Multidimensional washboard ratchet potentials for frustrated two-dimensional Josephson junctions arrays on square lattices

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Abstract

In this work, we derive an analytical procedure that allows us to write the multidimensional washboard ratchet potential (MDWBP) \( U_f \) for a two-dimensional Josephson junction array. The array has an applied perpendicular magnetic field. The magnetic field is given in units of the quantum flux per plaquette or frustration of the form \( f = \frac{M}{N} \Phi_0 \), where \( \Phi_0 \) is the flux quantum. The derivation is done under the assumption that the checkerboard pattern ground state or unit cell of a two-dimensional Josephson junction array is preserved under current biasing. The resistively and capacitively shunted Josephson junction model with a white noise term describes the dynamics for each junction in the array. The multidimensional potential is the unique expression of the collective effects that emerge from the array in contrast to the single junction. The first step in the procedure is to write the equation for the phases for the unit cell. In doing this, one takes into account the constraints imposed for the gauge invariant phases due to frustration. Second, and the key idea of the procedure, is to perform a variable transformation from the original systems of stochastic equations to a system of variables where the condition for the equality of mixed second partial happens. This is achieved via Poincaré’s theorem for differential forms. In this way, we find to a nonlinear matrix equation (equation (9) in the text), that permits us to find the new coordinate variables \( x_f \) where the potential exists. The transformation matrix also permits the correct transformation of the original white noise terms of each junction to the intensities in the \( x_f \) variables. The commensurate symmetries of the ground state pinned vortex lattice leads to discrete symmetries to the part of the
washboard potential that does not contain a tilt due to the external bias current
(equation (11) in the text). In this work we apply the procedure for the
important cases \( f = \frac{1}{2}, \frac{1}{3} \). For \( f = \frac{1}{2} \), we show that previously efforts for
finding the potential are restricted, leading to a reduced dimension of the
potential. The correct potential is given in equation (21). We examine this
issue in detail. New physics emerge when currents are applied in the \( x \) and \( y \)
directions, in particular, we confirm analytically previous numerical work for
\( f = \frac{1}{2} \), concerning the border of stationary states, a landmark of the potential.
For \( f = \frac{1}{3} \), we give a generalization of previous work, in which we include
both the currents in the \( x \) and \( y \) directions as well the noise terms. We find the
MDWBP realizes tilted ratchets analogous to a combustion motor.

Keywords: Josephson junction arrays, multidimensional potentials, magnetic
field frustration

(Some figures may appear in colour only in the online journal)

1. Introduction

The resistive behavior of a two-dimensional Josephson array (TDJJA) with a given frustration
parameter (the ratio of the perpendicular magnetic field to the flux quantum per plaquette,
\( f = \Phi/\Phi_0 \), where \( \Phi_0 = \hbar c/2e \)), has been a matter of study [1–6] for decades. When the
external bias is zero, a mean field approach based quantum interference method can be used to
obtain phase diagrams (see [7] and references therein), in which a localization without dis-
order [8, 9] is exploited. A description of dynamics at any temperature requires knowledge of
the origin of dissipation. In superconducting wire networks, near \( T_c \), this is tantamount for
using the generalized Ginzburg-Landau equations for each wire element [10, 11]. In
Josephson junction arrays, the resistively and capacitively shunted Josephson junction model
(RCSJ) [12] describes each junction. The model contains a tilted washboard potential, that
permits to obtain qualitative and quantitative understanding of the dynamics [13–21]. Usually
these arrays are made such that charging effects due to small capacitance can be ignored [13].
In fact, recently interesting questions in relation to switching rates, thermal hopping and
retrapping statistics were studied using the analogy of the RCSJ model with the dynamics of
Brownian particle in a tilted washboard potential [14, 15]. The important question is whether
a similar study can be done for TDJJA, albeit the high dimensional stochastic equations
do not exist. On the other hand, TDJJA constitutes natural systems where the realization of
ratchet behavior [22–24], in particular, TDJJA ratchets with an asymmetric washboard
potential, are fabricated [25] in the overdamped regime. Numerical studies for \( f = 1/2 \) [26],
explained some features of the experiments, but qualitative understanding without detailed
knowledge of the ratchet landscape is not possible. In fact, many researchers, when pursuing
numerical work, infer the existence of a multidimensional washboard potential in analogy for
a single Josephson junction. Others [27] used forced uniaxial tilted washboard one-dimen-
sional potentials, to model a complex multidimensional physical arrangement [28]. In this
work, we fulfill the need for such a multidimensional washboard potential for TDJJA with
frustration. In the next section we describe qualitatively how we proceed in order to obtain the
MDWBP. Then in section 3, we show how the equations of motion transform when one
carries out a variable transformation. In section 4, we illustrate how to construct the MDWBP.
and show explicitly how we obtain equation (9). We also discuss the special limits of the equations of motion. In section 5, we apply the procedure to the case \( f = 1/2 \) and give analytical results as far they can derived directly from the knowledge of the potential. We also demonstrate that previous results which claim the finding of the potential for this case \([36]\) are misleading. The case \( f = 1/3 \) is presented in the Appendix. Finally in chapter 7, we give the conclusions.

2. TDJJA with magnetic fields

When a large \( L \times L \) network is subject to a uniform external driving current injected along one edge of the array and removed at the opposite edge, from energy balance arguments one expects that the ground state configuration is preserved. Frustration dictates that the net circulation of the vector potential in a given sense around a plaquette is given by \( f_{n} \mod 2 \), where \( n_{j} \) is an integer which defines the vorticity \([12]\) and \( f_{j} \) is the total flux in unit of the flux quantum in plaquette \( j \). Neglecting macroscopic screening effects ensures equal frustration for all plaquette \( f_{j} = f = \frac{M}{N} \). In \([29]\), for example, the dynamics are analyzed for \( f = \frac{1}{3} \) in the overdamped regime at \( T = 0 \) and \( T \gg 0 \); analogous numerical studies are completed in \([6, 30–32]\) also in the overdamped regime. Recently, interest for the properties of TDJJA near incommensurability have also been studied experimentally, in which the frustration value \( f = \frac{2}{3} \) appears to be of relevance \([1, 2]\). Consequently, in this work we calculate the multidimensional washboard potential for TDJJA for frustrations \( f = \frac{1}{2}, \frac{1}{3} \). We manage to find the multidimensional potential for the cases
However, as the complexity and extension of the calculations for rational frustrations \( f = \frac{m}{N} \) grow with \( N \), these cases will be published in another work. The implementation of the procedure goes through the following steps: (1) write the equations for the ground state configuration or basic cell unit (this unit constitutes an \( N \times N \) array as illustrated in figure 1 [33, 34]); (2) identify the variables in order to define a system of stochastic differential equations with diagonal isotropic masses; (3) check that the cross derivatives of the resulting potential are not equal; (4) define a coordinate transformation to a set of new variables; (5) in the newly defined variables, establish the necessary condition for the existence of a potential by invoking Poincaré’s theorem for differential forms; and (6) obtain a non-linear matrix equation (equation (9) below), whose solution leads us to the potential (equation (6) below). The multidimensional potentials give the opportunity for studying analogous theoretical and experimental questions posed for a single Josephson junction [14, 15, 20, 21, 53]. We discuss that matter, along the working out of the theory, below and in the conclusions.

