Corona Theorem for $H^\infty$ on Coverings of Riemann Surfaces of Finite Type

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Abstract

In this paper continuing our work started in [Br1]-[Br3] we prove the corona theorem for the algebra of bounded holomorphic functions defined on an unbranched covering of a Caratheodory hyperbolic Riemann surface of finite type.

1. Introduction.

1.1. Let $X$ be a complex manifold and let $H^\infty(X)$ be the Banach algebra of bounded holomorphic functions on $X$ equipped with the supremum norm. We assume that $X$ is Caratheodory hyperbolic, that is, the functions in $H^\infty(X)$ separate the points of $X$. The maximal ideal space $\mathcal{M} = \mathcal{M}(H^\infty(X))$ is the set of all nonzero linear multiplicative functionals on $H^\infty(X)$. Since the norm of each $\phi \in \mathcal{M}$ is $\leq 1$, $\mathcal{M}$ is a subset of the closed unit ball of the dual space $(H^\infty(X))^\ast$. It is a compact Hausdorff space in the Gelfand topology (i.e., in the weak * topology induced by $(H^\infty(X))^\ast$). Further, there is a continuous embedding $i : X \hookrightarrow \mathcal{M}$ taking $x \in X$ to the evaluation homomorphism $f \mapsto f(x)$, $f \in H^\infty(X)$. The complement to the closure of $i(X)$ in $\mathcal{M}$ is called the corona. The corona problem is: given $X$ to determine whether the corona is empty. For example, according to Carleson’s celebrated Corona Theorem [C] this is true for $X$ being the open unit disk in $\mathbb{C}$. (This was conjectured by Kakutani in 1941.) Also, there are non-planar Riemann surfaces for which the corona is non-trivial (see, e.g., [JM], [G], [BD], [L] and references therein). The general problem for planar domains is still open, as is the problem in several variables for the ball and polydisk. (In fact, there are no known examples of domains in $\mathbb{C}^n$, $n \geq 2$, without corona.) At present, the strongest corona theorem for planar

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domains is due to Jones and Garnett [GJ]. It states that the corona is empty for any Denjoy domain, i.e., a domain of the form \( \mathbb{C} \setminus E \) where \( E \subset \mathbb{R} \).

The corona problem has the following analytic reformulation, see, e.g., [Ga]:

A collection \( f_1, \ldots, f_n \) of functions from \( H^\infty(X) \) satisfies the corona condition if
\[
1 \geq \max_{1 \leq j \leq n} |f_j(x)| \geq \delta > 0 \quad \text{for all} \quad x \in X. \tag{1.1}
\]

The corona problem being solvable (i.e., the corona is empty) means that the Bezout equation
\[
f_1g_1 + \cdots + f_ng_n \equiv 1 \tag{1.2}
\]
has a solution \( g_1, \ldots, g_n \in H^\infty(X) \) for any \( f_1, \ldots, f_n \) satisfying the corona condition.

We refer to \( \max_{1 \leq j \leq n} \|g_j\|_\infty \) as a “bound on the corona solutions”. (Here \( \| \cdot \|_\infty \) is the norm on \( H^\infty(X) \).)

The present paper is concerned with the corona problem for coverings of Riemann surfaces of finite type. Let us recall that a Riemann surface \( Y \) is of finite type if the fundamental group \( \pi_1(Y) \) is finitely generated. Our main result extends the class of Riemann surfaces for which the corona theorem is true:

**Theorem 1.1** Let \( r : X \to Y \) be an unbranched covering of a Carathéodory hyperbolic Riemann surface of finite type \( Y \). Then \( X \) is Carathéodory hyperbolic and for any \( f_1, \ldots, f_n \in H^\infty(X) \) satisfying (1.1) there are solutions \( g_1, \ldots, g_n \in H^\infty(X) \) of (1.2) with the bound \( \max_{1 \leq j \leq n} \|g_j\|_\infty \leq C(Y, n, \delta) \).

This result, in a sense, completes our work started in [Br1]-[Br3] on the corona problems on coverings of certain Riemann surfaces. Similarly to [Br1]-[Br3] the methods used in the present paper are based on \( L^2 \) cohomology technique on complete Kähler manifolds and Cartan’s \( A \) and \( B \) theorems for coherent Banach sheaves on Stein manifolds.

**Remark 1.2**  (1) Note that the assumption of the Carathéodory hyperbolicity of \( Y \) cannot be removed: It follows from the results of Lárusson [L] and the author [Br3] that for any integer \( n \geq 2 \) there are a compact Riemann surface \( S_n \) and its regular covering \( r_n : \tilde{S}_n \to S_n \) such that

(a) \( \tilde{S}_n \) is a complex submanifold of an open Euclidean ball \( \mathbb{B}_n \subset \mathbb{C}^n \);

(b) the embedding \( i : \tilde{S}_n \hookrightarrow \mathbb{B}_n \) induces an isometry \( i^* : H^\infty(\mathbb{B}_n) \to H^\infty(\tilde{S}_n) \).

In particular, (b) implies that the maximal ideal spaces of \( H^\infty(\tilde{S}_n) \) and \( H^\infty(\mathbb{B}_n) \) coincide. Thus the corona problem is not solvable for \( H^\infty(S_n) \).

(2) Under the assumptions of the theorem, let \( U \hookrightarrow X \) be a domain such that the embedding induces an injective homomorphism of the corresponding fundamental groups and \( r(U) \subset \subset Y \). Then as was shown in [Br1] and [Br2] in this case the following extension of Theorem 1.1 is valid:

**Theorem 1.3** Let \( A = (a_{ij}) \) be an \( n \times k \) matrix, \( k < n \), with entries in \( H^\infty(U) \). Assume that the family of minors of order \( k \) of \( A \) satisfies the corona condition. Then there is an \( n \times n \) matrix \( \tilde{A} = (\tilde{a}_{ij}) \), \( \tilde{a}_{ij} \in H^\infty(U) \), so that \( \tilde{a}_{ij} = a_{ij} \) for \( 1 \leq j \leq k, 1 \leq i \leq n \), and \( \det \tilde{A} = 1 \).
The proof of the theorem is based on a Forelli type theorem on projections in $H^\infty$ discovered in [Br1] and a Grauert type theorem for “holomorphic“ vector bundles on maximal ideal spaces (which are not usual manifolds) of certain Banach algebras proved in [Br2]. In a forthcoming paper we prove a result similar to Theorem 1.3 for matrices with entries in $H^\infty(X)$ with $X$ satisfying assumptions of Theorem 1.1. The techniques are necessarily more complicated than those used in this paper.

(3) The remarkable class of Riemann surfaces $X$ for which a Forelli type theorem is valid was introduced by Jones and Marshall [JM]. The definition is in terms of an interpolating property for the critical points of the Green function on $X$. For such $X$ the corona problem is solvable, as well. Moreover, every $X$ from this class is of Widom type, see [W] for the corresponding definition. (Roughly speaking, this means that the topology of $X$ grows slowly as measured by the Green function.) It is an interesting open question whether the surfaces $X$ in Theorem 1.1 are also of Widom type.

(4) Similarly to [JM] and [Br1] our proof of Theorem 1.1 uses the Carleson Corona Theorem for the open unit disk.

1.2. In this part we formulate some results used in the proof of Theorem 1.1. First, we recall the following

**Definition 1.4** Let $X$ be a complex manifold. A sequence $\{x_j\}_{j \in \mathbb{N}} \subset X$ is called interpolating for $H^\infty(X)$ if for every bounded sequence of complex numbers $a = \{a_j\}_{j \in \mathbb{N}}$ there is an $f \in H^\infty(X)$ so that $f(x_j) = a_j$ for all $j$. The constant of interpolation for $\{x_j\}_{j \in \mathbb{N}}$ is defined as

$$\sup_{\|a\|_{\infty} \leq 1} \inf \{\|f\|_{\infty} : f \in H^\infty(X), f(x_j) = a_j, \, j \in \mathbb{N}\}$$

(1.3)

where $\|a\|_{\infty} := \sup_{j \in \mathbb{N}} |a_j|$.

