Local constraints can globally shatter Hilbert space: a new route to quantum information protection

Vedika Khemani$^1$ and Rahul M. Nandkishore$^2$

$^1$Department of Physics, Harvard University, Cambridge, MA 02138, USA
$^2$Department of Physics and Center for Theory of Quantum Matter, University of Colorado, Boulder, CO 80309

We show how local constraints can globally ‘shatter’ Hilbert space into subsectors, leading to an unexpected dynamics with features reminiscent of both many body localization and quantum scars. A crisp example of this phenomenon is provided by a ‘fractonic circuit’ - a model of quantum circuit dynamics in one dimension constrained to conserve both charge and dipole moment. We show how the Hilbert space of the fractonic circuit dynamically fractures into disconnected emergent subsectors within a particular charge and dipole symmetry sector. A large number of the emergent subsectors, exponentially many in the size of the system, have dimension one and exhibit strictly localized quantum dynamics—even in the absence of spatial disorder and in the presence of temporal noise. Exponentially large localized subspaces can be proven to exist for any one dimensional fractonic circuit with finite spatial range, and provide a potentially new route for the robust storage of quantum information. Other emergent subsectors display non-trivial dynamics and may be constructed by embedding finite sized non-trivial blocks into the localized subspace. The shattering of a particular symmetry sector into a distribution of dynamical subsectors with varying sizes leads to the coexistence of high and low entanglement states, reminiscent of quantum scars. We discuss the detailed pattern of fracturing and its implications. We also discuss other mechanisms for similarly shattering Hilbert space.

I. INTRODUCTION

This paper lies at the intersection of two interesting streams of contemporary research: the study of quantum information and its protection, and the study of nonequilibrium dynamics in quantum systems. A major theme of interest in the former line of work is the quest for new models or phases of matter that protect quantum information (see [1] and references contained therein). While examples of thermally stable self correcting quantum memories exist in four or more dimensions, such as the toric code [2], the effects of finite temperature are more debilitating in lower dimensions. For example, while topological phases in two dimensions are known to possess a “logical subspace” into which information may be encoded in a manner immune to local perturbations, this protection is only afforded at zero temperature [2]. The recent discovery of ‘fracton’ phases of quantum matter [3–9] has opened a new direction in this quest. Fracton phases in three spatial dimensions also have a protected logical subspace (immune to local perturbations at zero temperature [10]), but of a size which grows exponentially in the linear extent of the system—unlike conventional topological phases, where the logical subspace has constant dimension. Nevertheless, it is believed that three dimensional ‘fractonic’ phases do thermalize at non-zero temperatures, such that they do not constitute ideal quantum memories at finite temperature [11, 12].

A parallel major theme of investigation involves the study of thermalization in isolated quantum many body systems. The cornerstone of this understanding is the Eigenstate Thermalization Hypothesis (ETH) [13–15]. The strong form of the ETH holds that all many-body energy eigenstates are thermal (the weak form of the ETH, in which only almost all eigenstates are thermal, is known to not be sufficient to guarantee thermalization [16]). In recent years, interest has grown in systems that violate the ETH. One well known class of ‘counterexamples’ to ETH is provided by integrable systems, which possess an extensive number of conserved quantities, and thermalize instead to a generalized Gibbs ensemble [17]. The other well studied counter-example is many-body localization (MBL) [18–21], driven by disorder, in which case essentially all eigenstates are non-thermal, characterized by an extensive number of emergent local integrals of motion [22, 23]. As a result, MBL systems never reach thermal equilibrium and retain local memory of their initial conditions for arbitrarily late times — a feature that can preserve quantum information even at finite energy-densities, and could potentially have use in developing new technologies such as quantum memories.

Still more recently, a new type of counterexample to strong ETH has been observed, which now goes under the name of "quantum many-body scars" [24–33]. Quantum many-body scars are loosely defined as a small number of non-thermal eigenstates (measure zero in the thermodynamic limit) embedded into an otherwise thermal spectrum. The presence of these “scar” states can lead to distinct signatures in quench experiments if the initial states have high overlap with the scar states, as was recently observed in an experiment on Rydberg atoms [34]. While it is known how scars may be embedded, by hand, into a thermal many-body spectrum in certain special classes of Hamiltonians [25], there is still little understanding of the general principles that could give rise to scars [27, 31, 35–38], whether the phenomenon survives weak perturbations [27], and whether and when scars should be expected to arise in generic Hamiltonians.
the thermodynamic limit, making this a very active area of research.

While the origin of scars, in general, is still largely unexplained, several known models with “exact” or “perfect” scars (in particular, those in [25, 32, 33]) can be understood as examples of the construction in Ref [25]. A useful perspective on Ref [25], connecting it to this present work, is to note that the construction therein leads to scars via a dynamical fracturing of the Hilbert space into disconnected emergent subsectors, even in the absence of explicit conservation laws. If one or a few of these emergent subspaces can be spanned using only a small number of low-entanglement basis states (that constitute a vanishing fraction of the full Hilbert space), then eigenstates living in these subspaces necessarily have low entanglement. These will violate strong ETH when they coexist at the same energy densities as the other, thermal, eigenstates. Again, barring the relatively fine-tuned setup in [25], general conditions that lead to emergent Hilbert space fracture are not known. As an aside, we note that while the scarred model of the Rydberg experiment does not have an obviously fractured Hilbert space, recent work has shown that the model is very close in parameter space to a proximate integrable point [27]. Integrable systems do have (explicitly) fractured Hilbert spaces due to the presence of extremely many explicit conservation laws. It is possible that weakly breaking integrability, in a certain manner, could cause most integrable subspaces to mix while preserving a few, thereby leading to an apparent fracturing and the appearance of anomalous eigenstates that live in the “unmixed” sectors. Physics analogous to fracture has also been discussed in prior models with kinetic constraints, including in classical settings, e.g. [39, 40].

In this manuscript we marry together two lines of research by demonstrating how certain local constraints can give rise to a dramatic fracturing of Hilbert space into exponentially many emergent dynamical subsectors — whence the word “shatter”. Moreover, we prove the existence of an exponentially large (in volume) number of subspaces which have dimension exactly equal to one, and hence have trivial, i.e. strictly localized, dynamics. Our models are constructed in one spatial dimension and, unlike two dimensional topological phases or three dimensional fracton phases, the localized subspace is not limited to zero temperature and is robust to both temporal noise and local perturbations that respect the constraints. This raises the tantalizing possibility that quantum information may be robustly encoded in these subspaces without any loss. Unlike conventional MBL, the ‘localization’ here involves only a measure zero fraction of the full spectrum, and requires neither disorder nor energy conservation. This localization relies instead on a new phenomenon that we uncover – namely, that local constraints can shatter Hilbert space into disconnected dynamical subsectors, each labelled by an emergent quantum number, giving rise to ergodicity breaking, ETH violation, and (in certain regions of Hilbert space) localized dynamics. This provides an intriguing new route to quantum information protection.

