A note on the Bethe ansatz solution of the
supersymmetric t-J model

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Abstract

The three different sets of Bethe ansatz equations describing the Bethe
ansatz solution of the supersymmetric t-J model are known to be equiva-
ient. Here we give a new, simplified proof of this fact which relies on the
properties of certain polynomials. We also show that the corresponding
transfer matrix eigenvalues agree.

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1 Introduction

The supersymmetric $t$-$J$ model, defined by the Hamiltonian

$$HP_0 = P_0 \left\{ - \sum_{j=1}^{L} (c_{j,a}^\dagger c_{j+1,a} + c_{j+1,a}^\dagger c_{j,a}) + 2 \sum_{j=1}^{L} \left( S_{j}^\alpha S_{j+1}^\alpha - \frac{n_j n_{j+1}}{4} + n_j \right) \right\} P_0 ,$$

(1)

is the familiar $t$-$J$ model of solid state physics (for details see e.g. [1]) at a certain ratio of hopping strength and coupling, $t/J = \frac{1}{2}$. For this ratio the Hamiltonian is invariant under the action of the Lie super algebra $gl(1\mid 2)$ [4, 10]. Its one-dimensional version is solvable by Bethe ansatz [8] and is related to the lattice gas models of Lai and Sutherland [7, 9]. An algebraic Bethe ansatz solution was obtained by Essler and Korepin in [2] (see also [3]).

A peculiarity of the model is the non-uniqueness of the form of the Bethe ansatz solution. In fact, three different sets of Bethe ansatz equations and three different expressions for the eigenvalue of the associated transfer matrix were obtained in [2]. The authors of [2] then used a technique [11] based on the residue theorem in order to show the mutual equivalence of the three sets of Bethe ansatz equations.

In a recent article [6] we reconsidered the Bethe ansatz solution of the supersymmetric $t$-$J$ model. Our intention was to give an example of how to apply the Bethe ansatz solution [6] of the so-called $gl(1\mid 2)$ generalized model [5]. We showed how the different gradings leading to the different forms of the Bethe ansatz equations relate to different choices of reference states for the Bethe ansatz. We found that if we count the different ways to construct the eigenvectors, then there are six different algebraic Bethe ansatz solutions of the supersymmetric $t$-$J$ model which are pairwise related by spin-flip transformations. From our considerations it also followed that the different expressions for the Bethe ansatz eigenvalues must agree. Here we give a simple direct proof of this fact, which is based on the properties of polynomials rather than on integration in the complex plane. Our technique also allows us to give a new proof of the equivalence of the Bethe ansatz equations. It seems to be widely applicable and applies, for instance, also to showing the relation between the eigenstates in the repulsive and attractive cases of the Hubbard model [11].

2 Equivalence of the three different forms of the Bethe ansatz equations

We show that the three different forms of the Bethe ansatz equations for the supersymmetric $t$-$J$ model associated with the gradings $(-,+,\cdot)$, $(-,\cdot,\cdot)$ and $(\cdot,\cdot,-)$ are mutually equivalent.

Let us start with the case $(-,+,\cdot)$. Then the Bethe ansatz equations for twisted
boundary conditions (twist angles $\phi_+$ and $\phi_-$) are

\[
\left( \frac{\lambda_\ell - \frac{i}{2}}{\lambda_\ell + \frac{i}{2}} \right)^L = e^{-i\phi_+} \prod_{j=1}^M \frac{\lambda_\ell - \omega_j - \frac{i}{2}}{\lambda_\ell - \omega_j + \frac{i}{2}}, \quad \ell = 1, \ldots, N, \quad (2a)
\]

\[
\prod_{\ell=1}^N \frac{\omega_j - \lambda_\ell - \frac{i}{2}}{\omega_j - \lambda_\ell + \frac{i}{2}} = e^{-i\phi_-}, \quad j = 1, \ldots, M, \quad (2b)
\]

where $\phi_+ = \phi_+ + \phi_-$. We shall further assume that $N \geq M$. Equations (2) turn into equations (4.3) of Essler and Korepin for $\phi_+ = \phi_+ = 0$ if we identify $\lambda_{k}^{(1)} = \omega_k$, $\lambda_\ell = \lambda_\ell$, $N_\uparrow = M$, and $N_\downarrow + N_\uparrow = N$.

