Black hole spectra in holography: consequences for equilibration of dual gauge theories

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Abstract

Energy gap in the spectrum of equilibrium states of interacting system, along with ability to continuously excite the system inside this gap, leads to violation of the ergodicity hypothesis. We explore this in the framework of gauge theory/string theory correspondence. We study the spectrum of static black holes in Pilch-Warner geometry. These black holes are holographically dual to equilibrium states of strongly coupled $SU(N) \mathcal{N} = 2^*$ gauge theory plasma on $S^3$ in the planar limit. We find that there is no energy gap in the black hole spectrum. Thus, there is a priory no obstruction for equilibration of arbitrary low-energy states in the theory via a small black hole gravitational collapse. The latter is contrasted with phenomenological examples of holography with dual four-dimensional CFTs having non-equal central charges in the stress-energy tensor trace anomaly.

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1 Introduction and summary

Consider an interacting system in a finite volume. Suppose that the theory is gapless — there are arbitrarily low-energy excitations. If a generic state in a theory equilibrates, there can not be a gap in the spectrum of equilibrium states in the theory. Note that the reverse is not necessarily true: the mere fact that available equilibrium states exist does not imply that they are attractors of long-time dynamical evolution. The simplest example of this phenomenon is the famous Fermi-Pasta-Ulam (FPU) problem [1].

The problem of dynamical evolution and (non-)equilibration/(non-)thermalization in interacting relativistic quantum field theories is a difficult one. It is particularly so if the theory in question is strongly coupled. Luckily, we have at our disposal the gauge theory/string theory correspondence [2, 3]: certain relativistic gauge theories in four space-time dimensions at strong coupling are equivalent to classical theories of gravity in five-dimensional space-time, obtained as low-energy limits of compactifications of string theory with fluxes\(^1\). To make discussion more specific, we begin with the best

\(^1\)The ‘holographic principle’ underlying the correspondence is actually more general, and can be
studied example of the correspondence: the holographic duality between $N = 4$ supersymmetric $SU(N)$ Yang-Mills (SYM) theory in the 't Hooft limit\(^2\), and type IIB supergravity in $AdS_5 \times S^5$ with RR five-form flux. The background space of the gauge theory is taken to be $S^3$, with the same radius $\ell$ as that of $S^5$, $L$. What are the candidates for the SYM equilibrium states in the gravitational dual?\(^3\) The answer depends on the energy\(^4\) $\delta E$ of a state over the vacuum energy $E_{\text{vacuum}}$,

$$E_{\text{vacuum}} = \frac{3(N^2 - 1)}{16\ell},$$

compared to the $AdS_5$ curvature scale $L$, $L^4 = g_{YM}^2 N \ell_s^4 = 4\pi g_s N \ell_s^4$, with $\ell_s$ and $g_s$ being the string length and the coupling constant. For

$$0 < \frac{\delta E}{E_{\text{vacuum}}} \lesssim \frac{\lambda^{5/4}}{N^2},$$

the Hilbert space of string theory is identified with the Fock space of supergravity modes in $AdS_5$ — the gas of massless particles in five dimensions — resulting in the entropy scaling

$$S(\delta E) \sim (\delta EL)^{4/5}. \quad \text{(1.3)}$$

The upper limit in (1.2) represents the correspondence point where the entropy of the gas of supergravity modes becomes comparable to that of the gas of free strings, which we assume to have Hagedorn density of states with

$$S(\delta E)_{\text{string}} \sim \delta E \ell_s \sim (\delta EL) \lambda^{-1/4}. \quad \text{(1.4)}$$

As the energy of the state $\delta E$ is increased, gravitational interactions in the free string gas cause its gravitational collapse. To estimate the Horowitz-Polchinski correspondence point [4] we recall black holes\(^5\) in $AdS_5$. Introducing $\epsilon = \delta E/E_{\text{vacuum}}$, up to higher-derivative supergravity $O(\lambda^{-3/2})$ and string loop $O(N^{-2})$ corrections,

$$S(\epsilon) = \frac{\pi N^2}{2^{3/2}} \left( \sqrt{1 + \epsilon} - 1 \right)^{3/2}, \quad (T\ell)^2 = \frac{1}{2\pi^2} \frac{1 + \epsilon}{\sqrt{1 + \epsilon} - 1}, \quad \text{(1.5)}$$

applied to engineer phenomenological models of gauge/gravity correspondence. As we encounter below, phenomenological models of the correspondence must be used with caution.

\(^2\)The gauge coupling $g_{YM}^2 \to 0$ and the number of colors $N \to \infty$ with the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$ kept fixed.

\(^3\)We restrict attention to $SO(6)$ invariant sector of the theory, i.e., addressing this question in the effective five-dimensional setting.

\(^4\)We work in microcanonical ensemble.

\(^5\)We review this in section 5.
Figure 1: The entropy versus energy of equilibrium string theory states in $AdS_5$.

where $S$ and $T$ are the entropy and the temperature of the AdS-Schwarzschild black hole. Note that the minimal Hawking temperature is achieved at $\epsilon = 3$; black holes with $\epsilon < 3$, the small black holes, have a negative specific heat, while black holes with $\epsilon > 3$, the large black holes, have positive specific heat. Furthermore,

$$S(\epsilon) = \begin{cases} 
\sim \epsilon^{3/2}, & \epsilon \ll 1, \\
\sim \epsilon^{3/4}, & \epsilon \gg 1.
\end{cases} \quad (1.6)$$

The black hole “size”, $\ell_{BH}$, is determined from its horizon area,

$$\left( \frac{\ell_{BH}}{L} \right)^3 \equiv \frac{A_{\text{horizon}}}{L^3} = 2^{-1/2} \pi^2 (\sqrt{1 + \epsilon} - 1)^{3/2}, \quad (1.7)$$
resulting in

$$\lim_{{\ell_{BH}/L \to 0}} \epsilon = 0. \quad (1.8)$$

Thus, there is no gap in the spectrum of black holes in $AdS_5$, at least in the supergravity approximation.

It is straightforward to verify that, given the energy $\delta E$, small black holes become entropically favorable when

$$\lambda^{-1/2} \sim \frac{\delta E}{E_{\text{vacuum}}}. \quad (1.9)$$
The summary of the structure of the Hilbert space of string theory in $AdS_5$ is presented in figure 1. Notice that in the planar limit, the ‘free graviton’ region is removed; in addition, in the supergravity approximation $\lambda \to \infty$, the region corresponding to the ‘gas of free strings’ is removed as well. It is expected (assuming that the system equilibrates at late times) that a generic initial condition, well approximated by supergravity modes, would end in AdS black hole: in the supergravity approximation, the decay of the supergravity modes into string states is suppressed; also, the evaporation of thermodynamically unstable small black holes is suppressed as well.

The holographic correspondence is formulated for ten dimensional supergravity, i.e., for $AdS_5 \times S^5$. The inclusion of the five-sphere modifies the Hilbert space structure (see [5,6]) — most notably there is a possibility of small Schwarzschild black holes, localized on $S^5$. Similar to their five-dimensional counterparts, the ten-dimensional black holes are thermodynamically unstable, and have entropy comparable to small black holes in $AdS_5$ when $\epsilon \sim 1$. It was argued in [7] that a small black hole in $AdS_5$ suffers Gregory-Laflamme instability resulting in its $S^5$ localization. Here, we adopt the five-dimensional perspective, as questions of holographic renormalization and dynamical evolution of non-equilibrium states of a dual gauge theory are technically easier in the effective five-dimensional gravitational bulk setting. However, one should be aware of the limitations imposed by the five-dimensional perspective: one is constrained to study the sector of the theory endowed with the symmetries of the internal manifold in its Kaluza-Klein reduction. As a result, spontaneous breaking of these symmetries ($SO(6)_R \to SO(5)_R$ as in $S^5$ black hole localization) is difficult to see.

The picture outlined above for the example of holographic duality between planar $\mathcal{N} = 4$ SYM at strong coupling and $AdS_5$ supergravity is reasonable to expect to hold in general gravity-scalar dynamical system in $AdS_{d+1}$ (Einstein gravity with a negative cosmological constant in $d+1$ space-time dimensions), without any reference to holography. Specifically, since there is no gap in the spectrum of black holes in $AdS_{d+1}$ for $d \geq 3$, one would expect that a generic initial condition, no matter how small is its excitation energy over the vacuum, would result in the gravitational collapse — a formation of a small black hole. From this perspective, the initial broad claims by Bizon and Rostworowski (BR) [9] that

\[ AdS_{d+1} \text{ (for } d \geq 3 \text{ ) is unstable against black hole formation under arbitrarily small } \]

6See [8] for a boundary gauge theory perspective.

7The quote is from the slides by P. Bizon at Strings 2014 conference [10].
are not surprising. Of course, as we already alluded to earlier, the absence of the energy gap in the black hole spectrum is necessary but not sufficient condition for the gravitational collapse at vanishingly small energies. In fact, the BR claim was shown to be incorrect dynamically \[11\], and the evolution of small energy perturbations about the AdS is akin to FPU problem \[12\].

We now return to the necessary condition for equilibration at low-energies: the absence of the energy gap in the black hole spectrum. It is known that there is a gap in the spectrum of black holes in \textit{AdS}_3 (see for example \[3\]) and in Einstein-Gauss-Bonnet (EGB) gravity with a negative cosmological constant \[13, 14\]. In both cases, as in \[9\], it is possible to prepare an initial condition of arbitrary small energy over the vacuum energy. From the gravitational perspective, dynamical evolution can either result in formation of naked singularity or remain globally regular in time, if initial energy is below the threshold for the black hole formation. Numerical simulations for these models are reported in \[15,16\] — but it is probably too early to settle the outcome. Irrespectively, the existence of the energy gap in the spectrum of equilibrium states, along with ability to continuously excite the system 'inside the gap', if taken seriously within the holographic correspondence, is quite fascinating: it leads to violation of the ergodicity hypothesis, as it implies that states available during the dynamical evolution are not the same as in statistical averaging. Note that this violation is more severe than the one encountered in the FPU paradox. Presumably, faithful examples of gauge theory/string theory correspondence for \(d \geq 3\) space-time dimensional boundary should not do this\(^8\).

