On solutions related to FitzHugh-Rinzel type model

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Abstract

The FitzHugh-Rinzel system, a three dimensional non linear model able to
describe bursting phenomena, is analyzed. The system is reduced to a nonlinear
integro differential equation and the fundamental solution $H(x,t)$ is explicitly
determined. The initial value problem in all of the space is analyzed and the
solution is achieved by means of an integral equation involving function $H(x,t)$.
Moreover, for particular value’s constants that characterize the model’s kinetic,
explicit solutions of the FitzHugh-Rinzel system have been achieved.

1 Introduction

The FitzHugh-Rinzel (FHR) system [1–6] is a three dimensional model deriving
from the FitzHugh-Nagumo model [7–18] to incorporate bursting phenomena of
nerve cells. Generally, in many cell types, bursting oscillations are characterized
by a variable of the system that changes periodically from an active phase of rapid
spike oscillations to a silent phase during which the membrane potential only changes
slowly [8]. These phenomena are becoming increasingly important as it is being de-
tected in many scientific fields. Indeed, phenomena of bursting have been observed
as electrical behaviours in many nerve and endocrine cells such as hippocampal and
thalamic neurons, mammalian midbrain and pancreatic in β− cells. (see, f.i. [3]
and references therein). Also in the cardiovascular system, bursting oscillations are
generated by the electrical activity of cardiac cells that excite the heart membrane
to produce the contraction of ventricles and auricles [19] . Furthermore, bursting
oscillations represent a topic of potential interest in many fields of electromechanical
applications such as devices [20,46–48,51]. Recent studies proved that the develop-
ment of this field helps studying the restoration of synaptic connections. [20] Indeed,
seems that nanoscale memristor devices have potential to reproduce the behaviour
of a biological synapse [21,22]. This would lead in future, in case of traumatic lesions,
to the introduction of electronic synapses to connect neurons directly.

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1.1 Mathematical considerations, state of the art and aim of the paper

The FitzHugh-Rinzel type system considered is the following:

\[
\begin{align*}
&\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - w + y + f(u) \\
&\frac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u) \\
&\frac{\partial y}{\partial t} = \delta(-u + k - dy).
\end{align*}
\]

(1.1)

The first two equations in \((u, w)\) refer to the Fitzugh-Nagumo model where the second order term with \(D > 0\) represents the diffusion contribution. It can be associated to the axial current in the axon and it derives from the Hodgkin-Huxley (HH) theory for nerve membranes. Indeed, in (HH) model, if \(d\) represents the axon diameter and \(r_i\) is the resistivity, the spatial variation in the potential \(V\) gives the term \((d/4r_i)V_{xx}\) from which term \(Du_{xx}\) derives. [9].

To these two equations, an additional equation in \(y\) is considered and in this way model (1.1) can be consider as two time-scale slow-fast system with two fast variables \((u, w)\) and one slow variable \(y\). However, if for instance, \(\varepsilon = \delta\), system can be considered as a two time-scale with one fast variable \(u\) and two slow variables \((w, y)\). Otherwise, if \(\delta\) and \(\varepsilon\) have significant difference, it can also be considered as a three-time-scale system with the fast variable \(u\), the intermediate variable and the slow variable. [24]

Moreover, \(\beta > 0, \ d > 0, c, k,\) are arbitrary constants that characterize the model’s kinetic with \(0 < \varepsilon, \delta \ll 1\). Appropriate class of functions \(f(u)\) depends on the reaction kinetics of the model [7,8] and generally one has:

\[
f(u) = u(a - u)(u - 1) \quad (0 < a < 1),
\]

(1.2)

As consequence, it results

\[
f(u) = -au + \varphi(u) \quad \text{with} \quad \varphi(u) = u^2(a + 1 - u) \quad 0 < a < 1
\]

(1.3)

Then the system (1.1) becomes
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= D \frac{\partial^2 u}{\partial x^2} - au - w + y + \varphi(u) \\
\frac{\partial w}{\partial t} &= \varepsilon (-\beta w + c + u) \\
\frac{\partial y}{\partial t} &= \delta (-u + k - dy).
\end{aligned}
\] (1.4)

Indicating by means of (1.5) the initial values, from (1.4) it follows:

\[
\begin{aligned}
u(x,0) &= u_0, \quad w(x,0) = w_0 \quad y(x,0) = y_0, \quad (x \in \mathbb{R})
\end{aligned}
\] (1.5)

Consequently, denoting by (1.7) the source term, then the problem (1.4)-(1.5) can be modified into the following initial value problem \( P \) :

\[
\begin{aligned}
u_t - Du_{xx} + au + \int_0^t [\varepsilon e^{-\varepsilon\beta(t-\tau)} + \delta e^{-\delta d(t-\tau)}]u(x,\tau) d\tau &= F(x,t,u) \\
u(x,0) &= u_0(x) \quad x \in \mathbb{R}.
\end{aligned}
\] (1.8)

As for the state of art, mathematical considerations (see f.i. [26,27] and references therein) let us permit to assert that the knowledge of the fundamental solution \( H(x,t) \) related to linear parabolic operator \( L \) :

\[
Lu = \varphi(u) - \frac{c}{\beta} (1 - e^{-\varepsilon\beta t}) + \frac{k}{d} (1 - e^{-\delta dt})
\] (1.9)
let to determine the solution of $P$. Indeed, if $F(x, t, u)$ verifies appropriate assumptions, through the fixed point theorem, solution can be expressed by means of an integral equation [26].