Let $r : X \to Y$ be an unbranched covering of a Caratheodory hyperbolic Riemann surface of finite type $Y$. Let $K \subset Y$ be a compact subset.

**Theorem 1.5** For every $x \in K$ the sequence $r^{-1}(x) \subset X$ is interpolating for $H^\infty(X)$ with the constant of interpolation bounded by a number depending on $K$ and $Y$ only.

To formulate our next result used in the proof we will assume that $Y$ is equipped with a hermitian metric $h_Y$ with the associated $(1,1)$-form $\omega_Y$. Then we equip $X$ with the hermitian metric $h_X$ induced by the pullback $r^*\omega_Y$ of $\omega_Y$ to $X$. Now, if $\eta$ is a smooth differential $(0,1)$-form on $X$, by $|\eta|_z$, $z \in X$, we denote the norm of $\eta$ at $z$ defined by the hermitian metric $h_X^*$ on the fibres of the cotangent bundle $T^*X$ on $X$. We say that $\eta$ is bounded if

$$\|\eta\| := \sup_{z \in X} |\eta|_z < \infty,$$

(1.4)
Theorem 1.6 Let $\eta$ be a smooth bounded $(0,1)$-form on $X$ with support $\text{supp } \eta$ satisfying $r(\text{supp } \eta) \subset K$ for some compact subset $K \subset \subset Y$. Then the equation $\bar{\partial}f = \eta$ has a smooth bounded solution $f$ on $X$ such that

$$||f||_{L^\infty} := \sup_{z \in X} |f(z)| \leq C||\eta||$$

(1.5)

with $C$ depending on $K$, $Y$ and $h_Y$ only.

Remark 1.7 The methods used in the proofs of Theorems 1.5, 1.6 can be applied also to prove similar results for holomorphic $L^p$-functions on unbranched coverings of certain Stein manifolds. We present these results in a forthcoming paper.

2. Auxiliary Results.

In this part we collect some auxiliary results used in the proofs.

2.1. Let $Y$ be a Caratheodory hyperbolic Riemann surface of finite type. According to the theorem of Stout [St, Th.8.1], there exist a compact Riemann surface $R$ and a holomorphic embedding $\phi : Y \to R$ such that $R \setminus \phi(Y)$ consists of finitely many closed disks with analytic boundaries together with finitely many isolated points. Since $Y$ is Caratheodory hyperbolic, the set of the disks in $R \setminus \phi(Y)$ is not empty. Also, without loss of generality we may and will assume that the set of isolated points in $R \setminus \phi(Y)$ is not empty, as well. (For otherwise, $\phi(Y)$ is a bordered Riemann surface and the required results follow from [Br1].) We will naturally identify $Y$ with $\phi(Y)$. Also, we set

$$R \setminus Y := \left( \bigcup_{1 \leq i \leq k} D_i \right) \cup \left( \bigcup_{1 \leq j \leq l} \{x_j\} \right) \quad \text{and} \quad \tilde{Y} := Y \cup \left( \bigcup_{1 \leq j \leq l} \{x_j\} \right) \quad (2.1)$$

where each $D_i$ is biholomorphic to the open unit disk $\mathbb{D} \subset \mathbb{C}$ and these biholomorphisms are extended to diffeomorphisms of the closures $\overline{D_i} \to \overline{\mathbb{D}}$.

According to these definitions, $\tilde{Y}$ is a bordered Riemann surface, and there is a bordered Riemann surface $\tilde{Y}_1 \subset \subset R$ with $\tilde{Y} \subset \subset \tilde{Y}_1$ and $\pi_1(\tilde{Y}_1) \cong \pi_1(\tilde{Y})$.

2.2. Next, we introduce a complete Kähler metric on $Y_1 := \tilde{Y}_1 \setminus \left( \bigcup_{1 \leq j \leq l} \{x_j\} \right)$. To this end we consider an open cover $U = (U_j)_{0 \leq j \leq l}$ of $R$ such that for $1 \leq j \leq l$ $U_j \subset \subset \tilde{Y}$ is an open coordinate disk centered at $x_j$ and $U_0$ is a bordered Riemann surface which intersects each $U_j$, $1 \leq j \leq l$, by a set biholomorphic to an open annulus. Let $\{\rho_j\}_{0 \leq j \leq l}$ be a smooth partition of unity on $R$ subordinate to the cover $U$. By $z$ we denote a complex coordinate in $U_j$, $1 \leq j \leq l$, such that $z(x_j) = 0$ and $|z| < 1$. We set $f_j := \rho_j |z|^2$, $1 \leq j \leq l$. Then $f_j$ is a smooth nonnegative function on $R$ with $\text{supp } f_j \subset U_j$.

Now, we consider the positive smooth function

$$f := \frac{\rho_0}{2} + \sum_{j=1}^l f_j \quad (2.2)$$
on $R$ and determine the $(1,1)$-form $\tilde{\omega}$ on $R$ by

$$
\tilde{\omega} := -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\log f)^2.
$$

(2.3)

Since by the definition $0 < f < 1$, the form $\omega$ is well defined. Also, in an open neighbourhood of $x_j$, $1 \leq j \leq l$, the form $\omega$ is equal to

$$
\omega_P := \frac{\sqrt{-1}}{\pi} \frac{dz \wedge d\bar{z}}{|z|^2(\log |z|^2)^2} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\log |z|^2)^2.
$$

(2.4)

In the natural identification $U_j \setminus \{x_j\} = \mathbb{D} \setminus \{0\}$, $\omega_P$ coincides with the $(1,1)$-form of the Poincaré metric on the punctured disk.

Let $\omega_R$ be a Kähler $(1,1)$-form on the compact Riemann surface $R$ from section 2.1. Since $\bar{Y}_1$ is a bordered Riemann surface in $R$, there is a smooth plurisubharmonic function $f_R$ defined in a neighbourhood of the closure of $\bar{Y}_1$ such that

$$
\omega_R = \sqrt{-1} \cdot \partial \bar{\partial} f_R \quad \text{on} \quad \bar{Y}_1.
$$

(2.5)

Further, since $\bar{Y}_1$ is a Stein manifold, by the Narasimhan theorem [N] there is a holomorphic embedding $i : \bar{Y}_1 \hookrightarrow \mathbb{C}^3$ of $\bar{Y}_1$ as a closed complex submanifold of $\mathbb{C}^3$. By $\omega_e$ we denote the $(1,1)$-form on $\bar{Y}_1$ obtained as the pullback by $i$ of the Euclidean Kähler form $\sqrt{-1}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$ on $\mathbb{C}^3$ (here $z_1, z_2, z_3$ are complex coordinates on $\mathbb{C}^3$). Clearly, $\omega_e$ is a Kähler form on $\bar{Y}_1$ (i.e., it is positive on $\bar{Y}_1$ and $d$-closed).

**Lemma 2.1** There is a positive number $c_1$ depending on $Y_1$, $\omega_R$ and $\tilde{\omega}$ such that the $(1,1)$-form

$$
\omega := \tilde{\omega} + c_1(\omega_R + \omega_e)
$$

(2.6)

is a complete Kähler form on $Y_1$.

**Proof.** Since $\omega_R$ is a Kähler form on $R$, by the definition of $\tilde{\omega}$ there is a constant $c_1 > 0$ depending on $Y_1$, $\omega_R$ and $\tilde{\omega}$ such that

$$
\tilde{\omega} > -c_1 \omega_R \quad \text{on} \quad Y_1.
$$

Thus the form $\omega = \tilde{\omega} + c_1(\omega_R + \omega_e)$ is positive (and $d$-closed) on $Y_1$. Its completeness means that the path metric $d$ on $Y_1$ induced by $\omega$ is complete. Let us check this fact.

