A symmetry-breaking quantum phase transition far from equilibrium

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We study the current-carrying steady-state of a transverse field Ising chain coupled to magnetic thermal reservoirs and obtain the non-equilibrium phase diagram as a function of the magnetization potential of the reservoirs. Upon increasing the magnetization bias we observe a discontinuous jump of the magnetic order parameter that coincides with a divergence of the correlation length. For steady-states with a non-vanishing conductance, the entanglement entropy at zero temperature displays a bias dependent logarithmic correction that violates the area law and differs from the well-known equilibrium case. Our findings show that out-of-equilibrium conditions allow for novel critical phenomena not possible at equilibrium.

I. INTRODUCTION

Non-equilibrium phases of quantum matter in open systems is a topical issue of immediate experimental relevance. However, a theoretical framework for the description of out-of-equilibrium strongly-correlated systems is at present incomplete and requires the further development of reliable techniques for non-equilibrium conditions. The influence of a non-thermal drive on phase boundaries and quantum critical points (QCP) is of particular interest.

An important class of non-equilibrium states are current-carrying steady-states (CCSS) that emerge in the long-time limit of systems coupled to reservoirs which are held at different thermodynamic potentials. These states are characterized by a steady flow of otherwise conserved quantities, such as energy, spin or charge. They can be realized in solid-state devices6–8 and have recently also became available in cold atomic setups4.

For Markovian processes, substantial progress has been made due the discovery of exact solutions for boundary driven Lindblad dynamics5–8 allowing for the characterization of certain non-equilibrium phases and phase transitions. In these cases, however, the Markovian condition substantially simplifies the dynamics. As a result, its validity is confined to extreme non-equilibrium conditions (e.g., large bias) that cannot be connected to thermal equilibrium9,10. Non-thermal steady-states in Luttinger liquids have also been studied11–13, but the results are less general than their equilibrium counterparts. Other methods to study CCSS include, looking at the asymptotic dynamics in pairs of semi-infinite quantum wires following quenches of the hopping connecting the pairs14–18. Bethe ansatz-based approaches19,20 that exploit the properties of integrable systems, hybrid approaches involving Lindblad dynamics21 and more phenomenological approximations based on Boltzmann kinetic equations22,23.

Another guiding element is the occurrence of scaling and criticality, which signal the absence of intrinsic energy scales and make the system particularly susceptible to any non-equilibrium drive24–30. Phase-transitions under non-equilibrium conditions31–40 were shown to allow intrinsic non-equilibrium universal properties, not seen at equilibrium. Nevertheless, a systematic approach describing CCSS is not available and exact solutions therefore must serve as a guiding principle.

In this letter we discuss an order-disorder symmetry breaking transition induced by non-equilibrium conditions in one of such exactly-solvable models, i.e., a spin chain that admits an exact solution by a mapping to a non-interacting fermionic system. Besides presenting the phase diagram and a characterization of various non-equilibrium phases, we identify a remarkable mixed-order quantum phase transition, where a discontinuous jump of the order parameter occurs in the presence of a divergent correlation length. The coexistence of such defining features of first- and second-order phase transitions implies the emergence a universality class specific to non-equilibrium conditions, for which an effective field-theoretic description is yet to be developed.
II. MODEL

The model we consider is depicted in Fig. 1-(a) and consists of an Ising spin chain of length $L$ and an applied transverse field $h$. The corresponding Hamiltonian is

$$H_C = -J \sum_{r=1}^{L-1} \sigma^x_r \sigma^x_{r+1} - h \sum_{r=1}^{L} \sigma^z_r,$$

(1)

