Critical Behavior of CP$^1$ at $\theta = \pi$, Haldane’s Conjecture and the Universality Class

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Using an approach to analyze the $\theta$ dependence of systems with a $\theta$-term we recently proposed, the critical behavior of CP$^1$ at $\theta = \pi$ is studied. We find a region outside the strong coupling regime where Haldane’s conjecture is verified. The critical line however does not belong to the universality class of the Wess-Zumino-Novikov-Witten model at topological coupling $k = 1$ since it shows continuously varying critical exponents.

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Two dimensional CP$^{N-1}$ models are of great interest in high energy physics. They are toy models for testing nonperturbative and topological properties in a confining, asymptotically free quantum field theory and share also with QCD, the theory of the strong interaction between particles, a well defined topological charge, possess instanton solutions, have $\theta$-vacua and admit a 1/N expansion. Understanding the role of the $\theta$ parameter in QCD and its connection with the “strong CP problem” is one of the major challenges for high energy theorists [1]. Unfortunately, Euclidean lattice gauge theory, our main nonperturbative tool for QCD studies, has not been able to help us for two reasons: (i) the difficulties in defining the topological charge operator on the lattice [2] and (ii) the imaginary contribution to the action coming from the $\theta$ term that prevents the applicability of the importance sampling method. CP$^{N-1}$ models regularized on a lattice have a well defined topological charge but share with QCD point (ii) and this means that, from the point of view of numerical calculations, the complex action problem is as severe as in QCD. Indeed most properties of CP$^{N-1}$ models have been obtained in the context of the 1/N expansion [3, 4], showing a qualitative behavior as a function of $\theta$ similar to that of the massive Schwinger model at weak coupling [5]. The cusp in the free energy density at $\theta = \pi$ signals a first order phase transition at this value of $\theta$ with spontaneous breaking of CP, in agreement also with the strong coupling results [6]. This behavior, which is expected to persist for finite values of $N$, at least until $N = 4$, is common to most of the models the $\theta$ dependence of which is known, and was conjectured for non-abelian gauge theories by ’t Hooft [7] in 1981. Some time ago, using different arguments and under very general assumptions, we also argued that a singular behavior in $\theta$ is to be expected in systems with a $\theta$-term [8]. For $N \leq 3$ dislocations break the asymptotic scaling (continuum limit) [9] and the $\theta$ dependence becomes more intriguing in these cases.

However high energy physics is not the only field where topological structures and the $\theta$ term play a fundamental role. In condensed matter physics Haldane showed [10] that chains of quantum spins with antiferromagnetic interactions, in the semiclassical limit of large but finite $S$, are related to the two-dimensional $O(3)$ nonlinear sigma model at coupling $g = 2/[S(S + 1)]^{1/2}$ and topological term $\theta = 0, \pi$ (integer, half-integer spin). While integer spin chains have a mass gap, half-integer chains should be gapless. Haldane conjectured that the $O(3)$ non linear sigma model presents a second order phase transition at $\theta = \pi$, keeping its ground state CP symmetric. Based on a partial summation of the strong coupling expansion, Affleck [11] argued that the two-dimensional $O(3)$ non linear sigma model has indeed a gapless phase at $\theta = \pi$ and weak coupling. Furthermore Affleck and Haldane [12] also argued that the critical theory for generic half-integer spin antiferromagnets is the Wess-Zumino-Novikov-Witten model at topological coupling $k = 1$, with the following scaling law for the mass gap [13] near the phase transition point:

$$m(\theta) \propto (\pi - \theta)^{2/3} \ln[(\pi - \theta)^{-1/2}]$$

(1)

In 1995 Bietenholz, Pochinsky and Wiese [14] performed a simulation of the two-dimensional $O(3)$ nonlinear sigma model with a $\theta$-term in order to verify Haldane’s conjecture. They used an efficient Wolff cluster algorithm [15] on a triangular lattice with a constraint in the action which preserves the $O(3)$ symmetry, and simulated the system at a fixed value of the coupling (temperature) $g = \infty$ which, due to the constraint in the action, should be outside the strong coupling regime. By measuring the topological and magnetic susceptibilities and using finite size scaling theory, the authors of [14] found a second order phase transition at $\theta = \pi$ confirming Haldane’s conjecture, and a finite size scaling in relatively good agreement with the assumption that the critical exponents of the phase transition are those of the $k = 1$ Wess-Zumino-Novikov-Witten model [16].

