Gap Universality of Generalized Wigner and $\beta$-Ensembles

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Abstract

We consider generalized Wigner ensembles and general $\beta$-ensembles with analytic potentials for any $\beta \geq 1$. The recent universality results in particular assert that the local averages of consecutive eigenvalue gaps in the bulk of the spectrum are universal in the sense that they coincide with those of the corresponding Gaussian $\beta$-ensembles. In this article, we show that local averaging is not necessary for this result, i.e. we prove that the single gap distributions in the bulk are universal. In fact, with an additional step, our result can be extended to any potential $C^4(\mathbb{R})$.

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1 Introduction

The fundamental vision that random matrices can be used as basic models for large quantum systems was due to E. Wigner [58]. He conjectured that the eigenvalue gap distributions of large random matrices were universal ("Wigner surmise") in the sense that large quantum systems and random matrices share the same gap distribution functions. The subsequent works of Dyson, Gaudin and Mehta clarified many related issues regarding this assertion and a thorough understanding of the Gaussian ensembles has thus emerged (see the classical book of Mehta [45] for a summary). There are two main categories of random matrices: the invariant and the non-invariant ensembles. The universality conjecture, which is also known as the Wigner-Dyson-Gaudin-Mehta (WDGM) conjecture, asserts that for both ensembles the eigenvalue gap distributions are universal up to symmetry classes. For invariant ensembles, the joint distribution function of the eigenvalues can be expressed explicitly in terms of one dimensional particle systems with logarithmic interactions (i.e., log-gases) at an inverse temperature $\beta$. The values $\beta = 1, 2, 4$, correspond to the classical orthogonal, unitary and symplectic ensembles, respectively. Under various conditions on the external potential, the universality for the classical values $\beta = 1, 2, 4$ was proved, via analysis on the corresponding orthogonal polynomials, by Fokas-Its-Kitaev [36], Deift et. al. [14, 17, 18], Bleher-Its [6], Pastur-Shcherbina [48, 49] and in many consecutive works, see e.g. [15, 16, 44, 50, 57]. For nonclassical values of $\beta$ there is no matrix ensemble behind the model, except for the Gaussian cases [21] via tridiagonal matrices. One may still be interested in the local correlation functions of the log-gas as an interacting particle system. The orthogonal polynomial method is not applicable for nonclassical values of $\beta$ even for the Gaussian case. For certain special potentials and even integer $\beta$, however, there are still explicit formulas for correlation functions [37]. Furthermore, for general $\beta$ in the Gaussian case the local statistics were described very precisely with a different method by Valkó-Virág [55, 56]. The universality for general $\beta$-ensembles was established only very recently [7, 8] by a new method based on dynamical methods using Dirichlet form estimates from [26, 27]. This method is important for this article and we will discuss it in more details later on. All previous results achieved by this method, however, required in their statement to consider a local average of consecutive gaps. In the current paper we will prove universality of each single gap in the bulk.

Turning to the non-invariant ensembles, the most important class is the $N \times N$ Wigner matrices characterized by the independence of their entries. In general, there is no longer an explicit expression for the joint distribution function for the eigenvalues. However, there is a special class of ensembles, the Gaussian divisible ensembles, that interpolate between the general Wigner ensembles and the Gaussian ones. For these ensembles, at least in the special Hermitian case, there is still an explicit formula for the joint distribution of the eigenvalues based upon the Harish-Chandra-Itzykson-Zuber integral. This formula was first put into a mathematically useful form by Johansson [42] (see also the later work of Ben Arous-Péché [5]) to prove the universality of Gaussian divisible ensembles with a Gaussian component of size order one. In [24], the size of the Gaussian component needed for proving the universality was greatly reduced to $N^{-1/2+\epsilon}$. More importantly, the idea of approximating Wigner ensembles by Gaussian divisible ones was
first introduced and, after a perturbation argument, this resulted in the first proof of universality for Hermitian ensembles with general smooth distributions for matrix elements. The smoothness condition was later on removed in [51, 25].

In his seminal paper [22], Dyson observed that the eigenvalue distribution of Gaussian divisible ensembles is the same as the solution of a special system of stochastic differential equations, commonly known now as the Dyson Brownian motion, at a fixed time $t$. For short times, $t$ is comparable with the variance of the Gaussian component. He also conjectured that the time to “local equilibrium” of the Dyson Brownian motion is of order $1/N$, which is then equivalent to the universality of Gaussian divisible ensembles with a Gaussian component of order slightly larger than $N^{-1/2}$. Thus the work [24] can be viewed as proving Dyson’s conjecture for the Hermitian case. This method, however, completely tied with an explicit formula that is so far restricted to the Hermitian case.

A completely analytic approach to estimate the time to local equilibrium of the Dyson’s Brownian motion was initiated in [26] and further developed in [27, 31, 30], see [34] for a detailed account. In these papers, Dyson’s conjecture in full generality was proved [31] and universality was established for generalized Wigner ensembles for all symmetric classes. The idea of a dynamical approach in proving universality turns out to be a very powerful one. Dyson Brownian motion can be viewed as the natural gradient flow for Gaussian $\beta$ log-gases (we will often use the terminology $\beta$ log-gases for the $\beta$-ensembles to emphasize the logarithmic interaction). The gradient flow can be defined with respect to all $\beta$ log-gases, not just the Gaussian ones. Furthermore, one can consider gradient flows of local log-gases with fixed “good boundary conditions”. Here “local” refers to Gibbs measures on $N^a$, $0 < a < 1$, consecutive points of a log-gas with the locations of all other points fixed. By “good boundary conditions” we mean that these external points are rigid, i.e. their locations are close to their classical locations given by the limiting density of the original log-gas. Using this idea, we have proved the universality of general $\beta$-ensembles in [7, 8] for analytic potential.

The main conclusion of these works is that the local gap distributions of either the generalized Wigner ensembles (in all symmetry classes) or the general $\beta$-ensembles are universal in the bulk of the spectrum (see [33] for a recent review). The dynamical approach based on Dyson’s Brownian motion and related flows also provides a conceptual understanding for the origin of the universality. For technical reasons, however, these proofs apply to averages of consecutive gaps, i.e. cumulative statistics of $N^a$ consecutive gaps were proven to be universal. Averaging the statistics of the consecutive gaps is equivalent to averaging the energy parameter in the correlation functions. Thus, mathematically, the results were also formulated in terms of universality of the correlation functions with averaging in an energy window of size $N^{-1+\varepsilon}$.

The main goal of this paper is to remove the local averaging in the statistics of consecutive gaps in our general approach using Dyson Brownian motion for both invariant and non-invariant ensembles. We will show that the distribution of each single gap in the bulk is universal, which we will refer to as the single gap universality or simply the gap universality whenever there is no confusion. The single gap universality was proved for a special class of Hermitian Wigner matrices with the property that the first four moments of the matrix elements match those of the standard Gaussian random variable [52] and no other results have been known before. In particular, the single gap universality has not been proved even for the Gaussian orthogonal ensemble (GOE).

The gap distributions are closely related to the correlation functions which were often used to state the universality of random matrices. These two concepts are equivalent in a certain average sense. However, there is no rigorous relation between correlation functions at a fixed energy and single gap distributions. Thus our results on single gap statistics do not automatically imply the
universal of the correlation functions at a fixed energy which was rigorously proved only for Hermitian Wigner matrices [24, 51, 25, 34].

The removal of a local average in the universality results proved in [26, 29, 32] is a technical improvement in itself and its physical meaning is not especially profound. Our motivation for taking seriously this endeavor is due to that the single gap distribution may be closely related to the distribution of a single eigenvalue in the bulk of the spectrum [39] or at the edge [53, 54]. Since our approach does not rely on any explicit formula involving Gaussian matrices, some extension of this method may provide a way to understand the distribution of an individual eigenvalue of Wigner matrices. In fact, partly based on the method in this paper, the edge universality for the $\beta$-ensembles and generalized Wigner ensembles was established in [9].

The main new idea in this paper is an analysis of the Dyson Brownian motion via parabolic regularity using the De Giorgi-Nash-Moser idea. Since the Hamiltonians of the local log-gases are convex, the correlation functions can be re-expressed in terms of a time average of certain random walks in random environments. The connection between correlation functions of general log-concave measures and random walks in random environments was already pointed out in the work of Helffer and Sjöstrand [41] and Naddaf and Spencer [46]. This connection was used as an effective way to estimate correlation functions for several models in statistical physics, see, e.g. [3, 20, 38, 40, 19], as well as to remove convexity assumptions in gradient interface models [12, 13].

In this paper we observe that the single gap universality is a consequence of the Hölder regularity of the solutions to these random walk problems. Due to the logarithmic interaction, the random walks are long ranged and their rates may be singular. Furthermore, the random environments themselves depend on the gap distributions, which were exactly the problems we want to analyze! If we view these random walks as (discrete) parabolic equations with random coefficients, we find that they are of divergence form and are in the form of the equations studied in the fundamental paper by Caffarelli, Chan and Vasseur [11]. The main difficulty to apply [11] to gain regularity is that the jump rates in our settings are random and they do not satisfy the uniform upper and lower bounds required in [11]. In fact, in some space-time regime the jump rates can be much more singular than were allowed in [11]. For controlling the singularities of these coefficients, we prove an optimal level repulsion estimate for the local log-gases. With these estimates, we are able to extend the method of [11] to prove Hölder regularity for the solution to these random walks problems. This shows that the single gap distributions are universal for local log-gases with good boundary conditions, which is the key result of this paper.

For $\beta$-ensembles, it is known that the rigidity of the eigenvalues ensures that boundary conditions are good with high probability. Thus we can apply the local universality of single gap distribution to get the single gap universality of the $\beta$-ensembles. We remark, however, that the current result holds only for $\beta \geq 1$ in contrast to $\beta > 0$ in [7, 8], since the current proof heavily relies on the dynamics of the gradient flow of local log-gases. For non-invariant ensembles, a slightly longer argument using the local relaxation flow is needed to connect the local universality result with that for the original Wigner ensemble. This will be explained in Section 6.

In summary, we have recast the question of the single gap universality for random matrices, envisioned by Wigner in the sixties, into a problem concerning the regularity of a parabolic equation in divergence form studied by De Giorgi-Nash-Moser. Thanks to the insight of Dyson and the important progress by Caffarelli-Chan-Vasseur [11], we are able to establish the WDGM universality conjecture for each individual gap via De Giorgi-Nash-Moser’s idea. We now introduce our models rigorously and state the main results.
\section{Main results}

We will have two related results, one concerns the generalized Wigner ensembles, the other one the general $\beta$-ensembles. We first define the generalized Wigner ensembles. Let $H = (h_{ij})_{i,j=1}^{N}$ be an $N \times N$ hermitian or symmetric matrix where the matrix elements $h_{ij} = \overline{h}_{ji}$, $i \leq j$, are independent random variables given by a probability measure $\nu_{ij}$ with mean zero and variance $\sigma_{ij}^2 \geq 0$:

\begin{equation}
\mathbb{E} h_{ij} = 0, \quad \sigma_{ij}^2 := \mathbb{E}|h_{ij}|^2. \tag{2.1} \label{aver}
\end{equation}

The distribution $\nu_{ij}$ and its variance $\sigma_{ij}^2$ may depend on $N$, but we omit this fact in the notation. We also assume that the normalized matrix elements have a uniform subexponential decay,

\begin{equation}
P(|h_{ij}| > x\sigma_{ij}) \leq \theta_1 \exp(-x^\theta_2), \quad x > 0, \tag{2.2} \label{subexp}
\end{equation}

with some fixed constants $\theta_1, \theta_2 > 0$, uniformly in $N, i, j$. In fact, with minor modifications of the proof, an algebraic decay

\begin{equation}
P(|h_{ij}| > x\sigma_{ij}) \leq C_M x^{-M} \tag{2.2} \label{subexp}
\end{equation}

with a large enough $M$ is also sufficient.

\textbf{Definition 2.1 ([29])} The matrix ensemble $H$ defined above is called generalized Wigner matrix if the following assumptions hold on the variances of the matrix elements (2.1)

(A) For any $j$ fixed

\begin{equation}
\sum_{i=1}^{N} \sigma_{ij}^2 = 1. \tag{2.3} \label{sum}
\end{equation}

(B) There exist two positive constants, $C_{\inf}$ and $C_{\sup}$, independent of $N$ such that

\begin{equation}
\frac{C_{\inf}}{N} \leq \sigma_{ij}^2 \leq \frac{C_{\sup}}{N}. \tag{2.4} \label{1.3C}
\end{equation}

Let $\mathbb{P}$ and $\mathbb{E}$ denote the probability and the expectation with respect to this ensemble.

We will denote by $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ the eigenvalues of $H$. In the special case when $\sigma_{ij}^2 = 1/N$ and $h_{ij}$ is Gaussian, the joint probability distribution of the eigenvalues is given

\begin{equation}
\mu = \mu_{G}^{(N)}(d\lambda) = \frac{e^{-N^\beta \mathcal{H}(\lambda)}}{Z_\beta} d\lambda, \quad \mathcal{H}(\lambda) = \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{N} \sum_{i<j} \log |\lambda_j - \lambda_i|, \tag{2.5} \label{H}
\end{equation}

The value of $\beta$ depends on the symmetry class of the matrix; $\beta = 1$ for GOE, $\beta = 2$ for GUE and $\beta = 4$ for GSE. Here $Z_\beta$ is the normalization factor so that $\mu$ is a probability measure.

It is well known that the density or the one point correlation function of $\mu$ converges, as $N \to \infty$, to the Wigner semicircle law

\begin{equation}
\rho(x) := \frac{1}{2\pi} \sqrt{(4 - x^2)^+}. \tag{2.6} \label{scdef}
\end{equation}

We use the notation $\gamma_j$ for the $j$-th quantile of this density, i.e. $\gamma_j$ is defined by

\begin{equation}
\frac{j}{N} = \int_{-2}^{\gamma_j} \rho_G(x)dx. \tag{2.7} \label{defgamma}
\end{equation}
We now define a class of test functions. Fix an integer $n$. We say that $O = O_N : \mathbb{R}^n \to \mathbb{R}$, a possibly $N$-dependent sequence of differentiable functions, is an $n$-particle observable if

$$O_\infty := \sup_N \| O_N \|_\infty < \infty, \quad \text{supp} O_N \subset [-O_{\text{supp}}, O_{\text{supp}}]^n$$

with some finite $O_{\text{supp}}$, independent of $N$, but we allow $\| O_N' \|_\infty$ to grow with $N$. For any integers $A < B$ we also introduce the notation $[A,B] := \{A, A+1, \ldots, B\}$.

Our main result on the generalized Wigner matrices asserts that the local gap statistics in the bulk of the spectrum are universal for any general Wigner matrix, in particular they coincide with those of the Gaussian case.

**Theorem 2.2 (Gap universality for Wigner matrices)** Let $H$ be a generalized Wigner ensemble with subexponentially decaying matrix elements, (2.2). Fix positive numbers $\alpha, O_\infty, O_{\text{supp}}$ and an integer $n \in \mathbb{N}$. There exists an $\varepsilon > 0$ and $C > 0$, depending only on $\alpha, O_\infty$ and $O_{\text{supp}}$ such that for any $n$-particle observable $O = O_N$ satisfying (2.8) we have

$$\left| \mathbb{E} - \mathbb{E}' \right| O(N(\lambda_j - \lambda_{j+1}), N(\lambda_j - \lambda_{j+2}), \ldots, N(\lambda_j - \lambda_{j+n})) \leq CN^{-\varepsilon} \| O' \|_\infty$$

for any $j \in [\alpha N, (1 - \alpha)N]^n$ and for any sufficiently large $N \geq N_0$, where $N_0$ depends on all parameters of the model, as well as on $n, \alpha, O_\infty$ and $O_{\text{supp}}$.

More generally, for any $k, m \in [\alpha N, (1 - \alpha)N]^n$ we have

$$\left| \mathbb{E}O((N \varrho_k(\lambda_k - \lambda_{k+1}), (N \varrho_k)(\lambda_k - \lambda_{k+2}), \ldots, (N \varrho_k)(\lambda_k - \lambda_{k+n})) - \mathbb{E}'O((N \varrho_m(\lambda_m - \lambda_{m+1}), (N \varrho_m)(\lambda_m - \lambda_{m+2}), \ldots, (N \varrho_m)(\lambda_m - \lambda_{m+n}))) \right| \leq CN^{-\varepsilon} \| O' \|_\infty,$$

where the local density $\varrho_k$ is defined by $\varrho_k := \varrho(\gamma_k)$.

It is well-known that the gap distribution of Gaussian random matrices for all symmetry classes can be explicitly expressed via a Fredholm determinant provided that a certain local average is taken, see [14, 15, 16]. The result for a single gap, i.e. without local averaging, was only achieved recently in the special case of the Gaussian unitary ensemble (GUE) by Tao [52] (which then easily implies the same results for Hermitian Wigner matrices satisfying the four moment matching condition). It is not clear if a similar argument can be applied to the GOE case.

We now define the $\beta$-ensembles with a general external potential. Let $\beta > 0$ be a fixed parameter. Let $V(x)$ be a real analytic potential on $\mathbb{R}$ that grows faster than $(2 + \varepsilon) \log |x|$ at infinity and satisfies

$$\inf_{\mathbb{R}} V'' > -\infty.$$  

(2.11)

Consider the measure

$$\mu = \mu^{(N)}_{\beta, V}(d\lambda) = \frac{e^{-N\beta \mathcal{H}(\lambda)}}{Z_\beta} d\lambda, \quad \mathcal{H}(\lambda) = \frac{1}{2} \sum_{i=1}^{N} V(\lambda_i) - \frac{1}{N} \sum_{i < j} \log |\lambda_j - \lambda_i|.$$  

(2.12)

Since $\mu$ is symmetric in all its variables, we will mostly view it as a measure restricted to the cone

$$\Sigma^{(N)} := \{ \lambda : \lambda_1 < \lambda_2 < \ldots < \lambda_N \} \subset \mathbb{R}^N.$$  

(2.13)

\footnote{In fact, $V \in C^4(\mathbb{R})$ is sufficient, see Remark 5.1.}
Note that the Gaussian measure (2.5) is a special case of (2.12) with \( V(\lambda) = \lambda^2/2 \). In this case we use the notation \( \mu_G \) for \( \mu \).

Let

\[
\varrho^{(N)}_j(\lambda) := \mathbb{E}^{\mu} \frac{1}{N} \sum_{j=1}^{N} \delta(\lambda - \lambda_j)
\]

denote the density, or the one-point function, of \( \mu \). It is well known [2, 10] that \( \varrho^{(N)}_1 \) converges weakly to the equilibrium density \( \varrho = \varrho_V \) as \( N \to \infty \). The equilibrium density can be characterized as the unique minimizer (in the set of probability measures on \( \mathbb{R} \) endowed with the weak topology) of the functional

\[
I(\nu) = \int V(t) d\nu(t) - \iint \log|t-s|d\nu(t)d\nu(s). \tag{2.14}
\]

In the case, \( V(x) = x^2/2 \), the minimizer is the Wigner semicircle law \( \varrho = \varrho_G \), defined in (2.6), where the subscript \( G \) refers to the Gaussian case. In the general case we assume that \( \varrho = \varrho_V \) is supported on a single compact interval, \([A, B]\) and \( \varrho \in C^2(A, B) \). Moreover, we assume that \( V \) is regular in the sense that \( \varrho \) is strictly positive on \((A, B)\) and vanishes as a square root at the endpoints, see (1.4) of [8]. It is known that these condition are satisfied if, for example, \( V \) is strictly convex.

For any \( j \leq N \) define the classical location of the \( j \)-th particle \( \gamma_{j,V} \) by

\[
\frac{j}{N} = \int_A^{\gamma_{j,V}} \varrho_V(x)dx, \tag{2.15}
\]

and for the Gaussian case we have \([A, B] = [-2, 2]\) and we use the notation \( \gamma_{j,G} = \gamma_j \) for the corresponding classical location, defined in (2.7). We set

\[
\varrho^V_j := \varrho_V(\gamma_{j,V}), \quad \text{and} \quad \varrho^G_j := \varrho_G(\gamma_{j,G}) \tag{2.16}
\]

to be the limiting density at the classical location of the \( j \)-th particle. Our main theorem on the \( \beta \)-ensembles is the following.

**Theorem 2.3 (Gap universality for \( \beta \)-ensembles)** Let \( \beta \geq 1 \) and \( V \) be a real analytic potential with (2.11) such that \( \varrho_V \) is supported on a single compact interval, \([A, B]\), \( \varrho_V \in C^2(A, B) \), and that \( V \) is regular. Fix positive numbers \( \alpha, O_{\infty}, O_{\text{supp}} \), an integer \( n \in \mathbb{N} \) and an \( n \)-particle observable \( O = O_N \) satisfying (2.8). Let \( \mu = \mu_V = \mu^{(N)}_V \) be given by (2.12) and let \( \mu_G \) denote the same measure for the Gaussian case. Then there exist an \( \varepsilon > 0 \), depending only on \( \alpha, \beta \) and the potential \( V \), and a constant \( C \) depending on \( O_{\infty} \) and \( O_{\text{supp}} \) such that

\[
\begin{align*}
|\mathbb{E}^{\mu_V} O((N\varrho^V_k)(\lambda_k - \lambda_{k+1}), (N\varrho^V_k)(\lambda_k - \lambda_{k+2}), \ldots, (N\varrho^V_k)(\lambda_k - \lambda_{k+n})) & - \mathbb{E}^{\mu_G} O((N\varrho^G_m)(\lambda_m - \lambda_{m+1}), (N\varrho^G_m)(\lambda_m - \lambda_{m+2}), \ldots, (N\varrho^G_m)(\lambda_m - \lambda_{m+n}))| \\
& \leq C N^{-\varepsilon} \|O'\|_{\infty}
\end{align*} \tag{2.17}
\]

for any \( k, m \in \lfloor \alpha N, (1 - \alpha)N \rfloor \) and for any sufficiently large \( N \geq N_0 \), where \( N_0 \) depends on \( V, \beta \), as well as on \( n, \alpha, O_{\infty} \) and \( O_{\text{supp}} \). In particular, the distribution of the rescaled gaps w.r.t. \( \mu_V \) does not depend on the index \( k \) in the bulk.

\(^2\)In fact, \( V \in C^4(\mathbb{R}) \) is sufficient, see Remark 5.1.
Theorem 2.3, in particular, asserts that the single gap distribution in the bulk is independent of the index $k$. The special GUE case of this assertion is the content of [52] where the proof uses some special structures of GUE.

The proofs of both Theorems 2.2 and 2.3 rely on the uniqueness of the gap distribution for a localized version of the equilibrium measure (2.5) with a certain class of boundary conditions. This main technical result will be formulated in Theorem 4.1 in the next section after we introduce the necessary notations. An orientation of the content of the paper will be given at the end of Section 4.1.

We remark that Theorem 2.3 is stated only for $\beta \geq 1$; on the contrary, the universality with local averaging in [7, 8] was proved for $\beta > 0$. The main reason is that the current proof relies heavily on the dynamics of the gradient flow of local log-gases. Hence the well-posedness of the dynamics is crucial which is available only for $\beta \geq 1$. On the other hand, in [7, 8] we use only certain Dirichlet form inequalities (see, e.g. Lemma 5.9 in [7]), which we could prove with an effective regularization scheme for all $\beta > 0$. For $\beta < 1$ it is not clear if such a regularization can also be applied to the new inequalities we will prove here.

3 Outline of the main ideas in the proof

For the orientation of the reader we briefly outline the three main concepts in the proof without any technicalities.

1. Local Gibbs measures and their comparison

The first observation is that the macroscopic structure of the Gibbs measure $\mu^{(N)}_{\beta, V}$, see (2.12), heavily depends on $V$ via the density $\varrho_V$. The microscopic structure, however, is essentially determined by the logarithmic interaction alone, the local density plays only the role of a scaling factor. Once the measure is localized, its dependence on $V$ is reduced to a simple linear rescaling. This gives rise to the idea to consider the local Gibbs measures, defined on $K$ consecutive particles (indexed by a set $I$) by conditioning on all other $N - K$ particles. The frozen particles, denoted collectively by $y = \{y_j\}_{j \notin I}$, play the role of the boundary conditions. The potential of the local Gibbs measure $\mu_y$ is given by $\frac{1}{2}V_y(x) = \frac{1}{2}V(x) - \frac{1}{N} \sum_{j \notin I} \log |x - y_j|$. From the rigidity property of the measure $\mu$, see [7], the frozen particles are typically very close to their classical locations determined by the appropriate quantiles of the equilibrium density $\varrho_V$. Moreover, from the Euler-Lagrange equation of (2.14) we have $V(x) = 2 \int \log |x - y| \varrho_V(y)dy$. These properties, together with the choice $K \ll N$ ensure that $V_y$ is small away from the boundary. Thus, apart from boundary effects, the local Gibbs measure is independent of the original potential $V$. In particular, its gap statistics can be compared with that of the Gaussian ensemble after an appropriate rescaling. For convenience, we scale all local measures so that the typical size of their gaps is one.

2. Random walk representation of the covariance

The key technical difficulty is to estimate the boundary effects which is given by the correlation between the external potential $\sum_i V_y(x_i)$ and the gap observable $O(x_i - x_{i+1})$ (for simplicity we look at one gap only). We introduce the notation $\langle X; Y \rangle := \mathbb{E}XY - \mathbb{E}X \mathbb{E}Y$ to denote the covariance of two random variables $X$ and $Y$. Following the more customary statistical physics terminology, we will refer to $\langle X; Y \rangle$ as correlation. Due to the long range of the logarithmic interaction, the two-point correlation function $\langle \lambda_i; \lambda_j \rangle$ of a log-gas decays only logarithmically in $|i - j|$, i.e. very
slowly. What we really need is the correlation between a particle \( \lambda_i \) and a gap \( \lambda_i - \lambda_{i+1} \) which decays faster, as \(|i - j|^{-1}\), but we need quite precise estimates to exploit the gap structure.

For any Gibbs measure \( \omega(dx) = e^{-\beta H(x)}dx \) with strictly convex Hamiltonian, \( H'' \geq c > 0 \), the correlation of any two observables \( F \) and \( G \) can be expressed as

\[
\langle F(x); G(x) \rangle_\omega = \frac{1}{2} \int_0^\infty ds \int d\omega(x) E_x [\nabla G(x(s)) U(s, x(\cdot)) \nabla F(x)] . \tag{3.1} \]

Here \( E_x \) is the expectation for the (random) paths \( x(\cdot) \) starting from \( x(0) = x \) and solving the canonical SDE for the measure \( \omega \):

\[
dx(s) = dB(s) - \beta \nabla H(x(s)) ds
\]

and \( U(s) = U(s, x(\cdot)) \) is the fundamental solution to the linear system of equations

\[
\partial_s U(s) = -U(s) A(s), \quad A(s) := \beta H''(x(s)) \tag{3.2}
\]

with \( U(0) = I \). Notice that the coefficient matrix \( A(s) \), and thus the fundamental solution, depend on the random path \( x(s) \).

If \( G \) is a function of the gap, \( G(x) = O(x_j - x_{j+1}) \), then (3.1) becomes

\[
\langle F(x); O(x_j - x_{j+1}) \rangle_\omega = \frac{1}{2} \int_0^\infty ds \int d\omega(x) \sum_{i \in I} E_x \left[ O'(x_j - x_{j+1}) (U_{i,j}(s) - U_{i,j+1}(s)) \partial_i F(x) \right]. \tag{3.3}
\]

We will estimate the correlation (3.3) by showing that for a typical path \( x(\cdot) \) the solution \( U(s) \) is Hölder-regular in a sense that \( U_{i,j}(s) - U_{i,j+1}(s) \) is small if \( j \) is away from the boundary and \( s \) is not too small. The exceptional cases require various technical cutoff estimates.

3. Hölder-regularity of the solution to (3.2)

We will apply (3.3) with the choice \( \omega = \mu_\gamma \) and with a function \( F \) representing the effects of the boundary conditions. For any fixed realization of the path \( x(\cdot) \), we will view the equation (3.2) as a finite dimensional version of a parabolic equation. The coefficient matrix, the Hessian of the local Gibbs measure, is computed explicitly. It can be written as \( A = B + \mathcal{W} \), where \( \mathcal{W} \geq 0 \) is diagonal, \( B \) is symmetric with quadratic form

\[
\langle u, B(s) u \rangle = \frac{1}{2} \sum_{i,j \in I} B_{ij}(s) (u_i - u_j)^2, \quad B_{ij}(s) := \frac{\beta}{(x_i(s) - x_j(s))^2}.
\]

After rescaling the problem and writing it in microscopic coordinates where the gap size is of order one, for a typical path and large \( i - j \) we have

\[
B_{ij}(s) \sim \frac{1}{(i - j)^2} \tag{3.4}
\]

by rigidity. We also have a lower bound for any \( i \neq j \)

\[
B_{ij}(s) \gtrsim \frac{1}{(i - j)^2}, \tag{3.5}
\]
at least with a very high probability. If a matching upper bound were true for any $i \neq j$, then (3.2) would be the discrete analogue of the general equation

$$\partial_t u(t, x) = \int K(t, x, y)[u(t, y) - u(t, x)]dy, \quad t > 0, \quad x, y \in \mathbb{R}^d$$

(3.6)

considered by Caffarelli-Chan-Vasseur in [11]. It is assumed that the kernel $K$ is symmetric and there is a constant $0 < s < 2$ such that the short distance singularity can be bounded by

$$C_1|x - y|^{-d-s} \leq K(t, x, y) \leq C_2|x - y|^{-d-s}$$

(3.7)

for some positive constants $C_1, C_2$. Roughly speaking, the integral operator corresponds to the behavior of the operator $|p|^s$, where $p = -i\nabla$. The main result of [11] asserts that for any $t_0 > 0$, the solution $u(t, x)$ is $\varepsilon$-Hölder continuous, $u \in C^\varepsilon((t_0, \infty), \mathbb{R}^d)$, for some positive exponent $\varepsilon$ that depends only on $t_0$, $C_1$, $C_2$. Further generalizations and related local regularity results such as weak Harnack inequality can be found in [35].

Our equation (3.2) is of this type with $d = s = 1$, but it is discrete and in a finite interval $I$ with a potential term. The key difference, however, is that the coefficient $B_{ij}(t)$ in the elliptic part of (3.2) can be singular in the sense that $B_{ij}(t)|i - j|^2$ is not uniformly bounded when $i, j$ are close to each other. In fact, by extending the reasoning of Ben Arous and Bourgade [4], the minimal gap $\min_i(x_{i+1} - x_i)$ for GOE is typically of order $N^{-1/2}$ in the microscopic coordinates we are using now. Thus the analogue of the uniform upper bound (3.7) does not even hold for a fixed $t$.

The only control we can guarantee for the singular behavior of $B_{ij}$ with a large probability is the estimate

$$\sup_{0 \leq s \leq \sigma} \sup_{0 \leq M \leq CK \log K} \frac{1}{1 + s} \int_0^s \frac{1}{M} \sum_{i \in I : |i - Z| \leq M} B_{i, i+1}(s)ds \leq CK^\rho$$

(3.8)

with some small exponent $\rho$ and for any $Z \in I$ far away from the boundary of $I$. This estimate essentially says that the space-time maximal function of $B_{i, i+1}(t)$ at a fixed space-time point $(Z, 0)$ is bounded by $K^\rho$. Our main generalization of the result in [11] is to show that the weak upper bound (3.8), together with (3.4) and (3.5) (holding up to a factor $K^\xi$) are sufficient for proving a discrete version of the Hölder continuity at the point $(Z, 0)$. More precisely, in Theorem 9.8 we essentially show that there exists a $q > 0$ such that for any fixed $\sigma \in [K^c, K^{1-c}]$, the solution to (3.2) satisfies

$$\sup_{|j - Z| + |j' - Z| \leq \sigma^{1-\alpha}} |U_{i, j}(\sigma) - U_{i, j'}(\sigma)| \leq CK^\xi \sigma^{-1-\frac{3}{q} + \alpha},$$

(3.9)

with any $\alpha \in [0, 1/3]$ if we can guarantee that $\rho$ and $\xi$ are sufficiently small. The exponent $q$ is a universal positive number and it plays the role of the Hölder regularity exponent. In fact, to obtain Hölder regularity around one space-time point $(Z, \sigma)$ as in (3.9), we need to assume the bound (3.8) around several (but not more than $(\log K)^{C}$) space-time points, which in our applications can be guaranteed with high probability.

Notice that $U_{i, j}(\sigma)$ decays as $\sigma^{-1}$, hence (3.9) provides an additional decay for the discrete derivative. In particular, this guarantees that the $ds$ integration in (3.3) is finite in the most critical intermediate regime $s \in [K^c, CK \log K]$.

The proof of Theorem 9.8 is given in Section 10. In this section we also formulate a Hölder regularity result for initial data in $L^\infty$ (Theorem 10.1), which is the basis of all other results. Readers interested in the pure PDE aspect of our work are referred to Section 10 which can be read independently of the other sections of the paper.
4 Local equilibrium measures

4.1 Basic properties of local equilibrium measures

Fix two small positive numbers, $\alpha, \delta > 0$. Choose two positive integer parameters $L, K$ such that

$$L \in \left[ \alpha N, (1 - \alpha)N \right], \quad N^\delta \leq K \leq N^{1/4}. \quad \text{(4.1)}$$

We consider the parameters $L$ and $K$ fixed and often we will not indicate them in the notation. All results will hold for any sufficiently small $\alpha, \delta$ and for any sufficiently large $N \geq N_0$, where the threshold $N_0$ depends on $\alpha, \delta$ and maybe on other parameters of the model. Throughout the paper we will use $C$ and $c$ to denote positive constants which, among others, may depend on $\alpha, \delta$ and on the constants in (2.2) and (2.4), but we will not emphasize this dependence. Typically $C$ denotes a large generic constant, while $c$ denotes a small one whose values may change from line to line. These constants are independent of $K$ and $N$, which are the limiting large parameters of the problem, but they may depend on each other. In this case we will use $C_0, C_1, \ldots$ and $c_0, c_1, \ldots$ etc. to denote specific constants in order to be able to refer to them along the proof.

For convenience, we set $K := 2K + 1$.

Denote $I = I_{L,K} := [L - K, L + K]$ the set of $K$ consecutive indices in the bulk. We will distinguish the inside and outside particles by renaming them as

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) := (y_1, \ldots y_{L-K-1}, x_{L-K}, \ldots, x_{L+K}, y_{L+K+1}, \ldots y_N) \in \Xi^{(N)}.$$

Note that the particles keep their original indices. The notation $\Xi^{(N)}$ refers to the simplex (2.13). In short we will write

$$\mathbf{x} = (x_{L-K}, \ldots x_{L+K}), \quad \text{and} \quad \mathbf{y} = (y_1, \ldots y_{L-K-1}, y_{L+K+1}, \ldots y_N).$$

These points are always listed in increasing order, i.e. $\mathbf{x} \in \Xi^{(K)}$ and $\mathbf{y} \in \Xi^{(N-K)}$. We will refer to the $y$'s as the external points and to the $x$'s as internal points.

We will fix the external points (often called boundary conditions) and study the conditional measures on the internal points. We first define the local equilibrium measure (or local measure in short) on $\mathbf{x}$ with boundary condition $\mathbf{y}$ by

$$\mu_\mathbf{y}(d\mathbf{x}) := \mu_\mathbf{y}(\mathbf{x})d\mathbf{x}, \quad \mu_\mathbf{y}(\mathbf{x}) := \mu(\mathbf{y}, \mathbf{x}) \left[ \int \mu(\mathbf{y}, \mathbf{x})d\mathbf{x} \right]^{-1}. \quad \text{eq:muydx}$$

where $\mu = \mu(\mathbf{y}, \mathbf{x})$ is the (global) equilibrium measure (2.12) (we do not distinguish between the measure $\mu$ and its density function $\mu(\mathbf{y}, \mathbf{x})$ in the notation). Note that for any fixed $\mathbf{y} \in \Xi^{(N-K)}$, all $x_j$ lie in the open configuration interval, denoted by

$$J = J_\mathbf{y} := (y_{L-K-1}, y_{L+K+1}).$$

Define

$$\bar{y} := \frac{1}{2}(y_{L-K-1} + y_{L+K+1})$$
to be the midpoint of the configuration interval. We also introduce
\[ \alpha_j := \bar{y} + \frac{j - L}{K + 1}|J|, \quad j \in I_{L,K}, \]  
(4.4)

to denote the \( K \) equidistant points within the interval \( J \).

For any fixed \( L, K, y \), the equilibrium measure can also be written as a Gibbs measure,
\[ \mu_y = \mu_y^{(N)} = Z_y^{-1}e^{-N\beta \mathcal{H}_y}, \]  
(4.5)

with Hamiltonian
\[ \mathcal{H}_y(x) := \sum_{i \in I} \frac{1}{2}V_y(x_i) - \frac{1}{N} \sum_{i,j \in I} \log|x_j - x_i|, \]
\[ V_y(x) := V(x) - \frac{2}{N} \sum_{j \notin I} \log|x - y_j|. \]  
(4.6)

Here \( V_y(x) \) can be viewed as the external potential of a \( \beta \)-log-gas of the points \( \{x_i : i \in I\} \) in the configuration interval \( J \).

Our main technical result, Theorem 4.1 below, asserts that, for \( K, L \) chosen according to (4.1), the local gap statistics are essentially independent of \( V \) and \( y \) as long as the boundary conditions \( y \) are regular. This property is expressed by defining the following set of “good” boundary conditions with some given positive parameters \( \nu, \alpha \):
\[ \mathcal{R}_{L,K} = \mathcal{R}_{L,K}(\nu, \alpha) := \{y : |y_k - \gamma_k| \leq N^{-1+\nu}, \quad \forall k \in [\alpha N, (1 - \alpha) N] \setminus I_{L,K}\} \]  
(4.7)
\[ \cap \{y : |y_k - \gamma_k| \leq N^{-4/15+\nu}, \quad \forall k \in [N^{3/5+\nu}, N - N^{3/5+\nu}]\} \]
\[ \cap \{y : |y_k - \gamma_k| \leq 1, \quad \forall k \in [1, N] \setminus I_{L,K}\}. \]

In Section 5 we will see that this definition is tailored to the previously proven rigidity bounds for the \( \beta \)-ensemble, see (5.4). The rigidity bounds for the generalized Wigner matrices are stronger, see (6.1), so this definition will suit the needs of both proofs.

**Theorem 4.1 (Gap universality for local measures)** Fix \( L, \bar{L} \) and \( K = 2K + 1 \) satisfying (4.1) with an exponent \( \delta > 0 \). Consider two boundary conditions \( y, \bar{y} \) such that the configuration intervals coincide,
\[ J = (y_{L-K-1}, y_{L+K+1}) = (\bar{y}_{\bar{L}-K-1}, \bar{y}_{\bar{L}+K+1}). \]  
(4.8)

We consider the measures \( \mu = \mu_{y, \beta, V} \) and \( \bar{\mu} = \mu_{\bar{y}, \beta, \bar{V}} \) defined as in (4.5), with possibly two different external potentials \( V \) and \( \bar{V} \). Let \( \xi > 0 \) be a small constant. Assume that \( |J| \) satisfies
\[ |J| = \frac{K}{N g(\bar{y})} + O\left(\frac{K^\xi}{N}\right). \]  
(4.9)

Suppose that \( y, \bar{y} \in \mathcal{R}_{L,K}(\xi^2 \delta/2, \alpha/2) \) and that
\[ \max_{j \in I_{L,K}} \left| E^{\mu_y} x_j - \alpha_j \right| + \max_{j \in I_{L,K}} \left| E^{\bar{\mu}_{\bar{y}}} x_j - \alpha_j \right| \leq CN^{-1}K^\xi \]  
(4.10)
holds. Let the integer number $p$ satisfy $|p| \leq K - K^{1-\xi^*}$ for some small $\xi^* > 0$. Then there exists $\xi_0 > 0$, depending on $\delta$, such that if $\xi, \xi^* \leq \xi_0$ then for any $n$ fixed and any $n$-particle observable $O = O_N$ satisfying (2.8) with fixed control parameters $O_\infty$ and $O_{\text{supp}}$, we have

\begin{equation}
\left| \mathbb{E}^{\mu_y} O(N(x_{L+p} - x_{L+p+1}), \ldots N(x_{L+p} - x_{L+p+n})) - \mathbb{E}^{f_i \mu} O(N(x_{L+p} - x_{L+p+1}), \ldots N(x_{L+p} - x_{L+p+n})) \right| \leq CK^{-\varepsilon} \|O\|_\infty
\end{equation}

for some $\varepsilon > 0$ depending on $\delta, \alpha$ and for some $C$ depending on $O_\infty$ and $O_{\text{supp}}$. This holds for any $N \geq N_0$ sufficiently large, where $N_0$ depends on the parameters $\xi, \xi^*, \alpha, \beta,$ and $C$ in (4.10).

In the following two theorems we establish rigidity and level repulsion estimates for the local log-gas $\mu_y$ with good boundary conditions $y$. While both rigidity and level repulsion are basic questions for log gases, our main motivation to prove these theorems is to use them in the proof of Theorem 4.1. The current form of the level repulsion estimate is new, a weaker form was proved in (4.11) of [7]. The rigidity estimate was proved for the global equilibrium measure $\mu$ in [7]. From this estimate, one can conclude that $\mu_y$ has a good rigidity bound for a set of boundary conditions with high probability w.r.t. the global measure $\mu$. However, we will need a rigidity estimate for $\mu_y$ for a set of $y$’s with high probability with respect to some different measure, which may be asymptotically singular to $\mu$ for large $N$. For example, in the proof for the gap universality of Wigner matrices such a measure is given by the time evolved measure $f_{t\mu}$, see Section 6. The following result asserts that a rigidity estimate holds for $\mu_y$ provided that $y$ itself satisfies a rigidity bound and an extra condition, (4.12), holds. This provides explicit criteria to describe the set of "good" $y$’s whose measure w.r.t. $f_{t\mu}$ can then be estimated with different methods.

**Theorem 4.2 (Rigidity estimate for local measures)** Let $L$ and $K$ satisfy (4.1) with $\delta$ the exponent appearing in (4.1). Let $\xi, \alpha$ be any fixed positive constants. For $y \in \mathcal{R}_{L,K}(\xi \delta/2, \alpha)$ consider the local equilibrium measure $\mu_y$ defined in (4.5) and assume that

\begin{equation}
\left| \mathbb{E}^{\mu_y} x_j - \alpha_j \right| \leq CN^{-1} K^\xi, \quad j \in I = I_{L,K},
\end{equation}

is satisfied. Then there are positive constants $C, c$, depending on $\xi$, such that for any $k \in I$ and $u > 0$,

\begin{equation}
\mathbb{P}^{\mu_y} \left( |x_k - \alpha_k| \geq u K^\xi N^{-1} \right) \leq C e^{-cu^2}.
\end{equation}

Now we state the level repulsion estimates which will be proven in Section 7.2.

**Theorem 4.3 (Level repulsion estimate for local measures)** Let $L$ and $K$ satisfy (4.1) and let $\xi, \alpha$ be any fixed positive constants. Then for $y \in \mathcal{R}_{L,K} = \mathcal{R}_{L,K}(\xi^2 \delta/2, \alpha)$ we have the following estimates:

i) Weak form of level repulsion For any $s > 0$ we have

\begin{equation}
\mathbb{P}^{\mu_y} [x_{i+1} - x_i \leq s/N] \leq C (Ns)^{\beta+1}, \quad i \in [L - K - 1, L + K]
\end{equation}

and

\begin{equation}
\mathbb{P}^{\mu_y} [x_{i+2} - x_i \leq s/N] \leq C (Ns)^{2\beta+1} \quad i \in [L - K - 1, L + K - 1].
\end{equation}
ii) [Strong form of level repulsion] Suppose that there exist positive constants $C, c$ such that the following rigidity estimate holds for any $k \in I$:

$$\mathbb{P}^{y} \left( |x_k - \alpha_k| \geq CK^\xi N^{-1} \right) \leq C \exp \left( -K^c \right).$$

Then there exists a small constant $\theta$, depending on $C, c$ in (4.16), such that for any $s \geq \exp(-K^\theta)$, we have

$$\mathbb{P}^{y} \left[ x_{i+1} - x_i \leq s/N \right] \leq C \left( K^\xi s \log N \right)^{3\beta+1}, \quad i \in \llbracket L - K - 1, L + K \rrbracket$$

and

$$\mathbb{P}^{y} \left[ x_{i+2} - x_i \leq s/N \right] \leq C \left( K^\xi s \log N \right)^{2\beta+1}, \quad i \in \llbracket L - K - 1, L + K - 1 \rrbracket.$$
is the distance to the boundary of $J$ and

$$d_-(x) := d(x) + \varrho(\bar{y}) N^{-1} K^\xi, \quad d_+(x) := \max\{|x-a|, |x-b|\} + \varrho(\bar{y}) N^{-1} K^\xi.$$  

The following lemma, proven in Appendix A, asserts that “good” boundary conditions $y$ give rise to regular external potential $V_y$.

**Lemma 4.5** Let $L$ and $K$ satisfy (4.1) and $\delta$ is the exponent appearing in (4.1). Then for any $y \in \mathcal{R}_{L,K}(\xi\delta/2, \alpha/2)$ the external potential $V_y$ (4.6) on the configuration interval $J_y$ is $K^\xi$-regular:

\begin{align*}
|J_y| &= \frac{K}{N \varrho(\bar{y})} + O\left(\frac{K^\xi}{N}\right), \quad (4.22) \\
V_y'(x) &= \varrho(\bar{y}) \frac{d_+(x)}{d_-(x)} + O\left(\frac{K^\xi}{Nd(x)}\right), \quad x \in J_y, \quad (4.23) \\
V_y''(x) &\geq \inf V'' + \frac{c}{d(x)}, \quad x \in J_y. \quad (4.24)
\end{align*}

**Remark.** The proofs of Theorems 4.1, 4.2, 4.3 and 7.3 do not use the explicit form of $V_y$ and $J_y$; they depend only on the property that $V_y$ on $J_y$ is regular.

