Coupled macro-spin model with two variables for polarity reversals in the Earth and the Sun

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Abstract

Geomagnetism is extremely complex and the straightforward magnetohydrodynamics (MHD) simulations are still unable to reveal the whole dynamics of it. Recently, a simple macro-spin model for geomagnetism has been suggested. This model is based on the idea that the whole geomagnetism is described by interactions with many local dynamo elements (macro-spins). This model can reproduce some complex features of geomagnetism by solving a simple set of ordinary differential equations with a minimal number of variables. In this paper, we complete this macro-spin model by considering the full set of variables in three dimensions. Utilizing this model, we can address several basic features of geomagnetism which could not be described in the previous minimal model: migration of the North and South Magnetic Poles, a precise comparison of the detailed surface distribution of magnetic fields with the observed data, etc. Moreover, by applying this model to the solar magnetism, we can reproduce the periodic polarity reversals and the power index of the power spectrum from this complete spin model. Finally, we elucidate the statistical properties of the pole migration.
1. Introduction

We still have unsolved basic problems in geomagnetism. One of them is the long-time dynamics of the polarity reversal which took place 332 times in the last $1.6 \times 10^8$ years (Fig. 1) [1, 2]. The first simple model of the rotation disk [3] can explain polarity reversal, however, the actual geomagnetism is far from such a too simplified disk. Today, geomagnetism has been studied by many numerical simulations of MHD [4-6], and some dynamo experiments [7]. These simulations could reproduce some polarity reversals, however, the number of reversals was far less than observations and the values of the parameters used in the numerical simulations are far from the real values inside the Earth [4]. These simulations have not fully revealed the Physical reasons for the polarity reversal [6,8].

Recently, the macro-spin model for the polarity reversal has been proposed [9] by one of the present authors. This model is based on the idea that the electric current winds around the vortex structures, generated by convective structures in the outer iron core of Earth. The idea of the model is to identify this winding electric current as a macroscopic spin. The behavior of the whole dynamo mechanism is described by the cooperative interactions among these macroscopic spins. Although this original spin model could reproduce some features of geomagnetism, the spin motion was restricted within a plane, thus it has only one set of variables $\{\theta_i\}_{i=1,N}$, where $N$ is the total number of spins. This spin model is considered to apply to the Sun and other celestial objects in general.

In this paper, we first improve the original macro-spin model so that each spin can freely move in all directions $(\theta, \phi)$ in three dimensions, as should be. Then we numerically calculate this new model to reproduce both the polarity reversals and the migration of the North/ South Magnetic Poles. Thus we demonstrate the advantage of this new macro-spin model which is much more general than the original model [9].

2. Long-range Coupled Spin (LCS) model with two variables

The macro-spin model has two types. One is the short-range coupled spin (SCS) model where macro-spins interact with only their neighboring spins [10]. In this SCS model, friction and random force terms are necessary for the polarity reversal (Langevin-type equation). Another is the long-range coupled spin (LCS) model where the spins interact with all the other spins. In this paper, we adopt the latter LCS model since it does not require friction nor random force and therefore more fundamental. We newly introduce the longitudinal angle $\phi$ for the macro-spin model as well as the original latitudinal angle $\theta$. Then, the $i$-th spin $\vec{S}_i$ is described as $\vec{S}_i = (\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i)$, $(i = 1, 2, \cdots, N)$, where $N$ is the total number of spins. Thus, the kinetic energy $T$ of the system becomes.

$$T \equiv \frac{1}{2} \sum_{i=1}^{N} \left( \frac{d\vec{S}_i}{dt} \right)^2 = \frac{1}{2} \sum_{i=1}^{N} \left( \dot{\theta}_i^2 + \dot{\phi}_i^2 \sin^2 \theta_i \right). \quad (1)$$

Using the same form for the potential energy $V$ as in the previous study [9],

$$V = \mu \sum_{i=1}^{N} (\vec{\Omega} \cdot \vec{S}_i)^2 + \frac{\lambda}{2N} \sum_{i,j=1}^{N} (\vec{S}_i \cdot \vec{S}_j). \quad (2)$$

