Wellposedness and Convergence of Solutions to a Class of Forced Non-diffusive Equations with Applications

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Abstract. This paper considers a family of non-diffusive active scalar equations where a viscosity type parameter enters the equations via the constitutive law that relates the drift velocity with the scalar field. The resulting operator is smooth when the viscosity is present but singular when the viscosity is zero. We obtain Gevrey-class local well-posedness results and convergence of solutions as the viscosity vanishes. We apply our results to two examples that are derived from physical systems: firstly a model for magnetostrophic turbulence in the Earth’s fluid core and secondly flow in a porous media with an “effective viscosity”.

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1. Introduction

Active scalar equations arise in many areas of fluid dynamics, with the most classical being the two dimensional Euler equation for an incompressible, inviscid flow in vorticity form. Another much studied active scalar equation is the surface quasi-geostrophic equation (SQG) introduced by Constantin et al. [5] as a two dimensional analogue of the three dimensional Euler equation (c.f [8–10,17,22]). The physics of an active scalar equation is encoded in the constitutive law that relates the transport velocity vector $u$ with the scalar field $\theta$. This law produces a differential operator that when applied to the scalar field determines the velocity. The singular or smoothing properties of the operator are closely connected with the mathematical properties of the active scalar equation. In this present paper we study the following class of non-diffusive active scalar equations in $\mathbb{T}^d \times (0,\infty) = [0,2\pi]^d \times (0,\infty)$ with $d \geq 2$:

\[
\begin{align*}
\partial_t \theta + u^\nu \cdot \nabla \theta &= S, \\
u_j^\nu &= \partial_x T_{ij}^\nu(\theta^\nu), \theta^\nu(x,0) = \theta_0(x)
\end{align*}
\]

(1.1)

where $\nu \geq 0$. Here $\theta_0$ is the initial condition and $S = S(x)$ is a given smooth function that represents the forcing of the system.

Our motivation for addressing such a class of active scalar equations comes from two rather different physical systems that under particular parameter regimes give rise to systems of the form (1.1). The first example comes from MHD and a model proposed by Moffatt and Loper [19], Moffatt [21] for magnetostrophic turbulence in the Earth’s fluid core. Under the postulates in [19], the governing equation reduces to a three dimensional active scalar equation for a temperature field $\theta$

\[
\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta + S
\]

(1.2)

where the constitutive law is obtained from the linear system

\[
e_3 \times u = -\nabla P + e_2 \cdot \nabla b + \theta e_3 + \nu \Delta u,
\]

(1.3)

\[
0 = e_2 \cdot \nabla u + \Delta b,
\]

(1.4)

\[
\nabla \cdot u = 0, \nabla \cdot b = 0.
\]

(1.5)
This system encodes the vestiges of the physics in the problem, namely the Coriolis force, the Lorentz force and gravity. Vector manipulations of (1.3–1.5) give the expression
\[
\{[\nu \Delta^2 - (e_2 \cdot \nabla)^2] + (e_3 \cdot \nabla)^2 \Delta \} u = \frac{\nu \Delta^2 - (e_2 \cdot \nabla)^2}{\nu} \nabla \times (e_3 \times \nabla \theta) + (e_3 \cdot \nabla) \Delta (e_3 \times \nabla \theta).
\]  
(1.6)

Here \((e_1, e_2, e_3)\) denote Cartesian unit vectors. The explicit expression for the components of the Fourier multiplier symbol \(\hat{M}^\nu\) as functions of the Fourier variable \(k = (k_1, k_2, k_3) \in \mathbb{Z}^3\) are obtained from the constitutive law (1.6) to give
\[
\hat{M}_1^\nu(k) = [k_2 k_3 |k|^2 - k_1 k_3 (k_2^2 + \nu |k|^4)] D(k)^{-1},
\]  
(1.7)
\[
\hat{M}_2^\nu(k) = [-k_1 k_3 |k|^2 - k_2 k_3 (k_2^2 + \nu |k|^4)] D(k)^{-1},
\]  
(1.8)
\[
\hat{M}_3^\nu(k) = [(k_1^2 + k_2^2) (k_2^2 + \nu |k|^4)] D(k)^{-1},
\]  
(1.9)
where
\[
D(k) = |k|^2 k_3^2 + (k_2^2 + \nu |k|^4)^2.
\]  
(1.10)

The nonlinear equation (1.2) with \(u\) related to \(\theta\) via (1.6) is called the magnetogeostrophic (MG) equation and its mathematical properties have been studied in a series of papers including [11–16]. In the magnetostrophic turbulence model the parameters \(\nu\), the nondimensional viscosity, and \(\kappa\), the nondimensional thermal diffusivity, are extremely small. The behavior of the MG equation is dramatically different when the parameters \(\nu\) and \(\kappa\) are present (i.e. positive) or absent (i.e. zero). The limit as either or both parameters vanish in highly singular. Since both parameters multiply a Laplacian term, their presence is smoothing. However \(\kappa\) enters (1.2) in a parabolic heat equation role whereas \(\nu\) enters via the constitutive law (1.6). The mathematical properties of the MG equation have been determined in various settings of the parameters via an analysis of the Fourier multiplier symbol \(\hat{M}^\nu\) given by (1.7–1.10).

We note that the Fourier multiplier symbols \(\hat{M}^0\) given by (1.7–1.10) with \(\nu = 0\) are not bounded in all regions of Fourier space [15]. More specifically in “curved” regions where \(k_3 = O(1), k_2 = O(|k_1|^{1/2})\) the symbols are unbounded as \(|k_1| \to \infty\) with \(|\hat{M}^0(k)| \leq C |k|\) for some positive constant \(C\). Thus when \(\nu = 0\) the relation between \(u\) and \(\theta\) is given by a singular operator of order 1. The implications of this fact for the inviscid MG equation are summarized in the survey article by Friedlander et al. [11]. In particular, when \(\kappa > 0\) the inviscid but thermally dissipative MG equation is globally well-posed. In contrast when \(\nu = 0\) and \(\kappa = 0\), the singular inviscid MG\(^0\) equation is ill-posed in the sense of Hadamard in any Sobolev space. In a recent paper [14] Friedlander and Suen examine the limit of vanishing viscosity in the case when \(\kappa > 0\). They prove global existence of classical solutions to the forced MG\(^\nu\) equations and obtain strong convergence of solutions as the viscosity vanishes. In this present paper we turn to the case where \(\kappa = 0\) and examine the MG\(^\nu\) system, without the benefit of thermal diffusion, both when \(\nu > 0\) and in the case \(\nu = 0\) where the operator MG\(^0\) is singular of order 1. We obtain Gevrey-class local well-posedness and convergence of solutions as \(\nu \to 0\). The precise statements of the theorems are given in Sect. 2 in the context of a general class of non-diffusive active scalar equations that includes the MG equations with \(\kappa = 0\).

The second example of a physical system which can be modeled by an active scalar equation where a small smoothing parameter enters into the constitutive law comes from flow in a porous medium. The incompressible porous media Brinkman equation with an “effective viscosity” \(\nu\) is derived via a modified Darcy’s Law as suggested by Brinkman [4]. The 2D equation relating the velocity \(u\), the density \(\theta\) and the pressure \(P\) is given in non-dimensional form by
\[
\frac{\rho}{\rho_0} \frac{D}{D} \frac{\nu \Delta u}{\nu_0} - e_2 \theta + \nu \Delta u = \nabla P = -e_2 \theta + \nu \Delta u
\]  
(1.11)

\[
\nabla \cdot u = 0
\]  
(1.12)
which produces the constitutive law

\[ u = (1 - \nu\Delta)^{-1}[\nabla \cdot (-\Delta)^{-1}e_2 \cdot \nabla \theta - e_2 \theta] \]

\[ = (1 - \nu\Delta)^{-1}R^\perp R_1 \theta \]  \hspace{1cm} (1.13)

where \( R = (R_1, R_2) \) is the vector of Riesz transforms. The 2D components of the Fourier multiplier symbol corresponding to (1.13) are

\[
\frac{1}{1 + \nu(k_1^2 + k_2^2)} \left( \frac{k_1k_2}{k_1^2 + k_2^2}, \frac{-k_1^2}{k_1^2 + k_2^2} \right) \]

Again there is a dramatic difference in the operator between the two cases \( \nu > 0 \) and \( \nu = 0 \). In the first case the operator is smoothing of order 2 and in the second case the operator is singular of order zero. The IPMB active scalar example is a two dimensional nonlinear equation for \( \theta \) given by

\[ \partial_t \theta + (u \cdot \nabla) \theta = 0 \]  \hspace{1cm} (1.15)

coupled with the constitutive law (1.13).

The well known IPM equations, i.e. (1.13)–(1.15) without the effective viscosity \( \nu \), have been studied in a number of papers, c.f [6,7]. As we observed when \( \nu = 0 \) the operator in (1.13) is a singular integral operator of order zero. This is also the case for the SQG equations. However the SQG operator is odd where as the IPM operator is even, a property that it shares with the MG operator. Implications for well/ill posedness due to the odd/even structure of the operator in an active scalar equation are explored in [10,12,18]. In this present paper we study the system (1.13)–(1.15) in the limit of vanishing viscosity. We obtain results analogous to those for the MG \( \nu \) system in the limit of vanishing viscosity. The principal difference is that the MG \( \nu = 0 \) operator is singular of order 1 where as the IPM operator is singular of order zero. In the “smoother” IPM case our convergence results are valid in Sobolev spaces rather than the Gevrey-class convergence results for the MG equation.

2. Main Results for a General Class of Active Scalar Equations

We now return to the abstract formulation of our problem in the setting of the following system of active scalar equations parameterized by a “viscosity” parameter \( \nu \).

\[
\begin{cases}
\partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu = S, \\
u^\nu_j = \partial_x T^\nu_{ij}[\theta^\nu], \theta^\nu(x, 0) = \theta_0(x)
\end{cases}
\]

where \( \mathbb{T}^d \times (0, \infty) = [0, 2\pi]^d \times (0, \infty) \) with \( d \geq 2 \). We assume that

\[ \int_{\mathbb{T}^d} \theta^\nu(x, t)dx = \int_{\mathbb{T}^d} S(x) = 0, \]

for all \( t \geq 0 \). \( \{T^\nu_{ij}\}_{\nu \geq 0} \) is a sequence of operators\(^1\) which satisfy:

A1 \( \partial_i \partial_j T_{ij}^\nu f = 0 \) for any smooth functions \( f \) for all \( \nu \geq 0 \).

A2 \( T^\nu_{ij} : L^\infty(\mathbb{T}^d) \to BMO(\mathbb{T}^d) \) are bounded for all \( \nu \geq 0 \).

A3 For each \( \nu > 0 \), there exists a constant \( C_\nu > 0 \) such that for all \( 1 \leq i, j \leq d \),

\[ |\widehat{T^\nu_{ij}}(k)| \leq C_\nu |k|^{-3}, \forall k \in \mathbb{Z}^d. \]

A4 For each \( 1 \leq i, j \leq d \),

\[ \lim_{\nu \to 0} \sum_{k \in \mathbb{Z}^d} |\widehat{T^\nu_{ij}}(k) - \widehat{T^0_{ij}}(k)|^2|\hat{g}(k)|^2 = 0 \]

for all \( g \in L^2 \).

\(^1\)For simplicity, we sometime write \( T^0_{ij} = T^\nu_{ij} \big|_{\nu=0} \) and \( u^\nu = \partial_x T^\nu[\theta^\nu] \).
Moreover, we further assume that \( \{T_{ij}^\nu\}_{\nu \geq 0} \) satisfy either one of the following assumption:

A5\(_1\) There exists a constant \( C_0 > 0 \) independent of \( \nu \), such that for all \( 1 \leq i, j \leq d \),
\[
\sup_{\nu \in (0, 1)} \sup_{k \in \mathbb{Z}^d} |\hat{T}_{ij}^\nu(k)| \leq C_0;
\]
\[
\sup_{k \in \mathbb{Z}^d} |\hat{T}_{ij}^0(k)| \leq C_0.
\]

A5\(_2\) There exists a constant \( C_0 > 0 \) independent of \( \nu \), such that for all \( 1 \leq i, j \leq d \),
\[
\sup_{\nu \in (0, 1)} \sup_{k \in \mathbb{Z}^d} |k_i\hat{T}_{ij}^\nu(k)| \leq C_0;
\]
\[
\sup_{k \in \mathbb{Z}^d} |k_i\hat{T}_{ij}^0(k)| \leq C_0.
\]

Here are some remarks regarding the assumptions on \( T_{ij}^\nu \):

- The assumption A1 implies that \( u^\nu \) is divergence-free for all \( \nu \geq 0 \).
- The assumption A3 is needed for obtaining global-in-time wellposedness for (2.1) for the case \( \nu > 0 \), which implies that \( \partial_x T^\nu \) are operators of smoothing order 2 for \( \nu > 0 \).
- The assumption A5\(_2\) is stronger than assumption A5\(_1\). In particular, assumption A5\(_2\) implies that \( \partial_x T^\nu \) are operators of zero order.
- The assumption A4 is needed for obtaining convergence of solutions as \( \nu \to 0 \), and is consistent with the one given in [14].
- All the assumptions A1–A4 and A5\(_1\) are consistent with the case for the magnetogeostrophic (MG) equations, while assumptions A1–A4 and A5\(_2\) are consistent with the case for incompressible porous media Brinkmann (IPMB) equations.

