SEMI-INARIANT PICTURES AND TWO CONJECTURES ON MAXIMAL GREEN SEQUENCES

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Abstract. We use semi-invariant pictures to prove two conjectures about maximal green sequences. First: if $Q$ is any acyclic valued quiver with an arrow $j \to i$ of infinite type then any maximal green sequence for $Q$ must mutate at $i$ before mutating at $j$. Second: for any quiver $Q'$ obtained by mutating an acyclic valued quiver $Q$ of tame type, there are only finitely many maximal green sequences for $Q'$. Both statements follow from the Rotation Lemma for reddening sequences and this in turn follows from the Mutation Formula for the semi-invariant picture for $Q$.

INTRODUCTION

This paper proves two conjectures about maximal green sequences.

Conjecture 1. Given an acyclic valued quiver $Q$ with an arrow $j \to i$ of infinite type, i.e., with $ab \geq 4$, any maximal green sequence mutates at the target $i$ before the source $j$.

Conjecture 2. If the valued quiver $Q$ is mutation equivalent to an acyclic quiver of tame type, then $Q$ has only finitely many maximal green sequences.

The original first step of the proof of the Conjecture 1 came from the semi-invariant picture, using the fact that the lines are labeled by $c$-vectors and the normal orientation on the lines determines the sign of the $c$-vector, hence visualizing green mutations as crossing the lines always in the direction of normal orientation as illustrated on the following example.
The double arrow \( j \Rightarrow i \) creates infinite families of walls. A maximal green sequence is a path going from the unbounded region to the center which only goes inward at each wall. The dashed line is the maximal green sequence \((i, k, j, i)\). The dotted lines show that maximal green sequences cannot mutate \( j \) before \( i \).

Conjecture 2 is known when \( Q \) is acyclic \[3\]. The main step in proving both Conjecture 1 and the extension of the known cases to the more general cases of Conjecture 2 is our Rotation Lemma (Theorem 2.2.2), which comes from the theory of semi-invariant pictures as developed in \[7\].

**Theorem 0.0.1 (Rotation Lemma).** Let \((k_0, k_1, \ldots, k_{m-1})\) be a reddening sequence on \( Q \) with associated permutation \( \sigma \). Then the sequence \((k_1, k_2, \ldots, k_{m-1}, \sigma^{-1}(k_0))\) is a reddening sequence on \( \mu_{k_0} Q \) with the same permutation \( \sigma \) and with the same number of red mutations as the original sequence.

This Lemma, together with Corollary \[3.3.4\] implies Conjecture 1: If there were a maximal green sequence which mutates \( j \) before \( i \) then rotation of that sequence will produce a maximal green sequence on a quiver \( Q' \) so that mutation at \( j \) is the first mutation. Since the mutation sequence which gives \( Q' \) from \( Q \) does not mutate \( i \) or \( j \) and \( Q \) is acyclic, there will still be an arrow of infinite type \( j \to i \) in \( Q' \). So, the first mutation on \( Q' \) cannot be at \( j \). This contradiction proves the conjecture.

Note that Conjecture 1 is not true if the word “acyclic” is removed. However, the Rotation Lemma which holds for any valued quiver, not necessarily acyclic, together with Corollary \[3.3.4\] implies the following more general version of the conjecture: (Corollary \[3.3.2\]) Consider any maximal green sequence on any valued quiver \( Q \). Then, at each step, the mutation is at a vertex of the mutated quiver \( Q' \) which is not the source of any arrow of infinite type.

To prove Conjecture 2, we need to strengthen the theorem of \[3\] to the following theorem.

**Theorem 0.0.2.** Let \( Q \) be any acyclic tame quiver and let \( r \geq 0 \). Then there are at most finitely many reddening sequences on \( Q \) with at most \( r \) red mutations.
To see that this theorem implies the conjecture, choose a fixed mutation sequence \((j_1, \ldots, j_r)\) from the acyclic tame quiver \(Q\) to any quiver \(Q'\). For each maximal green sequence \((k_0, \ldots, k_{m-1})\) on \(Q'\) we associate a reddening sequence \((j_r, \ldots, j_1, j_r, k_0, \ldots, k_{m-1})\) on \(Q'\) with exactly \(r\) red mutations. Then, by applying the Rotation Lemma \(r\) times, we see that

\[
(j_1, \ldots, j_r, k_0, \ldots, k_{m-1}, \sigma^{-1}(j_r), \ldots, \sigma^{-1}(j_1))
\]

is a reddening sequence on \(Q\) with exactly \(r\) red mutations. By the theorem, there are at most finitely many such sequences. Therefore, \(Q'\) has at most finitely many maximal green sequences.

Maximal green sequences are considered in physics (under the name “finite chambers”) when studying the BPS spectrum of a quantum field theory with extended supersymmetry, see [1] [13] and references therein. In [1] an explicit method is given to construct maximal green sequences for all quivers stemming from a triangulation of a surface. They cut the surface into manageable pieces and then glue the quiver together according to a rule that implicitly uses Conjecture 1: the setup is such that automatically the maximal green sequence would first mutate the target of a multiple arrow. In section 4.2 of [13], Conjecture 1 is explicitly formulated by saying if there are two green vertices connected by double arrows, then one cannot mutate the source in order to obtain a finite chamber. The Rotation Lemma is also suggested by Figure 11 in [13].

1. Preliminaries

We will use the following notation and definitions throughout the paper. Let \(Q\) be a valued quiver, with valuations on arrows as \(i \xrightarrow{d_{ij}} j\) and \(f_i\) for each vertex \(i \in Q_0\), where \(d_{ij}f_j = d_{ji}f_i\). The associated Euler matrix \(E = (E_{ij})\) of this quiver is given by \(E_{ii} = f_i\) for all \(i \in Q_0\) and \(E_{ij} = -d_{ij}\) for each arrow \(i \xrightarrow{d_{ij}} j\) in \(Q_1\). The associated skew-symmetrizable matrix \(B\) is defined as \(DB = E^t - E\) where \(D\) is the diagonal matrix with entries \(f_i\) on the diagonal.

**Example 1.0.3.** We use the following example to illustrate some of the definitions:

\[
3 \xrightarrow{3,1} 2 \xrightarrow{1,1} 1 \text{ and } f_3 = 3, \ f_2 = 1, \ f_1 = 1.
\]

The Euler matrix \(E\), the exchange matrix \(B\) and the diagonal matrix \(D\) in this case will be:

\[
E = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -3 & 3
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & -3 \\
0 & 1 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 3
\end{bmatrix}.
\]

2. Rotation Lemma for Reddening Sequences

In order to prove the Rotation Lemma we need precise formulas which relate sequence of \(c\)-matrices obtained by mutations \(\mu_{k_0}, \mu_{k_1}, \mu_{k_2}, \ldots, \mu_{k_m}\) of a skew-symmetrizable matrix \(B\) and the sequence of \(c\)-matrices obtained by mutations \(\mu_{k_1}, \mu_{k_2}, \ldots, \mu_{k_m}\) of the skew-symmetrizable matrix \(\mu_{k_0}B\). In both cases the initial \(c\)-matrix is the identity matrix \(I = I_n\).

2.1. Mutation formula. In Theorem 2.1.8 we give a formula which relates \(c\)-vectors of a skew-symmetrizable exchange matrix and a once-mutated exchange matrix. For the proof we use the associated \(g\)-matrices, sign-consistency of \(g\)-vectors and sign-coherence of \(c\)-vectors, recently shown in full generality in [5]. Instead of using the definition of \(g\)-matrix and \(g\)-vectors, we will use the result of Nakanishi-Zelevinsky that the \(g\)-matrix related to the \(c\)-matrix \(C\) is
given as \((G^t)^{-1} = DCD^{-1}\). We consider cluster algebras which satisfy sign-coherence of \(c\)-vectors, i.e., each \(c\)-vector either has all coordinates \(\geq 0\) or has all coordinates \(\leq 0\). From this it follows by \([11]\) that \(g\)-vectors are sign-consistent, i.e., the rows of \(g\)-matrix \(G\) are sign coherent. We will also use the following simple lemma.

**Lemma 2.1.1.** If the \(j\)-th column of a \(c\)-matrix \(C\) is \(\pm e_k\) then \(f_k = f_j\). In particular, if \(C\) is a permutation matrix or negative permutation matrix then \(CD = DC\).