3. Equations of motion

For the array in figure 1 there are junctions [31, 33, 34], whose dynamics are given by the RCSJ model [12]:

\[
\frac{d^2 \gamma_{ij}}{d\tau^2} + \frac{d\gamma_{ij}}{d\tau} + \frac{dU(\gamma_{ij})}{d\gamma_{ij}} = \frac{2 k_B T}{E_j} \xi_{ij}(\tau)
\]

where the tilted one-dimensional ‘washboard’ potential is \( U(\gamma_{ij}) = \left[ -\cos(\gamma_{ij}) - \frac{I_\circ}{k_B T} \gamma_{ij} \right] \), and where \( \gamma_{ij} \) is the gauge-invariant phase difference, \( \gamma_{ij} = \gamma_i - \gamma_j - (2e/h) \int_{l_c}^l A \cdot dl \), where \( A \) is the vector potential, \( I_\circ \) is the critical current assumed equal for all junctions, \( I_i \) is the current injected in the \( x \)-direction, and \( I_j \), the injected current in the \( y \) direction. One has \( \tau = (2eI_\circ R/h)/t \) as the dimensionless time, \( \beta_i = (2eI_\circ CR^2/h) \) is the Stewart–McCumber parameter, \( R \) is the shunt resistance, \( C \) is the capacitance, \( h \) is the Planck’s constant and \( e \) is the electron charge. The stochastic term describes white noise with intensity \( \sqrt{\frac{2k_B T}{E_j}} \), \( \langle \xi_{ij}(\tau) \rangle = 0, \langle \xi_{ij}(\tau)\xi_{nj}(\tau') \rangle = \frac{2k_B T}{E_j} \delta(\tau - \tau') \), \( k_B \) is the Boltzmann constant, and \( T \) is the temperature, and \( E_j = (h/2e)I_\circ \) is the Josephson coupling energy [12]. A bookkeeping counting allows us to establish the number of independent equations for the unit cell. First, periodic boundary conditions imply that the phases in opposite places in figure 1 are equal. One has an \( N^2 \) plaquette, and in each of them the sum of the gauge invariant phases around each plaquette must be \( \sum_{\text{plaquette}} \gamma_i = 2\pi f (\text{mod} 2\pi) = 2\pi (f - n_j) \). When going around a contour, vorticity is given by \( \sum_{\text{contour}} \gamma_i = 2\pi \sum_{\text{enclosed cells}} (f - n_j) \); ground state symmetries constrain further the number of phases. First, the circulation around the perimeter of the unit cell, should be zero, in order to avoid size scale dependent energy terms Secondly, the circulation around a contour formed from plaquette in any column or any row should also be zero. For \( f = \frac{1}{N} \Phi_0 \), the zero circulation can always be achieved in any contour around a column or row by putting has one vortex in a selected plaquette (circulation = \(-2\left(\frac{N-1}{N}\right)\)) and zero vortices in the other \((N-1)\) plaquette (circulation = \(2\left(\frac{N-1}{N}\right)\) on each). The particular pattern configurations of the minimal energy, i.e., the \( n_j \) were found in [33] for the frustrations in which we are interested. In addition, one finds that the only different phases are...
the ones in the first column and the first row. The internal phases are just suitable combinations of these phases, i.e. there are $2N$ different phases. We calculate them from the current conservation in the $(N - 1)$ internal nodes in the second row (symbols A, B...) and current conservation in the $x$ and $y$ directions. We have also $(N + 1)$ current equations for $2N$ unknowns. We use then the $(N - 1)$ relations from the plaquette circulation to eliminate the remaining $(N - 1)$ phases. One finally has $(N + 1)$ current equations for the $(N + 1)$ independent variables $y_i$; they make the functions $g_j(y_1, ..., y_{N+1})$ (primes are derivatives with respect the dimensionless time, $\beta_1$ is the Stewart-McCumber parameter):

$$\beta_1 y''_j + y'_j + g_j(y_1, ..., y_{N+1}) = 0; \ j = 1, ..., N + 1$$

(2)

$$g_j(y_1, ..., y_{N+1}) = \frac{1}{2} \sum_{k=1}^{2N} \omega_{jk} \sin \Phi_k(y_j) - \frac{I_a}{2L} \delta_{j1} - \frac{I_a}{2L} \delta_{j2} + \sqrt{\frac{2kBT}{E_j}} \left[ \sum_{m=1}^{N} \delta_{jm} \sum_{l=1}^{N} \xi_l(\tau) + \sum_{m=3}^{N+1} \delta_{jm} \sum_{l=1}^{4} \xi_l(\tau) \right]$$

(3)

$y_1$ is the sum of the phases in the left side, and $y_2$ is the sum of the phases in the upper side in figure 1. The rest of the variables $y_i$, $i = 3, ..., N + 1$ are chosen in consistence with the form of equation (2). Furthermore, $\delta_{jm}$ is the Kroenecker delta. The original phases $\Phi_k, k = 1, ..., 2N$ are now the function of the $(N + 1)$ new variables $y_i$ and the $2N \times (N + 1)$ matrix $\omega_{jk}$, with entries $-1, 0, 1$ giving the presence of the functions $\sin \Phi_k(y_j)$. In equation (3), for the variable $y_1$, there are $2N$ independent noise terms, similar for the variable $y_2$. For the each of the remaining variables $y_i, i = 3, ... , N + 1$, there are four independent noise terms. One finds $\partial g_j / \partial y_1 = \partial g_j / \partial y_1$, i.e. $g_j$, are not the derivatives of a potential [35]. One searches for a new set of variables $x$, and a transformation is carried out through an $(N + 1) \times (N + 1)$ matrix $D$, $x = Dy$, and corresponding inverse transformation $y = D^{-1}x$. Multiplying equation (2) with $D_{ij}$, one obtains a new system of stochastic differential equations:

$$\frac{\beta_1}{x_i} \sum_{j=1}^{N+1} D_{ij} y''_j + \sum_{j=1}^{N+1} D_{ij} y'_j + \sum_{j=1}^{N+1} D_{ij} g_j(y_1, y_2, ..., y_{N+1}) = 0$$