In the case of Einstein-Gauss-Bonnet gravity, the energy gap \(\epsilon_{\text{gap}}\) is related to the Gauss-Bonnet coupling \(\lambda_{\text{GB}}\) as \[14\]

\[
\frac{\delta E}{|E_{\text{vacuum}}|} \geq \epsilon_{\text{gap}} = \frac{4(1 - \beta^2)}{6\beta^2 - 5} \times \begin{cases} 1, & \lambda_{\text{GB}} > 0, \\ -(2\beta^2 - 1)^2, & \lambda_{\text{GB}} < 0, \end{cases}
\]

(1.10)

where\(^9\)

\[
\beta^2 = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\lambda_{\text{GB}}}, \quad \lambda_{\text{GB}} \leq \frac{1}{4}.
\]

(1.11)

\(^8\)The apparent violation of ergodicity in \textit{AdS}_3/\textit{CFT}_2 holography might not be problematic as the dual field theory is integrable.

\(^9\)The upper bound on \(\lambda_{\text{GB}}\) comes from the requirement of the asymptotic AdS geometry.
Note that $\epsilon_{\text{gap}}$ vanishes as $\lambda_{\text{GB}} \to 0$. The Einstein-Gauss-Bonnet gravity with the negative cosmological constant can be (phenomenologically) interpreted within general holographic framework as a gravitation dual to a four-dimensional CFT with non-equal central charges $c \neq a$ in the trace anomaly of the stress-energy tensor:

$$\frac{c-a}{c} = \frac{4(1 - \beta^2)}{2\beta^2 - 1}.$$  \hfill (1.12)

Turns out that the holographic interpretation of the EGB gravity puts additional constraints on allowed values of $\lambda_{\text{GB}}$, further reducing $\epsilon_{\text{gap}}$: causality of the holographic hydrodynamics requires that \cite{17}

$$\frac{-7}{36} \leq \lambda_{\text{GB}} \leq \frac{9}{100} \implies \epsilon_{\text{gap}} \leq \begin{cases} 1, & \lambda_{\text{GB}} > 0, \\ \frac{16}{27}, & \lambda_{\text{GB}} < 0. \end{cases}$$  \hfill (1.13)

Recently, it was pointed out \cite{18} that pure EGB gravity with a negative cosmological constant can not arise as a low-energy limit of a gauge theory/string theory correspondence — the difference of central charges (1.12) is bounded by $\Delta_{\text{gap}}^{-2}$, where $\Delta_{\text{gap}}$ is the dimension of the lightest single particle operators with spin $J > 2$ in the holographically dual conformal gauge theory. Inclusion of these higher-spin states in pure EGB gravity might introduce additional equilibrium states inside the gap of the black hole spectrum\textsuperscript{11}, completely eliminating $\epsilon_{\text{gap}}$ and thus resolving the ergodicity puzzle.

Can we compute the spectrum in black holes in some nontrivial example of gauge theory/string theory correspondence and calculate $\epsilon_{\text{gap}}$? The purpose of this paper is to present such computation in the context of $\mathcal{N} = 2^*$ holography between mass deformed $\mathcal{N} = 4$ SYM and the holographic Pilch-Warner (PW) renormalization group (RG) flow \cite{19–21}.

From the gauge theory perspective, $SU(N) \, \mathcal{N} = 2^*$ gauge theory is obtained from the parent $\mathcal{N} = 4$ SYM by giving a mass to $\mathcal{N} = 2$ hypermultiplet in the adjoint representation. In $R^{3,1}$ space-time, the low-energy effective action of the theory can be computed exactly \cite{22}. The theory has quantum Coulomb branch vacua $\mathcal{M}_C$, parameterized by the expectation values of the complex scalar $\Phi$ in the $\mathcal{N} = 2$ vector multiplet, taking values in the Cartan subalgebra of the gauge group,

$$\Phi = \text{diag}(a_1, a_2, \cdots, a_N), \quad \sum_i a_i = 0,$$  \hfill (1.14)

\textsuperscript{10}$\epsilon_{\text{gap}}$ is unbounded as $\lambda_{\text{GB}} \to -\infty$ and $\lambda_{\text{GB}} \to 5/36$.

\textsuperscript{11}Integrating out massive $J > 2$ spin states could produce new higher-curvature contributions, in addition to the Gauss-Bonnet term.
resulting in complex dimension of the moduli space

\[ \dim_{\mathbb{C}} \mathcal{M}_C = N - 1. \]  

(1.15)

In the large-\(N\) limit, and for strong ’t Hooft coupling, the holographic duality reduces to the correspondence between the gauge theory and type IIb supergravity. Since supergravities have finite number of light modes, one should not expect to see the full moduli space of vacua in \(\mathcal{N} = 2\) examples of gauge/gravity correspondence. This is indeed what is happening: the PW flow localizes on a semi-circle distribution of (1.14) with a linear number density \([20]\),

\[ \text{Im}(a_i) = 0, \quad a_i \in [-a_0, a_0], \quad a_0^2 = \frac{m^2 g_Y^2 M N}{4\pi^2}, \]

(1.16)

where \(m\) is the hypermultiplet mass. This holographic localization can be deduced entirely from the field theory perspective \([23]\), using the \(S^4\)-supersymmetric localization techniques \([24]\). To summarize, \(\mathcal{N} = 2^*\) holography is a well-understood nontrivial example of gauge/gravity correspondence that passes a number of highly nontrivial tests \([20, 23, 25]\).

We would like to compactify the background space of the \(\mathcal{N} = 2^*\) strongly coupled gauge theory on \(S^3\) of radius \(\ell\) — in a dual gravitational picture we prescribe the boundary condition for the non-normalizable component of the metric in PW effective action to be that of \(R \times S^3\). This is in addition to specifying non-normalizable components (corresponding to \(m\) in (1.16)) for the two PW scalars, dual to the mass deformation operators of dimensions \(\Delta = 2\) and \(\Delta = 3\) of the gauge theory hypermultiplet mass term. Thus, we produced a holographic example of a strongly interacting system in a finite volume. The single dimensionless parameter\(^{12}\), so far, is \(m\ell\). We proceed to construct regular solutions of the PW effective gravitational action with the prescribed boundary condition, interpreting them as vacua of \(S^3\)-compactified strongly coupled \(\mathcal{N} = 2^*\) gauge theory. Using the standard holographic renormalization technique\(^{13}\) we compute the vacuum energy of the theory as a function of \(m\ell\), \(E_{\text{vacuum}} = E_{\text{vacuum}}(m\ell)\). We do not verify in this work whether described \(S^3\)-compactifications preserve any supersymmetry; thus, it is important to check the stability of the vacuum solutions. Previously,

\(^{12}\)\(\mathcal{N} = 2^*\) theory in Minkowski space-time has a scale associated with the Coulomb branch moduli distribution (1.16). Once the theory is compactified on the \(S^3\) the moduli space is lifted.

\(^{13}\)For the model in hand this was developed in [26].
careful analysis of the $S^4$-compactified PW holographic flows of [27] pointed to the discrepancy in the free energy of the solutions, compared with the localization prediction in [23]. This discrepancy was resolved by identifying a larger truncation [25] (BEFP), where it was pointed out that preservation of the $S^4$-supersymmetry necessitates turning on additional bulk scalar fields. Stability of the PW embedding inside BEFP was discussed in [28]. We verify here that $S^3$-compactified PW vacua are stable within BEFP truncation. Having constructed vacuum solutions, we move to the discussion of the black hole spectrum. We construct regular Schwarzschild black hole solutions in PW effective action, and compute $\delta E \equiv \delta E(m\ell, \ell_{BH}/L) \equiv E - E_{\text{vacuum}}(m\ell)$. We argue that, as in BR setting, there is no obstruction of initializing arbitrary low-energy excitations over the vacuum. Thus, one would expect no gap in the energy spectrum of PW black hole solutions, realizing equilibrium configurations of the strongly coupled $\mathcal{N} = 2^*$ gauge theory in the planar limit. Indeed, we find that

$$\lim_{\ell_{BH}/L \to 0} \frac{\delta E(m\ell, \ell_{BH}/L)}{E_{\text{vacuum}}(m\ell = 0)} = 0.$$  

(1.17)

It is tempting to speculate that the result (1.17) should hold for all bona fide cases of holography between the gauge theory and string theory in $d \geq 3$ space-times dimensions.

The rest of the paper is organized as follows. In the next section we review the PW effective action and its embedding within a larger BEFP truncation. In section 3 we construct gravitational dual to vacuum states of $\mathcal{N} = 2^*$ gauge theory on $S^3$. Stability of the latter states within BEFP truncation is discussed in section 4. In section 5 we study the spectrum of black holes in PW effective action.

2 PW/BEFP effective actions

We begin with description of the PW effective action [19]. The action of the effective five-dimensional supergravity including the scalars $\alpha$ and $\chi$ (dual to mass terms for the bosonic and fermionic components of the hypermultiplet respectively) is given by

$$S = \int_{M_5} d\xi^5 \sqrt{-g} \mathcal{L}_{\text{PW}}$$

$$= \frac{1}{4\pi G_5} \int_{M_5} d\xi^5 \sqrt{-g} \left[ \frac{1}{4} R - 3(\partial \alpha)^2 - (\partial \chi)^2 - P \right],$$  

(2.1)

Of course, BEFP can itself be consistently truncated to PW.
where the potential\(^{15}\)

\[
\mathcal{P} = \frac{1}{16} \left[ \frac{1}{3} \left( \frac{\partial W}{\partial \alpha} \right)^2 + \left( \frac{\partial W}{\partial \chi} \right)^2 \right] - \frac{1}{3} W^2,
\]

(2.2)
is a function of \(\alpha\) and \(\chi\), and is determined by the superpotential

\[
W = -e^{-2\alpha} - \frac{1}{2} e^{4\alpha} \cosh(2\chi).
\]

(2.3)

In our conventions, the five-dimensional Newton’s constant is

\[
G_5 \equiv \frac{G_{10}}{2^5 \text{vol}_5} = \frac{4\pi}{N^2}.
\]

(2.4)

Supersymmetric vacuum of \(\mathcal{N} = 2^*\) gauge theory in Minkowski space-time is given by

\[
ds_5^2 = e^{2A} (-dt^2 + d\vec{x}^2) + dr^2, \quad \rho = \rho(r) \equiv e^{\alpha(\rho)}, \quad \chi = \chi(r),
\]

