SINGLE SPOT IDEALS OF CODIMENSION 3 AND LONG BOURBAKI SEQUENCES

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Abstract. Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ be a polynomial ring. A single spot ideal $I \subset S$ is a graded ideal whose local cohomology $H^i_m(S/I)$, $i < \dim S/I$ and $m = (x_1, \ldots, x_n)$, only has non-trivial value $N$, a finite length module, at $i = \text{depth } S/I$. We consider characterization of single spot ideals in terms of (long) Bourbaki sequences. The codimension 2 case has been fairly well investigated. In this paper, we focus on the codimension 3 case.

Introduction

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$ with the standard grading and let $m = (x_1, \ldots, x_n)$. All the modules and ideals in this paper are graded. A finitely generated $S$-module $M$ is called a generalized Cohen-Macaulay (CM) module if the local cohomology module $H^i_m(M)$ has finite length for all $i < \dim(M)$. A ring $R$ is a generalized CM ring if it is a generalized CM $R$-module. An ideal $I \subset S$ is called a generalized CM ideal if $S/I$ is a generalized CM ring. If a generalized CM module $M$ satisfies $\dim M = \dim S$, it is called maximal.

In this paper, we are interested in generalized CM ideals. In particular, single spot ideals. An ideal $I \subset S$ is called a single spot ideal of type $(t, N)$ where $t = \text{depth } S/I$ and $N$ is a finite length $S$-module if the local cohomology only has a non-trivial value $N$ at dimension $t$, i.e.,

$$H^i_m(S/I) = \begin{cases} 0 & \text{for all } i < \dim S/I \text{ with } i \neq t \\ N & \text{if } i = t \end{cases}$$

Let $I \subset S$ be a generalized CM ideal of codim $I = r$ ($r \geq 2$). Then by Corollary 1.3 [4] we have a long Bourbaki sequence

$$0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_1 \rightarrow M \rightarrow I \rightarrow 0$$

with $S$-free modules $F_i$ ($i = 1, \ldots, r-1$) and $M$ is a maximal generalized CM module whose local cohomology is as follows

$$H^i_m(M) \cong \begin{cases} H^i_m(I) \cong H^{i-1}_m(S/I) & \text{if } i < n-r+1 \\ 0 & \text{if } i = n-r+1 \end{cases}$$

(1)

In this sense, the ideal $I$ is approximated by $M$. Notice that the value of $H^i_m(M)$ for $i = n-r+2, \ldots, n-1$ are irrelevant to this approximation. However, the construction given in the proof of Corollary 1.3 [4] (and also Lemma 1.3 [1] in a slightly different situation) always makes the module $M$ such that $H^i_m(M) = 0$ for $i = n-r+2, \ldots, n-1$. In this paper, we are interested in long Bourbaki sequences with approximation modules $M$ such that $H^i_m(M)$ ($i = n-r+2, \ldots, n-1$)
are not always trivial, and study the case of codimension 3, namely the case of $H_{m}^{n-1}(M) = N$ where $N$ is a non-trivial finite length module. Notice that in the case of codimension 2 we always have $H_{m}^{n-1}(M) = 0$.

First of all, we will give a characterization of a maximal generalized CM module $M$ whose local cohomology is

$$H_{m}^{i}(M) = \begin{cases} 
K & \text{if } i = t + 1 \\
N & \text{if } i = n - 1 \\
0 & \text{if } i < n - 2, \ i \neq t + 1
\end{cases}$$

in terms of the first syzygy of $M$. See Theorem 1.1. Then we consider the special case of $M = E_{t+1} \oplus E_{n-1}(d)$ where $E_{j}$ denotes the $j$th syzygy module of the field $K$ over $S$. We will use the notation $M(d)$, for a graded module $M$ and $d \in \mathbb{Z}$, such that $M(d)_{i} = M_{d+i}$ (the $d$ + $i$th component of $M$) for all $i \in \mathbb{Z}$. Our question is how we can construct a long Bourbaki sequence

$$0 \rightarrow F \rightarrow G \rightarrow E_{t+1} \oplus E_{n-1}(d) \rightarrow I(c) \rightarrow 0$$

of non-trivial type. Here a trivial type construction is as follows. First construct a long Bourbaki sequence

$$0 \rightarrow F' \xrightarrow{f} G' \xrightarrow{f} E_{t+1} \xrightarrow{\phi} I(c) \rightarrow 0$$

according to the method given in the proof of Corollary 1.3 [4] (or Lemma 1.3 [1]). Then make the direct sum

$$0 \rightarrow F' \oplus K_{n}(d) \xrightarrow{f \oplus \delta_{0}} G' \oplus K_{n-1}(d) \xrightarrow{g \oplus \delta_{n-1}} E_{t+1} \oplus E_{n-1}(d) \xrightarrow{\phi} I(c) \rightarrow 0$$

where $(K_{\ast}, \partial_{\ast})$ is the Koszul complex of the sequence $x_{1}, \ldots, x_{n}$ over $S$. We will denote a base $x_{i_{1}} \wedge \cdots \wedge x_{i_{k}}$ ($1 \leq i_{1} < \cdots < i_{k} \leq n$) of the Koszul complex of the sequence $x_{1}, \ldots, x_{n}$ by $e_{i_{1}, \ldots, i_{k}}$ or $e_{i_{1} \cdots i_{k}}$. Notice that, in the trivial type Bourbaki sequence $E_{n-1}(d)$ does not contribute to $I$ via $\phi$.

It is well known that a (short) Bourbaki sequence $0 \rightarrow F \rightarrow M \rightarrow I \rightarrow 0$ is constructed by finding 'graded basic elements' in $M$. See [2, 5] for the standard basic element theory and [1, 6] for graded version. However, there is no comparative notion for long Bourbaki sequences. We give a simple answer to this problem in the case of $E_{t+1} \oplus E_{n-1}(d)$ (and $M = E_{t+1}$). We will give a characterization of long Bourbaki sequences [3] in terms of elemets from $K_{t+1} \oplus K_{n-1}$ (from $K_{t+1}$) and from $K_{n-t-1} \oplus K_{1}$ (from $K_{n-t-1}$) satisfying certain conditions, which suggests a construction of the long Bourbaki sequences. See Theorem 2.2 and 2.3. In particular, non-trivial type construction is characterized by an additional condition on the elements from $K_{t+1} \oplus K_{n-1}$ (Theorem 2.7).

However, the existence of a long Bourbaki sequence [3] only means that $I$ is a single spot ideal of codimension less than or equal to 3. We give a numerical condition to assure codim $I = 3$. See Theorem 3.5. Finally, we give some examples.

For a module $M \neq K$, we will denote the $j$th syzygy module over $S$ by $\Omega_{j}(M)$. Also we use two kinds of duals, $(-)^{\vee} = \text{Hom}_{S}(-, K)$ and $(-)^{*} = \text{Hom}_{S}(-, S(-n))$. 

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\[ 0 \rightarrow F \rightarrow G \rightarrow E_{t+1} \oplus E_{n-1}(d) \rightarrow I(c) \rightarrow 0 \]

\[ 0 \rightarrow F' \xrightarrow{f} G' \xrightarrow{f} E_{t+1} \xrightarrow{\phi} I(c) \rightarrow 0 \]

\[ 0 \rightarrow F' \oplus K_{n}(d) \xrightarrow{f \oplus \delta_{0}} G' \oplus K_{n-1}(d) \xrightarrow{g \oplus \delta_{n-1}} E_{t+1} \oplus E_{n-1}(d) \xrightarrow{\phi} I(c) \rightarrow 0 \]
1. Approximation Modules of Single Spot Ideals of type \((t, K(-c))\)

In this section, we consider approximation modules \(M\) of codimension 3 single spot ideals \(I \subset S\) of type \((t, K(-c))\) in long Bourbaki sequences

\[0 \rightarrow F \rightarrow G \rightarrow M \rightarrow I(c) \rightarrow 0.\]

If we restrict ourselves to the case of \(H^m_{n-1}(M) = 0\), we have \(M = E_t \oplus H\) for some free \(S\)-module \(H\) according to Herzog, Takayama [4] and Amasaki [1]. We will now consider the general case.

**Theorem 1.1.** Let \(M\) be a maximal generalized CM module over \(S\) and consider its first syzygy:

\[0 \rightarrow \Omega_1(M) \rightarrow F \rightarrow M \rightarrow 0\]

and let \(g_1, \ldots, g_t\) be a minimal set of generators of \(\Omega_1(M)\) whose degrees are \(a_1, \ldots, a_t\). Also let \(N\) be a finite length module over \(S\), which may be 0. Then the following are equivalent.