**Proof.** Let \(G\) be the corresponding \(g\)-matrix. Then \(G\) is an integer matrix with determinant \(\pm 1\) and \((G^t)^{-1} = DCD^{-1}\). The \(j\)-th column of this matrix is \(\pm f_k f_j^{-1} e_k\). This is an integer which divides \(\det C = \pm 1\). Therefore, \(f_k = f_j\). \(\square\)

Let \(B = B_0\) be a skew-symmetrizable exchange matrix and let \(\tilde{B}_0 = \begin{bmatrix} B_0 \\ I_n \end{bmatrix}\) be the extended exchange matrix with the initial \(c\)-matrix \(C_0 = I = I_n\). Consider the two sequences of mutations, both closely related to the matrix \(B\):

\[
\tilde{B}_0 = \begin{bmatrix} B_0 \\ I \end{bmatrix} \overset{\mu_{k_2}}{\longrightarrow} \begin{bmatrix} B_1 \\ C_1 \end{bmatrix} \overset{\mu_{k_3}}{\longrightarrow} \begin{bmatrix} B_2 \\ C_2 \end{bmatrix} \overset{\mu_{k_4}}{\longrightarrow} \cdots \overset{\mu_{k_{m-1}}}{\longrightarrow} \begin{bmatrix} B_m \\ C_m \end{bmatrix},
\]

\[(*)\]

\[
\tilde{B}_1 = \begin{bmatrix} B_1 \\ I \end{bmatrix} \overset{\mu_{k_2}}{\longrightarrow} \begin{bmatrix} B_2' \\ C_2' \end{bmatrix} \overset{\mu_{k_3}}{\longrightarrow} \begin{bmatrix} B_3' \\ C_3' \end{bmatrix} \overset{\mu_{k_4}}{\longrightarrow} \cdots \overset{\mu_{k_{m-1}}}{\longrightarrow} \begin{bmatrix} B_m' \\ C_m' \end{bmatrix}.
\]

For each \(s \geq 0\) we will express \(c\)-matrix \(C'_s\) in terms of \(c\)-matrix \(C_s\). For this we will need the following matrices \(X_j^+\), \(X_j^-\), which essentially perform mutations on \(c\)-matrices. We state without proof some of the basic properties of these matrices.

**Definition 2.1.2.** Let \(B\) be an \(n \times n\) skew-symmetrizable matrix. For \(\varepsilon = \pm\) define \(X_j^\varepsilon\) to be the matrix equal to the identity matrix \(I_n\) except for its \(j\)-th row which is given by

\[
(X_j^{\varepsilon})_{jk} = \begin{cases} -1 & \text{if } k = j \\ [\varepsilon b_{jk}]_+ & \text{otherwise} \end{cases}
\]

where \([x]_+ = \max(0, x)\).

**Example 2.1.3.** For the quiver \(\begin{array}{c} 3 \\ \rightarrow \end{array}\) with matrix \(B\), the matrices \(X_2^+\) and \(X_2^-\) are:

\[
B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{bmatrix}, \quad X_2^+ = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_2^- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.
\]

We have the following easy observation.

**Lemma 2.1.4.** Let \(B\) be an \(n \times n\) skew-symmetrizable matrix. Then:

1. \(X_j^+ X_j^+ = I_n = X_j^- X_j^-\) for \(j \in \{1, 2, \ldots, n\}\).
2. \(X_j^+ X_j^- = I_n + J_j B\), where \(J_j\) is the diagonal matrix with \(d_{jj} = 1\) and \(d_{ii} = 0\) for all \(i \neq j\). \(\square\)

**Lemma 2.1.5.** Let \(B\) be a skew-symmetrizable matrix, \(\tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}\) and \(j \in \{1, \ldots, n\}\). Let \(\mu_j B\) and \(\mu_j C\) be the mutated matrices \(B\) and \(C\) in the direction of the \(j\)-th column. Then:

1. \(\mu_j C = CX_j^+\) if the \(j\)-th column of the matrix \(C\) has entries \(\geq 0\) and \(\mu_j C = CX_j^-\) if the \(j\)-th column of the matrix \(C\) has entries \(\leq 0\).
Consider the two sequences of mutations as in

\[ \mu_j I = X_j^+ \text{ for } j \in \{1, \ldots, n\}. \]

(3) The matrices \( G, \mu_j G \) differ only in their \( j \)-th columns.

**Proof.** (1) follows directly from the definition of \( \mu_j C \). (2) follows from (1). (3) follows from (1) since

\[ D^{-1}(\mu_j G)^t D = (\mu_j C)^{-1} = (CX_j)^{-1} = (X_j)^{-1}C^{-1} = X_j^+C^- \]

which differs from \( C^{-1} = D^{-1}G^t D \) only in its \( j \)-th row.

We need to introduce some more notation. For any fixed \( B_k \), by sign consistency of \( g \)-vectors \([11]\), the \( k \)-th coordinate of the corresponding \( g \)-vectors all have the same sign, i.e. the \( k \)-th row of the \( g \)-matrix \( G \) is sign coherent for each \( k \in \{1, \ldots, n\} \).

**Definition 2.1.6.** Let \( k \in \{1, 2, \ldots, n\} \). Define \( H_k^+ \) to be the set of all \( c \)-matrices \( C \) so that the corresponding \( g \)-vectors have \( k \)-th coordinate \( \geq 0 \). Let \( H_k^- \) be the other \( c \)-matrices (whose \( g \)-vectors have nonpositive \( k \)-th coordinates). We call \( H_k^+, H_k^- \) the \( k \)-hemispheres.

The following simple lemma is very useful for the rest of the paper. The lemma is true for any \( k \in \{1, 2, \ldots, n\} \), but the main application will be for \( k = k_0 \), i.e., for the \( c \)-vector of the first mutation in the above sequence of mutations.

**Lemma 2.1.7.** Let \( C \) be a \( c \)-matrix and \( G \) the corresponding \( g \)-matrix.

1. If any \( c \)-vector in matrix \( C \) is \( e_k \) (resp. \( -e_k \)) then the \( k \)-th row in \( G \) has all entries \( \geq 0 \) (resp. \( \leq 0 \)), hence matrix \( C \) is in \( H_k^+ \) (resp. \( H_k^- \)).

2. The matrices \( C \) and \( \mu_j C \) are in different \( k \)-hemispheres if and only if the \( j \)-th vector in \( C \) is \( \pm e_k \).

**Proof.** (1) Suppose that the \( j \)-th column of the matrix \( C \) is \( \pm e_k \). Using the equation \( G^t DC = D \) from \([11]\), we see that the \( (k, j) \) entry of matrix \( G \) must be \( \pm 1 \). Therefore, by sign-consistency of \( g \)-vectors, the \( k \)-th row of \( G \) must have the same sign. This proves (1).

(2) Furthermore, the \( j \)-th column of \( \mu_j C \) will be \( \mp e_k \) and the sign of the \( k \)-th row of \( \mu_j G \) will be the opposite of that of \( G \). This proves the implication \( (\Leftarrow) \). Conversely, suppose that \( C, \mu_j C \) are in opposite \( k \)-hemispheres. Then the \( k \)-th rows of \( G, \mu_j G \) have opposite sign. By Lemma 2.1.3(3), \( G, \mu_j G \) differ only in their \( j \)-th columns. So, the \( k \)-th rows of \( G \) and \( \mu_j G \) can have only one nonzero entry in position \((k, j)\). Then the \( j \)-th row of \( G^{-1} \) has only one nonzero entry in position \((j, k)\). So, \( C = (DG^{-1}D^{-1})^t \) has only one nonzero entry in its \( j \)-th column in position \((k, j)\), i.e., the \( j \)-th column of \( C \) is \( \pm e_k \).

**Theorem 2.1.8** (Mutation formula). Let \( B = B_0 \) be a skew-symmetrizable exchange matrix. Consider the two sequences of mutations as in \([11]\), both closely related to the matrix \( B \). Then:

\[ C'_s = X_{k_0}^{\varepsilon(s)} C_s \]

where

\[ \varepsilon(s) = \begin{cases} + & \text{if } C_s \in H_{k_0}^- \\ - & \text{if } C_s \in H_{k_0}^+ \end{cases} \]

**Remark 2.1.9.** This formula can be derived from the mutation formula for \( g \)-matrices in \([5, 12]\), but we restate it and reprove it in a form more convenient for proving the conjectures.

**Proof.** The proof will be by induction on \( s \) for the following two statements:

(a) For each \( k \) let \( c_k, c_k' \) denote the \( k \)-th columns of the matrices \( C_s \) and \( C'_s \). Then \( c_k, c_k' \) have the same sign unless \( c_k = \pm e_{k_0} \) in which case \( c_k' = -c_k \).

