TAU FUNCTIONS, HODGE CLASSES AND DISCRIMINANT LOCI ON
MODULI SPACES OF HITCHIN’S SPECTRAL COVERS

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Abstract. We define two tau functions, $\tau$ and $\hat{\tau}$, on moduli spaces of spectral covers of $GL(n)$ Hitchin’s systems. Analyzing the properties of $\tau$, we express the divisor class of the universal Hitchin’s discriminant in terms of standard generators of the rational Picard group of the moduli spaces of spectral covers with variable base. The function $\hat{\tau}$ is used to compute the divisor of canonical 1-forms with multiple zeros.

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1. Introduction

Yang-Mills equations have a deep connection to the theory of integrable systems: most of soliton equations are dimensional reductions of the self-dual Yang-Mills equation (SDYM). In the pioneering work [17][18], N. Hitchin proposed a dimensional reduction of SDYM by splitting 4-dimensional space into the product of a Riemann surface $\Sigma$ and the real plane $\mathbb{R}^2$, where the gauge fields are assumed to be independent of coordinates on $\mathbb{R}^2$. As a result of such dimensional reduction, one arrives at the class of finite-dimensional completely integrable systems, called Hitchin’s systems; we refer to Atiyah’s book [2] (Sect. 6.3) for an introduction to the topic and to the original papers [17][18] and reviews [10][11] for more detailed description of the subject. Hitchin’s systems, as well as their generalizations to the meromorphic case [20], provide the most general class of integrable systems associated to Riemann surfaces of an arbitrary genus.
Let \( \Sigma \) be a Riemann surface (smooth complex curve) of genus \( g \geq 1 \). The Hamiltonians of Hitchin’s system are encoded in the so-called spectral cover \( \hat{\Sigma} \) which is an \( n \)-sheeted cover of \( \Sigma \) defined by the equation in \( T^* \Sigma \)

\[
\hat{\Sigma} = \{(x, v) \in \Sigma \times T^* \Sigma \mid P_n(v) = 0\}
\tag{1.1}
\]

where

\[
P_n(v) = v^n + Q_1 v^{n-1} + \cdots + Q_{n-1} v + Q_n,
\]

\(Q_k\) is a holomorphic \( k\)-differential on \( \Sigma \), and \( v \) is considered as a holomorphic 1-form on \( \hat{\Sigma} \). In the framework of [17] the equation (1.1) is given by the characteristic polynomial \( P_n(v) = \det(\Phi - vI) \) of the so-called Higgs field \( \Phi \) on \( \Sigma \).

For the most general case of \( GL(n) = GL(n, \mathbb{C}) \) Hitchin’s systems all differentials \( Q_k \) from (1.1) are arbitrary; in the case of \( SL(n) \) systems \( Q_1 = 0 \). In this paper we mainly consider the generic case of \( GL(n) \) systems although most of the formulas are applicable to the \( SL(n) = SL(n, \mathbb{C}) \) case without modification.

The branch points of the cover \( \hat{\Sigma} \) are the zeros of the discriminant \( W \) of \( P_n(v) \) that coincides with the resultant of \( P_n \) and \( P'_n \) up to a sign:

\[
W = \text{Discr} (P_n) = (-1)^{\frac{n(n-1)}{2}} \text{Res} (P_n, P'_n).
\tag{1.2}
\]

It is easy to verify that the discriminant \( W \) is a holomorphic \( n(n-1) \)-differential on \( \Sigma \). Thus, the number \( m \) of zeros of \( W \), counted with multiplicities, equals

\[
m = n(n-1)(2g-2),
\tag{1.3}
\]

and the Riemann-Hurwitz formula gives the genus \( \hat{g} \) of \( \hat{\Sigma} \):

\[
\hat{g} = n^2(g-1) + 1
\tag{1.4}
\]

so that \( m = 2(\hat{g} - 1 - n(g-1)) \). When all zeros of \( W \) are simple, all branch points of the cover \( \hat{\pi} : \hat{\Sigma} \rightarrow \Sigma \) are also simple. In the simplest case of (1.1) \( n = 2 \); in particular, for \( SL(2) \) we have \( Q_1 = 0 \) and the equation (1.1) takes the form

\[
v^2 + Q_2 = 0.
\tag{1.5}
\]

where \( Q_2 \) is a holomorphic quadratic differential. The cover defined by (1.5) is sometimes called a “canonical cover” [1, 12]. The genus of (1.5) equals \( 4g - 3 \) (assuming that all zeros of \( Q_2 \) are simple) and the dimension of the moduli space of curves (1.5) equals \( 6g - 6 \) when the base curve \( \Sigma \) is also allowed to vary. Since the space of covers (1.5) forms an open subspace in the cotangent bundle \( T^* \mathcal{M}_g \) of the moduli space \( \mathcal{M}_g \) of curves of genus \( g \), it possesses a canonical symplectic structure. This symplectic structure, including a natural system of period, or homological, Darboux coordinates was studied in detail in the recent paper [8].

An immediate generalization of (1.5) is given by the family of \( \mathbb{Z}_n \)-invariant covers

\[
v^n + Q_n = 0,
\tag{1.6}
\]
where $Q_n$ is a holomorphic $n$-differential and $Z_n = \mathbb{Z}/n\mathbb{Z}$. The genus of the covering (1.6) is also given by (1.3), but this case is far from being generic since all ramification points of (1.6) are of order $n$. The moduli space of $Z_n$-covers (1.6) was studied in [29], see also [4].

The goal of this paper is to extend some of the results about the moduli spaces of $Z_n$-covers to the moduli spaces of Hitchin’s generic $GL(n)$ covers (1.1).

In particular, we generalize the theory of tau functions (which can be considered as a higher genus generalizations of Dedekind’s eta function) to the moduli spaces of Hitchin’s covers. For moduli of $Z_n$-curves this was done in [28] (for $n = 2$) and in [29] (for $n > 2$), using the approach developed earlier in [25, 26, 27].

In particular, the Bergman tau functions (called so due to their close ties to the Bergman projective connection) allowed to find new relations in the Picard groups of these moduli spaces.

We also notice that the tau functions we discuss here can be interpreted as determinants of appropriate $\bar{\partial}$-operators in the spirit of [37]; they also have close relations to conformal field theory [31], isomonodromic deformations [21, 35, 22, 33] and Frobenius manifolds [13, 14, 23, 24].

