Representations of Cuntz algebras, loop groups and wavelets

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Abstract. A theorem of Glimm states that representation theory of an NGCR $C^*$-algebra is always intractable, and the Cuntz algebra $\mathcal{O}_N$ is a case in point. The equivalence classes of irreducible representations under unitary equivalence cannot be captured with a Borel cross section. Nonetheless, we prove here that wavelet representations correspond to equivalence classes of irreducible representations of $\mathcal{O}_N$, and they are effectively labeled by elements of the loop group, i.e., the group of measurable functions $A : \mathbb{T} \to U_N(\mathbb{C})$. These representations of $\mathcal{O}_N$ are constructed here from an orbit picture analysis of the infinite-dimensional loop group.

1. Introduction

Recall the Cuntz algebra with $N$ generators $s_0, \ldots, s_{N-1}$ is the $C^*$-algebra $\mathcal{O}_N$ on the relations

\begin{equation}
  s_i^* s_j = \delta_{ij} 1, \quad \sum_{i=0}^{N-1} s_i s_i^* = 1.
\end{equation}

Cuntz [Cun77] showed that it is simple and infinite. (We will consider $N$ finite only, $N = 2, 3, \ldots$) By a theorem of [Gli60] there does not exist a Borel section parameterizing the irreducible representations of $\mathcal{O}_N$. Hence we shall restrict to special representations and consider the questions of irreducibility and decomposition. In particular we show that the elements in the loop group, i.e., all measurable maps

\[ A : \mathbb{T} \rightarrow U_N(\mathbb{C}), \]

do parameterize the equivalence classes of wavelet representations. The wavelets in turn correspond 1-to-1 to representations

\[ \pi^{(A)} \in \text{Rep}(\mathcal{O}_N, L^2(\mathbb{T})). \]

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where $T = \mathbb{R}/2\pi\mathbb{Z}$ is the 1-torus. We review joint research papers \cite{BEJ00, BrJo97, BrJo98, BrJo99, BrJo00} and the solo paper \cite{Jor00}. See also \cite{Dau92}.

If $A$ is given, we show that the operators
$$S_i^{(A)} = \pi^{(A)}(s_i)$$
are weighted shift operators on $L^2(T)$. Our papers \cite{BrJo97, BJKW00} indicate generalizations to $T^d$, $d > 1$, but we restrict to $d = 1$ here.

2. A Hilbert Module

Let $(X, \mu)$ be a measure space, and we assume that $\mu$ is a probability measure on $X$. Let $\sigma : X \to X$ be a measurable $N$-to-$1$ self map, and let $N$ be given, and fixed.

It will further be assumed that there is a probability measure $\mu$ on $X$ such that
$$1 \sum_{i=0}^{N-1} \mu \circ \sigma_i^{-1} = \mu,$$
where $\sigma_i : X \to X$ is some chosen sections for $\sigma$, i.e., satisfying
$$\sigma \circ \sigma_i = \text{id}_X, \quad i = 0, \ldots, N - 1.$$  

The analysis below refers to such a measure $\mu$. If $\sigma$ is expansive, $\mu$ is known to exist and be unique \cite{BJO99}. If $X = T$, and $\sigma(z) = z^N$, then $\mu$ is the usual Haar measure on $T$.

Let
$$A_\sigma := \{ f \circ \sigma : f \in L^\infty(X) \}.$$  

We will also assume that $L^\infty(X)$ and its subalgebras act by multiplication on the Hilbert space $\mathcal{H}_\mu = L^2(X, \mu)$.

**Lemma 2.1.** (a) A system of measurable functions
$$m_i : X \to \mathbb{C}, \quad i = 0, \ldots, N - 1,$$
forms an orthonormal basis for $\mathcal{H}_\mu$ as an $A_\sigma$-Hilbert module if and only if there are sections $\sigma_i : X \to X$, i.e., $\sigma \circ \sigma_i = \text{id}_X$ such that the $N \times N$ matrix
$$M_m := (m_i \circ \sigma_j)_{i,j=0}^{N-1}$$
is unitary,

i.e., defines
$$M_m : X \to U_N(\mathbb{C}).$$

(b) If $m_0 \in L^\infty(X)$ is given such that
$$\sum_{y: \sigma(y) = x} |m_0(y)|^2 \equiv 1 \ a.a. \ x \in X,$$
then there is a measurable selection $m_1, \ldots, m_{N-1}$ such that the combined system $m_0, \ldots, m_{N-1}$ satisfies (2.4).

