ON THE SYMMETRIC POWERS OF CUSP FORMS ON GL(2) OF ICOSAHEDRAL TYPE

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0. Introduction

In this Note, we prove three theorems. Throughout, $F$ will denote a number field with absolute Galois group $G_F = \text{Gal}(F/F)$, and the adele ring $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$. When $\rho$ is an irreducible continuous 2-dimensional $\mathbb{C}$ representation of $G_F$, one says that it is icosahedral, resp. octahedral, resp. tetrahedral, resp. dihedral when the projective image of $\rho(G_F)$ is $A_5$, resp. $S_4$, resp. $A_4$, resp. $D_{2m}$ for some $m \geq 1$. Such $\rho$ is said to be modular if and only if there exists a cuspidal automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of $GL_2(\mathbb{A}_F)$ such that $L(s, \rho) = L(s, \pi_f)$. Modularity is unknown (in general) only when $\rho$ is icosahedral, in which case $\rho$ is rational over $\mathbb{Q}(\sqrt{5})$. Denoting by $\tau$ the nontrivial automorphism of $\mathbb{Q}(\sqrt{5})$, we will say that $\rho$ is strongly modular if both $\rho$ and $\rho^\tau$ are modular. When $F$ is totally real and $\rho$ totally odd, which is the primary case of interest, modularity implies strong modularity (see below). In the Theorem below, $\text{sym}^m(\rho)$ denotes the symmetric $m$-th power of $\rho$, i.e., the composition of $\rho$ with the symmetric $m$-th power representation of $GL_2(\mathbb{C})$ into $GL_{m+1}(\mathbb{C})$.

The first main result is the following.

Theorem A. Let

$$\rho : G_F \to GL_2(\mathbb{C})$$

be a continuous irreducible, icosahedral representation which is strongly modular, i.e., for which there exists a cuspidal automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of $GL_2(\mathbb{A}_F)$, such that $L(s, \rho) = L(s, \pi_f)$. Then there exists a cuspidal representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $GL_6(\mathbb{A}_F)$ such that

$$L(s, \text{sym}^5(\rho)) = L(s, \Pi_f)$$
When $F = \mathbb{Q}$, many odd icosahedral representations $\rho$ of $\mathcal{G}_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, have been shown to be modular by R. Taylor, et al (BDST2001, Ta98, Ta2002, BS2002); The first example had been given by Buhler (Bu78). In these cases the associated $\pi$ is generated by a holomorphic newform $\phi$ of weight 1, and $\phi^\tau$ is again a holomorphic newform of weight 1. By a theorem of Deligne and Serre (D-S74), $\phi^\tau$ is associated to a 2-dimensional representation $\rho^\tau$, which must be isomorphic to $\rho^\tau$ by the Chebotarev density theorem. Hence modularity implies strong modularity for odd icosahedral representations of $\mathcal{G}_\mathbb{Q}$. (The situation is the same when the base field $\mathbb{Q}$ is replaced by a totally real field $F$ as long as $\rho$ is totally odd, and this is due to the analog of the Deligne-Serre theorem due to Wiles (Wiles88).)

By base change (AC, La80) we then get modular icosahedral representations of $\mathcal{G}_K$ for any cyclic extension $K$ of $\mathbb{Q}$. So our theorem applies to these cases with no hypothesis.

Given any 2–dimensional irreducible icosahedral representation $\rho$ of $\mathcal{G}_F$, one sees that $\text{sym}^m(\rho)$ is irreducible if and only if $m \leq 5$ (see Section 1), and the strong Artin conjecture, which is a part of the Langlands philosophy, predicts the existence of a cuspidal automorphic representation $\Pi_m$ of $GL_{m+1}(\mathbb{A}_F)$ for $m \leq 5$ with the same $L$–functions as $\text{sym}^m(\rho)$. When $\rho$ is strongly modular, the cuspidality of $\text{sym}^2(\rho)$ has been known for a long time by the work of Gelbart and Jacquet (GeJ79), and certain major recent works of H. Kim and F. Shahidi (KSh2002-1, KSh2002-2, K2001) establish this for $m = 3$ and 4. In fact, it is known by Kim (K2002) when $F = \mathbb{Q}$ and $\rho$ odd that every $\text{sym}^m(\rho)$ is attached to an automorphic form on $GL(m + 1)$. Briefly, for $m = 5$, the reason for this is that $\text{sym}^5(\rho)$ is twist equivalent to $\rho \otimes \text{sym}^2(\rho')$ where $\rho'$ is a Galois conjugate of $\rho$ (see Section 1).

Our main contribution here is to show that this $\Pi$ here is indeed cuspidal on $GL(6)/F$. We prove it in two different ways. One is to study the poles of the $L$–function, and the other, which is perhaps of independent interest, is to prove the following cuspidality criterion for the Kim–Shahidi automorphic transfer from $GL(2) \times GL(3)$ to $GL(6)$ (KSh2002-2), $(\pi, \eta) \mapsto \pi \boxtimes \eta$, when $\eta$ is a twist of the symmetric square of a cusp form on $GL(2)$. More precisely, we prove the following:

**Theorem B.** Let $\pi$, $\pi'$ be two cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$, and let $\Pi = \pi \boxtimes \text{sym}^2(\pi')$ be the associated isobaric automorphic representation of $GL_6(\mathbb{A}_F)$. 
Then $\Pi$ is cuspidal if and only if one of the following conditions hold:

1. $\pi = I_K^F(\chi)$ is dihedral for some quadratic extension field $K$ of $F$, $\pi'_K$, the base change of $\pi'$ to $K$, is not dihedral, and
   
   \[ \text{sym}^2(\pi'_K) \not\cong \text{sym}^2(\pi'_K) \otimes \chi^{-1}(\chi \circ \theta) \]
   where $\theta$ is the nontrivial automorphism of $K/F$;

2. $\pi$ is not dihedral, $\pi'$ is tetrahedral, or not of solvable polyhedral type, and $\text{Ad}(\pi)$ and $\text{Ad}(\pi')$ are not equivalent.

3. $\pi$ is not dihedral, $\pi'$ is octahedral, and $\text{Ad}(\pi)$ and $\text{Ad}(\pi')$ are not equivalent or twist equivalent by $\mu$ where $\mu$ is the global character corresponding to the class field $K$ which is a quadratic extension field of $F$ such that $\pi'_K$ is tetrahedral.