The cleanest setting for illustrating this phenomenon is the ‘fractonic random circuit’ introduced in [41]. This is a model of quantum circuit dynamics [42–48] constrained to conserve both a $U(1)$ charge and its dipole moment. It was argued in [41] that, in certain settings, such a fractonic circuit can give rise to localization despite the lack of energy conservation or any attendant locator expansion. However, while [41] examined mixed state dynamics (specifically operator growth), here we examine instead the pure state dynamics of this system, including the nature of eigenstates in a time-periodic Floquet version of the fractonic circuit model. This analysis reveals the ‘shattering’ of Hilbert space in this model into an exponentially large number of dynamical subsectors within a given charge and dipole symmetry sector, giving rise to a breakdown of ergodicity and a violation of ETH as conventionally understood. We identify, in particular, an exactly localized subspace (exponentially large in system size) which can be labeled by state-dependent emergent local integrals of motion, and into which quantum information may be robustly encoded. This localization does not require spatial disorder, and survives temporal noise. The robustness of the results to spatial perturbations respecting the constraints is evidenced by the fact that the unitary gates generating the dynamics may be chosen randomly with respect to Haar measure and are not fine tuned in any way. We prove the existence of such a robust, exponentially large, localized subspace for fractonic circuits with any finite spatial range of gates. We also identify additional sectors with non-trivial dynamics that may be constructed by embedding non-trivial blocks into the localized background. We examine in detail the structure of the fracturing of Hilbert space and its implications, which include the co-existence, in a single symmetry sector with a particular charge and dipole moment, of both high and low entanglement states, reminiscent of quantum scarring. We comment on some of the implications for dynamics.

While the ‘fractonic’ random circuit provides a clean example of local constraints shattering Hilbert space, we also point out that not all local constraints act in this way. Nevertheless, the ‘shattering’ of Hilbert space is not particular to fractonic circuits, and we also discuss some other (not obviously fractonic) examples of circuits which also exhibit a shattered Hilbert space. We conclude with a discussion of some open directions.

II. THE MODEL

We work with the model of quantum circuit dynamics introduced in [41]. The Hilbert space consists of a one dimensional chain of $S = 1$ quantum spins of length $L$, acted upon by local unitary gates which locally conserve both charge ($Q = \text{total } S^z$) and dipole moment ($P = \sum_j j S_j^z$, where $j$ is a site label). The simplest re-
alization of these rules is provided by circuits with three site unitary gates, which take the form of $27 \times 27$ matrices as shown in Fig. 1. The charge and dipole moment conservation lead to a block diagonal structure in the gates. Notably, there are only four non-trivial two by two ‘blocks,’ each of which is a random unitary drawn independently from the Haar measure on $U(2)$, while the rest of the matrix is diagonal (pure $U(1)$ phase). We will begin our analysis with a discussion of this simple circuit with three site gates, but we will prove that the key results are robust for any finite gate size (while also flagging some special features that do depend on gate size). We note that while [41] considered a circuit that was random in both space and time, this is not important for our purposes - our results hold just as well if the circuit is uniform in space (translation invariant), and/or if it is periodically repeated in time (Floquet). In all that follows, we work with a circuit that is translation invariant, since this makes more dramatic our central result (localization). We also work with a circuit that is stroboscopically repeated with period 3, since this allows us to meaningfully discuss eigenstates. However, we emphasize that our basic results require neither translation invariance in space, nor periodicity in time.

This circuit has only two symmetries: charge conservation and dipole moment, and the ‘symmetry sectors’ of the theory are correspondingly labelled by just two quantum numbers: charge $Q$ and dipole $P$. In the Floquet version of the model, three staggered “layers” of the circuit are chosen independently, but the layers are then repeated in time. The time evolution operator for one Floquet period is given by $U^F = U_3 U_2 U_1$, where

$$U_n = \begin{cases} \prod_i U_{3i, 3i+1, 3i+2}^n & \text{if } n = 0 \\ \prod_i U_{3i-1, 3i, 3i+1}^n & \text{if } n = 1 \\ \prod_i U_{3i-2, 3i-1, 3i}^n & \text{if } n = 2, \end{cases}$$

where the gates $U^1, U^2$ and $U^3$ are chosen at random for a given realization, but remain fixed throughout the run corresponding to that realization. We work throughout with open boundary conditions. In certain layers of the circuit, there may be sites near the boundary that are acted on trivially (pure phase) but the Floquet operator as a whole acts non-trivially on every site.

We can now perform a numerical exact diagonalization of the Floquet operator, and extract the spectrum of eigenstates within each symmetry sector. For each eigenstate $|\psi\rangle$ we construct a density matrix $\rho = |\psi\rangle \langle \psi |$, and extract the half-chain entanglement entropy $S_{ent}$ according to $S_{ent} = -\text{Tr}_B \rho \log \rho$, where the trace is over half the chain. In Fig. 2 we plot the entanglement entropy of the eigenstates for a system of size $L = 13$, in total charge $Q = 0$ sector, as a function of dipole moment $P$. We note that the states with maximal charge have $Q = \pm L$, so $Q = 0$ corresponds to the middle of the many body spectrum, where we could expect the ETH to apply in a translation invariant and not conventionally integrable model. In every symmetry sector $(Q, P)$ we find a combination of low and high entanglement eigenstates, in sharp contrast to the usual expectations from eigenstate thermalization, and strongly reminiscent of the quantum scar phenomenon. As we will show, this apparent violation of the ETH arises from the shattering of Hilbert space.

### III. SHATTERING OF HILBERT SPACE

We now demonstrate how the local constraints fracture Hilbert space, giving rise to an exponentially large number of emergent dynamical subsectors. By contrast, note that the twin conservation laws of charge and dipole moment only lead to $O(L^3)$ explicit symmetry sectors, labeled by the values of charge and dipole moment ranging from $Q = \{-L, \cdots, L\}$ and $P = \{-L(L-1)/2, \cdots, L(L-1)/2\}$.
A. Localized eigenstates

In this section, we show how fractonic circuits have exponentially many exactly localized inert states, labeled by state dependent local integrals of motion (despite the absence of spatial randomness). These constitute emergent subsectors of dimension exactly one. Notably, these inert states are product states of charge (i.e. product states of $S^z$), so these are exceptionally simple, physically realizable states. These states are eigenstates of the Floquet fractonic circuit with zero entanglement, while they are left invariant by circuits that are random (i.e. non-repeating) in time, thereby also demonstrating robustness to temporal noise.