The main results that we prove in this present work are stated in the following theorems:

**Theorem 2.1** (Wellposedness in Sobolev space in the case \( \nu > 0 \)). Let \( \theta_0 \in W^{s,d} \) for \( s \geq 0 \) and \( S \) be a \( C^\infty \)-smooth source term. Then for each \( \nu > 0 \), under the assumptions A1–A3 and A5\(_i\) for \( i = 1 \) or 2, we have:

- if \( s = 0 \), there exists unique global weak solution to (2.1) such that
  \[
  \theta^\nu \in BC((0, \infty); L^d),
  \]
  \[
  u^\nu \in C((0, \infty); W^{2,d}).
  \]  

In particular, \( \theta^\nu(\cdot, t) \to \theta \) weakly in \( L^d \) as \( t \to 0^+ \). Here BC stands for bounded continuous functions.
- if \( s > 0 \), there exists a unique global-in-time solution \( \theta^\nu \) to (2.1) such that \( \theta^\nu(\cdot, t) \in W^{s,d} \) for all \( t \geq 0 \). Furthermore, for \( s = 1 \), we have the following single exponential growth in time on \( \|\nabla \theta^\nu(\cdot, t)\|_{L^d} \):  
  \[
  \|\nabla \theta^\nu(\cdot, t)\|_{L^d} \leq C\|\nabla \theta_0\|_{L^d} \exp \left( C\left( t\|\theta_0\|_{W^{1,d}} + t^2\|S\|_{L^\infty} + t\|S\|_{W^{1,d}} \right) \right),
  \]  
where \( C > 0 \) is a constant which depend only on \( \nu \) and the spatial dimension \( d \).

**Theorem 2.2** (Gevrey-class global wellposedness in the case \( \nu > 0 \)). Fix \( s \geq 1 \). Let \( \theta_0 \) and \( S \) be of Gevrey-class \( s \) with radius of convergence \( \tau_0 > 0 \). Then for each \( \nu > 0 \), under the assumptions A1–A3 and A5\(_i\) for \( i = 1 \) or 2, there exists a unique Gevrey-class \( s \) solution \( \theta^\nu \) to (2.1) on \( \mathbb{T}^d \times [0, \infty) \) with radius of convergence at least \( \tau = \tau(t) \) for all \( t \in (0, \infty) \), where \( \tau \) is a decreasing function satisfying
\[
\tau(t) \geq \tau_0 e^{-C \left( \|e^{\tau_0 \theta_0} \|_{L^2} + 2\|e^{\tau_0 \theta_0} S\|_{L^2} \right) t}.
\]  
Here \( C > 0 \) is a constant which depends on \( \nu \) but independent of \( t \).
Theorem 2.3 (Gevrey-class local wellposedness in the case $\nu = 0$). Fix $s \geq 1$, $r > \frac{d}{2} + \frac{3}{2}$ and $K_0 > 0$. Let $\theta^0(\cdot, 0) = \theta_0$ and $S$ be of Gevrey-class $s$ with radius of convergence $\tau_0 > 0$ and satisfy
\[
\|A^r e^{r_0 A^\frac{1}{2}} \theta^0(\cdot, 0)\|_{L^2} \leq K_0, \quad \|A^r e^{r_0 A^\frac{1}{2}} S\|_{L^2} \leq K_0.
\]

For $\nu = 0$, under the assumptions $A1$–$A2$ and $A5_1$, there exists $\bar{T}, \bar{\tau} > 0$ and a unique Gevrey-class $s$ solution $\theta^0$ to (2.1) defined on $\mathbb{T}^d \times [0, \bar{T}]$ with radius of convergence at least $\bar{\tau}$. In particular, there exists a constant $C = C(K_0) > 0$ such that for all $t \in [0, \bar{T}]$,
\[
\|A^r e^{r_0 A^\frac{1}{2}} \theta^0(\cdot, t)\|_{L^2} \leq C.
\]
Moreover, if the assumption $A3$ holds as well, then we have
\[
\|A^r e^{r_0 A^\frac{1}{2}} \theta^0(\cdot, t)\|_{L^2} \leq C, \forall \nu > 0,
\]
where $\theta^\nu$ are Gevrey-class $s$ solutions to (2.1) for $\nu > 0$ as described in Theorem 2.2.

Theorem 2.4 (Local wellposedness in Sobolev space in the case $\nu = 0$ and Property $A5_2$ holds). For $d \geq 2$, we fix $s > \frac{d}{2} + 1$. Assume that $\theta_0, S \in H^s(\mathbb{T}^d)$ have zero-mean on $\mathbb{T}^d$. Then for $\nu = 0$, under the assumption $A1$–$A2$ and $A5_2$, there exists a $T > 0$ and a unique smooth solution $\theta^0$ to (2.1) such that $\theta^0 \in L^\infty(0, T; H^s(\mathbb{T}^d))$.

Theorem 2.5 (Convergence of solutions as $\nu \to 0$). Depending on the assumptions $A5_1$ and $A5_2$, we have the following cases:

- Assume that the hypotheses and notations of Theorem 2.3 are in force. Under the assumptions $A3$–$A4$, if $\theta^\nu$ and $\theta^0$ are Gevrey-class $s$ solutions to (2.1) for $\nu > 0$ and $\nu = 0$ respectively with initial datum $\theta_0$ on $\mathbb{T}^d \times [0, \bar{T}]$ with radius of convergence at least $\bar{\tau}$ as described in Theorem 2.3, then there exists $T < \bar{T}$ and $\tau = \tau(t) < \bar{\tau}$ such that, for $t \in [0, T]$, we have
\[
\lim_{\nu \to 0} \|\langle A^r e^{r_0 A^\frac{1}{2}} \theta^\nu - A^r e^{r_0 A^\frac{1}{2}} \theta^0(\cdot, t)\|_{L^2} = 0.
\]
- Assume that the hypotheses and notations of Theorem 2.4 are in force. Under the assumptions $A3$–$A4$, for $d \geq 2$ and $s > \frac{d}{2} + 1$ and $t \in [0, T]$, we have
\[
\lim_{\nu \to 0} \|\theta^\nu - \theta^0(\cdot, t)\|_{H^{s-1}} = 0.
\]

3. Preliminaries

We introduce the following notations. We say $(\theta, u)$ is a weak solution to (2.1) if they solve the system in the weak sense, that means for all $\phi \in C_0^\infty(\mathbb{T}^d \times (0, \infty), \mathbb{R}^d)$, we have
\[
\int_0^\infty \int_{\mathbb{T}^d} (\partial_t \phi + u \cdot \nabla \phi) \theta(x, t)dxdt + \int_{\mathbb{T}^d} \phi(x, 0) \theta_0(x)dx = \int_0^\infty \int_{\mathbb{T}^d} \phi S(x)dxdt.
\]
$W^{s,p}$ is the usual inhomogeneous Sobolev space with norm $\| \cdot \|_{W^{s,p}}$. For simplicity, we write $\| \cdot \|_{L^p} = \| \cdot \|_{L^p(\mathbb{T}^d)}$, $\| \cdot \|_{W^{s,p}} = \| \cdot \|_{W^{s,p}(\mathbb{T}^d)}$, etc. unless otherwise specified. We also write $H^s = W^{s,2}$.

We define $\| \cdot \|_{L,L}$ to be the Log-Lipschitz norm given by
\[
\|f\|_{L,L} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|(1 + |\log |x - y||)}.
\]
As in [16,20], for $s \geq 1$, the Gevrey-class $s$ is given by
\[
\bigcup_{\tau > 0} D(A^r e^{r_0 A^\frac{1}{2}}),
\]
for any $r \geq 0$, where
\[
\|\Lambda e^{\tau \Lambda^{1/2}} f\|^2_{L^2} = \sum_{k \in \mathbb{Z}^d} |k|^{2r} e^{2r|k|^2} |\hat{f}(k)|^2,
\]
where $\tau = \tau(t) > 0$ denotes the radius of convergence and $\Lambda = (-\Delta)^{1/2}$.

We also recall the following facts from the literature (see for example Azzam–Bedrossian [1], Bahouri–Chemin–Danchin [2] and Ziemer [24]):

\[
\|f\|_{L^1} \leq C\|\nabla f\|_{BMO},
\]
\[
\|f\|_{BMO} \leq C\|f\|_{W^{1,d}},
\]
\[
\|f\|_{L^\infty} \leq C\|f\|_{W^{2,d}},
\]
and for $q > d$, there are constants $C(q) > 0$ such that
\[
\|f\|_{L^\infty} \leq C(q)\|f\|_{W^{1,q}}.
\]
If $k > l$ and $k - \frac{d}{p} > l - \frac{d}{q}$, then we have
\[
\|f\|_{W^{1,q}} \leq C\|f\|_{W^{k,p}}.
\]

4. The Non-diffusive Active Scalar Equations

In this section, we study the non-diffusive equations (2.1) for $\nu \geq 0$. Depending on the values of $\nu$, we subdivide it into two cases, namely $\nu > 0$ and $\nu = 0$.

4.1. The Case Where $\nu > 0$

In this subsection we study the non-diffusive equations (2.1) for $\nu > 0$. First, given $\nu > 0$ and initial datum $\theta_0 \in W^{s,d}$ with $s > 0$, we prove that (2.1) has a unique global-in-time solution $\theta_\nu \in W^{s,d}$.

We begin with the following lemma which gives a priori bounds on $\theta_\nu$.

**Lemma 4.1.** Let $\nu > 0$ and $\theta_0 \in W^{s,p}$ for $s > 0$ and $p > 1$, and let $S = S(x)$ be a $C^\infty$-smooth source term. Then we have
\[
\|\theta_\nu(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p} + t\|S\|_{L^\infty},
\]
and we also have a priori bounds on $\theta_\nu$
\[
\|\theta_\nu(\cdot,t)\|_{W^{s,p}} \leq C\|\theta_0\|_{W^{s,p}} \exp\left(C \int_0^t \|\nabla u_\nu(\cdot,\tilde{t})\|_{L^\infty} d\tilde{t} + Ct\|S\|_{W^{s,p}}\right).
\]
Here $C > 0$ is a constant which depends on $p$ and the spatial dimension $d$.

**Proof.** The assertion (4.1) follows from standard energy estimates. And for (4.2), we let $\Delta_k$’s be the dyadic blocks for $k \in \{-1\} \cup \mathbb{N}$. Applying $\Delta_k$ on (2.1),
\[
(\partial_t + u \cdot \nabla)(\Delta_k \theta_\nu) = R_k + \Delta_k S,
\]
where $R_k = u \cdot \nabla \Delta_k \theta_\nu - \Delta_k (u \cdot \nabla \theta_\nu)$. Since
\[
\|\Delta_k S(\cdot,t)\|_{L^p} + \|R_k(\cdot,t)\|_{L^p} \\
\leq C C_k(t) \left[2^{-ks}\|\nabla u_\nu(\cdot,t)\|_{L^\infty} \|\theta_\nu(\cdot,t)\|_{W^{s,p}} + \|S\|_{W^{s,p}}\right]
\]
with $\|C_k(\cdot,t)\|_{L^p} = 1$, which implies (4.2).

**Proof of Theorem 2.1.** We divide it into several cases.
**Case 1.** \( s = 0. \) First, we observe that, by the assumption A3, \( \partial_x, T^\nu_{ij} \) is a smoothing operator of degree 2 for \( \nu > 0. \) Hence with the help of the Fourier multiplier theorem (see Stein [23]), given \( p > 1, \) there exists some constant \( C = C(p,d) > 0 \) such that

\[
\|u^\nu(\cdot,t)\|_{W^{2,p}} \leq C\|\theta^\nu(\cdot,t)\|_{L^p}.
\] (4.3)

Together with (4.1), for \( p > 1, \) if \( \theta_0 \in L^p \) and \( S \in C^\infty, \) then we have

\[
\|u^\nu(\cdot,t)\|_{W^{2,p}} \leq C (\|\theta_0\|_{L^p} + t\|S\|_{L^\infty}),
\] (4.4)

where \( t\|S\|_{L^\infty} \) is replaced by \( \int_0^\infty \|S(\cdot,\tilde{t})\|_{L^\infty} d\tilde{t}. \) Next, using embedding theorems (3.1)–(3.3) and (4.4) for \( p = d, \) we have

\[
\|u^\nu(\cdot,t)\|_{L^\infty} \leq C\|u^\nu(\cdot,t)\|_{W^{2,d}}
\]

\[
\leq C (\|\theta_0\|_{L^p} + t\|S\|_{L^\infty}),
\] (4.5)

and

\[
\|u^\nu(\cdot,t)\|_{L,L,L} \leq C\|\nabla u^\nu(\cdot,t)\|_{BMO}
\]

\[
\leq C\|\nabla u^\nu(\cdot,t)\|_{W^{1,d}}
\]

\[
\leq C\|u^\nu(\cdot,t)\|_{W^{2,d}}
\]

\[
\leq C (\|\theta_0\|_{L^p} + t\|S\|_{L^\infty}),
\] (4.6)

which shows that both \( \|u^\nu(\cdot,t)\|_{L,L,L} \) and \( \|u^\nu(\cdot,t)\|_{L^\infty} \) are bounded in terms of \( \theta_0 \) and \( S. \) A bound on the Log-Lipschitzian norm of \( u \) is essential to assure the existence and uniqueness of the flow map, and hence the existence and uniqueness of the solution.