(4)

$$f_i = \sum_{k=1}^{2N} a_{ik} \sin \Phi_k(x_i) - \frac{I_a}{2L} D_{i1} - \frac{I_a}{2L} D_{i2} + \sqrt{\frac{2kBT}{E_j}} \left[ \sum_{m=1}^{N} D_{im} \sum_{l=1}^{N} \xi_l(\tau) + \sum_{m=3}^{N+1} D_{im} \sum_{l=1}^{4} \xi_l(\tau) \right]$$

(5)

where $a_{ik} = \frac{1}{2} \sum_{j=1}^{N+1} D_{ij} \omega_{jk}$. One needs to find a defining equation for the matrix $D$, such that in the new variables the cross derivatives are equal. This is achieved in the next section.

4. Construction of the potential

We define a 1–form by $F = f dx_i, i = 1, 2, ..., N + 1$, and force the condition $dF = 0$, i.e. the 1-form is closed. We invoke Poincaré’s theorem and look for a 0–form $U$, such that $dU = F$, implying $d(dU) = 0$, i.e. the 1–form is exact [35], i.e., we obtain the potential. We define $\Omega = \sqrt{\frac{2kBT}{E_j}}$. For the form of $f_i$, we suggest the following Ansatz for the 0–form:
therefore, from equation (5), the necessary condition for \( dU = F \) is given by: \( a_{ik} x_k = \frac{\partial \Phi_k}{\partial x_i} \),

\[
d (dU) = \sum_{i=1}^{N+1} \sum_{m=1}^{2N} \frac{\partial}{\partial x^m_i} (dU) dx^m \wedge dx^i = 0,
\]

A necessary and sufficient condition for \( d(dU) = dF = 0 \) is the equality of mixed partial derivatives: \( \frac{\partial^2 \Phi_k}{\partial x_i \partial x_m} = \frac{\partial^2 \Phi_k}{\partial x_m \partial x_i} \). We use the last relation in obtaining \( D \). Note that last condition can be written as \( a_{ik} a_{km} = a_{mk} a_{ik} \). By noting that \( \Phi_{0k,k} = \Phi_{0k,k} D^{-1} \), one obtains:

\[
\frac{1}{2} D^T D = \Phi_{0k,k} \omega^T (\omega \omega^T)^{-1}
\]

where \( \Phi_{0k,k} = \frac{\partial \Phi_k}{\partial y} \) represents the matrix of the derivatives of phases with respect to the variables \( y_j \). After solving equation (9) for \( D \), we read out the potential (equation (6)) and the equations of motions (equations (4)–(5)). Equivalently, the functions \( f_i(x_j) \), which define the equations of motion (equation (4)), can be found by differentiating the potential:

\[
f_i(x_1, \ldots, x_{N+1}) = \frac{\partial U}{\partial x_i}; \quad i = 1, \ldots, N + 1.
\]

The potential has in general a periodic part with period \( \omega \) and a linear tilt, i.e.,

\[
U(\vec{r}) = U_0(\vec{r}) - \vec{g} \cdot \vec{r} \\
\vec{g} = (g_1, g_2, 0, \ldots, 0) \\
U_0(\vec{r}) = U_0(\vec{r} + a_i \vec{e}_i) = U_0(\vec{r} + \vec{a}).
\]

Therefore one has:

\[
U(x_1 + \delta x_1, x_2 + \delta x_2, \ldots, x_{N+1} + \delta x_{N+1}) = U(x_1, \ldots, x_{N+1}) + \delta U_1 + \delta U_2
\]

where \( \delta x_i \) is the period in direction \( i \), and \( \delta U_{1,2} \) is the increment of the potential due to the applied currents. In a regime when the noise terms and dissipation can be neglected, one obtains a Hamiltonian system with

\[
H = \sum_{i=1}^{N+1} \frac{\dot{q}_i^2}{2 \beta_i} + U(x_1, \ldots, x_{N+1}).
\]

The first term is the kinetic energy, \( \frac{\dot{q}_i^2}{2 \beta_i} \), the dot represents the time derivative. The Stewart and McCumber parameter can be written in the form:
\[ \frac{\beta_c}{2} = \frac{4E_J}{E_C} \frac{R^2}{R_Q^2} \]  

(14)

where \( E_J \) is the Josephson coupling energy already previously defined, \( E_C = \frac{e^2}{2C} \) is the charging energy, and \( R_Q = \frac{\hbar}{e^2} \) is the quantum resistance. Equation (13) is the generalization of the one junction case (see equation (4) in [12, 13]) as the potential is a multidimensional one. Quantization of equation (12) is an standard task, the relevance of which is given for the case when \( E_C \geq E_J \) and \( R \geq R_Q \). On the other hand, the existence of minima of the potential warrant the stability of the quantum system. We calculate the border of stability as a function of the applied currents for the first example that we discuss in the next section. Without noise but maintaining the dissipation terms, one can analyze the flow properties of the associated first order system:

\[ \beta_c \dot{X} = \Xi \]  

(15)

\[ \Xi' = -\frac{\Xi}{\beta_c} - \nabla U(x_1, \ldots, x_{N+1}) \]  

(16)

where \( X = (x_1, \ldots, x_{N+1}) \), \( \Xi = (\xi_1, \ldots, \xi_{N+1}) \) [41], in this case the dynamical system is phase space contracting [42]. For \( \beta_c = 0 \), one has the overdamped limit,

\[ X' = -\nabla U(x_1, \ldots, x_{N+1}) \]  

(17)

a gradient system with at most fixed points [44]. Proper interplay of nonlinearities and noise in the systems we derive below happens in the underdamped regime.

