The Ramanujan Machine: Automatically Generated Conjectures on Fundamental Constants

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Abstract

Fundamental mathematical constants like $e$ and $\pi$ are ubiquitous in diverse fields of science, from abstract mathematics and geometry to physics, biology and chemistry. Nevertheless, for centuries new mathematical formulas relating fundamental constants have been scarce and usually discovered sporadically. In this paper we propose a novel and systematic approach that leverages algorithms for deriving mathematical formulas for fundamental constants and help reveal their underlying structure. Our algorithms find dozens of well-known as well as previously unknown continued fraction representations of $\pi$, $e$, and the Riemann zeta function values. Two conjectures produced by our algorithm, along with many others, are:

\[
\frac{e}{e - 2} = 4 - \frac{1}{5 - \frac{2}{7 - \frac{3}{9 - \frac{4}{11 - \ddots}}}}, \quad \frac{4}{3\pi - 8} = 3 - \frac{1 \cdot 1}{6 - \frac{2 \cdot 3}{9 - \frac{3 \cdot 4}{12 - \ddots}}}
\]

We present two algorithms that proved useful in finding conjectures: a variant of the Meet-In-The-Middle (MITM) algorithm and a Gradient Descent (GD) tailored to the recurrent structure of continued fractions. Both algorithms are based on matching numerical values and thus they conjecture formulas without providing proofs and without requiring any prior knowledge on any underlying mathematical structure. This approach is especially attractive for fundamental constants for which no mathematical structure is known, as it reverses the conventional approach of sequential logic in formal proofs. Instead, our work supports a different conceptual approach for research: computer algorithms utilizing numerical data to unveil mathematical structures, thus trying to play the role of intuition of great mathematicians of the past, providing leads to new mathematical research.

Code available at: \url{http://www.ramanujanmachine.com/} or \url{https://github.com/AnonGit90210/RamanujanMachine}
1 Introduction

Fundamental mathematical constants such as $e$, $\pi$, the golden ratio $\varphi$, and many others play an instrumental part in diverse fields such as geometry, number theory, calculus, fundamental physics, biology, and ecology\(^1\). Throughout history simple formulas of fundamental constants symbolized simplicity, aesthetics, and mathematical beauty\(^2\). A couple of well-known examples include Euler’s identity $e^{i\pi} + 1 = 0$ or the continued fraction representation of the Golden ratio:

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}.$$

The discovery of such Regular Formulas (RFs)\(^1\) was often sporadic and considered an act of mathematical ingenuity or profound intuition. One prominent example is Gauss’ ability to see meaningful patterns in numerical data that led to new fields of analysis such as elliptic and modular functions and to the hypothesis of the Prime Number Theorem. He is even famous for saying “I have the result, but I do not yet know how to get it”\(^5\), which emphasizes the role of identifying patterns and RFs in data as enabling acts of mathematical discovery.

In a different field but in a similar manner, Johannes Rydberg’s discovery of his formula of hydrogen spectral lines\(^6\), resulted from his data analysis of the spectral emission by chemical elements: $\lambda^{-1} = R_H(n_1^2 - n_2^2)$, where $\lambda$ is the emission wavelength, $R_H$ is the Rydberg constant, $n_1$ and $n_2$ are the upper and lower quantum energy levels respectively. This insight, emerging directly from identifying patterns in the data, had profound implications on modern physics and quantum mechanics.

Unlike measurements in physics and all other sciences, most mathematical constants can be calculated to an arbitrary precision\(^2\) (number of digits) with an appropriate formula, thus providing an absolute ground truth. In this sense, mathematical constants contain an unlimited amount of data (e.g. the infinite sequence of digits in an irrational number), which we use as a ground truth for finding new RFs. Since the fundamental constants are universal and ubiquitous in their applications, finding such patterns can reveal new mathematical structures with broad implications, e.g. the Rogers-Ramanujan continued fraction (which has implications on modular forms) and the Dedekind $\eta$ and $j$ functions\(^8\)\(^9\).

Consequently, having systematic methods to derive new RFs can help research in many fields of science.

In this paper, we establish a novel method to learn mathematical relations between constants and we present a list of conjectures found using this method. While the method can be leveraged for many forms of RFs, we demonstrate its potential with equations of the form of polynomial continued fractions (PCFs)\(^10\) in which the partial numerators and denominators follow closed-form polynomials:

$$x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \ldots}}.$$

where $a_n, b_n \in \mathbb{Z}$ for $n = 1, 2, \ldots$ are partial numerators and denominators respectively. PCFs have been of interest to mathematicians for centuries and still are today, e.g. William Broucker’s $\pi$ representation\(^11\) or\(^11\)\(^12\)\(^13\). More on PCFs in Appendix Section F where we present two proofs of PCF formulas.

We demonstrate our approach by finding identities between a PCF and the value of a rational function at a fundamental constant. For simplicity, enumeration and expression aesthetics, we limit ourselves to integer polynomials on both sides of the equality. We propose two search algorithms: The first algorithm uses the Meet-In-The-Middle (MITM) algorithm to a relatively small precision in order to reduce the search space and eliminate mismatches. It increases the precision with a larger number of PCF iterations on the remaining hits to validate them as conjectured RFs, and is therefore called MITM-RF. The second algorithm uses an optimization-based method, which we call Descent&Repel, converging to integer lattice points that define conjectured RFs.

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1 By regular formulas we refer to any mathematical expression or equality that is infinite in nature but can be encapsulated using a finite expression.
2 Exceptions for this are constructions such as the Chaitin’s constant\(^7\).
Our MITM-RF algorithm was able to produce several novel conjectures, for example:

\[
\frac{4}{3\pi - 8} = 3 - \frac{1 \cdot 1}{6 - \frac{2 \cdot 3}{9 - \frac{3 \cdot 5}{12 - \frac{4 \cdot 7}{15 - \frac{5 \cdot 9}{18 - \frac{6 \cdot 11}{21 - \ldots}}}}}}
\]

\[
\frac{e}{e - 2} = 4 - \frac{1}{5 - \frac{2}{6 - \frac{3}{7 - \frac{4}{8 - \frac{5}{9 - \frac{6}{10 - \ldots}}}}}}
\]

\[
\frac{2}{\pi + 2} = 0 + \frac{1 \cdot (3 - 2 \cdot 1)}{3 + \frac{2 \cdot (3 - 2 \cdot 2)}{6 + \frac{3 \cdot (3 - 2 \cdot 3)}{9 + \frac{4 \cdot (3 - 2 \cdot 4)}{12 + \frac{5 \cdot (3 - 2 \cdot 5)}{15 + \ldots}}}}
\]

\[
\frac{1}{e - 2} + 1 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{3}{1 + \frac{1}{1 + \ldots}}}}}
\]

**Results 1-4:** A sample of automatically generated conjectures for mathematical formulas of fundamental constants, as generated by our proposed Ramanujan Machine by applying the MITM-RF algorithm. To the best of our knowledge, these results are previously-unknown conjectures. Both results for \(\pi\) converge exponentially and both results for \(e\) converge super-exponentially. See Table 3 in the Appendix Section A for additional results from our algorithms along with their convergence rates, which we separate to known formulas and to unknown formulas (to the best of our knowledge), and discuss their proofs.

