Strictly non-proportional geodesically equivalent metrics have $h_{\text{top}}(g) = 0$

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1. Definition and main results

Definition 1. Two ($C^{\infty}$-smooth) Riemannian metrics $g$ and $\bar{g}$ on a manifold $M^n$ are said to be **geodesically equivalent** if their geodesics coincide as unparameterized curves. They are strictly non-proportional at $x \in M^n$, if the polynomial $\det(g|_{x - t\bar{g}})$ has only simple roots.

The question of whether two different metrics can have the same geodesics is natural and, therefore, classical. The first examples are due to E. Beltrami [B], a local descriptions of geodesically equivalent metrics was understood by U. Dini [Di] and T. Levi-Civita [LC]. We will recall Levi-Civita’s Theorem in Section 2.1. For more historical details, see the surveys [Mi, Am], or/and the introductions to the papers [M1, M4].

The main result of our paper is (for definition and properties of $h_{\text{top}}$ we refer to [Bo, KH, Ma]):

**Theorem 1.** Suppose the Riemannian metrics $g$ and $\bar{g}$ on a closed connected manifold $M^n$ are geodesically equivalent and strictly non-proportional at least at one point. Then the topological entropy $h_{\text{top}}(g)$ of the geodesic flow of $g$ vanishes.

The condition that the metrics are strictly non-proportional is important: for example, the product metric on a closed product manifold $M = M_1 \times M_2$ admits a family $g_1 + t\bar{g}_2$ of non-proportional metrics (but not strictly non-proportional if $\dim M > 2$) with the same geodesics. But if at least one factor has fundamental group with positive exponential growth (for instance if $M_1$ is hyperbolic), then by the Dinaburg Theorem any geodesic flow on $M$ has $h_{\text{top}}(g) > 0$.

Vanishing of the topological entropy of a $C^{\infty}$-smooth flow implies a lot of dynamical restrictions. For example, the ball volume grows sub-exponentially with its radius (Manning’s inequality [Ma]), the number of geodesic arcs joining two generic points grows sub-exponentially with its maximal length (Mañé’s formula [Ma]) and the volume of a compact submanifold propagated by the geodesic flow also changes sub-exponentially (Yomdin’s Theorem [Y]), see also [P2].

Probably even more interesting are topological restrictions implied by $h_{\text{top}}(g) = 0$. The subexponential growth of $\pi_1(M^n)$ (Dinaburg’s Theorem [D]) is not very intriguing under the assumptions of Theorem 1 since it is known [MB] that in this case the fundamental group is virtually abelian. But the restriction coming from the Gromov-Paternain Theorem [G, P1] and from [PP1] are new, nontrivial and interesting: Namely in the simply connected case the manifold $M^n$ is **rationally elliptic**, i.e. $\pi_1(M^n) \otimes \mathbb{Q}$ is finite-dimensional. This is a very restrictive property since by the results of [EHII, Pal] a rationally elliptic manifold $M^n$ enjoys the following properties:

1. $\dim \pi_1(M^n) \otimes \mathbb{Q} \leq n$, $\dim H_i(M^n, \mathbb{Q}) \leq 2^{n-1}$, $\dim H_i(M^n, \mathbb{Q}) \leq \frac{1}{2} \left( \begin{array}{c} n \cr i \end{array} \right)$ ($i = 1, \ldots, n-1$),

2. The Euler characteristic $\chi(M^n)$ satisfies $2^n - n + 1 \geq \chi(M^n) \geq 0$. Moreover, $\chi(M^n) > 0$ if $H_{\text{odd}}(M^n, \mathbb{Q}) = 0$.

A manifold $M$ with finite $\pi_1(M)$ is called **rationally hyperbolic**, if its universal cover is not rationally elliptic. Thus, as a consequence of Theorem 1 we get
Corollary 1. A rationally hyperbolic closed manifold $M^n$ does not admit two geodesically equivalent Riemannian metrics $g$ and $\bar{g}$ which are strictly non-proportional at least at one point.

Rational hyperbolithity means nothing in dimensions less than 4, since all closed 4-manifolds with finite fundamental group are rational-elliptic. Note that the topology of closed 2- and 3-manifolds admitting non-proportional geodesically equivalent metrics is completely understood: In dimension 2, such manifolds are homeomorphic to the sphere, the projective plane, the torus or the Klein bottle [MT2]. In dimension 3, such manifolds are homeomorphic to lens spaces or to Seifert manifolds with zero Euler number [M2].

Starting from dimension 4, almost all simply-connected manifolds are rationally hyperbolic. For example, in dimension 4, up to homeomorphism, there exist infinitely many simply-connected closed manifolds, and only five of them are rationally elliptic: $S^4$, $S^2 \times S^2$, $CP^2$, $CP^2 \# CP^2$ and $CP^2 \# CP^2$. It is possible to construct geodesically equivalent metrics on $S^4$ and $S^2 \times S^2$ that are strictly non-proportional at least at one point. We conjecture here that these two are the only closed simply-connected 4-manifolds admitting strictly non-proportional geodesically equivalent metrics. In dimension 5, a closed rational-elliptic manifold has rational homotopy type of $S^2 \times S^3$ or $S^5$ (there are infinitely many homotopy types for simply-connected 5-manifolds). By recent results of [PP1] (see Theorem E there), a closed manifold admitting a metric with zero topological entropy is $S^5$, $S^3 \times S^2$, $SU(3)/SO(3)$ or the nontrivial $S^3$-bundle over $S^2$. We conjecture that $S^3 \times S^2$ and $S^5$ are the only closed simply-connected connected 5-manifolds admitting geodesically equivalent metrics which are strictly non-proportional at least at one point.

In Section 5 we announce the restrictions on the topology of non-simply-connected manifolds (admitting geodesically equivalent metrics which are strictly non-proportional at least at one point) that follows from Corollary 1.

Now let us comment the proof of Theorem 1. The main ingredients are Theorems 2, 3 and Corollary 2, which imply that the geodesic flow of $g$ is Liouville-integrable.

Precisely the same integrable systems were recently actively studied in mathematical physics, in the framework of the theory of separation of variables. Depending on the school, they are called L-systems [Be], Benenti-systems [IMM] and quasi-bi-hamiltonian systems [CST].

But Liouville integrability does not immediately imply vanishing of the topological entropy; counterexamples can be found in [BT1, BT2, Bu1, Bu2, K, KT]. If the singularities of the integrable system behave sufficiently good (non-degenerate in the sense of Williamson-Vey-Eliasson-Ito [E, I], see [P1], or the Taimanov conditions [T]), or if the system has a lot of symmetries (for example, as in collective integrability [BP, P1]), then $h_{top}(g) = 0$. But for other situations nothing is known (at least if $n > 2$, see [P0]), even if the integrals are real-analytic or polynomial in momenta.

It is worth mentioning that geodesically equivalent metrics are usually not real-analytic: Levi-Civita’s Theorem from Section 2.1 shows the existence of an infinite-dimensional space of non-analytic $C^\infty$-perturbations in the class of geodesically-equivalent metrics. Also the set of singular points of the constructed integrals for the corresponding Hamiltonian system can be quite complicated. For instance, the projection of the singularities in $TM^n$ to the base $M^n$ is surjective for $n > 2$ and its restriction to a singular Liouville fiber can have image which is locally the product of the Cantor set and the $(n - 1)$-dimensional disk.

The logic of our proof for Theorem 1 is as follows:

1. We show that the topological entropy is supported on the singularities, which we describe.
2. We show that dynamics on them can be considered as a subsystem of the geodesic flow
   - on a lower-dimensional closed submanifold
   - admitting geodesically equivalent metrics which are strictly non-proportional at least at one point.

Therefore we can apply induction by the dimension.
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2 Geometry behind the geodesic equivalence

In what follows we always assume that the manifold $M^n$ is connected and that the Riemannian metrics $g$ and $\bar{g}$ on $M^n$ are geodesically equivalent and strictly non-proportional at least at one point.

