New multidimensional partially integrable generalization of \(S\)-integrable \(N\)-wave equation

A.I. Zenchuk
Center of Nonlinear Studies of L.D.Landau Institute for Theoretical Physics
(International Institute of Nonlinear Science)
Kosygina 2, Moscow, Russia 119334
E-mail: zenchuk@itp.ac.ru

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Abstract

This paper develops a modification of the dressing method based on inhomogeneous linear integral equation with integral operator having nonempty kernel. Method allows one to construct the systems of multidimensional Partial Differential Equations (PDEs) having differential polynomial structure in any dimension \(n\). Associated solution space is not full, although it is parametrized by certain number of arbitrary functions of \((n-1)\) variables. We consider 4-dimensional generalization of the classical \((2+1)\)-dimensional \(S\)-integrable \(N\)-wave equation as an example.

1 Introduction

Completely integrable nonlinear Partial Differential Equations (PDEs) became an attractive field of research after the discovery of complete integrability of the Korteweg-de Vries equation [1]. \(S\)-integrable [2, 3, 4, 5, 6, 7, 8] and \(C\)-integrable [9, 10, 11, 12, 13, 14] systems are mostly remarkable among multidimensional nonlinear PDEs. The dressing method is one of the promoted methods for constructing and solving \(S\)-integrable PDEs. After the original version of the dressing method [2] several important modifications have been developed [3, 4, 5, 15, 16, 17] (see also [7, 8]). All of them are based either on a Riemann-Hilbert or on a \(\bar{\partial}\)-problem [18, 19], which are uniquely solvable linear integral equations for some matrix function \(U(\lambda; x)\) depending on spectral parameter \(\lambda\), where independent variables of nonlinear PDEs appear as a set of additional parameters \(x = (x_1, x_2, \ldots)\).

This paper is based on the modification of the \(\bar{\partial}\)-problem introduced in [20, 21]. It was shown [21], that classical \(S\)-integrable PDEs, some types of \(C\)-integrable PDEs and some types of "mixed" PDEs (i.e. equations having \(S\)- and \(C\)-integrable systems as the particular cases) can be studied by the dressing method based on the following integral equation, which is equivalent to the \(\bar{\partial}\)-problem [4, 5]:

\[
\Phi(\lambda; x) = \int \Psi(\lambda, \mu; x)U(\mu; x)d\Omega(\mu) \equiv \Psi(\lambda, \mu; x) \ast U(\mu; x) \equiv \hat{\Psi}U. \tag{1}
\]
Here $\lambda$ and $\mu$ are spectral variables, $U$ is the unknown matrix function. The matrix functions $\Phi$ and $\Psi$ are defined by some extra conditions, which fix their dependence on the additional parameters $x_i$. $\Omega$ is some largely arbitrary scalar measure in the $\mu$-space. Apart from $\Omega$, all the functions appearing in this paper are $Q \times Q$ matrix functions.

To describe the classical completely integrable equations, kernel of the integral operator $\Psi$ must be trivial, i.e. $\dim \ker \Psi = 0$. It was shown in [22] that this requirement is too restrictive. More general dressing algorithm must be based on the integral operator with $\dim \ker \Psi > 0$. So, some examples of nonlinear PDEs corresponding to $\dim \ker \Psi = 1$ have been derived in [22]. Generalization $\dim \ker \Psi > 0$ removes any formal restriction on dimensionality of the constructed nonlinear PDEs, although derived systems have restricted solution spaces. Namely, $n$-dimensional nonlinear system admit at most $(n - 2)$-dimensional solution space, $n \geq 1$. The increase of dimensionality of nonlinear PDEs in comparison with classical $S$-integrable equations happens due to the presence of the external dressing function in the algorithm, whose dimensionality has no restriction. Remember that the classical dressing algorithm uses only the internal dressing functions whose dimensionalities are strongly restrictive and usually equal 1. However, the new version of the dressing algorithm has an important common feature with the classical algorithm. Namely, derivatives $\Psi_{x_j}^s(\lambda, \mu; x)$ are separated functions of the spectral parameters $\lambda$ and $\mu$. This fact fixes dimensionalities of the internal dressing functions mentioned above. This property has been removed in [23] (i.e. $\Psi_{x_j}^s(\lambda, \mu; x)$ are not separated function of spectral parameters there) keeping $\dim \ker \Psi = 0$. But there is serious obstacles for construction of the explicit solutions, which have not been found in [23].

