LAPLACIAN SIMPLICES

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Abstract. This paper initiates the study of the Laplacian simplex $T_G$ obtained from a finite graph $G$ by taking the convex hull of the columns of the Laplacian matrix for $G$. Basic properties of these simplices are established, and then a systematic investigation of $T_G$ for trees, cycles, and complete graphs is provided. Motivated by a conjecture of Hibi and Ohsugi, our investigation focuses on reflexivity, the integer decomposition property, and unimodality of Ehrhart $h^*$-vectors. We prove that if $G$ is a tree, odd cycle, complete graph, or a whiskering of an even cycle, then $T_G$ is reflexive. We show that while $T_{K_n}$ has the integer decomposition property, $T_{C_n}$ for odd cycles does not. The Ehrhart $h^*$-vectors of $T_G$ for trees, odd cycles, and complete graphs are shown to be unimodal. As a special case it is shown that when $n$ is an odd prime, the Ehrhart $h^*$-vector of $T_{C_n}$ is given by $(h^*_0, \ldots, h^*_n-1) = (1, \ldots, 1, n^2 - n + 1, 1, \ldots, 1)$. We also provide a combinatorial interpretation of the Ehrhart $h^*$-vector for $T_{K_n}$.

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1. Introduction

1.1. Motivation. Let $G$ be a finite graph on the vertex set $\{1, 2, \ldots, n\}$. There are many profitable ways to associate a polytope to $G$. One well-known example is the edge polytope of $G$, obtained by taking the convex hull of the vectors $e_i + e_j$ for each edge $\{i, j\}$ in $G$, where $e_i$ denotes the $i$th standard basis vector in $\mathbb{R}^n$. Equivalently, the edge polytope is the convex hull of the columns of the unsigned vertex-edge incidence matrix of $G$. Many geometric, combinatorial, and algebraic properties of edge polytopes have been established over the past several decades, e.g. \cite{17, 18, 26, 28}. Another well-known matrix associated with a graph $G$ is the Laplacian $L(G)$ (defined in Section 2).

Our purpose in this paper is to study the analogue of the edge polytope obtained by taking the convex hull of the columns of $L(G)$, resulting in a lattice simplex that we call the Laplacian simplex of $G$ and denote $T_G$.

While to our knowledge the simplex $T_G$ has not been previously studied, there has been recent research regarding graph Laplacians from the perspective of polyhedral combinatorics and integer-point enumeration. For example, M. Beck and the first author investigated hyperplane arrangements defined by graph Laplacians with connections to nowhere-harmonic colorings and inside-out polytopes \cite{4}. A. Padrol and J. Pfeifle explored Laplacian Eigenpolytopes \cite{20} with a focus on the effect of graph operations on the associated polytopes. The first author, R. Davis, J. Doering, A. Harrison, J. Noll, and C. Taylor studied integer-point enumeration for polyhedral cones constrained by graph Laplacian minors \cite{8}. In a recent preprint \cite{10}, A. Dall and J. Pfeifle analyzed polyhedral decompositions of the zonotope defined as the Minkowski sum of the line segments from the origin to each column of $L(G)$ in order to give a polyhedral proof of the Matrix-Tree Theorem.

Beyond the motivation of studying $T_G$ in order to develop a Laplacian analogue of the theory of edge polytopes, our primary motivation in this paper is the following conjecture (all undefined terms are defined in Section 2).

Conjecture 1.1 (Hibi and Ohsugi \cite{19}). If $P$ is a lattice polytope that is reflexive and satisfies the integer decomposition property, then $P$ has a unimodal Ehrhart $h^*$-vector.

The cause of unimodality for $h^*$-vectors in Ehrhart theory is mysterious. Schepers and van Langenhoven \cite{28} have raised the question of whether or not the integer decomposition property alone is sufficient to force unimodality of the $h^*$-vector for a lattice polytope. In general, the interplay of the qualities of a lattice polytope being reflexive, satisfying the integer decomposition property, and having a unimodal $h^*$-vector is not well-understood \cite{6}. Thus, when new families of lattice polytopes are introduced, it is of interest to explore how these three properties behave for that family. Further, lattice simplices have been shown to be a rich source of examples and have been the subject of several recent investigations, especially in the context of Conjecture 1.1 \cite{7, 9, 21, 24}.

1.2. Our Contributions. After reviewing necessary background in Section 2, we introduce and establish basic properties of Laplacian simplices in Section 3. We show that several graph-theoretic operations produce reflexive Laplacian simplices (Theorem 3.14 and Proposition 3.4). We prove that if $G$ is a tree, odd cycle, complete graph, or the whiskering of an even cycle, then $T_G$ is reflexive (Proposition 4.1, Theorem 5.1, Proposition 5.4 and Theorem 6.1). As a result of a general investigation of the structure of $h^*$-vectors for odd cycles (Theorem 5.10), we show that if $n$ is odd then $T_{C_n}$ does not have the integer decomposition property (Corollary 5.11). On the other hand, we show that $T_{K_n}$ does have the integer decomposition property since it admits a regular unimodular triangulation (Corollary 6.3). We prove that for trees, odd cycles, and complete graphs, the $h^*$-vectors of their Laplacian simplices are unimodal (Corollary 4.2, Theorem 5.7, and Corollary 6.5). Additionally, we provide a combinatorial interpretation of the $h^*$-vector for $T_{K_n}$ (Proposition 6.9) and we determine that, when $n$ is an odd prime, the $h^*$-vector of $T_{C_n}$ is given by $(h^*_0, \ldots, h^*_{n-1}) = (1, \ldots, 1, n^2 - n + 1, 1, \ldots, 1)$ (Theorem 5.10).
2. Background

2.1. Reflexive Polytopes. A lattice polytope of dimension $d$ is the convex hull of finitely many points in $\mathbb{Z}^n$, which together affinely span a $d$-dimensional hyperplane of $\mathbb{R}^n$. Two lattice polytopes are unimodularly equivalent if there is a lattice preserving affine isomorphism mapping them onto each other. Consequently we consider lattice polytopes up to affine automorphisms of the lattice. The dual polytope of a full dimensional polytope $\mathcal{P}$ which contains the origin in its interior is

$$\mathcal{P}^* := \{ x \in \mathbb{R}^d \mid x \cdot y \leq 1 \text{ for all } y \in \mathcal{P} \}.$$

Duality satisfies $(\mathcal{P}^*)^* = \mathcal{P}$. A $d$-polytope formed by the convex hull of $d + 1$ vertices is called a $d$-simplex.

Definition 2.1. A lattice polytope $\mathcal{P}$ is called reflexive if it contains the origin in its interior, and its dual $\mathcal{P}^*$ is a lattice polytope.

Any lattice translate of a reflexive polytope is also called reflexive. The following generalization of reflexive polytopes was introduced in [15]. A lattice point is its dual $\mathcal{P}^*$.

Definition 2.2. A lattice polytope $\mathcal{P}$ is $\ell$-reflexive if, for some $\ell \in \mathbb{Z}_{>0}$, the following conditions hold:

(i) $\mathcal{P}$ contains the origin in its (strict) interior;
(ii) The vertices of $\mathcal{P}$ are primitive;
(iii) For any facet $F$ of $\mathcal{P}$ the local index $\ell_F = \ell$.

We refer to $\mathcal{P}$ as a reflexive polytope of index $\ell$. The reflexive polytopes of index 1 are precisely the reflexive polytopes in Definition 2.2.

2.2. Ehrhart Theory. For $t \in \mathbb{Z}_{>0}$, the $t^{th}$ dilate of $\mathcal{P}$ is given by $t\mathcal{P} := \{ tp \mid p \in \mathcal{P} \}$. One technique used to recover dilates of polytopes is coning over the polytope. Given $\mathcal{P} = \text{conv} (v_1, \ldots, v_m) \subseteq \mathbb{R}^n$, we lift these vertices into $\mathbb{R}^{n+1}$ by appending 1 as their last coordinate to define $w_1 = (v_1, 1), \ldots, w_m = (v_m, 1)$. The cone over $\mathcal{P}$ is

$$\text{cone}(\mathcal{P}) = \{ \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m \mid \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0 \} \subseteq \mathbb{R}^{n+1}.$$

For each $t \in \mathbb{Z}_{>0}$ we recover $t\mathcal{P}$ by considering $\text{cone}(\mathcal{P}) \cap \{ z_{n+1} = t \}$. To record the number of lattice points we let $L_\mathcal{P}(t) = |t\mathcal{P} \cap \mathbb{Z}^n|$. In [12], Ehrhart proved that $L_\mathcal{P}(t)$, called the Ehrhart polynomial of $\mathcal{P}$, is a polynomial in degree $d = \dim(\mathcal{P})$ with generating function

$$\text{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_\mathcal{P}(t) z^t = \frac{h_0^* z^d + h_1^* z^{d-1} + \cdots + h_d^* z + h_0^*}{(1 - z)^{d+1}}.$$

The above is referred to as the Ehrhart series of $\mathcal{P}$. We call $h^*(\mathcal{P}) = (h_0^*, h_1^*, \ldots, h_d^*)$ the $h^*$-vector or $\delta$-vector of $\mathcal{P}$. The Euclidean volume of a polytope $\mathcal{P}$ is $\text{vol}(\mathcal{P}) = \frac{1}{d!} \sum_{i=0}^d h_i^*$. The normalized volume is given by $\text{dvol}(\mathcal{P}) = \frac{1}{d!} \sum_{i=0}^d h_i^*$. Stanley proved the $h^*$-vector of a convex lattice $d$-polytope satisfies $h_0^* = 1$ and $h_i^* \in \mathbb{Z}_{\geq 0}$ [25]. Note that if $\mathcal{P}$ and $\mathcal{Q}$ are lattice polytopes such that $\mathcal{Q}$ is the image of $\mathcal{P}$ under an affine unimodular transformation, then their Ehrhart series are equal.

A vector $x = (x_0, x_1, \ldots, x_d)$ is unimodal if there exists a $j \in [d]$ such that $x_i \leq x_{i+1}$ for all $0 \leq i < j$ and $x_k \geq x_{k+1}$ for all $j \leq k < d$. A major open problem in Ehrhart theory is to determine properties of $\mathcal{P}$ that imply unimodality of $h^*(\mathcal{P})$ [6]. For the case of symmetric $h^*$-vectors, Hibi established the following connection to reflexive polytopes.

Theorem 2.3 (Hibi [14]). A $d$-dimensional lattice polytope $\mathcal{P} \subseteq \mathbb{R}^d$ containing the origin in its interior is reflexive if and only if $h^*(\mathcal{P})$ satisfies $h_i^* = h_{d-i}^*$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$. 

Thus, when investigating symmetric $h^*$-vectors, reflexive polytopes (and, more generally, Gorenstein polytopes) are the correct class to work with. As indicated by Conjecture 1.1, the following property has been frequently correlated with unimodality, and is interesting in its own right.

