Acoustic properties of a layered porous elastic structure saturated with a Maxwell fluid

B N Starovoitov\textsuperscript{1} and V N Starovoitov\textsuperscript{1,2}
\textsuperscript{1} Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia
\textsuperscript{2} Novosibirsk State University, Novosibirsk, Russia
E-mail: botagoz@hydro.nsc.ru, starovoitov@hydro.nsc.ru

Abstract. We consider a one-dimensional layered structure that consists of an elastic material and a Maxwell fluid. It is assumed that the width of the layers is small compared with the size of the domain, where the problem is being solved. By employing theNguetseng two-scale convergence technique, we have homogenized the equations that describe the mechanical system. For the homogenized equation, the limit cases of the Maxwell fluid with the small viscosity and with the small relaxation time are considered. The question of existence of the travelling wave solution for the homogenized equation is investigated.

1. Statement of the problem
In this paper, we consider a periodic layered structure that consists of alternating elastic and fluid layers. Our mail goal is to derive a homogenized macroscopic equations when the thickness of the layers tends to zero. A similar problem was investigated in [1]. Unlike that paper, we consider a Maxwell fluid instead of the usual viscous one. Besides that, we homogenize the non-stationary problem and then investigate the existence of the harmonic waves.

Suppose that the mechanical system is periodic in the direction of the variable $x$ and does not depend on other variables. Denote by $h$ the period of the structure and assume that the strips $0 < x < x_h$ and $x_h < x < h$ are occupied by the solid and by the fluid, respectively. Due to the periodicity, the same is true for the strips $kh < x < x_h + kh$ and $x_h + kh < x < (k+1)h$, where $k$ is an arbitrary integer number. Let us write down the governing equations. Since we have supposed that the system depends only on the spatial variable $x$ and on the time $t$, the corresponding equations will contain only these two variables. Besides that, we assume that all vector functions, such as the velocity and the displacement, will have only one non-zero component directed along the variable $x$.

The displacement $u_S$ in the elastic part of the continuum satisfies the following equation:

$$\rho_S \partial^2_t u_S = (\lambda_S + 2\mu_S) \partial_x^2 u_S,$$

(1)

where $\rho_S = \text{const}$ is the density, $\lambda_S$ and $\mu_S$ are the Lame coefficients. The external balk forces are neglected, which is suitable for possible acoustic applications.

The Maxwell fluid is a continuum with the stress tensor of the form $-p_F + \sigma_F$, where $p_F$ is the pressure. The tensor $\sigma_F$ in our case has only one non-zero component denoted again by $\sigma_F$ and defined as follows:

$$(1 + \tau_s \partial_t)\sigma_F = 2\mu_F \partial_x v_F,$$

where $\tau_s = \text{const}$ is the relaxation time, $\mu_F$ is a constant, and $v_F$ is the fluid velocity.
where \( v_F \) is the velocity of the fluid, \( \mu_F = \text{const} \) its viscosity, and \( \tau_s \) is a constant that has the dimension of time and is usually called the relaxation time. Originally, in the classical derivation of the Maxwell constitutive law, \( \tau_s = 2\mu_F / E_F \), where \( E_F \) is the elastic modulus.

Since we intend to describe acoustic processes, we assume that the fluid is slightly compressible and the governing equations are linear:

\[
\rho_F \partial_t v_F = -\partial_x p_F + \partial_x \sigma_F, \\
\partial_t p_F = -\gamma \partial_x v_F,
\]

where \( \rho_F = \text{const} \) is the density and \( \gamma \) is a positive constant that characterizes the compressibility of the fluid.

On the interface that separates the solid and fluid phases, we impose the usual continuity conditions for the velocity and for the internal forces:

\[
v_F(t, x) = \partial_t u_S(t, x), \\
-p_F(t, x) + \sigma_F(t, x) = \mu_S \partial_x u_S(t, x)
\]

for \( x = kh \) and \( x = x_h + kh \) with an arbitrary integer \( k \) and for all \( t \).

