CRITICAL POINTS OF EISENSTEIN SERIES

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ABSTRACT. For any even integer \( k \geq 4 \), let \( E_k \) be the normalized Eisenstein series of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \). Also let \( D \) be the closure of the standard fundamental domain of the Poincaré upper half plane modulo \( \text{SL}_2(\mathbb{Z}) \). F. K. C. Rankin and H. P. F. Swinnerton-Dyer showed that all zeros of \( E_k \) in \( D \) are of modulus one. In this article, we study the critical points of \( E_k \), that is to say the zeros of the derivative of \( E_k \). We show that they are simple. We count those belonging to \( D \), prove that they are located on the two vertical edges of \( D \) and produce explicit intervals that separate them. We then count those belonging to \( \gamma D \), for any \( \gamma \in \text{SL}_2(\mathbb{Z}) \).

1. Introduction and statement of the results

1.1. The Eisenstein series

Let \( k \) be an even integer \( \geq 4 \). For any point \( z \) in the Poincaré upper half-plane \( \mathcal{H} \), let

\[
G_k(z) := \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} (cz + d)^{-k}.
\]

We have

\[
G_k(z) = 2\zeta(k)E_k(z),
\]

where \( E_k \) is the normalized Eisenstein series of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \). Its Fourier expansion is given by

\[
E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,
\]

where \( q := e^{2\pi iz} \), \( \sigma_{k-1}(n) \) is the sum of the \((k-1)\)-th powers of the positive divisors of \( n \) and \( B_k \) is the Bernoulli number of index \( k \).

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We recall in subsection 1.2 the results of Rankin and Swinnerton-Dyer concerning the location of zeros of \( E_k \), or what amounts to the same of \( G_k \). We then state in subsections 1.3 and 1.4, the results we obtain concerning the location of the critical points of \( E_k \), i.e. of the zeros of the derivative \( E'_k \) of \( E_k \).

1.2. Results of F.K.C. Rankin and H.P.F. Swinnerton-Dyer

Since the function \( E_k \) is a modular form of weight \( k \) for the group \( SL_2(\mathbb{Z}) \), the set of its zeros in \( \mathfrak{H} \) is stable under the action of \( SL_2(\mathbb{Z}) \), and the zeros of a same orbit have the same multiplicity. We therefore content ourselves with describing those belonging to a fundamental domain of \( \mathfrak{H} \) modulo \( SL_2(\mathbb{Z}) \).

Let us denote by \( D \) the closure in \( \mathfrak{H} \) of the standard fundamental domain of \( \mathfrak{H} \) modulo \( SL_2(\mathbb{Z}) \), i.e. the set of points \( z \) in \( \mathfrak{H} \) such that \( |\text{Re}(z)| \leq \frac{1}{2} \) and \( |z| \geq 1 \). Rankin and Swinnerton-Dyer proved in [4] that all the zeros of \( E_k \) in \( D \) are of modulus 1, and more precisely that :

\( a) \) The points \( e^{\pi i/3} \) and \( e^{2\pi i/3} \) are simple zeros of \( E_k \) when \( k \equiv 4 \mod 6 \), zeros of multiplicity 2 when \( k \equiv 2 \mod 6 \) and are not zeros of \( E_k \) when \( k \equiv 0 \mod 6 \).

\( b) \) The point \( i = e^{\pi i/2} \) is a simple zero of \( E_k \) when \( k \equiv 2 \mod 4 \) and is not a zero of \( E_k \) when \( k \equiv 0 \mod 4 \).

\( c) \) The zeros of \( E_k \) lying on the open circular arc consisting of the \( e^{i\theta} \), with \( \frac{\pi}{3} < \theta < \frac{\pi}{2} \), are simple. Their number is \( \lfloor \frac{k}{12} \rfloor \) when \( k \not\equiv 2 \mod 12 \) and \( \lfloor \frac{k}{12} \rfloor - 1 \) when \( k \equiv 2 \mod 12 \). Here \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \). The zeros located on the open circular arc consisting of the \( e^{i\theta} \), with \( \frac{\pi}{2} < \theta < \frac{2\pi}{3} \), are symmetric to the previous ones with respect to the imaginary axis, and are simple as well.

Remark 1. - These results are consistent with the following fact : since \( E_k \) is a modular form of weight \( k \) which does not vanish at infinity, the weighted number of its zeros in \( \mathfrak{H} \) modulo \( SL_2(\mathbb{Z}) \) (counted with multiplicities, with weight \( \frac{1}{2} \) for those in the orbit of \( e^{\pi i/2} \), \( \frac{1}{3} \) for those in the orbit of \( e^{i\pi/3} \), and 1 otherwise) has to be equal to \( \frac{k}{12} \).

Remark 2. - It follows from [3] that the function \( t \mapsto E_k(\frac{1}{2} + it) \) is real-valued and analytic for \( t > 0 \), and tends to 1 when \( t \) tends to \( +\infty \). Further, it does not vanish for \( t > \frac{\sqrt{3}}{2} \) by the results stated above. Hence we have \( E_k(\frac{1}{2} + it) > 0 \) for all \( t > \frac{\sqrt{3}}{2} \).

1.3. Zeros of the derivative of \( E_k \) in \( D \)

In this article, we are interested in the zeros of the derivative of \( E_k \). This derivative \( E'_k \) is not a modular form, but only a quasi-modular form. The set
of its zeros is no longer stable under the action of $\text{SL}_2(\mathbb{Z})$. Thus in order to get precise information about these zeros, we need to count them not only in $D$, but also in $\gamma D$ for any $\gamma \in \text{SL}_2(\mathbb{Z})$. We state in this subsection three results about the zeros located in $D$, and in the next subsection, those about the zeros located in $\gamma D$, for any $\gamma \in \text{SL}_2(\mathbb{Z})$.

**Theorem 1.**— The zeros of $E'_k$ in $D$ are simple, and have real part either $\frac{1}{2}$ or $-\frac{1}{2}$. Those with real part $\frac{1}{2}$ are the translates by $\frac{1}{2}$ of those with real part $-\frac{1}{2}$.

The second assertion follows from the fact that $E'_k(z + 1) = E'_k(z)$. So let us restrict our attention to the zeros of $E'_k$ in $D$ with real part $\frac{1}{2}$, i.e. located on the closed half-line $\frac{1}{2} + i\sqrt{\frac{3}{2}}, +\infty[$. Their number is given by the following theorem:

**Theorem 2.**—

- a) The point $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a zero of $E'_k$ if and only if $k \equiv 2 \mod 6$.

- b) The number of zeros of $E'_k$ in the open half-line $\frac{1}{2} + i\sqrt{\frac{3}{2}}, +\infty[$ is $\left\lfloor \frac{k-4}{6} \right\rfloor$. Assertion b) can be rephrased by saying that the number of zeros of $E'_k$ in $\frac{1}{2} + i\sqrt{\frac{3}{2}}, +\infty[$ is $\left\lfloor \frac{k-4}{6} \right\rfloor$ when $k \equiv 4 \mod 6$ and $\left\lfloor \frac{k-4}{6} \right\rfloor - 1$ otherwise. The following theorem gives more precise information about their locations by producing open intervals that separate their imaginary parts:

**Theorem 3.**— Let $M := \left\lfloor \frac{k}{6} \right\rfloor$. For $1 \leq m \leq M$, let $t_m := \frac{1}{2} \cot(\frac{m\pi}{k+1})$. We have therefore

$$t_1 > \cdots > t_M > \frac{\sqrt{3}}{2}.$$ 

For each integer $m$ such that $1 \leq m \leq M - 1$, $E'_k$ has a unique zero in the set $\left\lfloor \frac{1}{2} + i \right\rfloor t_{m+1}, t_m[$. Moreover, when $k \equiv 4 \mod 6$ and $k \neq 4$, $E'_k$ has a unique zero in $\left\lfloor \frac{1}{2} + i \right\rfloor t_M$. We shall then prove, in Lemma 2 of subsection 2.3, by a suitable application of the residue theorem, that the sum of the multiplicities of the zeros of $E'_k$ with modulus $> 1$ and with real part in $]-\frac{1}{2}, \frac{1}{2}[$ is equal to $\left\lfloor \frac{k-4}{6} \right\rfloor$.

Let us briefly outline the plan of the proofs of the above three theorems, which will be given in full detail in sections 2 and 3. We shall first prove, in Lemma 1 of subsection 2.2, that $E'_k$ has no zeros of modulus 1 in $D$ when $k \not\equiv 2 \mod 6$, and that its only zeros of modulus 1 in $D$ are simple zeros at $e^{\pi i/3}$ and $e^{2\pi i/3}$ when $k \equiv 2 \mod 6$. This among other things implies assertion a) of Theorem 2. We shall then prove, in Lemma 2 of subsection 2.3, by a suitable application of the residue theorem, that the sum of the multiplicities of the zeros of $E'_k$ with modulus $> 1$ and with real part in $]-\frac{1}{2}, \frac{1}{2}[$ is equal to $\left\lfloor \frac{k-4}{6} \right\rfloor$. The restriction of $E'_k$ to the half-line $\frac{1}{2} + i[0, +\infty[$ is purely imaginary. We shall prove in Corollary of Proposition 1 of subsection 3.1 that, with the notations of Theorem 3, $iE'_k$ takes non-zero values of opposite signs at $\frac{1}{2} + t_m$ and $\frac{1}{2} + it_{m+1}$ for $1 \leq m \leq M - 1$, and also at $\frac{1}{2} + \sqrt{\frac{3}{2}}$ and $\frac{1}{2} + it_M$ when $k \equiv 4 \mod 6$ and
$k \neq 4$. This yields us a total of $\left\lfloor \frac{k-4}{6} \right\rfloor$ disjoint open intervals on the half-line $\left[ \frac{1}{2} + i \frac{\sqrt{3}}{2}, +\infty \right]$ in each of which $E'_k$ has at least one zero. But then necessarily each of these intervals can contain only one zero of $E'_k$, this zero must be simple, and there cannot be any other zero of $E'_k$ whose modulus is $> 1$ and whose real part belongs to $\left] - \frac{1}{2}, \frac{1}{2} \right]$. This proves simultaneously Theorem 1, part b) of Theorem 2 and Theorem 3.

1.4. Zeros of $E'_k$ in $\gamma D$, for $\gamma \in \text{SL}_2(\mathbb{Z})$

**Theorem 4.**— All the zeros of $E'_k$ in $\mathcal{F}$ are simple.

We shall prove this theorem in section 4. It is a consequence of much more general results (see subsection 4.4, Theorem 6 and subsection 4.5, Theorem 7) on the multiplicity of zeros of quasi-modular forms for $\text{SL}_2(\mathbb{Z})$ with algebraic Fourier coefficients, these results being themselves deduced from a theorem of algebraic independence of G. V. Chudnovsky.

Let us call trivial zeros of $E'_k$ those zeros of $E'_k$ which are also zeros of $E_k$. By subsection 1.2, there exist such zeros only when $k \equiv 2 \mod 6$, and these are then the elements of the orbit of $e^{\pi i/3}$ under the action of $\text{SL}_2(\mathbb{Z})$. Let us denote by $Z(E'_k)$ the set of non-trivial zeros of $E'_k$. It is translation invariant by the elements of $\mathbb{Z}$ since $E'_k$ is periodic with period 1.

We are interested in the description of the points of $Z(E'_k)$ belonging to $\gamma D$, when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\text{SL}_2(\mathbb{Z})$. When $c = 0$, $\gamma$ is of the form $\pm \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, $\gamma D$ is the translate of $D$ by $r$, and the description of $Z(E'_k) \cap \gamma D$ can be deduced from the results in subsection 1.3. For $c \neq 0$, we have the following results:

**Theorem 5.**— a) If $|d| < |c|$, the set $Z(E'_k) \cap \gamma D$ is empty.

b) If $|d| \geq |c| > 0$, the set $Z(E'_k) \cap \gamma D$ has cardinality $\left\lfloor \frac{k+2}{6} \right\rfloor$. It is contained in the interior of $\gamma D$ when $|d| \neq |c|$. When $|d| = |c|$, it is contained in the boundary of $\gamma D$, and more precisely in $\gamma C$, where $C$ is the open arc of the unit circle consisting of the points $e^{i\theta}$, where $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$.

Note that when $|d| = |c|$, $\gamma$ is of the form $\pm \left( \begin{pmatrix} r+1 & 1 \\ 1 & 1 \end{pmatrix} \right)$ or $\pm \left( \begin{pmatrix} r & -r-1 \\ 1 & 1 \end{pmatrix} \right)$ with $r \in \mathbb{Z}$, and $\gamma C$ is then the open vertical segment $r + \frac{1}{2} + i \left[ \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2} \right]$.

Theorem 5 will be proved in subsection 5.6. Here is a consequence:

**Corollary.**— The number of zeros of $E'_k$ in $\mathcal{F}$ with real part $\frac{1}{2}$ is $1 + 2 \left\lfloor \frac{k-2}{6} \right\rfloor$. More precisely, $\left\lfloor \frac{k-4}{6} \right\rfloor$ of them have an imaginary part strictly greater than $\frac{\sqrt{3}}{2}$, $\left\lfloor \frac{k+2}{6} \right\rfloor$ of them have an imaginary part strictly between $\frac{\sqrt{3}}{6}$ and $\frac{\sqrt{3}}{2}$, none have an imaginary part strictly less than $\frac{\sqrt{3}}{6}$, and when $k \equiv 2 \mod 6$, one has an imaginary part equal to $\frac{\sqrt{3}}{6}$ and another one an imaginary part equal to $\frac{\sqrt{3}}{2}$.
The number of zeros of $E'_k$ in the set $\frac{1}{2} + i \frac{\sqrt{3}}{2}, +\infty[ is \left[ \frac{k-4}{6} \right]$ by Theorem 2, b).

The set $\frac{1}{2} + i \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}$ is equal to $\left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) C$, with the notations of Theorem 5. By Theorem 5 b), it contains $\left[ \frac{k+2}{6} \right]$ zeros of $E'_k$, and its endpoints $\frac{1}{2} + i \frac{\sqrt{3}}{2}$ and $\frac{1}{2} + i \frac{\sqrt{3}}{2}$ are zeros of $E'_k$ if and only if $k \equiv 2 \bmod 6$.

Finally, the set $\frac{1}{2} + i \sqrt{3} \frac{2}{6}, \frac{\sqrt{3}}{2}$ is the image by the homography $z \mapsto \frac{z}{2z+1}$ of the half line $-\frac{1}{2} + i \frac{\sqrt{3}}{2}, +\infty[. Then by Theorem 5 a), it does not contain any zero of $E'_k$.

Thus the total number of zeros of $E'_k$ with real part $\frac{1}{2}$ is $\left[ \frac{k-4}{6} \right] + \left[ \frac{k+2}{6} \right] = \frac{k-3}{3}$ when $k \equiv 0 \bmod 6$, $\left[ \frac{k-4}{6} \right] + \left[ \frac{k+2}{6} \right] + 2 = \frac{k+1}{3}$ when $k \equiv 2 \bmod 6$, $\left[ \frac{k-4}{6} \right] + \left[ \frac{k+2}{6} \right] = \frac{k-1}{3}$ when $k \equiv 4 \bmod 6$, hence is $1 + 2\left[ \frac{k-2}{6} \right]$ in each of the three cases.

The corollary of Theorem 5 gives complete information on the number and location of zeros of $E'_k$ with real part $\frac{1}{2}$. Partial results about these zeros had previously been obtained by R. Balasubramanian and S. Gun (see [1], th. 1.2 and th. 1.5).

2. Counting the zeros of $E'_k$ in the standard fundamental domain

As before, $k$ is an even integer $\geq 4$ and $E_k$ is the normalized Eisenstein series of weight $k$ for $\text{SL}_2(\mathbb{Z})$.

2.1. Variation of the argument of a function

Let $g$ be a continuous function on a closed bounded interval $[a, b]$ of $\mathbb{R}$, with values in $\mathbb{C}$. Since $[a, b]$ is simply connected, there exists a continuous function $h$ from $[a, b]$ to $\mathbb{R}$ such that $h(t)$ is an argument of $g(t)$ for every $t \in [a, b]$. Such a function is called a continuous argument of $g$. It is unique up to addition of a constant function on $[a, b]$ with values in $2\pi \mathbb{Z}$. The real number $h(b) - h(a)$ does not depend on the choice of $h$, and is called the variation of the argument of $g$ along $[a, b]$.

Remark 3. – When $g$ is of class $C^1$, the variation of its argument along $[a, b]$ is the imaginary part of $\int_a^b g'(t) \frac{dt}{g(t)}$.

Remark 4. – Let $c$ be a point of $[a, b]$. The variation of the argument of $g$ along $[a, c]$ is the sum of the variations of the argument of $g$ along $[a, c]$ and along $[c, b]$. 
Remark 5. – Let $[\alpha, \beta]$ be another closed bounded interval of $\mathbb{R}$ and let $u : [\alpha, \beta] \to [a, b]$ be a continuous function which maps $\alpha$ to $a$ and $\beta$ to $b$. The variation of the argument of $g$ along $[a, b]$ is equal to that of $g \circ u$ along $[\alpha, \beta]$.

More generally, let $\gamma : [a, b] \to \mathbb{C}$ be a continuous map and let $\varphi$ be a continuous function on $\gamma([a, b])$, with values in $\mathbb{C}^*$. We define variation of the argument of $\varphi$ along the oriented curve $\gamma$ to be the variation of the argument of $\varphi \circ \gamma$ along $[a, b]$. By Remark 5, it is invariant by a continuous change of parametrization of the oriented curve.

