Minimizing Sum-MSE Implies Identical Downlink and Dual Uplink Power Allocations

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Abstract

In the multiuser downlink, power allocation for linear precoders that minimize the sum of mean squared errors under a sum power constraint is a non-convex problem. Many existing algorithms solve an equivalent convex problem in the virtual uplink and apply a transformation based on uplink-downlink duality to find a downlink solution. In this letter, we analyze the optimality criteria for the power allocation subproblem in the virtual uplink, and demonstrate that the optimal solution leads to identical power allocations in the downlink and virtual uplink. We thus extend the known duality results and, importantly, simplify the existing algorithms used for iterative transceiver design.

Index Terms

MIMO systems, optimization methods, least mean square methods

I. INTRODUCTION

In the multiuser multiple-input, multiple-output (MIMO) downlink, linear transmit/receive processing to minimize the sum of mean squared errors (sum-MSE) under a sum power constraint is a well-studied problem. When formulated as a precoder design problem, with implicit minimum-MSE (MMSE) receive matrices, the sum-MSE is a non-convex function of the downlink precoders. A standard approach used to solve the problem is to find an equivalent formulation...
in the virtual uplink, wherein the roles of transmitter and receiver are exchanged \cite{[1]-[4]} . In the virtual uplink, the receiver is the Wiener filter and the power allocation subproblem is convex. The equivalence of the downlink and virtual uplink problems are enabled by an uplink-downlink duality result for the MSE of each data stream. Duality results for linear precoding systems were first presented for signal-to-interference-plus-noise ratios (SINR) in \cite{[5]} with single-antenna receivers. This work was later extended to MSEs and systems with multiple receive antennas in \cite{[1], [2]} and subsequently generalized in \cite{[6]}. The algorithms we focus on are based on iterating between the downlink and virtual uplink. Crucially, a common feature is a transformation of the resulting power allocation in the virtual uplink to the downlink, while achieving the same MSE in each stream in both systems. This transformation requires solving a matrix equation in each iteration.

In this letter, we use the Karush-Kuhn-Tucker (KKT) conditions for the power allocation subproblem in the virtual uplink to show that at the optimal point, the powers allocated to each data stream in both the downlink and virtual uplink are identical. This result extends the known dualities in the multiuser MIMO case. Importantly, this also eliminates the need for the uplink-to-downlink power transformation, significantly simplifying the algorithms used.

This paper is organized as follows: Section II describes the system model and existing algorithms for minimizing the sum-MSE using uplink-downlink duality. In Section III we present the KKT conditions for the virtual uplink power allocation subproblem, and use the resulting expressions to prove the equality of the downlink and virtual uplink power allocations. Section IV wraps up the paper with some conclusions.

II. BACKGROUND

A. System Model with Linear Precoding

In the linear precoding system, illustrated in Fig. 1 a base station with $M$ antennas transmits to $K$ decentralized mobile users over flat wireless channels; user $k$ has $N_k$ receive antennas. The channel between the transmitter and user $k$ is represented by the $N_k \times M$ matrix $H_k^H$, and the overall $N \times M$ channel matrix is $H$, with $H = [H_1, \ldots, H_K]$. User $k$ receives $L_k$ data symbols $x_k = [x_{k1}, \ldots, x_{kL_k}]^T$ from the base station, and the vector $x = [x_1^T, \ldots, x_K^T]^T$ comprises independent symbols with unit average energy ($\mathbb{E}[xx^H] = I_L$ (the $L \times L$ identity matrix), where $L = \sum_{k=1}^K L_k$). User $k$’s data streams are precoded by the $M \times L_k$ transmit
filter $\tilde{U}_k = [\tilde{u}_{k1}, \ldots, \tilde{u}_{kL_k}]$, where $\tilde{u}_{kj}$ is the precoding beamformer for stream $j$ of user $k$ with $\|\tilde{u}_{kj}\| = 1$. These individual precoders are combined in the $M \times L$ global transmitter precoder matrix $\tilde{U} = [\tilde{U}_1, \ldots, \tilde{U}_K]$. Power is allocated to user $k$’s data streams in the vector $p_k = [p_{k1}, \ldots, p_{kL_k}]^T$ and $P_k = \text{diag}[p_k]$; we define the diagonal downlink power allocation matrix as $P = \text{diag}\{[p_1^T, \ldots, p_K^T]\}$. Based on this model, user $k$ receives a length-$N_k$ vector $y_k^{DL} = H_k^H \tilde{U} \sqrt{P} x + n_k$, where the superscript $^{DL}$ indicates the downlink, and $n_k \sim \mathcal{CN}(0, \sigma^2 I_{N_k})$ consists of zero-mean white Gaussian noise. To estimate its $L_k$ symbols $x_k$, user $k$ applies the $L_k \times N_k$ receive filter $V_k^H$, yielding the estimated symbols $\hat{x}_k^{DL} = V_k^H H_k^H \tilde{U} \sqrt{P} x + V_k^H n_k$.

