Gromov’s macroscopic dimension conjecture

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In this note we construct a closed 4–manifold having torsion-free fundamental
group and whose universal covering is of macroscopic dimension 3. This yields a
counterexample to Gromov’s conjecture about the falling of macroscopic dimension.

57R19; 57R20

1 Introduction

The following definition was given by M Gromov [2]:

Definition 1.1 Let V be a metric space. We say that \( \dim \varepsilon V \leq k \) if there is a \( k \)–
dimensional polyhedron \( P \) and a proper uniformly cobounded map \( \phi: V \to P \) such that
\( \text{Diam}(\phi^{-1}(p)) \leq \varepsilon \) for all \( p \in P \). A metric space \( V \) has macroscopic \( \dim_{mc} V \leq k \) if
\( \dim \varepsilon V \leq k \) for some possibly large \( \varepsilon < \infty \). If \( k \) is minimal, we say that \( \dim_{mc} V = k \).

Gromov also stated the following questions which, for convenience, we state in the
form of conjectures:

C1 Let \((M^n, g)\) be a closed Riemannian \( n \)–manifold with torsion-free fundamental
group, and let \((\tilde{M}^n, \tilde{g})\) be the universal covering of \( M^n \) with the pullback metric. Suppose
that \( \dim_{mc}(\tilde{M}^n, \tilde{g}) < n \). Then \( \dim_{mc}(\tilde{M}^n, \tilde{g}) < n - 1 \).

In [1] we proved C1 for the case \( n = 3 \).

Evidently, the following conjecture would imply C1 (see also (C) of Section 2):

C2 Let \( M^n \) be a closed \( n \)–manifold with torsion-free fundamental group \( \pi \) and let
\( f: M^n \to B\pi \) be a classifying map to the classifying space \( B\pi \). Suppose that \( f \) is
homotopic to a mapping into the \( (n - 1) \)–skeleton of \( B\pi \). Then \( f \) is in fact homotopic
to a mapping into the \( (n - 2) \)–skeleton of \( B\pi \).

In this note we show that both conjectures fail for \( n \geq 4 \).

We always assume that universal covering are equipped with the pullback metrics.

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Framed cobordism, Pontryagin manifolds and classification of mappings to the sphere

Let $M$ be a smooth compact manifold possibly with a boundary and let $(N, v)$ and $(N', w)$ be closed $n$–submanifolds in the interior of $M$ with trivial normal bundles and framings $v$ and $w$, respectively.

**Definition 1.2** Two framed submanifolds $(N, v)$ and $(N', w)$ are *framed cobordant* if there exists a cobordism $X \subset M \times [0, 1]$ between $N$ and $N'$ and a framing $u$ of $X$ such that

\[
u(x, t) = (v(x), 0) \quad \text{for } (x, t) \in N \times [0, \varepsilon),
\]

\[
u(x, t) = (w(x), 1) \quad \text{for } (x, t) \in N' \times (1 - \varepsilon, 1].
\]

**Remark 1.3** If $(N', w) = \emptyset$ we say $(N, v)$ is *framed cobordant to zero*.

Now let $f : M \to S^p$ be a smooth mapping and $y \in S^p$ be a regular value of $f$. Then $f$ induces the following framing of the submanifold $f^{-1}(y) \subset M$. Choose a positively oriented basis $\nu = (v^1, \ldots, v^p)$ for the tangent space $T(S^p)_y$. Notice that for each $x \in f^{-1}(y)$ the differential $df_x : T_xM \to T(S^p)_y$ vanishes on the subspace $Tf^{-1}(y)_x$ and isomorphically maps its orthogonal complement $Tf^{-1}(y)_x^\perp$ onto $T(S^p)_y$. Hence there exists a unique vector

$$w_i^j \in Tf^{-1}(y)_x^\perp \subset TM_x$$

which is mapped by $df_x$ to $v_i$. So we have an induced framing $w = f^*\nu$ of $f^{-1}(y)$.

**Definition 1.4** This framed manifold $(f^{-1}(y), f^*\nu)$ will be called the *Pontryagin manifold* associated with $f$.

