ON SETS WITH SMALL SUMSET IN THE CIRCLE

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Abstract. We prove results on the structure of a subset of the circle group having positive inner Haar measure and doubling constant close to the minimum. These results go toward a continuous analogue in the circle of Freiman’s $3k - 4$ theorem from the integer setting. An analogue of this theorem in $\mathbb{Z}_p$ has been pursued extensively, and we use some recent results in this direction. For instance, obtaining a continuous analogue of a result of Serra and Zémor, we prove that if a subset $A$ of the circle is not too large and has doubling constant at most $2 + \varepsilon$ with $\varepsilon < 10^{-4}$, then for some integer $n > 0$ the dilate $n \cdot A$ is included in an interval in which it has density at least $1/(1 + \varepsilon)$. Our arguments yield other variants of this result as well, notably a version for two sets which makes progress toward a conjecture of Bilu. We include two applications of these results. The first is a new upper bound on the size of $k$-sum-free sets in the circle and in $\mathbb{Z}_p$. The second gives structural information on subsets of $\mathbb{R}$ of doubling constant at most $3 + \varepsilon$.

1. Introduction

A result of Freiman from 1959 [12], often called the $3k - 4$ theorem, states that if $A$ is a set of integers such that the sumset $A + A$ satisfies $|A + A| \leq 3|A| - 4$, then $A$ is contained in an arithmetic progression of length $|A + A| - |A| + 1$. This theorem motivated the search for analogues in other settings, especially in groups $\mathbb{Z}_p$ of integers with addition modulo a prime $p$. Treatments of the latter direction include [13, 16, 22, 25, 30]. Part of the difficulty in finding a fully satisfactory $\mathbb{Z}_p$-analogue of the $3k - 4$ theorem is that the statement has to involve more assumptions than in the integer setting, in particular to avoid certain counterexamples that occur in $\mathbb{Z}_p$ when $A + A$ is too large. In [30], Serra and Zémor proposed the following conjecture and proved a result towards it (namely [30, Theorem 3], which we also recall below).

Conjecture 1.1. Let $p$ be a prime, let $r$ be a non-negative integer, and let $A \subset \mathbb{Z}_p$ satisfy

$$|A + A| = 2|A| + r - 1 \leq \frac{3p}{2} + |A| - 2, \text{ and } r \leq |A| - 3.$$ 

Then $A$ is included in an arithmetic progression of length $|A| + r$.

By an interval in $\mathbb{Z}_p$ we mean an arithmetic progression of difference 1. For a subset $A$ of an abelian group and an integer $n$, we denote by $n \cdot A$ the image of $A$ under the

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homomorphism $x \mapsto n x$ (for $A \subset \mathbb{Z}_p$ and $n \in \mathbb{Z}_p$ we also use $n \cdot A$ to denote the image of $A$ under $x \mapsto n x$). The conclusion of Conjecture 1.1 can be rephrased as follows: there exists $n \in \mathbb{Z}_p \setminus \{0\}$ and an interval $I \subset \mathbb{Z}_p$ such that $n \cdot A \subset I$ and $|I| \leq |A| + r$.

Freiman’s $3k - 4$ theorem has an extension applicable to two possibly different sets $A$, $B$. A $\mathbb{Z}_p$-analogue of this extension has also been proposed, namely the so-called r-critical pair conjecture. A version of this conjecture appeared\(^1\) in [17] and was proved for small sets in [3] [15]. We recall the following more recent version [16, Conjecture 19.2].

**Conjecture 1.2.** Let $p$ be a prime, let $r$ be a non-negative integer, and let $A, B$ be non-empty subsets of $\mathbb{Z}_p$ with $|A| \geq |B|$ and satisfying

$$|A + B| = |A| + |B| + r - 1 \leq \frac{1}{2}(p + |A| + |B|) - 2, \quad \text{and} \quad r \leq |B| - 3. \quad (1)$$

Then there exist intervals $I, J, K \subset \mathbb{Z}_p$ and $n \in \mathbb{Z}_p \setminus \{0\}$ such that $n \cdot A \subset I$, $n \cdot B \subset J$, $n \cdot (A + B) \subset K$, and $|I| \leq |A| + r$, $|J| \leq |B| + r$, $|K| \geq |A| + |B| - 1$.

Note that this extends Conjecture 1.1 in particular in that the conclusion here concerns not only $A, B$ but also the third set $A + B$.

The following equivalent version of Conjecture 1.2, appearing for instance in [16, Conjecture 19.5], is notable for its symmetry.

**Conjecture 1.3.** Let $p$ be a prime, let $r$ be a non-negative integer, and let $A_1, A_2, A_3$ be subsets of $\mathbb{Z}_p$ satisfying the following conditions:

$$|A_1|, |A_2|, |A_3| > r + 2, \quad |A_1| + |A_2| + |A_3| > p - r, \quad |A_1 + A_2 + A_3| < p. \quad (2)$$

Then there exist intervals $I_1, I_2, I_3 \subset \mathbb{Z}_p$ and $n \in \mathbb{Z}_p \setminus \{0\}$ such that $n \cdot A_j \subset I_j$ and $|I_j| \leq |A_j| + r$ for $j = 1, 2, 3$.

Considering analogues of the $3k - 4$ theorem in the continuous setting of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ goes back at least to the paper [14] from 1973 by Freiman, Judin, and Moskvin. Conjecture 1.3 has a natural analogue in this setting. In this paper, we obtain the following result toward this continuous analogue.

**Theorem 1.4.** Let $\rho \in (0, c)$ where $c = 3.1 \cdot 10^{-1549}$. Let $A_1, A_2, A_3$ be subsets of $\mathbb{T}$ satisfying the following conditions:

$$\mu(A_1), \mu(A_2), \mu(A_3) > \rho, \quad \mu(A_1) + \mu(A_2) + \mu(A_3) > 1 - \rho, \quad \mu(A_1 + A_2 + A_3) < 1. \quad (3)$$

Then there exist closed intervals $I_1, I_2, I_3 \subset \mathbb{T}$ and $n \in \mathbb{N}$ such that $n \cdot A_j \subset I_j$ and $\mu(I_j) \leq \mu(A_j) + \rho$ for $j = 1, 2, 3$.

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\(^1\)Note that [17] Conjecture 1] appeared before Conjecture 1.1 but in the case $A = B$ it was recognized only later as the likely optimal conjecture, in [30], thanks to an example given in that paper.
Here and throughout this paper, we denote by \( \mathbb{N} \) the set of positive integers and by \( \mu \) the inner Haar measure on \( T \), thus for any set \( A \subset T \) we have that \( \mu(A) \) is the supremum of the Haar measures of closed sets included in \( A \). We use the inner Haar measure, rather than the Haar measure, in order to deal with non-measurable sets and with the fact that the sumset of two measurable sets can be non-measurable.

The conjecture mentioned just before Theorem 1.4 puts forward that this theorem holds for every \( \rho \in (0, 1) \).

As we will show, the argument that establishes the equivalence between Conjectures 1.2 and 1.3 can be adapted to the continuous setting, by incorporating several additional technicalities, to show that Theorem 1.4 implies the following result.

**Theorem 1.5.** Let \( \rho \in (0, c) \) where \( c = 3.1 \cdot 10^{-1549} \). Let \( A, B \subset T \) satisfy
\[
\mu(A + B) = \mu(A) + \mu(B) + \rho < \frac{1}{2}(1 + \mu(A) + \mu(B)), \quad \text{and} \quad \rho < \mu(B) \leq \mu(A).
\]
Then there exist intervals \( I, J, K \subset T \), with \( I \) closed, \( K \) open, \( n \in \mathbb{N} \), such that \( n \cdot A \subset I \), \( n \cdot B \subset J \), \( K \subset n \cdot (A + B) \), and \( \mu(I) \leq \mu(A) + \rho, \mu(J) \leq \mu(B) + \rho, \mu(K) \geq \mu(A) + \mu(B) \).

This theorem makes progress toward an analogue of Conjecture 1.2 for \( T \), analogue which originated in work of Bilu on the so-called \( \alpha + 2\beta \) inequality in the torus (see [2, Conjecture 1.2]). We detail this in Remark 2.18 in Section 2, after having proved Theorem 1.5.

The bound \( c = 3.1 \cdot 10^{-1549} \) in Theorems 1.4 and 1.5 comes from a result in \( \mathbb{Z}_p \) due to Grynkiewicz, as we explain in Section 2. In the symmetric case, i.e. when \( A = B \), a better bound had been given by the result of Serra and Zémor toward Conjecture 1.1 (result recalled as Theorem 2.4 below). We prove the following \( T \)-analogue of this result.

**Theorem 1.6.** Let \( 0 \leq \varepsilon \leq 10^{-4} \). Let \( A \subset T \) satisfy \( \mu(A) > 0 \) and
\[
\mu(A + A) = (2 + \varepsilon) \mu(A) < \frac{1}{2} + \mu(A).
\]
Then there exist intervals \( I, K \subset T \), with \( I \) closed, \( K \) open and \( n \in \mathbb{N} \), such that \( n \cdot A \subset I \), \( K \subset n \cdot (A + A) \), \( \mu(I) \leq \mu(A + A) - \mu(A) \) and \( \mu(K) \geq 2\mu(A) \).

Apart from their relation to continuous analogues of the \( 3k - 4 \) theorem and Bilu’s conjecture, the theorems above are motivated by the following applications.

The first application concerns the problem of determining the supremum of measures of Borel sets \( A \subset T \) such that the cartesian power \( A^3 \) contains no triple \( (x, y, z) \) solving the equation \( x + y = kz \), where \( k \geq 3 \) is a fixed integer. This is an analogue in \( T \) of a problem which goes back to Erdős (see [7]) and which has been treated in several works, first in the integer setting (see in particular [11, 7]) and then also in the continuous setting...
of an interval in $\mathbb{R}$ \cite{21, 23}. The above-mentioned supremum is seen to be at most 1/3 by a simple application of Raikov’s inequality from \cite{21} (see also \cite{20} Theorem 1). Our result, discussed in Section 3, improves on this upper bound using Theorem 1.6; see Theorem 3.1. Via a correspondence established in \cite{5} between this problem in $\mathbb{T}$ and a similar problem in $\mathbb{Z}_p$, Theorem 3.1 implies a similar result in $\mathbb{Z}_p$; see Remark 3.4.