(2.5)

with

\[
e^A = k\rho^2 \sinh(2\chi), \quad \rho^6 = \cosh(2\chi) + \sinh^2(2\chi) \ln \frac{\sinh(\chi)}{\cosh(\chi)}, \quad \frac{dA}{dr} = -\frac{1}{3} W,
\]

(2.6)

where the single integration constant \(k\) is related to the hypermultiplet mass \(m\) according to [20]

\[
k = mL = 2m.
\]

(2.7)

The BEFP effective action [25] is given by

\[
S_{BEFP} = \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \, \mathcal{L}_{BEFP}
\]

\[
= \frac{1}{4\pi G_5} \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \left[ R - 12 \frac{(\partial \eta)^2}{\eta^2} - 4 \frac{(\partial \vec{X})^2}{(1 - \vec{X}^2)^2} - \mathcal{V} \right],
\]

(2.8)

with the potential

\[
\mathcal{V} = -\left[ \frac{1}{\eta^4} + 2\eta^2 \frac{1 + \vec{X}^2}{1 - \vec{X}^2} - \eta^8 \frac{(X_1)^2 + (X_2)^2}{(1 - \vec{X}^2)^2} \right],
\]

(2.9)

where \(\vec{X} = (X_1, X_2, X_3, X_4, X_5)\) are five of the scalars and \(\eta\) is the sixth. The symmetry of the action reflects the symmetries of the dual gauge theory: the two scalars \((X_1, X_2)\)

\(^{15}\)We set the five-dimensional supergravity coupling to one. This corresponds to setting the radius \(L\) of the five-dimensional sphere in the undeformed metric to 2.
form a doublet under the $U(1)_R$ part of the gauge group, while $(X_3, X_4, X_5)$ form a triplet under $SU(2)_V$ and $\eta$ is neutral. The PW effective action is recovered as a consistent truncation of (2.8) with

$$X_2 = X_3 = X_4 = X_5 = 0,$$

provided we identify the remaining BEFP scalars $(\eta, X_1)$ with the PW scalars $(\alpha, \chi)$ as follows

$$e^\alpha \equiv \eta, \quad \cosh 2\chi = \frac{1 + (X_1)^2}{1 - (X_1)^2}.$$  

(2.11)

Note that once $m \neq 0$ (correspondingly $X_1 \neq 0$), the $U(1)_R$ symmetry is explicitly broken; on the contrary, $SU(2)_V$ remains unbroken in truncation to PW.

3 Holographic duals to $\mathcal{N} = 2^*$ vacuum states on $S^3$

We derive bulk equations of motion and specify boundary conditions representing gravitational dual to vacuum states of strongly coupled $\mathcal{N} = 2^*$ gauge theory on $S^3$. We assume that the vacua are $SO(4)$-invariant. We argue that there is no obstruction of exciting these vacua by arbitrarily small perturbations of the bulk scalar fields $\alpha$ and $\chi$. We review holographic renormalization of the theory and compute the vacuum energy. Next, we solve static gravitational equations perturbatively in the mass deformation parameter $m\ell \ll 1$ — this would serve as an independent check for the general $O(m\ell)$ numerical solutions. We conclude with the plot representing $\epsilon \equiv E_{\text{vacuum}}(m\ell)/E_{\text{vacuum}}^{\mathcal{N}=4}$,

$$E_{\text{vacuum}}^{\mathcal{N}=4} \equiv E_{\text{vacuum}}(m\ell = 0) = \frac{3N^2}{16\ell},$$

as a function of $m\ell$. Interestingly, while the vacuum energy of the $\mathcal{N} = 4$ SYM is positive, it is negative$^{16}$ for $\mathcal{N} = 2^*$ gauge theory once $m\ell \gtrsim 0.87$.

3.1 Equations of motion and the boundary conditions

We consider the general time-dependent $SO(4)$-invariant ansatz for the metric and the scalar fields:

$$ds^2 = \frac{4}{\cos^2 x} \left( -A e^{-2\delta}(dt)^2 + \frac{(dx)^2}{A} + \sin^2 x (d\Omega_3)^2 \right),$$

(3.2)

\footnote{Prior to imposing causality constraints in EGB gravity, its vacuum energy becomes negative once $\lambda_{\text{GB}} > 5/36$. Vacuum energy of a different nonconformal gauge theory on $S^3$ was also observed to be negative in [29].}
where \((d\Omega_3)^2\) is a metric on a unit\(^{17}\) round \(S^3\), and \(\{A, \delta, \alpha, \chi\}\) being functions of a radial coordinate \(x\) and time \(t\). Introducing

\[
\Phi_\alpha \equiv \partial_x \alpha, \quad \Phi_\chi \equiv \partial_x \chi, \quad \Pi_\alpha \equiv \frac{e^\delta}{A} \partial_t \alpha, \quad \Pi_\chi \equiv \frac{e^\delta}{A} \partial_t \chi, \quad (3.3)
\]

we obtain from (2.1) the following equations of motion:

- the evolution equations, \(\dot{\alpha} = \partial_t \alpha\), \(\dot{\chi} = \partial_t \chi\),

\[
\begin{align*}
\dot{A} &= Ae^{-\delta} \Pi_\alpha, \\
\dot{\Phi}_\alpha &= (Ae^{-\delta} \Pi_\alpha)_x, \\
\dot{\Phi}_\chi &= (Ae^{-\delta} \Pi_\chi)_x, \\
\dot{\Pi}_\alpha &= \frac{1}{\tan^3 x} (\tan^3 x Ae^{-\delta} \Phi_\alpha)_x - \frac{2}{3 \cos^2 x} e^{-\delta} \partial^P \\
\dot{\Pi}_\chi &= \frac{1}{\tan^3 x} (\tan^3 x Ae^{-\delta} \Phi_\chi)_x - \frac{2}{\cos^2 x} e^{-\delta} \partial^P \\
\end{align*}
\]

- the spatial constraint equations,

\[
\begin{align*}
A_x &= \frac{2 + 2 \sin^2 x}{\sin x \cos x} (1 - A) - 2 \sin(2x)A \left( \Phi_\alpha^2 + \Pi_\alpha^2 + \frac{1}{3} \Phi_\chi^2 + \frac{1}{3} \Pi_\chi^2 \right) \\
&\quad - 4 \tan x \left( 1 + \frac{4}{3} P \right), \\
\delta_x &= -2 \sin(2x) \left( \Phi_\alpha^2 + \Pi_\alpha^2 + \frac{1}{3} \Phi_\chi^2 + \frac{1}{3} \Pi_\chi^2 \right), \\
\end{align*}
\]

- and the moment constraint equation,

\[
A_t + 4 \sin(2x) A e^{-\delta} \left( \Phi_\alpha \Pi_\alpha + \frac{1}{3} \Phi_\chi \Pi_\chi \right) = 0. \quad (3.6)
\]

It is straightforward to verify that the spatial derivative of (3.6) is implied by (3.4) and (3.5); thus it is sufficient to impose this equation at a single point. As \(x \to 0_+\), the momentum constraint implies that \(A(0, t)\) is a constant\(^{18}\), and as \(x \to \frac{\pi}{2}_-\) the latter constraint is equivalent to the conservation of the boundary stress-energy tensor (see \(3.2\) for details).

The general non-singular solution of (3.4), (3.5) at the origin takes form

\[
\begin{align*}
A(t, x) &= 1 + \mathcal{O}(x^2), \quad \delta(t, x) = d_0^b(t) + \mathcal{O}(x^2), \quad \\
\alpha(t, x) &= \alpha_0^b(t) + \mathcal{O}(x^2), \quad \chi(t, x) = \chi_0^b(t) + \mathcal{O}(x^2). \\
\end{align*}
\]

\(^{17}\)We set \(\ell = 1\); the \(\ell\) dependence can be easily recovered from dimensional analysis.

\(^{18}\)In fact, the non-singularity of \(A(t, x)\) in this limit automatically solves (3.6).
It is completely characterized by three time-dependent functions:

\[
\{d_0^h, \alpha_0^h, \chi_0^h\}. \tag{3.8}
\]

At the outer boundary \(x = \frac{\pi}{2}\) we introduce \(y \equiv \cos^2 x\) so that we have

\[
A = 1 + y \left( \frac{2}{3} c_{1,0} + y^2 \left( a_{2,0}(t) + \left( \frac{2}{3} c_{1,0}(c_{1,0} + 1) + 8\rho_{1,1}^2 + 16\rho_{1,1}\rho_{1,0}(t) \right) \ln y + 8\rho_{1,1}^2 \ln^2 y \right) + O(y^3 \ln^3 y) \right).
\]

\[
\delta = y \left( \frac{1}{3} c_{1,0} + y^2 \left( \frac{1}{2} c_{2,0}(t) - \frac{1}{36} c_{1,0}^2 + 4\rho_{1,0}^2(t) - \frac{1}{8} c_{1,0} + 2\rho_{1,1}^2 + 4\rho_{1,0}(t)\rho_{1,1} \right) \ln y + 4\rho_{1,1}^2 \ln^2 y \right) + O(y^3 \ln^3 y),
\]

\[
e^\alpha = 1 + y \left( \rho_{1,0}(t) + \rho_{1,1} \ln y \right) + y^2 \left( \frac{1}{12} c_{1,0}^2 + \rho_{1,0}(t) - 3\rho_{1,1} c_{1,0} + 6\rho_{1,1} \right)
- 4\rho_{1,0}(t)\rho_{1,1} + \frac{4}{3} c_{1,0}\rho_{1,0}(t) + \frac{3}{2} \rho_{1,1}^2 + \frac{1}{4} \partial_t^2 \rho_{1,0}(t) + \left( \frac{4}{3} \rho_{1,1} c_{1,0} + \rho_{1,1} \right)
- 4\rho_{1,1}^2 + 3\rho_{1,0}(t)\rho_{1,1} \right) \ln y + \frac{3}{2} \rho_{1,1}^2 \ln^2 y \right) + O(y^3 \ln^3 y),
\]

\[
cosh 2\chi = 1 + y \left( c_{1,0} + y^2 \left( c_{2,0}(t) + \left( \frac{1}{2} c_{1,0} + \frac{2}{3} c_{1,0}^2 \right) \ln y \right) \right) + O(y^3 \ln^2 y), \tag{3.9}
\]

where we explicitly indicated time-dependence, i.e.,

\[
\frac{d}{dt} c_{1,0} = 0, \quad \frac{d}{dt} \rho_{1,1} = 0. \tag{3.10}
\]