Assume, on the contrary, that $d$ is not complete. This means that there is a sequence $\{w_n\} \subset Y_1$ convergent either to the boundary of $\bar{Y}_1$ or to one of the points $x_j$, $1 \leq j \leq l$, such that $\{d(o, w_n)\}$ is bounded (for a fixed point $o \in Y_1$). Then, since $\omega \geq \omega_e$, the sequence $\{i(w_n)\} \subset \mathbb{C}^3$ is bounded. This implies that $\{w_n\}$ cannot converge to the boundary of $\bar{Y}_1$. Thus it converges to one of $x_j$. But since $\omega \geq \omega_P$ near $x_j$, the latter is impossible because the Poincaré metric on the punctured disk is complete. \quad \Box

2.3. According to our construction the embedding $Y \hookrightarrow Y_1$ induces an isomorphism of the corresponding fundamental groups. By the covering homotopy theorem
this implies that for any unbranched covering \( r : X \to Y \) there is an unbranched covering \( r_1 : X_1 \to Y_1 \) and an embedding \( j : X \hookrightarrow X_1 \) such that \( r_1 \circ j = r \) and \( j_* : \pi_1(X) \to \pi_1(X_1) \) is an isomorphism. Without loss of generality we identify \( j(X) \) with \( X \). Then \( r := r_1|_X \). Now the form \( r^*_1 \omega \) with \( \omega \) determined by (2.6) is a complete Kähler form on \( X_1 \).

Let \( TY_1 \) be the complex tangent bundle on \( Y_1 \) equipped with the hermitian metric induced by the Kähler form \( \omega_R \). Since \( TY_1 \) is the restriction to \( Y_1 \) of the tangent bundle \( TR \) on \( R \) with the hermitian metric defined by \( \omega_R \), the curvature \( \Theta_{Y_1} \) of \( TY_1 \) satisfies

\[
\Theta_{Y_1} \geq -c_2 \omega_R
\]

for some \( c_2 > 0 \) depending on \( Y_1 \) and \( \omega_R \). In turn, the curvature \( \Theta_{X_1} := r^*_1 \Theta_{Y_1} \) of the tangent bundle \( TX_1 \) on \( X_1 \) equipped with the hermitian metric induced by \( r^*_1 \omega_R \) satisfies

\[
\Theta_{X_1} \geq -c_2 r^*_1 \omega_R.
\]

Next, by the definition of \( \omega_e \) there is a smooth plurisubharmonic function \( g \) on \( \tilde{Y}_1 \) such that \( \omega_e = \sqrt{-1} \cdot \partial \bar{\partial} g \) (as such \( g \) one takes the pullback by \( i \) of the function \( |z|^2 := |z_1|^2 + |z_2|^2 + |z_3|^2 \) on \( \mathbb{C}^3 \)).

Let \( E_0 := X_1 \times \mathbb{C} \) be the trivial holomorphic line bundle on \( X_1 \). We equip \( E_0 \) with the hermitian metric \( e^{h_1-(c_1+c_2)g_1} \) where \( h_1 = r^*_1 h := \frac{\log(f^2)}{2\pi} \) with \( f \) from (2.2) and \( g_1 := r^*_1 (g + f \tilde{r}) \), see (2.5). (This means that for \( z \times v \in E \) the square of its norm in this metric equals \( e^{h_1(z)-(c_1+c_2)g_1(z)}|v|^2 \) where \( |v| \) is the modulus of \( v \in \mathbb{C} \).) Then by (2.8) the curvature \( \Theta_E \) of the bundle \( E := E_0 \otimes TX_1 \) satisfies

\[
\Theta_E := -\sqrt{-1} \cdot \partial \bar{\partial} \log e^{h_1-(c_1+c_2)g_1} + \Theta_{X_1} \geq r^*_1 \omega.
\]

2.4. Let \( X \) be a complete Kähler manifold of dimension \( n \) with a Kähler form \( \omega \) and \( E \) be a hermitian holomorphic vector bundle on \( X \) with curvature \( \Theta \). Let \( L^p,q(X, E) \) be the space of \( L^2 \) \( E \)-valued \( (p, q) \)-forms on \( X \) with the \( L^2 \) norm, and let \( W^p,q(X, E) \) be the subspace of forms such that \( \partial \eta \) is \( L^2 \). (The forms \( \eta \) may be taken to be either smooth or just measurable, in which case \( \partial \eta \) is understood in the distributional sense.) The cohomology of the resulting \( L^2 \) Dolbeault complex \((W^{p, q}, \overline{\partial})\) is the \( L^2 \) cohomology

\[
H^{p, q}_{(2)}(X, E) = Z^{p, q}_{(2)}(X, E)/B^{p, q}_{(2)}(X, E),
\]

where \( Z^{p, q}_{(2)}(X, E) \) and \( B^{p, q}_{(2)}(X, E) \) are the spaces of \( \overline{\partial} \)-closed and \( \overline{\partial} \)-exact forms in \( L^{p, q}_{(2)}(X, E) \), respectively.

If \( \Theta \geq \epsilon \omega \) for some \( \epsilon > 0 \) in the sense of Nakano, then the \( L^2 \) Kodaira-Nakano vanishing theorem, see [D], [O], states that

\[
H^{n, r}_{(2)}(X, E) = 0 \quad \text{for} \quad r > 0.
\]

Moreover, for \( \eta \in Z^{n, r}_{(2)}(X, E) \), \( r > 0 \), there is a form \( \tilde{\eta} \in W^{n, r-1}_{(2)}(X, E) \) such that \( \overline{\partial}\tilde{\eta} = \eta \) and

\[
||\tilde{\eta}||_2 \leq \frac{1}{\epsilon} ||\eta||_2.
\]
see [D, Rem. 4.2]. Here symbols \( \| \cdot \|_2 \) denote the corresponding \( L^2 \) norms.

We can apply this result to the bundle \( E \) from section 2.3 with \( X_1 \) equipped with the complete Kähler form \( r_1^* \omega \). Then from (2.9) we obtain

**Proposition 2.2** For every \( \eta \in W^{1,1}_2(X_1, E) \) there is \( \tilde{\eta} \in W^{1,0}_2(X_1, E) \) such that \( \bar{\partial} \tilde{\eta} = \eta \) and

\[
\| \tilde{\eta} \|_2 \leq \| \eta \|_2. \tag{2.12}
\]

2.5. Let \( T^* X_1 \) be the cotangent bundle on \( X_1 \) equipped with the hermitian metric induced by \( r_1^* \omega \). We consider the hermitian line bundle \( V := E \otimes T^* X_1 \) equipped with the tensor product of the corresponding hermitian metrics. Then from Proposition 2.2 we obtain

\[
H^{0,1}_2(X_1, V) \cong H^{1,0}_2(X_1, E) = 0 \tag{2.13}
\]

Moreover, for every \( \eta \in W^{0,1,1}_2(X_1, V) \) there is \( F \in W^{0,0,0}_2(X_1, V) \) such that \( \bar{\partial} F = \eta \) and

\[
\| F \|_2 \leq \| \eta \|_2. \tag{2.14}
\]

Further, there is a canonical isomorphism \( I : X_1 \times \mathbb{C} \to V \) defined in local coordinates \( z \) on \( X_1 \) by the formula

\[
I(z \times v) := v \cdot 1 \otimes \frac{\partial}{\partial z} \otimes dz. \tag{2.15}
\]

(Clearly this definition does not depend on the choice of local coordinates.) In what follows we identify \( V \) with \( X_1 \times \mathbb{C} \) by \( I \).