The second application provides new results about the structure of subsets of $\mathbb{R}$ of doubling less than 4. We discuss this in Section 4. Essentially, if a closed set $A \subset [0, 1]$ has doubling constant at most $3 + \varepsilon$, then modulo 1 it has doubling constant at most $2 + \varepsilon$, and so Theorem 1.6 can be used to obtain information on the structure of $A$; see Theorem 4.1. In particular, under a special case of the conjecture of Bilu mentioned above \cite[Conjecture 1.2]{2}, we obtain a version of \cite[Theorem 6.2]{10} with effective bounds; see Corollary 4.6.

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2. Proofs of the main results

As mentioned in the introduction, the analogues in $\mathbb{Z}_p$ of Theorems 1.3 and 1.6 are known.

Indeed, Theorem 1.3 is a $\mathbb{T}$-analogue of the following result.

**Theorem 2.1.** Let $p$ be a prime, and let $r$ be an integer with $0 \leq r \leq cp - 1.2$ where $c = 3.1 \cdot 10^{-1549}$. Let $A_1, A_2, A_3$ be subsets of $\mathbb{Z}_p$ satisfying the following conditions:

$$|A_1|, |A_2|, |A_3| > r + 2, \quad |A_1| + |A_2| + |A_3| > p - r, \quad |A_1 + A_2 + A_3| < p. \quad (4)$$

Then there exist intervals $I_1, I_2, I_3 \subset \mathbb{Z}_p$ and a non-zero $n \in \mathbb{Z}_p$ such that $n \cdot A_j \subset I_j$ and $|I_j| \leq |A_j| + r$, for $j = 1, 2, 3$.

For a subset $A$ of an abelian group $G$ we denote by $A^c$ the complement $G \setminus A$. Theorem 2.1 can be deduced from the following result of Grynkiewicz (see \cite[Theorem 21.8]{16}).

**Theorem 2.2.** Let $p$ be a prime, and let $r$ be an integer with $0 \leq r \leq cp - 1.2$ where $c = 3.1 \cdot 10^{-1549}$. Let $A, B$ be subsets of $\mathbb{Z}_p$ satisfying the following conditions:

$$|A|, |B|, |C| > r + 2, \quad |A + B| \leq |A| + |B| + r - 1, \quad (5)$$

where $C = -(A + B)^c$. Then there exist intervals $I, J, K \subset \mathbb{Z}_p$ and a non-zero $n \in \mathbb{Z}_p$ such that $n \cdot A \subset I$, $n \cdot B \subset J$, $n \cdot C \subset K$, and $|I| \leq |A| + r$, $|J| \leq |B| + r$, $|K| \leq |C| + r$.

**Lemma 2.3.** Theorem 2.2 implies Theorem 2.1.
Proof. Starting from the assumptions in Theorem 2.1 note that since $|A_1 + A_2 + A_3| < p$ we may assume (modulo translating $A_3$, which does not affect the theorem) that $0 \notin A_1 + A_2 + A_3$. Hence $A_3 \subset -(A_1 - A_2)^c = -(A_1 + A_2)^c$. Let $A = A_1$, $B = A_2$, $C = -(A_1 + A_2)^c$, and note that $|A|, |B|, |C| > r + 2$. Moreover, from $|A_1| + |A_2| + |A_3| > p - r$ we deduce that $|A + B| \leq |A| + |B| + r - 1$. Let $s \leq r$ be such that $|A + B| = |A| + |B| + s - 1$. Applying Theorem 2.2 with $s$, we obtain intervals $I_1 = I, I_2 = J, I_3 = K$ and $n \in \mathbb{Z}_p \setminus \{0\}$ such that $n \cdot A_j \subset I_j$ and $|I_j| \leq |A_j| + s \leq |A_j| + r$ for $j = 1, 2$. Moreover $n \cdot A_3 \subset n \cdot C \subset I_3$, and $|I_3| \leq |C| + s = p - |A_1 + A_2| + s = p - |A_1| - |A_2| + 1 \leq |A_3| + r$, so we obtain the conclusion of Theorem 2.1. 

One can also deduce Theorem 2.2 from Theorem 2.1 in a straightforward way; we leave this to the reader, and in any case the main ideas in this deduction will be used in the continuous setting in Subsection 2.3 to prove Theorem 1.5.

In the case $A = B$ of Theorem 2.2 (the symmetric case), the following result of Serra and Zémor toward their Conjecture 1.1 provided a better bound for $r$ than in Theorem 2.2 (see [30, Theorem 3]).

**Theorem 2.4.** Let $p$ be a prime greater than $2^{94}$, let $0 \leq \varepsilon \leq 10^{-4}$, and let $A \subset \mathbb{Z}_p$ satisfy

$$|A + A| = (2 + \varepsilon)|A| - 1 \leq \min \{3|A| - 4, \frac{p}{2} + |A| - 2\}. \quad (6)$$

Then there is an interval $I \subset \mathbb{Z}_p$ and $n \in \mathbb{Z}_p \setminus \{0\}$ such that $n \cdot A \subset I$ and $|I| \leq |A + A| - |A| + 1$.

In this section we prove Theorems 1.4, 1.5 and 1.6. Inspired by arguments of Bilu from [2], we deduce the first and the third of these theorems from their discrete versions, i.e. Theorems 2.1 and 2.4 respectively. In the process, we also deduce Theorem 1.5 using Theorem 1.4.

Let $\frac{1}{p}\mathbb{Z}_p$ denote the subgroup of $\mathbb{T}$ isomorphic to $\mathbb{Z}_p$. We use the following notation for discrete approximations of sets in $\mathbb{T}$.

**Definition 2.5.** For any set $A \subset \mathbb{T}$ and prime $p$, we define the set

$$A_p = A \cap \frac{1}{p}\mathbb{Z}_p.$$ 

To prove Theorem 1.4 first we focus on sets in $\mathbb{T}$ that are unions of finitely many intervals. We refer to such sets as simple sets. In Subsection 2.2 we show that if $A_1, A_2, A_3$ are open simple sets and satisfy the conditions of Theorem 1.4 then their discrete approximations $A_{j,p}$ obey the conclusion of Theorem 2.1. However, this is not enough to deduce directly that the conclusion of Theorem 1.4 holds for the original sets $A_j$, because the integer $n$
provided by Theorem 2.1 is not a priori bounded in any way that would ensure that the sets \( n \cdot A_j \) are contained in suitably small intervals the way their discrete approximations are. To ensure this additional fact, in the next subsection we use the Fourier transform on \( \mathbb{Z}_p \) to bound the integer \( n \). Finally, in Subsection 2.3 we obtain Theorems 1.4 and also 1.5 and 1.6 by generalizing from simple sets to arbitrary sets.

2.1. On the \( n \)-diameter of simple sets.

Given a set \( A \subset \mathbb{T} \) and an integer \( n \), we define the \( n \)-diameter of \( A \) by
\[
D_n(A) = \inf \{ \mu(I) : I \subset \mathbb{T} \text{ a closed interval such that } n \cdot A \subset I \}. \tag{7}
\]
For a set \( B \subset \mathbb{Z}_p \) and \( n \in \mathbb{Z}_p \), we define similarly the \( n \)-diameter of \( B \) by
\[
D_n(B) = \min \{ |I|/p : I \subset \mathbb{Z}_p \text{ an interval such that } n \cdot B \subset I \}
\]

We prove the following result concerning the \( n \)-diameter of simple sets in \( \mathbb{T} \).

**Proposition 2.6.** Let \( A \subset \mathbb{T} \) be a union of at most \( m \) intervals with \( \mu(A) > 0 \), and suppose that \( n \in \mathbb{Z} \) satisfies \( D_n(A) < \min \left( \frac{1}{2}, \frac{\mu(A)}{1-2/\pi} \right) \). Then \( |n| \leq \frac{m}{2 (\mu(A)-(1-2/\pi)D_n(A))} \).

For \( s \in \mathbb{Z}_p \) we denote by \( |s|_p \) the absolute value of the unique integer in \( (-\frac{p}{2}, \frac{p}{2}) \) congruent to \( s \) modulo \( p \). We deduce Proposition 2.6 from the following discrete version, which is in fact the main result from this subsection that we use in the sequel.

**Proposition 2.7.** Let \( B \subset \mathbb{Z}_p \) be a union of at most \( m \) intervals with \( \beta > 0 \), and suppose that \( n \in \mathbb{Z}_p \) satisfies \( D_n(B) < \min \left( \frac{1}{2}, \frac{\beta}{1-2/\pi} \right) \). Then \( |n|_p \leq \frac{m}{2 (\beta-(1-2/\pi)D_n(B))} \).

Proposition 2.6 follows by applying Proposition 2.7 to \( B = A \) for primes \( p \to \infty \).

To obtain Proposition 2.7, we use the following result concerning the Fourier coefficients of a subset of \( \mathbb{Z}_p \) that is the union of at most \( m \) disjoint intervals, to the effect that these Fourier coefficients decay in a useful way.

For \( f : \mathbb{Z}_p \to \mathbb{C} \), let \( \hat{f} \) denote the Fourier transform \( \widehat{\mathbb{Z}_p} \cong \mathbb{Z}_p \to \mathbb{C} \) defined by \( \hat{f}(s) = \frac{1}{p} \sum_{j \in \mathbb{Z}_p} f(j) e^{2\pi i sj/p} \). We write \([m]\) for the set of integers \( \{1, \ldots, m\} \).

