) \to (z, z)} P_x \nabla_{(y)}^\mu Z_n(x, y) = \lim_{(x, y) \to (z, z)} \nabla_{(x)}^\mu P_y Z_n(x, y) = \delta_D k_D \nabla_{(x)}^\mu U_{D/2}(z, z) \]

(14)

with

\[ k_D \equiv (-1)^{D/2+1} \frac{(2D^2 + 1)D}{2^D \pi D/2 \Gamma(D/2)} . \]

(15)
(c) Using the point-splitting prescription to compute $\langle \hat{\phi}(z)P\hat{\phi}(z)\rangle_\omega$ and $\langle P(\hat{\phi}(z))\hat{\phi}(z)\rangle_\omega$,

\[ \langle \hat{\phi}(z)P\hat{\phi}(z)\rangle_\omega \overset{\text{def}}{=} \lim_{(x,y)\to(z,z)} P_\omega \left[ G_\omega(x,y) - Z_n(x,y) \right] = -\delta_D c_D U_{D/2}(z,z), \]

(16)

\[ \langle P(\hat{\phi}(z))\hat{\phi}(z)\rangle_\omega \overset{\text{def}}{=} \lim_{(x,y)\to(z,z)} P_\omega \left[ G_\omega(x,y) - Z_n(x,y) \right] = -\delta_D c_D U_{D/2}(z,z). \]

(17)

In particular $\langle \hat{\phi}(z)P\hat{\phi}(z)\rangle_\omega = \langle P(\hat{\phi}(z))\hat{\phi}(z)\rangle_\omega$.

Proof. See the Appendix B.

Remark. With our conventions, when $D$ is even, the anomalous quantum correction to the trace of the stress-energy tensor is $-2c_D U_{D/2}(z,z)/(D + 2)$ (and coincides with the conformal anomaly if $V \equiv 0$, $\xi = \xi_D$ in (2)). Notice that the coefficients $U_k(z, z)$ do not depend on either $\omega$ and the scale $\lambda$ used in the definition of $Z_n$. We conclude that $\langle \hat{\phi}(z)P\hat{\phi}(z)\rangle_\omega$ (i) does not depend on the scale $\lambda$, (ii) does not depend on $\omega$ and (iii) is proportional to the anomalous quantum correction to the trace of the stress-energy tensor.

2.5 The improved point-splitting procedure. Let us show that the point-splitting procedure (11) produces a renormalized stress-energy tensor which is well behaved and in agreement with Wald’s four axioms (straightforwardly generalized to the case $V' \neq 0$ when necessary) for a particular value of $\eta$ uniquely determined.

Theorem 2.1. Let $\omega$ be a Hadamard quantum state of a field $\varphi$ on a smooth globally-hyperbolic $D$-dimensional ($D \geq 2$) spacetime $(M, g)$ with field operator (2). If $D^{(n)}_{\mu\nu}(x, y)$ is given by (10), consider the symmetric tensor field and the scalar field locally defined by

\[ z \mapsto \langle \hat{T}^{(n)}_{\mu\nu}(z)\rangle_\omega \overset{\text{def}}{=} \lim_{(x,y)\to(z,z)} D^{(n)}_{\mu\nu}(x, y)[G_\omega(x,y) - Z_n(x,y)], \]

(18)

\[ z \mapsto \langle \hat{\varphi}^2(z)\rangle_{\omega, \lambda} \overset{\text{def}}{=} \lim_{(x,y)\to(z,z)} [G_\omega(x,y) - Z_n(x,y)], \]

(19)

where, respectively, $n \geq 3$ and $n > 0$. The following statements hold.

(a) Both $z \mapsto \langle \hat{T}^{(n)}_{\mu\nu}(z)\rangle_\omega$ and $z \mapsto \langle \hat{\varphi}^2(z)\rangle_{\omega, \lambda}$ are smooth and do not depend on $n$. Moreover, if (and only if) $\eta = \eta_D \overset{\text{def}}{=} D[2(D + 2)]^{-1}$, they satisfy the analogue of (11) for all spacetimes

\[ \nabla^\mu (\hat{T}^{(\eta_D)}_{\mu\nu}(z))_\omega = -\frac{1}{2} \langle \hat{\varphi}^2(z)\rangle_\omega \nabla_\nu V'(z). \]

(20)

(b) Concerning the trace of $\langle \hat{T}^{(\eta_D)}_{\mu\nu}(z)\rangle_\omega$, it holds

\[ g^{\mu\nu}(z) \langle \hat{T}^{(\eta_D)}_{\mu\nu}(z)\rangle_\omega = \left[ \frac{\xi_D - \xi}{4\xi_D - 1} \Delta - V(x) \right] \langle \hat{\varphi}^2(z)\rangle_\omega - \delta_D \frac{2c_D}{D + 2} U_{D/2}(z, z), \]

(21)

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The term on the last line, does not depend on the scale $\lambda > 0$ used to define $Z_n$ and coincides with the conformal anomaly for $\xi = \xi_D$, $V \equiv 0$.

(c) If $D$ is even, $\eta \in \mathbb{R}$, $Q_{\eta,\eta_D}(z) \overset{\text{def}}{=} \delta_D(\eta - \eta_D)c_D U_{D/2}(z, z)$, it holds

$$\langle T_{\mu\nu}^{(\eta_D)}(z) \rangle_{\omega} = \langle T_{\mu\nu}^{(\eta)}(z) \rangle_{\omega} + g_{\mu\nu}(z)Q_{\eta,\eta_D}(z),$$

(22)

(d) Changing the scale $\lambda \rightarrow \lambda' > 0$ one has, with obvious notation,

$$\langle T_{\mu\nu}^{(\eta_D)}(z) \rangle_{\omega,\lambda} - \langle T_{\mu\nu}^{(\eta_D)}(z) \rangle_{\omega,\lambda'} = \delta_D \ln \left( \frac{\lambda'}{\lambda} \right) t_{\mu\nu}(z)$$

(23)

where the smooth symmetric tensor field $t$ is independent from either the quantum state, $\lambda$ and $\lambda'$, is conserved for $V' \equiv 0$ and it is built up, via standard tensor calculus, by employing the metric and the curvature tensors at $z, m, \xi, V'(z)$ and their covariant derivatives at $z$.

(e) If $(M, g)$ is the $(D = 4)$ Minkowski spacetime, $V' \equiv 0$ and $\omega$ is the Minkowski vacuum, there is $\lambda > 0$ such that $\langle T_{\mu\nu}^{(\eta_D)}(z) \rangle_{\omega,\lambda} = 0$ for all $z \in M$. If $m = 0$ this holds for every $\lambda > 0$.

Proof. See the Appendix B.

Def.2.1 (Quantum averaged stress-energy tensor and field fluctuation). Let $\omega$ be a Hadamard quantum state of a field $\varphi$ in a smooth globally-hyperbolic $D$-dimensional ($D \geq 2$) spacetime $(M, g)$ with field operator (2). Referring to Theorem 2.1, the tensor field defined in local coordinates by $z \mapsto \langle T_{\mu\nu}(z) \rangle_{\omega} \overset{\text{def}}{=} \langle \hat{T}_{\mu\nu}^{(\eta_D)}(z) \rangle_{\omega}$ and the scalar field $z \mapsto \langle \hat{\varphi}^2(z) \rangle_{\omega}$, are respectively said the quantum averaged stress-energy tensor in the state $\omega$ and the quantum field fluctuation of the state $\omega$.

Remarks. (1) The point-splitting renormalization defined above turns out to be in agreement with four Wald’s axioms. This can be realized by following the same discussion, developed in [7] concerning the standard point-splitting prescription and using the theorem above.

(2) The need of adding a term to the classical stress-energy tensor to fulfill the conservation requirement can be heuristically explained as follows. As in [7], let us assume that there is some functional of the metric corresponding to the one-loop effective action:

$$S_{\omega}[g] \overset{\text{def}}{=} \ln \int Dg \varphi e^{-iS[\varphi, g]},$$

where $S$ is the classical action associated with $P$, and $\omega$ enters the assignment of the integration domain. In this context, the averaged stress-energy tensor is defined as

$$\langle \hat{T}_{\mu\nu}(z) \rangle_{\omega} = -\frac{2}{\sqrt{-g(z)}} \frac{\delta S_{\omega}[g]}{\delta g_{\mu\nu}(z)},$$

where the functional derivative is evaluated at the actual metric of the spacetime. The conservation of the left-hand side is equivalent to the (first order) invariance under diffeomorphisms of
$S_\omega[g]$. The relevant point is that the measure $\mathcal{D}_g\varphi$ in general must be supposed to depend on the metric $[^{17}\,^{9},^{18}]$. Changing $g$ into $g'$ by a diffeomorphism, one gets, assuming the invariance of $S_\omega[g]$ and making explicit the dependence of the measure on the metric

$$0 = \int \mathcal{D}_g\varphi \left[ \nabla^\mu \frac{2}{\sqrt{-g(z)}} \frac{\delta J[\varphi, g, g']}{\delta g^{\mu\nu}(z)} |_{g'=g} - i\nabla^\mu T_{\mu\nu}(z) \right] e^{-iS[\varphi, g]}$$

where $J[\varphi, g, g']\mathcal{D}_g\varphi = \mathcal{D}_g\varphi \varphi$, $J[\varphi, g, g] = 1$. The conserved quantity is a term corresponding to the classical stress-energy tensor added to a further term depending on the functional measure

$$\nabla^\mu \left[ \langle \hat{\mathcal{J}}_{\mu\nu}(\eta=0)(z) \rangle_\mu + i\frac{2e^{iS_\omega[g]}}{\sqrt{-g(z)}} \int \mathcal{D}_g\varphi \frac{\delta J[\varphi, g, g']}{\delta g^{\mu\nu}(z)} |_{g'=g} e^{-iS[\varphi, g]} \right] = 0.$$

Therefore the found term $\eta DG_{\mu\nu}(z)<\varphi(z)P\varphi(z)\rangle_\omega$ added to the classical stress-energy tensor should be related to the second term in the brackets above.

(3) The functional approach can be implemented via Wick rotation in the case of a static spacetime with compact Cauchy surfaces for finite temperature $(1/\beta)$ states and provided $V$ does not depend on the global Killing time. Within that context, the Euclidean section turns out to be compact without boundary and $G^{(1)}_\beta$ has to be replaced with the unique Green function $G_\beta$, with Euclidean Killing temporal period $\beta$, of the operator obtained by Wick rotation of $P$.

One expects that the following identity holds

$$-\frac{2}{\sqrt{-g(z)}} \frac{\delta S_{E,\lambda}(g)}{\delta g^{\mu\nu}(z)} = \lim_{(x,y)\to(z,z)} \frac{\partial}{\partial \theta(x,y)} \langle T_{\mu\nu}(z) \rangle_{(z)ab}(x,y) [G_\beta(x,y) - Z_n(x,y)]$$

where we have replaced Lorentzian objects by corresponding Euclidean ones and $a,b$ denote tensor indices in a Euclidean manifold. In a sense, (24) can actually be rigorously proven as stated in the theorem below. Indeed, the point-splitting procedure in the right hand side can be implemented also in the Euclidean case because the parametrices $Z_{\lambda,n}$ ($\lambda$ being the length scale used in the definition of the parametrices) can be defined also for Euclidean metrics using the same definition given above, omitting $\theta(s(x,y))$ in (8) and dropping $\mid$ in the logarithm in (7). On the other hand, the left-hand side of (24) may be interpreted, not depending on the right-hand side, as an Euclidean $\zeta$-function regularized stress-energy tensor $\langle \mathcal{T}_{\mu\nu}(z) \rangle_{(z)\beta,\mu,\nu}$, which naturally introduces an arbitrary mass scale $\mu$ (see (20) where $\sigma(x,y)$ indicates $s(x,y)/2$). We remind the reader that, in the same hypotheses, it is possible to define a $\zeta$-function regularization of the field fluctuation, $\langle \hat{\varphi}^2(z) \rangle_{(z)\beta,\mu,\nu}$ (see (20) and references therein).

**Theorem 2.2.** Let $(M, g)$ be a smooth spacetime endowed with a global Killing time-like vector field normal to a compact Cauchy surface and a Klein-Gordon operator $P$ in (2), where $V'$ does not depend on the Killing time. Consider a compact Euclidean section of the spacetime $(M_\beta, g_E)$ obtained by (a) a Wick analytic continuation with respect to the Killing time and (b) an identification of the Euclidean time into Killing orbits of period $\beta > 0$. Let $G_\beta$ be
the unique (except for null-modes ambiguities) Green function of the Euclidean Klein-Gordon operator defined on $C^\infty(M_\beta)$ obtained by analytic continuation of $P$. It holds

$$
\langle T_{ab}(z) \rangle^{(K)} \beta,\mu^2 = \langle T_{ab}(z) \rangle_{\beta,\lambda}
$$  \hspace{1cm} (25)

$$
\langle \varphi^2(z) \rangle^{(K)} \beta,\mu^2 = \langle \varphi^2(z) \rangle_{\omega,\lambda}
$$  \hspace{1cm} (26)

where $\lambda = c\mu^{-2}$, $c > 0$ being some constant and the right-hand sides, and the right-hand sides of the (25) and (26) are defined as in Def.2.1 using $G_\beta$ in place of $G^{(1)}_{\omega}$ and the Euclidean parametrix.

