Langevin Equation on Fractal Curves

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Abstract

We analyse a random motion of a particle on a fractal curve, using Langevin approach. This involves defining a new velocity in terms of mass of the fractal curve, as defined in recent work. The geometry of the fractal curve, hence plays an important role in this analysis. A Langevin equation with a particular noise model is thus proposed and solved using techniques of the newly developed $F^{\alpha}$-Calculus.

1 Introduction

Diffusion in disordered media is a topic of immense interest [1, 13]. There has been a lot of activity in this area recently. Anomalous diffusion has been studied e.g in [3, 4, 5, 6, 7]. In many of these approaches anomalous diffusion originates due to nonlocality in space or time.

Fractional Langevin equations have been used to describe FBM in several references. In [8] a theoretical method for subdiffusion based on generalized Langevin equation with fractional Gaussian noise is discussed. In [9] anomalous transport process is described by a Langevin equation with Levy noise and corresponding Fokker-Planck equation containing fractional derivative in space is discussed. In [10] subdiffusion is established on the basis of an extension of conventional Langevin dynamics to include long-tailed trapping events. Different methods have been used to solve generalized Langevin equations [11, 12]. A Langevin equation for unbranched cracks in 3-D is proposed in [13].

Here we propose a Langevin equation for particles moving on media, which can be modelled as fractal curves. This equation is local in space and time, while noise is taken to be Levy distributed in general and special cases for gaussian and cauchy distributed noise are discussed. This Langevin equation on fractal curves involves generalization of velocity for motion of particle on a fractal curve and geometrical concepts for a fractal curve from the recently developed Calculus on fractal curves [15]. Newly defined fractal integrals and methods of solving these integrals is also borrowed from this work. Hence this is a new formulation for studying nonequilibrium phenomena or anomalous diffusion on disordered...
media which can be modelled as fractal curves. Solutions of this Langevin equation in presence of a regular Levy stable noise are discussed. We give a method of solving this Langevin equation, by using the $F^{\alpha}$-Calculus developed in [15].

2 Motion of a particle on a fractal curve

Let a fractal curve $F$ be embedded in 3-D space, so that, motion of a particle on such a fractal curve, can be described by Newton’s laws. For simplicity we actually consider the case of a fractal curve embedded in $R^2$ (e.g a von Koch curve). We consider a particle performing random motion on $F$, then such a motion in the continuum limit (in presence of a noise) can be described by a Langevin equation.

Consider the construction of a von-Koch like curve, carried out recursively, only to a finite stage say $n$. The motion on such a curve, which is made up of broken line segments, is then described by Newton’s law. The only component of force $f$ along these straight line segments is then relevant. The velocity can be quantified in terms of total distance travelled from the initial point along the curve. Since this distance scales locally according to the $\alpha^{th}$ power of Euclidean distance, a more appropriate quantity to define the velocity (in the limiting case as $n \to \infty$) would be the change in the values of $S^\alpha_F$, the mass accumulated upto a point on the curve (see the description below) with time as the particle moves on the fractal path. With this motivation we now define ‘$\alpha$-velocity’ of the particle as follows:

$$v^{(\alpha)}(t) = \lim_{t \to t'} \frac{S^\alpha_F(u(t)) - S^\alpha_F(u(t'))}{t - t'}$$

$$= \frac{d}{dt}S^\alpha_F(u(t))$$

where $S^\alpha_F(u(t))$ is the rise function as defined in [15], which gives the mass of the fractal curve $F$, covered upto a certain point on $F$ in time $t$. We use the notation $\theta$ to label a point on the fractal curve $F$ and $J(\theta) = S^\alpha_F(u)$ for the mass of the fractal accumulated upto point $\theta$ as in earlier chapters.

A similar construction of velocity was introduced in another context [10]. The crucial difference between that construction and the $v^{(\alpha)}(t)$ considered here is that, we consider the derivative of $S^\alpha_F$, rather than just $\alpha^{th}$ power of space variable itself.

We now propose the Langevin equation in the overdamped case as:

$$\frac{dJ(\theta(t))}{dt} = \eta(t)$$

where $\eta(t)$ is the noise in the system. The formal solution of the above
equation is given by:
\[ J(\theta(t)) = \int_0^t \eta(t') dt' \quad (2) \]

### 2.1 Model of noise

In this section we completely follow the noise and corresponding renormalization scheme introduced in [17]. Therefore, we give a brief summary of their model.