**Spaces of coverings with fixed base.** Let $\Sigma$ be a smooth curve of genus $g$ and denote by $M_{\Sigma}$ the moduli space of $GL(n)$ spectral covers of the form (1.1). Then

$$M_{\Sigma} = \bigoplus_{j=1}^{n} H^0(\Sigma, K_{\Sigma}^{\otimes j})$$

(1.7)

where $K_{\Sigma} = T^*\Sigma$ is the canonical line bundle on $\Sigma$, and

$$\dim M_{\Sigma} = \hat{g} = n^2(g-1) + 1$$

(1.8)

(recall that $\dim H^0(\Sigma, K) = g$ and $\dim H^0(\Sigma, K_{\Sigma}^{\otimes j}) = (2j-1)(g-1)$ for $j \geq 2$).

There is a natural coordinate system on $M_{\Sigma}$ given by the $a$-periods of $v$:

$$P_j = \int_{\hat{a}_j} v$$

(1.9)

where \{\hat{a}_j, \hat{b}_j\}_{j=1}^{\hat{g}}$ is a canonical sympletic basis in $H_1(\hat{\Sigma}, \mathbb{Z})$.

We consider the following two codimension 1 loci in $M_{\Sigma}$ – the locus $D_W$ of sets $Q_1, \ldots, Q_n$ of differentials such that the discriminant $W$ of $P_n(v)$ has multiple zeroes, and the locus $D_V$ of sets $Q_1, \ldots, Q_n$ such that the Abelian differential $v$ on $\hat{\Sigma}$ has multiple zeroes. The locus $D_W$ is called the **Hitchin’s discriminant locus**, whereas we call $D_V$ the locus of **degenerate spectral covers**. For a generic point in $M_{\Sigma}$, that is a point in the complement $M_{\Sigma} \setminus (D_W \cup D_V)$, all zeros of the discriminant $W$ and all zeros of the Abelian differential $v$ on $\hat{\Sigma}$ are simple.

Consider also the space $M_{\Sigma}^W = M_{\Sigma} \setminus D_W$ of spectral covers with simple branch points and the space $M_{\Sigma}^V = M_{\Sigma} \setminus D_V$ of covers with simple zeros of $v$.

**Spaces of covers with variable base.** Let $\overline{M}_g$ be the Deligne-Mumford compactification of the moduli space of curves $M_g$, let $\nu : \overline{\mathcal{T}}_g \to \overline{M}_g$ be the universal curve, and let
\( \omega_g = \omega_{\overline{g}/\mathcal{M}_g} \) be the relative dualizing sheaf. Put
\[
\overline{\mathcal{M}} = \bigoplus_{j=1}^{n} \Omega_g^{(j)},
\]
where \( \Omega_g^{(j)} = R^0 \nu_* \omega_g^{\otimes j} \) is the direct image of the \( j \)th power of \( \omega_g \). We have
\[
\dim \mathcal{M} = \dim \mathcal{M}^\Sigma + 3g - 3 = (n^2 + 3)(g - 1) + 1
\]
(1.10)

There is also a natural forgetful map \( \overline{\mathcal{M}} \to \mathcal{M}_g \) such that the fiber over \( \Sigma \in \mathcal{M}_g \) coincides with \( \mathcal{M}_\Sigma \) (the fibers over nodal curves are described in detail in [29]). Denote by \( D_W \) and \( D_v \) respectively the unions of loci \( D_{\Sigma W} \) and \( D_{\Sigma v} \) as \( \Sigma \) varies over the entire moduli space \( \mathcal{M}_g \).

There is a natural action of \( \mathbb{C}^* \) on \( \overline{\mathcal{M}} \) that fiberwise looks like \( Q_j \mapsto \epsilon Q_j, \epsilon \in \mathbb{C}^*, j = 1, \ldots, n \), and respects the codimension one loci \( D_W \) and \( D_v \). After projectivization both \( PD_W \) and \( PD_v \) become divisors in \( \widetilde{P\mathcal{M}} \). There is a natural forgetful map \( h : \widetilde{P\mathcal{M}} \to \mathcal{M}_g \).

The main goal of this paper is to express the class of divisor \( [PD_W] \) in terms of the standard generators of the rational Picard group \( \text{Pic}(P\overline{\mathcal{M}}) \otimes \mathbb{Q} \). We also express the class \( [PD_v] \) via the natural divisorial classes on \( P\overline{\mathcal{M}} \).

1.1. Components of the universal discriminant locus. Assume that two simple zeros \( x_1 \) and \( x_2 \) of \( W \) coalesce to a double zero on \( \Sigma \). To describe possible deformations of the cover \( \hat{\Sigma} \), choose a system of generators \( \{\alpha_i, \beta_i\}_{i=1}^{g} \), \( \{\gamma_i\}_{i=1}^{m} \) of \( \pi_1(\Sigma \setminus \{x_j\}_{j=1}^{m}, x_0) \) satisfying the standard relation
\[
\prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \gamma_1 \ldots \gamma_m = id.
\]
(1.13)

The covering \( \hat{\Sigma} \) defines the group homomorphism \( G : \pi_1(\Sigma \setminus \{x_j\}_{j=1}^{m}) \to S_n \), the permutation group of \( n \) elements. Let \( s_1 = G(\gamma_1) \) and \( s_2 = G(\gamma_2) \). When all zeros of \( W \) (i.e. branch points of the covering \( \hat{\Sigma} \)) are simple, both \( s_1 \) and \( s_2 \) are simple permutations. As \( x_2 \to x_1 \), the covering \( \hat{\Sigma} \) degenerates to a covering \( \hat{\Sigma}_0 \) whose structure depends on the type of the product \( s_1 s_2 \).

Consider a neighborhood \( U \) of \( \Sigma \) containing both \( x_1 \) and \( x_2 \), and introduce a local coordinate \( z \) in \( U \). Let \( z_1 = z(x_1) \) and \( z_2 = z(x_2) \). There are three patterns of local behavior of \( x_1 \) and \( x_2 \) that correspond to three different components of \( D_W \). We will use the terminology of [34]:

1. The “boundary” \( D_W^{(b)} \).

In this case \( s_1 s_2 \) is a trivial permutation. In the limit \( z_1 \to z_2 \) the spectral cover \( \hat{\Sigma} \) acquires a node (double point) while approaching the (Deligne-Mumford) boundary of \( M_g \). Since the order of points \( x_1 \) and \( x_2 \) is irrelevant, a transversal local coordinate on \( \mathcal{M} \) near \( D_W^{(b)} \) can be chosen as
\[
t_b = (z_1 - z_2)^2.
\]
(1.14)
2. The “Maxwell stratum” $D_W^{(m)}$.