**Theorem 1.5** (Milnor [3]) If $y'$ is another regular value of $f$ and $\nu'$ is a positively oriented basis for $T(S^p)_{y'}$, then the framed manifold $(f^{-1}(y), f^*\nu)$ is framed cobordant to $(f^{-1}(y'), f^*\nu')$.

**Theorem 1.6** (Milnor [3]) Two mappings from $(M, \partial M)$ to $(S^p, s_0)$ are smoothly homotopic if and only if the associated Pontryagin manifolds are framed cobordant.
2 The construction of an example

Consider a circle bundle $S^3 \times S^1 \to S^2 \times S^1$ obtained by multiplying the Hopf circle bundle $S^3 \to S^2$ by $S^1$. Take also the trivial circle bundle $T^4 = S^1 \times T^3 \to T^3$ and produce a connected sum

$$M^4 = S^3 \times S^1 \#_{s^1} T^4$$

of these circle bundles along small tubes consisting of the circle fibers equipped with natural trivialization. Clearly

(A) $M^4$ is the total space of the circle bundle

$$p : M^4 \to M^3 = S^2 \times S^1 \# T^3;$$

(B) $\pi_1(M^4) = \pi_1(M^3)$. Denote this group by $\pi$;

(C) $B\pi = S^1 \vee T^3$ and $\dim_{mc} M^4 \leq 3$. Indeed, the classifying map $f : M^4 \to B\pi$ can be lifted to the proper cobounded (by $\text{Diam}(M^4)$) map $\tilde{f} : M^4 \to \tilde{B}\pi$ of the universal coverings;

(D) the classifying map $f : M^4 \to B\pi$ can be defined as the composition

$$M^4 \xrightarrow{p} S^2 \times S^1 \# T^3 \xrightarrow{f_1} S^2 \times S^1 \vee T^3 \xrightarrow{f_2} S^1 \vee T^3,$$

where $f_1$ is a quotient map which maps a separating sphere $S^2$ to a point, and $f_2$ is the mapping which coincides with the projection onto the generating circle of $S^2 \times S^1$ and is the identity on $T^3$–component.

Let $g : S^1 \vee T^3 \to S^3$ be a degree one map which maps $S^1$ to a point. Then the following composition $J = g \circ f_2 \circ f_1 : M^3 \to S^3$ also has degree one.

Theorem 2.1 The mapping $f : M^4 \to B\pi$ is not homotopic into the 2–skeleton of $B\pi$.

Proof Let $\pi : E \to M^3$ be a two-dimensional vector bundle associated with the circle bundle $p : M^4 \to M^3$. Let $E_0$ denote $E$ without zero section $s : M^3 \hookrightarrow E$ and $j : M^4 \hookrightarrow E_0$ be a unit circle subbundle of $E$.

The following diagram is homotopically commutative:

$$\begin{array}{ccc}
M^4 & \xhookrightarrow{i} & E_0 \\
\downarrow p & & \downarrow \text{embedding} \\
M^3 & \xhookrightarrow{s} & E
\end{array}$$

Obviously, $j$ and $s$ are homotopy equivalences.
Recall that we have the Thom isomorphism (see Milnor and Stasheff [4])

\[ \Phi: H^k(M^3; \Lambda) \to H^{k+2}(E, E_0; \Lambda) \]

defined by

\[ \Phi(x) = (\pi^* x) \cup u, \]

where \( \Lambda \) is a ring with unity, and \( u \) denotes the Thom class.

The Thom class \( u \) has the following properties [4]:

(a) If \( e \) is the Euler class of \( E \) then we have the Thom–Wu formula

\[ \Phi(e) = u \cup u. \]

(b) \( s^*(u) = e. \)

Let

\[ M_p = M^4 \times I/(x \times 1 \sim p(x)) \]

be the cylinder of the map \( p: M^4 \to M^3 \). Then we have natural embeddings

\[ i_1: M^4 \to M^4 \times 0 \subseteq M_p \quad \text{and} \quad i_2: M^3 \to M^3 \times 1 \subseteq M_p \]

and a natural retraction \( r: M_p \to M_3 \). It is easy to see that \( M_p \) is just a \( D^2 \)-bundle associated to the circle bundle \( p: M^4 \to M^3 \) and \( r|_{M^4} = p \).

