On spectral Petrov-Galerkin method for solving optimal control problem governed by a two-sided fractional diffusion equation

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Abstract: In this paper, we investigate a spectral Petrov-Galerkin method for an optimal control problem governed by a two-sided space-fractional diffusion-advection-reaction equation. Taking into account the effect of singularities near the boundary generated by the weak singular kernel of the fractional operator, we establish the regularity of the problem in weighted Sobolev space. Error estimates are provided for the presented spectral Petrov-Galerkin method and the convergence orders of the state and control variables are determined. Furthermore, a fast projected gradient algorithm with a quasi-linear complexity is presented to solve the resulting discrete system. Numerical experiments show the validity of theoretical findings and efficiency of the proposed fast algorithm.

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1 Introduction

Optimal control problems with partial differential equation constraints have wide applications in science and engineering fields, such as biology, ecology, economic, finance, etc., see \cite{3, 13, 22, 24, 29}. In recent two decades, fractional diffusion equations (FDEs) have attracted increasing attention due to the capability of modeling complex physical phenomena with anomalous diffusion or long-time memory, such as contaminant transport in ground water flow \cite{6}, anomalous transport in biology \cite{18}, viscoelasticity \cite{28}, etc. As a consequence, both theory and numerical methods to optimal control problems governed by FDEs have gradually aroused more and more researchers’ interest. Various numerical methods have been investigated for the optimal control problems with FDE constraints, e.g. finite element methods \cite{4, 5, 10, 17, 23, 33, 38, 39, 41}, spectral Galerkin methods \cite{32, 34, 35, 40}, collocation methods \cite{25, 26, 36}, etc.

In this paper, we consider the following optimal control problem governed by a space-
fractional diffusion-advection-reaction equation:

\[
\min_{q \in U_{ad}} J(u, q) = \frac{1}{2} \| u - u_d \|^2_{L^2(\Omega)} + \frac{\gamma}{2} \| q \|^2_{L^2(\Omega)}
\]  

(1.1)

subject to

\[
\begin{cases}
\mathcal{L}_0^\alpha u + \lambda_1 Du + \lambda_2 u = f(x) + q(x), & x \in \Omega := (0, 1), \\
u(0) = 0, & x \in \partial \Omega,
\end{cases}
\]  

(1.2)

where \( U_{ad} \) is an admissible set defined by

\[
U_{ad} = \{ q \in L^2(\Omega) : \int_\Omega q(x) dx \geq 0 \},
\]  

(1.3)

\( D \) denotes the first-order derivative with respect to \( x \), \( f \) and \( u_d \) are given functions, \( \gamma, \lambda_1 \) and \( \lambda_2 \) are constants, \( \gamma > 0, \lambda_1 \neq 0, \lambda_2 \geq 0 \). \( \mathcal{L}_0^\alpha u := -[\theta \, 0^{D^\alpha}_x + (1 - \theta) \, x^{D^\alpha}_1] \) is a general two-sided fractional operator with \( \alpha \in (1, 2) \) and \( \theta \in [0, 1] \). Here \( 0^{D^\alpha}_x \) and \( x^{D^\alpha}_1 \) are left and right Riemann-Liouville fractional derivatives [30], respectively, defined by

\[
0^{D^\alpha}_x u(x) = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{d x^2} \int_0^x \frac{u(s)}{(x - s)^{\alpha-1}} ds, \quad x > 0,
\]  

(1.4)

\[
x^{D^\alpha}_1 u(x) = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{d x^2} \int_x^1 \frac{u(s)}{(s - x)^{\alpha-1}} ds, \quad x < 1.
\]  

(1.5)

Due to the existence of weak singular kernel of fractional derivatives, exact solutions of model equation (1.2) and its variants usually exhibit singularities near the boundary. Even given smooth inputs, it leads to the low regularity of the problem in standard Sobolev space. In order to compensate for the weak singularities of solutions, weighted Sobolev spaces were employed in [14, 21]. For the model equation (1.2) with \( \theta = 1/2 \), Hao and Zhang gave sharp regularity estimates of solutions in weighted Sobolev space [21]. The work was extended by Ervin in [14] to the general case, i.e. \( \theta \in [0, 1] \). Based on the regularity results, a spectral Petrov-Galerkin method with higher convergence order was obtained in [37] for (1.2). Above works motivate us to analyze the regularity of the solution to the optimal control problem with FDE constraints (1.1)-(1.3), and then present a spectral Petrov-Galerkin method and investigate its convergence order in weighted Sobolev space.

When taking \( \theta = 1/2 \), the operator \( \mathcal{L}_{1/2}^\alpha \) is equivalent to the integral fractional Laplacian operator (sometimes called Riesz fractional Laplacian). Zhang and Zhou [40] considered a
spectral Galerkin approximation to optimal control problem with integral fractional Laplacian equation constraints. They derived the first-order optimality condition and studied the regularity of solution and the convergence order of approximation in weighted Sobolev space. Recently Wang et al. [32] considered the spectral Galerkin approximation for the optimal control problem governed by fractional diffusion-advection-reaction equations with integral fractional Laplacian (i.e. (1.1)-(1.3) for $\theta = 1/2$). There are also some works on numerical methods for optimal control problems with a general two-sided space FDE constraints in standard Sobolev space. For example, a fast Gradient project method was exploited in [11] to solve the discrete systems derived from two finite difference methods; a finite element Galerkin method was developed in [10] when the constraint is a random two-sided space-fractional diffusion equation.

The aim of this work is to present an efficient spectral Petrov-Galerkin method based on the regularity analysis in weighted Sobolev space for the optimal control problem (1.1)-(1.3) with $\theta \in [0, 1]$. To this end, we perform the following three steps.

- Firstly, we construct the regularity of (1.1)-(1.3) in weighted Sobolev space.

  Due to the asymmetry of the fractional operator $\mathcal{L}_\theta^\alpha$ for $\theta \neq 1/2$, the regularity of state $u$ and adjoint state $z$ are analyzed in two different weighted Sobolev spaces based on the coupled optimality system (3.3)-(3.5). To overcome this gap, the regularity connection between $(1-x)^{-\sigma} x^{-\sigma^*} u$ and $u$ is established in the weighted Sobolev space $H^p_{\omega, \sigma^*} (\Omega)$, see Lemma 3.6, where $\sigma$ and $\sigma^*$ are constants defined by (2.3) depending on $\alpha$ and $\theta$. The regularity results of the state variable $u$, the adjoint state variable $z$ and the control variable $q$ are given in Theorem 3.9.

- Secondly, we present a spectral Petrov-Galerkin method to (1.1)-(1.3) and give its error estimation.

  Based on the regularity results obtained in the first step, we adopt weighted Jacobi polynomials as basis in the spectral Petrov-Galerkin method to recover accuracy of numerical solution from the boundary singularity, and estimate the errors of approximations to $u$ in $L^2_{\omega, \sigma, \sigma^*}$-norm, $z$ in $L^2_{\omega, \sigma^*, -\sigma}$-norm and $q$ in $L^2$-norm, see Theorem 4.5.

- Finally, we provide a fast projected gradient algorithm with a quasi-linear complexity to solve the resulting discrete system.

  Compared to the projected gradient method for solving the linear optimization system obtained from the spectral Petrov-Galerkin method, which requires $O(N^2)$ storage and $O(N^3)$ computational complexity, see [22, 27, 32], we follow the idea in [19] and present a fast projected gradient method with linear storage $O(N)$ and quasilinear computational cost $O(N \log^2 N)$, see Algorithm 1.

The rest of the paper is organized as follows. In Section 2, we introduce some necessary notations and properties of Jacobi polynomials and weighted Sobolev spaces. In Section 3, we derive the first-order optimality condition and study the regularity of solutions for the optimal control problem (1.1)-(1.3). Based on the regularity results, a spectral Petrov-Galerkin method is presented and its error estimate is given in Section 4. In Section 5, a fast iteration algorithm is presented to solve the discrete systems. Numerical examples given in this section verify our theoretical findings and show that the proposed spectral Petrov-Galerkin method with the fast algorithm is efficient.
2 Preliminary

In this section, we will give some basic definitions and properties of Jacobi polynomials and weighted Sobolev spaces.

2.1 Jacobi polynomials

For $\gamma, \beta > -1$, $n \in \mathbb{N}$ and $t \in [-1, 1]$, $P_n^{\gamma, \beta}(t)$ is the classical Jacobi polynomial of degree $n$. Let $t = 2x - 1$, then the domain of the Jacobi polynomials $P_n^{\gamma, \beta}(t)$ is transformed to $[0, 1]$.

Following this idea, we introduce

$$Q_n^{\gamma, \beta}(x) = P_n^{\gamma, \beta}(2x - 1), \quad x \in [0, 1].$$

• Orthogonality. The Jacobi polynomials $Q_n^{\gamma, \beta}(x)$ are mutually orthogonal with respect to the weight $(1 - x)^{\gamma}x^\beta$: for $\gamma, \beta > -1$,

$$\int_0^1 (1 - x)^{\gamma}x^\beta Q_n^{\gamma, \beta}(x)Q_m^{\gamma, \beta}(x)dx = \delta_{mn}\|Q_n^{\gamma, \beta}\|_{\omega^{\gamma, \beta}}^2,$$

where $\delta_{mn}$ is the Kronecker symbol and

$$\|Q_n^{\gamma, \beta}\|_{\omega^{\gamma, \beta}}^2 = \frac{1}{2n + \gamma + \beta + 1} \cdot \frac{\Gamma(n + \beta + 1)\Gamma(n + \gamma + 1)}{\Gamma(n + 1)\Gamma(n + \gamma + \beta + 1)} := h_n^{\gamma, \beta}.$$

• Properties of Jacobi polynomials.

Lemma 2.1 ([13]). For the $n$-th order Jacobi polynomials $Q_n^{\sigma^*, \sigma^*}(x)$ and $Q_n^{\sigma, \sigma}(x)$, $x \in [0, 1]$, it holds that

$$\mathcal{L}_\theta^\alpha \left[(1 - x)^\sigma x^{\sigma^*} Q_n^{\sigma, \sigma^*}(x)\right] = \lambda_{\theta, n}^{\sigma} Q_n^{\sigma^*, \sigma^*}(x),$$

in which

$$\lambda_{\theta, n}^{\sigma} = -\frac{\sin(\pi\alpha)}{\sin(\pi\sigma^*) + \sin(\pi\sigma)} \frac{\Gamma(n + 1 + \alpha)}{\Gamma(n + 1)},$$

$\sigma^* = \alpha - \sigma$ and $\sigma$ is determined by

$$\theta = \frac{\sin(\pi(\alpha - \sigma))}{\sin(\pi(\alpha - \sigma)) + \sin(\pi\sigma)}.$$
Remark 2.2. To ensure \((2.3)\) uniquely solvable, we constrain \(\sigma, \sigma^* \in (0, 1]\). Note that in Lemma 2.1, 
\(\sigma = \sigma^* = \alpha/2\) for \(\theta = 1/2\) and \(\sigma = 1, \sigma^* = \alpha - 1\) for \(\theta = 1\).

