Fundamental Limits of Caching: Improved Bounds with Coded Prefetching

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Abstract

We consider a cache network in which a single server is connected to multiple users via a shared error free link. The server has access to a database with $N$ files of equal length $F$, and serves $K$ users each with a cache memory of $MF$ bits. A novel centralized coded caching scheme is proposed for scenarios with more users than files $N \leq K$ and cache capacities satisfying $\frac{K}{F} \leq M \leq \frac{N}{F}$. The proposed scheme outperforms the best rate-memory region known in the literature if $N \leq K \leq \frac{N^2}{M}$.

I. INTRODUCTION

Content caching techniques are recently increasing attention to combat peak hour traffic in content delivery services. The basic idea is simple. If contents are made available at user terminals during low traffic periods, then the peak rate can be reduced. However, content requests are unknown to the server and thus content caching at user memories must be carefully chosen in order to be useful regardless of the contents requested during peak hours. The simplest caching scheme consists of storing each file partially at each user memory. Then, the server transmits the remaining requested data uncoded [1], [2]. For single user caching systems, this strategy is optimal. However, for multi-user systems, the seminal work in [3] by Mohammad-Ali and Niesen shows that important gains can be obtained by a new coded caching strategy. Specifically, there authors show that, besides the local caching gain that is obtained by placing contents at user caches before they are requested, it is possible to obtain a global caching gain by creating broadcast opportunities. This is, by carefully choosing the content caches at different users, and using network coding techniques it is possible to transform the initial multi-cast network, where every user is requesting a different file, into a broadcast network, where every user requests exactly the same “coded” file, obtaining the new global caching gain.

The fundamental caching scheme developed in [3] was latter extended to more realistic situations, including a decentralized scheme in [4], non-uniforms demands in [5], and online coded caching in [6]. In addition, new schemes pushing further the fundamental limits of caching systems have appeared in [7]–[13].

The work here proposed investigates the fundamental achievable rate for the particular situation where there are more users than files, and the caching memories at users are small compared to the number of files in the system. Besides its theoretical relevance, this situation can be readily found in the real world. For instance, global content delivery services such a Netflix serve a few multimedia contents to millions of users across the world. In addition, it was shown in [5] that a near optimal caching strategy consists in dividing the files into groups with similar popularity, and then applying the coded caching strategy to each group separately. Since the amount of users in each groups remains the same, when there are many groups, the cache size dedicated to each group is small as well as the number of files per user in each group.

The rest of this paper is organized as follows. In Section II we present the system model together with the more relevant previous results. In Section III, we summarize the main results of this paper. Section IV describes the caching scheme proposed, first, by providing a detailed example and, then extending it to the general case. Finally, conclusions are drawn in Section V.

II. SYSTEM MODEL AND PREVIOUS RESULTS

We consider a communication system with one server connected to $K$ users, denoted as $U_1, ..., U_K$, through a shared, error-free link. There is a database at the server with $N$ files, each of length $F$ bits, denoted as $W_1, ..., W_N$. Each user is equipped with a local cache of capacity $MF$ bits and is assumed to request only one full file. Here, we consider the special case where $M \in \left[0, \frac{N}{F}\right]$ and there are more users than files $N \leq K$. For convenience, we define parameter $g = \frac{N}{MF}$.

We consider the communication model introduced in [3]. The caching system operates in two phases: the placement phase and the delivery phase. In the placement phase, users have access to the server database, and each user fills their cache. As in [11], we allow coding in the prefetching phase. Then, each user $U_k$ requests a single full file $W_{d_k}$ where $d = (d_1, ..., d_K)$ denotes the demand vector. We denote the number of distinct request in $d$ as $N_c(d)$. In the delivery phase, only the server has access to the database. After being informed of the user demands, the server transmits a signal $X$ of size $RF$ bits over the shared link to satisfy all user requests simultaneously. The signal $X$ is a function of the demand vector $d$, all the files in the data base, and the content in the user caches. The rate $R$ is referred to as the rate of the shared link and is a function of the demand vector $d$. Using the local cache content and the received signal $X$, each user $U_k$ reconstructs its requested file $W_{d_k}$.

For a caching system $(M, N, K)$, given a particular demand $d$, we define a communication rate $R$ is achievable if and only if there exists a message $X$ of length $RF$ bits such that every user $U_k$ is able to reconstruct its desired file $W_{d_k}$.
We denote as $R(d)$ the achievable rate for a particular demand $d$. Then, the rate needed for the worst demand is given by $R^* = \max_{d \in D} R(d)$.

For the caching system described, there is a trade-off between the memory $M$ at users, and the worst demand rate $R^*$. Observe, that if users have no caching capacity $M = 0$, the server needs to send the full requested files and thus, the worst demand rate is $R^* = N$. Instead, if users can have a complete copy of the server’s database $M = N$, then no information needs to be transmitted from the server $R^* = 0$. We are interested in characterizing this rate memory trade-off. To that end, we define the rate-memory pair $(R^*, M)$ and the rate-memory function $R^*(M)$.