More precisely, to prove the existence of flow map, we consider the standard mollifier \( \theta \in C_0^\infty, \) and we set \( \theta_0^{(n)} = \theta_n * \theta_0 \) for \( n \in \mathbb{N} \) and \( \theta_n(x) = n^d \theta(nx). \) By a standard argument, given \( \nu > 0, \) we can obtain a sequence of global smooth solution \( (\theta^\nu(n), u^\nu(n)) \) to (2.1) with \( u^\nu(n) = \partial_x T^\nu_{ij}[\theta^\nu(n)]. \) Define \( \psi_n(x,t) \) to be the flow map given by

\[
\partial_t \psi_n(x,t) = u^\nu(n)(\psi_n(x,t),t).
\] (4.7)

One can show (for example in [3]) that

\[
\|\psi_n(\cdot,t)\|_s \leq C \exp \left( \int_0^t \|u^\nu(n)(\cdot,\tilde{t})\|_{L,L,L} d\tilde{t} \right),
\] (4.8)

where \( C > 0 \) is independent of \( \nu \) and \( n, \) and the norm \( \| \cdot \|_s \) is given by

\[
\|\psi\|_s = \sup_{x \neq y} \Phi(\|\psi(x) - \psi(y)\|, |x - y|)
\]

with

\[
\Phi(r,s) = \begin{cases} 
\max\{\frac{1+|\log(s)|}{1+|\log(r)|}, 1+\frac{|\log(s)|}{1+|\log(r)|}\}, & \text{if } (1-s)(1-r) \geq 0, \\
(1+|\log(s)|)(1+|\log(r)|), & \text{if } (1-s)(1-r) \leq 0.
\end{cases}
\]

Using (4.6) (with \( u^\nu \) replaced by \( u^\nu(n) \)) and (4.8), we obtain

\[
|\psi_n(x_1,t) - \psi_n(x_2,t)| \leq \alpha(t)|x_1 - x_2|^{\beta(t)}
\] (4.9)

for all \( (x_1,t), (x_2,t) \in \mathbb{R}^d \times \mathbb{R}^+, \) where \( \alpha(t), \beta(t) \) are some continuous functions which depends on \( \theta_0 \) and \( S. \) Furthermore, for \( t_1, t_2 \geq 0, \) using (4.5) (with \( u^\nu \) replaced by \( u^\nu(n) \)),

\[
|\psi_n(x,t_1) - \psi_n(x,t_2)| \leq C|t_2 - t_1|(\|u^\nu(n)(\cdot,t_1)\|_{L^\infty} + \|u^\nu(n)(\cdot,t_2)\|_{L^\infty})
\]

\[
\leq C|t_2 - t_1|(\|\theta_0\|_{L^p} + \max\{t_1, t_2\}\|S\|_{L^\infty}).
\] (4.10)

Applying the estimates (4.9) and (4.10), we see that the family \( \{\psi_n\}_{n \in \mathbb{N}} \) is bounded and equicontinuous on every compact set in \( \mathbb{R}^d \times \mathbb{R}^+. \) By Arzela–Ascoli theorem, it implies the existence of a limiting trajectory.
ψ(x, t) as n → ∞. Performing the same analysis for \{ψ_n^{-1}\}, where ψ_n^{-1} is the inverse of ψ_n, we see that ψ(x, t) is a Lebesgue measure preserving homeomorphism with

\[
\|ψ^{-1}(\cdot, t)\| = \|ψ(\cdot, t)\| \leq C \exp \left( \int_0^t C \left( \|θ_0\|_{L^d} + \bar{t} \|S\|_{L^∞} \right) d\bar{t} \right) 
\]

\[
\leq C \exp \left( C \left( \|θ_0\|_{L^d} + t^2 \|S\|_{L^∞} \right) \right), \forall t ≥ 0.
\]

Define θ\nu(x, t) = θ₀(ψ⁻¹(x, t)) and u\nu = ∂₁T\nu₂[T\nu]. The rest of the proof then follows from the one given in [13], which shows that (θ\nu, u\nu) is a weak solution to (2.1).

To show that (θ\nu, u\nu) is unique, let T > 0 and ν > 0, and suppose that (θ\nu₁, u\nu₁) and (θ\nu₂, u\nu₂) solve (2.1) on \mathbb{T}^d × [0, T] with θ\nu₁(⋅, 0) = θ\nu₂(⋅, 0) = θ₀. Following the similar argument given in [2], there exists a constant C > 0 such that for all δ ∈ (0, 1) and k ∈ {−1} ∪ \mathbb{N}, we have

\[
\|Δ_k(θ\nu₁ - θ\nu₂)(⋅, t)\|_{L^∞} 
\]

\[
\leq 2^{kδ}(k + 1)C(\|u\nu₁(⋅, t)\|_{L^∞} + \|u\nu₂(⋅, t)\|_{L^∞}) \|θ\nu₁ - θ\nu₂\|(⋅, t)\|_{B^{-δ}_{∞,∞}},
\]

for all t ∈ [0, T], where \|⋅\|_{L^∞} = \|⋅\|_{L^∞} + \|⋅\|_{L^L}. We define

\[
\bar{t} = \sup \left\{ t \in [0, T] : C \int_0^t (\|u\nu₁(⋅, \bar{t})\|_{L^∞} + \|u\nu₂(⋅, \bar{t})\|_{L^∞}) d\bar{t} ≤ \frac{1}{2} \right\},
\]

then by the bounds (4.4) and (4.5), \bar{t} is well-defined. We let

\[
δ_{\bar{t}} = C \int_0^{\bar{t}} (\|u\nu₁(⋅, \bar{t})\|_{L^∞} + \|u\nu₂(⋅, \bar{t})\|_{L^∞}) d\bar{t}.
\]

Using Theorem 3.28 in [2], for all k ≥ −1 and t ∈ [0, \bar{t}],

\[
2^{-kδ_{\bar{t}}}\|Δ_k(θ\nu₁ - θ\nu₂)(⋅, t)\|_{L^∞} \leq \frac{1}{2} \sup_{t \in [0, \bar{t}]} \|θ\nu₁ - θ\nu₂\|(⋅, t)\|_{B^{-δ}_{∞,∞}}.
\]

Summing over k and taking supremum over [0, \bar{t}], we conclude that θ\nu₁ = θ\nu₂ on [0, \bar{t}]. By repeating the argument a finite number of times, we obtain the uniqueness on the whole interval [0, T].

**Case 2.** s > 0. We only need a priori bounds on θ\nu. In view of (4.2) with p = d, it remains to obtain a bound on \|Δu\nu(⋅, t)\|_{L^∞}. We claim

\[
\|Δu\nu(⋅, t)\|_{L^∞} \leq C(\|θ₀\|_{W^{s,d}} + \bar{t} \|S\|_{L^∞}), \forall t ≥ 0.
\]

We subdivide into two subcases.

**Case 2(a)** 0 < s < 1. Define q = \frac{d}{1−s}. Then q > d and using the embedding that W^{s,d} ⊂ L^q, we have

\[
\|θ₀\|_{L^q} \leq C\|θ₀\|_{W^{s,d}}.
\]

Therefore using (3.4), (4.1) and (4.3) with p = d, we conclude that

\[
\|Δu\nu(⋅, t)\|_{L^∞} \leq C(q)\|u\nu(⋅, t)\|_{W^{2,q}} 
\]

\[
\leq C(\|θ₀\|_{W^{s,d}} + \bar{t} \|S\|_{L^∞}).
\]

**Case 2(b)** s ≥ 1. Using (4.3), we take p = 2d, which gives

\[
\|Δu\nu(⋅, t)\|_{W^{1,2d}} \leq C\|θ\nu(⋅, t)\|_{L^{2d}}.
\]

On the other hand, we apply (4.1), and the embeddings W^{1,2d} ⊂ L^∞ and W^{s,d} ⊂ W^{\frac{s}{2},d} ⊂ L^{2d} to get

\[
\|Δu\nu(⋅, t)\|_{L^∞} \leq C\|Δu\nu(⋅, t)\|_{W^{1,2d}} 
\]

\[
\leq C\|θ\nu(⋅, t)\|_{L^{2d}} 
\]

\[
\leq C(\|θ₀\|_{L^{2d}} + \bar{t} \|S\|_{L^∞}).
\]
\[
\leq C \left( \|\theta_0\|_{W^{\frac{2}{d},d}} + t\|S\|_{L^\infty} \right) \\
\leq C \left( \|\theta_0\|_{W^{\frac{2}{d},d}} + t\|S\|_{L^\infty} \right).
\]

We substitute the above estimates on \( \|\nabla u^\nu(\cdot,t)\|_{L^\infty} \) into (4.2) with \( p = d \) and obtain the desired \textit{a priori} required bounds on \( \theta^\nu \), namely
\[
\|\theta^\nu(\cdot,t)\|_{W^{r,d}} \\
\leq C \|\theta_0\|_{W^{r,d}} \exp \left( C \int_0^t \|\nabla u^\nu(\cdot,\bar{t})\|_{L^\infty} d\bar{t} + Ct\|S\|_{W^{r,d}} \right) \\
\leq C \|\theta_0\|_{W^{r,d}} \exp \left( C \int_0^t \left( \|\theta_0\|_{W^{r,d}} + \bar{t}\|S\|_{L^\infty} \right) d\bar{t} + Ct\|S\|_{W^{r,d}} \right) \\
\leq C \|\theta_0\|_{W^{r,d}} \exp \left( C \left( t\|\theta_0\|_{W^{r,d}} + t^2\|S\|_{L^\infty} + t\|S\|_{W^{r,d}} \right) \right). \tag{4.12}
\]

Finally, the single exponential growth in time on \( \|\nabla \theta^\nu(\cdot,t)\|_{L^d} \) follows readily from (4.12) and Poincaré inequality. We finish the proof of Theorem 2.1. \( \square \)

\textbf{Remark 4.2.} For \( r > 0 \), if we take \( s > \max\{\frac{2r+2-d}{2}, r\} \), then by the Sobolev embedding theorem (3.5) and the bound (4.12), we have
\[
\|\theta^\nu(\cdot,t)\|_{H^r} \leq C \|\theta^\nu(\cdot,t)\|_{W^{r,d}} \\
\leq C \|\theta_0\|_{W^{r,d}} \exp \left( C \left( t\|\theta_0\|_{W^{r,d}} + t^2\|S\|_{L^\infty} + t\|S\|_{W^{r,d}} \right) \right),
\]

where \( H^r = W^{r,2} \). Hence Theorem 2.1 implies that there exists a unique global-in-time \( H^r \) solution \( \theta^\nu \) to (2.1) whenever \( \theta_0 \in W^{s,d} \) and \( S \in C^\infty \) for \( r > 0 \) and \( s > \max\{\frac{2r+2-d}{2}, r\} \).

Next we study the Gevrey-class \( s \) solutions to (2.1) for \( \nu > 0 \) when the initial datum \( \theta_0 \) and forcing term \( S \) are in the same Gevrey-class. Recall that for \( \nu > 0 \), there exists a constant \( C_\nu > 0 \) such that for all \( 1 \leq i, j \leq d, \)
\[
\|T^\nu_{ij}(k)\| \leq C_\nu |k|^{-3}, \forall k \in \mathbb{Z}^d,
\]
which gives 2-orders of smoothing
\[
\|u^\nu\|_{H^2} \leq C_\nu \|\theta^\nu\|_{L^2}. \tag{4.13}
\]

To prove the global-in-time existence as claimed in Theorem 2.2, we give the estimates of \( \theta^\nu \) as follows. We take \( L^2 \)-inner product of (2.1) with \( e^{2\tau \Lambda^\frac{1}{2}} \theta^\nu \) and obtain
\[
\frac{1}{2} \frac{d}{dt} \|e^{\Lambda^\frac{1}{2}} \theta^\nu\|_{L^2}^2 - \tau \|\Lambda^\frac{1}{2} e^{\tau \Lambda^\frac{1}{2}} \theta^\nu\|_{L^2}^2 \\
= -\langle u^\nu \cdot \nabla \theta^\nu, e^{2\tau \Lambda^\frac{1}{2}} \theta^\nu \rangle + \langle S, e^{2\tau \Lambda^\frac{1}{2}} \theta^\nu \rangle \\
\leq \left| -\langle u^\nu \cdot \nabla \theta^\nu, e^{2\tau \Lambda^\frac{1}{2}} \theta^\nu \rangle \right| + \|e^{\tau \Lambda^\frac{1}{2}} S\|_{L^2}^2 \|e^{\tau \Lambda^\frac{1}{2}} \theta^\nu\|_{L^2}. \tag{4.14}
\]

The following lemma gives the estimates on the term \(-\langle u^\nu \cdot \nabla \theta^\nu, e^{2\tau \Lambda^\frac{1}{2}} \theta^\nu \rangle \).

\textbf{Lemma 4.3.} For \( \nu > 0 \) and \( s \geq 1 \), we have
\[
\left| -\langle e^{\tau \Lambda^\frac{1}{2}}(u^\nu \cdot \nabla \theta^\nu), e^{\tau \Lambda^\frac{1}{2}} \theta^\nu \rangle \right| \leq C\tau \|e^{\tau \Lambda^\frac{1}{2}} \theta^\nu\|_{L^2} \|\Lambda^\frac{1}{2} e^{\tau \Lambda^\frac{1}{2}} \theta^\nu\|_{L^2}^2, \tag{4.15}
\]
where \( C = C(\nu) > 0 \) depends on \( \nu \).

