We study Landau–Ginzburg orbifolds \((f,G)\) with \(f = x_1^n + \cdots + x_N^n\) and \(G = S \times G^d\), where \(S \subseteq S_N\) and \(G^d\) is either the maximal group of scalar symmetries of \(f\) or the intersection of the maximal diagonal symmetries of \(f\) with \(SL_N(\mathbb{C})\). We construct a mirror map between the corresponding phase spaces and prove that it is an isomorphism restricted to a certain subspace of the phase space when \(n = N\) is a prime number. When \(S\) satisfies the parity condition of Ebeling–Gusein-Zade, this subspace coincides with the full space. We also show that two phase spaces are isomorphic for \(n = N = 5\).

Keywords: mirror symmetry, nonabelian symmetry group, singularity theory

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1. Introduction

Initiated in the late 1980s by physicists (cf. [1]–[3]), much effort has been given to the study of the so-called Landau–Ginzburg orbifolds. These are the pairs \((f,G)\) with \(f = f(x) \in \mathbb{C}[x_1, \ldots, x_N]\) having only isolated critical points and \(G\) being a group of symmetries of \(f(x)\), namely, a group of elements \(g \in \text{End}(\mathbb{C}^N)\) such that \(f(g \cdot x) = f(x)\). These pairs turned out to play an important role in mirror symmetry. Two vector spaces can be associated with each such pair: an \(A\)-vector space and a \(B\)-vector space, both being of the same dimension, but different in construction.

The mirror symmetry conjecture states that there is a dual pair \((\tilde{f}, \tilde{G})\) such that its \(A\)-vector space (resp. \(B\)-vector space) is isomorphic to the \(B\)-vector space (resp. \(A\)-vector space) of \((f,G)\). Such an isomorphism is called the mirror map. All the other mirror symmetry results can only be obtained after a good mirror map is established.
In this paper, we focus on the Fermat polynomials
\[ f = x_1^n + \cdots + x_N^n. \]
We then have \( \tilde{f}(x) = f(x) \), but the construction of the dual group \( \tilde{G} \) is more involved.

The B-vector space is given by the Hochschild cohomology \( \text{HH}^*(f, G) \) of the category of \( G \)-equivariant matrix factorizations of \( f \). We let \( f^g \) denote the restriction of \( f \) to \( \text{Fix}(g) \), the eigenvalue-1 subspace of \( g \) in \( \mathbb{C}^N \), and set \( \mathcal{A}_{f,g} := \text{Jac}(f^g) \), the Jacobian algebra of \( f^g \). We have (cf. [4])
\[
\text{HH}^*(f, G) \cong (\mathcal{A}_{f,G})^G \text{ for } \mathcal{A}_{f,G} := \bigoplus_{g \in G} \mathcal{A}_{f,g}.
\]
The action of \( v \in G \) is such that \( v^*: \mathcal{A}_{f,u} \to \mathcal{A}_{f,vuv^{-1}} \) for any \( u \in G \). The subspaces \( \mathcal{A}_{f,u} \) are called narrow sectors if \( \text{Fix}(u) = 0 \) and broad sectors otherwise.

The A-vector space is constructed via the Fan–Jarvis–Ruan–Witten (FJRW) theory, which is only defined for the groups \( G \) acting diagonally (cf. [5], see also [6] for some examples). But it is expected to have the same vector space structure as \( \text{HH}^*(f, G) \) above. This allows regarding the mirror map as an involutive vector space isomorphism \( \text{HH}^*(f, G) \cong \text{HH}^*(\tilde{f}, \tilde{G}) \).

1.1. Diagonal symmetry groups. We consider the maximal group of diagonal symmetries
\[
G_f^d := \{ g = (g_1, \ldots, g_N) \in (\mathbb{C}^*)^N \mid f(g \cdot x) = f(x) \}.
\]
We call \( G \) a diagonal symmetry group if \( G \subseteq G_f^d \). For such groups, the notion of a dual group \( \tilde{G} \) was introduced by Berglund–Hübsch–Henningson in [7], [8] and the mirror map was found by Krawitz in [9]. The mirror map of Krawitz always interchanges broad and narrow sectors.

Especially important are the groups \( SL_f \) and \( \langle J \rangle \) such that
\[
\begin{align*}
SL_f & := \left\{ g = (g_1, \ldots, g_N) \in G_f^d \mid \prod_{i=1}^N g_i = 1 \right\}, \\
J & := (e^{2\pi \sqrt{-1}/n}, \ldots, e^{2\pi \sqrt{-1}/n}).
\end{align*}
\]
These groups are dual to each other.

The diagonal symmetry group \( G \) is always abelian, which simplifies the computations significantly. In particular, the \( G \)-action does not mix the sectors of \( \mathcal{A}_{f,G}^d \). For \( f \) being an invertible polynomial, the corresponding FJRW and Hochschild cohomology rings were computed in [10]–[13].

1.2. Fermat quintic with a nonabelian group of symmetries. In [1], the authors introduced two functions \( q_t, q_c : \mathcal{A}_{f,G} \to \mathbb{Q} \) that provide the bigrading of \( \text{HH}^*(f, G) \). This bigrading was used in [14] to obtain the Hodge numbers \( h^{p,q}(f, G) \) for a B-vector space. For \( N = n = 5 \) and for \( G \) preserving the volume form, Mukai has shown (Theorem 7.1 loc. cit.) that \( h^{p,q}(f, G) = h^{5-p,q}(X) \) for \( X \) being the mirror Calabi–Yau quintic of \( (f, G) \). This approach was extended further in [15], [16] by Ebeling and Gusein-Zade, who considered the mirror pairs \( (f, S \rtimes G^d) \), \( (\tilde{f}, S \rtimes \tilde{G}^d) \) with \( \langle J \rangle \subseteq G^d \subseteq SL_f \) from the standpoint of the Hodge theory of Milnor fibers. They have shown that for two mirror Calabi–Yau quintics \( X \) and \( \tilde{X} \) of these pairs, the equality \( h^{1,1}(X) = h^{3,1}((\tilde{X}) \) holds if and only if the following parity condition (PC) holds:
\[
\text{for any } T \subseteq S \text{ holds } \dim(\mathbb{C}^N)^T \equiv N \mod 2.
\]
In particular, \( S \) can only satisfy PC (2) if \( S \subseteq A_N \). This condition was conjectured to be necessary for mirror symmetry to hold.
1.3. Our results. We focus on the case of prime $N$ and nonabelian symmetry groups $G = S \ltimes SL_f$ and $\tilde{G} = S \ltimes \langle J \rangle$ with $S \subseteq S_N$. The corresponding Hochschild cohomology groups were computed in [17] via the technique of [4].

We introduce the mirror map $\tau: HH^*(f, S \ltimes SL_f) \to HH^*(f, S \ltimes \langle J \rangle)$. It coincides with the mirror map of Krawitz on $A^\dagger_{f,u}$ for $u \in G_d$.

It is easy to see that there is no mirror map for $n \neq N$ because the dimensions do not match (see Example 1 in Sec. 3.1). For $n = N$, we show the following theorem (also see Theorem 3).

**Theorem 1.** The map $\tau$ establishes an isomorphism $A^\text{stable}_{f,S \ltimes SL_f} \to A^\text{stable}_{f,S \ltimes \langle J \rangle}$ between certain subspaces of $HH^*(f, S \ltimes SL_f)$ and $HH^*(f, S \ltimes \langle J \rangle)$. In particular, these subspaces coincide with the whole vector spaces if the group satisfies Ebeling–Gusein-Zade condition (2). Under this isomorphism, we have $q_\ell(X) = q_\ell(\tau(X))$ and $q_\ell(X) = N - 2 - q_\ell(\tau(X))$ for any homogeneous $X \in A^\text{stable}_{f,S \ltimes SL_f}$.

It is important to note that our mirror map establishes the isomorphism between the whole spaces $HH^*(f, S \ltimes SL_f)$ and $HH^*(f, S \ltimes \langle J \rangle)$ in some examples where the PC does not hold.

In the special case $N = n = 5$, we can formulate an even stronger result (see Theorem 4 and its corollary).

**Theorem 2.** For $N = 5$, there is a vector space isomorphism

$$HH^*(f, S \ltimes SL_f) \to HH^*(f, S \ltimes \langle J \rangle)$$

for any group $S \subseteq S_5$. Moreover, for $X$ and $\tilde{X}$ being the respective mirror quintics of $(f, S \ltimes SL)$ and $(f, S \ltimes \langle J \rangle)$, we have the following relation between the Hodge numbers:

$$h^{1,1}(X) + h^{2,1}(X) = h^{1,1}(\tilde{X}) + h^{2,1}(\tilde{X}).$$

This theorem involves another mirror map, which generalizes the mirror map $\tau$ and also the mirror map of Krawitz, but does not always send the narrow sectors to broad ones and vice versa.

