Logarithmic $N = 1$ superconformal field theories

M. Khorrami$^{1,3,*}$, A. Aghamohammadi$^{2,3}$, A. M. Ghezelbash$^{2,3}$

1 Department of Physics, Sharif University of Technology P.O. Box 11365-9161, Tehran, Iran.
2 Department of Physics, Alzahra University, Tehran 19834, Iran.
3 Institute for Studies in Theoretical Physics and Mathematics, P.O.Box 5531, Tehran 19395, Iran.
* mamwad@netware2.ipm.ac.ir

Abstract

We study the logarithmic superconformal field theories. Explicitly, the two–point functions of $N = 1$ logarithmic superconformal field theories (LSCFT) when the Jordan blocks are two (or more) dimensional, and when there are one (or more) Jordan block(s) have been obtained. Using the well known three-point functions of $N = 1$ superconformal field theory (SCFT), three–point functions of $N = 1$ LSCFT are obtained. The general form of $N = 1$ SCFT's four–point functions is also obtained, from which one can easily calculate four–point functions in $N = 1$ LSCFT.
1 Introduction

It has been shown by Gurarie [1], that conformal field theories (CFT’s) whose correlation functions exhibit logarithmic behaviour, can be consistently defined. It is shown that if in the OPE of two local fields, there exist at least two fields with the same conformal dimension, one may find some special operators, known as logarithmic operators. As discussed in [1], these operators with the ordinary operators form the basis of a Jordan cell for the operators $L_i$. In some interesting physical theories, for example dynamics of polymers [2], the WZNW model on the $GL(1,1)$ super-group [3], WZNW models at level 0 [4, 5, 6], percolation [7], the Haldane-Rezayi quantum Hall state [8], and edge excitation in fractional quantum Hall effect [9], one can naturally find logarithmic terms in correlators. Recently the role of logarithmic operators have been considered in study of some physical problems such as: $2D$–magnetohydrodynamic turbulence [10, 11, 12], $2D$–turbulence [13, 14], $c_{p,1}$ models [15, 16], gravitationally dressed CFT’s [17], and some critical disordered models [18, 19]. They play a role in the so called unifying $W$ algebra [20] and in the description of normalizable zero modes for string backgrounds [1]. Logarithmic conformal field theories for $D$ dimensional case ($D > 2$) has also been studied [21].

The basic properties of logarithmic operators are that they form a part of the basis of the Jordan cell for $L_i$’s, and in the correlator of such fields there is a logarithmic singularity [1] [18]. It has been shown that in rational minimal models two fields with the same dimensions, don’t occur [11]. In [23] assuming conformal invariance two– and three–point functions for the case of one or more logarithmic fields in a block, and one or more sets of logarithmic fields have been explicitly calculated. Regarding logarithmic fields formally as derivatives of ordinary fields with respect to their conformal dimension, $n$–point functions containing logarithmic fields have been calculated in terms of those of ordinary fields. These have been done when conformal weights belong to a discrete set. In [24], logarithmic conformal
field theories with continuous weights have been considered. The first study of a logarithmic superconformal field theory was carried out in [4, 5]. The WZNW model for SU(2) at the level \( k \) is equivalent to the bosonic sector of the supersymmetric WZNW model at level \( k + 2 \) [26, 27]. In [5] using this equivalence and the results of the WZNW model for SU(2) at level \( k = 0 \), conformal blocks and OPE’s of the supersymmetric SU\(_2\)(2) have been obtained. For this supersymmetric case, some OPE’s contain logarithmic terms.

We want to study the general form of correlation functions of any LSCFT. To do this one should know the general form of correlation functions of any SCFT. The general form of three–point functions of SCFT’s has been obtained in [25]. So, at first we construct the general form of four–point functions of SCFT’s. Super–four–point functions for the supersymmetric WZNW models are constructed in [27], in the special case of SU(N) and O(N) symmetry, using superconformal and super–Kac-Moody Ward identities. It can be easily seen that these results are in agreement with our general results.

