A Quadratic Regularization for the Multi-Attribute Unit-Demand Envy-Free Pricing Problem

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Abstract

We consider a profit-maximizing model for pricing contracts as an extension of the unit-demand envy-free pricing problem: customers aim to choose a contract maximizing their utility based on a reservation price and multiple price coefficients (attributes). Classical approaches suppose that the customers have deterministic utilities; then, the response of each customer is highly sensitive to price since it concentrates on the best offer. To circumvent the intrinsic instability of deterministic models, we introduce a quadratically regularized model of customer’s response, which leads to a quadratic program under complementarity constraints (QPCC). This provides an alternative to the classical logit approach, still allowing to robustify the model, while keeping a strong geometrical structure. In particular, we show that the customer’s response is governed by a polyhedral complex, in which every polyhedral cell determines a set of contracts which is effectively chosen. Moreover, the deterministic model is recovered as a limit case of the regularized one. We exploit these geometrical properties to develop a pivoting heuristic, which we compare with implicit or non-linear methods from bilevel programming, showing the effectiveness of the approach. Throughout the paper, the electricity provider problem is our guideline, and we present a numerical study on this application case.

1 Introduction

1.1 Context

For a company, the question of determining the correct prices of its products is crucial: a compromise has to be found between having enough consumers buying products and setting prices that are sufficiently important to cover the production cost. Profit-maximization models have been extensively studied. They consist in maximizing the seller profit taking in account the customer behavior.

The special structure of these problems can be generally cast into the bilevel framework, that have been extensively studied in the last decades [Bar13; Dem+15]. As detailed in Kleinert et al. [Kle+21], two classical approaches consist in reformulating the problem as a single-level one, either using strong-duality or the KKT conditions, to express the optimality of the lower decision and constrain the upper problem. Formulations based on on KKT conditions lead to Mathematical Programs with Complementarity Constraints (MPCC), a class of optimization problems whose interest has been growing in recent years [Dus+20] and particularly in the energy sector [Af¸s+16; Ale+19; Aus+20; ARR21].

The unit-demand envy-free pricing problem is a specific case. We consider a finite number of customers (or segments of customers) who are supposed to buy precisely one product, among the ones maximizing their utility. Moreover, products are available in unlimited supply. Guruswami et al. showed that this problem is APX-hard (even on a restricted class of instances), see [Gur+05]. Shiodia, Tunçel, and Myklebust developed Mixed-Integer Programming (MIP) formulations, along

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with valid cuts and heuristics \cite{STM11}. They also robustified the model to ensure that each customer faces a unique maximum utility. Fernandes et al. compare in \cite{Fer+16} several MIP formulations and reinforce them with new valid cuts.

Choice models of a probabilistic nature have also been considered. Then, the value of the lower level objective determines the probability distribution of the customer’s choice. The most studied case concerns the logit model \cite{Tra09, McF74}. Li and Huh suppose in \cite{LH11} that the population is homogeneous, meaning that there is only one segment. They reformulate the problem as a concave maximization problem by a market-share transformation. Shao and Kleywegt extend in \cite{SK20} this approach to the case of multiple price attributes. Logit pricing models with multiple consumers segments have only been studied very recently: Li et al. formalize in \cite{Li+19} the product pricing under the Mixed Multinomial Logit (MMNL), and develop algorithms to find good solutions. Hohberger applies such models to the revenue management case study of the German long-distance railway network \cite{Hoh20}.

The latter works always suppose a discrete population, or an a priori discretization of a continuous population. Sun, Su and Chen \cite{SSC17} show convergence of the discretized model to the continuous limit case when the number of customers goes to infinity. To this end, they define a quadratic primal-dual regularized version of the model to overcome its ill-posedness.

\subsection{Contribution}

We first introduce the multi-attribute unit-demand envy-free pricing problem that we model by a bilinear bilevel formulation. We first show that the tie-breaking rule does not play any role in this problem i.e., the optimistic and pessimistic versions share the same optimal value, under a technical assumption. Then, we derive from the unifying bilevel framework two distinct MIP formulations, using either strong-duality or KKT conditions, and we correlate them to formulations from the literature \cite{Fer+16}.

The numerical results reveal an intrinsic instability of the deterministic model: a little perturbation of the prices can make the profit plummet. To overcome this issue, we consider regularized models. The first one is based on the classical logit model, for which customers have now probabilities to choose an option, and so the response is not binary. The logit model involves a parameter quantifying the rationality of the customers. We prove asymptotic results for this model (convergence to the value of deterministic optimistic or pessimistic models when the rationality of the customers increases). In a special case, we interpret the deviation of the logit model from its deterministic analogue as a moral hazard.

However, the logit model is hard to solve optimally. In particular, the reduction to a convex optimization problem that holds for a homogeneous population (single segment of customers) is no longer true for the heterogeneous populations considered here. Hence, we introduce a new model, based on a quadratic regularization of customer’s response, which also involves a “rationality” parameter. This is more realistic than deterministic models for the class of applications to energy we have in mind, since in the presence of near-ties (contracts with similar utilities), customer’s response distributes among the best contracts (rather than concentrating on a single one). The development of this model is the main contribution of this work. In particular, we give a closed-form expression of the lower response and highlight its polyhedral structure. We show that this response is governed by a polyhedral complex, in which each open cell determines a set of contracts which are effectively chosen. Besides, we show that this model has the same good theoretical properties as the logit model (stability and convergence to the unregularized case) and provide
metric estimates showing that the responses of the two models are close. The main interest of quadratic regularization, then, lies in computational tractability. We show that it reduces to a convex Quadratic Program with Complementary Constraints (QPCC). Powerful methods based on mixed or semidefinite programming allow one to solve instances of significant size of QPCC with optimality guarantees, although problems of this kind are generally difficult. In fact, to complete this study, we prove the APX-Hardness of our quadratic model reusing the transformation introduced for the deterministic case in [Gur+05].

Our study is inspired by several works. We adopt the viewpoint of Gilbert, Marcotte and Savard [GMS15] in that we consider the MMNL model [Li+19; Hoh20] as a regularized version of its deterministic analog [STM11; Fer+16]. They look at a related problem that studies the toll pricing optimization, and demonstrate, among other things, asymptotic convergence of the logit regularization to the deterministic model. Besides, we link the MIP formulations that we derive from the bilevel framework to the ones of [Fer+16]. We obtain them by standard and generic reformulations, as suggested by Kleinert et al. in their review of bilevel programming [Kle+21]. By comparison with all these works, the main novelty is the introduction of the quadratic regularized model and the evidences that it has the same good features as the logit model, in terms of economic realism and robustness, while being computationally more tractable. The description of the customers choices as a polyhedral complex is inspired by the study of Baldwin and Klemperer [BK19], and we recover their results as a limit case when the regularization term vanishes. The numerical tests were achieved on the problem of electricity pricing, based on realistic customer’s data and French contract types. We succeeded in solving instances of consequent size, and the pivoting heuristic we designed has demonstrated its ability to find near-optimal solution in a reasonable time, competing with general solvers (CMA-ES [Han06], Knitro [Art20] and FilterMPEC [FL04]). Finally, our quadratic regularization differs from the one used by Sun, Su and Chen [SSC17] in that our version is primal feasible and aims to catch the consumers behavior.

The paper is organized as follows: we first define the deterministic multi-attribute unit-demand envy-free pricing problem and present some properties of the model. We then define the MMNL behavior before introducing our quadratic relaxation. We finally give numerical results to highlight the relevance of our model in the context of electricity pricing.

2 Preliminaries

2.1 Notation

In the sequel, we denote by $\Delta_N$ the simplex of $\mathbb{R}^N$, and by $\|x\|_N$ the Euclidean norm associated with the canonical scalar product $\langle x, y \rangle_N$ on $\mathbb{R}^N$. For any polyhedron $Q$, Vert($Q$) denotes the set of vertices of $Q$. Moreover, for any optimization problem $(P)$, the value $v(P) \in \mathbb{R} \cup \{\pm \infty\}$ denotes its optimal value (that can be infinite if $(P)$ is infeasible or unbounded).

2.2 Model

We suppose that a company has $W$ different types of contracts and that a market study has distinguished beforehand $S$ customer segments, where customers of a given segment have approximately the same behavior. Given a segment $s \in [S] := \{1, \ldots, S\}$ and a product $w \in [W]$, the reservation price $R_{sw}$ is the maximum price that customers of this segment are willing to spend on $w$. In the classical product pricing model, the items to sell are only characterized by a price (determined by
the company) and each customer faces the same price. In our setting, we consider the multi-attribute case where the price of each contract $w$ is determined by a finite number $H > 1$ of variables (or attributes), denoted by $x^h_w$. For instance, in the French electricity market, the invoice of a customer depends on at least two variables, representing a fixed and a variable portion, the former depending on the subscribed power of the customer and the latter depending on his electricity consumption \cite{Com04}. Moreover, in the peak/off-peak contract, the variable portion distinguishes between the peak and off-peak consumption. Then, the invoice is determined by at least three variables. The following assumption captures such contracts.

**Assumption 2.1.** The price $\theta_{sw}(x)$ the customer segment $s$ will pay for contract $w$ is a linear form:

$$\theta_{sw}(x) := \langle E_{sw}, x_w \rangle_H,$$

where $E_{sw} = (E^h_{sw})_{h \in H} \in \mathbb{R}^H_0$. Besides, the price coefficients $x^h_w$ are supposed to be in a non-empty polytope $X$ given by a Cartesian product

$$X = \bigtimes_{w \in [W]} X_w,$$

where $X_w \subset \mathbb{R}^H$, for all $w \in [W]$.

In the electricity market context, $E_{sw}$ represents the electricity consumption of the customers of segment $s$ who chose the contract $w$. It depends on $h$ (the period of the day) and on the contract type $w$. This is realistic, since the notion of peak and off-peak period can vary along the contracts, and since customers adapt their electricity consumption depending on their choice of contract. The situation in which the invoice price $\theta_{sw}(x)$ is affine in the energy consumption, instead of being linear as in (1), reduces to the latter case by adding to the set $H$ an extra element $h = 0$, with $E^0_{sw} = 1$ for all $s, w$. Besides, assuming that the polytope $X$ is a Cartesian product means that there is no constraint coupling the prices of the different contracts.

We also make classical assumptions:

**Assumption 2.2 (Unit-Demand).** Each customer purchases exactly one contract.

The utility or surplus of segment $s$ for contract $w$ is the difference between the reservation price and the invoice, i.e., $R_{sw} - \theta_{sw}(x)$. The disutility is the opposite of the utility.

**Assumption 2.3 (Envy-free).** There is no limitation on the maximum number of customers able to purchase the same contract and so each customer chooses a contract maximizing his utility.

**Assumption 2.4 (No-purchase option).** Consumers have the option not to purchase any contract, or in a competitive environment, to choose a contract from a competitor.

The no-purchase option may be explained a bit more: in a competitive environment, customers can choose one of the competing contracts but we assume here that the competition is static, meaning that competitors do not react to the company prices. In this way, the contracts from different competitors can be aggregated in a unique contract of a virtual competitor. Moreover, since utility is defined up to an additive constant, we set the utility of the no-purchase option to be 0.

Finally, when a segment $s$ chooses a contract $w$, the company has to fulfill the service (in the case of an electricity provider, it has to supply electricity).
Assumption 2.5. The cost for the company to fulfill the service \( w \) for segment \( s \) is denoted by \( C_{sw} \), and we suppose that they have the same structure as the bills \( \theta \) i.e.,

\[
\forall s \in [S], \ w \in [W], \ C_{sw} = \langle E_{sw}, \check{C}_w \rangle_{H}
\]

where \( \check{C}_w = (\check{C}^h_w)_{h \in H} \in \mathbb{R}^H \).

In this way, \( \check{C}_w \) represents the unitary production cost at period \( h \) for the contract type \( w \). Note that this cost depends on the contract: for instance, a “green electricity” contract may induce a higher production costs than a classical contract. A fixed cost (not proportional to the consumption) can be incorporated in this model by introducing the dummy period \( h = 0 \) with a unit virtual consumption \( E_{sw}^0 = 1 \), as explained above.

To model the customers behavior of segment \( s \), we define the variables \( y_s \in \mathbb{R}^W \) such that

\[
\forall s \in [S], \ w \in [W], \ y_{sw} = \begin{cases} 1 & \text{if segment } s \text{ chooses } w, \\ 0 & \text{otherwise.} \end{cases} \tag{2}
\]

To make explicit the no-purchase option, we introduce a variable \( y_{s0} \) and denote the extended choice vector for segment \( s \) by

\[
\bar{y}_s := (y_{s0}, y_s) \in \mathbb{R} \times \mathbb{R}^W = \mathbb{R}^{W+1}.
\]

We shall think of an element \( \bar{y}_s \in \Delta_{W+1} \) as a relaxed choice of segment \( s \). When \( \bar{y}_s \) is a vertex of \( \Delta_{W+1} \), \( y_s = (y_{sw})_{w \in W} \) determines the behavior of segment \( s \), according to (2). The no-purchase option corresponds to \( y_{s0} = 1 \). For a price strategy \( x \in X \), the customers behavior is defined by the solution set mapping \( \Psi \) defined as

\[
\Psi(x) := \arg\min_{\bar{y} \in (\Delta_{W+1})^S} \left\{ \sum_{s \in [S]} \langle \theta_s(x) - R_s, y_s \rangle_W \right\}. \tag{3}
\]

Note that the scalar product that appears in the objective is on \( \mathbb{R}^W \) since the no-purchase option induces a zero utility for any customer.