(b) \( C'_s = X_{k_0}^{\varepsilon(s)} C_s \) where \( \varepsilon(s) \) is as defined above.
(s = 1) (a1) By definition $C_1 = \mu_{k_0} C_0 = \mu_{k_0} I_n = X_{k_0}^+$ by Lemma 2.1.5 (2). Notice that matrix $X_{k_0}^+$ has $-1$ in the $(k_0, k_0)$ entry, positive diagonal entries and only positive entries of the $k_0$-th row of matrix $B$, hence all but $k_0$-th columns are positive, while $k_0$-th column is $-e_{k_0}$.

The matrix $C'_1 = I_n$, hence all columns are positive. Therefore (a1) holds.

(b1) $C'_1 = I_n = X_{k_0}^+ X_{k_0}^{-1} X_{k_0}^+$ follows from Lemma 2.1.3 (1) and Lemma 2.1.5 (2).

Assume ($a_s$) and ($b_s$) hold.

**Claim:** ($b_{s+1}$) holds, i.e. $C'_{s+1} = X_{k_0}^{e(s+1)} C_{s+1}$.

Proof of the claim: We know that $C'_s = X_{k_0}^{e(s)} C_s$. Consider $C'_{s+1} = \mu_{k_s} C'_s$ and $C_{s+1} = \mu_{k_s} C_s$.

**Case 1:** The $k_s$-th columns in both $C_s$ and $C'_s$ are positive. Then, by Lemma 2.1.5 (1) it follows that $C'_{s+1} = \mu_{k_s} C'_s = C'_s X_{k_s}^+$ and $C_{s+1} = \mu_{k_s} C_s = C_s X_{k_s}^+$. Therefore $C'_{s+1} = \mu_{k_s} C'_s = C'_s X_{k_s}^+ = X_{k_0}^{e(s)} C_s X_{k_s}^+ = X_{k_0}^{e(s)} C_{s+1}$. We only need to show that $e(s) = e(s+1)$. To see that, we notice that by the induction hypothesis and assumption that they have the same sign, the $k_s$-th columns of $C_s$ and $C'_s$ are not $\pm e_{k_0}$. Therefore, by Lemma 2.1.7, $C_{s+1}$ and $C_s$ are in the same $k_0$-hemisphere. Hence $e(s+1) = e(s)$ by the definition of $e$. Therefore $C'_{s+1} = X_{k_0}^{e(s+1)} C_{s+1}$.

**Case 2:** The $k_s$-th columns in both $C_s$ and $C'_s$ are negative. The proof is the same as in case 1, using the matrix $X_{k_s}^{-1}$.

**Case 3:** The $k_s$-th columns in $C_s$ and $C'_s$ have opposite signs. By induction hypothesis ($a_s$), this means that the $k_s$-th columns of $C'_s$ and $C_s$ are $e_{k_0}$ and $-e_{k_0}$ (or conversely $-e_{k_0}$ and $e_{k_0}$).

In matrix form this condition is:

$$C_s J_{k_s} = \pm J_{k_0} P_\tau$$

where $J_{k_s}$ is as in Lemma 2.1.3 and $P_\tau$ is the permutation matrix of the transposition $\tau = (k_0, k_s)$. ($P_\tau D = DP_\tau$ by Lemma 2.1.1) Since $e_{k_0}$ has entries $\geq 0$ and $-e_{k_0}$ has entries $\leq 0$ it follows by Lemma 2.1.7 that:

$$C_s \in H_{k_0}^+, \quad C'_s \in H_{k_0}^-, \quad C_{s+1} = \mu_{k_s} C_s \in H_{k_0}^-, \quad C'_{s+1} = \mu_{k_s} C'_s \in H_{k_0}^+.$$ 

Applying now Lemma 2.1.5 it follows that:

$$C'_{s+1} = \mu_{k_s} C'_s = C'_s X_{k_s}^- \quad \text{and} \quad C_{s+1} = \mu_{k_s} C_s = C_s X_{k_s}^-.$$ 

By the induction hypothesis ($b_s$) and the fact that $C_s \in H_{k_0}^+$ it follows that

$$C'_s = X_{k_0}^{e(s)} C_s = X_{k_0}^{e(s)} C_j \quad \text{and therefore} \quad C'_{s+1} = C'_s X_{k_s}^- = X_{k_0}^{e(s)} C_s X_{k_s}^-.$$ 

In order to prove ($b_{s+1}$) in this case and using the fact that $C_{s+1} \in H_{k_0}^-$, we want to show:

$$C'_{s+1} = X_{k_0}^{e(s+1)} C_{s+1} = X_{k_0}^{e(s)} C_s X_{k_s}^{-1}.$$ 

From the above formulas, it is enough to show:

$$X_{k_0}^{e(s)} X_{k_0}^{-1} X_{k_s} C_s X_{k_s}^{-1} = X_{k_0}^{e(s)} C_s X_{k_s}^{-1}.$$ 

Using Lemma 2.1.4 (1) and (2), it will be enough to show:

$$X_{k_0}^{e(s)} X_{k_0}^{-1} C_s X_{k_s}^- \quad \text{or, equivalently,} \quad (I_n + J_{k_0} B_{0}) C_s = C_s (I_n + J_{k_0} B_{0}).$$ 

Since $I_s C_s = C_s I_n$, it suffices to show that $J_{k_0} B_0 C_s = C_s J_{k_0} B_0$. But, we have $C_s J_{k_s} = J_{k_0} P_\tau$. So,

$$C_s J_{k_0} D^{-1} C_s D = C_s J_{k_0} D^{-1} J_{k_0} C_s D = J_{k_0} P_\tau D^{-1} P_\tau J_{k_0} D = J_{k_0} D^{-1} J_{k_0} D = J_{k_0}.$$ 

Multiplying both sides by $B_0 C_s$ and using the equation $B_s = D^{-1} C_s D B_0 C_s$ from [11] we get:

$$C_s J_{k_s} B_s = C_s J_{k_s} D^{-1} C_s D B_0 C_s = J_{k_0} B_0 C_s.$$
as required. That was the last step which now implies \((b_{s+1})\), i.e., \(C_{s+1} = X_{k_0}^e(s+1)C_{s+1}\).

Claim: Assuming \((a_s)\) and \((b_s)\) the statement \((a_{s+1})\) holds.

The proof of this claim uses the fact that \((b_{s+1})\) holds from above, i.e. \(C_{s+1} = X_{k_0}^e(s+1)C_{s+1}\). The matrix \(X_{k_0}^e(s+1)\) has nonnegative entries in all rows different from \(k_0\)-th. Hence each row different from \(k_0\)-th row in the matrices \(C_{s+1}'\) and \(C_{s+1}\) will have the same sign. Because of sign coherence of c-vectors, this means that each pair of corresponding columns in \(C_{s+1}'\) and \(C_{s+1}\) which are \(\neq \pm e_{k_0}\) must have the same sign. If the \(k\)-th column vector in \(C_{s+1}'\) is \(\pm e_{k_0}\) then the \(k\)-th column vector in \(C_{s+1}\) will be \(\mp e_{k_0}\) since the \((k_0, k_0)\) entry in \(X_{k_0}^e(s+1)\) is \(-1\). Therefore \((a_{s+1})\) holds.

\[\square\]

2.2. Rotation Lemma. We will show that the Rotation Lemma follows from the Mutation Formula in Theorem 2.1.8.

Lemma 2.2.1. For any vertex \(k\) of \(Q\), every reddening sequence for \(Q\) will mutate the \(c\)-vector \(+\epsilon_k\) one more time than it mutates the \(c\)-vector \(-\epsilon_k\).

Proof. Consider the sequence of \(c\)-matrices \(I_n = C_0, \ldots, C_m = -P_\sigma\) of a reddening sequence on \(Q\). The first \(c\)-matrix \(C_0 = I_n\) lies in \(H^+_k\) for every \(k\) and the last \(c\)-matrix lies in \(H^-_k\) for every \(k\). So, during the mutation sequence it must pass from the positive to the negative side of the hyperplane \(H_k\) one more time than it goes from the negative to the positive side. By the mutation formula, the first event occurs when the mutated \(c\)-vector is \(+\epsilon_k\). The second event occurs when the mutated \(c\)-vector is \(-\epsilon_k\). The lemma follows.

\[\square\]

Theorem 2.2.2 (Rotation Lemma). Let \(B\) be a skew-symmetrizable matrix and let \((k_0, k_1, \ldots, k_{m-1})\) be the vertex labeling of an \(r\)-reddening sequence for \(B\) with associated permutation \(\sigma\) and let \(B' = \mu_{k_0} B\). Then

\((k_1, k_2, \ldots, k_{m-1}, \sigma^{-1}(k_0))\)

is an \(r\)-reddening sequence for \(B'\) with the same associated permutation \(\sigma\).

