Relativistic hodograph equation for a two-dimensional stationary isentropic hydrodynamical motion

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Abstract

We derive a relativistic hodograph equation for a two-dimensional stationary isentropic hydrodynamical motion. For the case of stiff matter, when the velocity of sound coincides with the light speed, the singularity in this equation disappears and the solutions become regular in all hodograph plane.

The non-relativistic problem for two-dimensional stationary isentropic hydrodynamical motion was formulated and solved by Chaplygin in 1902 (see [1]). The solution was based on a transformation from the physical (coordinate) plane \(x_1, x_2\) to the velocity plane \(v_1, v_2\). This transformation is called hodograph transformation. The beauty of this Chaplygin’s approach consists in the fact that it allows to undertake a transition from two non-linear hydrodynamical equations to one linear partial derivatives equation for a potential in the hodograph plane.

Below we formulate and solve general stationary two-dimensional problem for the relativistic hydrodynamics. We use some results from the work by Khalatnikov (1954) [2], where the general approach to the relativistic hydrodynamics was formulated and the general one-dimensional non-stationary Chaplygin problem was solved.
To begin with let us introduce the variables: the relativistic velocity \( u_i \) such that
\[
\begin{align*}
\frac{u_i u^i}{c^2} &= u_1^2 + u_2^2 - u_4^2 = -1, \\
\end{align*}
\]
the enthalpy per particle \( w \) and the particle density \( n \) \cite{1, 2}. From the quasi-potentiality condition \cite{2} in the relativistic hydrodynamics
\[
w u_i = \frac{\partial \varphi}{\partial x^i},
\]
follows that for the potential \( \varphi \):
\[
d\varphi = w u_1 dx_1 + w u_2 dx_2.
\]
Let us make a Legendre transformation to the hodograph plane, going from the variables \( x_1 \) and \( x_2 \) to the variables \( u_1 \) and \( u_2 \), and introducing the potential
\[
\chi = \varphi - x_1 w u_1 - x_2 w u_2.
\]
For the potential motion we write instead of the relativistic Euler equation its first integral, i.e. the Bernouilli equation \cite{3, 2}
\[
w u_4 = \text{const}.
\]
In the hodograph plane we introduce the “angles” \( \eta \) and \( \theta \) such that
\[
\begin{align*}
\begin{align*}
&u_1 = \sinh \eta \cos \theta, \\
&u_2 = \sinh \eta \sin \theta, \\
&u_3 = \cosh \eta.
\end{align*}
\end{align*}
\]
In these variables the differential of the potential \( \chi \) is
\[
d\chi = -w[(x_1 \cos \theta + x_2 \sin \theta) \cosh \eta d\eta \\
+(-x_1 \sin \theta + x_2 \cos \theta) \sinh \eta d\theta] \\
-(x_1 \cos \theta + x_2 \sin \theta) \sinh \eta dw.
\]
Taking into account the Bernouilli relation \cite{5} and omitting the normalization factor we have
\[
\begin{align*}
\frac{\partial \chi}{\partial \eta} &= -\frac{1}{\cosh^2 \eta} (x_1 \cos \theta + x_2 \sin \theta), \\
\frac{\partial \chi}{\partial \theta} &= -\tanh \eta (-x_1 \sin \theta + x_2 \cos \theta).
\end{align*}
\]
\(^1\)We choose the velocity of light equal to zero.
The absolute value of the spatial velocity $v$ is

$$v = \tanh \eta.$$  \hfill (9)

The inverse transformation to the physical space is

$$x_1 = -\left(\cos \theta \frac{\partial \chi}{\partial v} - \frac{\sin \theta}{v} \frac{\partial \chi}{\partial \theta}\right),$$
$$x_2 = -\left(\sin \theta \frac{\partial \chi}{\partial v} + \frac{\cos \theta}{v} \frac{\partial \chi}{\partial \theta}\right).$$ \hfill (10)

The relation between potentials $\varphi$ and $\chi$ is

$$\varphi = \chi - v \frac{\partial \chi}{\partial v}.$$ \hfill (11)

Note that these relations have the same form as their non-relativistic analogs [1].