**Definition 2.4.** A lattice polytope $P \subseteq \mathbb{R}^n$ has the **integer decomposition property** if, for every integer $t \in \mathbb{Z}_{>0}$ and for all $p \in tP \cap \mathbb{Z}^n$, there exists $p_1, \ldots, p_t \in P \cap \mathbb{Z}$ such that $p = p_1 + \cdots + p_t$. We will frequently say that $P$ is IDP when $P$ possesses this property.

It is well-known that if $P$ admits a unimodular triangulation, then $P$ is IDP; we will use this fact when analyzing complete graphs.

2.3. Lattice Simplices. Simplices play a special role in Ehrhart theory, as there is a method for computing their $h^*$-vectors that is simple to state (though not always to apply).

**Definition 2.5.** Given a lattice simplex $P \subseteq \mathbb{R}^{n-1}$ with vertices $\{v_i\}_{i \in [n]}$, the **fundamental parallelipiped** of $P$ is the subset of cone $(P)$ defined by

$$\Pi_P := \left\{ \sum_{i=1}^{n} \lambda_i(v_i, 1) \mid 0 \leq \lambda_i < 1 \right\}.$$ 

Further, $|\Pi_P \cap \mathbb{Z}^n|$ is equal to the determinant of the matrix whose $i$th row is given by $(v_i, 1)$.

**Lemma 2.6** (see Chapter 3 of [5]). Given a lattice simplex $P$,

$$h^*_i(P) = |\Pi_P \cap \{ x \in \mathbb{Z}^n \mid x_n = i \}|.$$ 

Using the notation from Definition 2.5, let $A$ be the matrix whose $i$th row is $(v_i, 1)$. One approach to determine $h^*_i(P)$ in this case is to recognize that finding lattice points in $\Pi_P$ is equivalent to finding integer vectors of the form $\lambda \cdot A$ with $0 \leq \lambda_i < 1$ for all $i$. Cramer’s rule implies the $\lambda \in \mathbb{Q}^n$ that yield integer vectors will have entries of the form

$$\lambda_i = \frac{b_i}{\det A} < 1$$

for $b_i \in \mathbb{Z}_{\geq 0}$. In particular, if $x = \frac{1}{\det(A)} b \cdot A \in \mathbb{Z}^n$, then $b_i = \det A(i, x)$ where $A(i, x)$ is the matrix obtained by replacing the $i$th row of $A$ by $x$. Since $A(i, x)$ is an integer matrix, $\det A(i, x) \in \mathbb{Z}$. Notice that for any $\lambda$, the last coordinate of $\lambda A$ is $\langle \lambda, 1 \rangle = \sum_{i=1}^{n} \frac{b_i}{\det A}$. Thus, we have

$$\Pi_P \cap \mathbb{Z}^n = \mathbb{Z}^n \cap \left\{ \frac{1}{\det A} b \cdot A \mid 0 \leq b_i < \det(A), b_i \in \mathbb{Z}, \sum_{i=1}^{n} b_i \equiv 0 \mod \det(A) \right\}.$$ 

One profitable method for determining the lattice points in $\Pi_P$ is to find the det$(A)$-many lattice points in the right-hand set above, by first considering all the $b$-vectors that satisfy the given modular equation.

2.4. Graph Laplacians. Let $G$ be a connected graph with vertex set $V(G) = [n] := \{1, 2, \ldots, n\}$ and edge set $E(G)$. The **Laplacian matrix** $L$ of a graph $G$ is defined to be the difference of the degree matrix and the $\{0, 1\}$-adjacency matrix of a graph. Thus, $L$ has rows and columns indexed by $[n]$ with entries $a_{ii} = \deg i$, $a_{ij} = -1$ if $\{i, j\} \in E(G)$, and 0 otherwise. We let $\kappa$ denote the number of spanning trees of $G$. The following facts are well-known [2].

**Proposition 2.7.** The Laplacian matrix $L$ of a connected graph $G$ with vertex set $[n]$ satisfies the following:

(i) $L \in \mathbb{Z}^{n \times n}$ is symmetric.
(ii) Each row and column sum of $L$ is 0.
(iii) $\ker \mathbb{R} L = \langle 1 \rangle$ and $\im \mathbb{R} L = \langle 1 \rangle^\perp$
(iv) $\text{rk } L = n - 1$.
(v) (The Matrix-Tree Theorem [16]) Any cofactor of $L$ is equal to $\kappa$.

In this paper we often refer to a submatrix of $L$ defined by restricting to specified rows and columns. For $S, T \subseteq \{0, \ldots, n-1\}$, define $L(S \mid T)$ to be the matrix with rows from $L$ indexed by $\{0, \ldots, n-1\} \setminus S$ and columns from $L$ indexed by $\{0, \ldots, n-1\} \setminus T$. Equivalently, $L(S \mid T)$ is obtained from $L$ by the deletion of rows indexed by $S$ and columns indexed by $T$. For simplicity, we define $L(i)$ to be the matrix obtained by deleting the $i$th column of $L$, that is, $L(i) := L(\emptyset \mid i) \in \mathbb{Z}^{n \times (n-1)}$.

3. The Laplacian Simplex of a Finite Graph

3.1. Definition and Basic Properties. Assume that $G$ is a connected graph with Laplacian matrix $L$. Consider $L(i) \in \mathbb{Z}^{n \times (n-1)}$. It is a straightforward exercise to show the rank of $L(i)$ is $n-1$. We recognize the rows of $L(i)$ as points in $\mathbb{Z}^{n-1}$ and consider their convex hull, $\conv(L(i)^\mathsf{T})$, where $\conv(A)$ refers to the convex hull of the columns of the matrix $A$. Notice the rows of $L(i)$ form a collection of $n$ affinely independent lattice points, which makes $\conv(L(i)^\mathsf{T})$ an $n-1$ dimensional simplex.

**Proposition 3.1.** The lattice simplices $\conv(L(i)^\mathsf{T})$ and $\conv(L(j)^\mathsf{T})$ are unimodularly equivalent for all $i, j \in \{0, \ldots, n-1\}$.

**Proof.** Notice the matrices $L(i)$ and $L(j)$ differ by only one column when $i \neq j$. In particular we can write $L(i) \cdot U = L(j)$ where $U \in \mathbb{Z}^{n \times (n-1)}$ has columns $c_k$ for $1 \leq k \leq n-1$ defined to be

$$c_k = \begin{cases} e_\ell & \text{column } k \text{ in } L(j) \text{ is column } \ell \text{ in } L(i) \\ (-1, -1, \ldots, -1)^T & \text{column } k \text{ in } L(j) \text{ is not among columns of } L(i) \end{cases}$$

where $e_\ell$ is the vector with a 1 in the $\ell$th entry and 0 else.

Notice $U$ has integer entries and $\det U = \pm 1$, as computed by expanding along the column with all entries equal to $-1$. This shows $U$ is a unimodular matrix. Further, $U$ maps the vertices of $\conv(L(i)^\mathsf{T})$ onto the vertices of $\conv(L(j)^\mathsf{T})$. Thus $\conv(L(i)^\mathsf{T})$ and $\conv(L(j)^\mathsf{T})$ are unimodularly equivalent lattice polytopes.

Given a fixed graph $G$, we choose a representative for this equivalence class of lattice simplices to be used throughout, unless otherwise noted. Let $B = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n\}$ be the standard basis for the orthogonal complement of the all-ones vector $1 \in \mathbb{R}^n$, where $e_i \in \mathbb{R}^n$ is the standard basis vector that contains a 1 in the $i$th entry and 0 else. Then $B$ is a basis of the column space of $L$. Define $L_B \in \mathbb{Z}^{n \times (n-1)}$ to be the representation of the matrix $L$ with respect to the basis $B$. In practice, $L_B$ can be computed using the matrix multiplication $L_B = L \cdot A$ where $A$ is the upper triangular $(n \times (n-1))$ matrix with entries

$$a_{ij} = \begin{cases} 1 & i \leq j \leq n-1 \\ 0 & \text{else} \end{cases}.$$

**Example 3.2.** Given the cycle $C_5$ of length five, we have

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \quad \quad L_B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -2 \end{bmatrix}.$$

This brings us to the object of study in this paper.

**Definition 3.3.** For a connected graph $G$, the $n-1$ dimensional lattice simplex

$$T_G := \conv(L_B)^\mathsf{T} \subseteq \mathbb{R}^{n-1}$$

is called the Laplacian Simplex associated to the graph $G$. 
Proposition 3.4. Let $G$ be a connected graph on $n$ vertices.

(i) $T_G$ is a representative of the equivalence class \{conv $(L(i)^T)$\}_{i \in [n]}.$

(ii) $T_G$ has normalized volume equal to $n \cdot \kappa.$

(iii) $T_G$ contains the origin in its interior.

(iv) $h_i^*(T_G) \geq 1$ for all $0 \leq i \leq n - 1.$

Proof. (i) Notice we can write $L(n) \cdot A(n \mid \emptyset) = L_B$ where $A$ is the matrix defined in equation (1). Let $U := A(n \mid \emptyset).$ Then $U$ is the upper diagonal matrix of all ones so that $\det U = 1.$ This implies $T_G$ is unimodularly equivalent to conv $(L(n)^T).$ By Proposition 3.1, the result follows.

(ii) Since $T_G$ is a simplex, the normalized volume of $T_G$ is equal to

$$\left| \det [L_B \mid 1] \right| = \left| \sum_{i=1}^{n} (-1)^{i+n}M_{in} \right| = \left| \sum_{i=1}^{n} C_{in} \right|,$$

where $M_{i,n}$ is a minor of $[L_B \mid 1]$, $C_{i,n}$ is the corresponding cofactor, and the determinant is expanded along the appended column of ones. The relation $L(n) \cdot U = L_B$ yields $L(i \mid n) \cdot U = L_B(i \mid \emptyset).$ Then for each cofactor,

$$C_{i,n} = (-1)^{i+n} \det L_B(i \mid \emptyset) = (-1)^{i+n} \det (L(i \mid n) \cdot U) = (-1)^{i+n} \det L(i \mid n) \det U = (-1)^{i+n} \det L(i \mid n) = \tilde{C}_{i,n} = \kappa$$

where $\tilde{C}_{i,n}$ is the cofactor of $L$, and the last equality is a result of the Matrix Tree Theorem. Summing over all $i \in [n]$ yields the desired result.

