ON THE SPECTRAL PROPERTIES OF THE LANDAU HAMILTONIAN PERTURBED BY A MODERATELY DECAYING MAGNETIC FIELD

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ABSTRACT. The Landau Hamiltonian, describing the behavior of a quantum particle in dimension 2 in a constant magnetic field, is perturbed by a magnetic field with power-like decay at infinity and a similar electric potential. We describe how the spectral subspaces change and how the Landau levels split under this perturbation.

1. Introduction

The paper is devoted to the study of the spectrum of the Schrödinger and Pauli operators in the plane, with nonzero constant magnetic field perturbed by a smooth magnetic and electric fields that decay power-like at infinity. It continues the paper [15] by the authors, where the case of a compactly supported perturbation was considered.

The Landau Hamiltonian describing the motion of the quantum particle in two dimensions under the influence of the constant magnetic field is one of the classical models in quantum physics. The spectrum (found first in [4], see also [9]) consists of eigenvalues with infinite multiplicity lying at the points of an arithmetic progression. These eigenvalues are traditionally called Landau levels (LL) and the corresponding spectral subspaces are called Landau subspaces.

A natural question arises, what happens with the spectrum of the Landau Hamiltonian under the perturbation by a weak electrostatic potential or/and magnetic field. One should expect that the Landau levels split, and the problem consists in describing quantitatively this splitting as well as in studying the behavior of the spectral subspaces under the perturbation.

The case of the perturbation by an electric potential $V$, moderately, power-like decaying at infinity, was first studied by Raikov in [11]. The case of a fast decaying (or compactly supported) electric potential was dealt with much later, in [13], see also [10]. It was found that the character of the Landau levels splitting depends essentially on the rate of decay of $V$ and the asymptotics of the eigenvalues in clusters can be expressed in the terms of $V$, quasi-classically or not.
When the magnetic field is also perturbed, the situation becomes more complicated since is not the magnetic field itself but its potential that enters in the quantum Hamiltonian. So, the perturbation of the operator turns out to be fairly strong even for a compactly supported perturbation of the field and it may even be not relatively compact if the perturbation goes to zero at infinity not sufficiently fast. Iwatsuka [6] proved that the invariance of the essential spectrum still takes place, so Landau levels are the only possible limit points of eigenvalues lying in the gaps between them. Further on, in [3], [7], [8], [12] the character of the splitting of the lowest Landau level was investigated. For compactly supported magnetic field perturbation and electric potential the splitting of all Landau levels was studied in [15]. The main result of [15] was the description of the spectral subspaces of the perturbed operator corresponding to the clusters around Landau levels. It was found that these subspaces change fairly strongly, and a rather exact approximation for these subspaces was found in the terms of modified creation and annihilation operators. At the same time, although the perturbation of the operator may be very strong, the splitting of eigenvalues is super-exponentially weak, just like it is in the case of a perturbation by a compactly supported electric field only.

In the present paper we continue the study of the Landau Hamiltonian with a perturbed magnetic field. Now we consider the case of the perturbation decaying moderately, power-like, at infinity. For the spectral subspaces the results similar to the ones in [15] hold. As for the asymptotics of the eigenvalues in clusters, we obtain more complete results, proving estimates and, under some natural conditions, the power-like asymptotic formulas for eigenvalues. The problem of the splitting of Landau levels under moderately decaying perturbations of the magnetic field has been considered in [5], see Theorem 11.3.17 there, where even the remainder term of the asymptotics of the eigenvalues in clusters was found. Powerful methods of microlocal analysis are used in [5]. Our methods are more elementary, give the approximation for the spectral subspaces, moreover, we do not need the rather restrictive 'hyperbolicity' condition (11.3.49) imposed in [5].

The paper is heavily based upon the methods and results of the papers [7] and [15]. Following the structure of the latter paper, we refer rather shortly to the fragments that should be repeated, word for word or with minor changes only, in order to be able to concentrate on important differences. For convenience of references, the numbering of sections here coincides with that in [15].

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2. Magnetic Schrödinger and Pauli operators

2.1. The unperturbed operators. We will denote the points in the plane \( \mathbb{R}^2 \) by \( x = (x_1, x_2) \); it is convenient to identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) by setting \( z = x_1 + ix_2 \). So, the Hilbert space \( L_2(\mathbb{R}^2) \) with Lebesgue measure (which will be denoted by \( dx \)) is identified with \( L_2(\mathbb{C}) \). The derivatives are denoted by by \( \partial_k = \partial_{x_k} \) and we set, as usual, \( \partial = (\partial_1 + i\partial_2)/2, \bar{\partial} = (\partial_1 - i\partial_2)/2. \)

The constant magnetic field is denoted by \( B^o > 0 \). The corresponding magnetic potential is \( A^o(x) = (A^o_1, A^o_2) = \frac{B^o}{2}(-x_2, x_1) \). Then the (unperturbed) magnetic Schrödinger operator in \( L_2(\mathbb{R}^2) \) is

\[
H^o = -(\nabla + iA^o)^2. \tag{2.1}
\]

The Pauli operator describing the motion of a spin-\( \frac{1}{2} \) particle acts in the space of two-component vector functions, it has the diagonal form, \( P^o = \text{diag}(P^o_+, P^o_-) \), where \( P^o_\pm = H^o \pm B^o \).

The spectrum of the Schrödinger operator is described by the classi-
cal construction originating in [4]. For the complex magnetic potential \( A^c = A^o_1 + iA^o_2 \) the creation and annihilation operators are introduced,

\[
\bar{Q}^c = -2i\partial - \overline{A^o}, \quad Q^c = -2i\bar{\partial} - A^o. \tag{2.2}
\]

These operators can be also expressed by means of the scalar potential, the function \( \Psi^o(z) = \frac{B^o}{4\pi}|z|^2 \), solving the equation \( \Delta \Psi^o = B^o \):

\[
Q^o = -2ie^{-\Psi^o}\partial e^{\Psi^o}, \quad \bar{Q}^c = -2ie^{\Psi^o}\bar{\partial}e^{-\Psi^o}.
\]

The operators \( \bar{Q}^c, Q^c \) satisfy the following basic relations

\[
[Q^c, \bar{Q}^c] = 2B^o. \tag{2.3}
\]

\[
P^c_\pm = Q^c\bar{Q}^c, \quad P^o_\pm = \overline{Q^c}Q^c, \quad H^c = Q^c\bar{Q}^c - B^o = \overline{Q^c}Q^c + B^o. \tag{2.4}
\]

The spectrum of \( H^o \) is described in the following way. The equation \( P^c_- u = 0, u \in L_2 \) is equivalent to \( Q^o u = e^{-\Psi^o}\bar{\partial}(e^{\Psi^o} u) = 0 \).

This means that \( f = e^{\Psi^o} u \) is an entire analytical function in \( \mathbb{C} \), such that after being multiplied by \( e^{-\Psi^o} \) it belongs to \( L_2 \). The space of such functions \( f \) is called Fock or Segal-Bargmann space \( \mathcal{F} \).

So, the null subspace of the operator \( P^o_- \), i.e., its spectral subspace corresponding to the eigenvalue \( \Lambda_0 = 0 \), is \( \mathcal{L}_0 = e^{-\Psi^o} \mathcal{F} \). After this, by the commutation relations (2.3), (2.4), \( \mathcal{L}_q = \overline{Q^c}\mathcal{L}_0 \) are the spectral subspaces of \( P^o_- \) with eigenvalues \( \Lambda_q = 2qB^o \), \( q = 0, 1, \ldots, \) called Landau levels, and the spectra of \( H^c, P^o_\pm \) consist, respectively, of \( \Lambda_q + B^o \) and \( \Lambda_q + 2B^o \). The operators \( \bar{Q}^c, Q^c \) act between Landau subspaces

\[
\bar{Q}^c : \mathcal{L}_q \mapsto \mathcal{L}_{q+1}, \quad Q^c : \mathcal{L}_q \mapsto \mathcal{L}_{q-1}, \quad Q^o : \mathcal{L}_0 \mapsto 0, \tag{2.5}
\]

and are, up to constant factors, isometries of Landau subspaces.
The spectral projection $P^q_0$ of $P_0$ corresponding to the eigenvalue $\Lambda_q = 2qB^0$ can be thus expressed as

$$P^q_0 = C_q^{-1} \overline{Q^q_0} P^q_0 Q^q_0, \quad C_q = q!(2B^0)^q, \quad q = 0, 1, \ldots$$

(2.6)

2.2. The perturbed operator. We introduce the convenient class of functions. A function $F(x)$ is said to belong to the class $S_{\beta}$, $\beta < 0$, if

$$F(x) = O(|x|^\beta), \quad \partial_x^k F(x) = O(|x|^\beta - \delta), \quad |x| \to \infty, \quad k = 1, 2, \ldots$$

(2.7)

for some $\delta \in (0, -\beta)$. The particular value of $\delta$ is irrelevant. So, if $F \in S_{\beta}$ then $\partial_j F$ can be considered as a function in $S_{\beta - \frac{\delta}{2}}$ (with $\frac{\delta}{2}$ acting as $\delta$).