Proof. We use the same notation as in the mutation formula. The reddening sequence \((k_0, \ldots, k_{m-1})\) gives the mutation sequences

\[
\begin{array}{cccc}
B & I_n & B_0 & C_0 \\
\mu_{k_0} & \mu_{k_1} & \mu_{k_2} & \ldots \\
C_1 & C_2 & \ldots & C_m \\
\end{array}
\]

By Nakanishi-Zelevinsky we have:

\[DB_s = C'_sDB_0C_s\]

Since \((k_0, k_1, \ldots, k_{m-1})\) is a reddening sequence we have \(C_m = -P\) where \(P = P_\sigma\) is the permutation matrix given by permuting the rows of \(I_n\) by \(\sigma\). By Lemma 2.1.1 we have that \(P\) commutes with \(D\). Thus, \(B_m = D^{-1}P^tDB_0P = P^tB_0P\). Let \(j = \sigma^{-1}(k_0)\) so that the \(j\)-th column of \(P\) is the unit vector \(\epsilon_{k_0}\). By the mutation formula we get:

\[
\tilde{B}'_m = \begin{bmatrix} B'_m \\ C'_m \end{bmatrix} = \begin{bmatrix} P^tB_0P \\ -C_1P \end{bmatrix}.
\]

The \(c\)-matrix \(C_1\) is equal to the identity matrix \(I_n\) except for its \(k_0\)-th row which is given by

\[(C_1)_{k_0\ell} = \begin{cases} -1 & \text{if } \ell = k_0 \\ |b_{k_0\ell}| & \text{otherwise} \end{cases} \]
where \([x]_+ = \max(0, x)\). Multiplication by \(-P_\sigma\) gives the matrix \(-C_1P_\sigma\) which is equal to \(-P_\sigma\) except for its \(j\)-th row \((j = \sigma^{-1}(k_0))\) where

\[
-(C_1P_\sigma)_{j\ell} = \begin{cases} 1 & \text{if } \ell = j \\ -[b_{\sigma(j)\sigma(\ell)}]_+ & \text{otherwise} \end{cases}
\]

But, \(b_{\sigma(j)\sigma(\ell)} = b_{k_0\sigma(\ell)}\) is exactly the \((j, \ell)\) entry of the matrix \(B_m = P_\sigma^t B_0 P_\sigma\). When \(b_{\sigma(j)\sigma(\ell)} > 0\) we add \(b_{\sigma(j)\sigma(\ell)}\) times the \(j\)-th column of \(C'_m = -C_1P_\sigma\) (which is \(e_{k_0}\)) to its \(\ell\)th column

\[
-C_1P_\sigma e_\ell = -e_{\sigma(\ell)} - [b_{k_0\sigma(\ell)}]_+ e_{k_0}
\]

to get \(-e_{\sigma(\ell)}\). Then we change the sign of the \(j\)-th column to produce the \(c\)-matrix \(C'_{m+1} = -P_\sigma\). This proves that \(\langle k_1, \ldots, k_{m-1}, \sigma^{-1}(k_0)\rangle\) is a reddening sequence for \(B'\) with the same associated permutation \(\sigma\).

It remains to show that this new reddening sequence has the same number of red mutations as the original reddening sequence.

Let \(r \geq 0\) be the number of red mutations in the first sequence. This includes \(p\) mutations at \(c\)-vector \(-e_{k_0}\) and \(q = r - p\) mutations at other negative \(c\)-vectors. By the lemma, there will be exactly \(p + 1\) mutations at the positive \(c\)-vector \(e_{k_0}\). The first mutation will be one of these. Of the remaining \(m - 1\) mutations in the first mutation sequence, exactly \(p\) will be at the \(c\)-vector \(e_{k_0}\) and exactly \(p\) will be at the \(c\)-vector \(-e_{k_0}\).

By the mutation formula, the sign of the mutation \(C_s \xrightarrow{\mu_s} C_{s+1}\) will be the same as the sign of the mutation \(C'_s \xrightarrow{\mu'_s} C'_{s+1}\) if the \(c\)-vector being mutated is not equal to \(\pm e_{k_0}\). This means that both mutation sequences have the same number \(q\) of mutations at negative \(c\)-vectors not equal to \(-e_{k_0}\). The mutation formula also tells us that, if the \(c\)-vector being mutated in \(C_s\) is \(\pm e_{k_0}\), then the \(c\)-vector being mutated in \(C'_s\) will be the negative of that vector. Thus, the \(p\) red mutations at \(-e_{k_0}\) for \(C_s\) will become \(p\) green mutations for \(C'_s\) and vice versa. The number of red mutations for the rotated sequence will thus be \(p + q = r\). (The last mutation is at the positive \(c\)-vector \(e_{k_0}\).) This completes the proof of the Rotation Lemma.

\[\Box\]

**Proposition 2.2.3.** Let \(\langle k_0, \ldots, k_{m-1}\rangle\) be an \(r\)-reddening sequence for \(B\) and let \(k\) be maximal so that \(k_k\) is mutation on vector \(e_{k_0}\). Let \(c_0, \ldots, c_{m-1}\) be the \(c\)-vector labeling of the same mutation sequence. (So, \(c_k = e_{k_0}\).) Let \(c'_1, \ldots, c'_m\) be the \(c\)-vector labeling of the rotated reddening sequence for \(B'\). Then \(c_s = C_1c'_s = X_{k_0}^+ c'_s\) for all \(k < j < m\).

**Proof.** Since a reddening sequence must end in \(H_{k_0}^m\), it cannot leave the region after entering it for the last time. Therefore, for \(j > k\), the \(c\)-matrix \(C_s\) must remain in the negative part \(H_{k_0}^-\) of the hyperplane \(H_{k_0}\). The mutation formula then gives \(c_s = C_1c'_s = X_{k_0}^+ c'_s\) as claimed. \[\Box\]

2.3. **Greg Muller’s example.** This section provides some explanations why our Rotation Lemma does not contradict Greg Muller’s example.

At first sight, the Rotation Lemma might seem contradictory to Greg Muller’s recent preprint [10] where he provides examples showing that the existence of a maximal green sequence is not invariant under quiver mutation. However, as 0-reddening sequences are maximal green sequences, the Rotation Lemma shows: Given a maximal green sequence \((k_0, k_1, \ldots, k_{m-1})\) on \(Q\) with associated permutation \(\sigma\),

\[(k_1, k_2, \ldots, k_{m-1}, \sigma^{-1}(k_0))\]

is a maximal green sequence on \(\mu_{k_0}Q\) with the same permutation \(\sigma\). So the existence of maximal green sequences, and their respective length, is in fact preserved under mutation for
those quivers that appear along the maximal green sequence. This yields a restriction on which quivers in the mutation class of $Q$ can appear along the maximal green sequence for $Q$, an effect that can already be illustrated for a quiver $Q$ of type $A_3$:

We reproduce Figure 1 from [3] of the oriented mutation graph for the linear oriented quiver $Q$ of type $A_3$. Maximal green sequences are the oriented paths starting in the vertex encircled in gray and ending in the black encircled vertex. In particular, there are maximal green sequences of lengths 3, 4, 5 and 6.

![Figure 3. The oriented mutation graph of $Q : 1 \to 2 \to 3$.](image)

On the other hand, the cyclic oriented simple graph $Q'$ with three vertices yields the oriented mutation graph shown in Figure 2. We see that the only maximal green sequences are of lengths 4 or 5 in this case, even if the quivers $Q$ and $Q'$ are mutation equivalent. So it is certainly not the case that every maximal green sequence of $Q$ yields a maximal green sequence (of the same length) of the mutated quiver $Q'$. The structural result implied by the Rotation Lemma is: The cyclically oriented quiver $Q'$ cannot occur along a maximal green sequence for $Q$ of length 3 or 6. In fact, the only quivers appearing along these maximal green sequences are the acyclic quivers of type $A_3$, and those admit sequences of length 3 and 6. Thus the membership to certain maximal green sequences yields a finer subdivision of the mutation class of $Q$.