Now, we can deduce the equation for the potential $\chi$ using the continuity equation

$$\frac{\partial}{\partial x_i} (nu^i) = \frac{\partial}{\partial x_1} (nu_1) + \frac{\partial}{\partial x_2} (nu_2) = 0,$$ \hfill (12)

or, in terms of the variables $w, v$ and $\theta$

$$\frac{\partial}{\partial x_1} \left(\frac{n}{w} v \cos \theta\right) + \frac{\partial}{\partial x_2} \left(\frac{n}{w} v \sin \theta\right) = 0.$$ \hfill (13)

To get the equation for $\chi$, we should make in the preceding equation (13) the transition to the variables $v$ and $\theta$:

$$\frac{\partial^2 \chi}{\partial \theta^2} + \frac{v^2 (1 - v^2)}{1 - \frac{v^2}{c^2}} \frac{\partial^2 \chi}{\partial v^2} + v \frac{\partial \chi}{\partial v} = 0,$$ \hfill (14)

where the sound velocity $c$ is defined as [2]

$$c^2 = \frac{n}{w} \frac{\partial w}{\partial n}.$$ \hfill (15)

The relativistic hodograph equation (14) together with Eqs. (10) play a role of equations of motion. Thus, the problem of solution of nonlinear equations of motion is reduced to the solution of the linear equation for $\chi$ in the hodograph plane. However, the boundary conditions for Eq. (14) are nonlinear (for details see [1]).
Let us pay a special attention to the crucial difference with respect to the non-relativistic case [1]

\[ \frac{\partial^2 \chi}{\partial \theta^2} + \frac{v^2}{1 - \frac{v^2}{c^2}} \frac{\partial^2 \chi}{\partial v^2} + v \frac{\partial \chi}{\partial v} = 0, \]

which consists in the appearance of the factor \((1 - v^2)\) in front of the term \(\frac{\partial^2 \chi}{\partial v^2}\). Both these equations have a singularity at \(v = c\) which corresponds to a transition to the supersonic regime. The supersonic regime brings also another problem, which is connected with the possible vanishing of the Jacobian

\[ \frac{\partial (x_1, x_2)}{\partial (\theta, v)} = \frac{1}{v} \left[ \left( \frac{\partial^2 \chi}{\partial v \partial \theta} - \frac{1}{v} \frac{\partial \chi}{\partial \theta} \right)^2 + \frac{v^2 (1 - v^2)}{1 - \frac{v^2}{c^2}} \left( \frac{\partial^2 \chi}{\partial v^2} \right)^2 \right], \]

which in the subsonic regime is always positive. Nullification of this Jacobian on some “limiting” line \(v = v(\theta)\) in the supersonic regime makes the velocity \(v\) complex on one side of this limiting line [1]. It signifies that the appearance of shock waves is unavoidable in this regime.

It is extremely interesting that these problems are absent for the relativistic motion of a fluid with the stiff equation of state when pressure coincides with energy density and the sound velocity coincides with the speed of light \((c = 1)\). In this case the factors \((1 - v^2)\) and \(\left(1 - \frac{v^2}{c^2}\right)\) cancel each other. Naturally, the supersonic regime seems to be impossible for the stiff fluid, because such a regime would be also a superluminal one. Note, however, that even considering supersonic/superluminal regime, we always have real solutions of the Chaplygin equation and the potential and velocity cannot become complex.

The consideration presented above can be treated as an introduction to the further study of two-dimensional motion in the relativistic hydrodynamics, which could have astrophysical and cosmological applications.

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Appendix. Some general remarks concerning the relativistic hydrodynamics

One-dimensional relativistic non-stationary Chaplygin problem was studied in paper by Khalatnikov [2]. This study was undertaken for further
development of the Landau-Fermi theory of high-energy multiple particle production. Here, we would like to mention only its main results and write down the relativistic non-stationary one-dimensional Chaplygin equation.

Starting with the quasi-potentiality condition (2) and introducing the velocity components as
\[ u_1 = \sinh \eta, \quad u_4 = \cosh \eta, \]
we undertake the Legendre transformation
\[ \chi = \varphi - w u_1 x_1 - w u_4 x_4. \]
Acting as above we come to the equation
\[ \frac{1}{n} \frac{\partial n}{\partial w} \left( \frac{\partial \chi}{\partial w} - \frac{1}{w} \frac{\partial^2 \chi}{\partial \eta^2} \right) + \frac{\partial^2 \chi}{\partial w^2} = 0. \]
Using the expression (15) for the sound velocity and introducing a new variable \( y = \ln w \), one gets the following equation
\[ c^2 \frac{\partial^2 \chi}{\partial y^2} + (1 - c^2) \frac{\partial \chi}{\partial y} - \frac{\partial^2 \chi}{\partial \eta^2} = 0. \]
This equation represents a solution of the one-dimensional non-stationary problem in relativistic hydrodynamics. As before it should be accompanied by the choice of the special boundary conditions and by formulas, which express the time and spatial coordinate variable as functions of the potential \( \chi \):
\[ t = e^{-y} \left( \frac{\partial \chi}{\partial y} \cosh \eta - \frac{\partial \chi}{\partial \eta} \sinh \eta \right), \]
\[ x_1 = e^{-y} \left( \frac{\partial \chi}{\partial y} \sinh \eta - \frac{\partial \chi}{\partial \eta} \cosh \eta \right). \]

Again, in the stiff matter case \( c = 1 \), Eq. (21) simplifies essentially and becomes a simple one-dimensional wave equation.

References
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