We also provide many examples in Secs. 3.1 and 6.3.

2. Preliminaries and notation

For any $u \in S_N \ltimes G_d^\text{f}$, we set $u = \sigma \cdot g$ assuming that $\sigma \in S_N$ and $g \in G_d^\text{f}$. Let $\text{Fix}(u)$ be the eigenvalue-1 subspace of $\mathbb{C}^N_\text{f}$ of $u$ and $I^p_u$ be the set of all indices $k$ such that $u \cdot x_k \neq x_k$. The restriction of $f$ to $\text{Fix}(u)$, $f^u = f|_{\text{Fix}(u)}$, is again a Fermat-type polynomial.

Let $\sigma = \prod_{a=1}^p \sigma_a$ be the decomposition into nonintersecting cycles. We let $|\sigma_a|$ denote the length of the cycle $\sigma_a$. We also allow $\sigma_a$ to be of length 1, such that we always have $\sum_{a=1}^p |\sigma_a| = N$. There exists a unique set $g_1, \ldots, g_p$ of $G_d^\text{f}$ elements such that $g_a$ acts nontrivially only on $I^p_u$ and $\sigma \cdot g = \prod_{a=1}^p g_a$. We call the product $\sigma \cdot g = \prod_{a=1}^p g_a$ the generalized cycle decomposition of $u$.

A generalized cycle $\sigma_a g_a$ is said to be special if $\det(g_a) = 1$ and nonspecial otherwise. It is clear that $\text{Fix}(\sigma_a g_a) \cap \mathbb{C}^{I^p_u}_{g_a} = 0$ for a nonspecial cycle, where by $\mathbb{C}^{I^p_u}_{g_a}$ we mean the subspace of $\mathbb{C}^N$ spanned by standard basis vectors with indices in $I^p_u$. For a special cycle, we have $\dim \text{Jac}(f^\sigma g_a|_{\mathbb{C}^{I^p_u}_{g_a}}) = n - 1$.

We let $[\phi(x)]$ denote the class of the polynomials $\phi(x)$ in $\text{Jac}(f^\sigma g_a)$. Let $\tilde{x}_{i_a}$ be a $\sigma_a g_a$-invariant linear combination of $x_\bullet$ with indices in $I^p_u g_a$ such that $\text{Jac}(f^\sigma g_a|_{I^p_u g_a})$ has the basis $[\tilde{x}_{i_a}^k]$, $k = 0, 1, \ldots, n - 2$. We set

$$A'_{\sigma_a g_a} := ([1], [\tilde{x}_{i_a}], \ldots, [\tilde{x}_{i_a}^{n-2}])\xi_{\sigma_a g_a},$$

where $\xi_{\sigma_a g_a}$ denotes the formal letter associated with $\sigma_a g_a$. The elements of $A'_{\sigma_a g_a}$ are denoted by $[\phi(x)]\xi_{\sigma_a g_a}$ in what follows.
In particular, for \( g_a = \text{id} \), we have \( \hat{x}_{i_a} = \sum_i x_i \), where the summation is taken over \( i \in I_{x_a} \). We also use the above notation for nonspecial cycles, assuming that \( \left[ \hat{x}_{i_a}^0 \right] \xi_{\sigma_a g_a} = [1] \xi_{\sigma_a g_a} \). For \( u \in G \) with the generalized cycle decomposition \( u = \prod_{a=1}^p \sigma_a g_a \), we have

\[
\mathcal{A}_{f,u}^J = \bigotimes_{a=1}^p \mathcal{A}_{g_a}^{J_a}.
\]

We fix \( \zeta := e^{2\pi \sqrt{-1}/n} \) and \( t_k \in G_f^d \) with \( k = 1, \ldots, N \) by

\[
t_k: (x_1, \ldots, x_N) \rightarrow (x_1, \ldots, \zeta_k x_k, \ldots, x_N).
\]

Then the Fermat-type polynomial maximal diagonal symmetries group \( G_f^d \) is generated by \( t_1, \ldots, t_N \).

We also set \( SL_f := \{ g \in G_f^d \mid \det(g) = 1 \}, \quad J = t_1 \ldots, t_N \). The groups \( S \ltimes SL_f \) and \( S \ltimes \langle J \rangle \) with \( S \subseteq S_N \) are of special importance in this paper.

2.1. The phase space. In what follows, we need to consider \( HH^*(f,G) \) as a vector space for the fixed \( S \subseteq S_N \) and different \( G_d \subseteq G_f^d \). For simplicity, we consider the space \( \mathcal{A}_{J} := \bigoplus_{u \in S \ltimes G_f^d} \mathcal{A}_{f,u}^J \). We call its direct summand \( \mathcal{A}_{f,u}^J \) the \( u \)th sectors.

For any \( G \subseteq S_N \ltimes G_f^d \), we let \( \mathcal{A}_{f,G} \) denote the phase space of \( G \), which is the subspace of \( \mathcal{A}_{J} \) defined as follows. Let \( C^G \) stand for the set of representatives of the conjugacy classes of \( G \). We set

\[
\mathcal{A}_{f,G} = \bigoplus_{u \in C^G} (\mathcal{A}_{f,u}^J)^{Z(u)},
\]

where the action of \( v \in Z(u) \) on \( \mathcal{A}_{f,u}^J \) is computed as follows. Let \( \lambda_k^u \) and \( \lambda_k^v \) be the eigenvalues of \( u \) and \( v \) computed in their common eigenvectors basis. For \( X = \{ \phi(x) \} \xi_u \in \mathcal{A}_{f,u}^J \) and \( v \in Z(u) \), we have

\[
v^* (X) = \prod_{k=1, \ldots, N, \lambda_k^u \neq 1} \frac{1}{\lambda_k^v} \left[ \phi(v \cdot x) \right] \xi_u = \text{det}^{-1}(v_{\text{Fix}(u)}) \left[ \phi(v \cdot x) \right] \xi_u.
\]

This is a particular case of the \( G \)-action of \( HH^*(f,G) \). Moreover, we have (cf. Proposition 42 in [17])

\[
HH^*(f,G) \cong \mathcal{A}_{f,G}.
\]

This isomorphism allows us to regard \( HH^*(f,G) \) with the different groups \( G \) as subspaces of \( \mathcal{A}_{J} \).

For any \( G_1, G_2 \subseteq S_N \ltimes G_f^d \), we have the natural inclusion \( i_1: \mathcal{A}_{f,G_1} \rightarrow \mathcal{A}_{J} \) and the projections \( \pi_2: \mathcal{A}_{J} \rightarrow \mathcal{A}_{f,G_2} \). In what follows, we consider the maps \( \psi: \mathcal{A}_{f,G_1} \rightarrow \mathcal{A}_{f,G_2} \) defined in terms of \( \psi: \mathcal{A}_{J} \rightarrow \mathcal{A}_{J} \) as \( \psi := \pi_2 \circ \psi \circ i_1 \).

With respect to the generalized cycle decomposition \( u = \prod_{a=1}^p \sigma_a g_a \), we have the relation \( \xi_u = \prod_{a=1}^p \xi_{\sigma_a g_a} \) between the generators of different vector spaces \( \mathcal{A}_{f,G_1} \) and \( \mathcal{A}_{f,G_2} \). This extends to the product of arbitrary \( X_1 = \{ \phi_1 \} \xi_u \) and \( X_2 = \{ \phi_2 \} \xi_v \), viewed as \( \mathcal{A}_{J} \)-elements, given by \( X_1 X_2 = \{ \phi_1 \phi_2 \} \xi_{uv} \) if \( I_u \cap I_v = \emptyset \). This is not to be confused with the \( \cup \)-product on the Hochschild cohomology as in [17].

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2.2. Bigrading. Let \( \mathbb{C}[x_1, \ldots, x_N] \) be graded by setting \( \deg(x_k) = 1/n \). We extend this grading to \( A_{\text{tot}} \) as follows. For any \( u \in S_N \ltimes G_f \), let \( \lambda_1, \ldots, \lambda_N \in \mathbb{C} \) be the eigenvalues of the linear transformation \( x \mapsto ux \). We can assume that \( \lambda_k = e^{2\pi \sqrt{-1} a_k} \) for some \( a_k \in \mathbb{Q} \cap [0,1) \). We set

\[
\text{age}(u) := \sum_{k=1}^{N} a_k.
\]

For the inverse element \( u^{-1} \), we then have

\[
\text{age}(u) + \text{age}(u^{-1}) = N - \dim \text{Fix}(u) = d_u.
\]

For any homogeneous \( p \in \mathbb{C}[x] \), we consider the element \( |p| \xi_u \in A_{\text{tot}} \). For the element \( |p| \xi_u \), we define the left charge \( q_l \) and the right charge \( q_r \) as

\[
(q_l, q_r) = \left( \frac{\deg p - d_u}{n} + \text{age}(u), \frac{\deg p - d_u}{n} + \text{age}(u^{-1}) \right).
\]

This definition endows \( A_{\text{tot}} \) with the structure of a \( \mathbb{Q} \)-bigraded vector space. This is exactly the bigrading introduced in [1], [14]. It follows immediately that \( q_r(\xi_u) + q_r(\xi_v) = q_r(\xi_{uv}) \) for \( u, v \in G \) such that \( I_u^c \cap I_v^c = \emptyset \).