Moreover, as for operator $L$, according to [14], in the case in which kernel is expressed by means of an unique exponential function, many properties and inequalities have already been achieved.

And it is important to observe that system 1.8, as soon as $\varepsilon \equiv \delta$ and $\beta \equiv d$, is reduced to

$$
\begin{aligned}
&u_t - Du_{xx} + au + \int_0^t \varepsilon e^{-2\varepsilon \beta (t-\tau)} u(x, \tau) d\tau = F(x, t, u) \\
&u(x, 0) = u_0(x) \quad x \in \mathbb{R},
\end{aligned}
$$

(1.10)

that it can immediately report to the case already studied in [14].

In this paper, after determining explicitly the fundamental solution $H(x, t)$, whose kernel involves two exponential functions, the initial value problem in all of the space is analyzed and the solution is deduced by means of an integral equation. Moreover, using a method of travelling wave, for particular values of constants $\varepsilon, \delta, d, \beta$, solutions of the FitzHugh-Rinzel system have been explicitly determined.

2 Fundamental solution and firstly results

Indicating by $T$ is an arbitrary positive constant, let us consider the initial-value problem 1.8 defined in all of the space $\Omega_T$

$$
\Omega_T = \{(x, t) : x \in \mathbb{R}, \quad 0 < t \leq T\}.
$$

So that, let us denote by

$$
\hat{u}(x, s) = \int_0^\infty e^{-st} u(x, t) dt, \quad \hat{F}(x, s) = \int_0^\infty e^{-st} F(x, t, u(x, t)) dt,
$$

the Laplace transform with respect to $t$. If $\hat{H}(x, s)$ represents the $L_t$ transforms of fundamental solution $H(x, t)$, from 1.8 it follows:

$$
\hat{u}(x, s) = \int_\mathbb{R} \hat{H}(x - \xi, s) [u_0(\xi) + \hat{F}(\xi, s)] d\xi,
$$

(2.11)

Therefore, if $H(x, t)$ represents the inverse $L_t$ transforms of $\hat{H}(x, s)$, formally it follows that
\[ u(x,t) = \int_\mathbb{R} H(x - \xi, t) u_0(\xi) \, d\xi + \int_0^t d\tau \int_\mathbb{R} H(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] \, d\xi. \]

So that, if \( J_1(z) \) denotes the Bessel function of first kind and order 1, let us consider the following functions:

\[
H_1(x, t) = e^{\frac{x^2}{2\sqrt{\pi Dt}}} e^{-at} +
\]

\[
-\frac{1}{2} \int_0^t \frac{e^{-\sqrt{\frac{y}{t-y}} - a}}{\sqrt{t-y}} \frac{\sqrt{\pi} e^{-\beta \varepsilon (t-y)}}{\sqrt{\pi D}} J_1(2 \sqrt{\varepsilon y (t-y)}) \, dy
\]

\[
H_2 = \int_0^t H_1(x, y) e^{-\delta d (t-y)} \sqrt{\frac{\delta}{t-y}} J_1(2 \sqrt{\delta y (t-y)}) \, dy
\]

Besides, letting

\[
r^2 = s + a + \frac{\varepsilon}{s + \beta \varepsilon}
\]

\[
\sigma^2 = s + a + \frac{\delta}{s + \delta d} + \frac{\varepsilon}{s + \beta \varepsilon},
\]

denoting by

\[ H = H_1 - H_2, \]

the following theorem holds:

**Theorem 2.1.** In the half-plane \( \Re s > \max(-a, -\beta \varepsilon, -\delta d) \) the Laplace integral \( \mathcal{L}_t H \) converges absolutely for all \( x > 0 \), and it results:

\[
\mathcal{L}_t H \equiv \hat{H} = \int_0^\infty e^{-st} H(x, t) \, dt = \frac{1}{\sqrt{D}} \frac{e^{\frac{|x|}{\sqrt{D} \sigma}}}{2 \sigma}.
\]

Moreover, the function \( H \) has the same basic properties of the fundamental solution of the heat equation, that is:

\[ H(x, t) \in C^\infty \quad t > 0, \quad x \in \mathbb{R}. \]

For fixed \( t > 0 \), \( H \) and its derivatives are vanishing exponentially fast as \( |x| \) tends to infinity.

