Balls minimize trace constants in $BV$

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Abstract

Balls are shown to have the smallest optimal constant, among all admissible Euclidean domains, in Poincaré type boundary trace inequalities for functions of bounded variation with vanishing median or mean value.

1 Introduction and main results

A branch of mathematical research which bridges analysis and geometry is concerned with variational problems for quantities of geometric-analytic nature associated with sets from some prescribed collection. Typically, the relevant quantities are, in turn, expressed as a supremum or infimum of some functional, defined on each set, which has often a physical meaning. A prototypical result in this area is the standard isoperimetric inequality in the Euclidean space $\mathbb{R}^n$. Further classical issues amount to so called isoperimetric problems of mathematical physics, and include inequalities for eigenvalues of elliptic operators and isocapacitary inequalities. Most of these problems were originally stated as conjectures. Some of them have been solved in the last century via methods of the modern calculus of variations. Their solution has led to such results as Polya’s theorem on De Saint Venant’s conjecture on cylindrical beams with the highest torsional rigidity, Szegö’s theorem on Poincaré’s conjecture on the body of largest electrostatic capacity,
Faber and Krahn’s theorem on Lord Rayleigh’s conjecture on the lowest principal frequency of vibrating clamped membranes. Other conjectures, including the minimizing property of the ball for the first eigenvalue in a fourth-order eigenvalue problem modeling the vibration of an elastic clamped plate (Szegő conjecture), or the minimizing property of the disk for the capacity in the family of convex sets in the three-dimensional space with prescribed surface area (Pólya-Szegő conjecture), are still open, or are only known is special cases. We do not even attempt an exhaustive bibliography on these topics. Let us just refer to the monographs and surveys [AB, GGS, H, Ka, Ke, Ta] for an account of results and techniques in this field.

A class of quantities associated with open sets in $\mathbb{R}^n$, whose maximization has traditionally attracted the attention of specialists in functional and geometric analysis, is that of the sharp constants in Sobolev-Poincaré type inequalities. Of course, in many instances these constants can be interpreted as eigenvalues of an associated Euler equation. An overview of results and problems in this connection can be found e.g. in [BrV]. The present contribution falls within this line of investigations, and focuses a minimization problem for the optimal constants in Poincaré type inequalities for functions of bounded variation.

Assume that $\Omega$ is a domain, namely a bounded connected open set in $\mathbb{R}^n$, $n \geq 2$. It is well known that if the boundary $\partial \Omega$ of $\Omega$ is sufficiently regular, then a linear operator if defined on the space $BV(\Omega)$ of functions of bounded variation in $\Omega$, which associates with any function $u \in BV(\Omega)$ its (suitably defined) boundary trace $\tilde{u} \in L^1(\partial \Omega)$. Here, $L^1(\partial \Omega)$ denotes the Lebesgue space of integrable functions on $\partial \Omega$ with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. Moreover, there exists a constant $C$, depending on $\Omega$, such that

\begin{equation}
\inf_{c \in \mathbb{R}} \|\tilde{u} - c\|_{L^1(\partial \Omega)} \leq C\|Du\|(\Omega)
\end{equation}

for every $u \in BV(\Omega)$, where $\|Du\|(\Omega)$ stands for the total variation over $\Omega$ of the total variation of the distributional gradient $Du$ of $u$ [Ma3, Theorem 9.6.4].

A property of $L^1$ norms ensures that the infimum in (1.1) is attained when $c$ agrees with a median of $\tilde{u}$ on $\partial \Omega$, given by

$$\text{med}_{\partial \Omega} \tilde{u} = \sup\{t \in \mathbb{R} : \mathcal{H}^{n-1}(\{\tilde{u} > t\}) > \mathcal{H}^{n-1}(\partial \Omega)/2\}$$

(see e.g. [CP, Lemma 3.1]) Thus, inequality (1.1) is equivalent to

\begin{equation}
\|\tilde{u} - \text{med}_{\partial \Omega} \tilde{u}\|_{L^1(\partial \Omega)} \leq C_{\text{med}}(\Omega)\|Du\|(\Omega)
\end{equation}

for every $u \in BV(\Omega)$, where $C_{\text{med}}(\Omega)$ denotes the optimal – smallest possible – constant which renders (1.2) true.

An other customary Poincaré type trace inequality holds, when $\text{med}_{\partial \Omega} \tilde{u}$ is replaced with the mean value $\bar{u}_{\partial \Omega}$ of $\tilde{u}$ over $\partial \Omega$, defined as

$$\bar{u}_{\partial \Omega} = \frac{1}{\mathcal{H}^{n-1}(\partial \Omega)} \int_{\partial \Omega} \tilde{u} \, d\mathcal{H}^{n-1}(x).$$

The relevant inequality reads

\begin{equation}
\|\tilde{u} - \bar{u}_{\partial \Omega}\|_{L^1(\partial \Omega)} \leq C_{\text{mv}}(\Omega)\|Du\|(\Omega)
\end{equation}

for every $u \in BV(\Omega)$, where we have denoted by $C_{\text{mv}}(\Omega)$ the optimal constant in (1.3).

Observe that, in the light of the above discussion, one has that

\begin{equation}
C_{\text{med}}(\Omega) \leq C_{\text{mv}}(\Omega)
\end{equation}
for every domain $\Omega$. Also, note that both $C_{\text{med}}(\Omega)$ and $C_{\text{mv}}(\Omega)$ are invariant under dilations of $\Omega$, and hence they only depend on the shape of $\Omega$, but not on its size.

A minimal regularity assumption for inequalities (1.2) and (1.3) to hold is that $\Omega$ be an admissible domain, in the sense that
\[
H^{n-1}(\partial \Omega) < \infty, \quad H^{n-1}(\partial \Omega \setminus \partial^M \Omega) = 0,
\]
and
\[
\min\{H^{n-1}(\partial^M E \cap \partial \Omega), H^{n-1}(\partial \Omega \setminus \partial^M E)\} \leq C H^{n-1}(\partial^M E \cap \Omega)
\]
for some positive constant $C$ and every measurable set $E \subset \Omega$. Here, $\partial^M$ denotes the subset of the topological boundary, called the essential boundary in geometric measure theory. A local version of (1.5) is, in fact, a necessary condition for the trace of $BV$ functions to be well defined on $\partial \Omega$ [AG]. Standard instances of admissible domains are provided by the Lipschitz domains, namely bounded open sets whose boundary is locally the graph of a Lipschitz function of $(n-1)$ variables.

In this paper we address the problem of minimizing the trace constants $C_{\text{med}}(\Omega)$ and $C_{\text{mv}}(\Omega)$, as $\Omega$ ranges in the class of all admissible domains $\Omega$ in $\mathbb{R}^n$. Heuristically speaking, domains with stretched shapes, such as sharp outward peaks or narrow passages, tend to have large values of $C_{\text{med}}(\Omega)$ and $C_{\text{mv}}(\Omega)$. One is thus led to guess that these constants attain their minimum value when $\Omega$ is a ball, in a sense the most rounded domain.

In the two-dimensional case, the minimum problem for $C_{\text{med}}(\Omega)$ also arises in connection with questions of different nature. The minimizing property of the disk for $C_{\text{med}}(\Omega)$ in classes of admissible domains is known, and has been independently established in [KS] [ES2] [EGK].

The higher-dimensional case appears to be open in the existing literature. A lower estimate for $C_{\text{med}}(\Omega)$, when $n \geq 3$, is given in [ES2] Theorem 6. This estimate, however, depends on the geometry of $\partial \Omega$, and seems not to yield the solution to the minimum problem for $C_{\text{med}}(\Omega)$ in any obvious way.

Our results confirm the above guess in any dimension $n$, and also point out a singular phenomenon as far as the uniqueness of minimizers is concerned.

Let us first consider $C_{\text{med}}(\Omega)$. In this regard, we have that the ball is the only minimizer for $C_{\text{med}}(\Omega)$ in any dimension $n \geq 2$.

**Theorem 1.1** Let $\Omega$ be an admissible domain in $\mathbb{R}^n$, $n \geq 2$. Then
\[
C_{\text{med}}(\Omega) \geq \sqrt{\pi} \frac{n}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.
\]

Moreover, equality holds in (1.6) if and only if $\Omega$ is equivalent to a ball, up to a set of $H^{n-1}$ measure zero.

Our approach to Theorem 1.1 relies upon a characterization of $C_{\text{med}}(\Omega)$ as a genuinely geometric quantity associated with $\Omega$, namely the optimal constant $C$ in (1.5). Indeed, [Ma3] Theorem 9.5.2 tells us that
\[
C_{\text{med}}(\Omega) = \sup_{E \subset \Omega} \frac{\min\{H^{n-1}(\partial^M E \cap \partial \Omega), H^{n-1}(\partial \Omega \setminus \partial^M E)\}}{H^{n-1}(\partial^M E \cap \Omega)},
\]
where the supremum is extended over all measurable sets $E \subset \Omega$ with positive Lebesgue measure. Note that, in particular, the constant appearing on the right-hand side of (1.6) equals $\frac{\omega_n}{2\omega_{n-1}}$, where $\omega_n = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})}$, the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The fact that $C_{\text{med}}(\Omega) = \frac{\omega_n}{2\omega_{n-1}}$ when $\Omega$ is a ball was proved in [BM], [Ma3] Theorem 9.5.2 and Corollary 9.4.4/3.
also [BS and [Es2]). The supremum in (1.7) is attained at a half-ball in this case. Moreover, characteristic functions of half-balls yield equality in (1.2).

