Research Article

Research on a Cournot–Bertrand Game Model with Relative Profit Maximization

Yi-min Huang,1 Qiu-xiang Li,2 Yan-yan Guo,1 and Yu-hao Zhang3

1School of Management & Economics, North China University of Water Resources and Electric Power, Zhengzhou 450046, China
2Institute of Management Science and Engineering, Henan University, Kaifeng 475004, China
3School of Information Management and Engineering, Shanghai University of Finance and Economics, Shanghai 200433, China

Correspondence should be addressed to Qiu-xiang Li; lqxkycg@163.com and Yan-yan Guo; guoyyy9621@stu.ncwu.edu.cn

Received 10 October 2019; Revised 14 April 2020; Accepted 20 April 2020; Published 10 June 2020

Academic Editor: Lucia Valentina Gambuzza

Copyright © 2020 Yi-min Huang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper considers a Cournot–Bertrand game model based on the relative profit maximization with bounded rational players. The existence and stability of the Nash equilibrium of the dynamic model are investigated. The influence of product differentiation degree and the adjustment speed on the stability of the dynamic system is discussed. Furthermore, some complex properties and global stability of the dynamic system are explored. The results find that the higher degree of product differentiation enlarges the stable range of the dynamic system, while the higher unit product cost decreases the stable range of price adjustment and increases the one of output adjustment; period cycles and aperiodic oscillation (quasi-period and chaos) occur via period-doubling or Neimark–Sacker bifurcation, and the attraction domain shrinks with the increase of adjustment speed values. By selecting appropriate control parameters, the chaotic system can return to the stable state. The research of this paper is of great significance to the decision-makers’ price decision and quantity decision.

1. Introduction

Oligarchic market is one type of market structures controlled by several players who produce homogeneous products and take actions on the basis of their rival reactions for seeking profit maximization. Oligopoly game is extensively studied by many scholars; especially, in recent years, the dynamics of the oligopoly game has been extensively studied as a branch of mathematical economics.

A large significant number of previous articles have analyzed the stability of discrete dynamic oligarchic models as well as the dynamic behaviors with players having the naive rule, adaptive rule, and bounded rational expectation rule, respectively. The dynamic Cournot game was studied by many researchers, such as Kopel [1], Agiza [2], Bischl et al. [3], Bischl and Lamantia [4], Agiza and Elsadany [5], Fanti and Gori [6], Fanti et al. [7], Zhu et al. [8], Gori et al. [9], and Elsadany [10]. Most of the aforementioned works analyzed the adjustment process of quantity of the players with bounded rationality and found the periodic cycles and chaotic behaviors occurred in the Cournot model when the adjustment speed is faster than a certain value.

Many scholars researched on the Bertrand game and explored its complex features using nonlinear dynamic theory. Ma and Guo [11] analyzed the impacts of information on the dynamical price game, including the changes of the stability region and attraction basin of the Nash equilibrium point. Yi and Zeng [12] developed a duopoly Bertrand model with quadratic function and analyzed the stability region, chaotic behaviors, and chaos attractors of the Bertrand model. Elsadany and Awad [13] investigated a duopolistic Bertrand competition market with environmental taxes when the public firm is privatized and not privatized. The dynamical behaviors of the models are studied by numerical simulation and period-doubling bifurcation and chaos appeared. In addition, the research about the dynamic Stackelberg game was also studied by many scholars, such as Shi et al. [14], Peng and Lu [15], Peng et al. [16], and Askar [17].

In recent years, the Cournot–Bertrand game received more attention by some scholars. Tremblay and Tremblay
[18] established a Cournot–Bertrand duopoly model with differentiated products and discussed the Nash equilibrium of the model. Askar [19] presented a Cournot–Bertrand model based on a nonlinear price function and analyzed the stability of the Nash equilibrium point. Naimzada and Tramontana [20] considered a dynamic Cournot–Bertrand duopoly model with product differentiation and studied the role of the best dynamic response for the stability of the equilibrium point. Wang and Ma [21] constructed a mixed duopoly game model with limited information, and the existence and stability of the Nash equilibrium point are investigated. Ma and Pu [22] researched the chaotic behaviors the Cournot–Bertrand duopoly model using non-linear dynamics theory. Different from the above works, the Cournot–Bertrand game model developed in this paper is based on relative profit maximization of policymakers.