Lemma 2.3 \([14]\). For Jacobi polynomials \(Q_n^{\gamma,\beta}(x)\) with \(\gamma, \beta > -1\), it holds that

\[
D^k \left[ Q_n^{\gamma,\beta}(x) \right] = \frac{\Gamma(n + k + \gamma + \beta + 1)}{n + \gamma + \beta + 1} Q_{n-k}^{\gamma+k,\beta+k}(x)
\]

and

\[
D^k \left[ (1 - x)^{\gamma+k} x^{\beta+k} Q_{n-k}^{\gamma+k,\beta+k}(x) \right] = \frac{(-1)^k n!}{(n-k)!} (1 - x)^{\gamma} x^{\beta} Q_n^{\gamma,\beta}(x), \tag{2.4}
\]

in which \(D^k\) denotes \(\frac{d^k}{dx^k}\) and \(k = 1, 2, \cdots\).

2.2 Weighted Sobolev spaces

- \(L_0^{\gamma,\beta}(\Omega)\). Denote \(\omega^{\gamma,\beta}(x) = (1 - x)^{\gamma} x^{\beta}\), \(\gamma, \beta > -1\). Then

\[
L_0^{\gamma,\beta}(\Omega) = \{ v : \int_\Omega \omega^{\gamma,\beta}(x) v^2(x) dx < \infty \}
\]

equipped with the following inner product and norm

\[
(u,v)_{\omega^{\gamma,\beta}} = \int_\Omega \omega^{\gamma,\beta} uv dx, \quad \|u\|_{\omega^{\gamma,\beta}} = \sqrt{(u,u)_{\omega^{\gamma,\beta}}}.
\]

- \(H_0^{s,\omega^{\gamma,\beta}}(\Omega)\). The weighted Sobolev space with non-negative integer \(s\) is defined as (see \([1,10]\))

\[
H_0^{s,\omega^{\gamma,\beta}}(\Omega) = \left\{ v : D^k v(x) \in L_0^{\gamma+k,\beta+k}(\Omega), k = 0, 1, \cdots, s \right\},
\]

equipped with the norm

\[
\|v\|_{H_0^{s,\omega^{\gamma,\beta}}} = \left( \sum_{k=0}^{s} \|D^k v\|_{\omega^{\gamma+k,\beta+k}}^2 \right)^{1/2}, \quad |v|_{H_0^{s,\omega^{\gamma,\beta}}} = \|D^s v\|_{\omega^{\gamma+s,\beta+s}}.
\]

For \(s \in \mathbb{R}^+\), \(H_0^{s,\omega^{\gamma,\beta}}(\Omega)\) can be defined by interpolation via the K-method \([1]\). For \(s < 0\), it is defined by the (weighted) \(L^2\) duality.
• $H^s_{(\eta)}(J)$. Let $N_0 = \mathbb{N} \cup \{0\}$ and $J = (0, \frac{3}{4})$, for $s \geq 0$, $s = \lfloor s \rfloor + \nu$, $0 \leq \nu < 1$, where $\lfloor s \rfloor$ is the integer part of $s$,

$$H^s_{(\eta)}(J) = \{ v : v(x) \text{ is measurable and } \|v\|_{H^s_{(\eta)}(J)} < \infty \},$$

in which the norm $\| \cdot \|_{H^s_{(\eta)}(J)}$ is defined by

$$\|v\|^2_{H^s_{(\eta)}(J)} = \begin{cases} \sum_{k=0}^{\lfloor s \rfloor} \|D^k v\|^2_{L^2_{(\eta+k)}(J)}, & s \in N_0, \\ \sum_{k=0}^{\lfloor s \rfloor} \|D^k v\|^2_{L^2_{(\eta+k)}(J)} + \|v\|^2_{H^s_{(\eta)}(J)}, & s \in \mathbb{R}^+ \setminus N_0, \end{cases}$$

and

$$\|v\|^2_{L^2_{(\eta)}(J)} = \int_J x^\eta v^2(x) dx, \quad \|v\|^2_{H^s_{(\eta)}(J)} = \iint_{\Lambda^*} x^{\eta+\nu} \left[ D^{[s]}_\eta v(x) - D^{[s]}_\eta v(y) \right]^2 |x-y|^{1+2\nu} dxdy$$

where

$$\Lambda^* = \left\{ (x,y) : \frac{2}{3} x < y < \frac{3}{2} x, \ 0 < x < \frac{1}{2} \right\} \cup \left\{ (x,y) : \frac{3}{2} x - \frac{1}{2} < y < \frac{2}{3} x + \frac{1}{3}, \ \frac{1}{2} \leq x < \frac{3}{4} \right\}.$$

By the definition of $H^s_{\omega,\gamma,\beta}(\Omega)$ and $H^s_{(\eta)}(J)$, the following result can be readily obtained.

**Lemma 2.4** ([14, 27]). A function $v \in H^s_{\omega,\gamma,\beta}(\Omega)$ if and only if $v \in H^s_{(\beta)}(J)$ and $\hat{v} \in H^s_{(\gamma)}(J)$, where $\hat{v}(x) := v(1-x)$.

### 3 The optimal control problem

In this section, we derive the first-order optimality condition and analyze the regularity of the optimal control problem (1.1)-(1.3) in weighted Sobolev space. We define the bilinear form $A : H^\frac{\alpha}{2}_0(\Omega) \times H^\frac{\alpha}{2}_0(\Omega) \to \mathbb{R}$ as

$$A(u, v) := \theta(0 D_\alpha^2 u, x D_1^\frac{\alpha}{2} v) + (1 - \theta)(x D_1^\frac{\alpha}{2} u, 0 D_\alpha^2 v) + \lambda_1(Du, v) + \lambda_2(u, v).$$
Then the weak formulation of state equation (1.1) is defined as

\[ A(u, v) = (f + q, v), \forall v \in H^\alpha_0(\Omega), \]

from [12], which admits a unique solution \( u \in H^\alpha_0(\Omega) \). Therefore, we can formulate the optimal control problem (1.1)-(1.3) as:

\[
\begin{align*}
\text{min}_{q \in U_{ad}} & \quad J(u, q) = \frac{1}{2} \|u - u_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2} \|q\|^2_{L^2(\Omega)} \\
\text{subject to} & \quad A(u, v) = (f + q, v), \forall v \in H^\alpha_0(\Omega) 
\end{align*}
\]  

(3.1)

subject to

\[
A(u, v) = (f + q, v), \forall v \in H^\alpha_0(\Omega). \quad (3.2)
\]

Let \( \hat{J}(q) := J(u(q), q) \). With this notation, the optimal control problem (3.1)-(3.2) can be written as a reduced optimization problem:

\[
\text{min}_{q \in U_{ad}} \hat{J}(q).
\]

Note that the admissible set \( U_{ad} \) is closed and convex, and cost functional \( \hat{J} \) is strictly convex, then it admits a unique solution by a routine argument [24, 29]. The existence and uniqueness of the solution to the optimal control problem follows this result.

3.1 The first-order optimality condition

**Lemma 3.1** ([12]). Suppose that \( u, v \in L^2(\Omega) \) and vanish on boundary \( \partial \Omega \), then it holds that

\[
(L^\alpha_0 u, v) = (u, L^\alpha_1 v).
\]

**Theorem 3.2.** Suppose that \( q \in U_{ad} \) is an optimal control for the problem (1.1)-(1.3) and \( u \) is the associated state variable. Then there exists an adjoint state variable \( z \), such that \((u, z, q)\) satisfies the following optimality conditions

\[
\begin{align*}
\begin{cases}
L^\alpha_0 u + \lambda_1 Du + \lambda_2 u &= f(x) + q(x), \quad x \in \Omega, \\
u(x) &= 0, \quad x \in \partial \Omega,
\end{cases} \quad (3.3)
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
L^\alpha_1 z - \lambda_1 Dz + \lambda_2 z &= u(x) - u_d(x), \quad x \in \Omega, \\
z(x) &= 0, \quad x \in \partial \Omega,
\end{cases} \quad (3.4)
\end{align*}
\]
and the variational inequality

\[
\int_{\Omega} (\gamma q + z)(v - q) dx \geq 0, \quad v \in U_{ad}.
\]  

(3.5)

Proof. We denote by \(f'(w)(h)\) the Frechét derivative of \(f\) at \(w\) in the direction \(h\). The first-order necessary (and, owing to the convexity of cost functional \(\hat{J}\), also sufficient) optimality condition takes the form

\[
\hat{J}'(q)(v - q) = \int_{\Omega} u'(q)(v - q)(u - u_d) dx + \int_{\Omega} \gamma q(v - q) dx \geq 0, \quad \forall v \in U_{ad},
\]

(3.6)

In view of the state equation, \(\hat{u} := u'(q)(v - q)\) satisfies

\[
\begin{cases}
\mathcal{L}_\alpha \hat{u} + \lambda_1 D\hat{u} + \lambda_2 \hat{u} = v(x) - q(x), & x \in \Omega, \\
\hat{u}(x) = 0, & x \in \partial \Omega.
\end{cases}
\]

Introducing the following adjoint state equation

\[
\begin{cases}
\mathcal{L}_{1-\theta}^\alpha z - \lambda_1 Dz + \lambda_2 z = u(x) - u_d(x), & x \in \Omega, \\
z(x) = 0, & x \in \partial \Omega.
\end{cases}
\]

Then by using Lemma 3.1 we have

\[
\int_{\Omega} \hat{u}(u - u_d) dx = \int_{\Omega} \hat{u}(\mathcal{L}_{1-\theta}^\alpha z - \lambda_1 Dz + \lambda_2 z) dx
\]

\[
= \int_{\Omega} (\mathcal{L}_\alpha \hat{u} + \lambda_1 D\hat{u} + \lambda_2 \hat{u}) z dx = \int_{\Omega} (v - q) z dx.
\]

With this, (3.6) becomes

\[
\hat{J}'(q)(v - q) = \int_{\Omega} (\gamma q + z)(v - q) dx \geq 0, \quad v \in U_{ad}.
\]

Remark 3.3. According to [3], the variational inequality (3.5) is equivalent to the following condition

\[
\gamma q = \max\{0, \tilde{z}\} - z,
\]

(3.7)

in which \(\tilde{z} = \frac{1}{|\Omega|} \int_{\Omega} z(x) dx\) and \(|\Omega|\) is the interval size.
3.2 Regularity for optimal control problem in weighted Sobolev space

Lemma 3.4 ([14, 27]). Suppose $n \leq s < n + 1$, $n \in \mathbb{N}_0$, $p \geq 0$, $\mu > -1$ and $\psi \in H^s_{(\mu)}(J)$. If

$$0 \leq \vartheta \leq s, \quad \zeta + 2p \geq \mu, \quad \zeta + 2p - \vartheta > -1, \quad \zeta + 2p + \vartheta \geq \mu + s,$$

then $t^p \psi \in H^s_{(\zeta)}(J)$. Moreover, there exists a positive constant $C$ independent of $\psi$, such that

$$\|t^p \psi\|_{H^s_{(\zeta)}(J)} \leq C \|\psi\|_{H^s_{(\mu)}(J)}.$$

Lemma 3.5 ([14]). Let $s \geq 0$, $\phi \in H^s_{(\mu)}(J)$, $\mu > -1$, and $g \in C^s([s])$, then

$$\|g \phi\|_{H^s_{(\mu)}(J)} \leq C \|g\|_{C^s([s])} \|\phi\|_{H^s_{(\mu)}(J)},$$

where $\lceil s \rceil$ is the smallest integer greater than $s$.