The multi-attribute unit-demand envy-free pricing problem can now be expressed as the following bilinear bilevel model

\[
\max_{x \in X, \bar{y}} \sum_{s \in [S]} \rho_s \langle \theta_s(x) - C_s, y_s \rangle_W \quad \text{s.t. } (x, \bar{y}) \in \text{gph} \, \Psi \tag{\text{o-BP}}
\]

In the model, \( \rho_s \) stands for the weight of segment \( s \) in terms of company’s profit. The label (\text{o-BP}) refers to the optimistic nature of this bilevel problem: if the lower level problem has several optimal solutions, the upper level optimizer takes into account the most favorable of these optimal solutions, see [Dem+15] for instance.

Remark. Because all the segments react independently, we can aggregate all their actions under the same problem. Hence, the minimization in the lower level problem [3] is made over the Cartesian product of simplices. The vertices of each of these simplices represent the possible decisions of a given segment.
The following result justifies the minimization over relaxed choices in \((o-BP)\).

**Proposition 2.1.** There exists an optimal solution of \((o-BP)\) with integer lower values \(y\).

**Proof.** We denote by \((x^*, \bar{y}^*)\) an optimal solution, which exists because \(\text{gph } \Psi\) is compact and non-empty (from Assumption 2.1). The argmin set \(\Psi(x^*)\) is a face of the Cartesian product of simplices \((\Delta_{W+1})^S\), since it arises from the minimization of a linear objective on this product. So it is a non-empty integer polyhedron. Moreover, there exists an extreme point of \(\Psi(x^*)\), denoted by \(\hat{y}\), such that \(F(x^*, \hat{y}) = F(x^*, \bar{y}^*)\) owing to the linearity in \(y\) of the upper objective. To conclude, \((x^*, \hat{y})\) is also an optimal solution and \(\hat{y}\) is integer as it is an extreme point of \(\Psi(x^*)\).

Problem \((o-BP)\) is a very specific bilinear bilevel problem with a quite simple lower problem (minimization over the simplex, without integrity constraints). However, despite its apparent simplicity, this model is APX-hard since it includes as a special case the unit-demand envy-free pricing model, which was shown to be APX-hard \([Gur+05]\).

The problem \((o-BP)\) is a profit-maximization problem: in fact, we can define the optimistic leader profit function \(\pi^{\text{opt}}\) for a given price strategy \(x\) as

\[
\pi^{\text{opt}}(x) := \sum_{s \in [S]} \rho_s \sum_{w \in [W]} (\theta_{sw}(x) - C_{sw}) y_{sw}^{\text{opt}}(x)
\]

where \(y_{sw}^{\text{opt}}(x)\) is the optimistic lower response (which is binary, see (2)). The problem \((o-BP)\) is therefore the maximization of the function \(\pi^{\text{opt}}\) over \(X\).

The optimistic profit function \(\pi^{\text{opt}}\) is piecewise linear (the profit is linear for a given customers distribution \(y\), and the possible customers distribution lies in a discrete set). However, \(\pi^{\text{opt}}\) is in general discontinuous at prices inducing ties (multiple minimum disutilities for a segment), see Fig. 3.

### 2.3 Tie-breaking rules

**Proposition 2.2** (Degeneracy). Let \((x^*, \bar{y}^*)\) be an optimal solution of \((o-BP)\) and suppose that all the contracts are chosen by at least one segment (otherwise the contract is not useful). If \(x^* \in \text{Int}(X)\), then there are at least \(W\) segments that face ties i.e. for all \(w \in [W]\), there exists \(s \in [S]\) such that \(y_{sw}^{*} = 1\) and

\[
\theta_{sw}(x^*) - R_{sw} = \min \left\{ 0, \{\theta_{sw'}(x^*) - R_{sw'}\}_{(w' \neq w)} \right\}.
\]

**Proof.** Suppose that

\[
\exists w \in [W], y_{sw}^{*} = 1 \Rightarrow \theta_{sw}(x^*) - R_{sw} < \min \left\{ 0, \{\theta_{sw'}(x^*) - R_{sw'}\}_{(w' \neq w)} \right\}.
\]

Then, one could increase the price of the contract \(w\) by a little amount – which is possible because \(x^* \in \text{Int}(X)\) – and keep the same customers response, contradicting the optimality of the leader’s decision.

Proposition 2.2 proves that solutions always contain ties between invoices for some customers. Therefore, the relevance of the optimistic hypothesis has to be discussed. To this end, we consider...
two other versions of the problem \((o-BP)\). First let us consider the pessimistic version in which customers having ties are supposed to choose the worst invoice in terms of leader’s profit:

\[
sup_{x \in X} \min_{s} \sum_{s \in [S]} \rho_s \langle \theta_s(x) - C_s, y_s \rangle_W \tag{p-BP}
\]

\[
s.t. \quad \bar{y}_s \in \arg\min_{y'_s \in \Delta_{W+1}} \langle \theta_s(x) - R_{sw}, y'_s \rangle_W, \forall s
\]

**Remark.** The existence of a solution is not guaranteed. Dempe gives in [Dem02, Theorem 3.3] a pessimistic problem such that the optimum is not attained.

We also introduce an intermediary model between optimistic and pessimistic, called uniform model, defined as

\[
sup_{x \in X} \sum_{s \in [S]} \rho_s \langle \theta_s(x) - C_s, y_s \rangle_W \tag{u-BP}
\]

\[
s.t. \quad y_{sw} = 1_{w \in \Phi_s(x)/|\Phi_s(x)|}, \forall s, w
\]

where \(\Phi_s(x) := \arg\min \left\{ 0, \{\theta_{sw}(x) - R_{sw}\}_{w \in [W]} \right\} \) denotes the set of minimum disutilities among the \(W + 1\) possibilities. In this model, when a tie occurs, customers have no preference on the contracts and spread their choice on the different possibilities (of course, the lower response cannot be binary anymore). We then define the uniform leader profit function as

\[
\pi_{uni}(x) := \sum_{s \in [S]} \sum_{w \in [W]} (\theta_{sw}(x) - C_{sw}) y_{sw}^{uni}(x) \tag{5}
\]

where \(y_{sw}^{uni}\) stands for the uniform lower response. The three responses (optimistic, pessimistic and uniform) only differ when a tie between two contracts appears. Using an idea of Gilbert, Marcotte and Savard [GMS15], the following proposition establishes the equality of the three versions in terms of optimal value, assuming a condition on the costs.

**Assumption 2.6 (No-profit option).** We say that the bilevel models \((o-BP)\), \((p-BP)\) and \((u-BP)\) allow the no-profit option if we can set the price to be equal to \(\bar{C}\) i.e., \(\bar{C} \in X\).

In the electricity provider context, this condition is realistic: setting prices equal to \(\bar{C}\) yields a public service type policy (with no benefit), in which the company aims to cover exactly its costs.

**Theorem 2.3** (Indifference to tie-breaking rule). The inequalities \(v(o-BP) \geq v(u-BP) \geq v(p-BP)\) always hold. Moreover, as soon as the models allow the no-profit option,

\[
v(o-BP) = v(u-BP) = v(p-BP)
\]

**Proof.** The inequalities \(v(o-BP) \geq v(u-BP) \geq v(p-BP)\) are immediate. Let \((x^*, \bar{y}^*)\) be an optimistic optimal solution. For any \(\delta > 0\), we consider the perturbed price matrix \(x^\delta\) defined by

\[
\forall w \in [W], h \in [H], (x^\delta)^h_w = \frac{1}{1 + \delta} \left( (x^*)^h_w + \delta \bar{C}_w^h \right)
\]

This new price matrix lies in the polytope \(X\), since it a barycenter of \(x^* \in X\) and \(\bar{C} \in X\). Suppose that for a given segment \(s\), there is a tie between contract \(w_1\) and \(w_2\) i.e.,

\[
\theta_{sw_1}(x^*) - R_{sw_1} = \theta_{sw_2}(x^*) - R_{sw_2}
\]

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In this optimistic problem, the segment \( s \) chooses between \( w_1 \) and \( w_2 \) the contract that maximizes the profit of the leader i.e., the contract with the highest value \( \theta_{sw}(x^*) - C_{sw} \). Without loss of generality, we suppose that it is \( w_1 \).

The new price policy \( x^\delta \) is constructed in order to break the tie while keeping the same choice of contract: from the definition of \( x^\delta \), one can obtain that for any segment \( s \) and any contract \( w \)
\[
\theta_{sw}(x^\delta) - \theta_{sw}(x^*) = -\frac{\delta}{1+\delta} (\theta_{sw}(x^*) - C_{sw}).
\]
Then,
\[
\left[\theta_{sw1}(x^\delta) - R_{sw1}\right] - \left[\theta_{sw2}(x^\delta) - R_{sw2}\right]
= \left[\theta_{sw1}(x^\delta) - \theta_{sw1}(x^*)\right] - \left[\theta_{sw2}(x^\delta) - \theta_{sw2}(x^*)\right]
= -\frac{\delta}{1+\delta} ([\theta_{sw1}(x^*) - C_{sw1}] - [\theta_{sw2}(x^*) - C_{sw2}]) \leq 0.
\]

Note that the only possibility to conserve the tie between \( w_1 \) and \( w_2 \) with price strategy \( x^\delta \) is when both contracts yield the same profit for the leader. Therefore, for any \( \delta \) sufficiently close to 0, \((x^\delta, \bar{y}^*)\) is a pessimistic solution with objective \( \frac{1}{1+\delta} v(o-BP) \). Hence, \( v(p-BP) \geq \frac{1}{1+\delta} v(o-BP) \), leading to \( v(p-BP) \geq v(o-BP) \) when \( \delta \to 0^+ \).

Theorem 2.3 proves that under the Assumption 2.6 the tie-breaking rule at the lower level does not affect the optimal value. In consequence, considering the optimistic behavior does not introduce any bias.

2.4 Single-level reformulation

2.4.1 Using the classical KKT transformation

The most common way to express the optimality of the lower problem as a system of inequalities is to use the Karush-Kuhn-Tucker conditions. Applying this idea to \((o-BP)\) leads to the following property

**Proposition 2.4.** We can reformulate \((o-BP)\) as
\[
\max_{x \in X, \bar{y}} \sum_{s \in [S]} \rho_s \mu_s + \rho_s \langle R_s - C_s, y_s \rangle_W
\]
\[
s.t. \quad 0 \leq y_{sw} \perp \theta_{sw}(x) - R_{sw} - \mu_s \geq 0, \forall s, w
\]
\[
0 \leq y_{s0} \perp \mu_s \leq 0, \forall s
\]
\[
\bar{y}_s \in \Delta_{W+1}, \forall s
\]

**Proof.** The lower problem of \((o-BP)\) is linear, the KKT conditions are therefore necessary and sufficient optimality conditions. Replacing the lower problem by the KKT conditions gives
\[
\max_{x \in X, \bar{y}} \sum_{s \in [S]} \rho_s \left[\theta_s(x) - C_s, y_s\right]_W
\]
\[
s.t. \quad \theta_{sw}(x) - R_{sw} - \lambda_{sw} - \mu_s = 0, \forall s, w
\]
\[
0 \leq y_{sw} \perp \lambda_{sw} \geq 0, \forall s, w
\]
\[
0 \leq y_{s0} \perp \mu_s \leq 0, \forall s
\]
\[
\bar{y}_s \in \Delta_{W+1}, \forall s
\]
For $s \in [S]$, from the constraints, if $y_{sw} > 0$ then $\lambda_{sw} = 0$ and then $\mu_s = \theta_{sw} - R_{sw}$. Thus,
\[
\sum_{w \in [W]} \theta_{sw}(x)y_{sw} = \mu_s \sum_{w \in [W]} y_{sw} + \sum_{w \in [W]} R_{sw}y_{sw} = \mu_s - \mu_s y_{s0} + \sum_{w \in [W]} R_{sw}y_{sw}.
\]
Finally, the complementarity constraints ensure that $\mu_s y_{s0} = 0$.

\[\square\]

Remark. The classical KKT reformulation preserves the global optimal solution set but can introduce new local solutions, see [Dem+15]. This remark holds for Proposition 2.4.

To numerically solve this formulation, we usually replace the complementarity constraints by Big-M constraints introducing new binary variables. Using Proposition 2.1, we provides a compact formulation in which the lower variables $y_s$ are the only binary variables:

\[
\max \sum_{s \in [S]} \rho_s \mu_s + \rho_s \langle R_s - C_s, y_s \rangle_W
\]
\[
\text{s.t. } 0 \leq \theta_{sw}(x) - R_{sw} - \mu_s \leq M_{sw}(1 - y_{sw}), \forall s, w
\]
\[
0 \leq -\mu_s \leq M_{s0}(1 - y_{s0}), \forall s
\]
\[
\bar{y}_s \in \text{Vert}(\Delta_{W+1}), \forall s
\]

(6)

Here, the set of vertices Vert($\Delta_{W+1}$) is known and is equal to \( \{ y \in \{0,1\}^{W+1} | \sum_{w=0}^{W} y_w = 1 \} \). The Big-M parameters $M_{sw} > 0$ must be chosen to be sufficiently large to prevent cutting any optimal solution (this is always possible owing to the compactness of $X$).

The MIP formulation (6) generalized the (U) formulation introduced by Fernandes et al. [Fer+13] that applies in the single-attribute case: the variables $\mu_s$ express the disutilities of each segment $s$.

2.4.2 Using the strong-duality condition

We next present an alternative formulation exploiting strong-duality, following an idea of Kleinert et al. [Kle+21] and the references therein. It uses the dual of the lower problem, and expresses the equality of the primal and dual objectives.

