ON GIBBS STATES OF MECHANICAL SYSTEMS WITH SYMMETRIES

by

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Abstract. — Gibbs states for the Hamiltonian action of a Lie group on a symplectic manifold were studied, and their possible applications in Physics and Cosmology were considered, by the French mathematician and physicist Jean-Marie Souriau. They are presented here with detailed proofs of all the stated results. Using an adaptation of the cross product for pseudo-Euclidean three-dimensional vector spaces, we present several examples of such Gibbs states, together with the associated thermodynamic functions, for various two-dimensional symplectic manifolds, including the pseudo-spheres, the Poincaré disk and the Poincaré half-plane.

Résumé. — Les états de Gibbs sur une variété symplectique associés à l'action hamiltonienne d'un groupe de Lie sur cette variété ont été étudiés par le mathématicien et physicien Jean-Marie Souriau, qui en a aussi considéré des applications en Physique et en Cosmologie. Ils sont décrits ici avec la preuve détaillée de tous les résultats présentés. Grâce à une adaptation du produit vectoriel aux espaces vectoriels pseudo-euclidiens de dimension 3, plusieurs exemples de tels états de Gibbs sont déterminés, ainsi que les fonctions thermodynamiques qui leur sont associées, pour diverses variétés symplectiques de dimension 2, notamment les pseudo-sphères, le disque de Poincaré et le demi-plan de Poincaré.

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1. Introduction

The French mathematician and physicist Jean-Marie Souriau (1922–2012) considered, first in [32], then in his book [33], Gibbs states on a symplectic manifold built with the moment map of the Hamiltonian action of a Lie group, and the associated thermodynamic functions. In several later papers [34, 35, 36], he developed these concepts and considered their possible applications in Physics and in Cosmology. A partial translation in English of these three papers, made by Frédéric Barbaresco, is available at [7].

Recently, under the name Souriau’s Lie groups thermodynamics, these Gibbs states and the associated thermodynamic functions were considered by several scientists, notably by Frédéric Barbaresco, for their possible applications in today very fashionable scientific topics, such as geometric information theory, deep learning and machine learning [3, 4, 5, 6, 8, 26, 27]. Although including these topics in a reasearch program seems to be, nowadays, a good way to obtain a public funding, I am not going to speak about them, since they are far from my field of knowledge. I will rather stay on Gibbs states and their possible applications in classical and relativistic Mechanics.

Long before the works of Souriau, Gibbs states associated to a Hamiltonian Lie group action were considered by the American scientist Josiah Willard Gibbs (1839–1903). In his book [12] published in 1902, he clearly described Gibbs states in which the components of the total angular momentum (which
are the components of the moment map of the action of the group of rotations
on the phase space of the considered system) appear, on the same footing as
the Hamiltonian. He even considered Gibbs states involving conserved quan-
tities more general than those associated with the Hamiltonian action of a Lie
group. In this domain, Souriau’s main merits do not lie, in my opinion, in the
consideration of Gibbs states for the Hamiltonian action of a Lie group, a not
so new idea, but rather in the use of the manifold of motions of a Hamiltonian
system instead of the use of its phase space, and his introduction, under the
name of Maxwell’s principle, of the idea that a symplectic structure should
exist on the manifold of motions of systems encountered as well in classical
Mechanics as in relativistic Physics. He therefore considered Gibbs states for
Hamiltonian actions, on a symplectic manifold, of various Lie groups, including
the Poincaré group, often considered in Physics as a group of symmetries for
isolated relativistic systems. He was well aware of the fact that Gibbs states
for the Hamiltonian action of the full considered groups may not exist, which
led him to carefully discuss the physical meaning and the possible applications
of Gibbs states associated to the action of some of their subgroups.

Section 2 begins with a reminder about some concepts used in statistical
mechanics, notably the concepts of statistical states and of entropy, and about
the use of Hamiltonian vector fields in Mechanics. Gibbs states in the special
case in which the only conserved quantity considered is the Hamiltonian, and
the associated thermodynamic functions, are then briefly discussed. Their
physical interpretation as states of thermodynamic equilibrium is discussed.
The relation of the real parameter $\beta$ used to index statistical states with the
temperature is explained.

The notion of manifold of motions of a Hamiltonian dynamical system is
presented in Section 3. Then Gibbs states for the Hamiltonian action of a Lie
group on a symplectic manifold are discussed, with full proofs of all the stated
results. Most of these proofs can be found in Souriau’s book [33], which some
readers may find difficult to access. A good English translation of this book
is available, which faithfully preserves the language and the notations of the
author.

In section 4, some examples of Gibbs states are presented, together with
the associated thermodynamic functions. The main tool used in this section
is an adaptation of the well known cross product for three-dimensional ori-
ented, pseudo-Euclidean vector spaces. Remarkably, according to [38], the
cross product of two elements of a three-dimensional, oriented, Euclidean vector space appeared for the first time in the lecture notes *Elements of Vector Analysis* [12], privately written in 1881 for students in physics by Gibbs, one of the most important founders of statistical mechanics.

The readers will find at the beginning of sections 3 and 4 a more detailed presentation of the contents of these sections.

2. Some concepts used in statistical mechanics

2.1. The birth of statistical mechanics. — In his book *Hydrodynamica* published in 1738, Daniel Bernoulli (1700–1782) considered fluids (gases as well as liquids) as made of a very large number of moving particles. He explained that the pressure in the fluid is the result of collisions of the moving particles against the walls of the vessel in which it is contained, or against the probe which measures the pressure.

Daniel Bernoulli’s idea remained ignored by most scientists for more than one hundred years. It is only in the second half of the XIX-th century that some scientists, notably Rudolf Clausius (1822–1888), James Clerk Maxwell (1831–1879) and Ludwig Eduardo Boltzmann (1844–1906), considered Bernoulli’s idea as reasonable. As soon as 1857, Clausius began the elaboration of a *kinetic theory of gases* aiming at the explanation of macroscopic properties of gases (such as temperature, pressure and other thermodynamic properties), starting from the equations which govern the motions of the moving particles. Around 1860, Maxwell determined the probability distribution of the moving particles velocities in a gas in thermodynamic equilibrium. For a gas not in thermodynamic equilibrium, an evolution equation for this probability distribution was obtained by Boltzmann in 1872. Using probabilistic arguments about the way in which collisions of particles can occur, Boltzmann introduced a quantity, denoted by $H^{(1)}$ which, as a function of time, always monotonically decreases. Boltzmann’s $H$ function is now identified with the opposite of the *entropy* of the gas. On this basis, Josiah Willard Gibbs (1839–1903) laid the foundations of a new branch of theoretical physics, which he called *statistical mechanics* [13].

In the first half of the XX-th century, scientists understood that the motions of molecules in a material body do not perfectly obey Newton’s laws of classical

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1. In Boltzmann’s mind, this letter was probably the Greek boldface letter Êta rather than the Latin letter H.
mechanics, and that the laws of quantum mechanics should be used instead. The basic concepts of statistical mechanics established by Gibbs were general enough to remain valid in this new framework, and to be used for liquids or solids as well as for gases.

2.2. Statistical states and entropy. — In this subsection, after a reminder of some well known facts about the use of Hamiltonian vector fields in classical mechanics and about symplectic manifolds, the important concept of a statistical state is presented and the definition of its entropy is given.

2.2.1. The use of Hamiltonian vector fields in classical mechanics

Let us recall how the evolution with time of the state of a material body is mathematically described by a dynamical system, in the framework of classical mechanics. The physical time $T$ is a one-dimensional real, oriented affine space, identified with $\mathbb{R}$ once a unit and an origin of time are chosen. The set of all possible kinematic states of the body is a symplectic manifold $(M, \omega)$, very often a cotangent bundle, traditionally called the phase space of the system.

For an isolated system, a smooth real-valued function $H$, defined on $M$, called a Hamiltonian for the system, determines all its possible evolutions with time. Let indeed $X_H$ be the unique smooth vector field, defined on $M$, which satisfies the equality

$$i(X_H)\omega = -dH.$$  

It is called the Hamiltonian vector field admitting the function $H$ as a Hamiltonian. Any possible evolution with time of the system is described by a smooth curve $t \mapsto \varphi(t)$, defined on an open interval in $\mathbb{R}$, with values in $M$, which is a maximal integral curve of the differential equation, called Hamilton's equation, in honour of the Irish mathematician William Rowan Hamilton (1805–1865),

$$\frac{d\varphi(t)}{dt} = X_H(\varphi(t)).$$  

The Hamiltonian $H$ is a first integral of this differential equation: it means that for each smooth curve $t \mapsto \varphi(t)$, solution of this differential equation, $H(\varphi(t))$ is a constant.

More generally, when the system is not isolated, its Hamiltonian $H$ is a smooth function defined on $\mathbb{R} \times M$ (or on an open subset of $\mathbb{R} \times M$) since it may depend on time. The Hamiltonian vector field $X_H$ which admits such a function as Hamiltonian is still, for each time $t \in \mathbb{R}$, determined by equation (*) above, in the righthand side of which the differential $dH$ must be calculated, for each $t \in \mathbb{R}$, as its partial differential with respect to the variable $x \in M$,
the time \( t \in \mathbb{R} \) being considered as fixed. Therefore \( X_H \) is a \textit{time-dependent vector field} on \( M \), i.e., a smooth map, defined on some open subset of \( \mathbb{R} \times M \), with values in the tangent bundle \( TM \), such that for each fixed \( t \in \mathbb{R} \), the map \( x \mapsto X_H(t, x) \) is an usual smooth vector field defined on some open subset of \( M \). Any possible evolution with time of the system is still described by a smooth curve \( t \mapsto \varphi(t) \), which is a maximal integral curve of the differential equation (**) above, which now must be written as

\[
\frac{d\varphi(t)}{dt} = X_H(t, \varphi(t)),
\]

in order to indicate that \( X_H \) may depend on \( t \in \mathbb{R} \) as well as on \( \varphi(t) \in M \). In this case the Hamiltonian \( H \) is no more a first integral of this differential equation.

\textbf{2.2.2. The Liouville measure on a symplectic manifold}

Let \( (M, \omega) \) be a \( 2n \)-dimensional symplectic manifold. Let \( (U, \varphi) \) be an admissible chart of \( M \). For each \( x \in M \), we set

\[
\varphi(x) = (q^1, \ldots, q^n, p_1, \ldots, p_n) \in \varphi(U) \subset \mathbb{R}^{2n}.
\]

The chart \( (U, \varphi) \) is said to be \textit{canonical}, or to be a \textit{Darboux chart}, if the local expression of \( \omega \) in \( U \) is

\[
\omega = \sum_{i=1}^n dp_i \wedge dq^i.
\]

The local coordinates \( q^1, \ldots, q^n, p_1, \ldots, p_n \) in this chart are called \textit{canonical coordinates} or \textit{Darboux coordinates}. The famous \textit{Darboux theorem}, so named in honour of the French mathematician Gaston Darboux (1842–1917), asserts that any point in \( M \) is an element of the domain of a canonical chart. By using this theorem, one can prove the existence of a unique positive measure on the Borel \( \sigma \)-algebra \(^2\) of \( M \), called the \textit{Liouville measure}, in honour of the French mathematician Joseph Liouville (1809–1882) and denoted by \( \lambda_\omega \), such that for any measurable subset \( A \) of \( M \) contained in the domain \( U \) of a canonical chart \( (U, \varphi) \) of \( M \), such that \( \varphi(A) \) is a bounded subset of \( \mathbb{R}^{2n} \),

\[
\lambda_\omega(A) = \int_{\varphi(A)} dq^1 \ldots dq^n dp_1 \ldots dp_n.
\]

\(^2\) The \( \sigma \)-algebra of a topological space \( M \) is the smallest family of subsets of \( M \) which contains all open subsets and is stable by complementation and by intersections of countable subfamilies. It is so named in honour of the French mathematician Émile Borel (1871–1956).
The Liouville measure is invariant by symplectomorphisms, which means that its direct image $\Phi_* \lambda_\omega$ by any symplectomorphism $\Phi : M \to M$ is equal to $\lambda_\omega$.

2.2.3. Definitions. — Let $(M, \omega)$ be a symplectic manifold and $\lambda_\omega$ its Liouville measure.

1. A statistical state on $M$ is a probability measure $\mu$ on the Borel $\sigma$-algebra of $M$. The statistical state $\mu$ is said to be continuous (respectively, smooth) when it can be written as $\mu = \rho \lambda_\omega$, where $\rho$ is a continuous function (respectively, a smooth function) defined on $M$. The function $\rho$ is then said to be the probability density (or simply the density) of the statistical state $\mu$ with respect to $\lambda_\omega$.

2. Let $\mu$ be a statistical state on $M$ and $f$ be a function, defined on $M$, which takes its values in $\mathbb{R}$ or in a finite-dimensional vector space. When $f$ is integrable on $M$ with respect to the measure $\mu$, its integral is called the mean value of $f$ in the statistical state $\mu$, and denoted by $E_\mu(f)$. When the statistical state $\mu$ is continuous, with the continuous function $\rho$ as probability density with respect to $\lambda_\omega$, the mean value of $f$ in the statistical state $\mu$ is, by a slight abuse of notations, denoted by $E_\rho(f)$. Its expression is

$$E_\rho(f) = \int_M f(x) \rho(x) \lambda_\omega(dx).$$

2.2.4. Comments about the use of statistical states

When the considered dynamical system, determined by the Hamiltonian vector field $X_H$, is made of a large number $N$ of moving particles, the dimension of the symplectic manifold $(M, \omega)$ which represents the set of all its possible kinematic states is very large: at least $6N$, and even more when the particles are not treated as material points. A perfect knowledge of each element of $M$ is not possible, which explains the use of statistical states in classical Mechanics. In this framework, when the state of the considered system at a given time $t_0$ is mathematically described by a statistical state $\mu$, it means that instead of looking at the evolution in time of a unique system whose kinematical state at time $t_0$ is a given element $x_0 \in M$, one is going to look at the evolution in time of a whole family of similar systems. The evolution with time of each of these systems is described by the differential equation determined by $X_H$, and its kinematical states at time $t_0$ can be any point in the support of $\mu$.

3. The support of a measure $\mu$ defined on the Borel $\sigma$-algebra of a topological state $M$ is the closed subset of $M$, complementary to the open subset made by points contained in an open subset $U$ of $M$ such that $\mu(U) = 0$. 

When, instead of classical mechanics, quantum mechanics is used for the mathematical description of the evolution with time of the state of a physical system, the use of statistical states is not due to an imperfect knowledge of the initial state of the system: it is mandatory. Informations about the evolution with time of the state of a system given by quantum mechanics are indeed always probabilistic. By nature, quantum mechanics is always statistical.

2.2.5. Examples
1. Let $x_1, x_2, \ldots, x_N$ be $N$ pairwise distinct points in $M$, and $k_1, k_2, \ldots, k_N$ be real numbers satisfying $k_i > 0$ for all $i \in \{1, \ldots, N\}$ and $\sum_{i=1}^{N} k_i = 1$. For each $i \in \{1, \ldots, N\}$, let $\delta_{x_i}$ be the Dirac measure at $x_i$, whose value $\delta_{x_i}(A)$ for a measurable subset $A$ of $M$ is 0 when $x_i \notin A$ and 1 when $x_i \in A$. The measure $\mu = \sum_{i=1}^{N} k_i \delta_{x_i}$ is a statistical state, which is neither continuous, nor smooth. The mean value of a function $f$ in the statistical state $\mu$ is $\sum_{i=1}^{N} k_i f(x_i)$.

For each $i \in \{1, \ldots, N\}$, the measure $\delta_{x_i}$ is a statistical state in which the kinematical state of the system is the point $x_i$, with a probability 1. One can say that $\delta_{x_i}$ is a state in the usual sense. In the statistical state $\mu$, the kinematical state of the system is a random variable which can take each value $x_i$ with the probability $k_i$.

2. Still under the same assumptions, for each $i \in \{1, \ldots, N\}$, let $U_i$ be a neighbourhood of $x_i$ and $\varphi_i$ be a positive valued, smooth function, with compact support contained in $U_i$, satisfying the equality $\int_M f(x) \lambda_\omega(dx) = 1$. The measure $\nu$ whose probability density with respect to the Liouville measure $\lambda_\omega$ is $\rho_\nu = \sum_{i=1}^{N} k_i \varphi_i$ is a smooth statistical state, which can be considered as a smooth approximation of the discrete statistical state $\mu$ considered above. Such smooth approximations of non-smooth statistical states were extensively used by the founder of geostatistics, the French mathematician and geologist Georges Matheron (1930–2000) [24].

2.2.6. Remark. — Let $\mu$ be a continuous statistical state on the symplectic manifold $(M, \omega)$ and $\rho$ its probability density with respect to the Liouville measure $\lambda_\omega$. For each measurable subset $A$ of $M$, we have

$$\mu(A) = \int_A \rho(x) \lambda_\omega(dx), \text{ so for } A = M, \mu(M) = \int_M \rho(x) \lambda_\omega(dx) = 1.$$ 

The function $\rho$ therefore takes its values in $\mathbb{R}^+$ and is integrable on $M$ with respect to the Liouville measure.
2.2.7. Evolution with time of a statistical state. — Let \((M, \omega)\) be a symplectic manifold, \(H \in C^\infty(M, \mathbb{R})\) be a smooth Hamiltonian on \(M\) which does not depend on time and \(X_H\) be the associated Hamiltonian vector field on \(M\). We denote by \(\Phi^{X_H}t\) the reduced flow\(^4\) of \(X_H\). If, at a time \(t_0\), the state of the dynamical system described by \(X_H\) is a perfectly defined point \(x_0 \in M\), the state of the system, at any other time \(t_1\) at which it exists, is the point \(x_1 = \Phi^{X_H}_{t_1-t_0}(x_0)\).