2.2. Zeros of modulus 1 of $E'_k$ in $D$

Lemma 1.– When $k \not\equiv 2 \mod 6$, $E'_k$ has no zeros of modulus 1 in $D$. When $k \equiv 2 \mod 6$, its only zeros of modulus 1 in $D$ are simple zeros at $e^{\pi i/3}$ and $e^{2\pi i/3}$.

For all $z \in \mathcal{H}$, we have $E_k(-\bar{z}) = \overline{E_k(z)}$, and hence

$$E'_k(-\bar{z}) = -\overline{E'_k(z)}.$$  

(4)

Since $E_k$ is a modular form of weight $k$ for $SL_2(\mathbb{Z})$, we also have $E_k(-\frac{1}{z}) = z^kE_k(z)$, and hence

$$E'_k(-\frac{1}{z}) = z^{k+2}E'_k(z) + kz^{k+1}E_k(z).$$  

(5)

If $z$ is a zero of $E'_k$ of modulus 1, $-\bar{z}$ is equal to $-\frac{1}{z}$, and we then deduce from (4) and (5) that $E_k(z) = 0$. Hence $z$ is a multiple zero of $E_k$. But, by subsection 1.2, $E_k$ does not have multiple zeros in $D$ unless $k \equiv 2 \mod 6$, and these are then zeros of order 2 of $E_k$, hence simple zeros of $E'_k$, located at $e^{\pi i/3}$ and $e^{2\pi i/3}$.

2.3. Other zeros of $E'_k$ in $D$

We have determined in subsection 2.2 the zeros of $E'_k$ in $D$ which are of modulus 1. Let us now look at the other zeros of $E'_k$ in $D$, namely those of modulus $> 1$. Since those with real part $\frac{1}{2}$ are translates by 1 of those with real part $-\frac{1}{2}$, we will only be interested in those zeros whose real part is distinct from $-\frac{1}{2}$. The following lemma allows us to count them:

Lemma 2.– The sum of the multiplicities of those zeros of $E'_k$ with modulus $> 1$ and real part in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is equal to $\left[\frac{k-4}{6}\right]$.

When $\text{Im}(z)$ tends to $+\infty$, $E'_k(z)$ is equivalent to $-\frac{4k\pi i}{3k}e^{2\pi iz}$. Therefore we can choose a real number $T > 1$ such that all zeros of $E'_k$ have an imaginary part

\[e^{\pi i/3}, e^{2\pi i/3}, e^{\pi i/3}, e^{2\pi i/3}, \ldots, e^{\pi i/3}, e^{2\pi i/3}, \ldots\]
strictly less than $T$. By the residue theorem, the sum considered in Lemma 2 is equal, for $\varepsilon > 0$ sufficiently small, to the integral

$$I := \frac{1}{2\pi i} \int_{\gamma_{T,\varepsilon}} \frac{E_k''(z)}{E_k'(z)} \, dz,$$

where $\gamma_{T,\varepsilon}$ is the contour of integration obtained as follows: one modifies the oriented boundary of $D_T = \{ z \in D \mid \text{Im}(z) \leq T \}$ by going around the zeros of $E_k'$ lying on the boundary of $D_T$ by arcs of circles of radius $\varepsilon$ passing outside of $D_T$ for the zeros with modulus $> 1$ and real part $\frac{1}{2}$, and passing inside $D_T$ for all the others (see Figure 1 below\(^{(1)}\)).

![Figure 1](image)

Figure 1. The contour of integration $\gamma_{T,\varepsilon}$ when $k \equiv 2 \mod 6$.

The computation of the integral $I$ will be the subject of the remaining parts of this section. We will show that $I = \left\lfloor \frac{k-4}{6} \right\rfloor$ by distinguishing three cases according to the congruence class of $k$ modulo 6 (see Lemmas 3, 4 and 5). Lemma 2 then will follow.

\(^{(1)}\) The authors thank Sunil Naik, a doctoral student at the Institute of Mathematical Sciences in Chennai, who carried out the drawing of all figures in this paper.
2.4. Reduction of the computation of the integral $I$ to a variation of argument

The contribution to the integral $I$ of the upper horizontal oriented segment (joining $\frac{1}{2} + iT$ to $-\frac{1}{2} + iT$) is independent of $T$ since $E_k$ is periodic with period 1. This contribution is equal to $-1$: this is seen by letting $T$ go to $+\infty$, and observing that $\frac{E_k''(z)}{E_k'(z)}$ tends to $2\pi i$ when $\text{Im}(z)$ goes to $+\infty$.

The contributions to the integral $I$ of the vertical segments contained in the line with abscissa $\frac{1}{2}$ and of those contained in the line with abscissa $-\frac{1}{2}$ are opposite of each other, since $E_k$ is periodic with period 1 and these segments have opposite orientations.

Similarly, the contributions to the integral $I$ of the semicircles of radius $\varepsilon$ centered at the zeros of $E_k'$ of modulus $> 1$ and real part $\frac{1}{2}$ are opposite to those of the semicircles of radius $\varepsilon$ centered at the zeros of $E_k'$ of modulus $> 1$ and real part $-\frac{1}{2}$.

When $k \not\equiv 2 \mod 6$, $E_k'$ has no zeros of modulus 1 in $D$ by Lemma 1. The only remaining contribution to the integral $I$ is then that of the arc of the unit circle joining $e^{2\pi i/3}$ to $e^{\pi i/3}$. It can be written as $-\frac{A}{2\pi}$, where

$$A := \int_\pi^{2\pi/3} \frac{E_k''(e^{i\theta})}{E_k'(e^{i\theta})} e^{i\theta} \, d\theta.$$  \hfill (7)

We have $I = -1 - \frac{A}{2\pi}$. Since $I$ is an integer, $A$ is a real number. It follows that $A$ is the variation of the argument of $\theta \mapsto E_k'(e^{i\theta})$ along the interval $[\frac{\pi}{3}, \frac{2\pi}{3}]$ (see 2.1, Remark 3).

When $k \equiv 2 \mod 6$, it follows from Lemma 1 that the only zeros of $E_k'$ in $D$ of modulus 1 are simple zeros at $e^{\pi i/3}$ and $e^{2\pi i/3}$. The function $\frac{E_k''(z)}{E_k'(z)}$ is equivalent to $\frac{1}{z-e^{\pi i/3}}$ when $z$ approaches $e^{\pi i/3}$. The contribution to $I$ of the circular arc of radius $\varepsilon$ centered at $e^{\pi i/3}$ tends to $-\frac{1}{6}$ when $\varepsilon$ tends to 0. The same holds for the contribution to $I$ of the circular arc of radius $\varepsilon$ centered at $e^{2\pi i/3}$. Hence we have $I = -\frac{4}{3} - \frac{A}{2\pi}$, where

$$A := \lim_{\eta \to 0^+} \int_{\pi/3 + \eta}^{2\pi/3 - \eta} \frac{E_k''(e^{i\theta})}{E_k'(e^{i\theta})} e^{i\theta} \, d\theta.$$  \hfill (8)

We see as before that $A$ is real and is equal to the limit, when $\eta > 0$ tends to 0, of the variation of the argument of $\theta \mapsto E_k'(e^{i\theta})$ along the interval $[\frac{\pi}{3} + \eta, \frac{2\pi}{3} - \eta]$. 

2.5. **Behaviour of** $E'_{k}$ **on the unit circle**

For any point $z$ in $\mathbb{H}$ of modulus 1, we have

$$E_k(z) = E_k(-\bar{z}) = E_k(-\frac{1}{z}) = z^kE_k(z).$$

Hence, for all $\theta \in ]0, \pi[$, we can write

$$E_k(e^{i\theta}) = e^{-ki\theta/2}f_k(\theta),$$

where $f_k(\theta)$ is a real number. The function $f_k$ defined in this manner on $]0, \pi[$ is real analytic. By differentiating, we get

$$E'_k(e^{i\theta}) = -ie^{-(k+2)i\theta/2}g_k(\theta),$$

where

$$g_k(\theta) := f'_k(\theta) - \frac{ki}{2}f_k(\theta).$$

**Remark 6.** It follows from formulas (9) and (10) that we have

$$f_k(\pi - \theta) = (-1)^{k/2}f_k(\theta)$$

for $\theta \in ]0, \pi[$. So we have $f'_k(\pi - \theta) = -(1)^{k/2}f'_k(\theta)$, and $g_k(\pi - \theta)$ is symmetric of $g_k(\theta)$ with respect to the horizontal axis when $k \equiv 2 \mod 4$, and to the vertical axis when $k \equiv 0 \mod 4$.

It follows from Lemma 1 of subsection 2.2 that $g_k$ has no zeros in the interval $[\pi/3, 2\pi/3]$ when $k \not\equiv 2 \mod 6$, and has zeros only at the endpoints of this interval when $k \equiv 2 \mod 6$. We then deduce from (11) that

$$A = -\frac{k + 2}{6}\pi + B,$$

where $B$ is the variation of the argument of $g_k$ along $[\pi/3, 2\pi/3]$ when $k \not\equiv 2 \mod 6$, and the limit, when $\eta > 0$ tends to 0, of the variation of the argument of $g_k$ along $[\pi/3 + \eta, 2\pi/3 - \eta]$ when $k \equiv 2 \mod 6$.

2.6. **Computation of the integral I when** $k \equiv 4 \mod 6$

**Lemma 3.** When $k \equiv 4 \mod 6$, the integral I is equal to $\frac{k-4}{6}$.

Suppose that $k \equiv 4 \mod 6$. It follows from subsection 1.2 that the function $f_k$ has only simple zeros in the interval $[\pi/3, 2\pi/3]$, and that they can be written as

$$\theta_0 < \theta_1 < \cdots < \theta_N < \theta_{N+1}$$

where $\theta_0 = \pi/3$, $\theta_{N+1} = 2\pi/3$, and $N$ is equal to $2[\frac{k}{12}]$ when $k \equiv 4 \mod 12$ and to $2[\frac{k}{12}] + 1$ when $k \equiv 10 \mod 12$, and hence to $\frac{k-4}{6}$ in both of these cases.

The points $\theta_j$ are those points in the interval $[\pi/3, 2\pi/3]$, where the complex-valued function $g_k = f'_k - \frac{ki}{2}f_k$ crosses the real line. Since $\theta_j$ is a simple zero of
\( f_k \), the real number \( g_k(\theta_j) = f'_k(\theta_j) \) is non-zero. If it is positive, \( g_k \) crosses the real line at \( \theta_j \) from upper half-plane to lower half-plane. If it is negative, \( g_k \) crosses the real line at \( \theta_j \) from lower half-plane to upper half-plane. Therefore \( g_k(\theta_j) \) and \( g_k(\theta_{j+1}) \) have opposite signs for \( 0 \leq j \leq N \), and more precisely:

\( a) \) If \( g_k(\theta_j) < 0 \), \( g_k([\theta_j, \theta_{j+1}]) \) is contained in the closed upper half-plane, and \( g_k \) has a continuous argument along \([\theta_j, \theta_{j+1}]\) taking the values \( \pi \) at \( \theta_j \) and 0 at \( \theta_{j+1} \);

\( b) \) If \( g_k(\theta_j) > 0 \), \( g_k([\theta_j, \theta_{j+1}]) \) is contained in the closed lower half-plane, and \( g_k \) has a continuous argument along \([\theta_j, \theta_{j+1}]\) taking the values 0 at \( \theta_j \) and \( -\pi \) at \( \theta_{j+1} \).

We observe that the variation of the argument of \( g_k \) along each interval \([\theta_j, \theta_{j+1}]\), where \( 0 \leq j \leq N \), is equal to \( -\pi \). Hence the variation \( B \) of the argument of \( g_k \) along the interval \([\frac{\pi}{3}, \frac{2\pi}{3}]\) is equal to \( -(N+1)\pi = -\frac{k+2}{6} \pi \) (subsection 2.1, Remark 4). It follows from formula (13) that \( A = -\frac{k+2}{6} \pi + B = -\frac{k+2}{3} \pi \), and then by subsection 2.4 that \( I = -1 - \frac{A}{2\pi} = \frac{k-4}{6} \).

**Remark 7.** – Although it is not required for the paper, let us point out that for \( 0 \leq j \leq N+1 \), the sign of \( g_k(\theta_j) \) is equal to \((-1)^{N+1-j}\). Since we already know that consecutive terms of this sequence have opposite signs, it is sufficient to show that \( g_k(\theta_{N+1}) = g_k(\frac{2\pi}{3}) \) is positive. Now, by subsection 1.2, the real analytic function \( t \mapsto E_k(-\frac{1}{2} + it) \) has a simple zero at \( \frac{\sqrt{3}}{2} \) and is strictly positive for \( t > \frac{\sqrt{3}}{2} \). It follows that its derivative at the point \( \frac{\sqrt{3}}{2} \) is strictly positive. This derivative is \( iE'_k(e^{2\pi i/3}) \). It is equal to \( g_k(\frac{2\pi}{3}) \) by formula (11), since \( k \equiv 4 \mod 6 \).

![Figure 2](image-url)

*Figure 2. The oriented curve \( g([\frac{\pi}{3}, \frac{2\pi}{3}]) \), where \( g = g_{16} \).*
2.7. Computation of the integral \( I \) when \( k \equiv 0 \pmod{6} \)

**Lemma 4.**—When \( k \equiv 0 \pmod{6} \), the integral \( I \) is equal to \( \frac{k-6}{6} \), and so to \( \left[ \frac{k-4}{6} \right] \).

Suppose that \( k \equiv 0 \pmod{6} \). It follows from subsection 1.2 that the zeros of \( f_k \) in the interval \( \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right] \) are simple, distinct from \( \frac{\pi}{3} \) and \( \frac{2\pi}{3} \), and that their number \( N \) is equal to \( 2\left[ \frac{k}{12} \right] \) when \( k \equiv 0 \pmod{12} \) and to \( 2\left[ \frac{k}{12} \right] + 1 \) when \( k \equiv 6 \pmod{12} \), i.e. to \( \frac{k}{6} \) in both cases. Let us write these zeros as

\[
\theta_1 < \cdots < \theta_N.
\]

One proves as in subsection 2.6 that the \( g_k(\theta_j) \) are real, non-zero, of alternating signs, and that the variation of the argument of \( g_k \) along \( [\theta_j, \theta_{j+1}] \) is \(-\pi\) for \( 1 \leq j \leq N - 1 \).

The integer \( N = \frac{k}{6} \) is non-zero. If \( g_k(\theta_N) > 0 \) (resp. \( g_k(\theta_N) < 0 \)), the set \( g_k([\theta_N, \frac{2\pi}{3}]) \) is contained in the open lower (resp. upper) half-plane. In both of these cases, the variation of the argument of \( g_k \) along the interval \( [\theta_N, \frac{2\pi}{3}] \) is strictly between \(-\pi\) and 0. We show in a similar way (or we deduce from Remark 6) that the variation of the argument of \( g_k \) along the interval \( [\frac{\pi}{3}, \theta_1] \) is also strictly between \(-\pi\) and 0.

Summing up these contributions, we see that the variation \( B \) of the argument of \( g_k \) along the interval \( [\frac{\pi}{3}, \frac{2\pi}{3}] \) is strictly between \(-(N+1)\pi\) and \(-(N-1)\pi\), i.e. between \(-\frac{k+6}{6}\pi\) and \(-\frac{k-6}{6}\pi\). Since \( A = -\frac{k+2}{6}\pi + B \) by formula (13) and \( I = -1 - \frac{A}{2\pi} \) by subsection 2.4, \( I \) is strictly between \( \frac{k-8}{6} \) and \( \frac{k-2}{6} \). We know that \( I \) is an integer and by hypothesis we have \( k \equiv 0 \pmod{6} \). It follows that \( I = -\frac{k-6}{6} \) (and consequently \( A = -\frac{k}{3}\pi \) and \( B = -\frac{k-2}{3}\pi \)).

**Remark 8.**—Although it is not required, let us list few additional details.

By subsection 1.2, the real analytic function \( t \mapsto E_k(\frac{1}{2} + it) \) is strictly positive for \( t > \frac{\sqrt{3}}{2} \) and is non-zero at \( \frac{\sqrt{3}}{2} \). We hence have \( E_k(e^{2\pi i/3}) = E_k(e^{\pi i/3}) > 0 \). Therefore, using formula (10), we have \( f_k(\frac{2\pi}{3}) > 0 \), the sign of \( f_k(\frac{\pi}{3}) \) is \((-1)^{k/6} \), and the sign of \( g_k(\theta_j) \) is \((-1)^{N-j} \) for \( 1 \leq j \leq N \).

**Remark 9.**—The restriction of \( E_k' \) to the half-line \(-\frac{1}{2} + i[0, +\infty[ \) is purely imaginary. We therefore deduce from formula (11) that \( g_k(\frac{\pi}{3}) \) belongs to \( e^{2\pi i/3} \mathbb{R} \), and in fact to \( e^{-\pi i/3} \mathbb{R}^+ \) since its imaginary part is strictly negative (Remark 8). It follows that the variation of the argument of \( g_k \) along the interval \( [\theta_N, \frac{2\pi}{3}] \) is \(-\pi\). We show in a similar way (or deduce from Remark 6) that the variation of the argument of \( g_k \) along the interval \( [\frac{\pi}{3}, \theta_1] \) is also equal to \(-\pi\). This provides a new proof, more direct than the previous one, of the fact that the variation of the argument of \( g_k \) along the interval \( [\frac{\pi}{3}, \frac{2\pi}{3}] \) is equal to \(-(N-1)\pi - \frac{2\pi}{3} \), i.e. to \(-\frac{k-2}{3}\pi \), and so of Lemma 4.
2.8. Computation of the integral $I$ when $k \equiv 2 \mod 6$

**Lemma 5.**— When $k \equiv 2 \mod 6$, the integral $I$ is equal to $\frac{k-8}{6}$, and so to $\left[ \frac{k-4}{6} \right]$.