**Proposition 2.5.** We can reformulate \((\text{o-BP})\) as

\[
\max_{x \in X, \bar{y}} \sum_{s \in [S]} \rho_s \mu_s + \rho_s \langle R_s - C_s, y_s \rangle_W
\]
\[
\text{s.t. } \langle \theta_{sw}(x) - R_{sw}, y_s \rangle_W \leq \theta_{sw}(x) - R_{sw}, \forall s, w
\]
\[
\langle \theta_{sw}(x) - R_{sw}, y_s \rangle_W \leq 0, \forall s
\]
\[
\bar{y}_s \in \Delta_{W+1}, \forall s
\]

(\(\text{o-SDC}\))

**Proof.** For a segment $s$, the dual of the lower problem is expressed as

\[
\max_{\mu_s \in \mathbb{R}} \left\{ \mu_s \left| \begin{array}{c}
\mu_s \leq \theta_{sw}(x) - R_{sw}, \forall w \\
\mu_s \leq 0
\end{array} \right. \right\}.
\]

Due to strong duality, the primal objective is lower than the dual one and observing that the dual variable can be eliminated gives us the result. \[\square\]
Remark. In our special case, the approach by strong duality leads to the same formulation as the Optimal Value Transformation\cite{Dem15}. In fact, being lower than all possible values is here equivalent to being lower than the extreme points of the simplex, which is exactly what the strong-duality reformulation gives us.

The formulation \((\alpha\text{-SDC})\) does not contain complementarity constraints but there is no free-lunch: the difficulty is recast in the bilinear terms \(\theta_{sw}(x)y_{sw}, \forall s, w\). These terms are non-convex and they are often relaxed into lifting variables that have to verify the McCormick inequalities. Here, thanks to Proposition 2.1, we can suppose that the lower variables are binary, and thus the McCormick relaxation becomes exact. Formulation \((\alpha\text{-SDC})\) is therefore equivalent to

\[
\begin{align*}
\max_{x \in X, y} & \quad \sum_{s \in [S]} \rho_s \sum_{w \in [W]} \nu_{sw} - C_{sw}y_{sw} \\
\text{s.t.} & \quad \sum_{w' \in [W]} \nu_{sw'} - R_{sw'}y_{sw'} \leq \theta_{sw}(x) - R_{sw}, \forall s, w \\
& \quad \sum_{w' \in [W]} \nu_{sw'} - R_{sw'}y_{sw'} \leq 0, \forall s \\
& \quad \nu_{sw} \leq \theta_{sw}(x), \forall s, w \\
& \quad \nu_{sw} \leq R_{sw}y_{sw}, \forall s, w \\
& \quad \nu_{sw} \geq \theta_{sw}(x) - \max_{x \in X} \{\theta_{sw}(x)\}(1 - y_{sw}), \forall s, w \\
& \quad \bar{y}_s \in \text{Vert}(\Delta_{W+1}), \nu_s \in \mathbb{R}^W_{\geq 0}, \forall s
\end{align*}
\]

Once again, the compactness of \(X\) ensures that the Big-M value \(\max_{x \in X} \{\theta_{sw}(x)\}\) is sufficiently large to keep the validity of the reformulation. Note that the lifting variable \(\nu_{sw}\) represents the product \(\theta_{sw}(x)y_{sw}\) which cannot exceed \(R_{sw}\).

The MIP formulation of equation (7) extends the formulation HLMS, cited in \cite{Fer16} and initially developed by Heilporn et al \cite{Hei10}, to the multi-attribute case.

3 Logit regularization

The formulation \((\alpha\text{-BP})\) models customers reactions as deterministic behaviors. It relies on two assumptions:

(i) customers have perfect rational and deterministic behavior,

(ii) parameters such as reservation prices and costs are perfectly known.

Both assumptions can be discussed: not only real customers are not purely rational agents in that they can choose a contract that does not maximize the utility, but also a segment is the aggregation of quasi-similar customers, not strictly identical ones. Therefore in reality, when a segment faces two very close disutilities, customers of this segment are likely to spread themselves over the two possibilities. Besides, the reservation prices and costs are estimations obtained by analysis on the market but cannot be known exactly. Hence, assuming lower response to be binary as in the optimistic model can be quite unrealistic and may lead to an unachievable optimum. This can be avoided by Logit modeling \cite{Tra09} which captures the probabilistic nature of customers’ choice by adding a Gumbell uncertainty.

Previously, consumers were supposed to choose a contract minimizing their deterministic disutility i.e., each segment \(s \in [S]\) selects \(w^* \in \{0\ldots W\}\) such as \(V_{sw^*} = \min_{w \in \{0\ldots W\}} V_{sw}\) where
\( V_{sw} := \theta_{sw}(x) - R_{sw} \) for all \( w \in [W] \) and \( V_{s0} := 0 \). We now suppose that their disutilities are defined as

\[ U_{sw} := \beta V_{sw} + \varepsilon_{sw}, \forall s, w \]

where \( \varepsilon_{sw} \) is a Gumbell random variable and \( \beta \geq 0 \) is an inverse temperature in the sense of physics. As a consequence, the lower response is expressed as

\[ y_{sw} = \mathbb{P}(U_{sw} \leq U_{sw}^\prime, \forall w^\prime \neq w], \forall s, w . \tag{8} \]

Hence, a customer has a probability to choose a contract which is not the optimal one in terms of deterministic utility.

The calculation of the probability \( y_{sw} \) arising in equation (8) is done in [Tra09] and it has an explicit form. Replacing the deterministic lower response by this expression of \( y_{sw} \) leads to the following Mixed Multinomial Logit model:

\[
\max_{x \in X, y} \sum_{s \in [S]} \rho_s \langle \theta_s(x) - C_s, y_s \rangle_W \\
\text{s.t. } y_{sw} = \frac{e^{-\beta V_{sw}}}{1 + \sum_{w^\prime \in [W]} e^{-\beta V_{sw}^\prime}}, \forall s, w, W_{sw} = \theta_{sw}(x) - R_{sw}, \forall s, w
\]

### Mixed Multinomial Logit model:

\[
\max_{x \in X, y} \sum_{s \in [S]} \rho_s \langle \theta_s(x) - C_s, y_s \rangle_W \\
\text{s.t. } y_{sw} = \frac{e^{-\beta V_{sw}}}{1 + \sum_{w^\prime \in [W]} e^{-\beta V_{sw}^\prime}}, \forall s, w, W_{sw} = \theta_{sw}(x) - R_{sw}, \forall s, w
\]

**Remark.** The ‘1’ in the denominator corresponds to the no-purchase option.

Equivalently, \( \beta-BP \) can be recast into the following bilevel program

\[
\max_{x \in X, \bar{y}} \sum_{s \in [S]} \rho_s \langle \theta_s(x) - C_s, y_s \rangle_W \\
\text{s.t. } \bar{y}_s \in \arg \min_{\bar{y}^\prime_s \in \Delta_{W+1}} \left\{ \langle \theta_s(x) - R_s, \bar{y}^\prime_s \rangle_W + \frac{1}{\beta} \langle \log(\bar{y}^\prime_s), \bar{y}^\prime_s \rangle_{W+1} \right\}, \forall s, w
\]

This model highlights that the logit expression is the optimum of a strictly convex minimization problem (the property was already pointed out in [Fis80] and [GMS15]). The objective function is the deterministic objective function to which we add the entropic regularization term \( \beta^{-1} \langle \log(\bar{y}^\prime_s), \bar{y}^\prime_s \rangle_{W+1} \), attracting the lower response to the center of the simplex \( \Delta_{W+1} \).

The model \( \beta-BP \) is intrinsically defined as a single-level problem since the lower response for any segment \( s \) is unique and analytically known. For a given price strategy \( x \), we define the leader profit function \( \pi^{log}(x; \beta) \) as

\[
\pi^{log}(x; \beta) := \sum_{s \in [S]} \rho_s \sum_{w \in [W]} (\theta_{sw}(x) - C_{sw}) y^{log}_{sw}(x; \beta)
\]

where \( y^{log} \) stands for the logit lower response. This objective function \( \pi^{log} \) is in general neither concave nor convex, see [Li+19].

We can think of the logit model as a regularization of the deterministic case, so it is of interest to look at the convergence for \( \beta \to +\infty \), expecting that the regularized optimum converges to the deterministic optimum. Such a study has been done by Gilbert, Marcotte and Savard [GMS15] in the context of toll pricing. In our setting, we prove asymptotic results without requiring equality between the optimistic and pessimistic versions.
Proposition 3.1. For a fixed price strategy $x \in X$, $\lim_{\beta \to +\infty} y^\log(x; \beta) = y^\uni(x)$. Moreover,
\[
v(u-BP) \leq \liminf_{\beta \to +\infty} v(\beta-BP) \leq \limsup_{\beta \to +\infty} v(\beta-BP) \leq v(o-BP)
\]
and the equalities occur under the no-profit option.

Proof. See Appendix A.1.

The last proposition confirms us that $(\beta-BP)$ is a valid regularization in that it consists on a smooth approximation of $(o-BP)$ for sufficiently big $\beta$ value. Nonetheless, we want to go further in the analysis by obtaining indications on the evolution of the optimal value as $\beta$ grows. To do so, we study the simpler case where there is a unique segment (homogeneous population) and unconstrained prices. This leads to a pricing model under the standard Mixed Multinomial Logit (MNL) customer behavior:

\[
v_\beta := \max_{\theta, y} \left\{ \langle \theta - C, y \rangle_W \mid y_w = \frac{e^{-\beta(\theta_w - R_w)}}{1 + \sum_{w' \in [W]} e^{-\beta(\theta_{w'} - R_{w'})}}, \forall w \right\}.
\]

In [LH11], Li et al. deeply study the model defined in (11) and provide in particular a characterization of its optimal solution. We reuse this property in the next proposition to show an asymptotic result. To this end, we set $v_\infty := \max_w (R_w - C_w)$ and $\#v_\infty$ the cardinality of the latter argmax.

Proposition 3.2 (Customers behavior). For the standard MNL model defined in equation (11),

(i) $v_\beta = \frac{1}{\beta} W_0(W/e) + o \left( \frac{1}{\beta} \right)$; where $W_0$ denotes the Lambert function [Cor+96].

(ii) if $v_\infty > 0$ then $v_\beta = v_\infty - \frac{\ln(v_\infty)}{\beta} + \frac{\ln(\#v_\infty)-1}{\beta} + o \left( \frac{1}{\beta} \right)$.

Proof. See Appendix A.2.

In Figure 1, we draw the optimal value on a simple case, along with the first-order asymptotic expansions found in proposition 3.2. The result for small $\beta$ values is quite intuitive: with customers randomly reacting, the company can impose very high price in such a way that there is always some consumers buying its products. Hence, the company’s profit becomes infinite as $\beta \to 0$. The result for large $\beta$ values is not so evident and can be interpreted as a moral hazard: the randomness in the followers decision negatively impacts the leader revenue. For the company, having deterministic customers is more beneficial since the price can be adjusted to perfectly fit the population behavior.

Establish a similar property in the general case of a heterogeneous population is far more complicated because we are not able to express the optimal value as a function of $\beta$. However, our numerical results tend to confirm that this moral hazard remains in the general case.

4 Quadratic regularization

In the case of a homogeneous population and unconstrained prices, Li et al. [Li+19] express the problem in terms of lower variables to obtain a concave maximization problem. If we add bounds on prices and consider multi-attribute utilities, Shao et Kleywegt [SK20] show another concave
transformation that keeps tractability in the resolution. However, with heterogeneous segments as it is the case here, no tractable transformation is known, and only local optimum of \((q_{\beta^{-BP}})\) can generally be found. This motivates us to look at a new convex penalization, replacing the entropy penalization term in (9) by a quadratic one.

\[
\max_{x \in X, y} \sum_{s \in [S]} \rho_s \langle \theta_s(x) - C_s, y_s \rangle_W \\
\text{s.t. } \bar{y}_s \in \arg\min_{\bar{y}'_s \in \Delta_{W+1}} \left\{ \langle \theta_s(x) - R_s, y'_s \rangle_W + \frac{1}{\beta} \langle \bar{y}'_s - 1, \bar{y}'_s \rangle_{W+1} \right\}, \forall s
\]

The quadratic term \(\beta^{-1} \langle \bar{y} - 1, \bar{y} \rangle\) is chosen so that it vanishes at any vertex of the simplex \(\Delta_{W+1}\). The following result shows that two perhaps more intuitive quadratic terms lead to the same optimum.

**Proposition 4.1.** The two following penalization are equivalent to the one in \((q_{3\beta^{-BP}})\):

(i) \(\frac{1}{\beta} \left\| \bar{y}_s - \frac{1}{W+1} \right\|_{W+1}^2 \) (uniform law attractor),

(ii) \(\frac{1}{\beta} \left\| \bar{y}_s \right\|_{W+1}^2\).

**Proof.** \(\left\| \bar{y}_s - \alpha \right\|_{W+1}^2 - \langle \bar{y}_s - 1, \bar{y}_s \rangle_{W+1} = 2(1 - \alpha) \left( y_{s0} + \sum_{w \in [W]} y_{sw} \right) + \alpha = 2 - \alpha.\)

The two objective functions are equal up to a constant for valid lower responses, thus the argmins are the same.

---

Figure 1: Optimal value of (11) according to \(\beta\).

The solid line represents the optimal value of the example. The asymptotic results found in 3.2 are drawn with dotted lines.
The first item suggests that our new penalization acts as an attractor to the uniform law whose intensity is inversely proportional to $\beta$. The bigger $\beta$ is, the more customers will uniformly spread their choices on all the possibilities.

