DISCRETE MULTICHANNEL SCATTERING WITH STEP-LIKE POTENTIAL

ISAAC ALVAREZ-ROMERO AND YURI LYUBARSKII

Abstract. We study direct and inverse scattering problem for systems of interacting particles, having web-like structure. Such systems consist of a finite number of semi-infinite chains attached to the central part formed by a finite number of particles. We assume that the semi-infinite channels are homogeneous at infinity, but the limit values of the coefficients may vary from one chain to another.

1. Introduction

The aim of this article is to study the direct and inverse problems for small oscillations near equilibrium position for a system of particles \( \mathcal{A} = \{\alpha, \beta, \ldots\} \) which interact with each other and perhaps with an external field. The interaction is described by the matrix

\[
\mathcal{L} = (L(\alpha, \beta))_{\alpha, \beta \in \mathcal{A}}.
\]

We say that the particles \( \alpha \) and \( \beta \) interact with each other if \( L(\alpha, \beta) \neq 0 \) and we assume that each particle interacts with at most a finite number of its neighbours: \( \#\{\beta, L(\alpha, \beta) \neq 0\} < \infty \) for each \( \alpha \in \mathcal{A} \).

Our the system has a "web-like" structure: it includes a finite set of "channels", i.e. semi-infinite chains of particles attached to a "central part" formed by a finite number of interacting particles.

Given a set of particles \( \mathcal{X} \) we denote by \( \mathcal{M}(\mathcal{X}) \) and \( l^2(\mathcal{X}) \) the spaces of all functions on \( \mathcal{X} \) and square summable functions on \( \mathcal{X} \) respectively.

The matrix \( \mathcal{L} \) is related to the Hessian matrix of the potential energy near equilibrium position, so we always assume that all \( L(\alpha, \beta) \) are real and

\[
\mathcal{L} > 0,
\]

here \( \mathcal{L} \) is considered as an operator in \( l^2(\mathcal{A}) \). In what follows we do not distinguish between matrices and the corresponding linear operators.

After separation of variables one arrives to the spectral problem

\[
\mathcal{L}\xi = \lambda\xi; \quad \xi = \{\xi(\alpha)\} \in l^2(\mathcal{A}).
\]

which can be considered as a discrete version of spectral problems for quantum graphs, see \([3, 6, 8, 17]\) as well as later articles \([9, 10, 11]\).
The web-like structure of the system allows us to treat the spectral problem (1.1) as a scattering problem: the points of continuous spectra correspond to frequencies of incoming and outgoing waves which are propagating along the channels; the points of discrete spectra correspond to proper oscillations of the system. The spectral data includes transmission and reflection coefficients: we consider waves incoming along one of the channels and observe (at infinite ends of the channels) how do such waves come through the system. The direct problem is to determine the spectral data through the characteristics of the system. The inverse problem deals with recovering of characteristics of the system from the spectral data. This is a classical setting of the scattering problem, see e.g. [16].

This work is a continuation of [12] which considers the case of the same wave propagation speed along all channels. Now we assume that each channel has its own speed. The problem then becomes more complicated: different points of continuous spectra may have different multiplicities; we do not have a single scattering matrix for the whole spectra; besides the generalized eigenfunctions may exponentially decay along some of the channels. These eigenfunctions need to be treated specially, it does not make sense to observe the phase of a wave which decays exponentially at infinity, just their amplitudes can be taken into account. Respectively, for such waves, only absolute values of the transmission coefficients can be included to spectral data.

As in [12] the problem can be reduced to a system of difference Schrödinger equations on semi-infinite discrete string with the initial conditions related to the way the channels are attached to the central part. In our setting the potentials have different limits for different equations in this system. In the classical case of infinite string this corresponds to a step-like potential, such problem for continuous case has been studied for example in [2, 4, 7], for the Jacobi operators the case of step-like quasi-periodic potential has been considered in [5]. We modify techniques of these articles, especially those in [4], in order to make it applicable for the graph setting.

The scattering data are determined as the set of scattering coefficients, eigenvalues, and also as normalization constants of the corresponding eigenfunctions. These scattering data are associated with the data which can be measured in an experiment at infinite ends of channels.

We solve the direct problem, i.e., description of scattering data from physical characteristic of the system, and (under additional assumptions) the inverse problem, i.e., reconstruction the characteristics of the channels from given spectral data, that is, we reconstruct the matrix $L$ on the channels. In order to make this reconstruction we reduce the problem to a discrete analog of Marchenko equation, which is known to have unique solution and actually admits numeric implementation.

In this article we do not discuss possibility of reconstruction of data which is related to the central part of the graph. In order to have such reconstruction, one needs to demand additional sparsity conditions of the central part. We refer the reader to
the recent book [15], which contains examples of such conditions.

The article is organised as follows: in the next section we describe the system, derive the boundary condition which allows us to treat the problem as a system of equations on a string. In section 3 we consider the characteristics of the channels, introduce the Jost solutions and also describe the spectral data related to the continuous spectra. In section 4 we collect some known results, see e.g. [13, 14, 18], as well as some new ones about solutions of finite difference equations. These results will be used in the sequel. In particular in section 5 we construct the special solutions, i.e. solutions which correspond to wave incoming along one of the channels. Using these solutions we study the structure of discrete spectra, this is done in section 6.

As it was already mentioned, no scattering matrix can be defined for the whole spectra, however the scattering coefficients possess some symmetry which plays the crucial role in our construction. This symmetry is described in section 7. In section 8 we return to discrete spectra and connect each eigenvalue of the problem with normalized matrix of eigenfunctions which gives the energies, completing the set of spectral data.

In section 9 we collect all previous results and finally obtain equations of the inverse scattering problem. Section 10 contains some concluding remarks.

Acknowledgments. We thank V. A. Marchenko, who suggested the problem, and also for numerous fruitful discussions.

2. Geometry of the system and the boundary condition

We consider the systems $\mathcal{A}$ which have a "web-like" structure. Namely $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ where $\mathcal{A}_1$ is a central part, and $\mathcal{A}_0$ is a union of a finite number of semi-infinite channels. For such web-like systems the inverse spectral problem can be treated as an inverse scattering problems: sending a wave along one of the channels and observing how does it pass through the system we are trying to reconstruct the characteristics of the system, i.e. the values $L(\alpha, \beta)$.

Definition A sequence of particles $\sigma = \{\alpha(p)\}_{p=0}^\infty$ is a channel if, for $p > 0$ the particle $\alpha(p)$ interacts with the particles $\alpha(p-1)$ and $\alpha(p+1)$ only (and, perhaps with the external field), while $\alpha(0)$ interacts with $\alpha(1)$ and some other particles in $\mathcal{A}$ which do not belong to $\sigma$.

We will use the following notation:
- the set of all channels is $\mathcal{C}$, the channels will be denoted by $\sigma, \nu, \gamma$ etc.;
- the particles in $\sigma \in \mathcal{C}$ are $\sigma(0)$, $\sigma(1)$, $\sigma(2)$, ..., the point $\sigma(0)$ is called the attachment point of $\sigma$;
- $\Gamma := \{\sigma(0)\}_{\sigma \in \mathcal{C}}$, $\mathcal{A}_0 := \cup_{\sigma \in \mathcal{C}} \cup_{k=1}^\infty \sigma(k)$, $\mathcal{A}_1 := \mathcal{A} \setminus \mathcal{A}_0$.
- For $\sigma \in \mathcal{C}, k = 1, 2, \ldots$ we also denote
  \[ - b_{\sigma}(k-1) = L(\sigma(k-1), \sigma(1)), \quad a_{\sigma}(k) = L(\sigma(k), \sigma(1)) \]  (2.1)
so equation (1.1) on the channel $\sigma$ takes the form:
\[ - b_{\sigma}(k-1)\xi(\sigma(k-1)) + a_{\sigma}(k)\xi(\sigma(k)) - b_{\sigma}(k)\xi(\sigma(k+1)) = \lambda \xi(\sigma(k)). \]  (2.2)

We assume that the number of particles in the central part as well as the number of channels is finite:
\[ M := \sharp \mathcal{A}_1 < \infty, \quad \sharp \mathcal{C} < \infty \]
Also (for simplicity) we assume $\sigma(0) \neq \nu(0)$, $\sigma, \nu \in \mathcal{C}$, $\sigma \neq \nu$.

2.1. **Boundary conditions.** Let $\xi \in \mathcal{M}(\mathcal{A})$ be a solution to (1.1). Then it meets (2.2) and also, for each $\alpha \in \mathcal{A}_1$,

$$
\lambda \xi(\alpha) - \sum_{\beta \in \mathcal{A}_1} L(\alpha, \beta) \xi(\beta) = \sum_{\beta \in \mathcal{A}_0} L(\alpha, \beta) \xi(\beta)
$$

The only pairs $(\alpha, \beta) \in \mathcal{A}_1 \times \mathcal{A}_0$ for which $L(\alpha, \beta) \neq 0$ are of the form $(\sigma(0), \sigma(1))$, $\sigma \in \mathcal{C}$, so with account of (2.1) this relation can be written as

$$
\lambda \xi(\alpha) - \sum_{\beta \in \mathcal{A}_1} L(\alpha, \beta) \xi(\beta) = \begin{cases} 
-b_\nu(0) \xi(\nu(1)), & \alpha = \nu(0) \in \Gamma, \\
0, & \alpha \in \mathcal{A}_1 \setminus \Gamma
\end{cases}
$$

(2.3)

Consider the matrix

$$
\mathcal{L}_1 = (L(\alpha, \beta))_{\alpha, \beta \in \mathcal{A}_1}.
$$

Being a truncation of $\mathcal{L}$, the matrix $\mathcal{L}_1$ also is strictly positive. Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_M$ and $p_1, \ldots, p_M$ be its eigenvalues and the corresponding normalised eigenvectors. We may choose $p_j$’s to be real-valued. For $\lambda \notin \{\lambda_j\}_{j=1}^M$ the operator $\mathcal{L}_1 - \lambda I$ is invertible and

$$
(\mathcal{L}_1 - \lambda I)^{-1} = \mathcal{R}(\lambda) = (r(\alpha, \beta; \lambda))_{\alpha, \beta \in \mathcal{A}_1}, \quad r(\alpha, \beta; \lambda) = \sum_{j=1}^M \frac{p_j(\alpha)p_j(\beta)}{\lambda_j - \lambda}.
$$

Relation (2.3) can be then written as

$$
\xi(\alpha) = \sum_{\nu \in \mathcal{C}} r(\alpha, \nu(0); \lambda)b_\nu(0)\xi(\nu(1)); \quad \alpha \in \mathcal{A}_1.
$$

(2.4)

In particular for $\alpha = \sigma(0)$, $\sigma \in \mathcal{C}$ we obtain

$$
\xi(\sigma(0)) = \sum_{\nu \in \mathcal{C}} r(\sigma(0), \nu(0); \lambda)b_\nu(0)\xi(\nu(1)).
$$

(2.5)

After introducing vector notations

$$
\tilde{\xi}(k) := (\xi_{\sigma(k)})_{\sigma \in \mathcal{C}}, \quad k = 0, 1, \ldots, \quad \mathcal{R}(\lambda) = (r(\sigma(0), \nu(0); \lambda))_{\sigma, \nu \in \mathcal{C}}
$$

we can rewrite (2.3) as

$$
\tilde{\xi}(0) = \mathcal{R}(\lambda)\text{diag}\{b_\nu(0)\}_{\nu \in \mathcal{C}} \tilde{\xi}(1).
$$

(2.6)

This relation connects the values of solution $\xi$ on $\mathcal{A}_0$ and on $\Gamma$. It plays the role of the boundary condition for vector-valued scattering problem. The following statement holds.