In this case $s_1s_2$ is a product of two cycles of length 2, i.e. the ramification points of $\hat{\Sigma}$ remain simple, but two of them correspond to the same critical value $x_1 \in \Sigma$. Then, since a point in the Maxwell stratum splits into two simple critical values in two ways, a transversal local coordinate on $\mathcal{M}$ near $D_W^{(m)}$ can be chosen as

$$t_m = z_1 - z_2.$$  \hspace{1cm} (1.15)

3. The “caustic” $D_W^{(c)}$.

In this case $s_1s_2$ is a cycle of length 3, i.e. as $x_2 \to x_1$ the cover $\hat{\Sigma}$ acquires a ramification point of order 3. It can be decomposed into a product of two transpositions in 3 different ways, a transversal local coordinate on $\mathcal{M}$ near $D_W^{(c)}$ can be chosen as

$$t_c = (z_1 - z_2)^{2/3}.$$  \hspace{1cm} (1.16)

The transversal local coordinates $t_b$, $t_m$, and $t_c$ can be specified further as follows. Let us choose the coordinate $z \in U$ in such a way that the discriminant $W$ is given by

$$W = (z - z_1)(z - z_2)dz^{n(n-1)}.$$  \hspace{1cm} (1.17)

Put $w = W^{1/(n-1)}$; then, up to a multiplicative constant,

$$\int_{x_1}^{x_2} w \sim (z_1 - z_2) \frac{n(n-1)}{n(n-1)+2} \left(\int_{x_1}^{x_2} w\right)^{n(n-1)+2}.$$  \hspace{1cm} (1.18)

As one can see from the above considerations, the universal Hitchin’s discriminant locus $D_W$ splits into 3 components:

$$[D_W] = [D_W^{(b)}] + 2[D_W^{(m)}] + 3[D_W^{(c)}].$$  \hspace{1cm} (1.19)

This splitting respects the action of $\mathbb{C}^*$ on $\overline{\mathcal{M}}$ and descends to the projectivizations of these divisors.

2. Tau functions on spaces of Abelian and higher order differentials

Here we summarize previously known results from $[25, 27, 28, 29]$.  

2.1. Preliminaries. For a Torelli marked Riemann surface $\Sigma$ of genus $g$ introduce the canonical bidifferential $B(x, y)$, $x, y \in \Sigma$, which has the quadratic pole with biresidue 1 on the diagonal and vanishing a-periods. The bidifferential $B$ is expressed via the the prime-form $E(x, y)$ as follows: $B(x, y) = d_x d_y \log E(x, y)$ (see $[15, 36]$ for details). Consider a basis of holomorphic differentials $v_i$ on $\Sigma$ normalized by $\int_{a_\alpha} v_\beta = \delta_{\alpha\beta}$. The period matrix $\Omega$ of $\Sigma$ is given by: $\Omega_{ij} = \int_{b_i} v_j$. 
In a local coordinate $\xi$ near the diagonal $\{x = y\} \subset \Sigma \times \Sigma$, the bidifferential $B(x, y)$ has the expansion

$$B(x, y) = \left( \frac{1}{(\xi(x) - \xi(y))^2} + \frac{S_B(\xi(x))}{6} + O((\xi(x) - \xi(y))^2) \right) d\xi(x) d\xi(y),$$

(2.1)

where $S_B$ is a projective connection on $\Sigma$ called the Bergman projective connection.

If two canonical bases of cycles on $\Sigma$, $\{a_\alpha', b_\alpha'\}_{\alpha=1}^g$ and $\{a_\alpha, b_\alpha\}_{\alpha=1}^g$ are related by a matrix

$$\sigma = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2g, \mathbb{Z}),$$

(2.2)

then the corresponding canonical bidifferentials are related as follows (p. 21 of [15]):

$$B^\sigma(x, y) = B(x, y) - 2\pi \sqrt{-1} \sum_{i, j=1}^g (C\Omega + D)^{-1}_{ij} v_i(x) v_j(y).$$

(2.3)

The Abel map is defined by $A^\alpha_{x_0}(x) = \int_{x_0}^x v_i$. Let us also define

$$C(x) = \frac{1}{W(x)} \left( \sum_{i=1}^g v_i(x) \frac{\partial}{\partial v} \right)^g \Theta(v; \Omega) \big|_{v=K^x},$$

(2.4)

where

$$W(x) = \det \left( v_i^{(j-1)}(x) \right)_{i, j=1}^g$$

(2.5)

is the Wronskian determinant of the basic holomorphic differentials, $\Theta$ is the theta function and $K^x$ is the vector of Riemann constants with base point $x$. The expression (2.4) is a multi-valued $g(1 - g)/2$-differential on $\Sigma$ which does not have any zeros or poles [16]. In the case of genus 1 the $x$-dependence in (2.4) drops out and $C(x)$ turns into $\Theta'((\Omega + 1)/2)$.

**2.2. Spaces of holomorphic Abelian differentials.** Denote by $\mathcal{H}_g$ the moduli space of pairs $(\Sigma, v)$ where $\Sigma$ is a Riemann surface of genus $g$ and $v$ is a holomorphic differential on $\Sigma$; clearly $\dim \mathcal{H}_g = 4g - 3$. The space $\mathcal{H}_g$ can be stratified according to multiplicities of zeros of $v$: for any partition $[k_1, \ldots, k_M]$ of $2g - 2$ denote by $\mathcal{H}_g(k_1, \ldots, k_M)$ the moduli space of pairs such that multiplicities of zeros $y_1, \ldots, y_M$ of $v$ are given by $\{k_i\}_{i=1}^M$. Then $\dim \mathcal{H}_g(k_1, \ldots, k_M) = 2g + M - 1$ and a system of period, or homological, coordinates on $\mathcal{H}_g(k_1, \ldots, k_M)$ can be obtained by integrating $v$ over a system of generators in the relative homology group $\mathcal{H}_1(\Sigma, \{y_i\}_{i=1}^M)$, see [32] for details. A natural choice of generators in this homology group is

$$\{s_1, \ldots, s_{2g+M-1}\} = \{a_1, \ldots, a_g, b_1, \ldots, b_g, l_2, \ldots, l_M\}$$

(2.6)

where $\{a_j, b_j\}_{j=1}^g$ is a canonical basis of cycles on $\Sigma$ and $l_j$ is a path connecting the $y_1$ with $y_j$.

