A method for computing the Perron-Frobenius root for primitive matrices

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Abstract

For a nonnegative matrix, the eigenvalue with the maximum magnitude or Perron-Frobenius root exists and is unique if the matrix is primitive. It is shown that for a primitive matrix $A$, there exists a positive rank one matrix $X$ allowing to have the Hadamard product $B = A \circ X$, where the row (column) sums of matrix $B$ are the same and equal to the Perron-Frobenius root. An iterative algorithm is presented to obtain matrix $B$ without an explicit knowledge of $X$. The convergence rate of this algorithm is similar to that of the power method but it uses less computational load.

Keywords: Nonnegative primitive matrix; Perron-Frobenius root; Hadamard product

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1 Introduction

Let $A$ a square matrix of order $n$. An eigenvalue $\lambda$ and an eigenvector $\mathbf{x} \neq 0$ of $A$ satisfy [1] page 490

$$A\mathbf{x} = \lambda \mathbf{x} \quad (1)$$

The eigenvalues of matrix $A$ are the roots ($\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$) of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I_n) \quad (2)$$

where $I_n$ is the identity matrix of order $n$.

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For two matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) with the same number of rows and columns, their Hadamard product is a matrix of elementwise products:

\[
A \odot B = (a_{ij}b_{ij})
\]  

(3)

For scalars \( \alpha \) and \( \beta \):

\[
\alpha A \odot \beta B = \alpha \beta (A \odot B)
\]  

(4)

If \( A \) and \( B \) are rank one matrices, i.e. \( A = uv^T \) and \( B = xy^T \) then

\[
A \odot B = (uv^T) \odot (xy^T) = (u \odot x)(v \odot y)^T.
\]  

(5)

Many properties for Hadamard product are given in [2][3, chapter 5].

The matrix \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) is positive if \( a_{ij} > 0 \), and nonnegative if \( a_{ij} \geq 0 \). Perron [4] shows that a positive matrix \( A \) has only one positive real-valued eigenvalue \( \lambda \) with a maximum magnitude [1, page 667]. This eigenvalue noted \( \rho(A) \) is called the Perron root and the associated vector, Perron vector. Nonnegative matrices are frequently encountered in real life applications. Frobenius [5] extended the Perron’s work on positive matrices to nonnegative matrices. A nonnegative matrix \( A \) has a dominant eigenvalue, if it is irreducible, i.e. \( (I_n + A)^{n-1} \) is a positive matrix [6, page 508][1, page 672]. The dominant eigenvalue of an irreducible matrix is unique if it is primitive [6, page 516][1, page 674]. A nonnegative matrix is primitive if \( A^m \) is a positive matrix for some non null \( m \) [6, page 516][1, page 678]. Wielandt, [7], showed that a nonnegative square matrix \( A \) of order \( n \) is primitive if \( A^{n^2-2n+2} \) is a positive matrix [6, page 520]. To verify primitivity of a nonnegative matrix using Frobenius or Wielandt formula leads to huge calculations especially when \( n \) is high. It is shown in [6, page 521] that only some power calculations of the matrix are necessary.

This paper is on the calculation of the Perron-Frobenius (PF) root. The power method is generally used to obtain the eigenvalue with the maximum modulus and associated eigenvector [6 page 523], [8 page 330], [1 page 533]. With the power method, a vector non orthogonal to the unknown PF vector is chosen. Then the components of this vector are iteratively modified until convergence, i.e. there is no significant change in the component values from an iteration to the next. The convergence rate of the power method depends on the ratio of the second eigenvalue to the first [8 page 330], [1 page 533]. More iterations will be required when the modulus of the second highest eigenvalue is close to that of the first. Here, an iterative algorithm is proposed for calculating the PF root for primitive matrices. This algorithm is based on successive improvement of bounds for the PF root. There are
many research works on localization of the PF root for nonnegative matrices \cite{9, 10, 11, 12, 13, 14}. Frobenius carried out the following bounds \cite[page 492]{6}:

\[
\min_{i=1,\ldots,n} \{ r_i(A) \} \leq \rho(A) \leq \max_{i=1,\ldots,n} \{ r_i(A) \} \tag{6}
\]

\[
\min_{j=1,\ldots,n} \{ c_j(A) \} \leq \rho(A) \leq \max_{j=1,\ldots,n} \{ c_j(A) \} \tag{7}
\]

where \( r_i(A) = \sum_{j=1}^{n} a_{ij} \) and \( c_j(A) = \sum_{i=1}^{n} a_{ij} \) are the row and column sums, respectively. In (6) and (7), equalities occur when \( \rho(A) \) is equal to the row or column sums. The column sums of the matrix in (8) are both equal to 3.