Sketch of proof. The left-hand side of (25) coincides with

$$
\langle \hat{T}_{ab}^{(D)}(z) \rangle_{\omega,\lambda} + g_{ab}(z)Q_{\nu_D,\eta_D}(z)
$$

where $\nu_D = (D - 2)/(2D)$, as shown in Theorem 4.1 of [20] provided (using $\hbar = c = 1$) $\lambda$ coincides with $\mu^{-1}$ with a suitable positive constant factor. (The smooth term $W$ added to the parametrix which appears in the cited theorem can be completely re-absorbed in the logarithmic part of the parametrix as one can directly show). (22) holds true also in the Euclidean case as one can trivially show and thus the thesis is proven. The proof of (26) is similar. $\square$

3 The stress-energy operator in terms of local Wick products.

[1, 2, 3] contain very significant progress in the definition of perturbative quantum field theory in curved spacetime. Those works take advantage from the methods of microlocal analysis [21] and the wave front set characterization of the Hadamard requirement found out by Radzikowski [14]. In [1] it is proven that, in the Fock space generated by a quasifree Hadamard state, a definition of Wick polynomials (products of field operators evaluated at the same event) can be given with a well-defined meaning of operator-valued distributions. That is obtained by the introduction of a normal ordering prescription with respect to a chosen Hadamard state. In the subsequent paper [2], it is shown that quantum field theory in curved spacetime gives rise to “ultraviolet divergences” which are of the same nature as in Minkowski spacetime. This result is achieved by a suitable generalization of the Epstein-Glaser method of renormalization in Minkowski spacetime used to analyze time ordered products of Wick polynomial, involved in the perturbative construction of interacting quantum field theory. However the performed analysis shows that quantities which appear at each perturbation order in Minkowski spacetime as renormalized coupling constant are replaced, in curved spacetime, by functions whose dependence upon the spacetime points can be arbitrary. In [3] generalizing the content of [1] and using ideas of [1, 2], it is found that such ambiguity can be reduced to finitely many degrees of freedom by imposing a suitable requirement of covariance and locality (which is an appropriate replacement of the condition of Poincaré invariance in Minkowski spacetime). The key-step is a precise notion of local, covariant quantum field. In fact, a definition of local Wick products of
field operators in agreement with the given definition of local covariant quantum field is stated. Imposing further constrains concerning scaling behavior, appropriate continuity properties and commutation relations, two uniqueness theorem are presented about local Wick polynomials and their time-ordered products. The only remaining ambiguity consists of a finite number of parameters. Hollands and Wald also sketch a proof of existence of local Wick products of field operators in [3]. The found local Wick products make use of the Hadamard parametrix only and turn out to be independent from any preferred Hadamard vacuum state. In principle, by means of a straightforward definition to local Wick products of differentiated field, these local Wick products may be used to define a well-behaved notion of stress-energy tensor operator. However, as remarked in [3] such a definition would produce a non conserved stress-energy tensor. In this section, after a short review of the relevant machinery developed in [3], we prove how such a problem can be overcome generalizing ideas of Section 2.

3.1. Normal products and the algebra $W(M, g)$. From now on, referring to a globally hyperbolic spacetime $(M, g)$ equipped with a Klein-Gordon operator (2), we assume dim$(M) = 4$ and $V' \equiv 0$ in [2]. In the following, for $n = 1, 2, \ldots$, $D(M^n)$ denotes the space of smooth compactly-supported complex functions on $M^n$ and $D_n(M) \subset D(M^n)$ indicates the subspace containing the functions which are symmetric under interchange of every pair of arguments.

In the remaining part of the work we make use of some mathematical tools defined in microlocal analysis. (See chapter VIII of [21] concerning the notion of wave front set and [14] concerning the microlocal analysis characterization of the Hadamard requirement.) Preserving the usual seminorm-induced topology on $D(M)$, all definitions and theorems about distributions $u \in D'(M)$ (chapter VI of [21]) can straightforwardly be re-stated for vector-valued distributions and in turn, partially, for operator-valued distributions on $D(M)$. That is, respectively, continuous linear maps $v : D(M) \to \mathcal{H}$, $\mathcal{H}$ being a Hilbert space, and continuous linear maps $A : D(M) \to \mathcal{A}$, $\mathcal{A}$ being a space of operators on $\mathcal{H}$ (with common domain) endowed with the strong Hilbert-space topology. The content of Chapter VIII of [21] may straightforwardly be generalized to vector-valued distributions.

In this part we consider quasifree states $\omega$. Referring to 2.2, this means that the $n$-point functions are obtained by functionally differentiating with respect to $f$ the formal identity

$$\omega(e^{i\varphi(f)}) = e^{-\frac{1}{2}\omega(\varphi(f)\varphi(f))}.$$ 

In that case there is a GNS Hilbert space $\mathcal{H}_\omega$ which is a bosonic Fock space, $\Omega_\omega \in \mathcal{H}_\omega$ is the vacuum vector therein, operators $\hat{\varphi}(f)$ are essentially self-adjoint on $D_\omega$ if $f \in D(M)$ is real and Weyl’s relations are fulfilled by the one-parameter groups generated by operators $\hat{\varphi}(f)$.

Let us introduce normal Wick products defined with respect to a reference quasifree Hadamard state $\omega$, [2, 3]. Fix a GNS triple for $\omega$, $(\mathcal{H}_\omega, \Pi_\omega, \Omega_\omega)$ and consider the algebra of operators with domain $D_\omega, A_\omega(M, g)$ (see 2.2). From now on, we write $\hat{\varphi}$ instead of $\hat{\varphi}_\omega$ whenever it does not give rise to misunderstandings. For $n \geq 1$, define the symmetric operator-valued linear map, $\hat{W}_{\omega,n} : D_n(M) \rightarrow A_\omega(M, g)$, given by the formal symmetric kernel

$$\hat{W}_{\omega,n}(x_1, \ldots, x_n) \overset{\text{def}}{=} \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n)_\omega.$$
where the result of the formal functional derivative is supposed to be symmetrized, and thus only the symmetric part of \( \omega \), i.e., \( G_{\omega}^{(1)} \), takes place in (27). \( \hat{\varphi}(x) \) is the formal kernel of \( \hat{\varphi}(= \hat{\varphi}_{\omega}) \), \( \omega(x, y) \) is the formal kernel of \( \omega \). Finally define \( \hat{W}_{\omega, 0} \equiv I \) the unit of \( \mathcal{A}_{\omega}(M, g) \).

The operators \( \hat{W}_{\omega,n}(h) \) can be extended (or directly defined) \( \hat{W}_{\omega,n}(h) \) to a dense invariant subspace of \( \mathcal{H}_\omega \), the “microlocal domain of smoothness” \( \mathcal{F}(\mathcal{D}(M)) \), \( D_\omega \supset \mathcal{D}_\omega \), which is contained in the self-adjoint extension of every operator \( \hat{\varphi}(f) \) smeared by real \( f \in \mathcal{D}(M) \).

From now on we assume that every considered operator is defined on \( \mathcal{D}_\omega \).

\( D_\omega \) enjoys two relevant properties. (a) Every map \( h \mapsto \hat{W}_{\omega,n}(h), h \in \mathcal{D}_n(M) \), defines a symmetric operator-valued distribution. (b) Those operator-valued distributions may give rise to operators which can be interpreted as products of field operators evaluated at the same event.

This is because every \( \hat{W}_{\omega,n} \) can be smeared by a suitable class of distributions and, in particular, \( \hat{W}_{\omega,n}(f \delta_n) \) can be interpreted as \( \hat{\varphi}^{\prime n} \omega(f) \) if \( f \in \mathcal{D}(M) \) and \( \delta_n \) is the distribution:

\[
\int_M h(x_1, \ldots, x_n) \delta_n(x_1, \ldots, x_n) \mu_g(x_1) \cdots \mu_g(x_n) = \int_M h(x, x, \ldots, x) \mu_g(x).
\]

Let us summarize the proof of this remarkable result following \( \mathcal{B} \). By Lemma 2.2 in \( \mathcal{B} \), if \( \Psi \in D_\omega \), the wave front set of the vector-valued distributions \( t \mapsto \hat{W}_{\omega,n}(t) \Psi, WF(\hat{W}_{\omega,n}(\cdot) \Psi) \) \( \mathcal{B}[2][1] \), is contained in the set

\[
\mathcal{F}_n(M, g) \equiv \{(x_1, k_1, \ldots, x_n, k_n) \in (T^*M)^n \setminus \{0\} | k_i \in V_{x_i}^-, i = 1, \ldots, n \},
\]

\( V_{x}^{+/-} \) denoting the set of all nonzero time-like and null co-vectors at \( x \) which are future/past directed. Theorem 8.2.10 in \( \mathcal{B}[2][1] \) states that if the wave front sets of two distributions \( u, v \in \mathcal{D}(N), N \) being any manifold, satisfy \( WF(u) + WF(v) \not\subset \{0\} \), then a pointwise product between \( u \) and \( v \), \( u \circ v \) can be unambiguously defined giving rise to a distribution of \( \mathcal{D}^\prime(N) \). The theorem can be straightforwardly generalized to vector-valued distributions. In our case we are allowed to define the product between a distribution \( t \) and a vector-valued distribution \( \hat{W}_{\omega,n}(\cdot) \Psi \) provided \( WF(t) + \mathcal{F}_n(M, g) \not\subset \{0\} \). To this end define

\[
\mathcal{E}_n'(M, g) \equiv \{ t \in \mathcal{D}_n'(M) | supp \ t \ \text{is compact}, WF(t) \subset \mathcal{G}_n(M, g) \}
\]

where

\[
\mathcal{G}_n(M, g) \equiv T^*M \setminus \left( \bigcup_{x \in M} (V_{x}^+) \cup \bigcup_{x \in M} (V_{x}^-) \right).
\]

It holds \( WF(t) + \mathcal{F}_n(M, g) \not\subset \{0\} \) for \( t \in \mathcal{E}_n'(M, g) \). By consequence the product, \( t \circ \hat{W}_{\omega,n} \Psi \), of the distributions \( t \) and \( \hat{W}_{\omega,n}(\cdot) \Psi \) can be defined for every \( \Psi \in D_\omega \) and it is possible to show that\(^4\)

\( \therefore \) Weyl's commutation relations, and thus bosonic commutation relations on \( D_\omega \), are preserved.
\( (t \circ \hat{W}_{\omega,n} \Psi) (f) \in D_\omega \) for every \( f \in D_n(M) \). In turn, varying \( \Psi \in D_\omega \), one straightforwardly gets a well-defined \textit{operator-valued} distribution \( t \circ \hat{W}_{\omega,n} \).

Summarizing: if \( t \in E'_n(M, g) \), \( n \in \mathbb{N} \), it is well-defined an operator-valued symmetric distribution \( D_n(M) \ni f \mapsto (t \circ \hat{W}_{\omega,n}) (f) \), with values defined in the dense invariant domain \( D_\omega \).

To conclude we notice that if \( t \in E'_n(M, g) \), \( \hat{W}_{\omega,n} \) can be \textit{smeared} by \( t \) making use of the following definition. Since, for all \( \Psi \in D_{\omega} \), \( \text{supp} (t \circ \hat{W}_{\omega,n}) \subset \text{supp} (t) \) take \( f \in D_n(M) \) such that \( f(x_1, \ldots , x_n) = 1 \) for \( (x_1, \ldots , x_n) \in \text{supp} t \) and define the operator, with domain \( D_\omega \),

\[
\hat{W}_{\omega,n}(t) \overset{\text{def}}{=} (t \circ \hat{W}_{\omega,n}) (f) .
\]

It is simply proven that the definition does not depend on the used \( f \) and the new smearing operation reduces with the usual one for \( t \in D_\omega \), \( \hat{W}_{\omega,n}(t) \). Finally, since \( f\delta_n \in E'_n(M, g) \) if \( f \in D(M) \), the following operator-valued distribution is well-defined on \( D_\omega \),

\[
f \mapsto : \hat{\varphi}^n(f) : \omega \overset{\text{def}}{=} \hat{W}_{\omega,n}(f\delta_n) ,
\]

\( : \hat{\varphi}^n(f) : \omega \) is called \textit{normal ordered product of} \( n \) \textit{field operators with respect to} \( \omega \). Generalized normal ordered \textit{Wick products of} \( k \) \textit{fields}, \( : \hat{\varphi}^{n_1}(f_1) \cdots \hat{\varphi}^{n_k}(f_k) : \omega \) are similarly defined.

