Higher order Hochschild cohomology of schemes

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Abstract: We show that Higher Hochschild complex [Pir00] associated to a connected pointed simplicial set commutes with localization of commutative algebras over a field of characteristic zero. Then, we define in two ways higher order Hochschild cohomology of schemes over a field of characteristic zero. Originally, we can take the hyperext functor of the sheaf associated to Higher Hochschild presheaf. We obtain a Hodge decomposition for higher order Hochschild cohomology of smooth algebraic varieties over a field of characteristic zero which generalizes Pirashvili’s Hodge decomposition. We can also define the Higher Hochschild cohomology of order $d$ of a separated scheme by taking the ext functor of its structure presheaf over the Higher Hochschild presheaf of order $d - 1$. These two definitions are really close to those of Swan [Swa96] for classical Hochschild cohomology, but our tools are model categories and derived functors. We also generalize the equivalence of Swan’s definitions to any separated schemes over a field.

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1 Introduction

The Hochschild complex $C(A)$ of an associative algebra $A$ is the chain complex

$$
\ldots \rightarrow a_0 \otimes \cdots \otimes a_i + 1 \otimes \cdots \otimes a_n \rightarrow a_{n+1} \otimes \cdots \otimes a_n \rightarrow \cdots 
$$

It is closely related to the simplicial circle $S^1$ with in degree $n$ a copy of $A$ for each $n$-simplex. The idea of Loday [Lod98, 6.4] and Pirashvili [Pir00] was to replace the circle by any degreewise finite simplicial set $K$ when $A$ is a commutative algebra in order to build a new simplicial complex $C(K, A)$ and then to obtain a new (co)homology functor $H(K, A)$. In Section 3, we will use skeleton of simplicial sets and derived tensor product [GTZ14] to show that it commutes with localization in homology if the simplicial set $K$ is connected.

**Theorem 3.2.2.** Let $K$ be a connected pointed simplicial set and $S$ a multiplicative system of an algebra $A$. The canonical morphism of CDGA $S^{-1}C(K, A) \rightarrow C(K, S^{-1}A)$ is a quasi-isomorphism.

Pirashvili was particularly interested in the case of simplicial $d$-spheres $S^d$ and generalized the $\lambda$-decomposition of Loday [Lod98, 6.4.5] to a Hodge decomposition depending only on the parity of $d$

$$
H_n(S^d, A) \cong \bigoplus_{i+j=n} H^{(i)}_{i+j}(A)
$$

when $A$ is over a field of characteristic zero [Pir00, 2.5].

On the other hand, Swan defined in three different ways the Hochschild cohomology of a scheme $X$ [Swa96]. Following an idea of Grothendieck and Loday, he used the sheaf $C_X$ associated to the presheaf $U \mapsto C(O_X(U))$ and the hyperext functor $\text{Ext}_{O_X}(C_X, \quad)$ [Swa96, 2]. This construction can easily be done with the simplicial complexes $C(K, A)$. In Section 4, we will be interested in the cohomological functor $H_{[d]}(X, \quad)$ corresponding to the $d$-spheres $S^d$. We will show that there exists a Hodge decomposition for smooth varieties and that the functor $H_{[d]}(X, \quad)$ and this decomposition are the same as Pirashvili’s for affine schemes.
**Theorem 4.1.5.** Let $d$ be a positive integer, $X$ a separated smooth scheme over a field of characteristic zero and $\mathcal{F}$ an $\mathcal{O}_X$-module. There is a natural isomorphism

$$H^n_{[d]}(X, \mathcal{F}) \cong \bigoplus_{p+j=d} H^p(X, D^j_X \otimes \mathcal{F})$$

where $D^j_X$ is the dual sheaf of $\Omega^j_X$ if $d$ is odd and of $\text{Sym}^d_{\mathcal{O}_X} \Omega^1_X$ if $d$ is even.

If the ground ring is a field then the Hochschild cohomology of $A$ is given by the ext functor $\text{Ext}_{A^e}(A, -)$ [Lod98, 1.5]. Guided by this fact, Gerstenhaber and Schack defined the Hochschild cohomology of a diagram of algebras $\mathcal{O}$ by the ext functor $\text{Ext}_{\mathcal{O}^e}(\mathcal{O}, -)$ [GS88, 21.0]. Swan took the structure sheaf of a separated scheme $X$ affine open sets as a diagram of algebras to get another definition of the Hochschild cohomology of $\mathcal{O}$ [Swa96, 3]. In Section 5, we will adapt this definition for the higher order and we will show that it gives us the same functor $H^n_{[d]}(X, -)$ on the quasi-coherent sheaves.

**Theorem 5.2.3.** Let $d$ be a positive integer, $X$ a separated scheme over a field of characteristic zero with structure presheaf $\mathcal{O}$ and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module thought of as a preasheaf on $X$. There is a natural isomorphism

$$H^n_{[d]}(X, \mathcal{F}) \cong H^n(\mathbb{R}\text{Hom}_{C^{[d-1]}(\mathcal{O})}(\mathcal{O}, \mathcal{F}))$$

where $C^{[d-1]}(\mathcal{O})$ is the presheaf of CDGA on $X$ given by $U \mapsto C(S^{d-1}, \mathcal{O}_X(U))$.

This same remark led Swan to give a more simple definition of the Hochschild cohomology of a scheme $X$ with diagonal morphism $\delta : X \to X \times X$ by taking the ext functor $\text{Ext}_{\mathcal{O}_{X \times X}}(\delta_* \mathcal{O}_X, \delta_* -)$ [Swa96, 1]. He showed that this functor is the same as Grothendieck-Loday’s for quasi-projective schemes over a field. In Section 6, we will generalize this fact to any separated scheme over a field.

**Theorem 6.1.4.** Let $X$ be a separated scheme over a field with diagonal morphism $\delta : X \to X \times X$ and $\mathcal{F}$ an $\mathcal{O}_X$-module. There is a natural isomorphism

$$\text{Ext}_{\mathcal{O}_X}^n(C_X, \mathcal{F}) \cong \text{Ext}_{\mathcal{O}_{X \times X}}^n(\delta_* \mathcal{O}_X, \delta_* \mathcal{F})$$

### 2 Notations and conventions

1. **Homological algebra.** Our complexes are homologically graded, but we can talk about the cohomology of a complex $C$ with the convention $H^n(C) = H_{-n}(C)$. We denote by $\text{Ch}(\mathcal{A})$ the category of complexes in an abelian category $\mathcal{A}$ and $\text{Hom}_{\text{Ch}(\mathcal{A})}$ the hom complex in $\text{Ch}(\mathcal{A})$ [Bou80, 5.1]. If $\mathcal{A}$ is a Grothendieck category [Hov01, 1] then $\text{Ch}(\mathcal{A})$ has an injective model structure which is cofibrantly generated, proper and such that the weak equivalences are the quasi-isomorphisms and the cofibrations are the monomorphisms [Hov01, 2.2]. We denote by $\text{Vect}$ the category of *vector spaces* over a base field $k$ of characteristic zero, $\otimes = \otimes_k$, $\text{Ch} = \text{Ch}(\text{Vect})$ and we call complexes in $\text{Vect}$ *chain complexes*.

2. **Simplicial objects.** Let $\Delta$ be the category with objects the finite sets $[n] = \{0, \ldots, n\}$ for all $n \in \mathbb{N}$ and with morphisms the non-decreasing applications. A simplicial object of a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$ and a cosimplicial object of $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. The category of simplicial objects of $\mathcal{C}$ is denoted by $s\mathcal{C}$. A *finite* simplicial set is a simplicial object of the category of finite sets $\text{Fin}$. We denote by $\text{Fin}'$ the category of finite
pointed sets. For any $d \in \mathbb{N}$, let $\Delta^d = \text{Hom}_\Delta(\emptyset, [d])$ the standard $d$-simplex, $\partial \Delta^d \subset \Delta^d$ the simplicial subset generated by the non-degenerated simplices $\sigma \neq 1_{[d]}$ its border and $\ast = \Delta^0$ the point. We define the $d$-sphere by the simplicial pushout $S^d = \ast \cup_{\partial \Delta^d} \Delta^d$. The category $sSet$ of simplicial sets has a Kan model structure which is cofibrantly generated, proper and such that the cofibrations are injections [Hov99, 3.6.5, 3.2.2] [Hir03, 13.1.13].

3. CDGA. A monoid in a monoidal category $(\mathcal{M}, \otimes, 1)$ is an object $A$ with two morphisms $A \otimes A \to A$ and $1 \to A$ and an $A$-module is an objet $M$ of $\mathcal{M}$ with a morphism $A \otimes M \to M$. They have to satisfy usual diagram relations [Mac98, VII, 3, 4]. We denote by $A$-mod the category of $A$-modules, $\text{Sym}_A^n$ the $n$th symmetric power over $A$ and $\Lambda^n_A$ the $n$th exterior power over $A$ for any $n \in \mathbb{N}$. We call commutative monoids in $(\text{Vect}, \otimes, k)$ algebras and we denote by Alg their category. For any algebra $A$, we denote by $\text{Spec}(A)$ the set of prime ideals of $A$, $\Omega^n_A$ the $A$-module of differentials of $A$ [Bou07, III, 10.11] and $\Omega^1_A = \Lambda^n_A \Omega^n_A$ for any $n \in \mathbb{N}$. For any multiplicative system $S$ of $A$, we denote by $S^{-1}A$ the fraction ring of $A$ defined by $S$ [Bou85, II, 1.1] and $S^{-1}M = S^{-1}A \otimes_A M$ for any $A$-module $M$. If $S = \{1, s, s^2, \ldots, s^n, \ldots\}$ we use the notations $A_s = S^{-1}A$ and $M_s = S^{-1}M$. We call commutative monoids in $(\text{Ch}, \otimes, k)$ CDGA and we denote by CDGA their category.

Since $k$ is a field of characteristic zero, it is a cofibrantly generated model category such that the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections [Hin97, 4.1.1, 4.2.5.2]. Any simplicial algebra $A : \Delta^{op} \to \text{Alg}$ can be thought as a CDGA $A = (A_n)_{n \in \mathbb{N}}$ with differential $d_n = \sum_{i=0}^n (-1)^i d_i$ [Wei94, 8.2.1] and multiplication the composition of the shuffle product [Mac65, VIII, 8.8] and the multiplications of $A$.

4. Sheaves. All the sheaves on a topological space $X$ take values in $\text{Vect}$. For any sheaf $\mathcal{F}$ on $X$, we use notation $\Gamma \mathcal{F} = \mathcal{F}(X)$. A ringed space $X$ is a topological space with a sheaf $\mathcal{O}_X$ with values in $\text{Alg}$. We denote by $\text{Hom}_{\mathcal{O}_X}$ the hom sheaf [Gro57, 4.1], $\text{Ext}_{\mathcal{O}_X}^n$ the ext sheaf [Gro57, 4.2] and $\mathcal{F}' = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ the dual sheaf of an $\mathcal{O}_X$-module $\mathcal{F}$. If it is not confusing then we can use the notation $\otimes = \otimes_{\mathcal{O}_X}$. By convention, all schemes are over $\text{Spec}(k)$. We denote by $\times$ the product of schemes over $\text{Spec}(k)$ [Gro60, 3.2.1], $\Omega^n_X$ the $\mathcal{O}_X$-module of 1-differentials of $X$ over $\text{Spec}(k)$ [Gro67, 16.3.1] and $\Omega^n_X = \Lambda^n_{\mathcal{O}_X} \Omega^n_X$ for any $n \in \mathbb{N}$.

## 3 Higher Hochschild functor

In all of this section, if nothing else is specified, $k$ is a field of characteristic zero. Recall that algebras are associatives and commutatives over $k$ with identity.