### 3. Proof of Theorem 1.5.

Below by \( A, B, C, c \) etc we denote constants depending on characteristics related to the sets \( Y, x \in Y \) and \( K \subset \subset Y \) but not on coverings \( X \) of \( Y \). (We will briefly say that they depend on \( Y, x \) and \( K \) only.) These constants may change from line to line and even in a single line.

3.1. Let \( x \in K \subset \subset Y \). We must check that the sequence \( r^{-1}(x) \subset X \) is interpolating for \( H^\infty(X) \) with the constant of interpolation bounded by a number depending on \( K \) and \( Y \) only. Fix a neighbourhood \( \hat{Y} \) of the closure of \( Y \) in \( Y_1 \) such that \( \hat{Y} \) is relatively compact in \( \hat{Y}_1 \), see section 2.1 for the corresponding definitions. First we will prove that \( r^{-1}(x) \) is interpolating for the space \( L^2_{\mathcal{O}}(\hat{X}_1; r_1^* \omega_R) \) of holomorphic \( L^2 \)-functions on the covering \( \hat{X}_1 := r_1^{-1}(\hat{Y}) \) of \( \hat{Y} \) with norm defined by integration with respect to \( r_1^* \omega_R \) (recall that \( X \subset X_1 \) and \( r_1|_X = r \)).

**Proposition 3.1** Let \( a \) be an \( l^2 \)-function on \( r^{-1}(x) \) with norm \( \| a \|_2 \). Then there is a function \( f \in L^2_{\mathcal{O}}(\hat{X}_1; r_1^* \omega_R) \) such that \( f|_{r^{-1}(x)} = a \) and

\[
\| f \|_2 \leq c \| a \|_2
\]

with \( c \) depending on \( K \) and \( Y \) only.
Proof. Using the fact that the closure of $\tilde{Y}_1$ in $R$ possesses a Stein neighbourhood and applying some basic results of the theory of Stein manifolds (see, e.g., [GR]) one obtains easily that there are a holomorphic function $\phi_x$ with a simple zero at $x$ defined in a fixed neighbourhood of the closure of $\tilde{Y}_1$, a simply connected coordinate neighbourhood $U_x \subset Y$ of $x$ with a complex coordinate $z$, $z(x) = 0$ and $|z| < 1$ on $U_x$, and positive numbers $A$ and $r$, $0 < r < 1$, depending on $x$ and $Y$ only such that

1. $\sup_{y \in \tilde{Y}_1} |\phi_x(y)| \leq A$;

2. $|\phi_x(y)| \geq \frac{1}{A}$ for all $y \in U_{x,r} := \{z \in U_x : |z| \geq r\}$;

3. $\phi_x(y) \neq 0$ for all $y \in U_x \setminus \{x\}$

(For a construction of such $\phi_x$ see, e.g., [Br2, Cor. 1.8].)

Further, there is a $C^\infty$-function $\rho_x$, $0 \leq \rho_x \leq 1$, on $U_x$ such that $\rho_x$ is equal to 1 on $U_x \setminus U_{x,r}$ and 0 outside $U_x$, and

$$|d\rho_x|_{z;\omega} \leq B \quad \text{for all} \quad z \in Y_1 \quad (3.1)$$

where $\{|\cdot|_{z;\omega} : z \in Y_1\}$ is the hermitian metric on the fibres of the cotangent bundle $T^*Y_1$ on $Y_1$ determined by the form $\omega$ (see (2.6)), and the constant $B$ depends on $x$ and $Y$ only.

Since $U_x$ is simply connected, $r_1^{-1}(U_x)$ is biholomorphic to $U_x \times S$ where $S$ is the fibre of $r_1$. In what follows without loss of generality we will identify these sets. Then $a$ is an $l^2$-function of $\{x\} \times S$. We extend $a$ to a locally constant function $\hat{a}$ on $U_x \times S$ by the formula

$$\hat{a}(z, s) := a(x, s) \quad \text{for all} \quad (z, s) \in U_x \times S. \quad (3.2)$$

Let us consider a $(0, 1)$ form $\hat{\eta}$ on $X_1$ determined by

$$\hat{\eta}(w) = \begin{cases} \frac{\hat{a}(z, s)d\rho_x(z)}{\phi_x(z)} & \text{if} \quad w = (z, s) \in U_x \times S \\ 0 & \text{if} \quad w \notin U_x \times S. \end{cases} \quad (3.3)$$

Next, we fix a noncompact neighbourhood $O$ of the closure $\tilde{Y}_1$ in $R$. Since $O$ is a one-dimensional Stein manifold, it is homotopically equivalent to a one-dimensional CW-complex (see, e.g., [GR]). In particular, any continuous vector bundle on $O$ is topologically trivial. Then by the Grauert theorem [Gr], any holomorphic vector bundle on $O$ is also trivial. Applying this to the bundle $T^*R|_O$ we find a nowhere vanishing holomorphic section $\lambda$ of $T^*R|_O$. (By the definition $\lambda$ is a holomorphic 1-form on $O$.) Then $\lambda^{-1} = \frac{1}{\lambda}$ is a nowhere vanishing holomorphic section of the
tangent bundle $TO$ on $O$. Moreover, there is a positive constant $C$ depending on $\bar{Y}_1$ such that
\[
\frac{1}{C} \leq |\lambda|_{z;\omega_R} \leq C, \quad \frac{1}{C} \leq |\lambda^{-1}|_{z;\omega_R} \leq C \quad \text{for all} \quad z \in \bar{Y}_1 \tag{3.4}
\]
where symbols $| \cdot |_{z;\omega_R}$ denote the corresponding norms on $T^*\bar{Y}_1$ and $T\bar{Y}_1$ at $z \in \bar{Y}_1$ determined by the form $\omega_R$.

Continuing the proof of the proposition consider the $V$-valued $(0,1)$-form $\eta := \hat{\eta} \wedge r_1^*\lambda \otimes r_1^*\lambda^{-1}$ on $X_1$. By $N_w$ and $N_w'$ we denote the hermitian norms on $V$ and $E$ (see section 2.3) at $w \in X_1$. Then we have
\[
N_w(\eta) := N_w'(\hat{\eta} \otimes r_1^*\lambda^{-1}) \cdot |r_1^*\lambda|_{w;\omega_R} = |\hat{\eta}|_{w;\omega_R} \cdot |r_1^*\lambda|_{w;\omega_R} \cdot e^{h_1(w)-(c_1+c_2)g_1(w)}. \tag{3.5}
\]
Here $|\hat{\eta}|_{w;\omega_R}$ and $|r_1^*\lambda|_{w;\omega_R}$ are determined by the form $r_1^*\omega$ and $|r_1^*\lambda^{-1}|_{w;\omega_R}$ is determined by the form $r_1^*\omega_R$. Since $\omega_R$ is equivalent to $\omega$ on $U_x \subset Y$, from (3.4) we obtain for all $w \in r_1^{-1}(U_x)$:
\[
|r_1^*\lambda|_{w;\omega_R} := |\lambda|_{r_1(w);\omega} \leq C_1 \quad \text{and} \quad |r_1^*\lambda^{-1}|_{w;\omega_R} := |\lambda^{-1}|_{r_1(w);\omega_R} \leq C \tag{3.6}
\]
where $C_1$ depends on $C$ and on the constant of the equivalence of $\omega_R$ and $\omega$ on $U_x$. Also, by the definition of $h_1$ and $g_1$, see section 2.3, there is a constant $C_2 > 0$ depending on $h_1, g_1, U_x$ and $Y_1$ such that
\[
e^{h_1(w)-(c_1+c_2)g_1(w)} \leq C_2 \quad \text{for all} \quad w \in r_1^{-1}(U_x). \tag{3.7}
\]
From here, (3.6), (3.5), (3.1)-(3.3) and the definition of $\phi_x$ we obtain
\[
N_w(\eta) \leq \bar{C}|\bar{a}(w)|. \tag{3.8}
\]
where $\bar{C} := ABCC_1C_2$, $\bar{a} := \bar{a} \cdot \chi_{r_1^{-1}(U_x)}$ and $\chi_{r_1^{-1}(U_x)}$ is the characteristic function of $r_1^{-1}(U_x)$. Using (3.8) we estimate the $L^2$ norm of $\eta$ as follows:
\[
||\eta||_2 := \left( \int_{w \in X_1} N_w^2(\eta)(r_1^*\omega)(w) \right)^{1/2} \leq \tilde{C}||a||_2 \left( \int_{y \in U_x} \omega(y) \right)^{1/2} \leq \tilde{C}||a||_2. \tag{3.9}
\]
Here $\tilde{C}$ depends on $x$ and $Y$ only.