The second item gives us a more geometrical interpretation: for $W + 1$ disutilities $V_{s0}, \ldots, V_{sW}$, the follower response of a given segment $s$ can be written as

$$\arg\min_{\Delta_{W+1}} \left\{ \sum_{w=0}^{W} V_{sw}y_{sw} + \frac{1}{\beta}y_{sw}^2 \right\} = \arg\min_{\Delta_{W+1}} \left\| y_s - \left( -\frac{\beta}{2}V_s \right) \right\|_{W+1}$$

$$= \text{Proj}_{\Delta_{W+1}} \left( -\frac{\beta}{2}V_s \right).$$

(12)

Here again, the disutility $V_{sw}$ of a segment $s$ stands for a certain $\theta_{sw}(x) - R_{sw}$ in the problem $(q\beta-BP)$. The response can be understood as a projection on the simplex of a specific vector whose intensity varies proportionally to $\beta$.

**Proposition 4.2.** The quadratic lower response $y^{\text{quad}}(x; \beta)$, solution of the lower problem in $(q\beta-BP)$, is a continuous function of the price variables $x$.

**Proof.** From equation (12), the lower response is a projection of the disutility vector $(\theta_{sw}(x) - R_{sw})_{w \in [W]}$. Recall that the projection on a closed convex set is Lipschitz of constant one in the Euclidean norm, a fortiori, it is continuous. Moreover, $\theta_{sw}$ is a linear form. \(\square\)

### 4.1 Lower Response and Leader’s Profit

In the logit model, the lower response of a segment $s$ is analytically known and is defined by the logit expression. To better understand the customer behavior, we aim to find an explicit calculation of the lower response for a segment $s$ that faces disutilities $V_{s0}, \ldots, V_{sW}$. We assume that these disutilities are sorted in ascending order. The lower response $y$ that satisfies (12) is the solution of the KKT conditions expressed as:

$$V_{sw} + \frac{2}{\beta}y_{sw} - \lambda_{sw} - \mu_s = 0, \quad w \in \{0 \ldots W\}$$

$$0 \leq y_{sw} \perp \lambda_{sw} \geq 0, \quad w \in \{0 \ldots W\}$$

$$y_s \in \Delta_{W+1}, \lambda_s \in \mathbb{R}^{W+1}, \mu_s \in \mathbb{R}$$

These conditions are necessary and sufficient because we study a convex minimization problem where the Slater’s condition holds. In the sequel, we analyze the KKT system (13) to characterize the customer response.

**Lemma 4.3 (Monotonicity).** If $y$ satisfies (13), the sequence $(y_{sw})_{w=0..W}$ is decreasing for disutilities sorted in ascending order.

**Proof.** We consider $V_{sw1} \leq V_{sw2}$. If $y_{sw2} = 0$, there is nothing to prove, the inequality $y_{sw1} \geq y_{sw2}$ is automatically satisfied. If however $y_{sw2} > 0$, $\lambda_{sw2} = 0$ by complementarity, and therefore $V_{sw1} + \frac{2}{\beta}y_{sw1} - \lambda_{sw1} = V_{sw2} + \frac{2}{\beta}y_{sw2}$. Since $V_{sw2} - V_{sw1} \geq 0$ and $\lambda_{sw1} \geq 0, \frac{2}{\beta}(y_{sw1} - y_{sw2}) \geq 0$. Therefore, in any case, $\forall w_1, w_2, V_{sw1} \leq V_{sw2} \Rightarrow y_{sw1} \geq y_{sw2}$. \(\square\)

We are now able to exhibit a procedure to compute the solution of the system (13) and so the solution of (12):

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Proposition 4.4 (Lower response algorithm). For any segment $s$, let the sequence $(c_{sw})_{w \in [W]}$ be

$$c_{sw} := \frac{1}{w} \left[ \frac{2}{\beta} + \sum_{w'}^{w-1} V_{sw'} \right]$$

and let the index $\tau$ be defined as $\tau = \min \{ w \in [W], V_{sw} \geq c_{sw} \}$. Then, the sequence $(c_{sw})$ verifies the following property:

$$V_{sw} \geq c_{s\tau} \text{ for } w \geq \tau$$
$$V_{sw} < c_{s\tau} \text{ for } w < \tau$$

(14)

Moreover, the solution $(y_s, \lambda_s, \mu_s)$ of (13) can be expressed as follows

(i) $y_{sw} = \frac{\beta}{2} \left[ c_{s\tau} - V_{sw} \right]$, $w < \tau$,
   $y_{sw} = 0$, $w \geq \tau$,

(ii) $\lambda_{sw} = 0$, $w < \tau$,
     $\lambda_{s\tau} = V_{s\tau} - c_{s\tau}$,
     $\lambda_{sw} = \lambda_{s,w-1} + V_{sw} - V_{s,w-1}$, $w > \tau$,

(iii) $\mu_s = c_{s\tau}$.

The index $\tau$ is therefore the index from which the probability $y$ becomes zero.

Proof. The first property on $(c_{sw})$ comes with the ascending sort of $V_s$ and the definition of $\tau$: $V_{s\tau} \geq c_{s\tau}$ and therefore $V_{sw} \geq c_{s\tau}$ for $w \geq \tau$. Besides, by minimality of $\tau$, $V_{s,\tau-1} < c_{s,\tau-1}$. Using the definition of $(c_{sw})$, for all $w < \tau$, $V_{sw} \leq V_{s,\tau-1} = c_{s\tau} - \frac{\tau-1}{\tau}(c_{s,\tau-1} - V_{s,\tau-1}) < c_{s\tau}$.

Concerning the second part of the proposition, one can first remark that solution of (13) is unique since it is a projection on the simplex, see (12). The procedure returns a certain $(y_s, \lambda_s, \mu_s)$ which is feasible for (13):

$\diamond$ $y_{sw}$ is nonnegative for all $w$: the result is immediate with (14).
$\diamond$ $\lambda_{sw}$ is nonnegative for all $w$: it comes from (14) that $\lambda_{s\tau} \geq 0$, and $\lambda_{s,w+1} \geq \lambda_{sw}$ (disutilities are sorted).
$\diamond$ By construction, $\sum_{w=0}^{\tau} y_{sw} = 1$ and complementarity constraints are satisfied.

The solution we obtain is therefore the unique solution of (13).

From the explicit calculation of the lower response, one can observe the following property

Corollary 4.5 (Soft threshold). If $y_s$ satisfies (13), the first disutility is chosen with probability 1 if and only if the difference between any other disutility and the one chosen is higher than $2/\beta$, i.e.,

$$y_{s0} = 1 \text{ and } \forall w > 0, y_{sw} = 0 \iff \forall w > 0, V_{sw} \geq V_{s0} + \frac{2}{\beta}.$$ 

Proof. From the last proposition, the condition $\forall w > 0, V_{sw} \geq V_{s0} + \frac{2}{\beta}$ is equivalent to $V_{s1} \geq V_{s0} + \frac{2}{\beta}$ which means that $\tau = 1$.

Coming back to problem $(q\beta-BP)$, we summarize the properties of the lower response in the following corollary.
Corollary 4.6 (Lower response of \((q\beta-BP)\)). For a price strategy \(x\) and a given \(\beta\), the quadratic lower response \(y_{sw}^{\text{quad}}(x;\beta)\) for a segment \(s\) can be computed by the following algorithm:

1. Compute \(V_{sw}(x) := \theta_{sw}(x) - R_{sw}\) for all \(w \in [W]\) and \(V_{s0} = 0\),
2. Reindex the utility so that they are sorted in the ascending order,
3. Calculate the solution \(y\) defined in Proposition 4.4,
4. The value \(y_{sw}^{\text{quad}}(x;\beta)\) is the component of \(y\) that corresponds to the disutility \(V_s\) initially indexed by \(w\).

As pointed out in equation (12), the lower response can be viewed as a projection on the simplex. Five algorithms to compute the projection are provided in \([\text{Con16}]\). The first one, applied to the projection \(\text{Proj}_{\Delta W+1}\left(-\frac{\beta}{2} V_s\right)\), allows us to recover the response found in Proposition 4.4. Other algorithms are faster but do not contain such a clear interpretation that customers select disutilities with the lowest values.

The threshold that appears in Corollary 4.5 suggests a link with the work of Shioda et al \([\text{STM11}]\), where they constrain the price strategy to ensure a minimal gap between the lowest disutility and the other ones. Our result shows a soft threshold effect at a finite rationality \((\beta < \infty)\).

We allow the variables \(y_{sw}\) to be fractional values, but they will concentrate on a unique contract per segment if the disutilities are sufficiently separated. This effect only occurs asymptotically \((\beta = \infty)\) in the logit model.

For a given price strategy \(x\), the leader profit function \(\pi^{\text{quad}}(x;\beta)\) is then defined as

\[
\pi^{\text{quad}}(x;\beta) := \sum_{s \in [S]} \rho_s \sum_{w \in [W]} (\theta_{sw}(x) - C_{sw}) y_{sw}^{\text{quad}}(x;\beta) \tag{15}
\]

where \(y_{sw}^{\text{quad}}(x;\beta)\) is defined as explained in Corollary 4.6. The problem \((q\cdot BP)\) is therefore the maximization of the function \(\pi^{\text{quad}}\) over \(X\).

As we done for the logit formulation, we can easily prove the pointwise convergence of the quadratic leader profit to its uniform analog, as well as the convergence of the optimal value:

**Proposition 4.7.** For a fixed price strategy \(x \in X\), \(\lim_{\beta \to +\infty} y_{sw}^{\text{quad}}(x;\beta) = y_{sw}^{\text{uni}}(x)\) and therefore \(\lim_{\beta \to +\infty} \pi^{\text{quad}}(x;\beta) = \pi^{\text{uni}}(x)\) . Moreover,

\[
v(u-BP) \leq \liminf_{\beta \to +\infty} v(q\beta-BP) \leq \limsup_{\beta \to +\infty} v(q\beta-BP) \leq v(o-BP)
\]

and the equalities occur under the no-profit option.

**Proof.** The proof follows exactly the same arguments as the ones we used for the logit version, see Appendix A.1. \(\square\)

### 4.2 Price complex

Baldwin and Klemperer \([\text{BK19}]\) have introduced a geometric approach to analyze the response of agents to prices, in a discrete choice model. They showed that the deterministic response is governed by a polyhedral complex: all prices in a given cell yield the same response. Here, we
Proposition 4.8 (Characterization of the price complex cells). For any pattern \( A \in \mathcal{A}^\beta \) and any \( \beta > 0 \), the UPR \( X(A; \beta) \) is defined as \( X(A; \beta) = X^0(A; \beta) \cap X^1(A; \beta) \) where

\[
X^0(A; \beta) = \left\{ x \in X \mid \forall s, w, \text{ if } A_{sw} = 0, \quad \left| A_s \right| V_{sw}(x) \geq 2\beta^{-1} + \sum_{w' : A_{sw'} = 1} V_{sw'}(x) \right\}, \tag{16a}
\]

\[
X^1(A; \beta) = \left\{ x \in X \mid \forall s, w, \text{ if } A_{sw} = 1, \quad \left| A_s \right| V_{sw}(x) < 2\beta^{-1} + \sum_{w' : A_{sw'} = 1} V_{sw'}(x) \right\}. \tag{16b}
\]

where \( \left| A_s \right| \) corresponds to the number of active contracts for \( s \) and \( V_{sw}(x) \) is defined as in Corollary \[4.6\].

As a consequence, \( X(A; \beta) = X^0(A; \beta) \cap X^1(A; \beta) \) where \( X^1(A; \beta) := cl \left( X^1(A; \beta) \right) \), obtained by weakening the inequalities \( (16b) \).
Proof. Given a pattern $A \in \mathcal{A}^\beta$ and a $\beta > 0$, we can assume w.l.o.g. that for any segment $s$ the disutilites are sorted in ascending order so that the active contracts are the first $|A_s|$ ones.

$\diamond$ First, we consider a price strategy $x \in X(A; \beta)$. Using the notation of Proposition 4.4, $\tau = |A_s|$ and equation (14) gives us exactly the equation (16).

$\diamond$ Reciprocally, we suppose that (16) is satisfied i.e.,
\[
V_{sw} < c_{s,|A_s|} \text{ for } w < |A_s|,
\]
\[
V_{sw} \geq c_{s,|A_s|} \text{ for } w \geq |A_s|.
\]

Then, $\tau = |A_s|$ and $x \in X(A; \beta)$.

\[\square\]

Corollary 4.9. The collection of price complex cells constitutes a $|X|$-dimensional polyhedral complex, and the $|X|$-cells are closures of UPRs i.e., $\overline{X}(A; \beta)$ for some pattern $A \in \mathcal{A}^\beta$.

Proof. It is clear that the collection of price complex cells covers the space $X$. Besides, from the definition of a cell, the intersection of two cells $P$ and $P'$ is again a price complex cell or is empty. Finally, Proposition 4.8 gives us a characterization of the cells with linear inequalities, therefore the intersection of $P$ with another $P'$ is then characterized by the same inequalities as $P$ but with some of them saturated. Hence, the intersection is a common face of $P$ and $P'$.

We now study the asymptotic behavior of the price complex ($\beta \to \infty$) and show how it embeds in the deterministic complex introduced by Baldwin and Klemperer [BK19]. To this end, we first denote by $\overline{X}(A; \infty)$ the polytope defined by the same inequalities as in $X(A; \beta)$ setting $\beta^{-1} = 0$ (idem for $\overline{X}^0$ and $\overline{X}^1$), and by $\mathcal{A}^\infty$ the set of patterns inducing a non empty $\overline{X}(A; \infty)$.