Lemma 3.6. If $\omega^{-\sigma,-\sigma^*} u \in H^s_{\omega^\sigma,s^*}(\Omega)$ and $s \leq \min\{5\sigma + 1, 5\sigma^* + 1\}$, then $u \in H^\vartheta_{\omega^\sigma,s^*}(\Omega)$ for

$$\vartheta = \min\{s, 3\sigma + 1 - \varepsilon, 3\sigma^* + 1 - \varepsilon\},$$

with arbitrarily small $\varepsilon > 0$.

Proof. According to Lemma 2.4,

$$\omega^{-\sigma,-\sigma^*} u \in H^s_{\omega^\sigma,s^*}(\Omega) \iff \omega^{-\sigma,-\sigma^*} u = (1 - x)^{-\sigma} x^{-\sigma^*} u \in H^s_{(\sigma^*)}(J) \text{ and } \hat{\omega}^{-\sigma,-\sigma^*} \hat{u} = x^{-\sigma} (1 - x)^{-\sigma^*} \hat{u} \in H^s_{(\sigma^*)}(J).$$

• On one hand, from $(1 - x)^{-\sigma} x^{-\sigma^*} u \in H^s_{(\sigma^*)}(J)$ and Lemma 3.3 (obviously $(1 - x)^\sigma \in C^\infty(J)$) we know that

$$x^{-\sigma^*} u = (1 - x)^\sigma \cdot (1 - x)^{-\sigma} x^{-\sigma^*} u \in H^s_{(\sigma^*)}(J).$$

According to Lemma 3.4 let $s = s$, $\mu = p = \zeta = \sigma^*$, we have $\vartheta = \min\{s, 3\sigma^* + 1 - \varepsilon\}$. In fact,

$$\begin{cases} 0 \leq \vartheta \leq s, \\ \vartheta < 3\sigma^* + 1, \\ \vartheta + 2\sigma^* \geq s, \end{cases}$$

which results in

$$u \in H^\min\{s, 3\sigma^* + 1 - \varepsilon\}_{(\sigma^*)}(J). \quad (3.8)$$
On the other hand, from \( x^{-\sigma}(1-x)^{-\sigma^*} \hat{u} \in H^s_{(\sigma)}(J) \) and Lemma 3.5 we know that

\[
x^{-\sigma} \hat{u} = (1-x)^{\sigma^*} \cdot (1-x)^{-\sigma^*} x^{-\sigma} \hat{u} \in H^s_{(\sigma)}(J).
\]

According to Lemma 3.4, let \( s = s, \mu = p = \zeta = \sigma \), we have \( \vartheta = \min\{s, 3\sigma + 1 - \varepsilon\} \). Indeed,

\[
\begin{cases}
0 \leq \vartheta \leq s, \\
\vartheta < 3\sigma + 1, \\
\vartheta + 2\sigma \geq s,
\end{cases}
\]

which implies

\[
\hat{u} \in H^\min(s, 3\sigma + 1 - \varepsilon)(J).
\]

Putting (3.8) and (3.9) together and according to Lemma 2.4, it can be deduced that

\[
u \in H^\min(s, 3\sigma + 1 - \varepsilon, 3\sigma^* + 1 - \varepsilon)(\Omega).
\]

\( \square \)

**Lemma 3.7.** For the state equation (3.3) with right hand term \( \tilde{f} \in H^r_{\omega^*, \sigma}(\Omega), r \geq 0 \), there exists a unique solution \( u \) such that

\[
\omega^{-\sigma, -\sigma^*} u \in H^\min(r + \alpha, s)(\Omega), \]

\( u \in H^\min(r + \alpha, s)(\Omega), \)

in which \( s = 2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon. \)

**Proof.** The well-posedness and regularity of the state equation (3.3) has been already established in [14]: For the state equation (3.3) with right hand term \( f \in H^r_{\omega^*, \sigma}(\Omega) \), we have

\[
\omega^{-\sigma, -\sigma^*} u \in H^\min(r + \alpha, s)(\Omega).
\]

It is easy to verify that

\[
\min\{r + \alpha, s\} < \min\{5\sigma + 1, 5\sigma^* + 1\},
\]

so all the assumptions in Lemma 3.6 are satisfied, which leads to

\[
u \in H^\min(r + \alpha, s, 3\sigma + 1 - \varepsilon, 3\sigma^* + 1 - \varepsilon)(\Omega).
\]
According to $$\sigma^* + \sigma = \alpha$$ and $$\sigma^*$$, $$\sigma \in (0, 1]$$ in Remark 2.2, we know that $$3 \min \{\sigma, \sigma^*\} + 1 - \varepsilon > s$$, which implies

$$u \in H_{\omega^\sigma, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega).$$

A similar argument can be applied to the adjoint state equation to derive the following result.

**Lemma 3.8.** For the adjoint state equation (3.4) with right hand term $$g \in H_{\omega^\sigma, \sigma^*}^r(\Omega)$$, $$r \geq 0$$, there exists a unique solution $$z$$ such that

$$\omega^{-\sigma^*,-\sigma} z \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega),$$

in which $$s = 2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon$$.

**Theorem 3.9.** Assume that $$(u, z, q)$$ is the solution of the optimality system (1.1)-(1.3), if $$f \in H_{\omega^\sigma^*, \sigma^*}^r(\Omega)$$, $$u_d \in H_{\omega^\sigma^*, \sigma^*}^r(\Omega)$$ and $$q \in L^2(\Omega)$$, $$r \geq 0$$, then the regularity of the state $$u$$, adjoint state $$z$$ and control $$q$$ satisfy

$$u \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega), \quad z \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega), \quad q \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega),$$

respectively. Moreover, we have

$$\omega^{-\sigma^*,-\sigma} u \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega), \quad \omega^{-\sigma^*,-\sigma} z \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega), \quad \omega^{-\sigma^*,-\sigma} q \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega)$$

where $$s = 2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon$$.

**Proof.** Note that $$f \in H_{\omega^\sigma^*, \sigma^*}^r(\Omega)$$ and $$q \in L^2(\Omega) \subset H_{\omega^\sigma^*, \sigma^*}^0(\Omega)$$, thus $$f + q \in H_{\omega^\sigma^*, \sigma^*}^0(\Omega)$$ in equation (3.3). According to $$\alpha \in (1, 2)$$ and $$\sigma^*, \sigma \in (0, 1]$$ we observe

$$\alpha < s = 2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon < 2\alpha,$$

then by Lemma 3.7 it follows that

$$u \in H_{\omega^\sigma^*, \sigma^*}^{\min\{\alpha, s\}}(\Omega) = H_{\omega^\sigma^*, \sigma^*}^\alpha(\Omega).$$

Besides, for $$u_d \in H_{\omega^\sigma^*, \sigma^*}^r(\Omega)$$, $$u - u_d \in H_{\omega^\sigma^*, \sigma^*}^{\min\{\alpha, r\}}(\Omega)$$ in equation (3.4). By using Lemma 3.8 and noting that $$s < 2\alpha$$, we have

$$\omega^{-\sigma^*,-\sigma} z \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, 2\alpha, s\}}(\Omega), \quad \omega^{-\sigma^*,-\sigma} z \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega), \quad z \in H_{\omega^\sigma^*, \sigma^*}^{\min\{r+\alpha, s\}}(\Omega).$$
From (3.7), \( q \) has the same regularity as \( z \), i.e.

\[
\omega^{-\sigma,-\sigma} q \in H^{\min\{r+\alpha,s\}}_{\omega^{\sigma,-\sigma}}(\Omega), \quad q \in H^{\min\{r+\alpha,s\}}_{\omega^{\sigma,-\sigma}}(\Omega).
\]

Then for \( f + q \in H^{\min\{r,s\}}_{\omega^{\sigma,-\sigma}}(\Omega) \) in (3.3), by Lemma 3.7 we can obtain that

\[
\omega^{-\sigma,-\sigma} u \in H^{\min\{r+\alpha,s\}}_{\omega^{\sigma,-\sigma}}(\Omega), \quad u \in H^{\min\{r+\alpha,s\}}_{\omega^{\sigma,-\sigma}}(\Omega).
\]

This completes the proof. \( \square \)

Remark 3.10. Note that the regularity of solution \((u, z, q)\) given by Theorem 3.9 can not be further improved.

4 Spectral Petrov-Galerkin method

In this section, we consider a spectral Petrov-Galerkin method for the space fractional optimal control problem (1.1)-(1.3) and present its error estimates.

