Matter Collineations of Some Static Spherically Symmetric Spacetimes

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Abstract

We derive matter collineations for some static spherically symmetric spacetimes and compare the results with Killing, Ricci and Curvature symmetries. We conclude that matter and Ricci collineations are not, in general, the same.

Keywords: Isometries, Collineations

1 Introduction

There has been a recent literature [1-11, and references therein] which shows a significant interest in the study of various symmetries. These symmetries arise in the exact solutions of Einstein field equations given by

\[
G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab},
\]

where \( G_{ab} \) are the components of the Einstein tensor, \( R_{ab} \) are the components of Ricci tensor and \( T_{ab} \) are the components of matter (energy-momentum) tensor, \( R \) is the Ricci scalar and \( \kappa \) is the gravitational constant.

The well known connection between Killing vectors (KVs) and constants of the motion [12,13] has encouraged the search for general relations between collineations and conservation laws [9,14]. Curvature and the Ricci tensors

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are the important quantities which play a significant role in understanding the geometric structure of spacetime. A basic study of curvature collineations (CCs) and Ricci collineations (RCs) has been carried out by Katzin, et al [15] and a complete classification of CCs and RCs has been obtained by Qadir, et al [4-7,9].

In this paper we shall analyze the properties of a vector field along which the Lie derivative of the energy-momentum tensor vanishes. Such a vector is called matter collineation. Since the energy-momentum tensor represents the matter part of the Einstein field equations and gives the matter field symmetries. Thus the study of matter collineations (MCs) seems more relevant from the physical point of view.

The motivation for studying MCs can be discussed as follows. When we find exact solutions to the Einstein’s field equations, one of the simplifications we use is the assumption of certain symmetries of the spacetime metric. These symmetry assumptions are expressed in terms of isometries expressed by the spacetimes, also called Killing vectors which give rise to conservation laws [16,17]. Symmetries of the energy-momentum tensor provide conservation laws on matter fields. These symmetries are called matter collineations. These enable us to know how the physical fields, occupying in certain region of spacetimes, reflect the symmetries of the metric [18]. In other words, given the metric tensor of a spacetime, one can find symmetry for the physical fields describing the material content of that spacetime. There is also a purely mathematical interest of studying the symmetry properties of a given geometrical object, namely the Einstein tensor, which arises quite naturally in the theory of General Relativity. Since it is related, via Einstein field equations, to the material content of the spacetime, it has an important role in this theory.

There is a growing interest in the study of MCs [10,19,20 and references therein]. Carot, et al [19] has discussed MCs from the point of view of the Lie algebra of vector fields generating them and, in particular, he discussed spacetimes with a degenerate $T_{ab}$. Hall, et al [20], in the discussion of RC and MC, have argued that the symmetries of the energy-momentum tensor may also provide some extra understanding of the subject which has not been provided by Killing vectors, Ricci and Curvature collineations. Keeping this point in mind we address the problem of calculating MCs for some static spherically symmetric spacetimes using some special techniques in this paper. It is hoped that some particular methods can be developed to solve partial differential equations involving in matter collineation equations for general
spacetimes. This would enable one to obtain a complete classification of general spacetimes according to their MCs.

The distribution of the paper follows. In the next section, we give some basic definitions and write down the matter collineation equations. In section three we calculate MCs by solving matter collineation equations for some particular spacetimes. Final section carries a discussion of the results obtained.

2 Some Basic Facts and Matter Collineation Equations

Let \((M, g)\) be a spacetime, \(M\) being a Hausdorff, simply connected, four dimensional manifold, and \(g\) a Lorentz metric with signature (+,−,−,−).

A vector \(\xi\) is called a MC if the Lie derivative of the energy-momentum tensor along that vector is zero. That is,

\[ \mathcal{L}_\xi T = 0, \]

where \(T\) is the energy-momentum tensor and \(\mathcal{L}_\xi\) denotes the Lie derivative along \(\xi\) of the energy-momentum tensor \(T\). This equation, in a torsion-free space in a coordinate basis, reduces to a simple partial differential equation,

\[ T_{ab,c}\xi^c + T_{ac}\xi^c_b + T_{bc}\xi^c_a = 0, \quad a, b, c = 0, 1, 2, 3. \]

Collineations can be proper or improper. A collineation of a given type is said to be proper if it does not belong to any of the subtypes. When we solve matter collineation equations, solutions representing proper collineations can be found. However, in order to be related to a particular conservation law, and its corresponding constants of the motion, the properness of the collineation type must be known.