For any fixed \( \delta > 0 \), uniformly for all \( |x| \geq \delta \), it results \( \lim_{t \downarrow 0} H(x, t) = 0. \)
Proof. Since for all real $z$ one has $|J_1(z)| \leq 1$, the Fubini-Tonelli theorem assures that
\[
L_t \left( \frac{1}{2} \sqrt{\frac{\varepsilon}{t}} \int_0^t e^{-\frac{\varepsilon^2}{4Dy}} e^{-\beta \varepsilon (t-y)} \sqrt{\frac{\varepsilon}{\pi D}} J_1 \left( 2 \frac{\sqrt{\varepsilon y (t-y)}}{y} \right) dy \right) = \frac{\sqrt{\varepsilon}}{2 \sqrt{\pi D}} \int_0^\infty e^{-(s+a)y} e^{-\frac{s^2}{4Dy}} dy \int_0^\infty e^{-(s+\beta \varepsilon + \epsilon y) t} J_1 \left( 2 \frac{\sqrt{\varepsilon y (t-y)}}{t} \right) dt
\]
and being
\[
\int_0^\infty e^{-pt} \sqrt{\frac{\varepsilon}{t}} J_1 \left( 2 \frac{\sqrt{\varepsilon t}}{t} \right) dt = 1 - e^{-c/p} \quad (\Re p > 0),
\]
(2.16)

it results:
\[
\hat{H}_1(x,s) = \frac{1}{2 \sqrt{\pi D} \sqrt{s+a}} e^{-x \sqrt{s+a}}
\]
Besides, since Fubini-Tonelli theorem and 2.15 it results:
\[
\hat{H}_2 = \hat{H}_1 - \frac{1}{2 \sqrt{\pi D} \sqrt{s+a}} \int_0^\infty e^{-\frac{s^2}{4Dy} - \frac{(s+a+\beta \varepsilon + \epsilon y) y}{\sqrt{y}}} \frac{dy}{\sqrt{y}} = \frac{1}{2 \sqrt{D} \sqrt{s+a}} e^{-\frac{x^2}{r}}
\]
from which
\[
\hat{H}(x,s) = \frac{1}{2 \sqrt{\pi D} \sigma} e^{-\frac{x^2}{\sigma}}
\]
is deduced.

Others properties can be deduced according to theorems proved in [14, 26]. \qed

Now, let us indicate:
\[
\varphi(x,t) = \frac{e^{-x^2/4Dt}}{2 \sqrt{\pi Dt}} e^{-a t};
\]
\[
\psi_{\varepsilon}(y, t-y) = \frac{\sqrt{\varepsilon y} e^{-\beta \varepsilon (t-y)}}{\sqrt{t-y}} J_1 \left( 2 \frac{\sqrt{\varepsilon y (t-y)}}{y} \right)
\]
(2.20)
\[
\psi_{\delta}(y, t-y) = \frac{\sqrt{\delta y} e^{-\delta \delta (t-y)}}{\sqrt{t-y}} J_1 \left( 2 \frac{\sqrt{\delta y (t-y)}}{y} \right)
\]
it results:

\[(2.21) \quad H_1(x,t) = \varphi(x,t) - \int_0^t \varphi(x,y) \psi_\varepsilon(y,t-y) \, dy\]

\[(2.22) \quad H_2(x,t) = \int_0^t H_1(x,y) \psi_\delta(y,t-y) \, dy,\]

and indicating by

\[(2.23) \quad g_1(x,t) * g_2(x,t) = \int_0^t g_1(x,t-\tau)g_2(x,\tau) \, d\tau,\]

the convolution with respect to \(t\), for \(t > 0\), as proved in [14], it results:

\[(2.24) \quad (\partial_t + a - D\partial_{xx})H_1 = -\varepsilon e^{-\varepsilon \beta t} * H_1(x,t) = -\varepsilon K_\varepsilon\]

where \(K_\varepsilon\) is given by

\[(2.25) \quad K_\varepsilon(x,t) = \frac{1}{2\sqrt{\pi D}} \int_0^t e^{-\frac{x^2}{4D\tau}} -a y - \beta \varepsilon (t-y) \, J_0(2\sqrt{\varepsilon y(t-y)}) \, dy.\]

So the following theorem holds:

**Theorem 2.2.** For \(t > 0\), it results \(LH = 0\), i.e.

\[(2.26) \quad H_t - DH_{xx} + aH + \int_0^t \left[ \varepsilon e^{-\varepsilon \beta(t-\tau)} + \delta e^{-\delta d(t-\tau)} \right] H(x,\tau) \, d\tau = 0.\]

**Proof.** Let us consider that:

\[(2.27) \quad (\partial_t + a - D\partial_{xx})H_2 = H_1(x,t)\psi_\delta(t,t) + \int_0^t H_1(x,y) \left[ \partial_t \psi_\delta(y,t) + a\psi_\delta(y,t) \right] \]

\[- \int_0^t D \psi_\delta(y,t) \partial_{xx} H_1(x,y) \, dy\]

So, since 2.24 one has

\[(2.28) \quad (\partial_t + a - D\partial_{xx})H_2 = \int_0^t \left[ H_1(x,y) \partial_t \psi_\delta(y,t) - \psi_\delta(y,t)\partial_y H_1(x,y) \right] dy +

