Differential models for B-type open-closed topological Landau-Ginzburg theories

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Abstract: We propose a family of differential models for B-type open-closed topological Landau-Ginzburg theories defined by a pair \((X,W)\), where \(X\) is any non-compact Calabi-Yau manifold and \(W\) is any holomorphic complex-valued function defined on \(X\) whose critical set is compact. The models are constructed at cochain level using smooth data, including the twisted Dolbeault algebra of polyvector-valued forms and a twisted Dolbeault category of holomorphic factorizations of \(W\). We give explicit proposals for cochain level versions of the bulk and boundary traces and for the bulk-boundary and boundary-bulk maps of the Landau-Ginzburg theory. We prove that most of the axioms of an open-closed TFT (topological field theory) are satisfied on cohomology and conjecture that the remaining axioms are also satisfied.

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1. Introduction

Classical oriented open-closed topological Landau-Ginzburg theories of type B are classical field theories parameterized by pairs \((X,W)\) and defined on compact oriented surfaces with
corners, where \( X \) is a non-compact Kählerian manifold and \( W : X \to \mathbb{C} \) is a non-constant holomorphic function. The general construction of such theories was given\(^1\) in [1,2]. As shown in loc. cit., such classical field theories determine a collection of topological D-branes, which arise as objects parameterizing the boundary contribution to the classical action. The Lagrangian construction of [1,2] shows that the topological D-branes are described by special Dolbeault factorizations, defined as pairs \((S, \mathfrak{D})\) formed of a smooth complex vector superbundle \( S \) on \( X \) and a “special Dolbeault superconnection” \( \mathfrak{D} \) defined on \( S \) and which squares to \( \text{Wid}_E \). As implicit in [2] and explained in Appendix B, such objects naturally form a certain \( \mathbb{Z}_2 \)-graded differential graded (DG) category \( \text{DF}_\infty(X, W) \), whose morphisms are bundle-valued differential forms and whose differentials are induced by the superconnections. The classical action on a compact oriented surface \( \Sigma \) with corners depends on the choice of a special Dolbeault factorization \((E_i, D_i)\) for each boundary component of \( \Sigma \), on the choice of a morphism of \( \text{DF}_\infty(X, W) \) at each corner of \( \Sigma \) as well as on the choice of a Kähler metric on \( X \) and of an “admissible” Hermitian metric \( h_i \) on each \( E_i \).

It is widely expected that such theories admit a non-anomalous quantization when \( X \) is a Calabi-Yau manifold in the sense that the canonical line bundle of \( X \) is holomorphically trivial. On general grounds, a physically acceptable quantization procedure for the Calabi-Yau case must produce, among other structures, a quantum oriented open-closed topological field theory in the sense axiomatized in reference [3] — a mathematical object which, as shown in loc. cit., can be described equivalently by a certain algebraic structure called a TFT datum of signature equal to the mod \( 2 \) reduction \( \mu \in \mathbb{Z}_2 \) of the complex dimension of \( X \) (see Section 2 for the precise definition and terminology). Using a non-rigorous method of “partial localization” of the path integral, reference [2] argued that, when \( X \) is Calabi-Yau and the critical set of \( W \) is compact, the TFT datum of quantum B-type topological Landau-Ginzburg theories can be constructed as the cohomology of a family of smooth differential models built using a dg-algebra \((\text{PV}(X), \delta_W)\) of polyvector-valued forms defined on \( X \) and of the dg-category \( \text{DF}_\infty(X, W) \) mentioned above. The “bulk differential” \( \delta_W \) is obtained from the Dolbeault differential on polyvector-valued forms by adding an operator proportional to the contraction with \( \partial W \).

In this paper, we reconsider the proposal of [2] from a mathematical perspective. Using the Koszul-Malgrange correspondence, we show that \( \text{DF}_\infty(X, W) \) is equivalent with a dg-category \( \text{DF}(X, W) \) of holomorphic factorizations of \( W \). A holomorphic factorization of \( W \) is a pair \((E, D)\), with \( E \) a holomorphic vector superbundle defined on \( X \) and \( D \) an odd holomorphic section of \( \text{End}(E) \) which squares to \( \text{Wid}_E \). The morphisms of \( \text{DF}(X, W) \) are smooth bundle-valued differential forms while the differentials are obtained from the Dolbeault operator by adding a correction dependent on \( D \). Using this equivalent description of the dg-category of topological D-branes, we give a simplified and rigorous construction of the differential models initially proposed in [2] and study their mathematical properties.

The differential models constructed in this paper provide a cochain level realization of the constitutive blocks of the TFT datum, from which the TFT datum can be recovered by passing to cohomology. These differential models depend explicitly on metric data, namely on the choice of a Kähler metric \( G \) on \( X \) and of an “admissible” Hermitian metric \( h_\alpha \) on the holomorphic vector superbundle \( E \) underlying each holomorphic factorization \( a = (E, D) \) of \( W \). They also depend on the choice of a “localization parameter” \( \lambda \in \mathbb{R}_{\geq 0} \). More precisely, the bulk and boundary traces \( \text{Tr} \) and \( \text{tr} = (\text{tr}_a)_{a \in \text{ObDF}(X, W)} \) and the bulk-boundary and boundary-bulk maps \( e = (e_\alpha)_{a \in \text{ObDF}(X, W)} \) and \( f = (f_a)_{a \in \text{ObDF}(X, W)} \) of the TFT datum (see Section 2) have cochain level realizations which depend on \( \lambda \) and on the metrics \( G \) and \( h_\alpha \), such that the maps induced\(^1\) References [1,2] considered only the case of oriented surfaces with boundary. However, the construction of loc. cit. extends easily to oriented surfaces with corners, leading in the boundary sector to the differential graded category \( \text{DF}_\infty(X, W) \) discussed in Appendix B.
by these on cohomology recover $\text{Tr}$, $\text{tr}$ and $e$, $f$. We prove that the cohomological bulk and boundary traces $\text{Tr}$ and $\text{tr}$ constructed in this manner are independent of the choice of $\lambda$, $G$ and $h_a$ and that the cohomological bulk-boundary maps $e_a$ and cohomological boundary-bulk maps $f_a$ are independent on the choice of $\lambda$ and $h_a$ and depend only on the Kähler class of $G$. We also show that most of the axioms of a TFT datum are satisfied on cohomology:

**Theorem 1.1** Suppose that the critical set $Z_W$ of $W$ is compact. Then the cohomology algebra $\text{HPV}(X,W)$ of $(\text{PV}(X),\delta_W)$ is finite-dimensional over $\mathbb{C}$ while the total cohomology category $\text{HDF}(X,W)$ of $\text{DF}(X,W)$ is Hom-finite over $\mathbb{C}$. Moreover, the system:

$$(\text{HPV}(X,W),\text{HDF}(X,W),\text{Tr},\text{tr},e)$$

obeys all defining properties of a TFT datum of signature equal to the mod 2 reduction of $d \text{def} = \dim_{\mathbb{C}} X$, up to non-degeneracy of the bulk and boundary traces and to the topological Cardy constraint.

In view of the path integral arguments of [2] and of experience with the case when $X$ is Stein and $W$ has finite critical set [4], we make the following conjecture, whose proof we will address in a separate publication:

**Conjecture 1.2** Suppose that $Z_W$ is compact. Then $(\text{HPV}(X,W),\text{HDF}(X,W),\text{Tr},\text{tr},e)$ is a TFT datum and hence it defines a quantum open-closed TFT in the axiomatic sense of [3]. Moreover, the boundary-bulk maps of this TFT datum are induced by the cochain level boundary-bulk maps, and the same is true for the corresponding bulk-boundary maps.

The paper is organized as follows. Section 2 recalls the algebraic description of (non-anomalous and oriented) quantum open-closed two-dimensional topological field theories (TFTs) which was derived in [3] using the definition of the latter as certain monoidal functors from the category of two-dimensional oriented cobordisms with corners to the category of finite-dimensional $\mathbb{Z}_2$-graded vector spaces over $\mathbb{C}$. Section 3 discusses the twisted Dolbeault dg-algebra $(\text{PV}(X),\delta_W)$ of polyvector-valued forms defined on $X$, which was proposed in [2] as an “off-shell” model for the algebra of bulk observables of B-type Landau-Ginzburg TFTs. When the critical set of $W$ is compact, we show that the cohomology $\text{HPV}(X,W)$ of this algebra is finite-dimensional over $\mathbb{C}$ and that it is isomorphic with the cohomology $\text{HPV}_c(X,W)$ of the dg-subalgebra $\text{PV}_c(X)$ obtained by restricting to polyvector-valued forms with compact support. Section 4 discusses the dg-category $\text{DF}(X,W)$ of holomorphic factorizations of $W$, which (as shown in Appendix B) provides an equivalent model for the dg-category of topological D-branes. When the critical set of $W$ is compact, we show that the total cohomology category $\text{HDF}(X,W)$ of this dg-category is Hom-finite over $\mathbb{C}$ and that it is isomorphic with the total cohomology category $\text{HDF}_c(X,W)$ of a non-full dg-subcategory $\text{DF}_c(X,W)$ obtained by restricting to morphisms of compact support. Sections 5 and 6 discuss cochain level models for the bulk and boundary traces as well as the maps which they induce on cohomology. The cochain level traces can always be defined on $\text{PV}_c(X)$ and $\text{DF}_c(X,W)$ and the cochain level boundary traces satisfy a strict cyclicity condition. When the critical set of $W$ is compact, the cochain level traces induce cohomological traces on $\text{HPV}(X,W)$ and $\text{HDF}(X,W)$, which in particular make the latter into a pre-Calabi-Yau $\mathbb{Z}_2$-graded category. Section 7 discusses a dg-algebra which provides an off-shell model for the algebra of “disk observables” of the B-type Landau-Ginzburg theory. Sections 8 and 9 discuss cochain level boundary-bulk and bulk-boundary maps between $\text{DF}_c(X,W)$ and $\text{PV}_c(X)$, showing that they satisfy a strict adjointness condition with respect to the cochain level bulk
and boundary traces and that the cohomological maps which they induce between HDF\(_c(X,W)\) and HPV\(_c(X,W)\) depend on metric data only through the Kähler class of \(G\). When the critical set of \(W\) is compact, these induce cohomological maps between HDF\((X,W)\) and HPV\((X,W)\) which satisfy the adjointness condition of a TFT datum. Section 10 gives the proof of Theorem 1.1. Appendix A gives various expressions in local coordinates and establishes equivalence of our model for the off-shell bulk algebra with another description found in the literature. Appendix B discusses the dg-category DF\(_\infty(X,W)\) of special Dolbeault factorizations and establishes its equivalence with the dg-category DF\((X,W)\) of holomorphic factorizations used in the present paper.

1.1. Notations and conventions.

I. The notation \(A \overset{\text{def}}{=} B\) means that \(A\) is defined to equal \(B\). The notation \(A := B\) means that \(A\) is a simplified notation for \(B\), usually obtained by omitting indices which indicate the dependence of \(B\) on other quantities, when the latter is clear from the context.

II. Numbers. The imaginary unit is denoted by \(i\) and the field of integers modulo two is denoted by \(\mathbb{Z}_2 \overset{\text{def}}{=} \mathbb{Z}/2\mathbb{Z}\). The mod 2 reduction of any integer \(n \in \mathbb{Z}\) is denoted by \(\hat{n} \in \mathbb{Z}_2\).

III. Rings and categories. Let Vect\(_\mathbb{C}\) denote the category of vector spaces over \(\mathbb{C}\) and vect\(_\mathbb{C}\) denote the full subcategory of finite-dimensional \(\mathbb{C}\)-vector spaces. For any ring \(R\) (not necessarily commutative) and any element \(r \in R\), we denote by \(\hat{r} : R \rightarrow R\) the operator of left multiplication with \(r\):

\[
\hat{r}(x) \overset{\text{def}}{=} rx, \quad \forall x \in R.
\]

IV. Graded categories. Let \(\Lambda\) be an Abelian group and \(R\) be a commutative ring. By a \((\Lambda\text{-graded}) R\text{-linear} dg\text{-category}\) we mean a \((\Lambda\text{-graded}) linear dg\text{-category}\) defined over \(R\). Given a \(\Lambda\text{-graded} R\text{-linear} category \(\mathcal{A}\), let \(\mathcal{A}^0\) denote the \(R\text{-linear} category\) obtained from \(\mathcal{A}\) by keeping the same objects and keeping only those morphisms of \(\mathcal{A}\) which have degree zero. Given a \(\Lambda\text{-graded} R\text{-linear} dg\text{-category} \(\mathcal{C}\), let \(H(\mathcal{C})\) denote its total cohomology category, defined as the \(\Lambda\text{-graded} R\text{-linear} category\) having the same objects as \(\mathcal{C}\) and morphism spaces given by the total cohomology \(R\text{-modules}\):

\[
\text{Hom}_{H(\mathcal{C})}(a,b) \overset{\text{def}}{=} H(\text{Hom}_\mathcal{C}(a,b),d_{a,b}) = \bigoplus_{\lambda \in \Lambda} H^\lambda(\text{Hom}_\mathcal{C}(a,b),d_{a,b}),
\]

where \(d_{a,b}\) is the differential on the \(R\text{-module} \text{Hom}_\mathcal{C}(a,b)\). In this case, we use the notation \(H^0(\mathcal{C}) = H(\mathcal{C})^0\).

V. Complex manifolds. All manifolds considered are smooth, paracompact, connected and of non-zero dimension and all vector bundles considered are smooth. For a complex manifold \(X\), we use the following notations:

1. Given a global section \(s\) of any vector bundle \(S\) defined on \(X\), an open subset \(U \subset X\) and a section \(\sigma\) of \(S|_U\), the notation \(s =_U \sigma\) means \(s|_U = \sigma\).
2. Given any holomorphic vector bundle \(E\) defined on \(X\), let:

\[
\Omega(X,E) \overset{\text{def}}{=} \bigoplus_{i,j=0}^d \Omega^{i,j}(X,E)
\]
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Let $\mathcal{A}^j(X, E) \overset{\text{def}}{=} \Omega^{0,j}(X, E)$ and:

$$\mathcal{A}(X, E) \overset{\text{def}}{=} \bigoplus_{j=0}^d \mathcal{A}^j(X, E).$$

Let $\mathcal{D}_E : \Omega(X, E) \to \Omega(X, E)$ denote the Dolbeault operator of $E$, which satisfies $\mathcal{D}_E^2 = 0$ as well as $\mathcal{D}_E(\Omega^{i,j}(X, E)) \subset \Omega^{i,j+1}(X, E)$. Let $\partial$ and $\mathcal{D}$ denote the Dolbeault operators of $X$. We have $\mathcal{D} = \mathcal{D}_\mathcal{O}_X$ and $\mathcal{A}(X) := \mathcal{A}(X, \mathcal{O}_X) = \bigoplus_{j=0}^d \mathcal{A}^j(X)$, where $\mathcal{A}^j(X) := \mathcal{A}^j(X, \mathcal{O}_X) = \Omega^{0,j}(X) := \Omega^{0,j}(X, \mathcal{O}_X)$.

3. Let $TX$ and $T^\ast X$ denote the holomorphic and anti-holomorphic tangent bundles of $X$. Let $T^\ast X := (TX)^\vee$, $T X := (\bar{T} X)^\vee$ denote the holomorphic and anti-holomorphic cotangent bundles of $X$. Let $T_X X$ denote the tangent bundle of the underlying real manifold and $T X$ denote the complexification of $T_X X$. We have $T X = T X \oplus \bar{T} X$. The map $\text{Re} : T X \to T_X X$ which takes the real part gives a canonical isomorphism of real vector bundles between $T X$ and $T_X X$ [5].

4. Let $\mathcal{V}^\infty_b(X)$ denote the exact category of complex vector bundles defined on $X$. Let $\mathcal{C}^\infty_\mathcal{O}$ denote the sheaf of smooth complex-valued functions defined on open subsets of $X$ (notice that this sheaf is not coherent when $X$ has positive dimension) and $\mathcal{S}_\infty(X)$ denote the Abelian category of sheaves of $\mathcal{C}^\infty_\mathcal{O}$-modules. Let $\mathcal{V}^\infty_b(X)$ denote the full subcategory of $\mathcal{S}_\infty(X)$ whose objects are the locally-free sheaves of finite rank. Let $\Gamma_b : \mathcal{V}^\infty_b(X) \to \mathcal{S}_\infty(X)$ be the functor which sends a complex vector bundle $S$ to the corresponding locally-free sheaf of $\mathcal{C}^\infty_\mathcal{O}$-modules $S := \Gamma_b(S)$ consisting of locally-defined smooth sections of $S$ and sends a morphism of vector bundles into the corresponding morphism of sheaves. Then $\Gamma_b$ embeds $\mathcal{V}^\infty_b(X)$ as a non-full subcategory of $\mathcal{V}^\infty_b(X)$ having the same objects as the latter. This allows us to view the objects of $\mathcal{V}^\infty_b(X)$ as complex vector bundles defined on $X$. With this identification, the morphisms of $\mathcal{V}^\infty_b(X)$ are those morphisms of $\mathcal{V}^\infty_b(X)$ whose kernel and cokernel are locally-free sheaves. Let $\mathcal{C}^\infty_b(X) := \mathcal{C}^\infty_\mathcal{O}(X)$ denote the commutative $\mathbb{C}$-algebra of complex-valued smooth functions defined on $X$ and $\Gamma_b(X, S) := \Gamma_b(S)(X) = S(X)$ denote the $\mathcal{C}^\infty_b(X)$-module of globally-defined smooth sections of a complex vector bundle $S$. For any two complex vector bundles $S_1$ and $S_2$ defined on $X$, we have:

$$\text{Hom}_{\mathcal{V}^\infty_b(X)}(S_1, S_2) = \Gamma_b(X, \text{Hom}_{\mathcal{C}^\infty_\mathcal{O}}(S_1, S_2)) = \text{Hom}_{\mathcal{C}^\infty_\mathcal{O}}(S_1, S_2)(X),$$

where $\text{Hom}_{\mathcal{C}^\infty_\mathcal{O}}(S_1, S_2) \overset{\text{def}}{=} S_1^\vee \otimes S_2$ denotes the vector bundle of morphisms from $S_1$ to $S_2$ and $\text{Hom}_{\mathcal{C}^\infty_\mathcal{O}}(S_1, S_2) \overset{\text{def}}{=} S_1^\vee \otimes_{\mathcal{C}^\infty_\mathcal{O}} S_2$ is the inner Hom sheaf of the corresponding locally-free sheaves of $\mathcal{C}^\infty_\mathcal{O}$-modules $S_i := \Gamma_b(S_i)$.

5. Let $\mathcal{V}^b(X)$ denote the exact category of holomorphic vector bundles defined on $X$. Let $\mathcal{O}_X$ denote the sheaf of holomorphic complex-valued functions defined on open subsets of $X$ (which is coherent by Oka’s coherence theorem) and $\mathcal{C}^\infty_b(X)$ denote the Abelian category of coherent sheaves of $\mathcal{O}_X$-modules. Let $\mathcal{V}^b(X)$ denote the full subcategory of $\mathcal{C}^\infty_b(X)$ whose objects are the locally-free sheaves of finite rank. Let $\Gamma : \mathcal{V}^b(X) \to \mathcal{C}^\infty_b(X)$ be the functor which sends a holomorphic vector bundle $E$ to the corresponding locally-free sheaf of $\mathcal{O}_X$-modules $\mathcal{E} := \Gamma_b(E)$ consisting of locally-defined holomorphic sections of $E$ and sends a morphism of holomorphic vector bundles to the corresponding morphism of sheaves. Then $\Gamma$ identifies $\mathcal{V}^b(X)$ with a non-full subcategory of $\mathcal{V}^b(X)$. 
This allows us to view the objects of $VB(X)$ as holomorphic vector bundles defined on $X$. With this identification, the morphisms of $Vb(X)$ are those morphisms of $VB(X)$ whose kernel and cokernel are locally-free sheaves. Let $O(X) := \mathcal{O}_X(X)$ denote the commutative $\mathbb{C}$-algebra of complex-valued holomorphic functions defined on $X$. For any holomorphic vector bundle $E$ defined on $X$, let $\Gamma(X, E) := \Gamma(E)(X) = \mathcal{E}(X)$ denote the $O(X)$-module of globally-defined holomorphic sections of $E$ (which coincides with the zeroth sheaf cohomology $H^0(\mathcal{E})$ of the corresponding locally-free sheaf $\mathcal{E} := \Gamma(E)$). For any two holomorphic vector bundles $E_1$ and $E_2$ defined on $X$, we have:

$$\text{Hom}_{VB(X)}(E_1, E_2) = \Gamma(X, Hom(E_1, E_2)) = Hom(\mathcal{E}_1, \mathcal{E}_2)(X),$$

where $\text{Hom}(E_1, E_2) = E_1^\vee \otimes E_2$ denotes the holomorphic vector bundle of morphisms from $E_1$ to $E_2$ and $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2) = \mathcal{E}_1^\vee \otimes_{\mathcal{O}_X} \mathcal{E}_2$ denotes the sheaf inner Hom of the corresponding locally-free sheaves of $\mathcal{O}_X$-modules $\mathcal{E}_i := \Gamma(E_i)$.

6. As a general point of terminology inspired by the physics literature, “off-shell” refers to an object defined at cochain level while “on-shell” refers to an object defined at cohomology level.

**Remark 1.1.** Unlike the categories $Vb_\infty(X)$ and $Vb(X)$ (which are only $\mathbb{C}$-linear), the categories $VB_\infty(X)$ and $VB(X)$ are respectively $C^\infty(X)$-linear and $O(X)$-linear.

### 2. Algebraic description of quantum two-dimensional oriented open-closed TFTs

A (non-anomalous) quantum two-dimensional oriented open-closed topological field theory (TFT) can be defined axiomatically [3, 6, 7] as a symmetric monoidal functor from a certain symmetric monoidal category $\text{Cob}_2^{\text{ext}}$ of labeled 2-dimensional oriented cobordisms with corners to the symmetric monoidal category $\text{vect}_C$ of finite-dimensional super-vector spaces defined over $\mathbb{C}$. The objects of the category $\text{Cob}_2^{\text{ext}}$ are finite disjoint unions of circles and segments while the morphisms are oriented cobordisms with corners between such, carrying appropriate labels on boundary components. By definition, the *closed sector* of such a theory is obtained by restricting the monoidal functor to the subcategory of $\text{Cob}_2^{\text{ext}}$ whose objects are disjoint unions of circles and whose morphisms are ordinary cobordisms (without corners). It was shown in [3] that such a functor can be described equivalently by an algebraic structure which we shall call a *TFT datum*. We start by describing a simpler algebraic structure, which forms part of any such datum.

#### 2.1. Pre-TFT data

**Definition 2.1** A pre-TFT datum is an ordered triple $(\mathcal{H}, \mathcal{T}, e)$ consisting of:

A. A finite-dimensional unital and supercommutative superalgebra $\mathcal{H}$ defined over $\mathbb{C}$ (called the *bulk algebra*), whose unit we denote by $1_\mathcal{H}$

B. A $\text{Hom}$-finite $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category $\mathcal{T}$ (called the category of topological D-branes), whose composition of morphisms we denote by $\circ$ and whose units we denote by $1_a \in \text{Hom}_\mathcal{T}(a, a)$ for all $a \in \text{Ob}\mathcal{T}$

C. A family $e = (e_a)_{a \in \text{Ob}\mathcal{T}}$ consisting of even $\mathbb{C}$-linear maps $e_a : \mathcal{H} \to \text{Hom}_\mathcal{T}(a, a)$ defined for each object $a$ of $\mathcal{T}$ (the map $e_a$ is called the *bulk-boundary map* of $a$)

such that the following conditions are satisfied:
1. For any object $a \in \text{Ob} \mathcal{T}$, the map $e_a$ is a unital morphism of $\mathbb{C}$-superalgebras from $\mathcal{H}$ to the endomorphism algebra $(\text{End}_\mathcal{T}(a), \circ)$, where $\text{End}_\mathcal{T}(a) \overset{\text{def.}}{=} \text{Hom}_\mathcal{T}(a, a)$.

2. For any two objects $a, b \in \text{Ob} \mathcal{T}$ and for any $\mathbb{Z}_2$-homogeneous elements $h \in \mathcal{H}$ and $t \in \text{Hom}_\mathcal{T}(a, b)$, we have:

$$e_b(h) \circ t = (-1)^{\deg h \cdot \deg t} t \circ e_a(h) .$$

Given a pre-TFT datum $(\mathcal{H}, \mathcal{T}, e)$, the objects of the category $\mathcal{T}$ are called topological D-branes. Elements $h \in \mathcal{H}$ are called on-shell bulk states. For any topological D-branes $a, b \in \text{Ob} \mathcal{T}$, elements $t \in \text{Hom}_\mathcal{T}(a, b)$ are called on-shell boundary states from $a$ to $b$.

**Definition 2.2** Let $\mathcal{T}$ be a $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category. The total endomorphism algebra of $\mathcal{T}$ is the associative $\mathbb{C}$-superalgebra whose underlying $\mathbb{Z}_2$-graded vector space is defined through:

$$\text{End}(\mathcal{T}) \overset{\text{def.}}{=} \bigoplus_{a, b \in \text{Ob} \mathcal{T}} \text{Hom}_\mathcal{T}(a, b)$$

and whose multiplication is given by the obvious $\mathbb{C}$-bilinear extension of the composition of morphisms of $\mathcal{T}$.

The total endomorphism algebra $\text{End}(\mathcal{T})$ is unital with unit $1_\mathcal{T} \overset{\text{def.}}{=} \oplus_{a \in \text{Ob} \mathcal{T}} 1_a$. Notice that $\text{End}(\mathcal{T})$ is infinite-dimensional if $\mathcal{T}$ has an infinity of objects, even when $\mathcal{T}$ is Hom-finite.

**Definition 2.3** Let $(\mathcal{H}, \mathcal{T}, e)$ be a triplet satisfying conditions A., B. and C. of Definition 2.1. The total bulk-boundary map of $(\mathcal{H}, \mathcal{T}, e)$ is the even $\mathbb{C}$-linear map $e : \mathcal{H} \to \text{End}(\mathcal{T})$ defined through:

$$e(h) \overset{\text{def.}}{=} \bigoplus_{a \in \text{Ob} \mathcal{T}} e_a(h) , \ \forall h \in \mathcal{H} .$$

The following statement is obvious:

**Proposition 2.4** Let $(\mathcal{H}, \mathcal{T}, e)$ be a triplet satisfying conditions A., B. and C. above. Then $(\mathcal{H}, \mathcal{T}, e)$ is a pre-TFT datum iff the total bulk-boundary map $e$ of $(\mathcal{H}, \mathcal{T}, e)$ is a unital morphism of $\mathbb{Z}_2$-graded algebras whose image lies in the supercenter of the total endomorphism algebra $(\text{End}(\mathcal{T}), \circ)$.

2.2. Calabi-Yau and pre-Calabi-Yau supercategories.

**Definition 2.5** A Calabi-Yau supercategory of parity $\mu \in \mathbb{Z}_2$ is a pair $(\mathcal{T}, \text{tr})$, where:

A. $\mathcal{T}$ is a $\mathbb{Z}_2$-graded and $\mathbb{C}$-linear Hom-finite category

B. $\text{tr} = (\text{tr}_a)_{a \in \text{Ob} \mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\text{tr}_a : \text{Hom}_\mathcal{T}(a, a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree $\mu$

such that the following conditions are satisfied:

1. For any two objects $a, b \in \text{Ob} \mathcal{T}$, the $\mathbb{C}$-bilinear pairing $(\cdot, \cdot)_{a, b} : \text{Hom}_\mathcal{T}(a, b) \times \text{Hom}_\mathcal{T}(b, a) \to \mathbb{C}$ defined through:

$$\langle t_1, t_2 \rangle_{a, b} \overset{\text{def.}}{=} \text{tr}_b(t_1 \circ t_2) , \ \forall t_1 \in \text{Hom}_\mathcal{T}(a, b) , \ \forall t_2 \in \text{Hom}_\mathcal{T}(b, a)$$

is non-degenerate.
2. For any two objects \(a, b \in \text{Ob} \mathcal{T}\) and any \(\mathbb{Z}_2\)-homogeneous elements \(t_1 \in \text{Hom}_\mathcal{T}(a, b)\) and \(t_2 \in \text{Hom}_\mathcal{T}(b, a)\), we have:

\[
\langle t_1, t_2 \rangle_{a, b} = (-1)^{\deg t_1 \deg t_2} \langle t_2, t_1 \rangle_{b, a} \quad .
\]

(2.1)

If only condition 2. above is satisfied, we say that \((\mathcal{T}, \text{tr})\) is a pre-Calabi-Yau supercategory of parity \(\mu\).