Now we introduce the perturbation $b \in C^\infty(\mathbb{R}^2)$ of the magnetic field and set $B = B^0 + b$. We suppose that

$$b \in S_{\beta} \text{ for some } \beta < -2.$$  

(2.8)

Under the condition (2.8), the upper estimate for the counting function of the eigenvalues in the clusters will be proved, as well as the approximate representation of the spectral subspaces, following [15]. For obtaining asymptotic formulas, additional conditions will be imposed on $b$ and eventually on the electric perturbation.

Let $\psi$ be a scalar potential for the field $b$, a solution of the equation $\Delta \psi = b$. Of course, $\psi$ is defined up to a harmonic summand, the choice of $\psi$ corresponds to the choice of gauge. The magnetic potential $a = (a_1, a_2) = (-\partial_2 \psi, \partial_1 \psi)$, $\text{curl } a = b$, is thus determined up to a gradient, and the complete scalar and vector magnetic potentials are

$$\Psi = \Psi^0 + \psi, \quad A = A^0 + a.$$  

(2.9)

We define the perturbed magnetic Schrödinger operator as in (2.1), with $A^0$ replaced by $A$: $H = -(\nabla + iA)^2$, and the components of Pauli operator as $P_\pm = H \pm B$. It is easy to observe that the difference between $H$ and $H^0$ contains an operator of multiplication by $A^0 \cdot a$, the latter function does not decay at infinity, and thus looks like being not a relatively compact perturbation of $H^0$. However, thanks to the special form of this term, the perturbation is still relatively compact if $a \to 0$ at infinity, as was noticed by Besch [1]. In our case, under the conditions imposed on $b$, the scalar potential grows at most logarithmically, and the vector potential $a$ decays as $|x|^{-1}$ at infinity.

Now let us look at the algebraic structure related with the perturbed operators. The perturbed creation and annihilation operators are defined similarly to (2.2): $Q = -2i\partial - \overline{A}$, $\overline{Q} = -2i\overline{\partial} - A$, where $A$ is the complex magnetic potential, $A = A^0 + (a_1 + ia_2)$. The commutation relations for $Q, \overline{Q}$ have the form

$$[Q, \overline{Q}] = 2B = 2B^0 + 2b.$$  

(2.10)
and $P_\pm$ and $H$ satisfy
\begin{align}
P_+ &= Q\overline{Q}, \quad P_- = \overline{Q}Q, \quad P_+ - P_- = 2B = 2B^* + 2b, \\
H &= Q\overline{Q} - B = \overline{Q}Q + B.
\end{align}
(2.11)
(2.12)

The relations (2.10)-(2.12) contain variable functions on the right-hand side and the spectra of Schrödinger and Pauli operators do not determine each other any more. The only information that one can obtain immediately, is the description of the lowest point of the spectrum of $P_-$. Since, again, $Q = Q^*$, the equation $P_- u = 0$ is equivalent to $Qu = 0$, or $\bar{\partial}(\exp(\Psi)u) = 0$. So the function $f = u \exp \Psi$ is an entire analytical function such that $u = \exp(-\Psi)f \in L^2$. The space of entire functions with this property is, obviously, infinite-dimensional, it contains at least all polynomials in $z$ variable, although it does not necessarily coincide with the Fock space. We denote the null-space of $P_-$, the space of zero modes, by $H_0$. It is infinite-dimensional; complex polynomials times $\exp(-\Psi)$ form a dense set in $H_0$. The lowest Landau level $\Lambda_0$ is an isolated point in the spectrum of $P_-$. As it follows from the relative compactness of the perturbation, by Weyl’s theorem, the essential spectrum of the perturbed operator $P_-$ consists of the same Landau levels $\Lambda_q$, and the eigenvalues in the gaps may only have $\Lambda_q$ as their limit points. This latter fact was established much earlier by Iwatsuka [6], and the constructions in [15] can be considered as the extension of the approach in [6].

Now we add a perturbation by the electric potential. Let $V(x)$ be a real valued function in $S_\beta$, $\beta < -2$. We introduce the operators
\begin{equation}
H(V) = H + V, \quad P_\pm(V) = P_\pm + V.
\end{equation}
(2.13)

Since the operator of multiplication by $V$ is relatively compact with respect to $H, P_\pm$, the operators (2.13) have the same essential spectra as the respective unperturbed ones ($\Lambda_0$ ceases to be an isolated point of the spectrum of $P_-(V)$). In this paper we are going to study the distribution of the eigenvalues of $H(V), P_\pm(V)$ near $\Lambda_q$.

2.3. Some resolvent and commutator estimates. In the course of our proof we will need some boundedness property of the resolvent of the operators $P_+, P_-, H$ and their spectral projections. These properties, in a slightly less general form, were established in [7], Lemma 1.4.

We denote by $R_{\pm}(z)$ the resolvent of the operators $P_\pm$ and by $\Pi_j$, $j = 1, 2$, the operators $\Pi_j = i\partial_j + (A_j + a_j)$.

Proposition 2.1. Suppose that $V, b$ belong to $S_\nu$, $\nu < -2$. Let $P$ be the spectral projection of the operator $P_\pm$ or $H$ corresponding to some bounded isolated piece of the spectrum. Then for any real $l$ the following operators are bounded: $\langle x \rangle^{-\nu + \delta - l}[V, P]\langle x \rangle^l, \langle x \rangle^{-\nu + \delta - l}[b, P]\langle x \rangle^l, \langle x \rangle^{-l}R_\pm\langle x \rangle^l, \langle x \rangle^{-l}\Pi_jR_\pm\langle x \rangle^l$. 
The proof in [7] is given for the class $S_\nu$ with $\delta = 1$, for the projection $P$ corresponding to the lowest Landau level and for positive $l$ only. In our formulation, the proposition is proved in an analogous way.

Next we establish the estimates for eigenvalues and singular numbers of some compact operators. The estimates will be needed further on, in the process of proving the required eigenvalue asymptotics.

For a compact operator $T$ we, as usual, denote by $n(\lambda, T)$ the distribution function of the singular numbers (s-numbers) of $T$, i.e., the quantity of s-numbers of $T$ that are bigger than $\lambda$. If the operator is self-adjoint, the distribution functions for the positive and negative eigenvalues of $T$ are denoted by $n_\pm(\lambda, T)$. The operator $T$ can be dropped from the notation if this does not cause misunderstanding.

**Proposition 2.2.** Let $V$ be a function in $S_\nu, \nu < 0$. Consider the operator $X = X_N(V) = VP_+^{-N}$. Then for $N$ sufficiently large,

$$n(\lambda, X) = O(\lambda^{\frac{5}{2}}), \quad \lambda \to 0. \quad (2.14)$$

Moreover, if $K$ is a compact operator then

$$n(\lambda, KX), \quad n(\lambda, XK) = o(\lambda^{\frac{5}{2}}), \quad \lambda \to 0. \quad (2.15)$$

Compared with Proposition 5.1 below, the above estimate shows that for $N$ large enough, the operator $X_N(V)$ admits the same spectral estimates as the Toeplitz type operator, with $P_+^{-N}$ replaced by the spectral projection of $P_\pm$.