Asymptotic expansion (3.9) is completely characterized by two constants\footnote{Prescribing time dependence to these coefficients amounts to study quantum quenches in \(\mathcal{N} = 2^*\) gauge theory [30].} \(\{\rho_{1,1}, c_{1,0}\}\) and three time-dependent functions

\[
\{a_{2,0}, \rho_{1,0}, c_{2,0}\}, \tag{3.11}
\]

constraint by (3.6) to satisfy

\[
0 = \frac{d}{dt} \left( a_{2,0} - 8\rho_{1,0}^2(t) - 16\rho_{1,0}(t)\rho_{1,1} - \frac{2}{3} c_{2,0}(t) \right). \tag{3.12}
\]
The non-normalizable coefficients $\rho_{1,1}$ and $c_{1,0}$ are related to the mass deformation parameters of the dual gauge theory. Following [31], the precise relation can be established by matching the asymptotics (3.9) with the supersymmetric PW RG flow (2.6),

$$\{\rho_{1,1}, c_{1,0}\}^{PW} = k^2 \left\{ \frac{1}{48}, \frac{1}{8} \right\} = m^2 \left\{ \frac{1}{12}, \frac{1}{2} \right\}.$$ (3.13)

A specific relation between the non-normalizable coefficients of the bulk scalars $e^\alpha$ and $\cosh 2\chi$, \textit{i.e.},

$$c_{1,0} = 6\rho_{1,1},$$ (3.14)

realizes $\mathcal{N} = 2$ supersymmetry of the boundary gauge theory in the UV. As in [31], it is possible to study the theory with explicitly broken supersymmetry, \textit{i.e.},

$$\rho_{1,1} \equiv \frac{1}{48} (m_b L)^2 \neq \frac{1}{6} \times c_{1,0} \equiv \frac{1}{6} \times \frac{1}{8} (m_f L)^2,$$ (3.15)

where $m_b$ and $m_f$ are the masses of the bosonic and the fermionic components of the $\mathcal{N} = 2$ hypermultiplet of the boundary gauge theory.

A non-equilibrium state of the gauge theory can be specified with the following initial/boundary conditions:

$$\alpha(0, x) = \alpha^{\text{init}}(x), \quad \chi(0, x) = \chi^{\text{init}}(x), \quad \Phi_\alpha(0, x) = \Phi_\alpha^{\text{init}} = \frac{d\alpha^{\text{init}}}{dx},$$ (3.16)

$$\Phi_\chi(0, x) = \Phi_\chi^{\text{init}} = \frac{d\chi^{\text{init}}}{dx}, \quad \Pi_\alpha(0, x) = \Pi_\alpha^{\text{init}}(x), \quad \Pi_\chi(0, x) = \Pi_\chi^{\text{init}}(x),$$

and as $y \equiv \cos^2 x \to 0$,

$$\alpha^{\text{init}}(y) = \rho_{1,1} y \ln y + \mathcal{O}(y), \quad \cosh (2\chi^{\text{init}}(y)) = 1 + y c_{1,0} + \mathcal{O}(y^2 \ln y),$$

$$\Pi_\alpha^{\text{init}}(y) = \mathcal{O}(y), \quad \Pi_\chi^{\text{init}}(y) = \mathcal{O}(y^{3/2}),$$ (3.17)

$$A(0, x) = 1 + \frac{\cos^4 x}{\sin^2 x} \exp \left( -\frac{2}{3} \int_0^x d\xi \sin(2\xi) \left( (\Pi_\alpha^{\text{init}}(\xi))^2 + (\Phi_\alpha^{\text{init}}(\xi))^2 \right) \right) \times g(x),$$

$$g(x) = -\frac{4}{3} \int_0^x d\xi \tan^3 \xi \exp \left( \frac{2}{3} \int_0^\xi d\eta \sin(2\eta) \left( (\Pi_\alpha^{\text{init}}(\eta))^2 + (\Phi_\alpha^{\text{init}}(\eta))^2 \right) \right) \times \left( \frac{4\Pi_\alpha^{\text{init}}(\xi) + 3}{\cos^2 \xi} + (\Pi_\alpha^{\text{init}}(\xi))^2 \right)$$

$$+ (\Phi_\alpha^{\text{init}}(\xi))^2 + 3 \left( \Pi_\alpha^{\text{init}}(\xi) \right)^2 + 3 \left( \Phi_\alpha^{\text{init}}(\xi) \right)^2 \right) \right) \times \left( \frac{4\Pi_\alpha^{\text{init}}(\xi) + 3}{\cos^2 \xi} + (\Pi_\alpha^{\text{init}}(\xi))^2 \right)$$ (3.18)

$$\mathcal{P}^{\text{init}}(\xi) = \mathcal{P} \left( \alpha^{\text{init}}(\xi), \chi^{\text{init}}(\xi) \right),$$

14
\[ \delta(0, x) = -\frac{2}{3} \int_0^x d\xi \sin(2\xi) \left( (\Pi^\text{init}_c(\xi))^2 + (\Phi^\text{init}_c(\xi))^2 + 3 (\Pi^\text{init}_\alpha(\xi))^2 + 3 (\Phi^\text{init}_\alpha(\xi))^2 \right), \]

(3.19)

where we explicitly solved for \( A(0, x) \) and \( \delta(0, x) \) using constraint equations (3.5). Notice that while \( A(0, x) \) and \( \delta(0, x) \) are free from the singularities given arbitrary profiles (3.16), a large amplitude initial conditions might cause \( A(0, x) \) to vanish for some \( 0 < x_0 < \frac{\pi}{2} \), i.e., \( A(0, x_0) = 0 \), — this corresponds to 'putting a black hole in the initial data'. Clearly, initial conditions arbitrarily small perturbed about static gravitational solutions without a horizon (see below) are well defined. In particular, as in [9], one can consider perturbations with

\[
\alpha^\text{init} = \alpha^v, \quad \chi^\text{init} = \chi^v, \quad \Pi^\text{init}_{\alpha, \chi} = \lambda \, \pi_{\alpha, \chi}(x), \quad \lambda \to 0, \quad (3.20)
\]

where the superscript \( v \) stands for a static (vacuum) solution and \( \lambda \) characterizes an overall amplitude of the perturbation with given initial profiles \( \pi_\alpha \) and \( \pi_\chi \).

The \( SO(4) \)-invariant vacua of strongly coupled \( \mathcal{N} = 2^* \) gauge theory correspond to static solutions of (3.4)-(3.6). To avoid unnecessary cluttering of the formulas, we omit the superscript \( v \), use a radial coordinate \( y \equiv \cos^2 x \), and introduce

\[
A(t, y) = a(y), \quad \delta(t, y) = d(y), \quad e^{\alpha(t, y)} = \rho(y), \quad \cosh(2\chi(t, y)) = c(y). \quad (3.21)
\]

We find then

\[
0 = c'' - \frac{c'(c')^2}{c^2 - 1} + c' \left( \frac{a'}{a} - d' \right) - \frac{(y + 1)c'}{y(1 - y)} - \frac{\rho^2(c^2 - 1)(\rho^6 c - 4)}{4(1 - y)y^2 a},
\]

\[
0 = \rho'' - \frac{\rho'\rho'}{\rho} + \rho' \left( \frac{a'}{a} - d' \right) - \frac{(y + 1)\rho'}{y(1 - y)} - \frac{(c^2 - 1)\rho^9}{12(1 - y)y^2 a} - \frac{1 - \rho^6 c}{6\rho^3 y^2 a(1 - y)},
\]

\[
0 = d' - \frac{2y(1 - y)(c')^2}{3(c^2 - 1)} - \frac{8(1 - y)y\rho'^2}{\rho^2},
\]

\[
0 = a' - (y - y^2)a \left( \frac{8(\rho')^2}{\rho^2} + \frac{2(c')^2}{3(c^2 - 1)} \right) + \frac{(y - 2)a + y}{y(1 - y)} - \frac{(c^2 - 1)\rho^8 - 8\rho^2 c}{6y} + \frac{2}{3y\rho^2},
\]

(3.22)

where \( \frac{d}{dy} \). The boundary conditions as \( y \to 0 \) are as in (3.9), once we neglect the
time dependence. At the origin, using $z \equiv 1 - y$ we have

$$a = 1 + \left(-1 + \frac{1}{3(\rho_0^h)^4} - \frac{(\rho_0^h)^8}{12} ((c_0^h)^2 - 1) + \frac{2c_0^h(\rho_0^h)^3}{3}\right) z + \mathcal{O}(z^2),$$

$$d = d_0^h + \mathcal{O}(z^2),$$

$$\rho = \rho_0^h + \left(\frac{(\rho_0^h)^9}{24} ((c_0^h)^2 - 1) + \frac{1 - (\rho_0^h)^6c_0^h}{12(\rho_0^h)^3}\right) z + \mathcal{O}(z^2),$$

$$c = c_0^h + \frac{1}{8}(\rho_0^h)^2 ((c_0^h)^2 - 1) \left(c_0^h(\rho_0^h)^6 - 4\right) z + \mathcal{O}(z^2).$$  \hspace{1cm} (3.23)

We consider geometries with $\mathcal{N} = 2$ supersymmetry in the ultraviolet, so we impose the constraint (3.13). Having fixed $m$, the complete set of normalizable coefficients in the UV/IR is given by:

$$\{a_{2,0}, \rho_{1,0}, c_{2,0}, \rho_0^h, c_0^h, d_0^h\}. \hspace{1cm} (3.24)$$

Note that the six integration constants (3.24) is exactly what is needed to uniquely fix a solution of a coupled system of two second-order and two first-order ODEs.

### 3.2 Holographic renormalization and the vacuum energy

Holographic renormalization of RG flows in PW geometry was discussed in [26]. Here we apply the analysis for the gravitational solutions dual to vacua of $\mathcal{N} = 2^*$ gauge theory on $S^3$.