In this article, we construct two–point functions of \( N = 1 \) LSCFT when the Jordan blocks are two or more dimensional, and when there are one or more Jordan blocks. For \( n \)–point functions (\( n \geq 3 \)), the logarithmic correlation functions can be obtained through formal differentiation of their analogues in ordinary SCFT’s with respect to the superconformal weights. Using the well known three-point functions of SCFT’s, one can easily obtain three–point functions of LSCFT’s. We also obtain the general form of \( N = 1 \) SCFT’s four–point functions, which can be used for calculating four–point functions in \( N = 1 \) LSCFT.
2 Quasi–superprimary operators

A superprimary operator with conformal weight \( \Delta \), is an operator satisfying

\[
[L_n, \Phi(z, \theta)] = [z^{n+1} \partial + (n+1)(\Delta + \frac{1}{2} \theta \delta) z^n] \Phi,
\]

and

\[
[G_r, \Phi(z, \theta)] = \{z^{r+1/2} \partial - \theta[z^{r+1/2} \partial + (2r + \Delta) z^{r-1/2}]\} \Phi.
\]

Here \( L_n \)'s and \( G_r \)'s are the generators of the super Virasoro algebra satisfying

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},
\]

\[
[L_m, G_r] = (\frac{m}{2} - r)G_{m+r},
\]

and

\[
\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}.
\]

Throughout this article, the subscripts of \( G \)'s are half-integers, so that we are studying the Neveu–Schwarz sector. The superfield \( \Phi \) is written as

\[
\Phi(z, \theta) = \phi(z) + \theta \psi(z),
\]

and

\[
\partial := \frac{\partial}{\partial z}, \quad \delta := \frac{\partial}{\partial \theta}.
\]

These, in fact, define a superprimary chiral operator. One can similarly define a complete superprimary operator \( \Phi(z, \bar{z}, \theta, \bar{\theta}) \) through

\[
\Phi(z, \bar{z}, \theta, \bar{\theta}) = \phi(z, \bar{z}) + \theta \psi(z) + \bar{\theta} \bar{\psi}(\bar{z}) + \theta \bar{\theta} F(z, \bar{z}),
\]
relations (1) and (2), and obvious analogous relations with $\bar{L}_n$’s and $\bar{G}_r$’s.

Now, suppose that the component field $\phi(z)$ has a logarithmic counterpart (or quasi–primary field) $\phi'(z)$:

$$[L_n, \phi'(z, \theta)] = [z^{n+1} \partial + (n + 1)\Delta]z^n\phi' + (n + 1)z^n\phi(z).$$  \hspace{1cm} (9)

Our aim is show that $\phi'$ is the first component of a superfield, which is the formal derivative of the superfield $\Phi(z, \theta)$. To do so, define the fields $\psi_r'$ through

$$[G_r, \phi'(z)] =: z^{r+1/2}\psi'_r(z).$$  \hspace{1cm} (10)

Acting on both sides with $L_m$ using the Jacobi identity, and using (9), (1), and (4), we have

$$[L_m, \psi'_r(z)] = (\frac{m}{2} - r)z^m(\psi'_{m+r} - \psi'_r) + [z^{m+1} \partial + (m + 1)(\Delta + \frac{1}{2})z^m]\psi'_r + (m + 1)z^m\psi.$$  \hspace{1cm} (11)

Demanding

$$[L_{-1}, \psi'_r(z)] = \partial \psi'_r(z),$$  \hspace{1cm} (12)

it is seen that

$$\psi'_r = \begin{cases} 
\psi', & r \geq -1/2 \\
\psi'', & r \leq -3/2.
\end{cases}$$  \hspace{1cm} (13)

Then, equating $[L_1, \psi'_{-5/2}]$ and $[L_1, \psi'_{-3/2}]$, we arrive at

$$\psi'' = \psi'.$$  \hspace{1cm} (14)