We obtain the total Lagrangian $L = T - V$ for this system. In the above, the angular velocity vector $\vec{\Omega}$ is parallel to the Earth rotational axis and we choose $\vec{\Omega} = (0,0,1)$. The first term of the potential is based on the idea of the
Taylor column which appears in the rotating fluid by the Coriolis force. In particular, the relevant Taylor column is the anti-cyclone which rotates opposite to the planetary spin. The electric current more strongly winds around the anti-cyclone and enhances the coherent magnetic field [11]. The macro-spin in our model represents the generated magnetic field by this winding current. Spins can flip their directions to yield the overall polarity flip. Thus the first term $\sum_{i=1}^{N}(\vec{\Omega} \cdot \vec{S}_i)^2$ represents the tendency of the spins to align parallel to the rotational axes due to the Coriolis force. A larger value of $-\mu$ ($\mu$ negative) means the dominance of the Coriolis force over the other effects and the Taylor–Proudman effect becomes more prominent.

On the other hand, the negative parameter $\lambda$ in the second term of the potential measures the strength of the interaction between the element spins. It may reflect the complicated topology of the fluid flow and the electric current across the neighboring spins [12].

This coupled spin model is based on the idea that the whole dynamo mechanism is described by global interactions of multiple local elements (macro-spins). In the case of Earth, the local elements are considered to be associated with the special topology created by the Taylor cells in the liquid iron core.

It is also possible to adopt the above Lagrangian from a much fundamental point of view. In general, a Lagrangian should be a scalar including the square of the time derivative. In our case of classical rotating spins, the Lagrangian should be composed of the vectors $\vec{\Omega}$ and $\vec{S}_i$ ($1 \leq i \leq N$). Under these conditions, the above form of Lagrangian is the simplest.

![Fig. 1. Geomagnetic polarity reversal record by observation in the last $1.6 \times 10^8$ years. This record has 332 times polarity reversal. In the vertical axes, +1.0 means the present geomagnetic polarity, and −1.0 the opposite [1].](https://academic.oup.com/ptep/advance-article/doi/10.1093/ptep/ptab062/6294412)

By the least action principle, we obtain the following equations of motion for $\theta_i$ and $\phi_i$ from our Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_i'}\right) = \frac{\partial L}{\partial \theta_i},$$

(3)

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \phi_i'}\right) = \frac{\partial L}{\partial \phi_i}.$$  

(4)

In our LCS model, friction and random force terms are not necessary for the polarity reversal. However, recently, the Langevin-type equation in the LCS model has been analyzed [13]. We numerically solve the set of equations (3) and (4) and study the dynamics of geomagnetism.

We define the “magnetization vector” $\vec{M}(t)$ as an average of the spins:
\[ \vec{M}(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \vec{s}_i, \quad (5) \]

and its projection onto the rotational axis \( \vec{\Omega} = (0,0,1) \) as “magnetization” \( M(t) \):

\[ M(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \vec{\Omega} \cdot \vec{s}_i = \frac{1}{N} \sum_{i=1}^{N} \cos \theta_i, \quad (6) \]

which can be a natural indicator of polarity. The inversion of the sign of \( M(t) \) indicates the polarity reversal.

3. Numerical results and their comparison with the observed data

Our spin model is a conservative system but highly chaotic reflecting the non-linearity of our Lagrangian. Therefore the relevant parameters would be the total energy \( E \) the only conserved quantity in our system as well as the explicit parameters \( \mu \) and \( \lambda \). However, the simultaneous multiplication of an arbitrary factor \( \gamma \) to \( \mu \) and \( \lambda \) simply corresponds to the change of the time scale as \( t \rightarrow \gamma^{-1/2} t \). Thus the parameter set can be chosen to be the energy \( E \), the ratio \( \mu/\gamma \), and the time duration. We have played around with many sets of parameters and have found a variety of the flipping dynamics of \( M(t) \). They may describe the variety of magnetic fields of planets in our solar system and many exoplanetary systems if the basic mechanism were the same.

We consider the geomagnetism and set our initial conditions so that all \( \theta_i \) are within the range of \( \pm 3\pi/5 \) around the North pole and the all \( \theta_i \) are equally separated configuration, except section 6.2 and after. This fixes the system energy \( E \). Further setting the fiducial parameters as \( N = 9, \mu = -10, \) and \( \lambda = -20, \) we have obtained 361 reversals within \( 2 \times 10^5 \) calculation time. The value \( N = 9 \) has been chosen from the roughness of the estimated west-moving pattern on the core-mantle boundary surface (Jackson, 2003) [14] so that the roughness of the magnetic field inferred from \( N = 9 \) is almost consistent with this estimation. The values of parameters \( \mu \) and \( \lambda \) are about ten times larger than those in Nakamichi et al. (2012) (\( \mu = -1 \), and \( \lambda = -1.8 \)), but the ratio \( \mu\lambda \) is about the same: we have kept parameters almost consistent with our previous work.