The gravitational action (2.1) evaluated on a static solution (3.22) diverges — this divergence is a gravitational reflection of a standard UV divergence of the free energy in the interacting boundary gauge theory. It is regulated by cutting off the radial coordinate integration at $y = y_c \ll 1$. It is straightforward to verify that the regularized Euclidean gravitational Lagrangian, $\mathcal{L}_{\text{reg}}^E$, is a total derivative,

$$\mathcal{L}_{\text{reg}}^E = \frac{1}{4\pi G_5^{\text{vol}}} \int_1^{y_c} dy \frac{d}{dy} \left(\frac{4(1 - y)^2 e^{-d}}{y^2} (a + 2yad' - ya')\right) \left|^{y_c}_{1}\right.$$  \hspace{1cm} (3.25)

where in the second equality, using (3.23), we observe that the only contribution comes from the upper limit of integration. Regularized Lagrangian (3.25) has to be supple-
mented with contributions coming from the familiar Gibbons-Hawking term, $\mathcal{L}_{GH}^E$, 

$$S_{GH}^E = -\frac{1}{8\pi G_5} \int_{\partial \mathcal{M}_5} d\xi^4 \sqrt{h_E} \nabla_\mu n^\mu \equiv \int d\tau_E \mathcal{L}_{GH}^E,$$

$$\mathcal{L}_{GH}^E = \text{vol}(\Omega_3) \left[ \frac{4(1 - y)e^{-d}}{y^2} (a(y - 4) - 2d'y(1 - y)a + a'y(1 - y)) \right] |_{y_c},$$

and the counterterm Lagrangian $^{20}$, $\mathcal{L}_{counter}^E$,

$$S_{counter}^E \equiv \int d\tau_E \mathcal{L}_{counter}^E,$$

$$\mathcal{L}_{counter}^E = \frac{\text{vol}\Omega_3}{4\pi G_5} \sqrt{h_E} \left[ \frac{3}{4} + \frac{1}{4} R_4 + \frac{1}{2} \chi^2 + 3\alpha^2 - \frac{3}{2} \frac{\alpha^2}{\ln \epsilon_c} + \ln \epsilon_c \left( -\frac{1}{3} \chi^2 R_4 - \frac{2}{3} \chi^4 \right) + \frac{1}{6} \chi^4 \right] |_{y_c},$$

where $R_4 \equiv R_4(h_E)$ is the Ricci scalar constructed from $h_E$, and $\epsilon_c$ parameterizes conformal anomaly terms in terms of the $g_{tE}tE$ metric component,

$$R_4 = \frac{3y}{2(1 - y)}, \quad \epsilon_c \equiv \sqrt{g_{tE}tE} = \frac{2\sqrt{ae^{-d}}}{\sqrt{y}}.$$

The renormalized Lagrangian $\mathcal{L}_{renom}^E$, finite in the limit $y_c \to 0$, is identified with the free energy $\mathcal{F}$ of the boundary gauge theory,

$$\mathcal{F} = \mathcal{L}_{renom}^E = \lim_{y_c \to 0} \left( \mathcal{L}_{reg}^E + \mathcal{L}_{GH}^E + \mathcal{L}_{counter}^E \right),$$

$$= \frac{\text{vol}\Omega_3}{4\pi G_5} \frac{3}{2} \left( 1 + c_{1,0}^2 \left( \frac{4}{9} - \frac{16}{9} \ln 2 \right) + c_{1,0} \left( -\frac{4}{3} - \frac{8}{3} \ln 2 \right) + 64\rho_{1,1}^2 \ln 2 \right. \left. + \left\{ 64\rho_{1,1}\rho_{1,0} + \frac{8}{3} c_{2,0} + 32\rho_{1,0}^2 - 4a_{2,0} \right\} \right)$$

$$= \frac{3N^2}{16\ell} \left( 1 + \frac{(ml)^4}{9} - \frac{2}{3}(1 + 2 \ln 2)(ml)^2 + \left\{ 32\rho_{1,0}^2 + \frac{16}{3}(ml)^2 \rho_{1,0} + \frac{8}{3} c_{2,0} - 4a_{2,0} \right\} \right),$$

where in the second line we used the asymptotic expansion (3.9) and expressed the last line in terms of gauge theory variables using (2.4) and (3.13) and restoring the size $\ell$ of the $S^3$. Several comments are in order:

- For static gravitational solutions without Schwarzschild horizon (as discussed here),

---

$^{20}$We keep only the counterterms relevant for the $R \times S^3$ background geometry of the gauge theory.
the free energy $\mathcal{F}$ must coincide with the energy $E$ of the boundary stress-energy tensor. We explicitly verified that, indeed,

$$\mathcal{F} = E \equiv E_{\text{vacuum}}(m\ell) .$$

The latter is identified with the vacuum energy of $\mathcal{N} = 2^*$ gauge theory on $S^3$.

- In a limit when all the (non-)normalizable coefficients vanish we recover the vacuum energy of the $\mathcal{N} = 4$ SYM (3.1).

- It is easy to extend discussion for general $SO(4)$-invariant non-equilibrium states of $\mathcal{N} = 2^*$ gauge theory — the final answer is as (3.29), except with $\{\rho_{1,0}, c_{2,0}, a_{2,0}\}$ now being functions of time. Note that

$$\frac{d\mathcal{E}}{dt} \propto \frac{d}{dt} \left( 4 \left\{ 16\rho_{1,1}\rho_{1,0}(t) + \frac{2}{3}c_{2,0}(t) + 8\rho_{1,0}^2(t) - a_{2,0}(t) \right\} \right) = 0 ,$$

according to (3.12). That is, the boundary gauge theory energy conservation is enforced by the bulk momentum constraint (3.6).

### 3.3 Vacuum states for $m\ell \ll 1$

In preparation to the full numerical solution of (3.22), we discuss here its perturbative solution for $\rho_{1,1} \ll 1$. We introduce

$$c = \cosh(2\lambda\chi_1(y) + \mathcal{O}(\lambda^3)) , \quad \rho = e^{\lambda^2\alpha_2(y) + \mathcal{O}(\lambda^4)} ,$$

$$a = 1 + \lambda^2a_2(y) + \mathcal{O}(\lambda^4) , \quad d = \lambda^2d_2(y) + \mathcal{O}(\lambda^2) ,$$

where $\lambda$ is a small parameter. Substituting (3.32) into (3.22) we find

$$0 = \chi''_1 - \frac{1+y}{y(1-y)}\chi'_1 + \frac{3}{4y^2(1-y)}\chi_1 ,$$

$$0 = \alpha''_2 - \frac{1+y}{y(1-y)}\alpha'_2 + \frac{1}{y^2(1-y)}\alpha_2 ,$$

$$0 = a'_2 - \frac{2-y}{y(1-y)}a_2 - \frac{8}{3}y(1-y)(\chi'_1)^2 + \frac{2}{y}(\chi_1)^2 ,$$

$$0 = d'_2 - \frac{8}{3}y(1-y)(\chi'_1)^2 .$$

Solutions to (3.33) must satisfy boundary conditions corresponding to (3.9) and (3.23). We can solve equation for $\alpha_2$ analytically,

$$\alpha_2 = \rho_{1,1,(2)} \frac{y\ln y}{1-y} ,$$
where $\rho_{1,1,(2)}$ is the non-normalizable integration coefficient. The remaining equations in (3.33) are solved with “shooting method” developed in [32]. In particular, given the asymptotic expansions in the UV, $y \to 0_+$,

\[
\begin{align*}
\chi_1 &= y^{1/2} \left( 1 + y \left( \chi_{1,0,(1)} + \frac{1}{4} \ln y \right) + O(y^2 \ln y) \right), \\
a_2 &= \frac{4}{3} y + y^2 \left( a_{2,0,(2)} + \frac{4}{3} \ln y \right) + O(y^3 \ln^2 y), \\
d_2 &= \frac{2}{3} y + y^2 \left( -\frac{1}{4} + 2 \chi_{1,0,(1)} + \frac{1}{2} \ln y \right) + O(y^3 \ln^2 y),
\end{align*}
\]  

and in the IR, $z \to 0_+$,

\[
\begin{align*}
\chi_1 &= \chi^h_{0,(1)} \left( 1 - \frac{3}{8} z + O(z^2) \right), \\
a_2 &= (\chi^h_{0,(1)})^2 \left( z - \frac{5}{8} z^2 + O(z^3) \right), \\
d_2 &= d^h_{0,(2)} - \frac{3}{16} (\chi^h_{0,(1)})^2 z^2 + O(z^3),
\end{align*}
\]  

we find numerically,

\[
\begin{array}{|c|c|c|c|}
\hline
\chi_{1,0,(1)} & a_{2,0,(2)} & \chi^h_{0,(1)} & d^h_{0,(2)} \\
0.0568528 & -0.363452 & 0.785398 & 0.199266 \\
\hline
\end{array}
\]  

(3.37)

To compare with the full numerical solution, we identify, to order $O(\lambda^2)$,

\[
\begin{align*}
\rho_{1,1} &= \rho_{1,1,(2)} \lambda^2, & c_{1,0} &= 2 \lambda^2, & \rho_{1,0} &= 0, & c_{2,0} &= 4 \chi_{1,0,(1)} \lambda^2, \\
a_{2,0} &= a_{2,0,(2)} \lambda^2, & \rho^h_0 &= 1 - \rho_{1,1,(2)} \lambda^2, & c^h_0 &= 1 + 2 (\chi^h_{0,(1)})^2 \lambda^2, & d^h_0 &= d^h_{0,(2)} \lambda^2.
\end{align*}
\]  

(3.38)

Note that $\mathcal{N} = 2$ supersymmetry in the UV at $O(\lambda^2)$ leads to (see (3.14))

\[
\rho_{1,1,(2)} = \frac{1}{3}.
\]  

(3.39)

From (3.29),

\[
\epsilon \equiv \frac{E_{\text{vacuum}}}{E_{\text{N=4}}} = 1 + \left( \frac{32}{3} \chi_{1,0,(1)} - 4 a_{2,0,(2)} - \frac{8}{3} (1 + 2 \ln 2) \right) \lambda^2 + O(\lambda^4)
\]  

\[
\begin{align*}
&= 1 + \left( \frac{8}{3} \chi_{1,0,(1)} - a_{2,0,(2)} - \frac{2}{3} (1 + 2 \ln 2) \right) (m \ell)^2 + O((m \ell)^4).
\end{align*}
\]  

(3.40)
3.4 Gravitational solution and $E_{\text{vacuum}}$ for general $m\ell$

Using the shooting method of [32], we solve (3.22) and determine the normalizable coefficients (3.24) as a function of $m\ell \equiv (12\rho_{1,1})^{1/2}$. The results of the computations for small values of $\rho_{1,1}$ are collected for numerical test in figure 2. The solid curves are obtained from numerical solution of full nonlinear equations (3.22), and the dashed lines represent perturbative prediction (3.38) with (3.37).