Given a quasifree Hadamard state \( \omega \) and a GNS representation, \( \mathcal{W}_\omega(M, g) \) is the \(*\)-algebra generated by \( I \) and the operators \( \hat{W}_{\omega,n}(t) \) for all \( n \in \mathbb{N} \) and \( t \in E'_n(M, g) \) with involution given by \( \hat{W}_{\omega,n}(t)^* \overset{\text{def}}{=} \hat{W}_{\omega,n}(t)^{\dagger} \mid_{D_\omega} (= \hat{W}_{\omega,n}(\overline{t})) \). \( A_\omega(M, g) \) turns out to be a sub \(*\)-algebra of \( \mathcal{W}_\omega(M, g) \) since one finds that \( \hat{\varphi}_\omega(f) = : \hat{\varphi}(f) : \omega \) for \( f \in D(M) \).

Different GNS triples for the same \( \omega \) give rise to unitary equivalent algebras \( \mathcal{W}_\omega(M, g) \) by GNS’s theorem. However, if \( \omega, \omega' \) are two quasifree Hadamard states, \( \mathcal{W}_\omega(M, g), \mathcal{W}_{\omega'}(M, g) \) are isomorphic (not unitary in general) under a canonical \(*\)-isomorphism \( \alpha_{\omega,\omega'} : \mathcal{W}_\omega(M, g) \rightarrow \mathcal{W}_{\omega'}(M, g) \), as shown in Lemma 2.1 in \cite{3}. These \(*\)-isomorphisms also satisfy, \( \alpha_{\omega',\omega} \circ \alpha_{\omega,\omega'} = \alpha_{\omega',\omega} \) and \( \alpha_{\omega,\omega'}(\hat{\varphi}_\omega(t)) = \hat{\varphi}_{\omega'}(t) \), but in general, for \( n > 1 \), \( \alpha_{\omega,\omega'}(\hat{\varphi}^n(t) \omega) \neq \hat{\varphi}^n(t) \omega \).

One can define an abstract \(*\)-algebra \( \mathcal{W}(M, g) \), isomorphic to each \(*\)-algebra \( \mathcal{W}_\omega(M, g) \) by \(*\)-isomorphisms \( \alpha_{\omega} : \mathcal{W}(M, g) \rightarrow \mathcal{W}_\omega(M, g) \) such that, if \( \omega, \omega' \) are quasifree Hadamard states, \( \alpha_{\omega'} \circ \alpha_{\omega}^{-1} = \alpha_{\omega,\omega'} \). As above \( \mathcal{A}(M, g) \) is \(*\)-isomorphic to a sub \(*\)-algebra of \( \mathcal{W}(M, g) \) and \( \alpha_{\omega}(\hat{\varphi}(t)) = : \hat{\varphi}(t) \omega \). Elements \( \mathcal{W}_{\omega,n}(t) \) and \( : \varphi^n(f) : \omega \) are defined in \( \mathcal{W}(M, g) \) via (23).

\[ \text{3.2. Local Wick products. Following \cite{3}, a quantum field in one variable} \ \Phi \ \text{is an assignment which associates with every globally hyperbolic spacetime} \ (M, g) \ \text{a distribution} \ \Phi[g] \ \text{taking values in the algebra} \ \mathcal{W}(M, g). \ \Phi, \ \text{is said local and covariant \cite{3}} \ \text{if it satisfies the following}\]

\[ \textbf{Locality and Covariance requirement: For any embedding} \ \chi \ \text{from a spacetime} \ (N, g') \ \text{into another spacetime} \ (M, g) \ \text{which is isometric (thus} \ g' = \chi^* g) \ \text{and causally preserving}, \ \text{it holds} \]

\[ \text{\footnote{It can be shown using the continuity of the product with respect to the Hörmander pseudo topology and theorem 6.2.3 of \cite{2} which assures that each distribution is the limit in that pseudo topology of a sequence of smooth functions and the fact that the convergence in the pseudo topology implies the usual convergence in} \mathcal{D}' \text{.}} \]

\[ \text{\footnote{That is} \chi \text{preserves the time orientation and} \ J^+(p) \cap J^-(q) \subset \chi(N) \ \text{if} \ p, q \in \chi(N).} \]
\[ i_\chi(\Phi[g'](f)) = \Phi[g](f \circ \chi^{-1}) \quad \text{for all } f \in \mathcal{D}(N). \]

Above \( i_\chi : \mathcal{W}(N, g') \to \mathcal{W}(M, g) \) is the injective \(*\)-algebra homomorphism such that if \( \omega \) is a quasifree Hadamard state on \((M, g)\) and \( \omega'(x, y) = \omega(\chi(x), \chi(y)) \), we have,

\[ i_\chi(W_{\omega, n}(t)) = W_{\omega, n}(t \circ \chi^{-1}) \quad \text{for all } n \in \mathbb{N}, \ t \in \mathcal{E}'_n(N, g'). \tag{29} \]

where \( \chi^{-1} \) is defined on \( \chi(N) \) and \( (t \circ \chi^{-1})(x_1, \ldots, x_n) = t(\chi^{-1}(x_1), \ldots, \chi^{-1}(x_n)) \).

The generalization to (locally and covariant) quantum field in \( n \)-variables is straightforward.

It is worth stressing that the notion of local covariant field is not trivial. For instance, any assignment of the form \( (M, g) \mapsto \omega(M, g) \) where \( \omega(M, g) \) are quasifree Hadamard states, does not define a local covariant quantum field by the map \( (M, g) \mapsto : \varphi^2 \cdot \omega(M, g) [3] \).

In [3], Hollands and Wald sketched a proof of existence of local and covariant quantum fields in terms of local Wick products of field operators. Let us review the construction of these Wick products also making some technical improvements.

As \( M \) is strongly causal \([24, 10]\), there is a topological base of open sets \( N \) such that each \( N \) is contained in a convex normal neighborhood, each inclusion map \( i : N \to M \) is causally preserving and each \( N \) is globally hyperbolic with respect to the induced metric. We call causal domains these open neighborhoods \( N \).

Let \( N \subset M \) be a causal domain. The main idea to built up local Wick products \([3]\) consists of a suitable use of the Hadamard parametrix which is locally and covariantly defined in the globally hyperbolic spacetime \((N, g \mid N)\) in terms of the metric \([3]\). In fact, it is possible to define a suitable distribution \( H \in \mathcal{D}'(N \times N) \) such that, every distribution \( H - Z_n \mid_{N \times N} \) is a function of \((x, y) \in N \times N\) which is smooth for \( x \neq y \) and with vanishing derivatives for \( x = y \) up to the order \( n \). Then define the elements of \( \mathcal{W}(N, g \mid N) \), \( W_{H,0} \overset{\text{def}}{=} 1 \) and \( W_{H,n} \) given by (27) with \( \omega \) replaced by \( H \) and \( \varphi \) replaced by \( \varphi \in \mathcal{W}(N, g \mid N) \). These distributions enjoy the same smoothness properties of \( W_{\varphi, n} \) for every quasifree Hadamard state \( \omega \) because \( G^{(1)}(\omega) \mid_{N \times N} - H = (G^{(1)}(\omega) - Z_n) \mid_{N \times N} - (H - Z_n) \mid_{N \times N} \) is smooth on \( N \times N \) and all of its derivative (of any order) must vanish at \( x = y \) since \( n \) is arbitrary. In particular every \( W_{H,n} \) can be smeared by distributions of \( \mathcal{E}'_n(N, g \mid N) \). The local Wick products (on \( N \)) found by Hollands and Wald in [3] are the elements of \( \mathcal{W}(N, g \mid N) \) of the form, with \( f \in \mathcal{D}(N) \),

\[ : \varphi^n(f) : H \overset{\text{def}}{=} W_{H,n}(f \delta_n), \]

A few words on the construction of \( H \) are necessary. \( H \) is given as follows

\[ H \overset{\text{def}}{=} \Re(H^{(+)}) \tag{30}. \]

Using definitions and notation as in the Appendix A, the distribution \( H^{(+)} \in \mathcal{D}'(N \times N) \) is defined, in the sense of the \( e \)-prescription, by a re-arrangement of the kernel of \( Z^{(+)}_n \) \([55]\) with \( D = 4 \), i.e.,

\[ \beta_4^{(1)} \frac{U(x, y)}{s_{e,T}(x, y)} + \beta_4^{(2)} V^{(\infty)}(x, y) \ln \frac{s_{e,T}(x, y)}{\lambda^2}. \tag{31} \]
(Similarly to $Z_n^{(+)}$, $H^{(+)}$ does not depend on the choice of the temporal coordinate $T$.) Above

$$V^{(\infty)}(x,y) \overset{\text{def}}{=} \sum_{k=0}^{+\infty} \frac{1}{2^{k-1}k!} \hat{U}_{k+1}(x,y) \psi \left( \frac{s(x,y)}{\alpha_k} \right) s^k(x,y).$$

$\psi : \mathbb{R} \to \mathbb{R}$ is some smooth map with $\psi(x) = 1$ for $|x| < 1/2$ and $\psi(x) = 0$ for $|x| > 1$ and $\alpha_k > 0$ for all $k \in \mathbb{N}$. The series above converges to a smooth function which vanishes with all of its derivatives at $x = y$, provided the reals $\alpha_k$’s tend to zero sufficiently fast (see [23]).

From now on we omit the restriction symbol $|_N$ and $|_{N \times N}$ whenever these are implicit in the context. Our aim to extend the given definitions to the whole manifold $M$ (and not only $N$) and generalize to differentiated field the notion of local Wick products. We have a preliminary proposition.

**Proposition 3.1.** Referring to the given definitions, the sub $*$-algebra of $\mathcal{W}(N, \mathfrak{g})$, $\mathcal{W}_H(N, \mathfrak{g})$, generated by $\mathcal{W}_{H,n}(t)$, $t \in E_n^i(N, \mathfrak{g})$, $n = 0, 1, \ldots$ (a) $\mathcal{W}_H(N, \mathfrak{g})$ coincides with $\mathcal{W}(N, \mathfrak{g})$ it-self and (b) is naturally $*$-isomorphic to the sub $*$-algebra of $\mathcal{W}(M, \mathfrak{g})$ whose elements are smeared by distributions with support in $N^n$. In this sense $\mathcal{W}_{H,n}(t) \in \mathcal{W}(M, \mathfrak{g})$, $n \in \mathbb{N}$.

**Proof.** (a) Fix a Hadamard state $\omega$ in $(N, \mathfrak{g})$ and generate $\mathcal{W}(N, \mathfrak{g})$ by elements $W_{\omega,n}(t)$. Define the $*$-isomorphism $\alpha : \mathcal{W}_H(N, \mathfrak{g}) \to \mathcal{W}(N, \mathfrak{g})$ as in the proof of Lemma 2.1 in [8] with $d \overset{\text{def}}{=} G^{(1)}_\omega - H$. (The reality of $d$ is assured by the fact that $H$ is real.) $\alpha$, in fact, is the identity map in $\mathcal{W}_H(N, \mathfrak{g})$. (b) It is a direct consequence of the existence of the natural injective $*$-homomorphism defined in Lemma 3.1 in [8]. $\square$

In order to define local Wick products of field operators, consider $n$ linear differential operators $K_i$, acting on functions of $\mathcal{D}(M)$, with the form

$$K_i \overset{\text{def}}{=} a_i(0) + \nabla a_i(1) + \nabla^2 a_i(2) + \cdots + \nabla^L a_i(L),$$

where $a_i(0) \in C^\infty(M; \mathbb{C})$ and, for $k > 0$, $a_i(ik)$ is a smooth complex contravariant tensor field of order $k$ defined on $M$. $\nabla^k a : \mathcal{D}(M) \to \mathcal{D}(M)$ is defined, in each local chart, by

$$\nabla^k a(x) = a^{\mu_1\cdots\mu_k}(x) \nabla_{\mu_1}(x) \cdots \nabla_{\mu_k}(x).$$

$t_n[K_1, \ldots, K_n, f] \in \mathcal{D}'(M^n)$ is the compactly supported in $N^n$ distribution with formal kernel

$$t_n[K_1, \ldots, K_n, f](x_1, \ldots, x_n) \overset{\text{def}}{=} t_{K_n(x_n)} t_{K_{n-1}(x_{n-1})} \cdots t_{K_1(x_1)} f(x_1) \delta_n(x_1, \ldots, x_n),$$

where the right-hand side is supposed to be symmetrized in $x_1, \ldots, x_n$. Above $f \in \mathcal{D}(N)$, $\mathcal{D}(N)$ being identified with the subspace of $\mathcal{D}(M)$ containing the functions with support in $N$. The transposed operator $t K_i$ is defined as usual by "covariant" integration by parts with respect to $K_i$ [23]. As a general result, $WF(\partial u) \subset WF(u)$ and $WF(hu) \subset WF(u)$ if $h$ is smooth.
By consequence, for every \( f \in \mathcal{D}(\mathbb{N}) \) and operators \( K_i, t_n[K_1, \ldots, K_n, f] \in \mathcal{E}'_n(M, g) \) because \( \text{WF}(t_n[K_1, \ldots, K_n, f]) \subset \text{WF}(f\delta_n) \subset \{(x_1, k_1; \ldots; x_n, k_n) \in T^*M^n \setminus \{0\} \mid \sum_i k_i = 0\} \) which is a subset of \( G_n(M, g) \). This result enables us to state the following definition.