**Theorem 2.1.** (See theorem 1 in [12]) Let $\lambda \notin \{\lambda_j\}_{j=1}^M$. A function $\xi \in \mathcal{M}(\mathcal{A}_0 \cup \Gamma)$ admits prolongation to a function $\xi \in \mathcal{M}(\mathcal{A})$ satisfying (1.1) if and only if it satisfies (2.2) and also the boundary condition (2.4). The prolongation is unique and can be defined by (2.4).
3. Characteristics of the channels and spectral data

3.1. Jost solutions. We assume that the channels are asymptotically homogeneous at infinity. Namely, for each \(\sigma \in \mathcal{C}\) there exist \(b_\sigma\) and \(a_\sigma\) such that

\[
a_\sigma(k) \to a_\sigma, \quad b_\sigma(k) \to b_\sigma \quad \text{as} \quad k \to \infty.
\]

Moreover

\[
\sum_{k=1}^{\infty} k \{|a_\sigma(k) - a_\sigma|^2 + |b_\sigma(k) - b_\sigma|^2\} < \infty.
\]  \hspace{1cm} (3.1)

This relation provides existence of Jost solutions on each channel \(\sigma \in \mathcal{C}\). It is well known, see e.g. [15, 18], that for each \(\sigma \in \mathcal{C}\) there is a family \(\{e_\sigma(k, \theta)\}_{k=1}^{\infty}\) of functions holomorphic inside the open disk \(\mathbb{D}\), continuous up to the boundary and, for each \(\theta \in \mathbb{T}\) and \(k \geq 1\),

\[
-b_\sigma(k-1)e_\sigma(k-1, \theta) + a_\sigma(k)e_\sigma(k, \theta) - b_\sigma(k)e_\sigma(k+1, \theta) = \lambda_\sigma(\theta)e_\sigma(k, \theta), \hspace{1cm} (3.2)
\]

where

\[
\lambda_\sigma(\theta) = a_\sigma - b_\sigma \left(\theta + \theta^{-1}\right). \hspace{1cm} (3.3)
\]

In addition

\[
e_\sigma(k, \theta) = \theta^k (1 + o(1)) \quad \text{as} \quad k \to \infty
\]

uniformly with respect to \(\theta \in \mathbb{T}\).

The functions \(e_\sigma(k, \theta)\) admit the representation

\[
e_\sigma(k, \theta) = c_\sigma(k) \sum_{m \geq k} a_\sigma(k, m) \theta^m, \quad k = 0, 1, \ldots. \hspace{1cm} (3.4)
\]

Here

\[
c_\sigma(k) = \prod_{p=k}^{\infty} \frac{b_\sigma(p)}{b_\sigma(p)}, \quad a_\sigma(k, k) = 1 \quad \text{and} \quad \lim_{k \to \infty} \sum_{m \geq k+1} |a_\sigma(k, m)| = 0
\]

For \(k \geq 1\) the coefficients \(a_\sigma(k), b_\sigma(k)\) can be expressed through the coefficients of the functions \(e_\sigma(k, \theta)\)

\[
\frac{a_\sigma(k) - a_\sigma}{b_\sigma} = a_\sigma(k-1, k) - a_\sigma(k, k+1);
\]

\[
\frac{b_\sigma^2(k)}{b_\sigma^2} = \frac{a_\sigma(k) - a_\sigma}{b_\sigma} a_\sigma(k, k+1) + a_\sigma(k, k+2) - a_\sigma(k-1, k+1) + 1.
\]

These relations can be obtained by substituting representation \((3.4)\) in \((3.2)\) and then comparing the coefficients with the same powers of \(\theta\).

In this article we assume that the coefficients \(a_\sigma(k), b_\sigma(k)\) approach their limit values faster than \((3.1)\). Namely we assume that, for some \(\epsilon > 0\),

\[
\sum_{k=1}^{\infty} (1 + \epsilon)^k \{|a_\sigma(k) - a_\sigma|^2 + |b_\sigma(k) - b_\sigma|^2\} < \infty.
\]

Under this condition the functions \(e_\sigma(k, \theta)\) admit holomorphic prolongation to some vicinity of the unit disk \(\mathbb{D}\), see e.g. chapter 10 in [18]. This allows us to avoid extra technicalities which are necessary in the general case.
3.2. Spectra, scattering data corresponding to the continuous spectra.

As the parameter $\theta$ runs through the unit circle $\mathbb{T}$, the corresponding value $\lambda_\sigma(\theta)$ defined by (respective $\sigma$ runs through the segment

$$I_\sigma = [a_\sigma - 2b_\sigma, a_\sigma + 2b_\sigma].$$

We assume for the simplicity that $\cup_{\sigma \in \mathcal{C}}(a_\sigma - 2b_\sigma, a_\sigma + 2b_\sigma)$ is a connected set and choose $a, b \in \mathbb{R}$ so that

$$I = \cup_{\sigma \in \mathcal{C}} I_\sigma = [a - 2b, a + 2b].$$

We need the Zhukovskii mappings $W, W_\sigma : \mathbb{D} \to \mathbb{C}$:

$$W : \omega \mapsto a - b(\omega + \omega^{-1}), \quad W_\sigma : \theta \mapsto a_\sigma - b_\sigma(\theta + \theta^{-1}),$$

(3.5)

$$\theta_\sigma(\omega) = W_\sigma^{-1} \circ W(\omega).$$

(3.6)

Further we choose the set $J_\sigma \subset [-1, 1]$, so that $\theta_\sigma$ maps $\mathbb{D}$ onto $\mathbb{D}_\sigma := \mathbb{D} \setminus J_\sigma$ and let

$$T_{\sigma^+}^\sigma = \theta_\sigma^{-1}(\mathbb{T}), \quad T_{\sigma^-}^\sigma = \theta_\sigma^{-1}(J_\sigma) = \mathbb{T} \setminus T_{\sigma^+}^\sigma.$$  

(3.7)

The set $J_\sigma$ consists of one or two segments attached to the points $\pm 1$. The functions $e(k, \theta_\sigma(\omega))$ are holomorphic in $\mathbb{D}$. Moreover, for $\omega \in T_{\sigma^+}^\sigma$, the functions $e(k, \theta_\sigma(\omega)^{-1})$ are well defined.

By a special solution which correspond to $\sigma \in \mathcal{C}$ we mean the function $\psi^\sigma(\alpha, \omega)$, $\omega \in \mathbb{D}$, $\alpha \in \mathcal{A}$ with the following properties:

1. For each $\alpha \in \mathcal{A}_0$ the function $\psi^\sigma(\alpha, \cdot)$ is holomorphic in a vicinity of $\overline{\mathbb{D}}$, except, perhaps, a finite set $\mathcal{O} \subset \overline{\mathbb{D}}$ where it has poles.

2. For each $\omega \in \mathbb{D} \setminus \mathcal{O}$, the function $\psi^\sigma(\cdot, \omega) \in \mathcal{M}(\mathcal{A})$ meets the equation

$$\mathcal{L}\psi^\sigma(\cdot, \omega) = \lambda \psi^\sigma(\cdot, \omega), \quad \lambda = a + b(\omega + \omega^{-1}).$$

3. For $\gamma \neq \sigma$ there exist functions $s_{\sigma\gamma}(\omega)$, meromorphic in a vicinity of $\overline{\mathbb{D}}$ and a function $s_{\sigma\gamma}(\omega)$ continuous on $T_{\sigma^+}^\sigma$ such that the function $\psi^\sigma$ has the following representation along the channels:

$$\psi^\sigma(\gamma(k), \omega) = \begin{cases} e_\sigma(k, \theta_\sigma(\omega)^{-1}) + s_{\sigma\gamma}(\omega)e_\gamma(k, \theta_\gamma(\omega)), & \gamma = \sigma; \\ s_{\gamma\sigma}(\omega)e_\gamma(k, \theta_\gamma(\omega)), & \gamma \neq \sigma, \quad \omega \in T_{\sigma^+}^\sigma, \end{cases}$$

(3.8)

and for $\omega \in \mathbb{D}$

$$\psi^\sigma(\gamma(k), \omega) = \begin{cases} \theta_\sigma(\omega)^{-k}(1 + o(1)), & \text{as } k \to \infty, \quad \gamma = \sigma; \\ s_{\gamma\sigma}(\omega)e_\gamma(k, \theta_\gamma(\omega)), & \gamma \neq \sigma, \quad \omega \in \mathbb{D} \setminus T_{\sigma^+}^\sigma. \end{cases}$$

The scattering coefficients, $s_{\sigma\gamma}(\omega)$ are the elements of the scattering matrix, corresponding to the problem \((\mathcal{L}, \mathcal{D})\). We mention that so far they are defined on different arcs $T_{\sigma^+}^\sigma \subset \mathbb{T}$. In section 5 we will prove existence and uniqueness of such special solutions for each $\sigma \in \mathcal{C}$.

The solution $\psi^\sigma$ corresponds to the wave incoming along the channel $\sigma$ and distributing through the whole system. This wave is well-defined if $\omega \in T_{\sigma^+}^\sigma$ (respectively $\lambda \in I_\sigma$), for these values the equation (2.2) along the channel $\sigma$ admits two independent bounded solutions corresponding to in- and out-coming waves. For $\omega \in T_{\sigma^+}^\sigma \cap T_{\gamma^+}^\gamma$ the wave incoming along $\sigma$ generates an outcome wave along the
channel $\gamma$, this wave can be observed at infinity. For $\omega \in T_{\sigma}^* \cap T_{\gamma}^-$ the wave incoming along $\sigma$ generates an exponentially decaying wave $\gamma$, observation of phase of such wave at infinity is virtually impossible, one can measure the absolute values only. These reasoning explain the definition of the spectral data.

**Definition 3.1.** By continuous spectral data corresponding to the channel $\sigma \in C$ we mean the set of functions

$$S_{\sigma} = \{s_{\gamma\sigma}(\omega) : \omega \in T_{\sigma}^* \cap T_{\gamma}^+; \gamma \in C\} \cup |s_{\gamma\sigma}(\omega)| : \omega \in T_{\sigma}^* \cap T_{\gamma}^-; \gamma \in C\}
$$

By full continuous spectral data we mean

$$S = \bigcup_{\sigma \in C} S_{\sigma}.
$$

Later in sections 6 and 8 we will discuss discrete spectrum, which corresponds to eigenfunctions of the operator $\mathcal{L} : l^2(A) \to l^2(A)$.

4. Properties of solutions of finite difference equations

We need to establish some properties of solutions to (1.1) as well as solutions to difference equations along the channels. In this section we collect known (see e.g. [13, 14, 18]) properties of solutions of the finite-difference equation as well as some new statements related to solutions of (1.1)

$$-b(k-1)x(k-1) + a(k)x(k) - b(k)x(k+1) = \lambda x(k), \quad k = 1, 2, \ldots . \quad (4.1)$$

with real coefficients $a(k)$, $b(k)$. We also assume that $b(k) > 0$ in order that the Jost solutions will be well defined, see e.g. [18].