(iii) Note the sum of all rows of $L_B$ is 0, and $L_B$ has no column with all entries equal to 0. It follows that $(0,0, \ldots, 0) \in \mathbb{Z}^n$ is in the interior of $T_G.$

(iv) Observe each column in $L_B$ sums to 0. Consider lattice points of the form

$$p_i = \left( \frac{i}{n}, \frac{i}{n}, \ldots, \frac{i}{n} \right) \cdot [L_B \mid 1] = (0,0, \ldots, 0, i) \in \mathbb{Z}^{1 \times n}$$

for each $0 \leq i < n.$ Then $p_i \in \Pi_{T_G} \cap \{ x \in \mathbb{Z}^n \mid x_n = i \}$ implies $h_i^*(T_G) \geq 1$ for each $0 \leq i \leq n - 1.$

Example 3.5. The simplex $T_{C_5}$ is obtained as the convex hull of the columns of the transpose of

$$L_B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -2 \end{bmatrix}.$$ 

The determinant of $L_B$ with a column of ones appended is easily computed to be 25. By applying Lemma 2.6 to $T_{C_5}$, it is straightforward to verify that $h^*(T_{C_5}) = (1, 1, 21, 1, 1).$

In the proof of (ii) in Proposition 3.4 above, we showed the minor obtained by deleting the $i^{th}$ row of $L_B$ is equal to the minor obtained by deleting the $n^{th}$ column and the $i^{th}$ row of $L$ for some $i \in [n]$, i.e., $\det L_B(i \mid \emptyset) = \det L(i \mid n)$ for any $i \in [n].$ The second minors of $L_B$ and $L$ are related in the following manner, which we will need in subsequent sections.
Lemma 3.6. Let $i, k \in [n]$ and $j \in [n-1]$ such that $i \neq k$. Then
\[
\det L_B(i, k \mid j) = \det L(i, k \mid j, n) + \det L(i, k \mid j+1, n).
\]
In the case $j = n - 1$, $\det L_B(i, k \mid n - 1) = \det L(i, k \mid n - 1, n)$.

Proof. Recall $L_B = L \cdot A$ where $A$ is the $n \times (n - 1)$ upper diagonal matrix defined in equation \(\Box\). It follows $L_B(i, k \mid j) = L(i, k \mid \emptyset) \cdot A(j)$. Apply the Cauchy-Binet formula to compute the determinant
\[
\det L_B(i, k \mid j) = \sum_{S \subseteq [[n-2]]} \det L(i, k \mid \emptyset)_{[n-2],S} \det A(j)_{S, [n-2]}
\]
\[
= \det L(i, k \mid \emptyset)_{[n-2],[n] \setminus \{j,n\}} \det A(j)_{[n] \setminus \{j,n\} , [n-2]}
+ \det L(i, k \mid \emptyset)_{[n-2],[n] \setminus \{j+1,n\}} \det A(j)_{[n] \setminus \{j+1,n\} , [n-2]}
= \det L(i, k \mid j, n) + \det L(i, k \mid j+1, n).
\]

The only nonzero terms in the sum arise from choosing $(n - 2)$ linearly independent rows in $A$. Based on the structure of $A$, there are only two ways to do this unless we are in the case $j = n - 1$ in which there is exactly one way. $\square$

The following is a special case of a general characterization of reflexive simplices using cofactor expansions.

Theorem 3.7. For a connected graph $G$ with Laplacian matrix $L$, $T_G$ is reflexive if and only if for each $i \in [n]$, $\kappa$ divides
\[
\sum_{k=1}^{n-1} C_{kj} = \sum_{k=1}^{n-1} (-1)^{k+j} M_{kj}
\]
for each $1 \leq j \leq n - 1$. Here $C_{kj}$ is the cofactor and $M_{kj}$ is the minor of the matrix $L_B(i \mid \emptyset) \in \mathbb{Z}^{(n-1) \times (n-1)}$.

Proof. We show $T_G$ is reflexive by showing the vertices of its dual polytope are lattice points. By [20] Theorem 2.11, the hyperplane description of the dual polytope is given by $T_G^* = \{ x \in \mathbb{R}^{n-1} \mid L_B \cdot x \leq 1 \}$. Each intersection of $(n - 1)$ hyperplanes will yield a unique vertex of $T_G^*$ since any first minor of $L_B$ is nonzero. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of $T_G^*$. Each $v_i$ satisfies
\[
L_B(i \mid \emptyset) v_i = 1
\]
for $i \in [n]$. Reindex the rows of $L_B(i \mid \emptyset)$ in increasing order by $[n - 1]$. We can write
\[
v_i = L_B(i \mid \emptyset)^{-1} \cdot 1 = \frac{1}{\det L_B(i \mid \emptyset)^{CT}} \cdot 1
\]
where $C^T$ is the $(n - 1) \times (n - 1)$ matrix whose whose $(j, k)$ entry is the $(k, j)$ cofactor of $L_B(i \mid \emptyset)$, which we denote as $C_{kj}$. Since $\det L_B(i \mid \emptyset) = \det L(i \mid n) = \pm \kappa$, each vertex is of the form
\[
v_i = \frac{1}{\pm \kappa} \left( \sum_{k=1}^{n-1} C_{k1}, \sum_{k=1}^{n-1} C_{k2}, \ldots, \sum_{k=1}^{n-1} C_{k(n-1)} \right)^T,
\]
which is a lattice point if and only if $\kappa$ divides each coordinate. $\square$

Remark 3.8. Apply Lemma 3.6 to Proposition 3.7 to yield a condition on the second minors of $L$ when determining if $T_G$ is reflexive. Notice
\[
(C^T)_{jk} = C_{kj}
\]
\[
= (-1)^{k+j} \det L_B(i, k \mid j)
\]
\[
= (-1)^{k+j} (\det L(i, k \mid j, n) + \det L(i, k \mid j+1, n)),
\]
which shows for a given $v_i$, its $\ell$th coordinate has the form

$$
\frac{1}{\pm \kappa} \sum_{k=1}^{n-1} C_{k\ell} = \frac{1}{\pm \kappa} \sum_{k=1}^{n-1} (-1)^{k+\ell} \left( \det L(i, k | \ell, n) + \det L(i, k | \ell + 1, n) \right).
$$

**Remark 3.9.** Computing alternating sums of second minors of Laplacian matrices can be challenging. Thus, we often verify reflexivity by explicitly computing the vertices of $T_G$ via ad hoc methods.

### 3.2. Graph Operations and Laplacian Simplices

**Proposition 3.10.** Let $G$ be a connected graph on $n$ vertices such that the following cut is possible. Partition $V(G)$ into vertex sets $A$ and $B$ such that all edges between $A$ and $B$ are incident to a single vertex $x \in A$; label those edges $\{e_1, \ldots, e_k\}$. Additionally suppose $x$ has a leaf with adjacent vertex $y \in A$. Form a new graph $G'$ by moving the edges $\{e_1, \ldots, e_k\}$ previously incident to $x$ to be incident to $y$. Then $G'$ has vertex set $V(G)$, and edge set $(E(G) \setminus \{e_1, \ldots, e_k\}) \cup \{(y, v_i) : i = 1, \ldots, k\}$ where $e_i = \{x, v_i\} \in E(G)$. Then $T_G \cong T_{G'}$.

**Proof.** Label the vertices of $G$ with $[n]$. Observe $G'$ has the same labels since $V(G) = V(G')$. We refer to each vertex by its label for simplicity. Let $N_G(i)$ be the set of neighbors of vertex $i$ in $G$, that is, $N_G(i) := \{j \in V(G) \mid (i, j) \in E(G)\}$. Let $L$ be the Laplacian matrix of $G$ and $L'$ be the Laplacian matrix of $G'$. We describe row operations that take each row $r_i \in L$ to row $r'_i \in L'$. For each $i \in V(G), \ 1 \leq i \leq n$, we have the following cases.

Consider $i \in A$ such that $i \neq x, y$. Then $N_G(i) = N_{G'}(i)$, so we set $r'_i = r_i$ since the $i$th row is the same in $L$ and $L'$. Then $r'_i \in L'$.

Consider $i \in B \setminus N_{G'}(x)$. Again, $N_G(i) = N_{G'}(i)$, so we set $r'_i = r_i$ and have $r'_i \in L'$.

Consider $i \in B \cap N_{G'}(x)$. The degree of $i$ is constant in $G$ and $G'$, but $\{i, x\} \in E(G)$ becomes $\{i, y\} \in E(G')$ in the described algorithm. Set $r'_i = r_i - r_y$ to reflect the change in incident edges of $i$ from $G$ to $G'$. Since $y \in V(G)$ is a leaf, $r'_i$ now has a 0 in the $x$th coordinate, a $-1$ in the $y$th coordinate, and all remaining coordinates are unchanged. Then $r'_i \in L'$.

Consider $i = x$. Set $r'_x = r_x + \sum_{j \in B} r_j$. Observe $N_G(x) \setminus N_{G'}(x) = \{v_1, \ldots, v_k\}$. Then adding $\sum_{\ell=1}^k r_v$ decreases the $x$th coordinate of $r_x$ by $k$, which is the new degree of vertex $x \in V(G')$. Adding the other rows does contribute to the $x$th coordinate of $r'_x$ since those vertices are not adjacent to $x \in V(G)$; however, we must add all rows corresponding to $j \in B$ to obtain a 0 in all coordinates indexed by $j \in B$. Notice the coordinates indexed by the vertices in $A$ remain fixed. Then $r'_x \in L'$.

Finally consider $i = y$. Set $r'_y = (k + 1) r_y - \sum_{j \in B} r_j$. The $y$th coordinate of $r'_y$ is $k + 1$, which is the degree of $y$ in $V(G')$. Observe $N_G(y) \setminus N_{G'}(y) = \{v_1, \ldots, v_k\}$. Then subtracting $\sum_{\ell=1}^k r_v$ from $(k + 1) r_y$ ensures the $x$th coordinate of $r'_y$ is $-1$. We subtract all rows corresponding to $j \in B$ from $(k + 1) r_y$ to obtain a $-1$ in all coordinates of $r'_y$ indexed by $\{v_i\}_{i=1}^k$. Then $r'_y \in L'$.

It is straightforward to verify that the collection of row operations described above is a unimodular transformation of the Laplacian matrix and thus can be represented by the multiplication of unimodular matrix $U \in \mathbb{Z}^{n \times n}$ such that $U \cdot L = L'$. It follows that $U \cdot L(n) = L'(n)$. Thus $\text{conv}(L(n)^T) = \text{conv}(L'(n)^T)$, and we have shown $T_G \cong T_{G'}$.

**Example 3.11.** In the figure below, the graph on the left is the wedge of $K_5$ and $C_5$ with a leaf, and the graph on the right is the bridge of $K_5$ and $C_5$ with the appropriate labels.
In the graph on the left, let $A = \{1, 2, 3, 4, 9, 10\}$, let $x = 9$ and let $y = 10$. It is straightforward to verify that with this assignment, the graphs above are related via Proposition 3.10 and thus their respective Laplacian simplices are lattice equivalent.

Remark 3.12. It is not obvious which graph operations, aside from the transformations detailed in the proof of Proposition 3.10 and those found in Proposition 4.4, will result in unimodularly equivalent Laplacian simplices. It would be interesting to investigate this phenomenon further.