It will be more convenient for us to rewrite the equations for the fluid in terms of the displacement \( u_F \). Since in the framework of the linear approach \( v_F = \partial_t u_F \), we after some calculation obtain that

\[
\rho_F \partial_t^2 u_F = \gamma \partial_x^2 u_F + \frac{2\mu_F}{\tau_s} \partial_x^2 (u_F - J_t u_F) - \partial_x F_F, \tag{2}
\]

where

\[
J_t u = \frac{1}{\tau_s} \int_0^t e^{(\tau-t)/\tau_s} u(\tau) \, d\tau
\]

and

\[
F_F(t, x) = e^{-t/\tau_s} G_F^0(x) + \partial_t^0(x), \quad G_F^0 = \frac{2\mu_F}{\tau_s} \partial_x u_F^0 - \sigma_F^0, \quad p_F^0 = p_F^0 + \gamma \partial_x u_F^0.
\]

The upper index 0 means the initial value of the corresponding quantity at \( t = 0 \).

Equations (1) and (2) can be written as one equation. To do this, we introduce the function \( \chi_h \) that is equal to 1 in the fluid part of the domain and to 0 in the solid. The subscript \( h \) shows that this function corresponds to the problem with the layers width \( h \). The function

\[
u_h(t, x) = u_F(t, x) \chi_h(x) + u_S(t, x)(1 - \chi_h(x))
\]

satisfies the following equation:

\[
\varrho_h \partial_t^2 u_h = \partial_x \left( (\lambda_h + 2\mu_h) \partial_x u_h \right) - \partial_x \left( 2\mu_h \chi_h \partial_x J_t u_h \right) - \partial_x F_h, \tag{3}
\]

where

\[
\varrho_h = \varrho_F \chi_h + \varrho_S(1 - \chi_h), \quad \lambda_h = \gamma \chi_h + \lambda_S(1 - \chi_h), \quad \mu_h = \frac{\mu_F}{\tau_s} \chi_h + \mu_S(1 - \chi_h), \quad F_h = \chi_h F_F.
\]

Equation (3) is understood in the distributional sense. This equation should be supplemented by the initial data \( u_h(0, x) = u_h^0(x) \), \( \partial_t u_h(0, x) = u_h^1(x) \) and by boundary conditions. Up to now, we have not specified the domain, where the problem is considered. It can be any finite interval of the real axis. We denote this interval by \( \Omega \) and, without loss of generality, assume that \( \Omega = (0, L) \), \( L > 0 \). We suppose that \( u_h \) satisfies the homogeneous boundary conditions:
$u_h(t, 0) = u_h(t, L) = 0$ for all $t$. In other case, we can make them homogeneous by changing the unknown function $u_h$ and the right-hand side $F_h$. This problem will be referred as Problem $A_h$.

It is not difficult to prove that, under natural conditions on the initial data, Problem $A_h$ has a unique weak solution $u_h \in L^2(0, T; H_0^1(\Omega))$ such that $\partial_t u_h \in L^\infty(0, T; L^2(\Omega))$. Here, $T$ is an arbitrary positive number, $L^p$ and $H_0^1$ are usual Lebesgue and Sobolev function spaces, respectively. This problem, however, is not appropriate for applications since its numerical realisation demands a lot of computer resources. The reason for this is that (3) is an equation with rapidly oscillating coefficients. Therefore, it would be better to homogenize the equation, i.e., to replace it by a new one whose coefficients do not oscillate and whose solution is close in some sense to the original one.

2. Homogenization of the problem

Suppose that $h/L = o(1)$, i.e., the domain $\Omega$ contains a large number of the layers. Without loss of generality, we can assume that $L$ is finite and $h$ is small. Generally speaking, we would have to rewrite the problem in the dimensionless form in which only the ratio of $h$ and $L$ as well as that of the other physical parameters makes sense. However, we will not do this since the dimensionless equation looks exactly the same. One can simply assume that all the quantities in (3) are dimensionless.