Next, according to (2.14) there is $F \in W_{2,0}^0(X_1, V)$ such that $\mathcal{J}F = \eta$ and
\[
||F||_2 \leq ||\eta||_2 \leq \tilde{C}||a||_2. \tag{3.10}
\]
We regard $F$ as a function on $X_1$, see section 2.5.

Observe that since $\bar{Y}_1( \subset Y_1)$ is relatively compact in $\bar{Y}_1$, $g_1$ is bounded on $\bar{X}_1$ and $h_1$ is bounded from below on $\bar{X}_1$, see section 2 for the corresponding definitions.
Moreover, $\omega_R \leq \tilde{c} \cdot \lambda \wedge \overline{\lambda}$ on $\tilde{Y}_1$ for some $\tilde{c}$ depending on $\tilde{Y}_1$. These facts, (3.4) and (3.5) imply

$$
\left( \int_{\tilde{X}_1} |F|^2 \cdot r_1^* \omega_R \right)^{1/2} \leq C \left( \int_{w \in \tilde{X}_1} [N'_w(F \otimes r_1^* \lambda^{-1})]^2 (r_1^* \lambda \wedge r_1^* \overline{\lambda})(w) \right)^{1/2} = 
$$

$$
C \left( \int_{w \in \tilde{X}_1} N_w^2(F \cdot r_1^* \lambda \otimes r_1^* \lambda^{-1}) (r_1^* \omega)(w) \right)^{1/2} \leq (3.11)
$$

$$
C \left( \int_{w \in \tilde{X}_1} N_w^2(F \cdot r_1^* \lambda \otimes r_1^* \lambda^{-1}) (r_1^* \omega)(w) \right)^{1/2} := C||F||_2
$$

with $C$ depending on $Y$. Let us consider the function $f := \tilde{a} \cdot \rho_x - F \cdot r_1^* \phi_x$ on $\tilde{X}_1$. Then according to (3.10) and (3.11)

$$
\left( \int_{\tilde{X}_1} |f|^2 r_1^* \omega_R \right)^{1/2} \leq c||a||^2
$$

where $c$ depends on $Y$ and $x$ only. Moreover, from (3.3) it follows that $f$ is holomorphic and $f|_{r_1^{-1}(x)} = a$.

This shows that $r^{-1}(x)$ is interpolating for $L_2^2(\tilde{X}_1; r_1^* \omega_R)$.

To complete the proof of the proposition it remains to prove that for $x \in K \subset Y$ the constant of interpolation of $r^{-1}(x)$ with respect to $L_2^2(\tilde{X}_1; r_1^* \omega_R)$ is bounded by a number depending on $K$ and $Y$ only.

Let us consider the restriction map $R_x : L_2^2(\tilde{X}_1; r_1^* \omega_R) \to L_2^2(\tilde{X}_1; r_1^* \omega_R)|_{r^{-1}(x)}$.

**Lemma 3.2** $R_x$ maps $L_2^2(\tilde{X}_1; r_1^* \omega_R)$ continuously onto $l^2(r^{-1}(x))$. Moreover, the norm $||R_x||$ of $R_x$ is bounded by a constant depending on $Y$, $x$.

**Proof.** We will consider the coordinate neighbourhood $U_x \subset Y$ from the proof of Proposition 3.1 with a complex coordinate $z$ so that $z(x) = 0$, $|z| < 1$ on $U_x$. Also, we naturally identify $r_1^{-1}(U_x)$ with $U_x \times S$ where $S$ is the fibre of $r_1$. By definition there is a constant $C$ depending on $Y$, $U_x$ and $\omega_R$ such that

$$
\sqrt{-1} \cdot dz \wedge d\overline{z} \leq C \omega_R \quad \text{on} \quad U_x.
$$

(3.12)

Let $f \in L_2^2(\tilde{X}_1; r_1^* \omega_R)$. Then on $r^{-1}(x) = \{x\} \times S$ we have by the mean-value property for subharmonic functions

$$
\sum_{s \in S} |f(x, s)|^2 \leq \sum_{s \in S} \left( \frac{1}{\pi} \int_{U_x} |f(z, s)|^2 \sqrt{-1} \cdot dz \wedge d\overline{z} \right) \leq 
$$

$$
\frac{C}{\pi} \int_{U_x} \left( \sum_{s \in S} |f(z, s)|^2 \right) \omega_R(z) = \frac{C}{\pi} \int_{r_1^{-1}(U_x)} |f|^2 r_1^* \omega_R \leq \frac{C}{\pi} ||f||^2_{L_2^2}.
$$

(3.13)

This shows that $R_x$ maps $L_2^2(\tilde{X}_1; r_1^* \omega_R)$ continuously into $l^2(r^{-1}(x))$. Also, $R_x$ is surjective according to the first part of Proposition 3.1 proved above. □
Now, since \( R_x : L^2_0(\overline{X}_1; r_1^* \omega_R) \to l^2(r^{-1}(x)) \) is a linear continuous surjective map of Hilbert spaces, there is a linear continuous map \( T_x : l^2(r^{-1}(x)) \to L^2_0(\overline{X}_1; r_1^* \omega_R) \) such that \( R_x \circ T_x = id \). Let \( \{e_s\}_{s \in S}, e_s(x,t) = 0 \) for \( t \neq s \) and \( e_s(x,s) = 1 \), be an orthonormal basis of \( l^2(r^{-1}(x)) \). We set
\[
h_s := T_x(e_s) \in L^2_0(\overline{X}_1; r_1^* \omega_R).
\]
Then for a sequence \( a = \{a_s\}_{s \in S} \in l^2(S) \) we have
\[
h_a := \sum_{s \in S} a_s h_s \in L^2_0(\overline{X}_1; r_1^* \omega_R) \quad \text{and} \quad ||h_a||_2 \leq c ||a||_2. \tag{3.14}
\]
Further, for each \( y \in U_x \) by \( L_y : l^2(r^{-1}(y)) \to l^2(r^{-1}(x)) \) we denote the natural isomorphism that sends \( a(y,s) \in l^2(r^{-1}(y)) \) to \( a(x,s) \in l^2(r^{-1}(x)) \). Let us consider the map \( S_y := R_y \circ T_x \circ L_y : l^2(r^{-1}(y)) \to l^2(r^{-1}(y)) \) determined by the formula
\[
[S_y(a)](y,t) := \sum_{s \in S} a_s h_s(y,t), \quad (y,t) \in r^{-1}(y). \tag{3.15}
\]
Here \( a(y,t) = \sum_{s \in S} a_s e_s(y,t), t \in S \), and \( e_s(y, \cdot) \) are determined similarly to \( e_s(x, \cdot) \). Identifying \( a \in l^2(r^{-1}(y)) \) with \( \{a_s\}_{s \in S} \in l^2(S) \) we can regard, according to \( (3.15) \), \( \{S_y\}_{y \in U_x} \) as a family of bounded linear operators \( l^2(S) \to l^2(S) \) depending holomorphically on \( y \in U_x \). According to \( (3.14) \) and Lemma 3.2 there is a constant \( c' \) depending on \( U_x \) and \( Y \) such that
\[
||S_y|| \leq c' \quad \text{for all} \quad y \in U_x.
\]
Moreover, by our construction \( S_z = I \) where \( I : l^2(S) \to l^2(S) \) is the identity operator. Identifying \( U_x \) with \( \mathbb{D} \) by the coordinate \( z \), we obtain by the Cauchy integral formula for bounded holomorphic on \( \mathbb{D} \) functions:
\[
S_z := I + \sum_{k=1}^{\infty} S_k z^k \quad \text{for some} \quad S_k : l^2(S) \to l^2(S), \quad ||S_k|| \leq c'.
\]
In particular, for \( |z| < \frac{1}{2c'+4} \) we have
\[
\left| \sum_{k=1}^{\infty} S_k z^k \right| \leq c' \frac{|z|}{1-|z|} < \frac{2}{3}.
\]
Thus for every \( y \in U_x, |z(y)| < \frac{1}{2c'+4} \), the inverse operator \( S_y^{-1} \) exists and its norm is bounded by \( \frac{1}{|1-2/3|} = 3 \). Finally we set
\[
\hat{T}_y := T_x \circ L_y \circ S_y^{-1}, \quad y \in \hat{U}_x := \{y \in U_x : |z(y)| < \frac{1}{2c'+4}\}. \tag{3.16}
\]
Then by the definition we have
\[
R_y \circ \hat{T}_y = id \quad \text{for all} \quad y \in \hat{U}_x.
\]
This shows that \( \{ \tilde{T}_y : L^2(r^{-1}(y)) \to L^2_0(\tilde{X}_1; r_1^*\omega_R) : y \in \tilde{U}_x \} \) is a family of interpolation operators depending holomorphically on \( y \) such that \( \|\tilde{T}_y\| \leq 3c \). Taking a finite open cover of \( K \subset \subset Y \) by the sets \( \tilde{U}_x, x \in K \), and considering on these sets the interpolation operators \( \tilde{T}_y, y \in \tilde{U}_x \), we obtain that for every \( x \in K \) the constant of interpolation of \( r^{-1}(x) \) with respect to \( L^2_0(\tilde{X}_1; r_1^*\omega_R) \) is bounded by a number depending on \( K \) and \( Y \) only.