2.1 Integrability and Levi-Civita’s Theorem

A Riemannian metric $g$ determines the map $\gamma_g : TM \to T^*M$ with the inverse $\gamma^g : T^*M \to TM$. Consider the $(1,1)$-tensor (automorphism field) $L : TM \to TM$ given by the formula

$$L = \left(\det(\gamma^g \circ \gamma_g)\right)^{-\frac{1}{n+1}} \cdot (\gamma^g \circ \gamma_g).$$

In local coordinates, $L_i = \sqrt[n+1]{\frac{\det(g)}{\det(\bar{g})}} g_{ij} \bar{g}^{j3}$. This tensor $L$ determines the family $S_t \in C^\infty(T^*M \otimes TM)$, $t \in \mathbb{R}$, of $(1,1)$-tensors

$$S_t := \det(L - t Id) \cdot (L - t Id)^{-1}.$$  

Remark 1. Although $(L - t Id)^{-1}$ is not defined for $t \in \text{Sp}(L)$, the tensor $S_t$ is well-defined for every $t \in \mathbb{R}$. In fact, it is the adjoint matrix of $(L - t Id)$. Thus by the Laplace main minors formula, $S_t$ is a polynomial in $t$ of degree $n - 1$ with coefficients being $(1,1)$-tensors.

The isomorphism $\gamma^g$ allows us to identify the tangent and cotangent bundles of $M^n$. This identification allows us to transfer the natural Poisson structure and the Hamiltonian system $H(x,p) = \frac{1}{2}p \cdot \gamma^g(p)$ from $T^*M^n$ to $TM^n$.

Theorem 2 (MT1). If $g$, $\bar{g}$ are geodesically equivalent, then, for every $t_1, t_2 \in \mathbb{R}$, the functions

$$I_{t_i} : TM^n \to \mathbb{R}, \quad I_{t_i}(v) := g(S_{t_i}(v), v)$$

are commuting integrals for the geodesic flow of $g$.

Since $L$ is self-adjoint with respect to both $g$ and $\bar{g}$, the spectrum $\text{Sp}(L)$ is real at every point $x \in M^n$. Denote it by $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$. Every eigenvalue $\lambda_i(x)$ is at least continuous functions on $M^n$, and is smooth near the points where it is a simple eigenvalue.

Theorem 3 (MT1). Let $(M^n, g)$ be a geodesically complete connected Riemannian manifold. Let a Riemannian metric $\bar{g}$ on $M^n$ be geodesically equivalent to $g$. Then, for every $i \in \{1, \ldots, n-1\}$ and for all $x, y \in M^n$, the following holds:

1. $\lambda_i(x) \leq \lambda_{i+1}(y)$.
2. If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.
3. If $\lambda_i(x) = \lambda_j(y)$ for a certain $j \neq i$, then there exists $z \in M^n$ such that $\lambda_i(z) = \lambda_j(z)$.

Corollary 2 (MT3). Let $(M^n, g)$ be a connected Riemannian manifold. Suppose a Riemannian metric $\bar{g}$ on $M^n$ is geodesically equivalent to $g$ and is strictly non-proportional to $g$ at least at one point. Then, for every mutually-different $t_1, t_2, \ldots, t_n \in \mathbb{R}$, the integrals $I_{t_i}$ are functionally independent almost everywhere, i.e. the differentials $dI_{t_i}$ are linearly independent a.e. in $TM$.  

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Let us describe the local form of the integrals $I_t$. For every $x \in M^n$ consider coordinates in $T_x M^n$ such that the metric $g$ is given by the diagonal matrix diag$(1, 1, \ldots, 1)$ and the tensor $L$ is given by the diagonal matrix diag$(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then the tensor $\mathbf{2}$ reads:

$$S_t = \det(L - t \, \text{Id})(L - t \, \text{Id})^{-1} = \text{diag}(\Pi_1(t), \Pi_2(t), \ldots, \Pi_n(t)),$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \defeq \prod_{j \neq i} (\lambda_j - t).$$

Hence, for every $\xi = (\xi_1, \ldots, \xi_n) \in T_x M^n$, the polynomial $I_t(x, \xi)$ is given by

$$I_t = \xi_1^2 \Pi_1(t) + \xi_2^2 \Pi_2(t) + \cdots + \xi_n^2 \Pi_n(t). \quad (4)$$

For further use, let us consider the one parameter family of functions

$$I'_t \defeq \frac{d}{dt}(I_t).$$

For every fixed $t \in \mathbb{R}$ this function is an integral of the geodesic flow for $g$.

Let us now formulate (a weaker version of) the classical Levi-Civita’s Theorem.

**Theorem 4 (Levi-Civita [LC]).** Consider two Riemannian metrics on an open subset $U^n \subset M^n$ and the tensor $L$ given by $\mathbf{7}$. Suppose the spectrum $\text{Sp}(L)$ is simple at every point $x \in U^n$.

Then the metrics are geodesically equivalent on $U^n$ if and only if around each point $x \in U^n$ there exist coordinates $x_1, x_2, \ldots, x_n$ in which the metrics have the following model form:

$$\begin{align*}
\frac{ds^2}{g} &= |\Pi_1(\lambda_1)|dx_1^2 + |\Pi_2(\lambda_2)|dx_2^2 + \cdots + |\Pi_n(\lambda_n)|dx_n^2, \\
\frac{ds^2}{\bar{g}} &= \rho_1 |\Pi_1(\lambda_1)|dx_1^2 + \rho_2 |\Pi_2(\lambda_2)|dx_2^2 + \cdots + \rho_n |\Pi_n(\lambda_n)|dx_n^2,
\end{align*} \quad (5)$$

where the functions $\rho_i$ are given by

$$\rho_i \defeq \frac{1}{\lambda_1 \lambda_2 \cdots \lambda_i},$$

and $\lambda_i = \lambda_i(x_i)$ are smooth functions of one variable.

**Definition 2.** The above coordinates will be called **Levi-Civita coordinates** and the neighborhoods where the coordinates are defined will be called **Levi-Civita charts**.

In Levi-Civita coordinates the tensor $L$ is diagonal $\text{diag}(\lambda_1, \ldots, \lambda_n)$, so the notations in the Levi-Civita Theorem are compatible with those in the beginning of the section.

**Corollary 3 ([M1], [BM]).** Suppose the Riemannian metrics $g$, $\bar{g}$ are geodesically equivalent on $M$. Then, the Nijenhuis torsion of the tensor $L$ given by $\mathbf{11}$ vanishes: $N_L = 0$.

If the metrics are strictly non-proportional at least at one point, Corollary 3 follows from the above version of Levi-Civita’s theorem. In the general case, Corollary 3 follows from the original version of Levi-Civita’s Theorem [LC] and was proven in [M1] and [BM].

Combining formulae 3 and 11, we see that in the Levi-Civita coordinates the function $I_t$ is given by

$$I_t = \sum_i |\Pi_i(\lambda_i(x))| |\Pi_i(t)| \xi_i^2 \quad (7)$$

In particular, the function $I_{\lambda_i(x)}$ as the function on the cotangent bundle is equal to $(-1)^{i-1}p_i^2$.  

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2.2 Distributions of eigenvectors: submanifolds $M_A$

We begin with investigation of the set of points from the Levi-Civita charts, the union of which is the open dense set

$$\text{Reg}(M) = \{ x \in M : \lambda_i(x) \neq \lambda_j(x) \text{ for } i \neq j \}.$$ 

This set can be represented as the intersection $\text{Reg}(M) = \cap_A \text{Reg}_A(M)$ by all (proper) subsets $A \subset \{1, 2, \ldots, n\}$, where we denote

$$\text{Reg}_A(M) = \{ x \in M : \forall i \in A \forall j \notin A \lambda_i(x) \neq \lambda_j(x) \}.$$ 

At every point $x \in \text{Reg}_A(M)$ denote by $D_A(x)$ the subspace of $T_xM^n$ spanned by the eigenspaces with the eigenvalues $\lambda_i$, where $i \in A$. Since the eigenvalues $\lambda_i$ for $i \in A$ do not bifurcate with the eigenvalues $\lambda_j$ for $j \notin A$, $D_A$ is a smooth distribution on $\text{Reg}_A(M)$. By Corollary 3 it is integrable. We will denote by $M_A(x)$ its integral submanifold containing $x \in \text{Reg}_A(x) \subset M^n$.