These parameter values are also determined to reproduce the two observed geomagnetic data: the polarity-reversal number in the last \( 1.6 \times 10^8 \) years and pole migration length in the last \( 1.6 \times 10^3 \) years (from 200 to 1800 AD). These numerical calculations were performed by utilizing Mathematica 12 platform. In Figs. 2 and 3, the time series and the power spectrum of \( M(t) \) are presented. In Fig. 4, the power spectrum of observed data is shown [1, 2]. Comparing these results and observed data, we obtain the following results.
Fig. 2. The time series of $M(t)$ in the macro-spin model of two variables with the parameters $N = 9$, $\mu = -10$, and $\lambda = -20$. In this case, there appears 361 times reversal in $2 \times 10^5$ calculation time. We see a lot of random and rapid reversals.

(i) Unit calculation time: $8.5 \times 10^2$ years.

(ii) The average time of polarity flipping: $4 \times 10^3$ years (observation: $(2 - 3) \times 10^3$ years).

(The polarity flipping time, the necessary time for a polarity flip should be properly defined, for example, by fitting $M(t)$ around the flipping region using the function $\tanh(t/\tau)$. However, in this paper, we simply evaluate the time scale $\tau$ by an eyeball fitting since the flipping region is often quite noisy and therefore the above fitting is not effective.)

(iii) Variety of chron: $(0.06 - 4.7) \times 10^6$ years (observation: $(0.1 - 6) \times 10^6$ years).

(iv) The power index of the power spectrum of $M(t)$: $-0.31$ in the low-frequency side and $-1.80$ in the high-frequency side (observation: $-0.76$ and $-1.77$, respectively).

(v) The mean pole migration length every 50 years 0.1 radian (observation: 0.098 radian [15]).

Thus, the calculation of the adopted parameters in our model can reproduce, to some extent, characteristic features of the real geomagnetic behaviors. Moreover, we can describe the moment of polarity change in three dimensions as our model contains two variables. In Fig. 5, snapshots of a polarity flipping event are shown wherein each snapshot, all the directions of the nine macro-spins are shown on top of the average magnetization $M(t)$. The observational data are from Table 1 in Clement (2004) [16].

Migration dynamics of the magnetic poles are compared in Fig. 6 (a) and (b), where (a) is the observation and (b) is
from our calculation. Both are projected on a unit sphere. The thick solid line is the time series of the pole trajectory. Fig. 6 (a) shows the trajectory of the observed paleomagnetic data: the migration of the North Magnetic Pole [14]. Fig. 6 (b) shows the sequence of the normalized magnetization vector \( \hat{M}(t) \equiv \bar{M}(t)/|\bar{M}(t)| \) in the numerical calculation of our model. In both figures, each segment corresponds to 50 years of migration. Circles in Fig. 6 represent the latitude lines every ten degrees viewed from the north. Our numerical simulation well describes the observed migration data in their randomness and the spatial extensions.

On the other hand, in a much longer time scale, the spin model cannot describe the preferred route of migration. There have been many paleomagnetic observations that report the existence of the preferred route of the migration from one pole to another [17] reflecting the topological characteristics of the Earth. Our spin model is statistically isotropic as shown in Fig. 6(b), and does not reproduce the preferred route.

Fig. 3. The power spectrum of \( M(t) \) in the 3D macro-spin model. The spectrum is represented by two powers: \(-0.31\) in low-frequency region and \(-1.80\) in the high-frequency region.

Fig. 4. The power spectrum of the observed data [1,2]. This data can also be represented by two power laws: \(-0.77\) in the low-frequency region and \(-1.8\) in the high-frequency region.
Fig. 5. A snapshot at the moment of polarity flipping. The black arrow from the center of the sphere represents “magnetization” \( \vec{M}(t) \). Almost all spins are moving from the top to the bottom together.

Fig. 6. (a) A trajectory of migration of the North Magnetic Pole that based on observed paleomagnetic data from 200 to 1800 AD. Each segment between the black dots corresponds to the 50 years interval. (b) A typical trajectory of migration of pole in our model \((t = 990 - 992)\), corresponds to 1700 years. The trajectory is drawn as the sequence of short arrows each of them represents the movement of the magnetization vector \( \vec{M}(t) \) within 0.059 calculation time or equivalently 50 years. We see the random magnetic pole migration around the north pole in (b) is similar to observation data in (a).