Let \( a : L^2_{\omega^{\sigma,-\sigma}}(\Omega) \times H^\alpha_{\omega^{\sigma,-\sigma}}(\Omega) \to \mathbb{R} \), \( b : L^2_{\omega^{\sigma,-\sigma}}(\Omega) \times H^\alpha_{\omega^{\sigma,-\sigma}}(\Omega) \to \mathbb{R} \),

\[
a(u, v) := (u, \mathcal{L}_{1-\beta}(\omega^{\sigma,-\sigma} v)) - \lambda_1(u, D(\omega^{\sigma,-\sigma} v)) + \lambda_2(u, v)_{\omega^{\sigma,-\sigma}},
\]

\[
b(z, w) := (z, \mathcal{L}_0^\alpha (\omega^{\sigma,-\sigma} w)) + \lambda_1(z, D(\omega^{\sigma,-\sigma} w)) + \lambda_2(z, w)_{\omega^{\sigma,-\sigma}}.
\]

We consider the spectral Petrov-Galerkin ultra-weak formulation for the optimal control problem (1.1)-(1.3): Given \( f \in H^r_{\omega^{\sigma,-\sigma}}(\Omega) \), \( u_d \in H^r_{\omega^{\sigma,-\sigma}}(\Omega) \) with \( r \geq 0 \), to find \((u, q)\) in

\( L^2_{\omega^{\sigma,-\sigma}}(\Omega) \times U_{\text{ad}} \) such that

\[
\min_{q \in U_{\text{ad}}} J(u, q) = \frac{1}{2} ||u - u_d||^2 + \frac{\gamma}{2} ||q||^2
\]

subject to

\[
a(u, v) = (f + q, v)_{\omega^{\sigma,-\sigma}}, \quad \forall v \in H^\alpha_{\omega^{\sigma,-\sigma}}(\Omega).
\]

Then the spectral Petrov-Galerkin ultra-weak formulation of the first-order optimality condition is: Given \( f \in H^r_{\omega^{\sigma,-\sigma}}(\Omega) \), \( u_d \in H^r_{\omega^{\sigma,-\sigma}}(\Omega) \) with \( r \geq 0 \), to find \((u, z, q)\) in

\( L^2_{\omega^{\sigma,-\sigma}}(\Omega) \times \)
\[ L^2_{\omega^{*},\sigma}(\Omega) \times U_{ad} \] such that

\[
\begin{cases}
a(u,v) = (f + q,v)_{\omega^{*},\sigma}, \forall v \in H^\alpha_{\omega^{*},\sigma}(\Omega), \\
b(z,w) = (u - u_d,w)_{\omega^{*},\sigma}, \forall w \in H^\alpha_{\omega^{*},\sigma}(\Omega), \\
(\gamma q + z,v - q) \geq 0, \forall v \in U_{ad}.
\end{cases}
\]

We introduce the finite dimensional spaces

\[ U_N = \{ u | u = \omega v, v \in W_N \}, \quad W_N = \text{span}\{Q_m^{\sigma^{*}}\}_{m=0}^N \subset H^\alpha_{\omega^{*},\sigma}(\Omega), \]
\[ Z_N = \{ z | z = \omega v, v \in V_N \}, \quad V_N = \text{span}\{Q_m^{\sigma^{*}}\}_{m=0}^N \subset H^\alpha_{\omega^{*},\sigma}(\Omega). \]

The spectral Petrov-Galerkin scheme for the optimal control problem is: Given \( f \in H^r_{\omega^{*},\sigma}(\Omega) \), \( u_d \in H^r_{\omega^{*},\sigma}(\Omega) \) with \( r \geq 0 \), to find \( (u_N,q_N) \in U_N \times U_{ad} \) such that

\[
\min_{q_N \in U_{ad}} J(u_N,q_N) = \frac{1}{2}\|u_N - u_d\|^2 + \frac{\gamma}{2}\|q_N\|^2
\]
subject to

\[
a(u_N,v_N) = (f + q_N,v_N)_{\omega^{*},\sigma}, \forall v_N \in V_N.
\]

Similar to the infinite dimensional problem (1.1)-(1.3), the above discrete problem also admits a unique solution.

In order to obtain the discrete first-order optimality condition of the optimal control problem, we define a Lagrange functional as

\[ L(u_N,z_N,q_N) := J(u_N,q_N) + (f + q_N,z_N) - (u_N,L^{\alpha}_1z_N) + \lambda_1(u_N,Dz_N) - \lambda_2(u_N,z_N). \]

Then, the discrete optimality conditions can be deduced by computing the Fréchet derivatives as follows:

\[
\frac{\partial L(u_N,z_N,q_N)}{\partial u_N}(w_N) = 0, \forall w_N \in U_N \quad \text{and} \quad \frac{\partial L(u_N,z_N,q_N)}{\partial q_N}(v - q_N) \geq 0, \forall v \in U_{ad},
\]

and the following equations can be obtained

\[
(L^{\alpha}_1z_N,w_N) - \lambda_1(Dz_N,w_N) + \lambda_2(z_N,w_N) = (u_N - u_d,w_N), \forall w_N \in U_N,
\]
\[
(\gamma q_N + z_N,v - q_N) \geq 0, \forall v \in U_{ad}.
\]
Therefore, the discrete first-order optimality condition can be summarized as: Given $f \in H^r_{\omega^*,\sigma}(\Omega)$, $u_d \in H^r_{\omega^*,\sigma}(\Omega)$ with $r \geq 0$, to find $(u_N, z_N, q_N) \in U_N \times Z_N \times U_{ad}$ such that

$$
\begin{align*}
& a(u_N, v_N) = (f + q_N, v_N)_{\omega^*,\sigma}, \quad \forall v_N \in V_N, \quad (4.2a) \\
& b(z_N, w_N) = (u_N - u_d, w_N)_{\omega^*,\sigma}, \quad \forall w_N \in W_N, \quad (4.2b) \\
& (\gamma q_N + z_N, v - q_N) \geq 0, \quad \forall v \in U_{ad}. \quad (4.2c)
\end{align*}
$$

To establish the error estimates of this problem, we need to consider the following auxiliary problem:

$$
\begin{align*}
& a(u_N(q), v_N) = (f + q, v_N)_{\omega^*,\sigma}, \quad \forall v_N \in V_N, \quad (4.3a) \\
& b(z_N(q), w_N) = (u_N(q) - u_d, w_N)_{\omega^*,\sigma}, \quad \forall w_N \in W_N, \quad (4.3b) \\
& b(z_N(u), w_N) = (u - u_d, w_N)_{\omega^*,\sigma}, \quad \forall w_N \in W_N. \quad (4.3c)
\end{align*}
$$

**Lemma 4.1** ([19, 37]). There exists a constant $C > 0$, such that for $N$ sufficiently large,

$$
\sup_{0 \neq v_N \in V_N} \frac{a(u_N, v_N)}{\|v_N\|_{H^0_{\omega^*,\sigma}}} \geq C \|u_N\|_{\omega^*,\omega}, \quad \forall u_N \in U_N, u_N \neq 0. \quad (4.4)
$$

**Lemma 4.2** ([19, 37]). Assume that $u$ and $u_N$ are solutions of the state equation (4.1a) and its discrete counterpart with right hand term $\tilde{f}$, respectively. If $\tilde{f} \in H^r_{\omega^*,\sigma}(\Omega)$, $r \geq 0$, then there exists a number $N_0 > 0$, such that when $N > N_0$, we have the following error estimate

$$
\|u - u_N\|_{\omega^*,\omega} \leq C N^{-m} |\omega^*,\omega| u|_{H^m_{\omega^*,\omega}}, \quad (4.5)
$$

where $m = \min\{r + \alpha, 2\alpha + \min(\sigma, \sigma^*) - 1 - \epsilon\}$.

An analogous argument can be applied to the adjoint state equation to obtain the Lemmas as follows.

**Lemma 4.3.** There exists a constant $C > 0$, such that for $N$ sufficiently large,

$$
\sup_{0 \neq w_N \in W_N} \frac{b(z_N, w_N)}{\|w_N\|_{H^0_{\omega^*,\sigma}}} \geq C \|z_N\|_{\omega^*,\omega}, \quad \forall z_N \in Z_N, z_N \neq 0. \quad (4.6)
$$

**Lemma 4.4.** Assume that $z$ and $z_N$ are solutions of the adjoint state equation (4.1b) and its discrete counterpart with right hand term $g$, respectively. If $g \in H^r_{\omega^*,\sigma}(\Omega)$, $r \geq 0$, then there exists a number $N_0 > 0$, such that when $N > N_0$, we have the following error estimates

$$
\|z - z_N\|_{\omega^*,\omega} \leq C N^{-m} |\omega^*,\omega| z|_{H^m_{\omega^*,\omega}}, \quad (4.7)
$$
where \( m = \min\{r + \alpha, 2\alpha + \min(\sigma, \sigma^*) - 1 - \varepsilon\}. \)

**Theorem 4.5.** Assume that \((u, z, q)\) and \((u_N, z_N, q_N)\) are solutions of (4.1) and (4.2), respectively. If \( f \in H^r_{\omega^*, \sigma}(\Omega), u_d \in H^r_{\omega^*, \sigma}(\Omega) \) with \( r \geq 0 \), we have the following error estimate

\[
\|u - u_N\|_{\omega^*, \sigma} + \|z - z_N\|_{\omega^*, \sigma} + \|q - q_N\| \leq CN^{-m},
\]

where \( m = \min\{r + \alpha, 2\alpha + \min(\sigma, \sigma^*) - 1 - \varepsilon\}. \)

**Proof.** The errors can be expressed by

\[
u - u_N = u - u_N(q) + u_N(q) - u_N, \\
z - z_N = z - z_N(u) + z_N(u) - z_N.
\]

From (4.1), (4.3a) and (4.3c), we know that \( u_N(q) \) and \( z_N(u) \) are the spectral Petrov-Galerkin approximation of \( u \) and \( z \), respectively. By the estimates (4.5), (4.7) and Theorem 3.9, we have

\[
\|u - u_N(q)\|_{\omega^*, \sigma^*} \leq CN^{-m}, \tag{4.9}
\]

\[
z - z_N(u)\|_{\omega^*, \sigma} \leq CN^{-m}, \tag{4.10}
\]

where \( m = \min\{r + \alpha, 2\alpha + \min(\sigma, \sigma^*) - 1 - \varepsilon\} \) determined by the regularity of state and adjoint state in Theorem 3.9.