We consider a Levy distributed noise \( p(\eta) \) (e.g. as given in [9, 17]) of the form
\[ p(\eta) = \mu \eta_0^\mu \eta^{-1-\mu} \quad (3) \]
where \( \eta_0 \) is a lower cut off, introduced to ensure normalization of the distribution \( p(\eta) \).

Its Fourier transform is given by
\[ p(k) = \langle e^{-ik\eta} \rangle = \int d\eta \exp(-ik\eta)p(\eta) = \exp(-D\eta_0^\mu |k|^{\mu}) \quad (4) \]
where \( D \) is a dimensionless geometric factor, \( 0 < \mu < 2 \) is the scaling index.

In equation (1), \( \eta(t) \) is the instantly correlated Levy white noise at a particular instant of time. The microscopic steps \( \eta_i \) with distribution \( p(\eta_i) \) are discrete. The corresponding difference equation for Langevin equation can be written in terms of discrete time steps \( \Delta = t/n \), where \( n \) is the number of steps or divisions on the time axis.

The Langevin equation is thus discretized as
\[ \frac{J(\theta_{n+1}) - J(\theta_n)}{\Delta} = \eta_n \quad (5) \]
where \( J(\theta_n) = J(\theta(t_n)) \) and \( \eta_n = \eta(t_n) \).

Now, we choose the expression for noise as a stable Levy process of the form given in Fourier space as in [17]
\[ P(k,t) = \exp[-D\eta_0^\mu \Delta^{\mu-1} |k|^{\mu}t] \quad (6) \]
To keep the coefficient \( D \) fixed and eliminate time step \( \Delta \), the cut off \( \eta_0 \) has to be renormalized by, \( \eta_0^\mu \Delta^{\mu-1} = 1 \). Consider equation (5), and take appropriate as follows
\[ \frac{\langle J(\theta_{n+1}) - J(\theta_n) \rangle_\theta}{\Delta} = \langle \eta_n \rangle_\eta \quad (7) \]
Explicitly evaluating the moment \( \langle \eta_n \rangle_\eta \) by using the expression for \( p(\eta) \) above gives a finite value for \( 1 < \mu < 2 \), which is
\[ \frac{\langle J(\theta_{n+1}) - J(\theta_n) \rangle_\theta}{\Delta} = \text{const.} \eta_0 \quad (8) \]
Hence, we can see that as \( \Delta \to 0, \eta_0 \to \infty \) i.e. the cut-off moves to infinity, which hold for \( \eta_0^\mu \Delta^{\mu-1} = 1 \) in this range for \( \mu \). For \( 0 < \mu < 1 \), in order for the renormalization \( \eta_0^\mu \Delta^{\mu-1} = 1 \), to hold, \( \eta_0 \to 0 \) for \( \Delta \to 0 \). This renormalization is same as mentioned above in [17] for ordinary Langevin equation.

Further, we denote \( D\eta_0^\mu = D_1 \).
2.2 Solution of the Langevin equation

We define the function \( \delta_F^\alpha \) which is analogous to dirac delta function with respect to \( F^\alpha \)-integrals. Thus

\[
\delta_F^\alpha(\theta(t) - \theta(t')) = \delta(J(\theta(t)) - J(\theta(t')))
\]

The formal solution of the Langevin equation (11) is given by equation (2). The associated distribution can now be found as follows.

Let \( p(\theta, t) \) denote the probability distribution for a particle located at point \( \theta \) on \( F \) at time \( t \). Thus

\[
p(\theta_0, t) = \langle \delta_F^\alpha(\theta_0 - \theta(t)) \rangle _{\theta} \tag{9}
\]

Now the Fourier Transform as defined in appendix is given by

\[
\hat{f}(\psi) = \int_{C(-\infty, \infty)} e^{-iJ(\psi)J(\theta)} f(\theta) d_F^\alpha \theta 
\]

and the inverse Fourier Transform as

\[
f(\theta) = \frac{1}{2\pi} \int_{C(-\infty, \infty)} e^{iJ(\psi)J(\theta)} \hat{f}(\psi) d_F^\alpha \psi \tag{11}
\]

then, \( \delta_F^\alpha \) can be defined in terms of \( F^\alpha \) integral (this can be obtained easily by applying conjugacy (15) to the \( \delta \)-function in ordinary case) as:

\[
\delta_F^\alpha(\theta) = \frac{1}{2\pi} \int_{C(-\infty, \infty)} \exp(iJ(\psi) J(\theta)) d_F^\alpha \psi 
\]

thus equation (9) becomes

\[
p(\theta, t) = \frac{1}{2\pi} \int d_F^\alpha \psi \exp(iJ(\psi)[J(\theta) - J(\theta'(t))]) \hat{p}(\psi, t) \tag{12}
\]

or

\[
p(\theta, t) = \int d_F^\alpha \psi p(\theta') \left( \frac{1}{2\pi} \int d_F^\alpha \psi \exp(iJ(\psi)[J(\theta) - J(\theta'(t))]) \right)
\]