Exactly localized eigenstates may be constructed in the thermodynamic limit using an inductive method. We demonstrate this for the circuit with range three unitary gates. For system size $L = 3$, there is only one gate acting, and there are exactly 19 product states (in the charge basis) which have trivial dynamics, and are hence localized - these are the 19 states acted upon by trivial blocks of the constrained random unitary in Fig. 1 (e.g. the state |00+). These states do not mix with the rest of the Hilbert space, and are hence ‘inert,’ lying in a subsector with dimension one. Meanwhile, if a state is inert in a system of size $L$, then it will remain inert when an additional degree of freedom is added if the final two degrees of freedom of the $L$ site system and the additional degree of freedom collectively form one of the ‘inert’ configurations of an $L = 3$ site system. This is because the only “new” dynamics in the presence of the additional spin comes from the addition of a single three site unitary gate acting on the three spins formed by the added spins and the two penultimate spins of the length $L$ chain. Importantly, for any inert state of an $L$ site system, there is at least one choice of spin state for the added spin (and sometimes more than one), which leaves the resulting state in the $L + 1$ site system also inert. Specifically, an inert state in a system of size $L$ remains inert upon addition of another degree of freedom if the conditions tabulated in Table I are satisfied. Now let $N_{ab}(L)$ be the number of inert states in a system of size $L$, in which the final two sites have $S^z$ eigenvalues $a$ and $b$ respectively. The total number of inert states for a system of size $L$ is obtained by summing $N_{ab}(L)$ over all choices $ab$. Using Table 1, we can see that these quantities obey the recursion relations

\[
\begin{align*}
N_{++} & = 100100100100, \\
N_{+0} & = 100000000000, \\
N_{+-} & = 1000000000000, \\
N_{0+} & = 0100100100100, \\
N_{00} & = 0100100100100, \\
N_{0-} & = 00100100100100, \\
N_{-0} & = 0000000000001, \\
N_{--} & = 00000000000001,
\end{align*}
\]

TABLE I. For the fractonic circuit with three site gates, if an inert state in a system of size $L$ has the final two sites in the states shown in the left column, then it remains inert upon addition of another spin if the new spin is in the corresponding state shown in the right column.

This matrix can be diagonalized and its eigenvalues and eigenvectors, combined with the known values for $N_{ab}(3)$ can be used to exactly determine the number of inert states for any $L$. However, asymptotically at large $L$, the growth will be controlled by the largest eigenvalue of this matrix, $\lambda$, i.e. the dimension of the Hilbert space grows asymptotically as $|\lambda|^L$, where $\lambda$ is the largest eigenvalue. The matrix in question has only one real, positive eigenvalue with norm greater than one, $\lambda = 2.2$ which tells us that the dimension of the localized subspace grows asymptotically as $2.2^L$.

We therefore conclude that in the thermodynamic limit there are approximately $2.2^L$ inert states, each of which exists in its own emergent subsector, undergoes trivial (pure phase) dynamics, and does not mix with the rest of the Hilbert space. This is verified by exact numerical counting of the number of inert states in systems up to sizes $L=15$, and shown in Fig. 3(a). These constitute an exponentially large ‘localized subspace’ into which information may be robustly encoded. Each of these inert states can be labeled by state-dependent local integrals of motion corresponding to the local values of charge and dipole moment. Note that this type of localization does not require disorder - indeed it occurs even in a circuit that is translationally invariant in the thermodynamic limit. It even survives temporal noise, as long as the constraints are obeyed.

We note that the key feature of fractonic circuits that leads to this exponentially growing inert subspace is the existence of multiple pathways or choices for getting new inert states upon adding spins to inert states of a given size. By contrast, in a system with only charge conservation, the only choices for building inert states require ++ to be followed by +, or -- to be followed by −. This, however, gives exactly two inert states due to a lack of exponential branching arising from multiple pathways.

We now show that the existence of an exponentially large localized subspace is not an artifact of the simplicity of the three site gate circuit, but rather is required by fractonic constraints. To see this, one can firstly verify via an asymptotically exact counting (see Appendix) that the analogous fractonic circuit with four site gates also has an
exponentially large localized subspace, with asymptotic dimension \( \sim 1.8^L \) in the thermodynamic limit, again numerically verified in Fig. 3(a). Exact analytical calculations for larger gate sizes rapidly become tedious, but it is straightforward to show that an exponentially large exactly localized subspace survives for any finite gate size \( N \). To show this, note that any pattern that interconverts between locally ‘all plus’ and locally ‘all minus,’ with domain walls between ‘all plus’ and ‘all minus’ regions at least \( N \) sites apart, must be inert. These are states of the form \( \left| + + + + + - - - - - - - - - + + \cdots \right\rangle \). This follows because every gate acting on such a state straddles either zero or one domain walls. If it straddles zero domain walls, then it acts locally on a block with extremal charge, which is obviously inert. If it straddles one domain wall, then it acts on a block with extremal dipole moment given its charge, and this must also be inert. The inexactness of the latter kind of block follows because it is made up of only + and – charged sites, and the only charge conserving moves that one can make are (i) to reshuffle + and – charges and (ii) to delete + and – charges in pairs and replace them by zeros. However, if every + charge is to the right of any – charge (or vice versa) then any such move necessarily changes the dipole moment, and so is forbidden.

One can then straightforwardly lower bound the size of the exactly localized subspace for circuits with gate-size \( N \) by dividing the system up into blocks of length \( N \), and allowing each block to be either ‘all plus’ or ‘all minus.’ This yields an inert subspace of dimension at least \( 2^{L/N} = c^L \), where \( c = 2^{1/N} \). This is exponentially large in system size for any finite gate size \( N \), and cleanly illustrates how simultaneously conserving charge and dipole moment leads to the emergence of exponentially large localized subspaces. Note that this bound is not tight, since for \( N = 3 \) it predicts a localized subspace of dimension at least \( 1.25^L \), whereas a more careful counting gives a localized subspace of dimension \( 2.2^L \). Nevertheless, it is sufficient to establish the existence of an exponentially large robust localized subspace for any finite gate size.