The relative performance or relative profits as a competition mechanism have paid attention by some literature studies. There are cases in point, on the one hand, even if a person earns big money, he is not happy enough and may be disappointed if his brother/sister or close friend earns bigger money. On the other hand, even if he is very poor and his neighbor is poorer, he may be consoled by that fact [23]. The firm’s relative profit is the difference between its absolute profit and that of its competitor. Pursuing the relative profit or utility is considered to be rational based on human nature. The firms consider their relative profits rather than absolute profits as the business objective is closer to reality.

Miller and Pazgal [24] argued that business people are likely to be inherently competitive, caring more about the relative position and status than about absolute profits, and researched the equilibria of profit maximizing owners selecting different types of managers with various types of competition (a variety of attitudes toward relative performance). The equilibrium with the quantity setting behaviors and price setting behaviors of the duopoly game under relative profit maximization in the static state was explored in depth by Satoh and Tanaka [23], Hattori and Tanaka [25], Tanaka [26, 27], and so on. Some scholars have studied the game model with relative profit maximization under dynamic decision-making. Fanti et al. [7] developed the dynamics of the nonlinear Cournot duopoly game with managerial delegation and bounded rational players. Furthermore, the on-off intermittency and blow-out bifurcations as well as coexistence of attractors are observed. Lu [28] studied the influence of the relative profit maximization on endogenous timing in the Stackelberg game with public and private oligopoly. Elsadany [10] developed a dynamic Cournot duopoly model with relative profit maximization and externality cost functions. Askar [17] investigated the dynamic characteristic of a duopoly game of two firms based on their relative profit maximization.

To the best of our knowledge, little literature has studied the Cournot–Bertrand game with relative profit maximization using nonlinear dynamic theory. It is a very interesting item to study the dynamic characteristic of the Cournot–Bertrand game model based on the relative profit maximization.

Our theoretical contribution is as follows. The first contribution is to construct a dynamic Cournot–Bertrand game model, in which one firm adopts price as his decision variable and the other adopts quantity as his decision variable based on the relative profit maximization. The second contribution is to study the influence of parameters changing on the stability and profitability of the dynamic Cournot–Bertrand game model and to obtain some interesting phenomenon, such as existing period-doubling bifurcation and Neimark–Sacker bifurcation.

This paper is organized as follows. In Section 2, the dynamic Cournot–Bertrand model with the relative profit maximization is given. In Section 3, the stability of Nash equilibrium points of the model is analyzed. Numerical simulations are carried out to present the complex features of the dynamic system in Section 4. Section 5 gives the global stability analysis of the dynamic system using basins of attraction. In Section 6, the variable feedback control method is adopted to remove the chaos of the model. Finally, some conclusions are concluded in Section 7.

2. Model Construction

This paper considers two firms (firm 1 and firm 2) in the same market with differentiation products ($x_1$ and $x_2$) with linear inverse demand function and linear cost function. Firm 1 regards the quantity of the products as his strategy variable, while firm 2 considers the price of the products as the strategy variable. That is to say, firm 1 competes in product quantity ($q_1$) and firm 2 competes in product price ($p_2$). Both firms will take actions based on the information from their rival and market for seeking relative profit maximization. This paper mainly focuses on contributing to construct a dynamic Cournot–Bertrand game model based on relative profit maximization and investigate the dynamic behaviors of players and the complex properties of the system.

The inverse demand functions of products are as follows (for more details, refer to [29, 30]):

\begin{align*}
p_1 &= 1 - q_1 - d q_2, \\
p_2 &= 1 - q_2 - d q_1,
\end{align*}

(1)

where $p_1$, $p_2$, denote the retail prices of products from firm 1 and firm 2, respectively, and $q_1$ and $q_2$ are the product quantities of firm 1 and firm 2. Moreover, $d \in [0, 1]$ denotes the degree of product differentiation between two firms. When $d$ approaches one, the products of two firms become less differentiated; when $d = 1$, the two products are completely homogeneous; and when $d = 0$, the structure of the market transfers to the monopolist from the duopoly. The cost functions are in linear form:

\begin{align*}
C_i(q_i) &= c q_i, \quad i = 1, 2,
\end{align*}

(2)

where $c > 0$ is the unit product cost.