+ H_1(x,t)\psi_\delta(t,t) - \varepsilon \int_0^t K_\varepsilon(x,y)\psi_\delta(y,t) \, dy\]
Besides, it results:

\[
\int_0^t \psi_\delta(y, t) \partial_y H_1(x, y) \, dy = H_1(x, t) \psi_\delta(t, t) - \int_0^t H_1(x, y) \partial_y \psi_\delta(y, t) \, dy
\]

and one has:

\[
(\partial_t + a - D \partial_{xx}) H_2 = \int_0^t H_1(x, y) \left[ \partial_t \psi_\delta(y, t) + \partial_y \psi_\delta(y, t) \right] \, dy
\]

(2.30)

\[-\varepsilon \int_0^t K_\varepsilon(x, y) \psi_\delta(y, t) \, dy\]

where it results:

(2.31)

\[\partial_t \psi_\delta(y, t) + \partial_y \psi_\delta(y, t) = \delta e^{-\delta d (t-y)} J_0(2 \sqrt{\delta y(t-y)})\]

Denoting by

\[K_\delta(x, t) \equiv \int_0^t e^{-\delta d (t-y)} H_1(x, y) J_0(2 \sqrt{\delta y(t-y)}) \, dy\]

one has:

\[(\partial_t + a - D \partial_{xx}) H_2 = \delta K_\delta - \varepsilon \int_0^t K_\varepsilon(x, t-y) \psi_\delta(y, t) \, dy\]

from which, since \(\varepsilon K_\varepsilon(x, y) = \varepsilon H_1(x, y) * e^{-\varepsilon \beta y}\), and by means of Fubini - Tonelli theorem, it possible to prove that it results:

(2.32)

\[(\partial_t + a - D \partial_{xx}) H_2 = \delta K_\delta - \varepsilon e^{-\varepsilon \beta t} * H_2(x, t)\]

On the other side, the convolution \(e^{-\delta d t} * H(x, t)\) is given by

\[e^{-\delta d t} * H(x, t) = e^{-\delta d t} * H_1(x, t) - \int_0^t H_1(r, y) \, dy \int_y^t e^{-\delta d (t-\tau)} \psi_\delta(y, \tau) \, d\tau\]

with

(2.33)

\[\int_y^t e^{-\delta d (t-\tau)} \psi_\delta(y, \tau) \, d\tau = e^{-\delta d (t-y)} \int_y^t \sqrt{\frac{\delta y}{\tau - y}} J_1(2 \sqrt{\delta y(\tau-y)}) \, d\tau = \]

8
\[ e^{-\delta d (t-y)} \left[ 1 - J_0 \left( 2 \sqrt{\delta y(t-y)} \right) \right]. \]

As consequence, one has

\[ e^{-\delta d t} * H = K_{\delta}. \]  

(2.34)

So, from 2.24, 2.32, 2.34, theorem holds.  

\[ \square \]

3 Solution related to the (FHR) problem

To give the solution by means of the integral expression, let us consider

\[ e^{-\varepsilon \beta t} * H_2 = \int_0^t H_1(x, y) \, dy \int_0^t e^{-\beta \varepsilon (t-\tau)} \psi_\delta(y, \tau) \, d\tau = \]

(3.35)

\[ \int_0^t H_1(x, y) \, dy \quad \int_0^t \quad e^{(\beta \varepsilon - \delta d) \tau} \quad \frac{\sqrt{\delta y}}{\sqrt{\tau - y}} \quad J_1 \left( 2 \sqrt{\delta y(t-y)} \right) \quad d\tau = \]

\[ e^{-\beta \varepsilon t} * H_1(x, t) - \int_0^t e^{-\delta d(t-y)} H_1(x, y) \quad J_0 \left( 2 \sqrt{\delta y(t-y)} \right) \quad dy + \]

(3.36)

\[ (\varepsilon \beta - \delta d) \quad \int_0^t \quad e^{\delta d y} \quad e^{-\beta \varepsilon t} \quad \varphi(r, y) \quad dy \quad \int_0^t \quad e^{(-\delta d + \varepsilon \beta) \tau} \quad J_0 \left( 2 \sqrt{\delta y(\tau - y)} \right) \quad d\tau \]

where

\[ (\varepsilon \beta - \delta d) \quad \int_0^t \quad e^{\delta d y} \quad e^{-\beta \varepsilon t} \quad \varphi(r, y) \quad dy \quad \int_0^t \quad e^{(-\delta d + \varepsilon \beta) \tau} \quad J_0 \left( 2 \sqrt{\delta y(\tau - y)} \right) \quad d\tau = \]

\[ = (\varepsilon \beta - \delta d) \quad \int_0^t \quad \varphi(r, y) \quad dy \quad \int_0^t \quad e^{-\delta d(\tau - y)} \quad e^{-\varepsilon \beta (t-\tau)} \quad J_0 \left( 2 \sqrt{\delta y(\tau - y)} \right) \quad d\tau = \]

\[ (\varepsilon \beta - \delta d) \quad \int_0^t \quad d\tau \quad \int_0^\tau \quad \varphi(r, y) \quad e^{-\delta d(\tau - y)} \quad e^{-\varepsilon \beta (t-\tau)} \quad J_0 \left( 2 \sqrt{\delta y(\tau - y)} \right) \quad dy = \]

\[ (\varepsilon \beta - \delta d) e^{-\beta \varepsilon t} \quad \int_0^t \quad \varphi(x, y) \quad e^{-\delta d(t-y)} \quad J_0 \left( 2 \sqrt{\delta y(t-y)} \right) \quad dy \]