We emphasize, again, that these localized states imply the presence of exponentially many zero entanglement states that are left inert by the dynamics, and that are eigenstates of the Floquet operator in a time periodic system. By contrast, we note that a recent paper examining the presence of scars in the same Floquet fractonic circuit as the one considered by us erroneously claimed that there is only one low entanglement “scarred” eigenstate in each \((Q, P)\) sector, for a total of \( O(L^3) \) scar states [49]. One distinction between our works is that Ref. [49] used periodic boundary conditions which does not commute with dipole conservation and allows superpositions of degenerate eigenstates in different \( P \) sectors. However, our recursive construction makes clear that there will still be exponentially many inert low entanglement eigenstates even with closed boundaries. What structure may be obtained by superposing degenerate eigenstates belonging to different \((Q, P)\) symmetry sectors is an interesting open question.
B. Larger subsectors

We now turn to the construction of emergent dynamical subspaces of dimension greater than one, which do not mix with the rest of the Hilbert space. A simple example for a circuit with three site gates is provided by a configuration of the form \(\cdots 0 + 0 \cdot \cdots\), where in each case the \(\cdots\) denote inert configurations with a ++ next to the non-trivial block. This subspace has non-trivial dynamics only over three sites in real space, and has Hilbert space dimension two. The total charge within this restricted region of real space is then independently a local integral of the motion, even though the circuit is in principle allowed to spatially move charge. Importantly, this local integral of motion is state dependent - a single charge immersed in a sea of zeros can move freely by emitting dipoles, whereas a charge blockaded on both sides by inert configurations ending in ++ cannot leave a restricted region of real space. Multiple analogous “active” blocks with locally non-trivial dynamics may trivially be introduced into an otherwise ‘inert’ background, each block “shielded” by ++ on either end. The size of the active blocks may also be varied in size. Such constructions manifestly exist for any finite gate size, since there is always a localized subspace into which finite non-trivial blocks may be embedded, with appropriate shielding. For example, for a circuit with four site gates, +++++ would suffice to ‘shield’ a 0 + 0 region. These are not the only examples (e.g. all charges could be reversed), but they suffice to make the point that non-trivial blocks can always be embedded into otherwise inert regions.

We therefore expect the Hilbert space within each symmetry sector to ‘shatter’ into numerous emergent subsectors. This ‘shattering’ may be straightforwardly resolved by numerically extracting the ‘connectivity’ of the Floquet operator, within a particular symmetry sector. In Fig. 4 we show this shattering quantitatively, for a twelve site system in the sector with \(Q = 0\) and \(P = 0\) and three-site gates. The sectors with exactly one state correspond to the ‘inert’ states (localized subspace) discussed above, but as one can see, there is a distribution of emergent subspaces of a wide variety of sizes. In Figure 3(b), we show the full distribution of emergent subsector sizes for circuits with gate size \(N = 3, 4\) in a system of size \(L = 13\). We see that, for three site gates, the frequency of subsectors of a particular size decreases polynomially with the dimension of the subsector. For four site gates there is an initial polynomial decrease followed by a saturation, in that beyond a certain subsector size, further increases in subsector size do not seem to translate into a decrease in frequency. Notably, the maximum size of the emergent subsectors for four site gates is much larger than that for three site gates.

In fact, the largest subsector for three-site gates is numerically observed to contain exactly \((\binom{L-1}{(L-1)/2})\) states, which asymptotically scales as \(2^L\). This is a vanishing fraction of the largest \((Q, P)\) sector, which scales as \(3^L\) (upto polynomial in \(L\) corrections). This indicates a strongly constrained dynamics, which is only ever able to connect a vanishing fraction of the full Hilbert space, also shown quantitatively in Figure 3(c). By contrast, the figure shows that the largest subsector with longer range gates has the same size as the largest symmetry sector, and thus the dynamics can access much larger parts of the Hilbert space. We next turn to an important feature of three site gates which may be responsible for this distinction.

C. Bottlenecks with three site gates

We now comment on a special feature of the fractonic circuit with three site gates, namely the existence of local integrals of motion that can act as ‘bottlenecks’ regardless of what larger state they are embedded into.

A simple example of such a bottleneck is provided by a local pattern of the form +++++ (or the charged reversed version). If such a (finite size) pattern is embedded into a larger state that is non-trivial everywhere to the left and the right, then the outer two + charges can move away (by absorbing dipoles), but crucially these ‘outer’ charges perfectly screen the inner charges from dipoles that could make them move. The inner + charges must always be adjacent either to another + charge, or to a 0, and thus any three-site gate acting on or across the two inner charges must necessarily be trivial (pure phase).
As a result, the inner two charges are perfectly localized regardless of what larger state they are embedded into, and act as a ‘bottleneck’ that cuts the chain in two. The two halves can then be separately labeled by values of charge and dipole moment that are conserved in each half. Likewise, the presence of these bottlenecks at multiple locations can break up the chain into effectively much smaller segments, and the charge and dipole moment of each segment is separately conserved.

This fragmentation of the chain via bottlenecks has important consequences for both dynamics, which look localized within fragmented regions, and for the entanglement of eigenstates (discussed in the next section), which can be lower than expected because the different fragments are unentangled.

This ‘bottleneck’ phenomenon does not survive with longer range gates, since longer range gates allow for dipole pair creation from the vacuum (e.g. $0000 \rightarrow - + + -$), such that the outer charges cannot perfectly screen the inner charges, if embedded into a large non-trivial region.

D. Entanglement of Eigenstates

We now re-examine our results on the mid-cut entanglement entropy of eigenstates, armed with our understanding of Hilbert space fracture. The first point is that there are emergent dynamical subsectors of varying sizes even within a single $(Q,P)$ sector, and the ‘thermal’ value for eigenstates in a given dynamical subsector will be controlled by the size of the subsector [50]. This naturally leads to a broad distribution of high and low entanglement states within a symmetry sector, as was observed in Fig. 2.

To examine this more quantitatively, in Fig 5(a), we plot the average entanglement entropy of each emergent dynamical subsector against the thermal (Page) value for that subsector in a system of length $L = 15$ with three-site gates. We consider all eigenstates in all subsectors in the $Q = \{0,1\}$ sectors (with all possible $P$ values). The data is averaged over 100 independent circuit realizations. The Page value is computed by explicitly examining the $S^z$ basis states that span a given subsector, and using these to extract $D_L$ and $D_R$, the dimension of the Hilbert spaces in the left and right halves of the chain for that subsector. Because of the constraints, these depend on the exact basis states that form the subspace and could be different for different subsectors of the same size. Because some of the subsector sizes are very small, we use the exact expression for the Page value [50]

$$S_{\text{Page}} = \sum_{k=m+1}^{mn} \frac{k}{2} - \frac{m-1}{2m},$$

where $m = \min[D_L, D_R]$ and $n = \max[D_L, D_R]$; this reduces to the more familiar form

$$S_{\text{Page}} \sim \log(n) - \frac{m}{2n}$$

for $1 \leq m \leq n$.