The condition that \(\text{tr}_a\) has degree \(\mu\) amounts to the requirement:

\[
\text{tr}_a(h) = 0 \quad \text{unless} \quad h \in \text{Hom}^\mu_{\mathcal{T}}(a, a) ,
\]

where \(\text{Hom}^\mu_{\mathcal{T}}(a, a)\) denotes the homogeneous component of \(\text{Hom}_\mathcal{T}(a, a)\) lying in \(\mathbb{Z}_2\)-degree \(\mu\).

Considering the total endomorphism algebra \((\text{End}(\mathcal{T}), \circ)\) as above, \(\text{tr}\) induces a \(\mathbb{C}\)-linear map of degree \(\mu\):

\[
\text{tr} : \text{End}(\mathcal{T}) \to \mathbb{C} ,
\]

defined as follows for \(t \in \text{Hom}_\mathcal{T}(a, b)\):

\[
\text{tr}(t) \overset{\text{def}}{=} \begin{cases} \text{tr}(t) &, \text{if } a = b \\ 0 &, \text{if } a \neq b \end{cases} .
\]

This map satisfies:

\[
\text{tr}(t_1 \circ t_2) = (-1)^{\deg t_1 \deg t_2} \text{tr}(t_2 \circ t_1) ,
\]

for any \(\mathbb{Z}_2\)-homogeneous elements \(t_1, t_2 \in \text{End}(\mathcal{T})\).

2.3. TFT data. As shown in [3], quantum two-dimensional topological field theories of the type considered above can be identified with the following algebraic structure:

**Definition 2.6** A TFT datum of parity \(\mu \in \mathbb{Z}_2\) is a system \((\mathcal{H}, \mathcal{T}, e, \text{Tr}, \text{tr})\), where:

A. \((\mathcal{H}, \mathcal{T}, e)\) is a pre-TFT datum

B. \(\text{Tr} : \mathcal{H} \to \mathbb{C}\) is an even \(\mathbb{C}\)-linear map (called the bulk trace)

C. \(\text{tr} = (\text{tr}_a)_{a \in \text{Ob} \mathcal{T}}\) is a family of \(\mathbb{C}\)-linear maps \(\text{tr}_a : \text{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}\) of \(\mathbb{Z}_2\)-degree \(\mu\) (called the boundary traces)

such that the following conditions are satisfied:

1. \((\mathcal{H}, \text{Tr})\) is a supercommutative Frobenius superalgebra. This means that the pairing induced by \(\text{Tr}\) on \(\mathcal{H}\) is non-degenerate, in the sense that the condition \(\text{Tr}(hh') = 0\) for all \(h' \in \mathcal{H}\) implies \(h = 0\).

2. \((\mathcal{T}, \text{tr})\) is a Calabi-Yau supercategory of parity \(\mu\).

3. The following relation, called the topological Cardy constraint, is satisfied for all \(a, b \in \text{Ob} \mathcal{T}\):

\[
\text{Tr}(f_a(t_1)f_b(t_2)) = \text{str}(\Phi_{ab}(t_1, t_2)) , \quad \forall t_1 \in \text{Hom}_{\mathcal{T}}(a, a) , \forall t_2 \in \text{Hom}_{\mathcal{T}}(b, b) ,
\]

where \(\text{str}\) denotes the supertrace on the finite-dimensional \(\mathbb{Z}_2\)-graded vector space \(\text{End}_\mathbb{C}(\text{Hom}_{\mathcal{T}}(a, b))\) and:
(a) The $\mathbb{C}$-linear map $f_a : \text{Hom}_T(a,a) \to \mathcal{H}$ of $\mathbb{Z}_2$-degree $\mu$ (which is called the boundary-bulk map of $a$) is defined as the adjoint of the bulk-boundary map $e_a : \mathcal{H} \to \text{Hom}_T(a,a)$ with respect to the non-degenerate traces $\text{Tr}$ and $\text{tr}_a$, being determined by the relation:

$$\text{Tr}(hf_a(t)) = \text{tr}_a(e_a(h) \circ t), \quad \forall h \in \mathcal{H}, \forall t \in \text{Hom}_T(a,a).$$

(b) For any $a,b \in \text{Ob}T$ and any $t_1 \in \text{Hom}_T(a,a)$ and $t_2 \in \text{Hom}_T(b,b)$, the $\mathbb{C}$-linear map $\Phi_{ab}(t_1,t_2) : \text{Hom}_T(a,b) \to \text{Hom}_T(a,b)$ is defined through:

$$\Phi_{ab}(t_1,t_2)(t) \overset{\text{def}}{=} t_2 \circ t \circ t_1, \quad \forall t \in \text{Hom}_T(a,b).$$

Remark 2.1. The topological Cardy constraint in the form given above was first derived in [3, Subsection 4.7].

2.4. B-type topological Landau-Ginzburg theories. A quantum open-closed B-type topological Landau-Ginzburg theory is a particular quantum two-dimensional field theory defined on compact oriented surfaces with corners, which is conjecturally associated to pairs $(X,W)$, where $X$ is a non-compact Calabi-Yau manifold and $W : X \to \mathbb{C}$ is a non-constant complex-valued holomorphic function defined on $X$. At the classical (i.e. non-quantum) level, such theories admit a Lagrangian description whose general construction for compact oriented surfaces with boundary was given in [1,2]. That construction extends immediately to compact oriented surfaces with corners, as outlined in Appendix B. As explained in [2], a (non-rigorous) procedure of “partial path integral localization” allows one to extract an explicit family of smooth differential models for the off-shell state spaces of this theory, whose cohomology conjecturally recovers the TFT datum of the open-closed TFT defined by the on-shell unintegrated correlators. In the next sections, we give an equivalent mathematical description of that model, referring the reader to Appendix B for the precise relation to the superconnection language used in [2]. Let us first define the data parameterizing such theories. By a Kählerian manifold we mean a complex manifold which admits at least one Kähler metric.

Definition 2.7 A Landau-Ginzburg (LG) pair of dimension $d$ is a pair $(X,W)$, where:

1. $X$ is a non-compact Kählerian manifold of complex dimension $d$ which is Calabi-Yau in the sense that the canonical line bundle $K_X \overset{\text{def}}{=} \wedge^d T^*X$ is holomorphically trivial.
2. $W : X \to \mathbb{C}$ is a non-constant complex-valued holomorphic function defined on $X$.

The signature $\mu(X,W)$ of a Landau-Ginzburg pair $(X,W)$ is defined as the mod 2 reduction of the complex dimension of $X$:

$$\mu(X,W) \overset{\text{def}}{=} \hat{d} \in \mathbb{Z}_2.$$

Definition 2.8 The critical set of $W$ is the set:

$$Z_W \overset{\text{def}}{=} \{ p \in X | (\partial W)(p) = 0 \}$$

of critical points of $W$.

\footnote{The case of oriented surfaces without boundary was studied much earlier in [8,9].}
3. The off-shell bulk algebra

Let \((X, W)\) be a Landau-Ginzburg pair with \(\dim \mathbb{C} X = d\). In this subsection, we discuss a dg-algebra \((\mathcal{PV}(X, W), \delta_W)\) of polyvector-valued forms. When the critical set of \(W\) is compact, the path integral argument of [2] shows that the total cohomology algebra of this dg-algebra can be interpreted as the bulk algebra of the B-type open-closed topological Landau-Ginzburg theory defined by \((X, W)\).

3.1. The bicomplex of smooth differential forms. Let \(\mathcal{T} X = TX \oplus \overline{TX}\) denote the complexified tangent bundle of the real manifold underlying \(X\) and \(\mathcal{T}^* X \overset{\text{def}}{=} (\mathcal{T} X)^\vee\) denote the complexified cotangent bundle. Let \(\wedge \mathcal{T}^* X \overset{\text{def}}{=} \bigoplus_{k=0}^d \wedge^k \mathcal{T}^* X\) denote the complexified exterior bundle of \(X\). Let \(\Lambda^\ast \mathcal{T}^* X \overset{\text{def}}{=} \bigoplus_{k=0}^d \Lambda^k \mathcal{T}^* X\) denote the holomorphic and antiholomorphic exterior bundles. Let:

\[
\Omega(X) \overset{\text{def}}{=} \Gamma_\infty(X, \wedge \mathcal{T}^* X)
\]
denote the Grassmann algebra of \(X\), i.e. the \(C^\infty(X)\)-algebra of smooth inhomogeneous differential forms defined on \(X\), with multiplication given by the wedge product. Let:

\[
\Omega^{i,j}(X) \overset{\text{def}}{=} \Gamma_\infty(X, \wedge^i \mathcal{T}^* X \otimes \wedge^j \overline{\mathcal{T}}^* X)
\]
denote the \(C^\infty(X)\)-submodule of \(\Omega(X)\) consisting of smooth differential forms of type \((i,j)\). Set \(\Omega^{i,j}(X) = 0\) for \(i \not\in \{0, \ldots, d\}\) or \(j \not\in \{0, \ldots, d\}\). We view \(\Omega(X)\) as a unital bigraded \(C^\infty(X)\)-algebra using the decomposition:

\[
\Omega(X) = \bigoplus_{i,j=0}^d \Omega^{i,j}(X) .
\]
The rank grading of \(\Omega(X)\) is the total \(\mathbb{Z}\)-grading induced by this bigrading, which corresponds to the decomposition:

\[
\Omega(X) = \bigoplus_{k=0}^{2d} \Omega^k(X) \quad \text{where} \quad \Omega^k(X) \overset{\text{def}}{=} \bigoplus_{i+j=k} \Omega^{i,j}(X) .
\]
Let \(d\) be the de Rham differential and \(\partial\) and \(\overline{\partial}\) be the Dolbeault differentials of \(X\). We have:

\[
d = \partial + \overline{\partial}
\]
and:

\[
d^2 = \partial^2 = \overline{\partial}^2 = \partial \overline{\partial} + \overline{\partial} \partial = 0 .
\]
In particular, \((\Omega(X), \partial, \overline{\partial})\) is a bicomplex of \(\mathbb{C}\)-vector spaces whose total differential equals \(d\). When \(\Omega(X)\) is endowed with the rank grading, the triplet \((\Omega(X), \wedge, d)\) is a unital \(\mathbb{Z}\)-graded supercommutative dg-algebra over \(\mathbb{C}\). Notice that the differential \(\overline{\partial}\) is \(O(X)\)-linear.

3.2. The dg-algebra of \((0,\ast)\)-forms. Let:

\[
\mathcal{A}^\ast(X) \overset{\text{def}}{=} \Omega^{0,j}(X) = \Gamma_\infty(X, \wedge^j \overline{\mathcal{T}}^* X)
\]
be the $C^\infty(X)$-module of smooth differential forms of type $(0, j)$ defined on $X$. Consider the $\mathbb{Z}$-graded $C^\infty(X)$-module:

$$\mathcal{A}(X) \overset{\text{def.}}{=} \Gamma_\infty(X, \wedge^j T X) = \bigoplus_{j=0}^d \mathcal{A}^j(X)$$

consisting of all differential forms which have degree zero with respect to the first grading of $\Omega(X)$. Then $(\mathcal{A}(X), \wedge, \bar{\partial})$ is a $\mathbb{Z}$-graded $O(X)$-linear dg-algebra.

3.3. The algebra of polyvector fields. Let $\wedge TX \overset{\text{def.}}{=} \bigoplus_{i=0}^d \wedge^i TX$ be the bundle of holomorphic polyvectors. Consider the $C^\infty(X)$-module:

$$\mathcal{B}^i(X) \overset{\text{def.}}{=} \Gamma_\infty(X, \wedge^i TX)$$

and the $\mathbb{Z}$-graded $C^\infty(X)$-module:

$$\mathcal{B}(X) \overset{\text{def.}}{=} \Gamma_\infty(X, \wedge TX) = \bigoplus_{i=0}^d \mathcal{B}^i(X)$$

Then $\mathcal{B}(X)$ becomes a unital associative and supercommutative $\mathbb{Z}$-graded $C^\infty(X)$-algebra when endowed with the multiplication induced by the wedge product of polyvector fields.

3.4. The twisted Dolbeault algebra of polyvector-valued forms. Consider the $C^\infty(X)$-module $^3$:

$$\text{PV}(X) \overset{\text{def.}}{=} \mathcal{A}(X, \wedge TX) \simeq \mathcal{A}(X) \otimes_{C^\infty(X)} \mathcal{B}(X)$$

endowed with the unital and associative $C^\infty(X)$-linear multiplication determined uniquely by the condition:

$$(\rho_1 \otimes P_1)(\rho_2 \otimes P_2) = (-1)^{|ij|}(\rho_1 \wedge \rho_2) \otimes (P_1 \wedge P_2)$$

for all $\rho_1 \in \mathcal{A}(X), \rho_2 \in \mathcal{A}^j(X), P_1 \in \mathcal{B}^i(X), P_2 \in \mathcal{B}(X)$. For any $i = -d, \ldots, 0$ and $j = 0, \ldots, d$, let:

$$\text{PV}^{i,j}(X) \overset{\text{def.}}{=} \mathcal{A}^i(X, \wedge^{|j|} TX) \simeq \mathcal{A}^i(X) \otimes_{C^\infty(X)} \mathcal{B}^{|j|}(X)$$

denote the space of smooth $(0, j)$-forms valued in the holomorphic vector bundle $\wedge^{|j|} TX$. Set $\text{PV}^{i,j}(X) = 0$ for $i \not\in \{-d, \ldots, 0\}$ or $j \not\in \{0, \ldots, d\}$. With these conventions, the decomposition:

$$\text{PV}(X) \overset{\text{def.}}{=} \bigoplus_{i=-d}^0 \bigoplus_{j=0}^d \text{PV}^{i,j}(X)$$

makes $\text{PV}(X)$ into a unital associative $\mathbb{Z} \times \mathbb{Z}$-graded $C^\infty(X)$-algebra with multiplication given by the wedge product, whose grading is concentrated in bidegrees $(i, j)$ satisfying $i \in \{-d, \ldots, 0\}$ and $j \in \{0, \ldots, d\}$. Notice that $\text{PV}(X)$ is a unital supercommutative $C^\infty(X)$-algebra when endowed with the total grading, which we denote by $\text{deg}$.

3 In some references, elements of $\text{PV}(X)$ are called “polyvector fields”, though such language does not match the traditional terminology used in differential geometry. Strictly speaking, a smooth polyvector field of type $(\ast, 0)$ is a smooth section of the bundle $\wedge TX$ and not a differential form valued in that bundle.
Definition 3.1 The canonical grading of $PV(X)$ is the total $\mathbb{Z}$-grading of the bigrading introduced above, i.e. the $\mathbb{Z}$-grading whose homogeneous components are defined through:

$$PV^k(X) \overset{\text{def.}}{=} \bigoplus_{i+j=k} PV^{i,j}(X) \quad (k \in \mathbb{Z}) .$$

The reduced grading of $PV(X)$ is the $\mathbb{Z}_2$-grading defined as the mod 2 reduction of the canonical grading:

$$PV^0(X) \overset{\text{def.}}{=} \bigoplus_{k=\text{ev}} PV^k(X),$$

$$PV^1(X) \overset{\text{def.}}{=} \bigoplus_{k=\text{odd}} PV^k(X) .$$

Notice that $PV^k(X) = 0$ unless $k \in \{-d, \ldots, d\}$. We have:

$$PV^{-d}(X) = PV^{-d,0}(X) = B^d(X), \quad PV^d(X) = PV^{0,d}(X) = A^d(X) .$$

Let $\overline{\partial} := \overline{\partial}_{\wedge TX} : PV(X) \to PV(X)$ denote the Dolbeault operator of the holomorphic vector bundle $\wedge TX$. This is an $O(X)$-linear odd derivation of $PV(X)$ which preserves the first $\mathbb{Z}$-grading and has degree +1 with respect to the second $\mathbb{Z}$-grading:

$$\overline{\partial}(PV^{i,j}(X)) \subset PV^{i,j+1}(X) .$$

Let $\iota_W : PV(X) \to PV(X)$ be the unique $C^\infty(X)$-linear map which satisfies the following conditions:

- $\iota_W$ has bidegree $(1,0)$:

$$\iota_W(PV^{i,j}(X)) \subset PV^{i+1,j}(X) . \quad (3.1)$$

- $\iota_W$ is an odd derivation of $PV(X)$ with respect to the canonical grading:

$$\iota_W(\omega \eta) = (\iota_W \omega) \eta + (-1)^k \omega (\iota_W \eta) , \quad \forall \omega \in PV^k(X) , \forall \eta \in PV(X) . \quad (3.2)$$

- $\iota_W$ coincides with the contraction with the holomorphic 1-form $-i\partial W \in \Gamma(X, T^*X) \subset \Omega^{1,0}(X)$ when restricted to the submodule $PV^{-1,0}(X) = B^1(X) = \Gamma_\infty(X, TX)$.

Both $\overline{\partial}$ and $\iota_W$ have degree +1 with respect to the canonical grading and they are odd $O(X)$-linear derivations of the algebra $PV(X)$, when the latter is endowed with the reduced grading. The derivation $\iota_W$ is also $C^\infty(X)$-linear.

Definition 3.2 Let $(X, W)$ be a Landau-Ginzburg pair. The twisted differential determined by $W$ on $PV(X)$ is the operator $\delta_W : PV(X) \to PV(X)$ defined through:

$$\delta_W \overset{\text{def.}}{=} \overline{\partial} + \iota_W .$$

The twisted differential is $O(X)$-linear. It has degree +1 with respect to the canonical grading and it is odd with respect to the reduced grading. We have:

$$\delta_W^2 = \overline{\partial}^2 = (\iota_W)^2 = \iota_W \circ \overline{\partial} + \overline{\partial} \circ \iota_W = 0 ,$$

which shows that $(PV(X), \overline{\partial}, \iota_W)$ is a bicomplex of $O(X)$-modules.
Definition 3.3 The twisted Dolbeault algebra of polyvector-valued forms of the Landau-Ginzburg pair \((X, W)\) is the supercommutative \(\mathbb{Z}\)-graded \(O(X)\)-linear dg-algebra \((\text{PV}(X), \delta_W)\), where \(\text{PV}(X)\) is endowed with the canonical \(\mathbb{Z}\)-grading. The cohomological twisted Dolbeault algebra of \((X, W)\) is the supercommutative \(\mathbb{Z}\)-graded \(O(X)\)-linear algebra defined through:

\[
\text{HPV}(X, W) \overset{\text{def}}{=} H(\text{PV}(X), \delta_W) .
\]

Proposition 3.4 Assume that the critical set \(Z_W\) is compact. Then \(\text{HPV}(X, W)\) is finite-dimensional over \(\mathbb{C}\).

Proof. Using Dolbeault resolutions shows that the \(\mathbb{Z}\)-graded \(O(X)\)-module \(\text{HPV}(X, W)\) is isomorphic with the hypercohomology of the sheaf Koszul complex (see [4]):

\[
0 \to \wedge^d T X \overset{\iota_W}{\to} \wedge^{d-1} T X \overset{\iota_W}{\to} \ldots \overset{\iota_W}{\to} O_X \to 0 .
\]

The latter is exact outside \(Z_W\), so its sheaf cohomology is concentrated on the critical set, which is compact. This implies that the hypercohomology is finite-dimensional. \(\square\)

When \(Z_W\) is compact, the path integral argument of [2] shows that the bulk algebra \(\mathcal{H}\) of the open-closed topological field theory defined by the un-integrated correlators of on-shell observables of the B-type Landau-Ginzburg theory of the pair \((X, W)\) can be identified with the \(\mathbb{Z}_2\)-graded \(\mathbb{C}\)-superalgebra obtained from \(\text{HPV}(X, W)\) by restricting scalars from \(O(X)\) to \(\mathbb{C}\) and reducing the canonical grading modulo 2.

3.5. The reduced contraction determined by a holomorphic volume form. Recall that the canonical line bundle \(K_X \overset{\text{def}}{=} \wedge^d T^* X\) is holomorphically trivial, so it admits nowhere-vanishing holomorphic sections (called holomorphic volume forms). Let \(\Omega \in \Gamma(X, K_X) \subset \Omega^{d,0}(X)\) be a holomorphic volume form.

Definition 3.5 The reduced contraction with \(\Omega\) is the \(C^\infty(X)\)-linear map \(\Omega_{\cdot, 0} : \text{PV}(X) \to \mathcal{A}(X)\) defined through:

\[
\Omega_{\cdot, 0}\omega = \begin{cases} 0 , & \text{if } \omega \notin \text{PV}^{-d,*}(X) \\ \Omega_{\cdot, \omega} , & \text{if } \omega \in \text{PV}^{-d,*}(X) \end{cases} .
\]

We have:

\[
\Omega_{\cdot, 0}(\text{PV}^{i,j}(X)) = \begin{cases} 0 , & \text{if } i \neq -d \\ \mathcal{A}^j(X) , & \text{if } i = -d \end{cases} .
\]

Since \(\Omega\) is nowhere-vanishing, the restricted map \(\Omega_{\cdot, 0} : \text{PV}^{-d,*}(X) = \mathcal{A}^*(X, \wedge^d T X) \to \mathcal{A}^*(X)\) is an isomorphism of \(C^\infty(X)\)-modules. Since \(\Omega\) is holomorphic while \(\iota_W\) decreases polyvector rank, the reduced contraction satisfies:

\[
\Omega_{\cdot, 0}(\delta_W \omega) = \Omega_{\cdot, 0}(\partial \omega) = (-1)^d \partial(\Omega_{\cdot, 0}\omega) , \quad \forall \omega \in \text{PV}(X) .
\] (3.3)

Thus \(\Omega_{\cdot, 0}\) is a map of complexes of \(O(X)\)-modules from \((\text{PV}(X), \delta_W)\) to \((\mathcal{A}(X), \partial)\) which reduces polyvector rank by \(d\) without changing the form rank. This map has \(\mathbb{Z}\)-degree \(+d\) when \(\text{PV}(X)\) is endowed with the canonical \(\mathbb{Z}\)-grading and \(\mathbb{Z}_2\)-degree \(\mu = d\) when \(\text{PV}(X)\) is endowed with the reduced grading.
Remark 3.1. Picking a holomorphic volume form \( \Omega \), the operator \( \iota_\Omega = \Omega \partial \) of contraction with \( \Omega \) gives an isomorphism of \( C^\infty(X) \)-modules:

\[
\iota_\Omega : \PV^{i,j}(X) \xrightarrow{\sim} \Omega^{d+i,j}(X) \, .
\]

This can be used to transport the holomorphic Dolbeault differential \( \partial \) of \( \Omega(X) \) to a differential \( \partial_\Omega : \PV(X) \rightarrow \PV(X) \) which satisfies:

\[
\partial_\Omega(\PV^{i,j}(X)) \subset \PV^{i+1,j}(X) \, .
\]

Consider the bracket \( \{\cdot, \cdot\}_\Omega : \PV(X) \times \PV(X) \rightarrow \PV(X) \) defined through:

\[
\{\omega, \eta\}_\Omega \overset{\text{def}}{=} \partial_\Omega(\omega \wedge \eta) - (\partial_\Omega \omega) \wedge \eta - (-1)^{\deg \omega} \omega \wedge \partial_\Omega \eta \, . \tag{3.4}
\]

Then \( (\PV(X), \{\cdot, \cdot\}_\Omega, \partial_\Omega) \) is a Batalin-Vilkovisky algebra. One can check (see Appendix A) that we have:

\[
\iota_W = -i(W, \cdot)_\Omega \, , \tag{3.5}
\]

so the operator \( \delta_W \) coincides with the operator denoted by \( \overline{\partial}_f \) in [10] provided that one takes \( f = -W \). Notice that \( \iota_W \) is independent of \( \Omega \), so the formulation in terms of \( \iota_W \) (which was already used in [2]) is more natural. In particular, the dg-algebra \( (\PV(X), \delta_W) \) and its cohomology \( \HPV(X, W) \) are independent of \( \Omega \).

3.6. Compact support version. Let:

\[
\PV_c(X) \subset \PV(X)
\]

denote the \( C^\infty(X) \)-submodule of \( \PV(X) \) consisting of compactly-supported polyvector-valued forms. Then \( \PV_c(X) \) is a subalgebra of the bidifferential algebra \( (\PV(X), \overline{\partial}, \iota_W) \). In particular \( \PV_c(X) \) when endowed with the canonical \( \mathbb{Z} \)-grading is a dg-subalgebra of the \( O(X) \)-linear dg-algebra \( (\PV(X), \delta_W) \).

Definition 3.6 The compactly supported twisted Dolbeault algebra of polyvector-valued forms is the \( \mathbb{Z} \)-graded \( O(X) \)-linear dg-algebra \( (\PV_c(X), \delta_W) \), where \( \PV_c(X) \) is endowed with the canonical \( \mathbb{Z} \)-grading.

Let \( \HPV_c(X, W) \overset{\text{def}}{=} \H(\PV_c(X), \delta_W) \) be the total cohomology algebra of the dg-algebra \( (\PV_c(X), \delta_W) \). Then \( \HPV_c(X, W) \) is a \( \mathbb{Z} \)-graded \( O(X) \)-linear non-unital associative and supercommutative algebra when endowed with the canonical \( \mathbb{Z} \)-grading. Let \( i_* : \HPV_c(X, W) \rightarrow \HPV(X, W) \) denote the morphism of \( \mathbb{Z} \)-graded \( O(X) \)-algebras induced on cohomology by the inclusion \( i : \PV_c(X) \rightarrow \PV(X) \).

Proposition 3.7 Assume that the critical set \( Z_W \) is compact. Then the inclusion \( i : \PV_c(X) \hookrightarrow \PV(X) \) is a homotopy equivalence (and hence a quasi-isomorphism) of differential \( \mathbb{Z} \)-graded \( O(X) \)-modules from \( (\PV_c(X), \delta_W) \) to \( (\PV(X), \delta_W) \).

