A statistical analysis of the nuclear structure uncertainties in $\mu D$

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Abstract. The charge radius of the deuteron (D), was recently determined to three times the precision compared with previous measurements using the measured Lamb shift in muonic deuterium ($\mu D$). However, the $\mu D$ value is 5.6 $\sigma$ smaller than the world averaged CODATA-2014 value [1]. To shed light on this discrepancy we analyze the uncertainties of the nuclear structure calculations of the Lamb shift in $\mu D$ and conclude that nuclear theory uncertainty is not likely to be the source of the discrepancy.

Keywords: muonic atoms, spectroscopy, two-photon exchange, uncertainty quantification, statistical analysis

1 Introduction

The two-photon exchange (TPE) contribution is a crucial ingredient in the precision determination of the charge radius from Lamb shift (LS) measurements in muonic atoms. The charge radius is extracted from the measurements of the 2S-2P energy splitting $\Delta E_{LS}$ through

$$\Delta E_{LS} = \delta_{\text{QED}} + \delta_{\text{TPE}} + \delta_{FS}(r_d^2),$$

valid up to fifth order in $(Z\alpha)$, where $Z$ is the charge number of the nucleus and $\alpha$ is the fine structure constant. The term $\delta_{\text{QED}}$ denote the quantum electrodynamical (QED) corrections, $\delta_{\text{TPE}}$ are the nuclear structure corrections dominated by the two-photon exchange, and $\delta_{FS}(r_d^2)$ is the finite size correction proportional
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to the deuteron charge radius $r_d$. The bottle-neck in the precise determination of $r_d$ are the nuclear structure corrections. In this work, we overview the process of the uncertainty quantification of $\delta_{\text{TPE}}$ in $\mu D$ using nucleon-nucleon (NN) potentials at various orders (from LO to N$^4$LO) in chiral effective field theory (EFT).

2 Analysis of uncertainties

To quantify the total theoretical uncertainties of $\delta_{\text{TPE}}$, all relevant uncertainty sources must be identified and estimated [2,3]. These various sources are:

- $\sigma_{\text{stat}}$: uncertainties arising from the spread of the low-energy constants (LECs) $\tilde{\alpha}$ in the nuclear potential;
- $\sigma_{\text{TMax}}$: uncertainties from the maximum lab energy $T_{\text{Lab}}^{\text{Max}}$ used in the fits of the NN potential;
- $\sigma_{\Delta}$: uncertainty due to the truncation of the chiral order;
- $\sigma_{\Lambda}$: uncertainty from the variations of the the cut-off $\Lambda$ in the NN potentials;
- $\sigma_{\eta}$: uncertainty due to the expansion (on a parameter known as $\eta$) which we use in relating $\delta_{\text{TPE}}$ to the nuclear response functions;
- $\sigma_{J}$: uncertainties from systematic approximations in the electromagnetic operators $J^\mu(x)$;
- $\sigma_{\text{N}}$: uncertainties due to single nucleon physics;
- $\sigma_{Z\alpha}$: uncertainties arising from higher ($Z\alpha$) corrections.

For an observable $A$, the statistical uncertainties $\sigma_{\text{stat}}(A)$ induced by variations in the LECs $\tilde{\alpha}$ of the NN potential are calculated around their optimal values $\tilde{\alpha}_0$ by assuming that the LECs follow a multivariate Gaussian probability distribution. Under these conditions the leading approximation to $\sigma_{\text{stat}}(A)$ will be given by

$$\sigma_{\text{stat}}^2(A) = \langle A^2 \rangle - \langle A \rangle = J_A \text{Cov}(\tilde{\alpha}_0) J_A^T, \quad (2)$$

where Cov($\tilde{\alpha}_0$) represents the covariance matrix of the LECs at the optimum, and $J_A$ is the Jacobian vector of $A$ with respect to the LECs,

$$J_{A,i} = \left( \frac{\partial A}{\partial \tilde{\alpha}_i} \right)_{\tilde{\alpha} = \tilde{\alpha}_0}. \quad (3)$$

The systematic uncertainties $T_{\text{Lab}}^{\text{Max}}$ arise from the energy span in the NN scattering data used to fit the LECs. This uncertainty was estimated from the N$^k$LO$_{\text{sim}}$ potentials ($k = 0, 1, 2$) [4] by varying the maximum lab energies of the fit from 125 MeV to 290 MeV and their uncertainties $\sigma_{T_{\text{Lab}}^{\text{Max}}}$ where found to dominate over the statistical uncertainties $\sigma_{\text{stat}}$.