Let $A$ be an algebra. The Loday functor [Lod98, 6.4.4]

$$L(A, A) : \text{Fin} \to \text{Alg}$$

associates to each finite set $K$ the tensor product $A^\otimes K$ and to each map $f : K \to L$ the morphism of algebras $f_* : A^\otimes K \to A^\otimes L$ given by

$$f_* \left( \bigotimes_{x \in K} a_x \right) = \bigotimes_{y \in L} \left( \prod_{f(x) = y} a_x \right)$$

The idea of Pirashvili [Pir00, 2.1] is to build a simplicial algebra $C(K, A)$ associated to any finite simplicial set $K$ by taking the composition

$$C(K, A) : \Delta^{op} \xrightarrow{K} \text{Fin} \xrightarrow{L(A, A)} \text{Alg}$$

4
which induces a CDGA with homology denoted by $H(K, A)$. Remark that if $A$ is a CDGA then one can define $C(K, A)$ by taking the total complex of $(A^\otimes K^p)_q$ \cite[3.1]{GTZ}. We can also define $C(K, A)$ if $K$ is not finite by taking the colimit of $C(L, A)$ for all finite simplicial subsets $L \subset K$. Thus, we defined the Higher Hochschild functor

$$C : \text{sSet} \times \text{CDGA} \to \text{CDGA}$$

It preserves weak equivalences and homotopy colimits of simplicial sets \cite[3.2]{GTZ}.

**Example 3.0.1.** Let $A$ be a CDGA. For any diagram of simplicial set

$$X \leftarrow Z \to Y$$

we have a natural weak equivalence of CDGA

$$C(X \cup_Z^i Y, A) \simeq C(X, A) \underbrace{\otimes \cdots \otimes}_{C(Z, A)} C(Y, A)$$

Because we want to work with schemes, we will only study $C(K, A)$ with CDGA concentrated in degree 0, i.e. algebras. The following example is important to give a $A$-module structure on $C(K, A)$.

**Example 3.0.2.** Let $A$ be an algebra. The chain complex $C(\ast, A)$ is given by

$$\cdots \rightarrow 0 \rightarrow A \rightarrow 1 \rightarrow A \rightarrow 0$$

so $C(\ast, A)$ deformation retract of $A$ as a CDGA. In particulary, if $K$ is a pointed simplicial set then $C(K, A)$ is a CDGA over $A$:

$$A \rightarrow C(\ast, A) \rightarrow C(K, A)$$

### 3.1 Higher order Hochschild Homology

We first introduce the higher order Hochschild complex and its homology.

**Definition 3.1.1.** Let $d$ be a positive integer and $A$ an algebra. We define

$$C^{[d]}(A) = C(S^d, A)$$

the Higher Hochschild complex of order $d$ of $A$ and

$$H^{[d]}(A) = H(S^d, A)$$

the Higher Hochschild homology of order $d$ of $A$.

We will use the notations $C(A) = C^{[1]}(A)$ and $HH(A) = H^{[1]}(A)$ for the Hochschild complex of $A$ and its homology.

One can compute the first higher order Hochschild homology groups.

**Example 3.1.2.** For any positive integer $d$, there is a natural isomorphism

$$H^{[d]}_q(A) \cong \begin{cases} A & \text{if } q = 0 \\ 0 & \text{if } 0 < q < d \\ \Omega^1_A & \text{if } q = d \end{cases}$$
This can be proved using the \textit{Hodge decomposition} of Pirashvili \cite[2.5, 1.15]{Pirashvili} in characteristic zero, but also by hand in \textbf{any characteristic}. The computation depends on the parity of $d$ but the result is the same. Recall our simplicial model $S^d$ for the sphere

$$
\cdots, S^d_{d+1} = \{\ast, s_0\sigma, \ldots, s_d\sigma\}, \ S^d_d = \{\ast, \sigma\}, \ S^d_{d-1} = \{\ast\}, \ \ldots, \ S^d_0 = \{\ast\}
$$

To describe the differential, just look at the fibers over $\{\sigma\}$ by the face maps $d_i$:

$$d_i^{-1}\{\sigma\} = \begin{cases} 
\{s_0\sigma\} & \text{if } i = 0 \\
\{s_{i-1}\sigma, s_i\sigma\} & \text{if } 0 < i \leq d \\
\{s_d\sigma\} & \text{if } i = d + 1
\end{cases}
$$

\textbf{$d$ odd :} The chain complex $C^{[d]}(A)$ is given by

$$
\cdots \xrightarrow{\partial} A \otimes A^{\otimes d+1} \xrightarrow{d} A \otimes A \xrightarrow{0} A \xrightarrow{1} \cdots \xrightarrow{1} A \xrightarrow{0} A
$$

so we just have to show that $H^d_d(A) \cong \Omega^1_A$. Here we can use the \textit{Leibniz rule} definition of $\Omega^1_A$ \cite[1.1.9]{Lodder}. The boundaries of degree $d$ are generated in $A \otimes A$ by \textit{Leibniz rule} elements of the form $1 \otimes xy - y \otimes x - x \otimes y$:

$$d(1 \otimes x_0 \otimes \cdots \otimes x_d) = \left( \prod_{j \neq 0} x_j \right) \otimes x_0 + \sum_{i=1}^{d} (-1)^i \left( \prod_{j \neq i-1, i} x_j \right) \otimes x_{i-1}x_i + \left( \prod_{j \neq d} x_j \right) \otimes x_d$$

Conversely, the \textit{Leibniz rule} elements of the form $1 \otimes xy - x \otimes y - x \otimes y$ are boundaries:

$$d(1 \otimes x \otimes y \otimes 1 \otimes \cdots \otimes 1) = y \otimes x - 1 \otimes xy + x \otimes y + \sum_{i=3}^{d+1} (-1)^i xy \otimes 1$$

$$= y \otimes x - 1 \otimes xy + x \otimes y$$

\textbf{$d$ even :} The chain complex $C^{[d]}(A)$ is given by

$$
\cdots \xrightarrow{\partial} A \otimes A^{\otimes d+1} \xrightarrow{d} A \otimes A \xrightarrow{\mu} A \xrightarrow{0} \cdots \xrightarrow{1} A \xrightarrow{0} A
$$

where $\mu$ is the multiplication of $A$, so we just have to show that $H^d_d(A) \cong \Omega^1_A$. Here we can use the isomorphism $\Omega^1_A \cong \ker(\mu)/\ker(\mu)^2$ \cite[1.3.7]{Lodder}. Since $\ker(\mu)$ is generated by the elements of the form $1 \otimes x - x \otimes 1$, the boundaries of degree $d$ in $A \otimes A$ are in $\ker(\mu)^2$:

$$d(1 \otimes x_0 \otimes \cdots \otimes x_d) = \left( \prod_{j \neq 0} x_j \right) \otimes x_0 + \sum_{i=1}^{d} (-1)^i \left( \prod_{j \neq i-1, i} x_j \right) \otimes x_{i-1}x_i - \left( \prod_{j \neq d} x_j \right) \otimes x_d$$

$$= \sum_{i=1}^{d} (-1)^{i-1} \left( \prod_{j \neq i-1, i} x_j \right) (x_i \otimes x_{i-1} - 1 \otimes x_{i-1}x_i + x_{i-1} \otimes x_i)$$

$$= \sum_{i=1}^{d} (-1)^{i-1} \left( \prod_{j \neq i-1, i} x_j \right) (1 \otimes x_{i-1} - x_{i-1} \otimes 1)(x_i \otimes 1 - 1 \otimes x_i)$$

since $\sum_{i=1}^{d} (-1)^{i-1} \left( \prod_{j \neq i-1, i} x_j \right) (x_{i-1}x_i \otimes 1) = \left( \prod_{j=0}^{d} x_j \right) \sum_{i=1}^{d} (-1)^{i-1}(1 \otimes 1) = 0$.

Conversely, $\ker(\mu)^2$ is generated by boundaries:

$$d(1 \otimes x \otimes y \otimes 1 \otimes \cdots \otimes 1) = y \otimes x - 1 \otimes xy + x \otimes y + \sum_{i=3}^{d+1} (-1)^i xy \otimes 1$$

$$= (1 \otimes x - x \otimes 1)(y \otimes 1 - 1 \otimes y)$$
An important consequence of this computation is that any $A$-module can be thought of as a $C^{[d]}(A)$-module using the morphism of CDGA

$$C^{[d]}(A) \to A$$

given by the natural projection of a non-negatively graded CDGA over its homology of degree 0.

To compute all degrees of the higher order Hochschild homology groups, one can use with more hypotheses the following theorem of Pirashvilli which can be found in [Pir00].

Pirashvilli’s Theorem 3.1.3. Let $d$ be a positive integer and $A$ a smooth algebra. There is a natural isomorphism

$$H^{[d]}_{q}(A) \cong \begin{cases} 
\Omega_A^q & \text{if } q = jd \& d \text{ is odd} \\
\text{Sym}_A^q \Omega_A^1 & \text{if } q = jd \& d \text{ is even} \\
0 & \text{if } d \nmid q
\end{cases}$$

3.2 Localization

If $R$ is a commutative graded algebra (CGA for short) and $S$ is a multiplicative system of the algebra $R_0 \subset R$ then we define the localization $S^{-1}R = (S^{-1}R_0) \otimes_{R_0} R$ which is also a CGA. If $R$ is in fact a CDGA, then $S^{-1}R$ naturally has a differential. It gives rise to an exact functor

$$R{-}\text{mod} \to (S^{-1}R){-}\text{mod}$$

given by the flat base change $(S^{-1}R) \otimes_R$ which we simply denote by $S^{-1}$.

It is well known that Hochschild homology commutes with localization. Here we give the same result about Higher Hochschild homology associated to connected pointed simplicial set.

Examples 3.2.1. 1. The circle. By definition, we have $S^1 \cong * \cup^h_{|S^0} \Delta^1 \cong * \cup^h_{|S^0} *$ so Examples 3.0.1 and 3.0.2 show that the Hochschild complex of an algebra $A$ is

$$C(A) \cong A \otimes_{A^e} A$$

with $A^e = A \otimes A$. This well known weak equivalence exists in any characteristic [Lod98 1.1.13]. If $S$ is a multiplicative system of $A$ then we have $S^{-1}A \cong (S^{-1}A)^e \otimes_{A^e} A$ and so

$$S^{-1}C(A) \cong (S^{-1}A) \otimes_{A^e} A \cong (S^{-1}A) \otimes_{(S^{-1}A)^e} (S^{-1}A) \cong C(S^{-1}A)$$

since localization is a flat base change.

2. The standard simplices. Let $n \in \mathbb{N}$. Any vertex of $\Delta^n$ gives rise to a weak equivalence of simplicial set $* \to \Delta^n$. We have $A \cong C(\Delta^n, A)$ for any algebra $A$ by Example 3.0.2 [Pir00 2.4]. Thus, if $S$ is a multiplicative system of $A$ then the commutative triangle

$$\xymatrix{ S^{-1}A \ar[r] & S^{-1}C(\Delta^n, A) \ar[r] & C(\Delta^n, S^{-1}A) }$$

and Property 2-out-of-3 [Hov99 1.1.3] shows that $S^{-1}C(\Delta^n, A) \cong C(\Delta^n, S^{-1}A)$. 7
3. **Two points.** For any algebra $A$, we have $C(* \sqcup *, A) \cong C(*, A)^e$ [GTZ14, 3.1] so by Exemple 3.0.2 we have

$$S^{-1}C(* \sqcup *, A) \cong S^{-1}A \otimes A \not\cong (S^{-1}A)^e \cong C(* \sqcup *, S^{-1}A)$$

[GTZ14, 3.1].