Firstly, we give the error estimate of \( u_N \). Subtracting (4.3a) from (4.2a), it follows

\[
a(u_N - u_N(q), v_N) = (u_N - u_N(q), L^\alpha_{1-\theta}(\omega^*, \sigma v_N) - \lambda_1 D(\omega^*, \sigma v_N) + \lambda_2 \omega^*, \sigma v_N)
\]

\[
= (q_N - q, v_N)\omega^*, \sigma, \forall v_N \in V_N. \tag{4.11}
\]

By (4.4) and the Cauchy-Schwarz inequality we have

\[
C\|u_N - u_N(q)\|_{\omega^*, \sigma^*} \leq \sup_{0 \neq \varepsilon N \in V_N} \frac{a(u_N - u_N(q), v_N)}{\|v_N\|_{\omega^*, \sigma}} \leq \sup_{0 \neq \varepsilon N \in V_N} \frac{\|q - q_N\|_{\omega^*, \sigma} \|v_N\|_{\omega^*, \sigma}}{\|v_N\|_{\omega^*, \sigma}} \leq \|q - q_N\| \tag{4.12}
\]

This combined with (4.9) implies

\[
\|u - u_N\|_{\omega^*, \sigma^*} \leq CN^{-m} + C\|q - q_N\|. \tag{4.13}
\]

Secondly, The error estimate of \( z_N \) can be obtained in a similar way. Substracting (4.3c) from (4.2b) gives

\[
b(z_N - z_N(u), w_N) = (u_N - u, w_N)_{\omega^*, \sigma^*}, \forall w_N \in W_N. \tag{4.14}
\]
Then by (4.6) and the Cauchy-Schwarz inequality we have

\[ C \| z_N - z_N(u) \|_{\omega^{-\sigma, -\sigma}} \leq \sup_{0 \neq w_N \in W_N} \frac{b(z_N - z_N(u), w_N)}{\| w_N \|_{H^{\omega, \sigma^*}}} \]
\[
\leq \sup_{0 \neq w_N \in W_N} \frac{\| u - u_N \|_{\omega^\sigma} \| w_N \|_{\omega^\sigma}}{\| w_N \|_{H^{\omega, \sigma^*}}} \leq \| u - u_N \|_{\omega^{-\sigma, -\sigma}^*} \quad (4.15)\]

Combining with (4.10) and (4.13), we get

\[ \| z - z_N \|_{\omega^{-\sigma, -\sigma}} \leq C N^{-m} + C \| q - q_N \|. \quad (4.16)\]

Finally, we determine the error estimate of \( q_N \). By subtracting (4.2b) from (4.3b) we have

\[ b(z_N(q) - z_N, w_N) = (z_N(q) - z_N, \mathcal{L}^{\omega^\sigma}_{\theta}(w_N) + \lambda_1 D(z_N, w_N) + \lambda_2 \omega^\sigma w_N) \]
\[
= (u_N(q) - u_N, w_N), \quad \forall w_N \in W_N. \quad (4.17)\]

It follows from (3.5) with \( v = q_N \) and (4.2c) with \( v = q \) that

\[ \gamma \| q - q_N \|^2 = \int_{\Omega} \gamma q (q - q_N) dx - \int_{\Omega} \gamma q_N (q - q_N) dx \]
\[ \leq \int_{\Omega} z (q_N - q) dx + \int_{\Omega} z_N (q - q_N) dx \]
\[ = \int_{\Omega} (z - z_N(q)) (q_N - q) dx + \int_{\Omega} (z_N(q) - z)(q_N - q) dx. \]

By taking \( \omega^\sigma w_N = z_N(q) - z_N \) in (4.11) and \( \omega^\sigma w_N = u_N - u_N(q) \) in (4.11), we have

\[ \int_{\Omega} (z_N(q) - z_N)(q_N - q) dx \]
\[ = (u_N - u_N(q), \mathcal{L}^{\omega^\sigma}_{\theta}(z_N(q) - z_N) - \lambda_1 D(z_N(q) - z_N) + \lambda_2(z_N(q) - z_N)) \]
\[ = (z_N(q) - z_N, \mathcal{L}^{\omega^\sigma}_{\theta}(u_N - u_N(q)) + \lambda_1 D(u_N - u_N(q)) + \lambda_2(u_N - u_N(q))) \]
\[ = -(u_N - u_N(q), u_N - u_N(q)) \leq 0. \]

So we have

\[ \gamma \| q - q_N \|^2 \leq \int_{\Omega} (z - z_N(q)) (q_N - q) dx \]
\[
\leq \|z - z_N(q)\|_{\omega^{-\sigma^*,-\sigma}} \|q - q_N\|_{\omega^{\sigma^*,\sigma}}
\]

\[
\leq \|z - z_N(q)\|_{\omega^{-\sigma^*,-\sigma}} \|q - q_N\|
\]

Considering the above inequality and 
\[z - z_N(q) = z - z_N(u) + z_N(u) - z_N(q),\]
we get
\[
\|q - q_N\| \leq C \left( \|z - z_N(u)\|_{\omega^{-\sigma^*,-\sigma}} + \|z_N(u) - z_N(q)\|_{\omega^{-\sigma^*,-\sigma}} \right). \tag{4.18}
\]

Subtracting (4.3b) from (4.3c), it follows
\[
b(z_N(u) - z_N(q), w_N) = (u - u_N(q), w_N), \forall w_N \in W_N.
\]

Similar to the above derivation, by using (4.5), (4.6) and Theorem 3.9, we have
\[
\|z_N(u) - z_N(q)\|_{\omega^{-\sigma^*,-\sigma}} \leq C \|u - u_N(q)\|_{\omega^{-\sigma^*,-\sigma}} \leq CN^{-m}. \tag{4.19}
\]

Substituting (4.10) and (4.19) into (4.18), it is obtained that
\[
\|q - q_N\| \leq CN^{-m}. \tag{4.20}
\]

Combining the estimates (4.13), (4.16) and (4.20) lead us to attain the conclusion (4.8). \qed

5 Numerical experiments

In this section, we present a fast iteration algorithm for solving the discrete optimality conditions (4.2) produced by the spectral Petrov-Galerkin method and give some numerical examples to verify the theoretical results.

5.1 Numerical implementation

Substituting \( u_N = \omega^{\sigma^*,\sigma} \sum_{n=0}^{N} \hat{u}_n Q_n^{\sigma^*,\sigma}, \ z_N = \omega^{\sigma^*,\sigma} \sum_{n=0}^{N} \hat{z}_n Q_n^{\sigma^*,\sigma} \) into the discrete optimality conditions (4.2) and taking \( v_N = Q_m^{\sigma^*,\sigma}, \ w_N = Q_m^{\sigma^*,\sigma}, \ m = 0, 1, \ldots, N, \) we obtain the following linear system by using (2.2) and (2.4)

\[
\begin{cases}
(S - \lambda_1 D + \lambda_2 M)U = F, \tag{5.1a} \\
(S + \lambda_1 \hat{D} + \lambda_2 M^T)Z = G, \tag{5.1b} \\
\gamma q_N = \max\{0, \hat{z}_0 h_0^{\sigma^*,\sigma}\} - z_N. \tag{5.1c}
\end{cases}
\]

where
\[
S = \text{diag}(\lambda_{1-\theta,0} h_0^{\sigma^*,\sigma}, \lambda_{1-\theta,1} h_1^{\sigma^*,\sigma}, \ldots, \lambda_{1-\theta,N} h_N^{\sigma^*,\sigma}).
\]

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\[ U = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_N)^T, \quad Z = (\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_N)^T, \]

\[ F = (f_0, f_1, \ldots, f_N)^T, \quad G = (g_0, g_1, \ldots, g_N)^T, \]

in which \( f_m = (f + q_N, Q_m^{\sigma^*, \sigma})_{\omega^*, \sigma}, \ g_m = (u_N - u_d, Q_m^{\sigma^*, \sigma})_{\omega^*, \sigma}, \ m = 0, 1, \ldots, N. \)

Moreover, \( M, D \) and \( \hat{D} \) are the \((N + 1) \times (N + 1)\) matrices with entries

\[ M_{mn} = (\omega^{\alpha, \alpha} Q_m^{\sigma^*, \sigma}, Q_m^{\sigma^*, \sigma}), \]
\[ D_{mn} = -(m + 1)(\omega^{\alpha-1, \alpha-1} Q_m^{\sigma^*, \sigma}, Q_{m+1}^{\sigma^*, \sigma-1}), \]
\[ \hat{D}_{mn} = -(m + 1)(\omega^{\alpha-1, \alpha-1} Q_m^{\sigma^*, \sigma}, Q_{m+1}^{\sigma^*, \sigma-1}). \]

The above linear optimization system (5.1) can be solved by the projected gradient method, see [22, 26, 32] for more details, which requires \( O(N^2) \) storage and \( O(N^3) \) computational complexity. To improve the computational complexity, following the idea in [19] we present a fast projected gradient method with linear storage \( O(N) \) and quasilinear computational cost \( O(N \log^2 N) \).

Denote \( A = S - \lambda_1 D + \lambda_2 M, \ B = S + \lambda_1 \hat{D} + \lambda_2 M^T, \) we use the fixed-point iteration

\[ U^{m+1} = U^m + P^{-1}(F - AU^m), \tag{5.2} \]
\[ Z^{m+1} = Z^m + \hat{P}^{-1}(G - BZ^m), \tag{5.3} \]

where the preconditioners \( P = S - \lambda_1 K + \lambda_2 Q \) and \( \hat{P} = S + \lambda_1 \hat{K} + \lambda_2 Q \) with

\[ Q = \text{diag}(h_0^{\alpha, \alpha}, h_1^{\alpha, \alpha}, \ldots, h_N^{\alpha, \alpha}), \tag{5.4} \]

\( K \) and \( \hat{K} \) are two tridiagonal matrices with

\[ K_{n,n+1} = D_{n,n+1}, \quad K_{n+1,n} = D_{n+1,n}, \]
\[ \hat{K}_{n,n+1} = \hat{D}_{n,n+1}, \quad \hat{K}_{n+1,n} = \hat{D}_{n+1,n}, \ n = 0, 1, \ldots, N, \]

and other entries are zero. In each iteration, without forming the matrices \( A \) and \( B \), we compute the matrix-vector products \( AU \) and \( BZ \) by using the fast polynomial transforms and the fast matrix-vector product [31] for Toeplitz-dot-Hankel matrices (can be written as a Hadamard product of a Toeplitz and Hankel matrix).
Let \( Q^{\gamma,\beta}_k = (Q^{\gamma,\beta}_0, Q^{\gamma,\beta}_1, \cdots, Q^{\gamma,\beta}_k)^T, \ k \geq 0. \) we have the following transformation formulas of Jacobi polynomials

\[
Q^{\gamma,\beta}_k = C^{\gamma \rightarrow \sigma,\beta}_k Q^{\sigma,\beta}_k, \quad Q^{\sigma,\delta}_k = C^{\sigma,\delta \rightarrow \beta}_k Q^{\sigma,\beta}_k.
\] (5.5)

Here \( C^{\gamma \rightarrow \sigma,\beta}_k \) and \( C^{\sigma,\delta \rightarrow \beta}_k \) are \((k+1) \times (k+1)\) lower triangular conversion matrices and can be decomposed into matrices

\[
C^{\gamma \rightarrow \sigma,\beta}_k = \overline{D}_1(T_1 \circ H_1)\overline{D}_2, \quad C^{\sigma,\delta \rightarrow \beta}_k = \overline{D}_3(T_2 \circ H_2)\overline{D}_4,
\] (5.6)

where \( \overline{D}_i, \ i = 1,2,3,4, \) are diagonal matrices, \( T_i, \ i = 1,2 \) are lower triangular Toeplitz matrices, \( H_i, \ i = 1,2 \) are Hankel matrices and ‘\( \circ \)’ is the Hadamard matrix product. We refer to [19] for explicit entries of these matrices.