So, it results:
\[ e^{-\varepsilon \beta t} * H_2 = e^{-\varepsilon \beta t} * H_1 - K_\delta + (\varepsilon \beta - \delta d)e^{-\beta \varepsilon t} * K_\delta \]

Now let us indicate by

\[ f_1(x, t) * f_2(x, t) = \int_{\mathbb{R}} f_1(\xi, t) f_2(x - \xi, t) \, d\xi \]

the convolution with respect to the space \( x \), and let

\[ H \otimes F = \int_0^t d\tau \int_{\mathbb{R}} H(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] \, d\xi. \]

So that, from (??) (??) one has:

\[
\begin{cases}
H \otimes (w_0 e^{-\beta \varepsilon t}) = w_0 \ast [K_\delta - (\varepsilon \beta - \delta d)e^{-\beta \varepsilon t} * K_\delta] \\
H \otimes (y_0 e^{-\delta dt}) = y_0 \ast K_\delta
\end{cases}
\]

where

\[ w_0 \ast (\varepsilon \beta - \delta d)e^{-\beta \varepsilon t} * K_\delta = (\varepsilon \beta - \delta d) w_0 \otimes (e^{-\beta \varepsilon t} K_\delta) \]

Besides, to evaluate \( \int_0^t d\tau \int_{\mathbb{R}} H(x - \xi, \tau) \, d\xi \), it takes into account that if

\[ K_\varepsilon(x, t) = \int_0^t e^{-\varepsilon \beta(t-y)} e^{-\frac{y^2}{\pi Dy}} e^{-a y} \frac{J_0(2 \sqrt{\varepsilon y(t - y)})}{\sqrt{\pi Dy}} \, dy \]

since 2.32, ??, one has

\[ \int_0^t d\tau \int_{\mathbb{R}} d\xi \int_0^\tau \varphi(x - \xi, y) \psi_\varepsilon(y, \tau) \, dy = \]

\[ = \int_0^t d\tau \int_{\mathbb{R}} \varphi(\xi, \tau) d\xi - \int_{\mathbb{R}} K_\varepsilon(\xi, t) d\xi - \beta \varepsilon e^{-\beta \varepsilon t} \otimes K_\varepsilon \]

and

\[ \int_0^t d\tau \int_{\mathbb{R}} d\xi \int_0^\tau \varphi(x - \xi, y) \psi_\delta(y, \tau) \, dy = \]

\[ = \int_0^t d\tau \int_{\mathbb{R}} \varphi(\xi, \tau) d\xi - \int_{\mathbb{R}} K_\delta(\xi, t) d\xi - \delta d e^{-\delta dt} \otimes K_\delta, \]
Further, because
\[
\int_{\mathbb{R}} \frac{e^{-\frac{|x-\xi|^2}{4\varepsilon t}}}{2\sqrt{\pi \varepsilon t}} d\xi = 1,
\]
it results:
\[
\int_{0}^{t} d\tau \int_{\mathbb{R}} \varphi(x-\xi,t) d\xi = \frac{2}{a} - \frac{2e^{-at}}{a}
\]
it results:
\[
2 \int_{0}^{t} d\tau \int_{\mathbb{R}} H(x-\xi,\tau), d\xi = \beta \varepsilon e^{-\beta \varepsilon t} \otimes K_\varepsilon + \delta \varepsilon d e^{-\delta dt} \otimes K_\delta + \int_{\mathbb{R}} (K_\delta + K_\varepsilon) d\xi
\]
As consequences, from 2.16, one has:
\[
u(x,t) = u_0 \ast H + \varphi \otimes H + \delta y_0 \ast K_\delta - w_0 \ast K_\delta + (-\varepsilon \beta - \delta d) w_0 \otimes e^{-\beta \varepsilon t} K_\delta +
\]
\[
\frac{-c}{\beta} + \frac{k}{d} \int_{0}^{t} d\tau \int_{\mathbb{R}} H(x-\xi,\tau) d\xi,
\]
and this formula, together with (1.6), allows to obtain also \( v(x,t) \) and \( y(x,t) \) in terms of the data.
Indeed, if we observe that
\[
K_{\varepsilon \varepsilon}(x,t) \equiv \int_{0}^{t} e^{-\beta \varepsilon (t-\tau)} K_\varepsilon(x,\tau) d\tau =
\]
\[
= \int_{0}^{t} e^{-\frac{x^2}{4\varepsilon y} - a\ y - \beta \varepsilon (t-y)} \sqrt{\frac{t-y}{\varepsilon y}} J_1(2\sqrt{\varepsilon y (t-y)}) dy
\]
\[
K_{\varepsilon \delta}(x,t) \equiv \int_{0}^{t} e^{-\beta \varepsilon (t-\tau)} K_\delta(x,\tau) d\tau =
\]
\[
= \int_{0}^{t} e^{-\frac{x^2}{4\varepsilon y} - a\ y - \beta \varepsilon (t-y)} \sqrt{\frac{t-y}{\varepsilon y}} J_1(2\sqrt{\varepsilon y (t-y)}) dy
\]
by means of \((1.6)\) it results:

\begin{equation}
(3.47)
\begin{aligned}
w &= w_0 e^{-\varepsilon \beta t} + \frac{\varepsilon}{\beta} (1 - e^{-\varepsilon \beta t}) + \\
&\quad \varepsilon \int_0^t e^{-\varepsilon \beta (t-\tau)} u(x, \tau) \, d\tau
\end{aligned}
\end{equation}

\begin{equation}
(3.48)
\left\{
\begin{aligned}
w &= w_0 e^{-\varepsilon \beta t} + \frac{\varepsilon}{\beta} (1 - e^{-\varepsilon \beta t}) + \varepsilon \int_0^t e^{-\varepsilon \beta (t-\tau)} u(x, \tau) \, d\tau \\
y &= y_0 e^{-\delta dt} + \frac{k}{d} (1 - e^{-\delta dt}) - \delta \int_0^t e^{-\delta d (t-\tau)} u(x, \tau) \, d\tau
\end{aligned}
\right.
\end{equation}

\section{Explicit solutions}

To find exact solutions related to partial differential equations a lot of methods there exist \([12, 14, 29, 31-37]\) and see, f. i. \([11, 28]\) (and references therein), too.

In the case of system 1.1, redefining \(\tilde{x} = \frac{1}{\sqrt{D}} x\) and than remove the superscripts henceforth, we may assume \(D = 1\) without loss of generality.

Besides, assuming \(\varepsilon \beta = \delta d = -1\), let

\[ z = x - t, \]

one has:

\begin{equation}
(4.49)
\begin{aligned}
u_{zzz} + 2u_{zz} + (1 - a)u_z - 3u^2u_z + 2(a + 1)u u_z + \\
&\quad (\varepsilon + \delta - a)u - u^3 + (a + 1)u^2 + \varepsilon c - \delta k = 0
\end{aligned}
\end{equation}

Let us assume

\begin{equation}
(4.50)
u(z) = A f(z) + B
\end{equation}

where

\begin{equation}
(4.51)f(z) = \sqrt{v} \, \text{tanh} (\sqrt{v} (z - z_0)),
\end{equation}

is solution of Riccati equation:

\[ f_z + f^2 - v = 0. \]
Since

\[
\begin{align*}
 u_z &= -A f^2(z) + A v \\
 u_{zz} &= 2A f^3 - 2A f(z) v \\
 u_{zzz} &= -6Af^4(z) + 8A f^2(z) v - 2A v^2 \\
 uu_z &= -A f^3(z) - Abf^2 + A^2 v f + Abv \\
 u^2 u_z &= -A^3 f^4 - 2A^2 b f^3 - Ab^2 f^2 + A^3 v f^2 + 2A^2 v b f + Ab^2 v
\end{align*}
\]

(4.52)

to satisfy 4.49, it must be \( \varepsilon + \delta = 0 \) and \( A^2 = 2 \). So, denoting by

\[ b = \varepsilon c - \delta k = -\delta(c + k) \]

according to values of constant \( b \), many solutions can be achieved. For example, if

\[ b < -0.0237194 \quad \text{or} \quad b > 0.390367, \]

according that \( A_{1,2} = \pm \sqrt{2} \), letting

\[ \gamma_{1,2} = \left( \sqrt{2916 b^2 - 756 \sqrt{2} b - 27 - 54 b \mp 7 \sqrt{2}} \right)^{1/3}, \]

it does have:

\[ B_{1,2} = \frac{1}{6} \left( \gamma_{1,2} + \frac{5}{\gamma_{1,2}} + 3 \mp 2 \sqrt{2} \right) \]

Consequently it results:

\[ v_{1,2} = \frac{\gamma_{1,2}^2}{24} + \frac{25}{24 \gamma_{1,2}^2} - \frac{\gamma_{1,2}}{6 \sqrt{2}} \pm \frac{5}{6 \sqrt{2} \gamma_{1,2}} + \frac{13}{24} \quad a + 1 = 3 B_{1,2} + \frac{A_{1,2}}{2} \]

Moreover, if \( b = 0 \), for instance it possible to choose \( B = 1/2 \) and \( A = -\sqrt{2} \) and consequently one has: \( v \simeq 0, 125 \).

References

[1] J. Rinzel, J. B. Keller *Traveling Wave Solutions of a Nerve Conduction Equation* Biophysical Journal, Volume 13, Issue 12, 1313-1337,(1973)

[2] A. Yadav, A. K. Swami, A. Srivastava Bursting and Chaotic Activities in the Nonlinear Dynamics of FitzHugh-Rinzel Neuron Model International Journal of Engineering Research and General Science Volume 4, Issue 3, 2016 2091-2730
[3] R. Bertram, T. Manish J. Butte, T. Kiemel and A. Sherman Topological and phenomenological classification of bursting oscillations Bulletin of Mathematical Biology, Vol. 57, No. 3, pp. 413-39, 1995

[4] J. Wojcik, A. Shilnikov Voltage Interval Mappings for an Elliptic Bursting Model in Nonlinear Dynamics New Directions Theoretical Aspects Gonzalez-Aguilar H; Ugalde E. (Eds.) (2015) 219 p Springer

[5] Rinzel J. A Formal Classification of Bursting Mechanisms in Excitable Systems, in Mathematical Topics in Population Biology, Morphogenesis and Neurosciences, Lecture Notes in Biomathematics, Springer-Verlag, New York, Vol. 71, pp. 267281, 1987.