Proof. This is an extension to the global case of [10, Lemma 2.3] (see also [11, Lemma 3.1]); we give the proof in detail for clarity and completeness. Let \( \delta := \delta_W = \overline{\partial} + \iota_W \) and \( \delta_c := \delta|_{\PV_c(X)} \). Let \( I := \id_{\PV(X)} \) and \( I_c := \id_{\PV_c(X)} \). We will construct maps:

\[
\pi : \PV(X) \rightarrow \PV_c(X) \text{ and } \mathcal{R} : \PV(X) \rightarrow \PV(X)
\]
such that \( \pi \) is \( O(X) \)-linear (in fact, \( C^\infty(X) \)-linear) and such that:

\[
I - i \circ \pi = [\delta, \mathcal{R}] \quad \text{and} \quad I_c - \pi \circ i = [\delta_c, \mathcal{R}_c] ,
\]

where \( \mathcal{R} \) preserves the subspace \( \text{PV}_c(X) \) and \( \mathcal{R}_c := \mathcal{R}|_{\text{PV}_c(X)} \). Let \( X_0 \equiv X \setminus Z_W \) and \( W_0 := W|_{X_0} \). To construct \( \mathcal{R} \), pick a Hermitian metric \( G \) on \( X \) and consider the following smooth section of the holomorphic tangent bundle of \( X_0 \):

\[
s \overset{\text{def}}{=} \frac{i}{||\partial W||_G^2} \text{grad}_G \bar{W} \in \Gamma_\infty(X_0, TX_0) = \text{PV}^{-1,0}(X_0) ,
\]

where \( || \cdot ||_G \) denotes the norm induced by \( G \) on \( \wedge T^*X \), \( \bar{W} : X \to \mathbb{C} \) denotes the complex conjugate of \( W \) and \( \text{grad}_G \bar{W} \in \Gamma_\infty(X, TX) \) denotes the gradient of \( \bar{W} \) taken with respect to \( G \) (see Subsection 5.4 for details). The section \( s \) satisfies (cf. equation (5.7)):

\[
l_W(s) = 1_{X_0} ,
\]

where \( 1_{X_0} \in C^\infty(X_0) \) is the unit function of the open subset \( X_0 \subset X \). For any element \( \eta \in \text{PV}(X_0) \), let \( \hat{\eta} : \text{PV}(X_0) \to \text{PV}(X_0) \) denote the operator of multiplication from the left with \( \eta \) in the algebra \( \text{PV}(X_0) \):

\[
\hat{\eta}(\omega) \overset{\text{def}}{=} \eta \omega , \quad \forall \omega \in \text{PV}(X_0) .
\]

Let \( I_0 \overset{\text{def}}{=} \text{id}_{\text{PV}(X_0)} \). Relation (3.7) implies:

\[
[\delta, \hat{s}] = I_0 + \bar{\mathcal{S}}s ,
\]

where \([\cdot, \cdot] \) is the graded commutator of operators acting in \( \text{PV}(X_0) \):

\[
[A, B] = A \circ B - (-1)^{\deg A \deg B} B \circ A , \quad \forall A, B : \text{PV}(X_0) \to \text{PV}(X_0) .
\]

Since \( \bar{\mathcal{S}}s \in \text{PV}^{-1,1}(X_0) \), the operator \( \bar{\mathcal{S}}s \) is nilpotent. Hence the operator \([\delta, \hat{s}] \) is invertible on \( \text{PV}(X_0) \) and we have:

\[
[\delta, \hat{s}]^{-1} = (I_0 + \bar{\mathcal{S}}s)^{-1} = \sum_{k=0}^{d} (-1)^k (\bar{\mathcal{S}}s)^k = \hat{S} ,
\]

where:

\[
S \overset{\text{def}}{=} (1 + \bar{\mathcal{S}}s)^{-1} = \sum_{k=0}^{d} (-1)^k (\bar{\mathcal{S}}s)^k \in \text{PV}^0(X_0) .
\]

This allows us to define the following operator on the space \( \text{PV}(X_0) \):

\[
\mathcal{R} \overset{\text{def}}{=} \hat{s} \circ [\delta, \hat{s}]^{-1} = \hat{s} \circ \hat{S} = \hat{R} ,
\]

where:

\[
R \overset{\text{def}}{=} sS = s(1 + \bar{\mathcal{S}}s)^{-1} = s \sum_{k=0}^{d} (-1)^k (\bar{\mathcal{S}}s)^k \in \text{PV}^{-1}(X_0) .
\]

Since \( \mathcal{R} \) is the operator of left multiplication with \( R \), it preserves the subspace \( \text{PV}_c(X_0) \). Applying the operator \([\delta, \cdot] \) to the relation:

\[
[\delta, \hat{s}] \circ \hat{S} = I_0
\]
\[ [\delta, \hat{S}] = 0 \]

since \([\delta, [\delta, \hat{s}]] = 0\) and \([\delta, \hat{s}]\) is invertible on \(PV(X_0)\). Thus:

\[ [\delta, R] = [\delta, \hat{s} \circ \hat{S}] = [\delta, \hat{s}] \circ \hat{S} = I_0 . \tag{3.8} \]

Since the critical set \(Z_W\) is compact and of codimension at least one (recall that \(W\) is not constant), there exists a relatively compact open neighborhood \(U\) of \(Z_W\). Let \(U_1\) be an open subset of \(X\) such that \(Z_W \subset U_1 \subset \overline{U_1} \subset U\) and \(\rho \in C^\infty_c(X)\) be a smooth compactly-supported function which equals 1 on \(U_1\) and vanishes outside \(U\). The desired operators \(R\) and \(\pi\) are defined as:

\[
R \overset{\text{def}}{=} (1 - \rho)\hat{R} = (1 - \rho)R, \quad \pi \overset{\text{def}}{=} \rho I + (\overline{\partial} \rho)\hat{R} = \rho I + (\overline{\partial} \rho)R , \tag{3.9}
\]

where \(1_X \in C^\infty(X)\) is the unit function of \(X\). We have:

\[ R_c := R|_{PV_c(X)} = (1 - \rho)R|_{PV_c(X_0)} . \]

Noticing that \(\delta \rho = \overline{\partial} \rho\), we compute:

\[ [\delta, R] = -(\overline{\partial} \rho)R + (1 - \rho)[\delta, R] = -(\overline{\partial} \rho)R + (1 - \rho)I_0 = -(\overline{\partial} \rho)R + (1 - \rho)I = I - i \circ \pi , \tag{3.10} \]

where we used (3.8) and noticed that \((1 - \rho)I_0 = (1 - \rho)I\), since \(1 - \rho\) vanishes on \(Z_W\). This shows that the first relation in (3.6) is satisfied. On the other hand, we have:

\[ \pi \circ i = \pi|_{PV_c(X)} = \rho I_c + (\overline{\partial} \rho)R_c , \]

so restricting (3.10) to \(PV_c(X)\) gives the second relation in (3.6). Since \(\pi\) is \(C^\infty(X)\)-linear, relations (3.6) imply that \(\pi\) is an \(O(X)\)-linear map of complexes and that \(i\) is a homotopy equivalence having \(\pi\) as an inverse up to the homotopies provided by \(R\) and \(R_c\). □

4. The dg-category of topological D-branes

Let \((X, W)\) be a Landau-Ginzburg pair with \(\dim_{\mathbb{C}} X = d\). In this subsection, we describe a \(\mathbb{Z}_2\)-graded dg-category \(DF(X, W)\) whose objects are so-called holomorphic factorizations of \(W\) (which can be viewed as a certain global version of matrix factorizations) but whose morphisms are bundle-valued differential forms; the differentials on the Hom spaces are certain deformations of the Dolbeault differential. The path integral arguments of [2] imply that the total cohomology category of \(DF(X, W)\) can be interpreted as the category of topological D-branes of the B-type Landau-Ginzburg model defined by \((X, W)\). The precise relation with the superconnection formalism used in loc. cit. is explained in Appendix B.

4.1. A category of holomorphic vector superbundles. We start by describing a category of holomorphic vector superbundles which is equivalent with the \(\mathbb{Z}_2\)-graded \(O(X)\)-linear category of finite rank sheaves of locally-free \(O_X\)-supermodules defined on \(X\). Recall the categories \(VB(X)\) and \(VB_\infty(X)\) defined in Subsection 1.1.
Definition 4.1 A holomorphic vector superbundle on $X$ is a $\mathbb{Z}_2$-graded holomorphic vector bundle defined on $X$, i.e. a complex holomorphic vector bundle $E$ endowed with a direct sum decomposition $E = E^0 \oplus E^1$, where $E^0$ and $E^1$ are holomorphic sub-bundles of $E$.

Definition 4.2 The $\mathbb{Z}_2$-graded $O(X)$-linear category $\text{VB}^s(X)$ is defined as follows.

- The objects are the holomorphic vector superbundles on $X$.
- Given two holomorphic vector superbundles $E$ and $F$ on $X$, let:

$$\text{Hom}_{\text{VB}^s(X)}(E,F) \overset{\text{def}}{=} \Gamma(X,\text{Hom}(E,F))$$

be the $O(X)$-module of holomorphic sections of the bundle $\text{Hom}(E,F) \overset{\text{def}}{=} E^\vee \otimes F$, endowed with the $\mathbb{Z}_2$-grading with homogeneous components:

$$\text{Hom}_0^{\text{VB}^s(X)}(E,F) \overset{\text{def}}{=} \Gamma(X,\text{Hom}(E^0,F^0)) \oplus \Gamma(X,\text{Hom}(E^1,F^1))$$

$$\text{Hom}_1^{\text{VB}^s(X)}(E,F) \overset{\text{def}}{=} \Gamma(X,\text{Hom}(E^0,F^1)) \oplus \Gamma(X,\text{Hom}(E^1,F^0))$$

- The composition of morphisms is induced from that of $\text{VB}(X)$.

Remark 4.1. Given a holomorphic vector superbundle $E = E^0 \oplus E^1$, the locally-free sheaf of $O_X$-modules $E \overset{\text{def}}{=} \Gamma(E)$ defined by $E$ is $\mathbb{Z}_2$-graded by the decomposition $E = E^0 \oplus E^1$, where $E^0 := \Gamma(E^0)$ and $E^1 := \Gamma(E^1)$ are the locally-free sheaves corresponding to the holomorphic vector bundles $E^0$ and $E^1$. We have:

$$\text{Hom}_0^{\text{VB}^s(X)}(E,F) \overset{\text{def}}{=} \text{Hom}(E^0,F^0) \oplus \text{Hom}(E^1,F^1)$$

$$\text{Hom}_1^{\text{VB}^s(X)}(E,F) \overset{\text{def}}{=} \text{Hom}(E^0,F^1) \oplus \text{Hom}(E^1,F^0)$$

where $\text{Hom}$ denotes the outer Hom of sheaves of $O_X$-modules. Thus $\text{VB}^s(X)$ can be identified with the $\mathbb{Z}_2$-graded $O(X)$-linear category of $\mathbb{Z}_2$-graded locally-free sheaves of $O_X$-modules.

Let $E$ and $F$ be two holomorphic vector superbundles on $X$. The graded direct sum of $E$ with $F$ is the direct sum $E \oplus F$ of the underlying holomorphic vector bundles, endowed with the $\mathbb{Z}_2$-grading given by:

$$(E \oplus F)^\kappa = E^\kappa \oplus F^\kappa \quad \forall \kappa \in \mathbb{Z}_2$$

The graded tensor product of $E$ with $F$ is the ordinary tensor product $E \otimes F$ of the underlying holomorphic vector bundles, endowed with the $\mathbb{Z}_2$-grading given by:

$$(E \otimes F)^0 \overset{\text{def}}{=} (E^0 \otimes F^0) \oplus (E^1 \otimes F^1)$$

$$(E \otimes F)^1 \overset{\text{def}}{=} (E^0 \otimes F^1) \oplus (E^1 \otimes F^0)$$

The graded dual $E$ is the holomorphic vector superbundle whose underlying holomorphic vector bundle is the ordinary dual $E^\vee$ of $E$, endowed with the $\mathbb{Z}_2$-grading given by:

$$(E^\vee)^\kappa \overset{\text{def}}{=} (E^\kappa)^\vee \quad \forall \kappa \in \mathbb{Z}_2$$

The holomorphic vector superbundle of morphisms from $E$ to $F$ is the holomorphic vector superbundle $\text{Hom}(E,F) \overset{\text{def}}{=} E^\vee \otimes F$, where $E^\vee$ is the graded dual and $\otimes$ is the graded tensor product.
Thus $\text{Hom}(E, F)$ is the usual bundle of morphisms between the underlying holomorphic vector bundles, endowed with the $\mathbb{Z}_2$-grading given by:

$$
\begin{align*}
\text{Hom}^0(E, F) &\overset{\text{def}}{=} \text{Hom}(E^0, F^0) \oplus \text{Hom}(E^1, F^1) \\
\text{Hom}^1(E, F) &\overset{\text{def}}{=} \text{Hom}(E^0, F^1) \oplus \text{Hom}(E^1, F^0)
\end{align*}
$$

When $F = E$, we set $\text{End}(E) \overset{\text{def}}{=} \text{Hom}(E, E)$, $\text{End}^\alpha(E) \overset{\text{def}}{=} \text{Hom}^\alpha(E, E)$ etc.

4.2. Holomorphic factorizations of $W$. We next introduce holomorphic factorizations of $W$ and two natural dg-categories constructed with such objects, which we denote by $\text{F}_\infty(X, W)$ and $\text{F}(X, W)$. The total cohomology category $\text{HF}(X, W)$ of $\text{F}(X, W)$ can be viewed as a naive candidate for the category of topological D-branes of the Landau-Ginzburg theory defined by the pair $(X, W)$. In general, however, the category $\text{HF}(X, W)$ differs from the correct category $\text{HDF}(X, W)$ which results from the path integral arguments of [2] and is described in the next subsection.

**Definition 4.3** A holomorphic factorization of $W$ is a pair $a = (E, D)$, where $E = E^0 \oplus E^1$ is a holomorphic vector superbundle on $X$ and $D \in \Gamma(X, \text{End}^1(E))$ is a holomorphic section of the bundle $\text{End}^1(E) = \text{Hom}(E^0, E^1) \oplus \text{Hom}(E^1, E^0) \subset \text{End}(E)$ which satisfies the condition $D^2 = \text{Wid}_E$.

**Remark 4.2**. Let $a = (E, D)$ be a holomorphic factorization of $W$. Decomposing $E = E^0 \oplus E^1$, the condition that $D$ is odd implies that $D$ has the form:

$$
D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}, \tag{4.1}
$$

where $u \in \Gamma(X, \text{Hom}(E^0, E^1))$ and $v \in \Gamma(X, \text{Hom}(E^1, E^0))$. The condition $D^2 = \text{Wid}_E$ amounts to the relations:

$$
u \circ u = \text{Wid}_{E^0}, \quad u \circ v = \text{Wid}_{E^1}.$$

**Definition 4.4** The smooth dg-category of holomorphic factorizations of $W$ is the $\mathbb{Z}_2$-graded $C^\infty(X)$-linear dg-category $\text{F}_\infty(X, W)$ defined as follows:

- The objects are the holomorphic factorizations of $W$.
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$ of $W$, we set:

$$
\text{Hom}_{\text{F}_\infty(X, W)}(a_1, a_2) = \Gamma_\infty(X, \text{Hom}(E_1, E_2)),
$$

endowed with the $\mathbb{Z}_2$-grading with homogeneous components:

$$
\text{Hom}^\alpha_{\text{F}_\infty(X, W)}(a_1, a_2) \overset{\text{def}}{=} \Gamma_\infty(X, \text{Hom}^\alpha(E_1, E_2)), \quad \forall \kappa \in \mathbb{Z}_2
$$

and with the differentials (called defect differentials) $\partial_{a_1, a_2}$ determined uniquely by the condition:

$$
\partial_{a_1, a_2}(f) \overset{\text{def}}{=} D_2 \circ f - (-1)^\kappa f \circ D_1, \quad \forall f \in \Gamma_\infty(X, \text{Hom}^\kappa(E_1, E_2)), \quad \forall \kappa \in \mathbb{Z}_2.
$$

- The composition of morphisms is induced by that of $\text{VB}_\infty(X)$. 

The smooth cohomological category of holomorphic factorizations is the $\mathbb{Z}_2$-graded $C^\infty(X)$-linear category $\text{HF}_\infty(X,W) \overset{\text{def}}{=} \text{H}(F_\infty(X,W))$.

**Definition 4.5** The holomorphic dg-category of holomorphic factorizations of $W$ is the $\mathbb{Z}_2$-graded $O(X)$-linear dg-category $F(X,W)$ defined as follows:

- The objects are the holomorphic factorizations of $W$.
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$ of $W$, we set:
  \[ \text{Hom}_{F(X,W)}(a_1, a_2) = \Gamma(X, \text{Hom}(E_1, E_2)) \]

endowed with the $\mathbb{Z}_2$-grading with homogeneous components:

\[ \text{Hom}^c_{F(X,W)}(a_1, a_2) \overset{\text{def}}{=} \Gamma(X, \text{Hom}^\kappa(E_1, E_2)) \]

and with the differentials $\partial_{a_1,a_2}$ determined uniquely by the condition:

\[ \partial_{a_1,a_2}(f) = D_2 \circ f - (-1)^\kappa f \circ D_1 \quad \forall f \in \Gamma(X, \text{Hom}^\kappa(E_1, E_2)) \]

- The composition of morphisms is induced by that of $\text{VB}(X)$.

The holomorphic cohomological category of holomorphic factorizations is the $\mathbb{Z}_2$-graded $O(X)$-linear category $\text{HF}(X,W) \overset{\text{def}}{=} \text{H}(F(X,W))$.

Notice that $F(X,W)$ is a non-full dg-subcategory of the $O(X)$-linear dg-category obtained from $F_\infty(X,W)$ by restriction of scalars.

**Remark 4.3.** Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$ of $W$, write $D_i = \begin{bmatrix} 0 & v_i \\ u_i & 0 \end{bmatrix}$ as in (4.1), where $u_i \in \Gamma(X, \text{Hom}(E_1^0, E_1^i))$ and $v_i \in \Gamma(X, \text{Hom}(E_1^i, E_1^0))$. Then an even morphism $f \in \text{Hom}^0_{F(X,W)}(a_1, a_2)$ has the form $f = \begin{bmatrix} f_{00} & 0 \\ 0 & f_{11} \end{bmatrix}$ with $f_{00} \in \Gamma(X, \text{Hom}(E_1^0, E_2^0))$, $f_{11} \in \Gamma(X, \text{Hom}(E_1^1, E_2^1))$ and we have:

\[ \partial_{a_1,a_2}(f) = D_2 \circ f - f \circ D_1 = \begin{bmatrix} 0 & v_2 \circ f_{11} - f_{00} \circ v_1 \\ u_2 \circ f_{11} - f_{00} \circ u_1 & 0 \end{bmatrix}. \] (4.2)

On the other hand, an odd morphism $g \in \text{Hom}^1_{F(X,W)}(a_1, a_2)$ has the form $g = \begin{bmatrix} 0 & g_{10} \\ g_{01} & 0 \end{bmatrix}$ with $g_{10} \in \Gamma(X, \text{Hom}(E_1^1, E_2^0))$, $g_{01} \in \Gamma(X, \text{Hom}(E_1^0, E_2^1))$ and we have:

\[ \partial_{a_1,a_2}(g) = D_2 \circ g + g \circ D_1 = \begin{bmatrix} v_2 \circ g_{10} + g_{10} \circ u_1 & 0 \\ 0 & u_2 \circ g_{10} + g_{01} \circ v_1 \end{bmatrix}. \] (4.3)

**4.3. The twisted Dolbeault category of holomorphic factorizations.** We are now ready to describe the $\mathbb{Z}_2$-graded dg-category $DF(X,W)$ which results from the path integral arguments of [2] and whose total cohomology category $\text{HDF}(X,W)$ is the topological D-brane category of the B-type Landau-Ginzburg theory defined by $(X,W)$. We refer the reader to Appendix B for the precise relation with the superconnection formalism used in loc. cit. The category $DF(X,W)$ is closely related to the Dolbeault category of holomorphic vector superbundles, so we start by introducing the latter.
The Dolbeault category of holomorphic vector superbundles. For any holomorphic vector superbundle $E = E^0 \oplus E^1$ on $X$, the $C^\infty(X)$-module of smooth sections $\Gamma_\infty(X,E)$ is $\mathbb{Z}_2$-graded with homogeneous components $\Gamma_\infty^k(X,E) \overset{\text{def}}{=} \Gamma_\infty(X,E^k)$. Accordingly, the $C^\infty(X)$-module $\mathcal{A}(X,E) \overset{\text{def}}{=} \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X,E)$ has an induced $\mathbb{Z} \times \mathbb{Z}_2$-grading. This bigrading corresponds to the decomposition:

$$\mathcal{A}(X,E) = \bigoplus_{k=0}^{\infty} \bigoplus_{\kappa \in \mathbb{Z}_2} \mathcal{A}^k(X,E^\kappa)$$

where:

$$\mathcal{A}^k(X,E^\kappa) = \mathcal{A}^k(X) \otimes_{C^\infty(X)} \Gamma_\infty(X,E^\kappa)$$

The $\mathbb{Z}$-grading is called the rank grading of $\mathcal{A}(X,E)$ and corresponds to the decomposition:

$$\mathcal{A}(X,E) = \bigoplus_{i=0}^{d} \mathcal{A}^i(X,E)$$

where $\mathcal{A}^i(X,E) \overset{\text{def}}{=} \mathcal{A}^i(X) \otimes_{C^\infty(X)} \Gamma_\infty(X,E)$. The $\mathbb{Z}_2$-grading is called the bundle grading of $\mathcal{A}(X,E)$ and corresponds to the decomposition:

$$\mathcal{A}(X,E) = \mathcal{A}(X,E^0) \oplus \mathcal{A}(X,E^1)$$

where $\mathcal{A}(X,E^\kappa) \overset{\text{def}}{=} \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X,E^\kappa)$. For any $\alpha \in \mathcal{A}^k(X,E)$, let $\text{rk} \alpha \overset{\text{def}}{=} k \in \mathbb{Z}$. For any $\alpha \in \mathcal{A}(X,E^\kappa)$, let $\sigma(\alpha) \overset{\text{def}}{=} \kappa \in \mathbb{Z}_2$. The total grading of $\mathcal{A}(X,E)$ is the $\mathbb{Z}_2$-grading given by the decomposition:

$$\mathcal{A}(X,E) = \mathcal{A}(X,E^0) \oplus \mathcal{A}(X,E^1)$$

where:

$$\mathcal{A}(X,E^0) \overset{\text{def}}{=} \bigoplus_{i=\text{even}} \mathcal{A}^i(X,E^0) \bigoplus \bigoplus_{i=\text{odd}} \mathcal{A}^i(X,E^1)$$

$$\mathcal{A}(X,E^1) \overset{\text{def}}{=} \bigoplus_{i=\text{even}} \mathcal{A}^i(X,E^1) \bigoplus \bigoplus_{i=\text{odd}} \mathcal{A}^i(X,E^0)$$

For any $\alpha \in \mathcal{A}(X,E^\kappa)$, we set $\text{deg} \alpha = \kappa \in \mathbb{Z}_2$. For all $\rho \in \mathcal{A}^k(X)$ and all $s \in \Gamma_\infty(M,E^\kappa)$, we have:

$$\text{deg}(\rho \otimes s) = k + \kappa \in \mathbb{Z}_2 \quad \text{(4.4)}$$

**Definition 4.6** The Dolbeault category $\mathcal{D}^\bullet(X)$ of holomorphic vector superbundles is the $C^\infty(X)$-linear $\mathbb{Z} \times \mathbb{Z}_2$-graded category defined as follows:

- The objects are the holomorphic vector superbundles defined on $X$.
- The $C^\infty(X)$-modules of morphisms are given by:

$$\text{Hom}_{\mathcal{D}^\bullet(X)}(E,F) = \mathcal{A}(X,\text{Hom}(E,F)) = \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X,\text{Hom}(E,F))$$

endowed with the obvious $\mathbb{Z} \times \mathbb{Z}_2$-grading.
- The $C^\infty(X)$-bilinear composition of morphisms $\circ : \mathcal{A}(X,\text{Hom}(F,G)) \times \mathcal{A}(X,\text{Hom}(E,F)) \to \mathcal{A}(X,\text{Hom}(E,G))$ is determined uniquely through the condition:

$$\text{(4.5)}$$

for all pure rank forms $\rho, \eta \in \mathcal{A}(X)$ and all pure $\mathbb{Z}_2$-degree elements $f \in \Gamma_\infty(X,\text{Hom}(F,G))$ and $g \in \Gamma_\infty(X,\text{Hom}(E,F))$. 

\[ (\rho \otimes f) \circ (\eta \otimes g) = (-1)^{\sigma(f) \text{rk} \eta}(\rho \wedge \eta) \otimes (f \circ g) \]
The twisted Dolbeault category of holomorphic factorizations. Consider two holomorphic factorizations \( a_1 = (E_1, D_1) \) and \( a_2 = (E_2, D_2) \) of \( W \). Then the space \( \mathcal{A}(X, \text{Hom}(E_1, E_2)) \) carries two natural differentials:

- The Dolbeault differential \( \overline{\partial} := \overline{\partial}_{a_1, a_2} = \overline{\partial}_{\text{Hom}(E_1, E_2)} \) determined on \( \mathcal{A}(X, \text{Hom}(E_1, E_2)) \) by the holomorphic structure of the vector bundle \( \text{Hom}(E_1, E_2) \). This differential is \( O(X) \)-linear, preserves the bundle grading and has degree +1 with respect to the rank grading.

- The defect differential, i.e. the \( C^\infty(X) \)-linear endomorphism \( \partial := \partial_{a_1, a_2} \) of \( \mathcal{A}(X, \text{Hom}(E_1, E_2)) \) determined uniquely by the condition:

\[
\partial_{a_1, a_2} (\rho \otimes f) = (-1)^{|\rho|} \rho \otimes (D_2 \circ f) - (-1)^{|\rho|+|f|} \rho \otimes (f \circ D_1)
\]

for all pure rank forms \( \rho \in \mathcal{A}(X) \) and all pure \( \mathbb{Z}_2 \)-degree elements \( f \in \Gamma_{\infty}(X, \text{Hom}(E_1, E_2)) \). Notice that \( \partial \) squares to zero because \( D_i \) square to \( W \text{Id}_{E_i} \). Moreover, this differential preserves the rank grading and is odd with respect to the bundle grading.

Since \( D_i \) are holomorphic, the Dolbeault and defect differentials anticommute. Thus:

\[
\overline{\partial}^2 = \partial^2 = \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0 ,
\]

and hence \( (\mathcal{A}(X, \text{Hom}(E_1, E_2)), \overline{\partial}_{a_1, a_2}, \partial_{a_1, a_2}) \) is a \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded bicomplex.

**Definition 4.7** The twisted differential \( \delta := \delta_{a_1, a_2} \) of the ordered pair \((a_1, a_2)\) is the total differential of the bicomplex \( (\mathcal{A}(X, \text{Hom}(E_1, E_2)), \overline{\partial}_{a_1, a_2}, \partial_{a_1, a_2}) \):

\[
\delta_{a_1, a_2} \overset{\text{def.}}{=} \overline{\partial}_{a_1, a_2} + \partial_{a_1, a_2} .
\]

Notice that the twisted differential is odd with respect to the total \( \mathbb{Z}_2 \)-grading.

**Definition 4.8** The twisted Dolbeault category of holomorphic factorizations of \( (X, W) \) is the \( \mathbb{Z}_2 \)-graded \( O(X) \)-linear dg-category \( \text{DF}(X, W) \) defined as follows:

- The objects of \( \text{DF}(X, W) \) are the holomorphic factorizations of \( W \).

- Given two holomorphic factorizations \( a_1 = (E_1, D_1) \) and \( a_2 = (E_2, D_2) \), define:

\[
\text{Hom}_{\text{DF}(X,W)}(a_1, a_2) \overset{\text{def.}}{=} \mathcal{A}(X, \text{Hom}(E_1, E_2)) ,
\]

endowed with the total \( \mathbb{Z}_2 \)-grading and with the twisted differentials \( \delta_{a_1, a_2} \).

- The composition of morphisms coincides with that of \( \mathcal{D}^\bullet(X) \) (see (4.5)).

It is easy to check that \( \text{DF}(X, W) \) is indeed a dg-category. As a \( \mathbb{Z}_2 \)-graded category, \( \text{DF}(X, W) \) coincides with the \( O(X) \)-linear category obtained from \( \mathcal{D}^\bullet(X) \) by restriction of scalars. Let:

\[
\text{HDF}(X, W) \overset{\text{def.}}{=} \text{H}(\text{DF}(X, W))
\]

denote its total cohomology category, which is \( \mathbb{Z}_2 \)-graded and \( O(X) \)-linear. This can also be viewed as a \( \mathbb{Z}_2 \)-graded \( \mathbb{C} \)-linear category by restriction of scalars.

**Proposition 4.9** Assume that the critical set \( Z_W \) is compact. Then \( \text{HDF}(X, W) \) is Hom-finite as a \( \mathbb{C} \)-linear category.
\[
\delta_1 := \delta_{a_1,a_1} \quad , \quad \delta_2 := \delta_{a_2,a_2} \quad , \quad \delta := \delta_{a_1,a_2} .
\]

The space \( \operatorname{Hom}_{\mathcal{D}}(\mathcal{O}_X)(a_1,a_2) \) coincides with the space of global sections of the coherent sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{H}_{a_1,a_2} \) defined by \( \mathcal{H}_{a_1,a_2}(U) \equiv \mathcal{H}(\mathcal{A}(U, \mathcal{O}_X(E), E), \delta) \) for any open subset \( U \subset X \). Since \( W \) is non-constant, the critical set \( Z_W \) is a closed proper subset of \( X \). Choose a point \( x \in X \setminus Z_W \) and, without loss of generality, we can assume that \( \partial W(x) \neq 0 \). By shrinking \( U \) small enough such that \( E_i|_U \) are holomorphically trivializable. Picking holomorphic trivializations of \( E_i|_U \) allows us to identify \( D_i|_U \) with matrices whose entries are complex-valued holomorphic functions defined on \( U \) and we tacitly do so below; this allows us to define the holomorphic partial derivatives \( \partial_i D_i = \frac{\partial D_i}{\partial \bar{z}^i} \). Since \( x \) does not belong to \( Z_W \), at least one of the partial derivatives \( \partial_i W \) do not vanish at \( x \) and, without loss of generality, we can assume that \( \partial_i W(x) \neq 0 \). By shrinking \( U \) if necessary, we can further assume that \( \partial_i W \) doesn’t vanish at any point of \( U \). Then the relation \( D_2^2 = \text{Id}_{E_2} \) implies:

\[
[D_2, \partial_1 D_2] = D_2(\partial_1 D_2) + (\partial_1 D_2)D_2 = (\partial_1 W)\text{Id}_{E_2} ,
\]

which gives:

\[
\text{id}_{E_2} = [D_2, f] = \delta_2(f) ,
\]

where \( f \equiv \frac{1}{\partial \bar{z}^i} \partial_i D_2 \in \Gamma(U, \mathcal{A}(E_2)) \subset \mathcal{A}(U, \mathcal{O}_X(E_2)) \) and we used the fact that \( f \) is holomorphic. For any \( \alpha \in \mathcal{A}(U, \mathcal{O}_X(E_2)) \) such that \( \delta \alpha = 0 \), we have:

\[
\alpha = \text{id}_{E_2} \alpha = \delta_2(f)\alpha = \delta(f\alpha) .
\]

Thus \( \mathcal{H}_{a_1,a_2}(U) = \mathcal{H}(\mathcal{A}(U, \mathcal{O}_X(E_1,E_2)), \delta) = 0 \). This shows that \( \mathcal{H}_{a_1,a_2} \) is supported on the critical set \( Z_W \). Thus \( \operatorname{Hom}_{\mathcal{D}}(\mathcal{O}_X)(a_1,a_2) = \mathcal{H}_{a_1,a_2}(X) \) is finite-dimensional since \( Z_W \) is compact.