(c) A function system $\{m_i\}_{i=0}^{N-1}$ satisfies the conditions in (a) if and only if the system of operators
$$S_i^{(m)} f := \sqrt{N} m_i f \circ \sigma, \quad f \in \mathcal{H}_\mu,$$
defines a representation of $\mathcal{O}_N$, i.e., $S^{(m)} \in \text{Rep}(\mathcal{O}_N, \mathcal{H}_\mu)$. (Note the right-hand side in (2.4) is the function $\sqrt{N} m_i(x)f(\sigma(x))$.)
**Definition 2.2.** The system \(\{m_i\}_{i=0}^{N-1}\) is called a quadrature mirror filter system (QMF) by analogy to the example \(N=2\), in which case \(m_0 \sim (a_k)\) serves as the low pass filter, and \(m_1 \sim (b_k)\) as the high pass filter. It is the orthogonality relations (2.4) which motivate the name QMF, and the use of filters in wavelet theory is further fleshed out in \[\text{Dau92}\].

**Proof of Lemma 2.1.** Most of the details are contained in the paper \[\text{BrJo97}\], and others will be in a later more detailed paper. But we sketch here the argument for orthogonality in the Hilbert module. Let the functions \(\{m_i\}_{i=0}^{N-1}\) be as stated in the lemma. Orthogonality refers to

\[
\langle A_m m_i \mid A_m m_j \rangle_{\mathcal{H}_m} = 0
\]

whenever \(i \neq j\). Hence we must calculate, for \(f \in L^\infty(X)\):

\[
\int_X m_i(x)f(\sigma(x))m_j(x) \, d\mu(x) = \frac{1}{N} \int_X f(x) \sum_{y: \sigma(y) = x} m_i(y)m_j(y) \, d\mu(x),
\]

using (2.4), and the sum under the integral vanishes by (2.4) if \(i \neq j\).

It is important that the selection result mentioned in (b) is generally not possible in the category of continuous functions. (See \[\text{Kad84}\].)

Conversely, if \(\{m_i\}_{i=0}^{N-1} \subset L^\infty(X)\) is given such that (2.6) defines a representation of \(\mathcal{O}_N\) on \(\mathcal{H}_m\), then the closed subspaces \(S_i^{(m)}\mathcal{H}_m\) are the submodules in a corresponding orthogonal \(A_m\)-module decomposition, \(\mathcal{H}_m = \sum_{i=0}^{N-1} [A_m m_i]\), i.e., the \(L^2(\mu)\)-closure of \(m_i A_m\) is \(S_i^{(m)}\mathcal{H}_m\) for each \(i\). Specifically, from (1.1), we get the identity \(\mathcal{H}_m \to f = \sum_i S_i^{(m)} S_i^{(m)}* f = \sum_i m_i k_i \circ \sigma\), where \(k_i = \sqrt{N} S_i^{(m)}* f\).

### 3. Comparing Representations

**Lemma 3.1.** Let \(G_N(X)\) be the group of measurable maps

\[
A: X \to U_N(C).
\]

Then \(G_N(X)\) acts transitively on the systems \(\{m_i\}_{i=0}^{N-1}\) of functions from Lemma 2.1.

**Proof.** Let \(\{m_i\}\) be given as in Lemma 2.1, and let \(A \in G_N(X)\). Set

\[
n_i(x) := \sum_{j=0}^{N-1} A_{i,j}(\sigma(x)) m_j(x).
\]

Then \(\{n_i\}\) satisfies the same orthogonality relations. For

\[
\sum_k n_i(\sigma_k(x))n_j(\sigma_k(x)) = \sum_k \sum_{l,l'} \bar{A}_{i,l}(x) \bar{m}_i(\sigma_l(x)) A_{j,l'}(x) m_j(\sigma_l(x))
= \sum_{l,l'} \delta_{l,l'} \bar{A}_{i,l}(x) A_{j,l}(x) = \sum_{l,l'} \bar{A}_{i,l}(x) A_{j,l}(x) = \delta_{i,j},
\]

proving the assertion. We used the fact that \(\sigma(\sigma_k(x)) = x\), see (2.2).

Conversely, if \(S_i^{(m)}\) and \(S_i^{(n)}\) are given representations as described in (2.6) of Lemma 2.1, then

\[
(S_i^{(m)} S_i^{(m)}) \in M_N(L^\infty(X)) = M_N(C) \otimes L^\infty(X).
\]
For it follows from the Cuntz relations (1.1) that the matrix in (3.2) is unitary, and a computation shows that its matrix entries are multiplication operators. Indeed,

\[(S^m_i)^* f(x) = \frac{1}{\sqrt{N}} \sum_{y: \sigma(y) = x} m_i(y) f(y), \quad \text{for } x \in X \text{ and for } f \in H_\mu.\]

Hence

\[(S^{(m)}_i)^* S^{(m)}_j f(x) = \sum_{y: \sigma(y) = x} n_i(y)m_j(y) f(x),\]

and so

\[A_{i,j}(x) := \sum_{y: \sigma(y) = x} m_i(y)m_j(y)\]

defines an element of \(G_N(X)\), and it satisfies (3.1) by its very construction. This proves transitivity.

