Numerical simulation of stress-strain state of electrophoretic shell molds

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Abstract. In the foundry engineering, castings obtained in one-piece non-gas-generating high-refractory electrophoretic shell molds (ShM) by investment patterns (IP) have an increased rejects percentage associated with low deformation resistance and crack resistance of the SM at different stages of their formation and manufacturing. Crack resistance of the ShM based on IP depends mainly on their stress-strain state (SSS) at various stages of mold forming. SSS decrease significantly improves their crack resistance and decreases their rejects percentage of castings occurring due to clogging and surface defects. In addition, the known methods of decreasing the SSS are still poorly understood. Thus, current research trends are to determine SSS at each stage of ShM forming and develop the ways to decrease it. Theoretical predicting of crack formation in multiple-layer axisymmetric shell molds is given in the work [1], and SSS of multiple-layer axisymmetric shell molds is given in the work [2]. Monolayer electrophoretic ShM had a lack of concern in this field, thus it became an argument for the present work Mathematical Model of ShM SSS

1. Engineering Formulation of the Problem
Mathematical model of ShM SSS is based on heat transfer equations that allow calculation of distribution of thermal fields within the mold and casting, and, based on equations of continuum mechanics, calculation of distribution of stresses and strains in the structure of the ShM during solidification of the casting. To construct a mathematical model, the numerical method developed by Professor V I Odinokov, has been used. Ceramic shell mold, single-layer, one-piece. Curvilinear orthogonal coordinates around the perimeter of the area are shown on the Figure 1 (a), (b). Cross-section of the casting, axially symmetrically shaped in the vertical plane, has a view shown in Figure 1. Where I-I is the axis of rotation. The mold is exposed from inside to ferrostatic pressure of molten metal – $P_1$. During simulation, two options of filling the mold were considered. In option one, casting process and forming casting formation to a preheated mold were simulated – the case of casting to the ShMs obtained by investment patterns. In option two, casting process and forming casting formation to a cold shell mold were simulated – the case of casting to the shell molds obtained by permanent models.

During development of the mathematical process model, it was taken into consideration that the simulated system at the initial stage is a two-component: liquid metal – mold; as time goes, due to change in temperature fields, the system evolves into a three-component one – liquid metal I – $(LM)$, solid metal II – $(SM)$, mold III. It is modeling of interaction of these three components in temperature field changing over time at a pressure $P_1$ that is an aim of this paper.
2. Mathematical Formulation of the Problem

An axially symmetric body of revolution is considered. Movement is considered being slow. There is a three-component system. Deformable medium, solidified metal (area II), and mold (area III) are isotropic materials. The process is transient. Using the theory of small elastic-plastic deformations and Euler coordinate system, we can write a system of equations for each of the areas. Namely, we have

Area I:

\[ \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma = P_1, \]
\[ \dot{\vartheta} = a_1 \Delta \vartheta. \]  

(1)

Areas II, III:
\[ \sigma_{y,j} = 0, \]
\[ \sigma_y - \sigma \delta_y = 2G_p \varepsilon^*; \quad \varepsilon^*_y = \varepsilon_y - \frac{1}{3} \varepsilon \delta_y; \quad \varepsilon = \varepsilon_u, \]  
\[ \varepsilon_u = 3k_p \sigma + 3\alpha_p \left( \theta - \theta^*_p \right); \quad \varepsilon_y = 0.5\left( U_{i,j} + U_{j,i} \right) \]  
\[ \dot{\theta} = a^*_p \Delta \theta. \]

where \( U_i \) are displacements, \( \varepsilon_{ij} \) are strains, \( \sigma \) is hydrostatic stress, \( G_p \) (\( p = II, III \)) is shear modulus of the metal (\( p = II \)) and mold (\( p = III \)) (\( p = III \)), \( \delta_{ij} \) is Kronecker symbol, \( k_p \) are volume compressibility coefficients, \( \alpha_p \) are linear expansion coefficients, \( a^*_p \) are thermal diffusivity coefficients, (\( p = II, III \)), \( \theta_n \) is current temperature, \( \tau \) is time, \( \theta^*_p \) are initial temperatures in the areas (\( p = II, III \)).