## 5 Gap universality for $\beta$-ensembles: proof of Theorem 2.3

### 5.1 Rigidity bounds and its consequences

The aim of this section is to use Theorem 4.1 to prove Theorem 2.3. In order to verify the assumptions of Theorem 4.1, we first recall the rigidity estimate w.r.t. $\mu$ defined in (2.12). Recall that $\gamma_k = \gamma_k, V$ denotes the classical location of the $k$-th point (2.15). For the case of convex potential, in Theorem 3.1 of [7] it was proved that for any fixed $\alpha > 0$ and $\nu > 0$, there are constants $C_0, c_1, c_2 > 0$ such that for any $N \geq 1$ and $k \in \llbracket \alpha N, (1-\alpha)N\rrbracket$,

$$\mathbb{P}^\mu \left( |\lambda_k - \gamma_k| > N^{-1/\nu} \right) \leq C_0 \exp \left( - c_1 N^{\nu} \right). \quad (5.1)$$

The same estimate holds also for the non-convex case, see Theorem 1.1 of [8], by using a convexification argument.

Near the spectral edges, a somewhat weaker control was proven for the convex case, see Lemma 3.6 of [7] which states that for any $\nu > 0$ there are $C_0, c_1, c_2 > 0$ such that

$$\mathbb{P}^\mu \left( |\lambda_k - \gamma_k| > N^{-4/15+\nu} \right) \leq C_0 \exp \left( - c_1 N^{\nu} \right) \quad (5.2)$$

for any $N^{3/5+\nu} \leq k \leq N - N^{3/5+\nu}$, if $N \geq N_0(\nu)$ is sufficiently large. We can choose $C_0, c_1, c_2$ to be the same in (5.1) and (5.2). Combining this result with the convexification argument in [8], one can show that the estimate (5.2) holds also for the non-convex case.

Finally, we have a very weak control that holds for all points (see (1.7) in [8]): for any $C > 0$ there are positive constants $C_0, c_1$ and $c_2$ such that

$$\mathbb{P}^\mu \left( |\lambda_k - \gamma_k| > C \right) \leq C_0 \exp \left( - c_1 N^{\nu} \right). \quad (5.3)$$

Given $C$, we can choose the positive constants $C_0, c_1, c_2$ to be the same in (5.1), (5.2) and (5.3).
The set \( \mathcal{R}_{L,K} \) in (4.7) was exactly defined as the set of events that these three rigidity estimates hold. From (5.1), (5.2) and (5.3) we have
\[
\mathbb{P}^\mu (\mathcal{R}_{L,K}(\nu, \alpha)) \geq 1 - C_0 \exp (-c_1 N^{c_2}) \tag{5.4}
\]
for any \( \nu > 0, \alpha > 0 \) with some positive constants \( C_0, c_1, c_2 \) that depend on \( \nu \) and \( \alpha \).

**Remark 5.1** The real analyticity of \( V \) in this paper is used only to obtain the rigidity results (5.1) (5.2) and (5.3) using earlier results from [7, 8]. After the first version of the current work appeared in the ArXiv, jointly with P. Bourgade we proved the following stronger rigidity result (Theorem 2.4 of [9]) For any \( \beta > 0, \xi > 0 \) and \( V \in C^4(\mathbb{R}) \), regular with equilibrium density supported on a single interval \([A, B]\), there is a \( c > 0 \) and \( N_0 \) such that
\[
\mathbb{P}^\mu \left( |\lambda_k - \gamma_k| > N^{-\frac{2}{3} + \xi (k)^{-\frac{1}{3}}} \right) \leq e^{-N^c}, \quad \forall k \in [1, N] \tag{5.5}
\]
holds for any \( N \geq N_0 \). This result allows us to relax the original real analyticity condition to \( V \in C^4(\mathbb{R}) \). It would also allow us to redefine the set \( \mathcal{R}_{L,K} \) in (4.7) to the more transparent set appearing in (4.13), but this generalization does not affect the rest of the proof.

**Lemma 5.2** Let \( L \) and \( K \) satisfy (4.1) and \( \delta \) is the exponent appearing in (4.1). Then for any small \( \xi \) and \( \alpha \) there exists a set \( \mathcal{R}^* = \mathcal{R}_{L,K,\mu}(\xi^2 \delta/2, \alpha/2) \subset \mathcal{R}_{L,K}(\xi^2 \delta/2, \alpha/2) \) such that
\[
\mathbb{P}^\mu(\mathcal{R}^*) \geq 1 - C_0 \exp \left( - \frac{1}{2} c_1 N^{c_2} \right) \tag{5.6}
\]
with the constants \( C_0, c_1, c_2 \) from (5.1). Moreover, for any \( y \in \mathcal{R}^* \) we have
\[
|\mathbb{E}^{\nu y} x_k - \alpha_k| \leq C N^{-1} K^\xi, \quad k \in I_{L,K}, \tag{5.7}
\]
where \( \alpha_k \) was defined in (4.4).

*Proof.* For any \( \nu > 0 \) define
\[
\mathcal{R}_{L,K,\mu}^*(\nu, \alpha) := \left\{ y \in \mathcal{R}(\nu, \alpha) : \mathbb{P}^{\nu y} (|x_k - \gamma_k| > N^{1+\nu}) \leq \exp \left( - \frac{1}{2} c_1 N^{c_2} \right), \quad \forall k \in I_{L,K} \right\} \tag{5.8}
\]
with the \( \nu \)-dependent constants \( c_1, c_2 > 0 \) from (5.4). Note that \( \mathcal{R}^* \), unlike \( \mathcal{R} \), depends on the underlying measure \( \mu \) through the family of its conditional measures \( \mu_y \). Applying (5.4) for \( \nu = \xi^2 \delta/2 \) and setting \( \mathcal{R} = \mathcal{R}_{L,K}(\xi^2 \delta/2, \alpha/2), \mathcal{R}^* = \mathcal{R}_{L,K,\mu}^*(\xi^2 \delta/2, \alpha/2) \), we have
\[
\mathbb{P}^\mu(\mathcal{R}^*) \geq 1 - C_0 \exp \left( - \frac{1}{2} c_1 N^{c_2} \right)
\]
with some \( C_0, c_1, c_2 \). Now if \( y \in \mathcal{R}^* \), then
\[
|\mathbb{E}^{\nu y} x_k - \gamma_k| \leq C_0 e^{-c_1 N^{c_2}} + C N^{-1} K^\xi, \quad k \in I_{L,K}. \tag{5.9}
\]
In order to prove (5.7), it remains to show that \( |\alpha_k - \gamma_k| \) is bounded by \( C N^{-1} K^\xi \) for any \( k \in I_{L,K} \). To see this, we can use that \( \varphi \in C^1 \) away from the edge, thus
\[
\varphi(x) = \varphi(\bar{y}) + O(x - \bar{y})
\]
(recall that \( \bar{y} \) is the midpoint of \( J \)). By Taylor expansion we have

\[
k - (L - K - 1) = N \int_{y_L - K - 1}^{y_{L - K}} \varrho = N \int_{y_L - K - 1}^{y_{L - K}} \varrho + O(N^{\xi \delta / 2}) = N|\gamma_k - y_{L - K - 1}| \varrho(\bar{y}) + O(N|J|^2 + N^{\xi \delta / 2}),
\]

i.e.

\[
\gamma_k = y_{L - K - 1} + \frac{k - L + K + 1}{N \varrho(\bar{y})} + O(N^{-1} K^\xi).
\]

(5.10)

Here we used that \( J = J_x \) satisfies (4.22) according to Lemma 4.5, since \( y \in \mathcal{R}_{L, K}(\xi^2 \delta / 2, \alpha/2) \subset \mathcal{R}_{L, K}(\xi \delta / 2, \alpha) \). Comparing (5.10) with the definition of \( \alpha_k \), (4.4), using (4.22) and the fact that \( \bar{y} - y_{L - K - 1} = \frac{1}{2}|J| \), we have

\[
|\alpha_k - \gamma_k| \leq CN^{-1} K^\xi.
\]

(5.11)

Together with (5.9) this implies (5.7) and this completes the proof of Lemma 5.2. \( \square \)

### 5.2 Completing the proof of Theorem 2.3

We first notice that it is sufficient to prove Theorem 2.3 for the special case \( m = N/2 \), i.e. when the local statistics for the Gaussian measure is considered at the central point of the spectrum. Indeed, once Theorem 2.3 is proved for any \( V \) and \( m = N/2 \), then with the choice \( V(x) = x^2 / 2 \) we can use it to establish that the local statistics for the Gaussian measure around any fixed index \( k \) in the bulk coincide with the local statistics in the middle. So from now on we assume \( m = N/2 \), but we carry the notation \( m \) for simplicity.

Given \( k \) and \( m = N/2 \) as in (2.17), we first choose \( L, \bar{L}, K \), satisfying (4.1) (maybe with a smaller \( \alpha \) than given in Theorem 2.3), so that \( k = L + p, m = \bar{L} + p \) hold for some \( |p| \leq K/2 \). In particular

\[
|\bar{L} - N/2| \leq K.
\]

(5.12)

For brevity, we use \( \mu = \mu_V \) and \( \bar{\mu} = \mu^G \) in accordance with the notation of Theorem 4.1.

We consider \( y \in \mathcal{R}_{L, K, \mu}(\xi^2 \delta / 2, \alpha) \) and \( \bar{y} \in \mathcal{R}_{L, K, \bar{\mu}}(\xi^2 \delta / 2, \alpha) \), where \( \delta \) is the exponent appearing in (4.1). We omit the arguments and recall that

\[
\mu(\mathcal{R}_{L, K, \mu}^*) \geq 1 - C_0 \exp\left(-\frac{1}{2} c_1 N^{c_2}\right), \quad \bar{\mu}(\mathcal{R}_{L, K, \bar{\mu}}^*) \geq 1 - C_0 \exp\left(-\frac{1}{2} c_1 N^{c_2}\right)
\]

(5.13)

with some positive constants.

**Proposition 5.3** With the above choice of the parameters and for any \( y \in \mathcal{R}_{L, K, \mu}(\xi^2 \delta / 2, \alpha) \) and \( \bar{y} \in \mathcal{R}_{L, K, \bar{\mu}}(\xi^2 \delta / 2, \alpha) \), we have

\[
\mathbb{E}^{\mu_y} O((N \varrho^V_{L+p})(x_{L+p} - x_{L+p+1}), \ldots, (N \varrho^V_{L+p})(x_{L+p} - x_{L+p+n})) - \mathbb{E}^{\bar{\mu}_{\bar{y}}} O((N \varrho^G_{L+p})(x_{L+p} - x_{L+p+1}), \ldots, (N \varrho^G_{L+p})(x_{L+p} - x_{L+p+n})) \leq CK^{-\varepsilon} \|O'\|_\infty,
\]

(5.14)

where \( \varepsilon \) is from Theorem 4.1.
Theorem 2.3 follows immediately from (5.13) and this proposition.

The rest of this section is devoted to the proof of Proposition 5.3.

**Proof of Proposition 5.3.** We will apply Theorem 4.1, but first we have to bring the two measures onto the same configuration interval \( J \) to satisfy (4.8). This will be done in three steps. First, using the scale invariance of the Gaussian log-gas we rescale it so that the local density approximately matches with that of \( \mu_V \). This will guarantee that the two configuration intervals have almost the same length. In the second step we adjust the local Gaussian log-gas \( J \) so that \( J \) has exactly the correct length. Finally, we shift the two intervals so that they coincide. This allows us to apply Theorem 4.1 to conclude the local statistics are identical.

The local densities \( \varrho_V \) around \( \gamma_{L+p,V} \) and \( \varrho_G \) around \( \gamma_{L+p,G} \) may considerably differ. So in the first step we rescale the Gaussian log-gas so that

\[
\varrho_V(\gamma_{L+p,V}) = \varrho_G(\gamma_{L+p,G}), \tag{5.15}
\]

To do that, recall that we defined the Gaussian log-gas with the standard \( V(x) = x^2/2 \) external potential, but we could choose \( V_s(x) = s^2x^2/2 \) with any fixed \( s > 0 \) and consider the Gaussian log-gas

\[
\mu^s_G(\lambda) \sim \exp\left( -N\beta H_s(\lambda) \right), \quad H_s(\lambda) := \frac{1}{2} \sum_{i=1}^{N} V_s(\lambda_i) - \frac{1}{N} \sum_{i<j} \log |\lambda_j - \lambda_i|.
\]

This results in a rescaling of the semicircle density \( \varrho_G \) to \( \varrho^s_G(x) := s\varrho_G(sx) \) and \( \gamma_{i,G} \) to \( \gamma^s_{i,G} := s^{-1}\gamma_{i,G} \) for any \( i \), so \( \varrho_G(\gamma_{i,G}) \) gets rescaled to \( \varrho^s_G(\gamma^s_{i,G}) = s\varrho_G(\gamma_{i,G}) \). In particular, \( \varrho_G(\gamma_{L+p,G}) \) is rescaled to \( s\varrho_G(\gamma_{L+p,G}) \), and thus choosing \( s \) appropriately, we can achieve that (5.15) holds (keeping the left hand side fixed). Set

\[
O_s(x) := O\left((N\varrho^s_G(\gamma^s_{m,G}))(x_m - x_{m+1}), \ldots, (N\varrho^s_G(\gamma^s_{m,G}))(x_m - x_{m+n})\right), \quad m = \bar{L} + p,
\]

and notice that \( O_s(x) = O(sx) \). This means that the local gap statistics \( E^{\mu_G}O_s \) is independent of the scaling parameter \( s \), since the product \( (N\varrho^s_G(\gamma^s_{m,G}))(x_m - x_{m+n}) \) (notation defined in (2.16)) is unchanged under the scaling. So we can work with the rescaled Gaussian measure. For notational simplicity we will not carry the \( s \) parameter further and we just assume that (5.15) holds with the original Gaussian \( V(x) = x^2/2 \).

We have now achieved that the two densities at at some points of the configuration intervals coincide, but the lengths of these two intervals still slightly differ. In the second step we match them exactly. Since \( y \in R_{L,K}(\xi\delta/2,\alpha) \) and \( \tilde{y} \in R_{\tilde{L},K}(\xi\delta/2,\alpha) \); from (4.22) in Lemma 4.5 we see that

\[
|J_y| = |y_{L+K+1} - y_{L-K-1}| = \frac{K}{N\varrho_V(y)} + O(N^{-1}K^\xi) \tag{5.16}
\]

\[
|J_{\tilde{y}}| = |\tilde{y}_{L+K+1} - \tilde{y}_{L-K-1}| = \frac{K}{N\varrho_G(\tilde{y})} + O(N^{-1}K^\xi). \tag{5.17}
\]

Since \( \varrho_V \) is \( C^1 \), we have for any \( |j| \leq K \),

\[
|\varrho_V(\tilde{y}) - \varrho_V(\gamma_{L+j,V})| \leq C|\tilde{y} - \gamma_{L+j,V}|
\]

\[
\leq C|\tilde{y} - y_{L,V}| + C|\gamma_{L+j,V} - \gamma_{L,V}| + O(N^{-1}K^\xi) \leq CKN^{-1},
\]

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and similarly for \( \rho_G(\overline{y}) \).

Using (5.16), (5.15) and that the densities are separated away from zero, we easily obtain that

\[
s := \left| \frac{J_x}{\overline{J_y}} \right| \quad \text{satisfies} \quad s = s_y, \overline{y} = 1 + O(K^{-1+\xi}). \tag{5.18}
\]

Note that this \( s \) is different from the scaling parameter in the first step but it will play a similar role so we use the same notation. For each fixed \( y, \overline{y} \) we can now scale the conditional Gaussian log-gas \( \mu_{\overline{y}} \) by a factor \( s \), i.e. change \( \overline{y} \) to \( sy \), so that after rescaling \( |J_y| = |J_{sy}| \).

We will now show that this rescaling does not alter the gap statistics:

**Lemma 5.4** Suppose that \( s \) satisfies (5.18) and let \( \mu = \mu_G \) be the Gaussian log-gas. Then we have

\[
\left| \left[ \mathbb{E}^{\mu_G} - \mathbb{E}^{\mu_{\overline{y}}} \right] \mathcal{O}(x) \right| \leq CK^{-1+\xi} \tag{5.19}
\]

with

\[
\mathcal{O}(x) := O\left((N\delta_m^G(x_m - x_{m+1}), \ldots, (N\delta_m^G(x_m - x_{m+n})) \right)
\]

for any \( L - K \leq m \leq \overline{L} + K - n \) (note that the observable is not rescaled).

**Proof.** Define the Gaussian log-gas

\[ \mu_{\overline{y}}^s \sim e^{-N^{\gamma} H^s_{\overline{y}}} \]

with \( H^s_{\overline{y}} \) defined exactly as in (4.6) but \( V_y(x) \) is replaced with

\[ V^s_y(x) = V_s(x) = \frac{2}{N} \sum_{j \notin I} \log |x - y_j|, \quad V_s(x) = \frac{1}{2} s^2 x^2, \quad \overline{I} := [\overline{L} - K, \overline{L} + K]. \]

Then by scaling

\[ \mathbb{E}^{\mu_{\overline{y}}^s} \mathcal{O}(x) = \mathbb{E}^{\mu_{\overline{y}}^s} \mathcal{O}(x/s) = \mathbb{E}^{\mu_{\overline{y}}^s} \mathcal{O}(x) + O(\|O'\|_{\infty}|s - 1|), \tag{5.20} \]

where in the last step we used that the observable \( O \) is a differentiable function with compact support. The error term is negligible by (5.18) and (4.1).

In order to control \( \mathbb{E}^{\mu_{\overline{y}}^s} - \mathbb{E}^{\mu_{\overline{y}}} \mathcal{O}(x) \), it is sufficient to bound the relative entropy \( S(\mu_{\overline{y}}^s | \mu_{\overline{y}}) \). However, for any \( y \in R_{L,K} \) we have

\[
\mathcal{H}^s_y \geq \min_{x \in \mathbb{R}} \frac{1}{N} \sum_{j \notin I} \frac{1}{|x - y_j|^2} \geq \frac{cN}{K} \tag{5.21}
\]

with a positive constant. Applying this for \( \overline{y} \), we see that \( \mu_{\overline{y}} \) satisfies the logarithmic Sobolev inequality (LSI)

\[
S(\mu_{\overline{y}} | \mu_{\overline{y}}) \leq CK \frac{1}{N} D(\mu_{\overline{y}} | \mu_{\overline{y}}),
\]

where

\[
S(\mu | \omega) := \int \left( \frac{d\mu}{d\omega} \log \frac{d\mu}{d\omega} \right) d\omega, \quad D(\mu | \omega) := \frac{1}{2N} \int |\nabla \sqrt{\frac{d\mu}{d\omega}}|^2 d\omega
\]

is the relative entropy and the relative Dirichlet form of two probability measures. Therefore

\[
S(\mu_{\overline{y}} | \mu_{\overline{y}}) \leq \frac{CK}{N^2} \mathbb{E}^{\mu_{\overline{y}}} \sum_{i \in I} |NV'_{\overline{y}}(x_i) - NV'(x_i)|^2 = CK(s^2 - 1)^2 \mathbb{E}^{\mu_{\overline{y}}} \sum_{i \in I} x_i^2 \leq CK^4 N^{-2} (s - 1)^2.
\]

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In the last step we used (5.12) which, by rigidity for the Gaussian log-gas, guarantees that $|x_i| \leq CK/N$ with very high probability for any $i \in \tilde{I}$. Together with (5.20) and (5.18) we obtain (5.19) and this proves Lemma 5.4.

Summarizing, we can from now on assume that (5.15) holds and that $y, \bar{y}$ satisfy $|Jy| = |\bar{y}|$. By a straightforward shift we can also assume that $Jy = J\bar{y}$ so that the condition (4.8) of Theorem 4.1 is satisfied. The condition (4.9) has already been proved to hold in Lemma 4.5. Condition (4.10) follows from the definition of the sets $R_{L,K,\mu}^*$ and $R_{L,K,\bar{\mu}}^*$, see Lemma 5.2. Thus all conditions of Theorem 4.1 are verified. Finally, we remark that the multiplicative factors $\varrho_{L+p}^V$ and $\varrho_{L+p}^G$ in (5.14) coincide by (5.15) and (2.16). Then Theorem 4.1 (with an observable $O$ rescaled by the common factor $\varrho_{L+p}^V = \varrho_{L+p}^G$) implies Proposition 5.3.

6 Gap universality for Wigner matrices: proof of Theorem 2.2

In our recent results on the universality of Wigner matrices [26, 29, 32], we established the universality for Gaussian divisible matrices by establishing the local ergodicity of the Dyson Brownian motion (DBM). By local ergodicity we meant an effective estimate on the time to equilibrium for local average of observables depending on the gap. In fact, we gave an almost optimal estimate on this time. Then we used the Green function comparison theorem to connect Gaussian divisible matrices to general Wigner matrices. The local ergodicity of DBM was done by studying the flow of the global Dirichlet form. The estimate on the global Dirichlet form in all these works was sufficiently strong so that it implied the “ergodicity for locally averaged observables” without having to go through the local equilibrium measures. In the earlier work [28], however, we used an approach common in the hydrodynamical limits by studying the properties of local equilibrium measures. Since by Theorem 4.1 we now know the local equilibrium measures very well, we will now combine the virtue of both methods to prove Theorem 2.2. To explain the new method we will be using, we first recall the standard approach to the universality from [26, 29, 32] that consists of the following three steps:

i) rigidity estimates on the precise location of the eigenvalues.

ii) Dirichlet form estimates and local ergodicity of DBM.

iii) Green function comparison theorem to remove the small Gaussian convolution.

In order to prove the single gap universality, we will need to apply a similar strategy for the local equilibrium measure $\mu_y$. However, apart from establishing rigidity for $\mu_y$, we will need to strengthen Step ii). The idea is to use Dirichlet form estimates as in the previous approach, but we then use these estimates to show that the “local structure” after the evolution of the DBM for a short time is characterized by the local equilibrium $\mu_y$ in a strong sense, i.e. without averaging. Since Theorem 4.1 provides a single gap universality for the local equilibrium $\mu_y$, this proves the single gap universality after a short time DBM evolution and thus obtain the strong form of the Step ii) without averaging the observables. Notice that the key input here is Theorem 4.1 which contains an effective estimate on the time to equilibrium for each single gap. We will call this property the strong local ergodicity of DBM. In particular, our result shows that the local averaging taken in our previous works is not essential.
We now recall the rigidity estimate which asserts that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of a generalized Wigner matrix follow the Wigner semicircle law $\rho_G(x)$ (2.6) in a very strong local sense. More precisely, Theorem 2.2. of [31] states that the eigenvalues are near their classical locations, $\{\gamma_j\}_{j=1}^N$, (2.7), in the sense that

$$\mathbb{P}\left\{ \exists j : |\lambda_j - \gamma_j| \geq (\log N)^\zeta \left[ \min\left(j, N - j + 1\right)\right]^{-1/3} N^{-2/3} \right\} \leq C \exp \left[ -c(\log N)^{\phi \zeta} \right]$$

(6.1) \text{rigidity}

for any exponent $\zeta$ satisfying

$$A_0 \log \log N \leq \zeta \leq \frac{\log(10N)}{10 \log \log N}$$

where the positive constants $C, \phi, A_0$, depend only on $C_{inf}, C_{sup}, \theta_1, \theta_2$, see (2.2), (2.4). In particular, for any fixed $\alpha > 0$ and $\nu > 0$, there are constants $C_0, c_1, c_2 > 0$ such that for any $N \geq 1$ and $k \in [\alpha N, (1 - \alpha) N]$, we have

$$\mathbb{P}\left( |\lambda_k - \gamma_k| > N^{-1+\nu} \right) \leq C_0 \exp \left( -c_1 N^{c_2} \right)$$

(6.2) \text{bulk rigi1}

and (6.1) also implies

$$\mathbb{E} \sum_{k=1}^N (\lambda_k - \gamma_k)^2 \leq N^{-1+2\nu}$$

(6.3) \text{12dist}

for any $\nu > 0$. The constants $C_0, c_1, c_2$ may be different from the ones in (5.1) but they play a similar role so we keep their notation. With a slight abuse of notation, we introduce the set $\mathcal{R}_{L,K} = \mathcal{R}_{L,K}(\xi, \alpha)$ from (4.7) in the generalized Wigner setup as well, just $\gamma_k$ denote the classical localizations with respect to the semicircle law, see (2.7). In particular (6.1) implies that for any $\xi, \alpha > 0$

$$\mathbb{P}(\mathcal{R}_{L,K}(\xi, \alpha)) \geq 1 - C_0 \exp \left( -c_1 N^{c_2} \right)$$

(6.4) \text{muR1}

holds with some positive constants $C_0, c_1, c_2$, analogously to (5.4). We remark that the rigidity bound (6.1) for the generalized Wigner matrices is optimal (up to logarithmic factors) throughout the spectrum and it gives a stronger control than the estimate used in the intermediate regime in the second line of the definition (4.7). For the forthcoming argument the weaker estimates are sufficient, so for notational simplicity we will not modify the definition of $\mathcal{R}$.

The Dyson Brownian motion (DBM) describes the evolution of the eigenvalues of a flow of Wigner matrices, $H = H_t$, if each matrix element $h_{ij}$ evolves according to independent (up to symmetry restriction) Brownian motions. The dynamics of the matrix elements are given by an Ornstein-Uhlenbeck (OU) process which leaves the standard Gaussian distribution invariant. In the Hermitian case, the OU process for the rescaled matrix elements $v_{ij} := N^{1/2} h_{ij}$ is given by the stochastic differential equation

$$dv_{ij} = d\beta_{ij} - \frac{1}{2} v_{ij} dt, \quad i,j = 1, 2, \ldots N,$$

(6.5) \text{zij}

where $\beta_{ij}$, $i < j$, are independent complex Brownian motions with variance one and $\beta_{ii}$ are real Brownian motions of the same variance. The real symmetric case is analogous, just $\beta_{ij}$ are real Brownian motions.
Denote the distribution of the eigenvalues \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \) of \( H_t \) at time \( t \) by \( f_t(\lambda) \mu(d\lambda) \) where the Gaussian measure \( \mu \) is given by (2.5). The density \( f_t = f_{t,N} \) satisfies the forward equation

\[
\partial_t f_t = \mathcal{L} f_t,
\]

where

\[
\mathcal{L} = \mathcal{L}_N := \sum_{i=1}^N \frac{1}{2N} \partial_i^2 + \sum_{i=1}^N \left( -\frac{\beta}{4} \lambda_i + \frac{\beta}{2N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) \partial_i, \quad \partial_i = \frac{\partial}{\partial \lambda_i}.
\]

with \( \beta = 1 \) for the real symmetric case and \( \beta = 2 \) in the complex hermitian case. The initial data \( f_0 \) given by the original generalized Wigner matrix.

Now we define a useful technical tool that was first introduced in [26]. For any \( \tau > 0 \) denote by \( W = W^\tau \) an auxiliary potential defined by

\[
W^\tau(\lambda) := \sum_{j=1}^N W^\tau_j(\lambda_j), \quad W^\tau_j(\lambda) := \frac{1}{2\tau} (\lambda_j - \gamma_j)^2,
\]

i.e. it is a quadratic confinement on scale \( \sqrt{\tau} \) for each eigenvalue near its classical location, where the parameter \( \tau > 0 \) will be chosen later.

**Definition 6.1** We define the probability measure \( d\mu^\tau := Z_{\tau}^{-1} e^{-N\beta H^\tau} \), where the total Hamiltonian is given by

\[
\mathcal{H}^\tau := \mathcal{H} + W^\tau.
\]

Here \( \mathcal{H} \) is the Gaussian Hamiltonian given by (2.5) and \( Z_{\tau} = Z_{\mu^\tau} \) is the partition function. The measure \( \mu^\tau \) will be referred to as the relaxation measure with relaxation time \( \tau \).

Denote by \( Q \) the following quantity

\[
Q := \sup_{0 \leq t \leq 1} \frac{1}{N} \int \sum_{j=1}^N (\lambda_j - \gamma_j)^2 f_t(\lambda) \mu(d\lambda).
\]

Since \( H_t \) is a generalized Wigner matrix for all \( t \), (6.3) implies that

\[
Q \leq N^{-2+2\nu}
\]

for any \( \nu > 0 \) if \( N \geq N_0(\nu) \) is large enough.

Recall the definition of the Dirichlet form w.r.t. a probability measure \( \omega \)

\[
D^\omega(\sqrt{g}) := \sum_{i=1}^N D^\omega_i(\sqrt{g}), \quad D^\omega_i(\sqrt{g}) := \frac{1}{2N} \int |\partial_i \sqrt{g}|^2 d\omega = \frac{1}{8N} \int |\partial_i \log g|^2 g d\omega,
\]

and the definition of the relative entropy of two probability measures \( g \omega \) and \( \omega \)

\[
S(g \omega|\omega) := \int g \log g d\omega.
\]

Now we recall Theorem 2.5 from [33]:


Theorem 6.2  For any $\tau > 0$ and consider the local relaxation measure $\mu^\tau$. Set $\psi := \frac{d\tau}{d\mu}$ and let $g_t := f_t/\psi$. Suppose there is a constant $m$ such that
\[ S(f_t \mu^\tau | \mu^\tau) \leq C N^m. \] (6.12)

Then for any $t \geq \tau N^{\epsilon'}$ the entropy and the Dirichlet form satisfy the estimates:
\[ S(g_t \mu^\tau | \mu^\tau) \leq C N^2 Q^{\tau^{-1}}, \quad D_{\mu^\tau}(\sqrt{g_t}) \leq C N^2 Q^{\tau^{-2}}, \] (6.13)

where the constants depend on $\epsilon'$ and $m$.

Corollary 6.3  Fix $a > 0$ and let $\tau \geq N^{-a}$. Under the assumptions of Theorem 6.2, for any $t \geq \tau N^{\epsilon'}$ the entropy and the Dirichlet form satisfy the estimates:
\[ D_{\mu^\tau}(\sqrt{f_t}) \leq C N^2 a^{2+2\nu}, \] (6.14)

Furthermore, if the initial data of the DBM, $f_0$, is given by a generalized Wigner ensemble, then
\[ D_{\mu^\tau}(\sqrt{f_t}) \leq C N^2 a^{2+2\nu}. \] (6.15)

Proof. Since $g_t = f_t/\psi$, we have
\[
D_{\mu^\tau}(\sqrt{f_t}) = \sum_{i=1}^{N} \frac{1}{8N} \int |\partial_i \log g_t + \partial_i \log \psi|^2 f_t d\mu \\
\leq \frac{1}{4N} \sum_{i=1}^{N} \int |\partial_i \log g_t|^2 f_t d\mu + \frac{1}{4N} \sum_{i=1}^{N} \int |\partial_i \log \psi|^2 f_t d\mu \\
\leq 2D_{\mu^\tau}(\sqrt{g_t}) + 2N^2 Q^{\tau^{-2}}.
\]

Thus (6.14) follows from Theorem 6.2. Finally, (6.15) follows from (6.14) and (6.11).

Define $f_y$ to be the conditional density of $f_{\mu}$ given $y$ w.r.t. $\mu_y$, i.e. it is defined by the relation $f_y \mu_y = (f_{\mu})_y$. For any $y \in \mathbb{R}_{L,K}$ we have the convexity bound (5.21). Thus we have the logarithmic Sobolev inequality
\[ S(f_y \mu_y | \mu_y) \leq C \frac{K}{N} \sum_{i \in I} D_{i}^{\mu_y}(\sqrt{f_y}) \] (6.16)

and the bound
\[ \int d\mu_y |f_y - 1| \leq \sqrt{S(f_y \mu_y | \mu_y)} \leq C \sqrt{\frac{K}{N} \sum_{i \in I} D_{i}^{\mu_y}(\sqrt{f_y})}. \] (6.17)

To control the Dirichlet forms $D_i$ for most external configurations $y$, we need the following Lemma.

Lemma 6.4  Fix $a > 0$, $\nu > 0$, and $\tau \geq N^{-a}$. Suppose the initial data $f_0$ of the DBM is given by a generalized Wigner ensemble. Then, with some small $\epsilon' > 0$, for any $t \geq \tau N^{\epsilon'}$ there exists a set $\mathcal{G}_{L,K} \subset \mathcal{R}_{L,K}$ of good boundary conditions $y$ with
\[ \mathbb{P}^{f_0 \mu}(\mathcal{G}_{L,K}) \geq 1 - C N^{-\epsilon'}, \] (6.18)
such that for any \( y \in G_{L,K} \) we have

\[
\sum_{i \in I} D_i^{\mu_y} (\sqrt{f_t}) \leq CN^{3\epsilon' + 2a + 2\nu}, \quad f_{t,y} = (f_t)_y, \quad I = I_{L,K}, \tag{6.19}
\]

and for any bounded observable \( O \)

\[
\left| [\mathbb{E}^{f_t,\mu_y} - \mathbb{E}^{\mu_y}] O(x) \right| \leq CK^{1/2} N^{2\epsilon' + a + \nu - 1/2}. \tag{6.20}
\]

Furthermore, for any \( k \in I \) we also have

\[
|\mathbb{E}^{f_t,\mu_y} x_k - \gamma_k | \leq CN^{-1 + \nu}. \tag{6.21}
\]

**Proof.** In this proof, we omit the subscript \( t \), i.e. we use \( f = f_t \). By definition of the conditional measure and by (6.15), we have

\[
\mathbb{E}^{f \mu} \sum_{i \in I} D_i^{\mu_y} (\sqrt{f}) = \sum_{i \in I} D_i^{\mu} (\sqrt{f}) \leq D^{\mu} (\sqrt{f}) \leq CN^{2a + 2\nu}.
\]

By Markov inequality, (6.19) holds for all \( y \) in a set \( G_{L,K}^1 \) with \( \mathbb{P}^{f \mu} (G_{L,K}^1) \geq 1 - CN^{-3\epsilon'} \). Without loss of generality, by (6.4) we can assume that \( G_{L,K}^1 \subset \mathcal{R}_{L,K} \). The estimate (6.20) now follows from (6.19) and (6.17).

Similarly, the rigidity bound (6.2) with respect to \( f \mu \) can be translated to the measure \( f_y \mu_y \), i.e. there exists a set \( G_{L,K}^2 \), with

\[
\mathbb{P}^{f_y \mu_y} ( G_{L,K}^2 ) \geq 1 - C_0 \exp \left( - \frac{1}{2} c_1 N^{c_2} \right),
\]

such that for any \( y \in G_{L,K}^2 \) and for any \( k \in I \), we have

\[
\mathbb{P}^{f_y \mu_y} ( |x_k - \gamma_k | \geq N^{-1 + \nu} ) \leq \exp \left( - \frac{1}{2} c_1 N^{c_2} \right).
\]

In particular, we can conclude (6.21) for any \( y \in G_{L,K}^2 \). Setting \( G_{L,K} := G_{L,K}^1 \cap G_{L,K}^2 \) we proved the lemma.

---

**Lemma 6.5** Fix \( a > 0, \nu > 0 \), and \( \tau \geq \tau_{\nu} \). Suppose the initial data \( f_0 \) of the DBM is given by a generalized Wigner ensemble. Then, with some small \( \epsilon' > 0 \), for any \( t \geq \tau N^{\epsilon'} \), \( k \in I \) and \( y \in G_{L,K} \), we have

\[
| \mathbb{E}^{\mu_y} x_k - \mathbb{E}^{f_t,\mu_y} x_k | \leq KN^{-3/2 + \nu + a + 2\epsilon'}. \tag{6.22}
\]

In particular, if the parameters chosen such that

\[
KN^{-3/2 + \nu + a + 2\epsilon'} \leq N^{-1} K^\xi, \quad \text{and} \quad N^{-1 + \nu} \leq N^{-1} K^\xi
\]

with some small \( \xi > 0 \), then

\[
| \mathbb{E}^{\mu_y} x_k - \alpha_k | \leq CN^{-1} K^\xi, \quad k \in I, \tag{6.23}
\]

where \( \alpha_k \) is defined in (4.4). In other words, the analogue of (4.10) is satisfied.
Notice that if we apply (6.20) with the special choice $O(x) = x_k$ then the error estimate would be much worse than (6.22). We wish to emphasize that (6.23) is not an obvious fact although we know that it holds for $y$ with high probability w.r.t. the equilibrium measure $\mu$. The key point of (6.23) is that it holds for any $y \in \mathcal{G}_{L,K}$ and thus with "high probability" w.r.t $f_t\mu$!

**Proof.** Once again, we omit the subscript $t$. The estimate (6.23) is a simple consequence of (6.22), (6.21) and (5.11). To prove (6.22), we run the reversible dynamics

$$\partial_t q_s = \mathcal{L}_y q_s \tag{6.24}$$

starting from initial data $q_0 = f_y$, where the generator $\mathcal{L}_y$ is the unique reversible generator with the Dirichlet form $D^{\mu_y}$, i.e.,

$$-\int f \mathcal{L}_y g \, d\mu_y = \sum_{i \in I} \frac{1}{2N} \int \nabla_i f \cdot \nabla_i g \, d\mu_y.$$

Recall that from the convexity bound (5.21), $\tau_K = K/N$ is the time to equilibrium of this dynamics. After differentiation and integration we get,

$$|\mathbb{E}^{\mu_y} x_k - \mathbb{E}^{f_y \mu_y} x_k| = \left| \int_0^{K' \tau_K} du \frac{1}{2N} \int (\partial_u q_u) d\mu_y \right| + O(\exp(-cK')).$$

From the Schwarz inequality with a free parameter $R$, we can bound the last line by

$$\frac{1}{N} \int_0^{K' \tau_K} du \int \left( R(\partial_u \sqrt{q_u})^2 + R^{-1} \right) d\mu_y + O(\exp(-cK')).$$

Dropping the trivial subexponential error term and using that the time integral of the Dirichlet form is bounded by the initial entropy $\mu$, we can bound the last line by

$$RS(f_y | \mu_y) + \frac{K' \tau_K}{NR}.$$ Using the logarithmic Sobolev inequality for $\mu_y$ and optimizing the parameter $R$, we can bound the last term by

$$|\mathbb{E}^{\mu_y} x_k - \mathbb{E}^{f_y \mu_y} x_k| \leq \tau_K R \sum_{i \in I} D_i^{\mu_y} (\sqrt{f_y}) + \frac{K' \tau_K}{NR} + O(\exp(-cK'))$$

$$\leq \frac{K' \tau_K}{N^{1/2}} \left( \sum_{i \in I} D_i^{\mu_y} (\sqrt{f_y}) \right)^{1/2} + O(\exp(-cK')). \tag{6.25}$$

Combining this bound with (6.19), we obtain (6.22).

**Lemma 6.6** Fix $n > 0$, $\alpha > 0$ and $\tau \geq N^{-\alpha}$. Then for sufficient small $\alpha$ there exist positive $\varepsilon$ and $\varepsilon'$ such that for any $t \geq \tau N^{\varepsilon'}$, for any $n$ and for any $n$-particle observable $O$ we have

$$\left| \mathbb{E}^{f_t \mu} - \mathbb{E}^{\mu} \right| O(N(x_j - x_{j+1}), N(x_j - x_{j+2}), \ldots, N(x_j - x_{j+n})) \leq C N^{-\varepsilon} \|O'\|_{\infty}, \tag{6.26}$$

for any $j \in [\alpha N, (1 - \alpha) N]$ and for any sufficiently large $N$. 

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Proof. We will apply Lemma 6.4, and we choose \( L = j \). Since \( K \leq N^{1/4} \), the right hand side of (6.20) is smaller than \( N^{-\epsilon} \). Then we have

\[
\left| \mathbb{E}_{x_1}^{L_1} \mathbb{E}_{x_2}^{L_2} - \mathbb{E}_{x_1}^{L_1} \mathbb{E}_{x_2}^{L_2} \right| \leq C N^{-\epsilon},
\]

for all \( y \in G_{L,K} \) with the probability of \( G_{L,K} \) satisfying (6.18). Choose any \( \tilde{y} \in R^* \) defined in Lemma 5.2. We now apply Theorem 4.1 with both \( \mu_y \) and \( \mu_x \) given by local Gaussian \( \beta \)-ensemble. Thus the estimate (4.10) is guaranteed by (5.7) and (6.23). Since \( y, \tilde{y} \in R = R_{L,K}(\xi^2/2, \alpha) \) and Lemma 4.5 guarantees (4.9), we can apply Theorem 4.1 so that

\[
\left| \mathbb{E}_{x_1}^{L_1} \mathbb{E}_{x_2}^{L_2} - \mathbb{E}_{x_1}^{L_1} \mathbb{E}_{x_2}^{L_2} \right| \leq C N^{-\epsilon} \left\| O' \right\|_\infty,
\]

for all \( y \in G_{L,K} \) and \( \tilde{y} \in R^* \). Since \( \mathbb{E}_y(R^*) \geq 1 - N^{-\epsilon} \), see (5.6), we have thus proved that

\[
\left| \mathbb{E}_{x_1}^{L_1} \mathbb{E}_{x_2}^{L_2} - \mathbb{E}_{x_1}^{L_1} \mathbb{E}_{x_2}^{L_2} \right| \leq C N^{-\epsilon} \left\| O' \right\|_\infty.
\]

for all \( y \in G_{L,K} \). From (6.27), (6.29) and the probability estimate (6.18) for \( G_{L,K} \), with possibly reducing \( \epsilon \) so that \( \epsilon \leq \epsilon' \), we obtain that

\[
\left| \mathbb{E}_{x_1}^{L_1} \mathbb{E}_{x_2}^{L_2} - \mathbb{E}_{x_1}^{L_1} \mathbb{E}_{x_2}^{L_2} \right| \leq C N^{-\epsilon} \left\| O' \right\|_\infty.
\]

This proves Lemma 6.6.

Recall that \( H_t \) is the generalized Wigner matrix whose matrix elements evolve by independent OU processes. Thus in Lemma 6.6 we have proved that the local statistics of \( H_t \), for \( t \geq N^{-2\epsilon+\epsilon'} \), is the same as the corresponding Gaussian one for any initial generalized matrix \( H_0 \). Finally, we need to approximate the generalized Wigner ensembles by Gaussian divisible ones. The idea of approximation first appeared in [24] via a “reverse heat flow” argument and was also used in [51] via a four moment theorem. We will follow the Green function comparison theorem of [29, 32] and in particular, the result in [43] since these results were formulated and proved for the generalized Wigner matrices.

Theorem 1.10 from [43] implies that if the first four moments of two generalized Wigner ensembles \( H^w \) and \( H^w \) are the same then

\[
\lim_{N \to \infty} \left[ \mathbb{E}_y^w - \mathbb{E}_x^w \right] \mathbb{P} (N(x_j - x_{j+1}), N(x_j - x_{j+2}), \ldots, N(x_j - x_{j+n})) = 0,
\]

provided that one of the ensembles, say \( H^w \), satisfies the following level repulsion estimate: For any \( \lambda > 0 \), there is an \( \alpha_0 > 0 \) such that for any \( \alpha \) satisfying \( 0 < \alpha \leq \alpha_0 \) there exists a \( \nu > 0 \) such that

\[
\mathbb{P} (N(E - N^{-1-\alpha}, E + N^{-1-\alpha}) \geq 2) \leq N^{-\alpha - \nu}
\]

for all \( E \in [-2 + \kappa, 2 - \kappa] \), where \( N(a, b) \) denotes the number of eigenvalues in the interval \((a, b)\). Although this theorem was stated with the assumption that all four moments of the matrix elements of the two ensembles match exactly, it in fact only requires that the first three moments match exactly and the differences of the fourth moments are less than \( N^{-\epsilon'} \) for some small \( \epsilon' > 0 \). The relaxation of the fourth moment assumption was carried out in details in [29, 32] and we will not repeat it here.

We now apply (6.31) with the choice \( H^w \) being the generalized Wigner ensemble for which we wish to prove the universality and \( H^w = H_t \) with \( t = N^{-\epsilon'} \) for some small \( \epsilon' \). The necessary
Lemma 6.6. This completes the proof of (2.9) showing that the local gap statistics with the same for different gap labels (2.10) follows now directly from Theorem 2.3 that, in particular, compares the local gap statistics $H$ and $N$ than $\alpha(2.10)$. The level repulsion bound for GOE/GUE is well-known; it also follows from part ii) of H $\|\alpha(see (5.4)). Finally we choose $O$ with an observable $\nu$ for $|x| > 1$. We can take for example $\theta(x) = (x - 1)^2 1_{x > 1} + (x + 1)^2 1_{x < -1}$ in the following.

For any $m \in [\alpha N, (1-\alpha)N]$ and any integer $1 \leq M \leq \alpha N$, we denote $I^{(m,M)} = [m-M, m+M]$ estimate on the level repulsion (6.32) follows from the gap universality and the rigidity estimate for $H_t$. More precisely, for any energy $E$ in the bulk, choose the index $k$ such that $|\gamma_k - E| \leq C/N$. Then from the rigidity estimate (6.1), we have for any $c > 0$ that

$$
P^w \left(N(E - N^{-1-\alpha}, E + N^{-1-\alpha}) \geq 2 \right) \leq \sum_{j: |j-k| \leq N^{\alpha}} P^w \left(\lambda_{j+1} - \lambda_j \leq N^{-1-\alpha}\right) + e^{-N^\epsilon}$$

$$\leq \sum_{j: |j-k| \leq N^{\epsilon}} \left[P^\mu \left(\lambda_{j+1} - \lambda_j \leq N^{-1-\alpha}\right) + CN^{-\epsilon}\right] + e^{-N^\epsilon}$$

$$\leq CN^{\epsilon} N^{-(\beta+1)\alpha + \epsilon} + CN^{\alpha - \epsilon}.$$  

Here in the first inequality we used the rigidity (6.1) and in the second inequality we used (6.30) with an observable $O$ that is a smoothed version of the characteristic function on scale $N^{-\alpha}$, i.e. $\|O\|_\infty \leq CN^{\alpha}$. In the last step we used the level repulsion bound for GOE/GUE for $\beta = 1$ or 2, respectively. The level repulsion bound for GOE/GUE is well-known; it also follows from part ii) of Theorem 4.3 and the fact that (4.18) holds for all $y \in R_{L,K}$, i.e. with a very high probability (see (5.4)). Finally we choose $a_0 \leq \epsilon/4$. Then for any $\alpha < a_0$, there exist small exponents $\nu, c, \xi$ such that $\nu + c + \xi < \alpha$. This proves (6.32) for the ensemble $H_t$.

Following [32], we construct an auxiliary Wigner matrix $H_0$ such that the first three moments of $H_t$ and the original matrix $H^y$ are identical while the differences of the fourth moments are less than $N^{-c''}$ for some small $c'' > 0$ depending on $c'$ (see Lemma 3.4 of [32]). The gap statistics of $H^y$ and $H^w = H_t$ coincide by (6.31) and the gap statistics of $H_t$ coincides with those of GUE/GOE by Lemma 6.6. This completes the proof of (2.9) showing that the local gap statistics with the same gap-label $j$ is identical for the generalized Wigner matrix and the Gaussian case. The proof of (2.10) follows now directly from Theorem 2.3 that, in particular, compares the local gap statistics for different gap labels ($k$ and $m$) in the Gaussian case. This completes the proof Theorem 2.2.