Considering that the strategic variables are different between two firms, the inverse demand function of firm 1 and the demand function of firm 2 are given, respectively:
\[ p_1 = 1 - d - (1 - d^2)q_1 + dp_2, \]
\[ q_2 = 1 - p_2 - dq_1. \]

Then, the absolute profit functions of two firms are as follows:
\[ \pi_1 = q_1 \left( 1 - d - (1 - d^2)q_1 + dp_2 \right) - cq_1, \]
\[ \pi_2 = p_2 \left( 1 - p_2 - dq_1 \right). \]

Based on previous assumptions, the relative profits of the players are defined as the difference between their absolute profits and those of other players. The relative profits of firm 1 and firm 2 are represented by \( \Phi_1 \), \( \Phi_2 \), respectively, given by
\[ \Phi_1 = \pi_1 - \pi_2 = q_1 \left( 1 - d - (1 - d^2)q_1 + dp_2 \right) - cq_1 - p_2 \left( 1 - p_2 - dq_1 \right), \]
\[ \Phi_2 = \pi_2 - \pi_1 = p_2 \left( 1 - p_2 - dq_1 \right) - q_1 \left( 1 - d - (1 - d^2)q_1 + dp_2 \right) - cq_1. \]

The first partial derivative of relative profit maximization of firm 1 with respect to \( q_1 \) is
\[ \frac{\partial \Phi_1}{\partial q_1} = 1 - d - c - cd + 2dp_2 - 2q_1 + 2d^2q_1. \]

Similarly, the first condition of the relative profit maximization of firm 2 with respect to \( p_2 \) is given by
\[ \frac{\partial \Phi_2}{\partial p_2} = 1 + c - 2p_2 - 2dq_1. \]

The second condition of the relative profit maximization of firm 1 and firm 2 is easily calculated as follows:
\( (\partial^2 \Phi_1/\partial q_1^2) = 2(1 - d - 1) < 0 \) and \( (\partial^2 \Phi_2/\partial p_2^2) = -2 < 0 \). Then, the relative profit functions of two firms are concave, and unique maximum value exists.

According to the actual market, the players in the market cannot get all the information they need in collecting and processing. Hence, both firms express bounded rational behaviors when making strategy decisions of their own. The optimal actions of two firms based on their rival’s reaction are derived, respectively:
\[ q_1 (t + 1) = \arg \max \Phi_1 (q_1 (t), p_2 (t)), \]
\[ p_2 (t + 1) = \arg \max \Phi_2 (q_1 (t), p_2 (t)). \]

For achieving the relative profit maximization, the firms will choose the optimal actions based on the marginal relative profit. If the marginal relative profit is positive (negative) in the current time period, the firms will increase (decrease) their quantity (price) in the next time period. Therefore, the discrete dynamical adjustment mechanism of the Cournot–Bertrand game model is described by
\[ \begin{aligned}
q_1 (t + 1) &= q_1 (t) + aq_1 (t) \left( 1 - d - c - cd + 2dp_2 - 2q_1 + 2d^2q_1 \right), \\
p_2 (t + 1) &= p_2 (t) + \beta p_2 (t) \left( 1 + c - 2p_2 - 2dq_1 + c \right).
\end{aligned} \]

Consequently, the time evolution of the Cournot–Bertrand game with bounded rational expectation is built. The next section analyzes the stability and complex features of dynamic system (12).

3. Stability Analysis of the Dynamic System (12)

In this section, we emphasize on analyzing the stability of the Nash equilibrium points of dynamic system (12). Letting \( q_1 (t + 1) = q_1 (t) \) and \( p_2 (t + 1) = p_1 (t) \), four equilibrium solutions are computed as follows:
\[ E_1 = (0, 0), \]
\[ E_2 = \left( 0, \frac{1 + c}{2} \right), \]
\[ E_3 = \left( \frac{1 - d - c - cd}{1 - d^2}, 0 \right), \]
\[ E_4 = (q_1^*, p_2^*) = \left( \frac{1 - c}{2}, \frac{1 + c - d + cd}{2} \right). \]

In order to ensure that the equilibrium solutions are nonnegative, conditions should be satisfied as follows:
Complexity

\[
\begin{align*}
1 - c &> 0, \\
1 - d - c - cd &> 0, \\
1 + c - d + cd &> 0.
\end{align*}
\]  
(14)

Lemma 1. \(E_1, E_2, \) and \(E_3\) are the unstable equilibrium points, and \(E_4\) is a unique Nash equilibrium solution of dynamic system (12).

**Proof.** The Jacobian matrix of dynamic system (12) takes the following form:

\[ J(q_1, p_2) = \begin{bmatrix} 1 + aA & 2\alpha dq_1 \\ -\beta d p_2 & 1 + \beta B \end{bmatrix}, \]  
(15)

where \(A = 1 - d - c - cd + 2dp_2 - 4q_1 + 4d^2q_1\) and \(B = 1 - 4p_2 - 2dq_1 + c\).