We next make use of the notion of Painlevé-Kuratowski limits of sets. We refer to [RW09, Chapter 4] for background on this notion, including the definition and properties of upper and lower limits.

Lemma 4.10. Let us consider two sequences of polyhedra $(P^+_\beta)$ and $(P^-_\beta)$ defined as
\[
P^\pm_\beta := \left\{ x \in X : Ax \leq b \pm \beta^{-1}e \right\}
\]
where $e$ is the all-ones vector. We consider also the limit case $P := \{ x \in X : Ax \leq b \}$. Then, $P^+_\beta \to P$ and $\lim_\beta P^-_\beta \subseteq P$. Moreover, if $\text{Int}(P) \neq \varnothing$, $P^-_\beta \to P$.

Proof. Throughout the proof, we consider a sequence $(\beta_n)$ converging to $\infty$, and the notation $P^\pm_n$ has to be understood as $P^\pm_{\beta_n}$.

The two monotone sequences have a limit: $\lim_n P^+_n = \bigcap_n P^+_n$ and $\lim_n P^-_n = \bigcup_n P^-_n$, see [RW09, Exercise 4.3], it remains to prove that this limit coincides with $P$. Two first inclusions come with the definition of the sequences: $\lim_n P^-_n \subseteq P$ and $P \subseteq \lim_n P^+_n$.

Let us consider $x \notin P$. If $x \notin X \setminus P$, then there exists a row $i$ such that $A_ix = b_i + \epsilon$ where $\epsilon > 0$. Therefore, for $\beta_n \geq \epsilon^{-1}$, $x \notin P^+_n$. Otherwise, if $x \notin X$, $x$ can not be in any $P^+_n$. In any case, $x \notin P \Rightarrow x \notin \lim_n P^+_n$, and therefore $\lim_n P^+_n \subseteq P$. 

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We now assume that \( \text{Int}(P) \neq \emptyset \). For any given \( x \in P \), let us define the sequence \( x_n := \text{Proj}_{P_n}(x) \). Since the \( P_n \) are decreasing, the distance \( \|x_n - x\| \) is a decreasing sequence bounded from below by 0 and converges to a distance \( d \geq 0 \). Suppose now that \( d > 0 \), then for any unitary vector \( u, x + du \notin P_n, n \in \mathbb{N} \). Besides, there exists \( 0 \leq d' \leq d \) and a unitary vector \( v \) such that \( x + d'v \in \text{Int}(P) \). Defining \( y = x + d'v \), we obtain that \( y \in \text{Int}(P) \) and \( y \notin P_n, n \in \mathbb{N} \). As it belongs to the interior of \( P \), \( Ay \leq b - \epsilon e, \epsilon > 0 \) and for any \( \beta_n \geq \epsilon^{-1}, y \in P_n^{-} \). This yields a contradiction: \( d \) must be equal to 0, and therefore \( x_n \to x \). To conclude, for any \( x \in P \), we can exhibit a sequence of points \( x_n \in P_n^{-} \) converging to \( x \), so \( P \subseteq \lim_{n \to \infty} P_n^{-} \).

**Proposition 4.11 (Convergence).** For any pattern \( A \), \( \limsup_{\beta \to \infty} \overline{X}(A; \beta) \subseteq \overline{X}(A; \infty) \). Moreover, if \( \text{Int}(\overline{X}(A; \infty)) \neq \emptyset \),

\[
\overline{X}(A; \beta) \to \overline{X}(A; \infty) .
\]

**Proof.** Using Lemma 4.10, one can obtain the following inclusions:

\[
\limsup_{\beta} \overline{X}(A; \beta) = \limsup_{\beta} \left( \overline{X}^0(A; \beta) \cap \overline{X}^1(A; \beta) \right) 
\subseteq \lim_{\beta} \overline{X}^0(A; \beta) \cap \lim_{\beta} \overline{X}^1(A; \beta) \subseteq \overline{X}^0(A; \infty) \cap \overline{X}^1(A; \infty) .
\]

Moreover, if \( \text{Int}(\overline{X}(A; \infty)) \neq \emptyset \), then \( \lim_{\beta} \overline{X}^0(A; \beta) = \overline{X}^0(A; \infty) \), see Lemma 4.10. Besides, \( \overline{X}^0(A; \infty) \) and \( \overline{X}^1(A; \infty) \) cannot be separated, and therefore \( \overline{X}^0(A; \beta) \cap \overline{X}^1(A; \beta) \to \overline{X}^0(A; \infty) \cap \overline{X}^1(A; \infty) \).

**Lemma 4.12.** For any pattern \( A \in A^\infty \), the asymptotic cell \( \overline{X}(A; \infty) \) can be equivalently defined by the following system

\[
\forall s, w, w', \text{ if } A_{sw} = A_{sw'} = 1, \quad V_{sw}(x) = V_{sw'}(x), \\
\text{ if } A_{sw} = 1 \text{ and } A_{sw'} = 0, \quad V_{sw'}(x) \geq V_{sw}(x) . \tag{17}
\]

**Proof.** We first define the mean active disutility for a segment \( s \) as \( \bar{V}_s = \frac{1}{|A_s|} \sum_{w' \neq A_{sw'} = 1} V_{sw'} \). Then, we know by (16b) that for any active contract \( w, V_{sw} - \bar{V}_s \leq \frac{2}{\beta} \). Denoting by \( V^+ \) and \( V^- \) the extreme disutilities of active contracts, we obtain \( 0 \leq V^+ - V^- \leq \frac{2}{\beta} \). At the limit, active contracts share a same disutility, equal to \( \bar{V}_s \). Besides, we also know from (16a) that for any inactive contract \( w, V_{sw} \geq \frac{2}{\beta} + \bar{V}_s \). At the limit, any inactive contract has disutilities greater than the active contracts.

**Lemma 4.13.** For any mixed pattern \( A \), there exist \( k > 1 \) pure patterns \( A^1, \ldots, A^k \) such that

\[
\overline{X}(A; \infty) = \bigcap_{1 \leq i \leq k} \overline{X}(A^i; \infty) .
\]

**Proof.** Suppose that for a given segment \( s, |A_s| = k \), then we can construct patterns \( A^1, \ldots, A^k \) such that \( A^i \) is a copy of \( A \) where the row \( s \) is replaced by 1 on the \( i \)th active contract, and 0 everywhere else. From the characterization (17), we obtain that \( \overline{X}(A; \infty) = \bigcap_{1 \leq i \leq k} \overline{X}(A^i; \infty) \). Each pattern \( A^i \) has pure strategy for segment \( s \). If there still exist mixed strategies for other segment, we can start again the transformation until all the patterns are pure.

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At the limit \( \beta = \infty \), each mixed pattern is a face of some pure patterns. The pure patterns are therefore sufficient to describe any cell.

**Theorem 4.14** (Asymptotic cells and UPRs). Let \( A^1, \ldots, A^k \) be \( k \) pure patterns, then

\[
x \in P = \bigcap_{1 \leq i \leq k} X(A^i; \infty) \iff \{ A^1, \ldots, A^k \} \subseteq \Psi(x).
\]

where \( \Psi(x) \) is the set of optimistic best responses, defined in \( \text{[3]} \). Moreover, for any pure pattern \( A \),

\[
\text{Int}(X(A; \infty)) = \{ x \in X : \{ A \} = \Psi(x) \}.
\]

**Proof.** The equivalence is a direct consequence of the Lemma \([4.12]\). The equality also arises from this lemma: the set \( \{ x \in X : \{ A \} = \Psi(x) \} \) is characterized by \([17]\) with strict inequalities.

Theorem \( 4.14 \) establishes a link with the approach of Baldwin and Klemperer: the cell definition we introduce in Definition \( 2 \) is equivalent to \([BK19, \text{Definition 2.5}]\) for \( \beta = \infty \) and generalizes the price complex to relaxed choices. Moreover, Baldwin and Klemperer define unique demand region (UDR) where the set \( \Psi(x) \) has a unique element, and Theorem \( 4.14 \) proves that any pure UPR converges to the corresponding UDR. To illustrate Proposition \( 4.8 \), Figure \( 2 \) shows the complex

![Price complex in a simple case.](image)

Figure 2: Price complex in a simple case.

For \( \beta = \infty \) (deterministic case, solid line), the three cells correspond to the the choice of a unique contract (the rectangles indicate the choice). For \( \beta < \infty \), each line “splits” to create intermediate cells (mixed strategies). Pure strategies correspond to white zones, strategies mixing two contracts correspond to light gray zones and the strategy mixing the three contracts corresponds to the dark gray zone.

cells for a single customer making a choice among two contracts from the company and one from a competitor. The deterministic complex was depicted in \([BK19, \text{Figure 1}]\) or in \([Eyt18]\) for bilevel models, and Figure \( 2 \) illustrates the generalization of the price complex to relaxed choices: note that new types of full-dimensional cells, representing choices concentrated on several contracts, appear.
We have showed that the logit profit function has no good convexity properties in our context of a heterogeneous population. Thanks to the properties of the lower response and the notion of polyhedral complex, we can prove that its quadratic analog is more structured:

**Lemma 4.15.** For $K \geq N$, the function $J : x \in \mathbb{R}^N \mapsto \sum_{i=1}^{N} x_i^2 - \frac{1}{K} \left( \sum_{i=1}^{N} x_i \right)^2$ is convex.

**Proof.** The Hessian $H$ of the function $J$ is $H_{ij} = -2/K$ for $i \neq j$ and $H_{ii} = 2 - 2/K$. Using the Gershgorin circle theorem, any spectral value $\lambda_i$ has to verify $|\lambda_i - (2 - 2/K)| \leq \sum_{j \neq i} 2/K$. Therefore, $\lambda_i \geq 2 - 2N/K$ and we deduce that all spectral values of $H$ are nonnegative. \(\square\)

**Theorem 4.16 (Profit decomposition).** The quadratic leader profit function $\pi^{quad}(x; \beta)$ is continuous. Moreover, the problem $\{q\beta-BP\}$ is equivalent to the following problem

$$
\max_{A \in A^\beta} \left\{ \varphi(A; \beta) := \max_{x \in X(A; \beta)} \pi^{quad}(x; \beta) \right\}
$$

where $\pi^{quad}(x; \beta)$ is concave on each price complex cell $X(A; \beta)$, defined in Proposition 4.8.

**Proof.** The continuity of the lower response (Proposition 4.2) suffices to ensure the continuity of $\pi^{quad}$. The difficulty lies in the concave foundation.

Because the profit function is a sum over the segments, we may assume that there is only one segment $s$. Let us consider a feasible pattern $A \in A^\beta$. On the cell $X(A; \beta)$ associated with this pattern, the profit function is expressed as

$$
J^A_s(x) := \sum_{w \in [W] \mid A_{sw} = 1} (\theta_{sw}(x) - C_{sw})y_{sw}^{quad}(x; \beta).
$$

To keep compact notation, we define $W^A_s := \{w \in [W] \mid A_{sw} = 1\}$, and once again, we use the notation $V_{sw} := \theta_{sw}(x) - R_{sw}$ for $w \in [W]$ and $V_{s0} = 0$. Using Corollary 4.6, we can rewrite the function $J$ as

$$
J^A_s(x) = \frac{\beta}{2} \sum_{w \in W^A_s} (V_{sw} + R_{sw} - C_{sw})(c_{s|A_s} - V_{sw})
$$

$$
= \frac{\beta}{2} \sum_{w \in W^A_s} (R_{sw} - C_{sw})(c_{s|A_s} - V_{sw}) + \frac{\beta}{2} \left[ \sum_{w \in W^A_s} V_{sw}^2 - c_{s|A_s} \sum_{w \in W^A_s} V_{sw} \right]
$$

$$
= L - \frac{\beta}{2} \left[ \sum_{w \in W^A_s} V_{sw}^2 - \frac{1}{|A_s|} \left( \sum_{w \in W^A_s} V_{sw} \right)^2 \right]
$$

where $L := \frac{1}{|A_s|} \sum_{w \in W^A_s} V_{sw} + \frac{\beta}{2} \sum_{w \in W^A_s} (R_{sw} - C_{sw})(c_{s|A_s} - V_{sw})$ is the linear part of the objective.

The set $W^A_s$ has a cardinality of $|A_s|$ or $|A_s| - 1$ depending on if the no-purchase option appears in the first $|A_s|$ disutilities. Therefore, by Lemma 4.15, $J^A_s$ is concave in $V_s$, and thus is concave in $x$ since the functions $\theta$ are linear.

Finally, explore $X(A; \beta), A \in A^\beta$ is sufficient to cover the whole space $X$. \(\square\)

Theorem 4.16 paves the way to enumerative scheme resolutions because it shows that the problem can be polynomially solved on each cells of the polyhedral complex, and if all the cells are explored it gives a global optimum. Nonetheless, it could be very cumbersome (especially for low $\beta$ values).
4.3 QPCC Reformulation

As in the deterministic case, the model can be recast into a single-level program with complementarity constraints using the KKT conditions. Moreover, we are able to replace the bilinear terms using manipulations on the constraints:

**Theorem 4.17.** The problem \((q\beta-BP)\) is equivalent to the following concave QPCC problem

\[
\begin{align*}
\max_{x \in X, \mu \in \mathbb{R}^S, \bar{y}, z} & \sum_{s \in [S]} \rho_s \mu_s + \rho_s \langle R_s - C_s, y_s \rangle_W - 2\beta^{-1} \rho_s \|\bar{y}_s\|_{W+1}^2 \\
\text{s.t.} & \quad 0 \leq y_{sw} \perp \theta_{sw}(x) - R_{sw} + 2\beta^{-1} y_{sw} - \mu_s \geq 0, \forall s, w \\
& \quad 0 \leq y_{s0} \perp 2\beta^{-1} \bar{y}_s - \mu_s \geq 0, \forall s \\
& \quad \bar{y}_s \in \Delta_{W+1}, \forall s
\end{align*}
\]

\((q\beta-QPCC)\)

Proof. The KKT optimality condition have been detailed in (13). One can remark that the variable \(\lambda\) can be removed to obtain the KKT system of \((q\beta-QPCC)\). We then reformulate the objective by using the constraints: for a given \(s \in [S]\),

\[
\langle \theta_s(x), y_s \rangle_W = \langle \mu_s e_W + R_s, y_s \rangle_W - 2\beta^{-1} \|y_s\|_W^2
\]

\[
= \mu_s - \mu_s y_{s0} + \langle R_s, y_s \rangle_W - 2\beta^{-1} \|y_s\|_W^2.
\]

Finally, the objective in \((q\beta-QPCC)\) is obtained using the complementarity constraint on the no-purchase option: \(\mu_s y_{s0} = 2\beta^{-1} y_{s0}^2\).