7 Rigidity and level repulsion of local measures

7.1 Rigidity of $\mu_y$: proof of Theorem 4.2

We will prove Theorem 4.2 using a method similar to the proof of Theorem 3.1 in [7]. Theorem 3.1 of [7] was proved by a quite complicated argument involving induction on scales and the loop equation. The loop equation, however, requires analyticity of the potential and it cannot be applied to prove Theorem 4.2 for a local measure whose potential $V_y$ is not analytic. We note, however, that in [7] the loop equation was used only to estimate the expected locations of the particles. Now this estimate is given as a condition by (4.12) and thus we can adapt the proof in [7] to the current setting. For later application, however, we will need a stronger form of the rigidity bound, namely we will establish that the tail of the gap distribution has a Gaussian decay. This stronger statement requires some modifications to the argument from [7] which therefore we partially repeat here. We now introduce the notations needed to prove Theorem 4.2.

Let $\theta$ be a continuously differentiable nonnegative function with $\theta = 0$ on $[-1, 1]$ and $\theta'' \geq 1$ for $|x| > 1$. We can take for example $\theta(x) = (x - 1)^2 1_{x > 1} + (x + 1)^2 1_{x < -1}$ in the following.

For any $m \in \lbrack \alpha N, (1-\alpha)N \rbrack$ and any integer $1 \leq M \leq \alpha N$, we denote $I^{(m,M)} = [m-M, m+M]$
and $M = |I^{(m,M)}| = 2M + 1$. Let $\eta := \xi/3$. For any $k, M$ with $|k - L| \leq K - M$, define

$$\phi^{(k,M)}(x) := \sum_{i < j, i,j \in I^{(k,M)}} \theta \left( \frac{N(x_i - x_j)}{MK^{2\eta}} \right).$$  \hspace{1cm} (7.1)  \hspace{1cm} \text{def:phi}

Let

$$\omega^{(k,M)} = Z_{k,M} \mu e^{-\phi^{(k,M)}},$$

where $Z_{k,M}$ is a normalization constant. Choose an increasing sequence of integers, $M_1 < M_2 < \ldots < M_A$ such that $M_1 = K^\xi$, $M_A = CK^{1-2\eta}$ with a large constant $C$, and $M_\gamma/M_{\gamma-1} \sim K^\eta$ (meaning that $cK^\eta \leq M_\gamma/M_{\gamma-1} \leq CK^\eta$). We can choose the sequence such that $A \leq CN^{-1}$. We set $\omega := \omega^{(k,M)}$ and we study the rigidity properties of the measures $\omega_A, \omega_{A-1}, \ldots, \omega_1$ in this order. Note that $\mu = \omega_A$ since $y \in R(L,K) = R(L,K)(\xi\delta/2, \alpha)$ guarantees that $|x_i - x_j| \leq |y| \leq CK/N$, see (4.22), thus for $M = M_A = CK^{1-2\eta}$ the argument of $\theta$ in (7.1) is smaller than 1, so $\phi \equiv 0$ in this case. We also introduce the notation

$$x^{[M]}_k := \frac{1}{2M + 1} \sum_{j=k-M}^{k+M} x_j.$$

**Definition 7.1** We say that $\mu_y$ has exponential rigidity on scale $\ell$ if there are constants $C, c$, such that the following bound holds

$$\mathbb{P}^{\mu_y}( |x_k - \alpha_k| \geq \ell + uK^{\xi}N^{-1} ) \leq Ce^{-cu^2}, \quad u > 0,$$  \hspace{1cm} (7.2)  \hspace{1cm} \text{rigg}

for any $k \in I$.

First we prove that $\mu_y$ has exponential rigidity on scale $M_AN^{-1}$. Starting from $\gamma = A$, by the Herbst bound and the logarithmic Sobolev inequality for $\mu_y$ with LSI constant of order $K/N$ (6.16), we have for any $k \in [L - K + M_A, L + K - M_A]$ that

$$\mathbb{P}^{\mu_y} \left( |x^{[M_A]}_k - \mathbb{E}^{\mu_y} x^{[M_A]}_k| \geq \frac{b}{\sqrt{M_A}} \right) \leq e^{-c(N/K)N^2}, \quad b \geq 0,$$  \hspace{1cm} (7.3)

i.e.

$$\mathbb{P}^{\mu_y} \left( |x^{[M_A]}_k - \mathbb{E}^{\mu_y} x^{[M_A]}_k| \geq \frac{uK^\eta}{N} \right) \leq Ce^{-cu^2}.$$  \hspace{1cm} (7.4)  \hspace{1cm} \text{A1}

Using the estimate (6.23) we have that

$$\left| \mathbb{E}^{\mu_y} x^{[M_A]}_k - \alpha^{[M_A]}_k \right| \leq CN^{-1}K^\xi.$$

Thus we obtain

$$\mathbb{P}^{\mu_y} \left( |x^{[M_A]}_k - \alpha^{[M_A]}_k| \geq CN^{-1}K^\xi + \frac{uK^\eta}{N} \right) \leq Ce^{-cu^2}.$$  \hspace{1cm} (7.5)  \hspace{1cm} \text{Aav}

Since $x^{[M]}_{k-M} \leq x_k \leq x^{[M]}_{k+M}$ and the $\alpha_k$’s are regular with spacing of order $1/N$, we get

$$x_k - \alpha_k \leq x^{[M]}_{k+M} - \alpha^{[M]}_{k-M} \leq x^{[M]}_{k+M} - \alpha^{[M]}_{k+M} + CMN^{-1}$$
and we also have a similar lower bound. Thus

$$\mathbb{P}^{\mu_2\gamma}\left( |x_k - \alpha_k| \geq C M_A N^{-1} + \frac{u K^\gamma}{N} \right) \leq C e^{-c u^2}$$  \hspace{1cm} (7.6)  

for any $k \in [L-K+2M, L+K-2M]$, where we used that $M_A \geq K^\xi$. If $k \in [L-K, L-K+2M]$, then

$$x_k - \alpha_k \leq x_{L-K+2M} - \alpha_{L-K+2M} + C M_A N^{-1}$$

and

$$x_k - \alpha_k \geq y_{L-K-1} - \alpha_k \geq -C M_A N^{-1}.$$ 

Thus we have the estimate

$$|x_k - \alpha_k| \leq |x_{L-K+2M} - \alpha_{L-K+2M}| + C M_A N^{-1}.$$ 

Since (7.6) holds for the difference $x_{L-K+2M} - \alpha_{L-K+2M}$, we have that it holds for $x_k - \alpha_k$ as well (with at most an adjustment of $C$) for any $k \in [L-K, L-K+2M]$. Similar argument holds for $k \in [L + K - 2M, L + K]$. Thus we proved (7.6) for all $k \in [L-K, L+K]$, i.e. we showed exponential rigidity on scale $M_A N^{-1}$.

Now we use an induction on scales and we show that if

(i) for any $k \in [L-K + M_\gamma, L+K-M_\gamma]$ we have

$$\mathbb{P}^{\mu_2\gamma}\left( |x_k^{[M_\gamma]} - \alpha_k^{[M_\gamma]}| \geq u K^\xi N^{-1} \right) \leq C e^{-c u^2}, \quad u \geq 0;$$  \hspace{1cm} (7.7)  

(ii) exponential rigidity holds on some scale $M_\gamma N^{-1}$,

$$\mathbb{P}^{\mu_2\gamma}\left( |x_k - \alpha_k| \geq C M_\gamma N^{-1} + u K^\xi N^{-1} \right) \leq C e^{-c u^2}, \quad k \in I, \quad u \geq 0;$$  \hspace{1cm} (7.8)  

(iii) we have the entropy bound

$$S(\mu_2\gamma | \omega_\gamma) \leq C e^{-c M_\gamma^2 K^{-5\eta}},$$  \hspace{1cm} (7.9)  

then (i)–(iii) also hold with $\gamma$ replaced by $\gamma - 1$ as long as $M_{\gamma-1} \geq K^\xi$. The iteration can be started from $\gamma = A$, since (7.7) and (7.8) were proven in (7.5) and in (7.6) (even with a better bound), and (7.9) is trivial for $\gamma = A$ since $\omega_A = \mu_2 x_\gamma$.

We first notice that on any scale $M_\gamma$, the bound (7.7) implies (7.8) by the same argument as we concluded (7.6) for any $k \in I$ from (7.5). So we can focus on proving (7.7) and (7.9) on the scale $M_{\gamma-1}$.

To prove (7.9) on scale $M_{\gamma-1}$, notice that (7.8), with the choice $u = M_\gamma K^{-\xi}$, implies

$$\mathbb{P}^{\mu_2\gamma}\left( |x_k - \alpha_k| \geq C M_\gamma N^{-1} \right) \leq C e^{-c M_\gamma^2 K^{-2\eta}}, \quad k \in I.$$  \hspace{1cm} (7.10)  

Since

$$\theta\left( \frac{N(x_i - x_j)}{M_{\gamma-1} K^{2\eta}} \right) = 0$$

unless $|x_i - x_j| \geq C M_{\gamma-1} N^{-1} K^{2\eta} \geq C M_{\gamma} N^{-1} K^{\eta}$, we have that the scale $C M_\gamma N^{-1}$ is by a factor $K^\eta$ smaller than the scale of $x_i - x_j$ built into the definition of $\phi(k, M_{\gamma-1})$, see (7.1). But for
\(i, j \in I^{(k, M_{\gamma-1})}\) we have \(|x_i - x_j| \leq |x_i - \alpha_i| + |x_j - \alpha_j| + CM_{\gamma-1} N^{-1}\). Thus \(\phi^{(k, M_{\gamma-1})} = 0\) unless we are on the event described in (7.10) at least for one \(k\). Moreover, \(|\nabla \phi^{(k, M_{\gamma-1})}(x)| \leq N^{C}\) for any configuration \(x\) in \(J\). Thus, following the argument in Lemma 3.15 of [7], via the logarithmic Sobolev inequality for \(\mu_y\), we get
\[
S(\mu_y | \omega_{\gamma - 1}) \leq CKN^{-1} \mathbb{E}^{\mu_y} |\nabla \phi^{(k, M_{\gamma-1})}|^2 \leq CN^C e^{-c M^2 K^{-2\xi}} \leq C e^{-c M_{\gamma-1} K^{-5\eta}}. \tag{7.11}
\]
Here we used that the prefactor \(N^C\) can be absorbed in the exponent by using that \(M^2 K^{-2\xi} - M_{\gamma-1} K^{-5\eta} \geq K^{2\xi-5\eta} = K^{\eta} \geq N^{\eta}\). Here we have used \(\xi = 3\eta\) and \(M_{\gamma-1} \geq K^{\xi}\). We will not need it here, but we note that the same bound on the opposite relative entropy,
\[
S(\omega_{\gamma - 1} | \mu_y) \leq C e^{-c M_{\gamma-1} K^{-5\eta}},
\]
is also correct. Thus (7.9) for \(\gamma - 1\) is proved.

Now we focus on proving (7.7) on the scale \(M_{\gamma-1}\). Set \(1 \leq M' \leq M \leq K\) and fix an index \(k \in I\) such that \(|k - L| \leq K - M\). We state the following slightly generalized version of Lemma 3.14 of [7]

**Lemma 7.2** For any integers \(1 \leq M' \leq M \leq K\), \(k \in [L - K + M, L + K - M]\) and \(k' \in [k - M + M', k + M - M']\), we have
\[
\mathbb{P}^{\omega^{(k, M)}} \left( |\lambda^{[M']}_{k'} - \lambda^{[M]}_{k'} - \mathbb{E}^{\omega^{(k, M)}} (\lambda^{[M']}_{k'} - \lambda^{[M]}_{k'})| > \frac{u K^{2\eta}}{N} \sqrt{\frac{M}{M'}} \right) \leq C e^{-c u^2}. \tag{7.12}
\]

Compared with Lemma 3.14 of [7], we first note that \(N^\xi\) in Lemma 3.14 [7] is changed to \(K^{2\eta}\) due to that \(\phi^{(k, M)}(x)\) in (7.1) is defined with a \(K^{2\eta}\) factor instead of \(N^\xi\). Furthermore, here we allowed the center at the scale \(M'\) to be different from \(k\). The only condition is that the interval \([k' - M', k' + M'] \subset [k - M, k + M]\). The proof of this lemma is identical to that of Lemma 3.14 of [7].

In particular, for any \(\gamma = 2, 3, \ldots A\) and with \(M' = M_{\gamma-1}\) and \(M = M_{\gamma} \leq K\), \(M_{\gamma-1} \leq K^{\eta} M_{\gamma-1}\) and with any choice of \(k_{\gamma} \in [L - K + M_{\gamma}, L + K - M_{\gamma}]\), \(k_{\gamma-1} \in [L - K + M_{\gamma-1}, L + K - M_{\gamma-1}]\), so that \([k_{\gamma-1} - M_{\gamma-1}, k_{\gamma} + M_{\gamma}-1] \subset [k_{\gamma} - M_{\gamma}, k_{\gamma} + M_{\gamma}]\), we get
\[
\mathbb{P}^{\omega_{\gamma}} \left( |x^{[M_{\gamma-1}]}_{k_{\gamma-1}} - x^{[M_{\gamma}]}_{k_{\gamma}} - \mathbb{E}^{\omega_{\gamma}} (x^{[M_{\gamma-1}]}_{k_{\gamma-1}} - x^{[M_{\gamma}]}_{k_{\gamma}})| > \frac{u K^{5\eta/2}}{N} \right) \leq C e^{-c u^2}. \tag{7.12}
\]

The entropy bound (7.9) and the boundedness of \(x_k\) imply that
\[
|\mathbb{E}^{\omega_{\gamma}} x_k - \mathbb{E}^{\mu_y} x_k| \leq C \sqrt{S(\mu_y | \omega_{\gamma})} \leq C e^{-c M^2 K^{-5\eta}};
\]
where \(M^2 K^{-5\eta} \geq K^{2\xi-5\eta} \geq K^{\eta}\). We can combine it with (4.12) to have
\[
|\mathbb{E}^{\omega_{\gamma}} x_k - \alpha_k| \leq CK^{\xi}/N.
\]
The measure \(\omega_{\gamma}\) in (7.12) can also be changed to \(\mu_y\) at the expense of an entropy term \(S(\mu_y | \omega_{\gamma})\). Using (7.9), we thus have
\[
\mathbb{P}^{\mu_y} \left( |x^{[M_{\gamma-1}]}_{k_{\gamma-1}} - x^{[M_{\gamma}]}_{k_{\gamma}} - (\alpha^{[M_{\gamma-1}]}_{k_{\gamma-1}} - \alpha^{[M_{\gamma}]}_{k_{\gamma}})| \geq CK^{\xi} N^{-1} + \frac{u K^{5\eta/2}}{N} \right) \leq C e^{-c u^2} + C e^{-c M^2 K^{-5\eta}}. \tag{7.13}
\]
Combining it with (7.7) and recalling $\xi = 3\eta$, we get

$$
\mathbb{P}^\mu_y \left( \left| x_k^{[M_y]} - \alpha_k^{[M_y]} \right| \geq CK^\xi N^{-1} + \frac{uK^\xi}{N} \right) \leq Ce^{-cu^2} + Ce^{-cM_y^2K^{-5n}}. \tag{7.14}
$$

This gives (7.7) on scale $M_y$ if $u \leq cM_yK^{-5n/2}$ with a small constant $c$. Suppose now that $u \geq cM_yK^{-5n/2}$, which, in particular, means that $u \geq cK^{-n/2}$. Then, by (7.8), we have

$$
\mathbb{P}^\mu_y \left( \left| x_k^{[M_y]} - \alpha_k^{[M_y]} \right| \geq CK^\xi N^{-1} + \frac{uK^\xi}{N} \right)
\leq \mathbb{P}^\mu_y \left( \left| x_k^{[M_y]} - \alpha_k^{[M_y]} \right| \geq CM_yN^{-1} + (1 - CK^{-n/2})uK^\xi \right)
\leq \sum_{k \in I} \mathbb{P}^\mu_y \left( |x_k - \alpha_k| \geq CM_yN^{-1} + (1 - CK^{-n/2})uK^\xi \right)
\leq CK e^{-c(1 - CK^{-n/2})u^2} \leq Ce^{-cu^2}.
$$

This proves (7.7) for $\gamma - 1$. Note that the constants slightly deteriorate at each iteration step, but the number of iterations is finite (of order $1/\eta = 3/\xi$), so eventually the constants $C, c$ in (4.13) may depend on $\xi$. In fact, since the deterioration is minor, one can also prove (4.13) with $\xi$-independent constants, but for simplicity of the presentation we did not follow the change of these constants at each step.

After completing the iteration, from (7.8) for $\gamma = 1$, $M_1 = K^\xi$, we have

$$
\mathbb{P}^\mu_y \left( |x_k - \alpha_k| \geq CK^\xi N^{-1} + uK^\xi N^{-1} \right) \leq Ce^{-cu^2}, \quad k \in I;
$$

This concludes (4.13) for $u \geq 1$. Finally, (4.13) is trivial for $u \leq 1$ if the constant $C$ is sufficiently large. This completes the proof of Theorem 4.2.

### 7.2 Level repulsion estimates of $\mu_y$: proof of Theorem 4.3

We now prove the level repulsion estimate, Theorem 4.3, for the local log-gas $\mu_y$ with good boundary conditions $y$. There are two key ideas in the following argument. We first recall the weak level repulsion estimate (4.11) in [7], which in the current notation asserts

$$
\mathbb{P}^\mu_y (x_L - K - y_L - K - 1 \leq s/N) \leq CN s
$$

for any $s > 0$, and similar estimates may be deduced for internal gaps. Compared with (4.14), this estimate does not contain any $\beta$ exponent, moreover, in order to obtain (4.17), the $N$ factor has to be reduced to $K^\xi$ (neglecting the irrelevant logarithmic factor). Our first idea is to run this proof for a local measure with only $K^\xi$ particles to reduce the $N$ factor to $K^\xi$. The second idea involves introducing some auxiliary measures to catch some of the $\beta$ related factors. We first introduce these two auxiliary measures which are slightly modified versions of the local equilibrium measure:

$$
\mu_0 := \mu_{y,0} = Z_0 (x_L - K - y_L - K - 1)^{-\beta} \mu_y; \quad \mu_1 := \mu_{y,1} = Z_1 W^{-\beta} \mu_y, \tag{7.15}
$$

$$
W = (x_L - K - y_L - K - 1)(x_L - K + 1 - y_L - K - 1), \tag{7.16}
$$

where $Z_0, Z_1$ are chosen for normalization. In other words, we drop the term $(x_L - K - y_L - K - 1)^\beta$ from the measure $\mu_y$ in $\mu_0$ and we drop $W^\beta$ in $\mu_1$. To estimate the upper gap, $y_{L+K+1} - x_L + K$, we...
Lemma 7.3 Let $L$ and $K$ satisfy (4.1) and consider the local equilibrium measure $\mu_y$ defined in (4.5).

i) Let $\xi, \alpha$ be any fixed positive constants and let $y \in \mathcal{R}_{L,K}(\xi, \alpha)$. Then for any $s > 0$ we have

$$P^{\mu_y}[x_{L-K} - y_{L-K-1} \leq s/N] \leq C (K s \log N)^{\beta + 1}, \quad (7.17)$$

and

$$P^{\mu_y}[x_{L-K+1} - y_{L-K-1} \leq s/N] \leq C (K s \log N)^{2\beta + 1}. \quad (7.18)$$

ii) Let $y$ be arbitrary with the only condition that $|y_i| \leq C$ for all $i$. Then for any $s > 0$ we have the weaker estimate

$$P^{\mu_y}[x_{L-K} - y_{L-K-1} \leq s/N] \leq \left(\frac{C s K^{\beta + 1}}{|J_{\alpha}|}\right), \quad (7.19)$$

$$P^{\mu_y,j}[x_{L-K+1} - y_{L-K-1} \leq s/N] \leq \left(\frac{C s K^{2\beta + 1}}{|J_{\alpha}|}\right), \quad j = 0, 1. \quad (7.20)$$

To prove Lemma 7.3, we first prove estimates even weaker than (7.17)–(7.20) for $\mu_y$ and $\mu_{y,j}$.

Lemma 7.4 Let $L$ and $K$ satisfy (4.1).

i) Let $\xi, \alpha$ be any fixed positive constants and let $y \in \mathcal{R}_{L,K} = \mathcal{R}_{L,K}(\xi, \alpha)$, then we have for any $s > 0$

$$P^{\mu_y}(x_{L-K} - y_{L-K-1} \leq s/N) \leq C K s \log N, \quad (7.21)$$

$$P^{\mu_y,j}(x_{L-K} - y_{L-K-1} \leq s/N) \leq C K s \log N, \quad j = 0, 1. \quad (7.22)$$

ii) Let $y$ be arbitrary with the only condition that $|y_i| \leq C$ for all $i$. Then for any $s > 0$ we have the weaker estimate

$$P^{\mu_y}(x_{L-K} - y_{L-K-1} \leq s/N) \leq \frac{C s K}{|J_{\alpha}|}, \quad (7.23)$$

$$P^{\mu_y,j}(x_{L-K} - y_{L-K-1} \leq s/N) \leq \frac{C s K}{|J_{\alpha}|}, \quad j = 0, 1. \quad (7.24)$$

Proof. We will prove (7.21), the same proof with only change of notations works for (7.22) case as well. We will comment on this at the end of the proof.

For notational simplicity, we first shift the coordinates by $S$ such that in the new coordinates $\tilde{y} = 0$, i.e. $y_{L-K-1} = -y_{L+K+1}$ and $J$ is symmetric to the origin. With the notation $a := -y_{L-K-1}$ and $I = [L - K, L + K]$, we first estimate the following quantity, for any $0 \leq \varphi \leq c$ (with a small constant)

$$Z_{\varphi} := \int \ldots \int_{a - a \varphi}^{a + a \varphi} dx \prod_{i < j} (x_i - x_j)^{\beta} e^{-N \varphi \sum_{j} V_{\alpha}(S + x_j)}$$

$$= (1 - \varphi)^{K + \beta K(1 - 1/2)} \int \ldots \int_{a} dw \prod_{i < j} (w_i - w_j)^{\beta} e^{-N \varphi \sum_{j} V_{\alpha}(S + (1 - \varphi)w_j)},$$
where we set
\[ w_j := (1 - \varphi)^{-1} x_{L+j}, \quad dx = \prod_{|j| \leq K} dx_{L+j} \quad dw = \prod_{|j| \leq K} dw_j. \] (7.25)

By definition,
\[ e^{-N\hat{H}_N(S+(1-\varphi)w_j)} = e^{-N\hat{H}_N(S+(1-\varphi)w_j)} \prod_{k \leq L-K-1} ((1 - \varphi)w_j - y_k)^\beta \prod_{k \geq L+K+1} (y_k - (1 - \varphi)w_j)^\beta. \] (7.26)

For the smooth potential \( V \), we have
\[ |V(S + (1 - \varphi)w_j) - V(S + w_j)| \leq C|\varphi w_j| \leq \frac{CK\varphi}{N} \] (7.27)
with a constant depending on \( V \), where we have used \( |w_j| \leq a \leq CK/N \) which follows from \( |J_y| \leq CK/N \) due to \( y \in \mathcal{R}_{L,K} \), see (4.22).

Using \( (1 - \varphi)w_j - y_k \geq (1 - \varphi)(w_j - y_k) \) for \( L - 2K \leq k \leq L - K - 1 \) and the identity
\[ (1 - \varphi)w_j - y_k = (w_j - y_k)\left[1 - \frac{\varphi w_j}{w_j - y_k}\right] \]
for any \( k \), we have
\[ \prod_{k \leq L-K-1} ((1 - \varphi)w_j - y_k)^\beta \geq (1 - \varphi)^{\beta K} \prod_{k \leq L-K-1} (w_j - y_k)\prod_{n \leq L-2K} \left[1 - \frac{\varphi w_j}{w_j - y_n}\right]^{\beta}, \] (7.28)
and a similar estimate holds for \( k \geq L + K + 1 \). After multiplying these estimates for all \( j = 1, 2, \ldots, K \), we thus have the bound
\[ \frac{Z_\varphi}{Z_0} \geq \left[ e^{-C\beta K\varphi} (1 - \varphi)^{\beta K} \min_{|w| \leq a} \left( \prod_{k \leq L-2K} \left[1 - \frac{\varphi w_j}{w_j - y_k}\right]^{\beta} \prod_{k > L+2K} \left[1 - \frac{\varphi w_j}{y_k - w}\right]^{\beta} \right) \right]^K. \] (7.29)

Recall that \( y \in \mathcal{R}_{L,K} \), i.e. we have the rigidity bound for \( y \) with accuracy \( N^{-1}K^6 \ll K/N \ll a \), see (4.7), i.e. \( y_k \)'s are regularly spaced on scale \( a \) or larger. Combining this with \( |w| \leq a \leq CK/N \), we have
\[ \sum_{k \leq L-2K} \frac{\varphi w_j}{w_j - y_k} \leq C\varphi K \log N. \] (7.30)
Hence
\[ \prod_{k \leq L-2K} \left[1 - \frac{\varphi w_j}{w_j - y_k}\right]^{\beta} \geq 1 - C\varphi K \log N, \] (7.31)
and similar bounds hold for the \( k \geq L + 2K \) factors. Thus for any \( \varphi \leq c \) we get
\[ \frac{Z_\varphi}{Z_0} \geq 1 - C(\beta K^2 + K^2 \log N) \varphi \geq 1 - CK^2 \varphi (\log N). \]

Now we choose \( \varphi := s/(aN) \) and recall \( a \sim K/N \). Therefore the \( \mu_y \)-probability of \( x_{L+1} - y_L \geq a\varphi = s/N \) can be estimated by
\[ \mathbb{P}^{\mu_y}(x_{L-K} - y_{L-K-1} \geq s/N) \geq \frac{Z_\varphi}{Z_0} \geq 1 - CK s(\log N). \]
for all $sK \log N$ sufficiently small. If $sK \log N$ is large, then (7.21) is automatically satisfied. This proves (7.21).

In order to prove (7.23), we now drop the assumption $y \in \mathcal{R}_{L,K}$ and replace it with $|y_i| \leq C$. Instead of (7.28), we now have

$$
\prod_{k \leq L-K-1} ((1-\varphi)w_j - y_k)^\beta \geq (1-\varphi)^{\beta N} \prod_{k \leq L-K-1} (w_j - y_k)^\beta,
$$

(7.32)

and a similar estimate holds for $k \geq L + K + 1$. We thus have the bound

$$
\mathbb{P}^{\mu_{y}}(x_{L-K} - y_{L-K-1} \geq s/N) \geq \left[ e^{-C\beta K\rho(1-\varphi)^{\beta N}} \right]^K \geq 1 - C\varphi NK.
$$

(7.33)

With the choice $\varphi := s/(|J_1|N)$ this proves (7.23).

The proof of (7.22) and (7.24) for $\mu_{y,0}$ is very similar, just the $k = L - K - 1$ factor is missing from (7.26) in case of $j = -K$. For $\mu_{y,1}$, two factors are missing. These modifications do not alter the basic estimates. This concludes the proof of Lemma 7.4.

Proof of Lemma 7.3. Recalling the definition of $\mu_0$ and setting $X := x_{L-K} - y_{L-K-1}$ for brevity, we have

$$
\mathbb{P}^{\mu_{y}}[X \leq s/N] = \frac{\mathbb{E}^{\mu_0}[1(X \leq s/N)X^\beta]}{\mathbb{E}^{\mu_0}[X^\beta]},
$$

(7.34)

From (7.22) we have

$$
\mathbb{E}^{\mu_0}[1(X \leq s/N)X^\beta] \leq C(s/N)^\beta Ks \log N
$$

and with the choice $s = cK^{-1}(\log N)^{-1}$ in (7.22) we also have

$$
\mathbb{P}^{\mu_0}(X \geq \frac{c}{NK \log N}) \geq 1/2
$$

with some positive constant $c$. This implies that

$$
\mathbb{E}^{\mu_0}[X^\beta] \geq \frac{1}{2} \left( \frac{c}{NK \log N} \right)^\beta.
$$

We have thus proved that

$$
\mathbb{P}^{\mu_{y}}[X \leq s/N] \leq C(s/N)^\beta Ks \log N (NK \log N)^\beta = C (Ks \log N)^{\beta+1},
$$

(7.35)

i.e. we obtained (7.17).

For the proof of (7.18), we similarly use

$$
\mathbb{P}^{\mu_{y}}[x_{L-K+1} - y_{L-K-1} \leq s/N] = \frac{\mathbb{E}^{\mu_1}[1(x_{L-K+1} - y_{L-K-1} \leq s/N)W^\beta]}{\mathbb{E}^{\mu_1}[W^\beta]},
$$

(7.36)

From (7.22) we have

$$
\mathbb{E}^{\mu_1}[1(x_{L-K+1} - y_{L-K-1} \leq s/N)W^\beta] \leq (s/N)^{2\beta} \mathbb{E}^{\mu_1}[x_{L-K} - y_{L-K-1} \leq s/N] \leq C(s/N)^{2\beta} Ks \log N.
$$
By the same inequality and with the choice \( s = cK^{-1}(\log N)^{-1} \), we have

\[
\mathbb{P}^y \left( W \geq \frac{c}{(NK \log N)^2} \right) \geq 1/2
\]

with some positive constant \( c \). This implies that

\[
\mathbb{E}^y[W^\beta] \geq \frac{1}{2} \left( \frac{c}{(NK \log N)^2} \right)^\beta.
\]

We have thus proved that

\[
\mathbb{P}^y[x_{L-K+1} - y_{L-K-1} \leq s/N] \leq C(\beta)(s/N)^{2\beta}Ks \log N \left((NK \log N)^2\right) = C \left(Ks \log N\right)^{2\beta+1},
\]

which proves (7.18). Finally, (7.19) and (7.20) can be proved using (7.23) and (7.24). This completes the proof of Lemma 7.3.

**Proof of Theorem 4.3.** For a given \( i \), define the index set

\[
\tilde{I} := [\max(i - K^\xi, L - K - 1), \min(i + K^\xi, L + K + 1)]
\]

to be the indices in a \( K^\xi \) neighborhood of \( i \). We further condition the measure \( \mu_y \) on the points

\[
z_j := x_j \quad j \in \tilde{I} := I_{L,K} \setminus \tilde{I}
\]

and we let \( \mu_{y,z} \) denote the conditional measure on the remaining \( z \) variables \( \{x_j : j \in \tilde{I}\} \). Setting \( L' = i, K' = K^\xi \), from the rigidity estimate (4.16) we have \( (y, z) \in \mathcal{R} = \mathcal{R}_{L',K'}(\xi^2 \delta/2, \alpha) \) with a very high probability w.r.t. \( \mu_y \). We will now apply (7.17) to the measure \( \mu_{y,z} \) with a new \( \delta' = \delta \xi \) and \( K' = K^\xi \). This ensures that the condition \( N^{\delta'} \leq K' \) is satisfied and by the remark after (4.1), the change of \( \delta \) affects only the threshold \( N_0 \). We obtain

\[
\mathbb{P}^{\mu_{y,z}}[x_i - x_{i+1} \leq s/N] \leq C \left(K^\xi s \log N\right)^{\beta+1}
\]

with a high probability in \( z \) w.r.t. \( \mu_y \). The subexponential lower bound on \( s \), assumed in part ii) of Theorem 4.3, allows us to include the probability of the complement of \( \mathcal{R} \) in the estimate, we thus have proved (4.17). Similar argument but with (7.17) replaced by (7.18) yields (4.18).

To prove the weaker bounds (4.14), (4.15) for any \( s > 0 \), we may assume that \( L - K \leq i \leq L; i > L \) is treated similarly. Since \( y \in \mathcal{R}_{L,K} \), we have \( |J_y| \geq cK/N \). We consider two cases, either \( x_i - y_{L-K-1} \leq c'K/N \) or \( x_i - y_{L-K-1} \geq c'K/N \) with \( c' < c/2 \). In the first case, we condition on \( x_{L-K}, \ldots, x_i \) and we apply (7.23) to the measure \( \nu_i = \mu_{y,x_{L-K},\ldots,x_i} \). The configuration interval of this measure has length at least \( cK/(2N) \), so we have

\[
\mathbb{P}^{\nu_{i}}(x_{i+1} - x_i \leq s/N) \leq \frac{CKs}{cK/(2N)} \leq CNS.
\]

In the second case, \( x_i - y_{L-K-1} \geq c'K/N \), we condition on \( x_{i+1}, x_{i+2}, \ldots, x_{L+K} \). The corresponding measure, denoted by \( \nu_2 = \mu_{y,x_{i+1},\ldots,x_{L+K}} \), has a configuration interval of length at least \( c'K/N \). We can now have the estimate (7.39) for \( \nu_2 \). Putting these two estimates together, we have proved (4.14). Finally (4.15) can be proved in a similar way. This completes the proof of Theorem 4.3.
8 Proof of Theorem 4.1

8.1 Comparison of the local statistics of two local measures

In this section, we start to compare gap distributions of two local log-gases on the same configuration interval but with different external potential and boundary conditions. We will express the differences of gap distributions between two measures in terms of random walks in time dependent random environments. From now on, we use microscopic coordinates and we relabel the indices so that the coordinates of \( x_j \) are \( j \in I = \{-K, \ldots, 0, 1, \ldots, K\} \), i.e. we set \( L = \tilde{L} = 0 \) in the earlier notation. This will have the effect that the labelling of the external points \( y \) will not run from 1 to \( N \), but from some \( L^- \leq 0 \) to \( L^+ > 0 \) with \( L_+ - L_- = N \). The important input is that the index set \( I \) of the internal points is macroscopically separated away from the edges, i.e. \( |L_\pm| \geq \alpha N \).

The local equilibrium measures and their Hamiltonians will be denoted by the same symbols, \( \mu_y \) and \( \mathcal{H}_y \), as before, but with a slight abuse of notations we redefine them now to the microscopic scaling. Hence we have two measures \( \mu_y = e^{-\beta H_y}/Z_y \) and \( \mu_{\tilde{y}} = e^{-\beta \tilde{H}_{\tilde{y}}}/Z_{\tilde{y}} \), defined on the same configuration interval \( J = J_y = J_{\tilde{y}} \) with center \( \bar{y} \), which, for simplicity, we assumed \( \bar{y} = 0 \). The local density at the center is \( \rho(0) > 0 \).

The Hamiltonian is given by

\[
\mathcal{H}_y(x) := \sum_{i \in I} \frac{1}{2} V_y(x_i) - \sum_{i,j \in I, i < j} \log |x_j - x_i| \\
V_y(x) := NV(x/N) - 2 \sum_{j \not\in I} \log |x - y_j|,
\]

and \( \tilde{\mathcal{H}}_{\tilde{y}} \) is defined in a similar way with \( V \) in (8.1) replaced with another external potential \( \tilde{V} \).

Recall also the assumption that \( V'', \tilde{V}' \geq -C \) (2.11). We will need the rescaled version of the bounds (4.22), (4.23) and (4.24), i.e.

\[
|J_y| = K \rho(0) + O(K^\xi), \quad \text{(8.2) lengthresc} \\
V_y'(x) = \rho(0) \log \frac{d_+}{d_-(x)} + O \left( \frac{K^\xi}{d(x)} \right), \quad x \in J, \quad \text{(8.3) Vby1resc} \\
V_y''(x) \geq \frac{\inf V''}{N} + \frac{c}{d(x)}, \quad x \in J, \quad \text{(8.4) Vby2secresc}
\]

where \( d(x) := \min \{|x - y - K - 1|, |x - yK + 1|\} \) (8.5) ddef is the distance to the boundary and we redefined \( d_\pm(x) \) as

\[
d_-(x) := d(x) + \rho(0)K^\xi, \quad d_+(x) := \max \{|x - y - K - 1|, |x - yK + 1|\} + \rho(0)K^\xi.
\]

The rescaled version of Lemma 4.5 states that (8.2), (8.3) and (8.4) hold for any \( y \in \mathcal{R}_{L,K}(\xi\delta/2, \alpha/2) \), where the set \( \mathcal{R}_{L,K} \), originally defined in (4.7), is expressed in microscopic coordinates.

We also rewrite (4.10) in the microscopic coordinate as

\[
|E^\mu x_j - \alpha_j| + |E^{\tilde{\mu}} x_j - \alpha_j| \leq CK^\xi,
\]

where

\[
\alpha_j := \frac{j}{K + 1} |J| \quad \text{(8.7) aldefnew}
\]
is the rescaled version of the definition given in (4.4), but we keep the same notation.

The Dirichlet form is also redefined; in microscopic coordinates it is now given by

$$D^{\mu_y}(\sqrt{g}) = \sum_{i \in I} D_i^{\mu_y}(\sqrt{g}) = \frac{1}{2} \sum_{i \in I} \int |\partial I \sqrt{g}|^2 d\mu_y.$$  

(8.8) Dirdef

Due to the rescaling, the LSI from (6.16) now takes the form, for \( y \in \mathcal{R}_{L,K}, \)

$$S(y; \mu_y | \mu_y) \leq C K D^{\mu_y}(\sqrt{g}).$$  

(8.9) lsim

Define the interpolating measures

$$\omega_{y, \tilde{y}}^r = Z_r e^{-\beta r(\tilde{V}_y(x) - \tilde{V}_y(x))} \mu_y, \quad r \in [0, 1],$$  

(8.10) omd

so that \( \omega_{y, \tilde{y}}^1 = \tilde{\mu}_y \) and \( \omega_{y, \tilde{y}}^0 = \mu_y \) (\( Z_r \) is a normalization constant). This is again a local log-gas with Hamiltonian

$$\mathcal{H}_y^{r, \tilde{y}}(x) = \frac{1}{2} \sum_{i \in I} V_{y, \tilde{y}}^r(x_i) - \sum_{i<j} \log |x_i - x_j|$$  

(8.11) Hyy

and external potential

$$V_{y, \tilde{y}}^r(x) := (1-r)V_y(x) + r\tilde{V}_y(x)$$

$$V_y(x) := NV(x/N) - 2 \sum_{j \not\in I} \log(x - y_i),$$

$$\tilde{V}_y(x) := N\tilde{V}(x/N) - 2 \sum_{j \not\in I} \log(x - \tilde{y}_i).$$

The Dirichlet for \( D^\omega \) w.r.t. the measure \( \omega = \omega_{y, \tilde{y}}^r \) is defined similarly to (8.8).

For any bounded smooth function \( Q(x) \) with compact support we can express the difference of the expectations w.r.t. two different measures \( \mu_y \) and \( \tilde{\mu}_y \) as

$$\mathbb{E}_{\tilde{\mu}_y} Q(x) - \mathbb{E}_{\mu_y} Q(x) = \int_0^1 \frac{d}{dr} \mathbb{E}_{\omega_{y, \tilde{y}}^r} Q(x) dr = \int_0^1 \beta \langle h_0(x); Q(x) \rangle_{\omega_{y, \tilde{y}}^r} dr,$$  

(8.12) rcorr

where

$$h_0 = h_0(x) = \sum_{i \in I} (V_y(x_i) - \tilde{V}_y(x_i))$$  

(8.13) h0def

and \( \langle f; g \rangle_\omega := \mathbb{E} f g - (\mathbb{E} f)(\mathbb{E} g) \) denotes the correlation. From now on, we will fix \( r \). Our main result is the following estimate on the gap correlation function.

**Theorem 8.1** Consider two smooth potentials \( V, \tilde{V} \) with \( V'', \tilde{V}'' \geq -C \) and two boundary conditions, \( y, \tilde{y} \in \mathcal{R}_{L=0,K}(\xi^2\delta/2, \alpha) \), with some sufficiently small \( \xi \), such that \( J = J_y = J_{\tilde{y}} \). Assume that (8.6) holds for both boundary conditions \( y, \tilde{y} \). Then, in particular, the rescaled version of the rigidity bound (4.13) and the level repulsion bounds (4.17), (4.18) hold for both \( \mu_y \) and \( \tilde{\mu}_y \) by Theorem 4.2 and Theorem 4.3.

Fix \( \xi^* > 0 \). Then there exist \( \varepsilon > 0 \) and \( C > 0 \), depending on \( \xi^* \), such that for any sufficiently small \( \xi \), for any \( 0 \leq r \leq 1 \) and for \( |p| \leq K^{1 - \xi^*} \) we have

$$|\langle h_0; O(x_p - x_{p+1}, \ldots x_p - x_{p+n}) \rangle_{\omega_{y, \tilde{y}}^r}| \leq K^C \xi K^{-\varepsilon} \|O'\|_\infty$$  

(8.14) eq:cor

for any \( n \)-particle observable \( O \), provided that \( K \geq K_0(\xi, \xi^*, n) \) is large enough.
Notice that this theorem is formulated in terms of \( K \) being the only large parameter; \( N \) disappeared. We also remark that the restriction \(|p| \leq K^{1-\xi} \) can be easily relaxed to \(|p| \leq K - K^{1-\xi} \) with an additional argument conditioning on set \( \{x_i \in I \setminus \tilde{I}\} \) to ensure that \( p \) is near the middle of the new index set \( \tilde{I} \). We will not need this more general form in this paper.

First we complete the proof of Theorem 4.1 assuming Theorem 8.1. The measure \( \mu_\gamma \) is to show that both the rigidity and level repulsion estimates hold w.r.t. the measure \( \mu_\gamma \).

**Proof of Theorem 4.1.** The family of measures \( \omega^r_{\gamma, \tilde{\gamma}} \), \( 0 \leq r \leq 1 \), interpolate between \( \mu_\gamma \) and \( \tilde{\mu}_\gamma \). So we can express the right hand side of (4.11), in the rescaled coordinates and with \( L = \tilde{L} = 0 \) as

\[
\left| \mathbb{E}^{\mu_\gamma} - \mathbb{E}^{\tilde{\mu}_\gamma} \right| O(x_p - x_{p+1}, \ldots x_p - x_{p+n}) \right| \leq \int_0^1 dr \frac{d}{dr} \mathbb{E}^{\omega^r_{\gamma, \tilde{\gamma}}} O(x_p - x_{p+1}, \ldots x_p - x_{p+n}).
\]

(8.15)

Using (8.12) and (8.14) we obtain that this difference is bounded by \( K^{C_2} K^{-\xi} \). Choosing \( \xi \) sufficiently small so that \( K^{C_2} K^{-\xi} \leq K^{-\xi/2} \), we obtain (4.11) (with \( \epsilon/2 \) instead of \( \epsilon \)). This completes the proof of Theorem 4.1.

In the rest of the paper we will prove Theorem 8.1. The main difficulty is due to the fact that the correlation function of the points, \( \langle x_i; x_j \rangle_\omega \), decays only logarithmically. In fact, for the GUE, Gustavsson proved that (Theorem 1.3 in [39])

\[
\langle x_i; x_j \rangle_{GUE} \sim \log \frac{N}{|i-j| + 1},
\]

and a similar formula is expected for \( \omega \). Therefore, it is very difficult to prove Theorem 8.1 based on this slow logarithmic decay. We notice that, however, the correlation function of the type

\[
\langle g_1(x_i); g_2(x_j - x_{j+1}) \rangle_\omega
\]

decays much faster in \(|i-j|\) due to that the second factor \( g_2(x_j - x_{j+1}) \) depends only on the difference. Correlations of the form \( \langle g_1(x_i - x_{i+1}); g_2(x_j - x_{j+1}) \rangle_\omega \) decay even faster. The fact that observables of differences of particles behave much nicer was a basic observation in our previous approach [26, 29, 32] of universality.

The measure \( \omega = \omega^r_{\gamma, \tilde{\gamma}} \) is closely related to the measures \( \mu_\gamma \) and \( \tilde{\mu}_\gamma \). Our first task in Section 8.2 is to show that both the rigidity and level repulsion estimates hold w.r.t. the measure \( \omega \). Then we will rewrite the correlation functions in terms of a random walk representation in Proposition 9.1. The decay of correlation functions will be translated into a regularity property of the corresponding parabolic equation, whose proof will be the main content of Section 10. Section 9 consists of various cutoff estimates to remove the singularity of the diffusion coefficients in the random walk representations. We emphasize that these cutoffs are critical at \( \beta = 1 \); we do not know if our argument can be extended to \( \beta < 1 \).

### 8.2 Rigidity and level repulsion of the interpolating measure \( \omega^r_{\gamma, \tilde{\gamma}} \)

In this section we establish rigidity and level repulsion results for the interpolating measure \( \omega^r_{\gamma, \tilde{\gamma}} \), similar to the ones established for \( \mu_\gamma \) in Section 7 and stated in Theorems 4.2 and 4.3.

**Lemma 8.2** Let \( L \) and \( K \) satisfy (4.1) and \( \gamma, \tilde{\gamma} \in \mathcal{R}_{L,K}(\xi^2\delta^2, \alpha) \). With the notation \( \omega = \omega^r_{\gamma, \tilde{\gamma}} \) there exist constants \( C, \theta_3, C_2 \) and \( C_3 \) such that the following estimates hold:

i) [Rigidity bound]

\[
P^\omega(\left| x_j - y_i \right| \geq CK^{C_2^2}) \leq C e^{-K^{\theta_3}}, \quad i \in I.
\]

(8.18)
Corollary 8.3
Under the assumptions of Lemma 8.2, for any $p < \beta + 1$ we have
\[
\mathbb{E}^{\omega} \frac{1}{|x_i - x_{i+1}|^p} \leq C_p K^{C_{\alpha}} \xi, \quad i \in [L - K - 1, L + K],
\] (8.24) expinv
and for any $p < 2\beta + 1$
\[
\mathbb{E}^{\omega} \frac{1}{|x_i - x_{i+2}|^p} \leq C_p K^{C_{\alpha}} \xi, \quad i \in [L - K - 1, L + K - 1].
\] (8.25) expinvsec

The key to translate the rigidity estimate of the measures $\mu_\gamma$ and $\mu_{\tilde{\gamma}}$ to the measure $\omega = \omega_{\gamma, \tilde{\gamma}}$ is to show that the analogue of (8.6) holds for $\omega$.