We let \( h^{p,q}(f,G) \) be the dimension of the space of bidegree-\((p,q)\) elements of \( A_{f,G} \).

3. Mirror map

In this section, we define the mirror map \( \tau : A_{\text{tot}} \to A_{\text{tot}} \). It is used in Theorem 3 below to set up a mirror isomorphism. We first consider the following examples. We do not give all computations explicitly, referring the interested reader to the Examples section in [17].

3.1. Examples. Example 1 shows that there is no mirror map if \( N \neq n \). In Example 2, we consider the case where there is a mirror map interchanging broad and narrow sectors. Example 3 deals with the symmetry group that does not satisfy the PC of Ebeling–Gusein-Zade, but the mirror map nevertheless exists. Example 4 represents the situation where a mirror map exists but does not always interchange broad and narrow sectors as it does for diagonal symmetry groups. Examples 2 and 3 are particular cases of Theorem 3, and Example 4 is a particular case of Theorem 4.

In all the examples in what follows, we consider the groups \( G = S \ltimes SL_f \) and \( \tilde{G} = S \ltimes \langle J \rangle \) with different \( S \subseteq S_N \). There is a decomposition

\[
A_{f,S \ltimes SL_f} = A_{SL,d} \oplus A_{SL,s}, \quad A_{f,S \ltimes \langle J \rangle} = A_{\langle J \rangle,d} \oplus A_{\langle J \rangle,s}
\]

for \( A_{SL,d} \) and \( A_{\langle J \rangle,d} \) being the direct sums of all \( u \)th sectors of \( A_{f,S \ltimes SL_f} \) and \( A_{f,S \ltimes \langle J \rangle} \) such that \( u \in \text{id} \cdot SL_f \) and \( u \in \text{id} \cdot \langle J \rangle \).

Example 1. Let \( N = 3 \) and \( n = 4 \), \( S = S_3 \). We have

\[
A_{S_3 \ltimes SL_f,s} = \mathbb{C}[[x_1 + x_2 + x_3]^2 \xi_{(1,2,3)}],
\]

\[
A_{S_3 \ltimes \langle J \rangle,s} = \mathbb{C}[[x_1 + x_2 + x_3]^2 \xi_{(1,2,3)}],
\]

showing that there is no mirror map between \( A_{f,S_3 \ltimes SL_f} \) and \( A_{f,S_3 \ltimes \langle J \rangle} \) if \( N \neq n \).
Example 2. Let $N = n = 5$ and $S = \langle (1, 2)(3, 4) \rangle \subset S_5$. We set

$$\phi_{a, b, c}(x) = (x_1 + x_2)^a (x_3 + x_4)^b x_5^c, \quad a, b, c \geq 0.$$  

We have

$$\mathcal{A}_{(J), s} = \bigoplus_{k=1}^4 \mathbb{C} \langle \xi_{(1, 2)(3, 4), J^k} \rangle \bigoplus_{0 \leq a, b, c \leq 3, a + b + c = 2} \mathbb{C} \langle [\phi_{a, b, c}] \xi_{(1, 2)(3, 4)} \rangle, \quad \mathcal{A}_{SL, s} = \bigoplus_{0 \leq a, b, c \leq 3, a + b = 2} \mathbb{C} \langle [\phi_{a, b, c}] \xi_{(1, 2)(3, 4)} \rangle \bigoplus_{1 \leq a, b \leq 4} \mathbb{C} \langle \xi_{(1, 2)(3, 4)} t_1 t_3 t_5^{a-b} \rangle. \quad (6)$$

The vector space isomorphism $\mathcal{A}_{f, S \times SL_f} \rightarrow \mathcal{A}_{f, S \times (J)}$ is

$$[H^k] \xi_{id} \mapsto \xi_{J^{k-1}}, \quad \xi_{(1, 2)(3, 4)} t_1 t_3 t_5 \mapsto [\phi_{a-1, b-1, c-1}] \xi_{(1, 2)(3, 4)}, \quad [\phi_{a, b, c}] \xi_{(1, 2)(3, 4)} \mapsto \xi_{(1, 2)(3, 4)} t^{a-b}. \quad (7)$$

This map interchanges bidegree-(1, 2) classes with bidegree-(1, 1) classes and bidegree-(2, 1) classes with bidegree-(2, 2) classes. The group considered can be diagonalized, but this requires changing the polynomial $f$, and therefore we then no longer have the relation $\tilde{f} = f$.

Example 3. Let $N = n = 5$, $S = \langle (1, 2, 3), (1, 2) \rangle \subset S_5$. We set

$$\phi_{a, b, c}(x) = (x_1 + x_2 + x_3)^a x_4^b x_5^c, \quad a, b, c \geq 0.$$  

We have

$$\mathcal{A}_{(J), s} = \bigoplus_{k=1}^4 \mathbb{C} \langle \xi_{(1, 2, 3), J^k} \rangle \bigoplus_{0 \leq a, b, c \leq 3, a + b + c = 2} \mathbb{C} \langle [\phi_{a, b, c}] \xi_{(1, 2, 3)} \rangle, \quad \mathcal{A}_{SL, s} = \bigoplus_{a + b = 2} \mathbb{C} \langle [\phi_{a, b, b}] \xi_{(1, 2, 3)} \rangle \bigoplus_{1 \leq a, b \leq 4} \mathbb{C} \langle \xi_{(1, 2, 3)} t_1 t_3 t_5^{a-b} \rangle. \quad (8)$$

The vector space isomorphism $\mathcal{A}_{f, S \times SL_f} \rightarrow \mathcal{A}_{f, S \times (J)}$ is

$$[H^k] \xi_{id} \mapsto \xi_{J^{k-1}}, \quad \xi_{(1, 2, 3)} t_1 t_3 t_5 \mapsto [\phi_{a-1, b-1, c-1}] \xi_{(1, 2, 3)}, \quad [\phi_{a, b, b}] \xi_{(1, 2, 3)} \mapsto \xi_{(1, 2, 3)} t^{a-b}. \quad (9)$$

This map interchanges bidegree-(1, 2) classes with bidegree-(1, 1) classes, and bidegree-(2, 1) classes with bidegree-(2, 2) classes.

Example 4. Let $N = n = 5$, $S = S_5$. We set

$$\phi_{a, b, c}(x) := (x_1 + x_2 + x_3)^a (x_4 x_5^b + x_4 x_5^b), \quad \psi_{a, b}(x) := (x_1 + x_2 + x_3)^a (x_4 x_5)^b.$$  

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We have $A_{(j),s} = A^{(1)}_{(j),s} \oplus A^{(2)}_{(j),s}$, $A_{SL,s} = A^{(1)}_{SL,s} \oplus A^{(2)}_{SL,s}$ with

$$
\begin{align*}
A^{(1)}_{(j),s} &= \mathbb{C} \langle \phi_{0,0,2} | \xi(1,2,3), | \phi_{0,1,0} | \xi(1,2,3), | \phi_{2,2,3} | \xi(1,2,3), | \phi_{1,3,0} | \xi(1,2,3) \rangle, \\
A^{(2)}_{(j),s} &= \mathbb{C} \langle \psi_{0,1} | \xi(1,2,3), | \psi_{1,3} | \xi(1,2,3), | \psi_{2,0} | \xi(1,2,3), | \psi_{3,2} | \xi(1,2,3) \rangle, \\
A^{(1)}_{SL,s} &= \mathbb{C} \langle \xi(1,2,3) t_1, t_2, t_3, t_4, 2^\xi \xi(1,2,3) t_1 t_2, 2^\xi \xi(1,2,3) t_2 t_3, (1,2,3) t_1 t_2 t_3, (1,2,3) t_2 t_3 t_4 \rangle, \\
A^{(2)}_{SL,s} &= \mathbb{C} \langle \xi(1,2,3) t_1, t_2, t_3, t_4, 2^\xi \xi(1,2,3) t_1 t_2, 2^\xi \xi(1,2,3) t_2 t_3, (1,2,3) t_1 t_2 t_3, (1,2,3) t_2 t_3 t_4 \rangle.
\end{align*}
$$

(10)

We see that $A_{(j),d}$ contains only narrow sectors and $A_{(j),s}$ contains only broad sectors; $A_{SL,d}$ contains only broad sectors while $A_{SL,s}$ is a direct sum of both broad and narrow sectors.