Recall that $\text{Ad}(\pi) \cong \text{sym}^2(\pi) \otimes \omega^{-1}_\pi$ where $\omega_\pi$ is the central character of $\pi$. Also, note that if $\pi'$ is octahedral, then $\mu$ is exactly the quadratic character such that

\[ \text{sym}^3(\pi') \cong \text{sym}^3(\pi') \otimes \mu \]

We say that a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ is called not of solvable polyhedral type ([RaWa2001]) if it is not dihedral, tetrahedral or octahedral. It is a theorem of Kim–Shahidi that if $\pi$ is not of solvable polyhedral type then $\text{sym}^m(\pi)$ is cuspidal for $m \leq 4$ ([KSh2002-2], [K2001]).

Theorem B is proved in Section 2 below and the two proofs of Theorem A will be given in Section 3.

Next, Recall that, a Landau–Siegel zero of an $L$–function with a functional equation and Euler product is a real zero of this $L$–function close to $s = 1$ (see [HRa95] and [Ra99]). Of course, the Generalized Riemann Hypothesis (GRH) implies the nonexistence of Landau–Siegel zeros for nice $L$–functions. Unfortunately, this is obtained for only a few cases. ([HRa95], [RaWa2001]).

Our third main result which will be proved in Section 4 is the following, where we mean by a cusp form on $GL(2)/F$ of strongly icosahedral type a cuspidal automorphic representation $\pi$ of $GL_2(\mathbb{A}_F)$ attached to a strongly modular icosahedral representation of $G_F$: 
**Theorem C.** Let $\pi$ be a cusp form on $GL(2)/F$ of strongly icoshedral type, and $\chi$ a idele character of $K$. Then $L(s, \text{sym}^m(\pi) \otimes \chi)$ has no Landau–Siegel zero unless $\text{sym}^m(\pi) \otimes \chi$ has a constituent of a trivial or quadratic character $Q$. If this happens, $m$ is even, $Q = \omega_m^{m/2} \chi^{m+1}$ and there is at most one Landau–Siegel zero which should come from the $L$–function of this character. When $F = \mathbb{Q}$ and $\pi$ is self dual, $L(s, \text{sym}^m(\pi))$ has no Landau–Siegel zero at all.

**Remark.** If $m < 12$, then the exceptional case will not happen so that $L(s, \text{sym}^m(\pi) \otimes \chi)$ has no Landau–Siegel zero. One can show by comparing central characters that if $\text{sym}^m(\pi)$ has a character as its constituent, then $m$ is even and this character should be $\omega_m^{m/2}$.

This theorem needs more precise structure theory (Theorem D in Section 3) of icoshedral representations (see Section 1 & 3, Bu78, FH91). The point (see also Bu78, K2002) is that, each twist of $\text{sym}^m(\pi)$ is an isobaric sum of the twists of the following (where $\pi^\tau$ is the Galois conjugate of $\pi$ by $\tau$ which is the nontrivial element of $\text{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$):

$$1, \pi, \pi^\tau, \text{sym}^2(\pi), \text{sym}^2(\pi^\tau),$$

(({A}))

$$\text{sym}^3(\pi), \pi \boxtimes \pi^\tau, \text{sym}^4(\pi), \text{sym}^5(\pi).$$

So it suffices to show the nonexistence of Landau–Siegel zero for the twist $L$–functions of ((A)). It is well known (HRa95, Stk74) that $L(s, \chi)$ has no Landau–Siegel zero unless $\chi$ is trivial or quadratic; The nonexistence of Landau–Siegel zero for the twist $L$–functions for $\pi$ or $\pi'$ is obtained from HRa95. From HRa95 and Ba97, we also obtain the same for the twist $L$–functions for $\text{sym}^2(\pi)$ and $\text{sym}^2(\pi^\tau)$. From RaWa2001, we get the same things for $\pi \times \pi^\tau$. Furthermore, if $\pi$ is self dual or is twist equivalent to a self dual automorphic representation, $L(s, \text{sym}^4(\pi))$ has no Landau–Siegel zero (RaWa2001).

So we get almost everything except for $\text{sym}^m(\pi) \otimes \chi$ for $m = 3, 4$ or $5$. This is finally done by using a useful criterion first formulated in HRa95 by D. Ramakrishnan and J. Hoffstein (also developed in RaWa2001), the modularity for $\text{sym}^m(\pi)$ (K2002), and Theorem A.

This Note was inspired by a talk of H. Kim based on K2002 at an MSRI conference in Banff during 2001. Of course, without the breakthrough works by Kim and Shahidi on the functoriality on $GL(2) \times GL(3)$, $\text{sym}^3$ (KSh2002-2) and $\text{sym}^4$ (K2001), we cannot get these results. Also, We would like to thank to D. Ramakrishnan and F. Shahidi.
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1. Structure Theory for Icosahedral Representations

In this section, we lay some facts about icosahedral representations. Recall that a Galois representation \( \rho : \mathcal{G}_F \to GL_2(\mathbb{C}) \) is said to be icosahedral if its image in \( PGL_2(\mathbb{C}) \) is isomorphic to \( A_5 \).

Let \( \tilde{A}_5 \) denote the nontrivial central extension of \( A_5 \) by \( \mathbb{Z}/2\mathbb{Z} \). It is unique since \( H^2(A_5, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \). In fact, \( \tilde{A}_5 \cong SL_2(\mathbb{F}_5) \).

Table 1. Character Table for \( SL_2(\mathbb{F}_5) \).