One may wonder whether the conjectures discovered by this work are indeed mathematical identities or merely mathematical coincidences that break down once enough digits are calculated. However, the method employed in this work makes it fairly unlikely for the conjectures to break down. For an enumeration space of \(10^9\) and result accuracy of more than 50 digits, the probability of finding a random match is smaller than \(10^{-40}\). This accuracy still does not imply a proof, as examples of remarkable mathematical coincidences that hold true to high degree (albeit tens of orders of magnitude lower than ours) of approximation indeed exist [14]. Nevertheless, the aesthetic nature of the conjectures, their resemblance to other known formulas, the recreation of known truths by our algorithms, and the aforementioned minuscule probability, make us believe that many (if not all) of the new conjectures are indeed truths awaiting a rigorous proof by the mathematical community. Indeed, since the original version of the paper and the project website went online, several of the conjectures have been proven by contributions from the community. In the past, the development of such proofs sometimes led to new discoveries, such as the consequences on number theory of the proof of Fermat’s last theorem [15]. We believe and hope that proofs of new computer-generated conjectures on fundamental constants will lead to new discoveries in the future.

After discovering dozens of PCFs, we observed empirically that there is a relationship between the ratio of the polynomial order of \(a_n\) and \(b_n\) and the rate at which the formula converges as a function of the number of iterations. This relationship is proven in the Appendix Section C.

In contrast to the method we present, many known RFs of fundamental constants were discovered by conventional proofs, i.e. sequential logical steps derived from known properties of these constants. For example, several RFs of \(e\) and \(\pi\) were generated using the Taylor expansion of the exponent and the trigonometric functions and using Euler’s continued fraction formula [10], which connects an infinite sum and an infinite PCF. In our work, we aim to reverse this process, finding new RFs for the fundamental constants using their numerical data alone, without any prior knowledge about their mathematical structure. Each RF may enable reverse-engineering of the mathematical structure that produces the RF, and in certain cases where the proof uses new techniques, may also provide new insight on the field. Our approach is especially powerful in cases of empirical constants, such as the Feigenbaum constant from chaos theory (Table 2), which are derived numerically from simulations and have no analytic representation.
Given the success of our approach in finding new RFs for fundamental constants, there are many additional avenues for more advanced algorithms and future research. Inspired by worldwide collaborative efforts in mathematics such as the Great Internet Mersenne Prime Search (GIMPS) we launch the initiative www.RamanujanMachine.com dedicated for finding new RFs for fundamental constants. The general community can donate computational time to find RFs, propose mathematical proofs for conjectured RFs, or suggest new algorithms for finding them (see Appendix Section D). Since its inception, The Ramanujan Machine initiative has already yielded fruit, and several of the conjectures posed by our algorithms have already been proven through contributions by colleagues from the world-wide mathematics community (since progress is still being made, the full acknowledgements will be available in the website, and in future updated versions of the paper).

2 Related Work

The process of mathematical research is complex, nonlinear, and often leverages abstract mathematical intuition, all of which are difficult to express and study thoroughly. Respecting this fact, one may think in an oversimplified manner about mathematical research as being separated to two main steps: conjecturing and proving (as in Fig. 1).

While both steps received some attention in the literature, it is the latter step that was studied more extensively in the computer science literature and is known as Automated Theorem Proving (ATP), which focuses on proving existing conjectures. In ATP, algorithms already proved many theorems such as the Four Color Theorem, the Robbins’ problem of optimal stopping, the Lorenz attractor problem, the Kepler Conjecture on the density of sphere packing, as well as proving a conjectured identity for \( \zeta(4) \), and other recent results, including applications of machine learning for ATP.

Our work focuses on automating the first step of the process, generation of new conjectures, which we refer to as Automated Conjecture Generation (ACG). Early work on ACG started with and included important contributions such as the Automated Mathematician and EURISKO, which envisioned the use of computers for the entire process of scientific discovery. Notable work by Fajtlowicz has found new conjectures in graph theory and matrix theory, while other works on the topic used heuristics to discover new mathematical or physical concepts, rules, inequalities, or statements, and were applied in a wide range of fields of mathematics (e.g. number theory) and in various natural sciences. ACG has also appeared as part of a combined approach supporting efforts in ATP. A particularly noteworthy algorithm in this context is PSLQ, which has been used to find formulas "by a combination of inspired guessing and extensive searching".

Our work differs from the others in a few manners. Namely, we present an end-to-end ACG, allowing...
for a fully automatic process including redundancy and false-positive removal without user input. This way, we can validate our conjectures to arbitrary precision using numerical data as the ground truth for conjecturing. Most importantly, our conjectures focus on formulas for **fundamental constants**.

Proposing conjectures is often times more significant than proving them. For this reason some of the most original mathematicians and scientists are known for their famous **unsolved conjectures** rather than for their solutions to other problems, like Fermat’s last theorem, Hilbert’s problems, Landau’s problems, Hardy-Littlewood prime tuple conjecture, Birch-Swinnerton-Dyer conjecture, and of course the Riemann Hypothesis [15, 42, 43, 44, 45]. Maybe the most famous example is Ramanujan, who posed dozens of conjectures involving fundamental constants and considered them to be revelations from one of his goddesses [46]. In our work, **we aim to automate the process of conjectures generation** and demonstrate it by providing new conjectures for **fundamental constants**.

By analyzing mathematical relationships of fundamental constants that are aesthetic and concise, the Ramanujan Machine can eventually extend known works of great mathematicians such as Gauss, Riemann, and Ramanujan himself to help us discover new mathematics.

### 3 The Meet-In-The-Middle-RF Algorithm

*Figure 2: The Meet-In-The-Middle (MITM-RF) method: first we enumerate RHS to a low precision (10 digits), values are stored in a hash-table. Then we enumerate LHS to a low precision and search for matches. The matches are reevaluated to a higher precision and compared again. The process is repeated until a specified, arbitrary decimal precision is reached, thus reducing false positives. The final results are then posed as new conjectures.*

Given a fundamental constant $c$ (e.g. $c = \pi$), our goal is to learn a set of four polynomials $(\alpha, \beta, \gamma, \delta)$:

$$
\frac{\gamma(c)}{\delta(c)} = f_i \left( \text{PCF}(\alpha, \beta) \right)
$$

(3)
for \( \{f_i\} \) a given set of functions (e.g. \( f_1(x) = x, \ f_2(x) = \frac{1}{x}, \ldots \)), where PCF(\( \alpha, \beta \)) means the polynomial continued fraction with the partial numerator and denominator \( a_n = \alpha(n); b_n = \beta(n) \) respectively as defined in Eq. (2). \( \alpha, \beta, \gamma \) and \( \delta \) are integer polynomials.