During cooling the liquid metal, provided that the metal temperature is \( \theta_M \leq \theta_K \) (\( \theta_K \) is crystallization temperature), thickness of the solidified layer is determined based on solution of interphase transfer equation:

\[ \frac{d\theta}{dn} \big|_1 - \frac{d\theta}{dn} \big|_2 = \frac{d\Delta}{d\tau} L \rho \]  

where \( \theta_1 \) and \( \theta_2 \) are temperatures, respectively, in solid and liquid phases, \( \lambda_1 \) and \( \lambda_2 \) are thermal conductivity coefficients in the respective phases, \( \Delta \) is thickness of crust, \( L \) is latent heat of fusion, \( \rho \) is density, \( n \) is normal line to the boundary of the two phases.

Crystallization process time, \( \tau_* \), is divided into small steps \( \Delta \tau_n \) (\( n \) is a time step number). At each time step \( \Delta \tau_n \), thickness of the solid phase \( \Delta_n \) (\( \Delta = \sum_1^n \Delta_n \)) is calculated. Thus, it is assumed that temperature in the solid phase along thickness \( \Delta_n \) varies linearly and temperature gradient in the liquid phase is zero. In view of this, solution of equation (3) gives the following relation for determining the thickness of crystallized crust \( \Delta_n \) at time step \( \Delta \tau_n \)

\[ \Delta_n = C \sqrt{\tau}, \quad C = \sqrt{\frac{2 \Delta \theta \lambda}{\rho L}}, \]  

where \( \Delta \theta \) is differential temperature in the solid phase near solidification front.

2.1. Numerical Diagram for Solving the Problem

To solve system (2), the numerical method described in paper [3] was used. According to this method, the strain area is divided into a finite number of orthogonal curvilinear elements (Figure 2, a).
In case of axial symmetry (Figure 1, a), the following conditions take place:

\[ \sigma_{31} = \sigma_{32} = 0; \quad \epsilon_{13} = \epsilon_{23} = 0; \quad U_3 = 0 \]  

(5)

In accordance with paper [3] and due to (5), equations (2) and values \( \epsilon_\phi \) can be written as

\[
\begin{align*}
S_{13} &\Delta S_{12} \left( \sigma_{11} - \sigma_{22} \right) + S_{12} \Delta S_{13} \left( \sigma_{11} - \sigma_{33} \right) + 0.5 \Delta \sigma_{11} S_{12} S_{13} + \\
&+ 0.5 \Delta \sigma_{12} S_{21} S_{23} + \left( S_{21} \Delta S_{23} + 2 S_{23} \Delta S_{21} \right) \sigma_{12} = 0, \\
S_{23} &\Delta S_{21} \left( \sigma_{22} - \sigma_{33} \right) + S_{32} \Delta S_{23} \left( \sigma_{22} - \sigma_{11} \right) + 0.5 \Delta \sigma_{22} S_{32} S_{23} + \\
&+ 0.5 \Delta \sigma_{23} S_{12} S_{13} + \left( S_{12} \Delta S_{13} + 2 S_{13} \Delta S_{12} \right) \sigma_{21} = 0;
\end{align*}
\]  

(6)

Figure 2. Strain area: a – area division into finite orthogonal elements; b – shear strains.
\[
\sigma_{i1} - \sigma_{i2} = 2G_p (e_{i1} - e_{i2}),
\]
\[
\sigma_{i2} - \sigma_{i3} = 2G_p (e_{i2} - e_{i3}),
\]
\[
\sigma_{i2} = G_p \cdot e_{i2};
\]
\[
e_{i1} + e_{i2} + e_{i3} = 3k_p \sigma + 3\alpha_p (\theta_a - \theta_p'),
\]

where \(\sigma_y = 0.5(\sigma_y + \sigma_y'), \Delta \sigma_y = \sigma_y' - \sigma_y, (i, j = 1,2,3)\).