We now take into account $C_{mv}(\Omega)$. The next result shows that the ball minimizes $C_{mv}(\Omega)$ as well, and it is the unique minimizer provided that $n \geq 3$. Interestingly enough, unlike $C_{med}(\Omega)$, disks are not the only minimizers of $C_{mv}(\Omega)$ if $n = 2$.

**Theorem 1.2** Let $\Omega$ be an admissible domain in $\mathbb{R}^n$. If $n \geq 3$, then

$$C_{mv}(\Omega) \geq \sqrt{\pi} \frac{n}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)},$$

and the equality holds in (1.8) if and only if $\Omega$ is equivalent to a ball, up to a set of $H^{n-1}$ measure zero.

If $n = 2$, then

$$C_{mv}(\Omega) \geq 2,$$

and the equality holds in (1.9) if $\Omega$ is a disk. However there exist open sets $\Omega$, that are not equivalent to a disc, for which equality yet holds in (1.9).

Also the proof of Theorem 1.2 makes use of a geometric characterization of $C_{mv}(\Omega)$. This is provided by [Ci3, Theorem 1.1], and reads

$$(1.10) \quad C_{mv}(\Omega) = \frac{2}{\mathcal{H}^{n-1}(\partial \Omega)} \sup_{E \subset \Omega} \frac{\mathcal{H}^{n-1}(\partial M E \cap \partial \Omega) \mathcal{H}^{n-1}(\partial \Omega \setminus \partial M E)}{\mathcal{H}^{n-1}(\partial M E \cap \partial \Omega)},$$

where the supremum is extended over all measurable sets $E \subset \Omega$ with positive Lebesgue measure. The value of $C_{mv}(\Omega)$ when $\Omega$ is a ball has recently been shown to coincide with the right-hand side of either (1.8) or (1.9), according to whether $n \geq 3$ or $n = 2$ [Ci3 Theorem 1.2]. In the former case, $C_{mv}(\Omega) = C_{med}(\Omega)$, and the supremum in (1.10) is achieved if $E$ is a half-ball. In the latter case, however, $C_{mv}(\Omega) > C_{med}(\Omega)$, and no maximizer exists on the right-hand side of (1.10). The supremum is approached along any sequence of circular segments – intersections of a disk with a half-plane – whose measure tends to zero. Accordingly, if $n \geq 3$, equality holds in (1.3) provided that $u$ is the characteristic function of a half-ball, whereas, if $n = 2$, equality never holds in (1.3), although the constant $C_{mv}(\Omega) = 2$ is sharp, as demonstrated by any sequence of characteristic functions of circular segments whose measure tends to zero. The lack of a maximizer in (1.10) in the case when $\Omega$ is a disk allows, in a sense, for slight deformations of $\Omega$ which do not affect $C_{mv}(\Omega)$. As will be shown in the proof of Theorem 1.2 a family of domains $\Omega$ for which $C_{mv}(\Omega)$ agrees with that of a disk consists in (nearly circular) stadium-shaped sets.

**Remark 1.3** Inequalities (1.2) and (1.3) hold, in particular, for every function $u$ in the Sobolev space $W^{1,1}(\Omega) \subset BV(\Omega)$. Of course, in this case $\|Du\|(\Omega)$ can be replaced with $\|\nabla u\|_{L^1(\Omega)}$, where $\nabla u$ denotes the weak gradient of $u$. Let us emphasize that the constants $C_{med}(\Omega)$ and $C_{mv}(\Omega)$ are optimal in the resulting Poincaré trace inequalities in $W^{1,1}(\Omega)$ as well. Indeed, any function $u \in BV(\Omega)$ can be approximated by a sequence of functions $u_k \in W^{1,1}(\Omega)$ in such a way that

$$\tilde{u}_k = \tilde{u} \quad \text{and} \quad \lim_{k \to \infty} \|\nabla u_k\|_{L^1(\Omega)} = \|Du\|(\Omega).$$

The existence of the sequence $\{u_k\}$ follows, for instance, from [Gi, Theorem 1.17 and Remark 1.18]. Thus, Theorems 1.1 and 1.2 also hold if $C_{med}(\Omega)$ and $C_{mv}(\Omega)$ are interpreted as the optimal constants in the trace inequalities (1.2) and (1.3) for $u \in W^{1,1}(\Omega)$. 
Optimal trace constants, and related shape optimization problems, are the subject of various contributions, besides those already mentioned above. Estimates for the constant $C$ in the Sobolev type trace inequality
\[ \|\tilde{u}\|_{L^1(\partial\Omega)} \leq C(\|Du\|_{(\Omega)} + \|u\|_{L^1(\Omega)}) \]
for $u \in BV(\Omega)$ are provided in [AMR]. The inequality
\[ (1.11) \quad \|u\|_{L^{\frac{n}{n-1}}(\Omega)} \leq \frac{\Gamma(1 + \frac{n}{p})^\frac{1}{n}}{\sqrt{n\pi}}(\|Du\|_{(\Omega)} + \|\tilde{u}\|_{L^1(\partial\Omega)}) \]
for $u \in W^{1,1}(\Omega)$, and hence for $u \in BV(\Omega)$, where $\frac{\Gamma(1 + \frac{n}{p})^\frac{1}{n}}{\sqrt{n\pi}}$ is the optimal constant, was proved in [Ma1] (see also [Ma2, Ma3]). Versions of inequality (1.11) for functions in the Sobolev space $W^{1,p}(\Omega)$, with $p > 1$, can be found in [MV1, MV2]. The paper [Ci2] contains a Poincaré trace inequality, with sharp constant, for functions in the limiting Sobolev space $W^{1,n}(\Omega)$. The optimal constant in the trace inequality for functions in $W^{1,p}(\Omega)$, when $\Omega$ is a half-space, was exhibited in [Es1] for $p = 2$, and in [Na] for any $p \in (1, n)$; the case $p = 1$ is easy, as observed in [Ci3]. A related Hardy-type trace inequality, with sharp constant, in a half-space is established in [DDM]; an improved inequality, with remainder terms, is the object of [AFV]. Related issues about optimal constants in Sobolev trace inequalities are discussed in [BGP, Ro].

Let us finally mention that questions of a similar nature for mean-value Poincaré type inequalities for functions in $BV(\Omega)$ and $W^{1,1}(\Omega)$, involving norms of $u$ in the whole of $\Omega$ instead of trace norms, are treated in [BoV, BrV, Ci1, EFKNT]. Contributions on optimal Poincaré inequalities in Sobolev spaces $W^{1,p}(\Omega)$, with $p > 1$, include [BK, DGS, DN, ENT, FNT, GW, Le].

## 2 A Cauchy formula for sets of finite perimeter

After recalling a few basic definitions and properties from geometric measure theory, in this section we establish a version for sets of finite perimeter of the classical Cauchy formula which expresses the perimeter of an $n$-dimensional convex set in terms of the measure of its $(n - 1)$-dimensional projections.

Let $E$ be a measurable set in $\mathbb{R}^n$. The upper and lower densities $\overline{D}(E, x)$ and $\underline{D}(E, x)$ of $E$ at a point $x \in \mathbb{R}^n$ are defined as
\[ \overline{D}(E, x) = \limsup_{r \to 0} \frac{\lambda^n(E \cap B_r(x))}{\lambda^n(B_r(x))} \quad \text{and} \quad \underline{D}(E, x) = \liminf_{r \to 0} \frac{\lambda^n(E \cap B_r(x))}{\lambda^n(B_r(x))}, \]
respectively. Here, $\lambda^n$ denotes the (outer) Lebesgue measure, and $B_r(x)$ the ball centered at $x$, with radius $r$. When $\overline{D}(E, x)$ and $\underline{D}(E, x)$ agree, their common value is called the density of $E$ at $x$ and is denoted by $D(E, x)$. For each $\alpha \in [0, 1]$, the set $E^\alpha = \{x \in R^n : D(E, x) = \alpha\}$ is called the set of points of density $\alpha$ with respect to $E$, and is a Borel set. The set $E^1$ of points of density 1 with respect to $E$ agrees with $E$, up to sets of Lebesgue measure zero. The essential boundary of $E$, defined as
\[ \partial^M E = R^n \setminus (E^0 \cup E^1), \]
is also a Borel set. Observe that $\partial^M E \subset \partial E$.