### 4.4. The extended supertrace.

Let \( \alpha = (E, D) \) be a holomorphic factorization of \( W \) and set \( \delta_\alpha := \delta_{a,a} \). Let \( \operatorname{str} : \Gamma_\infty(X, \mathcal{A}(E)) \to C^\infty(X) \) denote the fiberwise supertrace, which is an even map when \( \mathcal{C} \) is viewed as a \( \mathbb{Z}_2 \)-graded \( \mathbb{C} \)-algebra concentrated in degree zero. \(^4\) Noticing that \( \mathcal{A}(X, \mathcal{A}(E)) \cong \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X, \mathcal{A}(E)) \) is a left module over \( \mathcal{A}(X) \), we extend the fiberwise supertrace to a left \( \mathcal{A}(X) \)-linear map denoted by the same symbol:

\[
\operatorname{str} \equiv \text{id}_{\mathcal{A}(X)} \otimes_{C^\infty(X)} \operatorname{str} : \mathcal{A}(X, \mathcal{A}(E)) \to \mathcal{A}(X) .
\]

For any \( \rho \in \mathcal{A}(X) \) and any \( s \in \Gamma_\infty(X, \mathcal{A}(E)) \), we have:

\[
\operatorname{str}(\rho \otimes s) = \rho \operatorname{str}(s)
\]

and the extended supertrace satisfies:

\[
\operatorname{str}(\alpha \beta) = (-1)^{\deg \alpha \deg \beta} \operatorname{str}(\beta \alpha) , \quad \forall \alpha, \beta \in \mathcal{A}(X, \mathcal{A}(E)) = \mathcal{A}(X, \mathcal{A}(E)) . \tag{4.6}
\]

The extended supertrace is a morphism of complexes of \( \mathcal{O}(X) \)-modules from \( \mathcal{A}(X, \mathcal{A}(E)) \) to \( \mathcal{A}(X, \mathcal{A}(E)) \), when the latter is endowed with the mod 2 reduction of its \( \mathbb{Z}_2 \)-grading:

\[
\operatorname{str}(\delta \alpha) = \overline{\mathcal{J}} \operatorname{str}(\alpha) , \quad \forall \alpha \in \mathcal{A}(X, \mathcal{A}(E)) \tag{4.7}
\]

and satisfies the relation:

\[
\operatorname{str}(\delta \alpha) = 0 , \quad \forall \alpha \in \mathcal{A}(X, \mathcal{A}(E)) . \tag{4.8}
\]

Notice that \( \operatorname{str}(\omega) \) vanishes unless \( \omega \in \mathcal{A}(X, \mathcal{A}(E)) \) is even with respect to the bundle grading.

\(^4\) This means that \( \operatorname{str}(s) = 0 \) for \( s \in \Gamma_\infty(X, \mathcal{A}(E)) \).
4.5. Compact support version.

**Definition 4.10** The compactly-supported twisted Dolbeault category of holomorphic factorizations of \( W \) is the \( O(X) \)-linear \( \mathbb{Z}_2 \)-graded dg-category \( \text{DF}_c(X, W) \) defined as follows:

- The objects of \( \text{DF}_c(X, W) \) are the holomorphic factorizations of \( W \).
- Given two objects \( a_1 = (E_1, D_1) \) and \( a_2 = (E_2, D_2) \), set:
  \[
  \text{Hom}_{\text{DF}_c(X, W)}(a_1, a_2) \overset{\text{def}}{=} \{ \alpha \in \text{Hom}_{\text{DF}(X, W)}(a_1, a_2) \mid \text{supp}(\alpha) = \text{compact} \}
  \]
- The differentials and composition of morphisms are induced from \( \text{DF}(X, W) \).

Notice that \( \text{DF}_c(X, W) \) is a non-full subcategory of \( \text{DF}(X, W) \). Let:

\[
\text{HDF}_c(X, W) \overset{\text{def}}{=} \text{H}(\text{DF}_c(X, W))
\]

denote the total cohomology category of \( \text{DF}_c(X, W) \); this is a \( \mathbb{Z}_2 \)-graded \( O(X) \)-linear category. Let \( j : \text{DF}_c(X, W) \to \text{DF}(X, W) \) be the inclusion functor and \( j_* : \text{HDF}_c(X, W) \to \text{HDF}(X, W) \) denote the functor induced by \( j \) between the total cohomology categories.

**Proposition 4.11** Assume that the critical set \( Z_W \) is compact. Then the inclusion functor \( j : \text{DF}_c(X, W) \to \text{DF}(X, W) \) is a quasi-equivalence of \( \mathbb{Z}_2 \)-graded \( O(X) \)-linear dg-categories.

**Proof.** This is an extension to the global case of [12, Proposition 3.3]; the proof is similar to that of Proposition 3.7. Let \( a_1 = (E_1, D_1) \) and \( a_2 = (E_2, D_2) \) be two holomorphic factorizations of \( W \). Let:

\[
\delta := \delta_{a_1, a_2} \quad , \quad \delta_c := \delta_{|\text{Hom}_{\text{DF}_c(X, W)}(a_1, a_2)} \\
I := \text{id}_{\text{Hom}_{\text{DF}(X, W)}(a_1, a_2)} \quad , \quad I_c := \text{id}_{\text{Hom}_{\text{DF}_c(X, W)}(a_1, a_2)}. \tag{4.9}
\]

Let \( \delta_i := \delta_{a_i} \), \( \partial_i := \partial_{a_i} = [D_i, \cdot] \), \( \overline{\partial}_i := \overline{\partial}_{a_i} = \overline{\partial}_{\text{End}(E_i)} \) and \( \overline{\delta} := \delta_{a_1, a_2} \), \( \overline{\partial} := \overline{\partial}_{a_1, a_2} = \overline{\partial}_{\text{Hom}(E_1, E_2)} \). Let \( j := j_{a_1, a_2} : \text{Hom}_{\text{DF}_c(X, W)}(a_1, a_2) \to \text{Hom}_{\text{DF}(X, W)}(a_1, a_2) \) denote the inclusion morphism. We will construct maps:

\[
\pi := \pi_{a_1, a_2} : \text{Hom}_{\text{DF}(X, W)}(a_1, a_2) \to \text{Hom}_{\text{DF}_c(X, W)}(a_1, a_2) \\
\mathcal{R} := \mathcal{R}_{a_1, a_2} : \text{Hom}_{\text{DF}(X, W)}(a_1, a_2) \to \text{Hom}_{\text{DF}_c(X, W)}(a_1, a_2) \tag{4.10}
\]

such that \( \pi \) is a map of complexes and such that:

\[
I - j \circ \pi = [\delta, \mathcal{R}] \\
I_c - \pi \circ j = [\delta_c, \mathcal{R}_c], \tag{4.11}
\]

where \( \mathcal{R} \) preserves the subspace \( \text{Hom}_{\text{DF}_c(X, W)}(a_1, a_2) \) and \( \mathcal{R}_c \overset{\text{def}}{=} \mathcal{R}_{|\text{Hom}_{\text{DF}_c(X, W)}(a_1, a_2)} \). Let \( X_0 := X \setminus Z_W \) and \( W_0 := W|_{X_0} \). To construct \( \mathcal{R} \), pick a Hermitian metric \( G \) on \( X \) and a Hermitian metric \( h \) on \( E \) such that \( h_{E^\theta \otimes E^\theta} \overset{\text{def}}{=} 0 \)\(^5\) and let \( \langle \cdot, \cdot \rangle \) denote the Hermitian metric induced by \( G \) and \( h \) on \( \wedge T^* X \) and \( \| \cdot \| \) denote the corresponding norm. We trivially extend the former to a pairing:

\[
\langle \cdot, \cdot \rangle : (\wedge T^* X) \times (\wedge T^* X \otimes \text{End}(E)) \to \text{End}(E) \tag{4.5}
\]

\(^5\) This is a so-called admissible metric on \( E \), see Subsection 6.3.
Let $\partial_E^0$ be the operator defined in (6.3) and $\partial$ be the induced differential of the graded associative algebra $\Omega(X, \text{End}(E_2))$ (see Subsection 6.3). Differentiating the relation $D_2^2 = \text{Wid}_{E_2}$ gives:

$$D_2 \circ (\partial D_2) + (\partial D_2) \circ D_2 = (\partial W)\text{id}_{E_2} .$$

This implies that the following relation holds on $X_0$:

$$\mathfrak{d}_2(s) = [D_2, s] = D_2|_{X_0} \circ s + s \circ D_2|_{X_0} = \text{id}_{E_2}|_{X_0} , \quad (4.12)$$

where:

$$s \overset{\text{def}}{=} \frac{\langle \partial W, \partial D_2 \rangle}{||\partial W||^2} \in \Gamma_\infty(X_0, \text{End}^1(E_2)) \subset \text{Hom}_{DF}(X_0, W_0)(a_2, a_2) .$$

For any element $\beta \in \text{Hom}_{DF}(X_0, W_0)(a_2, a_2)$, let $\hat{\beta} : \text{Hom}_{DF}(X_0, W_0)(a_1, a_2) \to \text{Hom}_{DF}(X_0, W_0)(a_1, a_2)$ denote the operator of left composition with $\beta$ in the dg-category $DF(X_0, W_0)$ (see the commutative diagram (4.13)):

$$\hat{\beta}(\alpha) \overset{\text{def}}{=} \beta \alpha \in \text{Hom}_{DF}(X_0, W_0)(a_1, a_2) , \quad \forall \alpha \in \text{Hom}_{DF}(X_0, W_0)(a_1, a_2) . \quad (4.13)$$

Relation (4.12) implies:

$$[\mathfrak{d}, s] = \mathfrak{d}_2(s) = \text{id}_{E_2} = I_0 ,$$

where $I_0 \overset{\text{def}}{=} \text{id}_{\text{Hom}_{DF}(X_0, W_0)(a_1, a_2)}$. Since $\overline{\partial}_2s \in \mathcal{A}^1(X, \text{End}^1(E_2))$ has rank one, the operator $[\overline{\partial}, s] = \overline{\partial}_2s$ is nilpotent on the space $\text{Hom}_{DF}(X_0, W_0)(a_1, a_2)$. This implies that the operator:

$$[\delta, s] = I_0 + [\overline{\partial}, s] = I_0 + \overline{\partial}_2s$$

is invertible on the same space. We have:

$$[\delta, s]^{-1} = (I_0 + \overline{\partial}_2s)^{-1} = \hat{S} ,$$

where:

$$S \overset{\text{def}}{=} (\text{id}_{E_2} + \overline{\partial}_2s)^{-1} = \sum_{k=0}^{d} (-1)^k (\overline{\partial}_2s)^k \in \mathcal{A}(X, \text{End}(E_2))^0 = \text{Hom}_{DF}(X_0, W_0)(a_2)^0$$

and $\text{id}_{E_2}^{(0)} \in \Gamma_\infty(X_0, \text{End}(E_2))$ is the identity endomorphism of $E_2|_{X_0}$. Consider the following operator defined on $\text{Hom}_{DF}(X_0, W_0)(a_1, a_2)$:

$$\mathcal{R} \overset{\text{def}}{=} \hat{s} \circ \hat{S} = \hat{R} ,$$

where:

$$R \overset{\text{def}}{=} sS .$$

Applying the operator $[\delta, \cdot]$ to the relation:

$$[\delta, s] \circ \hat{S} = I_0$$
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gives:
\[ [\delta, \hat{S}] = 0 \]
since \([\delta, [\delta, \hat{s}]] = 0 \) and \([\delta, \hat{s}] \) is invertible. Thus:
\[ [\delta, \mathcal{R}] = [\delta, \hat{s}] \circ \hat{S} = I_0 \]  \hspace{1cm} (4.14)
Since the critical set \( Z_W \) is compact, we can find a relatively compact open neighborhood \( U \) of \( Z_W \). Let \( U_1 \) be an open subset of \( X \) such that \( Z_W \subset U_1 \subset \overline{U_1} \subset U \) and let \( \rho \in \mathcal{C}^\infty(X) \) be a smooth function which is identically 1 on \( U_1 \) and vanishes outside \( U \). The operator \( \mathcal{R} \) is now defined as:
\[ \mathcal{R} \overset{\text{def}}{=} (1_X - \rho) \mathcal{R} \]
where \( 1_X \in \mathcal{C}^\infty(X) \) is the unit function of \( X \). Notice that \( \mathcal{R} \) is odd with respect to the total \( \mathbb{Z}_2 \)-grading of \( \text{Hom}_{\mathcal{D}F(X,W)}(a_1, a_2) \) and that \( \mathcal{R} \) preserves the subspace \( \text{Hom}_{\mathcal{D}F_c(X,W)}(a_1, a_2) \). We now define \( \pi \overset{\text{def}}{=} \rho I + (\overline{\partial}\rho) \mathcal{R} \). Relation (4.14) implies:
\[ [\delta, \mathcal{R}] = I - j \circ \pi \]
showing that the first of equations (4.11) is satisfied. Restricting this to \( \text{Hom}_{\mathcal{D}F_c(X,W)}(a_1, a_2) \) shows that the second equation in (4.11) also holds, which implies that \( j_{a_1, a_2} \) is a quasi-isomorphism. Since \( a_1 \) and \( a_2 \) are arbitrary, this gives the conclusion. \( \Box \)

5. Off-shell bulk traces and the bulk flow

In this section, we describe the off-shell models for the bulk trace which result from the path integral analysis of [2]. Let \((X, W)\) be a Landau-Ginzburg pair with \( \text{dim}_\mathbb{C} X = d \) and \( \Omega \) be a holomorphic volume form on \( X \).

5.1. The Serre trace induced by \( \Omega \) on \( \mathcal{A}_c(X) \).

**Definition 5.1** The Serre trace induced by \( \Omega \) on \( \mathcal{A}_c(X) \) is the \( \mathbb{C} \)-linear map \( \int_\Omega : \mathcal{A}_c(X) \to \mathbb{C} \) defined through:
\[ \int_\Omega \rho \overset{\text{def}}{=} \int_X \Omega \wedge \rho \quad , \quad \forall \rho \in \mathcal{A}_c(X) \]  \hspace{1cm} (5.1)
Let \( \mathbb{C}[0] \) denote the complex of vector spaces over \( \mathbb{C} \) whose only non-trivial term equals \( \mathbb{C} \) and sits in degree zero.

**Proposition 5.2** The Serre trace \( \int_\Omega \) is a map of complexes of degree \(-d\) from \((\mathcal{A}_c(X), \overline{\partial})\) to \( \mathbb{C}[0] \).

**Proof.** Since \( \Omega \) has type \((d,0)\), we have \( \int_\Omega \rho = 0 \) unless \( \rho \in \mathcal{A}_c^d(X) \), so \( \int_\Omega \) has degree \(-d\). Since \( \Omega \) is holomorphic, we have:
\[ \int_\Omega (\partial \rho) = \int_X \Omega \wedge \partial \rho = (-1)^d \int_X \overline{\partial}(\Omega \wedge \rho) = (-1)^d \int_X d(\Omega \wedge \rho) = 0 \quad , \quad \forall \rho \in \mathcal{A}_c(X) \]
where we noticed that \( \partial(\Omega \wedge \rho) = 0 \) for degree reasons. This shows that \( \int_\Omega \) is a map of complexes. \( \Box \)
5.2. The canonical off-shell bulk trace induced by $\Omega$ on $\text{PV}_c(X)$.

**Definition 5.3** The canonical off-shell bulk trace induced by $\Omega$ on $\text{PV}_c(X)$ is the $\mathbb{C}$-linear map

$$
\text{Tr}_B := \text{Tr}_B^\Omega : \text{PV}_c(X) \to \mathbb{C}
$$

defined through:

$$
\text{Tr}_B^\Omega(\omega) = \int_X \Omega \wedge (\Omega \lrcorner \omega) \quad \forall \omega \in \text{PV}_c(X) .
$$

(5.2)

Since $\Omega$ has bidegree $(d,0)$, we have:

$$
\text{Tr}_B^\Omega(\omega) = \int_X \Omega \wedge (\Omega \lrcorner 0 \omega) = \int_X (\Omega \lrcorner 0 \omega) ,
$$

(5.3)

where $\Omega \lrcorner 0$ is the reduced contraction introduced in Definition 3.5. Thus $\text{Tr}_B^\Omega = \int_\Omega \circ (\Omega \lrcorner 0)$. We also have:

$$
\text{Tr}_B^\Omega(\omega) = 0 \text{ unless } \omega \in \text{PV}^{-d,d}_c(X) .
$$

5.3. Cohomological bulk traces. In the following, we simplify notation by omitting to indicate dependence of the traces on $\Omega$.

**Proposition 5.4** For any $\eta \in \text{PV}_c(X)$, we have:

$$
\text{Tr}_B(\delta_W \eta) = \text{Tr}_B(\bar{\partial} \eta) = \text{Tr}_B(\iota_W \eta) = 0 .
$$

In particular, $\text{Tr}_B$ descends to $\text{HPV}_c(X,W)$.

**Proof.** For any $\eta \in \text{PV}_c(X)$, we have:

$$
\text{Tr}_B(\bar{\partial} \eta) = \int_X \Omega \wedge [\Omega \lrcorner (\bar{\partial} \eta)] = \int_X \bar{\partial} [\Omega \wedge (\Omega \lrcorner \eta)] = \int_X d [\Omega \wedge (\Omega \lrcorner \eta)] = 0 ,
$$

where in the third equality the Dolbeault differential $\bar{\partial}$ can be replaced with the de Rham differential $d$ because $\Omega \wedge (\Omega \lrcorner \eta)$ either vanishes or has type $(d,*)$. The last equality follows from the Stokes theorem since $\eta$ (and hence also $\Omega \wedge (\Omega \lrcorner \eta)$) has compact support. To prove the relation $\text{Tr}_B(\iota_W \eta) = 0$, notice that $\iota_W \eta$ has polyvector rank at most $d - 1$, which implies $\Omega \lrcorner (\iota_W \eta) = 0$ and hence $\text{Tr}_B(\iota_W \eta) = 0$ by (5.3). Finally, the relation $\text{Tr}_B(\delta_W \eta) = 0$ follows by adding the two relations established above. $\Box$

**Definition 5.5** The cohomological bulk trace induced by $\Omega$ on $\text{HPV}_c(X,W)$ is the $\mathbb{C}$-linear map

$$
\text{Tr}_c := \text{Tr}_c^\Omega : \text{HPV}_c(X,W) \to \mathbb{C}
$$

defined by $\text{Tr}_c^\Omega = \text{Tr}_c^\Omega \circ i^{-1} : \text{HPV}_c(X,W) \to \mathbb{C}$ obtained by composing $\text{Tr}_c$ with the inverse of the linear isomorphism $i_* : \text{HPV}_c(X,W) \to \mathbb{C}$.

When the critical set of $W$ is compact, Proposition 3.7 implies that the inclusion map $i : \text{PV}_c(X) \to \text{PV}(X)$ induces an isomorphism of $\mathbb{Z}$-graded $\text{O}(X)$-modules $i_* : \text{HPV}_c(X,W) \cong \text{HPV}(X,W)$ (moreover, $\text{HPV}(X,W)$ is finite-dimensional over $\mathbb{C}$ by Proposition 3.4). This allows us to transfer $\text{Tr}_c$ to a trace on $\text{HPV}(X,W)$:

**Definition 5.6** Assume that the critical set $Z_W$ is compact. In this case, the cohomological bulk trace induced by $\Omega$ on $\text{HPV}(X,W)$ is the $\mathbb{C}$-linear map

$$
\text{Tr} := \text{Tr}_c^\Omega : \text{HPV}(X,W) \to \mathbb{C}
$$

obtained by composing $\text{Tr}_c$ with the inverse of the linear isomorphism $i_* : \text{HPV}_c(X,W) \cong \text{HPV}(X,W)$ induced on cohomology by the inclusion map.
5.4. The bulk flow determined by a Kähler metric on \( PV(X) \). Let \( G \) be a Kähler metric on \( X \) and \( \nabla : \Gamma_\infty(X, TX) \to \Omega^1(X, TX) \) be its Levi-Civita connection. Let \( G^c \) denote the complexification of \( G \) (which is a complex-valued and fiberwise non-degenerate pairing on \( TX \)) and let \( h_G \) denote the Hermitian metric induced by \( G \) on \( TX \). We have: \( G^c|_{TX \otimes TX} = G^c|_{TX \otimes TX} = 0 \) while \( h_G \) restricts to Hermitian metrics on \( TX \) and \( TX \). We also have \( h_G(s_1, s_2) = G^c(s_1, s_2) \) for all \( s_1, s_2 \in \Gamma_\infty(X, TX) \). Identifying \( TX \) with \( TX \) through the isomorphism of real vector bundles \( \text{Re} : TX \to TX \), the Levi-Civita connection \( \nabla \) can be viewed as a connection on \( TX \).

Since \( G \) is Kähler, \( \nabla \) is \( \mathbb{C} \)-linear as a connection on \( TX \) and coincides \([5, \text{Theorem 3.13}]\) with the Chern connection of the Hermitian vector bundle \((TX, h_G)\). Hence \( \nabla \) can be viewed as the unique Hermitian connection on \((TX, h_G)\) which satisfies:

\[
\nabla^{0,1} = \overline{\partial}_{TX}|_{\Gamma_\infty(X, TX)} .
\]

(5.4)

Let \( \overline{W} : X \to \mathbb{C} \) denote the complex conjugate of the function \( W \). Let \( \text{grad}_G \overline{W} \in \Gamma_\infty(X, TX) = PV^{-1,0}(X) \) denote the gradient of \( \overline{W} \) with respect to \( G^c \), which is defined through the relation:

\[
(\overline{\partial} \overline{W})(s) = G^c(\text{grad}_G \overline{W}, s) = h_G(\text{grad}_G \overline{W}, s), \quad \forall s \in \Gamma_\infty(X, TX) .
\]

(5.5)

Let:

\[
\text{Hess}_G(\overline{W}) \overset{\text{def}}{=} \nabla(\text{grad}_G \overline{W}) \in \Omega^1(X, TX)
\]

denote the Hessian operator of \( \overline{W} \). Let:

\[
H_G \overset{\text{def}}{=} \text{Hess}^{0,1}_G(\overline{W}) = \nabla^{0,1}(\text{grad}_G \overline{W}) = \overline{\partial}_{TX}(\text{grad}_G \overline{W}) \in \mathcal{A}^1(X, TX) = PV^{-1,1}(X)
\]

(5.6)

denote the \((0, 1)\)-part of the Hessian operator. Let \( \hat{h}_G \) denote the Hermitian metric induced by \( h_G \) on the bundle \( T^*X \) and:

\[
||\partial W||^2_{\hat{G}} \overset{\text{def}}{=} \hat{h}_G(\partial W, \partial W) = h_G(\text{grad}_G \overline{W}, \text{grad}_G \overline{W}) = (\partial W)(\text{grad}_G \overline{W}) \in \mathcal{A}^0(X) = PV^{0,0}(X)
\]

(5.7)

denote the squared norm of the holomorphic 1-form \( \partial W \in \Gamma(X, T^*X) \subset \Omega^{1,0}(X) \). Notice that \( H_G \) is a nilpotent element of the algebra \( PV(X) \), which allows us to define its exponential. For any \( \lambda \in [0, +\infty) \), we have:

\[
e^{-i\lambda H_G} = \sum_{p=0}^d \frac{1}{p!}(-i\lambda)^p(H_G)^p \in PV^0(X) ,
\]

where the expansion of the exponential reduces to the first \( d + 1 \) terms because \( H_G \) belongs to \( PV^{-1,1}(X) \) and hence \((H_G)^p \) belongs to \( PV^{-p,0}(X) \), while \( PV^{-p,p}(X) = 0 \) for \( p > d \).

**Definition 5.7** The bulk flow generator determined by the Kähler metric \( G \) is the element:

\[
L_G \overset{\text{def}}{=} ||\partial W||^2_{\hat{G}} + iH_G \in PV^{0,0}(X) \oplus PV^{-1,1}(X) \subset PV^0(X) .
\]

Notice that \( L_G \) has degree zero with respect to the canonical \( \mathbb{Z} \)-grading of \( PV(X) \).

**Proposition 5.8** We have:

\[
L_G = \delta_W v_G ,
\]

where:

\[
v_G \overset{\text{def}}{=} i \text{grad}_G \overline{W} \in \Gamma_\infty(X, TX) = PV^{-1,0}(X) .
\]
In particular, we have:
\[ \lambda \]
this gives:
\[ \lambda \]
For any \( \lambda \), the unique solution of (5.12) which vanishes at \( \delta \).
Proposition 5.9 implies that the
\[ \lambda \]
which implies the conclusion.

Consider the polyvector-valued form:
\[ e^{-\lambda L_G} \] def. \[ e^{-\lambda \|\partial W\|_G^2} e^{-1\lambda H_G} \in PV^0(X) \] .

**Proposition 5.9** For any \( \lambda \in \mathbb{R}_{\geq 0} \), we have:
\[ e^{-\lambda L_G} = 1 - \delta W S_G(\lambda) \] ,
where:
\[ S_G(\lambda) = v_G \int_0^\lambda dt e^{-t L_G} \in PV^{-1}(X) \] .
In particular, we have:
\[ \delta W e^{-\lambda L_G} = 0 \] .

**Proof.** For any \( x \in X \), the map \( \mathbb{R}_{\geq 0} \ni \lambda \mapsto e^{-\lambda L_G(x)} \in (\wedge T^*\lambda M) \) is smooth and gives the unique solution of the linear equation:
\[ \frac{d}{d\lambda} f(\lambda) = -L_G(f(x) f(\lambda)) \text{ with } f : \mathbb{R}_{\geq 0} \rightarrow (\wedge T^*\lambda M) \] 
which satisfies the initial condition \( f(0) = 1 \). Since \( \delta W L_G = 0 \), the function \( \lambda \mapsto [\delta W(e^{-\lambda L_G})](x) \) is the unique solution of (5.12) which vanishes at \( \lambda = 0 \); thus \( \delta W e^{-\lambda L_G} = 0 \). Since \( L_G = \delta W v_G \), this gives:
\[ \frac{d}{d\lambda} e^{-\lambda L_G} = -\delta W(v_G e^{-\lambda L_G}) \]
which implies the conclusion. \( \square \)

Proposition 5.9 implies that the \( \delta W \)-cohomology class of \( e^{-\lambda L_G} \) coincides with the unit of the algebra \( HPV(X, W) \). Let \( L_G \) denote the operator of left multiplication with the element \( L_G \) in the algebra \( PV(X) \).

**Definition 5.10** The bulk flow determined by the Kähler metric \( G \) is the semigroup \( (U_G(\lambda))_{\lambda \geq 0} \) generated by \( L_G \). Thus \( U_G(\lambda) \) is the even \( C^\infty(X) \)-linear endomorphism of \( PV(X) \) defined through:
\[ U_G(\lambda)(\omega) \] def. \[ e^{-\lambda L_G} \omega \] , \( \forall \omega \in PV(X) \) .
Notice that \( U_G(\lambda) \) has degree zero with respect to the canonical \( \mathbb{Z} \)-grading of \( PV(X) \) and that it preserves the subspace \( PV_c(X) \). We have \( U_G(0) = \text{id}_{PV(X)} \) and \( U_G(\lambda_1) U_G(\lambda_2) = U_G(\lambda_1 + \lambda_2) \) for all \( \lambda_1, \lambda_2 \geq 0 \).