It is easy to guess the vector space isomorphism $A_{f,S_b \ltimes SL,f} \rightarrow A_{f,S_b \ltimes \psi(j)}$ in this case:

$$
\begin{align*}
[H^k] \xi_{id} &\mapsto \xi_{j^{k-1}}, \\
[\xi(1,2,3) t_1 t_2 t_3 t_4] \mapsto [\phi_{a-1,b-1,c-1}] \xi(1,2,3), \\
[\psi_{a,b}] \xi(1,2,3) &\mapsto [\psi_{a,b}] \xi(1,2,3).
\end{align*}
$$

(11)

The first two lines here interchange degree-(1, 2) and (2, 1) classes with the respective degree-(1, 1) and (2, 2) classes. However, the last line in (11) sets up the isomorphism $A^{(2)}_{(j),s} \rightarrow A^{(2)}_{SL,s}$ mapping the degree-(1, 1) and (2, 2) classes again to the degree-(1, 1) and (2, 2) classes.

3.2. Definition of the mirror map. For any $u \in S \ltimes G_{f}^d$, let $u = \prod_{a=1}^{p} \sigma_a g_a$ be its generalized cycle decomposition. An arbitrary element $X \in A'_{f,u}$ has the form

$$
X = \prod_{a=1}^{p} \langle \tilde{x}_{t_a}^{r_a} | \xi_{\sigma_a g_a} \rangle
$$

with $r_a = 0$ if $\sigma_a g_a$ is nonspecial and $0 \leq r_a \leq N - 2$ if $\sigma_a g_a$ is special.

We set

$$
\tau(X) := \prod_{a=1}^{p} \tau(\langle \tilde{x}_{t_a}^{r_a} | \xi_{\sigma_a g_a} \rangle),
$$

with the product in the right-hand side understood in the sense of Sec. 2. The map $\tau$ is defined on the generalized cycles as follows.

CASE 1: $u = \sigma_1 \cdot g_1$ is a nonspecial cycle. We assume $g_1 = \prod_{p} t_{d_1,p}^c$, where $p$ runs over $f_{g_1}^c$. We then set $d_1 := \sum_{p} d_{1,p} \mod N$ with $1 \leq d_1 \leq N$. Set

$$
\tau(\langle \xi_{\sigma_1, g_1} \rangle) := \langle \tilde{x}_{t_1}^{d_1-1} | \xi_{\sigma_1} \rangle.
$$

(12)

CASE 2: $u = \sigma_1 \cdot g_1$ is a special cycle. For any $0 \leq r_1 \leq N - 2$, we set

$$
\tau(\langle \tilde{x}_{t_1}^{r_1} | \xi_{\sigma_1, g_1} \rangle) := \xi_{\sigma_1, h};
$$

(13)

for $h = (\prod_{a=1}^{p} t_{a})^{k_1}$ with $a$ running over $f_{g_1}^c$ and $k_1 \in \{1, \ldots, N - 1\}$ being a unique integer such that $r_1 + 1 \equiv k_1 |\sigma_1| \mod N$. 

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3.3. Mirror map of Krawitz. The mirror map τ generalizes the mirror map of Krawitz. Namely, he defines (cf. [9]) \( \tau_{k_i} : A_{f,S SL_f} \rightarrow A_{f,(j)} \) by the rule
\[
\prod_{a=1}^{p} x_{p}^{r_{a}} \xi_{id} \mapsto \xi_{\prod_{a=1}^{p} t_{p}^{r_{a}+1}}, \quad \xi_{\prod_{a=1}^{p} t_{p}^{r_{a}}} \mapsto \prod_{a=1}^{p} x_{p}^{d_{a}-1} \xi_{id}.
\]

We now show that our mirror map coincides with the mirror maps of Krawitz on \( A'_{f,u} \) for \( u = \text{id} \cdot g \). The generalized cycle decomposition of such elements is given by length-1 cycles \( (a) \), and hence we have \( u = \prod_{a=1}^{N} (a)t_{a}^{d_{a}} \) for \( g = \prod_{a} t_{a}^{d_{a}} \). We have \( x_{a} = \tilde{x}_{a} \) if \( d_{a} \equiv 0 \).

When \( g = \text{id} \), we have
\[
\tau\left( \prod_{a=1}^{p} (x_{a}^{r_{a}}) \xi_{id} \right) = \prod_{a=1}^{p} \tau((\tilde{x}_{a}^{r_{a}}) \xi_{id}) = \prod_{a=1}^{p} (\lfloor 1 \rfloor \xi_{\tilde{x}_{a}^{r_{a}+1}}) = [1] \xi_{\prod_{a=1}^{p} t_{a}^{r_{a}+1}}.
\]

For \( d_{a} \neq 0 \), for all \( a \) we have
\[
\tau((\lfloor 1 \rfloor \xi_{\prod_{a=1}^{p} t_{a}^{d_{a}}}) = \prod_{a=1}^{p} \tau((\lfloor 1 \rfloor \xi_{d_{a}}) = \prod_{a=1}^{p} [\tilde{x}_{a}^{d_{a}-1}] \xi_{id}.
\]

4. The vector space structure

In this section, we consider prime \( N = n \) and arbitrary \( S \subseteq S_N \). The aim of this section is to describe the structure of the vector spaces \( A_{f,S \times SL_f} \) and \( A_{f,S \times (j)} \).

According to the definition, we should consider the sets of representatives of the conjugacy classes \( C_{S \times SL_f} \) and \( C_{S \times (j)} \). Let \( C_{S} \) be some set of representatives of the conjugacy classes of \( S \). We have

\[
C_{S \times (j)} = \{ \sigma \cdot J^k \mid \sigma \in C_{S}, k = 0,1,\ldots,N-1 \}.
\]

The map \( S \times SL_f \rightarrow S \) is compatible with the conjugation action. We choose representatives in \( C_{S \times SL_f} \) in a way compatible with the choice of \( C_{S} \). The set \( C_{S \times SL_f} \) is described in Proposition 3 below.

As regards the bases of \( A_{f,S \times SL_f} \), \( A_{f,S \times (j)} \), we show the following proposition.

**Proposition 1.** Let \( 0 \leq r_{a} < N - 2 \), \( 1 \leq d_{a} < N - 1 \) and \( 1 \leq q \leq N - 1 \).

1. The basis of \( A_{f,S \times SL_f} \) can be chosen to consists of the narrow vectors \( \prod_{a} \xi_{a}^{d_{a}} \) and the broad vectors \( \prod_{a} \tilde{x}_{a}^{r_{a}} \xi_{a} \).

2. The basis of \( A_{f,S \times (j)} \) can be chosen to consists of the narrow vectors \( \prod_{a} \xi_{a}^{q} \) and the broad vectors \( \prod_{a} \tilde{x}_{a}^{r_{a}} \xi_{a} \).

We note that this proposition does not guarantee that all vectors of the specified form appear in the bases of \( A_{f,S \times SL_f} \) and \( A_{f,S \times (j)} \). The proof is given in the subsections that follow.

4.1. The group \( G = S \times SL_f \).

**Proposition 2.** For any \( g \in SL_f \) and length-\( N \) cycle \( \sigma \), we have \( (A'_{f,\sigma \cdot g})^{Z(\sigma \cdot g)} = 0 \).

**Proof.** The element \( \sigma \cdot g \) is special with a 1-dimensional fixed locus. We have \( A'_{f,\sigma \cdot g} \cong \mathbb{C}(\tilde{x}_{1}^{p}) \xi_{\sigma \cdot g} \) for \( p = 0,1,\ldots,N-2 \). We also have \( J \in G \) and \( J \in Z(\sigma \cdot g) \). The action of \( J \) gives

\[
J^{*}((\tilde{x}_{1}^{p}) \xi_{\sigma \cdot g}) = \zeta_{N}^{p+1} \cdot (\tilde{x}_{1}^{p}) \xi_{\sigma \cdot g},
\]

because \( \det(J) = 1 \). The vector is invariant under the action \( J \) if and only if \( p+1 \equiv 0 \mod N \), which never holds for the range of \( p \) as specified.
It is obvious that for a cycle \( \sigma \) of length \( N - 1 \), an element \( \sigma \cdot g \) is conjugate to \( \sigma \cdot t_1^d \cdot t_2^d \) in the group \( G \) for \( d_1 + d_2 \equiv 0 \) mod \( N \) and \( i_1 \in I_{1}^c \), \( i_2 \notin I_{1}^c \). This can be generalized to the following statement.