It is easily verified from the definition of essential boundary that, if $E$ and $F$ are measurable subsets of $R^n$, then
\[ (2.1) \quad \partial^M(E \cup F) \cup \partial^M(E \cap F) \subset \partial^M E \cup \partial^M F. \]
Note also that, if $E$ is any measurable set and $A$ is an open set, then
\begin{equation}
\partial^M E \cap A \subset \partial^M (A \cap E).
\end{equation}
Equation (2.2) follows from the fact that, since $A$ is open,
\begin{equation}
\mathcal{L}^n(E \cap B_r(x)) = \mathcal{L}^n((A \cap E) \cap B_r(x)) \quad \text{if } x \in A,
\end{equation}
provided that $r$ is sufficiently small.

Let $\Omega$ be an open set. If $E$ and $F$ are measurable subsets of $\Omega$ such that $E \subset F$ (up to sets of zero Lebesgue measure), then
\begin{equation}
\partial^M E \cap \partial^M \Omega \subset \partial^M F \cap \partial^M \Omega.
\end{equation}
This is an easy consequence of the definition of essential boundary, and of the fact that, if $x \in \partial^M E \cap \partial^M \Omega$, then
\begin{equation}
\limsup_{r \to 0} \frac{\mathcal{L}^n(\Omega \cap B_r(x))}{\mathcal{L}^n(B_r(x))} \geq \limsup_{r \to 0} \frac{\mathcal{L}^n(F \cap B_r(x))}{\mathcal{L}^n(B_r(x))} \geq \limsup_{r \to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))} > 0
\end{equation}
and
\begin{equation}
1 > \liminf_{r \to 0} \frac{\mathcal{L}^n(\Omega \cap B_r(x))}{\mathcal{L}^n(B_r(x))} \geq \liminf_{r \to 0} \frac{\mathcal{L}^n(F \cap B_r(x))}{\mathcal{L}^n(B_r(x))} \geq \liminf_{r \to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))}.
\end{equation}
As a consequence of [Ma3, Lemma 9.4.2],
\begin{equation}
\mathcal{H}^{n-1}(\partial^M \Omega \cap \partial^M (\Omega \setminus E)) = \mathcal{H}^{n-1}(\partial^M \Omega \setminus \partial^M E)
\end{equation}
for every measurable subset $E$ of $\Omega$.

The space $BV(\Omega)$ consists of those functions $u \in L^1(\Omega)$ whose first-order distributional gradient $Du$ is a vector-valued Radon measure with finite total variation $\|Du\|(\Omega)$. The space $BV(\Omega)$ is a Banach space endowed with the norm given by $\|u\|_{L^1(\Omega)} + \|Du\|(\Omega)$ for $u \in BV(\Omega)$. The boundary trace $\tilde{u}$ of a function $u \in BV(\Omega)$ can be defined for $\mathcal{H}^{n-1}$-a.e. $x \in \partial \Omega$ as
\begin{equation}
\tilde{u}(x) = \lim_{r \to 0} \frac{1}{\mathcal{L}^n(B_r(x) \cap \Omega)} \int_{B_r(x) \cap \Omega} u(y) \, dy,
\end{equation}
see [Ma3 Corollary 9.6.5]. Note that this limit actually exists for $\mathcal{H}^{n-1}$-a.e. $x \in \partial \Omega$. As recalled in Section 1, one has that $\tilde{u} \in L^1(\partial \Omega)$ for every function $u \in BV(\Omega)$. Moreover, $L^1(\partial \Omega)$ cannot be replaced with any smaller Lebesgue space independent of $u$. Alternative definitions of the boundary trace of a function of bounded variation are available in the literature. One depends on the upper and lower approximate limits of the extension of $u$ by 0 outside $\Omega$ [Zi Definition 5.10.5]. Another one involves the rough trace [Ma3 Section 9.5.1]. Both of them coincide with $\tilde{u}$, up to subsets of $\partial \Omega$ of $\mathcal{H}^{n-1}$-measure zero.

Traces of functions $u$ from the Sobolev space $W^{1,1}(\Omega)$ are more classically defined on the boundary of a Lipschitz domain $\Omega$ as the limit of the restrictions to $\partial \Omega$ of approximating sequences of smooth functions on $\Omega$. This definition also yields a function on $\partial \Omega$ which agrees with $\tilde{u}$, up to subsets of $\partial \Omega$ of $\mathcal{H}^{n-1}$-measure zero.

A measurable set $E \subset \mathbb{R}^n$ is said to be of finite perimeter relative to $\Omega$ if $D\chi_E$ is a vector-valued Radon measure in $\Omega$ with finite total variation in $\Omega$. The perimeter of $E$ relative to $\Omega$ is defined as
\begin{equation}
P(E; \Omega) = \|D\chi_E\|(\Omega).
\end{equation}
A result in geometric measure theory tells us that $E$ is of finite perimeter in $\Omega$ if and only if $\mathcal{H}^{n-1}(\partial^M E \cap \Omega) < \infty$; moreover,

$$P(E; \Omega) = \mathcal{H}^{n-1}(\partial^M E \cap \Omega)$$

[Fe] Theorem 4.5.11]. When $\Omega = \mathbb{R}^n$, we denote $P(E; \Omega)$ simply by $P(E)$, and call it the perimeter of $E$. Thus,

$$P(E) = \mathcal{H}^{n-1}(\partial^M E).$$

Let $E$ be a set of finite perimeter in $\mathbb{R}^n$. Then the derivative $\nu^E$ of the vector-valued measure $D\chi_E$ with respect to its total variation $|D\chi_E|$ exists, and satisfies $|\nu^E(x)| = 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^M E$. The vector $\nu^E(x)$ is called the generalized inner normal to $E$ at $x$.

Given $\nu \in S^{n-1}$, denote by $\nu^\perp$ the hyperplane which contains 0 and is orthogonal to $\nu$. Given a measurable set $E \subset \mathbb{R}^n$, and $z \in \nu^\perp$, we define

$$E'_z = \{ r \in \mathbb{R} : z + r\nu \in E \}.$$

We also define the essential projection of $E$ on $\nu^\perp$ as

$$\Pi_\nu(E)^+ = \{ z \in \nu^\perp : L^1(E'_z) > 0 \}.$$

Of course, the essential projection of $E$ agrees with its standard projection if $E$ is open.

If, $E$ is a bounded measurable set, we set, for $\nu \in S^{n-1}$ and $z \in \Pi_\nu(E)^+$,

$$\phi_{E,\nu}(z) = \inf_{E'_z} E'_z,$$

and, according to [Gr] p. 233], we call the illuminated portion of $E$ along $\nu$ the set

$$I_\nu(E) = \{ z + \phi_{E,\nu}(z)\nu : z \in \Pi_\nu(E)^+ \}.$$

The classical Cauchy formula tells us that, if $G$ is a convex set, then

$$P(G) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathcal{H}^{n-1}(\Pi_\nu(G)^+) \, d\nu ,$$

see [BZ] Equation (32), Section 19] or [Sch] Equation (5.3.27)]. This formula was employed in the approach of [EGK].

A version of (2.11) for sets of finite perimeter is the content of the following result.

**Theorem 2.1** Let $G$ be a set of finite perimeter and finite Lebesgue measure in $\mathbb{R}^n$. Then

$$P(G) = \frac{1}{2\omega_{n-1}} \int_{S^{n-1}} \left( \int_{\nu^\perp} \mathcal{H}^0((\partial^M G')_z) \, d\mathcal{H}^{n-1}(z) \right) d\mathcal{H}^{n-1}(\nu) .$$

In particular,

$$P(G) \geq \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathcal{H}^{n-1}(\Pi_\nu(G)^+) \, d\nu .$$

Moreover, the following facts are equivalent:

(i) The equality holds in (2.13);
(ii) $G$ is equivalent to a convex set, up to sets of Lebesgue measure zero;
(iii) $G^1$ is convex.
The discussion of the case of equality in \((2.13)\) in the proof of Theorem \(2.1\) makes use of the next lemma, a slight extension of a result of G. Alberti, reported in the survey paper [Fu, Lemma 4.12].

**Lemma 2.2** Let \(G\) be a measurable set in \(\mathbb{R}^n\), \(n \geq 2\), such that, for \(\mathcal{H}^{n-1}\) a.e. \(\nu \in \mathbb{S}^{n-1}\), and for \(\mathcal{H}^{n-1}\) a.e. \(z \in \Pi_\nu(G)^+\), the set \(G_z\) is equivalent to an interval. Then \(G\) is convex.

**Proof.** Set \(F = G^1\). Observe that, since \(G\) and \(F\) are equivalent (up to sets of Lebesgue measure zero), then, as a consequence of Fubini’s theorem, the set \(F\) satisfies the same assumptions as \(G\), namely (2.14) for \(\mathcal{H}^{n-1}\) a.e. \(\nu \in \mathbb{S}^{n-1}\), and for \(\mathcal{H}^{n-1}\) a.e. \(z \in \Pi_\nu(F)^+\), the set \(F_z\) is equivalent to an interval.