Let us assume that \(\mu(t_0)\) is the statistical state of such a system at a given time \(t_0\). We assume, for simplicity, that \(\mu(t_0)\) is smooth and we denote by \(\rho(t_0)\) its probability density with respect to the Liouville measure \(\lambda_\omega\). Let \(t_1\) be another time at which the considered system still exists. The reduced flow \(\Phi^{X_H}t\) of the Hamiltonian vector field \(X_H\) is such that \(\Phi^{X_H}_{t_1-t_0}\) is a symplectic diffeomorphism of an open subset of \(M\) onto another open subset of this manifold, whose inverse is \(\Phi^{X_H}_{t_0-t_1}\). The statistical state of the system at time \(t_1\) is therefore smooth, with a probability density \(\rho(t_1)\) with respect to \(\lambda_\omega\), related to \(\rho(t_0)\) by the equation

\[
\rho(t_1) = \rho(t_0) \circ \Phi^{X_H}_{t_1-t_0}.
\]

In other words, for any \(x \in M\),

\[
\rho(t_1, x) = \rho(t_0, \Phi^{X_H}_{t_0-t_1}(x)).
\]

2.2.8. Definition. — Let \(\rho\) be the probability density, with respect to the Liouville measure \(\lambda_\omega\), of a continuous statistical state on the symplectic manifold \((M, \omega)\). The entropy of this statistical state, denoted by \(s(\rho)\), is defined as follows. With the convention that when \(x \in M\) is such that \(\rho(x) = 0\), we set \(\log \left( \frac{1}{\rho(x)} \right) \rho(x) = 0\), we can consider \(x \mapsto \log \left( \frac{1}{\rho(x)} \right) \rho(x)\) as a continuous function well defined on \(M\), taking its values in \(\mathbb{R}\). When this function is integrable on \(M\) with respect to the Liouville measure \(\lambda_\omega\), we set

\[
s(\rho) = \int_M \log \left( \frac{1}{\rho(x)} \right) \rho(x) \lambda_\omega(dx) = -\int_M \log(\rho(x)) \rho(x) \lambda_\omega(dx).
\]

\(^4\) The full flow, or in short the flow, of a smooth vector field \(X\), which may depend on time, defined on \(\mathbb{R} \times M\) (or on a subset of \(\mathbb{R} \times M\)) is the map \(\Psi^X\), defined on an open subset of \(\mathbb{R} \times \mathbb{R} \times M\), taking its values in \(M\), such that for each \(t_0 \in \mathbb{R}\) and each \(x_0 \in M\), the maximal solution \(\varphi\) of the differential equation determined by \(X\) which satisfies \(\varphi(t_0) = x_0\) is the map \(t \mapsto \Psi^X(t, t_0, x_0)\). When \(X\) does not depend on time, \(\Psi^X(t, t_0, x_0)\) only depends on \(t-t_0\) and \(x_0\). So instead of the full flow \(\Psi^X\), one can use the reduced flow \(\Phi^X\), defined on an open subset of \(\mathbb{R} \times M\) by the equality \(\Phi^X(t, x_0) = \Psi^X(t, 0, x_0)\). One often write \(\Phi^X_t(x_0)\) to emphasize the fact that \(\Phi^X\) is a diffeomorphism between two open subsets of \(M\).
Otherwise, we set

\[ s(\rho) = -\infty. \]

The map \( \rho \mapsto s(\rho) \) so defined on the set of all continuous probability densities on \( M \) is called the entropy functional.

### 2.2.9. Comments about entropy

1. The concept of entropy is due to Rudolf Clausius, who used it to formulate precisely the second principle of thermodynamics.

2. The entropy of a real system in Physics is always positive. The third law of thermodynamics states that the entropy of a system in thermodynamic equilibrium, when its state of minimal energy is unique, decreases towards 0 when its absolute temperature decreases towards 0 degree Kelvin. Physicists therefore consider as an unacceptable anomaly the fact that the entropy functional can take negative values, and are scandalized at the sight of \(-\infty\) as a possible value of entropy. Indeed, such a value is in clear conflict with Heisenberg’s principle of uncertainty. The von Neumann entropy \(^{(5)}\), used in quantum statistical mechanics, is always positive, and the entropy defined in 2.2.8 is only its imperfect classical approximation.

3. In his famous paper \([31]\), written during the second world war and published in 1948, the American mathematician, electrical engineer and cryptographer Claude Elwood Shannon (1916–2001) laid the foundations of information theory. He defined in this paper a concept of entropy whose opposite can be used as measurement of the information contained in a message, and considered its evolution when the message is transmitted through a telecommunications channel. Curiously enough, by reference to Boltzmann’s works, the notation he used for his entropy is the letter \( H \), although he observed that his entropy’s expression is similar to the expression of the opposite of Boltzmann’s H-function.

For a random variable \( X \) which can take \( N \) possible values \( x_i \), respectively with the probabilities \( k_i \) \(^{(6)}\) \((1 \leq i \leq n)\), the \( k_i \) satisfying \( k_i \geq 0 \) and \( \sum_{i=1}^{N} k_i = 1 \),

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5. In quantum statistical mechanics, the von Neumann entropy of a state mathematically described by a density matrix \( \rho \) is the trace of \(-\rho \ln \rho\). It was defined and extensively used by the Hungarian-American universal scientist John von Neumann (1903–1957).

6. The notation used by Shannon for the probability of \( x_i \) is \( p_i \), \( 1 \leq i \leq N \). Here I use \( k_i \) instead to avoid any risk of confusion with the Darboux coordinates \( p_i \) in a canonical chart of a symplectic manifold.
Shannon defined its entropy $H(X)$ by stating

$$H(X) = \sum_{i=1}^{N} \log \left( \frac{1}{k_i} \right) k_i = -\sum_{i=1}^{N} (\log k_i) k_i,$$

with the usual convention $0 \log 0 = 0$. In Appendix 2 of his above cited paper, page 49, he proved that up to multiplication by a strictly positive constant, his entropy is the only function which satisfies the following three very reasonable requirements.

— The function $H$ must continuously depend on the probabilities $k_i$, $1 \leq i \leq N$.

— When the $k_i$ are all equal to $1/N$ the function $N \mapsto H(1/N, \ldots, 1/N)$ ($N$ terms) must increase monotonically with $N$.

— When some possible values of the random variable $X$ are obtained as the result of two successive choices, the value of $H(X)$ must be equal to the weighted sum of the individual values of $H$. For example, for a random variable $X$ with the three possible values: $x_1$ with probability $k_1 = 1/2$, $x_2$ with probability $k_2 = 1/3$ and $x_3$ with probability $k_3 = 1/6$, the values $x_1$, $x_2$ and $x_3$ can be obtained in two steps. In the first step, a first trial is done in which one looks at the value taken by a random variable $Y$ with two possible values, $y_1$ and $y_2$, both obtained with probability $1/2$. In the second step, if the value taken by $Y$ is $y_1$, one states that the value taken by $X$ is $x_1$; if the value taken by $Y$ is $y_2$, one looks at the value taken by a random variable $Z$ with two possible values, $z_1$ with probability $2/3$ and $z_2$ with probability $1/3$. If the value taken by $Z$ is $z_1$, one states that the value taken by $X$ is $x_2$, and if the value taken by $Z$ is $z_2$, one states that the value taken by $X$ is $x_3$. The equality that the function $H$ is required to satisfy is

$$H \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{6} \right) = H \left( \frac{1}{2}, \frac{1}{2} \right) + \frac{1}{2} H \left( \frac{2}{3}, \frac{1}{3} \right).$$

Interested readers are referred to Alain Chenciner’s paper [10] for a more detailed account of Claude Shannon’s works and their influence on today’s science.

4. The American physicist Edwin Thompson Jaynes (1922–1998) observed, in [16, 17] (see also [39]), that the definition 2.2.8 of entropy for a continuous statistical state of probability density $\rho$ with respect to the Liouville measure,

$$s(\rho) = \int_M \log \left( \frac{1}{\rho(x)} \right) \rho(x) \lambda_\omega(dx) = -\int_M \log (\rho(x)) \rho(x) \lambda_\omega(dx),$$
is not a correct adaptation of Shannon’s entropy for a discrete statistical state which can take \( N \) distinct values \( x_i \), with the respective probabilities \( k_i \),

\[
H(X) = -\sum_{i=1}^{N} (\log k_i) k_i , \quad \text{with } k_i \geq 0 \text{ for all } i \in \{1, \ldots, N\} \text{ and } \sum_{i=1}^{N} k_i = 1 .
\]

While \( H(X) \) is always a dimensionless number satisfying \( H(X) \geq 0 \), and \( H(X) = 0 \) if and only if there exists only one integer \( i \in \{1, \ldots, N\} \) such that \( k_i = 1 \), all other \( k_j \), for \( j \neq i \), being equal to 0, the above expression of \( s(\rho) \) depends on the chosen units. Indeed in this expression, while \( \rho(x) \lambda_\omega(dx) \) is dimensionless, \( \rho(x) \), as well as \( \lambda_\omega(dx) \) are not dimensionless. A change of the units (of length, time and mass) changes the value of the the term \( \log(\rho(x)) \) by addition of a constant, which can be either positive or negative. Therefore when one uses definition 2.2.8, the sign of entropy does not have any physical meaning. In the above cited papers of Jaynes, the author considered problems in statistics more general than those encountered in classical statistical mechanics, in which the Liouville measure may not be available. He proposed to replace, in the expression of the entropy \( s(\rho) \), the term \( \log \left( \frac{1}{\rho(x)} \right) \)

by \( \log \left( \frac{m(x)}{\rho(x)} \right) \), where \( m(x) \) is the probability density of a reference statistical state with respect to which the entropy \( s(\rho) \) is evaluated. Of course the probability densities \( m(x) \) and \( \rho(x) \) must be taken with respect to the same measure. In the framework of classical statistical mechanics, this measure is the Liouville measure \( \lambda_\omega \), so the correction proposed by Jaynes can be written

\[
s_{\text{Jaynes}}(\rho) = \int_M \log \left( \frac{m(x)}{\rho(x)} \right) \rho(x) \lambda_\omega(dx) .
\]

Probably because he considered problems in which the Liouville measure did not appear, Jaynes did not clearly state how \( m(x) \) should be chosen, although he recommended the use of a probability density invariant by the group of automorphisms of the considered measurable space. Therefore it seems that in the framework of classical statistical mechanics, when the support \( W \) of \( \rho \) is of finite \( \lambda_\omega \)-measure, one should use the following probability density:

\[
m(x) = \begin{cases} 
\frac{1}{\lambda_\omega(W)} & \text{when } x \in W, \\
0 & \text{when } x \notin W .
\end{cases}
\]

\(^7\) The support of \( \rho \) is the closure of the subset of \( M \) made of points \( x \in M \) such that \( \rho(x) \neq 0 \).
With this choice of $m$, $s(\rho)$ and $s_{\text{Jaynes}}(\rho)$ are related by

$$s_{\text{Jaynes}}(\rho) = s(\rho) - \log(\lambda_\omega(W)).$$

The corrected entropy $s_{\text{Jaynes}}(\rho)$ proposed by Jaynes is dimensionless. It differs from the entropy $s(\rho)$ of definition 2.2.8 only by a constant, which depends on the units chosen for time, length and mass, and can take negative as well as positive values.

In calculus of variations, it may be useful to consider infinitesimal variations of $\rho$ whose support does not always remain contained in the support of $\rho$. Instead of the support of $\rho$, one should take for $W$, in the above formula, an open subset of $M$ of finite $\lambda_\omega$-measure which contains the support of $\rho$.

5. For a better understanding of how the entropies of continuous and discrete statistical states are related, let us consider the process of discretization of a continuous statistical state. As above, we assume that the support $W$ of the probability density $\rho$ is of finite $\lambda_\omega$-measure. For simplicity\(^8\), we moreover assume that $\rho_{\text{max}} = \sup_{x \in M} \rho(x)$ too is finite and that, for each real $r$ satisfying $0 \leq r \leq \rho_{\text{max}},$

$$\lambda_\omega(\{ x \in W \mid \rho(x) = r \}) = 0.$$  

For each $r \geq 0$, let us set

$$G(r) = \lambda_\omega(\{ x \in W \mid 0 \leq \rho(x) \leq r \}).$$

Then $G$ is a continuous and monotonically increasing function which takes all values in the closed interval $[0, \lambda_\omega(W)]$. Let $N$ be an integer satisfying $N > 2$. There exist $N$ real numbers $r_i^N$, $1 \leq i \leq N$, such that for each $i \in \{1, \ldots, N\}$

$$G(r_i^N) = \frac{i\lambda_\omega(W)}{N}.$$  

We set

$$V_1^N = \{ x \in W \mid 0 \leq \rho(x) \leq r_1 \},$$

and, for each $i \in \{2, \ldots, N\}$,

$$V_i^N = \{ x \in W \mid r_{i-1} < \rho(x) \leq r_i \}.$$  

The $V_i^N$ are measurable subsets of $M$ which satisfy, for $1 \leq i, j \leq N$,

$$\lambda_\omega(V_i^N) = \frac{\lambda_\omega(W)}{N}, \quad V_i^N \cap V_j^N = \emptyset \text{ if } i \neq j, \quad \bigcup_{i=1}^N V_i^N = W.$$  

\(^8\) These assumptions could probably be avoided with the use of more sophisticated concepts in integration theory, such as the Stieltjes integral, so named in honour of the Dutch mathematician Thomas Joannes Stieltjes (1856–1892).
Now we set, for each \( i \in \{1, \ldots, N\} \),
\[
k_i^N = \int_{V_i^N} \rho(x) \lambda_\omega(\text{d}x), \quad \rho_i^N = \frac{Nk_i^N}{\lambda_\omega(W)}.
\]

We have
\[
0 \leq k_i^N \leq 1 \text{ for each } i \in \{1, \ldots, N\}, \quad \sum_{i=1}^{N} k_i^N = 1.
\]

Let \( \rho^N \) be the function defined on \( M \) by
\[
\rho^N(x) = \begin{cases} 
\frac{Nk_i^N}{\lambda_\omega(W)} & \text{if } x \in V_i^N, \ 1 \leq i \leq N, \\
0 & \text{if } x \notin \bigcup_{i=1}^{N} V_i^N = W.
\end{cases}
\]

The function \( \rho^N \) is everywhere \( \geq 0 \) on \( M \), and only takes \( N \) distinct non-zero values. It is a discrete approximation of the probability density \( \rho \), which satisfies
\[
\int_{M} \rho^N(x) \lambda_\omega(\text{d}x) = \sum_{i=1}^{N} k_i^N = 1.
\]

The function \( \rho^N \) is therefore the probability density of a statistical state on \( M \). Although it is not continuous, we can use 2.2.8 to calculate \( s(\rho^N) \). We obtain
\[
s(\rho^N) = \sum_{i=1}^{N} k_i^N (\log k_i^N) + \log \left(\frac{\lambda_\omega(W)}{\lambda_\omega(W)}\right) - \log N.
\]

We observe that the term \( \sum_{i=1}^{N} k_i (\log k_i) \) is the Shannon entropy \( H(X^N) \) of a random variable \( X^N \) which can take \( N \) distinct values, for example the values \( 1, \ldots, N \), with the respective probabilities \( k_1^N, \ldots, k_N^N \). So we can write
\[
H(X^N) = s(\rho^N) - \log(\lambda_\omega(W)) + \log N = s_{\text{Jaynes}}(\rho^N) + \log N.
\]

When \( N \to +\infty \), \( s_{\text{Jaynes}}(\rho^N) \to s_{\text{Jaynes}}(\rho) \) and \( \log N \to +\infty \). The above equality proves that when \( N \to +\infty \), the Shannon entropy of the discrete approximation, by a random variable \( X^N \) which can take \( N \) distinct non-zero values, of the continuous statistical state of probability density \( \rho \), does not remain bounded and increases as fast as \( \log N \).

During the years 1950–1960, several scientists, notably Edwin Thompson Jaynes cited above (see also [14, 15] by the same author) and the American mathematician George Whitelaw Mackey (1916–2006) [18], proposed the use of information theory in thermodynamics.
Interested readers are referred to Roger Balian’s paper [1], in which they will find a clear account of the use of probability concepts in physics and of information theory in quantum mechanics.

2.2.10. Proposition. — On a symplectic manifold \((M,\omega)\), we consider a smooth Hamiltonian \(H \in C^\infty(M,\mathbb{R})\) which does not depend on time. Let \(X_H\) be the associated Hamiltonian vector field on \(M\). Let \(\rho(t_0)\) be the probability density of a smooth statistical state of the dynamical system determined by \(X_H\) at a time \(t_0\). The probability density \(\rho(t_1)\) of the statistical state of the system at any other time \(t_1\) at which the system still exists is such that

\[
s(\rho(t_1)) = s(\rho(t_0)).
\]

In other words, the entropy of the statistical state of the system remains constant as long as this statistical state exists.

Proof. — As seen in 2.2.7, \(\rho(t_1) = \rho(t_0) \circ \Phi^{X_H}_{t_1-t_0}\), so for each \(x \in M\),

\[
\rho(t_0, x) = \rho(t_1, \Phi^{X_H}_{t_1-t_0}(x)).
\]

When \(s(\rho(t_0)) \neq -\infty\), we can write

\[
s(\rho(t_0)) = \int_M \log \left( \frac{1}{\rho(t_0, x)} \right) \rho(t_0, x) \lambda_\omega(dx)
\]

\[
= \int_M \log \left( \frac{1}{\rho(t_1, \Phi^{X_H}_{t_1-t_0}(x))} \right) \rho(t_1, \Phi^{X_H}_{t_1-t_0}(x)) \lambda_\omega(dx)
\]

\[
= \int_M \log \left( \frac{1}{\rho(t_1, y)} \right) \rho(t_1, y) \lambda_\omega(dy)
\]

\[
= s(\rho(t_1)),
\]

where we have used the change of integration variable \(y = \Phi^{X_H}_{t_1-t_0}(x)\) and the invariance of the Liouville measure by symplectomorphism (2.2.2). When \(s(\rho(t_0)) = -\infty\), the same calculation leads to a divergent integral for the expression of \(s(\rho(t_1))\), which therefore is equal to \(-\infty\).

2.3. Gibbs states for a Hamiltonian system. — In this subsection, \(H\) is a smooth Hamiltonian which does not depend on time, defined on a symplectic manifold \((M,\omega)\), and \(X_H\) is the associated Hamiltonian vector field. The Gibbs states defined here are built with the Hamiltonian \(H\) as the only conserved
quantity. Their main properties are briefly indicated. Gibbs state for the Hamiltonian action of a Lie group are considered in Section 3

2.3.1. Proposition. — Under the assumptions and with the notations of 2.3, let \( \rho \) be the probability density, with respect to the Liouville measure \( \lambda_\omega \), of a smooth statistical state on \( M \). We assume that \( \rho \) is such that the integrals which define the entropy \( s(\rho) \) (definition 2.2.8) and the mean value \( \mathcal{E}_\rho(H) \) of the Hamiltonian \( H \) (definition 2.2.3) are convergent and can be differentiated under the sign \( \int \) with respect to infinitesimal variations of \( \rho \). The entropy function \( s \) is stationary at \( \rho \) with respect to smooth infinitesimal variations of \( \rho \) which leave fixed the mean value of \( H \) if and only if there exists a real \( \beta \in \mathbb{R} \) such that, for every \( x \in M \),

\[
\rho(x) = \frac{1}{P(\beta)} \exp(-\beta H(x)) , \quad \text{with} \quad P(\beta) = \int_M \exp(-\beta H(x)) \lambda_\omega(dx).
\]

Proof. — Let \( \tau \mapsto \rho_\tau \) be a smooth infinitesimal variation of \( \rho \) which leaves fixed the mean value of \( H \). Since \( \int_M \rho_\tau(x) \lambda_\omega(dx) \) and \( \int_M \rho_\tau(x) H(x) \lambda_\omega(dx) \) do not depend on \( \tau \), it satisfies, for all \( \tau \in ]-\varepsilon,\varepsilon[ \),

\[
\int_M \frac{\partial \rho(\tau,x)}{\partial \tau} \lambda_\omega(dx) = 0, \quad \int_M \frac{\partial \rho(\tau,x)}{\partial \tau} H(x) \lambda_\omega(dx) = 0.
\]

Moreover an easy calculation leads to

\[
\frac{ds(\rho_\tau)}{d\tau} \big|_{\tau=0} = - \int_M \frac{\partial \rho(\tau,x)}{\partial \tau} \bigg|_{\tau=0} (1 + \log(\rho(x))) \lambda_\omega(dx).
\]

By a well known result in calculus of variations, this implies that the entropy functional is stationary at \( \rho \) with respect to smooth infinitesimal variations of \( \rho \) which leave fixed the mean value of \( H \), if and only if there exist two real constants \( \alpha \) and \( \beta \), the Lagrange multipliers, such that, for every \( x \in M \),

\[
1 + \log(\rho(x)) + \alpha + \beta H(x) = 0,
\]

which leads to

\[
\rho(x) = \exp(-1 + \alpha - \beta H(x)).
\]

By writing that \( \int_M \rho(x) \lambda_\omega(dx) = 1 \), we see that \( \alpha \) is determined by \( \beta \):

\[
\exp(1 + \alpha) = P(\beta) = \int_M \exp(-\beta H(x)) \lambda_\omega(dx).
\]
2.3.2. Definitions. — Let $\beta \in \mathbb{R}$ be a real which satisfies the conditions of proposition 2.3.1. The smooth statistical state whose probability density, with respect to the Liouville measure $\lambda_\omega$, is

$$\rho_\beta(x) = \frac{1}{P(\beta)} \exp(-\beta H(x)), \quad x \in M,$$

with

$$P(\beta) = \int_M \exp(-\beta H(x)) \lambda_\omega(dx),$$

is called the {Gibbs state} associated to (or indexed by) $\beta$. The function $P$ of the real variable $\beta$ is called the {partition function} of the dynamical system determined by the Hamiltonian vector field $X_H$.