To minimize the sum-MSE in the multiuser MIMO downlink, we use the virtual uplink, also illustrated in Fig. [1] where each matrix is replaced by its conjugate transpose. In this transformed system, we imagine transmissions from mobile user $k$ that propagate via the transpose channel $H_k^H$ to the base station. The transmit and receive filters for user $k$ become $V_k$ and $U_k^H$ respectively, with normalized precoding beamformers; i.e., $||\tilde{v}_{kj}|| = 1$. Power is allocated to user $k$’s data streams as $q_k = [q_{k1}, \ldots, q_{kL_k}]^T$, with $Q_k = \text{diag}[q_k]$ and $Q = \text{diag}\{[q_1^T, \ldots, q_K^T]\}$. The received symbol vector at the base station and the estimated symbol vector for user $k$ are $y_k^{UL} = \sum_{i=1}^{K} H_i \bar{V}_i \sqrt{Q_i} x_i + n$ and $x_k^{UL} = \sum_{i=1}^{K} U_k^H H_i \bar{V}_i \sqrt{Q_i} x_i + U_k^H n$, respectively, with zero-mean white Gaussian noise $n \sim \mathcal{CN}(0, \sigma^2 I_M)$.

### B. Minimum Sum-MSE Multiuser MIMO Linear Precoding

1) Convex Minimum Sum-MSE Precoder Design: The MSE matrix for user $k$ in the downlink using arbitrary precoder and decoder matrices can be written as

$$E_k^{DL} = \mathbb{E} \left[ (\hat{x}_k^{DL} - x_k) (\hat{x}_k^{DL} - x_k)^H \right]$$

$$= V_k^H J_k V_k - V_k^H H_k^H \tilde{U}_k - \tilde{U}_k^H H_k V_k + I_{L_k},$$

where $J_k = H_k^H \bar{U} P \bar{U}^H H_k + \sigma^2 I$, $\tilde{U}_k = \bar{U}_k \sqrt{P_k}$, and data and noise terms are assumed to be independent. The individual MSE terms are minimized using the MMSE receiver $V_k^* = \tilde{U}_k^H H_k J_k^{-1}$. The resulting MMSE matrix is $E_k^{DL} = I_{L_k} - \tilde{U}_k^H H_k J_k^{-1} H_k^H \tilde{U}_k$, and the minimum sum-MSE for any choice of $\tilde{U}_k$ is $\text{SMSE}^{DL} = \sum_{k=1}^{K} \text{tr} \left[ E_k^{*DL} \right]$.

The problem of finding the sum-MSE minimizing precoders and power allocations in the downlink under a sum power constraint $\text{tr} \left[ P \right] \leq P_{\text{max}}$ is non-convex when MMSE receivers $V_k^*$ are defined as a function of $\tilde{U}$ due to the cross-coupling introduced by the presence of all $\tilde{U}_i$. 

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terms in every $J_k$. The authors of [6] demonstrate hidden convexity in the downlink sum-MSE minimization problem by optimizing receive matrices (using closed form MMSE precoders) and applying a modified cost function. However, this problem is more commonly solved via transformation to the virtual uplink, which gives rise to several equivalent problems that can be solved using convex optimization. The set of virtual uplink minimum sum-MSE precoders and power allocations \{$(\bar{V}_k, q_k)$, $k = 1, \ldots, K$\} can be found jointly, by finding the optimum covariance matrices $R_k = \bar{V}_k^H Q_k \bar{V}_k$ and applying Cholesky or eigen-decomposition [4]. An alternative approach finds the optimum precoders $\bar{V}_k$ and power allocations $q_k$ in an iterative manner [2], [4]. The convexity of these problems originates from the decoupling of users in the virtual uplink. The MMSE matrix $E^{UL}_{k}$ for user $k$ is found using the MMSE receiver $U^*_{k} =  \tilde{V}_k^H H_k^H J^{-1}$, 

$$E^{UL}_{k} = I_{L_k} - \tilde{V}_k^H H_k^H J^{-1} H_k \tilde{V}_k,$$  

(2)

with $\tilde{V}_k = \bar{V}_k \sqrt{Q_k}$ and $J = \sum_{k=1}^{K} H_k \bar{V}_k \bar{V}_k^H H_k^H + \sigma^2 I_M$. The resulting minimum sum-MSE is

$$\text{SMSE}^{UL} = \sum_{k=1}^{L_k} L_k - \text{tr} \left[ J^{-1} \sum_{k=1}^{K} H_k \bar{V}_k \bar{V}_k^H H_k^H \right]$$