**Proposition 5.11** For any \( \lambda \in [0, +\infty) \), the endomorphism \( U_G(\lambda) \) is homotopy equivalent with \( \text{id}_{PV(X)} \). In particular, we have:
\[ \delta W \circ U_G(\lambda) = U_G(\lambda) \circ \delta W \] .
Thus \( U_G(\lambda) \) preserves the subspaces \( \ker(\delta W) \) and \( \text{im}(\delta W) \) and it induces the identity endomorphism of \( HPV(X, W) \) on the cohomology of \( \delta_W \).
Proof. Relation (5.10) implies:
\[ U_G(\lambda) = \text{id}_{PV(X)} - [\delta_W, \hat{S}_G(\lambda)] \ , \]
where \( \hat{S}_G(\lambda) : PV(X) \to PV(X) \) is the \( C^\infty(X) \)-linear operator of left multiplication in the algebra \( PV(X) \) with the element \( S_G(\lambda) \in PV^{-1}(X) \) defined in (5.9):
\[ \hat{S}_G(\lambda)(\omega) \overset{\text{def}}{=} S_G(\lambda)\omega \ , \ \forall \omega \in PV(X) \ . \]

Notice that \( \hat{S}_G(\lambda) \) is an operator of degree \(-1\) with respect to the canonical \( \mathbb{Z} \)-grading of \( PV(X) \). The remaining statements are now obvious. \( \square \)

5.5. Tempered traces on \( PV_{c}(X) \).

Definition 5.12 For any \( \lambda \geq 0 \), the \( \lambda \)-tempered trace induced by \( G \) and \( \Omega \) on \( PV_{c}(X) \) is the \( \mathbb{C} \)-linear map \( \text{Tr}^{(\lambda)} := \text{Tr}^{(\lambda),\Omega,G} : PV_{c}(X) \to C^\infty(X) \) defined through:
\[ \text{Tr}^{(\lambda),\Omega,G} \overset{\text{def}}{=} \text{Tr}_{B}^{\Omega} \circ U_G(\lambda) \ . \]

This map has degree zero with respect to the canonical \( \mathbb{Z} \)-grading of \( PV_{c}(X) \).

We have \( \text{Tr}^{(0),\Omega,G} = \text{Tr}_{B}^{\Omega} \) (which is independent of \( G \)) and:
\[ \text{Tr}^{(\lambda),\Omega,G}(\omega) = \text{Tr}_{B}^{\Omega}(e^{-\lambda L_{G}}\omega) \]
for all \( \omega \in PV_{c}(X) \). In the following, we simplify notation by omitting to indicate the dependence of the traces on \( \Omega \) and \( G \).

Proposition 5.13 For any \( \omega \in PV_{c}^{i,j}(X) \), we have:
\[ \text{Tr}^{(\lambda)}(\omega) = 0 \text{ unless } i + j = 0 \]
and:
\[ \text{Tr}^{(\lambda)}(\omega) = \frac{(-1)^{d-j}}{(d-j)!} \int_X \Omega \wedge ([H_{G}]^{d-j}\omega) e^{-\lambda ||\partial W||^2_{G}} , \text{ when } \omega \in PV_{c}^{-i,j}(X) \ . \quad (5.14) \]

Proof. For any \( \omega \in PV_{c}^{i,j}(X) \), we have:
\[ \Omega_{\omega}(e^{-\lambda L_{G}}\omega) = e^{-\lambda ||\partial W||^2_{G}} \sum_{p=0}^{d} \frac{(-1)^{p}}{p!} \Omega_{\omega}([H_{G}]^{p}\omega) \ , \]
with \( [H_{G}]^{p}\omega \in PV^{i-p,j+p}(X) \) and \( \Omega_{\omega}([H_{G}]^{p}\omega) \in PV^{d+i-p,j+p}(X) \). Thus \( \Omega \wedge (\Omega_{\omega}([H_{G}]^{p}\omega)) \in \Omega^{2d+i-p,j+p}(X) \) and \( \int_X \Omega \wedge (\Omega_{\omega}([H_{G}]^{p}\omega)) \) vanishes unless \( 2d + i - p = j + p = d \), which requires \( p = d - j \) and \( i + j = 0 \). Thus \( \text{Tr}^{(\lambda)}(\omega) \) vanishes unless \( i + j = 0 \), in which case (5.14) holds. \( \square \)

Proposition 5.14 Let \( \omega \in PV_{c}(X) \). Then the following statements hold for any \( \lambda \geq 0 \):

1. If \( \omega = \delta_W \eta \) for some \( \eta \in PV_{c}(X) \), then \( \text{Tr}^{(\lambda)}(\omega) = 0 \).
2. If \( \delta_W \omega = 0 \), then \( \text{Tr}^{(\lambda)}(\omega) \) does not depend on \( \lambda \) or \( G \) and coincides with \( \text{Tr}_B(\omega) \):
\[
\text{Tr}^{(\lambda)}(\omega) = \text{Tr}^{(0)}(\omega) = \text{Tr}_B(\omega) .
\]

In particular, the map induced by \( \text{Tr}^{(\lambda)}(\omega) \) on \( \text{HPV}_c(X,W) \) coincides with \( \text{Tr}_c \).

Proof. Follows immediately from (6.1) upon using relation (4.6).

1. Since \( \delta_W(e^{-\lambda L_G}) = 0 \) (see (5.11)), we have:
\[
\text{Tr}^{(\lambda)}(\delta_W \eta) = \text{Tr}_B \left[ e^{-\lambda L_G} \delta_W \eta \right] = \text{Tr}_B \left[ \delta_W (e^{-\lambda L_G} \eta) \right] = 0 ,
\] where in the last equality we used Proposition 5.4.

2. Notice that \( \text{Tr}^{(\lambda)}(\omega) \) is differentiable with respect to \( \lambda \). Since \( L_G = \delta_W v_G \) and \( \delta_W \omega = 0 \), we have:
\[
\frac{d}{d\lambda} \text{Tr}^{(\lambda)}(\omega) = \frac{d}{d\lambda} \text{Tr}_B \left[ e^{-\lambda L_G} \omega \right] = -\text{Tr}_B \left[ e^{-\lambda L_G} \delta_W(v_G) \omega \right] = -\text{Tr}^{(\lambda)}[\delta_W(v_G) \omega] ,
\] which vanishes by point 1. This shows that \( \text{Tr}^{(\lambda)}(\omega) \) does not depend on \( \lambda \) or \( G \) and hence (5.15) holds.

\( \square \)

6. Off-shell boundary traces and boundary flows

6.1. The canonical off-shell boundary traces induced by \( \Omega \) on \( \text{DF}_c(X,W) \). Let \( a = (E,D) \) be a holomorphic factorization of \( W \). Let \( \delta_a := \delta_{a,a} \) and \( \vartheta_a := \vartheta_{a,a} \) denote the twisted and defect differentials on \( \text{End}_{\text{DF}(X,W)}(a) \). Let \( \overline{\delta}_a := \overline{\delta}_{a,a} = \overline{\delta}_{\text{End}(E)} \) denote the Dolbeault operator of \( \text{End}(E) \). We have:
\[
\delta_a = \overline{\delta}_a + \vartheta_a , \quad \vartheta_a = [D, \cdot] ,
\] where \([\cdot, \cdot]\) denotes the graded commutator.

Definition 6.1 The canonical off-shell boundary trace induced by \( \Omega \) on \( \text{End}_{\text{DF}_c(X,W)}(a) \) is the \( \mathbb{C} \)-linear map \( \text{tr}^{B}_{a} := \text{tr}^{B,\Omega}_{a} : \text{End}_{\text{DF}_c(X,W)}(a) \to \mathbb{C} \) defined through:
\[
\text{tr}^{B}_{a} \Omega(\alpha) = \int_X \Omega \wedge \text{str}(\alpha) = \int_\Omega \text{str}(\alpha) ,
\]
for all \( \alpha \in \text{End}_{\text{DF}_c(X,W)}(a) = \mathcal{A}_c(X,\text{End}(E)) \), where \( \text{str} \) denotes the extended supertrace of Subsection 4.4.

We have:
\[
\text{tr}^{B,\Omega}_{a}(\alpha) = 0 \text{ unless } \alpha \in \mathcal{A}^{d}(X,\text{End}^{d}(E)) \subset \mathcal{A}_c(X,\text{End}(E))^\mu ,
\] where \( \mu = \hat{d} \) is the signature of \( (X,W) \). In the following, we simplify notation by omitting to indicate dependence of the traces on \( \Omega \).

Proposition 6.2 For any holomorphic factorizations \( a_1 \) and \( a_2 \) of \( W \), we have:
\[
\text{tr}^{B}_{a_2} (\alpha \beta) = (-1)^{\deg_a \deg_B} \text{tr}^{B}_{a_1} (\beta \alpha) ,
\]
when \( \alpha \in \text{Hom}_{\text{DF}_c(X,W)}(a_1, a_2) \) and \( \beta \in \text{Hom}_{\text{DF}_c(X,W)}(a_2, a_1) \) have pure total \( \mathbb{Z}_2 \)-degree.

Proof. Follows immediately from (6.1) upon using relation (4.6). \( \square \)

Remark 6.1. Relation (6.2) is an off-shell (cochain level) version of the cyclicity condition (2.1).
6.2. Cohomological boundary traces.

**Proposition 6.3** For any $\alpha \in \text{End}_{\text{DF},c}(X,W)(a)$, we have:

$$\text{tr}_a^B(\delta_a \alpha) = \text{tr}_a^B(\overline{\delta}_a \alpha) = \text{tr}_a^B(\mathcal{O}_a \alpha) = 0.$$  

In particular, $\text{tr}_a^B$ descends to $\text{End}_{\text{HDF},c}(X,W)(a) = H(A_c(X, \text{End}(E)), \delta_a)$.

**Proof.** We have:

$$\text{tr}_a^B(\partial_a \alpha) = \int_X \Omega \wedge \text{str}(\partial_a \alpha) = \int_X \Omega \wedge [\partial \text{str}(\alpha)] = (-1)^d \int_X [\Omega \wedge \text{str}(\alpha)] = 0,$$

where we used (4.7) and (4.8) and the fact that $\Omega \wedge \text{str}(\alpha)$ has type $(d,*)$. The relation $\text{tr}_a^B(d_a \alpha) = 0$ follows from (4.8), while the relation $\text{tr}_a^B(\delta_a \alpha) = 0$ follows by combining these two properties.

**Definition 6.4** The cohomological boundary trace induced by $\Omega$ on $\text{End}_{\text{HDF},c}(X,W)(a)$ is the $\mathbb{C}$-linear map $\text{tr}_a^c := \text{tr}_a^c,\Omega : \text{End}_{\text{HDF},c}(X,W)(a) \to \mathbb{C}$ induced by $\text{tr}_a^B,\Omega$ on $\text{End}_{\text{HDF},c}(X,W)(a)$.

When the critical set $Z_W$ is compact, Proposition 4.11 implies that the inclusion induces a strictly surjective and fully-faithful functor $j_* : \text{HDF}_c(X,W) \to \text{HDF}(X,W)$ (moreover, $\text{HDF}(X,W)$ is Hom-finite over $\mathbb{C}$ by Proposition 4.9). This allows us to transport $\text{tr}_a^c$ to a trace on $\text{End}_{\text{HDF}(X,W)}(a)$:

**Definition 6.5** Assume that the critical set $Z_W$ is compact. Then the cohomological boundary trace induced by $\Omega$ on $\text{End}_{\text{HDF}(X,W)}(a)$ is the $\mathbb{C}$-linear map $\text{tr}_a^c \overset{\text{def}}{=} \text{tr}_a^c,\Omega : \text{End}_{\text{HDF}(X,W)}(a) \to \mathbb{C}$, where $j_* : \text{End}_{\text{HDF}_c(X,W)}(a) \to \text{End}_{\text{HDF}(X,W)}(a)$ is the linear isomorphism induced by the inclusion functor.

Let $\text{tr}_a^c \overset{\text{def}}{=} (\text{tr}_a^c)_{a \in \text{ObHDF}_c(X,W)}$ and $\text{tr} \overset{\text{def}}{=} (\text{tr}_a)_{a \in \text{ObHDF}(X,W)}$. When the critical set $Z_W$ is compact, Proposition 6.2 and the remarks above imply that $(\text{HDF}_c(X,W), \text{tr}^c)$ and $(\text{HDF}(X,W), \text{tr})$ are equivalent pre-Calabi-Yau supercategories.

6.3. Hermitian holomorphic factorizations. Let $E = E^0 \oplus E^1$ be a holomorphic vector superbundle on $X$.

**Definition 6.6** A Hermitian metric $h$ on $E$ is called admissible if the sub-bundles $E^0$ and $E^1$ of $E$ are $h$-orthogonal:

$$h|_{E^0 \times E^1} = 0.$$ 

**Definition 6.7** A Hermitian holomorphic factorization of $W$ is a triplet $a = (E, h, D)$, where $a = (E, D)$ is a holomorphic factorization of $W$ and $h$ is an admissible Hermitian metric on $E$. 

Let us fix a Hermitian holomorphic factorization \( \mathbf{a} = (E, h, D) \) of \( W \) with underlying holomorphic factorization \( a = (E, D) \). Let \( \nabla := \nabla_a \) denote the Chern connection of the Hermitian holomorphic vector bundle \((E, h)\). This is the unique \( h \)-compatible \( \mathbb{C} \)-linear connection on \( E \) which satisfies:
\[
\nabla_{a}^{0,1} = \partial E|_{\Gamma_{\infty}(X, E)} ,
\]
where \( \partial E \) is the Dolbeault operator of \( E \). Let \( \Omega(X, E) := \Omega(X) \otimes_{C^\infty(X)} \Gamma_{\infty}(X, E) \) and \( \partial E^{h} : \Omega(X, E) \to \Omega(X, E) \) be the unique \( \mathbb{C} \)-linear operator which satisfies the Leibniz rule:
\[
\partial E^{h}(\rho \otimes s) = (\partial \rho) \otimes s + (-1)^{k} \rho \wedge \nabla_{a}^{1,0}(s) \quad (6.3)
\]
for all \( \rho \in \Omega^{k}(X) \) and all \( s \in \Gamma_{\infty}(X, E) \). We have \( \partial^{h}_{E}(\Omega^{i+j}(X, E)) \subset \Omega^{i+1+j}(X, E) \) for all \( i, j \) and:
\[
\partial E^{h}|_{\Gamma_{\infty}(X, E)} = \nabla_{a}^{1,0} .
\]

Let \( F_{a} \in \Omega^{1,1}(X, \text{End}^{\delta}(E)) = A^{1}(X, T^{*} X \otimes \text{End}^{\delta}(E)) \) denote the curvature of \( \nabla_{a} \). Since \( F_{a} \) has type \( (1, 1) \), we have \((\nabla_{a}^{1,0})^{2} = (\nabla_{a}^{0,1})^{2} = 0 \) and \( \nabla_{a}^{1,0} \nabla_{a}^{1,0} + \nabla_{a}^{0,1} \nabla_{a}^{0,1} = F_{a} \). These relations imply:
\[
(\partial E^{h})^{2} = \overline{\partial E}^{2} = 0 , \quad \partial E^{h} \overline{\partial E} + \overline{\partial E} \partial E^{h} = F_{a} ,
\]
where \( \overline{F}_{a} \) denotes the operator of left multiplication in the associative algebra \( \Omega(X, \text{End}(E)) \).

Let \( \partial_\mathbf{a} : \Omega(X, \text{End}(E)) \to \Omega(X, \text{End}(E)) \) denote the differential induced by \( \partial E^{h} \) on the graded associative algebra \( \Omega(X, \text{End}(E)) \). We have:
\[
\partial_{\mathbf{a}}^{2} = 0 \quad , \quad \partial_{\mathbf{a}} \overline{\partial}_{\mathbf{a}} + \overline{\partial}_{\mathbf{a}} \partial_{\mathbf{a}} = [F, \cdot] \quad (6.4)
\]
where \( \overline{\partial}_{\mathbf{a}} = \overline{\partial}_{\text{End}(E)} \).

**Definition 6.8** The boundary flow generator of \( \mathbf{a} = (E, h, D) \) determined by the Kähler metric \( G \) is defined through:
\[
L^{G}_{\mathbf{a}} := ||\partial W||^{2}_{G} \text{id}_{E} + H_{G, \mathbf{a}}(\partial_{\mathbf{a}} D + F) \in A^{0}(X, \text{End}^{\delta}(E)) \oplus A^{1}(X, \text{End}^{\delta}(E)) \oplus A^{2}(X, \text{End}^{\delta}(E)) .
\]

**Proposition 6.9** We have:
\[
L^{G}_{\mathbf{a}} = \partial_{\mathbf{a}}^{*} v^{G}_{\mathbf{a}} ,
\]
where:
\[
v^{G}_{\mathbf{a}} := \text{grad}_{G} \overline{\partial}_{\mathbf{a}} + F \in A^{0}(X, \text{End}^{\delta}(E)) \oplus A^{1}(X, \text{End}^{\delta}(E)) .
\]

**Proof.** The decomposition \( \partial_{\mathbf{a}} = \overline{\partial}_{\mathbf{a}} + \partial_{\mathbf{a}} \) gives:
\[
\partial_{\mathbf{a}}(v^{G}_{\mathbf{a}}) = \overline{\partial}_{\mathbf{a}}(v^{G}_{\mathbf{a}}) + \partial_{\mathbf{a}}(v^{G}_{\mathbf{a}}) .
\]

Since \( H_{G} = \overline{\partial}_{TX}(\text{grad}_{G} \overline{\partial}) \), we have:
\[
\overline{\partial}_{\mathbf{a}}(v^{G}_{\mathbf{a}}) = H_{G, \mathbf{a}}(\partial_{\mathbf{a}} D + F) - \text{grad}_{G} \overline{\partial}[F, D] \quad (6.5),
\]
where we used the second relation in (6.4) and the fact that \( \overline{\partial}_{\mathbf{a}} D = \overline{\partial}_{\mathbf{a}} F = 0 \). On the other hand, we have:
\[
\partial_{\mathbf{a}}(v^{G}_{\mathbf{a}}) = -\text{grad}_{G} \overline{\partial}([D, \partial_{\mathbf{a}} D] + [D, F]) = ||\partial W||^{2}_{G} \text{id}_{E} + \text{grad}_{G} \overline{\partial}[F, D] \quad (6.6),
\]
where we used the relation \( [D, \partial_{\mathbf{a}} D] = D \partial_{\mathbf{a}} D - (\partial_{\mathbf{a}} D) D = -\partial W \text{id}_{E} \) (which follows by applying \( \partial_{\mathbf{a}} \) to the identity \( D^{2} = W \text{id}_{E} \)) and the relation \( \text{grad}_{G} \overline{\partial} \partial W = (\partial W)(\text{grad}_{G} \overline{\partial}) = ||\partial W||^{2}_{G} \) (see (5.7)). Adding (6.5) and (6.6) gives the conclusion. \( \square \)
Notice that $L^G_a \in \text{End}^0_{DF(X,W)}(a)$ and $e^G_a \in \text{End}^1_{DF(X,W)}(a)$. Also notice that $H_{G,J}(\partial_a D + F)$ is nilpotent, which allows us to define its exponential. For any $\lambda \geq 0$, we have:

$$e^{-\lambda H_{G,J}(\partial_a D + F)} = \sum_{k=0}^{d} \frac{(-\lambda)^k}{k!} [H_{G,J}(\partial_a D + F)]^k \in \text{End}^0_{DF(X,W)}(a),$$

where the series reduces to the first $d+1$ terms because $H_{G,J}(\partial_a D + F)$ belongs to $\mathcal{A}^1(X, \text{End}^1(E)) \oplus \mathcal{A}^2(X, \text{End}^0(E))$ and hence $[H_{G,J}(\partial_a D + F)]^k$ vanishes for $k > d$. Define:

$$e^{-\lambda L^G_a} \overset{\text{def.}}{=} e^{-\lambda \|\partial W\|^2_G} e^{-\lambda H_{G,J}(\partial_a D + F)} \in \text{End}^0_{DF(X,W)}(a).$$

Proposition 6.10 For any $\lambda \geq 0$, we have:

$$\lambda L^G_a \overset{\text{def.}}{=} \lambda \|\partial W\|^2_G, \quad e^{-\lambda L^G_a} \in \text{End}^0_{DF(X,W)}(a). \quad (6.7)$$

where:

$$S^G_a(\lambda) \overset{\text{def.}}{=} \int_{0}^{\lambda} d\tau e^{-\lambda L^G_a} \in \text{End}^1_{DF(X,W)}(a). \quad (6.8)$$

In particular, we have:

$$\delta_a(e^{-\lambda L^G_a}) = 0. \quad (6.9)$$

Proof. The proof is almost identical to that of Proposition 5.9. $\square$

Let $\hat{L}^G_a$ denote the operator of left multiplication with the element $L^G_a$ in the algebra $\text{End}^0_{DF(X,W)}(a) = \mathcal{A}(X, \text{End}(E))$.

Definition 6.11 The boundary flow of $a = (E, h, D)$ determined by the Kähler metric $G$ is the semigroup $(U^G_a(\lambda))_{\lambda \geq 0}$ generated by $L^G_a$. Thus $U^G_a(\lambda)$ is the even $C^\infty(X)$-linear endomorphism of $\text{End}^\lambda_{DF(X,W)}(a)$ defined through:

$$U^G_a(\lambda)(\alpha) \overset{\text{def.}}{=} e^{-\lambda L^G_a} \alpha, \quad \forall \alpha \in \text{End}^\lambda_{DF(X,W)}(a).$$

Notice that $U^G_a(\lambda)$ is even with respect to the canonical $\mathbb{Z}_2$-grading and that it preserves the subspace $\text{End}^\lambda_{DF(X,W)}(a) = \mathcal{A}^\lambda(X, \text{End}(E))$. We have $U^G_a(0) = \text{id}_{\text{End}^\lambda_{DF(X,W)}(a)}$ and $U^G_a(\lambda_1)U^G_a(\lambda_2) = U^G_a(\lambda_1 + \lambda_2)$ for all $\lambda_1, \lambda_2 \geq 0$.

Proposition 6.12 For any $\lambda \geq 0$, the endomorphism $U^G_a(\lambda)$ is homotopy equivalent with $\text{id}_{\text{End}^\lambda_{DF(X,W)}(a)}$. In particular, we have:

$$\delta_a \circ U^G_a(\lambda) = U^G_a(\lambda) \circ \delta_a. \quad (6.10)$$

Hence $U^G_a(\lambda)$ preserves the subspaces $\ker(\delta_a)$ and $\text{im}(\delta_a)$ and it induces the identity endomorphism of $\text{End}^\lambda_{DF(X,W)}(a)$ on the cohomology of $\delta_a$.

Proof. Relation (6.7) implies:

$$U^G_a(\lambda) = \text{id}_{\text{End}^\lambda_{DF(X,W)}(a)} - [\delta_a, S^G_a(\lambda)],$$

where $\hat{S}^G_a(\lambda)$ is the $C^\infty(X)$-linear operator of left multiplication with the element $S^G_a(\lambda)$ defined in (6.8) in the algebra $\text{End}^\lambda_{DF(X,W)}(a)$, which is odd with respect to the canonical $\mathbb{Z}_2$-grading. This implies all statements in the proposition. $\square$
6.5. Tempered traces on $DF_c(X, W)$.

**Definition 6.13** Let $\lambda \in \mathbb{R}_{\geq 0}$. The $\lambda$-tempered trace of $a = (E, h, D)$ induced by $\Omega$ and $G$ is the $\mathbb{C}$-linear map $\text{tr}_a^{(\lambda)} := \text{tr}_{a}^{(\lambda),\Omega,G} : \text{End}_{DF_c(X, W)}(a) \to \mathbb{C}$ defined through:

$$\text{tr}_a^{(\lambda),\Omega,G} \text{ def. } = \text{tr}_a^{B,\Omega} \circ U_a^G(\lambda) \, .$$

We have $\text{tr}_a^{(0),\Omega,G} = \text{tr}_a^{B,\Omega}$ (which is independent of $G$ and $h$) and:

$$\text{tr}_a^{(\lambda),\Omega,G}(a) = \text{tr}_a^{B,\Omega}(e^{-\lambda L_a^G} a) \, , \quad \forall \alpha \in \text{End}_{DF_c(X, W)}(a) \, . \quad (6.11)$$

In the following, we simplify notation by omitting to indicate the dependence of the traces on $\Omega$ and $G$.

**Proposition 6.14** Let $\alpha \in \text{End}_{DF_c(X, W)}(a)$. Then the following statements hold for any $\lambda \geq 0$:

1. If $\alpha = \delta_a \beta$ for some $\beta \in \text{End}_{DF_c(X, W)}(a)$, then $\text{tr}_a^{(\lambda)}(\alpha) = 0$.
2. If $\delta_a \alpha = 0$, then $\text{tr}_a^{(\lambda)}(\alpha)$ does not depend on $\lambda$ or on the metrics $G$ and $h$:

$$\text{tr}_a^{(\lambda)}(\alpha) = \text{tr}_a^{(0)}(\alpha) = \text{tr}_a^{B}(\alpha) \, . \quad (6.12)$$

In particular, the map induced by $\text{tr}_a^{(\lambda)}$ on $\text{End}_{DF_c(X, W)}(a)$ coincides with $\text{tr}_a^c$.

**Proof.**

1. Since $e^{-\lambda L_a^G}$ is even and $\delta_a$-closed, we have:

$$\text{tr}_a^{(\lambda)}(\delta_a \beta) = \text{tr}_a^{B} \left[ e^{-\lambda L_a^G} \delta_a \beta \right] = \text{tr}_a^{B} \left[ \delta_a (e^{-\lambda L_a^G} \beta) \right] = 0 \, , \quad (6.13)$$

where in the last equality we used Proposition 6.3.

2. Notice that $\text{tr}_a^{(\lambda)}$ is differentiable with respect to $\lambda$. Since $L_a^G = \delta_a v_a^G$ while $\alpha$ is $\delta_a$-closed, we have:

$$\frac{d}{d\lambda} \text{tr}_a^{(\lambda)}(\alpha) = -\text{tr}_a^{(\lambda)}[\delta_a (\text{tr}_a^{(\lambda)}(\alpha))] = -\text{tr}_a^{(\lambda)}[\delta_a (v_a^G \alpha)] \, ,$$

which vanishes by point 1. This implies (6.12).

\square

**Proposition 6.15** Let $a_1 = (E_1, h_1, D_1)$ and $a_2 = (E_2, h_2, D_2)$ be two Hermitian holomorphic factorizations of $W$ with underlying holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$. Let $\alpha \in \text{Hom}_{DF_c(X, W)}(a_1, a_2)$ and $\beta \in \text{Hom}_{DF_c(X, W)}(a_2, a_1)$ have pure total $\mathbb{Z}_2$-degree and satisfy $\delta_{a_1,a_2} \alpha = \delta_{a_2,a_1} \beta = 0$. Then:

$$\text{tr}_{a_2}^{(\lambda),G}(\alpha \beta) = (-1)^{\deg \alpha \deg \beta} \text{tr}_{a_1}^{(\lambda),G}(\beta \alpha) \, .$$

**Proof.** Since $\delta_{a_1,a_2} \alpha = \delta_{a_2,a_1} \beta = 0$, we have $\delta_{a_2}(\alpha \beta) = 0$ and $\delta_{a_1}(\beta \alpha) = 0$. Thus $\text{tr}_{a_2}^{(\lambda),G}(\alpha \beta) = \text{tr}_{a_2}^{B}(\alpha \beta)$ and $\text{tr}_{a_1}^{(\lambda),G}(\beta \alpha) = \text{tr}_{a_1}^{B}(\beta \alpha)$ by Proposition 6.14. This implies the conclusion upon using Proposition 6.2. \hspace{1cm} \square
7. The disk algebra

7.1. The dg-algebra $PV(X, \text{End}(E))$. Let $a = (E, D)$ be a holomorphic factorization of $W$ and set $\delta_a := \delta_{a,a} = \delta_{\text{End}(E)}$, $\delta_a := \delta_{a,a} = [D, \cdot]$ and $\delta_a := \delta_{a,a} = \delta_a + \delta_a$. Consider the $\mathbb{Z}_2$-graded unital associative $C^\infty(X)$-algebra:

$$PV(X, \text{End}(E)) \overset{\text{def}}{=} \mathcal{A}(X, \wedge TX \otimes \text{End}(E)) \simeq PV(X) \otimes_{C^\infty(X)} \Gamma_\infty(X, \text{End}(E)),$$

where $\otimes_{C^\infty(X)}$ denotes the graded tensor product and $PV(X)$ in the right hand side is endowed with the canonical $\mathbb{Z}_2$-grading. The isomorphism of $\mathbb{Z}_2$-graded $C^\infty(X)$-algebras $\mathcal{A}(X, \text{End}(E)) \simeq \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X, \text{End}(E))$ implies:

$$PV(X, \text{End}(E)) \simeq PV(X) \hat{\otimes}_{\mathcal{A}(X)} \mathcal{A}(X, \text{End}(E)) = PV(X) \hat{\otimes}_{\mathcal{A}(X)} \text{End}_{DF(X,W)}(a).$$

The unit of $PV(X, \text{End}(E))$ is the identity endomorphism $id_E$ of $E$.