We next provide in Theorem 3.14 an operation on graphs that preserves reflexivity of Laplacian simplices. We will require the following lemma.

Lemma 3.13. Let $A \in \mathbb{Z}^{k \times k}$. If $(\det A)$ divides $mC_{ki}$ for each $i \in [k]$, where $C_{ki}$ is the cofactor of $A$, and $Ax = 1$ has an integer solution $x \in \mathbb{Z}^k$, then $Aw = [1, \ldots, 1, 1 + m]^T$ has an integer solution $w \in \mathbb{Z}^k$.

Proof. Notice we can write

$$Aw = A(x + y) = Ax + Ay = \begin{bmatrix} 1 & \vdots & \vdots & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \vdots & 1 & 1 + m \\ \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ m \\ \end{bmatrix}.$$ 

Solving the system $Ay = [0, \ldots, 0, m]^T$ yields

$$y = A^{-1} \cdot \begin{bmatrix} 0 \\ \vdots \\ m \\ \end{bmatrix} = \frac{1}{\det A} C^T \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \end{bmatrix} = \frac{m}{\det A} \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kk} \\ \end{bmatrix}$$

in which $C_{ki}$ is the cofactor of $A$. The above is an integer for each $i \in [k]$ by assumption. Set $w_j = x_j + y_j \in \mathbb{Z}$, and the result follows. \qed

We apply Lemma 3.13 when considering a connected graph $G$ on $m = n$ vertices with $A = L_B(i \mid \emptyset)$ for any $i \in [n]$. Here $\det L_B(i \mid \emptyset) = \pm \kappa$. Observe in this case the condition $Ax = 1$ for all $i \in [n]$ is equivalent to $T_G$ being a reflexive Laplacian simplex.

Theorem 3.14. Let $G$ and $G'$ be graphs with vertex set $[n]$ such that $T_G$ and $T_{G'}$ are reflexive. Suppose $\kappa_G$ divides $nM_{ij}$ and $\kappa_{G'}$ divides $nM'_{ij}$ for all $i, j \in [n - 1]$, where $M_{ij} = \det L_B(i, n \mid j)$ with $L$ as the Laplacian matrix of $G$, and $M'_{ij}$ is defined similarly. Let $H$ be the graph formed by $G$ and $G'$ with $V(H) = V(G) \cup V(G')$ and $E(H) = E(G) \cup E(G') \cup \{i, i'\}$ where $i \in V(G)$ and $i' \in V(G')$. Then $T_H$ is reflective.

Proof. To show $T_H$ is reflective, we show $T_H^*$ is a lattice simplex. Label the vertices of $H$ such that $V(G) = [n], V(G') = [2n] \setminus [n]$. Let $L_B, L_B(G), \text{ and } L_B(G')$ be the Laplacian matrices with basis
Example 3.11. Equivalent to the Laplacian simplex associated to the bridge of \( G \) as a consequence of Proposition 3.10 shows the Laplacian simplex associated to the wedge of \( G \) and \( G' \) is lattice polytope. Thus, the wedge of \( G \) and \( G' \) is reflexive if \( G \) and \( G' \) satisfy the conditions in Theorem 3.14.

The following proposition shows that Theorem 3.14 applies to graphs such as the one given in Example 3.11.
**Proposition 3.16.** If $G = C_{2k+1}$ and $G' = K_{2k+1}$, then the bridge graph between these is reflexive.

**Proof.** For cyclic graphs on $n$ vertices, the number of spanning trees is $n$. This and Lemma 3.17 show that both cyclic graphs and complete graphs satisfy the condition $\kappa$ divides $|V(G)| \cdot M_{ij}$, as described in Lemma 3.13. We show in later sections that $T_{K_n}$ and $T_{C_{2k+1}}$ are reflexive Laplacian simplices. \[ \square \]

**Lemma 3.17.** For all $n \geq 1$, $G = K_n$ satisfies the conditions of Lemma 3.13; that is, for each $i \in [n-1]$, $\kappa$ divides $nM_{nj}$ for each $1 \leq j \leq n-1$. Here $M_{nj} = \det L_B(i, n \mid j, n)$.

**Proof.** It is sufficient to show for each $1 \leq i, j \leq n-1$, $\kappa$ divides $nM_{ij}$ where $M_{ij} = \det L(i, n \mid j, n)$. By Lemma 3.6 this implies the result. For $G = K_n$, recall Cayley’s formula yields $\kappa = n^{n-2}$. Then we must show $n^{n-3}$ divides $M_{ij}$.

There are two cases to consider. Suppose $i = j$. Then using row operations on $L(i, n \mid i, n) \in \mathbb{Z}^{(n-2) \times (n-2)}$ which preserve the determinant, we have

\[
M_{ii} = \det \begin{pmatrix}
(n-1) & -1 & \cdots & \cdots & -1 \\
-1 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & -1 & \ddots \\
-1 & \cdots & \cdots & -1 & (n-1)
\end{pmatrix}
= \det \begin{pmatrix}
2 & 2 & \cdots & \cdots & 2 \\
-1 & (n-1) & -1 & \cdots & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & -1 & \ddots \\
-1 & \cdots & \cdots & -1 & (n-1)
\end{pmatrix}
= 2 \det \begin{pmatrix}
1 & 1 & \cdots & \cdots & 1 \\
-1 & (n-1) & -1 & \cdots & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & -1 & \ddots \\
-1 & \cdots & \cdots & -1 & (n-1)
\end{pmatrix}
= 2n^{n-3}.
\]

In the case $i \neq j$, $L(i, n \mid j, n) \in \mathbb{Z}^{(n-2) \times (n-2)}$ contains exactly one row and one column with all entries of $-1$. Without loss of generality we have

\[
= 2 \det \begin{pmatrix}
1 & 1 & \cdots & \cdots & 1 \\
0 & n & 0 & \cdots & 0 \\
\vdots & 0 & n & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & n
\end{pmatrix}
= 2n^{n-3}.
\]
We leave it as an exercise for the reader to show that $S_k$ has only one spanning tree. Consider the case where $G = P_k$, a path on $k$ vertices. Label the vertices along the path with the elements of $[k]$ in increasing order. Then $L$ and consequently $L_B$ have the form

$$L = \begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \vdots \\ 0 & -1 & 2 & -1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & 2 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}, \quad L_B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \vdots \\ 0 & -1 & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & -1 \end{bmatrix}$$

Observe that multiplication by the lower triangular matrix of all ones yields

$$L_B \cdot \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & 1 \\ -1 & \cdots & \cdots & -1 & -1 \end{bmatrix}.$$

Since the lower triangular matrix is an element in $GL_{k-1}(\mathbb{Z})$, it follows that $T_P$ is lattice equivalent to

$$S_{k-1}(1) := \text{conv} \left( e_1, e_2, \ldots, e_{k-1}, -\sum_{i=1}^{k-1} e_i \right).$$

We leave it as an exercise for the reader to show that $S_{k-1}(1)$ is the unique reflexive $(k-1)$-polytope of minimal volume. This extends to all trees as follows.

**Proposition 4.1.** Let $G$ be a tree on $n$ vertices. Then $T_G$ is unimodularly equivalent to $S_{n-1}(1)$.

**Proof.** Let $G$ be a tree on $n$ vertices. Then $T_G$ is a simplex that contains the origin in its strict interior and has normalized volume equal to $n$, since $G$ has only one spanning tree. Consider the
triangulation of $T_G$ that consists of creating a pyramid at the origin over each facet. Since $G$ is a tree,
\[
\text{vol}(T_G) = \sum_{\text{facet}} \text{vol}(F) = 1 \cdot n = n.
\]
There are $n$ facets, so each must have $\text{vol}(F) = 1$. Applying a unimodular transformation to $n-1$ of the vertices of $T_G$, we can assume that the vertices of $T_G$ are the $n$ standard basis vectors and a single integer vector in the strictly negative orthant (so that the origin is in the interior of $T_G$). Because the normalized volume of the pyramid over each facet is equal to 1, it follows that the final vertex is $-\mathbf{1}$.

**Corollary 4.2.** The $h^*$-vector of the Laplacian simplex for any tree is $(1,1,\ldots,1)$, hence is unimodal.

**Corollary 4.3.** Let $G$ be a tree on $n$ vertices with Laplacian matrix $L_B$. Then there exists $U \in GL_{n-1}(\mathbb{Z})$ such that
\[
L_B \cdot U = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1 \\
-1 & \cdots & \cdots & -1
\end{bmatrix}
\]

The next proposition asserts that attaching an arbitrary tree with $k$ vertices to a graph on $n$ vertices yields a lattice isomorphism between the resulting Laplacian simplex and the Laplacian simplex obtained by attaching any other tree with $k$ vertices at the same root.

**Proposition 4.4.** Let $G$ be a connected graph on $n$ vertices, and let $v$ be a vertex of $G$. Let $G'$ be the graph obtained from $G$ by attaching $k$ vertices such that $G'$ restricted to the vertex set $\{v\} \cup [k]$ forms a tree, call it $T$. The edges of $G'$ are the edges from $G$ along with any edges among the vertices $\{v\} \cup [k]$. Let $P$ be the graph obtained from $G$ by attaching $k$ vertices such that $P$ restricted to the vertex set $\{v\} \cup [k]$ forms a path. Then $T_{G'} \cong T_P$.

**Proof.** The reduced Laplacian matrix associated to $T_{G'}$ is the following $(n+k) \times (n+k-1)$ matrix:
\[
\begin{bmatrix}
L_B(G) & 0 \\
0 & L_B(T)
\end{bmatrix}
\]

Here $L_B(T) \in \mathbb{Z}^{(k+1) \times k}$ is the Laplacian matrix for $T$, the tree on $(k+1)$ vertices. Let $U \in GL_k(\mathbb{Z})$ be the matrix such that $L_B(T) \cdot U$ gives the matrix with vertex set $S_k(1)$ as in Corollary 4.3. Then we have
For any set of \( k \) vertices we attach to a vertex \( v \in V(G) \) to obtain a tree on the vertex set \( \{v\} \cup [k] \), we get a corresponding unimodular matrix \( U \) such that the above multiplication holds. The determinant of the \( (n - 1 + k) \times (n - 1 + k) \) transformation matrix is equal to the determinant of \( U \), which is \( \pm 1 \). Then \( T_{G'} \) is lattice equivalent to \( T_P \) for any such \( G' \).

**Remark 4.5.** It follows from Theorem 3.14 that bridging a tree to a graph \( G \) with \( T_G \) reflexive and \( L(G) \) satisfying the appropriate division condition on minors will result in a new reflexive Laplacian simplex. Further, Proposition 4.4 shows that the equivalence class of the resulting reflexive simplex is independent of the choice of tree used in the attachment.