**Proposition 3.** Let \( u = \sigma \cdot g \in G \) be an element with the cycle decomposition \( \sigma = \prod_{a=1}^{m} \sigma_a \) with \( m \geq 2 \). Let \( i_a \) be the first index of \( I_a^c \). Then for some \( d_a \in \mathbb{Z} \), \( u \) is conjugate to \( \prod_{a=1}^{m} \sigma_a t_i^d \) in the group \( G \).

**Proof.** We consider \( \sigma_a = (i_1, \ldots, i_p) \) and \( g_a = t_i^{d_1}, \ldots, t_i^{d_p} \). For \( p = 1 \), there is nothing to prove. For \( p \geq 2 \), it is sufficient to consider the bigger group \( S \ltimes G_f^d \), because it is generated by its center and \( G \). Now we can conjugate by \( h = t_i^{d_1 + d_2 + \cdots + d_p} t_i^{d_1 + d_2 + \cdots + d_p} \) to obtain the desired result.

The following corollary makes a clear distinction between broad and narrow sectors for the group \( S \ltimes SL_f \).

**Corollary 1.** Let \( X \in A_S \ltimes SL_f \) be a nonzero element with \( X = [\phi(x)] \xi_u \). Then \( [\phi(x)] \) is nonconstant only if \( u \) is conjugate to \( \prod_{a=1}^{m} \sigma_a \). In particular, for every such \( X \), the conjugacy class of \( u \) contains exactly one \( v = wu w^{-1} \) such that \( [\phi(w \cdot x)] \xi_v \in A_S \).

**Proof.** We set \( u = \sigma \cdot g \) with \( \sigma \in S \), \( g \in G^d \). If \( [\phi(x)] \) is nonconstant, then at least one of the cycles of \( \sigma \) is special. Let this be \( \sigma_1 \). We assume further that there is a non special cycle \( \sigma_2 \) in the cycle decomposition of \( \sigma \).

Let \( d \) be such that \(|\sigma_1| + d |\sigma_2| \equiv 0 \mod N \). We consider the element

\[
 h = \left( \prod_i t_i \right) \left( \prod_j t_j \right)^d \in SL_f 
\]

with \( i \) running over \( I_{\sigma_1}^c \) and \( j \) running over \( I_{\sigma_2}^c \). Then \( h \) commutes with \( u \). We have \( h^* [x_i^{r_a}] \xi_u = \zeta_N^{r_a \chi} \xi_u \).

Because \( h \) fixes other variables and \( r \) varies between 0 and \( n - 2 \), there are no invariant elements in the sector.

This completes proof of Case 1 in Proposition 1.

**Remark.** The conjugacy classes of \( S_N \) are indexed by partitions of \( N \). In particular, any length-\( N \) cycle of \( S_N \) is conjugate to \( \sigma = (1, \ldots, N) \). We conclude that the representatives of the conjugacy classes of \( S_N \ltimes SL_f \) are given by two partitions of \( n = N \) of the same length. The first one is given by the cycle type of \( \sigma \) and the second, by the set of exponents \( d_\bullet \).

4.2. The group \( \tilde{G} = S \ltimes \langle J \rangle \). Let \( u = \sigma \cdot J^q \) and \( \sigma = \prod_{a=1}^{p} \sigma_a \) be the cycle decomposition. If \( q \equiv 0 \), all \( \sigma_a \) are special. If \( q \not\equiv 0 \), for \( n = N \) being a prime number, either \( p = 1 \) and \( \sigma = \sigma_1 \) is special again or all \( \sigma_a \) are non-special.

For our group, \( J \in Z(u) \). We consider the action of this element on \( A_f^u \). Let \( X = [\phi(x)] \xi_{\sigma \cdot J^q} \) and \( Y = [x_i^{r_a}] \xi_{\sigma \cdot J^q} \) with some \( q \not\equiv 0 \). Up to a constant multiple, for some \( 0 \leq r_a \leq N - 2 \) we have

\[
 X = \prod_{a=1}^{p} [x_i^{r_a}] \xi_{\sigma_a} \quad \Rightarrow \quad J^*(X) = \zeta_N^{\sum_a (r_a + 1)} \cdot X, \\
 Y = [x_i^{r_a}] \xi_{\sigma, J^q} \quad \Rightarrow \quad J^*(Y) = \zeta_N^{r_a + 1} \cdot Y.
\]

Therefore, \( X \) is \( J \)-invariant only if \( \sum_a (r_a + 1) \in NZ \), and \( Y \) is never \( J \)-invariant. Similarly, we have that \( \xi_{\sigma, J^q} \) is \( J \)-invariant for any \( q \not\equiv 0 \). This gives the following proposition and completes proof of Case 2 in Proposition 1.
Proposition 4. Let $X := |φ(x)|ξ_{σ,g} ∈ A'_{S_N,J}$. If $X$ is $(J)$-invariant, then either $|φ(x)|$ is a constant or $g = id$.

4.3. Symmetric group action. For any $u = σ · g ∈ G ⊆ S_N ∼ G_f$, we have $u ∈ Z(u)$ and for any $X = |φ(x)|ξ_u$, we have $|φ(u · x)| = |φ(x)|$. But the action of $u$ on the generator $ξ_u$ is not necessarily trivial. It was proved in Corollary 39 in [17] (or can be deduced from Eq. (4)) that

$$u^*(ξ_u) = (−1)^{sgn(σ)}ξ_u.$$  

We consider the $(u)$-invariant subspace of $A'_{f,u}$ such that $(A'_{f,u})^{(u)} = 0$ if $σ$ is odd and $A'_{f,u}$ otherwise. It follows that no sector of $u = σ · g$ with odd $σ$ appears in $A_{f,G}$. However, such elements $u$, if they exist in the group, can still affect the conjugacy class decomposition and also the centralizers of other group elements, contributing nontrivially to $A_{f,G}$.

In the next proposition, we consider the group-action approach to the PC of Ebeling–Gusein-Zade.

Proposition 5. Let $u,v ∈ A_N$ be even commuting permutations. For $T := ⟨u,v⟩$,

1) the PC holds if and only if $u^*(ξ_v) = ξ_v$ and $v^*(ξ_u) = ξ_u$;

2) if the PC does not hold, we have $u^*(ξ_v) = −ξ_v$ or $v^*(ξ_u) = −ξ_u$.

Proof. Let $u$ and $v$ share a common eigenbasis $x_1, ..., x_N$. Let

$$u(x_k) = λ'_k x_k, \quad v(x_k) = λ''_k x_k,$$

$$λ'_1 = ... = λ'_{p} = 1, \quad λ''_1 = ... = λ''_{q} = 1.$$  

We have

$$v^*(ξ_u) = \frac{1}{λ'_{p+1} ... λ'_{N}}ξ_u = \frac{1}{λ''_{p+1} ... λ''_{N}}ξ_u, \quad q ≤ p,$$

$$v^*(ξ_u) = \frac{1}{λ''_{q+1} ... λ''_{N}}ξ_u = \frac{1}{λ'_{q+1} ... λ'_{N}}ξ_u, \quad q > p.$$  

(17)

It follows that in both cases,

$$v^*(ξ_u) = \frac{λ'_1 ... λ'_{q}}{det(v)}ξ_u = \frac{det(v|\text{Fix}(u))}{det(v)}ξ_u,$$  

(18)

where $v|\text{Fix}(u)$ stands for the restriction of $v$ to the fixed locus of $u$. Let $q > p$. The action of $v$ on $\text{Fix}(u)$ is given by a permutation of the set $\{1, ..., q\}$. Therefore, both determinants assumed are either $+1$ or $−1$. We have

$$det(v) = (−1)^{N − dim \text{Fix}(v)}, \quad det(v|\text{Fix}(u)) = (−1)^{dim \text{Fix}(u) − dim \text{Fix}(v)},$$

This completes the proof. ■

5. Mirror isomorphism

We saw in Example 1 in Sec. 3.1 that the condition $N = n$ is necessary for mirror symmetry to hold. We assume it to hold in what follows, and assume further that $N$ is prime. We fix some $S ⊆ S_N$.

Let $G ⊂ S ∼ G_f$. We call the $u$th sector of $A_{f,G}$ stable if

$$σ^*(ξ_u) = ξ_u \quad \text{for any} \quad σ ∈ Z(u) ∩ S ⊂ G.$$  

(19)
We let $A^\text{stable}_{f,G} \subseteq A_{f,G}$ denote the direct sum of all stable sectors $A_{f,u}$. It is important to note that the stability property refers to the generator of the $u$th sector rather than an arbitrary element $|\phi(x)|\xi_u$ of it.

