Lie-algebraic discretization of differential equations

by

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ABSTRACT

A certain representation for the Heisenberg algebra in finite-difference operators is established. The Lie-algebraic procedure of discretization of differential equations with isospectral property is proposed. Using \( sl_2 \)-algebra based approach, (quasi)-exactly-solvable finite-difference equations are described. It is shown that the operators having the Hahn, Charlier and Meixner polynomials as the eigenfunctions are reproduced in present approach as some particular cases. A discrete version of the classical orthogonal polynomials (like Hermite, Laguerre, Legendre and Jacobi ones) is introduced.

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Discretization is one of the most powerful tools of solving continuous theories. It appears in different forms in various physics sciences leading to discrete versions of differential equations of classical mechanics, lattice field theories etc. However, once the discretization is chosen as an approach to a problem we meet a hard problem of ambiguity – there exist infinitely-many different ways of discretization having the same continuous limit. The goal of present paper is to demonstrate a certain discretization scheme for differential equations with exceptional properties – eigenvalues remain unchanged (isospectrality), eigenfunctions are modified in a simple manner etc.

1. Two operators \( a \) and \( b \) obeying the commutation relation

\[
[a, b] \equiv ab - ba = 1, \tag{1}
\]

with the identity operator in the r.h.s. define the Heisenberg algebra. The Heisenberg algebra plays the central role in many branches of theoretical and mathematical physics. A standard representation of (1) in the action on the real line is the coordinate-momentum one:

\[
a = \frac{d}{dx}, \quad b = x. \tag{2}
\]

Our goal is to build the one-parametric representation of (1) on the line in terms of finite-difference operators having (2) as a limiting case.

Let us introduce the finite-difference operators \( \mathcal{D}_\pm \) possessing the property of the translation covariance

\[
\mathcal{D}_+ f(x) = \frac{f(x + \delta) - f(x)}{\delta} \equiv \frac{e^{\delta d/dx} - 1}{\delta} f(x) \tag{3}
\]

and

\[
\mathcal{D}_- f(x) = \frac{f(x) - f(x - \delta)}{\delta} \equiv \frac{1 - e^{-\delta d/dx}}{\delta} f(x) \tag{4}
\]

and \( \mathcal{D}_+ \rightarrow \mathcal{D}_- \), once \( \delta \rightarrow -\delta \). It is worth noting that

\[
\mathcal{D}_+ - \mathcal{D}_- = \delta \mathcal{D}_- \mathcal{D}_+. \tag{5}
\]

Now define the so-called quasi-monomial

\[
x^{(n+1)} = x(x - \delta)(x - 2\delta)\ldots(x - n\delta) = \delta^{n+1} \frac{\Gamma\left(\frac{x}{\delta} + 1\right)}{\Gamma\left(\frac{x}{\delta} - n\right)}, \tag{6}
\]
which in the limit $\delta \to 0$ becomes the ordinary monomial $x^{n+1}$. The operators $D_{\pm}$ (3)--(4) act on quasi-monomials as follows

$$D_{+}x^{(n)} = nx^{(n-1)}, \quad x(1 - \delta D_{-})x^{(n)} = x^{(n+1)}.$$

(7)

So the operator $D_{+}$ plays the same role in action on the quasi-monomial $x^{(n)}$ as the operator of differentiation in action on monomial $x^n$: it converts $n$th quasi-monomial to $(n-1)$th with the coefficient of proportionality equals to $n$. The operator $x(1 - \delta D_{-})$ acts similarly to the operator of multiplication $x$ on monomials: it converts $n$th quasi-monomial to $(n+1)$th quasi-monomial. In the limit $\delta \to 0$ those operators become $\frac{d}{dx}$ and $x$, respectively. Straightforward calculation shows that the commutator of the operators $D_{+}$ and $x(1 - \delta D_{-})$ equals to one! So the operators

$$a = D_{+},$$
$$b = x(1 - \delta D_{-})$$

(8)

form one-parametric representation of the Heisenberg algebra (1). It is easy to show that (8) belongs to the point canonical transformations.