We know that every KV is an MC, but the converse is not always true. As given by Carot et al. [19], if \(T_{ab}\) is non-degenerate, \(\text{det}(T_{ab}) \neq 0\), the Lie algebra of the MCs is finite dimensional. If \(T_{ab}\) is degenerate, i.e., \(\text{det}(T_{ab}) = 0\), we cannot guarantee the finite dimensionality of the MCs.
3 Solution of the Matter Collineation Equations

We shall solve the MC equations for Minkowski, Einstein/anti-Einstein, de Sitter/anti-de Sitter, Schwarzschild, Riessner-Nordstrom and some Bertotti-Robinson like metrics. However, for Minkowski and Schwarzschild spacetimes, the Ricci tensor is zero, which implies the vanishing of the energy-momentum tensor. Hence the MC Eqs.(3) are satisfied for any arbitrary values of $\xi^a$. Thus in Minkowski and Schwarzschild spacetimes every $\xi^a$ is a MC.

In de Sitter/anti-de Sitter spacetimes, the energy-momentum tensor is related to the metric tensor by

$$T_{ab} = \frac{3}{\kappa D^2} g_{ab},$$

(4)

where $D$ is a constant. If we take the Lie derivative on both sides of this equation and assume the vanishing of Lie derivative, the symmetries of the energy-momentum tensor turns out to be identical to that of the metric tensor. Thus the MCs for these spacetimes are equal to that of KVs and these are ten.

For Reissner-Nordstrom spacetime, the energy-momentum tensor is proportional to the Ricci tensor. Thus the MCs become the same as the four RCs.

We explicitly solve the MC equations for the Einstein metric. However, we would suffice to give the results for Bertotti-Robinson like metrics as the same procedure will be applicable for solving MC equations of these metrics.

For the Einstein metric

$$ds^2 = dt^2 - \frac{1}{1 - r^2/D^2} dr^2 - r^2 d\Omega^2$$

(5)

the non-zero components of the energy-momentum tensor are given by

$$T_{00} = \frac{3}{\kappa D^2}, \quad T_{ij} = \frac{1}{\kappa D^2} g_{ij}, \quad (i, j = 1, 2, 3).$$

(6)

Substituting these values in Eqs.(3), we have

$$\xi^0 = A(r, \theta, \phi),$$

(7)
\[3(1 - r^2/D^2)\xi_0^3 - \xi_0^1 = 0,\]  
\[3\xi_0^2 - r^2\xi_0^3 = 0,\]  
\[3\xi_0^3 - r^2 \sin^2 \theta \xi_0^3 = 0,\]  
\[r\xi_0^2 + D^2(1 - r^2/D^2)\xi_0^1 = 0,\]  
\[\xi_0^1 + r\xi_0^2 = 0,\]  
\[\xi_0^1 + r \cot \theta \xi_0^2 + r\xi_0^3 = 0,\]  
\[\xi_0^1 + r^2(1 - r^2/D^2)\xi_0^2 = 0,\]  
\[\xi_0^1 + r^2(1 - r^2/D^2)\sin^2 \theta \xi_0^3 = 0,\]  
\[\xi_0^2 + \sin^2 \theta \xi_0^3 = 0.\]  

Solving Eqs. (11)-(16), we have

\[\xi_0^1 = (1 - r^2/D^2)^{1/2}[B(t) \cos \theta + B_0(t, \phi) \sin \theta],\]  
\[\xi_0^2 = \frac{(1 - r^2/D^2)^{1/2}}{r}[-B(t) \sin \theta + B_0(t, \phi) \cos \theta] + C_1(t, \phi),\]  
\[\xi_0^3 = \frac{(1 - r^2/D^2)^{1/2}}{r \sin \theta}B_2\phi(t) + \cot \theta C_1\phi(t, \phi) + E(t, \phi),\]

where \(B_3\) is a function of time only while \(B_2, C_1\) and \(E\) are functions of \(t\) and \(\phi\). Using these equations together with Eqs. (7)-(10), we finally arrive at the following results