Suppose that $k \equiv 2 \mod 6$. It follows from subsection 1.2 that $f_k$ has zeros of order 2 at $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, that the zeros of $f_k$ in the interval $\left[ \frac{\pi}{3}, \frac{2\pi}{3} \right]$ are simple, and that their number $N$ is equal to $2k$ when $k \equiv 8 \mod 12$ and to $2k - 1$ when $k \equiv 2 \mod 12$, i.e. to $\frac{k-8}{6}$ in both the cases. Let us write the latter ones as

(16) \[ \theta_1 < \cdots < \theta_N. \]

One proves as in subsection 2.6 that the $g_k(\theta_j)$ are real, non-zero, of alternating signs, and that the variation of the argument of $g_k$ along $[\theta_j, \theta_{j+1}]$ is $-\pi$ for $1 \leq j \leq N - 1$.

When $N \neq 0$ (i.e. $k \neq 8$), let us distinguish two cases:

a) If $g_k(\theta_N) > 0$, $g_k(\left[ \theta_N, \frac{2\pi}{3} \right])$ is contained in the closed lower half-plane. Let $h$ be the continuous argument of $g_k = f'_k - \frac{ki}{2}f_k$ in $[\theta_N, \frac{2\pi}{3}]$ which takes the value 0 at $\theta_N$. When $\theta < \frac{2\pi}{3}$ tends to $\frac{2\pi}{3}$, $f_k(\theta)$ is equivalent to $c(\theta - \frac{2\pi}{3})^2$ for some positive real number $c$, $f'_k(\theta)$ is equivalent to $2c(\theta - \frac{2\pi}{3})$, hence the real and imaginary parts of $g_k(\theta)$ are strictly negative and their quotient $\frac{\text{Im} g_k(\theta)}{\text{Re} g_k(\theta)}$ tends to 0, which implies that $h(\theta)$ tends to $-\pi$.

b) If $g_k(\theta_N) < 0$, $g_k(\left[ \theta_N, \frac{2\pi}{3} \right])$ is contained in the closed upper half-plane. Let $h$ be the continuous argument of $g_k = f'_k - \frac{ki}{2}f_k$ in $[\theta_N, \frac{2\pi}{3}]$ which takes the value $\pi$ at $\theta_N$. When $\theta < \frac{2\pi}{3}$ tends to $\frac{2\pi}{3}$, $f_k(\theta)$ is equivalent to $c(\theta - \frac{2\pi}{3})^2$ for some negative real number $c$, $f'_k(\theta)$ is equivalent to $2c(\theta - \frac{2\pi}{3})$, hence the real
and imaginary parts of $g_k(\theta)$ are strictly positive and their quotient $\frac{\text{Im} g_k(\theta)}{\text{Re} g_k(\theta)}$ tends to 0, which implies that $h(\theta)$ tends to 0.

We observe that in both the previous cases, the limit when $\eta > 0$ tends to 0 of the variation of the argument of $g_k$ along the interval $[\theta_N, \frac{2\pi}{3} - \eta]$ is equal to $-\pi$. Similarly we prove (or we deduce from Remark 6) that the limit when $\eta > 0$ tends to 0 of the variation of the argument of $g_k$ along the interval $[\frac{\pi}{3} + \eta, \theta_1]$ is equal to $-\pi$.

Finally, when $N = 0$ (i.e. $k = 8$), $g_k([\frac{\pi}{3}, \frac{2\pi}{3}])$ is either contained in the closed lower half-plane or in the closed upper half-plane. In the first (resp. the second) case, we check as above that there is a continuous argument $h$ of $g_k$ along the interval $[\frac{\pi}{3}, \frac{2\pi}{3}]$ which has limit 0 at $\frac{\pi}{3}$ and $-\pi$ at $\frac{2\pi}{3}$ (resp. $\pi$ at $\frac{\pi}{3}$ and 0 at $\frac{2\pi}{3}$). The limit when $\eta > 0$ tends to 0 of the variation of argument of $g_k$ along the interval $[\frac{\pi}{3} + \eta, \frac{2\pi}{3} - \eta]$ is therefore equal to $-\pi$.

Summing up all these contributions, we see that, in all cases, the limit $B$ when $\eta > 0$ tends to 0 of the variation of the argument of $g_k$ along the interval $[\frac{\pi}{3} + \eta, \frac{2\pi}{3} - \eta]$ is equal to $-(N + 1)\pi$, i.e. to $-\frac{k-2}{6}\pi$. Since $A = -\frac{k^2}{6}\pi + B$ by formula (13) and $I = -\frac{4}{3} - \frac{A}{2\pi}$ by subsection 2.4, we have $A = -\frac{k}{5}\pi$ and $I = \frac{k-8}{6}$.

Remark 10.— Although it is not required for the paper, let us add few more clarifications. When $k \equiv 2 \bmod 6$, $E_k$ has by subsection 1.2 a zero of order 2 at $e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Hence there exists $c \in \mathbb{C}$ such that $E_k(z)$ is equivalent to $-c(z - e^{2\pi i/3})^2$ when $z$ tends to $e^{2\pi i/3}$. Since $E_k(-\frac{1}{2} + it) > 0$ for $t > \frac{\sqrt{3}}{2}$ (see subsection 1.2, Remark 2), $c$ is a positive real number. But then $E_k(e^{i\theta})$ is equivalent to $ce^{4\pi i/3}(\theta - \frac{2\pi}{3})^2$ when $\theta$ tends to $\frac{2\pi}{3}$, and $f_k(\theta)$ to $c(\theta - \frac{2\pi}{3})^2$ by formula (10). We then deduce from the above discussion that $g_k(\theta_N) > 0$ (if $N \neq 0$), so that $g_k(\theta_i)$ has sign $(-1)^{N-i}$ for $1 \leq i \leq N$. 
3. Signs changes of $E'_k$

3.1. Statement of results

In this section, $k$ denotes an integer $\geq 2$ (not necessarily even) and $h_k$ the function defined on the Poincaré upper half-plane by

$$h_k(z) := \sum_{n=1}^{\infty} \frac{n^k q^n}{(1 - q^n)^2},$$

where $q = e^{2\piiz}$. Note that we have

$$h_k(z) = \sum_{n=1}^{\infty} n^k \left( \sum_{m=1}^{\infty} mq^{mn} \right) = \sum_{n=1}^{\infty} n\sigma_{k-1}(n)q^n.$$  

It follows by differentiating formula \([3]\) that, when $k$ is an even integer $\geq 4$,

$$E'_k(z) = -\frac{4\pi ki}{B_k} h_k(z).$$

For $1 \leq m \leq \left[ \frac{k+1}{6} \right]$, let $t_m := \frac{1}{2} \cot\left( \frac{m\pi}{k+1} \right)$. This section is devoted to prove the following proposition.

**Proposition 1.**— For any integer $m$ between 1 and $\left[ \frac{k+1}{6} \right]$, $h_k\left( \frac{1}{2} + it_m \right)$ is a non-zero real number whose sign is $(-1)^m$.

Note that the inequalities $1 \leq m \leq \left[ \frac{k+1}{6} \right]$ imply that $t_m \geq \frac{\sqrt{3}}{2}$. In subsection 3.2, we prove Proposition 1 when $3m^2 \leq k$ using the expression \([17]\) of...
the function $h_k$. We will give in subsection 3.3 a new expression of $h_k$ and we shall use it to prove Proposition 1 when $3m^2 \geq k + 1$ in subsection 3.4.

**Corollary.**— Suppose that $k$ is an even integer $\geq 4$ and let $M := \lceil \frac{k}{6} \rceil$.

a) For any integer $m$ between 1 and $M - 1$, $iE'_k$ takes non-zero real values of opposite signs at $\frac{1}{2} + t_m$ and $\frac{1}{2} + it_{m+1}$.

b) When $k \equiv 4 \mod 6$ and $k \neq 4$, $iE'_k$ takes at $\frac{1}{2} + i\sqrt{3} / 2$ and $\frac{1}{2} + it_M$ non-zero real values of opposite signs.

Assertion a) follows from Proposition 1, using formula (19).

When $k \equiv 4 \mod 6$ and $k \neq 4$, we have $k = 6M + 4$ and $M \geq 1$. The sign of $h_k(\frac{1}{2} + it_M)$ is $(-1)^M$ by Proposition 1. The sign of $B_k$ is $-1$ or 1 depending on whether $k$ is congruent to 0 or 2 modulo 4; it is hence equal to $(-1)^{M-1}$. We then deduce from formula (19) that $iE'_k$ takes at $\frac{1}{2} + i\sqrt{3} / 2$ and $\frac{1}{2} + it_M$ non-zero real values of opposite signs.

3.2. **Proof of Proposition 1 when $3m^2 \leq k$**

Throughout this subsection, we shall assume that $m$ is an integer such that $1 \leq m \leq \lceil \frac{k+1}{6} \rceil$ and $3m^2 \leq k$. We put $t_m := \frac{1}{2}\cot\left(\frac{m\pi}{k+1}\right)$, $z_m := \frac{1}{2} + it_m$ and $q_m := e^{2\pi iz_m} = -e^{-2\pi t_m}$.

**Lemma 6.**— We have $|q_m| \leq e^{-\frac{k+1}{m} + \frac{4m}{k+1}}$.

For all $x \in [0, \frac{\pi}{2}]$, we have

\begin{equation}
\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \left( \frac{1}{x-n\pi} + \frac{1}{x+n\pi} \right) = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2\pi^2 - x^2}
\end{equation}

and hence

\begin{equation}
\cot x \geq \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2\pi^2 - \frac{x^2}{4}} = \frac{1}{x} - \frac{2x}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} \right) = \frac{1}{x} - \frac{4x}{\pi^2}.
\end{equation}

Applying (21) with $x = \frac{m\pi}{k+1}$, which belongs to the interval $[0, \frac{\pi}{6}]$, we obtain

\begin{equation}
\cot\left(\frac{m\pi}{k+1}\right) \geq \frac{k+1}{m\pi} - \frac{4m}{(k+1)\pi},
\end{equation}

and we conclude by noting that $|q_m| = e^{-2\pi t_m} = e^{-\pi \cot\left(\frac{m\pi}{k+1}\right)}$.

**Lemma 7.**— We have $\frac{m+1}{m} \leq e^{\frac{1}{m} - \frac{1}{2m(k+1)}}$.
Indeed for any \( x \in [0, 1] \), we have
\[
(23) \quad -\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \leq x + \frac{1}{2} \sum_{n=2}^{\infty} x^n = x + \frac{x^2}{2(1 - x)}.
\]
Putting \( x = \frac{1}{m+1} \) in (23), we obtain
\[
(24) \quad \log\left(\frac{m+1}{m}\right) \leq \frac{1}{m+1} + \frac{1}{2m(m+1)} = \frac{1}{m} - \frac{1}{2m(m+1)}.
\]

From now on, we shall put \( u_n := \frac{n^k q_m^n}{(1-q_m^n)^2} \) for all \( n \geq 1 \). Hence we have
\[
h_k(\frac{1}{2} + it_m) = \sum_{n=1}^{\infty} u_n \text{ by formula (17)}.
\]

**Lemma 8.** We have \( |u_{n+1}| < \frac{|u_n|}{2} \), for \( n \geq m \).

Suppose that \( n \geq m \). We have
\[
(25) \quad \frac{|u_{n+1}|}{|u_n|} = \left(\frac{n+1}{n}\right)^k |q_m| \left(\frac{1 - q_m^n}{1 - q_m^{n+1}}\right)^2
\]
\[
\leq \left(\frac{m+1}{m}\right)^k |q_m| \left(\frac{1 + |q_m|}{1 - |q_m|^2}\right)^2
\]
\[
= \left(\frac{m+1}{m}\right)^k \frac{|q_m|}{(1 - |q_m|)^2}.
\]
Using the upper bounds of \( |q_m| \) and of \( \frac{m+1}{m} \) obtained in Lemmas 6 and 7, and the inequality \( k \geq 3m^2 \), we get
\[
(26) \quad \left(\frac{m+1}{m}\right)^k |q_m| \leq e^{\frac{4m}{3m^2+1} - \frac{1}{m} - \frac{k}{2m(m+1)}} \leq e^{\frac{4m}{3m^2+1} - \frac{1}{m} - \frac{3m}{2(m+1)}} \leq e^{-3/4},
\]
where the last inequality follows from the fact that
\[
\frac{4m}{3m^2+1} - \frac{1}{m} - \frac{3m}{2(m+1)} = \frac{3}{2} + \frac{3}{2} \leq \frac{1}{6} - \frac{3}{2} + \frac{1}{2} = -\frac{5}{6} < -\frac{3}{4}
\]
is equal to \(-\frac{3}{4}\) when \( m = 1 \), and is bounded by
\[
\frac{4}{3m} - \frac{1}{m} - \frac{3m}{2(m+1)} = \frac{1}{3m} - \frac{3}{2} + \frac{3}{2} \leq \frac{1}{6} - \frac{3}{2} + \frac{1}{2} = -\frac{5}{6} < -\frac{3}{4}
\]
when \( m \geq 2 \).

Since \( t_m \geq \frac{\sqrt{3}}{2} \), we have \( |q_m| = e^{-2\pi t_m} \leq e^{-\pi \sqrt{3}} \), and we therefore deduce from (25) and (26) that
\[
(27) \quad \frac{|u_{n+1}|}{|u_n|} \leq e^{-3/4} < \frac{1}{2}.
\]

**Lemma 9.** We have \( |u_{n-1}| < \frac{|u_n|}{2} \) for \( 2 \leq n \leq m \).
Suppose that $2 \leq n \leq m$. We have

\begin{equation}
\frac{|u_{n-1}|}{|u_n|} = \left(\frac{n-1}{n}\right)^k |q_m|^{-1} \left(\frac{1-q_m^n}{1-q_m^{n-1}}\right)^2 \leq \left(\frac{m-1}{m}\right)^k |q_m|^{-1} \left(1+\frac{|q_m|^2}{1-|q_m|}\right)^2.
\end{equation}

Now we have $\log\left(\frac{m-1}{m}\right) \leq -\frac{1}{m} - \frac{1}{2m^2}$ and $|q_m|^{-1} = e^{\pi \cot \left(\frac{m\pi}{k+1}\right)} \leq e^{\frac{k+1}{m}}$, from where it follows

\begin{equation}
\left(\frac{m-1}{m}\right)^k |q_m|^{-1} \leq e^\frac{1}{m} \cdot \frac{k}{2m^2} \leq e^\frac{1}{m} \cdot \frac{3}{2} \leq e^{-1}
\end{equation}
as $k \geq 3m^2$ and $m \geq 2$. We also have $|q_m| = e^{-2\pi t_m} \leq e^{-\pi \sqrt{3}}$. We deduce that

\begin{equation}
\frac{|u_{n-1}|}{|u_n|} \leq \frac{1}{e} \left(1 + \frac{|q_m|^2}{1-|q_m|}\right)^2 \leq \frac{1}{e} \left(1 + e^{-2\pi \sqrt{3}}\right)^2 < \frac{1}{2}.
\end{equation}

We now complete the proof of Proposition 1 when $k \geq 3m^2$. The sign of $u_n$ is $(-1)^n$ for all $n \geq 1$. We deduce from Lemma 8 that $\frac{u_m}{2} + \sum_{n=m+1}^{\infty} u_n$ is an alternating series of real numbers whose sum is non-zero and has sign $(-1)^m$. Similarly Lemma 9 implies that the finite sum $\frac{u_m}{2} + \sum_{n=1}^{m-1} u_n$ is real, non-zero and has sign $(-1)^m$. Hence $h_k(\frac{1}{2} + it_m) = \sum_{n=1}^{\infty} u_n$ is a non-zero real number whose sign is $(-1)^m$.

### 3.3. A new expression of the function $h_k$.

**Lemma 10.** For any $z \in H$, we have

\begin{equation}
h_k(z) = (2\pi)^{-k-1} k! \sum_{c=1}^{\infty} \sum_{d \in \mathbb{Z}} c \left(\frac{cz+d}{i}\right)^{-k-1}.
\end{equation}

For any $z \in H$, we have

\begin{equation}
\sum_{d \in \mathbb{Z}} (z+d)^{-2} = \frac{\pi^2}{\sin^2(\pi z)} = (2\pi i)^2 \frac{q}{(1-q)^2} = (2\pi i)^2 \sum_{n=1}^{\infty} nq^n,
\end{equation}

where $q = e^{2\pi iz}$. By differentiating $k-1$ times with respect to $z$, we obtain

\begin{equation}
(-1)^{k-1} k! \sum_{d \in \mathbb{Z}} (z+d)^{-k-1} = (2\pi i)^{k+1} \sum_{n=1}^{\infty} n^k q^n.
\end{equation}
This equality, applied to \(cz\) where \(c\) is an integer \(\geq 1\), can be written as

\[
(2\pi)^{-k}k! \sum_{d \in \mathbb{Z}} \left( \frac{cz + d}{i} \right)^{-k-1} = \sum_{n=1}^{\infty} n^k q^{nc}.
\]

Thus the right-hand side of (31) is equal to \(\sum_{c=1}^{\infty} \sum_{n=1}^{\infty} cn^k q^{nc}\), i.e. to \(h_k(z)\) by formula (18).