One of possible interpretations of (8) is to identify the finite-difference operator $a$ with momentum in discretized form, while $b$ becomes the conjugated coordinate (in a complete agreement to the limit $\delta \to 0$).

2. Consider the eigenvalue problem for some linear differential operator

$$L\left[\frac{d}{dx}, x\right] \varphi(x) = \lambda \varphi(x)$$

(9)

having polynomial eigenfunctions. Now let us replace in the equation (9) the derivative $\frac{d}{dx}$ and $x$ by the operators $a$ and $b$, respectively, obeying the commutation relation (1). Then we arrive at

$$L[a, b] \varphi(b) = \lambda \varphi(b)$$

(10)

In order to make sense to (10), one should add the definition of the vacuum $|0>$:

$$a|0> = 0.$$

(11)
Then the equation (10) has a meaning of the operator eigenvalue problem in the Fock space with the vacuum (11). It is evident that the eigenvalue problem (10)–(11) has the same eigenvalues and the same eigenfunctions (with replacement of $x \rightarrow b$) as the original problem (9).

So taking different realizations of the algebra (1) and considering a certain element of the universal enveloping algebra of the algebra (1), we get different eigenvalue problems with the property of isospectrality. Below we will consider isospectral problems emerging from the representations (2) and (8). We name such a way of discretization the *Lie-algebraic discretization*.

Now we have to specify the operators $L$ in (10) having polynomial eigenfunctions. In [1] it was proven that $L$ has a certain amount of polynomial eigenfunctions if and only if, $L$ is the superposition of the element of the universal enveloping algebra of the $sl_2$-algebra taken in the finite-dimensional irreducible representation

$$
J_n^+ = b^2a - nb
$$

$$
J_n^0 = ba - \frac{n}{2}
$$

$$
J_n^- = a
$$

(12)

where $n$ is a non-negative integer, and the operator $B(b)a^{n+1}$, where $B(b)$ is any operator of $b$. The dimension of this representation is equal to $(n+1)$ and hence $(n+1)$ eigenfunctions have a form of polynomial of degree $n$. These operators $L$ are named *quasi-exactly-solvable*. Moreover, one can prove that $L$ possesses infinite-many polynomial eigenfunctions if and only if $L$ is a polynomial in generators $J^0 \equiv J_0^0$, $J^- \equiv J_0^-$ only. Those operators $L$ are named *exactly-solvable*.

3. In order to exploit the representation (8) let us firstly define the vacuum $|0>$. The condition (11) in the explicit form is

$$
f(x + \delta) = f(x) .
$$

(13)

Any periodic function with the period $\delta$ is the solution of this equation, however, without the loss of generality we can make the choice

$$
f(x) = 1 .
$$

(14)
With this vacuum, it is easy to see that
\[ b^n|0 > = [x(1 - \delta D_-)]^n|0 > = x^{(k)} \] (15)
where the relation
\[ [xe^{-\delta \frac{d}{dx}}]^n = x^{(n)}e^{-n\delta \frac{d}{dx}}. \]
was used for derivation (15). So, starting from (10)–(11), we obtain the eigenvalue problem for the finite-difference operator
\[ L[D_+, x(1 - \delta D_-)]\tilde{\varphi}(x) = \lambda \tilde{\varphi}(x) \] (16)
where the solutions \( \tilde{\varphi}(x) \) are related to the solutions \( \varphi(x) \) of (9). To clarify this relationship, let us assume that
\[ \varphi(x) = \sum \alpha_k x^k \] (17)
is a certain solution of the equation (9). The transition from (9) to (10) implies the replacement of \( x \) by \( b \). Then taking into account (15) we come to the conclusion that each monomial in (17) should be replaced by the quasi-monomial. Finally, we get that the corresponding solution of the equation (16) has the form
\[ \tilde{\varphi}(x) = \sum \alpha_k x^{(k)} \] (18)
The equation (16) has the same eigenvalues as the original equation (9), while the eigenfunctions are modified by replacing the monomials by quasi-monomials. It is evident, that such a procedure preserves the degree of polynomial.