A priori one might have thought that the existence of these multiple subsectors with a broad distribution of sizes would be sufficient to explain the co-existence of high and low entanglement states within a symmetry sector. Indeed, the eigenstate entanglement does broadly track the Page value for the appropriate subsector, as shown in Fig 5(a). However, the figure also shows the existence of a broad distribution of entanglement entropies even after resolving by subsector size. There even exist states with strictly zero entanglement in subsectors with dimension greater than one. Thus, the ‘shattering’ of Hilbert space is part of the explanation for the broad distribution of entanglement entropies, but it is not the whole picture.

We believe that a key part of the explanation for the broad distribution of entanglement entropies even after resolving by subsector size is the bottleneck phenomenon...
discussed in Sec.III C. In particular, the states with zero entanglement entropy (which are not in the strictly localized subspace) have been explicitly verified to contain a ‘bottleneck’ motif at the midpoint of the chain, which prevents development of any entanglement across this motif, which happens to overlap the entanglement cut. The existence of such ‘bottleneck’ motifs at positions away from the entanglement cut is also at least partially responsible for the existence of a broad distribution of entanglement entropies, even after resolving by subsector size, since the effective number of entangling degrees of freedom get reduced when the chain is ‘cut’. Whether the combination of Hilbert space fracture and bottlenecks is sufficient to fully explain the results, or whether additional physics is also at play (e.g. the ‘hydrodynamic’ picture of [41]) would be an interesting topic for future work. Along these lines, it is also worth noting that “fracturing” is a basis dependent statement. For example, a purely diagonal Hamiltonian in the $S^z$ basis does have a fractured Hilbert space in the $x$ basis, but this is not apparent in the $z$ basis. All our statements on fracture have been with respect to the naive $z$ basis, but it is possible that a different basis choice may reveal further subsectors of smaller size, which may improve the agreement with the Page value. Examining this more systematically is an interesting direction for future work.

Finally, we note that the entanglement entropy in the subsector of largest size does appear to well approximate the thermal Page value, and this agreement gets better with increasing system size (Fig. 5(b)). While we don’t yet have a prescriptive way of constructing all $(\lfloor (L−1)/2 \rfloor)$ basis states that span the largest subsector, we have numerically verified that these basis states do not contain any bottlenecks of the kind described in III C.

E. Implications for dynamics

Finally, we turn to the implications of our results for dynamics starting from different initial states. While we have proven the existence of an exponentially large localized subspace, this subspace is still a measure zero fraction of the entire Hilbert space in the thermodynamic limit. While initial conditions that have high overlap with this localized subspace will clearly exhibit localization, initial conditions chosen randomly in Hilbert space will have vanishing overlap with the localized subspace. We now discuss the implications of Hilbert space shattering for the dynamics from random initial conditions. Dynamics from random initial conditions is expected to be highly sensitive to the degree of shattering. In Fig. 3(c) we examine what fraction of the states in a symmetry sector are contained in the emergent subsector of largest size. For three site gates, the largest emergent subsector is observed to contain a vanishing fraction of the states in the thermodynamic limit, consistent with our analytic estimates. (Recall that the largest subsector contained $\sim 2^L$ states, whereas the Hilbert space dimension is $3^L$). In contrast, for longer range gates a non-zero fraction (almost exactly equal to one) of the Hilbert space is contained in the emergent subsector of largest size, and this does not change with changing system size.

These differences may have interesting implications for the dynamics from randomly chosen initial product states (which are not in the $z$ basis and are not confined to any particular symmetry sector). For example, for three site gates, the largest subsector of Hilbert space has dimension $\sim 2^L$. The late-time entanglement entropy should therefore be dominated this subsector and scale as $L \ln 2$. Meanwhile, the entire Hilbert space has dimension $3^L$, and so the thermal or ‘Page’ entanglement entropy for the full Hilbert space is of order $L \ln 3$. We would therefore expect that for a circuit with three site gates, a random initial condition should exhibit entanglement entropy growth saturating to a value approximately equal to $\ln 2 S_{Page} \approx 0.63 S_{Page}$. However, for a circuit with four site gates, the largest subsector size and the Hilbert space dimension scale similarly, as $3^L$, and one might expect dynamics starting from random initial conditions to lead to entanglement entropy growth saturating close to the Page value.

To test this intuition, in Fig. 6, we show the growth of entanglement entropy for both three and four site gates, starting from an initial condition that is a random product state. Note that a random product state (not in the $z$ basis) is a superposition of multiple symmetry sectors and subsectors. For three site gates, the entanglement entropy is observed to saturate to a clearly subthermal

![FIG. 6. Figure showing the dynamics of entanglement starting from a random product state (not in the $z$ basis). Entanglement entropy is given as a ratio of the ‘Page’ value for a random product state in a Hilbert space of dimension $3^L$. For three site gates, the entanglement entropy saturates to well below its Page value, consistent with expectations given the strong fracturing of Hilbert space. For four site gates, the saturating value of entanglement entropy is much closer to the Page value, with a slow upward drift with increasing system size.](image_url)
value of order $0.6S_{Page}$, consistent with our expectations. Meanwhile, for four site gates the saturation value for the entanglement entropy is clearly higher, much closer to the Page value, with a slow upward drift with increasing system size. Whether the saturating value of entanglement entropy actually reaches the Page value in the thermodynamic limit is not clear from the present numerics. A more extensive investigation of pure state dynamics starting from random initial conditions, and how this depends on gate range, would be an interesting problem for future work.

IV. SHATTERING (AND ITS ABSENCE) IN NON-FRACTONIC CIRCUITS

We now point out that not all local constraints shatter Hilbert space in the manner discussed above. A good and topical example of a constrained system without a shattered Hilbert space is provided by the Rydberg chain. The dynamics of the Rydberg chain [34] is believed to be well approximated by the PXP model [26, 51, 52], which acts on a constrained Hilbert space of a chain of spin $S = 1/2$ variables (which may be in state $0$ or $1$), with no two adjacent spins up. That is, the Hilbert space for three sites consists of the states $(0,0,0)$, $(0,0,1)$, $(1,0,0)$ and $(0,1,0)$ only. Moreover, this model only mixes the states $(0,0,0)$ and $(0,1,0)$, so that for a system with $L = 3$ there are exactly three inert states, $(1,0,0)$, $(1,0,1)$, and $(0,0,1)$. Meanwhile if $N_{ab}(L)$ is the number of inert states in a system of size $L$ ending in $(a,b)$, then we obtain the recursion relation

$$\begin{pmatrix} N_{10} \\ N_{01} \\ N_{00} \end{pmatrix}_{L+1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} N_{10} \\ N_{01} \\ N_{00} \end{pmatrix}_L$$

All eigenvalues of this matrix are zero. In sharp contrast to the fractonic circuit, therefore, the Rydberg chain does not exhibit an obvious exponentially large localized subspace or an attendant shattered Hilbert space within the constrained subspace. Nevertheless, this system does have “scars” in this subspace, whose origin is not yet understood. These could still be caused by emergent Hilbert space fracture in some non-obvious basis, either due to emergent subsystem symmetries [32] or proximity to integrability [27] or some such mechanism.