So we have one welldefined field $\psi'$ satisfying

$$[G_r, \phi'(z)] = z^{r+1/2}\psi'(z),$$  \hspace{1cm} (15)

and

$$[L_m, \psi'(z)] = [z^{m+1} \partial + (m + 1)(\Delta + \frac{1}{2})z^m]\psi' + (m + 1)z^m\psi.$$  \hspace{1cm} (16)
To the end, one can calculate \( \{G_r, \psi'(z)\} \) through the Jacobi identity. The result is

\[
\{G_r, [G_r, \phi']\} = \frac{1}{2}\{[G_r, G_r], \phi']\,
\]

\[
= [L_{2r}, \phi'],
\]

\[
= \{z^{2r+1} \partial + (2r + 1)\Delta z^{2r}\} \phi' + (2r + 1)z^{2r}\phi,
\]

or

\[
\{G_r, \psi'\} = \{z^{r+1/2} \partial + (2r + 1)\Delta z^{r-1/2}\} \phi' + (2r + 1)z^{r-1/2}\phi.
\]

Now, combining \( \phi' \) and \( \psi' \) in the superfield

\[
\Phi'(z, \theta) := \phi'(z) + \theta \psi'(z),
\]

and using the action of super Virasoro generators on component fields \( \phi' \) and \( \psi' \) to write the action of super Virasoro generators on the superfield \( \Phi' \), we arrive at

\[
[L_n, \Phi'(z, \theta)] = \{z^{n+1} \partial + (n + 1)(\Delta + \frac{1}{2} \theta \delta)z^n\} \Phi' + (n + 1)z^n\Phi,
\]

and

\[
[G_r, \Phi'(z, \theta)] = \{z^{r+1/2} \partial - \theta[z^{r+1/2} \partial + (2r + 1)\Delta z^{r-1/2}]\} \Phi' - (2r + 1)z^{r-1/2}\theta \Phi.
\]

It is easy to see that (20) and (21) are formal derivatives of (1) and (2) with respect to \( \Delta \):

\[
\Phi' = \frac{d\Phi}{d\Delta}
\]

We call the field \( \Phi' \) a quasi–superprimary field, and the two superfields \( \Phi \) and \( \Phi' \) a two dimensional Jordanian block of quasi–primary fields. This has an obvious generalisation to \( m \) dimensional Jordanian blocks:

\[
[L_n, \Phi^{(i)}] = \{z^{n+1} \partial + (n + 1)(\Delta + \frac{1}{2} \theta \delta)z^n\} \Phi^{(i)} + (n + 1)z^n\Phi^{(i-1)}, \quad 1 \leq i \leq m - 1,
\]
and

\[ [G_r, \Phi^{(i)}] = \{z^{r+1/2} - \theta[z^{r+1/2} \partial + (2r+1) \Delta z^{-1/2}]} \} \Phi^{(i)} - (2r+1) z^{r-1/2} \theta \Phi^{(i-1)}, \quad 1 \leq i \leq m-1, \quad (24) \]

where \( \Phi^{(0)} \) is a superprimary field. It is easy to see that (23) and (24) are satisfied through the formal relation

\[ \Phi^{(i)} = \frac{1}{i!} d^i \Phi^{(0)}. \quad (25) \]

### 3 Green functions of the Jordanian blocks

Consider two Jordanian blocks of quasi–superprimary fields \( \Phi_1 \) and \( \Phi_2 \), with the same conformal weight \( \Delta \), which are \( p \)- and \( q \)-dimensional, respectively. Invariance of \( \langle \Phi_1^{(i)}(z_1, \theta_1) \Phi_2^{(j)}(z_2, \theta_2) \rangle \) with respect to \( L_{-1} \) and \( G_{-1/2} \) shows that the correlator depends only on

\[ z_{12} := z_1 - z_2 - \theta_1 \theta_2. \quad (26) \]