where $J$ is the Ising coupling (we have fixed $J = 1$ as reference scale) and $\sigma^{x,y,z}$ are the Pauli matrices acting on site $r$. At the left and right boundaries, for $r = 1 \equiv r_L$ and $r = L \equiv r_R$ respectively, the chain is coupled to two zero-temperature magnetic reservoirs, with Hamiltonian $H_{R,L}$. Each reservoir is characterized by a set of gapless magnetic excitations within an energy bandwidth $J_t$, ($l = L, R$), and its total magnetization, $M_l = \sum_{r \in \Omega_l} \sigma^z_r$, is a good quantum number, i.e. $[H_l, M_l] = 0$, with average value set by a magnetic potential $m_l$. For concreteness we consider $H_l = -J_t \sum_{r \in \Omega_l} (\sigma^x_r \sigma^x_{r+1} + \sigma^y_r \sigma^y_{r+1}) - m_l M_l$ with $\Omega_L = \{-\infty, ..., -1, 0\}$ and $\Omega_R = \{L + 1, ..., \infty\}$. The total Hamiltonian of the chain plus reservoirs is given by $H = H_C + \sum_{l=L,R} (H_l + H_{C-l})$, where $H_{C-l} = -J_t' \left( \sigma^x_{r_l} \sigma^x_{r_l'} + \sigma^y_{r_l} \sigma^y_{r_l'} \right)$, with $r_l' = 0$ and $r_l' = L + 1$, describes the chain-reservoirs coupling. In what follows we will focus on the properties of the system in the steady-state that forms in the infinity time limit ($t \to \infty$) after the chain-reservoir couplings $J_t'$ are turned on at $t = 0$.

III. NON-EQUILIBRIUM ORDER-DISORDER PHASE TRANSITION

The ground-state of $H_C$ in Eq. (1) has a continuous phase transition for $h = \pm 1$ that separates a $\mathbb{Z}_2$ symmetry broken state from a paramagnetic one. The symmetry-broken state can be characterized by an order parameter $\phi = \lim_{h_x \to 0} \lim_{L \to \infty} \langle \sigma^{x}_{r} \rangle, \forall r$, with $h_x$ a magnetic field along $x$ that explicitly breaks the $\mathbb{Z}_2$ symmetry. $\phi$ vanishes as $|\phi| \approx (1 - h^2)^{1/8}$ as the transition point is approached from the ordered side, i.e. $|h| \to 1$, with the critical exponent $\beta = 1/8$. The correlation length diverges as $\xi \sim (1 - h^2)^{-\nu}$ with $\nu = 1$. This phase transition is in the universality class of the 2d classical Ising model and thus the QCP is described by a $\phi^4$ theory.

Our primary concern in this letter is the steady-state phase diagram that emerges far from equilibrium when $J_t' \neq 0$. Fig. 1-(b) shows the non-equilibrium phase diagram as a function of the left and right magnetic potentials. We consider the case $|h| < 1$, for which the equilibrium phase is ordered. Interestingly, the ordered state survives a non-vanishing coupling to the reservoirs for $|m_{l,R}| < m_1$, for $m_1 = 2 (-h + 1) > 0$. The order parameter $\phi$ along the red segment in Fig. 1-(b) is depicted in Fig. 1-(d). Within the ordered phase, $\phi$ does not depend on $m_R$ while $|m_R| \neq m_1$. At $|m_R| = m_1$, $\phi$ drops discontinuously to zero as $L \to \infty$, and this limit is approached as $\phi(|m_R| > m_1) \sim L^{-1/2}$. Fig. 1-(c) shows $\xi$ computed for the disordered phase where $|m_R| > m_1$. The divergence of $\xi \propto |m_R - m_1|^{-\nu}$ for $m_R \to m_1$ from the disordered phase is compatible with that of a power-law divergence with exponent $\nu = 1/2$.

Our results imply that the discontinuous vanishing of $\phi$ at $|m_R| = m_1$ in the $L \to \infty$ limit, a characteristic feature of a discontinuous equilibrium phase transition, is accompanied by a divergent correlation length, a hallmark of continuous phase transitions. Therefore, such a behaviour could not be accommodated within an equilibrium effective description. Below, some immediate implications of this significant finding will be further substantiated and analyzed. In particular, we will present the order-disorder transition in the context of a detailed description of the model and its other interesting non-equilibrium properties.