Now let us proceed to concrete realization of idea of the Lie-algebraic discretization for differential equations with polynomial solutions. It is quite obvious that there exist two different approaches:

- In the first approach, we start from a certain differential equation with known spectra and apply above-described Lie-algebraic discretization. Generically, a second-order differential equation becomes a finite-difference equation relating the unknown function in more than three points or, in other words, more than the three-point finite-difference equations.
• The second approach is different: we look for the differential equations leading to the three-point finite-difference equations under the Lie-algebraic discretization. In other words, what are the elements of the universal enveloping \( sl_2 \)-algebra in the representation (8), (12) producing the three-point finite-difference equations.

4. The first approach. In \([1]\) (see also \([2]\)) it is shown that the most general, second-order exactly-solvable differential operator on the real line possessing infinitely-many polynomial eigenfunctions has the form of the hypergeometric operator

\[
E_2(\frac{d}{dx}, x) = -Q_2(x)\frac{d^2}{dx^2} + Q_1(x)\frac{d}{dx} + Q_0
\]  

(19)

where \(Q_k(x)\) are polynomials of the \(k\)th order with arbitrary coefficients:

\[
Q_2 = a_0x^2 + a_1x + a_2, \quad Q_1 = b_0x + b_1, \quad Q_0 = c_0.
\]

The operator (19) is equivalent to the operator appearing from the most general quadratic polynomial in the generators \(J^0, J^-\) (see (12)) defined through the representation (2) \([1]\). Performing the above-mentioned procedure of the Lie algebraization: (9) \(\rightarrow\) (10) \(\rightarrow\) (16), we arrive at the following exactly-solvable finite-difference operator isospectral with (19)

\[
\tilde{E}_2 \equiv E_2[D_+, x(1 - \delta D_-)] =
\]

\[-\tilde{a}_2e^{2\delta \frac{d}{dx}} + [-\tilde{a}_1x + (2\tilde{a}_2 + \tilde{b}_1)]e^{\delta \frac{d}{dx}} +
\]

\[-\tilde{a}_0x(x - \delta) + (2\tilde{a}_1 + \tilde{b}_0)x - (\tilde{a}_2 + \tilde{b}_1) +
\]

\[2\tilde{a}_0x(x - \delta) - (\tilde{a}_1 + \tilde{b}_0)x]e^{-\delta \frac{d}{dx}} - \tilde{a}_0x(x - \delta)e^{-2\delta \frac{d}{dx}},
\]

(20)

where \(\tilde{a}_i = \frac{a_i}{\delta}, \tilde{b}_i = \frac{b_i}{\delta}\). The corresponding eigenvalue problem has the form

\[-\tilde{a}_2\varphi(x + 2\delta) + [-\tilde{a}_1x + (2\tilde{a}_2 + \tilde{b}_1)]\varphi(x + \delta) +
\]

\[-\tilde{a}_0x(x - \delta) + (2\tilde{a}_1 + \tilde{b}_0)x - (\tilde{a}_2 + \tilde{b}_1)]\varphi(x) +
\]

\[2\tilde{a}_0x(x - \delta) - (\tilde{a}_1 + \tilde{b}_0)x]\varphi(x - \delta) - \tilde{a}_0x(x - \delta)\varphi(x - 2\delta) = \lambda \varphi(x),
\]

(21)
It is worth noting that in contrary to a naive expectation, although it is started from the second-order differential operator, the finite-difference operator connects to the function in five different points: \((x + 2\delta, x + \delta, x, x - \delta, x - 2\delta)\).