Periods of $v$ along the cycles (2.6) give a system of local coordinates on the stratum $\mathcal{H}_g(k_1, \ldots, k_M)$:

$$P_{s_i} = \int_{s_i} v, \quad i = 1, \ldots, 2g + M - 1.$$  

(2.7)
The dual basis of cycles in $H_1(\Sigma \setminus \{y_i\})$ is defined by
\[
\{s_1^*, \ldots, s_{2g-M+1}^*\} = \{-b_1, \ldots, -b_g, a_1, \ldots, a_g, c_1, \ldots, c_M\}
\]
(2.8)
where $c_j$ is a small positively oriented circle around $y_j$, so that $s_i^* \circ s_j = \delta_{ij}$ (the symbol $\circ$ denotes here the intersection pairing of 1-cycles).

The differential $v$ gives rise to a natural coordinate on $\Sigma$. Pick a fundamental polygon $\Sigma_0$ of $\Sigma$ and put
\[
z(x) = \int_{y_1}^x v.
\]
(2.9)
The (multivalued) coordinate $z$ is defined on $\Sigma$ everywhere except the zeros $y_i$; near $y_i$ the local coordinate, called distinguished, is given by
\[
\zeta_i(x) = \left(\int_{y_i}^x v\right)^{1/(k_i+1)}.
\]
(2.10)

Tau functions on strata of moduli spaces of holomorphic abelian differentials were introduced in [25], by generalizing the notion of isomonodromic Jimbo-Miwa tau function for Riemann-Hilbert problems [33, 24]. The tau function $\tau(\Sigma, v)$ is defined on the stratum $H_g(k_1, \ldots, k_M)$ by the system
\[
\frac{\partial \ln \tau(\Sigma, v)}{\partial P_{s_i}} = -\frac{1}{2\pi \sqrt{-1}} \int_{s_i^*} B_{\text{reg}}^r v, \quad i = 1, \ldots, 2g + M - 1
\]
(2.11)
where
\[
B_{\text{reg}}^r(x) = \left.(B(x, y) - \frac{v(x)y(y)}{\int_{y}^x v^2})\right|_{y=x}.
\]
(2.12)

Introduce two vectors $r$ and $s$ such that
\[
A_x((v)) + 2K^x + \Omega r + s = 0.
\]
(2.13)
Put
\[
E(x, y_i) = \lim_{y \rightarrow y_i} E(x, y)\sqrt{d\zeta_i(y)},
\]
(2.14)
and
\[
E(y_i, y_j) = \lim_{x \rightarrow y_i, y \rightarrow y_j} E(x, y)\sqrt{d\zeta_i(x)\sqrt{d\zeta_j(y)}}
\]
(2.15)
where $\zeta_i$ is the distinguished local parameter (2.10) on $\Sigma$ near $y_i$. Then the solution of the system (2.11) looks as follows (see [25] for the proof):
\[
\tau(\Sigma, v) = C^{2/3}(x) \left(\frac{v(x)}{\prod_{i=1}^{M} E^{k_i}(x, y_i)}\right)^{(g-1)/3} \left(\prod_{i<j} E(y_i, y_j)^{k_i k_j}\right)^{1/6} e^{-\frac{2\pi}{6} \langle \Omega r, r \rangle - \frac{2\pi}{3} \langle r, K^x \rangle}
\]
(2.16)
Under the change of Torelli marking of $\Sigma$ given by symplectic matrix (2.2), $\tau(\Sigma, v)$ transforms as follows:
\[
\tau(\Sigma, v) \rightarrow \rho \det(C\Omega + D) \tau(\Sigma, v),
\]
(2.17)
where $\rho$ is a root of unity of degree depending on the multiplicities $k_j$.

Another important property of the tau function is its behavior under the action of $\mathbb{C}^*$:

$$\tau(\Sigma, \epsilon v) = \epsilon^{\frac{1}{24} \sum_{j=1}^{M} \frac{k_j(k_j+2)}{2j+1}} \tau(\Sigma, v).$$

(2.18)

The tau function can be used for obtaining relations between divisors in the rational Picard group of the strata on the moduli space of Abelian differentials.

For the main stratum $\mathcal{H}_g(1,\ldots,1)$ these relations were found in [27]. Namely, let $\mathcal{P}_g(1,\ldots,1)$ be the projectivization of $\mathcal{H}_g(1,\ldots,1)$ respect to the action of $\mathbb{C}^*$. Let $L$ be the tautological line bundle associated to the projection $\mathcal{H}_g(1,\ldots,1) \to \mathcal{P}_g(1,\ldots,1)$, and denote by $\phi = c_1(L)$ its first Chern class. Furthermore, denote by $\lambda$ the pullback to $\mathcal{P}_g(1,\ldots,1)$ of the Hodge class on $\mathcal{M}_g$. Then we have the following relation in the rational Picard group $\text{Pic}(\mathcal{P}_g(1,\ldots,1)) \otimes \mathbb{Q}$ of the compactification of $\mathcal{P}_g(1,\ldots,1)$:

$$\lambda = \frac{g-1}{4} \phi + \frac{1}{24} \delta_{\text{deg}} + \frac{1}{12} \delta_0 + \frac{1}{8} \sum_{j=1}^{\lfloor g/2 \rfloor} \delta_j.$$

(2.19)

Here $\delta_{\text{deg}}$ is the divisor of Abelian differentials with multiple zeroes, and $\delta_j$, $j = 0,\ldots,\lfloor g/2 \rfloor$, are the pullbacks of the classes of the Deligne-Mumford boundary divisors on $\mathcal{M}_g$; see [27] for details.\footnote{This relation has later received pure algebraic proofs by D. Zvonkine (unpublished) and by D. Chen [9].}

### 2.3. Spaces of holomorphic $N$-differentials.

The above result was extended further to the spaces of $N$-differentials in [28] (for $N = 2$) and [29] (for $N > 2$).

Let $\mathcal{M}_g^N$ be the moduli space of equivalence classes of pairs $(\Sigma, W)$ where $W$ is a holomorphic $N$-differential on $\Sigma$ (both $\Sigma$ and $W$ are allowed to vary here). We refer to [29, 4] for a precise definition of the space $\mathcal{M}_g^N$ and its compactification $\overline{\mathcal{M}}_g^N$.