As stated before, the sign problem has restricted very much the lattice non-perturbative investigations of sys-
tems with a $\theta$-term. A few years ago we introduced two alternative schemes [16, 17] to analyze the $\theta$ dependence in these systems. The first approach [16] was based on a precise determination of the probability distribution function of the topological charge whereas the second one [17], originally inspired by analytical properties of the Ising model with an imaginary magnetic field, is very efficient to determine the critical behavior at $\theta = \pi$ in cases in which a continuous transition at this value of $\theta$ shows up. The two approaches were successfully tested in several integrable models and, after this consistency checks, applied to the analysis of the $\theta$ dependence of the $CP^3$ model. Indeed reference [18] contains the first full re-
sult of $CP$ models, showing in particular the spontaneous symmetry breaking at $\theta = \pi$ in the continuum limit. Here we want to use the second of the previously cited approaches [17] in order to analyze the phase structure in $\theta$ of $CP^3$ which, as well known, is equivalent to the non linear $O(3)$ sigma model. For this reason in the following we will briefly summarize the main algorithm steps.

We have adopted for the action the standard “auxiliary U(1) field” formulation

$$S_g = 2\beta \sum_{n,\mu}(\bar{z}_{n+\mu}z_nU_{n,\mu} + \bar{z}_n z_{n+\mu}\bar{U}_{n,\mu} - 2)$$

where $z_n$ is a 2-component complex scalar field that satisfies $\bar{z}_n z_n = 1$ and $U_{n,\mu}$ is a U(1) “gauge field”. The topological charge operator is defined directly from the $U(1)$ field:

$$S_\theta = i\frac{\theta}{2\pi} \sum_p \log(U_p)$$

where $U_p$ is the product of the U(1) field around the plaquette and $-\pi < \log(U_p) \leq \pi$.

Our numerical approach uses as input the results from numerical simulations of $CP^1$ at imaginary values of $\theta = -ih$ (h real) [17]. The action to simulate $S = S_g + S_\theta = -ih$ is then real and local. This implies that the main computational cost is practically equivalent to standard simulations of $CP^1$ model at $\theta = 0$. This is important since it allows to perform large volume simulations.

We use $x(h)$, the topological charge density, and the variable $x = \cosh \frac{\theta}{2}$ to define the quantity $y(z) = x(h)/\tanh \frac{x}{2}$. Using the transformation $y(x) = y(e^{\frac{\theta}{2}z})$ and plotting $y(x)/y$ against $x$ one gets typically a smooth function for small $y$ [17]. Hence we can expect a simple extrapolation to $y \to 0$ (i.e. in the region corresponding to real $\theta$) to be reliable and thus obtain $x(\theta)$ (of course the result has to be independent of the specific value of $\lambda$). In the same spirit the effective exponent $\gamma = \frac{4}{3}\log(y(x)/y)$ as a function of $y$ will give the dominant power of $y(z)$ as a function of $z$ near $z = 0$ or, equivalently, the behavior of $x(\theta)$ for $\theta \to \pi$. If $\gamma = 1$ CP symmetry is spontaneously broken at $\theta = \pi$, values between 1 and 2 indicate a second order phase transition with a divergent susceptibility and so on [17]. Our approach assumes implicitly that the theory has not phase transitions at $\theta < \pi$ and in particular that $CP$ symmetry is realized at $\theta = 0$, the last being a necessary condition for the theory to be well defined at $\theta \neq 0$ [13].

Fig. 1 shows the goodness of our method when applied to two models, the analytical solution of which is known [17]. The first one is the one-dimensional Ising model within an imaginary magnetic field, which breaks spontaneously $CP$ at $\theta = \pi$. The second one is a toy model with only one effective degree of freedom. This model, which resembles very much the free instanton gas model, has the partition function $Z_V(\theta) = (1 + A\cos \theta)^V$ for $V$ degrees of freedom. We have plotted in Fig. 1 the order parameter $x(\theta)$ against $\theta$ for both models. The lines stand for the exact analytical solution whereas the symbols correspond to the values extracted from our approach.