Recall that the Thom space \( (T(E), \infty) \) is the one point compactification of \( E \). Denote \( T(E) \) by \( T \). Clearly, \( T \) is homeomorphic to the quotient space \( M_p/M^4 \) and

\[ H^*(T, \infty; \Lambda) \cong H^*(E, E_0; \Lambda) \quad (1) \]

is a ring isomorphism (see Milnor and Stasheff [4] for more details).

If \( g \circ f : M^4 \to S^3 \) is nullhomotopic then we can extend the map \( J: M_3 \times 1 \to S^3 \) to a mapping \( G: T \to S^3 \). This means that the composition

\[ M^3 \xrightarrow{i_2} M_p \xrightarrow{\text{quotient}} \xrightarrow{G} S^3 \]

has degree 1 and \( G^*: H^3(S^3, s_0; \Lambda) \to H^3(T, \infty; \Lambda) \) is nontrivial.

Let \( a \in H^*(E, E_0; \Lambda) \) denote a class corresponding to the class \( G^*(\bar{s}) \) by isomorphism \( (1) \), where \( \bar{s} \) is a generator of \( H^3(S^3, \Lambda) \).

Let us consider the following exact sequence of pair:

\[ H^3(E, E_0; \Lambda) \xrightarrow{\xi} \xrightarrow{\psi} H^3(E; \Lambda) \xrightarrow{\psi} H^3(E_0; \Lambda) \]
Since $E$ is homotopy equivalent to $M^3$, we have $H^i(E; \Lambda) = H^i(M^3; \Lambda)$. Clearly $s^*\xi(a) = J^*(\bar{s})$. (Note that $J^*(\bar{s})$ is a generator of $H^3(M^3; \Lambda)$).

Let us note that $e \mod 2$ is equal to the Stiefel–Whitney class $w_2$ which is nonzero. Indeed, the restriction of $E$ onto the embedded sphere $i: S^2 \subset M^3$ is the vector bundle associated with the Hopf circle bundle, and so $i^*w_2 \neq 0$. By the Thom construction above there exists a class $z \in H^1(M^3; \mathbb{Z}_2)$ such that $\Phi(z) = a$. Thus

$$s^*\xi(a) = z \cup w_2 = \{\text{generator of } H^3(M^3; \mathbb{Z}_2)\}.$$ 

Recall the basic properties of Steenrod squares [6, 4]:

1. For each $n, i$ and $Y \subset X$ there exists an additive homomorphism
   $$Sq^i: H^n(X, Y; \mathbb{Z}_2) \to H^{n+i}(X, Y; \mathbb{Z}_2).$$

2. If $f: (X, Y) \to (X', Y')$ is a continuous map of pairs, then
   $$Sq^i \circ f^* = f^* \circ Sq^i.$$

3. If $a \in H^n(X, Y; \mathbb{Z}_2)$, then $Sq^0(a) = a$, $Sq^n(a) = a \cup a$ and $Sq^i(a) = 0$ for $i > n$.

4. We have Cartan’s formula:
   $$Sq^k(a \cup b) = \sum_{i+j=k} Sq^i(a) \cup Sq^j(b).$$

5. $Sq^1 = w_1: H^{m-1}(M; \mathbb{Z}_2) \to H^m(M; \mathbb{Z}_2)$, where $M$ is a closed smooth manifold and $w_1$ is the first Stiefel–Whitney class of the tangent bundle $TM$. This follows from the coincidence of the class $w_1$ with the first Wu class $v_1$ [4]. It is well known that $w_1 = 0$ if $M$ is an orientable manifold.

Let us show that $Sq^2(\Phi(z)) \neq 0$. Using the properties above, it is easy to see that $Sq^1(z) = Sq^2(z) = 0$. Using the Thom–Wu formula (1), we have

$$Sq^2(\Phi(z)) = \pi_z \cup Sq^2(u) = \pi_z \cup u \cup u = \pi_z \cup \Phi(w_2) = \Phi(z \cup w_2) \neq 0.$$ 

Whence $0 = G^*(Sq^2(\bar{s})) = Sq^2(G^*(\bar{s})) \neq 0$. This contradiction implies that the composition $g \circ f: M^4 \to S^3$ is not homotopic to zero and $f: M^4 \to B\pi$ can not be deformed into the 2–skeleton of $B\pi$. $\square$

**Corollary 2.2** The Pontryagin manifold $(p^{-1}(m), p^*(w))$ is not cobordant to zero, where $(m, w)$ is any framed point of $M^3$. 