This completes the proof of the proposition. \( \square \)

3.2. Let us prove now that \( r^{-1}(x) \) is interpolating for \( H^\infty(X) \) with the constant of interpolation bounded by a number depending on \( K \) and \( Y \) only.

We will use the interpolation operators \( \hat{T}_y, y \in \tilde{U}_x \subset \subset U_x \), of the previous section. As before we set

\[
h_{s,y} := \hat{T}_y(e_s(y, \cdot)) \in L^2_0(\tilde{X}_1; r_1^*\omega_R).
\]

Then the family of functions \( \{ h_{s,y} : s \in S, y \in \tilde{U}_x \} \) depends holomorphically on \( y \).

Now for a sequence \( a = \{ a_s \}_{s \in S} \in L^2(S) \) we have

\[
h_{a,y} := \sum_{s \in S} a_s h_{s,y} \in L^2_0(\tilde{X}_1; r_1^*\omega_R) \quad \text{and} \quad \|h_{a,y}\|_2 \leq 3c|a|_2.
\] (3.17)

From here and Lemma 3.2 it follows that

\[
\sum_{z \in r^{-1}(w)} |h_{a,y}(z)|^2 \leq c|a|_2^2, \quad w \in Y, \ y \in \tilde{U}_x,
\] (3.18)

with \( c \) depending on \( w, x \) and \( Y \).

To continue the proof we require an extension of Lemma 3.2.

For its formulation we fix a holomorphic function \( \phi \) defined in a neighbourhood of the closure of \( Y_1 \subset \subset R \) having simple zeros at all points \( x_j, 1 \leq j \leq l \), see (2.1), and nonzero outside these points. (Such \( \phi \) exists, e.g., by [Br2, Cor. 1.8].) As before, by \( R_w : L^2_0(\tilde{X}_1; r_1^*\omega_R) \to L^2(r^{-1}(w)), w \in Y, \) we denote the restriction map (here \( r := r_1|_X \)).

**Lemma 3.3** There is a constant \( A > 0 \) depending on \( Y \) such that

\[
\|R_w\| \leq \frac{A}{|\phi(w)|}.
\]

**Proof.** Let \( U_j \subset \subset \tilde{Y} \) be a coordinate neighbourhood of \( x_j \) with a complex coordinate \( z \) so that \( z(x_j) = 0, |z| < 1 \) on \( U_j \). We have

\[
\sqrt{-1} \cdot dz \wedge d\bar{z} \leq C\omega_R \quad \text{on} \quad U_j
\]

for some \( C \) depending on \( \omega_R, z \) and \( Y \). Thus for any \( f \in L^2_0(\tilde{X}_1; r_1^*\omega_R) \) its restriction \( f|_{r_1^{-1}(U_j)} \) belongs to the \( L^2 \) space on \( r_1^{-1}(U_j) \) defined by integration with respect to the form \( r_1^*(\sqrt{-1} \cdot dz \wedge d\bar{z}) \), and the \( L^2 \) norm of the restriction is bounded by \( C\|f\|_2 \).

Next, for a point \( w \in \tilde{U}_j \setminus \{ x_j \}, \tilde{U}_j := \{ z \in U_j : |z| < 1/2 \} \), we set \( d := |z(w)| \) and \( D_w = \{ y \in U_j : |z(y) - z(w)| < d \} \). Then \( D_w \subset U_j \setminus \{ x_j \} \) and so \( r_1^{-1}(D_w) \) is
naturally identified with \( D_w \times S \). In this identification we have by the mean-value property for subharmonic functions:

\[
\sum_{s \in S} |f(w, s)|^2 \leq \frac{1}{\pi d^2} \int_{D_w} \left( \sum_{s \in S} |f(z, s)|^2 \right) \sqrt{-1} \cdot dz \wedge d\sigma \leq C \pi d^2 \int_{\tilde{U}_z} \sqrt{\int_{r^{-1}(D_w)} |f|^2 r_1^* \omega} \leq C \pi d^2 \int_{\tilde{U}_z} \sqrt{\int_{r^{-1}(D_w)} |f|^2}.
\]

From here and the fact that the function \( \phi/z \) is bounded on \( U_j \) we obtain that there is a constant \( c_j > 0 \) depending on \( Y \) such that

\[
||f||_{2^{-1}(w)} \leq \frac{c_j}{||\phi(w)||^2} ||f||_2 \quad \text{for all } w \in \tilde{U}_j. \tag{3.19}
\]

This proves the required inequality for \( w \in \bigcup_{1 \leq j \leq l} \tilde{U}_j \).

The remaining part \( K := Y \setminus (\bigcup_{1 \leq j \leq l} \tilde{U}_j) \) is a relatively compact subset of \( \tilde{Y} \setminus (\bigcup_{1 \leq j \leq l} \{x_j\}) \), see section 3.1 for the definition of \( \tilde{Y} \). Then the required estimate on \( K \) follows from (3.13) and the fact that \( |\phi| \) is bounded on \( K \). We leave the details to the reader. \( \Box \)

From this lemma we obtain the following improvement of (3.18):

\[
\sum_{z \in r^{-1}(w)} |h_{a,y}(z)|^2 \leq C \frac{C}{||\phi(w)||^2} ||a||_2^2, \quad w \in Y, \ y \in \tilde{U}_x, \tag{3.20}
\]

with \( C \) depending on \( Y \) and \( x \) only. This and the definition of \( h_{a,y} \), see (3.17), imply

\[
\sum_{s \in S} |h_{s,y}(z)|^2 \leq C \frac{C}{||\phi(w)||^2}, \quad z \in r^{-1}(w), \ w \in Y, \ y \in \tilde{U}_x. \tag{3.21}
\]

Let us continue the proof of the theorem. Consider the holomorphic function

\[
f_y := \frac{\phi^2}{\phi^2(y)}, \quad y \in \tilde{U}_x,
\]

defined in a neighbourhood of the closure of \( \tilde{Y}_1 \). Here \( \phi \) is the same as in Lemma 3.3 and \( \tilde{U}_x \subset \subset Y \) is the coordinate neighbourhood of \( x \in K \) defined by (3.16). Then \( f_y \) has double zeros at all \( x_j, 1 \leq j \leq l \), is nonzero outside these points, and \( f_y(y) = 1 \).