\[\xi_0^0 = \alpha_0,\]  
\[\xi_0^1 = (1 - r^2/D^2)^{1/2}[\alpha_0 \cos \theta + (\alpha_2 \cos \phi + \alpha_3 \sin \phi) \sin \theta],\]  
\[\xi_0^2 = \frac{(1 - r^2/D^2)^{1/2}}{r}[-\alpha_1 \sin \theta + (\alpha_2 \cos \phi + \alpha_3 \sin \phi) \cos \theta] + \alpha_4 \cos \phi + \alpha_5 \sin \phi,\]  
\[\xi_0^3 = \frac{(1 - r^2/D^2)^{1/2}}{r \sin \theta}(-\alpha_2 \sin \phi + \alpha_3 \cos \phi) + \cot \theta(-\alpha_4 \sin \phi + \alpha_5 \cos \phi) + \alpha_6.\]
Thus the MCs are given as

\[
\begin{align*}
\xi &= \alpha_0 \frac{\partial}{\partial t} + \alpha_1 \left[(1 - r^2/D^2)^{1/2} \cos \theta \frac{\partial}{\partial r} - \frac{(1 - r^2/D^2)^{1/2}}{r} \sin \theta \frac{\partial}{\partial \theta}\right] + \alpha_2 \left[(1 - r^2/D^2)^{1/2} \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{(1 - r^2/D^2)^{1/2}}{r} \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \frac{(1 - r^2/D^2)^{1/2}}{r \sin \theta} \sin \phi \frac{\partial}{\partial \phi}\right] + \alpha_3 \left[(1 - r^2/D^2)^{1/2} \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{(1 - r^2/D^2)^{1/2}}{r \sin \theta} \sin \theta \sin \phi \frac{\partial}{\partial \phi}\right] + \alpha_4 \left[\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}\right] + \alpha_5 \left[\sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi}\right] + \alpha_6 \frac{\partial}{\partial \phi}. \tag{24}\end{align*}
\]

It follows that the MCs are seven given by

\[
\begin{align*}
\xi^{(1)} &= \frac{\partial}{\partial t}, \tag{25}\\
\xi^{(2)} &= (1 - r^2/D^2)^{1/2} \cos \theta \frac{\partial}{\partial r} - \frac{(1 - r^2/D^2)^{1/2}}{r} \sin \theta \frac{\partial}{\partial \theta}, \tag{26}\\
\xi^{(3)} &= (1 - r^2/D^2)^{1/2} \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{(1 - r^2/D^2)^{1/2}}{r} \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \frac{(1 - r^2/D^2)^{1/2}}{r \sin \theta} \sin \phi \frac{\partial}{\partial \phi}, \tag{27}\\
\xi^{(4)} &= (1 - r^2/D^2)^{1/2} \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{(1 - r^2/D^2)^{1/2}}{r} \cos \theta \sin \phi \frac{\partial}{\partial \phi} + \frac{(1 - r^2/D^2)^{1/2}}{r \sin \theta} \cos \phi \frac{\partial}{\partial \phi}, \tag{28}\\
\xi^{(5)} &= \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \tag{29}\\
\xi^{(6)} &= \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi}. \tag{30}
\end{align*}
\]
\[\xi_{(7)} = \frac{\partial}{\partial \phi}.\] (31)

which are also the generators of a group \(G_7\). We see that \(\xi_{(5)}, \xi_{(6)}, \xi_{(7)}\) are the KVs associated with spherical symmetry of the Einstein metric. Also, it is to be noted that MCs \(\xi_{(5)}, \xi_{(6)}, \xi_{(7)}\) are the improper. Following the same procedure we can evaluate MCs for the anti-Einstein metric which will be the same as for the Einstein spacetime except for the difference that the factor \(D^2\) will be replaced by \(-D^2\).

There are also three (there may be more) Bertotti-Robinson like space-times available in the literature [21] which are static spherically symmetric spacetimes. For the first metric given by

\[ds_{I}^2 = (B + r)^2 dt^2 - dr^2 - a^2 d\Omega^2,\] (32)

where \(B\) and \(a\) are constants, the only non-zero energy-momentum components are

\[T_{00} = \frac{(B + r)^2}{\kappa a^2}, \quad T_{11} = -\frac{1}{\kappa a^2}.\] (33)

Using Eq.(33) in Eq.(3), it follows that

\[\xi^0 = -\frac{1}{B + r}(\alpha_0 e^t - \alpha_1 e^{-t}) + \alpha_2,\] (34)
\[\xi^1 = \alpha_0 e^t + \alpha_1 e^{-t},\] (35)
\[\xi^2 = \xi^2(x^a),\] (36)
\[\xi^3 = \xi^3(x^a).\] (37)