**Theorem 3.2.2.** Let $K$ be a connected pointed simplicial set and $S$ a multiplicative system of an algebra $A$. The canonical morphism of CDGA

$$S^{-1}C(K, A) \to C(K, S^{-1}A)$$

is a quasi-isomorphism.

**Proof** Consider the skeleton of $K$

$$K^0 \subset K^1 \subset \cdots \subset K^n \subset K^{n+1} \subset \cdots \subset K$$

with $K^n \subset K$ the simplicial subset generated by the non-degenerate simplices $\sigma \in K_p$ for $p \leq n$. We have two weak equivalences of simplicial sets

$$K^n \cong K^{n-1} \sqcup_{(\partial \Delta^n) \cup \Sigma_n} (\Delta^n) \cup \Sigma_n \quad \& \quad K \cong \text{hocolim}_{n \in \mathbb{N}} K^n$$

with $\Sigma_n$ the set of non-degenerated $n$-simplices of $K$ [Hov99, 5.1.3]. Since localization and Higher Hochschild functor preserve homotopy colimits [GTZ14, 3.2], there is a factorization

$$S^{-1}C(K, A) \cong \text{hocolim}_{n \in \mathbb{N}} S^{-1}C(K^n, A) \to \text{hocolim}_{n \in \mathbb{N}} C(K^n, S^{-1}A) \cong C(K, S^{-1}A)$$

Thus, we just have to show that the canonical morphism of CDGA

$$S^{-1}C(K^n, A) \to C(K^n, S^{-1}A)$$

is a quasi-isomorphism from a certain rank. We work by induction on $n \geq 1$. For $n = 1$, the geometric realization [Hov99, 3.1] of the simplicial set $K^1$ is a connected graph [Hat02, 0.1], so there exist a subset $E \subset \Sigma_1$ and a weak equivalence of simplicial set $K^1 \cong (S^1)^{\cup \Sigma E}$ [Hat02, 0.7]. By Exemple 3.2.1.1 we have weak equivalences of CDGA

$$S^{-1}C(K^1, A) \cong S^{-1} \left( C(A)^{\otimes^l E} \right)$$

$$\cong \left( S^{-1}C(A) \right)^{\otimes^l_{S^{-1}A} E}$$

$$\cong \left( C(S^{-1}A) \right)^{\otimes^l_{S^{-1}A} E} \cong C(K^1, S^{-1}A)$$

Assume that $n > 1$ and choose a vertex $* \to \partial \Delta^n$. By Exemple 3.0.1 and induction, we have weak equivalences of CDGA

$$S^{-1}C(K^n, A) \cong S^{-1}C(K^{n-1}, A) \otimes C(\Delta^n, A)^{\otimes \Sigma_n}$$

$$\sim C(K^{n-1}, S^{-1}A) \otimes C(\Delta^n, A)^{\otimes \Sigma_n}$$

$$\sim C(K^{n-1}, S^{-1}A) \otimes (S^{-1}C(\Delta^n, A))^{\otimes \Sigma_n}$$
Here we used the flat base change
\[(A^\otimes n^1)^\otimes \Sigma_n \to (S^{-1}A \otimes A^\otimes n)\otimes \Sigma_n\]
and the commutative triangle of CDGA
\[
\begin{array}{ccc}
C(\partial \Delta^n, A)^\otimes \Sigma_n & \to & C(K^{n-1}, S^{-1}A) \\
\downarrow & & \downarrow \\
(S^{-1}C(\partial \Delta^n, A))^\otimes \Sigma_n & \to & \\
\end{array}
\]
This computation show that it is enough to show that [Theorem 3.2.2] is true for the simplicial sets $\Delta^d$ and $\partial \Delta^d$ for any integer $d \geq 2$ [Hin97, 3.3.2]. For $\Delta^d$, it is [Example 3.2.1.2].
For $\partial \Delta^d$, we can work by induction on $d \geq 2$ using the homotopy pushout $\partial \Delta^d \simeq \Delta^{d,0} \cup_{\Delta^{d-1}} \Delta^{d-1} \simeq * \cup_{\partial \Delta^{d-1}} *$ [Hov99, 3.1]. If $d = 2$ then $\partial \Delta^2 = * \cup_{\partial \Delta^1} *$ as in [Example 3.2.1.1]. Assume that $d \geq 2$ and choose a vertex $* \to \partial \Delta^{d-1}$. Since localization is a flat base change, we have by [Example 3.0.1] and induction weak equivalences of CDGA
\[
S^{-1}C(\partial \Delta^d, A) \simeq S^{-1}A \otimes_{C(\partial \Delta^{d-1}, A)} A \\
\simeq S^{-1}A \otimes_{C(\partial \Delta^{d-1}, S^{-1}A)} S^{-1}A \\
\simeq S^{-1}A \otimes_{C(\partial \Delta^{d-1}, S^{-1}A)} S^{-1}A \simeq C(\partial \Delta^d, S^{-1}A)
\]
This ends the proof.

**Corollary 3.2.3.** Le $d$ a positive integer and $S$ a multiplicative system of an algebra $A$. The canonical morphism of CDGA
\[
S^{-1}C[\Delta^d](A) \to C[\Delta^d](S^{-1}A)
\]
is a quasi-isomorphism.

**Remark 3.2.4** There is an other way to prove [Theorem 3.2.2]. Any pointed simplicial set is weak equivalent to a minimal pointed simplicial set [May82, 9]. Any connected minimal simplicial set $K$ has a single vertex [May82, 9.2] and one can show that $K$ is weak equivalent to the homotopy colimit of its pointed simplices $* \to \Delta^d \to K$. Since localization and Higher Hochschild complex preserve homotopy colimits, [Example 3.2.1.2] leads to the general case.

## 4 Grothendieck-Loday type definition

In all of this section, if nothing else is specified, $k$ is a field of characteristic zero. Recall that algebras are associative and commutative over $k$ with identity.

Let $X$ be a ringed space. The injective model structure on the category $\text{Ch}(\mathcal{O}_X)$ of complexes of $\mathcal{O}_X$-modules is a model structure such that the weak equivalences are the quasi-isomorphisms and the cofibrations are the monomorphisms [Hov01, 2]. The fibrant objects of $\text{Ch}(\mathcal{O}_X)$ are injective in the sense of Spaltenstein [Spa88, 1.5] [Hov99, 1.2.10] so the hom complex functor
\[
\text{Hom}_{\text{Ch}(\mathcal{O}_X)} : \text{Ch}(\mathcal{O}_X)^{\text{op}} \times \text{Ch}(\mathcal{O}_X) \to \text{Ch}
\]
has a right derived functor.
Let $K$ be a pointed simplicial set. We denote by $C(K, X)$ the sheaf of CDGA on $X$ associated to the presheaf of CGDA on $X$

$$U \mapsto C(K, \mathcal{O}_X(U))$$

One can define the $K$-cohomology of $X$ with coefficients in any complex of $\mathcal{O}_X$-modules $\mathcal{D}$, let $H(K, X, \mathcal{D})$, by taking the cohomology of $\mathbb{R}\text{Hom}_{\text{Ch}}(\mathcal{C}(K, X), \mathcal{D})$. In this paper, we are interested in the case where $K = S^d$ for any positive integer $d$ and $\mathcal{D}$ is an $\mathcal{O}_X$-module thought of as a complex concentrated in degree 0.

**Definition 4.0.1.** Let $d$ a positive integer and $X$ a ringed space. We define

$$\mathcal{C}^{[d]}_X = \mathcal{C}(S^d, X)$$

the Higher Hochschild sheaf of order $d$ of $X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define

$$H_{[d]}(X, \mathcal{F}) = H\left(\mathbb{R}\text{Hom}_{\text{Ch}}(\mathcal{C}^{[d]}_X, \mathcal{F})\right)$$

the Higher Hochschild cohomology of order $d$ of $X$ with coefficients in $\mathcal{F}$.

We will use the notations $\mathcal{C}_X = \mathcal{C}^{[1]}_X$ and $HH(X, \mathcal{F}) = H_{[1]}(X, \mathcal{F})$ for the Hochschild sheaf of $X$ and for the Hochschild cohomology of $X$ with coefficients in $\mathcal{F}$. When $X$ is a scheme, we recover the Grothendieck-Loday type definition [Swa96, 2]. In fact, if we choose an injective resolution of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{I}$, then we get a fibrant replacement $\mathcal{I}$ of the complex of $\mathcal{O}_X$-modules $\mathcal{F}$, since $\mathcal{I}$ is bounded above [Hov01, 2.12]. Hence the chain complex $\text{Hom}_{\text{Ch}}(\mathcal{C}_X, \mathcal{I})$ computes both the derived hom complex $\mathbb{R}\text{Hom}_{\text{Ch}}(\mathcal{C}_X, \mathcal{F})$ and the hyperext functor $\mathbb{E}\text{xt}_{\mathcal{O}_X}(\mathcal{C}_X, \mathcal{F})$.

**Example 4.0.2.** A non-separated scheme. Let $A$ be an algebra over a field of any characteristic and $s \in A$ an element such that the canonical morphism $A \to A_s$ is not surjective. We define $S = \text{Spec}(A)$, $U = D(s) = \{ p \in S : s \notin p \}$ and $X = S \cup_U S$ the affine scheme $S$ with doubled subvariety $Z = V(s) = S \setminus U$ [Gro60, I, 2.3.2]. By assumption, $X$ is not separated [Gro60 I, 5.5.6] and there is a diagram of schemes

$$1_S : S \xrightarrow{i} X \xrightarrow{p} S$$

since $\Gamma \mathcal{O}_X = \Gamma \mathcal{O}_S \times_{\mathcal{O}_S(U)} \Gamma \mathcal{O}_S = A \times_A A$. The relation $i^* p^* = j^* p^* = 1_{\mathcal{O}_S-\text{mod}}$ shows that the functor $p^*$ is exact and so the pair

$$p^* : \text{Ch}(\mathcal{O}_S) \xrightarrow{i} \text{Ch}(\mathcal{O}_X) : p_*$$

is a Quillen adjonction [Hov01 2.13]. Moreover, for any simplicial set $K$, we have weak equivalences of sheaves of CDGA on $X$

$$\mathcal{C}(K, X) \cong p^* \mathcal{C}(K, S) \simeq \mathbb{L} p^* \mathcal{C}(K, X)$$

Deriving the enriched adjonction $\text{Hom}_{\text{Ch}}(\mathcal{O}_S) \circ (p^* \times 1) \cong \text{Hom}_{\text{Ch}}(\mathcal{O}_X) \circ (1 \times p_*)$ [Hov99 1.3.7], we obtain a natural isomorphism

$$H^\ast(K, X, \mathcal{D}) \cong H^\ast(K, S, \mathbb{R} p_* \mathcal{D})$$

for any complex of $\mathcal{O}_X$-modules $\mathcal{D}$. In Section 6, we will compute with more precision the cohomology groups $HH^n(X, \mathcal{F})$ for any $\mathcal{O}_X$-module $\mathcal{F}$. It
4.1 The Hodge decomposition

The tools we are going to use are close to those of Swan [Swa96]. To get a Hodge decomposition of higher order Hochschild cohomology of schemes, we also refer to Pirashvili’s work [Pir00].

For any ringed space $X$ and any pointed simplicial set $K$, one can define the $K$-homology sheaf of $X$ by $\mathcal{H}(K, X) = H(C(K, X))$. We are particularly interested in the homology of the higher order Hochschild sheaf.

**Definition 4.1.1.** Let $d$ be a positive integer and $X$ a ringed space. We define

$$\mathcal{H}^{[d]}_X = H(C^{[d]}_X)$$

the **Higher Hochschild homology sheaf of order $d$ of $X$**.