[6] Zemlyanukhin A. I., Bochkarev A. V., Analytical Properties and Solutions of the FitzHugh Rinzel Model, Rus. J. Nonlin. Dyn., 2019, Vol. 15, no. 1, pp. 3-12

[7] Izhikevich E.M. : Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting. The MIT press. England (2007)

[8] Keener, J. P. Sneyd, J. Mathematical Physiology. Springer-Verlag, N.Y (1998)

[9] Murray, J.D. : Mathematical Biology I. Springer-Verlag, N.Y (2003)

[10] M. De Angelis, P. Renno Asymptotic effects of boundary perturbations in excitable systems arXiv:1304.3891

[11] N. K. Kudryashov, K. R. Rybka, A. Sboev Analytical properties of the perturbed FitzHugh-Nagumo model Applied Mathematics Letters 76 (2018) 142-147

[12] M De Angelis, P. Renno, On the FitzHugh-Nagumo model in “ WASCOM 2007” 14th Conference on Waves and Stability in Continuous Media”, World Sci. Publ., Hackensack, NJ, 2008 193-198.

[13] A. Dikansky Fitzhugh-Nagumo equations in a nonhomogeneous medium Discrete and continuous dynamical systems Supplement Volume 2005 pag 216-224

[14] De Angelis, M. Renno, P Existence, uniqueness and a priori estimates for a non linear integro - differential equation Ricerche di Mat. 57 95-109 (2008)

[15] Rionero, S. : A rigorous reduction of the $L^2$-stability of the solutions to a nonlinear binary reaction-diffusion system of PDE’s to the stability of the solutions to a linear binary system of ODE’s. J. Math. Anal. Appl. 319 no. 2, 377-397 (2006)

[16] Rionero, S.: A nonlinear $L^2$-stability analysis for two-species population dynamics with dispersal. Math. Biosci. Eng. 3 no. 1, 189-204 (2006)(electronic).
[17] P. Colli Franzone, L. Pavarino, S. Scacchi Mathematical Cardiac Electrophysiology Springer, 2014 - 397 pp

[18] G. Gambino, M. C. Lombardo, G. Rubino, M. Sammartino Pattern selection in the 2D FitzHugh-Nagumo model Ricerche di Matematica (2018) https://doi.org/10.1007/s11587-018-0424-6

[19] A. Quarteroni, A. Manzoni and C. Vergara The cardiovascular system: Mathematical modelling, numerical algorithms and clinical applications Acta Numerica (2017), pp. 365-590

[20] H. Simo, P. Woafo, Bursting oscillations in electromechanical systems, Mechanics Research Communications 38 (2011) 537-541

[21] E. Juzekaeva, A. Nasretdinov, S. Battistoni, T. Berzina, S. Iannotta, R. Khazipov, V. Erokhin, M. Mukhtarov, Coupling Cortical Neurons through Electronic Memristive Synapse Adv. Mater. Technol. 2019, 4, 1800350 (6)

[22] Fernando Corinto, Valentina Lanza, Alon Ascoli, and Marco Gilli Synchronization in Networks of FitzHugh-Nagumo Neurons with Memristor Synapses 20th European Conference on Circuit Theory and Design (ECCTD) IEEE 2011 DOI: 10.1109/ECCTD.2011.6043616

[23] Faghii R.T., Savla K., Dahleh M.A., and Brown E.N., Broad Range of Neural Dynamics from a Time-Varying FitzHugh-Nagumo Model and Its Spiking Threshold Estimation, IEEE Transactions on Biomedical Engineering, Vol. 59, No. 3, pp. 816-823, 2012.

[24] Wenxian Xie, Jianwen Xu, Jianwen Xu, Cai Li, Li Cai, Yanfei Jin, Yanfei Jin Dynamics and Geometric Desingularization of the Multiple Time Scale FitzHugh Nagumo Rinzel Model with Fold Singularity Communications in Nonlinear Science and Numerical Simulation, 63, p. 322-338. 2018

[25] A. Tonnelier, The McKean’s caricature of the FitzHugh-Nagumo model. I : The space-clamped system, SIAM Journal on Applied Mathematics 63, pp. 459-484 (2002)

[26] J. R. Cannon, The one-dimensional heat equation, Addison-Wesley Publishing Company (1984)

[27] De Angelis, E Maio, Mazziotti Existence and uniqueness results for a class of non linear models. In Mathematical Physics models and engineering sciences. (2008)

[28] N.K. Kudryashov Asymptotic and Exact Solutions of the FitzHugh-Nagumo Model Regul. Chaotic Dyn., 2018, vol 23, No 2, 152160
[29] Mohammadreza Foroutan, Jalil Manafian, Hamed Taghipour-Farshi. Exact solutions for Fitzhugh-Nagumo model of nerve excitation via Kudryashov method. Opt Quant Electron (2017) 49, 152.