Lemma 1. For $x \in \text{Reg}_A(M)$ the following statements hold:

1. The restrictions of $g$ and $\bar{g}$ to $M_A(x)$ are geodesically equivalent.
2. $g|_{M_A(x)}$ and $\bar{g}|_{M_A(x)}$ are strictly non-proportional at least at one point.
3. For $i \in A$ the $i^{th}$ eigenvector of $L$ (corresponding to $\lambda_i$) coincides with the respective eigenvector of the operator $L_A$, constructed via (8) for the metrics $g|_{M_A(x)}$ and $\bar{g}|_{M_A(x)}$.
4. There exists a universal along $M_A(x)$ constant $c$ (calculated explicitly in the proof) such that the part of $c \cdot \text{Sp}(L)$, corresponding to $A$, coincides with the spectrum of the operator $L_A$, constructed by the restricted to $M_A(x)$ metrics.
5. In particular, if an eigenvalue $\lambda_i$, $i \in A$ is constant, then the corresponding eigenvalue of the operator $L_A$, constructed for the restrictions of $g$ and $\bar{g}$ to $M_A(x)$, is constant on $M_A(x)$.

Proof: The distribution $D_A$ defines a foliation on $\text{Reg}_A(M)$ and on its open dense subset $\text{Reg}(M)$. Then it is sufficient to prove the first, third and the fourth statements of the lemma at the points of this subset. By Theorems 3, 4 in a neighborhood of every point $x \in \text{Reg}(M)$, there exist Levi-Civita coordinates such that the metrics $g$, $\bar{g}$ are given by formulas (7)–(9). In these coordinates, $M_A(x)$ is the coordinate plaque of the coordinate collection $x_\alpha$ with $\alpha \in A = \{\alpha_1, \ldots, \alpha_m\}$. Then the restrictions of the metrics to $M_A(x)$ are given by:

$$g|_{M_A} = \Pi_{\alpha_1}(\lambda_{\alpha_1})dx_{\alpha_1}^2 + \Pi_{\alpha_2}(\lambda_{\alpha_2})dx_{\alpha_2}^2 + \cdots + \Pi_{\alpha_m}(\lambda_{\alpha_m})dx_{\alpha_m}^2,$$

$$\bar{g}|_{M_A} = \rho_{\alpha_1}\Pi_{\alpha_1}(\lambda_{\alpha_1})dx_{\alpha_1}^2 + \rho_{\alpha_2}\Pi_{\alpha_2}(\lambda_{\alpha_2})dx_{\alpha_2}^2 + \cdots + \rho_{\alpha_m}\Pi_{\alpha_m}(\lambda_{\alpha_m})dx_{\alpha_m}^2.$$ 

Since $\lambda_j$ is constant on $M_A(x)$ for every $j \notin A$, every factor of $\Pi_{\alpha_i}$ of the form $\lambda_j - \lambda_{\alpha_i}$ can be “hidden” in $dx_{\alpha_i}^2$. We see that then the first metric is already in the Levi-Civita form, and the second metric becomes in the Levi-Civita’s form after multiplication by

$$C \overset{\text{def}}{=} \prod_{j \notin A} \lambda_j,$$ 

which is constant on $M_A(x)$. Hence, by Levi-Civita’s Theorem, the restrictions of the metrics to $M_A$ are geodesically equivalent.

Direct calculations show that in local coordinates the tensor $L_A$ is given by:

$$C^{1/(m+1)}\text{diag}(\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_m}).$$

(9)

The third and the fourth statements of the lemma follow.

Now let us prove the second statement. Suppose the restriction of the metrics are not strictly non-proportional at every point of a certain $M_A(x)$. Then, by Theorem 3 there exist $\alpha_1, \alpha_2 \in A$
such that \( \lambda_{a_1} \equiv \lambda_{a_2} \) on \( M_A(x) \). Consider the set \( B := \{1, \ldots, n\} \setminus A \). Take the union of all leaves \( M_B \) containing at least one point of \( M_A(x) \). Clearly, this union contains an open subset of \( M^n \). Since the eigenvalues \( \lambda_{a_1}, \lambda_{a_2} \) are constant along \( M_B \), in view of Corollary 3 and Theorem 5 at every point of this open subset we have \( \lambda_{a_1} = \lambda_{a_2} \), which contradicts Theorem 5. Lemma 4 is proven.

Lemma 2. Suppose the eigenvalue \( \lambda_i \) is not a constant. Take a point \( y \in M^n \) such that
\[
\max_{x \in M} \lambda_{i-1}(x) < \lambda_i(y) < \min_{x \in M} \lambda_{i+1}(x).
\]
(We assume by definition that \( \min_{x \in M} \lambda_{n+1}(x) = -\infty \) and \( \max_{x \in M} \lambda_0(x) = -\infty \).)
Let \( C(i) := \{1, 2, \ldots, n\} \setminus \{i\} \). Then, \( M_{C(i)}(y) \) is a closed submanifold.

The conditions that the eigenvalue is not constant and that \( \lambda_i \) is neither maximum nor minimum are important: one can construct counterexamples, if one of these conditions is omitted.

Proof of Lemma 2 Since \( \max_{x \in M} \lambda_{i-1}(x) < \lambda_i(y) < \min_{x \in M} \lambda_{i+1}(x) \), there exist \( c_{\text{small}}, c_{\text{big}} \in \mathbb{R} \) such that

1. \( c_{\text{small}} < \lambda_i(y) < c_{\text{big}} \),
2. at least one of the numbers \( c_{\text{small}}, c_{\text{big}} \) is a regular value of the function \( \lambda_i \),
3. the other number is not a critical value of \( \lambda_i \) (i.e. is either a regular value or is equal to \( \lambda_i \) at no point.)

Denote by \( N \) the connected component of the set
\[
\{ x \in M^n : \ c_{\text{small}} \leq \lambda_i(x) \leq c_{\text{big}} \},
\]
containing the point \( y \). Then \( N \subset \text{Reg}_{C(i)}(M) \) is a connected manifold with boundary. Therefore, \( D_{C(i)} \) is a smooth distribution on \( N \). Since it is integrable by Corollary 3 it defines a foliation. By Corollary 2 the function \( \lambda_i \) is constant on the leaves of the foliation. Then, every connected component of \( N \) is a leaf of the foliation.

At every \( x \in M^n \), consider the vector \( v_i \) satisfying
\[
\begin{align*}
\Pi_i(\lambda_i) &= L(v_i) = \lambda_i(x)v_i, \\
g(v_i, v_i) &= |\Pi_i(\lambda_i)|.
\end{align*}
\] (10)

By definition of \( N \), the function \( |\Pi_i(\lambda_i)| \) is nonzero and smooth at every point of \( N \). Thus \( v_i \) vanishes nowhere in \( N \). Hence, at least on the double-cover of \( N \), it is defined globally up to a sign and is smooth. The double-cover projection maps closed submanifolds into closed ones. Therefore, without loss of generality we can assume that the vector field \( v_i \) is globally defined already on \( N \).

Consider the flow of the vector field \( v_i \). It takes leaves to leaves. Indeed, it is sufficient to prove this almost everywhere, for instance in Levi-Civita charts. In Levi-Civita coordinates the leaves of the foliation are the plaques of the coordinates \( x_\alpha \), where \( \alpha \in C(i) \), and the vector field \( v_i \) is \( \pm \partial / \partial x_\alpha \), so the claim is trivial.

Since the leaves are \((n-1)\)-dimensional and the flow of \( v_i \) shuffles them, the flow acts transitively and all leaves are homeomorphic. Every connected component of the boundary of \( B \) is compact and is a leaf, whence all leaves are compact. In particular, \( M_{C(i)}(y) \) is compact. Lemma 2 is proven.