As showcased in Fig. 2 we start by enumerating over the two sides of Eq. (3) and successively generating many different integer polynomials for \( \alpha, \beta, \gamma, \delta \)\footnote{We delete instances which produce trivial results like \( \gamma = 3 \cdot \delta \) or \( \alpha = 0 \) and instances whose \( \beta \) polynomial has roots at natural numbers, which result in a finite PCF, necessarily representing a rational number.}. We calculate the Right-Hand-Side (RHS) of each instance up to a limited number of iterations and store the results in a hash-table. We continue by evaluating the Left-Hand-Side (LHS) up to a pre-selected decimal point. We attempt to then locate each result from the LHS in the hash-table with the RHS results, where successful attempts are considered as candidate solutions, and will be referred to as "hits".

Since the LHS and RHS calculations are performed up to a limited precision, several of the hits are bound to be false positives. We then eliminate these false positives by calculating the RHS and LHS to a higher precision to reduce the likelihood that the equality is coincidental as shown in Fig. 3.

A naive enumeration method is very computationally intensive with time complexity of \( O(MN) \), where \( M \) and \( N \) are the LHS and RHS space size respectively, and space complexity of \( O(1) \). Since calculating the RHS is more computationally costly, we store the RHS in the hash-table in order to significantly reduce computation time at the expense of space. This makes the algorithm’s time complexity \( O(M+N) \) and its space complexity \( O(N) \). Moreover, the hash-table of the PCF (RHS) can be saved and reused for further LHS enumerations, reducing future enumeration durations by a significant amount.

We also generalize the aforementioned algorithm to allow for \( \alpha \) and \( \beta \) to be interlaced sequences, i.e. they may consist of multiple integer polynomials. In a simple example of such a sequence even values of \( n \) are equal to one polynomial and odd values of \( n \) are equal to a different polynomial. For results and details see Appendix Section A and the MITM-RF code on www.RamanujanMachine.com.

Our proposed MITM algorithm discovered new PCFs other than those previously known. Seeing how successful our algorithm was despite being relatively simplistic, we believe there is still ample room for new results, which should follow by leveraging more sophisticated algorithms, thus discovering hidden
truths about fundamental constants that may be considered to be more exotic than $\pi$ and $e$, perhaps with formulas that are more complex than the PCFs used in this work (see Appendix Section D).

3.1 Other Fundamental Constants with the MITM-RF

\[
\frac{1}{\zeta(3)} = 0^3 + 1^3 - \frac{1^6}{1^3 + 2^3 - \frac{2^6}{2^3 + 3^3 - \frac{3^6}{3^3 + 4^3 - \frac{4^6}{4^3 + 5^3 + \ldots}}}}
\]

\[
\frac{5}{2\zeta(3)} = 2 + 0 \cdot 2 \cdot 4 + \frac{2 \cdot 1^5 \cdot 1}{2 + 1 \cdot 3 \cdot 7 + \frac{2 \cdot 2^5 \cdot 3}{2 + 2 \cdot 4 \cdot 10 + \frac{2 \cdot 3^5 \cdot 5}{2 + 3 \cdot 5 \cdot 13 + \frac{2 \cdot 4^5 \cdot 7}{2 + 4 \cdot 6 \cdot 16 + \ldots}}}
\]

Formulas 1 & 2: New formulas for $\zeta(3)$ can be discovered using prior knowledge incorporated into MITM. The top formula converges polynomially while to bottom converges exponentially. See Appendix Section B and the attached code for details.

We also studied other fundamental constants of more exotic nature than $\pi$ and $e$ and found two new PCFs for Apéry’s constant $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$. Note that the MITM-RF algorithm does not need to use any prior knowledge on the fundamental constant. However, there is a vast body of research on the properties of many fundamental constants from which various structures can be inferred. Hence when aiming for such a constant, one promising way to utilize such prior knowledge is to study other formulas of the fundamental constant, in attempt to find a common element and use that as a prior for the MITM-RF algorithm. Such an approach can reduce dramatically the enumeration space and the computational complexity, thus improving the chance for finding possible solutions. As a proof of concept of this approach for the Apéry constant, consider formulas 1 & 2.

4 Descent&Repel

We propose a GD optimization method and demonstrate its success in finding RFs, and compare it with the MITM-RF method. The MITM-RF method, although proven successful, is not trivially scalable. This issue can be targeted by either a more sophisticated variant or by switching to an optimization-based method, as is done by the following algorithm.

As explained in Section 3 we want to find integer solutions to Eq. (3). This can also be written as the following constrained optimization problem:

\[
\text{minimize } \mathcal{L} = \left\| \frac{\gamma(\pi)}{\delta(\pi)} - \text{PCF}(\alpha, \beta) \right\| \quad \text{s.t. } \{\alpha, \beta, \gamma, \delta\} \subset \mathbb{Z}[x]
\]

Solving this optimization problem with GD appears implausible, since we are only satisfied with global minima without any error and the solutions must be integers. However, we found an important feature of the loss landscape of the described problem that helped us develop a slightly modified GD that we name 'Descent&Repel' (Fig. 4, example of results in Table 1). Minima are not 0-dimensional points but rather $(d-1)$-dimensional manifolds with $d$ being the number of optimization variables, as would be expected given the single constraint. Moreover, we observed empirically that all minima are global and their errors
Figure 4: Schematic diagram of our Descent&Repel algorithm for finding new RFs for fundamental constants, by relying on GD optimization. The key observation that enables this method is that all minima are global ($L = 0$) and appear as $(d - 1)$-dimensional manifolds, where $d$ is the number of optimization variables. Starting with our initial conditions (in this example, consisting of 600 points) on a vertical line, we perform ordinary GD alternating with "Coulomb" repulsion between all the points. Finally, to arrive to grid points, we perform GD toward integral points and toward the minimum curves, alternately. Lastly, we check whether any point satisfies the equation.

are zero, therefore any GD process will result in a solution with $L = 0$. It is well known that any real number can be expressed as a simple continued fraction [47], and the aforementioned feature hints that this may also be true for PCFs with integer polynomials.