5. CYCLES

Let \( C_n \) denote the cycle with \( n \) vertices. In this section, we show that odd cycles are reflexive and have unimodal \( h^* \)-vectors, but fail to be IDP. We show that whiskering even cycles results in reflexive Laplacian simplices. Finally, we determine the \( h^* \)-vectors for \( T_{C_n} \) when \( n \) is an odd prime.

5.1. Reflexivity and Whiskering.

**Theorem 5.1.** For \( n \geq 3 \), the simplex \( T_{C_n} \) is reflexive if and only if \( n \) is odd. For \( k \geq 2 \), the simplex \( T_{C_{2k}} \) is 2-reflexive.

**Proof.** Let \( C_n \) be a cycle with vertex set \( [n] \) and vertices labeled cyclically. Then \( L \) and consequently \( L_B \) have the form (when rows and columns are suitably labeled)

\[
L = \begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & -1 \\
-1 & 2 & -1 & \ddots & 0 \\
0 & -1 & 2 & -1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & -1 & 2 & -1 \\
-1 & 0 & \cdots & 0 & -1 & 2
\end{bmatrix}
\]

\[
L_B = \begin{bmatrix}
2 & 1 & \cdots & 0 & -1 \\
-1 & 1 & 0 & \cdots & 0 \\
0 & -1 & 1 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 1 \\
-1 & -1 & \cdots & -1 & -2
\end{bmatrix}
\]

To show that \( T_{C_n} \) is reflexive, we show \( T_{C_n}^* = \{x \mid L_Bx \leq 1\} \) is a lattice polytope. Each intersection of \( (n - 1) \) of these facet hyperplanes will yield a unique vertex of \( T_{C_n}^* \), since the rank of \( L_B \) is \( n - 1 \). For each \( i \in [n] \), let \( v_i \in \mathbb{R}^{n-1} \) be the vertex that satisfies \( L_B(i \mid 0) \cdot v_i = 1 \). Solving the appropriate system of linear equations yields

\[
v_1 = \left( \frac{1 - n}{2}, \frac{3 - n}{2}, \frac{5 - n}{2}, \ldots, \frac{n - 5}{2}, \frac{n - 3}{2} \right) = \left( \frac{(2j - 1) - n}{2} \right)_{j=1}^{n-1}
\]

\[
v_i = \left( \frac{(2j + 1) + n - 2i}{2} \right)_{j=1}^{i-1}, \left( \frac{(2j + 1) - n - 2i}{2} \right)_{j=i}^{n-1}, \text{ for } 2 \leq i \leq n - 1
\]

\[
v_n = \left( \frac{3 - n}{2}, \frac{5 - n}{2}, \frac{7 - n}{2}, \ldots, \frac{n - 3}{2}, \frac{n - 1}{2} \right) = \left( \frac{(2j + 1) - n}{2} \right)_{j=1}^{n-1}
\]
These are the vertices of $T_{C_n}^*$. Note $v_i \in \mathbb{Z}^{n-1}$ only if $n$ is odd. Then $T_{C_n}$ is reflexive if and only if $n$ is odd.

For the even case, observe the coordinates of each vertex of $2 \cdot T_{C_{2k}}^*$ are relatively prime. Then each of these vertices is primitive. Thus, for $n = 2k$ each vertex of $T_{C_{2k}}^*$ is a multiple of $\frac{1}{2}$, which allows us to write

$$T_{C_{2k}} = \left\{ x \mid \frac{1}{2} \tilde{A}x \leq 1 \right\} = \left\{ x \mid \tilde{A}x \leq 2 \cdot 1 \right\}$$

where $\tilde{A} \in \mathbb{Z}^{n \times (n-1)}$. The facets of $T_{C_{2k}}$ have supporting hyperplanes $\langle r_i, x \rangle = 2$ where $r_i$ is the $i^{th}$ row of $\tilde{A}$. Thus $T_{C_{2k}}$ is a 2-reflexive Laplacian simplex. \hfill $\square$

**Example 5.2.** Below are the dual polytopes to $T_{C_n}$ for small $n$.

- $T_{C_3}^* = \text{conv} \left( (-1,0), (1,-1), (0,1) \right)$
- $T_{C_4}^* = \text{conv} \left( (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}, 1), (\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}), (-\frac{1}{2}, 1, \frac{3}{2}) \right)$
- $T_{C_5}^* = \text{conv} \left( (-2, -1, 0, 1), (2, -2, -1, 0), (1, 2, -2, -1), (0, 1, 2, -2), (-1, 0, 1, 2) \right)$

Although $T_{C_{2k}}$ is not reflexive, we show next that whiskering $C_{2k}$ results in a graph $W(C_{2k})$ such that $T_{W(C_{2k})}$ is reflexive. The technique of whiskering graphs has been studied previously in the context of Cohen-Macaulay edge ideals, see [11, Theorem 4.4] and [27].

**Definition 5.3.** To add a whisker at a vertex $x \in V(G)$, one adds a new vertex $y$ and the edge connecting $x$ and $y$. Let $W(G)$ denote the graph obtained by whiskering all vertices in $G$. We call $W(G)$ the whiskered graph of $G$. If $V(G) = \{x_1, \ldots, x_n\}$ and $E(G) = E$, then $V(W(G)) = V(G) \cup \{y_1, \ldots, y_n\}$ and $E(W(G)) = E \cup \{x_1 y_2, \ldots, x_n y_n\}$.

**Proposition 5.4.** $T_{W(C_n)}$ is reflexive for even integers $n \geq 2$.

*Proof.* $W(C_n)$ is a graph with vertex set $[2n]$ and $2n$ edges. Label the vertices of the cycle with $[n]$ in a cyclic manner. Label the vertices of each whisker with $i$ and $n+i$ where $i \in [n]$. The Laplacian matrix has the following form.

$$L = \begin{bmatrix}
L + I_n & -I_n \\
-I_n & I_n
\end{bmatrix}$$

Consequently if $A$ is the $n \times (n-1)$ matrix given by Equation (I), then

$$L_B = \begin{bmatrix}
L_B(C_n) + A & A^T \\
1 & \cdots & 1 \\
-A & -A^T \\
-1 & \cdots & -1
\end{bmatrix}.$$ 

We show $T_{W(C_n)}$ is reflexive by showing $T_{W(C_n)}^*$ is a lattice polytope. Each vertex of the dual is a solution to $L_B(i \mid 0)v_i = 1$. We consider the following cases.

**Case:** $1 \leq i \leq n$. Multiply both sides of $L_B(i \mid 0)v_i = 1$ by the $(2n-1) \times (2n-1)$ upper diagonal matrix with the following entries.
Since \( L \) corresponds to adding the two rows of \( L_B(i \mid \emptyset) \) that are indexed by the labels of a whisker in the graph. The last \( n \) rows will have an entry of 1 along the diagonal and an entry of -1 on the superdiagonal, which corresponds to subtracting consecutive rows in \( L_B(i \mid \emptyset) \) to achieve cancellation. We obtain the following system of linear equations.

\[
\begin{bmatrix}
L_B(C_n)(i \mid \emptyset) & 0 \\
-I_{n-1} & \begin{bmatrix} 0 & \vdots & I_{n-1} \\
0 & \ddots & \vdots \\
-1 & \cdots & -1 \end{bmatrix}
\end{bmatrix}
\]

Let \((v_i^*)_j\) denote the \(j\)th coordinate of the vertex \(v_i \in \mathbb{Q}^{n-1}\) of \(T_{C_n}^*\) described in Proposition 5.1. Then the vertex \(v_i\) of \(T_{W(C_n)}^*\) has the following form.

\[
(v_i)_j = \begin{cases} 
2(v_i^*)_j, & \text{if } 1 \leq j \leq n-1 \\
-1 - \sum_{k=1}^{n-1} 2(v_i^*)_k, & \text{if } j = n \\
2(v_i^*)_{j-n}, & \text{if } n+1 \leq j \leq 2n-1
\end{cases}
\]

Since \(2(v_i^*)_j \in \mathbb{Z}\) by Proposition 5.1 for \(1 \leq j \leq n-1\), then \(v_i \in \mathbb{Z}^{2n-1}\).

**Case:** \(n + 2 \leq i \leq 2n\). The strategy is to multiply the equality \(L_B(i \mid \emptyset)v_i = 1\) by the matrix that performs the following row operations. Let \(r_m \in \mathbb{Z}^{2n-1}\) denote the \(m\)th row of \(L_B(i \mid \emptyset)\). For each whisker with vertex labels \(\{m, n+m\}\), replace \(r_m\) with \(r_m + r_{m+n}\) for \(m \in [n]\). Row \(i-1\) will not have a row to add because the index of its whisker is the index of the deleted row. Since each column in \(L\) sums to 0, the negative sum of all the rows of \(L_B(i \mid \emptyset)\) is equal to the row removed. We recover the missing row by replacing \(r_{i-n}\) with \(-\sum_{k=1}^{n-1} r_k\) for \(r_k \in L_B(i \mid \emptyset)\). Then as in the previous case, we want to replace row \(r_k\) with \(r_k - r_{k+1}\) for \(n+1 \leq k \leq 2n-2\). Here \(r_{i-n}\) plays the role of the deleted \(r_i\). We obtain a similar system of linear equations found in the first case. The vertex \(v_i\) of \(T_{W(C_n)}^*\) has the following form.

\[
(v_i)_j = \begin{cases} 
2(v_i^*)_j, & \text{if } 1 \leq j \leq n-1 \\
-1 - \sum_{k=1}^{n-1} 2(v_i^*)_k, & \text{if } j = n \\
2(v_i^*)_{j-n}, & \text{if } n+1 \leq j \leq 2n-1 \text{ and } j \neq i-1, i \\
2(v_i^*)_j - 2n, & \text{if } j = i-1 \\
2(v_i^*)_j - 2n, & \text{if } j = i
\end{cases}
\]

Observe in the case \(i = 2n\), the last equality is not applicable since \(j \in [2n-1]\). Then \(v_i \in \mathbb{Z}^{2n-1}\).

**Case:** \(i = n + 1\). Here \((v_i)_{n-1} = (v_i)_n = -(2n-1) - \sum_{k=1}^{n-1} 2(v_i^*)_k \in \mathbb{Z}\) and the other coordinates are as described above. Then \(v_i \in \mathbb{Z}^{2n-1}\). \(\square\)
We extend Proposition 5.4 to a more general result, that whiskering a graph whose Laplacian simplex is 2-reflexive results in a graph whose Laplacian simplex is reflexive. Although even cycles are the only known graph type to result in 2-reflexive Laplacian simplices, we include the following result.

**Proposition 5.5.** If $G$ is a connected graph on $n$ vertices such that $T_G$ is 2-reflexive, then $T_{W(G)}$ is reflexive for all $n \geq 2$.