Let \(x_1, x_2 \in F\). Let \(\nu \in \mathbb{S}^{n-1}\), \(z \in \nu^\perp\), \(y_1, y_2 \in \mathbb{R}\) be such that \(x_1 = \hat{z} + y_1 \nu\), \(x_2 = \hat{z} + y_2 \nu\), with \(y_1 < y_2\). We have to show that any point \(\hat{x}\) of the form \(\hat{x} = \hat{z} + \tilde{y} \nu\), for some \(\tilde{y} \in (y_1, y_2)\), belongs to \(F\).

Fix an orthogonal system whose \(n\)-th axis has the same direction and orientation as \(\nu\). Given \(x \in \mathbb{R}^n\) and \(r > 0\), denote by \(Q_r(x)\) the cube, centered at \(x\) and with side-length \(2r\), whose sides are parallel to the coordinate axes of the relevant system. Since \(x_1, x_2\) have density 1 with respect to \(G\), they have density 1 with respect to \(F\) as well. One has that \(x \in F\) if and only if \(\mathcal{L}^n(F \cap Q_r(x)) = 1\) (see e.g. [Fu, Section 4.2]). Thus, for every \(\varepsilon > 0\), there exists \(r_\varepsilon > 0\) such that, if \(0 < r < r_\varepsilon\), then

\[
\mathcal{L}^n(F \cap Q_r(x_i)) > 1 - \varepsilon, \quad \text{for } i = 1, 2.
\]

By property (2.14), there exists a sequence \(\{\nu_k\} \subset \mathbb{S}^{n-1}\), such that \(\nu_k \to \nu\) as \(k \to \infty\), and \(F_{\nu_k}^\prime\) is equivalent to an interval for \(\mathcal{H}^{n-1}\)-a.e. \(z \in \Pi_{\nu_k}(F)^+\). From (2.15) and Fubini’s theorem we deduce that

\[
2^n r_n(1 - \varepsilon) < \mathcal{L}^n(F \cap Q_r(x_i)) = \int_{\Pi_{\nu_k}(F \cap Q_r(x_i))^+} \mathcal{L}^1((F \cap Q_r(x_i))^{\nu_k}) d\mathcal{H}^{n-1}(z) \\
\leq 2r a_k \mathcal{H}^{n-1}(\Pi_{\nu_k}(F \cap Q_r(x_i))^+) \quad \text{for } i = 1, 2,
\]

for some sequence \(\{a_k\}\) such that \(a_k \geq 1\), and \(a_k \to 1\) as \(k \to \infty\). Hence,

\[
\mathcal{H}^{n-1}(\Pi_{\nu_k}(F \cap Q_r(x_i))^+) \geq \frac{2^{n-1} r^{n-1}(1 - \varepsilon)}{a_k} \quad \text{for } i = 1, 2, \text{ and } k \in \mathbb{N}.
\]

Since \(\Pi_{\nu_k}(F \cap Q_r(x_1))^+\) and \(\Pi_{\nu_k}(F \cap Q_r(x_2))^+\) satisfy (2.17), and are contained in a set which converges to an \((n - 1)\)-dimensional cube in \(\nu^\perp\) of side-length \(2r\) as \(k \to \infty\), there exists another sequence \(\{b_k\}\) such that \(b_k \geq 1\), and \(b_k \to 1\) as \(k \to \infty\), such that

\[
\mathcal{H}^{n-1}(\Pi_{\nu_k}(F \cap Q_r(x_1))^+ \cap \Pi_{\nu_k}(F \cap Q_r(x_2))^+) > \frac{2^{n-1} r^{n-1}(1 - 2b_k \varepsilon)}{a_k}.
\]

Since \(\{\nu_k\}\) is chosen in such a way that (2.14) is satisfied with \(\nu = \nu_k\), we have that for \(\mathcal{H}^{n-1}\)-a.e. \(z \in \Pi_{\nu_k}(F \cap Q_r(x_1))^+ \cap \Pi_{\nu_k}(F \cap Q_r(x_2))^+\), the set \(F_{\nu_k}^\prime\) is an interval, and \(\mathcal{L}^1((F \cap Q_r(x_i))^{\nu_k}) > 0\) for \(i = 1, 2\). Thus, if \(r\) is sufficiently small, depending on \(y_1, y_2\) and \(\tilde{y}\), there exists a sequence \(\{A_{k,r}\}\) of polyhedra in \(\nu_k^\perp\) such that

\[
\mathcal{H}^{n-1}((\Pi_{\nu_k}(F \cap Q_r(x_1))^+ \cap \Pi_{\nu_k}(F \cap Q_r(x_2))^+) \setminus A_{k,r}) \to 0 \quad \text{as } k \to \infty,
\]
and
\[ (2.19) \quad L^1((F \cap Q_r(\widehat{x}))^\nu) > 2r \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } z \in \Pi_{\nu_k}(F \cap Q_r(x_1))^+ \cap \Pi_{\nu_k}(F \cap Q_r(x_2))^+ \cap A_{k,r}. \]

Coupling (2.18) with (2.19) tells us that
\[ (2.20) \quad L^n(F \cap Q_r(\widehat{x})) > 2r \left( \frac{2^{n-1}r^{n-1}(1-2b_k\varepsilon)}{a_k} - c_{k,r} \right) \]
for some sequence \( \{c_{k,r}\} \) such that \( c_{k,r} \geq 0 \) and \( c_{k,r} \to 0 \) as \( k \to \infty \) for every fixed (sufficiently small) \( r \). Passing to the limit in (2.20) as \( k \to \infty \) yields
\[ \liminf_{r \to 0} \frac{L^n(F \cap Q_r(\widehat{x}))}{2^n r^n} > (1-2\varepsilon), \]
whence
\[ \liminf_{r \to 0} \frac{L^n(F \cap Q_r(\widehat{x}))}{2^n r^n} > (1-2\varepsilon). \]

Owing to the arbitrariness of \( \varepsilon \), the last equation ensures that \( \widehat{x} \) has density 1 with respect to \( F \), and hence \( x \in F \). \qed

**Proof of Theorem 2.1** A special case of the coarea formula on rectifiable sets tells us that for each \( \nu \in \mathbb{S}^{n-1} \),
\[ (2.21) \quad \int_{\partial^M G} |\nu^G(x) \cdot \nu| d\mathcal{H}^{n-1}(x) = \int_{\nu_\perp} \mathcal{H}^0((\partial^M G)_z^\nu) d\mathcal{H}^{n-1}(z) \]
(see, for instance, [CCF, Theorem F]).

On integrating equation (2.21) with respect to \( \nu \) over \( \mathbb{S}^{n-1} \) yields
\[ (2.22) \quad \int_{\mathbb{S}^{n-1}} \int_{\partial^M G} |\nu^G(x) \cdot \nu| d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(\nu) = \int_{\mathbb{S}^{n-1}} \left( \int_{\nu_\perp} \mathcal{H}^0((\partial^M G)_z^\nu) d\mathcal{H}^{n-1}(z) \right) d\mathcal{H}^{n-1}(\nu). \]

By Fubini’s theorem
\[ (2.23) \quad \int_{\mathbb{S}^{n-1}} \int_{\partial^M G} |\nu^G(x) \cdot \nu| d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(\nu) = \int_{\partial^M G} \int_{\mathbb{S}^{n-1}} |\nu^G(x) \cdot \nu| d\mathcal{H}^{n-1}(\nu) d\mathcal{H}^{n-1}(x) \]
\[ = 2\omega_{n-1} \int_{\partial^M G} d\mathcal{H}^{n-1}(x) = 2\omega_{n-1} P(G), \]
where the second inequality holds since the integral
\[ \int_{\mathbb{S}^{n-1}} |\nu^G(x) \cdot \nu| d\mathcal{H}^{n-1}(\nu) \]
is clearly independent of \( G \) and \( x \), and equals \( 2\omega_{n-1} \). Equation (2.12) thus follows from (2.22) and (2.23).

Let us now focus on (2.13). We claim that, for every \( \nu \in \mathbb{S}^{n-1} \),
\[ (2.24) \quad \mathcal{H}^0((\partial^M G)_z^\nu) \geq 2 \quad \text{for } \mathcal{H}^{n-1}-\text{a.e. } z \in \Pi_{\nu}(G)^+. \]

Indeed, for every \( \nu \in \mathbb{S}^{n-1} \), there exists a Borel subset \( B_{G,\nu} \) of \( \Pi_{\nu}(G)^+ \) such that \( \mathcal{H}^{n-1}(\Pi_{\nu}(G)^+ \setminus B_{G,\nu}) = 0 \), and, for every \( z \in B_G \), the set \( G_z^\nu \) is of finite perimeter and measure in \( \mathbb{R} \), and
\[ (2.25) \quad (\partial^M G)_z^\nu = \partial^M(G_z^\nu) \].
(see e.g. [CCF, Theorem G]). Note that the fact that $\mathcal{L}^1(G^\nu_z) < \infty$ for $z \in B_{G,\nu}$ is a consequence of the assumption $\mathcal{L}^n(G) < \infty$, since, by Fubini’s theorem,

\[ \mathcal{L}^n(G) = \int_{\Pi_\nu(G)^+} \mathcal{L}^1(G^\nu_z) \, dz. \]

By the isoperimetric inequality in $\mathbb{R}$,

\[ \mathcal{H}^0(\partial^\mathcal{M}(G^\nu_z)) \geq 2 \]

for every $z \in B_{G,\nu}$, and hence for $\mathcal{H}^{n-1}$- a.e. $z \in \Pi_\nu(G)^+$. Moreover, the equality holds in (2.26) if and only if $G^\nu_z$ is equivalent to an interval. Thus, (2.24) follows from (2.24) and (2.26), and one has that the equality holds in (2.27) if and only if $G^\nu_z$ is equivalent to an interval.