We will use the notation $\mathcal{H}_X = \mathcal{H}^{[1]}_X$ for the Hochschild homology sheaf of $X$.

Just like Swan explained [Swa96] 2.4, when $X$ is a scheme and $K$ is connected, the sheaves $\mathcal{H}_q(K, X)$ are quasi-coherent such that on each affine open set $U$ of $X$, we have

$$\mathcal{H}(K, X)(U) = H(K, \mathcal{O}_X(U))$$

This is a consequence of Theorem 3.2.2 and the fact that sheafification commutes with homology [Gro57] 3.1 so that the sheaf $\mathcal{H}(K, X)$ is associated to the presheaf

$$U \mapsto H(K, \mathcal{O}_X(U))$$

One can compute the first higher order Hochschild homology sheaves.

**Example 4.1.2.** Let $X$ be a separated scheme over a field of any characteristic. For any positive integer $d$, there is a natural isomorphism

$$\mathcal{H}^{[d]}_{X,q} \cong \begin{cases} \mathcal{O}_X & \text{if } q = 0 \\ 0 & \text{if } 0 < q < d \\ \Omega^1_X & \text{if } q = d \end{cases}$$

It was constructed on any affine open set $U$ of $X$ in Example 3.1.2, since we always have

$$\Omega^1_X|_U \cong \Omega^1_U$$

[Gro67] IV, 17.1.3.(i), 17.2.4]. These natural isomorphisms are compatible with restriction, which allows us to patch them [Har77] II, Ex, 1.15.

We can also compute higher degrees in the smooth case.

**Theorem 4.1.3.** Let $d$ be a positive integer and $X$ a separated smooth scheme. There is a natural isomorphism

$$\mathcal{H}^{[d]}_{X,q} \cong \begin{cases} \Omega^1_X & \text{if } q = jd \& d \text{ is odd} \\ \text{Sym}^j \mathcal{O}_X \Omega^1_X & \text{if } q = jd \& d \text{ is even} \\ 0 & \text{if } d \nmid q \end{cases}$$

**Proof.** We use the same argument as for Example 4.1.2: this natural isomorphism is given on any affine open set $U$ of $X$ by Theorem 3.1.3 [Gro67] IV, 17.3.2.(ii)]. We just have to show that the restriction functor $|_U$ always commutes with exterior and symmetric powers: for any $\mathcal{O}_X$-module $\mathcal{F}$ and any positive integer $j$, there are colimit descriptions
\[ \bigwedge^j_{\mathcal{O}_X} \mathcal{F} = \underset{\sigma \in S_j}{\text{colim}} \left( \mathcal{F}^{\otimes j} \overset{\pm \sigma}{\rightarrow} \mathcal{F}^{\otimes j} \right) \quad \& \quad \text{Sym}^j_{\mathcal{O}_X} \mathcal{F} = \underset{\sigma \in S_j}{\text{colim}} \left( \mathcal{F}^{\otimes j} \overset{\sigma}{\rightarrow} \mathcal{F}^{\otimes j} \right) \]

where the sign ± is the signature of the permutation, and restriction functor \( |_{\sigma} \) commutes with colimits and tensor products [Gro60, 0.4.3.1, 4.3.3.1].

We study the properties of the higher order Hochschild homology sheaf because of the next proposition, which is inspired by the Hodge spectral sequence of Swan [Swa96, 2.3].

**Proposition 4.1.4.** Let \( X \) be a ringed space, \( K \) a pointed simplicial set and \( \mathcal{F} \) an \( \mathcal{O}_X \)-module. There is a natural spectral sequence

\[ E_2^{pq} = \text{Ext}^p_{\mathcal{O}_X} \left( (\mathcal{H}_q(K, X), \mathcal{F}) \right) \implies H^{p+q}(K, X, \mathcal{F}) \]

**Proof.** In the same way that we computed the Hochschild cohomology of order 1 of \( X \) with coefficients in \( \mathcal{F} \), we can choose an injective resolution of \( \mathcal{O}_X \)-modules \( \mathcal{F} \rightarrow \mathcal{I} \) to get a fibrant replacement \( \mathcal{I} \) of the complex of \( \mathcal{O}_X \)-modules \( \mathcal{F} \). Now we have a double complex \( E_2^{pq} = \text{Hom}_{\mathcal{O}_X} \left((C_\mathcal{I}(K, X), \mathcal{T}^p)\right) \) with total cohomology \( H(K, X, \mathcal{F}) \). If we filter it by columns [Wei94, 5.6.1] then we get the announced spectral sequence since the first page \( E_1 \) is the vertical cohomology which is preserved by the exact functor \( \text{Hom}_{\mathcal{O}_X} \) and the second page \( E_2 \) is the horizontal cohomology which computes the ext functors:

\[ E_1^{pq} = H^q \left( \text{Hom}_{\mathcal{O}_X} \left(C(K, X), \mathcal{T}^p\right) \right) \cong \text{Hom}_{\mathcal{O}_X} \left( \mathcal{H}_q(K, X), \mathcal{T}^p \right) \]

\[ E_2^{pq} \cong H^p \left( \text{Hom}_{\mathcal{O}_X} \left( \mathcal{H}_q(K, X), \mathcal{T} \right) \right) = \text{Ext}^p_{\mathcal{O}_X} \left( \mathcal{H}_q(K, X), \mathcal{F} \right) \]

Using the spectral sequence of [Proposition 4.1.4] for \( K = S^d \), we obtain the Hodge decomposition for higher order Hochschild cohomology of smooth varieties.

**Theorem 4.1.5.** Let \( d \) be a positive integer, \( X \) a separated smooth scheme and \( \mathcal{F} \) an \( \mathcal{O}_X \)-module. If \( d \) is odd then we have a natural isomorphism

\[ H^n_{[d]}(X, \mathcal{F}) \cong \bigoplus_{p+jd=n} H^p(X, \mathcal{T}_X^j \otimes \mathcal{F}) \]

where \( \mathcal{T}_X^j = (\Omega^j_X)^\vee \). If \( d \) is even then we have a natural isomorphism

\[ H^n_{[d]}(X, \mathcal{F}) \cong \bigoplus_{p+jd=n} H^p(X, S_X^j \otimes \mathcal{F}) \]

where \( S_X^j = (\text{Sym}^j_{\mathcal{O}_X} \Omega^1_X)^\vee \).

**Proof.** First, we show that the spectral sequence of [Proposition 4.1.4] degenerates at the page \( E_2 \). For \( d = 1 \), this was done by Swan [Swa96, 2.6], but this is also a consequence of Loday’s \( \lambda \)-decomposition [Lod98, 4.5.10]. It induces a decomposition \( C_X^{[j]} = \bigoplus_{i \geq 0} C^{(i)} \) such that \( H_q(C^{(i)}) \neq 0 \) if and only if \( q = i \) [Lod98, 4.5.12, 3.4.4]. Therefore the page \( E_2 \) is a direct sum of single columns and the differentials are zero. For \( d \geq 2 \), we can use [Theorem 4.1.3]. We have \( H^q_{X,q} 
eq 0 \) if and only if \( d \) divides \( q \) so at the page \( E_2 \), each non-zero row is surrounded by zero rows.
Hence the differentials are zero and we have a natural isomorphism

\[ H_{[d]}^n(O_X, \mathcal{F}) \cong \bigoplus_{p+q=n} \text{Ext}_{O_X}^p(H_{X,jd}^{[d]}, \mathcal{F}) = \bigoplus_{p+jd=n} \text{Ext}_{O_X}^p(H_{X,jd}^{[d]}, \mathcal{F}) \]

Now we want to compute \( \text{Ext}_{O_X}^p(H_{X,jd}^{[d]}, \mathcal{F}) \). We can use the usual Grothendieck spectral sequence \([\text{Gro57}, 2.4.1]\) of \( \text{Hom}_{O_X}(H_{X,jd}^{[d]}, \mathcal{F}) = \Gamma \circ \text{Hom}_{O_X}(H_{X,jd}^{[d]}, \mathcal{F}) \):

\[ F_{pq}^n = H^p(X, \text{Ext}_{O_X}^q(H_{X,jd}^{[d]}, \mathcal{F})) \implies \text{Ext}_{O_X}^{p+q}(H_{X,jd}^{[d]}, \mathcal{F}) \]

Since \( X \) is smooth, \( H_{X,jd}^{[d]} \) is locally free of finite rank \([\text{Gro67}, IV, 17.2.3]\) and we have \( \text{Ext}_{O_X}^q(H_{X,jd}^{[d]}, \mathcal{F}) = 0 \) for \( q > 0 \) \([\text{Gro57}, 4.2.3]\). Thus, the spectral sequence \( F \) induces a natural isomorphism

\[ \text{Ext}_{O_X}^p(H_{X,jd}^{[d]}, \mathcal{F}) \cong H^p(X, \text{Hom}_{O_X}(H_{X,jd}^{[d]}, \mathcal{F})) \]

Moreover, the natural morphism \((H_{X,jd}^{[d]})^\vee \otimes \mathcal{F} \to \text{Hom}_{O_X}(H_{X,jd}^{[d]}, \mathcal{F})\) is an isomorphism \([\text{Gro60}, 0, 5.4.2.1]\), which ends the proof.

**Example 4.1.6.** Let \( d \) be a positive integer, \( X \) a separated smooth scheme and \( \mathcal{F} \) an \( O_X \)-module. One can visualize the higher order Hochschild cohomology groups:

\[
\begin{align*}
H_{[d]}^0(X, \mathcal{F}) & \cong \Gamma \mathcal{F} \\
& \vdots \\
H_{[d]}^n(X, \mathcal{F}) & \cong H^n(X, \mathcal{F}) & 0 < n < d \\
& \vdots \\
H_{[d]}^d(X, \mathcal{F}) & \cong H^d(X, \mathcal{F}) \oplus \Gamma(\mathcal{T}_X \otimes \mathcal{F}) \\
& \vdots \\
H_{[d]}^n(X, \mathcal{F}) & \cong H^n(X, \mathcal{F}) \oplus H^{n-d}(X, \mathcal{T}_X \otimes \mathcal{F}) & d < n < 2d \\
& \vdots \\
H_{[d]}^{2d}(X, \mathcal{F}) & \cong H^{2d}(X, \mathcal{F}) \oplus H^d(X, \mathcal{T}_X \otimes \mathcal{F}) \oplus \Gamma(D_X^1 \otimes \mathcal{F}) \\
& \vdots
\end{align*}
\]
where $T_X = T_1^X = S_1^X = (\Omega_X^1)^\vee$ and $D_X^d$ is $T_X^d$ for $d$ odd and $S_X^d$ for $d$ even. Assuming that $d > \dim(X)$, we have for any integer $n \geq 0$ a natural isomorphism

$$H^n_{[d]}(X, F) \cong H^n(X, D_X^d \otimes F)$$

where $n = jd + p$ is the Euclidean division of $n$ by $d$ with remainder $p$ [Har77, III, 2.7].

### 4.2 Affine case

The definition of the $K$-cohomology of a scheme lead us to the following question. Does this definition be equivalent to the algebraic definition of the Higher Hochschild cohomology [Gin17, 3.9]? In other words, do we have an isomorphism

$$H(K, \text{Spec}(A), \widetilde{M}) \cong H(K, A, M)$$

for any simplicial set $K$ and any algebra $A$? Exemple 3.2.1.3 shows that generally the complexe of sheaves $C(K, \text{Spec}(A))$ is not weak equivalent to the complexe of sheaves associated to the Higher Hochschild complex $C(K, A)$.

Using Theorem 3.2.2, we will show that the $K$-cohomology of an affine scheme is isomorphic to the $K$-cohomology of the underlying algebra if the simplicial set $K$ is connected. We will also compare the spectral sequence of Proposition 4.1.4 and the spectral sequence of Pirashvili [Pir00, 2.4].