Given two functions $x = x(k)$, $y = y(k)$ on the integers we define their Wronskian $\{x, y\}$ as

$$\{x, y\}(k) = x(k)y(k+1) - x(k+1)y(k), \quad k = 0, 1, \ldots .
$$

1. Let $x, y$ be solutions to (4.1). Then, for all $N$,

$$b(N)\{x, y\}(N) - b(0)\{x, y\}(0) = 0 \quad (4.2)$$

and

$$b(N)\{x, \bar{x}\}(N) - b(0)\{x, \bar{x}\}(0) = (\lambda - \bar{\lambda}) \sum_{k=1}^{N} |x(k)|^2. \quad (4.3)$$

If in addition $\lambda = \lambda(\omega)$ and $x = x(k, \omega)$ are differentiable functions of a parameter $\omega$, then

$$b(N)\{\dot{x}, \bar{x}\}(N) - b(0)\{\dot{x}, \bar{x}\}(0) = \bar{\lambda} \sum_{k=1}^{N} |x(k)|^2 + (\lambda - \bar{\lambda}) \sum_{k=1}^{N} \dot{x}(k)x(k), \quad (4.4)$$

here and in what follows the dot denotes derivative with respect to $\omega$.

If in addition

$$\sum_{k=1}^{\infty} k(|b(k) - b| + |a(k) - a|) < \infty,
$$

and $e(\theta) = e(k, \theta)$ are the corresponding Jost solutions, relation (4.3) yields

$$b(0)\{e(\theta), \bar{e}(\theta)\}(0) = \begin{cases} b(\bar{\theta} - \theta), & |\theta| = 1; \\ b(\bar{\theta} - \theta)(|\theta|^{-2} - 1) \sum_{k=1}^{\infty} |e(k, \theta)|^2, & |\theta| < 1. \end{cases} \quad (4.5)$$
Lemma 4.1. Let \( \omega \in \mathbb{T} \cup (-1,1) \) and \( \sigma \in \mathcal{C} \) be such that \( \theta_{\sigma}(\omega) \in (-1,1) \). Then

\[
- b_{\sigma}(0) \{ \dot{e}_\sigma, (\theta_{\sigma}(\omega))e_\sigma(\theta_{\sigma}(\omega)) \} \sigma(0) = \lambda \sum_{k=1}^{\infty} e_\sigma(k, \theta_{\sigma}(\omega))^2. 
\] (4.6)

This is an immediate consequence of (4.4).

2. We will consider Wronskians, which correspond to various channels \( \sigma \in \mathcal{C} \). Given two functions \( \xi, \eta \in \mathcal{M}(\mathcal{A}) \) and \( \sigma \in \mathcal{C} \) we denote

\[
\{\xi, \eta\}_\sigma(k) := \{\xi(\sigma(k))\eta(\sigma(k+1)) - \xi(\sigma(k+1))\eta(\sigma(k))\}
\]

Remark. If both \( \xi \) and \( \eta \) are solutions of (1.1) it follows from (4.2) that the quantity \( b_{\sigma}(k)\{\xi, \eta\}_\sigma(k) \) depends on \( \sigma \) only.

Lemma 4.2. Let \( \xi, \eta \in \mathcal{M}(\mathcal{A}) \) be solutions to (1.1). Then

\[
\sum_{\sigma \in \mathcal{C}} b_{\sigma}(0) \{\xi, \eta\}_\sigma(0) = 0
\] (4.7)

Proof. For \( \alpha \in \mathcal{A}_1 \) we have

\[
\sum_{\beta \in \mathcal{A}_1} L(\alpha, \beta) \xi(\beta) + \sum_{\beta \in \mathcal{A}_0} L(\alpha, \beta) \xi(\beta) = \lambda \xi(\alpha),
\]

\[
\sum_{\beta \in \mathcal{A}_1} L(\alpha, \beta) \eta(\beta) + \sum_{\beta \in \mathcal{A}_0} L(\alpha, \beta) \eta(\beta) = \lambda \eta(\alpha),
\]

We multiply these relations by \( \eta(\alpha) \) and \( \xi(\alpha) \) respectively. Summation with respect to \( \alpha \in \mathcal{A}_1 \) gives

\[
\sum_{\alpha, \beta \in \mathcal{A}_1} L(\alpha, \beta) \xi(\beta) \eta(\alpha) + \sum_{\alpha \in \mathcal{A}_1, \beta \in \mathcal{A}_0} L(\alpha, \beta) \xi(\beta) \eta(\alpha) = \lambda \sum_{\alpha \in \mathcal{A}_1} \xi(\alpha) \eta(\alpha),
\]

\[
\sum_{\alpha, \beta \in \mathcal{A}_1} L(\alpha, \beta) \eta(\beta) \xi(\alpha) + \sum_{\alpha \in \mathcal{A}_1, \beta \in \mathcal{A}_0} L(\alpha, \beta) \xi(\alpha) \eta(\beta) = \lambda \sum_{\alpha \in \mathcal{A}_1} \xi(\alpha) \eta(\alpha).
\]

The right-hand sides in these relations coincide as well as the first terms in the left-hand sides (since \( L(\alpha, \beta) = L(\beta, \alpha) \)). Therefore

\[
\sum_{\alpha \in \mathcal{A}_1, \beta \in \mathcal{A}_0} L(\alpha, \beta) \{ \xi(\beta) \eta(\alpha) - \xi(\alpha) \eta(\beta) \} = 0.
\]

This proves the lemma since the only option for \( L(\alpha, \beta) \neq 0 \) for \( \alpha \in \mathcal{A}_1, \beta \in \mathcal{A}_0 \) is \( \alpha = \sigma(0), \beta = \sigma(1) \) for some \( \sigma \in \mathcal{C} \) and in this case \( L(\sigma(1), \sigma(0)) = -b_{\sigma}(0) \). \( \square \)

3. We need a special statement in order to calculate the energy of eigenfunctions of the operator \( L \).

Lemma 4.3. Let a function \( \xi(\alpha) = \xi(\alpha, \omega) \in \mathcal{M}(\mathcal{A}) \) be differentiable with respect to \( \omega \) in a neighborhood of \( \omega \in \mathbb{T} \) and \( \xi(\alpha) \) satisfy the equation

\[
\lambda(\omega)\xi(\alpha, \omega) = \sum_{\beta \in \mathcal{A}} L(\alpha, \beta) \xi(\beta, \omega).
\] (4.8)
Let also a function \( \eta(\alpha) \) meet this equation for \( \omega = \tilde{\omega} \). Then
\[
\hat{\lambda}(\hat{\omega}) \sum_{\alpha \in \mathcal{A}_1} \xi(\alpha, \hat{\omega})\eta(\alpha) = \sum_{\sigma \in \mathcal{C}} b_{\sigma}(0)\{\xi(\hat{\omega}), \eta\}_{\sigma}(0).
\]
In particular
\[
\hat{\lambda}(\hat{\omega}) \sum_{\alpha \in \mathcal{A}_1} |\xi(\alpha, (\hat{\omega}))|^2 = \sum_{\sigma \in \mathcal{C}} b_{\sigma}(0)\{\xi(\hat{\omega}), \xi(\hat{\omega})\}_{\sigma}(0).
\]
Proof. The statement uses the same idea as in lemma 2.1 in [12], we repeat here the construction:
Differentiate \( \xi(\alpha, \omega) \) with respect to \( \omega \):
\[
\hat{\lambda}(\omega)\xi(\alpha, \omega) + \lambda(\omega)\dot{\xi}(\alpha, \omega) = \sum_{\beta \in \mathcal{A}} L(\alpha, \beta)\dot{\xi}(\beta, \omega).
\]
Besides
\[
\lambda(\omega)\eta(\alpha) = \lambda(\hat{\omega})\eta(\alpha) = \sum_{\beta \in \mathcal{A}} L(\alpha, \beta)\eta(\beta).
\]
Combining these equations we obtain
\[
\hat{\lambda}(\hat{\omega})\xi(\alpha, \omega)\eta(\alpha) = \sum_{\beta \in \mathcal{A}} L(\alpha, \beta)\xi(\beta, \hat{\omega})\eta(\alpha) - \dot{\xi}(\alpha, \hat{\omega})\eta(\beta),
\]
and
\[
\hat{\lambda}(\hat{\omega}) \sum_{\alpha \in \mathcal{A}_1} \xi(\alpha, \omega)\eta(\alpha) = \sum_{\alpha \in \mathcal{A}_1} \sum_{\beta \in \mathcal{A}_1} L(\alpha, \beta)\xi(\beta, \hat{\omega})\eta(\alpha) - \dot{\xi}(\alpha, \hat{\omega})\eta(\beta) + \sum_{\alpha \in \mathcal{A}_1} \sum_{\beta \in \mathcal{A}_0} L(\alpha, \beta)\xi(\beta, \hat{\omega})\eta(\alpha) - \dot{\xi}(\alpha, \hat{\omega})\eta(\beta).
\]
The first summand in the right-hand side of this relation vanishes because it is anti-symmetric in \( \alpha \) and \( \beta \). In the second summand the only non-zero coefficient \( L(\alpha, \beta) \) appears in the case \( \alpha = \sigma(0), \beta = \sigma(1) \) for some \( \sigma \in \mathcal{C} \) and \( L(\sigma(0), \sigma(1)) = -b_{\sigma}(0) \).

5. CONSTRUCTION OF THE SPECIAL SOLUTIONS

Consider the diagonal matrices
\[
B := \text{diag}\{b_{\sigma}\}_{\sigma \in \mathcal{C}}, \quad B(0) := \text{diag}\{b_{\sigma}(0)\}_{\sigma \in \mathcal{C}};
\]
and also the matrix-functions in \( \mathcal{D} \)
\[
\mathcal{E}(k, \omega) := \text{diag}\{e(k, \theta_{\sigma}(\omega))\}_{\sigma \in \mathcal{C}}, \quad P(k, \omega) := \text{diag}\{p_{\sigma}(k, \omega)\}_{\sigma \in \mathcal{C}},
\]
here \( \{p_{\sigma}(k, \omega)\}_{k = 0}^{\infty} \) satisfy the equation on the channels:
\[
-b_{\sigma}(k - 1)p_{\sigma}(k - 1, \omega) + a_{\sigma}(k)p_{\sigma}(k, \omega) - b_{\sigma}(k)p_{\sigma}(k + 1, \omega) = \lambda(\omega)p_{\sigma}(k, \omega)
\]
for \( k = 1, 2, \ldots, \sigma \in \mathcal{C} \) and with boundary conditions
\[
p_{\sigma}(0, \omega) = 1, \quad p_{\sigma}(1, \omega) = 0, \quad \sigma \in \mathcal{C}
\]
The functions \( \mathcal{E}(k, \omega) \) and \( P(k, \omega) \) are holomorphic in a neighborhood of \( \mathcal{D} \), except, perhaps zero, where \( P(k, \omega) \) may have poles.
Moreover, it is well known, see [15, 18], that the Jost solutions form a fundamental system of solutions of the finite-difference equation (4.1). If \( |\theta| = 1 \) and \( |\theta| \neq 1 \), it follows from (4.5) that \( e(\theta), \tilde{e}(\theta) = e(\theta^{-1}) \) are independent solutions of (4.1).
and any other solution \( \{ x(k, \theta) \}_{k \geq 0} \) can be expressed as \( x(k, \theta) = m(\theta)e(k, \theta) + n(\theta)e(k, \theta^{-1}) \), with \( m(\theta), n(\theta) \) independent of \( k \). Thus, for \( \omega \in T^+_\omega \) the function \( p_\sigma(k, \omega) \) may be expressed in terms of the corresponding Jost solutions:

\[
p_\sigma(k, \omega) = \frac{b_\sigma(0)e_\sigma(k, \theta_\sigma(\omega))e_\sigma(1, \theta_\sigma(\omega)^{-1}) - e_\sigma(k, \theta_\sigma(\omega)^{-1})e_\sigma(1, \theta_\sigma(\omega))}{\theta_\sigma(\omega)^{-1} - \theta_\sigma(\omega)}.
\]

(5.2)

In order to construct the special solutions we need auxiliary operator-function \( T(\omega) : l^2(\mathcal{C}) \to l^2(\mathcal{C}) \) defined as

\[
T(\omega) = \mathcal{E}(0, \omega) - \mathcal{R}(\lambda(\omega))\mathcal{E}(1, \omega), \quad \omega \in \mathbb{D}.
\]

(5.3)

This function is holomorphic in a vicinity of \( \mathbb{D} \), except the set \( \mathcal{O} \) of poles of the function \( \mathcal{R}(\lambda(\omega)) \).