**Def 3.1 (Local Wick products of (differentiated) fields I).** Let \( N \) be a causal domain in a globally hyperbolic spacetime \( M \) with \( H \) defined in (30). The local Wick product of \( n \) (differentiated fields) generated by \( n \) operators \( K_i \) and \( f \in \mathcal{D}(M) \) with \( \text{supp} \ f \subset \mathbb{N} \) is

\[
:K_1\varphi \cdots K_n\varphi(f)_H \overset{\text{def}}{=} W_{H,n}(t_n[K_1, \ldots, K_n, f]) \in W(M, g).
\]

(34)

The definition can be improved dropping the restriction \( \text{supp} \ f \subset \mathbb{N} \) as follows. A preliminary lemma is necessary.

**Lemma 3.1.** Referring to Def 3.1, the following statements hold.

(a) The local Wick products on a causal domain \( N \subset M \), do not depend on the arbitrary terms \( \psi \) and \( \{\alpha_k\} \) used in the definition of \( H \) (but may depend on the length scale \( \lambda \)).

(b) If \( N' \subset M \) is another causal domain with \( N \cap N' \neq \emptyset \) and \( :K_1\varphi \cdots K_n\varphi(f)_H \) denote a local Wick product of differentiated fields operators defined on \( N' \) using the same length scale \( \lambda \) as in \( N \), then

\[
:K_1\varphi \cdots K_n\varphi(f)_H = :K_1\varphi \cdots K_n\varphi(f)_H'
\]

and for any choice of operators \( K_i \) and \( f \in \mathcal{D}(N \cap N') \).

**Proof.** See the Appendix B.

**Def. 3.2. (Local Wick products of (differentiated) fields II)** Referring to Def. 3.1, consider an open cover \( \{N_i\} \) of \( M \) made of causal domains with distributions \( H_i \) defined with the same scale length \( \lambda \). Take a smooth partition of the unity \( \{\chi_{ij}\} \), with \( \text{supp} \ \chi_{ij} \subset O_{ij} \subset N_i \), \( \{O_{ij}\} \) being a locally finite refinement of \( \{N_i\} \). The local Wick product of \( n \) (differentiated) fields generated by \( n \) operators \( K_i \) and \( f \in \mathcal{D}(M) \) is the element of \( W(M, g) \)

\[
:K_1\varphi \cdots K_n\varphi(f) : \overset{\text{def}}{=} \sum_{i,j} :K_1\varphi \cdots K_n\varphi(\chi_{ij}, f)_H : ,\]

(36)

**Remark.** Only a finite number of non vanishing terms are summed in the right-hand side of (36) as a consequence of the locally finiteness of the cover \( \{O_{ij}\} \) and the compactness of \( \text{supp} \ f \). Moreover, by (a) of Lemma 3.1 and the linearity on \( \mathcal{E}'_n(M, g) \) of the involved distributions, the given definition does not depend on the functions \( \psi \) and constants \( \{\alpha_k\} \) used in the definition of \( H \). By (b) of Lemma 3.1 the definition is independent from the choice of the cover and on the partition of the unity.

The (differentiated) local Wick products enjoy the following properties.
Proposition 3.2. Let \((M, g)\) be a four-dimensional globally hyperbolic spacetime with a Klein-Gordon operator \( \hat{\Delta} \) with \( V' \equiv 0 \). Given \( n > 0 \) operators \( K_i \), the following statements hold.

(a) Given \( a, b \in \mathbb{C} \), \( f, h \in \mathcal{D}(M) \)

\[
\forall \varphi \colon f \mapsto \varphi(f),
\]

\[
:\hat{K}_1 \varphi \cdots \hat{K}_n \varphi(f) :^* = K_1^* \varphi \cdots K_n^* \varphi(f),
\]

\[
:\hat{K}_1 \varphi \cdots \hat{K}_n \varphi(a f + b h) : = \alpha(b) \hat{K}_1 \varphi \cdots \hat{K}_n \varphi(f) : + b \hat{K}_1 \varphi \cdots \hat{K}_n \varphi(h) :.
\]

(b) If \( \omega \) is a quasifree Hadamard state on \( M \), define \( : \hat{K}_1 \varphi \cdots \hat{K}_n \varphi(f) : \in \mathcal{W}_\omega(M, g) \) with \( f \in \mathcal{D}(M) \), by Def 3.2 using the operators \( \hat{\omega}(h) \) of a GNS representation of \( \omega \). It holds

\[
:\hat{K}_1 \varphi \cdots \hat{K}_n \varphi(f) : = \alpha(f) \hat{K}_1 \varphi \cdots \hat{K}_n \varphi(f):
\]

Moreover, varying \( f \in \mathcal{D}(M) \), the left-hand side gives rise to an operator-valued distribution \( f \mapsto : \hat{K}_1 \varphi \cdots \hat{K}_n \varphi(f) : \) defined on the dense invariant subspace \( \mathcal{D}_\omega \).

(c) For \( f \in \mathcal{D}(M) \)

\[
:K_1 \varphi \cdots K_n \varphi(f): \subset :K_1 \varphi \cdots K_n \varphi(f):
\]

(d) If \( \omega, \omega' \) are Hadamard states on \( M \) and \( f \in \mathcal{D}(M) \),

\[
\alpha_{\omega, \omega'}(f) = :K_1 \varphi \cdots K_n \varphi(f) :.
\]

Remark. (d) does not hold for normal products defined w.r.t. any quasifree Hadamard state \( \omega \).

Sketch of proof. (a) is direct consequence of the given definitions, the reality of \( H \) and the linearity of all the involved distributions on \( \mathcal{E}_r(M, g) \). (b) The continuity with respect to the strong Hilbert-space topology is the only non trivial point. It can be shown as follows. Take a sequence of functions \( \{ f_j \} \subset \mathcal{D}(M) \) with \( f_j \to f \) in \( \mathcal{D}(M) \). In particular, this implies that there is a compact \( K \) with \( \text{supp} f_j, \text{supp} f \subset K \) for \( j > j_0 \). By Def.3.2, it is sufficient to prove the continuity when the supports of test functions belong to a common causal domain \( N \subset M \), i.e., \( \alpha_{\omega}(W_{H,n}(t_n[K_1, \ldots, K_n, f])) \Psi \to \alpha_{\omega}(W_{H,n}(t_n[K_1, \ldots, K_n, f])) \Psi \) if \( f_k \to f \) in \( \mathcal{D}(N) \) and \( \Psi \in \mathcal{D}_\omega \). By the definition of \( \mathcal{W}(M, g) \) and the isomorphism \( \alpha_{\omega} \) (see 3.1), it is sufficient to show that \( t_n[K_1, \ldots, K_n, f_j] \to t_n[K_1, \ldots, K_n, f] \) in the closed conic set \( \Gamma_n = \{(x_1, k_1; \ldots; x_n, k_n) \in T^n M^n \setminus \{0\} | \sum_i k_i = 0\} \) which contains the wave front set of all involved distributions, if \( f_j \to f \) in \( \mathcal{D}(N) \). The proof of the required convergence property is quite technical and it is proven in the Appendix B. (c) is a trivial consequence of the fact that \( \alpha_{\omega} \) is a \( * \)-isomorphism and the definition of the involution on \( \mathcal{W}(M, g) \). (d) is a trivial consequence of (42) and the identity \( \alpha_{\omega, \omega'} = \alpha_{\omega'} \circ \alpha_{\omega}^{-1} \).

We can state a generalized locality and covariance requirement. A differentiated quantum field in one variable \( \Phi \) is an assignment which associates with every globally hyperbolic
spacetime \((M, g)\) and every smooth contravariant tensor field on \(M\), \(A\) (with fixed order) a distribution \(\Phi[g, A]\) taking values in the algebra \(\mathcal{W}(M, g)\). \(\Phi\), is said \textbf{local and covariant} if it satisfies the following

**Locality and Covariance requirement for differentiated fields:** For any embedding \(\chi\) from a globally hyperbolic spacetime \((N, g')\) into another globally hyperbolic spacetime \((M, g)\) which is isometric and causally preserving it holds

\[
i_{\chi}(\Phi[g', A'](f)) = \Phi[g, A](f \circ \chi^{-1}),
\]

for all \(f \in \mathcal{D}(N)\) and all smooth vector fields \(A\) on \(M\), \(A'\) denoting \((\chi^{-1})_{*}A|_{\chi(N)}\). The generalization to (locally and covariant) quantum field in \(n\)-variables and depending on several smooth contravariant vector fields is straightforward.

We conclude this part by showing that the introduced differentiated local Wick polynomial are local and covariant.

**Theorem 3.1.** Take \(n \in \{1, 2, \ldots \}\) and, for every \(i \in \{1, \ldots, n\}\), take integers \(L_i = 0, 1, \ldots\). Let \(\Phi\) be the map which associates with every globally hyperbolic spacetime \((M, g)\) and every class smooth contravariant vector field on \(M\), \(\{a_{(ij)}\}_{i=1,\ldots,n,j=0,\ldots,L_i}\), the (abstract) distribution \(f \mapsto K_1\varphi \cdots K_n\varphi(f):\), where \(f \in \mathcal{D}(M)\) and each \(K_i\) being defined in (32) using the fields \(a_{(ij)}\). \(\Phi\) is a locally and covariant differentiated quantum field in one variable.

**Sketch of proof.** By Def.3.2 the proof reduces to check (43) making use of spacetimes \((N, g')\) and \((M, g)\) which are causal domains. In that case, if \(H'\) and \(H\) are the distributions (30) on \(N\) and \(M\) respectively, one finds \(H'(x, y) = H(x, \chi(y))\) (provided the length scale \(\lambda\) is the same in both cases). Representing generators \(W_{\omega, n}\) in terms of generators \(W_{H, n}\) as indicated in the proof of Proposition 3.1, one also gets that the injective\( \ast\)-algebra homomorphism \(i_{\chi} : \mathcal{W}(N, g') \rightarrow \mathcal{W}(M, g)\) (29) satisfies \(i_{\chi}(W_{H', n}(t)) = W_{H, n}(t \circ \chi^{-1})\). Referring to (33) and (12), we adopt the notation, \(t_n[g, a_{(ij)}, f] \stackrel{\text{def}}{=} t_n[K_1, \ldots, K_n, f]\). With the obtained results and using (34), (43) turns out to be equivalent to \(W_{H, n}(t_n[g', (\chi^{-1})_{*}a_{(ij)}, f] \circ \chi^{-1}) = W_{H, n}(t_n[g, a_{(ij)}, f \circ \chi^{-1}])\) for all \(n \in \mathbb{N}\), \(f \in \mathcal{D}(N)\) and all smooth tensor fields \(a_{(ij)}\) on \(M\). That identity holds because \(t_n[g, a_{(ij)}, f \circ \chi^{-1}] = t_n[g', (\chi^{-1})_{*}a_{(ij)}, f] \circ \chi^{-1}\) by the definition of distributions \(t_n[K_1, \ldots, K_n, f]\) (33) and \(g' = \chi^{-1}g\). \(\square\)

3.3. **The stress-energy tensor operator.** From now on, \(K_1\varphi(x) \cdots K_n\varphi(x):\) indicates the formal kernel of the one-variable distribution \(f \mapsto K_1\varphi \cdots K_n\varphi(f):\). Using that notation and interpreting \(h \in C^\infty(M)\) as a multiplicative operator, we also define

\[
h(x)\varphi^n(x) : \stackrel{\text{def}}{=} K_1\varphi(x) \cdots K_n\varphi(x): \quad \text{where } K_1 = h \text{ and } K_i = I \text{ if } i = 2, \ldots, n
\]

\[
h(x)\nabla_X\nabla_Y\varphi^2(x) : \stackrel{\text{def}}{=} 2: h(x)\varphi(x) \nabla_X\nabla_Y\varphi(x): + 2: h(x)\nabla_X\varphi(x) \nabla_Y\varphi(x):
\]

Let \(\{Z_{(a)}\}_{a=0,1,2,3}\) be a set of \textit{tetrad fields}, i.e., four smooth contravariant vector fields defined on \(M\) such that \(g(Z_{(a)}, Z_{(b)})(x) = \eta_{ab}\), where \(\eta_{ab} \stackrel{\text{def}}{=} \eta^{ab} \stackrel{\text{def}}{=} \epsilon_{a\beta}\delta_{ab}\) everywhere (there is no summation...
with respect to \(a\) with \(c_0 = -1\) and \(c_a = 1\) otherwise. Making use of fields \(Z_{(a)}\), we define

\[
:h(x)g(\nabla \varphi, \nabla \varphi)(x): \overset{\text{def}}{=} \sum_{a,b} \eta^{ab} :h(x)\nabla Z_{(a)} \varphi(x) \nabla Z_{(b)} \varphi(x):
\]

\[
:h(x)\varphi(x)\Delta \varphi(x): \overset{\text{def}}{=} \sum_{a,b} \eta^{ab} :h(x)\varphi(x) \nabla Z_{(a)}\nabla Z_{(b)} \varphi(x):
\]

\[= - \sum_{a,b} \eta^{ab} :h(x)\varphi(x) \nabla (\nabla Z_{(a)} Z_{(b)}) \varphi(x): .\]

These definitions do not depend on the choice of the tetrad fields and reduce to the usual ones if the field operators are replaced by classical fields. Finally,

\[
:h(x)\Delta \varphi^2(x): \overset{\text{def}}{=} 2 :h(x)g(\nabla \varphi, \nabla \varphi)(x): + 2 :h(x)\varphi(x)\Delta \varphi(x): .
\]

Theorem 2.1 strongly suggests the following definition.