By (2.24)

\[ \int_{\nu_\perp} \mathcal{H}^0((\partial^\mathcal{M} G^\nu_z) \, d\mathcal{H}^{n-1}(z) \geq 2\mathcal{H}^{n-1}(\Pi_\nu(G)^+) \quad \text{for every } \nu \in S^{n-1}. \]

Inequality (2.13) is a consequence of (2.12) and (2.27). As far as the case of equality in (2.13) is concerned, if (ii) holds, then the equality holds in inequalities (2.24)–(2.27), and hence also in (2.13), whence (i) follows. The fact that (iii) implies (ii) is a consequence of the equivalence of $G$ and $G^1$ up to sets of Lebesgue measure zero. It remains to show that (i) implies (iii). Assume that (i) holds. Then equality holds in (2.27) for $\mathcal{H}^{n-1}$- a.e. $\nu \in S^{n-1}$. Hence, for $\mathcal{H}^{n-1}$- a.e. $\nu \in S^{n-1}$, equality also holds in (2.26) for $\mathcal{H}^{n-1}$ a.e. $z \in \Pi_\nu(G)^+$. Thus, for $\mathcal{H}^{n-1}$- a.e. $\nu \in S^{n-1}$, the set $G^\nu_z$ equivalent to an interval for $\mathcal{H}^{n-1}$- a.e. $z \in \Pi_\nu(G)^+$. Property (iii) hence follows via Lemma 2.2.

3 Proofs of the main results

We begin by accomplishing the proof of Theorem 1.1.

Proof of Theorem 1.1 For any given $\nu \in S^{n-1}$, there exist two sequences of open half-spaces $\{H^+_{\nu,i}\}_{i \in \mathbb{N}}$ and $\{H^-_{\nu,i}\}_{i \in \mathbb{N}}$ such that, for $i \in \mathbb{N}$,

\[ \pm \nu \text{ is the outer unit normal to } H^\pm_{\nu,i} \text{ on } \partial H^\pm_{\nu,i}, \]

\[ H^+_{\nu,i} \subset H^+_{\nu,i+1}, \]

\[ \overline{H^+_{\nu,i}} \cap \overline{H^-_{\nu,i}} = \emptyset, \]

\[ \mathcal{H}^{n-1}(\partial \Omega \cap \partial^\mathcal{M}(H^\pm_{\nu,i} \cap \Omega)) \leq P(\Omega)/2, \]

and, on defining $H^+_{\nu} = \bigcup_i H^+_{\nu,i}$ and $H^-_{\nu} = \bigcup_i H^-_{\nu,i}$,

\[ \partial H^+_{\nu} = \partial H^-_{\nu}. \]

Notice that here, and in similar occurrences below, the use of just $\partial \Omega$ instead of $\partial^\mathcal{M} \Omega$ is allowed by the fact that $\Omega$ is an admissible domain.

One has that

\[ \mathcal{H}^{n-1}(\partial \Omega \cap \partial^\mathcal{M}(H^+_{\nu} \cap \Omega)) + \mathcal{H}^{n-1}(\partial \Omega \cap \partial^\mathcal{M}(H^-_{\nu} \cap \Omega)) \]

\[ = \mathcal{H}^{n-1}(\partial \Omega \cap \partial^\mathcal{M}(H^+_{\nu} \cap \Omega)) + \mathcal{H}^{n-1}(\partial \Omega \setminus \partial^\mathcal{M}(H^+_{\nu} \cap \Omega)) = P(\Omega). \]
Observe that the first equality in (3.2) follows from (2.1) applied with \( E = H^+_\nu \cap \Omega \), since \( \Omega \setminus E = (H^-_\nu \cap \Omega) \cup (\partial H^-_\nu \cap \Omega) \), and hence \( \Omega \setminus E = (H^-_\nu \cap \Omega) \), up to sets of Lebesgue measure zero. The second equality in (3.2) holds since \( \mathcal{H}^{n-1} \) is a measure on Borel sets.

Let us define
\[
H_\nu = H^+_\nu.
\]

We claim that
\[
(3.3) \quad \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H_\nu \cap \Omega)) = P(\Omega)/2 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \nu \in S^{n-1}.
\]

Owing to (3.2), equation (3.3) only fails if
\[
(3.4) \quad \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^+_\nu \cap \Omega)) \neq \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^-_\nu \cap \Omega)).
\]

Thus, in order to prove our claim, it suffices to show that (3.4) can only hold for \( \nu \) in a countable subset of \( S^{n-1} \). Assume that (3.4) is on force for some \( \nu \in S^{n-1} \). Then

\[
(3.5) \quad \lim_{i \to \infty} \min \left\{ \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^+_\nu,i \cap \Omega)), \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^-_\nu,i \cap \Omega)) \right\} \\
\leq \min \left\{ \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^+_\nu \cap \Omega)), \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^-_\nu \cap \Omega)) \right\} < P(\Omega)/2,
\]

where the first inequality holds owing to equation (2.6) with \( E = H^+_\nu,i \cap \Omega \) and \( F = H^+_\nu \cap \Omega \), and the second one by (3.1) and (3.4). The following chain holds:

\[
(3.6) \quad P(\Omega) = \mathcal{H}^{n-1}(\partial \Omega \cap H^+_\nu) + \mathcal{H}^{n-1}(\partial \Omega \cap H^-_\nu) + \mathcal{H}^{n-1}(\partial \Omega \cap \partial H^+_\nu) \\
= \lim_{i \to \infty} \left( \mathcal{H}^{n-1}(\partial \Omega \cap H^+_\nu,i) + \mathcal{H}^{n-1}(\partial \Omega \cap H^-_\nu,i) \right) + \mathcal{H}^{n-1}(\partial \Omega \cap \partial H^+_\nu) \\
\leq \lim_{i \to \infty} \left( \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^+_\nu,i \cap \Omega)) + \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^-_\nu,i \cap \Omega)) \right) + \mathcal{H}^{n-1}(\partial \Omega \cap \partial H^+_\nu) \\
\leq \lim_{i \to \infty} \min \left\{ \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^+_\nu,i \cap \Omega)), \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M(H^-_\nu,i \cap \Omega)) \right\} + P(\Omega)/2 + \mathcal{H}^{n-1}(\partial \Omega \cap \partial H^+_\nu) < P(\Omega) + \mathcal{H}^{n-1}(\partial \Omega \cap \partial H^+_\nu).
\]

Note that the first equality in (3.6) holds since \( \mathcal{H}^{n-1} \) is a measure when restricted to Borel sets, and hence it is (countably) additive on disjoint Borel sets, the second equality again relies upon the fact that \( \mathcal{H}^{n-1} \) is a measure on Borel sets, and \( \partial \Omega \cap H^+_\nu,i \cap \partial \Omega \cap H^-_\nu,i \), the first inequality is a consequence of the inclusion
\[
(3.7) \quad \partial^M \Omega \cap H^+_\nu,i \subset \partial^M \Omega \cap \partial^M(H^+_\nu,i \cap \Omega),
\]

which, in turn, follows from (2.2) applied with \( A = H^+_\nu,i \), and the second and third inequality are consequences of (3.1) and (3.5), respectively. Equation (3.6) implies that \( \mathcal{H}^{n-1}(\partial \Omega \cap \partial H^+_\nu) > 0 \). Since \( \Omega \) has finite perimeter, the latter inequality can hold at most for \( \nu \) in a countable subset of \( S^{n-1} \). Hence, our claim follows.

Now, by (2.1), \( \partial^M(H_\nu \cap \Omega) \subset \partial^M H_\nu \cup \partial^M \Omega = \partial H_\nu \cup \partial^M \Omega \). Thus, \( \Omega \cap \partial^M(H_\nu \cap \Omega) \subset \Omega \cap (\partial H_\nu \cup \partial^M \Omega) = (\partial H_\nu \cap \Omega) \cup (\partial^M \Omega \cap \Omega) = \partial H_\nu \cap \Omega \). Hence,
\[
\mathcal{H}^{n-1}(\Omega \cap \partial^M(H_\nu \cap \Omega)) \leq \mathcal{H}^{n-1}(\partial H_\nu \cap \Omega).
\]

Since \( \nu \) is orthogonal to the hyperplane \( \partial H_\nu \),
\[
\mathcal{H}^{n-1}(\partial H_\nu \cap \Omega) \leq \mathcal{H}^{n-1}(\Pi^+_\nu(\Omega)).
\]
Altogether, we obtain
\[ \mathcal{H}^{n-1}(\Pi_\nu(\Omega)^+) \geq \mathcal{H}^{n-1}(\Omega \cap \partial^M(H_\nu \cap \Omega)) \] for \( \mathcal{H}^{n-1}\)-a.e. \( \nu \in \mathbb{S}^{n-1} \).