The integrals can be interchanged and

\[
p(\theta, t) = \frac{1}{2\pi} \left[ \int d_F^\alpha \psi \int d_F^\alpha \theta' p(\theta') \exp(iJ(\psi) J(\theta) \exp(-iJ(\psi) J(\theta'(t))) \right]
\]

or

\[
p(\theta, t) = \frac{1}{2\pi} \int d_F^\alpha \psi \exp(iJ(\psi) J(\theta) \exp(-iJ(\psi) J(\theta'(t)))) \hat{p}(\psi, t) \tag{13}
\]

Hence we can see that

\[
p(\theta, t) = \frac{1}{2\pi} \int d_F^\alpha \psi \exp(iJ(\psi) J(\theta)) \hat{p}(\psi, t)
\]
Thus, taking the inverse Fourier transform of the above,

\[ \tilde{p}(\psi, t) = \int d^D \theta \exp(-iJ(\psi)J(\theta))p(\theta, t) \]

or

\[ \tilde{p}(\psi, t) = (\exp(-iJ(\psi)J(\theta)))_\theta \]

Substituting the value of \( J(\theta) \) from equation (2)

\[ \tilde{p}(\psi, t) = \langle \exp(-iJ(\psi)\int_0^t \eta(t')dt') \rangle_\eta \quad (14) \]

Discretizing the integral in the above equation

\[ \tilde{p}(\psi, t) = \prod_{n=0}^{N} \langle \exp[-iJ(\psi)\eta(t_n)\Delta] \rangle \text{ where } t_n = n\Delta \quad (15) \]

such that the interval [0, t] is divided into \( N \) equal parts and \( \Delta = (t - 0)/N \).

We now assume a model of noise \( \eta(t) \) which is given by (4).

Comparing equation (15) with the noise model (4) above we can write

\[ \tilde{p}(\psi, t) = \prod_{n=0}^{N} \exp[-D_1|J(\psi)\Delta|^\mu] \quad (16) \]

Using the renormalization \( D_1 \Delta^{\mu-1} \rightarrow D \) and reintroducing the integral in the above equation:

\[ \tilde{p}(\psi, t) = \exp[-D|J(\psi)|^\mu \int_0^t dt'] \]

\[ \tilde{p}(\psi, t) = \exp[-(D|J(\psi) = S^2(k)|^\mu t)] \quad (17) \]

For \( \mu = 2 \) we get

\[ \tilde{p}(\psi, t) = \exp[-(DS_0^2(k)^2t)] \quad (18) \]

The Fourier transform for the above equation gives

\[ p(\theta, t) = \frac{1}{\sqrt{2\pi Dt}} \exp\left(-\frac{S_0^2(w)^2}{2Dt}\right) \quad (19) \]

We see that the equation (19) is the same as that obtained for the solution of diffusion equation on fractal curve as discussed in [18].

3 Results

In this paper we have proposed a Langevin equation for random motion of a particle on a fractal curve. The Langevin equation we propose is local in space and time, while noise is taken to be Levy distributed. Special case for
gaussian distributed noise is discussed. This Langevin equation on fractal curves involves defining velocities and geometrical concepts for a fractal curve from the Calculus on fractal curves developed in [15]. Our consideration demonstrates that the framework we have introduced is suitable for studying nonequilibrium phenomena or anomalous diffusion on fractally disordered media, which can be modelled as fractal curves (such as backbone of a percolating cluster, polymer chain etc.)

For gaussian distributed white noise we have shown that the probability distribution of the space variable obtained from the Langevin equation is the same as that obtained by solving a Fokker-Planck equation on a fractal curve, as done in [18]. Hence the two approaches are equivalent as in ordinary space, for this special case.