Thus far, our discussion of circuits exhibiting shattering has been particular to circuits with ‘fractonic’ constraints (viz. conservation of charge and dipole moment). However, not obviously fractonic circuits displaying a similar shattering of Hilbert space may also be constructed. For example, consider a circuit made out of local two spin gates acting on a one dimensional chain of $S = 1$ spins. If this two site gate is constrained so that it acts trivially on the states $|0+\rangle$, $|+0\rangle$, $|0-\rangle$, and $|0-\rangle$, then it may be readily verified, through methods similar to those employed for the fractonic circuit, that there is an exponentially large space of inert states displaying trivially localized dynamics. For a chain of size $L = 2$, there are then exactly four inert states. Meanwhile, if $N_{\beta}(L)$ is the number of inert states ending in $\beta$ in a system of size $L$, then this quantity obeys the recursion relation

$$\begin{pmatrix} N_+ \\ N_0 \\ N_- \end{pmatrix}_L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} N_+ \\ N_0 \\ N_- \end{pmatrix}_L$$

The matrix in the recursion relation has eigenvalues $\pm \sqrt{2}$ and zero. The dimension of the degenerate subspace thus grows asymptotically as $\sqrt{2}^L$, providing a concrete example of a not obviously fractonic circuit with an exponentially large localized subspace. The mechanism again involves the existence of “multiple” pathways for extending inert states when new sites are added. However, in the absence of a physical principle giving rise to this particular circuit architecture, analogous to the ‘fractonic’ constraints of charge and dipole moment conservation, it is unclear how this circuit should be generalized to gates of longer range, and hence the question of whether this ‘shattering’ survives in the presence of longer range gates is ill posed. Nevertheless, ‘shattering’ may be produced by similar constructions in circuits involving gates of larger size - a sufficient condition is that there should exist at least two locally inert patterns which can be combined together in an inert fashion.

A fruitful perspective on which types of circuits produce ‘shattering’ of Hilbert space is provided by recursion relations of the form discussed above. For a circuit acting on a system with local Hilbert space dimension $q$, and random $N$ site gates, the recursion relation is governed by a square matrix of size $q^{N-1}$. The entries in this matrix can only be $0$ or $1$ - and at least two of the entries must be zeros, otherwise the circuit acts trivially on every possible state (which is a trivial shattering, say by diagonal matrices). Every such matrix with an eigenvalue larger than $1$ specifies a circuit with an exponentially large inert subspace. From this it follows that there are no spin $1/2$ chains with only two site gates that realize a shattered Hilbert space (in the obvious $z$ basis)- spin $S = 1$ and two site gates is the minimal case necessarily to realize such shattering.

V. DISCUSSION

We have shown how a local ‘fractonic’ constraint can ‘shatter’ Hilbert space into a huge number of emergent dynamical subsectors, leading to the emergence of exponentially large localized subspaces in which the localization is robust to temporal noise, does not require disorder, and is characterized by state dependent emergent local integrals of motion. The shattering leads to the coexistence, within a particular symmetry sector, of both high and low entanglement states similar to systems with
many-body scars, thereby violating ETH as conventionally defined. Moreover the gates in the circuit may be chosen randomly subject to the constraints, so the model is not at all fine tuned. This large localized subspace could have an obvious application as a protected quantum memory. The key results have been shown to be robust for fractonic (i.e. charge and dipole conserving) circuits with any finite gate size.

We note that insofar as the time evolution operator within each symmetry sector further ‘block diagonalizes’ into subsectors, the analysis we have presented is reminiscent of the construction for scars by Shiraishi and Mori [25]. However, whereas the projective structure in [25] is introduced by hand, here it emerges naturally as a result of imposing ‘fractonic’ constraints (viz. conservation of charge and dipole moment respectively).

We have pointed out that the Rydberg chain, perhaps the best studied model in the context of ‘quantum scars,’ does not have an (obviously) fractured Hilbert space and hence not all constraints lead to fracture. However, we have also provided examples of not obviously fractonic circuits that exhibit shattering. What physical principles underlie these circuits - beyond the fractonic constraints discussed herein - would be an interesting topic for future work. We note that our general construction of circuits exhibiting shattering bears a striking resemblance to cellular automata [53], a connection that may be worth deeper exploration.

Thus far we have focused on circuit dynamics. However, Hamiltonian dynamics is of greater relevance for the study of physical systems. Insofar as Hamiltonian dynamics is ‘more constrained’ than circuit dynamics, being required to conserve energy, we expect that it should be if anything ‘more localized.’ Indeed we do believe that our results should carry over mutatis mutandis to Hamiltonian systems realizing fracton phases in one spatial dimension. The existing fracton literature does not contain any examples of Hamiltonians realizing robust fracton phases in one dimension. However, we are not aware of any ‘no-go’ theorems in this regard either, and the rich physics to be found in one dimensional fracton problems (detailed in part in the present manuscript) provides a good motivation for searching for such Hamiltonians. We believe such Hamiltonian extensions will be discussed in two forthcoming works [54, 55].

Additionally, it would be worth understanding to what extent the physics discussed herein carries over to higher dimensions. The connections to cellular automata alluded to above may prove particularly fruitful in this regard. It would also be interesting to explore how this microscopic understanding of pure state dynamics discussed here relates to mixed state dynamics of the form discussed in [41].

The broadest physical class of theories involving local constraints are of course gauge theories, and ‘fractonic’ phases are known to be describable as gauge theories of ‘higher rank’ [7]. It would be interesting to explore the possibility of Hilbert space shattering in gauge theories more generally, to clarify whether there are other types of gauge theories (beyond the ‘fractonic’ ones discussed herein) which exhibit such shattering. This may also connect to recent works on ergodicity breaking in gauge theories [35–37, 56, 57].

Finally, we note that thus far our discussion has assumed that the constraints are applied as hard constraints, which cannot be violated. However, local constraints usually come from energetics, and are typically not ‘hard’ but rather ‘soft’ i.e. constraints can be violated, at the cost of paying a large energy penalty. What happens to the phenomenon of Hilbert space fracture when the constraints are softened? Presumably at the longest times the connectivity of the Hilbert space is restored, as is ergodicity and ETH, but there may well be some interesting intermediate time dynamics. Investigation of this issue would also be a fruitful topic for future work.