Thus, $h$ is a small parameter in Problem $A_h$. In order to describe the homogenized continuum, one have to find equations for the principal term $u$ of the expansion $u_h = u + O(h)$ as $h \to 0$. To do this, we employ the notion of two-scale convergence introduced in [2] and further developed in a number of papers (see, e.g., [3, 4]). We denote by $\Sigma$ the interval $[0, 1]$ and say that a function is $\Sigma$-periodic if it has the period 1. A sequence of functions $\{v_h\}$ in $L^2([0, T] \times \Omega)$ is said to be two-scale converging to a function $v \in L^2([0, T] \times \Omega \times \Sigma)$ if

$$\lim_{h \to 0} \int_0^T \int_\Omega v_h(t, x) \phi(t, x, x/h) \, dx \, dt = \int_0^T \int_\Sigma v(t, x, \xi) \phi(t, x, \xi) \, d\xi \, dx \, dt$$

for every smooth function $\phi(t, x, \xi)$ that is $\Sigma$-periodic in $\xi$. We denote this fact by $v_h \rightharpoonup v$.

Let us define the $\Sigma$-periodic function $\chi(\xi)$ that is equal to 0 in (0, $x_h$/h) and to 1 in ($x_h$/h, 1). It is not difficult to see that $\chi_h(x) = \chi(x/h)$ and $\chi(\xi) = \chi_h(h\xi)$. Similarly, we define the $\Sigma$-periodic functions: $\phi(\xi) = \phi_h(h\xi)$ and $\lambda(\xi) = \lambda_h(h\xi)$. These functions can be also defined exactly as in equation (3) but with $\chi_h$ replaced by $\chi$. It is not difficult to see that $\chi_h \rightharpoonup \chi(\xi)$, $\phi_h \rightharpoonup \phi(\xi)$, and $\lambda_h \rightharpoonup \lambda(\xi)$. As for the limit of $\mu_h$, we will consider two cases. In the first one, the viscosity $\mu_r$ of the Maxwell fluid is a finite positive constant. The second case deals with a weakly viscous Maxwell whose viscosity is small. This means that $\mu_r = \mu_s + o(1)$ as $h$ is close to 0, $\mu_s$ is a positive constant in the first case and $\mu_s = 0$ in the second one. Thus, $\mu_h \to \mu$, where

$$\mu(\xi) = \frac{\mu_s}{\tau_s} \chi(\xi) + \mu_s (1 - \chi(\xi))$$

is a $\Sigma$-periodic function.

Let us turn to the initial data. Suppose that the sequences $\{u^1_h\}$, $\{\partial_x u^0_h\}$, $\{p^0_r\}$, and $\{\sigma^0_r\}$ are bounded in $L^2(\Omega)$ uniformly in $h$. The functions $p^0_r$ and $\sigma^0_r$ are assumed to be extended by zero to the solid part of the domain $\Omega$. Then there exist functions $u^1 = u^1(x, \xi)$, $\sigma^0 = \sigma^0(x, \xi)$, $p^0 = p^0(x, \xi)$, $u^0 = u^0(x)$, and $\bar{u}^0 = \bar{u}^0(x, \xi)$ that are the two-scale limits of the corresponding sequences. Notice that $\partial_x u^0_h \rightharpoonup \partial_x u^0 + \partial_\xi \bar{u}^0$. This implies that $F_h \rightharpoonup F$, where

$$F(t, x, \xi) = \chi(\xi)(e^{-t/\tau_s} G^0(x, \xi) + P^0(x, \xi)),$$

$$G^0 = \frac{2\mu_s}{\tau_s} (\partial_x u^0 + \partial_\xi \bar{u}^0) - a^0, \quad P^0 = p^0 + \gamma \partial_x u^0 + \gamma \partial_\xi \bar{u}^0.$$
By using the two-scale convergence technique, it is possible to prove that there exist functions $u = u(t, x)$ and $\bar{u} = \bar{u}(t, x, \xi)$ such that, up to a subsequence, $u_h \rightarrow u$, $\partial_t u_h \rightarrow \partial_t u$, and $\partial_x u_h \rightarrow \partial_x u + \partial_\xi \bar{u}$. Notice that $u$ does not depend on $\xi$ and $\bar{u}$ is $\Sigma$-periodic in $\xi$. The functions $u$ and $\bar{u}$ satisfy the following integral identities:

$$\int_0^T \int_\Sigma (\phi \partial_t u \partial_t \phi - (\lambda + 2\mu)(\partial_x u + \partial_\xi \bar{u}) \partial_x \phi + 2\mu \chi (\partial_x J_t u + \partial_\xi J_t \bar{u}) \partial_\xi \phi) \, d\xi \, dx \, dt = - \int_\Omega \phi \, u^0 \, d\xi \, dx \, dt - \int_0^T \int_\Sigma F \partial_\xi \phi \, d\xi \, dx \, dt,$$  \hspace{0.5cm} (4)

$$\int_\Sigma ((\lambda + 2\mu)(\partial_x u + \partial_\xi \bar{u}) - 2\mu \chi (\partial_x J_t u + \partial_\xi J_t \bar{u}) - F) \partial_\xi \tilde{\phi} \, d\xi = 0 \hspace{0.5cm} (5)$$

for arbitrary smooth functions $\phi = \phi(t, x)$ and $\tilde{\phi} = \tilde{\phi}(t, x, \xi)$ that are equal to zero at $t = T$ and for $x \in \partial \Omega$. In equation (4), $\phi^0 = \phi|_{t=0}$. We note also that $\mu \chi = \chi_{\mu}/\tau$. Equation (5) is called cell equation and satisfied for almost all $(t, x) \in [0, T] \times \Omega$.

Although it is possible to deal with equations (4)–(5), this system is too complicated and says nothing about the mechanical properties of the homogenized continuum. Our next goal is to exclude the function $\bar{u}$ and the variable $\xi$ from equation (4). To do this, we find an explicit expression for the function $\bar{u}$ from the cell equation and substitute it into (4). This procedure is based on the technique developed in [5].

Let us rewrite (5) as an equation in the Hilbert space $H = H^1_\Sigma(\Sigma)/\mathbb{R}$ with the inner product $(u, v) = \int_\Sigma \partial_x u \partial_x v \, d\xi$. The subscript $\#$ means that it is a space of $\Sigma$-periodic functions. Let us define bounded self-adjoint operators $A : H \rightarrow H$ and $B : H \rightarrow H$ such that

$$(Au, v) = \int_\Sigma (\lambda + 2\mu) \partial_x u \partial_x v \, d\xi, \quad (Bu, v) = \int_\Sigma 2\mu \chi \partial_\xi u \partial_\xi v \, d\xi$$

for all $u, v \in H$. We define also functions $a, b, f \in H$ such that

$$(a, v) = \int_\Sigma (\lambda + 2\mu) \partial_\xi v \, d\xi, \quad (b, v) = \int_\Sigma 2\mu \chi \partial_\xi v \, d\xi, \quad (f, v) = \int_\Sigma F \partial_\xi v \, d\xi$$

for all $v \in H$. Equation (5) is equivalent to the following equation in the space $H$:

$$A\bar{u} - \mathcal{J}_t B\bar{u} = -(a - b J_t) \partial_x u + f. \hspace{0.5cm} (6)$$