Appendix 1

Review of Calculus on Fractal Curves

For a set $F$ and a subdivision $P_{[a,b]}, a < b, [a, b] \subset [a_0, b_0]$ let $w : [a, b] \to F$, then we define the mass function as follows:

$$
\gamma^\alpha(F, a, b) = \lim_{\delta \to 0} \inf_{P_{[a,b]}:|P| \leq \delta} \sum_{i=0}^{n-1} \frac{|w(t_{i+1}) - w(t_i)|^\alpha}{\Gamma(\alpha + 1)}
$$

(20)

where $\cdot$ denotes the euclidean norm on $\mathbb{R}^n$, $1 \leq \alpha \leq n$ and $P_{[a,b]} = \{a = t_0, \ldots, t_n = b\}$.

The staircase function, which gives the mass of the curve upto a certain point on the fractal curve $F$ is defined as

$$
S^\alpha_F(u) = \begin{cases} 
\gamma^\alpha(F, p_0, u) & u \geq p_0 \\
-\gamma^\alpha(F, u, p_0) & u < p_0 
\end{cases}
$$

(21)

where $u \in [a_0, b_0]$.

A point on the curve $w(u) \equiv \theta$ and $S^\alpha_F(u) \equiv J(\theta)$. The $F^\alpha$ derivative is defined as:

$$
(D^\alpha_F f)(\theta) = F \lim_{\theta' \to \theta} \frac{f(\theta') - f(\theta)}{J(\theta') - J(\theta)}
$$

(22)

The $F^\alpha$-integral is also defined and is denoted by

$$
\int_{C(a,b)} f(\theta) d^\alpha_F \theta
$$

(23)

Appendix 2

The Fourier Transform
From the definition of conjugacy (15)

\[ \phi[f](S_\alpha^\phi(u)) = f(w(u)) \]  

(24)

The Fourier Transform on the real line, for a function \( g(v) \), is defined by

\[ g(v) = \int_{-\infty}^{\infty} \tilde{g}(y) \exp(-ivy)dy \]  

(25)

and the inverse Fourier Transform is

\[ \tilde{g}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(v) \exp(ivy)dy \]  

(26)

In the case of a parametrizable fractal curve \( F \), which is obtained by a fractalizing transformation on an interval of the real line, we propose that the Fourier space can also be obtained by the same fractalizing transformation on the interval of a real line. The interval may be \((-\infty, \infty)\).

Let \( \tilde{g} = \phi[\tilde{f}] \) and \( g = \phi[f] \) also \( v = S_\alpha^\phi(k) \) and \( y = S_\alpha^\phi(u) \).

We use the notation \( J(\theta) = S_\alpha^\phi(u) \) and \( J(\psi) = S_\alpha^\phi(k) \), where \( \theta = w(u) \) and \( \psi = w(k) \).

Then taking Fourier transform of the LHS of equation (24) one can write

\[ \tilde{\phi}[f](v = J(\psi)) = \int_{-\infty}^{\infty} \phi[f](y = J(\theta)) \exp(-iyv)dy \]

\[ = \int_{C(-\infty, \infty)} f(\theta) \exp(-iJ(\theta)v) d_\alpha^\phi \theta \]  

(27)

where

\[ C(-\infty, \infty) = \lim_{a \to -\infty, b \to \infty} C(a, b) \]

and

\[ \phi[\tilde{f}](v = J(\psi)) = \phi[\int_{-\infty}^{\infty} f(y = J(\theta)) \exp(-iyv)dy] \]

\[ = \int_{C(-\infty, \infty)} f(\theta) \exp(-iJ(\theta)v) d_\alpha^\phi \theta \]  

(28)

Comparing equations (27) and (28), we can write

\[ \tilde{\phi}[f] = \phi[\tilde{f}] \]

Also one can define the action of \( \phi \) in Fourier space as

\[ \phi[\tilde{f}](v = J(\psi)) = \tilde{f}(\psi) \]  

(29)

Now using conjugacy one can rewrite equation (27) as

\[ \tilde{f}(\psi) = \int_{C(-\infty, \infty)} f(\theta) \exp(-iJ(\theta)J(\psi)) d_\alpha^\phi \theta \]  

(30)
Similarly, inverse transform of the above can be obtained from equation (26), which can be written as

\[
f(\theta) = \frac{1}{2\pi} \int_{C(-\infty,\infty)} \tilde{f}(\psi) \exp(iJ(\theta)J(\psi))d\psi
\]

(31)

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