Denote

\[
\Delta_\sigma(\omega) = \begin{cases} 
b_\sigma, \\
b_\sigma(|\theta_\sigma(\omega)|^{-2} - 1) \sum_{k=1}^{\infty} |e_\sigma(k, \theta_\sigma(\omega))|^2, 
\end{cases}
\]

and

\[
\Delta(\omega) = \text{diag}\{\Delta_\sigma(\omega)\}_{\sigma \in \mathcal{C}}, \quad \Phi(\omega) = \text{diag}\{\bar{\theta}_\sigma(\omega) - \theta_\sigma(\omega)\}_{\sigma \in \mathcal{C}}.
\]

(5.4)

The operator \( \Delta(\omega) \) is positive uniformly with respect \( \omega \in \mathbb{D} \), i.e., for some \( C > 0 \),

\[
\langle \Delta(\omega)x, x \rangle \geq C||x||^2, \quad x \in l^2(\mathcal{C}), \quad \omega \in \mathbb{D}.
\]

Besides relation (4.5) now reads

\[
B(0) \{ \mathcal{E}(0, \omega)\mathcal{E}(1, \omega)^* - \mathcal{E}(0, \omega)^*\mathcal{E}(1, \omega) \} = \Phi(\omega)\Delta(\omega),
\]

(5.5)

Lemma 5.1. (See lemma 3.1 in [12]) The following inequality holds for all \( \theta \in \mathbb{D} \setminus \mathcal{O} \), \( x = (x_\sigma)_{\sigma \in \mathcal{C}} \in l^2(\mathcal{C}) \)

\[
|\langle \mathcal{E}(1, \omega)^* B(0)T(\omega)x, x \rangle| \geq 3|\langle \mathcal{E}(1, \omega)^* B(0)T(\omega)x, x \rangle| \geq C \sum_{\sigma \in \mathcal{C}}|\bar{\theta}_\sigma(\omega) - \theta_\sigma(\omega)||x_\sigma|^2.
\]

(5.6)

Proof. We have

\[
2\Im\langle \mathcal{E}(1, \omega)^* B(0)T(\omega)x, x \rangle = |\langle \mathcal{E}(1, \omega)^* B(0)T(\omega)x, x \rangle - \langle T(\omega)^* B(0)\mathcal{E}(1, \omega)x, x \rangle|
\]

and by (5.3)

\[
\mathcal{E}(1, \omega)^* B(0)T(\omega) - T(\omega)^* B(0)\mathcal{E}(1, \omega) = B(0)(\mathcal{E}(0, \omega)\mathcal{E}(1, \omega)^* - \mathcal{E}(0, \omega)^*\mathcal{E}(1, \omega) + (B(0)\mathcal{E}(1, \omega))^* (\mathcal{R}(\lambda(\omega))^* - \mathcal{R}(\lambda(\omega))))(B(0)\mathcal{E}(1, \omega))
\]

The lemma now follows from (5.4) - (5.5) and from the fact that

\[
\mathcal{R}(\lambda(\omega))^* - \mathcal{R}(\lambda(\omega)) = b(\bar{\omega} - \omega) (|\omega|^{-2} - 1) \sum_{l=1}^{M} \frac{p_l(\sigma(0))p_l(\nu(0))}{|\lambda_l - \lambda(\omega)|^2} \Delta_1(\omega),
\]

here \( \Delta_1(\theta) \) is a non-negative operator.

Corollary 5.1. Operators \( T(\omega) \) are invertible for all non-real \( \omega \) in the open disk \( \mathbb{D} \setminus \{ \mathcal{O} \cup \{ \pm 1 \} \} \).
Proof. Indeed, fix an \( \omega \in \mathbb{D} \setminus [-1, 1] \) and denote \( \delta(\omega) = \inf_{\sigma} \{ |\Im \theta_{\sigma}(\omega)| \} > 0 \). Relation (5.6) now reads

\[
\|T(\omega)x\| > C\delta(\omega)\|x\|,
\]

which yields invertibility of \( T(\omega) \). \( \square \)

Since \( T(\omega) \) is an analytic function, we can now claim that \( T(\omega) \) is invertible in a vicinity of \( \mathbb{D} \), except perhaps at a finite set of points.

Consider the analytic matrix functions:

\[
D(\omega) := \text{diag}\{\theta_{\sigma}(\omega)^{-1} - \theta_{\sigma}(\omega)\}
\]

and

\[
U(k, \omega) = \left[ -P(k, \omega) + \mathcal{E}(k, \omega)T(\omega)^{-1} \right] BB(0)^{-1}D(\omega)\mathcal{E}(1, \omega)^{-1}. \quad (5.7)
\]

**Lemma 5.2.** Consider the vector function on \( A_0 \cup \Gamma \)

\[
\psi_{\sigma}^{\gamma}(k, \omega) = (\psi_{\sigma}^{\gamma}(k, \omega))_{\gamma \in \mathbb{C}} := U(k, \omega)n_{\sigma}.
\]

This function satisfies the boundary condition (2.6) and, for \( \omega \in T^+_\sigma \), admits representation (3.7) along the channels. Thus by (2.4) it may be prolonged to a special solution of (1.1).

**Proof.** Representation (3.7) for \( \omega \in T^+_\sigma \) is just a consequence of (5.7) and (5.2).

In order to prove that \( \psi_{\sigma}^{\gamma}(k, \omega) \) meets the boundary condition for \( \omega \in T^+_\sigma \), we prove that this condition is met by the whole matrix-function \( U(k, \omega) \):

\[
U(0, \omega) = \mathcal{R}(\lambda(\omega))B(0)U(1, \omega), \quad \omega \in \mathbb{D} : \quad (5.9)
\]

This implies that \( \psi_{\sigma}^{\gamma}(k, \omega) \) also meets the boundary condition. Relation (5.9) is straightforward: after factoring out the inessential factor \( BB(0)^{-1}D(\omega)\mathcal{E}(1, \omega)^{-1} \) in the definition of (5.7) it becomes

\[
-I + \mathcal{E}(0, \omega)T(\omega)^{-1} = \mathcal{R}(\lambda(\omega))B(0)\mathcal{E}(1, \omega)T(\omega)^{-1},
\]

which is just the definition of \( T(\omega) \).

So far the special solution is not defined at a point \( \omega_0 \in T^+_\sigma \) in case \( \omega_0 \) is a pole of \( U(k, \omega) \). This case will be considered in the following sections: in section 6 we prove that all poles of \( U(k, \omega) \) are simple, later in section 8 we show that, if \( \omega_0 \in T^+_\sigma \), the function \( U(k, \omega)n_{\sigma} \) is continuos at \( \omega_0 \) even if \( \omega_0 \) is a pole of \( U(k, \omega) \). In particular

\[
\text{Res}_{\omega_0} u_{\nu,\sigma}(\omega) = 0, \quad \nu \in \mathbb{C} \quad (5.10)
\]

\( \square \)

6. Singularities of \( U(k, \omega) \).

The matrix function \( U(k, \omega) \) is analytic in a vicinity of \( \overline{\mathbb{D}} \), in particular it has a finite number of poles which belong to \( \overline{\mathbb{D}} \).

**Lemma 6.1.** There is a finite set \( \Omega \subset \overline{\mathbb{D}} \) such that all poles of the matrix function \( U(k, \omega) \in \overline{\mathbb{D}} \) belong to \( \Omega \cup \{0\} \). In the origin \( U(k, \omega) \) has pole of order \( k \), all poles in \( \Omega \) are simple.
Proof. The proof follows the pattern of Lemma 4.1 in [12]. We rewrite \((5.7)\) as

\[
U(k, \omega) = -P(k, \omega)BB(0)^{-1}D(\omega)E(1, \omega)^{-1} + \mathcal{E}(k, \omega)T(\omega)^{-1}BB(0)^{-1}D(\omega)E(1, \omega)^{-1}.
\]  

(6.1)

Therefore all poles of \(U(k, \omega)\) in \(\mathbb{C} \setminus \{0\}\) belong to the finite set \(\Omega \cup \{0\}\) where \(\Omega\) includes all poles of \(T(\omega)^{-1}\) and \(E(1, \omega)^{-1}\) in \(\mathbb{C}\).

That at the origin the functions \(U(0, \omega)\) and \(U(1, \omega)\) have poles of order 0 and 1 respectively follows from the initial conditions for \(P\). For \(k \geq 2\) the main contribution to singularity of \(U(k, \omega)\) at zero comes from the first term in the right-hand side of \((6.1)\) because \(P(k, \omega)\) is a polynomial of degree \(k - 2\) with respect to \(\lambda = a - b(\omega + \omega^{-1})\).

It now suffices to study the singularities of the second term in the right-hand side in \((6.1)\). Let \(\Omega\) be the set of all such singularities in \(\mathbb{C}\). These singularities come from the poles of \(T(\omega)^{-1}\), that is, when \(\det(T(\omega)) = 0\) and also from the zeros of \(\det(E(1, \omega)) = \prod_{\sigma \in \mathcal{C}} e_\sigma(1, \theta_\sigma(\omega))\).

Actually \(U(k, \omega)\) cannot have poles outside \(\mathbb{T} \cup (-1, 1)\). This is due to the poles of \(T(\omega)^{-1}\) located in \(\mathbb{T} \cup (-1, 1)\) and \(e_\sigma(1, \theta_\sigma(\omega)) = 0\) only if \(\theta_\sigma(\omega) \in (-1, 1)\), see \((4.5)\), and thus \(\omega \in \mathbb{T} \cup (-1, 1)\).

Let \(\tilde{\omega} \in (-1, 0) \cup (0, 1)\). We are going to use \((5.6)\) as \(\omega\) approaches \(\tilde{\omega}\). We have \(|\theta_\sigma(\omega) - \theta_\sigma(\tilde{\omega})| \approx |\omega - \tilde{\omega}|\), so for any \(y \in l^2(\mathcal{C})\) relation \((5.6)\) with

\[
x = T(\omega)^{-1}BB(0)^{-1}D(\omega)E(1, \omega)^{-1}y
\]

gives

\[
|\langle (1, \omega)^*E(1, \omega)^{-1}BD(\omega)y, T(\omega)^{-1}BB(0)^{-1}D(\omega)E(1, \omega)^{-1}y \rangle| \geq C|\omega - \tilde{\omega}|\|T(\omega)^{-1}BB(0)^{-1}D(\omega)E(1, \omega)^{-1}y\|^2.
\]

Since \(E(1, \omega)^*E(1, \omega)^{-1}\) is unitary and also \(|\det(D(\omega))|\) stays bounded from below near \(\tilde{\omega}\), the Schwartz inequality gives

\[
\|T(\omega)^{-1}BB(0)^{-1}D(\omega)E(1, \omega)^{-1}y\| \leq \text{Const} \frac{\|y\|}{|\omega - \tilde{\omega}|} \text{ as } \omega \to \tilde{\omega},
\]

this is possible only in case \(\tilde{\omega}\) is a simple pole of \(T(\omega)^{-1}BB(0)^{-1}D(\omega)E(1, \omega)^{-1}\).