We chose the variables of the optimization problem as the coefficients of the $\alpha, \beta, \gamma, \delta$ polynomials in Eq. (4). The algorithm is initialized with a large set of points, specifically in the examples we present all initial conditions were set on a line, as is showcased in Fig. 4. The algorithm is then constructed of three main parts; GD, 'Repel', lattice GD.

First (GD), we iteratively perform GD for each point, with $L = \left\| \frac{\pi}{\delta} \alpha - \beta \right\|$, thus, $x_t = x_{t-1} - \mu \partial L$. Second ('Repel'), to increase the search space we force all the points to push off one another via a Coulomb-like repulsion, $\frac{C}{|a-b|^2}$. The repel mechanism is used to increase the search space and thus the probability of a match. Third (Lattice GD), to enforce the constraint of integer results, we alternate the optimization between the original loss Eq. (4) and an integer loss term that scales like the square of the difference between the value and its closest integer (round). This method allows us to find only points that have $L = 0$ on both losses, meaning an integer solution to our optimization problem.
Table 1: RFs for $\pi$ and $e$ found in a proof-of-concept run of the Descent&Repel algorithm.

| Convergence      | Known / New | Formula | Polynomials |
|------------------|-------------|---------|-------------|
| Exponential      | known       | $\frac{\pi}{2} = 1 + \frac{1}{3 + \frac{1}{5 + \frac{1}{7 + \ldots}}} = a_n = 1 + 2n$, $b_n = n^2$ |
| Super-Exponential| new and proven | $e = 3 + \frac{1}{3 + \frac{1}{5 + \frac{1}{7 + \ldots}}} = a_n = 3 + n$, $b_n = -n$ |

5 Discussion

5.1 Hypotheses Generation

Our results so far point to new interesting questions about fundamental constants; for example, we found many more continued fractions for $e$ than for other constants we tested, despite a smaller space tested for it with our algorithm. Why does it seem that some constants have more RFs compared to others? More generally, which fundamental constants can even be expressed with polynomial PCFs? Could there be constants (also in Section 4) for which RFs do not exist at all? It is intriguing that the research approach we propose with the Ramanujan Machine not only finds conjectures about RFs of fundamental constants, but also finds conjectures about the intrinsic mathematical structures of these constants.

A conjecture about the mathematical structure of PCFs that emerged from this research and that we observed and later proved concerns the rate of convergence of a PCF as a function of the degrees of the $\alpha, \beta$ polynomials. When $\frac{\deg(\beta)}{\deg(\alpha)} > 2$, then the convergence is polynomial in the PCF depth. When the ratio is smaller than 2, then the convergence is super-exponential. When the ratio is precisely 2, then the convergence can be exponential, depending on more subtle conditions (see Appendix Section C for details). This result allowed us to further improve the MITM-RF algorithm.

We propose a systematic way of generating a space of candidate RF conjectures, generalizing beyond the examples that we explored above. To establish new candidate mathematical conjectures, we envision harvesting the scientific literature (e.g., arXiv.org containing over 1.5M papers) as was done in [48] and generalizing RFs with machine learning algorithms such as clustering methods. The rich dataset available online should provide a strong ground truth for candidate RFs, which can be explored using algorithms similar to the ones described in this work. Such approach may discover many new mathematical conjectures that go far beyond PCFs and can be explored in a future work.

5.2 Applications

New RF conjectures could have intriguing applications. Fast converging PCFs and other identities are being utilized for efficient calculation of different constants, for example, one of the most efficient historic methods to compute $\pi$ was based on a formula by Ramanujan [49]. More generally, new RFs could help us calculate other constants faster, like the super-exponential convergence that was demonstrated above for $e$. Another potential application of new RFs is for proving intrinsic properties of fundamental constants. An example is Apéry’s proof that $\zeta(3)$ is irrational, done by representing it as a PCF [50], which led to similar proofs for other constants.

5.3 The Universality of Fundamental Constants

We have so far only provided the groundwork for a far more comprehensive study into fundamental constants and their underlying mathematical structure. With our proposed algorithms and their extensions, we were able to find RFs for the constants $\pi$, $e$, and $\zeta(3)$. Table 1 presents a selection of additional fundamental constants of particular interest to our approach. For part of them, e.g. Feigenbaum constants, no RF is known. We also list a few examples of constants with intrinsic connections to the theory of PCF. Potentially the most interesting constants for further research are the ones coming from other fields, like number theory (not so ironically, some of them are also named after Ramanujan) and various fields of
| Field                                      | Name                        | Decimal Expansion        |
|-------------------------------------------|-----------------------------|--------------------------|
| Related to Continued Fractions            | Lévy’s constant             | \( \gamma = 3.275822 \ldots \) |
|                                           | Khinchin’s constant          | \( K_0 = 2.685452 \ldots \) |
| Physics                                   | First Feigenbaum constant   | \( \delta = 4.669201 \ldots \) |
|                                           | Second Feigenbaum constant  | \( \alpha = 2.502907 \ldots \) |
|                                           | Laplace Limit               | \( \lambda = 0.662743 \ldots \) |
| Number Theory                             | Twin Prime constant         | \( \Pi_2 = 0.660161 \ldots \) |
|                                           | Meissel – Mertens constant  | \( M = 0.261497 \ldots \) |
|                                           | Landau–Ramanujan constant   | \( \Lambda = 0.764223 \ldots \) |
| Combinatorics                             | Euler–Mascheroni constant   | \( \gamma = 0.577215 \ldots \) |
|                                           | Catalan’s constant           | \( G = 0.915965 \ldots \) |
|                                           | ...                         | ...                      |

Table 2: A sample of fundamental constants from different fields, which are all relevant targets for our method, a wider list is available in [1] and [9]. For all of these, new RF conjectures will point to deep underlying connections. There are thousands more constants for which enough numerical data exists and our method is applicable. With further improvement on our suggested approaches, along with new algorithms provided by the community, we expect that such expressions will be found. Note, that some constants in the table like the Feigenbaum constants have no analytical expression what-so-ever, and so far can only be computed using numerical simulation. Therefore, having a RF for them will reveal a hidden truth not only about the constant, but also about the entire field to which it relates.

physics. With such constants, any new RF can point to a new hidden connection between fields of science. With further improvements and new algorithms, applied on the thousands of fundamental constants in the literature, we expect many new RFs to be found.