A. Previous Results

For the special case considered here $M \in [0, \frac{N}{K}]$ and $N \leq K$, the best known rate-memory function in the literature can be obtained by memory sharing between four achievable rate-memory pairs: the trivial rate-memory pair $(N, 0)$, the rate-memory pairs obtained in [11] for $M = \frac{N}{K}$

\[(R_{CFL}^*, M_{CFL}) = \left(N - \frac{N}{K}, \frac{1}{K}\right),\]  

the rate-memory pair obtained by the schemes proposed in [7] and [12] for $M = \frac{N}{K}$

\[(R_{GBC}^*, M_{GBC}) = \left(N - \frac{N(N + 1)}{2K}, \frac{N}{K}\right),\]  

and the rate-memory pairs obtained in [13]

\[(R_{MDS}^*, M_{MDS}) = \left(\frac{N(K - t)}{K}, \frac{t[(N - 1)K - N]}{K(K - 1)}\right), t = 0, 1, ..., K.\]  

The lower convex envelope of all these rate-memory pairs, provides the best rate-memory function in the literature.

Our scheme is mainly motivated by the scheme described in [11], which makes use of coded prefetching at users’ cache. As shown in [11], the scheme achieving [1] is optimal for $M = \frac{N}{K}$. The schemes proposed in [7] and [12] assume uncoded prefetching. They are essentially the same at $M = \frac{N}{K}$. The scheme proposed in [12] was shown to be optimal among all the uncoded prefetching schemes. Finally, the scheme developed in [13] makes use of non binary codes, in particular distant separable (MDS) codes and rank metric codes to obtain the rate-memory pairs in [3], which are shown to be optimal at certain points. There have been other coded prefetching schemes proposed in the literature, see [10] and [8] but either they do not improve the current best known rate-memory trade-off or they apply to other situations. Despite all this efforts, the optimal rate-memory trade-off for a caching systems remains an open problem. There have been also efforts to obtain theoretical lower bounds on the delivery rate. The cut-set bound was studied in [3]. A tighter lower bound was obtained in [14]. Through a computational approach a lower-bound for the special case $N = K = 3$ is derived in [15]. Finally, in [9] yet another lower-bound is proposed.

III. Main Result

The following theorem presents the delivery rate obtained by the proposed coded caching scheme for a particular demand $d$.

**Theorem 1.** For a caching problem with $K$ users and $N$ files, $K \geq N$, local cache size of $M$ files at each user, and parameter $g = \frac{N}{MK}$: Given a particular demand $d$, the delivery rate

\[R = \frac{KN_c(d) \left(\frac{N - 1}{g - 1} \right) - N_c(d) \left(\frac{N_c(d) + 1}{g + 1} \right) \left(\frac{N_c(d) - 1}{g - 1} \right)}{K \left(\frac{N - 1}{g - 1} \right)}\]  

is achievable for $g \in \{1, ..., N]\). Furthermore, for $g \in \{1, N\]$ equals the lower convex envelope of its values at $g \in \{1, ..., N\}$.

We prove this result and the next corollary in the following section by describing the new caching scheme.

**Corollary 1.1.** For a caching problem with $K$ users and $N$ files, $K \geq N$, local cache size of $M$ files at each user, and parameter $g = \frac{N}{MK}$, the delivery rate-memory pairs

\[(R^*, M) = \left(N - \frac{N(N + 1)}{K(g + 1)}, \frac{N}{Kg}\right)\]  

are achievable for $g \in \{1, ..., N\}$. Furthermore, for $g \in \{1, N\]$ the rate-memory pairs in the the lower convex envelope of its values at $g \in \{1, ..., N\}$ are achievable.
We can see that the cut set lower bound is achieved by the rate-memory function in Corollary 1.1. This is K ≤ N^2 + 1/2.

Corollary 1.3. Let R_{CB}(M) denote the rate-memory function obtained for the cut set lower bound. The rate difference between the cut set rate and the rate-memory pairs in Corollary 1.1 is

\[ R_{CB}(M) - R^* (M) = \frac{N}{K} \left( \frac{N}{g} - \frac{N + 1}{g + 1} \right) = \frac{N}{K} \left( \frac{N}{g + 1} \right) \]

Observe that, as first reported in [11] Theorems 3 and 4], for g = N, M = \frac{1}{K} the cut set lower bound is achieved by the proposed strategy. From the second equality (6), given that K \left( \frac{N - 1}{g - 1} \right) is the number of subfile partitions in our scheme, we can see that \left( \frac{N}{g + 1} \right) is the distance in number of subfile transmissions to the cut set bound.

Corollary 1.4. Let R_{STC}(M) denote the lower bound on the rate-memory function presented in [14]. For K = N and M = \frac{N}{(N - 1)K} = \frac{1}{N - 1} this lower bound is achievable by the rate-memory function in Corollary 1.1. This is R^* \left( \frac{1}{N - 1} \right) = R_{STC} \left( \frac{1}{N - 1} \right).