5. Potential for \( f = 1/2 \)

Consider the ground state 2 \times 2 superlattice unit cell for \( f = 1/2 \) in figure 2 (see figure 1 in [33]); following section 3, one first obtains three equations from current conservation in the \( x, y \) direction and from the central node:

\[ \beta_c (\gamma - \alpha) + (\gamma' - \alpha') + (\sin \gamma - \sin \alpha) = I_x/I_c \]  

\[ \beta_c (\kappa - \beta) + (\kappa' - \beta') + (\sin \kappa - \sin \beta) = I_y/I_c \]  

\[ \beta_c (\gamma'' + \alpha'' - \beta'' - \kappa'') + (\gamma' + \alpha' - \beta' - \kappa') + (\sin \gamma + \sin \alpha - \sin \beta - \sin \kappa) = 0. \]  

(18)
Then, secondly, the quantization condition for the \( n = 0 \) plaquette is given by 
\[
\beta + \kappa + \alpha + \gamma = \pi,
\]
which allows the elimination of the \((\alpha + \gamma)\) variables. We arrive at the choice of variables \( y = (y_1, y_2, y_3) \) as:
\[
y_1 = \frac{1}{2}(\gamma - \alpha), \\
y_2 = \frac{1}{2}(\kappa - \beta), \\
y_3 = - (\beta + \kappa).
\]
We follow equations (3)–(5) for \( N = 2 \). The \( \Phi_k \) variables are 
\( \Phi_k(y) = (\Phi_1 = \alpha, \Phi_2 = \beta, \Phi_3 = \gamma, \Phi_4 = \kappa) \), and the matrix \( \omega \), which can be read from equation (18), is given by:
\[
\omega = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
1 & -1 & 1 & -1
\end{pmatrix}
\]
one has:
\[
\Phi_{\partial y_k} = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
1/2 & -1/2 & 1/2 & -1/2
\end{pmatrix}
\]
First one proves that \( \partial \Phi_k / \partial y_i = \partial \Phi_i / \partial y_j \), and proceeding to calculate the right hand side of equation (9), one obtains:
\[
D^T D = \begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
This implies,
\[
D = \begin{pmatrix}
\sqrt{2} & 0 & 0 \\
0 & \sqrt{2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
and its inverse,
\[
D^{-1} = \begin{pmatrix}
1/\sqrt{2} & 0 & 0 \\
0 & 1/\sqrt{2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
The new variables, \( x = Dy \) are:
\[
x_1 = \sqrt{2} y_1; \quad x_2 = \sqrt{2} y_2; \quad x_3 = y_3.
\]
We define \( x = x_1, y = x_2, z = x_3 \) and obtain with equation (5), the potential \( U \),
\[
U(x, y, z, \Omega, I_3, I_c) = - \frac{4}{k=1} \cos \Phi_k(x, y, z) + \sqrt{2} \left( \Omega \sum_{l=1}^2 \xi_l(\tau) - \frac{I_3}{I_c} \right) x + \frac{\sqrt{2}}{2} \left( \Omega \sum_{l=3}^4 \xi_l(\tau) - \frac{I_3}{I_c} \right) y + \frac{\sqrt{2}}{2} \left( \Omega \sum_{l=5}^8 \xi_l(\tau) \right) z.
\]
Figure 3. \( U(x, y, \pi/2) \) from equation (13), \( \Omega = 0, I_x = 0.292893, I_y = 0.0 \).

Figure 4. \( U(x, 0, z) \) from equation (13), \( \Omega = 0, I_x = 0.8485, I_y = 0.0 \).
**Figure 5.** $U(x, 0, z)$ from equation (13), $\Omega = 0$, $I_\chi = 0.4142$, $I_\psi = 0.0$.

**Figure 6.** $U(x, y, -\pi/2)$ from equation (13), $\Omega = 0$, $I_\chi = 0.828429$, $I_\psi = 0.0$. 
Figure 7. $U(x, y, -\pi)$ from equation (13), $\Omega = 0, I_\chi = 0.849942$, $I_\nu = 0.3999$.

Figure 8. $U(x, y, -\pi/4)$ from equation (13), $\Omega = 0, I_\chi = 0.8499$, $I_\nu = 0.3999$. 
The phases $\Phi_k(x, y, z)$ in equation (18) are given by:

$$
\begin{align*}
\Phi_1 &= \alpha = \frac{1}{2}(\pi - \sqrt{2}x + z) \\
\Phi_2 &= \beta = -\frac{1}{2}(\sqrt{2}y + z) \\
\Phi_3 &= \gamma = \frac{1}{2}(\pi + \sqrt{2}x + z) \\
\Phi_4 &= \kappa = \frac{1}{2}(\sqrt{2}y - z).
\end{align*}
$$

From equation (11) one has $U(\vec{r}) = U_0(\vec{r}) - \vec{g} \cdot \vec{r}$, with $\vec{r} = x_i \hat{r}_i$, $i = 1, 2, 3$, $|\vec{r}| = 1$, $\vec{g} = \frac{1}{\sqrt{2}} \vec{r}_1 + \frac{1}{\sqrt{2}} \vec{r}_2$, and symmetry period $\vec{d} = a_i \hat{r}_i = 4\pi \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$ (see equation (11)). Figures 3–8 show the projection of the potential $U(x, y, z)$ for some specific values of the variables $y$ or $z$.

With the knowledge of $D$ the corresponding equations of motion (equations (4) and (5) can be straightforwardly written, or alternatively, the functions $f_i(x, y, x)$ can be obtained using equation (10):
For $\Omega = 0$, there is a stationary time-independent regime. Stable solutions in this regime correspond to local minima of the potential. In this case, for given $I_\chi$, $I_\upsilon$, one manipulates equation (23), and obtain a relation which defines the $x$ variable:

$$\beta_\chi x'' + x' - \frac{2}{\sqrt{2}} \sin \left( \frac{z}{2} \right) \sin \left( \frac{x}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \left( \frac{I_\chi}{I_0} + \Omega \sum_{l=1}^{2} \xi_l \right)$$

$$\beta_\chi y'' + y' + \frac{2}{\sqrt{2}} \cos \left( \frac{z}{2} \right) \sin \left( \frac{y}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \left( \frac{I_\upsilon}{I_0} + \Omega \sum_{l=1}^{2} \xi_l \right)$$

$$\beta_\chi z'' + z' + \cos \left( \frac{z}{2} \right) \cos \left( \frac{x}{\sqrt{2}} \right) + \sin \left( \frac{z}{2} \right) \cos \left( \frac{y}{\sqrt{2}} \right) = \frac{\Omega}{2} \sum_{l=1}^{4} \xi_l. \tag{23}$$