As in the deterministic case, we can replace the complementarity constraints in \((q\beta-BP)\) by Big-M constraints to obtain a mixed-integer quadratic problem (MIQP). However, the introduction of binary variables is unavoidable since the existence of a solution with integer lower response is no longer true:

\[
\begin{align*}
\max_{x \in X, \mu \in \mathbb{R}^S, \bar{y}, z} & \sum_{s \in [S]} \rho_s \mu_s + \rho_s \langle R_s - C_s, y_s \rangle_W - 2\beta^{-1} \rho_s \|\bar{y}_s\|_{W+1}^2 \\
\text{s.t.} & \quad 0 \leq \theta_{sw}(x) - R_{sw} + 2\beta^{-1} y_{sw} - \mu_s \leq M_{sw}(1 - z_{sw}), \forall s, w \\
& \quad 0 \leq 2\beta^{-1} \bar{y}_s - \mu_s \leq M_{s0}(1 - z_{s0}), \forall s \\
& \quad y \leq z \\
& \quad \bar{y}_s \in \Delta_{W+1}, z \in \{0, 1\}^{W+1}, \forall s
\end{align*}
\]

\((19)\)

QPCC problems have been recently studied, using conic relaxations [Den+17; ZX19] or logical Benders [BMP13; Jar+20]. In the latter, they introduce the notion of complementarity piece defined by a valuation of the binary vector \(z\). The complementarity pieces of \((q\beta-QPCC)\) coincide with the cells \(X(A; \beta)\) of the price complex (16): admissible valuations of \(z\) define feasible patterns, and vice versa.

5 Comparisons between relaxations

Quadratic and logit regularizations share a parameter \(\beta\), interpreted as a rationality parameter. It will be convenient to replace the regularization parameter \(\beta\) in the quadratic model by \(\beta' = \frac{2\beta^2}{1 - 2}\).
\(\beta e/4\), leaving the value \(\beta\) in the logit model. In fact, the minimum of \(\frac{1}{\beta} y(y - 1)\) is \(-\frac{1}{4\beta}\) whereas the minimum of \(\frac{1}{\beta} y \log(y)\) is \(-\frac{1}{e\beta}\), and so this choice of \(\beta\) equalizes the minimal intensity of the regularization term. To have a better intuition on the differences and similarities between the logit and quadratic regularization, we study a simple case where there is one single-attribute contract and five customers. We provide in Fig. 3 the leader profit as a function of the contract price for multiple configurations (the optimistic version, the quadratic version and the logit version for two values of \(\beta\)). The behavior of the deterministic and logit profit have been already compared in an

![Figure 3: Comparison of profit functions \(\pi^{opt}\), \(\pi^{log}\) and \(\pi^{quad}\)](image)

other context in [GMS15]. We now include the quadratic model in this comparison. The following properties of profit functions can be identified:

- The deterministic profit is piecewise linear but contains discontinuities that arise when two contracts share the same minimal disutilities (here between the only contract and the no-purchase option). As proved in Proposition 2.2, the optimal profit is always attained at such a frontier price, leading to an instability: for this specific case, the optimal deterministic profit is higher than 11 and is achieved for \(x = 3.7\). Nevertheless, a price of \(x = 3.71\) induces a profit lower than 4.

- The logit regularization smooths the deterministic profit function while maintaining its global shape for \(\beta\) large enough. Nonetheless, the function is non-convex and we can observe for \(\beta = 0.8\) two local maxima.

- The quadratic regularization and its logit analog share the same behavior: in fact, the shape is very similar for both values of \(\beta\). The difference lies in the structure of the quadratic model: the profit function is piecewise concave, see Theorem 4.16.

To complete the comparison, we make explicit in Appendix C estimates of the distance between customers response in the logit and quadratic model.
Figure 4: Comparison between the two quadratic regularizations

The profit $\pi^{\text{quad}}$ is drawn for $\beta = 4$ and $\beta = 20$ and the version of [SSC17] ($\pi^{\text{SSC}}$) is drawn for $\varepsilon = 0.1$ and $\varepsilon = 0.01$.

Sun, Su and Chen [SSC17] have already used a quadratic regularization for the multi-product pricing problem. As we do in this paper, they provide a closed-form formula and look at the leader profit function. The main difference between the two quadratic regularizations lies in the fact that our version is only a primal regularization whereas the regularization of [SSC17] is of a primal-dual nature. This leads to distinct regularized solutions – in particular, the lower responses that we obtain still lie in the simplex. Figure 4 illustrates the profit obtained with the two versions, reusing Example 2 provided in [SSC17].

6 Heuristics

In the previous sections, we proved geometrical properties for the quadratic regularization, and in particular Theorem 4.17 provides a direct formulation which allows us to find a global optimum via MIQP techniques. However, such methods are workable up to a limited instance size, above which a good optimality gap cannot be obtained in reasonable time. Therefore, a heuristic has been designed, taking advantage of the structure highlighted in Theorem 4.16.

6.1 Local descent

Theorem 4.16 shows that finding a good solution consists of finding a good cell of the polyhedral complex. Therefore, a local search can be performed by iterating on the cells/patterns and solving at each iteration the quadratic program associated with the cell.

Proposition 4.8 describes a cell as a system of $S \times (W + 1)$ linear inequalities and the neighboring cells are characterized by the same system with an inequality having its sign reversed. Therefore, exploring all the possible neighbors requires the resolution of $S \times (W + 1)$ quadratic programs. To
curtail the exploration, we introduce a narrow neighborhood which selects only a few neighbors that are good candidates in the search for better solutions.

To this aim, for a given feasible pattern \( A \), suppose that \( x_A \) is the solution of the inner maximization problem of (18) i.e., \( \varphi(A; \beta) = \pi_{\text{quad}}(x_A; \beta) \), then for any segment \( s \), we select two inequalities on which we will pivot:

(i) the inequality \((s, w)\) where \( w \) is the active contract with the greater disutility for \( s \) (i.e., with the lowest positive probability),

(ii) and the inequality \((s, w)\) where \( w \) is the non-active contract with the lowest disutility for \( s \).

These two inequalities constitute for segment \( s \) the boundary between being active and non-active.

Although this strategy does not explore the whole neighborhood (only explores \( 2S \) possibilities), pivoting on the selected inequalities is likely to produce feasible neighbors. Numerically, there are two possibilities for exploring these neighbors:

1. computing \( \varphi(\cdot; \beta) \) for each of the \( 2S \) neighboring patterns by solving \( 2S \) quadratic programs, see (18), and returning the best pattern \( A' \) with its value \( \varphi(A'; \beta) \) (could be the initial pattern if no improvement was made),

2. or solving (19) where the only unfixed binary variables \( z \) are the \( 2S \) variables indexed by the selected inequalities (the other variables \( z \) are equal to the current pattern values) and returning the pattern \( A' \) obtained by the solver with its value \( \varphi(A'; \beta) \).

One has to notice that for a strict equivalence between the two options, we should add in the second one the constraint that a unique \( z \) variable changes its value. Nonetheless, we observe in the tests that forgetting this constraint is still tractable and this leads to a larger exploration.

This exploration phase will be called \texttt{exploreGoodNeighbors} and returns the best feasible pattern (could be the initial one if no better solution was found) and the optimal value.

With this subroutine, a local descent can be performed as follows

\textbf{Algorithm 1} Local Descent

\begin{algorithm}
\begin{algorithmic}[1]
\Require \( A \) (initial pattern)
\While{true}
\State \( A', \varphi \leftarrow \text{exploreGoodNeighbors}(A) \)
\If{\( A' = A \)}
\State \textbf{break}
\EndIf
\State \( A \leftarrow A' \)
\EndWhile
\State \Return \( A, \varphi \)
\end{algorithmic}
\end{algorithm}

This local search always terminates because we progress through patterns that have better and better optimal values. Since the number of patterns is finite, we are sure to stop. In our data set, it happens in less than 20 iterations.
6.2 Random teleporting

The local search ends up with a local optimum in the sense that there is no neighbor (achievable by `exploreGoodNeighbors`) that produces a better solution. To visit more space and possibly obtain greater optimum, a larger neighborhood is mandatory. To this end, when a local search is ended, we choose indices \((s, w)\) to unfix as follow:

(i) \(\gamma^S\) segments are fully unfixed: for such a segment \(s\), indices \((s, w)\) are selected for all \(w \in 0 \ldots W\) i.e., the whole row \(s\) in the pattern,

(ii) \(\gamma^W\) contracts are fully unfixed: for such a contract \(s\), indices \((s, w)\) are selected for all \(s \in [S]\) i.e. the whole column \(w\) in the pattern,

(iii) each index \((s, w)\) has a probability \(\sigma \in [0, 1]\) of being unfixed.

We then solve (19) where the unfixed binary variables \(z\) corresponds to the selected indices and return the pattern \(A'\) obtained by the solver with its value \(\varphi(A'; \beta)\). When the algorithm achieves \(j_{\text{max}}\) successive iterations without any improvement, it returns the best solution found. Figure 5 makes iterations of both `localDescent` and `randomTeleporting` visual, and Section 6.2 describes the combination of the local descent with the random teleporting iterations.

![Figure 5: Pattern evolution during the heuristic.](image)

On subfigure (a), for each segment/row two indices are selected: the least active (orange) and the first non-active (green) ones. On subfigure (b), a segment \(s\) is selected and the binary variables associated to \(s\) are unfixed, same for a contract \(w\). Besides, some other random binary variables are unfixed.

6.3 Implicit resolution via CMA-ES

Another way to solve the model \([q\beta-BP]\) is from the profit-maximization point of view, considering directly the nonsmooth problem

\[
\max_{x \in X} \pi^{\text{quad}}(x; \beta)
\]  

(20)
Algorithm 2 localDescent+RandomTeleporting

Require: $A$ (initial pattern)

1: $k, j \leftarrow 0, 0$
2: while true do
3:   $A', \varphi \leftarrow \text{randomTeleporting}(A)$
4:   if $A' = A$ then
5:     $j \leftarrow j + 1$
6:   else
7:     $j \leftarrow 0$
8:   end if
9:   $A', \varphi \leftarrow \text{localDescent}(A')$
10:  $A \leftarrow A'$
11: if $j < j_{\text{max}}$ then
12:   break
13: end if
14: end while

where the function $\pi^{\text{quad}}(\cdot; \beta)$ is defined in (15). Taking advantage of the lower response uniqueness to end in a nonsmooth problem – where lower variables are functions of the upper ones – constitutes the basis of implicit methods for bilevel problems, see [KLM20]. Implicit methods require an oracle able to evaluate the objective function for any given point. Therefore, the explicit calculation of the lower response given by Corollary 4.6 turns out to be essential in order to design the oracle. Powerful algorithms are already available, and we focus on Covariance matrix adaptation evolution strategy (CMA-ES) [Han06] which has demonstrated its efficiency, see for instance [Han+10].

In our problem the search space $X$ has a reasonable dimension ($W \times H$). Therefore, we can expect CMA-ES to find good solutions. For the numerical tests, we used a library available in C++ that implements this algorithm [Ben21].

7 Numerical results

In the numerical tests, we focus on electricity pricing: a company has $W = 10$ different contracts that need to be optimized, each one depending on $H = 3$ coefficients (peak/off-peak/fixed part). These contracts mimic the existing French contracts, and are listed in the Table 1. Concerning the customers, a thousand of load curves (obtained by the SMACH simulator of EDF, see [Hur+15]) represents the power consumption of the entire French population, taking into account different household compositions, locations, and electrical equipments. To construct our set of instances, we use the $k$-means algorithm to obtain $S$ clusters (segments), where $S$ varies along with the instances. In this way, customers that have similar consumptions and contract preferences are aggregated in the same cluster.

The pre and post processing algorithms are implemented in Python 3.7, whereas the optimization methods are implemented in C++ for numerical efficiency. Besides, we use Cplex v12.10 [Cpl09] as a MIQP solver and the tests are performed on a laptop Intel Core i7 @2.20GHz × 12. However, in order to have easily replicable results, we constrained Cplex to run only on 4 threads.
| Offer       | Description                                                                 |
|-------------|-----------------------------------------------------------------------------|
| LowCost     | Low cost offers (digital-only customer services)                            |
| Green       | Higher costs, but preferred by some segments (higher reservation price)     |
| Week-End    | Week-end consumption priced as an off-peak period                           |
| Week-End++  | Customers add a day to the week-end as an off-peak period                   |
| Green+WE    | A Green offer, but the week-end is considered as an off-peak period         |

Table 1: Contracts used in the instances

Each offer has a base load version (no price difference between peak and off-peak periods) and a version with different prices at peak and off-peak periods, making a total of 10 contracts.