Notice that the operator $B$ is degenerate and its inverse exists only in the image of this operator $R(B)$ which is a closed subspace of $H$. Since $b \in R(B)$, there exists a unique $w \in R(B)$ such that $B w = b$. Using this function we can write down the solution of equation (6) explicitly:

$$\bar{u}(t) = -\partial_x u A^{-1} a + \int_0^t \partial_x u M_{t-s} (w - A^{-1} a) \, ds + A^{-1} f + \int_0^t M_{t-s} A^{-1} f \, ds,$$

where

$$M_t = \frac{1}{\tau} A^{-1} B e^{-t/\tau (I - A^{-1} B)}$$

and $I$ is the identity operator in $H$. As a consequence, we find that

$$\mathcal{J}_t \bar{u} = -w \partial_x J_t u + \int_0^t \partial_x u N_{t-s} (w - A^{-1} a) \, ds + \int_0^t N_{t-s} A^{-1} f \, ds,$$
where
\[ N_t = \frac{1}{\tau_s} e^{-t/\tau_s (I-A^{-1}B)}. \]

Substitution of these expressions into (4) gives the following integral identity:
\[ \int_0^T \int_\Omega \left( \partial_t u \partial_t \phi - \alpha \partial_x u \partial_x \phi + \beta \partial_x (J_t u) \partial_x \phi \right) \, dx \, dt \]
\[ + \int_0^T \int_0^t \eta(t-s) \partial_x u(s) \, ds \, \partial_x \phi \, dx \, dt = - \int_\Omega \varrho u_0^I \phi^0 \, dx \, dt + \int_0^T \int_\Omega \zeta \partial_x \phi \, dx \, dt, \quad (7) \]

where \( \theta = \int \chi(\xi) \, d\xi \) is the porosity and
\[ \varrho_0 = \int \varrho(\xi) \, d\xi = \varrho \theta + \varrho_s(1-\theta), \quad \alpha_0 = \lambda_0 + 2\mu_0 - (a,A^{-1}a), \quad \beta_0 = \frac{2\theta \mu_s}{\tau_s} - (b,w), \]
\[ \lambda_0 = \int \lambda(\xi) \, d\xi = \gamma \theta + \lambda_s(1-\theta), \quad \mu_0 = \int \mu(\xi) \, d\xi = \frac{\mu_s}{\tau_s} \theta + \mu_s(1-\theta), \]
\[ \eta(t) = -(a, M_t(w - A^{-1}a)) + (b, N_t(w - A^{-1}a)), \]
\[ \zeta(t) = (a, A^{-1}f + \int_0^t M_{t-s}A^{-1}f \, ds) - (b, \int_0^t N_{t-s}A^{-1}f \, ds) - F_0, \]
\[ \varrho_0^I(x) = \int \varrho(\xi) \, u^I(x,\xi) \, d\xi, \quad F_0(t,x) = \int F(t,x,\xi) \, d\xi. \]

Notice that \( \varrho_0, \alpha_0, \) and \( \beta_0 \) are constants while \( \eta(t) \) and \( \zeta(t,x) \).

Equation (7) can be also rewritten in the following form:
\[ \varrho_0 \partial_t^2 u - \alpha_0 \partial_x^2 u + \beta_0 \partial_x^2 J_t u = -\partial_x^2 \int_0^t \eta(t-s) \, u(s) \, ds - \partial_x \zeta \quad (8) \]

that should be understood in the distributional sense. This equation describes the behavior of the homogenized continuum. The terms with the coefficients \( \alpha_0 \) and \( \beta_0 \) in this equation are related to the stresses. The integral term in the right-hand side represents a memory effect. This term arises due to the homogenization technique and is standard for non-stationary problems (see, e.g., [5,6]). The kernel \( \eta \), however, decreases rapidly, so, this memory is very short. The function \( \zeta \) comes from the initial data and, generally speaking, from the boundary conditions.

3. Parameters of the homogenized continuum

In this section, we calculate the coefficients and the functions that are present in equation (8).