4. Application to the Sun

The previous study \[9\] has shown that the macro-spin model may be applied not only to geomagnetism but also to solar magnetism. This generality would be fully verified if the inside convection currents form macroscopic eddy structures. We now simply improve the description of solar magnetism by using our new macro-spin model.

We first have to estimate the number of macroscopic spins in the case of the Sun. We use the scaling law which was found in our previous study to estimate it extrapolating from the case of Earth. We previously found that ten celestial objects, in the mass range over 8-digits, are almost on a straight line in the diagram of \( d / (m^{1/2} R^3 \ (2 \rho_0 \Omega / \sigma)^{1/2}) \) against the mass \( m \), where \( d \), \( m \), \( R \), \( \rho_0 \), and \( \sigma \) are, respectively, magnetic dipole moment, the mass of the celestial object, the radius, the average mass density, and the electric conductivity of the
outer core. This scaling law and the analogy from MHD equations lead to the relation between the spin number and the mass of the celestial object,

\[ N \gamma^2 \propto m^{1/2}, \tag{7} \]

where \( \gamma \) is the ratio of the radius of the Taylor cell and that of the convective region. If this parameter \( \gamma \) were constant for various objects and if we use \( N = 9 \) for Earth, then the number of spins in the Sun is determined to be \( N_{\text{sun}} \approx 5 \times 10^3 \).

However, an immediate caution must be made for the use of the above scaling (7). The Taylor-Proudman balance, which was needed for the scaling (7), may not be guaranteed for the Sun because the solar Coriolis force is weak owing to the slow self-rotation. Nevertheless, several empirical scaling relations are reported (Christen (2010) [18], Brun (2015) [19]) in a similar form as (7). Therefore, reserving the serious physical justification, we use the scaling relation (7) in this paper.

On the other hand, the number of spins for the Sun can alternatively be estimated directly from the observed structure of the solar surface. Because the macro-spins are related to the vortex structure, it will be natural to identify the supergranule as a macro-spin. Simply assuming that the constant size supergranules are fully packed with the solar convective region, we can roughly estimate the total spin number as \( 35000 \) for the supergranule diameter \( 3 \times 10^4 \) km. However, the vortex structure may hierarchically continue inward increasing vortex size, and this estimated number may reduce.

Despite the estimated number of spins is \( N_{\text{sun}} \approx 5 \times 10^3 \) or more, it is technically difficult for us to perform the numerical calculation of the two-parameter LCS model with this spin number. Reserving the full calculation to our future publications, we now estimate typical properties of the solar magnetism using a very limited number of spins.

We prepare the initial condition, as the fiducial setting for solar magnetism, that all macro-spin’s latitude angle \( \theta_i \) are within a range of \( \pm \pi/2 \) around the pole, and set the spin number \( N = 300 \) with the parameters \( \mu = -0.3 \) and \( \lambda = -2 \). Then the numerical calculation of our macro-spin model yields the time series of \( M(t) \) as in Fig. 7 and the power spectrum of \( M(t) \) as in Fig. 8. In particular, an apparent periodic pattern with irregular amplitude modulation as in Fig. 7 would be a general property of this model. Although the other parameter choices also yield similar results, a systematic analysis will be reported elsewhere, including the locality of the interactions.

We naturally expect that the short main period in Fig. 7 corresponds to the 11 years, a typical period of solar activity. The appearance of the amplitude modulation in the same graph is interesting: The spin model naturally describes the intermittent activity of the solar magnetism. This intermittency will be related to the various activities of the Sun beyond the solar magnetism. This intermittency is also reflected in the power spectrum in Fig.8. We observe the increase of the signal in the low-frequency region on top of the peak around \( \omega = 2 \), which represents the 11-year periodic activity in Fig. 8. The numerical fit of this power spectrum yields a power law with an index of \( -0.99 \) within the corresponding time domain of 550-years to a 1.1 year. This index value is almost consistent with the observed value \( -1.1 \) for the time variation of sunspot number within the time domain of 21-years to 2 months [20], smaller than ours. It is further interesting that the observation of the solar wind also shows the power index \( -1 \) within the time domain of a 4 days to 3.5 hours [21], much smaller than the above time ranges. If the origin of these activities, solar magnetism, sunspots, and the solar wind, were the solar dynamo mechanism, it would be much interesting. This common index value \( -1 \) reminds us of universal 1/f fluctuations or the flicker noise.
The “1/f noise” has been observed everywhere in nature in a wide range of time domain, including the interval between heartbeats, candle flame fluctuation, the swaying train, sound of a stream, sunlight shining through the leaves of trees, fireflies, cell firing intervals of electrical signals from neurons in living organisms [22]. The Sun may belong to such a general category, and the macro-spin model describes such general fluctuation correctly.