| Conj classes | \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) | \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) | \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) | \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \) | \( \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \) | \( \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \) | \( \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \) | \( \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix} \) |
|---|---|---|---|---|---|---|---|---|---|
| Size | 1 | 12 | 12 | 12 | 12 | 30 | 20 | 20 |
| \( U \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( V \) | 5 | 5 | 0 | 0 | 0 | 0 | 1 | -1 | -1 |
| \( W \) | 6 | 6 | 1 | 1 | -1 | -1 | 0 | 0 | 0 |
| \( X_1 \) | 4 | -4 | -1 | -1 | 1 | 1 | 0 | 1 | -1 |
| \( X_2 \) | 4 | 4 | -1 | -1 | -1 | -1 | 0 | 1 | 1 |
| \( W' \) | 3 | 3 | \( \frac{1+\sqrt{5}}{2} \) | \( \frac{1-\sqrt{5}}{2} \) | \( \frac{1+\sqrt{5}}{2} \) | \( \frac{1-\sqrt{5}}{2} \) | -1 | 0 | 0 |
| \( W'' \) | 3 | 3 | \( \frac{1-\sqrt{5}}{2} \) | \( \frac{1+\sqrt{5}}{2} \) | \( \frac{1-\sqrt{5}}{2} \) | \( \frac{1+\sqrt{5}}{2} \) | -1 | 0 | 0 |
| \( X' \) | 2 | -2 | \( \frac{1+\sqrt{5}}{2} \) | \( \frac{1-\sqrt{5}}{2} \) | \( \frac{1+\sqrt{5}}{2} \) | \( \frac{1-\sqrt{5}}{2} \) | 0 | 1 | -1 |
| \( X'' \) | 2 | -2 | \( \frac{1-\sqrt{5}}{2} \) | \( \frac{1+\sqrt{5}}{2} \) | \( \frac{1-\sqrt{5}}{2} \) | \( \frac{1+\sqrt{5}}{2} \) | 0 | 1 | -1 |

From Table 1 ([Bu78, NS80, FH91]), we see that there exist two self dual irreducible 2–dimensional representation of \( \tilde{A}_5 \), namely \( X' \) and \( X'' \) which are rational over \( \mathbb{Q}(\sqrt{5}) \). Furthermore, \( X' \) and \( X'' \) are conjugate by \( \tau \) which is the nontrivial automorphism of \( \mathbb{Q}(\sqrt{5})/\mathbb{Q} \). We use symbol \( \rho_{ico} \) for one of them, namely \( X' \). Hence we denote \( \rho_{ico}^{\tau} \) as \( X'' \).

Also, the irreducible representations of \( SL_2(\mathbb{F}_5) \) are the following: (For a proof the assertions, use the character table.)

- The trivial representation \( U \);
- The 2–dimensional representations \( \rho_{ico} = X' \), and \( \rho_{ico}^{\tau} = X'' \), rational over \( \mathbb{Q}(\sqrt{5}) \);
In this case, $G$ is an icosahedral representation of $G_F$, and $G$ denotes $\rho(G)$, then $G$ is generated by its commutator subgroup $G_0$ and its center $Z(G) \cong \mu_{2m}$, which is a group of roots of unity of order $2m$. Furthermore, $G_0$ is isomorphic to $\tilde{A}_5$ with center $\{ \pm I \}$, and $G \cong (G_0 \times \mu_{2m})/\{ \pm (I, 1) \}$. Hence each irreducible representation $\Lambda$ of $G$ can be expressed (uniquely) as $(\Lambda_0, \mu)$ where $\Lambda_0 = \Lambda|_G$ is an irreducible of $G_0$, and $\mu = \Lambda|_{\mu_{2m}}$ is a character of $\mu_{2m}$, and such that $\Lambda_0(-I) = \mu(-1)I$. Furthermore each such pair $(\Lambda_0, \mu)$ gives an irreducible representation of $G$.

**Remark:** Note that if $\rho$ is self-dual of degree 2, $m = 1$, then $\rho$ is either the standard representation or its Galois conjugation by $\tau \in Aut(C)$ sending $\sqrt{5}$ to $-\sqrt{5}$. Identify $G_0$ with $\tilde{A}_5 \cong SL_2(F_5)$, $\rho$ is $\rho_{ico}$ or $\rho_{ico}^\tau$.

**Proof of Proposition 1.1.**

First consider the case when $\det(G) = 1$, i.e. $\text{det}g = 1$ for all $g \in G$. In this case, $G$ is a covering group of $A_5$ of degree $n$, where $n = \#Z(G)$, while $Z(G) \subset Z(GL_2(C))$. As $\det(G) = 1$, $Z(G) \subset \{ \pm I \}$. More relations for the representations of $SL_2(F_5)$ (see also [K2002]):

- The symmetric $5$-the power of $\rho_{ico}$, namely $\text{sym}^5(\rho_{ico})$, which is of dimension 6, is equivalent to $W \cong \text{sym}^2(\rho_{ico}) \otimes \rho_{ico}^\tau \cong \text{sym}^2(\rho_{ico}^\tau) \otimes \rho_{ico}$; Also $\text{sym}^5(\rho_{ico}^\tau) \cong \text{sym}^5(\rho_{ico}^\tau) \cong W$;
- The symmetric $6$-the power of $\rho_{ico}$, namely $\text{sym}^6(\rho_{ico})$, is not irreducible, and is equivalent to $\text{sym}^2(\rho_{ico}^\tau) \oplus (\rho_{ico} \otimes \rho_{ico}^\tau)$;
- The symmetric $7$-the power of $\rho_{ico}$, namely $\text{sym}^7(\rho_{ico})$, is not irreducible either, and is equivalent to $\rho_{ico}^\tau \oplus \text{sym}^5(\rho_{ico})$.

For the general icosahedral representation, we have the following proposition:
Furthermore, since \( A_5 \) has no irreducible representation of dimension 2 (see [FH91]), \( Z(G) \) cannot be trivial. Thus \( G \) is a nonsplit central extension of \( A_5 \) by \( \mathbb{Z}/2\mathbb{Z} \). Thus \( G \cong \tilde{A}_5 \) (see the definition of \( \tilde{A}_5 \) at the beginning of this section).

In general case, all elements of \( G_0 = (G, G) \), which is the commutator group of \( G \), have determinant 1, and the image of \( G_0 \) in \( PGL_2(\mathbb{C}) \) is the same as the one of \( G \), and is also isomorphic to \( A_5 \) since \( (A_5, A_5) = A_5 \). We conclude that \( G = \langle G_0, Z(G) \rangle \cong (G_0 \times \mu_{2m})/\{ \pm (1, 1) \} \) where \( Z(G) \cong \mu_{2m} \) for some \( m \). From the discussion of the case \( \det G = 1 \), we have \( G_0 \cong \tilde{A}_5 \). The proof of the rest assertions of the proposition is then straightforward.

\[ \square \]

**Corollary 1.2.** If \( \Lambda_1, \Lambda_2 \) are two representations of \( G \) whose restrictions to \( G_0 \cong \tilde{A}_5 \) are equivalent, then they are twist equivalent by a character. In fact, if \( \Lambda_i = (\Lambda_0, \mu_i) \), then they are twist equivalent by \((1, \mu_2 \mu_i^{-1})\) which is a character of \( G \) factoring through \( \mu_{2m}/\{ \pm 1 \} \).

\[ \square \]

The following corollary describes all irreducible representations of \( G \) and some relations.