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Proof Indeed, from Theorem 2.1 and Theorem 1.6 it follows that if \( s \in S^3 \) is a regular point of \( g \circ f : M^4 \to S^3 \), then the Pontryagin manifold \( (f^{-1}(g^{-1}(s)), f^*(g^*(v))) \) is not cobordant to zero, where \( v \) is a framing at \( s \). Thus the Pontryagin manifold \( (p^{-1}(m), p^*(w)) \) for \( (m, w) = (J^{-1}(s), (J^*(v)) \) is also not cobordant to zero. Now the statement follows from Theorem 1.5 and regularity of the map \( p : M^4 \to M^3 \).

3 The main theorem

Definition 3.1 A metric space is called uniformly contractible (UC) if there exists an increasing function \( Q : \mathbb{R}_+ \to \mathbb{R}_+ \) such that each ball of radius \( r \) contracts to a point inside a ball of radius \( Q(r) \).

It is well known that the universal covering of a compact \( K(\tau, 1) \) space is UC (see Gromov [2] for more details).

Denote by \( \rho \) the distance function on \( \widetilde{B}_\pi \).

Lemma 3.2 Let \( \widetilde{f} : \widetilde{M}^4 \to \widetilde{B}_\pi \) be a lifting of a classifying map to the universal coverings. If \( \dim_{mc} \widetilde{M}^4 \leq 2 \), then there exists a short homotopy \( \widetilde{F} : \widetilde{M}^4 \times I \to \widetilde{B}_\pi \) of \( \widetilde{f} \) such that \( \widetilde{F}(x, 0) = \widetilde{f}(x) \) and \( \widetilde{F}(x, 1) \) is a through mapping

\[
\widetilde{F}(x, 1) : \widetilde{M}^4 \to P^2 \to \widetilde{B}_\pi,
\]

where \( P^2 \) is a 2–dimensional polyhedron and “short homotopy” means that we have \( \rho(\widetilde{f}(x), \widetilde{F}(x, t)) \leq \text{const} \) for each \( x \in \widetilde{M}^4 \), \( t \in I \).

Proof Let \( h : \widetilde{M}^4 \to P \) be a proper cobounded continuous map to some 2–dimensional polyhedron \( P \). Using a simplicial approximation of \( h \), we can suppose that \( h \) is a simplicial map between such triangulations of \( \widetilde{M}^4 \) and \( P \), that the preimage of the star of each vertex is uniformly bounded (recall that \( h \) is proper). Since \( \widetilde{f} \) is a quasi-isometry, the \( \widetilde{f} \)–image \( \widetilde{f}(h^{-1}(\text{St}(v))) \) of the preimage of the star of each vertex \( v \in P \) is bounded by some constant \( d \). Let \( M_h \) be the cylinder of \( h \) with natural triangulation consisting of the triangulations of \( \widetilde{M}^4 \) and \( P \) and the triangulations of the simplices \( \{v_0, \ldots, v_k, h(v_k), \ldots, h(v_p)\} \), where \( \{v_0, \ldots, v_p\} \) is a simplex in \( \widetilde{M}^4 \) with \( v_0 < v_1 < \ldots, < v_p \) [5].

Consider the map \( \tilde{f}_0 : (M_h)^0 \to \widetilde{B}_\pi \) from 0–skeleton \( (M_h)^0 \) of \( M_h \) which coincides with \( \widetilde{f} \) on the lower base of \( (M_h)^0 \) and with the composition \( \tilde{f} \circ t_0 \) on the upper base of \( (M_h)^0 \), where \( t_0 : (P)^0 \to \widetilde{M}^4 \) is a section of \( h \) defined on the 0–skeleton \( (P)^0 \) of \( P \).
Since $\tilde{B}_1$ is uniformly contractible, we can extend $\tilde{f}_0$ to $M_h$ using the function $Q$ of the definition of UC-spaces as follows:

By the construction above, $\tilde{f}_0$–image of every two neighbouring vertexes of $M_h$ lies into a ball of radius $d$. Therefore we can extend the map $\tilde{f}_0$ to a mapping $\tilde{f}_1: (M_h)^1 \to \tilde{B}_1$ such that $\rho(\tilde{f}(x),\tilde{f}_1(x,t)) \leq d$, $x \in (M_h)^0$. The $\tilde{f}_1$–image of the boundary of arbitrary 2–simplex of $M_h$ lies into a ball of radius $3d$. So we can extend $\tilde{f}_1$ to a mapping $\tilde{f}_2: (M_h)^2 \to \tilde{B}_1$ so that $\rho(\tilde{f}(x),\tilde{f}_2(x,t)) \leq 4Q(3d)$, $x \in (M_h)^1$. Similarly, continue $\tilde{f}_2$ to mappings $\tilde{f}_3, \ldots, \tilde{f}_5$ defined on skeletons $(M_h)^3, \ldots, (M_h)^5 = M_h$ respectively, so that $\rho(\tilde{f}(x),\tilde{f}_5(x,t)) \leq c$, where $c$ is a constant.

**Main Theorem** $\dim_{mc} \tilde{M}^4 = 3$.

**Proof** Let $q: \tilde{B}_1 \to \tilde{B}_1/(\tilde{B}_1 \setminus D^3) \cong S^3$ be a quotient map, where $D^3$ is an embedded open 3–dimensional ball.

Suppose that $\dim_{mc} \tilde{M}^4 \leq 2$ and let $h: \tilde{M}^4 \to P$ be a proper cobounded continuous map to some 2–dimensional polyhedron $P$ as in Lemma 3.2. It is not difficult to find a compact smooth submanifold with boundary $W \subset \tilde{M}^4$ such that $W$ contains a ball of arbitrary fixed radius $r$. Since $\tilde{f}$ is a quasi-isometry, using Lemma 3.2 we can choose $r$ big enough such that $D^3 \subset \tilde{f}(W)$ and $\tilde{F}(\partial W \times I) \cap D^3 = \emptyset$, where $\tilde{F}$ denotes the short homotopy from Lemma 3.2. Thus we have a homotopy

$$q \circ \tilde{F}: (W, \partial W) \times I \to (S^3, s_0)$$

which maps $\partial W \times I$ into the base point $s_0$. Since $\dim P = 2$, from Lemma 3.2 it follows that $q \circ \tilde{F}(x, 0)$ is homotopic to zero. Therefore $q \circ \tilde{F}(x, 0) = q \circ \tilde{f}$ is homotopic to zero (and $q \circ \tilde{f}$ is smoothly homotopic to zero [3]). Let $(s, v)$ be a framed regular point in $S^3$ for the map $q \circ \tilde{f}$. Then the Pontryagin manifold

$$(\tilde{f}^{-1} \circ q^{-1}(s), \tilde{f}^* q^*(v))$$

must be cobordant to zero (see Theorem 1.6). Let $(\tilde{\Omega}, w)$ be a framed nullcobordism which is embedded in $W \times I$ with the boundary $(\tilde{f}^{-1} \circ q^{-1}(s), \tilde{f}^* q^*(v))$.

Consider the covering map $\tau: \tilde{M}^4 \times I \to M^4 \times I$. Then $\tau(\tilde{f}^{-1} \circ q^{-1}(s), \tilde{f}^* q^*(v))$ is an immersed framed submanifold of $M^4$ which coincides with the Pontryagin manifold $(p^{-1}(m), p^*(\nu))$ of some framed point $(m, \nu) \in M^3$. And $\tau(\tilde{\Omega}, w)$ is an immersed framed submanifold of $M^4 \times I$. Using the Whitney Embedding Theorem [7], we can make a small perturbation of $\tau(\tilde{\Omega}, w)$ identically on the small collar of the boundary to obtain a framed nullcobordism with the boundary $\tau(\tilde{f}^{-1} \circ q^{-1}(s), \tilde{f}^* q^*(v))$. But this is impossible by Corollary 2.2.

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Remark 3.3  By similar arguments one can prove that
\[ \dim_{mc}(\tilde{M}^4 \times T^p) = p + 3. \]

Question  Does \( M^4 \times T^p \) admit a PSC–metric for some \( p \)?

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