We conclude this section by illustrating in Fig. 1 the rate-memory function for the proposed scheme and for the state of the art (SOTA). We consider the case N = 10 files and K = 15 users. In addition to the best state of the art rate-memory function, we provide the rate-memory function recently discovered in [12], the previously introduced rate-memory pairs (R_{MDS}, M_{MDS}), (R_{CFL}, M_{CFL}), and (R_{GBC}, M_{GBC}) as well as the rate-memory pair \left( R^*, \frac{N}{N - 1} \right) found in [10] by using a coded prefetching scheme. We also include in this figure for comparison, the cut set lower bound and the information-theoretical lower bound obtained in [14]. We observe that, for the special situation considered here, the new proposed scheme obtains a significant improvement with respect to the previous best SOTA.

IV. PROPOSED CACHING SCHEME

We describe the proposed scheme, first with a particular example. Then, we provide a detailed description of the general case.

A. Example

Consider a caching system with N = 3 files, K = 6 users and a caching capacity of MF bits with M = \frac{1}{4}, which corresponds to choosing g = 2. For this particular case, the best known coded caching scheme obtains the rate-memory pair \left( \frac{2g}{2} + \frac{1}{3}, \frac{1}{4} \right) by memory sharing between the rate-memory pair \left( R_{MDS}, M_{MDS} \right) = \left( R^*, \frac{N}{N - 1} \right) found in [2] with

| Subfile assignment | Coded cached subfiles |
|-------------------|-----------------------|
| W_{1,1}^{(1)} ⊕ W_{1,2}^{(1)} | W_{1,1}^{(1)} ⊕ W_{1,2}^{(1)} |
| W_{2,1}^{(1)} ⊕ W_{2,2}^{(1)} | W_{2,1}^{(1)} ⊕ W_{2,2}^{(1)} |
| W_{2,1}^{(1)} ⊕ W_{2,2}^{(1)} | W_{2,1}^{(1)} ⊕ W_{2,2}^{(1)} |
| W_{2,1}^{(1)} ⊕ W_{2,2}^{(1)} | W_{2,1}^{(1)} ⊕ W_{2,2}^{(1)} |

TABLE I: Subfile assignment and prefetching at user U_j, for the proposed coded caching scheme when K = 6, N = 3 and M = \frac{1}{4}, i.e. g = 2.

Remark 1. The rate-memory function in (5) coincides with (2) for g = N, M = \frac{1}{K} and with (1) for g = 1, M = \frac{N}{K}. For g = 1, M = \frac{N}{K}, our scheme is essentially the same as the one described in [12] and [7]. However, the scheme proposed in [11] to achieve (1) differs slightly from the one considered here for K < N. Indeed, while [11] divides each file into NK subfiles, our scheme requires only K subfiles per file.

The next three corollaries compare the proposed scheme with the scheme presented in [13], the cut set lower bound derived in [3], and the lower bound obtained in [14]. These results are proved in the Appendices.

Corollary 1.2. For a caching problem with K users and N files, K ≥ N, and local cache size of \frac{1}{K} ≤ M ≤ \frac{N}{K}, a sufficient condition for the proposed scheme to outperform the rate-memory region obtained by the lower convex envelope of the rate-memory pairs in (5), is

\[ K ≤ \frac{N^2 + 1}{2}. \]
Fig. 1: Rate-memory functions of the proposed scheme (New Strategy) compared with existing schemes and lower bounds in the literature for $N = 10$ and $K = 15$.

t = 1, and the rate-memory pair $(R_{MDS}^*, M_{MDS}) = \left(2, \frac{7}{15}\right)$ from (3) with $t = 2$. Here, we show that the rate-memory pair $(\frac{28}{72}, \frac{1}{4})$ is achievable. The best known outer bound \[14\] obtains the rate-memory pair $(\frac{27}{32}, \frac{1}{4})$.

The prefetching strategy is illustrated in Table I. We break each file $W_f, f = \{1, 2, 3\}$ into $K \left(\frac{N - 1}{g - 1}\right) = 6 \times 2 = 12$ subfiles of equal size. Next, two different subfiles for each file are assigned to every user. Each subfile of file $W_f$ assigned to user $U_i$ is further associated to one of the two sets of $g = 2$ subfiles $A \in \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ which contain file index $f$, this is $f \in A$, and denoted as $W_f^{(i)}$. Finally, each user $U_i, i = \{1, \ldots, 6\}$ computes the binary sum of the $g$ subfiles assigned to each of the 3 possible sets $A$, and stores the resultant coded subfile. In the following, these stored coded subfiles are referred to as coded cached subfiles.