2.3 Bifurcation of eigenvalues: submanifolds \( \text{Sing}_i^d \)

The spectrum \( \text{Sp}(L) \) is simple in \( \text{Reg}(M) \), i.e. almost everywhere in \( M^n \). But at certain points the multiplicity of some \( \lambda_i \) can become greater than one. Such points will be called the bifurcation points of \( \lambda_i \). By Theorem 5 the following types of bifurcations of the eigenvalue \( \lambda_i \) are possible.
Case 1: The eigenvalues $\lambda_i$ and $\lambda_{i+1}$ are not constant and there exists $x \in M$ such that $\lambda_i(x) = \lambda_{i+1}(x)$. Denote $\bar{\lambda}_i = \max \lambda_i(x) = \min \lambda_{i+1}(x)$. Let us consider the set
\[
\text{Sing}_i^1 \overset{\text{def}}{=} \{ x \in M^n : (\lambda_i(x) - \bar{\lambda}_i)(\lambda_{i+1}(x) - \bar{\lambda}_i) \leq 0 \}.
\]
This set was studied in [M1] (see Theorem 6 there). It was shown that $\text{Sing}_i^1$ is a connected closed totally geodesic submanifold of codimension one. The restrictions of the metrics to it are strictly non-proportional at least at one point. Note that not all points of $\text{Sing}_i^1$ are points of bifurcation of the eigenvalues $\lambda_i, \lambda_{i+1}$.

Case 2: There exists $x \in M$ and $i \in \{2, \ldots, n-1\}$ such that $\lambda_{i-1}(x) = \lambda_{i+1}(x)$. In this case, the eigenvalue $\lambda_i$ is constant. Let us consider the set
\[
\text{Sing}_i^2 \overset{\text{def}}{=} \{ x \in M^n : (\lambda_{i-1}(x) - \lambda_i)(\lambda_{i+1}(x) - \lambda_i) = 0 \}.
\]
This set was also studied in [M1] (see Theorem 6 there). It was shown that $\text{Sing}_i^2$ is a connected closed totally geodesic submanifold of codimension two. The restrictions of the metrics to it are strictly non-proportional at least at one point. Moreover, the set of the points $x \in \text{Sing}_i^2$ such that $\lambda_{i-1}(x) = \lambda_{i+1}(x)$ is nowhere dense in $\text{Sing}_i^2$.

Case 3a: The eigenvalue $\lambda_i$ is constant, there exists $x \in M$ such that $\lambda_i = \lambda_{i+1}(x)$ and there exists no $y$ such that $\lambda_{i-1}(y) = \lambda_i$.

Case 3b: The eigenvalue $\lambda_i$ is constant, there exists $x \in M$ such that $\lambda_i = \lambda_{i-1}(x)$ and there exists no $y$ such that $\lambda_{i+1}(y) = \lambda_i$.

In Cases 3a, 3b, let us consider respectively the sets
\[
\text{Sing}_i^3 = \{ x \in M^n : \lambda_i = \lambda_{i+1}(x) \} \quad \text{or} \quad \text{Sing}_i^3 = \{ x \in M^n : \lambda_i = \lambda_{i-1}(x) \}.
\]

The next lemma shows that, similar to Cases 1 and 2, $\text{Sing}_i^3$ is a submanifold of codimension 2 and the restrictions of the metrics to $\text{Sing}_i^3$ are geodesically equivalent and strictly non-proportional at least at one point. Note that, contrast to the previous cases, the set $\text{Sing}_i^3$ is not necessary connected.

**Lemma 3.** Under assumptions of Cases 3a or 3b, the set $\text{Sing}_i^3$ is a

1. totally geodesic
2. closed submanifold of codimension 2.
3. Moreover, the restrictions of the metrics to $\text{Sing}_i^3$ are strictly non-proportional at least at one point.

Here we will proof that $\text{Sing}_i^3$ is a closed submanifold of codimension 2 such that the restrictions of the metrics to it are strictly non-proportional at least at one point. The first statement of the lemma, namely that $\text{Sing}_i^3$ is totally geodesic, will follow immediately from Theorem 6 see Remark 2. Before Theorem 5 Lemma 5 will be used only once, namely in the proof of Theorem 5. Since the proof of Theorem 5 does not require Theorem 5 no logical loop appears.

**Proof of statements 2,3 of Lemma 5** We consider Case 3a, the other case is completely analogous. By definition, the set $\text{Sing}_i^3$ is closed and, therefore, compact.

Let us show that locally $\text{Sing}_i^3$ is a submanifold of codimension 2. Let $A = \{i, i + 1\}$. Take a point $x_0$ such that $\lambda_i = \lambda_{i+1}(x_0)$. Then $x_0 \in \text{Reg}_A(M)$ and we can consider the set $M_{A}(x_0)$. By Lemma 1 the restrictions of the metrics to $M_{A}(x_0)$ are geodesically equivalent and strictly non-proportional at least at one point. Since $M_{A}(x_0)$ is two-dimensional, the set of points, where these restrictions are proportional, is discrete [2]. In view of Lemma 1 the restrictions of the metrics are proportional at $x_0$. Then in a small neighborhood of $x_0$, there exists no other point $x \in M_{A}(x_0)$ such that $\lambda_i = \lambda_{i+1}(x)$. Denote by $B$ the set $\{1, 2, \ldots, n\} \setminus A$. For every point $x$ of a small neighborhood of $x_0$ in $M_{A}(x_0)$, consider the set $M_{B}(x)$. It is a submanifold of codimension two. Since the eigenvalues $\lambda_i, \lambda_{i+1}$ are constant along $M_{B}$, in a small neighborhood of $x_0$ the set $\text{Sing}_i^3$ coincides with $M_{B}(x_0)$. Thus it is a submanifold of codimension 2.
By the second statement of Lemma 1, the restrictions of the metrics to \( \text{Sing}_j \) are strictly non-proportional at least at one point. The 2\textsuperscript{nd} and 3\textsuperscript{rd} statements of Lemma 3 are proven.

Let us note that for a fixed \( i \) only one of the submanifolds \( \text{Sing}_j \), \( j = 1, 2, 3 \), can be non-empty.

3 Description of singular points

Consider some mutually-different numbers \( t_1, \ldots, t_n \in \mathbb{R} \) and the respective integrals \( I_{t_1}, \ldots, I_{t_n} \). Consider the Poisson action of the the group \( (\mathbb{R}^n, +) \) on \( TM^n \): an element \( (a_1, \ldots, a_n) \in \mathbb{R}^n \) acts by time-one shift along the Hamiltonian vector field of the function \( a_1 I_{t_1} + \ldots + a_n I_{t_n} \). Since the functions are commuting integrals, the action is well-defined, smooth, symplectic, preserves the integrals \( I_t \) and the Hamiltonian of the geodesic flow, see §49 of [A] for details.

A point \( (x, \xi) \in TM \) is called singular if the differentials \( dI_{t_1}, \ldots, dI_{t_n} \) are linearly dependent at \( (x, \xi) \). An orbit of the action is called singular if it has a singular point. All points of a singular orbit are singular and have the same coefficients of the linear dependence.

Although the Poisson action depends on the choice of constants \( t_1, \ldots, t_n \), the property of \( (x, \xi) \) being singular does not depend on the choice of \( t_i \) as far as these numbers are all different.

3.1 Singular points in Levi-Civita coordinates

The next theorem describes singular points that lie over a Levi-Civita chart \( U^n \subset \text{Reg}(M^n) \). Fix a point \( x \in \text{Reg}(M^n) \) and denote by \( \lambda_1, \ldots, \lambda_n \) the constants \( \lambda_1(x), \ldots, \lambda_n(x) \) respectively.

**Theorem 5.** Let the metrics \( g \) and \( \bar{g} \) be given by formulas (4)-(6) in a neighborhood \( U^n \subset M^n \). If the point \( (y, \xi) = (x_1, \ldots, x_m, \xi_1, \ldots, \xi_m) \in T \text{Reg}(M^n) \) is singular, then there exists \( i \in \{1, \ldots, n\} \) such that \( dI_{\lambda_i} = 0 \). Then \( I_{\lambda_i}(x, \xi) = 0 \) and at least one of the following statements holds:

1. The derivative \( \frac{\partial \lambda_i(x)}{\partial x_j} \) vanishes at \( x \).
2. The function \( I'_{\lambda_i} \) vanishes at \( (x, \xi) \).

Moreover, if \( M_{C(i)}(y) \) is compact, the whole geodesic passing through \( y \) with the velocity vector \( \xi \) is contained in \( M_{C(i)}(y) \), where \( C(i) \) is the same as in Lemma 4.