Reached equations (6) formulate by themselves equations of balance, equations (7) – equations of state (or the Hook's law), and (8) - mass-conservation equation.

\[
\begin{align*}
\varepsilon_{i1} &= \frac{2\Delta U_1}{S_{21}} + \frac{2\Delta S_{21}}{S_{12}}; \\
\varepsilon_{i2} &= \frac{2\Delta U_2}{S_{32}} + \frac{2\Delta S_{32}}{S_{12}}; \\
\varepsilon_{i3} &= \frac{2\Delta U_3}{S_{31}} + \frac{2\Delta S_{31}}{S_{32}},
\end{align*}
\]

where \(U_i = U_i^1 + U_i^2; \Delta U_i = U_i^2 - U_i^1, (i = 1,2)\).

In this case, \(S_{ij}^1, (i,k = 1,2,3; j = 1,2)\) is arc value of the midline located on the face \(j\) along the coordinate \(\alpha_i\) and perpendicular to the coordinate \(\alpha_j\). Values of \(S_{ij}^1\) are calculated as averages of arc values of the face boundary. Designations of \(S_y = S_y^1 + S_y^2, \Delta S_y = S_y^2 - S_y^1\) are introduced. Location of \(\sigma_y\) and \(U_j\) on the element faces is shown in Figure 2.

Equations (6), (9) are written considering that

\[
\frac{\partial U_i}{\partial x_3} = 0, \quad \frac{\partial \sigma_{ij}}{\partial x_3} = 0; \quad i = 1,2,3;
\]

for bodies of revolution, the following takes place

\[
\Delta S_{31} = 0; \quad \Delta S_{32} = 0; \quad \Delta \bar{U}_i = 0, \quad \Delta \bar{U}_3 = 0; \quad U_3 = 0;
\]

on surface \(x_1, x_2\) : \(S_1^+ - S_1^- = 0\); on surface \(x_2, x_3\) : \(S_2^+ - S_2^- = 0\), shear values \(\varepsilon_{ij}\) \((i \neq j)\) can be written for node (0) in the following form

\[
\varepsilon_{i12} = \frac{2\Delta \bar{U}_1}{S_i} - 0.5\Delta \bar{U}_2 \frac{S_2^+ - S_2^-}{S_2} + \frac{2\Delta \bar{U}_3}{S_3^+ - S_3^-} - 0.5\bar{U}_1 \frac{S_1^+ - S_1^-}{S_1},
\]

where \(S_i = S_i^1 + S_i^2; \Delta \bar{U}_i = \bar{U}_i^2 - \bar{U}_i^1\) (Fig 2, b); \(S_1^+ = S_1^{1+} + S_1^{2+}, S_1^- = S_1^{1-} + S_1^{2-}\); values \(\bar{U}_i^j\) are calculated as averages of values \(U_i\) on faces of the element. Paper [3] proves that the difference analogue of system (6)–(8) taking into account (9), in case of presence of initial and boundary conditions, is determinable. Dimensionality of system (6)–(8) is significantly reduced upon implementation of the following operations:

1. Differences \((\sigma_y - \sigma_y')\) in equations (6) can be expressed via (7).

2. Mass conservation equation can be rewritten in the recurrent form taking into account expressions (9) in the form of:

\[
U_i^2 = U_i^1 + [A],
\]

where \([A]\) is operator not containing \(U_i^2\); sense of area rotation by \(x_1(\rightarrow)\), by \(x_2(\uparrow)\).
3. Shear expressions $\varepsilon_g$ ($i \neq j$) are determined by the inner grid nodes in accordance with formulas (10); $i = 1, j = 2$.

4. $\sigma_g$ ($i \neq j$) are determined by the inner grid nodes of equations of state: $\sigma_{12}^0 = G_{p}^{0} \varepsilon_{12}^{0}$.

5. $\sigma_g$ are determined by external grid nodes from the boundary conditions, and on the contact surfaces – from the friction law.

6. $\sigma_g$ are determined on faces of the elements as the averages of values $\sigma_g$ in nodes of the faces.