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A Additional Results by the MITM-RF Algorithm

In this section we show a sample of polynomial continued fractions (PCFs) that were all found by our MITM-RF algorithm, and are listed here in addition to the ones presented in the main text. Our MITM-RF algorithm was able to reproduce previously known and proven results, along with new, previously unknown RFs. Some of these results have been proven by the general mathematics community since being announced on www.RamanujanMachine.com while others are still regarded as conjectures.

| Convergence | Novelty          | Formula          | Polynomials       |
|--------------|------------------|------------------|-------------------|
| Super        | Known            | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 1, \ b_n = 1\) |
|              | New and proven   | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 1, \ b_n = n\) |
|              | New and unproven | \(e = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 3 + n, \ b_n = n\) |
| Exponential  | Known            | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 1 + n, \ b_n = 1 + n\) |
|              | New and unproven | \(e = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 3 + 2n, \ b_n = n(n + 2)\) |
|              | New and unproven | \(e = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 1 + 3n, \ b_n = n(3 - 2n)\) |
|              | New and unproven | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 1 + 3n, \ b_n = n(1 - 2n)\) |
|              | New and unproven | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 1 - 3n, \ b_n = n(3 - 2n)\) |
|              | New and unproven | \(e = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 2 + 3n, \ b_n = n(1 - 2n)\) |
| Polynomial   | Known            | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 1 + 2n, \ b_n = n^2\) |
|              | Known            | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 2, \ b_n = (2n - 1)^2\) |
|              | Known            | \(e = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 6, \ b_n = (2n - 1)^2\) |
|              | Known            | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 3, \ b_n = -2n(2n + 1)\) |
|              | Known            | \(e = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\ldots}}}}}}}}\) | \(a_n = 1, \ b_n = -n(n - 1)\) |

Table 3: (Conjectures 5-8) Sample of automatically generated conjectures for mathematical formulas of fundamental constants, as generated by our proposed Ramanujan Machine by applying the Meet-In-The-Middle Regular Formula (MITM-RF) algorithm. We mark which results are previously unknown conjectures to the best of our knowledge. For each RF we provide the convergence rates and polynomials. We also emphasize that the novelty of our work is in the concept of computer-generated conjectures, and in the specific algorithms we developed. We distinguish these points of novelty from the novelty of the generated results, where some can be known: i.e., we have found this result in the literature, and therefore it serves as a proof-of-concept for the Ramanujan Machine, but it is not considered new. Results could also be new and unproven: i.e., we have not found this result in the literature, we consider it as a new conjecture, until proven or until an equivalent form that is unknown to us is found. Finally, results can be new and proven: i.e., colleagues have suggested simple proofs after the conjecture was published on the website, and therefore it is not considered as a conjecture any longer.

\(^4\)New to the best of our knowledge. Some of these results might have already been found and published or be simple generalizations of previously known results. Regardless of their novelty, the Ramanujan Machine produced these results without any prior knowledge.
B Structured MITM-RF

From previously known representations for Apéry’s constant as infinite sums, and by deriving PCFs from infinite sums using Euler’s continued fraction identity, we found formulas for Apéry’s constant, and noted that they are commonly constructed with high degree polynomials as partial numerators and partial denominators (with the $b_n$ polynomial having double the degree of the $a_n$ polynomial). Yet, they can be transformed into sparse polynomials. Then, by enumerating only on sparse integer polynomials in the MITM algorithm, we were able to find the RFs for Apéry’s constant (Table 4).

\[
\frac{1}{\zeta(3)} = 0^2 + 1^3 - \frac{1^6}{1^3 + 2^3} - \frac{1^6}{2^3 + 3^3} - \frac{2^6}{3^3 + 4^3} - \ldots
\]

\[
\frac{5}{2\zeta(3)} = 2 + \frac{2^3 \cdot 1^3}{2 + 1 \cdot 3 \cdot 7} + \frac{2^3 \cdot 1^3}{2 + 1 \cdot 4 \cdot 10} + \frac{2^3 \cdot 1^3}{2 + 1 \cdot 5 \cdot 13} + \ldots
\]

\begin{align*}
\text{Polynomials} & : \quad a_n = n^3 + (n + 1)^3 \\
& : \quad b_n = -n^6
\end{align*}

Table 4: Two Apéry PCF and their polynomials, matching formulas 1 & 2 in the main text. The top converges with polynomial rate, and the bottom converges with exponential rate.

C PCF Convergence Rate

The method detailed in Section 3 requires estimating the expected accuracy from finite approximation of PCFs. In this section we characterize the convergence rate of the PCFs, as well as a trick that improves this convergence rate for the exponential case.

For two sets of numbers $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ we define the polynomial continued fraction (PCF) generated by them as

\[
[a_0; (b_1, a_1), (b_2, a_2), \ldots] := a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \ldots}}
\]

and the partial PCF as

\[
\eta_n := [a_0; (b_1, a_1), (b_2, a_2), \ldots, (b_n, a_n)]
\]

if the limit exists, we define:

\[
\eta := \lim_{n \to \infty} \eta_n
\]

We also define the tail:

\[
\tau_n := [a_n; (b_{n+1}, a_{n+1}), (b_{n+2}, a_{n+2}), \ldots]
\]

From there it follows that:

\[
\eta = \frac{p(\tau_n)}{q(\tau_n)}
\]

where $p, q$ are polynomials of degree $n - 1$ whose coefficients depend on $\{a_i\}_{i=0}^{n-1}$, $\{b_i\}_{i=1}^{n}$. Specifically, for $a_i, b_i \in \mathbb{Z}$ we have $p, q \in \mathbb{Z}[x]$. 

15
It was shown (Jones & Thron, 1982) that the partial PCF $\eta_n$ can be computed as a series of Matrix-Vector multiplications:

$$
\begin{pmatrix}
 p_0 \\
 p_{-1}
\end{pmatrix}
:=
\begin{pmatrix}
 a_0 \\
 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
 q_0 \\
 q_{-1}
\end{pmatrix}
:=
\begin{pmatrix}
 1 \\
 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
 p_{n+1} & q_{n+1} \\
 p_n & q_n
\end{pmatrix}
:=
\begin{pmatrix}
 a_n & b_n \\
 1 & 0
\end{pmatrix}
\begin{pmatrix}
 p_n & q_n
\end{pmatrix}
$$

$$
\begin{align*}
p_{n+1} &= a_np_n + b_n p_{n-1} \\
q_{n+1} &= a_nq_n + b_nq_{n-1}
\end{align*}
$$

From which $\eta_n$ can be calculated like so:

$$
\eta_n = \frac{p_n}{q_n}
$$

In the following sections we discuss PCFs with integer polynomials for $a_n$ and $b_n$:

$$
a(x), b(x) \in \mathbb{Z}[x]
$$

$$
a_n = a(n) \\
b_n = b(n)
$$

which we will abbreviate as PCFs.