The situation becomes more dramatic in Greg Muller’s example where we consider the acyclic quiver $Q_{2,1,0} : 1 \Rightarrow 2 \to 3$. It does admit the following maximal green sequences:
(2, 3, 1, 2) which we already considered in the introduction, as well as (2, 3, 2, 1) and the minimal sink reflection sequence (3, 2, 1). The quivers appearing along these maximal green sequences are $Q_{2,1,0}$ and its source-sink reflections, as well as $Q_{2,1,2}^\circ$, where we denote by $Q_{a,b,c}$ the cyclic quiver with $a$ arrows from 1 to 2, $b$ arrows from 2 to 3 and $c$ arrows from 3 to 1. These are exactly the quivers in the mutation class of $Q_{2,1,0}$ that admit maximal green sequences. The mutation class of $Q_{2,1,0}$ is infinite, and the minimal quiver in this class not having a maximal green sequence is the quiver $Q_{2,3,2}$ discussed in Corollary 2.3.3 of [10]. However, as illustrated in Figures 11 and 18 of [10], the quiver $Q_{2,3,2}$ has a 1-reddening sequence of length 6 passing through $Q_{2,1,0}$.

3. The Target before Source Conjecture

In the general form of Conjecture 1 we prove, it will be convenient to introduce the following notion.

**Definition 3.0.1.** The maximal green tail of a reddening sequence $(k_0, k_1, \ldots, k_{m-1})$, is the subsequence $(k_\ell, k_{\ell+1}, k_{m-1})$ where $k_{\ell-1}$ is the last red mutation.

Note that the maximal green tail of a reddening sequence need not itself be a maximal green sequence, as illustrated in the following example.

**Example 3.0.2.** In the Kronecker quiver $2 \to 1$ the sequence $(1, 2, 1, 1)$ is a reddening sequence. Its maximal green tail is the single mutation (1) which is green, but not maximal.

**Definition 3.0.3.** For an arrow $\alpha: j \to i$ of a quiver $Q$, denote by $Q[\alpha]$ the rank 2 quiver consisting of the single arrow $\alpha: j \to i$.

An arrow $\alpha: j \to i$ with valuation $(d_{ji}, d_{ij})$ is of infinite type if $d_{ji}d_{ij} \geq 4$, or equivalently if $Q[\alpha]$ is representation infinite.

3.1. Recursion for rank 2 preinjective roots. In order to prove the Theorem, we need to relate the $c$-vectors of the reddening sequence to the preinjective roots of the rank 2 quiver $Q[\alpha]$ where $\alpha$ is an infinite type arrow of $Q$. Throughout this subsection we fix an infinite type arrow $\alpha: j \to i$ of $Q$. For simplicity, set $a = d_{ji}$ and $b = d_{ij}$.

The preinjective roots of $Q[\alpha]$ are linearly ordered by position in the Auslander-Reiten quiver. Denote by $q_t$ the root of $Q$ obtained by extending the $t$-th preinjective root of $Q[\alpha]$ by zero on vertices $k \neq i, j$. In particular we have $q_0 = e_j$.

In order to give a uniform description of the vectors $q_t$, we introduce the following family of polynomials.
**Definition 3.1.1.** Define a family of Chebyshev-like polynomials $U_n(x, y)$ for $n \geq -1$ by $U_{-1}(x, y) = 0$, $U_0(x, y) = 1$ and for $n \geq 1$ by the recursion

$$U_n(x, y) = x U_{n-1}(y, x) - U_{n-2}(x, y).$$

**Remark 3.1.2.** The ordinary Chebyshev polynomials (of the second kind) $U_n(x)$ are recovered from the $U_n(x, y)$ by the substitution $x, y \mapsto x/2$. The normalization factor $\frac{1}{2}$ is chosen to simplify the following formula for the roots $q_t$.

**Lemma 3.1.3.** The roots $q_t$ of $Q$ have coordinates $q_t(i) = U_{t-1}(b, a)$, $q_t(j) = U_t(a, b)$ and $q_t(k) = 0$ for $k \neq j, i$.

**Proof.** Number the vertices of $Q$ so that $i = 1$ and $j = 2$. With this numbering, the Auslander-Reiten translate $\tau$ of $Q$ is given on dimension vectors by

$$\tau = \begin{bmatrix} -1 & b & \ast \\ -a & ab - 1 & \ast \\ \ast & \ast & \ast \end{bmatrix}.$$  

In the quiver $Q[\alpha]$ the injective roots are $\dim I_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\dim I_1 = \begin{bmatrix} 1 \\ a \end{bmatrix}$, so the lemma is true for $t = 0, 1$. For $t \geq 2$ the roots $q_t$ are related by $q_t = \tau q_{t-2}$. Thus the coordinates of the $q_t$ satisfy the simultaneous recursion

$$q_t(i) = b q_{t-2}(j) - q_{t-2}(i)$$
$$q_t(j) = (ab - 1) q_{t-2}(j) - a q_{t-2}(i) = a q_t(j) - q_{t-2}(j).$$

By induction we have

$$q_t(i) = b U_{t-2}(a, b) - U_{t-3}(b, a) = U_{t-1}(b, a)$$
$$q_t(j) = a U_{t-1}(b, a) - U_{t-2}(a, b) = U_t(a, b)$$

proving the lemma. \[ \square \]

Denote by $q_t'$ for $t \geq 0$ the extension of the preinjective roots of $Q'[\alpha']$ to $Q' = \mu_j Q$ where $\alpha' : i \to j$. Denote by $q_{-1}$ the vector with $q_{-1}(i) = -1$ and all other coordinates 0. The following lemma explains the relationship between the roots $q_t$ of $Q$ to the roots $q_t'$ of $Q'$.

**Lemma 3.1.4.** For every $t \geq 0$ one has $q_{t-1} = X_j^+ q_t$ where $X_j^+$ is as in Definition 2.1.2.

**Proof.** Upon numbering the vertices of $Q$ so that $1 = i$ and $2 = j$, the matrix $X_j^+$ is given by

$$X_j^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -1 & \ast \\ 0 & 0 & I_{n-2} \end{bmatrix}$$

and so

$$X_j^+ q_t = \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & \ast \\ 0 & 0 & I_{n-2} \end{bmatrix} \begin{bmatrix} U_{t-1}(b, a) \\ U_t(a, b) \\ 0 \end{bmatrix} = \begin{bmatrix} U_{t-1}(b, a) \\ a U_{t-1}(b, a) - U_t(a, b) \\ 0 \end{bmatrix} = \begin{bmatrix} U_{t-1}(b, a) \\ U_{t-2}(a, b) \\ 0 \end{bmatrix} = q_{t-1}$$

provided that $t > 0$. When $t = 0$ one calculates

$$X_j^+ q_t = \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & \ast \\ 0 & 0 & I_{n-2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = q_{-1}$$

proving the lemma. \[ \square \]
3.2. Proof of Target before Source Conjecture. To prove the Target before Source Conjecture we need the following technical lemma.

Lemma 3.2.1. Suppose $Q$ is a quiver, $\alpha : j \to i$ is a fixed arrow and $k$ is a reddening sequence. Consider the extended sequence \( k' = (j, j, k_0, k_1, \ldots, k_{m-1}) \), and let \( k'' \) be reddening sequence in \( Q' = \mu_j Q \) given by rotating \( k' \). If the $c$-vector \( e_j \) occurs a vertex \( k_\ell \) in tail\((k')\) preceding the $c$-vector \( e_i \), then:

1. \( C_\ell \in H^+_{i} \cap H^+_j \)
2. \( C'_\ell \in H^+_{i} \cap H^-_j \)
3. the $c$-vector \( e_i \) occurs before the $c$-vector \( e_j \) in tail\((k'')\).

Proof. Let \( k_p \) be the vertex in tail\((k)\) where \( c_{k_p} = e_i \) and \( p > \ell \).

1. By Lemma 2.1.7 the $c$-matrix \( C_\ell \in H^+_{j} \) and \( C_p \in H^+_{j} \). However, all of the mutations in tail\((k)\) are green and so the $c$-matrices cannot go from \( H^-_{j} \) to \( H^+_{j} \). Since \( \ell < p \), we have \( C_\ell \in H^+_{j} \) as claimed.