IV. METHODOLOGY

The full Hamiltonian, $H$, has a useful mapping to a fermionic model through the so-called Jordan-Wigner mapping, where $\sigma^x_r \rightarrow \hat{c}^\dagger_r \hat{c}_r$, where $c^\dagger/r$ creates/annihilates a spinless fermion at site $r$. This leads to a Kitaev chain in contact with two reservoirs at chemical potentials $\mu_{L,R} = 2m_{l,R}$, and with the topological phase corresponding to the ordered phase of the original spin model. The transformed Hamiltonian is quadratic and is given by $H_C = \frac{i}{2} \Psi^\dagger H C \Psi$, with $\Psi^\dagger = (c^\dagger_{l_1}, \ldots, c^\dagger_L, c_L, \ldots, c^\dagger_{L_1})$, and where $H_C$ is a $2L \times 2L$ Hermitian matrix respecting the particle-hole symmetry condition $S^{-1} H_C^\dagger S = -H_C$ with $S = \tau^z \otimes 1_{L \times L}$ and where $\tau^z$ interchanges particle and hole subspaces. In the fermionic representation, any correlation function of the chain can be described in terms of the retarded $G^R(t,t')$, $-i \Theta(t-t') \left\{ \Psi(t), \Psi^\dagger(t') \right\}$ and Keldysh components $G^K(t,t') = -i \left\{ \Psi(t), \Psi^\dagger(t') \right\}$ of the single-particle Green’s function. In the steady-state, the Dyson equation becomes to $G^{R-1}(\omega) = \omega - H_C - \Sigma^K(\omega)$ and $G^K(\omega) = G^R(\omega) \Sigma^K(\omega) G^A(\omega)$, with $G^A(\omega) = G^R(\omega)$ and where the self-energies $\Sigma^{R,K}(\omega) = \Sigma^{L,K}(\omega) + \Sigma^{R,K}_R(\omega)$ are imposed by the reservoirs. For the reservoirs, $\Sigma^{K}_R(\omega) = \left[ \Sigma^{K}_L(\omega) - \Sigma^{K}_L(\omega) \right] [1 - 2n_{p_{l,R}}(\omega)]$ holds with $n_{p_{l,R}}(\omega)$ being the Fermi-function for reservoir $l$, which is a manifestation of the equilibrium fluctuation dissipation relation for reservoir $l$. In the following, we make the simplifying assumption that the bandwidth of the reservoirs, $J_t$, is much larger than all other energy scales, which renders $\Sigma^{R}_L(\omega) = -i (\gamma_{l} + \bar{\gamma}_{l})$ frequency-independent, where $\gamma_{l} = \Gamma_l \langle | \gamma_l \rangle \langle \gamma_l |$ and $\bar{\gamma}_{l} = \Gamma_l \langle | \bar{\gamma}_l \rangle \langle \bar{\gamma}_l |$, and where $| \gamma \rangle$ and $| \bar{\gamma} \rangle = S | r \rangle$ are single-particle and...
hole states. $\Gamma_l = \pi J_l^2 D_l$ is the hybridization energy scale and $D_l$ is the local density of states of reservoir $l$. We therefore can define the non-Hermitian single-particle operator $K = H_l - i \sum_{l=L,R} (\gamma_l + \bar{\gamma}_l)$, with eigenvalues $\lambda_n$ and corresponding right and left eigenvectors $|\alpha\rangle$ and $\langle \bar{\alpha}|$, through which the Green function $G^R$ is simply given by $G^R(\omega) = \sum_{\alpha} |\alpha\rangle (\omega - \lambda_\alpha)^{-1} \langle \bar{\alpha}|$. Equal-time observables can thus be obtained from the single-particle density matrix, defined as $\chi = \langle \Psi | \chi | \Psi \rangle$:

$$\chi = \frac{1}{2} + \sum_{l=L,R} \sum_\alpha \langle \alpha | \beta \rangle \times$$

$$\langle \bar{\alpha}| \gamma_l I_l (\lambda_\alpha, \lambda_\beta) - \bar{\gamma}_l I_l (-\lambda_\alpha, -\lambda_\beta) \rangle \langle \beta |$$

$$= \frac{1}{2} \int \frac{d\omega}{2\pi} G^K(\omega) + 1$$

where $I_l(z, z') = -\frac{1}{\pi} \frac{g(z-2\nu_l) - g(z'-2\nu_l)}{z-z'}$ with $g(z) = \ln(-\text{sgn}(\Im(z))/z)$. The energy drain of left reservoir $L$ is $J_e = -i \langle [H_l, H_R] \rangle$, which equals the steady-state energy current in any cross section along the chain and thus can be obtained as a function of $\chi$. Explicitly the energy flow can be obtained as $J_e = -\frac{1}{2} \text{Tr}[J_{\rho} \chi] \forall \rho$, with

$$J_{\rho} = -2i\hbar J \left( (1 + S) |r - 1\rangle \langle r | (1 + S) - \text{H.c.} \right).$$

The linear and non-linear thermal conductivity as well as other thermoelectric properties of the chain are determined by $J_e$.