The dimension of $\mathcal{M}_g^N$ is the sum of $3g-3$ and $(2N-1)(g-1)$, i.e.

$$\dim \mathcal{M}_g^N = 2(N+1)(g-1).$$

(2.20)

The space $\mathcal{M}_g^N$ has an open subset $\mathcal{M}_g^{N,0}$, that consists of equivalence classes of pairs $(\Sigma, W)$, where $\Sigma$ is a smooth curve, and $W$ has only simple zeroes. The complement $\overline{\mathcal{M}}_g^N \setminus \mathcal{M}_g^{N,0}$ is the union of $\lfloor g/2 \rfloor + 2$ divisors that we denote by $D_{\text{deg}}, D_0,\ldots, D_{\lfloor g/2 \rfloor}$, where $D_{\text{deg}}$ is the divisor of degenerate $N$-differentials (i.e. having multiple zeroes), and $D_i$ ($i = 0,\ldots,\lfloor g/2 \rfloor$) are the pullbacks of the components of the Deligne-Mumford boundary of $\overline{\mathcal{M}}_g$.

A natural $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_g^N$ is given by multiplication $W \to \epsilon W$, $\epsilon \in \mathbb{C}^*$. Denote by $\mathcal{L}$ the tautological line bundle associated with the canonical projection $\overline{\mathcal{M}}_g^N \to \overline{\mathcal{M}}_g$ and put $\psi = c_1(\mathcal{L}) \in \text{Pic}(\overline{\mathcal{M}}_g^N) \otimes \mathbb{Q}$.

Denote by $\lambda$ the Hodge class on $\overline{\mathcal{M}}_g$ (i.e. the pullback of the Hodge class from the moduli space of curves $\overline{\mathcal{M}}_g$), and consider the classes of boundary divisors $\delta_i$, $i = 0,\ldots,\lfloor g/2 \rfloor$, in
Pic($\mathfrak{M}_g^N$) $\otimes \mathbb{Q}$. Then the rational Picard group Pic($\mathfrak{M}_g^N$) $\otimes \mathbb{Q}$ is freely generated by the classes $\psi, \lambda, \delta_0, \ldots, \delta_{[g/2]}$.

To each pair $(\Sigma, W)$ one can naturally associate a canonical cyclic branched cover $p : \tilde{\Sigma} \to \Sigma$ of degree $N$, where

$$\tilde{\Sigma} = \{(x, w) \in \Sigma \times T_x^* \Sigma \mid w^N = W\}.$$  \hfill (2.21)

When all zeros of $W$ are simple, the cover $\tilde{\Sigma}$ is smooth and its genus is $\tilde{g} = N^2(g - 1) + 1$. The cover $\tilde{\Sigma}$ is invariant with respect to the natural $\mathbb{Z}/N\mathbb{Z}$-action $(x, w) \mapsto (x, \rho^k w)$ where $\rho = e^{2\pi \sqrt{-1}/N}$. Denote by $f : \tilde{\Sigma} \to \tilde{\Sigma}$ the automorphism of $\tilde{\Sigma}$ corresponding to $k = 1$. By definition, the holomorphic 1-form $w$ satisfies $f^*w = \rho w$.

The group $H_1(\tilde{\Sigma}, \mathbb{C})$ can be decomposed into the eigenspaces of the automorphism $f$,

$$H_1(\tilde{\Sigma}, \mathbb{C}) = \bigoplus_{k=0}^{N-1} S_k,$$  \hfill (2.22)

where $\dim S_0 = 2g$ and the dimensions of $S_k$ are independent of $k$ and given by

$$\dim S_k = (N + 1)(2g - 2), \quad k = 1, \ldots, N - 1.$$  \hfill (2.23)

The differential $w$ has non-vanishing periods only over the cycles in $S_1$; these periods can be used as local coordinates on the moduli space $\mathfrak{M}_g^{N,0}$:

$$\tilde{P}_i = \int_{\tilde{s}_i} w, \quad i = 1, \ldots, (N + 1)(2g - 2).$$  \hfill (2.24)

where

$$\tilde{s}_1, \ldots, \tilde{s}_{(N+1)(2g-2)}$$  \hfill (2.25)

is a basis of the eigenspace $S_1$.

For any two cycles $s_1 \in S_l$ and $s_2 \in S_k$ we have $s_1 \circ s_2 = 0$ unless $k + l = N$. The spaces $S_k$ and $S_{N-k}$ are therefore dual to each other with respect to the standard intersection pairing (the space $S_0$ can be identified with $H_1(\Sigma)$, and, therefore, it is dual to itself).

Therefore, one can introduce a set of cycles dual to (2.25) which form a basis in the space $S_{N-1}$:

$$\tilde{s}_1^*, \ldots, \tilde{s}_{(N+1)(2g-2)}^*, \quad \tilde{s}_i^* \circ \tilde{s}_j = \delta_{ij}.$$  \hfill (2.26)

Now assume that all zeros of $W$ are simple, i.e.

$$(W) = \sum_{i=1}^{N(2g-2)} x_i.$$  \hfill (2.27)

Then the distinguished local coordinate on $\Sigma$ in a neighbourhood of the point $x_i$ is given by

$$\zeta_i(x) = \left( \int_{x_i}^{x} v \right)^{N/(N+1)}.$$  \hfill (2.28)
In terms of these coordinates we define

\[
E(x, x_k) = \lim_{y \to x_k} E(x, \zeta(y)) \sqrt{d\zeta_k(y)}.
\]

\[
E(x_k, x_l) = \lim_{x \to x_k, y \to x_l} E(x, y) \sqrt{d\zeta_k(x)} \sqrt{d\zeta_l(y)}.
\]

We choose two vectors \( r, s \in \frac{1}{n} \mathbb{Z}^g \) that satisfy the condition

\[
\frac{1}{N} A_x((W)) + 2K^x = \Omega r + s.
\]  

(2.29)

The tau function on the space \( \mathcal{M}_g^{N,0} \) is defined by

\[
\tau(\Sigma, W) = C^{2/3}(x) e^{-\frac{\pi}{6} (\Omega r, r, \Omega s)} \left( \prod_{i=1}^{m} E(x, x_i) \right)^{\frac{4N}{3}} \prod_{i<j} E(x_i, x_j)^{\frac{1}{6N}}
\]  

(2.30)

see [29] for details.

The tau function (2.30) satisfies the following system of equations with respect to the periods of \( w \) (2.24):

\[
\frac{\partial \ln \tau(\Sigma, W)}{\partial \hat{P}_i} = -\frac{1}{2\pi \sqrt{-1} N} \int_{\Sigma} B_{w}^{\text{reg}} \frac{w}{w} \]

(2.31)

where

\[
B_{w}^{\text{reg}}(x) = \left( B_{w}(x, y) - \frac{w(x) w(y)}{(f_x^y)^2} \right) \bigg|_{y=x}.
\]  

(2.32)

The tau function (2.30) has properties similar to those of (2.16):

- Under the change (2.2) of a Torelli marking of \( \Sigma \) the tau function (2.30) transforms as follows:

\[
\tau(\Sigma, W) \to \rho \det(C\Omega + D) \tau(\Sigma, W),
\]  

(2.33)

where \( \rho \) is a root of unity of order \( 48(N+1) \).