2.3.3. Proposition. — Let $\beta \in \mathbb{R}$ be a real which satisfies the conditions of proposition 2.3.1. The probability density $\rho_\beta$ of the corresponding Gibbs state (definition 2.3.2) remains invariant under the flow of the Hamiltonian vector field $X_H$.

Proof. — Since the Hamiltonian $H$ does not depend on time, it is a first integral of the differential equation determined by $X_H$, i.e., it keeps a constant value on each integral curve of $X_H$. Therefore $\rho_\beta$ keeps a constant value on each integral curve of $X_H$.

2.3.4. Some properties of Gibbs states. — We have seen (2.3.1) that the entropy functional $s$ is stationary at each Gibbs state with respect to all infinitesimal variations of its probability density which leave invariant the mean value of the Hamiltonian $H$. A stronger result holds: given any Gibbs state of probability density $\rho_\beta$, on the set of all continuous statistical states whose probability density $\rho$ is such that $E_\rho(H) = E_{\rho_\beta}(H)$, the entropy functional $s$ reaches its only strict maximum at the Gibbs state of probability density $\rho_\beta$.

When the set $\Omega$ of reals $\beta$ for which a Gibbs state indexed by $\beta$ exists is not empty, this set is an open interval $]a, b[$ of $\mathbb{R}$, where either $a \in \mathbb{R}$, or $a = -\infty$, either $b \in \mathbb{R}$ and $b > a$, or $b = +\infty$. When in addition $H$ is bounded from below, i.e., when there exists $m \in \mathbb{R}$ such that, for any $x \in M$, $m \leq H(x)$, the open interval $\Omega$ is unbound on the right side, i.e., $\Omega = ]a, +\infty[$ where either $a \in \mathbb{R}$, or $a = -\infty$.

We have already defined on $\Omega$ the partition function $P$ (2.3.2). Other functions can be defined on $\Omega$ as follows. For each $\beta \in \Omega$, the entropy $s(\rho_\beta)$ exists of course, and one can prove that the mean value $E_{\rho_\beta}(H)$ (definition 2.2.3) of
the Hamiltonian $H$, in the Gibbs state indexed by $\beta$, exists too, as well as $E_{\rho_\beta}(H^2)$ and $E_{\rho_\beta}\left((H - E_{\rho_\beta}(H))^2\right)$. So we can set

$$S(\beta) = s(\rho_\beta), \ E(\beta) = E_{\rho_\beta}(H), \ \beta \in \Omega.$$ 

The functions $P$ (partition function), $E$ (mean value of the Hamiltonian, considered by physicists as the energy) and $S$ (entropy) so defined are of class $C^\infty$ on $\Omega$ and satisfy, for any $\beta \in \Omega$,

$$P(\beta) > 0, \ E(\beta) = -\frac{1}{P(\beta)} \frac{dP(\beta)}{d\beta} = \frac{d(-\log P(\beta))}{d\beta},$$
$$\frac{dE(\beta)}{d\beta} = \frac{d^2(-\log P(\beta))}{d\beta^2} = -E_{\rho_\beta}\left((H - E_{\rho_\beta}(H))^2\right),$$

$$S(\beta) = \log P(\beta) + \beta E(\beta) = \beta \frac{d(-\log P(\beta))}{d\beta} - (-\log P(\beta)), $$
$$\frac{dS(\beta)}{d\beta} = \beta \frac{dE(\beta)}{d\beta}.$$

The above expression of $\frac{dE(\beta)}{d\beta}$ proves that $\beta \mapsto E(\beta)$ is a non-increasing function. When the Hamiltonian $H$ is not a constant, for each $\beta \in \Omega$, the continuous function defined on $M \mapsto (H(x) - E_{\rho_\beta}(H))^2$ takes its values in $\mathbb{R}^+$ and is not always equal to $0$. Its mean value $E_{\rho_\beta}\left((H - E_{\rho_\beta}(H))^2\right)$ is therefore $>0$, which proves that $\beta \mapsto E(\beta)$ is a strictly decreasing function on $\Omega$. The map $E$ is open, and is a diffeomorphism of $\Omega$ onto its image $\Omega^*$.

The above expression of $S(\beta)$ shows that the functions $\beta \mapsto -\log (P(\beta))$ and $\beta \mapsto S(\beta)$ are Legendre transforms of each other. They are indeed linked by the same relation as that which, in calculus of variations, links a hyper-regular Lagrangian with the associated energy. Here the hyper-regular “Lagrangian”, defined on $\Omega$, is $\beta \mapsto -\log P(\beta)$, the Legendre map is the diffeomorphism $E: \Omega \mapsto \Omega^*$, the “energy”, defined on $\Omega$, is $\beta \mapsto S(\beta)$, and the “Hamiltonian”, defined on $\Omega^*$, is $S \circ E^{-1}$. By using the above expression of $\frac{dS(\beta)}{d\beta}$, we can write

$$E^{-1}(e) = \frac{d(S \circ E^{-1}(e))}{de}, \ e \in \Omega^*.$$ 

As soon as 1869, the Legendre transform was used in thermodynamics by the French scientist François Massieu [2, 21, 22, 23].
The results stated here without proof are proven below, in a more general setting, for Gibbs states (3.1.3, 3.1.6, 3.2.5).

2.3.5. Gibbs states, temperatures and thermodynamic equilibria

Let us now assume that the dynamical system determined by the Hamiltonian vector field \( X_H \) mathematically describes the evolution with time of a physical system, an object of the real world. Physicists consider each Gibbs state of the considered dynamical system, indexed by some \( \beta \in \mathbb{R} \), as the mathematical description of a state of thermodynamic equilibrium of the corresponding physical system, and \( \beta \) as a quantity related to the absolute temperature \( T \) of the physical system by the equality

\[
\beta = \frac{1}{kT} ,
\]

where \( k \) is a constant which depends on the chosen units, called Boltzmann’s constant. This identification of Gibbs states with thermodynamic equilibria is justified by the following property. Let us consider two similar physical systems, mathematically described by two Hamiltonian systems, whose Hamiltonians are, respectively, \( H_1 \) defined on the symplectic manifold \( (M_1, \omega_1) \) and \( H_2 \) defined on the symplectic manifold \( (M_2, \omega_2) \). We first assume that they are independent and both in a Gibbs state. We denote by \( \rho_{1, \beta_1} \) and \( \rho_{2, \beta_2} \) the probability densities of the Gibbs states, indexed by the reals \( \beta_1 \) and \( \beta_2 \), in which these two systems are, respectively. Let \( E_{1}(\beta_1) \) and \( E_{2}(\beta_2) \) be the corresponding mean values of their Hamiltonians. Let us now assume that the two systems are coupled in a way allowing an exchange of energy between them. For example, the corresponding objects of the real world can be two vessels containing a gas, separated by a wall allowing a transfer of heat between them. Coupled together, they make a new physical system, mathematically described by a Hamiltonian system on the symplectic manifold \( (M_1 \times M_2, \omega_{\text{new}} = p_1^* \omega_1 + p_2^* \omega_2) \), where \( p_1 : M_1 \times M_2 \to M_1 \) and \( p_2 : M_1 \times M_2 \to M_2 \) are the canonical projections. The Hamiltonian of this new system can be made as close to \( H_1 \circ p_1 + H_2 \circ p_2 \) as one wishes, by making very small the coupling between the two systems. We can therefore consider \( H_1 \circ p_1 + H_2 \circ p_2 \) as a reasonable approximation of the Hamiltonian of the new system. When the two subsystems are in the Gibbs states indexed, respectively, by \( \beta_1 \) and by \( \beta_2 \), the new system made of these two coupled subsystems is in the statistical state of probability density \( \rho_{1, \beta_1} \circ p_1 + \rho_{2, \beta_2} \circ p_2 \), and its entropy is \( S_1(\beta_1) + S_2(\beta_2) \). If \( \beta_1 \neq \beta_2 \), the new system is not in a Gibbs state. Let us indeed assume, for example, that \( \beta_1 < \beta_2 \). If a transfer of energy between the
two subsystems occurs, in which the energy of the first subsystem decreases while the energy of the second subsystem increases by an equal amount, the modified Gibbs state of the first subsystem becomes indexed by \( \beta'_1 > \beta_1 \) and that of the second subsystem by \( \beta'_2 < \beta_2 \) since, as seen in 2.3.4, for \( i = 1 \) as well as for \( i = 2 \), we have \( \frac{dE_i(\beta'_i)}{d\beta'_i} < 0 \). As long as \( \beta'_1 < \beta'_2 \), such an energy transfer between the two subsystems results in an increase of the entropy of the total new system, until \( \beta'_1 = \beta'_2 = \beta_n \), which indexes the Gibbs state of the new system for a mean value of its Hamiltonian \( E_1(\beta_1) + E_2(\beta_2) \). We have of course \( \beta_1 < \beta_n < \beta_2 \), which proves that when the state of the new system evolves from its initial state towards its Gibbs state, the energy flow goes from the subsystem whose Gibbs initial state is indexed by the smaller \( \beta_1 \) towards the subsystem whose initial Gibbs state is indexed by the larger \( \beta_2 \). This result is in agreement with everyday’s experience, since equality (∗) implies that when \( \beta > 0 \), a smaller value of \( \beta \) corresponds to a higher temperature.

2.3.6. Evolution towards a thermodynamic equilibrium

In the real world, the state of an approximately isolated system often evolves with time towards a state of thermodynamic equilibrium. When such a state is approximately reached, it remains approximately stationary, with small fluctuations. Let us mathematically modelize this evolution as the variation with time of the statistical state of a Hamiltonian dynamical system, whose smooth Hamiltonian \( H \) is defined on a very high-dimensional symplectic manifold \( (M,\omega) \). Proposition 2.3.3 above, which states that Gibbs states do not change with time, seems to be in reasonably good agreement with the identification of Gibbs states with thermodynamic equilibria, although it does not explain fluctuations which are experimentally observed in thermodynamic equilibria. On the contrary, proposition 2.2.10 above, which states that as long as any smooth statistical state exists, its entropy remains constant, is in clear disagreement with the behaviour of isolated systems in the real world. The mathematical description of the evolution with time of a real physical system by the dynamical system determined by the Hamiltonian vector field of a smooth Hamiltonian which does not depend on time, together with definition 2.2.8 of the entropy, can be used only for reversible systems. It cannot be used to describe the evolution with time of some statistical states towards the corresponding Gibbs state.
3. Gibbs states for Hamiltonian actions of Lie groups

The general definition of a Gibbs states is very natural: it amounts to introduce, in the definition of a statistical state, not only the Hamiltonian, but other conserved quantities too, on the same footing as the Hamiltonian. One may even forget the Hamiltonian and consider only the moment map of the Hamiltonian action.

This idea is already present in the book published by Gibbs in 1902 ([13], chapter I, page 42 and the following pages), the conserved quantities other than the Hamiltonian being the components of the total angular momentum.

Following an idea first proposed around 1809 by Joseph Louis Lagrange (1736–1813), Jean-Marie Souriau defined statistical states on the manifold of motions of a Hamiltonian dynamical system, instead of on its phase space. This approach allows a more natural treatment on the same footing of both the Hamiltonian and other conserved quantities, because the action of the group of translations in time, which may act only locally on the phase space, always acts globally on the space of motions. The concept of manifold of motions and its properties are presented in subsection 3.1 below. Gibbs states for a Hamiltonian action of a Lie group on a symplectic manifold are then defined (3.1.4), together with generalized temperatures and partition functions, and their main property (maximality of entropy) is proven (3.1.6). Thermodynamic functions associated to a Gibbs state (mean value of the moment map and entropy) are maps defined on the set of generalized temperatures (subsection 3.2). The expressions of their differentials lead to the definition, on the set of generalized temperatures, of a remarkable Riemannian metric, linked to the Fisher-Rao metric of statisticians. The adjoint action on the set of generalized temperatures is considered in subsection 3.3, in which the Riemannian metric induced on each adjoint orbit is expressed in terms of a symplectic cocycle.

3.1. Symmetries and statistical states. — In this subsection we consider the dynamical system determined on a symplectic manifold \((M, \omega)\), as explained in 2.2.1, by a Hamiltonian vector field \(X_H\) whose smooth Hamiltonian \(H\), defined on \(\mathbb{R} \times M\) or on one of its open subsets, may depend on time.

3.1.1. The manifold of motions of a Hamiltonian system

Jean-Marie Souriau called motion of the dynamical system determined by a Hamiltonian vector field \(X_H\) any maximal solution \(\varphi: t \mapsto \varphi(t)\) of Hamilton’s
differential equation
\[ \frac{d\varphi(t)}{dt} = X_H(t, \varphi(t)) \, . \]

The manifold of motions of the system, denoted by Mot\((X_H)\), is simply the set of all motions, i.e., the set of all maximal solutions \(\varphi\) of the above differential equation. It always has the structure of a smooth symplectic manifold. For each \(t_0 \in \mathbb{R}\), the map \(h_{t_0}\) which associates, to each motion \(\varphi : t \mapsto \varphi(t)\) whose interval of definition contains \(t_0\), the point \(h_{t_0}(\varphi) = \varphi(t_0) \in M\), is indeed, when the subset made of motions defined on an interval of \(\mathbb{R}\) which contains \(t_0\) is not empty, a bijection of this subset of Mot\((X_H)\) onto an open subset of \(M\).

This simple fact allows the definition of a topology and a structure of smooth manifold on Mot\((X_H)\) such that, for each \(t_0 \in \mathbb{R}\), the map \(h_{t_0} : \varphi \mapsto h_{t_0}(\varphi) = \varphi(t_0)\) is a diffeomorphism of the open subset of Mot\((X_H)\) made of motions defined on an interval of \(\mathbb{R}\) which contains \(t_0\), onto an open subset of \(M\). Since the reduced flow of \(X_H\) is made of symplectomorphisms, the pull-back \(h_{t_0}^*\omega\) of the symplectic form \(\omega\) does not depend on \(t_0\), therefore determines globally a symplectic form \(\omega_{\text{Mot}(X_H)}\) on the manifold of motions Mot\((X_H)\).

The manifold Mot\((X_H)\) may be a non-Hausdorff manifold, although any of its elements has an open Hausdorff neighbourhood symplectomorphic to an open subset of \(M\).

We assume now, until the end of this section, that the Hamiltonian \(H\) does not depend on time. For a given motion \(\varphi \in \text{Mot}(X_H)\), the value of \(H(\varphi(t_0))\) does not depend on the choice of \(t_0\) in the interval on which \(\varphi\) is defined. Therefore there exists a real-valued, smooth function \(H_{\text{Mot}}\), defined on \(\text{Mot}(X_H)\), such that, for each \(\varphi \in \text{Mot}(X_H)\) and any \(t_0\) in the interval on which \(\varphi\) is defined, \(H_{\text{Mot}}(\varphi) = H(\varphi(t_0))\). Since, for each \(t_0 \in \mathbb{R}\), the function \(H_{\text{Mot}}\), restricted to the open subset made of motions whose interval of definition contains \(t_0\), is the pull-back \(h_{t_0}^*H\) of the Hamiltonian \(H\), the Hamiltonian vector field \(X_{\text{Hmot}}\) on \(\text{Mot}(X_H)\), restricted to this open subset of \(\text{Mot}(X_H)\), is the inverse image \(h_{t_0}^*(X_H)\) of the Hamiltonian vector field \(X_H\) defined on \(M\).

For any \(s \in \mathbb{R}\) and any motion \(\varphi : ]a, b[ \to M \in \text{Mot}(X_H)\), defined on the interval \(]a, b[ \subset \mathbb{R}\), let \(\Phi_{\text{Mot}}(s, \varphi)\) be the parametrized curve, defined on the

---

9. We recall that a topological space is said to be Hausdorff when for each pair of distinct elements \(x\) and \(y\) of this space, there exist neighbourhoods \(U\) of \(x\) and \(V\) of \(y\) such that \(U \cap V = \emptyset\). This property is so named in honour of the German mathematician Felix Hausdorff (1868–1942), an important founder of topology and set theory, who after losing his Professor position at the university of Bonn, was driven to suicide by the Nazi regime.
interval $]a-s, b-s[ \subset \mathbb{R}$, with values in $M$,  

$$\Phi_{\text{Mot}}(s, \varphi)(t) = \varphi(t+s), \quad t \in ]a-s, b-s[.$$  

One can easily see that $\Phi_{\text{Mot}}(s, \varphi) \in \text{Mot}(X_H)$ and that the map  

$$\Phi_{\text{Mot}} : \mathbb{R} \times \text{Mot}(X_H) \to \text{Mot}(X_H)$$  

is a smooth action on the left of the additive Lie group $\mathbb{R}$ on the manifold of motions $\text{Mot}(X_H)$. The infinitesimal generator of this action is the vector field on $\text{Mot}(X_H)$, temporarily denoted by $Z$, defined by the equality  

$$Z(\varphi) = \frac{d \Phi_{\text{Mot}}(s, \varphi)}{ds} \bigg|_{s=0}, \quad \varphi \in \text{Mot}(X_H).$$  

For any real $t_0$ which belongs to the open interval on which $\varphi$ is defined, we have  

$$\text{Th}_{t_0}(Z(\varphi)) = \frac{d}{ds} \left( h_{t_0} \left( \Phi_{\text{Mot}}(s, \varphi) \right) \right) \bigg|_{s=0} = \frac{d\varphi(t_0 + s)}{ds} \bigg|_{s=0} = X_{H_{\text{Mot}}}(\varphi(t_0)).$$  

This result proves that the infinitesimal generator $Z$ of the action $\Phi_{\text{Mot}}$ is the Hamiltonian vector field $X_{H_{\text{Mot}}}$. Being generated by the flow of a Hamiltonian vector field, the action $\varphi$ is therefore Hamiltonian. It admits the Hamiltonian $H_{\text{Mot}}$ as a moment map (with the usual convention in which the Lie algebra of the additive Lie group $\mathbb{R}$ is identified with $\mathbb{R}$ with the zero bracket and its dual is too identified with $\mathbb{R}$, the pairing by duality being the usual product of reals).