### C.1 PCF Error Bound

Taking the determinant of the above linear equation, we can deduce the following expression for the matrix determinant:

$$
p_{n+1}q_n - q_{n+1}p_n = (-1)^n \prod_{i=1}^{n} b_i
$$

$$
\eta_{n+1} - \eta_n = (-1)^n \frac{\prod_{i=1}^{n} b_i}{q_{n+1}q_n}
$$

Let $k$ be such that $b_i$ is positive for all $i > k$, $a_i$ keep a constant sign, and $q_j, j > k$ be dominated by either $a_i$ or $b_i$, thus either keeping sign or switching sign, thus $q_jq_{j+1}, j > k$ having a constant sign. Such $k$ exists for example for polynomial $a, b$ ($b$ with a positive leading coefficient), where one of the polynomials dominates the $p, q$ series from some point. We get the following Leibniz series:

$$
\sum_{i=k}^{\infty} \eta_{i+1} - \eta_i
$$

Since:

$$
\eta_{n+1} = \eta_k + \sum_{i=k}^{n} \eta_{i+1} - \eta_i = \eta_k + \sum_{i=k}^{n} (-1)^i \frac{\prod_{j=1}^{i} b_j}{q_{i+1}q_i}
$$

the following relation is achieved:

$$
\forall n \geq k \quad \eta_{2n+\kappa} \leq \lim_{n \to \infty} \eta_{2n+\kappa} \leq \lim_{n \to \infty} \eta_{2n+\kappa-1} \leq \eta_{2n+\kappa-1}
$$

where $\kappa \in \{0, 1\}$, depending on the sign of $\prod_{i=1}^{k} b_i$ and $k \mod 2$. Hence we get that $\exists \eta$ and

$$
|\eta - \eta_n| \leq \left| \frac{\prod_{i=1}^{n} b_i}{q_{n+1}q_n} \right|
$$
C.2 1-Periodic PCF

A PCF is called $k$-periodic if $\forall n \in \mathbb{N} \ a_n = a_{n+k}, \ b_n = b_{n+k}$. A 1-periodic PCF is one of the form:

$$a + \frac{b}{a + \frac{b}{a + \ldots}}$$

hence for $(w_n) = (p_n)$ or $(w_n) = (q_n)$:

$$\begin{pmatrix} w_{n+1} \\ w_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix}$$

For $a^2 > -4b$, we find that the promoter matrix is real-diagonalizable

\begin{align*}
eigvals \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} &= \left\{ \frac{a \pm \sqrt{a^2 + 4b}}{2} \right\} = \{ \lambda_\pm \} \\
eigvecs \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} &= \left\{ \begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix} \right\} = \{ v_\pm \}
\end{align*}

\[ \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \kappa_+ v_+ + \kappa_- v_- \]
\[ \begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \lambda_+^n \kappa_+ v_+ + \lambda_-^n \kappa_- v_- \]

And from there, the decomposition for $p, q$ is:

\[ \begin{pmatrix} p_0 \\ p_{-1} \end{pmatrix} = \begin{pmatrix} a_0 \\ 1 \end{pmatrix} = \frac{\lambda_+}{\sqrt{a^2 + 4b}} v_+ - \frac{\lambda_-}{\sqrt{a^2 + 4b}} v_- \]
\[ \begin{pmatrix} q_0 \\ q_{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{v_+ - v_-}{\sqrt{a^2 - 4b}} \]

Thus

\[ \frac{p_{n-1}}{q_{n-1}} = \frac{1}{\sqrt{a^2 + 4b}} \left( \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+^{n} - \lambda_-^{n}} \right) = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+^{n} - \lambda_-^{n}} \]

For $a > 0$ we get that $|\lambda_+| > |\lambda_-| \geq 0$, hence:

\[ \frac{p_{n-1}}{q_{n-1}} = \lambda_+ \frac{1 - \left( \frac{\lambda_-}{\lambda_+} \right)^{n+1}}{1 - \left( \frac{\lambda_-}{\lambda_+} \right)^{n}} \]

\[ \lim_{n \to \infty} \eta_n = \lambda_+ \]

While in the case $a < 0$ we have $|\lambda_-| > |\lambda_+| \geq 0$, which in turn results in:

\[ \lim_{n \to \infty} \eta_n = -\lambda_- \]

Note that provided $\exists \lim_{n \to \infty} \eta_n$ then: $\eta = a + \frac{b}{q}$, yielding a quadratic equation with the same results.
C.3 Types Of Convergence

Not every continued fraction converges. In the case it does, its rate of convergence is either: exponential, super-exponential, or sub-exponential (which seems to be at a polynomial rate, however it is yet to be proven). When the continued fraction does not converge, it may oscillate between a set of values or “converge” to a discrete oscillating cycle, meaning that for a \(k\)-oscillation with values \(\{a_i\}_{i=0}^{k-1}\), we have \(\lim_{n \to \infty} |\eta_n - a_{n \mod k}| = 0\).

In the following parts, we analyze the PCFs behaviour with regard to its defining polynomials \(a, b\). We’ll use the following notation:

\[
\begin{align*}
    d_a &= \deg(a) \\
    d_b &= \deg(b) \\
    a(x) &= \sum_{j=1}^{d_a} a_j x^j \\
    b(x) &= \sum_{i=0}^{d_b} \beta_j x^i
\end{align*}
\]

For an easier analysis of the PCFs behavior, we use the equivalence transformation and define its semi-canonical form \(^5\)

\[
\forall n \in \mathbb{N} \quad c_n := \frac{b_n}{a_{n-1} a_n} \\
\]

\[
a_0 \left(1 + \frac{b_1}{a_0 a_1} \frac{b_2}{a_1 a_2} \right) =: [a_0; (c_1, c_2, \ldots)]
\]

From there it follows:

\[
\eta_n = [a_0; (c_1, \ldots, c_n)]
\]

Unless stated otherwise we’ll regard only the main part of the above PCF:

\[
1 + \frac{b_1}{a_0 a_1} \frac{b_2}{a_1 a_2} = 1 + \frac{c_1}{1 + \frac{c_2}{1 + \ldots}}
\]

We now recognize 3 distinct cases. In the first case, denoted as the exponential case we have:

\[
\begin{align*}
    d_b &= 2d_a \\
    \downarrow \\
    \lim_{n \to \infty} c_n &= \beta_{d_b} a_{d_a} \\
\end{align*}
\]

The second case, denoted as the super-exponential case we have:

\[
\begin{align*}
    d_b &< 2d_a \\
    \downarrow \\
    \lim_{n \to \infty} c_n &= 0 \\
\end{align*}
\]

And finally, the third case, denoted as the sub-exponential or the polynomial case we have:

\[
\begin{align*}
    d_b &> 2d_a \\
    \downarrow \\
    \lim_{n \to \infty} c_n &= \text{sign} (\beta_{d_b}) \cdot \infty
\end{align*}
\]

\(^5\)This is well defined, as we’re examining the tail’s behavior. Therefore neglect \(n\)’s for which \(a_n = 0\), as they are finite.
For all cases, from some point, \( c_n \approx \frac{\beta_{da}}{d_a^2} n^{d_b-2d_a} \), meaning that \( \text{sign}(c_n) = \text{sign}(\beta_{da}) \). Thus \( |q_n| = |q_{n-1} + c_n q_{n-2}| \) is monotonically increasing.