**Proof.** Consider a very special case first. Let $V \in S_\nu, \nu \in (-\frac{1}{2}, 0)$ and $N \in (-\nu, \frac{1}{4})$. We will obtain the s-numbers estimates of the operator $VP_+^{-N}$. These numbers are majorated by the s-numbers of the analogous operator with $V$ replaced by $V_\nu = \langle x \rangle^{\nu/2}$. The semigroup generated by $P_\nu - N$, by the diamagnetic inequality, is dominated by the semigroup generated by $(1 - \Delta)^{-\frac{N}{2}}$, therefore, by the results of [14] (see Theorems 1, 4 and Sect.5.4 there), these singular numbers are majorated by the s-numbers of the operator $T_\nu = V_\nu (1 - \Delta)^{-\frac{N}{2}}$. By the Fourier transform, this operator is unitary equivalent to $(1 - \Delta)^{\nu} \langle \xi \rangle^{-\frac{N}{2}}$, and for the latter operator the required eigenvalue estimate is given by Cwikel's theorem in [2].

For the general case, again for $V = V_\nu, \nu < 0$, we take $N$ so large that $-\nu/N < \frac{1}{4}$. We represent $V$ in the form $V = \tilde{V}^{N+1}, \quad \tilde{V} = V^{\frac{1}{N+1}}$. Now in the expression $\tilde{V}^{N+1}P_+^{-N}$ we leave one copy of $\tilde{V}$ in the first (utmost left) position and start moving the remaining copies to the right, commuting them with copies of $P_+^{-1}$ in such way that finally there will be only one entry of $\tilde{V}$ or its derivatives between two copies of $P_+^{-1}$. As a result, we arrive to a collection of summands, each being the product of operators of the form $WP_+^{-\frac{3}{2}}, \quad P_+^{-\frac{1}{2}}W$, and, possibly, some more bounded operators of the form considered in Proposition
2.1, where $W \in S_{\nu/N}$ is $\tilde{V}$ or some of its derivatives. For the operators $W \mathbf{P}_{\pm}^{-\frac{1}{2}}, \mathbf{P}_{\pm}^{-\frac{1}{2}}W$ we can apply the estimate found in the first part of the proof and obtain the required inequality using the Weyl inequality for the s-numbers of the product of operators. The second statement follows from the first one and, again, the Weyl inequality.

3. Approximate spectral subspaces

In this section, under the condition that $b$ satisfies (2.8), we construct the approximate spectral subspaces of the operators (2.13). This is done in the same way as in [15], so we just briefly describe the construction and pinpoint the main differences.

First of all, we consider the null subspace $H_0$. In [15] it is shown that for $b \in C^\infty_0, H_0$ possesses a dense subspace of rapidly decaying functions. The same reasoning proves this property for our case. It is here, that the condition $\beta < -2$ implying that $V, b \in L_1$ is essential.

Let $\delta_q = (\Lambda_q - \gamma, \Lambda_q + \gamma), q = 0, 1, 2, \ldots, \gamma < B^\circ$, be intervals of the same size on the real axis, centered at the Landau levels $\Lambda_q = 2qB^\circ$. We choose the size of $\delta_q$ in such way that neither of these intervals has the eigenvalues of $\mathbf{P}_-$ at its endpoints. Moreover, since the lowest LL $\Lambda_0 = 0$ is an isolated point of the spectrum of $\mathbf{P}_-$, we can choose the size of the intervals in such way that $\delta_0$ contains only this point of spectrum. We denote by $H_q$ the spectral subspace of $\mathbf{P}_-$ corresponding to the interval $\delta_q$ and by $P_q$ the corresponding spectral projection. Since, by [6], the spectrum of $\mathbf{P}_-$ is discrete between Landau levels, the change of $\delta_q$ leads only to a finite-rank perturbation of $P_q$. As usual, the spectral projection $P_q$ can be be expressed by means of the integration of the resolvent of $\mathbf{P}_-$ along a closed contour $\Gamma_q$ in the complex plane, not passing through the eigenvalues of $\mathbf{P}_-$ and containing inside only those eigenvalues that lie in $\delta_q$. Again, using the discreteness of the spectrum of $\mathbf{P}_-$ between the Landau levels, we can choose these contours so that they are obtained from $\Gamma_0$ by the shift along the real axis in the complex plane, $\Gamma_q = \Gamma_0 + 2qB^\circ$.

Now we are going to establish several properties of the subspaces $H_q$, projections $P_q$ and some related operators. First, note the simple fact following directly from the spectral theorem.

**Proposition 3.1.** For any $q = 0, 1, \ldots$, and any polynomial $p(\lambda)$ the operator $p(\mathbf{P}_-)P_q$ is bounded, moreover $(p(\mathbf{P}_-) - p(\Lambda_q))P_q$ is compact.

In fact, by the spectral theorem the nonzero spectrum of the operator $p(\mathbf{P}_-)P_q$ consists of the points $p(\lambda_j)$ where $\lambda_j$ are all points of spectrum of $\mathbf{P}_-$ in $\delta_q$ and thus all $p(\Lambda_j)$ live in a bounded interval. Moreover, $p(\Lambda_j)$ may only have $p(\Lambda_q)$ as their limit point by Iwatsuka’s theorem.

The following lemma will enable us later to prove a much stronger compactness property.
Lemma 3.2. Let each of $T_j$, $j = 1, \ldots, N$ be one of operators $Q$ or $\overline{Q}$. Then for some constants $C, C'$, for any $u$ in the domain of the operator $P_N^N$,
\[ \|T_1T_2 \ldots T_N u\| \leq C(P_N^N u, u) + C'\|u\|^2. \] (3.1)

The proof of Lemma 3.2 for the case $b \in C_b^\infty$ can be found in [15] Section 7, in our case the proof goes exactly in the same way.

Now we can establish the compactness property.

Proposition 3.3. Let $T_1, \ldots, T_N$ be a collection of operators, each being $Q$ or $\overline{Q}$, and let $h_j$, $j = 0, \ldots, N$ be functions with all derivatives bounded, $T = h_0 T_1 h_1 \ldots T_N h_N$. Then for any $q$ and for any polynomial $p(\lambda)$ the operators $T (p(P_-) - p(\Lambda_q)) P_q T$ are compact.

Proof. By commuting functions $h_j$ and operators $T_j$ (moving all functions to the left), we transform the left operator $T$ to the sum of terms of the form $\tilde{h}_\kappa T_\kappa$ where $\tilde{h}_\kappa$ are bounded functions and $T_\kappa$ is a product of no more than $N$ operators $\overline{Q}, Q$. Similarly, in $T$ that stands to the right of $P_q$, we move all functions to the utmost right positions, to get the representation of $T$ as a sum of terms $T_\kappa \tilde{h}_\kappa$, being the product of a bounded smooth function $\tilde{h}_\kappa$ and no more than $N$ operators $\overline{Q}, Q$.

For each of the terms arising in this way in the decomposition of $T (p(P_-) - p(\Lambda_q)) P_q T$, we can write
\[ \tilde{h}_\kappa T_\kappa (p(P_-) - p(\Lambda_q)) P_q T_\kappa \tilde{h}_\kappa = [\tilde{h}_\kappa T_\kappa (P_\kappa^N + 1)^{-1}] \times [(P_\kappa^N + 1)(p(P_-) - p(\Lambda_q)) P_q (P_\kappa^N + 1)] \times [(P_\kappa^N + 1)^{-1} T_\kappa \tilde{h}_\kappa]. \] (3.2)

In (3.2), the first factor in brackets is bounded by Lemma 3.2, and the middle factor is compact by Proposition 3.1. The last factor in brackets is also bounded, by Lemma 3.2 applied to the adjoint operator. \[ \square \]

Now we describe the main construction of the paper, the approximate spectral subspaces of the perturbed operator. It is sufficient to consider the operator $P_- (V)$. In fact, by (2.10), (2.11), (2.12),
\[ H(V) = P_-(V + b) + B^e, P_+(V) = P_-(V + 2b) + 2B^e, \] (3.3)
and thus these operators differ from $P_- (V)$ by a shift and by the electric type perturbations $b, 2b$. We find the approximate spectral subspaces of $P_-$. Adding an electric perturbation will then be an easier task.