The reader will observe that while the flow of the vector field $X_H$ does not always determine a Hamiltonian action of $\mathbb{R}$ on the symplectic manifold $(M, \omega)$, but only a local Hamiltonian action, except when all the motions are defined for all $t \in \mathbb{R}$, the flow of $X_{H_{\text{Mot}}}$ always determines a Hamiltonian action of $\mathbb{R}$ on $(\text{Mot}(X_H), \omega_{\text{Mot}(X_H)})$. However, the price paid for obtaining better properties of the flow of a Hamiltonian vector field is the fact that $\text{Mot}(X_H)$ can be a non-Hausdorff manifold. For this reason, some important results, for example the theorem which asserts the unicity, for a given initial condition, of a maximal solution of a smooth differential equation, can no more be used.

In all what follows, the notation $(M, \omega)$ will be used to denote as well the phase space as the space of motions of the dynamical system determined by a smooth Hamiltonian which does not depend on time, according to the context in which it is used.
3.1.2. Definitions. — Let \( g \) be a real, finite-dimensional Lie algebra which acts on a connected symplectic manifold \((M, \omega)\) by a Hamiltonian action \( \varphi : g \to A^1(M) \). Let \( J : M \to g^* \) be a moment map of the action \( \varphi \). For each \( \beta \in g \), we consider the integral
\[
\int_M \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx),
\]
where \( \lambda_\omega \) is the Liouville measure on \( M \).

1. The above integral (*) is said to be \textit{normally convergent} when there exists an open neighbourhood \( U \) of \( \beta \) in \( g \) and a function \( f : M \to \mathbb{R}^+ \), integrable on \( M \) with respect to the Liouville measure \( \lambda_\omega \), such that for any \( \beta' \in U \), the following inequality
\[
\exp(-\langle J(x), \beta' \rangle) \leq f(x)
\]
is satisfied for all \( x \in M \).

2. When \( \beta \in g \) is such that the integral (*) above is normally convergent, \( \beta \) is said to be a \textit{generalized temperature}. The subset of \( g \) made of generalized temperatures will be denoted by \( \Omega \).

3. When the set \( \Omega \) of generalized temperatures is not empty, the \textit{partition function} associated to the Hamiltonian action \( \varphi \) is the function \( P \) defined on \( \Omega \) by the equality
\[
P(\beta) = \int_M \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx), \quad x \in M, \quad \beta \in \Omega \subset g.
\]

3.1.3. Proposition. — The assumptions and notations are those of 3.1.2. The set \( \Omega \) of generalized temperatures does not depend on the choice of the moment map \( J \) of the Hamiltonian action \( \varphi \). When it is not empty, this set is an open convex subset of the Lie algebra \( g \), the partition function \( P \) is of class \( C^\infty \) and its differentials of all orders can be calculated by differentiation under the integration sign \( \int \).

Proof. — When \( \beta \) is a generalized temperature, definition 3.1.2 implies that there exists a neighbourhood \( U \) of \( \beta \) whose all elements are generalized temperatures. When it is not empty, the set \( \Omega \) of generalized temperatures is therefore open. When the moment map \( J \) is replaced by another moment map \( J' \), the difference \( J' - J \) is a constant. The replacement of \( J \) by \( J' \) has no effect.
on the eventual normal convergence of the above integral (*), therefore $\Omega$ does not depend on the choice of the moment map $J$.

Let $\beta_0$ and $\beta_1$ be two distinct elements in $\Omega$ (assumed to be non-empty), $U_0$ and $U_1$ be neighbourhoods, respectively of $\beta_0$ and $\beta_1$, $f_0$ and $f_1$ be the positive functions, defined on $M$ and integrable with respect to the Liouville measure, greater or equal, respectively, than the functions $x \mapsto \exp(-\langle J(x), \beta'_0 \rangle)$ and $x \mapsto \exp(-\langle J(x), \beta'_1 \rangle)$ for all $\beta'_0 \in U_0$ and $\beta'_1 \in U_1$. For any $\lambda \in [0,1]$, $U_\lambda = \{(1-\lambda)\beta'_0 + \lambda\beta'_1 \ | \ \beta'_0 \in U_0, \ \beta'_1 \in U_1\}$ is a neighbourhood of $\beta_\lambda = (1-\lambda)\beta_0 + \lambda\beta_1$. The function $f_\lambda = (1-\lambda)f_0 + \lambda f_1$ is integrable on $M$. For any $\beta'_\lambda \in U_\lambda$, it is greater or equal to the function $x \mapsto \exp(-\langle J(x), \beta'_\lambda \rangle)$. Therefore $\beta_\lambda \in \Omega$, which proves the convexity of $\Omega$.

For each $x \in M$ fixed, the $k$-th differential of $\exp(-\langle J(x), \beta \rangle)$ with respect to $\beta$ is

$$D^k \left( \exp(-\langle J, \beta \rangle) \right) = (-1)^k J^\otimes k(x) \exp(-\langle J(x), \beta \rangle),$$

where $J^\otimes k(x) = J(x) \otimes \cdots \otimes J(x) \in (\mathfrak{g}^*)^\otimes k$. Let us recall that $(\mathfrak{g}^*)^\otimes k$ is canonically isomorphic with the space $L^k(\mathfrak{g},\mathbb{R})$ of $k$-multilinear forms on $\mathfrak{g}$. Let us choose any norm on $\mathfrak{g}$. We take on $L^k(\mathfrak{g},\mathbb{R})$ the sup norm. For any $x \in M$, we have

$$\|J^\otimes k(x)\| = \sup_{X_i \in \mathfrak{g}, \|X_i\| \leq 1, 1 \leq i \leq k} |\langle J(x), X_1 \rangle \cdots \langle J(x), X_k \rangle|. $$

Let $\beta \in \Omega$ be a generalized temperature. It follows from the definition of a generalized temperature that there exist a real $\varepsilon > 0$ and a non-negative function $f$ defined on $M$, integrable with respect to the Liouville measure and greater than the function $x \mapsto \exp(-\langle J(x), \beta' \rangle)$ for any $\beta' \in \mathfrak{g}$ satisfying $\|\beta' - \beta\| \leq \varepsilon$. Let $\beta'' \in \mathfrak{g}$ be such that $\|\beta'' - \beta\| \leq \frac{\varepsilon}{2}$. For all $X_i \in \mathfrak{g}$ satisfying $\|X_i\| \leq 1$, with $1 \leq i \leq k$, and any $x \in M$, we have

$$J^\otimes k(x)(X_1, \ldots, X_k) = \langle J(x), X_1 \rangle \cdots \langle J(x), X_k \rangle. $$

Taking into account the inequality, valid for all $i \in \{1, \ldots, k\}$,

$$|\langle J(x), X_i \rangle| \leq \frac{2k}{\varepsilon} \exp \left( \frac{\varepsilon}{2k} |\langle J(x), X_i \rangle| \right),$$

we can write

$$|J^\otimes k(x)(X_1, \ldots, X_k)| \exp(-\langle J(x), \beta'' \rangle) \leq \left( \frac{2k}{\varepsilon} \right)^k \exp \left( -\langle J(x), \beta'' + \frac{\varepsilon}{2k} (\eta_1 X_1 + \cdots + \eta_k X_k) \rangle \right),$$
where the terms $\eta_i, 1 \leq i \leq k$, all equal either to 1 or to $-1$, are chosen in such a way that $\langle J(x), \eta_i X_i \rangle \leq 0$. For each $i \in \{1, \ldots, k\}$, $|X_i| \leq 1$, therefore
\[
\left\| \beta - \left( \beta'' + \frac{\varepsilon}{2k} (\eta_1 X_1 + \cdots + \eta_k X_k) \right) \right\| \leq \left\| \beta - \beta'' \right\| + \frac{\varepsilon k}{2k} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
We see that
\[
\left\| J \otimes k(x_1, \ldots, x_k) \right\| \exp(-\langle J(x), \beta'' \rangle) \leq f(x).
\]
By taking the upper bound of the left hand side when the $X_i$ take all possible values among elements in $g$ whose norm is smaller than or equal to 1,
\[
\left\| J \otimes k(x) \right\| \exp(-\langle J(x), \beta'' \rangle) \leq f(x).
\]
The integral
\[
\int_M D^k \left( \exp(-\langle J(x), \beta'' \rangle) \right) \lambda_\omega(dx)
\]
is therefore normally convergent. It follows that the partition function $P$ is of class $C^\infty$, and that its differentials of all orders can be calculated by differentiation under the sign $\int$.

3.1.4. **Definition.** — Let $\beta \in \Omega$ be a generalized temperature. The statistical state on $M$ whose probability density, with respect to the Liouville measure $\lambda_\omega$, is expressed as
\[
\rho_\beta(x) = \frac{1}{P(\beta)} \exp(-\langle J(x), \beta \rangle), \quad x \in M,
\]
is called the *Gibbs state* associated to (or indexed by) $\beta$.

3.1.5. **Gibbs states of subgroups of the Galilei group**

The Galilei group, so named in honour of the Italian scientist Galileo Galilei (1564–1642), is the group of symmetries of the mathematical model of space-time used in classical (non relativistic) mechanics. It is a ten-dimensional Lie group diffeomorphic to $SO(3) \times \mathbb{R}^7$. The Lie group of symmetries of any isolated mechanical system must contain the Galilei group as a Lie subgroup. In his book [33], Souriau has proven that for any mechanical system made of a set of material objects whose total mass is non-zero, the set of generalized temperatures, for the action of the Galilei group on the manifold of motions, is empty. There is therefore no Gibbs state for these systems. However, Gibbs states for subgroups of the Galilei group do exist and have interesting interpretations in physics and in cosmology [34].

The interested reader will find more results about the Galilei group and its central extension, Bargmann’s group, in [28, 29, 30].
### 3.1.6. Proposition.

For any generalized temperature $\beta \in \Omega$, the integral below
\[
\mathcal{E}_{\rho_\beta}(J) = \frac{1}{P(\beta)} \int_M J(x) \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx)
\]
is convergent. This integral defines the mean value $\mathcal{E}_{\rho_\beta}(J)$ of the moment map $J$ in the Gibbs state indexed by $\beta$. Moreover, for any other continuous statistical state with a probability density $\rho_1$ with respect to the Liouville measure $\lambda_\omega$, such that $\mathcal{E}_{\rho_1}(J)$ exists and is equal to $\mathcal{E}_{\rho_\beta}(J)$, the entropy functional $s$ satisfies the inequality $s(\rho_1) \leq s(\rho_\beta)$, and the equality $s(\rho_1) = s(\rho_\beta)$ occurs if and only if $\rho_1 = \rho_\beta$.

**Proof.** — The normal convergence (which implies the usual convergence) of the integral which defines $\mathcal{E}_{\rho_\beta}(J)$ follows from Proposition 3.1.3. Let $\rho_1$ be the probability density, with respect to $\lambda_\omega$, of another continuous statistical state such that $\mathcal{E}_{\rho_1}(J)$ exists and is equal to $\mathcal{E}_{\rho_\beta}(J)$. The function, defined on $\mathbb{R}^+$,
\[
z \mapsto h(z) = \begin{cases} z \log \left( \frac{1}{z} \right) & \text{if } z > 0 \\ 0 & \text{if } z = 0 \end{cases}
\]
being convex, the straight line tangent to its graph at one of its point $(z_0, h(z_0))$ is always above this graph. Therefore, for all $z > 0$ and $z_0 > 0$, the following inequality holds:
\[
h(z) \leq h(z_0) - (1 + \log z_0)(z - z_0) = z_0 - z(1 + \log z_0).
\]
With $z = \rho_1(x)$ and $z_0 = \rho_\beta(x)$, for any $x \in M$, this inequality becomes
\[
h(\rho_1(x)) = \rho_1(x) \log \left( \frac{1}{\rho_1(x)} \right) \leq \rho_\beta(x) - (1 + \log \rho_\beta(x)) \rho_1(x).
\]
By integrating on $M$ both sides of the above inequality, we get, since $\rho_\beta$ is the probability density of the Gibbs state indexed by $\beta$,
\[
s(\rho_1) \leq 1 - 1 - \int_M \rho_1(x) \log \rho_\beta(x) \lambda_\omega(dx) = s(\rho_\beta).
\]
We have proven the inequality $s(\rho_1) \leq s(\rho_\beta)$. If $\rho_1 = \rho_\beta$, of course $s(\rho_1) = s(\rho_\beta)$. Conversely, let us now assume that $s(\rho_1) = s(\rho_\beta)$. The functions $\varphi_1$ and $\varphi$, defined on $M$, whose expressions are
\[
\varphi_1(x) = \rho_1(x) \log \left( \frac{1}{\rho_1(x)} \right), \quad \varphi(x) = \rho_\beta(x) - (1 + \log \rho_\beta(x)) \rho_1(x), \quad x \in M,
\]
are continuous, except, maybe, the function $\varphi$ at points $x$ where $\rho_\beta(x) = 0$ and $\rho_1(x) \neq 0$. For the Liouville measure $\lambda_\omega$, the subset of $M$ made of these
points is of measure 0, since $\varphi$ is integrable. The functions $\varphi$ and $\varphi_1$ satisfy the inequality $\varphi_1 \leq \varphi$, are integrable on $M$ and their integrals are equal. Their difference $\varphi - \varphi_1$, everywhere $\geq 0$ on $M$ and with an integral equal to 0, is therefore everywhere equal to 0. Therefore, for any $x \in M$,

$$\rho_1(x) \log \left( \frac{1}{\rho_1(x)} \right) = \rho_\beta(x) - (1 + \log \rho_\beta(x)) \rho_1(x).$$

(\ast)

For any $x \in M$ such that $\rho_1(x) \neq 0$, we can divide both sides of the above equality by $\rho_1(x)$. We get

$$\frac{\rho_\beta(x)}{\rho_1(x)} - \log \left( \frac{\rho_\beta(x)}{\rho_1(x)} \right) = 1.$$

The function $z \mapsto z - \log z$ reaches its minimum at only one point $z > 0$, the point $z = 1$, and its minimum is equal to 1. So for all $x \in M$ such that $\rho_1(x) > 0$, $\rho_1(x) = \rho_\beta(x)$. At points $x \in M$ such that $\rho_1(x) = 0$, equality (\ast) proves that $\rho_\beta(x) = 0$. Therefore $\rho_1 = \rho_\beta$ everywhere on $M$. \hfill \Box

3.1.7. **Proposition.** — We now assume that $\mathfrak{g}$ is the Lie algebra of a Lie group $G$ which acts on the symplectic manifold $(M,\omega)$ by a Hamiltonian action $\Phi$, and that $\varphi$ is the Lie algebra action associated to $\Phi$. The Gibbs state indexed by any generalized temperature $\beta \in \Omega$ is invariant by the restriction of the action $\Phi$ to the one-parameter subgroup $\{ \exp(\tau\beta) \mid \tau \in \mathbb{R} \}$ of $G$.

**Proof.** — The orbits of the action of this subgroup on $M$ are the integral curves of the Hamiltonian vector field which admits the function $x \mapsto \langle J(x), \beta \rangle$ as Hamiltonian. This function is therefore constant on each orbit of this one-parameter subgroup. The expression of the probability density $\rho_\beta$ of the Gibbs state indexed by $\beta$ proves that this probability density too is constant on each orbit of the action of $\{ \exp(\tau\beta) \mid \tau \in \mathbb{R} \}$.

\hfill \Box

3.2. **Thermodynamic functions.** — In this section, the map $\Phi : G \times M \to M$ is a Hamiltonian action of a Lie group $G$ on a connected symplectic manifold $(M,\omega)$ and $J : M \to \mathfrak{g}^\ast$ is a moment map of this action. It is assumed that the open subset $\Omega \subset \mathfrak{g}$ of generalized temperatures is not empty. In 3.1.2, we have defined the partition function $P$, whose expression is

$$P(\beta) = \int_M \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx), \quad \beta \in \Omega.$$

On the set $\Omega$ of generalized temperatures, we define below other thermodynamic functions whose expressions can be derived from that of $P$. 
3.2.1. Definitions. — Assumptions and notations here are those of 3.2.

1. The mean value of the moment map $J$ is the function, denoted by $E_J$, defined on $\Omega$ and taking its values in the dual vector space $g^*$ of the Lie algebra $g$, whose expression is

$$E_J(\beta) = \mathcal{E}_{\rho_\beta}(J) = \frac{1}{P(\beta)} \int_M J(x) \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx), \quad \beta \in \Omega.$$ 

2. The entropy function is the function, denoted by $S$, defined on $\Omega$ and taking its values in $\mathbb{R} \cup \{-\infty\}$, which associates, to each generalized temperature $\beta \in \Omega$, the entropy of the Gibbs state indexed by $\beta$:

$$S(\beta) = s(\rho_\beta) = \int_M \rho_\beta(x) \log \left( \frac{1}{\rho_\beta(x)} \right) \lambda_\omega(dx), \quad \beta \in \Omega.$$ 

3.2.2. Proposition. — For each generalized temperature $\beta \in \Omega$, the values at $\beta$ of the thermodynamic functions mean value of $J$ and entropy (defined in 3.2.1) are given by the formulae

$$E_J(\beta) = -\frac{1}{P(\beta)} DP(\beta) = -D(\log P)(\beta),$$

$$S(\beta) = \log P(\beta) + \langle E_J(\beta), \beta \rangle = \log P(\beta) - \langle D(\log P)(\beta), \beta \rangle.$$ 

Proof. — Proposition 3.1.3 states that the partition function $P$ is of class $C^\infty$ and that its differentials of all orders can be obtained by differentiation under the sign $\int$. Therefore

$$DP(\beta) = -\int_M J \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx) = -P(\beta) E_J(\beta),$$

which proves the indicated expresions of $E_J(\beta)$. Since for each $x \in M$, we have

$$\rho_\beta(x) = \frac{\exp(-\langle J(x), \beta \rangle)}{P(\beta)},$$

$$\rho_\beta(x) \log \frac{1}{\rho_\beta(x)} = \frac{1}{P(\beta)} \exp(-\langle J(x), \beta \rangle) \left( \langle J(x), \beta \rangle + \log P(\beta) \right).$$

By integration over $M$ of both members of this equality with respect to $\lambda_\omega$, we obtain the indicated expression of $S(\beta)$. 

3.2.3. Proposition. — The thermodynamic functions $E_J$ (mean value of $J$) and $S$ (entropy) are of class $C^\infty$ on $\Omega$. The first differential of $E_J$ is the function, defined on $\Omega$ and taking its values in the space of linear applications
of \( g \) in its dual vector space \( g^\ast \), whose expression is
\[
\langle DE_J(\beta)(X), Y \rangle = -D^2(\log P)(\beta)(X,Y)
\]
\[
= -\frac{1}{P(\beta)} \int_M \langle J(x) - E_J(\beta), X \rangle \langle J(x) - E_J(\beta), Y \rangle \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx),
\]
with \( X \) and \( Y \in g \). For each \( \beta \in \Omega \), \( DE_J(\beta) \) can be considered as a bilinear, symmetric form on \( g \).