3.4. Proof of Proposition 1 when \(3m^2 \geq k + 1\)

In this subsection, we assume that \(m\) is an integer such that \(1 \leq m \leq \left\lceil \frac{k+1}{6} \right\rceil\) and \(3m^2 \geq k + 1\). This implies that we have \(m \geq 2\) and \(k \geq 11\). As before, we let \(t_m := \frac{1}{2} \cot \left( \frac{m\pi}{k+1} \right)\) and \(z_m := \frac{1}{2} + it_m\).

It follows from Lemma 10 that the sum

\[
\sum_{c=1}^{\infty} \sum_{d \in \mathbb{Z}} c \left( \frac{cz_m + d}{i} \right)^{-k-1}
\]

is real. We shall show that it is non-zero and that its sign is \((-1)^m\). This, along with Lemma 10, will then imply Proposition 1 in the case considered.

Let \(\theta_m := \frac{m\pi}{k+1}\). Note that we have

\[
z_m = \frac{1}{2} (1 + i \cot \theta_m) = \frac{ie^{-i\theta_m}}{2 \sin \theta_m} = ie^{-i\theta_m} |z_m|,
\]

\[
z_m - 1 = \frac{1}{2} (-1 + i \cot \theta_m) = \frac{ie^{i\theta_m}}{2 \sin \theta_m} = ie^{i\theta_m} |z_m|.
\]

The contribution to the sum (35) of the terms indexed by the ordered pairs \((c, d)\) equal to \((1, 0)\) and \((1, -1)\) is therefore

\[
\left( \frac{z_m}{i} \right)^{-k-1} + \left( \frac{z_m - 1}{i} \right)^{-k-1} = 2 \cos((k+1)\theta_m)|z_m|^{-k-1} = 2(-1)^m |z_m|^{-k-1}.
\]

It is real and its sign is \((-1)^m\). To conclude, it will therefore suffice to prove that the sum of the absolute values of the other terms of (35) is strictly less than \(2|z_m|^{-k-1}\).

Lemma 11. For any \(x \in [0, \frac{1}{9}]\), we have \(\frac{1+9x}{1+x} \geq e^{\alpha x}\), where \(\alpha = 9 \log \left( \frac{9}{5} \right)\).

The functions \(x \mapsto x + 1\) and \(x \mapsto e^{\alpha x}\) are convex and increasing in the interval \([0, \frac{1}{9}]\), hence the function \(x \mapsto (1 + x)e^{\alpha x} - (1 + 9x)\) is convex in this interval. Since it takes the value 0 at both of its endpoints, it is less than or equal to 0 in the interval considered.
Lemma 12.— For any integer $d \geq 1$ and any $x \in [0, \frac{1}{4d-1}]$, we have

\begin{equation}
1 + (2d + 1)^2 x \leq 1 + 9x \quad \text{and} \quad 1 + (2d + 1)^2 x \geq \left( \frac{1 + 9x}{1 + x} \right)^d.
\end{equation}

The first inequality follows from the identity

\begin{equation}
1 + (2d + 1)^2 x \leq 1 + 9x \quad \text{and} \quad 1 + (2d - 1)^2 x \leq \left( \frac{1 + 9x}{1 + x} \right) x \quad \text{for} \quad x \in [0, \frac{1}{4d-1}].
\end{equation}

We prove the second inequality by induction on $d$. It is clear when $d = 1$.

Suppose that $d \geq 2$. If $x$ belongs to the interval $[0, \frac{1}{4d-5}]$, it belongs to the
interval $[0, \frac{1}{4d-1}]$ and hence we have $1 + (2d - 1)^2 x \geq \left( \frac{1 + 9x}{1 + x} \right) x$ by the induction hypothesis. By multiplying this inequality by the first inequality of (39), we obtain the second inequality.

For any integer $j \geq 0$, let $A_j$ be the set of integers $d \geq 1$ such that $d + \frac{1}{2}$ lies in the set $[jt_m, (j + 1)t_m]$. Its cardinality is bounded by

\begin{equation}
t_m + 1 = \frac{1}{2} \cot \left( \frac{m\pi}{k+1} \right) + 1 \leq \frac{k+1}{2m\pi} + 1 \leq \frac{1}{2\pi} \sqrt{3(k+1)} + 1 \leq \sqrt{k+1}
\end{equation}

since $m \geq \sqrt{\frac{k+1}{3}}$ and $k+1 \geq 12$.

Lemma 13.— We have $\sum_{d \in A_0} \frac{|z_m + d|^{-k-1}}{|z_m|^{-k-1}} < \frac{1}{6000}$.

Let $d \in A_0$. We have

\begin{equation}
\left( \frac{|z_m + d|}{|z_m|} \right)^2 = \left( \frac{d + \frac{1}{2} + t_m^2}{(\frac{1}{2} + t_m^2)} \right)^2 = 1 + (2d + 1)^2 \tan^2 \theta_m.
\end{equation}

We have by hypothesis $d + \frac{1}{2} < t_m$, hence $\tan^2 \theta_m < \frac{1}{(2d + 1)^2}$. Thus $\tan^2 \theta_m$ is bounded above by $\frac{1}{4d-1}$ and also by $\frac{1}{9}$. We now obtain from Lemmas 11 and 12 that

\begin{equation}
\left( \frac{|z_m + d|}{|z_m|} \right)^2 \geq \left( \frac{1 + 9 \tan^2 \theta_m}{1 + \tan^2 \theta_m} \right) d \geq e^{d\alpha \tan^2 \theta_m} \geq e^{d\alpha \theta_m^2},
\end{equation}

where $\alpha = 9 \log(\frac{2}{9})$, and consequently that

\begin{equation}
\left( \frac{|z_m + d|}{|z_m|} \right)^{-k-1} \leq e^{-d\alpha \frac{k+1}{2} \theta_m^2} = e^{-\frac{d\alpha m^2}{2(k+1)}} \leq e^{-\frac{d\alpha x^2}{6}},
\end{equation}

since by hypothesis $3m^2 \geq k+1$. This implies that

\begin{equation}
\sum_{d \in A_0} \frac{|z_m + d|^{-k-1}}{|z_m|^{-k-1}} \leq \sum_{d=1}^{\infty} e^{-\frac{d\alpha x^2}{6}} = \frac{1}{e^{\frac{\alpha x^2}{6}} - 1} < \frac{1}{6000}.
\end{equation}
Lemma 14. — We have $\sum_{d=1}^{\infty} \frac{|z_m + d|^{-k-1}}{|z_m|^{-k-1}} < \frac{5}{16}$ and $\sum_{d \leq -2} \frac{|z_m + d|^{-k-1}}{|z_m|^{-k-1}} \leq \frac{5}{16}$.

If $d \in A_j$, we have $|z_m + d|^2 = (d + \frac{j}{2})^2 + t_m^2 \geq (j^2 + 1)t_m^2$. Also since $t_m \geq \frac{\sqrt{3}}{2}$, we have

$$|z_m|^2 = \frac{1}{4} + t_m^2 \leq \frac{4}{3}t_m^2.$$  
(46)

It follows that $\frac{|z_m + d|^2}{|z_m|}$ is bounded below by $\frac{3(j^2 + 1)}{4}$, hence by $\frac{3}{2}$ when $j = 1$ and by $\frac{3j^2}{4}$ when $j \geq 2$. Consequently, using (41),

$$\sum_{d \in A_1} \frac{|z_m + d|^{-k-1}}{|z_m|^{-k-1}} \leq \left(\frac{3}{2}\right)^{-(k+1)/2} \sqrt{k + 1} \leq \left(\frac{3}{2}\right)^{-6} \sqrt{12},$$  
(47)

where the last inequality follows from the inequality $k + 1 \geq 12$ and the fact that the function $x \mapsto x(\frac{3}{2})^{-x}$ is decreasing for $x \geq 12$. Similarly we have for $j \geq 2$,

$$\sum_{d \in A_j} \frac{|z_m + d|^{-k-1}}{|z_m|^{-k-1}} \leq \left(\frac{3j^2}{4}\right)^{-6} \sqrt{12},$$  
(48)

Using Lemma 13 and the inequalities (47) and (48), we obtain

$$\sum_{d=1}^{\infty} \frac{|z_m + d|^{-k-1}}{|z_m|^{-k-1}} \leq \frac{1}{6000} + \left(\frac{3}{2}\right)^{-6} \sqrt{12} + \left(\frac{3}{4}\right)^{-6} \sqrt{12}(1 - 1) < \frac{5}{16}.$$  
(49)

This proves the first assertion of Lemma 14. The second assertion is deduced from the first one by noticing that $|z_m + d| = |z_m - d - 1|$ for any integer $d \geq 1$.

Lemma 15. — Let $c$ be an integer $\geq 2$. We have

$$\sum_{d \in \mathbb{Z}} c \frac{|cz_m + d|^{-k-1}}{|z_m|^{-k-1}} < c^{1-k}(\frac{\sqrt{3}}{2})^{-k-1} + 3.$$  
(50)

For any $r \in \mathbb{Z}$, let $C_r$ be the set of integers $d \in \mathbb{Z}$ for which the real part of $cz_m + d$ belongs to the interval $[c(r - \frac{1}{2}), c(r + \frac{1}{2})]$. The cardinality of $C_r$ is $c$. For all $d \in C_r$, using (46), we have

$$c|cz_m + d|^{-k-1} \leq \begin{cases} c^{-k}t_m^{-k-1} \leq c^{-k}(|z_m|\frac{\sqrt{3}}{2})^{-k-1} & \text{if } r = 0, \\ c^{-k}|r|^{-k-1} \leq c^{-k}|z_m + |r| - 1|^{-k-1} & \text{if } r \neq 0. \\ \end{cases}$$  
(51)
We then deduce from (51) and Lemma 14 that
\[
\sum_{d \in \mathbb{Z}} c \left| \frac{cz_m + d}{|zm|^{k-1}} \right|^{k-1} \leq c^{1-k} \left( \frac{\sqrt{3}}{2} \right)^{k-1} + 2 \sum_{d=0}^{\infty} \left| \frac{zm + d}{|zm|^{k-1}} \right|^{k-1} \leq c^{1-k} \left( \frac{\sqrt{3}}{2} \right)^{k-1} + 2 \left( 1 + \frac{5}{16} \right).
\]
This completes the proof of Lemma 15.

We deduce from Lemmas 14 and 15 that the sum of the terms \( \frac{c|cz_m + d|^{k-1}}{|zm|^{k-1}} \), extended to all tuples \((c, d) \in \mathbb{Z}^2\) distinct from \((1, -1)\) and \((1, 0)\) and for which \(c \geq 1\), is bounded above by
\[
\frac{5}{8} + \sum_{c \geq 2} c^{1-k} \left( \frac{\sqrt{3}}{2} \right)^{k-1} + 3 \leq \frac{5}{8} + \sum_{c \geq 2} c^{-10} \left( \frac{\sqrt{3}}{2} \right)^{-12} + 3 = \frac{5}{8} + \left( \frac{\sqrt{3}}{2} \right)^{-12} + 3 \left( \zeta(10) - 1 \right) < \frac{2}{3}.
\]
This completes the proof of Proposition 1, when \(3m^2 \geq k + 1\).

4. On the multiplicity of zeros of quasi-modular forms with algebraic Fourier coefficients

4.1. The algebra of quasi-modular forms for \(\text{SL}_2(\mathbb{Z})\)

The normalized Eisenstein series \(E_4\) and \(E_6\) have the Fourier expansions
\[
E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,
\]
\[
E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n.
\]
They are algebraically independent over \(\mathbb{C}\) and the \(\mathbb{C}\)-algebra of modular forms for \(\text{SL}_2(\mathbb{Z})\) is none other than \(\mathbb{C}[E_4, E_6]\). We have \(E_4^3 - E_6^2 = 1728\Delta\), where the modular discriminant \(\Delta\) is defined by
\[
\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.
\]
Formula (3) which defines Eisenstein series when \(k \geq 4\) retains a meaning when \(k = 2\) and allows us to define a function \(E_2\) on the Poincaré half-plane,
periodic with period 1, by its Fourier expansion

\begin{equation}
E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,
\end{equation}

where as usual \( q = e^{2\pi i z} \). We have \( E_2 = \frac{D\Delta}{\Delta} \), where \( D \) is the differential operator \( \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq} \). It follows that \( E_2 \) is not modular, but it satisfies for every element \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of \( \text{SL}_2(\mathbb{Z}) \) the relation

\begin{equation}
E_2\left( \frac{az + b}{cz + d} \right) = (cz + d)^2 E_2(z) + \frac{12c(cz + d)}{2\pi i}.
\end{equation}

The functions \( E_2, E_4, E_6 \) are algebraically independent over \( \mathbb{C} \). We call \( \text{quasi-modular forms} \) for \( \text{SL}_2(\mathbb{Z}) \) the elements of the \( \mathbb{C} \)-algebra \( \mathbb{C}[E_2, E_4, E_6] \) generated by these three functions. This algebra is graded by the weight, agreeing that \( E_k \) is of weight \( k \) for \( k \in \{2, 4, 6\} \).

Let \( X, Y, Z \) be three indeterminates. For any polynomial \( P \in \mathbb{C}[X, Y, Z] \), let \( \psi_P := P(E_2, E_4, E_6) \). The map \( P \mapsto \psi_P \) is an isomorphism from the polynomial algebra \( \mathbb{C}[X, Y, Z] \) to the algebra \( \mathbb{C}[E_2, E_4, E_6] \) of quasi-modular forms for \( \text{SL}_2(\mathbb{Z}) \). The Fourier coefficients of \( \psi_P \) belong to a subfield \( K \) of \( \mathbb{C} \) if and only if \( P \) belongs to \( K[X, Y, Z] \). The function \( \psi_P \) is a quasi-modular form of weight \( k \) if and only if the polynomial \( P \) is isobaric of weight \( k \), when assigning \( X, Y, Z \) the weights 2, 4, 6 respectively.

### 4.2. Derivations of the algebra of quasi-modular forms

Ramanujan noticed that one has

\begin{equation}
D E_2 = \frac{1}{12}(E_2^2 - E_4), \quad DE_4 = \frac{1}{3}(E_2 E_4 - E_6), \quad DE_6 = \frac{1}{2}(E_2 E_6 - E_4^2).
\end{equation}

Consequently, \( D \) defines a graded derivation of degree 2 of the graded algebra \( \mathbb{C}[E_2, E_4, E_6] \) of quasi-modular forms for \( \text{SL}_2(\mathbb{Z}) \).

Let us denote by \( \frac{\partial}{\partial E_2} \) the derivation of the algebra \( \mathbb{C}[E_2, E_4, E_6] \) which, under the isomorphism \( P \mapsto \psi_P \), corresponds to the derivation \( \frac{\partial}{\partial X} \) of the algebra of polynomials \( \mathbb{C}[X, Y, Z] \). It is graded of degree \(-2\).

The two derivations \( \frac{\partial}{\partial E_2} \) and \( D \) stabilise \( K[E_2, E_4, E_6] \) for any subfield \( K \) of \( \mathbb{C} \). Their bracket \( \left[ \frac{\partial}{\partial E_2}, D \right] = \frac{\partial}{\partial E_2} D - D \frac{\partial}{\partial E_2} \) is the derivation of the algebra \( \mathbb{C}[E_2, E_4, E_6] \) which maps any quasi-modular form \( \psi \) of weight \( k \) to \( \frac{k}{12} \psi \) : it suffices to check this when \( \psi \) is one of the functions \( E_2, E_4, E_6 \), and it then follows from formula (59).
Lemma 16.— Let \( r \) be an integer \( \geq 1 \). The bracket \([\frac{\partial}{\partial E_2}, D^r] \) maps any quasi-modular form \( \psi \) of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \) to \( \frac{(k+r-1)r}{12}D^{r-1}\psi \).

We shall argue by induction on \( r \). When \( r = 1 \), our assertion has already been proved. Suppose that \( r \geq 2 \). It follows from the induction hypothesis that, for any quasi-modular form \( \psi \) of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \), we have

\[
[\frac{\partial}{\partial E_2}, D^r] \psi = \left[ \frac{\partial}{\partial E_2}, D^{r-1} \right] D^r \psi + \frac{k}{12} D^{r-1}\psi = \frac{(k+r-1)r}{12}D^{r-1}\psi
\]

Lemma 17.— Let \( r \) and \( j \) be integers such that \( 0 \leq j \leq r \). For any modular form \( f \) of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \), we have

\[
\left( \frac{\partial}{\partial E_2} \right)^j D^r f = \left( \prod_{i=1}^{j} \frac{(k+r-i)(r-i+1)}{12} \right) D^{r-j} f.
\]

Here, as usual, an empty product is considered to be equal to \( 1 \).

We prove this by induction on \( j \). When \( j = 0 \), it is clear. Suppose that \( j \geq 1 \). By the induction hypothesis, we have

\[
\left( \frac{\partial}{\partial E_2} \right)^j D^r f = \frac{\partial}{\partial E_2} \left( \left( \frac{\partial}{\partial E_2} \right)^{j-1} D^r f \right)
\]

\[
= \left( \prod_{i=1}^{j-1} \frac{(k+r-i)(r-i+1)}{12} \right) \frac{\partial}{\partial E_2} D^{r-j+1} f
\]

As \( f \) belongs to the algebra \( \mathbb{C}[E_4, E_6] \) of modular forms, we have \( \frac{\partial f}{\partial E_2} = 0 \). Thus \( \frac{\partial}{\partial E_2} D^{r-j+1} f \) is equal to \([\frac{\partial}{\partial E_2}, D^{r-j+1}] f \), hence to \( \frac{(k+r-j)(r-j+1)}{12} D^{r-j} f \) by Lemma 16. This completes the proof of Lemma 17.