The subspaces approximating $H_q$ will be defined as:
\[ G_0 = H_0, G_q = \overline{Q}^q G_0, q = 1, 2, \ldots. \] (3.4)

So we mimic the construction of the eigenspaces of the unperturbed Landau Hamiltonian, see (2.5), in the same way as it was done in [15], by applying the creation operators to the space of zero modes.

Of course, since we apply the unbounded operator $\overline{Q}$, we must show that we never leave the space $L_2$, and moreover, that the subspaces $G_q$ are closed. Both these properties, as well as some other results will be
based upon the important Proposition 3.4 (an analogy of Proposition 3.4 in [15]).

The essential difference is that now, when the perturbation does not have compact support, we have to trace the rate of decay of different terms arising in the process of transformations. This analysis enables us to single out the leading terms in the resulting expansions.

**Proposition 3.4.** Let $q > 0$.

1. There exists a function $Z_q[b] \in S_\beta$ depending only on $q$, $B^\circ$, and $b$ such that for any $u \in H_0$,

\[ \|\overline{Q}^q u\|^2 = C_q^2 \|u\|^2 + (Z_q[b]u, u), \quad C_q = q!(2B^\circ)^q. \]  

(3.5)

The function $Z_q[b]$ is a polynomial in $b$ and its derivatives up to the order $2q - 2$ with coefficients depending on $B^\circ$. The term linear in $b$ and not containing derivatives equals $C'_q B^\circ q - 1 b$, $C'_q = 2^q q^q q!$. Moreover,

\[ Z_q[b] - C'_q B^\circ q - 1 b = O(|x|^{\beta - \delta}) \]  

(3.6)

at infinity.

2. Let $U(x)$ be a function in $S_\beta$, $\beta < -2$. There exists a function $\mathcal{X}_q[b, U] \in S_\beta$ depending only on $q$, $B^\circ$, $b$, and $U$ such that for any $u \in H_0$,

\[ (U \overline{Q}^q u, \overline{Q}^q u) = (\mathcal{X}_q[b, U]u, u). \]  

(3.7)

The function $\mathcal{X}_q[b, U]$ is expressible as an order $2q$ linear differential operator acting on $U$, with coefficients depending polynomially on $b$, its derivatives, and $B^\circ$, moreover,

\[ \mathcal{X}_q[b, U] - C'_q B^\circ q U = O(|x|^{\beta - \delta}) \]  

(3.8)

at infinity.

**Proof.** The combinatorial part of the proof, consisting of multiple commuting of the creation and annihilation operators with functions and with each other is exactly the same as in [15]. What remains to be checked are the estimates (3.6) and (3.8). These estimates follow from the fact that all terms in the expressions $Z_q[b]$, $\mathcal{X}_q[b, U]$, except the leading ones, contain either derivatives of $b$ or $U$, or products of these functions, and therefore decay at infinity not slower than $|x|^{\beta - \delta}$. □

When applying Proposition 3.4 and similar results, we need a certain compactness property. Such facts were used persistently in [13], [10], but for the case of a constant magnetic field only.

**Lemma 3.5.** Let $W(x)$ be a function in $S_\nu$, $\nu < 0$. Let $\mathcal{L}$ be an arbitrary differential operator having the form

\[ \mathcal{L} = f_1 T_1 f_2 T_2 \ldots T_m, \]  

(3.9)
where each of $T_j$ is one of operators $Q, \overline{Q}$ and $f_j$ are functions, with all derivatives bounded. Then the quadratic form

$$w[u] = \int W(x)|\mathcal{L}u|^2dx$$  \hspace{1cm} (3.10)

is compact in the space $\mathcal{H}_0$.

Proof. Let $u = e^{-\psi}h$ be a function in $\mathcal{H}_0$, so $h(z)$ is an analytical function. We write the quadratic form (3.10) as

$$w[u] = (W\mathcal{L}u, \mathcal{L}u). \hspace{1cm} (3.11)$$

Now we move all the operators $\overline{Q}$ in $\mathcal{L}$ from the second factor in (3.11) to the first one and from the first factor to the second one; thus they turn into $Q$. In the process of commuting these $\overline{Q}$ with $Q$ and the functions $W$ and $f_j$, some derivatives of these functions appear; the function $W$ goes to zero at infinity, together with derivatives, the derivatives of $f_j$ are bounded. Then, by means of commuting the operators $Q$ with the functions, we move all entries of $Q$ in the first and in the second factors in (3.11) to utmost right position, where they vanish since $Qu = 0$ for $u \in \mathcal{H}_0$. The only remaining term in the form $w[u]$ will be

$$w[u] = (W_1u, u), \hspace{1cm} (3.12)$$

where $W_1$ is a function (a combination of $W, f_j, b$ and their derivatives) tending to zero at infinity. Now take $\varepsilon > 0$ and represent $W_1$ as $W_1 = W_{1,\varepsilon} + W_{1,\varepsilon}'$ so that $|W_{1,\varepsilon}'| < \varepsilon$ and $W_{1,\varepsilon}$ has compact support. For $(W_{1,\varepsilon}u, u)$, we have the estimate by $\varepsilon ||u||^2$, so the corresponding operator has norm not greater than $\varepsilon$. For $(W_{1,\varepsilon}u, u)$, we take some $R$ such that the support of the function $W_{1,\varepsilon}$ lies inside the circle $C_R$ with radius $R$ centered in the origin. For each $r \in (R, 2R)$ we write the Cauchy representation for an analytical function $h(z)$:

$$h(z) = (2\pi i)^{-1} \int_{C_r} h(\zeta)(z - \zeta)^{-1}d\zeta. \hspace{1cm} (3.13)$$

for some fixed function $\xi(r) \in C^\infty_0(R, 2R)$, $\int \xi(r)dr = 1$, we multiply (3.13) by $\xi(r)$ and integrate in $r$ from $R$ to $2R$. This gives the integral representation of $h(x)$, $|x| < R$, in the form $h(x) = \int_{R < |y| < 2R} K(x, y)h(y)dy$, with smooth bounded kernel $K(x, y)$. After applying $\mathcal{L}$ in $x$ variable, we obtain the representation for $\mathcal{L}u = \mathcal{L}(e^{-\psi}h)$:

$$\mathcal{L}u(x) = \int_{R < |y| < 2R} e^{-\psi(x)}K(x, y)e^{\psi(y)}u(y)dy = (K\mathcal{L}u)(x).$$

The integral operator $|W_{1,\varepsilon}|^{1/2}K\mathcal{L}$ has a bounded kernel with compact support and therefore is compact in $L_2$, and thus the quadratic form $w[u]$ can be written as $w[u] = (\text{sign} W_{1,\varepsilon}|W_{1,\varepsilon}|^{1/2}K\mathcal{L}u, |W_{1,\varepsilon}|^{1/2}K\mathcal{L}u)$ and therefore it is compact. Now we see that the quadratic form (3.10)
can be for any \( \varepsilon \) represented as the sum of a compact form and a form with norm less than \( \varepsilon \), and this proves the required compactness. \( \square \)

Now we are able to justify our construction of the spaces \( G_q \).

**Proposition 3.6.** The sets \( G_q \) defined in (3.4) are closed subspaces in \( L_2 \).

**Proof.** The fact that \( G_q \subset L_2 \) follows directly from Proposition 3.4. Next, the relation (3.5) can be written as

\[
(P_0Q^q \overline{Q}^q u, u) = C_q(u, u) + (P_0Z_q[b]u, u); \quad u = P_0u \in \mathcal{H}_0. \tag{3.14}
\]

In the second term in (3.14), by Lemma 3.5, the operator \( P_0Z_q[b] \) is compact in \( \mathcal{H}_0 \), and therefore we can understand (3.14) as showing that the operator \( C_q^{-1}P_0Q^q \) is a left parametrix for \( Q^q : \mathcal{H}_0 \to L_2 \). This implies that the range of \( Q^q \) is closed. \( \square \)

The null space of \( Q \) and therefore of \( Q^q \) is zero. Consider the operator \( Q^q \) as acting from \( G_0 = \mathcal{H}_0 \) to \( G_q \). This is a bounded invertible operator, therefore the inverse, that we denote by \( Q^{-q} \), is a bounded operator from \( G_q \) to \( G_0 \). It is a compact perturbation of \( Q^q \).