### 7.1 Evolution with the clustering size

![Figure 6: Numerical results for five methods (Opt-KKT, Opt-SD, Quad-Cplex, Quad-H, Quad-CMA).](image)

The upper graph shows the objective value and the lower graph shows the resolution time for a segments number $S$ varying between 5 to 50 and a $\beta$ fixed to 0.5. For heuristic methods, five tries have been done and vertical lines indicate the least and the greatest value. The final gap obtained with the quadratic method is represented with a yellow zone (between the best solution and the best upper bound).

Figure 6 shows the performances of the five methods listed in Table 2.

The numerical tests highlight the combinatorial explosion induced by the direct resolution of the quadratic model with CPLEX for a finite $\beta$ (method Quad-Cplex). The critical size seems to be around 30 segments on our data set. In contrast, the optimistic value is very fast to obtain up to 50 segments, either with Opt-SD or Opt-KKT. This emphasizes the need of heuristics to rapidly
obtain good solutions of the quadratic model.

The method CMA-ES is rather suitable for very large instances. In fact, the algorithm explores the domain $X$ which does not depend on the number of segments $S$, and the time to compute the lower response (by Proposition 4.4) linearly increases in $S$. The overall resolution time of CMA-ES has therefore an affine growth in the number of segments. Besides, the best solution found by CMA-ES seems to edge closer to optimum as the size grows. Increasing the number of segments dwindles the weight of each one in the objective, that tends to smooth the profit function and, as a consequence, facilitates CMA-ES in the resolution.

The great power of Quad-H is to systematically find very good solutions (no large variance of the optimal value), even for large instances. Of course, this is only possible because we exploit the special geometry of our problem (as opposed to a generic algorithm like CMA-ES). Concerning resolution time, Quad-H is also faster. However, Quad-H becomes computationally more expensive as the number of segments increases, since it involves the randomTeleporting step (solution of a MIQP problem).

Finally, this numerical study gives us an a posteriori way to know how many segments are needed to accurately represent the population. After 30 segments the objective value seems to reach a plateau: using more segments does not seem to add a useful information (at least in terms of optimal value).

### 7.2 Evolution with the rationality parameter

The influence of $\beta$ on the optimal solution is emphasized by Fig. 7. The objective (upper graph) and Figure 1 share a very similar shape. The moral hazard effect that was proven on a simple logit model remains in the quadratic case – at least in this instance. The middle graph shows the mean disutility -- defined as $\sum_{s \in [S]} \rho_s \langle \theta_s(x) - R_s, y_s \rangle_W$. It almost vanishes as $\beta \to \infty$: the company succeeds in providing to each segment a contract that is a few cents cheaper than the no-purchase option (competitors offer). Moreover, in this data set, customers take advantage of a small unpredictability ($\beta < \infty$) since the mean disutility approaches 0 from below.

The computational time is maximal for intermediate values of the rationality parameter ($\beta \simeq 10^{-1}$). Indeed, the quadratic regularization degenerates to the deterministic best response when $\beta \to \infty$, making the problem less difficult. Moreover, for very small values of $\beta$, the problem is also degenerate since very high prices are still chosen by customers leading to trivial optimal policies of the leader.

### 7.3 Comparison with NLP solvers

Non-linear programming (NLP) constitutes a third alternative – with implicit methods and combinatorial methods – in the resolution of complementarity problems. A generic complementarity
constraint $0 \leq x \perp y \geq 0$ can be viewed as the set of constraints

$$x \geq 0, \ y \geq 0, \ x^T y \leq 0 .$$

Unfortunately, the non-linear inequality in (21) violates the Mangasarian-Fromovitz constraint qualification (MFCQ) at any feasible point, see [Dus+20]. To circumvent this issue, a smooth function $\phi$ can be introduced to regularize the complementarity constraint into

$$x \geq 0, \ y \geq 0, \ \phi(x, y) \leq 0 .$$

This function belongs to the class of NCP functions, see [Ley06] for a description of the most widely used functions. Solvers have been designed/adapted to deal with these reformulations, see [KLM20] for a recent practical survey. For the numerical tests, we focus on two solvers:

(i) Knitro [Art20], which is a powerful commercial solver, able to recognize if the problem contains complementarity constraints to reformulate them as a non-linear inequalities,

(ii) filterMPEC [FL04], which is an extension a Sequential Quadratic Programming (SQP) solver designed to solve MPECs. The theoretical material is described in [Ley06]. Note that we keep the scalar product form ($\text{compl Frm} = 1$) in all the resolutions.

Both solvers are available through the free optimization platform NEOS [CMM98].
The upper graph shows the objective value and the lower graph shows the resolution time for a segments number $S$ varying between 5 to 50 and a $\beta$ fixed to 0.5. For heuristic methods, five tries have been done and vertical lines indicate the least and the greatest value. The final gap obtained with the quadratic method is represented with a yellow zone (between the best solution and the best upper bound).

Figure 8 compares the results obtained by Knitro and filterMPEC with our heuristic. We still display the value returned by Quad-Cplex to bound the optimality gap. The whole graph is computed with instances that slightly differ from the ones on Figure 6: the polytope $X$ only contains the bounds on prices and not any other constraint. In fact, the solution returned by NLP methods violates the constraints by an $\epsilon$ and if the polytope $X$ were more complicated than a box, it would require a finer post-processing to reconstruct a valid price vector $x$ that exactly respects the inequalities/equalities of $X$.

The two NLP solvers are very fast to return a solution, either Knitro or filterMPEC, even if the time cannot be considered as a uniform indicator since the calculations were achieved on NEOS servers whereas Quad-H was run on a personal computer. On these instances, Quad-H always returns better solutions. In fact, only Clarke-Stationary points can be ensured by NLP solvers, see [KLM20] and the references therein. Of the two solvers, Knitro seems to be the fastest, but we run it on 4 threads whereas filterMPEC uses a SQP algorithm which is difficult to parallelize.
8 Conclusion

We explored an extension of the unit-demand envy-free pricing problem, in which the customer invoice is determined by multiple price coefficients. We first analyzed a bilevel programming model, assuming a fully deterministic behavior of customers (every customer takes only one contract, maximizing her utility). This is inspired by known models in the case of a unique price coefficient. Such bilevel problems reduce to mixed linear programming, allowing one to solve optimally instances of intermediate size. However, the assumption of deterministic behavior is not realistic, at least for the class of electricity pricing problems that motivate this work. We developed an alternative model, based on a quadratic regularization, which combines tractability and realism. We demonstrated that the lower response map of this quadratic model is characterized by a polyhedral complex, and using this geometrical property, we designed a heuristic which showed its efficiency in terms of optimality and time on our data set. In particular, we could solve instances of substantial size (10 contracts, 50 segments) in a reasonable time with an accuracy of less than 3%.

Several extensions may be considered to further improve the realism. In particular, throughout the paper, competitors are supposed not to adjust their prices to the strategy of the company (static competition), and customers are supposed to immediately react to the prices (no switching cost). Modeling such features would lead to dynamic games.

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A Logit regularization – Proofs

A.1 Proof of Proposition 3.1

\[ \limsup_{\beta \to +\infty} v(\beta \cdot BP) \leq v(o \cdot BP): \]

Let \((\beta_n)_{n \in \mathbb{N}} \in \mathbb{R}^N\) such that \(\lim_{n \to +\infty} \beta_n = +\infty\) and \((z_n)_{n \in \mathbb{N}} \in (X \times \Delta_{W+1})^N\) the sequence of \(z_n = (x_n, \tilde{y}_n)\) solutions of \((\beta_n \cdot BP)\). By compactness of \(X\) and \(\Delta_{W+1}\), \((z_n)_{n \in \mathbb{N}}\) has accumulation points.

Let \((x^*, \tilde{y}^*)\) be one of these points, we must show that \(\tilde{y}^* \in \Psi(x^*)\). We take \(s \in [S]\) and two contracts \(w_1\) et \(w_2\) such that

\[ \theta_{sw_2}(x^*) - R_{sw_2} = \gamma + \theta_{sw_1}(x^*) - R_{sw_1}, \gamma > 0. \]

By definition of accumulation points, there exists a sub-sequence \((x_k, \tilde{y}_k)_{k \in K}\) converging to \((x^*, \tilde{y}^*)\) and by continuity of \(\theta\), \(\exists k_1 \in \mathbb{N}, \forall k \geq k_1, \theta_{sw_2}(x_k) - R_{sw_2} \geq \theta_{sw_1}(x_k) - R_{sw_1} + \frac{1}{2}\gamma.\)

Therefore, using the definition of logit probabilities, \(\forall k \geq k_1, 0 \leq (\tilde{y}_k)_{sw_2} \leq e^{-\frac{1}{2}\beta_k \gamma}.\)

Because \(\beta_k\) goes to infinity, we can conclude that \((\tilde{y}^*)_{sw_2} = 0\) and since only minimum disutilities have positive probabilities, \(\tilde{y}^* \in \Psi(x^*).\)

To conclude, all adherent points \((x^*, \tilde{y}^*)\) of the sequence are feasible solution of \((o \cdot BP)\).

\[ v(u-BP) \leq \liminf_{\beta \to +\infty} v(\beta \cdot BP): \]
Because we can’t be sure that the supremum of the uniform problem \( u\text{-BP} \) is attained, we take a sequence \( (x_n, \tilde{y}_n)_{n \in \mathbb{N}} \in (X \times \Delta_{W+1})^\mathbb{N} \) of solutions of the uniform model. We denoted by \( v_n \) their objective value which converges to \( v(u\text{-BP}) \).

Let \( \varepsilon > 0 \) be given, from the convergence, \( \exists n_1 \in \mathbb{N}, \forall n \geq n_1, v_n \geq v(u\text{-BP}) - \varepsilon / 2 \).

Besides, from each \( x_n \) we construct \( \tilde{y}_n(\beta) \) such that \( \tilde{y}_n(\beta) = \logit(x_n; \beta) \) and \( \lim_{\beta \to +\infty} \tilde{y}_n(\beta) = \bar{y}_n \).

For all \( \beta \), \( (x_n, \tilde{y}_n(\beta)) \) is a valid solution for \( (\beta\text{-BP}) \) and its objective function converges to \( v_n \) when \( \beta \to +\infty \) i.e.,

\[
\exists \beta_n \in \mathbb{R}, \forall \beta \geq \beta_n, v(\beta\text{-BP}) \geq v_n - \varepsilon / 2 .
\]

In particular, for all \( \beta \geq \beta_{n_1} \), \( v(\beta\text{-BP}) \geq v(u\text{-BP}) - \varepsilon \).

### A.2 Proof of Proposition 3.2

We know by [LH11, Theorem 2] that \( \beta v_\beta \) satisfies

\[
\beta v_\beta e^{\beta v_\beta} = \sum_{w \in [W]} e^{-1 + \beta(R_w - C_w)} .
\]

and so \( \beta v_\beta = W_0(f(\beta)) \), where \( f(\beta) = \sum_{w \in [W]} e^{-1 + \beta(R_w - C_w)} \).

The result for the first item comes naturally.

For the second, because we suppose \( v_\infty > 0 \), we have \( \lim_{\beta \to +\infty} f(\beta) = +\infty \).

An elementary calculation shows that

\[
\ln(f(\beta)) = \beta v_\infty - 1 + \ln(\#v_\infty) + o(1) .
\]

and it follows also \( \ln\ln(f(\beta)) = \ln(\beta v_\infty) + o(1) \).

From the properties of the Lambert function,

\[
W_0(f(\beta)) = \ln(f(\beta)) - \ln\ln(f(\beta)) + O\left(\frac{\ln\ln(f(\beta))}{\ln(f(\beta))}\right) .
\]

We therefore obtain that \( \beta v_\beta = \beta v_\infty - \ln(\beta v_\infty) - 1 + \ln(\#v_\infty) + o(1) \). The result is obtained by dividing by \( \beta \).

### B Complexity

Guruswami et al. [Gur+05] proved that the deterministic model is APX-hard (see [Pas09] for a description of this class). Using this result, we prove that the quadratic case is also APX-hard:

**Proposition B.1.** The problem \( (q\beta\text{-BP}) \) is APX-hard, even in the single-attribute setting and without price constraints.

**Proof.** Reusing the same polynomial transformation as in [Gur+05], we claim the existence of a sufficiently large parameter \( \beta \) \((\beta \geq 8(n + m))\) such that the quadratic optimal value is not far from the deterministic one i.e., \(|v(q\beta\text{-BP}) - v(o\text{-BP})| \leq 1 / 4\).

First, it can be noticed that the optimal prices can’t be any values: for any product,
If the price is in $\left[ 1 - \frac{2}{\beta}, 1 - \frac{2}{\beta} \right]$, then customers having a null reservation price for the contract will have no chance to purchase it and customers having reservation price of 1 or 2 will purchase it with probability 1. So the company has more interest in setting the price at $1 - \frac{2}{\beta}$.

With the same logic, if the price is in $\left[ 1 + \frac{2}{\beta}, 2 - \frac{2}{\beta} \right]$, then the company has more interest in setting the price at $2 - \frac{2}{\beta}$.

If the price is less than $\frac{2}{\beta}$, the profit made by the company with this contract is less than $1/4$, so setting the price to $1 - \frac{2}{\beta}$ is more beneficial.

Finally, a price greater than $2 + \frac{2}{\beta}$ doesn’t make any profit.