**Def. 3.3. (The stress-energy tensor operator)** Let \((M, g)\) be a four-dimensional smooth globally-hyperbolic spacetime equipped with a Klein-Gordon operator \(\Box\) with \(V' \equiv 0\). Let \(X,Y\) be a pair of smooth vector fields on \(M\). The **stress-energy tensor operator** with respect to \(X,Y\) and \(f \in \mathcal{D}(M)\), \(T_{X,Y}(f):\), is defined by the formal kernel

\[
:T_{X,Y}(x): \overset{\text{def}}{=} \nabla_X \varphi(x) \nabla_Y \varphi(x): - \frac{1}{2} g_{X,Y}(x)g(\nabla \varphi, \nabla \varphi)(x): - \frac{m^2}{2} g_{X,Y}(x)\varphi^2(x):
\]

\[= - \frac{1}{2} g_{X,Y}(x) R(x) \varphi^2(x) : + \xi \left( R_{X,Y}(x) - \frac{1}{2} g_{X,Y}(x) R(x) \right) \varphi^2(x): + \xi : g_{X,Y}(x) \Delta \varphi^2(x): - \xi : g_{X,Y}(x) \nabla_X \nabla_Y \varphi^2(x): + \frac{1}{3} : g_{X,Y}(x) \varphi(x) P \varphi(x): , \quad \text{(44)}
\]

where \(g_{X,Y}(x) \overset{\text{def}}{=} g(X,Y)(x)\), \(R_{X,Y}(x) \overset{\text{def}}{=} R(X,Y)(x)\), \(R\) being the Ricci tensor and

\[
g_{X,Y}(x) \varphi(x) P \varphi(x): \overset{\text{def}}{=} - : g_{X,Y}(x) \varphi(x) \Delta \varphi(x): + : g_{X,Y}(x) (R(x) + m^2) \varphi^2(x): . \quad \text{(45)}
\]

**Remarks.** (1) We have introduced the, classically vanishing, term \( : g_{X,Y}(x) \varphi(x) P \varphi(x): \). Its presence is crucial to obtain the conservation of the stress-energy tensor operator using the analogous property of the point-splitting renormalized stress-energy tensor as done in the proof of the theorem below.

(2) The given definition depends on the choice of a length scale \(\lambda\) present in the distribution \(H\) used to define the local Wick products of fields operators.

To conclude our analysis we analyze the interplay between the above-introduced stress-energy tensor operator and the point-splitting procedure discussed in the Section 2. Concerning
the issue of the conservation of the stress-energy tensor, we notice in advance that, if \( T \) is a second-order covariant symmetric tensor field,

\[
- \int_M (\nabla \cdot T)_X d\mu_g = \int_M f T_{\nabla \otimes X} d\mu_g + \sum_{a,b} \eta^{ab} \int_M \left\{ T_{Z_{(a)} \cdot X} \nabla \cdot (f Z_{(b)}) + f T_{X, \nabla Z_{(a)} Z_{(b)}} \right\} d\mu_g \tag{46}
\]

for all \( f \in \mathcal{D}(M) \) and all smooth contravariant vector fields \( X \) on \( M \). Above \((\nabla \cdot T)_X = (\nabla \mu T_{\mu \nu} X^\nu) \) and \( T_{\nabla \otimes X} = T_{\mu \nu} \nabla^\mu X^\nu \) in the abstract index notation. Therefore the conservation requirement \( \nabla \cdot T = 0 \) is equivalent to the requirement that the right-hand side of (46) vanishes for all \( f \in \mathcal{D}(M) \) and smooth contravariant vector fields \( X \) on \( M \).

We have a following conclusive theorem where, if \( \nu \) is a quasifree Hadamard state, \( :T_{\nu, \nabla X Y}(f)\): , \( :\varphi^2(f)\): respectively represent \( :T_{X, Y}(f)\): and \( :\varphi^2(f)\): in \( \mathcal{W}_\nu(M, g) \) in the sense of (b) in Proposition 3.2.

**Theorem 3.2.** Let \((M, g)\) be a four-dimensional smooth globally-hyperbolic spacetime equipped with a Klein-Gordon operator \((\square)\) with \( V' = 0 \). Let \( \lambda > 0 \) be the scale length used to define local Wick products of fields operators and \( \{ Z_{(a)} \}_{a=1,...,4} \) a set of tetrad fields. Considering the given definitions, the statements below hold for every \( f \in \mathcal{D}(M) \).

(a) For every \( h \in C^\infty(M), :h(x) \varphi(x) P \varphi(x)\): does not depend on \( \lambda \) and turns out to be a smooth function. In particular if \( U_2(x, x) \) is defined as in the Appendix A,

\[
:h \varphi P \varphi(f) : = \frac{3}{2 \pi^2} \left( \int_M h(x) U_2(x, x) f(x) d\mu_g(x) \right) 1 \tag{47}
\]

(b) Take a quasifree Hadamard state \( \nu \) and let \( \omega \) be any (not necessarily quasifree) Hadamard state represented by \( \Psi_\omega \in D_\nu \subset \mathcal{H}_\nu \) in a GNS representation of \( \nu \). For every pair of contravariant vector fields \( X, Y \), it holds

\[
\langle \Psi_\omega, :T_{\nu, \nabla X Y}(f) : \Psi_\omega \rangle_\nu = \int_M \langle \hat{T}_{XY}(z) \rangle_\omega f(z) d\mu_g(z), \tag{48}
\]

\[
\langle \Psi_\omega, :\varphi^2(f) : \Psi_\omega \rangle_\nu = \int_M \langle \hat{\varphi}^2(z) \rangle_\omega f(z) d\mu_g(z), \tag{49}
\]

\( \langle \hat{T}_{XY}(z) \rangle_\omega = \langle \hat{T}_{\nu, \nabla}(z) \rangle_\omega X^\mu(z) Y^\nu(z) \) and \( \langle \hat{\varphi}^2(z) \rangle_\omega \) denoting the fields obtained by the point-splitting procedure Def.2.1.

(c) The stress-energy tensor operator is conserved, i.e., for every contravariant vector field \( X \) on \( M \) it holds

\[
:(\nabla \cdot T)_X (f) : = 0 , \tag{50}
\]

where, following (44),

\[
:(\nabla \cdot T)_X (f) : \overset{\text{def}}{=} - :T_{\nabla \otimes X}(f) : - \sum_{a,b} \eta^{ab} \left\{ :T_{Z_{(a)} \cdot X} (\nabla \cdot (f Z_{(b)})) : + :T_{X, \nabla Z_{(a)} Z_{(b)}} (f) : \right\} . \tag{51}
\]
The trace of the stress-energy tensor operator satisfies

$$
\sum_{a,b} \eta^{ab} : T_{Z(a),Z(b)}(f) : = \frac{6\xi - 1}{2} : \Delta \varphi^2(f) : - (m^2 + \xi R)\varphi^2(f) : + \frac{1}{2\pi^2} \int_{M} U_2(x,x)f(x)d\mu_g(x) \quad 1.
$$

If $0 < \lambda' \neq \lambda$, with obvious notation,

$$
: T_{X,Y}(f) \chi(\lambda) - T_{X,Y}(f) \chi(\lambda') = \ln \left( \frac{\lambda'}{\lambda} \right) \int_{M} t_{X,Y}(x)f(x)d\mu_g(x) \quad 1,
$$

where the smooth, symmetric, conserved tensor field $t$ is that introduced in \(23\).

**Proof.** We start by proving \(13\) which is the simplest item. It is obvious by Def.3.2 that we may reduce to consider $f \in \mathcal{D}(N)$ where $N \subset M$ is a causal domain. We have

$$
\langle \Psi_\omega, : \varphi^2(v)(f) : \Psi_\omega \rangle_\nu = \lim_{j \to \infty} \langle \Psi_\omega, : \varphi^2(v)(s_j) : \Psi_\omega \rangle_\nu,
$$

where $\{s_j\} \subset \mathcal{D}(N^2)$ is a sequence of smooth functions which converge to $t_2(I, I, f) = f \delta_2$ in the Hörmander pseudo topology in a closed conic set in $N \times (\mathbb{R}^4 \setminus \{0\})$ containing $WF(f \delta_2)$. Such a sequence does exist by Theorem 8.2.3 of \[21\]. Above we have used the continuity of the scalar product as well as the continuity of the map $t_2(I, I, f) \mapsto : \varphi^2(v)(f) : \Psi_\omega$ since $\Psi_\omega \in \mathcal{D}_\nu$. On the other hand we may choose each $s_j$ of the form $\sum_j c_j h_j \otimes h'_j$, where the sum is finite, $c_j \in \mathbb{C}$ and $h_j, h'_j \in \mathcal{D}(N)$. This is because, using Weierstrass’ theorem on uniform approximation by means of polynomials in $\mathbb{R}^m$, it turns out that the space of finite linear combinations $h \otimes h'$ as above is dense in $\mathcal{D}(N \times N)$ in its proper seminorm-induced topology (viewing $N$ as a subset of $\mathbb{R}^4$ because of the presence of global coordinates). We leave the trivial details to the reader.

With that choice one straightforwardly finds

$$
\langle \Psi_\omega, : \varphi^2(v)(s_j) : \Psi_\omega \rangle_\nu = (G_{\omega_j}^{(1)} - H)(s_j) = s_j(G_{\omega_j}^{(1)} - H),
$$

where we have used the fact that both $G_{\omega_j}^{(1)} - H$ and $s_j$ are smooth. Since the convergence in the Hörmander pseudotopology imply the convergence in $\mathcal{D}'(N)$, we finally get

$$
\langle \Psi_\omega, : \varphi^2(v)(f) : \Psi_\omega \rangle_\nu = \lim_{j \to \infty} s_j(G_{\omega_j}^{(1)} - H) = \int_{N \times N} (G_{\omega_j}^{(1)} - H)(x,y)f(x)d\mu_g(x)d\mu_g(y)
$$

The achieved result can be re-written in a final form taking \[19\] into account and noticing that $Z_n - H$ is $C^n(N \times N)$, it vanishes with all of the derivatives up to the order $n$ for $x = y$ and $n$ may be fixed arbitrarily large. By this way we get

$$
\langle \Psi_\omega, : \varphi^2(v)(f) : \Psi_\omega \rangle_\nu = \int_M (\varphi^2(x))_\omega f(x)d\mu_g(x)
$$
which is nothing but our thesis. Using the same approach one may prove \[^{(48)}\] as well as

\[
\langle \Psi_\omega, h\hat{\varphi}_\nu P\hat{\varphi}_\nu(f) \rangle = \int_M h(x)\langle \hat{\varphi}(x)P\hat{\varphi}(x) \rangle_\omega f(x)d\mu_g(x),
\]

where \( f \in \mathcal{D}(N) \) and \( h \in C^\infty(M) \). In other words, by Lemma 2.1,

\[
\langle \Psi_\omega, h\hat{\varphi}_\nu P\hat{\varphi}_\nu(f) \rangle = -\int_M h(x)c_4 U_2(x,x)f(x)d\mu_g(x).
\]

The right hand side does not depend on \( \Psi_\omega \) which, it being Hadamard as \( \nu \) (but not necessarily quasifree), may range in the dense subspace of the Fock space \( \mathcal{H}_\omega \) containing \( n \)-particle states with smooth modes \(^3\). Finally, using the fact that the Hilbert space is complex one trivially gets the operator identity on \( D_\omega \)

\[
:\!h\hat{\varphi}_\nu P\hat{\varphi}_\nu(f)\! = -\int_M h(x)c_4 U_2(x,x)f(x)d\mu_g(x)I.
\]