**V. RESULTS**

Fig. 2-(a) shows the current $J_e$ as a function of the magnetic potentials, where we have taken $\hbar = 0.2$ as an example. Evidently, the energy transport is able to discriminate between the different phases labeled in the phase diagram of Fig. 1-(b). The non-conducting phases NC and O, with $J_e = 0$ arise around the condition $m_l = m_R$. Note, however, that this condition does not correspond to equilibrium for the fermionic system away from $m = 0$. This is due to that fact that the non-interacting p-wave superconductor does not conserve the number of particles which in the spin representation translates to the non-conservation of the total magnetization. A conducting phase, C, characterized by a non-zero conductance $\delta_m$, $J_e \neq 0$, for $l = R$ or O arises for $|m_l| \in (m_1, m_2)$, with $m_2 = 2 + h > 0$. A set of phases we will refer to as current-saturated, or CS, arise for $|m_l| > m_2$ and are characterized by a finite current, $J_e \neq 0$, and a vanishing conductance $\delta_m J_e = 0$ for $l = R$ or O.

We now turn to the analysis of the two-point correlation function, $C_{\rho_{\alpha \beta}} = \langle \sigma_{\alpha}^r \sigma_{\beta}^r \rangle - \langle \sigma_{\alpha}^r \rangle \langle \sigma_{\beta}^r \rangle$, which can also be found in terms of $\chi$. To this end, we have extended the equilibrium expressions to general non-equilibrium conditions. For example:

$$C_{\rho_{\alpha \beta}} = \text{det} \left[ \left( 2 \chi_{[\rho-r]} - 1 \right)^2 \right]$$

where, for $r > r' + 1$, $\chi_{[\rho-r]}$ is a 2 $(r-r')$ matrix obtained as the restriction of $\chi$ to the subspace in which $P_{\rho-r} = \sum_{\gamma_{r+1}} \langle \mu \mu' | \rho \rangle | \nu \nu' \rangle$, with $| \nu \nu' \rangle = (|r \pm i\rangle) / \sqrt{2}$. A derivation of Eq. (4) is given in the supplementary material (SM).

Except for $C_{\rho_{\alpha \beta}}$ in the ordered phase, O, all the other components of $C_{\rho_{\alpha \beta}}$ for $\alpha, \beta = x, y$, decay exponentially. In Fig. 1-(c) was obtained fitting the $C_{\rho_{\alpha \beta}}$, for $r > r' + 1$, $\chi_{[\rho-r]}$ is a 2 $(r-r')$ matrix obtained as the restriction of $\chi$ to the subspace in which $P_{\rho-r} = \sum_{\gamma_{r+1}} \langle \mu \mu' | \rho \rangle | \nu \nu' \rangle$, with $| \nu \nu' \rangle = (|r \pm i\rangle) / \sqrt{2}$. A derivation of Eq. (4) is given in the supplementary material (SM).

We now turn to the entropy content of the non-equilibrium state. The entropy of a subsystem $S_l$, here taken to be a segment of the chain of length $l$, is given by $S_{Sl} = -\text{Tr}[\bar{\rho}_S \ln \bar{\rho}_S]$, where $\bar{\rho}_S$, is the single-particle density matrix restricted to $S_l$. In the limit $l \rightarrow \infty$, the entropy behaves as $S_{Sl} = l_0 l + c_0 \log(l + 1).

Ground states of gapped systems in equilibrium obey the area law, i.e. $l_0 = c_0 = 0$, while gapless fermions and spin chains show a universal logarithmic violation of the area law with $c_0 = 1/3$. This result is a consequence of the violation of the area law in 1+1 conformal theories in which case $c_0 = c/3$, where $c$ is the central charge. For a non-equilibrium Fermi-gas, it was shown that both $l_0$ and $c_0$ can be non-zero and $s$ depends on the system-reservoir coupling and is a non-analytic function of the bias $B$.