(i) For \(t \leq n - 4\), we have

\[H^i_m(M) = \begin{cases} K & \text{if } i = t + 1 \\ 0 & \text{if } i \leq n - 2, i \neq t + 1 \\ N & \text{if } i = n - 1 \end{cases}\]

(ii) (a) \(\Omega_1(M) \cong \bigoplus_{i=1}^t S(-a_i)/E_{t+3}\), and

(b) \(\Omega_1(M)^* \cong N^\vee + F^*/M^*\)

**Proof.** We first prove \((i)\) to \((ii)\). Let \(F_*\) be a minimal free resolution of \(M\) over \(S\):

\[F_* : 0 \rightarrow F_{n-t-1} \xrightarrow{\varphi_{n-t-1}} F_{n-t-2} \rightarrow \cdots \rightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \rightarrow 0.\]

By taking the dual, we have

\[0 \rightarrow F_0^* \xrightarrow{\varphi_0^*} F_1^* \xrightarrow{\varphi_1^*} F_2^* \rightarrow \cdots \rightarrow F_{n-t-2}^* \xrightarrow{\varphi_{n-t-2}^*} F_{n-t-1}^* \rightarrow 0.\]

Then by local duality the \(j\)th cohomology of this complex is

\[\text{Ext}^j_S(M, S(-n)) \cong H_m^{n-j}(M)^\vee = \begin{cases} N^\vee & \text{if } j = 1 \\ 0 & \text{if } j \geq 2, j \neq n - t - 1 \\ K & \text{if } j = n - t - 1 \end{cases}\]

for \(j \geq 1\). Thus

\[0 \rightarrow \text{Im} \varphi_2^* \rightarrow F_2^* \rightarrow \cdots \rightarrow F_{n-t-1}^* \rightarrow K \rightarrow 0\]

is exact and \(F_2^*\) to \(F_{n-t-1}^*\) part is a begining of a minimal free resolution of \(K\), which is isomorphic to a begining of the Koszul complex \((K_*, \partial_*)\) of the sequence \(x_1, \ldots, x_{n-1}\). Namely,

\[F_{n-t-1}^* \cong K_0, \ldots, F_2^* \cong K_{n-t-3} \quad \text{and} \quad \text{Im} \varphi_2^* \cong E_{n-t-2}.\]

On the other hand, we have \(N^\vee \cong \ker \varphi_2^*/\text{Im} \varphi_1^*\) and \(E_{n-t-2} \cong \text{Im} \varphi_2^* \cong F_1^*/\ker \varphi_2^*.\) Now set \(U := \text{Coker} \varphi_1^* = F_1^*/\ker \varphi_1^*\). Then

\[U/N^\vee \cong (F_1^*/\text{Im} \varphi_1^*)/(\ker \varphi_2^*/\text{Im} \varphi_1^*) \cong F_1^*/\ker \varphi_2^* = E_{n-t-2}.\]
Thus we have
\[ 0 \longrightarrow N^\vee \longrightarrow U \longrightarrow E_{n-t-2} \longrightarrow 0 \]
Taking the dual, we have
\[ 0 \longrightarrow E^*_{n-t-2} \longrightarrow U^* \longrightarrow (N^\vee)^* \]
Since \( N \) has finite length, \( N^\vee \) has also finite length by Matlis duality, so that \((N^\vee)^* = 0\). Also \( E^*_{n-t-2} \cong E_{t+3} \) by selfduality of Koszul complex. Thus we have \( U^* \cong E^*_{t+3} \).
Then by dualizing the exact sequence
\[
\begin{array}{c}
0 \longrightarrow M^* \longrightarrow F_0^* \xrightarrow{\varphi_1^*} F_1^* \longrightarrow U^* \longrightarrow 0
\end{array}
\]
we have
\[
0 \longrightarrow E_{t+3} \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0.
\]
This proves \((ii)(a)\). Now from the short exact sequence
\[ 0 \longrightarrow \Omega_1(M) \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 0 \]
we have the long exact sequence
\[ 0 \longrightarrow M^* \xrightarrow{\varphi^*} F^* \longrightarrow \Omega_1(M)^* \longrightarrow N^\vee \longrightarrow 0 \]
since we have \( \text{Ext}_S^1(M, S(-n)) \cong H_{m-1}^n(M)^n = N^\vee \) by local duality. This proves \((ii)(b)\).
Next we prove \((ii)\) to \((i)\). By \((ii)(a)\) we have a \( S \)-free resolution of \( M \):
\[ 0 \longrightarrow K_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{t+4}} K_{t+3} \xrightarrow{\partial_{t+3}} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0 \]
where \( F_0 \) and \( F_1 \) are \( S \)-free modules. By taking the dual, we have the complex
\[ 0 \longrightarrow M^* \xrightarrow{\varphi^*_0} F_0^* \xrightarrow{\varphi^*_1} F_1^* \xrightarrow{\partial^*_0} K_{t+3}^* \xrightarrow{\partial^*_1} \cdots \xrightarrow{\partial^*_n} K_n^* \longrightarrow 0 \]
Then by local duality and selfduality of Koszul complex we compute
\[
H^i_m(M) \cong \text{Ext}^i_S(M, S(-n))^\vee = \begin{cases} 
K & \text{if } i = t + 1 \\
0 & \text{if } i \leq n - 2, \ i \neq t + 1
\end{cases}
\]
Now by dualizing the exact sequence
\[ K_{t+3} \xrightarrow{\partial_{t+3}} F_1 \xrightarrow{\varphi_1} \text{Ker} \varphi_0 \longrightarrow 0 \]
we have
\[ 0 \longrightarrow (\text{Ker} \varphi_0)^* \longrightarrow F_1^* \xrightarrow{\partial^*_0} K_{t+3}^* \]
so that we have \( \Omega_1(M)^* = (\text{Ker} \varphi_0)^* \cong \text{Ker}(\partial^*_0) \). Then by the condition \((ii)(b)\) we compute
\[
\begin{align*}
H_{m-1}^n(M)^\vee & \cong \text{Ext}^1_S(M, S(-n)) = \text{Ker} \partial^*_0 / \text{Im} \varphi_1^* \\
& \cong \Omega_1(M)^* / (F_0^*/\varphi_0^*(M^*)) \\
& = N^\vee
\end{align*}
\]
as required. \( \square \)
Corollary 1.2. Let $M$ be a maximal generalized CM module satisfying Theorem (i). Then its minimal free resolution is in the form of

$$0 \to K_n \xrightarrow{\partial_n} K_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_{t+1}} K_{t+3} \xrightarrow{\partial_{t+3}} F_1 \xrightarrow{\varphi} F_0 \to M \to 0.$$  

Notice that $F_1$ is a $S$-free module containing a submodule isomorphic to $E_{t+3}$.

Example 1.3. Let $M = E_{t+1} \oplus E_{n-1}$. Then $H^i_m(M)$ is as in Theorem (i) with $N = K$. Since $M$ has a minimal free resolution $0 \to K_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{t+1}} K_{t+3} \xrightarrow{\partial_{t+3}} K_{t+2} \oplus K_n \xrightarrow{\partial_{t+2} \oplus \partial_n} K_{t+1} \oplus K_{n-1} \xrightarrow{\partial_{t+1} \oplus \partial_{n-1}} M \to 0$, we have $\Omega_1(M) = E_{t+2} \oplus E_n \cong G/E_{t+3}$ where $G = K_{n+2} \oplus E_n$. Thus we have Theorem (ii)(a). On the other hand, we have $\Omega_1(M)^* = E_{t+2} \oplus E_n^* \cong E_{n-t-1} \oplus S$ by selfduality of Koszul complex and $E_n \cong S(-n)$. Again by selfduality we have $(K_{t+1} \oplus K_{n-1})^*/M^* = (K_{t+1} \oplus K_{n-1})^*/(E_{t+1} \oplus E_{n-1})^* = (K_{n-t-1} \oplus K_{1})/(E_{n-t} \oplus E_2) = (K_{n-t-1} \oplus E_{n-1}) \oplus (K_{1} \oplus E_2) = E_{n-t-1} \oplus E_1 = E_{n-t} \oplus m$. Thus $\Omega_1(M)^*/(K_{t+1} \oplus K_{n-1})^*/M^* \cong S/m \cong K \cong K^\vee$ and we obtain Theorem (ii)(b).