7. The first equation (6) can be rewritten in the form of a recurrence:

$$
\sigma_{11}^1 = \sigma_{11}^0 + [B],
$$

where $[B]$ is operator not containing $\sigma_{11}^0$; sense of area rotation $x, x_s, \psi$.

8. From the system of equations – the second equation in system (6) and equation $\sigma_{22} - \sigma_{11} = 2\lambda(\varepsilon_{11} - \varepsilon_{22})$ – values $\sigma_{22}^i$ and $\sigma_{22}^f$ are defined for the element and the equations having the following are made:

$$
F_j = (\sigma_{22}^j)_{ij} - (\sigma_{22}^f)_{ij} = 0
$$

for internal faces, where $J$ is an index of the element by coordinate $x_2$.

Thus, if variables $X = \{U, U_1(\xi_1,\eta_1), \sigma_{11}(\xi_1,\eta_1)\}$ are considered as independent, then along a sequence of $(1+7)$, the dependent variables can be determined via $X$ ($x_i^*$ is the final value of the coordinate $x$ along the curved area).

Equivalent system of equations consists of:

$$
F_1 = (U^2_1 - U_1^*)_{\xi_1 = \eta_1} = 0, \quad F_2 = (\sigma_{11}^1 - \sigma_{11}^f)_{\xi_1 = 0} = 0, \quad F_3 = 0,
$$

where $U_1^*$ are known from the boundary conditions displacements $U_1$ on the boundary of the area ($x_1 = x_1^*$); $\sigma_{11}^1$ are known from boundary conditions stresses $\sigma_{11}$ on the boundary of the area ($x_1 = 0$). Moreover, number of equations $F_1 = 0$ equals the number of unknown variables $\sigma_{11}(\xi_1,\eta_1)$, number of equations $F_2 = 0$ equals the number of unknown variables $U_1(\xi_1,\eta_1)$.

In order to solve the thermal conductivity equation, a numerical method developed in paper [3] is used. According to this method, for each internal $k$-th element (Figure 2, a), a system of thermal conductivity is written in a difference form, and iterative procedure is made that, given that the heat flux on $x_1$ is zero, is represented by iterative formula

$$
\theta_k = \frac{\theta_k^* + t_{12} \theta_k^* + t_{12} \theta_k^* + t_{12} \theta_k^* + t_{12} \theta_k^*}{1 + t_{12} + t_{12} + t_{12} + t_{12}}, \quad (11)
$$

where

$$
t_{12} = \frac{2(\lambda_k + \lambda_k^*)}{S_{21} + S_{21}^*} \frac{\Delta \tau}{C_k \gamma k V_k^2}; \quad t_{11} = \frac{2(\lambda_k + \lambda_k^*)}{S_{21} + S_{21}^*} \frac{\Delta \tau}{C_k \gamma k V_k^2};
$$

$$
t_{22} = \frac{2(\lambda_k + \lambda_k^*)}{S_{12} + S_{12}^*} \frac{\Delta \tau}{C_k \gamma k V_k^2}; \quad t_{21} = \frac{2(\lambda_k + \lambda_k^*)}{S_{12} + S_{12}^*} \frac{\Delta \tau}{C_k \gamma k V_k^2};
$$

$$
F_j^i = S_k^i \cdot S_k^j; \quad i \neq k \neq p; \quad i,k,p = 1,2,3; \quad j = 1,2; \quad V_e = \frac{S_{13} S_{12} (S_{21} + S_{31})}{16}.
$$
\( \theta_k^* \) is the average temperature in the k-th element at the beginning of the time step; \( \Delta \tau, \lambda_k, \theta_k, C_k, \gamma_k \) are averages of thermal conductivity, temperature, heat capacity and specific gravity in the k-th element at the end of the time step; \( \Delta \tau; \lambda_j, \theta_j (i = 1, 2) \) are, respectively, thermal conductivity and temperature coefficients in the element following the k-th element for the coordinate \( x_i \) in the negative direction; \( \lambda_j, \theta^+_j \) are equivalent parameters in the positive direction \( x_j \).

\[ S^+_{21} = S^+_{21} + S^+_{22}; \quad S^-_{21} = S^-_{21} + S^-_{21}; \]

\( S^+_{2j} (i \neq j; i, j = 1, 2) \) is arc length of the \( S^+_i \)-th element following the k-th element in the positive direction for the coordinate \( x_j \); \( S^-_{2j} \) is the same, but in the negative direction on \( x_j \); \( \theta^+_j \) is temperature of the element following the k-th element in the direction \( x_j \); \( \theta^-_j \) is the same, but in the negative direction \( x_j \).