Remark 11.— Lemma 17 remains valid when we replace \( f \) by the quasi-modular form \( E_2 \) (with \( k = 2 \) in this case). The proof is analogous to that of Lemma 17, except that we replace there the relation \( \frac{\partial f}{\partial E_2} = 0 \) by \( \frac{\partial E_2}{\partial E_2} = 1 \).

4.3. On the simplicity of zeros of certain quasi-modular forms

As in subsection 4.1, let \( \psi_P = P(E_2, E_4, E_6) \) for any polynomial \( P \in \mathbb{C}[X, Y, Z] \). We denote by \( \overline{\mathbb{Q}} \) the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \).

Proposition 2.— Let \( P \) and \( Q \) be two irreducible elements of the unique factorization domain \( \overline{\mathbb{Q}}[X, Y, Z] \) which are not associates of each other. The functions \( \psi_P \) and \( \psi_Q \) have no common zeros in the Poincaré upper half-plane \( \mathfrak{H} \).
Let us argue by contradiction, by assuming that there exists one common zero. We denote it by \( a \). The map \( u : \mathbb{R} \to \psi_\mathbb{R}(a) \) is a ring homomorphism from \( \mathbb{Q}[X, Y, Z] \) to \( \mathbb{C} \). Its kernel \( p \) is a prime ideal of \( \mathbb{Q}[X, Y, Z] \). It contains \( P \) and \( Q \) which are two irreducible elements of \( \mathbb{Q}[X, Y, Z] \), not associates of each other. The height of \( p \) is therefore at least 2. Consequently, the ring \( \mathbb{Q}[E_2(a), E_4(a), E_6(a)] \) which is the image of \( u \) has Krull dimension at most 1. Since it is a \( \mathbb{Q} \) algebra of finite type, its Krull dimension is the transcendence degree of its fraction field, i.e. of \( \mathbb{Q}(E_2(a), E_4(a), E_6(a)) \). Now this transcendence degree is at least 2 by a theorem of G. V. Chudnovsky \(^{(2)} \).

**Corollary 1.** Let \( P \) and \( Q \) be two non zero elements of the unique factorization domain \( \mathbb{Q}[X, Y, Z] \) without common irreducible factors. The functions \( \psi_P \) and \( \psi_Q \) have no common zeros in \( \mathcal{H} \).

**Example 1.** Let \( f \) be a quasi-modular form whose Fourier coefficients are algebraic over \( \mathbb{Q} \). If \( f \) vanishes at \( i \), \( f \) is a multiple of \( E_6 \) in the ring of quasi-modular forms. If it vanishes at \( e^{\pi i/3} \), it is a multiple of \( E_4 \).

**Corollary 2.** Let \( P \) be a non zero element of the unique factorization domain \( \mathbb{Q}[X, Y, Z] \). Let \( a \) be a zero of \( \psi_P \) in \( \mathcal{H} \) and \( e \) be its order. There exists an irreducible factor \( R \) of \( P \), unique up to multiplication by a scalar, such that \( \psi_R(a) = 0 \). The function \( \psi_R \) has a simple zero at \( a \) and the \( R \)-adic valuation of \( P \) is \( e \).

The existence of \( R \) is clear. Its uniqueness up to multiplication by a scalar follows from Proposition 2. It follows from Proposition 2 that \( e = v_R(P)e' \), where \( e' \) is the order of \( \psi_R \) at \( a \) and \( v_R(P) \) is the \( R \)-adic valuation of \( P \). Recall that \( D \) denotes the differential operator \( \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq} \). There exists a polynomial \( Q \in \mathbb{Q}[X, Y, Z] \) such that \( D\psi_P = \psi_Q \). The order at \( a \) of \( D\psi_P \) is \( e - 1 \). By Proposition 2, it is equal to \( v_R(Q)e' \). But then we have \( (v_R(P) - v_R(Q))e' = 1 \). It implies that \( e' = 1 \). We then have \( v_R(P) = e \).

**Corollary 3.** Let \( P \) be an element of \( \mathbb{Q}[X, Y, Z] \) without multiple factors. The quasi-modular form \( \psi_P \) has only simple zeros in \( \mathcal{H} \).

This is an immediate consequence of Corollary 2.

**Corollary 4.** The quasi-modular form \( E_2 \) has only simple zeros in \( \mathcal{H} \).

This result follows from Corollary 3 by taking \( P = X \).

\(^{(2)} \) The theorem of Chudnovsky was stated in a different manner in terms of periods and quasi-periods of elliptic curves (see \([2]\), Chapter 7, Theorem 3.1); for the equivalence of the two formulations, see for example in \([6]\), the equivalence of Theorem 3 and its Corollary 1.
Corollary 4 can also be deduced from the first equality of \[59\], as remarked by H. Saber and A. Sebbar in [5], p. 1786. Much more precise information about the zeros of $E_2$ can be found in [3] and [7].

4.4. Multiple zeros of iterated derivatives of modular forms

**Theorem 6.** Let $f$ be a non-zero modular form of weight $k > 0$ for $SL_2(\mathbb{Z})$ whose Fourier coefficients are algebraic over $\mathbb{Q}$, and let $r$ be an integer $\geq 1$. If the $r$-th derivative of $f$ has a zero at a point $a$ in $\mathcal{H}$ of order $e \geq 2$, then $f$ has a zero of order $e + r$ at this point.

By hypothesis, the function $D^r f = (2\pi i)^{-r} \frac{\partial^r f}{\partial z^r}$ has a zero of order $e \geq 2$ at $a$. Let $P$ be the element of $\overline{\mathbb{Q}}[X, Y, Z]$ such that $\psi_P = D^r f$, with the notations of subsection 4.1. By Corollary 2 of Proposition 2, there exists an irreducible factor $R$ of $P$ in $\mathbb{Q}[X, Y, Z]$, unique up to multiplication by a scalar, such that $\psi_R(a) = 0$. Further $\psi_R$ has a simple zero at $a$ and we have $v_R(P) = e$. We distinguish two cases:

\begin{itemize}
  \item[a)] The polynomial $R$ is of degree $\geq 1$ in $X$.
  
  The polynomial $R$ is then also irreducible when we consider it as an element of the principal ideal domain $\overline{\mathbb{Q}}[Y, Z][X]$, and we have $v_R(\frac{\partial P}{\partial X}) = e - 1$. Now we have
  \[\psi_{\frac{\partial P}{\partial X}} = \frac{\partial}{\partial E_2} D^{r-1}f = \frac{(k + r - 1)r}{12} D^r f\]
  
  by Lemma 17, hence $D^{r-1}f$ has a zero of order $e - 1$ at $a$. It implies that $D^r f$ has a zero of order $e - 2$ at $a$, and this contradicts our hypothesis. Hence case a) cannot occur.
  
  \item[b)] The polynomial $R$ belongs to $\overline{\mathbb{Q}}[Y, Z]$.
  
  In this case, if we consider $P$ as a polynomial in $X$ with coefficients in $\overline{\mathbb{Q}}[Y, Z]$, $R$ divides the coefficient of each power of $X$ in $P$. Consequently, for every integer $j \geq 0$, $R$ divides the polynomial $\frac{\partial^j P}{\partial X^j}$, and therefore $\frac{\partial}{\partial E_2} D^j f$ vanishes at $a$. It follows from Lemma 17 that $D^{r-j} f(a) = 0$ for $0 \leq j \leq r$. Thus $f$ and its iterated derivatives of order $\leq r$ vanish at $a$. Since the $r$-th derivative of $f$ has a zero of order $e$ at $a$, $f$ has a zero of order $e + r$ at $a$.
\end{itemize}

4.5. Simplicity of the zeros of iterated derivatives of Eisenstein series

**Theorem 7.** Let $k$ be an even integer $\geq 2$. For any integer $r \geq 1$, the $r$-th derivative of the function $E_k$ has only simple zeros in the Poincaré upper half-plane.
When \( k \geq 4 \), this follows from Theorem 6 applied to the modular form \( f = E_k \): indeed, the zeros of \( E_k \) in \( \mathfrak{H} \) have order \( \leq 2 \) by subsection 1.2.

When \( k = 2 \), Theorem 6 remains true for \( f = E_2 \), although \( E_2 \) is not a modular form: the proof is identical, by appealing to Remark 11 of subsection 4.2. We conclude by noting that the zeros of \( E_2 \) in \( \mathfrak{H} \) are simple, by Corollary 4 of Proposition 2.

5. Zeros of \( E'_k \) in \( \gamma D \), for \( \gamma \in SL_2(\mathbb{Z}) \)

5.1. Derivatives of modular functions

Let \( k \) be an even integer \( \geq 2 \). Let us denote by \( \mathcal{M}_k \) the vector space of meromorphic functions \( f \) on the Poincaré upper half-plane \( \mathfrak{H} \) such that

\[
f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z),
\]

holds in \( \mathfrak{H} \), for any \( \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in SL_2(\mathbb{Z}) \). This implies that

\[
f'( \frac{az + b}{cz + d} ) = (cz + d)^{k+2} f'(z) + kc(cz + d)^{k+1} f(z)
\]

holds in \( \mathfrak{H} \), for any \( \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in SL_2(\mathbb{Z}) \).

For every non-zero function \( f \in \mathcal{M}_k \), let \( \varphi_f \) denote the meromorphic function on \( \mathfrak{H} \) defined by

\[
\varphi_f(z) := z + k f'(z).
\]

Since we have \( f'(-\frac{1}{z}) = z^{k+2} f'(z) + k z^{k+1} f(z) \) by formula (62), we have

\[
\varphi_f(z) = z^{-k-1} \frac{f'(-\frac{1}{z})}{f'(z)}
\]
in \( \mathfrak{H} \). To the best of our knowledge, the functions \( \varphi_f \) have been introduced as well as investigated in this context by H. Saber and A. Sebbar in \([5]\). These functions have the following important equivariance property:

**Lemma 18.**— Let \( f \) be a non-zero element of \( \mathcal{M}_k \). We have

\[
\varphi_f \left( \frac{az + b}{cz + d} \right) = \frac{a \varphi_f(z) + b}{c \varphi_f(z) + d}
\]
in \( \mathfrak{H} \), for any \( \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in SL_2(\mathbb{Z}) \).
Formula \((65)\) follows from formula \((63)\) when \((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\) and from formula \((64)\) when \((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\). These two matrices generate the group \(\text{SL}_2(\mathbb{Z})\). The lemma follows.

**Remark 12.** One can prove the identity \((65)\) by a direct computation: it may be useful for example if one tries to generalize Lemma 18 when \(\text{SL}_2(\mathbb{Z})\) is replaced by any Fuchsian group. Indeed, it follows from formulae \((61)\) and \((62)\) that the following equalities of meromorphic functions hold in \(\mathbb{H}\):

\[
\varphi_f(\frac{az + b}{cz + d}) = \frac{az + b}{cz + d} + \frac{1}{(cz + d)(\varphi_f(z) - z) + c}
\]

\[
= \frac{az + b}{cz + d} + \varphi_f(z) - z
\]

\[
= \frac{((az + b)c + 1)\varphi_f(z) + (az + b)d - z}{(cz + d)(c\varphi_f(z) + d)}
\]

\[
= \frac{(ac + ad)\varphi_f(z) + bcz + bd}{(cz + d)(c\varphi_f(z) + d)} = \frac{a\varphi_f(z) + b}{c\varphi_f(z) + d}.
\]

5.2. Zeros of derivatives of modular functions

In this subsection, \(k\) denotes an even integer \(\geq 2\) and \(f\) a non-zero element of \(\mathcal{M}_k\). The zeros of the derivative \(f'\) of \(f\) in \(\mathfrak{F}\) are of two different types:

a) The multiple zeros of \(f\): a zero of \(f\) of multiplicity \(e \geq 2\) is a zero of \(f'\) of multiplicity \(e - 1\). We shall call them the trivial zeros of \(f'\). The set of these zeros is stable under the action of \(\text{SL}_2(\mathbb{Z})\).

b) The zeros of \(f'\) which are not zeros of \(f\). We shall call them the non trivial zeros of \(f'\) and shall denote the set of those zeros of \(f'\) by \(Z(f')\). If \(\tau\) is an element of \(Z(f')\) and \((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})\) an element of \(\text{SL}_2(\mathbb{Z})\) with \(c \neq 0\), \(\frac{ar + b}{cr + d}\) cannot be a zero of \(f'\) by formula \((62)\).

In particular, it follows from b) that in the upper half plane \(\mathfrak{F}\) a fixed point of a matrix \((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{SL}_2(\mathbb{Z})\) with \(c \neq 0\) cannot belong to \(Z(f')\). In other words, \(Z(f')\) does not meet the orbits of \(i\) and of \(e^{\pi i/3}\) under the action of \(\text{SL}_2(\mathbb{Z})\).

**Lemma 19.**— The poles of \(\varphi_f\) are the points of \(Z(f')\). Moreover, the order of a point \(\tau \in Z(f')\) as a pole of \(\varphi_f\) is also its order as a zero of \(f'\).

Let \(\tau \in \mathfrak{F}\). We deduce from formula \((63)\) the following: if \(\tau\) is neither a pole of \(f\) nor a zero of \(f'\), \(\varphi_f\) is holomorphic at \(\tau\); if \(\tau\) is a pole of order \(e\) of \(f\), it is a pole of order \(e + 1\) of \(f'\), hence \(\varphi_f\) is holomorphic at \(\tau\); if \(\tau\) is a zero of
order $e \geq 2$ of $f$, it is a zero of order $e - 1$ of $f'$, hence $\varphi_f$ is holomorphic at $\tau$; finally, if $\tau$ is not a zero of $f$ but is a zero of order $e \geq 1$ of $f'$, it is a pole of order $e$ of $\varphi_f$. This proves the lemma.

**Lemma 20.** Let $\tau \in \mathfrak{H}$, and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\text{SL}_2(\mathbb{Z})$ such that $c \neq 0$. In order for $\gamma \tau = \frac{a\tau + b}{c\tau + d}$ to be a non trivial zero of $f'$, it is necessary and sufficient that $\tau$ is a zero of $\varphi_f + \frac{d}{c}$, and then the order of $\gamma \tau$ as a zero of $f'$ is the same as the order of $\tau$ as a zero of $\varphi_f + \frac{d}{c}$.

Indeed, for $\gamma \tau$ to be a non trivial zero of $f'$, it is necessary and sufficient that it is a pole of $\varphi_f$ (Lemma 19), i.e. $\tau$ is a pole of $\frac{a\varphi_f + b}{c\varphi_f + d}$ (Lemma 18), i.e. a zero of $\varphi_f + \frac{d}{c}$. Then the order of $\gamma \tau$ as a zero of $f'$ is equal to the order of $\gamma \tau$ as a pole of $\varphi_f$, i.e. of $\tau$ as a pole of $\frac{a\varphi_f + b}{c\varphi_f + d}$, and also of $\tau$ as a zero of $\varphi_f + \frac{d}{c}$.

**Remark 13.** One can also deduce Lemma 20 from the formula

$$ (cz + d)^{-k-1} \frac{f'(\frac{az + b}{cz + d})}{f'(z)} = cz + d + kc \frac{f(z)}{f'(z)} = c\varphi_f(z) + d, $$

which follows from formula (62).

### 5.3. Derivative of the function $\varphi_f$

In this subsection, $k$ denotes an even integer $\geq 2$ and $f$ a non-zero element of $\mathcal{M}_k$. By differentiating (63), we get

$$ \varphi'_f(z) = k + 1 - k\frac{f(z)f''(z)}{f'(z)^2} = \frac{F_f(z)}{f'(z)^2}, $$

in $\mathfrak{H}$, where

$$ F_f(z) = (k + 1)f'(z)^2 - kf(z)f''(z). $$

We have the following noteworthy fact:

**Lemma 21.** The function $F_f$ is an element of $\mathcal{M}_{2k+4}$.

The function $F_f$ is meromorphic in $\mathfrak{H}$. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, we have

$$ f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z), $$

$$ f'\left(\frac{az + b}{cz + d}\right) = (cz + d)^{k+2} f'(z) + kc(cz + d)^{k+1} f(z), $$

$$ f''\left(\frac{az + b}{cz + d}\right) = (cz + d)^{k+4} f''(z) + 2(k + 1)c(cz + d)^{k+3} f'(z) + k(k + 1)c^2(cz + d)^{k+2} f(z), $$

$$ f'''\left(\frac{az + b}{cz + d}\right) = (cz + d)^{k+6} f'''(z) + 6k(k + 1)c^2(cz + d)^{k+5} f''(z) + 2k(k + 1)c^3(cz + d)^{k+4} f'(z) + k(k + 1)c^4(cz + d)^{k+3} f(z), $$

$$ f^{(4)}\left(\frac{az + b}{cz + d}\right) = (cz + d)^{k+8} f^{(4)}(z) + 24k(k + 1)c^2(cz + d)^{k+7} f'''(z) + 24k(k + 1)c^3(cz + d)^{k+6} f''(z) + 8k(k + 1)c^4(cz + d)^{k+5} f'(z) + 3k(k + 1)c^5(cz + d)^{k+4} f(z), $$

and so on.
Remark 14. — If \( f \) is a modular form of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \), \( F_f \) is a cuspform of weight \( 2k + 4 \) for \( \text{SL}_2(\mathbb{Z}) \): this follows from Lemma 21, as the constant terms of the Fourier expansions of \( f' \) and \( f'' \) are both equal to 0.