Now taking, for instance, a concrete operator (19) with the Hermite polynomials \(H_k(x)\) as the eigenfunctions \((a_0 = a_1 = b_1 = 0, a_2 = -1, b_0 = -2)\):

\[
h\left(\frac{d}{dx}, x\right) = \frac{d^2}{dx^2} - 2x \frac{d}{dx},
\]

or, equivalently,

\[
h = J^- J^- - 2J^0
\]

and performing the Lie algebraization (taking into account (5)), we arrive at the isospectral finite-difference operator

\[
\tilde{h} = D^2_+ - 2xD_-
\]

The corresponding eigenvalue problem has the form

\[
\frac{1}{\delta^2}\varphi(x + 2\delta) - \frac{2}{\delta^2}\varphi(x + \delta) - \frac{1}{\delta}(2x - \frac{1}{\delta})\varphi(x) + \frac{2x}{\delta}\varphi(x - \delta) = \lambda\varphi(x),
\]

with the eigenvalues \(\lambda_k = 2k\) and the eigenfunctions \(H_{(k)}(x) = \sum_{i=0}^{k} c_i x^{(i)}\), where \(k = 0, 1, 2 \ldots\) and \(c_i\) are the standard Hermite polynomials coefficients. We name these polynomials \(H_{(k)}(x)\) the discrete Hermite polynomials. In similar way one can construct the discrete Laguerre, Legendre, Jacobi polynomials. The results will be presented elsewhere.

Analogously, one can take the general second-order quasi-exactly-solvable operator

\[
T_2\left(\frac{d}{dx}, x\right) = -P_4(x)\frac{d^2}{dx^2} + P_3(x)\frac{d}{dx} + P_2(x)
\]

where \(P_k(x)\) are polynomials of the \(k\)th order depending on ten free parameters one of which is a non-negative integer, \(n\) (see (12); for more details see [1]). It leads to the following isospectral, quasi-exactly-solvable, finite-difference operator
\[ \tilde{T}_2 \equiv T_2[D_+, x(1 - \delta D_-)] = \]
\[ \tilde{\alpha}_2 e^{2\delta \frac{d}{dx}} + \tilde{P}_{1,1}(x) e^{\delta \frac{d}{dx}} + \]
\[ \tilde{P}_{0,2}(x) + x \tilde{P}_{-1,2}(x) e^{-\delta \frac{d}{dx}} + x^{(2)} \tilde{P}_{-2,2}(x) e^{-2\delta \frac{d}{dx}} + \]
\[ x^{(3)} \tilde{P}_{-3,1}(x) e^{-3\delta \frac{d}{dx}} + \tilde{\alpha}_{-4} x^{(4)} e^{-4\delta \frac{d}{dx}}, \]  
where \( \alpha \)'s are parameters and \( \tilde{P}_{j,k}(x) \) are polynomials of the kth order. It is worth emphasizing that the quasi-exactly-solvable finite-difference equation corresponding to (25) relates unknown function to seven different points: \((x + 2\delta, x + \delta, x, x - \delta, x - 2\delta, x - 3\delta, x - 4\delta)\).

5. *The second approach.* Standard second-order finite-difference equation relates an unknown function at three points and has the form

\[ A(x) \varphi(x + \delta) - B(x) \varphi(x) + C(x) \varphi(x - \delta) = \lambda \varphi(x), \]  
where \( A(x), B(x), C(x) \) are some functions. One can pose a natural problem: what are the most general coefficient functions \( A(x), B(x), C(x) \) for which the equation (26) admits infinitely-many polynomial eigenfunctions? Basically, the answer is presented in \[1\]: any operator with above property can be represented as a polynomial in the generators:

\[ J^0 = \frac{x}{\delta} (1 - e^{-\delta \frac{d}{dx}}) \]
\[ J^- = \frac{1}{\delta} (e^{\delta \frac{d}{dx}} - 1) \]  
(27)