In full nonlinear numerical analysis we constructed vacua for $0 < m\ell \lesssim 8.5$. The vacuum energy of the $\mathcal{N} = 2^*$ gauge theory on $S^3$ relative to $\mathcal{N} = 4$ SYM Casimir
energy is given by

\[ \epsilon \equiv \frac{E_{\text{vacuum}}(m\ell)}{E_{\text{vacuum}}^{\mathcal{N}=4}} = 1 + \frac{1}{4} \left( \frac{m\ell}{\rho_1} \right)^4 - \frac{2}{9} \left( 1 + 2 \ln 2 \right) (m\ell)^2 \]

\[ + \left\{ 32\rho_1^2 + \frac{16}{3} (m\ell)^2 \rho_{1,0} + \frac{8}{3} c_{2,0} - 4a_{2,0} \right\}. \]  

(3.41)

It is presented in figure 3. The vertical red line indicates the mass scale \( m_0 \ell \),

\[ \epsilon(m_0\ell) = 0 \implies m_0\ell \approx 0.87031, \]  

(3.42)

at which the vacuum energy of the \( \mathcal{N} = 2^* \) gauge theory vanishes and becomes negative

for even larger value of \( m\ell \).

4 Stability of \( \mathcal{N} = 2^* \) vacuum states within BEFP

In the previous section we constructed gravitational solutions within PW effective

action, identified as vacua of the \( \mathcal{N} = 2^* \) gauge theory on \( S^3 \). While the complete

stability analysis of these solutions is beyond the scope of this paper, here we would

like to analyze their stability within BEFP effective action.

Effective action describing the fluctuations of an arbitrary PW static solution within
BEFP has been constructed in [28],
\[
\delta \mathcal{L} \equiv \mathcal{L}_{BEFP} - \mathcal{L}_{PW} + \mathcal{O}(X^4) \equiv \delta \mathcal{L}_2 + \delta \mathcal{L}_V ,
\]
\[
\delta \mathcal{L}_2 = -(1 + c)^2 (\partial X_2)^2 - \frac{1 + c}{4} \left( (c^2 + 1) \rho_6^{4/3} - 4(1 + c) \rho_6^{1/3} + \frac{4(\partial c)^2}{c^2 - 1} \right) (X_2)^2 ,
\]
\[
\delta \mathcal{L}_V = -(1 + c)^2 (\partial \bar{X}_V)^2 - \frac{1 + c}{4} \left( (c^2 - 1) \rho_6^{4/3} - 4(1 + c) \rho_6^{1/3} + \frac{4(\partial c)^2}{c^2 - 1} \right) (\bar{X}_V)^2 ,
\]
(4.1)

where \( \rho_6 = \rho^6 \) and \( \bar{X}_V = (X_3, X_4, X_5) \) (see section 2 for more details). Note that \( \delta \mathcal{L} \) is \( SU(2)_V \) invariant; as a result it is enough to consider a spectrum of only one of \( \bar{X}_V \) components. In what follows we choose the latter to be \( X_3 \).

Introducing
\[
X_2 = e^{-i\omega t} F_2(y) \Omega_s(S^3) , \quad X_3(t, y) = e^{-i\omega t} F_3(y) \Omega_s(S^3) ,
\]
(4.2)

where \( \Omega_s(S^3) \) are \( S^3 \) Laplace-Beltrami operator eigenfunctions with eigenvalues \( s = l(l + 2) \) for integer \( l \),
\[
\Delta_{S^3} \Omega_s(S^3) = -s \Omega_s(S^3) = -l(l + 2) \Omega_s(S^3) ,
\]
(4.3)

we find from (4.1) the following equations of motion
\[
0 = F_2'' + F_2' \left( \frac{2cc'}{c + 1} + \frac{(c^2 - 1)\rho^8}{6ay} - \frac{4c\rho^2}{3ay} + \frac{2y - 1}{y(y - 1)} + \frac{1}{a(y - 1)} - \frac{2}{3a\rho^4y} \right) + \frac{F_2}{4y(1 - y)a} \left( \frac{e^{2d} \omega^2}{a} - \frac{s}{1 - y} \right) + F_2 \left( \frac{(c')^2}{(1 - c^2)(c + 1)} + \frac{\rho^2(\rho^6 c - 4)}{4ay^2(y - 1)} \right) ,
\]
(4.4)

\[
0 = F_3'' + F_3' \left( \frac{2cc'}{c + 1} + \frac{(c^2 - 1)\rho^8}{6ay} - \frac{4c\rho^2}{3ay} + \frac{2y - 1}{y(y - 1)} + \frac{1}{a(y - 1)} - \frac{2}{3a\rho^4y} \right) + \frac{F_3}{4y(1 - y)a} \left( \frac{e^{2d} \omega^2}{a} - \frac{s}{1 - y} \right) + F_3 \left( \frac{(c')^2}{(1 - c^2)(c + 1)} + \frac{\rho^2(\rho^6 (c - 1) - 4)}{4ay^2(y - 1)} \right) .
\]
(4.5)

The radial wavefunctions \( F_{2,3} \) must be regular at the origin, i.e., \( z \to 0_+ \),
\[
F_2 = z^{l/2} f_2^h (1 + \mathcal{O}(z)) , \quad F_3 = z^{l/2} f_3^h (1 + \mathcal{O}(z)) ,
\]
(4.6)

and normalizable as \( y \to 0_+ \),
\[
F_2 = y^{3/2} \left( 1 + y \left( \frac{s}{8} - \frac{1}{2} c_{1,0} + \frac{9 - \omega^2}{8} \right) + \mathcal{O}(y^2 \ln y) \right) ,
\]
\[
F_3 = y \left( 1 + y \left( \frac{s}{4} + \frac{4 - \omega^2}{4} + 4\rho_{1,1} - 2\rho_{1,0} - \frac{1}{6} c_{1,0} - 2\rho_{1,1} \ln y \right) + \mathcal{O}(y^2 \ln y) \right) .
\]
(4.7)
Figure 4: Low energy states in the spectrum of BEFP fluctuations about PW vacua: \(\{n, l\} = \{(0, 0); (0, 1); (1, 0)\}\) (blue, red, green). See section 4.

Note that we set the normalizable coefficient of \(F_2, 3\) in the UV to one.

When both scalars of the PW flow are set to zero, (4.4)-(4.7) corresponds to fluctuations of gravitational modes dual to dimension-3 (for \(F_2\)) and dimension-2 (for \(F_3\)) operators of the \(\mathcal{N} = 4\) SYM on \(S^3\). In this case the equations can be solved analytically. We find,

\[
F^{SYM}_{2,\{n,l\}} = y^{3/2}(1 - y)^{l/2} {}_2F_1\left(-n, 3 + n + l; l + 2; 1 - y\right),
\]

(4.8)

\[
\omega^{SYM}_{2,\{n,l\}} = 3 + 2n + l,
\]

\[
F^{SYM}_{3,\{n,l\}} = y(1 - y)^{l/2} {}_2F_1\left(-n, 2 + n + l; l + 2; 1 - y\right),
\]

(4.9)

\[
\omega^{SYM}_{3,\{n,l\}} = 2 + 2n + l,
\]

where \(\{n, l\}\) are non-negative integers. For supersymmetric PW flows (3.14) we have to resort to numerics. The results of the numerical analysis are presented in figure 4. We look at the states with \(\{n, l\} = \{(0, 0); (0, 1); (1, 0)\}\) for both \(F_2\) and \(F_3\) radial functions. Over the range of parameters discussed, the embedding of PW flows within BEFP effective action is stable.

5 Black hole spectrum in PW effective action

We begin with the metric ansatz and the boundary conditions representing regular Schwarzschild black hole solutions in PW effective action with the \(S^3\) horizon. We explain how the normalizable coefficients of the gravitational solution encode the thermodynamic properties of the black holes: the temperature \(T_{BH}\), the energy \(E_{BH}\), the
entropy $S_{BH}$ and the free energy $F_{BH}$. We define the size $\ell_{BH}$ of a black hole as in (1.7). We compute excitation energy $\Delta(\ell_{BH}/L, (m\ell))$,

$$
\Delta(\ell_{BH}/L, (m\ell)) = \frac{E_{BH}(\ell_{BH}/L, m\ell) - E_{vacuum}(m\ell)}{E_{vacuum}^{N=4}},
$$

(5.1)
as a function of $\ell_{BH}/L$, but for select values of $m\ell$:

- perturbatively in $m\ell$, to order $O((m\ell)^2)$;
- for $\rho_{1,1} = \frac{1}{12}(m\ell)^2 = \{1, 1.5, 2, \ldots, 5.5, 5.8\}$ (the last value corresponds to the largest value of $m\ell$ for which we computed $E_{vacuum}$);

and present a strong numerical evidence that

$$
\lim_{\ell_{BH}/L \to 0} \Delta(\ell_{BH}/L, (m\ell)) = 0.
$$

(5.2)

Thus, we conclude that there is no gap in the spectrum of black holes in PW geometry; correspondingly, there is no gap in $SO(4)$-invariant equilibrium states of the $N = 2^*$ gauge theory on $S^3$ in the planar limit and for large 't Hooft coupling, as there is no energy gap for generic $SO(4)$-invariant excitations in this theory.

### 5.1 Metric ansatz and the boundary conditions for black holes in PW

Recall that the vacuum solutions of section 3 were obtained within metric ansatz (3.2),

$$
ds_5^2 \bigg|_{vacuum} = \frac{4}{\cos^2 x} \left(-ae^{-2d}(dt)^2 + \frac{(dx)^2}{a} + \sin^2 x(d\Omega_3)^2\right)
= \frac{4}{y} \left(-ae^{-2d}(dt)^2 + \frac{(dy)^2}{4y(1 - y)a} + (1 - y)(d\Omega_3)^2\right),
$$

(5.3)

where in the second line we recalled the radial coordinate $y = \cos^2 x$, $y \in [0, 1]$. Regularity at the origin ($y \to 1_-$) required that the metric functions $a$ and $d$ remain finite and non-zero. Notice that the three-sphere shrinks to zero size in this limit.