In paper [3] the convergence of the iterative procedure (11) was proved. The initial conditions of the problem:

- \( \Delta |_{t=0} = 0 \) is lack of a solid metal phase;
- \( \theta^+_i |_{t=0} = \theta_0 \) is temperature of the cast liquid metal;
- \( \theta^+_m |_{t=0} = \theta^* \) is initial mold temperature; \( \theta''_n \) is sand temperature; \( \theta''_m \) is metal crystallization temperature. The boundary conditions of the problem: at the symmetry axis

\[ U_1 = 0, \quad \sigma_{21} = 0, \quad q_n = 0. \]

on surfaces \( S_1, S_3 \)

\[ \sigma_1|_{S_1} = -P; \quad \sigma_{12}|_{S_1} = 0; \quad U_1|_{S_1} = 0; \quad \sigma_{21}|_{S_1} = 0; \quad \sigma_{22}|_{S_1} = 0, \]

\[ \sigma_{12}|_{S_1} = -\psi \frac{V_x}{V^*} \cos(n_u x_1); \quad q_n|_{S_1} = \alpha^* (\theta_n - \theta''_n), \]

where \( V_x \) is the sliding velocity of the mold material relative to sand, \( V^* \) is normalizing speed, \( \psi \) is the parameter characterizing the friction conditions between the mold and sand, \( q_n|_{S_1} \) is heat flux density normal to the surface \( S_3 \), \( \alpha^* \) is heat transfer coefficient between the ceramic mold and sand.

### 2.2. Problem Solution Algorithm

1. Cooling time \( \tau^* \) is divided into a finite number of steps: \( \tau^* = \sum \Delta \tau_n \), \( n \) – a time step number.
2. The area of interest is divided into a finite number of orthogonal elements.
3. The initial and boundary conditions are given for the elements forming the area of interest, as well as constants of physical-mechanical properties of materials.
4. Arc lengths of elements \( S^i_k \) \((i, k = 1, 2; i \neq k; j = 1, 2) \) are calculated.
5. Temperature field at time step \( \Delta \tau_n \) is determined by numerical solution of the thermal conductivity equation using the iterative formula (11) in case of presence of initial and boundary conditions for a given time step.
6. If the temperature in the area (I) at the surface \( S_2 \) \( \theta|_{S_2} \leq \theta''_n \), then thickness of crystallized crust \( \Delta_n \) is calculated by the formula (4). If \( \theta|_{S_2} > \theta''_n \), then operation from clause 7 is performed.
7. System of equations (2) is solved with the difference analogues (6), (7), (9) and developed technique described in paper [3]. Stress $\sigma_{ij}$ and displacement $U_i, (i, j = 1,2)$ fields are determined.

8. Time step is made. If $\sum \Delta \tau_n < \tau^*$, then operation from clause 4 is performed. If $\sum \Delta \tau_n = \tau^*$, calculating process is finished.

3. Conclusion
Formulas reached in the work [3] for the optional element limited by orthogonal area were used in this work. Specific feature of previous works in this field is their connectivity to multiple-layer shell molds but present is on simulation on monolayer electrophoretic ShM. Problem on process simulation was solved. The numerical scheme and problem algorithm were developed. At the present time a range of works on process flow optimization in the fields of machinery, metallurgy and foundry engineering is elaborated, for example steel pouring process into different structure types of multiple-layer shell molds. The solutions of these problems are based on technology specified in this work. Proven numerical approach was used for the solution of formulated equation system.

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