For $\Omega = 0$, there is a stationary time-independent regime. Stable solutions in this regime correspond to local minima of the potential. In this case, for given $I_\chi$, $I_\upsilon$, one manipulates equation (23), and obtain a relation which defines the $x$ variable:

$$\frac{\sin(\sqrt{2}x)}{\sqrt{2}(1 + \sqrt{1 - \left( \frac{z}{2} \right)^2 \sin^2(\sqrt{2}x))} + \cos^2 \left( \frac{y}{2} \right)} = I_\chi, \tag{24}$$

$z(I_\chi)$ is obtained from the relation $\tan(\frac{z}{2}) = -\cos(\frac{x}{\sqrt{2}})/\cos(\frac{y}{\sqrt{2}})$ and $y(I_\chi)$ from $\sin(\frac{x}{\sqrt{2}})/\sin(\frac{y}{\sqrt{2}}) = \left( \frac{1}{2} \right)^2$. For $I_\chi = 0$, there is a critical current, at the value of which the local minima and maxima merge, i.e., at the critical current the stable and unstable fixed points coalesce into one. This critical fixed point can be obtained by maximizing equation (24) with $I_\upsilon = 0$, one obtains: $x^{\text{crit}} = \frac{1}{\sqrt{2}} \cos^{-1} \left( \frac{1}{\sqrt{2}} - 3 \right)$, $y^{\text{crit}} = 0$, $z^{\text{crit}} = -2 \sin^{-1} \sqrt{\frac{1}{\sqrt{2}}}$ per cell, in agreement with the calculation in [38]. Identical results are obtained, as one should expect by symmetry, if one puts $I_\chi = 0$ and $I_\upsilon = 0$. For $I_\chi < I_\chi^{\text{crit}}$, the local minima of the potential occurs always at $y = 0$; an example is shown in figure 3. Also local maxima of the potential happens at $y = 0$. For $I_\chi > I_\chi^{\text{crit}}$, however, the potential has no local minimum, which implies the non-existence of fixed points. Recalling that $y = 0$ implies $\kappa = \beta$, if one uses this constraint in the potential (equation (21)), one obtains another system with one dimension reduced. This system exists in the projection of the potential to the line $y = 0$. Next we show this statement. Suppose now that the terms representing stochastic forces are irrelevant ($\Omega \approx 0$) and take the imposed transversal current $I_\upsilon$ to be zero (see figure 1). If one assumes one could pin the variable $y$ to the value zero at any current (see figure 2), in this case, the potential reduces to:

![Figure 10. A 3 × 3 Josephson junctions array under a magnetic field $f = \frac{1}{3} \theta_0$.](image-url)
\[ U(x, 0, z) = -\cos \alpha - 2 \cos \beta - \cos \gamma - \frac{I_x}{\sqrt{2I_c}} \]  

where \( \alpha, \beta \) and \( \gamma \) are given by

\[ \Phi_1 = \alpha = \frac{1}{2} (\pi - \sqrt{2} x_1 + x_3) \]  
\[ \Phi_2 = \beta = -\frac{1}{2} x_3 \]  
\[ \Phi_3 = \gamma = \frac{1}{2} (\pi + \sqrt{2} x_1 + x_3). \]  

This potential can be rewritten using trigonometric identities and the sum and the difference of the equations (26) and (28)

\[ U(\alpha, \gamma) = -2 \cos \left( \frac{\alpha + \gamma}{2} \right) \cos \left( \frac{\alpha - \gamma}{2} \right) - 2 \sin \left( \frac{\alpha + \gamma}{2} \right) + \frac{I}{I_c} \left( \frac{\alpha - \gamma}{2} \right). \]  

Now we introduce the scaled sum and difference variables \( \xi \equiv (\alpha + \gamma)/\sqrt{2} \) and \( \eta \equiv (\alpha - \gamma)/2 \), this allows the potential to be written as

\[ U = 2 \left[ -\cos \left( \frac{\xi}{\sqrt{2}} \right) \cos \left( \eta \right) - \sin \left( \frac{\xi}{\sqrt{2}} \right) + \frac{I}{2I_c} \eta \right] = 2U(\xi, \eta). \]  

One obtains the system:

\[ \beta_c \eta'' + \eta' + \frac{\partial U(\xi, \eta)}{\partial \eta} \]  
\[ \beta_c \phi'' + \phi' + \frac{\partial U(\xi, \eta)}{\partial \phi}. \]  

This system was used for simulations in [36]. Forcing \( y = 0 \) for \( I_0 > I_{\text{crit}}^{\text{out}} \) however, is incompatible with the phase flow properties of equation (23) [41]. One knows that for currents greater than the critical one, the line \( y = 0 \) is neither local nor global attracting. Instead, there will be limit cycles and fluctuations in the variable \( y \), i.e., voltage fluctuations in the y-direction and we let the current go to zero. The second equation in the set of equation (18) remains valid in this limit. One does not arrive in the special case where each term in parenthesis is zero. The form of the equation persists. As shown above, if one forces each term in parenthesis to be zero, one obtains a spurious result consisting in a system with a faulty reduced dimensionality. Therefore, the correct equations of motions for \( f = \frac{1}{2} \) is given by equation (23) and the correct potential is given by equation (21).

