BRST Cohomology of $N = 2$ Super-Yang-Mills Theory in $4D$

A. Tanzini$^{a,b}$, O. S. Ventura$^{a,b}$, 
L.C.Q.Vilar$^b$, S.P. Sorella$^b$

$^a$CBPF, Centro Brasileiro de Pesquisas Físicas 
Departamento de Campos e Partículas 
Rua Xavier Sigaud 150, 22290-180 Urca 
Rio de Janeiro, Brazil.

$^b$UERJ, Universidade do Estado do Rio de Janeiro 
Departamento de Física Teórica, Instituto de Física 
Rua São Francisco Xavier, 524 
20550-013, Maracanã, Rio de Janeiro, Brazil.

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Abstract

The BRST cohomology of the $N = 2$ supersymmetric Yang-Mills theory in four dimensions is discussed by making use of the twisted version of the $N = 2$ algebra. By the introduction of a set of suitable constant ghosts associated to the generators of $N = 2$, the quantization of the model can be done by taking into account both gauge
invariance and supersymmetry. In particular, we show how the twisted $N = 2$ algebra can be used to obtain in a straightforward way the relevant cohomology classes. Moreover, we shall be able to establish a very useful relationship between the local gauge invariant polynomial $tr\phi^2$ and the complete $N = 2$ Yang-Mills action. This important relation can be considered as the first step towards a fully algebraic proof of the one-loop exactness of the $N = 2$ $\beta$-function.
1 Introduction

The $N=2$ super-Yang-Mills field theory has a long history and many interesting properties, both at the perturbative and nonperturbative level, as for instance the one-loop exactness of its $\beta$-function.

After Witten’s work [1], $N=2$ super-Yang-Mills became known to be deeply related to the topological Yang-Mills theory (TYM). In fact, as pointed out by [1], the TYM action can be seen as the twisted version of the euclidean $N=2$ supersymmetric Yang-Mills theory [2]. The twisted supersymmetric charges of $N=2$ then lead to a scalar, a vector and a self-dual tensor charge, which are the symmetry generators of the twisted action. The final relationship of the twisted $N=2$ theory with TYM is done by identifying the $R$-charge of the twisted fields and generators with the ghost number. This means that the matter fields of $N=2$ acquire the status of ghost fields and the nilpotent scalar operator obtained from the twist of the supersymmetric generators is interpreted as a BRST-like operator. The effect of this identification results in the independence of the theory from any scale, as the BRST cohomology becomes completely trivial. In fact, it was shown that TYM can be fully obtained from the gauge fixing of a surface term (the Pontryagin index) [3, 4]. Furthermore, the Witten observables of the TYM can be recovered through the formalism of the equivariant cohomology [5, 6].

More recently, an important progress in the algebraic quantization of supersymmetric field theories in the Wess-Zumino gauge has been achieved. This goal has been accomplished by collecting the gauge symmetry and all the supersymmetry generators into an extended BRST operator, as shown by [7] in the case of $N=4$ Yang-Mills. Moreover, the case of $N=2$ super-Yang-Mills has also been successfully worked out, with the result that its BRST cohomology is in fact nontrivial, allowing for a nonvanishing beta function [8]. However, a fully algebraic proof of the nonrenormalization properties of the $N=2$ $\beta$-function is still lacking.

Our attitude in this paper will be to take into further analysis the twisted version of the conventional $N=2$ supersymmetric euclidean Yang-Mills theory, without identifying the $R$-charge with the ghost number. The twisting mechanism has the advantage of clarifying the topological structure which
underlies the $N = 2$ super-Yang-Mills field theory. This will allow us to analyse the BRST structure of the $N = 2$ super-Yang-Mills within the framework of the descent equations, providing a better understanding of the finiteness properties displayed by the model.

At this stage, we will refer to this version as the twisted theory (TSYM) and leave the denomination of TYM for the cohomological theory obtained after the aforementioned identification. It is worth mentioning here that the renormalization of Witten’s topological Yang-Mills has been discussed by the authors [9, 10], who have been able to show that the topological character of TYM is preserved at the quantum level.

Notice that the classical action for the $N = 2$ twisted theory will be just Witten’s TYM action, but now with all fields interpreted as gauge or matter fields. In this way, the quantization of the twisted theory can proceed along the same line of the quantization of the supersymmetric models in the Wess-Zumino gauge [7, 8]. As we shall see, this procedure will require to take into account the full set of symmetries (gauge and supersymmetry). The contact with the cohomological formulations of TYM [3, 4] will then be established.

It is worth underlining that the requirement of analyticity in the constant ghosts will play a fundamental role in order to obtain nontrivial cohomology classes. In fact, we shall be able to show that one can move from a nontrivial theory to the cohomological TYM by giving up of this analyticity condition. In this perturbative approach, we will have the possibility to show that the unique invariant counterterm of the $N = 2$ theory can be fully characterized by analysing the solutions of the set of descent equations corresponding to the integrated BRST cohomology. As a consequence, we shall be able to show that the full $N = 2$ Yang-Mills action turns out to be related to the invariant polynomial $tr \phi^2$. This will allow us to guess on the origin of the one-loop exactness of the $\beta$-function in $N = 2$ supersymmetric Yang-Mills theory [11].

The paper is organized as follows. In the next section we present a simple review of the twisting mechanism, making the connection of TYM with $N = 2$ super-Yang-Mills. In Section three we proceed with the quantization of the twisted theory. Section four is devoted to the renormalization of the model. Finally, we shall make contact with the results already existing in the literature and we shall draw a possible path toward the algebraic proof of the nonrenormalization theorem for the $N = 2 \beta$-function.
2 The Twisted Action

Following [4], the TSYM classical action is given by

\[ S_{\text{TSYM}} = \frac{1}{g^2} \text{tr} \int d^4x \left( \frac{1}{2} F_{\mu\nu}^+ F^{+\mu\nu} + \frac{1}{2} \bar{\phi} \{ \psi^\mu, \psi_\mu \} ight. 
- \chi^{\mu\nu} (D_\mu \psi_\nu - D_\nu \psi_\mu)^+ + \eta D_\mu \psi^\mu - \frac{1}{2} \bar{\phi} D_\mu D^\mu \phi 
- \frac{1}{2} \phi \{ \chi^{\mu\nu}, \chi_{\mu\nu} \} - \frac{1}{8} [\phi, \eta] \eta - \frac{1}{32} [\phi, \bar{\phi}] [\phi, \bar{\phi}] \left) , \right. \tag{2.1} \]

where \( g \) is the unique coupling constant and \( F_{\mu\nu}^+ \) is the self-dual part of the Yang-Mills field strength. The three fields \((\chi_{\mu\nu}, \psi_\mu, \eta)\) in the expression (2.1) are anticommuting, with \( \chi_{\mu\nu} \) self-dual, and \((\phi, \bar{\phi})\) are commuting complex scalar fields, \( \bar{\phi} \) being assumed to be the complex conjugate of \( \phi \). Of course, TSYM being a gauge theory, is left invariant by the gauge transformations

\[
\delta^g A_\mu = -D_\mu \epsilon, \\
\delta^g \lambda = [\epsilon, \lambda], \quad \lambda = \chi, \psi, \eta, \phi, \bar{\phi}. \tag{2.2}
\]

