Radiative transitions of electrons between Landau levels in a moderately strong magnetic field

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Abstract. We investigate the processes of radiative electron transitions between Landau levels in a moderately strong magnetic field. Under such conditions, it is necessary to take into account transitions in which both the initial and final electrons can be in states corresponding to arbitrary Landau levels. The results obtained can be used in calculating the efficiency of electron-positron plasma generation under the conditions of a Kerr black hole accretion disk.

1. Introduction
An external magnetic field makes possible the processes forbidden in vacuum, for example, electron transitions between Landau levels with a photon emission. An essential factor is the magnitude of the field on which the calculation technique depends. In most cases, two asymptotics are used: first, the approximation of a crossed (relatively weak) field, and secondly, the limit of a strong field, when it exceeds significantly the critical value \( B_e = \frac{m_e^2}{e} \simeq 4.4 \times 10^{13} \text{G} \), and electrons can only occupy the ground Landau level. However, conditions are possible in which none of these limits is appropriate. For instance, when a field is strong, \( B \sim B_e \), but the superstrong field limit, \( B \gg B_e \), is not realized, it is necessary to take into account transitions in which both the initial and final electrons can be in states corresponding to arbitrary Landau levels. Such a set of parameters corresponds to the conditions in accretion disks of Kerr black holes, considered by specialists as the most probable source of a short cosmological gamma-ray burst. One more example when a strong magnetic field could possibly manifest itself are the experiments at modern colliders, e.g. non-central collisions of heavy ions. This leads to the need to use a special calculation technique, which we call the limit of a moderately strong magnetic field. An overview of the methods and results of calculations can be found e.g. in Refs. [1–3].

The purpose of this paper is to study the processes \( e(\ell) \rightarrow e(n) + \gamma \) in physical conditions corresponding to a moderately strong magnetic field. We compare also the probabilities of different polarization channels with the results obtained earlier in the superstrong field limit. The calculation technique is based on the use of exact solutions of the Dirac equation for an electron in a magnetic field, which are the eigenfunctions of the covariant magnetic polarization operator \( \hat{\mu}_z [4,5] \):

\[
\hat{\mu}_z = m_e \Sigma_z - i\gamma_0\gamma_5[\Sigma \times P]_z,
\]
where $\mathbf{A} = \gamma_0 \mathbf{\gamma} \gamma_5 \cdot \mathbf{P} = -i \mathbf{\nabla} + e \mathbf{A}$. We take the frame where the field is directed along the $z$ axis, and the Landau gauge where the four-potential is: $A^\nu = (0, 0, xB, 0)$. In this approach, the electron wave functions have the form

$$\Psi_{\rho,n}^s(X) = \frac{1}{\sqrt{4\varepsilon_n M_n (\varepsilon_n + M_n)(M_n + m_e) \ell_Y \ell_Z}} e^{-i(\varepsilon_n t - p_0 y - p_z z)} U_n^s(\xi), \quad (2)$$

where $X^\nu = (t, x, y, z)$, $\varepsilon_n = \sqrt{M_n^2 + p_n^2}$, $M_0 = \sqrt{m_e^2 + 2\beta^2}$, $\beta = \frac{eB}{\sqrt{\beta}}$, $\xi = \sqrt{\beta}(x + py/\beta)$. The functions $\Psi_{\rho,n}^s(X)$ satisfy the equation: $\hat{\mu}_z \Psi_{\rho,n}^s(X) = s M_n \Psi_{\rho,n}^s(X) (s = \pm 1)$. The bispinors $U_n^s(\xi)$ in Eq. (2) take the form:

$$U_n^-(\xi) = \begin{pmatrix} -i\sqrt{2\beta} p_z V_{n-1}(\xi) \\ (\varepsilon_n + M_n)(M_n + m_e) V_{n-1}(\xi) \\ -i\sqrt{2\beta} (\varepsilon_n + M_n) V_{n-1}(\xi) \\ -p_z (M_n + m_e) V_{n-1}(\xi) \end{pmatrix}, \quad U_n^+(\xi) = \begin{pmatrix} (\varepsilon_n + M_n)(M_n + m_e) V_{n-1}(\xi) \\ -i\sqrt{2\beta} p_z V_n(\xi) \\ p_z (M_n + m_e) V_{n-1}(\xi) \\ i\sqrt{2\beta} (\varepsilon_n + M_n) V_n(\xi) \end{pmatrix}. \quad (3)$$

Here, $V_n(\xi)$ are the well-known normalized harmonic oscillator functions, which are expressed in terms of the Hermite polynomials $H_n(\xi)$. The advantage of choosing the Dirac equation solutions for an electron in a magnetic field in the form (2), (3) is that the partial amplitudes corresponding to different electron polarization states and calculated by direct multiplication of the bispinors have an explicit Lorentz invariant structure.