It is thus sufficient to calculate \( \langle \phi_1^{(i)}(z_1) \phi_2^{(j)}(z_2) \rangle \) to obtain the correlator of the superfields. From (23), however, we have

\[ \langle \phi_1^{(i)}(z_1) \phi_2^{(j)}(z_2) \rangle = \begin{cases} (z_1 - z_2)^{-2\Delta} \sum_{k=0}^{i+j-n} \frac{(-2)^k}{k!} a_{n-k} [\log(z_1 - z_2)]^k, & i + j \geq \max(p, q) \\ 0, & i + j < \max(p, q). \end{cases} \quad (27) \]

as \( \phi_1 \) and \( \phi_2 \) are Jordanian blocks of quasi–primary fields. From (27) we have

\[ \langle \Phi_1^{(i)}(z_1, \theta_1) \Phi_2^{(j)}(z_2, \theta_2) \rangle = \begin{cases} (z_{12})^{-2\Delta} \sum_{k=0}^{i+j-n} \frac{(-2)^k}{k!} a_{n-k} [\log(z_{12})]^k, & i + j \geq \max(p, q) \\ 0, & i + j < \max(p, q). \end{cases} \quad (28) \]

It is also obvious that such a correlator is nonzero, only if the weights of the blocks are identical.

The three point functions of Jordanian blocks can be easily obtained by formal differentiation of the three point functions of the primary superfield [23]. The general form of the latter has been obtained in
\[ < \Phi_1(z_1, \theta_1)\Phi_2(z_2, \theta_2)\Phi_3(z_3, \theta_3) > = \prod_{i<j}(z_{ij})^{\Delta - 2\Delta_i - 2\Delta_j} (a + bW), \] (29)

where

\[ \Delta := \sum_i \Delta_i, \] (30)

\[ W := \frac{\theta_1 z_{23} - \theta_2 z_{13} + \theta_4 z_{12} + \theta_1 \theta_2 \theta_3}{\sqrt{z_{12} z_{13} z_{23}}}, \] (31)

and \( a, b \) are two undetermined constants (\( b \) is Grassman valued). To obtain \( < \Phi_1^{(i)} \Phi_2^{(j)} \Phi_3^{(k)} > \), one only needs to perform differentiation \( i \) times with respect to \( \Delta_1 \), \( j \) times with respect to \( \Delta_2 \), and \( k \) times with respect to \( \Delta_3 \). In this process, \( a \) and \( b \) are also treated formally as functions of \( \Delta \)'s, so that each differentiation introduces two more undetermined constants.

The process of obtaining four point functions is exactly the same. First, one must obtain the general four point functions of super primary fields. To do so, one observes that the combination

\[ Y := \prod_{i<j}(z_{ij})^{\Delta/3 - \Delta_i - \Delta_j}, \] (32)

satisfies the equations obtained through the action of the OSP(2\( |1 \)) subalgebra of the super Virasoro algebra \( (L_{\pm 1}, L_0, G_{\pm 1/2}) \) on the correlator \( < \Phi_1\Phi_2\Phi_3\Phi_4 > \). One must now find all OSP(2\( |1 \)) invariants functions constructed from \( z_i \)'s and \( \theta_i \)'s. The first OSP(2\( |1 \)) invariant is the obvious modification of the anharmonic ratio \( x \), defined as

\[ x := \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \] (33)

It is easy to see that

\[ X := \frac{z_{12} z_{34}}{z_{13} z_{24}} \] (34)

is OSP(2\( |1 \)) invariant. The zeroth order term of \( X \) with respect to \( \theta_i \)'s is \( x \). Other OSP(2\( |1 \)) invariants are obtained from combinations odd with respect to Grassman variables. These are analogues of \( W \), eq.
There are four combinations, but not all of them are independent. One can show that the cubic terms in \( \theta \)'s are determined as linear combinations of the linear terms, and that two linear terms in \( \theta \)'s are determined in terms of the other two terms. The coefficients of these expansions are functions of \( X \). The last \( \text{OSP}(2|1) \) invariant is an even function of the Grassman variables, without a term of zeroth Grassman order. One can find it to be

\[
V := \frac{\theta_1 \theta_2 z_{34}}{z_{13} z_{24}} + \frac{\theta_3 \theta_4 z_{12}}{z_{13} z_{24}} + \frac{\theta_1 \theta_4 z_{23}}{z_{13} z_{24}} - \frac{\theta_1 \theta_3}{z_{13}} - \frac{\theta_2 \theta_4}{z_{24}} + \frac{3 \theta_1 \theta_2 \theta_3 \theta_4}{z_{13} z_{24}}.
\]