Denote \( u_N = (1-x)^\sigma x^\sigma \tilde{u}_N, \) by using (5.6) we have

\[
\tilde{u}_N = \sum_{n=0}^{N} \hat{u}_n Q^{\sigma,\sigma}_n = (Q^\alpha_0)^T \hat{U} = (Q^\alpha_0)^T (C^{\sigma,\sigma \rightarrow \sigma}_N)^T (C^{\sigma \rightarrow \alpha,\sigma}_N)^T U := (Q^\alpha_0)^T U^\alpha,
\] (5.7)

where \( U^\alpha = (\hat{u}_0^\alpha, \hat{u}_1^\alpha, \cdots, \hat{u}_N^\alpha)^T \) can be obtained by the fast Jacobi-to-Jacobi polynomial transform [31]. Note that \( (Q^\alpha_N)^T U^\alpha = \sum_{n=0}^{N} \hat{u}_n^\alpha Q^{\alpha,\alpha}_n. \)

In the following we show the algorithm for computing \( AU = (S - \lambda_1 D + \lambda_2 M)U. \) Note that \( S \) is a diagonal matrix, so we just need consider \( DU \) and \( MU. \) For the fast matrix-vector product of \( MU, \) we have

\[
MU = \int_\Omega \omega^{\alpha,\alpha} Q^{\sigma^*,\sigma}_N (Q^{\sigma,\sigma^*}_N)^T d\omega U
\]

\[
= \int_\Omega \omega^{\alpha,\alpha} C^{\sigma^*,\sigma \rightarrow \alpha}_N C^{\sigma^*,\alpha,\sigma^*}_N Q^{\alpha,\alpha}_N (C^{\sigma \rightarrow \alpha,\sigma^*}_N C^{\sigma,\alpha,\sigma^*}_N)^T d\omega U
\] (5.8)

\[
= C^{\sigma^*,\sigma \rightarrow \alpha}_N C^{\sigma^*,\alpha,\sigma^*}_N Q \ (C^{\alpha,\alpha,\sigma^*}_N)^T (C^{\sigma^*,\alpha,\sigma^*}_N)^T U
\]

\[
= C^{\sigma^*,\sigma \rightarrow \alpha}_N C^{\sigma^*,\alpha,\sigma^*}_N Q U^\alpha.
\]

By (5.6), \( C^{\sigma^*,\sigma \rightarrow \alpha}_N \) and \( C^{\sigma^*,\alpha,\sigma^*}_N \) both are Toeplitz-dot-Hankel matrices, so \( MU \) can be computed without forming a matrix by using the fast matrix-vector product [31].
To fulfill the fast matrix-vector product for $DU$, we denote
\[
\hat{Q} = \text{diag}(h_0^{\alpha,\alpha}, h_1^{\alpha,\alpha}, \cdots, h_N^{\alpha,\alpha}, 0),
\] (5.9)
\[
U^{\alpha-1} = (\hat{u}_0^{\alpha-1}, \hat{u}_1^{\alpha-1}, \cdots, \hat{u}_N^{\alpha-1})^T,
\] (5.10)
\[
\Lambda = -\text{diag}(1, 2, \cdots, N + 1),
\] (5.11)
and introduce a $(N+2) \times (N+1)$ matrix $W$ defined as
\[
W = \int_{\Omega} \omega^{\alpha-1,\alpha} Q_{n+1}^{\sigma^* - 1, \sigma^* - 1} (Q_{\sigma^*}^{\sigma,\sigma^*})^T dx
\]
\[
= \int_{\Omega} \omega^{\alpha-1,\alpha} C_{N+1}^{\sigma^* - 1, \sigma^* - 1} \rightarrow \alpha - 1 \rightarrow \alpha - 1 (C_{\sigma^*}^{\sigma^* - 1, \sigma^* - 1} C_{\sigma^*}^{\sigma^* - 1, \sigma^* - 1} \rightarrow \alpha - 1 \rightarrow \alpha - 1 (Q_{\sigma^*}^{\sigma^* - 1, \sigma^* - 1} \rightarrow \alpha - 1 \rightarrow \alpha - 1)^T dx
\]
\[
= C_{N+1}^{\sigma^* - 1, \sigma^* - 1} \rightarrow \alpha - 1 \rightarrow \alpha - 1 \rightarrow \alpha - 1 \hat{Q}(C_{\sigma^*}^{\sigma^* - 1, \sigma^* - 1})^T (C_{\sigma^*}^{\sigma^* - 1, \sigma^* - 1})^T.
\]
Similarly, we have
\[
\hat{u}_N = \sum_{n=0}^{N} \hat{u}_n Q_{\sigma,\sigma^*}^{\sigma^* - 1, \sigma^* - 1} = (Q_{N}^{\sigma,\sigma^*})^T U = (Q_{N}^{\alpha-1,\alpha-1})^T (C_{\sigma}^{\alpha-1,\sigma^* - 1} \rightarrow \alpha - 1 \rightarrow \alpha - 1 (C_{\sigma}^{\sigma^* - 1, \sigma^* - 1})^T U
\]
\[
:= (Q_{N}^{\alpha-1,\alpha-1})^T U^{\alpha-1} = \sum_{n=0}^{N} \hat{u}_n^{\alpha-1} Q_{\sigma,\sigma^*}^{\sigma^* - 1, \sigma^* - 1}.
\]
Therefore, $WU$ also can be obtained without forming a matrix by using the fast matrix-vector product combined with the fast Jacobi-to-Jacobi polynomial transform. For the advection term, we can write
\[
DU = \Lambda \hat{W} U,
\] (5.12)
where $\hat{W}$ denotes the sub-matrix of $W$ formed by removing its first row, and $\hat{W}U$ can be obtained conveniently by removing the first element of vector $WU$.

In summary, for the fast calculation of $AU$, we give
\[
AU = SU - \lambda_1 \Lambda \hat{W} U + \lambda_2 C_{\sigma^*}^{\sigma^* - 1, \sigma^* - 1} C_{\sigma^*}^{\sigma^* - 1, \sigma^* - 1} \rightarrow \alpha - \alpha Q U^{\alpha},
\] (5.13)
where $\Lambda$, $Q$, $C_{\sigma^*}^{\sigma^* - 1, \sigma^* - 1}$, $U^{\alpha}$ and $\hat{W}U$ are given above, see (5.11), (5.4), (5.5), (5.7) and (5.12), respectively.

Fast calculation of matrix-vector product $BZ$ in the fixed-point iteration (5.3) can be fulfilled by an analogous way. We omit the details here for brevity. Moreover, the right hand side $F$ can also be obtained in a similar manner, see (5.15) in next subsection for more details.
The fast projected gradient method can be summarized as follows:

**Algorithm 1** Fast projected gradient algorithm

1: Solving the system (5.1) with \( N = 8 \) by the general projected gradient method combined with a direct solver to get the initial guess \((U^0, Z^0, q_N)\).
2: Setting error = 1, and given the tolerance \( \varepsilon \).
3: while error > \( \varepsilon \) do
4: Solving the state equation in (5.1a) by the fixed-point iteration (5.2):
   \[ U^{m+1} = U^m + P^{-1}(F - AU^m), \]
   where \( AU^m \) can be computed by (5.13).
5: Solving the adjoint state equation in (5.1b) by the fixed-point iteration (5.3):
   \[ Z^{m+1} = Z^m + \hat{P}^{-1}(G - BZ^m), \]
   where \( BZ^m \) can be computed in a similar manner to \( AU^m \).
6: Computing the control variable \( q_N \):
   \[ q_N^{\text{new}} = \max\{0, \frac{\hat{z}_0 h_0^{\ast,\sigma}}{\gamma} - \frac{z_N}{\gamma}\}, \]
7: Calculate the error by error := \( \text{norm}(q_N - q_N^{\text{new}}, \text{inf})/\text{norm}(q_N, \text{inf}) \).
8: Update the control variable \( q_N = q_N^{\text{new}} \).
9: end while

5.2 Numerical results

In the optimal control problem (1.1)-(1.2) with \( \gamma = 1 \), we take
\[ f = (1 - x)\beta x^{\beta} \sin x, \quad u_d = (1 - x)\beta x^{\beta} \cos x. \] (5.14)

The right hand side \( f_m \) and \( g_m \) can be computed as \( (MU)_m \). Specifically, we evaluate \( \sin x \) and \( \cos x \) at the Chebyshev collocation points \( x_i \) \((1 \leq i \leq M, \ M \geq N)\), and by the inverse fast Chebyshev transform [31] we can obtain the approximations
\[ f \approx \omega^{\beta,\beta} \sum_{n=0}^{M} \hat{f}^{-1/2} Q^{-1/2,-1/2}, \quad u_d \approx \omega^{\beta,\beta} \sum_{n=0}^{M} \hat{g}^{-1/2} Q^{-1/2,-1/2}. \] (5.15)

Then invoking the fast Jacobi-to-Jacobi polynomial transform separately on \( f \) and \( q_N \) in \( f_m \) \((u_d \text{ and } u_N \text{ in } g_m)\), the right hand terms \( F \) and \( G \) in (5.1) can be observed in an analogous treatment as in (5.8).

In the computation, we take \( \lambda_1 = \lambda_2 = 1 \) and measure the errors in the following sense:
\[ E_N^{a,b}(p) = \frac{\|P_N - P_Nr\|_{\omega^{a,b}}}{\|P_Nr\|_{\omega^{a,b}}}, \]
where $a, b > -1$, $p_N$ is the numerical solution and $p_{N_r}$ is the reference solution computed by the same solver but with a very fine resolution, $N_r = 2^{14}$. In Table 1 cited from [20], we present the numerical values of $(\sigma, \sigma^*)$ corresponding to different $\alpha$ and $\theta$, which were calculated by using Newton’s method with a tolerance $10^{-16}$.

**Example 5.1.** Take $\beta = 0$ in (5.14) and $\theta = 0.7$ in (1.2). Note that $f \in H^{2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon}(\Omega)$, $u_d \in H^{2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon}(\Omega)$.