Let $X = \text{Spec}(A)$ be an affine scheme. Recall the functor

$$\sim : \text{A-mod} \rightarrow \mathcal{O}_X-\text{mod}$$

which associates to each $A$-module $M$ a quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ [Gro60, I, 1.3]. The following Quillen adjunction is the main tool.

**Proposition 4.2.1.** Let $X = \text{Spec}(A)$ be an affine scheme. The pair

$$\sim : \text{Ch}(A) \xrightarrow{\sim} \text{Ch}(\mathcal{O}_X) : \Gamma$$

is a Quillen adjunction between injective model structures. Let $C$ be a complex of $A$-modules and $M$ a $A$-module. There is a natural weak equivalence of chain complexes

$$\mathbb{R}\text{Hom}_{\text{Ch}(\mathcal{O}_X)}(\widetilde{C}, \widetilde{M}) \simeq \mathbb{R}\text{Hom}_{\text{Ch}(A)}(C, M)$$

**Proof.** It is a standard fact that this pair is an adjunction [Har77, II, Ex., 5.3]. In order to show that it is a Quillen adjunction, we just have to show that the functor $\sim$ preserves cofibrations and trivial cofibrations [Hov99, 1.3.4], which is a consequence of its exactness [Har77, II, 5.2]. Deriving the enriched adjunction $\text{Hom}_{\text{Ch}(\mathcal{O}_X)} \circ (\sim \times 1) \cong \text{Hom}_{\text{Ch}(\mathcal{O}_X)} \circ (1 \times \Gamma)$ [Hov99, 1.3.7], we obtain the announced weak equivalence because functor $\sim$ is its left derived functor since all objects are cofibrant in $\text{Ch}(A)$, and we have a natural weak equivalence of chain complexes of $A$-modules $M \simeq \mathbb{R}\Gamma\widetilde{M}$ since quasi-coherent $\mathcal{O}_X$-modules have no sheaf cohomology on $X = \text{Spec}(A)$ [Gro60, III, 1.3.1].

Before getting to the main theorem of this section, we have to compare Definition 4.0.1 with a quasi-coherent version.
**Lemma 4.2.2.** Let $K$ a connected pointed simplicial set and $X = \text{Spec}(A)$ an affine scheme. The canonical morphism of sheaves of CDGA on $X$

\[
\hat{C}(K, A) \rightarrow C(K, X)
\]

is a quasi-isomorphism.

**Proof.** This morphism is given on open subset $D(s) = \{ p \in X : s \notin p \}$ of $X$ by

\[
C(K, A)_s \rightarrow C(K, A_s) \rightarrow C(K, X)(D(s))
\]

Theorem 3.2.2 shows that the presheaves of CDGA on $X$

\[
D(s) \mapsto C(K, A)_s \quad \text{and} \quad D(s) \mapsto C(K, A_s)
\]

are quasi-isomorphic. Thus, the associated sheaves are quasi-isomorphic \cite[3.1]{Gro57}.

**Theorem 4.2.3.** Let $K$ be a connected pointed simplicial set, $X = \text{Spec}(A)$ an affine scheme and $\mathcal{F} = \hat{M}$ a quasi-coherent $\mathcal{O}_X$-module. There is a natural isomorphism

\[
H^n(K, X, \mathcal{F}) \cong H^n(K, M)
\]

**Proof.** Lemma 4.2.2 and Proposition 4.2.1 give us two natural weak equivalences

\[
\mathbb{R}\text{Hom}_{\text{Ch}(\mathcal{O}_X)}(\hat{C}(K, X), \hat{M}) \cong \mathbb{R}\text{Hom}_{\text{Ch}(\mathcal{O}_X)}(\hat{C}(K, A), \hat{M}) \cong \mathbb{R}\text{Hom}_{\text{Ch}(A)}(C(K, A), M)
\]

and the derived hom complex $\mathbb{R}\text{Hom}_{\text{Ch}(A)}(C(K, A), M)$ can be computed by the chain complex $\text{Hom}_{\text{Ch}(A)}(C(K, A), M)$. In fact, for any integer $n \geq 0$, $A^{\otimes K_n}$ is a vector space, so $A^{\otimes K_n}$ is a free $A$-module. This implies that $C(K, A)$ is a cofibrant complex of $A$-modules for the projective model structure \cite[2.3.6]{Hov99}.

**Remark 4.2.4.** Projective schemes. Let $X = \text{Proj}(A)$ be a projective scheme associated to a graded algebra $A = \bigoplus_{n \in \mathbb{N}} A_n$ \cite[II, 2]{Gro60}. As in Lemma 4.2.2, for any connected pointed simplicial set $K$, the canonical morphism of sheaves of CDGA on $X$

\[
\hat{C}(K, A) \rightarrow C(K, X)
\]

is a quasi-isomorphism, where $C_q(K, A)$ is the $\mathcal{O}_X$-module associated to the graded $A$-module $C_q(K, A) = \bigoplus_{n \in \mathbb{N}} C_q(K, A_n)$ for any integer $q \geq 0$. In fact, this morphism is given on open subset $D_+(s) = \{ p \in X : s \notin p \}$ of $X$ by

\[
(C(K, A)_s)_{0} \rightarrow C(K, (A_s)_{0}) \rightarrow C(K, X)(D_+(s))
\]

so we just have to check that Hochschild differentials preserve the graduation of $A$, since they already commute with localization : for any map $f : L \rightarrow L'$, we have

\[
\left| f_\ast \left( \bigotimes_{x \in L} a_x \right) \right| = \left| \bigotimes_{y \in L'} \left( \prod_{f(x) = y} a_{f(x)} \right) \right| = \left| \sum_{x \in L} \left| a_x \right| \right| = \left| \bigotimes_{x \in L} a_x \right|
\]

where $|a|$ is the degree of an homogeneous element $a$ of any graded algebra. As in Proposition 4.2.1 we also have a Quillen adjunction

\[
\sim : \text{Ch}(A - \text{grmod}) \leftrightarrow \text{Ch}(\mathcal{O}_X) : \Gamma_\ast
\]

but in general, we don’t have an analogue of the weak equivalence between the derived hom complexes as in \cite[2.2.2]{Gro60} since quasi-coherent $\mathcal{O}_X$-modules may have sheaf cohomology on $X$ \cite[III, 2.2.2]{Gro60}. 

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Following Theorem 4.2.3, we want to compare the spectral sequences approaching $K$-cohomology of affine schemes. Let $\text{Vect}_{\text{Fin'}}$ be the category of contravariant functors over the category of pointed finite sets with values in the category of vector spaces. Pirashvili associated to any functor $F : (\text{Fin'})^\text{op} \to \text{Vect}$ and any pointed finite simplicial set $K$ a cosimplicial vector space $F(K)$ by taking the composition

$$F(K) : \Delta \xrightarrow{K} (\text{Fin'})^\text{op} \xrightarrow{F} \text{Vect}$$

and then a spectral sequence approaching its cohomology

$$E_2^{pq} = \text{Ext}^p_{\text{Vect}_{\text{Fin'}}} \left( J_q(H(K)), F \right) \Longrightarrow H^{p+q}(F(K))$$

where $J$ is the graded version of the dual Loday functor [Pir00, 1.7] and $H(K)$ is the homology of $K$ with coefficients in $k$. We are particularly interested in the case where $K$ is connected and $F$ is the functor $H(A, M) = \text{Hom}_A(L(A, A), M)$ for an algebra $A$ and an $A$-module $M$ so that the cohomology of $F(K)$ is $K$-cohomology of $A$ with coefficients in $M$ [Gin17, 3.9].

Let $\text{Ch}_{\text{Fin'}}$ be the category of contravariant functors over the category of pointed finite sets with values in the category of chain complexes. It is a model category such that the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms [Hir03, 11.6.1]. The cofibrant objects of $\text{Ch}_{\text{Fin'}}$ are projective in the sense of Spaltenstein [Spa88, 1.4] [Hov99, 1.2.10] so the hom complex functor

$$\text{Hom}_{\text{Ch}_{\text{Fin'}}} : (\text{Ch}_{\text{Fin'}})^\text{op} \times \text{Ch}_{\text{Fin'}} \to \text{Ch}$$

has a right derived functor. To get an isomorphism between the spectral sequence of Proposition 4.1.4 and the Pirashvili’s spectral sequence, we will use another Quillen adjunction.

**Proposition 4.2.5.** Let $A$ be an algebra. The pair

$$\otimes_{\text{Fin'}} L(A, A) : \text{Ch}_{\text{Fin'}} \xrightarrow{\text{Ch}(A)} \text{Ch}(A) : H(A, \quad)$$

is a Quillen adjunction. Let $F : (\text{Fin'})^\text{op} \to \text{Vect}$ be a functor and $M$ a $A$-module. There is a natural weak equivalence of chain complexes

$$\mathcal{R}\text{Hom}_{\text{Ch}(A)} \left( F \otimes_{\text{Fin'}} L(A, A), M \right) \simeq \mathcal{R}\text{Hom}_{\text{Ch}_{\text{Fin'}}} \left( F, H(A, M) \right)$$

**Proof.** It is a standard fact that this pair is an adjunction [Mac98 IX, 5, 6]. In order to show that it is a Quillen adjunction, we just have to show that functor $H(A, \quad)$ preserves fibrations and trivial fibrations [Hov99, 1.3.4], which is a consequence of its exactness. In fact, we saw that for any positive integer $n$, $A^{\otimes n-1}$ is a vector space, so $A^{\otimes n}$ is a free $A$-module. Deriving the enriched adjunction $\text{Hom}_{\text{Ch}(A)} \circ (\otimes_{\text{Fin'}} L(A, A) \times 1) \simeq \text{Hom}_{\text{Ch}_{\text{Fin'}}} \circ (1 \times H(A, \quad))$ [Hov99, 1.3.7], we obtain the announced weak equivalence because $H(A, \quad) = \mathcal{R}H(A, \quad)$ since all object of $\text{Ch}(A)$ are fibrant.

Remark that we could have formulated a more general adjunction with an arbitrary functor $G : \text{Fin'} \to \text{Vect}$ instead of $L(A, A)$, so that derived functor $\otimes_{\text{Fin'}}^G$ always exists. For any pointed finite simplicial set $L$, Pirashvili defined a functor

$$h_L : (\text{Fin'})^\text{op} \to s\text{Vect}$$
which associates to each pointed finite set $K$ the free vector space over $\Hom_{\Fin'}(K, L)$. The simplicial structure of $h_L(K)$ is naturally given by the simplicial structure of $L.$ Pirashvili showed that we have a natural isomorphism of simplicial vector spaces

$$h_L \otimes_{\Fin'} G \cong G(L)$$

[Pir00, 1.5]. We can compute the homology of $G(L)$ with this formula.

**Lemma 4.2.6.** Let $q$ be a non-negative integer, $K$ a pointed finite simplicial set and $F : \Fin' \to \Vect$ a functor. There is a natural weak equivalence of chain complexes

$$H_q(h_K) \otimes_{\Fin'} F \approx H_q(F(K))$$

**Proof.** $H_q(h_K) \otimes_{\Fin'} F \approx H_q\left(h_K \otimes_{\Fin'} F\right) = H_q(h_K \otimes_{\Fin'} F) \approx H_q(F(K))$ [Pir00 1.1].