**Lemma 2.8.** Let \( J_1, J_2, \ldots, J_m \) be pairwise disjoint intervals in \( \mathbb{Z}_p \), and let \( B = \bigcup_{j \in [m]} J_j \). Let \( s \) be a non-zero element of \( \widehat{\mathbb{Z}_p} \cong \mathbb{Z}_p \). Then we have
\[
|\hat{1}_B(s)| \leq \frac{m}{2|s|_p}. \tag{8}
\]
Proof. We first estimate the Fourier coefficients of a single interval $J \subset \mathbb{Z}_p$, by the following standard calculation. Supposing that $J = \{a, a + 1, \ldots, a + (t - 1)\}$, for every non-zero $s \in (-\frac{p}{2}, \frac{p}{2})$ we have

$$|\hat{1}_J(s)| = \left| \frac{1}{p} \sum_{j=0}^{t-1} e^{2\pi i \frac{aj}{p}} \right| \leq \frac{1}{p} \left| \frac{1 - e^{2\pi i \frac{at}{p}}}{1 - e^{2\pi i \frac{a}{p}}} \right| \leq \frac{1}{p} \frac{2}{\left| 1 - e^{2\pi i \frac{a}{p}} \right|} \quad (9)$$

Letting $\|\theta\|_T$ denote the distance from $\theta \in \mathbb{R}$ to the nearest integer, and using the standard estimate $|1 - e^{2\pi i \theta}| \geq 4\|\theta\|_T$ for $\|\theta\|_T < 1/2$, we deduce that

$$|\hat{1}_J(s)| \leq \frac{1}{2|s|_p}. \quad (10)$$

Now, since $1_B = 1_{J_1} + \cdots + 1_{J_m}$, we deduce (8) by linearity of the Fourier transform, the triangle inequality, and applying (10) to each interval $J_i$. \hfill \Box

An immediate consequence of this lemma is that for such a set $B$ the large Fourier coefficients can only occur at bounded frequencies, in the following sense.

Corollary 2.9. Let $J_1, \ldots, J_m \subset \mathbb{Z}_p$ be pairwise disjoint intervals, let $B = \bigcup_{i \in [m]} J_i$, and let $\gamma > 0$. If $s \in \mathbb{Z}_p$ satisfies $|\hat{1}_B(s)| \geq \gamma$, then $|s|_p \leq \frac{m}{2\gamma}$.

Proof. We may assume that $s \neq 0$, and then by (8) we have $\frac{m}{2|s|_p} \geq |\hat{1}_B(s)| \geq \gamma$, whence the result follows. \hfill \Box

We shall combine this corollary with the following result.

Lemma 2.10. Let $B \subset \mathbb{Z}_p$, let $n \in \mathbb{Z}_p \setminus \{0\}$, and let $I$ be an interval in $\mathbb{Z}_p$ such that $n \cdot B \subset I$ and $|I| < p/2$. Then

$$|\hat{1}_B(n)| > \frac{1}{p} (|B| - (1 - \frac{2}{p})|I|). \quad (11)$$

This lemma yields a positive lower bound for $|\hat{1}_B(n)|$ when $|B|/|I| > 1 - \frac{2}{p} \approx 0.364$.

Proof. We have

$$\hat{1}_B(n) = \frac{1}{p} \sum_{j \in \mathbb{Z}_p} 1_B(j) e^{2\pi i \frac{nj}{p}} = \frac{1}{p} \sum_{j} 1_B(n^{-1}j) e^{2\pi i \frac{j}{p}} = \frac{1}{p} \sum_{j} 1_{nB}(j) e^{2\pi i \frac{j}{p}}$$

$$= \frac{1}{p} \sum_{j} 1_I(j) e^{2\pi i \frac{j}{p}} + \frac{1}{p} \sum_{j} (1_{nB}(j) - 1_I(j)) e^{2\pi i \frac{j}{p}}.$$

The last sum here has magnitude at most $\frac{1}{p}|I \setminus n \cdot B|$. We may assume that $I = \{0, 1, \ldots, (t - 1)\}$ for some $t < p/2$. Hence

$$|\hat{1}_B(n)| \geq \frac{1}{p} \left( \left| \sum_{j \in I} e^{2\pi i \frac{j}{p}} \right| - |I \setminus n \cdot B| \right) = \frac{1}{p} \left( \left| \frac{1 - e^{2\pi i \frac{t}{p}}}{1 - e^{2\pi i \frac{1}{p}}} \right| + |B| - |I| \right)$$
where we have used the same calculation as in (9). Using the estimates
\[ 4\|\theta\|_T \leq |1 - e^{2\pi i \theta}| \leq 2\pi |\theta|_T \quad \text{for } \|\theta\|_T < 1/2, \]
we obtain \[ \frac{|1 - e^{2\pi i \theta}|}{|1 - e^{2\pi i \theta}|} \geq \frac{4\pi/2\pi}{2\pi/2\pi} = \frac{2}{\pi}|I|, \]
and the result follows. \qed

Proof of Proposition 2.7. By definition of \( D_n(B) \) there is an interval \( I \subset \mathbb{Z}_p \) satisfying
\[ |I| = D_n(B) < \frac{1}{2} \] and \( n \cdot B \subset I \). Lemma 2.10 gives us \( |\hat{1}_B(n)| > \frac{1}{p}(|B| - (1 - \frac{2}{\pi})|I|) \), and this lower bound is positive by our assumptions. Combining this with Corollary 2.9 we obtain \( |n|_p \leq \frac{m}{2(\beta - (1 - 2/\pi)D_n(B))} \), as claimed. \qed

Remark 2.11. Some restriction on the size of \( D_n(B) \) is necessary in Lemma 2.10 and in Proposition 2.7. Indeed, if \( B = I = \{0, \ldots, t - 1\} \) with \( t = p(1 - \theta) \) and \( 0 < \theta < 1/2 \), then \( D_1(B) = |B|/p = 1 - \theta \). In this case, with \( n = 1 \) we have
\[ |\hat{1}_B(n)| = |\hat{1}_I(1)| = \frac{1}{p} \left| \frac{1 - e^{2\pi i \frac{1}{2} \theta}}{1 - e^{2\pi i \frac{1}{2} \theta}} \right| = \frac{1}{p} \frac{\sin \left( \frac{\pi}{p} \right)}{\sin \left( \frac{\pi}{p} \right)} = \frac{\sin(\pi\theta)}{\pi \sin(\pi\theta)}. \]
For \( p \) large, this is very close to \( \frac{1}{\pi} \sin(\pi\theta) < 1/\pi \), whereas \( \frac{1}{p}(|B| - (1 - \frac{2}{\pi})|I|) = \frac{2}{\pi}(1 - \theta) > 1/\pi \). This shows that (11) can fail if \( D_n(B) > 1/2 \).

Proposition 2.7 fails for this set \( B \) also if \( \beta = 1 - \theta > \pi/4 \), since in this case we have \( D_1(B) = \beta, m = 1, \) and yet \( 1 > \frac{\pi}{2\pi} = \frac{1}{2(\beta - (1 - 2/\pi)D_1(B))}. \)

2.2. Proof of the main result for simple open sets.

In this subsection we establish Theorem 1.4 for simple open sets as follows.

Proposition 2.12. Let \( \rho \in (0, c) \) where \( c = 3.1 \cdot 10^{-1549} \). For each \( j \in [3] \) let \( A_j \subset \mathbb{T} \) be a union of at most \( m \) pairwise disjoint open intervals, and suppose that
\[ \min_j \mu(A_j) > \rho, \quad \mu(A_1) + \mu(A_2) + \mu(A_3) > 1 - \rho, \quad \mu(A_1 + A_2 + A_3) < 1. \]
Then there exists a positive integer \( n \leq \frac{2m}{\min_j \mu(A_j)} \) and closed intervals \( I_1, I_2, I_3 \subset \mathbb{T} \) such that \( n \cdot A_j \subset I_j \) and \( \mu(I_j) \leq \mu(A_j) + \rho \) for \( j \in [3] \).

Proof. Fix any \( \delta > 0 \) satisfying
\[ \delta < \min \left\{ \frac{1}{2} \left( \mu(A_1) + \mu(A_2) + \mu(A_3) - (1 - \rho) \right), 1 - \mu(A_1 + A_2 + A_3), \rho/10 \right\}. \]
For \( p \) sufficiently large, we can assume that \( A_{j,p} \) is the union of at most \( m \) intervals in \( \frac{1}{p} \mathbb{Z}_p \).
Let \( \mu_p \) denote the discrete measure \( \frac{1}{p} \sum_{j=0}^{p-1} \delta_{j/p} \) on \( \mathbb{T} \), where \( \delta_{j/p} \) is a Dirac \( \delta \) measure at \( j/p \). We have that \( \mu_p \) converges weakly to the Haar probability measure on \( \mathbb{T} \) as \( p \to \infty \), and so \( \frac{1}{p} |A_{j,p}| = \mu_p(A_{j,p}) \to \mu(A_j) \) and \( \mu_p((A_1 + A_2 + A_3)_p) \to \mu(A_1 + A_2 + A_3) \) as \( p \to \infty \).
Note that the inner Haar measure and the Haar measure of simple sets in $\mathbb{T}$ coincide and that a sumset of simple sets is a simple set. In particular, for $p$ sufficiently large (depending on the sets $A_j$ and $\delta$) we have

$$|\mu_p(A_{j,p}) - \mu(A_j)| \leq \delta/3, \quad |\mu_p((A_1 + A_2 + A_3)_p) - \mu(A_1 + A_2 + A_3)| \leq \delta. \quad (13)$$

By our assumptions in (12), the fact that $A_{1,p} + A_{2,p} + A_{3,p} \subset (A_1 + A_2 + A_3)_p$, and our choice of $\delta$ and $p$, we then have

$$\mu_p(A_{j,p}) > \rho - \delta/3, \quad \sum_{j \in [3]} \mu_p(A_{j,p}) > 1 - \rho + \delta, \quad \mu_p(A_{1,p} + A_{2,p} + A_{3,p}) < 1. \quad (14)$$