2. Results

Using the standard computation technique, see e.g. Refs. [1,2], we found a set of formulas for the probabilities of different polarization channels $e_\ell^\pm(n) \rightarrow e_\ell^\pm(n) + \gamma_1$, where the upper symbol \pm of the initial and final electrons corresponds to two polarization states, which are determined by the eigenvalues of the magnetic polarization operator. The upper character $\lambda = 1, 2$ of the photon corresponds to one of the two its polarizations (see Ref. [1], p. 50).

The formulas obtained are designated as $W_{\ell n}^{(\lambda ss')}$. The results are expressed by single integrals being convenient for further analysis. The probabilities for the channels $e_\ell (s) \rightarrow e_\ell (s') + \gamma_1$ with the photon polarization $\lambda = 1$ can be presented in the unified form

$$W_{\ell n}^{(lss')} = \frac{\alpha m_e^2 b}{4 \xi_\ell} \int_0^{\rho_1} d\rho \rho^{\ell_n - 1} e^{-\rho} \sqrt{(1 - ss')\rho_1 + (1 + ss')\rho_2 - 2\rho}
\sqrt{(1 + ss')\rho_1 + (1 - ss')\rho_2 - 2\rho}
\times \left[ \frac{n!}{(\ell - 1)!} \left(1 + \frac{m_e}{M_\ell} \right) \left(1 - s' m_e M_n^{-1} \right) \right]^{\ell_n - 1} \rho^{\ell_n - 1} \rho \right)^2. \quad (4)$$

For the probabilities of the channels $e_\ell (s) \rightarrow e_\ell (s') + \gamma_2$ with the photon polarization $\lambda = 2$ one obtains

$$W_{\ell n}^{(2ss')} = \frac{\alpha m_e^2 b}{4 \xi_\ell} \left(1 - s\ell' \frac{m_e}{M_\ell} \right) \int_0^{\rho_1} d\rho \rho^{\ell_n - 1} e^{-\rho} \sqrt{(1 + ss')\rho_1 + (1 - ss')\rho_2 - 2\rho}
\sqrt{(1 - ss')\rho_1 + (1 + ss')\rho_2 - 2\rho}
\times \left[ \frac{n!}{(\ell - 1)!} \left(1 - s\ell' \frac{m_e}{M_\ell} \right) \left(1 - s' \frac{m_e}{M_\ell} \right) \right]^{\ell_n - 1} \rho^{\ell_n - 1} \rho \right)^2. \quad (5)$$

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Here, the following notations are used
\[ \rho_1 = (\ell - n) \frac{M_\ell - M_n}{M_\ell + M_n}, \quad \rho_2 = (\ell - n) \frac{M_\ell + M_n}{M_\ell - M_n}, \quad (6) \]

\( \alpha \) is the fine structure constant, \( \varepsilon_\ell \) is the initial electron energy, \( b = B/B_e, \quad M_n = m_e \sqrt{2nb + 1} \), \( L_n^s(\rho) \) are the generalized Laguerre polynomials with a condition \( L_{n-1}^s(\rho) = 0, \quad \rho = q_{\perp}^2/(2\beta) \), \( q_{\perp} = \sqrt{q_x^2 + q_y^2} \) is the photon momentum projection on the plane orthogonal to the magnetic field direction.

For illustration, we give here the probabilities for four polarization channels \( e_\ell^{(\pm)} \to e_n^{(\mp)} + \gamma^{(\lambda)} \).

In this case, according to Eq. (6):
\[ \rho_1 = (M_1 - m_e)/(M_1 + m_e), \quad \rho_2 = (M_1 + m_e)/(M_1 - m_e). \]

The probabilities can be represented as:
\[ W_{10}^{(1-\cdots-1)} = \frac{\alpha m_e^2 b}{2 \varepsilon_1} \left( 1 - \frac{m_e}{M_1} \right) \int_0^{p_1} d\rho e^{-\rho} \sqrt{\frac{p_2 - \rho}{\rho_1 - \rho}}; \quad W_{10}^{(2-\cdots-2)} = \rho_2 W_{10}^{(1+\cdots+1)}; \quad (7) \]
\[ W_{10}^{(1+\cdots+1)} = \frac{\alpha m_e^2 b}{2 \varepsilon_1} \left( 1 + \frac{m_e}{M_1} \right) \int_0^{p_1} d\rho e^{-\rho} \sqrt{\frac{\rho_1 - \rho}{\rho_2 - \rho}}; \quad W_{10}^{(2+\cdots+2)} = \rho_1 W_{10}^{(1-\cdots-1)}. \quad (8) \]