### 4. Approximate Spectral Projections

In this section we prove that the subspaces \( G_q \) are very good approximations to the spectral subspaces \( H_q \) of the operator \( P_- \), and to the spectral subspaces of \( P_- (V) \). Closeness of subspaces will be measured by closeness of orthogonal projections onto them. Recall that the projection onto \( H_q \) is denoted by \( P_q \). Let \( Q_q \) be the projection onto \( G_q \).

**Theorem 4.1.** The projections \( P_q \) and \( Q_q \) are close: for any \( N \), and any collection of the operators \( T_j \), \( j = 1, \ldots, N \), each of \( T_j \) being \( Q \) or \( Q \), the operator \( T(P_q - Q_q)T \), is compact, \( T = T_1T_2 \ldots T_N \).

In justifying the theorem, we need two technical lemmas, both concerning the properties of products of many copies of the resolvents of \( P_+ \) and \( P_- \), the creation and annihilation operators \( \overline{Q} \) and \( Q \), functions \( h_j \) with all derivatives bounded, and, possibly, the spectral projection \( P_q \). In such product, we assign order 1 to \( \overline{Q} \) and \( Q \), order \(-2\) to the resolvent, order 0 to functions and projections. The order of the product is defined as the sum of orders of factors.

**Lemma 4.2.** Let \( \mathcal{A} \) be the product of creation, annihilation operators, resolvents, and functions \( h_j \), have negative order, and let at least one of the functions \( h_j \) belong to \( S_\nu \), \( \nu < 0 \). Then \( \mathcal{A} \) is compact.

**Lemma 4.3.** Let \( \mathcal{A} \) be the product of creation, annihilation operators, resolvents, functions and the projection \( P_q \). Then \( \mathcal{A} \) is bounded. If, moreover, at least one of \( h_j \) belongs to \( S_\nu \), \( \nu < 0 \), then \( \mathcal{A} \) is compact.
The proof of the earlier versions of these lemmas, with the $S_\nu$-condition replaced by the compactness of the support of one of the function is given in [15]. In the present formulation, the proof goes exactly in the same way, only using the new version of Lemma 3.2 and Lemma 3.1, fit for the relaxed conditions for the functions.

The proof of the theorem goes in the following way. We construct an intermediate operator $\mathcal{S}_q$ with range in $\mathcal{G}_q$ and prove that $\mathcal{S}_q$ is close both to $P_q$ and $Q_q$. For $q > 0$ we define the operator $\mathcal{S}_q$ as

$$\mathcal{S}_q = C_q^{-1}\overline{Q}^q P_0 Q^q, \quad C_q = q!(2B^q)^q.$$  

(4.1)

So, our expression for the approximate spectral projection is just a natural modification of the exact formula (2.6) for the unperturbed operator. Equivalently, the operator $\mathcal{S}_q$ can be described by the formula

$$\mathcal{S}_q = C_q^{-1}\mathcal{G}\mathcal{G}^*, \quad \mathcal{G} = \overline{Q}^q P_0;$$

in Proposition 3.4 this operator is shown to be bounded.

The proof of Theorem 4.1 will consist of two parts, showing that $\mathcal{S}_q$ is close to $P_q$ and showing that it is close to $Q_q$. The second part is proved exactly like in [15], since it is based upon Lemma 4.3 only. As for the first part, we need a more detailed information of the difference $\mathcal{S}_q - P_q$. This information is given in the following statement.

**Proposition 4.4.** The operator $\mathcal{S}_q$ is close to the projection $P_q$. Moreover, the difference $\mathcal{S}_q - P_q$ has the the form

$$\mathcal{S}_q - P_q = C_q(P_- - \Lambda_q)P_q + Z_q,$$  

(4.2)

where $Z_q$ is such an operator that $\langle x \rangle^{-\beta+\delta}T Z_q T''$ is bounded for any $T, T'$ being finite products of creation and annihilation operators.

So, the improvement, compared with Proposition 4.4 in [15], consists, first, in the separation of the leading term in the difference, the operator $(P_- - \Lambda_q)P_q$, and, secondly, in the establishing the improved smallness of the remainder term $Z_q$.

**Proof.** Combinatorically, the reasoning goes in the same way as in [15], with a natural replacement of the auxiliary results requiring the compactness of the support of the perturbation and singling out the leading term. We remind the main structure of the reasoning, omitting the details. The case $q = 1$ is considered first; this case contains all typical features. The higher Landau levels, $q > 1$, can be taken care of in the same way as in [15], by the induction on $q$.

Recall that $R_\pm(\zeta)$ denotes the resolvent of the operator $P_\pm$. The projection $P_0$ can be expressed via Riesz integral

$$P_0 = (2\pi i)^{-1}\int_{\Gamma_0} R_-(\zeta) d\zeta,$$
where $\Gamma_0$ is the closed curve defined, together with curves $\Gamma_q$, in Sect.3. We are going to transform the expression for the resolvent $R_-(\zeta)$ using the commutation relations (2.10), (2.11), (2.12). After this, the crucial observation is that the integral of $R_+^k$, $k \geq 2$ along $\Gamma_q$ vanishes. This enables us to dispose of terms that before integration were not weak enough. We start by writing

$$R_-(\zeta) = (P_+ - 2B^\circ - 2b - \zeta)^{-1} = (P_+ - 2B^\circ - \zeta)^{-1} -$$

$$- (P_+ - \zeta)^{-1}(2b)(P_+ - 2B^\circ - \zeta)^{-1} = R_+(2B^\circ + \zeta) - Z(\zeta)$$

(4.3)

We multiply (4.3) by $\overline{Q}$ from the left and by $Q$ from the right, as (4.1) requires. For the first term we use that

$$\overline{Q}R_+(2B^\circ + \zeta)Q = \overline{Q}(Q\overline{Q} - 2B^\circ - \zeta)^{-1}Q =$$

$$\overline{Q}Q(\overline{Q}Q - 2B^\circ - \zeta)^{-1} = P_- R_-(2B^\circ + \zeta).$$

Integration gives

$$\int_{\Gamma_0} \overline{Q}(P_- - 2B^\circ - \zeta)^{-1}Q d\zeta = P_- \int_{\Gamma_1} (P_- - \zeta)^{-1}d\zeta = (2\pi i)P_- P_1.$$  

So, we have

$$S_1 - P_1 = \Lambda_1^{-1}(P - \Lambda_1)P_1 + Z_1,$$

(4.5)

where $Z_1$ is the contour integral of the term $Z(\zeta)$ in (4.3).

As explained in Section 2, the nonzero eigenvalues of $P_- P_1$ may converge only to $\Lambda_1$, and thus $(P_- - \Lambda_1)P_1$ is compact by Proposition 3.3. Moreover, this compactness is preserved after the multiplication by any product of creation and annihilation operators.

Now we consider the remainder term, $Z(\zeta)$ in (4.3). We are going to show that for any operators of the type $T, T'$ the integral of

$$\langle x \rangle^{-\beta + \delta} T Z(\zeta) T'$$

along $\Gamma_0$ is bounded. Due to the arbitrariness of $T, T'$ this, of course, implies the compactness required by the theorem.

Let $N$ be some fixed, sufficiently large integer. We apply the resolvent formula (4.3) $2N$ times to the first factor in $Z(\zeta)$. This operation will produce terms of order $-4, -6, \ldots$, containing factors $R_-(\zeta)$ and $b$, and the remainder term of order $-2N$ containing these factors with, additionally, one factor $R_+(2B^\circ + \zeta)$. This last remainder term, for $N$ sufficiently large, is compact by Lemma 4.2, even after the multiplication by $T, T'$. We will study its spectral properties below.