Lemma 8.4
Let $L$ and $K$ satisfy (4.1) and $\gamma, \tilde{\gamma} \in R_{L,K}(\xi \delta/2, \alpha)$. Consider the local equilibrium measure $\mu_\gamma$ defined in (4.6) and assume that (4.10) is satisfied. Let $\omega_{\gamma, \tilde{\gamma}}$ be the measure defined in (8.10). Then there exists a constant $C$, independent of $\xi$, such that
\[
\mathbb{E}^{\omega_{\gamma, \tilde{\gamma}}} |x_j - \alpha_j| \leq CK^{C_\xi}.
\] (8.26) Exm3

Proof of Lemma 8.4. We first prove the following estimate on the entropy.

Lemma 8.5
Suppose $\mu_1$ is a probability measure and $\omega = Z^{-1} e^{\beta} d\mu_1$ for some function $g$ and normalization $Z$. Then we can bound the entropy by
\[
S := S(\omega | \mu_1) = \mathbb{E}^{\omega} g - \log \mathbb{E}^{\mu_1} e^{\beta} \leq \mathbb{E}^{\omega} g - \mathbb{E}^{\mu_1} g.
\] (8.27) level
Consider two probability measures $d\mu_i = Z_i^{-1} e^{-H_i} dx$, $i = 1, 2$. Denote by $g$ the function
\[
g = r(H_1 - H_2), \quad 0 < r < 1
\] (8.28) secondlevel
and set $\omega = Z^{-1} e^{\beta} d\mu_1$ as above. Then we can bound the entropy by
\[
\min(S(\omega | \mu_1), S(\omega | \mu_2)) \leq \left[ \mathbb{E}^{\mu_2} - \mathbb{E}^{\mu_1} \right] (H_1 - H_2).
\] (8.29) secondlevel11

Note that in (8.18) we state only the weaker form of the rigidity bound, similar to (4.16). It is possible to prove the strong form of rigidity with Gaussian tail (4.13) for $\omega$, but we will not need it in this paper.

The level repulsion bounds will mostly be used in the following estimates which trivially follow from (8.19)–(8.22):

Corollary 8.3
Under the assumptions of Lemma 8.2, for any $s > 0$ we have
\[
\mathbb{P}^{\omega} (x_{i+1} - x_i \leq s) \leq C (Ns)^{\beta + 1}, \quad i \in [L - K - 1, L + K], \quad s > 0,
\] (8.19) level
\[
\mathbb{P}^{\omega} (x_{i+2} - x_i \leq s) \leq C (Ns)^{2\beta + 1}, \quad i \in [L - K - 1, L + K - 1], \quad s > 0,
\] (8.20) secondlevel
\[
\mathbb{P}^{\omega} (x_{i+1} - x_i \leq s) \leq C (K^{C_{\alpha}} s)^{\beta + 1}, \quad i \in [L - K - 1, L + K],
\] (8.21) level11
\[
\mathbb{P}^{\omega} (x_{i+2} - x_i \leq s) \leq C (K^{C_{\alpha}} s)^{2\beta + 1}, \quad i \in [L - K - 1, L + K - 1],
\] (8.22) secondlevel11
\[
\text{iv) Logarithmic Sobolev inequality}
\]
\[
S(g\omega | \omega) \leq CKD^{\omega}(\sqrt{g}).
\] (8.23) lsi22
Proof. The first inequality is a trivial consequence of the Jensen inequality

\[ S = \mathbb{E}^{\alpha_{\mu}} g - \log \mathbb{E}^{\mu_{1}} e^{g} \leq \mathbb{E}^{\alpha_{\mu}} g - \mathbb{E}^{\mu_{1}} g. \]

The entropy inequality yields that

\[ \mathbb{E}^{\alpha_{\mu}} g \leq r \log \mathbb{E}^{\mu_{1}} e^{g / r} + rS. \]  

(8.30)

By the definition of \( g \), we have

\[ \log \mathbb{E}^{\mu_{1}} e^{g / r} = - \log \int e^{-g / r} d\mu_{2} \leq \mathbb{E}^{\mu_{2}} g / r. \]

Using this inequality and (8.30) in (8.27), we have proved

\[ \text{the entropy inequality yields that} \]

\[ S \leq \frac{r}{1 - r} \left[ \mathbb{E}^{\mu_{2}} - \mathbb{E}^{\mu_{1}} \right] (H_{1} - H_{2}). \]  

(8.31)

We can assume that \( r \leq 1 / 2 \leq 1 - r \) since otherwise we can switch the roles of \( H_{1} \) and \( H_{2} \). Hence (8.29) holds and this concludes the proof of Lemma 8.5.

We now apply this lemma with \( \mu_{2} = \tilde{\mu}_{\tilde{\gamma}} \) and \( \mu_{1} = \mu_{\gamma} \) to prove that

\[ \min \left\{ S(\omega'_{\gamma, \tilde{\gamma}} | \mu_{\gamma}), S(\omega'_{\gamma, \tilde{\gamma}} | \tilde{\mu}_{\tilde{\gamma}}) \right\} \leq K^C \xi \]  

(8.32)

To see this, by definition of \( g \) and the rigidity estimate (4.13), we have

\[ \mathbb{E}^{\mu_{2}} g - \mathbb{E}^{\mu_{1}} g = \frac{r}{2} \left[ \mathbb{E}^{\mu_{2}} - \mathbb{E}^{\mu_{1}} \right] \sum_{i \in I} \left[ V_{\gamma}(x_{i}) - \tilde{V}_{\gamma}(x_{i}) \right] \]

\[ = \frac{r}{2} \left[ \mathbb{E}^{\mu_{2}} - \mathbb{E}^{\mu_{1}} \right] \sum_{i \in I} \int_{0}^{1} ds \left[ V_{\gamma}(s \alpha_{i} + (1 - s)x_{i}) - \tilde{V}_{\gamma}(s \alpha_{i} + (1 - s)x_{i}) \right](x_{i} - \alpha_{i}) \]

\[ = \left[ \mathbb{E}^{\mu_{2}} + \mathbb{E}^{\mu_{1}} \right] O \left( \sum_{i \in I} \sup_{s \in [0,1]} \frac{K^{C}}{d(s \alpha_{i} + (1 - s)x_{i}) \mid x_{i} - \alpha_{i} \mid} \right) \leq K^{C} \xi. \]  

(8.33)

In the first step we used that the leading term \( V_{\gamma}(\alpha_{i}) - \tilde{V}_{\gamma}(\alpha_{i}) \) in the Taylor expansion is deterministic, so it vanishes after taking the difference of two expectations. In the last step we used that with a very high \( \mu_{1} \)- or \( \mu_{2} \)-probability \( d(s \alpha_{i} + (1 - s)x_{i}) \sim d(\alpha_{i}) \) are equidistant up to an additive error \( K^{C} \) if \( i \) is away from the boundary, i.e., \(-K + K^{C} \xi \leq i \leq K - K^{C} \xi \), see (4.13). For indices near the boundary, say \(-K \leq i \leq -K + K^{C} \xi \), we used \( d(s \alpha_{i} + (1 - s)x_{i}) \geq c \min\{1, d(x_{-K})\} \).

Noticing that \( d(x_{-K}) = x_{-K} - y_{-K} - 1, \) the level repulsion bound (4.17) (complemented with the weaker bound (7.17) that is valid for all \( s > 0 \)) guarantees that the short distance singularity \( |d(x_{-K})|^{-1} \) has an \( \mathbb{E}^{\mu_{1,2}} \) expectation that is bounded by \( CK^{C} \xi \).

We now assume that (8.32) holds with the choice of \( S(\omega'_{\gamma, \tilde{\gamma}} | \mu_{\gamma}) \) for simplicity of notation. By the entropy inequality, we have

\[ \mathbb{E}^{\alpha_{\tilde{\gamma}}} e^{x_{i} - \alpha_{i}} \leq \log \mathbb{E}^{\mu_{\gamma}} e^{x_{i} - \alpha_{i}} + K^{C} \xi. \]  

(8.34)

From the Gaussian tail of the rigidity estimate (4.13), we have

\[ \log \mathbb{E}^{\mu_{\gamma}} e^{x_{i} - \alpha_{i}} \leq K^{C} \xi. \]  

(8.35)
Using this bound in (8.34) we have proved (8.26) and this concludes the proof of Lemma 8.4. □

Proof of Lemma 8.2. Given (8.26), the proof of (8.18) follows the argument in the proof of Theorem 4.2, applying it to $\xi^2$ instead of $\xi$. Once the rigidity bound (8.18) is proved, we can follow the proof of Theorem 4.3 to obtain all four level repulsion estimates, (8.19)–(8.22), analogously to the proofs of (4.14), (4.15), (4.17) and (4.18), respectively. The log $N$ factor can be incorporated into $K^{C_3}$. 

Finally, to prove (8.23), let $L_\omega$ be the reversible generator given by the Dirichlet form

$$-\int f L_\omega f d\omega_{\tilde{y},\tilde{y}} = \frac{1}{2} \sum_{|j| \leq K} (\partial_j f)^2 d\omega_{\tilde{y},\tilde{y}}. \tag{8.36}$$

Thus for the Hamiltonian $H = H_{\tilde{y},\tilde{y}}$ of the measure $\omega = \omega_{\tilde{y},\tilde{y}}$ (see (8.11)), we have

$$\langle v, \nabla^2 H(x) v \rangle = \frac{1}{2} \sum_i \left[ (1-r)V_{\tilde{y}}''(x_i) + r\tilde{V}_{\tilde{y}}''(x_i) \right] v_i^2 + \sum_{i<j} \frac{(v_i - v_j)^2}{(x_i - x_j)^2} \geq \frac{c}{K} \sum_i v_i^2, \tag{8.37}$$

by using (8.4) and $d(x) \leq CK$ for good boundary conditions. Thus LSI takes the form

$$S(g_\omega | \omega) \leq CK D^2 (\sqrt{g}). \tag{8.38}$$

This completes the proof of Lemma 8.2. □

The dynamics given by the generator $L_\omega$ with respect to the interpolating measure $\omega = \omega_{\tilde{y},\tilde{y}}$ can also be characterized by the following SDE

$$dx_i = dB_i + \beta \left[ -\frac{1}{2} (V_{\tilde{y}}')'(x_i) + \frac{1}{2} \sum_{j \neq i} \frac{1}{(x_i - x_j)^2} \right] dt, \tag{8.39}$$

where $(B_{-K}, B_{-K+1}, \ldots, B_K)$ is a family of independent standard Brownian motions. With a slight abuse of notations, when we talk about the process, we will use $P_\omega$ and $E_\omega$ to denote the probability and expectation w.r.t. this dynamics with initial data $\omega$, i.e., in equilibrium. This dynamical point of view gives rise to a representation for the correlation (8.14) in terms random walks in random environment.

Starting from Section 9 we will focus on proving Theorem 8.1. The proof is based on dynamical idea and it will be completed in Section 9.7.

9 Local statistics of the interpolating measures: Proof of Theorem 8.1

9.1 Outline of the proof of Theorem 8.1

Theorem 8.1 will be proved by the following main steps. We remind the readers that the boundary conditions $y, \tilde{y}$ are in the good sets and we have chosen $L = 0$ for convenience. For simplicity, we assume that $n = 1$, i.e. we consider a single gap observable $O(x_p - x_{p+1})$.

Step 1. Random walk representation. The starting point is a representation formula for the correlation $\langle h_0, O(x_p - x_{p+1}) \rangle_\omega$. For any smooth observables $F(x)$ and $Q(x)$ and any time $T > 0$ we
have the following representation formula for the time dependent correlation function (see (9.21) for the precise statement):

\[
\mathbb{E}^\omega Q(x) F(x) - \mathbb{E}^\omega Q(x(0)) F(x(T)) = \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \mathbb{E}^\omega \sum_{b \in I} \partial_b Q(x(0)) \langle \nabla F(x(S)), v^b(S, x(\cdot)) \rangle \, dS \, dS.
\]  

(9.1) 

Here the path \( x(\cdot) \) is the solution of the reversible stochastic dynamics with equilibrium measure \( \omega \), (8.39). We use the notation \( \mathbb{E}^\omega \) also for the expectation with respect to the path measure starting from the initial distribution \( \omega \) and \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^K \), recalling that \( |I| = 2K + 1 = K \). Furthermore, for any \( b \in I \) and for any fixed path \( x(\cdot) \), the vector \( v^b(t) = v^b(t, x(\cdot)) \in \mathbb{R}^K \) is the solution to the equation

\[
\partial_t v^b(t) = -A(t)v^b(t), \quad t \geq 0, \quad v^b(0) = \delta_{bj}.
\]  

(9.2) 

The matrix \( A(t) \) depends on time through the path \( x(t) \) and it is given by

\[
A(t) := \beta \nabla^2 H^\omega_{x, y}(x(t)).
\]

(9.3) 

From (8.11), it is of the form \( A(t) = \tilde{A}(x(t)) = \tilde{B}(x(t)) + \tilde{W}(x(t)) \) with \( \tilde{W}(x(t)) \geq 0 \). The matrix elements of \( \tilde{B} \) is given by:

\[
[B(x)v]_j = -\sum_{k \neq j} \tilde{B}_{jk}(x)(v_k - v_j), \quad [\tilde{B}_{jk}(x)] = \frac{\beta}{(x_j - x_k)^2}, \quad j \neq k.
\]  

(9.4) 

Furthermore, \( A(t) \geq CK^{-1}, (8.37) \), and the time to equilibrium for the \( x(t) \) process is of order \( K \) (Corollary 9.2). Applying this representation to \( O(x_p - x_{p+1}) \) and cutting off the time integration at \( C_1K \log K \) with some large constant \( C_1 \), we will have (see (9.39))

\[
\langle h_0, O(x_p - x_{p+1}) \rangle \omega = \frac{1}{2} \int_0^{C_1K \log K} \int_{\mathbb{R}^d} \mathbb{E}^\omega \left[ \frac{\partial_b h_0(x)O(x_p - x_{p+1})(v^b_p(\sigma) - v^b_{p+1}(\sigma))}{\nabla^2 H^\omega_{x, y}(x(t))} \right] \, d\sigma + O\left( \|O\|_{\infty}K^{-2} \right),
\]  

(9.5) 

It is easy to check that \( \partial_b h_0 \) satisfies the estimate with some small \( \xi' \) (see (9.72))

\[
|\partial_b h_0(x)| \leq \frac{K^\xi'}{\min(|x_b - K|, |x_b + K|) + 1}.
\]  

(9.6) 

**Step 2. Cutoff of bad sets.** Setting \( T := [0, C_1K \log K] \), we define the “good set” of paths (see (9.41)) for which the rigidity estimate holds uniformly in time:

\[
G := \left\{ \sup_{s \in T} \sup_{|j| \leq K} |x_j(s) - \alpha_j| \leq K^{\xi'} \right\},
\]  

(9.7) 

where \( \xi' \) is a small parameter to be specified later and \( \alpha_j \) is the classical location given by (8.7). For any \( Z \in I \) and \( \sigma \in T \) we also define the following event that the gaps between particles near \( Z \) are not too small in an appropriate average sense by

\[
Q_{\sigma, Z} := \left\{ \sup_{s \in T} \sup_{1 \leq M \leq K} \frac{1}{1 + |s - \sigma|} \int_s^\sigma \int_{\mathbb{R}^d} \mathbb{E}^\omega \left[ \frac{1}{M} \sum_{i \in I: |i - Z| \leq M} \frac{1}{|x_i(\sigma) - x_{i+1}(\sigma)|^2} \right] \leq K^\rho \right\}.
\]  

(9.8)
where \( \rho > 0 \) is a small parameter to be specified later. By convention we set \( x_i(a) = y_i \) whenever \( |i| > K \). We will need that the gaps are not too small not only near \( Z \) but also near the boundary so we define the new good set

\[
\tilde{Q}_{\sigma,Z} := Q_{\sigma,Z} \cap Q_{\sigma,-K} \cap Q_{\sigma,K}.
\]

(9.8)

Finally, we need to control the gaps not just around one time \( \sigma \) but around a sequence of times that dyadically accumulate at \( \sigma \). The significance of this stronger condition will only be clear in the proof of our version of the De Giorgi-Nash-Moser bound in Section 10. We define

\[
\tilde{Q}_{\sigma,Z} := \bigcap_{\tau \in \Xi} \tilde{Q}_{\sigma+\tau,Z},
\]

(9.9)

where

\[
\Xi := \{- K \cdot 2^{-m} (1 + 2^{-k}) : 0 \leq k, m \leq C \log K\}.
\]

(9.10)

We will choose \( Z \) near the center of the interval \( I \) and show in (9.42) and (9.43) that the bad events are small in the sense that

\[
P_\omega(G^c) \leq C e^{-K^\theta}
\]

(9.11)

with some \( \theta > 0 \), and

\[
P_\omega(\tilde{Q}^c_{\sigma,Z}) \leq C R C^4 \xi^{-\rho}
\]

(9.12)

for each fixed \( Z \in I \) and fixed \( \sigma \in \mathcal{T} \) where \( \xi \) is introduced in Theorem 8.1. Notice that while the rigidity bound (9.11) holds with a very high probability, the control on small gaps (9.12) is much weaker due to the power-law behavior of the level repulsion estimates.

Our goal is to insert the characteristic functions of the good sets into the expectation in (9.4). More precisely, we will prove in (9.74) that

\[
| \langle h_0; O(x_p - x_{p+1}) \rangle \omega | \leq \frac{1}{2} \| O' \|_\infty \int_0^{C_1 K \log K} \sum_{b \in I} E^{\mu} \left[ \tilde{Q}_{\sigma,Z} \mathcal{G} \{ \partial_b h_0(x) \} | v_b^b(\sigma) - v_p^{b+1}(\sigma) | \right] d\sigma + O \left( \| O' \|_\infty K^{-\rho/6} \right).
\]

(9.13)

(With a slight abuse of notations we use \( \mathcal{G} \) and \( \tilde{Q}_{\sigma,Z} \) also to denote the characteristic function of these sets.) To prove this inequality, we note that the contribution of the bad set \( \tilde{Q}^c \) can be estimated by (9.11). To bound the contribution of the bad set \( \tilde{Q}^c_{\sigma,Z} \), the estimate (9.12) alone is not strong enough due to the time integration in (9.13). We will need a time-decay estimate for the solution \( v^b(\sigma) \). On the good set \( \mathcal{G} \), the matrix element \( B_{jk} \) satisfies

\[
B_{jk}(s) = \frac{\beta}{(x_j(s) - x_k(s))^2} \geq \frac{b}{(j - k)^2}, \quad 0 \leq s \leq \sigma, \quad j \neq k.
\]

(9.14)

with \( b = \beta K^{-2\xi'} \). With this estimate, we will show in (9.65) that, for any \( 1 \leq p \leq q \leq \infty \), the following decay estimate for the solution to (9.2) holds:

\[
\| v(s) \|_q \leq (sb)^{-\left(\frac{1}{p} - \frac{1}{q} \right)} \| v(0) \|_p, \quad 0 < s \leq \sigma.
\]

(9.15)

This allows us to prove (9.13).

**Step 3. Cutoff of the contribution from near the center.** From (9.5), \( \partial_b h_0(x) \) decays as a power law when \( x_b \) moves away from the boundary of \( J \), i.e., when the index \( b \) moves away from \( \pm K \).
With the decay estimate (9.15), it is not difficult to show that the contribution of $b$ in the interior, i.e., the terms with $|b| \leq K^{1-\varepsilon}$ for some $c > 0$ in the sum in (9.13), is negligible.

Step 4. Finite speed of propagation. We will prove that in the good set $\mathcal{G} \cap \bar{Q}_{\sigma,Z}$ the dynamics (9.2) satisfies the finite speed of propagation estimate

$$|v_p^J(s)| \leq \frac{CK^{\varepsilon+1/2}}{|p-b|^{1/2}}.$$

(9.16) [finite]

for some small constant $c$ (see (9.80)). This estimate is not optimal, but it allows us to cutoff the contribution in (9.13) for time $\sigma \leq K^{1/4}$ for $b$ away from the center, i.e., $K \geq |b| \geq K^{1-\varepsilon}$. In this step we use that $|p| \leq K^{1-\varepsilon}$ ($\xi^*$ is some small constant) and the exponents are chosen such that $|p-b| \geq cK^{1-\varepsilon}$.

Step 5. Parabolic regularity with singular coefficients. Finally, we have to estimate the r.h.s of (9.13) in the regime $K^{1/4} \leq \sigma \leq C_1 K \log K$ and for $|p| \leq K^{1-\varepsilon}$ with the choice $Z = p$. This estimate will work uniformly in $b$. We will show that for all paths in $\mathcal{G} \cap \bar{Q}_{\sigma,p}$, any solution to (9.2) satisfies the Hölder regularity estimate in the interior, i.e., for some constants $\alpha, q > 0$,

$$\sup_{|j-p|+|j'-p| \leq \sigma^{1-\alpha}} |v_j(\sigma) - v_{j'}(\sigma)| \leq CK^{\xi\sigma^{-\frac{1}{2}q\alpha}}.$$

(9.17) [HC]

Notice that the regularity depends on the time $\sigma$ and that is why we need the short time cutoff in the previous step. This estimate (9.17) allows us to complete the proof that $\langle h_0 : O(x_p-x_{p+1}) \rangle_\omega \to 0$ as $K \to \infty$. The Hölder estimate will be stated as Theorem 9.8 and the entire Section 10 will be devoted to its proof.

9.2 Random Walk Representation

First we will recall a general formula for the correlation functions of the process (8.39) through a random walk representation, see (9.23) below. This equation in a lattice setting was given in Proposition 2.2 of [20] (see also Proposition 3.1 in [38]). The random walk representation already appeared in the earlier paper of Naddaf and Spencer [46], which was a probabilistic formulation of the idea of Helffer and Sjöstrand [41].

In this section we will work in a general setup. Let $J \subset \mathbb{R}$ be an interval and $I$ an index set with cardinality $|I| = K$. Consider a convex Hamilton function $\mathcal{H}(x)$ on $J^K$ and let $x(s)$ be the solution to

$$dx_i = dB_i + \beta \partial_i \mathcal{H}(x) dt, \quad i \in I,$$

(9.18) [SDEgen]

with initial condition $x(0) = x \in J_{x,J}^I$, where $\{B_i : i \in I\}$ is a family of independent standard Brownian motions. The parameter $\beta > 0$ is introduced only for consistency with our applications. Let $\mathbb{E}_\omega$ denote the expectation with respect to this path measure. With a slight abuse of notations, we will use $\mathbb{P}_\omega$ and $\mathbb{E}_\omega$ to denote the probability and expectation with respect to the path measure of the solution to (9.18) with initial condition $x$ distributed by $\omega$. We assume that $\mathbb{P}_\omega(x(t) \in J^K) = 1$, i.e. the Hamiltonian confines the process to remain in the interval $J$. The corresponding invariant measure is $d\omega = Z^{-1}_\omega e^{-\beta \mathcal{H}(x)} dx$ with generator $\mathcal{L}_\omega = -\frac{1}{2} \Delta + \frac{\beta}{2} \nabla \mathcal{H} \cdot \nabla$ and Dirichlet form

$$D^\omega(f) := \frac{1}{2} \int |\nabla f|^2 d\omega = -\int f \mathcal{L}_\omega f d\omega.$$
For any fixed path \(x(\cdot) := \{x(s) : s \geq 0\}\) we define the operator (\(K \times K\) matrix)
\[
A(s) := \tilde{A}(x(s)),
\]  
where \(\tilde{A} := \beta H''\) and we assume that the Hessian matrix is positive definite, \(H''(x) \geq c > 0\).

**Proposition 9.1** Assume that the Hessian matrix is positive definite
\[
\inf_x H''(x) \geq \tau^{-1}
\]  
with some constant \(\tau > 0\). Then for any functions \(F, G \in C^1(J^K) \cap L^2(d\omega)\) and any time \(T > 0\) we have
\[
\mathbb{E}^\omega [F(x) G(x)] - \mathbb{E}^\omega [F(x(0)) G(x(T))] = \frac{1}{2} \int_0^T d\omega \int K \partial_h F(x) \mathbb{E}_x [\partial_a G(x(S)) v^b_a(S, x(\cdot))].
\]  
(9.21) 

Here for any \(S > 0\) and for any path \(\{x(s) \in J^K : s \in [0, S]\}\), we define \(v^b(t) = v^b(t, x(\cdot))\) as the solution to the equation
\[
\partial_t v^b(t) = -A(t) v^b(t), \quad t \in [0, S], \quad v^b_a(0) = \delta_{ba}.
\]  
(9.22) 

The dependence of \(v^b\) on the path \(x(\cdot)\) is present via the dependence \(A(t) = \tilde{A}(x(t))\). In other words, \(v^b_a(t)\) is the fundamental solution of the heat semigroup \(\partial_s + A(s)\).

Furthermore, for the correlation function we have
\[
\langle F; G \rangle_\omega = \frac{1}{2} \int_0^{\infty} d\omega \int K \partial_h F(x) \mathbb{E}_x [\partial_a G(x(S)) v^b_a(S, x(\cdot))]
\]  
(9.23) 
\[
= \frac{1}{2} \int_0^{\infty} d\omega \int K \partial_h F(x) \mathbb{E}_x [\partial_a G(x(S)) v^b_a(S, x(\cdot))] + O(K^{-\epsilon A})
\]  
(9.24) 

for any constant \(A > 0\).

**Proof.** This proposition in the lattice setting was already proved in [20, 38, 46]; we give here a proof in the continuous setting. Let \(G(t, x)\) be the solution to the equation \(\partial_t G = \mathcal{L} G\) with initial condition \(G(0, x) := G(x)\). By integrating the time derivative, we have
\[
\mathbb{E}^\omega F(x) G(x) - \mathbb{E}^\omega F(x(0)) G(x(T)) = -\int_0^T d\omega \frac{d}{d\omega} \mathbb{E}^\omega [F e^{S \mathcal{L} G}]
\]  
(9.25) 
\[
= -\int_0^T d\omega \mathbb{E}^\omega [F \mathcal{L} e^{S \mathcal{L} G}]\]
\[
= \frac{1}{2} \int_0^T d\omega \mathbb{E}^\omega \langle \nabla F(x), \nabla G(S, x) \rangle.
\]  
(9.27) 

where \(\langle \cdot, \cdot \rangle\) denotes the scalar product in \(\mathbb{R}^K\).
Taking the gradient of the equation $\partial_t G = \mathcal{L}^\omega G$ and computing the commutator $[\nabla, \mathcal{L}^\omega]$ yields the equation
\[
\partial_t \nabla G(t, x) = \mathcal{L}^\omega [\nabla G(t, x)] - \tilde{A}(x) [\nabla G(t, x)]
\] (9.28)
for the $x$-gradient of $G$. Setting $u(t, x) := \nabla G(t, x)$ for brevity, we have the equation
\[
\partial_t u(t, x) = \mathcal{L}^\omega u(t, x) - \tilde{A}(x) u(t, x)
\] (9.29)
with initial condition $u(0, x) = u_0(x) := \nabla G(x)$.

Notice that $\tilde{A}$ is a matrix and $\mathcal{L}^\omega$ acts on the vector $u$ as a diagonal operator in the index space, i.e., $[\mathcal{L}^\omega u(t, x)]_i = \mathcal{L}^\omega [u(t, x)]_i$. The equation (9.29) can be solved by solving an equations (9.33) over the indices with coefficients that depend on the path generated by the operator $\mathcal{L}^\omega$ and then by taking expectation over the paths starting at $x$. To obtain such a representation, we start with the time-dependent Feynman-Kac formula:
\[
u(x_0, x_0) = \mathbb{E}_x \left[ \exp \left( - \int_0^\sigma \tilde{A}(x(s)) ds \right) u_0(x(\sigma)) \right], \quad \sigma > 0,
\] (9.30)
where
\[
\tilde{\mathbb{E}} \left( - \int_0^\sigma \tilde{A}(x(s)) ds \right) := 1 - \int_0^\sigma \tilde{A}(x(s_1)) ds_1 + \int_{0 \leq s_1 < s_2 \leq \sigma} \tilde{A}(x(s_1)) \tilde{A}(x(s_2)) ds_1 ds_2 + \ldots
\] (9.31)
is the time-ordered exponential. To prove that (9.30) indeed satisfies (9.29), we notice from the definition (9.31) that
\[
u(x_0, x_0) = \mathbb{E}_x \tilde{\mathbb{E}} \left( - \int_0^\sigma \tilde{A}(x(s)) ds \right) u_0(x(\sigma))
\] (9.32)
\[
= \mathbb{E}_x u_0(x(\sigma)) - \int_0^\sigma \mathbb{E}_x \tilde{\mathbb{E}} A(x(s_1)) \mathbb{E}_x \tilde{\mathbb{E}} \left( \int_{s_1}^\sigma \tilde{A}(x(s)) ds \right) u_0(x(\sigma)) ds_1.
\]
Using that the process is stationary in time, we have
\[
u(x_0, x_0) = \mathbb{E}_x u_0(x(\sigma)) - \int_0^\sigma \mathbb{E}_x \tilde{A}(x(s_1)) u_0(x(\sigma-s_1), x(s_1)) ds_1
\]
\[
= \mathbb{E}_x u_0(x(\sigma)) - \int_0^\sigma \mathbb{E}_x \tilde{A}(x(\sigma-s_1)) u(s_1, x(\sigma-s_1)) ds_1
\]
\[
= e^{\sigma \mathcal{L}} u_0(x) - \int_0^\sigma [e^{(\sigma-s_1) \mathcal{L}} \tilde{A}(\cdot) u(s_1, \cdot)](x) ds_1.
\]
Differentiating this equation in $\sigma$ we obtain that $u$ defined in (9.30) indeed satisfies (9.29).

For any fixed path $\{x(s) : s > 0\}$, the time-ordered exponential in (9.30)
\[
\mathcal{U}(t) = \mathcal{U}(t; x(\cdot)) := \tilde{\mathbb{E}} \exp \left( - \int_0^t \tilde{A}(x(s)) ds \right)
\]
satisfies the matrix evolution equation
\[
\partial_t \mathcal{U}(t) = -\mathcal{U}(t) \tilde{A}(t), \quad \mathcal{U}(0) = I,
\]
with initial condition $u(0, x) = u_0(x) := \nabla G(x)$.
which can be seen directly from (9.31). Let \( \mathbf{v}^b(t) \) be the transpose of the \( b \)-th row of the matrix \( \mathbf{U}(t) \), then the equation for the column vector \( \mathbf{v}^b(t) \) reads

\[
\partial_t \mathbf{v}^b(t) = -\mathbf{A}(t)\mathbf{v}^b(t), \quad \mathbf{v}^b_0(0) = \delta_{ab}.
\]  (9.33)

Thus taking the \( b \)-th component of (9.30) we have

\[
u_b(\sigma, x) = \partial_h G(\sigma, x) = E_x \left[ \mathbf{U}(\sigma) \nabla G(x(\sigma)) \right]_b = \sum_a E_x [\partial_a G(x(\sigma))v_a^b(\sigma)],
\]

and plugging this into (9.25), we obtain (9.21) by using that \( E_x[\cdot] = \int E_x[\cdot] \omega(dx) \).

Formula (9.24) follows directly from (9.21) and from the fact that \( \mathbf{H}'' \geq \tau^{-1} \) implies a spectral gap of order \( \tau \), in particular,

\[
\left| E_{\omega} [F(x(0))G(x(T))] - E_{\omega}[F] E_{\omega}[G] \right| \leq e^{-cT/\tau} \| F \|_{L^2(\omega)} \| G \|_{L^2(\omega)}.
\]

Finally, (9.23) directly follows from this, by taking the \( T \to \infty \) limit.

Now we apply our general formula to the gap correlation function on the left hand side of (8.14). For brevity of the formulas, we consider only the single gap case, \( n = 1 \); the general case is a straightforward extension. The gap index \( p \in I, p \neq K \), is fixed, later we will impose further conditions on \( p \) to separate it from the boundary. The index set is \( I = [-K,K] \), the Hamiltonian in (9.18) is given by \( \mathbf{H}_{xyz} \) and (9.18) takes the form of (8.39). It is well known [1] that due to the logarithmic interaction in the Hamiltonian, \( \beta \geq 1 \) implies that the process \( x(t) = (x_{-K}(t), \ldots, x_K(t)) \) preserves the initial ordering, i.e., \( x_{-K}(t) \leq \ldots \leq x_K(t) \) and \( x_i(t) \in J \) for every \( i \in I \). The matrix \( \mathbf{A} \) is given by \( \mathbf{A} := \mathbf{B} + \mathbf{W} \) where \( \mathbf{B} \) and \( \mathbf{W} \) are the following \( x \)-dependent matrices acting on vectors \( x \in \mathbb{R}^K \):

\[
[\mathbf{B}(x)v]_j = -\sum_k \mathbf{B}_{jk}(x)(v_k - v_j), \quad \mathbf{B}_{jk}(x) = \frac{\beta}{(x_j - x_k)^2} \geq 0
\]  (9.34)

and

\[
[\mathbf{W}(x)v]_j := \mathbf{W}_j(x)v_j
\]  (9.35)

with the function

\[
\mathbf{W}_j(x) := \frac{\beta}{2} \left\{ \sum_{|k| \geq K+1} \left[ \frac{1-r}{(x_j - y_k)^2} + \frac{r}{(x_j - y_k)^2} \right] + \frac{1-r}{N} V'' \left( \frac{x_j}{N} \right) + \frac{r}{N} V'' \left( \frac{x_j}{N} \right) \right\}.
\]  (9.36)

Here \( r \in [0,1] \) is a fixed parameter which we will omit from the notation of \( \mathbf{W} \).

For any fixed path \( x(\cdot) \), define the following time-dependent operators (matrices) on \( \mathbb{R}^K \)

\[
\mathbf{A}(s) := \mathbf{A}(x(s)), \quad \mathbf{B}(s) := \mathbf{B}(x(s)), \quad \mathbf{W}(s) := \mathbf{W}(x(s)),
\]  (9.37)

where \( \mathbf{W} \) is a multiplication operator with the \( j \)-th diagonal \( W_j(s) = \mathbf{W}_j(x_j(s)) \) depending only the \( j \)-th component of the process \( x(s) \). Clearly \( \mathbf{A}(s) = \mathbf{B}(s) + \mathbf{W}(s) \). We also define the associated
(time dependent) quadratic forms which we denote by the corresponding lower case letters, in particular

\[ b(s)[u, v] := \sum_{i \in I} u_i [B(s)v]_i = \frac{1}{2} \sum_{k, j \in I} B_{jk}(s)(u_k - u_j)(v_k - v_j) \]

\[ \nu(s)[u, v] := \sum_{i \in I} u_i [W(s)v]_i = \sum_{i} u_i W_i(s) v_i \]

\[ a(s)[u, v] := b(s)[u, v] + \nu(s)[u, v]. \]  \hfill (9.38)

With these notations we can apply Proposition 9.1 to our case and we get:

**Corollary 9.2** Let \( h_0 \) be given by (8.13), let \( O = O_N : \mathbb{R} \rightarrow \mathbb{R} \) be an observable for \( n = 1 \), see (2.8), and assume that \( y, \tilde{y} \in \mathcal{R}_{L=0,K}(\xi^2 \delta/2, \alpha) \), in particular \( A(s) \) given in (9.37) satisfies \( A(s) \geq \tau^{-1} \) with \( \tau = CK \) by (8.37). Then with a large constant \( C_1 \) and for any \( p \in I \), \(-K \leq p \leq K - 1 \), we have

\[ (h_0; O(x_p - x_{p+1}))_\omega \leq \frac{1}{2} \int_0^{C_1 K \log K} \mathrm{d} \sigma \int \sum_{b \in I} \partial_b h_0(\mathbf{x}) \mathbb{E}_\mathbf{x} \left[ O'(x_p - x_{p+1})(v^b_p(\sigma) - v^b_{p+1}(\sigma)) \right] \omega(\mathrm{d} \mathbf{x}) + O\left( \|O'\|_\infty K^{-2} \right), \]  \hfill (9.39)

where \( v^b(s) = v^b(s, \mathbf{x}(\cdot)) \) solves (9.22) with \( A(s) \) given in (9.37).

**Proof.** If \( h_0 \) were a smooth function, then (9.39) directly followed from (9.24). The general case is a simple cutoff argument using that \( h_0 \in L^2(d \omega) \) and

\[ \mathbb{E}^\omega |\partial_b h_0| \leq \mathbb{E}^\omega \left[ |(V_Y)'(x_p)| + |(V_{\tilde{Y}})'(x_p)| \right] \leq \sum_{j \notin I} \mathbb{E}^\omega \left[ \frac{1}{|y_j - x_p|} + \frac{1}{|y_j - x_p|} \right] + C \leq C K^{(C_3 + 1)\xi}. \]  \hfill (9.40)

Here we used (8.24) and that \( y, \tilde{y} \in \mathcal{R}_{L=0,K} = \mathcal{R}_{L=0,K}(\xi^2 \delta/2, \alpha) \) are regular on scale \( K^{\xi^2} \leq K^\xi \), so the summation is effectively restricted to \( K^\xi \) terms. \( \square \)

The representation (9.39) expresses the correlation function in terms of the discrete spatial derivative of the solution to (9.22). To estimate \( v^b_p(\sigma, \mathbf{x}(\cdot)) - v^b_{p+1}(\sigma, \mathbf{x}(\cdot)) \) in (9.39), we will now study the Hölder continuity of the solution \( v^b(s, \mathbf{x}(\cdot)) \) to (9.22) at time \( s = \sigma \) and at the spatial point \( p \). For any fixed \( \sigma \) we will do it for each fixed path \( \mathbf{x}(\cdot) \), with the exception of a set of “bad” paths that will have a small probability.

Notice that if all points \( x_i \) were approximately regularly spaced in the interval \( J \), then the operator \( B \) has a kernel \( B_{ij} \sim (i - j)^{-2} \), i.e. it is essentially a discrete version of the operator \( |p| = \sqrt{-A} \). Hölder continuity will thus be the consequence of the De Giorgi-Nash-Moser bound for the parabolic equation (9.22). However, we need to control the coefficients in this equation, which depend on the random walk \( \mathbf{x}(\cdot) \).

For the De Giorgi-Nash-Moser theory we need both upper and lower bounds on the kernel \( B_{ij} \). The rigidity bound (8.18) guarantees a lower bound on \( B_{ij} \), up to a factor \( K^{-C_2 \xi^2} \geq K^{-\xi} \). The level repulsion estimate implies certain upper bounds on \( B_{ij} \), but only in an average sense. In the next section we define the good set of paths that satisfy both requirements.
9.3 Sets of good paths

From now on we assume the conditions of Theorem 8.1. In particular we are given some \( \xi > 0 \) and we assume that the boundary conditions satisfy \( y, \tilde{y} \in \mathcal{R}_{L=0,K} = \mathcal{R}_{L=0,K}(\xi^2 \delta/2, \alpha) \) and (8.6) with this \( \xi \). We define the following “good sets”:

\[
G := \left\{ \sup_{0 \leq s \leq C_1 K \log K} \sup_{|j| \leq K} |x_j(s) - \alpha_j| \leq K^{\xi'} \right\},
\]

(9.41)

where

\[
\xi' := (C_2 + 1) \xi^2,
\]

with \( C_2 \) being the constant in (8.18) and \( \alpha_j \) is given by (8.7). We recall the definition of the event \( \tilde{Q}_{\sigma,Z} \) for any \( Z \in I \) and \( \sigma \in T = [0, C_1 K \log K] \) from (9.9).

**Lemma 9.3** There exists a positive constant \( \theta \), depending on \( \xi' = (C_2 + 1) \xi^2 \), such that

\[
P_\omega(G^c) \leq C e^{-K^\theta}.
\]

(9.42)

Moreover, there is a constant \( C_4 \), depending on the constant \( C_2 \) in (8.18) and on \( C_3 \) in (8.21), (8.22) such that for any \( \xi \) and \( \rho \) small enough, we have

\[
P_\omega(\tilde{Q}_{\sigma,Z}^c) \leq C K^{C_4 \xi - \rho}
\]

(9.43)

for each fixed \( Z \in I \) and fixed \( \sigma \in T \).

**Proof.** From the stochastic differential equation of the dynamics (8.39) we have

\[
|x_i(t) - x_i(s)| \leq C|t - s| + \int_s^t \left[ \sum_{j \neq i} \frac{1}{|x_j(a) - x_i(a)|} + \sum_{j \in I_c} \frac{1}{|y_j - x_i(a)|} \right] da + |B_i(t) - B_i(s)|.
\]

(9.44)

Using (8.24) and that \( x(\cdot) \) is invariant under \( \omega \), we have the bound

\[
\mathbb{E}^\omega \left[ \int_s^t \sum_{j \neq i} \frac{1}{|x_j(a) - x_i(a)|} \right]^{3/2} \leq CK^3|t - s|^{3/2} \max_{i \in I} \mathbb{E}^\omega \frac{1}{|x_i - x_i|^{3/2}} \leq CK^{3+C_3 \xi}|t - s|^{3/2}.
\]

(9.45)

This implies for any fixed \( s < t \leq C_1 K \log K \) and for any \( R > 0 \) that

\[
P_\omega \left[ \int_s^t \sum_{|j| > K} |x_j(a) - x_i(a)| \geq R \right] \leq CK^{3+C_3 \xi}|t - s|^{3/2} R^{-3/2}.
\]

(9.46)

A similar bound holds for the second summation in (9.44); the summation over large \( j \) can be performed by using that \( y \) is regular, \( y \in \mathcal{R}_{L=0,K} \).

Set a parameter \( q \leq cR \) and choose a discrete set of increasing times \( \{s_k : k \leq C_1 K \log K/q\} \) such that

\[
0 = s_0 < s_1 \leq s_2 \leq \ldots \leq C_1 K \log K, \quad \text{and} \quad |s_k - s_{k+1}| \leq q.
\]

(9.47)
From standard large deviation bounds on the Brownian motion increment $B_i(t) - B_i(s)$ and from (9.44), we have the stochastic continuity estimate
\[ \mathbb{P}^\omega \left( \sup_{s,t \in [s_k,s_{k+1}], |i| \leq K} |x_i(s) - x_i(t)| \geq R \right) \leq Ke^{-CR^2/q} + CK^4q^{3/2}R^{-3/2} \]
for any fixed $k$. Taking sup over $k$, and overestimating $C_1 K \log K \leq K^2$, we have
\[ \mathbb{P}^\omega \left( \sup_{0 \leq s,t \leq C_1 K \log K, |i-s| \leq q, |i| \leq K} |x_i(s) - x_i(t)| \geq R \right) \leq K^3 q^{-1} e^{-CR^2/q} + CK^6q^{1/2}R^{-3/2} \] (9.48)
for any positive $q$ and $R$ with $q \leq cR$.
From the rigidity bound (8.18) we know that for some $\theta_3 > 0$ and for any fixed $k$ we have,
\[ \mathbb{P}^\omega \left\{ |x_j(s_k) - \alpha_j| \geq CK^{C_2 \xi^2} \right\} \leq Ce^{-K^{\theta_3}}, \quad j \in I. \] (9.49)
Choosing $R = K^{\xi}/2$ and $q = \exp(-K^{\theta_3}/2)$, and using that $CK^{C_2 \xi^2} \leq K^{\xi}/2$ with the choice of $\xi'$, we have
\[ \mathbb{P}^\omega (\xi') \leq Ce^{-K^{\theta_3}}K^3 q^{-1} + K^3 q^{-1} e^{-CR^2/q} + CK^6q^{1/2}R^{-3/2} \leq C \exp(-K^{\theta_3}/3), \] (9.50)
for sufficiently large $K$, and this proves (9.42) with $\theta = \theta_3/3$.

We will now prove (9.43). The number of intersections in the definition of $\tilde{Q}_{\sigma,Z}$ is only a $(\log K)$-power, so it will be sufficient to prove (9.43) for one set $Q^\ell$. We will consider only the set $Q^\ell_{\sigma,Z}$ and only for $Z = 0$ and $\sigma = 0$. The modification needed for the general case is only notational. We start the proof by noting that for $s > 0$
\[ \frac{1}{1 + s'} \int_0^{s'} \! da \frac{1}{M'} \sum_{i=-M'}^{M'} \frac{1}{|x_i(a) - x_{i+1}(a)|^2} \] (9.51)
holds for any $s' \in [s/2, s]$ and $M' \in [M/2, M]$. Hence it is enough to estimate the probability
\[ \mathbb{P}^\omega \left\{ \frac{1}{1 + s} \int_0^s \! da \frac{1}{M} \sum_{i=-M}^M \frac{1}{|x_i(a) - x_{i+1}(a)|^2} \geq K^p \right\} \] (9.52)
for fixed dyadic points $(s,M) = \{(2^{-p_1}K^2,2^{-p_2}K)\}$ in space-time for each integer $p_1, p_2 \leq C \log K$. Since the cardinality of the set of these dyadic points is just $C(\log K)^2$, it suffices to estimate (9.52) only for a fixed $s,M$.

The proof is different for $\beta = 1$ and $\beta > 1$. In the latter case, from (8.24) we see that the random variable in (9.52) has expectation $CK^{C_3 \xi}$. Thus the probability in (9.52) is bounded by $CK^{C_3 \xi - C_3}$, so (9.43) holds in this case with $C_4$ slightly larger than $C_3 + 1$ to accommodate the log $K$ factors.

In the case $\beta = 1$ the random variable in (9.52) has a logarithmically divergent expectation. To prove (9.43) for $\beta = 1$, we need to regularize the interaction on a very small scale of order $K^{-C}$.
with a large constant $C$. This regularization is a minor technical detail which does not affect other parts of this paper. We now explain how it is introduced, but for simplicity we will not carry it in the notation in the subsequent sections.

