Active Vibration Absorber for Superharmonic Resonance Condition

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Abstract. The present paper, analyses vibration suppression of a nonlinear single degree of freedom (SDOF) spring-mass-damper system under hard harmonic excitation by attaching a piezoelectric based active dynamic vibration absorber (ADVA). The host structure consists of spring with cubic nonlinear stiffness and negligible damping. The acceleration feedback is used to provide the actuating force through ADVA on the vibrating host structure. The hard harmonic excitation on the host structure induces superharmonic resonance condition i.e. when excitation frequency nearly matches one-third of the natural frequency of the host structure. The optimal tuning ratio and damping ratio of the vibration absorber are considered to depend upon the actuating force so that one can actively tune the operating frequency of the ADVA. The method of multiple scales (MMS) is used to obtain the reduced equations to study the frequency response of the system. The analysis is carried out by studying the effect of active force and cubic nonlinear stiffness on the responses of the host structure. The results obtained by MMS are compared numerically showing good agreement.

1. Introduction
To absorb the vibration of vibrating structures generally, vibration absorbers are used. These vibration absorbers consist of an auxiliary mass and a spring [1] which is attached to the host vibrating structure to attenuate its vibration. The vibration absorber is tuned to the natural frequency of the host vibrating structure to completely attenuate the vibration of the host system but only at the resonating frequency and also induces two large amplitudes of vibration near to the resonating frequency. This type of vibration absorber which is known as a tuned mass damper (TMD). The TMD minimizes the amplitudes of two peaks near the resonance frequency of operation but the vibration of the host structure still remains [2]. The design of the TMDs utilizes numerous optimization methods such as $H_2 / H_\infty$ optimization to obtain the optimum tuning/frequency ratio and damping ratio for the absorber to minimize these peaks amplitude in the host structure response [3-5]. Unlike vibration absorber without damper in optimal TMDs, the tuning ratio of the absorber is kept less/more [6] than the natural frequency of the host structure for minimal equal peaks in the response amplitude of host structure. The vibration suppression by the optimized TMD is still a broad area of research as many different types of design models [7, 8], use of smart/active materials [9] and control algorithms [10] are developed to minimize the vibration of the host structure for a wider operating frequency range with the minimal weight of the auxiliary mass [11]. The host structure under external disturbance often exhibits nonlinearity in the stiffness due to prolonged use and irregular structures. For this reason, passive TMDs are not very
effective in suppressing vibration for a broader range of frequency of operation [12]. The use of smart and semi-active materials in the TMDs outshine the passive TMDs in terms of minimizing vibration and mass ratio between auxiliary mass and the host vibrating structure mass [13, 14]. Nonlinear analysis of active vibration absorber with various perturbation methods are widely used to study the frequency response of the system for various resonance conditions [15, 16].

In the present paper nonlinear analysis of an active vibration absorber attached to a nonlinear SDOF host system under hard harmonic excitation is studied by using the perturbation Technique: method of multiple scales (MMS). The analysis is carried out by considering acceleration feedback of the host structure to investigate frequency and time response of the host structure for the superharmonic resonance condition. It is shown that with the application of active force the vibration of the host system minimizes and the nonlinear system behaves as a linear system without any nonlinear phenomena in the frequency response curve. In the proposed model also, the natural frequency of the absorber can be tuned actively to suppress the vibration of the host system for various other resonance conditions which is limited in the case for a passive vibration absorber. In this work, only the superharmonic resonance condition is considered.

2. System description
A non-linear SDOF host system having mass, damper, stiffness, and cubic nonlinear stiffness \( m_1 \), \( c_1 \), \( k_1 \) and \( k_{11} \) respectively is subjected harmonic force excitation with amplitude \( F \) and excitation frequency \( \omega \) as shown in figure 1. To suppress the vibration the host structure a piezoelectric based active dynamic vibration absorber is used with mass, damper, and stiffness \( m_2 \), \( c_2 \) and \( k_2 \) respectively. The active part of the absorber consists of a PZT actuator with stiffness \( k_p \). The acceleration feedback control law for the host structure is used to provide voltage to the PZT actuator. The nominal displacement control law for the host structure is used to provide voltage to the PZT actuator. The natural frequency of the absorber can be tuned actively to suppress the vibration of the host system for various other resonance conditions which is limited in the case for a passive vibration absorber. In this work, only the superharmonic resonance condition is considered.