**Corollary 1.3.** Each irreducible representation of \( G \) is twist equivalent to one of the following:

\[ 1, \Lambda_{ico}, \Lambda'_{ico}, \text{sym}^2(\Lambda_{ico}), \text{sym}^2(\Lambda'_{ico}), \text{sym}^3(\Lambda_{ico}), \text{sym}^4(\Lambda_{ico}), \text{sym}^5(\Lambda_{ico}), \Lambda_{ico} \otimes \Lambda'_{ico} \]

where \( \Lambda_{ico}, \Lambda'_{ico} \) are two irreducible representations of \( G \) whose restrictions to \( G_0 \cong \tilde{A}_5 \) are \( \rho_{ico} \) and \( \rho'_{ico} \). Furthermore, \( \text{sym}^m(\Lambda_{ico}) \) and \( \text{sym}^m(\Lambda'_{ico}) \) are twist equivalent for \( m = 3, 4 \) and \( 5 \); \( \text{sym}^2(\Lambda'_{ico} \otimes \Lambda_{ico}), \text{sym}^2(\Lambda_{ico} \otimes \Lambda'_{ico}), \text{sym}^5(\Lambda_{ico}) \) and \( \text{sym}^5(\Lambda'_{ico}) \) are twist equivalent.

**Proof.** By Proposition 1.1 and Corollary 1.2 the first part is easy. For the rest part, restrict all representations involved to \( G_0 \), and apply Corollary 1.2. We carry out the proof of the twist equivalence of \( \text{sym}^6(\Lambda_{ico}) \) and \( \Lambda'_{ico} \otimes \text{sym}^2(\Lambda_{ico}) \) here, while the other relations are totally similar to deal with.

Without loss of generality, say the restriction of \( \Lambda_{ico} \) to \( G_0 \) is \( \rho_{ico} \). Then the restrictions of both sides are \( \text{sym}^5(\rho_{ico}) \) and \( \rho'_{ico} \otimes \text{sym}^2(\rho_{ico}) \) respectively. They are equivalent from the discussion in this section.
Applying Corollary 1.2, we get the twist equivalence of $\text{sym}^5(\Lambda_{ico})$ and $\Lambda'_{ico} \otimes \text{sym}^2(\Lambda_{ico})$.

Before concluding this section, we want to point out the following:

**Lemma 1.4.** If $m$ is even, then $\Lambda_{ico}$ and $\Lambda'_{ico}$ are not even twist equivalent to a self dual representation; Furthermore, there is no self dual 2–dimensional irreducible representation of $G$.

**Proof.** Each 2–dimensional representation $\rho$ of $G$ is written as $(\rho_0, \mu)$. Since $\rho_0$ is self dual, $\bar{\rho} = (\rho_0, \bar{\mu})$. If $\rho$ is real, $\mu$ must be also real, hence trivial or quadratic. However, as $\mu(-1)I = \rho_0(-I) = -I$ (since $\rho_0$ is either the standard representation or its Galois conjugation by $\tau$), and $\mu$ is of order divisible by 4, hence $\mu$ cannot be real.

2. Cuspidality Criterion for $\pi \boxtimes \text{sym}^2(\pi')$

In this section, we will prove Theorem B. Before this Note, no cuspidality criterion for the automorphic tensor product ($[KSh2002-1]$) on $GL(2) \times GL(3)$ was known.

Throughout this section, $\pi$ and $\pi'$ will be two cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$. First we will deal with the simpler case when $\pi$ is dihedral.

**Lemma 2.1.** Assume that $\pi = I^F_K(\chi)$ for some global character $\chi$ of $C_K$ where $K$ is a quadratic extension field of $F$. Let $\Pi = \pi \boxtimes \text{sym}^2(\pi')$. Then

$$\Pi = I^F_K(\text{sym}^2(\pi'_K) \otimes \chi)$$

where $\pi'_K$ is the base change of $\pi'$ to $K$.

Hence $\Pi$ is cuspidal if and only if $\pi'_K$ is not dihedral, and

$$\text{sym}^2(\pi'_K) \ncong \text{sym}^2(\pi'_K) \otimes \chi^{-1}(\chi \circ \theta)$$

**Remark:** As $\pi$ is cuspidal, $\chi \neq \chi \circ \theta$. If $\pi'_K$ is not dihedral or tetrahedral, then of course this lemma applies, as $\text{sym}^2(\pi_K)$ will not admit a nontrivial self twist.

**Proof.** The first statement is clear by the reciprocity law of automorphic inductions and base changes ($[AC, Cl86, HH95, La80]$).
Let $\eta_0 = \text{sym}^2(\pi'_K) \otimes \chi$. From Mackey’s criterion $\Pi = I^F_K(\eta_0)$ is cuspidal if and only if $\eta_0$ is cuspidal and $\eta_0 \not\sim \eta_0 \otimes \chi^{-1}(\chi \circ \theta)$ where $\theta$ is the nontrivial automorphism of $K/F$. Note that $\eta_0$ is cuspidal if and only if $\pi'_K$ is not dihedral. Hence the second statement is true.

□

From now on, we assume that $\pi$ is not dihedral. Of course, if $\pi'$ is dihedral, then $\text{sym}^2(\pi')$ is not cuspidal and so is $\pi \otimes \text{sym}^2(\pi')$. Thus, Theorem 3 is finally reduced to the case when $\pi$ and $\pi'$ are both nondihedral.

Let’s recall a useful theorem about Rankin–Selberg $L$–functions and isobaric decompositions ([JS90], [JPSS83], [La79-1], [La79-2]),

**Lemma 2.2. (Jacquet–Shalika, Langlands)**

1. Let $\Pi, \tau$ be two automorphic representations of $GL_n(\mathbb{A}_F)$ and $GL_m(\mathbb{A}_F)$ respectively. Assume that $\tau$ is cuspidal. Then the order of the pole of $L(s, \Pi \otimes \tilde{\tau})$ is the same of the multiplicity of $\tau$ occurring in the isobaric sum decomposition of $\Pi$.

2. Let $\Pi$ be an isobaric automorphic representation of $GL_n(\mathbb{A}_F)$. Then $L(s, \Pi \times \tilde{\Pi})$ has a pole of order $m = \sum_i m_i^2$ at $s = 1$, where $\Pi = \boxplus_i m_i \pi_i$ be the isobaric decomposition of $\Pi$, and $\pi_i$ are inequivalent cuspidal representations of smaller degree.