[30] M. De Angelis, E. Mazziotti, Non linear travelling waves with diffusion. Rend. Acc. Sei Fis Mat Napoli, vol LXXIII (2006) pp 23-36.

[31] Cherniha, R. Pluikhin, O.: New conditional symmetries and exact solutions of nonlinear reaction-diffusion-convection equations. J. Phys. A 40, (33), 10049-10070 (2007).

[32] Li H., Guoa Y.: New exact solutions to the Fitzhugh Nagumo equation. Applied Mathematics and Computation 180, 2, 524-528 (2006).

[33] Shih M., Momoniat E. Mahomed F.M.: Approximate conditional symmetries and approximate solutions of the perturbed Fitzhugh Nagumo equation, J. Math Physics 46, 023505 1-10 (2005).

[34] S. Johnson, P. Suarez, and A. Biswas. New Exact Solutions for the Sine-Gordon Equation in 2+1 Dimensions. Computational Mathematics and Mathematical Physics, 52, 1 98-104 (2012).

[35] Wei-Xiong Chen and Ji Lin. Some New Exact Solutions of (1+2) Dimensional Sine-Gordon Equation, Abstract and Applied Analysis, vol 2014, article ID 645456, 8 pages (2014).

[36] Aktosun, Demontis, van der Mee, Exact solutions for the sine-Gordon equation, Journal of Math Physics 51, 123521 (2010).

[37] Jhan-Luis Alvis-Zuniga, Ana-Magnolia Marin-Ramirez and Ruben-Dario Ortiz-Ortiz. Solutions for the Perturbed Sine-Gordon Equation. Journal of Engineering and Applied Sciences, 11, 2294-2297, (2016).

[38] F. De Angelis, An internal variable variational formulation of viscoplasticity, Comput. Methods Appl. Mech. Engrg. 190 (2000) 35-54.

[39] G. Alfano, F. De Angelis, L. Rosati. General solution procedures in elasto visco plastic, Comput. Methods Appl. Mech. Engrg., 190 (2001) 5123-5147.

[40] F. De Angelis, A variationally consistent formulation of nonlocal plasticity, Int. Journal for Multiscale Computational Engineering, 5 (2) (2007) 105-116.

[41] F. De Angelis, On the structural response of elasto/viscoplastic materials subject to time-dependent loadings, Structural Durability and Health Monitoring, 8 (4) (2012) 341-358.

[42] F. De Angelis, Computational issues and numerical applications in rate-dependent plasticity, Advanced Science Letters, 19 (8) (2013) 2359-2362.
[43] F. De Angelis, R.L. Taylor, An efficient return mapping algorithm for elasto-plasticity with exact closed form solution of the local constitutive model, Engineering Computations, 32 (8) (2015) 2259-2291.

[44] F. De Angelis, R.L. Taylor, A nonlinear finite element plasticity formulation without matrix inversions, Finite Elements in Analysis and Design, 112 (2016) 11-25.

[45] F. De Angelis, D. Cancellara, Multifield variational principles and computational aspects in rate plasticity, Computers and Structures, 180 (2017) 27-39.

[46] D. Cancellara, F. De Angelis, Nonlinear dynamic analysis for multi-storey RC structures with hybrid base isolation systems in presence of bi-directional ground motions, Composite Structures, 154 (2016) 464492.

[47] D. Cancellara, F. De Angelis, A base isolation system for structures subject to extreme seismic events characterized by anomalous values of intensity and frequency content, Composite Structures, 157 (2016) 285302.

[48] D. Cancellara, F. De Angelis, Assessment and dynamic nonlinear analysis of different base isolation systems for a multi-storey RC building irregular in plan, Computers and Structures, 180 (2017) 7488.

[49] F. De Angelis, D. Cancellara, L. Grassia, A. DAmore, The influence of loading rates on hardening effects in elasto/viscoplastic strain-hardening materials Assessment and dynamic nonlinear analysis of different base isolation systems for a multi-storey RC building irregular in plan, Mechanics of Time-Dependent Materials, 22 (4) (2018) 533-551.

[50] F. De Angelis, Extended formulations of evolutive laws and constitutive relations in non-smooth plasticity and viscoplasticity, Composite Structures, 193 (2018) 35-41.

[51] D. Cancellara, F. De Angelis, Dynamic assessment of base isolation systems for irregular in plan structures: response spectrum analysis vs nonlinear analysis, Composite Structures, 215 (2019) 98-115.

[52] E.Juzekaeva, A. Nasretdinov, S. Battistoni, T. Berzina, S. Iannotta, R. Khazipov, V. Erokhin, M. Mukhtarov, Coupling Cortical Neurons through Electronic Memristive Synapse Adv. Mater. Technol. 2019, 4, 1800350 (6) DOI: 10.1002/admt.201800350