Actually, the assumption that \( M_{C(i)}(y) \) is compact is not necessary: Theorem 5 remains true, if we replace this condition by the condition that \( y \not\in \text{Sing}_1 \). Our stronger assumption makes the proof shorter.

**Proof of Theorem 5.** Suppose the point \( (y, \xi) \) is singular. Then, there exist constants \( (\mu_1, \ldots, \mu_n) \neq (0, \ldots, 0) \) such that at \( (y, \xi) \) it holds:

\[
\mu_1 dI_{\lambda_1} + \cdots + \mu_n dI_{\lambda_n} = 0.
\]

We will show that for every \( i \) such that \( \mu_i \neq 0 \) the differential \( dI_{\lambda_i} \) vanishes at \( (y, \xi) \). For every \( j \in \{1, \ldots, n\} \) consider the function \( I_{\lambda_j}(x) := (I_{t_j}(x, \eta))_{|\eta=\lambda_j(x)} \). In a small neighborhood of \( y \), the function \( \lambda_j \) is smooth. Hence the function \( I_{\lambda_j}(x) \) is smooth as well. At the point \( (y, \xi) \) we have:

\[
dI_{\lambda_j}(y) = dI_{\lambda_j} + I'_{\lambda_j} \cdot d\lambda_j.
\]

We will work on the cotangent bundle to \( M^n \). As we explained in Section 2.1 the function \( I_{\lambda_j}(x) \) is equal to \( (-1)^{j-1} p_j^2 \) and its differential has coordinates

\[
(0, \ldots, 0, 2 \cdot (-1)^{j-1} \cdot p_j, 0, \ldots, 0).
\]

Since the function \( \lambda_j \) depends on \( x_j \) only, its differential is

\[
(0, \ldots, 0, \frac{\partial \lambda_j}{\partial x_j}, 0, \ldots, 0).
\]
Thus $dI_{\lambda_j}$ at $(y, \xi)$ is given by

$$
(0, \ldots, 0, I_k^{\lambda_j} \cdot \frac{\partial \lambda_j}{\partial x_j}, 0, \ldots, 0, 2 \cdot (-1)^{j-1}, p_j, 0, \ldots, 0).
$$

We see that the differentials $dI_{\lambda_j}$ do not combine: If $\mu_i \neq 0$, then $dI_{\lambda_i} = 0$. Therefore, $p_i = 0$ (i.e. $\xi_i = 0$), which is equivalent to $I_k^{\lambda_j}(x, \xi) = 0$, and at least one of the following holds: $\frac{\partial \lambda_j}{\partial x_j}(x) = 0$ or $I_k^{\lambda_j}(x, \xi) = 0$. The first part of the theorem is proven.

Now let us show that the geodesic $\gamma$ such that $(\gamma(0), \dot{\gamma}(0)) = (y, \xi)$ is contained in $M_{C(i)}(y)$. Since $M_{C(i)}(y)$ is compact, it is sufficient to prove that at almost every point of the geodesic the velocity vector of the geodesic is contained in $D_{C(i)}$. Since $\text{Sing}_I$ are totally geodesic submanifolds, the geodesic $\gamma$ intersect them transversally, and it is sufficient to prove that the velocity vector of the geodesic lies in $D_{C(i)}$ in Levi-Civita’s charts.

Since $I_{\lambda_i}$ is an integral and $dI_{\lambda_i} = 0$ at $(y, \xi)$, we obtain that $dI_{\lambda_i}$ vanishes at every point $(\gamma(t), \dot{\gamma}(t))$. Then, as we explained above, in the Levi-Civita chart, the component $\xi_i$ equals zero, so that the velocity vector of the geodesic lies in $D_{C(i)}$. Finally, the geodesic stays in $M_{C(i)}$ forever. Theorem 5 is proven.

### 3.2 Removable singularities

Our next goal is to show that certain singular points are artificially singular: if we use a finite cover and choose the integrals appropriate, they become regular.

Suppose the eigenvalue $\lambda_i$ is constant. From the proof of Theorem 5 it follows that for every $x \in \text{Reg}_{\{i\}}(M)$ and $\xi \in D_{C(i)}(x) \subset T_xM^n$ the differential $dI_{\lambda_i}$ vanishes at $(x, \xi)$.

We will show that this singularity is **removable**, in the sense that on an appropriate finite cover we can find a linear in velocities function $J_i$ such that $J_i^2 = (-1)^{i-1}I_{\lambda_i}$. This relation immediately implies that $J_i$ commutes with the functions $I_i$. Since $I_{\lambda_i}$ is an integral, $J_i$ is an integral as well. Since it is linear in velocities, it corresponds to a Killing vector field. We will show that this Killing vector field is nonzero at $x$, which automatically implies that the differential of this integral does not vanish at $(x, \xi)$.

In the Levi-Civita coordinates $I_{\lambda_i} = (-1)^{i-1}p_i^2$ and we can put $J_i = \pm p_i$. Clearly, in the Levi-Civita coordinate system, $J_i(\eta) := g(v_i, \eta)$, where $v_i = \pm \frac{\partial}{\partial x_i}$.

Note that the vector field $\frac{\partial}{\partial x_i}$ satisfies conditions (11), and that near every regular point every vector field satisfying (11) is the vector field $\frac{\partial}{\partial x_i}$ of a certain Levi-Civita coordinate system.

Thus, in order to show that (at least on a finite cover) there exists a smooth function $J_i$ such that it is linear in velocities and such that $J_i^2 = (-1)^{i-1}I_{\lambda_i}$, it is sufficient to prove

**Theorem 6.** Suppose $\lambda_i$ is constant. Then at least on a double cover of $M^n$ there exists a smooth vector field $v_i$ satisfying (11) at every point $x \in M^n$.

**Remark 2.** Conditions (11) imply that the zeros of $v_i$ coincide with $\cup_{j=2,3} \text{Sing}_j$. Since $v_i$ is a Killing vector field, $\text{Sing}_j$ is a totally-geodesic submanifold.

**Proof of Theorem 6** First we show that at least on the double-cover there exists a continuous vector field $v_i$ with the required properties. In order to do this, it is sufficient to prove the following semi-local statement:

**Proposition (S).** Locally near every point $x$ there exist precisely two continuous vector fields $v_i$ satisfying (11).

If $\lambda_{i-1}(x) \neq \lambda_i \neq \lambda_{i+1}(x)$, then $y \in \text{Reg}_{\{i\}}(M)$. Then, $\Pi_i(\lambda_i) \neq 0$. Hence, $v_i \neq 0$ in a small neighborhood of $x$ and the statement (S) is trivial.

Let us consider $x \in \text{Sing}_j$, where $j = 2$ or 3, and prove the statement in a small disk neighborhood $U^n \ni x$. 


First of all, if a vector field $v_i$ satisfies (1), then the vector field $-v_i$ satisfies (1) as well. Since Sing$_i$ is nowhere dense, the fields do not coincide. Therefore we obtain at least two different required vector fields.

Next, there exist no more than two such vector fields. Indeed, such a vector field $v_i$ must vanish along Sing$_i$, since $\Pi_i(\lambda_i)$ equals zero there, and it is non-zero in the complement. This complement is connected, because Sing$_i$ has codimension 2 (by proven part of Lemma 8 and as we explained in Section 2.3), and the claim follows.

At last, let us prove that such continuous field $v_i$ exists in the small disk neighborhood $U^n \ni x$. Since $U^n \setminus$ Sing$_i$ is connected, we can define $v_i$ in one of two possible ways at some point $x_0$ and extend by continuity along paths in $U^n \setminus$ Sing$_i$. We need to show that the result is well-defined.

In order to do this we connect two paths $\phi_0, \phi_1$ from $x_0$ to $x_1$ in $U^n \setminus$ Sing$_i$ by a homotopy $\phi_r$ in $U^n$. The paths and the homotopy can be assumed smooth. Since Sing$_i$ has codimension 2, we can perturb homotopy and make it to be transversal to Sing$_i$. Thus, the intersection of Image$_{\phi_r}$ with Sing$_i$ is a finite set $\{(t_k, \tau_k)\} \in [0, 1] \times [0, 1]$ and it suffices to consider only one point of intersection $y_0 = \phi_{\tau_0}(t_0) = \phi(t_0, \tau_0) \in$ Sing$_i$. If we can find the required field $v_i$ on a transversal 2-dimensional disk at $y_0$, we are done.