In particular, $m = 1$ if and only if $\Pi$ is cuspidal.

□

Now, we analyze $L(s, \Pi \otimes \tilde{\Pi})$ where $\Pi = \pi \otimes \text{sym}^2(\pi')$. Let $\omega$ and $\omega'$ be the central characters of $\pi$ and $\pi'$. Denote $\text{Ad}(\pi) = \text{sym}^2(\pi) \otimes \omega^{-1}$, $\text{Ad}(\pi') = \text{sym}^2(\pi') \otimes \omega'^{-1}$ and $A^4(\pi') = \text{sym}^4(\pi) \otimes \omega'^{-2}$. Note that $\text{Ad}(\pi), \text{Ad}(\pi')$ and $A^4(\pi')$ are self dual.

Hence, we have

$$\pi \boxtimes \tilde{\pi} = 1 \boxplus \text{Ad}(\pi)$$

and

$$\text{sym}^2(\pi') \boxtimes \text{sym}^2(\tilde{\pi}') = \text{Ad}(\pi') \boxtimes \text{Ad}(\pi') = 1 \boxplus \text{Ad}(\pi') \boxplus A^4(\pi')$$

Thus

$$\Pi \boxtimes \tilde{\Pi} = \pi \boxtimes \tilde{\pi} \boxtimes \text{sym}^2(\pi') \boxtimes \text{sym}^2(\tilde{\pi}')$$

$$= (1 \boxplus \text{Ad}(\pi)) \boxplus (1 \boxplus \text{Ad}(\pi') \oplus A^4(\pi'))$$
Hence
\[ L(s, \Pi \times \tilde{\Pi}) = \zeta_F(s)L(s, Ad(\pi))L(s, Ad(\pi'))L(s, A^4(\pi')) \]
\[ \cdot L(s, Ad(\pi) \times Ad(\pi'))L(s, Ad(\pi) \times A^4(\pi')) \]
Thus by Lemma 2.2, \( \Pi \) is cuspidal if and only if the order of the pole of \( L(\Pi \times \tilde{\Pi}) \) at \( s = 1 \) is 1, if and only if the other \( L \)-factors above other than \( \zeta_F(s) \) are holomorphic at \( s = 1 \). These lead to the following lemma: (Note that now \( Ad(\pi) \) and \( Ad(\pi') \) are cuspidal since \( \pi \) and \( \pi' \) are assumed to be nondihedral.)

**Lemma 2.3.** If \( \pi \) and \( \pi' \) are not dihedral, then \( \Pi = \pi \boxtimes \text{sym}^2(\pi') \) is cuspidal if and only if all the following hold:

1. \( Ad(\pi) \) and \( Ad(\pi') \) are not equivalent.
2. \( A^4(\pi') \) does not have the trivial character as a constituent.
3. \( A^4(\pi') \) does not have \( Ad(\pi) \) as a constituent.

\[ \square \]

**Lemma 2.4.** If \( \pi' \) is not of solvable polyhedral type, then (2) and (3) of Lemma 2.3 hold.

**Proof.** From [K2001], \( \text{sym}^4(\pi') \) is cuspidal and so is \( A^4(\pi') \). Thus (2) and (3) of Lemma 2.3 hold.

\[ \square \]

**Lemma 2.5.** (1) If \( \pi' \) is tetrahedral, then
\[ A^4(\pi') \cong Ad(\pi') \oplus \eta \oplus \eta^2 \]
where \( \eta \) is a cubic character such that
\[ \text{sym}^2(\pi') \cong \text{sym}^2(\pi') \otimes \eta \]
(2) If \( \pi' \) is octahedral, then
\[ A^4(\pi') \cong Ad(\pi') \otimes \mu \oplus \pi_0 \]
where \( \mu \) is a quadratic character such that
\[ \text{sym}^3(\pi') \cong \text{sym}^3(\pi') \otimes \mu \]
and \( \pi_0 = I_K^F(\chi_0) \) is some cuspidal dihedral automorphic representation of \( GL_2(\mathbb{A}_F) \) where \( K \) is the class field of \( \mu \).

**Proof.** See [Tu81], Theorem 3.3.7 of [KSh2002-2], and Section 5 of [RaWa2001].

\[ \square \]

**Proof of Theorem B**
The case when $\pi$ is dihedral is dealt with in Lemma 2.1. Now assume that $\pi$ and $\pi'$ are nondihedral.

First we prove the necessity. If $\Pi = \pi \boxtimes \text{sym}^2(\pi')$ is cuspidal, then Lemma 2.3 and 2.4 apply. Hence $\text{Ad}(\pi)$ and $\text{Ad}(\pi')$ are not equivalent. If $\pi'$ is octahedral with $\mu$ and $K$ described in Lemma 2.5, then $A^4(\pi')$ does not contain $\text{Ad}(\pi)$ as a constituent. Note that, $\text{Ad}(\pi') \otimes \mu$ is a constituent of $A^4(\pi')$ hence it is not equivalent to $\text{Ad}(\pi)$. Then the necessity is done.

Now the sufficiency. Assume first that $\text{Ad}(\pi)$ and $\text{Ad}(\pi')$ are not equivalent. (2) and (3) of Lemma 2.3 hold when $\pi'$ is not of solvable polyhedral type. Then in this case, Lemma 2.3 applies, hence $\Pi$ is cuspidal.

If $\pi'$ is tetrahedral, then from Lemma 2.3 the cuspidal constituents of $A^4(\pi')$ are $\text{Ad}(\pi')$ and two cubic characters. Hence (2) of Lemma 2.3 hold, and (1) and (3) are equivalent. So the sufficiency in this case is proved.

Finally, we deal with the case when $\pi'$ is octahedral. From Lemma 2.3 the only cuspidal constituent of $A^4(\pi')$ are $\text{Ad}(\pi') \otimes \mu$ and $I^F_K(\chi_0)$. So (2) of Lemma 2.3 hold. Thus if $\text{Ad}(\pi')$ and $\text{Ad}(\pi)$ are not equivalent or twist equivalent by $\mu$, then (1) and (3) hold, thus Lemma 2.3 applies. The sufficiency in this case is also obtained.

Done.