The leading terms in $Z(\zeta)$, having orders $-4, -6, \ldots$, will be transformed by repeatedly commuting $b$ and $R_-(\zeta)$ and then the resulting commutants again with $R_-(\zeta)$ and so on. Under commuting $R_-(\zeta)$ with a function, with $Q$, or with $\overline{Q}$, this factor $R_-(\zeta)$ moves to the left or to the right, and one more product in the sum composing $T Z(\zeta) T'$ arises, of the order lower by 1. In this commuting procedure we aim for collecting the factors $R_-(\zeta)$ together. As soon as we obtain a term with all $R_-(\zeta)$ standing together, we leave it alone and do not transform any more. After sufficiently many commutations, we arrive at a
collection of terms of negative order, smaller than $-N$, in $T \mathcal{H}(\zeta)T'$. All of which are compact by Lemma 4.2. The terms of order $-N$ or higher will have the form $G_1 R_-(\zeta)^k G_2$ with $k > 1$ and some operators $G_1, G_2$. These terms vanish after integration along $\Gamma_0$.

Let us consider the structure of all remaining operators of order lower than $-N$ again. They will contain $b$ and its derivatives as factors. The only term where only one factor $b$ is present, as it follows from (4.3), will be the one of the form $2b(\mathcal{P} - \zeta)^{-2}$ obtained by commuting the first and second factors in $(\mathcal{P} - \zeta)(2b)(\mathcal{P} - \zeta)$. The integral of this term along the contour vanishes. All the remaining terms will contain at least two entries of $b$ or at least one derivative of $b$. By Proposition 2.1, used with proper $l$, this means that each such term is bounded as acting into the weighted space with weight $\langle x \rangle^{-\beta + \delta}$. This boundedness property is preserved after the contour integration.

The detailed combinatorics in the above reasoning as well as the inductive procedure enabling us to pass from $q = 1$ to an arbitrary $q$, is explained in [15]. □

Thus the operator $\mathcal{S}_q$ is close to the projection $P_q$. Together with the closeness of $\mathcal{S}_q$ to $Q_q$, this reasoning proves Theorem 4.1.

Now we add a perturbation by a smooth electric potential $V(x) \in S_\beta$. Similar to [15] under such perturbation the spectral subspaces ‘almost’ do not change. Note, first of all, that the perturbation of $\mathcal{P}$ by $V$ is relatively compact, therefore, again, the spectrum of the operator $\mathcal{P}(V) = \mathcal{P} + V$ between Landau levels is discrete. We can change the contours $\Gamma_q$ a little, so that they do not pass through the eigenvalues of $\mathcal{P}(V)$. We denote by $\mathcal{H}_q^V$ the spectral subspaces of $\mathcal{P}(V)$ corresponding to the spectrum inside $\Gamma_q$, by $P_q^V$ the corresponding spectral projections, and by $R_q^V(\zeta)$ the resolvent of $\mathcal{P}(V)$.

**Proposition 4.5.** The projections $P_q^V$ and $Q_q$ are close in the sense used in Theorem 4.1. Moreover, $\langle x \rangle^{-\beta + \delta} T(P_q^V - Q_q)T'$ is bounded for any finite products $T, T'$ of creation and annihilation operators.

**Proof.** We will prove that the projection $P_q^V$ is close to $P_q$, then the result will follow from Theorem 4.1.

We use the representation of projections $P_q^V$ and $Q_q$ by means of resolvents and subtract:

$$
\langle x \rangle^{-\beta + \delta} P_q^V - P_q = -(2\pi i)^{-1} \langle x \rangle^{-\beta + \delta} \int_{\Gamma_q} R_-(\zeta)V R_q^V(\zeta) d\zeta = (4.6)
$$

$$(2\pi i)^{-1} \sum_{k=1}^{2N-1} \int_{\Gamma_q} \langle x \rangle^{-\beta + \delta}(-R_-(\zeta)V)^k R_-(\zeta) d\zeta
$$

$$
+ (2\pi i)^{-1} \int_{\Gamma_q} \langle x \rangle^{-\beta + \delta}(-R_-(\zeta)V)^N R_q^V(\zeta)(-VR_-(\zeta))^N d\zeta.
$$
The last term in (4.6) is bounded and stays bounded after the multiplication by the creation and annihilation operators, as soon as $N$ is large enough, by Lemma 4.2. With the leading terms in (4.6), we can perform the same procedure as when proving Proposition 4.4. We commute the resolvent with $V$ and with the terms arising by commutation and so on, aiming to collect the resolvents together all the time. We arrive at a number of terms of sufficiently negative order, thus bounded before the integration, and terms with all entries of the resolvent collected together, and thus vanishing after the integration. The boundedness property with weight follows again, as in the proof of Proposition 4.4, from the fact that after the commutation, all surviving terms will contain at least two entries of $V$ or an entry of a derivative of $V$. □

5. Spectrum of Toeplitz-type operators

We move on to the study of the splitting of Landau levels of our operators. Similar to [13], [10], [15], [7], the properties of the eigenvalues of the perturbed operators are determined by the properties of the spectrum of certain Toeplitz-type operators.

Usually, by Toeplitz operator one understands an operator of the form $T(W) = PW$, where $P$ is the orthogonal projection onto some subspace $G$ in $L_2$ and $W$ is the operator of multiplication by some function. Alternatively, if we consider the Toeplitz operator as acting in $G$ it can be written as $T(W) = PW$. Usually, the subspace $G$ consists of functions, related with analytical ones, for example, Hardy or Bergman spaces.

In [13], [10] such operators were considered, with $G$ being one of Landau subspaces and the electric potential $V$ acting as $W$. We will study, as in [15], this latter kind of Toeplitz type operators, with the space $H_0$ of zero modes of the perturbed Pauli operator acting as $G$ and some differential operator acting as $W$. The result below differs essentially from the one in [15].

Let $V, b$, be real functions in $S_\beta$, $\beta < -2$. We consider the Toeplitz-type operator in $H_0$:

$$\mathfrak{I}_0u = \mathfrak{I}_0(V)u = P_0Q^q(P_\Lambda - \Lambda q + V)\overline{Q^q}u, \ u \in H_0. \tag{5.1}$$

This operator, by (2.10), corresponds to the quadratic form

$$t^V[u] = ((P_\Lambda - \Lambda q + V)\overline{Q^q}u, \overline{Q^q}u) = (P_\Lambda - \Lambda q + 1)||Q^{q+1}u||^2$$

$$-\Lambda q + 1||Q^q u||^2 + ((V - 2b)\overline{Q^q}u, \overline{Q^q}u), \ u \in H_0. \tag{5.2}$$

We are going to study the spectrum of $\mathfrak{I}_0(V)$. We denote by $\lambda_n^\pm = \lambda_n^\pm(\mathfrak{I}_0(V))$ positive, resp., negative eigenvalues of the operator $\mathfrak{I}_0(V)$. The distribution functions $n_\pm(\lambda), \lambda > 0$, are defined as $n_\pm(\lambda) = n_\pm(\lambda; \mathfrak{I}_0(V)) = \#\{n : \pm\lambda_n^\pm > \lambda\}$. The s-numbers $s_n$ of the operator $\mathfrak{I}_0(V)$ are just the absolute values of $\lambda_n^\pm$ ordered
non-increasingly, and their distribution function equals \( n(\lambda, \Sigma_0(V)) = n_+(\lambda, \Sigma_0(V)) + n_-(\lambda, \Sigma_0(V)) \).