\[ \ddot{z}_1 + \omega_2^2 z_2 + \epsilon \dot{z}_2 \dot{z}_1 + \epsilon \dot{z}_2^2 - \epsilon^2 \mu \dot{z}_1 - \epsilon^2 \dot{z}_2^2 \dot{z}_2 = F \cos \Omega \tau \]  

(1)

\[ \ddot{z} + \Omega_2^2 z_2 + \epsilon \dot{z}_2^2 = -\left( \alpha / \mu \right) \dot{z}_1 + 2 \epsilon \Omega \dot{z}_2 \dot{z}_2 + \Omega_2^2 z_2 \]  

(2)

where

\[ \epsilon, \mu \] are controller gain. The mathematical model of the two DOF system is described by the following nonlinear coupled equations using the non-dimensional displacement \( z_1 (x_1 / x_o) \) and \( z_2 (x_2 / x_o) \), non-dimensional time \( \tau = \omega t \), \( \omega = \sqrt{k_1 / m_1} \) and with bookkeeping parameter \( \epsilon \), where \( x_o \) is arbitrary reference length and \( \Omega \) non-dimensional external frequency of excitation.
\[ \frac{\dot{z}_1}{x_o} = \frac{x_1}{x_o}, \quad \frac{\dot{z}_2}{x_o} = \frac{x_2}{x_o}, \quad \ddot{z}_2 = \dot{z}_2 - \dot{z}_1, \quad \alpha = \frac{k_2 k_n d_2 V}{k_1 x_o}, \quad \omega_h = \sqrt{\frac{k_1}{m_1}}, \quad \frac{\dot{k}_1}{m_1} = \frac{\ddot{k}_1}{\epsilon (1-\alpha)}, \quad F_1 = F / (1-\alpha), \]

\[ \mu = \frac{m_2}{m_1}, \quad \Omega = \frac{\omega_o}{\omega_h}, \quad \Omega = \frac{\omega_o}{\omega_h}, \quad \bar{\Omega} = \frac{2(\xi_1 + \mu \xi_3 \Omega_3)}{\epsilon (1-\alpha)}, \quad \bar{\Omega} = \frac{2 \xi_1 \mu \Omega_3}{\epsilon (1-\alpha)}, \quad \bar{\xi}_1 = \frac{c_1}{2m_1 \omega_h}, \quad \bar{\xi}_2 = \frac{c_2}{2m_2 \omega_h}, \quad \bar{\xi}_2 = \frac{2 \xi_2 \Omega_2}{\epsilon (1-\alpha)} \]

2.1. Approximate frequency response by MMS

The first-order method of multiple scales (MMS) is used to obtain the approximate solutions of the governing nonlinear coupled Eq. (1) and (2) by assuming \( z_i = z_{i0} (\tau_0 + \tau_1) + \varepsilon z_{i1} (\tau_0, \tau_1) + ... \) and \( z_2 = z_{20} (\tau_0, \tau_1) + \varepsilon z_{21} (\tau_o, \tau_1) + ... \), where the time scale is described as \( \tau_a = \varepsilon^a \tau \), with \( n = 0, 1 \). The first and second-order differential operators for different time scales can be written as \( d / d \tau = D_0 + \varepsilon D_1 + ... \) and \( d^2 / d \tau^2 = D_0^2 + 2 \varepsilon D_0 D_1 + ... \) [16]. The above-mentioned expressions \( z_i \) and \( z_2 \) are substituted into Eq. (1) and (2) and by collecting the same power of \( \varepsilon \) the following partial differential equations are obtained.

\[ \varepsilon^0: \quad \left( D_0^2 + \alpha^2 \right) z_{i0} = F_1 \cos(\Omega \tau_0) \] (3a)
\[ \varepsilon^1: \quad \left( D_0^2 + \alpha^2 \right) z_{i1} = -\left( \alpha / \mu \right) D_0 \dot{z}_{i0} + 2 \xi_2 \Omega_2 D_0 \dot{z}_{i0} + \Omega_2^2 \dot{z}_{i0} \] (3b)
\[ \varepsilon^1: \quad \left( D_0^2 + \alpha^2 \right) z_{i2} = -2 D_0 D_1 \dot{z}_{i0} - \xi_2 D_0 \dot{z}_{i0} - k \dot{z}_{i0}^3 \] (4a)
\[ \varepsilon^1: \quad \left( D_0^2 + \alpha^2 \right) z_{i2} = -2 D_0 D_1 \dot{z}_{i0} - \xi_2 D_0 \dot{z}_{i0} - \alpha (\mu) \left( 2 D_0 D_1 \dot{z}_{i0} + D_0^2 \dot{z}_{i0} \right) \] (4b)