Taking \(E_1 = (0, 0)\) into equation (15),

\[ J(E_1) = \begin{bmatrix} 1 + a(1 - d - c - cd) & 0 \\ -\beta(1 + c) & 1 + \beta(1 + c) \end{bmatrix}. \]  
(16)

The two eigenvalues of matrix \(J(E_1)\) are \(\lambda_1 = 1 + a(1 - d - c - cd)\) and \(\lambda_2 = 1 + \beta(1 + c)\). According to equation (16), it is easy to know \(|\lambda_1| > 1\) and \(|\lambda_2| > 1\). Hence, \(E_1\) is a repelling node of dynamic system (12).

At \(E_2\), the Jacobian matrix is given as follows:

\[ J(E_2) = \begin{bmatrix} 1 + a(1 - c) & 0 \\ -\beta d (1 + c) & 1 + \beta (1 + c) \end{bmatrix}. \]  
(17)

Two eigenvalues of matrix \(J(E_2)\) are obtained: \(\lambda_1 = 1 + a(1 - c) > 1\) and \(\lambda_2 = 1 - \beta(1 + c) < 1\). Therefore, \(E_2\) is a saddle point. By the same way, we can prove that \(E_3\) is a saddle point, and \(E_4\) is the only positive Nash equilibrium solution. This completes the proof.

Then, the Jacobian matrix at the positive Nash equilibrium point \(E_4\) is

\[ J(E_4) = \begin{bmatrix} 1 + aA_1 & 2\alpha dq_1^* \\ -\beta dp_2^* & 1 + \beta A_2 \end{bmatrix}, \]  
(18)

where \(A_1 = 1 - d - c - cd + 2dp_2^* - 4q_1^* + 4d^2q_1^*\) and \(A_2 = 1 - 4p_2^* - 2dq_1^* + c\).

Let \(P(\lambda) = \lambda^2 - MA + N\) be the characteristic polynomial of \(J(E_4)\) and \(\Delta = M^2 - 4N\) be its discriminant with \(M = 2 + aA_1 + \beta A_2\) and \(N = (1 + aA_1)(1 + \beta A_2) + 4\alpha\beta d^2 q_1^* p_2^*\).

According to Jury’s conditions, the dynamic system (12) is asymptotically stable if and only if it satisfies the following conditions:

\[
\begin{align*}
1 - N &> 0, \\
P(1) &= 1 - M + N > 0, \\
P(-1) &= 1 + M + N > 0.
\end{align*}
\]  
(19)

More precisely, for inequality (19), when \(|N| > 1\), the Hopf (in the continuous system) or Neimark–Sacker bifurcation occurs; when \(P(1) < 0\), it means one of the real eigenvalues is bigger than \(-1\), and then the tangent, pitch, or transcritical bifurcation will take place; and when \(P(-1) < 0\), it denotes one of the real eigenvalues is smaller than \(-1\), and then the flip (period-doubling) bifurcation will be raised. In addition, if the aforementioned three conditions are not fulfilled at the same time, then the dynamic system (12) will lose its stability and become unstable, so the conditions above defined the surfaces in the parameter space on which kinds of bifurcations will appear.

However, solving the inequality equation (19) is very complicated. Next, we give the stable region of the dynamic system (12) through the parameter basins. According to the current situation, we take the parameter values as follows: \(q_1 = 0.2, p_2 = 0.1, c = 0.1,\) and \(d = 0.2\).

The parameter basin map is also called 2D bifurcation map, which can clearly show the path of the system from the stable state to chaotic state. The parameter basins are drawn in Figure 1 with \(d = 0.2\), in which different colors represent different period cycles, for example, red (stable), blue (2-period), green (3-period), yellow (4-period), orange (5-period), light blue (7-period), brown (8-period), magenta (chaos), and white (divergence). From Figure 1, when the values of the adjustment speed of two firms are in the stable region (red region), the dynamic system (12) finally stabilizes at the Nash equilibrium point after many iterations. When the values of the adjustment speeds of two firms escape from the stable region (red region), the Nash equilibrium point will become unstable, period-doubling cycles or Neimark–Sacker bifurcations occur and even evolve into chaos finally. Furthermore, it is important that the player will drop out the market when the adjustment speed is greater than a certain value (white region).

Figure 2 gives the parameter basins of dynamic system (12) when \(d\) and \(c\) take different values. Comparing the sizes of stability regions (red region) in Figures 1 and 2, we find that higher product differentiation enlarges the stable region of the dynamic system (12), and the higher unit product cost decreases the stable region of price adjustment and increases the stable region of output adjustment. So, two firms should adjust the parameters according to the actual market conditions so that the dynamic system (12) is in a stable state.