Finally, we introduce

\[
F_{s,y}(w) := h_{s,y}(w) \cdot (r_1^* f_y)(w) \quad \text{for all } w \in X, \ s \in S, \ y \in \tilde{U}_x. \tag{3.22}
\]

According to (3.21) we have

\[
\sum_{s \in S} |F_{s,y}(z)| \leq C' \quad \text{for all } z \in X, \ y \in \tilde{U}_x, \tag{3.23}
\]

with \( C' := \frac{C}{||\phi(y)||^2} \) depending on \( x, Y \) only. Moreover,

\[
F_{s,y}(y, t) = \delta_{st}. \tag{3.24}
\]
where \( \delta_{st} = 0 \) for \( s \neq t \) and \( \delta_{ss} = 1 \).

Using the functions \( F_{s,y} \), \( s \in S \), let us prove that \( r^{-1}(y) \) is interpolating for \( H^{\infty}(X) \) for all \( y \in \hat{U}_x \).

In fact, for \( a = \{a_s\}_{s \in S} \in l^{\infty}(S) \) and \( y \in \hat{U}_x \) consider the function

\[
[L_y(a)](z) := \sum_{s \in S} a_s F_{s,y}(z), \quad z \in X.
\] (3.25)

According to (3.23) we have

\[
\sup_{z \in X} |[L_y(a)](z)| \leq ||a||_{l^{\infty}} \cdot \sup_{z \in X} \left( \sum_{s \in S} |F_{s,y}(z)| \right) \leq C'||a||_{l^{\infty}}.
\]

Thus \( L_y \) is a linear continuous operator from \( l^{\infty}(r^{-1}(y)) \) to \( H^{\infty}(X) \) depending holomorphically on \( y \in \hat{U}_x \) with the norm bounded by a number depending on \( x \) and \( Y \) only. Also, from (3.24) we obtain

\[
[L_y(a)](y, t) := \sum_{s \in S} a_s F_{s,y}(y, t) = \sum_{s \in S} a_s \delta_{st} = a_t =: a(y, t).
\] (3.26)

That is, \( L_y(a) |_{r^{-1}(y)} = a \). Therefore \( r^{-1}(y) \), \( y \in \hat{U}_x \), is an interpolating sequence for \( H^{\infty}(X) \) with the constant of interpolation depending on \( Y \) and \( x \) only. Taking a finite open cover of \( K \) by sets \( \hat{U}_x \) and considering the corresponding interpolation operators \( L_y \) on \( \hat{U}_x \) we obtain that the constant of interpolation of each \( r^{-1}(x) \), \( x \in K \), is bounded by a number depending on \( K \) and \( Y \) only.

The proof of Theorem 1.5 is complete. \( \square \)

4. Proof of Theorem 1.6.

Let \( \eta \) be a smooth bounded \((0,1)\)-form on \( X \) with \( r(\text{supp} \ \eta) \subset K \) for some compact \( K \subset \subset Y \). We must find a smooth function \( f \) on \( X \) such that

\[
\bar{\partial} f = \eta \quad \text{and} \quad ||f||_{L^{\infty}} := \sup_{z \in X} |f(z)| \leq C||\eta||
\] (4.1)

with \( C \) depending on \( K, Y \) and a hermitian metric \( h_Y \) used in the definition of the norm of \( \eta \), see (1.4).

Without loss of generality we may assume that \( \eta \) has compact support. Indeed, let \( \{X_i\}_{i \in \mathbb{N}}, \ X_i \subset \subset X \), be an exhaustion of \( X \) by relatively compact open domains. Let \( \{\chi_i\}_{i \in \mathbb{N}} \) be a family of smooth functions on \( X \) such that \( \chi_i = 1 \) on a sub-domain \( Z_i \subset \subset X_i \), 0 outside \( X_i \) and \( 0 \leq \chi_i \leq 1, \ i \in \mathbb{N} \). Assume also that \( \{Z_i\}_{i \in \mathbb{N}} \) forms an exhaustion of \( X \), as well. Now, we set \( \eta_i := \chi_i \cdot \eta \). Then \( \{\eta_i\} \) converges to \( \eta \) uniformly on compact subsets of \( X \) and \( ||\eta_i|| \leq ||\eta|| \) for all \( i \). If we will find smooth functions \( f_i \) on \( X \) satisfying the corresponding conditions (4.1), then a standard normal family argument will give us a subsequence \( \{f_{i_k}\}_{k \in \mathbb{N}} \) of \( \{f_i\} \) converging uniformly on compact subsets of \( X \) to a smooth function \( f \) satisfying (4.1). Thus it suffices to prove the theorem for the forms \( \eta \) with compact supports.
Next, consider a finite open cover \((U_i)_{1 \leq i \leq n}\) of \(K \subset Y\) by sets \(U_i := \hat{U}_x, x_i \in K\), defined by (3.16). By definition \((U_i)_{1 \leq i \leq n}\) also covers a neighbourhood \(N \subset Y\) of \(K\). Now we consider a finite open cover \((U_i)_{n+1 \leq i \leq m}\) of \(R \setminus N\) (where \(R\) is a compact Riemann surface from (2.1) containing \(Y\)) by coordinate disks \(U_i\) such that \(U_i \cap K = \emptyset\) for all \(n+1 \leq i \leq m\). Let \(\{\rho_i\}_{1 \leq i \leq m}\) be a smooth partition of unity subordinate to the cover \((U_i)_{1 \leq i \leq m}\) of \(R\). Then since \(r(\text{supp } \eta) \subset K, U_i \cap K = \emptyset\) for all \(n+1 \leq i \leq m\), and \(\text{supp } \rho_i \subset U_i\) for \(1 \leq i \leq n\),

\[
\eta = \sum_{i=1}^{m} (r_i^* \rho_i) \eta = \sum_{i=1}^{n} (r_i^* \rho_i) \eta.
\]

By the definition each \(\eta_i := (r_i^* \rho_i) \eta\) is a smooth \((0,1)\)-form with compact support such that \(r(\text{supp } \eta_i) \subset U_i\). It suffices to prove the theorem for such forms \(\eta_i\), i.e., to find smooth functions \(f_i\) such that \(\partial f_i = \eta_i\) and \(\|f_i\|_{L^\infty} \leq C_i \|\eta_i\| \leq C_i \|\eta\|\) with \(C_i\) depending on \(U_i, Y\) and \(h_Y\). Then \(f := \sum_{i=1}^{n} f_i\) satisfies the required statement of the theorem.

Thus without loss of generality we may assume that \(r(\text{supp } \eta) \subset \hat{U}_x := U\) for some \(x \in K\) and \(\eta\) has compact support. As before we identify \(r^{-1}(U)\) with \(U \times S\) where \(S\) is the fibre of \(r\). Then there is a finite subset \(S_\eta \subset S\) such that \(\text{supp } \eta \subset U \times S_\eta\). In a complex coordinate \(z\) on \(U\) the form \(\eta\) is written as

\[
\eta(z, s) = g(z, s) \, d\bar{z}, \quad (z, s) \in U \times S_\eta, \quad \text{and } \eta = 0 \text{ outside } U \times S_\eta.
\]

By the hypothesis of the theorem we have

\[
|\eta| := \sup_{z \in U, s \in S_\eta} |g(z, s)| \leq c \|\eta\| \quad (4.2)
\]

for some \(c\) depending on \(U\) and \(h_Y\) only.