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
3 & 0 & 3 \\
0 & 2 & 0
\end{pmatrix}
\]  

The eigenvalues of this imprimitive matrix are: 3, \(-3\) and 0. This example shows that equality in (6) or (7) can occur for an imprimitive matrix.

Let \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) a vector with only positive values, \( x_i > 0 \), and \( D_x \) a diagonal matrix formed with \( x \). The matrix \( B \) defined by:

\[
B = D_x^{-1} A D_x
\]  

is diagonally similar to \( A \) \cite{15}, and we have:

**Lemma 1.1.** For matrices \( A \) and \( B \) in (9)

a) if \( A \) is irreducible, then \( B \) is irreducible,

b) if \( A \) is primitive, then \( B \) is primitive,

c) \( A \) and \( B \) have the same eigenvalues

Proof. a) if \( A \) is irreducible then \((I_n + A)^{n-1}\) is a positive matrix. From (9), we have:

\[
I_n + B = I_n + D_x^{-1} A D_x = D_x^{-1} (I_n + A) D_x
\]  

\[
(I_n + B)^{n-1} = D_x^{-1} (I_n + A)^{n-1} D_x
\]

\( D_x \) has only positive values and \((I_n + A)^{n-1}\) is a positive matrix. The matrix \((I_n + B)^{n-1}\) is then positive that implies irreducibility of the matrix \( B \).

b) if \( A \) is a primitive matrix then there exists an integer \( m \) such that \( A^m \) is positive. Using (9), we have \( B^m = D_x A^m D_x \), that implies \( B^m \) is positive and the result follows.
c) using (9), we can write:

\[ A = D_x B D_x^{-1} \]  \hspace{1cm} (12)

\[ A - \lambda I_n = D_x B D_x^{-1} - \lambda I_n = D_x (B - \lambda I_n) D_x^{-1} \]  \hspace{1cm} (13)

Hence, \( A - \lambda I_n \) and \( B - \lambda I_n \) have the same determinant (characteristic polynomial).

Using an improvement of bounds in (6) and (7) by Minc [16], relation (9) and the uniqueness of eigenvalue with a maximum modulus for a primitive matrix, an iterative algorithm is proposed to obtaining the PF root.

2 Methods

Nonnegative primitive matrices arise in many fields: economic growth, population models, Markov chains, etc.

**Lemma 2.1.** Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) a nonnegative matrix. If matrix \( A \) has a row (column) with only zero entries, then \( A \) cannot be a primitive matrix.

*Proof.* The components of the power two of \( A \), noted \( C \), can be expressed as:

\[ c_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}, \quad i, j = 1, \ldots, n. \]  \hspace{1cm} (14)

If entries in row \( i \) (column \( j \)) of \( A \) are zero, then entries in the row \( i \) (column \( j \)) of \( C \) will be equal to zero too. By induction, row \( i \) (column \( j \)) of any power of \( A \) will have zero entries and the Frobenius primitivity test will fail.

For a primitive matrix, from the Lemma 2.1 and relations (6)–(7), we have the following two observations. The minimum value of the row (column) sums for a primitive matrix is greater than zero. The maximum value of the row (column) sums for a primitive matrix is greater than or equal to the PF root.

Let us note \( D_r \) and \( D_c \) diagonal matrices formed with the row sums \( r = (r_1(A), r_2(A), \ldots, r_n(A)) \) and the column sums \( c = (c_1(A), c_2(A), \ldots, c_n(A)) \) of \( A \), respectively. The Frobenius bounds (6) and (7) have been improved by Minc [16, page 27]:

\[ \min_{i=1, \ldots, n} \{ r_i(D_r^{-1} A D_r) \} \leq \rho(A) \leq \max_{i=1, \ldots, n} \{ r_i(D_r^{-1} A D_r) \} \]  \hspace{1cm} (15)

\[ \min_{j=1, \ldots, n} \{ c_j(D_c^{-1} A D_c) \} \leq \rho(A) \leq \max_{j=1, \ldots, n} \{ c_j(D_c^{-1} A D_c) \} \]  \hspace{1cm} (16)
In (15) and (16), equalities hold when the row or column sums are the same and correspond to the PF root. Using the row sums relation, (15) allows to write:

\[ D^{-1}AD = \begin{pmatrix} a_{11} & \frac{r_2}{r_1}a_{12} & \frac{r_3}{r_1}a_{13} & \cdots & \frac{r_n}{r_1}a_{1n} \\ \frac{r_1}{r_2}a_{21} & a_{22} & \frac{r_3}{r_2}a_{23} & \cdots & \frac{r_n}{r_2}a_{2n} \\ \frac{r_1}{r_3}a_{31} & \frac{r_2}{r_3}a_{32} & a_{33} & \cdots & \frac{r_n}{r_3}a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{r_1}{r_n}a_{n1} & \frac{r_2}{r_n}a_{n2} & \frac{r_3}{r_n}a_{n3} & \cdots & a_{nn} \end{pmatrix} \]

(17)

where \( X \) is a positive matrix formed with:

\[ x_{ij} = \frac{r_j(A)}{r_i(A)} ; \quad i, j = 1, 2, \ldots, n \]

(19)

The unicity of the PF root for a primitive matrix and (18) suggest that the components of the matrix \( X \) can be chosen to have the same row sums for \( A \circ X \). In the case of a second order nonnegative matrix (\( n = 2 \)), we have:

\[ A \circ X = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \circ \begin{pmatrix} 1 & \frac{r_2}{r_1} & \frac{r_3}{r_1} & \cdots & \frac{r_n}{r_1} \\ \frac{r_1}{r_2} & 1 & \frac{r_3}{r_2} & \cdots & \frac{r_n}{r_2} \\ \frac{r_1}{r_3} & \frac{r_2}{r_3} & 1 & \cdots & \frac{r_n}{r_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{r_1}{r_n} & \frac{r_2}{r_n} & \frac{r_3}{r_n} & \cdots & 1 \end{pmatrix} \]

(20)

To have a value for \( x \), \( a_{12} \) and \( a_{21} \) should be nonzero. Relation (21) allows to have a second order equation which resolution leads to a value for \( S \):

\[ S^2 - (a_{11} - a_{22})S + a_{11}a_{22} - a_{12}a_{21} = 0 \]

(22)

The solution of (22) with the maximum modulus is:

\[ S = \frac{a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2} \]

(23)
A second order nonnegative matrix $A$ is primitive if $a_{12}$ and $a_{21}$ are both nonzero, on the one hand. On the other hand, at least $a_{11}$ or $a_{22}$ should be nonzero. Hence, for a second order primitive matrix, explicit expressions can be obtained for matrix $X$ (parameter $x$) and the PF root, (23). However, a direct search for components of the matrix $X$ in (18) becomes difficult when $n > 2$. Compared to the Frobenius bounds (6) and (7), the bounds in (15) and (16) are based on a modification of the initial matrix. This process can be repeated to further sharpen the bounds. From the unicity of the PF root for a primitive matrix, a repetitive improvement of the Minc bounds will lead to equalities of the row (column) sums. The main result of this paper is the following.

**Theorem 2.1.** Let $A$ be a primitive matrix of order $n$. There exists a positive rank one matrix $X$ of the form

$$X = \begin{pmatrix}
1 & x_2/x_1 & x_3/x_1 & \ldots & x_n/x_1 \\
x_1/x_2 & 1 & x_3/x_2 & \ldots & x_n/x_2 \\
x_1/x_3 & x_2/x_3 & 1 & \ldots & x_n/x_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1/x_n & x_2/x_n & x_3/x_n & \ldots & 1
\end{pmatrix}$$

(24)

such that the matrix $B = A \circ X$ is similar to $A$. In addition, the row (column) sums of $B$ are the same and equal to the Perron-Frobenius root of $A$.