5.4. Properties of \( \varphi_f : \) the real case

In this subsection, \( k \) denotes an even integer \( \geq 2 \). We shall say that an element \( f \) of \( \mathcal{M}_k \) is real if we have \( f(it) \in \mathbb{R} \cup \{ \infty \} \) for every \( t > 0 \). It is equivalent to saying that
\[
\varphi_f(-\overline{z}) = \overline{\varphi_f(z)}
\]
holds in \( \mathfrak{H} \).

Example 2. — If there exists a real number \( T \geq 0 \) such that \( f \) is holomorphic in the half-plane \( \{ z \in \mathbb{C} \mid \text{Im}(z) > T \} \), then \( f \) admits in this half-plane a Fourier expansion of the form \( f(z) = \sum_{n \in \mathbb{Z}} a_n(f)q^n \), where \( q = e^{2\pi iz} \), and \( f \) is a real element of \( \mathcal{M}_k \) if and only if all the coefficients \( a_n(f) \) are real.

Proposition 3. — Let \( f \) be a non-zero real element of \( \mathcal{M}_k \) and let \( \tau \) be a point of \( \mathfrak{H} \).

a) If the real part of \( \tau \) belongs to \( \frac{1}{2}\mathbb{Z} \) and \( \tau \) is not a pole of \( \varphi_f \), then the real part of \( \varphi_f(\tau) \) is equal to that of \( \tau \).

b) If the modulus of \( \tau \) is 1, the modulus of \( \varphi_f(\tau) \) is 1; in particular \( \tau \) is not a pole of \( \varphi_f \).

Let \( \sigma \in \frac{1}{2}\mathbb{Z} \). Since \( f \) is periodic of period 1, one deduces from (70) that all the values taken by \( f \) on \( \sigma + i\mathbb{R}_+ \) lie in \( \mathbb{R} \cup \{ \infty \} \). Hence all the values taken by \( f' \) on this half-line lie in \( i\mathbb{R} \cup \{ \infty \} \), and the same holds for the function \( z \mapsto \varphi_f(z) - z \). This proves assertion a).

One deduces from (70) that \( f'(-\overline{z}) = -\overline{f'(z)} \) holds in \( \mathfrak{H} \), and therefore also \( \varphi_f(-\overline{z}) = -\overline{\varphi_f(z)} \). To prove b), it suffices to consider the case where \( \tau \) is not a pole of \( \varphi_f \), as the general case then follows by continuity. Then one concludes by noticing that
\[
|\varphi_f(\tau)|^2 = \varphi_f(\tau)\overline{\varphi_f(\tau)} = -\varphi_f(\tau)\varphi_f(-\overline{\tau}) = -\varphi_f(\tau)\varphi_f(-\frac{1}{\tau}) = 1
\]
by Lemma 18.

Corollary 1. — Let \( f \) be a non-zero real element of \( \mathcal{M}_k \). Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an element of \( \text{SL}_2(\mathbb{Z}) \) such that \( c \neq 0 \), \( |c| \neq |d| \) and \( |c| \neq 2|d| \).
a) The function \( \varphi_f \) does not take the value \( -\frac{d}{c} \) at any point of the boundary of \( D \) in \( \mathcal{H} \).

b) The function \( f' \) has no non trivial zeros on the boundary of \( \gamma D \) in \( \mathcal{H} \).

According to our hypotheses, \( -\frac{d}{c} \) is distinct from \(-1, 1, -\frac{1}{2}, \frac{1}{2}\). Hence, by Proposition 3, \( \varphi_f \) cannot take the value \( -\frac{d}{c} \) at a point of \( \mathcal{H} \) whose real part is \( \frac{1}{2} \) or \(-\frac{1}{2} \), or whose modulus is 1. In particular, it cannot take the value \( -\frac{d}{c} \) at a point of the boundary of \( D \) in \( \mathcal{H} \). This proves assertion \( a) \).

Assertion \( b) \) follows from \( a) \) by Lemma 20.

5.5. Properties of \( \varphi_f \) when \( f \) is an Eisenstein series

In this subsection, \( k \) denotes an even integer \( \geq 4 \) and \( E_k \) the normalized Eisenstein series of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \). To simplify the notation, we shall denote by \( \varphi_k \) the meromorphic function \( \varphi_{E_k} \). We hence have

\[
\varphi_k(z) = z + k \frac{E_k(z)}{E'_k(z)} = z^{-k-1} \frac{E'_k(-\frac{1}{2}z)}{E'_k(z)}.
\]

The derivative of \( \varphi_k \) is given, using formulae (67) and (68), by

\[
\varphi'_k(z) = \frac{F_k(z)}{E'_k(z)^2},
\]

where

\[
F_k(z) = (k+1)E'_k(z)^2 - kE_k(z)E''_k(z).
\]

It follows from Remark 14 of subsection 5.3 that \( F_k \) is a cusp form of weight \( 2k + 4 \) for \( \text{SL}_2(\mathbb{Z}) \). It has a simple zero at the cusp \( \infty \), as is seen from the Fourier expansion of the right hand-side of (74). The set of its zeros in \( \mathcal{H} \) is therefore stable under the action of \( \text{SL}_2(\mathbb{Z}) \) and the weighted number of its zeros modulo \( \text{SL}_2(\mathbb{Z}) \) (counted with their multiplicities, with weight \( \frac{1}{2} \) for those in the orbit of \( e^{\pi i/2}, \frac{1}{3} \) for those in the orbit of \( e^{\pi i/3} \), and 1 for the others) is equal to \( \frac{2k+4}{12} - 1 = \frac{k-4}{6} \).

a) Behaviour of \( \varphi_k \) on the half-line \( \frac{1}{2} + i\left[ \frac{\sqrt{3}}{2} , +\infty \right] \)

By Lemma 19, the poles of \( \varphi_k \) lying on the half-line \( \frac{1}{2} + i\left[ \frac{\sqrt{3}}{2} , +\infty \right] \) are the non trivial zeros of \( E'_k \) lying on this half-line. It follows from Theorems 1 and 2 that their number is \( n = \left\lfloor \frac{k-4}{6} \right\rfloor \). Moreover they belong to the open half-line \( \frac{1}{2} + i\left[ \frac{\sqrt{3}}{2} , +\infty \right] \) and are simple zeros of \( E'_k \), hence simple poles of \( \varphi_k \). Let us write them

\[
\frac{1}{2} + ib_1, \ldots, \frac{1}{2} + ib_n,
\]
with \( b_1 > \cdots > b_n > \frac{\sqrt{3}}{2} \), and let us fix the convention that \( b_0 := +\infty \) and \( b_{n+1} := \frac{\sqrt{3}}{2} \).

For any real number \( t > \frac{\sqrt{3}}{2} \) distinct from \( b_1, \cdots, b_n \), the complex number \( \varphi_k(\frac{1}{2} + it) \) has real part \( \frac{1}{2} \) by Proposition 3, a), hence can be written as \( \frac{1}{2} + iv_k(t) \), with \( v_k(t) \in \mathbb{R} \). The variations of the function \( v_k \) are described in the following two lemmas.

**Lemma 22.**— a) We have \( v_k(\frac{\sqrt{3}}{2}) = -\frac{\sqrt{3}}{2} \) when \( k \equiv 0 \text{ mod } 6 \) and \( v_k(\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2} \) when \( k \not\equiv 0 \text{ mod } 6 \).

b) We have \( \lim_{t \to +\infty} v_k(t) = (-1)^{\frac{k}{2}} \infty \).

c) We have \( \lim_{t \to b_{m}} v_k(t) = (-1)^{\frac{1}{2}+m} \infty \) for \( 1 \leq m \leq n \).

d) We have \( \lim_{t \to b_{m}} v_k(t) = (-1)^{\frac{1}{2}+m-1} \infty \) for \( 1 \leq m \leq n \).

It follows from Proposition 3, b) that \( \varphi_k(\frac{1}{2} + i\frac{\sqrt{3}}{2}) \) has modulus 1. Hence \( v_k(\frac{\sqrt{3}}{2}) \) must be equal to either \( \frac{\sqrt{3}}{2} \) or \( -\frac{\sqrt{3}}{2} \). It follows from formula (72) that we have \( v_k(\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2} \) if and only if \( \frac{E_k}{E_k'} \) vanishes at the point \( \frac{1}{2} + i\frac{\sqrt{3}}{2} \); according to subsection 1.2, this is the case if and only if \( k \not\equiv 0 \text{ mod } 6 \). This proves assertion a).

One deduces from formula (3) that when \( t \) tends to \( +\infty \), \( E_k(\frac{1}{2} + it) \) tends to 1 and \( E_k'(\frac{1}{2} + it) \) is equivalent to \( \frac{4\pi ik}{B_k} e^{-2\pi t} \). It then follows from (72) that \( v_k(t) \) is equivalent to \( -\frac{B_k}{4\pi} e^{2\pi t} \), hence tends to \( (-1)^{\frac{k}{2}} \infty \) when \( t \) tends to \( +\infty \). This proves assertion b).

For \( t > \frac{\sqrt{3}}{2} \), we have \( E_k(\frac{1}{2} + it) \in \mathbb{R}^* \) (subsection 1.2, Remark 2) and \( iE_k'(\frac{1}{2} + it) \in \mathbb{R} \). Moreover the function \( t \mapsto iE_k'(\frac{1}{2} + it) \) vanishes only at the points \( b_1, \cdots, b_n \), and has simple zeros at these points. It therefore keeps a constant sign on each interval \( ]b_{m+1}, b_m[ \), where \( 0 \leq m \leq n \) (with \( b_0 = +\infty \) and \( b_{n+1} = \frac{\sqrt{3}}{2} \)), and these signs are alternatively positive and negative. This sign is \( (-1)^{\frac{k}{2}} \) on \( ]b_1, b_0[ \) by the previous discussion, and is therefore \( (-1)^{\frac{k}{2}+m} \) on \( ]b_{m+1}, b_m[ \) for \( 0 \leq m \leq n \). It follows that for \( 1 \leq m \leq n \), the imaginary part of \( \frac{E_k}{E_k'}(\frac{1}{2} + it) \) tends to \( (-1)^{\frac{k}{2}+m} \infty \) when \( t < b_m \) tends to \( b_m \), and to \( (-1)^{\frac{k}{2}+m-1} \infty \) when \( t > b_m \) tends to \( b_m \). The same then holds for \( v_k(t) \) by formula (72). This proves assertions c) and d).

**Lemma 23.**— a) In each interval \( ]b_m, b_{m-1}[, \) where \( 1 \leq m \leq n \), there is a unique point \( c_m \) at which the derivative \( v_k' \) of \( v_k \) vanishes. Moreover, \( v_k' \) does not vanish at any point of the interval \( ]\frac{\sqrt{3}}{2}, b_n[ \).

b) The zeros of \( \varphi_k \) in \( \mathbb{S} \) are all simple. They are the elements of the orbits under the action of \( \text{SL}_2(\mathbb{Z}) \) of the points \( \frac{1}{2} + ic_m \), where \( 1 \leq m \leq n \), together with those of the orbit of \( e^{\pi i/3} \) when \( k \equiv 0 \text{ mod } 6 \).
Let \( m \) be an integer such that \( 1 \leq m \leq n \). The function \( v_k \) is differentiable (and even real analytic) in the interval \([b_m, b_{m-1}]\), and it has the same infinite limit at both endpoints of this interval by Lemma 22. Its derivative \( v_k' \) must therefore vanish at least one point \( c_m \) of this interval. This proves the existence of \( c_m \) in assertion \( a \).

The derivative \( \varphi_k' \) of \( \varphi_k \) then vanishes at the point \( \frac{1}{2} + ic_m \). Hence the modular function \( F_k \) also vanishes at this point by formula (73). We have thus produced \( n = \lfloor \frac{k-4}{6} \rfloor \) zeros of \( F_k \) which are pairwise inequivalent modulo \( SL_2(\mathbb{Z}) \), and also not equivalent to \( e^{\pi i/2} \) or to \( e^{\pi i/3} \). Since the weighted number of zeros of \( F_k \) modulo \( SL_2(\mathbb{Z}) \) (counted with their multiplicities, with weight \( \frac{1}{2} \) for those in the orbit of \( e^{\pi i/2} \), \( \frac{1}{3} \) for those in the orbit of \( e^{\pi i/3} \), and 1 for the others) is equal to \( \frac{k-4}{6} \), the zeros of \( F_k \) found previously are all simple, and there are no other zeros of \( F_k \) modulo \( SL_2(\mathbb{Z}) \), except the points of the orbit of \( e^{\pi i/3} \) which are simple zeros of \( F_k \) when \( k \equiv 0 \mod 6 \) and zeros of multiplicity 2 of \( F_k \) when \( k \equiv 2 \mod 6 \). When \( k \equiv 0 \mod 6 \), these latter points are simple zeros of \( \varphi_k' \); when \( k \equiv 2 \mod 6 \), they are simple zeros of \( E_k' \) by subsection 1.2 and hence they are not zeros of \( \varphi_k' \) by formula (73). We have thus proved assertion \( b \).

The uniqueness in \( a \) as well as the last assertion of \( a \) follow from \( b \).

**Remark 15.**—Assertion \( a \) of Lemme 23 implies that the function \( v_k \) is strictly monotonic in the intervals \([b_m, c_m]\) and \([c_m, b_{m-1}]\), for \( 1 \leq m \leq n \), as well as in the interval \([\sqrt[3]{2}, b_n]\). We shall prove in Proposition 4 of subsection 5.6 that \( \varphi_k \) does not take the value \( \frac{1}{2} \) in \( \mathcal{D} \). It follows that \( v_k \) does not vanish at any point of the intervals \([b_m, b_{m-1}]\), where \( 1 \leq m \leq n + 1 \), and therefore keeps a constant sign on each of them. This sign is \((-1)^{\frac{3}{2} + m - 1}\) by Lemma 22.

\( b \) Behaviour of \( \varphi_k \) on the half-line \(-\frac{1}{2} + i[\sqrt[3]{2}, +\infty[\)

We have \( \varphi_k(z + 1) = \varphi_k(z) + 1 \) in \( \mathcal{S}_5 \). Hence, with the notations of \( a \), the poles of \( \varphi_k \) lying on the half-line \(-\frac{1}{2} + i[\sqrt[3]{2}, +\infty[\) are \(-\frac{1}{2} + ib_1, \ldots, -\frac{1}{2} + ib_n \), and they are simple. Moreover, for any real number \( t \geq \sqrt[3]{2} \) distinct from \( b_1, \ldots, b_n \), we have

\[
(75) \quad \varphi_k(-\frac{1}{2} + it) = \varphi_k(\frac{1}{2} + it) - 1 = -\frac{1}{2} + iv_k(t),
\]

and the variations of \( v_k \) have been studied in \( a \).

\( c \) Behaviour of the function \( \theta \mapsto \varphi_k(e^{i\theta}) \) in the interval \([\pi/3, 2\pi/3]\)

All values of the function \( \theta \mapsto \varphi_k(e^{i\theta}) \) in the interval \([\pi/3, 2\pi/3]\) are of modulus 1 by Proposition 3, \( b \) of subsection 5.4. Its value at \( \pi/3 \) is equal to \( e^{-\pi i/3} \) when \( k \equiv 0 \mod 6 \) and to \( e^{\pi i/3} \) when \( k \not\equiv 0 \mod 6 \) by Lemma 22, \( a \).
Hence there exists a unique continuous function $w_k : \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \to \mathbb{R}$ such that
\begin{equation}
\varphi_k(e^{i\theta}) = e^{i w_k(\theta)}
\end{equation}
for $\theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ and
\begin{equation}
w_k\left(\frac{\pi}{3}\right) = \begin{cases} 
-\frac{\pi}{3} & \text{when } k \equiv 0 \text{ mod } 6, \\
\frac{\pi}{3} & \text{when } k \not\equiv 0 \text{ mod } 6.
\end{cases}
\end{equation}

Lemma 24. — The function $w_k$ is strictly increasing in $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ and we have
\begin{equation}
w_k\left(\frac{2\pi}{3}\right) = \begin{cases} 
\frac{(k-2)\pi}{3} & \text{when } k \equiv 0 \text{ mod } 6, \\
\frac{k\pi}{3} & \text{when } k \equiv 2 \text{ mod } 6, \\
\frac{(k+4)\pi}{3} & \text{when } k \equiv 4 \text{ mod } 6.
\end{cases}
\end{equation}

The function $w_k$ is real analytic in the interval $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. We get by differentiating (76)
\begin{equation}
e^{i\theta} \varphi'_k(e^{i\theta}) = w'_k(\theta) e^{i w_k(\theta)}
\end{equation}
for $\theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. It follows from Lemma 23, b) that $\varphi'_k(e^{i\theta})$ does not vanish at any point of the open interval $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. Therefore the function $w'_k$ does not vanish at any point of this interval, and keeps a constant sign on this interval. It follows that the function $w_k$ is strictly monotonic in the closed interval $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$.

Let us denote by $C$ the open circular arc consisting of the points $e^{i\theta}$, where $\theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. The function $E'_k$ does not vanish at any point of $C$ by Lemma 1 of subsection 2.2. For every $\tau \in C$, we have
\begin{equation}
\varphi_k(\tau) = \tau^{-k-1} E'_k\left(-\frac{1}{\tau}\right) = \tau^{-k-1} E'_k\left(-\frac{1}{\tau}\right) = -\tau^{-k-1} E'_k(\tau)
\end{equation}
by formulae (72) and (4).