For any $y, \tilde{y} \in R_{L,K}$ satisfying (4.8) and $\varepsilon > 0$, we define the extension $\omega^\varepsilon := \omega^\varepsilon_{y,\tilde{y}}$ of the measure $\omega = \omega^0_{y,\tilde{y}}$ (see (8.10)) from the simplex $J^K \cap \Xi(k)$ to $R^K$ by replacing the singular logarithm with a $C^2$-function. For $x \in R^K$ and $a := |J| \sim K$ we set

$$H_\varepsilon(x) := \frac{1}{2} \sum_{i \in I} U^\varepsilon_i(x_i) - \sum_{i<j} \log_{ae}(x_j - x_i) \quad \text{and} \quad U^\varepsilon(x) := U^\varepsilon_{y,\tilde{y}}(x) = (1 - r)V^\varepsilon(x) + r\tilde{V}^\varepsilon(x)$$

where we define

$$V^\varepsilon_{y}(x) := NV(x/N) - 2 \sum_{k<K} \log_{ae}(x - y_k) - 2 \sum_{k>K} \log_{ae}(y_k - x), \quad (9.53)$$

We remark that the same regularization for a different purpose was introduced in Appendix A of [23]. It is easy to check that $\log_{ae}(x) \in C^2(R)$, is concave, and satisfies

$$\lim_{\varepsilon \to 0} \log_{ae}(x) = \begin{cases} \log x & \text{if } x > 0 \\ -\infty & \text{if } x \leq 0. \end{cases}$$

Furthermore, we have the lower bound

$$\partial^2 \log_{ae}(x) \geq \begin{cases} -\frac{1}{\varepsilon^2} & \text{if } x > \varepsilon \\ -\frac{1}{\varepsilon} & \text{if } x \leq \varepsilon. \end{cases} \quad (9.55)$$

We then define

$$\omega^\varepsilon(dx) := Z_\varepsilon^{-1} e^{-\beta H_\varepsilon(x)}dx, \quad \text{on } R^K, \quad \text{where } Z_\varepsilon := \int e^{-\beta H_\varepsilon(x)}dx.$$ 

Notice that on the support of $\omega^\varepsilon$ the particles do not necessarily keep their natural order and they are not confined to the interval $J$. We recall that $\omega^\varepsilon_{y,\tilde{y}} = \mu_y$ and $\omega^\varepsilon_{y,\tilde{y}} = \tilde{\mu}_{\tilde{y}}$ so these definitions also regularize the initial local measures in Theorem 4.1.

In order to apply the proof of Theorem 4.1 to $\omega^\varepsilon$, we need two facts. First that $\omega$ and $\omega^\varepsilon$ are close in entropy sense, i.e.

$$S(\omega_0|\omega_\varepsilon) \leq CK^C\varepsilon^2 \quad (9.56)$$

Using this entropy bound with $\varepsilon = K^{-C'}$ for a sufficiently large $C'$, we see that the measures $\mu_y$ and $\tilde{\mu}_{\tilde{y}}$ can be replaced with their regularized versions $\tilde{\mu}_y^\varepsilon, \tilde{\mu}_{\tilde{y}}^\varepsilon$ both in the condition (4.10) and in the statement (4.11). We can now use the argument of Section 8 with the regularized measures.

The second fact is that rigidity and level repulsion estimates given in Lemma 8.2 also hold for the regularized measure $\omega^\varepsilon$. In fact, apart from the rigidity in the form of (8.18), we also need the following weaker level repulsion bound:

$$P_{\omega^\varepsilon}(x_{i+1} - x_i \leq s) \leq CK^Cs^2, \quad i \in [L - K - 1, L + K], \quad s \geq K^C\varepsilon. \quad (9.57)$$
Using (9.55), this bound easily implies
\[
\mathbb{E}^{\omega} \log^2(x_{i+1} - x_i) \leq C K^\xi |\log \varepsilon|.
\] (9.58) \[\text{gapexp}\]

Thus the regularized version of the random variable in (9.52) has a finite expectation and we obtain (9.43) also for \( \beta = 1 \).

With these comments in mind, these two facts can be proved following the same path as the corresponding results in Section 7. The only slight complication is that the particles are not ordered, but for \( \varepsilon = K^{-C'} \) the regularized potential strongly suppresses switching order. More precisely, we have
\[
P^{\omega'}(x_{i+1} - x_i \leq -Ma\varepsilon) \leq e^{-cM^2}
\] (9.59) \[\text{trunc11}\]
for any \( M \geq K^3 \). This inequality follows from the integral
\[
\int_{-\infty}^{M_{\text{max}}} e^{\log a} v \, dv \leq (a\varepsilon)^2 \int_{-\infty}^M e^{-cu^2} \, du \leq e^{-cM^2},
\]
since for \( M \geq K^3 \) all other integrands in the measure \( \omega^\varepsilon \) can be estimated trivially at the expense of a multiplicative error \( K^{C'K^2} \) that is still negligible when compared with the factor \( \exp(-cM^2) \).

The estimate (9.59) allows us to restrict the analysis to \( x_{i+1} \geq x_i - K^{-C''} \) with some large \( C'' \). This condition replaces the strict ordering \( x_{i+1} \geq x_i \) that is present in Section 7. This replacement introduces irrelevant error factors that can be easily estimated. This completes the proof of Lemma 9.3.

In the rest of the paper we will work with the regularized measure \( \omega^\varepsilon \) but for simplicity we will not carry this regularization in the notation.

### 9.4 Restrictions to the good paths

#### 9.4.1 Restriction to the set \( \mathcal{G} \)

Now we show that the expectation (9.39) can be restricted to the good set \( \mathcal{G} \) with a small error. We just estimate the complement as
\[
\int \sum_{b \in I} |\partial_b h_0(x)| \mathbb{E}^{\omega} \left[ |O'(x_p - x_{p+1})||v_p^b(\sigma) - v_{p+1}^b(\sigma)| \right] \omega(\text{d}x) \\
\leq C\|O'\|_{\infty} \int \mathbb{E}^{\omega} \sum_{b} |\partial_b h_0(x)| \mathbb{G}^c \left[ |v_p^b(\sigma)| + |v_{p+1}^b(\sigma)| \right].
\]

Since \( A \geq 0 \) as a \( K \times K \) matrix, the equation (9.22) is contraction in \( L^2 \). Clearly \( A \) is a contraction in \( L^1 \) as well, hence it is a contraction in any \( L^q, 1 \leq q \leq 2 \), by interpolation. By the Hölder inequality and the \( L^q \)-contraction for some \( 1 < q < 2 \), we have for each fixed \( b \in I \) that
\[
\mathbb{E}^{\omega} |\partial_b h_0(x)| \mathbb{G}^c |v_p^b(\sigma)| \leq \left[ \mathbb{E}^{\omega} \mathbb{G}^c \right]^{q/(q-1)} \left[ \mathbb{E}^{\omega} |\partial_b h_0(x)||v_p^b(\sigma)|^q \right]^{1/q} \\
\leq \left[ \mathbb{P}^{\omega} \mathbb{G}^c \right]^{q/(q-1)} \left[ \mathbb{E}^{\omega} |\partial_b h_0(x)|^q \sum_{i \in I} |v_i^p(0)|^q \right]^{1/q} \\
\leq CK_3 \xi e^{-cK^a} \leq e^{-cK^a}
\]

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with some $\theta_4 > 0$. Here we used (9.42) for the first factor. The second factor was estimated by (8.24) (recall the definition of $h_0$ from (8.13)). After the summation over $b$, we get

$$ E^\omega \sum_b |\partial_b h_0(x)| G^c \left[ O'(x_p - x_{p+1}) \right] |v_p^b(\sigma) - v_{p+1}^b(\sigma)| \leq Ce^{-cK^{\theta_4} \|O'\|_\infty}. $$

Therefore, under the conditions of Corollary 9.2, and using the notation $E^\omega$ for the process, we have

$$ \left| \langle h_0; O(x_p - x_{p+1}) \rangle \right| \leq \frac{1}{2} \|O'\|_\infty \int_0^{C_1 K \log K} \mathbb{E}^\omega \left[ |G| \partial_b h_0(x) \left[ (v_p^b(\sigma) - v_{p+1}^b(\sigma)) \right] \right] \, d\sigma + O\left( \|O'\|_\infty K^{-2} \right), $$

where $v^b$ is the solution to (9.22), assuming that the constant $C_1$ in the upper limit of the integration is large enough.

### 9.4.2 Restriction to the set $\tilde{Q}$ and the decay estimates

The complement of the set $\tilde{Q}_{\sigma, Z}$ includes the “bad” paths for which the level repulsion estimate in an average sense does not hold. However, the probability of $\tilde{Q}_{\sigma, Z}$ is not very small, it is only a small negative power of $K$, see (9.43). This estimate would not be sufficient against the time integration of order $C_1 K \log K$ in (9.60); we will have to use an $L^1 - L^\infty$ decay property of (9.22) which we now derive. Denote the $L^p$-norm of a vector $u = \{u_j : j \in I\}$ by

$$ \|u\|_p = \left( \sum_{j \in I} |u_j|^p \right)^{1/p}. $$

**Proposition 9.4** Consider the evolution equation

$$ \partial_s u(s) = -A(s)u(s), \quad u(s) \in \mathbb{R}^I = \mathbb{R}^K $$
and fix $\sigma > 0$. Suppose that for some constant $b$ we have

$$ B_{jk}(s) \geq \frac{b}{(j-k)²}, \quad 0 \leq s \leq \sigma, \quad j \neq k, $$

and

$$ W_j(s) \geq \frac{b}{d_j}, \quad d_j := |j| - K + 1, \quad 0 \leq s \leq \sigma. $$

Then for any $1 \leq p \leq q \leq \infty$ we have the decay estimate

$$ \|u(s)\|_q \leq (sb)^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \|u(0)\|_p, \quad 0 < s \leq \sigma. $$

**Proof.** We consider only the case $b = 1$, the general case follows from scaling. We follow the idea of Nash and start from the $L^2$-identity

$$ \partial_s \|u(s)\|_2^2 = -2a(s)[u(s), u(s)]. $$
For each $s$ we can extend $u(s) : I \to \mathbb{R}^K$ to a function $\tilde{u}(s) : \mathbb{Z}$ by defining $\tilde{u}_i(s) = u_j(s)$ for $|j| \leq K$ and $\tilde{u}_j(s) = 0$ for $j > |K|$. Dropping the time argument, we have, by the estimates (9.63) and (9.64) with $b = 1$,

$$2\alpha(u, u) \geq \sum_{i,j \in \mathbb{Z}} \frac{(\tilde{u}_i - \tilde{u}_j)^2}{(i-j)^2} \geq c\|\tilde{u}\|_4^4\|\tilde{u}\|_2^{-2}, \quad (9.67)$$

with some positive constant. Where, in the second step, we used the Gagliardo-Nirenberg inequality for the discrete operator $\sqrt{-\Delta}$, see (B.5) in the Appendix B with $p = 4, s = 1$. Thus we have

$$a[u, u] \geq c\|u\|_4^4\|u\|_2^{-2}, \quad (9.68)$$

and the energy inequality

$$\partial_s\|u\|_2^2 \leq -c\|u\|_4^4\|u\|_2^{-2} \leq -c\|u\|_4^4\max\|u\|_1^{-1}, \quad (9.69)$$

using the Hölder estimate $\|u\|_2 \leq \|u\|_1^{1/3}\|u\|_2^{2/3}$. Integrating this inequality from 0 to $s$ we get

$$\|u(s)\|_2 \leq Cs^{-1/2}\|u(0)\|_1, \quad (9.70)$$

and similarly we also have $\|u(2s)\|_2 \leq Cs^{-1/2}\|u(s)\|_1$. Since the previous proof uses only the time independent lower bounds (9.63), (9.64), we can use duality in the time interval $[s, 2s]$ to have

$$\|u(2s)\|_\infty \leq Cs^{-1/2}\|u(s)\|_2.$$  

Together with (9.70) we have

$$\|u(2s)\|_\infty \leq Cs^{-1}\|u(0)\|_1.$$  

By interpolation, we have thus proved (9.65).

In the good set $G$ (see (9.41)), the bounds (9.63) and (9.64) hold with $b = cK^{-\xi'}$. Hence from the decay estimate (9.65), for any fixed $\sigma, Z$, we can insert the other good set $\tilde{Q}_{\sigma, Z}$ into the expectation in (9.60). This is obvious since the contribution of its complement is bounded by

$$\int_0^{C_1K\log K} \frac{d\sigma}{\sigma} \sum_{b \in I} E^\omega \tilde{Q}_{\sigma, Z}|\partial_b h_0(x)|(v^b_p(\sigma) + v^b_{p+1}(\sigma))$$

$$\leq CK^{-\xi'} \int_0^{C_1K\log K} \frac{d\sigma}{\sigma} E^\omega \left[ G\left( \sum_{b \in I} |\partial_b h_0(x)|^{1+\xi'} \tilde{Q}_{\sigma, Z} \right) \right]$$

$$\leq CK^{-2\xi'} \int_0^{C_1K\log K} \frac{d\sigma}{\sigma} E^\omega \left[ 1 + d(x_K(\sigma))^{-1} + d(x_{-K}(\sigma))^{-1} \right] \tilde{Q}_{\sigma, Z}$$

$$\leq CK^{-2\xi'} \int_0^{C_1K\log K} \frac{d\sigma}{\sigma} E^\omega \left[ \left[ 1 + d(x_K(\sigma))^{-1} + d(x_{-K}(\sigma))^{-1} \right]^{2/3} \right]^{2/3} \left[ P^\omega (\tilde{Q}_{\sigma, Z}) \right]^{1/3}$$

$$\leq CK^{-2\xi'} (C_1 K \log K) \xi K^{C_3\xi} K^{(C_4\xi - \rho)/3}, \quad (9.71)$$

where in the first line we used a Hölder inequality with exponents $1 + \xi$ and its dual, then we used the decay estimate (9.65) with $q = \infty, p = 1 + \xi$ in the second line. The purpose of taking a Hölder inequality with a power slightly larger than one was to avoid the logarithmic singularity in the $d\sigma$ integration at $\sigma \sim 0$. In the third line we split the sum into two parts and used the bound

$$|\partial_b h_0(x)| \leq |(V_x)'(x_j) - (V_x)'(x_j)| \leq \frac{K^\xi}{d(x_0)}, \quad (9.72)$$

init
that follows from (8.3) (with $\xi$ replaced by $\xi^2$ since $y, \bar{y} \in \mathcal{R}_{L,K}(\xi^2\delta/2, \alpha/2)$). Recall that $d(x)$ is the distance to the boundary, see (8.5). For indices away from the boundary, $|b| \leq K - CK\xi$, we have $|d(x_b)| \geq K^{-\xi} \min\{|b - K|, |b + K|\}$ on the set $\mathcal{G}$ that guarantees the finiteness of the sum.

For indices near the boundary we just estimated every term with the worst one, i.e. with the term $b = \pm K$. We used a Hölder inequality in the fourth line of (9.71) and computed the expectation by using (8.24) in the last line. Hence we have proved the following proposition:

**Proposition 9.5** Suppose that

$$\rho \geq 12\xi^2 + 6(C_4 + C_3 + 1)\xi$$

(9.73)

holds with $C_3$ and $C_4$ defined in (8.21) and in (9.43), respectively. Then for any fixed $Z, p \in I$ with $p \neq K$, we have

$$\left|\langle h_0; \Omega(x_p - x_{p+1})\rangle\omega\right| \leq \frac{1}{2}||O'||\infty \int_0^{C_1K\log K} \sum_{b \in I} \mathbb{E}^\omega\left[\bar{\mathcal{Q}}_{\sigma,Z}\mathcal{G}|\partial_b h_0(x)|\left|v^b_p(\sigma) - v^b_{p+1}(\sigma)\right|\right]d\sigma + O\left(||O'||\infty K^{-\rho/6}\right).$$

(9.74)

### 9.5 Short time cutoff and finite speed of propagation

The Hölder continuity of the parabolic equation (9.22) emerges only after a certain time, thus for the small $\sigma$ regime in the integral (9.74) we need a different argument. Since we are interested in the Hölder continuity around the middle of the interval $I$ (note that $|p| \leq K^{1-\xi}$ in Theorem 8.1), and the initial condition $\partial_b h_0$ is small if $b$ is in this region, a finite speed of propagation estimate for (9.22) will guarantee that $v^b_\sigma$ is small if $\sigma$ is not too large.

From now on, we fix $\sigma \leq C_1K\log K$, $|Z| \leq K/2$ and a path $\mathbf{x}(\cdot)$, and assume that $\mathbf{x}(\cdot) \in \mathcal{G} \cap \bar{\mathcal{Q}}_{\sigma,Z}$. In particular, thanks to the definition of $\mathcal{G}$ and the regularity of the locations $\alpha_j$, the time dependent coefficients $B_{ij}(s)$ and $W_i(s)$ of the equation (9.22) satisfy (9.63) and (9.64) with $b = K^{1-\xi}$.

We split the summation in (9.74). Fix a positive constant $\theta_5 > 0$. The contribution of the indices $|b| \leq K^{1-\theta_5}$ to (9.74) is bounded by

$$\int_0^{C_1K\log K} \mathbb{E}^\omega\bar{\mathcal{Q}}_{\sigma,Z}\mathcal{G} \sum_{|b| \leq K^{1-\theta_5}} |\partial_b h_0(x)| \left|v^b_p(\sigma) + v^b_{p+1}(\sigma)\right|d\sigma$$

(9.75)

where we neglected the $v^b_{p+1}$ term for simplicity since it can be estimated exactly in the same way. From the second to the third line we used that

$$|\partial_b h_0(x)| \leq \frac{K^\xi}{\min\{|b - K|, |b + K|\} + 1} \leq CK^\xi - 1, \quad |b| \leq K^{1-\theta_5},$$

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holds on the set $G$ from (9.72) and from the rigidity bound provided by $G$. Arriving at the last line of (9.75) we used the $L^1 \to L^\infty$ decay estimate (9.65) and we recall that the singularity $\sigma \sim 0$ can be cutoff exactly as in (9.71), i.e. by considering a power slightly larger than 1 in the first line. Note that the set $\tilde{Q}_{\sigma,Z}$ played no role in this argument.

Together with (9.74) and with the choice $\theta_5 > \rho$ (9.76) and recalling $\rho \geq 4 \xi'$ from (9.73), we have

$$\left| \langle h_0 O(x_p - x_{p+1}) \rangle \right| \leq \frac{1}{2} \| O' \|_\infty \int_0^{C_1 K \log K} \sum_{|b| > K^{1 - \theta_5}} E^\omega \left[ \tilde{Q}_{\sigma,Z} \partial_b h_0(x) \| v^b_p(\sigma) - v^b_{p+1}(\sigma) \| \right] d\sigma + O\left( \| O' \|_\infty K^{-\rho/6} \right).$$

The following lemma provides a finite speed of propagation estimate for the equation (9.22) which will be used to control the short time regime in (9.77). This estimate is not optimal, but it is sufficient for our purpose. The proof will be given in the next section.

**Lemma 9.6 [Finite Speed of Propagation Estimate]** Fix $b \in I$ and $\sigma \leq C_1 K \log K$. Consider $v^b(s)$, the solution to (9.22) and assume that the coefficients of $A$ satisfy $W_i(s) \geq K^{-\xi'}, B_{ij}(s) \geq K^{-\xi'} |i - j|^2$, $0 \leq s \leq \sigma$, (9.78) where $d_i := \min\{|i + K|, |i - K|\} + 1$. Assume that the bound $\sup_{0 \leq s \leq \sigma} \sup_{0 \leq M \leq K} \frac{1}{1 + s} \int_0^s \frac{1}{M} \sum_{i \in I, |i - Z| \leq M} \sum_{j \in I, |j - Z| \leq M} B_{ij}(s) ds \leq CK^{\rho_1}$, (9.79) is satisfied for some fixed $Z$, $|Z| \leq K/2$. Then for any $s > 0$ we have the estimate $|v^b_p(s)| \leq \frac{C K^{\rho_1 + 2\xi' + 1/2} \sqrt{s + 1}}{|p - b|}$, (9.80)

**9.6 Proof of the Finite Speed of Propagation Estimate, Lemma 9.6**

Let $1 \ll \ell \ll K$ be a parameter to be specified later. Split the time dependent operator $A = A(s)$ defined in (9.37) into a short range and a long range part, $A = S + R$, with

$$(Su)_j := - \sum_{k : |j - k| \leq \ell} B_{jk}(u_k - u_j) + W_j u_j$$

and

$$(Ru)_j := - \sum_{k : |j - k| > \ell} B_{jk}(u_k - u_j).$$

Note that $S$ and $R$ are time dependent. Denote by $U_S(s_1, s_2)$ the semigroup associated with $S$ from time $s_1$ to time $s_2$, i.e.

$$\partial_{s_2} U_S(s_1, s_2) = -S(s_2) U_S(s_1, s_2)$$

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for any $s_1 \leq s_2$, and $U_S(s_1, s_1) = I$; the notation $U_A(s_1, s_2)$ is analogous. Then by the Duhamel formula

$$v(s) = U_S(0, s)v_0 + \int_0^s U_A(s', s)R(s')U_S(0, s')v_0ds'.$$

(9.83)

Notice that for $\ell \gg K^{\ell'}$ and for $x(\cdot)$ in the good set $G$ (see (9.41)), we have

$$\|Rv\|_1 = \sum_{|j| \leq K} \sum_{k:|j-k| \geq \ell} \frac{1}{(x_j - x_k)^2} u_k \leq C\ell^{-1}\|v\|_1,$$

(9.84)

or more generally,

$$\|Rv\|_p \leq C\ell^{-1}\|v\|_p, \quad 1 \leq p \leq \infty.$$  

(9.85)

Recall the decay estimate (9.65) for the semigroup $U_A$ that is applicable by (9.78). Hence we have, for $s \geq 2$,

$$\int_0^s \|U_A(s', s)R(s')U_S(0, s')v_0\|_\infty \, ds' \leq K^{\ell'} \int_0^s (s - s')^{-1} \|R(s')U_S(0, s')v_0\|_1 \, ds' \leq K^{\ell'} \ell^{-1}(\log s)\|v_0\|_1,$$

where we used that $U_S$ is a contraction on $L^1$. The non-integrable short time singularity for $s'$ very close to $s$, $|s - s'| \leq K^{-C}$, can be removed by using the $L^p \to L^\infty$ bound (9.65) with some $p > 1$, invoking a similar argument in (9.71). In this short time cutoff argument we used that $U_S$ is an $L^p$ contraction for any $1 \leq p \leq 2$ by interpolation, and that the rate of the $L^p \to L^\infty$ decay of $U_A$ are given in (9.65).

$$\|v(s) - U_S(0, s)v_0\|_\infty \leq \ell^{-1}(\log s)K^{\ell'} \leq C\ell^{-1}(\log K)K^{\ell'},$$

(9.86)

where we have used that $x(\cdot)$ is in the good set $G$ and that $s \leq C_1 K \log K$.

We now prove a cutoff estimate for the short range dynamics. Let $r(s) := U_S(0, s)v_0$ and define

$$f(s) = \sum_j \phi_j r_j^2(s), \quad \phi_j = e^{\frac{|j-b_i|}{\theta}}$$

(9.87)

with some parameter $\theta \geq \ell$ to be specified later. Recall that $b$ is the location of the initial condition, $v_0 = \delta_b$. In particular, $f(0) = 1$.

Differentiating $f$ and using $W_j \geq 0$, we have

$$f'(s) = \partial_s \sum_j \phi_j r_j^2(s) \leq 2 \sum_j \phi_j \sum_{k:|j-k| \leq \ell} r_j(s)B_{kj}(s)(r_k - r_j)(s)$$

$$= \sum_{|j-k| \leq \ell} B_{kj}(s)(r_k - r_j)(s) [r_j(s)\phi_j - r_k(s)\phi_k]$$

$$= \sum_{|j-k| \leq \ell} B_{kj}(s)(r_k - r_j)(s) \phi_j [r_j - r_k](s)$$

$$+ \sum_{|j-k| \leq \ell} B_{kj}(s)(r_k - r_j)(s) [\phi_j - \phi_k] r_k(s).$$

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In the second term we use Schwarz inequality and absorb the quadratic term in \( r_k - r_j \) into the first term that is negative. Assuming \( \ell \leq \theta \), we have \( \phi_k^{-2} |\phi_j - \phi_k|^2 \leq C\ell^2 / \theta^2 \) for \( |j-k| \leq \ell \). Thus
\[
 f'(s) \leq C \sum_{|j-k| \leq \ell} B_{kj}(s)\phi_k^{-1} |\phi_j - \phi_k|^2 r_k^2(s) \leq C\theta^{-2} \ell^2 \left( \sum_{k', \ell, |j-k'| \leq \ell} B_{k', j}(s) \right) \sum_k \phi_k r_k^2(s).
\]

From a Gromwall argument we have
\[
 f(s) \leq \exp \left[ C\theta^{-2} \ell^2 \int_0^s \sum_{k, j: |j-k| \leq \ell} B_{kj}(s')ds' \right] f(0).
\] (9.88)

From the assumption (9.79) with \( M = K \) and any \( Z \), we can bound the integration in the exponent by
\[
 \int_0^s \sum_{k, j: |j-k| \leq K} B_{kj}(s')ds' \leq K^{1+\rho_1}(s+1).
\] (9.89)

Thus we have
\[
 \sum_j e^{(j-b)/\theta} r_j^2(s) = f(s) \leq \exp \left[ \theta^{-2}\ell^2 K^{1+\rho_1}(s+1) \right] f(0) \leq C,
\] (9.90)

provided that we choose
\[
 \theta = \ell K^{(\rho_1+1)/2}/\sqrt{s+1}.
\] (9.91)

In particular, this shows the following exponential finite speed of propagation estimate for the short range dynamics
\[
 r_j(s) \leq C \exp \left( -\frac{|j-b|}{\ell K^{(\rho_1+1)/2}/\sqrt{s+1}} \right).
\] (9.92)

Now we choose
\[
 \ell = |p-b| K^{-\xi'-\rho_1-1/2}/(s+1)^{-1/2}
\]
so that \( e^{(p-b)/\theta} \geq \exp \left( K^{\xi'} \right) \). Using this choice in (9.90) and (9.86) to estimate \( v^b_p(s) \), we have thus proved that
\[
 |v^b_p(s)| \leq \ell^{-1} (\log K) K^{\xi'} + C e^{-K^{-\xi'}} \leq \frac{K^{2\xi'} + (\rho_1+1)/2 \sqrt{s+1}}{|p-b|}.
\] (9.93)

This concludes the proof of Lemma 9.6.

### 9.7 Completing the proof of Theorem 8.1

In this section we complete the proof of Theorem 8.1 assuming a discrete version of the De Giorgi-Nash-Moser Hölder regularity estimate for the solution (9.2) (Theorem 9.8 below).

Notice that on the set \( G \cap \tilde{Q}_{\rho, \sigma, Z} \) the conditions of Lemma 9.6 are satisfied, especially (9.79) with the choice
\[
 \rho_1 := \rho + \xi',
\] (9.94)

follows from the definition (9.7) since for the summands with \( |i-j| \geq K^{\xi'} \) in (9.99) we can use \( B_{ij} \leq C|\alpha_i - \alpha_j|^{-2} \leq C|i-j|^{-2} \). Thus we can use (9.80) to estimate the short time integration regime in (9.77). Setting
\[
 \theta_5 := \min \left\{ \frac{\xi^*}{2}, \frac{1}{100} \right\},
\] (9.95)
we obtain, for any $|Z| \leq 2K^{1-\xi^*}$ and $|p| \leq K^{1-\xi^*}$
\[
\int_{0}^{K^{1/4}} \sum_{|b| > K^{1-\theta_5}} \mathbb{E}^\omega \left[ \tilde{Q}_{\sigma,Z} G |\partial_b h_0(x)| v^b_p(\sigma) - v^b_{p+1}(\sigma) \right] d\sigma \leq C \int_{0}^{K^{1/4}} \mathbb{E}^\omega \left[ \tilde{Q}_{\sigma,Z} G \sum_{|b| > K^{1-\theta_5}} |\partial_b h_0(x)| \right] d\sigma \leq CK^{2\xi^* + \rho_1 + 1/2 + \frac{\theta_5}{4} - (1 - \theta_5)} \mathbb{E}^\omega \left[ \tilde{Q}_{\sigma,Z} G \sum_{|b| > K^{1-\theta_5}} |\partial_b h_0(x)| \right] \leq CK^{4\xi^* + \rho_1 + \theta_5 - \frac{1}{4}} \mathbb{E}^\omega \left[ \tilde{Q}_{\sigma,Z} G \left( \frac{1}{d(x_K)} + \frac{1}{d(x-K)} \right) \right] \leq CK^{4\xi^* + \rho_1 + C_3\xi + \theta_5 - \frac{1}{4}} \leq K^{-\frac{1}{100}}
\]
provided that
\[
4\xi^* + \rho_1 + C_3\xi \leq \frac{1}{100}. \tag{9.97}
\]
In the third line above we used (9.80) together with $|p - b| \geq \frac{1}{2}K^{1-\theta_5}$. This latter bound follows from $|b| > K^{1-\theta_5}$ and $|p| \leq K^{1-\xi^*}$ and from the choice $\theta_5 < \xi^*$. In the fourth line we used (9.72) and that on the set $\mathcal{G}$ we have
\[
\sum_j \frac{1}{d(x_j)} \leq (\log K)K^{\xi^*} \left[ \frac{1}{d(x_K)} + \frac{1}{d(x-K)} \right].
\]
Moreover, in the last step we used (8.24). This completes the estimate for the small $\sigma$ regime. Notice that the set $\tilde{Q}_{\sigma,Z}$ did not play a role in this argument.

After the short time cutoff (9.96), we finally have to control the regime of large time and large $b$-indices, i.e.
\[
\int_{K^{1/4}}^{C_4 K \log K} \sum_{|b| > K^{1-\theta_5}} \mathbb{E}^\omega \left[ \tilde{Q}_{\sigma,Z} G |\partial_b h_0(x)| \right] d\sigma \leq \sum_{|b| > K^{1-\theta_5}} \mathbb{E}^\omega \left[ \tilde{Q}_{\sigma,Z} G |\partial_b h_0(x)| \right] d\sigma \leq (\log K)K^{\xi^*} \left[ \frac{1}{d(x_K)} + \frac{1}{d(x-K)} \right].
\]
from (9.77). We will exploit the Hölder regularity of the solution $\psi^b$ to (9.22). We will assume that the coefficients of $\mathcal{A}$ in (9.22) satisfy a certain regularity condition.

**Definition 9.7** The equation
\[
\partial_t \psi(t) = -\mathcal{A}(t)\psi(t), \quad \mathcal{A}(t) = \mathcal{B}(t) + \mathcal{W}(t), \quad t \in \mathcal{T} \tag{9.98}
\]
is called **regular** at the space-time point $(Z, \sigma) \in I \times \mathcal{T}$ with exponent $\rho$, if
\[
\sup_{s \in \mathcal{T}} \sup_{1 \leq M \leq K} \left[ \frac{1}{1 + |s - \sigma|} \right] \int_s^\sigma \left[ \frac{1}{M} \sum_{i,j} \sum_{|j| \leq M} B_{ij}(u) du \right] \leq K^\rho. \tag{9.99}
\]
Furthermore, the equation is called **strongly regular** at the space-time point $(Z, \sigma) \in I \times \mathcal{T}$ with exponent $\rho$ if it is regular at all points $(Z) \times \{ \Xi + \sigma \}$, where we recall the definition of $\Xi$ from (9.10):
\[
\Xi = \left\{ -K \cdot 2^{-m} (1 + 2^{-k}) \mid 0 \leq k, m \leq C \log K \right\}.
\]
Fix a $Z \in I$, $|Z| \leq K/2$ and a $\sigma \in \mathcal{T}$. Recall that on the set $\mathcal{G} \cap Q_{\sigma,Z}$ the regularity at $(p, \sigma)$ with exponent $p_1$ from (9.94) follows from (9.7). Analogously, on the event $\mathcal{G} \cap \tilde{Q}_{\sigma,Z}$, the strong regularity at $(Z, \sigma)$ with a slightly increased exponent $p_1$ holds.

We formulate the partial Hölder regularity result for the equation (9.98). We collect the following facts on the coefficients $B_{ij}(s)$ and $W_i(s)$ that follow from $x(\cdot) \in \mathcal{G}$.

\begin{align}
B_{ij}(s) &\geq \frac{K^{-\xi}}{|i-j|^2}, \quad W_i(s) \geq \frac{K^{-\xi}}{d_i} \quad \text{for any } s \in \mathcal{T}, \ i, j \in I. \quad (9.100) \\
W_i(s) &\leq \frac{K^{\xi}}{d_i}, \quad \text{for any } s \in \mathcal{T}, \ d_i \geq K^{C\xi}. \quad (9.101)
\end{align}

and

\[ \frac{1}{C(i-j)^2} \leq B_{ij}(s) \leq \frac{C}{(i-j)^2} \quad \text{for any } s \in \mathcal{T}, \ |i-j| \geq \tilde{C}K^{\xi}. \quad (9.102) \]

\textbf{Theorem 9.8} There exists a universal constant $q > 0$ with the following properties. Let $v(t) = v^b(t)$ be a solution to (9.98) for any choice of $b \in I$, with initial condition $v^b_0(0) = \delta_b$. Let $Z \in I$, $|Z| \leq K/2$ and $\sigma \in [K^{\alpha_1}, C_1K \log K]$ be fixed, where $c_3 > 0$ is an arbitrary positive constant. There exist positive constants $\xi_0$, $p_0$ (depending only on $c_3$) such that if the coefficients of $A$ satisfy (9.100), (9.101), (9.102) with some $\xi' \leq \xi_0$ and the equation is strongly regular at the point $(Z, \sigma)$ with an exponent $p_1 \leq p_0$ then for any $\alpha \in [0,1/3]$ we have

\[ \sup_{|j-Z|+|j'-Z| \leq \sigma_1^{1-\alpha}} |v_j(\sigma) - v_{j'}(\sigma)| \leq CK^{\xi'} \sigma_1^{-1-\frac{3}{4}q}, \quad \sigma_1 := \min\{\sigma, K^{1-c_3}\}, \quad (9.103) \]

where $v = v^b$ for any choice of $b$. The constant $C$ in (9.103) depends only on $c_3$.

Theorem 9.8 follows directly from the slightly more general Theorem 10.2 presented in Section 10 and it will be proved there.

Armed with Theorem 9.8, we now complete the proof of Theorem 8.1. As we already remarked, the conditions of Theorem 9.8 are satisfied on the set $Q_{\sigma,Z} \cap \mathcal{G}$ with some $p_0, \xi_0$ small universal constants. For any $|p| \leq K^{-1-\xi'}$ fixed, we choose $Z = p$ (in fact, we could choose any $Z$ with $|Z - p| \leq C$). Using (9.72), we have, for the large time integration regime in (9.77),

\[ \int_{K^{1/4}}^{C_1K \log K} \sum_{|b| > K^{1-\xi_5}} E^\omega \left[ \tilde{Q}_{\sigma,p} \partial_b h_0(x) \left| v^b_p(\sigma) - v^b_{p+1}(\sigma) \right| \right] d\sigma \]

\[ \leq CK^{\xi'} \int_{K^{1/4}}^{C_1K \log K} E^\omega \left[ \tilde{Q}_{\sigma,p} \sum_{|b| > K^{1-\xi_5}} \frac{1}{d(x_b)} \left| v^b_p(\sigma) - v^b_{p+1}(\sigma) \right| \right] d\sigma \]

\[ \leq CK^{2\xi'} \int_{K^{1/4}}^{C_1K \log K} \sigma^{-1-\frac{4}{3}q} E^\omega \left[ \tilde{Q}_{\sigma,p} \sum_{|b| > K^{1-\xi_5}} \frac{1}{d(x_b)} \right] d\sigma \]

\[ \leq CK^{3\xi' + p_1 + C_3\xi - \frac{3}{2}q}. \]

In the third line we used Theorem 9.8 with $c_3 = 1/4$ and $\alpha = 1/3$. In the last line of (9.104) we used a similar argument as in the last step of (9.96).
Finally, from (9.77), (9.96) and (9.104) and ρ₁ = ρ + ξ' we have

\[ |(h₀; O(x_p - x_{p+1}))| ≤ C∥O′∥_{∞}K^{4ξ' + p + C_3ξ - \frac{1}{4}q} + O(K^{-\frac{1}{6}}) + O(K^{-\rho/6}). \tag{9.105} \]

For a given ξ* > 0, recall that we defined \( θ₃ := \min\{\frac{ξ*}{100}, \frac{1}{100'}\} \) and we now choose

\[ ρ := \min\left\{ \frac{θ₃}{100'}, \frac{θ₃}{2} \right\} = \min\left\{ \frac{q}{100}, \frac{ξ*}{4}, \frac{1}{200} \right\}. \tag{9.106} \]

in particular (9.76) is satisfied. Since q > 0 is a universal constant, it is then clear that for any sufficiently small ξ all conditions in (9.97) and (9.73) on the exponents ξ, ξ' = (C₂ + 1)ξ² and ρ₁ = ρ + ξ' can be simultaneously satisfied. Thus we can make the error term in (9.105) smaller than \( K^{Cξ}K^{-ρ/6} \). With the choice of \( ε = ρ/6 \), where ρ is from (9.106), we proved Theorem 8.1. □

Although the choices of parameters seem to be complicated, the underlying mechanism is that there is a universal positive exponent \( q \) in (9.103). This exponent provides an extra smallness factor in addition to the natural size of \( v_j(σ) \), which is \( σ^{-1} \) from the \( L^1 \to L^∞ \) decay. As (9.103) indicates, this gain comes from a Hölder regularity on the relevant scale. The parameters \( ξ, ξ' \) and \( ξ* \) can be chosen arbitrarily small (without affecting the value of q). These parameters govern the cutoff levels in the regularization of the coefficients of \( A \). There are other minor considerations due to an additional cutoff for small time where we have to use a finite speed estimate. But the arguments for this part are of technical nature and most estimates are not optimized. We just worked out estimates sufficient for the purpose of proving Theorem 8.1. The choices of exponents related to the various cutoffs do not have intrinsic meanings.

As a guide to the reader, our choice of parameters, roughly speaking, are given by the following rule: We first fix a small parameter \( ξ* \). Then we choose the cutoff parameter \( θ₃ \) to be slightly smaller than \( ξ*, (9.95) \). The exponent \( ρ \) in (9.7) has a lower bound by \( ξ \) and \( ξ' \) in (9.73). On the other hand, \( ρ \) will affect the cutoff bound and so we have the condition \( ρ < θ₃ \) (i.e., (9.76)). So we choose \( ρ ≪ ξ* \) and make \( ξ, ξ' \) very small so that the lower bound requirement on \( ρ \) is satisfied. Finally, if the parameter \( ξ* \leq q/100 \), we can use the gain from the Hölder continuity to compensate all the errors which depend only on \( ξ, ξ', ξ* \).

10 A discrete De Giorgi-Nash-Moser estimate

In this section we prove Theorem 9.8, which is a Hölder regularity estimate for the parabolic evolution equation

\[ \partialₜu(s) = -A(s)u(s). \tag{10.1} \]

where \( A(s) = B(s) + W(s) \) are symmetric matrices defined by

\[ [B(s)u]_j = -\sum_{k\neq j} B_{jk}(s)(u_k - u_j), \quad [W(s)u]_i = W_i(s)u_i \tag{10.2} \]

and \( B_{ij}(s) \geq 0 \). Here \( I = \{-K, -K + 1, \ldots, K\} \) and \( u ∈ C^I \). We will study this equation in a time interval \( T ⊂ \mathbb{R} \) of length \( |T| = σ \) and we will assume that \( σ ∈ [K^{C₃}, CK \log K] \). The reader can safely think of \( σ = CK \log K \). In the applications we set \( T = [0, σ] \), but we give some definitions more generally. The reason is that traditionally in the regularity theory for parabolic equations one sets the initial condition \( u(-σ) \) at some negative time \( -σ < 0 \) and one is interested
in the regularity of the solution \( u(s) \) around \( s = 0 \). In this case \( \mathcal{T} \) starts at \(-\sigma\), so in this section \( \mathcal{T} = [-\sigma, 0] \). This convention is widely used for parabolic equations and in particular in [11]. Later on in our application, we will need to make an obvious shift in time.

We will now state a general Hölder continuity result, Theorem 10.1, concerning the deterministic equation (10.1) over the finite set \( I \) and on the time interval \( \mathcal{T} = [-\sigma, 0] \). Theorem 10.1 will be a local Hölder continuity result around an interior point \( Z \in I \) separated away from the boundary.

We recall the definition of strong regularity from Definition 9.7. The following conditions on \( \mathcal{A} \) will be needed that are characterized by two parameters \( \xi, \rho > 0 \).

\( \text{(C1)}_\rho \) The equation (10.1) is strongly regular with exponent \( \rho \) at the space-time point \((Z, 0)\).

\( \text{(C2)}_\xi \) Denote by \( d_i = d_i^I := \min\{|i + K + 1|, |1 + K - i|\} \) the distance of \( i \) to the boundary of \( I \).

For some large constants \( C, \tilde{C} \geq 10 \), the following conditions are satisfied:

\[
B_{ij}(s) \geq \frac{K^{-\xi}}{|i - j|^2}, \quad \text{for any } s \in \mathcal{T}, \quad d_i \geq \frac{K}{C}, \quad d_j \geq \frac{K}{C}, \tag{10.3}
\]

\[
W_i(s) \leq \frac{K^\xi}{d_i}, \quad \text{if } \quad d_i \geq K^{C\xi}, \quad s \in \mathcal{T}, \tag{10.4}
\]

\[
\frac{1}{C(i - j)^2} \leq B_{ij}(s) \leq \frac{C}{(i - j)^2}, \quad \text{if } \quad |i - j| \geq \tilde{C}K^\xi \quad \text{and } s \in \mathcal{T}. \tag{10.5}
\]

**Theorem 10.1 (Parabolic regularity with singular coefficients)** There exists a universal constant \( q > 0 \) such that the following holds. Consider the equation (10.1) on the time interval \( \mathcal{T} = [-\sigma, 0] \) with some \( \sigma \in [K^{-c_1}, K^{1-c_3}] \), where \( c_3 > 0 \) is a positive constant. Fix \( |Z| \leq K/2 \) and \( \alpha \in [0, 1/3] \). Suppose that \( \text{(C1)}_\rho \) and \( \text{(C2)}_\xi \) hold with some exponents \( \xi, \rho \) small enough depending on \( c_3 \). Then for the solution \( u \) to (10.1) we have

\[
\sup_{|j - Z| + |j' - Z| \leq \sigma^{1-\alpha}} |u_j(0) - u_{j'}(0)| \leq C\sigma^{-q\alpha} \|u(-\sigma)\|_\infty. \tag{10.6}
\]

The constant \( C \) in (10.6) depends only on \( c_3 \) and is uniform in \( K \). The result holds for any \( K \geq K_0 \), where \( K_0 \) depends on \( c_3 \).

We remark that the upper bound \( \sigma \leq K^{1-c_3} \) is not an important condition, it is imposed only for convenience to state (10.6) with a single scaling parameter. More generally, for any \( \sigma \geq K^{c_3} \) it holds that

\[
\sup_{|j - Z| + |j' - Z| \leq \sigma^{1-\alpha}} |u_j(0) - u_{j'}(0)| \leq C\sigma_1^{-q\alpha} \|u(-\sigma)\|_\infty. \tag{10.7}
\]

where \( \sigma_1 := \min\{\sigma, K^{1-c_3}\} \). If \( \sigma \geq K^{1-c_3} \), then (10.7) immediately follows by noticing that \( \|u(-\sigma_1)\|_\infty \leq \|u(-\sigma)\|_\infty \) and apply (10.6) with \( \sigma_1 = K^{1-c_3} \) instead of \( \sigma \).

To understand why Theorem 10.1 is a Hölder regularity result, we rescale the solution so that it runs up a time of order one. I.e., for a given \( \sigma \ll 1 \) we define the rescaled solution

\[
U(T, X) := u_{[\sigma X] + Z}(T\sigma), \quad \sigma \gg 1,
\]

(where \([\cdot]\) denoted the integer part). Then the bound (10.6) says that

\[
\sup_{|X| + |Y| \leq \varepsilon} |U(0, X) - U(0, Y)| \leq C\varepsilon^{q\alpha} \|U(-1, \cdot)\|_\infty, \quad \varepsilon \in [\sigma^{-1/3}, 1].
\]

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Thus, in the macroscopic coordinates \((T, X)\) the Hölder regularity for \(U\) holds around \((0, 0)\) from order one scales down to order \(\sigma^{-1/3}\) scales. Note that Hölder regularity holds only at one space-time point, since the strong regularity condition \((C1)_\rho\) was centered around a given space-time point \((Z, 0)\) in microscopic coordinates.

Notice that by imposing the regularity condition we only require the time integration of the singularity of \(B_{ij}\) is bounded. Thus we substantially weaken the standard assumption in parabolic regularity theory on the supremum bound on the ellipticity.

Theorem 10.1 is a Hölder regularity result with \(L^\infty\) initial data. Combining it with the decay estimate Proposition 9.4, we get a Hölder regularity result with \(L^1\) initial data. However, for the application of the decay estimate, we need to strengthen the condition (10.3) to

\[
B_{ij}(s) \geq \frac{K^{-\xi}}{|i-j|^2}, \quad W_i(s) \geq \frac{K^{-\xi}}{d_i}, \quad \text{for any } s \in \mathcal{T}, \ i, j \in I. \tag{10.8}
\]

Let \((C2)^\xi_k\) be the condition identical to \((C2)^\xi\) except that (10.3) is replaced with (10.8).

**Theorem 10.2** There exists a universal constant \(q > 0\) such that the following holds. Consider the equation (10.1) on the time interval \(\mathcal{T} = [-\tau - \sigma, 0]\) with some \(\tau > 0\) and \(\sigma \in [K^{c_3}, K^{1-c_3}]\), where \(c_3 > 0\) is a positive constant. Fix \(|Z| \leq K/2\) and \(\alpha \in [0, 1/3]\). Suppose that \((C1)_\rho\) and \((C2)^\xi\) hold with some small exponents \(\xi, \rho\) depending on \(c_3\). Then for the solution \(u\) to (10.1) we have

\[
\sup_{|j-Z|+|j'-Z| \leq \sigma^{1-\alpha}} |u_j(0) - u_{j'}(0)| \leq C K^{\xi_1} \sigma^{-q_0} \tau^{-1} \|u(-\tau - \sigma)\|_1. \tag{10.9}
\]

The constant \(C\) in (10.6) depends only on \(c_3\) and is uniform in \(K\). The result holds for any \(K \geq K_0\), where \(K_0\) depends on \(c_3\).

**Proof.** We can apply Proposition 9.4 with \(b = K^{-\xi}, p = 1, q = \infty\) on the time interval \([-\tau - \sigma, -\sigma]\). Then (9.65) asserts that

\[
\|u(-\sigma)\|_\infty \leq K^{\xi_1} \tau^{-1} \|u(-\tau - \sigma)\|_1
\]

and (10.9) follows from (10.6).