(3)

which follows from $\text{tr}[AB] = \text{tr}[BA]$, linearity of the trace operator, and the definition of $J$. Minimizing the sum-MSE thus only requires minimization of $\text{tr}[J^{-1}]$, which is convex for both the power allocation subproblem in $q_k$ and the joint precoder design problem in covariance matrices $R_k$. In this letter, we consider the former optimization problem, which is formally stated as

$$(q_1^*, \ldots, q_L^*) = \arg \min_{q_1, \ldots, q_L} \left[ \sum_{l=1}^{L} q_l \hat{h}_l^H (q_l + \sigma^2 I_M)^{-1} \right]$$

(4)

$$\text{s.t. } q_l \geq 0, \quad l = 1, \ldots, L, \quad \sum_{l=1}^{L} q_l \leq P_{\text{max}},$$

where we have defined the effective channel $\tilde{H} = [H_1 \bar{V}_1, \ldots, H_K \bar{V}_K] = [\hat{h}_1, \ldots, \hat{h}_L]$. Note that the columns in $\tilde{H}$ refer to the effective channel vectors for each individual data stream $l = 1, \ldots, L$. 

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2) Uplink-Downlink Duality: The duality results in [2], [4] show that any set of MSEs that are achievable in the virtual uplink are also achievable in the downlink under the same sum power constraint. This duality can be satisfied by factoring the downlink and virtual uplink receive beamforming vectors for each data stream \( l \) as

\[
    v_l^H = p_l^{-\frac{1}{2}} \beta_l \bar{v}_l^H \quad \text{and} \quad u_l^H = q_l^{-\frac{1}{2}} \beta_l \bar{u}_l^H,
\]

where \( \beta_l > 0 \). Any set of feasible MSEs \( \varepsilon = \text{diag}\{\varepsilon_1, \ldots, \varepsilon_L\} \) can then be achieved by the downlink and virtual uplink power allocations

\[
    p = \sigma^2 (\varepsilon - D - \beta^2 \Psi^T)^{-1} \beta^2 1_L, \quad q = \sigma^2 (\varepsilon - D - \beta^2 \Psi)^{-1} \beta^2 1_L,
\]

where \( 1_L \) is the length-\( L \) vector consisting of all ones, \( \beta = \text{diag}\{\beta_1, \ldots, \beta_L\} \), \( D \) is diagonal with

\[
    [D]_{l,l} = |\beta_l \bar{h}_l^H \bar{u}_l|^2 - 2 \beta_l \Re [\bar{h}_l^H \bar{u}_l] + 1,
\]

where \( \Re [\cdot] \) denotes the real part of a complex number and

\[
    [\Psi]_{ij} = \begin{cases} 
    |\bar{h}_i^H \bar{u}_j|^2 & i \neq j \\
    0 & i = j 
    \end{cases}.
\]

III. Equality of Downlink and Uplink Power Allocations

Based on (5), we see that \( \Psi = \Psi^T \) is a sufficient condition for the equality of \( p \) and \( q \). We now proceed to prove that this transpose symmetry indeed applies for arbitrary virtual uplink precoders \( \bar{V}_k \) as long as the optimum power allocation \( q^* = [q_1^*, \ldots, q_L^*] \) satisfying (4) and the corresponding MMSE receive beamformers \( u_l^* \) are used.

A. KKT Conditions for MMSE Precoding

From the objective and constraint functions in (4), the Lagrangian is

\[
    \mathcal{L}(q, \mu) = \text{tr} \left[ \left( \sum_{l=1}^{L} q_l \bar{h}_l \bar{h}_l^H + \sigma^2 I_M \right)^{-1} \right] + \mu_{\text{sum}} \left( \sum_{l=1}^{L} q_l - P_{\text{max}} \right) - \sum_{l=1}^{L} \mu_l q_l,
\]
and the resulting KKT conditions are

\[
\begin{bmatrix}
\hat{h}_1^H J^{-2} \hat{h}_1 \\
\vdots \\
\hat{h}_L^H J^{-2} \hat{h}_L
\end{bmatrix} + \mu_{\text{sum}} \mathbf{1}_L - \sum_{l=1}^{L} \mu_l \mathbf{e}_l = \mathbf{0}_L \\
\sum_{l=1}^{L} q_l \leq P_{\text{max}}, \quad q_l \geq 0 \\
\mu_{\text{sum}} \geq 0, \mu_l \geq 0 \\
\mu_{\text{sum}} \left( \sum_{l=1}^{L} q_l - P_{\text{max}} \right) = 0, \quad \mu_l q_l = 0.
\]