**Definition 7.1** The total twisted differential $\Delta_a$ on $PV(X, \text{End}(E))$ is the odd $O(X)$-linear differential:

$$\Delta_a \overset{\text{def}}{=} \delta_W \hat{\otimes}_{\mathcal{A}(X)} id_{\mathcal{A}(X, \text{End}(E))} + id_{PV(X)} \hat{\otimes}_{\mathcal{A}(X)} \delta_a$$

induced on $PV(X, \text{End}(E))$ by the differentials $\delta_W$ of $PV(X)$ and $\delta_a$ of $\text{End}_{DF(X,W)}(a)$.

It is easy to see that $\Delta_a$ is an odd derivation of $PV(X, \text{End}(E))$ which squares to zero. To simplify notation, we write $\delta_W$ instead of $\delta_W \hat{\otimes}_{\mathcal{A}(X)} id_{\mathcal{A}(X, \text{End}(E))}$ and $\delta_a$ instead of $id_{PV(X)} \hat{\otimes}_{\mathcal{A}(X)} \delta_a$. Then $(PV(X, \text{End}(E)), \delta_W, \delta_a)$ is a $\mathbb{Z} \times \mathbb{Z}_2$-graded bicomplex when endowed with the $\mathbb{Z}$-grading induced by the canonical $\mathbb{Z}$-grading of $PV(X)$ and with the $\mathbb{Z}_2$-grading induced by the bundle grading of $E$; the $\mathbb{Z}_2$-grading of $PV(X, \text{End}(E))$ is the total $\mathbb{Z}_2$-grading induced by this bigrading. Moreover, $\Delta_a$ is the total differential of this bicomplex.

**Definition 7.2** The off-shell disk algebra of the holomorphic factorization $a = (E, D)$ is the $O(X)$-linear $\mathbb{Z}_2$-graded unital dg-algebra $(PV(X, \text{End}(E)), \Delta_a)$. The cohomological disk algebra of $a$ is the $\mathbb{Z}_2$-graded $O(X)$-linear algebra $HPV(X, a)$ defined as the total cohomology algebra of the off-shell disk algebra:

$$HPV(X, a) \overset{\text{def}}{=} H(PV(X, \text{End}(E)), \Delta_a).$$

**Remark 7.1.** Let $\delta_{\wedge TX \otimes \text{End}(E)} : PV(X, \text{End}(E)) \to PV(X, \text{End}(E))$ be the Dolbeault operator of the holomorphic vector bundle $\wedge TX \otimes \text{End}(E)$. Then:

$$\delta_{\wedge TX \otimes \text{End}(E)} = \delta_{\wedge TX} \hat{\otimes}_{\mathcal{A}(X)} id_{\mathcal{A}(X, \text{End}(E))} + id_{PV(X)} \hat{\otimes}_{\mathcal{A}(X)} \delta_{\text{End}(E)} \quad (7.1)$$

and hence:

$$\Delta_a = \delta_{\wedge TX \otimes \text{End}(E)} + \iota_W \hat{\otimes}_{\mathcal{A}(X)} id_{\mathcal{A}(X, \text{End}(E))} + id_{PV(X)} \hat{\otimes}_{\mathcal{A}(X)} \delta_a \quad (7.2)$$

For simplicity, we denote $\delta_{\wedge TX} \hat{\otimes}_{\mathcal{A}(X)} id_{\mathcal{A}(X, \text{End}(E))}$ by $\delta_\theta$ and $id_{PV(X)} \hat{\otimes}_{\mathcal{A}(X)} \delta_{\text{End}(E)}$ by $\delta_a$. Similarly, we denote $\iota_W \hat{\otimes}_{\mathcal{A}(X)} id_{\mathcal{A}(X, \text{End}(E))}$ and $id_{PV(X)} \hat{\otimes}_{\mathcal{A}(X)} \delta_a$ by $\iota_W$ and $\delta_a$. With these notations, combining (7.1) and (7.2) gives a decomposition of $\Delta_a$ as a sum of four odd differentials which mutually anticommute:

$$\Delta_a = \delta_\theta + \delta_a + \iota_W + \delta_a.$$
7.2. The disk extended supertrace. Notice that PV(\(X, \text{End}(E)\)) is a left module over the superalgebra PV(\(X\)) (when the latter is endowed with the canonical \(\mathbb{Z}_2\)-grading) as well as a right module over the superalgebra \(\Gamma_\infty(X, \text{End}(E))\). We extend the fiberwise supertrace \(\text{str} : \Gamma_\infty(X, \text{End}(E)) \to C^\infty(X)\) to a left PV(\(X\))-linear map denoted by the same symbol and called the disk extended supertrace:

\[
\text{str} \overset{\text{def.}}{=} \text{id}_{PV(X)} \otimes_{C^\infty(X)} \text{str} : PV(X, \text{End}(E)) \to PV(X) .
\]

For any \(\omega \in PV(X)\) and any \(s \in \Gamma_\infty(X, \text{End}(E))\), we have:

\[
\text{str}(\omega \otimes_{C^\infty(X)} s) = \omega \text{str}(s)
\]

and the disk extended supertrace satisfies:

\[
\text{str}(\delta_a \mathfrak{z}) = \delta_W \text{str}(\mathfrak{z}) , \quad \forall \mathfrak{z} \in PV(X, \text{End}(E)) .
\]

The disk extended supertrace is an even map of \(\mathbb{Z}_2\)-graded complexes from \((PV(X, \text{End}(E)), \Delta_a)\) to \((PV(X), \delta_W)\), when the latter is endowed with the canonical \(\mathbb{Z}_2\)-grading:

\[
\text{str}(\Delta_a \mathfrak{z}) = \delta_W \text{str}(\mathfrak{z}) , \quad \forall \mathfrak{z} \in PV(X, \text{End}(E))
\]

and, with the simplified notations introduced above, it satisfies:

\[
\text{str}(\delta_W \mathfrak{z}) = \delta_W \text{str}(\mathfrak{z}) \quad \text{and} \quad \text{str}(\theta_a \mathfrak{z}) = 0 , \quad \forall \mathfrak{z} \in PV(X, \text{End}(E)) .
\]

7.3. The extended reduced contraction induced by a holomorphic volume form. Given a holomorphic volume form \(\Omega\) on \(X\), we extend the reduced contraction of Definition 3.5 to a right \(\Gamma_\infty(X, \text{End}(E))\)-linear map of \(\mathbb{Z}_2\)-degree \(\mu\):

\[
\Omega_{\cdot,0} \overset{\text{def.}}{=} (\Omega_{\cdot,0} \otimes_{C^\infty(X)} \text{id}_{\Gamma_\infty(X, \text{End}(E))}) : PV(X, \text{End}(E)) \to \text{End}_{DF(X,W)}(a) = \mathcal{A}(X, \text{End}(E)) .
\]

For any \(\omega \in PV(X)\) and \(s \in \Gamma_\infty(X, \text{End}(E))\), we have:

\[
\Omega_{\cdot,0}(\omega \otimes_{C^\infty(X)} s) = (\Omega_{\cdot,0} \omega) \otimes_{C^\infty(X)} s .
\]

With these definitions, the following relation is satisfied (see the commutative diagram (7.9)):

\[
\text{str}(\Omega_{\cdot,0} \mathfrak{z}) = \Omega_{\cdot,0} \text{str}(\mathfrak{z}) , \quad \forall \mathfrak{z} \in PV(X, \text{End}(E)) .
\]

Lemma 7.3 For any \(\mathfrak{z} \in PV(X, \text{End}(E))\), we have:

\[
\delta_a(\Omega_{\cdot,0} \mathfrak{z}) = (-1)^d \Omega_{\cdot,0}(\Delta_a \mathfrak{z}) .
\]

Thus \(\Omega_{\cdot,0}\) is a map of complexes of \(\mathbb{Z}_2\)-degree \(\mu = \hat{d}\) from \((PV(X, \text{End}(E)), \Delta_a)\) to \((\text{End}_{DF(X,W)}(a), \delta_a)\).

Proof. It is enough to prove the restriction of this relation to an open subset \(U \subset X\) supporting complex coordinates \(z^1, \ldots, z^d\). Write \(\Omega = \varphi(z)dz^1 \wedge \cdots \wedge dz^d\), where \(\varphi \in \mathcal{O}_X(U)\). Expand \(\mathfrak{z} = \sum_{k=0}^d \sum_{1 \leq i_1 < \cdots < i_k \leq d} \theta^{i_k} \cdots \theta^{i_1} \mathfrak{3}_{i_1 \cdots i_k}\), with coefficients \(\mathfrak{3}_{i_1 \cdots i_k} \in \mathcal{A}(U, \text{End}(E))\). Then

\[
\Omega_{\cdot,0} \mathfrak{z} = \varphi(z)\mathfrak{3}_{1 \cdots d} .
\]

where we used the relations \(dz^1 \partial_j = dz_j \partial_{z^1} = \delta_{ij}\). On the other hand, we have:

\[
\Delta_a \mathfrak{z} = \sum_{k=0}^d \sum_{1 \leq i_1 < \cdots < i_k \leq d} \left[ (-1)^k \theta^{i_k} \cdots \theta^{i_1} (\delta_a \mathfrak{3}_{i_1 \cdots i_k}) - i \sum_{s=1}^k (-1)^{k-s} (\partial_{z^s} W) \theta_{i_k} \cdots \theta_{i_s+1} \theta_{i_{s-1}} \cdots \theta_{i_1} \mathfrak{3}_{i_1 \cdots i_k} \right]
\]
where we noticed that $\Delta_a(\theta_i) = \iota_W(\theta_i) = -i\partial_i W$. The reduced contraction of $\Omega$ with every term in this sum vanishes for degree reasons with the single exception of the contraction with the term $(-1)^d \theta_d \cdots \theta_1 (\delta_\alpha \delta_1 \ldots \delta_d)$, which equals $(-1)^d \varphi(z) \delta_\alpha \delta_1 \ldots \delta_d$. Thus:

$$
\Omega_{\omega}(\Delta_a \delta) = (-1)^d \varphi(z) \delta_\alpha \delta_1 \ldots \delta_d \quad .
$$

(7.7)

Comparing (7.6) and (7.7) gives the conclusion. □

Consider the degree $\mu$ map of $\mathbb{Z}_2$-graded complexes $\lambda_\Omega : (\text{PV}(X, \text{End}(E)), \Delta_a) \to (\mathcal{A}(X), \overline{\partial})$ defined through:

$$
\lambda_\Omega \overset{\text{def}}{=} (\Omega_{\omega}) \circ \text{str} = \text{str} \circ (\Omega_{\omega}) \quad ,
$$

(7.8)

where we used relation (7.5). Combining everything, we have the following commutative diagram of $\mathbb{Z}_2$-graded complexes, where the $\mathbb{Z}_2$-degree of each map is indicated in square brackets:

$$
\begin{array}{ccc}
\text{(PV}(X, \text{End}(E)), \Delta_a) & \overset{\lambda_\Omega}{\longrightarrow} & (\text{PV}(X, W), \delta_W) \\
\text{(PV}(X, W), \delta_W) & \overset{\text{str}}{\longrightarrow} & (\text{PV}(X, \text{End}(E)), \delta_{\omega}) \\
(\mathcal{A}(X), \overline{\partial}) & \overset{\text{str}}{\longrightarrow} & (\text{End}_{\mathcal{D}(X)}(a), \delta_{\omega})
\end{array}
$$

(7.9)

7.4. The extended disk algebra. The $C^\infty(X)$-module $\text{PV}_e(X, \text{End}(E)) \overset{\text{def}}{=} \Omega(X, \wedge \mathcal{O} \otimes \text{End}(E))$ carries a natural structure of $C^\infty(X)$-superalgebra which makes it isomorphic with the graded tensor product:

$$
\text{PV}_e(X, \text{End}(E)) \simeq \Omega^{*,0}(X) \hat{\otimes} C^\infty(X) \text{PV}(X, \text{End}(E))
$$

, where $\Omega^{*,0}(X) = \Gamma_{\infty}(X, \wedge T^* X) = \bigoplus_{k=0}^d \Omega^{k,0}(X)$. We trivially extend $\Delta_a$ to an odd $\mathcal{O}(X)$-linear differential on this algebra which we denote by the same symbol:

$$
\Delta_a \overset{\text{def}}{=} \text{id}_{\Omega^{*,0}(X)} \hat{\otimes} C^\infty(X) \Delta_a \quad .
$$

For any $\rho \in \Omega^{i,0}(X)$ and any $\delta \in \text{PV}(X, \text{End}(E))$, we have:

$$
\Delta_a(\rho \otimes \delta) = (-1)^i \rho \otimes (\Delta_a \delta) \quad .
$$

**Definition 7.4** The extended disk algebra is the unital and $\mathbb{Z}_2$-graded $\mathcal{O}(X)$-linear $\text{dg}$-algebra $(\text{PV}_e(X, \text{End}(E)), \Delta_a)$. The cohomological extended disk algebra is the unital $\mathbb{Z}_2$-graded $\mathcal{O}(X)$-linear algebra $\text{HPV}_e(X, a)$ defined as the total cohomology algebra of the extended disk algebra:

$$
\text{HPV}_e(X, a) \overset{\text{def}}{=} \text{H}(\text{PV}_e(X, \text{End}(E)), \Delta_a) \quad .
$$

Let $\Omega$ be a holomorphic volume form on $X$. For any element $s \in \Omega^{1,*}(X, \wedge \mathcal{O} \otimes \text{End}(E))$, let $s^d \in \Omega^{d,*}(X, \wedge \mathcal{O} \otimes \text{End}(E))$ denote the $d$-th power of $s$ computed in this superalgebra. Since $\Omega$
is a nowhere-vanishing section of $\wedge^d T^*X$, there exists a unique element $\det_G s \in \text{PV}(X, \text{End}(E))$ such that:

$$s^d = (-1)^{\frac{d(d-1)}{2}} \Omega \otimes (\det_G s)$$.

This gives a map $\det_G : \Omega^{1,*}(X, \wedge TX \otimes \text{End}(E)) \to \text{PV}(X, \text{End}(E))$. For $\rho \in \Omega^{1,0}(X)$ and $\omega \in \text{PV}(X, \text{End}(E))$, we have:

$$\det_G (\rho \otimes \omega) = (\det_G \rho) \omega^d,$$

where $\det_G \rho \in C^\infty(X)$ is defined through the relation:

$$\rho^d = (\det_G \rho) \Omega \in \Omega^{d,0}(X).$$

In a complex coordinate chart $(U, (z^1, \ldots, z^d))$ such that $\Omega = \varphi(z) dz^1 \wedge \ldots \wedge dz^d$ (where $f \in O_X(U)$ does not vanish on $U$), we can expand $s = \sum_{i=1}^d dz^i \otimes s_i$ with $s_i \in \text{PV}(U, \text{End}(E))$ and we have:

$$\det_G s = \sum_{i=1}^d \frac{1}{\varphi(z)} \epsilon^{i_1 \ldots i_d} s_{i_1} \ldots s_{i_d}, \quad (7.10)$$

where $\epsilon^{i_1 \ldots i_d}$ is the Levi-Civita symbol and we use implicit summation over repeated indices, the multiplication being taken in the algebra $\text{PV}(U, \text{End}(E))$.

7.5. The twisted curvature. The natural isomorphism $\text{End}(TX) \simeq T^*X \otimes TX$ maps the identity endomorphism $\text{id}_{TX} \in \Gamma(X, \text{End}(TX))$ of the holomorphic tangent bundle $TX$ into a holomorphic section $\theta \in \Gamma(X, T^*X \otimes TX) \subset \Omega^{1,0}(X, TX)$. Let $G$ be a Kähler metric on $X$ and $\omega_G \in \Omega^{1,1}(X)$ be the Kähler form of $G$. Let $a = (E, h, D)$ be a Hermitian factorization of $W$. Define:

$$V_a^G \overset{\text{def}}{=} \partial_a D + F_a - \omega_G \text{id}_E \in \Omega^{1,0}(X, \text{End}^1(E)) \oplus \Omega^{1,1}(X, \text{End}^0(E)),$$

where $F_a \in \Omega^{1,1}(X, \text{End}^0(E))$ is the curvature of the Chern connection of $(E, h)$ and $\partial_a$ was defined in Subsection 6.3.

**Definition 7.5** The twisted curvature of the Hermitian holomorphic factorization $a$ determined by $G$ is defined through:

$$A_a^G \overset{\text{def}}{=} \theta \otimes \text{id}_E + i V_a^G \in \Omega^{1,0}(X, TX \otimes \text{End}^d(E)) \oplus \Omega^{1,0}(X, \text{End}^1(E)) \oplus \Omega^{1,1}(X, \text{End}^0(E)).$$

**Remark 7.2.** We have:

$$A_a^G \in \Omega^{1,*}(X, \wedge TX \otimes \text{End}(E)) \simeq \Omega^{1,0}(X) \otimes C^\infty(X) \text{PV}(X, \text{End}(E)).\quad (7.12)$$

Choose local complex coordinates $z^1, \ldots, z^d$ defined on an open subset $U \subset X$. Using the isomorphism in (7.12), we have the following expansions, where juxtaposition in the second equality denotes multiplication in the algebra $\text{PV}(X, \text{End}(E))$:

$$\theta = \sum_{i=1}^d dz^i \otimes \theta_i,$$

$$F = \sum_{i=1}^d dz^i \otimes \left( \sum_{j=1}^d dz^j F_{ij} \right),$$

$$\omega_G = \sum_{i=1}^d dz^i \otimes \left( \sum_{j=1}^d iG_{ij} dz^j \right).$$
Here:
\[ \theta_i \overset{\text{def}}{=} \partial_i := \frac{\partial}{\partial z_i} \in \Gamma(U, TX) \subset A^0(U, TX) = PV^{-1,0}(U) \]
and \( G_{ij} \in C^\infty(U), \ F_{ij} \in \Gamma(U, \text{End}^\delta(E)) \). We also have \( V_i^G = U \sum_{i=1}^d dz^i \otimes V_i \) and \( A_i^G = U \sum_{i=1}^d dz^i \otimes A_i \), where:
\[ V_i = U \nabla_i^{1,0}D + \sum_{j=1}^d dz^j \otimes (F_{ij} - iG_{ij}i\delta_E) \in A^0(U, \text{End}^\delta(E)) \oplus A^1(U, \text{End}^\delta(E)) \subset PV^1(U, \text{End}(E)) \]
\[ A_i = U \theta_i \otimes i\delta_E + iV_i \in PV^1(U, \text{End}(E)) \quad . \tag{7.13} \]

### 7.6. The disk kernel of a Hermitian holomorphic factorization.

**Definition 7.6** The disk kernel of the Hermitian holomorphic factorization \( \mathfrak{a} = (E, h, D) \) determined by \( \Omega \) and by the Kähler metric \( G \) is the element \( \Pi_{\mathfrak{a}} := \Pi_{\mathfrak{a}}^{G,G} \in PV(X, \text{End}(E)) \) defined through the relation:
\[ \Pi_{\mathfrak{a}}^{G,G} = \frac{1}{\det\Omega} \det\Omega A_i^G \tag{7.14} \]

If \( (U, (z^1, \ldots, z^d)) \) is a complex coordinate chart with \( \Omega = U \varphi(z) dz^1 \wedge \ldots \wedge dz^d \), relations (7.13) and (7.10) give:
\[ \Pi_{\mathfrak{a}} = U \frac{1}{\det\varphi(z)} \epsilon^{i_1 \ldots i_d} (\theta_{i_1}i\delta_E + iV_{i_1}) \ldots (\theta_{i_d}i\delta_E + iV_{i_d}) \quad . \tag{7.15} \]

Notice that \( A_i \) have odd total \( \mathbb{Z}_2 \)-degree and hence:
\[ \deg \Pi_{\mathfrak{a}} = \mu = \hat{d} \in \mathbb{Z}_2 \quad . \]

**Lemma 7.7** The following relations hold in any complex coordinate chart \( (U, (z^1, \ldots, z^d)) \) on \( X \):
\[ \delta_a V_i = (\partial_i W) \otimes i\delta_E \quad , \tag{7.16} \]
\[ \delta_W \theta_i = -i\delta_i W \quad , \tag{7.17} \]
\[ \Delta_a (A_i) = 0 \quad . \tag{7.18} \]

**Proof.**

1. Since \( D \) is holomorphic, we have \( \overline{\partial}_a D = 0 \). Since \( \omega_G \) and \( F_a \) are closed \((1,1)\)-forms, we also have \( \overline{\partial}_a \omega_G = \overline{\partial}_a F_a = 0 \). Thus \( \overline{\partial}_a V_a^G = \overline{\partial}_a \partial_a D = [F, D] \) (see (6.4)) and:
\[ \delta_a V_a^G = \overline{\partial}_a V_a^G + [D, V_a^G] = [F, D] + [D, \overline{\partial}_a D + F] = [D, \overline{\partial}_a D] = (\partial W) i\delta_E \quad , \]
where in the second relation we used the condition \( D^2 = W i\delta_E \), which implies \( [D, \overline{\partial}_a D] = (\partial W) i\delta_E \). Thus (7.16) holds.

2. Since \( \theta_i \) is holomorphic, we have:
\[ \delta_W \theta_i = \iota_W (\partial_i) = \iota_W (\partial_i) = -i\partial_i W \quad . \]

3. Using relations (7.16) and (7.17), we compute:
\[ \Delta_a (A_i) = \Delta_a (\theta_i + iV_i) = (\delta_W \theta_i) i\delta_E + i\delta_a V_i = 0 \quad . \]
\[ \square \]
7.7. The twisted Atiyah class. Recall that the Atiyah class \[13\]:

\[ A(E) \in H^{1,1}(X, \text{End}(E)) \simeq H^1(T^* X \otimes \text{End}(E)) \]

of a holomorphic vector bundle \(E\) on \(X\) coincides \[14\], Proposition 4.3.10\) with the \(\mathcal{F}_{\text{End}(E)}\)-cohomology class of the curvature of the Chern connection \(F \in \Omega^{1,1}(X, E)\) determined by any Hermitian connection \(h\) on \(E\). In this subsection, we introduce a “twisted” version of the Atiyah class for holomorphic factorizations. Let \(a = (E, h, D)\) be a Hermitian holomorphic factorization of \(W\) with underlying holomorphic factorization \(a = (E, D)\).

**Proposition 7.8** We have:

\[ \Delta_a A_a = 0 \quad , \quad (7.19) \]

and:

\[ \Delta_a \Pi_a = 0 \quad . \quad (7.20) \]

Moreover, the \(\Delta_a\)-cohomology class \([\Pi_a]_{\Delta_a} \in \text{HPV}(X, a)\) depends only on \(\Omega\) and on the \(\Delta_a\)-cohomology class \([A_a]_{\Delta_a} \in \text{HPV}_e(X, a)\).

**Proof.** Both statements regarding \(A_a\) follow from (7.18) upon using the fact that \(\Delta_a\) is a graded derivation. The fact that \([\Pi_a]_{\Delta_a}\) depends only on \(\Omega\) and \([A_a]_{\Delta_a}\) follows immediately using \(\Delta_a\)-closure of \(A_a\) and the definition of \(\Pi_a\). \(\square\)

**Definition 7.9** The twisted Atiyah class of \(a\) induced by \(G\) is the \(\Delta_a\)-cohomology class \([A^G_a]_{\Delta_a} \in \text{HPV}_e(X, a)\) of \(A^G_a\).

**Definition 7.10** We say that the \(\partial \overline{\partial}\)-lemma holds for (1, 1)-forms on \(X\) if any (1, 1)-form \(\omega \in \Omega^{1,1}(X)\) which is both \(\partial\) and \(\overline{\partial}\)-closed as well as \(d\)-exact can be written as \(\omega = \partial \overline{\partial} \varphi\) for some smooth complex-valued function \(\varphi \in C^\infty(X)\).

Even though \(X\) is Kählerian, it need not satisfy the \(\partial \overline{\partial}\)-lemma for (1, 1) forms since it is non-compact. The \(\partial \overline{\partial}\)-lemma holds for (1, 1)-forms on \(X\) iff the natural map \(H^{1,1}_{BC}(X) \to H^2_{dR}(X, \mathbb{C})\) is injective, where \(H^{1,1}_{BC}(X)\) and \(H^2_{dR}(X, \mathbb{C})\) denote the corresponding Bott-Chern and de Rham cohomology groups of \(X\).

**Proposition 7.11** The twisted Atiyah class \([A^G_a]_{\Delta_a}\) of a Hermitian holomorphic factorization \(a = (E, D, h)\) is independent of the choice of admissible metric \(h\) on \(E\). Moreover, when the \(\partial \overline{\partial}\)-lemma holds for (1, 1)-forms on \(X\), the twisted Atiyah class \([A^G_a]_{\Delta_a}\) depends only on the Kähler class \([\omega_G] \in H^{1,1}(X)\) of \(G\) and on the Atiyah class \(A(E) \in H^{1,1}(X, \text{End}(E))\) of the holomorphic vector bundle \(E\).

**Proof.** We first show that the twisted Atiyah class does not depend on the choice of \(h\). Let \(\nabla\) be the Chern connection of \((E, h)\) and \(F \in \Omega^{1,1}(X, \text{End}(E))\) denote its curvature. Let \(h'\) be another admissible Hermitian metric on \(E\) and \(F' \in \Omega^{1,1}(X, \text{End}(E))\) be the curvature of the Chern connection \(\nabla'\) of \((E, h')\). Let \(\overline{\partial} := \overline{\partial}_{\text{End}(E)} = \overline{\partial}_a\) and \(\partial = \partial_a, \partial' := \partial_{a'}\), where \(a'\) denotes the Hermitian holomorphic factorization \((E, h, D)\). We have \(\nabla' - \nabla = \nabla'^{1,0} - \nabla^{1,0} = S\) for some \(S \in \Omega^{1,0}(X, \text{End}(E))\). This implies \(\partial' E = \partial E + S\). In turn, this gives:

\[ \partial' = \partial + [S, \cdot] \quad , \quad F' = F + \overline{\partial} S \quad , \]
where \([\cdot, \cdot]\) denotes the graded commutator. Let \(a' \overset{\text{def}}{=} (E, h', D)\) and \(A := A^G, A' := A'^G\).

Relation (7.11) gives:

\[
A' = A + [S, D] + \partial \bar{\partial} (S - \rho \text{id}_E) = A + \delta_a B = A + \Delta_a B ,
\]

where \(B \overset{\text{def}}{=} S - \rho \text{id}_E \in \Omega^{1,0}(X, \text{End}^\delta(E))\). Thus \([A']_{\Delta_a} = [A]_{\Delta_a}\).

Let us now assume that the \(\partial \bar{\partial}\)-lemma holds for \((1, 1)\)-forms on \(X\). Let \(G'\) be another Kähler metric on \(X\) whose Kähler form \(\omega' := \omega_{G'}\) belongs to the same de Rham cohomology class as \(\omega := \omega_{G}\). Since \(\omega' - \omega\) has type \((1, 1)\), the conditions \(d \omega = d \omega' = 0\) imply that \(\omega' - \omega\) is both \(\partial\)- and \(\bar{\partial}\)-closed. Since \(\omega' - \omega\) is \(d\)-exact and the \(\partial \bar{\partial}\)-lemma holds for \((1, 1)\)-forms on \(X\), there exists \(\varphi \in C^\infty(X)\) such that:

\[
\omega' - \omega = \partial \bar{\partial} \varphi .
\]

Setting \(\rho := \partial \varphi \in \Omega^{1,0}(X)\), this gives:

\[
\omega' = \omega + \partial \rho .
\]

\(\square\)

Remark 7.3. Propositions 7.8 and 7.11 imply that the \(\Delta_a\)-cohomology class \([\Pi_a]_{\Delta_a} \in \text{HPV}(X, a)\) depends only on the holomorphic volume form \(\Omega\), on the Kähler class \([\omega_G]\) and on the Atiyah class \(A(E)\), provided that the \(\partial \bar{\partial}\)-lemma holds for \((1, 1)\)-forms on \(X\).

8. Off-shell boundary-bulk maps

Definition 8.1 The canonical off-shell boundary-bulk map of the Hermitian holomorphic factorization \(a = (E, h, D)\) determined by \(\Omega\) and by the Kähler metric \(G\) is the \(C^\infty(X)\)-linear map \(f_a^B := f_a^{B, \Omega, G} : \text{End}_{\text{DF}(X,W)}(a) \to \text{PV}(X)\) defined through:

\[
f_a^{B, \Omega, G}(\alpha) \overset{\text{def}}{=} \text{str}(\Pi_a^{\Omega, G} \alpha) , \quad \forall \alpha \in \text{End}_{\text{DF}(X,W)}(a) = \mathcal{A}(X, \text{End}(E)) .
\]

Notice that \(f_a^B\) has total \(\mathbb{Z}_2\)-degree \(\mu\).

Proposition 8.2 We have:

\[
\delta_W \circ f_a^B = (-1)^d f_a^B \circ \delta_a .
\]

In particular, \(f_a^B\) descends to an \(O(X)\)-linear map from \(\text{End}_{\text{DF}(X,W)}(a)\) to \(\text{HPV}(X, W)\).