By Def.3.2. such an identity holds true also for \( f \in \mathcal{D}(M) \), then

\[
: h\varphi P\varphi(f) : = \alpha^{-1}_\nu (:\!h\hat{\varphi}_\nu P\hat{\varphi}_\nu(f)\!) = \int_M h(x)c_4 U_2(x,x)f(x)d\mu_g(x),
\]

because \( \alpha \) is an algebra isomorphism. We have proven the item (a). The items (e) and (d) may be proven analogously starting from \(^{(48)}\), in particular (d) is a direct consequence of (b) in Theorem 2.1. Let us prove the conservation of the stress-energy tensor operator \(^{(50)}\). To this end, we notice that \(^{(48)}\) together with the item (a) of Theorem 2.1 for \( V' \equiv 0 \) by means of the procedure used to prove \(^{(50)}\) implies the operator identity on \( \mathcal{H}_\nu \)

\[
:\!\hat{T}_\nu \otimes X(f)\! = \sum_{a,b} \eta^{ab} \left\{ : \!\hat{T}_\nu Z_{(a)}Y (\nabla \cdot (fZ_{(b)})) : + : \!\hat{T}_\nu X,\nabla Z_{(a)} Z_{(b)}(f) : \right\} = 0, \tag{54}
\]

for any Hadamard state \( \omega \), \( X,Y \) smooth vector fields and \( Z_{(a)} \) tetrad fields. This identity entails \(^{(50)}\) by applying \( \alpha^{-1}_\nu \) on both sides. \( \square \)

4 Summary and final comments.

We have shown that a definition of stress-energy tensor operator in curved spacetime is possible in terms of local Wick products of field operators only by adding suitable terms to the classical form of the stress energy tensor. Such a definitions seems to be quite reasonable and produces results in agreement with well-known regularization procedures of averaged quantum observables. The added terms \( : h\varphi P\varphi(f) : \) in the form of the stress-energy tensor operator enjoy three remarkable properties. (1) They classically vanish, (2) they are written as local Wick products of field operators, (3) they are in a certain sense *universal*, i.e., they do not depend on the scale
λ and belong to the commutant of the algebra $W(M, g)$.
The issue whether or not it could be possible to define a stress-energy tensor operator free from these terms is related to the issue of the existence of Hadamard singular bidistributions, defined locally and somehow "determined by the local geometry only". The positiveness seems not to be a requirement strictly necessary at this level. The appearance of the so-called conformal anomaly shared by the various regularization techniques and related to the presence of the found terms could be in contrast to the existence of such local bisolutions. However, no proof, in any sense, exists in literature to the knowledge of the author.

As a final comment we suggest that the universal terms $h \varphi P \varphi(f)$: or similar terms may be useful in studying other conservation laws within the approach based on local Wick products, e.g., conserved currents in (non-)Abelian gauge theories and related anomalies.

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Appendix A.

If $(M, g)$ is a smooth Riemannian or Lorentzian manifold, an open set $C \subset M$ is said a normal convex neighborhood if there is an open set $W \subset TM$, $W = \{(q, v) | q \in C, v S_q \}$, $S_q \subset T_q M$ being a starshaped open neighborhood of the origin, such that $\exp |W: (q, v) \mapsto \exp q v$ is a diffeomorphism onto $C \times C$. It is clear that $C$ is connected and there is only one geodesic segment joining any pair $q, q' \in C$, completely contained in $C$, i.e., $t \mapsto \exp q (t((\exp q)^{-1} q')) t \in [0,1]$. Moreover if $q \in C$, $\{e_\alpha| q \} \subset T_q M$ is a basis, $t = t^\alpha e_\alpha| q \mapsto \exp q (t^\alpha e_\alpha| q)$, $t \in S_q$ defines a set of coordinates on $C$ centered in $q$ which is called normal Riemannian coordinate system centered in $q$. In $(M, g)$ as above, $s(x, y)$ indicates the squared geodesic distance of $x$ from $y$: $s(x, y) \overset{def}{=} g_x(\exp x^{-1} y, \exp x^{-1} y)$. By definition $s(x, y) = s(y, x)$ and $s$ turns out to be smoothly defined on $C \times C$ if $C$ is a convex normal neighborhood. The class of the convex normal neighborhood of a point $p \in M$ defines a fundamental system of neighborhoods of $p$.

With the signature $(-, +, \cdots, +)$, we have $s(x, y) > 0$ if the points are space-like separated, $s(x, y) < 0$ if the points are time-like related and $s(x, y) = 0$ if the points are light related. In Euclidean manifolds $s$ defined as above is everywhere nonnegative.

The distribution $Z_n^{(+)}$ is defined by the following integral kernel in the sense of the usual $\epsilon \to 0^+$ prescription.

$$
\beta_D^{(1)} \frac{U(x, y)}{s_\epsilon T^{D/2-1}(x, y)} + \beta_D^{(2)} V^{(n)}(x, y) \ln \frac{s_\epsilon T(x, y)}{\lambda^2} \quad \text{if } D \text{ is even,} \tag{55}
$$

$$
\beta_D^{(1)} \frac{T^{(n)}(x, y)}{s_\epsilon T^{D/2-1}(x, y)} \quad \text{if } D \text{ is odd}. \tag{56}
$$

$s^k(x, y) \overset{def}{=} (s(x, y))^k$. The cut branch in the logarithm is fixed along the negative real axis, moreover $s_\epsilon T \overset{def}{=} s(x, y) + i\epsilon(T(x) - T(y)) + \epsilon^2$, $T$ being any global temporal function defined on
used in the expansions above, satisfying the differential equations on the neighborhood \(C\) if

\[
D\equiv 2 - |D|.
\]

If \(\Theta\) is odd, \((57)\) and Appendix A3 of \([16]\).

\[
\beta_D^{(1)} \equiv (-1)^{\frac{D+1}{2}} \frac{\pi^{\frac{D-2}{2}}}{2\Gamma(\frac{D-2}{2})} \quad \text{for } D \text{ odd,}
\]

\[
\beta_D^{(2)} \equiv (-1)^{\frac{D}{2}} \frac{2^{1-D}}{\pi^{\frac{D}{2}}\Gamma(\frac{D}{2})}.
\]

\(U, V^{(n)}, T^{(n)}\) admit the following expansions in powers of \(s(x, y)\). If \(D\) is even

\[
U(x, y) \equiv \Theta_D \sum_{k=0}^{(D-4)/2} \frac{1}{(4-D)k} U_k(x, y) s^k(x, y),
\]

\[
V^{(n)}(x, y) \equiv \left(2\left(\frac{D}{2} - 1\right)\right) \sum_{k=0}^{n} \frac{1}{2^k k!} U_{\frac{D}{2}+k-1}(x, y) s^k(x, y),
\]

where \(\Theta_2 = 0\) and \(\Theta_D = 1\) if \(D > 2\), and

\[
T^{(n)}(x, y) \equiv \sum_{k=0}^{n+(D-3)/2} \frac{1}{(4-D)k} U_k(x, y) s^k(x, y),
\]

if \(D \geq 3\) is odd. \((\alpha|0) \equiv 1\) and \((\alpha|k) \equiv \alpha(\alpha + 2) \cdots (\alpha + 2k - 2)\). For any open convex normal neighborhood \(C\) in \(M\) there is exactly one sequence of \(C^\infty(C \times C)\) real valued functions \(U_k\), used in the expansions above, satisfying the differential equations on \(C \times C\):

\[
P_x U_{k-1}(x, y) + g_x((\nabla_x) s(x, y), U_k(x, y)) + (M(y, x) + 2k) U_k(x, y) = 0,
\]

with the initial conditions \(U_{-1}(x, y) = 0\) and \(U_0(x, x) = 1\). The function \(M\) is defined as \(M(x, y) \equiv \frac{1}{2} \Delta_x s(x, y) - D\), with \(D\) is the dimension of the manifold. The proof of existence and uniqueness is trivial using normal coordinates centered in \(x\). The coefficients \(U_k(x, y)\) can be defined, by the same way, also if the metric is Euclidean. They coincide, barring numerical factors, with the so called Hadamard-Minakshisundaram-DeWitt-Seeley coefficients. If \(C', C\) are convex normal neighborhoods and \(C' \subset C\), the restriction to \(C'\) of each \(U_k\) defined in \(C\) coincides with the corresponding coefficient directly defined on \(C'\). There is a wide literature on coefficients \(U_k\), in relation with heat-kernel theory and \(\zeta\)-function regularization technique [21].

As in 2.2 \(Z(x, y)\) indicates the kernel of \(\text{Re}(Z_{+}(\cdot))\) which is smooth for \(s(x, y) \neq 0\) in every convex normal neighborhood \(C \ni x, y\). It is possible to show that the coefficients are symmetric, i.e., if \(x, y \in C\), \(U_n(x, y) = U_n(y, x)\) [19] and thus, since \(s(x, y) = s(y, x)\), it also holds

\[
Z_n(x, y) = Z_n(y, x)
\]

for any \(n \geq 0\) and \(s(x, y) \neq 0\).
The recurrence relations (62) have been obtained by requiring that the sequence $Z_n$ defines a local $y$-parametrized “approximated solution” of $P_s S(x, y) = 0$ [25]. That solution is exact if one takes the limit $Z = \lim_{n \to \infty} Z_n$ of the sequence provided the limit exist. This happens in the analytic case, but in the smooth general case the sequence may diverge. Actually, in order to produce an approximated/exact solution for $D$ is even, a smooth part $W$ has to be added to $Z$, $S = Z + W$, and also $W$ can be expanded in powers of $s$ [25]. Differently from the expansion of $Z$ which is completely determined by the geometry and the operator $P$, the expansion of $W$ depends on its first term $W_0$ (corresponding to $s^0$) and there is no natural choice of $W_0$ suggested by $P$ and the local geometry. Finally if $D$ is even and for whatever choice of $W_0$ there is no guarantee for producing a function $Z + W$ (provided the limits of corresponding sequences exist) which is solution of field equations in both arguments: This is because in general $W(x, y) \neq W(y, x)$ also if $W_0(x, y)$ is symmetric [3].

Appendix B.

Referring to 2.2, the properties (b) and (c) respectively imply the relations in $M \times M$

\begin{align*}
G^{(1)}_\omega(x, y) &= G^{(1)}_\omega(y, x), \quad \text{(63)} \\
P_x G^{(1)}_\omega(x, y) &= P_y G^{(1)}_\omega(x, y) = 0, \quad \text{(64)}
\end{align*}

which hold when $x \neq y$ are not light-like related. These relations are useful in the following.

**Proof of Lemma 2.1.** (a) In the following we take advantage of the identity, where $f$ and $g$ are $C^2$ functions, $P(fg) = fPg - (\Delta f)g - 2g(\nabla f, \nabla g)$ . Suppose $D > 2$ even, using the identity above and the definition of $Z_n$, one finds, for either $x, y$ time-like related or space-like separated,

\[
P_x Z_n(x, y) = \beta_D^{(1)} \left( \frac{P_x U(x, y)}{s^{D/2 - 1}} - U(x, y) \Delta(x)s^{1-D/2} - 2g_x(\nabla_x s^{1-D/2}, \nabla_x U(x, y)) \right) \\
+ \beta_D^{(2)} \left( \frac{P_x V^{(n)}(x, y))}{2} \log \frac{|x|}{\lambda} - V^{(n)}(x, y) \Delta(x) \log \frac{|x|}{\lambda} - 2g_x(\nabla_x s, \nabla_x V^{(n)}(x, y)) \right).
\]

Using (62) for $n \geq 1$ we have

\[
P_x Z_n(x, y) = -\beta_D^{(2)} \left[ -\left( -|s| \right) P_x V^{(n)}(x, y) + (\Delta(x) \log \frac{|x|}{\lambda} + V^{(n)}(x, y) + 2g_x(\nabla_x s, \nabla_x V^{(n)}(x, y)) \right]
\]

where

\[
V^{(n)}(x, y) \overset{\text{def}}{=} \sum_{k=1}^{n} V_k(x, y)s^k(x, y)
\]

with

\[
V_k(x, y) \overset{\text{def}}{=} \left( \frac{1}{2^{k-1}} - 1 \right) \frac{1}{2k!} U_{D+2k-1}(x, y).
\]
Expanding the derivatives and using (62) once again, if \( n \geq 1 \), one gets that, \(-\left(\beta_2^{(2)}\right)^{-1} P_x Z_n(x, y)\) equals

\[
\begin{align*}
&\frac{s}{s} \left(\frac{\Delta(x) s(x, y)}{s} - \frac{4}{s} V_1(x, y) + 2 \frac{g_x(\nabla_x s(x, y), \nabla_x s(x, y))}{s} V_1(x, y) + 2 g_x(\nabla_x s, \nabla_x V_1(x, y)) \right) \\
&\quad + |s|^n O_{1, n}(x, y) \ln \frac{|s|}{\lambda^2} + |s|^{n-1} O_{2, n}(x, y) + |s|^{n-1/2} O_{3, n}(x, y) ,
\end{align*}
\]

where \( O_{k, n} \) are smooth in a neighborhood of \((z, z)\) and the last two terms appear for \( n > 1 \) only. Using \( g_x(\nabla_x s(x, y), \nabla_x s(x, y)) = 4s(x, y) \), one finds

\[
-\left(\beta_2^{(2)}\right)^{-1} P_x Z_n(x, y) = (\Delta(x) s(x, y) - 4) V_1(x, y) + 8 V_1(x, y) + 2 g_x(\nabla_x s, \nabla_x V_1(x, y))
\]

and thus, since \(\Delta(x) s(x, y) \to 2D \) and \(\nabla_x s(x, y) \to 0\) as \((x, y) \to (z, z)\),

\[
\lim_{(x, y) \to (z, z)} P_x Z_n(x, y) = c_D U_{D/2}(z, z) ,
\]

which is a part of (12) for \( D \) even. \( Z_n(x, y) = Z_n(y, x) \) implies the remaining identity in (12). If \( D = 2 \), \( x, y \) are either time-like related or space-like separated and \( n \geq 1 \), the proof is essentially the same. One directly finds

\[
P_x Z_n(x, y) = -\beta_2^{(2)} \left[ (-P_x V^{(n)}(x, y)) \ln \frac{|s|}{\lambda^2} + V^{(n)}(x, y) \Delta(x) \ln \frac{|s|}{\lambda^2} + 2 g_x(\nabla_x s, \nabla_x V^{(n)}(x, y)) \right] ,
\]

with

\[
V^{(n)}(x, y) \overset{\text{def}}{=} \sum_{k=0}^{n} V_k(x, y) s^k(x, y)
\]

and

\[
V_k(x, y) \overset{\text{def}}{=} \left( 2 \frac{D}{2} - 1 \right) \frac{1}{2^k k!} U_{D+2k-1}(x, y) .
\]