2. Since \( C_\ell \in H^+_{j} \), we have \( C'_\ell = X_j^+ C_\ell \) where

\[
X_j^+ = \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & * \\ 0 & 0 & I_{n-2} \end{bmatrix}
\]

(assuming \( i = 1 \), \( j = 2 \) for simplicity). So, \( (G'_\ell)^t = D(C'_\ell)^{-1}D^{-1} = DC^{-1}_\ell X_j^{-1}D^{-1} = G'_\ell DX_j^{-1}D^{-1} \). By (1), columns \( i \) and \( j \) of \( G'_\ell \) are positive. So, columns \( i \) and \( j \) of \( (G'_\ell)^t \) are positive and negative, respectively. This is equivalent to (2).

3. By (2) \( C'_\ell \in H^-_{j} \). Since all mutations in tail\((k'')\) are green, the last $c$-matrix is in \( H^-_{j} \). By Lemma 2.1.7 there is a \( q > \ell \) so that the mutation \( k_q \) of \( C'_\ell \) is at $c$-vector \( e_i \). The last mutation of \( k'' \) is at the $c$-vector \( e_j \) by rotation, proving the lemma.

We now turn to the main theorem of this section.

Theorem 3.2.2. Suppose that \( Q \) is a valued quiver having an infinite type arrow \( \alpha : j \to i \) and \( k = (k_0, k_1, \ldots, k_{m-1}) \) is a reddening sequence. Then in the $c$-vector sequence of the maximal green tail \( (k) \), the simple root \( e_i \) must occur before the simple root \( e_j \).

Proof. Suppose \( k \) is a reddening sequence for \( Q \) in which \( e_j \) occurs before \( e_i \) in tail\((k)\). Let \( k_\ell \) be the first vertex in tail\((k)\) with corresponding $c$-vector \( e_j \).

We claim that for each integer \( s \geq 0 \) there is a subsequence

\( (k_{t_0}, k_{t_1}, \ldots, k_{t_s}) \)

of tail\((k)\) with \( k_{t_0} = k_\ell \) and corresponding $c$-vectors \( c_{t_\ell} = q_t \) for \( 0 \leq t \leq s \). This provides a contradiction, as the sequence \( k \) is finite.

The claim is proven by induction on \( s \). The statement holds when \( s = 0 \) since \( k_\ell = k_{t_0} = e_j = q_0 \) by definition. Suppose that the claim holds for some \( s \). Consider the extended reddening sequence \( k' = (j, j, k_0, k_1, \ldots, k_{m-1}) \) as in Lemma 3.2.1.

By induction, there is a subsequence \( (k_{0t}, \ldots, k_{st}) \) of tail\((k)\) = tail\((k')\), with corresponding $c$-vectors \( q_t \) for \( 0 \leq t \leq s \). By Lemma 3.1.3 the subsequence \( (k_1, k_2, \ldots, k_s) \) of the tail of the rotated sequence \( k'' \) has associated $c$-vectors \( q'_0, q'_1, \ldots, q'_{s-1} \). Moreover, by Lemma 3.2.1...
$e'_k = e_i$ and mutation at $e_j$ occurs after $k_1$ so by induction there is a vertex $k_{t+1}$ of tail($k''$) with corresponding $c$-vector $q'_s$. Since the matrix $X_j^+X_j^- = I$, the $c$-vector of $Q$ corresponding to $k_{t+1}$ of the unrotated sequence $k''$ is $X_j^+q'_s = q_{s+1}$. Thus by induction, the claim holds.

3.3. Proof of Proposition 3.3.4. The Target before Source Conjecture derives its name from Proposition 3.3.3 which is a corollary of Theorem 3.2.2. We will need two lemmas.

Lemma 3.3.1. Any maximal green sequence mutates at each simple root $e_k$ exactly once.

Proof. A maximal green sequence crosses each hyperplane $H_k$. By Lemma 2.1.2, crossing $H_k$ amounts to mutating at simple roots $\pm e_k$. Since only green mutations are being performed, all of these mutations must be at $+e_k$. The maximal green sequence starts on the $+$-side of all hyperplanes, end on the $-$-side of all hyperplanes, and so must mutate each $e_k$. □

Corollary 3.3.2. Consider any maximal green sequence on any valued quiver $Q$. Then, at each step, the mutation is at a vertex of the mutated quiver $Q'$ which is not the source of any arrow of infinite type.

Proof. If this occurs, use the Rotation Lemma to make $Q'$ the initial quiver. Then we have an arrow of infinite type $j \to i$ and the first mutation is at $c$-vector $e_j$. There must be a mutation at $e_i$ later by the lemma above and this gives a contradiction. □

Lemma 3.3.3. Suppose $Q$ is acyclic having an arrow $\alpha: j \to i$ of infinite type. If $k_1, k_2, \ldots, k_s$ is any sequence of vertices with each $k_i \neq i, j$, then $Q' = \mu_{k_s} \circ \mu_{k_{s-1}} \circ \cdots \circ \mu_{k_1}Q$ has an arrow $\alpha': j \to i$ of infinite type.

Proof. Let $T = P_1 \oplus \cdots \oplus P_n$ be the projective cluster in the cluster category $\mathcal{C}_Q$ of $Q$, and let $T'$ be the cluster-tilting object given by $\mu_{k_s} \circ \mu_{k_{s-1}} \circ \cdots \circ \mu_{k_1}T$ (cf., [2]). The quiver $Q'$ is the Gabriel quiver of the cluster tilted algebra $B = \text{End}_{\mathcal{C}_Q}(T')^{\text{op}}$, and the valuation $(d'_{ji}, d''_{ij})$ of $\alpha': j \to i$ in $Q'$ is given by $d'_{ji} = \dim F_i \text{Irr}_B(P_i, P_j)$ and $d''_{ij} = \dim F_j \text{Irr}_B(P_i, P_j)$ where $\text{Irr}_B(P_i, P_j)$ denotes the space of irreducible $B$-linear maps from $P_i \to P_j$.

Since $\text{Irr}_B(P_i, P_j)$ is the quotient of $\text{Irr}_{\mathcal{C}_Q}(P_i, P_j)$ by the ideal of morphisms $P_i \to P_j$ factoring through objects in the cluster $T'$ not equal to $P_j$ or $P_i$, the natural map $\text{Irr}_B(P_i, P_j) \to \text{Irr}_B(P_i, P_j)$ is surjective. Hence $d'_{ji}d''_{ij} \geq d'_{ji}d_{ij}$. In particular, if $\alpha: j \to i$ is infinite type in $Q$, $\alpha': i \to j$ is infinite type in $Q'$. □

Proposition 3.3.4. If $Q$ is acyclic with an infinite type arrow $\alpha: j \to i$, any maximal green sequence mutates at the vertex $i$ before the vertex $j$.

Proof. Suppose that the first occurrence of $j$ precedes the first occurrence of $i$ in the maximal green sequence of $Q$. Rotate the sequence to form a maximal green sequence of a quiver $Q'$ having $i$ as the first mutation. By Lemma 3.3.3 the quiver $Q'$ still has an infinite type arrow $\alpha': i \to j$.

The first mutation of the rotated sequence occurs at the $c$-vector $e_j$. By Lemma 3.3.1 the rotated sequence eventually mutates at the $c$-vector $e_i$. Since $Q'$ has an infinite type arrow $\alpha': i \to j$, this contradicts Theorem 3.2.2 proving the proposition. □

4. Finite number of reddening sequences

In this section we prove the following theorem.

Theorem 4.0.5. If $Q$ is a quiver which is mutation equivalent to an acyclic tame quiver then $Q$ has at most finitely many $r$-reddening sequences for every $r \geq 0$. In particular, $Q$ has at most finitely many maximal green sequences.
By the Rotation Lemma, it suffices to prove the theorem in the case when $Q$ is any acyclic tame (valued) quiver. The proof uses domains of semi-invariants and the easy observation that every cluster contains at least one preprojective or preinjective component. We begin with the basic definitions and an outline of the proof.

4.1. Definitions and outline of proof. Let $\Lambda$ be a fixed tame hereditary algebra which is finite dimensional over a field $K$. We recall that Auslander-Reiten translation $\tau$ is given on nonprojective roots by:

$$\tau \beta = -E^{-1}E^i\beta$$

where $E$ is the Euler matrix of $\Lambda$. If $\pi_i$ denotes the dimension vector of the $i$-th projective module then $-\tau \pi_i$ is the dimension vector of the $i$-th injective module. The matrix $-E^{-1}E^i$ is invertible and we have Auslander-Reiten duality: $\langle \alpha, \tau \beta \rangle = -\langle \beta, \alpha \rangle$ and $\tau$ is an isometry: $\langle \tau \alpha, \tau \beta \rangle = \langle \alpha, \beta \rangle$.