**Proof.** If $T_G$ is 2-reflexive, then each vertex $v_i$ of $T_G^*$ satisfies $2v_i \in \mathbb{Z}^{n-1}$ for each $1 \leq i \leq n$. As in the proof of Proposition 5.4, we can find descriptions of the vertices of $T_{W(G)}^*$ in terms of the coordinates from vertices of $T_G^*$ to show they are lattice points. The result follows.

Given a graph $G$ with $T_G$ reflexive, we have already seen that attaching a tree on $|V(G)|$ vertices to obtain a new graph $G'$ on $2 \cdot |V(G)|$ vertices results in the reflexive Laplacian simplex $T_{G'}$. Whiskering a graph also preserves the reflexivity of $T_G$, as seen in the following result.

**Proposition 5.6.** If $G$ is a connected graph on $n$ vertices such that $T_G$ is reflexive, then $T_{W(G)}$ is reflexive for all $n \geq 1$.

**Proof.** If $T_G$ is reflexive, then vertices of $T_G^*$ are integer and satisfy $L_B(i | \emptyset)v_i = 1$ for all $1 \leq i \leq n$. Observe $2v_i \in \mathbb{Z}^{n-1}$ satisfies $L_B(i | \emptyset)2v_i = 2 \cdot 1$. Following the proof technique in Proposition 5.4, we can find descriptions of the vertices of $T_{W(G)}^*$ in terms of the coordinates from vertices of $T_G^*$ to show they are lattice points.

5.2. $h^*$-Unimodality. For odd $n$, our proof of the following theorem can be interpreted as establishing the existence of a weak Lefschetz element in the quotient of the semigroup algebra associated to cone $(T_{C_n})$ by the system of parameters corresponding to the ray generators of the cone. This proof approach is not universally applicable, as there are examples of reflexive IDP simplices with unimodal $h^*$-vectors for which this proof method fails [7].

**Theorem 5.7.** For odd $n$, $h^*(T_{C_n})$ is unimodal.

**Proof.** Recall from Lemma 2.6 that $h_i^*(T_{C_n})$ is the number of lattice points in $\Pi_{T_{C_n}}$ at height $i$. Theorem 5.4 shows $h_i^*(T_{C_n})$ is symmetric. Our goal is to prove that for $i \leq [n/2]$ we have $h_i^* \leq h_{i+1}^*$. This will show that $h^*(T_{C_n})$ is unimodal.

While $\kappa = n$ for $C_n$, we will freely use both $\kappa$ and $n$ to denote this quantity, as it is often helpful to distinguish between the number of spanning trees and the number of vertices. Lattice points in the fundamental parallelepiped of $T_{C_n}$ can be described as follows:

$$\mathbb{Z}^n \cap \left\{ \frac{1}{\kappa n} \sum_{i=1}^{n} b_i \cdot [L_B | 1], 0 \leq b_i < \kappa n, b_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} b_i \equiv 0 \mod \kappa n \right\}.$$  

We will use the modular equation above extensively in our analysis. Denote the height of a lattice point in $\Pi_{T_{C_n}}$ by

$$h(b) := \frac{\sum_{i=1}^{n} b_i}{n \kappa} \in \mathbb{Z}_{\geq 0}.$$  

We first show that every lattice point in $\Pi_{T_{C_n}}$ arising from $b$ satisfies

$$\frac{(k-j+1)(b_1-b_n)}{\kappa n} + \frac{b_j-b_{k+1}}{\kappa n} \in \mathbb{Z}$$

for each $1 \leq j < k \leq n - 1$. Since the lattice point lies in $\Pi_{T_{C_n}}$, we have the following constraint equations:

$$\frac{b_1-b_n+b_j-b_{k+1}}{\kappa n} \in \mathbb{Z}$$
for each \(1 \leq i \leq n - 1\). Summing any consecutive set of these equations where \(1 \leq j \leq k \leq n - 1\) yields
\[
\sum_{i=j}^{k} \left( \frac{b_1 - b_n}{\kappa n} + \frac{b_i - b_{i+1}}{\kappa n} \right) \in \mathbb{Z}.
\]

The result follows.

Thus, each vector \(b\) corresponding to an integer point in \(\Pi_{T_{C_n}}\) satisfies \(\kappa \mid (b_1 - b_n)\), which follows from setting \(j = 1\) and \(k = n - 1\). We next claim that every lattice point in \(\Pi_{T_{C_n}}\) arises from \(b \in \mathbb{Z}^n\) such that \(b_i \equiv b_j \mod (\kappa)\) for each \(1 \leq i, j \leq n\). To prove this, let \(B = \frac{b_1 - b_n}{\kappa}\) which is divisible by \(\kappa\). For \(1 \leq i \leq n - 1\), our constraint equation becomes \(\frac{B}{n} + \frac{b_i - b_{i+1}}{\kappa n} = C\) for some \(C \in \mathbb{Z}\). Then \(\frac{b_i - b_{i+1}}{\kappa n} = Cn - B \in \mathbb{Z}\) holds for each \(i\). The result follows.

**First Major Claim:** For \(n\) odd, any lattice point in \(\Pi_{T_{C_n}}\) arises from \(b \in \mathbb{Z}^n\) such that \(b_i \equiv 0 \mod (\kappa)\) for each \(1 \leq i \leq n\).

To prove this, let \(b_i = m_i \kappa + \alpha\) such that \(0 \leq m_i < \kappa\) and \(0 \leq \alpha < \kappa\). Constraint equations yield
\[
\frac{b_1 - b_n + b_i - b_{i+1}}{\kappa n} = \frac{m_1 - m_n + m_i - m_{i+1}}{n} \in \mathbb{Z}
\]
using \(\kappa = n\). Summing all \(n - 1\) integer expressions with linear coefficients yields
\[
\sum_{i=1}^{n} m_i = \frac{n(n-1)}{2} m_1 + \sum_{i=1}^{n-1} m_i - (n-1)m_n - \frac{n(n-1)}{2} m_n,
\]
which is divisible by \(n\). Call the resulting sum \(An\) for some \(A \in \mathbb{Z}\). Finally, notice the last constraint equation (corresponding to \(h(b)\)) can be written
\[
\sum_{i=1}^{n} \frac{b_i}{\kappa n} = \frac{\sum_{i=1}^{n} m_i + \alpha}{n}
\]
\[
= \frac{m_n + An - \frac{n(n-1)}{2} m_1 + (n-1)m_n + \frac{n(n-1)}{2} m_n + \alpha}{n} \in \mathbb{Z}.
\]

Then \(n\) odd implies \(n\) divides \(\frac{n(n-1)}{2}\) so that \(n\) divides \(\alpha\). Since \(0 \leq \alpha < n\), then \(\alpha = 0\) as desired.

**Second Major Claim:** Consider \(T_{C_n}\) for odd \(n\). Suppose \(h(b) < \frac{n-1}{2}\). If \(p \in \Pi_{T_{C_n}} \cap \mathbb{Z}^n\), then \(p + (0, \ldots, 0, 1)^T \in \Pi_{T_{C_n}} \cap \mathbb{Z}^n\).

To establish this, it suffices to prove that for every \(p = \frac{1}{n^2} b \cdot [L_B | \mathbb{1}] \in \Pi_{T_{C_n}} \cap \mathbb{Z}^n\) such that \(h(b) < \frac{n-1}{2}\), we have \(b_i < n(n-1)\) for each \(i\). This would imply
\[
p + (0, \ldots, 0, 1)^T = \frac{1}{n^2} (b + n\mathbb{1}) \cdot [L_B | \mathbb{1}] \in \Pi_{T_{C_n}} \cap \mathbb{Z}^n,
\]
providing an injection from the lattice points in \(\Pi_{T_{C_n}}\) at height \(i\) to those at height \(i+1\). Constraint equations yield, using the same notation as in the proof of our first major claim, that
\[
-m_{j-1} + 2m_j - m_{j+1} \in n\mathbb{Z}
\]
for each \(1 \leq j \leq n\). Note that this comes from subtracting the two integers
\[
\frac{m_1 + m_j - m_{j+1} - m_n}{n} - \frac{m_1 + m_{j-1} - m_j - m_n}{n} = \frac{2m_j - (m_{j-1} + m_{j+1})}{n} \in \mathbb{Z}
\]
for each \(2 \leq j \leq n - 1\), as well as
\[
\frac{2m_1 - m_2 - m_n}{n}, \frac{-(m_1 + m_{n-1} - 2m_n)}{n} \in \mathbb{Z}.
\]
For a contradiction, suppose there exists a $j$ such that $b_j = n(n-1)$. Then $m_j = n - 1$. Constraints on the other variables $m_i$ imply

$$0 \leq \frac{2(n-1) - (m_{j-1} + m_{j+1})}{n} \leq 1 \implies 2(n-1) - (m_{j-1} + m_{j+1}) = 0 \text{ or } n.$$ 

Case 1: If the above is 0, then

$$2(n-1) = m_{j-1} + m_{j+1} \implies m_{j-1} = m_{j+1} = n - 1.$$ 

Apply these substitutions on other constraint equations to yield $m_i = n - 1$ for all $1 \leq i \leq n$. Then

$$h(b) = \frac{\sum_{i=1}^{n} m_i}{n} = \frac{n(n-1)}{n} = n - 1 > \frac{n-1}{2},$$

which is a contradiction.

Case 2: If the above is 1, then $n - 2 = m_{j-1} + m_{j+1}$. Adding subsequent constraint equations yields

$$(-m_j + 2m_{j-1} - m_{j-2}) + (-m_j + 2m_{j+1} - m_{j+2}) = -2m_j + 2(m_{j-1} + m_{j+1}) - (m_{j-2} + m_{j+2})$$

$$= -2(n-1) + 2(n-2) - (m_{j-2} + m_{j+2})$$

$$= -2 - (m_{j-2} + m_{j+2})$$

Since the above is in $n\mathbb{Z}$, it is equal to either $-2n$ or $-n$.

Case 2a: If the above is equal to $-2n$, then $m_{j-2} = m_{j+2} = n - 1$. Then

$$-m_{j-3} + 2m_{j-2} - m_{j-1} = -m_{j-3} + m_{j+1} \in n\mathbb{Z} \implies m_{j-3} = m_{j+1}.$$ 

A similar argument shows $m_{j+3} = m_{j-1}$. Continuing in this way shows $m_{j+k} = m_{j+1}$ for remaining $m_i$. Then for each of the $\frac{n-3}{2}$ pairs, $m_{j-k} + m_{j+k} = n - 2$ where $k \in \{1, 2, 3, \ldots, \frac{n-1}{2}\}$. But then

$$h(b) = \frac{\sum_{i=1}^{n} m_i}{n} = \frac{n - 1 + 2(n-1) + \frac{n-3}{2}(n-2)}{n}$$

$$= \frac{n + 1}{2},$$

which is a contradiction.