Note added: While we were finalizing our manuscript, we learnt about related work by P. Sala, T. Rakovszky, R. Verresen, M. Knap and F. Pollmann which will appear in the same arXiv posting.

ACKNOWLEDGMENTS

We would like to thank Anushya Chandran, Jason Iaconis, Chris Lauren, Sanjay Moudgalya, Shriya Pai, Abhinav Prem, Michael Pretko, and Sagar Vijay for useful conversations and for collaborations on related work. We also acknowledge useful conversations with David Huse, Tibor Rakovszky, Pablo Sala, Michael Knap and Frank Pollmann. VK is supported by the Harvard Society of Fellows and the William F. Milton Fund. This material is based in part upon work supported by the Air Force Office of Scientific Research under award number FA9550-17-1-0183 (RN). Both authors also acknowledge the hospitality of the KITP, where part of this work was conducted, during a visit to the program “Dynamics of Quantum Information” . The KITP is supported in part by the National Science Foundation under Grant No. NSF PHYS-1748958.

Appendix A: Localized subspace for fractonic circuit with four site gates

In this Appendix we provide an explicit calculation of the localized subspace for the fractonic circuit with four site gates. In this case the gates are matrices of rank $3^4 = 81$, with structure as detailed in Table I of [41]. Note however that there is a typo in the charge zero block of that table, in that configurations such as $+00-$ and $-00+$ should be inert, whereas $+-+-$ should mix freely with $+0-0$ and $0+0-$, but not with $+00-$. With this typo corrected, we note that in a chain of size $L = 4$ there are twenty six trivial states. If a state is inert in an $L$ site system, then the addition of another site will leave
The largest eigenvalue of the above matrix has magnitude 1.8, leading us to conclude that the dimension of the localized subspace grows asymptotically as $1.8^L$, in agreement with Figure 3(a) and again, faster than the lower bound of $2^{L/4} \sim 1.2^L$. Similar analyses may be carried through for any finite range of gates in the fractonic circuit, but the analysis rapidly becomes tedious and so we do not pursue it here.

[1] Benjamin J. Brown, Daniel Loss, Jiannis K. Pachos, Chris N. Self, and James R. Wootton, “Quantum memories at finite temperature,” Rev. Mod. Phys. 88, 045005
A.Yu. Kitaev, “Fault-tolerant quantum computation by anyons,” Annals of Physics 303, 2 – 30 (2003).

Claudio Chamon, “Quantum glassiness in strongly correlated clean systems: An example of topological overprotection,” Phys. Rev. Lett. 94, 040402 (2005).

Jeongwan Haah, “Local stabilizer codes in three dimensions without string logical operators,” Phys. Rev. A 83, 042330 (2011).

Sagar Vijay, Jeongwan Haah, and Liang Fu, “A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations,” Phys. Rev. B 92, 235136 (2015).

Sagar Vijay, Jeongwan Haah, and Liang Fu, “Fracton topological order, generalized lattice gauge theory, and duality,” Phys. Rev. B 94, 235157 (2016).

Michael Pretko, “Subdimensional particle structure of higher rank u(1) spin liquids,” Phys. Rev. B 95, 115139 (2017).

Michael Pretko, “Generalized electromagnetism of subdimensional particles: A spin liquid story,” Phys. Rev. B 96, 035119 (2017).

R. M. Nandkishore and M. Hermele, “Fractons,” ArXiv e-prints (2018), arXiv:1803.11196 [cond-mat.str-el].

Isaac H. Kim and Jeongwan Haah, “Localization from superselection rules in translationally invariant systems,” Phys. Rev. Lett. 116, 027202 (2016).

Karthik Siva and Beni Yoshida, “Topological order and memory time in marginally-self-correcting quantum memory,” Phys. Rev. A 95, 032324 (2017).

Abhinav Prem, Jeongwan Haah, and Rahul Nandkishore, “Glassy quantum dynamics in translation invariant fracton models,” Phys. Rev. B 95, 155133 (2017).

J. M. Deutsch, “Quantum statistical mechanics in a closed system,” Phys. Rev. A 43, 2046–2049 (1991).

Marcos Rigol, Vanja Dunjko, and Maxim Olshanii, “Thermalization and its mechanism for generic isolated quantum systems,” Nature 452, 854–858 (2008).

Mark Srednicki, “Chaos and quantum thermalization,” Phys. Rev. E 50, 888-901 (1994).

Giulio Biroli, Corinna Kollath, and Andreas M. Läuchli, “Effect of rare fluctuations on the thermalization of isolated quantum systems,” Phys. Rev. Lett. 105, 250401 (2010).

Lev Vidmar and Marcos Rigol, “Generalized gibbs ensemble in integrable lattice models,” Journal of Statistical Mechanics: Theory and Experiment 2016, 064007 (2016).

I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, “Interacting electrons in disordered wires: Anderson localization and low-t transport,” Phys. Rev. Lett. 95, 206603 (2005).

D.M. Basko, I.L. Aleiner, and B.L. Altshuler, “Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states,” Annals of Physics 321, 1126 – 1205 (2006).

Rahul Nandkishore and David A. Huse, “Many-body localization and thermalization in quantum statistical mechanics,” Annual Review of Condensed Matter Physics 6, 15–38 (2015), https://doi.org/10.1146/annurev-conmatphys-031214-014726.

Dmitry A. Abanin, Ehud Altman, Immanuel Bloch, and Maksym Serbyn, “Many-body localization, thermalization, and entanglement,” arXiv e-prints , arXiv:1804.11065 (2018), arXiv:1804.11065 [cond-mat.dis-nn].

David A. Huse, Rahul Nandkishore, and Vadim Oganesyan, “Phenomenology of fully many-body-localized systems,” Phys. Rev. B 90, 174202 (2014).

Maksym Serbyn, Z. Papic, and Dmitriy A. Abanin, “Local conservation laws and the structure of the many-body localized states,” Phys. Rev. Lett. 111, 127201 (2013).

Sanjay Moudgalya, Stephan Rachel, B. Andrei Bernevig, and Nicolas Regnault, “Exact excited states of nonintegrable models,” Phys. Rev. B 98, 235155 (2018).

Naoto Shiraishi and Takashi Mori, “Systematic construction of counterexamples to the eigenstate thermalization hypothesis,” Phys. Rev. Lett. 119, 030601 (2017).

CJ Turner, AA Michailidis, DA Abanin, M Serbyn, and Z Papic, “Weak ergodicity breaking from quantum many-body scars,” Nature Physics (2018).