\textbf{Proof.} The proof is reminiscent of the one given in [20]. Since \( \nabla \cdot u^\nu = 0 \) we have
\[
\langle u^\nu \cdot \nabla e^{\tau \Lambda^\frac{1}{2}} \theta^\nu, e^{\tau \Lambda^\frac{1}{2}} \theta^\nu \rangle = 0,
\]
which gives
\[
\left| \langle u' \cdot \nabla \theta, e^{2 \tau \Lambda^{\frac{1}{2}} \theta} \rangle \right| = \left| \langle u' \cdot \nabla \theta, e^{2 \tau \Lambda^{\frac{1}{2}} \theta} \rangle - \langle u' \cdot \nabla e^{2 \tau \Lambda^{\frac{1}{2}} \theta}, e^{2 \tau \Lambda^{\frac{1}{2}} \theta} \rangle \right|
= \left| i(2 \pi)^d \sum_{j+k=l} (\hat{u}_j \cdot k)(\hat{\theta}_k \cdot \hat{\theta}_l) e^{\tau|l|^{\frac{1}{2}}} (e^{\tau|l|^{\frac{1}{2}} - e^{\tau|k|^{\frac{1}{2}}}) \right|.
\]

We make use of the inequality $e^x - 1 \leq x e^x$ for $x \geq 0$ and the triangle inequality $|k + j|^{\frac{1}{2}} \leq |k|^{\frac{1}{2}} + |j|^{\frac{1}{2}}$ to obtain
\[
\left| e^{\tau|l|^{\frac{1}{2}}} - e^{\tau|k|^{\frac{1}{2}}} \right| \leq C \tau \frac{|j|}{|k|^{1-\frac{1}{2}} + |l|^{1-\frac{1}{2}}} e^{\tau|l|^{\frac{1}{2}}} e^{\tau|k|^{\frac{1}{2}}}.
\]

Hence we have
\[
\left| \langle u' \cdot \nabla \theta, e^{2 \tau \Lambda^{\frac{1}{2}} \theta} \rangle \right|
\leq C \tau \sum_{j+k=l} |k| |\hat{u}_j| e^{\tau|j|^{\frac{1}{2}}} |\hat{\theta}_k| |e^{\tau|k|^{\frac{1}{2}}} |\hat{\theta}_l| e^{\tau|l|^{\frac{1}{2}}} \frac{|j|}{|k|^{1-\frac{1}{2}} + |l|^{1-\frac{1}{2}}}
\leq C \tau \sum_{j+k=l} |j| |\hat{u}_j| e^{\tau|j|^{\frac{1}{2}}} |\hat{\theta}_k| |e^{\tau|k|^{\frac{1}{2}}} |\hat{\theta}_l| e^{\tau|l|^{\frac{1}{2}}} \frac{1}{|k|^{\frac{1}{2}} + |l|^{\frac{1}{2}}}
\leq C \tau \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2} \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2} \|L\|_{L^2}^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta'} \theta'\|_{L^2}
+ C \tau \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2}^{\frac{1}{2}} \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2}
\leq C \tau \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2} \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2} \|L\|_{L^2}^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta'} \theta'\|_{L^2}
+ C \tau \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2}^{\frac{1}{2}} \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2},
\]
where we have used the fact that $\sum_{j \neq 0, k \in \mathbb{Z}} |j|^{-2} < \infty$. Using the property (4.13), we have
\[
\|L^{2+\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} u\|_{L^2} \leq C \nu \|L^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2}
\]
and
\[
\|L^{2+\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} u\|_{L^2} \leq C \nu \|L^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2}.
\]
Hence there is $C = C(\nu) > 0$ such that
\[
\left| -\langle e^{\tau \Lambda^{\frac{1}{2}} (u' \cdot \nabla \theta')}, e^{\tau \Lambda^{\frac{1}{2}} \theta} \rangle \right| \leq C \tau \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2} \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2}^{2} + C \tau \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2}^{2},
\]
which finishes the proof of (4.15).

To complete the proof of Theorem 2.2, we apply (4.15) on (4.14) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2}^{2} - \tau \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2}^{2}
\leq C \tau \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2} \|\Lambda^{\frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta} \theta'\|_{L^2} + \|e^{\tau \Lambda^{\frac{1}{2}} S}\|_{L^2} \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2}.
\]
Choose $\tau > 0$ such that
\[
\tau + C \tau \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2} = 0,
\]
then we have
\[
\frac{1}{2} \frac{d}{dt} \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2}^{2} \leq \|e^{\tau \Lambda^{\frac{1}{2}} S}\|_{L^2} \|e^{\tau \Lambda^{\frac{1}{2}} \theta'}\|_{L^2},
\]
which gives
\[
\|e^{\tau(t)\Lambda^{\frac{1}{2}} \theta'}(t)\|_{L^2} \leq \|e^{\tau_0 \Lambda^{\frac{1}{2}} \theta_0}\|_{L^2} + 2\|e^{\tau_0 \Lambda^{\frac{1}{2}} S}\|_{L^2}.
\]
Hence $\tau(t)$ satisfies
\[
\tau(t) \geq \tau_0 e^{-C\left(\|e^{\tau_0 \Delta^\frac{1}{2}} \theta_0\|_{L^2} + 2\|e^{\tau_0 \Delta^\frac{1}{2}} S\|_{L^2}\right)t}
\]
and the proof of Theorem 2.2 is complete.

**Remark 4.4.** We notice that for the “diffusive” case, i.e. for the following system when $\kappa > 0$:
\[
\begin{align*}
\partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu &= \kappa \Delta \theta^\nu, \\
u^\nu_j &= \partial_x \left(T^0_{ij}[\theta^\nu], \theta^\nu(x,0) = \theta_0(x),
\end{align*}
\]  
(4.16)
\]
one can obtain global-in-time existence of solution Gevrey class $s \geq 1$ with lower bound on $\tau(t)$ that does not vanish as $t \to \infty$. To see it, we apply the previous analysis on (4.16) to obtain
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|e^{\tau \Delta^\frac{1}{2}} \theta^\nu\|_{L^2}^2 - \tilde{\tau} \|e^{\tau \Delta^\frac{1}{2}} \theta^\nu\|_{L^2}^2 &\geq C \tau \|e^{\tau \Delta^\frac{1}{2}} \theta^\nu\|_{L^2}^2 \\
&\leq C \tau \|e^{\tau \Delta^\frac{1}{2}} \theta^\nu\|_{L^2}^2 \|e^{\tau \Delta^\frac{1}{2}} \theta^\nu\|_{L^2}^2.
\end{align*}
\]
Choosing $\tau > 0$ such that
\[
\tilde{\tau} + C \tau \|e^{\tau \Delta^\frac{1}{2}} \theta^\nu\|_{L^2} = 0,
\]
then we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|e^{\tau \Delta^\frac{1}{2}} \theta^\nu\|_{L^2}^2 &\leq \kappa \|e^{\tau \Delta^\frac{1}{2}} \theta^\nu\|_{L^2}^2 \leq 0.
\end{align*}
\]
Hence we obtain
\[
\begin{align*}
\|e^{\tau(t) \Delta^\frac{1}{2}} \theta^\nu(t)\|_{L^2} &\leq \|e^{\tau_0 \Delta^\frac{1}{2}} \theta_0\|_{L^2} e^{-\frac{\tilde{\tau}}{2}},
\end{align*}
\]
and
\[
\begin{align*}
\tau(t) &\geq \tau(0) e^{-C\|e^{\tau_0 \Delta^\frac{1}{2}} \theta_0\|_{L^2} \int_0^t \|e^{\tau \Delta^\frac{1}{2}} \theta_0\|_{L^2} ds} \geq e^{-2C\|e^{\tau_0 \Delta^\frac{1}{2}} \theta_0\|_{L^2}}.
\end{align*}
\]
Observe that the lower bound $e^{-\frac{2C}{\kappa}\|e^{\tau_0 \Delta^\frac{1}{2}} \theta_0\|_{L^2}}$ tends to zero as $\kappa \to 0$.

**4.2. The Case Where $\nu = 0$**

In this subsection we study the non-diffusive equations (2.1) for $\nu = 0$. Recall that we consider the following active scalar equation
\[
\begin{align*}
\partial_t \theta^0 + u^0 \cdot \nabla \theta^0 &= S, \\
u^0_j &= \partial_x \left(T^0_{ij}[\theta^0], \theta^0(x,0) = \theta_0(x),
\end{align*}
\]  
(4.17)
\]
where $T^0_{ij}$ is an operator which satisfies assumptions A1–A2 and either A5$_1$ or A5$_2$. Based on the assumptions A5$_1$ and A5$_2$, we consider the following two cases separately:

**4.2.1. When A5$_1$ is in Force.** Different from the case for $\nu > 0$, as it was proved in [16], the Eq. (4.17) is *ill-posed* in the sense of Hadamard, which means that the solution map associated to the Cauchy problem for (4.17) is not Lipschitz continuous with respect to perturbations in the initial datum around a specific steady profile $\theta_0$, in the topology of a certain Sobolev space $X$. Nevertheless, as pointed out in [16], it is possible to obtain the local existence and uniqueness of solutions to (4.17) in spaces of real-analytic functions, owing to the fact that the derivative loss in the nonlinearity $u^0 \cdot \nabla \theta^0$ is of order at most one (both in $u^0$ and in $\nabla \theta^0$).

In the present work, we extend the results of [16] to the case of Gevrey-class solutions. We first state and prove the following proposition which gives the Gevrey-class local wellposedness for (4.17).
Proposition 4.5. Fix $s \geq 1$ and $K_0 > 0$. Let $\theta_0$ and $S$ be of Gevrey-class $s$ with radius of convergence $\tau_0 > 0$ and

$$\|\Lambda r e^{\tau_0 \Lambda r^2} \theta_0(\cdot, 0)\|_{L^2} \leq K_0, \quad \|\Lambda r e^{\tau_0 \Lambda r^2} S\|_{L^2} \leq K_0,$$

where $r > \frac{d}{2} + \frac{3}{2}$. There exists $T_* = T_*(\tau_0, K_0) > 0$ and a unique Gevrey-class $s$ solution on $[0, T_*)$ to the initial value problem associated to (4.17).

Proof. The idea of the proof follows by a similar argument given in [16]. For $r > \frac{d}{2} + \frac{3}{2}$, we define

$$\|\theta^0\|^2_{r, r} = \|\Lambda r e^{\tau_0 \Lambda r^2} \theta_0\|^2_{L^2} = \sum_{k \in \mathbb{Z}^d} |k|^{2r^*} e^{2r|k|^\frac{1}{2}} |\widehat{\theta}_0(k)|^2$$

We take $L^2$-inner product of (4.17) with $\Lambda r e^{2r \Lambda r^2} \theta_0$ and obtain

$$\frac{1}{2} \frac{d}{dt} \|\theta^0\|^2_{r, r} - \tau \|\Lambda r \theta_0\|^2_{r, r} = \langle u^0 \cdot \nabla \theta_0, \Lambda r e^{2r \Lambda r^2} \theta_0 \rangle - \langle \Lambda r e^{\tau_0 \Lambda r^2} S, \Lambda r e^{\tau_0 \Lambda r^2} \theta_0 \rangle. \quad (4.19)$$

Write $R = \langle u \cdot \nabla \theta_0, \Lambda r e^{2r \Lambda r^2} \theta_0 \rangle$, then it can rewritten as

$$R = i(2\pi)^d \sum_{j+k=i, l, \ell \in \mathbb{Z}^d} \widehat{u}^0(j) \cdot k \widehat{\theta}_0(k) |l|^{2r} e^{2r|l|^2} |\widehat{\theta}_0(-l)|.$$

Using the assumption A2 that $|\widehat{u}^0(j)| \leq C|j||\widehat{\theta}_0(j)|$ and the fact $|l|^2 = |j+k|^2 \leq |j|^2 + |k|^2$ for $s \geq 1$ and $|j|, |k| \geq 1$, we have

$$R \leq C \sum_{j+k=i, l, \ell \in \mathbb{Z}^d} |j||k||l|^2 e^{r|l|^2} |\widehat{\theta}_0(k)| e^{r|k|^2} |\widehat{\theta}_0(l)| e^{r|l|^2}$$

$$\leq C \sum_{j+k=i, l, \ell \in \mathbb{Z}^d} (|j|^{2r} + |k|^{2r} + |l|^{2r}) |\widehat{\theta}_0(j)| e^{r|j|^2} |\widehat{\theta}_0(k)| e^{r|k|^2} |\widehat{\theta}_0(l)| e^{r|l|^2}$$

$$\leq C \|\Lambda \theta_0\|^2_{r, r} \sum_{j+k=i, l, \ell \in \mathbb{Z}^d} |j|^2 |\widehat{\theta}_0(j)| e^{r|j|^2}$$

$$\leq C \|\Lambda \theta_0\|^2_{r, r} \|\theta_0\|_{r, r},$$

where the last inequality follows since $r > \frac{d}{2} + \frac{3}{2}$. Hence we obtain from (4.19) that

$$\frac{1}{2} \frac{d}{dt} \|\theta^0(\cdot, t)\|^2_{r, r} \leq (\dot{\tau}(t) + C \|\theta^0(\cdot, t)\|_{r, r} \|\Lambda \theta_0\|^2_{r, r} + \|S\|_{r, r} \|\theta^0(\cdot, t)\|_{r, r}). \quad (4.20)$$