Given the above prefetching scheme, we illustrate our proposed delivery strategy for a representative demand scenario, where users $U_1$ and $U_2$ request file $W_1$, users $U_3$ and $U_4$ request file $W_2$, and users $U_5$ and $U_6$ request file $W_3$. We show later in this section, by addressing the general case, that this indeed corresponds to the worst possible demand. We represent this demand by the demand vector $d = [1, 1, 2, 2, 3, 3]$. Then $W_d^{(i)}$ returns the file requested by user $U_i$.

First, observe that, users all together, store all subfiles XORed at some coded cached subfile, and that every subfile is only XORed at one of the coded cached subfiles. Having this in mind, the delivery scheme proposed divides the coded cached subfiles into three types and obtains the subfiles XORed at each of the coded cached subfiles types separately.

Coded cached subfiles Type I: First, consider those coded cached subfiles XORing a subfile requested by some user together with subfiles that are not requested by any user. For the specific demand considered in this example, there are none of these coded cached subfiles, as every file is requested by some user. If there would be some, the server would simply broadcast, one by one, the subfiles requested by users XORed in every coded cached subfile Type I.

Coded cached subfiles Type II: Second, consider those coded cached subfiles XORing a subfile requested by the user which stores them. For our specific example, there are two of such coded cached subfiles at each user. The delivery scheme for all the subfiles XORed at these coded cached subfiles is represented in Table I. The delivery scheme is divided into two consecutive phases, and each phase is further divided into a broadcast and computation task:

Phase I: For each of the coded cached subfiles Type II, the server first broadcasts (forth column), one by one, all the subfiles XORed at each coded cached subfile except for the one requested by the user in which the coded cached subfile is stored. Users obtain directly from this broadcasted transmission, all their requested subfiles which are XORed at the coded
user leaders can compute the requested subfiles (fifth column) which are XORed at coded cached subfiles (third column), each user computes the requested subfiles (fifth column) which are XORed at each of the coded cached subfiles.

**Phase 2:** Observe that, after Phase 1, every set of users with the same demand has computed a different and non-overlapping subset of requested subfiles. In order to all the users with a common demand share their computed subfiles in Phase 1, the server arbitrarily selects one user leader for each subset of users with the same requested file. Let $\mathcal{U} = \{1, 2, 3\}$ denote the set of user leaders, then for every user $i \notin \mathcal{U}$ and every subset $\mathcal{A}$ of $g$ files including $W_{d(i)}$, the server transmits (seventh column) the binary sum

$$W_{d(u)}^i \oplus W_{d(i)}^i$$

where $u \in \mathcal{U}$ is the user leader with the same request $d(u) = d(i)$. For clarity, we underline the subfiles obtained in Phase 1 at user leaders. Finally (eighth column), given that each of the broadcasted coded subfiles XOR a subfile already available at the user leader together with a subfile already available at a non-user leader in the same demand group, user leaders can compute the remaining requested subfiles. Every other user, would first compute the subfile belonging to the user leader and then use it to obtain the rest of the requested subfiles in their demand group.

**Coded cached subfiles Type III:** Finally, consider those coded cached subfiles which XOR together subfiles not requested by the user storing them. The delivery of the subfiles in coded cached subfiles Type III is illustrated in Table III. In our specific example, there is only one of such coded cached subfiles at each user cache (third column). Again, the delivery scheme is partitioned into two consecutive phases:

| User | Request | Coded cached subfiles | Phase 1 | Phase 2 |
|------|---------|-----------------------|---------|---------|
| $U_1$ | $W_1$ | $W_1^{(1)} \oplus W_1^{(2)}$ | $W_1^{(1)} \oplus W_1^{(2)}$ | $W_1^{(1)} \oplus W_1^{(2)}$ |
| $U_2$ | $W_2$ | $W_2^{(1)} \oplus W_2^{(2)}$ | $W_2^{(1)} \oplus W_2^{(2)}$ | $W_2^{(1)} \oplus W_2^{(2)}$ |
| $U_3$ | $W_3$ | $W_3^{(1)} \oplus W_3^{(2)}$ | $W_3^{(1)} \oplus W_3^{(2)}$ | $W_3^{(1)} \oplus W_3^{(2)}$ |
| $U_4$ | $W_4$ | $W_4^{(1)} \oplus W_4^{(2)}$ | $W_4^{(1)} \oplus W_4^{(2)}$ | $W_4^{(1)} \oplus W_4^{(2)}$ |
| $U_5$ | $W_5$ | $W_5^{(1)} \oplus W_5^{(2)}$ | $W_5^{(1)} \oplus W_5^{(2)}$ | $W_5^{(1)} \oplus W_5^{(2)}$ |
| $U_6$ | $W_6$ | $W_6^{(1)} \oplus W_6^{(2)}$ | $W_6^{(1)} \oplus W_6^{(2)}$ | $W_6^{(1)} \oplus W_6^{(2)}$ |