Using \( V_0 = U_0 \) \((D = 2)\) and (62) for \( k = 0 \), one gets (66) once again. If \( D \) is odd, \( n \geq 1 \) and \( x, y \) are either space-like separated or time-like related, (62) entails

\[
P_x Z_n(x, y) = \theta(s(x, y)) |s(x, y)|^{n-1/2} O_n(x, y) ,
\]

where \( O_n \) is smooth in a neighborhood of \((z, z)\). Therefore,

\[
\lim_{(x, y) \to (z, z)} P_x Z_n(x, y) = 0 .
\]

which is a part of (12) for \( D \) odd the other part is a trivial consequence of the symmetry as above. Notice that the proof shows also that the limit is uniform in the three treated cases.
because $|s(x, y)|$ uniformly tends to 0 as $(x, y) \rightarrow (z, z)$.

(b) The proof follows a very similar procedure as that used in the proof of (a). One obtains that

$$\lim_{(x,y) \rightarrow (z,z)} P_z \nabla^\mu \nabla_\mu Z_n(x, y)$$

equals

$$- \beta_D^{(2)} \left[ (2D - 4) \nabla^{\mu}_y V_1(x, y) + 8 \nabla^{\mu}_y V_1(x, y) - 4 \nabla^{\mu}_z V_1(x, y) \right]_{x = y = z}.$$  \( \text{(68)} \)

(One has to differentiate \( \text{(66)} \) with respect to $\nabla^\mu$ and use $\nabla^\mu \nabla_\mu s(x, y)|_{x = y = z} = 0$ and $\nabla^\mu \nabla_\mu s(x, y)|_{x = y = z} = -2g_{\mu\nu}(z)$.) Finally one notices that $V_1$ is proportional to $U_{D/2}$ and, since $U_n(x, y) = U_n(y, x)$, it also holds $\nabla^\alpha \nabla_\alpha V_1(z, z) = 2\nabla^\alpha \nabla_\alpha V_1(x, y)|_{x = y = z}$.

Using that in \( \text{(68)} \) the thesis (b) arises. For $D = 2$ and $D$ odd the proof is the same with trivial modifications. (c) The proof directly follows from (a) and \( \text{(54)} \). \( \square \)

Proof of Theorem 2.1. (a) \( (x, y) \mapsto G^{(1)}_{\omega}(x, y) - Z_n(x, y) \) is $C^m$ in a whole neighborhood of $z$ (also for light-like related arguments). Since we want to apply the second order operator $D^{(\eta)}_{(z)\mu\nu}$ to it, we need to fix $n \geq 2$, however we also want to derive the obtained stress-energy tensor and thus we need $n \geq 3$. With $n \geq 3$ the map above is $C^m$ at least and $z \mapsto (\hat{T}^{(\eta)}_{\mu\nu}(z))_\omega$ is $C^{m-1}$. Finally, since the latter map do not depend on $n$ it must be $C^\infty$ also if $n \geq 3$ is finite. The independence from $n$ is a consequence of $\lim_{(x,y) \rightarrow (z,z)} \Delta_{n,n'}(x, y) = 0$ where $\Delta_{n,n'}(x, y) \defeq Z_n(x, y) - Z_{n'}(x, y)$ which holds true for any pair $n, n' \geq 3$ as the reader may straightforwardly check and prove by induction. The remaining part of (a) may be proven as follows. For any $C^3$ function \( (x, y) \mapsto \Gamma(x, y) \) symmetric under interchange of $x$ and $y$ we have the identity

$$\nabla^\mu_{(z)} \left( D^{(\eta)}_{(z)\mu\nu} \Gamma(x, y) \right)_{x = y = z} = -P_z \nabla^\mu_{(y)\nu} \Gamma(x, y)|_{x = y = z} + \eta \nabla^\mu_{(z)\nu} \left( P_z \Gamma(x, y) \right)|_{x = y = z} - \frac{1}{2} \Gamma(z, z) \nabla^\nu V'(x).$$  \( \text{(69)} \)

Indeed, if $\varphi$ does not satisfy the field equation, \( \text{(3)} \) reads,

$$\nabla^\alpha T_{\alpha\beta}(x) = -(P\varphi)(x) \nabla_\beta \varphi(x) - \frac{1}{2} \varphi^2(x) \nabla_\beta V'(x),$$  \( \text{(70)} \)

Such an identity can be obtained by using the form of the stress-energy tensor and the symmetry of $\Gamma(x, y) = \varphi(x) \varphi(y)$ only. So it holds true for each symmetric sufficiently smooth map $(x, y) \mapsto \Gamma(x, y)$. The proof of \( \text{(39)} \) is nothing but the proof of \( \text{(70)} \) taking the added term proportional to $\eta$ into account. Then put $\Gamma(x, y) = G^{(1)}_{\omega}(x, y) - Z_n(x, y)$ into \( \text{(68)} \) with $n \geq 3$, this is allowed because $Z_n$ is symmetric by construction (see Appendix A) and $G^{(1)}_{\omega}$ satisfies \( \text{(39)} \). The first line of right-hand side of \( \text{(68)} \) reduces to

$$\lim_{(x,y) \rightarrow (z,z)} P_z \nabla^\mu_{(y)\nu} Z_n(x, y) - \eta \nabla^\mu_{(z)\nu} \lim_{(x,y) \rightarrow (z,z)} P_z Z_n(x, y)$$

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because of (64). Both terms above can be computed by Lemma 2.1 finding
\[
\nabla^\mu (\hat{T}^{(q)}_{\mu\nu}(z))_\omega = \delta_D(k_D - \eta c_D)\nabla_\nu U_{D/2}(z, z) - \frac{1}{2}(\hat{\varphi}^2(z))_\omega \nabla_\nu V'(z).
\]
The former term in the right hand side vanishes if and only if \( \eta = k_D/c_D \), i.e., \( \eta = \eta_D \). This concludes the proof of (a). (b) Directly from the form of the stress-energy tensor, one finds that, if \( P\varphi \neq 0 \), (53) reads
\[
g_{\alpha\beta}(x)T^{\alpha\beta}(x) = \left[ \frac{\xi_D - \xi}{4\xi_D - 1} \Delta - V(x) \right] \varphi^2(x) + \left( 1 - \frac{D}{2} \right) (P\varphi)(x)\varphi(x).
\]
The same results holds if \( \varphi(x)\varphi(y) \) is replaced by any sufficiently smooth symmetric function \( \Gamma \). Therefore, if \( \Gamma \) is as above, similarly to (71) we get
\[
g_{\alpha\beta}'(z)T^{(q)}_{\alpha\beta}(z)|_{x=y=z} = \left[ \frac{\xi_D - \xi}{4\xi_D - 1} \Delta - V(z) \right] \Gamma(z, z) + \left( 1 - \frac{D}{2} + \eta D \right) (P\Gamma(x, y))|_{x=y=z}.
\]
If \( \eta = \eta_D \) and \( \Gamma(x, y) = G^{(1)}_{\omega}(x, y) - Z_{\omega}(x, y) \), using (12) and Lemma 2.1, we get the identity in (b). \(-2c_D/(D+2)U_{D/2}(z, z)\) is the conformal anomaly for \( V \equiv 0 \) and \( m = 0 \), \( \xi = \xi_D \) because it coincides with the heat-kernel coefficient \( a_{D/2}(z, z)/(4\pi)^{D/2} \) [20]. This can be seen by direct comparison of recursive equations defining both classes of coefficients (see references in [20]). This concludes the proof of (b). (c) The proof is direct by employing the given definitions.

(d) For \( D \) odd the proof of the thesis is trivial. Hence assume \( D \) even. In that case, with obvious notation, \( Z_{\lambda, n}(x, y) - Z_{\lambda', n}(x, y) = 2 \ln \left( \frac{\lambda'}{\lambda} \right) \sum_{k=0}^{n} c_k s^k(x, y)U_{k-1+D/2}(x, y) \), where \( c_k \) are numerical coefficients defined above. \( \lim_{(x, y) \to (z, z)} T^{(q)}_{\alpha\beta}(z)(Z_{\lambda} - Z_{\lambda'}) \) is a polynomial of coefficients \( U_k(z, z) \) and their derivatives. These coefficients do not depend on the state are proportional to heat-kernel ones and thus are built up as indicated in the thesis [3]. Notice that the obtained tensor field \( t \) must be conserved because is the difference of two conserved tensor fields if \( V' \equiv 0 \). (e) For \( m = 0 \) the proof of the thesis is trivial because \( Z_{\omega} \) in Minkoski spacetime does not contain the logarithmic term and does not depend on both \( \lambda \) and \( n \), moreover, if \( \omega \) is Minkowski vacuum \( G^{(1)}_{\omega}(x, y) = Z_{\omega}(x, y) \) and thus the renormalized stress-energy tensor vanishes. Let us consider the case \( m > 0 \). In that case, the smooth kernels of the two-point function is given by, in the sense of the analytic continuation if \( s(x, y) < 0 \),
\[
G^{(+)}_{\omega}(x, y) = \lim_{\epsilon \to 0^+} \frac{4m}{(4\pi)^2 \sqrt{s_c,T(x, y)} K_1 \left( m \sqrt{s_c,T(x, y)} \right)}
\]
with \( s_{\epsilon,T}(x, y) \equiv s(x, y) + 2i(T(x) - T(y)) + \epsilon^2 \) where \( \epsilon \to 0^+ \) indicates the path to approach the branch cut of the squared root along the negative real axis if \( s(x, y) < 0 \). \( T \) indicates any
global time coordinate increasing toward the future. $K_1$ is a modified Bessel function. The corresponding Hadamard function can be expandend as

$$G^{(1)}_{\omega}(x, y) = \frac{4}{(4\pi)^2} + \frac{m^2}{(4\pi)^2} \left\{ 1 + \frac{m^2s}{8} \right\} \ln \left( \frac{e^{2\gamma m^2s}}{4} \right) + s^2f(s) + s^2g(s)$$

where $f$ and $g$ are smooth functions and $\gamma$ is Euler-Mascheroni’s constant. Similarly

$$Z_{\lambda, \beta}(x, y) = \frac{4}{(4\pi)^2} + \frac{m^2}{(4\pi)^2} \left\{ 1 + \frac{m^2s}{8} + C_2s^2 + C_3s^3 \right\} \ln \frac{s}{\lambda^2},$$

where $C_2$ and $C_3$ are constants. Therefore

$$G^{(1)}_{\omega} - Z_{\lambda, \beta}(x, y) = \frac{m^2}{(4\pi)^2} \left\{ 1 + \frac{m^2s}{8} \right\} \ln \left( \frac{\lambda^2e^{2\gamma m^2}}{4} \right) - \frac{m^2}{(4\pi)^2} \left[ 1 + \frac{5m^2s}{16} \right]$$

$$+ s^2 \left[ h(s) + k(s) \ln s \right],$$

where $h$ and $k$ are smooth functions. Trivial computations lead to

$$\langle \hat{\mathcal{T}}_{\mu\nu}^{(n)}(z) \rangle_{\omega, \lambda} = -\frac{m^4}{3(4\pi)^2} \left[ \ln \left( \frac{\lambda^2e^{2\gamma m^2}}{4} \right) - \frac{7}{4} \right] g_{\mu\nu}(z).$$