The differential of the entropy function \( S \) at each \( \beta \in \Omega \) is an element of \( g^\ast \) whose expression is
\[
\langle DS(\beta), X \rangle = \langle DE_J(\beta)(X), \beta \rangle, \quad X \in g.
\]

Proof. — According to 3.2.2, for each \( \beta \in \Omega \), \( E_J(\beta) = -D(\log P)(\beta) \). Therefore, for all \( X \) and \( Y \in g \),
\[
\langle DE_J(\beta)(X), Y \rangle = -D^2(\log P)(\beta)(X,Y),
\]
which shows that \( DE_J(\beta) \) can be considered as a bilinear, symmetric form on \( g \). Since \( DP(\beta) \) can be obtained by differentiation under the sign \( \int \),
\[
D(\log P)(\beta) = \frac{1}{P(\beta)} DP(\beta) = -\frac{1}{P(\beta)} \int_M J(x) \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx).
\]
By a second differentiation under the sign \( \int \), we therefore obtain, for all \( X \) and \( Y \in g \),
\[
D^2(\log P)(\beta)(X,Y) = \frac{1}{P(\beta)} \int_M \langle J(x), X \rangle \langle J(x), Y \rangle \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx)
\]
\[
+ \frac{1}{(P(\beta))^2} DP(\beta)(Y) \int_M \langle J(x), X \rangle \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx).
\]
Let us replace \( DP(\beta)(Y) \) and \( \int_M \langle J(x), X \rangle \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx) \), in the right hand side of this equality, by their expressions
\[
DP(\beta)(Y) = -P(\beta) \langle E_J(\beta), Y \rangle,
\]
\[
\int_M \langle J(x), X \rangle \exp(-\langle J(x), \beta \rangle) \lambda_\omega(dx) = P(\beta) \langle E_J(\beta), X \rangle.
\]
We obtain
\[ D^2(\log P)(\beta)(X,Y) = \frac{1}{P(\beta)} \int_M \langle J(x),X \rangle \langle J(x),Y \rangle \exp(-\langle J(x),\beta \rangle) \lambda_\omega(dx) \]
\[
- \langle E_J(\beta),Y \rangle \langle E_J(\beta),X \rangle \\
= \frac{1}{P(\beta)} \int_M \langle J(x) - E_J(\beta),X \rangle \langle J(x) - E_J(\beta),Y \rangle \exp(-\langle J(x),\beta \rangle) \lambda_\omega(dx),
\]

The expression of \( \langle DE_J(\beta)(X),Y \rangle \) given in the statement follows.

By differentiation of the expression of \( S(\beta) \) given in 3.2.2, and using the equality \( DP(\beta) = -P(\beta)E_J(\beta) \), we obtain, for any \( X \in g \),
\[
\langle DS(\beta),X \rangle = \frac{1}{P(\beta)} DP(\beta)(X) + \langle E_J(\beta),X \rangle + \langle DE_J(\beta)(X),\beta \rangle \\
= \langle DE_J(\beta)(X),\beta \rangle.
\]

3.2.4. Theorem. — For all \( \beta \in \Omega \), \( X \) and \( Y \in g \), let
\[
\Gamma(\beta)(X,Y) = -\langle DE_J(\beta)(X),Y \rangle = D^2(\log P)(\beta)(X,Y).
\]
The map \( \Gamma \) so defined is a \( C^\infty \) bilinear, symmetric differential form defined on \( \Omega \) such that, for each \( \beta \in \Omega \) and \( X \in g \),
\[
\Gamma(\beta)(X,X) \geq 0.
\]
Moreover, if \( X \in g \) is such that \( x \mapsto \langle J(x),X \rangle \) is not a constant function,
\[
\Gamma(\beta)(X,X) > 0.
\]

When, in addition, the Hamiltonian action \( \Phi : G \times M \to M \) is effective (it means that for any \( X \in g \), \( X \neq 0 \), the function \( x \mapsto \langle J(x),X \rangle \) is not a constant on \( M \)), \( \Gamma \) is a Riemannian metric on \( \Omega \). Moreover, the map \( E_J : \Omega \to g^* \) is injective, its image is an open subset \( \Omega^* \) of \( g^* \), and considered as valued in \( \Omega^* \), \( E_J \) is a diffeomorphism of the set \( \Omega \) of generalized temperatures onto the open subset \( \Omega^* \) of \( g^* \).

Proof. — The first assertions follow from the the expression of \( \langle DE_J(\beta)(X),Y \rangle \) given in Proposition 3.2.3. When \( X \in g \) is such that the function \( x \mapsto \langle J(x),X \rangle \) is not a constant on \( M \), the function
\[
x \mapsto \langle J(x) - E_J(\beta),X \rangle \exp(-\langle J,\beta \rangle)
\]
is continuous, with values \( \geq 0 \) and not everywhere equal to \( 0 \) on \( M \). Its integral with respect to the Liouville measure is therefore strictly positive, which proves that \( \Gamma(\beta)(X,X) > 0 \).
When, in addition, the action $\Phi$ effective, for any $\beta \in \Omega$ and any non-zero $X \in \mathfrak{g}$, $\Gamma(\beta)(X,X) > 0$. The map $\Gamma$ is therefore an Riemannian metric on $\Omega$. For all $\beta \in \Omega$ and $Y \in \mathfrak{g}\{0\}$, we have $\langle DE_J(\beta)(Y), Y \rangle < 0$, which implies that $DE_J(\beta)$ is invertible. The map $E_J : \Omega \to \mathfrak{g}^*$ is therefore open. This map cannot take the same value at two distinct points $\beta_1$ and $\beta_2 \in \Omega$, since this would imply
\[
\langle E_J(\beta_1), (\beta_2 - \beta_1) \rangle = \langle E_J(\beta_2), (\beta_2 - \beta_1) \rangle.
\]
The real-valued function
\[
\lambda \mapsto \left\langle E_J((1 - \lambda)\beta_1 + \lambda\beta_2), (\beta_2 - \beta_1) \right\rangle,
\]
would be well defined on $[0,1]$ since $\Omega$ is convex, smooth in $]0,1[\$, and would take the same value for $\lambda = 0$ and $\lambda = 1$. Its derivative with respect to $\lambda$, whose value is $\left( DE_J(\lambda\beta_1 + (1 - \lambda)\beta_2)\right)(\beta_2 - \beta_1)$, would vanish for some $\lambda \in ]0,1[$, which would contradict the effectiveness of $\Phi$. Being open and injective, the map $E_J : \Omega \to \mathfrak{g}^*$ is a diffeomorphism of $\Omega$ onto its image $\Omega^*$, which is an open subset of $\mathfrak{g}^*$. \hfill $\square$

3.2.5. Remarks. — Theorem 3.2.4 leads to the following observations.

1. In the language of Probability theory, $-\langle DE_J(\beta)(X), X \rangle$ is the variance, in other words the square of the standard deviation of the random variable $\langle J, X \rangle$, for the probability law $\rho_{\beta \lambda \omega}$ on $M$.

2. For each generalized temperature $\beta \in \Omega$, the Gibbs state indexed by $\beta$ is the probability law on $M$, absolutely continuous with respect to the Liouville measure $\lambda_\omega$, of probability density
\[
\rho_{\beta} = \frac{1}{P(\beta)} \exp(-\langle J, \beta \rangle).
\]
The open subset $\Omega$ of $\mathfrak{g}$, in which live the generalized temperatures $\beta$ which index a family of probability laws defined on $M$, is called by statisticians a statistical manifold. The Fisher-Rao metric, so named in honour of the British statistician and genetician Ronald Aylmer Fisher (1890–1962) and the Indian statistician Calyampudi Radhakrishna Rao (born in 1920, emeritus Professor at the Indian Statistics Institute and at the Pennsylvania State University) is a Riemannian metric, defined on some statistical manifolds, which is used to evaluate the distance between probability laws. Frédéric Barbaresco [5] observed that the Riemannian metric $\Gamma$ defined by Jean-Marie Souriau on $\Omega$ is nothing else than the Fisher-Rao metric when, as indicated above, $\Omega$ is considered as a statistical manifold.
3. Under the assumptions of 3.2.4, the equality

\[ S(\beta) = \langle D(-\log P)(\beta), \beta \rangle - (-\log P)(\beta) \]

proves that each of the two functions \(-\log P : \Omega \to \mathbb{R}\) and \(S \circ E_J^{-1} : \Omega^* \to \mathbb{R}\) is the Legendre transform of the other, just as a hyper-regular Lagrangian \(L : TM \to \mathbb{R}\) and the associated Hamiltonian \(H : T^*M \to \mathbb{R}\), defined, respectively, on the tangent bundle \(TM\) and on the cotangent bundle \(T^*M\) to some smooth manifold \(M\). Here the Legendre map \(E_J : \Omega \to \Omega^*\). This map and its inverse \((E_J)^{-1} : \Omega^* \to \Omega\) are expressed by formulae similar to those which express the Legendre map \(TM \to T^*M\) and its inverse \(T^*M \to TM\) in calculus of variations,

\[ E_J = D(-\log P), \quad (E_J)^{-1} = D(S \circ E_J^{-1}). \]

The moment map \(J\) of the Hamiltonian action \(\Phi\) is not unique: it is well known that for any constant \(\mu \in \mathfrak{g}\), \(J + \mu\) is too a moment map of \(\Phi\). Proposition 3.2.6 below indicates the effect of such a change on the thermodynamic functions \(P\), \(E_J\) and \(S\).

**3.2.6. Proposition.** — Let \(\mu \in \mathfrak{g}^*\) be a constant. When the moment map \(J\) of the Hamiltonian action \(\Phi\) is replaced by \(J_1 = J + \mu\), the set \(\Omega\) of generalized temperatures does not change. The thermodynamic functions \(P\), \(E_J\) and \(S\) are replaced, respectively, by \(P_1\), \(E_{J_1}\) and \(S_1\), whose expressions are

\[ P_1(\beta) = \exp(-\langle \mu, \beta \rangle)P(\beta), \quad E_{J_1}(\beta) = E_J(\beta) + \mu, \quad S_1(\beta) = S(\beta). \]

For each \(\beta \in \Omega\), the associated Gibbs state, its probability density \(\rho_\beta\) with respect to the Liouville measure \(\lambda_\omega\) and the bilinear, symmetric form \(\Gamma\) (theorem 3.2.4) are not changed.

**Proof.** — The stated results follow from the equality

\[ \exp(-\langle J + \mu, \beta \rangle) = \exp(-\langle \mu, \beta \rangle) \exp(-\langle J, \beta \rangle). \]

3.3. Generalized temperatures and adjoint action. — As in the previous section, \(\Phi : G \times M \to M\) is a Hamiltonian action of a connected Lie group \(G\) on a connected symplectic manifold \((M, \omega)\) and \(J : M \to \mathfrak{g}^*\) is a moment map of this action. The set of generalized temperatures is assumed to be a non-empty subset \(\Omega\) of the Lie algebra \(\mathfrak{g}\). As seen in 3.2.6, \(\Omega\) does not depend on the choice of the moment map \(J\). We moreover assume that \(\Phi\) is effective, which implies (theorem 3.2.4) that \(E_J\) is a diffeomorphism of \(\Omega\) onto an open subset \(\Omega^*\) of \(\mathfrak{g}^*\), and that the bilinear, symmetric form \(\Gamma\) is a Riemannian metric on \(\Omega\). By considering the adjoint action of \(G\) on \(\Omega\), we
prove below that \( \Omega \) is a union of adjoint orbits (proposition 3.3.1) and that the Riemannian metric induced by \( \Gamma \) on each of these orbits can be expressed in terms of a symplectic cocycle of the Lie algebra \( \mathfrak{g} \) (theorem 3.3.4).

The next proposition proves that \( \Omega \) is a union of adjoint orbits and indicates the variations of the thermodynamic functions \( P, E,J \) and \( S \) on each adjoint orbit contained in \( \Omega \).

3.3.1. Proposition. — The set \( \Omega \) of generalized temperatures is a union of orbits of the adjoint action of the Lie group \( G \) on its Lie algebra \( \mathfrak{g} \). Let \( \theta : G \to \mathfrak{g}^* \) be the symplectic cocycle of \( G \) (see, for example, \([19]\)) such that, for each \( g \in G \)

\[
J \circ \Phi_g = \text{Ad}_{g^{-1}}^* J + \theta(g).
\]

For any \( \beta \in \Omega \) and any \( g \in G \), we have

\[
P(\text{Ad}_g \beta) = \exp \left( -\langle \theta(g^{-1}), \beta \rangle \right) P(\beta) = \exp \left( -\langle \text{Ad}_g^* \theta(g), \beta \rangle \right) P(\beta),
\]

\[
E_J(\text{Ad}_g \beta) = \text{Ad}_{g^{-1}}^* E_J(\beta) + \theta(g),
\]

\[
S(\text{Ad}_g \beta) = S(\beta).
\]

Proof. — Let us assume that the integral which defines \( P(\text{Ad}_g \beta) \) is convergent. We can write

\[
P(\text{Ad}_g \beta) = \int_M \exp \left( -\langle J(x), \text{Ad}_g \beta \rangle \right) \lambda_\omega(dx)
\]

\[
= \int_M \exp \left( -\langle \text{Ad}_g^* J(x), \beta \rangle \right) \lambda_\omega(dx)
\]

\[
= \int_M \exp \left( -\langle J \circ \Phi_{g^{-1}}(x) - \theta(g^{-1}), \beta \rangle \right) \lambda_\omega(dx)
\]

\[
= \exp \left( -\langle \theta(g^{-1}), \beta \rangle \right) \int_M \exp \left( -\langle J \circ \Phi_{g^{-1}}(x), \beta \rangle \right) \lambda_\omega(dx).
\]

The change of integration variable \( y = \Phi_{g^{-1}}(x) \) in the last integral leads to

\[
\int_M \exp \left( -\langle J \circ \Phi_{g^{-1}}(x), \beta \rangle \right) \lambda_\omega(dx) = \int_M \exp \left( -\langle J(y), \beta \rangle \right) \Phi_g^* \lambda_\omega(dy) = P(\beta),
\]

since \( \Phi_g^* \lambda_\omega = \lambda_\omega \), the Liouville measure being invariant by symplectomorphisms. Moreover, \( \theta(g^{-1}) = -\text{Ad}_g^* \theta(g) \) (see for example \([19]\)), so we can write

\[
P(\text{Ad}_g \beta) = \exp \left( -\langle \text{Ad}_g^* \theta(g), \beta \rangle \right) P(\beta).
\]

By reversing the above calculation step by step, we prove that the normal convergence of the integral which defines \( P(\beta) \) implies the normal convergence of the integral which defines \( P(\text{Ad}_g \beta) \). We therefore have proven that \( \Omega \) is a
union of adjoint orbits of $G$, as well as the expression of $P(\text{Ad}_g \beta)$ in terms of $P(\beta)$ and $\theta$ given in the statement.

Since $E_J(\beta) = -D(\log P)(\beta)$, $E_J(\text{Ad}_g \beta) = -D(\log P)(\text{Ad}_g \beta)$. To calculate the right hand side of this equality, we observe that for each $\delta \in g$ and each real $s$,

$$
D(\log P)(\text{Ad}_g \beta)(\delta) = \frac{d}{ds} (\log P(\text{Ad}_g(\beta + s \delta))) |_{s=0}
$$

Using the expression of $P(\text{Ad}_g \beta)$ obtained above, we have

$$
\log P(\text{Ad}_g(\beta + s \text{Ad}_g^{-1} \delta)) = -\langle \text{Ad}_g^* \theta(g), \beta + s \text{Ad}_g^{-1} \delta \rangle + \log P(\beta + s \text{Ad}_g^{-1} \delta).
$$

Taking the derivative with respect to $s$, then setting $s = 0$, we get

$$
D \log P(\text{Ad}_g \beta)(\delta) = -\langle \text{Ad}_g^* \theta(g), \text{Ad}_g^{-1} \delta \rangle + D \log P(\beta)(\text{Ad}_g^{-1} \delta)
$$

$$
= -\langle \theta(g), \delta \rangle + D \log P(\beta)(\text{Ad}_g^{-1} \delta)
$$

$$
= -\langle \theta(g) + \text{Ad}_g^{-1} E_J(\beta), \delta \rangle,
$$

where we have used the already obtained equality $D \log P(\beta) = -E_J(\beta)$.

Therefore,

$$
E_J(\text{Ad}_g \beta) = \text{Ad}_g^* E_J(\beta) + \theta(g).
$$

Finally,

$$
S(\text{Ad}_g \beta) = \log P(\text{Ad}_g \beta) - \langle D \log P(\text{Ad}_g \beta), \text{Ad}_g \beta \rangle
$$

$$
= -\langle \text{Ad}_g^* \theta(g), \beta \rangle + \log P(\beta) + \langle \text{Ad}_g^* E_J(\beta) + \theta(g), \text{Ad}_g \beta \rangle
$$

$$
= \log P(\beta) + \langle E_J(\beta), \beta \rangle
$$

$$
= S(\beta).
$$

\[ \square \]

3.3.2. Remark. — The equality

$$
E_J(\text{Ad}_g \beta) = \text{Ad}_g^* E_J(\beta) + \theta(g)
$$

states that the map $E_J : \Omega \to \Omega^*$ is equivariant with respect to the adjoint action $\Phi$ of $G$ on $g$, restricted to the open subset $\Omega$ of $g$, and its affine action $a_\theta$ on $g^*$:

$$
a_\theta(g, \xi) = \text{Ad}_g^* \xi + \theta(g), \quad g \in G, \quad \xi \in g^*,
$$

restricted to the open subset $\Omega^*$ of $g^*$. This result is not surprising, since it is well known (see, for example, [19]) that the moment map $J$ itself is equivariant with respect to the action $\Phi$ of $G$ on $M$ and its affine action $a_\theta$ on $g^*$: it states that the equivariance of $J$ implies the equivariance of its mean value.
3.3.3. Proposition. — Let $\Theta = T_\epsilon \theta : \mathfrak{g} \rightarrow \mathfrak{g}^*$ be the 1-cocycle of the Lie algebra $\mathfrak{g}$ associated to the symplectic 1-cocycle $\theta$ of the Lie group $G$ (see, for example, [19]). For any $\beta \in \Omega$ and any $X \in \mathfrak{g}$,
\[
\langle E_J(\beta), [X, \beta] \rangle = \langle \Theta(X), \beta \rangle,
\]
\[
DE_J(\beta)([X, \beta]) = -\text{ad}_X^* E_J(\beta) + \Theta(X).
\]

Proof. — Let us set $g = \exp(\tau X)$ in the expression of $P(\text{Ad}_g \beta)$ given in proposition 3.3.1, then take the derivative with respect to $\tau$ and set $\tau = 0$. Using the well known equalities $\theta(e) = 0$ and $T\epsilon \theta = \Theta$, we obtain
\[
DP(\beta)([X, \beta]) = \frac{d}{d\tau} \left( \exp(-\langle \text{Ad}_{\exp(\tau X)}^* \theta(\exp(\tau X)), \beta \rangle) P(\beta) \right) \bigg|_{\tau=0}
= -\langle \Theta(X), \beta \rangle P(\beta),
\]
which proves the first assertion, since $DP(\beta) = -P(\beta)E_J(\beta)$.

Similarly, let us set $g = \exp(\tau X)$ in the expression of $E_J(\text{Ad}_g \beta)$ given in proposition 3.3.1, then take the derivative with respect to $\tau$ and set $\tau = 0$. We obtain
\[
DE_J(\beta)([X, \beta]) = \frac{d}{d\tau} \left( \text{Ad}_{\exp(-\tau X)}^* E_J(\beta) + \theta(\exp(\tau X)) \right) \bigg|_{\tau=0}
= -\text{ad}_X^* E_J(\beta) + \Theta(X).
\]

3.3.4. Theorem. — Let us set, for each generalized temperature $\beta \in \Omega$,
\[
J_\beta = J - E_J(\beta),
\]
and, for each $g \in G$,
\[
\theta_\beta(g) = \theta(g) - E_J(\beta) + \text{Ad}_{g^{-1}}^* E_J(\beta).
\]
The map $J_\beta$ is the unique moment map of the Hamiltonian action $\Phi$ whose mean value, for the generalized temperature $\beta$, is equal to 0. The map $\theta_\beta$ is the symplectic cocycle of the Lie group $G$, cohomologous to $\theta$, associated to the moment map $J_\beta$. It depends on $\beta$ but not on the choice of $J$.