We define the polynomial

\[
p(z) = \prod_{n=1}^N (z - \lambda_n - \frac{i}{2}) - e^{-i\phi_+} \prod_{n=1}^N (z - \lambda_n + \frac{i}{2}). \quad (3)
\]

Let us fix a solution $\{ \{ \lambda_\ell \}_{\ell=1}^N, \{ \omega_j \}_{j=1}^M \}$ of (2). Then (2b) implies that $p(\omega_j) = 0$ for $j = 1, \ldots, M$. $p$ is a polynomial of degree $N \geq M$. Besides the zeros $\omega_j$ it has $N - M$ additional zeros $\mu_k$, $k = 1, \ldots, N - M$. Thus, $p(z)$ can be represented as

\[
p(z) = (1 - e^{-i\phi_+}) \prod_{j=1}^M (z - \omega_j) \prod_{k=1}^{N-M} (z - \mu_k). \quad (4)
\]

Comparing (3) and (4) we obtain

\[
p(\lambda_\ell - \frac{i}{2}) = \prod_{n=1}^N (\lambda_\ell - \lambda_n - i) = (1 - e^{-i\phi_+}) \prod_{j=1}^M (\lambda_\ell - \omega_j - \frac{i}{2}) \prod_{k=1}^{N-M} (\lambda_\ell - \mu_k - \frac{i}{2}),
\]

\[
p(\lambda_\ell + \frac{i}{2}) = -e^{-i\phi_+} \prod_{n=1}^N (\lambda_\ell - \lambda_n + i) = (1 - e^{-i\phi_+}) \prod_{j=1}^M (\lambda_\ell - \omega_j + \frac{i}{2}) \prod_{k=1}^{N-M} (\lambda_\ell - \mu_k + \frac{i}{2}).
\]

Dividing the latter two equations we arrive at the identity

\[
e^{i\phi_+} \prod_{n=1}^N \frac{\lambda_\ell - \lambda_n - i}{\lambda_\ell - \lambda_n + i} \prod_{k=1}^{N-M} \frac{\lambda_\ell - \mu_k + \frac{i}{2}}{\lambda_\ell - \mu_k - \frac{i}{2}} = \prod_{j=1}^M \frac{\lambda_\ell - \omega_j - \frac{i}{2}}{\lambda_\ell - \omega_j + \frac{i}{2}} \quad \quad (5)
\]

which, when inserted into the right hand side of (2a), implies

\[
\left( \frac{\lambda_\ell - \frac{i}{2}}{\lambda_\ell + \frac{i}{2}} \right)^L = e^{-i\phi_+} \prod_{n=1}^N \frac{\lambda_\ell - \lambda_n - i}{\lambda_\ell - \lambda_n + i} \prod_{k=1}^{N-M} \frac{\lambda_\ell - \mu_k + \frac{i}{2}}{\lambda_\ell - \mu_k - \frac{i}{2}}, \quad \ell = 1, \ldots, N. \quad (6a)
\]

Moreover, since the $\mu_k$ are zeros of $p(z)$ we conclude with (3) that

\[
\prod_{\ell=1}^N \frac{\mu_k - \lambda_\ell - \frac{i}{2}}{\mu_k - \lambda_\ell + \frac{i}{2}} = e^{-i\phi_+}, \quad k = 1, \ldots, N-M. \quad (6b)
\]
Equations (6) are the Bethe ansatz equations for \((-,-,+)\) grading. Upon identifying \(\lambda_\ell = \lambda_\ell, \tilde{\lambda}^{(1)}_\ell = \mu_k\) and setting \(\phi_\ell = \phi_c = 0\) they turn into equation (3.73) of [2]. The polynomial \(p(z)\) is fixed for fixed \(\{\lambda_\ell\}_{\ell=1}^N\). Thus, every solution of (2) gives a solution of (6) and vice versa.

We may now apply the same polynomial trick to equation (2a). Let us define

\[
q(z) = (z - \frac{i}{2})^L \prod_{n=1}^M (z - \omega_n + \frac{i}{2}) - e^{-i\Phi} (z + \frac{i}{2})^L \prod_{n=1}^M (z - \omega_n - \frac{i}{2}). \tag{7}
\]

The polynomial \(q\) has \(L + M\) zeros, the \(\lambda_\ell, \ell = 1,\ldots,N\), are among them. Let us denote the remaining zeros by \(\rho_k, k = 1,\ldots,L - N + M\) (we assume that \(N \leq L\)). Then

\[
q(z) = (1 - e^{-i\Phi}) \prod_{\ell=1}^N (z - \lambda_\ell) \prod_{k=1}^{L-N+M} (z - \rho_k). \tag{8}
\]