2. LONG BOURBAKI SEQUENCES WITH APPROXIMATION MODULE $E_{t+1} \oplus E_{n-1}(d)$

In the last chapter, we considered approximation modules $M$ satisfying the condition of Theorem (i). We now focus on a special case of $M = E_{t+1} \oplus E_{n-1}(d)$, and investigate the long Bourbaki sequences.

We will use the following well known result frequently without referring it. First of all, we will give a proof for the readers’ convenience.

**Lemma 2.1.** For the $t$-th syzygy module of $K$ over $S$, we have

$$\text{rank } E_t = \binom{n-1}{t-1}$$

**Proof.** Let $K_i$ be the $i$th Koszul complex. Then we have an exact sequence

$$0 \to K_n \xrightarrow{\partial_n} K_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_t} K_t \to E_t \to 0$$

so that

$$\text{rank}(E_t) = \sum_{i=t}^{n} (-1)^{i-t} \text{rank}(K_i) = (-1)^t \sum_{i=t}^{n} (-1)^i \binom{n}{i} = (-1)^{t-1} \sum_{i=0}^{t-1} (-1)^i \binom{n}{i}.$$  

Now set $\alpha(n, t) := \text{rank}(E_t)$. By a straightforward calculation, we have $\alpha(n, t) = \alpha(n-1, t) = \alpha(n-1, t-1)$. Thus we have

$$\alpha(n, t) = \alpha(n-1, t) + \alpha(n-1, t-1)$$

$$= (-1)^{t-1} \sum_{i=0}^{t-1} (-1)^i \binom{n-1}{i} + (-1)^{t-2} \sum_{i=0}^{t-2} (-1)^i \binom{n-1}{i} = \binom{n-1}{t-1}.$$  

as required. \qed

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2.1. Characterization of long Bourbaki Sequences. For $J, K \subset [n] = \{1, 2, \ldots, n\}$ with $J \cap K = \emptyset$ we define $\sigma(J, K) = (-1)^i$ where $i = \sharp \{(j, k) \in J \times K \mid j > k\}$.
Then we have $x_J \wedge x_K = \sigma(J, K)x_{J \cup K}$.

Now a long Bourbaki sequence with approximation module $E_{t+1} \oplus E_{n-1}(d)$ is characterized by suitable sequences from $K_{t+1} \oplus K_{n-1}$ and its dual. Namely,

**Theorem 2.2.** Following are equivalent.

(i) We have a long Bourbaki sequence

$$0 \longrightarrow \bigoplus_{i=1}^{p} S(-a_i) \longrightarrow \bigoplus_{i=1}^{q} S(-b_i) \longrightarrow E_{t+1} \oplus E_{n-1}(d) \longrightarrow I(c) \longrightarrow 0 \quad \text{(exact)}$$

where $I \subset S$ is a graded ideal.

(ii) We have $\beta_1, \ldots, \beta_q \in K_{t+1} \oplus K_{n-1}(d) \backslash E_{t+2} \oplus E_n(d)$ and $\varphi = (a, b)$ with $a \in A$ and $b \in B$, where

$$A = \langle \sum_{j=1}^{n-t} (-1)^{j+1} \sigma(L \backslash \{i_j\}, ([n] \setminus L) \cup \{i_j\}) x_{i_1} e_{[n] \setminus L \cup \{i_j\}}^* \mid L = \{i_1, \ldots, i_{n-t}\} \subset [n] \rangle$$

$$B = \langle (-1)^i x_{i_j} e_{[n] \setminus \{i_j\}}^* \rangle - (-1)^i x_{i_j} e_{[n] \setminus \{i_j\}}^* \rangle \mid 1 \leq i < j \leq n \rangle,$$

such that

(a) $\varphi : K_{t+1} \oplus K_{n-1}(d) \rightarrow S(-n)$ is a degree 'n + c' homomorphism and $\text{Ker}(\varphi) = \langle \beta_1, \ldots, \beta_q \rangle + E_{t+2} \oplus E_n(d)$, and

(b) we have the following diagram, with $p = q - n + 2 - \binom{n-1}{t}$

$$\begin{array}{cccc}
0 & 0 & & \\
& & \downarrow & \\
0 & \longrightarrow & \text{Ker Res}(\beta) & \longrightarrow & \text{Ker } \beta & \longrightarrow & 0 \\
& & \downarrow & \\
0 & \longrightarrow & \bigoplus_{i=1}^{p} S(-a_i) & \longrightarrow & \bigoplus_{i=1}^{q} S(-b_i) \\
& \downarrow_{\text{Res}(\beta)} & & \downarrow_{\beta} & \\
0 & \longrightarrow & \langle \beta_1, \ldots, \beta_q \rangle \cap E_{t+2} \oplus E_n(d) & \longrightarrow & \langle \beta_1, \ldots, \beta_q \rangle \\
& & \downarrow & \\
& & 0 & 0 & \\
\end{array}$$

where $\beta(g_i) = \beta_i$ for all $i$ with $g_1, \ldots, g_q$ the free basis of $\bigoplus_{i=1}^{q} S(-b_i)$, and $\text{Res}(\cdot)$ denotes the restriction of maps.

In this case, we have $I = \varphi(K_{t+1} \oplus K_{n-1}(d))(-c)$.
Proof. We first prove (ii) to (i). First notice that, by the selfduality of Koszul complex, we have

\[ E_i \cong E_{n-i+1}^* = \partial_i^* E_{n-i+1}^* \]

\[ = \left\{ \sum_{k=1}^{i} (-1)^{k+1} \sigma(J \setminus \{ j_k \}, [n] - (J \setminus \{ j_k \})) x_{j_k} e_{[n]- (J \setminus \{ j_k \})} : J = \{ j_1, \ldots, j_i \} \subset [n] \right\}. \]

See \[3\] Chapter 1.6. Thus \( \text{A} = \partial_{t+1}^* E_{t+1}^* \) and \( \text{B} = \partial_{n-1}^* E_{n-1}^* \). Then, there exists \( \bar{a} \in E_{t+1}^* \) and \( \bar{b} \in E_{n-1}^* \) such that \( a = \bar{a} \circ \partial_{t+1} \) and \( b = \bar{b} \circ \partial_{n-1} \). Then by (a) we have the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \langle \beta_1, \ldots, \beta_q \rangle + E_{t+2} \oplus E_n(d) & \rightarrow & K_{t+1} \oplus K_{n-1}(d) & \rightarrow \phi \rightarrow S(c) \\
\delta & \downarrow & \downarrow & & \downarrow & \\
0 & \rightarrow & \langle \bar{\beta}_1, \ldots, \bar{\beta}_q \rangle & \rightarrow & E_{t+1} \oplus E_{n-1}(d) & \rightarrow \phi \rightarrow S(c)
\end{array}
\]

(5)

where \( \phi = (\bar{a}, \bar{b}) \) and \( \bar{\beta} := \partial_{t+1} \oplus \partial_{n-1}(d) \). On the other hand, we have

\[ \text{Ker} \bar{\beta} \circ \beta = \left\{ \sum_{i=1}^{q} h_i g_i \mid \bar{\beta}(\sum_{i=1}^{q} h_i \beta_i) = 0, h_i \in S \right\} \]

\[ = \left\{ \sum_{i=1}^{q} h_i g_i \mid \sum_{i=1}^{q} h_i \beta_i \in \text{Ker} \bar{\beta} = E_{t+2} \oplus E_n(d), h_i \in S \right\} \]

\[ = \left\{ \sum_{i=1}^{q} h_i g_i \mid \beta(\sum_{i=1}^{q} h_i g_i) \in \langle \beta_1, \ldots, \beta_q \rangle \cap E_{t+2} \oplus E_n(d) \right\} \]

\[ \cong \bigoplus_{i=1}^{p} S(-a_i) \]

where the last isomorphism is by (b). Notice that let \( u \in \bigoplus_{i=1}^{q} S(-b_i) \) be such that \( \beta(u) \in \langle \beta_1, \ldots, \beta_q \rangle \cap E_{t+2} \oplus E_n(d) \). Then \( u \) must be in \( \bigoplus_{i=1}^{p} S(-a_i) \). In fact, by (b) we can choose \( v \in \bigoplus_{i=1}^{p} S(-a_i) \) such that \( \text{Res}(\beta)(v) = \beta(u) \). Thus \( u - v \in \text{Ker} \beta \cong \text{Ker} \text{Res}(\beta) \), and we have \( u + v \in \text{Ker} \text{Res}(\beta) \subset \bigoplus_{i=1}^{p} S(-a_i) \) as required. Then by (3) we obtain