Let $\Theta_\beta : \mathfrak{g} \rightarrow \mathfrak{g}^*$ be the symplectic cocycle of the Lie algebra $\mathfrak{g}$ associated to the symplectic 1-cocycle $\theta_\beta$ of the Lie group $G$ (see, for example, [19]). Its expression is
\[
\Theta_\beta(X) = T_\epsilon \theta_\beta(X) = \Theta(X) - \text{ad}_X^* E_J(\beta).
\]
The map $\Theta_\beta$ is the unique symplectic 1-cocycle of the Lie algebra $\mathfrak{g}$ which is cohomologous to $\Theta$ and satisfies the equality
\[
\Theta_\beta(\beta) = 0.
\]
Let $X$ and $Y$ be two elements in $\mathfrak{g}$, considered as two elements of $T_{\beta}\Omega$, in other words as two vectors tangent to $\Omega$ at its point $\beta$. Let us moreover assume that $X$ is tangent to the adjoint orbit of $\beta$ at its point $\beta$. There exists $X_1 \in \mathfrak{g}$ such that $X = [\beta, X_1]$. When evaluated on the pair of tangent vectors $(X, Y)$, the Riemannian metric $\Gamma$ can be expressed as

$$\Gamma(\beta)(X, Y) = \langle \Theta_{\beta}(X_1), Y \rangle.$$  

If $Y$ too is tangent to the adjoint orbit of $\beta$ at its point $\beta$, there exists $Y_1 \in \mathfrak{g}$ such that $Y = [\beta, Y_1]$, and we have the two equalities, which express the Riemannian metric induced by $\Gamma$ on the adjoint orbit of $\beta$,

$$\Gamma(\beta)(X, Y) = \langle \Theta_{\beta}(X_1), Y \rangle = \langle \Theta_{\beta}(Y_1), X \rangle. $$

Proof. — Since $\Theta$, being a symplectic cocycle, is skew-symmetric, we have, for each $X \in \mathfrak{g}$, $\langle \Theta(\beta), X \rangle = -\langle \Theta(X), \beta \rangle$. Using the equalities proven in 3.3.3, we obtain

$$\langle \Theta_{\beta}(\beta), X \rangle = \langle \Theta(\beta), X \rangle - \langle \text{ad}_{\beta}^* E_J(\beta), X \rangle = -\langle \Theta(X), \beta \rangle - \langle E_J(\beta), [\beta, X] \rangle = -\langle E_J(\beta), [X, \beta] \rangle - \langle E_J(\beta), [\beta, X] \rangle = 0.$$  

Other statements about $J_{\beta}, \theta_{\beta}$ and $\Theta_{\beta}$ easily follow from well known properties of moment maps of Hamiltonian actions (see for example [19]).

Using theorem 3.2.4 and proposition 3.3.3, we obtain, for all $\beta \in \Omega, X_1$ and $Y \in \mathfrak{g}$, with $X = [X_1, \beta]$,

$$\Gamma(\beta)([X_1, \beta], Y) = -\langle DE_J(\beta)([X_1, \beta]), Y \rangle = \langle \text{ad}_{[X_1, \beta]}^* E_J(\beta) + \Theta(X_1), Y \rangle.$$  

According to proposition 3.2.6, the bilinear form $\Gamma$ does not depend on the choice of the moment map $J$, so we can replace $J$ by $J_{\beta}$ in the right hand side of the above equality. Of course we have to replace too $E_J$ by $E_J_{\beta}$ and $\Theta$ by $\Theta_{\beta}$. The map $J_{\beta}$ was chosen so that $E_{J_{\beta}}(\beta) = 0$, so we obtain

$$\Gamma(\beta)([X_1, \beta], Y) = \langle \Theta_{\beta}(X_1), Y \rangle.$$  

When we both have $X = [X_1, \beta]$ and $Y = [Y_1, \beta]$, with $X_1$ and $Y_1 \in \mathfrak{g}$, we can exchange the parts played by $X$ and $Y$ and write

$$\Gamma(\beta)(X, Y) = \Gamma(\beta)([X_1, \beta], [Y_1, \beta]) = \Gamma(\beta)([Y_1, \beta], [X_1, \beta]) = \Gamma(\beta)([Y_1, \beta], X) = \langle \Theta_{\beta}(Y_1), X \rangle.$$  

$\square$
4. Examples of Gibbs states

This section describes several examples of Gibbs states and the associated thermodynamic functions. It begins with a subsection (4.1) in which some properties of oriented three-dimensional Euclidean or pseudo-Euclidean vector spaces are recalled. A remarkable isomorphism of such a vector space onto the Lie algebra of its group of symmetries is defined (4.1.1). This isomorphism, which can be expressed in terms of the Hodge star operator (4.1.2), is well known and often used in mechanics when the considered vector space is properly Euclidean, maybe a little less well known when it is pseudo-Euclidean. With its use, the considered vector space can be endowed both with a Lie algebra structure and a Lie-Poisson structure (4.1.3). The coadjoint orbits of its group of symmetries can be considered as submanifolds of this vector space (4.1.4), a property used in the following subsection (4.2) for the determination of Gibbs states on several two-dimensional symplectic manifolds: the two-dimensional sphere (4.2.1), the two-dimensional pseudo-sphere (4.2.2), the Poincaré disk (4.2.3) and the Poincaré half-plane (4.2.4). In 4.2.5, it is proven that on a two-dimensional symplectic vector space, there is no Gibbs state for the action of the linear symplectic group. Finally, the Gibbs states and the associated thermodynamic functions for the action, on an Euclidean affine space, of the group of its displacements, is determined in 4.2.6.

4.1. Three-dimensional oriented vector spaces. — In what follows, $\zeta$ is a real integer whose value is either $+1$ or $-1$, and $F$ is a three-dimensional real vector space endowed with a scalar product $F \times F \to \mathbb{R}$, denoted by $(v, w) \mapsto v \cdot w$, with $v$ and $w \in F$, whose signature is $(+, +, +)$ when $\zeta = 1$ and $(+, +, -)$ when $\zeta = -1$. This scalar product is Euclidean when $\zeta = 1$ and pseudo-Euclidean when $\zeta = -1$. A basis $(e_x, e_y, e_z)$ of $F$ is said to be orthonormal when

$$e_x \cdot e_x = e_y \cdot e_y = 1, \ e_z \cdot e_z = \zeta, \ e_x \cdot e_y = e_y \cdot e_z = e_z \cdot e_x = 0.$$ 

When $\zeta = -1$, the vector space $F$ is called a three-dimensional Minkowski vector space. A non-zero element $v \in F$ is said to be space-like when $v \cdot v > 0$, time-like when $v \cdot v < 0$ and light-like when $v \cdot v = 0$. The subset of $F$ made of non-zero time-like or light-like elements has two connected components. A temporal orientation of $F$ is the choice of one of these two connected components, whose elements are said to be directed towards the future.

Both when $\zeta = 1$ and when $\zeta = -1$, we will assume in what follows that an orientation of $F$ in the usual sense is chosen, and when $\zeta = -1$, we will...
assume that a temporal orientation of F is chosen too. The orthonormal bases of F used will always be chosen positively oriented and, when \( \zeta = -1 \), their third element \( e_z \) will be chosen time-like and directed towards the future. Such bases of F will be called admissible bases.

We denote by \( G \) the subset of \( \text{GL}(F) \) made of linear automorphisms \( g \) of F which transform any admissible basis \((e_x, e_y, e_z)\) of F into an admissible basis \((g(e_x), g(e_y), g(e_z))\). Elements \( g \) of \( G \) preserve the scalar product in F, i.e., they are such that, for any pair \((v, w) \in F \times F\),

\[
g(v) \cdot g(w) = v \cdot w.
\]

Moreover, they preserve the orientation of F and, when \( \zeta = -1 \), its temporal orientation. The subset \( G \) of \( \text{GL}(F) \) is the group of symmetries of F, endowed with its scalar product, its orientation and, when \( \zeta = -1 \), its temporal orientation. It is a connected Lie group isomorphic to the rotation group \( \text{SO}(3) \) when \( \zeta = 1 \), and to the restricted three-dimensional Lorentz group \( \text{SO}(2,1) \) when \( \zeta = -1 \). Its Lie algebra, which will be denoted by \( \mathfrak{g} \), is therefore isomorphic to \( \text{so}(3) \) when \( \zeta = 1 \), and to \( \text{so}(2,1) \) when \( \zeta = -1 \).

Some useful properties of the vector space F, of its symmetry group G and of the Lie algebra \( \mathfrak{g} \) are recalled below. The interested reader will find their detailed proofs in [20] or, for most of them, in the very nice book [25].

### 4.1.1. A remarkable Lie algebras isomorphism

Let \((e_x, e_y, e_z)\) be an admissible basis of F, in the sense indicated in 4.1. For any triple \((a,b,c) \in \mathbb{R}^3\), let \( j(ae_x + be_y + ce_z) \) be the linear endomorphism of F whose matrix, in the basis \((e_x, e_y, e_z)\), is

\[
\text{matrix of } j( ae_x + be_y + ce_z ) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -\zeta b & \zeta a & 0 \end{pmatrix}.
\]

The map \( j \) does not depend on the admissible basis of F used for its definition. This property follows from the fact that \( j \) can be expressed in terms of the Hodge star operator, as explained below in 4.1.2. It is linear and injective, and its image is the Lie algebra \( \mathfrak{g} \), considered as a vector subspace of the vector space \( \mathcal{L}(F, F) \) of linear endomorphisms of F. There exists a unique bilinear and skew-symmetric map, defined on \( F \times F \) and with values in F, denoted by \((v, w) \mapsto v \times w\), such that, for all \( v \) and \( w \in F\),

\[
j(v \times w) = [j(v), j(w)] = j(v) \circ j(w) - j(w) \circ j(v).
\]
The bilinear map \((v, w) \mapsto v \times w\) will be called the **cross product** on \(F\). The map \(j : F \to \mathfrak{g}\) is a Lie algebras isomorphism of \(F\) (endowed with the cross product as composition law) onto the Lie algebra \(\mathfrak{g}\), whose composition law is the commutator of endomorphisms. Its transpose \(j^T : \mathfrak{g}^* \to F^*\), defined by the equality
\[
\langle j^T(\xi), v \rangle = \langle \xi, j(v) \rangle, \quad \xi \in \mathfrak{g}^*, \; v \in F,
\]
is therefore an isomorphism of the dual vector space \(\mathfrak{g}^*\) of the Lie algebra \(\mathfrak{g}\) onto the dual vector space \(F^*\) of \(F\).

When \(\zeta = 1\), the cross product is the well known **cross product** \((v, w) \mapsto v \times w\) on the Euclidean oriented three-dimensional vector space \(F\), and the map \(j : F \to \mathfrak{g} \equiv \mathfrak{so}(3)\) is the isomorphism of \(F\) onto the Lie algebra \(\mathfrak{g} \equiv \mathfrak{so}(3)\) of its Lie group of symmetries \(G \equiv \text{SO}(3)\), very often used in mechanics (see for example [33]). These remarkable properties of oriented Euclidean three-dimensional vector spaces therefore still hold for oriented pseudo-Euclidean three-dimensional vector spaces, the usual cross product being replaced with the cross product defined above.

Both when \(\zeta = 1\) and when \(\zeta = -1\), for all \(g \in G\), \(v\) and \(w \in F\),
\[
j(g(v)) = \text{Ad}_g(j(v)), \quad g(v) \times g(w) = g(v \times w).
\]
The first above equality expresses the fact that the map \(j\) is equivariant with respect to the natural action of \(G\) on \(F\) and its adjoint action on its Lie algebra \(\mathfrak{g}\). The second expresses the fact that the action of the group of symmetries \(G\) preserves the cross product.

We denote by \(\text{scal}\) the linear map defined by the equality
\[
\langle \text{scal}(u), v \rangle = u \cdot v \quad \text{for all} \; u \text{ and } v \in F,
\]
where, in the left hand side, \(\langle \text{scal}(u), v \rangle\) denotes the pairing by duality of \(\text{scal}(u) \in F^*\) with \(v \in F\). The map \(\text{scal}\) is an isomorphism of \(F\) onto its dual vector space \(F^*\), which satisfies, for any admissible basis \((e_x, e_y, e_z)\) of \(F\),
\[
\text{scal}(e_x) = \varepsilon_x, \quad \text{scal}(e_y) = \varepsilon_y, \quad \text{scal}(e_z) = \zeta \varepsilon_z,
\]
where \((\varepsilon_x, \varepsilon_y, \varepsilon_z)\) is the basis of \(F^*\) dual of the basis \((e_x, e_y, e_z)\) of \(F\).

The isomorphism \(\text{scal} : F \to F^*\) satisfies, for all \(g \in G\) and \(v \in F\),
\[
\text{scal}(g(v)) = (g^{-1})^T(\text{scal}\; v),
\]
where \((g^{-1})^T : F^* \to F^*\) is the linear automorphism of \(F^*\) transpose of the linear automorphism \(g^{-1}\) of \(F\). This equality expresses the fact that \(\text{scal}\) is
equivariant with respect to the natural action of $G$ on $F$ and its contragredient action on the left on $F^*$, $(g, \eta) \mapsto (g^{-1})^T(\eta)$, with $g \in G$, $\eta \in F^*$.

Therefore the map $(j^{-1})^T \circ \text{scal} : F \to g^*$ is a linear isomorphism which satisfies, for all $g \in G$ and $v \in F$,

$$(j^{-1})^T \circ \text{scal}(g(v)) = \text{Ad}_{g^{-1}}^*((j^{-1})^T \circ \text{scal}(v)),$$

which expresses the fact that the isomorphism $(j^{-1})^T \circ \text{scal}$ is equivariant with respect to the natural action of $G$ on $F$ and its coadjoint action on the left on $g^*$, $(g, \xi) \mapsto \text{Ad}_{g^{-1}}^*\xi$, with $g \in G$, $\xi \in g$.

In what follows the vector space $F$ will be identified either with the Lie algebra $g$ by means of the isomorphism $j$, or with the dual vector space $g^*$ by means of the isomorphism $(j^{-1})^T \circ \text{scal}$. We will write simply $F \equiv g$ when $F$ is identified with $g$ and $F \equiv g^*$ when it is identified with $g^*$, without writing explicitly the isomorphism used for this identification. The natural action of $G$ on $F$ will therefore be identified with its adjoint action on $g$ when $F \equiv g$ and with its coadjoint action on the left on $g^*$ when $F \equiv g^*$.

4.1.2. Expression of the map $j$ in terms of the Hodge star operator

For any oriented $n$-dimensional real vector space endowed with a nondegenerate scalar product with any signature, the Hodge star operator, introduced by the British mathematician W. V. D. Hodge (1903–1975) is a linear automorphism of the vector space $\bigwedge V = \bigoplus_{k=0}^{n} \bigwedge^k V$ which, for each integer $k$ satisfying $0 \leq k \leq n$, maps $\bigwedge^k V$ onto $\bigwedge^{n-k} V$, with, by convention, $\bigwedge^0 V = \mathbb{R}$ (see for example [37] or [11], page 281). For the three-dimensional vector space $F$ considered here, the Hodge star operator satisfies, for any admissible basis $(e_x, e_y, e_z)$ of $F$, the following equalities:

$$\ast(1) = e_x \wedge e_y \wedge e_z,$$

and conversely $\ast(e_x \wedge e_y \wedge e_z) = \zeta$,

$$\ast(e_x) = e_y \wedge e_z,$$

and conversely $\ast(e_y \wedge e_z) = \zeta e_x$,

$$\ast(e_y) = e_z \wedge e_x,$$

and conversely $\ast(e_z \wedge e_x) = \zeta e_y$,

$$\ast(e_z) = \zeta e_x \wedge e_y,$$

and conversely $\ast(e_x \wedge e_y) = e_z$.

By using these formulae, one easily can check that the isomorphism $j : F \to g$ is expressed in terms of the Hodge star operator as follows. For any triple $(a, b, c) \in \mathbb{R}^3$,

$$j(ae_x + be_y + ce_z) = \ast(\zeta ae_x + \zeta be_y + \zeta ce_z).$$

This result immediately implies that the isomorphism $j$ does not depend on the choice of the admissible basis used for its definition. When I first introduced $j$
when $\zeta = -1$ in [20], I was not aware of its expression in terms of the Hodge star operator. With a better choice of conventions for the definition of $j$, its expression in terms of the Hodge star operator could be made more natural.

4.1.3. The pseudo-Riemannian or Riemannian metric, the Lie algebra and the Lie-Poisson structures of $F$

Since, as explained at the end of 4.1.1, we have both $F \equiv g$ and $F \equiv g^*$, the vector space $F$ is endowed with a Lie algebra structure for which its identification with $g$ is a Lie algebra isomorphism, and with a Lie-Poisson structure for which its identification with $g^*$ is a Poisson diffeomorphism. Let $(e_x, e_y, e_z)$ be an admissible basis of $F$, and let $x, y$ and $z$ be the coordinate functions on $F$ in this admissible basis.

As seen in 4.1.1, the composition law of the Lie algebra structure of $F$ is the cross product $(v, w) \mapsto v \times w$. The non-zero brackets of ordered pairs of elements of the considered admissible basis are

$$e_x \times e_y = -e_y \times e_x = \zeta e_z, \quad e_y \times e_z = -e_z \times e_y = e_x, \quad e_z \times e_x = -e_x \times e_z = e_y.$$ 

For the Lie-Poisson structure of $F$, the non-zero brackets of ordered pairs of coordinate functions are

$$\{x, y\} = -\{y, x\} = z, \quad \{y, z\} = -\{z, y\} = \zeta x, \quad \{z, x\} = -\{x, z\} = \zeta y,$$

and the expression of the Poisson bivector $\Lambda_F$, in these coordinates, is

$$\Lambda_F(x, y, z) = z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \zeta x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \zeta y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.$$ 

Still in the coordinates functions considered, the expression of the pseudo-Riemannian or Riemannian metric on $F$ determined by its scalar product is

$$d\mathbf{s}_F^2(x, y, z) = dx^2 + dy^2 + \zeta dz^2.$$ 

4.1.4. The coadjoint orbits of $G$ as submanifolds of $F$

Since $F \equiv g^*$, the coadjoint orbits of $G$ can be considered as submanifolds of $F$. So considered they are the connected components of submanifolds of $F$ defined as $\{v \in F | v \cdot v = \text{Constant}\}$, with any possible Constant $\in \mathbb{R}$. In other words, with the coordinate functions $x, y$ and $z$ in an admissible basis $(e_x, e_y, e_z)$ of $F$, coadjoint orbits are connected components of submanifolds of $F$ determined by an equation $x^2 + y^2 - \zeta z^2 = \text{Constant}$, for some Constant $\in \mathbb{R}$. The singleton $\{0\}$, whose unique element is the origin of $F$, is a zero-dimensional coadjoint orbit. All other coadjoint orbits are two-dimensional.