It is easily checked that the kinetic terms in the action (2.1) corresponding to the fields \((\chi, \psi, \eta, \phi, \bar{\phi})\) are nondegenerate, so that these fields have well defined propagators. The only degeneracy is that related to the pure Yang-Mills term \( F_{\mu\nu}^+ F^{+\mu\nu} \). Therefore, from eq.(2.2) one is led to interpret \((\chi, \psi, \eta, \phi, \bar{\phi})\) as ordinary matter fields, in spite of the unconventional tensorial character of \((\chi_{\mu\nu}, \psi_\mu)\). We assign to \((A, \chi, \psi, \eta, \phi, \bar{\phi})\) the dimensions \((1, 3/2, 3/2, 3/2, 1, 1)\) and \(\mathcal{R}\)-charges \((0, -1, 1, -1, 2, -2)\), so that the TSYM action (2.1) has vanishing total \(\mathcal{R}\)-charge. Let us emphasize once more that we do not identify the \(\mathcal{R}\)-charge with the ghost number, so that we avoid the cohomological interpretation which leads to TYM theory. The action (2.1) has to be regarded just as the twisted version of \(N = 2\) super-Yang-Mills theory.

For a better understanding of this point, let us briefly review the twisting procedure of the \(N = 2\) supersymmetric algebra in flat euclidean space-time [4]. In the absence of central extension, the \(N = 2\) supersymmetry in the Wess-Zumino gauge is characterized by 8 charges \((Q^i_\alpha, \bar{Q}^i_\dot{\alpha})\) obeying the
following relations
\[
\begin{align*}
\{ Q^i_\alpha, \overline{Q}_j\dot{\alpha} \} &= \delta^i_j \partial_{\alpha \dot{\alpha}} + \text{gauge transf.} + \text{eqs. mot.}, \\
\{ Q^i_\alpha, Q^j_\beta \} &= \{ \overline{Q}^i_\alpha, \overline{Q}^j_\beta \} = \text{gauge transf.} + \text{eqs. mot.} .
\end{align*}
\]

(2.3)
where \((\alpha, \dot{\alpha}) = 1, 2\) are the spinor indices, \((i, j) = 1, 2\) the internal \(SU(2)\) indices labelling the different charges of \(N = 2\), and \(\partial_{\alpha \dot{\alpha}} = (\sigma^\mu)_{\alpha \dot{\alpha}} \partial_\mu\), \(\sigma^\mu\) being the Pauli matrices. The special feature of \(N = 2\) is that both spinor and internal indices run from 1 to 2, making it possible to identify the index \(i\) with one of the two spinor indices \((\alpha, \dot{\alpha})\). This corresponds to substitute the \(SU(2)_L\) factor of the Lorentz symmetry group of the theory with the diagonal subgroup \(SU(2)'_L = \text{diag}(SU(2)_L \times SU(2)_A)\). It is precisely this identification which defines the twisting procedure [12]. Identifying therefore the internal index \(i\) with the spinor index \(\alpha\), we can construct now the following twisted generators \((\delta, \delta_\mu, \delta_{\mu \nu})\),
\[
\delta = \frac{1}{\sqrt{2}} \varepsilon^{\alpha \beta} Q_\beta \alpha , \quad \delta_\mu = \frac{1}{\sqrt{2}} Q_{\alpha \beta} (\sigma^\mu)_{\beta \dot{\alpha}} , \quad \delta_{\mu \nu} = \frac{1}{\sqrt{2}} (\sigma_{\mu \nu})^{\alpha \beta} Q_\beta \alpha = -\delta_{\nu \mu} .
\]

(2.4)
Notice that the generators \(\delta_{\mu \nu}\) are self-dual due to the fact that the matrices \(\sigma_{\mu \nu}\) are self-dual in euclidean space-time. In terms of these generators, the \(N = 2\) susy algebra reads now
\[
\begin{align*}
\delta^2 &= \text{gauge transf.} + \text{eqs. of motion} , \\
\{ \delta, \delta_\mu \} &= \partial_\mu + \text{gauge transf.} + \text{eqs. of motion} , \\
\{ \delta_\mu, \delta_\nu \} &= \text{gauge transf.} + \text{eqs. of motion} , \\
\{ \delta, \delta_{\mu \nu} \} &= \{ \delta_{\mu \nu}, \delta_\rho \} = \text{gauge transf.} + \text{eqs. of motion} , \\
\{ \delta_\mu, \delta_{\rho \sigma} \} &= -\varepsilon_{\mu \rho \sigma \nu} \partial_\nu - g_{[\mu \rho} \partial_{\sigma]} + \text{gauge transf.} + \text{eqs. of motion} .
\end{align*}
\]

(2.5)
(2.6)
The algebraic structure realized by the generators \((\delta, \delta_\mu)\) in eq.(2.3) is typical of the topological models [14, 15]. In this case the vector charge \(\delta_\mu\), usually called vector supersymmetry, is known to play an important role in the derivation of the ultraviolet finiteness properties of the topological models and in the construction of their observables [14].
Let us now turn to the relationship between Witten’s TYM and the $N = 2$ Yang-Mills theory, and show, in particular, that TYM has the same field content of the $N = 2$ Yang-Mills theory in the Wess-Zumino gauge. The minimal $N = 2$ supersymmetric pure Yang-Mills theory is described by a multiplet which, in the Wess-Zumino gauge, contains a gauge field $A_\mu$, two spinors $\psi^i_\alpha, i = 1, 2$, their conjugate $\bar{\psi}^i_\dot{\alpha}$, and two scalars $\phi, \bar{\phi}$ ($\bar{\phi}$ being the complex conjugate of $\phi$). All these fields are in the adjoint representation of the gauge group. We proceed by applying the previous twisting procedure to the $N = 2$ Wess-Zumino vector multiplet $(A_\mu, \psi^i_\alpha, \overline{\psi}^i_\dot{\alpha}, \phi, \bar{\phi})$. Identifying then the internal index $i$ with the spinor index $\alpha$, it is very easy to see that the spinor $\overline{\psi}^i_\dot{\alpha}$ can be related to an anticommuting vector $\psi_\mu$, i.e.