**TABLE II:** Delivery scheme of requested subfiles XORed at coded cached subfiles type II.

| User | Request | Coded cached subfiles | Phase 1 | Phase 2 |
|------|---------|-----------------------|---------|---------|
| $U_1$ | $W_1$ | $W_1^{(1)}$ | $W_1^{(1)}$ | $W_1^{(1)}$ |
| $U_2$ | $W_2$ | $W_2^{(1)}$ | $W_2^{(1)}$ | $W_2^{(1)}$ |
| $U_3$ | $W_3$ | $W_3^{(1)}$ | $W_3^{(1)}$ | $W_3^{(1)}$ |
| $U_4$ | $W_4$ | $W_4^{(1)}$ | $W_4^{(1)}$ | $W_4^{(1)}$ |
| $U_5$ | $W_5$ | $W_5^{(1)}$ | $W_5^{(1)}$ | $W_5^{(1)}$ |
| $U_6$ | $W_6$ | $W_6^{(1)}$ | $W_6^{(1)}$ | $W_6^{(1)}$ |

**TABLE III:** Delivery scheme of requested subfiles XORed at coded cached subfiles type III.
**Phase 1:** Select a set of user leaders \( \mathcal{U} = \{1, 3, 5\} \), for simplicity we consider the one used for coded cached subfiles Type II. Next, for every subset \( \mathcal{V} \subseteq \mathcal{U} \) of \( g + 1 = 3 \) user leaders (in our particular example there is only one of such subsets \( \mathcal{V} = \{1, 2, 3\} \)), there is an associated set of file requests \( \mathcal{B}(\mathcal{V}) = \{W_1, W_2, W_3\} \). The servers arbitrarily selects one subfile, (the underlined subfiles in the second column), each from a different file in \( \mathcal{B}(\mathcal{V}) \), from the subfiles XORed at each of the coded cached subfiles Type III of the user leaders in \( \mathcal{V} \). Next, for every other subfile XORed at coded cached subfiles Type III of all users (leaders or not), the server broadcasts the binary sum of such subfile and the one selected (underlined) that belongs to the same file (see third column). Finally, using these coded broadcasted subfiles, together with their own coded cached subfiles Type III, each user computes the binary sum of all the selected (underlined) subfiles belonging to files distinct from their requested file (forth column).

**Phase 2:** Next, the server broadcasts the binary sum of all the selected/underlined subfiles (fifth column). Using this coded broadcasted subfile together with the coded subfile computed in Phase 1 (third column) each user computes the selected/underlined subfile belonging to its requested file (sixth column). Finally, each user obtains their remaining requested subfiles (eight column) by computing the binary sum of the recently computed subfile (sixth column) and each of the coded broadcasted subfiles in Phase 1 that XOR two subfiles belonging to its requested file (seventh column).

**B. General Scheme**

Next, we generalize the proposed centralized coded caching scheme. Consider a scenario with \( K \) users, \( N \) files of size \( F \) bits and a cache capacity at users of \( M = \frac{N}{gK} \). To achieve the rate \( R \) stated in Theorem 1, we present a prefetching and delivery scheme for \( g \in \{0, 1, \ldots, N\} \), since for general \( \frac{1}{K} \leq M \leq \frac{N}{g} \), the minimum rate can be achieved by memory sharing.

**Prefetching scheme:** We partition each file, \( W_f, f \in \{1, \ldots, N\} \) into \( K \left( \frac{N - 1}{g - 1} \right) \) non-overlapping subfiles of equal size. First, we assign non-overlapping subsets of \( \binom{N - 1}{g} \) different subfiles of each file \( W_f \) to each user \( U_i, i \in \{1, \ldots, K\} \). Next, consider all \( \binom{N}{g} \) possible subsets \( A \subseteq \{1, \ldots, N\} \) of \( g \) different files \( |A| = g \). Observe that there are \( \binom{N - 1}{g - 1} \) subsets \( A \) which include a particular file \( W_f \), i.e. satisfying \( f \in A \). Each of the \( \binom{N - 1}{g - 1} \) subfiles of file \( W_f \) assigned to user \( U_i \) is then assigned to a different subset \( A \) satisfying \( f \in A \). We identify this subfile as \( W_f^{(i)} \). Given this partition, at user \( U_i \), for every subset of files \( A \), we compute the binary sum of the \( g \) subfiles assigned to subset \( A \), and store the coded cached subfile

\[
Z_A^{(i)} = \oplus_{f \in A} W_f^{(i)}.
\]

Because each user stores \( \binom{N}{g} \) coded cached subfiles, and each subfile has \( F \binom{N - 1}{g - 1} \) bits, the required cache load at each user equals \( MF = \frac{\binom{N}{g}}{K \binom{N - 1}{g - 1}} F = \frac{N}{gK} F \) bits, which satisfies the memory constraint imposed.

**Delivery scheme:** As pointed out in the example, all subfiles can be found only once at some code cached subfile. The delivery scheme proposed divides the coded cached subfiles into three types and obtains the requested subfiles XORed at each of the coded cached subfile types separately.