Let $O$ be any two-dimensional coadjoint orbit. On suitably chosen open subsets of $O$, one can use as coordinates two of the three coordinate functions
x, y and z associated with an admissible basis \((e_x, e_y, e_z)\) of \(F\), the third coordinate being on the chosen subset a smooth function of the other two coordinates. Another possible choice of coordinates, which seems the most convenient, is made of the third coordinate \(z\) and the angular coordinate \(\varphi\), defined by the equalities

\[
x = \sqrt{x^2 + y^2} \cos \varphi, \quad y = \sqrt{x^2 + y^2} \sin \varphi.
\]

The symplectic form \(\omega_O\) of the coadjoint orbit \(O\) admits the four equivalent expressions in terms of these coordinates:

\[
\omega_O = \frac{1}{z(x,y)} \, dx \wedge dy = \frac{\zeta}{x(y,z)} \, dy \wedge dz = \frac{\zeta}{y(z,x)} \, dz \wedge dx = \zeta \, d\varphi \wedge dz.
\]

Each of the first three expressions of \(\omega_O\) is valid on open subsets of \(O\) on which the coordinate considered as a smooth function of the other two coordinates is non-zero: \(z(x,y)\) for the first, \(x(y,z)\) for the second and \(y(z,x)\) for the third expression. The fourth expression is valid on the dense open subset of \(O\) on which the angular coordinate \(\varphi\) can be locally defined, i.e., on the complementary subset of the set of points in \(O\) where both \(x = 0\) and \(y = 0\). When \(\zeta = 1\) this occurs only at two points of each two-dimensional coadjoint orbit. When \(\zeta = -1\), it occurs nowhere on some two-dimensional coadjoint orbits (the one sheeted hyperboloids denoted below by \(H_R\) and the light cones with their apex removed denoted below by \(C^+\) and \(C^-\)), and at a single point for other coadjoint orbits (the pseudo-spheres denoted below by \(P^+_R\) and \(P^-_R\)).

For this reason the coordinate system made of \(z\) and \(\varphi\) is the most convenient for the determination of Gibbs states. With these coordinates, the expression of the Liouville measure \(\lambda_{\omega_O}\) is

\[
\lambda_{\omega_O}(dv) = dz \, d\varphi, \quad v \in O \text{ with coordinates } (z, \varphi).
\]

When \(\zeta = 1\), all two-dimensional coadjoint orbits are spheres centered on the origin 0 of \(F\). Their radius can be any real \(R > 0\). We will denote by \(S_R\) the sphere of radius \(R\) centered on 0. It should be observed that the symplectic form on the coadjoint orbit \(\omega_{S_R}\) is not the area form on this sphere, since it is proportional to \(R\), not to \(R^2\). The area form of \(S_R\) is \(R \omega_{S_R}\).

When \(\zeta = -1\), there are three kinds of two-dimensional coadjoint orbits, described below.

— The orbits, denoted by \(P^+_R\) and \(P^-_R\), whose respective equation is

\[
z = \sqrt{R^2 + x^2 + y^2} \text{ for } P^+_R \text{ and } z = -\sqrt{R^2 + x^2 + y^2} \text{ for } P^-_R, \quad \text{with } R > 0.
\]
They are called *pseudo-spheres* of radius $R$. Each one is a sheet of a two-sheeted two-dimensional hyperboloid with the $z$ axis as revolution axis. They are said to be *space-like* submanifolds of $\mathbf{F}$, since all their tangent vectors are space-like vectors.

— The orbits, denoted by $H_R$, defined by the equation

$$x^2 + y^2 = z^2 + R^2 , \quad \text{wit } R > 0.$$ 

Each of these orbits is a single-shetted hyperboloid with the $z$ axis as revolution axis. The tangent space at any point to such an orbit is a two-dimensional Minkowski vector space.

— The two orbits, denoted by $C^+$ and $C^-$, defined respectively by

$$z^2 = x^2 + y^2 \text{ and } z > 0 , \quad z^2 = x^2 + y^2 \text{ and } z < 0 .$$

They are the cones in $\mathbf{F}$ (without their apex, the origin 0 of $\mathbf{F}$), made of light-like vectors directed, respectively, towards the future and towards the past.

### 4.2. Gibbs states on some symplectic manifolds

In this subsection assumptions and notations are those of 4.1.

#### 4.2.1. Gibbs states on two-dimensional spheres

We assume here that $\zeta = 1$. The Lie group $G$ is therefore isomorphic to $\text{SO}(3)$ and its Lie algebra $\mathfrak{g}$ is isomorphic to $\text{so}(3)$. Let us consider the sphere $S_R$ of radius $R > 0$ centered on the origin 0 of the vector space $\mathbf{F} \equiv \mathfrak{g}^*$ (identified with the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, as explained at the end of 4.1.1). This sphere is a coadjoint orbit, and the moment map of the Hamiltonian action of $G$ on it is its canonical injection into $\mathbf{F} \equiv \mathfrak{g}^*$. Let $\beta \in \mathfrak{g} \equiv \mathbf{F}$. Let us choose an admissible basis $(e_x, e_y, e_z)$ of $\mathbf{F}$ such that $e_z$ and $\beta$ are parallel and directed in the same direction. Therefore we have $\beta = \beta e_z$, with $\beta \geq 0$.

For each $r = xe_x + ye_y + ze_z \in S_R$, we have

$$\langle J(r), \beta \rangle = r \cdot \beta = \beta z .$$

As explained in 4.1.4, on the dense open subset of $S_R$ complementary to the union of the two poles $\{-Re_z, Re_z\}$, we can use the coordinate system $(z, \varphi)$
and write
\[ \int_{S_R} \exp(-\langle J(r), \beta \rangle) \lambda_{\omega_{S_R}}(dr) = \int_0^{2\pi} \left( \int_{-R}^R \exp(-\beta z) dz \right) d\varphi \]
\[ = \begin{cases} 
4\pi R & \text{if } \beta = 0, \\
\frac{4\pi \sinh(R\beta)}{\beta} & \text{if } \beta > 0.
\end{cases} \]

Since \( S_R \) is compact, the above integral is always normally convergent. The open subset \( \Omega \) of generalized temperatures is the whole Lie algebra \( \mathfrak{g} \). The partition function \( P \) and the probability density \( \rho_\beta \) of the Gibbs state indexed by \( \beta \) are expressed as
\[ P(\beta) = \begin{cases} 
\frac{4\pi \sinh(R\beta)}{\beta} & \text{if } \beta > 0, \\
4\pi R & \text{if } \beta = 0,
\end{cases} \]
\[ \rho_\beta(r) = \begin{cases} 
\frac{\beta \exp(-\beta z)}{4\pi \sinh(R\beta)} & \text{if } \beta > 0, \\
\frac{1}{4\pi R} & \text{if } \beta = 0,
\end{cases} \quad \text{with } r = xe_x + ye_y + ze_z \in S_R. \]

When \( \beta > 0 \), the thermodynamic functions mean value of \( J \) and entropy are expressed as
\[ E_J(\beta) = \frac{1 - R\beta \coth(R\beta)}{\beta^2} \beta, \]
\[ S(\beta) = 1 + \log \left( \frac{4\pi \sinh(R\beta)}{\beta} \right) - R\beta \coth(R\beta). \]

4.2.2. **Gibbs states on two-dimensional pseudo-spheres and other \text{SO}(2,1) coadjoint orbits**

We assume here that \( \zeta = -1 \). The Lie group \( G \) is therefore isomorphic to \( \text{SO}(2,1) \) and its Lie algebra \( \mathfrak{g} \) is isomorphic to \( \mathfrak{so}(2,1) \). For each coadjoint orbit \( \mathcal{O} \) of \( G \), we must determine whether the integral, which defines a function of the variable \( \beta \in \mathfrak{g} \),
\[ \int_{\mathcal{O}} \exp(-\langle J(r), \beta \rangle) \lambda_{\omega_{\mathcal{O}}} dr \]
is normally convergent.

Let us first assume that \( \mathcal{O} \) is the pseudo-sphere \( P_R^+ \) defined in 4.1.4, for some real number \( R > 0 \). When the vector \( \beta \in \mathfrak{F} \equiv \mathfrak{g} \) is time-like, we choose an admissible basis \( (e_x, e_y, e_z) \) of \( \mathfrak{F} \) such that \( e_z \) and \( \beta \) are parallel.
We can therefore write $\beta = \beta e_z$, with $\beta \in \mathbb{R}$. We have now, for each $r = xe_x + ye_y + ze_z \in \mathcal{O} \subset \mathbf{F} \equiv \mathbf{F}^*$,

$$\langle J(r), \beta \rangle = r \cdot \beta = \zeta z \beta = -z \beta,$$

since $\zeta = -1$. We can choose $(z, \varphi)$ as coordinates on the dense open subset of $\mathcal{O}$ complementary to the singleton \{Re_z\}, so the above integral (*) becomes

$$\int_{0}^{2\pi} \left( \int_{R}^{+\infty} \exp(\beta z) \, dz \right) \, d\varphi.$$

This integral is normally convergent if and only if $\beta < 0$, in other words if and only if the time-like vector $\beta$ is directed towards the past.

Still with $\mathcal{O} = P^+_R$, let us now assume that the vector $\beta$ is space-like. We choose an admissible basis $(e_x, e_y, e_z)$ of $\mathbf{F}$ such that $e_x$ and $\beta$ are parallel. We therefore have $\beta = \beta e_x$, with $\beta \in \mathbb{R}$, $\beta \neq 0$. With $(z, \varphi)$ as coordinate system, the above integral (*) is expressed as

$$\int_{0}^{2\pi} \left( \int_{R}^{+\infty} \exp\left( -\beta \cos \varphi \sqrt{z^2 - R^2} \right) \, dz \right) \, d\varphi.$$

This integral is always divergent, as well when $\beta < 0$ as when $\beta > 0$, since $-\beta \cos \varphi > 0$ for many values of $\varphi$, using the fact that for $z > 0$ large enough $\sqrt{z^2 - R^2} \equiv z$.

The subset $\Omega$ of $\mathbf{F} \equiv \mathfrak{g}$ of generalized temperatures contains all time-like vectors in $\mathbf{F}$ directed towards the past, no time-like vector directed towards the future and no space-like vector. Since it is open, it cannot contain the origin of $\mathbf{F}$, nor light-like vectors. Therefore $\Omega$ is exactly the subset of $\mathbf{F}$ made of time-like vectors directed towards the past. The partition function $P$ and the probability density $\rho_\beta$ associated to a time-like vector $\beta$ directed towards the past are expressed as

$$P(\beta) = \frac{2\pi}{\|\beta\|} \exp(-\|\beta\| R), \quad \beta \in \mathbf{F}, \ \beta \text{ time-like directed towards the past},$$

$$\rho_\beta(\mathbf{r}) = \frac{\|\beta\| \exp\left( -\|\beta\| (z(\mathbf{r}) - R) \right)}{2\pi \|\beta\|^{z(\mathbf{r}) - R}}, \quad \mathbf{r} \in P^+_R,$$

where we have set $\|\beta\| = \sqrt{-\beta \cdot \beta}$, since $\beta \cdot \beta < 0$.

The thermodynamic functions mean value of $J$ and entropy are

$$E_J(\beta) = -\frac{1 + R\|\beta\|}{\|\beta\|^2} \beta,$$

$$S(\beta) = 1 + \log \frac{2\pi}{\|\beta\|}.$$
Similarly, one can prove that on the pseudo-sphere $P_R^-$, the open subset $\Omega$ of generalized temperatures is the subset of $F$ made of time-like vectors directed towards the future, and that the probability density of Gibbs states and the corresponding thermodynamic functions are given by the same formulae as those indicated above, of course with the appropriate sign changes.

By similar calculations, one can prove that on the other two-dimensional coadjoint orbits of $G$ denoted by $H_R$, $C^+$ and $C^-$, there are no Gibbs states, since on these orbits the subset $\Omega$ of generalized temperatures is empty.

4.2.3. Gibbs states on the Poincaré disk. — Assumptions and notations here are still those of 4.1, with $\zeta = -1$. The choice of any admissible basis $(e_x, e_y, e_z)$ of $F$ determines, for each $R > 0$, a diffeomorphism $\psi_R$ of the pseudo-sphere $P_R^+$ onto the Poincaré disk $D_P$, subset of the complex plane $\mathbb{C}$ whose elements $w$ satisfy $|w| < 1$. Its expression is

$$\psi_R(r) = \frac{x + iy}{R + \sqrt{R^2 + x^2 + y^2}}, \quad r = x e_x + y e_y + \sqrt{R^2 + x^2 + y^2} e_z \in P_R^+.$$ 

It is composed [20] of the stereographic projection of the pseudo-sphere $P_R^+$ on the two-dimensional vector subspace of $F$ generated by $e_x$ and $e_y$,

$$r = x e_x + y e_y + \sqrt{R^2 + x^2 + y^2} e_z \mapsto \frac{R}{R + \sqrt{R^2 + x^2 + y^2}} (x e_x + y e_y)$$

with the map

$$(u e_x + v e_y) \mapsto w = \frac{u + iv}{R}.$$ 

The pseudo-sphere $P_R^+$ is endowed both with the Riemannian metric induced by that of $F$, and with its symplectic form of coadjoint orbit of the Lie group $G \equiv SO(2,1)$ ($F$ being identified with the dual vector space $g^*$ of the Lie algebra $g \equiv so(2,1)$). The Poincaré disk $D_P$ is therefore endowed with a Riemannian metric $ds_{D_P}^2$ and with a symplectic form $\omega_{D_P}$ for which the map $\psi_R$ is both an isometry and a symplectomorphism. Their expressions are

$$ds_{D_P}^2(w) = \frac{4R^2}{(1 - |w|^2)^2} dw d\overline{w} = \frac{4R^2}{(1 - |w|^2)^2} (dw_r^2 + dw_{im}^2),$$

$$\omega_{D_P}(w) = \frac{2iR}{(1 - |w|^2)^2} dw \wedge d\overline{w} = \frac{4R}{(1 - |w|^2)^2} dw_r \wedge dw_{im},$$

$w_r$ and $w_{im}$ being the real and the imaginary parts of $w = w_r + iw_{im}$, respectively. In this expression, the choice of the real number $R > 0$ plays the part of the choice of a unit of length on the Poincaré disk $D_P$. 
Moreover, the Lie group $G \equiv \text{SO}(2, 1)$ acts on the symplectic manifold $(D_P, \omega_{D_P})$ by a Hamiltonian action for which the diffeomorphism $\psi_R$ is equivariant, $P^+_R$ being identified with a coadjoint orbit on which $G$ acts by its coadjoint action on the left. The moment map $J_{D_P}$ of this Hamiltonian action of $G$ on the Poincaré disk is the inverse $\psi_R^{-1}$ of $\psi_R$, considered of course as a map defined on $D_P$ and taking its values in $\mathbb{F} \equiv \mathfrak{g}^*$. Its expression is

$$J_{D_P}(w) = \frac{R}{1 - |w|^2} (2w_x e_x + 2w_{im} e_y + (1 + |w|^2)e_z).$$

In $P^+_R$, the open subset $\Omega$ of generalized temperatures, determined in 4.2.2, is the set of time-like vectors directed towards the past. Let $\beta$ be one of its elements. Let us choose the admissible basis $(e_x, e_y, e_z)$ such that $\beta = \beta e_z$, with $\beta \in \mathbb{R}$, $\beta < 0$. By using the symplectomorphism $\psi_R$, the Gibbs state on $D_P$ indexed by the generalized temperature $\beta$ can be easily deduced from the corresponding Gibbs state on $P^+_R$. We have indeed

$$\rho_\beta(r)drd\phi = \frac{2R|\beta||w|}{\pi(1 - |w|^2)^2} \exp\left(-\frac{2R|\beta||w|^2}{1 - |w|^2}\right) d|w|d\phi, \quad r \in P^+_R,$$

where we have set

$$|w| = \sqrt{w_x^2 + w_{im}^2}, \quad \text{therefore} \quad w = w_x + iw_{im} = |w| \exp(i\phi).$$

The probability density of the Gibbs state indexed by $\beta$ on the Poincaré disk $D_P$, with respect to the measure $d|w|d\phi$ associated to the polar coordinate system $(|w|, \phi)$, is therefore

$$\rho_\beta(|w|, \phi) = \frac{2R|\beta||w|}{\pi(1 - |w|^2)^2} \exp\left(-\frac{2R|\beta||w|^2}{1 - |w|^2}\right).$$

The associated thermodynamic functions (mean value of the moment map $E_J$ and entropy $S$) are the functions of the generalized temperature $\beta$ whose expressions are given in 4.2.2.

Instead of the Lie group $G \equiv \text{SO}(2, 1)$, the Lie group $\text{SU}(1, 1)$ is very often used as group of symmetries of the Poincaré disk $D_P$. It is the group of complex $2 \times 2$ matrices which can be written as

$$A = \begin{pmatrix} a & b \\ b & \overline{a} \end{pmatrix}, \quad \text{with} \quad a \text{ and } b \in \mathbb{C}, \quad |a|^2 - |b|^2 = aa - \overline{b}b = 1. \quad (*)$$

This group acts on the Poincaré disk $D_P$ by Möbius transformations, so called in honour of the German mathematician August Ferdinand Möbius (1790–1868). We recall that the Möbius transformation determined by a complex...
2 \times 2 matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), with \( a, b, c \) and \( d \in \mathbb{C} \) satisfying \( ad - bc \neq 0 \), is the map \( U_A : \mathbb{C} \to \mathbb{C} \), with \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \),

\[
U_A(w) = \begin{cases} 
aw + b \\
\frac{cw + d}{w} & \text{if } w \in \mathbb{C} \text{ and } cw + d \neq 0, \\
\infty & \text{if } w \in \mathbb{C} \text{ and } cw + d = 0, \\
\frac{a}{c} & \text{if } w = \infty \text{ and } c \neq 0, \\
\infty & \text{if } w = \infty \text{ and } c = 0.
\end{cases}
\]

The Möbius transformations \( U_A \) and \( U_{A'} \) determined by the two matrices \( A \) and \( A' \) are equal if and only if \( A' = \lambda A \) for some \( \lambda \in \mathbb{C} \), \( \lambda \neq 0 \). When \( A \) and \( A' \in SU(1, 1) \), \( U_{A'} = U_A \) if and only if \( A' = \pm A \). The Möbius transformation \( U_A \), determined by \( A \in SU(1, 1) \), restricted to the Poincaré disk \( D_P \), is a diffeomorphism of \( D_P \) onto itself, and the map \( SU(1, 1) \times D_P \to D_P \) so defined is a holomorphic left action of \( SU(1, 1) \) on \( D_P \). There exists a surjective Lie groups homomorphism \( \Phi \) of \( SU(1, 1) \) onto \( SO(2, 1) \) whose kernel is the discrete group \( \{ 1, -1 \} \) (where 1 stands for the unit \( 2 \times 2 \) matrix and -1 for the opposite matrix). For each complex \( 2 \times 2 \) matrix \( A \in SU(1, 1) \), expressed as indicated by the formulae (*) above, \( \Phi(A) \) is the real \( 3 \times 3 \) matrix (see for example [20])

\[
\Phi(A) = \begin{pmatrix}
(a^2 + \overline{a}^2 + (b^2 + \overline{b}^2) & a^2 - \overline{a}^2 - (b^2 - \overline{b}^2) & -(ab + \overline{a\overline{b}}) \\
2i & a^2 + \overline{a}^2 - (b^2 + \overline{b}^2) & -\overline{a\overline{b}} \\
-(ab + \overline{a\overline{b}}) & -\overline{a\overline{b}} & (aa + b\overline{b})
\end{pmatrix}.
\]

The Lie algebras of the Lie groups \( SU(1, 1) \) and \( SO(2, 1) \) are therefore isomorphic. They can both be identified with the vector space \( F \), as well as their dual vector spaces. The action of \( SU(1, 1) \) on the Poincaré disk \( D_P \) by Möbius transformations can therefore be identified with the Hamiltonian action of \( G \equiv SO(2, 1) \) discussed above, and admits \( J_{D_P} \) as moment map.