Thus we confirm the intermittent almost periodic behavior of the three-dimensional coupled spin model, although a similar behavior was already found in the previous tentative two-dimensional model. To confirm this behavior in our spin model, we need to extend the simulation time as well as the number of the spins; both of them require us to improve the present numerical methods.

Fig. 7. The time series of $M(t)$ in the macro-spin model with two variables for the Sun. The parameters are $N = 300$, $\mu = -0.3$, $\lambda = -2$. The quasi-periodic behavior of reversal is new in the present case and different from the behavior in geomagnetism. We can identify the short reversing period corresponds to the well-known 11 years period, and the period of weak magnetic field intensity corresponds to the time when solar magnetic activity was weak.

In our present 3D model, the macro-spins can have any direction while they were constrained in an artificial plane in the previous 2D model. Therefore present 3D spin model will make it possible for us to study the whole directions of all spins. Figure 9 shows a snapshot of the directions of all the spins in our model at a moment when the overall polarity is fixed. There exist a lot of macro-spins pointing to the opposite direction to the overall polarity in the figure. This inhomogeneous patchy distribution of polarities may be related to the recent observation that both positive and negative magnetic field patches are observed in the north polar region of the Sun [23]. The macro-spin model may reveal such an interesting local structure of the solar magnetic field if we further extend our spin model including local interactions as well as long-range coupling. This insight is new in our model which was not achieved in the previous 2D model.
Fig. 8. The power spectrum of $M(t)$ in the macro-spin model with two variables for the Sun (solid line) and the power index (dashed line). The peak around $\omega = 2$ corresponds to 11 years period and the power index is almost $-1$.

Fig. 9. A snapshot of all the 3D macro-spins projected on a unit sphere. The arrow from the center of the sphere (looks like an inverted triangle) represents “magnetization” $\vec{M}(t)$. This overall polarity is downward in this figure while there are a lot of spins that point to the opposite direction of the polarity.
Fig. 10. The histograms of the probability distribution of the migration length within a fixed time scale $\Delta t$ in our model. (a) $\Delta t = 0.24$ (about 100 years). This histogram is similar to the normal distribution. (b) $\Delta t = 2.4$ (about $10^3$ years). (c) $\Delta t = 24$ (about $10^4$ years). This histogram has a long tail on the right side. (d) $\Delta t = 240$ (about $10^5$ years). Another peak appears on the right side of the histogram at around 1.7, probably reflecting the occasional polarity reversals.

5. Statistical properties of pole migration

As we have seen in the last section, our 3D spin model can describe, to some extent, the local structure and dynamics of geomagnetism. We reserve a precise description of the local geomagnetism which requires both the local interaction and the long-distance interaction as well as the spatial configuration of the spins. In this section, we explore the local description within the present 3D LCS model. This may correspond to the smallest limit of the ratio of dynamo-region and the radius of Earth.

A migration of the geomagnetic poles may be a good indicator of the dynamics of the local geomagnetic structure and is also directly related to the observations as well. We now explore some specific features of the migration of the geomagnetic poles in this section based on our 3D spin model.

5.1. Straight-line distance

We investigate the migration length of the geomagnetic pole by the numerical simulation and examine the distribution functions in our model.
In Fig. 10, the histograms of the migration length in our model are shown. Each histogram represents the probability of the migration length within the fixed time interval $\Delta t$. We first consider the continuous trajectory $\vec{M}(t)$ on a unit sphere. Then we discretize this trajectory by the time interval $\Delta t$. Then the migration length $\Delta l$ between the discrete time $t$ and $t + \Delta t$ is defined as

$$\Delta l (t) = |\vec{M}(t + \Delta t) - \vec{M}(t)|.$$  

(8)

Such many lengths $\Delta l(t)$ along with the original continuous trajectory yield a statistical ensemble of migration lengths and the histogram of this ensemble is calculated. We take the time interval $\Delta t = 0.24, 2.4, 24$ and 240 for Fig.10 (a), (b), (c) and (d), respectively.