**Proof of Theorem 9.8.** To avoid confusion between the roles of \(\sigma\) in this proof we denote the \(\sigma\) in the statement of Theorem 9.8 by \(\sigma'\). We will apply Theorem 10.2 and we choose \(\sigma \) and \(\tau\) such that \(\sigma' = \sigma + \tau\). We also shift the time by \(\sigma'\) so that the initial time is zero and the final time \(\sigma + \tau = \sigma'\). The conditions \((C1)_\rho\) and \((C2)^\xi\) directly follow from (9.100), (9.101), (9.102) and from strong regularity at \((Z, \sigma')\) but \(\varrho_1\) and \(\xi_1\) are replaced by \(\varrho\) and \(\xi\) for simplicity of notations. Given \(\sigma' \in [K^{c_3}, K_1 K \log K]\), we consider two cases. If \(\sigma' \leq K^{1-c_3}\), we apply Theorem 10.2 with \(\sigma = \tau = \sigma'/2\). Then \(\|u(-\tau - \sigma)\|_1\) becomes \(\|\varphi\|_1 = 1\) on the right hand side of (10.9) and (9.103) follows. If \(\sigma' \geq K^{1-c_3}\), then we apply Theorem 10.2 with \(\sigma = \frac{\sigma'}{2} K^{1-c_3}\) and \(\tau := \sigma' - \sigma\). In this case \(\tau\) is comparable with \(\sigma'\) and \(\sigma' \leq \sigma^{3/2}\) and (9.103) again follows.

The rest of this section is devoted to the proof of Theorem 10.1. Our strategy follows the approach of [11]; the multiscale iteration scheme and the key cutoff functions (10.24, 10.25) are also the same as in [11]. The main new feature of our argument is the derivation of the local energy estimate, Lemma 10.6, for parabolic equation with singular coefficients satisfying \((C1)_\rho\) and \((C2)^\xi\). The proof of Lemma 10.6 will proceed in two steps. We first use the condition \((C1)_\rho\)
and the argument of the energy estimate in [11] to provide a bound in $L_t^\infty(L^2)\cap L^2_t(L^\infty)$ on the solution to (10.1) (part (i) of Lemma 10.6). Along this proof we also prove an energy dissipation estimate which can be translated into the statement that the energy is small for most of the times. Using a new Sobolev type inequality (Proposition B.4) designed to deal with weak ellipticity we can improve the bound in $L_t^\infty(L^2)\cap L^2_t(L^\infty)$ in space for most of the times to obtain part (ii) of Lemma 10.6. Finally, we run the argument again to improve the $L_t^\infty(L^2)$ estimate for short times (part (iii) of Lemma 10.6) that is needed to close the iteration scheme. Besides this proof, the derivation of the second De Giorgi estimate (Lemma 10.7) is also adjusted to the weaker condition (C1),

We warn the reader that the notations of various constants in this section will follow [11] as much as possible for the sake of easy comparison with the paper [11]. The conventions of these constants will differ from the ones in the previous sections, and, in particular, we will restate all conditions.

10.1 Hölder regularity

Define for any set $S$ and any real function $f$ the oscillation $\text{Osc}_S f := \sup_S f - \inf_S f$.

**Theorem 10.3** There exists a universal positive constant $q$ with the following property. For any two thresholds $1 < \vartheta_1 < \vartheta_0$ there exist two positive constants $\xi, \rho$, depending only on $\vartheta_1$ and $\vartheta_0$ such that the following hold:

Set $\mathcal{M} := 2^{-\tau_0}K$ where $\tau_0 \in \mathbb{N}$ is chosen such that $\vartheta := \log K / \log \mathcal{M} \in [\vartheta_1, \vartheta_0]$. Suppose that (10.1) satisfies (C1) and (C2) with some $Z \in [-K/2, K/2]$. Suppose $u$ is a solution to (10.1) in the time interval $\mathcal{T} = [-3\mathcal{M}, 0]$. Assume that $u$ satisfies

$$\sup_{t \in [-3\mathcal{M}, 0]} \max_i |u_i(t)| \leq \ell$$  \hspace{1cm} (10.10) \hspace{1cm} \text{supbound}

for some $\ell$. Then for any $\alpha \in [0, 1/3]$ there is a set $\mathcal{G} \subset [-\mathcal{M}^{1-\alpha}, 0]$ such that

$$\text{Osc}_{\mathcal{G}^*}(u) \leq 4\ell \mathcal{M}^{-q\alpha}, \quad Q^{(\alpha)*} := \mathcal{G} \times \mathbb{Z}, \quad Q^{(\alpha)} := [-\mathcal{M}^{1/2}, 0] \times \mathbb{Z},$$  \hspace{1cm} (10.11) \hspace{1cm} \text{holder}

with

$$|[-\mathcal{M}^{1-\alpha}, 0] \setminus \mathcal{G}| \leq \mathcal{M}^{1/4},$$

i.e. the oscillation of the solution on scale $\mathcal{M}^{1-\alpha}$ (and away from the edges of the configuration space) is smaller than $4\ell \mathcal{M}^{-q\alpha}$ for most of the times. Moreover,

$$\text{Osc}_{Q^{(\alpha)}}(u) \leq C \ell \mathcal{M}^{-q\alpha}, \quad Q^{(\alpha)} := [-\mathcal{M}^{1/2}, 0] \times \mathbb{Z},$$  \hspace{1cm} (10.12) \hspace{1cm} \text{sureholder}

i.e. the oscillation is controlled for all times near 0. These results hold for any $K \geq K_0$ sufficiently large, where the threshold $K_0$ as well as the constant $C$ in (10.12) depend only on the parameters $\vartheta_0, \vartheta_1$.

We remark that the constant $q$ plays the role of the Hölder exponent and it depends only on $\varepsilon_0$ from Lemma 10.6. This will be explained after Lemma 10.8 below.

**Proof of Theorem 10.1.** With Theorem 10.3, we now complete the proof of Theorem 10.1. Given $\sigma \in [K^{c_3}, K^{1-c_3}]$, define $\mathcal{M} := 2^{-\tau_0}K$ with some $\tau_0 \in \mathbb{N}$ such that $\sigma/6 \leq \mathcal{M} \leq \sigma/3$. Choosing

$$\vartheta_1 := 1 + \frac{1}{2c_3}, \quad \vartheta_0 := \frac{2}{c_3},$$

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clearly \( \vartheta = \log K / \log M \in [\vartheta_1, \vartheta_0] \). Then (10.6) follows from (10.12) at time \( t = 0 \) using that \( \sigma^{1-\alpha} \leq 3M^{1-\alpha} \).

The proof of Theorem 10.3 will be a multiscale argument. On each scale \( n = 0, 1, 2, \ldots, n_{\text{max}} \) we define a space-time scale \( M_n := \nu^n M \) and a size-scale \( \ell_n := \zeta^n \ell \) with some \( \nu, \zeta < 1 \) scaling parameters to be chosen later. The initial scales are \( M_0 = M \) and \( \ell_0 = \ell \). For notational convenience we assume that \( \nu \) is of the form \( \nu = 2^{-j_0} \) for some \( j_0 > 0 \) integer. We assume that \( \nu \leq \frac{\zeta 10}{10} \), and eventually \( \zeta \) will be very close 1, while \( \nu \) will be very close to 0. The corresponding space-time box on scale \( n \) is given by

\[
Q_n := [-M_n, 0] \times [Z - M_n, Z + M_n].
\]

We will sometimes use an enlarged box

\[
\hat{Q}_n := [-3M_n, 0] \times [Z - \hat{M}_n, Z + \hat{M}_n], \quad \hat{M}_n := LM_n
\]

with some large parameter \( L \) that will always be chosen such that \( \nu \leq \frac{1}{2^L} \) and thus \( \hat{Q}_n \subset Q_{n-1} \). We stress that the scaling parameters \( \nu, \zeta, L \) will be absolute constants, independent of any parameters in the setup of Theorem 10.3.

The smallest scale is given by the relation \( M_{n_{\text{max}}} \sim M^{1-\alpha} \), i.e. \( n_{\text{max}} = \alpha \log M / \log \nu \). In particular, since \( \alpha \leq 1/3 \), all scales arise in the proofs will be between \( M^{2/3} \) and \( M \):

\[
M^{2/3} \leq M_n \leq M = M_0, \quad \forall n = 0, 1, \ldots, n_{\text{max}}.
\]

The following statement is the main technical result that will immediately imply Theorem 10.3. In the application we will need only the second part of this technical theorem, but its formulation is tailored to its proof that will be an iterative argument from larger to smaller scales.

There will be several exponents in this theorem, but the really important one is \( \chi \), see explanation around (10.20) later. The exponents \( \xi \) and \( \rho \) can be chosen arbitrarily small and the reader can safely neglect them at first reading.

**Theorem 10.4 (Staircase estimate)** There exist positive parameters \( \nu, \zeta, L \), satisfying

\[
\nu < \min\{\zeta^{10}/10, 1/(2L)\}
\]

with the following property.

For any two thresholds \( 1 < \vartheta_1 < \vartheta_0 \) there exist three positive constants \( \chi, \xi \), and \( \rho \) depending only on \( \vartheta_1 \) and \( \vartheta_0 \) (given explicitly in (10.85) and (10.86) later) such that under the setup and conditions of Theorem 10.3, for any scale \( n = 0, 1, 2, \ldots, n_{\text{max}} \) there exists a decreasing sequence of sets \( G_n \subset [-3M_n, 0] \) of “good” times, \( G_n \subset G_{n-1} \subset \ldots \), with

\[
|G_n| \leq C \sum_{m=0}^{n-1} \frac{M_m^{1/4}}, \quad G_n^c := [-3M_n, 0] \setminus G_n,
\]

such that we have the following two estimates:

i) [Staircase estimate] Define the constant \( \bar{u}_n \) by

\[
\sup_{Q_n^c} |u - \bar{u}_n| = \frac{1}{2} \text{Osc}_{Q_n^c} (u),
\]

\[
\text{Osc}_{Q_n^c} (u) := \sup_{Q_n^c} u - \inf_{Q_n^c} u.
\]
where
\[ Q^*_n := ([{-M_n, 0}] \cap G_n) \times [Z - M_n, Z + M_n] \]
and for any \( m < n \) define
\[ S_{m,n} := \sum_{j=m}^{n-1} |\bar{u}_j - \bar{u}_{j+1}|. \quad (10.16) \]

Then
\[ |u_i(t) - \bar{u}_n| \leq \Psi^{(n)}_i(t) \quad \forall t \in [-3M_n, 0], \quad \forall i \quad (ST)_n \]
where \( \Psi^{(n)} \) is a function on \([-3M_n, 0] \times I \) defined by
\[ \Psi^{(n)}_i(t) := \Lambda^{(n)}_i \cdot 1(t \in G_n) + \Phi^{(n)}_i(t) \cdot 1(t \in G_n^c) \]
with
\[ \Lambda^{(n)}_i := 1(\widehat{\mathcal{M}}_0 \leq |i - Z|) \cdot \ell_0 + \sum_{m=0}^{n-1} 1(\widehat{\mathcal{M}}_{m+1} \leq |i - Z| \leq \widehat{\mathcal{M}}_m) \cdot \left[ \ell_m + S_{m,n} \right] + 1(|i - Z| \leq \widehat{\mathcal{M}}_n) \cdot \ell_n \]
and
\[ \Phi^{(n)}_i(t) := C_\Phi \cdot 1(\widehat{\mathcal{M}}_0 \leq |i - Z|) \cdot \ell_0 + C_\Phi \sum_{m=0}^{n-1} 1(\widehat{\mathcal{M}}_{m+1} \leq |i - Z| \leq \widehat{\mathcal{M}}_m) \cdot \left[ \ell_m \left( 1 + \sqrt{\frac{|t| + M^{1/2}_m M^{1/2}_m}{M_n^2}} + S_{m,n} \right) \right] + C_\Phi \cdot 1(|i - Z| \leq \widehat{\mathcal{M}}_n) \cdot \ell_n \left( 1 + \sqrt{\frac{|t| + M^{1/2}_m M^{1/2}_m}{M_n^2}} \right) \]
with some fixed constant \( C_\Phi \). The subscript \( \Phi \) in \( C_\Phi \) indicates that this specific constant controls the functions \( \Phi^{(n)} \).

ii) [Oscillation estimate] For the good times we have
\[ \frac{1}{2} \text{Osc}_{Q_{n+1}^*}(u) \leq \zeta_{n+1} \ell_n, \quad (OSC)_n \]
i.e. \((10.19)\) asserts that in the smaller box \( Q_{n+1}^* \subset Q^*_n \) the oscillation is reduced from \( \ell_n \) to \( \ell_{n+1} \).

All statements hold for any \( K \geq K_0 \) sufficiently large, where the threshold \( K_0 \) as well as the constant \( C_\Phi \) depend only on the universal constants \( \nu, \zeta, L \) and on the parameters \( \vartheta_0, \vartheta_1, \varphi, \rho \).

Here the time-independent profile \( \Lambda^{(n)}_i \) is the “good” staircase function, representing the control for most of the times (“good times”). The function \( i \to \Lambda^{(n)}_i \) is a stepfunction that monotonically increases in \( |i - Z| \) at a rate approximately
\[ \Lambda^{(n)}_i \sim \ell_n \left( \frac{|i - Z|}{M_n} \right), \quad |i - Z| \gg M_n, \]
where
\[ q = \frac{\log \zeta}{\log \nu} \]  
(10.20)

is a small positive exponent. Note that this exponent is the same as the final Hölder exponent in Theorems 10.3 and 10.1.

For the "bad times" (the complement of good times), a larger control described by \( \Phi^{(n)}(t) \) holds. This weaker control is time dependent and deteriorates with larger \(|t|\). The exponent \( \chi \) in the definition of \( \Phi \), see (10.18), will be essentially equal to \( q \) (modulo some upper cutoff, see (10.85) later). The factor \( M_n^{1/2} \) on scale \( n \) expresses how much the estimate deteriorates for "bad times" compared with the estimate at "good times".

The bound (10.17) for good times \( t \in G_n \) with the control function \( \Lambda^{(n)} \) directly follows from (10.19) and (10.10). The new information in (10.17) is the weaker estimate expressed by \( \Phi^{(n)} \) that holds for all times. Note that \( \Lambda^{(n)} \) follows from (10.17). We also remark that (10.19) implies \( |\bar{u}_n - \bar{u}_{n+1}| \leq \ell_n \) and thus

\[ S_{m,n} = \sum_{j=m}^{n-1} |\bar{u}_j - \bar{u}_{j+1}| \leq \sum_{j=m}^{n-1} \ell_j \leq \frac{\ell_m}{1 - \zeta}, \]  
(10.22)

which gives an estimate for the effect of the shifts in the definition of \( \Lambda^{(n)} \) and \( \Phi^{(n)} \). Moreover, the uniform bound (10.10) shows that for any \( n \)

\[ |\bar{u}_n| \leq \ell_0 = \ell. \]  
(10.23)

**Proof of Theorem 10.3.** Without loss of generality we can assume that \( \mathcal{M}^{-\alpha} \leq \nu^2 \), otherwise \( \mathcal{M}^{-\alpha} \geq \zeta^2 \geq 2/3 \), so (10.11) immediately follows from (10.10). For \( \mathcal{M}^{-\alpha} \leq \nu^2 \), the estimate (10.11) directly follows from (10.19), by choosing \( n \geq 1 \) such that \( M_{n+2} \leq 3\mathcal{M}^{1-\alpha} \leq M_{n+1} \), i.e. \( \nu^{n+2} \leq 3\mathcal{M}^{-\alpha} \leq \nu^{n+1} \). Then \( \ell_{n+1} = \ell_{n+1}^{n+1} \leq 2\mathcal{M}^{-\alpha} \) with \( q \) defined in (10.20). The set \( G \) in Theorem 10.3 will be just \( G_{n+1} \cap [-M^{1-\alpha}, 0]. \) The proof of (10.12) follows from (10.17) noting that for \(|t| \leq \mathcal{M}^{1/2} \leq M_n^{1/4} \) (see (10.13)) the terms

\[ \sqrt{|t| + M^{1/2} M_n^{1/2}} \ll 1, \quad m = 0, 1, 2 \ldots, n, \]

are all negligible and simply we have

\[ \Phi^{(n)}(t) \leq C \Phi_L^{(n)}, \quad |t| \leq \mathcal{M}^{1/2}. \]

Thus (10.12) follows exactly as (10.11). This completes the proof. \( \square \)

For the rest of the section we will prove Theorem 10.4. We will iteratively check the main estimates, \( (ST)^n_0 \) and \( (OSC)^n_0 \) from scale to scale. For \( n = 0 \), the bound \( (ST)^0 \) is given by (10.10). In Section 10.2 we prove for any \( n \) that \( (ST)^n_0 \) implies \( (OSC)^n_0 \). In Section 10.3 we prove that \( (ST)^n_0 \) and \( (OSC)^n_0 \) imply \( (ST)^{n+1}_0 \). From these two statements it will follow that \( (ST)^n_0 \) and \( (OSC)^n_0 \) hold for any \( n \). Sections 10.4 and 10.5 contain the proof of two independent results (Lemma 10.6 and 10.7) formulated on a fixed scale, that are used in Section 10.2. These are the generalizations of the first and second De Giorgi lemmas in [11], adjusted to our situation where no supremum bound is available on the coefficients \( B_{ij}(s) \), we have control only in a certain average sense.

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10.2 Proof of \((ST)_n \implies (OSC)_n\)

For any real number \(a\), we will use the notation \(a_+ = \max(a, 0) \geq 0\) and \(a_- = \min(a, 0) \leq 0\), in particular \(a = a_+ + a_-\). Fix a large integer number \(M\) and a center \(Z \in I\) with \(dZ \geq K/2\) (recall that \(d_i\) was defined above (10.3); it is the distance of \(i\) to the boundary). For any \(\ell > 0\) and \(\lambda \in (0,1/10)\) define the functions

\[
\psi_i = \psi^{(M,Z,t)}_i := \ell \left( \left| \frac{i - Z}{M} \right|^{1/2} - 1 \right)_+,
\]

\[
\bar{\psi}_i = \bar{\psi}^{(M,Z,t,\lambda)}_i := \ell \left( \left| \frac{i - Z}{M} \right| - \lambda^{-4} \right)^{1/4} - 1 \right)_+.
\]

Notice that \(\psi_i = 0\) if \(|i - Z| \leq M\) and \(\bar{\psi}_i = 0\) if \(|i - Z| \leq M\lambda^{-4}\). Here \(\ell\) will play the role of the typical size of \(u - \psi\). One could scale out \(\ell\) completely, but we keep it in. We also define the scaled versions of these functions for any \(n \geq 0\):

\[
\psi_i^{(n)} := \psi^{(M_n,Z_n,\ell_n)}_i, \quad \bar{\psi}_i^{(n)} := \bar{\psi}^{(M_n,Z_n,\ell_n,\lambda)}_i.
\]

**Proposition 10.5** Suppose that for some \(n \geq 0\) we know \((ST)_j\) for any \(j = 0,1,\ldots,n\). Then \((OSC)_n\) holds. Furthermore, we have

\[
\sum_i \left( u_i(t) - \bar{u}_n - \ell_n - \psi_i^{(n)} \right)^2 \leq C \left( \left| t \right| + \frac{M^{1/2}}{M_n} \right) M_n \ell_n^2, \quad t \in [-M_n,0].
\]

**Proof of Proposition 10.5.** With a small constant \(\lambda \in (10L^{-1/4},1)\) and a large integer \(k_0\), to be specified later, define the rescaled and shifted functions

\[
v_i^{(n,k)}(t) := \ell_n + \lambda^{-2k} \left( u_i(t) - \bar{u}_n - \ell_n \right), \quad k = 0,1,2,\ldots,k_0.
\]

In particular, from \((ST)_n\) we have

\[
v_i^{(n,k)}(t) \leq \ell_n + \lambda^{-2k} \left( \psi_i^{(n)}(t) - \ell_n \right), \quad t \in [-3M_n,0].
\]

We will show that with an appropriate choice \(k = k(n)\), \(v = v^{(n,k+1)}\) satisfies a better upper bound than (10.28) which then translates into a decrease in the oscillation of \(u\) on scale \(n\). The improved upper bound on \(v\) will follow from applying two basic lemmas from parabolic regularity theory, traditionally called the first and the second De Giorgi lemmas. The second De Giorgi lemma asserts that going from a larger to a smaller space-time regime, the maximum of \(v_i(t)\) decreases in an average sense. The first De Giorgi lemma enhances this statement to a supremum bound for \(v_i(t)\) that is strictly below the maximum of \(v_i(t)\) on a larger space-time regime. This is equivalent to the reduction of the oscillation of \(v\).

In the next section we first state these two basic lemmas, then we continue the proof of Proposition 10.5. The proofs of the De Giorgi lemmas are deferred to Sections 10.4 and 10.5.

10.2.1 Statement of the generalized De Giorgi lemmas

Both results will be formulated on a fixed space-time scale \(M\) and with a fixed size-scale \(\ell\). We fix a center \(Z \in I\) with \(|Z| \leq K/2\). Recall the definition of \(\bar{\psi} = \bar{\psi}^{(M,Z,\ell)}\) from (10.24). The first De Giorgi lemma is a local dissipation estimate:
Lemma 10.6 There exists a small positive universal constant $\varepsilon_0$ with the following properties. Consider the parabolic equation (10.1) on the time interval $\mathcal{T} = [-\sigma, 0]$ with some $\sigma \in [K^{c_3}, K^{1-c_3}]$ and let $u$ be a solution. Define $v := u - \bar{u}$ with some constant shift $\bar{u} \in \mathbb{R}$. Fix small positive constants $\kappa, \xi, \rho, \chi$ and a large constant $\bar{\vartheta}_0$ such that

\begin{equation}
10\vartheta_0(\xi + \rho) \leq \kappa \leq \frac{1}{1000}, \quad \kappa + 10\vartheta_0(\xi + \rho) \leq \chi \leq \frac{1}{1000}
\end{equation}

holds. Let $M$ be defined by $M := K^{1/\varepsilon}$ with some $\varepsilon \in [1 + 2\kappa, \bar{\vartheta}_0]$. We assume that the matrix elements of $A = B + W$ satisfy (10.3), (10.4), (10.5) with exponent $\xi$ and that (10.1) is regular with exponent $\rho$ at the space-time points $(Z, t)$, $t \in \Xi_0$, where

\begin{equation}
\Xi_0 := \{-M \cdot 2^{-m}(1 + 2^{-k}) : 0 \leq m, k \leq C \log M\}.
\end{equation}

Assume

\begin{equation}
|\bar{u}| \leq C\ell K^{1-\xi} M^{-1},
\end{equation}

\begin{equation}
\left[\frac{1}{M^2} \int_{-2M}^{0} dt \sum_i (v_i(t) - \psi_i_2^1)\right]^{1/2} \leq \varepsilon \ell \ell, \quad \psi_i = \psi_i^{(M,Z,\ell)},
\end{equation}

\begin{equation}
\sup_{t \in [-2M, 0]} \sup \{|i - Z| : v_i(t) > \psi_i\} \leq M^{1+\kappa},
\end{equation}

\begin{equation}
\sup_{t \in [-2M, 0]} \max \{v_i(t) : |i - Z| \leq M^{1+\kappa}\} \leq C\ell M^{3/2},
\end{equation}

and there exists a set $G^* \subset [-2M, 0]$ with $|[-2M, 0] \setminus G^*| \leq CM^{1/4}$ such that

\begin{equation}
\sup_{t \in [-2M, 0]/G^*} \max \{v_i(t) : |i - Z| \leq M^{1+\kappa}\} \leq C\ell M^{3/4}.
\end{equation}

Then, for any sufficiently large $K \geq K_0(\vartheta_0)$, we have the following statements:

i) We have

\begin{equation}
\sup_{t \in [-M, 0]} \sum_i \left(v_i(t) - \psi_i - \frac{\ell}{3}\right)^2 \leq CM^\chi \ell^2.
\end{equation}

ii) There exists a set $G \subset [-M, 0]$ of “good” times such that

\begin{equation}
\sup_{t \in G} v_i(t) \leq \frac{\ell}{2} + \psi_i, \quad \forall i, \quad \text{and} \quad \{|-M, 0] \setminus G\| \leq CM^{1/4}.
\end{equation}

iii) For any $\widetilde{M}$ with $M^{2\chi} \ll \widetilde{M} \leq \frac{1}{2} M$ we have

\begin{equation}
\sup_{t \in [-M, 0]} \sum_i \left(v_i(t) - \psi_i - \frac{2\ell}{3}\right)^2 \leq C\left(\frac{\widetilde{M}}{M}\right) M^{3/2} \ell^2.
\end{equation}

These results hold for any $K \geq K_0$, where the threshold $K_0$ and the constants in (10.36)–(10.38) may depend on $\chi, \kappa, \xi, \rho, \vartheta_0$ and on the constants $C$ and $\widetilde{C}$ in (10.3), (10.4), (10.5).
For the orientation of the reader we mention how the various exponents will be chosen in the application. The important exponents are $\kappa$ and $\chi$; they will be related by $\kappa = \frac{3\chi}{4}$, see (10.77) later (actually, the really important relation is that $\kappa < \chi$). The exponents $\xi,\rho$ will be chosen much smaller; the reader may neglect them at first reading.

Notice that (10.36) is off from the optimal bound by a factor $M^\chi$. However, (10.37) shows that for most of the times, this factor is not present, while (10.38) shows that this factor is reduced if the time interval is shorter. We remark that precise coefficients of $\ell$ in the additive shifts appearing in (10.36)–(10.38) are not important; instead of $\frac{1}{2} > \frac{2}{5} > \frac{1}{3}$ essentially any three numbers between 0 and 1 with the same ordering could have been chosen.

The second De Giorgi lemma is a local decrease of oscillation on a single scale. As before, we are given three parameters, $M, Z, \ell$. Define a new function $F$ by

$$F_i = F_i^{(M,Z,\ell)} := \ell \cdot \max \left\{ -1, \min \left( 0, \frac{|i - Z|}{M} - 81 \right) \right\} \quad (10.39)$$

for any $M, Z, \ell$. Notice that $-\ell \leq F_i \leq 0$, furthermore $F_i = 0$ if $|i - Z| \leq 9M$ and $F_i = -\ell$ if $|i - Z| \geq 9M$. We also introduce a new parameter $\lambda \in (0, 1/10)$. Recalling the definition of $\tilde{\psi}$ from (10.25), we also define three cutoffs, all depending on all four parameters, $M, Z, \ell, \lambda$

$$\varphi_i^{(0)} := \ell + \tilde{\psi}_i + F_i$$
$$\varphi_i^{(1)} := \ell + \tilde{\psi}_i + \lambda F_i$$
$$\varphi_i^{(2)} := \ell + \tilde{\psi}_i + \lambda^2 F_i.$$  

Notice that

$$\varphi_i^{(0)} \leq \varphi_i^{(1)} \leq \varphi_i^{(2)} \leq \ell + \tilde{\psi}_i,$$  

and when $|i - Z| \geq 9M$ all inequalities become equalities. Notice that $\varphi_i^{(0)} = 0$ if $|i - Z| \leq 8M$.

**Lemma 10.7** Consider the parabolic equation (10.1) on the time interval $T = [-\sigma, 0]$ with some $\sigma \in [K^{c_3}, K^{1-c_3}]$ and let $u$ be a solution. Define $v := u - \bar{u}$ with some constant shift $\bar{u} \in \mathbb{R}$. Fix small positive constants $\kappa_1, \kappa_2, \xi, \rho$ and a large constant $\vartheta_0$ such that

$$\kappa_1 + \kappa_2 + 10\vartheta_0 (\xi + \rho) \leq \frac{1}{1000}, \quad (10.41)$$

Let $M$ be defined by $M := K^{1/\vartheta}$ with some $\vartheta \in [1 + 2\kappa_1, \vartheta_0]$. We assume that the matrix elements of $A = B + W$ satisfy (10.3), (10.4), (10.5) with exponent $\xi$ and that (10.1) is regular with exponent $\rho$ at the space-time points $(Z,t)$, $t \in \Xi_0$, where $\Xi_0$ was given in (10.30).

For any $\delta > 0$ and $\mu > 0$ there exist $\gamma > 0$ and $\lambda \in (0, 1/8)$ such that whenever

$$|\bar{u}| \leq C\lambda K^{1-\xi} M^{-1}, \quad (10.42)$$
and the shifted solution \( v(t) = u(t) - \bar{u} \) satisfies the following five properties:

\[
\exists G \subset [-3M, 0], \quad \left| [-3M, 0] \setminus G \right| \leq CM^{1/4}, \quad \text{s.t.} \quad v_i(t) \leq \ell + \bar{\psi}_i, \quad t \in G, \quad \forall i,
\]

\[
\sup_{t \in [-3M, 0]} \max \left\{ |i - Z| : v_i(t) > \ell + \bar{\psi}_i \right\} \leq M^{1+\kappa_1},
\]

\[
\sup_{t \in [-3M, 0]} \sup \left\{ v_i(t) : |i - Z| \leq M^{1+\kappa_1} \right\} \leq \ell M^{\kappa_2},
\]

\[
\frac{1}{M^2} \int_{-3M}^{-2M} 1(t \in G) \cdot \# \left\{ |i - Z| \leq M : v_i(t) < \varphi_i^{(0)} \right\} dt \geq \mu,
\]

\[
\frac{1}{M^2} \int_{-3M}^{0} 1(t \in G) \cdot \# \left\{ v_i(t) > \varphi_i^{(2)} \right\} dt \geq \delta,
\]

then

\[
\frac{1}{M^2} \int_{-3M}^{0} 1(t \in G) \cdot \# \left\{ i : \varphi_i^{(0)} < v_i(t) < \varphi_i^{(2)} \right\} dt \geq \gamma.
\]

This conclusion holds for any \( K \geq K_0 \) where the threshold \( K_0 \) depends on all parameters \( \vartheta_0, \kappa_1, \kappa_2, \xi, \rho, \delta, \mu \) and the constants in the conditions (10.3), (10.4), (10.5).

We remark that the choices of \( \gamma \) and \( \lambda \) are explicit, one may choose

\[
\gamma := c\delta^3, \quad \lambda := c\delta^6\mu
\]

with a small absolute constant \( c \).

This lemma asserts that whenever the a substantial part of the function \( v \) increases from \( \varphi_i^{(0)} \) to \( \varphi_i^{(2)} \) in time of order \( M \), then there is a time interval of order \( M \) so that a substantial part of \( v \) lies between \( \varphi_i^{(0)} \) and \( \varphi_i^{(2)} \).

### 10.2.2 Verifying the assumptions of Lemma 10.7

We will apply Lemma 10.7 to the function \( v = v^{(n,k)} \) given in (10.27) with the choice \( M = M_n, \ell = \ell_n \). The following lemma collects the necessary information on \( v = v^{(n,k)} \) to verify the assumptions in Lemma 10.7. The complicated relations among the parameters, listed in (10.50) and (10.51) below, can be simultaneously satisfied; their appropriate choice will be given in Section 10.2.4.

**Lemma 10.8** Assume that \((ST)_n\) holds, see (10.17). Suppose that in addition to the previous relations \( \nu < \min \{ \zeta^{10}/10, 1/(2L) \} \) and \( \lambda \geq 10L^{-1/4} \) among the parameters, the following further relations also hold:

\[
10 \leq (1 - \zeta)\lambda^{2k_0} \zeta L^{1/4}, \quad \chi + 10d_0(\xi + \rho) \leq \frac{1}{1000}, \quad 100d_0(\xi + \rho) \leq \chi \leq \frac{\log \zeta}{\log \nu},
\]

\[
\psi \in [1 + 2\chi, \vartheta_0], \quad 1 - \frac{1}{2}\lambda^{2(k_0 + 1)} \leq \zeta < 1.
\]

Then for any \( v_i^{(n,k)}(t) \) with \( k \leq k_0 \), defined in (10.27) and satisfying (10.28), we have the following
three bounds:

\[ \sup_{t \in \mathcal{G}_n} \sup_{k \leq k_0} v_i^{(n,k)}(t) \leq \ell_n + \psi_i^{(n)}, \quad \text{(10.52)} \]

\[ \sup_{k \leq k_0} \sup_{t \in [-3M_n,0]} \max \\{ |i - Z| : v_i^{(n,k)}(t) > \ell_n + \psi_i^{(n)} \} \leq M_n^{1 + 3\chi/4}, \quad \text{(10.53)} \]

\[ \sup_{k \leq k_0} \sup_{t \in [-3M_n,0]} \sup \{ v_i^{(n,k)}(t) : |i - Z| \leq M_n^{1 + 3\chi/4} \} \leq C\ell_n M_n^{\chi/2}. \quad \text{(10.54)} \]

For the shift in (10.27) we have the bound

\[ |\ell_n - \lambda^{-2k(\bar{u}_n + \ell_n)}| \leq C\lambda\ell_n K^{1-\xi} M_n^{-1}. \quad \text{(10.55)} \]

The constants C may depend on all parameters in (10.50), (10.51).

We remark that the factor 3/4 in the exponent in (10.53) can be improved to 2/3 + \epsilon' for any \epsilon' > 0, but what is really important for the proof is that it is strictly smaller than 1, since this will translate into the crucial \kappa < \chi condition in (10.34).

**Proof of Lemma 10.8.** All four estimates follow by direct calculations from the definition of \Psi_i^{(n)}(t) and from the relations (10.50), (10.51) among the parameters. Based upon (10.28), the estimate (10.52) amounts to checking

\[ \Lambda_i^{(n)} \leq \ell_n + \lambda^{2k_0} \ell_n \left( \left| \frac{i - Z}{M_n} \right| - \lambda^{-4} \right)^{1/4} \quad \text{(10.56)} \]

For \left| i - Z \right| \leq \tilde{M}_n we immediately have \Lambda_i^{(n)} = \ell_n and thus (10.56) holds. For \tilde{M}_{m+1} \leq \left| i - Z \right| \leq \tilde{M}_m (with some m \leq n - 1) we can use (10.22), to have that \Lambda_i^{(n)} \leq 2(1 - \zeta)^{-1} \ell_m. The right hand side of (10.56) is larger than

\[ \ell_n + \lambda^{2k_0} \ell_n \left[ \left( \frac{\tilde{M}_{m+1}}{M_n} \right) - \lambda^{-4} \right]^{1/4} - 1 \]

which is larger than \ell_n (1 + \frac{1}{2} \lambda^{2k_0} L^{1/4} L_{\rho}^{(m-n)/4}). Now (10.56) follows from the first inequality in (10.50) and from \nu \leq \zeta^{10}/10.

For the proof of (10.53), starting from (10.28), it is sufficient to check that

\[ \Phi_i^{(n)}(t) \leq \ell_n + \lambda^{2k_0} \ell_n \left[ \left( \frac{i - Z}{M_n} \right) - \lambda^{-4} \right]^{1/4} - 1 \quad \text{(10.57)} \]

for any \left| i - Z \right| \geq \frac{1}{2} M_n^{1 + 3\chi/4} and \ t \in [-3M_n,0]. On the left hand side we can use the largest time \left| t \right| = 3M_n \geq M_n^{1/2}. Considering the regime \tilde{M}_m \leq \left| i - Z \right| \leq \tilde{M}_{m-1} with \ M_m = M_n^{1 + \beta} for some 0 < \beta < \frac{1}{2}, we see that

\[ \text{l.h.s. of (10.57)} \leq 2\ell_m \left( \frac{M_m}{M_n} \right)^{1/2} M_n^{1/2}, \quad \text{r.h.s. of (10.57)} \geq \frac{1}{2} \lambda^{2k_0} \ell_n \left( \frac{M_m}{M_n} \right)^{1/4}, \]

Using \chi \leq \frac{\log \tilde{G}}{\log \lambda} from (10.50), we have

\[ \frac{\ell_m}{\ell_n} \leq \left( \frac{M_m}{M_n} \right)^{\chi} \]

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Recalling that $M_m = M_n^{1+\beta}$, we see that for small $\chi$ (10.58) is satisfied if $\beta > \frac{2\chi}{\log M}$ (and $M_n$ is sufficiently large depending on all constants $\lambda, \nu, L, k_0, \nu, \zeta$). This is guaranteed if $\beta \geq 3\chi/4$ since we assumed $\chi \leq 1/1000$. This proves (10.53).

For the proof of (10.54) we notice that

$$\max \left\{ \Phi_i^{(n)}(t) : |i-Z| \leq M_n^{1+3\chi/4} \right\} \leq C\Phi(\ell_m + M_n^{1/2}\ell_n) \leq CM_n^{1/2}\ell_n$$

for any $t \in [-2M_n,0]$, where $m < n$ is defined by $\tilde{M}_{m+1} \leq M_n^{1+3\chi/4} \leq \tilde{M}_m$. The first inequality in (10.59) follows from (10.18); the second one is a consequence of

$$\frac{\ell_m}{\ell_n} = \left( \frac{M_m}{M_n} \right)^{\frac{\log \xi}{\log \nu}} \leq \left( \frac{M_m}{M_n} \right)^{\frac{10}{74}} \leq M_n^{1/10}$$

by $|\log \xi| \leq \frac{1}{10} |\log \nu|$. Then (10.54) directly follows from (10.28) and (10.59).

Finally, (10.55) follows from the facts that $|a_n| \leq \ell = \ell_0$, $K^{1-\mu} \geq \tilde{M} = M_0$ (using $\nu \geq 1 + 2\xi$) and that $\ell_0/\ell_n \leq M_0/M_n$. This completes the proof of Lemma 10.8.

10.2.3 Completing the proof of Proposition 10.5

We now continue the proof of Proposition 10.5. Set

$$F_i^{(n)} := F_i^{(M_n, Z, \ell_n)},$$

where $F$ is given in (10.39) and we define further cutoff functions:

$$\varphi_i^{(0),(n)} := \ell_n + \bar{\psi}_i^{(n)} + F_i^{(n)}$$
$$\varphi_i^{(1),(n)} := \ell_n + \bar{\psi}_i^{(n)} + \lambda F_i^{(n)}$$
$$\varphi_i^{(2),(n)} := \ell_n + \bar{\psi}_i^{(n)} + \lambda^2 F_i^{(n)}.$$

Throughout this section $n$ is fixed, so we will often omit this from the notation. In particular $\ell = \ell_n$, $M = M_n$, $\bar{u} = \bar{u}^{(n)}$, $\varphi_i^{(k)} = \varphi_i^{(n,k)}$, $F = F^{(n)}$, $\bar{\psi} = \bar{\psi}^{(n)}$, $\varphi_i^{(a)} = \varphi_i^{(a),(n)}$ for $a = 0, 1, 2$, $\mathcal{G} = \mathcal{G}_n$ etc. At the end of the proof we will add back the superscripts.

From the definitions of these cutoff functions, we have

$$\varphi_i^{(0)} \leq \varphi_i^{(1)} \leq \varphi_i^{(2)} \leq \ell + \bar{\psi}_i,$$  \hspace{1cm} (10.61)  \textbf{alleq1}

and when $|i-Z| \geq 9M$ all inequalities become equalities. Notice that $\varphi_i^{(0)} = 0$ if $|i-Z| \leq 8M$.

Choose a small constant $\mu \in (0, 1/10)$, say

$$\mu := \frac{1}{100}.$$  \hspace{1cm} (10.62)  \textbf{muchoice}

Without loss of generality, we can assume

$$\frac{1}{M^2} \int_{-M}^{-2M} \# \left\{ i : |i-Z| \leq M, u_i(t) - \bar{u} < \varphi_i^{(0)} \right\} dt \geq \mu$$  \hspace{1cm} (10.63)  \textbf{441}
Therefore, we can bound the last integral in (10.68) by
\[ \frac{1}{M^2} \int_{-2M}^{-3M} 1(t \in \mathcal{G}) \cdot \# \{ i : |i - Z| \leq M, v_i^{(k)}(t) < \varphi_i^{(0)} \} dt \geq \mu, \] (10.64)
since the set of \( i \) indices in (10.64) is increasing in \( k \) for any \( t \in \mathcal{G} \) and \( v^{(0)} = u - \bar{u} \).

Assuming that the parameters satisfy (10.50) and (10.51), we can now use the conclusions (10.52)-(10.55) in Lemma 10.8. These bounds together with (10.64) allow us to apply Lemma 10.7 to \( v^{(k)} = v^{(n,k)} \) with the choice
\[ \kappa_1 := \frac{3}{4} \chi, \quad \kappa_2 := \frac{1}{2} \chi, \quad \delta := \frac{\varepsilon_0}{100}, \] (10.65)
where \( \varepsilon_0 > 0 \) is a universal constant which was determined in Lemma 10.6. Notice that with these choices (10.41) follows from (10.50) and \( \vartheta \in [1 + 2\kappa_1, \vartheta_0] \) follows from \( \vartheta \in [1 + 2\chi, \vartheta_0] \). Thus the application of Lemma 10.7 yields that there exist a \( \lambda \) (introduced explicitly in the construction of the cutoffs \( \varphi^{(a)} \) and used also in (10.25) and (10.28)) and a \( \gamma > 0 \) (see (10.49) for their explicit values) such that if
\[ \frac{1}{M^2} \int_{-2M}^{0} 1(t \in \mathcal{G}) \cdot \# \{ i : v_i^{(k)}(t) > \varphi_i^{(2)} \} dt > \delta \] (10.66)
then
\[ \frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \{ i : \varphi_i^{(0)} < v_i^{(k)}(t) < \varphi_i^{(2)} \} dt \geq \gamma. \] (10.67)

Therefore
\[ \frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \{ i : v_i^{(k)}(t) > \varphi_i^{(2)} \} dt \leq \frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \{ i : v_i^{(k)}(t) > \varphi_i^{(0)} \} dt - \gamma. \] (10.68)

Notice that, by (10.52) and \( F_i = 0 \) if \( |i - Z| \geq 9M \), for any \( k \leq k_0 \) the inequality \( v_i^{(k)}(t) > \varphi_i^{(0)} \) (for \( t \in \mathcal{G} \)) can hold only if \( |i - Z| \leq 9M \). Assuming \( |i - Z| \leq 9M, t \in \mathcal{G} \) and \( v_i^{(k)}(t) > \varphi_i^{(0)} \), we have
\[ \frac{1}{\lambda^2} (v_i^{(k-1)}(t) - \ell) + \ell = v_i^{(k)}(t) > \varphi_i^{(0)}. \] (10.69)

Since \( |i - Z| \leq 9M \leq \lambda^{-4} M \), we have, together with (10.61) and that \( \tilde{\psi}_i = 0 \) in this regime, that
\[ v_i^{(k-1)}(t) \geq \lambda^2 (\tilde{\psi}_i + F_i) + \ell \geq \varphi_i^{(2)}. \] (10.70)

Therefore, we can bound the last integral in (10.68) by
\[ \frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \{ i : v_i^{(k)}(t) > \varphi_i^{(0)} \} dt \leq \frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \{ i : |i - Z| \leq 9M, v_i^{(k-1)}(t) > \varphi_i^{(2)} \} dt. \] (10.71)
We have thus proved that
\[
\frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \left\{ i : v_i^{(k)}(t) > \varphi_i^{(2)} \right\} dt \\
\leq \frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \left\{ i : |i - Z| \leq 9M, v_i^{(k-1)}(t) > \varphi_i^{(2)} \right\} dt - \gamma.
\] (10.72)

Iterating this estimate \( k \) times, we get
\[
\frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \left\{ i : v_i^{(k)}(t) > \varphi_i^{(2)} \right\} dt \\
\leq \frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \left\{ i : |i - Z| \leq 9M, v_i^{(0)}(t) > \varphi_i^{(2)} \right\} dt - k\gamma,
\]
which becomes negative if \( k\gamma \geq 100 \). Setting
\[
k_0 := \frac{100}{\gamma},
\] (10.73)
thus there is a \( k < k_0 \) such that (10.66) is violated, i.e.,
\[
\frac{1}{M^2} \int_{-3M}^{0} 1(t \in \mathcal{G}) \cdot \# \left\{ i : v_i^{(k)}(t) > \varphi_i^{(2)} \right\} dt \leq \delta.
\] (10.74)

From now on let \( k = k(n) \) denote the smallest index so that (10.74) holds (recall that the underlying \( n \) dependence was omitted from the notation in most of this section). Furthermore, since \( \varphi_i^{(0)} = 0 \) for \( |i - Z| \leq 8M \), we have
\[
\frac{1}{M^2} \int_{-2M}^{0} 1(t \in \mathcal{G}) \cdot \# \left\{ i : |i - Z| \leq 8M, v_i^{(k+1)}(t) > 0 \right\} dt \\
= \frac{1}{M^2} \int_{-2M}^{0} 1(t \in \mathcal{G}) \cdot \# \left\{ i : |i - Z| \leq 8M, v_i^{(k+1)}(t) > \varphi_i^{(0)} \right\} dt \\
\leq \frac{1}{M^2} \int_{-2M}^{0} 1(t \in \mathcal{G}) \cdot \# \left\{ i : v_i^{(k)}(t) > \varphi_i^{(2)} \right\} dt \leq \delta = \frac{\varepsilon_0^2}{100},
\] (10.75)
where we have used (10.71) in the last inequality.

Armed with (10.75), our goal is to apply Lemma 10.6 with \( M = M_n \) to \( v = v^{(n,k(n)+1)} \) with the value \( k = k(n) \) determined after (10.74). Clearly \( v \) is of the form
\[
v = \lambda^{-2k-2} u + [\ell_n - \lambda^{-2k-2}(\bar{u}_n + \ell_n)],
\] (10.76)
i.e. it is a solution to (10.1) (namely \( \lambda^{-2k-2} u \)) shifted by \( [\lambda_n - \lambda^{-2k-2}(\bar{u}_n + \ell_n)] \). The value \( \kappa \) in Lemma 10.6 will be set to
\[
\kappa := \frac{3}{4} \chi
\] (10.77)
and the set \( \mathcal{G}^* \) in Lemma 10.6 will be chosen as \( \mathcal{G}^* := \mathcal{G}_n \) (for \( n = 0 \) we set \( \mathcal{G}^* = [-3M_0, 0] \), i.e. at the zeroth step of the iteration every time is “good”, see (10.10)). The choice \( \kappa = 3\chi/4 \) together with the constraints on \( \chi \) in (10.50) guarantee that the relations in (10.29) hold. We need to check
five conditions (10.31), (10.32), (10.33), (10.34) and (10.35). The sixth condition, the regularity at \((Z, t)\) for \(t \in \Xi_0\) follows automatically from (C1) since \(M = M_n = M^{\nu} = 2^{-\nu}M^\nu K\) with an integer \(\tau_0\) and \(\nu\) itself is a negative power of 2, thus \(\Xi_0 \subset \Xi\), see (9.10). The first condition (10.31) for the shift in (10.76) was verified in (10.55).