### 4.2.4. Gibbs states on the Poincaré half-plane.

The Möbius transformation \( U_M \) determined by the matrix \( M = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \), restricted to the Poincaré disk \( D_P \), is the map

\[
w \mapsto \xi = U_M(w) = \frac{i(-w + 1)}{w + 1}, \quad w \in D_P = \{ w \in \mathbb{C} \mid |w| < 1 \}.
\]
Its image is the half plane
\[ \Pi_P = \{ \xi = \xi_r + i \xi_{im} \in \mathbb{C} \mid \xi_{im} > 0 \} , \]
where \( \xi_r \) and \( \xi_{im} \) are respectively the real and the imaginary parts of the complex number \( \xi \). Endowed with the Riemannian metric and the symplectic form for which \( U_M : D_P \to \Pi_P \) is both an isometry and a symplectomorphism, \( \Pi_P \) is called the Poincaré half-plane. The expressions of its Riemannian metric \( ds_{\Pi_P}^2 \) and of its symplectic form \( \omega_{\Pi_P} \) are
\[
ds_{\Pi_P}^2(\xi) = \frac{R^2}{\xi_{im}^2} (d\xi_r^2 + d\xi_{im}^2),
\]
\[
\omega_{\Pi_P}(\xi) = \frac{R}{\xi_{im}^2} d\xi_r \wedge d\xi_{im}.
\]
A matrix
\[
A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \text{SU}(1,1), \text{ with } a, b \in \mathbb{C}, \mid a \mid^2 - \mid b \mid^2 = \bar{a}a - \bar{b}b = 1,
\]
acts on the Poincaré disk \( D_P \) by the Möbius transformation \( U_A \). Since the Möbius transformation \( U_M \) determined by the matrix \( M \), restricted to \( D_P \), is a diffeomorphism of \( D_P \) onto the Poincaré half-plane \( \Pi_P \), the corresponding action of \( A \) on \( \Pi_P \) is the Möbius transformation determined by the matrix
\[
MAM^{-1} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{pmatrix}
\]
\[
= \begin{pmatrix} a + \bar{a} - b - \bar{b} & -i(a - \bar{a} + b - \bar{b}) \\ i(a - a - b + \bar{b}) & a + a + b + \bar{b} \end{pmatrix}
\]
\[
= 2 \begin{pmatrix} a_r - b_r & a_{im} + b_{im} \\ -a_{im} + b_{im} & a_r + b_r \end{pmatrix}.
\]
The Möbius transformations determined by \( MAM^{-1} \) and by \((1/2)MAM^{-1}\) being equal, we are led to consider the map \( \Sigma \), defined on \( \text{SU}(1,1) \), taking its values in the set of real \( 2 \times 2 \) matrices, which associates to each matrix \( A \in \text{SU}(1,1) \) the matrix
\[
\Sigma(A) = \begin{pmatrix} a_r - b_r & a_{im} + b_{im} \\ -a_{im} + b_{im} & a_r + b_r \end{pmatrix}, \text{ with } A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \text{SU}(1,1).
\]
Observing that \( \det(\Sigma(A)) = |a|^2 - |b|^2 = 1 \), we see that the map \( \Sigma \) is a Lie groups isomorphism of \( \text{SU}(1,1) \) onto \( \text{SL}(2,\mathbb{R}) \). The Lie group \( \text{SL}(2,\mathbb{R}) \) therefore acts on the Poincaré half-plane \( \Pi_P \) by a Hamiltonian action. As for the action of \( \text{SU}(1,1) \) on the Poincaré disk \( D_P \), the dual vector space of
the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ can be identified with the vector space $\mathcal{F}$. With this identification, the expression of the moment map of the Hamiltonian action of $\text{SL}(2, \mathbb{R})$ on the Poincaré half-plane $\Pi_P$ is

$$J_{\Pi_P}(\xi) = \frac{R}{2\xi_{\text{im}}} \left( (1 - |\xi|^2)\mathbf{e}_x + 2\xi_r \mathbf{e}_y + (1 + |\xi|^2)\mathbf{e}_z \right).$$

The set $\Omega$ of generalized temperatures for the Hamiltonian action of $\text{SL}(2, \mathbb{R})$ on the Poincaré half-plane $\Pi_P$ is, as for the action of $\text{SU}(1,1)$ on the Poincaré disk $\mathcal{D}_P$, the set of time-like elements in $\mathcal{F} \equiv \mathfrak{sl}(2, \mathbb{R})$ directed towards the past. Let $\beta$ be one of its elements. We choose an admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathcal{F}$ such that $\beta = \beta \mathbf{e}_z$, with $\beta < 0$. Proceeding as for the Poincaré disk, we see that the probability density $\rho_\beta$ of the Gibbs state on $\Pi_P$ indexed by $\beta$, with respect to the measure $d\xi_r d\xi_{\text{im}}$, is, when expressed with the coordinate system $(\xi_r, \xi_{\text{im}})$,

$$\rho_\beta(\xi_r, \xi_{\text{im}}) = \frac{R|\beta|}{2\pi \xi_{\text{im}}^2} \exp \left( -\frac{R|\beta|((\xi_{\text{im}} - 1)^2 + \xi_r^2)}{2\xi_{\text{im}}} \right).$$

The associated thermodynamic functions (mean value of the moment map $E_J$ and entropy $S$) are the functions of the generalized temperature $\beta$ whose expressions are given in 4.2.2.

4.2.5. There is no Gibbs state on a two-dimensional symplectic vector space

We consider the plane $\mathbb{R}^2$ (coordinates $u, v$), endowed with the symplectic form $\omega = du \wedge dv$. The symplectic group $\text{Sp}(\mathbb{R}^2, \omega)$ is the group $\text{SL}(2, \mathbb{R})$ of real $2 \times 2$ matrices with determinant 1. As seen in 4.2.4, its Lie algebra, as well as its dual vector space, can be identified with the vector space $\mathcal{F}$, once an admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathcal{F}$ is chosen. The infinitesimal generators of the action of $\text{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$ are the three Hamiltonian vector fields

$$X_{\mathbb{R}^2}(u, v) = \frac{1}{2} \left( v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v} \right), \text{ whose Hamiltonian is } H_{X_{\mathbb{R}^2}}(u, v) = \frac{u^2 - v^2}{4},$$

$$Y_{\mathbb{R}^2}(u, v) = \frac{1}{2} \left( v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} \right), \text{ whose Hamiltonian is } H_{Y_{\mathbb{R}^2}} = \frac{-uv}{2},$$

$$Z_{\mathbb{R}^2}(u, v) = \frac{1}{2} \left( v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} \right), \text{ whose Hamiltonian is } H_{Z_{\mathbb{R}^2}} = \frac{-u^2 + v^2}{4}.$$  

The infinitesimal generators $X_{\mathbb{R}^2}, Y_{\mathbb{R}^2}$ and $Z_{\mathbb{R}^2}$ are the images, by the action on $\mathbb{R}^2$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \equiv \mathcal{F}$, of $\mathbf{e}_x, \mathbf{e}_y$ and $\mathbf{e}_z$, respectively. We therefore obtain a moment map $J_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathcal{F} \equiv \mathfrak{sl}(2, \mathbb{R})^*$ of this action by writing
\[ \langle J_{\mathbb{R}^2}(u,v), e_x \rangle = H_{X_{\mathbb{R}^2}}(u,v), \quad \langle J_{\mathbb{R}^2}(u,v), e_y \rangle = H_{Y_{\mathbb{R}^2}}(u,v), \quad \langle J_{\mathbb{R}^2}(u,v), e_z \rangle = H_{X_{\mathbb{R}^2}}(u,v). \]

So we have

\[ J_{\mathbb{R}^2}(u,v) = H_{X_{\mathbb{R}^2}}(u,v)\varepsilon_x + H_{Y_{\mathbb{R}^2}}(u,v)\varepsilon_y + H_{Z_{\mathbb{R}^2}}(u,v)\varepsilon_z, \]

where \((\varepsilon_x, \varepsilon_y, \varepsilon_z)\) is the basis of \(F^*\) dual of the basis \((e_x, e_y, e_z)\) of \(F\). With the identification of \(F\) with its dual \(F^*\) by means of the scalar product on \(F\) of signature \((+, +, -)\), we have \(\varepsilon_x = e_x, \varepsilon_y = e_y, \varepsilon_z = -e_z\). Therefore

\[ J_{\mathbb{R}^2}(u,v) = H_{X_{\mathbb{R}^2}}(u,v)\varepsilon_x + H_{Y_{\mathbb{R}^2}}(u,v)\varepsilon_y - H_{Z_{\mathbb{R}^2}}(u,v)\varepsilon_z = \frac{u^2 - v^2}{4}e_x - \frac{uv}{2}e_y + \frac{u^2 + v^2}{4}e_z. \]

By observing that

\[ \left( \frac{u^2 - v^2}{4} \right)^2 + \left( \frac{uv}{2} \right)^2 - \left( \frac{u^2 + v^2}{4} \right)^2 = 0 \quad \text{and} \quad \frac{u^2 + v^2}{4} \geq 0, \]

we see that the \(J_{\mathbb{R}^2}(\mathbb{R}^2)\) is the union of two coadjoint orbits of \(SL(2,\mathbb{R})\): a zero-dimensional orbit, the singleton \(\{0\}\) (where 0 stands for the origin of \(F\)), and a two-dimensional orbit, the cone \(C^+\) of light-like elements in \(F\) directed towards the future. We have seen above (4.2.2) that no Gibbs state can exist on \(C^+\). Therefore no Gibbs state can exist on a two-dimensional symplectic vector space, for the natural action of the linear symplectic group.

4.2.6. The Gibbs states and thermodynamic functions on an affine Euclidean and symplectic plane for the group of its displacements

As in the preceding section, we consider the plane \(\mathbb{R}^2\) (coordinates \(u, v\)) endowed with the symplectic form \(\omega = du \wedge dv\). Moreover we endow it with its usual Euclidean metric, and consider the action of its group of displacements (rotations and translations), denoted by \(E(2,\mathbb{R})\). In matrix notations, an element of \(\mathbb{R}^2\) of coordinates \((u,v)\) is represented by the column vector \(\begin{pmatrix} u \\ v \\ 1 \end{pmatrix}\) and an element \(g(\varphi,x,y)\) of \(E(2,\mathbb{R})\) by a matrix \( \begin{pmatrix} \cos \varphi & -\sin \varphi & x \\ \sin \varphi & \cos \varphi & y \\ 0 & 0 & 1 \end{pmatrix} \) depending on the three real parameters \(\varphi, x\) and \(y\). The action of \(E(2,\mathbb{R})\) on \(\mathbb{R}^2\) is expressed as the product of matrices

\[ \begin{pmatrix} \cos \varphi & -\sin \varphi & x \\ \sin \varphi & \cos \varphi & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} u \cos \varphi - v \sin \varphi + x \\ u \sin \varphi + v \cos \varphi + y \\ 1 \end{pmatrix}. \]
We denote by \((e_r, e_x, e_y)\) the basis of the Lie algebra \(\mathfrak{e}(2, \mathbb{R})\) whose elements, identified with the corresponding matrices, are

\[
e_r = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The corresponding fundamental vector fields on \(\mathbb{R}^2\) are the Hamiltonian vector fields, generators of the action of \(E(2, \mathbb{R})\),

\[
(e_r)_{\mathbb{R}^2}(u, v) = -v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v}, \quad \text{whose Hamiltonian is } H(e_r)_{\mathbb{R}^2}(u, v) = \frac{u^2 + v^2}{2},
\]

\[
(e_x)_{\mathbb{R}^2}(u, v) = \frac{\partial}{\partial u}, \quad \text{whose Hamiltonian is } H(e_x)_{\mathbb{R}^2}(u, v) = -v,
\]

\[
(e_y)_{\mathbb{R}^2}(u, v) = \frac{\partial}{\partial v}, \quad \text{whose Hamiltonian is } H(e_y)_{\mathbb{R}^2}(u, v) = u.
\]

Proceeding as in 4.2.5, we obtain the expression of the moment map

\[
J_{\mathbb{R}^2}(u, v) = \frac{u^2 + v^2}{2} e_r - y e_x + xe_y,
\]

where \((e_r, e_x, e_y)\) is the basis of \(\mathfrak{e}(2, \mathbb{R})^*\) dual of the basis \((e_r, e_x, e_y)\) of \(\mathfrak{e}(2, \mathbb{R})\).

An element \(\beta = \beta_r e_r + \beta_x e_x + \beta_y e_y\) in \(\mathfrak{e}(2, \mathbb{R})\) is a generalized temperature if, considered as a function of \(\beta_r, \beta_x\) and \(\beta_y\), the integral

\[
\int_{\mathbb{R}^2} \exp\left(-\langle J(u, v), \beta \rangle\right) \lambda_\omega = \int_{\mathbb{R}^2} \exp\left(-\frac{u^2 + v^2}{2} \beta_r + v \beta_x - u \beta_y \right) du dv \quad (*)
\]

is normally convergent. Clearly, a necessary condition for the normal convergence of this integral is

\[
\beta_r > 0. \quad (**)
\]

When this condition is satisfied, we can write

\[
-\frac{u^2 + v^2}{2} \beta_r + v \beta_x - u \beta_y = \frac{\beta_x^2}{2 \beta_r} - \frac{\beta_x}{2} \left[u + \frac{\beta_y}{\beta_r}\right]^2 + \left(-\frac{\beta_x}{\beta_r}\right)^2.
\]

By using on the plane \(\mathbb{R}^2\), instead of \((u, v)\), the polar coordinates \((\rho, \psi)\), determined by

\[
u' = u + \frac{\beta_y}{\beta_r} = \rho \cos \psi, \quad v' = v - \frac{\beta_x}{\beta_r} = \rho \sin \psi,
\]

we see that when (***) is satisfied, the integral (*) above is normally convergent. Condition (**) is therefore both necessary and sufficient for the normal convergence of (*). The set \(\Omega\) of generalized temperatures, for the action of \(E(2, \mathbb{R})\) on \((\mathbb{R}^2, \omega)\), is made of elements \(\beta = \beta_r e_r + \beta_x e_x + \beta_y e_y \in \mathfrak{e}(2, \mathbb{R})\).
which satisfy Condition (***) above. The expression of the partition function $P$ is then

$$P(\beta) = \exp \left( \frac{\beta^2_x + \beta^2_y}{2\beta_r} \right) \int_0^{2\pi} \left( \int_0^{+\infty} \exp \left( -\frac{\beta_r \rho^2}{2} \right) \rho \, d\rho \right) \, d\psi$$

$$= \frac{\pi(\beta^2_x + \beta^2_y)}{\beta_r^2}, \quad \beta = \beta_r e_r + \beta_x e_x + \beta_y e_y \in \mathfrak{e}(2, \mathbb{R}).$$

The expression of the probability density $\rho_\beta$ of the Gibbs state indexed by $\beta \in \mathfrak{e}(2, \mathbb{R})$, with respect to the Liouville measure $dudv$, is

$$\rho_\beta(u, v) = \exp \left( \frac{\beta^2_x + \beta^2_y}{2\beta_r} \right) \exp \left( -\frac{\beta_r (u^2 + v^2)}{2} \right).$$

The expressions of the thermodynamic functions $E_J(\beta)$ (mean value of the moment map) and $S$ (entropy) are

$$E_J(\beta) = \frac{2}{\beta_r} \varepsilon_r - \frac{2\beta_x}{\beta^2_x + \beta^2_y} \varepsilon_x - \frac{2\beta_y}{\beta^2_x + \beta^2_y} \varepsilon_y,$$

$$S(\beta) = \log P(\beta) = \log \pi + \log(\beta^2_x + \beta^2_y) - \log(\beta_r^2).$$

The expressions of the symplectic Lie group cocycle $\theta : \mathfrak{e}(2, \mathbb{R}) \to \mathfrak{e}(2, \mathbb{R})^*$ and of the symplectic Lie algebra cocycle $\Theta : \mathfrak{e}(2, \mathbb{R}) \times \mathfrak{e}(2, \mathbb{R}) \to \mathbb{R}$ associated to the moment map $J_{\mathbb{R}^2}$ can be determined by using the formulae

$$\theta(g) = J \circ \Phi_g - \text{Ad}^*_g \circ J, \quad \Theta(X, Y) = \langle T_e \theta(X), Y \rangle,$$

where $g \in \mathfrak{e}(2, \mathbb{R})$, $X$ and $Y \in \mathfrak{e}(2, \mathbb{R})$, $\Phi_g : \mathbb{R}^2 \to \mathbb{R}^2$ being the affine isometry of $\mathbb{R}^2$ determined by the action of $g \in \mathfrak{e}(2, \mathbb{R})$. Although they are not necessary for the determination of Gibbs states, they are indicated below.

$$\theta(g_{(\varphi, x, y)}) = \frac{x^2 + y^2}{2} \varepsilon_r - y \varepsilon_x + x \varepsilon_y,$$

$$\Theta(r_1 e_r + x_1 e_x + y_1 e_y, r_2 e_r + x_2 e_x + y_2 e_y) = x_1 y_2 - y_1 x_2.$$

4.2.7. Remark. — The fact that generalized temperatures are elements of the Lie algebra $\mathfrak{e}(2, \mathbb{R})$ whose component $\beta_r$ on $e_r$ is strictly positive may seem surprising, since there is apparently no reason explaining why clockwise and counter-clockwise rotations have different properties. I believe that it follows from the choice of $du \wedge dv$ as a symplectic form on the plane $\mathbb{R}^2$, endowed with coordinates $u$ and $v$. This choice automatically implies the choice of an orientation of this plane: the Hamiltonian vector field which admits $(u^2 + v^2)/2$ as Hamiltonian is indeed the infinitesimal generator of counter-clockwise rotations around the origin. Replacing $du \wedge dv$ by its opposite $dv \wedge du$ would
have as consequence the replacement of this vector field by its opposite, which is the infinitesimal generator of clockwise rotations.

5. Final comments and thanks

We have given a few examples of Gibbs states for the Hamiltonian action of a non-commutative Lie group on a symplectic manifold, even when the considered symplectic manifold is non-compact. However, we encountered too several examples in which no Gibbs state can exist, the set of generalized temperatures being empty. All our examples are relative to two-dimensional symplectic manifolds. It seems interesting to look now at higher-dimensional symplectic manifolds.

Although I studied Jean-Marie Souriau’s book [33] in my youth, my knowledge of his works in statistical mechanics were rather superficial. I owe to Frédéric Barbaresco, who led me to look again at this book more in depth, my interest in Gibbs states.

Roger Balian and Alain Chenciner were kind enough to look at this work. Their numerous helpful remarks and constructive criticisms, their respective helps to better understand the foundations of quantum statistical mechanics on one hand and Shannon’s paper [31] on the other hand, were invaluable.

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