4. Numerical Simulations

In this section, the numerical simulation is carried out in depth to analyze the influence of the price speed adjustment and the degree of differentiated products on the stability of the Nash equilibrium point of the dynamic system (12). Moreover, the dynamic complex behaviors of the dynamic system (12) are analyzed by simulations, such as bifurcations, chaos, the largest Lyapunov exponent (LLE), strange attractors, and sensitive dependence on initial conditions.

4.1. The Effect of the Adjustment Speed on the Stability of the Dynamic System. The dynamic characteristics of the system can be well shown by the one-dimensional bifurcation diagram and the LLE with one-parameter changing when keeping other parameters fixed. Figure 3
shows the bifurcation diagram and the corresponding LLE with \( \alpha \) varying from 0 to 3.2 when \( \beta = 1 \). Along with the increase of \( \alpha \) from 0 to 2.355, the dynamic system (12) is in the stable state; when \( \alpha > 2.355 \), the dynamic system (12) loses its stability and periodic-doubling occurs, including period-2 and period-4. Finally, system (5) falls into chaos through flip bifurcations. The LLE can verify the appearing of chaos when most of the LLEs exceed zero.

Similarly, the bifurcation diagram and the corresponding LLE with respect to \( \beta \) are displayed in Figure 4. The first bifurcation occurs at \( \beta = 2.243 \), and then dynamic
Figure 3: (a) Bifurcation diagram and (b) LLE with respect to $\alpha$ when $\beta = 1$.

Figure 4: (a) Bifurcation diagram and (b) LLE with respect to $\beta$ when $\alpha = 1$.

Figure 5: Bifurcation diagram of the dynamic system (12) with respect to $\alpha$: (a) $\beta = 2$ and (b) $\beta = 2.3$. 

**Complexity**
system (12) becomes chaotic finally through period-doubling bifurcations, and the positive LLEs indicate the appearing of chaos.

As can be seen from Figure 5(a), the bifurcation diagram with respect to $\alpha$ is quite different from Figure 3 when $\beta = 2$. The dynamic system (12) remains in the stable state when $\alpha < 2.290$, with $\alpha$ increasing, and when exceeds 2.290, the dynamic system (12) becomes unstable, Neimark–Sacker bifurcation appears and then enters into the chaotic state at last. As shown in Figure 5(b), the dynamic behaviors of dynamic system (12) become more complicated when $\beta$ is fixed at 2.3, and the Nash equilibrium point is in a 2-period state as $\alpha$ varies from 0 to 1.342; when $1.342 < \alpha < 1.981$, the dynamic system (12) returns to the stable state from the period-2 state, and then the system undergoes Neimark–Sacker bifurcation and enters into chaos when $\alpha > 1.981$. In addition, as one can find that there is an intermittent odd cycle (period-7) in the chaos in Figures 5(a) and 5(b), it implies that the so-called topological chaos is created.

From the perspective of economics, when the adjustment speed of firms exceeds a certain value, the Nash equilibrium of the system will lose its stability, and the system becomes unpredictable and disorder; it is hard for firms to deal with
the complex scenario like that. Hence, firms should keep the adjustment speed in a proper range in order to avoid reducing efficiency of the market and profits of themselves.

Some strange attractors of the system are plotted in Figure 6 with respect to \((q_1, p_2)\) in the two-dimension map under different parameter combination values of the adjustment speed of firms. The limit cycle and coexistence of several attractors are displayed in Figures 6(a)–6(c) when \(\alpha \approx 2.5\) and \(\beta\) has different values. The strange attractors with the fractal structure are exhibited in Figure 6(d) which implies existence of chaotic behaviors.

An important feature of the chaotic system is the sensitivity to initial values. Here, keeping \(p_2\) unchanged and \(\alpha = 2.5, \beta = 2.615\), Figure 7 shows the differences of the evolution process of the dynamic system (12) when the initial value of \(q_1\) only changes, 0.001. We can see that the values of \(q_1\) and \(p_2\) have no difference in the first 71 time iterations, but after that, the values of \(q_1\) and \(p_2\) show great difference. That is to say, the small difference of the initial value will cause great deviation after much iteration, which provides us with the enlightenment that decision-makers should be more cautious in choosing the initial values of decision variables.