Let $\tau(t)$ be decreasing and satisfy

$$\dot{\tau} + 4CK_0 = 0,$$

with initial condition $\tau(0) = \tau_0$, then we have $\dot{\tau}(t) + C \|\theta^0(\cdot, t)\|_{r, r} < 0$, and from (4.20) that

$$\|\theta^0(\cdot, t)\|_{r, r} \leq \|\theta^0(\cdot, 0)\|_{r, r} + 2\|S\|_{r, r} = 3K_0$$

as long as $\tau(t) > 0$. Hence it implies the existence of a Gevrey-class $s$ solution $\theta^0$ on $[0, T_*)$, where the maximal time of existence of the Gevrey-class $s$ solution is given by $T_* = \frac{\tau_0}{4C}$. \hfill \Box

Proof of Theorem 2.3. By choosing $\bar{T} = \frac{T}{2}$ and $\bar{\tau} = \tau(\bar{T})$, where $T_*$, $\tau(t)$ are as defined in Proposition 4.5, if $\theta(\cdot, 0) = \theta_0$ and $S$ be of Gevrey-class $s$, both with radius of convergence at least $\tau_0$ and satisfy (4.18), then there exists a unique Gevrey-class $s$ solution $\theta^0$ to (2.1) for $\nu = 0$ defined on $[0, \bar{T}]$ with radius of convergence at least $\bar{\tau}$. The time $\bar{T}$ and radius on convergence $\bar{\tau}$ should only depend on $C_0$ as described in assumption A3, hence they can be chosen independent of $\nu$ and the proof of Proposition 4.5 also applies to (2.1) for $\nu > 0$. The bounds (2.7)–(2.8) follow immediately from (4.21). \hfill \Box

Remark 4.6. By uniqueness, for $\nu > 0$, the Gevrey-class $s$ solution $\theta^\nu$ as obtained in Theorem 2.2 coincides with the one as obtained in Theorem 2.3 on $\mathbb{T}^d \times [0, \bar{T}]$. 


4.2.2. When A5₂ is in Force. Contrary to the previous case, when assumption A5₂ is in force, the operator \( \partial_x T^0 \) becomes a zero order operator with \( \partial_x T^0 : L^2 \rightarrow L^2 \) being bounded. Following the idea given in [11], we show that under the assumptions A1–A2 and A5₂, the equation (4.17) is locally wellposed in Sobolev space \( H^s \) for \( s > \frac{d}{2} + 1 \), thereby proving Theorem 2.4.

Before we give the proof of Theorem 2.4, we recall the following proposition from [11]:

**Proposition 4.7.** Suppose that \( s > 0 \) and \( p \in (1, \infty) \). If \( f, g \in S \), then

\[
\| \Lambda^s(fg) - f \Lambda^s g \|_{L^p} \leq C \left( \| \nabla f \|_{L^{p_1}} \| \Lambda^{s-1} g \|_{L^{p_2}} + \| \Lambda^s f \|_{L^{p_3}} \| g \|_{L^{p_4}} \right),
\]

(4.22)

where \( \Lambda = (-\Delta)^{\frac{j}{2}} \), \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \), and \( p, p_2, p_3 \in (1, \infty) \).

**Proof of Theorem 2.4.** For simplicity, we denote \( \theta^0 \) and \( u^0 \) by \( \theta \) and \( u \) respectively. We subdivide the proof into three steps.

**Step 1.** We consider the sequence of approximations \( \{ \theta_n \}_{n \geq 1} \) given by the solutions of

\[
\begin{align*}
\partial_t \theta_1 &= S \\
\theta_1(\cdot, 0) &= \theta_0.
\end{align*}
\]

(4.23)

and

\[
\begin{align*}
\partial_t \theta_n + u_{n-1} \cdot \nabla \theta_n &= S \\
\theta_{n-1} &= \partial_x T^0[\theta_{n-1}] \\
\theta_n(\cdot, 0) &= \theta_0.
\end{align*}
\]

(4.24)

System (4.23) can be solved easily and for all \( T > 0 \), we also have the bound

\[
\| \Lambda^s \theta_1 \|^2_{L^\infty(0,T;L^2)} \leq \| \Lambda^s \theta_0 \|^2_{L^2} + T \| S \|^2_{H^s},
\]

where \( \Lambda = (-\Delta)^{\frac{j}{2}} \). For the system (4.24), we consider the linear approximated system

\[
\begin{align*}
\partial_t \theta^\varepsilon + v \cdot \nabla \theta^\varepsilon - \varepsilon \Delta \theta^\varepsilon &= S \\
\theta^\varepsilon(\cdot, 0) &= \theta_0,
\end{align*}
\]

and the details follow from Theorem A1 in [11]. This shows that there exists a unique solution \( \theta_n \in L^\infty(0,T;H^s) \) of (4.24).

**Step 2.** Next we show that \( \{ \theta_n \}_{n \geq 0} \) is bounded. We fix a time \( T \) (to be chosen later) such that

\[
T < \frac{\| \Lambda^s \theta_0 \|^2_{L^2}}{\| S \|^2_{H^s}}.
\]

Assume that

\[
\| \Lambda^s \theta_j \|^2_{L^\infty(0,T;L^2)} \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2},
\]

(4.25)

for \( 1 \leq j \leq n - 1 \). By A5₂, we have

\[
\| \Lambda^s u_{n-1}(\cdot, t) \|^2_{L^2} \leq \| \Lambda^s \theta_{n-1}(\cdot, t) \|^2_{L^2},
\]

for all \( t > 0 \). Apply \( \Lambda^s \) on (4.24) and take inner product with \( \Lambda^s \theta_n \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{T^d} |\Lambda^s \theta_n|^2 + \int_{T^d} \Lambda^s \theta_n \cdot \Lambda^s (u_{n-1} \cdot \nabla \theta_n) = \int_{T^d} \Lambda^s \theta_n \cdot \Lambda^s S.
\]

(4.26)
The term \( \int_{T^d} \Lambda^s \theta_n \cdot \Lambda^s (u_{n-1} \cdot \nabla \theta_n) \) can be rewritten as
\[
\int_{T^d} \Lambda^s \theta_n \cdot \Lambda^s (u_{n-1} \cdot \nabla \theta_n) \\
= \int_{T^d} \Lambda^s \theta_n \cdot \left( \Lambda^s (u_{n-1} \cdot \nabla \theta_n) - u_{n-1} \cdot \Lambda^s (\nabla \theta_n) \right) + \int_{T^d} \Lambda^s \theta_n \cdot u_{n-1} \Lambda^s (\nabla \theta_n),
\]
Upon integration by parts, the term \( \int_{T^d} \Lambda^s \theta_n \cdot u_{n-1} \Lambda^s (\nabla \theta_n) \) vanishes since \( \nabla \cdot u_{n-1} = 0 \). Using (4.22) for \( f = u_{n-1}, \ g = \nabla \theta_n, \ p = 2, \ p_1 = \infty, \ p_2 = 2, \ p_3 = 2, \ p_4 = \infty \), and applying the assumption \( A5_2 \), we have
\[
\left| \int_{T^d} \Lambda^s \theta_n \cdot \left( \Lambda^s (u_{n-1} \cdot \nabla \theta_n) - u_{n-1} \cdot \Lambda^s (\nabla \theta_n) \right) \right| \\
\leq \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s (u_{n-1} \cdot \nabla \theta_n) - u_{n-1} \cdot \Lambda^s (\nabla \theta_n) \|_{L^2} \\
\leq C \| \Lambda^s \theta_n \|_{L^2} \left( \| \nabla u_{n-1} \|_{L^\infty} \| \Lambda^s \theta_n \|_{L^2} + \| \Lambda^s u_{n-1} \|_{L^2} \| \nabla \theta_n \|_{L^\infty} \right) \\
\leq C \| \Lambda^s \theta_n \|_{L^2} \left( \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} + \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} \right) \\
\leq 2C \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2}. \tag{4.27}
\]
Hence we obtain from (4.26) and (4.27) that
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^s \theta_n (\cdot, t) \|_{L^2}^2 \\
\leq 2C \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} + \| \Lambda^s \theta_n \|_{L^2} \| \Lambda^s S \|_{L^2}. \tag{4.28}
\]
Applying the bound (4.25) on \( \theta_n \), we have
\[
\frac{d}{dt} \| \Lambda^s \theta_n (\cdot, t) \|_{L^2} \leq 4C \| \Lambda^s \theta_0 \|_{L^2} \| \Lambda^s \theta_n \|_{L^2} + 2 \| \Lambda^s S \|_{L^2}, \tag{4.29}
\]
and hence by integrating (4.29) over \( t \) and choosing \( T \) small enough, (4.25) also holds for \( j = n \).

**Step 3.** Finally, we show that \( \{ \theta_n \}_{n \geq 0} \) is a Cauchy sequence. Denote the difference of \( \theta_n \) and \( \theta_{n-1} \) by
\[
\tilde{\theta}_n = \theta_n - \theta_{n-1}.
\]
It follows from (4.24) that \( \tilde{\theta}_n \) satisfies
\[
\partial_t \tilde{\theta}_n + u_{n-1} \cdot \nabla \tilde{\theta}_n + \tilde{u}_{n-1} \cdot \nabla \theta_{n-1} = 0, \tag{4.30}
\]
where \( \tilde{u}_{n-1} = \partial_x T^0 \tilde{\theta}_{n-1} \). Apply \( \Lambda^{-1} \) on (4.30) and take inner product with \( \Lambda^{-1} \tilde{\theta}_n \),
\[
\frac{1}{2} \frac{d}{dt} \int_{T^d} |\Lambda^{-1} \tilde{\theta}_n|^2 \\
+ \int_{T^d} \Lambda^{-1} \tilde{\theta}_n \cdot \Lambda^{-1} (u_{n-1} \cdot \nabla \tilde{\theta}_n) + \int_{T^d} \Lambda^{-1} \tilde{\theta}_n \cdot \Lambda^{-1} (\tilde{u}_{n-1} \cdot \nabla \theta_{n-1}) = 0. \tag{4.31}
\]
The term \( \int_{T^d} \Lambda^{-1} \tilde{\theta}_n \cdot \Lambda^{-1} (u_{n-1} \cdot \nabla \tilde{\theta}_n) \) can be estimated as follows.
\[
\left| \int_{T^d} \Lambda^{-1} \tilde{\theta}_n \cdot \Lambda^{-1} (u_{n-1} \cdot \nabla \tilde{\theta}_n) \right| \\
\leq \left| \int_{T^d} \Lambda^{-1} \tilde{\theta}_n \cdot (\Lambda^{-1} (u_{n-1} \cdot \nabla \tilde{\theta}_n) - u_{n-1} \cdot \Lambda^{-1} (\nabla \tilde{\theta}_n) \right| \\
+ \left| \int_{T^d} \Lambda^{-1} \tilde{\theta}_n \cdot u_{n-1} \cdot \Lambda^{-1} (\nabla \tilde{\theta}_n) \right| \\
\leq C \| \Lambda^{-1} \tilde{\theta}_n \|_{L^2} \left( \| \nabla u_{n-1} \|_{L^\infty} \| \Lambda^{-2} \nabla \tilde{\theta}_n \|_{L^2} + \| \Lambda^{-1} u_{n-1} \|_{L^6} \| \nabla \tilde{\theta}_n \|_{L^3} \right)
\]
\[ \leq C\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} (\|\Lambda^s\theta_{n-1}\|_{L^2}\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} + \|\Lambda^s\theta_{n-1}\|_{L^2}\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2}) \]
\[ \leq 2C\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2}^2 \|\Lambda^s\theta_{n-1}\|_{L^2}, \]

where we used (4.22) for \( f = u_{n-1}, g = \nabla\tilde{\theta}_n \), \( p = 2, p_1 = \infty, p_2 = 2, p_3 = 6, p_4 = 3 \) and the assumption A5_2.