Let $r$ be an integer in $((\rho - \delta)p, (\rho - 2\delta/3)p)$. For $p \geq 6/\delta$ sufficiently large we can apply Theorem 2.1 to the sets $A_{j,p}$ with this integer $r$. This yields intervals $I_{j,p} \subset \frac{1}{p}\mathbb{Z}_p$ and a non-zero integer $n \in (-\frac{1}{p}, \frac{1}{p})$ such that $n \cdot A_{j,p} \subset I_{j,p}$ and $\mu_p(I_{j,p}) \leq \mu_p(A_{j,p}) + r/p \leq \mu(A_j) + \rho - \delta/3$. For every $x$ in the simple open set $A_j$, there exists $y \in A_{j,p}$ such that $\|x - y\|_T \leq \frac{1}{3p}$, which implies that $\|nx - ny\|_T \leq \frac{|n|}{3p}$. Hence

$$n \cdot A_j \subset n \cdot A_{j,p} + \left[ -\frac{|n|}{2p}, \frac{|n|}{2p} \right] \subset I_{j,p} + \left[ -\frac{|n|}{2p}, \frac{|n|}{2p} \right].$$

Let $I_j$ be the closed interval $I_{j,p} + \left[ -\frac{|n|}{2p}, \frac{|n|}{2p} \right]$. We have

$$\mu(I_j) \leq \mu_p(I_{j,p}) + \frac{|n|}{p} \leq \mu(A_j) + \rho - \delta/3 + \frac{|n|}{p}. \quad (15)$$

We have $\min_j \mu(A_j) < \frac{1}{3}$, by the third inequality in (12) and Raikov’s inequality [20, Theorem 1]. Therefore, supposing without loss of generality that $\min_j \mu(A_j) = \mu(A_1)$, we have $\mu_p(I_{1,p}) < 1/2$. We also have $\mu_p(I_{1,p}) < \frac{\mu_p(A_{1,p})}{1 - 2/\pi}$. By Proposition 2.7, we conclude that $|n| \leq \frac{m}{2(\mu_p(A_{1,p}) - (1 - 2/\pi)(2\mu(A_1) - \delta/3))}$. Since $\mu_p(I_{1,p}) < 2\mu(A_1) - \delta/3$ and $\delta < \rho/10$, we have

$$|n| \leq \frac{m}{2\mu(A_1) - \delta/3 - (1 - 2/\pi)(2\mu(A_1) - \delta/3)} \leq \frac{m}{2((\frac{4}{\pi} - 1)\mu(A_1) - \frac{2\delta}{5p})} \leq \frac{2m}{\mu(A_1)}.$$ 

Since we can take $p > \frac{6m}{\delta p}$, we have $\frac{2m}{\mu(A_1)} \leq \frac{\delta p}{3}$, and so from (15) we have $\mu(I_j) \leq \mu(A_j) + \rho$. Finally, as $n \cdot A_j$ and $-n \cdot A_j$ are both included in suitable intervals, we can have $n > 0$. \qed

### 2.3. From simple sets to arbitrary sets.

In this subsection we deduce Theorem 1.4 using Proposition 2.12. To that end, we first prove Theorem 1.4 for closed sets.
Proposition 2.13. Let $\rho \in (0, c)$ where $c = 3.1 \cdot 10^{-1549}$. Let $A_1, A_2, A_3$ be closed subsets of $\mathbb{T}$ satisfying the following conditions:

$$\mu(A_1), \mu(A_2), \mu(A_3) > \rho, \quad \sum_{j \in [3]} \mu(A_j) > 1 - \rho, \quad \mu(A_1 + A_2 + A_3) < 1.$$

Then there exist closed intervals $I_1, I_2, I_3 \subset \mathbb{T}$ and a positive integer $n$ such that $n \cdot A_j \subset I_j$ and $\mu(I_j) \leq \mu(A_j) + \rho$ for $j \in [3]$.

Proof. Fix any $\varepsilon > 0$ with $\varepsilon < \min\{\mu(A_1) + \mu(A_2) + \mu(A_3) - (1 - \rho), 1 - \mu(A_1 + A_2 + A_3)\}$. For $\delta > 0$ let $I_\delta$ denote the open interval $(-\delta, \delta)$ in $\mathbb{T}$. We have $A = \bigcap_{\delta > 0}(A + I_\delta)$ and $A_1 + A_2 + A_3 = \bigcap_{\delta > 0}(A_1 + A_2 + A_3 + I_{3\delta}) = \bigcap_{\delta > 0}(A_1 + I_\delta + A_2 + I_\delta + A_3 + I_\delta)$. Let $\delta > 0$ be sufficiently small so that $\forall j \in [3], \mu(A_j + I_\delta) \leq \mu(A_j) + \varepsilon$, and $\mu(A_1 + A_2 + A_3 + I_{3\delta}) \leq \mu(A_1 + A_2 + A_3) + \varepsilon < 1$.

By compactness of each set $A_j$, there exists a set $A_j'$ that is the union of finitely many translates of $I_\delta$ such that $A_j \subset A_j' \subset A_j + I_\delta$. The simple open sets $A_1', A_2', A_3'$ satisfy the inequalities in (12) with initial parameter $\rho - \varepsilon$. Therefore, by Proposition 2.12 applied to these sets with this parameter, we obtain a positive integer $n$ and closed intervals $I_1, I_2, I_3 \subset \mathbb{T}$ with $n \cdot A_j \subset n \cdot A_j' \subset I_j$, and $\mu(I_j) \leq \mu(A_j') + \rho - \varepsilon \leq \mu(A_j) + \rho$. \hfill \Box

From this we shall deduce Theorems 1.4 and 1.5. To do so we use the following lemma based on ideas of Bilu from [2].

Lemma 2.14. Let $C \subset A \subset \mathbb{T}$, where $C$ is a closed set, and let $X$ be a finite set of integers. Then for every $\varepsilon > 0$ there exists a closed set $E$ with $C \subset E \subset A$ such that $\mu(E) = \mu(C)$ and $n \cdot A \subset n \cdot E + [-\varepsilon, \varepsilon]$ for all $n \in X$.

Proof. For each $n \in X$, since $A$ is totally bounded, there is a finite subset $E(n, \varepsilon) \subset A$ such that $A \subset E(n, \varepsilon) + [-\varepsilon, \varepsilon]$, and so $n \cdot A \subset n \cdot E(n, \varepsilon) + [-\varepsilon, \varepsilon]$. The set $E = C \cup \bigcup_{n \in X} E(n, \varepsilon)$ satisfies the claim in the lemma. \hfill \Box

We shall combine this with the following special case of [2, Lemma 4.2.1].

Lemma 2.15. Let $B$ be a Haar measurable subset of $\mathbb{T}$ with $\mu(B) > 0$, and let $\lambda < 1$. Then there exist only finitely many integers $n$ such that $\mu(n \cdot B) \leq \lambda$.

We can now prove the main result.

Proof of Theorem 1.4. We assume that $\mu(A_j) > \rho$, that $\mu(A_1) + \mu(A_2) + \mu(A_3) > 1 - \rho$, and that $\mu(A_1 + A_2 + A_3) < 1$. Fix an arbitrary $\delta$ satisfying

$$0 < \delta < \min\left\{\frac{1}{2}(\mu(A_1) + \mu(A_2) + \mu(A_3) - (1 - \rho)), \rho\right\}.$$
We may assume that $\mu(A_1) = \min_{j \in [3]} \mu(A_j)$. Let $C_1$ be a closed subset of $A_1$ such that $\mu(C_1) > \mu(A_1) - \delta/3 > 0$. Let $X$ be the set of integers $n$ such that $\mu(n \cdot C_1) \leq \mu(A_1) + \rho$. Since $\mu(A_1 + A_2 + A_3) < 1$, by Raikov’s inequality we have $\mu(A_1) < 1/3$, and so we can certainly apply Lemma 2.15 to deduce that $X$ is finite. Let $A'_1$ be the closed subset of $A_1$ obtained by applying Lemma 2.14 to $C_1 \subset A_1$ with $\varepsilon = \delta/2$. Similarly, for $j = 2, 3$ let $A'_j$ be a closed subset of $A_j$ such that $\mu(A'_j) \geq \mu(A_j) - \delta/3$ and $n \cdot A_j \subset n \cdot A'_j + [-\frac{\delta}{2}, \frac{\delta}{2}]$ for all $n \in X$. We then have $\mu(A'_j) > \rho - \delta/3$ for every $j$, we have $\mu(A'_1 + A'_2 + A'_3) < 1$, and

$$\mu(A'_1) + \mu(A'_2) + \mu(A'_3) > \mu(A_1) + \mu(A_2) + \mu(A_3) - \delta > 1 - (\rho - \delta).$$

By Proposition 2.13 applied to the sets $A'_j$ with initial parameter $\rho - \delta$, there exist closed intervals $I'_j$ such that $\mu(I'_j) \leq \mu(A'_j) + \rho - \delta \leq \mu(A_j) + \rho - \delta$, and a positive integer $n$ such that $n \cdot A'_j \subset I'_j$, for $j = 1, 2, 3$. In particular we must have $n \in X$, and then by our choice of the sets $A'_j$ we have $n \cdot A_j \subset I'_j + [-\frac{\delta}{2}, \frac{\delta}{2}]$. Letting $I_j$ be the closed interval $I'_j + [-\frac{\delta}{2}, \frac{\delta}{2}]$ for each $j$, we have $\mu(I_j) \leq \mu(A_j) + \rho$, and the result follows.

We now deduce Theorem 1.5 from Proposition 2.13.

**Proof of Theorem 1.5.** Let $A, B \subset \mathbb{T}$ satisfy the assumptions in the theorem, namely

$$\mu(A + B) = \mu(A) + \mu(B) + \rho < \frac{1}{2}(1 + \mu(A) + \mu(B)), \quad \rho < \mu(B) \leq \mu(A), \quad \rho < c. \quad (17)$$

Note that this implies that $1 - \mu(A + B) > \rho$. One could want to deduce from this that $\mu((-A + B)^c) > \rho$ and then apply Theorem 1.4 to $A, B, -(A + B)^c$, but this raises several technical difficulties (in particular the behaviour of the inner Haar measure $\mu$ relative to taking complements). It is convenient to prove the result first for closed sets.