- \( \tau(\Sigma, \mu W) \) is quasi-homogeneous with respect to the action of \( \mathbb{C}^* \):

\[
\tau(\Sigma, \mu W) = e^{\kappa} \tau(\Sigma, W), \quad \mu \in \mathbb{C}^*,
\]  

(2.34)

with

\[
\kappa = \frac{(2N+1)(g-1)}{6N(N+1)}.
\]  

(2.35)

These properties, together with the asymptotics of \( \tau(\Sigma, W) \) near \( D_{\text{deg}} \) and the components of the Deligne-Mumford boundary, imply the following expression for the Hodge class on the space \( \overline{\mathcal{M}}_g^N \) (Theorem 3.9 of [29]):

\[
\lambda = \frac{(g-1)(2N+1)}{6N(N+1)} \psi + \frac{1}{12N(N+1)} \delta_{\text{deg}} + \frac{1}{12} \delta.
\]  

(2.36)
3. THE DIVISOR CLASS OF THE UNIVERSAL HITCHIN’S DISCRIMINANT

Consider the following vector bundles on the moduli spaces of curves and their pullbacks to $P\overline{M}$:

- The Hodge vector bundle $\Lambda \to \overline{M}_g$. The fiber of $\Lambda$ over a smooth curve $\Sigma$ is the $g$-dimensional vector space of holomorphic 1-forms (Abelian differentials) on $\Sigma$. This bundle naturally lifts to $P\overline{M}$, and we put $\lambda = c_1(\det \Lambda)$.

- The Hodge vector bundle $\hat{\Lambda} \to \overline{M}_{\hat{g}}$. The fiber of $\hat{\Lambda}$ over a smooth spectral cover $\hat{\Sigma}$ is the $\hat{g}$-dimensional vector space of holomorphic 1-forms on $\hat{\Sigma}$. This bundle also lifts to $P\overline{M}$, and, similarly, we put $\hat{\lambda} = c_1(\det \hat{\Lambda})$.

- The tautological line bundle $L$. The line bundle $L$ is associated with the natural action of $C^*$ on $\mathcal{M}$ by $Q_k \mapsto \epsilon^k Q_k$, $\epsilon \in C^*$ (3.1).

Denote by $P\mathcal{M}$ the projectivization of $\mathcal{M}$ with respect to the action (3.1). The fibers of the projection $P\mathcal{M} \to \mathcal{M}$ are weighted projective spaces $\mathcal{M}_\Sigma/C^*$, where $\mathcal{M}_\Sigma = \bigoplus_{j=1}^n H^0(\Sigma, K_{\Sigma} \otimes j)$, see (1.7). The bundle $L$ extends to the compactification $P\overline{M}$, and we put $\phi = c_1(L)$.

Remark 3.1. Rigorously speaking, all these objects should be understood in a proper sense (that is, as sheaves on smooth algebraic stacks). However, abusing the language, we will continue calling them vector bundles.

If the base curve $\Sigma$ has nodes, the differentials $Q_j$ may have poles up to order $j$ at each node. If $Q_j$ has poles of the maximal order $j$ at the two intersecting branches of $\Sigma$ with equal or opposite $j$-residues depending on the parity of $j$; see Section 1.1 of [29] or [4] for details. Therefore, the discriminant $W$ can have poles of order up to $n(n-1)$ at the nodes (in case of poles of order $n(n-1)$ the residues must be equal, since $n(n-1)$ is always even).

For a point $(\Sigma; \{Q_k\}_{k=1}^n) \in \mathcal{M}$ consider the cyclic $\mathbb{Z}/N\mathbb{Z}$-cover $\hat{\Sigma}$ of $\Sigma$ given by (2.21) with $N = n(n-1)$, and consider the decomposition (2.22) of the homology group $H_1(\hat{\Sigma}, \mathbb{C})$. Choose a set of $(n(n-1) + 1)(2g - 2)$ linearly independent cycles $\tilde{s}_1, \ldots, \tilde{s}_{2(n^2-n+1)(g-1)}$ (3.2) in the subspace $\mathcal{S}_1$. As in (2.26), consider the cycles dual to $\tilde{s}_1^*, \ldots, \tilde{s}_{2(n^2-n+1)(g-1)}^*$

\[
\tilde{s}_i^* \circ \tilde{s}_j = \delta_{ij} \tag{3.3}
\]

(they form a basis in the space $\mathcal{S}_{n(n-1)-1}$. Generally speaking, the periods of $w$ with respect to the basis (3.2) do not provide a coordinate system on $\mathcal{M}$ since $\dim \mathcal{M} = (n^2+3)(g-1)+1$ is smaller than $\dim \mathcal{S}_1$ for $N \geq 3$.

The tau function $\tau$ on the space $\mathcal{M}$ can be defined by the system of equations

\[
d \log \tau(\Sigma, W) = -\frac{1}{2\pi \sqrt{-1}n(n-1)} \sum_{i=1}^{2(n^2-n+1)(g-1)} \left( \int_{\tilde{s}_i^*} B_{w, \Sigma}^{\Sigma, g} \right) d \left( \int_{\tilde{s}_i} w \right). \tag{3.4}
\]
The solution \( \tau(\Sigma, W) \) of (3.31) is given by the formula (2.30), where \( N = n(n - 1) \) and \( x_i, i = 1, \ldots, n(n - 1)(2g - 2) \) are the zeroes of \( W \).

Since under the rescaling \( Q_k \mapsto e^k Q_k, \epsilon \in \mathbb{C}^* \), the discriminant \( W \) transforms as \( W \mapsto e^{n(n-1)} W \), the tautological line bundles \( L \) and \( L \) associated with the \( \mathbb{C}^* \)-actions on \( \mathcal{M}_g \) and \( \mathcal{M} \) respectively are related by \( L \simeq L^{n(n-1)} \), and \( \psi = c_1(L) = n(n - 1)c_1(L) = n(n - 1)\phi \).