On the other hand, using Proposition 2.1 in [11], the term \( \int_{T^d} \Lambda^{s-1}\tilde{\theta}_n \cdot \Lambda^{s-1}(\tilde{u}_{n-1} \cdot \nabla\theta_{n-1}) \) can be estimated by

\[ \left| \int_{T^d} \Lambda^{s-1}\tilde{\theta}_n \cdot \Lambda^{s-1}(\tilde{u}_{n-1} \cdot \nabla\theta_{n-1}) \right| \]
\[ \leq C\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} (\|\tilde{u}_{n-1}\|_{L^\infty}\|\Lambda^{s-1}\nabla\theta_{n}\|_{L^2} + \|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2}\|\nabla\theta_{n}\|_{L^\infty}) \]
\[ \leq C\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} (\|\Lambda^{s-1}\tilde{u}_{n-1}\|_{L^2}\|\Lambda^s\theta_{n}\|_{L^2} + \|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2}\|\Lambda^s\theta_{n}\|_{L^2}) \]
\[ \leq 2C\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} \|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} \|\Lambda^s\theta_{n}\|_{L^2}. \]

Hence we deduce from (4.31) that

\[ \frac{1}{2} \frac{d}{dt} \|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2}^2 \]
\[ \leq 2C\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2}^2 \|\Lambda^s\theta_{n-1}\|_{L^2} \|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} + \|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} \|\Lambda^s\theta_{n}\|_{L^2}. \]

Using (4.25) on (4.32),

\[ \frac{d}{dt} \|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} \leq 4\sqrt{2}C (\|\Lambda^s\theta_0\|_{L^2}\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} + \|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} \|\Lambda^s\theta_0\|_{L^2}). \]

Integrating (4.33) over \( t \) and choosing \( T \) small enough, we obtain

\[ \sup_{t \in [0,T]} \|\Lambda^{s-1}\tilde{\theta}_n(\cdot,t)\|_{L^2} \leq \frac{1}{2} \sup_{t \in [0,T]} \|\Lambda^{s-1}\tilde{\theta}_{n-1}(\cdot,t)\|_{L^2}. \]

Thus \( \theta_n \) is Cauchy in \( L^\infty(0,T;H^{s-1}) \) with \( \theta_n \) converges strongly to \( \theta \) in \( L^\infty(0,T,H^{s-1}) \). Since we assume that \( s > \frac{d}{2} + 1 \), this also implies that the strong convergence occurs in a Hölder space relative to \( x \) as \( n \rightarrow \infty \), hence the limiting function \( \theta \) is a solution of (4.17). Uniqueness of \( \theta \) follows by the same argument given in [11] and we omit the details. It finishes the proof of Theorem 2.4. \( \square \)

5. Convergence of Solutions as \( \nu \rightarrow 0 \)

In this section, we address the convergence of solutions to (2.1) as \( \nu \rightarrow 0 \) under the assumption A4 and give the proof of Theorem 2.5. Depending on the assumptions A5_1 and A5_2, we can address the convergence of solutions in two cases respectively:

- As we discussed before, it was proved in [16] that under the assumption A5_1, the Eq. (2.1) for \( \nu = 0 \) is ill-posed in the sense of Hadamard over \( L^2 \). Hence we focus on the case for Gevrey-class solutions \( \theta^\nu \) to (2.1). By Theorem 2.3, given Gevrey-class \( s \) initial datum \( \theta_0 \) and forcing \( S \), there exists \( \bar{T}, \bar{\tau} > 0 \) and a unique Gevrey-class solution \( \theta^\nu \) to (2.1) defined on \([0,\bar{T}]\) with radius of convergence at least \( \bar{\tau} \) for all \( \nu \geq 0 \). A natural question is the following: will the Gevrey-class solutions \( \theta^\nu \) converge as \( \nu \rightarrow 0 \)? The answer is affirmative and is presented in Theorem 2.5, which shows that the Gevrey-class solutions \( \theta^\nu \) converges to \( \theta^0 \) in some Gevrey-class norm as \( \nu \rightarrow 0 \).

- On the other hand, when assumption A5_2 is in force, by Theorem 2.4, the Eq. (2.1) for \( \nu = 0 \) is locally wellposed in Sobolev space \( H^s \) for \( s > \frac{d}{2} + 1 \). For sufficiently smooth initial data \( \theta_0 \) and forcing term \( S \), we aim at showing that \( \|\theta^\nu(\cdot,t)\|_{H^s} \rightarrow 0 \) as \( \nu \rightarrow 0 \) for \( s > \frac{d}{2} + 1 \) and \( t \in [0,\bar{T}] \). Such result is parallel to the one proved in [14], in which the authors proved that if \( \theta^\nu, \theta^0 \) are \( C^\infty \) smooth classical solutions of the diffusive system (4.16) for \( \nu > 0 \) and \( \nu = 0 \) respectively with initial datum \( \theta_0 \in L^2 \) and forcing term \( S \in C^\infty \), then \( \|\theta^\nu(\cdot,t)\|_{H^s} \rightarrow 0 \) as \( \nu \rightarrow 0 \) for \( s \geq 0 \) and \( t > 0 \).
Remark 5.1. In the diffusive system (4.16) studied in [14] there is no smoothing assumption imposed on \{T_{ij}^\nu\}_{\nu \geq 0}$ when \( \nu > 0 \). The main reason for the difference is that the diffusive term \( \kappa \Delta \theta^\nu \) present in (4.16) is sufficient to smooth out the solution \( \theta^\nu \) for all \( \nu \geq 0 \).

**Proof of Theorem 2.5.** We divide the proof into two cases:

**Case 1.** When A5$_1$ is in force. Fix \( s \geq 1 \) and \( r > \frac{d}{2} + \frac{3}{2} \). Throughout this proof, \( C > 0 \) is a generic constant which depends on \( C_0, \theta_0, S, s, r, d, \bar{T}, \tilde{r} \) and is independent of \( \nu \). Let \( \theta^\nu, \theta^0 \) be the Gevrey-class \( s \) solutions to \( (2.1) \), on \([0, \bar{T}]\) as obtained in Theorem 2.3. We define \( \phi^\nu = \theta^\nu - \theta^0 \) and write

\[
\| \phi^\nu \|_{T,R}^2 = \| \Lambda \phi^\nu \|_{T,R}^2 = \| \Lambda \phi^\nu \|_{L^2} = \sum_{k \in \mathbb{Z}^d} |k|^{2r} e^{2\tau |k|^\frac{1}{2}} |\phi^\nu(k)|^2.
\]

Then \( \phi^\nu \) satisfies the following equation on \([0, \bar{T}]\)

\[
\partial_t \phi^\nu + (u^\nu - u^0) \cdot \nabla \phi^\nu + u^\nu \cdot \nabla \phi^\nu = \theta^\nu \cdot \nabla \theta^\nu = 0,
\]

where \( u^0 = \partial_x T_{ij}^\nu[\theta^0] \) for all \( i, j \). From (5.1), we have the a priori estimate

\[
\frac{1}{2} \frac{d}{dt} \| \phi^\nu \|_{T,R}^2 \geq \tilde{\tau} \| \Lambda \phi^\nu \|_{T,R}^2 + \mathcal{R}_1 + \mathcal{R}_2.
\]

Using Plancherel’s theorem, the nonlinear term \( \mathcal{R}_1 \) can be written as

\[
\mathcal{R}_1 = i(2\pi)^d \sum_{j+k=1} (\hat{u}^\nu - \hat{u}^0)(j) \cdot k \hat{\theta}^0(k) |l|^{2r} e^{2\tau |l|^\frac{1}{2}} \hat{\phi}^\nu(l) e^{\tau |l|^\frac{1}{2}}.
\]

The term \( \mathcal{R}_1 \) can be estimated as follows.

\[
\mathcal{R}_1 \leq C \sum_{j+k=1} |j| k(|j| + |k|) (\hat{\theta}^0(k) |e^{\tau |j|/2}| \hat{\phi}^\nu(k) |e^{\tau |k|/2}| |l|^{2r} e^{\tau |l|/2} |\phi^\nu(l) e^{\tau |l|/2}|
\]

\[
\leq C \| \Lambda \phi^\nu \|_{T,R} \| \Lambda \phi^\nu \|_{T,R} \| \phi^\nu \|_{T,R}
\]

\[
+ C \sum_{j+k=1} |j|^{\frac{3}{2}} k |\hat{\theta}^0(k) |e^{\tau |j|/2}| \hat{\phi}^\nu(k) |e^{\tau |k|/2}| |l|^{\frac{1}{2}} \phi^\nu(l) e^{\tau |l|/2}|
\]

\[
\leq C \| \Lambda \phi^\nu \|_{T,R} \| \Lambda \phi^\nu \|_{T,R} \| \phi^\nu \|_{T,R}
\]

\[
+ C \| \Lambda \phi^\nu \|_{T,R} \| \Lambda \phi^\nu \|_{T,R} \| \phi^\nu \|_{T,R}
\]

\[
\times \left( \sum_{j \in \mathbb{Z}^d} j^{d+3} |\hat{\theta}^0(j) |e^{2\tau |j|/2}| |\hat{\phi}^\nu(j) |e^{2\tau |j|/2}| |(\hat{T}^\nu - \hat{T}^0)(j) |e^{2\tau |j|/2}\right)^\frac{1}{2} \left( \sum_{j \in \mathbb{Z}^d} j^{d-3} \right)^\frac{1}{2}
\]

\[
\leq C \| \Lambda \phi^\nu \|_{T,R} \| \Lambda \phi^\nu \|_{T,R} \| \phi^\nu \|_{T,R}.
\]
Following the proof of Theorem 2.4, shrinking the time and using the bounds (2.7) and (2.8), there exists

\[ \text{such that, for all } j \in \mathbb{Z}^d \]

\[ \sum_{j \in \mathbb{Z}^d} |j|^{d+3} |\hat{\theta}_0(j)|^2 e^{2\tau|j|^\frac{1}{2}} |((\hat{T}^\nu - \hat{T}^0)(j)|^2 \leq \frac{1}{2} C \left| \frac{d}{dt} \right| \phi^\nu \right|_{\tau,r}^2 + C \sum_{j \in \mathbb{Z}^d} |j|^{d+3} |\hat{\theta}_0(j)|^2 e^{2\tau|j|^\frac{1}{2}} |((\hat{T}^\nu - \hat{T}^0)(j)|^2.

Using the bounds (5.3) and (5.4) on (5.2), we obtain

\[ \frac{d}{dt} \phi^\nu \right|_{\tau,r}^2 \leq \left( \hat{\nu} + C \phi^\nu \right)_{\tau,r} + C \left| \frac{d}{dt} \theta^0 \right|_{\tau,r}^2 \left| \frac{d}{dt} \phi^\nu \right|_{\tau,r} \]

\[ + C \left| \frac{d}{dt} \phi^\nu \right|_{\tau,r} + C \sum_{j \in \mathbb{Z}^d} |j|^{d+3} |\hat{\theta}_0(j)|^2 e^{2\tau|j|^\frac{1}{2}} |((\hat{T}^\nu - \hat{T}^0)(j)|^2.

Choose \( \tau = \tau(t) \leq \hat{\nu} \) such that

\[ \left\{ \begin{array}{l}
\hat{\nu} + C \phi^\nu \right|_{\tau,r} + C \left| \frac{d}{dt} \theta^0 \right|_{\tau,r}^2 < 0,
\tau < \hat{\nu},
\end{array} \right. \]

then using the bounds (2.7) and (2.8), there exists \( T < \hat{T} \) such that for \( t \in [0,T] \), we have

\[ \left| \frac{d}{dt} \phi^\nu \right|_{\tau,r}^2 \leq C \phi^\nu \right|_{\tau,r}^2 + C \sum_{j \in \mathbb{Z}^d} |j|^{d+3} |\hat{\theta}_0(j)|^2 e^{2\tau|j|^\frac{1}{2}} |((\hat{T}^\nu - \hat{T}^0)(j)|^2.

Integrating the above with respect to \( t \), for \( t \in [0,T] \), we obtain

\[ \left| \frac{d}{dt} \phi^\nu \right|_{\tau,r}^2 \leq e^{CT} C \sum_{j \in \mathbb{Z}^d} |j|^{d+3} |\hat{\theta}_0(j)|^2 e^{2\tau|j|^\frac{1}{2}} |((\hat{T}^\nu - \hat{T}^0)(j)|^2.

Since \( \left| \phi^\nu \right|_{\tau,r} < \infty \) with \( r > \frac{d}{2} + \frac{3}{2} \), it implies \( \sum_{j \in \mathbb{Z}^d} |j|^{d+3} |\hat{\theta}_0(j)|^2 e^{2\tau|j|^\frac{1}{2}} \left| (\hat{T}^\nu - \hat{T}^0)(j) \right| < \infty \), and hence by the assumption A4,

\[ \lim_{\nu \to 0} \sum_{j \in \mathbb{Z}^d} |j|^{d+3} |\hat{\theta}_0(j)|^2 e^{2\tau|j|^\frac{1}{2}} \left| (\hat{T}^\nu - \hat{T}^0)(j) \right| = 0. \]

Therefore the result (2.9) follows.

**Case 2.** When A5.2 is in force. Fix \( s > \frac{d}{2} + 1 \), let \( \theta^\nu, \theta^0 \) be the \( H^s \) to (2.1) on \([0,T]\) as obtained in Theorem 2.4. We define \( \phi^\nu = \theta^\nu - \theta^0 \), then \( \phi^\nu \) satisfies (5.1) on \([0,T]\).