Let us resume our results for $CP^1$. This model is equivalent, at $\beta = 0$ (strong coupling), to two-dimensional compact $QED$ with topological charge. $CP$ symmetry is therefore spontaneously broken at $\theta = \pi$ in this limit. This result remains qualitatively unchanged when moving to larger $\beta$ values, until $\beta \sim 0.5$. For $\beta > 0.5$ $CP$ symmetry is recovered at $\theta = \pi$ and this result seems to hold for any $\beta$. For $\beta$ values between 0.5 and 1.5 the order parameter $x(\theta)$ vanishes at $\theta = \pi$ as $(\pi - \theta)^\epsilon(\beta)$ with $\epsilon(\beta)$ varying continuously between 0 and 1. This implies a divergent topological susceptibility $\chi_t = \frac{dx(\theta)}{d\theta}$ at $\theta = \pi$ for $0.5 < \beta < 1.5$ and we get therefore a line of second order phase transition points with continuously varying critical exponents. For $\beta > 1.5$ we find $\epsilon(\beta) = 1$.

Numerical evidence of these results is given in figures 2-5.

The details of the simulations are: we used lattices $40^2$ for $\beta \leq 1.0$, $100^2$ for $1.0 \leq \beta \leq 1.5$ and $200^2$ for $\beta \geq 1.5$ checking absence of finite volume effects; this choice has been taken looking at existing data for the correlation

![FIG. 1: Order parameter in Ising model (full symbols) and a toy model discussed in the text (open symbols); the lines are analytical results](image-url)
length at $\theta = 0$ [20] ($\xi < 3$ for $\beta \leq 1.0$, $3 < \xi < 20$ for $1.0 < \beta < 1.5$). For each value of $\beta$ we have performed simulations for 25-40 values of $h$ with statistics exceeding 500000 MC iterations each in order to evaluate $x(h)$.

In Fig. 2 we report the order parameter as a function of $\theta$ for $\beta = 1.0$, hence in the region where $0 < \epsilon < 1$; in Fig. 3 the same data are plotted in a log-log scale and, to give the feeling of the accuracy in the determination of the critical exponent, a fit with the function $x(\theta) = \text{const} + a(\pi - \theta)^{\beta}$ is also shown. The fitted parameters are: \text{const}=0, \ a=0.441(1) and $\epsilon(1.0) = 0.736(1)$. Equivalent graphs have been obtained for the other values of $\beta$ in the range 0.6 - 1.5 while, for $\beta \leq 0.5$, we have a different behaviour of the order parameter, approaching a non zero value at $\theta = \pi$. For the two larger values of $\beta$, namely 1.5 and 1.6, we obtain $\epsilon = 1$ and an order parameter indistinguishable from a $\sin(\theta)$ function between $\theta = 0$ and $\theta = \pi$.

Starting from the same raw data (the simulations at imaginary \(\theta\)) we can do a consistency check performing a different analysis using the effective exponent \(\gamma_\lambda\). We are interested in the effective exponent in the $y \to 0$ limit and, in Fig. 4, we can see the \(\gamma_\lambda\) extrapolation for $\beta = 1.0$ ($L = 100$ data, lower curve) and for $\beta = 1.6$ ($L = 200$ data, upper curve). For the $\beta = 1.0$ case extrapolations obtained using two different fitting functions, namely \(\gamma_\lambda(y) = \exp(a + by + cy^2)\) and $a + by + cy^2$ are shown; the first function is able to slightly better reproduce the data in the full $\beta$ range with respect to the simple polynomial form. The small difference in the extrapolated values does not change qualitatively the result.

For the $\beta = 1.6$ data there is essentially no need for extrapolation, being very clear that the results point to the value $\gamma = 2$ in the $y = 0$ limit. Data shown in the figure are those relative to $\lambda = 0.5$ but exactly the same conclusions can be reached using other values of $\lambda$. It is important to notice how, moving to the region of smaller $\beta$, the same fitting functions continue to describe very well the data and have been used to extrapolate the effective exponent at zero $y$.