For the second condition (10.32), with the notation \(G := [-3M, 0] \setminus \mathcal{G}\) we write

\[
\frac{1}{M^2} \int_{-2M}^{0} \sum_{i} (v_i^{(k+1)}(t) - \psi_i)^2 dt \leq \frac{1}{M^2} \int_{-2M}^{0} 1(t \in \mathcal{G}) \cdot \sum_{i} (v_i^{(k+1)}(t) - \psi_i)^2 dt + \frac{|G^c|}{M^2} \sup_{t \in [-2M, 0]} \sum_{i} (v_i^{(k+1)}(t) - \psi_i)^2. \tag{10.78}
\]

In the first term we use that

\[v_i^{(n,k+1)}(t) \leq \ell_n + \tilde{\psi}_i(n), \quad t \in \mathcal{G},\]

from (10.52) (we reintroduced the superscript \(n\)). Since \(\ell_n + \tilde{\psi}_i(n) \leq \psi_i(n)\) if \(|i| - Z| \geq 8M\), we can see that the summation in the first term on the right hand side of (10.78) is restricted to \(|i| - Z| \leq 8M\), and for these \(i\)’s we have \(v_i^{(n,k+1)}(t) \leq \ell_n\) since \(\tilde{\psi}_i(n) = 0\). We can therefore apply (10.75) and we get

\[
\frac{1}{M^2} \int_{-2M}^{0} \sum_{i} (v_i^{(n,k+1)}(t) - \psi_i(n))^2 dt \leq 4\delta\ell_n^2 + \frac{|G^c|}{M^2} \sup_{t \in [-2M, 0]} \sum_{i} (v_i^{(n,k+1)}(t) - \psi_i(n))^2. \tag{10.79}
\]

To estimate the second term, we use (10.28) and \(\Psi^{(n)} \leq \Phi^{(n)}\) to note that

\[
\psi_i^{(n)} \leq v_i^{(n,k+1)}(t) \quad \Rightarrow \quad \psi_i^{(n)} \leq \ell_n + \lambda^{-2k-2} \left(\Phi_i^{(n)}(t) - \ell_n\right), \quad t \in [-3M, 0]. \tag{10.80}
\]

Suppose first that \(|i - Z| \geq M_n^{1+3\chi/4}\). In this case (10.57) holds, thus (10.80) would imply

\[
\psi_i^{(n)} = \ell_n \left(\left|\frac{i}{M_n}\right|^{1/2} - 1\right) \leq \lambda^{-2k_0} \lambda^{-\lambda^{-4}} \left[\left|\frac{i}{M_n}\right| - \lambda^{-4}\right] \leq \lambda^{-2k_0} \ell_n + \ell_n \left[\left|\frac{i}{M_n}\right| - \lambda^{-4}\right],
\]

but this is impossible for \(|i - Z| \geq M_n^{1+3\chi/4}\) if \(M_n\) is large enough. In particular, this verifies (10.33). We therefore conclude that the summation in the second term on the right hand side of (10.79) is restricted to \(|i - Z| \leq M_n^{1+3\chi/4}\). For these values we have

\[
v_i^{(n,k+1)}(t) \leq \ell_n + \lambda^{-2k} \left(\Phi_i^{(n)}(t) - \ell_n\right) \leq C\lambda^{-2k} \ell_n M_n^{\chi/2}, \tag{10.81}
\]

(the first inequality is from (10.28), the second is from (10.59)). This verifies (10.34), recalling the choice of \(\kappa = 3\chi/4\).

Inserting these information into (10.79), we have

\[
4\delta\ell_n^2 + \frac{|G^c|}{M^2} \sup_{t \in [-2M, 0]} \sum_{i} (v_i^{(n,k+1)}(t) - \psi_i(n))^2 \leq 4\delta\ell_n^2 + C\lambda^{-2k} \ell_n M_n^{-1/2} \leq \varepsilon_n^2 \ell_n^2,
\]

where we used \(|G^c| \leq C M_n^{1/4} \leq M_n^{1/2}\) from (10.14) and (10.13). In the last step we used the choice \(\delta = \varepsilon_n^2/100\). This verifies (10.32).
Finally, we verify (10.35) with the previously mentioned choice $G^* := G_n$. Let $i$ such that $|i - Z| \leq M_n^{1+3\varepsilon/4}$ and $t \in G_n$. Then from (10.28) we have

$$v_i^{(n,k+1)}(t) = \lambda^{-2k-2} A_i^{(n)} + \ell_n (1 - \lambda^{-2k-2}) \leq C\ell_n \leq CM_n^{1/10}\ell_n$$

where $m$ is chosen such that $\tilde{M}_{m+1} \leq M_n^{1+3\varepsilon/4} \leq \tilde{M}_m$ and in the last step we used (10.60).

Thus we can apply Lemma 10.6 to $v = v^{(n,k+1)}$ and from (10.37) we get the existence of a set of times, denoted by $G'_{n} \subset [-M_n, 0]$, such that

$$\sup_{t \in G_{n}'} v_i^{(n,k+1)}(t) \leq \frac{\ell_n}{2}, \quad \forall |i - Z| \leq M_n$$

and

$$\left| [-M_n, 0] \setminus G'_{n} \right| \leq CM_n^{1/4}.$$  

Defining $G_{n+1} := G_n \cap G'_{n} \cap [-3M_{n+1}, 0]$ and using $\tilde{M}_{n+1} \leq M_n$, we obtain that

$$\sup_{t \in G_{n+1}} v_i^{(n,k+1)}(t) \leq \frac{\ell_n}{2}, \quad |i - Z| \leq \tilde{M}_{n+1}$$

and

$$\left| G'_{n+1} \right| \leq CM_n^{1/4} + |G'_{n}| \leq C \sum_{m=0}^{n} M_m^{1/4}$$

where we used the measure of $G'_{n}$ from (10.14).

Recalling the definition of $G'_{n}$ from (10.27), from (10.82) we have

$$u_i(t) - \bar{u}_n \leq \ell_n \left(1 - \frac{1}{2} \lambda^{2(k+1)}\right) \leq \ell_n \left(1 - \frac{1}{2} \lambda^{2(k_0+1)}\right) \leq \ell_n + \ell_{n+1} \quad i \in \tilde{Q}_{n+1} \text{ and } t \in G_{n+1},$$

where we recall that $k \leq k_0$ and (10.51). Repeating the argument for $-u$ instead of $u$, we obtain a similar lower bound on $u_i(t) - \bar{u}_n$. Since $Q'_{n+1} \subset G_{n+1} \times [Z - \tilde{M}_{n+1}, Z + \tilde{M}_{n+1}]$, and this proves (10.19) for $u$, i.e. (OSC)\(_n\).

The application of Lemma 10.6 also yields (see (10.38)) that for $t \in [-M_n, 0]$ we have

$$\sum_{i} \left( u_i^{(n,k+1)}(t) - \left(\frac{2}{5} \ell_n + \psi_i^{(n)}\right)\right)^2 \leq C \left(\frac{|t| + M_1^{1/2}}{M_n}\right) M_n^\lambda \ell_n^2,$$

which implies, by (10.27) and an elementary algebra, the second statement in Proposition 10.5 (the constant $C$ in (10.26) includes a factor $\lambda^{-2k} \leq \lambda^{-2k_0}$).

### 10.2.4 Summary of the choice of the parameters

Finally we present a possible choice of the parameters that were used in the proof of Proposition 10.5. Especially, we need to satisfy the complicated relations (10.50), (10.51).

Lemma 10.6 gives an absolute constant $\varepsilon_0$. Then we choose $\delta = \varepsilon_0^2/100$, $\gamma = c\delta^3$, $\lambda = c\delta^6\mu$ (with a small constant $c$), $k_0 = 100/\gamma$ and $\mu = 1/100$. These choices can be found in (10.65), (10.49), (10.73) and (10.62), respectively.
Having $\lambda, k_0$ determined, we define
\[ \zeta := 1 - \frac{1}{2} \lambda^{2(k_0 + 1)}, \quad L := \lambda^{-16(k_0 + 1)}, \quad \nu =: 2\lambda^{16(k_0 + 1)}. \]
If needed, reduce $\lambda$ so that $|\log \zeta|/|\log \nu| \leq 1/10$. Note that five numbers, $\lambda, k_0, \zeta, \nu, L$ are absolute positive constants (meaning that they do not depend on any input parameters in Theorem 10.3). In particular, they determine the absolute constant $q$ (10.20), which is the final Hölder exponent.

Next we set
\[ \chi := \min \left\{ \frac{|\log \zeta|}{|\log \nu|}, \frac{\nu_1 - 1}{2}, \frac{1}{2000} \right\} \] \hfill (10.85) \textbf{[chicho]} and then choose the exponents $\xi, \rho$ as
\[ \xi := \rho := \frac{\chi}{200d_0}. \] \hfill (10.86) \textbf{[xiro]}

Finally, $M = M_0$ (or, equivalently $K_0$) has to be sufficiently large depending on all these exponents.

It is easy to check that this choice of the parameters satisfies all the relations that were used in the proof of Proposition 10.5. This completes the proof of Proposition 10.5. \hfill $\square$

10.3 Proof of $(ST)_n + (OSC)_n \implies (ST)_{n+1}$

**Proposition 10.9** Suppose that for some $n$ integer $(ST)_n$ and $(OSC)_n$ hold. Then $(ST)_{n+1}$ also holds.

**Proof.** For $t \in \mathcal{G}_{n+1} \subset \mathcal{G}_n$ we have
\[ |u_i(t) - \bar{u}_{n+1}| \leq \ell_{n+1}, \quad |i - Z| \leq M_n \]
by $(OSC)_n$. Since $\bar{M}_{n+1} \leq M_n$ (as $\nu \leq \frac{1}{M}$), we immediately get $|u_i(t) - \bar{u}_{n+1}| \leq \Lambda_i^{(n+1)}$ for $|i - Z| \leq \bar{M}_{n+1}$. For $\bar{M}_{n+1} \leq |i - Z| \leq \bar{M}_1$ we just use
\[ |u_i(t) - \bar{u}_{n+1}| \leq |u_i(t) - \bar{u}_{n}| + |\bar{u}_{n+1} - \bar{u}_{n}| \leq \Lambda_i^{(n)} + |\bar{u}_{n+1} - \bar{u}_{n}| \leq \Lambda_i^{(n+1)} \]
where the last estimate is from the definition of $\Lambda$. For $|i - Z| \geq \bar{M}_1$ we have the trivial bound $\ell_0$.

Now we need to check the case $t \in [-3M_{n+1}, 0] \setminus \mathcal{G}_{n+1}$. For $\bar{M}_{n+1} \leq |i - Z| \leq \bar{M}_1$, from $(ST)_n$ we have
\[ |u_i(t) - \bar{u}_{n+1}| \leq |u_i(t) - \bar{u}_{n}| + |\bar{u}_{n+1} - \bar{u}_{n}| \leq \Phi_i^{(n)}(t) + |\bar{u}_{n+1} - \bar{u}_{n}| \leq \Phi_i^{(n+1)}(t), \]
where the last inequality is just from the definition of $\Phi$. Finally, if $|i - Z| \leq \bar{M}_{n+1}(\leq M_n)$, we use (10.26)
\[ |u_i(t) - \bar{u}_{n+1}| \leq |u_i(t) - \bar{u}_{n}| + |\bar{u}_{n+1} - \bar{u}_{n}| \leq 2\ell_n + \ell_n \sqrt{C |t| + L^{1/2}} M_n^{\chi/2} \]
since in this regime $\psi_i^{(n)} = 0$. The constant $C$ is from (10.26). The right hand side is bounded by
\[ C \Phi \ell_{n+1} \left( 1 + \sqrt{\frac{|t| + L^{1/2}}{M_{n+1}} M_n^{\chi/2}} \right), \]
by using that $\ell_n/\ell_{n+1} = \zeta^{-1} \leq \nu^{-1/10} = (M_n/M_{n+1})^{1/10}$ and choosing $C_\Phi$ large enough. This completes the proof of Proposition 10.9. \hfill $\square$
10.4 Proof of Lemma 10.6 (first De Giorgi lemma)

Assume for notational simplicity that $Z = 0$ and we set

$$\psi_i^\ell := \psi_i + \ell.$$ 

Using that $v$ solves the equation

$$\partial_t v_i(s) = - [A(s)v(s)]_i - W_i(s)\bar{u},$$ 

by direct computation we have

$$\partial_t \frac{1}{2} \sum_i |v_i - \psi_i^\ell|^2_+ = - \sum_{ij}(v_i - \psi_i^\ell)W_{ij}(v_i - v_j) - \sum_i(v_i - \psi_i^\ell)_+W_i(v_i + \bar{u}).$$

(10.88) by direct computation we have

Recall that $B_{ij}$ depends on time, but we will omit this from the notation. Since $W_i \geq 0$, the last term can be bounded by

$$- \sum_i(v_i - \psi_i^\ell)_+W_i(v_i + \bar{u}) \leq - \sum_i(v_i - \psi_i^\ell)_+W_i(v_i - \psi_i^\ell) - \bar{u} \sum_i(v_i - \psi_i^\ell)_+W_i$$

$$\leq - \bar{u}[v - \psi^\ell)_+,(v - \psi^\ell)_+] + |\bar{u}| \sum_i(v_i - \psi_i^\ell)_+W_i.$$ 

In the first term on the right hand side of (10.88) we can symmetrize and then add and subtract $\psi^\ell$ to $v$ we get

$$- \sum_{ij}(v_i - \psi_i^\ell)_+B_{ij}(v_i - v_j) = - b[v - \psi^\ell)_+, v]$$

$$= - b[v - \psi^\ell)_+, v - \psi^\ell) - b[v - \psi^\ell)_+, \psi^\ell] - b[v - \psi^\ell), \psi^\ell].$$

Since $B_{ij} \geq 0$ and $|a_+ - b_+||a_- - b_-| \geq 0$ for any real numbers $a, b$, for the cross-term we have $b[v - \psi^\ell)_+, \psi^\ell] \geq 0$. Thus the last equation is bounded by

$$\leq - b[v - \psi^\ell)_+, v - \psi^\ell) - b[v - \psi^\ell), \psi^\ell].$$

(10.89) by direct computation we have

Using the definition of $a$ (9.38), we have thus proved that

$$\partial_t \frac{1}{2} \sum_i |v_i - \psi_i^\ell|^2_+ \leq - a[v - \psi^\ell)_+, v - \psi^\ell)_+] - b[v - \psi^\ell)_+, \psi^\ell] + |\bar{u}| \sum_i(v_i - \psi_i^\ell)_+W_i.$$ 

(10.90) by direct computation we have

Decompose the first error term into

$$b[v - \psi^\ell)_+, \psi^\ell] = \Omega_1 + \Omega_2 + \Omega_3,$$

where $\Omega_1 := \frac{1}{2} \sum_{|i - j| \geq M} B_{ij}[\psi_i^\ell - \psi_j^\ell]|(v_i - \psi_i^\ell)_+ - (v_j - \psi_j^\ell)_+| \cdot 1(\max\{d_i^\ell, d_j^\ell\} \geq K/3)$

and $\Omega_2$ and $\Omega_3$ are defined in the same way except that the summation is restricted to $\bar{C}K^\ell \leq |i - j| \leq M$ for $\Omega_2$ and $|i - j| \leq \bar{C}K^\ell$ for $\Omega_3$, where $\bar{C}$ is the constant from (10.5). Notice that we inserted the characteristic function $1(\max\{d_i^\ell, d_j^\ell\} \geq K/3)$ for free, since (10.33) together with
$|Z| \leq K/2$ and $M^{1+\kappa} \ll K$ (from $\vartheta \geq 1 + 2\kappa$) guarantees that $(v_i - \psi_i^\ell)_+ = 0$ unless $d_i^\ell \geq K/3$. Thus the summation over $i, j$ can be restricted to index pairs, where at least one of them is far away from the boundary. Recall from (10.5) that in the regime $|i - j| \geq M$ we have $B_{ij} \leq C|i - j|^{-2}$ since $M \geq \tilde{C}K^\xi$. Moreover, we have

$$|\psi_i^\ell - \psi_j^\ell| \leq \ell M^{-1/2}|i - j|^{1/2}. \quad (10.91)$$

Altogether we have

$$|\Omega_1| \leq \ell M^{-1/2} \sum_{|i - j| \geq M} \frac{1}{|i - j|^{3/2}} [(v_i - \psi_i^\ell)_+ + (v_j - \psi_j^\ell)_+] \leq \ell M \sum_i (v_i - \psi_i^\ell)_+. $$

For $\Omega_2$, by symmetry of $B_{ij}$, we can rewrite it as

$$-\Omega_2 := - \sum_{\tilde{C}K^\xi \leq |i - j| \leq M, \psi_i^\ell \leq \psi_j^\ell} B_{ij} [\psi_i^\ell - \psi_j^\ell] [(v_i - \psi_i^\ell)_+ - (v_j - \psi_j^\ell)_+] \leq - \sum_{\tilde{C}K^\xi \leq |i - j| \leq M, \psi_i^\ell \leq \psi_j^\ell} B_{ij} [\psi_i^\ell - \psi_j^\ell] [(v_i - \psi_i^\ell)_+ - (v_j - \psi_j^\ell)_+] \cdot 1(v_i - \psi_i^\ell > 0) \leq \frac{1}{4} \sum_{\tilde{C}K^\xi \leq |i - j| \leq M} B_{ij} [(v_i - \psi_i^\ell)_+ - (v_j - \psi_j^\ell)_+]^2 + 4 \sum_{\tilde{C}K^\xi \leq |i - j| \leq M} B_{ij} |\psi_i^\ell - \psi_j^\ell|^2 \cdot 1(v_i - \psi_i^\ell > 0).$$

The first term is bounded by $\frac{\ell}{2} b[(v - \psi^\ell)_+, (v - \psi^\ell)_+]$ and can be absorbed in the first term on the r.h.s. of (10.89). By the simple estimate $|\psi_i^\ell - \psi_j^\ell| \leq C\ell |i - j|/M$ and (10.5), the second term is bounded by

$$4 \sum_{\tilde{C}K^\xi \leq |i - j| \leq M} B_{ij} |\psi_i^\ell - \psi_j^\ell|^2 \cdot 1(v_i - \psi_i^\ell > 0) \leq C\ell^2 M^{-1} \sum_i 1(v_i - \psi_i^\ell > 0), \quad (10.92)$$

where we again used that the summation over $i$ is restricted to $d_i^\ell \geq K/3$. Thus

$$-\Omega_2 \leq \frac{1}{2} b[(v - \psi^\ell)_+, (v - \psi^\ell)_+] + C\ell^2 M^{-1} \sum_i 1(v_i - \psi_i^\ell > 0) \leq \frac{1}{2} a[(v - \psi^\ell)_+, (v - \psi^\ell)_+] + C\ell^2 M^{-1} \sum_i 1(v_i - \psi_i^\ell > 0)$$

using that $b \leq a$.

A similar estimate is performed for $\Omega_3$, but in the corresponding last term we use that

$$|\psi_i^\ell - \psi_j^\ell| \leq CK^\xi (\ell/M)$$

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for $|i-j| \leq \hat{C} K^\xi$. Thus we have

$$
-\Omega_3 \leq \sum_{|i-j| \leq \hat{C} K^\xi} B_{ij} |\psi_i^\ell - \psi_j^\ell|^2 \cdot 1(v_i - \psi_i^\ell > 0)
$$

$$
\leq C K^{2\xi} (\ell/M)^2 \sum_{|i-j| \leq \hat{C} K^\xi} 1(v_i - \psi_i^\ell > 0) B_{ij}
$$

$$
\leq C \frac{K^{3\xi} \ell^2}{M^2} \sum_i 1(v_i - \psi_i^\ell > 0) [B_{i,i+1} + B_{i,i-1}] \tag{10.93}
$$

Here we just overestimated sums by $\hat{C} K^\xi$. The conclusion of the energy estimate is

$$
\frac{1}{2} \partial_t \sum_i |v_i - \psi_i^\ell|^2 \leq -\frac{1}{2} a((v-v^\ell)_+ + (v-v^\ell)_+) + |\bar{u}| \sum_i (v_i - \psi_i^\ell)_+ W_i + \frac{C \ell}{M} \sum_i (v_i - \psi_i^\ell)_+ + \frac{C \ell^2}{M} \sum_i 1(v_i - \psi_i^\ell > 0) + \Omega_4, \tag{10.94}
$$

$$
\Omega_4 := C \frac{K^{3\xi} \ell^2}{M^2} \sum_i 1(v_i - \psi_i^\ell > 0) [B_{i,i+1} + B_{i,i-1}] \tag{10.95}
$$

Due to (10.33), we can assume that the summations in (10.94) over $i$ are restricted to $|i| \leq M^{1+k}$. In this regime we have $d_i \geq \epsilon K$ thanks to $M^{1+k} \leq K/2$, therefore $W_i \leq CK^{-1+\xi}$ by (10.4). Using the bound (10.31), we see that the error term $|\bar{u}| \sum_i (v_i - \psi_i^\ell)_+ W_i$ can be absorbed into the first error term in line (10.94).

Let $T_k := -M(1+2^{-k})$, $\ell_k := \frac{\ell}{2}(1-2^{-k}) \geq \frac{\ell}{2}$ where $k = 1, 2, \ldots, C \log M$. We claim that

$$
\int_\tau^t \Omega_4 ds \leq \frac{1}{M^{1+\kappa}} \sum_{|i| \leq M^{1+\kappa}} [B_{i,i+1} + B_{i,i-1}] (s) \leq C [(t-\tau) + 1] K^{3\xi+\rho \ell^2} M^{\kappa-1} \tag{10.96}
$$

for any integer $k \leq C \log M$ and for any pairs $(t, \tau) \in [T_k, 0] \times [T_k-1, T_k-1 + 2^{-k-1} M]$. The estimate (10.96) holds because

$$
\int_\tau^t [\ldots] ds \leq \int_{T_k-1}^t [\ldots] ds \leq 8 |t-\tau| + 1,
$$

where we used that the point $(T_k-1, Z = 0)$ is regular, see (10.30).

Define

$$
U_k = \sup_{t \in [T_k, 0]} \frac{1}{M \ell_k^2} \sum_i (v_i - \psi_i^\ell)^2_+(t) + \frac{1}{M \ell_k^2} \int_{T_k}^0 a((v-v^\ell)_+, (v-v^\ell)_+)(s) ds. \tag{10.97}
$$

Integrating (10.94) from $\tau$ to $t$ with $\tau \in [T_k-1, T_k-1 + 2^{-k-1} M] = [T_k-1, T_k - 2^{-k-1} M]$ and $t \in [T_k, 0]$, we have from (10.96)

$$
\sum_i (v_i - \psi_i^\ell)^2_+(t) + \int_\tau^t a((v-v^\ell)_+, (v-v^\ell)_+)(s) ds
$$

$$
\leq \sum_i (v_i - \psi_i^\ell)^2_+(\tau) + C \int_\tau^t \left[ \frac{\ell_k}{M} \sum_i (v_i - \psi_i^\ell)_+(s) + \frac{\ell_k^2}{M} \sum_i 1(v_i - \psi_i^\ell > 0)(s) \right] ds
$$

$$
+ C [(t-\tau) + 1] K^{3\xi+\rho \ell^2} M^{\kappa-1}. \tag{10.98}
$$
Taking the average over \( \tau \in [T_{k-1}, T_{k-1} + 2^{-k-1}M] \) and using that in this regime \( 2^{-k-1}M \leq t - \tau \leq M \), we have

\[
\sum_i [v_i - \psi_i^{-\ell k}]^2(t) + \int_{T_k}^t a[(v - \psi^{\ell k})_+, (v - \psi^{\ell k})_+](s) ds \\
\leq C \frac{2^{k+1}}{M} \int_{T_k}^t \sum_i [v_i - \psi_i^{\ell k}]^2(s) ds \\
+ C \int_{T_k}^t \left[ \frac{\ell_k}{M} \sum_i (v_i - \psi_i^{\ell k})_+ + \frac{\ell_k^2}{M} \sum_i 1 (v_i - \psi_i^{\ell k} > 0) \right] ds + CK^{3\xi + \rho} l^2 M^\kappa.
\]

Dividing through by \( M\ell_k^2 \) and taking the supremum over \( t \in [T_k, 0] \), for \( k \geq 1 \) we have

\[
U_k \leq C \frac{2^{k+1}}{M^2} \int_{T_k}^t \sum_i \left[ \frac{1}{\ell_k} [v_i - \psi_i^{\ell k}]^2 + \frac{1}{\ell_k} (v_i - \psi_i^{\ell k})_+ + 1 (v_i - \psi_i^{\ell k} > 0) \right] (s) ds \\
+ M^\kappa \frac{CK^{3\xi + \rho}}{M}. \tag{10.99}
\]

The first three integrands have the same scaling dimensions as \( v^2 / \ell^2 \). One key idea is to estimate them in terms of the \( L^4 \)-norm of \( v \) and then using the Sobolev inequality. It is elementary to check these three integrands can be bounded by the \( L^4 \)-norm of \((v - \psi^{\ell k})_+\), by using that if \( v_i \geq \psi_i^{\ell k} \), then

\[
\sum_i (v_i - \psi_i^{\ell k})_+ \leq \sum_i (v_i - \psi_i^{\ell k})_+ \cdot 1 (v_i - \psi_i^{\ell k-1} > 2^{-(k+2)} \ell) \tag{10.100}
\]

\[
\leq (2^{k+1})^4 \ell_k^{4-3} \sum_i (v_i - \psi_i^{\ell k-1})_+^4, \\
\sum_i 1 (v_i - \psi_i^{\ell k} > 0) \leq (2^{k+2})^4 \ell_k^{4-4} \sum_i (v_i - \psi_i^{\ell k-1})_+^4, \\
\sum_i [v_i - \psi_i^{\ell k}]^2 \leq (2^{k+2})^2 \ell_k^{2-2} \sum_i (v_i - \psi_i^{\ell k-1})_+^4.
\]

We now use the local version of Proposition B.4 from Appendix B; we first verify its conditions. Set

\[
\mathcal{I} := [-2K/3, 2K/3], \quad \hat{\mathcal{I}} := [-3K/4, 3K/4]. \tag{10.101}
\]

Clearly \( f_i := (v_i - \psi_i^{\ell k-1})_+ \) is supported in \( \mathcal{I} \); this follows from \(|Z| \leq K/2\), (10.33) and that \( M^{1+\kappa} \leq M^{(\theta+1)/2} \ll M^\theta = K \). By the lower bounds on \( B_{ij}(s) \) in (10.3) and (10.5) (with \( C \geq 4 \) in (10.5)) to guarantee that the lower bound holds for any \( i, j \in \hat{\mathcal{I}} \) the conditions (B.18), (B.19) hold with the choice \( b = K^{-\xi}, a = \hat{C}^{-1} K^{-\xi} \) and \( r = C \), where \( C \) and \( \hat{C} \) are constants from (10.5).

From (B.20) we then have

\[
\sum_i (v_i - \psi_i^{\ell k-1})_+ \leq C \left[ \sum_i (v_i - \psi_i^{\ell k-1})_+^2 \right] a[(v - \psi^{\ell k-1})_+, (v - \psi^{\ell k-1})_+] + \frac{1}{K} \sum_i (v_i - \psi_i^{\ell k-1})_+^2 \\
+ CK^{4\xi} \max_i (v_i - \psi_i^{\ell k-1})_+^4 \tag{10.102}
\]
(we omitted the time variable \( s \in T \)). The last term can be estimated by using (10.33) and (10.34) as
\[
\max_i (v_i(t) - \psi_t^{f_{k-1}})_+ \leq \max \{ v_i(t) : |i| \leq M^{1+\kappa} \} \leq \mathcal{C}M^{\kappa/2}
\]
for any \( t \in [-2M, 0] \). For \( t \in \mathcal{G}^* \) we have the stronger bound from (10.35)
\[
\max_i (v_i(t) - \psi_t^{f_{k-1}})_+ \leq \max \{ v_i(t) : |i| \leq M^{1+\kappa} \} \leq \mathcal{C}tM^{\kappa/8}, \quad t \in \mathcal{G}^*.
\]
Inserting these estimates, (10.100) and (10.102) into (10.99), splitting the time integration into \( \mathcal{G}^* \) and its complement, we have proved that for \( k \geq 2 \)
\[
U_k \leq C(2^{k+2})^5 \frac{1}{M^2 \ell_k} \int_{T_{k-1}}^0 ds \left[ \sum_i (v_i - \psi_t^{f_{k-1}})_+^2(s) \right]
\]
\[
\times \left[ a \{(v - \psi_t^{f_{k-1}})_+, (v - \psi_t^{f_{k-1}})_-\}(s) + \frac{1}{K} \sum_i (v_i - \psi_t^{f_{k-1}})_+^2(s) \right]
\]
\[
+ \frac{1}{M} \left[ \mathcal{C}M^\kappa K^{3\xi+\rho} + 32^k M^{\kappa/2} K^{4\xi} + 32^k M^{2\kappa-1} K^{4\xi} \right]|[-T_{k-1}, 0] \setminus \mathcal{G}^*]
\]
\[
\leq 32^k \left[ C_1 U_{k-1}^2 + M^{-1+\kappa} K^{-\rho} \right], \quad (10.104) \]
recalling that \(|[-T_{k-1}, 0] \setminus \mathcal{G}^*| \leq CM^{1/4}, K = M^\rho \leq M^{\delta_0} \) and \( \chi \geq \kappa + 10(\xi + \rho)\delta_0 \). We also used that \(|T_k| \leq K \).

For \( k = 1 \), we estimate the integrands in (10.99) by \( L^2 \)-norms. We have the following general estimates for any \( \ell' < \ell'' \)
\[
\sum_i (v_i - \psi_t^{f_{\ell''}})_+ \leq \sum_i (v_i - \psi_t^{f_{\ell''}})_+ \leq \frac{1}{\ell'' - \ell'} \sum_i (v_i - \psi_t^{f_{\ell'}})_+^2, \quad (10.105) \]
\[
\sum_i 1(v_i - \psi_t^{f_{\ell''}} > 0) \leq \frac{1}{(\ell'' - \ell')^2} \sum_i (v_i - \psi_t^{f_{\ell'}})_+^2.
\]
We use (10.105) with \( \ell'' = \ell_1 \) and \( \ell' = 0 \) in (10.99), this implies that
\[
U_1 \leq \frac{C}{\ell_1^2 M^2} \int_{-2M}^0 ds (v_i - \psi_t^{f_{\ell}})_+^2(s) + CM^{-1+\chi}.
\]
Without loss of generality, we assume that \( C_1 \geq 2 \), where \( C_1 \) is the constant in (10.104). Now choose the universal constant \( \varepsilon_0 \) in (10.32) so small and \( M \) big enough so that this last inequality implies
\[
U_1 \leq \frac{1}{32^6 C_1}, \quad (10.106)
\]
Choose \( k_* \) such that \( 32^{k_*+2} C_1 = K^\rho \), i.e. \( k_\star \) is of order \( \rho \log K \geq \rho \log M \). Then from (10.104) for any \( k \leq k_* \) we have the recursive inequality
\[
B_k \leq B_{k-1}^2 + M^{-1+\chi}, \quad \text{with} \quad B_k := 32^{k+2} C_1 U_k.
\]
By a simple induction, this recursion implies
\[
B_{k+1} \leq (2B_k)^{k-1} + 2M^{-1+\chi}.
\]
Together with the initial estimate (10.106) we obtain that $B_{k+1} \leq 4M^{-1+x}$ for any integer $k$ with $100\log\log M \leq k \leq k_1$, in particular we can apply it to $k' = 100\log\log M$ and obtain $U_{k'} \leq CM^{-1+x}$. Notice that $U_k$ is decreasing in $k$ as it can be seen from the monotonicity in the definition of $U_k$ (10.97) and from the fact that $T_k$ and $\ell_k$ increase. Thus

$$U_k \leq CM^{-1+x} \quad (10.107)$$

for any $k \geq 100\log\log M$. Taking $k \to \infty$, we find from the $L^2$-norm term in $U_k$ that (10.36) in Lemma 10.6 holds.

For the proof of (10.37), we notice that the estimate (10.107) together with the monotonicity also implies that

$$\frac{1}{M^{\ell^2}} \int_{-M}^{0} a[(v - \psi^{\ell/3})_+, (v - \psi^{\ell/3})_+] ds \leq CM^{-1+x}$$

from the dissipation term in the definition of $U_k$.

Set

$$G := \left\{ t \in [-M, 0] : a[(v - \psi^{\ell/3})_+, (v - \psi^{\ell/3})_+] (t) \leq M^{-1/4} \ell^2 \right\}$$

then clearly

$$||[-M, 0] \setminus G|| \leq CM^{1/4}.$$  

We now use a Sobolev inequality (B.5) from Appendix B, with the choice of $p = 4$, $s = 1$ and $f_i := (v_i - \psi^{\ell/3})_+$. We recall the definitions of $I$ and $\hat{I}$ from (10.101) and that $f_i = (v_i - \psi^{\ell/3})_+$ is supported in $\hat{I}$ by (10.33). Thus

$$\sum_i f_i^4 \leq C \sum_i f_i^2 \left[ \sum_{i \neq j \in I} \frac{|f_i - f_j|^2}{|i - j|^2} + 2 \sum_{i \in I} |f_i|^2 \sum_{j \notin I} \frac{1}{|i - j|^2} \right] \leq C K^{2s} \|f\|^2 a[f, f] + \frac{C}{K} \|f\|^2,$$

where we used the lower bound on $B_{ij}$ in (10.3). Thus

$$\sum_i (v_i - \psi_{i+3/3})^4 \leq C K^{2s} \sum_i (v_i - \psi_{i+3/3})^2 a[(v - \psi^{\ell/3})_+, (v - \psi^{\ell/3})_+] + \frac{C}{K} \left[ \sum_i (v_i - \psi_{i+3/3})^2 \right]^2. \quad (10.108)$$

This implies that for any $t \in G$ and any $i$

$$(v_i(t) - \psi_{i+3/3})_+ \leq \| (v(t) - \psi^{\ell/3})_+ \|_4 \quad (10.109)$$

$$\leq C K^{s/2} \left[ \sum_i (v_i(t) - \psi_{i+3/3})^2 \right]^{1/4} (M^{-1/4} \ell^2)^{1/4} + \frac{C}{K^{1/4}} \left[ \sum_i (v_i(t) - \psi_{i+3/3})^2 \right]^{1/2}$$

$$\leq CM^{-1/20} \ell,$$

where we used (10.36) in the last step and the fact that $\chi \geq 10\xi \vartheta_0$ together with (10.29). This proves (10.37).

For the proof of (10.38), we first notice that it is sufficient to consider the case when $\tilde{M}$ is of the form $\tilde{M} = 2^{-m} M$, $m = 1, 2 \ldots C \log M$. We now repeat the proof of (10.36) but with $\ell_k, k \geq 1$, replaced by

$$\hat{\ell}_k = \frac{2\ell}{5} (1 - 2^{-k-2}) \quad (10.110)$$

in the definition of $\psi^{\ell_k}$ and working in the time interval of scale $\tilde{M}$.  

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Set \( \hat{T}_k := -\tilde{M}(1 + 2^{-k}) \). Define
\[
\hat{U}_k = \sup_{t \in [\hat{T}_k, 0]} \frac{1}{M \ell_k^2} \sum_i (v_i - \psi_i^{\hat{k}})_+(t) + \frac{1}{M \ell_k^2} \int_{\hat{T}_k}^0 a[(v - \psi^{\hat{k}}_i)_+, (v - \psi^{\hat{k}}_i)_+](s)ds.
\]

The previous proof is unchanged up to (10.96), the integral of
\[
\hat{\Omega}_4(s) := \frac{CK^{3\xi_1^2}}{M^2} \sum_i 1(v_i(s) - \psi_i^{\hat{\ell}}) > 0)[B_{i,i+1}(s) + B_{i,i-1}(s)]
\]
is still estimated by (cf. (10.96))
\[
\int_{\hat{T}_k}^t \hat{\Omega}_4(s)ds \leq C[(t - \tau) + 1]K^{3\xi_1+\rho\hat{\ell}^2}M^{-1+\kappa} \leq C[(t - \tau) + 1]\hat{\ell}^2K^{3\xi_1+\rho}M^{-1+\kappa}
\]
for \( \tau \in [\hat{T}_{k-1}, \hat{T}_{k-1} + 2^{-k-1}\hat{M}] = [\hat{T}_{k-1}, \hat{T}_{k} - 2^{-k-1}\hat{M}] \) and \( t \in [\hat{T}_k, 0] \). Here we used (10.29).

Similarly to (10.98), we integrate (10.94) (with \( \hat{\ell} \) replacing \( \ell \)) from \( \tau \) to \( t \)
\[
\sum_i [v_i - \psi_i^{\hat{\ell}}]_+^2(t) + \int_{\hat{T}_k}^t a[(v - \psi_i^{\hat{\ell}})_+, (v - \psi_i^{\hat{\ell}})_+](s)ds
\]
\[
\leq \sum_i [v_i - \psi_i^{\hat{\ell}}]_+^2(\tau) + C \int_{\hat{T}_k}^t \left[ \frac{\ell_k}{M} \sum_i (v_i - \psi_i^{\hat{\ell}})_+(s) + \frac{\hat{\ell}_k^2}{M} \sum_i 1(v_i - \psi_i^{\hat{\ell}} > 0)(s) \right]ds + C[(t - \tau) + 1]\hat{\ell}^2K^{3\xi_1+\rho}M^{-1+\kappa}. \tag{10.111} \]

Taking the average over \( \tau \in [\hat{T}_{k-1}, \hat{T}_{k-1} + 2^{-k-1}\hat{M}] = [\hat{T}_{k-1}, \hat{T}_{k} - 2^{-k-1}\hat{M}] \) and using that in this regime \( 2^{-k-1}\hat{M} \leq t - \tau \leq \hat{M} \), we have
\[
\sum_i [v_i - \psi_i^{\hat{\ell}}]_+^2(t) + \int_{\hat{T}_k}^t a[(v - \psi_i^{\hat{\ell}})_+, (v - \psi_i^{\hat{\ell}})_+](s)ds \leq C\frac{2^{k+1}}{M} \int_{\hat{T}_{k-1}}^{\hat{T}_k-2^{-k-1}\hat{M}} \sum_i [v_i - \psi_i^{\hat{\ell}}]_+^2(s)ds + C \int_{\hat{T}_k}^t \left[ \frac{\ell_k}{M} \sum_i (v_i - \psi_i^{\hat{\ell}})_+(s) + \frac{\hat{\ell}_k^2}{M} \sum_i 1(v_i - \psi_i^{\hat{\ell}} > 0)(s) \right]ds + C\hat{\ell}^2M^{3\xi_1+\rho}M^{-1+\kappa}.
\]

Dividing through by \( M\ell_k^2 \) and taking supremum over \( t \in [\hat{T}_k, 0] \), for \( k \geq 1 \) and using \( \hat{M} \leq M \), we have, as in (10.99),
\[
\hat{U}_k \leq C\frac{2^{k+1}}{MM} \int_{\hat{T}_{k-1}}^{0} \sum_i \left[ \frac{1}{\ell_k^2} [v_i - \psi_i^{\hat{\ell}}]_+^2 + \frac{1}{\ell_k} (v_i - \psi_i^{\hat{\ell}})_+ + 1(v_i - \psi_i^{\hat{\ell}} > 0) \right](s)ds + C\hat{\ell}^2M^{3\xi_1+\rho}M^{-2+\kappa}. \tag{10.112} \]

Using the bounds (10.100) and Proposition B.4 as in (10.102)–(10.104), instead of (10.104) we
get

$$
\hat{U}_k \leq \frac{(2k+2)^5}{MM^2} \int_{-T_{k-1}}^0 ds \left[ \sum_i (v_i - \psi_{i,k-1})^2_+(s) \right] a[(v - \psi_{k-1})_+(v - \psi_{k-1})_+(s)
+ C\tilde{M}M^{-2+\chi}K^{-\rho}
\leq 32k \left[ C_1 M^{\tilde{U}_{k-1}} + C\tilde{M}M^{-1+k}K^{-\rho} \right], \quad k \geq 2.
$$

(10.113)

Similarly to the proof of (10.107), this new recurrence inequality has the solution

$$
\hat{U}_k \leq C\tilde{M}M^{-2+\chi}
$$

(10.114)

for any sufficiently large $k$, as long as the recursion can be started, i.e. if we knew

$$
\hat{U}_1 \leq \frac{\tilde{M}}{M}.
$$

(10.115)

For $k = 1$ the estimate (10.112) together with (10.105) (with $\ell_1$ replacing $\ell_1$) becomes

$$
\hat{U}_1 \leq \frac{C}{MM} \int_{-2\tilde{M}}^0 \sum_i \left[ 1/\ell_1^2 \right] \left[ (v_i - \psi_{i,1})^2_+ + 1/\ell_i (v_i - \psi_{i,1})_+ + 1 (v_i - \psi_{i,1} > 0) \right] (s) ds + C\tilde{M}M^{-2+\chi}
\leq \frac{C}{MM} \int_{-2\tilde{M}}^0 \sum_i \left[ 1/\ell_1^2 \right] \left[ (v_i(s) - \psi_{i,1}^{1/3})^2_+ \right] ds + C\tilde{M}M^{-2+\chi}
\leq \frac{CM^\chi}{M} + C\tilde{M}M^{-2+\chi}.
$$

In the second step we used (10.105) with $\ell'' = \ell_1$ and $\ell' = \ell/3$ noting that $\hat{\ell}_1 = \frac{20}{3} \ell > \frac{1}{3} \ell$. In the last step we used (10.36) and $2\tilde{M} \leq M$. Thus (10.115) is satisfied if $\tilde{M} \gg M^\chi$.

Finally, taking $k \to \infty$ in (10.114) implies (10.38). This completes the proof of Lemma 10.6.

10.5 Proof of Lemma 10.7 (second De Giorgi lemma)

Set $Z = 0$ for simplicity. Since the statement is stronger if $\mu$ and $\delta$ are reduced, we can assume that they are small positive numbers, e.g. we can assume $\mu, \delta < 1/8$. We are looking for a sufficiently small $\lambda$ so that there will be a positive $\gamma$ with the stated properties. The key ingredient of the proof is an energy inequality (10.123) including a new dissipation term which was dropped in the proof of Lemma 10.6. Most of this section closely follows the argument in [11]; the main change is that we need split time integrations into “good” and “bad” times. The argument [11] applies to the good times. The bad times have a small measure, so their contribution is negligible.

10.5.1 Dissipation with the good term

Let $-3M \leq T_1 < T_2 < 0$. For any $t \in [-3M, 0]$, define

$$
\theta_i(t) := 1(|i| \leq 9M) \cdot 1(t \in G) + 1(|i| \leq M^{1+\kappa_1}) \cdot 1(t \notin G),
$$

(10.116)
We use the calculation (10.89)–(10.90) (with cutoff \( \varphi^{(1)} \) instead of \( \psi^\ell \)) but we keep the “good” \( b((v-\varphi^{(1)})_+, (v-\varphi^{(1)})_-) \geq 0 \) term that was estimated trivially in (10.89) and we drop the (positive) potential term in \( a \). We have

\[
\frac{1}{2} \sum_{i} |v_i(t) - \varphi_i^{(1)}|^2_+ \bigg|_{t=T_2}^{T_2} + \int_{T_1}^{T_2} b((v(t) - \varphi^{(1)})_+, (v(t) - \varphi^{(1)})_+) dt \tag{10.117}
\]

\[
\leq - \int_{T_1}^{T_2} b((v(t) - \varphi^{(1)})_+, (v(t) - \varphi^{(1)})_-) dt - \int_{T_1}^{T_2} b((v(t) - \varphi^{(1)})_+, \varphi^{(1)}) dt + |\bar{u}\int_{T_1}^{T_2} \sum_i (v_i(t) - \varphi_i^{(1)})_+ W_i \theta_i dt|
\]

Notice that we inserted the characteristic function \( \theta_i(t) \) using the fact that (10.43) and (10.40) imply \( v_i(t) \leq \varphi_i^{(1)} \) for \( |i| \geq 9M \), \( t \in \mathcal{G} \), and \( v_i(t) \leq \varphi_i^{(1)} = \bar{\psi}_i \) for \( |i| \geq M^{1+\kappa_1} \) and for all \( t \in [-3M, 0] \) i.e. \( v_i - \varphi^{(1)} = (v_i - \varphi^{(1)}) \theta_i \) for any time. Moreover, \( v_i(t) - \varphi_i^{(1)} \leq \lambda \ell \) for \( t \in \mathcal{G} \) and \( |i| \leq 9M \).

The last error term in (10.117) is estimated trivially; in the regime \( |i| \leq M^{1+\kappa_1} \) we have \( W_i \leq CK^{-1+\kappa} \) and then from (10.45), \( |\mathcal{G}| \leq CM^{1/4} \) and (10.42) we have

\[
|\bar{u}\int_{T_1}^{T_2} \sum_i (v_i(t) - \varphi_i^{(1)})_+ W_i \theta_i dt| \leq \lambda^2 \ell^2 (T_2 - T_1) + C\lambda^2 \ell \lambda^{1+\kappa_2} |\mathcal{G}| \leq \lambda^2 \ell \lambda^{1+\kappa_2} (T_2 - T_1) + \lambda^{-1} M^{1/2}
\]

after splitting the integration regime into “good” times \( \mathcal{G} \) and “bad” times \( \mathcal{G}^c := [-3M, 0] \setminus \mathcal{G} \). We also used (10.41).