As $f$ and $u_d$ are analytic, we have

$$
\omega^{-\sigma, -\sigma^*} u \in H^{2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon}(\Omega), \quad \omega^{-\sigma^*, -\sigma} z \in H^{2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon}(\Omega)
$$

by Theorem 3.9 From Theorem 4.5 the convergence order of numerical solutions $(u_N, z_N, q_N)$ is expected to be $2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon$. We test the performance of the spectral Petrov-Galerkin method and the fast projected gradient method for different values of $\alpha$. In Table 2, we measure the convergence orders and errors of state $u_N$ in $\| \cdot \|_{\omega^{-\sigma, -\sigma^*}}$ norm, adjoint state $z_N$ and control $q_N$ in $\| \cdot \|_{\omega^{-\sigma^*, -\sigma}}$ norm. In Table 3, we show the convergence orders and errors of numerical solutions in standard $L^2$-norm.

Data in Table 2 show that the convergence orders of $u_N$ in $\| \cdot \|_{\omega^{-\sigma, -\sigma^*}}$ norm and $z_N$ in $\| \cdot \|_{\omega^{-\sigma^*, -\sigma}}$ norm are $2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon$, as expected from Theorem 4.5 In Table 3, we observe that the convergence orders of control $q_N$ in standard $L^2$-norm are better than expected $2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon$, while its convergence orders in $\| \cdot \|_{\omega^{-\sigma^*, -\sigma}}$ norm are numerically close to $2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon$, see Table 2. Moreover, numerical results in Tables 2 and 3 also show that the convergence orders and accuracy of numerical solutions $(u_N, z_N, q_N)$ in standard $L^2$-norm are higher than these in weighted $L^2$-norm with negative index, which numerically indicate that it is meaningful to consider the spectral Petrov-Galerkin method for the optimal control problem in weighted Sobolev space.

In the last two rows of Tables, we list the iterative numbers and CPU time required for the fast projected gradient algorithm with different values of $N$ and $\alpha$. We observe that the CPU time increases roughly as $O(N \log^2 N)$ and the iteration numbers are independent on truncation numbers $N$. However, the iteration numbers decrease with $\alpha$: when $\alpha = 1.2$, iteration number is 51 while the number is 9 for $\alpha = 1.8$, which suggests the need of searching for better preconditioner $P$.

**Table 1:** The numerical values of $(\sigma, \sigma^*)$ corresponding to different $\alpha$ and $\theta$ (cited from [20]).

| $\theta$ | $\alpha=1.2$ | $\alpha=1.4$ | $\alpha=1.6$ | $\alpha=1.8$ |
|-----------|--------------|--------------|--------------|--------------|
| 0.5       | (0.6000, 0.6000) | (0.7000, 0.7000) | (0.8000, 0.8000) | (0.9000, 0.9000) |
| 0.7       | (0.8829, 0.3171) | (0.8602, 0.5398) | (0.8900, 0.7100) | (0.9411, 0.8589) |
| 1.0       | (1.0000, 0.2000) | (1.0000, 0.4000) | (1.0000, 0.6000) | (1.0000, 0.8000) |

**Example 5.2.** Take $\beta = -0.4$ in (5.14) and $\theta = 0.5, 0.7, 1$ in (1.2). Note that

$$
f \in H^{2\beta + \min\{\sigma, \sigma^*\} + 1 - \varepsilon}(\Omega), \quad u_d \in H^{2\beta + \min\{\sigma, \sigma^*\} + 1 - \varepsilon}(\Omega),$$

by Lemma B.4 in [20].
Table 2: Convergence orders and errors of the spectral Petrov-Galerkin method with $\theta = 0.7$, $f = \sin x$ and $u_d = \cos x$ in weighted norm. The expected convergence order is $2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon$ (Theorems 3.9 and 4.5).

| $\alpha$ | $N$ | $E_N^{0,0}(u)$ | order | $E_N^{0,0}(z)$ | order | $E_N^{0,0}(q)$ | order | iter | CPU(s) |
|--------|-----|----------------|-------|----------------|-------|----------------|-------|------|--------|
| 1.2    | 128 | 3.74e-05       |       | 5.61e-05       |       | 5.61e-05       |       | 51   | 6.70   |
|       | 256 | 7.46e-06       | 2.33  | 1.15e-05       | 2.28  | 1.15e-05       | 2.28  | 51   | 7.46   |
|       | 512 | 1.50e-06       | 2.27  | 2.51e-06       | 2.20  | 2.51e-06       | 2.20  | 51   | 10.16  |
|       | 1024| 3.38e-07       | 2.20  | 5.83e-07       | 2.10  | 5.83e-07       | 2.10  | 51   | 27.25  |
| Expected order |    | 1.72 |       | 1.72 |       |       |       |      |        |
| 1.4    | 128 | 1.56e-06       |       | 2.41e-06       |       | 2.41e-06       |       | 15   | 1.17   |
|       | 256 | 3.05e-07       | 2.35  | 4.78e-07       | 2.33  | 4.78e-07       | 2.33  | 15   | 1.50   |
|       | 512 | 6.00e-08       | 2.35  | 9.46e-08       | 2.34  | 9.46e-08       | 2.34  | 15   | 2.14   |
|       | 1024| 1.18e-08       | 2.35  | 1.87e-08       | 2.34  | 1.87e-08       | 2.34  | 15   | 6.25   |
| Expected order |    | 2.34 |       | 2.34 |       |       |       |      |        |
| 1.6    | 128 | 8.47e-08       |       | 1.09e-07       |       | 1.09e-07       |       | 12   | 0.68   |
|       | 256 | 1.15e-08       | 2.89  | 1.15e-05       | 2.88  | 1.15e-05       | 2.88  | 12   | 0.97   |
|       | 512 | 1.54e-09       | 2.90  | 2.51e-06       | 2.90  | 1.99e-09       | 2.90  | 12   | 1.34   |
|       | 1024| 2.65e-10       | 2.91  | 5.85e-07       | 2.90  | 2.66e-10       | 2.90  | 12   | 4.08   |
| Expected order |    | 2.91 |       | 2.91 |       |       |       |      |        |
| 1.8    | 128 | 3.02e-09       |       | 3.29e-09       |       | 3.29e-09       |       | 9    | 0.48   |
|       | 256 | 2.81e-10       | 3.42  | 3.07e-10       | 3.42  | 3.07e-10       | 3.42  | 9    | 0.69   |
|       | 512 | 2.58e-11       | 3.44  | 2.83e-11       | 3.44  | 2.83e-11       | 3.44  | 9    | 1.01   |
|       | 1024| 2.37e-12       | 3.45  | 2.60e-12       | 3.44  | 2.60e-12       | 3.44  | 9    | 3.04   |
| Expected order |    | 3.46 |       | 3.46 |       |       |       |      |        |

Table 3: Convergence orders and errors of the spectral Petrov-Galerkin method with $\theta = 0.7$, $f = \sin x$ and $u_d = \cos x$ in standard $L^2$-norm (higher than that in Table 2).

| $\alpha$ | $N$ | $E_N^{0,0}(u)$ | order | $E_N^{0,0}(z)$ | order | $E_N^{0,0}(q)$ | order | iter | CPU(s) |
|--------|-----|----------------|-------|----------------|-------|----------------|-------|------|--------|
| 1.2    | 64  | 8.44e-05       |       | 1.35e-04       |       | 1.35e-04       |       | 51   | 3.41   |
|       | 128 | 1.14e-05       | 2.89  | 1.90e-05       | 2.82  | 1.90e-05       | 2.82  | 51   | 6.68   |
|       | 256 | 1.62e-06       | 2.82  | 2.92e-06       | 2.71  | 2.92e-06       | 2.71  | 51   | 7.52   |
|       | 512 | 2.57e-07       | 2.65  | 5.19e-07       | 2.49  | 5.19e-07       | 2.49  | 51   | 10.10  |
| Expected order |    | 2.55 |       | 2.55 |       |       |       |      |        |
| 1.4    | 64  | 2.72e-07       |       | 3.79e-06       |       | 3.79e-06       |       | 15   | 0.55   |
|       | 128 | 3.72e-07       | 2.72  | 5.86e-07       | 2.69  | 5.86e-07       | 2.69  | 15   | 1.17   |
|       | 256 | 5.58e-08       | 2.74  | 8.87e-08       | 2.73  | 8.87e-08       | 2.73  | 15   | 1.39   |
|       | 512 | 8.25e-09       | 2.76  | 1.32e-08       | 2.75  | 1.32e-08       | 2.75  | 15   | 2.10   |
| Expected order |    | 2.79 |       | 2.79 |       |       |       |      |        |
| 1.6    | 64  | 1.74e-07       |       | 2.24e-07       |       | 2.24e-07       |       | 12   | 0.32   |
|       | 128 | 1.78e-08       | 3.29  | 1.90e-05       | 3.28  | 2.31e-08       | 3.28  | 12   | 0.70   |
|       | 256 | 1.77e-09       | 3.33  | 2.92e-06       | 3.33  | 2.29e-09       | 3.33  | 12   | 0.90   |
|       | 512 | 1.72e-10       | 3.36  | 5.19e-07       | 3.36  | 2.23e-10       | 3.36  | 12   | 1.33   |
| Expected order |    | 3.35 |       | 3.35 |       |       |       |      |        |
| 1.8    | 64  | 8.56e-09       |       | 9.35e-09       |       | 9.35e-09       |       | 9    | 0.23   |
|       | 128 | 5.98e-10       | 3.84  | 6.54e-10       | 3.84  | 6.54e-10       | 3.84  | 9    | 0.49   |
|       | 256 | 4.02e-11       | 3.90  | 4.39e-11       | 3.90  | 4.39e-11       | 3.90  | 9    | 0.66   |
|       | 512 | 2.65e-12       | 3.92  | 3.15e-12       | 3.80  | 3.15e-12       | 3.80  | 9    | 1.01   |
By Theorem 3.9, we have

\[ \omega^{-\sigma,-\sigma^*} u \in H^{\min\{2\beta + \alpha + 1, 2\alpha - 1\} + \min\{\sigma, \sigma^*\} - \varepsilon}(\Omega), \]

and \[ \omega^{-\sigma^*,-\sigma} z \in H^{\min\{2\beta + \alpha + 1, 2\alpha - 1\} + \min\{\sigma, \sigma^*\} - \varepsilon}(\Omega). \]

According to Theorem 4.5, we expect that the convergence order of state \( u_N \) in \( || \cdot ||_{\omega^{-\sigma,-\sigma^*}} \) norm, adjoint state \( z_N \) in \( || \cdot ||_{\omega^{-\sigma^*,-\sigma}} \) norm and control \( q_N \) in standard \( L^2 \) norm will be \( \min\{2\beta + \alpha + 1, 2\alpha - 1\} + \min\{\sigma, \sigma^*\} - \varepsilon \). In Tables 4-6, we test the convergence orders of numerical solutions and examine the performance of the fast projected gradient algorithm for \( \theta = 0.5, 0.7 \) and 1, respectively.