One can show that the most general polynomial in the generators (27) leading to (26) is

\[ \tilde{E} = A_1 J^0 J^0 (J^- + \frac{1}{\delta}) + A_2 J^0 J^- + A_3 J^0 + A_4 J^- + A_5 \]  
(28)

and in explicit form
The spectral problem corresponding to the operator (29) is given by

\[ (A_4 \delta + \frac{A_2}{\delta^2} x + \frac{A_1}{\delta^3} x^2)e^{\delta \frac{d}{dx}} + \]

\[ + [A_5 - \frac{A_4}{\delta} + (\frac{A_1}{\delta^2} + 2\frac{A_2}{\delta^2} - \frac{A_3}{\delta})x - 2\frac{A_1}{\delta^3} e^2] \]

\[ + [-\frac{A_1}{\delta^2} - \frac{A_2}{\delta^2} + \frac{A_3}{\delta})x + \frac{A_1}{\delta^3} x^2]e^{-\delta \frac{d}{dx}} \]  

(29)

The spectral problem corresponding to the operator (29) is given by

\[ (\frac{A_4}{\delta} + \frac{A_2}{\delta^2} x + \frac{A_1}{\delta^3} x^2)f(x + \delta) + \]

\[ -[-A_5 + \frac{A_4}{\delta} - (\frac{A_1}{\delta^2} + 2\frac{A_2}{\delta^2} - \frac{A_3}{\delta})x + 2\frac{A_1}{\delta^3} x^2]f(x) \]

\[ + [-\frac{A_1}{\delta^2} - \frac{A_2}{\delta^2} + \frac{A_3}{\delta})x + \frac{A_1}{\delta^3} x^2]f(x - \delta) = \lambda f(x) . \]  

(30)

In general, this spectral problem has the Hahn polynomials \( h_k^{(\alpha,\beta)}(x, N) \) as the eigenfunctions (therein we follow the notations of [3]). Namely, these polynomials appear, if \( \delta = -1, A_5 = 0 \) and

\[ A_1 = -1, \ A_2 = N - \beta - 2, \ A_3 = -\alpha - \beta - 1, \ A_4 = (\beta + 1)(N - 1) . \]

Besides that, if

\[ A_1 = 1, \ A_2 = 2 - 2N - \nu, \ A_3 = 1 - 2N - \mu - \nu, \ A_4 = (N + \nu - 1)(N - 1) \]

the so-called analytically-continued Hanh polynomials \( \tilde{h}_k^{(\mu,\nu)}(x, N) \) appear, where \( k = 0, 1, 2 \ldots \).

So the equation (30) corresponds to the most general exactly-solvable finite-difference problem, while the operator (28) is the most general element of the universal enveloping \( sl_2 \)-algebra leading to (26). Hence the Hahn polynomials are related to the finite-dimensional representations of a certain cubic element of the universal enveloping \( sl_2 \)-algebra (for a general discussion see [2]).

Taking in (30) \( \delta = 1, A_5 = 0 \) and putting

\[ A_1 = 0, A_2 = -\mu, \ A_3 = \mu - 1, A_4 = \gamma \mu , \]
we reproduce the equation having the Meixner polynomials as the eigenfunctions. Furthermore, if

\[ A_1 = 0, A_2 = 0, A_3 = -1, A_4 = \mu , \]

the equation (30) corresponds to the equation with the Charlier polynomials as the eigenfunctions (for the definition of the Meixner and Charlier polynomials see e.g. [3]). For a certain particular choice of the parameters, one can reproduce the equations having Tschebyschov and Krawtchouk polynomials as the solutions.

Among the equations (26) there also exists quasi-exactly-solvable equations possessing a finite amount of polynomial eigenfunctions. All those equations are classified via the element of the universal enveloping \( sl_2 \)-algebra taken in the representation (8), (12)

\[
\tilde{T} = A_1 J_n^+ J_n^0 J_n^- + A_2 J_n^0 J_n^- J_n^- + A_3 J_n^0 J_n^- + A_4 J_n^- + A_5 \] (31)

(cf. (28)), where A’s are free parameters.

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