In the overdamped regime \( \beta_c = 0 \), for \( I_0 = 0 \), and \( I_\chi = 0 \), Fisher et al carried out numerical calculations for \( \Omega = 0 \) [43]. As equations of motion they used equation (18) for the gauge invariant phases \( \Phi_i \) in the overdamped regime (equation (17)). They found a regime with voltage zero, which they called a pinned regime. The border of this regime can be obtained analytically from the calculation of the maximum permitted value of \( I_\chi \) for a given \( I_0 \). Again, by maximizing equation (24) one finds a polynomial equation of degree six for the unknown \( \cos (\sqrt{2} x) \). For a given \( R \equiv I_c/I_\chi \), the solution of the polynomial equation allows to calculate the maximal value of \( I_\chi \), i.e., the border of stability of the pinned phase. For the special value \( R = 1 \), the algebraic equation reduces to one of degree four, one finds the solution \( x = y = \pi/2\sqrt{2}, \) and \( z = -\pi/2, \) which implies \( I_\chi^{\text{max}} = 1 \) per cell. When \( R = 0 \) one
has the case discussed above. Therefore, there is an island of stability between $I_{c1}^{\text{min}}$ and $I_{c1}^{\text{max}}$. In order to obtain an analytical equation for the border of stability, one finds first from equation (24) a polynomial equation of degree four for $y \equiv \cos(\sqrt{2}x)$,

$$y^4 + I_{c1}^2y^3 + [2[I_{c1}^2 - 1] + \frac{I_{c1}^4}{4}(1 - R^2)]y^2$$

$$+ [(I_{c1}^2 - 1)^2y + [(I_{c1}^2 - 1)^2 - \frac{I_{c1}^4}{4}(1 - R^2)] = 0.$$ (32)

One factorizes the $y = -1$ root, and obtains:

$$y^3 + [(I_{c1}^2 - 1)]y^2 + [(I_{c1}^2 - 1) + \frac{I_{c1}^4}{4}(1 - R^2)]y$$

$$+ [(1 - 2I_{c1}^2) + I_{c1}^4\left(1 - \frac{(1 - R^2)}{4}\right)] = 0.$$ (33)

This equation gives the value for given $I_c$ and $R$, the corresponding value of $x$, $y$, $z$. Third, one writes the discriminant $D(I_c, R)$, of this third degree equation and looks for its change in sign, i.e., one solves $D(I_c, R) = 0$ which for given $I_c$, is a polynomial equation of degree three for $R_{\text{min}}(I_c)$, the value of which defines the pinned phase border (the value of the branch of one of the real roots for $\cos(\sqrt{2}x)$ that evolves from $R = 1$ to the value $R_{\text{min}}$ defined as the value of $R$ where it transform into a complex root). Figure 9 shows the pinned phase (for clarity, notice that in the figure we denote $I_\chi$ the current in the $x$ direction, similar for the $y$ direction, in compliance with the notation in figure 2). This phase is a landmark property of the potential independent of $\beta_\chi$. Figure 6 in [43] shows a numerical calculation of this exact analytical result, the difference of factor two in the axis is because we use the notion of critical current per cell. Beyond the pinned border of the stationary regime, there are no local minima of the potential and only time dependent solutions exist. For finite temperatures, numerical simulation seems to be the only way to study this regime, however, qualitative understanding can be obtained from the potential. The kind of questions one can ask are included in [32].

The authors made numerical simulations of large arrays with frustration $f = 1/25$ in the overdamped limit. Their phase diagram temperature versus applied current (their figure 1 show various phases). The pinned phase, without a voltage, corresponds to the stationary regime shown in figure 5; it destabilizes for sufficiently large value of $\Omega$, transforming it in a phase with a finite time average voltage. The mechanism behind it is similar to the well known case of a single Josephson junction [15, 45], only that the barrier height $\Delta U$ has to be calculated from equation (10) with $\Omega = 0$ and the criterium that the scape rate turns significant when $\Delta U \approx \Omega$ [48]. In this way, tilting the potential asymmetrically, i.e. by applying currents below or above the $I_\chi = I_c$ line, it is clear that one direction destabilizes first, and then for larger $\Omega$, the other direction. This is only a qualitative picture, and a quantitative analysis needs the full nonlinear dynamics and particular properties of the potential in order to understand the final state after escaping. Proper use of a multidimensional Wiener process [46] is also required. On the other hand, the so called transverse depinning [32], viewed from our theory, constitutes the ratchet effect similar to the previous case. When $I_\chi \geq I_{\text{th}}$ and one turns $I_\chi \approx \epsilon$ on, one begins to tilt the potential in the $y$ direction. There are channels around $z = n^*\pi$, where the potential permits the particle of mass $\beta_\chi$ to slide almost freely down, whereas for example $z = \pi/4$ it is halted by a relatively big potential barrier (see figures 7–8), then at a sufficiently big value of $\Omega$, the particle begins to slide down the direction $y$, accompanied with a voltage. What we have at hand is an analogous molecular
motor [39, 47]. The numerical study of these scenarios, and also eventually the combination of a constant current and time dependent periodic current, is a matter of future research. This last scenario has been treated numerically in [31] for \( f = \frac{1}{2} \) at zero temperature \((\Omega = 0)\) in the overdamped regime \((\beta_c = 0)\).

6. Potential for \( f = 1/3 \)

In the appendix, we apply the general method (equations (5) and (9)), to that case and obtain the potential. In this way, we present a generalization of the results given in [37] which includes the noise terms in the potential variables. Deeper analytical and numerical work of that case is a challenging future task.