As we explained in Section 2.3 at almost every point $y \in$ Sing$_i$ we have $\lambda_{i-1}(y) \neq \lambda_{i+1}(y)$ (Actually, for $j = 3$ this is true at every point.) Thus, without loss of generality, we can assume that $\lambda_{i-1}(y_0) \neq \lambda_{i+1}(y_0)$.

Assume $\lambda_{i-1}(y_0) \neq \lambda_i = \lambda_{i+1}(y_0)$, the case $\lambda_{i-1}(y_0) = \lambda_i \neq \lambda_{i+1}(y_0)$ is completely analogous.

Let $A = \{i, i+1\}$. Then $y_0 \in$ Reg$_A(M)$. Consider the leaf $M_A(y_0)$. This is a 2-dimensional manifold transverse to Sing$_i$ at $y_0$. The homotopy can be perturbed to have the image locally coinciding with $M_A(y_0)$. Since $v_i \in D_A$, the problem, thanks to Lemma 11, is reduced to a local 2-dimensional question on $M_A(y_0)$.

Consider the restriction of the metrics to $M_A(y_0)$. Denote by $L_A$ the tensor (11) constructed for the restrictions of the metrics. We denote by $\lambda_A \leq \lambda'_A$ its eigenvalues. By Lemma 11 $\lambda_A$ is constant, $\lambda'_A$ is not. If there exists a (continuous) vector field $v_A$ on $M_A$ such that it vanishes precisely at $y_0$, such that it is eigenvector of $L_A$ with eigenvalue $\lambda_A$, and such that its length is $\sqrt{\lambda'_A - \lambda_A}$, we are done. Indeed, by Lemma 11 the vector field $v_i$ given by

$$\sqrt{C^{-1/3}} \prod_{\alpha \neq i, i+1} (\lambda_i - \lambda_\alpha) v_A,$$

where $C$ is given by (9), satisfies the conditions (11). Since

$$\sqrt{C^{-1/3}} \prod_{\alpha \neq i, i+1} (\lambda_i - \lambda_\alpha)$$

is a smooth positive function, the existence of $v_A$ implies the existence of $v_i$.

Let us prove the existence of such vector field $v_A$. At every $y \in M_A(y_0), y \neq y_0$, denote by $l_A$ the eigenspace of $L_A$ corresponding to $\lambda_A$. Let us show that that for every geodesic $\gamma$ on $M_A(y_0)$ passing through $y_0$ the velocity vector $\dot{\gamma}(t)$ is orthogonal (in the restriction of $g$) to $l_A$ at every $\gamma(t) \neq y_0$. Indeed, let $I_A^\gamma(t)$ be the one-parametric family of the integrals from Theorem 12 constructed for the restrictions of $g$ and $\bar{g}$ to $M_A(y_0)$. Consider the integral $I_A^{\gamma}|_x$. At the tangent plane to every point $z$ consider the coordinates such that the restriction of $g$ to $M_A(y_0)$ is given by diag(1, 1) and $L_A$ is diag($\lambda_A, \lambda'_A$). In this coordinates, the integral $I_A^\gamma(t)$ equals $(\lambda_A - t)\xi_1^2 + (\lambda'_A - t)\xi_2^2$, so that $I_A^{\gamma}|_x$ is equal to $(\lambda'_A - \lambda_A)\xi_2^2$. We see that the integral vanishes on every geodesic $\gamma$ passing through $y_0$. Because $\lambda'_A(z) \neq \lambda_A(z)$ for $z \neq y_0$, we obtain that the component $\xi_1$ of the velocity vector of $\gamma$ at $z$ vanishes, which means that the eigenvalue of $L_A$ corresponding to $\lambda_A$ is orthogonal to $\gamma$.

Clearly, in $M_A(y_0) \setminus y_0$ there exists a vector field of length 1 such that it is orthogonal to the geodesics passing through $y_0$, see Figure 11.
Multiplying this vector field by $\sqrt{\lambda_i - \lambda_{\perp}}$, we obtain a required vector field $v_A$ on $M_A(y_0) \setminus y_0$. We put $v_A = 0$ at point $y_0$. Since $\sqrt{\lambda_i - \lambda_{\perp}}$ converges to 0 when $x$ tends to $y_0$, the result is a required continuous vector field $v_A$ on $M_A(y_0)$. Therefore, there exists a vector field $v_i$ along $M_A(y_0)$ (satisfying (11)). Thus, the vector $v_i$ at $x_1$ does not depend on the choice of path connecting $x_0$ and $x_1$. Finally, $v_i$ is well-defined at the whole $U^n \setminus \text{Sing}_i^n$, and is at least continuous on it.

At the points of $U^n \cap \text{Sing}_i^n$ let us put $v_i$ equal to zero. Since $\Pi_i(\lambda_i)$ tends to 0 when $x$ approaches $\text{Sing}_i^n$, the vector field is continuous on $U^n$. Statement (S) is proven.

Then, at least on the double cover of $M^n$, there exists a continuous vector field $v_i$ satisfying (10). Without loss of generality, we can assume that the vector field $v_i$ is defined already on $M^n$.

Now let us prove that the vector field $v_i$ is actually smooth. Clearly, it is smooth on the compliment to $\text{Sing}_i^n$, because it coincides with the appropriate field $\frac{\partial}{\partial x_i}$ there. Denote by $F_t$ the flow of the vector field $v_i$ on $M^n \setminus (\text{Sing}_i^n \cup \text{Sing}_j^n)$. This flow is globally (=for every value of $t$) defined. Indeed, if $x \notin \text{Sing}_i^n \cup \text{Sing}_j^n$, then $\lambda_{i-1}(x) < \lambda_i < \lambda_{i+1}(x)$. Since $v_i$ is an eigenvector of $L$ with eigenvalue $\lambda_i$ and the Nijenhuis tensor $N_L$ vanishes (Corollary 4), for every $t$ we have: $\lambda_{i-1}(F_t(x)) = \lambda_{i-1}(x)$, $\lambda_{i+1}(F_t(x)) = \lambda_{i+1}(x)$. Therefore, the trajectory of the flow passing through $x$ never approaches the set $\text{Sing}_i^n \cup \text{Sing}_j^n$.

The function $J(\eta) := g(v_i, \eta)$ is a linear in velocities integral of the geodesic flow, which implies that $F_t$ acts by isometries on $M^n \setminus (\text{Sing}_i^n \cup \text{Sing}_j^n)$. Since $M^n \setminus (\text{Sing}_i^n \cup \text{Sing}_j^n)$ is everywhere dense in $M^n$, the map $F_t$ can be extended by completeness to act by isometries on the whole $M^n$. Thus, there exists a Killing vector field on $M^n$ coinciding with $v_i$ almost everywhere. Since every Killing vector field is smooth, the vector field $v_i$ is smooth. Theorem 6 is proven.

4 Proof of Theorem 1

We use induction by the dimension. If dimension of the manifold is $n < 2$, Theorem 1 is trivial. Assume that for every dimension less than $n$ Theorem 1 is true and consider $\dim M = n$.

Vanishing of the topological entropy for the lift of a dynamical system to a finite cover (of a closed manifold) implies vanishing of the topological entropy of the original system. Thus, we assume that already on $M^n$ for every constant eigenvalue $\lambda_i$ we can associate a global vector field $v_i$ from Theorem 6. Therefore for every constant $\lambda_i$ we globally define the integral $J_t$ such that its differential does not vanish over the points of $\text{Reg}(M^n)$, it commutes with all integrals $I_t$, it is functionally dependent with the integral $I_{\lambda_i}$.

By geodesic flow we will understand the restriction of the Hamiltonian system on $TM^n$ with the Hamiltonian $H(\xi) := g(\xi, \xi)$ to $T_1M^n = \{ \xi \in TM^n : H(\xi) = 1 \}$. The symplectic form on
$TM^n$ came from $T^*M^n$ via standard identification by $g$.