**Coded cached subfiles Type I:** First, consider those requested subfiles which are XORed at coded cached subfiles that also XOR subfiles not requested by any user. The server simply broadcast, one by one, all the requested subfiles in every coded cached subfile Type I. Suppose only \( N_e(d) \) distinct files out of the total \( N \) files are requested by all users. From the placement phase, we know that XORed at all the coded cached subfiles, there are \( \binom{N - 1}{g - 1} \) subfiles of each file at every user, from which there are \( \binom{N_e(d) - 1}{g - 1} \) requested subfiles XORed to other requested subfiles. Thus, there are \( \binom{N - 1}{g - 1} - \binom{N_e(d) - 1}{g - 1} \) distinct subfiles of every file at every user which are encoded together with some not requested subfile. Finally, since there are \( N_e(d) \) different requests and \( K \) users, the total number of subfile transmissions needed is

\[
T_I = KN_e(d) \left( \binom{N - 1}{g - 1} - \binom{N_e(d) - 1}{g - 1} \right).
\]

**Coded cached subfiles Type II:** Second, consider those coded cached subfiles which XOR a subfile requested by the user which stores them. Notice that, all requested subfiles still missing are encoded at some coded cache subfile together with \( g - 1 \)
other also requested subfiles. The delivery strategy for all the subfiles XORed at the coded cached subfiles Type II is divided into two consecutive phases. In the first phase, each user \( U_i \) obtains the subfiles \( W_{d(i),A}^{(i)} \) of their own demand \( d(i) \) encoded at coded cached subfiles \( Z_A^{(i)} \) in their own cache. In the second phase, every two distinct users \( U_i, U_j \) requesting the same file \( d(i) = d(j) \) share the subfiles obtained in Phase 1.

**Phase 1:** For each of the coded cached subfiles Type II, the server first broadcasts, one by one, all the subfiles XORed at \( Z_A^{(i)} \) in their own cache. This is, the server broadcast each of the \( g - 1 \) subfiles XORed to \( W_{d(i),A}^{(i)} \) in \( Z_A^{(i)} \). Given that there are \( \binom{N_e(d) - 1}{g - 1} \) of such subfiles at each user, we require

\[
K(g - 1) \left( \frac{N_e(d) - 1}{g - 1} \right) 
\]

subfile transmissions. Next, by computing the binary sum of \( Z_A^{(i)} \) and the broadcasted subfiles XORed in the coded cached subfile \( Z_A^{(i)} \), each user obtains the requested subfile \( W_{d(i),A}^{(i)} \). At the end of this phase, user \( U_i \) has obtained the \( \binom{N_e(d) - 1}{g - 1} \) requested subfiles which were cached at some coded cached subfile in its own cache. In addition, users obtain, directly, from the broadcasted subfiles XORed at \( Z_A^{(i)} \) in their own cache. In the second phase, every two distinct users \( U_i, U_j \) share the subfiles obtained in Phase 1, the server transmits the binary sum

\[
\sum_{\forall f \in d} (K - |K(f)|) = \binom{N_e(d) - 2}{g - 2} (N_e(d)K - K)
\]

where (8) follows from \( \binom{n}{m} = \binom{n - 1}{m - 1} \frac{n}{m} \).

**Phase 2:** After Phase 1, because all subfiles XORed at coded cached subfiles of different users are distinct, users with the same request have obtained a different subset of \( \binom{N_e(d) - 1}{g - 1} \) subfiles of their common requested file. Consider the set of users \( K(f) \) requesting file \( W_f \). In order to all users \( U_i \), requesting file \( W_f \), \( i \in K(f) \) share the subfiles obtained Phase 1, the server arbitrarily selects one user leader \( u \in K(f) \) per file \( W_f \). For every other user \( U_i \), \( i \in K(f)/u \) and every subset \( A \) of \( g \) requested files including \( W_{d(i)} \), the server transmits the binary sum

\[
Y_{f,A}^{(i,u)} = W_{f,A}^{(i)} \oplus W_{f,A}^{(u)}.
\]

Given that there are \( \binom{N_e(d) - 1}{g - 1} \) subsets \( A \) of \( g \) requested files, we require a total of

\[
(|K(f)| - 1) \left( \frac{N_e(d) - 1}{g - 1} \right)
\]

coded subfile transmissions per file.