(35)

So, the most general form of the four point function of superprimay fields is

\[
\langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle = Y(a + b_1 W_{234} + b_4 W_{123} + c V),
\]

(36)

where \( a, b_1, b_4, \) and \( c \) are undetermined functions of \( X \) (\( b_1 \) and \( b_4 \) are Grassman valued), and

\[
W_{ijk} := \frac{\theta_i z_{jk} - \theta_j z_{ik} + \theta_k z_{ij} + \theta_i \theta_j \theta_k}{\sqrt{z_{ij} z_{ik} z_{jk}}}.
\]

(37)

It is easily seen that the results obtained in [27] are in agreement with the general result [30].

Differentiating (36) with respect to the weights, one obtains the most general form of the four point functions of Jordanian blocks. Once again, one must treat the undetermined functions as functions of the weights and introduce new functions in each differentiation.
References

[1] V. Gurarie; Nucl. Phys. **B410**[FS] (1993) 535.

[2] H. Saleur; Yale Preprint YCTP-P38-91, 1991.

[3] L. Rozansky and H. Saleur; Nucl. Phys. **B376** (1992) 461.

[4] I.I. Kogan, N.E. Mavromatos; Phys. Lett.**B375** (1996) 111.

[5] J.S. Caux, I.I. Kogan, A. Lewis and A.M. Tsvelik; Nucl.Phys. **B489** (1997) 469.

[6] I.I. Kogan, A. Lewis; hep-th/9705240.

[7] J. Cardy; J. Phys. A, 25 (1992) L201.

[8] V. Gurarie, M.A.I. Flohr and C. Nayak, cond-mat/9701212

[9] X.G. Wen, Y.S. Wu and Y. Hatsugai; Nucl. Phys. **B422**[FS] (1994) 476.

[10] M.R. Rahimi Tabar and S. Rouhani; Annals of Phys. **246** (1996)446.

[11] M.R. Rahimi Tabar and S. Rouhani; Nouvo Cimento **B112** (1997) 1079.

[12] M.R. Rahimi Tabar and S. Rouhani; Europhys. Lett. **37** (1997) 447.

[13] M.A. I. Flohr; Nucl. Phys. **B482** (1996) 567.

[14] M.R. Rahimi Tabar and S. Rouhani; Phys. Lett. **A224** (1997) 331.

[15] M.R. Gaberdiel, H.G. Kausch; Nucl.Phys. **B447** (1996) 293.

[16] M.A. I. Flohr; Int. J. Mod. Phys. **A11** (1996) 4147.
[17] A. Bilal and I.I. Kogan; Nucl. Phys. B449 (1995) 569.

[18] J.S. Caux, I.I. Kogan and A.M. Tsvelik; Nucl.Phys. B466 (1996) 444.

[19] Z. Maassarani, D. Serban; Nucl. Phys. B489 (1997) 603.

[20] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, R. H"ubel; Phys. Lett. B332 (1994) 51, Int. J. Mod. Phy. A10 (1995) 2367.

[21] J. Ellis, N.E. Mavromatos and D.V. Nanopoulos; Int. J. Mod. Phys. A12 (1997) 2639.

[22] A. M. Ghazelbash, V. Karimipour; Phys. Lett. B402 (1997) 282.

[23] M.R. Rahimi Tabar, A. Aghamohammadi, M. Khorrami; Nucl. Phys. B497 (1997) 555.

[24] M. Khorrami, A. Aghamohammadi, M.R. Rahimi Tabar; to appear in Phys. Lett. B (1998).

[25] Z. Qiu; Nucl. Phys. B270 [FS16] (1986) 205.

[26] P. Di Vechia, G. Knizhnic, J. L. Petersen and P. Rossi, Nucl. Phys. B253 (1985) 701.

[27] J. Fuchs, Nucl. Phys. B286 (1987) 455.