According to our analysis, for a small time-interval $\Delta t$, the histogram turns out to be well described by a simple normal distribution. On the other hand, the longer the time-interval $\Delta t$, there gradually appears an extended tail on the right side of the main peak of the distribution, reflecting the effect from the rapid movement of $\vec{M}(t)$ associated with the polarity reversals and excursions. An extreme example is shown in Fig. 10 (d), where even another peak appears, at around 1.7, on the right side of the main peak.

![Fig. 11.](https://example.com/fig11.png)

The same as Fig 10, but for total integrated distance in our model. (a) $\Delta t = 0.24$ (about 100 years). The fit distribution function for this histogram is similar to normal distribution. (b) $\Delta t = 0.72$ (about 300 years). (c) $\Delta t = 16.8$ (about $7 \times 10^3$ years). This histogram has a long tail on the left side. (d) $\Delta t = 216$ (about $9 \times 10^4$ years). This histogram has a long tail on the right side.

5.2. Integrated distance
We now examine the statistics of the integrated migration distance $L$ in this subsection. $L$ is another characteristic for the migration of the geomagnetic poles. We define the integrated migration length $L_{n\tau}(t)$, fixing the base distance $\tau = 0.24$, as

$$L_{n\tau}(t) = \sum_{j=0}^{n-1} |\vec{M}(t + (j+1)\tau) - \vec{M}(t + j\tau)|.$$  

(9)

Therefore, $L_{n\tau}(t)$ means the total length between $\vec{M}(t + n\tau)$ and $\vec{M}(t)$, along the trajectory with fixed time step $\tau$. The histograms of $L_{n\tau}(t)$ for the cases $n = 1, 3, 70$ and 900 are shown in Fig. 11 (a), (b), (c) and (d), respectively.

According to our analysis, the peak of each histogram thus calculated simply shifts longer migration length for increasing $n$. The histogram seems to be normal except the last case $n=900$, in Fig. 11 (d), which shows a little bit elongated tail with a very small enhancement. This tail may simply reflect the previous analysis in Fig. 10 (d).

In the conclusion of this section, we cannot see any systematic behavior of the distribution functions for the migration length, although we expected to find any universal behavior reflecting the scaling law in the power spectrum Fig.3.

6. What determines the state of reversal

We have, so far, analyzed our model with fixed fiducial parameters to describe the geomagnetism. In this section, we examine the generality of our model examining a wide variety of parameter values. We also expect the variety of our spin model may be able to describe other types of planetary magnetism. Reserving the general argument on this planetary variety to our next publications, we here examine the basic parameter dependence on the dynamics of the polarity flip. We calculate the time evolution of the dipole moment $M(t)$ and the power spectrum of it when we change the parameter values of $\mu, \lambda, E$, fixing the number of spins as $N = 9$ in this paper.

6.1. Naïve $\mu, \lambda, parameter dependence

We first simply change the parameters $\mu, \lambda$, with fixed initial conditions (numerical values of the variables). Fig.12 describes the evolution of $M(t)$ for each $\mu$, fixing $\lambda = -18.4$ the fiducial value. Fig.13 describes the evolution of $M(t)$ for each $\mu$, fixing $\mu = -11$ the fiducial value. These graphs clearly show that the larger negative values of $\mu, \lambda$ more reduce the frequency of polarity flip. On the other hand, larger negative values of $\mu, \lambda$ means the reduction of the total energy, for the fixed initial values of the parameters. Actually, the total energies in Fig. 12 are, from top to bottom, 29.941, $-23.4194$, $-53.911$, $-99.6485$, and those in Fig.13 are 17.2664, $-13.6803$, $-53.911$, $-83.3104$.

To elucidate the effect of the total energy explicitly, in the below, we release the initial values of parameters but instead fix the total energy $E$. 
Fig. 12. The time series of \( M(t) \) in the macro-spin model for different values of \( \mu \), fixing \( \lambda = -18.4 \). The value of \( \mu \) we have chosen is 0, -7, -11, -17 from top to bottom. The total energy is reducing in this order, 29.941, -23.4194, -53.911, -99.6485, and the number of polarity reversal is decreasing.