When the lemma is applied to the example \(X = T, \sigma(x) = z^N, z \in T, \) and \(\sigma_k(x) = e^{2\pi i k/N} z^k, k = 0, \ldots, N - 1\), we conclude that the loop group (see Section 1) of measurable \(A: T \to U_N(\mathbb{C})\) acts transitively on the wavelet representations. Let \(n_k(z) = \frac{1}{N} z^k, k = 0, \ldots, N - 1\), and let \(\{m_k\}_{k=0}^{N-1}\) be an arbitrary \(m\)-system as in Lemma 2.1.

Then we have

**Corollary 3.2.** There is a 1-to-1 bijective correspondence between the loops \(A\) and the \(m\)-systems of Lemma 2.1 given as follows:

\[m_i(z) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} A_{i,j}(z^N) z^j\]

and

\[A_{i,j}(z) = \frac{1}{\sqrt{N}} \sum_{w \in T; w^N = z} m_i(w) w^{-j}.\]

**4. Wavelets**

Let \(\varphi \in L^2(\mathbb{R})\) be the compactly supported scaling function, i.e., a solution to the scaling identity

\[\varphi(x) = \sum_{k=0}^{Ng-1} a_k \varphi(Nx - k).\]

Then the wavelet generators \(\psi_1, \ldots, \psi_{N-1} \in L^2(\mathbb{R})\) are constructed from \(\varphi\) by use of Lemma 2.1(b) above, and standard wavelet tools from [Dau92]. The generators make the system

\[\{N^{j/2} \psi_i(N^j x - k); i = 0, 1, \ldots, N - 1, j, k \in \mathbb{Z}\}\]

into an orthonormal basis for \(L^2(\mathbb{R})\); except for a smaller variety of cases when the system is only a tight frame. The coefficients \(\{a_k\}\) represent a wavelet filter, and

\[m_0(z) = \sum_k a_k z^k.\]
Then define the operator
\( W : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}) \)
by
\[
W\xi = \sum_{k} \xi_k \varphi(x - k).
\]
(4.4)

The conditions on the wavelet filter \( \{a_k\} \) may now be restated in terms of \( m_0(z) \) in (4.2) as follows:
\[
\sum_{k=0}^{N-1} \left| m_0(ze^{i\frac{2\pi k}{N}}) \right|^2 = 1,
\]
(4.5)
and
\[
m_0(1) = 1, \quad \text{the low pass property.}
\]
(4.6)

Then \( W \) in (4.4) maps \( \ell^2(\mathbb{Z}) \) onto the resolution subspace \( \mathcal{V}_0 (\subset L^2(\mathbb{R})) \), and we note that
\[
U_N W = W S_0,
\]
(4.7)
where
\[
U_N f(x) = N^{-\frac{1}{2}} f \left( \frac{x}{N} \right), \quad f \in L^2(\mathbb{R}), \ x \in \mathbb{R}.
\]
(4.8)

We showed in [BrJo00] that there are functions \( m_1, \ldots, m_{N-1} \) such that the \( N \)-by-\( N \) complex matrix
\[
\left( m_j(e^{i\frac{2\pi k}{N}}) \right)_{j,k=0}^{N-1}
\]
is unitary for all \( z \in \mathbb{T} \). (See Lemma 2.1(b).) We define
\[
S_j f(z) = \sqrt{N} m_j(z) f(z^N), \quad f \in L^2(\mathbb{T}), \ z \in \mathbb{T}.
\]
(4.10)

The main result will be stated in the present section, but without proof. Instead the reader is referred to [Jor00] for the full proof, and for a detailed discussion of its implications. We noted above that the representation (4.10) given from a QMF system \( m_j = m_j^{(A)} \), via
\[
m_j^{(A)}(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_{j,k}(z^N) z^k,
\]
(4.11)
\[
A_{j,k}(z) = \frac{1}{\sqrt{N}} \sum_{w^N = z} w^{-k} m_j(w)
\]
is irreducible if and only if the subbands are optimal, in that they do not admit further reduction into a refined system of closed subspaces of \( L^2(\mathbb{R}) \).

**Theorem 4.1.** The representation
\[
S_j^{(A)} f(z) = \sqrt{N} m_j^{(A)}(z) f(z^N), \quad f \in L^2(\mathbb{T}), \ z \in \mathbb{T},
\]
(4.10)
is an irreducible representation of $O_N$ on $L^2(\mathbb{T})$ if and only if $A : \mathbb{T} \to U_N(\mathbb{C})$ does not admit a matrix corner of the form

\[
V \begin{pmatrix}
  z^{n_0} & 0 & & \\
  & z^{n_1} & & \\
  & & \ddots & \\
  0 & & & z^{n_{M-1}}
\end{pmatrix},
\]

(4.13)

for some $V \in U_M(\mathbb{C})$, and where $n_0, n_1, \ldots, n_{M-1} \in \{0, 1, 2, \ldots\}$. Moreover, two representations $\pi^{(A)}$ and $\pi^{(B)}$ defined from different loops $A, B$ are unitarily inequivalent unless $A \equiv B$ modulo a matrix corner of type (4.13).

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