Proof. For any \(\alpha \in \mathcal{A}(X, \text{End}(E))\), we have:

\[
f_a^B(\delta_a \alpha) = \text{str}[\Pi_a(\delta_a \alpha)] = (-1)^d \text{str}[\Delta_a(\Pi_a \alpha)] = (-1)^d \delta_W[\text{str}(\Pi_a \alpha)] = (-1)^d \delta_W f_a^B(\alpha) ,
\]

where in the last equality we noticed that \(\Delta_a \alpha = \delta_a \alpha\) and used (7.20) and (7.3) \(\square\)

Definition 8.3 The cohomological boundary-bulk map of \(a = (E, h, D)\) is the \(O(X)\)-linear map \(f_a := f_a^{\Omega, G} : \text{End}_{\text{HDF}(X,W)}(a) \to \text{HPV}(X, W)\) induced by \(f_a^{B, \Omega, G}\) on cohomology.
9. Off-shell bulk-boundary maps

9.1. Canonical off-shell bulk-boundary maps.

**Definition 9.1** The canonical off-shell bulk-boundary map of the Hermitian holomorphic factorization $\mathbf{a} = (E, h, D)$ determined by $\Omega$ and by the Kähler metric $G$ is the $C^\infty(X)$-linear map $\varepsilon_a^B := e_a^{B,\Omega, G} : \text{PV}(X) \to \text{End}_{\text{DF}(X,W)}(a)$ defined through:

$$
\varepsilon_a^{B,\Omega, G}(\omega) \overset{\text{def}}{=} \Omega_{\omega,0}(\omega H_a^{\Omega, G}) \quad , \quad \forall \omega \in \text{PV}(X) .
$$

Notice that $\varepsilon_a^B$ has total $\mathbb{Z}_2$-degree $\hat{0}$.

**Proposition 9.2** We have:

$$
\delta_a \circ \varepsilon_a^B = (-1)^d \varepsilon_a^B \circ \delta_W \quad .
$$

In particular, $\varepsilon_a^B$ descends to an $O(X)$-linear map from $\text{HPV}(X, W)$ to $\text{End}_{\text{DF}(X,W)}(a)$.

**Proof.** Let $\omega \in \text{PV}(X)$. Using Lemma 7.3, we compute:

$$
\delta_a(\varepsilon_a^B(\omega)) = \delta_a[\Omega_{\omega,0}(\omega H_a)] = (-1)^d \Omega_{\omega,0}(\Delta_a(\omega)H_a) = (-1)^d \Omega_{\omega,0}(\delta_W(\omega)H_a) = (-1)^d \varepsilon_a^B(\delta_W \omega) \quad ,
$$

where we used the fact that $H_a$ is $\Delta_a$-closed (see (7.20)) and noticed that $\Delta_a \omega = \delta_W \omega$. \qed

**Definition 9.3** The cohomological bulk-boundary map of $\mathbf{a} = (E, h, D)$ is the $O(X)$-linear map $\varepsilon_a := \varepsilon_a^{\Omega, G} : \text{HPV}(X, W) \to \text{End}_{\text{DF}(X,W)}(a)$ induced by $\varepsilon_a^{B,\Omega, G}$ on cohomology.

9.2. Tempered off-shell bulk-boundary maps.

**Definition 9.4** For any $\lambda \geq 0$ the $\lambda$-tempered off-shell bulk-boundary map of the Hermitian holomorphic factorization $\mathbf{a} = (E, h, D)$ determined by $\Omega$ and by the Kähler metric $G$ is the $C^\infty(X)$-linear map $\varepsilon_a^{(\lambda)} := e_a^{(\lambda),\Omega, G} : \text{PV}(X) \to \text{End}_{\text{DF}(X,W)}(a)$ defined through:

$$
\varepsilon_a^{(\lambda),\Omega, G} \overset{\text{def}}{=} U_a^G(-\lambda) \circ \varepsilon_a^{B,\Omega, G} \circ U_G(\lambda) \quad .
$$

Notice that $\varepsilon_a^{(\lambda)}$ has total $\mathbb{Z}_2$-degree $\hat{0}$. We have:

$$
\varepsilon_a^{(0)} = \varepsilon_a^B \quad (9.2)
$$

and:

$$
\varepsilon_a^{(\lambda)}(\omega) = e^{\lambda L_a^G} \Omega_{\omega,0} \left( e^{-\lambda L_a^G \omega} H_a \right) \quad , \quad \forall \omega \in \text{PV}(X) \quad .
$$

In what follows, we fix a holomorphic volume form $\Omega$ and a Kähler metric $G$ on $X$ and denote $\varepsilon_a^{(\lambda),\Omega, G}$ by $\varepsilon_a^{(\lambda)}$ and $e_a^{B,\Omega, G}$ by $e_a^B$. We also denote $L_a$, $v_G$ by $L$, $v$ and $L_a^G$, $v_a^G$ by $L_a$, $v_a$.

**Proposition 9.5** For any $\lambda \geq 0$, we have:

$$
\delta_a \circ \varepsilon_a^{(\lambda)} = (-1)^d \varepsilon_a^{(\lambda)} \circ \delta_W \quad .
$$

**Proof.** Follows immediately from Proposition 9.2 using relations (5.13) and (6.10). \qed
Proposition 9.6 Let $\lambda \geq 0$. For any $\omega \in \ker \delta_W$, we have:

$$e_a^{(\lambda)}(\omega) = e_a^B(\omega) \mod \im \delta_a \ . \tag{9.4}$$

In particular, the map induced by $e_a^{(\lambda)}$ on cohomology does not depend on $\lambda$ and coincides with the cohomological bulk-boundary map.

Proof. For any $\omega \in \PV(X)$ such that $\delta_W \omega = 0$, we compute:

$$\frac{d}{d\lambda} e_a^{(\lambda)}(\omega) = \frac{d}{d\lambda} \left[ e^{\lambda L} e_a B(e^{-\lambda L} \omega) \right] = L_a e_a^{(\lambda)}(\omega) - e_a^{(\lambda)}(L \omega) = (\delta_a v_a) e_a^{(\lambda)}(\omega) - e_a^{(\lambda)}((\delta_W v) \omega) \ ,$$

where we used the fact that $L_a = \delta_a v_a$ and $L = \delta_W v$. We have:

$$(\delta_a v_a) e_a^{(\lambda)}(\omega) = \delta_a \left[ v_a e_a^{(\lambda)}(\omega) \right] + v_a \delta_a e_a^{(\lambda)}(\omega) = \delta_a \left[ v_a e_a^{(\lambda)}(\omega) \right] ,$$

where we noticed that relation (9.3) implies:

$$\delta_a e_a^{(\lambda)}(\omega) = (-1)^d e_a^{(\lambda)}(\delta_W \omega) = 0 \ ,$$

since $\delta_W \omega = 0$. On the other hand, we have:

$$e_a^{(\lambda)}((\delta_W v) \omega) = e_a^{(\lambda)}(\delta_W (v \omega)) = (-1)^d \delta_a e_a^{(\lambda)}(v \omega) \ ,$$

where we once again used relation (9.3). Thus:

$$\frac{d}{d\lambda} e_a^{(\lambda)}(\omega) = \delta_a \left[ v_a e_a^{(\lambda)}(\omega) - (-1)^d e_a^{(\lambda)}(v \omega) \right] ,$$

which implies that $e_a^{(\lambda)}$ is independent of $\lambda$ modulo $\delta_a$-exact terms. This implies relation (9.4) upon recalling that $e_a^{(0)} = e_a^B$. $\square$

9.3. Independence of metric data.

Proposition 9.7 The cohomological bulk-boundary and boundary-bulk maps $e_a$ and $f_a$ of a Hermitian holomorphic factorization $a = (E, D, h)$ do not depend on the choice of the admissible Hermitian metric $h$ on $E$. Moreover, when the $\partial \bar{\partial}$-lemma holds for $(1,1)$-forms on $X$ then $e_a$ and $f_a$ depend only on $\Omega$, on the Kähler class $[\omega_G] \in \HH^1(X) \otimes G$ and on the Atiyah class $A(E) \in \HH^1(X, E \otimes \Omega)$. The holomorphic vector bundle $E$. Accordingly, we denote $f_a$ and $e_a$ by $f_a$ and $e_a$ (since they are independent of $h$).

Proof. Let $G$ and $G'$ be two Kähler metrics on $X$ having the same Kähler class and $h$ and $h'$ be two admissible Hermitian metrics on $a = (E, D)$. Let $a = (E, h, D)$ and $a' = (E, h', D)$. Let $f_B := f_a^{\partial \bar{\partial}, \Omega, \Gamma}$, $f_B' := f_a^{\partial \bar{\partial}, \Omega, \Gamma'}$ and $e_B := e_a^{\partial \bar{\partial}, \Omega, \Gamma}$, $e_B' := e_a^{\partial \bar{\partial}, \Omega, \Gamma'}$. Let $\Pi := \Pi_a^{\partial \bar{\partial}, \Omega, \Gamma}$ and $\Pi' := \Pi_a^{\partial \bar{\partial}, \Omega, \Gamma'}$. When $G' = G$ or when $G' \neq G$ but the $\partial \bar{\partial}$-lemma holds for $(1,1)$-forms on $X$, Proposition 7.11 implies:

$$\Pi' = \Pi + \Delta_a(T)$$
for some $T \in \Omega(X, \wedge TX \otimes \text{End}(E))$. This implies that the following relations hold for any $\alpha \in \ker \delta_a$ and any $\omega \in \ker \delta_W$ of pure $\mathbb{Z}_2$-degree:

$$f'_B(\alpha) = \text{str}(\Pi' \alpha) = f_B(\alpha) + \text{str}[\Delta_aT \alpha] = f_B(\alpha) + \text{str} \Delta_a(T \alpha) = f_B(\alpha) + \delta_W \text{str} \Delta_a(\alpha) \quad (9.5)$$

and:

$$e'_B(\omega) = \Omega_{\omega_0} [\omega \Pi'] = e_B(\omega) + \Omega_{\omega_0}[\omega \Delta_a \Pi] = e_B(\omega) + (-1)^{\text{deg}_W} \Omega_{\omega_0} \Delta_a(\omega \Pi) = e_B(\omega) + (-1)^{d_+ + \text{deg}_W} \delta_a[\Omega_{\omega_0}(\omega \Pi)] \quad , \quad (9.6)$$

where in the last equalities we used relation (7.3) and Lemma 7.3. Relations (9.5) and (9.6) imply that $f'_B$ and $e'_B$ induce the same maps on cohomology as $f_B$ and $e_B$, respectively. □

### 9.4. Adjointness relations.

Let $G$ be a fixed Kähler metric on $X$ and $a = (E, h, D)$ be a Hermitian holomorphic factorization of $W$ with underlying holomorphic factorization $a = (E, D)$. Let $f^B_a := f^B_{a, \Omega^G}$ and $e^B_a := e^B_{a, \Omega^G}$. For any $\omega \in PV(X)$ and any $\alpha \in \text{End}_{DF}(X,W)(a) = \mathcal{A}(X, \text{End}(E))$, we have:

$$\Omega_{\omega_0}[\omega f^B_a(\alpha)] = \text{str}[e^B_a(\omega) \alpha] = \lambda_\Omega(\omega \Pi_a \alpha) \ ,$$

where $\Pi_a := \Pi^\Omega_a$ and $\lambda_\Omega$ was defined in (7.8). This identity can be viewed as a local adjointness relation between $e^B_a$ and $f^B_a$. When either $\omega$ or $\alpha$ has compact support, applying $f_\Omega$ to the relation above gives the global adjointness relation:

$$\text{Tr}_B[\omega f^B_a(\alpha)] = \text{tr}_B[e^B_a(\omega) \alpha] = \int_\Omega \lambda_\Omega(\omega \Pi_a \alpha) \ .$$

The following result shows that similar adjointness relations are satisfied by the tempered traces and tempered bulk-boundary and boundary-bulk maps.

**Proposition 9.8** Let $\lambda \geq 0$. For any $\omega \in PV(X)$ and any $\alpha \in \text{End}_{DF}(X,W)(a)$, we have:

$$\Omega_{\omega_0}[e^{-\lambda \omega} f^B_a(\alpha)] = \text{str}[e^{-\lambda e^\alpha_a}_a(\omega) \alpha] = \lambda_\Omega(e^{-\lambda \omega} \Pi_a \alpha) \quad . \quad (9.7)$$

**Proof.** Using the definitions of $f^B_a$ and $e^\alpha_a$, we compute:

$$\Omega_{\omega_0}[e^{-\lambda \omega} f^B_a(\alpha)] = \Omega_{\omega_0}[e^{-\lambda \omega} \text{str}(\Pi_a \alpha)] = \lambda_\Omega(e^{-\lambda \omega} \Pi_a \alpha)$$

and:

$$\text{str}[e^{-\lambda e^\alpha_a}_a(\omega) \alpha] = \text{str} \left( \Omega_{\omega_0}[e^{-\lambda \omega} \Pi_a \alpha] \right) = \Omega_{\omega_0} \left[ e^{-\lambda \omega} \text{str}(\Pi_a \alpha) \right] = \lambda_\Omega(e^{-\lambda \omega} \Pi_a \alpha) \ ,$$

showing that (9.7) holds. □

**Proposition 9.9** Let $\lambda \geq 0$. For any $\omega \in PV(X)$ and any $\alpha \in \text{End}_{DF}(X,W)(a)$, we have:

$$\text{Tr}^{(\lambda)}(\omega f^B_a(\alpha)) = \text{tr}^{(\lambda)}(e^\alpha_a(\omega) \alpha) = \int_\Omega \lambda_\Omega(\omega \Pi_a \alpha) \ . \quad (9.8)$$

**Proof.** Follows immediately by applying $f_\Omega$ to both sides of relation (9.7). □
Corollary 9.10 Let \( a = (E, h, D) \) be a Hermitian holomorphic factorization of \( W \). For any \( \omega \in HPV_c(X, W) \) and any \( \alpha \in \text{End}_{HDF_c(X, W)}(a) \), we have:

\[
\text{Tr}_c(\omega f_a(\alpha)) = \text{tr}_a^c(e_a(\omega)\alpha)
\]

When the critical set \( Z_W \) is compact, we have:

\[
\text{Tr}(\omega f_a(\alpha)) = \text{tr}_a(e_a(\omega)\alpha), \quad \forall \omega \in HPV(X, W) \quad \forall \alpha \in \text{End}_{HDF(X, W)}(a).
\]

Proof. The first statement follows immediately from Proposition 9.9. This implies the second statement upon using the definitions of \( \text{Tr} \) and \( \text{tr}_a \) and Propositions 3.7 and 4.11. \( \square \)

10. Proof of the main theorem

Assume that \( W \) has compact critical set. Let \( f := (f_a)_{a \in \text{ObHDF}(X)} \) and \( e := (e_a)_{a \in \text{ObHDF}(X)} \). In this case, Propositions 3.7 and 4.11 allow us to transport traces \( \text{Tr}_c \) and \( \text{tr}_c := (\text{tr}_a^c)_{a \in \text{ObHDF}(X)} \) into traces \( \text{Tr} \) and \( \text{tr} := (\text{tr}_a)_{a \in \text{ObHDF}(X, W)} \) defined on \( HPV(X, W) \) and \( HDF(X, W) \) respectively. Proposition 3.4 shows that the algebra \( HPV(X, W) \) is finite-dimensional over \( \mathbb{C} \) while Proposition 4.9 shows that the category \( HDF(X, W) \) is Hom-finite over \( \mathbb{C} \). On the other hand, Proposition 6.2 implies that \( (HDF_c(X, W), \text{tr}_c) \) (and hence also \( (HDF(X, W), \text{tr}) \)) is a pre-Calabi-Yau category. Combining these results gives Theorem 1.1. Moreover, Corollary 9.10 implies that the cohomological bulk-boundary maps are adjoint to the cohomological boundary-bulk maps with respect to \( \text{Tr} \) and \( \text{tr} \).

In view of Theorem 1.1, proving Conjecture 1.2 amounts to showing that the cohomological bulk and boundary traces are non-degenerate and that the topological Cardy constraint is satisfied when the critical set of \( W \) is compact. We will address these questions in a separate paper. When \( X \) is Stein and the critical set of \( W \) is compact, the conjecture follows by adapting the results of [15, 16, 17].

A. Some expressions in local holomorphic coordinates

Choosing local holomorphic coordinates \( z = (z_1, \ldots, z_d) \) defined on an open subset \( U \subset X \), set \( \partial_i := \frac{\partial}{\partial z_i} \) and \( \bar{\partial}_i := \frac{\partial}{\partial \bar{z}_i} \). We have:

\[
\begin{align*}
TX|_U &= \text{Span}_\mathbb{C}\left\{\partial_1, \ldots, \partial_d\right\}, & \bar{TX}|_U &= \text{Span}_\mathbb{C}\left\{\bar{\partial}_1, \ldots, \bar{\partial}_d\right\},
T^*X|_U &= \text{Span}_\mathbb{C}\{dz_1, \ldots, dz_d\}, & \bar{T}^*X|_U &= \text{Span}_\mathbb{C}\{d\bar{z}_1, \ldots, d\bar{z}_d\}.
\end{align*}
\]

For any increasingly ordered subset \( \{t_1, \ldots, t_i\} \subset \{1, \ldots, d\} \) of size \( |I| = i \), define:

\[
dz_I := dz_{t_1} \wedge dz_{t_2} \wedge \cdots \wedge dz_{t_i}, \quad \partial_I := \partial_{t_1} \wedge \cdots \wedge \partial_{t_i}
\]

and their complex conjugates:

\[
d\bar{z}_I := d\bar{z}_{t_1} \wedge d\bar{z}_{t_2} \wedge \cdots \wedge d\bar{z}_{t_i}, \quad \bar{\partial}_I := \bar{\partial}_{t_1} \wedge \cdots \wedge \bar{\partial}_{t_i}.
\]

For \( \omega \in PV^{-i,j}(X) \) and \( \eta \in PV^{-k,l}(X) \), we can expand:

\[
\omega = \sum_{|I|=i, |J|=j} \omega_{I,J} dz_I \otimes \partial_J, \quad \eta = \sum_{|K|=k, |L|=l} \eta_{K,L} d\bar{z}_L \otimes \bar{\partial}_K.
\]
with $\omega^I, \eta^K \in C^\infty(X)$. Then:

$$\omega \wedge \eta = U \sum_{I,J,K,L} (-1)^i \omega^I \eta^K (dz_I \wedge d\bar{z}_L) \otimes (\partial_I \wedge \partial_K) ,$$

where $|I| = i$, $|J| = j$, $|K| = k$ and $|L| = l$. We also have:

$$\bar{\partial} \omega = U \sum_{|I| = i, |J| = j} ([\bar{\partial} \omega^I_J] \wedge d\bar{z}_J) \otimes \partial_I = \sum_{|I| = i, |J| = j} \sum_{r=1}^d (\bar{\partial}_r \omega^I_J)(dz_r \wedge d\bar{z}_J) \otimes \partial_I \in PV^{-i,j+1}(X) .$$

Let $\Omega$ be a holomorphic volume form on $X$. Then:

$$\Omega = U \varphi(z) dz^1 \wedge \ldots \wedge dz^d$$

for some $\varphi \in \mathcal{O}_X(U)$. The following local expansion holds for any $\omega \in PV^{-i,j}(X)$ [10]:

$$\partial_\Omega \omega = U \sum_{|I| = i, |J| = j} \sum_{r=1}^d [\partial_r (\varphi \omega^I_J)(dz^r) \otimes (dz_r \otimes \partial_I)] \in PV^{-i+1,j}(X) . \quad (A.1)$$

Let us prove relation (3.5). Using (A.1) in (3.4) and noticing that $\partial_\Omega W = 0$ due to degree reasons, we compute:

$$\{ W, \omega \}_W = U \partial_\Omega (W \omega) - (\partial_\Omega W) \wedge \omega - (-1)^{|W|} W \partial_\Omega \omega = \partial_\Omega (W \omega) - W \partial_\Omega \omega =$$

$$= \frac{1}{\varphi} \sum_{|I| = i, |J| = j} \sum_{r=1}^d \left[ \partial_r (\varphi W \omega^I_J)(dz^r) \otimes (dz_r \otimes \partial_I) - W \partial_r (\varphi \omega^I_J)(dz^r) \otimes (dz_r \otimes \partial_I) \right]$$

$$= \frac{1}{\varphi} \sum_{|I| = i, |J| = j} \sum_{r=1}^d (\varphi (\partial_r W) \omega^I_J)(dz^r) \otimes (dz_r \otimes \partial_I) = \sum_{|I| = i, |J| = j} \sum_{r=1}^d (\partial_r W)(dz^r) \otimes \omega .$$

Recall that:

$$\iota_W = -i \iota_\partial W = U \sum_{r=1}^d (\partial_r W)dz^r \otimes ,$$

where we used the expansion $\partial W = U \sum_{r=1}^d (\partial_r W)dz^r \otimes$. Comparing with the expression above gives (3.5).

**B. Superconnection formulation of Landau-Ginzburg TFT data**

The path integral argument of references [1, 2] is formulated using a certain class of superconnections, leading to a description of the category of topological D-branes which is equivalent with that given in Section 4. The equivalence of the two descriptions follows from the Koszul-Malgrange correspondence, as we explain in this appendix.

**B.1. The Koszul-Malgrange correspondence.** Recall that $VB_\infty(X)$ denotes the $C^\infty(X)$-linear category of finite rank locally-free sheaves of $C^\infty$-modules (whose objects we identify with smooth vector bundles defined on $X$), while $VB(X)$ denotes the $O(X)$-linear category of finite rank locally-free sheaves of $O_X$-modules (whose objects we identify with holomorphic vector bundles defined on $X$). The latter category has a well-known description in terms of smooth complex vector bundles endowed with integrable $(0, 1)$-connections [18, 19], as we recall below.
Given a complex vector bundle $S$ on $X$, a \((0,1)\)-connection on $S$ is a $\mathbb{C}$-linear map $D : \Gamma^\infty(X, S) = \mathcal{A}^0(X, S) \to \mathcal{A}^1(X, S)$ which satisfies the Leibniz rule:

$$D(fs) = (\partial f) \otimes s + f(Ds), \quad \forall f \in C^\infty(X), \forall s \in \Gamma^\infty(X, S).$$

A Dolbeault derivation is a $\mathbb{C}$-linear map $\overline{\partial} : \mathcal{A}(X, S) \to \mathcal{A}(X, S)$ which is homogeneous of degree one with respect to the rank grading of $\mathcal{A}(X, S)$ and satisfies:

$$\overline{\partial}(\rho \otimes s) = (\overline{\partial} \rho) \wedge s + (-1)^k \rho \otimes (\overline{\partial}s), \quad \forall \rho \in \mathcal{A}^k(X), \forall s \in \Gamma^\infty(X, S).$$

Any Dolbeault derivation on $S$ restricts to a $(0,1)$-connection. Conversely, any $(0,1)$-connection on $S$ extends uniquely to a Dolbeault derivation. This correspondence gives a bijection between $(0,1)$ connections on $S$ and Dolbeault derivations on $S$.

A $(0,1)$-connection on $S$ is called integrable if its Dolbeault derivation $\overline{\partial}$ is a differential (i.e. if it satisfies $\overline{\partial}^2 = 0$). A Dolbeault pair defined on $X$ is a pair $(S, \overline{\partial})$, where $S$ is a complex vector bundle on $X$ and $\overline{\partial}$ is an integrable Dolbeault derivation on $S$. Given two such pairs $(S_1, \overline{\partial}_1)$ and $(S_2, \overline{\partial}_2)$, a morphism of Dolbeault pairs from $(S_1, \overline{\partial}_1)$ to $(S_2, \overline{\partial}_2)$ is a morphism $f : S_1 \to S_2$ of $\text{VB}^\infty(X)$ such that $\overline{\partial}_2 \circ (\text{id}_{\mathcal{A}(X)} \otimes f) = (\text{id}_{\mathcal{A}(X)} \otimes f) \circ \overline{\partial}_1$. With these definitions, Dolbeault pairs over $X$ and morphisms of such form a category which we denote by $\mathcal{P}(X)$.

Given a Dolbeault pair $(S, \overline{\partial})$, the locally-free sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(S, \overline{\partial})$ defined by $\mathcal{O}_X(S, \overline{\partial})(U) \overset{\text{def}}{=} \ker \overline{\partial}|_{\Gamma^\infty(U, S)}$ for any open subset $U \subset X$ is called the sheaf of $\overline{\partial}$-holomorphic sections of $S$. This determines a holomorphic vector bundle $K(S, \overline{\partial})$ whose underlying complex vector bundle equals $S$. A morphism of Dolbeault pairs $f : (S_1, \overline{\partial}_1) \to (S_2, \overline{\partial}_2)$ induces a morphism between the corresponding sheaves of $\overline{\partial}$-holomorphic sections $K(f) : \mathcal{O}(S_1, \overline{\partial}_1) \to \mathcal{O}(S_2, \overline{\partial}_2)$, i.e. a morphism $K(f) \in \text{Hom}_{\text{VB}}(K(S_1, \overline{\partial}_1), K(S_2, \overline{\partial}_2))$ in the category $\text{VB}(X)$. This correspondence gives a functor $K : \mathcal{P}(X) \to \mathcal{VB}(X)$. Let $\Phi : \mathcal{P}(X) \to \text{VB}^\infty(X)$ be the functor which forgets the holomorphic structure and $\Psi : \mathcal{P}(X) \to \text{VB}^\infty(X)$ be the functor which forgets the Dolbeault derivation. We also have a functor $\mathcal{M} : \mathcal{VB}(X) \to \mathcal{P}(X)$ which sends a holomorphic vector bundle $E$ into the Dolbeault pair $\mathcal{M}(E) \overset{\text{def}}{=} (\Phi(E), \overline{\partial}_E)$ (where $\overline{\partial}_E$ is the Dolbeault derivation defined on $\Phi(E)$ by the holomorphic structure of $E$) and sends a morphism $f$ of $\text{VB}(X)$ into the underlying morphism $\mathcal{M}(f) \overset{\text{def}}{=} \Phi(f)$ of complex vector bundles. This functor satisfies $\Psi = \Phi \circ K$. The Koszul-Malgrange theorem [18,19] states that $\mathcal{K}$ and $\mathcal{M}$ are mutually quasi-inverse equivalences of categories between $\mathcal{P}(X)$ and $\text{VB}(X)$. In particular, the fiber of $\Phi$ at a complex vector bundle $S$ can be identified with the set of all integrable $(0,1)$ connections on $S$.

**B.2. Complex vector superbundles.**

**Definition B.1** A complex vector superbundle on $X$ is a complex $\mathbb{Z}_2$-graded vector bundle, i.e. a complex vector bundle $S$ endowed with a direct sum decomposition $S = S^0 \oplus S^1$, where $S^0$ and $S^1$ are complex linear sub-bundles of $S$.

**Definition B.2** Let $\text{VB}^\infty(X)$ denote the $\mathbb{Z}_2$-graded $C^\infty(X)$-linear category defined as follows:

- The objects are the complex vector superbundles defined on $X$.
- Given two complex vector superbundles $S$ and $T$ over $X$, the space of morphisms from $S$ to $T$ is the $C^\infty(X)$-module $\text{Hom}_\infty(S, T) \overset{\text{def}}{=} \Gamma^\infty(X, \text{Hom}(S, T))$, endowed with the $\mathbb{Z}_2$-grading
with homogeneous components:

\[
\begin{align*}
\text{Hom}_\infty^0(S, T) & \overset{\text{def}}{=} \Gamma_\infty(X, \text{Hom}(S^0, T^0)) \oplus \Gamma_\infty(X, \text{Hom}(S^1, T^1)) \\
\text{Hom}_\infty^1(S, T) & \overset{\text{def}}{=} \Gamma_\infty(X, \text{Hom}(S^0, T^1)) \oplus \Gamma_\infty(X, \text{Hom}(S^1, T^0))
\end{align*}
\]

- The composition of morphisms is induced by that of \(\text{VB}_\infty(X)\).