4.2. Effect of Degree of Product Differentiation and Unit Product Cost on System Stability. The bifurcation diagram and the corresponding LLE with respect to \(d\) are drawn in Figure 8 when \(\alpha\) and \(\beta\) are fixed at 1.5; the dynamic system (12) is in the stable state at first, and as \(d\) increases, we can see that the product quantity of firm 1 is unchanged until \(d = 0.935\), while the price of firm 2 is sharply decreased at the same time; then, a Neimark–Sacker bifurcation occurs when \(d > 0.935\), and the dynamic behaviors of players fall into the
quasi-periodic state; it is hard for the bounded rational players to predict the trajectory of its quantity or price and make a long competition strategy. In addition, the bifurcation diagram with respect to $d$ is shown in Figure 9 when $\alpha = 1.5$, $\beta = 2.4$. The dynamic system (12) enters the stable state from the chaotic state, and then enters the chaotic state when $d \in (0.25, 0.85)$, so it is an efficient way to remove chaos by controlling $d$ in a proper range of value. As $d > 0.877$, the dynamic system (12) exhibits Neimark–Sacker bifurcation and stays in the quasi-period state. Compared with Figure 8, the dynamic behaviors of the market become much more complicated in the chaotic state. The LLE plots in Figures 8 and 9 are used to verify existence of stable cycles, periodic cycles, and chaotic behaviors. One can obtain a better understanding of the effects of degree of product differentiation on stability of the Nash equilibrium point and complex characteristics of the dynamic system.

The bifurcation diagrams with respect to $c$ are drawn in Figure 10 when the dynamic system (12) is in the stable state and unstable state. From Figure 10(a), when $0 < c \leq 0.88$, the output of firm 1 decreases first and then goes into the unstable state; when $c > 0.88$, the output of firm 1 equals to zero; when $0 < c \leq 1$, the price of firm 2 increases first and then goes into the chaotic state through flip bifurcation. From Figure 10(b), we can find that when the dynamic system (12) is in the unstable state, the dynamic system (12) will go into chaos early with $c$ increasing.

From the above analysis, we can get the following conclusions: (1) when the dynamic system (12) is in the stable state, a higher substitution between products may destabilize the stability of the Nash equilibrium point and lead to appearance of aperiodic cycles. In addition, the product quantity of firm 1 remains in a certain value and unchanged, while the price of firm 2 is decreased with the increase in degree of product differentiation. It denotes that the player who takes price as the strategy variable is more sensitive to changes of degree of product differentiation, which is different from the classic Cournot–Bertrand game discussed in previous articles. When the dynamic system (12) is in the chaotic state and the degree of product differentiation tending to 0 or 1, periodic cycle and chaos will appear in system (12). (2) When the dynamic system (12) is in the stable state, as the unit product cost exceeds a certain value, firm 1 will withdraw from the market; when the dynamic system (12) is in the chaotic state, the increase of unit product cost of two firms will make the dynamic system (12) go into chaos early. So, two firms should control the degree of product differentiation and unit product cost so as to avoiding the market to go into a chaotic state.

![Figure 10: Bifurcation diagrams with respect to $c$ when $\alpha = 1.5$ and (a) $\beta = 1.5$ and (b) $\beta = 2.6$.](image10)

![Figure 11: Basins of attraction of the dynamic system (12). (a) $\alpha = \beta = 0.5$. (b) $\alpha = \beta = 1.5$.](image11)
5. Global Stability of the Dynamic System (12)

An effective method for analyzing the influences of parameter changing on the global stability is the basins of attraction in which there include attraction domain and escaping area. It is the set of initial conditions; if the initial variable is taken from the attraction domain, the system will emerge the same attractor after a series of iterations. If the initial variable is outside the basins of attraction, the system will fall into divergence at last.

By fixing the parameter values of the dynamic system (12) as mentioned above, the basins of attraction about \( q_1 \) and \( p_2 \) of the dynamic system (12) are shown in Figure 11 under different values of adjustment speed, in which the red region denotes the stable attraction domain and the white region denotes the escape area. We found that the attraction domain shrank with the increase of adjustment speed values.

Parameter’s value has a great influence on the global stability of the dynamic system. Figure 12 shows the basins of attraction of \( q_1 \) and \( p_2 \) with \( c \) and \( d \) having different values when \( \alpha = \beta = 1.5 \). Comparing Figures 11 and 12, we find that the basins of attraction decrease with the increase of unit product cost and degree of product differentiation.

From an economic point, the initial values of variables of two firms should be in basins of attraction in order to maintain market stability.