**Calculation of \( a, b, \) and \( f. \)**

Let us define a function \( q \in H \) as the unique solution of the following equation:
\[ \int_\Sigma \partial_\xi q \partial_\xi v \, d\xi = \int_\Sigma \chi \partial_\xi v \, d\xi \quad \text{for all} \quad v \in H. \]

It is not difficult to see that \( a = a_s q \) and \( b = 2q \mu_s / \tau_s \), where \( a_s = \gamma - \lambda_s - 2\mu_s + 2\mu_s / \tau_s \). The function \( f \) also can be expressed in terms of the function \( q \) but we omit this representation. The function \( q \) can be easily find:
\[ q(\xi) = \begin{cases} -(1-\theta)^2/2 + (1-\theta)/2 - \xi \theta, & \xi \in [0,x_h/h], \\ -(1-\theta)^2/2 - (1-\theta)/2 + \xi(1-\theta), & \xi \in [x_h/h,1]. \end{cases} \]
Functions in $H$ are defined up to an additive constant, for this reason, we have assumed that $\int_{\mathbb{R}} q \, d\xi = 0$. This condition will be fulfilled further for all functions from $H$. Besides that, we have taken into account that $x_h/h = 1 - \theta$.

**Calculation of $w$.**

The function $w$ is in $R(B)$. As it follows from Proposition 4.10 in [5], this function is affine on the interval $[0, x_h/h]$. As a consequence of the definition of this function, it is affine also on $[x_h/h, 1]$. Thus, it is not difficult to find that

$$w(\xi) = \begin{cases} \theta/2 - \xi \theta/(1 - \theta), & \xi \in [0, x_h/h], \\ \theta/2 - 1 + \xi, & \xi \in [x_h/h, 1]. \end{cases}$$

**Calculation of $\alpha_{\theta}$ and $\beta_{\theta}$.**

In order to calculate $\alpha_{\theta}$, we have to find $(a, A^{-1}a)$. Since $a = a_s q$, it is enough to find $(q, A^{-1}q)$. If $r = A^{-1}q$, then $r$ is the unique solution of the equation $Ar = q$. It is not difficult to find this solution: $r = q/r_s$, where $r_s = \theta(\lambda_s + 2\mu_s) + (1 - \theta)(\gamma + 2\mu_s/\tau_s)$. As a consequence, we have that $(q, A^{-1}q) = (q, r) = \theta(1 - \theta)/r_s$ and $(a, A^{-1}a) = a_s^2 \theta(1 - \theta)/r_s$. Thus,

$$\alpha_{\theta} = \left(\frac{\theta}{\gamma + 2\mu_s/\tau_s} + \frac{1 - \theta}{\lambda_s + 2\mu_s}\right)^{-1}.$$

As for $\beta_{\theta}$, it is not difficult to calculate that $\beta_{\theta} = 0$. It means that the term in equation (8) which corresponds to the Maxwell fluid vanishes, i.e., the homogenized continuum is purely elastic. Of course, the elastic modulus $\alpha_{\theta}$ of this continuum depends on parameters of the original Maxwell fluid such as $\gamma$, $\tau_s$, and $\mu_s$. Notice also that, in the multi-dimensional case, $\beta_{\theta}$ is a non-zero fourth-rank tensor.

**4. Asymptotic properties of the homogenized equation**

If we neglect the influence of the initial data, then equation (8) with the calculated in the previous section coefficients takes the form:

$$q_\theta \partial_t^2 u - \alpha_{\theta} \partial_s^2 u = -\partial_s^2 \int_0^t \eta(t - s) u(s) \, ds. \tag{9}$$

The left-hand side of this equation is the wave operator related to the elastic behavior of the homogenized continuum, however, (9) has no solutions of the travelling wave type. This is due to the memory term in the right-hand side of this equation. There are no travelling waves even for the weakly viscous fluid, i.e., for $\mu_s = 0$. The memory term is usual for homogenized non-stationary equations, but the one-dimensional case has special features. It is worth noting that the function $\eta$ is positive and rapidly tends to zero. For small $\tau_s$, it looks like the $\delta$-function concentrated at 0. Really, it is possible to prove that

$$\int_0^t \eta(t - s) u(s) \, ds - \eta_0 u(t) = o(1) \quad \text{in } H \quad \text{as } \tau_s \rightarrow 0,$$