**Theorem 4.2.7.** Let $K$ be a connected pointed simplicial set, $X = \Spec(A)$ an affine scheme and $\mathcal{F} = \widetilde{M}$ a quasi-coherent $\mathcal{O}_X$-module. The following spectral sequences are naturally isomorphic

$$\Ext_{\mathcal{O}_X}^p\left(\mathcal{H}_q(K, X), \mathcal{F}\right) \implies H^{p+q}(K, X, \mathcal{F})$$

$$\Ext_{\Vect_{\Fin'}}^p\left(\mathcal{J}_q(H(K)), \mathcal{H}(A, M)\right) \implies H^{p+q}(K, A, M)$$

**Proof.** [Lemma 4.2.2 and Proposition 4.2.1] give rise to natural isomorphisms

$$\Ext_{\mathcal{O}_X}^p\left(\mathcal{H}_q(K, X), \mathcal{F}\right) \cong \Ext_{\mathcal{O}_X}^p\left(\mathcal{H}_q(K, A), \tilde{M}\right) \cong \Ext_A^p\left(\mathcal{H}_q(K, A), M\right)$$

for any integers $p, q \geq 0.$ [Lemma 4.2.6 and Proposition 4.2.5] give us natural isomorphisms

$$\Ext_A^p\left(\mathcal{H}_q(K, A), M\right) = H^p\left(\mathbb{R}\Hom_{\Ch(A)}\left(\mathcal{H}_q(K, A), M\right)\right)$$

$$\cong H^p\left(\mathbb{R}\Hom_{\Ch(A)}\left(h_K \otimes_{\Fin'} \mathcal{L}(A, A), M\right)\right)$$

$$\cong H^p\left(\mathbb{R}\Hom_{\Fin'}\left(h_K, \mathcal{H}(A, M)\right)\right) = \Ext_{\Vect_{\Fin'}}^p\left(h_K, \mathcal{H}(A, M)\right)$$

for any integer $p, q \geq 0,$ which ends the proof since Pirashvili showed that the second spectral sequence is in fact isomorphic to the spectral sequence

$$\Ext_{\Vect_{\Fin'}}^p\left(h_K, \mathcal{H}(A, M)\right) \implies H^{p+q}\left(\Hom_{\Vect_{\Fin'}}\left(h_K, \mathcal{H}(A, M)\right)\right)$$

[Pir00, 1.6, 2.4].

**Corollary 4.2.8.** Let $d$ be a positive integer, $X = \Spec(A)$ an affine scheme and $\mathcal{F} = \tilde{M}$ a quasi-coherent $\mathcal{O}_X$-module. The following spectral sequences are naturally isomorphic

$$\Ext_{\mathcal{O}_X}^p\left(\mathcal{H}_{X,q}^d, \mathcal{F}\right) \implies H^{p+q}_{[d]}(X, \mathcal{F})$$

$$\Ext_{\Vect_{\Fin'}}^p\left(\mathcal{J}_q(H(S^d)), \mathcal{H}(A, M)\right) \implies H^{p+q}_{[d]}(A, M)$$
Gerstenhaber-Schack type definition

In all of this section, if nothing else is specified, $k$ is a field of characteristic zero. Recall that algebras are associative and commutative over $k$ with identity.

Let $X$ be a separated scheme. We call the contravariant functors over the category $\text{Aff}(X)$ of affine open sets of $X$ presheaves on $X$. One can also build a Higher Hochschild functor with presheaves by taking for any finite simplicial set $K$ and any presheaf of algebras $\mathcal{O}$ on $X$ the presheaf of CDGA on $X$

$$C(K, \mathcal{O}) : U \to C(K, \mathcal{O}(U))$$

We can define $C^{[d]}(\mathcal{O}) = C(S^d, \mathcal{O})$ and $H^{[d]}(\mathcal{O}) = H(C^{[d]}(\mathcal{O}))$ as in Definition 3.1.1 for any positive integer $d$. Example 3.1.2 shows that we always have an isomorphism of algebras $H^{[d]}(\mathcal{O}) \cong \mathcal{O}$ which allows us to think of any $\mathcal{O}$-module as a $C^{[d]}(\mathcal{O})$-module via the morphism of presheaves of CDGA on $X$

$$C^{[d]}(\mathcal{O}) \to \mathcal{O}$$

Following the Gerstenhaber-Schack type definition [Swa96, 3] and the weak equivalence of simplicial set $S^d \simeq \ast \cup h_{S^{d-1}} \ast$, we would like to define the Higher Hochschild cohomology of order $d$ of $X$ with coefficients in any $C^{[d-1]}(\mathcal{O})$-module $\mathcal{M}$ by taking the cohomology of $\mathbb{R}\text{Hom}_{C^{[d-1]}(\mathcal{O})}(\mathcal{O}, \mathcal{M})$ where $\mathcal{O}$ is the structure presheaf of $X$ and with the convention $C^{[0]}(\mathcal{O}) = \mathcal{O}^e = \mathcal{O} \otimes \mathcal{O}$. We therefore need to give a meaning to this derived hom complex.

Model structure of modules over a presheaf of CDGA

Throughout this paragraph, $k$ may be a field of any characteristic.

The category $\text{Ch}(X)$ of presheaves of chain complexes over a separated scheme $X$ is a closed symmetric monoidal category with the pointwise tensor product $\otimes$ and the hom presheaf $\mathcal{H}\text{om}$. It is also a cofibrantly generated model category such that the generating cofibrations are the maps $(S^{n-1})_U \to (D^n)_U$ and the generating trivial cofibrations are the maps $0 \to (D^n)_U$ [Har77 II, Ex. 1.19]. Remark that there is a natural isomorphism $\mathcal{H}\text{om}_{\text{Ch}(X)}((S^n)_U, D) \cong D_n(U)$ so that $(S^n)_U$ is a small object of $\text{Ch}(X)$ [Hov99 2.1.3].

The following construction can be done with any presheaf of DGA on a basis of open sets of a topological space. Because of our context, we prefer to work with a presheaf of CDGA on the affine open sets of a separated scheme.

Theorem 5.1.1. Let $\mathcal{A}$ be a presheaf of CDGA on a separated scheme $X$. The category $\mathcal{A}\text{-mod}$ of presheaves of $\mathcal{A}$-modules on $X$ is a cofibrantly generated monoidal model category such that the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms. Moreover, if $\mathcal{M} \to \mathcal{N}$ is a cofibration of $\mathcal{A}$-modules then $\mathcal{M}(U) \to \mathcal{N}(U)$ is a cofibration of $\mathcal{A}(U)$-modules for any affine open set $U$ of $X$.

Proof. The category $\mathcal{A}\text{-mod}$ can be thought of as the category of algebras over the monad $\mathcal{A} \otimes : \text{Ch}(X) \to \text{Ch}(X)$ [Mac98 VI]. This functor commutes with colimits, and any object of $\text{Ch}(X)$ is fibrant, so we just have to show that every $\mathcal{A}$-module has a path object to obtain the model structure [SS00, 3.1, A.3]. Let $I$ be the bounded below chain complex.
For any chain complex $C$, the hom complex $C^I = \text{Hom}(I, C)$ is a path object for $C$. Since weak equivalences and fibrations are defined pointwise in $\text{Ch}(X)$, the presheaf $M^I : U \mapsto M(U)^I$ is a path object for any $\mathcal{A}$-module $M$. For cofibrations, consider the product of section functors over affine open sets of $X$

$$\Gamma : \mathcal{A}^-\text{-mod} \to \prod_{U \in \text{Aff}(X)} (\mathcal{A}(U)^-\text{-mod})$$

It has a right adjoint which associates to each family $(M_U)_{U \in \text{Aff}(X)}$ the $\mathcal{A}$-module

$$U \mapsto \prod_{V \subset U} M_V$$

This right adjoint functor is exact, so $\Gamma$ is in fact a left Quillen functor $[\text{Hov99}, 1.3.4]$. 

Now we can define the derived tensor product and the derived hom complex over a presheaf of CDGA on a separated scheme $[\text{Hov99}, 4.3.1]$. The derived tensor product can be computed on any affine open set in the same way as the classical tensor product.

**Lemma 5.1.2.** Let $\mathcal{A}$ be a presheaf of CDGA on a separated scheme $X$, $\mathcal{M}$ and $\mathcal{N}$ two $\mathcal{A}$-modules. For any affine open set $U$ of $X$, we have

$$\left(\mathcal{M} \overset{L}{\otimes}_{\mathcal{A}} \mathcal{N}\right)(U) = \mathcal{M}(U) \overset{L}{\otimes}_{\mathcal{A}(U)} \mathcal{N}(U)$$

**Proof.** By Theorem 5.1.1 if $Q$ is a cofibrant replacement of the $\mathcal{A}$-module $\mathcal{M}$ then $Q(U)$ is a cofibrant replacement of the $\mathcal{A}(U)$-module $\mathcal{M}(U)$ and we have

$$\left(Q \otimes_{\mathcal{A}} \mathcal{N}\right)(U) = Q(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U)$$

We conclude this section by giving a derived base change lemma.

**Lemma 5.1.3.** Let $\mathcal{A} \to \mathcal{B}$ be a morphism of presheaves of CDGA on a separated scheme $X$, $\mathcal{M}$ a $\mathcal{A}$-module and $\mathcal{N}$ a $\mathcal{B}$-module. There is a natural weak equivalence of chain complexes

$$\mathbb{R}\text{Hom}_B\left(\mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} \mathcal{M}, \mathcal{N}\right) \simeq \mathbb{R}\text{Hom}_A(\mathcal{M}, \mathcal{N})$$

**Proof.** In fact, one can obtain a weak equivalence of presheaves of chain complexes on $X$ deriving the enriched adjunction $\mathcal{H}om_B \circ (\mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} 1) \cong \mathcal{H}om_A [\text{Mac98}, \text{IX, 5, 6}] [\text{Hov99}, 1.3.7]$, because the forgetful functor is its right derived functor since all objects are fibrant in $\mathcal{A}$-mod.

**Remark 5.1.4.** Let $\mathcal{A}$ be a presheaf of CDGA on a separated scheme $X$. One can see that if $\mathcal{M}$ is a cofibrant $\mathcal{A}$-module then $\mathcal{M}_x$ is a cofibrant $\mathcal{A}_x$-module for any $x \in X$. In fact, if $f : Y \to X$ is an affine map and $\mathcal{A} \to f_* \mathcal{B}$ a morphism of presheaves of CDGA on $X$, then the pair

$$f^* : \mathcal{A}^-\text{-mod} \xrightarrow{f_*} \mathcal{B}^-\text{-mod} : f_*$$

is a Quillen adjunction, since $f_*$ is always an exact functor $[\text{Hov99}, 1.3.4]$. We can apply this to $x : * \to X$ and $\mathcal{A} \to x_* \mathcal{A}_x$. 

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5.2 Link between both definitions

Section 5.1 gives rise to the following definition.

**Definition 5.2.1.** Let $d$ be a positive integer, $X$ a separated scheme with structure presheaf $\mathcal{O}$ and $\mathcal{F}$ a $\mathcal{O}$-module. We define

$$H_{[d]}(\mathcal{O}, \mathcal{F}) = H(\mathbb{R}\text{Hom}_{C^{[d-1]}(\mathcal{O})}(\mathcal{O}, \mathcal{F}))$$

the Higher Hochschild cohomology of order $d$ of $\mathcal{O}$ with coefficients in $\mathcal{F}$.

We recover the Gerstenhaber-Schack type definition for $d = 1$ [Swa96, 3]. Recall our convention $C[0](\mathcal{O}) = \mathcal{O}^e$ and choose a projective resolution of $\mathcal{O}^e$-module $P \to \mathcal{F}$ [Gro57, 1.10.1]. Since fibrations are epimorphisms in $\text{Ch}(\mathcal{O}^e)$, $P$ is a cofibrant remplacement of the complex of $\mathcal{O}^e$-module $\mathcal{F}$. Hence the chain complex $\text{Hom}_{\text{Ch}(\mathcal{O}^e)}(P, \mathcal{F})$ computes both the derived hom complex $\mathbb{R}\text{Hom}_{\text{Ch}(\mathcal{O}^e)}(\mathcal{O}, \mathcal{F})$ and the ext functor $\text{Ext}_{\mathcal{O}^e}(\mathcal{O}, \mathcal{F})$.