Based on the observation that \( \eta \) is a rational function of \( \tau_n \), it’s enough to show that the above claims for the convergence rate apply for the tail \( \tau_n \) for some \( n \).

C.3.1 Exponential

From some point \( c_n \approx \frac{\beta_{da}}{d_a^2} =: c \), therefore \( q_n \) increases as a generalized Fibonacci series. Specifically, it does not change sign. Therefore we can now refer to the 1-periodic case, as equivalent results can be derived here similarly to Section C.2 since:

\[
\begin{pmatrix}
q_{n+1} \\
q_n
\end{pmatrix} =
\begin{pmatrix}
1 & c \\
1 & 0
\end{pmatrix}^{n-k}
\begin{pmatrix} q_k \\ q_{k-1} \end{pmatrix}
\]

So with the same condition on the determinant, which here translate to:

\[
4c + 1 > 0
\]

\[
\Downarrow
\]

\[
4\beta_{da} > -\alpha_{d_a}^2
\]

We get that the PCF converges to \( \lambda_+ \) (WLOG, we assume that \( |\lambda_+| > |\lambda_-| \). The \( -\lambda_- \) case is similar), and therefore:

\[
\eta_n \approx \kappa \lambda_+ \frac{1 - \left(\frac{\lambda_-}{\lambda_+}\right)^{n+1}}{1 - \left(\frac{\lambda_-}{\lambda_+}\right)^n}
\]

where \( \kappa \) is a constant arising from the point \( k \) at which we assume \( c_i \approx c \). We then receive:

\[
|\eta - \eta_n| \approx \kappa \lambda_+ \left| 1 - \frac{1 - \left(\frac{\lambda_-}{\lambda_+}\right)^{n+1}}{1 - \left(\frac{\lambda_-}{\lambda_+}\right)^n} \right|
\]

\[
= \kappa \lambda_+ \left| \frac{\lambda_-}{\lambda_+} \right|^n \left| -1 + \left(\frac{\lambda_-}{\lambda_+}\right)^n \right|
\]

\[
\leq \kappa \lambda_+ \left| -1 + \left(\frac{\lambda_-}{\lambda_+}\right)^n \right|
\]

Since \( |\lambda_-| < |\lambda_+| \) we get an exponentially decreasing value, hence:

\[
|\eta - \eta_n| \leq \prod_{i=1}^n c_i \leq \kappa_3 \left| \frac{1}{4c + 2 + 2\sqrt{4c + 1}} \right|^{(n-k)}
\]

\[
\Downarrow
\]

\[
|\eta - \eta_n| \leq \kappa_3 \exp \left(-\kappa_4 n\right)
\]

Notice that this result is dependent on \( \kappa \), which makes it an upper bound for the error rather than the exact error (who’s calculation is equivalent to the calculation of \( \eta \)). With a more careful calculation the exponent parameters can be found to get a tighter bound on the error. In sake of brevity, these calculations are omitted.
C.3.2 Super-exponential

Again we assume WLOG that $\beta_d > 0$. Therefore for all $n \geq k$ for some $k$:

$$c_n \approx \frac{\beta_d}{\alpha_d} n^{d_b - 2d_a}$$

Also, since $\forall n > k : c_n > 0$. From there we get that $\forall n > k : q_n > q_k$. Which in turn results in:

$$|\eta - \eta_n| \leq \frac{\prod_{i=1}^{n} c_i}{q_{n+1} q_n} \leq \frac{\prod_{i=1}^{n} c_i}{q_k^n} \approx \kappa_1 \left( \frac{\beta_d}{\alpha_2 d_a} \right)^{n-k} \frac{n!}{k!} d_b - 2d_a$$

$$= \kappa_2 \left( \frac{\beta_d}{\alpha_2 d_a} \right)^{n-k} \left( \frac{n!}{k!} \right)^{d_b - d_a}$$

Since $\frac{\exp(n)}{n!}$ is decreasing super-exponentially, the desired result is obtained.

C.3.3 Sub-Exponential

The case satisfying the determinant constraint $4\beta_d > -\alpha^2 d_a$ can be seen as a limit of the exponential convergence case with $c \to \infty$, therefore the derived convergence is sub-exponential. We believe this sub-exponential convergence to be polynomial.

C.3.4 Tail Estimation

In the case of an exponentially converging PCF, we found that from some point the tail is approximately:

$$1 + \frac{c}{1 + \frac{c}{1 + \ldots}}$$

We calculated the convergence value of 1-periodic PCFs like this earlier. Therefore, we can improve a PCF calculation by substituting this tail at the final step. The accuracy improvement wasn’t analyzed, but empiric results display an improvement of a fixed number of digits (for any large $n$). This in turn allowed us to improve the complexity of the MITM-RF algorithm.

D Collaborative Algorithm-Enhanced Mathematics

The Ramanujan Machine in its most general sense can be seen as a methodology to generate conjectures on fundamental constants. The more computational power and the more time the algorithm runs on a selected space of parameters, the more conjectures it may generate. Moreover, since the Ramanujan Machine produces conjectures on fundamental constants but not their proofs, we realize that computational power as well as proving power (i.e. time spent by an intelligent being trying to prove or refute a conjecture) are key assets for making the Ramanujan Machine more prolific. It is the goal of this section to discuss how one may leverage these facts about the Ramanujan Machine methodology to inspire the wider community about mathematics and number theory.

We created the Ramanujan Machine as an open source project that is fully available to the community on [www.RamanujanMachine.com](http://www.RamanujanMachine.com). Soon, with our ongoing development, individuals around the world would be able to donate their computational power to the mission of discovering new mathematical structures and mathematical equations by downloading the Ramanujan Machine screen saver on the website. Similarly to SETI (Search for Extraterrestrial Intelligence), we plan to have the Ramanujan
Machine screen saver distribute via BOINC the various computational tasks to every computer in the network, so when a computer is idle, the Ramanujan Machine is initiated.

We believe this methodology can inspire the greater community about mathematics. In order to achieve this goal, the site www.RamanujanMachine.com includes an up-to-date record of some (and in a short time every) conjecture generated by the machine. When a specific computer in the network discovers a new conjecture, after verifying that the conjecture had not been discovered elsewhere, the owner of the laptop will receive the credit for contributing his or her computer power to discover the conjecture and the credit is maintained in a leadership board. Since the Ramanujan Machine is a conjecture-generating machine (similarly to much of the work of Ramanujan himself), we let the community suggest proofs for each conjecture, thus honorably claiming affiliation to Ramanujan’s legacy, and introducing an algorithm-enhanced approach for collaborative research.