For the present case, Eq. (1) attached to magnetic leads, the linear coefficient $l_0$ is shown in Fig. 2-(b) for all phases, the details of the calculation are given in SM. We find that $l_0$ does not vary with $m_1$ ($l = R, L$) away from the conducting phase, depending only on the values of $h$ and $\Gamma_I$ (not shown in the figure). Moreover, $l_0$ vanishes within the ordered phase. The coefficient $c_0$ is most easily extracted from the mutual information, $I(A, B) \equiv E(\tilde{\rho}_A) + E(\tilde{\rho}_B) - E(\tilde{\rho}_{A+B})$, of two adjacent segments $A$ and $B$ of total size $l$. Since $I(A, B) \approx c_0 [2 \log(l + 1) - 2 \log(2l)]$, this coefficient was obtained in this way and is depicted in Fig. 2-(c). We find
that $c_0$ is non-zero in the C phase and vanishes otherwise. As in the case of a Fermi gas, $c_0$ depends on the strength of the reservoir-system couplings. In the present case, we find that it also depends on the bias potentials away from $m_L = m_R = 0$.

**Excitation numbers**: In order to conceptualize these results we turn to the fermionic representation. In the infinite-volume limit, $L \to \infty$, boundary effects vanish and the state becomes translationally invariant. The Hamiltonian of the translationally invariant chain in its diagonal representation is given by, $H = \sum_k \varepsilon_k (\gamma_k^\dagger \gamma_k - 1/2)$, where the operators $(\gamma_k, \gamma_k^\dagger)^T = e^{i \theta_k \sigma_x} (c_k, c_k^\dagger)^T$ describe the Bogoliubov excitations, $\sin(2 \theta_k) = -2J \sin(k)/\varepsilon_k$ and $\varepsilon_k = 2\sqrt{(h + J \cos k)^2 + (J \sin k)^2}$. The excitation number $n_k \equiv \langle \gamma_k^\dagger \gamma_k \rangle_{S_i}$ within $S_i$ can be obtained from the numerically computed single-particle density matrix, $\chi$. The results are shown in Figs. 3-(a)-(d), where the parameters used are labelled by the symbols marked in Fig. 1-(b).

**VI. DISCUSSION**

We study a spin chain that can order magnetically, driven out of equilibrium by keeping the magnetization at the two ends of the chain fixed at different values. A set of non-equilibrium phases is observed and characterized according to the conductance and the scaling of the entanglement entropy. This model offers a remarkable example of an extended, strongly-interacting system that can be continuously tuned from equilibrium to non-equilibrium conditions and admits an exact solution through the generalization of the Jordan-Wigner mapping. Moreover, we demonstrated that upon increasing the reservoir magnetization a discontinuous jump of
the magnetic order parameter occurs that coincides with a divergence of the correlation length. At equilibrium, the first observation is a signature of a first-order transition, while the second is a hallmark of continuous transitions. While this seems reminiscent of the situation that can occur at the lower critical dimension and which has been discussed in long-ranged spin chains in the context of mixed-order transitions, there are notable differences. In the present case, the interaction is short-ranged, and, more importantly, a second-order phase transition is recovered at equilibrium. Thus, our findings exemplify that out-of-equilibrium conditions allow for novel critical phenomena which are not possible in equilibrium. This kind of phase transition also differs from those obtained for systems where dissipation is present in the bulk which induces a change of the dynamical critical exponent. Therefore, to our best knowledge, this transition belongs to a novel universality class for which an effective field theoretic description out of equilibrium is yet to be developed. The exactly solvable model presented here should prove useful in developing such a description which will elucidate the role of interactions, e.g., the presence of magnetization gradients across the chain.

From the point of view of 1D fermionic systems, such peculiar critical properties might provide alternative signatures of the topological transition. To address this question, it would be interesting to extend our study of criticality under strong nonequilibrium condition to concrete setups of semiconductor nanowires.