Evaluating \(q(z)\) at \(z = \omega_j - \frac{i}{2}\) and at \(z = \omega_j + \frac{i}{2}\) by using both forms, (7) and (8), of the polynomial and dividing the resulting equations by each other we obtain the identity

\[
e^{-i\Phi} \prod_{n=1}^M \frac{\omega_j - \omega_n - i}{\omega_j - \omega_n + i} e^{-i\Phi} \prod_{k=1}^{L-N+M} \frac{\omega_j - \rho_k + \frac{i}{2}}{\omega_j - \rho_k - \frac{i}{2}} = \prod_{\ell=1}^N \frac{\omega_j - \lambda_\ell - \frac{i}{2}}{\omega_j - \lambda_\ell + \frac{i}{2}}, \tag{9}
\]

which holds for \(j = 1,\ldots,M\). Inserting this identity into (2b) it follows that

\[
\prod_{k=1}^{L-N+M} \frac{\omega_j - \rho_k - \frac{i}{2}}{\omega_j - \rho_k + \frac{i}{2}} = e^{-i\Phi} \prod_{n=1}^M \frac{\omega_j - \omega_n - i}{\omega_j - \omega_n + i}, \quad j = 1,\ldots,M. \tag{10a}
\]

Since the \(\rho_k\) satisfy \(q(\rho_k) = 0\), we also have

\[
\left(\frac{\rho_k - \frac{i}{2}}{\rho_k + \frac{i}{2}}\right)^L = e^{-i\Phi} \prod_{j=1}^M \frac{\rho_k - \omega_j - \frac{i}{2}}{\rho_k - \omega_j + \frac{i}{2}}, \quad k = 1,\ldots,L - N + M. \tag{10b}
\]

Equations (10) are the Bethe ansatz equations for the grading \((+,-,-)\). We recover equation (3.49) of [2] setting \(\tilde{\lambda}_k = \rho_k, \lambda_j^{(1)} = \omega_j, \phi_x = \phi_c = 0, L - N + M = N_\ell, M = N_\downarrow\).

To sum up, we have shown the mutual equivalence of the three sets of Bethe ansatz equations (2), (6) and (10) by exploiting the properties of the polynomials \(p(z)\) and \(q(z)\) defined in equations (3) and (7).

### 3 Equivalence of the transfer matrix eigenvalues

The polynomials \(p(z)\) and \(q(z)\) also allow us to show that the eigenvalues corresponding to the three forms of the Bethe ansatz solutions agree.
The transfer matrix eigenvalue for the grading \((-, +, -)\) is

\[
\Lambda_{(-+-)}(\lambda) = -(\frac{\lambda - i}{\lambda + i})^L \prod_{\ell=1}^{N} \frac{\lambda - \lambda_\ell + \frac{i}{2}}{\lambda - \lambda_\ell - \frac{i}{2}} + \left( \frac{\lambda}{\lambda + i} \right)^L \left( \prod_{\ell=1}^{N} \frac{\lambda - \lambda_\ell + \frac{i}{2}}{\lambda - \lambda_\ell - \frac{i}{2}} - 1 \right) \prod_{j=1}^{M} \frac{\lambda - \omega_j - i}{\lambda - \omega_j},
\]  

(11)

where the Bethe ansatz roots \(\lambda_\ell\) and \(\omega_j\) are solutions of (2) with \(\phi_y = \phi_c = 0\). \(\Lambda_{(-+-)}\) agrees with the expression (4.4) of [2] if we set \(\lambda_j^{(1)} = \omega_j\), \(\tilde{\lambda}_\ell = \lambda_\ell\), \(N_\downarrow = M\) and \(N_h + N_\downarrow = N\).

It follows from equations (3) and (4) that

\[
\frac{p(\lambda - i)}{p(\lambda)} = \frac{\prod_{j=1}^{N} \lambda - \omega_j - i}{\prod_{k=1}^{M} \lambda - \mu_k} = \frac{e^{-i\phi_c} - \prod_{\ell=1}^{N} \frac{\lambda - \lambda_\ell - \frac{3i}{2}}{\lambda - \lambda_\ell - \frac{i}{2}}}{e^{-i\phi_c} - \prod_{\ell=1}^{N} \frac{\lambda - \lambda_\ell + \frac{i}{2}}{\lambda - \lambda_\ell - \frac{i}{2}} - 1}.
\]