Next, for every requested file \( W_f \), the user leader \( u \in K(f) \) computes the remaining requested subfiles by computing the binary sum of the subfile \( W_{f,A}^{(u)} \) obtained in Phase 1 with each of the broadcasted coded subfiles

\[
W_{f,A}^{(i)} = Y_{f,A}^{(i,u)} \oplus W_{f,A}^{(u)}
\]

for all \( i \in K(f)/u \) and all sets of \( g \) requested files \( A \), such that \( d(u) \in A \). Every other user \( U_j \), \( j \neq u \), requesting file \( W_f \), \( j \in K(f) \) first obtains the subfile of the user leader \( W_{f,A}^{(u)} \) by computing

\[
W_{f,A}^{(u)} = Y_{f,A}^{(j,u)} \oplus W_{f,A}^{(j)}
\]

and then uses \( W_{f,A}^{(u)} \) to obtain the rest of the subfiles by computing (11) for all \( i \in K(f)/u/j \) and all sets of \( g \) requested files \( A \), such that \( f \in A \). Because there are \( N_e(d) \) distinct requested files, the total number of coded subfile transmissions (9) is

\[
\sum_{\forall f \in d} (|K(f)| - 1) \left( \frac{N_e(d) - 1}{g - 1} \right) = (K - N_e(d)) \left( \frac{N_e(d) - 1}{g - 1} \right).
\]
Finally, adding together (7) and (12), the total number of subfile length transmissions needed to obtain the requested subfiles XORed in the coded cached subfiles Type II is

$$T_{II} = (Kg - N_e(d)) \left( \frac{N_e(d) - 1}{g - 1} \right).$$

Coded cached subfiles Type III: Finally, consider those coded cached subfiles which XOR together subfiles not requested by the user storing them. Specifically, consider any subset $V$ of $g + 1$ users with different demands. Let $B(V)$ denote the set of $g + 1$ demands of the users in $V$. For any user, $U_i$, $i \in V$ there is exactly one coded cached subfile

$$Z_{B(V) \setminus d(i)}^{(i)} = \oplus_{j \in B(V) \setminus d(i)} W_{r_V(j), B(V) \setminus d(i)}^{(i)}$$

which does not XOR a subfile of $d(i)$, but XORs together one subfile requested by each of the other $g - 1$ users in $V$. In this phase, all user obtain such subfiles. Observe that for coded cached subfiles Type III to exist, there might be at least $g + 1$ users with different demands, this is $N_e(d) \geq g + 1$. Again, the delivery scheme is partitioned into two consecutive phases:

**Phase 1:** First, the server selects a set of $N_e(d)$ user leaders $U$ each with a different demand. Next, for every subset $V \subseteq U$ of $g + 1$ user leaders, the server arbitrarily associates to each user, $U_j$, $j \in V$ a different file $r_V(j) \in B(V)$, such that $r_V(j) \neq d(j)$ and $r_V(i) \neq r_V(j)$ for all $i \neq j$, $i \in V$. Since $|B(V)| = g + 1$, this is always possible. The server then broadcasts the coded subfiles

$$Y_{V(i,j)}^V = W_{r_V(i), B(V) \setminus d(i)}^{(i)} \oplus W_{r_V(j), B(V) \setminus d(i)}^{(j)}$$

for all $j \in V \setminus i$ with $r_V(j) \neq d(i)$, $r_V(j) \neq r_V(i)$ and all $i \in V$. Because there are $\binom{N_e(d)}{g + 1}$ subsets $V \subseteq U$ of $g + 1$ users, this results in a total of $\binom{N_e(d)}{g + 1}(g - 1)(g + 1)$ subfile length transmissions. Next, for every other user, $U_i$, $i \notin U$ satisfying $d(i) \in B(V)$, the server broadcasts

$$Y_{V(i,j)}^V = W_{r_V(i), B(V) \setminus d(i)}^{(i)} \oplus W_{r_V(j), B(V) \setminus d(i)}^{(j)}$$

for all $j \in V$ with $r_V(j) \neq d(i)$. Because there are $K - N_e(d)$ not users leaders $i \notin U$, and there are subsets $V \subseteq U$ of $g + 1$ users satisfying $d(j) \in V$, this results in a total of $g(K - N_e(d)) \binom{N_e(d) - 1}{g}$ coded subfile transmissions.

Next, for all subsets of $g + 1$ users $V \subseteq U$, every user $U_i$, satisfying $d(i) \in B(V)$ computes

$$Z_{B(V) \setminus d(i)}^{(i)} \oplus_{j \in V \setminus i, r_V(j) \neq d(i)} Y_{V(i,j)}^V = \oplus_{j \in V, r_V(j) \neq d(i)} W_{r_V(j), B(V) \setminus d(i)}^{(i)} \oplus_{j \in V \setminus i, r_V(j) \neq d(i)} Y_{V(i,j)}^V$$

where for every $i \notin V$, we defined $r_V(i) = r_V(j)$ where $j \in V$ and $d(i) = d(j)$.

**Phase 2:** Finally, for each subset of $g + 1$ users leaders $V \subseteq U$, the server broadcasts the coded subfile

$$Y_V = \oplus_{j \in V} W_{r_V(j), B(V) \setminus d(j)}^{(j)}.$$
for all subsets of \( g + 1 \) users \( V \subseteq U \) and all users \( l \neq i \) satisfying \( d(i) = d(l) \).