Suppose, then, that $A, B$ are closed and satisfy (17). Fix any $\delta > 0$ satisfying

$$\mu(A) + \mu(B) + \rho + \delta < \frac{1}{2}(1 + \mu(A) + \mu(B)), \quad \delta < \rho/4, \quad \mu(B) > \rho + \delta, \quad \rho + \delta < c. \quad (18)$$

Note that the equality in (17) is equivalent to $\mu(A) + \mu(B) + \mu(- (A + B)^c) = 1 - \rho$, which together with the first inequality in (18) implies that $\mu(-(A + B)^c) > \rho + 2\delta$. In particular this, the last equality, and $\mu(B) > \rho + \delta$ together imply that

$$\mu(A) + 3(\rho + \delta) < 1; \quad \text{we have similarly } \mu(-(A + B)^c) + 3\rho + 2\delta < 1. \quad (19)$$

Now let $A_1 = A$, $A_2 = B$, and let $A'_3$ be a closed subset of $-(A_1 + A_2)^c$ satisfying $\mu(A'_3) > \mu(-(A_1 + A_2)^c) - \delta > \rho + \delta$. Let $X$ be the finite set of integers $n$ such that $\mu(n \cdot A_2) \leq \mu(A_2) + \rho + \delta$ (finite by Lemma 2.15). Since $\mu(A_2) + \rho + \delta < 1$ by (19), let $A_3$ be the closed set given by applying Lemma 2.14 to $A'_3 \subset -(A_1 + A_2)^c$ with $\varepsilon = \delta/2$. 


We now show that $A_1, A_2, A_3$ satisfy the conditions to apply Proposition 2.13 with initial parameter $\rho + \delta$. Firstly, as seen above, by construction we have $\mu(A_j) > \rho + \delta$ for $j = 1, 2, 3$. Secondly, we have

$$\mu(A_1) + \mu(A_2) + \mu(A_3) > \mu(A) + \mu(B) + \mu(-(A+B)^c) - \delta = 1 - (\rho + \delta).$$  \hspace{1cm} (20)

Finally, since $A_1 + A_2 + A_3$ is a closed set included in $A + B - (A+B)^c$, and since the latter set does not contain 0, the closed set $A_1 + A_2 + A_3$ must miss an entire open interval about 0, whence $\mu(A_1 + A_2 + A_3) < 1$.

We can now apply Proposition 2.13 to $A_1, A_2, A_3$ and thus obtain a positive integer $n$ and closed intervals $I_j$ such that $\mu(I_j) \leq \mu(A_j) + \rho + \delta$ and $n \cdot A_j \subset I_j$ for $j = 1, 2, 3$.

In particular $n$ must be in $X$, so by construction $n \cdot (A + B)^c$ is included in the closed interval $-I_3 + [-\frac{\delta}{2}, \frac{\delta}{2}]$, and therefore so is $(n \cdot (A + B))^c$. Let $I_{3,\delta} = -I_3 + [-\frac{\delta}{2}, \frac{\delta}{2}]$, thus $\mu(I_{3,\delta}) \leq \mu(A_3) + \rho + 2\delta$, and let $I_{j,\delta} := I_j$ for $j = 1, 2$.

Now, we repeat the above argument for each term of a decreasing sequence of positive numbers $\delta_m$ satisfying (18) and tending to 0 as $m \to \infty$. Note that although the integer $n = n(\delta_m)$ could vary as $m$ increases, we can assume that it is constant, by passing to a subsequence if necessary, since $X$ is finite. For $j = 1, 2, 3$ let $I_{j,m} = \bigcap_{k \leq m} I_{j,\delta_k}$. We have that $I_{j,m}$ is a closed interval for all $m$. Indeed, a priori the intersection of two intervals $I, J$ in $\mathbb{T}$ could be a union of two disjoint intervals, but this can occur only if $I \cup J = \mathbb{T}$, whereas here for $k < \ell$ we have by inclusion-exclusion that $\mu(I_{j,\delta_k} \cup I_{j,\delta_\ell}) \leq \mu(A_j) + 2\rho + 4\delta_k$, which is less than 1 by (19). Therefore $(I_{j,m})_m$ is a decreasing sequence of closed intervals, each including $n \cdot A_j$ for $j = 1, 2$ and $(n \cdot (A + B))^c$ for $j = 3$, whence the closed intervals $I_j = \bigcap_m I_{j,m}$ also include these sets respectively (in particular $n \cdot (A+B)$ includes the open interval $I_3^c$), and we have $\mu(I_j) \leq \mu(A_j) + \rho$.

This completes the proof of Theorem 1.5 for closed sets $A$ and $B$.

Now let $A, B$ be arbitrary subsets of $\mathbb{T}$ satisfying (17). Let $\delta > 0$ satisfy

$$\rho + 2\delta < c, \quad \delta < \frac{1}{3}(\mu(B) - \rho), \quad \mu(A) + \mu(B) + \rho + 2\delta < \frac{1}{2}(1 + \mu(A) + \mu(B) - 2\delta).$$

Then, let $A_1', A_2'$ be closed subsets of $A, B$ respectively with $\mu(A_1') > \mu(A) - \delta$ and $\mu(A_2') > \mu(B) - \delta$, and let $A_1, A_2$ be the closed sets obtained by applying Lemma 2.14 with $A_1' \subset A, A_2' \subset B$, with $X$ being the finite set of integers $n$ such that $\mu(n \cdot A'_1) \leq \mu(A) + \delta$.

There is then $\rho' \leq \rho + 2\delta$ such that

$$\mu(A_1 + A_2) = \mu(A_1) + \mu(A_2) + \rho' < \frac{1}{2}(1 + \mu(A_1) + \mu(A_2)), \quad \rho' < \min(\mu(A_1), \mu(A_2)).$$

Applying the result for closed sets we obtain closed intervals $I, J$, an open interval $K$, and a positive integer $n$ such that $n \cdot A \subset I + [-\frac{\delta}{2}, \frac{\delta}{2}]$ and this closed interval has measure at
most \( \mu(A) + \rho + 3\delta \), similarly \( n \cdot B \subset J + [-\frac{\delta}{2}, \frac{\delta}{2}] \) with \( \mu(J + [-\frac{\delta}{2}, \frac{\delta}{2}]) \leq \mu(B) + \rho + 3\delta \), and finally \( n \cdot (A + B) \supset n \cdot (A_1 + A_2) \supset K \) with \( \mu(K) \geq \mu(A_1) + \mu(A_2) \geq \mu(A) + \mu(B) - 2\delta \). Now letting \( \delta \to 0 \) in an argument similar to the one above for closed sets (taking a countable union of open intervals in the case of \( K \)), the result follows. \( \square \)

In the case \( A = B \), using Theorem 2.4 rather than Theorem 2.1 yields a better bound \( c \).

**Proof of Theorem 1.6.** Following a similar strategy as for Theorem 1.4, we can reduce to the case of \( A \) being a union of finitely many open intervals. We replace the condition \( \rho < c \) from Theorem 1.5 by \( \rho < \varepsilon \mu(A) \) with \( \varepsilon < 10^{-4} \) and use an argument similar to the proof of Proposition 2.12 to obtain a positive integer \( n \) and a closed interval \( I \) such that \( n \cdot A \subset I \). Now Theorem 2.4 does not give information on the structure of \( A + A \), so we need to proceed differently to find some interval \( K \) included in \( n \cdot (A + A) \). Write \( \hat{A} = n \cdot A \subset \mathbb{T} \). We have \( \mu(\hat{A}) \geq \mu(A) \) and, since \( \hat{A} \subset I \), we have

\[
\mu(\hat{A} + \hat{A}) \leq 2\mu(I) \leq 2(\mu(A + A) - \mu(A)) \leq (2 + 2\varepsilon)\mu(A) < 3\mu(A) \leq 3\mu(\hat{A}).
\]

We know that \( \hat{A} \) is included in an interval of length at most \( \mu(A + A) - \mu(A) < 1/2 \). The desired conclusion, i.e. that \( \hat{A} + \hat{A} \) contains a large interval, is not affected by translating \( \hat{A} \) in \( \mathbb{T} \), so we may suppose that \( \hat{A} \subset [0, 1/2) \), where we identify \( \mathbb{T} \) as a set with \([0, 1)\). Then the sum \( \hat{A} + \hat{A} \) behaves as a sum in \( \mathbb{R} \), and so \( \hat{A} \) can be treated as a subset of \( \mathbb{R} \) of doubling constant strictly less than 3. Theorem 1 from [27] then ensures the existence of an interval \( K \subset \hat{A} + \hat{A} \) of length at least \( 2\mu(\hat{A}) \geq 2\mu(A) \), which completes the proof. \( \square \)

**Remark 2.16.** Given the above deductions of Theorems 1.4, 1.5 and 1.6 from their counterparts in \( \mathbb{Z}_p \), any improvement of the bounds \( c \) in these discrete counterparts will immediately yield the same improvement in the continuous setting.

In [30] Serra and Zémor give an example in \( \mathbb{Z}_p \) to show that the condition \( |A + A| < \frac{|A|^2}{2} + |A| \) in Theorem 2.4 is necessary. We can adapt this to show that the condition \( \mu(A + A) < \frac{1}{2} + \mu(A) \) is also necessary for Theorem 1.6 to hold, as follows.

**Example 2.17.** Viewing \( \mathbb{T} \) as \([0, 1] \) with addition mod 1, consider the set

\[
A = \left( \frac{1}{4} - \delta, \frac{1}{2} \right] \cup \left( 1 - \delta, 1 \right] \subset \mathbb{T}, \text{ for an arbitrary fixed } \delta \in (0, \frac{1}{8}).
\]

We have \( \mu(A) = \frac{1}{4} + 2\delta \), and \( A + A = \left( \frac{1}{2} - 2\delta, 1 \right] \cup \left( \frac{1}{4} - 2\delta, \frac{1}{2} \right] = \left( \frac{1}{4} - 2\delta, 1 \right] \). Hence \( \mu(A + A) = \frac{3}{4} + 2\delta = \frac{1}{2} + \mu(A) < 3\mu(A) \). Moreover \( \mu(A + A) = 2\mu(A) + 2 \left( \frac{1}{8} - \delta \right) \) and \( \frac{1}{8} - \delta \) can be made arbitrarily small. However, we cannot include \( n \cdot A \) in a preimage of a closed interval \( I \) of measure \( \mu(A + A) - \mu(A) = \frac{1}{2} \), for any positive integer \( n \). Indeed, this is clear for \( n = 1 \), as \( A \) is not contained in an interval of length \( \frac{1}{2} = \mu(A + A) - \mu(A) \).
For $n \geq 2$, note that $\mu(n \cdot A) \geq \mu(n \cdot (\frac{1}{4} - \delta, \frac{1}{2}])$, and this is at least $\mu(2 \cdot (\frac{1}{4} - \delta, \frac{1}{2}])$ (in general, for any interval $J \subset \mathbb{T}$ and any integers $n \geq m > 0$, we have $\mu(nJ) \geq \mu(mJ)$). Since $\mu(2 \cdot (\frac{1}{4} - \delta, \frac{1}{2}]) = \mu((\frac{1}{2} - 2\delta, 1]) > \frac{1}{2}$, we must have $D_n(A) > \frac{1}{2}$.