$$\overline{\psi}^i_\dot{\alpha} \xrightarrow{\text{twist}} \psi^i_\alpha \xrightarrow{\text{twist}} \psi_\mu = (\sigma_\mu)^{\dot{\alpha}\alpha} \psi_\alpha.$$

Concerning now the fields $\psi^i_\beta$, we have $\psi^i_\beta \xrightarrow{\text{twist}} \psi_{(\alpha\beta)} = \psi_{(\alpha\beta)} + \psi_{[\alpha\beta]}$, $\psi_{(\alpha\beta)}$ and $\psi_{[\alpha\beta]}$ being respectively symmetric and antisymmetric in the spinor indices $\alpha, \beta$. To $\psi_{[\alpha\beta]}$ we associate an anticommuting scalar field $\eta$, while $\psi_{(\alpha\beta)}$ turns out to be related to an antisymmetric self-dual field $\chi_{\mu\nu}$ through

$$\psi_{[\alpha\beta]} \xrightarrow{\text{twist}} \eta = \varepsilon^{\alpha\beta}\psi_{[\alpha\beta]},$$

$$\psi_{(\alpha\beta)} \xrightarrow{\text{twist}} \chi_{\mu\nu} = \tilde{\chi}_{\mu\nu} = (\sigma_{\mu\nu})^{\alpha\beta}\psi_{(\alpha\beta)}.$$

Therefore, the twisting procedure allows to replace the $N = 2$ Wess-Zumino multiplet $(A_\mu, \psi^i_\alpha, \overline{\psi}^i_\dot{\alpha}, \phi, \bar{\phi})$ by the twisted multiplet $(A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, \phi, \bar{\phi})$ whose field content is precisely that of the TSYM action $\mathcal{S}_{\text{SYM}}$. The same holds for the $N = 2$ pure Yang-Mills action $\mathcal{S}_{\text{YM}}$:

$$\mathcal{S}_{\text{YM}}^N(A_\mu, \psi^i_\alpha, \overline{\psi}^i_\dot{\alpha}, \phi, \bar{\phi}) \xrightarrow{\text{twist}} \mathcal{S}_{\text{SYM}}(A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, \phi, \bar{\phi}).$$

Thus the TSYM comes in fact from the twisted version of the ordinary $N = 2$ Yang-Mills in the Wess-Zumino gauge. This important point, already underlined by Witten in his original work $[1]$, deserves a few clarifying remarks in order to make contact with the results on topological field theories obtained in the recent years.

The first observation is naturally related to the existence of further symmetries of the TSYM action $\mathcal{S}_{\text{SYM}}$. According to the previous analysis, we conclude that the TSYM will be left invariant by the twisted generators
In fact, it is easy to check that the twisted scalar generator $\delta$ corresponds to Witten’s fermionic symmetry $\delta_W$ [1]

$$
\begin{align*}
\delta_W A_\mu &= \psi_\mu, & \delta_W \psi_\mu &= -D_\mu \phi, & \delta_W \phi &= 0, \\
\delta_W \chi_{\mu\nu} &= F^{+}_{\mu\nu}, & \delta_W \bar{\phi} &= 2 \eta, & \delta_W \eta &= \frac{1}{2} [\phi, \bar{\phi}].
\end{align*}
$$

The action of the vector generator on the set of fields is given by

$$
\begin{align*}
\delta_\mu A_\nu &= \frac{1}{2} \chi_{\mu\nu} + \frac{1}{8} g_{\mu\nu} \eta, \\
\delta_\mu \psi_\nu &= F_{\mu\nu} - \frac{1}{2} F^{+}_{\mu\nu} - \frac{1}{16} g_{\mu\nu} [\phi, \bar{\phi}], \\
\delta_\mu \eta &= \frac{1}{2} D_\mu \bar{\phi}, \\
\delta_\mu \chi_{\sigma\tau} &= \frac{1}{8} (\varepsilon_{\mu\sigma\tau\nu} D^\nu \bar{\phi} + g_{\mu\sigma} D_\tau \bar{\phi} - g_{\mu\tau} D_\sigma \bar{\phi}), \\
\delta_\mu \phi &= -\psi_\mu, \\
\delta_\mu \bar{\phi} &= 0,
\end{align*}
$$

and

$$
\delta_W S_{TSYM} = \delta_\mu S_{TSYM} = 0.
$$

We underline here that the form of the TSYM action (2.11) is not completely specified by the fermionic symmetry $\delta_W$. In other words, (2.11) is not the most general gauge invariant action compatible with the $\delta_W$-invariance. Nevertheless, it turns out to be uniquely characterized by $\delta_\mu$. The conditions (2.11) fix all the relative numerical coefficients of the Witten’s action (2.1), allowing, in particular, for a single coupling constant. This feature will be recovered in the renormalizability analysis of the model. The last generator, $\delta_{\mu\nu}$, will reproduce, together with the operators $\delta_W, \delta_\mu$, the complete $N=2$ susy algebra (2.5), (2.6). The reasons why we do not actually take in further account the transformations $\delta_{\mu\nu}$ are due partly to the fact that, as previously remarked, the TSYM action is already uniquely fixed by the $(\delta_W, \delta_\mu)-$symmetries and partly to the fact that the generator $\delta_{\mu\nu}$ turns out to be trivially realized on the fields in terms of the $\delta_W$-transformations [14].

The second remark is related to the standard perturbative Feynman diagram computations. From the equivalence between $S^{N=2}_{YM}$ and $S_{TSYM}$ it is very tempting to argue that the values of quantities like the $\beta$-function

$$
(\delta, \delta_\mu, \delta_{\mu\nu}).
$$
should be the same when computed in the ordinary $N = 2$ Yang-Mills and in the twisted version. After all, at least at the perturbative level, the twisting procedure has the effect of a linear change of variables on the fields. The computation of the one loop $\beta$-function for the twisted theory has indeed been performed by R. Brooks et al. \[11\]. As expected, the result agrees with that of the untwisted $N = 2$ Yang-Mills.