Case 2b: If the above is equal to $-n$, then $m_{j-2} + m_{j+2} = n - 2$. Adding subsequent constraint equations as above yields $n - 2 - (m_{j-3} + m_{j+3})$. Since the above is in $n\mathbb{Z}$, it is equal to either $-2n$ or $-n$.

Case 2b(i): If the above is equal to $-n$, then $m_{j-3} = m_{j+3} = n - 1$. Following the same argument as Case 2a leads to the contradiction, $h(b) = \frac{n+1}{2}$.

Case 2b(ii): If the above is equal to $-2n$, then $m_{j-3} + m_{j+3} = n - 2$. Continuing in this manner yields $m_{j-k} + m_{j+k} = n - 2$ for all $k \in \{1, 2, \ldots, \frac{n-1}{2}\}$. But then

$$h(b) = \frac{n - 1 + \frac{(n-1)}{2}(n-2)}{n} = \frac{n-1}{2},$$

which is a contradiction. This concludes the proof of our second major claim.

The second claim implies that for $i \leq \lfloor n/2 \rfloor$, we have $h_i^* \leq h_{i+1}^*$. Thus, our proof is complete.
5.3. Structure of $h^*$-vectors. We next classify the lattice points in the fundamental parallelepiped for $T_{C_n}$ by considering the matrix $[L_B \mid 1]$ over the ring $\mathbb{Z}/\kappa\mathbb{Z}$. Let

$$[\tilde{L} \mid 1] := [L_B \mid 1] \mod \kappa.$$  

Recall that for a cycle we have $n = \kappa$.

**Lemma 5.8.** For $C_n$ with odd $n$ and corresponding reduced Laplacian matrix $[L_B \mid 1]$, we have

$$\ker_{\mathbb{Z}/\kappa\mathbb{Z}} [\tilde{L} \mid 1] = \{ x \in (\mathbb{Z}/\kappa\mathbb{Z})^n \mid x[L_B \mid 1] \equiv 0 \mod \kappa \} = (1^n, (0, 1, \ldots, n-1)).$$

**Proof.** Consider the second principal minor of $[L_B \mid 1]$ with the first and $n$th rows and columns deleted. The matrix $[L_B \mid 1](1, n \mid 1, n)$ is the lower diagonal matrix of the following form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

Then $\det [L_B \mid 1](1, n \mid 1, n) = 1$ implies there are $n - 2$ linearly independent columns, hence $\text{rk}_{\mathbb{Z}/\kappa\mathbb{Z}}[L_B \mid 1] \geq n - 2$.

Since the entries in each column of $[L_B \mid 1]$ sum to 0, then

$$1 \cdot [L_B \mid 1] = (0, 0, \ldots, 0) \equiv 0 \mod \kappa$$

implies $1 \in \ker_{\mathbb{Z}/\kappa\mathbb{Z}} [L_B \mid 1]$. Consider

$$(0, 1, \ldots, n-1) \cdot
\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ -1 & -1 & -1 & \cdots & -2 & 1 \end{pmatrix} = (-n, \ldots, -n, \frac{n(n-1)}{2}) \equiv 0 \mod \kappa.$$  

This shows $(0, 1, \ldots, n-1) \in \ker_{\mathbb{Z}/\kappa\mathbb{Z}}[L_B \mid 1]$. Since these two vectors are linearly independent, we have $\text{rk}_{\mathbb{Z}/\kappa\mathbb{Z}}[L_B \mid 1] \leq n - 2$.

Thus, the kernel is two-dimensional and we have found a basis. \hfill $\square$

**Theorem 5.9.** For odd $n \geq 3$, lattice points in $\Pi_{T_{C_n}}$ are of the form

$$\frac{(\alpha \mathbf{1} + \beta(0, 1, \ldots, n-1))}{\kappa} \mod \kappa \cdot [L_B \mid 1]$$

for all $\alpha, \beta \in \mathbb{Z}/\kappa\mathbb{Z}$. Thus, $h^*_T(G)$ is equal to the cardinality of

$$\left\{ \frac{(\alpha \mathbf{1} + \beta(0, 1, \ldots, n-1))}{\kappa} \mod \kappa \cdot [L_B \mid 1] \mid 0 \leq \alpha, \beta < \kappa - 1, \sum_{j=0}^{n-1} (\alpha + j\beta) \mod \kappa = i \right\}.$$  

**Proof.** Since $|\Pi_{T_{C_n}} \cap \mathbb{Z}^n| = \sum_{i=0}^{n-1} h^*_i(T_{C_n}) = n\kappa = n^2$, there are $n^2$ lattice points in the fundamental parallelepiped. Similarly, there are $n^2$ possible linear combinations of $\mathbf{1}$ and $(0, 1, 2, \ldots, n-1)$ in $\mathbb{Z}/\kappa\mathbb{Z}$. We show that each such linear combination yields a lattice point. Recall the sum of the coordinates down each of the first $n - 1$ columns of $[L_B \mid 1]$ is 0. Since

$$(\alpha \mathbf{1} + \beta(0, 1, \ldots, n-1)) \cdot [L_B \mid 1] \equiv 0 \mod \kappa$$
by Lemma 5.8 it follows that
\[(\alpha \mathbf{1} + \beta(0, 1, \ldots, n-1) \mod \kappa) \cdot [L_B \mid \mathbf{1}] \equiv \mathbf{0} \mod \kappa.\]

Then \[(\alpha \mathbf{1} + \beta(0, 1, \ldots, n-1)) \mod \kappa \cdot [L_B \mid \mathbf{1}]\] is a lattice point. Since we are reducing the numerators of the entries in the vector of coefficients modulo \(\kappa\) prior to dividing by \(\kappa\), it follows that each entry in the coefficient vector is greater than or equal to 0 and strictly less than 1, and hence the resulting lattice point is an element of \(\Pi_{\mathcal{T}_n}\).

\begin{proof}
Consider \(\mathcal{T}_n\) where \(n \geq 3\) is odd. Let \(n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}\) be the prime factorization of \(n\) where \(p_1 > p_2 > \cdots > p_k\). Then
\[h^*(\mathcal{T}_n) = (1, 1, \ldots, 1, h_m, h_{m+1}, \ldots, h_{\frac{m}{2}}, \ldots, h_{n-m-1}, h_{n-m}, 1, \ldots, 1)\]
where \(m = \frac{1}{2}(n - p_1^{a_1} \cdots p_k^{a_k-1})\) and \(h_m > 1\). Further, if \(\mathbb{Z}_n^*\) denotes the group of units of \(\mathbb{Z}_n\), we have that \(h^*_m(n-1)/2 \geq n \cdot |\mathbb{Z}_n^*| + 1\). In particular, if \(n\) is prime, we have
\[h^*(\mathcal{T}_n) = (1, 1, n^2 - n + 1, 1, \ldots, 1)\]

Proof. Keeping in mind that \(n = \kappa\) for \(\mathcal{T}_n\), denote the height of the lattice point
\[(\alpha \mathbf{1} + \beta(0, 1, \ldots, n-1)) \mod \frac{n}{\kappa} \cdot [L_B \mid \mathbf{1}]\]
in the fundamental parallelepiped by
\[h(\alpha, \beta) := \frac{1}{n} \sum_{j=0}^{n-1} ((\alpha + j\beta) \mod n).\]

Each \(\alpha \in \mathbb{Z}/n\mathbb{Z}\) paired with \(\beta = 0\) produces a lattice point at a unique height in \(\Pi_{\mathcal{T}_n}\), and thus each \(h^*_m \geq 1\). Let \(\mathbb{Z}_n^*\) denote the group of units of \(\mathbb{Z}_n\). If \(\beta \in \mathbb{Z}_n^*\), then \(\beta(0, 1, \ldots, n-1) \mod n\) yields a vector that is a permutation of \((0, 1, \ldots, n-1)\), and thus for any \(\alpha\) we have the height of the resulting lattice point is \((n-1)/2\), proving that \(h^*_m(n-1)/2 \geq n \cdot |\mathbb{Z}_n^*| + 1\). Thus, when \(n\) is an odd prime, it follows that
\[h^*(\mathcal{T}_n) = (1, 1, n^2 - n + 1, 1, \ldots, 1).\]

Now, suppose that \(\gcd(\beta, n) = \prod p_i^{b_i} \neq 1\). Then the order of \(\beta\) in \(\mathbb{Z}_n\) is \(\prod p_i^{a_i-b_i}\), and (after some reductions in summands modulo \(n\))
\[h(\alpha, \beta) = \frac{1}{n} \prod p_i^{b_i} \cdot \left(\prod_{j=0}^{\frac{n}{p_1^{a_1-b_1}-1}} \left((\alpha + j \prod p_i^{b_i}) \mod n\right)\right).\]

Thus, we see that for a fixed \(\beta\), the height is minimized (not uniquely) when \(\alpha = 0\). In this case, we have
\[h(0, \beta) = \frac{1}{n} \prod p_i^{b_i} \cdot \left(\prod_{j=0}^{\frac{n}{p_1^{a_1-b_1}-1}} \left(j \prod p_i^{b_i} \mod n\right)\right).\]
\[= \frac{n - \prod p_i^{b_i}}{2}.\]
This value is minimized when $\prod p_i^{b_i} = p_1^{a_1} \cdots p_k^{a_k-1}$, and this height is attained more than once by setting $\beta = p_1^{a_1} \cdots p_k^{a_k-1}$ and $\alpha = 0, 1, 2, \ldots, p_1^{a_1} \cdots p_k^{a_k-1} - 1$. \hfill $\Box$

**Corollary 5.11.** $T_{C_n}$ is not IDP for odd $n \geq 3$.

**Proof.** Theorem 5.10 yields $h_1^*(T_{C_n}) = 1$ for odd $n \geq 3$. It is known [5, Corollary 3.16] that for an integral convex $d$-polytope $P$, $h_1^*(P) = |P \cap \mathbb{Z}^n| - (d + 1)$. In this case,

$$|T_{C_n} \cap \mathbb{Z}^n| = h_1^*(T_{C_n}) + (n - 1) + 1 = n + 1$$

is the number of lattice points in $T_{C_n}$. In particular, the lattice points consist of the $n$ vertices of $T_{C_n}$ and the origin. Then $\Pi_{T_{C_n}} \cap \{x \mid x_n = 1\} \cap \mathbb{Z}^n = (0, 0, \ldots, 0, 1)$. If $T_{C_n}$ is IDP, then every lattice point in $\Pi_{T_{C_n}}$ is of the form $(0, \ldots, 0, 1) + \cdots + (0, \ldots, 0, 1)$, which is not true by Proposition 5.9. The result follows. \hfill $\Box$

6. Complete Graphs

The simplex $T_{K_n}$ is a generalized permutohedron, where a permutohedron $P_n(x_1, \ldots, x_n)$ for $x_i \in \mathbb{R}$ is the convex hull of the $n!$ points obtained from $(x_1, \ldots, x_n)$ by permutations of the coordinates. For $K_n$, the Laplacian matrix has diagonal entries equal to $n - 1$ and all other entries equal to $-1$. Then $\text{conv}(L(n)^T) = P_n(n-1, -1, \ldots, -1) \cong P_n(n, 1, \ldots, 1)$. Many properties of generalized permutohedra are known [22]. While some of the findings in this section follow from these general results, for the sake of completeness we will prove all results in this section from first principles.