**Remark 2.18.** The conjecture of Bilu mentioned in the introduction, namely [2 Conjecture 1.2], proposes (in its special case for $\mathbb{T}$) that if $A, B \subset \mathbb{T}$ with $\alpha = \mu(A) \geq \mu(B) = \beta$ satisfy $\mu(A+B) < \min(\alpha+2\beta, 1)$, then there exist closed intervals $I, J \subset \mathbb{T}$ and $n \in \mathbb{N}$ such that $n\cdot A \subset I, n\cdot B \subset J$, and $\mu(I) \leq \mu(A+B) - \mu(B)$, $\mu(J) \leq \mu(A+B) - \mu(A)$. Bilu proved that this conjecture holds under the additional condition that $\alpha/\tau \leq \beta \leq \alpha \leq c(\tau)$, where $c$ is some positive constant depending on $\tau \geq 1$; see [2 Theorem 1.4]. However, note that the conjecture itself does not hold for arbitrary $\alpha, \beta$. Indeed, Example 2.17 shows that the conjecture can fail for sets greater than $1/4$. These counterexamples can be ruled out by adding a condition to the conjecture, for instance that $\mu(A+B) < \frac{1}{2}(1 + \mu(A) + \mu(B))$. Thus, a plausible version of Bilu’s conjecture on $\mathbb{T}$, without a fixed upper restriction on $\mu(A)$, could be that Theorem 1.5 holds for every $\rho \in (0, 1)$. In another direction, one may try to find the largest upper bound on $\mu(A)$ under which Bilu’s conjecture holds (given Example 2.17 this bound must be at most $1/4$).

3. **Application to $k$-sum-free sets in $\mathbb{T}$**

A subset of an abelian group is said to be $k$-sum-free if it does not contain any triple $(x, y, z)$ solving the linear equation $x + y = kz$, where $k$ is a fixed positive integer.\(^2\) In the case $k = 1$ the corresponding sets are known simply as sum-free sets, and their study dates back to work of Schur from 1916 [29]. The case $k = 2$ concerns sets avoiding 3-term arithmetic progressions, and this topic includes Roth’s theorem from 1953 [28] as well as the numerous related later works, recent examples of which include [4, 9, 11, 28]. Note that this case differs in nature from the other cases, in that this is the only value of $k$ for which the linear equation in question is *translation invariant*, meaning that if $(x, y, z)$ is a solution then so is $(x + t, y + t, z + t)$ for every fixed element $t$ in the group.

For $k \geq 3$ the topic goes back at least to work of Erdős, who conjectured in particular that for large $n$ the odd numbers in $[n]$ form the unique 3-sum-free set of maximum size (see [7]). Chung and Goldwasser proved this conjecture in [7], and made an analogous conjecture about the maximum size of $k$-sum-free subsets of $[n]$ for $k \geq 4$, which was proved by Baltz, Hegarty, Knape, Larsson and Schoen in [11]. Chung and Goldwasser also initiated the study of $k$-sum-free sets in the continuous setting. In particular, in [8]\(^2\)The term *$k$-sum-free set* is used for instance in [1]. These sets should not be confused with sets free of solutions to the equation $a_1 + \cdots + a_k = b$, which have also been called $k$-sum-free sets (see [19]).
they determined the structure and measure of maximal $k$-sum-free Lebesgue measurable subsets of the interval $(0, 1]$ for $k \geq 4$. They then made a conjecture concerning the structure and measure of maximal 3-sum-free sets in this setting. Significant progress toward this conjecture was made by Matolcsi and Ruzsa in [21], and the conjecture was then fully proved by Plagne and the second named author in [23].

Here we initiate the study of $k$-sum-free sets in $\mathbb{T}$ by considering the problem of estimating the following quantity:

$$d_k(\mathbb{T}) = \sup\{\mu(A) : A \text{ is a Haar measurable } k\text{-sum-free subset of } \mathbb{T}\}.$$ 

Note that $A$ is $k$-sum-free if and only if $(A + A) \cap k \cdot A = \emptyset$, and since by Raikov’s inequality we have $\mu(A + A) \geq 2\mu(A)$, it follows that

$$3\mu(A) \leq \mu(A + A) + \mu(k \cdot A) = \mu((A + A) \cup k \cdot A) \leq 1,$$ 

so $\mu(A) \leq 1/3$. (21)

Given this, the problem of determining $d_1(\mathbb{T})$ is easily settled: in $\mathbb{T}$ viewed as $[0, 1)$ with addition mod 1, the interval $(\frac{1}{3}, \frac{2}{3})$ is a sum-free set of maximum measure 1/3.

For $k = 2$, it follows from the above-mentioned invariance of the equation $x + y = 2z$ that $d_2(\mathbb{T}) = 0$ (in fact any set $A \subset \mathbb{T}$ of measure $\alpha > 0$ must contain a positive measure $c(\alpha)$ of 3-term progressions; see for instance [6, Theorem 1.4])

Let us now focus on $k \geq 3$. Here we can improve on (21) as follows.

**Theorem 3.1.** Fix any $\varepsilon > 0$ for which Theorem 1.6 holds, and let $k \geq 3$ be an integer. Then $d_k(\mathbb{T}) \leq \max\{\frac{1}{3 + \varepsilon}, \frac{1 + k\varepsilon}{k + 2}\}$.

The greatest value of $\varepsilon$ currently available here is the one provided by Serra and Zémor in [30], namely $\varepsilon = 10^{-4}$. This gives us $d_k(\mathbb{T}) \leq \frac{1}{3 + 10^{-4}}$ for all $k \geq 3$.

We prove Theorem 3.1 in several steps.

For a set $X \subset \mathbb{T}$ and $n \in \mathbb{N}$, we denote by $n^{-1}X$ the set $\{t \in \mathbb{T} : nt \in X\}$. Note that $X \subset \mathbb{T}$ is $k$-sum-free if and only if $X \cap k^{-1}(X + X) = \emptyset$. The following lemma tells us that if $A$ is $k$-sum-free and has measure close to 1/3 then for some $n \in \mathbb{N}$ we must have $n \cdot A$ contained efficiently in an interval $I$ that is almost $k$-sum-free, in the sense that $I \cap k^{-1}(I + I)$ has small measure.

**Lemma 3.2.** Let $k \geq 3$ be an integer, let $A \subset \mathbb{T}$ be a $k$-sum-free Borel set, and let $\varepsilon \leq 10^{-4}$. Then either $\mu(A) \leq 1/(3 + \varepsilon)$ or there exists a closed interval $I \subset \mathbb{T}$ and a positive integer $n$ such that $A \subset n^{-1}I$, $\mu(I) \leq \mu(A)(1 + \varepsilon)$, and $\mu(I \cap k^{-1}(I + I))) \leq 2\varepsilon \mu(I)$.

**Proof.** If $\mu(A + A) \geq (2 + \varepsilon)\mu(A)$, then arguing as in (21) we deduce that $\mu(A) \leq 1/(3 + \varepsilon)$. We may therefore assume that $\mu(A + A) \leq (2 + \varepsilon)\mu(A)$. Applying Theorem 1.6 with $\varepsilon$, we obtain an interval $I$ with $\mu(I) \leq \mu(A + A) - \mu(A)$ and $n \in \mathbb{N}$ such that $A \subset n^{-1}I$. 


Letting $B = n^{-1}I$ and using that the map $x \mapsto nx$ is measure-preserving, we have $\mu(B) = \mu(I)$, and so

$$\mu(B \setminus A) = \mu(I) - \mu(A) \leq \varepsilon \mu(A) \leq \varepsilon \mu(I).$$

(22)

Note also that, since for every set $X \subset \mathbb{T}$ we have $n^{-1}(X + X) = n^{-1}X + n^{-1}X$, we have $\mu(B + B) = \mu(n^{-1}(I + I)) = 2\mu(I)$ and so $\mu(B + B) \leq \mu(A + A) + (1 + \varepsilon)\mu(A) \triangleq \mu(A + A) + \varepsilon \mu(A)$. Hence

$$\mu((B + B) \setminus (A + A)) \leq \varepsilon \mu(I).$$

(23)

Writing $(B + B) = (A + A) \cup ((B + B) \setminus (A + A))$, we have

$$B \cap k^{-1}(B + B) \subset \left[B \cap k^{-1}(A + A)\right] \cup k^{-1}[(B + B) \setminus (A + A)].$$

Writing $B = A \cup (B \setminus A)$, and using that $A$ is $k$-sum-free, we have $B \cap k^{-1}(A + A) \subset B \setminus A$. Hence

$$B \cap k^{-1}(B + B) \subset (B \setminus A) \cup k^{-1}[(B + B) \setminus (A + A)].$$

(24)

Combining (22), (23), (24), and the fact that $\mu(I \cap k^{-1}(I + I)) = \mu(B \cap k^{-1}(B + B))$, the result follows. \qed

Given this lemma, our goal now is to obtain a useful upper bound on the measure of an almost-$k$-sum-free interval.