The remaining part of the proof repeats literally the proof of Proposition 5.1 of [Br4]. We refer to this paper for details.

Consider the family of interpolation operators \(L_z : l^\infty(r^{-1}(y)) \rightarrow H^\infty(X)\) holomorphic in \(z \in U\) with norms bounded by a number \(C'\) depending on \(U\) and \(Y\) only, see (3.25). Let us define the \((0,1)\)-form \(\lambda\) on \(R\) with values in \(H^\infty(X)\) by the formula

\[
\lambda(z) := L_z(g(z, \cdot)) \, d\bar{z}, \quad z \in U, \quad \text{and } \lambda = 0 \text{ outside } U. \quad (4.3)
\]

Since \(\text{supp } \eta \subset U \times S_\eta\) and \(S_\eta\) is a finite subset of \(S\), the definition of \(L_z\) in (3.26) implies that \(\lambda\) is smooth. Using an integral formula we can solve the equation \(\overline{\partial} F = \lambda\) on a fixed neighborhood of the closure of \(Y\) in \(R\) to get a smooth solution \(F : Y \rightarrow H^\infty(X)\) satisfying

\[
\sup_{z \in Y, w \in X} |[F(z)](w)| \leq c' |\lambda| := c' \cdot \sup_{z \in Y, w \in X} |[L_z(g(z, \cdot))](w)|. \quad (4.4)
\]

with \(c'\) depending on \(Y\) only. Finally, we set

\[
f(w) := [F(r(w))](w), \quad w \in X. \quad (4.5)
\]
Since \( \{L_z\} \) are interpolation operators holomorphic in \( z \in U \) one has
\[
\overline{f}(w) := [\lambda(r(w))](w) = \eta(w), \quad w \in X.
\]
Moreover, from estimates (4.2), (4.4) and ||\( L_z \)|| \leq C' we get
\[
\sup_{w \in X} |f(w)| \leq C||\eta||
\]
where \( C := c' \cdot C' \cdot c \).

This completes the proof of the theorem. \( \square \)

**5. Proof of Theorem 1.1.**

Let \( r : X \to Y \) be an unbranched covering of a Caratheodory hyperbolic Riemann surface of finite type \( Y \). The fact that \( X \) is Caratheodory hyperbolic follows easily from Theorem 1.5 and the Caratheodory hyperbolicity of \( Y \). Let us prove now the corona theorem for \( H^\infty(X) \).

First we consider a finite open cover \( \mathcal{U} = \{U_j\}_{0 \leq j \leq l} \) of \( \overline{Y} := Y \cup (\cup_{1 \leq j \leq l}\{x_j\}) \) such that for \( 1 \leq j \leq l \) the set \( U_j \subset \subset \overline{Y} \) is an open coordinate disk centered at \( x_j \) and \( U_0 \) is a bordered Riemann surface which intersects each \( U_j, 1 \leq j \leq l \), by a set biholomorphic to an open annulus, see definition (2.1). By \( \{\rho_j\}_{0 \leq j \leq l} \) we denote a smooth partition of unity on \( \overline{Y} \) subordinate to the cover \( \mathcal{U} \). We set \( U_j^* := U_j \setminus \{x_j\}, \) \( 1 \leq j \leq l \). Then \( U_j^* \) is biholomorphic to a punctured open disk in \( \mathbb{C} \). Now, \( r^{-1}(U_j^*) \) is a disjoint union of sets biholomorphic to \( \mathbb{D} \) or to the punctured disk \( \mathbb{D}^* \) (because the fundamental group of \( U_j^* \) is \( \mathbb{Z} \)). Moreover, according to our construction \( \pi_1(U_0) \cong \pi_1(Y) \). Hence \( r^{-1}(U_0) \) is an open connected subset of \( X \).

Suppose now that a collection \( f_1, \ldots, f_n \) of functions from \( H^\infty(X) \) satisfies the corona condition (1.1). Since each connected component of \( r^{-1}(U_j^*) \) is biholomorphic to \( \mathbb{D} \) or \( \mathbb{D}^* \), according to the Carleson Corona Theorem, see, e.g., [Ga, Ch. VIII, Th. 2.1], there are a constant \( C_1(n, \delta) \) (with \( \delta \) from (1.1)) and functions \( g_1^j, \ldots, g_n^j \) from \( H^\infty(r^{-1}(U_j^*)), 1 \leq j \leq l, \) such that
\[
f_1g_1^j + \cdots + f_ng_n^j \equiv 1 \quad \text{on} \quad r^{-1}(U_j^*) \quad \text{and} \quad (5.1)
\]
\[
||g_k^j|| \leq C_1(n, \delta), \quad 1 \leq j \leq l, \quad 1 \leq k \leq n.
\]
Also, since \( U_0 \subset \subset Y \) is a bordered Riemann surface, according to [Br1, Cor. 1.6] there are a constant \( C_2(Y, n, \delta) \) and functions \( g_1^0, \ldots, g_n^0 \) from \( H^\infty(r^{-1}(U_0)) \) such that
\[
f_1g_1^0 + \cdots + f_ng_n^0 \equiv 1 \quad \text{on} \quad r^{-1}(U_0) \quad \text{and} \quad (5.2)
\]
\[
||g_k^0|| \leq C_2(Y, n, \delta), \quad 1 \leq k \leq n.
\]
We set
\[
h_k := \sum_{j=0}^{l} (r^*\rho_j)g_k^j, \quad 1 \leq k \leq n, \quad 0 \leq j \leq l. \quad (5.3)
\]
Since $\text{supp } \rho_j \subset U_j$, $0 \leq j \leq l$, $h_k$ are smooth functions on $X$ such that
\begin{equation}
\begin{aligned}
    f_1 h_1 + \cdots + f_n h_n &\equiv 1 \text{ on } X \text{ and } \\
    \|h_k\|_{L^\infty} &\leq C_3(Y, n, \delta), \quad 1 \leq k \leq n.
\end{aligned}
\end{equation}

Next we will use a standard construction based on the Koszul complex, see [Ga, Ch. VIII]. Namely, we write
\begin{align}
    g_j(z) &= h_j(z) + \sum_{k=1}^{n} a_{j,k}(z) f_k(z) \\
    a_{j,k}(z) &= b_{j,k}(z) - b_{k,j}(z) \quad \text{and} \\
    \partial h_{j,k} &= h_j \cdot \partial h_k =: \eta_{j,k}, \quad j \neq k.
\end{align}

According to (5.3) and (5.4) the smooth $(0,1)$-forms $\eta_{j,k}$ on $X$ satisfy
\begin{equation}
    r(\text{supp } \eta_{j,k}) \subset U_0 \quad \text{and} \quad \|\eta_{j,k}\| \leq C_4(Y, n, \delta) \quad \text{for all } j, k.
\end{equation}

where $\| \cdot \|$ is defined with respect to a fixed hermitian metric $h_Y$ on $Y$, see (1.5). Therefore by Theorem 1.6 there are smooth functions $b_{j,k}$ on $X$ satisfying equations (5.5) such that
\begin{equation}
    \|b_{j,k}\|_{L^\infty} \leq C_5(Y, n, \delta) \quad \text{for all } j, k.
\end{equation}

Then the functions $g_j$ on $X$ belong to $H^\infty(X)$ and satisfy
\begin{align}
    f_1 g_1 + \cdots + f_n g_n &\equiv 1 \quad \text{and} \\
    \|g_j\| &\leq C(Y, n, \delta) \quad \text{for all } j.
\end{align}

This completes the proof of Theorem 1.1. \hfill \Box

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