For an optimal solution, the price values can only be in $\left[ 1 - \frac{2}{\beta}, 1 + \frac{2}{\beta} \right] \cup \left[ 2 - \frac{2}{\beta}, 2 + \frac{2}{\beta} \right]$. Taking the optimal quadratic prices and rounding them to obtain a price vector of values 1 or 2 provides a price vector for the deterministic problem with a value closed to the quadratic optimum i.e., $v(\alpha-BP) \leq v(\beta-BP) - \frac{2}{\beta}(n + m)$.

For the converse, taking the optimal deterministic solution (we know that the prices can only be 1 or 2) and subtracting $\frac{2}{\beta}$ to each price gives a quadratic solution with objective value closed to the deterministic optimum i.e., $v(\beta-BP) \leq v(\alpha-BP) - \frac{2}{\beta}(n + m)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{Representation of the objective value for the transformation of [Gur+05]}
\end{figure}

Optimistic optimum is integer and the quadratic one lies in a small interval centered on it (hashed zones).

Computing the quadratic optimum for $\beta \geq 8(n + m)$ and rounding it gives us the deterministic optimum. Thus, the quadratic case is at least as hard as the deterministic case, which was proved to be APX-hard.

\begin{remark}
The structure of this specific instance allows us to exhibit a threshold from which the quadratic model is a sufficiently good approximation for the optimistic model. In a more general case, even if we have established the convergence of the quadratic model to the deterministic one, we are not able to provide such a threshold.
\end{remark}

\section{C Metric estimates to compare logit and quadratic regularization}

\begin{proposition}
Consider a segment $s$ facing $W + 1$ disutilities $V_0, \ldots, V_s$ sorted in ascending order. For a given $\beta > 0$, we denote by $(y_{sw}^{quad})_w$ the quadratic response (computed with a parameter $\beta' = \beta e/4$) and by $(y_{sw}^{log})_w$ its logit analog (computed with $\beta$). Then,

\begin{align}
\text{If } y_{sw}^{quad} &= 0, \text{ then } y_{sw}^{log} &\leq\gamma_w := \left(1 + \frac{s}{w}w\right)^{-1} \quad (\leq 1/9) \\
\text{Conversely, } y_{sw}^{log} &\leq \eta^W_w := \left(W + 1 + w(e^s - 1)\right)^{-1}, \text{ then } y_{sw}^{quad} &= 0
\end{align}
\end{proposition}
Proof. Suppose that \( y_{sw}^{quad} = 0 \), then from Proposition [4.4], \( V_{sw} \geq c_{sw} = \frac{1}{w} \left[ 8/e^\beta + \sum_{k=0}^{w-1} V_{sk} \right] \) and thus
\[
\exp \left( \frac{8}{we} - \beta V_{sw} \right) \leq \exp \left( -\frac{1}{w} \sum_{k=0}^{w-1} \beta V_{sk} \right) \leq \frac{1}{w} \sum_{k=0}^{w-1} e^{-\beta V_{sk}} ,
\]
where the latter inequality is obtained by convexity of the exponential. We then deduce that \( \gamma_w e^{-\beta V_{sw}} \leq \sum_{k=0}^{w} e^{-\beta V_{sk}} \). Using the logit expression gives us the desired result.

Suppose that \( y_{sw}^{log} \leq \eta \) for a given \( \eta \). We exploit the ascending sort on \( V \) in the logit expression to obtain
\[
\eta \geq \frac{e^{-\beta V_{sw}}}{\sum_{k=0}^{w-1} e^{-\beta V_{sk}} + \sum_{k=w}^{W} e^{-\beta V_{sk}}} \geq \frac{e^{-\beta V_{sw}}}{\sum_{k=0}^{w-1} e^{-\beta V_{s0}} + \sum_{k=w}^{W} e^{-\beta V_{sw}}} .
\]
Continuing the simplifications, \( \eta^{-1} \leq we^{-\beta(V_{s0} - V_{sw})} + (W - w + 1) \) and therefore
\[
V_{sw} \geq V_{s0} + \frac{1}{\beta} \log \left( \frac{\eta^{-1} - (W - w + 1)}{w} \right)
\]
Finally, taking \( \eta = \eta_{w}^{W} \) implies that \( V_{sw} \geq V_{s0} + \frac{8}{e^\beta} \), insuring that \( y_{sw}^{quad} = 0 \).

Figure 10 illustrates Proposition [C.1] and shows the logit and quadratic paths for a disutility vector \( V = (0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}) \). The trajectory shares the same start point (the simplex center for \( \beta = 0 \)) and the same end point (the vertex \( y = (1, 0, 0) \) for \( \beta \to +\infty \)). However, for the rest of the path

---

Figure 10: Logit and quadratic path on the simplex, as functions of \( \beta \)
the trajectories slightly deviate: we observe the sparsity effect of the projection operator in the behavior of the quadratic path whereas the logit trajectory always lies in the interior of the simplex.

References

[Afs+16] Sezin Afsar et al. “Achieving an optimal trade-off between revenue and energy peak within a smart grid environment”. In: Renewable Energy 91 (2016), pp. 293–301.

[Ale+19] Ekaterina Alekseeva et al. “A bilevel approach to optimize electricity prices”. In: Yugoslav Journal of Operations Research 29.1 (2019), pp. 9–30.

[ARR21] Arega Abate, Rossana Riccardi, and Carlos Ruiz. “Retailer-consumers model in electricity market under demand response”. May 2021. DOI: 10.13140/RG.2.2.36840.80647.

[Art20] Artelys. “KNITRO v12.3”. In: (2020). URL: https://www.artelys.com/solvers/knitro/

[Aus+20] Didier Aussel et al. “A trilevel model for best response in energy demand-side management”. In: European Journal of Operational Research 281.2 (2020), pp. 299–315.

[Bar13] Jonathan F Bard. Practical bilevel optimization: algorithms and applications. Vol. 30. New York: Springer Science & Business Media, 2013.

[Ben21] Emmanuel Benazera. libcmaes 0.9.8. https://github.com/CMA-ES/libcmaes 2021.

[BK19] Elizabeth Baldwin and Paul Klemperer. “Understanding preferences: “demand types”, and the existence of equilibrium with indivisibilities”. In: Econometrica 87.3 (2019), pp. 867–932.

[BMP13] Lijie Bai, John Mitchell, and Jong-Shi Pang. “On convex quadratic programs with linear complementarity constraints”. In: Computational Optimization and Applications 54 (Apr. 2013). DOI: 10.1007/s10589-012-9497-4

[CMM98] Joseph Czyzyk, Michael P Mesnier, and Jorge J Moré. “The NEOS server”. In: IEEE Computational Science and Engineering 5.3 (1998), pp. 68–75.

[Com04] Commission for Energy Regulation. “Electricity Tariff Structure Review: International Comparisons. An Information Paper”. 2004.

[Con16] Laurent Condat. “Fast Projection onto the Simplex and the l1 Ball”. In: Mathematical Programming, Series A 158.1 (July 2016), pp. 575–585.

[Cor+96] Robert Corless et al. “On the Lambert W Function”. In: Advances in Computational Mathematics 5 (Jan. 1996), pp. 329–359. DOI: 10.1007/BF02124750.

[Cpl09] IBM ILOG Cplex. “V12. 1: User’s Manual for CPLEX”. In: International Business Machines Corporation 46.53 (2009), p. 157.

[Dem+15] Stephan Dempe et al. Bilevel Programming Problems. Jan. 2015. ISBN: 978-3-662-45826-6. DOI: 10.1007/978-3-662-45827-3

[Dem02] Stephan Dempe. Foundations of Bi-Level Programming. Jan. 2002.
[Den+17] Zhibin Deng et al. “Globally solving Quadratic programs with convex objective and complementarity constraints via completely positive programming”. In: *Journal of Industrial and Management Optimization* 13 (June 2017), pp. 64–64. DOI: 10.3934/jimo.2017064

[Dus+20] Jean-Pierre Dussault et al. “On approximate stationary points of the regularized mathematical program with complementarity constraints”. In: *Journal of Optimization Theory and Applications* 186.2 (2020), pp. 504–522.

[Eyt18] Jean-Bernard Eytard. “A tropical geometry and discrete convexity approach to bilevel programming: application to smart data pricing in mobile telecommunication networks”. PhD thesis. Université Paris-Saclay (ComUE), 2018.

[Fer+13] Cristina G Fernandes et al. “The unit-demand envy-free pricing problem”. In: *arXiv preprint arXiv:1310.0038* (2013).

[Fer+16] Cristina G Fernandes et al. “The envy-free pricing problem, unit-demand markets and connections with the network pricing problem”. In: *Discrete Optimization* 22 (2016), pp. 141–161.

[Fis80] Caroline Fisk. “Some developments in equilibrium traffic assignment”. In: *Transportation Research Part B: Methodological* 14.3 (1980), pp. 243–255. DOI: https://doi.org/10.1016/0191-2615(80)90004-1

[FL04] Roger Fletcher and Sven Leyffer. “Solving mathematical programs with complementarity constraints as nonlinear programs”. In: *Optimization Methods and Software* 19.1 (2004), pp. 15–40. DOI: 10.1080/10556780410001654241

[GMS15] François Gilbert, Patrice Marcotte, and Gilles Savard. “A Numerical Study of the Logit Network Pricing Problem”. In: *Transportation Science* 49 (Jan. 2015), p. 150105061815001. DOI: 10.1287/trsc.2014.0560.

[Gur+05] Venkatesan Guruswami et al. “On profit-maximizing envy-free pricing.” In: *SODA*. Vol. 5. 2005, pp. 1164–1173.

[Han+10] Nikolaus Hansen et al. “Comparing results of 31 algorithms from the black-box optimization benchmarking BBOB-2009”. In: *Proceedings of the 12th annual conference companion on Genetic and evolutionary computation*. 2010, pp. 1689–1696.

[Han06] N. Hansen. “The CMA evolution strategy: a comparing review”. In: *Towards a new evolutionary computation. Advances on estimation of distribution algorithms*. Ed. by J.A. Lozano et al. New York: Springer, 2006, pp. 75–102.

[Hei+10] Géraldine Heilporn et al. “A parallel between two classes of pricing problems in transportation and marketing”. In: *Journal of Revenue and Pricing Management* 9 (Jan. 2010), pp. 110–125. DOI: 10.1057/rpm.2009.39

[Hoh20] Simon Hohberger. “Dynamic pricing under customer choice behavior for revenue management in passenger railway networks”. PhD thesis. Universität Mannheim, 2020. URL: https://madoc.bib.uni-mannheim.de/54035/

[Hur+15] Thomas Huraux et al. “Simulations multi-agents de l’activité humaine: application dans le contexte énergétique résidentiel français”. In: *Applications Pratiques de l’Intelligence Artificielle (APIA)* (2015).
[Jar+20] Francisco Jara-Moroni et al. “An enhanced logical benders approach for linear programs with complementarity constraints”. In: *Journal of Global Optimization* 77 (May 2020). DOI: 10.1007/s10898-020-00905-z.

[Kle+21] Thomas Kleinert et al. “A Survey on Mixed-Integer Programming Techniques in Bilevel Optimization”. working paper or preprint. 2021.

[KLM20] Youngdae Kim, Sven Leyffer, and Todd Munson. “Mpec methods for bilevel optimization problems”. In: *Bilevel Optimization*. New York: Springer, 2020, pp. 335–360.

[Ley06] Sven Leyffer. “Complementarity constraints as nonlinear equations: Theory and numerical experience”. In: 2 (Jan. 2006), pp. 169–208. DOI: 10.1007/0-387-34221-4_9.

[LH11] Hongmin Li and Woonghee Huh. “Pricing Multiple Products with the Multinomial Logit and Nested Logit Models: Concavity and Implications”. In: *Manufacturing and Service Operations Management* 13 (Oct. 2011), pp. 549–563. DOI: 10.1287/msom.1110.0344.

[Li+19] Hongmin Li et al. “Product-Line Pricing Under Discrete Mixed Multinomial Logit Demand”. In: *Manuf. Serv. Oper. Manag.* 21 (2019), pp. 14–28.

[McF74] D. McFadden. *Conditional logit analysis of qualitative choice behavior*. 1974.

[Pas09] Vangelis Th Paschos. “An overview on polynomial approximation of NP-hard problems”. In: *Yugoslav Journal of Operations Research* 19.1 (2009), pp. 3–40.

[RW09] R Rockafellar and Roger Wets. *Variational Analysis*. 2009. DOI: 10.1007/978-3-642-02431-3.

[SK20] Hongzhang Shao and Anton J. Kleywegt. *Tractable Constrained Optimization over Multiple Product Attributes under Discrete Choice Models*. 2020. arXiv: 2007.09193 [math.OC].

[SSC17] Hailin Sun, Che-Lin Su, and Xiaojun Chen. “SAA-regularized methods for multiproduct price optimization under the pure characteristics demand model”. In: *Mathematical Programming* 165.1 (2017), pp. 361–389.

[STM11] R. Shioda, L. Tunçel, and T. Myklebust. “Maximum utility product pricing models and algorithms based on reservation price”. In: *Computational Optimization and Applications* 48 (Mar. 2011), pp. 157–198. DOI: 10.1007/s10589-009-9254-5.

[Tra09] Kenneth Train. *Discrete Choice Methods With Simulation*. Vol. 2009. Jan. 2009. ISBN: 9780521766555. DOI: 10.1017/CBO9780511805271.

[XZ19] Jing Zhou and Zhijun Xu. “A simultaneous diagonalization based SOCP relaxation for convex quadratic programs with linear complementarity constraints”. In: *Optimization Letters* 13 (Oct. 2019). DOI: 10.1007/s11590-018-1337-8.