Since $T_1M^n$ is compact, the variational principle (see, for example, Theorem 4.5.3 of [KH]) holds, and we obtain

$$h_{\text{top}}(g) = \sup_{\mu \in \mathcal{B}} h_\mu(g).$$

Here $\mathcal{B}$ is the set of all invariant ergodic probability measures on $T_1M^n$ and $h_\mu$ is the entropy of an invariant measure $\mu$. Recall that a measure is called ergodic, if $\mu(B)(1 - \mu(B)) = 0$ for all $\mu$-measurable invariant Borel sets $B$.

Therefore, in order to prove Theorem [A] it is sufficient to prove that $h_\mu(g) = 0$ for all $\mu \in \mathcal{B}$. Fix one such measure and let $\text{Supp}(\mu)$ be its support (the set of $x \in M^n$ such that every neighborhood $U_i(x)$ has positive measure).

Since the measure is ergodic, its support lies on a level surface of every invariant continuous function. Then, $\text{Supp}(\mu)$ is included into a Liouville leaf $\Upsilon$ (Recall that a Liouville leaf is a connected component of the set $\{I_1 = c_1, \ldots, I_n = c_n\}$, where $c_1, \ldots, c_n$ are constants.)

Suppose a point $\xi \in \text{Supp}(\mu)$ is nonsingular, or is a removable singular point (in the sense that every $I_k$ such that $dI_k \neq 0$). Then, a small neighborhood $U(\xi)$ of $\xi$ in $\text{Supp}(\mu)$

- has positive measure in $\mu$,
- contains only points that are nonsingular or removable-singular.

We will show that these two conditions imply that the entropy of $\mu$ is zero.

By implicit function Theorem, $\Upsilon$ is $n$-dimensional near $\xi$. Denote by $O(\xi)$ the orbit of the Poisson action of $T^\times M^n$ containing $\xi$. Since it is also $n$-dimensional, in a small neighborhood of $\xi$ it coincides with $\Upsilon$. Thus, $U(\xi) \subset O(\xi)$.

The orbits of the Poisson action and the dynamic on them are well-studied (see, for example, §49 of [A]). There exists a diffeomorphism to $T^k \times \mathbb{R}^{n-k} = S^1 \times \ldots \times S^1 \times \mathbb{R} \times \ldots \times \mathbb{R}$

with the standard coordinates $\phi_1, \ldots, \phi_k \in (\mathbb{R} \mod 2\pi)$, $t_{k+1}, \ldots, t_n \in \mathbb{R}$ such that in these coordinates (the push-forward of) every trajectory of the geodesic flow is given by the formula

$$\begin{align*}
(\phi_1(\tau), \ldots, \phi_k(\tau), t_{k+1}(\tau), \ldots, t_n(\tau)) &= (\phi_1(0) + \omega_1 \tau, \ldots, \phi_k(0) + \omega_k \tau, t_{k+1}(0) + \omega_{k+1} \tau, \ldots, t_n(0) + \omega_n \tau),
\end{align*}$$

where the constants $\omega_1, \ldots, \omega_n$ are universal on $T^k \times \mathbb{R}^{n-k}$.

We see that if at least one of the constants $\omega_{k+1}, \ldots, \omega_n$ is not zero, every point of $U(\xi)$ is wandering in $\text{Supp}(\mu)$ (see §3 in Chapter 3 of [KH] for definition), which contradicts the invariance of the measure. Then, the entropy of $\mu$ is zero.

If all constants $\omega_{k+1}, \ldots, \omega_n$ are zero, the coordinates $t_{k+1}, \ldots, t_n$ are constants on the trajectories of the geodesic flow. Since $\mu$ is ergodic, they are constant on the points of $\text{Supp}(\mu)$. Then, $\text{Supp}(\mu)$ is (diffeomorphic to) the torus $T^k$ of dimension $k \leq k$, and the dynamics on $\text{Supp}(\mu)$ is (conjugate to) the linear flow on $T^k$. Then, the entropy of $\mu$ is zero, see for example Proposition 3.2.1 of [KH].

Now suppose that $\text{Supp}(\mu)$ contains only singular points which are not removable. If all of them belong to $\bigcup_{i,j} T\text{Sing}^j$, then (because the measure is ergodic) $\text{Supp}(\mu)$ is a subset of a certain $T\text{Sing}^j$. Since $\text{Sing}^j$ is totally geodesic, and since by induction hypothesis the topological entropy on $\text{Sing}^j$ is zero, the entropy of $\mu$ is also zero.

The last case is when $\text{Supp}(\mu)$ contains a singular point which is not removable and which does not belong to $\bigcup_{i,j} T\text{Sing}^j$. Then, since all $\text{Sing}^j$ are totally geodesic, and since there are finitely many of them, $\text{Supp}(\mu)$ contains a singular point $\xi$ which is not removable and such that its projection does not belong to $\bigcup_{i,j} \text{Sing}^j$. Then, the projection of a small neighborhood $U(\xi) \subset \text{Supp}(\mu)$ of $\xi$ does not contain points of $\bigcup_{i,j} \text{Sing}^j$. 

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From Theorems $5$ and $6$ it follows, that for certain $\lambda_i$ such that $\lambda_i$ is not constant the differentials of $I_{\lambda_i}$ vanish at $\xi$. Since the number of such $\lambda_i$ is finite, and since the measure is ergodic, we obtain that there exists $i$ such that

- $dI_{\lambda_i} = 0$ at every point of $\text{Supp}(\mu)$,
- the eigenvalue $\lambda_i$ satisfies the assumptions of Lemma $2$ (Otherwise the singularity is removable or $\xi$ lies in $\cup_i T\text{Sing}_i$.)

Hence, by Lemma $2$ for every point $y$ from the projection of $U(\xi)$ we have that $M_{C(i)}(y)$ is compact. Then, by Theorem $5$ for every $\eta \in U(\xi)$, the projection of the trajectory of the geodesic flow passing through $\eta$ stays on the corresponding $M_{C(i)}$. Since all $M_{C(i)}$ passing through the projection of $U(\xi)$ are compact and do not intersect one another, a trajectory staying in one $T_1M_{C(i)}$ never approaches another $T_1M_{C(i)}$. Thus, since $\mu$ is ergodic, all points of $\text{Supp}(\mu)$ belong to a certain $T_1M_{C(i)}(y)$. Then, the dynamics on $\text{Supp}(\mu)$ is a subsystem of the geodesic flow for the restriction of $g$ to $M_{C(i)}(y)$. (Indeed, if a geodesic of a metric lies on a submanifold, then it is a geodesic in the restriction of the metric to the submanifold.) Finally, by induction assumptions, the entropy of $\mu$ is zero.

Thus, for every ergodic probabilistic invariant measure $\mu$ its entropy is zero. Finally, the topological entropy is zero. Theorem $4$ is proven.

5 Topological restrictions for manifolds with infinite fundamental group: announcement

Theorem 7. Suppose the Riemannian metrics $g$ and $\bar{g}$ on a closed connected manifold $M^n$ are geodesically equivalent and strictly non-proportional at least at one point. Then some finite cover of $M^n$ is diffeomorphic to the product $Q^k \times T^{n-k}$ of a rational-elliptic manifold and the torus.

The proof of this theorem is lengthy and will appear elsewhere (for small dimensions, in view of Theorem $4$, Theorem $7$ follows from $[PP2]$). Here we sketch the proof only. It uses Corollary $4$ methods developed in $[M1, M3]$ and classical results of $[CG]$.