Hence, by (2.13),
\[ (3.8) \quad P(\Omega) \geq \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(\Pi_\nu(\Omega)^+) \, d\nu \geq \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(\Omega \cap \partial^M(H_\nu \cap \Omega)) \, d\nu. \]

Finally, from the definition of \( C_{\text{med}}(\Omega) \) and (3.8) we have that
\[ (3.9) \quad C_{\text{med}}(\Omega) \geq \frac{P(\Omega)}{2} \inf_{H \text{ half-space}} \frac{1}{\mathcal{H}^{n-1}(\Omega \cap \partial^M(H \cap \Omega))} \geq \frac{P(\Omega)}{2} \frac{1}{\omega_{n-1} \int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(\Omega \cap \partial^M(H_\nu \cap \Omega)) \, d\nu} \geq \frac{n \omega_n}{2 \omega_{n-1}} = \sqrt{\pi \frac{n}{2}} \Gamma\left(\frac{n+1}{2}\right), \]
whence (1.6) follows.

As for the equality case in (1.6), observe that if the equality holds in (1.6), then it also holds in the chain of inequalities (3.9), and hence in (3.8) as well. Hence, by Theorem 2.1, \( \Omega^1 \) is a convex set. The convexity of \( \Omega^1 \) implies that it is an open set. To verify this assertion, it suffices to show that \( \partial \Omega^1 \cap \Omega^1 = \emptyset \). The latter equality is in turn a consequence of the fact that, since \( \Omega^1 \) is convex, if \( x \in \partial \Omega^1 \) then
\[ D(\Omega^1, x) = D(\Omega^1 \setminus \{x\}) = D(\Omega^1, x) \in (0, 1), \]
and hence \( \partial \Omega^1 = \partial^M \Omega^1 = \partial^M \Omega^1 \). The openness of \( \Omega \) implies that \( \Omega \subset \Omega^1 \). We claim that
\[ (3.10) \quad \mathcal{H}^{n-1}(\Omega^1 \setminus \Omega) = 0. \]

Indeed, since \( \Omega \subset \Omega^1 \subset \overline{\Omega} \), we have that \( \overline{\Omega}^1 = \overline{\Omega} \). Thus, \( \Omega^1 \setminus \Omega \subset \partial \Omega \). Inasmuch as \( \Omega^1 \cap \partial^M \Omega = \emptyset \), one has \( \Omega^1 \setminus \Omega \subset \partial \Omega \setminus \partial^M \Omega \), and hence (3.10) follows since \( \Omega \) is admissible. It only remains to show that \( \Omega^1 \) is a ball. Let \( H_\nu \) denote the half-space defined as above, with \( \Omega \) replaced with \( \Omega^1 \). The convexity of \( \Omega^1 \) ensures that, for every \( \nu \in \mathbb{S}^{n-1} \), \( H_\nu \) agrees with the unique open half space such that \( \nu \) is the outer normal to \( \partial H_\nu \), and
\[ (3.11) \quad \mathcal{H}^{n-1}(\partial \Omega^1 \cap \partial H_\nu) = \mathcal{H}^{n-1}(\partial \Omega^1 \cap \partial^M(H_\nu \cap \Omega^1)) = P(\Omega^1) / 2. \]

Since the equality holds in the chain of inequalities (3.8) and (3.9),
\[ (3.12) \quad \mathcal{H}^{n-1}(\Omega^1 \cap \partial^M(H_\nu \cap \Omega^1)) = \mathcal{H}^{n-1}(\Pi_\nu(\Omega^1)^+) = P(\Omega) \frac{\omega_{n-1}}{\omega_n}, \quad \text{for every } \nu \in \mathbb{S}^{n-1}. \]

Note that (3.12) actually holds for every \( \nu \in \mathbb{S}^{n-1} \), since \( \mathcal{H}^{n-1}(\Omega^1 \cap \partial^M(H_\nu \cap \Omega^1)) \) and \( \mathcal{H}^{n-1}(\Pi_\nu(\Omega^1)^+) \) are continuous functions of \( \nu \), owing to the convexity of \( \Omega^1 \).

Our next step consists in proving that \( \Omega^1 \) is, in fact, strictly convex. To this purpose, it suffices to show that no segment is contained in \( \partial \Omega^1 \). Let us assume, by contradiction, that there exist a straight line intersecting \( \partial \Omega^1 \) in a whole segment \( \Sigma \). Denote by \( x_1 \) and \( x_2 \) the endpoints of \( \Sigma \). Set \( \nu = \frac{x_2 - x_1}{|x_2 - x_1|} \). Observe that \( \partial H_\nu \cap \Sigma \neq \emptyset \), otherwise \( \mathcal{H}^{n-1}(\Pi_\nu(\Omega^1)^+) > \mathcal{H}^{n-1}(\Omega^1 \cap \partial^M(H_\nu \cap \Omega^1)) \), thus contradicting (3.12). Let \( \tilde{y} \) be the point such that \( \partial H_\nu \cap \Sigma = \{\tilde{y}\} \), and let \( \tilde{z} \) be any point in \( \Sigma \), different from \( x_1, x_2 \), and \( \tilde{y} \). Let \( H \) be an open half-space such that \( \Sigma \subset \partial H \) and \( \Omega^1 \subset H \) (in
particular, \(\partial H\) is a support hyperplane to the convex set \(\Omega^1\), and let \(\mu \in \mathbb{S}^{n-1}\) be the outward unit normal vector to \(H\) on \(\partial H\). One has that
\[
\text{span}(\mu, \nu) \cap \mathbb{S}^{n-1} = \{\xi(\vartheta) : \xi(\vartheta) = \cos(\vartheta)\mu + \sin(\vartheta)\nu \text{ for some } \vartheta \in [0, 2\pi]\}.
\]
Given \(\vartheta \in [0, 2\pi]\), denote by \(H(\vartheta)\) the open half-space such that \(\tilde{z} \in \partial H(\vartheta)\), and \(\xi(\vartheta)\) outward unit normal vector to \(H(\vartheta)\) on \(\partial H(\vartheta)\). In particular, \(H(0) = H\), and
\[
(3.13) \quad \partial H(z) \text{ is parallel to } \partial H(\nu).
\]
Define the function \(m : [0, 2\pi] \to [0, \infty)\) as
\[
m(\vartheta) = \mathcal{H}^{n-1}(\partial \Omega^1 \cap H(\vartheta)) \quad \text{for } \vartheta \in [0, 2\pi],
\]
and observe that \(m\) is a continuous function satisfying
\[
m(\pi) = 0, \quad m(0) > \frac{P(\Omega^1)}{2}.
\]
Therefore there exists \(\hat{\vartheta} \in (0, \pi)\) such that \(m(\hat{\vartheta}) = \frac{P(\Omega^1)}{2}\). Since \(\hat{y} \neq \hat{z}\) we have that \(\hat{\vartheta} \neq \frac{\pi}{2}\), otherwise \(H(\nu)\) and \(H(z)\) would be distinct half-spaces (since they intersect \(\Sigma\) at different points), satisfying \((3.13)\) and \(\mathcal{H}^{n-1}(\partial \Omega^1 \cap H(\nu)) = \mathcal{H}^{n-1}(\partial \Omega^1 \cap H(\frac{\pi}{2})) = \frac{P(\Omega^1)}{2}\). Since \(H(\hat{\vartheta})\) fulfills \(\mathcal{H}^{n-1}(\partial \Omega^1 \cap H(\hat{\vartheta})) = \frac{P(\Omega^1)}{2}\), and \(\tilde{z} \in \partial H(\hat{\vartheta})\), one has that \(H(\hat{\vartheta}) = H(\xi(\hat{\vartheta}))\). Hence, there exists a support hyperplane to \(\Omega^1\) at \(\tilde{z}\) which is orthogonal to \(\partial H(\hat{\vartheta})\), otherwise the first equality in \((3.12)\) would fail for \(\nu = \xi(\hat{\vartheta})\). Such support hyperplane to \(\Omega^1\), being orthogonal to \(\partial H(\hat{\vartheta})\), would intersect \(\Sigma\) only at \(\tilde{z}\), but this is impossible, since any support hyperplane to \(\Omega^1\) at a point of \(\Sigma\) necessarily contains the whole of \(\Sigma\). The strict convexity of \(\Omega^1\) is thus established.
By the strict convexity of \(\Omega^1\), and the first equality in \((3.12)\),
\[
(3.14) \quad \mathcal{H}^{n-1}(I_{\nu}(\Omega^1)) = \mathcal{H}^{n-1}(\partial \Omega^1 \cap H(\nu)) = \frac{P(\Omega^1)/2}{2} \quad \text{for every } \nu \in \mathbb{S}^{n-1},
\]
where \(I_{\nu}(\Omega^1)\) denotes the illuminated portion of \(\Omega^1\), defined as in \((2.10)\). In particular,
\[
(3.15) \quad \mathcal{H}^{n-1}(I_{\nu}(\Omega^1)) = \mathcal{H}^{n-1}(I_{-\nu}(\Omega^1)) \quad \text{for every } \nu \in \mathbb{S}^{n-1}.
\]
Property \((3.15)\) implies, via \([Gr\) Theorem 5.5.11], that \(\Omega^1\) is centrally symmetric. Finally, on calling \(B\) the ball with the same perimeter as \(\Omega^1\), we infer from the second equality in \((3.12)\) that
\[
\mathcal{H}^{n-1}(\Pi_{\nu}(\Omega^1)^+) = \mathcal{H}^{n-1}(\Pi_{\nu}(B)^+) \quad \text{for every } \nu \in \mathbb{S}^{n-1}.
\]
Hence, owing to \([Gr\) Theorem 5.5.6], we conclude that \(\Omega^1\) is a ball.