We first show that, for \( t \in [0,T] \),

\[ \lim_{\nu \to 0} \left| \phi^\nu(\cdot,t) \right|_{L^2} = 0. \quad (5.5) \]

Following the proof of Theorem 2.4, shrinking the time \( T \) if necessary, there exists \( C = C(T, \theta_0, S) > 0 \) independent of \( \nu \) such that, for all \( \nu \geq 0 \),

\[ \sup_{0 \leq \tau \leq T} \left| \phi^\nu(\cdot,t) \right|_{H^s} \leq C. \quad (5.6) \]

We multiply (5.1) by \( \theta^\nu \) and integrate, for \( t \in [0,T] \),

\[ \frac{1}{2} \frac{d}{dt} \left| \phi^\nu(\cdot,t) \right|_{L^2}^2 = - \int (u^\nu - u^0) \cdot \nabla \theta^0 \cdot \phi^\nu(x,t) dx. \quad (5.7) \]
We estimate the right side of (5.7) as follows. Using Sobolev embedding theorem and the bound (5.6),
\[
\left| - \int (u^\nu - u^0) \cdot \nabla \theta^0 \cdot \phi^\nu(x,t) dx \right|
\leq \|(u^\nu - u^0)(\cdot,t)\|_{L^2} \|(\theta^0, \cdot, t)\|_{L^2} \|\nabla \theta^0(\cdot,t)\|_{L^\infty}
\leq C \|(u^\nu - u^0)(\cdot,t)\|_{L^2} \|\phi^\nu(\cdot,t)\|_{L^2}
\leq \frac{C}{2} \|(u^\nu - u^0)(\cdot,t)\|^2_{L^2} + \frac{C}{2} \|\phi^\nu(\cdot,t)\|^2_{L^2}.
\tag{5.8}
\]

We focus on the term \(\|(u^\nu - u)(\cdot,t)\|^2_{L^2}\) as in (5.8). Using Plancherel Theorem and assumption A5, for each \(j\),
\[
\|(u^\nu_j - u_j)(\cdot,t)\|^2_{L^2} = \sum_{k \in \mathbb{Z}^d} |(u^\nu_j - u_j)(k,t)|^2
= \sum_{k \in \mathbb{Z}^d} |(\hat{\partial}_x T^\nu_{ij} \hat{\theta}^\nu - \hat{\partial}_x T^0_{ij} \hat{\theta}^0)(k,t)|^2
\leq \sum_{k \in \mathbb{Z}^d} |\hat{\partial}_x T^\nu_{ij}|^2 |\hat{\phi}|^2(k,t) + \sum_{k \in \mathbb{Z}^d} |\hat{T}^\nu_{ij} - \hat{T}^0_{ij}|^2 |\nabla \hat{\theta}^0|^2(k,t)
\leq C_0 \|\phi^\nu(\cdot,t)\|^2_{L^2} + I(\nu, t),
\]
where \(I(\nu, t) = \sum_{k \in \mathbb{Z}^3} |\hat{T}^\nu_{ij} - \hat{T}^0_{ij}|^2 |\nabla \hat{\theta}^0|^2(k,t)\). Applying the above estimate on (5.8), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\phi^\nu(\cdot,t)\|^2_{L^2} \leq \frac{C}{2} \left( C_0 \|\phi^\nu(\cdot,t)\|^2_{L^2} + I(\nu, t) \right) + \frac{C}{2} \|\phi^\nu(\cdot,t)\|^2_{L^2}.
\]
For \(t \in [0, T]\), since \(\|\theta^0(\cdot,t)\|_{L^2} < \infty\), by assumption A4, we have
\[
\lim_{\nu \to 0} I(\nu, t) = 0.
\]
Hence taking \(\nu \to 0\) and using Grönwall’s inequality, we conclude that (5.5) holds for \(t \in [0, T]\).

Finally, we apply the Gagliardo–Nirenberg interpolation inequality and the bound (5.6) to obtain, for \(t \in [0, T]\),
\[
\|(\theta^\nu - \theta^0)(\cdot,t)\|_{H^{s-1}} \leq C(d) \|(\theta^\nu - \theta^0)(\cdot,t)\|_{L^2} \|(\theta^\nu - \theta^0)(\cdot,t)\|_{H^s}^{1-\gamma}
\leq C(d) C^{1-\gamma} \|(\theta^\nu - \theta^0)(\cdot,t)\|_{L^2} \gamma,
\]
where \(\gamma \in (0, 1)\) depends on \(s\) and \(C(d) > 0\) is a positive constant which depends on \(d\) but is independent of \(\nu\). By taking \(\nu \to 0\) and applying the \(L^2\)-convergence (5.5) just proved, we conclude that (2.10) holds for \(t \in [0, T]\) as well.

6. Applications to Physical Models

We now apply our results claimed in Sect. 2 to some physical models, namely the magnetogeostrophic (MG) equations and the incompressible porous media (IPMB) equations discussed in Sect. 1.

6.1. Magnetogeostrophic Equations

We first consider the following magnetogeostrophic (MG) equation in the domain \(T^3 \times (0, \infty) = [0, 2\pi]^3 \times (0, \infty)\) with periodic boundary conditions:
\[
\begin{aligned}
\partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu &= S, \\
u = M^\nu[\theta^\nu], \theta(x, 0) &= \theta_0(x)
\end{aligned}
\tag{6.1}
\]
via a Fourier multiplier operator \( M^\nu \) which relates \( \nu^\nu \) and \( \theta^\nu \). More precisely,

\[
u^\nu_j = M^\nu_j(\theta^\nu) = (\hat{M}^\nu \hat{\theta}^\nu)^\vee
\]

for \( j \in \{1, 2, 3\} \). The explicit expression for the components of \( \hat{M}^\nu \) as functions of the Fourier variable \( k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \) are given by (1.7)–(1.10) in Sect. 1. We write \( M^\nu_j = \partial_j T^\nu_{ij} \) for convenience. To apply the results from Sect. 2, it suffices to show that the sequence of operators \( \{T^\nu_{ij}\}_{\nu \geq 0} \) satisfy the assumptions A1–A4 and A5_1 given in Sect. 1. We first prove the following lemma for the MG equations.

**Lemma 6.1.** For each \( L > 0 \),

\[
\lim_{\nu \to 0} \sup_{k \in \mathbb{Z}^3 : k \neq 0, |k| \leq L} \frac{|\hat{M}^\nu(k) - \hat{M}^0(k)|}{|k|} = 0. \tag{6.2}
\]

**Proof.** We only give the details for \( \hat{M}^\nu_1 \), since the cases for \( \hat{M}^\nu_2 \) and \( \hat{M}^\nu_3 \) are similar. We fix \( L > 0 \), then for each \( k \in \mathbb{Z}^3 \setminus \{k = 0\} \) with \( |k| \leq L \), we have

\[
\frac{|\hat{M}^\nu_1(k) - \hat{M}^0_1(k)|}{|k|} = \frac{| - \nu k_1 k_3^2 |k|^6 + \nu k_1 k_2^2 k_3 |k|^4 - \nu^2 k_2 k_3 |k|^8 \nu + \nu^2 k_1 k_2^2 k_3 |k|^8 - 2 \nu k_3^2 k_3 |k|^6|}{(k^2 k_3^2 + \nu^2 |k|^8 + 2 \nu |k|^4 k_3^2 + k_3^2 |k|^2 + k_2^2)|k|}
\]

\[
\leq \frac{\nu |k_1| |k_3|^3 |k|^6}{|k|^5 k_3^2} + \frac{\nu |k_1| |k_2|^2 |k_3| |k|^4}{|k|^5 k_3^3} + \frac{\nu^2 |k_2| |k_3| |k|^4}{|k|^5 k_3^3} + \frac{2 \nu |k_2|^3 |k_3| |k|^4}{|k|^5 k_3^3}
\]

\[
\leq \nu L^1 + \nu L^2 + \nu^2 L^2 + 2 \nu L^2.
\]

Hence

\[
\lim_{\nu \to 0} \sup_{k \in \mathbb{Z}^3 : k \neq 0, |k| \leq L} \frac{|\hat{M}^\nu_1(k) - \hat{M}^0_1(k)|}{|k|} = 0.
\]

□

**Proposition 6.2.** Let \( M^\nu_j = \partial_j T^\nu_{ij} \), where \( M^\nu \) is given by (1.7)–(1.9). Then \( T^\nu_{ij} \) satisfy the assumptions A1–A4 and A5_1 given in Sect. 1.

**Proof.** The details for the proof can be found in [14] Lemma 5.1–5.2 and from the discussion in ([15], Section 4). For example, to show \( T^\nu_{ij} \) satisfy the assumption A3, we only give the details for \( \hat{M}^\nu_1 \) since the cases for \( \hat{M}^\nu_2 \) and \( \hat{M}^\nu_3 \) are almost identical. We fix \( \nu \in (0, 1] \) and consider the following cases:

**Case 1.** \( |k| > \nu^{-\frac{2}{3}} \). Then for each \( k \in \mathbb{Z}^3 \setminus \{k = 0\} \),

\[
\frac{|\hat{M}^\nu_1(k)|}{|k|} = \frac{|k_2 k_3| |k|^2 - k_1 k_3 (k_2^2 + \nu |k|^4)|}{|k||k|^2 k_3^2 + (k_2^2 + \nu |k|^4)^2}.
\]

Since \( k \neq 0 \), so \( |k| \geq |k_3| \geq 1 \) for \( j = 1, 2, 3 \), in particular \( |k|^{-1} < \nu^\frac{2}{7} \). Hence we obtain

\[
\frac{|\hat{M}^\nu_1(k)|}{|k|} \leq \frac{|k_3 k_3| |k|^2}{|k|^3 k_3^2} + \frac{|k_1 k_3| k_3^2}{|k|^3 k_3^3} + \frac{\nu |k_1 k_3| |k|^4}{\nu^2 |k|^8}
\]

\[
\leq \frac{1}{|k_3|} + \frac{1}{|k_3|} + \frac{1}{\nu |k|^2}
\]

\[
\leq 2 + \frac{\nu}{\nu} = 3.
\]
Case 2. \(|k| \leq \nu^{-\frac{1}{2}}\). Then for each \(k \in \mathbb{Z}^3/\{k = 0\}\),
\[
\left| \frac{M^\nu_1(k)}{k} \right| \leq \frac{|k_2k_3||k|^2}{|k|^3k_3^2} + \frac{|k_1k_3||k|^4}{|k|^3k_3^2} + \nu|k_1k_3||k|^4 \\
\leq \frac{1}{|k_3|} + \frac{1}{|k_3|} + \nu|k|^2 \\
\leq 2 + \nu \cdot (\nu^{-\frac{1}{2}})^2 = 3.
\]
Combining two cases, we have
\[
\sup_{\nu \in (0, 1)} \sup_{\{k \in \mathbb{Z}^3: k \neq 0\}} \left| \frac{M^\nu_1(k)}{k} \right| \leq 3,
\]
and hence assumption A3 holds for some \(C_0 > 0\) independent of \(\nu\), which means that
\[
\sup_{\nu \in (0, 1)} \sup_{\{k \in \mathbb{Z}^3: k \neq 0\}} \left| \frac{M^\nu_1(k)}{k} \right| \leq C_0. \tag{6.3}
\]
On the other hand, to show \(T^\nu_{ij}\) satisfy the assumption A4, Fix \(g\) with
\[
\|g\|_{L^2} < \infty
\]
and we claim that
\[
\lim_{\nu \to 0} \sum_{k \in \mathbb{Z}^d} |\widehat{T^\nu_{ij}(k)} - \widehat{T^0_{ij}(k)}|^2 |\widehat{g(k)}|^2 = 0. \tag{6.4}
\]
Let \(\varepsilon > 0\) be given. Then \(\sum_{k \in \mathbb{Z}^3} |\widehat{g(k)}|^2 < \infty\), so there exists \(L = L(\varepsilon) > 0\) such that \(\sum_{k \in \mathbb{Z}^3: |k| > L} |\widehat{g(k)}|^2 < \varepsilon\). Hence for \(1 \leq i, j \leq d\), we have
\[
\sum_{k \in \mathbb{Z}^d} |\widehat{T^\nu_{ij}(k)} - \widehat{T^0_{ij}(k)}|^2 |\widehat{g(k)}|^2 \\
\leq \sum_{k \in \mathbb{Z}^3: k \neq 0} \left| \frac{M^\nu(k) - M^0(k)}{|k|^2} |\widehat{g(k)}|^2 \\
= \sum_{k \in \mathbb{Z}^3: k \neq 0, |k| \leq L} \left| \frac{M^\nu(k) - M^0(k)}{|k|^2} |\widehat{g(k)}|^2 + \sum_{k \in \mathbb{Z}^3: k \neq 0, |k| > L} \left(\frac{|M^\nu(k)|^2 + |M^0(k)|^2}{|k|^2} \right) |\widehat{g(k)}|^2 \\
\leq \left( \sup_{\{k \in \mathbb{Z}^3: k \neq 0, |k| \leq L\}} \left| \frac{M^\nu(k) - M^0(k)}{|k|^2} \right| \right)^2 \|g\|^2_{L^2} + 2C_0^2 \varepsilon. \tag{6.5}
\]
where the last inequality follows by the bound (6.3). Using (6.2) in Lemma 6.1 and taking \(\nu \to 0\) on (6.5),
\[
\lim_{\nu \to 0} \sum_{k \in \mathbb{Z}^d} |\widehat{T^\nu_{ij}(k)} - \widehat{T^0_{ij}(k)}|^2 |\widehat{g(k)}|^2 \leq 2C_0^2 \varepsilon.
\]
Since \(\varepsilon > 0\) is arbitrary, (6.4) follows and therefore \(T^\nu_{ij}\) satisfy the assumption A4.

In view of Proposition 6.2, the abstract Theorem 2.1–2.3 and Theorem 2.5 may therefore be applied to the MG equations (6.1) in order to obtain the wellposedness and convergence of Gevrey-class solutions. More precisely, we have

**Theorem 6.3** (Wellposedness in Sobolev space for the MG equations). Let \(\theta_0 \in W^{s,3}\) for \(s \geq 0\) and \(S\) be a \(C^\infty\)-smooth source term. Then for each \(\nu > 0\), we have:
• if $s = 0$, there exists unique global weak solution to (6.1) such that
  \[ \theta^\nu \in BC((0, \infty); L^3), \]
  \[ u^\nu \in C((0, \infty); W^{2,3}). \]
  In particular, $\theta^\nu(\cdot, t) \rightarrow \theta_0$ weakly in $L^3$ as $t \rightarrow 0^+$.
• if $s > 0$, there exists a unique global-in-time solution $\theta^\nu$ to (6.1) such that $\theta^\nu(\cdot, t) \in W^{s,3}$ for all $t \geq 0$.
  Furthermore, for $s = 1$, we have the following single exponential in time on $\| \nabla \theta^\nu(\cdot, t) \|_{L^3}$:
  \[ \| \nabla \theta^\nu(\cdot, t) \|_{L^3} \leq C \| \nabla \theta_0 \|_{L^3} \exp \left( C \left( t \| \theta_0 \|_{W^{1,3}} + t^2 \| S \|_{L^\infty} + t \| S \|_{W^{1,3}} \right) \right), \]
  where $C > 0$ is a constant which depends only on some dimensional constants.