Thus we have

\[\xi = \alpha_0(-\frac{1}{B + r} \frac{\partial}{\partial t} + \frac{\partial}{\partial r})e^t + \alpha_1(\frac{1}{B + r} \frac{\partial}{\partial t} + \frac{\partial}{\partial r})e^{-t} + \alpha_2 \frac{\partial}{\partial t} + \xi^2(x^a) \frac{\partial}{\partial \theta} + \xi^3(x^a) \frac{\partial}{\partial \phi}.\] (38)

The other two Bertotti-Robinson like metrics are given by

\[ds_{II}^2 = \cos^2(c + \sqrt{\alpha} r) dt^2 - dr^2 - a^2 d\Omega^2;\] (39)
\[ds_{III}^2 = \cosh^2(c + \sqrt{\alpha} r) dt^2 - dr^2 - a^2 d\Omega^2;\] (40)

where \(c, \alpha\) and \(a\) are constants. All the diagonal energy-momentum tensor components survive for these metrics. The MCs in both cases turn out to be the same as the six KVs for these metrics.
Table 1: Comparison of KVs, RCs, CCs, and MCs for some specific Spherically Symmetric Spacetimes

| Metric            | KVs | RCs          | CCs           | MCs          |
|-------------------|-----|--------------|---------------|--------------|
| Minkowski         | 10  | Arbitrary    | Arbitrary     | Arbitrary    |
| De Sitter/anti    | 10  | 10           | 10            | 10           |
| Einstein/anti     | 7   | $6 + \xi^0(x^a)$ | $6 + \xi^0(t)$ | 7            |
| Bertotti-Robinson | 6   | $3 + \xi^0(x^a), \xi^1(x^a)$ | $3 + \xi^0(t, r), \xi^1(t, r)$ | $3 + \xi^2(x^a), \xi^3(x^a)$ |
| Bertotti-Robinson | 6   | 6            | 6             | 6            |
| Bertotti-Robinson | 6   | 6            | 6             | 6            |
| Schwarzschild     | 4   | Arbitrary    | 4             | Arbitrary    |
| Reissner-Nordstrom| 4   | 4            | 4             | 4            |

4 Conclusion

We have evaluated MCs for some specific static spherically symmetric spacetimes. The motivation behind this is to understand the distribution of matter symmetries as compared to the symmetries of the metric, Ricci and Curvature tensors [22]. We discuss the results obtained using the table given above:

It is to be noted that the metric tensor is non degenerate whereas the Ricci, Riemann and energy-momentum tensors are not necessarily non degenerate. When there is a degeneracy, it is possible to have arbitrary collineations. Thus KVs will always be definite but collineations can be indefinite. Further, if the relevant tensor vanishes, all vectors become collineations as for Minkowski space where every vector is a MC. Also, for vacuum spacetime every vector will be an MC as it is Ricci flat (e.g. Schwarzschild metric). From the table we see that for Einstein/anti-Einstein metric MCs are seven while RCs are six plus one arbitrary function of four variable. This shows that MCs and RCs are not, in general, the same. However, if $R = 0$ then MCs and RCs obviously coincide.

In the Bertotti-Robinson like metric given by Eq.(32), the MCs turn out to be indefinite in $\xi^2$ and $\xi^3$. The second and third components of the MCs are functions of all the spacetime coordinates. This result is different from KVs and even different from RCs where we have indefinite RCs and CCs in the temporal and radial components. Thus this type of Bertotti-Robinson like metric provides new information. It has six KVs, three plus arbitrary temporal and radial (functions of spacetime coordinates) RCs, three plus
arbitrary temporal and radial (functions of $t$ and $r$ only) CCs and three plus arbitrary second and third (functions of all spacetimes) MCs. In the other two Bertotti-Robinson like metrics, we do not get any information as the KVs, RCs, CCs and MCs are all the same.

It would be interesting to extend this idea of classifying spacetimes with some minimal symmetry group in terms of their MCs and compare the general classification with KVs, RCs and CCs. To this end, one has to consider the simplest spacetimes which is the class of all spherically symmetric spacetimes. Then one can extend this by removing the condition of staticity, moving forward to cylindrical and plane symmetry and then reducing the minimal symmetry group further. Finally, it would be worthwhile to make comparison of the results with other classification schemes, e.g., Petrov classification. It is hoped that some interesting feature will come out.
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