In case when \(\tilde{\omega} \in \mathbb{T}\), the reasoning goes in a similar way, it suffices to let \(\omega\) approach \(\tilde{\omega}\) in a way that \(|\theta_\sigma(\omega) - \theta_\sigma(\tilde{\omega})| \approx |\omega - \tilde{\omega}|\).

\[\square\]

7. Relation for the Scattering Coefficients

The special solution \(\psi^\sigma(\alpha, \omega)\) is now well-defined for all \(\omega \in \mathbb{T} \setminus \Omega\). Thus the non-diagonal scattering coefficients \(s_{\sigma \gamma}(\omega)\) \(\sigma \neq \gamma\) are also well-defined for all \(\omega \in \mathbb{T} \setminus \Omega\). Together with \(\psi^\sigma(\omega)\) they are analytic in a vicinity of \(\overline{\mathbb{D}}\).

The scattering coefficients \(s_{\sigma \sigma}(\omega)\) are so far well-defined for \(\omega \in T^+_\omega\) only. For \(\omega \in T^\omega_\rho\) the corresponding value \(\theta_\sigma(\omega)\) belongs to the unit disk. One can construct (similarly to how this is done in Theorem 1.4.1 in [1] for the continuous case) a real-valued solution \(e_\sigma^{(1)}(k, \theta_\sigma)\) of \((5.2)\) such that

\[
e_\sigma^{(1)}(k, \theta_\sigma) = \theta_\sigma^{-k}(1 + o(1)), \ k \to \infty.
\]

(7.1)
This choice is not unique since by adding any multiple of \( e_{\sigma} \) we obtain a solution which still meets this relation. However we fix some choice of functions \( e_{\sigma}^{(1)}(k, \theta_\sigma) \). The functions \( p_{\sigma} \) can be represented as
\[
p_{\sigma}(k, \omega) = \frac{b_{\sigma}(0) e_{\sigma}(k, \theta_\sigma(\omega)) e_{\sigma}^{(1)}(1, \theta_\sigma(\omega)) - e_{\sigma}^{(1)}(k, \theta_\sigma(\omega)) e_{\sigma}(1, \theta_\sigma(\omega))}{\theta_\sigma(\omega)^{-1} - \theta_\sigma(\omega)},
\]
which yields
\[\psi^\sigma(\sigma(k), \omega) = \theta_\sigma(\omega)^{-k}(1 + o(1)), \ k \to \infty\]
as it should be for a special solution.

**Lemma 7.1.** For each \( \sigma, \gamma \in \mathcal{C}, \ \sigma \neq \gamma \) we have
\[
b_\sigma(\theta_\gamma(\omega)^{-1} - \theta_\sigma(\omega)) s_{\gamma, \sigma}(\omega) = b_\sigma(\theta_\sigma(\omega)^{-1} - \theta_\sigma(\omega)) s_{\sigma, \gamma}(\omega), \ \omega \in \mathbb{T} \setminus \Omega
\]
In addition the scattering coefficients \( s_{\gamma, \sigma}(\omega) \) and \( s_{\sigma, \gamma}(\omega) \) are continuous up to \( \mathbb{T} \setminus \Omega \)

**Proof.** Relation (7.3) follows from (4.7) for \( \xi = \psi^\sigma(\omega), \ \eta = \psi^\gamma(\omega) \) if one takes into account that according (3.7) and (3.8) we have
\[
b_\nu(0)\{\psi^\sigma, \psi^\gamma\}_\nu(0) = \begin{cases} -b_\sigma(\theta_\gamma(\omega)^{-1} - \theta_\sigma(\omega)) s_{\gamma, \sigma}(\omega), & \nu = \sigma; \\ b_\gamma(\theta_\gamma(\omega)^{-1} - \theta_\sigma(\omega)) s_{\sigma, \gamma}(\omega), & \nu = \gamma; \\ 0, & \nu \neq \sigma, \gamma. \end{cases}
\]
Continuity of the scattering coefficients \( s_{\gamma, \sigma}(\omega) \) follows from the definition of \( U(k, \omega) \) and the fact that it has finitely many poles contained in \( \Omega \).

**Corollary 7.1.** Let \( \omega \in \mathbb{T} \), then
\[
\text{Res}_{\omega} u_{\nu, \sigma}(k, \omega) = \text{Res}_{\omega} u_{\nu, \sigma}(k, \omega) = 0, \ \ \omega \in T_{\sigma}^+ \cup T_{\nu}^+, \ \ \nu \in \mathcal{C}
\]

**Proof.** Denote
\[
A_{\nu, \sigma} = \text{det}(\bar{T}_{\nu, \sigma}(\omega)), \ \ \omega \in \mathbb{T}, \ \ \nu, \sigma \in \mathcal{C},
\]
where \( \bar{T}_{\nu, \sigma}(\omega) \) denotes the matrix which comes from \( T(\omega) \) once we have removed the \( \sigma \)-column and the \( \nu \)-row.

Let now \( \omega_0 \in \mathbb{T}, \ \lambda(\omega_0), \) be a regular point of \( \mathcal{R}(\lambda(\omega)) \) and \( \text{det}(T(\omega_0)) = 0 \). Then there exists \( \mathbf{x} = \{x_\sigma\} \in l^2(\mathcal{C}) \) such that \( T(\omega_0)\mathbf{x} = 0 \), thus the vector function \( \tilde{\xi}(k) = \mathcal{E}(k, \omega_0)\mathbf{x} \) satisfies the equations (2.1) on the channels as well as the boundary conditions (2.6). Hence it can be prolonged to a solution of the whole problem (1.1).

For \( \mathbf{x} \in l^2(\mathcal{C}) \), denote \( \text{supp} \ \mathbf{x} = \{\sigma \in \mathcal{C}; \ x_\sigma \neq 0\} \) and let \( \xi = \{\xi(\alpha)\}_{\alpha \in \mathcal{A}} \) be a solution to (1.1) with \( \lambda = \lambda(\omega_0) \) which is obtained by prolongation of \( \tilde{\xi}(k) \). Then \( \eta(\alpha) = \xi(\alpha) \) also solves this problem. Relation (4.7) yields
\[
0 = \sum_{\sigma \in \text{supp} \ \mathbf{x}} b_\sigma(0)\{\xi, \eta\}_\sigma(0) = \sum_{\sigma \in \text{supp} \ \mathbf{x}} b_\sigma |x_\sigma|^2(\bar{\theta}_\sigma(\omega_0) - \theta_\sigma(\omega_0))
\]
Therefore \( \omega_0 \in T_{\sigma}^+ \) for each \( \sigma \in \text{supp} \ \mathbf{x} \). Thus \( A_{\nu, \sigma}(\omega_0) = 0, \ \omega_0 \in T_{\nu}^+, \ \sigma \in \mathcal{C} \).

Since the poles of \( U(k, \omega) \) are simple and (4.5) implies that \( e_{\nu}(1, \theta_\nu(\omega)) \neq 0 \) if \( \omega \in T_{\nu}^+ \), we obtain
\[
\text{Res}_{\omega_0} u_{\nu, \sigma}(k, \omega) = 0
\]
A simple application of lemma 7.1 gives us \( \text{Res}_{\omega_0} u_{\nu, \sigma}(k, \omega) = 0 \) \( \Box \)
Corollary 7.2. Let $\omega \in T_\sigma^+ \cup T_\nu^+$, $\sigma, \nu \in \mathbb{C}$ and $\sigma \neq \nu$. Then
\[ s_{\nu\sigma}(\omega^{-1}) = \overline{s_{\nu\sigma}(\omega)} \]
In particular $|s_{\nu\sigma}(\omega)|^2 = s_{\nu\sigma}(\omega)s_{\nu\sigma}(\omega^{-1})$

Proof. We have $u_{\nu\sigma}(k, \omega) = e_\nu(k, \theta_\nu(\omega))s_{\nu\sigma}(\omega)$ and by construction of the matrix $U(k, \omega)$, we know that
\[ u_{\nu\sigma}(k, \omega^{-1}) = \overline{u_{\nu\sigma}(k, \omega)}, \quad \omega \in T_\sigma^+ \cup T_\nu^+ \]
and the corollary follows. \hfill \square

8. Discrete spectra of the operator $L$

Lemma 8.1. Let $\hat{\omega} \in \mathbb{D} \setminus \{0\}$ be a pole of $U(k, \omega)$. Then $\lambda(\hat{\omega}) = a - b(\hat{\omega} + \hat{\omega}^{-1})$ is an eigenvalue of (1.1). If in addition $\hat{\omega} \in T_\sigma^+$, then all elements $u_{\nu\sigma}(k, \omega)$, $\nu \in \mathbb{C}$ are regular at $\hat{\omega}$.

Proof. Let $\hat{\omega} \in \mathbb{D} \setminus \{0\}$ be a (simple) pole of $U(k, \omega)$. Denote
\[ a_\nu(\omega) = \frac{\theta_\nu(\omega) - \theta_\nu(\hat{\omega})^{-1}}{\theta_\nu(\omega) - \theta_\nu(\hat{\omega})} (\omega - \hat{\omega}), \quad \nu \in \mathbb{C}; \quad A(\omega) = \text{diag}\{a_\nu(\omega)\}_{\nu \in \mathbb{C}}. \]
We then have
\[ \text{Res}_{\hat{\omega}} U(k, \omega) = \lim_{\omega \to \hat{\omega}} U(k, \omega)A(\omega). \]
For each $\nu \in \mathbb{C}$ and for each $\omega \neq \hat{\omega}$ the $\nu$-th column of $U(k, \omega)A(\omega)$
\[ \tilde{\phi}'(k, \omega) = (\phi'(\sigma(k), \omega))_{\sigma \in \mathbb{C}} = U(k, \omega)A(\omega)n_\nu \]
can be prolonged into $A_1$ to a solution of (1.1) with $\lambda = \lambda(\omega)$ according (2.4):
\[ \phi'(\alpha, \omega) = \sum_{\gamma \in \mathbb{C}} r(\alpha, \gamma(0); \lambda)b_\nu(0)\phi'(\gamma(1), \omega); \quad \alpha \in A_1 \]
Since $U(k, \omega)$ has a simple pole at $\hat{\omega}$, there exists the limit
\[ \tilde{\phi}'(k, \hat{\omega}) = \lim_{\omega \to \hat{\omega}} \tilde{\phi}'(k, \omega) \]

Claim The vector $\tilde{\phi}'(k, \hat{\omega})$ also can be prolonged to a solution of the problem (1.1).
In case $\lambda(\hat{\omega})$ is not a pole of $\mathcal{R}(\lambda(\omega))$, the prolongation is straightforward. By (2.4), if $\lambda(\hat{\omega})$ is at the same time an eigenvalue of $L_1$ one can apply the same reasonings as in lemma 4.3 in [12]. We omit the details.

Let now
\[ T(\omega)^{-1} = (\tau_{\nu\nu}(\omega))_{\sigma, \nu \in \mathbb{C}}, \]
and denote
\[ h_{\sigma\nu}(\omega) = -\frac{b_\sigma}{b_\nu(0)} \frac{\theta_\nu(\omega) - \theta_\nu(\hat{\omega})^{-1}}{e_\nu(1, \theta_\nu(\omega))} (\omega - \hat{\omega})\tau_{\sigma\nu}(\omega). \] (8.1)
It follows from (5.7) that for $\sigma \neq \nu$ we have
\[ \text{Res}_{\hat{\omega}} u_{\sigma\nu}(k, \omega) = \phi'(\sigma(k), \hat{\omega}) = e_\sigma(k, \hat{\omega})h_{\sigma\nu}(\hat{\omega}) \] (8.2)
here we denote $m_{\sigma\nu}(\hat{\omega}) = h_{\sigma\nu}(\hat{\omega})$. 