First we observe that if an interval $I \subset \mathbb{T}$ is $k$-sum-free then $\mu(I) \leq 1/(k + 2)$. Indeed, we must have $k \cdot I$ disjoint from $I + I$, which implies that $\mu(I + I) + \mu(k \cdot I) \leq 1$, which in turn implies (since then $\mu(I + I)$ and $\mu(k \cdot I)$ are both less than 1) that $\mu(I + I) = 2\mu(I)$ and $\mu(k \cdot I) = k\mu(I)$, which implies our claim. Note that this upper bound $1/(k + 2)$ is attained by the interval $I = \left[\frac{2}{k^2 - 4}, \frac{k}{k^2 - 4}\right)$, which is indeed $k$-sum-free (a simple calculation shows that $(I + I)^c = k \cdot I$).

We now show that if an interval is almost $k$-sum-free, then its measure cannot be much larger than $1/(k + 2)$.

**Lemma 3.3.** Let $k$ be a positive integer, let $\delta \in [0, 1)$, and let $I$ be a closed interval in $\mathbb{T}$ such that $\mu(I \cap k^{-1}(I + I)) \leq \delta \mu(I)$. Then $\mu(I) \leq \frac{1 + k\delta/2}{k + 2}$. 

**Proof.** Since $\mu(I \cap k^{-1}(I + I)) < \mu(I)$, we must have $I + I \neq \mathbb{T}$. The sumset $I + I$ is then a closed interval of measure $2\mu(I)$, and $k^{-1}(I + I)$ is a union of $k$ copies of $I + I$, each copy shrunk by a factor of $1/k$, and the centers of the copies forming an arithmetic progression of difference $1/k$. The complement of $k^{-1}(I + I)$ consists of $k$ components, each being an open interval of measure $\frac{1 + \mu(I)}{k}$. Let $j$ be the number of these components that have non-empty intersection with $I$. Then $I$ must cover $j - 1$ of the intervals making
up $k^{-1}(I + I)$, so we have $(j - 1)^2 \frac{\mu(I)}{k} \leq \mu(I \cap k^{-1}(I + I)) \leq \delta \mu(I)$, whence $j \leq 1 + \frac{4k}{\varepsilon}$.

We therefore have

$$
\mu(I) = \mu(I \setminus k^{-1}(I + I)) + \mu(I \cap k^{-1}(I + I)) \leq j \frac{1 - 2 \mu(I)}{k} + \delta \mu(I) \leq (1 + \frac{4k}{\varepsilon}) \frac{1 - 2 \mu(I)}{k} + \delta \mu(I),
$$

whence $\mu(I)(1 - \delta)k \leq (1 + \frac{4k}{\varepsilon})(1 - 2 \mu(I))$. After rearranging, we find that this inequality is equivalent to $\mu(I) \leq \frac{1}{2 + \varepsilon} + \frac{\delta}{2 + 4/k}$, and the result follows.

**Proof of Theorem 3.1** Combining Lemma 3.2 with Lemma 3.3 we have that either $\mu(A) \leq \frac{1}{3 + \varepsilon}$ or there exist an interval $I$ and $n \in \mathbb{N}$ such that $\mu(I) \leq \frac{1 + k\varepsilon}{k+2}$ and $n \cdot A \subset I$, whence $\mu(A) \leq \mu(n \cdot A) \leq \mu(I)$, and the result follows.

**Remark 3.4.** There is an equivalence between determining $d_k(T)$ and determining the quantity $d_k(\mathbb{Z}_p) = \max \left\{ \frac{|A|}{p} : A \subset \mathbb{Z}_p \text{ is } k\text{-sum-free} \right\}$ asymptotically as the prime $p$ tends to infinity. More precisely, it follows from [3, Theorem 1.3] that $\lim_{p \to \infty} d_k(\mathbb{Z}_p)$ exists and equals $d_k(T)$. Theorem 3.1 therefore implies that this limit is at most $\max \left\{ \frac{1}{3 + \varepsilon}, \frac{1 + k\varepsilon}{k+2} \right\}$.

4. **APPLICATION TO SETS OF DOUBLING LESS THAN 4 IN $\mathbb{R}$**

We shall write $\lambda$ for the inner Lebesgue measure on $\mathbb{R}$. For a bounded set $A \subset \mathbb{R}$ we denote by $\text{diam}(A)$ the diameter $\sup(A) - \inf(A)$.

The main result of this section is the following theorem which, for a bounded set $A \subset \mathbb{R}$ having doubling-constant not much larger than 3, gives information on the structure of $A$ modulo $\text{diam}(A)$.

**Theorem 4.1.** Let $\varepsilon \in [0, 1)$ be such that Theorem 1.6 holds. Let $A$ be a closed subset of $[0, 1]$ satisfying $\lambda(A + A) \leq (3 + \varepsilon)\lambda(A)$, $\lambda(A) \in \left(0, \frac{1}{2(1+\varepsilon)}\right)$, and $\text{diam}(A) = 1$. Then there exists a positive integer $n \leq \frac{1}{3 + \varepsilon}$ such that $n \cdot A \mod 1$ is included in a closed interval $I \subset \mathbb{T}$ with $\mu(I) \leq (1 + \varepsilon)\lambda(A)$.

Below, when we use the notation $\lambda$ together with a sumset, then addition is meant to be in $\mathbb{R}$; when we use instead the notation $\mu$, addition is meant to be in $\mathbb{T}$.

**Remark 4.2.** Any progress on the upper bound for $\varepsilon$ in Theorem 1.6 would yield progress in Theorem 4.1. In particular, by [14, Lemma 2] (see also [2, Corollary 1.5]), we already know that for some small absolute constant $a_0$, if we add to Theorem 1.6 the assumption that $A \subset \mathbb{T}$ has $\mu(A) \leq a_0$, then the theorem holds for every $\varepsilon \in [0, 1)$. This implies that for any set $A \subset [0, 1]$ satisfying $\lambda(A) \leq a_0$ and $\lambda(A + A) \leq (3 + \varepsilon)\lambda(A) < 4\lambda(A)$, there is a positive integer $n \leq \frac{1 + k\varepsilon}{k+2}$ such that $n \cdot A \mod 1$ is included in an interval $I \subset \mathbb{T}$ with
Remark 4.4. If $\mu(I) \leq (1 + \varepsilon)\lambda(A)$. Furthermore, if Bilu’s conjecture \cite[Conjecture 1.2]{Bilu} holds in the symmetric case $A = B \subset \mathbb{T}$ with $\mu(A) \leq 1/4$, then Theorem 4.1 holds with the condition $\lambda(A) \in \left(0, \frac{1}{2(1+\varepsilon)}\right)$ replaced by $\lambda(A) \in (0, 1/4)$, and any $\varepsilon < 1$.

Remark 4.3. Theorem 4.1 can be generalized to any bounded set $A \subset \mathbb{R}$, provided we replace the assumption $\lambda(A) \in \left(0, \frac{1}{2(1+\varepsilon)}\right)$ with $\lambda(A) \in \left(0, \frac{\text{diam}(A)}{2(1+\varepsilon)}\right)$ and that the conclusion is stated modulo $\text{diam}(A)$ rather than modulo 1.

Remark 4.4. If $\varepsilon < 1/3$ (this is the case for the $\varepsilon$ for which we know that Theorem 1.6 holds), then $n = 1$. This means that under the hypothesis of Theorem 4.1 with $\varepsilon < 1/3$, the set $A$ is included in a union $I_1 \cup I_2$ of two intervals, $I_1$ being of the form $[0, a]$ and $I_2$ of the form $[1 - b, 1]$, with $a + b \leq \lambda(A + A) - \lambda(A)$.

Proof of Theorem 4.1. By our assumptions $A \subset [0, 1]$ is closed with $0, 1 \in A$. If $\mu$ is the inner Haar measure on $\mathbb{T}$ and $\tilde{A}$ denotes $A$ mod 1, we have

$$\lambda(A + A) = \mu(\tilde{A} + \tilde{A}) + \mu(\Sigma_2),$$

where $\Sigma_2 = \{x \in [0, 1) : x, x + 1 \in A + A\}$. Since $0, 1 \in A$, we have that $A \setminus \{1\}$ is a subset of $\Sigma_2$, whence $\mu(\Sigma_2) \geq \mu(\tilde{A}) = \lambda(A)$. Therefore $\lambda(A + A) \leq (3 + \varepsilon)\lambda(A)$ implies that $\mu(\tilde{A} + \tilde{A}) \leq (2 + \varepsilon)\mu(\tilde{A}) < \frac{1}{2} + \mu(\tilde{A})$. Theorem 1.6 applied to $\tilde{A}$ gives us a positive integer $n$ such that $n \cdot \tilde{A}$ is included in a closed interval $I \subset \mathbb{T}$ of length at most $(1 + \varepsilon)\mu(\tilde{A}) = (1 + \varepsilon)\lambda(A)$. We thus have $\tilde{A} \subset n^{-1}I$, and $n^{-1}I$ viewed as a subset of $[0, 1)$ is a disjoint union of intervals $\frac{i}{n} + J$ mod 1, $i = 0, \ldots, n - 1$, where $J$ is a closed interval in $\mathbb{T}$ with $n \cdot J = I$ and $J \cap [0, \frac{1}{n}) \neq \emptyset$. (Note that $J$ viewed as a subset of $[0, 1)$ could have a part in $[1 - \frac{1}{2n}, 1]$.) Therefore, there exist sets $A_0 \subset [0, \frac{1}{n})$, $A_i \subset (-\frac{1}{n}, \frac{1}{n})$ for $i \in [n - 1]$, and $A_n \subset (-\frac{1}{n}, 0]$, such that

$$A = \bigcup_{i=0}^{n} \left(\frac{i}{n} + A_i\right), \text{ and } \bigcup_{i=0}^{n} A_i \mod 1 \subset J \text{, so in particular } \lambda\left(\bigcup_{i=0}^{n} A_i\right) \leq (1 + \varepsilon)\frac{\lambda(A)}{n}.$$ 

It remains to find an upper bound for $n$. We write $\alpha = \lambda(A)$, and $\alpha_i = \lambda(A_i)$ for $0 \leq i \leq n$. Since $(1 + \varepsilon)\alpha < \frac{1}{2}$, we have that $A + A$ is a disjoint union of subsets of $\mathbb{R}$ of the form

$$A + A = \bigcup_{i=0}^{2n} \left(\frac{i}{n} + S_i\right) \text{ with } S_i = \bigcup_{k,l:k+l=i} (A_k + A_l).$$
In particular $A_i + A_i \subseteq S_{2i}$ and $A_i + A_{i+1} \subseteq S_{2i+1}$. This yields

$$\lambda(A + A) = \sum_{i=0}^{2n} \lambda(S_i) = \sum_{i=0}^{n} \lambda(S_{2i}) + \sum_{i=0}^{n-1} \lambda(S_{2i+1}) \geq \sum_{i=0}^{n} 2\alpha_i + \sum_{i=0}^{n-1} (\alpha_i + \alpha_{i+1}) = 4\alpha - (\alpha_0 + \alpha_n).$$

Now $\alpha_0 + \alpha_n \leq (1 + \varepsilon)\alpha/n$, since mod 1 the sets $A_0 \setminus \{0\}$ and $A_n$ are disjoint and their union is included in $J$. Hence $(3 + \varepsilon)\alpha \geq \lambda(A + A) \geq 4\alpha - (\alpha_0 + \alpha_n) \geq \alpha \left(4 - \frac{1 + \varepsilon}{n}\right)$, and this implies that $n \leq \frac{1 + \varepsilon}{1 - \varepsilon}$.