Furthermore, according to formulas (3.13), (3.15) of [29], the tau function \( \tau(\Sigma, W) \) has the following asymptotics when two zeros of \( W \) (say, \( x_1 \) and \( x_2 \)) coalesce:

\[
\tau(\Sigma, W) \sim \left( \int_{x_1}^{x_2} \right)^{\frac{1}{(n^2 - n + 1)(n^2 - n + 2)}} (1 + o(1)) .
\] (3.5)

Using the transformation properties (2.33) and (2.34) of \( \tau(\Sigma, W) \) and computing its divisor, we obtain the following relation in \( \text{Pic}(\mathcal{P} \mathcal{M}) \otimes \mathbb{Q} \)

\[
12n(n - 1)\lambda = \frac{1}{(n^2 - n + 1)} [PD_W] + \frac{2(2n^2 - 2n + 1)(g - 1)}{n^2 - n + 1} \psi + n(n - 1)\delta ,
\] (3.6)

where \( \delta \) is the pullback of the Deligne-Mumford boundary class relative to the projection \( \mathcal{P} \mathcal{M} \to \mathcal{M}_g \). Expressing the class of \( PD_W \) in terms of \( \phi, \lambda \) and the boundary class we get

**Theorem 3.2.** The class of the (projectivized) universal Hitchin's discriminant \( PD_W \) defined by (1.19) expresses in terms of the standard generators of \( \text{Pic}(\mathcal{P} \mathcal{M}) \otimes \mathbb{Q} \) as follows:

\[
\frac{1}{n(n - 1)} [PD_W] = (n^2 - n + 1)(12\lambda - \delta) - 2(2n^2 - 2n + 1)(g - 1)\phi .
\] (3.7)

4. **Divisor \( PD_v \) and the Hodge class \( \lambda \)**

There is a natural map \( \mathcal{M} \to \mathcal{H}_{\hat{g}} \) to the moduli space of holomorphic 1-forms that sends the point \( (\Sigma, \{ Q_i \}_{i=1}^n) \in \mathcal{M} \) to the point \( (\hat{\Sigma}, v) \in \mathcal{H}_{\hat{g}} \). Generically, all zeros of the differential \( v \) are simple, and there are \( 2\hat{g} - 2 \) of them that we denote \( y_1, \ldots, y_{2\hat{g} - 2} \). The number of periods of \( v \) over the cycles (2.6) in the relative homology group \( H_1(\hat{\Sigma}, \{ y_1, \ldots, y_{2\hat{g} - 2} \}) \) is equal to \( 2\hat{g} - 3 = 4n^2(g - 1) + 1 \) which is in general greater than \( \dim \mathcal{M} = (n^2 + 3)(g - 1) + 1 \).

Consider the set of generators \( s_j \) of the relative homology group \( H_1(\hat{\Sigma}, \{ y_1, \ldots, y_{2\hat{g} - 2} \}) \):

\[
\{ s_1, \ldots, s_{2\hat{g} - 3} \} = \{ \hat{a}_1, \ldots, \hat{a}_{\hat{g}}, \hat{b}_1, \ldots, \hat{b}_{\hat{g}}, l_2, \ldots, l_{2\hat{g} - 2} \} ,
\] (4.1)

where \( l_j \) is a simple path connecting \( y_i \) with \( y_j \).

The dual system of generators in the homology group \( H_1(\hat{\Sigma} \setminus \{ y_1, \ldots, y_{2\hat{g} - 2} \}) \) is

\[
\{ s_1^*, \ldots, s_{2\hat{g} - 3}^* \} = \{ -\hat{b}_1, \ldots, -\hat{b}_{\hat{g}}, \hat{a}_1, \ldots, \hat{a}_{\hat{g}}, c_2, \ldots, c_{2\hat{g} - 2} \} ,
\] (4.2)

where \( c_j \) is a small positively oriented circle around \( y_j \) such that \( s_j^* \circ s_j = \delta_{ij} \).

The class of the divisor of zeros of the differential \( v \) can be expressed in terms of the Hodge class \( \lambda = c_1(\Lambda) \) and the classes \( \psi \) and \( \delta \) using the tau function (2.16) on the moduli spaces of holomorphic Abelian differentials with simple zeros on the complex curves of genus \( \hat{g} \). The tau function \( \tau(\hat{\Sigma}, v) \) on the space of spectral covers (1.1) is defined by the explicit formula (2.16).
Formula (2.17) implies that $\tau(\hat{\Sigma}, v)$ transforms like follows under the change of Torelli marking of $\hat{\Sigma}$ given by $\begin{pmatrix} \tilde{C} & \tilde{D} \\ \tilde{B} & \tilde{A} \end{pmatrix} \in Sp(2\hat{g}, \mathbb{Z})$:

$$
\tau(\hat{\Sigma}, v) \rightarrow \rho \tau(\hat{\Sigma}, v) \det(\tilde{C} \tilde{\Omega} + \tilde{D}) \quad (4.3)
$$

where $\rho^{24} = 1$. By (2.18), under the rescaling $v \mapsto \epsilon v$, $\epsilon \in \mathbb{C}^*$, $\tau(\hat{\Sigma}, v)$ behaves like

$$
\tau(\hat{\Sigma}, \epsilon v) = \epsilon^{(\hat{g} - 1) / 4} \tau(\hat{\Sigma}, v) \quad (4.4)
$$

Notice that when the curve $\Sigma$ approaches the boundary of $\hat{\mathcal{M}}$, the cover $\hat{\Sigma}$ approaches a codimension $n - 1$ locus $D_0$ in the component $\delta_0$ of the Deligne-Mumford boundary of $\hat{\mathcal{M}}$. Then the formulas (4.3) and (4.4) combined with the asymptotics of $\tau(\hat{\Sigma}, v)$ near $\delta_0$ in $\hat{\mathcal{M}}$ (cf. Lemma 7 of [27]), imply the following

**Theorem 4.1.** The class of the (projectivized) divisor $PD_v$ of non-generic (i.e., for $v$ with multiple zeroes) $GL(n)$ spectral covers in $Pic(P\hat{\mathcal{M}}) \otimes \mathbb{Q}$ is given by

$$
[PD_v] = 24\hat{\lambda} - 6(\hat{g} - 1)\phi - 2n\delta \quad (4.5)
$$

Here $\hat{\lambda}$ is the Hodge class of $\hat{\mathcal{M}}$ pulled back to $P\hat{\mathcal{M}}$, $\phi$ is the tautological class associated with the projection $\hat{\mathcal{M}} \rightarrow P\hat{\mathcal{M}}$, and $\delta$ is the pullback to $P\hat{\mathcal{M}}$ of the Deligne-Mumford boundary of $\hat{\mathcal{M}}$.

The proof of the theorem follows almost verbatim the proof of Theorem 2 in [29].