### 3 Quantizing the Twisted Theory

Before facing the problem of quantizing the theory, let us spend a few words on the strategy which will be adopted in the following. As we have already seen in the previous section, the twisted algebra of the generators of $N = 2$ closes on the translations only on shell and up to gauge transformations, due to the use of the Wess-Zumino gauge. We are dealing therefore with an open algebra, whose quantization requires the introduction of an appropriate set of antifields. Instead of making use of the conventional Batalin-Fradkin-Vilkovisky (BFV) approach, we shall proceed here with a slightly different procedure, already successfully used in the case of $N = 4$ super Yang-Mills \[7\], and completely equivalent to the BFV method.

We shall first begin by looking at an extended BRST operator $Q$ which turns out to be nilpotent on shell and which will define the gauge-fixed action. The construction of the final Slavnov-Taylor equation (or master equation) will be thus achieved by adding to the gauge-fixed action a suitable set of antifields, including in particular terms which are quadratic in the antifields, as required by the BFV procedure in the case of open algebras.

In order to obtain the BRST operator $Q$ we introduce the Faddeev-Popov ghost field $c$ corresponding to the gauge transformations \(2\),

\[
\begin{align*}
  sA_\mu &= -D_\mu c, & sc &= c^2, & s\phi &= [c, \phi], \\
  s\psi_\mu &= \{c, \psi_\mu\}, & s\chi_{\mu\nu} &= \{c, \chi_{\mu\nu}\}, & s\eta &= \{c, \eta\}, \\
  s\bar{c} &= \{c, \bar{c}\},
\end{align*}
\]

We associate now to each generator entering the algebra \(2\), namely $\delta_W$, $\delta_\mu$ and $\partial_\mu$, the constant ghost parameters $(\omega, \varepsilon^\mu, \nu^\mu)$ respectively, defining, in this way, the extended BRST operator

\[
\begin{align*}
  sS_{SYM} &= 0, & s^2 &= 0.
\end{align*}
\]
Now, we need to define the action of the four generators $s, \delta_W, \delta_\mu$ and $\partial_\mu$ on the ghosts $(c, \omega, \varepsilon^\mu, v^\mu)$. Let us work out in detail the case of the two operators $s$ and $\delta_W$. Working out eq. (3.13) explicitly, one looks then for an operator $(s + \omega \delta_W)$ nilpotent on $(A_\mu, \psi_\mu, \eta, \phi, c, \omega)$ and nilpotent on shell on $\chi_{\mu\nu}$ [13]. After a little experiment, it is not difficult to convince oneself that these conditions are indeed verified by defining the action of $s$ and $\delta_W$ on the ghost $(c, \omega)$ as

$$s\omega = 0, \quad \delta_W \omega = 0, \quad \delta_W c = -\omega \phi.$$  
(3.15)

The above procedure can be now easily repeated in order to include in the game also the operators $\delta_\mu$ and $\partial_\mu$. The final result is that the extension of the operator $Q$ on the ghosts $(c, \omega, \varepsilon^\mu, v^\mu)$ is found to be

$$Q c = c^2 - \omega^2 \phi - \omega \varepsilon^\mu A_\mu + \frac{\varepsilon^2}{16} + v^\mu \partial_\mu c,$$
$$Q \omega = 0, \quad Q \varepsilon^\mu = 0, \quad Q v^\mu = -\omega \varepsilon^\mu,$$
(3.16)

with

$$Q^2 = 0 \quad \text{on} \quad (A, \phi, \bar{\phi}, \eta, c, \omega, \varepsilon, v),$$
(3.17)

and

$$Q^2 \psi_\sigma = \frac{g^2}{4} \omega \varepsilon^\mu \frac{\delta S_{TYM}}{\delta \chi^{\mu\sigma}}$$
$$+ \frac{g^2}{32} \varepsilon^\mu \varepsilon^\nu \left( g_{\mu\sigma} \frac{\delta S_{TYM}}{\delta \psi^\nu} + g_{\nu\sigma} \frac{\delta S_{TYM}}{\delta \psi^\mu} - 2 g_{\mu\nu} \frac{\delta S_{TYM}}{\delta \psi^\sigma} \right),$$
(3.18)

$$Q^2 \chi_{\sigma\tau} = -\frac{g^2}{2} \omega^2 \frac{\delta S_{TYM}}{\delta \chi^{\sigma\tau}}$$
$$+ \frac{g^2}{8} \omega \varepsilon^\mu \left( \varepsilon_{\mu\sigma\tau} \frac{\delta S_{TYM}}{\delta \psi^\nu} + g_{\mu\sigma} \frac{\delta S_{TYM}}{\delta \psi^\tau} - g_{\mu\tau} \frac{\delta S_{TYM}}{\delta \psi^\sigma} \right).$$
(3.19)
For the usefulness of the reader, we give in the tables I and II the quantum numbers and the Grassmanian characters of all the fields and constant ghosts. We observe that the grading is chosen to be the sum of the ghost number and of the $\mathcal{R}$-charge.

| $A_\mu$ | $\chi_{\mu\nu}$ | $\psi_\mu$ | $\eta$ | $\phi$ | $\bar{\phi}$ |
|-------|-----------------|----------|-------|------|----------|
| dim.  | 1               | 3/2      | 3/2   | 1    | 1        |
| $\mathcal{R}$-charge | 0 | -1 | 1 | -1 | 2 | -2 |
| gh-number | 0 | 0 | 0 | 0 | 0 | 0 |
| nature | comm. | ant. | ant. | ant. | comm. | comm. |

Table 1: Quantum numbers

| $c$ | $\omega$ | $\varepsilon_\mu$ | $v_\mu$ |
|-----|---------|------------------|--------|
| dim. | 0 | -1/2 | -1/2 | -1 |
| $\mathcal{R}$-charge | 0 | -1 | 1 | 0 |
| gh-number | 1 | 1 | 1 | 1 |
| nature | ant. | comm. | comm. | ant. |

Table 2: Quantum numbers

The construction of the gauge fixing term is now straightforward. We introduce an antighost $\bar{c}$ and a Lagrangian multiplier $b$ transforming as $[7, 8]$

$$Q\bar{c} = b + v_\mu \partial_\mu c,$$

$$Qb = \omega \varepsilon_\mu \partial_\mu \bar{c} + v_\mu \partial_\mu b. \tag{3.20}$$

Thus, for the gauge fixing action we get

$$S_{gf} = Q \int d^4x \text{tr} (\bar{c} \partial A) \tag{3.21}$$

$$= tr \int d^4x \left( b \partial A + \bar{c} \partial D c - \omega \bar{c} \partial \psi - \frac{\varepsilon^\nu}{2} \bar{\partial}^\mu \chi_{\nu\mu} - \frac{\varepsilon^\mu}{8} \bar{\partial} \partial_\mu \eta \right),$$

so that the gauge fixed action ($S_{TSYM} + S_{gf}$) is $Q$-invariant. The above equation means that the gauge fixing procedure has been worked out by taking into account not only the pure local gauge symmetry but also the additional nonlinear invariances $\delta_W$ and $\delta_\mu$.