If \( e_\nu(1, \theta_\nu(\hat{\nu})) \neq 0 \), representation (8.2) is valid for \( \sigma = \nu \) as well. Assume that \( e_\nu(1, \theta_\nu(\hat{\nu})) = 0 \). Then

\[
p_\nu(k, \hat{\nu})e_\nu(0, \theta_\nu(\hat{\nu})) = e_\nu(k, \theta_\nu(\hat{\nu})),
\]

(8.3)
because the expressions in both sides satisfy the same recurrence equation and the same initial conditions, and also it follows from lemma 4.1 that \( \dot{e}_\nu(1, \theta_\nu(\hat{\nu})) \neq 0 \) because \( e_\nu(1, \theta_\nu(\hat{\nu})) \) and \( \dot{e}_\nu(1, \theta_\nu(\hat{\nu})) \) cannot vanish simultaneously. Thus we again obtain (8.2), yet now

\[
m_{\nu\nu}(\hat{\nu}) = \frac{b_\nu}{b_\nu(0)} \frac{\theta_\nu(\hat{\nu}) - \theta_\nu(\hat{\nu})^{-1}}{e_\nu(0)} + h_{\nu\nu}(\hat{\nu}).
\]

Representation (8.2) is now valid for all \( \nu, \sigma \in \mathbb{C} \).

If \( \hat{\nu} \in \mathbb{T} \) it follows from (8.2) that \( \phi^{\nu}(\alpha, \hat{\nu}) \) is an eigenfunction of \( \mathcal{L} \) with \( \lambda(\hat{\nu}) \) as eigenvalue. If \( \hat{\nu} \in \mathbb{T} \), we may have \( \theta_\sigma(\hat{\nu}) \in \mathbb{T} \), i.e., \( \hat{\nu} \in T_\sigma^+ \) for some \( \sigma \in \mathbb{C} \). It follows from corollary 7.1 that for such \( \sigma \) we have \( m_{\sigma\nu}(\hat{\nu}) = 0 \), in particular

\[
\text{Res}_{\nu,\sigma}(k, \cdot) = \phi^{\nu}(\sigma(k), \hat{\nu}) = 0, \quad \hat{\nu} \in \mathbb{T}, \quad \theta_\sigma(\hat{\nu}) \in \mathbb{T}.
\]

and, again \( \phi^{\nu}(\alpha, \hat{\nu}) \) is an eigenfunction of \( \mathcal{L} \) with \( \lambda(\hat{\nu}) \) as eigenvalue. \( \square \)

Consider the matrix

\[
m(\hat{\nu}) = (m_{\sigma\nu}(\hat{\nu}))_{\sigma,\nu \in \mathbb{C}}
\]

Properties of \( m(\hat{\nu}) \) are summarized in the statement below

**Lemma 8.2.** Let \( \hat{\nu} \in \Omega \). Then

\[
\text{Res}U(k, \hat{\nu}) = \mathcal{E}(k, \hat{\nu})m(\hat{\nu}), \quad k = 0, 1, 2, \ldots
\]

(8.4)
The diagonal elements \( m(\nu, \nu; \hat{\nu}) \) satisfy

\[
\|\phi^{\nu}(\hat{\nu})\|^2 = \frac{b_\nu(1 - \theta_\nu(\hat{\nu})^{-2})}{b_\nu(1 - \omega^{-2})} \theta_\nu(\hat{\nu})m_{\nu\nu}(\hat{\nu}),
\]

(8.5)
where \( \phi^{\nu} = \phi^{\nu}(\alpha, \hat{\nu}) \) is the eigenvector of \( \mathcal{L} \), corresponding to the eigenvalue \( \lambda(\hat{\nu}) \) and such that

\[
\phi^{\nu}(\sigma(k), \hat{\nu}) = e_\nu(k, \theta_\sigma(\hat{\nu}))m(\sigma, \nu; \hat{\nu}) \sigma, \nu \in \mathbb{C}, \quad k \geq 0.
\]

(8.6)

**Proof.** Relations (8.4) and (8.5) are already established in lemma 8.1. It remains to prove (8.6).

We apply Lemma 4.3 with \( \xi(\alpha) = \phi^{\nu}(\alpha, \hat{\nu}) \):

\[
\dot{\lambda}(\hat{\nu}) \sum_{\alpha \in \mathcal{A}_1} |\phi^{\nu}(\alpha, \hat{\nu})|^2 = \sum_{\sigma \in \mathbb{C}} b_\sigma(0) \{\dot{\phi}^{\nu}(\hat{\nu}), \overline{\phi^{\nu}(\hat{\nu})}\}_{\sigma}(0)
\]

(8.7)
and calculate the Wronskians in the right-hand side of this equality.

We then have

\[
\phi^{\nu}(\sigma(k), \nu) = p_\nu(k, \omega)h_{\nu\nu}(\nu)\delta_{\sigma,\nu} + e_\nu(k, \omega)h_{\sigma\nu}(\nu),
\]

(8.8)
with

\[
h_{\nu}(\nu) = \frac{b_\nu}{b_\nu(0)} \frac{\theta_\nu(\hat{\nu}) - \theta_\nu(\hat{\nu})^{-1}}{e_\nu(1, \theta_\nu(\nu))}(\omega - \hat{\nu}),
\]
and \( h_{\sigma \nu} \) is already defined in \((8.1)\).

Let \( e_{\nu}(1, \theta_{\nu}(\hat{\omega})) \neq 0 \). Then, for all \( \sigma, \nu \in \mathbb{C} \),
\[
h_{\nu}(\hat{\omega}) = 0, \quad \phi^{
u}(\sigma(k), \hat{\omega}) = e_{\sigma}(k, \theta_{\sigma}(\hat{\omega}))h_{\sigma \nu}(\hat{\omega}), \quad (8.9)
\]
and
\[
m_{\sigma \nu}(\hat{\omega}) = h_{\sigma \nu}(\hat{\omega}), \quad m(\hat{\omega}) = (m_{\sigma \nu}(\hat{\omega}))_{\sigma, \nu \in \mathbb{C}} \quad (8.10)
\]
We use \((8.8), (8.9), (8.10), (8.4)\) and that \( \dot{p}_{k}(\omega) = 0 \) for \( k = 0, 1 \) as follows from \((8.1)\):
\[
\dot{\phi}^{
u}(\sigma(k), \omega) = p_{\nu}(k, \omega) \dot{h}_{\nu}(\omega)\delta_{\sigma, \nu} + e_{\nu}(k, \omega)h_{\sigma \nu}(\omega) + e_{\sigma}(k, \omega)\dot{h}_{\sigma \nu}(\omega), \quad k = 0, 1. \quad (8.11)
\]
Besides
\[
\dot{h}_{\nu}(\hat{\omega}) = \frac{b_{\nu} - \theta_{\nu}(\hat{\omega}) - \dot{\theta}_{\nu}(\hat{\omega})^{-1}}{b_{\nu}(0) - e_{\nu}(1, \theta_{\nu}(\hat{\omega}))}.
\]
Since \( e_{\nu}(1, \theta_{\nu}(\hat{\omega})) \neq 0 \), we also have \( h_{\nu}(\hat{\omega}) = 0 \) and, according to \((4.6)\) and \((8.2)\),
\[
b_{\sigma}(0)\{\dot{\phi}^{
u}(\hat{\omega}), \phi^{
u}(\hat{\omega})\}_{\sigma}(0) = b_{\nu}(\theta_{\nu}(\hat{\omega}) - \theta_{\nu}(\hat{\omega})^{-1})m_{\sigma \nu}(\hat{\omega})\delta_{\sigma, \nu} +
\]
\[
b_{\sigma}(0)\{\dot{e}_{\sigma}(\hat{\omega}), e_{\nu}(\theta_{\sigma}(\hat{\omega}))\}_{\sigma}(0)m_{\sigma \nu}(\hat{\omega}) =
\]
\[
b_{\nu}(\theta_{\nu}(\hat{\omega}) - \theta_{\nu}(\hat{\omega})^{-1})m_{\sigma \nu}(\hat{\omega})\delta_{\sigma, \nu} - \lambda_{\sigma}(\hat{\omega}) \sum_{k=1}^{\infty} |\phi^{
u}(\sigma(k), \theta_{\nu}(\hat{\omega})))|^{2}. \quad (8.12)
\]
We can now return to \((8.7)\) in order to obtain normalization condition \((8.6)\) for the matrix \( m \):

In the case \( e_{\nu}(1, \hat{\omega}) = 0 \) representation \((8.8)\) is still valid, yet
\[
h_{\nu}(\hat{\omega}) = \frac{b_{\nu} - \theta_{\nu}(\hat{\omega}) - \dot{\theta}_{\nu}(\hat{\omega})^{-1}}{b_{\nu}(0) - e_{\nu}(1, \theta_{\nu}(\hat{\omega}))}e_{\nu}(0, \hat{\omega}) \neq 0.
\]
Taking \((8.3)\) into account we again obtain \((8.2)\), yet now
\[
m_{\sigma \nu}(\hat{\omega}) = \frac{b_{\nu} - \theta_{\nu}(\hat{\omega}) - \dot{\theta}_{\nu}(\hat{\omega})^{-1}}{b_{\nu}(0)\{e_{\nu}(\hat{\omega}), \dot{e}_{\nu}(\hat{\omega})\}_{\nu}(0)} \delta_{\sigma, \nu} + h_{\sigma \nu}(\hat{\omega}).
\]
Relation \((8.11)\) is still valid and for \( \nu \neq \sigma \) we arrive to relation \((8.12)\).

For \( \sigma = \nu \) we have
\[
b_{\nu}(0)\{\dot{\phi}^{\nu}(\hat{\omega}), \phi^{\nu}(\hat{\omega})\}_{\nu}(0) = b_{\nu}(0)\{\dot{e}_{\nu}(\hat{\omega})h_{2}(\hat{\omega}), e_{\nu}(\hat{\omega})\}_{\nu}(0)m_{\nu \nu}(\hat{\omega}) =
\]
\[
b_{\nu}(0)\{\dot{e}_{\nu}(\hat{\omega}), e_{\nu}(\hat{\omega})\}_{\nu}(0)m(\nu, \nu; \omega) - b_{\nu}(\theta_{\nu}(\hat{\omega}) - \theta_{\nu}(\hat{\omega})^{-1})m_{\nu \nu}(\hat{\omega}),
\]
and we again arrive to \((8.12)\).

Now one can complete the proof in the same way as if \( e_{\nu}(1, \theta_{\nu}(\hat{\omega})) \neq 0 \).

\[ \square \]

Remark. The matrix \( \overline{m(\hat{\omega})} \) corresponds to the point of discrete spectra \( \lambda(\hat{\omega}) \). Its columns are normalized eigenfunctions. This normalization is defined by relation \((8.5)\) and is therefore unique.