### 5. Prym Class in $Pic(P\hat{\mathcal{M}}) \otimes \mathbb{Q}$

Let $y \in \Sigma$ be a generic point of the projection $\pi : \hat{\Sigma} \rightarrow \Sigma$, i.e. $|\pi^{-1}(y)| = n$. Denote by $\xi$ a local coordinate on $\Sigma$ in a small neighborhood $U$ of $y$. Then $\xi$ can be used as a local coordinate on each of the $n$ connected components of the preimage of $U$.

A holomorphic Abelian differential $u$ on $\hat{\Sigma}$ is called a Prym differential if

$$
\sum_{x \in \pi^{-1}(y)} \frac{u}{d\xi}(x) = 0 \quad (5.1)
$$

for any $y \in \Sigma$ that is not a branch point of $\hat{\Sigma}$. Then there is the following decomposition of the space of holomorphic differentials on $\hat{\Sigma}$:

$$
\Omega^1_{\hat{\Sigma}} = \Omega^1_{\Sigma} \oplus H^1_{Prym}(\hat{\Sigma}) \quad (5.2)
$$

If the cover (1.1) arises from an $SL(n)$ Hitchin’s system (i.e. if $Q_1 = 0$), then the sum of solutions of the equation (1.1) is zero. In this case $v \in H^1_{Prym}(\hat{\Sigma})$ is a Prym differential.

The vector bundle on $\hat{\mathcal{M}}$ with fiber $H^1_{Prym}(\hat{\Sigma})$ over $\hat{\Sigma}$ is called the Prym vector bundle. The Prym bundle naturally extends to $\hat{\mathcal{M}}$ and descends to the projectivization $P\hat{\mathcal{M}}$. The first Chern class of the determinant of the Prym vector bundle is called the Prym class and is denoted by $\lambda_P$. 


We define the Prym tau function $\tau_P$ as
\[
\tau_P = \frac{\tau(\hat{\Sigma}, v)}{\tau(\Sigma, W)} .
\]
(5.3)

It may be viewed as a section of a holomorphic line bundle on $P\mathcal{M}$. Computing its divisor and using Theorems 3.2 and 4.1 we get

**Corollary 5.1.** The Prym class $\lambda_P$ decomposes in $\text{Pic}(P\mathcal{M}) \otimes \mathbb{Q}$ into a linear combination of the classes $[PD_W]$, $[PD_v]$, the tautological class $\phi$ and the boundary class $\delta$ as follows:
\[
\lambda_P = \frac{1}{24} [PD_v] - \frac{1}{12n(n-1)(n^2-n+1)} [PD_W] + \frac{g-1}{3} \left( n^2 - \frac{2n^2 - 2n + 1}{3} \right) \phi + \frac{n-1}{12} \delta .
\]
(5.4)

### 6. GL(2) Spectral Covers

The equation (1.1) of the spectral cover $\hat{\Sigma}$ in the $GL(2)$ case looks like follows:
\[
v^2 + Q_1 v + Q_2 = 0 \quad (6.1)
\]
The discriminant is then $W = Q_1^2 - 4Q_2$, and the differential $v$ on $\hat{\Sigma}$ is
\[
v = \frac{1}{2} ( -Q_1 \pm \sqrt{W} ) ,
\]
(6.2)

where the choice of $\sqrt{W}$ is compatible with the involution $\hat{\Sigma} \to \hat{\Sigma}$. Generically, the differential $v$ has $4g - 4$ simple zeros at the branch points (since both $Q_1$ and $\sqrt{W}$, being lifted to $\hat{\Sigma}$, have simple zeros at the branch points), and $4g - 4$ more simple poles elsewhere.

If all $4g - 4$ zeros of $W$ are simple, then the genus of $\hat{\Sigma}$ equals to $g = 4g - 3$. The formula (3.6) for the class of the divisor $PD_W$ (which in this case coincides with $PD_W^{(b)}$) takes the form
\[
[PD_W] = 72\lambda - 20(g-1)\phi - 6\delta ,
\]
(6.3)
and for the divisor $PD_v$ by (4.5) we have
\[
[PD_v] = 24\hat{\lambda} - 24(g-1)\phi - 4\delta .
\]
(6.4)

### 7. Open Questions

The following questions arise naturally in connection with the subject of this work.

1. Using (4.5) one can express the class $[PD_v]$ in terms of the class $\phi$ and the class $(\hat{\Theta})$ of the divisor of zeros of the product of even theta constants (the “theta-null”) on $\hat{\Sigma}$. This relation follows from (4.5) and the expression of $(\hat{\Theta})$ in terms of $\hat{\lambda}$ given in Proposition 3.1 of [38].

2. The holomorphic $n(n-1)$-differentials which appear as discriminants of Hitchin’s $GL(n)$ covers are rather special: for a fixed $\Sigma$ the space of discriminants has dimension...
g + (g - 1)\sum_{k=2}^{n}2k - 1 = n^2(g - 1) + 1, while the space of all holomorphic \( n(n-1) \)-differentials has dimension \( 2n(n-1)(g-1) \). How to distinguish differentials that are discriminants among all holomorphic \( n(n-1) \)-differentials?

3. What is the connection, if any, between the divisors appearing in this work and the “critical loci” discussed in the recent paper of N.Hitchin [19]?

Remark 7.1. This paper was originally published in the volume [30] dedicated to the memory of L. D. Faddeev. More recently the paper [6] by M. Basok appeared containing more detailed information about various divisor classes discussed in this paper. More precisely, for \( n \geq 3 \) and \( g \geq 1 \) the following formulas were obtained in [6] for the classes of the “caustic”, the “Maxwell stratum” and the boundary components of the universal Hitchin’s discriminant (1.19):

\[
\begin{align*}
[P_D(c)] & = n(n-1)(n-2)\left(12\lambda - \delta - 4(g-1)\phi\right), \\
[P_D(m)] & = \frac{n(n-1)(n-2)(n-3)}{2} \left(12\lambda - \delta + 4(g-1)\phi\right), \\
[P_D(b)] & = n(n-1)\left((n+1)(12\lambda - \delta) - 2(g-1)(2n+1)\phi\right).
\end{align*}
\]

The expression for the class of the universal Hitchin’ discriminant \( D_W \) obtained by means of these formulas coincides with our formula (3.7).

Another formula derived in [6] relates the classes \( \lambda \) and \( \tilde{\lambda} \):

\[
\tilde{\lambda} = n(2n^2 - 1)\lambda - \frac{n(n-1)(4n+1)(g-1)}{6}\phi - \frac{n(n^2 - 1)}{6}\delta.
\]

In particular, this formula yields a relation between the classes \( [D_W] \) and \( [D_v] \) given by the formulas (3.7) and (4.5).

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