Time-periodic solutions of the Einstein’s field equations II: geometric singularities

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Abstract  In this paper, we construct several kinds of new time-periodic solutions of the vacuum Einstein’s field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. The singularities of these new time-periodic solutions are investigated and some new physical phenomena are discovered.

Keywords  Einstein’s field equations, time-periodic solution, Riemann curvature tensor, singularity, event horizon

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1 Introduction

This work is a continuation of our previous work [3]. As in [3], we still consider the time-periodic solutions of the following vacuum Einstein’s field equations

\[ G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \]  (1)

or equivalently,
\[ R_{\mu\nu} = 0, \]  (2)

where \( g_{\mu\nu} (\mu, \nu = 0, 1, 2, 3) \) is the unknown Lorentzian metric, \( R_{\mu\nu} \) is the Ricci curvature tensor, \( R \) is the scalar curvature and \( G_{\mu\nu} \) is the Einstein tensor.

It is well-known that the exact solutions of the Einstein’s field equations play a crucial role in general relativity and cosmology. Typical examples are the Schwarzschild solution and Kerr solution. Although many interesting and important solutions have been obtained (see, e.g., [1, 5]), there are still many fundamental open problems. One such problem is if there exists a “time-periodic” solution, which contains physical singularities such as the black hole, to the vacuum Einstein’s field equations. This paper continues the discussion of this problem.

The first time-periodic solution of the vacuum Einstein’s field equations was constructed by the first two authors in [3]. The solution presented in [3] is time-periodic, and describes a regular space-time,
which has vanishing Riemann curvature tensor but is inhomogenous, anisotropic and not asymptotically flat. In particular, this space-time does not contain any essential singularity, but contains some non-essential singularities which correspond to steady event horizons, time-periodic event horizons and has some interesting new physical phenomena.

In this paper, we focus on finding the time-periodic solutions, which contain geometric singularities (see Definition 1 below) to the vacuum Einstein’s field equations (1). We shall construct three kinds of new time-periodic solutions of the vacuum Einstein’s field equations (1) whose Riemann curvature tensors vanish, keep finite or go to the infinity at some points in these space-times respectively. The singularities of these new time-periodic solutions are investigated and new physical phenomena are found. Moreover, the applications of these solutions in modern cosmology and general relativity may be expected. In the forthcoming paper [4], we shall construct a time-periodic solution of the Einstein’s field equations with physical singularities (see also Definition 1 below), which describes a time-periodic universe with many new and interesting physical phenomena.

The paper is organized as follows. In Section 2 we present our procedure of finding new solutions of the vacuum Einstein’s field equations and introduce the concepts of “geometric singularity” and “physical singularity”. In Section 3 we construct three kinds of new time-periodic solutions of the vacuum Einstein’s field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. In this section, the singularities of these new time-periodic solutions are also investigated and some new physical phenomena are found and discussed. A summary and some discussions are given in Section 4.

2 Procedure of finding new solutions

We consider the metric of the following form,

\[
(g_{\mu\nu}) = \begin{pmatrix}
u & v & p & 0 \\ v & 0 & 0 & 0 \\ p & 0 & f & 0 \\ 0 & 0 & 0 & h
\end{pmatrix},
\]

(3)

where \(u, v, p, f\) and \(h\) are smooth functions of the coordinates \((t, x, y, z)\). It is easy to verify that the determinant of \((g_{\mu\nu})\) is given by

\[
g = \det(g_{\mu\nu}) = -v^2fh.
\]

Throughout this paper, we assume that \(g < 0\).

(H)

Without loss of generality, we may suppose that \(f\) and \(g\) keep the same sign, for example,

\[
f < 0\hspace{1em}\text{(resp. } f > 0\text{)}\hspace{1em}\text{and}\hspace{1em}h < 0\hspace{1em}\text{(resp. } g > 0\text{)}.
\]

(5)

In what follows, we solve the Einstein’s field equations (2) under the framework of the Lorentzian metric of the form (3).

By a direct calculation, we have the Ricci tensor

\[
R_{11} = -\frac{1}{2} \left\{ \frac{v_x}{v} \left( \frac{f_x}{f} + \frac{h_x}{h} \right) + \frac{1}{2} \left[ \left( \frac{f_x}{f} \right)^2 + \left( \frac{h_x}{h} \right)^2 \right] - \left( \frac{f_{xx}}{f} + \frac{h_{xx}}{h} \right) \right\}.
\]

(6)

It follows from (2) that

\[
\frac{v_x}{v} \left( \frac{f_x}{f} + \frac{h_x}{h} \right) + \frac{1}{2} \left[ \left( \frac{f_x}{f} \right)^2 + \left( \frac{h_x}{h} \right)^2 \right] - \left( \frac{f_{xx}}{f} + \frac{h_{xx}}{h} \right) = 0.
\]

(7)