In [10], Eberhard, Green and Manners prove the following result (see [10, Theorem 6.2]).

**Proposition 4.5.** Let $A \subseteq [0,1]$ be an open set with $\lambda(A - A) \leq 4\lambda(A) - \delta$. Then for some constant $c > 0$ depending on $\delta$ there is an interval $I$ of length $\lambda(I) \geq c$ such that $\lambda(A \cap I) \geq (\frac{1}{2} + \frac{1}{4})\lambda(I)$.

Here $\lambda(A - A)$ can be replaced with $\lambda(A + A)$ (see Remark (ii) after Theorem 6.2 in [10]). Theorem 4.1 above yields an effective version of this result when $\varepsilon$ is close to $\lambda(A)$.

**Corollary 4.6.** Let $A \subseteq [0,1]$ be a non-empty closed set with $\lambda(A + A) \leq 4\lambda(A) - \delta$ and $\lambda(A) < \frac{\text{diam}(A)}{4} + \frac{\delta}{4}$, for some $\delta > 0$. If $\delta > \lambda(A)(1 - \varepsilon)$, with $\varepsilon$ such that Theorem 4.6 holds, then there is an interval $I$ with $\lambda(I) \geq \min(\delta/4, \delta^2)$ such that $\lambda(A \cap I) \geq (\frac{1}{2} + \frac{1}{4})\lambda(I)$.

**Remark 4.7.** The size of $\delta$ is conditioned by Theorem 4.6. If Bilu’s conjecture holds for sets $A$ with $\mu(A) \leq \frac{1}{4}$, then Corollary 4.6 gives an effective version of [10, Theorem 6.2] for sets $A$ with $\lambda(A) \leq \frac{\text{diam}(A)}{4}$.

**Proof.** We first prove the result assuming that $\text{diam}(A) = 1$. We use the notation introduced in the proof of Theorem 4.1. Thus $A \subseteq [0,1]$ is a closed set with $0, 1 \in A$, with $\lambda(A + A) \leq 4\alpha - \delta$ and $\delta > \alpha (1 - \varepsilon)$, where $\alpha = \lambda(A)$.

First suppose that $\delta > \alpha$. Then $\lambda(A + A) \leq 4\lambda(A) - \delta < 3\lambda(A)$. Note that since $\lambda(A + A) \geq 2\alpha$, we have $\delta \leq 2\alpha$. Applying Theorem 4.1 with $\varepsilon = 0$, we obtain an interval $I \subseteq \mathbb{T}$ covering $A$ and with $\lambda(I) = \alpha$. Now as a subset of $\mathbb{R}$, the set $I$ is either an interval, in which case the conclusion holds, since $\lambda(I) \geq \alpha \geq \delta/2$ and $\lambda(A \cap I) = \lambda(I)$; or $I$ is a union of two intervals, one of which has measure at least $\lambda(I)/2 \geq \delta/4$, and then this interval satisfies the desired conclusion.

Let us now suppose that $\delta \leq \alpha$, and write $\delta = (1 - \varepsilon)\alpha$. Then we have $\lambda(A + A) \leq (3 + \varepsilon)\alpha$, and our assumption $\alpha < \frac{1}{4} + \frac{\delta}{2}$ also implies that
α < \frac{1}{2(1+\varepsilon)}. \) Therefore, by Theorem 4.4, there exists a positive integer \( n \leq \frac{1+\varepsilon}{1-\varepsilon} \) such that

\[
A = \bigcup_{i=0}^{n} \left( \frac{i}{n} + A_i \right) \quad \text{with} \quad \text{diam}_T \left( \bigcup_{i=0}^{n} A_i \right) \leq (1+\varepsilon)\frac{\alpha}{n},
\]

where for a set \( B \subset T \) we denote by \( \text{diam}_T(B) \) the infimum of the Haar measures of intervals in \( T \) that cover \( B \). Writing \( \tilde{A}_i = \begin{cases} A_i & \text{if } i \in [n-1], \\ A_0 \cup A_n & \text{if } i = 0 \end{cases} \) for \( n, \) there exists \( i \in [0, n-1] \) such that \( \mu(\tilde{A}_i) \geq \frac{\alpha}{n} \) and \( \text{diam}_T(\tilde{A}_i) \leq (1+\varepsilon)\frac{\alpha}{n}. \) There are now two cases.

If \( i \neq 0, \) then letting \( I \) be the interval \( [\inf(A_i), \sup(A_i)] \) in \( \mathbb{R}, \) we have

\[
\frac{\lambda(A \cap I)}{\lambda(I)} = \frac{\lambda(A_i)}{\text{diam}(A_i)} \geq \frac{1}{1+\varepsilon} = \frac{2}{2 - \delta / \alpha} \geq \frac{2 - \delta}{\alpha} \geq \frac{1}{2} + \frac{\delta}{4}.
\]

where for the last inequality we used that \( \alpha < \frac{1}{2(1+\varepsilon)} \leq \frac{1}{2}. \) We also have

\[
\lambda(I) \geq \frac{\alpha}{n} \geq \frac{\alpha}{1+\varepsilon} = \frac{\delta}{2-\delta/\alpha} \geq \frac{\delta}{2}.
\]

If \( i = 0, \) so \( \lambda(A_0) + \lambda(A_n) \geq \frac{\alpha}{n}, \) then let \( d_0 = \text{diam}(A_0), \) \( d_n = \text{diam}(A_n) \) and \( \alpha_0 = \lambda(A_0), \) \( \alpha_n = \lambda(A_n). \) If \( \frac{\alpha}{n} \geq \frac{1}{2} + \frac{\delta}{2} \) for both \( i = 0 \) and \( i = n, \) then we choose \( I = [0, d_0] \) if \( d_0 \geq d_n, \) and \( I = [-d_n, 0] \) if \( d_0 \leq d_n. \) We then have \( \lambda(I) \geq \frac{\alpha}{2n} \geq \frac{\delta}{4}, \) and \( \lambda(A \cap I) = \frac{\alpha}{n} \lambda(I) \geq (\frac{1}{\alpha} + \frac{\delta}{2}) \lambda(I), \) so the desired conclusion holds. Otherwise, suppose that \( \frac{\alpha}{2n} < \frac{1}{2} + \frac{\delta}{2}. \) Then

\[
\alpha_0 \geq \frac{\alpha}{n} - \alpha_n \geq \frac{\alpha}{n} - \left( \frac{1}{2} + \frac{\delta}{2} \right) d_n \geq \frac{\alpha}{n} - \left( \frac{1}{2} + \frac{\delta}{2} \right) \left( 1 + \varepsilon \right) \alpha \frac{1}{n} - d_0.
\]

Using that \( \varepsilon = 1 - \delta / \alpha, \) the last term above is seen to equal

\[
\frac{\alpha}{n} - \left( \frac{1}{2} + \frac{\delta}{2} \right) \left( \frac{2\alpha - \delta}{n} - d_0 \right) \geq \left( \frac{1}{2} + \frac{\delta}{2} \right) d_0 + \frac{\delta}{2n} (1 - 2\alpha + \delta)
\]

\[
> \left( \frac{1}{2} + \frac{\delta}{2} \right) d_0 + \frac{\delta}{4n},
\]

where the last inequality used that \( \alpha < \frac{1}{4} + \frac{\delta}{2}. \) This implies on one hand that \( \frac{\alpha}{2n} \geq \frac{1}{2} + \frac{\delta}{2}, \) and on the other hand that \( \alpha_0 \geq \left( \frac{1}{2} + \frac{\delta}{2} \right) \alpha_0 + \frac{\delta}{4n}, \) thus \( \alpha_0 \left( 1 - \frac{\delta}{2} \right) \geq \frac{\delta}{4n}, \) and so \( \alpha_0 \geq \frac{\delta}{2n} \geq \frac{\delta}{2} \frac{1+\varepsilon}{1+\varepsilon} \geq \frac{\delta}{2\varepsilon} \geq \delta^2. \) Choosing \( I = [0, d_0] \) yields the result. The case \( \frac{\alpha}{2n} < \frac{1}{2} + \frac{\delta}{2} \) is similar. This completes the proof in the case \( \text{diam}(A) = 1. \)

Finally, if \( \text{diam}(A) < 1, \) we may rescale the set in \( \mathbb{R} \) defining \( B = \frac{1}{\text{diam}(A)} A \) (and translate if necessary so that we may assume that \( 0, 1 \in B)). \) Applying the previous case to \( B, \) with parameter \( \delta / \text{diam}(A) \), we obtain an interval \( I_B \) satisfying \( \lambda(I_B) \geq \min \left( \delta / \text{diam}(A), (\delta / \text{diam}(A))^2 \right) \) and \( \lambda(B \cap I_B) \geq (\frac{1}{2} + \frac{\delta}{4 \text{diam}(A)}) \lambda(I_B). \) The interval \( I = \text{diam}(A) \cdot I_B \) satisfies the desired conclusion. \( \square \)
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