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Appendix A: Details of the derivations

1. Current

For a quadratic fermionic model the density matrix within a subsystem S can be written as \( \rho_S = \frac{\Omega_S}{2\pi} \), where \( \Omega_S = \frac{1}{2} \Psi^\dagger \Omega_S \Psi \) is quadratic in the fermionic fields, with \( \Omega_S \) a \( 2V_S \times 2V_S \) matrix respecting the particle-hole symmetry conditions, and where \( V_S \) is the number of sites of S. In terms of \( \Omega_S \), the single-particle matrix \( \chi = \langle \Psi \Psi^\dagger \rangle \)

\[
\chi = \frac{1}{1 + e^{\Omega_S}}.
\]

The mean value of an observable \( O = \frac{1}{2} \Psi^\dagger O \Psi \) of S, quadratic in \( \Psi \) and defined by the hermitian, particle-hole symmetric matrix \( O \), can be obtained as

\[
\langle O \rangle = \text{tr} \left[ \rho_S \frac{1}{2} \Psi^\dagger O \Psi \right] = -\frac{1}{2} \text{tr} [\Omega_S \chi].
\]

The expression for the energy current in the main text is obtained in this way.

2. Two-spin correlation function

By symmetry arguments, for finite \( L \), \( \langle \sigma_m^x \rangle = 0 \). Thus, the \( \langle \sigma_m^x \sigma_n^x \rangle \) correlation function, for \( m \) and \( n \) belonging to a subsystem S, can be written as

\[
C_{mn}^{xx} = \langle \sigma_m^x \sigma_n^x \rangle = \text{tr} \left[ e^{i\pi \sum_{j=0}^{m-1} c_j^\dagger (c_j^+ + c_j) (c_j^+ + c_j) \rho_S} \right].
\]

We now re-write Eq. (A2) in terms of the operators \( \Omega_1 = \frac{1}{2} \Psi^\dagger \Omega_1 \Psi \), with \( \Omega_1 = i \pi \sum_{j=0}^{m-1} \langle j | j - \frac{1}{2} \rangle \), \( A = \langle \sigma_m^x \sigma_n^x \rangle = \frac{1}{2} \Psi^\dagger \Omega_1 \Psi \), with \( \Omega_1 = \frac{1}{2} \Psi^\dagger \Omega_1 \Psi \). \( \langle r \pm \rangle = (|r\rangle \pm |r\rangle) / \sqrt{2} \), and \( \rho_S \) as in previous section. These definitions lead to:

\[
C_{mn}^{xx} = -ie^{\Omega_1} e^{i\frac{\pi}{2} (m-n+1)} T,
\]

with

\[
T = \frac{\text{tr} [e^{\Omega_1} e^{i\frac{\pi}{2} A} e^{\Omega_S}]}{\text{tr} e^{\Omega_S}},
\]

using that \( e^{i\frac{\pi}{2} A} = i A \).

We now use the Levitov-Lesovik formula\footnote{58,59} to evaluate the trace,

\[
T = \frac{\text{det} [1 + e^{\Omega_1} e^{i\frac{\pi}{2} A} e^{\Omega_S}]}{\text{det}[1 + e^{\Omega_S}]^2},
\]
and Eq. (A1), to write
\[ T = \det \left[ 1 + (1 - \chi) \left( -1 + e^{\Omega_1 e^{i\frac{\pi}{2} A}} \right) \right]^{\frac{1}{2}}. \]  
(A7)

This expression can be further simplified noting that \( e^{\Omega_1} = P - P \) with
\[ P = \sum_{j=n}^{m} \left( |i\rangle \langle i| + |i\rangle \langle j| \right) \]
and \( P = 1 - P \). Since \( AP = PA = A \) we can write
\[ T = \det \left[ \chi + (1 - \chi) (P - P e^{i\frac{\pi}{2} A}) \right]^{\frac{1}{2}}. \]  
(A8)

This expression can also be simplified noting that, since \( P \left[ \chi + (1 - \chi) (P - P e^{i\frac{\pi}{2} A}) \right] P = P \) and \( P \left[ \chi + (1 - \chi) (P - P e^{i\frac{\pi}{2} A}) \right] P = 0 \), the determinant is solely determined by the projection onto the subspace where \( P \) acts as the identity. We define the restriction of \( \chi \) and \( A \) to that subspace, spanned by the sites \( n \leq r \leq m \), as
\[ \tilde{\chi} = p^T \chi P \]
\[ \tilde{A} = p^T A p \]  
(A9)

where \( P = pp^T \) and \( p^T p = 1 \). We can now write
\[ T = \det \left[ \tilde{\chi} \left( e^{i\frac{\pi}{2} \tilde{A}} + 1 \right) \right]^{\frac{1}{2}}. \]  
(A10)