Fig. 13. The time series of \( M(t) \) in the macro-spin model for different values of \( \lambda \) fixing \( \mu = -11 \). The value of \( \lambda \) we have chosen is 0, -8, -18.4, -26 from top to bottom. The total energy is reducing in this order, 17.2664, -13.6803, -53.911, -83.3104, and the number of polarity reversal is decreasing.

6.2. Total energy dependence

From now on, we set the total energy \( E = T + V \) as one of the independent parameters of the calculations as well as \( \mu, \lambda \). Accordingly, we have to release the initial conditions. We set the initial value of the time derivative of spin latitude angle \( \dot{\theta}_i \) as \( \dot{\theta}_i = c \sqrt{2E - N(\lambda + 2\mu)} \xi \), where \( \xi \) takes the random values within \([-1, 1]\) and the constant \( c \) is determined so that the total energy matches the given value \( E \). All the other parameters are set zero, except very small value \( 10^{-5} \) to avoid the singularity associated with the numerical calculations.
Further, to describe the dynamical characteristic more elucidate and to extend the parameter range wider, we also examine the power index of the Fourier power spectrum of the time evolution $M(t)$, as well as the number of polarity reversals.

We first change the total energy $E$ from its fiducial value $-52$, fixing the others fiducial $\mu = -10$, $\lambda = -20$. According to our calculation, the number of reversals simply increases for increasing total energy $E$, as shown in Fig. 14.

![Number of reversals vs Total Energy](image)

Fig. 14. The number of reversals of $M(t)$ for various energy, fixing the parameters $\mu = -10$ and $\lambda = -20$. Calculation time is $1 \times 10^4$ for each.

The power index of the power spectrum of $M(t)$ mildly increases, from $-1.4$ to $-0.7$ for increasing $E$ as shown in Fig. 15. Since each calculation time in Fig. 15 is one-fourth of the previous calculation in Fig. 3, the power index in Fig. 15 reflects a slightly higher frequency (shorter-timescale) region.

![Power Index vs Total Energy](image)

Fig. 15. Same as Fig. 14 but the power index of the power spectrum of $M(t)$.

6.3. $\mu$ Parameter dependence

We next change the parameter $\mu$ from its fiducial value $-10$, fixing the others as fiducial values.
According to our calculation, reducing $\mu$ from its fiducial value $-10$, makes the number of reversals increase and the power index increase from $-1.5$ to $0$. On the other hand, increasing $\mu$ from its fiducial value $-10$, makes almost no change.

**Fig. 16.** The number of reversals of $M(t)$ for various $\mu$, fixing the parameters $\lambda = -20$ and $E = -52$. Calculation time is $1 \times 10^4$ for each.

**Fig. 17.** Same as Fig. 16, but the power index of the power spectrum of $M(t)$.

### 6.4. $\lambda$ Parameter dependence

We finally change the parameter $\lambda$ from its fiducial value $-20$, fixing the others as fiducial values. When we reduce the value of $\lambda$, the number of reversals is increasing while we increase $\lambda$, the number is almost constant as is shown in Fig. 18.
On the other hand, the power index of the power spectrum of $M(t)$ is less sensitive to the variation of $\lambda$, except the lowest $\lambda$, as shown in Fig.19.

We will summarize the generality of our model studied in this section. The behavior of the polarity reversal number along with the variation of the parameters $\mu, \lambda$ in 6.1 and that in 6.3-4 seems to be conflicting with each other: reducing $\mu, \lambda$ makes the reversal number reduce in 6.1 while it makes the reversal number increase in 6.3-4. However, these are consistent with each other and simply the matter of initial conditions. In 6.1, we set the initial values of all the variables, and therefore the total energy reduces for reducing $\mu, \lambda$. This naturally makes the reversal number reduce. On the other hand in the rest of the subsections, including 6.3-4, the total energy is fixed and only the minimum of the potential $V$ reduces for reducing $\mu, \lambda$, as $\min V = N(\mu + \lambda/2)$ while the maximum is always 0 ($\max V = 0$). We have also confirmed the fact that the average kinetic energy $K$ is almost half of the potential energy $|V|$ during calculations; some kind of virial theorem is established. These facts lead to the observed property that reducing $\mu, \lambda$ and therefore increasing $|V|$ makes the kinetic energy increase, thus the spins go over the potential barrier easier. Thus the reduction of $\mu, \lambda$ makes the reversal number increase.