We denote by \( V = V_q[V, b] \) the effective weight
\[
V[V, b] = C_q(V + 2q b), \quad C_q = q!(2B^2)^q,
\]
and set for any function \( W \)
\[
E_{\pm}(\lambda, W) = (2\pi)^{-1}B^\circ \text{meas} \{ x \in \mathbb{R}^2 : \pm W(x) > \lambda \}.
\]

**Proposition 5.1.** For the eigenvalue distribution function \( n_{\pm}(\lambda; \Sigma_0(V)) \) of the operator \( \Sigma_0(V) \) the following estimate holds:
\[
n_{\pm}(\lambda; \Sigma_0(V)) \leq C\lambda^2, \quad \lambda \to 0 \quad (5.4)
\]
Moreover, suppose that the function \( E_{\pm}(\lambda, V) \) (for one or for both signs) satisfies the estimate
\[
E_{\pm}(\lambda, V) \geq C'\lambda^2, \quad \lambda \to 0 \quad (5.5)
\]
and the regularity condition
\[
\lim_{\epsilon \to 0} \limsup_{\lambda \to 0} \frac{E_{\pm}(\lambda(1 - \epsilon), V)}{E_{\pm}(\lambda, V)} = 1. \quad (5.6)
\]
Then for the distribution function \( n_{\pm}(\lambda; \Sigma_0(V)) \) of the eigenvalues of \( \Sigma_0(V) \) the asymptotic formula holds
\[
n_{\pm}(\lambda, \Sigma_0(V)) \sim E_{\pm}(\lambda, V), \quad \lambda \to 0. \quad (5.7)
\]

The conditions of the form (5.5) and (5.6) are traditional in the study of asymptotic properties of operators with (possibly) non-power behavior of eigenvalues, see, e.g., [11], [7], [8]. The first of them indicates that the 'rate of decay' \( \beta \) is chosen sharply, so that various remainder terms are, in fact, weaker than the leading one. The second condition means that the function \( E_{\pm}(\lambda, V) \) grows sufficiently regularly. It enables the use of various kinds of perturbation techniques.

We note also that for \( q = 0 \) and \( V = 0 \), the effective weight \( V \) vanishes. This property corresponds to the fact that \( P_- \) has \( \mathcal{H}_0 \) as its space of zero modes.

**Proof.** By Proposition 3.4, the expression (5.2) can be transformed to
\[
v^V[u] = (Vu, u) + (vu, u), \quad u \in \mathcal{H}_0,
\]
where \( v \in S_{\beta - \delta} \) and \( V = V_q[V, b] \in S_\beta \). By Lemma 2.5 in [7], for the operator defined by this quadratic form, the asymptotic relation
\[
n_{\pm}(\lambda) \sim (2\pi)^{-1}B^\circ \text{meas} \{ x \in \mathbb{R}^2 : V(x) + v(x) \geq \lambda \}, \quad \lambda \to 0,
\]
holds, as soon as the measure on the right-hand side grows not slower than \( \lambda^{2\beta} \) and is regular, i.e., if the conditions (5.5), (5.6) are satisfied. This reasoning takes care of the second part of the Proposition. The first part follows from Proposition 2.3 in [7], without qualified lower estimate. The function \( v \), due to its decay, does not contribute to the main order of the eigenvalue estimates and asymptotics. \( \square \)
Now we can establish the spectral estimates and asymptotics for a similar Toeplitz operator on the spectral subspace corresponding to the cluster around the Landau level $\Lambda_q$ for an arbitrary $q$.

**Proposition 5.2.** Let $b, V \in S_\beta$, $\beta < -2$. Consider the Toeplitz-type operator

$$\mathfrak{T}_q(V) = P_q(\mathcal{P}_- - \Lambda_q + V)P_q.$$  \hspace{1cm} (5.8)

Then for the distribution function of $s$-numbers of the eigenvalues of $\mathfrak{T}_q(V)$ the estimate holds

$$n(\lambda, \mathfrak{T}_q(V)) = O(\lambda^{\frac{3}{2}}).$$  \hspace{1cm} (5.9)

If, moreover, the effective weight $V$ defined in (5.3) satisfies the conditions (5.5), (5.6) then the asymptotic formula

$$n_\pm(\lambda, \mathfrak{T}_q(V)) \sim E_\pm(\lambda, V + 2q b)$$  \hspace{1cm} (5.10)

holds.

**Proof.** By Proposition 4.4, the spectral projection $P_q$ can be approximated by $\mathfrak{S}_q = Q_q^{\beta}P_0Q^q$. We use (4.2) to obtain

$$\mathfrak{T}_q(V) = \mathfrak{S}_q(\mathcal{P}_- - \Lambda_q + V)\mathfrak{S}_q + (P_q(\mathcal{P}_- - \Lambda_q)P_q + Z_q)(\mathcal{P}_- - \Lambda_q + V)\mathfrak{S}_q + \mathfrak{S}_q(\mathcal{P}_- - \Lambda_q + V)(P_q(\mathcal{P}_- - \Lambda_q)P_q + Z_q) + (P_q(\mathcal{P}_- - \Lambda_q)P_q + Z_q)(\mathcal{P}_- - \Lambda_q + V)(P_q(\mathcal{P}_- - \Lambda_q)P_q + Z_q).$$  \hspace{1cm} (5.11)

We re-arrange (5.11) to separate the leading terms:

$$\mathfrak{T}_q(V) = \mathfrak{S}_q(\mathcal{P}_- - \Lambda_q + V)\mathfrak{S}_q + P_q(\mathcal{P}_- - \Lambda_q + V)P_q(\mathcal{P}_- - \Lambda_q + V)P_q\mathfrak{S}_q + \mathfrak{S}_q P_q(\mathcal{P}_- - \Lambda_q + V)P_q(\mathcal{P}_- - \Lambda_q + V)P_q + P_q(\mathcal{P}_- - \Lambda_q + V)P_q(\mathcal{P}_- - \Lambda_q + V)P_q + \mathbf{Y}.$$  \hspace{1cm} (5.12)

So, we have

$$\mathfrak{T}_q(V) = \mathfrak{S}_q(\mathcal{P}_- - \Lambda_q + V)\mathfrak{S}_q + \mathfrak{T}_q(V)^2\mathfrak{S}_q + \mathfrak{S}_q\mathfrak{T}_q(V)^2 + \mathfrak{T}_q(V)^3 + \mathbf{Y}.$$  \hspace{1cm} (5.12)

Let us consider the structure of the remainder operator $\mathbf{Y}$. On the one hand, it contains terms having the factor $Z_q$. By Proposition 4.4, the operator $(x)^{-\beta + \delta}(\mathcal{P}_+)^N Z_q$ is bounded for $N$ that can be chosen arbitrarily large. Thus, by Proposition 2.2, the $s$-numbers of $Z_q$ satisfy the estimate $n(\lambda, Z_q) = o(\lambda^{\frac{2}{3}})$, $\lambda \to 0$. Therefore all terms in $\mathbf{Y}$ containing $Z_q$ satisfy the same kind of $s$-numbers estimate.

The terms in $\mathbf{Y}$ not containing $Z_q$, contain as factors the function $V$, the projection $P_q$ and some compact operators that remain compact after the multiplication by any power of $\mathcal{P}_+$. For the operator $V(\mathcal{P}_+)^{-N}$, by Proposition 2.2, the estimate $n(\lambda, V(\mathcal{P}_+)^{-N}) = O(\lambda^{\frac{2}{3}})$ holds. After the multiplication by a compact operator, the $O$ symbol can be replaced by $o$ in this formula. As a result, we obtain

$$n(\lambda, \mathbf{Y}) = o(\lambda^{\frac{2}{3}}), \lambda \to 0.$$  \hspace{1cm} (5.13)
Now consider the second, third, and fourth terms on the right-hand side in (5.12). They contain the square of the operator \( \mathcal{T}_q(V) \) multiplied by some bounded operators. Therefore the rate of decay of s-numbers of these terms is faster than the rate of decay of the ones of \( \mathcal{T}_q(V) \). So, the estimate (5.9) or the asymptotic formula (5.10) will be proved as soon as we establish these formulas for the operator \( D = \mathcal{S}_q(P_\Lambda + V)\mathcal{S}_q \).