\[ 0 \rightarrow \bigoplus_{i=1}^{p} S(-a_i) \rightarrow \bigoplus_{i=1}^{q} S(-b_i) \rightarrow E_{t+1} \oplus E_{n-1}(d) \rightarrow \phi \rightarrow S(c) \quad \text{(exact)} \]

and since \( p = q - n + 2 - \binom{n-1}{t} \) we know that rank \( \text{Im} \psi = 1 \) so that we have

\[ 0 \rightarrow \bigoplus_{i=1}^{p} S(-a_i) \rightarrow \bigoplus_{i=1}^{q} S(-b_i) \rightarrow E_{t+1} \oplus E_{n-1} \rightarrow \phi \rightarrow I(c) \rightarrow 0 \quad \text{(exact)} \]

for the ideal \( I := \phi(E_{t+1} \oplus E_{n-1}(d))(-c) = \varphi(K_{t+1} \oplus K_{n-1}(d))(-c) \) as required.
Next we prove (i) to (ii). Given a long Bourbaki sequence
\[
0 \rightarrow \bigoplus_{i=1}^{p} S(-a_i) \xrightarrow{f} \bigoplus_{i=1}^{q} S(-b_i) \xrightarrow{g} E_{t+1} \oplus E_{n-1}(d) \xrightarrow{\phi} I(c) \rightarrow 0 \quad \text{(exact)}
\]

with a graded ideal \( I \subset S \). Then we have \( p = q - n + 2 + \binom{n-1}{t} \) since \( 1 = \text{rank } I(c) = \text{rank } E_{t+1} \oplus E_{n-1}(d) - q + p \). Also since \( I(c) \subset S(c) \) we have
\[
\phi \in \text{Hom}_S(E_{t+1} \oplus E_{n-1}(d), S) = \text{Hom}_S(E_{t+1} \oplus E_{n-1}(d), S(-n))(n + c)
\]

Thus there exists a unique \((a, b) \in \partial_{t+1}^* (E_{t+1}^*) \oplus \partial_{n-1}^* (E_{n-1}^*(-d))\) such that \( \phi \circ \bar{\partial} = (a, b) \). Now we set \( \varphi = (a, b) \). Then \( E_{t+2} \oplus E_n(d) + N = \bar{\partial}^{-1}(\text{Ker } \phi)(\subset K_{t+1} \oplus K_{n-1}(d)) \) for some module \( N(\neq 0) \). Let \( \{\beta_1, \ldots, \beta_q\} \) be a minimal set of generators of \( N \). Then we have the following diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \bigoplus_{i=1}^{q} S(-b_i) & \xrightarrow{g} & E_{t+1} \oplus E_{n-1}(d) & \xrightarrow{\phi} & I(c) & \rightarrow & 0 \\
& & \downarrow \beta & & \leftarrow \delta & & \\
0 & \rightarrow & \langle \beta_1, \ldots, \beta_q \rangle & \rightarrow & K_{t+1} \oplus K_{n-1}(d) & \xrightarrow{\varphi=(a,b)} & I(c) & \leftarrow \\
& & \downarrow & & & & \\
& & & & 0 & & \\
\end{array}
\]

Then we have obtained \( \beta_1, \ldots, \beta_q \in K_{t+1} \oplus K_{n-1}(d) \) and \( (a, b) \in A \times B \) satisfying the condition (a) and the numerical condition on \( p, q, t \) and \( n \) in (b). Also we have \( I = \varphi(K_{t+1} \oplus K_{n-1}(d))(-c) \).
Now since \( \text{Ker }g = \text{Ker}(\text{Res}(\partial) \circ \beta) = \text{Im }f = \bigoplus_{i=1}^{p} S(-a_i) \) we readily have the following diagram:

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \text{Ker }\beta \circ f & \longrightarrow & \text{Ker }\beta & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \bigoplus_{i=1}^{p} S(-a_i) & \longrightarrow & \bigoplus_{i=1}^{q} S(-b_i) & \longrightarrow & E_{t+1} \oplus E_{n-1}(d) \\
\downarrow & \beta & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \langle \beta_1, \ldots, \beta_q \rangle \cap E_{t+2} \oplus E_n(d) & \longrightarrow & \langle \beta_1, \ldots, \beta_q \rangle & \longrightarrow & E_{t+1} \oplus E_{n-1}(d) \\
& & & & \downarrow & \downarrow \\
& & & & 0 & \text{exact} \\
\end{array}
\]

Notice that since \( \text{Ker }\beta \subset \text{Ker}(\text{Res}(\partial) \circ \beta) = \text{Ker }g = \text{Im }f \) we have the exactness of the first row. Since \( \text{Im}(\beta \circ f) = \beta(\text{Ker}(\text{Red}(\partial) \circ \beta)) = \text{Ker}(\text{Res}(\partial)) = \langle \beta_1, \ldots, \beta_q \rangle \cap E_{t+2} \oplus E_n(d) \), we have a well-defined surjection

\[ \beta \circ f : \bigoplus_{i=1}^{p} S(-a_i) \longrightarrow \langle \beta_1, \ldots, \beta_q \rangle \cap E_{t+2} \oplus E_n(d). \]

Thus we obtained the diagram of \((b)\) as required. \( \square \)

For any codimension 3 single spot ideal of type \((t, K(-c))\), there exists a long Bourbaki sequence with approximation module \( E_{t+1} \oplus H \) where \( H \) is a \( S \)-free module \([4, 1]\). The following is the case of \( H = 0 \), which can be proved with the same idea as that of Theorem 2.2.

**Theorem 2.3.** Following are equivalent.

(i) We have a long Bourbaki sequence

\[ 0 \longrightarrow \bigoplus_{i=1}^{p} S(-a_i) \longrightarrow \bigoplus_{i=1}^{q} S(-b_i) \longrightarrow E_{t+1} \longrightarrow I(c) \longrightarrow 0 \quad \text{(exact)} \]

where \( I \subset S \) is a graded ideal.

(ii) We have \( \beta_1, \ldots, \beta_q \in K_{t+1} \setminus E_{t+2} \) and \( \varphi \in A \) where

\[
A = \left\{ \sum_{j=1}^{n-t} (-1)^{j+1} \sigma(L \setminus \{i_j\}, ([n] \setminus L) \cup \{i_j\}) x_{i_j} e^*_{([n] \setminus L) \cup \{i_j\}} \mid L = \{i_1, \ldots, i_{n-t}\} \subset [n] \right\}
\]

such that

(a) \( \varphi : K_{t+1} \to S(-n) \) defines a degree \( n + c \) homomorphism and \( \text{Ker }\varphi = \langle \beta_1, \ldots, \beta_q \rangle + E_{t+1} \), and
(b) we we have the following diagram, with \( p = q + 1 - \binom{n-1}{t} \)

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Ker } \text{Res}(\beta) & \rightarrow & \text{Ker } \beta & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \bigoplus_{i=1}^{p} S(-a_i) & \rightarrow & \bigoplus_{i=1}^{q} S(-b_i) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \langle \beta_1, \ldots, \beta_q \rangle \cap E_{t+2} & \rightarrow & \langle \beta_1, \ldots, \beta_q \rangle & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & & & & & 0
\end{array}
\]

where \( \beta(g_i) = \beta_i \) for all \( i \) with \( g_1, \ldots, g_q \) the free basis of \( \bigoplus_{i=1}^{q} S(-b_i) \).

In this case, we have \( I = \varphi(K_{t+1})(n-c) \).

**Corollary 2.4.** There is no codimension 3 single spot ideal \( I \) of type \((0, K(-c))\) with approximation module \( E_{t+1} \) if \( t = 0 \).