In particular, according to Proposition 5, we have $A^\text{stable}_{f,G} = A_{f,G}$ if $S$ satisfies condition (2). The converse is not true. In particular, these vector spaces coincide in Examples 2 and 3. In Example 4, we have $A^\text{stable}_{f,G} = A_{G,d} \oplus A^{(1)}_{G,s} \subseteq A_{f,G}$.

**Theorem 3.** The map $\tau$ establishes an isomorphism $A^\text{stable}_{f,S \ltimes SL_f} \to A^\text{stable}_{f,S \ltimes (J)}$. Under this isomorphism, we have $q_i(X) = q_i(\tau(X))$ and $q_i(X) = N - 2 - q_i(\tau(X))$ for any homogeneous $X \in A^\text{stable}_{f,S \ltimes SL_f}$.

**Proof.** See Secs. 5.1 and 5.2 for the first claim and Sec. 5.3 for the second claim.

To show that the mirror map defines an isomorphism, we need to consider the action of the centralizers of the elements $u = \sigma \cdot g \in S \ltimes SL_f$ and $v = \sigma J^k \in S \ltimes (J)$ with the same $\sigma \in S$.

5.1. Narrow sectors. Let $u \in S \ltimes SL_f$ be such that the $u$th sector is stable and narrow. We assume that $u$ decomposes into generalized cycles as $u = \prod_{a=1}^p \sigma_a t_{ia}^d$. Let $Y := \xi_u \in A'_{f,S \ltimes SL_f}$. Then

$$X := \tau(Y) = \prod_a \frac{[x_{ia}]^{-1}}{d_a} \xi_{\sigma_a} \in A'_{f,S \ltimes (J)}.$$

Because the $u$th sector is narrow, any $v \in Z(u) \subset S \ltimes SL_f$ gives $v^*(Y) = (\det(v))^{-1} Y = Y$, and we know that $Y$ is nonzero in $A'_{f,S \ltimes SL_f}$. We show that $X$ is also nonzero in $A'_{f,S \ltimes (J)}$.

The centralizer $Z(\prod_a \sigma_a) \subset S \ltimes (J)$ is generated by the element $J$ and all permutations $\sigma' \in S$ commuting with $\prod_a \sigma_a$. We have $J^*(X) = X$ if and only if $\sum_a d_a \equiv 0 \mod N$, which is equivalent to the condition $\prod_a t_{ia}^{d_a} \in SL_f$. The action of $\sigma'$ on $X$ is trivial because the $u$th sector is stable.

5.2. Broad sectors. Let $u \in S \ltimes SL_f$ be such that the $u$th sector is stable and broad. We assume $u$ that decomposes into generalized cycles as $u = \prod_{a=1}^p \sigma_a$. Let

$$X := \prod_a \frac{[x_{ia}]^{-1}}{d_a} \xi_{\sigma_a} \in A'_{f,S \ltimes SL_f}.$$

**Proposition 6.** Let $X = \prod_{a=1}^p \frac{[x_{ia}]^{-1}}{d_a} \xi_{\sigma_a}$ be nonzero in $A'_{f,S \ltimes SL_f}$ and $k_1, \ldots, k_p$ be as in Sec. 3.2. Then $k_a = k_b$ for all $1 \leq a, b \leq p$.

**Proof.** We take $a = 1$ and $b = 2$. Let

$$g_1 = \left( \prod_{a \in I_{k_1}} t_a^\alpha \right), \quad g_2 = \left( \prod_{a \in I_{k_2}} t_a^\beta \right).$$

For $\alpha | \sigma_1 | + \beta | \sigma_2 | \equiv 0 \mod N$, it then follows that $v = g_1 g_2 \in Z(\prod_a \sigma_a) \subset S \ltimes SL_f$.

We have $v^*(X) = \lambda_1^{r_1+1} \lambda_2^{r_2+1} \cdot X$ with $\lambda_1 = \zeta_N^\alpha, \lambda_2 = \zeta_N^\beta$. We conclude that $X$ is invariant with respect to $v$ if and only if $(r_1 + 1)\alpha + (r_2 + 1)\beta \equiv 0$. For $N$ being prime, this holds for the chosen $\alpha$ and $\beta$ if and only if $k_1 \equiv k_2$. □

It follows from the proposition above that $Y := \tau(X) = \xi_{\sigma_a \cdot J^k} \in A'_{f,S \ltimes (J)}$ is well defined. We have $J^*(Y) = (\det(J))^{-1} Y = Y$, and it remains to consider the action of symmetric part elements on both $X$ and $Y$. Every $S$-element $\sigma' \in Z(\prod_a \sigma_a J^k) \subset S \ltimes (J)$ gives $(\sigma')^* Y = \det |\sigma'|^{-1} Y = Y$. At the same time, we have $\sigma' \in Z(u)$ and $(\sigma')^* X = X$ because the sector is stable.
5.3. Bidegree. Let $X = \xi_u \in A_{\text{tot}}$ with narrow $u = \prod_{a=1}^p \sigma_a g_a$. We set $Y := \tau(X) \in A_{\text{tot}}$. Let $d_1, \ldots, d_p$ be the exponents associated with $g_1, \ldots, g_p$ as in Sec. 3.2. We then have

$$q_\ell(Y) = q_\ell(Y) = \frac{1}{N} \sum_{a=1}^p \frac{d_a - 1}{N} + \frac{N - p}{2} = \frac{1}{N} \sum_{a=1}^p \frac{d_a - 1}{N} + \frac{N - p}{2},$$

$$q_\ell(X) = \frac{N - p}{2} + \frac{p}{N} - \frac{1}{2}, \quad q_\ell(X) = \frac{N - p}{2} - \frac{p}{N} - 1.$$

This gives the desired statement for the map $\tau$, sending a narrow sector $X \in A'_{f, \Sigma}$ to $Y = \tau(X) \in A'_{f, \Sigma}$. Because $\tau$ is involutive, this also proves the case where $\tau$ maps a broad sector element of $A'_{f, \Sigma}$ into a narrow sector of $A'_{f, \Sigma}$. However, it is useful to provide the full proof in this case as well.

Let

$$X = \prod_{a=1}^p |x_a|^\sigma_a, \quad Y := \tau(X) = \prod_{a=1}^p \xi_{a,x_a}.$$

According to the definition of the mirror map, for some $l_a \geq 0$ we have $r_a + 1 = k \cdot |\sigma_a| - l_a N$. Moreover, we can assume that $l_a < |\sigma_a|$ due to our assumptions on $k$ and $r_a$.

We have

$$q_\ell(X) = q_\ell(X) = \sum_{a=1}^p \frac{r_a}{N} + \frac{N - p}{2} = \sum_{a=1}^p \frac{r_a + 1}{N} - 1.$$  

The biggrading of $Y$ is more complicated. The eigenvalues of $\prod_{a=1}^p \sigma_a J^k$ are

$$\lambda_{a,q} := \exp\left(\frac{2\pi}{|\sigma_a|} \left(\frac{q}{|\sigma_a|} + \frac{k}{N}\right)\right)$$

for $a = 1, \ldots, p$ and $q = 1, \ldots, |\sigma_a|$. To compute $\text{age}(\prod_{a=1}^p \sigma_a J^k)$, we note that the expression in the brackets is not always less than 1. We have

$$\lambda_{a,q} := \exp\left(\frac{2\pi}{|\sigma_a|} \left(q + l_a + \frac{r_a + 1}{N}\right)\right).$$

Because $r_a + 1 \leq N - 1$, we conclude that there are exactly $l_a$ values of $q$ such that the expression in brackets is greater than or equal to $|\sigma_a|$. This gives

$$\text{age}(\prod_{a=1}^p \sigma_a J^k) = \sum_{a=1}^p \left(\sum_{q=1}^{|\sigma_a|} q + |\sigma_a| \frac{k}{N} - l_a \right) = \sum_{a=1}^p \left(\frac{|\sigma_a| - 1}{2} + \frac{k|\sigma_a|}{N} - l_a \right).$$

As a result, we have

$$q_\ell(Y) = \frac{N - p}{2} + k - \sum_{a=1}^p l_a - 1 = q_\ell(X).$$

It is easy to see that $N - 2 = q_\ell(Y) + q_\ell(Y)$, what finishes the proof.
6. Mirror map for a Fermat quintic

The purpose of this section is to prove the following theorem.

**Theorem 4.** For \( N = 5 \), there is a vector space isomorphism \( \mathcal{A}_{f,S \ltimes SL} \rightarrow \mathcal{A}_{f,S \ltimes \langle J \rangle} \).