In close analogy to (5.3), to describe regular horizon black holes, we reparameterize the radial coordinate $y \to y_hy$, with a constant $0 < y_h < 1$, while keeping $y \in [0, 1]$. We further require that $a$ has a simple zero and $d$ remains finite as $y \to 1_-$:

$$
ds_5^2 \bigg|_{BH} = \frac{4}{y_hy} \left(-ae^{-2d}(dt)^2 + \frac{y_h(dy)^2}{4y(1 - yhy)a} + (1 - yy_h)(d\Omega_3)^2\right),
$$

(5.4)

$$
0 < y_h < 1, \quad y \in [0, 1], \quad \lim_{y \to 1_-} a = 0, \quad \lim_{y \to 1_-} a' = \text{finite} \neq 0, \quad \lim_{y \to 1_-} d = \text{finite}.
$$
Given (5.4),

\[
A_{\text{horizon}} = 16\pi^2 \left(1 - y_h^2\right)^{3/2} y_h^{3/2} \quad \Rightarrow \quad \frac{\ell_{\text{BH}}}{L} \equiv \frac{A_{\text{horizon}}^{1/3}}{L} = (2\pi^2)^{1/3} \left(1 - y_h^2\right)^{1/2} y_h^{1/2}. \tag{5.5}
\]

The equations of motion describing black holes (5.4) can be obtained from (3.22) with the simple change of variables \(y \rightarrow yy_h\),

\[
0 = c'' - \frac{c(c')^2}{c^2 - 1} + c' \left(\frac{(c^2 - 1)\rho^8}{6ay} - \frac{4c\rho^2}{3ay} + \frac{a(2yy_h - 1) + yy_h}{ya(yy_h - 1)} - \frac{2}{3ya\rho^4}\right) - \frac{\rho^2(c^2 - 1)(\rho^8c - 4)}{4(1 - yy_h)y^2a},
\]

\[
0 = \rho'' - \frac{(\rho')^2}{\rho} + \rho' \left(\frac{(c^2 - 1)\rho^8}{6ay} - \frac{4c\rho^2}{3ay} + \frac{a(2yy_h - 1) + yy_h}{ya(yy_h - 1)} - \frac{2}{3ya\rho^4}\right) - \frac{(c^2 - 1)\rho^9}{12(1 - yy_h)y^2a} - \frac{1 - \rho^8c}{6\rho^3y^2a(1 - yy_h)},
\]

\[
0 = d' - \frac{2y(1 - yy_h)(c')^2}{3(c^2 - 1)} - \frac{8(1 - yy_h)y(\rho')^2}{\rho^2},
\]

\[
0 = a' - (y - y^2yy_h)\rho \left(\frac{8(\rho')^2}{\rho^2} + \frac{2(c')^2}{3(c^2 - 1)}\right) + \frac{(yy_h - 2)a + yy_h}{y(1 - yy_h)} - \frac{(c^2 - 1)\rho^8 - 8\rho^2c}{6y} + \frac{2}{3ya\rho^4}. \tag{5.6}
\]

\[\text{\underline{21}}\text{We used the last two equations to algebraically eliminate } a' \text{ and } d' \text{ from the first two.}\]
The boundary conditions in the UV, i.e., $y \to 0_+$, specify the asymptotic expansion

$$a = 1 + y \left( \frac{2}{3} \hat{c}_{1,0} + y^2 \left( \hat{a}_{2,0} + \left( \frac{2}{3} \hat{c}_{1,0} + y_h \right) + 8 \hat{\rho}_{1,1} + 16 \hat{\rho}_{1,1} \hat{\rho}_{1,0} \right) \ln y \\
+ 8 \hat{\rho}_{1,1}^2 \ln^2 y \right) + \mathcal{O}(y^3 \ln^3 y),$$

$$d = y \left( \frac{1}{3} \hat{c}_{1,0} + y^2 \left( \frac{1}{2} \hat{c}_{2,0} - \frac{1}{36} \hat{c}_{1,0}^2 + 4 \hat{\rho}_{1,0}^2 - \frac{1}{8} \hat{c}_{1,0} y_h + 2 \hat{\rho}_{1,1} + 4 \hat{\rho}_{1,0} \hat{\rho}_{1,1} \\
+ \left( \frac{1}{4} \hat{c}_{1,0} y_h + \frac{1}{3} \hat{c}_{1,0} y_h \right) + 8 \hat{\rho}_{1,0} \hat{\rho}_{1,1} \right) \ln y + 4 \hat{\rho}_{1,1}^2 \ln^2 y \right) + \mathcal{O}(y^3 \ln^3 y),$$

$$\rho = 1 + y \left( \hat{\rho}_{1,0} + \hat{\rho}_{1,1} \ln y \right) + y^2 \left( \frac{1}{12} \hat{c}_{1,0} + \hat{\rho}_{1,0} y_h - 3 \hat{\rho}_{1,1} \hat{c}_{1,0} + 6 \hat{\rho}_{1,1}^2 \\
- 4 \hat{\rho}_{1,0} \hat{\rho}_{1,1} + \frac{4}{3} \hat{c}_{1,0} \hat{\rho}_{1,0} + 3 \hat{\rho}_{1,1} y_h + \left( \frac{4}{3} \hat{\rho}_{1,1} \hat{c}_{1,0} + \hat{\rho}_{1,1} y_h - 4 \hat{\rho}_{1,1} \right) \\
+ 3 \hat{\rho}_{1,0} \hat{\rho}_{1,1} \right) \ln y + \frac{3}{2} \hat{\rho}_{1,1}^2 \ln^2 y \right) + \mathcal{O}(y^3 \ln^3 y),$$

$$c = 1 + y \left( \hat{c}_{1,0} + y^2 \left( \hat{c}_{2,0} + \left( \frac{1}{2} \hat{c}_{1,0} y_h + \frac{2}{3} \hat{c}_{1,0} \right) \ln y \right) + \mathcal{O}(y^3 \ln^2 y).$$

In (5.7) the non-normalizable coefficients $\hat{\rho}_{1,1}$ and $\hat{c}_{1,0}$ are related to corresponding coefficients of the vacuum solution as

$$\hat{\rho}_{1,1} = y_h \rho_{1,1}, \quad \hat{c}_{1,0} = y_h c_{1,0},$$

(5.8)

to be further matched with the mass parameters $\{m_b, m_f\}$ of the dual gauge theory as in (3.15). The rest of the coefficients in (5.7) are normalizable. The asymptotic expansion in the IR, i.e., as $z = (1 - y) \to 0_+$ is different from the one in (3.23) — here it reflects the presence of a regular horizon (see (5.4)),

$$a = \frac{z}{6} \left( 1 - (\hat{c}_{0})^2 \right) (\hat{\rho}_0)^8 + 8 \hat{c}_{0} (\hat{\rho}_0)^2 + \frac{4}{(\hat{\rho}_0)^4} + 6 y_h + \mathcal{O}(z^2),$$

$$d = \hat{d}_0 + \mathcal{O}(z),$$

(5.9)

$$\rho = \hat{\rho}_0 + \mathcal{O}(z),$$

$$c = \hat{c}_0 + \mathcal{O}(z).$$

The full set of the non-normalizable coefficients is

$$\{ \hat{a}_{2,0}, \hat{\rho}_{1,0}, \hat{c}_{2,0}, \hat{\rho}_0, \hat{c}_0, \hat{d}_0 \}.$$  

(5.10)

Note that we have the correct number of non-normalizable coefficients to uniquely specify a solution of two second-order and two first-order ODEs given a choice of (5.8).
5.1.1 Perturbative black holes solutions

As in section 3.3, we can construct solutions to (5.6)-(5.9) perturbatively in \(m\ell\) to order \(\mathcal{O}((m\ell)^2)\).

We introduce
\[
c = \cosh(2\lambda \hat{\chi}_1(y) + \mathcal{O}(\lambda^3)), \quad \rho = e^{\lambda^2 \hat{a}_2(y) + \mathcal{O}(\lambda^4)},
\]
\[
a = \frac{(1 - y)(1 + y(1 - y_h))}{1 - y y_h} + \lambda^2 \hat{a}_2(y) + \mathcal{O}(\lambda^4), \quad d = \lambda^2 \hat{d}_2(y) + \mathcal{O}(\lambda^2),
\]
where \(\lambda\) is a small parameter. Substituting (3.32) into (3.22) we find

\[
0 = \hat{\chi}_1'' - \frac{\hat{\chi}_1'}{y(1 - y)} \left( 1 + y \left( \hat{\chi}_{1,0,(1)} + \frac{y_h}{4} \ln y + \mathcal{O}(y^2 \ln y) \right) \right) + \frac{3 \hat{\chi}_1}{4y^2(1 - y)(1 + y(1 - y_h))},
\]
\[
0 = \hat{\alpha}_2'' - \frac{\hat{\alpha}_2'}{y(1 - y)} \left( 1 + y \left( \hat{\alpha}_{1,0,(2)} + \ln y + \mathcal{O}(y^2 \ln y) \right) \right) + \frac{\hat{\alpha}_2}{y^2(1 - y)(1 + y(1 - y_h))},
\]
\[
0 = \hat{a}_2' - \frac{2 - y y_h}{y(1 - y y_h)} \hat{a}_2 - \frac{8}{3} y(1 - y)(1 + y(1 - y_h)) (\hat{\chi}_1')^2 + \frac{2}{y} (\hat{\chi}_1)^2,
\]
\[
0 = \hat{d}_2' - \frac{8}{3} y(1 - y y_h) (\hat{\chi}_1')^2.
\]

For the asymptotic expansions we have:

- as \(y \to 0_+\),
  \[
  \hat{\chi}_1 = y^{1/2} \left( 1 + y \left( \hat{\chi}_{1,0,(1)} + \frac{y_h}{4} \ln y \right) + \mathcal{O}(y^2 \ln y) \right),
  \]
  \[
  \hat{\alpha}_2 = \hat{\rho}_{1,1,(2)} \left( \hat{\alpha}_{1,0,(2)} + \ln y \right) y + \mathcal{O}(y^2 \ln y),
  \]
  \[
  \hat{a}_2 = \frac{4}{3} y^2 + y^2 \left( \hat{a}_{2,0,(2)} + \frac{4y_h}{3} \ln y \right) + \mathcal{O}(y^3 \ln^2 y),
  \]
  \[
  \hat{d}_2 = \frac{2}{3} y^2 + y^2 \left( - \frac{y_h}{4} + 2 \hat{\chi}_{1,0,(1)} + \frac{y_h}{2} \ln y \right) + \mathcal{O}(y^3 \ln^2 y),
  \]

- as \(z \to 0_+\)
  \[
  \hat{\chi}_1 = \hat{\chi}_{0,(1)} + \frac{3}{4(2 - y_h)} z + \mathcal{O}(z^2),
  \]
  \[
  \hat{\alpha}_2 = \hat{\rho}_{1,1,(2)} \left( \hat{\alpha}_{0,(2)} + \frac{1}{(2 - y_h)^2} z + \mathcal{O}(z^2) \right),
  \]
  \[
  \hat{a}_2 = 2(\hat{\chi}_{0,(1)})^2 z + \mathcal{O}(z^2),
  \]
  \[
  \hat{d}_2 = \hat{d}_{0,(2)} = \frac{3(1 - y_h)}{2(2 - y_h)^2(\hat{\chi}_{0,(1)})^2} z + \mathcal{O}(z^2).
  \]
Equations (5.12)-(5.13) have to be solved numerically for different values of \( y_h \).