In order to obtain the Slavnov-Taylor identity we first couple the nonlinear $Q$-transformations of the fields ($c, \phi, A, \psi, \bar{\phi}, \eta, \chi$) to a set of antifields ($c^*, \phi^*, A^*, \psi^*, \bar{\phi}^*, \eta^*, \chi^*$),
$$S_{\text{ext}} = tr \int d^4x \left( \Phi^* Q \Phi \right),$$  \hspace{1cm} (3.22)$$

where $\Phi^i$, $\Phi^{*i}$ represent all fields and respective antifields. Moreover, taking into account that the extended operator $Q$ is nilpotent only modulo the equations of motion of the fields $\psi_\mu$ and $\chi_{\mu\nu}$, we also introduce a term quadratic in the corresponding antifields $\psi^*_{\mu}$, $\chi^{*\mu\nu}$, i.e.

$$S_{\text{quad}} = tr \int d^4x \left( \frac{g^2}{8} \omega \chi^{*\mu\nu} \chi_{\mu\nu} - \frac{g^2}{4} \omega \chi^{*\mu\nu} \varepsilon_\mu \psi^*_\nu - \frac{g^2}{32} \varepsilon_\mu \varepsilon^\nu \psi^*_\mu \psi^*_\nu + \frac{3g^2}{32} \varepsilon^2 \psi^*_\mu \psi^*_\mu \right).$$  \hspace{1cm} (3.23)$$

The complete action

$$\Sigma = S_{\text{SYM}} + S_{gf} + S_{\text{ext}} + S_{\text{quad}},$$  \hspace{1cm} (3.24)$$

obeys the classical Slavnov-Taylor identity

$$S(\Sigma) = tr \int d^4x \left( \frac{\delta \Sigma}{\delta \Phi^*} \frac{\delta \Sigma}{\delta \Phi} + (b + \nu^\mu \partial_\mu c) \frac{\delta \Sigma}{\delta c} \right) + (\omega \varepsilon^\mu \partial_\mu \bar{c} + \nu^\mu \partial_\mu b) \frac{\delta \Sigma}{\delta b} - \omega \varepsilon^\mu \frac{\partial \Sigma}{\partial \nu^\mu} = 0.$$  \hspace{1cm} (3.25)$$

This equation will be the starting point for the analysis of the renormalizability of the model.

At this point, it is worthwhile to draw the attention to a particular feature of the complete action $\Sigma$ given by eq.(3.24). One should notice that, in this action, the global ghost parameter $\omega$ only appears analytically. We expect therefore that the physical sectors of the theory should be characterized by field polynomials strictly analytic in $\omega$. This information will be of great relevance when we come to the characterization of the possible counterterms and anomalies of the theory.

The Slavnov-Taylor identity (3.25) can be simplified by using the fact that the complete action $\Sigma$ is invariant under space-time translations. Indeed, the dependence of $\Sigma$ on the corresponding translation constant ghost $\nu^\mu$ turns out to be fixed by the following linearly broken Ward identity

$$\frac{\partial \Sigma}{\partial \nu^\mu} = \Delta^i = tr \int d^4x \left( c^* \partial_\mu c - \phi^* \partial_\mu \phi - A^{*\nu} \partial_\mu A_\nu + \psi^{*\nu} \partial_\mu \psi_\nu - \phi \partial_\mu \phi \right) + \eta^* \partial_\mu \eta + \frac{1}{2} \chi^{*\mu\sigma} \partial_\mu \chi_{\nu\sigma} \right).$$  \hspace{1cm} (3.26)$$

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This means that we can completely eliminate the global constant ghost $v^\mu$ without any further consequence. Introducing the action $\Sigma$ through

$$
\Sigma = \hat{\Sigma} + v^\mu \Delta_{cl}^\mu , \quad \frac{\partial \hat{\Sigma}}{\partial v^\mu} = 0 ,
$$

(3.27)

it is easily verified from (3.24) that $\hat{\Sigma}$ obeys the modified Slavnov-Taylor identity

$$
\mathcal{S}(\hat{\Sigma}) = \omega \varepsilon^\mu \Delta_{cl}^\mu .
$$

(3.28)

Besides (3.28), the classical action $\hat{\Sigma}$ turns out to be characterized by further additional constraints [13], namely the Landau gauge fixing condition, the antighost equation, and the linearly broken ghost Ward identity (typical of the Landau gauge), respectively

$$
\frac{\delta \hat{\Sigma}}{\delta b} = \partial A , \quad \frac{\delta \hat{\Sigma}}{\delta c} + \partial_{\mu} \frac{\delta \hat{\Sigma}}{\delta A^*_\mu} = 0 ,
$$

$$
\int d^4 x \left( \frac{\delta \hat{\Sigma}}{\delta c} + \left[ \tau , \frac{\delta \hat{\Sigma}}{\delta b} \right] \right) = \Delta_{cl}^c ,
$$

(3.29)

with $\Delta_{cl}^c$ a linear classical breaking

$$
\Delta_{cl}^c = \int d^4 x \left( [c , c^*] - [A , A^*] - [\phi , \phi^*] + [\psi , \psi^*] - \left[ \phi , \phi^* \right] + \left[ \eta , \eta^* \right] + \frac{1}{2} [\chi , \chi^*] \right) .
$$

(3.30)

Following the standard procedure, let us introduce the so called reduced action $\tilde{S}$ defined through the gauge fixing condition (3.29) as

$$
\tilde{\Sigma} = \tilde{S} + tr \int d^4 x b \partial A ,
$$

(3.31)

so that $\tilde{S}$ is independent from the Lagrangian multiplier $b$. Moreover, from the antighost equation (3.29) it follows that $\tilde{S}$ depends from the antighost $\overline{\tau}$ only through the combination $A^*_\mu + \partial_{\mu} \overline{\tau}$. From now on $A^*_\mu$ will stand for this combination. Accordingly, for the Slavnov-Taylor identity we get

$$
\mathcal{S}(\tilde{S}) = tr \int d^4 x \left( \frac{\delta \tilde{S}}{\delta \phi^i} \frac{\delta \tilde{S}}{\delta \phi^i} \right) = \omega \varepsilon^\mu \Delta_{cl}^\mu .
$$