To get a link between [Definition 4.0.1] and [Definition 5.2.1] for a separated scheme, we need to pass from a presheaf to a sheaf and vice versa. Denote by $+$ the sheafification functor and by $#$ the forgetful functor.

**Lemma 5.2.2.** Let $X$ be a separated scheme with structure presheaf $\mathcal{O}$. The pair

$$+ : \text{Ch}(\mathcal{O}) \longrightarrow \text{Ch}(\mathcal{O}_X) : #$$

is a Quillen adjunction.

**Proof.** It is a standard fact that this pair is an adjunction [Har77, II, 1.2]. In order to show that it is a Quillen adjunction, we just have to show that functor $+$ preserves cofibrations and trivial cofibrations [Hov99, 1.3.4], which is a consequence of [Remark 5.1.4 Hov99 2.3.9] and its exactness [Gro57, 3.1].

Now we can prove that [Definition 4.0.1] and [Definition 5.2.1] coincide for separated schemes and quasi-coherent sheaves.

**Theorem 5.2.3.** Let $d$ be a positive integer, $X$ a separated scheme with structure presheaf $\mathcal{O}$ and $\mathcal{F}$ an $\mathcal{O}_X$-module with no sheaf cohomology on affine open sets of $X$. There is a natural isomorphism

$$H^n_{[d]}(X, \mathcal{F}) \cong H^n_{[d]}(\mathcal{O}, \mathcal{F}^#)$$

**Proof.** Deriving the enriched adjunction $\text{Hom}_{\text{Ch}(\mathcal{O}_X)}(\mathcal{O}^e, \mathcal{F}) \cong \text{Hom}_{\text{Ch}(\mathcal{O})}(1, \mathcal{F}^#)$ [Hov99 1.3.7], we have a natural weak equivalence of chain complexes

$$\mathbb{R}\text{Hom}_{\text{Ch}(\mathcal{O}_X)}(C^{[d]}_X, \mathcal{F}) \simeq \mathbb{R}\text{Hom}_{\text{Ch}(\mathcal{O})}(C^{[d]}(\mathcal{O}), \mathcal{F}^#)$$

In fact, functors $+$ and $#$ can be derived by [Lemma 5.2.2] and we have a natural weak equivalence $L+ \simeq +$ by exactness [Gro57, 3.1]. Moreover, $C^{[d]}_X = C^{[d]}(\mathcal{O})^+$ and we have a natural weak equivalence of complexes of $\mathcal{O}_X$-modules $\mathcal{F} \simeq \mathcal{F}^#$ by hypothesis. [Lemma 5.1.2] and the weak equivalence of simplicial set $S^d \simeq * \cup_{S^{d-1}} *$ give rise to a natural weak equivalence of $\mathcal{O}$-modules

$$C^{[d]}(\mathcal{O}) \simeq \mathcal{O} \underset{C^{[d-1]}(\mathcal{O})}{\bigotimes} \mathcal{O}$$
and so we have natural weak equivalences of chain complexes

\[ \mathbb{R}\text{Hom}_{\text{Ch}(\mathcal{O})}(C^{[d]}(\mathcal{O}), \mathcal{F}^\#) \simeq \mathbb{R}\text{Hom}_{\text{Ch}(\mathcal{O})}(\mathcal{O} \otimes_{C^{[d-1]}(\mathcal{O})} \mathcal{O}, \mathcal{F}^\#) \simeq \mathbb{R}\text{Hom}_{C^{[d]}(\mathcal{O})}(\mathcal{O}, \mathcal{F}^\#) \]

by Lemma 5.1.3

Remark 5.2.4. Let \( X \) be a scheme and \( \mathcal{F} \) a quasi-coherent \( \mathcal{O}_X \)-module. We saw in the proof of Proposition 4.2.1 that \( \mathcal{F} \) has no sheaf cohomology on affine open sets of \( X \) [Har77 II, 5.4] [Gro60 III, 1.3.1]. Thus we can apply Theorem 5.2.3 when \( X \) is separated.

6 Swan definition of classical Hochschild cohomology

In all of this section, \( k \) is a field of any characteristic. Recall that algebras are associative and commutative over \( k \) with identity.

Swan defined the Hochschild cohomology of a scheme \( X \) with coefficient in an \( \mathcal{O}_X \)-module \( \mathcal{F} \) by \( \text{Ext}^{p+q}_{\mathcal{O}_X \times \mathcal{O}_X}(\delta_* \mathcal{O}_X, \delta_* \mathcal{F}) \) where \( \delta : X \to X \times X \) is the diagonal morphism of \( X \). He showed that this ext functor coincides with the hyperext functor \( \text{E}xt^{p+q}_{\mathcal{O}_X}(\mathcal{C}_X, \mathcal{F}) \) if \( X \) is a quasi-projective scheme [Swa96, 2.1]. We will generalize this fact and show that it is true for any separated scheme using derived tensor product of sheaves and derived inverse/direct image [Spa88, 6, C, D].

Swan’s definition comes with the usual Grothendieck spectral sequence [Gro57, 2.4.1]

\[ H^p(X \times X, \text{Ext}^q_{\mathcal{O}_X \times \mathcal{O}_X}(\delta_* \mathcal{O}_X, \delta_* \mathcal{F})) \implies \text{Ext}^{p+q}_{\mathcal{O}_X}(\delta_* \mathcal{O}_X, \delta_* \mathcal{F}) \]

We will show that it is isomorphic to the spectral sequence of Proposition 4.1.4 when, for example, \( X \) is a separated smooth scheme.

6.1 Strong Swan theorem

Let \( X \) be a ringed space. The tensor product of complexes of \( \mathcal{O}_X \)-modules can be derived by means of a left Spaltenstein flat resolution of either of the factors [Spa88 5.1, 5.9, 6.5]. There is a relation between derived tensor products of sheaves and presheaves.

Lemma 6.1.1. Let \( \mathcal{O} \) be a presheaf of algebras on a separated scheme \( X \), \( \mathcal{C} \) and \( \mathcal{D} \) two complexes of \( \mathcal{O}^+ \)-modules. There is a natural weak equivalence of complexes of \( \mathcal{O}^+ \)-modules

\[ (\mathcal{C}^\# \otimes_{\mathcal{O}^+} \mathcal{D}^\#)^+ \simeq \mathcal{C}^\# \otimes_{\mathcal{O}^+} \mathcal{D} \]

Proof. Let \( Q \) be a cofibrant replacement of the complex of \( \mathcal{O} \)-modules \( \mathcal{C}^\# \). The complex of \( \mathcal{O} \)-modules \( Q \otimes_{\mathcal{O}} \mathcal{D}^\# \) computes the derived tensor product \( \mathcal{C}^\# \otimes_{\mathcal{O}^+} \mathcal{D}^\# \). By Remark 5.1.4, \( Q_x \) is cofibrant in \( \text{Ch}(\mathcal{O}_x) \) for any \( x \in X \) so \( Q^+ \to C^{#+} \simeq C \) is a left Spaltenstein flat resolution over \( \mathcal{O}^+ \) [Hov01 5.5] [Spa88 5.3] and the complex of \( \mathcal{O}^+ \)-modules \( Q^+ \otimes_{\mathcal{O}^+} \mathcal{D} \) computes the derived tensor product \( \mathcal{C} \otimes_{\mathcal{O}^+} \mathcal{D} \). Finally, we have an isomorphism of complexes of \( \mathcal{O}^+ \)-modules

\[ (Q \otimes_{\mathcal{O}} \mathcal{D}^#)^+ \to (Q^{#+} \otimes_{\mathcal{O}^+} \mathcal{D}^#)^+ \to (Q^{#+} \otimes_{\mathcal{O}^+} \mathcal{D}^#)^+ = Q^+ \otimes_{\mathcal{O}^+} \mathcal{D} \]

since \((Q \otimes_{\mathcal{O}} \mathcal{D}^#)_x^+ \simeq Q^+_x \otimes_{\mathcal{O}^+_x} \mathcal{D}_x \) for any \( x \in X \).
Now let \( f : X \to Y \) be a morphism of ringed spaces. The functors \( f^* \) and \( f_* \) can also be derived by means of a certain type of resolution such that \( \mathbb{L}f^* \) and \( \mathbb{R}f_* \) come with derived enriched adjunctions [Spa88, 6.7]. Consider the case of an inclusion \( i : Z \to X \) of a closed set \( Z \) of \( X \). For any \( \mathcal{O}_Z \)-module \( \mathcal{F} \) and any \( x \in X \), there is a natural isomorphism of algebras

\[
(i_*\mathcal{F})_x \cong \begin{cases} 
\mathcal{F}_x & \text{if } x \in Z \\
0 & \text{if } x \notin Z
\end{cases}
\]

[Har77, II, Ex, 1.19] and so the counit \( i^{-1}i_* \to 1 \) is a natural isomorphism.

**Lemma 6.1.2.** Let \( i : X \to Y \) be a morphism of ringed spaces which is an homeomorphism onto a closed set and \( \mathcal{C} \) a complex of \( \mathcal{O}_X \)-modules. There is a natural weak equivalence of complexes of \( \mathcal{O}_X \)-modules

\[
\mathbb{L}i^*i_*\mathcal{C} \cong \mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{i^{-1}\mathcal{O}_Y} \mathcal{C}
\]

**Proof.** Let \( \mathcal{Q} \to i_*\mathcal{C} \) be a left Spaltenstein flat resolution over \( \mathcal{O}_Y \). The complex of \( \mathcal{O}_X \)-modules \( i^*\mathcal{Q} \) computes the derived functor \( \mathbb{L}i^*i_*\mathcal{C} \). Moreover, \( i^{-1}\mathcal{Q} \to i^{-1}i_*\mathcal{C} \cong \mathcal{C} \) is a left Spaltenstein flat resolution over \( i^{-1}\mathcal{O}_Y \) [Spa88, 5.3] and so the complex of \( \mathcal{O}_X \)-modules \( \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} (i^{-1}\mathcal{Q}) \) computes the derived tensor product \( \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{C} \). We can conclude the proof by definition of the inverse image functor \( i^*\mathcal{Q} = \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} (i^{-1}\mathcal{Q}) \).

**Proposition 6.1.3.** Let \( X \) be a scheme with diagonal morphism \( \delta : X \to X \times X \). There is a natural isomorphism of sheaves of algebras on \( X \)

\[
\delta^{-1}\mathcal{O}_{X \times X} \cong \mathcal{O}_X^e
\]

**Proof.** The morphism is induced for any open sets \( U \) and \( V \) of \( X \) by the restriction maps

\[
\mathcal{O}_X(U) \otimes \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) \otimes \mathcal{O}_X(U \cap V)
\]

Let \( U = \text{Spec}(A) \) be an affine open set of \( X \). We have to identify the following two presheaves on \( U \)

\[
D(r) \mapsto \colim_{D(s \otimes t) \ni \delta D(r)} (A_s \otimes A_t) \quad \& \quad D(r) \mapsto A_r^e
\]

where \( D(r) = \{ p \in U : r \notin p \} \) for any \( r \in A \). Remark that

\[
D(s \otimes t) \ni \delta D(r) \iff \sqrt{s} \otimes \sqrt{t} \ni r \iff \sqrt{s \otimes t} \ni r \iff D(s \otimes t) \ni D(r \otimes r) \ni \delta D(r)
\]

so the functor \( \colim_{D(s \otimes t) \ni \delta D(r)} \) is given by taking the value on \( D(r \otimes r) \) [Mac98, IX, 3, Ex, 1].