\[\square\]

\textbf{Remark 3.1} The proof of Theorem 1.1 considerably simplifies under the additional assumption that \(\Omega\) is convex. In this case, the hyperplane \(H_\nu\) can be defined via \((3.11)\) for every \(\nu \in \mathbb{S}^{n-1}\), and inequality \((1.6)\) follows from formula \((2.11)\) and the chain \((3.9)\). The characterization of balls as the only convex sets yielding the equality in \((1.6)\) can be established as in the last part of the above proof, on replacing \(\Omega^1\) just by \(\Omega\).

We conclude with a proof of Theorem 1.2.
Proof of Theorem 1.2 Assume that \( n \geq 3 \). From (1.4) we deduce that
\[
C_{mv}(\Omega) \geq C_{med}(\Omega) \geq \sqrt{\pi \frac{n(n+1)}{2}} \frac{n}{2(n+2)},
\]
namely (1.8). Moreover, the assertion concerning the case of equality in (1.8) follows from Theorem 1.1, since the equality in (1.8) implies the equality (1.6).

Assume now that \( n = 2 \). Denote by \( \hat{\Omega} \) the convex hull of \( \Omega \). By a standard result in the theory of convex bodies, there exists an extreme point \( x_0 \) for \( \hat{\Omega} \), and, necessarily, \( x_0 \in \partial \hat{\Omega} \cap \partial \Omega \). Moreover, as a consequence, for instance, of Straszewicz’s theorem \([\text{Sch}, \text{Theorem 1.4.7}]\), the point \( x_0 \) can be chosen in such a way that it is also an exposed point for \( \hat{\Omega} \), so that there exists an open half-plane \( H_0 \) such that
\[
\Omega \subset H_0,
\]
and
\[
\hat{\Omega} \cap \partial H_0 = \{x_0\}.
\]
Assume, without loss of generality, that \( x_0 = (0,0) \), and \( H_0 = \{(x,y) \in \mathbb{R}^2 : y > 0\} \), whence
\[
\Omega \subset \{(x,y) \in \mathbb{R}^2 : y > 0\}.
\]
Given \( \varepsilon > 0 \), consider the open set
\[
\Omega(\varepsilon) = \{(x,y) \in \Omega : y < \varepsilon\}.
\]
We claim that
\[
(3.16) \quad \mathcal{H}^1(\partial^M \Omega(\varepsilon) \cap \Omega) \leq \mathcal{H}^1(\partial^M \Omega(\varepsilon) \cap \partial \Omega) \quad \text{for } \varepsilon > 0,
\]
and
\[
(3.17) \quad \lim_{\varepsilon \to 0} \mathcal{H}^1(\partial \Omega \setminus \partial^M \Omega(\varepsilon)) = P(\Omega).
\]
Let us prove (3.16) first. Recall that \( \mathcal{H}^1(\partial \Omega \setminus \partial^M \Omega) = 0 \), since \( \Omega \) is an admissible domain. It is easily seen that
\[
(3.18) \quad \partial^M \Omega(\varepsilon) \cap \Omega = \{y = \varepsilon\} \cap \Omega \quad \text{for } \varepsilon > 0.
\]
Set \( e_2 = (0,1) \). By the coarea formula (2.21),
\[
(3.19) \quad \int_{\partial^M \Omega(\varepsilon)} \mid \nu^\Omega(x) \cdot e_2 \mid d\mathcal{H}^1(x) = \int_{\mathbb{R}} \mathcal{H}^0((\partial^M \Omega(\varepsilon))^{e_2}) dz.
\]
We have that
\[
(3.20) \quad \int_{\mathbb{R}} \mathcal{H}^0((\partial^M \Omega(\varepsilon))^{e_2}) dz \geq 2 \mathcal{H}^1(\Pi e_2(\Omega(\varepsilon))^{+}) \geq 2 \mathcal{H}^1(\{y = \varepsilon\} \cap \Omega) = 2 \mathcal{H}^1(\partial^M \Omega(\varepsilon) \cap \Omega).
\]
On the other hand,
\[
(3.21) \quad \int_{\partial^M \Omega(\varepsilon)} \mid \nu^\Omega(x) \cdot e_2 \mid d\mathcal{H}^1(x) \leq \mathcal{H}^1(\partial^M \Omega(\varepsilon)) = \mathcal{H}^1(\partial^M \Omega(\varepsilon) \cap \Omega) + \mathcal{H}^1(\partial^M \Omega(\varepsilon) \cap \partial \Omega).
\]
Inequality (3.16) follows from (3.20) and (3.21).
Consider next (3.17). Owing to our choice of \( x_0 \), one has that \( \partial^M \Omega \setminus \partial^M \Omega(\varepsilon) \) is an increasing family of sets as \( \varepsilon \searrow 0^+ \). Moreover,

\[
    \bigcup_{\varepsilon > 0} (\partial^M \Omega \setminus \partial^M \Omega(\varepsilon)) = \partial^M \Omega \setminus \Gamma,
\]

where \( \Gamma \) equals either \( \{x_0\} \) or \( \emptyset \), according to whether \( x_0 \) belongs to \( \partial^M \Omega \) or not. In order to verify (3.22), observe that

\[
    \partial^M \Omega \cap \{y < \varepsilon\} \subset \partial^M \Omega \cap \partial^M (\Omega \cap \{y < \varepsilon\}) = \partial^M \Omega \cap \partial^M \Omega(\varepsilon),
\]

where the inclusion holds by (3.7), with \( H_{\nu,i}^\pm \) replaced with \( \{y < \varepsilon\} \). Thus,

\[
    \partial^M \Omega \cap \partial^M \Omega(\varepsilon) = \partial^M \Omega \setminus (\partial^M \Omega \cap \partial^M \Omega(\varepsilon)) \subset \partial^M \Omega \setminus (\partial^M \Omega \cap \{y < \varepsilon\}).
\]

Now, \( \partial^M \Omega \cap \{y < \varepsilon\} \searrow \Gamma \) as \( \varepsilon \to 0^+ \), since \( \partial^M \Omega \subset \partial \Omega \) and \( \partial \Omega \cap \{y < \varepsilon\} \searrow \{x_0\} \). Hence, (3.22) follows.

Since \( H^1 \) is a measure on Borel sets, from (3.22) we obtain that

\[
    \lim_{\varepsilon \to 0^+} H^1(\partial^M \Omega \setminus \partial^M \Omega(\varepsilon)) = H^1(\partial^M \Omega \setminus \Gamma) = H^1(\partial^M \Omega).
\]

Inasmuch as \( \Omega \) is an admissible domain, equation (3.17) is a consequence of (3.23).

By (3.16) and (3.17)

\[
    C_{mv}(\Omega) \geq \lim_{\varepsilon \to 0^+} \frac{2}{\mathcal{H}^1(\partial \Omega)} \frac{\mathcal{H}^1(\partial \Omega(\varepsilon) \cap \partial^M \Omega) \mathcal{H}^1(\partial^M \Omega \setminus \partial^M \Omega(\varepsilon))}{\mathcal{H}^1(\partial^M \Omega(\varepsilon) \cap \Omega)}
\]

\[
    \geq \lim_{\varepsilon \to 0^+} \frac{2 \mathcal{H}^1(\partial \Omega \setminus \partial^M \Omega(\varepsilon))}{\mathcal{H}^1(\partial \Omega)} = 2,
\]

whence (1.9) follows.