(3.32)
As a consequence the linearized Slavnov-Taylor operator $B_{\tilde{S}}$ defined as

$$B_{\tilde{S}} = tr \int d^4x \left( \delta \tilde{S} \frac{\delta}{\delta \Phi^i} \frac{\delta}{\delta \Phi^*_i} + \frac{\delta}{\delta \Phi^*_i} \frac{\delta}{\delta \Phi^i} \right)$$

(3.33)

is not strictly nilpotent. Instead, we have

$$B_{\tilde{S}} B_{\tilde{S}} = \omega \varepsilon^\mu P_\mu ,$$

(3.34)

meaning that $B_{\tilde{S}}$ is nilpotent only modulo a total derivative. It follows then that $B_{\tilde{S}}$ becomes a nilpotent operator when acting on the space of the integrated local polynomials in the fields and antifields. This is the case, for instance, of the invariant counterterms and of the anomalies.

4 Renormalization of the Twisted Theory

We are now ready to discuss the renormalization of the twisted $N = 2$ Yang-Mills theory. The first task is that of characterizing the cohomology classes of the linearized Slavnov-Taylor operator which turn out to be relevant for the anomalies and the invariant counterterms. Let us recall that both anomalies and invariant counterterms are integrated local polynomials $\Delta^G$ in the fields, antifields, and in the global ghosts $(\omega, \varepsilon)$, with dimension four, vanishing $R$-charge and ghost number $G$ respectively one and zero. In addition, they are constrained by the consistency condition

$$B_{\tilde{S}} \Delta^G = 0 , \quad G = 0, 1 .$$

(4.35)

In order to characterize the integrated cohomology of $B_{\tilde{S}}$ we introduce the operator $N_\varepsilon = \varepsilon^\mu \partial / \partial \varepsilon^\mu$, which counts the number of global ghosts $\varepsilon^\mu$ contained in a given field polynomial. Accordingly, the functional operator $B_{\tilde{S}}$ displays the following $\varepsilon$-expansion

$$B_{\tilde{S}} = b_{\tilde{S}} + \varepsilon^\mu W_\mu + \frac{1}{2} \varepsilon^\mu \varepsilon^\nu W_{\mu\nu} ,$$

(4.36)

where, from eq.(3.34) the operators $b_{\tilde{S}}$, $W_\mu$, $W_{\mu\nu}$ are easily seen to obey the following algebraic relations
\[
\begin{align*}
\{b_\Sigma b_\Sigma \} &= 0, \quad \{b_\Sigma, W_\mu \} = \omega \mathcal{P}_\mu, \\
\{W_\mu, W_\nu \} + \{b_\Sigma, W_{\mu \nu} \} &= 0, \\
\{W_\mu, W_{\nu \rho} \} + \{W_\nu, W_{\rho \mu} \} + \{W_\rho, W_{\mu \nu} \} &= 0, \\
\{W_{\mu \nu}, W_{\rho \sigma} \} + \{W_{\mu \rho}, W_{\nu \sigma} \} + \{W_{\mu \sigma}, W_{\nu \rho} \} &= 0.
\end{align*}
\] (4.37)

From (4.37) we observe that the operator \(b_\Sigma\) is strictly nilpotent and that the vector operator \(W_\mu\) allows to decompose the space-time translations \(\mathcal{P}_\mu\) as a \(b_\Sigma\)-anticommutator, providing thus an off-shell realization of the algebra (2.5). It is just given by
\[
b_\Sigma = s + \omega \delta_W.
\] (4.39)

According to the general results on cohomology, the integrated cohomology of \(\mathcal{B}_\Sigma\) is isomorphic to a subspace of the integrated cohomology of \(b_\Sigma\) [10]. Since \(b_\Sigma\) is exactly nilpotent, one can pass from the integrated version of the Wess-Zumino consistency condition (4.35) to its local version, which leads to the following set of descent equations,
\[
\begin{align*}
b_\Sigma \Omega^G_4 + \omega \partial^\mu \Omega^G_2_{\mu} &= 0, \\
b_\Sigma \Omega^G_2_{\mu} + \omega \partial^\nu \Omega^G_3_{[\mu \nu]} &= 0, \\
b_\Sigma \Omega^G_3_{[\mu \nu]} + \omega \partial^\rho \Omega^G_2_{[\mu \nu \rho]} &= 0, \\
b_\Sigma \Omega^G_2_{[\mu \nu \rho]} + \omega \partial^\sigma \Omega^G_2_{[\mu \nu \rho \sigma]} &= 0, \\
b_\Sigma \Omega^G_2_{[\mu \nu \rho \sigma]} &= 0,
\end{align*}
\] (4.40)

where the cocycle \(\Omega^G_2\) has ghost number \(G\) and dimension \(D\).

The presence of the parameter \(\omega\) in front of all the derivatives in the above set of equations is a feature of the algebra given by (4.37). In fact, as we have commented before in the preceding section, we are interested in the characterization of those cohomologically nontrivial cocycles which are given by local field polynomials depending analytically on the parameter \(\omega\). In other words, a cocycle will be nontrivial if it is analytic in \(\omega\) and if it cannot be written as a \(b_\Sigma\)-variation of any local field polynomial analytic in \(\omega\). Now, using the algebra (4.37), one can see that the solutions \(\Omega^G_2\) of the descent equations (4.40) can be obtained by suitably applying the operator.
$\mathcal{W}_\mu$ on the nontrivial solutions of the local cohomology of $b_-$ in each level of the descent equations \cite{17}. Also, it is not difficult to show that the operator $\mathcal{W}_\mu$ preserves the analyticity in $\omega$ of the space where it acts upon, i.e. it transforms local polynomials analytic in $\omega$ into local polynomials analytic in $\omega$. Then, as the nontrivial solutions of the local cohomology of $b_-$ belong to this analytic space, it is assured that $\mathcal{W}_\mu$ will map such solutions into nontrivial solutions of the cohomology of $b_-$ modulo total derivatives. As a consequence, this latter cohomology will also be restricted to the space of field polynomials analytic in $\omega$.