Vedika Khemani, Chris R. Laumann, and Anushya Chandran, “Signatures of integrability in the dynamics of rydberg-blockaded chains,” Phys. Rev. B 99, 161101 (2019).

C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papic, “Quantum scarred eigenstates in a rydberg atom chain: Entanglement, breakdown of thermalization, and stability to perturbations,” Phys. Rev. B 98, 155134 (2018).

Cheng-Ju Lin and Olexi I. Motrunich, “Exact Strong-ETH Violating Eigenstates in the Rydberg-blockaded Atom Chain,” arXiv e-prints , arXiv:1810.00888 (2018), arXiv:1810.00888 [cond-mat.quant-gas].

Sanjay Moudgalya, Nicolas Regnault, and B Andrei Bernevig, “Entanglement of exact excited states of aklt models: Exact results, many-body scars and the violation of strong eth,” arXiv preprint arXiv:1806.09624 (2018).

Wen Wei Ho, Soonwon Choi, Hannes Pichler, and Mikhail D. Lukin, “Periodic orbits, entanglement, and quantum many-body scars in constrained models: Matrix product state approach,” Phys. Rev. Lett. 122, 040603 (2019).

Soonwon Choi, Christopher J. Turner, Hannes Pichler, Wen Wei Ho, Alexios A. Michailidis, Zlatko Papic, Maksym Serbyn, Mikhail D. Lukin, and Dmitriy A. Abanin, “Emergent SU(2) dynamics and perfect quantum many-body scars,” arXiv e-prints , arXiv:1812.05561 (2018), arXiv:1812.05561 [quant-ph].

Seulgi Ok, Kenny Choo, Christopher Mudry, Claudio Castelnovo, Claudio Chamon, and Titus Neupert, “Topological many-body scar states in dimensions 1, 2, and 3,” arXiv preprint arXiv:1901.01260 (2019).

H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletić, and M. D. Lukin, “Probing many-body dynamics on a 51-atom quantum simulator,” Nature (London) 551, 579–584 (2017), arXiv:1707.04344 [quant-ph].

Federica M Surace, Paolo P Mazza, Giuliano Giudici, Alessio Lerose, Andrea Gambassi, and Marcello Dalmonte, “Lattice gauge theories and string dynamics in rydberg atom quantum simulators,” arXiv preprint arXiv:1902.09551 (2019).

Jonghoon Park, Yoshihito Kuno, and Ilkko I. O. Khatamian, “Glassy dynamics from quark confinement: Atomic quantum simulation of gauge-higgs model on lattice,” arXiv preprint arXiv:1903.07297 (2019).
[37] Andrew JA James, Robert M Konik, and Neil J Robinson, “Nonthermal states arising from confinement in one and two dimensions,” arXiv preprint arXiv:1804.09990 (2018).

[38] Thomas Iadecola, Michael Schecter, and Shenglong Xu, “Quantum Many-Body Scars and Space-Time Crystalline Order from Magnon Condensation,” arXiv e-prints, arXiv:1903.10517 (2019), arXiv:1903.10517 [cond-mat.str-el].

[39] Zhihao Lan, Merlijn van Horssen, Stephen Powell, and Juan P. Garrahan, “Quantum slow relaxation and metastability due to dynamical constraints,” Phys. Rev. Lett. 121, 040603 (2018).

[40] B. Olmos, M. Müller, and I. Lesanovsky, “Thermalization of a strongly interacting 1D Rydberg lattice gas,” New Journal of Physics 12, 013024 (2010), arXiv:0907.4420 [cond-mat.quant-gas].

[41] Shriya Pai, Michael Pretko, and Rahul M Nandkishore, “Localization in fractonic random circuits,” arXiv preprint arXiv:1807.09776 (2018).

[42] Adam Nahum, Jonathan Ruhman, Sagar Vijay, and Jeongwan Haah, “Quantum entanglement growth under random unitary dynamics,” Phys. Rev. X 7, 031016 (2017).

[43] A. Nahum, S. Vijay, and J. Haah, “Operator Spreading in Random Unitary Circuits,” ArXiv e-prints (2017), arXiv:1705.08975 [cond-mat.str-el].

[44] A. Nahum, J. Ruhman, and D. A. Huse, “Dynamics of entanglement and transport in 1D systems with quenched randomness,” ArXiv e-prints (2017), arXiv:1705.10364 [cond-mat.dis-nn].

[45] V. Khemani, A. Vishwanath, and D. A. Huse, “Operator spreading and the emergence of dissipation in unitary dynamics with conservation laws,” ArXiv e-prints (2017), arXiv:1710.09835 [cond-mat.stat-mech].

[46] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, “Diffusive hydrodynamics of out-of-time-ordered correlators with charge conservation,” ArXiv e-prints (2017), arXiv:1710.09827 [cond-mat.stat-mech].

[47] C. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. Sondhi, “Operator hydrodynamics, OTOCs, and entanglement growth in systems without conservation laws,” ArXiv e-prints (2017), arXiv:1705.08910 [cond-mat.str-el].

[48] C. Jonay, D. A. Huse, and A. Nahum, “Coarse-grained dynamics of operator and state entanglement,” ArXiv e-prints (2018), arXiv:1803.00089 [cond-mat.stat-mech].

[49] Shriya Pai and Michael Pretko, “Manifestation of quantum many-body scars in fracton systems,” arXiv e-prints, arXiv:1903.06173 (2019), arXiv:1903.06173 [cond-mat.stat-mech].

[50] Don N. Page, “Average entropy of a subsystem,” Phys. Rev. Lett. 71, 1291–1294 (1993).

[51] P. Fendley, K. Sengupta, and S. Sachdev, “Competing density-wave orders in a one-dimensional hard-boson model,” Phys. Rev. B 69, 075106 (2004), cond-mat/0309438.

[52] Subir Sachdev, K. Sengupta, and S. M. Girvin, “Mott insulators in strong electric fields,” Phys. Rev. B 66, 075128 (2002).

[53] John Von Neumann et al., “The general and logical theory of automata,” 1951, 1–41 (1951).

[54] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, “to appear in the same arXiv posting as this manuscript,” ArXiv e-prints (2019).

[55] S. et al Moudgalya, “manuscript in preparation,” ArXiv e-prints (2019).

[56] Rahul M. Nandkishore and S. L. Sondhi, “Many-body localization with long-range interactions,” Phys. Rev. X 7, 041021 (2017).

[57] Marlon Brenes, Marcello Dalmonte, Markus Heyl, and Antonello Scardicchio, “Many-body localization dynamics from gauge invariance,” Phys. Rev. Lett. 120, 030601 (2018).