We will see in the next section that \( m_{\nu \nu}(\hat{\omega}) \), i.e. the energies of the normalized
9. Equations of the inverse scattering problem

In order to obtain equations of the inverse scattering problem we introduce the matrix function

$$\Delta_l(\omega) := \text{diag} \left\{ \theta_\nu(\omega)^{l-1} \frac{d\theta_\nu(\omega)}{d\omega} \right\}_{\nu \in \mathbb{C}}, \quad l \in \mathbb{Z} \quad (9.1)$$

and consider the integral

$$J(l, k) = (j_{\nu\sigma}(l, k))_{\nu,\sigma \in \mathbb{C}} = \frac{1}{2\pi i} \int_{\mathbb{T}} \Delta_l(\omega)U(k, \omega)d\omega.$$

Since $\mathbb{T}$ may contain (simple) poles of $U(k, \omega)$ this integral as well as all integrals in this section is considered in principal value. We will calculate this integral in two ways: through the residues, this would correspond to the contribution of the discrete spectra, and through the scattering coefficients, this would correspond to the contribution of the continuous spectra.

Comparing two different expressions for $J_{l,k}$ leads one to the equations of the inverse scattering problem.

Let as before $U(k, \omega) = (u_{\nu\sigma}(k, \omega))_{\nu,\sigma}$. Then

$$j_{\nu\sigma}(l, k) = \frac{1}{2\pi i} \int_{\mathbb{T}} \theta_\nu(\omega)^{l-1} u_{\nu\sigma}(k, \omega) \frac{d\theta_\nu(\omega)}{d\omega} d\omega = \frac{1}{2\pi i} \int_{\mathbb{T}} \theta_\nu(\omega)^{l-1} \psi_{\nu}(k, \omega) \frac{d\theta_\nu(\omega)}{d\omega} d\omega, \quad (9.2)$$

here $\psi(\cdot, \omega)$ is the special solution, defined in (5.8).

Assume first that $\sigma \neq \nu$. Then, according to (3.7) and (3.8)

$$j_{\nu\sigma}(l, k) = \frac{1}{2\pi i} \int_{\mathbb{T}} \theta_\nu(\omega)^{l-1} e_{\nu}(k, \theta_\nu(\omega)) s_{\sigma\nu}(\omega) \frac{d\theta_\nu(\omega)}{d\omega} d\omega,$$

Consider the function

$$q_{\nu\sigma}(n) = \frac{1}{2\pi i} \int_{\mathbb{T}} s_{\sigma\nu}(\omega) \theta_{\nu}(\omega)^{n-1} \frac{d\theta_{\nu}(\omega)}{d\omega} d\omega. \quad (9.3)$$

Relation (3.4) now yields

$$j_{\nu\sigma}(l, k) = c_{\nu}(k) \sum_{m \geq k} q_{\nu\sigma}(m + l) a_{\nu}(k, m), \quad (9.4)$$

this is the desired expression.

Remark. The functions $q_{\nu,\sigma}$ cannot be determined from the spectral data generally speaking. However relation (9.4) determines the structure of the equation of the inverse scattering problem. Later in section 9.2 we will use this structure to get rid of the functions which cannot be observed from the spectral data.
Let now \( \nu = \sigma \) and \( T(\omega)^{-1} = (\tau_{\sigma}(\omega))_{\sigma,\nu} \). Relations (5.7) (9.1) yield

\[
j_{\sigma\sigma}(k, l) = \frac{1}{2\pi i} \int_T \left\{ \left[ -p_{\sigma}(k, \omega) + e_{\sigma}(k, \theta) \tau_{\sigma}(\omega) \right] \frac{b_{\sigma}(\theta^{-1} - \theta)}{b_{\sigma}(0)e_{\sigma}(1, \theta)} \theta_{\sigma}(\omega)^{i-1} \frac{d\theta_{\sigma}(\omega)}{d\omega} \right\} d\omega = \frac{1}{2\pi i} \left( \int_{T^+_{\sigma}} + \int_{T^-_{\sigma}} \right) \left\{ \cdot \right\} = j_{\sigma\sigma}^+(k, l) + j_{\sigma\sigma}^-(k, l). \tag{9.5}\]

As \( \omega \) runs over \( T^+_{\sigma} \) the function \( \theta_{\sigma}(\omega) \) runs over the whole \( \mathbb{T} \). Let \( \omega(\theta_{\sigma}) \) be the inverse function. Relations (9.2) together with (5.7) yield

\[
j_{\sigma\sigma}^+(k, l) = \frac{1}{2\pi i} \int_T \left[ e_{\sigma}(k, \theta^{-1}) + s_{\sigma\sigma}(\omega(\theta_{\sigma})))e_{\sigma}(k, \theta_{\sigma}) \right] \theta_{\sigma}^{-1} d\theta_{\sigma}.
\]

Besides it follows from corollary (7.1) that \( s_{\sigma\sigma}(\omega) \) is bounded on \( \mathbb{T}^+_{\sigma} \).

Consider the Fourier series of \( s_{\sigma\sigma}(\omega(\theta_{\sigma}))) \):

\[
s_{\sigma\sigma}(\omega(\theta_{\sigma})) = \sum_{n=-\infty}^{\infty} \hat{s}_{\sigma}(n) \theta^{-n}; \quad \hat{s}_{\sigma}(n) = \frac{1}{2\pi i} \int_{\mathbb{T}} s_{\sigma\sigma}(\omega(\theta_{\sigma})) \theta_{\sigma}^{-n} d\theta_{\sigma}.
\]

Together with (3.4) this yields

\[
e_{\sigma}(k, \theta_{\sigma}^{-1}) + s_{\sigma\sigma}(\omega(\theta_{\sigma})))e_{\sigma}(k, \theta_{\sigma}) = c_{\sigma}(k) \sum_{m} \{ a_{\sigma}(k, m) + \sum_{n} a_{\sigma}(k, n) \hat{s}_{\sigma}(m + n)\} \theta_{\sigma}^{-m},
\]

and finally

\[
j_{\sigma\sigma}^-(k, l) = c_{\sigma}(k) \left[ a_{\sigma}(k, l) + \sum_{n=-\infty}^{\infty} a_{\sigma}(k, n) \hat{s}_{\sigma}(l + n) \right]. \tag{9.6}\]

We now study \( j_{\sigma\sigma}^-(k, l) \). Denote

\[
a_1(k, \omega) = -p_{\sigma}(k, \omega) \frac{b_{\sigma}(1, \theta)_{\sigma}^{-1} - \theta_{\sigma}}{b_{\sigma}(0)} e_{\sigma}(k, \theta)_{\sigma}^{-1} \theta_{\sigma}^{-1} \theta_{\sigma}^{-1} - \theta_{\sigma};
\]

\[
a_2(\omega) = \tau_{\sigma\sigma}(\omega) \frac{b_{\sigma}(1, \theta)_{\sigma}^{-1} - \theta_{\sigma}}{b_{\sigma}(0)} e_{\sigma}(k, \theta)_{\sigma}^{-1} \theta_{\sigma}^{-1} - \theta_{\sigma}.
\]

We then have

\[
\psi_{\sigma}^{\sigma}(k, \omega) = a_1(k, \omega) + e_{\sigma}(k, \theta)_{\sigma}(\omega) a_2(\omega),
\]

and

\[
j_{\sigma\sigma}^-(k, l) = \frac{1}{2\pi i} \int_{T^+_{\sigma}} a_1(k, \omega) \theta_{\sigma}(\omega)^{i-1} \frac{d\theta_{\sigma}(\omega)}{d\omega} d\omega + \frac{1}{2\pi i} \int_{T^-_{\sigma}} a_2(\omega) e_{\sigma}(k, \theta)_{\sigma}(\omega) \theta_{\sigma}(\omega)^{i-1} \frac{d\theta_{\sigma}(\omega)}{d\omega} d\omega.
\]

As \( \omega \) runs over \( T^+_{\sigma} \), the function \( \theta_{\sigma}(\omega) \) runs twice in the opposite directions over \( J_{\sigma} = \theta_{\sigma}(T^+_{\sigma}) \subset \mathbb{R} \) and the values \( \theta_{\sigma}(\omega) \) and \( \theta_{\sigma}(\omega^{-1}) \) coincide. Respectively \( e_{\sigma}(k, \theta)_{\sigma}(\omega), p_{\sigma}(k, \omega) \in \mathbb{R} \) and \( e_{\sigma}(k, \theta)_{\sigma}(\omega) = e_{\sigma}(k, \theta)_{\sigma}(\omega^{-1}) \), \( p_{\sigma}(k, \omega) = p_{\sigma}(k, \omega^{-1}) \), thus \( a_1(k, \omega) = a_1(k, \omega^{-1}) \). Besides \( a_2(\omega) = a_2(\omega^{-1}) \), since \( \psi_{\sigma}^{\sigma}(k, \omega) = \psi_{\sigma}^{\sigma}(k, \omega^{-1}) \) for \( \omega \in \mathbb{T} \).
Therefore
\[ j_{σσ}^-(k, l) = \frac{1}{2πi} \int_{Γ_σ} \left[ a_2(ω) - a_2(ω^{-1}) \right] e_σ(k, θ_σ)θ_σ^{l-1} \frac{dθ_σ}{dω} dω, \] (9.7)
where \( Γ_σ = T^-_σ \cap C_+ \).

The function \( a_1(k, ω) \) satisfies equation (2.2) on the channel \( σ \) and relations (7.1), (7.2) yield
\[ a_1(k, ω) = θ_σ(ω)^{-k} (1 + o(1)). \]

Therefore
\[ b_σ(0)\{ψ^σ(ω), ψ^σ(ω^{-1})\}_σ(0) = b_σ(θ_σ(ω)^{-1} - θ_σ(ω)) [a_2(ω) - a_2(ω^{-1})]. \]

On the other hand by (4.7)
\[ b_σ(0)\{ψ^σ(ω), ψ^σ(ω^{-1})\}_σ(0) = - \sum_{ν \in C \setminus \{σ\}} b_ν(0)\{ψ^σ(ω), ψ^σ(ω^{-1})\}_ν(0). \]

We have
\[ \{ψ^σ(ω), ψ^σ(ω^{-1})\}_ν(0) = 0, \ ω \in T^-_ν. \]

For \( ω \in T^+_ν \cap T^-_σ \) we obtain
\[ b_ν(0)\{ψ^σ(ω), ψ^σ(ω^{-1})\}_ν(0) = \]
\[ b_ν(0)\{s^σ_ν(ω) e_ν(θ_ν(ω)), s^νσ(ω^{-1}) e_ν(θ_ν(ω)^{-1})\}_ν(0) \]
\[ |s^σ_ν(ω)|^2 b_ν(0)\{ψ^ν(θ_ν(ω)), ψ^ν(θ_ν(ω)^{-1})\}_ν(0) = \]
\[ |s^σ_ν(ω)|^2 b_ν(θ_ν(ω)^{-1} - θ_ν(ω)) = - |s^νσ(ω)|^2 \frac{(θ_σ^1 - θ_σ)}{b_ν(θ_ν^{σ1} - θ_ν)}. \]

Now, for \( θ \in J_σ \), we define \( ω_σ(θ) = W^{-1} \circ W_σ(θ) \), here the functions \( W, W_σ \) are defined by (6.5) and the branch is chosen so that \( ω_σ(θ) \in Γ_σ \). Let
\[ N_σ(θ) = \{ ν; \ ν_σ(θ) \in T^+_ν \}, \ θ \in J_σ. \]

Denote
\[ Φ_σ(θ) = \sum_{ν \in N_σ(θ)} |s^σ_ν(ω_σ)|^2 \left( \frac{b_ν(θ_ν^{-1} - θ_ν)}{b_ν(θ_ν(ω_σ) - 1 - θ_ν(ω_σ))} \right), \ ω_σ(θ) \in T^-_σ. \]

We finally obtain
\[ a_2(ω) - a_2(ω^{-1}) = Φ_σ(θ_σ(ω)) \] (9.8)
this function is expressed via the spectral data.