We come now to the main theorem of this section. It generalizes a theorem of Swan which was initialy about quasi-projectif schemes [Swa96, 2.1].

**Theorem 6.1.4.** Let \( X \) be a separated scheme with diagonal morphism \( \delta : X \to X \times X \) and \( \mathcal{F} \) an \( \mathcal{O}_X \)-module. There is a natural isomorphism

\[
HH(X, \mathcal{F}) \cong \text{Ext}_{\mathcal{O}_{X \times X}}(\delta_*\mathcal{O}_X, \delta_*\mathcal{F})
\]
Proof. Denote by \( \mathcal{O} \) the structure presheaf of \( X \). By Lemma 5.1.2, Example 3.2.1.1, exactness of functor \( \mathbf{+} \), Lemma 6.1.1, Proposition 6.1.3 and Lemma 6.1.2, there are natural weak equivalences of complexes of \( \mathcal{O}_X \)-modules

\[
\mathcal{C}_X = C(\mathcal{O})^+ \simeq \left( \mathcal{O} \otimes_{\mathcal{O}^e} \mathcal{O} \right)^+ \simeq \mathcal{O}_X \otimes_{\mathcal{O}^e} \mathcal{O}_X \cong \mathcal{O}_X \otimes_{\delta^{-1} \mathcal{O}_X \otimes \mathcal{O}_X} \mathcal{O}_X \cong \mathbb{L} \delta^* \mathcal{O}_X
\]

with \( C(\mathcal{O}) : U \mapsto C(\mathcal{O}(U)) \). By exactness, there is a natural weak equivalence \( \delta_* \simeq \mathbb{R} \delta_* \). This allow us to write following natural weak equivalences of chain complexes

\[
\mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_X)}(\mathcal{C}_X, \mathcal{F}) \simeq \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_X)}(\mathbb{L} \delta^* \mathcal{O}_X, \mathcal{F})
\]

\[
\cong \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_X)}(\delta_* \mathcal{O}_X, \mathbb{R} \delta_* \mathcal{F})
\]

\[
\cong \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_X)}(\delta_* \mathcal{O}_X, \mathcal{F})
\]

[Spa88, 6.7].

In fact, Theorem 6.1.4 is true as soon as \( \delta_* \simeq \mathbb{R} \delta_* \). One can see in the next example that it is the obstruction of the veracity of Theorem 6.1.4.

Example 6.1.5. A non-separated case. Recall the scheme \( X = S \cup_U S \) with \( S = \text{Spec}(A) \) and \( U = D(s) \) as in Example 4.0.2. We saw that for any \( \mathcal{O}_X \)-module \( \mathcal{F} \), the cohomology groups \( HH^n(X, \mathcal{F}) \) are isomorphic to the cohomology groups of the derived hom complex \( \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_S)}(\mathcal{C}_X, \mathbb{R} p_* \mathcal{F}) \). Consider the following commutative square of schemes

\[
\begin{array}{ccc}
X & \xrightarrow{p} & S \\
\downarrow{\delta_X} & & \downarrow{\delta_S} \\
X \times X & \xrightarrow{\Delta p} & S \times S
\end{array}
\]

[Gro60, I, 3.2.1, 5.3.1]. There is a natural weak equivalence \( \mathbb{R} (\delta_*), \mathbb{R} p_* \simeq \mathbb{R} (\Delta p_*), \mathbb{R} (\delta_*), \mathbb{R} S \) [Spa88, 6.7] and since \( S \) is a separated scheme, we have as in Theorem 6.1.4 a weak equivalence of complexes of \( \mathcal{O}_S \)-modules \( \mathcal{C}_S \simeq \mathbb{L} (\delta^*)^* (\delta S), \mathcal{O}_S \). Thus, we have weak equivalences

\[
\mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_S)}(\mathcal{C}_X, \mathbb{R} p_* \mathcal{F}) \simeq \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_S)}(\mathbb{L} (\delta^*)^* (\delta S), \mathcal{O}_S, \mathbb{R} p_* \mathcal{F})
\]

\[
\cong \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_{S \times S})}(\mathbb{L} (\delta S), \mathcal{O}_S, \mathbb{R} (\delta_*), \mathbb{R} p_* \mathcal{F})
\]

\[
\cong \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_{S \times S})}(\mathbb{L} (\delta S), \mathcal{O}_S, \mathbb{R} (\Delta p)_*, \mathbb{R} (\delta_*), \mathcal{F})
\]

[Spa88, 6.7]. On the other hand, \( \text{Ext}^n_{\mathcal{O}_{S \times S}}(\mathcal{O}_S, \mathcal{F}) \) is the \( n \)-cohomology of the derived hom complex \( \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_S)}(\mathbb{L} (\delta S), \mathcal{O}_S, \mathbb{R} (\delta_*), \mathcal{F}) \). Since \( \delta S \) is an affine morphism [Gro60, II, 2.1.1] and functor \( (\Delta p)^* \) is exact, we have weak equivalences of complexes of \( \mathcal{O}_{S \times S} \)-modules

\[
(\delta S)_* \mathcal{O}_S \cong (\delta S)_* \mathbb{R} p^* \mathcal{O}_S \cong (\Delta p)_*(\delta S)_* \mathcal{O}_S \cong \mathbb{L}(\Delta p)^* (\delta S)_* \mathcal{O}_S
\]

[Swa96, 7.4] and so we have weak equivalences

\[
\mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_S)}(\mathbb{L} (\delta S), \mathcal{O}_S, \mathbb{R} (\Delta p)_*, \mathbb{R} (\delta_*), \mathcal{F})
\]

\[
\cong \mathbb{R} \text{Hom}_{\text{Ch}(\mathcal{O}_{S \times S})}(\mathbb{L} (\delta S), \mathcal{O}_S, \mathbb{R} (\Delta p)_*(\delta_*), \mathcal{F})
\]

[Swa96, 6.7].
6.2 Spectral sequences

Let $X$ be a scheme with diagonal morphism $\delta : X \to X \times X$. The composition $\text{Hom}_{O_{X \times X}}(\delta_* O_X, \ ) = \Gamma \circ \text{Hom}_{O_{X \times X}}(\delta_* O_X, \ )$ gives rise for any $O_X$-module $\mathcal{F}$ to the usual Grothendieck spectral sequence \[ E_2^{pq} = H^p(X \times X, \mathcal{E}xt^q_{O_{X \times X}}(\delta_* O_X, \delta_* \mathcal{F})) \implies \text{Ext}^{p+q}_{O_X}(\delta_* O_X, \delta_* \mathcal{F}) \]
Assuming for example that $X$ is a separated smooth scheme, one can show that this spectral sequence is isomorphic to the spectral sequence of Proposition 4.1.4.

**Theorem 6.2.1.** Let $X$ be a separated scheme with diagonal morphism $\delta : X \to X \times X$ and $\mathcal{F}$ an $O_X$-module. Assume that $\mathcal{E}xt^q_{O_X}(\mathcal{H}_{X,q}, \mathcal{F}) = 0$ for any integers $p > 0$ and $q \geq 0$. The following spectral sequences are naturally isomorphic

\[ \text{Ext}^p_{O_X}(\mathcal{H}_{X,q}, \mathcal{F}) \implies H^{p+q}(X, \mathcal{F}) \]

\[ H^p(X \times X, \mathcal{E}xt^q_{O_{X \times X}}(\delta_* O_X, \delta_* \mathcal{F})) \implies \text{Ext}^{p+q}_{O_X}(\delta_* O_X, \delta_* \mathcal{F}) \]

**Proof.** As in the proof of Theorem 4.1.5 we have by hypothesis a natural isomorphism

\[ \text{Ext}^p_{O_X}(\mathcal{H}_{X,q}, \mathcal{F}) \cong H^p(X, \text{Hom}_{O_X}(\mathcal{H}_{X,q}, \mathcal{F})) \cong H^p(X \times X, \delta_* \text{Hom}_{O_X}(\mathcal{H}_{X,q}, \mathcal{F})) \]

for any integers $p > 0$ and $q \geq 0$ since $\mathbb{R}\Gamma_{X \times X} \circ \delta_* \cong \mathbb{R}\Gamma_{X \times X} \circ \mathbb{R}\delta_*$.

Moreover, there is a natural isomorphism of $O_X$-modules

\[ \text{Hom}_{O_X}(\mathcal{H}_{X,q}, \mathcal{F}) \cong H^q(\mathbb{R}\text{Hom}_{\text{Ch}(O_X)}(\mathcal{C}_X, \mathcal{F})) \]

for any integer $q \geq 0$. To see this, choose an injective resolution of $O_X$-modules $\mathcal{F} \to I$. The complex of $O_X$-modules $\text{Hom}_{\text{Ch}(O_X)}(\mathcal{C}_X, I)$ computes the derived hom sheaf $\mathbb{R}\text{Hom}_{\text{Ch}(O_X)}(\mathcal{C}_X, \mathcal{F})$. Filtering it by columns \[ \text{E}^1 \] is the vertical cohomology which is preserved by exact functors $\text{Hom}_{O_X}(\mathcal{H}_{X,q}, \mathcal{F})$ and the second page $\text{E}_2$ is the horizontal cohomology which computes the ext sheaves

\[ \mathcal{E}_1^{pq} = H^q(\text{Hom}_{O_X}(\mathcal{C}_X, I^p)) \cong \text{Hom}_{O_X}(\mathcal{H}_{X,q}, I^p) \]

\[ \mathcal{E}_2^{pq} = H^p(\text{Hom}_{O_X}(\mathcal{H}_{X,q}, I)) = \mathcal{E}xt^p_{O_X}(\mathcal{H}_{X,q}, \mathcal{F}) \cong \begin{cases} \text{Hom}_{O_X}(\mathcal{H}_{X,q}, \mathcal{F}) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases} \]

This gives the announced isomorphism. Using exactness of functor $\delta_*$ and the proof of Theorem 6.1.4, we can end the proof with natural isomorphisms of $O_{X \times X}$-modules

\[ \delta_* \text{Hom}_{O_X}(\mathcal{H}_{X,q}, \mathcal{F}) \cong \delta_* H^q(\mathbb{R}\text{Hom}_{\text{Ch}(O_X)}(\mathcal{C}_X, \mathcal{F})) \]

\[ \cong H^q(\mathbb{R}\text{Hom}_{\text{Ch}(O_X)}(\mathbb{L}\delta^* O_X, \mathcal{F})) \]

\[ \cong H^q(\mathbb{R}\text{Hom}_{\text{Ch}(O_X)}(\delta_* O_X, \mathbb{R}\delta_* \mathcal{F})) \]

\[ \cong H^q(\mathbb{R}\text{Hom}_{\text{Ch}(O_X)}(\delta_* O_X, \delta_* \mathcal{F})) = \mathcal{E}xt^q_{O_X}(\delta_* O_X, \delta_* \mathcal{F}) \]

for any integer $q \geq 0$ \[ \text{Spa88} \ 6.7 \].

**Remark 6.2.2.** Let $X$ be a separated smooth scheme and $\mathcal{F}$ an $O_X$-module. For any positive integer $q$, the $O_X$-module $\mathcal{H}_{X,q}$ is locally free of finite rank \[ \text{Gro67} \ IV, 17.2.3 \] and so we have $\mathcal{E}xt^q_{O_X}(\mathcal{H}_{X,q}, \mathcal{F}) = 0$ for any $p > 0$ \[ \text{Gro57} \ 4.2.3 \]. Thus we can apply Theorem 6.2.1.
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