**Corollary 2.** Let \( h^{p,q}_{SL} := h^{p,q}(f, S \ltimes SL) \) and \( h^{p,q}_{\langle J \rangle} = h^{p,q}(f, S \ltimes \langle J \rangle) \). We have

\[
h^{1,1}_S L + h^{2,1}_S L = h^{1,1}_{\langle J \rangle} + h^{2,1}_{\langle J \rangle}.
\]

**Proof.** In both spaces, the subspaces of the \((0,0), (3,3), (3,0), \) and \((0,3)\) classes are all 1-dimensional. Both \( \mathcal{A}_{f,S \ltimes SL} \) and \( \mathcal{A}_{f,S \ltimes \langle J \rangle} \) have a pairing that respects the bidegree (cf. Theorem 32 in [17]). The statement now follows from the equality of the dimensions of \( \mathcal{A}_{f,S \ltimes SL} \) and \( \mathcal{A}_{f,S \ltimes \langle J \rangle} \). □

The proof of Theorem 4 occupies the rest of this section. To show the theorem, we first need to redefine the mirror map.

**6.1. Definition.** As in general case, we set

\[
\hat{\tau}(X) := \prod_{a=1}^{p} \hat{\tau}(\langle \hat{x}_{t_a}^{\sigma} \rangle \xi_{\sigma,g_a}), \quad X = \prod_{a=1}^{p} \langle \hat{x}_{t_a}^{\sigma} \rangle \xi_{\sigma,g_a}.
\]

We (re)define \( \hat{\tau} \) on generalized cycles.

**Case 1:** \( u = \sigma_1 \cdot g_1 \) is a nonspecial cycle. We also set \( g = \prod_p t_p^{d_1,p} \) with \( p \) running over \( I_g \) and let \( d_1 := \sum_p d_1,p \mod N \) with \( 1 \leq d_1 \leq N - 1 \). Then

\[
\hat{\tau}(\xi_{\sigma,g}) := \hat{\tau}_1(\xi_{\sigma,g}) + \hat{\tau}_2(\xi_{\sigma,g}), \quad \text{for} \quad \hat{\tau}_1(\xi_{\sigma,g}) := [\hat{x}_1^{d_1-1}] \cdot \xi_{\sigma}, \quad \hat{\tau}_2(\xi_{\sigma,g}) := \xi_{\sigma,h},
\]

for \( h = (\prod_a t_a)^{k_1} \) and \( k_1 \in \{1, \ldots, N - 1\} \), a unique integer such that \( d_1 \equiv k_1|\sigma| \mod N \).

**Case 2:** \( u = \sigma \cdot g \) is a special cycle. We necessarily have \( g = \text{id} \) or \( \sigma = \text{id} \). For any \( 0 \leq r_1 \leq N - 2 \), we set

\[
\hat{\tau}(\langle \hat{x}_{t_1}^{r_1} \rangle \xi_{\sigma,g}) := -\hat{\tau}_3(\langle \hat{x}_{t_1}^{r_1} \rangle \xi_{\sigma,g}) + \hat{\tau}_4(\langle \hat{x}_{t_1}^{r_1} \rangle \xi_{\sigma,g}) \quad \text{for} \quad \hat{\tau}_3(\langle \hat{x}_{t_1}^{r_1} \rangle \xi_{\sigma,g}) := [\hat{x}_{t_1}^{r_1}] \xi_{\sigma,g}, \quad \hat{\tau}_4(\langle \hat{x}_{t_1}^{r_1} \rangle \xi_{\sigma,g}) := \xi_{\sigma,h},
\]

with \( h = (\prod_a t_a)^{k_1} \) and \( k_1 \in \{1, \ldots, N - 1\}, \) a unique integer such that \( r_1 + 1 = k_1|\sigma| \mod N \).

**6.2. The properties of \( \hat{\tau} \).** The map \( \hat{\tau} \) mixes the special and nonspecial generators \( \xi_{\sigma,g_a} \). However, in \( \mathcal{A}_{f,S \ltimes SL} \) and \( \mathcal{A}_{f,S \ltimes \langle J \rangle} \) we have only broad or narrow elements. It is easy to verify that the map \( \hat{\tau} \) still generalizes the mirror map of Krawitz.

**Proposition 7.** For \( G \) being \( SL \) or \( \langle J \rangle \), let \( X, Y \in \mathcal{A}_{S \ltimes G} \) respectively be narrow or broad basis elements. In \( \mathcal{A}_{f,S \ltimes G} \), we have

\[
\hat{\tau}(X) = \hat{\tau}_1(X) + \hat{\tau}_2(X), \quad \hat{\tau}(Y) = -\hat{\tau}_3(Y) + \hat{\tau}_4(Y).
\]

**Proof** follows immediately from Proposition 1.

The following proposition refers to elements for which both \( \hat{\tau}_1 \neq 0, \hat{\tau}_2 \neq 0 \) or both \( \hat{\tau}_3 \neq 0, \hat{\tau}_4 \neq 0 \).
Proposition 8. Let $Y := \xi_v$ be a nonzero narrow element of $A_{f, SKSL_f}$. We have the following.

1. Both $\hat{\tau}_1(Y) \neq 0$ and $\hat{\tau}_2(Y) \neq 0$ in $A_{f, SK(J)}$ if and only if there exists a narrow nonzero $X$ in $A_{f, SKSL_f}$ such that $\hat{\tau}_1(Y) = \hat{\tau}_3(X)$ and $\hat{\tau}_2(Y) = \hat{\tau}_4(X)$.

2. We define a $\mathbb{C}$-linear map $K : A_{f, SKSL_f} \to A_{f, SKSL_f}$ by letting it act by $K(X) = Y$ and $K(Y) = -X$ on $X$ and $Y$ as in case 1 and by identity otherwise. Then the map

$$\tilde{\tau} := \frac{1}{2} (\hat{\tau} \ast K + \hat{\tau})$$

maps the basis elements of $A_{f, SKSL_f}$ to the basis elements of $A_{f, SK(J)}$.

Proof. 1. Let $\hat{\tau}_1(Y) \neq 0$, $\hat{\tau}_2(Y) \neq 0$ and $v = \sigma \cdot g$, and let $\sigma = \prod_{a=1}^{p} \sigma_a$ be the cycle decomposition.

We set $X := \hat{\tau}_1(Y)$. By construction, $X \in \mathcal{A}^f_{\text{a}}$ with $u = \sigma$. For $d_a$ as in Sec. 3.2, we have $X = \prod_{a=1}^{p} [x_{i_a}^{d_a - 1}] \xi_u$. Let $k_a|\sigma_a| = d_a R \text{ mod } N$. Because $\hat{\tau}_2(Y) \neq 0$, it follows that $k_1 \cdots k_p := \kappa$.

To show that $X$ is nonzero in $A_{f, SKSL_f}$, we consider the centralizer $Z(u)$ in $S \times (J)$. It is generated by the permutations $\sigma' \in S$ that commute with $\sigma$ and by the diagonal element $J$. In $S \times SL_f$, the centralizer $Z(v)$ is generated by the same set of $\sigma' \in S$ as above and by the diagonal elements $h \in SL_f$ that commute with $\sigma$. We set $J_a := \prod_t t_i$ with $i \in I_{\tilde{\kappa}_{\sigma_a}}$. For some $l_1, \ldots, l_p$ we then have $h = \prod_{a=1}^{p} J_{l_a}^{l_a}$. By Eq. (4), we have

$$h^*(X) = \prod_{a=1}^{p} \xi_{N}^{l_a} d_a X.$$  

Using the $\kappa$ above, we have

$$\sum_{a=1}^{p} l_a d_a \equiv \kappa \left( \sum_{a=1}^{p} l_a |\sigma_a| \right) \equiv 0 \text{ mod } N,$$

because $h \in SL_f$. The action of $\sigma'$ is the same on $X$ considered in both $A_{f, SKSL_f}$ and $A_{f, SK(J)}$. It is trivial because a nonspecial $\hat{\tau}_2(Y)$ also has $\sigma'$ in its centralizer and is nonzero.

For $\hat{\tau}_3(X) \neq 0$, $\hat{\tau}_4(X) \neq 0$, we set $Y := \hat{\tau}_3(X)$. This case is treated totally similarly.

2. The statement is obvious for $X$ and $Y$ such that $\hat{\tau}_1(X)$, $\hat{\tau}_2(X)$ and $\hat{\tau}_4(Y)$ are not simultaneously non-zero. If this does not hold, by case 1 above we have $\tilde{\tau}(Y) = \hat{\tau}_1(Y)$ and $\tilde{\tau}(X) = \hat{\tau}_4(X)$.

Proposition 9. The map $\tilde{\tau}$ is an isomorphism for $N = 5$.