7. Conclusions

In this work, we have developed a general method for finding a potential for current biased TDJJJA with frustration. We analyzed in some detail the frustration value \( f = 1/2 \) for which new analytical results are found. One important result is the analytical calculation of the pinned phase, as it has a landmark character deriving from the potential. We remark that this pinned phase was calculated numerically by Fisher et al [43], and that our analytical results confirm their numerical work. Also, we pointed to the previous efforts to find the potential in [36] (section 5). In this work, an assumption concerning the condition for the current in the \( y \) direction was used. It consisted in assuming that the phases in the \( y \) direction (see figures 1 and 2), cancel when \( I_y = 0 \), even in the case when the current in the \( x \) direction \( I_x \) has surpassed the critical current. We have proved that this condition is incorrect for currents greater than the critical current. In fact, we have written for the first time the correct equations for this case inclusive of the noise terms. We remark that it is not just necessary to include the noise terms (a matter that was not considered in [36]) in the potential variables, but also the transformation of these variables from the original system (equation (18)) to the systems defined by the transformation matrix \( D \) (equation (21)). In this sense, there is a need to make numerical simulations for this case with the correct equation of motion (equation (23)). In the overdamped regime, a rocking ratchet effect was found in [26], where an asymmetrical potential was engineered. For our potential, we conjecture that inertial effects \((\beta_c \neq 0)\) can produce a dynamical ratchet. We also expect the current reversal phenomenon to exist in our system [49]; in fact, our systems are prominent examples of ratchets in which inertia, dissipation and noise combine together with high dimensionality and chaos [37] similar to the theory of molecular motors [23, 24, 39, 47]. Potentials for symmetric values of \( f \) around \( f = 1/2 \), for example \( f = 1/4 \) and \( f = 3/4 \) [26], appear interesting and can also be found with the method. At temperatures \( T \approx 0 \), when dissipation can be neglected, i.e. neglecting quasi-particles degrees of freedom [50], our systems are Hamiltonian ones as explained (see equation (13)). In this case [52, 53], the potentials derived here can still be used in conjunction with quantum noise [55]. If the charging energy due to a small capacitance is comparable to the Josephson coupling energy \( E_J \) [13], the problem turns into a quantum mechanical one [12, 53]. With the aim of studying superconducting to non-superconducting transition for TDJJJA with current bias but no magnetic field, Porter and Stroud [51] write a Hamiltonian similar to our equation (13), in which the kinetic terms are charging energies, which in the units used in [12], is the energy of a particle of mass \( C(\hbar/2e)^2 \), and the potential is the sum of the one-dimensional washboard potentials of the junctions. The search is for local minima of an unknown multidimensional potential as a signal of the superconducting
phase. Due to the lack of dissipation and fluctuations the only possible transition is from superconducting to non-superconducting states. This task for the TDJJA with frustration can be studied as we have shown with the help of the multidimensional potentials derived here. Finally, we mention the potentially interesting question posed in [54, 14] concerning the nature of the fluctuations generated by a single Josephson junction. We believe the study of the same question for our multidimensional system is a relevant issue. We also point out the possible direct application of the ideas of this paper to the problem of the superconductor-insulator transitions in frustrated Josephson arrays and the their dual systems the nanowires arrays with frustration [56]. Generalization of the cases studied in [20, 21] is in progress.

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Appendix. Potential for \( f = 1/3 \)

Consider the ground state \( 3 \times 3 \) superlattice unit cell for \( f = 1/3 \) as in figure 10 [33, 37]. Like in the previous case \( (f = 1/2) \), we derive the equations of motion for this arrangement. First we write the flux quantization conditions from the plaquette labeled II and III in figure 10, second we write the conservation of charge conditions at nodes A and B. Then we write the equations of currents in \( x \) and \( y \) directions. Later we introduce new variables to obtain an isotropic mass tensor. From the resulting system of equations we read the matrices \( \omega \) and \( \Phi \), from the right hand side of equation (9) and find the matrix \( D \). Then we write the potential (equation (6) and the equations of motion (equations (4)–(5)). We have the following flux quantization conditions from the plaquette labeled II and III:

\[
\lambda + \gamma + \beta_0 + \delta = \frac{2\pi}{3} \tag{A.1}
\]

\[
\alpha + \beta - \delta - \lambda = \frac{2\pi}{3} \tag{A.2}
\]

From current conservation of charge, we obtain from nodes A and B:

\[
\beta_{k}(\gamma'' + \alpha'' - \beta'' - \beta''_0) + (\gamma' + \alpha' - \beta' - \beta'_0) + (\sin \gamma + \sin \alpha - \sin \beta - \sin \beta_0) = 0 \tag{A.3}
\]

\[
\beta_{k}(\beta'' - \alpha'' - \delta'' - \lambda'') + (\beta' - \alpha' + \delta' - \lambda') + (\sin \beta - \sin \alpha + \sin \delta - \sin \lambda) = 0. \tag{A.4}
\]

We impose the condition for the currents in the \( xy \) direction

\[
\beta_{k}(\gamma'' - \alpha'' - \delta'') + (\gamma' - \alpha' - \delta') + (\sin \gamma - \sin \alpha - \sin \delta) = I_{i}/I_{c} \tag{A.5}
\]

\[
\beta_{k}(\beta'' - \beta''_0 + \lambda'') + (\beta' - \beta'_0 + \lambda') + (\sin \beta - \sin \beta_0 + \sin \lambda) = I_{i}/I_{c}. \tag{A.6}
\]

The conditions (A.1) and (A.2) allow us to rewrite the equations (A.3) and (A.4) respectively

\[
2\beta_{k}(\gamma'' + \alpha'') + 2(\gamma' + \alpha') + (\sin \gamma + \sin \alpha - \sin \beta - \sin \beta_0) = 0 \tag{A.7}
\]

\[
2\beta_{k}(\beta'' - \lambda'') + 2(\beta' - \lambda') + (\sin \beta - \sin \alpha + \sin \delta - \sin \lambda) = 0. \tag{A.8}
\]
We choose the variables \( y = (y_1, y_2, y_3, y_4) = (x, y, z, u) \). The scaling in the definitions of the new variables is necessary to obtain a diagonal and isotropic mass tensor

\[
\begin{align*}
    x &= 1/2(\gamma - \alpha - \delta) \\
    y &= 1/2(\beta - \beta_0 + \lambda) \\
    z &= 1/2(\beta - \alpha + \beta - \beta_0) = \beta - \lambda - \pi/3 \\
    u &= 1/2(\gamma + \alpha - \beta - \beta_0) = \gamma + \alpha - 2\pi/3.
\end{align*}
\] (A.9)

Using the equations (A.1), (A.2) and (A.9), the gauge invariant phases 
\( \Phi = (\alpha, \beta, \beta_0, \gamma, \lambda, \delta) \) can be obtained

\[
\begin{align*}
    \alpha &= 1/3(u - z - 2x + \pi) \\
    \beta &= 1/3(2y - u + z + \pi) \\
    \beta_0 &= 1/3(-2y - 2u - z + \pi) \\
    \gamma &= 1/3(2u + z + 2x + \pi) \\
    \lambda &= 1/3(2y - u - 2z) \\
    \delta &= 1/3(u + 2z - 2x).
\end{align*}
\] (A.10)

So we can write this system in compact manner

\[
\beta y^0_j + y^0_j + g_j(y_1, y_2, y_3, y_4) = 0 \] (A.11)

with \( j = 1, 2, 3, 4; \ y = (y_1 = x, y_2 = y, y_3 = z, y_4 = u) \), \( \Phi_k = (\Phi_1 = \alpha, \Phi_2 = \beta, \Phi_3 = \beta_0, \Phi_4 = \gamma, \Phi_5 = \lambda, \Phi_6 = \delta) \) and

\[
g_j(y_1, \ldots, y_4) = \frac{1}{2} \sum_{k=1}^{6} \omega_{jk} \sin \Phi_k(y_1, \ldots, y_4) = \frac{I_1}{2\ell_c} \delta_{j1} - \frac{I_0}{2\ell_c} \delta_{j2}
\] (A.12)