6. Chaos Control

Generally, the chaos of the economic system is regarded as an obstacle for players to make a long business strategy and achieve their own business objective. In recent years, the methods in controlling bifurcation and chaos are proposed by many scholars such as the OGY method, the variable feedback control method, and the delay feedback control (DFC) method. Consequently, this paper uses the variable feedback control, which is widely applied in supply chain management and insurance market to delay and eliminate the chaos of dynamic system (12).

We assume that the discrete dynamic model of system (6) takes the following form:

\[
\begin{align*}
q_1(t+1) &= f_1[q_1(t), p_2(t)], \\
p_2(t+1) &= f_2[q_1(t), p_2(t)].
\end{align*}
\]  

Then, the controlled system can be described as

\[
\begin{align*}
q_1(t+1) &= f_1[q_1(t), p_2(t)] - kq_1(t), \\
p_2(t+1) &= f_2[q_1(t), p_2(t)] - kp_2(t),
\end{align*}
\]  

where \( k \) is the control parameter, which can be regarded as government regulation or the learning ability of the players.

Figure 13 shows the bifurcation diagram and the corresponding LLE in terms of \( k \).
period cycle occur twice; when $0 < k < 0.129$, then controlled system (21) undergoes chaos range, quasi-period range, and returns to a stable state with $k$ increasing. The corresponding LLE becomes negative when $k > 0.378$, which verifies that the controlled system (21) returns to the stable state. In Figure 14, the trajectory of $q_1$ and $p_2$ is plotted when $\alpha = 2.5$, $\beta = 2.615$ and $k = 0.38$; after many iterations, the control system (21) gradually returns to the stable state from the chaotic state.

7. Conclusion

This paper devotes to establish a dynamic Cournot–Bertrand game model based on relative profit maximization. The stability of the dynamic system is analyzed, and the influences of variable adjustment speeds, unit product cost, and product differentiated degree on complex characteristics of the dynamic system are investigated. The results showed that the higher degree of product differentiation enlarges the stable range of the dynamic system, while higher unit product cost decreases the stable range of price adjustment and increases the one of output adjustment. Some complex properties and global stability of the dynamic system are explored, and it is found that period cycles and aperiodic oscillation (quasi-period and chaos) occur via period-doubling or Neimark–Sacker bifurcation. In addition, the limit cycle and coexistence of several attractors will appear in some cases, and the attraction domain shrank with the increase of adjustment speed values.

The study of complexity of the Cournot–Bertrand game model based on relative profit objectives has theoretical and practical significance. Nonetheless, there are some factors that are not taken into account, for example, if the risk attitude of decision-makers is taken into account, the Cournot–Bertrand game model will be more realistic. Firms can consider dual-channel to expand market share. We believe that the viewpoint of this paper will lay a dynamic foundation for future research in these directions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

Huang Yi-min and Guo Yan-yan revised the paper, Li Qiu-xiang provided research methods, and Zhang Yu-hao wrote the original draft.

Acknowledgments

This research was supported by the National Social Science Foundation of China (no. 19FGLB067) and Henan Province Soft Science Research Plan Project (no. 192400410088).

References

[1] M. Kopel, “Simple and complex adjustment dynamics in Cournot duopoly models,” Chaos, Solitons & Fractals, vol. 7, no. 12, pp. 2031–2048, 1996.
[2] H. N. Agiza, “On the analysis of stability, bifurcation, chaos and chaos control of Kopel map,” Chaos, Solitons & Fractals, vol. 10, no. 11, pp. 1909–1916, 1999.
[3] G. I. Bisci, C. Mammana, and L. Gardini, “Multistability and cyclic attractors in duopoly games,” Chaos Solitons & Fractals, vol. 11, no. 4, pp. 543–564, 2000.
[4] G. I. Bisci and A. Naimzada, “Global analysis of a dynamic duopoly game with bounded rationality,” Advances in Dynamic Games and Applications, pp. 291–298, Birkhäuser, Boston, MA, USA, 2000.
[5] H. N. Agiza and A. A. Elsadany, “Nonlinear dynamics in the Cournot duopoly game with heterogeneous players,” Physica A Statistical Mechanics & Its Applications, vol. 320, no. 1–4, pp. 512–524, 2002.
[6] L. Fanti and L. Gori, “The dynamics of a differentiated duopoly with quantity competition,” Economic Modelling, vol. 29, no. 2, pp. 421–427, 2012.
[7] L. Fanti, L. Gori, and M. Sodini, “Nonlinear dynamics in a Cournot duopoly with relative profit delegation,” Chaos, Solitons & Fractals, vol. 45, no. 12, pp. 1469–1478, 2012.
[8] X. Zhu, W. Zhu, and L. Yu, “Analysis of a nonlinear mixed Cournot game with boundedly rational players,” Chaos, Solitons & Fractals, vol. 59, pp. 82–88, 2014.
[9] L. Goril, L. Guerrini, and M. A. Sodini, “Continuous time cournot duopoly with delays, Chaos, Solitons and Fractals: Applications in Science and Engineering,” An Interdisciplinary Journal of Nonlinear Science, vol. 34, 2015.
[10] A. A. Elsadany, “Dynamics of a Cournot duopoly game with bounded rationality based on relative profit maximization,” Applied Mathematics and Computation, vol. 294, pp. 253–263, 2017.
[11] J. Ma and L. Xie, “The comparison and complex analysis on dual-channel supply chain under different channel power structures and uncertain demand,” Nonlinear Dynamics, vol. 83, no. 3, pp. 1379–1393, 2016.
Q. G. Yi and X. J. Zeng, “Complex dynamics and chaos control of duopoly Bertrand model in Chinese air-conditioning market,” *Chaos, Solitons & Fractals*, vol. 76, pp. 231–237, 2015.