Remark. In fact, if $\pi'$ is octahedral, $K$ is the quadratic field extension such that $\pi'_K$ is tetrahedral, and $\chi_0$ is a cubic character of $C_K$ such that

$$\text{sym}^2(\pi'_K) \cong \text{sym}^2(\pi_K) \otimes \chi_0$$

then $A^4(\pi') = I^F_K(\chi_0)$.

3. Cuspidality of $\text{sym}^5(\pi)$ for $\pi$ Icosahedral

In this section, we prove Theorem A in two different ways.

Let $\rho$ be a strongly modular icosahedral representation of $G_F$ and $\pi$ the automorphic representation of $GL_2(A_F)$ associated with $\rho$. Then by the structure theory (Corollary 1.3), $\text{sym}^5(\rho)$ is twist equivalent to $\rho^* \otimes \text{sym}^2(\rho)$ and is irreducible. The automorphy of $\text{sym}^5(\rho)$ is known when $F = \mathbb{Q}$ and $\rho$ is odd ([K2002]). One immediately gets the same for our $\rho$, and we indicate how. By assumption, $\rho^*$ is modular, and since
sym²(ρ) is modular, sym²(ρ) ⊗ ρτ and hence sym⁵(ρ) is also modular by [K2002].

In view of this, Theorem A is a result of the following known proposition which is an analogue of Lemma 2.2 on the Galois side.

**Proposition 3.1.** If ρ is an irreducible Galois representation of \( G_F \), then \( L(s, \rho \otimes \rho^\vee) \) has a simple pole at \( s = 1 \). If \( \rho \) is modular, and \( \pi \) is the automorphic representation corresponding to \( \rho \), then \( \pi \) is cuspidal if and only if \( \rho \) is irreducible.

**Proof.** (cf. Tate [Tate84])

One knows that given any \( \mathbb{C} \)-representation \( \sigma \) of \( G_F \), we have

\[
-\text{ord}_{s=1} L(s, \sigma) = \dim_{\mathbb{C}} \text{Hom}_{G_F}(1, \sigma^\vee)
\]

Taking \( \sigma \) to be \( \rho \otimes \rho^\vee \), we see that the order of pole is given by

\[
\dim_{\mathbb{C}} \text{Hom}_{G_F}(1, \rho \otimes \rho^\vee) = \dim_{\mathbb{C}} \text{End}_{G_F}(\rho)
\]

which, by Schur’s lemma is 1 if and only if \( \rho \) is irreducible.

Hence the first statement is clear. In fact, for each Galois representation \( \Lambda = \sum \tau c_{\tau} \tau \) where \( \tau \) are inequivalent irreducible representations of \( G_F \), the order of pole of \( L(s, \rho \otimes \rho^\vee) \) at \( s = 1 \) is \( \sum C_{\tau}^2 \).

For the second part, we work with incomplete \( L \)-functions. Let \( S \) be a finite set of places of \( F \) containing archimedean ones and the ones where \( \rho \) (or \( \pi \)) is ramified. Consider

\[
L_S(s, \pi \times \bar{\pi}) = \prod_{v \in S} L(s, \pi_v \times \bar{\pi}_v)
\]
\[
L^S(s, \pi \times \bar{\pi}) = \prod_{v \not\in S} L(s, \pi_v \times \bar{\pi}_v)
\]
\[
L_S(s, \rho \otimes \rho^\vee) = \prod_{v \in S} L_v(s, \rho \otimes \rho^\vee)
\]
\[
L^S(s, \rho \otimes \rho^\vee) = \prod_{v \not\in S} L_v(s, \rho \otimes \rho^\vee)
\]

It is well known that each local \( L \)-factor \( L(s, \pi_v \times \bar{\pi}_v) \) is holomorphic and not vanishing at \( s = 1 \) hence the order of the pole of \( L^S(s, \pi \times \bar{\pi}) \) is the same as \( L(\pi \times \bar{\pi}) \) hence is 1 if and only if \( \pi \) is cuspidal from Lemma 2.2.

Furthermore, for any Galois representation \( \sigma \) and any nonarchimedean place \( v \) of \( F \), \( L_v(s, \sigma) = P(Np_v^{-s})^{-1} \) where \( P \) is a polynomial with all roots being of norm 1. Hence \( L_v(s, \sigma) \) is holomorphic and not vanishing at \( s = 1 \). Thus the order of the pole of \( L^S(s, \sigma) \) at \( s = 1 \) is exactly the
same as of $L(s, \sigma)$. Thus from the first statement of this proposition, $L^S(s, \rho \otimes \rho')$ has a simple pole if and only if $\rho$ is irreducible.

Finally, as $\rho$ is modular, we have for all $v \notin S$,

$$L(s, \pi_v \times \tilde{\pi}_v) = L_v(\rho \otimes \rho')$$

Hence

$$L^S(s, \pi \times \tilde{\pi}) = L^S(\rho \otimes \rho')$$

Thus, $\pi$ is cuspidal if and only if

$$-\text{ord}_{s=1} L^S(s, \pi \times \tilde{\pi}) = -\text{ord}_{s=1} L^S(s, \rho \otimes \rho') = 1$$

if and only if $\rho$ is irreducible.

□

Remark: In general we don’t know whether the following equality holds at ALL places $v$:

$$L(s, \pi_v \times \pi'_v) = L_v(s, \rho \otimes \rho')$$

where $\rho$ and $\rho'$ are two modular Galois representations with two automorphic representations $\pi$ and $\pi'$ associated to them respectively, although we’ve known this for those $v$ where $\rho_v$ and $\rho'_v$ are unramified. When $\rho$ and $\rho'$ are 2–dimensional, or one of $\rho$ and $\rho'$ is 2–dimensional and the other one is 3–dimensional, the automorphy of $\rho \otimes \rho'$ (Ra2000, KS2000–1) guarantees this for all $v$.

The second way to prove Theorem A is to apply the criterion established in the previous section. As we have seen, $\text{sym}^5(\rho)$ is twist equivalent to $\rho^\tau \otimes \text{sym}^2(\rho)$, consequently, $\text{sym}^5(\pi)$ is twist equivalent to $\pi^\tau \otimes \text{sym}^2(\pi)$, where $\pi^\tau = \pi \circ \tau$ is the Galois conjugation of $\pi$ by $\tau$. Thus the condition (2) of Theorem B holds hence this theorem applies. In fact, $Ad(\pi^\tau) = \text{sym}^2(\pi^\tau) \otimes \omega^{-1}_{\pi^\tau}$ and $Ad(\pi) = \text{sym}^2(\pi) \otimes \omega^{-1}_{\pi} = \text{sym}^2(\rho^\tau)$ and $\text{sym}^2(\rho)$ are not twist equivalent.