\( \delta_{ji} \) is the kronecker delta and,

\[
\omega = \begin{pmatrix}
-1 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & -1 & 1 \\
1 & -1 & -1 & 1 & 0 & 0
\end{pmatrix}
\]

\[
\Phi_{0j,k} = \begin{pmatrix}
-2/3 & 0 & 0 & 2/3 & 0 & -2/3 \\
0 & 2/3 & -2/3 & 0 & 2/3 & 0 \\
-1/3 & 1/3 & -1/3 & 1/3 & -2/3 & 2/3 \\
1/3 & -1/3 & -2/3 & 2/3 & -1/3 & 1/3
\end{pmatrix}.
\]

Using equation (9) in the main text one obtain matrix \( D \):

\[
D = \begin{pmatrix}
2/\sqrt{3} & 0 & 0 & 0 \\
0 & 2/\sqrt{3} & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1/\sqrt{3} & -1/\sqrt{3}
\end{pmatrix}
\]

\[
D^{-1} = \begin{pmatrix}
\sqrt{3}/2 & 0 & 0 & 0 \\
0 & \sqrt{3}/2 & 0 & 0 \\
0 & 0 & 1/2 & \sqrt{3}/2 \\
0 & 0 & 1/2 & -\sqrt{3}/2
\end{pmatrix}.
\]
From the inverse transformation $D^{-1}$ one writes:

\[
\begin{align*}
    y_1 &= \frac{\sqrt{3}}{2} x_1 \\
    y_2 &= \frac{\sqrt{3}}{2} x_2 \\
    y_3 &= \frac{1}{2} x_3 + \frac{\sqrt{3}}{2} x_4 \\
    y_4 &= \frac{1}{2} x_3 - \frac{\sqrt{3}}{2} x_4.
\end{align*}
\]

(A.13)

With equation (6) one reads the potential:

\[
U = -\sum_{k=1}^{6} \cos \Phi_k (x_1, x_2, x_3, x_4) + \frac{1}{\sqrt{3}} \left( \Omega \sum_{i=1}^{3} \xi_i (\tau) - \frac{I_1}{I_4} \right) x_1 + \frac{1}{\sqrt{3}} \left( \Omega \sum_{i=4}^{6} \xi_i (\tau) - \frac{I_1}{I_4} \right) x_2 \\
+ \frac{1}{2} \Omega \sum_{i=7}^{14} \xi_i (\tau) x_3 + \frac{1}{2} \sqrt{3} \Omega \left( \sum_{i=1}^{10} \xi_i (\tau) - \sum_{i=11}^{14} \xi_i (\tau) \right) x_4.
\]

(A.14)

Also the equations of motion in the new variables $x$ can be written just by reading the corresponding elements of the matrix $D$ as equation (5) dictates, or from the potential using equation (10). The gauge invariant phases as a function of the new variables $x = Dy$, in equation (5) and in equation (A.13) are:

\[
\begin{align*}
    \Phi_1 &= \alpha = 1/3 (-\sqrt{3} x_1 - \sqrt{3} x_4 + \pi) \\
    \Phi_2 &= \beta = 1/3 (\sqrt{3} x_2 + \sqrt{3} x_4 + \pi) \\
    \Phi_3 &= \beta_0 = 1/3 \left( -\sqrt{3} x_2 - \frac{3}{2} x_3 + \frac{\sqrt{3}}{2} x_4 + \pi \right) \\
    \Phi_4 &= \gamma = 1/3 \left( \frac{3}{2} x_3 - \sqrt{3} x_4 + \sqrt{3} x_1 + \pi \right) \\
    \Phi_5 &= \lambda = 1/3 \left( \sqrt{3} x_2 - \frac{3}{2} x_3 - \frac{\sqrt{3}}{2} x_4 \right) \\
    \Phi_6 &= \delta = 1/3 \left( \frac{3}{2} x_3 + \sqrt{3} x_4 - \sqrt{3} x_1 \right).
\end{align*}
\]

This potential can be written according to equation (11) as

\[
U(\vec{r}) = U_0(\vec{r}) - \hat{g} \cdot \vec{r}
\]

(A.15)

with $\vec{r} = x_i \hat{r}_i$, $i = 1, \ldots, 4$, $|\hat{r}_i| = 1$, $\hat{g} = \frac{1}{\sqrt{3}} \hat{r}_1 + \frac{1}{\sqrt{3}} \hat{r}_2$ and period (see equation (11)):

\[
\vec{a} = a_i \hat{r}_i = \frac{2}{\sqrt{3}} \pi (3, 3, 2, 6).
\]

(A.16)

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The functions \( f_i(x_1, x_2, x_3, x_4) \) from their definition in equation (4), are given by:

\[
\begin{align*}
    f_1 &= \frac{2}{\sqrt{3}} g_1 = \frac{1}{\sqrt{3}} (\sin \gamma - \sin \alpha - \sin \delta + \Omega \sum_{i=1}^{3} \xi_i(\tau) - I_i / I_c) \\
    f_2 &= \frac{2}{\sqrt{3}} g_2 = \frac{1}{\sqrt{3}} (\sin \beta - \sin \beta_0 + \sin \lambda + \Omega \sum_{i=4}^{6} \xi_i(\tau) - I_i / I_c) \\
    f_3 &= g_3 + g_4 = \frac{1}{2} (\sin \delta - \sin \lambda + \sin \gamma - \sin \beta_0 + \Omega \sum_{i=7}^{10} \xi_i(\tau)) \\
    f_4 &= \frac{1}{\sqrt{3}} (g_3 - g_4) = \frac{1}{2\sqrt{3}} (2 \sin \beta - 2 \sin \alpha + \sin \delta - \sin \lambda \\
    &\quad - \sin \gamma + \sin \beta_0 + \Omega \sum_{i=1}^{10} \xi_i(\tau) - \Omega \sum_{i=11}^{14} \xi_i(\tau)).
\end{align*}
\]

In this way, the equations of motion for the \( x_i \) variables (equation (4)) are completed. The border of the pinned phase, in analogy with the \( f = 1/2 \) case, and other questions of the dynamics will be accomplished in another work.

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