Posing $\lambda^2 = 4e^{-\gamma}m^{-2}$ the right-hand side vanishes. □

**Proof of Lemma 3.1.** (a) The thesis can be proved working in a concrete algebra $\mathcal{W}_\omega(N, g)$ using operators $:K_1\hat{\varphi} \cdots K_n\hat{\varphi}(f):_H \in \mathcal{W}_\omega(N, g)$ where $\omega$ is fixed quasifree Hadamard state and $\hat{\varphi} = \hat{\varphi}_\omega$. Notice that, for every $\Psi \in D_\omega$, $\hat{\mathcal{W}}_{H,n+1}(x_1, \ldots , x_{n+1})\Psi$ equals

$$\left( \hat{\mathcal{W}}_{H,n}(x_1, \ldots , x_n)\hat{\mathcal{W}}_{H,1}(x_{n+1})\Psi \right)_S - \left( \sum_l \hat{\mathcal{W}}_{H,n}(x_1, \ldots , \hat{x}_l, x_{n+1})\Psi H(x_l, x_{n+1}) \right)_S$$

where $S$ indicates the symmetrization with respect to all arguments and $\hat{x}_l$ indicates that the argument is omitted. We prove the thesis, which is true for $n = 1$, by induction. If $H'$ is defined as $H$ but with a different choice of $\psi$ and $\{\alpha_k\}$, for each $\Psi \in D_\omega$, the formula above and our inductive hypothesis imply that $\langle :K_1\hat{\varphi} \cdots K_{n+1}\hat{\varphi}(f):_H - :K_1\hat{\varphi} \cdots K_{n+1}\hat{\varphi}(f):_H' \rangle \Psi$ reduces to

$$\int_{M^n} \left( \sum_l \hat{\mathcal{W}}_{H,n}(x_1, \ldots , \hat{x}_l, x_{n+1})\Psi S(x_l, x_{n+1}) \right)_S^{t \delta_{n+1} (x_{n+1}) \cdots t \delta_k (x_k) f(x_l) S(x_l, x_{n+1})}$$

where $S(x, y)$ is a smooth function which vanishes for $x = y$ together with all of its derivatives. As $\hat{\mathcal{W}}_{H,n}(x_1, \ldots , \hat{x}_l, x_{n+1})\Psi$ is singular on the diagonal we cannot directly conclude that the integral vanishes. However, there is sequence of smooth vector-valued functions $\{U_j\}$ which
tends to the distribution in \((\cdots)_S\) in the sense of Hörmander pseudo topology in a closed conic set containing the wave front set of the distribution. It is simply proven that the smearing procedure of distributions by means of distributions defined in 3.1 is continuous in the sense of Hörmander pseudo topology. Therefore \((K_1\varphi \cdots K_n+1\varphi(f) : \mathcal{H}) = :K_1\varphi \cdots K_n+1\varphi(f) : \mathcal{H}')\) \(\Psi\) can be computed as the limit

\[
\int_{M^n} \left( \sum_l f_j(x_1, \ldots, x_{n+1}) S(x_l, x_{n+1}) \right) \left( K_{n+1} \cdots K_1 f_1 \delta_n(x_1, \ldots, x_{n+1}) \right)
\]

for \(j \to \infty\). However, as each \(U_n\) is regular, we can conclude that each term of the sequence above vanishes and this proves the thesis because \(\Psi \in D_0\) is arbitrary.

(b) By (a) we may assume that the distributions \(H\) and \(H'\) are constructed using the same function \(\psi\) and the same sequence of numbers \(\{\alpha_k\}\). Define \(L = N \cap N'\) and take \(f \in D(L)\). Since the convex normal neighborhoods define a topological base, \(L\) is the union of convex normal neighborhoods. In turn, it implies that the compact set \(\text{supp} f \subset L\) admits a finite covering \(\{U_i\}\) made of convex normal neighborhoods contained in \(L\) and thus both in \(N\) and \(N'\). Therefore, if \(x, y \in U_i\), the squared geodesic distance \(s(x, y)\) computed by viewing \(x, y\) as elements of \(N\) agrees with the analogue by viewing \(x, y\) as elements of \(N'\) (also if \(N \cap N'\) may not be convex normal).

By consequence \(H\) and \(H'\) make the same distribution on each \(U_i\). \(\{U_i\}\) is locally finite and thus there is a smooth partition of the unity \(\{\chi_i\}_j\) subordinate to \(\{U_i\}\). By linearity we have

\[
\hat{K_1\varphi \cdots K_n\varphi(f)} : \mathcal{H} = \sum_j \hat{K_1\varphi \cdots K_n\varphi(\chi_j f)} : \mathcal{H} = \sum_j \hat{K_1\varphi \cdots K_n\varphi(\chi_j f)} : \mathcal{H}'.
\]

which concludes the proof. \(\square\)

**Proof of part of (b) in Proposition 3.2.** We want to show that, if \(N \subset M\) is a causal domain then, \(f_j \to f\) in \(D(N)\) entails \(t_n[K_1, \ldots, K_n, f_j] \to t_n[K_1, \ldots, K_n, f]\) in the sense of Hörmander pseudo topology in the conic set \(\Gamma_n = \{(x_1, k_1; \ldots; x_n, k_n) \in T^*M^n \setminus \{0\} | \sum_i k_i = 0\}\) which contains the wave front set of all involved distributions. Since there is a coordinate patch covering \(N, \xi : N \to O,\) (normal Riemannian coordinates centered on some \(p \in N\)), we can make use of the \(\mathbb{R}^n\)-distribution definition of convergence (see Definition 8.2.2. in [21]). Therefore, in the following, \(f, f_j\) and \(t_n[K_1, \ldots, K_n, f], t_n[K_1, \ldots, K_n, f_j]\) have to be understood as distributions in \(D'(O)\), and \(D'(O')\) respectively, \(O\) being an open subset of \(\mathbb{R}^4\). Posing \(u_j = t_n[K_1, \ldots, K_n, f_j]\) and \(u = t_n[K_1, \ldots, K_n, f]\), we have to show that (1) \(u_j \to u\) in \(D'(O')\), and this is trivially true by the given definitions since \(f_n \to f\) in \(D(O)\), and, (2),

\[
\sup_{V} |k|^N |\hat{\psi u}(k) - \hat{\psi u_j}(k)| \to 0, \quad \text{as} \quad j \to +\infty
\]

for all \(N = 0, 1, 2, \ldots, \psi \in D(O)\) and \(V, \) closed conic set\(^7\) in \(\mathbb{R}^{4n}\) such that

\[
\Gamma_n \cap (\text{supp} \ \psi \times V) = \emptyset.
\]

\(\hat{\psi}\) denotes the Fourier transform of \(\psi\). From now on \(K\) denotes a generic vector in \(\mathbb{R}^{4n}\) of the form \((k_1, \ldots, k_n)\), \(k_i \in \mathbb{R}^4\) and, in components \(k_i = (k^1_i, k^2_i, k^3_i, k^4_i)\). We leave to the reader to

\(^7\)A conic set \(V \subset \mathbb{R}^m\) is a set such that if \(v \in V, \lambda v \in V\) for every \(\lambda > 0\).
show that, with the definitions given above, \( \hat{\psi}u_j \) and \( \hat{\psi}u \) are polynomials in the components of \( K \), whose coefficients smoothly depend on \( k_1 + \cdots + k_n \), i.e.,

\[
\hat{\psi}u_j(k) = \sum_{r_{11}, \ldots, r_{n4} \in \mathbb{N}} a_{jr_{11}, \ldots, r_{n4}} (k_1 + \cdots + k_n) \prod_{i=1}^{n} \prod_{m=1}^{4} (q_i^m)^{r_{im}}. \tag{75}
\]

An analogous identity concerning \( u \) and coefficients \( a_{r_{11}, \ldots, r_{n4}} (k_1 + \cdots + k_n) \) holds by omitting \( j \) in both sides. (Above \( 0 \in \mathbb{N} \) and only a finite number of functions \( a_{jr_{11}, \ldots, r_{n4}} \) and \( a_{r_{11}, \ldots, r_{n4}} \) differ from the null function.)

Moreover \( f_j \to f \) in \( \mathcal{D}(O) \) implies that, for every \( N \in \mathbb{N} \) and \( r_{im} \in \mathbb{N} \),

\[
\sup_{x \in \mathbb{R}^4} |x|^N |a_j \ r_{11}, \ldots, r_{n4} (x) - a_{r_{11}, \ldots, r_{n4}} (x)| \to 0 \tag{76}
\]

With the given notations, our thesis (73) reduces to

\[
\sup_{K \in \mathcal{V}} |K|^N \sum_{r_{11}, \ldots, r_{n4} \in \mathbb{N}} \left[ a_j \ r_{11}, \ldots, r_{n4} \left( \sum_{i=1}^{4} k_i \right) - a_{r_{11}, \ldots, r_{n4}} \left( \sum_{i=1}^{4} k_i \right) \right] \prod_{i=1}^{n} \prod_{m=1}^{4} (q_i^m)^{r_{im}} \to 0 \tag{77}
\]

as \( j \to +\infty \) for all \( N \in \mathbb{N} \) and \( V \) which is closed in \( \mathbb{R}^{4n} \) conic and such that

\[
V \cap \{ K \in \mathbb{R}^{4n} \setminus \{0\} \mid \sum_{i=1}^{n} k_i = 0 \} = \emptyset. \tag{78}
\]

We want to prove (77) starting from (76). Consider a linear bijective map \( A : K \to Q \in \mathbb{R}^{4n} \), where \( Q = (q_1, \ldots, q_n) \) and \( q_1 = p_1 + \cdots + p_n \). The functions \( x \mapsto b_{s_{11}, \ldots, s_{n4}} (x) \) and \( x \mapsto b_{js_{11}, \ldots, s_{n4}} (x) \) which arise when translating (73) (and the analog for \( u \)) in the variable \( Q \), i.e.,

\[
\hat{\psi}u_j(k) = \sum_{s_{11}, \ldots, s_{n4} \in \mathbb{N}} b_{js_{11}, \ldots, s_{n4}} (q_1) \prod_{i=1}^{n} \prod_{m=1}^{4} (q_i^m)^{s_{im}},
\]

are linear combinations of the functions \( x \mapsto a_{jr_{11}, \ldots, r_{n4}} (x) \) (with coefficients which do not depend on \( j \)) and vice versa, therefore (74) entails

\[
\sup_{x \in \mathbb{R}^4} |x|^N |b_{js_{11}, \ldots, s_{n4}} (x) - b_{s_{11}, \ldots, s_{n4}} (x)| \to 0 \tag{79}
\]

for every \( N \in \mathbb{N} \) and \( s_{im} \in \mathbb{N} \). Since linear bijective maps transform closed conic sets into closed conic sets and \( |Q| \leq ||A|||K| \), \( |K| \leq ||A^{-1}||Q| \), our thesis (77) is equivalent to

\[
\sup_{Q \in \mathcal{U}} |Q|^N \sum_{s_{11}, \ldots, s_{n4} \in \mathbb{N}} [b_{js_{11}, \ldots, s_{n4}} (q_1) - b_{s_{11}, \ldots, s_{n4}} (q_1)] \prod_{i=1}^{n} \prod_{m=1}^{4} (q_i^m)^{s_{im}} \to 0 \tag{80}
\]
as \( j \to +\infty \) for all \( N \in \mathbb{N} \) and \( U \) closed in \( \mathbb{R}^{4n} \) conic and such that
\[
U \cap \{ Q \in \mathbb{R}^{4n} \setminus \{0\} \mid q_1 = 0 \} = \emptyset . \tag{81}
\]

It is possible to show that if \( U \in \mathbb{R}^{4n} \) is a set which fulfills (81) and \( U \) is conic and closed in \( \mathbb{R}^{4n} \), then there is \( p \in \mathbb{N} \setminus \{0\} \) such that \( U \subset U_p \), where the closed set \( U_p \) is defined by
\[
U_p = \left\{ Q \in \mathbb{R}^{4n} \mid |q_1| \geq \frac{1}{p} \sqrt{|q_2|^2 + \cdots + |q_n|^2} \right\} ,
\]
and thus \( U_p \cap \{ Q \in \mathbb{R}^{4n} \setminus \{0\} \mid q_1 = 0 \} = \emptyset \) for every \( p \in \mathbb{N} \setminus \{0\} \). The proof is left to the reader (hint: \( U \) is conic and satisfies (81) then, reducing to a compact neighborhood of the origin of \( \mathbb{R}^{4n} \), one finds a sequence of points of \( U \) which converges to some point \( x \in \{ Q \in \mathbb{R}^{4n} \setminus \{0\} \mid q_1 = 0 \} \) this is not possible because \( \overline{U} = U \) and thus \( x \in U \cap \{ Q \in \mathbb{R}^{4n} \setminus \{0\} \mid q_1 = 0 \} \). If (81) holds on each \( U_p \), it must hold true on each conic closed set \( U \) which fulfills (81). The validity of (81) on each \( U_p \) is a direct consequence of (79) and the inequalities which holds on \( U_p \)
\[
|Q| \leq \sqrt{1 + p^2} |q_1| \quad \text{and} \quad |q_s^r| \leq |q_1|/p \quad \text{for } r = 2, 3, \ldots, n, s = 1, 2, 3, 4. \quad \square
\]

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