**Proof.** The row sums are used for the proof, the column sums can also be used in a similar way.

Let us note $A^{(0)}$ the initial matrix and its row sums vector as $r^{(0)}$. Relation (15) allows to write:

$$A^{(t)} = D^{-1}_{r^{(t-1)}} A^{(t-1)} D_{r^{(t-1)}} ; t = 1, 2, \ldots$$

(25)

From (18) and (19), the components of matrix $A^{(t)}$ and its row sums are:

$$a_{ij}^{(t)} = \frac{r_i^{(t-1)}(A^{(t-1)})}{r_i^{(t-1)}(A^{(t-1)})} a_{ij}^{(t-1)} ; i, j = 1, 2, \ldots, n$$

(26)

$$r_i^{(t)}(A^{(t)}) = \sum_{j=1}^{n} a_{ij}^{(t)} ; i = 1, 2, \ldots, n$$

(27)

At the convergence, the row sums $r_i^{(t)}(A^{(t)}), i = 1, 2, \ldots, n$, are expected to have the same value.
From (25) and the notation in (18), we can write:

\[
A^{(1)} = A^{(0)} \circ X^{(0)} = A \circ X^{(0)} \quad (28)
\]
\[
A^{(2)} = A^{(1)} \circ X^{(1)} = A \circ X^{(0)} \circ X^{(1)} \quad (29)
\]
\[\vdots\]
\[
A^{(t)} = A \circ X^{(0)} \circ X^{(1)} \circ \ldots \circ X^{(t-2)} \circ X^{(t-1)} \quad (30)
\]

If convergence occurs at iteration \(t\), then values of \(r_i^{(t)}(A^{(t)})\), \(i = 1, \ldots, n\), are the same and \(X^{(t)}\) is formed with only 1, see (19). Then, from (30) we have

\[
B = A^{(t)} = A \circ X \quad (31)
\]

where:

\[
X = X^{(0)} \circ X^{(1)} \circ \ldots \circ X^{(t-2)} \circ X^{(t-1)} \quad (32)
\]
\[
x_{ij} = \prod_{s=0}^{t-1} r_i^{(s)}(A^{(s)}) / r_j^{(s)}(A^{(s)}) ; \ i, j = 1, 2, \ldots, n \quad (33)
\]

Relation (31) is another form of (17), then matrix \(B\) is similar to matrix \(A\). Since the row sums of matrix \(B\) are the same, equalities occur in (6) and lead to the PF root.

The matrix \(X\) in (24) can be write as a product of two vectors:

\[
X = xy^T \quad (34)
\]

where \(x = (1, x_1/x_2, \ldots, x_1/x_n)^T\) and \(y = (1, x_2/x_1, \ldots, x_n/x_1)^T\), i.e. the first column and first row of the matrix \(X\), respectively. \(X\) is then a rank one matrix, and is also positive because formed with the row sums of a primitive matrix, see Lemma 2.1.

Using the definition (11) and the notation (34), we have:

\[
X^x = x(y^Tx) \quad (35)
\]

Since \(y^Tx\) is a scalar equal to \(n\), this value corresponds to the unique nonzero eigenvalue of \(X\).

### 2.1 Algorithm for calculation of the Perron-Frobenius root

Relations (26) and (27) allow to obtain an algorithm for computing the matrix \(B\) in Theorem 2.1. A convergence test is based on the difference
between the maximum and the minimum values of the row or column sums, i.e. the range value.

\[
\text{error} = \max_{i=1,\ldots,n} \left\{ r_i(t)(A(t)) \right\} - \min_{i=1,\ldots,n} \left\{ r_i(t)(A(t)) \right\} \quad (36)
\]

\[
\text{error} = \max_{j=1,\ldots,n} \left\{ c_j(t)(A(t)) \right\} - \min_{j=1,\ldots,n} \left\{ c_j(t)(A(t)) \right\} \quad (37)
\]

Algorithm (using row sums)

1. Initialization
   - set: \( t \leftarrow 0 \), \( a_{ij}^{(t)} \leftarrow a_{ij} \), calculate the row sums using (27)
   - set stopping rules: \( \text{eps} \) (the acceptable error), \( \text{maxIter} \) (the maximum number of iterations), compute the initial error value using (36)

2. while (error > eps and \( t < \text{maxIter} \))
   - update matrix: (26)
   - calculate row sums: (27)
   - compute error: (36)
   - increase iteration number: \( t \leftarrow t + 1 \)

Remark 2.1. As mentioned, Theorem 2.1 is also valid using column sums. In the implementation, one can compute the row and column sums and perform the next step using the sums where the initial error is the lowest.

Remark 2.2. From an iteration to the next, the error should decrease by an amount that depends on the convergence rate. Otherwise, we must stop the algorithm because the matrix does not seem to be primitive. This observation can be used as an indirect test for primitivity of a matrix.