We complete the proof by showing that there exist domains \( \Omega \) in \( \mathbb{R}^2 \), with a continuously differentiable boundary, which are different from a disk, such that

\[
    C_{mv}(\Omega) = 2.
\]

The sets \( \Omega \) which will be exhibited are stadium-shaped. Let us preliminarily observe that, if \( \Omega \) is a bounded convex domain in \( \mathbb{R}^2 \), then

\[
    C_{mv}(\Omega) = \frac{2}{\mathcal{H}^1(\partial \Omega)} \sup_{E = H \cap \Omega} \frac{\mathcal{H}^1(\partial E \cap \partial \Omega) \mathcal{H}^1(\partial \Omega \setminus \partial E)}{\mathcal{H}^1(\partial E \cap \Omega)}.
\]

Indeed, as a consequence of the results of [Ma3, Section 9.4.1],

\[
    C_{mv}(\Omega) = \frac{2}{\mathcal{H}^1(\partial \Omega)} \sup_{E = P \cap \Omega} \frac{\mathcal{H}^1(\partial E \cap \partial \Omega) \mathcal{H}^1(\partial \Omega \setminus \partial E)}{\mathcal{H}^1(\partial E \cap \Omega)}.
\]
To prove this claim, observe that, if \( \{E_i\}_{i=1,\ldots,k}, k \in \mathbb{N} \), are the connected components of a set \( E \) as in (3.26), then

\[
(3.27) \quad \frac{\mathcal{H}^1(\partial E \cap \partial \Omega) \mathcal{H}^1(\partial \Omega \setminus \partial E)}{\mathcal{H}^1(\partial E \cap \Omega)} = \frac{\mathcal{H}^1(\partial(\cup_{i=1}^k E_i) \cap \partial \Omega) (\mathcal{H}^1(\partial \Omega \setminus \partial(\cup_{i=1}^k E_i)))}{\mathcal{H}^1(\partial(\cup_{i=1}^k E_i) \cap \Omega)}
= \frac{\mathcal{H}^1(\cup_{i=1}^k(\partial E_i) \cap \partial \Omega) (\mathcal{H}^1(\partial \Omega \setminus (\cup_{i=1}^k \partial E_i)))}{\mathcal{H}^1(\cup_{i=1}^k(\partial E_i) \cap \Omega)}
= \frac{\sum_{i=1}^k \mathcal{H}^1(\partial E_i \cap \partial \Omega) (\mathcal{H}^1(\partial \Omega) - \sum_{i=1}^k \mathcal{H}^1(\partial E_i \cap \partial \Omega))}{\sum_{i=1}^k \mathcal{H}^1(\partial E_i \cap \Omega)}
\leq \max_i \frac{\mathcal{H}^1(\partial E_i \cap \partial \Omega) (\mathcal{H}^1(\partial \Omega) - \mathcal{H}^1(\partial E_i \cap \partial \Omega))}{\mathcal{H}^1(\partial E_i \cap \Omega)}
= \max_i \frac{\mathcal{H}^1(\partial E_i \cap \partial \Omega) \mathcal{H}^1(\partial \Omega \setminus \partial E_i)}{\mathcal{H}^1(\partial E_i \cap \Omega)}.
\]

Thus, on replacing, if necessary, \( E \) with one of its connected components, the supremum in (3.26) can be restricted to the class of sets \( E \) of the form \( \mathcal{P} \cap \Omega \) which are connected. Our next step consists in showing that we may also assume that \( \partial E \cap \Omega \) is connected. Indeed, given any connected set of the form \( E = \mathcal{P} \cap \Omega \) for some polygon \( \mathcal{P} \), denote by \( \{F_j\}_{j=1,\ldots,m}, m \in \mathbb{N} \), the connected components of \( \Omega \setminus E \). Since \( F_j \) and \( \Omega \setminus F_j \) are connected for every \( j = 1, \ldots, m \), we have that \( \partial F_j \cap \Omega \) is connected as well. Equation (2.6), and an analogous chain as in (3.27) tells us that

\[
\frac{\mathcal{H}^1(\partial E \cap \partial \Omega) \mathcal{H}^1(\partial \Omega \setminus \partial E)}{\mathcal{H}^1(\partial E \cap \Omega)} = \frac{\mathcal{H}^1(\partial(\Omega \setminus E) \cap \partial \Omega) \mathcal{H}^1(\partial(\Omega \setminus E) \setminus \partial(\Omega \setminus E))}{\mathcal{H}^1(\partial(\Omega \setminus E) \cap \Omega)}
= \frac{\mathcal{H}^1(\partial(\Omega \setminus E) \cap \partial \Omega) (\mathcal{H}^1(\partial(\Omega \setminus E) \cap \partial \Omega))}{\mathcal{H}^1(\partial(\Omega \setminus E) \cap \Omega)}
\leq \max_j \frac{\mathcal{H}^1(\partial F_j \cap \partial \Omega) (\mathcal{H}^1(\partial \Omega) - \mathcal{H}^1(\partial F_j \cap \partial \Omega))}{\mathcal{H}^1(\partial F_j \cap \Omega)}
= \max_j \frac{\mathcal{H}^1(\partial F_j \cap \partial \Omega) \mathcal{H}^1(\partial \Omega \setminus \partial F_j)}{\mathcal{H}^1(\partial F_j \cap \Omega)}.
\]

This proves that, on further replacing, if necessary, any connected set \( E \) of the form \( \mathcal{P} \cap \Omega \) with one of the connected components of \( \Omega \setminus E \), the supremum in (3.26) can be restricted to the class of sets \( E \) of the form \( \mathcal{P} \cap \Omega \), which are connected and such that \( \partial E \cap \Omega \) is also connected. Since \( \Omega \) is convex, the relevant supremum can be finally restricted to the class of sets \( E \) such that \( \partial E \cap \Omega \) is a straight segment. Hence, our claim follows.

Now, define

\[
(3.28) \quad S_{R,d} = \text{convex hull of two disks of equal radii } R, \text{ with centers at distance } d,
\]

a stadium-shaped domain, with semi-perimeter \( p = d + \pi R \). Let us introduce the curvilinear abscissa \( s \in \mathbb{R} \) on \( \partial S_{R,d} \), and the periodic parametrization \( \mathbb{R} \ni s \mapsto x(s) \in \mathbb{R}^2 \) of \( \partial S_{R,d} \) with respect to \( s \). Observe that, for any half-plane \( H \), the set \( \partial H \cap S_{R,d} \) (if not empty) is a chord whose terminals split \( \partial S_{R,d} \) into two parts of lengths \( a \) and \( 2p - a \), for some \( a \in (0,p] \). Thus, such a chord has terminals \( x(s) \) and \( x(s + a) \), for some \( s \), and its length is given by

\[
(3.29) \quad \ell_a(s) = |x(s + a) - x(s)|.
\]
Hence, owing to (3.25),

\[
C_{mv}(S_R,d) = \frac{1}{p} \sup_{0 < a \leq p} \frac{a(2p - a)}{\min_s \ell_a(s)}.
\]

For each \( a \in (0, p] \), the function \( \ell_a \) is continuously differentiable in \( \mathbb{R} \), and periodic of period \( p \). Moreover, \( s \) is a stationary point of \( \ell_a(s) \) if and only if

\[
(x(s + a) - x(s)) \cdot (x'(s + a) - x'(s)) = 0,
\]

where \( \cdot \) stands for scalar product in \( \mathbb{R}^2 \). Condition (3.31) entails that the tangent straight-lines to \( \partial S_{R,d} \) at \( x(s) \) and \( x(s + a) \) are either orthogonal to the chord \( x(s + a) - x(s) \) or they meet at a point which is equidistant from \( x(s) \) and \( x(s + a) \). It is easily verified that (3.31) is satisfied only if one of the following situations occurs:

(i) the chord is parallel to the flat parts of \( \partial S_{R,d} \);

(ii) the chord is orthogonal to the flat parts of \( \partial S_{R,d} \);

(iii) the chord has terminals belonging to the same half circle of \( \partial S_{R,d} \).

Elementary arguments show that, in case (i), the chord cannot be a minimizer of \( \ell_a(s) \), whereas a minimizer with property (iii) can always be chosen in such a way that it is orthogonal to the flat parts of \( \partial S_{R,d} \).

In conclusion, in order to seek for the minimum of \( \ell_a(s) \) in \( s \), we may restrict our analysis to those values of \( s \) such that the chord with terminals \( x(s) \) and \( x(s + a) \) is orthogonal to the flat parts of \( \partial S_{R,d} \). It is then easily seen that

\[
\min_s \ell_a(s) = \begin{cases} 
2R & \text{if } a \geq \pi R \\
2R \sin \frac{a}{2R} & \text{if } a < \pi R.
\end{cases}
\]

A straightforward computation now shows that, if \( 0 < d \leq (4 - \pi)R \), then the supremum in (3.30) is achieved in the limit as \( a \) goes to 0. Hence, \( C_{mv}(S_{R,d}) = 2 \).

\[\square\]

**Remark 3.2** An inspection of the proof of Theorem 1.2 shows that there do exist sets \( \Omega \) for which \( C_{mv}(\Omega) > 2 \). Indeed, one can verify that

\[
C_{mv}(S_{R,d}) = \frac{d + \pi R}{2R} > 2,
\]

provided that \( d > (4 - \pi)R \).

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