The solution of Eq. (3a) can be written as follows

\[ z_{i0} = A_i (\tau_0) e^{i\omega_{i0} t} + \Gamma e^{i\omega_{i0} t} + \text{cc} \] (5)

where \( A_i \) is unknown complex amplitude of the host structure for time \( \tau_i \), ‘cc’ represents complex conjugate of preceding terms and \( \Gamma = \frac{F}{2(\omega_0^2 - \Omega^2)} \). Substituting Eq. (5) into Eq. (4a) one can obtain many resonance conditions out of which here super-harmonic resonance condition is analysed i.e. when the forcing frequency \( \Omega \) is nearly equals to \( \omega_{i0} / 3 \), or \( 3 \Omega = \omega_0 + \varepsilon \sigma_{t1} \), where \( \sigma_{t1} \) is the detuning parameter. The secular terms from the resulting equation can be eliminated from the following equation

\[ 2 D_0 A_i \dot{\omega}_0 = -\xi_2 \dot{\omega}_0 A_i - 3 k \dot{A}_i A_i - 6 k A_i \Gamma^2 - k \Gamma^3 e^{i\gamma_{i1}} \] (6)

The complex amplitude function \( A_i \) of the host system can be written as \( A_i = \frac{1}{2} d_i e^{i\beta_i} \). Substituting \( A_i = \frac{1}{2} a_i e^{i\beta_i} \) and \( \gamma_i = \sigma_i \tau_1 - \beta_i \) in Eq. (6) and separating the real and imaginary parts from the resulting equation one may obtain the following autonomous reduced equations.

\[ a_i \dot{\gamma_i} = \sigma_i a_i - \frac{3 \bar{k} a_i}{8 \omega_i} - \frac{3 \bar{k} a_i \Gamma^2}{\omega_i} - \frac{k \Gamma^3 \cos \gamma_i}{\omega_i} \] (7)
The steady-state response of the system is studied by taking into account $a'_i = \gamma'_i = 0$ in Eq. (7) and (8). Solving the resulting equations the following amplitude response of the host vibrating structure as a function of detuning parameter $\sigma_i$ is obtained.

$$
\sigma_i = \frac{3ka_i}{8\omega_1} + \frac{9k\Gamma^2}{a_i\omega_i} \pm \sqrt{\left( \frac{k\Gamma^3}{a_i\omega_i^3} \right)^2 - \frac{\bar{x}_i^2}{4}}
$$

The stability of the steady-state response of the host vibrating structure is investigated by obtaining the eigenvalues of the Jacobian matrix ($J$), which is obtained from the following equation.

$$
J = \begin{bmatrix}
-\frac{9ka_i}{8\omega_i} + \frac{\sigma_i}{a_i} - \frac{3k\Gamma^2}{a_i\omega_i} & -\frac{\bar{x}_i}{2} \\
-\frac{\bar{x}_i}{2} & -\sigma_i + \frac{3ka_i^3}{8\omega_i} + \frac{3a_i\bar{k}\Gamma^2}{\omega_i}
\end{bmatrix}
$$

The system will be stable for both trivial and nontrivial responses, if the real part of all the eigenvalues of the Jacobian matrix is negative, which leads to the following equation for a stable system.

$$
\left( -\frac{9ka_i}{8\omega_i} + \frac{\sigma_i}{a_i} - \frac{3k\Gamma^2}{a_i\omega_i} \right) - \sigma_i + \frac{3ka_i^3}{8\omega_i} + \frac{3a_i\bar{k}\Gamma^2}{\omega_i} - \frac{\bar{x}_i^2}{4} > 0
$$

In the following section analysis of frequency and time response for the host, the structure is undertaken.