6.1. Reflexivity, Triangulations, and $h^*$-Unimodality.

**Theorem 6.1.** The simplices $T_{K_n}$ are reflexive for $n \geq 1$.

**Proof.** Observe $L_B$ is an $n \times (n - 1)$ integer matrix of the form

$$L_B = \begin{bmatrix}
(n-1) & (n-2) & (n-3) & \cdots & \cdots & 1 \\
-1 & (n-2) & (n-3) & \cdots & \cdots & 1 \\
-1 & -2 & (n-3) & \cdots & \cdots & \vdots \\
-1 & -2 & -3 & (n-4) & \cdots & \vdots \\
\vdots & \vdots & \vdots & -4 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & 1 \\
-1 & -2 & -3 & \cdots & \cdots & -(n-1)
\end{bmatrix}$$

To prove $T_{K_n}$ is reflexive, we show $T_{K_n} = \{ x \in \mathbb{R}^{n-1} \mid A x \leq 1 \}$ for some $A \in \mathbb{Z}^{n \times (n-1)}$. We claim that $A$ has the following form:

$$A = \begin{bmatrix}
-1 & 0 & 0 & \cdots & 0 \\
1 & -1 & 0 & \vdots \\
0 & 1 & -1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 1 & -1 \\
0 & \cdots & 0 & 1 
\end{bmatrix} \in \{0, \pm 1\}^{n \times (n-1)}.$$

Let $r_i$ be the $i^{th}$ row of $L_B$. Observe that $A(i \mid \emptyset) r_i = 1$ for each $1 \leq i \leq n$. Then $\{r_i\}_{i=1}^n$ is a set of intersection points of defining hyperplanes of $T_{K_n}$, taken $(n-1)$ at a time. Notice $r_k$ is reflexive for odd $n \\n$, and further, each matrix $A(i \mid \emptyset)$ has full rank. This implies $\{r_i\}_{i=1}^n$ is the set of unique intersection points. Thus $\{ x \mid A x \leq 1 \} = \text{conv}(r_1, r_2, \cdots, r_n) = T_{K_n}$ shows that $T_{K_n}$ is reflexive. \hfill $\Box$
Proposition 6.2. The simplex $T_{K_n}$ has a regular unimodular triangulation.

Proof. Since the matrix of the facet normals is a signed vertex-edge incidence matrix for a path, it is totally unimodular. Thus, it follows from [13, Theorem 2.4] that $T_{K_n}$ has a regular unimodular triangulation. \qed

Corollary 6.3. The simplex $T_{K_n}$ is IDP.

Proof. If $T_{K_n}$ admits a unimodular triangulation, it follows that $T_{K_n}$ is IDP because cone($T_{K_n}$) is a union of unimodular cones with lattice-point generators of degree 1. \qed

Theorem 6.4 (Athanasiadis [1]). Let $P$ be a $d$-dimensional lattice polytope with $h^*_P = (h^*_0, h^*_1, \ldots, h^*_d)$. If $P$ admits a regular unimodular triangulation, then $h^*_i \geq h^*_{d-i+1}$ for $1 \leq i \leq \lfloor (d+1)/2 \rfloor$,

$$h^*_{\lfloor (d+1)/2 \rfloor} \geq \cdots \geq h^*_{d-1} \geq h^*_d$$

and

$$h^*_i \leq \binom{h^*_i + i - 1}{i}$$

for $0 \leq i \leq d$.

Corollary 6.5. For each $n \geq 2$, $h^*(T_{K_n})$ is unimodal.

6.2. $h^*(T_{K_n})$ and Weak Compositions. The following is a classification of all lattice points in cone($T_{K_n}$).

Theorem 6.6. The lattice points at height $h$ in cone($T_{K_n}$) are in bijection with weak compositions of $h \cdot n$ of length $n$, where the height of the lattice point in the cone is given by the last coordinate of the lattice point.

Proof. Recall the $t$th dilate of the polytope $T_{K_n} \subset \mathbb{R}^n$ is given by

$$\text{cone}(T_{K_n}) \cap \{z \mid z_n = t\} = \{\lambda \cdot [L_B \mid 1] \mid \lambda \in \mathbb{R}_{\geq 0}^n, \sum_{i=1}^n \lambda_i = t\},$$

since the last coordinate of the lattice point is given by $\sum_{i=1}^n \lambda_i$. Notice each lattice point in cone($T_{K_n}$) corresponds uniquely to a lattice point in $tT_{K_n}$ where $t$ is the last coordinate of the point. Then the lattice points of $tT_{K_n}$ are all $x = \lambda \cdot [L_B \mid 1] \in \mathbb{Z}^n$ where $0 \leq \lambda_i = \frac{b_i}{kn}$ for $b_i \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=1}^n \lambda_i = t$. Define the map

$$\Phi : \{\text{length } n \text{ weak compositions of } tn\} \rightarrow \{\text{lattice points of } tT_{K_n}\}$$

by

$$c \mapsto \frac{1}{n} c \cdot [L_B \mid 1]$$

To show $\Phi(c)$ is a lattice point, consider

$$\Phi(c) = \frac{1}{n} [c_1, c_2, \ldots, c_n] \cdot \begin{bmatrix} (n-1) & (n-2) & (n-3) & \cdots & 1 & 1 \\ -1 & (n-2) & (n-3) & \cdots & 1 & 1 \\ -1 & -2 & (n-3) & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & -3 & \cdots & -(n-1) & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}.$$
Since c is a weak composition of tn, 0 ≤ \( \frac{c_i}{n} \) ≤ t for all i and \( \frac{1}{n} \sum_{i=1}^{n} c_i = t \). Multiplying the above expression yields \( x_i = (\sum_{j=1}^{i} c_j) - it \) for all 1 ≤ i ≤ n − 1 and \( x_n = t \). This implies \( x \in \mathbb{Z}^n \), which shows x is a lattice point in \( tT_{K_n} \).

To show \( \Phi \) is a bijection, we consider the inverse

\[
\Phi^{-1} : \{ \text{lattice points of } tT_{K_n} \} \rightarrow \{ \text{length } n \text{ weak compositions of } tn \}
\]

\[
x \mapsto nx \cdot [L_B \mid 1]^{-1}
\]

It can be shown that

\[
[L_B \mid 1]^{-1} = \frac{1}{n} \left[ \begin{array}{cccccc}
1 & -1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & -1 & \ddots & & \\
& \vdots & 0 & 1 & -1 & \ddots \\
& & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 & -1 \\
1 & \cdots & \cdots & 1 & 1
\end{array} \right].
\]

Thus
\[
c = nx \cdot [L_B \mid 1]^{-1}
\]

\[
= (x_1 + x_n, -x_1 + x_2 + x_n, -x_2 + x_3 + x_n, \ldots, -x_{n-2} + x_{n-1} + x_n, -x_{n-1} + x_n)
\]

\[
= (x_1 + t, -x_1 + x_2 + t, -x_2 + x_3 + t, \ldots, -x_{n-2} + x_{n-1} + t, -x_{n-1} + t).
\]

It remains to show that c is a weak composition of tn. First note that \( \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} t = tn \). Next we show each \( c_i \geq 0 \). This is equivalent to \( x_1 \geq -t, -x_{n-1} \geq -t \), and \( -x_i + x_{i+1} \geq -t \) for all \( 2 \leq i \leq (n-2) \). Recall from the hyperplane description of \( tT_{K_n} \) that x is a lattice point if it satisfies

\[
\begin{bmatrix}
-1 & 0 & \cdots & \cdots & 0 \\
1 & -1 & \ddots & & \\
0 & 1 & -1 & \ddots & \\
& \vdots & \ddots & \ddots & 0 \\
& & \ddots & 1 & -1 \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1}
\end{bmatrix}
\leq
\begin{bmatrix}
t \\
t \\
\vdots \\
t
\end{bmatrix}
\]

These inequalities show c is a weak composition of tn of length n. Note \( \Phi \circ \Phi^{-1}(x) = x \) and \( \Phi^{-1} \circ \Phi(c) = c \). Thus \( \Phi \) is a bijection.

**Corollary 6.7.** The Ehrhart polynomial of \( T_{K_n} \) is \( L_{T_{K_n}}(t) = \binom{tn+n-1}{n-1} \).

**Proof.** The number of weak compositions of tn of length n is \( \binom{tn+n-1}{n-1} \). Then the result follows directly from Theorem 6.6. \( \square \)

We next restrict \( \Phi \) to obtain a classification of the lattice points in the fundamental parallelepiped, \( \Pi_{T_{K_n}} \).

**Corollary 6.8.** The lattice points of \( \Pi_{T_{K_n}} \) are in bijection with weak compositions of \( hn \) of length \( n \) with each part of size strictly less than \( n \).

**Proof.** Every \( x \in \Pi_{T_{K_n}} \cap \mathbb{Z}^n \) is of the form \( x = \frac{1}{\kappa n} b \cdot [L_B \mid 1] \) such that \( 0 \leq \frac{b_i}{\kappa} < 1 \) for each \( i \in [n] \), i.e., \( 0 \leq \frac{b_i}{\kappa} < n \). Each coordinate of the lattice point has the form \( x_i = (\sum_{j=1}^{i} b_j \kappa) - ih \), which is an integer. It follows by induction on \( j \) that \( \kappa \) divides \( b_j \) for each \( 1 \leq j \leq n \). Then it follows from
\[ \frac{1}{\kappa} \sum_{i=1}^{n} b_i = hn \] that \( \left( \frac{1}{\kappa}b \right) \) is a weak composition of \( hn \) of length \( n \) with parts no greater than \( n - 1 \).

With each \( c \in \{ \text{length } n \text{ weak compositions of } tn \text{ with parts of size less than } n \} \), associate \( \kappa c = b \). This \( b \) will generate a lattice point in the fundamental parallelepiped. The result follows. \( \square \)

**Proposition 6.9.** For each \( n \geq 2 \), the \( h^* \)-vector of \( T_{K_n} \) is given by
\[ h^*(T_{K_n}) = (1, m_1, \ldots, m_n) \]
where \( m_i \) is the number of weak compositions of \( in \) of length \( n \) with parts of size less than \( n \).

**Proof.** From Lemma 2.6, \( h^*_i \) enumerates \(|\Pi_{T_{K_n}} \cap \{x_n = i\} \cap \mathbb{Z}^n|\). By Corollary 6.8 the result follows. \( \square \)

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