The numerical results confirm the theoretically predicted orders for state \( u_N \) and adjoint state \( z_N \) with different values of \( \theta \) and \( \alpha \). It is shown that the \( L^2 \)-errors of control \( q_N \) have higher convergence order than expected and its convergence order in \( || \cdot ||_{\omega^{-\sigma^*,-\sigma}} \) norm is close to \( \min\{2\beta + \alpha + 1, 2\alpha - 1\} + \min\{\sigma, \sigma^*\} - \varepsilon \).

Note that when \( \theta = 0.5, \sigma = \sigma^* = \alpha/2 \), the two-side fractional derivative operator \( L^2_\theta \) is equivalent to the integral fractional Laplacian in one-dimensional and the spectral Petrov-Galerkin method is reduced to the spectral Galerkin method in [32] for the optimal control problem (1.1)-(1.3) with \( \theta = 0.5 \). Table 4 shows that our fast projected gradient algorithm is several times faster than the fast algorithm presented in [32] for the same problem. Moreover, from Tables 4-6, we find that the iterative numbers and CPU time were greatly reduced for \( \theta = 0.5 \) and 1, which inspired us to search for a better preconditioner in the future work.

| \( \alpha \) | \( N \) | \( E^\omega_{\omega^{-\sigma,-\sigma^*}}(u) \) | order | \( E^\omega_{\omega^{-\sigma^*,-\sigma}}(z) \) | order | \( E^\omega_{\omega^{-\sigma^*,-\sigma^*}}(q) \) | order | \( E^\omega_{\sigma^*,\sigma} \) | order | iter | CPU(s) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.2 | 128 | 1.04e-03 | * | 7.17e-04 | * | 7.17e-04 | * | 2.33e-04 | * | 33 | 3.52 |
| | 256 | 2.96e-04 | 1.81 | 2.05e-04 | 1.81 | 2.05e-04 | 1.81 | 5.01e-05 | 2.22 | 33 | 4.01 |
| | 512 | 8.11e-05 | 1.87 | 5.60e-05 | 1.87 | 5.60e-05 | 1.87 | 1.05e-05 | 2.32 | 33 | 5.64 |
| | 1024 | 2.16e-05 | 1.91 | 1.49e-05 | 1.91 | 1.49e-05 | 1.91 | 1.91e-06 | 2.39 | 33 | 15.27 |
| Expected order | | | 2.0 | | | | 2.0 | | |
| 1.4 | 128 | 3.17e-06 | * | 6.10e-06 | * | 6.10e-06 | * | 1.07e-06 | * | 14 | 0.99 |
| | 256 | 6.51e-07 | 2.28 | 1.31e-06 | 2.21 | 1.31e-06 | 2.21 | 1.70e-07 | 2.66 | 14 | 1.34 |
| | 512 | 1.35e-07 | 2.27 | 2.80e-07 | 2.23 | 2.80e-07 | 2.23 | 2.63e-08 | 2.69 | 14 | 1.81 |
| | 1024 | 2.80e-08 | 2.26 | 5.90e-08 | 2.25 | 5.90e-08 | 2.25 | 4.03e-09 | 2.71 | 14 | 5.44 |
| Expected order | | | 2.3 | | | | 2.3 | | |
| 1.6 | 128 | 1.09e-06 | * | 1.26e-06 | * | 1.26e-06 | * | 2.22e-07 | * | 11 | 0.67 |
| | 256 | 1.85e-07 | 2.55 | 2.05e-06 | 2.55 | 2.16e-07 | 2.55 | 2.77e-08 | 3.01 | 11 | 0.96 |
| | 512 | 3.12e-08 | 2.57 | 5.60e-05 | 2.57 | 3.64e-08 | 2.57 | 3.36e-09 | 3.04 | 11 | 1.36 |
| | 1024 | 5.22e-09 | 2.58 | 1.49e-05 | 2.58 | 6.18e-09 | 2.58 | 4.03e-10 | 3.06 | 11 | 4.07 |
| Expected order | | | 2.6 | | | | 2.6 | | |
| 1.8 | 128 | 3.40e-07 | * | 2.76e-07 | * | 2.76e-07 | * | 4.86e-08 | * | 9 | 0.41 |
| | 256 | 4.65e-08 | 2.87 | 3.78e-08 | 2.87 | 3.78e-08 | 2.87 | 4.76e-09 | 3.34 | 9 | 0.66 |
| | 512 | 6.29e-09 | 2.89 | 5.13e-09 | 2.88 | 5.13e-09 | 2.88 | 4.61e-10 | 3.37 | 9 | 0.98 |
| | 1024 | 8.47e-10 | 2.89 | 6.92e-10 | 2.89 | 6.92e-10 | 2.89 | 4.42e-11 | 3.38 | 9 | 2.99 |
| Expected order | | | 2.9 | | | | 2.9 | | |
Table 5: Convergence orders and errors of the spectral Petrov-Galerkin method with $\theta = 0.7$, $f = (1 - x)^{-0.4} x^{-0.4} \sin x$ and $u_d = (1 - x)^{-0.4} x^{-0.4} \cos x$. The expected convergence order is $2\beta + \min\{\sigma, \sigma^*\} + \alpha + 1 - \varepsilon$ (Theorems 3.9 and 4.5).

| $\alpha$ | $N$ | $E_N^{\sigma - \sigma} (u)$ order | $E_N^{\sigma - \sigma} (z)$ order | $E_N^{\sigma - \sigma} (q)$ order | $E_N^{\theta} (q)$ order | iter | CPU(s) |
|---------|-----|--------------------------------|--------------------------------|--------------------------------|----------------|------|--------|
| 1.2     | 128 | 2.43e-05                      | *                              | 1.65e-04                      | *              | 5.10e-05 | * 51  | 6.77  |
|         | 256 | 5.47e-06                      | 2.16                           | 4.18e-05                      | 1.98           | 9.57e-06 | 2.41  | 51    | 7.50  |
|         | 512 | 1.54e-06                      | 1.83                           | 1.11e-05                      | 1.91           | 1.97e-06 | 2.28  | 51    | 10.15 |
|         | 1024| 4.77e-07                      | 1.69                           | 3.05e-06                      | 1.86           | 4.37e-07 | 2.17  | 51    | 27.55 |
|         |     | Expected order                | 1.72                           | 1.72                          |               |      |        |
| 1.4     | 128 | 2.41e-06                      | *                              | 2.49e-06                      | *              | 4.51e-07 | * 15  | 1.19  |
|         | 256 | 5.75e-07                      | 2.07                           | 5.50e-07                      | 2.18           | 7.38e-08 | 2.61  | 51    | 1.50  |
|         | 512 | 1.34e-07                      | 2.10                           | 1.22e-07                      | 2.17           | 1.21e-08 | 2.61  | 51    | 2.24  |
|         | 1024| 3.10e-08                      | 2.12                           | 2.69e-08                      | 2.18           | 1.95e-09 | 2.63  | 51    | 6.33  |
|         |     | Expected order                | 2.14                           | 2.14                          |               |      |        |
| 1.6     | 128 | 8.87e-07                      | *                              | 6.82e-07                      | *              | 1.38e-07 | * 12  | 0.69  |
|         | 256 | 1.52e-07                      | 2.54                           | 4.18e-05                      | 2.61           | 1.66e-08 | 3.06  | 12    | 0.96  |
|         | 512 | 2.60e-08                      | 2.55                           | 1.11e-05                      | 2.63           | 1.94e-09 | 3.10  | 12    | 1.37  |
|         | 1024| 4.44e-09                      | 2.55                           | 3.05e-06                      | 2.65           | 2.23e-10 | 3.12  | 12    | 4.14  |
|         |     | Expected order                | 2.51                           | 2.51                          |               |      |        |
| 1.8     | 128 | 3.14e-07                      | *                              | 2.25e-07                      | *              | 4.13e-08 | * 9   | 0.49  |
|         | 256 | 4.26e-08                      | 2.88                           | 3.01e-08                      | 2.90           | 3.98e-09 | 3.37  | 9     | 0.70  |
|         | 512 | 5.73e-09                      | 2.89                           | 3.99e-09                      | 2.92           | 3.77e-10 | 3.40  | 9     | 1.04  |
|         | 1024| 7.69e-10                      | 2.90                           | 5.25e-10                      | 2.93           | 3.53e-11 | 3.42  | 9     | 3.07  |
|         |     | Expected order                | 2.86                           | 2.86                          |               |      |        |

6 Conclusion

In this paper, we have investigated a spectral Petrov-Galerkin method for the optimal control problem governed by a two-sided space-fractional diffusion-advection-reaction equation. To compensate the weak singularities of solutions near the boundary, we have analyzed the regularity of the fractional optimal control problem in weighted Sobolev space. If the reg-
ularity index of $f$ and $u_d$ is $r$, $r \geq 0$, and $q \in L^2(\Omega)$, then the regularity index of state $u$ and adjoint state $z$ is $\min\{r + \alpha, s\}$, where $s = 2\alpha + \min\{\sigma, \sigma^*\} - 1 - \varepsilon$, $\varepsilon > 0$ is arbitrary small, $\sigma$ and $\sigma^*$ are constants defined as $[2,3]$ depending on the fractional order $\alpha$ and $\theta$. Based on the regularity results, we have provided error estimates of the spectral Petrov-Galerkin approximations to $u$ in $L^2_{\omega - \sigma, -\sigma^*}$-norm, $z$ in $L^2_{\omega - \sigma^*, -\sigma}$-norm and $q$ in $L^2$-norm, where the convergence order is consistent with the obtained regularity index.

A fast projected gradient algorithm with linear storage and quasi-linear complexity has been presented to solve the resulting discrete system efficiently. Numerical examples have verified the theoretical findings and shown the efficiency of the fast algorithm. For the smooth source term $f$ and desired state $u_d$ in Example 5.1 as well $f$, $u_d$ with low regularity in Example 5.2 numerical results have illustrated that our error estimate for state $u_N$ and adjoint state $z_N$ are optimal, while the convergence orders of control $q_N$ in standard $L^2$-norm are better than the theoretical expectation.

An improved error estimate for the control $q_N$ and a better preconditioner which can further improve the performance (iterative numbers and CPU time) of the fast algorithm can be considered in future work. Moreover, we are working at numerical methods for optimal control problems governed by time-fractional/space-fractional diffusion equations with additive noises.

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