**Proof.** Assume that there exists a codimension 3 single spot ideal \( I \subset S \) of type \((0, K(-c))\) fitting into a long Bourbaki sequence

\[
0 \rightarrow F \rightarrow G \rightarrow E_1 \rightarrow I(c) \rightarrow 0.
\]

Then by Theorem 2.3, there exist \( \beta_1, \ldots, \beta_q \in K_1 \setminus E_2 \) and \( (0 \neq) \varphi \in A \) such that \( \langle \beta_1, \ldots, \beta_q \rangle + E_2 = \text{Ker}(\varphi : K_1 \rightarrow S(-n)) \). Since \( A = \langle x_1e_1^* + \cdots + x_ne_n^* \rangle \) we must have \( \beta_1, \ldots, \beta_q \in E_2 \), a contradiction. \( \square \)

**Remark 2.5.** In fact, the above Corollary holds for any codimension \( \geq 2 \). Here is an outline of the proof. We consider a straightforward extension of Theorem 2.3(a) to any codimension \( r \). In this case we consider long Bourbaki sequences

\[
0 \rightarrow F_{r-1} \rightarrow F_{r-2} \rightarrow \cdots \rightarrow F_1 \rightarrow E_1 \xrightarrow{\varphi} I(c) \rightarrow 0.
\]

and \( \varphi \) is determined by a nonzero element from \( A = \langle x_1e_1^* + \cdots + x_ne_n^* \rangle \).

### 2.2. Long Bourbaki sequences of non-trivial type.

Let \( n \geq 4 \) and \( t \leq n-4 \) and consider a long Bourbaki sequence

\[
0 \rightarrow F \xrightarrow{f} G \xrightarrow{g} E_{t+1} \oplus E_{n-1}(d) \xrightarrow{\phi} I(c) \rightarrow 0
\]

with \( c \in \mathbb{Z} \) and \( S \)-free modules \( F \) and \( G \). From this sequence, we construct the following diagram where the second row is the minimal free resolution of \( M = E_{t+1} \oplus E_{n-1}(d) \) and the third row is the mapping cone \( C(\alpha, \beta) \) of a chain map \( \alpha \oplus \beta \),
which is a free resolution of $I(c)$.

\[(7)\]

\[
\begin{array}{c}
0 \longrightarrow F \xrightarrow{f} G \xrightarrow{g} \ker \phi \longrightarrow 0 \\
\alpha \downarrow \quad \beta \downarrow \quad \phi \downarrow
\end{array}
\]

\[
0 \xrightarrow{} K_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{n+4}} K_{t+3} \xrightarrow{\partial_{t+3}} K_{t+2} \oplus K_n(d) \xrightarrow{\partial_{t+2}} K_{t+1} \oplus K_{n-1}(d) \xrightarrow{\partial_{t+1}} M \longrightarrow 0
\]

\[
0 \xrightarrow{} K_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{n+4}} F \oplus K_{t+3} \xrightarrow{\zeta} K_{t+2} \oplus K_n(d) \xrightarrow{\rho} K_{t+1} \oplus K_{n-1}(d) \xrightarrow{\phi} I(c) \longrightarrow 0
\]

where

\[
\rho : K_{t+2} \oplus K_n(d) \oplus G \xrightarrow{(a, b, c)} K_{t+1} \oplus K_{n-1}(d) \quad \xrightarrow{(\partial_{t+2}(a), \partial_n(b)) - \beta(c)}
\]

and

\[
\zeta : K_{t+3} \oplus F \xrightarrow{(a, b)} K_{t+2} \oplus K_n(d) \oplus G \quad \xrightarrow{(\partial_{t+3}(a), 0, f(b)) - \alpha(b)}
\]

Let $p_1 : K_{t+1} \oplus K_{n-1}(d) \rightarrow K_{t+1}$ and $p_2 : K_{t+1} \oplus K_{n-1}(d) \rightarrow K_{n-1}(d)$ be the first and the second projections. From the diagram \((7)\) we know $\ker \phi = \im g = (\partial_{t+1} \oplus \partial_{n-1}) \circ \beta(G)$ and then by considering the ranks of the modules in the short exact sequence

\[(8)\]

\[
0 \longrightarrow \ker \phi \longrightarrow E_{t+1} \oplus E_{n-1}(d) \xrightarrow{\phi} I(c) \longrightarrow 0
\]

we have

\[(9)\]

\[
\rank(\ker \varphi) = n - 2 + \binom{n-1}{t}.
\]

On the other hand, we have

\[(10)\]

\[
(\partial_{t+1} \circ p_1 \circ \beta)(G) \oplus (\partial_{n-1} \circ p_2 \circ \beta)(G) \supset (\partial_{t+1} \oplus \partial_{n-1}) \beta(G) = \ker \phi.
\]

Thus we have

\[(11)\]

\[
\rank I_{t+1} + \rank I_{n-1} \geq n - 2 + \binom{n-1}{t}.
\]

where $I_{t+1} := (\partial_{t+1} \circ p_1 \circ \beta)(G)(\subseteq E_{t+1})$ and $I_{n-1} := (\partial_{n-1} \circ p_2 \circ \beta)(G)(\subseteq E_{n-1}(d))$.

Since $\rank E_{t+1} = \binom{n-1}{t}$ and $\rank E_{n-1}(d) = n - 1$ we know from \((11)\) that

\[
(\rank I_{t+1}, \rank I_{n-1}) = (\binom{n-1}{t} - 1, n-1), (\binom{n-1}{t}, n-1), \text{ or } (\binom{n-1}{t}, n-2).
\]

Under this situation, we have

**Lemma 2.6.** Following are equivalent.
Proof. We will prove (i) to (ii). We assume that for all \( i \) we have either \( \beta(m_i) \in K_{t+1} \) or \( \beta(m_i) \in K_{n-1}(d) \) and will deduce a contradiction. First of all, we have equality in (10), and then from (11) we have

\[
\text{rank } \cdot (n-1) + \text{rank } \cdot (n+1) - 2 + \binom{n-1}{t} = \text{rank } I_{n+1} + \text{rank } I_{n-1}.
\]

Thus, we have \((\text{rank } I_{n+1}, \text{rank } I_{n-1}) = ((n-1), (n-1)) \) or \((n-1), (n-2)) \). Also, since \( \text{Ker } I = I_{n+1} \oplus I_{n-1} \), we have by (12)

\[
I(c) \cong (E_{t+1}/I_{t+1}) \oplus (E_{n-1}(d)/I_{n-1})
\]

**case** \((\text{rank } I_{n+1}, \text{rank } I_{n-1}) = ((n-1), (n-1))\): Since we have \( \text{rank } E_{n-1}(d)/I_{n-1} = \text{rank } E_{n-1}(d) - \text{rank } I_{n-1} = 0 \), \( E_{n-1}/I_{n-1} \) is 0 or a torsion-module. But since \( I(c) \) is torsion-free, we must have \( E_{n-1}(d)/I_{n-1} \) by (12). Thus \( \text{Ker } I = I_{n+1} \oplus E_{n-1}(d) \) and then the Bourbaki sequence (6) must be of trivial-type

\[
0 \rightarrow F' \oplus K_n \overset{f' \oplus \partial_n}{\rightarrow} G' \oplus K_{n-1} \overset{\partial_n \partial_{n-1}}{\rightarrow} E_{t+1} \oplus E_{n-1}(d) \overset{\phi}{\rightarrow} I(c) \rightarrow 0
\]

where \( 0 \rightarrow F' \rightarrow G' \rightarrow I_{t+1} \rightarrow 0 \) is a \( S \)-free resolution of \( I_{t+1} \), a contradiction.

**case** \((\text{rank } I_{n+1}, \text{rank } I_{n-1}) = ((n-1), (n-2))\): In this case we have \( \text{rank } E_{t+1}/I_{t+1} = 0 \). Since \( E_{t+1}/I_{t+1} \subset I(c) \) and \( I(c) \) is torsion-free, we must have \( E_{t+1}/I_{t+1} = 0 \). Thus \( \text{Ker } I = E_{t+1} \oplus I_{n-1} \) and the Bourbaki sequence (6) is

\[
0 \rightarrow K_n \oplus U_n \overset{\partial_n \oplus \partial_n}{\rightarrow} \cdots \overset{\partial_n \oplus \partial_{n-1}}{\rightarrow} K_{t+1} \oplus U_{t+1} \overset{\partial_{t+1} \oplus \partial_{t+1}}{\rightarrow} E_{t+1} \oplus E_{n-1} \overset{\phi}{\rightarrow} I(c) \rightarrow 0.
\]

But then we must have \( t \geq n - 2 \), which contradicts to the assumption that \( t \leq n - 4 \).