We are interested in the solutions of the descent equations in the case of the invariant counterterms and of the gauge anomalies, corresponding respectively to the sectors of ghost number $G = 0, 1$. In the case of the gauge anomalies, one can show that there is no possible nontrivial solution for the local cohomology of $b_-$ with the correct quantum numbers at any level of the descent equations \cite{140}. It is important to mention that this result, already obtained in \cite{8} in the analysis of the $N = 2$ untwisted gauge theories, means that there is no possible extension of the nonabelian Adler-Bardeen gauge anomaly compatible with $N = 2$ supersymmetry.

In the case of the invariant counterterms, the analysis of the local cohomology of $b_-$ shows the existence of some nontrivial solutions in different levels of the descent equations. The first one has dimension 2 and is given by

$$\Delta = \frac{1}{2} \text{tr} \phi^2.$$  \hfill (4.41)

It is a solution of the local cohomology of $b_-$ for the last of the eqs.\(1.40\). The second term has dimension 3,

$$\Delta_{\mu\nu} = a \left( F_{\mu\nu} \phi + \frac{g^2 \omega}{2} \chi_{\mu\nu}^* \phi \right),$$  \hfill (4.42)

and it is a solution of the local cohomology of $b_-$ at the intermediate level of $\Omega^0_{3[\mu\nu]}$. As we will discuss in the following, this term is ruled out from the cohomology of the complete operator $\mathcal{B}_-$ by requiring invariance under the vector symmetry operator $\mathcal{W}_\mu$. Actually, it is possible to show that the same happens for the other nontrivial solutions of the descent equations \(1.40\), thus we do not report on them here.
Before analyzing in detail the consequences of the above results on the cohomology of the complete operator $\mathcal{B}_S$, let us discuss here the important issue of the analyticity in the constant ghosts. In fact, the requirement of analyticity in the ghosts $(\varepsilon_\mu, \omega)$, stemming from pure perturbative considerations, is one of the most important ingredients in the cohomological analysis that we are doing. It is an almost trivial exercise to show that both cocycles (4.41) and (4.42) can be expressed indeed as a pure $b_s$-variation, namely
\[
\Delta = \frac{1}{2} b_s tr \left( -\frac{1}{\omega^2} \phi \frac{1}{3\omega^4} c^3 \right),
\]
\[
\Delta_{\mu\nu} = ab_s tr \left( \frac{1}{\omega} \phi \chi_{\mu\nu} \right).
\]
These expressions illustrate in a very clear way the relevance of the analyticity requirement. It is apparent from the eq.(4.43) that the price to be payed in order to write $tr\phi^2$ as a pure $b_s$-variation is in fact the loss of analyticity in the ghost $\omega$.

In other words, as long as one works in a functional space whose elements are power series in the constant ghosts, the cohomology of $b_s$ is not empty. On the other hand, if the analyticity requirement is relaxed, the cohomology of $b_s$, and therefore that of the complete operator $\mathcal{B}_S$, becomes trivial, leading thus to the cohomological interpretation of Baulieu-Singer [4] and Labastida-Pernici [3]. One goes from the standard field theory point of view, of $N = 2$ super Yang-Mills, to the cohomological one, of the topological Yang-Mills theory, by simply setting $\omega = 1$, which of course implies that analyticity is lost. In addition, it is rather simple to convince oneself that setting $\omega = 1$ has the meaning of identifying the $\mathcal{R}$-charge with the ghost number, so that the fields $(\chi, \psi, \eta, \phi, \bar{\phi})$ acquire a nonvanishing ghost number given respectively by $(-1, 1, -1, 2, -2)$. They correspond now to the so called topological ghosts of the cohomological interpretation.

It is also interesting to point out that there is some relationship between the analyticity in the global ghosts and the so called equivariant cohomology proposed by [5, 6] in order to recover the Witten’s observables [18, 19]. Roughly speaking, the equivariant cohomology can be defined as the restriction of the BRST cohomology to the space of the gauge invariant polynomials which cannot be written as the BRST variation of local quantities which are independent from the Faddeev-Popov ghost $c$. Considering now the polynomial $tr\phi^2$, we see that it yields a nontrivial equivariant cocycle in the
cohomological interpretation (*i.e.* $\omega = 1$), due to the unavoidable presence of the Faddeev-Popov ghost $c$ on the right hand side of eq. (1.43). However, the nontriviality of the second cocycle (1.42) relies exclusively on the analyticity requirement.

In fact, it is not surprising at all to have found more than one independent solution of the local cohomology of $b_{\tilde{S}}$. One should remember that the operator $\delta_W$, which builds with $s$ the operator $b_{\tilde{S}}$ of eq. (4.39) is not sufficient to completely fix the coefficients of the TSYM action (2.1). One needs to impose the invariance under $\delta_{\mu}$ in order to completely specify (2.1). The reflection of this point is the existence of others cocycles in the descent equations for the operator $b_{\tilde{S}}$. At this level, this does not mean the existence of other $\beta$-functions, beyond that associated to the gauge coupling $g$. Our interest, at the end, is in the integrated cocycles invariant under $B_{\tilde{S}}$ (4.36). Then, our approach is to climb the descent equations (4.40), which will give us the final solution on the integrated cohomology of $b_{\tilde{S}}$, and afterwards demand invariance under $B_{\tilde{S}}$. By using this last requirement we will get rid of the terms coming from the cocycle (1.42), as well as of that coming from the other nontrivial cocycles.

In order to climb the descent equations, one can apply the operator $W_\mu$, and reach the solution at the upper level for $\Omega^0$. This solution, when integrated, can be written in the form

$$
\Omega^0 = \varepsilon^{\mu\nu\rho\tau} W_\mu W_\nu W_\rho W_\tau \int d^4 x \Delta + \mathcal{W}^{\mu\nu} W_\mu W_\nu \int d^4 x \Delta_{\mu\nu} + b_{\tilde{S}} - \text{variation}
$$

$$
= S_{TSYM} + a \Xi + b_{\tilde{S}} \tilde{\Omega}^{-1},
$$

(4.45)

where $\tilde{\Omega}^{-1}$ is an arbitrary integrated polynomial in the fields analytic in $\omega$ with ghost number $-1$, and

$$
\Xi = \int d^4 x \left( \frac{g^2 \omega}{4} F^{+\mu\nu} \chi^*_{\mu\nu} + \frac{g^4 \omega^2}{8} \chi_{\mu\nu} \chi^*_{\mu\nu} - \frac{1}{4} \chi^{\mu\nu}(D_\mu \psi_\nu - D_\nu \psi_\mu)^+ \right)
$$

$$
- \frac{1}{4} \phi \{ \chi^{\mu\nu}, \chi_{\mu\nu} \} - \frac{3}{4} \psi^{\mu} D_\mu \eta - \frac{3}{4} \omega g^2 \phi D^\mu \psi^*_\mu - \frac{3}{4} \omega g^2 \psi^*_\mu A^*_\mu
$$

$$
+ \frac{3}{16} \phi \{ \eta, \eta \} - \frac{3}{2} \omega^2 g^2 \phi c^* + \frac{3}{4} \omega^2 g^2 \phi \left[ \phi, \eta^* \right].
$$