The reduction of the polarity reversal number for reducing total energy $E$ in 6.2 seems to be natural. Although the polarity reversal, for small $E$, seems to vanish in our relatively short time calculation, reversal may happen with
longer time scales. This is because we cannot expect any discontinuity in the time scales for polarity reversals. If this is the general case, we may be able to conclude that the polarity reversal is inevitable. We will reserve this problem for our next publications including the cases of other planets of our Solar system.

On the other hand, the power-law behavior of the power spectrum of $M(t)$ seems to be general and the power index seems to be insensitive to the parameter change. This is another general property of our model.

If the planetary magnetism, possibly also the stellar magnetism, were commonly described by our spin model, then the polarity reversal and the power-law behavior may be the common general features of them.

7. **Conclusions and discussions**

We first summarize all the results obtained, as well as what we could not have achieved, by the 3D macro-spin model with two variables in this paper. Then we briefly describe the perspective of our project.

1. The 3D spin model could reproduce the main features of the geomagnetic and solar magnetic behaviors; power spectrum, various time scales, the randomness of polarity reversals, etc. For a large number of spins $N$, say more than hundred, a quasi periodicity appears on top of the low-frequency amplitude modulations. However, we could not obtain sufficiently long period superchrons, which are observed in the geomagnetic records, in our analysis this time. Probably reflecting this fact, the power index of the power spectrum of the magnetization $M(t)$ is slightly larger than the observation. A superchron is a very long period of fixed polarity more than $1 \times 10^7$ years [24]. To describe such intermittency, we may need to introduce the fluctuation of the energy flow from the inner planet to outward. Of course, we need to extend our model to allow such energy flow.

2. We could reproduce random migration of the magnetic pole and the mean migration length of the migration by introducing the longitudinal angle. However, in a precise comparison with observations, we need the precise definition of the magnetic pole; there are some different concepts.

3. The kinetic energy form of the macro-spin model with two variables studied in this paper is simpler than the previous study [25]. Thus, our model will be more suitable for extracting the essence in the dynamics of polarity reversal.

4. It was possible, to a very limited extent, to reproduce the inhomogeneous magnetization profile on the Solar surface. It could be compared with the observation [23]. However, to describe the realistic inhomogeneity of magnetic fields, we need to make the spins distribute on the convective spherical region of the Sun. We are now ready to set the locally and globally interacting spins uniformly on a sphere. We will report this study in our future publications.

5. Finally, we have investigated the statistical properties of the magnetic-pole migration. As a result, we find that the migration length of the geomagnetic pole does not follow a single unique distribution function. We should study further this distribution properly defining the correct magnetic pole in our future work.
As described above, our macro-spin model can be modified in several ways. By doing so, we can apply this model to the other planets and can describe the variety of their magnetic fields. At present, we are preparing the classification scheme of the variety of planetary magnetic fields in our solar system by a single parameter: the ratio of the radius of the planet and that of the dynamo-region inside. For this extension, the present 3D spin model is essential. This paper is just a first step toward such a systematic application of the spin model.

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APPENDIX 2-dimensional vs 3-dimensional macro-spin models

In our previous papers [9], [10], we have studied a 2D macro-spin model in which the spins were constrained on an artificial surface. We extended, in this paper, this original spin model to 3D in which all the spins can move freely without any constraint. Verification of the macro-spin model should be performed in this natural 3D model. In this appendix, we examine how extent present 3D model inherits the 2D model features. We aim to confirm the generality of the macro-spin model which is independent of the details of the model.

To compare the 2D and 3D spin models, we calculate the number of reversals and the power index in Figs. 20-21 in the 2D model. This should be compared with the results in the 3D model Figs.14-15.

![Fig. 20. The number of reversals of $M(t)$ in the macro-spin model of one variable with the parameters $N = 9$, $\mu = -10$, and $\lambda = -20$, on each total energy. We proceed $1 \times 10^4$ timesteps calculation of each case.](image-url)
Fig. 21. The power index of the power spectrum $M(t)$ in the macro-spin model of one variable with the parameters $N = 9$, $\mu = -10$, and $\lambda = -20$, on each total energy. We proceed $1 \times 10^4$ timesteps calculation of each case.

Comparing Fig.14 to Fig.20, and Fig15 to Fig.21, disregarding the lower extended energy regions, it turns out that both the number of reversals and the power index of $M(t)$ are very similar to each other in 2D and 3D spin models. Thus for the above properties, we conclude the generality of the macro-spin model which does not depend on the details even on the dimensionality.