We further note that
\[ e^{i\frac{\pi}{2} \tilde{A}} - 1 = -2\tilde{Q} \]
with \( \tilde{Q} = |n_+\rangle \langle n_+| + |m_-\rangle \langle m_-| \), thus
\[ T = \det \left[ 2\tilde{\chi} \left( 1 - \tilde{Q} \right) - 1 \right]^{\frac{1}{2}}. \]  
(A11)

Again, this expression can be simplified in a way similar to Eq. (A8) noting that \( \left( 1 - \tilde{Q} \right) \left[ 2\tilde{\chi} \left( 1 - \tilde{Q} \right) - 1 \right] \tilde{Q} = 0 \) and \( \tilde{Q} \left[ 2\tilde{\chi} \left( 1 - \tilde{Q} \right) - 1 \right] \tilde{Q} = -\tilde{Q} \). Thus, we can define \( P_{mn} = P (1 - Q) \), where \( Q \) is the extension of \( \tilde{Q} \) to the entire space. The explicit expression of \( P_{mn} \) is given in the main text. For \( \tilde{q} \) such that \( P_{mn} = \tilde{q}q^T \) and \( q^T \tilde{q} = 1 \), we obtain
\[ T = \det \left[ 2\tilde{q}^T \chi \tilde{q} - 1 \right]^{\frac{1}{2}} \]  
(A12)
and recover the expression
\[ C_{mn}^{xy} = \det \left[ i \left( 2\tilde{q}^T \chi \tilde{q} - 1 \right) \right]^{\frac{1}{2}} \]  
(A13)
given in the main text. \( C_{mn}^{yy} \) can be obtained in a similar fashion.

**Appendix B: Additional numerical results**

For completeness, the following provides a complementary set of numerical results to those given in the main text.

1. **Excitations numbers**

Fig. 4-(a) illustrates the excitation numbers \( n_k \) in all regions of the phase diagram, for the same set of parameters used in the main text: \( J = 1 \), \( h = 0.2 \), \( \Gamma_R = \Gamma_L \) = 0.1 or 0.1, and zero temperature. This choice of parameters yields \( m_1 = 2(-h+1) = 1.6 \) and \( m_2 = 2(h+1) = 2.4 \). We have used \( L = 500 \) for which finite size effects are negligible.

As noted in the main text, the asymmetry upon changing \( k \to -k \) of the conducting phases is enhanced by a larger value of the hybridization between the chain and the reservoirs. The panels of \( n_k \) follow the same order as the markers depicted in the phase diagram.

2. **Case \( h = 1/2 \)**

Here we expand on the special case of \( h = 1/2 \), briefly mentioned in the main text, which leads to a different universality class, *i.e.*, to different critical exponents. Figs. 5-(a) and (b) show the correlation length and order
Figure 5. Correlation length (a) and order parameter (b) for $h = 1/2$. The inset shows the scaling of $\xi$ near the transition at $m_R = \pm m_1$. The fittings to compute $\xi$ are exemplified in (c) for two points (A and B) marked in the first panel. The finite size scaling behaviour of the order parameter is shown in (d), for four points inside different disordered phases (marked by arrows in panel (b)).

The special behavior for $h = 1/2$ can be understood by analyzing its excitation numbers. Fig. 4-(b) illustrates $n_k$ in all regions of the phase diagram, under the same conditions of Fig. 4-(a). The difference appears on values of $m_R$ and $m_L$ for which the excitations raise continuously from zero, as we drive the system out of the ordered phase. Note, in fact, that when $m_R \to m_1$ the disordered phase is characterized by $n_{k=\pm \pi} = 0$, which corresponds to the anomalous exponent $\nu = 5/2$. On the other hand, at the $m_R \to -m_1$ phase boundary one has $n_{k=\pm \pi} \neq 0$, giving the same exponent $\nu = 1/2$ discussed in the main text.