Recalling the definition of \( \mathcal{S}_q \) in (4.1), we have

\[
(Du, u) = C_q^{-1} \langle Q q P_0 Q q u, (P_\Lambda + V)Q q P_0 Q q u \rangle, \quad u \in \mathcal{H}_q.
\]

We set here \( v = P_0 Q^q u, \quad v \in \mathcal{H}_0 \), to get

\[
(Du, u) = C_{q^{-1}} \langle Q q v, (P_\Lambda + V)Q q v \rangle, \quad v \in \mathcal{H}_0. \tag{5.14}
\]

The spectral estimate and, under the conditions (5.5), (5.6), the asymptotics for the operator defined by the latter quadratic form is given by Proposition 5.1, where the factor \( C_{q^{-1}} \) is responsible for the replacement of \( V \) by \( C_{q^{-1}} V = V + 2q b \). This result carries over to the quadratic form \( (Du, u) \) using the fact that \( u = C_q(1 + K)Q q v \) for some compact operator \( K \), which is explained in [15].

The operators of the above type will be used in the next Section in order to find the leading term in the eigenvalue asymptotics in clusters. Another type of Toeplitz operators will be needed in order to perform a block digitalization of the Pauli operator.

**Proposition 5.3.** Let \( W \) be a function in \( S_{\beta}, \beta < -2 \). Consider the operator \( \mathcal{T}'(W) = (1 - P_q)WP_q \). Then for the distribution function of singular numbers of \( \mathcal{T}'(W) \)

\[
n(\lambda, \mathcal{T}'(W)) = o(\lambda^{\frac{3}{2}}), \quad \lambda \to 0. \tag{5.15}
\]

**Proof.** The quadratic form of the operator \( \mathcal{T}'(W)^*\mathcal{T}'(W) \) equals

\[
(\mathcal{T}'(W)u, \mathcal{T}'(W)u) = ((1 - P_q)W u, (1 - P_q)W u) \tag{5.16}
\]

\[
= ([W, P_q]P_q u, [W, P_q]P_q u), \quad u \in \mathcal{H}_q.
\]

Consider the case \( q = 0 \) first. By Proposition 2.2 the operator \([W, P_0]\) can be represented as

\[
[W, P_0] = L \langle x \rangle^{\beta - \delta}, \tag{5.17}
\]

for some bounded operator \( L \). By Proposition 5.1, the operator \([W, P_0]\) has singular numbers with the required decay rate.

For an arbitrary \( q > 0 \) by Proposition 2.1,

\[
[W, P_q] = L \langle x \rangle^{\beta - \delta}, \tag{5.18}
\]

We use the approximation of the projection \( P_q \) found in Section 4:

\[
[W, P_q]P_q = L \langle x \rangle^{\beta - \delta} \mathcal{S}_q + L \langle x \rangle^{\beta - \delta}(P_q - \mathcal{S}_q), \tag{5.19}
\]

where \( \mathcal{S}_q \) is defined in (4.1). The spectral estimate for the first, leading term in (5.19) involves the projection \( P_0 \) and the task of estimating its
spherical numbers reduces to the already considered case \( q = 0 \), see (5.14). By Proposition 4.4, the second term in (5.19) can be written as \( L(x)^{\beta - \delta} P^\circ + K \) with as large \( N \) as needed and a compact operator \( K \). The required spectral estimate for this operator follows now from the second part of Proposition 2.2.

\[ \square \]

6. Perturbed eigenvalues

Now we are able to establish our main result about eigenvalue asymptotics and estimates for the perturbed Schrödinger and Pauli operators. For a self-adjoint operator \( L \), we denote by \( N(\lambda, \mu) = N(\lambda, \mu; L) \) the number of eigenvalues of \( L \) in the interval \((\lambda, \mu)\).

**Theorem 6.1.** Let \( V, b \in S_\beta, \beta < -2 \). Fix an integer \( q \geq 0 \) and let \( \lambda_\pm \) be some fixed real numbers, \( \lambda_\pm \geq \Lambda_q \), \( |\lambda_\pm - \Lambda_q| < B^\circ \), such that they are not the points of the spectrum of \( P_- + V \).

For \( P_- (V) \) the following estimates hold for \( \lambda \to 0^+ \)

\[
N(\Lambda_q + \lambda, \lambda_+; P_- (V)) = O(\lambda^2), \quad N(\Lambda_-, \Lambda_q - \lambda; P_- (V)) = O(\lambda^2).
\] (6.1)

If the effective weight \( V_q[V, b] = C_q(V + 2qb) \) or, what is equivalent, \( V + 2qb \) satisfies the conditions (5.5), (5.6) with the sign “+” then, asymptotically,

\[ N(\Lambda_q + \lambda, \lambda_+; P_- (V)) \sim E_+(\lambda, V + 2qb), \quad \lambda \to 0, \quad \lambda > 0. \] (6.2)

If the effective weight satisfies the conditions (5.5), (5.6) with the sign “-” then, asymptotically,

\[ N(\Lambda_-, \Lambda_q - \lambda; P_- (V)) \sim E_-(\lambda, V + 2qb), \quad \lambda \to 0, \quad \lambda > 0. \] (6.3)

Similar results hold for the operator \( P_+ (V) \) and for the Schrödinger operator \( H(V) \) with the following obvious modifications, corresponding to (3.3). For the Pauli operator \( P_+ (V) \) one should replace in the estimates of the form (6.1) and in the asymptotic relations of the form \( \Lambda_q \) by \( \Lambda_{q+1} \), \( \lambda_\pm \) by \( \lambda_\pm + 2B^\circ \) and \( V \) by \( V + 2b \). For the Schrödinger operator \( H(V) \) one should replace \( \Lambda_q \) by \( \Lambda_q + B^\circ \), \( \lambda_\pm \) by \( \lambda_\pm + B^\circ \) and \( V \) by \( V + b \).

**Proof.** We will use the following statement about the eigenvalue distribution of perturbed operators. If \( L_0, L_1 \) are two self-adjoint operators, moreover, \( L_1 \) is compact, then

\[
N(\mu_1, \mu_2; L_0 + L_1) \leq N(\mu_1 - \tau_1, \mu_2 + \tau_2; L_0) + n(\tau_1; L_1) + n(\tau_2, L_1)
\] (6.4)

for any interval \((\mu_1, \mu_2)\) and any positive numbers \( \tau_1, \tau_2 \). The proof of (6.4) can be found, e.g., in [11], Lemma 5.4.

We show now the upper asymptotic estimate in (6.2). For a fixed \( q \), we take as \( L_0 \) the operator

\[
L_0 = P_q (P_- + V) P_q + (1 - P_q) (P_- + V) (1 - P_q),
\] (6.5)
and as $L_1$ the operator

$$L_1 = (1 - P_q)V P_q + P_qV(1 - P_q),$$

so that $L_0 + L_1 = P_- + V$. We fix some $\epsilon > 0$, and apply (6.4) for $\mu_1 = \Lambda_q + \lambda, \lambda_+ = \tau_1 = \tau_2 = \epsilon \lambda$. The spectrum of $L_0$ is the union of the spectra of the summands in (6.5). The asymptotics of the eigenvalues of $P_q(\mathbf{P}_- + V)P_q$ is given by the Proposition 5.2. The second term in (6.5) contributes to the spectrum near $\Lambda_q$ only with finitely many points. Thus, $N(\Lambda_q + \mu_1 - \tau_1, \Lambda_q + \mu_2 + \tau_2; L_0) \sim E_+(\lambda(1 - \epsilon); V + 2qb)$. On the other hand, for the spectrum of the operator $L_1$, by Proposition 5.3, we have $n(\epsilon \lambda, L_1) = o((\epsilon \lambda)^{\frac{1}{2}})$. We substitute these asymptotic estimates into (6.4), divide by $E_+(\lambda, V + 2qb)$ and pass to lim sup as $\lambda \to 0$. We arrive at

$$\limsup_{\lambda \to 0} \frac{N(\Lambda_q + \lambda, \lambda_+; \mathbf{P}_- + V)}{E_+(\lambda(1 - \epsilon); V + 2qb)} \leq 1.$$ 

Due to the arbitrariness of $\epsilon$, by our assumptions, this implies the upper asymptotic estimate in (6.2).

All other upper estimates in the Theorem are established analogously. The lower asymptotic estimate in (6.2), (6.3) is established in the same way, just interchanging $L_0$ and $L_1$ in (6.4).

\[\square\]

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