A. A. Elsadany and A. M. Awad, “Dynamics and chaos control of a duopolistic Bertrand competitions under environmental taxes,” *Annals of Operations Research*, vol. 274, pp. 211–240, 2018.

L. Shi, Y. Le, and Z. Sheng, "Analysis of price Stackelberg duopoly game with bounded rationality," *Discrete Dynamics in Nature and Society*, vol. 2014, Article ID 428568, 8 pages, 2014.

Y. Peng and Q. Lu, “Complex dynamics analysis for a duopoly Stackelberg game model with bounded rationality,” *Applied Mathematics and Computation*, vol. 271, pp. 259–268, 2015.

Y. Peng, Q. Lu, and Y. Xiao, “A dynamic Stackelberg duopoly model with different strategies,” *Chaos, Solitons & Fractals*, vol. 85, pp. 128–134, 2016.

S. S. Askar, “Duopolistic Stackelberg game: investigation of complex dynamics and chaos control,” *Operational Research*, pp. 1–15, 2018.

C. H. Tremblay and V. J. Tremblay, “The Cournot-Bertrand model and the degree of product differentiation,” *Economics Letters*, vol. 111, no. 3, pp. 233–235, 2011.

S. S. Askar, “On Cournot-Bertrand competition with differentiated products,” *Annals of Operations Research*, vol. 223, no. 1, pp. 81–93, 2014.

A. K. Naimzada and F. Tramontana, “Dynamic properties of a Cournot-Bertrand duopoly game with differentiated products,” *Economic Modelling*, vol. 29, no. 4, pp. 1436–1439, 2012.

H. Wang and J. Ma, “Complexity analysis of a cournot-bertrand duopoly game model with limited information,” *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 287371, 6 pages, 2013.

J. Ma and X. Pu, “The research on Cournot-Bertrand duopoly model with heterogeneous goods and its complex characteristics,” *Nonlinear Dynamics*, vol. 72, no. 4, pp. 895–903, 2013.

A. Satoh and Y. Tanaka, "Irrelevance of conjectural variation in duopoly under relative profit maximization and consistent conjectures," *Mpra Paper*, vol. 10, pp. 101–115, 2014.

N. Miller and A. Pazgal, "Relative performance as a strategic commitment mechanism," *Managerial & Decision Economics*, vol. 23, no. 2, pp. 51–68, 2002.

M. Hattori and Y. Tanaka, "Incentive for adoption of new technology in duopoly under absolute and relative profit maximization," *Economics Bulletin*, vol. 34, no. 3, pp. 2051–2059, 2014.

Y. Tanaka, "Irrelevance of the choice of strategic variables in duopoly under relative profit maximization," *Economics and Business Letters*, vol. 2, no. 2, p. 75, 2013.

Y. Tanaka, "Equivalence of Cournot and Bertrand equilibria in differentiated duopoly under relative profit maximization with linear demand," *Economics Bulletin*, vol. 33, no. 2, pp. 1479–1486, 2014.

Y. Lu, "The relative-profit-maximization objective of private firms and endogenous timing in a mixed oligopoly," *The Singapore Economic Review*, vol. 56, no. 2, pp. 203–213, 2011.

N. Singh and X. Vives, "Price and quantity competition in a differentiated duopoly," *The RAND Journal of Economics*, vol. 15, no. 4, pp. 546–554, 1984.

A. Dixit, "Comparative statics for oligopoly," *International Economic Review*, vol. 27, no. 1, pp. 107–122, 1986.