The chiral truncation uncertainties $\sigma_{\Delta}$ originate from the calculation of an observable $A(p)$ at a finite order $\nu$, with associated momentum scale $p$. This observable is assumed to obey the same expansion as the underlying NN-force given by

$$A(p) = A_0 \sum_{\mu=0}^{\nu} c_\mu(p) Q^\mu, \quad (4)$$
where $A_0$ is the result at leading order, $Q$ is the expansion parameter, and $c_{\mu}(p)$ are observable and interaction specific coefficients assumed to be independent and of natural size. Assuming that the next higher-order term $\Delta_{\nu}^{(1)} \equiv c_{\nu+1}Q^{\nu+1}$ dominates the truncation uncertainty in the calculation of $A(p)$, then the Bayesian posterior $P(\Delta_{\nu}^{(1)})$ is given by

$$P(\Delta_{\nu}^{(1)}) = \frac{\int d\bar{\epsilon} P(c_{\nu+1}\bar{\epsilon})P(c_0|\bar{\epsilon})P(c_2|\bar{\epsilon})...P(c_{\nu}|\bar{\epsilon})P(\bar{\epsilon})}{\int d\bar{\epsilon} P(c_0|\bar{\epsilon})P(c_2|\bar{\epsilon})...P(c_{\nu}|\bar{\epsilon})P(\bar{\epsilon})},$$

where $P(c_{\mu}|\bar{\epsilon})$ is the distribution of $c_{\mu}$ conditioned on the scale parameter $\bar{\epsilon}$ and $P(\bar{\epsilon})$ is the prior. In this contribution we update the results in Ref. [2] by evaluating the 68% confidence intervals of the posteriors given in Eq. (5) that represent the chiral truncation uncertainty $\sigma_\Delta$. The posterior distributions $A_0\Delta_{\nu}^{(1)}$ from N^2LO to N^4LO for $\delta_{\text{TPE}}$ using the chiral potentials from Ref. [6] are given in Fig. 1 for the priors A, B, C from Table I in Ref. [5].

Along with chiral truncation uncertainties, the chiral NN-potentials carry a parameter $\Lambda$ that regulates the interactions. The systematic uncertainties $\sigma_\Lambda$ arising from the regulators was probed using multiple cut-off values in the calculations of $\delta_{\text{TPE}}$. These variations were found to be more significant than the uncertainties due to the chiral truncation.

The $\eta$-expansion arises from the calculation of $\delta_{\text{TPE}}$ as a power series of the dimensionless operator $\eta \ll 1$. In the work of Ref. [23], this expansion was carried out to sub-sub-leading order in $\eta$ and the truncation uncertainty $\sigma_\eta$ from higher order terms was determined to be 0.3%.

Uncertainties from approximations in the electromagnetic operators $J^{\mu}(x)$, were estimated from the dipole response functions of Arenhövel [17] that included MEC and relativistic corrections. Both of these effects were of the order 0.05%.

The uncertainties $\sigma_N$ from single nucleon contributions to the TPE are an input in our analysis and taken from Ref. [8,9] and Ref. [10]. Lastly, there was an estimated 1% uncertainty from higher order $(Z\alpha)^6$ corrections, that include the three photon exchange.
Table 1. The uncertainty breakdown of the $\delta_{\text{TPE}}$ at N^4LO

| Source | % Uncertainty | Uncertainty in meV |
|--------|---------------|--------------------|
| $\sigma_{\text{syst}}$ | +0.5 | +0.008 |
| | −0.6 | −0.011 |
| $\sigma_{\text{stat}}$ | 0.06 | ±0.001 |
| $\sigma_\eta$ | 0.3 | ±0.005 |
| $\sigma_N$ | 0.6 / 1.2 | ±0.0102 [8] / ±0.0198 [10] |
| $\sigma_{Z\alpha}$ | 1.0 | ±0.0172 |
| $\sigma_{\text{Total}}$ | 1.3 / 1.6-1.7 | +0.022 / +0.028 / −0.023 / −0.029 |

3 Results and Conclusions

The results of the analysis outlined in the previous section are summarized in Table 1. The systematic nuclear physics uncertainty $\sigma_{\text{syst}}$ is a combination of the $\sigma_\Delta$, $\sigma_J$ and $\sigma_{T\text{Max}}$ uncertainties, while $\sigma_{\text{Total}}$ is a quadrature sum of all items in Table 1. The calculation of $\sigma_\Delta$ through the explicit calculation of the 68% confidence interval of the Bayesian posteriors instead of the prescription in Ref. [6] increases the lower bound slightly in $\sigma_{\text{Total}}$ from -0.024 meV in Ref. [2] to -0.023 meV when using the $\sigma_N$ values of Ref. [8] since the values of $\sigma_\Delta$ at N^4LO for prior A are smaller when computed this way. The final value for the TPE correction was taken to be the average value of the calculations at N^4LO yielding $\delta_{\text{TPE}} = -1.715$ meV with the final uncertainty $\sigma_{\text{Total}}$. This value differs from the experimentally determined value from Ref. [1] of $\delta_{\text{TPE}} = -1.7638(68)$ meV by less than 2 $\sigma$, which is not significant. From Table 1 we find that the uncertainties arising from the nuclear model dependence, $\sigma_{\text{syst}}$ and $\sigma_{\text{stat}}$, are small in comparison to the higher order $\sigma_{Z\alpha}$ or $\sigma_N$ uncertainties which dominate the total uncertainty. It is therefore unlikely that any differences between the experimental and theoretical determinations of $\delta_{\text{TPE}}$ stem from models of the NN-forces.

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