Fig. 5 resumes all results for exponents $\epsilon$ and $\gamma$ as a function of $\beta$: it is clear from this figure that, considering the fate of CP symmetry at $\theta = \pi$, the system passes from a phase in which the symmetry is broken ($\beta < 0.5$) to a symmetric phase ($0.5 < \beta < 1.5$) where a second order phase transition with varying critical exponents is present, ending with a region in which CP symmetry is still satisfied but without a divergent correlation length.

The location of the point where the system becomes CP symmetric at $\theta = \pi$ can not be obtained from our data with a high accuracy since different analysis gives slightly different results but the existence of such a point is evident from both analysis presented here.

Nevertheless, in our opinion, the important physical fact is the existence of a line of second order transition points with continuously varying critical exponents. Using the hyperscaling hypothesis we can relate $\epsilon(\beta)$ with the critical exponent $\nu$ for the mass gap. Indeed, following Kadanoff, one can assume that the singular part of the free energy density near the phase transition points is
proportional to $\xi^{-2}$, $\xi$ being the inverse mass gap, and this implies that $m(\theta) \sim (\pi - \theta)^{\nu}$. As a consequence simple algebra gives the relation $\epsilon = 2\nu - 1$.

In conclusion we have analyzed the critical behavior of CP$^1$ at $\theta = \pi$ and found a region in the coupling $\beta$ where Haldane’s conjecture is verified. The critical line shows continuously varying critical exponents and does not belong therefore to the universality class of the Wess-Zumino-Novikov-Witten model at topological coupling $k = 1$. From the point of view of Quantum Field Theory these new critical points open the possibility for new non-gaussian fixed points with anomalous dimensions, where a non trivial continuum limit could be defined.

[1] See R.D. Peccei, e-print: [hep-ph/9807514](http://arxiv.org/abs/hep-ph/9807514) for a review.
[2] B. Alles, M. D’Elia, A. Di Giacomo and and R. Kirchner, Phys. Rev. **D58**, 114506 (1998).
[3] D’ Adda, P. Di Vecchia and M. Luscher, Nucl. Phys. **B146**, 63 (1978).
[4] E. Witten, Nucl. Phys. **B149**, 285 (1979).
[5] S. Coleman, Ann. Phys. (N.Y.) **101**, 239 (1976).
[6] N. Seiberg, Phys. Rev. Lett. **53** 637 (1984).
[7] G. ’t Hooft, Nucl. Phys. **B190**, 455 (1981).
[8] V. Azcoiti, A. Galante and V. Laliena, Prog. Theor. Phys. **109**, 843 (2003).
[9] M. Luscher, Nucl. Phys. **B200** [FS4], 61 (1982).
[10] F.D.M. Haldane, Phys. Lett. **93A**, 464 (1983); Phys. Rev. Lett. **50** 1153 (1983). For a review, see I. Affleck, J. Phys.: Condens. Matter **1**, 3047 (1989).
[11] I. Affleck, Phys. Rev. Lett. **66** 2429 (1991).
[12] I. Affleck and F.D.M. Haldane, Phys. Rev. **B36** 5291 (1987).
[13] I. Affleck, D. Gepner, H.J. Schulz and T. Ziman, J. Phys. A: Math. Gen. **22**, 511 (1989).
[14] W. Bietenholz, A. Pochinsky and W. Wiese, Phys. Rev. Lett. **75** 4524 (1995).
[15] U. Wolff, Phys. Rev. Lett. **62** 361 (1989).
[16] V. Azcoiti, , G. Di Carlo, A. Galante and V. Laliena, Phys. Rev. Lett. **89**, 141601 (2002).
[17] V. Azcoiti, , G. Di Carlo, A. Galante and V. Laliena, Phys. Lett. **B563**, 117 (2003).
[18] V. Azcoiti, , G. Di Carlo, A. Galante and V. Laliena, Phys. Rev. **D69**, 056006 (2004).
[19] V. Azcoiti and A. Galante, Phys. Rev. Lett. **83**, 1518 (1999).
[20] M. Campostrini, P. Rossi, and E. Vicari Phys. Rev. **D46**, 2647 (1992).