By definition (see subsection 2.1), the variation of the argument of the function $\theta \mapsto \varphi_k(e^{i\theta})$ along the interval $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ is $w_k\left(\frac{2\pi}{3}\right) - w_k\left(\frac{\pi}{3}\right)$. It follows from formula (80) that it is also equal to $-\frac{(k+1)\pi}{3} - 2A$ where, as in subsection 2.4, $A$ denotes the variation of the argument of $\theta \mapsto E'_k(e^{i\theta})$ along $\left[\frac{\pi}{3} + \eta, \frac{2\pi}{3} - \eta\right]$ when $k \not\equiv 2 \text{ mod } 6$, and the limit when $\eta > 0$ tends to 0 of the variation of the argument of $\theta \mapsto E'_k(e^{i\theta})$ along $\left[\frac{\pi}{3} + \eta, \frac{2\pi}{3} - \eta\right]$ when $k \equiv 2 \text{ mod } 6$.

The value of $A$ has been computed in subsections 2.6, 2.7 and 2.8. It is equal to $-\frac{(k+2)\pi}{3}$ when $k \equiv 4 \text{ mod } 6$ and to $-\frac{k\pi}{3}$ when $k \not\equiv 4 \text{ mod } 6$. We have therefore
\begin{equation}
w_k\left(\frac{2\pi}{3}\right) - w_k\left(\frac{\pi}{3}\right) = \begin{cases} 
\frac{(k+1)\pi}{3} + 2\frac{(k+2)\pi}{3} = \frac{(k+3)\pi}{3} & \text{when } k \equiv 4 \text{ mod } 6, \\
\frac{(k+1)\pi}{3} + 2\frac{k\pi}{3} = \frac{(k-1)\pi}{3} & \text{when } k \not\equiv 4 \text{ mod } 6.
\end{cases}
\end{equation}
One notes that $w_k\left(\frac{2\pi}{3}\right) > w_k\left(\frac{\pi}{3}\right)$ in both cases. The strictly monotonic function $w_k$ is therefore strictly increasing in $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. Finally, formula (78) follows from formulae (77) and (81).

Remark 16. – As $\varphi_k\left(-\frac{1}{2}\right) \varphi_k(z) = -1$, the function $\theta \mapsto w_k(\pi - \theta) + w_k(\theta)$ is constant on the interval $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. By formula (77) and (78), its value is $\frac{(k-3)\pi}{3}$ when $k \equiv 0 \mod 6$, $\frac{(k+1)\pi}{3}$ when $k \equiv 2 \mod 6$ and $\frac{(k+5)\pi}{3}$ when $k \equiv 4 \mod 6$.

5.6. Zeros of $E'_k$ in $\gamma D$, for $\gamma \in \text{SL}_2(\mathbb{Z})$

In this subsection, $k$ denotes an even integer $\geq 4$, $E_k$ the normalized Eisenstein series of weight $k$ for $\text{SL}_2(\mathbb{Z})$ and $D$ the closure of the standard fundamental domain of $\mathfrak{H}$ modulo $\text{SL}_2(\mathbb{Z})$. We denote by $\gamma$ a given element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\text{SL}_2(\mathbb{Z})$. We are interested in the number of zeros of $E'_k$ lying in $\gamma D$.

We already know that all the zeros of $E'_k$ in $\mathfrak{H}$ are simple (Theorem 7 of subsection 4.5). Counting those lying in $\gamma D$ with or without multiplicities therefore leads to the same result. As in subsections 1.4 and 5.2, we shall call trivial zeros of $E'_k$ those zeros of $E'_k$ which are also zeros of $E_k$. Such zeros exist only when $k \equiv 2 \mod 6$ and they are then the elements of the orbit of $e^{\pi i/3}$ under the action of $\text{SL}_2(\mathbb{Z})$; the set $\gamma D$ hence contains $2$ trivial zeros of $E'_k$ when $k \equiv 2 \mod 6$, namely $\gamma e^{\pi i/3}$ and $\gamma e^{2\pi i/3}$, and no trivial zeros otherwise.

As in subsections 1.4 and 5.2, we shall denote by $Z(E'_k)$ the set of non trivial zeros of $E'_k$ in $\mathfrak{H}$. We still have to count the number of points of the set $Z(E'_k) \cap \gamma D$. By subsection 5.2, $Z(E'_k)$ does not meet the orbits of $i$ and of $e^{\pi i/3}$ under the action of $\text{SL}_2(\mathbb{Z})$. There is therefore no need to assign weights to the points of $Z(E'_k)$ while counting them.

When $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $Z(E'_k) \cap \gamma D$ consists of $\left\lfloor \frac{k-4}{6} \right\rfloor$ points with real part $\frac{1}{2}$ and the same number with real part $-\frac{1}{2}$ (subsection 1.3, Theorems 1 and 2). More generally, when $c = 0$, the cardinality of $Z(E'_k) \cap \gamma D$ is $2\left\lfloor \frac{k-4}{6} \right\rfloor$, since $Z(E'_k)$ is translation invariant by $\mathbb{Z}$.

We shall therefore suppose from now on that $c \neq 0$. Let $\tau \in D$; in order for $\gamma \tau$ to belong to $Z(E'_k)$, it is necessary and sufficient that we have $\varphi_k(\tau) = -\frac{d}{c}$ (Lemma 20). The cardinality of $Z(E'_k) \cap \gamma D$ is therefore the number of roots of the equation $\varphi_k(z) = -\frac{d}{c}$ in $D$. Moreover, all these roots are simple (loc. cit.).

So Theorem 5 of subsection 1.4 is a consequence of the following proposition :

Proposition 4.– Let $\lambda$ be a rational number.

a) When $|\lambda| < 1$, the set $\varphi_k^{-1}(\lambda) \cap D$ is empty.

b) When $|\lambda| \geq 1$, the set $\varphi_k^{-1}(\lambda) \cap D$ has cardinality $\left\lfloor \frac{k+2}{6} \right\rfloor$. It is contained in the interior of $D$ when $|\lambda| \neq 1$, and in $C = \{e^{i\theta} \mid \theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]\}$ when $|\lambda| = 1$. 

We can always write \( \lambda \) in the form
\[
-\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\in \text{SL}_2(\mathbb{Z}),
\]
with \( c \neq 0 \). By the previous discussion, the roots of the equation \( \varphi_k(z) = \lambda \) in \( D \) are simple. The poles of \( \varphi_k \) in \( D \) are, with the notations of subsection 5.5, the points of the form \( \pm \frac{1}{2} + ib_m \), where \( 1 \leq m \leq n \) and \( n = \lfloor \frac{k-4}{6} \rfloor \).

1) The case where \( |\lambda| \) is distinct from \( 1 \) and \( \frac{1}{2} \)

In this case, the set \( \varphi_k^{-1}(\lambda) \cap D \) is contained in the interior of \( D \) (subsection 5.4, Corollary 1 of Proposition 3). When \( z \in D \) and \( \text{Im}(z) \) tends to \( +\infty \), \( E_k(z) \) tends to \( 1 \) and \( E_k'(z) \) is equivalent to \( -\frac{4\pi ik}{B_k} e^{2\pi iz} \) by formula (3), hence \( \varphi_k(z) \) is equivalent to \( \frac{\text{B}_k}{4\pi} e^{-2\pi iz} \) by formula (72). So there exists a real number \( T > 1 \), such that \( T > b_1 \) if \( n \geq 1 \) (i.e. \( k \geq 10 \)), and such that all the points of \( \varphi_k^{-1}(\lambda) \cap D \) have imaginary part \( < T \). For \( \varepsilon > 0 \), let \( D_{T,\varepsilon} \) denote the set of points in \( D \) with imaginary part \( \leq T \) and also at a distance \( \geq \varepsilon \) from the poles \( \pm \frac{1}{2} + ib_m \), where \( 1 \leq m \leq n \). It follows from the residue theorem that, when \( \varepsilon \) is sufficiently small, we have

\[
\text{Card}(\varphi_k^{-1}(\lambda) \cap D) = \frac{1}{2\pi i} \int_{\partial D_{T,\varepsilon}} \frac{\varphi_k'(z)}{\varphi_k(z) - \lambda} \, dz
\]

where \( \partial D_{T,\varepsilon} \) is the oriented boundary of \( D_{T,\varepsilon} \) (see Figure 5).

![Figure 5. The contour of integration \( \partial D_{T,\varepsilon} \).](image-url)
Formula (82) may also be written as
\begin{equation}
\text{Card}(\varphi_k^{-1}(\lambda) \cap D) = \frac{1}{2\pi} (\text{variation of the argument of } \varphi_k - \lambda \text{ along } \partial D_{T,\varepsilon}).
\end{equation}

To evaluate the right-hand side, which does not depend on \(T\) and \(\varepsilon\) (when \(T\) is sufficiently large and \(\varepsilon\) sufficiently small), we shall decompose the contour of integration \(\partial D_{T,\varepsilon}\) into finitely many parts, study the variation of the argument of \(\varphi_k - \lambda\) along each of them, then take the limit of this variation when successively \(T\) tends to \(+\infty\) and \(\varepsilon\) to 0, and finally sum up these various limits.

\(a)\) Contribution of the upper horizontal segment

When \(z \in D\) and \(\text{Im}(z)\) tends to \(+\infty\), we have seen that \(\varphi_k(z)\) is equivalent to \(\frac{ib_k}{4\varepsilon}e^{-2\pi iz}\), and the same then holds for \(\varphi_k(z) - \lambda\). It follows that the variation of the argument of \(\varphi_k - \lambda\) along the oriented horizontal segment joining \(\frac{1}{2} + iT\) to \(-\frac{1}{2} + iT\) tends to \(2\pi\) when \(T\) tends to \(+\infty\).

\(b)\) Contributions of the upper vertical segments when \(n \geq 1\)

Suppose \(n \geq 1\), i.e. \(k \geq 10\). When \(t\) varies in the interval \([b_1 + \varepsilon, T]\), \(\varphi_k(\frac{1}{2} + it)\) is moving on the line \(\frac{1}{2} + iR\), starting from the point \(\frac{1}{2} + iv_k(b_1 + \varepsilon)\) and ending at the point \(\frac{1}{2} + iv_k(T)\), where \(v_k\) is the function considered in subsection 5.5. By Lemma 22, \(v_k(T)\) when \(T\) tends to \(+\infty\), and \(v_k(b_1 + \varepsilon)\) when \(\varepsilon\) tends to 0, have the same infinite limit \((-1)^{\frac{k}{2}}\infty\). It follows that the variation of the argument of \(\varphi_k - \lambda\) along the oriented segment joining \(\frac{1}{2} + i(b_1 + \varepsilon)\) to \(\frac{1}{2} + iT\) tends to 0 when \(T\) tends to \(+\infty\) and then \(\varepsilon\) tends to 0.

The same holds for the variation of the argument of \(\varphi_k - \lambda\) along the oriented vertical segment joining \(-\frac{1}{2} + iT\) to \(-\frac{1}{2} + i(b_1 + \varepsilon)\).

\(c)\) Contributions of the vertical segments lying between two consecutive poles

Let \(m\) be an integer such that \(1 \leq m \leq n - 1\). A reasoning similar to that of \(b)\) shows that the variation of the argument of \(\varphi_k - \lambda\) along the oriented vertical segment joining \(\frac{1}{2} + i(b_m + \varepsilon)\) to \(\frac{1}{2} + i(b_m - \varepsilon)\) tends to 0 when \(\varepsilon\) tends to 0.

The same holds for the variation of the argument of \(\varphi_k - \lambda\) along the oriented vertical segment joining \(-\frac{1}{2} + i(b_m - \varepsilon)\) to \(-\frac{1}{2} + i(b_{m+1} + \varepsilon)\).

\(d)\) Contributions of the semicircles centered at the poles

Let \(m\) be an integer such that \(1 \leq m \leq n\). The function \(\varphi_k\) has a simple pole at the point \(z_m = \frac{1}{2} + ib_m\), hence there exists \(\alpha \in \mathbb{C}^\ast\) such that \(\varphi_k(z)\) is equivalent to \(\alpha(z - z_m)^{-1}\) when \(z\) tends to \(z_m\), and the same then holds for \(\varphi_k(z) - \lambda\). Therefore, the variation of the argument of \(\varphi_k - \lambda\) along the semicircle of radius \(\varepsilon\) centered at \(z_m\), joining \(z_m - i\varepsilon\) to \(z_m + i\varepsilon\) and clockwise oriented, tends to \(\pi\) when \(\varepsilon\) tends to 0.
Similarly, the variation of the argument of $\varphi_k - \lambda$ along the semicircle of radius $\varepsilon$ centered at $z'_m = -\frac{1}{2} + ib_m$, joining $z'_m + i\varepsilon$ to $z'_m - i\varepsilon$ and clockwise oriented, tends to $\pi$ when $\varepsilon$ tends to 0.

c) Remaining contribution when $n \geq 1$

Suppose $n \geq 1$, i.e. $k \geq 10$. The remaining contribution concerns the part $\delta_\varepsilon$ of $\partial D_{T, \varepsilon}$ obtained by moving successively along the oriented vertical segment joining $-\frac{1}{2} + i(b_n - \varepsilon)$ to $-\frac{1}{2} + i\sqrt{3} \frac{\varepsilon}{2}$, then along the clockwise oriented arc of the unit circle joining $e^{2\pi i/3}$ to $e^{\pi i/3}$ and finally along the oriented vertical segment joining $\frac{1}{2} + i\sqrt{3} \frac{\varepsilon}{2}$ to $\frac{1}{2} + i(b_n - \varepsilon)$.

When $z$ moves along $\delta_\varepsilon$, $\varphi_k(z)$ is successively moving:

— on the line $-\frac{1}{2} + iR$, starting from the point $-\frac{1}{2} + iv_k(b_n - \varepsilon)$ and ending at the point $-\frac{1}{2} + iv_k(\sqrt{3} \frac{\varepsilon}{2})$;

— on the unit circle, starting from the point $-\frac{1}{2} + iv_k(\sqrt{3} \frac{\varepsilon}{2})$ and ending at the point $\frac{1}{2} + iv_k(\sqrt{3} \frac{\varepsilon}{2})$, the variation of the argument of $\varphi_k$ during this move being equal to $w_k(\frac{\pi}{3}) - w_k(\frac{2\pi}{3})$, where $w_k$ is the function considered in Lemma 24;

— on the line $\frac{1}{2} + iR$, starting from the point $\frac{1}{2} + iv_k(\sqrt{3} \frac{\varepsilon}{2})$ and ending at the point $\frac{1}{2} + iv_k(b_n - \varepsilon)$.

According to Lemma 22, c), the limit when $\varepsilon$ tends to 0 of $v_k(b_n - \varepsilon)$ is $(-1)^{\frac{k}{6} + n} = (-1)^{\frac{k}{6} + \left \lfloor \frac{k}{6} \right \rfloor}$ $\infty$. By Lemma 22 and formula (81), we have the following table:

| $k \equiv 0 \mod 6$ | $k \equiv 2 \mod 6$ | $k \equiv 4 \mod 6$ |
|---------------------|---------------------|---------------------|
| $\lim_{\varepsilon \to 0} v_k(b_n - \varepsilon)$ | $v_k(\sqrt{3} \frac{\varepsilon}{2})$ | $w_k(\frac{\pi}{3}) - w_k(\frac{2\pi}{3})$ |
| $-\infty$ | $-\frac{\sqrt{3}}{2}$ | $-(k-1)\pi$ |
| $+\infty$ | $\frac{\sqrt{3}}{2}$ | $-(k-1)\pi$ |
| $+\infty$ | $\frac{\sqrt{3}}{2}$ | $-(k+3)\pi$ |

As an example, the trajectory of $\varphi_k(z)$ when $z$ varies along $\delta_\varepsilon$ is represented in figure 6 below, in the particular case where $k = 12$ (and hence $n = 1$). Double arrows in this picture indicate arcs on which $\varphi_k(z)$ passes twice.
Figure 6. Trajectory of $\varphi_k(z)$ when $z$ varies along $\delta_\varepsilon$, for $k = 12$.

One deduces from the previous description of the trajectory of $\varphi_k(z)$ when $z$ varies along $\delta_\varepsilon$ and from the above table that:

(i) when $|\lambda| > 1$, the variation of the argument of $\varphi_k - \lambda$ along $\delta_\varepsilon$ tends to 0 when $\varepsilon$ tends to 0;

(ii) when $|\lambda| < 1$ and $|\lambda| \neq \frac{1}{2}$, the limit when $\varepsilon$ tends to 0 of the variation of the argument of $\varphi_k - \lambda$ along $\delta_\varepsilon$ equals

$$\frac{\pi}{3} - \frac{(k-1)\pi}{3} = \frac{\pi}{3} - \frac{k\pi}{3}$$ when $k \equiv 0 \mod 6$, $\frac{\pi}{3} - \frac{(k-2)\pi}{3}$ when $k \equiv 2 \mod 6$ and $\frac{\pi}{3} - \frac{(k+3)\pi}{3} = -\frac{(k+2)\pi}{3}$ when $k \equiv 4 \mod 6$, i.e. to $-2\left[\frac{k+2}{6}\right]\pi$ in all these three cases.

$e'$) Remaining contribution when $n = 0$

Suppose $n = 0$, i.e. $k \leq 8$. The remaining contribution then concerns the part $\delta_T$ of $\partial D_{T,\varepsilon}$ analog to that considered in $e)$, except that $b_n - \varepsilon$ is now replaced by $T$. One then proves in the same way as in $e)$ that the limit, when $T$ tends to $+\infty$, of the variation of the argument of $\varphi_k - \lambda$ along $\delta_T$ is 0 when $|\lambda| > 1$, and is $-2\left[\frac{k+2}{6}\right]\pi$ when $|\lambda| < 1$ and $|\lambda| \neq \frac{1}{2}$.