where $\eta_0 = -(a, A^{-1}B(A - B)^{-1}(Aw - a)) + (b, (A - B)^{-1}(Aw - a))$. The constant $\eta_0$ can be calculated:

$$\eta_0 = \theta \frac{m^2_{a_s}}{n_{\theta}^2} \left(\frac{\tau_s}{2\mu_s} + \frac{1 - \theta}{n_{\theta}}\right)^{-1},$$
where \( m_s = \lambda_s + 2\mu_s \), \( n_\theta = (1 - \theta)\gamma + \theta m_s \). We see that \( \eta_\theta \) depends on the fraction \( 2\mu_s/\tau_s \). The same is true for \( \alpha_\theta \). We consider three cases of possible values of the ratio \( 2\mu_s/\tau_s \) where \( \tau_s \) tends to zero. In all these cases, equation (9) takes the form:

\[
g_\theta \partial_t^2 u - (\alpha_\theta - \eta_\theta) \partial_x^2 u = 0
\]

but the values of \( \alpha_\theta \) and \( \eta_\theta \) will be different.

1. \( \tau_s \to 0 \), \( \mu_s \to 0 \), and \( 2\mu_s/\tau_s \to 0 \). This is the case of usual weakly viscous fluid (not the Maxwell one). In this case

\[
\alpha_\theta = \left( \frac{\theta}{\gamma} + \frac{1 - \theta}{\lambda_s + 2\mu_s} \right)^{-1} \text{ and } \eta_\theta = 0.
\]

2. \( \tau_s \to 0 \), \( \mu_s \to 0 \), and \( 2\mu_s/\tau_s \to \kappa \in (0, +\infty) \). This is another weakly viscous fluid. In this case

\[
\alpha_\theta = \left( \frac{\theta}{\gamma + \kappa} + \frac{1 - \theta}{\lambda_s + 2\mu_s} \right)^{-1} \text{ and } \eta_\theta = \theta \frac{m_s^2}{n_\theta^2} \left( \frac{1}{\kappa} + \frac{1 - \theta}{n_\theta} \right)^{-1},
\]

3. If \( \tau_s \to 0 \) and \( 2\mu_s/\tau_s \to +\infty \), then

\[
\alpha_\theta = \frac{\lambda_s + 2\mu_s}{1 - \theta} = \frac{m_s}{1 - \theta} \text{ and } \eta_\theta = \theta \frac{m_s^2}{n_\theta},
\]

It is not difficult to see that \( \alpha_\theta > \eta_\theta \) in all three cases. This means that equation (10) has solutions of the travelling wave type in all three cases. The velocity of the waves is equal to \( \sqrt{(\alpha_\theta - \eta_\theta) / g_\theta} \). It is the sound velocity. Notice that the travelling waves exists also in the third case where the viscosity of the fluid is not equal to zero.

**Acknowledgments**

This work was partially supported by Russian Foundation for Basic Research (Grants 15-01-01091 A).

**References**

[1] Shelukhin V V and Isakov A E 2012 Elastic waves in layered media: two-scale hogenization approach Euro. Jnl of Applied Mathematics 23 691-707

[2] Nguetseng G 1989 Asymptotic analysis for a functional related to the theory of homogenization SIAM Journal on Mathematical Analysis 20(3) 608-23

[3] Allaire G 1992 Homogenization and two-scale convergence SIAM Journal on Mathematical Analysis 23 1482-18

[4] Zhikov V V 2004 On two-scale convergence Journal of Mathematical Sciences 120(3) 1328-52

[5] Hoffmann K-H, Botkin N D and Starovoitov V N 2005 Homogenization of interfaces between rapidly oscillating fine elastic structures and fluids SIAM Journal on Applied Mathematics 65(3) 983-1005

[6] Gilbert R P, Panchenko A and Vasilec A 2011 Homogenizing the acoustic of cancellous bone with an interstitial non-Newtonian fluid Nonlinear Analysis: Theory, Methods and Applications 74(4) 1005-18