Proof. We follow the steps of the argument in Sec. 5. Let $X$ and $Y$ be respective nonzero broad and narrow elements of $A_{f, SKSL_f}$. By Proposition 8, part 2, it suffices to show that for nonzero $X$ and $Y$, $\hat{\tau}_1(Y)$ and $\hat{\tau}_2(Y)$ are not simultaneously zero and $\hat{\tau}_3(X)$ and $\hat{\tau}_4(X)$ are not simultaneously zero. Moreover it suffices to consider the action of the symmetric part elements $\sigma' \in S$ on both sides.

Let $\hat{\tau}_1(Y) = 0$ in $A_{f, SK(J)}$. Then there is $\sigma' \in Z(Y)$ such that $\sigma' \in Z(\hat{\tau}_1(Y))$ and $(\sigma')^*(\hat{\tau}_1(Y)) = -\hat{\tau}_1(Y)$. In this case, we should have $\sigma = (i, j)(k, l)$ for some pairwise distinct $1 \leq i, j, k, l \leq 5$. Without a loss of generality, we assume that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then we should have $u = \sigma \cdot g = (1, 2)(3, 4)\xi_{\sigma_{l_{\sigma, d}}^{-2d}}$. Then we have $\hat{\tau}_2(Y) = \xi_{\sigma_{l_{\sigma, d}}^{-2d}}$, which is nonzero in $A_{f, SK(J)}$.

The case of $\hat{\tau}_3(X) = 0$ in $A_{f, SK(J)}$ is treated totally similarly.

Remark. It is natural to consider the map $\tilde{\tau}$ in a more general context. However, an isomorphism can no longer be established already for $N = n = 7$.  

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6.3. Examples. We follow the notation in Sec. 3.1. In particular, we use the spaces $A_{SL,s}$ and $A_{(j),s}$.

6.3.1. Klein 4-group. We consider $S = \langle \sigma_1, \sigma_2 \rangle \subset S_5$ with $\sigma_1 = (1,2)(3,4)$, $\sigma_2 = (1,3)(2,4)$ and set $\sigma_3 := \sigma_1 \sigma_2$. The spaces $A_{SL,s}$ and $A_{(j),s}$ are 24-dimensional. Let

\[
\begin{align*}
\phi_{a,b,c}^{(1)} & := ((x_1 + x_2)^a(x_3 + x_4)^b - (x_1 + x_2)^b(x_3 + x_4)^a)x_5^c, \\
\phi_{a,b,c}^{(2)} & := ((x_1 + x_3)^a(x_2 + x_4)^b - (x_1 + x_3)^b(x_2 + x_4)^a)x_5^c, \\
\phi_{a,b,c}^{(3)} & := ((x_1 + x_4)^a(x_2 + x_3)^b - (x_1 + x_4)^b(x_2 + x_3)^a)x_5^c.
\end{align*}
\]

We have

\[
A_{SL,s} = \bigoplus_{a,b=1,\ldots,4; \ a \leq b} \mathbb{C}(\xi_{\sigma_1 t_1^a t_5^b-a-b}) \bigoplus \mathbb{C}(\xi_{\sigma_2 t_2^a t_5^b-a-b}) \bigoplus \mathbb{C}(\xi_{\sigma_3 t_3^a t_5^b-a-b}),
\]

\[
A_{(j),s} = \bigoplus_{k=1}^3 \bigoplus_{a=1}^2 \mathbb{C}(\xi_{\sigma_k J^k}) \bigoplus \mathbb{C}(\phi_{1,0,1}^{(k)} \xi_{\sigma_k}, \phi_{2,0,0}^{(k)} \xi_{\sigma_k}, \phi_{3,1,3}^{(k)} \xi_{\sigma_k}, \phi_{3,2,2}^{(k)} \xi_{\sigma_k}).
\]

That the polynomials $\phi_{a,b,c}^{(k)}$ are skew symmetric under the action of $\sigma_l$ with $l \neq k$ reflects the fact that $\sigma_l^T(\xi_{\sigma_k}) = -\xi_{\sigma_k}$.

Because $A_{SL,s}$ only contains broad sectors, we only have to consider the maps $\tilde{\tau}_1$ and $\tilde{\tau}_2$. We compute

\[
\tilde{\tau}_1(\xi_{\sigma_1 t_1^a t_5^b}) = \begin{cases} 
\phi_{a-1,b-1,c-1}^{(1)} & \text{if } a \neq b, \\
0 & \text{if } a = b
\end{cases}
\]

\[
\tilde{\tau}_2(\xi_{\sigma_1 t_1^a t_5^b}) = \begin{cases} 
0 & \text{if } a \neq b, \\
\xi_{\sigma_1 J^c} & \text{if } a = b.
\end{cases}
\]

The last equality seems mysterious because it only depends on $c$. However, the mystery is resolved by the fact that we have the coincidence of the indices $a, b$ in this case and $a, b, c$ are all connected by the degree condition.

In this example, $K$ turns out to be the identity map and $\bar{\tau} = \hat{\tau}$.

6.3.2. Group $S \cong S_3 \times S_2$. We consider $S = \langle (1,2,3), (1,2), (4,5) \rangle \subset S_5$. The spaces $A_{SL,s}$ and $A_{(j),s}$ are 8-dimensional. Let

\[
\phi_{a,b,c} = (x_1 + x_2 + x_3)^a(x_4^b x_5^c + x_4^c x_5^b).
\]

We have

\[
A_{SL,s} = \mathbb{C}(\xi_{(1,2,3)t_1^a t_5^b}, \xi_{(1,2,3)t_4^a t_5^b}, \xi_{(1,2,3)t_4^a t_4^b}, \xi_{(1,2,3)t_1^a t_4^b}) \bigoplus \\
\bigoplus \mathbb{C}(\phi_{0,1,1}^{(1)} \xi_{(1,2,3)}, \phi_{0,2,0}^{(1)} \xi_{(1,2,3)}, \phi_{0,3,2}^{(1)} \xi_{(1,2,3)}, \phi_{1,3,3} \xi_{(1,2,3)}),
\]

\[
A_{(j),s} = \bigoplus \mathbb{C}(\phi_{0,0,2}^{(1)} \xi_{(1,2,3)}, \phi_{1,0,1} \xi_{(1,2,3)}, \phi_{0,2,3}^{(1)} \xi_{(1,2,3)}, \phi_{3,1,3} \xi_{(1,2,3)}) \bigoplus \\
\bigoplus \mathbb{C}(\phi_{0,1,1}^{(1)} \xi_{(1,2,3)}, \phi_{2,0,0} \xi_{(1,2,3)}, \phi_{3,2,3} \xi_{(1,2,3)}, \phi_{1,3,3} \xi_{(1,2,3)}).
\]

The mirror map gives

\[
\hat{\tau}_3((\phi_{a,b,c} \xi_{(1,2,3)}) = [\phi_{a,b,c} \xi_{(1,2,3)}], \hat{\tau}_4((\phi_{a,b,c} \xi_{(1,2,3)}) = 0, \ b \neq c,
\]

\[
\hat{\tau}_1((\xi_{(1,2,3)t_1^a t_5^b}) = [\phi_{a-1,b-1,c-1} \xi_{(1,2,3)}], \hat{\tau}_2((\xi_{(1,2,3)t_1^a t_5^b}) = 0.
\]

In this example, $K$ turns out to be the identity map and $\bar{\tau} = \hat{\tau}$. 

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6.3.3. Group $S \cong S_3$ revisited. To illustrate how the map $K$ works, we set $S := \{(1, 2, 3), (1, 2)\}$ as in Example 3 in Sec. 3.1. We have
\begin{align*}
\hat{\tau}([H^K]_1) &= \xi_{j^b}, \\
\hat{\tau}(\xi_{(1,2,3)}[t_1t_2t_3]) &= |\phi_{a-1,b-1,c}| \xi_{(1,2,3)}, \quad b \neq c, \\
\hat{\tau}(\xi_{(1,2,3)}[t_1t_2t_3]) &= |\phi_{a-1,b-1,1}| \xi_{(1,2,3)} + \xi_{(2,3)}, J^b, \\
\hat{\tau}(\xi_{(1,2,3)}[t_1t_2t_3]) &= -|\phi_{a,b,b}| \xi_{(1,2,3)} + \xi_{(1,2,3)}, J^{b-1}.
\end{align*}
Then
$$K(\xi_{(1,2,3)}[t_1t_2t_3]) = -|\phi_{a-1,b-1,1}| \xi_{(1,2,3)}; \quad K(\xi_{(1,2,3)}[t_1t_2t_3]) = \xi_{(1,2,3)} [t_1t_2t_3]$$
and $K = \text{id}$ on all other basis vectors.

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