$f$) Summing up the various contributions

By summing up the limits of the various previously described contributions, one gets the total variation of the argument of $\varphi_k - \lambda$ along $\partial D_{T,\varepsilon}$. It is equal to $2\pi + 2n\pi = 2\left[\frac{k+2}{6}\right]\pi$ when $|\lambda| > 1$, and to $2\pi + 2n\pi - 2\left[\frac{k+2}{6}\right]\pi = 0$ when
\[ |\lambda| < 1 \text{ and } |\lambda| \neq \frac{1}{2}. \] It then follows from (83) that we have
\[
(84) \quad \text{Card}(\varphi_k^{-1}(\lambda) \cap D) = \begin{cases} \\
[\frac{k+2}{6}] & \text{when } |\lambda| > 1, \\
0 & \text{when } |\lambda| < 1 \text{ and } |\lambda| \neq \frac{1}{2}. 
\end{cases}
\]

**Remark 17.** To get (84), key ingredients were Lemma 22 and formula (81). On the other hand, we did neither need the additional properties of \( v_k \) stated in Lemma 23 and in Remark 15, nor the fact that \( w_k \) is an increasing function as stated in Lemma 24.

2) **The case where** \( |\lambda| = \frac{1}{2} \)

If the equation \( \varphi_k(z) = \frac{1}{2} \) had a solution in the interior of \( D \), then for each rational number \( \lambda \) sufficiently close to \( \frac{1}{2} \), the equation \( \varphi_k(z) = \lambda \) would also have a solution, by the open mapping theorem. This would contradict (84).

If the equation \( \varphi_k(z) = \frac{1}{2} \) had a solution lying on the boundary of \( D \), this solution would lie on the open half-line \( \frac{1}{2} + i \sqrt{3}/2 \), \(+\infty\) [ by Proposition 3 of subsection 5.4. In that case, for each rational number \( \lambda \) sufficiently close to \( \frac{1}{2} \), the equation \( \varphi_k(z) = \lambda \) would have a solution in \( D \cup (1 + D) \), by the open mapping theorem, and hence either the equation \( \varphi_k(z) = \lambda \) or the equation \( \varphi_k(z) = \lambda - 1 \) would have a solution in \( D \). This again would contradict (84).

We have thus proved that the set \( \varphi_k^{-1}(\frac{1}{2}) \cap D \) is empty. One proves in the same way that the set \( \varphi_k^{-1}(-\frac{1}{2}) \cap D \) is empty, or one deduces it from the identity \( \varphi_k(-z) = -\varphi_k(z) \).

3) **The case where** \( |\lambda| = 1 \)

If the equation \( \varphi_k(z) = 1 \) had a solution in the interior of \( D \), then for each rational number \( \lambda < 1 \) sufficiently close to 1, the equation \( \varphi_k(z) = \lambda \) would also have a solution, by the open mapping theorem. This would contradict (84). The set \( \varphi_k^{-1}(1) \cap D \) is therefore contained in the boundary of \( D \). It follows from Proposition 3, a) of subsection 5.4 that it is in fact contained in the open circular arc \( C \) consisting of the points \( e^{i\theta} \), where \( \theta \in ]\frac{\pi}{3}, \frac{2\pi}{3}[ \).

By formula (76), we have \( \varphi_k(e^{i\theta}) = e^{i\omega_k(\theta)} \) for every \( \theta \in ]\frac{\pi}{3}, \frac{2\pi}{3}[ \). The cardinality of \( \varphi_k^{-1}(1) \cap D \) is therefore the number of elements \( \theta \in ]\frac{\pi}{3}, \frac{2\pi}{3}[ \) for which we have \( \omega_k(\theta) \in 2\pi\mathbb{Z} \). It follows from Lemma 24 and formula (77) that this number is equal to \( \frac{k}{6} \) when \( k \equiv 0 \mod 6 \), to \( \frac{k-2}{6} \) when \( k \equiv 2 \mod 6 \) and to \( \frac{k+2}{6} \) when \( k \equiv 4 \mod 6 \), hence to \( \lfloor \frac{k+2}{6} \rfloor \) in all these three cases.

One proves similarly, or one deduces from the identity \( \varphi_k(-z) = -\varphi_k(z) \), that \( \varphi_k^{-1}(-1) \cap D \) is contained in \( C \) and has \( \lfloor \frac{k+2}{6} \rfloor \) elements.
5.7. The real locus of $\varphi_k$

In this subsection, we keep the notations of subsections 5.5 and 5.6: $k$ is an even integer $\geq 4$, $E_k$ is the normalized Eisenstein series of weight $k$ for $\text{SL}_2(\mathbb{Z})$ and $\varphi_k$ is the meromorphic function defined by $\varphi_k(z) = z + kE_k(z)/E'_k(z)$ on the Poincaré upper half-plane $\mathfrak{H}$. We shall present a pictorial interpretation of most of the results stated in subsections 5.5 and 5.6.

For this purpose, let us consider the real locus of the function $\varphi_k$, i.e., the set $\varphi_k^{-1}(P_1(\mathbb{R}))$. It is stable under the action of $\text{SL}_2(\mathbb{Z})$, since we have

\begin{equation}
\varphi_k \left( \frac{az + b}{cz + d} \right) = \frac{a \varphi_k(z) + b}{c \varphi_k(z) + d}
\end{equation}

in $\mathfrak{H}$ for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ (Lemma 18). The study of this real locus can therefore be restricted to the set

\begin{equation}
R_k = \varphi_k^{-1}(P_1(\mathbb{R})) \cap D,
\end{equation}

where $D$ is the closure of the standard fundamental domain of $\mathfrak{H}$ modulo $\text{SL}_2(\mathbb{Z})$. As an example, $R_k$ is represented for $k = 16$ in Figure 7 below.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{The set $R_k = \varphi_k^{-1}(P_1(\mathbb{R})) \cap D$ for $k = 16$.}
\end{figure}

As $\varphi_k(-\overline{z}) = -\varphi_k(z)$ in $\mathfrak{H}$, the set $R_k$ is symmetric with respect to the imaginary axis. So we can restrict its study to $R_k^+ = R_k \cap D^+$, where $D^+$ is the
set of points in D with non-negative real part. The set \( R^+_k \) is described by the following proposition, which yields a “visual” explanation of Proposition 4:

**Proposition 5.**—

a) The set \( R^+_k \) does not meet the imaginary axis.

b) The set \( R^+_k \) has \( n = \left[ \frac{k-4}{6} \right] \) points with real part \( \frac{1}{2} \). They are the poles of \( \varphi_k \) located on the half-line \( \frac{1}{2} + i \mathbb{Z} \), i.e. the zeros of \( E'_k \) located on the open half-line \( \frac{1}{2} + i \mathbb{R} \), \( + \infty \). They are therefore the points denoted by \( \frac{1}{2} + ib_1, \ldots, \frac{1}{2} + ib_n \) in subsection 5.5, a), with \( b_1 > \ldots > b_n > \sqrt{3} \).

c) The set \( R^+_k \) has \( n + 1 = \left[ \frac{k+2}{6} \right] \) points of modulus 1. They are the points of \( D_+ \) where \( \varphi_k \) takes the value 1 or \( -1 \). Let us denote them by \( u_0, \ldots, u_n \), where \( u_j = \exp(i\alpha_j) \) and \( \frac{\pi}{6} > \alpha_0 > \ldots > \alpha_n > \frac{\pi}{3} \).

d) We have \( \varphi_k(u_j) = (-1)^{\frac{k}{2} + j + 1} \).

e) The connected component \( \Gamma_0 \) of \( u_0 \) in \( R^+_k \) is a real analytic curve, of which \( u_0 \) is an endpoint and which has an infinite branch asymptotic to the vertical line with abscissa \( \frac{1}{2} \). The map \( \varphi_k \) induces an analytic isomorphism from \( \Gamma_0 \) onto the interval \( [1, +\infty[ \) when \( \frac{k}{2} \) is odd, and onto the interval \( ] - \infty, -1] \) when \( \frac{k}{2} \) is even.

f) Let \( j \) be an integer such that \( 1 \leq j \leq n \). The connected component \( \Gamma_j \) of \( u_j \) in \( R^+_k \) is a real analytic curve, of which one endpoint is \( u_j \) and the other one is \( \frac{1}{2} + ib_j \). The map \( \varphi_k \) induces an analytic isomorphism from \( \Gamma_j \) onto the interval \( [1, +\infty[ \) when \( \frac{k}{2} + j \) is odd, and onto the interval \( ] - \infty, -1] \) when \( \frac{k}{2} + j \) is even, these intervals being identified with their canonical images in \( \mathbf{P}_1(\mathbb{R}) \).

g) The curve \( \Gamma_j \) is orthogonal to the unit circle at the point \( u_j \), for \( 0 \leq j \leq n \). It is orthogonal to the vertical line with abscissa \( \frac{1}{2} \) at the point \( \frac{1}{2} + ib_j \), for \( 1 \leq j \leq n \).

h) The curves \( \Gamma_j \), for \( 0 \leq j \leq n \), are pairwise disjoint and their union is \( R^+_k \).

By differentiating formula (3), we see that the function \( E'_k \) takes purely imaginary values and has no zeros on the half-line \( i]0, +\infty[ \). The same then holds for the function \( \varphi_k \) by formula (72). This proves assertion a).

Let \( \tau \) be a point on the half-line \( \frac{1}{2} + i \mathbb{Z} \), \( + \infty \], distinct from the poles \( \frac{1}{2} + ib_1, \ldots, \frac{1}{2} + ib_n \) of \( \varphi_k \). Then \( \varphi_k(\tau) \) has real part \( \frac{1}{2} \) by Proposition 3, a), and is distinct from \( \frac{1}{2} \) by Proposition 4, a), hence is not real. This proves assertion b).

Let \( \tau \) be a point of modulus 1 of \( R_k \). By Proposition 3, b), \( \varphi_k(\tau) \) has modulus 1; as it is a real number, it is equal to 1 or \( -1 \). Conversely, by Proposition 4, b), any point of \( D \) at which \( \varphi_k \) takes the value 1 or \( -1 \) has modulus 1, and there are \( 2(n + 1) \) such points. Their set is symmetric with respect to the imaginary axis since \( \varphi(-\overline{z}) = -\varphi(z) \), hence \( n + 1 \) among them belong to \( D^+ \). This proves assertion c).
By formula (76), we have \( \varphi_k(e^{i\theta}) = e^{iw_k(\theta)} \) for \( \theta \in \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right] \), where \( w_k \) is the function defined in subsection 5.5, c). It follows that \( \alpha_0, \ldots, \alpha_n \) are the points of the interval \( \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right] \), in descending order, at which the function \( w_k \) takes a value in \( \pi \mathbb{Z} \). Since the function \( w_k \) is strictly increasing (Lemma 24), the real numbers \( \varphi_k(u_j) = e^{iw_k(\alpha_j)} \) are alternatively equal to 1 and -1. It therefore suffices to prove assertion d) when \( j = 0 \). By the above, \( w_k(\alpha_0) \) is the largest element of \( \pi \mathbb{Z} \) which is smaller that \( w_k(\frac{\pi}{2}) \). It follows from Remark 16 that we have

\[
(87) \quad w_k \left( \frac{\pi}{2} \right) = \begin{cases} 
\frac{(k-3)\pi}{6} & \text{when } k \equiv 0 \text{ mod } 6, \\
\frac{(k+1)\pi}{6} & \text{when } k \equiv 2 \text{ mod } 6, \\
\frac{(k+5)\pi}{6} & \text{when } k \equiv 4 \text{ mod } 6,
\end{cases}
\]

and therefore

\[
(88) \quad w_k(\alpha_0) = \begin{cases} 
\frac{(k-6)\pi}{6} & \text{when } k \equiv 0 \text{ mod } 6, \\
\frac{(k-2)\pi}{6} & \text{when } k \equiv 2 \text{ mod } 6, \\
\frac{(k+2)\pi}{6} & \text{when } k \equiv 4 \text{ mod } 6.
\end{cases}
\]

In each of these three cases, we have \( \varphi_k(u_0) = e^{iw_k(\alpha_0)} = e^{3iw_k(\alpha_0)} = (-1)^{k^2+1} \). This proves d).

By Lemma 23, b), the zeros of \( \varphi_k' \) in \( D^+ \) are \( \frac{1}{2} + ic_1, \ldots, \frac{1}{2} + ic_n \), where \( c_1, \ldots, c_n \) are some real numbers satisfying the inequalities

\[
(89) \quad c_1 > b_1 > c_2 > b_2 > \ldots > c_n > b_n.
\]

The function \( \varphi_k' \) hence does not vanish at any point of \( R_k^+ \). Moreover the poles \( \frac{1}{2} + ib_1, \ldots, \frac{1}{2} + ib_m \) of \( \varphi_k \) are simple. It follows, by the local inversion theorem, that if \( \tau \) is a point of \( R_k^+ \), the germ at \( \tau \) of the set \( \varphi_k^{-1}(P_1(\mathbb{R})) \) is the germ at \( \tau \) of some real analytic curve. Furthermore, this curve is orthogonal to the vertical line with abscissa \( \frac{1}{2} \) at the point \( \tau \) if \( \tau \) is one of the points \( \frac{1}{2} + ib_j \), with \( 1 \leq j \leq n \), and is orthogonal to the unit circle at the point \( \tau \), if \( \tau \) is one of the points \( u_j \), with \( 0 \leq j \leq n \): this follows from the fact that a biholomorphic map is conformal, and from the fact that \( \varphi_k \) has values in \( \left( \frac{1}{2} + i\mathbb{R} \right) \cup \{ \infty \} \) on the half-line \( \frac{1}{2} + i\mathbb{R}_+^\ast \) and values in the unit circle on the set of points of \( \mathcal{H} \) with modulus 1 (Proposition 3).

Let us prove that the germ of \( R_k^+ \) at infinity in \( D^+ \) is the germ of a real analytic curve, asymptotic to the vertical line with abscissa \( \frac{1}{3} \). It is sufficient to prove that, for \( t > 0 \) sufficiently large, there exists a unique real number \( \sigma(t) \in [0, \frac{1}{2}] \) such that \( \sigma(t) + it \in R_k^+ \), and that \( \sigma(t) \) tends to \( \frac{1}{4} \) when \( t \) tends to \( +\infty \). Note that we have the equivalences

\[
(90) \quad \varphi_k(z) \sim \frac{iB_k}{4\pi} e^{-2\pi i z}.
\]
(91) \[ \varphi'_k(z) \sim \frac{B_k}{2} e^{-2\pi i z}, \]

when \( z \in D^+ \) approaches infinity. The relation (90) implies that, for any \( \varepsilon \in [0, \frac{1}{8}] \), there exists \( T(\varepsilon) > 0 \) such that, for \( t \geq T(\varepsilon) \), the function \( x \mapsto \text{Im} \varphi_k(x + it) \) does not vanish in the intervals \([0, \frac{1}{4} - \varepsilon]\) and \([\frac{1}{4} + \varepsilon, \frac{1}{2}]\) and has opposite signs in them, hence vanishes at at least one point of the interval \([\frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon]\). The relation (91) implies that, for \( t \) sufficiently large, the function \( x \mapsto \frac{d}{dx} \text{Im} \varphi_k(x + it) = \text{Im} \frac{d\varphi_k(x + it)}{dx} = \text{Im} \varphi'_k(x + it) \) takes non-zero values with constant sign on the interval \([\frac{1}{4}, \frac{3}{4}]\), hence the function \( x \mapsto \text{Im} \varphi_k(x + it) \) vanishes at at most one point of this interval. This completes the proof of our assertion about the germ of \( R_k^+ \) at infinity.

Let \( \Gamma \) be a connected component of \( R_k^+ \). We deduce from the previous two paragraphs that \( \Gamma \) is a real analytic curve, that \( \varphi_k \) induces an analytic isomorphism from \( \Gamma \) onto some connected subset of \( P_1(\mathbb{R}) \). This subset is disjoint from \( ]-\infty, -1[ \) by Proposition 4 and its boundary points can only be \( 1, -1 \) or \( \infty \). Hence either it is the canonical image in \( P_1(\mathbb{R}) \) of \([1, +\infty]\) or \([-\infty, -1]\), in which case \( \Gamma \) joins one of the points \( u_0, \ldots, u_n \) to one of the points \( \frac{1}{2} + ib_1, \ldots, \frac{1}{2} + ib_n \); or it is equal to \([1, +\infty]\) or \([-\infty, -1]\), in which case one of the points \( u_0, \ldots, u_n \) is an endpoint of \( \Gamma \), and \( \Gamma \) has an infinite branch asymptotic to the vertical line with abscissa \( \frac{1}{4} \).

It follows from the previous paragraph that the connected components \( \Gamma_j \) of the points \( u_j \), where \( 0 \leq j \leq n \), are pairwise disjoint and that their union is \( R_k^+ \). Since they do not intersect, necessarily \( \Gamma_j \) joins \( u_j \) to \( \frac{1}{2} + ib_j \) for \( 1 \leq j \leq n \) and \( \Gamma_0 \) has an infinite branch asymptotic to the vertical line with abscissa \( \frac{1}{4} \). We have thus proved assertions \( e), f), g) \) and \( h) \).

Remark 18. It follows from this proof that, for every \( \lambda \in \mathbb{R} \), the roots of the equation \( \varphi(z) = \lambda \) in \( D \) are simple, and assertions \( a) \) and \( b) \) of Proposition 4 hold for \( \lambda \).

Remark 19. A proof similar to that of Proposition 4 shows that, on each of the \( 2n + 3 \) connected components of the interior of \( D - R_k \), \( \varphi_k \) is injective.

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