Now we show (ii) to (i). Assume that (6) is of trivial type. Then we must have \( \beta(G) = p_1(\beta(G)) \oplus p_2(\beta(G)) \). From this we immediately obtain the required result. \( \square \)

From Lemma 2.6 we immediately have

**Theorem 2.7.** Under the situation of Theorem 2.6, the long Bourbaki sequence is of non-trivial type if and only if

(i) the condition Theorem 2.6 (ii) holds, and
(ii) the submodule \( N := \langle \beta_1, \ldots, \beta_q \rangle \) of \( K_{t+1} \oplus K_{n-1}(d) \) cannot be decomposed in the form of \( N = A \oplus B \) for some \((0 \neq) A \subset K_{t+1} \) and \((0 \neq) B \subset K_{n-1}(d)\)
3. Numerical Characterizations

Existence of long Bourbaki sequence as in Theorem 2.2 and 2.3 only implies that \( I \) is a single spot ideal of codimension at most 3. To assure that the codimension is exactly 3, we need additional condition. In this section, we give a numerical condition to assure \( \text{codim } I = 3 \) for long Bourbaki sequences with approximation modules \( E_{t+1} \oplus E_{n-1} \).

We assume \( n \geq 4 \) and \( t \leq n-4 \), and let \( I \subset S \) be a graded ideal fitting into a long Bourbaki sequence

\[
0 \rightarrow F \xrightarrow{f} G \xrightarrow{g} M \xrightarrow{\phi} I \rightarrow 0
\]

with \( M = E_{t+1} \oplus E_{n-1} \), \( F = \bigoplus_{i=1}^{p} S(-a_i) \) and \( G = \bigoplus_{i=1}^{q} S(-b_i) \) are \( S \)-free modules and \( c \in \mathbb{Z} \). As in Theorem 2.2 we have

\[
q = p + \binom{n-1}{t} + n - 2.
\]

Now from the sequence (13), we construct the mapping cone \( C(\alpha, \beta) \) as in (7). The cone gives a \( S \)-free resolution \( F_* \) of the residue ring \( S/I \).

\[
F_* : 0 \rightarrow F_{n-t} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I \rightarrow 0
\]

where

\[
\begin{align*}
F_0 & = S \\
F_1 & = K_{t+1}(-c) \oplus K_{n-1}(d-c) = S(-t-1-c)^{\beta_1} \oplus S(-n+1+d-c)^n \\
F_2 & = K_{t+2}(-c) \oplus K_n(d-c) \oplus G(-c) \\
& = S(-t-2-c)^{\beta_2} \oplus S(-n+d-c) \oplus \bigoplus_{i=1}^{q} S(-b_i-c) \\
F_3 & = K_{t+3}(-c) \oplus F(-c) = S(-t-3-c)^{\beta_3} \oplus \bigoplus_{i=1}^{p} S(-a_i-c) \\
F_i & = K_{t+i}(-c) = S(-t-i-c)^{\beta_i} \quad (4 \leq i \leq n-t)
\end{align*}
\]

with \( \beta_i = \binom{n}{i+

\begin{align*}
\text{Notice that this resolution is minimal if and only if matrix representations of } \alpha \text{ and } \beta \text{ only have their entries from } \mathfrak{m}. \\
\text{Now we compute the Hilbert series } H(\mathbb{Z}/I, \lambda) \text{ of } S/I. \text{ We have}
\end{align*}

\[
H(\mathbb{Z}/I, \lambda) = \frac{Q(\lambda)}{(1-\lambda)^n}
\]

with

\[
Q(\lambda) = \sum_{i,j} (-1)^i \beta_{i,j} \lambda^j = 1 - n\lambda^{n-1+c-d} + \lambda^{n+c-d} + \sum_{i=1}^{q} \lambda^{b_i-c} - \sum_{i=1}^{p} \lambda^{a_i+c} + (-1)^t \lambda^c \sum_{i=t+1}^{n} \binom{n}{i}(-1)^i \lambda^i
\]
where $\beta_{i,j}$ are as in $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$, $(i = 0, \ldots, n - t)$. (see Lemma 4.1.13 [3]).

Since we have $H_i^m(S/I) = H_{i+1}^m(M)$ for $0 \leq i \leq n - 4$ by [1], we know that $\dim S/I \geq n - 3$, i.e., $\text{codim} I \leq 3$. To assure that $\text{codim} I \geq 3$ we must have $Q(1) = Q'(1) = Q''(1) = 0$ (see Corollary 4.1.14(a) [3]).

**Proposition 3.1.** $Q(1) = 0$ holds for all $n, t, c$ and $p$.

**Proof.** We compute using (14)

$$Q(1) = \left(\binom{n-1}{t}\right) - (-1)^{t+1} \sum_{i=t+1}^{n} (-1)^i \binom{n}{i} = \left(\binom{n-1}{t}\right) - \text{rank} E_{t+1} = 0.$$ 

where the last equation follows from the Koszul resolution of $E_{t+1}$:

$$0 \rightarrow K_n \rightarrow K_{n-1} \rightarrow \cdots \rightarrow K_{t+1} \rightarrow E_{t+1} \rightarrow 0 \ (\text{exact}).$$

**Proposition 3.2.** $Q'(1) = 0$ holds if and only if

$$\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i = n^2 - (2 + d)n + c + d + \binom{n-2}{t-1} + \binom{n-1}{t} t.$$

**Proof.** We compute using (14)

$$Q'(1) = -n^2 + (2 + d)n - c - d + \sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i - (-1)^{t+1} \sum_{i=t+1}^{n} (-1)^i \binom{n}{i} i.$$

and we know

$$(-1)^{t+1} \sum_{i=t+1}^{n} (-1)^i \binom{n}{i} i = (-1)^{t} \sum_{i=0}^{t} (-1)^i \binom{n}{i} i = \binom{n-2}{t-1} + \binom{n-1}{t} t$$

where the last equation is given in Example 2.3 [4].

Now before we go further, we need to show a combinatorial equation.

**Lemma 3.3.**

$$(-1)^{t+1} \sum_{i=t+1}^{n} (-1)^i \binom{n}{i} i^2 = \binom{n-1}{t} (t+1)^2 - \binom{n-2}{t} (2t+1) - 2 \binom{n-3}{t-1}$$

**Proof.** Let $A = (-1)^{t+1} \sum_{i=t+1}^{n} (-1)^i i^2$. A straightforward computation using the binomial coefficient theorem and (16) shows that

$$0 = \sum_{i=2}^{n} (-1)^i \binom{n}{i} i(i-1)$$

$$= (-1)^{t+1} A + \sum_{i=2}^{t} (-1)^i \binom{n}{i} i(i-1) - (-1)^{t+1} \left[ \binom{n-2}{t-1} + \binom{n-1}{t} t \right]$$
Thus we have

\[ A = \binom{n-2}{t-1} + \binom{n-1}{t}t + (-1)^t \sum_{i=2}^{t} (-1)^i \binom{n}{i} i(i-1) \]  

Now we will compute the last term. Set

\[ \alpha(n, t) = (-1)^t \sum_{i=2}^{t} (-1)^i \binom{n}{i} i(i-1). \]

and we compute

\[
\begin{align*}
\alpha(n, t) + \alpha(n - 1, t - 1) &= 2t^2 \binom{n-1}{t} - 2(-1)^t \sum_{i=2}^{t} (-1)^i \binom{n-1}{i} i + 2(-1)^t(n - 1) - \alpha(n - 1, t).
\end{align*}
\]

Also we have

\[
\alpha(n - 1, t - 1) + \alpha(n - 1, t) = t(t-1) \binom{n-1}{t}.
\]

Thus by using (16)

\[
\begin{align*}
\alpha(n, t) &= t(t + 1) \binom{n-1}{t} - 2(-1)^t \sum_{i=2}^{t} (-1)^i \binom{n-1}{i} i + 2(-1)^t(n - 1) \\
&= t(t + 1) \binom{n-1}{t} - 2 \left[ t \binom{n-2}{t} + \binom{n-3}{t-1} + (-1)^t(n - 1) \right] \\
&\quad + 2(-1)^t(n - 1) \\
&= t(t + 1) \binom{n-1}{t} - 2t \binom{n-2}{t} - 2 \binom{n-3}{t-1}.
\end{align*}
\]

Substiting this into (17), we obtain the desired result. \( \square \)