The other error term in (10.117) will be estimated by a Schwarz inequality, here we use the identity

\[
b(f \theta, g) = \sum_{ij} (f_i \theta_i - f_j \theta_j) B_{ij}(g_i - g_j) = \sum_{ij} (f_i \theta_i - f_j \theta_j) (\theta_i + \theta_j - \theta_i \theta_j) B_{ij}(g_i - g_j)
\]

for any functions \( f \) and \( g \), so

\[
|b(f \theta, g)| \leq \frac{1}{2} \sum_{ij} (f_i \theta_i - f_j \theta_j)^2 B_{ij} + 2 \sum_{ij} \theta_i B_{ij}(g_i - g_j)^2
\]

i.e.

\[
|b((v(t) - \varphi^{(1)})_+ \theta, \varphi^{(1)})| \leq \frac{1}{2} b((v(t) - \varphi^{(1)})_+ \theta, (v(t) - \varphi^{(1)})_+ \theta) + 2 \sum_{ij} \theta_i B_{ij}(\varphi_i^{(1)} - \varphi_j^{(1)})^2.
\]

The first term will be absorbed in the quadratic term in the left of (10.117). By definition of \( \varphi^{(1)} \), for the second term we have to control

\[
\int_{T_1}^{T_2} \left[ \lambda^2 \sum_{i,j} (F_i - F_j)^2 B_{ij} + \sum_{i,j} (\bar{\psi}_i - \bar{\psi}_j)^2 B_{ij} \theta_i \right] dt, \tag{10.119}
\]

Since \( |F_i - F_j| \leq C \ell M^{-1} |i-j| \) and \( F_i - F_j \) is supported on \( |i|, |j| \leq 9M \), by splitting the summation to the regime \( |i-j| \leq K^2 \) and its complement, we can bound the first term by

\[
\int_{T_1}^{T_2} \lambda^2 \sum_{i,j} (F_i - F_j)^2 B_{ij} dt \leq \lambda^2 \ell^2 M^{-2} \int_{T_1}^{T_2} \sum_{|i|, |j| \leq 9M} |i-j|^2 B_{ij}(t)
\]

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\[
\leq \lambda^2 \ell^2 M^{-2} K^{3\xi} \int_{T_1}^{T_2} \sum_{|i| \leq 9M} B_{i,i+1}(t) dt + C \lambda^2 \ell^2 M^{-2} \int_{T_1}^{T_2} \sum_{|i|,|j| \leq 9M, |i-j| \geq M^{\xi}} \frac{|i-j|^2}{|i-j|^2}
\]

\[
\leq \lambda^2 \ell^2 M^{-2} K^{3\xi} \int_{-3M}^{0} \sum_{|i| \leq 9M} B_{i,i+1}(t) dt + C \lambda^2 \ell^2 (T_2 - T_1),
\]

where we have used \(B_{i,j} \leq B_{i,i+1}\) in the first regime and the upper bound in (10.5) in the other regime. By the regularity at \((Z,0) = (0,0)\) we can bound the last line by

\[
C \lambda^2 \ell^2 K^{3\xi + \rho} + C \lambda^2 \ell^2 (T_2 - T_1) \leq C \lambda^2 \ell^2 [(T_2 - T_1) + M^{1/2}]
\]

(we also used (10.41) and \(K \leq M^{\theta_0}\)).

For the second term in (10.119) and for \(t \in G\) we use that \(\tilde{\psi}_i \theta_i(t) = 0\) and the supports of \(\theta_i\) and \(\tilde{\psi}_j\) are separated by a distance of order \(M \gg K^{\xi}\). Thus we can use the upper bound in (10.5) to estimate the kernel:

\[
\int_{T_1}^{T_2} 1(t \in G) \sum_{i,j} (\tilde{\psi}_i - \tilde{\psi}_j)^2 B_{ij}(t) \theta_i(t) dt \leq C \int_{T_1}^{T_2} \sum_{|i| \leq 9M} \sum_{|j| \geq M^{\xi-4}} \frac{\tilde{\psi}_j^2}{|j|^{1/2}} dt
\]

\[
\leq C M (T_2 - T_1) \sum_{|j| \geq M^{\xi-4}} \tilde{\psi}_j^2 \leq C \lambda^2 \ell^2 (T_2 - T_1), \tag{10.120}
\]

where we have used \(\tilde{\psi}_j \sim \ell (j/M)^{1/4}\) for large \(j\). For times \(t \notin G\), we use

\[
(\tilde{\psi}_i - \tilde{\psi}_j)^2 \leq \frac{C \ell^2}{M^{1/2}} \frac{(i-j)^2}{|i|^{3/2} + |j|^{3/2}}
\]

to get

\[
\int_{T_1}^{T_2} 1(t \notin G) \sum_{i,j} (\tilde{\psi}_i - \tilde{\psi}_j)^2 B_{ij}(t) \theta_i(t) dt \leq \int_{T_1}^{T_2} 1(t \notin G) \sum_{|i| \leq M^{\xi+1}} \sum_{|j| \geq M^{\xi-4}} \frac{1}{|i|^{3/2} + |j|^{3/2}} dt
\]

\[
+ \int_{T_1}^{T_2} 1(t \notin G) \sum_{|i| \leq M^{\xi+1}} \sum_{|j| \geq M^{\xi-4}} B_{ij}(t) \frac{|i-j|^2}{|i|^{3/2} + |j|^{3/2}} \cdot 1(|i-j| \leq K^{\xi}) dt
\]

\[
\leq CM^{1+\kappa_1} |G|^2 \int_{M^{1/2}}^{\ell^2} \lambda M^{-1/2} + C K^{3\xi} \ell^2 \int_{-3M}^{0} \sum_{|i| \leq M^{\xi+1}} \sum_{|j| \geq M^{\xi-4}} B_{ij}(t) \frac{1(|i-j| \leq K^{\xi})}{|i|^{3/2} + |j|^{3/2}} dt
\]

\[
\leq CM^2 \ell^{2M^{\xi+1/4}} + C K^{3\xi} \ell^2 \int_{-3M}^{0} \sum_{|i| \leq M^{\xi+1}} B_{i,i+1}(t) dt
\]

\[
\leq C \lambda^2 \ell^2 M^{1/2}.
\]

Here we first separated the summations over \(i, j\) into \(|i-j| \geq K^{\xi}\) and its complement. Then in the first regime we used the upper bound in (10.5) and that the measure of the bad time is small, i.e.,
We have from (10.43) and (10.46) that

\[
\frac{1}{2} \sum_i [v_i(t) - \varphi_i^{(1)}]^2 \bigg|_{t=T_1}^{T_2} + \frac{1}{2} \int_{T_1}^{T_2} b[(v(t) - \varphi^{(1)})_+, (v(t) - \varphi^{(1)})_-] dt
\]

(10.122) [diss3]

\[
\leq - \int_{T_1}^{T_2} b[(v(t) - \varphi^{(1)})_+, (v(t) - \varphi^{(1)})_-] dt + C\ell^2 \lambda^2 [(T_2 - T_1) + M^{1/2}].
\]

Define

\[
H(t) = \sum_i (v_i(t) - \varphi_i^{(1)})^2.
\]

We have

\[
H(T_2) + \int_{T_1}^{T_2} b[(v(t) - \varphi^{(1)})_+, (v(t) - \varphi^{(1)})_-] dt \leq H(T_1) + C\ell^2 \lambda^2 [(T_2 - T_1) + M^{1/2}]
\]

(10.123) [goodterm]

for any \(-3M \leq T_1 < T_2 < 0\). Notice that \(b(f_+, f_-) \geq 0\) for any function \(f\). Since \(|v_i(t) - \varphi_i^{(1)}| \leq \lambda \theta_i\) for all \(t \in \mathcal{G}\), we also have

\[
H(t) \leq C\lambda^2 \ell^2 M, \quad t \in \mathcal{G}.
\]

(10.124) [Hupp]

10.5.2 Time slices when the good term helps

Let \(\Sigma \subset \mathcal{G}\) be the set of times that \(v(T)\) is substantially below \(\varphi^{(0)}\), i.e.,

\[
\Sigma := \left\{ t \in (-3M, -2M) \cap \mathcal{G} : \# \left\{ i : M : v_i(T) \leq \varphi_i^{(0)} \right\} \geq \frac{1}{4} \mu M \right\}.
\]

We have from (10.43) and (10.46) that

\[
|\Sigma| \geq \frac{1}{4} M \mu - CM^{1/4} \geq \frac{1}{5} M \mu.
\]

(10.125) [sigmabound]

By (10.123) (applied to \(T_1 = \min \Sigma, T_2 = -2M\)) and (10.124) (applied to \(t = T_1\)), we have

\[
C\lambda^2 \ell^2 M \geq \int_{\Sigma} b[(v(t) - \varphi^{(1)})_+, (v(t) - \varphi^{(1)})_-] dt
\]

\[
\geq - \int_{\Sigma} \sum_{ij} (v_i(t) - \varphi_i^{(1)})_+ B_{ij}(t)(v_j(t) - \varphi_j^{(1)})_- dt
\]

\[
\geq - cM^{-2} \int_{\Sigma} \sum_{ij} (v_i(t) - \varphi_i^{(1)})_+ (v_j(t) - \varphi_j^{(1)})_- dt,
\]

(10.126) [c2]

where we have used that for \(v_i(t) - \varphi_i^{(1)}\) is supported on \(|i| \leq 9M\) (for \(t \in \mathcal{G}\)) and

\[
B_{ij}(t) \geq cM^{-2}, \quad |i|, |j| \leq 9M,
\]

(10.127) [Klower]
with some positive constant \( \bar{c} \) (this follows from the lower bound in (10.5), where \(|i| \leq 9M\) and \(M \leq K/10\) guarantee that \(d_i \geq K/C\) holds, and \(K^5 \ll M\) guarantees that (10.5) can be used for the extreme points \(i = -9M, j = 9M\) and finally we used monotonicity \(B_{ij} \geq B_{-9M,9M}\) for any \(|i|, |j| \leq 9M\). For \(t \in \Sigma\) the number of \(j\)'s with \(|j| \leq M\) such that \(v_j(t) \leq \varphi_j^{(0)}\) is at least \(\frac{1}{5} \mu M\); for such \(j\)'s we have
\[
-(v_j(t) - \varphi_j^{(1)})_- \geq \varphi_j^{(1)} - \varphi_j^{(0)} \geq (1 - \lambda)\ell \geq \frac{\ell}{2}.
\]
Thus we can bound (10.127) by
\[
\geq cM^{-1} \int_\Sigma \sum_i (v_i(t) - \varphi_i^{(1)})_+ \, dt \geq cM^{-1} \int_\Sigma \sum_i (v_i(t) - \varphi_i^{(1)})_+^2 \, dt,
\]
where we have used that \((v_i(t) - \varphi_i^{(1)})_+ \leq \lambda \ell\) for \(t \in G\).

Altogether we have proved
\[
\int_\Sigma \sum_i (v_i(t) - \varphi_i^{(1)})_+^2 \, dt \leq C\lambda^3 \mu^{-1} \ell^2 M^2 \leq \lambda^{3 - \frac{1}{2}} \ell^2 M^2
\]
if \(\lambda\) is sufficiently small (depending on \(\mu\)). Thus there exists a subset \(\Theta \subset \Sigma\) such that
\[
|\Theta| \leq \lambda^{1/8} M,
\]
and we have
\[
\sum_i (v_i(t) - \varphi_i^{(1)})_+^2 \leq \lambda^{3 - \frac{1}{2}} \ell^2 M, \quad \forall t \in \Sigma \setminus \Theta.
\]
Choosing \(\lambda\) small and recalling (10.125) we see that
\[
\sum_i (v_i(t) - \varphi_i^{(1)})_+^2 \leq \lambda^{3 - \frac{1}{2}} \ell^2 M \tag{10.129} \]
holds on a set of times \(t\)'s in \(\Sigma \subset [-3M, -2M] \cap G\) of measure at least \(M\mu/8\). In particular this set of times is non-empty.

10.5.3 Finding the intermediate set

Since (10.47) is satisfied, there is a \(T_0 \in (-2M, 0) \cap G\) such that
\[
\# \left\{ i : (v_i(T_0) - \varphi_i^{(2)})_+ > 0 \right\} \geq \frac{1}{2} M\delta - CM^{1/4}, \tag{10.130} \]
and choose a \(T_1 \in \Sigma\) (then \(T_1 < T_0\)) such that
\[
H(T_1) = \sum_i (v_i(T_1) - \varphi_i^{(1)})_+^2 \leq \lambda^{3 - \frac{1}{2}} \ell^2 M \tag{10.131} \]
(such \(T_1\) exists by the conclusion of the previous section, (10.129)).

We also have
\[
H(T_0) = \sum_i (v_i(T_0) - \varphi_i^{(1)})_+^2 \geq \sum_i (\varphi_i^{(2)}(T_0) - \varphi_i^{(1)})_+^2 \cdot 1((v_i(T_0) - \varphi_i^{(2)})_+ > 0)
\]

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\[ \geq \sum_i \ell^2 (\lambda - \lambda^2) F_i^2 \cdot 1((vi(T_0) - \varphi_i^{(2)})_+ > 0) \geq C_F \frac{\lambda^2}{4} \ell^2 \delta^3 M \]  

(10.132)

with some positive constant \( C_F \). This follows from (10.130); notice first that the set in (10.130) must lie in \([-9M, 9M]\) (see (10.40) and (10.43)), and even if the whole set (10.130) is near the “corner” (i.e. close to \( i \sim \pm 9M \)), still the sum of these \( F_i \)'s is of order \( \delta^3 M \) since \( F_i \) is linear near the endpoints \( i = \pm 9M \).

Choose now \( \lambda \) small enough (depending on the fixed \( \delta \)) s.t.
\[ \lambda^3 - \frac{7}{8} \ell^2 M \leq \frac{1}{16} C_F \lambda^2 \ell^2 \delta^3 M. \]

Since \( H(T) \) is continuous and it goes from a small value \( H(T_1) \leq \frac{1}{16} C_F \lambda^2 \ell^2 \delta^3 M \) to a large value \( H(T_0) \geq \frac{1}{4} C_F \lambda^2 \ell^2 \delta^3 M \), the set of intermediate times
\[ D := \{ t \in (T_1, T_0) : \frac{1}{16} C_F \lambda^2 \ell^2 \delta^3 M < H(t) < \frac{1}{4} C_F \lambda^2 \ell^2 \delta^3 M \} \]

is non-empty.

**Lemma 10.10** The set \( D \) contains an interval of size at least \( c\delta^3 M \) with some positive constant \( c > 0 \). Moreover, for any \( t \in D \cap G \), we have
\[ \#\{ i : \varphi_i^{(2)} \leq v_i(t) \} \leq \frac{1}{2} \delta M. \]

**Proof.** By continuity, there is an intermediate time \( T' \in D \subset [T_1, T_0] \) such that \( H(T') = \frac{1}{8} C_F \lambda^2 \ell^2 \delta^3 M \). We can assume that \( T' \) is the largest such time, i.e.
\[ H(t) > \frac{1}{8} C_F \lambda^2 \ell^2 \delta^3 M \quad \text{for any} \quad t \in [T', T_0] \cap D. \]

(10.134)

Let \( T'' = T' + c\delta^3 M \) with a small \( c > 0 \). We claim that \( [T', T''] \subset D \). For any \( t \in [T', T''] \) we can use (10.123):
\[ H(t) \leq H(T'') + C \ell^2 \lambda^2 [(t - T') + M^{1/2}] \leq \frac{1}{8} C_F \lambda^2 \ell^2 \delta^3 M + Cc \ell^2 \lambda^2 \delta^3 M < \frac{1}{4} C_F \lambda^2 \ell^2 \delta^3 M \]

(10.135)

if \( c \) is sufficiently small. This means that as \( t \) runs through \( [T', T''] \), \( H(t) \) has not reached \( \frac{1}{4} C_F \lambda^2 \ell^2 \delta^3 M \), in particular \( [T', T''] \subset (T_1, T_0) \) since \( H(T_0) \) is already above this threshold. Combining then (10.135) with (10.134), we get \( [T', T''] \subset D \). This proves the first statement of the lemma.

For the second statement, we argue by contradiction. Suppose we have \( \#\{ i : \varphi_i^{(2)} \leq v_i(t) \} > \frac{1}{2} \delta M \) for some \( \tau \in D \cap G \). Going through the estimate (10.132) but \( T_0 \) replaced with \( \tau \), we would get \( H(\tau) \geq C_F \frac{\lambda^2}{4} \ell^2 \delta^3 M \), but this contradicts to \( \tau \in D \). This completes the proof of the lemma. \( \square \)

Define the exceptional set \( \mathcal{F} \subset D \cap G \) of times where \( v \) is below \( \varphi^{(0)} \), i.e.
\[ \mathcal{F} := \{ t \in D \cap G : \#\{ j : |j| \leq 8M : v_j(t) - \varphi_j^{(0)} \leq 0 \} \geq \mu M \}. \]

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This set is very small, since from (10.124) (applied to $t_{\text{max}} := \sup F \in \mathcal{G}$) we have

\[ C\lambda^2 \ell^2 M \geq - \int_{-3M}^{t_{\text{max}}} \sum_{i,j} (v_i(t) - \varphi_i^{(1)})(v_j(t) - \varphi_j^{(1)}) \ dt \]

\[ \geq - \int_{F} \sum_{|i|,|j| \leq 9M} (v_i(t) - \varphi_i^{(1)})(v_j(t) - \varphi_j^{(1)}) \ dt \]

\[ \geq - \bar{c} M^{-2} \int_{\mathcal{F}} \sum_{|i|,|j| \leq 9M} (v_i(t) - \varphi_i^{(1)})(v_j(t) - \varphi_j^{(1)}) \ dt \]

\[ \geq \frac{\bar{c}}{2M} \ell \mu \int_{\mathcal{F}} \sum_{|i| \leq 9M} (v_i(t) - \varphi_i^{(1)})^+ dt, \]

where we restricted the time integration to $\mathcal{F}$ in the first step, then we used (10.128) in the second step. In the third step we used that whenever $v_j(t) - \varphi_j^{(0)} \leq 0$ (see the definition of $\mathcal{F}$), then $-(v_i(t) - \varphi_i^{(1)}) \geq (1 - \lambda) \geq \frac{\ell}{2}$. By (10.43), $(v_i(t) - \varphi_i^{(1)}) \geq \ell \lambda$ and $(v_i(t) - \varphi_i^{(1)}) = 0$ if $|i| \geq 9M$ and $t \in \mathcal{G}$. Hence we can continue the above estimate

\[ C\lambda^2 \ell^2 M \geq \frac{c\mu}{2M\lambda} \int_{\mathcal{F}} \sum_{i} (v_i(t) - \varphi_i^{(1)})^2 dt = \frac{\bar{c}\mu}{2M\lambda} \int_{F} H(t) dt \geq \frac{cC_F}{32} M^2 \delta^3 \mu |\mathcal{F}|. \]

Here we used that $\mathcal{F} \subset D$ and that in $D$ we have a lower bound on $H(t)$. The conclusion is that

\[ |\mathcal{F}| \leq \frac{C\lambda}{\delta^3 \mu} M \]

with some fixed constant $C > 0$. Using that $|D| \geq c\delta^3 M$ from Lemma 10.10 and the smallness of $|\mathcal{G}|$, we thus have

\[ |\mathcal{F}| \leq \frac{|D \cap \mathcal{G}|}{2}, \quad |D \cap \mathcal{G}| \geq \frac{1}{2} c\delta^3 M \]

if $\lambda$ is sufficiently small, like

\[ \lambda \leq \frac{c\delta^3 \mu}{3}. \quad (10.136) \]

This means that $|D \setminus \mathcal{F}| \geq \frac{\lambda}{2} \delta^3 M$. Now we claim that for $t \in (D \cap \mathcal{G}) \setminus \mathcal{F}$ we have

\[ A(t) := \# \left\{ i : \varphi_i^{(0)} \leq v_i(t) \leq \varphi_i^{(2)} \right\} \geq \frac{M}{2}. \quad (10.137) \]

This is because $t \notin \mathcal{F}$ guarantees that the lower bound $\varphi_i^{(0)} \leq v_i(t)$ is violated not more than $\mu M \leq M/4$ times among the indices $|i| \leq 8M$. By (10.133), the upper bound $v_i(t) \leq \varphi_i^{(2)}$ is violated not more than $\frac{1}{2} \delta \delta M \leq M/4$ times.

Finally, integrating (10.137) gives

\[ \int_{-3M}^{0} \# \left\{ i : \varphi_i^{(0)} < v_i(t) < \varphi_i^{(2)} \right\} dt = \int_{-3M}^{0} A(t) dt \geq \frac{M}{2} |(D \cap \mathcal{G}) \setminus \mathcal{F}| \geq c\delta^3 M^2 \]

with some small $c > 0$, which implies (10.48) with

\[ \gamma := c\delta^3. \quad (10.138) \]

This proves Lemma 10.7.
A Proof of Lemma 4.5

First we show that on the set $R_{L,K}$, the length of the interval $J = J_y = (y_{L-K-1}, y_{L+K+1})$ satisfies
\[
|J| = |y_{L+K+1} - y_{L-K-1}| = |\gamma_{L+K+1} - \gamma_{L-K-1}| + O(N^{-1+\xi\delta/2}). \tag{A.1}
\]

Then we use the Taylor expansion
\[
\varrho(x) = \varrho(\bar{y}) + O(x - \bar{y})
\]
around the midpoint $\bar{y}$ of $J$. Here we used that $\varrho \in C^1$ away from the edge. Thus from (2.15)
\[
K + 1 = N \int_{y_{L-K-1}}^{y_{L+K+1}} \varrho = N \int_{y_{L-K-1}}^{y_{L+K+1}} \varrho + O(N^{\xi\delta/2}) = N|J|\varrho(\bar{y}) + O(N|J|^2) + O(N^{\xi\delta/2}), \tag{A.2}
\]
since the contribution of the second order term in the Taylor expansion is of order $N|J|^2$. Expressing $|J|$ from this equation and using (4.1), we arrive at (4.22).

Now we prove (4.23). We set
\[
U(x) := V(x) - \frac{2}{N} \sum_{j : |j-L| \geq K+K^\delta} \log |x - y_j|.
\]
The potential $U$ is similar to $V_y$, but the interactions with the external points near the edges of $J$ ($y_j$’s with $|j-L| < K + K^\delta$) have been removed and the external points $y_j$ away from the edges have been replaced by their classical value $\gamma_j$. In proving (4.23), we will first compare $V_y$ with an auxiliary potential $U$ and then compute $U'$.

First we estimate the difference $V'_y(x) - U'(x)$. We fix $x \in J$, and for definiteness, we assume that $d(x) = x - y_{L-K-1}$, i.e. $x$ is closer to the lower endpoint of $J$; the other case is analogous. We get (explanations will be given after the equation)
\[
|V'_y(x) - U'(x)| \leq \frac{1}{N} \sum_{K < |j-L| < K+K^\delta} \frac{1}{N} \sum_{j : |j-L| \geq K+K^\delta} \frac{|y_j - \gamma_j|}{N \sum_{j=N/2}^{L-K-K^\delta} 1 \sum_{j=L+K+K^\delta}^{L-K-K^\delta} 1 \sum_{j=N/2}^{L-K-K^\delta} 1 \sum_{j=L+K+K^\delta}^{L-K-K^\delta} 1}
\]
\[
\leq \frac{CK^\delta}{Nd(x)} + \frac{C N^{-1+\xi\delta/2} \log N}{d(x)} + CN^{-4/15+\epsilon}
\]
\[
\leq \frac{CK^\delta}{Nd(x)}.
\tag{A.3}
\]

Here for the first bulk sum, $j \in \left[ N\alpha/2, L - K - K^\delta \right]$, we used $|y_j - \gamma_j| \leq N^{-1+\xi\delta/2}$ from the...
We thus obtain

\[ d \geq y_{L-K-1} - \gamma_j \]

\[ \geq \gamma_{L-K-1} - \gamma_j - |y_{L-K-1} - \gamma_{L-K-1}| \]

\[ \geq cN^{-1}(L-K-1) - CN^{-1+\xi/2} \]

\[ \geq cN^{-1}(L-K-1-j) \]

with some positive constants \( c, c' \). This estimate allows one to sum up \(|x - \gamma_j|^{-1}\) at the expense of a log \( N \) factor. Similar estimate holds for \( j \geq L + K + K^\xi \). In the intermediate sum, \( j \in [N^{3/5+\varepsilon}, N\alpha/2] \), we used \(|y_j - \gamma_j| \leq CN^{-4/15+\varepsilon}\) and that \(|x - y_j|\) and \(|x - \gamma_j|\) are bounded from below by a positive constant since

\[ x - y_j \geq y_{L-K-1} - y_j \geq y_{\alpha N} - y_j \geq \gamma_{N\alpha} - \gamma_{N\alpha/2} + O(N^{-1+\xi/2}) \geq c \]

and similarly for \( x - \gamma_j \). Finally, very near the edge, e.g. for \( j \leq N^{3/5+\varepsilon} \), we just estimated \(|y_j - \gamma_j|\) by a constant. This explains (A.3).

Now we estimate \( U'(x) \). We use the fact that the equilibrium measure \( \rho = \rho_V \) satisfies the identity

\[ \frac{1}{2} V'(x) = \int \frac{\rho(y)}{x-y} \, dy \]

from the Euler-Lagrange equation of (2.14), see [2, 10]. Thus

\[ \frac{1}{2} |U'(x)| \leq |\Omega_1| + |\Omega_2| + |\Omega_3| \]

with

\[ \Omega_1 := \int_{C_{L-K-K^\xi}}^{C_{L+K+K^\xi}} \frac{\rho(y)}{x-y} \, dy \]

\[ \Omega_2 := \int_{A}^{C_{L-K-K^\xi}} \frac{\rho(y)}{x-y} \, dy - \frac{1}{N} \sum_{j=1}^{L-K-K^\xi} \frac{1}{x - \gamma_j} \]

\[ \Omega_3 := \int_{C_{L+K+K^\xi}}^{B} \frac{\rho(y)}{x-y} \, dy - \frac{1}{N} \sum_{j=L+K+K^\xi}^{N} \frac{1}{x - \gamma_j} \]

where \([A, B]\) is the support of the density \( \rho \).

To estimate \( \Omega_1 \), we use Taylor expansion

\[ \rho(y) = \rho(x) + O(|x-y|) \]

For definiteness we again assume that \( d(x) = x - y_{L-K-1} \), and use that on \( R_{L,K} \) we have

\[ \gamma_{L-K-K^\xi} \leq y_{L-K-1} \leq x \leq y_{L+K+1} \leq \gamma_{L+K+K^\xi} \]

We thus obtain

\[ \Omega_1 = \int_{C_{L-K-K^\xi}}^{C_{L+K+K^\xi}} \frac{\rho(y)+O(|x-y|)}{x-y} \, dy \]

\[ = \rho(x) \log \frac{\gamma_{L+K+K^\xi}-x}{x-\gamma_{L-K-K^\xi}} + O(K/N) \]

\[ = \rho(y) \log \frac{d_+(x)}{d_-(x)} + O(KN^{-1+\xi}). \quad (A.4) \]

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In the first step above we computed the leading term of the integral, while the other term was estimated trivially using that the integration length is \( \gamma_{L+K+K^\xi} - \gamma_{L-K-K^\xi} = O(K/N) \). In the second step we used that \( \varrho \in C^1 \) away the edge, i.e. \( \varrho(x) = \varrho(\bar{y}) + O(K/N) \). To estimate the logarithm, we used

\[
\begin{align*}
\gamma_{L+K+K^\xi} - x &= (\gamma_{L+K+K^\xi} - \gamma_{L+1}) + (\gamma_{L+1} - y_{L+K+1}) + (y_{L+K+1} - x) \\
&= \varrho(\bar{y}) N^{-1} K^\xi + O(N^{-1+\xi\delta/2}) + (y_{L+K+1} - x) \\
&= d_+(x) + O(N^{-1+\xi\delta/2})
\end{align*}
\]

and the similar relation

\[
x - \gamma_{L-K-K^\xi} = d_-(x) + O(N^{-1+\xi\delta/2}).
\]

Notice that the error term in (A.4) is smaller than the target estimate \( K^\xi/(Nd(x)) \) since \( d(x) \leq K/N \ll K^{-1+\xi}N^{-\xi} \).

Now we estimate the \( \Omega_2 \) term; \( \Omega_3 \) can be treated analogously. We can write (with the convention \( \gamma_0 = A \))

\[
|\Omega_2| = \left| \sum_{j=1}^{L-K-K^\xi} \int_{\gamma_{j-1}}^{\gamma_j} \varrho(y) \left[ \frac{1}{x-y} - \frac{1}{x-\gamma_j} \right] dy \right| \\
\leq C \sum_{j=1}^{L-K-K^\xi} (\gamma_j - \gamma_{j-1}) \int_{\gamma_{j-1}}^{\gamma_j} \varrho(y) \frac{dy}{(x-y)^2} \\
\leq CN^{-1} \int_{A+\kappa}^{A+K} \frac{dy}{(x-y)^2} + CN^{-2/3} \int_A^{A+\kappa} \frac{dy}{(x-y)^2} \\
\leq \frac{CN^{-1}}{d(x)}.
\]

In the first step we used that

\[
\int_{\gamma_{j-1}}^{\gamma_j} \varrho(y) = \frac{1}{N}
\]

from (2.15). In the second step we used that \( \gamma_j - \gamma_{j-1} = O_\kappa(N^{-1}) \) in the bulk, i.e. for \( \gamma_j \geq A + \kappa \), and \( \max_j (\gamma_j - \gamma_{j-1}) = O(N^{-2/3}) \) (the order \( N^{-2/3} \) comes from the fact that the density \( \rho \) vanishes as a square root at the endpoints). The parameter \( \kappa = \kappa_\alpha \) is chosen such that \( A + 2\kappa \leq y_{L-K-1} \) which can be achieved since \( L \geq \alpha N \) and \( y_{L-K-1} \) is close to \( \gamma_{L-K-1} \). In the very last step we absorbed the \( N^{-2/3} \) error term into \( (Nd(x))^{-1} \geq K^{-1} \gg N^{-2/3} \).

Finally we prove (4.24). Since \( |y_j - \gamma_j| \leq K^\xi/N \), it follows that \( |x - y_j| \sim |x - \gamma_j| \) for \( |x - \gamma_j| \geq K^\xi/N \). Thus we have

\[
V''_x(x) = V''(x) + \frac{2}{N} \sum_{j \notin I} \frac{1}{(x-y_j)^2} \geq \inf V'' + \frac{c}{N} \sum_{j \notin I} \frac{1}{(x-\gamma_j)^2} \geq \inf V'' + \frac{c}{d(x)},
\]

with some positive constant \( c \) (depending only on \( \alpha \)). In estimating the summation, we used that the sequence \( \gamma_k \) is regularly spaced with gaps of order \( 1/N \). This completes the proof of Lemma 4.5.
B Discrete Gagliardo-Nirenberg inequalities

Recall the integral formula for quadratic form of the operator $(-\Delta)^{s/2}$ in $\mathbb{R}$ for any $s \in (0, 2)$:

$$
\int_{\mathbb{R}} \phi(x) ((-\Delta)^{s/2} \phi)(x) \, dx = C(s) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{1+s}} \, dx \, dy,
$$

where $C(s)$ is an explicit positive constant, $C(1) = (2\pi)^{-1}$ and $\phi \in H^{s/4}(\mathbb{R})$. We have the following Gagliardo-Nirenberg type inequality in the critical case (see (1.4) of [47] with the choice of parameters $n = 1, p = 4$)

$$
\|\phi\|^4 \leq C \|\phi\|^2 \int_{\mathbb{R}} \phi(x) (\sqrt{-\Delta}) \phi(x) \, dx, \quad \phi : \mathbb{R} \to \mathbb{R}.
$$

We first give a slight generalization of this inequality:

**Proposition B.1** Let $p \in (2, \infty)$ and $s \in (1 - \frac{2}{p}, 2)$. Then we have

$$
\|\phi\|_p \leq C_{p, s} \|\phi\|_2^{1 - \frac{2s}{p}} \left[ \int_{\mathbb{R}} \phi(x) ((-\Delta)^{s/2} \phi)(x) \, dx \right]^{\frac{2s}{p}}
$$

with some constant $C_{p, s}$ with $\| \cdot \|_p = \| \cdot \|_{L_p(\mathbb{R})}$.

**Proof.** We follow the proof of Theorem 2 in [47]. Setting $q = p/(p-1)$ and using Hausdorff-Young and Hölder inequalities for any $\lambda > 0$ and $\alpha > 1 - \frac{2}{p}$

$$
\|\phi\|_p \leq C_p \|\phi\|_q \leq C_p \|\hat{\phi}\|_2 (\lambda + |\xi|)^{\alpha/q} \|\lambda + |\xi|\|^{-\alpha/q} \|\phi\|_2 \|\phi\|_{2q/(2-q)}
$$

$$
\leq C_{p, \alpha} \left( \lambda^{\alpha/q} \right) \|\phi\|_2 \left( \|\phi\|_{2q/(2-q)} \right)
$$

$$
\leq C_{p, \alpha} \|\phi\|_2 \left( \|\phi\|_{2q/(2-q)} \right)
$$

where in the last step we chose $\lambda = (\phi, |p|^{2\alpha/q}q/2\alpha)\|\phi\|^{-q/\alpha}$. We used $\langle \cdot, \cdot \rangle$ to denote the inner product in $L^2(\mathbb{R})$. Setting $s = 2\alpha/q$, we obtain (B.3).

Now we derive the discrete version of this inequality.

**Proposition B.2** Let $p \in (2, \infty)$ and $s \in (1 - \frac{2}{p}, 2)$. Then there exists a positive constant $C_{p, s}$ such that

$$
\|f\|_p \leq C_{p, s} \|f\|_2^{1 - \frac{2s}{p}} \left[ \sum_{i \neq j \in \mathbb{Z}} \frac{|f_i - f_j|^2}{|i-j|^{1+s}} \right]^{\frac{2s}{p}}
$$

holds for any function $f : \mathbb{Z} \to \mathbb{R}$, where $\|f\|_p = \|f\|_{L_p(\mathbb{Z})} = \left( \sum_i |f_i|^p \right)^{1/p}$.

**Proof.** Given $f : \mathbb{Z} \to \mathbb{R}$, let $\phi : \mathbb{R} \to \mathbb{R}$ be its linear interpolation, i.e. $\phi(i) := f_i$ for $i \in \mathbb{Z}$ and

$$
\phi(x) = f_i + (f_{i+1} - f_i)(x - i) = f_{i+1} - (f_{i+1} - f_i)(i+1-x), \quad x \in [i, i+1).
$$

It is easy to see that

$$
C_p^{-1} \|\phi\|_{L_p(\mathbb{R})} \leq \|f\|_{L_p(\mathbb{Z})} \leq C_p \|\phi\|_{L_p(\mathbb{R})}, \quad 2 \leq p \leq \infty,
$$

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with some constant $C_p$. We claim that
\[\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{1+s}} \, dx \, dy \leq C_s \sum_{i \neq j \in \mathbb{Z}} \frac{|f_i - f_j|^2}{|i - j|^{1+s}}, \tag{B.8} \]

with some constant $C_s$. Then (B.7) and (B.8) will yield (B.5) from (B.2).

To prove (B.8), we can write
\[\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{1+s}} \, dx \, dy = \sum_{i,j} \int_{i}^{i+1} \int_{j}^{j+1} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{1+s}} \, dx \, dy. \tag{B.9} \]

Using the explicit formula (B.6), we first compute the $i = j$ terms in (B.9):
\[\sum_i \int_i^{i+1} \int_i^{i+1} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{1+s}} \, dx \, dy \tag{B.10} \]

\[= \sum_i |f_i - f_{i+1}|^2 \int_i^{i+1} \int_i^{i+1} \frac{dx \, dy}{|x - y|^s} = C_s \sum_i |f_i - f_{i+1}|^2 / |i - (i + 1)|^{1+s} \]

with some explicit $C_s$. Next we compute the terms $|i - j| = 1$ in (B.9). We assume $j = i - 1$, the terms $j = i + 1$ are analogous:
\[\sum_i \int_i^{i+1} \int_{i-1}^{i} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{1+s}} \, dx \, dy \tag{B.11} \]

\[\leq \sum_i (f_{i+1} - f_i)^2 \int_i^{i+1} \int_{i-1}^{i} \frac{(x - i)^2}{|x - y|^{1+s}} \, dx \, dy + \sum_i (f_i - f_{i-1})^2 \int_i^{i+1} \int_{i-1}^{i} \frac{(i - y)^2}{|x - y|^{1+s}} \, dx \, dy, \]

where we used $\phi(x) = f_i + (f_{i+1} - f_i)(x - i)$ and $\phi(y) = f_i - (f_i - f_{i-1})(i - y)$. The above integrals are finite constants $C_s$, so we get
\[\sum_i \int_i^{i+1} \int_{i-1}^{i} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^2} \, dx \, dy \leq C_s \sum_i \frac{(f_{i+1} - f_i)^2}{(i + 1 - i)^{1+s}} + \frac{(f_i - f_{i-1})^2}{(i - (i - 1))^{1+s}}. \]

Finally, for the terms $|i-j| \geq 2$, we can just replace $(x - y)^{1+s}$ by $(i - j)^{1+s}$ in the right hand side of (B.9) and use simple Schwarz inequalities to get
\[\int_i^{i+1} \int_{j}^{j+1} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{1+s}} \, dx \, dy \leq C_s \frac{|f_i - f_j|^2 + |f_{i+1} - f_i|^2 + |f_{j+1} - f_j|^2}{|i - j|^{1+s}}. \]

After summing up we get
\[\sum_{|i-j| \geq 2} \int_i^{i+1} \int_{j}^{j+1} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{1+s}} \, dx \, dy \leq C_s \sum_{|i-j| \geq 2} \frac{|f_i - f_j|^2}{|i - j|^{1+s}} + C_s \sum_i \frac{|f_{i+1} - f_i|^2}{((i + 1) - i)^{1+s}}. \tag{B.12} \]

The estimates (B.10), (B.11) and (B.12) together yield (B.8).

With two fixed parameters $a, b > 0$, define the function
\[m(\xi) := |\xi| \cdot 1(|\xi| \leq a) + b|\xi|, \quad \xi \in \mathbb{R}. \tag{B.13} \]
We will consider the operator $T = m(\sqrt{-\Delta})$ defined by $m$ being its Fourier multiplier, i.e.

$$\hat{T}\phi(\xi) = m(\xi)\hat{\phi}(\xi).$$

**Proposition B.3** We have

$$\|\phi\|^4_4 \leq C\|\phi\|^2_2\|m(\sqrt{-\Delta})\phi\| + \frac{C}{ab^3}\|\phi\|^4_\infty. \tag{B.14}$$

**Proof.** Let $\chi \in C_0^\infty(\mathbb{R})$ be a symmetric cutoff function such that $0 \leq \chi \leq 1$, $\chi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\chi(\xi) = 0$ for $|\xi| \geq 1$. Set $\chi_a(\xi) = \chi(\xi/a)$. Split $\phi = \phi_1 + \phi_2$ into low and high Fourier modes, the decomposition is defined via their Fourier transforms,

$$\phi = \phi_1 + \phi_2, \quad \hat{\phi}_1(\xi) = \hat{\phi}(\xi)\chi_a(\xi), \quad \hat{\phi}_2(\xi) = \hat{\phi}(\xi)(1 - \chi_a(\xi)).$$

First we estimate the contribution from $\phi_1$. With the choice of $p = 4$, $s = 1$ in (B.3) we have

$$\|\phi_1\|_4 \leq C\|\phi_1\|_2^{1/2}\left[\int_{\mathbb{R}} |\hat{\phi}_1(\xi)|^2\|\xi||d\xi\right]^{1/4} \leq C\|\phi_1\|_2^{1/2}\langle\phi_1, m(\sqrt{-\Delta})\phi_1\rangle^{1/4} \leq C\|\phi\|_2^{1/2}\langle\phi, m(\sqrt{-\Delta})\phi\rangle^{1/4},$$

where we used that on the support of $\hat{\phi}_1$ we have $|\xi| \leq m(\xi)$ and in the last step we used $|\hat{\phi}_1| \leq |\hat{\phi}|$ pointwise.

For the contribution of $\phi_2$, with any $\delta > 0$ we have

$$\|\phi_2\|_4 \leq \|\phi_2\|^1_4 \|\phi_2\|^3_3 \leq \delta^{-4}\|\phi_2\|_\infty + \delta^4/3\|\phi_2\|_3.$$

In the first term we use the Littlewood-Paley inequality

$$\|\phi_2\|_\infty \leq C\|\phi\|_\infty,$$

where $C$ depends only on the choice of $\chi$ but is independent of $a$. In the second term we use (B.3) with $s = 2/3$, $p = 3$:

$$\|\phi_2\|_3 \leq C\|\phi_2\|^{1/2}\left[\int_{\mathbb{R}} |\hat{\phi}_2(\xi)|^2\|\xi|^{2/3}d\xi\right]^{1/4} \leq Cb^{-1/4}a^{-1/12}\|\phi_2\|^{1/2}\|\phi_2, m(\sqrt{-\Delta})\phi_2\|^{1/4},$$

where in the second step we used $|\xi|^2/3 \leq 2b^{-1}a^{-1/3}m(\xi)$ for all $|\xi| \geq a/2$, i.e. on the support of $\hat{\phi}_2$. Using $|\hat{\phi}_2| \leq |\hat{\phi}|$, we thus have

$$\|\phi_2\|_4 \leq C\delta^{-4}\|\phi\|_\infty + C\delta^{4/3}b^{-1/4}a^{-1/12}\|\phi\|^1_2\langle\phi, m(\sqrt{-\Delta})\phi\rangle^{1/4}.$$ 

Choosing $\delta = b^{3/16}a^{1/16}$, we obtain (B.14). \qed

Finally, we derive a discrete version and a localized discrete version of Proposition B.3.

**Proposition B.4** Let $\{B_{ij} : i \neq j \in \mathbb{Z}\}$ be a bi-infinite matrix of nonnegative numbers with $B_{ij} = B_{ji}$.

(i) [Global version] Assume that for some positive constants $a, b, r$ with $b \leq r \leq 1$, we have

$$B_{ij} \geq \frac{b}{|i - j|^r}, \quad \forall i \neq j \in \mathbb{Z}$$

(B.15)
and
\[ B_{ij} \geq \frac{r}{|i-j|^2}, \quad \forall i, j \in \mathbb{Z} \text{ with } |i-j| \geq a^{-1}. \] (B.16)

Then for any function \( f : \mathbb{Z} \to \mathbb{R} \) we have
\[ \|f\|_4^4 \leq \frac{C}{r} \|f\|_2^2 \sum_{i \neq j} B_{ij} |f_i - f_j|^2 + \frac{C}{a b^3} \|f\|_\infty^4. \] (B.17)

(ii) [Local version] Let \( I = [Z - L,Z + L] \subset \mathbb{Z} \) be a subinterval of length \( |I| = 2L + 1 \) around \( Z \in \mathbb{Z} \) and let \( \tilde{I} := [Z - (1+\tau)L,Z + (1+\tau)L] \subset \mathbb{Z} \) be a slightly larger interval, where \( \tau > 0 \). Assume that for some positive constants \( a, b, r \) with \( b \leq r \leq 1 \), we have
\[ B_{ij} \geq \frac{b}{|i-j|^2}, \quad \forall i, j \in \tilde{I} \] (B.18)
and
\[ B_{ij} \geq \frac{r}{|i-j|^2}, \quad \forall i, j \in \tilde{I} \text{ with } |i-j| \geq a^{-1}. \] (B.19)

Then for any function \( f : \mathbb{Z} \to \mathbb{R} \) with \( \text{supp}(f) \subset I \) we have
\[ \|f\|_4^4 \leq C \|f\|_2^2 \frac{1}{r} \sum_{i \neq j \in I} B_{ij} |f_i - f_j|^2 + \frac{C}{a b^3} \|f\|_\infty^4. \] (B.20)

**Proof.** Following the proof of Proposition B.2, for any \( f : \mathbb{Z} \to \mathbb{R} \) we define its continuous extension \( \phi \) by (B.6). Then the combination of (B.7) and (B.14) yields
\[ \|f\|_4^4 \leq C \|f\|_2^2 \langle \phi, m(\sqrt{-\Delta}) \phi \rangle + \frac{C}{a b^3} \|f\|_\infty^4, \]
where \( m \) is given in (B.13) and \( a, b \) will be determined later. We compute
\[ \langle \phi, m(\sqrt{-\Delta}) \phi \rangle \leq b \langle \phi, \sqrt{-\Delta} \phi \rangle + \langle \phi, \sqrt{-\Delta} \chi_{2a}^2(\sqrt{-\Delta}) \phi \rangle, \] (B.21)
where we used that \( 1(|\xi| \leq a) \leq \chi_{2a}^2(\xi) \) by the definition of \( \chi \) at the beginning of the proof of Proposition B.3. The first term is bounded by
\[ b \langle \phi, \sqrt{-\Delta} \phi \rangle = b \int_\mathbb{R} \int_\mathbb{R} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^2} \, dx \, dy \leq C b \sum_{i<j} \frac{|f_i - f_j|^2}{|i-j|^2} \leq \sum_{i<j} B_{ij} |f_i - f_j|^2 \] (B.22)
using (B.8) in the first estimate and (B.15) in the second one. In the second term in (B.21) we use the trivial arithmetic inequality
\[ |\xi| \chi_{2a}^2(\xi) \leq Q(\xi) \quad \text{with} \quad Q(\xi) := 100a(1 - e^{-|\xi|/a}). \]
Thus
\[ \langle \phi, \sqrt{-\Delta} \chi_{2a}^2(\sqrt{-\Delta}) \phi \rangle \leq \int_\mathbb{R} |\hat{\phi}(\xi)|^2 Q(\xi) \, d\xi = 50 \int_\mathbb{R} \int_\mathbb{R} \frac{|\phi(x) - \phi(y)|^2}{(x-y)^2 + a^{-2}} \, dx \, dy. \]
Mimicking the argument leading to (B.8), we can continue this estimate
\[ \langle \phi, \sqrt{-\Delta} \chi_{2a}^2(\sqrt{-\Delta}) \phi \rangle \leq C \sum_{i \neq j} \frac{|f_i - f_j|^2}{|i-j|^2 + a^{-2}} \leq \frac{C}{r} \sum_{i \neq j} B_{ij} |f_i - f_j|^2, \] (B.23)
where we used (B.16) in the last step. This completes the proof of (B.17).

The proof of (B.20) is very similar, just in the very last estimates of (B.22) and (B.23) we use that $f$ is supported in $\mathcal{I}$. E.g. in (B.22) we have

$$b \sum_{i<j} \frac{|f_i - f_j|^2}{|i-j|^2} = b \sum_{i<j \in \hat{\mathcal{I}}} \frac{|f_i - f_j|^2}{|i-j|^2} + 2b \sum_{i \in \mathcal{I}} |f_i|^2 \sum_{j \notin \hat{\mathcal{I}}} \frac{1}{|i-j|^2} \leq \sum_{i<j} B_{ij} |f_i - f_j|^2 + \frac{2}{L_T} \|f\|_2^2,$$

and the estimate in (B.23) is analogous.

\[ \square \]

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