### 3. Results and discussions

In this section, the effect of active control force and the nonlinear stiffness on the frequency response and time response of the primary system is studied for a mass ratio $\mu$ equal to 0.05 and zero damping ratio in the host structure. The frequency response of the system is obtained by solving Eq. (9) and these results are compared numerically by the time responses. The frequency ratio and the damping ratio of the absorber are considered as $\Omega_2 = \sqrt{(1-\alpha)/(1+\mu)^2}$ and $\xi_2 = \sqrt{3\mu/(8(1+\mu))}$ [18]. The amplitude of the external harmonic excitation on the host structure is considered to be hard i.e. $F_i$ equal to 1 and the reference amplitude $x_0$ is taken as 1. The effect of active force on the frequency response of the host structure is studied in figure 2 for the superharmonic resonance condition. In figure 2(a) frequency response of the host vibrating structure shows an increase in the response amplitude as the cubic

![Figure 2](image_url)
nonlinear stiffness increases with minimal applied control force i.e. when \( \alpha \) equal to 0.0001. The response of the host structure shows multiple solutions as cubic nonlinear stiffness increase beyond 0.1 showing both stable (upper branch) and unstable (lower branch) solutions. The cubic nonlinearity in the spring stiffness shifts the operating resonate frequency and induces higher amplitude in the host system. The amplitude of vibration of the host system reduces significantly once the active force is applied by the ADVA which is shown in figure 2(b). From figure 2(b) one can observe with the maximum active control force amplitude of the host structure shifted to the right side of the resonant operating frequency as the cubic nonlinear stiffness is increased and also the response shows linear behavior with no multiple solutions. This suggests the system becomes stable for a wider range of operating frequencies. The maximum amplitude of the host structure is found to be 0.0032 for cubic nonlinear stiffness 0.1, which is 95% less than the linear passive host structure. Figure 3 illustrates the comparison between the time responses of the system by with (blue) and without (black) the active force at frequency 1.05 by solving Eq. (1) and (2). In figure 3(a) time response of the host vibrating structure is studied with cubic nonlinear stiffness 0.1 while all other parameters are kept the same as in figure 2. From these figures, one can observe that with the active control force vibration of the host structure reduces from 0.5 to 0.005 at the

Figure 3. Time domain and phase portrait response with the active controlling force for \( k_{11} = 0.1 \) (red) and without active controlling force (black) of the (a, b) host structure and (c, d) the auxiliary system.

steady state for the super-harmonic resonance condition which is shown blue line. The phase portrait of the host structure is shown in figure 3(b), from which one can observe that with the active force the host system vibration settles down swiftly by the active force. Figure 3(c) and (d) represents the time and phase response of the auxiliary system. Here, the vibration reduction of the absorber by the active control force is not observed as significant as in the case of the host structure. But nevertheless, with the active force, the response amplitude of the auxiliary system remains the same as that of a passive absorber. From figure 3 it can be noticed that with the active force the vibration in the host structure reduces and quickly settles than the passive vibration absorber while the auxiliary system response remains the same as the passive absorber. In figure 4, time response and phase portrait for the primary structure is shown where the cubic nonlinear stiffness is considered equal to 0.5. Here, also the effectiveness of active force can be observed which reduces 92% of the vibration of the host vibrating system as shown in black and blue lines. However, with increase in active force above 0.08 the vibration of the host system is more.

Figure 4. Time response with different feedbacks, (blue) displacement, (black) velocity, and (red) acceleration feedback. (a, b) host structure and (c, d) the auxiliary system.

It may be also noted that the response amplitude for the frequency and the time are in good agreement with each other at the specified operating frequency. In figure 5 effects of various feedbacks such as
displacement (blue), velocity (black), and acceleration feedback (red) with the same controlling force on the time response of the host structure and absorber are studied. From these figures, one can observe that with acceleration feedback the vibration suppression is more than other feedbacks.

![Graph](image)

**Figure 5.** Time domain and phase portrait response with different feedbacks, (blue) displacement, (black) velocity and (red) acceleration feedback. (a, b) host structure and (c, d) the auxiliary system.

**4. Conclusions**

A piezoelectric based active dynamic vibration absorber is proposed to suppress the vibration of a nonlinear SDOF system subjected to hard harmonic excitation. The frequency response of the system is obtained from the reduced equation of MMS and compared with the numerical method. The analysis by frequency and time response shows that with applied active force by acceleration feedback the vibration of the host structure reduces and settles down quickly than without controlling force. The nonlinearity in the host system makes the system unstable for a wide operating frequency range however, with the active force the system behaves like that of a linear system. Moreover, the active control force not only provides counteracting force on the vibrating structure but also tune the frequency ratio actively. The passive vibration absorber is effective for primary resonance condition while when hard excitation acts on the nonlinear host system then for superharmonic resonance condition the passive system fails to suppress the vibration as frequency ratio is not tuned. This can be overcome by using active control force which can tune the frequency of the absorber for various resonance conditions. The proposed model is a fail-safe design and outshines passive vibration absorber for superharmonic resonance condition.

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