Remark 2.3. With the proposed algorithm, the diagonal entries of \( A \) remain unchanged, only the off-diagonal components are modified. The Gerschgorin discs [6, page 344] associated with a matrix allow to illustrate this. As example:

\[
A = \begin{pmatrix}
3 & \sqrt{3} \\
\sqrt{3} & 1
\end{pmatrix} ; \quad A^{(1)} = \begin{pmatrix}
3 & 1 \\
3 & 1
\end{pmatrix}
\]

The off-diagonal components of \( A \) are modified in such a way all discs cross the same highest point on the x-axis (4 for this example). To show this, let us write:

\[
A = D_A + P
\]
where $D_A$ is a diagonal matrix formed with the diagonal elements of $A$ and $P$ is matrix $A$ where the diagonal elements are set to zero. From (31), we have:

$$B = D_A + P \circ X$$

(40)

Hence, the row sums of matrix $B$ are given by:

$$r_i(B) = r_i(D_A) + r_i(P \circ X) = a_{ii} + p_i^T x_i.$$  

(41)

where $p_i$ and $x_i$ are vectors formed with row $i$ of matrices $P$ and $X$, respectively.

**Remark 2.4.** Compared to the power method, there is no initial vector to set. The results obtained using the power method are the PF root and the associated eigenvalue. Only the PF root is obtained using the proposed algorithm. However, since the row (column) sums of the matrix $B$ in (31) are the same, a vector $1$ formed with only ones is an eigenvector of $B (B^T)$. 

$$B1 = \lambda 1 = (A \circ X)1$$  

(42)

where $\lambda$ is the row sums equal to the PF root.

**Corollary 2.1.** Vectors $x$ and $y$ allowing to obtain the matrix $X$, see (34), are in the space spanned by the Perron-Frobenius vector of matrix $A$.

**Proof.** The matrix $A$ can be write as the sum of rank one matrices using its eigenvalues and eigenvectors:

$$A = U \Lambda U^{-1} = U \Lambda V^T$$

$$= \lambda_1 u_1 v_1^T + \lambda_2 u_2 v_2^T + \ldots \lambda_n u_n v_n^T$$

(44)

From (44), (34) and (5) we have:

$$A \circ X = \lambda_1 (u_1 \circ x)(v_1 \circ y)^T + \lambda_2 (u_2 \circ x)(v_2 \circ y)^T + \ldots \lambda_n (u_n \circ x)(v_n \circ y)^T$$  

(45)

Comparing (45) and (42), we deduce:

$$ (u_1 \circ x)(v_1 \circ y)^T1 = 1$$

(46)

$$ (u_2 \circ x)(v_2 \circ y)^T1 = 0$$

(47)

$$ \vdots$$

$$ (u_n \circ x)(v_n \circ y)^T1 = 0$$

(48)

where $0$ is a vector formed with only zeros. The inner product $(v_i \circ y)^T1$ is a scalar which value is zero for $i = 2, \ldots, n$, on the one hand. On the other
hand, this scalar is nonzero for $i = 1$: $(v_1 \circ y)^T 1 = \delta$. The value of the scalar $\delta$ allows to have $\delta(u_1 \circ x) = 1$. Using the expression of $x$, we have:

$$\delta u_1 = (1, x_2/x_1, x_3/x_1, \ldots, x_n/x_1)^T = y$$  \hspace{1cm} (49)

\[\square\]

**Corollary 2.2.** The convergence rate of the proposed algorithm is similar to that of the power method and depends on the magnitude of the second highest eigenvalue.

**Proof.** Using (43) and (44), an expression for power $k$ of matrix $A$ is

$$A^k = \lambda_1^k u_1 v_1^T + \lambda_2^k u_2 v_2^T + \ldots + \lambda_n^k u_n v_n^T \quad (50)$$