Now Theorem A is complete. Then we get a complete structure theory for strongly modular icosahedral representations.

Notation: Let $\pi$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ of strongly icosahedral type, i.e., $\pi$ is associated to a strongly modular icosahedral representation $\rho$ of $G_F$. Denote $M_{\text{ico}}(\pi)$ as the set of irreducible admissible representations generated by $\pi$ and $\pi^\tau$ via isobaric sums, Rankin–Selberg products, twists and symmetric powers, where $\pi^\tau$ is the cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ associated to $\rho^\tau$.
Theorem D. (1) all elements of $M_{\text{ico}}(\pi)$ are isobaric sums of the twists of the set $MG_{\text{ico}}(\pi)$ consisting of the following representations:

1. $\pi$, $\pi^\tau$, $\text{sym}^2(\pi)$, $\text{sym}^2(\pi^\tau)$,
2. $\text{sym}^3(\pi)$, $\pi \boxtimes \pi^\tau$, $\text{sym}^4(\pi)$, $\text{sym}^5(\pi)$.

Furthermore, each two elements in $MG_{\text{ico}}(\pi)$ are not twist equivalent. $\text{sym}^m(\pi)$ and $\text{sym}^m(\pi^\tau)$ are twist equivalent for $m = 3, 4$ and $5$. Also, $\text{sym}^5(\pi), \text{sym}^5(\pi^\tau), \pi^\tau \boxtimes \text{sym}^2(\pi)$ and $\pi \boxtimes \text{sym}^2(\pi^\tau)$ are twist equivalent.

(2) All elements in $MG_{\text{ico}}(\pi)$ are automorphic. As a corollary, all elements in $M_{\text{ico}}(\pi)$ are automorphic.

(3) All elements in $MG_{\text{ico}}(\pi)$ are cuspidal.

Remark: This theorem was first formulated by Kim in [K2002]. The proof except for the cuspidality of $\text{sym}^5(\pi)$ was also due to him.

Proof of Theorem D

Let $\rho = \rho_\pi$ be the odd icosahedral representation associated to $\pi$. Then $\rho$ and $\rho^\tau$ can be viewed as representations of $G$ which is the image of $\rho$. Hence $\rho = \Lambda_{\text{ico}}$ or $\Lambda_{\text{ico}}'$ (see Corollary 1.3), and all representations obtained from $\rho$ and $\rho^\tau$ via twists, direct sums, tensor products and symmetric powers are also viewed as representations of $G$. Then Corollary 1.3 applies, and thus (1) is proved.

For (2) and (3), since $\pi$ and $\pi^\tau$ are not of solvable polyhedral type, $\text{sym}^m(\pi)$ and $\text{sym}^m(\pi^\tau)$ are cuspidal for $m = 2, 3$ and $4$ ([Gal79], [KSh2002-2], [K2001]). Also, $\pi \boxtimes \pi^\tau$ is cuspidal ([Ra2000]), and $\pi^\tau \boxtimes \text{sym}^2(\pi)$ is automorphic ([KSh2002-2]) and cuspidal (Theorem A or Theorem B). Done.

Before we end this section, we would like to point out a result of H. Kim in [K2002] which asserts that $\text{sym}^4(\pi)$ is monomial, thus giving an example of non-normal quintic automorphic induction. Before this result, all known examples of automorphic induction were for solvable extension ([AC],[JPSS79], [Ha98] and [Tu81]).

Theorem 3.2. (H. Kim)

Suppose that $K$ is an $A_5$–extension of $\mathbb{Q}$ satisfying the criteria as in [BDST2001] or [La2002], and that $\pi$ be a cuspidal automorphic representation of strongly icosahedral type lifted from $K/\mathbb{Q}$. Let $E$ be a non–normal quintic extension of $\mathbb{Q}$ in $K$ such that $\text{Gal}(K/E)$ is $A_4$. Let $N$ be the unique cyclic cubic extension of $E$ in $K$. Let $\chi$ be the global character of $C_E$ attached to the cubic extension $N/E$.
Then $\Gamma_E(\chi)$ is equivalent to $A^4(\pi) = \text{sym}^4(\pi) \otimes \omega^{-2},$ hence is automorphic.

□

4. LANDAU–SIEGEL ZEROS OF $L(s, \text{sym}^m(\pi) \otimes \chi)$

In this section, the notations are the same as in the previous section. Let us first quote the following useful criterion which is always used for showing non–existence of Landau–Siegel zeros.

Proposition 4.1. ([HRa95])

Let $\pi$ be an isobaric automorphic representation of $GL_n(\mathbb{A}_F)$ with $L(s, \pi \times \bar{\pi})$ having a pole of order $r \geq 1$ at $s = 1$. Then there is an effective constant $c \geq 0$ depending on $n$ and $r$, such that $L(s, \pi \times \bar{\pi})$ has at most $r$ real zeros in the interval

$$J := \{ s \in \mathbb{C} \mid 1 - c/\log M(\pi \times \bar{\pi}) < \Re(s) < 1 \}.$$

Furthermore, if $L(s, \pi \times \bar{\pi}) = L_1(s)^k L_2(s)$ for some nice $L$–series $L_1(s)$ and $L_2(s)$ with $k > r$, and $L_2(s)$ holomorphic in $(t, 1)$ for some fixed $t \in (0, 1)$, then $L_1(s)$ has no zeros in $J$.

□

Proof of Theorem C

From Theorem D all $\text{sym}^m(\pi) \otimes \chi$ are automorphic. Thus it suffices to prove the nonexistence of Landau–Siegel zero of $L(s, \Pi \otimes \chi)$, where $\Pi$ is $\pi$, $\pi^*$, $\text{sym}^2(\pi)$, $\text{sym}^2(\pi^*)$, $\text{sym}^3(\pi)$, $\text{sym}^4(\pi)$, $\text{sym}^5(\pi)$, or $\pi \boxtimes \pi^*$. If $\Pi \otimes \chi$ is not self dual, then it has no Landau–Siegel zero ([HRa95]). So we need only to consider the case when $\Pi \otimes \chi$ is self dual.

$L(s, \pi \otimes \chi), L(s, \pi^* \otimes \chi)$ have no Landau–Siegel zero ([HRa95]).