For every $k \geq 1$, let $S_k$ be the finite set of positive roots (real Schur roots) for $\Lambda$ given by

$$S_k = \{ t^{-i} : 0 \leq i < k, \alpha \text{ projective, } \beta \text{ injective} \}.$$

For every $k \geq 1$ let $V_k, W_k$ be subsets of $\mathbb{R}^n$ defined by

$$V_k := \{ x \in \mathbb{R}^n : \langle x, \beta \rangle \geq 0 \text{ for all preinjective } \beta \in S_k \}$$

$$W_k := \{ x \in \mathbb{R}^n : \langle x, \alpha \rangle > 0 \text{ for some preprojective } \alpha \in S_k \}.$$

Then, the complement of $W_k$ is

$$\mathbb{R}^n \setminus W_k = \{ x \in \mathbb{R}^n : \langle x, \alpha \rangle \leq 0 \text{ for all preprojective } \alpha \in S_k \}.$$

For each cluster tilting object $T = \bigoplus T_i$ in the cluster category of $\Lambda$ we have the simplicial fan $R(T) = \{ \sum a_i \dim T_i : a_i \geq 0 \}$. Recall that, for distinct $T, T'$, the interiors of the regions $R(T), R(T')$ do not intersect. Also, $R(\mu_k T) \cap R(T)$ is a subset of semi-invariant domain

$$D(\beta_k) = \{ x \in \mathbb{R}^n : \langle x, \beta \rangle = 0, \langle x, \beta' \rangle \leq 0 \text{ for all real Schur subroots } \beta' \subset \beta \}.$$

Furthermore, the interiors of the regions $R(T)$ are disjoint from all $D(\beta)$. We observe that the condition $\langle x, \beta' \rangle \leq 0$ for all $\beta' \subset \beta$ is equivalent to the condition that $\langle x, \beta'' \rangle \geq 0$ for all quotient roots $\beta''$ of $\beta$.

**Proposition 4.1.1.** For every $k > 0$ and every cluster $T$, the interior of $R(T)$ is either contained in $V_k$ or is disjoint from $V_k$. Similarly with $V_k$ replaced with $W_k$.

**Proof.** It suffices to show that the boundary of $V_k$ (it closure minus its interior) is a union of $D(\beta)$’s. So, let $x \in \partial V_k$. Then $\langle x, \beta \rangle = 0$ for some preinjective $\beta$ in the finite set $S_k$. By definition of $V_k$, we have $\langle x, \gamma \rangle \geq 0$ for all preinjective $\gamma \in S_k$. But this includes all quotient roots of $\beta$. Therefore, $x \in D(\beta)$ proving the claim. By an analogous argument we see that $\partial W_k$ is also contained in a union of $D(\beta)$’s. The proposition follows. □

Since $V_k$ is on the positive side of $D(\beta)$ at each point on the boundary, and similarly for $W_k$, we get the following.

**Corollary 4.1.2.** $V_k$ and $W_k$ are inescapable, i.e., all walls are red. □

**Definition 4.1.3.** We say that a reddening sequence for $\Lambda$ meets $V_k \setminus W_k$ if there is a cluster $T$ in the sequence so that $R(T) \subset V_k \setminus W_k$. If this is not the case, the proposition above implies that the interior of each $R(T)$ in the sequence is disjoint from $V_k \setminus W_k$ and we say that the reddening sequence is disjoint from $V_k \setminus W_k$.

Theorem 4.1.4 follows from the following properties of reddening sequences.

1. (Finiteness) $\forall r, k$ only finitely many $r$-reddening sequences are disjoint from $V_k \setminus W_k$. 


Also, Ext(M⟨γ,β⟩) for any preprojective α Schur root disjoint from the interior of \( V^k \). And there are only finitely many regular roots in the tame case, we are done.

Proof. If \( \gamma \) is any preprojective root which is not in \( S_{k+1} \) then Hom(Mγ, Mα) = 0, so \( \langle \gamma, \alpha \rangle \leq 0 \) for any preprojective \( \alpha \in S_{k+1} \) and Ext(Mγ, Mα) ≠ 0 and thus \( \langle \gamma, \alpha \rangle \leq 0 \) for some \( \alpha \in S_k \). Also, Ext(Mγ, Mβ) = 0, so \( \langle \gamma, \beta \rangle \geq 0 \) for any preinjective \( \beta \) and Hom(Mγ, Mβ) ≠ 0. So, \( \langle \gamma, \beta \rangle > 0 \) for some preinjective \( \beta \). Thus, \( \gamma \) lies in the interior of \( V^k \).

Similarly, for any preinjective \( \gamma \) not in \( S_k \), \( \gamma \) lies in the interior of \( V^k \). So, any real Schur root disjoint from the interior of \( V^k \) lies in the finite set \( S_{k+1} \) or is regular. Since there are only finitely many regular roots in the tame case, we are done.

We know that the dimension vector of every component \( T_i \) of every cluster tilting object \( T \) is a real Schur root. And \( R(T) \) is spanned by \( \dim T_i \). So, the lemma above implies that \( V^k \) contains all but finitely many \( R(T) \).

Lemma 4.2.2. An \( r \)-reddening sequence passes through the same cluster at most \( r+1 \) times. In other words, the same extended exchange matrix \( B_s \) cannot occur more than \( r+1 \) times.

Proof. Suppose there is an \( r \)-reddening sequence \((k_0, \ldots, k_{m-1})\) which reaches the same extended exchange matrix \( B_s \) say \( t > r+1 \) times. Apply the Rotation Lemma to make one of these the first mutation. Since the c-matrix determines the g-matrix, these t c-matrices all lie on the same side of every hyperplane \( H_k \). Therefore, by the Mutation Formula, they change to the same c-matrices in the rotated \( r \)-reddening sequence. Since this extended exchange matrix is the first one, the c-matrix is the identity matrix. So, the mutation which gives each of the other \( t-1 \) occurrences of the same extended exchange matrix are all red. But \( t-1 \geq r+1 \) contradicting the assumption that there are only \( r \) red mutations in the sequence.

These two lemmas imply the following.

Proposition 4.2.3. For every \( r, k \) there are at most finitely many \( r \)-reddening sequences disjoint from \( V^k \).

4.3. Disjunction. We will show that \( r \)-reddening sequences are disjoint from \( V^k \) for sufficiently large \( k \).

We use the fact that, in the tame case, there is a unique null root \( \eta \) and \( \tau \eta = \eta \). We also use the following formula from [4].

Theorem 4.3.1. For any (connected) tame hereditary algebra there is a positive integer \( m \) and, for every positive root \( \alpha \) of \( \Lambda \), there is an integer \( \delta(\alpha) \) called the defect of \( \alpha \) so that

\[
\tau^m \alpha = \alpha + \delta(\alpha) \eta.
\]

Furthermore, \( \delta(\alpha) \) is positive, negative or zero depending on whether \( \alpha \) is preinjective, preprojective or regular, respectively.

Let \( H(\eta) \) be the hyperplane in \( \mathbb{R}^n \) given by

\[
H(\eta) = \{ x \in \mathbb{R}^n : \langle x, \eta \rangle = 0 \}
\]

and let \( D(\eta) \) be the set of all \( x \in H(\eta) \) so that \( \langle x, \alpha \rangle \leq 0 \) for all preprojective roots \( \alpha \).
Lemma 4.3.2. $\tau D(\eta) = D(\eta)$.

Proof. We will show that $\tau^{-1}D(\eta) = D(\eta)$. $\tau^{-1}D(\eta)$ is the set of all $x \in H(\eta)$ so that $\langle \tau x, \alpha \rangle \leq 0$ for all preprojective $\alpha$. Since $\langle \tau x, \alpha \rangle = \langle x, \tau^{-1}\alpha \rangle$, this condition is equivalent to the condition that $\langle x, \alpha \rangle \leq 0$ for $\alpha$ preprojective but not projective. So, $D(\eta) \subseteq \tau^{-1}D(\eta)$.

But, for projective $\alpha$ and $x \in \eta^{-1}D(\eta)$, we have:

$$\langle x, \alpha \rangle = \langle x, \tau^{-m}\alpha - \delta(\tau^{-m}\alpha)\eta \rangle = \langle x, \tau^{-m}\alpha \rangle \leq 0.$$ 

So, $x \in D(\eta)$. This shows $\tau^{-1}D(\eta) = D(\eta)$. □

Proposition 4.3.3. If $x \in H(\eta)$ and $m$ is as above, the following are equivalent.