**Theorem 6.4** (Gevrey-class global wellposedness for the MG equations). Fix $s \geq 1$. Let $\theta_0$ and $S$ be of Gevrey-class s with radius of convergence $\tau_0 > 0$. Then for each $\nu > 0$, there exists a unique Gevrey-class $s$ solution $\theta^\nu$ to (6.1) on $T^3 \times [0, \infty)$ with radius of convergence at least $\tau = \tau(t)$ for all $t \in [0, \infty)$, where $\tau$ is a decreasing function satisfying
  \[ \tau(t) \geq \tau_0 e^{-C \left( \| e^{r_0 \Lambda^{\frac{1}{2}}} \theta_0 \|_{L^2} + 2 \| e^{r_0 \Lambda^{\frac{1}{2}}} S \|_{L^2} \right) t}. \]
  Here $C > 0$ is a constant which depends on $\nu$ but independent of $t$.

**Theorem 6.5** (Gevrey-class local wellposedness for the MG equations). Fix $s \geq 1$, $r > 3$ and $K_0 > 0$. Let $\theta_0$ and $S$ be of Gevrey-class $s$ with radius of convergence $\tau_0 > 0$ and
  \[ \| \Lambda^r e^{r_0 \Lambda^{\frac{1}{2}}} \theta_0(\cdot, 0) \|_{L^2} \leq K_0, \quad \| \Lambda^r e^{r_0 \Lambda^{\frac{1}{2}}} S \|_{L^2} \leq K_0. \]
  There exists $\bar{T}, \bar{\tau} > 0$ and a unique Gevrey-class $s$ solution $\theta^0$ to (6.1) for $\nu = 0$ defined on $T^3 \times [0, \bar{T}]$ with radius of convergence at least $\bar{\tau}$. Moreover, there exists a constant $C = C(K_0) > 0$ independent of $\nu$ such that for all $t \in [0, \bar{T}]$,
  \[ \| \Lambda^r e^{r_0 \Lambda^{\frac{1}{2}}} \theta^\nu(\cdot, t) \|_{L^2} \leq C, \forall \nu > 0, \]
  \[ \| \Lambda^r e^{r_0 \Lambda^{\frac{1}{2}}} \theta^0(\cdot, t) \|_{L^2} \leq C. \]
  Here $\theta^\nu$ are Gevrey-class $s$ solutions to (6.1) for $\nu > 0$ as described in Theorem 6.4.

**Theorem 6.6** (Convergence of solutions as $\nu \rightarrow 0$ for the MG equations). Fix $s \geq 1$, $r > 3$ and $K_0 > 0$. Let $\theta_0$ and $S$ be of Gevrey-class $s$ with radius of convergence $\tau_0 > 0$ and satisfy the assumptions given in Theorem 6.5. If $\theta^\nu$ and $\theta^0$ are Gevrey-class $s$ solutions to (6.1) for $\nu > 0$ and $\nu = 0$ respectively with initial datum $\theta_0$ on $T^3 \times [0, \bar{T}]$ with radius of convergence at least $\bar{\tau}$ as described in Theorem 6.5, then there exists $T < \bar{T}$ and $\tau = \tau(t) < \bar{\tau}$ such that, for $t \in [0, T]$, we have
  \[ \lim_{\nu \rightarrow 0} \| (\Lambda^r e^{r_0 \Lambda^{\frac{1}{2}}} \theta^\nu - \Lambda^r e^{r_0 \Lambda^{\frac{1}{2}}} \theta^0)(\cdot, t) \|_{L^2} = 0. \]

### 6.2. Incompressible Porous Media Equation

Next we study the incompressible porous media Brinkmann (IPMB) equation. Specifically, we address the following active scalar equation in $T^2 \times [0, \infty)$ with periodic boundary conditions:

\[
\begin{aligned}
\partial_t \theta^\nu + (u^\nu \cdot \nabla) \theta^\nu &= 0, \\
u^\nu &= M^\nu[\theta^\nu], \theta^\nu(x, 0) = \theta_0(x),
\end{aligned}
\]
where the symbol of $M^\nu$ is given by (1.14) with
\[
\hat{M}^\nu_1 (k) = \frac{1}{1 + \nu(k_1^2 + k_2^2)} \left( \frac{k_1 k_2}{k_1^2 + k_2^2} \right),
\]
\[
\hat{M}^\nu_2 (k) = \frac{1}{1 + \nu(k_1^2 + k_2^2)} \left( \frac{-k_1^2}{k_1^2 + k_2^2} \right).
\]

We also write $M^\nu = \partial_i T^\nu_{ij}$ for convenience. To apply the results from Sect. 2, it suffices to show that the sequence of operators $\{T^\nu_{ij}\}_{\nu \geq 0}$ satisfy the assumptions A1–A4 and A52 given in Sect. 1.

**Proposition 6.7.** Let $M^\nu_{ij} = \partial_i T^\nu_{ij}$, where $M^\nu$ is given by (6.7)–(6.8). Then $T^\nu_{ij}$ satisfy the assumptions A1–A4 and A52 given in Sect. 1.

**Proof.** It suffices to check that $T^\nu_{ij}$ satisfy assumptions A3 and A4. To show that $T^\nu_{ij}$ satisfy A3, for each $k \in \mathbb{Z}^2 / \{k = 0\}$,
\[
\left| \hat{M}^\nu_{ij} (k) \right| = \frac{|k_1 k_2|}{|k|} \times \frac{1}{1 + \nu|k|^2} \leq 1,
\]
since $|k| \geq 1$. Similarly, $\left| \hat{M}^0_{ij} (k) \right| \leq 1$ and to see that $T^\nu_{ij}$ satisfies A52, similar to the case of MG equation, it suffice to show that for each $L > 0$,
\[
\lim_{\nu \to 0} \sup_{k \in \mathbb{Z}^2 : k \neq 0, |k| \leq L} \left| \frac{\hat{M}^\nu_{ij} (k) - \hat{M}^0_{ij} (k)}{|k|} \right| = 0.
\]

Fix $L > 0$ and for each $k \in \mathbb{Z}^2 / \{k = 0\}$ with $|k| \leq L$, we have
\[
\left| \frac{\hat{M}^\nu_{ij} (k) - \hat{M}^0_{ij} (k)}{|k|} \right| = \frac{\nu|k|^2}{(1 + \nu|k|^2)} \times \frac{|k_1 k_2|}{|k|^3} \leq \nu \times \frac{|k_1 k_2|}{|k|} \leq \nu L,
\]
hence
\[
\lim_{\nu \to 0} \sup_{k \in \mathbb{Z}^2 : k \neq 0, |k| \leq L} \left| \frac{\hat{M}^\nu_{ij} (k) - \hat{M}^0_{ij} (k)}{|k|} \right| = 0.
\]

By the same argument, we also have
\[
\lim_{\nu \to 0} \sup_{k \in \mathbb{Z}^2 : k \neq 0, |k| \leq L} \left| \frac{\hat{M}^\nu_{ij} (k) - \hat{M}^0_{ij} (k)}{|k|} \right| = 0
\]
and (6.9) follows. \hspace{1cm} \Box

Thanks to Proposition 6.7, the abstract Theorem 2.1–2.5 can be applied to the IPMB equations (6.6). More precisely, we have

**Theorem 6.8** (Wellposedness in Sobolev space for the IPMB equations). Let $\theta_0 \in W^{s,2}$ for $s \geq 0$. Then for each $\nu > 0$, we have:

- if $s = 0$, there exists unique global weak solution to (6.1) such that
  \[
  \theta^\nu \in BC([0, \infty); L^2),
  \]
  \[
  u^\nu \in C([0, \infty); W^{2,2}).
  \]

In particular, $\theta^\nu (\cdot, t) \to \theta_0$ weakly in $L^2$ as $t \to 0^+$. 

• if \( s > 0 \), there exists a unique global-in-time solution \( \theta^\nu \) to (6.1) such that \( \theta^\nu(\cdot,t) \in W^{s,2} \) for all \( t \geq 0 \).
  Furthermore, for \( s = 1 \), we have the following single exponential growth in time on \( \| \nabla \theta^\nu(\cdot,t) \|_{L^2} \):
  \[
  \| \nabla \theta^\nu(\cdot,t) \|_{L^2} \leq C \| \nabla \theta_0 \|_{L^2} \exp \left( C t \| \theta_0 \|_{W^{1,2}} \right),
  \]
  where \( C > 0 \) is a constant which depends only on some dimensional constants.

**Theorem 6.9** (Gevrey-class global wellposedness for the IPMB equations). Fix \( s \geq 1 \). Let \( \theta_0 \) be of Gevrey-class \( s \) with radius of convergence \( \tau_0 > 0 \). Then for each \( \nu > 0 \), there exists a unique Gevrey-class \( s \) solution \( \theta^\nu \) to (6.6) on \( \mathbb{T}^2 \times [0, \infty) \) with radius of convergence at least \( \tau = \tau(t) \) for all \( t \in [0, \infty) \), where \( \tau \) is a decreasing function satisfying
  \[
  \tau(t) \geq \tau_0 e^{-C t \| e^{\tau_0 A^{1/2}} \theta_0 \|_{L^2}}.
  \]
  Here \( C > 0 \) is a constant which depends on \( \nu \) but independent of \( t \).

**Theorem 6.10** (Local wellposedness in Sobolev space for the IPMB equations). Fix \( s > 2 \) and assume that \( \theta_0 \in H^s(\mathbb{T}^2) \) has zero-mean on \( \mathbb{T}^2 \). Then there exists a \( T > 0 \) and a unique smooth solution \( \theta^0 \) to (6.6) with \( \nu = 0 \) such that
  \[
  \theta^0 \in L^\infty(0, T; H^s(\mathbb{T}^2)).
  \]

**Theorem 6.11** (Convergence of solutions as \( \nu \to 0 \) for the IPMB equations). Assume that the hypotheses and notations of Theorem 6.10 are in force. For \( t \in [0, T] \), we have
  \[
  \lim_{\nu \to 0} \| (\theta^\nu - \theta^0)(\cdot, t) \|_{H^{s-1}} = 0.
  \]

**Remark 6.12.** The results given in Theorem 6.10 are consistent with those discussed in [6] and [7]. Furthermore, the abstract Theorem 2.4 can also be applied to the non-diffusive SQG equation to show local wellposedness in Sobolev spaces [22].

**Remark 6.13.** In [10], the authors studied the singular incompressible porous media (SIPM) equations set in \( \mathbb{T}^2 \times [0, \infty) \) with periodic boundary conditions, which are given by
  \[
  \begin{align*}
  \partial_t \theta + v \cdot \nabla \theta &= 0, & (6.10) \\
  u &= -\nabla(-\Delta)^{-1} \partial_x \Lambda^\beta \theta - (0, \Lambda^\beta \theta) = M^\beta[\theta]. & (6.11)
  \end{align*}
  \]
  The operator \( M^\beta \) in (6.11) is a pseudodifferential operator of order \( \beta \), in which the Fourier multiplier symbol can be computed explicitly as \( |k|^{\beta+1} |k|^{-2} \). It is proved in [10] that when \( 0 < \beta \leq 1 \) the SIPM equations are ill-posed in Sobolev spaces, however local well-posedness holds for certain patch type weak solutions.

It is straightforward to see that for the case \( 0 < \beta \leq 1 \), the system (6.10)–(6.11) satisfies the properties A1–A2 and A5 (1) (by taking \( \nu = 0 \)), so the abstract Theorem 2.3 also holds in analogy with those for the MG equations. More specifically, we obtain the following local-in-time Gevrey class existence theorem for the SIPM equations:

**Theorem 6.14** (Gevrey-class local wellposedness for the SIPM equations). Fix \( \beta \in (0, 1] \), \( s \geq 1 \), \( r > \frac{5}{2s} \) and \( K_0 > 0 \). Let \( \theta(x, 0) = \theta_0 \) be of Gevrey-class \( s \) with radius of convergence \( \tau_0 > 0 \) and satisfies
  \[
  \| \Lambda^r e^{\tau_0 A^{1/2}} \theta(\cdot, 0) \|_{L^2} \leq K_0.
  \]
  There exists \( \bar{T}, \bar{\tau} > 0 \) and a unique Gevrey-class \( s \) solution \( \theta \) to (6.10)–(6.11) defined on \( \mathbb{T}^2 \times [0, \bar{T}] \) with radius of convergence at least \( \bar{\tau} \). In particular, there exists a constant \( C = C(K_0) > 0 \) such that for all \( t \in [0, \bar{T}] \),
  \[
  \| \Lambda^r e^{\tau A^{1/2}} \theta(\cdot, t) \|_{L^2} \leq C.
  \]
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