$L(s, \text{sym}^2(\pi) \otimes \chi)$ has no Landau–Siegel zero. In fact, when $\text{sym}^2(\pi) \otimes \chi$ is self dual, its central character is either trivial or quadratic. Thus, the non–existence of Landau–Siegel zero follows from [HRa95] and [Ba97]. When $\chi$ is trivial, we can also get this from [GHL94].

$L(s, \text{sym}^m(\pi) \otimes \chi)$ has no Landau–Siegel zero for $m = 3, 4, 5$. This follows from the Lemma 4.2

(To be continued.)
Lemma 4.2. Let $\pi$ be a nondihedral automorphic representation of $GL(2)$ over $F$ such that $\text{sym}^{m+2}(\pi)$ and $\text{sym}^{m-2}(\pi)$ are automorphic and $\text{sym}^m(\pi)$ are cuspidal automorphic. Then $L(s, \text{sym}^m(\pi) \otimes \chi)$ has no Landau–Siegel zero for any Hecke character $\chi$ of $K$.

**Proof of Lemma 4.2.**
Denote $\omega = \omega_\pi$ as the central character of $\pi$.

If $\text{sym}^m(\pi) \otimes \chi$ is not self dual, then its $L$–function has no Landau–Siegel zero.

Now assume that $\text{sym}^m(\pi) \otimes \chi$ is self dual. Let $\Pi = 1 \boxplus (\text{sym}^m(\pi) \otimes \chi) \boxplus (\text{sym}^2(\pi) \otimes \omega^{-1})$, then $\Pi$ is self dual, and $\Pi$ is an isobaric sum of three cuspidal representations. Hence $L(s, \Pi \times \Pi)$ has a pole of order 3 at 1.

However,

$$L(s, \Pi \times \Pi) = \zeta_s(s)L(\text{sym}^m(\pi) \otimes \chi)^2L(s, \text{sym}^2(\pi) \otimes \omega^{-1})^2 \\
\times L(s, \text{sym}^2(\pi) \otimes \omega^{-1} \times \text{sym}^m(\pi) \otimes \chi)^2L(s, \text{sym}^2(\pi) \times \text{sym}^2(\pi) \otimes \omega^{-2}) \\
\times L(s, \text{sym}^m(\pi) \times \text{sym}^m(\pi) \otimes \chi^{-2})$$

$$= \zeta_s(s)L(\text{sym}^m(\pi) \otimes \chi)^4L(\text{sym}^2(\pi) \otimes \omega^{-1})^2 \\
\times L(\text{sym}^{m+2}(\pi) \otimes \chi^m \omega^{-1})^2L(\text{sym}^{m-2}(\pi) \otimes \chi\omega)^2 \\
\times L(s, \text{sym}^2(\pi) \times \text{sym}^2(\pi) \otimes \omega^{-2})L(s, \text{sym}^m(\pi) \times \text{sym}^m(\pi) \otimes \chi^{-2})$$

since

$$\text{sym}^m(\pi) \boxtimes \text{sym}^2(\pi) = \\
\text{sym}^m(\pi) \boxplus \text{sym}^{m+2}(\pi) \otimes \omega^{-1} \boxplus \text{sym}^{m-2}(\pi) \otimes \omega$$

Hence $L(s, \text{sym}^m(\pi) \otimes \chi)^4$ divides $L(s, \Pi \times \Pi)$, and the rest factors are all automorphic $L$–functions.

Thus, by Proposition 4.1 (also [HRa95]), $L(s, \text{sym}^m(\pi) \otimes \chi)$ has no Landau–Siegel zero.

\[ \square \]

**Remark:** The non-existence of Landau–Siegel zero of $L(s, \text{sym}^4(\pi))$ when $\pi$ is self dual is followed from Theorem B of [RaWa2001]. Unfortunately, when $\pi = \pi(\rho)$ is a form corresponding to an odd icosahedral representation, $\pi$ cannot be self dual.

**Proof of Theorem C (Continued).**

Finally, the nonexistence of Landau–Siegel zero for $\pi \boxtimes \pi^\tau \otimes \chi$ follows from Theorem A of [RaWa2001] since $\pi$, $\pi^\tau$ are not dihedral and not
twist equivalent. (Here the form $\pi \boxtimes \pi^\tau$ is automorphic on $GL(4)$ (see [Ra2000]).)

The proof of the remaining statements are also straightforward.

\[\square\]

**Proposition 4.3.** Let $\pi$ be a cusp form on $GL(2)$ of strongly icosahedral type. If a character $\omega'$ is an isobaric constituent of $\text{sym}^m(\pi)$, then $m$ is even and $\omega' = \omega_{\pi}^m/2$. Also, $\text{sym}^m(\pi)$ has no character as its constituent for $m < 12$. Hence $L(s, \text{sym}^m(\pi) \otimes \chi)$ has no Landau–Siegel zero.

**Proof.**

It is convenient to work on the Galois side. Let $\rho$ be an odd icosahedral representation. Want to prove that if $\chi$ is contained in $\text{sym}^m(\rho)$, then $m$ is even, and $\chi = \det \rho^m/2$. In fact, writing $\rho = (\rho_0, \mu)$ as in Proposition 1.1 we have $\text{sym}^m(\rho) = (\text{sym}^m(\rho_0), \mu^m)$, and each irreducible component should be $(1, \mu^m)$ as $G_0 \cong \tilde{A}_5$ has no nontrivial 1–dimensional representations (see Section 1). Thus $m$ is even, and $(1, \mu^m) = \det \rho^m/2$. Translating above to the automorphic side, and noticing that $\omega_{\pi}$ is the global character corresponding to $\det \rho$, we get the first statement.

For the second, we need to verify the assertion for $m = 6, 8$ and 10. We again work on the Galois representation side. We want to prove that $\text{sym}^m(\rho)$ has no constituent of character. By the structure theory in Section 1 it suffices to show that $\text{sym}^m(\rho_0)$ does not contain trivial representations. This is true since

$$B = (\text{sym}^{m/2}(\rho_0))^{\otimes 2} = 1 \oplus (\otimes_{k=1}^{m/2} \text{sym}^{2k}(\rho_0)),$$

which contains 1 as multiplicity 1 since $\text{sym}^{m/2}(\rho_0)$ is irreducible. Hence $\text{sym}^m(\rho_0)$ cannot contain 1.

\[\square\]

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