**Proposition 3.4.** \( Q''(1) = 0 \) holds if and only if

\[
\sum_{i=1}^{q} b_i^2 - \sum_{i=1}^{p} a_i^2 = n^3 - (3 + 2d)n^2 + (d^2 + 4d + 1)n - c^2 - d^2 + \binom{n-1}{t}(t + 1)^2 - \binom{n-2}{t}(2t + 1) - 2 \binom{n-3}{t-1}.
\]
Proof. We compute
\[
Q''(1) = c + d - c^2 + d^2 - 2cd + (4c - 3 - d^2 - 5d + 2cd)n + (4 - 2c + 2d)n^2 - n^3 \\
- (2c - 1) \left[ \binom{n-2}{t-1} + \binom{n-1}{t} \right] + \sum_{i=1}^{q} b_i^2 - \sum_{i=1}^{p} a_i^2 \\
+ (2c - 1) \left( \sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i \right) + (-1)^t \sum_{i=t+1}^{n} (-1)^i \binom{n}{i} i^2 \\
= c^2 + d^2 - (d^2 + 4d + 1)n + (3 + 2d)n^2 - n^3 \\
+ \sum_{i=1}^{q} b_i^2 - \sum_{i=1}^{p} a_i^2 + (-1)^t \sum_{i=t+1}^{n} (-1)^i \binom{n}{i} i^2
\]
where the last equation is by Proposition 3.2. Then by Lemma 3.3 we obtain the desired result. □

To summarize, we obtain

**Theorem 3.5.** Let \( n \geq 4 \) and \( t \leq n - 4 \). Assume that we have the following long Bourbaki sequence

\[
0 \rightarrow \bigoplus_{i=1}^{p} S(-a_i) \rightarrow \bigoplus_{i=1}^{q} S(-b_i) \rightarrow E_{t+1} \oplus E_{n-1}(d) \rightarrow I(c) \rightarrow 0
\]

with \( I \subset S \) a graded ideal and \( c \in \mathbb{Z} \). Then we have \( \text{codim} I \leq 3 \) and the equality holds if and only if

1. \( q = p + \binom{n-1}{t-1} + n - 2; \)
2. \( \sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i = n^2 - (2 + d)n + c + d + \binom{n-2}{t-1} + \binom{n-1}{t}t; \)
3. \( \sum_{i=1}^{q} b_i^2 - \sum_{i=1}^{p} a_i^2 = n^3 - (3 + 2d)n^2 + (d^2 + 4d + 1)n - c^2 - d^2 \)

\( (n-1)^t - \binom{n-1}{t} \binom{n-2}{t-1} - \binom{n-2}{t}(2t+1) - 2 \binom{n-3}{t-1} \)

4. **Examples**

**Example 4.1.** We first give an application of Theorem 2.3. Namely, a single spot ideal \( I \) with approximation module \( E_{t+1} \). Let \( t = 1 \) and \( n = 6 \). Then \( A = \)
\{A_1, A_2, A_3, A_4, A_5, A_6\} = \partial_2^*(E_2^*) \subset K_2^* \text{ where}
\begin{align*}
A_1 &= x_1e_{16}^* + x_2e_{26}^* + x_3e_{36}^* + x_4e_{46}^* + x_5e_{56}^* \\
A_2 &= -x_1e_{15}^* - x_2e_{25}^* - x_3e_{35}^* - x_4e_{45}^* + x_6e_{66}^* \\
A_3 &= x_1e_{14}^* + x_2e_{24}^* + x_3e_{34}^* - x_5e_{54}^* - x_6e_{66}^* \\
A_4 &= -x_1e_{13}^* - x_2e_{23}^* + x_4e_{34}^* + x_5e_{54}^* + x_6e_{66}^* \\
A_5 &= x_1e_{12}^* - x_3e_{23}^* + x_4e_{34}^* + x_5e_{54}^* - x_6e_{66}^* \\
A_6 &= x_2e_{12}^* + x_3e_{13}^* + x_4e_{14}^* + x_5e_{15}^* + x_6e_{16}^*.
\end{align*}

Now let \(a \in A\) and \(\beta_i (i = 1, \ldots, 6)\) be as follows:
\begin{align*}
a &= x_6A_1 - x_5A_2 + x_4A_3 \\
&= x_1x_4e_{14}^* + x_1x_5e_{15}^* + x_1x_6e_{16}^* + x_2x_4e_{24}^* + x_2x_5e_{25}^* + x_2x_6e_{36}^* + x_3x_4e_{34}^* + x_3x_5e_{35}^* \\
&\quad + x_3x_6e_{66}^* \\
\beta_1 &= e_{12}, \quad \beta_2 = e_{13}, \quad \beta_3 = e_{23}, \quad \beta_4 = e_{45}, \quad \beta_5 = e_{46}, \quad \beta_6 = e_{56}\end{align*}

Then we have
\[\text{Ker}(a : K_2 \oplus K_5 \to S) = \langle \beta_1, \ldots, \beta_6 \rangle + E_3\]

and, for the map \(\beta : \bigoplus_{i=1}^6 S(-2) \to K_2 \oplus K_5\) such that \(\beta(m_i) = \beta_i (i = 1, \ldots, 6)\)
where \(\{m_i\}\) is a free basis, we obtain the diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \langle x_3m_1 - x_2m_2 + x_1m_3, \\
x_6m_4 - x_5m_6 + x_4m_6 \rangle & \to & \langle m_1, \ldots, m_6 \rangle \\
\downarrow & & \downarrow \\
\text{Res}(\beta) & & \beta \\
\downarrow & & \downarrow \\
0 & \to & \langle \beta_1, \ldots, \beta_6 \rangle \cap E_3 & \to & \langle \beta_1, \ldots, \beta_6 \rangle \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

and a defines a degree 0 homomorphism from \(E_2\) to \(S\). Then we obtain the long Bourbaki sequence
\[
0 \to S^2(-3) \xrightarrow{f} S^6(-2) \xrightarrow{g} E_2 \xrightarrow{\varphi} I \to 0
\]
where

\[ f : \quad S^2(-3) = S_{n1} \oplus S_{n2} \quad \rightarrow \quad S^6(-2) = S_{m1} \oplus \cdots \oplus S_{m6} \]

\[ n1 \quad \rightarrow \quad x_3 m_1 - x_2 m_2 + x_1 m_3 \]

\[ n2 \quad \rightarrow \quad x_6 m_4 - x_5 m_5 + x_4 m_6 \]

\[ g : \quad S^6(-2) = S_{m1} \oplus \cdots \oplus S_{m6} \quad \rightarrow \quad \partial_2(\beta_i) \quad (i = 1, \ldots, 6) \]

\[ E_2 \quad \rightarrow \quad \partial_2(e_{ij}) \quad \text{for} \quad (i, j) = (1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (5, 6) \]

\[ \partial_2(e_{ij}) \quad \text{for} \quad (i, j) \neq (1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (5, 6) \]

and we obtain \( I = (x_1 x_4, x_1 x_5, x_1 x_6, x_2 x_4, x_2 x_5, x_2 x_6, x_3 x_4, x_3 x_5, x_3 x_6) = (x_1, x_2, x_3)(x_4, x_5, x_6) \), a codimension 3 single spot ideal of type \((1, K)\).

**Example 4.2.** We continue to consider the situation in Example 4.1. As an application of Theorem 2.2, we can see that the same ideal fits into a long Bourbaki sequence with approximation module \( E_{i+1} \oplus E_{n-1} = E_1 \oplus E_5 \). In this case, we must also consider \( B = \{ B_{ij} \mid 1 \leq i < j \leq 6 \} = \partial_5(E_5^*) \subset K_5^* \) where \( B_{ij} = (-1)^i x_j e_{[6]}^{*i} - (-1)^j x_i e_{[6]}^{*j} \). Then we set \( a \in A \) as in Example 4.1 and

\[ b = -x_1^2 x_2 x_4 B_{14} = x_1^2 x_2 x_4^* e_{23456} + x_1^2 x_2 x_4^* e_{12356} \in B. \]