(12)

Setting \(\phi_c = 0\) we obtain the identity

\[
\left( \prod_{\ell=1}^{N} \frac{\lambda - \lambda_\ell + \frac{i}{2}}{\lambda - \lambda_\ell - \frac{i}{2}} - 1 \right) \prod_{j=1}^{M} \frac{\lambda - \omega_j - i}{\lambda - \omega_j} = \left( 1 - \prod_{\ell=1}^{N} \frac{\lambda - \lambda_\ell - \frac{3i}{2}}{\lambda - \lambda_\ell - \frac{i}{2}} \right) \prod_{k=1}^{M} \frac{\lambda - \mu_k}{\lambda - \mu_k - 1}.
\]

(13)

This identity allows us to replace the \(\omega_j\) in equation (11) in favour of the Bethe ansatz roots \(\mu_k\) connected with \((-,-,+)\) grading. We obtain

\[
\Lambda_{(-+-)}(\lambda) = -(\frac{\lambda - i}{\lambda + i})^L \prod_{\ell=1}^{N} \frac{\lambda - \lambda_\ell + \frac{i}{2}}{\lambda - \lambda_\ell - \frac{i}{2}} + \left( \frac{\lambda}{\lambda + i} \right)^L \left( 1 - \prod_{\ell=1}^{N} \frac{\lambda - \lambda_\ell - \frac{3i}{2}}{\lambda - \lambda_\ell - \frac{i}{2}} \right) \prod_{k=1}^{M} \frac{\lambda - \mu_k}{\lambda - \mu_k - 1}.
\]

(14)

But the latter expression is precisely the expression for the eigenvalue \(\Lambda_{(-+-)}(\lambda)\) (which has to be compared with (3.74) of [2]). We may formulate this result in the following way:

\[
\Lambda_{(-+-)}(\lambda) \{ \lambda_\ell \}^{N}_{\ell=1}, \{ \omega_j \}^{M}_{j=1} = \Lambda_{(-+-)}(\lambda) \{ \lambda_\ell \}^{N}_{\ell=1}, \{ \mu_k \}^{N-M}_{k=1},
\]

(15)

where we exposed explicitly the dependence of the transfer matrix eigenvalues on the Bethe ansatz roots. The two sets of Bethe ansatz roots \(\{ \omega_j \}^{M}_{j=1}\) and \(\{ \mu_k \}^{N-M}_{k=1}\) are connected by the polynomial \(p(z)\), equation (3).

In a similar way \(\Lambda_{(-+-)}\) is related to \(\Lambda_{(+-+)}\) through the polynomial \(q(z)\). The proof should now be obvious and is left as an exercise to the reader.
References

[1] Assa Auerbach, *Interacting electrons and quantum magnetism*, Springer-Verlag, New York, 1994.

[2] F. H. L. Essler and V. E. Korepin, *Higher conservation laws and algebraic Bethe Ansätze for the supersymmetric t-J model*, Phys. Rev. B 46 (1992), 9147.

[3] A. Foerster and M. Karowski, *Algebraic properties of the Bethe ansatz for an spl(2,1)-supersymmetric t-J model*, Nucl. Phys. B 396 (1993), 611.

[4] D. Förster, *Staggered spin and statistics in the supersymmetric t-J model*, Phys. Rev. Lett. 63 (1989), 2140.

[5] F. Göhmann, *Algebraic Bethe ansatz for the gl(1|2) generalized model and Lieb-Wu equations*, Nucl. Phys. B 620 (2002), 501.

[6] F. Göhmann and A. Seel, *Algebraic Bethe ansatz for the gl(1|2) generalized model II: the three gradings*, preprint, cond-mat/0309135, 2003.

[7] C. K. Lai, *Lattice gas with nearest neighbor interaction in one dimension with arbitrary statistics*, J. Math. Phys. 15 (1974), 1675.

[8] P. Schlottmann, *Integrable narrow-band model with possible relevance to heavy Fermion systems*, Phys. Rev. B 36 (1987), 5177.

[9] B. Sutherland, *Model for a multi component quantum system*, Phys. Rev. B 12 (1975), 3795.

[10] P. B. Wiegmann, *Superconductivity in strongly correlated electronic systems and confinement versus deconfinement phenomenon*, Phys. Rev. Lett. 60 (1988), 821.

[11] F. Woynarovich, *Low-energy excited states in a Hubbard chain with on-site attraction*, J. Phys. C 16 (1983), 6593.