1. $x \in D(\eta)$.
2. $\langle x, \alpha \rangle \leq 0$ for all preprojective $\alpha \in S_m$.
3. $\langle x, \beta \rangle \geq 0$ for all preinjective $\beta \in S_m$.

Proof. Clearly, (1) implies (2). Since $-\tau^m$ sends the preprojective roots in $S_m$ to the preinjective roots in $S_m$, statement (2) about $x$ is equivalent to the following.

$$(3)' \langle \tau^m x, \beta \rangle \geq 0$$ for all preinjective $\beta \in S_m$.

Since $x \in D(\eta)$ if $\tau x \in D(\eta)$, (3)' is equivalent to (3). So, (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3)' $\Leftrightarrow$ (3).

Conversely, assume (2). Then, for any preprojective root $\alpha$ not in $S_m$, there is a positive integer $t$ so that $\tau^m \alpha$ is a preprojective root in $S_m$. Then

$$\langle x, \alpha \rangle = \langle x, \tau^{km}\alpha - k\delta(\alpha)\eta \rangle = \langle x, \tau^{km}\alpha \rangle \leq 0$$

by (2). So, $x \in D(\eta)$ proving that (1), (2), (3) are equivalent. □

Corollary 4.3.4. If $k \geq m$ then

1. $\mathcal{V}_k \cap H(\eta) = D(\eta)$.
2. $\mathcal{W}_k \cap H(\eta) = H(\eta) \setminus D(\eta)$.

Proof. (1) follows from the equivalence (1) $\Leftrightarrow$ (2) in the Proposition and (2) follows from the equivalence (1) $\Leftrightarrow$ (3) in the Proposition. □

Lemma 4.3.5. For every preprojective or preinjective root $\gamma$, there is a $k$ so that $\gamma \notin \mathcal{V}_k \setminus \mathcal{W}_k$.

Proof. Any preprojective $\gamma$ lies in $S_p$ for some $p$. Then $\langle \gamma, \gamma \rangle > 0$ and $\gamma \in W_p$. So $\gamma \notin \mathcal{V}_k \setminus \mathcal{W}_k$ for all $k \geq p$. Similarly, any preinjective $\gamma$ lies in $S_q$ for some $q$. Then $\tau \gamma \in S_{q+1}$ and $\langle \gamma, \tau \gamma \rangle < 0$. So, $\gamma \notin \mathcal{V}_k \setminus \mathcal{W}_k$ for all $k > q$. □

Lemma 4.3.6. Every cluster tilting object in the cluster category of mod-$\Lambda$ has at least one preprojective or preinjective summand.

Proof. The dimension vectors of the summands of any cluster tilting object are linearly independent. But regular roots all lie in the hyperplane $H(\eta)$. So, the summands of a cluster tilting object cannot all be regular. □

Lemma 4.3.7. Every reddening sequence is disjoint from $\mathcal{V}_k \setminus \mathcal{W}_k$ for sufficiently large $k$.

Proof. A reddening sequence consists of a finite sequence of cluster tilting objects each having at least one preprojective or preinjective summand. By Lemma 4.3.5, there is a $k$ so that none of these roots lies in $\mathcal{V}_k \setminus \mathcal{W}_k$. Then the reddening sequence stays in the complement of $\mathcal{V}_k \setminus \mathcal{W}_k$. □

Proposition 4.3.8. If a reddening sequence meets $\mathcal{V}_{rm} \setminus \mathcal{W}_{rm}$ then it has at least $r$ red mutations. So, every $r$-reddening sequence is disjoint from $\mathcal{V}_{(r+1)m} \setminus \mathcal{W}_{(r+1)m}$.
Proof. The statement is vacuously true for \( r = 0 \). So, suppose \( r \geq 1 \) and the statement holds for \( r - 1 \). Suppose given a reddening sequence with a cluster tilting object \( T_1 \) so that \( R(T_1) \subseteq \mathcal{V}_r \setminus \mathcal{W}_r \). There are two cases.

**Case 1:** \( R(T_1) \) lies on the negative side of \( D(\eta) \).

In the first case, the remainder of the reddening sequence somehow arrive at the positive side of \( H(\eta) \). By Lemma 13.7, the sequence is disjoint from some \( \mathcal{V}_k \setminus \mathcal{W}_k \) which contains \( D(\eta) \) by Corollary 4.3.8. So, the reddening sequence must pass through \( H(\eta) \setminus D(\eta) \) which is in \( \mathcal{W}_r \setminus \mathcal{V}_r \). Therefore, it must pass through one of the red walls \( D(\beta) \) of \( \mathcal{V}_r \) on the negative side of \( H(\eta) \). Let \( T_2, T_3 \) be the two cluster tilting objects in the reddening sequence with \( R(T_2) \subseteq \mathcal{V}_r, R(T_3) \nsubseteq \mathcal{V}_r \) and \( R(T_2) \cap R(T_3) \nsubseteq R(\beta) \cap \partial \mathcal{V}_r \).

Let \( x \in D(\beta) \cap \partial \mathcal{V}_r \) be a point in the interior of the wall separating \( R(T_2) \) and \( R(T_3) \). Then \( \langle x, \beta \rangle = 0 \) and \( \beta \) is a preinjective root in \( \mathcal{S}_r \). We claim that \( \beta \) does not lie in \( \mathcal{S}_{(r-1)m} \). Otherwise, \( \tau^m \beta = \beta + \delta(\beta) \eta \) lies in \( \mathcal{S}_r \) and we would have \( 0 \leq \langle x, \tau^m \beta \rangle = \langle x, \beta \rangle + \delta(\beta) \langle x, \eta \rangle < 0 \) using the fact that \( \delta(\beta) > 0 \) for preinjective \( \beta \) and \( \langle x, \eta \rangle < 0 \) in Case 1.

But \( x \in \mathcal{V}_r \setminus \mathcal{W}_r \subseteq \mathcal{V}_{(r-1)m} \setminus \mathcal{W}_{(r-1)m} \). Since \( \beta \nsubseteq \mathcal{S}_{(r-1)m} \), \( x \) does not lie on \( \partial \mathcal{V}_{(r-1)m} \).

So, \( x \) lies in the interior of \( \mathcal{V}_{(r-1)m} \). This implies that \( R(T_3) \) also lies in the interior of \( \mathcal{V}_{(r-1)m} \setminus \mathcal{W}_{(r-1)m} \) and on the negative side of \( D(\eta) \). By induction on \( r \), the rest of the reddening sequence has at least \( r - 1 \) red mutations. Since the mutation from \( T_2 \) to \( T_3 \) is red, the entire reddening sequence has at least \( r \) red mutations. This proves the proposition in Case 1.

**Case 2:** \( R(T_1) \) lies on the positive side of \( D(\eta) \).

In this case, we look at the part of the reddening sequence before \( T_1 \). By an analogous argument, there is a \( T_0 \) with \( R(T_0) \) in \( \mathcal{V}_{(r-1)m} \setminus \mathcal{W}_{(r-1)m} \) in the reddening sequence. We need at least one red mutation to get from \( T_0 \) to \( T_1 \) and, by induction on \( r \), we need \( r - 1 \) red mutations to get to \( T_0 \). This gives at least \( r \) red mutations in Case 2, just as in Case 1.

So, every reddening sequence which meets \( \mathcal{V}_r \setminus \mathcal{W}_r \) has at least \( r \) red mutations. \( \square \)

Propositions 4.2.3 and 4.3.8 conclude the proof of Theorem 4.0.5. By the Rotation Lemma we conclude the following.

**Theorem 4.3.9.** Let \( Q \) a valued quiver which is mutation equivalent to an acyclic valued quiver of tame representation type. Then, for any \( r \geq 0 \), \( Q \) admits only finitely many \( r \)-reddening sequences. In particular, \( Q \) has only finitely many maximal green sequences.

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Figure 5. (Proof of 4.3.8) Since reddening sequences cannot cross $D(\eta)$, we need to cross $r$ red walls (gray in figure) to escape from any interior point $x \in V_{rm}$ on the positive side of $D(\eta)$. We need to cross $r$ red walls (black in figure) to reach any interior point $y \in V_{rm} \setminus W_{rm}$ on the negative side of $D(\eta)$.

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