To compare with the full numerical solution, we identify, to order \( \mathcal{O}(\lambda^2) \),

\[
\hat{\rho}_{1,1} = \hat{\rho}_{1,1,(2)} \lambda^2, \quad \hat{c}_{1,0} = 2 \lambda^2, \quad \hat{\rho}_{1,0} = \hat{\rho}_{1,1,(2)} \hat{\alpha}_{1,0,(2)} \lambda^2, \quad \hat{c}_{2,0} = 4 \hat{\chi}_{1,0,(1)} \lambda^2,
\]

\[
\hat{a}_{2,0} = y_h - 1 + \hat{a}_{2,0,(2)} \lambda^2, \quad \hat{\rho}_0 = 1 + \hat{\rho}_{1,1,(2)} \hat{\alpha}_{0,(2)} \lambda^2,
\]

\[
\hat{c}_0^h = 1 + 2(\hat{\chi}_{0,(1)})^2 \lambda^2, \quad \hat{d}_0 = \hat{d}_{0,(2)} \lambda^2.
\]

(5.15)

Note that \( \mathcal{N} = 2 \) supersymmetry in the UV at \( \mathcal{O}(\lambda^2) \) leads to (see (3.14))

\[
\hat{\rho}_{1,1,(2)} = \frac{1}{3}.
\]

(5.16)

### 5.2 Thermodynamic properties of black holes in PW

Requiring that there is no conical singularity in the analytical continuation \( t \to it_E \) of the metric (5.4) as \( y \to 1_\pm \) we compute the Hawking temperature \( T_{BH} \) of the black hole using (5.9),

\[
T_{BH} = \frac{e^{-\hat{d}_0}}{12 \pi y_h^{1/2} (1 - y_h)^{1/2}} \left( (1 - y_h)(1 - (\hat{c}_0^h)^2)(\hat{\rho}_0^h)^8 + 8 \hat{c}_0^h (1 - y_h)(\hat{\rho}_0^h)^2 + 6 y_h \right)
\]

\[
+ \frac{4(1 - y_h)}{(\hat{\rho}_0^h)^4} \right) .
\]

(5.17)

The Bekenstein-Hawking entropy of the black hole is given by

\[
S_{BH} = \frac{A_{horizon}}{4 G_5} = \frac{4 \pi^2}{G_5} \frac{(1 - y_h)^{3/2}}{y_h^{3/2}}.
\]

(5.18)

The free energy \( F_{BH} \) can be computed following holographic renormalization procedure discussed in section 3.2. We find

\[
F_{BH} = \frac{3 \pi}{4 G_5} \left( 1 + \frac{\hat{c}_{1,0}^2}{y_h^2} \left( \frac{4}{9} \ln 2 + \frac{16}{9} \ln y_h \right) + \frac{\hat{c}_{1,0}}{y_h} \left( -\frac{4}{3} \ln 2 + \frac{4}{3} \ln y_h \right) + \frac{32 \hat{\rho}_{1,1}^2}{y_h^2} (2 \ln 2 - \ln y_h) + \frac{1}{y_h^2} \right) \left( 64 \hat{\rho}_{1,1} \hat{\rho}_{1,0} + \frac{8}{3} \hat{c}_{2,0} + 32 \hat{\rho}_{1,0}^2 - 4 \hat{a}_{2,0} \right)
\]

\[
- \frac{(1 - y_h) \pi e^{-\hat{d}_0}}{3 y_h^2 G_5} \left( (1 - y_h)(1 - (\hat{c}_0^h)^2)(\hat{\rho}_0^h)^8 + 8 \hat{c}_0^h (1 - y_h)(\hat{\rho}_0^h)^2 + 6 y_h + \frac{4(1 - y_h)}{(\hat{\rho}_0^h)^4} \right).
\]

(5.19)
The contribution in the last line in (5.19) comes from the lower limit of integration of the bulk contribution to the regularized free energy, (3.25); it equals precisely to \((-S_{BH}T_{BH})\). Computing the holographic stress-energy tensor, as described in [26] we find

\[
E_{BH} = \frac{3\pi}{4G_5} \left( 1 + \frac{\hat{c}_{1,0}}{y_h^2} \left( \frac{4}{9} - \frac{8}{9} \ln 2 + \frac{8}{9} \ln y_h \right) + \frac{\hat{c}_{1,0}}{y_h} \left( -\frac{4}{3} - \frac{8}{3} \ln 2 + \frac{4}{3} \ln y_h \right) \right)
\]

\[
+ 32 \hat{\rho}_{1,1} \left( 2 \ln 2 - \ln y_h \right) + \frac{1}{y_h^2} \left\{ 64 \hat{\rho}_{1,1} \hat{\rho}_{1,0} + \frac{8}{3} \hat{c}_{2,0} + 32 \hat{\rho}_{1,1}^2 - 4 \hat{a}_{2,0} \right\} \right) \right)
\]

\[
= \frac{3N^2}{16\ell} \left( 1 + \frac{(m\ell)^4}{9} - \frac{2}{3} (1 + 2 \ln 2 - \ln y_h)(m\ell)^2 + \frac{1}{y_h^2} \left\{ 32 \hat{\rho}_{1,1} + \frac{16}{3} (m\ell)^2 y_h \hat{\rho}_{1,0} + \frac{8}{3} \hat{c}_{2,0} - 4 \hat{a}_{2,0} \right\} \right),
\]

where in the last line we expressed the energy in terms of the dual gauge theory variables using (5.8) and (3.13). Notice that the basic thermodynamic relation,

\[
F_{BH} = E_{BH} - S_{BH}T_{BH},
\]

is satisfied automatically.

Using (5.15), from (5.20) we have

\[
\frac{E_{BH}}{E_{N=4}^{\text{vacuum}}} = 1 + \frac{4(1 - y_h)}{y_h^2} \left( \frac{8}{3y_h} \hat{\chi}_{1,0,(1)} - \frac{1}{y_h} \hat{a}_{2,0,(2)} - \frac{2}{3} (1 + 2 \ln 2 - \ln y_h) \right) (m\ell)^2 + O((m\ell)^4).
\]

\[
(5.22)
\]

5.3 \( \Delta(\ell_{BH}/L, (m\ell)) \)

We are now ready to present results for \( \Delta(\ell_{BH}/L, (m\ell)) \) as defined by (5.1).

To order \( O((m\ell)^2) \), using (3.40) and (5.22), we find

\[
\Delta = \frac{4(1 - y_h)}{y_h^2} + \Delta_2 (m\ell)^2 + O((m\ell)^4),
\]

\[
\Delta_2 = \Delta_2(y_h) = \frac{8}{3} \left( \frac{\hat{\chi}_{1,0,(1)}}{y_h} - \chi_{1,0,(1)} \right) - \left( \frac{\hat{a}_{2,0,(2)}}{y_h} - a_{2,0,(2)} \right) + \frac{2}{3} \ln y_h.
\]

Results of numerical computations of \( \Delta_2 \) are presented in figure 5. A solid line represents the data points, and the red dotted line is the best quadratic fit using the first 10% of data points:

\[
\Delta_2 \bigg|_{fit} = -0.0269118 \left( \frac{\ell_{BH}}{L} \right)^2.
\]

(5.24)
Our numerical results present a strong evidence that

\[ \lim_{\ell_{BH}/L \to 0} \Delta_2 = 0, \]

(5.25)
as a result, we see that \( \Delta \) vanishes in this limit to order \( O((m\ell)^2) \).

Using (3.29) and (5.20) we compute \( \Delta \) for \( \rho_{1,1} = \frac{1}{12}(m\ell)^2 = \{1, 1.5, 2, \ldots, 5, 5.5, 5.8\} \). The results are presented in the left panel of figure 6 (the top-to-bottom blue curves correspond to \( \rho_{1,1} \) variation \( 1 \rightarrow 5.8 \)). The green curve represents \( \Delta(m\ell = 0) \):

\[ \Delta(m\ell = 0) = \frac{2^{4/3}}{\pi^{4/3}} \left( \frac{\ell_{BH}}{L} \right)^2 + \frac{2^{2/3}}{\pi^{8/3}} \left( \frac{\ell_{BH}}{L} \right)^4. \]  

(5.26)
The right panel represents \( \Delta \) for the largest value of \( m\ell \) computed: \( m\ell = 8.34266 \), with the red dotted line indicating the best quadratic fit to the first 10% of data points:

\[ \Delta(m\ell = 8.34266) |_{fit} = 0.339765 \left( \frac{\ell_{BH}}{L} \right)^2. \]  

(5.27)
Note that for \( m\ell = 8.34266 \), \( \epsilon = -243.785 \), implying that for the smallest size black hole studied, \( \ell_{BH}/L = 0.085056 \),

\[ \frac{E_{BH} - E_{\text{vacuum}}}{E_{\text{vacuum}}} = 1.04285 \times 10^{-5}. \]  

(5.28)
We conclude that numerical results strongly suggest (5.2).

Figure 5: Solid line represents \( \Delta_2 \) as defined in (5.23). The dotted red line represents the best quadratic fit to the first 10% of data points, see (5.24).
Figure 6: Left panel: Black hole mass gap relative to $E_{\text{vacuum}}^{N=4}$, see (5.1), as a function of $\ell_{BH}/L$ for select values of $m\ell$. The green curve represents $\Delta(m\ell = 0)$. Right panel: $\Delta$ for the largest value of $m\ell$ computed, $m\ell = 8.34266$; the dotted red line represents the best quadratic fit to the first 10% of data points, see (5.27).

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