In $[M1]$, it was shown that if a manifold with non-proportional geodesically equivalent metrics has an infinite fundamental group, it admits a local product structure (= a new Riemannian metric and two orthogonal foliations of complementary dimensions $B_k$ and $B_{n-k}$ such that in a small neighborhood of almost every point all three object look as they come from the Riemannian product of two Riemannian manifolds). In $[MG]$ (see Lemma 2 there), it was shown that (assuming that the initial metrics $g$ and $\bar{g}$ are strictly non-proportional at least at one point), the restriction of the local-product metric to the leaves of the foliations admits a metric which is geodesically equivalent to it and strictly non-proportional to it at almost every point. By applying the same construction to the leaves, we obtain that $M^n$ admits a Riemannian metric $h$ and $m$ orthogonal foliations $B_{k_1}, B_{k_2}, ..., B_{k_m}$ of complementary dimension $k_1 + k_2 + ... + k_m = n$ such that

- the restriction of the metric $h$ to $B_{k_1}$ is flat,
- the leaves of $B_{k_2}, B_{k_3}, ..., B_{k_m}$ are compact and have finite fundamental group (this is actually the lengthy part of the proof; its proof it similar to the proof of Theorem 2 from $[M1]$), but one can not apply Theorem 2 from $[M1]$ directly and should essentially repeat all steps of its proof in a slightly different setting.)
- the restriction of $h$ to each of $B_{k_2}, B_{k_3}, ..., B_{k_m}$ admits a metric which is geodesically equivalent to it and is strictly non-proportional to it at least at one point.
- locally, in a neighborhood of every point, the metric $h$ and the foliations $B_{k_i}$ look as they (simultaneously) came from the direct product of $m$ Riemannian manifolds.

Then, by Corollary $4$, the universal cover of $B_{k_2} \times B_{k_3} \times \times B_{k_m}$ is rational elliptic, and Theorem $7$ follows from Theorem 9.2 of $[CG]$. 

13
6 Vanishing of the entropy pseudonorm: announcement

An action \( \Phi : (\mathbb{R}^n, +) \to \text{Diff}(W) \) determines the following entropy pseudonorm

\[ \rho_\Phi(v) := h_{\text{top}}(\Phi(v)). \]

The triangle inequality is based on the Hu’s formula [H].

In particular, for the Poisson action \( \Phi : (\mathbb{R}^n, +) \to \text{Symp}(W^{2n}, \omega) \) associated with a Liouville-integrable Hamiltonian system one gets a certain pseudonorm \( \rho_\Phi : \mathbb{R}^n \to \mathbb{R} \geq 0 \).

This pseudonorm is degenerate for most examples of integrable geodesic flows with positive entropy \( (W^{2n} = TM^n) \), but it is possible to construct a Liouville-integrable Hamiltonian system such that \( \rho_\Phi \) is a norm [K].

**Theorem 8.** Suppose the Riemannian metrics \( g \) and \( \bar{g} \) on a closed connected manifold \( M^n \) are geodesically equivalent and strictly non-proportional at least at one point. Let \( \Phi \) be the Poisson action constructed by the integrals \( I_{t_1}, \ldots, I_{t_n} \), where the numbers \( t_i \) are mutually different. Then, \( \rho_\Phi(v) = 0 \) for every \( v \in \mathbb{R}^n \).

The proof of this theorem will be published elsewhere.

References

[A] V. I. Arnold, *Mathematical methods of classical mechanics*, Nauka, Moscow; Engl. transl.: Graduate Texts in Mathematics, Springer (1989).

[Be] S. Benenti, *An outline of the geometrical theory of the separation of variables in the Hamilton-Jacobi and Schrödinger equations*, SPT 2002: Symmetry and perturbation theory (Cala Gonone), 10–17, World Sci. Publishing, River Edge, NJ, 2002.

[Bo] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. A.M.S. 153 (1971), 401–414.

[BP] L. T. Butler, G. P. Paternain, *Collective geodesic flows*, Ann. Inst. Fourier (Grenoble) 53 (2003), no. 1, 265–308.

[CST] M. Crampin, W. Sarlet, G. Thompson, *Bi-differential calculi, bi-Hamiltonian systems and conformal Killing tensors*, J. Phys. A 33 (2000), no. 48, 8755–8770.
[D] E.I. Dinaburg, *Connection between various entropy characteristics of dynamical systems*, Izv. Akad. Nauk SSSR, Ser. Mat. 35 (1971), 324–366; Engl. Transl. in Math. USSR Izv. 5 (1971), 337–378.

[Di] U. Dini, *Sopra un problema che si presenta nella teoria generale delle rappresentazioni geografiche di una superficie su un’altra*, Ann. Mat., ser. 2, 3(1869), 269–293.

[E] L.H. Eliasson, *Normal forms for Hamiltonian systems with Poisson commuting integrals. Elliptic case*, Comment. Math. Helv. 65 no.1, (1990), 4–35.

[FHT] Y. Felix, S. Halperin, J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, 205. Springer-Verlag, New York, 2001.

[G] M. Gromov, *Entropy, homology and semi-algebraic geometry*, Sém. Bourbaki, vol. 1985/86, Astérisque 145-146 (1987), 225–240.

[H] Y. Hu, *Some ergodic properties of commuting diffeomorphisms*, Ergod. Th. & Dynam. Sys. 13, no. 1 (1993), 73–100.

[IMM] A. Ibort, F. Magri, G. Marmo, *Bihamiltonian structures and Stäckel separability*, J. Geom. Phys. 33(2000), no. 3–4, 210–228.

[I] H. Ito, *Action–angle coordinates at singularities for analytic integrable systems*, Math. Z. 206 (1991), 363–407.

[KH] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Math. and its Appl. 54, Cambridge University Press, Cambridge (1995).

[KT] A. Knauf, I. A. Taimanov, *On the integrability of the n-centre problem*, preprint: ArXiv.org/math.DS/0401202.

[K] B. Kruglikov, *Examples of integrable sub-Riemannian geodesic flows*, Jour. Dynam. Contr. Syst. 8(2002), no. 3, 323–340.

[LC] T. Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche*, Ann. di Mat., serie 2, 24(1896), 255–300.

[Ma] R. Mañé, *Ergodic theory and differentiable dynamics*, Springer-Verlag (1987).

[Mn] A. Manning, *Topological entropy for geodesic flows*, Ann. of Math. (2), 110(1979), no. 3, 567–573.

[MT1] V. S. Matveev, P. J. Topalov, *Trajectory equivalence and corresponding integrals*, Regular and Chaotic Dynamics, 3 (1998) no. 2, 30–45.

[MT2] V. S. Matveev and P. J. Topalov, *Metric with ergodic geodesic flow is completely determined by unparameterized geodesics*, ERA-AMS, 6(2000), 98–104.

[MT3] V. S. Matveev, P. J. Topalov, *Quantum integrability for the Beltrami-Laplace operator as geodesic equivalence*, Math. Z. 238(2001), 833–866.

[M1] V. S. Matveev, *Hyperbolic manifolds are geodesically rigid*, Invent. Math. 151 (2003), 579-609.

[M2] V. S. Matveev, *Three-dimensional manifolds having metrics with the same geodesics*, Topology 42(2003) no. 6, 1371-1395.

[M3] V. S. Matveev, *Projectively equivalent metrics on the torus*, Diff. Geom. Appl. 20(2004), 251-265.

[M4] V. S. Matveev, *Projective Lichnerowicz-Obata conjecture*, preprint: ArXiv.org/math.DG/0407337.

[Mi] J. Mikes, *Geodesic mappings of affine-connected and Riemannian spaces. Geometry, 2.*, J. Math. Sci. 78(1996), no. 3, 311–333.

[P0] G. Paternain, *Entropy and completely integrable Hamiltonian systems*, Proc. Amer. Math. Soc. 113(1991), no. 3, 871–873.
[P1] G. Paternain, *On the topology of manifolds with completely integrable geodesic flows*, I: Ergod. Th. & Dynam. Sys. 12(1992), 109–121; II: Journ. Geom. Phys. 123(1994), 289–298.

[P2] G. Paternain, *Geodesic flows*, Birkhäuser (1999).

[PP1] G. P. Paternain, J. Petean, *Minimal entropy and collapsing with curvature bounded from below*, Invent. Math. 151 (2003), no. 2, 415–450.

[PP2] G. P. Paternain, J. Petean, *Zero entropy and bounded topology*, preprint: ArXiv.org/math.DG/0406051.

[Pa] A.V. Pavlov, *Estimates for the Betti numbers of rationally-elliptic spaces*, Siberian Math. J. 43(2002), no. 6, 1080–1085.

[T] I.A. Taimanov, *Topology of Riemannian manifolds with integrable geodesic flows*, Proc. Steklov Inst. Math. 205 (1995), 139–150.

[Y] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. 57 (1987), no. 3, 285–300.