(4.46)

Now, we need to impose the invariance of $\Omega^0$ under $B_{\tilde{S}}$ (4.36), which, in
particular, means invariance under $W_\mu$

$$W_\mu \Omega^0 = W_\mu \left( S_{SYM} + a \Xi + b_\sim \bar{\Omega}^{-1} \right) = 0 \ .$$  \hspace{1cm} (4.47)

Obviously, the action $S_{SYM}$ of eq.(2.1) is already invariant under $W_\mu$. Then, using the algebra (4.38), the equation (4.47) gives the consistency condition

$$a W_\mu \Xi = b_\sim \Lambda^{-1}_\mu \ ,$$  \hspace{1cm} (4.48)

where $\Lambda^{-1}_\mu$ is an arbitrary integrated polynomial in the fields analytic in $\omega$ with ghost number $-1$. This implies that either $W_\mu \Xi$ is a trivial cocycle analytic in $\omega$, or the coefficient $a$ has to vanish. One can show, by a straightforward calculation, that the only way to write $W_\mu \Xi$ as an exact cocycle is to lose the analyticity in $\omega$

$$W_\mu \Xi = -\frac{1}{\omega} b_\sim \left( \frac{3}{8} F_{\mu
u} D^\nu \bar{\phi} + \frac{3}{64} [\phi, \bar{\phi}] D_\mu \bar{\phi} - \frac{3}{8} \psi^{\mu \nu} [\chi_{\mu\nu}, \bar{\phi}] ight)$$

$$+ \frac{3}{32} \psi_\mu [\eta, \bar{\phi}] - \frac{1}{16} \omega g^2 \chi_{\mu\nu}^* D^\nu \bar{\phi} + \frac{1}{4} \omega g^2 \chi_{\mu\nu} A^{*\nu}$$

$$- \frac{3}{8} \omega g^2 F_{\mu\nu} \psi^{*\nu} - \frac{3}{64} \omega g^2 \psi_\mu^* [\phi, \bar{\phi}] - \frac{3}{4} \omega g^2 \psi_\mu c^* + \frac{1}{16} \omega g^4 \chi_{\mu\nu} \psi^{*\nu} \right) \ ,$$

i.e., the only allowed solution for the equation (4.48) is given by $a = 0$. Then, going back to eq.(4.45), we can see that our cohomological analysis finally leads us to the conclusion that the nontrivial part of the counterterm $\Omega^0$ can be written as the starting $N = 2$ super-Yang-Mills action modulo a trivial $b_\sim \ - term$

$$\Omega^0 = S_{SYM} + b_\sim \bar{\Omega}^{-1} = \varepsilon^{\mu\nu\rho\tau} W_\mu W_\nu W_\rho W_\tau \int d^4 x \frac{1}{2} tr \phi^2 + b_\sim \ - variation \ ,$$  \hspace{1cm} (4.49)

namely,

$$S_{SYM} \approx \varepsilon^{\mu\nu\rho\tau} W_\mu W_\nu W_\rho W_\tau \int d^4 x \frac{1}{2} tr \phi^2 \ ,$$  \hspace{1cm} (4.50)

up to trivial cocycles.

The fact that we were left with only one arbitrary coefficient in the nontrivial part of the counterterm (which is the global coefficient of $S_{SYM}$) means the presence of only one coupling in the theory, and consequently, of only one $\beta$-function for the twisted $N = 2$ Yang-Mills theory.
5 Conclusion

We have shown how the quantization of the twisted $N = 2$ super-Yang-Mills action can be done by taking into account the full $N = 2$ twisted supersymmetric algebra. Nontrivial cohomology classes are characterized by requiring analyticity in the twisted constant global ghosts of the $N = 2$ quantized theory. The analyticity requirement, following from perturbation theory, plays a crucial rôle as it defines a criterium in order to have a nontrivial cohomology.

We have also seen that the operator $b_s$ has a nonvanishing integrated analytic cohomology only in the sector of the invariant counterterms, coming from the nontrivial local elements given by (4.41) and (4.42). Finally, we have been able to show that the cohomology of $B_s$ in the sector of the invariant counterterms contains a unique element given in eq.(4.50). This result is in complete agreement with that found in the case of untwisted $N = 2$ YM [8].

Moreover, as pointed out by eq.(1.50), we emphasize that the origin of the $N = 2$ super-Yang-Mills action (2.1) can be traced back, modulo an irrelevant exact cocycle, to the invariant polynomial $tr\phi^2$, eq.(1.41). Let us also recall here that the explicit Feynman diagrams computation yields a nonvanishing value for the renormalization of the gauge coupling, meaning that the twisted version of $N = 2$ Yang-Mills possesses a nonvanishing $\beta$-function. Of course, the latter agrees with that of the pure $N = 2$ untwisted Yang-Mills [11]. Moreover, it is well known that the $\beta$-function of $N = 2$ Yang-Mills theory receives only one loop order contributions [27]. On the other hand it is known since several years that in the $N = 2$ susy gauge theories the Green's functions with the insertion of composite operators of the kind of the invariant polynomials of the form $tr\phi^n$ display remarkable finiteness properties and can be computed exactly, even when nonperturbative effects are taken into account [27]. It is natural therefore to guess that the finiteness properties of $tr\phi^2$ are at the origin of the absence of the higher order corrections for the gauge $\beta$-function of both twisted and untwisted $N = 2$ gauge theories.

Finally, it is worth mentioning that, recently, an algebraic proof of the nonrenormalization theorem of the $N = 2$ $\beta$-function has been achieved [22] along the lines of the present work. As expected, the relationship (1.50) between the $N = 2$ Yang-Mills action and the local gauge invariant operator $tr\phi^2$ turns out to play a pivotal role.

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