Also we set \( \beta_1, \ldots, \beta_6 \) to be the same as those in Example 4.1 and \( \beta_7 = x_1 x_2 x_4 e_{14} - e_{23456}, \beta_8 = x_1^2 x_2 e_{14} - e_{12356}, \beta_9 = e_{13456}, \beta_{10} = e_{12456}, \beta_{11} = e_{12356}, \beta_{12} = e_{12345}. \) Notice that \( \{ \beta_i \}_{i=1}^{12} \) satisfies the condition of Theorem 2.7. Then \( \varphi = (a, b) \) defines a degree 0 map on \( E_2 \oplus E_5 \), and we have

\[ \text{Ker } \varphi = \langle \beta_1, \ldots, \beta_{12} \rangle + E_3 \oplus E_6 \]

and the diagram

\[
\begin{array}{ccc}
0 & \quad \quad & 0 \\
\downarrow & \quad \quad & \downarrow \\
0 & \quad \quad & \downarrow \\
\quad \quad & \quad \quad & \quad \quad \\
\downarrow & \quad \quad & \downarrow \\
\beta & \quad \quad & \beta \\
\downarrow & \quad \quad & \downarrow \\
0 & \quad \quad & 0 \\
\end{array}
\]

Then we have a long Bourbaki sequence of non-trivial type

\[ 0 \rightarrow S^2(-3) \oplus S(-6) \xrightarrow{f} S^6(-2) \oplus S^6(-5) \xrightarrow{g} E_2 \oplus E_5 \xrightarrow{\varphi} I \rightarrow 0 \]
where

\[ f : \ S^2(-3) \oplus S(-6) = Sn_1 \oplus Sn_2 \oplus Sn_3 \quad \rightarrow \quad S^6(-2) = Sm_1 \oplus \cdots \oplus Sm_6 \]

\[ n_1 \quad \rightarrow \quad x_3m_4 - x_4m_8 + x_2m_9 \]
\[ n_2 \quad \rightarrow \quad -x_3m_{10} - x_5m_{11} + x_6m_{12} \]
\[ n_3 \quad \rightarrow \quad x_6m_4 - x_5m_5 + x_4m_6 \]

\[ g : \ S^6(-2) = Sm_1 \oplus \cdots \oplus Sm_6 \quad \rightarrow \quad \tilde{\partial}(\beta_i) \quad (i = 1, \ldots, 12) \]

\[ m_i \quad \rightarrow \quad \tilde{\partial} = \partial_2 \oplus \partial_5 \]

and

\[ \varphi \]
\[ E_2 \quad \rightarrow \quad I \]
\[ \partial_2(e_{ij}) \quad \rightarrow \quad x_i x_j \]
\[ \partial_2(e_{ij}) \quad \rightarrow \quad 0 \]
\[ \partial_5(e_{23456}) \quad \rightarrow \quad x_1^2 x_2 x_4^2 \]
\[ \partial_5(e_{12356}) \quad \rightarrow \quad x_1^2 x_3 x_4 \]
\[ \partial_5(e_{ijklm}) \quad \rightarrow \quad 0 \text{ otherwise} \]

and the ideal \( I \) is the same as that in Example 4.1. We can also check that this sequence satisfies the numerical condition in Theorem 3.5.

Example 4.3. By Corollary 2.4, we do not have a long Bourbaki sequence with an approximation module \( E_1 \) and a codimension 3 generalized CM ideal \( I \). However, there are long Bourbaki sequences with approximation modules \( E_1 \oplus E_5(d) \) for \( d \in \mathbb{Z} \), which is an application of Theorem 2.2. Let \( k = 1 \) and \( n = 6 \). Then

\[ A = \langle x_1 e_1^* + \cdots + x_6 e_6^* \rangle \]
\[ B = \langle B_{ij} = (-1)^i x_j e_{[6]i} - (-1)^j x_i e_{[6]j} : 1 \leq i < j \leq 6 \rangle. \]

Let \( \varphi = (a, b) \) be

\[ a = x_1^3 e_1^* + x_1^2 x_2 e_2^* + x_1^2 x_3 e_3^* + x_1^2 x_4 e_4^* + x_1^2 x_5 e_5^* + x_1^2 x_6 e_6^* \]
\[ b = x_5 x_6 e_{12346} + x_5 x_6 e_{12345} + x_5 x_6 e_{13456} + x_2 x_6 e_{12456} \]

Let \( \varphi = (a, b) \) be

\[ a = x_1^3 e_1^* + x_1^2 x_2 e_2^* + x_1^2 x_3 e_3^* + x_1^2 x_4 e_4^* + x_1^2 x_5 e_5^* + x_1^2 x_6 e_6^* \]
\[ b = x_5 x_6 e_{12346} + x_5 x_6 e_{12345} + x_5 x_6 e_{13456} + x_2 x_6 e_{12456} \]
and set $\beta_i \in K_1 \oplus K_5(1)$ to be as follows:

$$
\begin{align*}
\beta_1 &= -x_6 e_{12345} + x_5 e_{12346} \\
\beta_2 &= x_6 e_3 - x_1^2 e_{13456} \\
\beta_3 &= x_6 e_2 - x_1^2 e_{12456} \\
\beta_4 &= x_2 x_5 e_2 - x_1 e_{12345} \\
\beta_5 &= x_4 x_6 e_2 - x_1^2 e_{12346} \\
\beta_6 &= -x_6 e_{12346} + x_2^4 e_{12456} \\
\beta_7 &= e_{23456} \\
\beta_8 &= e_{12356}.
\end{align*}
$$

Notice that $\beta_i \notin E_2 \oplus E_6(1)$ for all $i$, i.e., the condition in Theorem 2.7 is satisfied. Then we can check

1. $\text{Ker(} \varphi : K_1 \oplus K_5(1) \to S(-6) = \langle \beta_1, \ldots, \beta_8 \rangle \oplus E_2 \oplus E_6(1) \text{ and } \varphi \text{ is a degree 8 homomorphism, and}$
2. the diagram

$$
\begin{array}{cc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \text{Ker Res}(\beta) \quad \longrightarrow \quad \text{Ker } \beta \quad \longrightarrow \quad 0 \\
\downarrow & \downarrow \\
0 & F' \quad \longrightarrow \quad G \\
\text{Res}(\beta) & \beta \\
\downarrow & \downarrow \\
0 & \langle \beta_1, \ldots, \beta_8 \rangle \cap E_2 \oplus E_6(1) \quad \longrightarrow \quad \langle \beta_1, \ldots, \beta_8 \rangle \\
\downarrow & \downarrow \\
0 & 0
\end{array}
$$

where

$$
\begin{align*}
G &= \langle m_1, \ldots, m_8 \rangle = S(-5) \oplus S^4(-6) \oplus S(-8) \oplus S^2(-4) \\
F' &= \left\langle -x_1^2 m_1 + x_6 m_4 - x_5 m_5, x_2^3 m_3 - x_6^4 m_5 + x_1^2 m_6, \\
&\quad -x_2^2 m_1 - x_3 m_2 + x_5 m_3 - x_1^3 m_7 + x_1^2 x_4 m_8 \right\rangle \\
\text{Ker } \beta &= \text{Ker Res}(\beta) = \langle -x_1^2 m_1 + x_6 m_4 - x_5 m_5, x_2^3 m_3 - x_6^4 m_5 + x_1^2 m_6 \rangle
\end{align*}
$$

Thus we have a long Bourbaki sequence

$$
0 \to F \xrightarrow{f} G \xrightarrow{g} E_1 \oplus E_5(1) \xrightarrow{\phi} I(2) \to 0
$$

where $g(m_i) = \beta_i$, $i = 1, \ldots, 8$, and $F = S(-10) \oplus S^2(-7) = \langle u, v, w \rangle$ with $f(u) = x_2^3 m_3 - x_6^4 m_4 + x_1^2 m_6$, $f(v) = -x_1^2 m_1 + x_6 m_4 - x_5 m_5$, and $f(w) = -x_1^2 m_1 - x_3 m_2 + x_3 m_3 - x_1^3 m_7 + x_1^2 x_4 m_8$. The map $\phi$ is as follows: $\phi(x_i) = x_i x_1^2$ ($i = 1, \ldots, 6$).
\[ \phi(\partial_5(e_{12345})) = x_3^5x_5, \quad \phi(\partial_5(e_{12346})) = x_2^5x_6, \quad \phi(\partial_5(e_{12356})) = 0, \quad \phi(\partial_5(e_{12456})) = x_2^5, \]

\[ \phi(\partial_5(e_{13456})) = x_3^5x_6, \quad \phi(\partial_5(e_{23456})) = 0. \]

The ideal is \( I = \text{Im} \varphi = x_1^2m + (x_2^5x_6, x_2^5x_5, x_3^5, x_2x_5^5). \) Finally we can check that this Bourbaki sequence satisfies the numerical condition of Theorem 3.5.

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