It is important to emphasize that the methodology introduced in this work can be expanded far beyond continued fractions, number theory or mathematics. The Ramanujan Machine is an example of a broader methodology that has three core elements in its pipeline, as shown in the main text (Fig. 1).

E Further Information about the Descent&Repel Method and Results

This section provides an additional example of the Descent&Repel optimization process (in Fig. 5), in addition to providing Table 5 with further information about the process presented in the main text (in Fig. 4). The parameters chosen for Fig. 4 illustrate the optimization steps relatively clearly, however without converging to any real solution. In Fig. 5 we present a similar illustration (Fig. 5), presenting the convergence to $e = 3 + \frac{-1}{5 + \frac{-1}{6 + \frac{-1}{7 + \ldots}}}$.

Below are the parameters required to reproduce the results in Fig. 4 and Fig. 5.

| Fig. Number | Parameters | Values |
|-------------|------------|--------|
| Fig. 4      | $a(n)$     | $n$    |
|             | $b(n)$     | $n^2 + ny + x$ |
|             | $x$ range  | $[-20, 20]$ |
|             | $y$ range  | $[-20, 20]$ |
|             | fraction depth | 10 |
|             | constant   | $\pi$ |
|             | initial points | 600, uniform at $y \in [-15, 15]$ and $x = 10$ |
| Fig. 5      | $a(n)$     | $n + x$ |
|             | $b(n)$     | $-n + y$ |
|             | $x$ range  | $[-30, 20]$ |
|             | $y$ range  | $[-40, 10]$ |
|             | fraction depth | 20 |
|             | constant   | $e - 3$ |
|             | initial points | 500, uniform at $y = -20$ and $x \in [-10, 5]$ |

Table 5: Execution settings required to reproduce the Descent&Repel maps (Fig. 4, Fig. 5). Here, $a, b$ are similar to the polynomials $\alpha, \beta$ which define the PCF, but the RF is of the form: $\frac{a_0 + \frac{\pi}{a_1 + \ldots}}{a_0 + \frac{\pi}{a_1 + \ldots}}$.  

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Figure 5: Descent&Repel illustration, as in Fig. 4. Here the showcased scenario is of that of the restoration of our previous result (found by the MITM-RF lgorithm) $e = 3 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \ldots}}}$ . The converging point is the one at $(x, y) = (4, -1)$.

F Proofs of Ramanujan Machine Results

Since launching the Ramanujan Machine initiative [www.RamanujanMachine.com](http://www.RamanujanMachine.com) and posting there the new results conjectured by our algorithms, the world-wide community has taken notice and (so far) two results have been proven. These exemplify the concept discussed in Section D. We show the proofs here.

F.1 Proof for $e = 3 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \ldots}}}$

The proof for the aforementioned continued fraction was presented in [51] (can also be proven with a variant of the Euler continued fraction). The proof relies on the following feature of PCFs [52]. Define the auxiliary series on the $a_n$ and $b_n$ polynomials:

$$A_n = \begin{cases} b_nA_{n-1} + a_nA_{n-2} & \text{if } n > 1 \\ b_0 & \text{if } n = 0 \\ 1 & \text{if } n = -1 \end{cases}, \quad B_n = \begin{cases} b_nB_{n-1} + a_nB_{n-2} & \text{if } n > 1 \\ 1 & \text{if } n = 0 \\ 0 & \text{if } n = -1 \end{cases}$$

The value of the PCF is the limit of the ratio between the two auxiliary sequences $PCF(\alpha, \beta) = \lim_{n \to \infty} \frac{A_n}{B_n}$. In the case of this PCF, the auxiliary series are:
\[ A_n = \begin{cases} 
(n + 3)A_{n-1} - nA_{n-2} & \text{if } n > 1 \\
3 & \text{if } n = 0 \\
1 & \text{if } n = -1 
\end{cases}, \quad B_n = \begin{cases} 
(n + 3)B_{n-1} + -nB_{n-2} & \text{if } n > 1 \\
1 & \text{if } n = 0 \\
0 & \text{if } n = -1 
\end{cases} \]

And by using induction and observing that \( A_n \) is a shifted version of sequence A001339 in [3] the author of [51] gets that:

\[ B_n = \frac{((n+1))!^2}{n!} = (n+1) \cdot (n+1)! \quad , \quad A_n = \sum_{k=0}^{n+1} (k+1)! \binom{n+1}{k} \]

From there, the authors calculate the limit of the ratio and conclude the proof:

\[ \lim_{n \to \infty} \frac{A_n}{B_n} = \lim_{n \to \infty} \frac{n + 2}{n + 1} \sum_{k=0}^{n+1} \frac{1}{k!} - \frac{1}{n + 1} \sum_{k=0}^{n} \frac{1}{k!} = e \]

We expect there to be a more general proof that covers any PCF in which both polynomials are 1st order, but it seems to require certain identities of the Lerch Transcendent (see page 475 in [53]) and requires further research.

### F.2 Proof for \( \frac{4}{\pi^2} = 3 + \frac{13}{5 + \frac{2}{8 + \frac{2}{11}}/} \)

By using Gauss’s continued fraction [52] we know that:

\[ \frac{2F_1(a + 1, b; c + 1; z)}{c_2F_1(a, b; c; z)} = \frac{1}{c + \frac{(a-c)b\z}{(c+1)+\frac{(b-c+1)(a+1)}{(c+2)+\frac{(c-c+1)(a+2)}{(c+3)+\frac{(c-c+1)(a+3)}{(c+4)+\ldots}}}} \]

From there, by substituting \( a = 0, b = \frac{1}{2}, c = \frac{3}{2}, z = -1 \) we arrive at the conjectured PCF:

\[ 3 \frac{2F_1(0, \frac{1}{2}; \frac{3}{2}; -1)}{2F_1(1, \frac{1}{2}; \frac{3}{2}; -1)} = 3 + \frac{1 \cdot 3}{5 + \frac{2 \cdot 4}{7 + \frac{3 \cdot 5}{9 + \ldots}}} \]

Note that the last step required using identities on the ratio of generalized hypergeometric functions. We are also attempting to use the Gauss’s continued fraction to prove any of the other conjectures found by the algorithms of the Ramanujan Machine and we will update the paper and the website with more proofs as they come and after we verify them. Each case requires different identities of generalized hypergeometric functions, and it remains to be seen whether such functions can prove all the cases discovered so far by the algorithms of the Ramanujan Machine. While most conjectures are expected to have relatively simple proofs drawing from properties of functions such as the hypergeometric functions, it is also likely that proving some conjectures will eventually require inventing new mathematical techniques.