The total number of transmissions in both phases is

\[
T_{III} = \left( \frac{N_e(d)}{g + 1} \right) ((g - 1)(g + 1) + (K - N_0) g \left( \frac{N_e(d) - 1}{g} \right)
\]

\[
= \left( \frac{N_e(d) - 1}{g} \right) g \left( K - \frac{N_e(d)}{g + 1} \right)
\]

\[
= \left( \frac{N_e(d) - 1}{g - 1} \right) (N_e(d) - g) \left( K - \frac{N_e(d)}{g + 1} \right)
\]

where (13) follows from \( \binom{n}{k} = \binom{n - 1}{k - 1} \frac{n}{k} \) and (14) follows from \( \binom{n}{k} = \binom{n}{k - 1} \frac{n + 1 - k}{k} \).

Finally, adding together all the subfile length transmissions required for the three coded cached subfile types, we obtain

\[
T = KN_e(d) \left( \frac{N - 1}{g - 1} \right) - N_e(d) \left( \frac{N_e(d) + 1}{g + 1} \right) \left( \frac{N_e(d) - 1}{g - 1} \right)
\]

which leads to the rate (4) stated in Theorem 1. Finally, given that for \( K \geq N_e(d) \), (15) increases monotonically as a function of \( N_e(d) \). Particularizing to \( N_e(d) = N \), we obtain the rate-memory function stated in Corollary 1.1.

V. CONCLUSIONS

In this work, we proposed a novel centralized coded caching scheme for the case where there are more users than files and users are equipped with small memories. The scheme uses coded prefetching and outperforms previously proposed schemes for moderate-high number of users, \( N \leq K \leq \frac{N^2 + 1}{2} \). Our current future work includes the extension of the proposed coded prefetching technique to larger memories and scenarios with more users than files, as well as, the extension of the proposed centralized technique to the decentralized setting.

APPENDIX A
PROOF OF COROLLARY 1.2

To prove this results, we first observe that by isolating parameter \( t \) in the MDS rate points, \( t = \frac{K}{N} (N - R_{MDS}^*) \) and substituting it into the \( M_{MDS} \) memory points, we can obtain the following memory-rate function

\[
M_{MDS}^{(ib)}(R) = \frac{(N - R)((N - R)(NK - K) + N(K - N))}{N^2 (K - 1)}.
\]

Due to the convexity of \( M_{MDS}^{(ib)}(R) \) function with respect to \( R \), we have that \( M_{MDS}^{(ib)}(R) \) represents a lower-bound on the MDS memory-rate region that can be obtained by the lower convex envelope of the memory-rate pairs 3.

Particularizing (16) to the rate points in 5, we can write \( M_{MDS}^{(ib)} \) as a function of \( g \) as

\[
M_{MDS}^{(ib)}(g) = \frac{N + 1}{g + 1} \frac{1}{K(K - 1)} \left( \frac{N^2 - 1}{g + 1} + K - N \right)
\]

for \( g = 1, \ldots, N \). Thus, a sufficient conditions for the new rate-memory pairs in 5 to improve the MDS rate-memory region is

\[
\frac{N}{Mg} \leq M_{MDS}^{(ib)}(g)
\]

which leads us to

\[
K \leq \frac{N^2 g + 1}{g + 1}.
\]

Finally, since \( \frac{N^2 g + 1}{g + 1} \) increases monotonically with \( g \), we have that a sufficient condition for our strategy to improve the MDS strategy in the complete region considered \( M \in [0, \frac{N}{R}] \) and \( N \leq K \), is

\[
K \leq \frac{N^2 + 1}{2}.
\]
The proof of Corollary [1,3] follows the lines of [11]. From [11], the cut set lower bound is given by

$$R_{CB}(M) = \max_{s \in \{1, \ldots, \min(N, K)\}} \frac{1}{s} \left( N - sM - \frac{\mu(N - ls)}{s + \mu} - (N - Kl) \right)$$

where $R^*(M)$ is the rate-memory tradeoff function presented in Corollary [1,1] for $g \in \{1, \ldots, N\}$.

The lower bound obtained in [14, Theorem 1] is given by

$$R_{STC}(M) = \max_{s \in \{1, \ldots, K\}, l \in \{1, \ldots, \left\lceil \frac{N}{s} \right\rceil\}} \frac{1}{l} \left( N - sM - \frac{\mu(N - ls)}{s + \mu} - (N - Kl) \right)$$

where $\mu = \min\left(\frac{N - ls}{s}, K - s\right) \forall s, l$. Observe that for $l = 1$ and $s = N - 1$, $K \geq N$, we have $\mu = \min(N - s, K - s) = N - s = 1$, and the objective in (17) reads $N - (N - 1)M - \frac{1}{N}$. Thus,

$$R_{STC}(M) \geq N - (N - 1)M - \frac{1}{N}$$

which particularized to $K = N$, $M = \frac{N}{(N-1)K}$, $g = N - 1$ leads us to

$$R_{STC} \left( \frac{N}{(N-1)K} \right) \geq N - 1 - \frac{1}{N}$$

$$= R^* \left( \frac{1}{N - 1} \right).$$

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