**Remark.** All functions \( |s^σ_ν(ω)| \) participating in (9.8) are well defined and continuous. In addition, \( s^νσ(ω^{-1}) = s^σ_ν(ω) \) for \( ω \in T^+_ν \).

Relation (9.7) now takes the form
\[ j_{σσ}^-(k, l) = \frac{1}{2πi} \int_{J_σ} e_σ(k, θ) Φ_σ(θ) θ^{l-1} dθ, \]
and with account (3.4) we obtain
\[ j_{\sigma\sigma}(k, l) = c_{\sigma}(k) \sum_n a_{\sigma}(k, n) q_{\sigma\sigma}(n + l), \]  
(9.9)
where
\[ q_{\sigma\sigma}(n) = \frac{1}{2\pi i} \int_J \Phi_{\sigma}(\theta) \theta^{n-1} d\theta. \]  
(9.10)
Combining (9.5), (9.6), and (9.9) we finally obtain
\[ j_{\sigma\sigma}(k, l) = c_{\sigma}(k) \left[ a_{\sigma}(k, l) + \sum_{n=-\infty}^{\infty} a_{\sigma}(k, n) (\tilde{s}_{\sigma}(l + n) + q_{\sigma\sigma}(n + l)) \right], \]  
(9.11)
the functions \( \tilde{s}_{\sigma}(\cdot) \) and \( q_{\sigma\sigma}(\cdot) \) are defined through the scattering data.

9.2. Let now \( \Theta(\omega) = \text{diag}\{\theta_{\sigma}(\omega)\}_{\sigma \in \mathcal{C}}. \)
We use the representation (3.4) for \( E(k, \omega), \)
\[ E(k, \omega) = C(k) \sum_{m=-\infty}^{\infty} A(k, m) \Theta(\omega)^m. \]
The coefficients \( C(k) \) and \( A(k, m) \) are diagonal matrices
\[ C(k) = \text{diag}\{c_{\sigma}(k)\}_{\sigma \in \mathcal{C}}, \quad A(k, m) = \text{diag}\{a_{\sigma}(k, m)\}_{\sigma \in \mathcal{C}}, \]
and also \( A(k, m) = 0 \) for \( m < k. \)
It follows from (9.4) and (9.11) that
\[ J(k, l) = \frac{1}{2\pi i} \int_T \Delta_l(\omega) U(k, \omega) d\omega = C(k) \left\{ A(k, l) + \sum_{l=-\infty}^{\infty} A(k, m) Z(l + m) \right\}, \]  
(9.12)
here the integral is taken as a principal value and the matrix \( Z(n) = (z_{\nu,\sigma}(n))_{\nu,\sigma \in \mathcal{C}} \)
is given by the relations
\[ z_{\nu,\sigma}(n) = q_{\nu,\sigma}(n), \quad \nu \neq \sigma, \]  
(9.13)
\[ z_{\sigma,\sigma}(n) = \tilde{s}_{\sigma}(n) + q_{\sigma,\sigma}(n), \]  
(9.14)
here \( \tilde{s}_{\sigma}(n) \) are the Fourier coefficients of the reflection coefficient \( s_{\sigma\sigma} \) with respect to \( \theta_{\sigma} \) and the function \( q_{\nu,\sigma} \) is defined in (9.3) and (9.10).
Let now \( \Omega \subset \hat{D} \) be the set of all poles of \( U(k, \omega). \) It follows from (5.7) that
\( U(k, \tilde{\omega}) = U(k, \hat{\omega}), \) thus for each \( \tilde{\omega} \in \Omega \cap T \) the point \( \tilde{\omega} \) also belongs to \( \Omega. \) Moreover, for such \( \tilde{\omega} \) we have \( \text{Res}_{\tilde{\omega}} U(k, \omega) = \text{Res}_{\hat{\omega}} U(k, \omega). \) Denote \( \Omega = \{ \tilde{\omega} \in \Omega : |\tilde{\omega}| < 1 \} \cup \{ \hat{\omega} \in T \cap \Omega : \Im \hat{\omega} > 0 \}. \) We use (8.4) and (9.1): 
\[ J(k, l) = \sum_{\omega \in \Omega} \Delta_l(\omega) E(k, \omega) m(\hat{\omega}) = C(k) \sum_m A(k, m) M(m + l), \]
where
\[ M(n) = \sum_{\omega \in \Omega} \text{diag} \left\{ \frac{d\theta_{\sigma}}{d\omega}(\hat{\omega}) \right\}_{\sigma \in \mathcal{C}} \Theta(\omega)^{n-1} \text{Re}(\hat{\omega}). \]  
(9.15)
By compare this to (9.12) and taking into account that $A(k,m) = 0$ for $m < k$ and $A(k,k) = I$, we obtain a system of equations

$$A(k,m) + \sum_{s=k}^{\infty} A(k,s) F(s + m) = 0, \quad m = k + 1, k + 2, \ldots$$

where

$$F(n) = Z(n) - M(n). \quad (9.16)$$

Since $A(k,k) = I$, this relation can be written as

$$F(k + m) + A(k,m) + \sum_{s=k+1}^{\infty} A(k,s) F(s + m) = 0, \quad k = 1, 2, \ldots, m > k. \quad (9.17)$$

The matrices $A(k,m)$ are diagonal. The diagonal elements of $F(n)$, can be expressed through the spectral data. The non-diagonal elements of the matrices $F(n)$ vanish, as follows from the lemma below.

**Lemma 9.1.** The matrices $F(n)$ are diagonal for all $n \geq 1$: $F(n) = \text{diag}\{f_\sigma(n)\}_{\sigma \in \mathbb{C}}$, and their diagonal elements $f_\sigma(n)$ are determined by $a_\sigma(k,m), \ m > k \geq \left\lfloor \frac{n-1}{2} \right\rfloor$ only.

This statement is proved in [12], (Lemma 5.1) and we omit the proof.

**Theorem 9.1.** The following properties hold for the systems under consideration.

1) Equations (9.17) split into a system of independent scalar equations

$$f_\nu(k + m) + a_\nu(k,m) + \sum_{s=k+1}^{\infty} a_\nu(k,s) f_\nu(s + m) = 0, \quad m \geq k + 1 \geq 1, \quad (9.18)$$

here $f_\nu(n)$ is defined in (9.16), (9.15), (9.14).

2) For each $k \geq 0$ equations (9.18) has unique solutions $a_\sigma(k,m)$.

**Proof:** The first statement follows directly from relation (9.16) which defines $F(n)$ and Lemma 9.1. It follows from the same lemma that the functions $f_\nu(n), n \geq 1$ are uniquely defined by the coefficients $b_\nu(k), a_\nu(k), k \geq 1$. Therefore (9.18) coincide with equations for inverse scattering problem for equation (2.2) with boundary condition $\xi(\nu(0)) = 0$. It is well-known (see for example [14]) that the later has unique solution.

$$\square$$

10. Concluding remarks

In case that the continuos spectra $I = \cup_{\sigma \in \mathbb{C}} [a_\sigma - 2b_\sigma, a_\sigma + 2b_\sigma]$ splits into a number of disjoints intervals, one can repeat the procedure separately for each connected component of $I$.

If, say, $I^{(0)}$ is a connected component of $I$ and $\sigma$ is a channel corresponding to this component, then each wave incoming along $\sigma$ generates decaying waves in all channels which correspond to other connected components of $I$. We omit the details.

So far we have discussed reconstruction of the part $A_0$ of the whole system, that is, the channels. This information is, generally speaking, insufficient for reconstruction the whole matrix $L$. However if the matrix $L_1$ corresponding to the "central" part of the system is sufficiently sparse and also we know the matrix $B(0) = \text{diag}\{b_\sigma(0)\}_{\sigma \in \mathbb{C}}$ which realizes connections between the channels and the
central part of the system, the whole matrix $\mathcal{L}$ can be recovered from the scattering data. We refer the reader to Chapter 11 in [15], where statements of such type are obtained.

References

[1] Z.S. Agranovich, V.A. Marchenko, The inverse problem of scattering theory, Gordon and Breach Science Publishers, New York-London 1963 xiii+291 pp.

[2] V.S. Buslaev, V.N. Fomin, An Inverse Scattering Problem for the One-Dimensional Schrödinger Equation on the Entire Axis, Vestnik Leningrad. Univ., 17, (1962), 56 – 64

[3] R. Carlson, Inverse eigenvalue problems on directed graphs, Trans. Amer. Math. Soc., 351 (1999), no. 10, 4069 – 4088.

[4] A. Cohen, T. Kappeler, Scattering and inverse scattering for steplike potentials in the Schrödinger equation. Indiana Univ. Math. J., 34 (1985), no. 1, 127 – 180.

[5] I. Egorova, J. Michor, G. Teschl, Scattering theory for Jacobi operators with general steplike quasi-periodic background, Zh. Mat. Fiz. Anal. Geom. 4-1, 33 – 62 (2008).

[6] Gerasimenko, N. I.; Pavlov, B. S. A scattering problem on noncompact graphs. (Russian) Teoret. Mat. Fiz. 74 (1988), no. 3, 345–359; translation in Theoret. and Math. Phys., 74 (1988), no. 3, 23D-240.

[7] F. Gesztesy, R. Nowel, and W. Pötz, One-Dimensional Scattering Theory for Quantum Systems with Nontrivial Spatial Asymptotics, *Differential and Integral Equations*, 10, (1997), 521 – 546.

[8] B. Gutkin, U. Smilansky, Can one hear the shape of a graph? J. Phys. A, 34:31 (2001), 6061-6068.

[9] P. Kuchment, Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs. J. Phys. A, 38 (2005), no. 22, 4887 – 4900.

[10] P. Kurasov, M. Nowaczyk, Inverse spectral problem for quantum graphs, J. Phys. A 38 (2005), 4901 – 4915.

[11] V. Kostrykin, R. Schrader, The inverse scattering problem for metric graphs and the traveling salesman problem, ArXiv, math-ph/0603010, pp.1 – 68, 2006.

[12] Yu. I. Lyubarskii, V. A. Marchenko, Direct and inverse problems of multichannel scattering (Russian). Funktsional. Anal. i Prilozhen. 41 (2007), 58 – 77, (Translation in Funct. Anal. Appl. 41 (2007), no. 2, 126 –142 )

[13] Yu. I. Lyubarskii, V.A. Marchenko, Inverse problem for small oscillations, *Spectral analysis, differential equations and mathematical physics: a festschrift in honor of Fritz Gesztesy’s 60th birthday*, 263 – 290, Proc. Sympos. Pure Math., 87, Amer. Math. Soc., Providence, RI, 2013.

[14] V.A. Marchenko, *Introduction to the theory of inverse problems of spectral analysis*, (Russian), Kharkov, 2005, pp.1-135.

[15] V.A. Marchenko, V.V. Slavin, Inverse problems it theory of small oscillations, (Russian), Kiev, Naukova Dumka, 2015, 219 pp.

[16] R. G. Newton, Scattering theory of waves and particles, Dover Publications, Inc., Mineola, NY, 2002, pp. xx+745

[17] B.S. Pavlov, M.D. Faddeev, A model of free electrons and the scattering problem. (Russian), *Teoret. Mat. Fiz.* 55 (1983), no. 2, 257 – 268.

[18] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, American Mathematical Society, Providence, RI, 2000, xvii+351 pp.

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway

*E-mail address*: isaac.romero@math.ntnu.no, isaacalrom@gmail.com

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway

*E-mail address*: yura@math.ntnu.no