Let \(\text{VB}_\infty^\oplus(X)\) be the non-full subcategory of \(\text{VB}_\infty^*(X)\) obtained by restricting to even morphisms. We have an obvious equivalence of categories:

\[
\text{VB}_\infty^\oplus(X) \simeq \text{VB}_\infty(X) \times \text{VB}_\infty(X)
\]

The graded direct sum of \(S\) with \(T\) is the direct sum \(S \oplus T\) of the underlying complex vector bundles, endowed with the \(\mathbb{Z}_2\)-grading given by:

\[
(S \oplus T)^\kappa = S^\kappa \oplus T^\kappa\quad \forall \kappa \in \mathbb{Z}_2
\]

The graded tensor product of \(S\) with \(T\) is the ordinary tensor product \(S \otimes T\) of the underlying complex vector bundles, endowed with the \(\mathbb{Z}_2\)-grading given by:

\[
(S \otimes T)^0 \overset{\text{def}}{=} (S^0 \otimes T^0) \oplus (S^1 \otimes T^1)
\]

\[
(S \otimes T)^1 \overset{\text{def}}{=} (S^0 \otimes T^1) \oplus (S^1 \otimes T^0)
\]

The graded dual of \(S\) is the complex vector superbundle whose underlying complex vector bundle is the ordinary dual \(S^\vee\) of \(S\), endowed with the \(\mathbb{Z}_2\)-grading given by:

\[
(S^\vee)^\kappa \overset{\text{def}}{=} (S^\kappa)^\vee\quad \forall \kappa \in \mathbb{Z}_2
\]

The graded bundle of morphisms from \(S\) to \(T\) is the bundle \(\text{Hom}(S, T) \overset{\text{def}}{=} S^\vee \otimes T\), where \(S^\vee\) is the graded dual and \(\otimes\) is the graded tensor product. Thus \(\text{Hom}(S, T)\) is the usual bundle of morphisms, endowed with the \(\mathbb{Z}_2\)-grading given by:

\[
\text{Hom}_\infty^0(S, T) \overset{\text{def}}{=} \text{Hom}(S^0, T^0) \oplus \text{Hom}(S^1, T^1)
\]

\[
\text{Hom}_\infty^1(S, T) \overset{\text{def}}{=} \text{Hom}(S^0, T^1) \oplus \text{Hom}(S^1, T^0)
\]

We have \(\text{Hom}_\infty^*(S, T) = \Gamma_\infty(X, \text{Hom}^*(S, T))\). When \(T = S\), we set \(\text{End}(S) \overset{\text{def}}{=} \text{Hom}(S, S)\) and \(\text{End}^\kappa(S) \overset{\text{def}}{=} \text{Hom}^\kappa(S, S)\) etc.

The module of \(S\)-valued \((0, \ast)\)-forms. For any complex vector superbundle \(S = S^0 \oplus S^1\) on \(X\), the \(C^\infty(X)\)-module of smooth sections \(\Gamma_\infty(X, S)\) is \(\mathbb{Z}_2\)-graded with homogeneous components \(\Gamma_\infty^\kappa(M, S) \overset{\text{def}}{=} \Gamma_\infty(M, S^\kappa)\). Accordingly, the \(C^\infty(X)\)-module \(\mathcal{A}(X, S) \overset{\text{def}}{=} \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X, S)\) has an induced \(\mathbb{Z} \times \mathbb{Z}_2\)-grading. The \(\mathbb{Z}\)-grading is called the rank grading and corresponds to the decomposition:

\[
\mathcal{A}(X, S) = \bigoplus_{k=0}^d \mathcal{A}^k(X, S)
\]

The \(\mathbb{Z}_2\)-grading is called the bundle grading and corresponds to the decomposition:

\[
\mathcal{A}(X, S) = \mathcal{A}(X, S^0) \oplus \mathcal{A}(X, S^1)
\]
For any $\alpha \in \mathcal{A}^k(X, S)$, let $\text{r}k\alpha \overset{\text{def}}{=} k$. For any $\alpha \in \mathcal{A}(X, S^\kappa)$, let $\sigma(\alpha) \overset{\text{def}}{=} \kappa \in \mathbb{Z}_2$. The total grading of $\mathcal{A}(X, S)$ is the $\mathbb{Z}_2$-grading given by the decomposition $\mathcal{A}(X, S) = \mathcal{A}(X, S)^0 \oplus \mathcal{A}(X, S)^1$, where:

\[
\mathcal{A}(X, S)^0 \overset{\text{def}}{=} \bigoplus_{i=\text{ev}} \mathcal{A}^i(X, S^0) \oplus \bigoplus_{i=\text{odd}} \mathcal{A}^i(X, S^1)
\]

\[
\mathcal{A}(X, S)^1 \overset{\text{def}}{=} \bigoplus_{i=\text{ev}} \mathcal{A}^i(X, S^1) \oplus \bigoplus_{i=\text{odd}} \mathcal{A}^i(X, S^0)
\]

For any $\alpha \in \mathcal{A}(X, S)^\kappa$, we set $\text{deg}\alpha = \kappa \in \mathbb{Z}_2$. For all $\rho \in \mathcal{A}^k(X)$ and all $s \in \Gamma(M, S^\kappa)$, we have:

\[
\text{deg}(\rho \otimes s) = \hat{k} + \kappa \in \mathbb{Z}_2.
\]

**B.3. $\mathbb{Z}_2$-graded Dolbeault pairs.**

**Definition B.3** Let $S$ be a complex vector superbundle. A Dolbeault derivation $\overline{\mathcal{D}}$ on $S$ is called compatible with the $\mathbb{Z}_2$-grading of $S$ if it preserves the bundle grading of $\mathcal{A}(X, S)$, which means that it satisfies:

\[
\overline{\mathcal{D}}(\mathcal{A}(X, S^\kappa)) \subset \mathcal{A}(X, S^\kappa), \quad \forall \kappa \in \mathbb{Z}_2.
\]

A compatible Dolbeault derivation decomposes as $\overline{\mathcal{D}} = \overline{\mathcal{D}}^0 \oplus \overline{\mathcal{D}}^1$, where $\overline{\mathcal{D}}^\kappa$ are Dolbeault derivations on $S^\kappa$ for each $\kappa \in \mathbb{Z}_2$. Conversely, given Dolbeault derivations $\overline{\mathcal{D}}^\kappa$ on $S^\kappa$, their direct sum is a compatible Dolbeault derivation on $S$.

**Definition B.4** A $\mathbb{Z}_2$-graded Dolbeault pair on $X$ is a pair $(S, \overline{\mathcal{D}})$, where $S$ is a complex vector superbundle on $X$ and $\overline{\mathcal{D}}$ is an integrable Dolbeault derivation on $S$ which is compatible with the $\mathbb{Z}_2$-grading of $S$.

**Definition B.5** The category $\mathcal{P}^\kappa(X)$ of $\mathbb{Z}_2$-graded Dolbeault pairs on $X$ is the $\mathcal{O}(X)$-linear category defined as follows:

- The objects are the $\mathbb{Z}_2$-graded Dolbeault pairs defined on $X$.
- Given two $\mathbb{Z}_2$-graded Dolbeault pairs $(S_1, \overline{\mathcal{D}}_1)$ and $(S_2, \overline{\mathcal{D}}_2)$ on $X$, the corresponding Hom space is the $C^\infty(X)$-module:

\[
\text{Hom}_{\mathcal{P}^\kappa(X)}((S_1, \overline{\mathcal{D}}_1), (S_2, \overline{\mathcal{D}}_2)) \overset{\text{def}}{=} \{ f \in \Gamma(X, \text{Hom}^\hat{\kappa}(S_1, S_2)) | \overline{\mathcal{D}}_2 \circ (\text{id}_{\mathcal{A}(X)} \otimes f) = (\text{id}_{\mathcal{A}(X)} \otimes f) \circ \overline{\mathcal{D}}_1 \}
\]

- The composition of morphisms is inherited from $\text{VB}^\kappa_{\mathcal{O}_X}(X)$.

Recall the Koszul-Malgrange functor $\mathcal{K}$ defined in Subsection B.1. Let $\mathcal{K}^\kappa : \mathcal{P}^\kappa(X) \to \text{VB}^\kappa_{\hat{\mathcal{O}_X}}(X)$ be the functor which:

- Sends a $\mathbb{Z}_2$-graded Dolbeault pair $(S, \overline{\mathcal{D}})$ into the holomorphic vector bundle $\mathcal{K}(S, \overline{\mathcal{D}})$, viewed as a holomorphic vector superbundle by endowing it with the grading $\mathcal{K}(S, \overline{\mathcal{D}})^\kappa \overset{\text{def}}{=} \mathcal{K}(S^\kappa, \overline{\mathcal{D}}^\kappa)$, where $\kappa \in \mathbb{Z}_2$.
- Sends a degree zero morphism $f : (S_1, \overline{\mathcal{D}}_1) \to (S_2, \overline{\mathcal{D}}_2)$ of $\mathbb{Z}_2$-graded Dolbeault pairs which intertwines $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ into $\mathcal{K}(f) \overset{\text{def}}{=} f$, where $f$ is viewed as a degree zero morphism of locally-free sheaves of $\mathcal{O}_X$-supermodules.
The Koszul-Malgrange theorem implies that $\mathcal{K}^s$ is an equivalence of $O(X)$-linear categories between $P^s(X)$ and $VB^{s,0}(X)$. This gives equivalences of $O(X)$-linear categories:

$$P^s(X) \simeq VB^{s,0}(X) \simeq VB(X) \times VB(X) \quad (B.2)$$

**B.4. Dolbeault superconnections.** The following definition is a variant of the concept of superconnection due to Quillen [20].

**Definition B.6** Let $S$ be a complex vector superbundle on $X$. A Dolbeault superconnection on $S$ is a $\mathbb{C}$-linear map $\mathcal{D} : \mathcal{A}(X,S) \to \mathcal{A}(X,S)$ such that:

1. $\mathcal{D}$ is homogeneous of odd degree with respect to the total $\mathbb{Z}_2$-grading of $\mathcal{A}(X,S)$, i.e.:
   $$\mathcal{D}(\mathcal{A}(X,S)^{\kappa}) \subset \mathcal{A}(X,S)^{\kappa+1}, \quad \forall \kappa \in \mathbb{Z}_2.$$

2. The following condition is satisfied for all $k = 0, \ldots, d$, all $\rho \in \mathcal{A}^k(X)$ and all $\alpha \in \mathcal{A}(X,S)$:
   $$\mathcal{D}(\rho \wedge \alpha) = (\partial \rho) \wedge \alpha + (-1)^k \rho \wedge (\mathcal{D}\alpha).$$

The difference of two Dolbeault superconnections defined on the same complex vector superbundle $S$ is an endomorphism of the $\mathcal{A}(X)$-supermodule $\mathcal{A}(X,S)$ which is odd with respect to the total grading of the latter, i.e. an element of $\mathcal{A}(X,\text{End}(S))^1$. Thus:

**Proposition B.7** The space of Dolbeault superconnections on a complex vector superbundle $S$ is an affine space modeled on the vector space $\mathcal{A}(X,\text{End}(S))^1$.

**Definition B.8** Let $S$ be a complex vector superbundle on $X$. A Dolbeault superconnection on $S$ is called flat if it satisfies the condition $\mathcal{D}^2 = 0$. A Dolbeault superpair is a pair $(S, \mathcal{D})$, where $S$ is a complex vector superbundle. A Dolbeault superpair is called flat if $\mathcal{D}$ is a flat Dolbeault superconnection on $S$.

**B.5. Diagonal, flat and special Dolbeault superconnections.**

**Definition B.9** A Dolbeault superconnection on a complex vector superbundle $S$ defined on $X$ is called diagonal if it is homogeneous of degree one with respect to the rank grading of $\mathcal{A}(X,S)$. A Dolbeault superpair $(S, \mathcal{D})$ is called diagonal if $\mathcal{D}$ is a diagonal Dolbeault superconnection.

Notice that a diagonal Dolbeault superconnection on $S$ is the same as a Dolbeault derivation on $S$ which is compatible with the $\mathbb{Z}_2$-grading of $S$. Diagonal and flat Dolbeault superpairs can be identified with $\mathbb{Z}_2$-graded Dolbeault pairs as follows. Any flat and diagonal Dolbeault superconnection on $S$ decomposes as $\mathcal{D} = \overline{\mathcal{D}}^0 \oplus \overline{\mathcal{D}}^1$, where $\overline{\mathcal{D}}^{\kappa}$ are integrable Dolbeault derivations on the complex vector sub-bundles $S^{\kappa}$. Thus $\mathcal{D}$ is a compatible and integrable Dolbeault derivation on $S$ and $(S, \mathcal{D})$ can be viewed as a $\mathbb{Z}_2$-graded Dolbeault pair. Conversely, any compatible and integrable Dolbeault derivation on $S$ can be viewed as a flat and diagonal Dolbeault superconnection. Accordingly, flat and diagonal Dolbeault superpairs can be identified with holomorphic vector superbundles.
B.6. Special Dolbeault factorizations. Let \((X, W)\) be a Landau-Ginzburg pair.

**Definition B.10** A Dolbeault superconnection \(\mathcal{D}\) on a complex vector superbundle \(S\) defined on \(X\) is called special if it can be written in the form:

\[
\mathcal{D} = \mathcal{D}_0 + \text{id}_{\mathcal{A}(X)} \otimes D, \tag{B.3}
\]

where \(\mathcal{D}_0\) is a diagonal Dolbeault superconnection on \(S\), \(D \in \Gamma_\infty(X, \text{End}^\mathbb{D}_\infty(S))\) is a globally-defined smooth odd endomorphism of \(S\) and \(\otimes\) is the tensor product taken over \(C_\infty(X)\). In this case, \(\mathcal{D}_0\) and \(D\) are uniquely determined by \(\mathcal{D}\) and are called respectively the diagonal and off-diagonal parts of \(\mathcal{D}\), while the decomposition \((B.3)\) is called the canonical decomposition of \(\mathcal{D}\).

A Dolbeault superpair \((S, \mathcal{D})\) is called special if \(\mathcal{D}\) is a special Dolbeault superconnection on \(S\).

**Definition B.11** A special Dolbeault factorization of \(W\) is a special Dolbeault superpair \((S, \mathcal{D})\) such that \(\mathcal{D}^2 = W\text{id}_{\mathcal{A}(X,S)}\).

Special Dolbeault factorizations can be identified with holomorphic factorizations as follows. Let \((S, \mathcal{D})\) be a special Dolbeault factorization of \(W\) with diagonal and off-diagonal parts \(\mathcal{D}_0\) and \(D\). Separating ranks in the condition \(\mathcal{D}^2 = W\text{id}_{\mathcal{A}(X,S)}\) gives the system:

\[
\begin{align*}
\mathcal{D}_0^2 &= 0 \\
\mathcal{D}_0 \circ (\text{id}_{\mathcal{A}(X)} \otimes D) + (\text{id}_{\mathcal{A}(X)} \otimes D) \circ \mathcal{D}_0 &= 0 \\
D^2 &= \text{Wid}_S .
\end{align*}
\]

In particular, \(\mathcal{D}_0\) is a flat diagonal Dolbeault superconnection on \(S\) which we view as a compatible and integrable Dolbeault derivation as explained above. Let \(E \overset{\text{def}}{=} K^* (S, \mathcal{D}_0)\) be the holomorphic vector superbundle defined by \((S, \mathcal{D}_0)\). Then \(\mathcal{D}_0\) coincides with the (compatible) Dolbeault operator \(\overline{\mathcal{D}}_E\) of \(E\) and the second condition above amounts to:

\[
\overline{\mathcal{D}}_E^\text{ad} (D) = 0 , \tag{B.4}
\]

where \(\overline{\mathcal{D}}_E^\text{ad} = \overline{\mathcal{D}}_{\text{End}(E)}\) is the Dolbeault derivation of the holomorphic vector bundle \(\text{End}(E)\). The condition above means that \(D\) is a holomorphic section of \(E\). Thus \((E, D)\) is a holomorphic factorization of \(W\). By the Koszul-Malgrange correspondence, any holomorphic factorization of \(W\) can be obtained in this manner.

B.7. The twisted category of special Dolbeault factorizations. Let \(\mathcal{D}_\infty^s(X)\) denote the \(C_\infty(X)\)-linear \(\mathbb{Z} \times \mathbb{Z}_2\)-graded category whose objects are the complex vector superbundles defined on \(X\) and whose morphism spaces are given by:

\[
\text{Hom}_{\mathcal{D}_\infty^s}(S_1, S_2) = \mathcal{A}(X, \text{Hom}(S_1, S_2)) ,
\]

where the \(C_\infty(X)\)-bilinear composition of morphisms \(\circ : \mathcal{A}(X, \text{Hom}(S_2, S_3)) \times \mathcal{A}(X, \text{Hom}(S_1, S_2)) \to \mathcal{A}(X, \text{Hom}(S_1, S_3))\) is determined uniquely through the condition:

\[
(\rho \otimes f) \circ (\eta \otimes g) = (-1)^{\sigma(f)\mu_\eta} (\rho \wedge \eta) \otimes (f \circ g)
\]

for all pure rank forms \(\rho, \eta \in \mathcal{A}(X)\) and all pure degree elements \(f \in \Gamma(X, \text{Hom}(S_2, S_3))\) and \(g \in \Gamma(X, \text{Hom}(S_1, S_2))\). Given two special Dolbeault factorizations \((S, \mathcal{D}_i)\) \((i = 1, 2)\) of \(W\) (where \(\mathcal{D}_i\) have diagonal and off-diagonal parts denoted respectively \(\mathcal{D}_{i0}\) and \(D_i\)), the space \(\mathcal{A}(X, \text{Hom}(S_1, S_2))\) carries two natural differentials:
• The Dolbeault differential \( \overline{\partial} := \overline{\partial}(S_1, D_1), (S_2, D_2) \) is the unique \( O(X) \)-linear map which satisfies:

\[
\overline{\partial} \alpha = D_{20} \circ \alpha - (-1)^{\deg \alpha} \circ D_{10}
\]

for elements \( \alpha \in A(X, \text{Hom}(S_1, S_2)) \) of pure total degree. This differential preserves the bundle grading and has degree +1 with respect to the rank grading.

• The defect differential is the \( C^\infty(X) \)-linear endomorphism \( \partial := \partial(S_1, D_1), (S_2, D_2) \) of \( A(X, \text{Hom}(S_1, S_2)) \) which is determined uniquely by the condition:

\[
\partial(\rho \otimes f) = (-1)^{rk \rho} \rho \otimes (D_2 \circ f) - (-1)^{rk \rho + \sigma(f)} \rho \otimes (f \circ D_1)
\]

for all pure rank forms \( \rho \in A(X) \) and all pure degree elements \( f \in \Gamma_\infty(X, \text{Hom}(S_1, S_2)) \). This endomorphism squares to zero because \( D_i \) squares to \( W_i \text{id}_{S_i} \). Moreover, this differential preserves the rank grading and is odd with respect to the bundle grading.

Since \( D_1 \) and \( D_2 \) are holomorphic, these two differentials anticommute. Thus:

\[
\overline{\partial}^2 = \partial^2 = \partial \circ \partial + \partial \circ \overline{\partial} = 0
\]

The twisted differential \( \delta := \delta(S_1, D_1), (S_2, D_2) \) is the total differential of this bicomplex:

\[
\delta \overset{\text{def.}}{=} \overline{\partial} + \partial
\]

The twisted differential is odd with respect to the total \( \mathbb{Z}_2 \)-grading.

**Definition B.12** The twisted category \( \text{DF}_\infty(X, W) \) of special Dolbeault factorizations is the \( \mathbb{Z}_2 \)-graded \( C^\infty(X) \)-linear dg-category defined as follows:

• The objects are the special Dolbeault factorizations of \( W \).

• For any two special Dolbeault factorizations \( (S_1, D_1) \) and \( (S_2, D_2) \) of \( W \), the space of morphisms from \( (S_1, D_1) \) to \( (S_2, D_2) \) is the \( C^\infty(X) \)-module:

\[
\text{Hom}_{\text{DF}_\infty(X, W)}((S_1, D_1), (S_2, D_2)) \overset{\text{def.}}{=} A(X, \text{Hom}(S_1, S_2))
\]

endowed with the total grading and with the twisted differential \( \delta(S_1, D_1), (S_2, D_2) \).

• The composition of morphisms is that of \( \text{DF}_\infty(X) \).

**B.8. Equivalence of \( \text{DF}_\infty(X, W) \) with \( \text{DF}(X, W) \).** Recall the twisted Dolbeault category \( \text{DF}(X, W) \) of holomorphic factorizations introduced in Subsection 4. Let \( \mathbb{K} : \text{DF}_\infty(X, W) \rightarrow \text{DF}(X, W) \) be the functor defined as follows:

• Given a special Dolbeault factorization \( (S, \mathcal{D}) \) of \( W \), let \( \mathbb{K}(S, \mathcal{D}) \overset{\text{def.}}{=} (K^*(S, \mathcal{D}_0), D), \) where \( \mathcal{D}_0 \) and \( D \) are the diagonal and off-diagonal parts of \( \mathcal{D} \).

• Given \( \alpha \in \text{Hom}_{\text{DF}_\infty(X, W)}((S_1, D_1), (S_2, D_2)) = A(X, \text{Hom}(S_1, S_2)) \), let \( \mathbb{K}(\alpha) \overset{\text{def.}}{=} \alpha \), viewed as an element of \( \text{Hom}_{\text{DF}(X, W)}(K^*(S_1, D_{10}), K^*(S_2, D_{20})) = A(X, \text{Hom}(K^*(S_1, D_{10}), K^*(S_2, D_{20})) \).

The proof of the following statement is now obvious:

**Proposition B.13** The dg-functor \( \mathbb{K} \) is an equivalence of dg-categories between the twisted category \( \text{DF}_\infty(X, W) \) of special Dolbeault factorizations and the twisted Dolbeault category \( \text{DF}(X, W) \) of holomorphic factorizations.
B.9. Extension of the boundary coupling to compact oriented surfaces with corners. For completeness, we outline briefly how the Lagrangian formulation of [2] extends from compact oriented surfaces with boundary to compact oriented surfaces with corners. Let $\Sigma$ be a compact oriented surface with corners. Let $\text{Int} \Sigma$ denote the interior of $\Sigma$ and $C \overset{\text{def}}{=} \Sigma \setminus \text{Int} \Sigma$ denote the topological frontier. Let $(C_s)_{s=1,\ldots,n}$ denote the connected components of $C$, where $n \overset{\text{def}}{=} \text{Card}_\pi(C)$. Notice that each $C_s$ is homeomorphic with a circle. Let $V_s \subset C_s$ denote the set of those corners of $\Sigma$ which lie on $C_s$. Giving $C_s$ the orientation induced from that of $\Sigma$, we pick an element $v_s^0 \in V_s$ and enumerate $V_s$ starting from $v_s^0$ increasingly with respect to the orientation of $C_s$. This gives an enumeration $V_s = v_s^0, \ldots, v_s^{m_s-1}$, where $m_s \overset{\text{def}}{=} \text{Card} V_s$. For any $k \in \mathbb{Z}$, let $[k]_s \overset{\text{def}}{=} [k \mod m_s] \in \{0, \ldots, m_s - 1\}$; thus $[-1]_s = m_s - 1$ and $[m_s]_s = 0$.

For any $i \in \{0, \ldots, m_s - 1\}$, let $I_i^s$ denote that connected component (an open segment) of the set $C_s \setminus V_s$ which starts at $v_s^i$ and ends at $v_s^{[i+1]}_s$ with respect to the orientation of $C_s$. For every $i \in \{0, \ldots, m_s - 1\}$, pick a special Dolbeault factorization $(S^i_s, \mathcal{D}^i_s)$ of $W$ and an element $\omega^i_s \in \text{Hom}_{\text{DF-loc}}(X,W)((S^{[i]}_s, \mathcal{D}^{[i]}_s), (S^i_s, \mathcal{D}^i_s)) = \mathcal{A}(X,\text{Hom}(S^{[i]}_s, S^i_s))$. Now pick admissible Hermitian metrics $h^i_s$ on the complex vector superbundles $S^i_s$. Then the boundary coupling of the B-type topological Landau-Ginzburg model defined on $\Sigma$ is constructed as in Subsection 4.2. of [2], except that the superconnection factor $\mathcal{U}_s$ given in [2, eq. (4.16)] is replaced by:

$$\mathcal{U}_s \overset{\text{def}}{=} \text{str} \left[ \left( P e^{-\int_0^{m_s-1} d\tau M^{m_s-1}_s(\tau)} \right) \omega^{m_s-1}_s \cdots \left( P e^{-\int_{1}^{1} d\tau M^{1}_s(\tau)} \right) \omega^{1}_s \left( P e^{-\int_{0}^{0} d\tau M^{0}_s(\tau)} \right) \omega^{0}_s \right], \quad (B.5)$$

where $P$ is the path ordering symbol and $M^i_s(\tau)$ is given by equation [2, eq. (4.17)] in terms of the special Dolbeault factorization $(S^i_s, \mathcal{D}^i_s)$ and of the metrics $h^i_s$ (the dagger in that formula becomes the Hermitian conjugate taken with respect to $h^i_s$). Of course, the bundle-form values $\omega^i_s$ must be interpreted super-geometrically using the first identification given in [2, eq. (3.22)]. Notice that the quantities denoted in loc. cit. by $F$ and $G$ are denoted in this paper by $v$ and $u$ (cf. [2, eq. (4.13)] and equation (4.1)). An example is drawn in Figure B.1.

![Fig. B.1. Example of a compact surface with three corners and connected frontier. In this example, we have $n = 1$ and we do not indicate the index $s$.

Using (B.5), it is not hard to see that the path integral arguments of [2] extend to the case of compact oriented surfaces with corners, thereby leading to the conclusion that the total cohomology category $\text{HDF}_\infty(X,W)$ provides a mathematical model of the category of topological D-branes of the B-type topological Landau-Ginzburg theory defined by the LG pair $(X,W)$ (where we assume that $X$ is Calabi-Yau). Proposition B.13 allows us to identify the latter with the total cohomology category $\text{HDF}(X,W)$ of the Dolbeault category of holomorphic factorizations. This identification justifies the statement made in Section 4 that $\text{HDF}(X,W)$ provides a mathematical model for the category of topological D-branes which agrees with the construction of [2].
Acknowledgements. This work was supported by the research grant IBS-R003-S1.

References

1. Lazaroiu, C. I.: On the boundary coupling of topological Landau-Ginzburg models. JHEP 05, 037 (2005)
2. Herbst, M., Lazaroiu, C. I.: Localization and traces in open-closed topological Landau-Ginzburg models. JHEP 05, 044 (2005)
3. Lazaroiu, C. I.: On the structure of open-closed topological field theories in two dimensions. Nucl. Phys. B 603, 497–530 (2001)
4. Babalic, M., Doryn D., Lazaroiu, C. I., Tavakol, M.: On B-type open-closed Landau-Ginzburg theories defined on Calabi-Yau Stein manifolds. arXiv:1610.09813 [math.DG]
5. Voisin, C.: Hodge Theory and Complex Algebraic Geometry, I. Cambridge Studies in Advanced Mathematics 76, Cambridge (2002)
6. Moore, G., Segal, G.: D-branes and K-theory in 2D topological field theory. arXiv:hep-th/0609042
7. Lauda, A. D., Pfeiffer, H.: Open-closed strings: Two-dimensional extended TQFTs and Frobenius algebras. Topology Appl. 155 (7), 623–666 (2008)
8. Vafa, C.: Topological Landau-Ginzburg models. Mod. Phys. Lett. A 6, 337–346 (1991)
9. Labastida, J. M. F., Llatas, P. M.: Topological Matter in Two Dimensions. Nucl. Phys. B 379, 220–258 (1992)
10. Li, C., Li, S., Saito, K.: Primitive forms via polyvector fields. arXiv:1311.1659v3 [math.AG]
11. Chiang, H.-L., Li, M.-L.: Virtual residue and an integral formalism. arXiv:1508.02769v2 [math.AG]
12. Shklyarov, D.: Calabi-Yau structures on categories of matrix factorizations. J. Geometry Phys. 119, 193–207 (2017)
13. Atiyah, M. F.: Complex Analytic Connections in Fibre Bundles. Trans. Amer. Math. Soc. 85, 181–207 (1957)
14. Huybrechts, D.: Complex geometry: An introduction. Springer (2005)
15. Dyckerhoff, T.: Compact generators in categories of matrix factorizations. Duke Math. J. 159 (2), 223–274 (2011)
16. Dyckerhoff, T., Murfet, D.: The Kapustin-Li formula revisited. Adv. Math. 231 (3-4), 1858–1885 (2012)
17. Polishchuk, A., Vaintrob, A.: Chern characters and Hirzebruch-Riemann-Roch formula for matrix factorizations. Duke Math. J. 161 (10), 1863–1926 (2012)
18. Koszul, J.-L., Malgrange, B.: Sur certaines structures fibrées complexes. Archiv der Mathematik 9 (1), 102–109 (1958)
19. Pali, N.: Faisceaux ∂-cohérents sur les variétés complexes. Math. Ann. 336 (3), 571–615 (2006)
20. Quillen, D.: Superconnections and the Chern character. Topology 24 (1), 89–95 (1985)