Here, \( \mathbf{0}_L \) is the length-\( L \) all-zeroes vector, and \( \mathbf{e}_l \) is the standard basis vector with a single one in the \( l \)th position and zeroes elsewhere. The gradient in the stationarity condition follows from the identity \( \partial J^{-1} = -J^{-1} (\partial J) J^{-1} \) and the linearity of the trace operator. Thus,

\[
\frac{\partial \text{tr} \left[ J^{-1} \right]}{\partial q_l} = \text{tr} \left[ -J^{-1} \frac{\partial J}{\partial q_l} J^{-1} \right] = -\text{tr} \left[ J^{-1} \hat{h}_l^H \hat{h}_l^H J^{-1} \right] = -\hat{h}_l^H J^{-2} \hat{h}_l.
\]

\( \text{(9)} \)

\( \text{B. Conditions for Equality under Optimal Power Allocation} \)

Having solved (4) for an arbitrary set of virtual uplink precoders \( \bar{v}_l \), we then find the MMSE receive beamformers \( u_\star^l = J^{-1} \hat{h}_l \sqrt{q_\star^l} \). With the associated virtual uplink stream MSEs \( \varepsilon_l \) and scalars \( \beta_l = \sqrt{q_\star^l \| u_\star^l \|} \), we can then use (5) to find the downlink power allocation \( p \) that achieves the same MSEs for each data stream.

In the case where the optimal power allocation results in one or more inactive streams \( S_I = \{ l \in (1, \ldots, L) \mid q_\star^l = 0 \} \), this algorithm fails since \( u_\star^l = 0 \) for \( l \in S_I \). However, the same MSEs can be achieved for these inactive streams in the downlink by setting \( p_l = 0 \). The power allocation \( p \) for the set of active streams \( S_A = \{ l \in (1, \ldots, L) \mid q_\star^l > 0 \} \) can then be found by following the specified procedure after deleting the rows and columns from \( \beta \), \( D \), and \( \Psi \) corresponding to the inactive streams.

The coupling matrix \( \Psi \) is a real matrix whose off-diagonal entries \( [\Psi]_{ij} \) contain squared magnitudes of the end-to-end channel gains from transmitted symbol \( x_j \) to the decoded symbol.
\( \tilde{x}_i \). We observe that \( \Psi = \Psi^T \) is satisfied when

\[
\frac{\tilde{h}_i^H u_j^\star}{\|u_j^\star\|} = \frac{u_i^H \tilde{h}_j}{\|u_i^\star\|},
\]

or equivalently,

\[
\frac{\tilde{h}_i^H J^{-1} \tilde{h}_j \sqrt{q_j^*}}{\sqrt{q_j^* \tilde{h}_j^H J^{-2} \tilde{h}_j}} = \frac{\sqrt{q_j^* \tilde{h}_j^H J^{-2} \tilde{h}_i}}{\sqrt{q_i^* \tilde{h}_i^H J^{-2} \tilde{h}_i}}
\]

(12)

The power allocation terms \( q_i^* \) and \( q_j^* \) cancel out, and numerators are equal; thus, an equivalent expression for the sufficient condition for \( p = q \) is

\[
\tilde{h}_i^H J^{-2} \tilde{h}_i = \tilde{h}_j^H J^{-2} \tilde{h}_j \quad \forall i, j \in S_A.
\]

(13)

We rewrite the individual terms in (9) as \( \tilde{h}_l^H J^{-2} \tilde{h}_l = (\mu_{\text{sum}} - \mu_l) \). Due to the complementary slackness condition \( \mu_l q_l = 0 \), the dual variables \( \mu_l \) are zero for all active streams \( l \in S_A \) with \( q_l > 0 \). Thus, it follows that

\[
\tilde{h}_l^H J^{-2} \tilde{h}_l = \mu_{\text{sum}} \quad \forall l \in S_A;
\]

(14)

that is, (13) is satisfied, \( \Psi = \Psi^T \), and the downlink and virtual uplink power allocations \( p \) and \( q \) that achieve the same minimum sum-MSE are identical.

C. Discussion

The equality result presented in Section III-B was shown to apply for arbitrary \( \tilde{v}_l \), as long as the optimum power allocation and MMSE receivers are used. It follows that it also applies to the optimum covariance-based design, when covariance matrices for each stream are normalized as \( \tilde{R}_l = q_l^* \tilde{R}_l \) and \( \tilde{R}_l = \tilde{v}_l \tilde{v}_l^H \). This result implies that the virtual uplink to downlink transformation stage can be omitted from algorithms using both iterative and joint designs based on a virtual-uplink solution [1]–[4], thus allowing for simplified implementations.

IV. Conclusions

In this letter, we have proven that the optimum power allocations for the downlink and virtual uplink are identical when minimizing the sum-MSE under a sum power constraint. With this proof, we extend the known results in a well studied problem. Importantly, our result simplifies existing iterative algorithms, eliminating the solution of a matrix equation in each iteration.
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Fig. 1. Processing for user $k$ in downlink and virtual uplink.