It can be seen from the formulas that the integrands in two of the four probabilities, \( W_{10}^{(1-\cdots-1)} \) and \( W_{10}^{(2+\cdots+2)} \), have square-root singularities. It is interesting to compare the formulas (7)-(8) with the results of calculation performed in the limit of a superstrong field, \( B \gg B_e \), see Ref. [1], p. 140, where it was found that all four probabilities coincide. This result is confirmed by taking the indicated superstrong field limit, when \( \rho_1 \simeq \rho_2 \simeq 1 \), in formulas (7)-(8). However, this assertion is valid only in an asymptotically strong field. For instance, as the numerical analysis shows, at \( B = 10B_e \) the probabilities of the polarization channels \( W_{10}^{(1-\cdots-1)} \) and \( W_{10}^{(1+\cdots+1)} \) differ by more than a factor of three. Thus, in a situation of a moderately strong magnetic field, exact formulas of the form (7)-(8) should be used for the analysis of the processes \( e_\ell^{(\pm)} \to e_n^{(\mp)} + \gamma^{(\lambda)} \).

In figures 1, 2, the field dependence is shown of some process probabilities averaged over the initial and summarized over the final polarization states.

**Figure 1.** First-to-ground (dashed line), second-to-ground (dotted), second-to-first (solid) and third-to-first (dash-dotted line) Landau level transitions for the photon mode \( \lambda = 1 \).

**Figure 2.** First-to-ground (dashed line), second-to-ground (solid), second-to-first (dotted) and third-to-first (dash-dotted line) Landau level transitions for the photon mode \( \lambda = 2 \).

It is interesting to evaluate the electron lifetimes in excited Landau levels with respect to the transitions discussed, in order to compare it with a typical time of the strong magnetic field.
existence, formed in non-central collisions of heavy ions, see e.g. Ref. [6], which is estimated as $\sim 1 \text{ fm}/c$, where $c$ is the speed of light. This evaluation could be useful for an analysis of the similar quark radiative transitions. From our formulas (4), (5), we obtain the electron lifetime $\tau_1$ in the first Landau level to be $4.4 \times 10^4 \text{ fm}/c$ for $B = 10 B_e$ and $1.2 \times 10^4 \text{ fm}/c$ for $B = 100 B_e$. Similarly, the electron lifetime $\tau_2$ in the second Landau level is $3.5 \times 10^4 \text{ fm}/c$ for $B = 10 B_e$ and $1.0 \times 10^4 \text{ fm}/c$ for $B = 100 B_e$. Thus, the approximation of a constant magnetic field is inapplicable in this situation.

In figure 3, the photon spectra over the transverse momentum magnitude (squared and dimensionless) are presented where the above-mentioned square-root singularities can be seen. In addition, we have calculated the average energy of the produced photons $\omega_{av}$ with respect to the initial electron energy. Its magnetic field dependence is shown in figure 4.

![Photon spectra](image1)

**Figure 3.** Photon spectra for the second-to-first Landau level transitions with field values $b = 1$ (thin lines) and $b = 10$ (thick lines): dashed lines - $\lambda = 1, s = s' = -1$, solid lines - $\lambda = 2, s = s' = -1$, dotted lines - $\lambda = 1, s = +1, s' = -1$, dash-dotted lines - $\lambda = 2, s = +1, s' = -1$.

![Magnetic field dependence](image2)

**Figure 4.** Magnetic field dependence of the produced photons average energy for the first-to-ground (dashed line), second-to-ground (solid line), third-to-ground (dotted line) and second-to-first (dash-dotted line) Landau level transitions.

3. Conclusion

On the basis of the exact solutions of the Dirac equation for an electron in a magnetic field, the radiative electron transitions between Landau levels are analysed. The formulas for the transition probabilities are obtained in the form of single integrals being convenient for further analysis. The plots for the field dependence of the transition probabilities, the photon spectra, and the magnetic field dependence of the produced photons average energy are presented.

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References

[1] Kuznetsov A and Mikheev N 2013 *Electroweak Processes in External Active Media* (Berlin, Heidelberg: Springer-Verlag)
[2] Kuznetsov A V, Rumyantsev D A and Savin V N 2014 *Int. J. Mod. Phys. A* **29** 1450136
[3] Kuznetsov A V, Rumyantsev D A and Savin V N 2016 *J. Phys.: Conf. Ser.* **675** 032019
[4] Sokolov A A and Ternov I M 1968 *Synchrotron Radiation* (Oxford: Pergamon)
[5] Melrose D B and Parle A J 1983 *Aust. J. Phys.* **36** 755
[6] Skokov V V, Illarionov A Yu and Toneev V D 2009 *Int. J. Mod. Phys. A* **24** 5925