This expression allows to have another one similar to (45). Then, the row sums at the first step of the algorithm using $A^k$ are:

$$(A^k \circ X(0))^T 1 = \lambda_1^k \delta_1 z_1 + \lambda_2^k \delta_2 z_2 + \ldots + \lambda_n^k \delta_n z_n$$

$$= \lambda_1^k \delta_1 \left( z_1 + \frac{\delta_2}{\delta_1} \left( \frac{\lambda_2}{\lambda_1} \right)^k z_2 + \ldots + \frac{\delta_n}{\delta_1} \left( \frac{\lambda_n}{\lambda_1} \right)^k z_n \right) \quad (51)$$

$$= \lambda_1^k \delta_1 \left( z_1 + \frac{\delta_2}{\delta_1} \delta_2 \left( \frac{\lambda_2}{\lambda_1} \right)^k z_2 + \ldots + \frac{\delta_n}{\delta_1} \delta_n \left( \frac{\lambda_n}{\lambda_1} \right)^k z_n \right) \quad (52)$$

where $\delta_i = (v_i \circ y^{(0)})^T 1$, $z_i = (u_i \circ x^{(0)})$, $i = 1, 2, \ldots, n$, $x^{(0)} = \left( 1, \frac{r^{(0)}_1}{r^{(0)}_2}, \frac{r^{(0)}_2}{r^{(0)}_3}, \ldots, \frac{r^{(0)}_n}{r^{(0)}_1} \right)^T$

and $y^{(0)} = \left( 1, \frac{r^{(0)}_1}{r^{(0)}_2}, \frac{r^{(0)}_2}{r^{(0)}_3}, \ldots, \frac{r^{(0)}_n}{r^{(0)}_1} \right)^T$.

From corollary 2.1 vector $(A \circ X)^T 1 \in \text{span} \{ (A^k \circ X^{(0)})^T 1 \}$ then,

$$\text{dist } (\text{span}(A \circ X) 1, \text{span}(z_1)) = O \left( \frac{\lambda_2}{\lambda_1} \right)^k \text{ } (53)$$

\[\square\]

**Remark 2.5.** The proposed algorithm alters the input matrix $A$ which is preserved by the power method. At each iteration, the total numbers of multiplications and additions of the matrix-vector multiplication by the power method are equal to the total number of operations for the proposed algorithm. Hence, using the power method, the additional arithmetic operations used for calculating the eigenvector and the eigenvalue are extra computational load compared to the proposed algorithm.
3 Results and conclusions

The error level was arbitrarily set to $1.0 \times 10^{-8}$ for results presented. For the example in [8], the proposed algorithm converges after one iteration, the power method needs two. A second example is:

$$A = \begin{pmatrix}
2 & 1 & 0 \\
0.5 & 3 & 2 \\
1 & 2 & 4
\end{pmatrix}$$

The range values for the row and the column sums are 4 and 2.5 respectively. The algorithm is performed using the column sums. Figure 1 presents the

![Figure 1: Gerschgorin’s discs: thin plot lines for $A$ and $A(t)$ before convergence, bold plot lines for $A(t)$ at convergence.](image)

Gerschgorin discs for all iterations. The PF root for this example is 5.739952, the proposed algorithm and the power method require 17 and 19 iterations, respectively. Figure 1 shows that the major modifications of the off-diagonal elements of the matrix $A$ are done during the first five iterations.

For all of the tests performed, the algorithm proposed and the power method have close number of iterations. Worse results, in term of the number of iterations, were obtained using a tridiagonal matrix. Let $T(n; c, a, b)$ a tridiagonal matrix of order $n$, where $a$ is the value for the diagonal components, $b$ is the value for the upper diagonal components and $c$ is the value for the under diagonal components. An explicit expression relating eigenvalues
of matrix $T$ is available [17]:

$$\lambda_k = a + 2\sqrt{bc} \cos \frac{k\pi}{n+1}$$  \hspace{1cm} (55)

The ratio $\lambda_2/\lambda_1$ for $T$ is near 1 when $n$ is high. For $n = 50$, $a = 3$, $b = 2$ and $c = 1$ the first two eigenvalues of matrix $T$ are: 5.823063 and 5.806989. The proposed algorithm took 5,890 iterations to calculate the PF root. The power method did better by using 5,159 iterations.

Except the cases where the modulus of the second eigenvalue is near to that of the first, the algorithm proposed converges after few iterations, especially when the first eigenvalue is largely dominant. For a normal matrix of order $n = 1,600$ obtained from an image, the proposed and the power methods converged after 11 and 10 iterations, respectively.

With a convergence rate similar to that of the classic power method, the proposed algorithm for computing the PF root is computationally less demanding. But, it applies to only primitive matrices. However, it can be used as a low cost primitivity test compared to the matrix power calculations involved in the Frobenius and Wielandt tests.

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The $R$ code of the algorithm is available upon request.

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