Another problem appeared in [22] is a complicated "block" structure of the derived nonlinear PDEs, which is not typical for the equations of mathematical physics.

Combination of basic ideas of both refs. [22] and [23] enriches the solution space and improves structure of the system of nonlinear PDEs derived in this paper. The dimensionality $n$ of the resulting system of nonlinear PDEs has no restriction and its solution space is parametrized by certain number of arbitrary functions of $(n - 1)$ variables, although the full integrability is not achieved yet. Nevertheless, this is a progress in comparison with the results of [22], where the solution space of the derived system of nonlinear PDEs is parametrized by arbitrary functions of $(n - 2)$ variables. Structure of nonlinear PDEs is also improved: system of equations has differential polynomial structure.

Algorithm represented in this paper is based on the following modification of the eq.(1):

$$\Phi^{(sk)}(\lambda; x) = \sum_{n=1}^{Q} \int T^{(n)} \Psi^{(s)}(\lambda, \mu; x)U^{(nk)}(\mu; x)d\Omega(\mu) = \equiv \sum_{n=1}^{Q} \sum_{s=1}^{Q} \Psi^{(sn)}(\lambda, \mu; x) \ast U^{(nk)}(\mu; x) = \equiv \sum_{n=1}^{Q} \Psi^{(sn)}(\lambda, \mu; x) \ast U^{(nk)}(\mu; x) \equiv \sum_{n=1}^{Q} \hat{\Psi}^{(sn)}U^{(nk)},$$

which must be solved for the matrix spectral functions $U^{(nm)}$. Here $T^{(n)}$ are constant diagonal matrices, functions $\Phi^{(s)}$ and $\Psi^{(s)}$ are called the internal dressing functions and will be specified below. We write superscripts inside of parenthesis in order to distinguish them from the power notations.
It was observed that the integral equation (2) may be simplified taking

\[ T^{(n)}_\alpha = \delta^{(n)}_\alpha = \begin{cases} 
1, & n = \alpha \\
0, & n \neq \alpha
\end{cases} \]  

(3)

and the diagonal function \( \Phi^{(sm)} \) in all examples studied in this paper. So that the integral equation reads:

\[
\Phi^{(sk)}_\beta \delta_{\alpha\beta}(\lambda; x) = \sum_{n,\gamma=1}^{Q} \delta^{(n)}_\alpha \Psi^{(s)}_{\alpha\gamma}(\lambda, \mu; x) \ast U^{(nk)}_{\gamma\beta} \Rightarrow
\]

\[
\begin{cases}
\sum_{\gamma=1}^{Q} \Psi^{(s)}_{\beta\gamma}(\lambda, \mu; x) \ast U^{(\beta k)}_{\gamma\beta}(\mu; x) = \Phi^{(sk)}_\beta(\lambda; x), & \alpha = \beta \\
\sum_{\gamma=1}^{Q} \Psi^{(s)}_{\alpha\gamma}(\lambda, \mu; x) \ast U^{(\alpha k)}_{\gamma\beta}(\mu; x) = 0, & \alpha \neq \beta 
\end{cases}
\]

(5)

Hereafter (except Sec.3) double subscript (usually Greek) means the element of the appropriate matrix and single subscript denotes the nonzero element of the appropriate diagonal matrix unless different is specified.

At first glance, following the philosophy of the dressing method based on the integral equation, integral equations for \( U^{(\beta k)}_{\gamma\beta}(\lambda; x) \) and \( U^{(\alpha k)}_{\gamma\beta}(\lambda; x) \), \( \alpha \neq \beta \) are decoupled. This might result in appropriate decoupling of the system of nonlinear PDEs. Moreover, the integral eqs. (5a) and (5b) with different values of the indexes \( \alpha \) and \( \beta \) might be decoupled as well. However, we will show that general situation is different and the above statements are partially correct. Namely, we will obtain an example of the complete system of nonlinear PDEs written for the fields produced by \( U^{(1k)}_{\gamma1}(\lambda; x) \) and \( U^{(1k)}_{\gammaQ}(\lambda; x) \), \( k, \gamma = 1, \ldots, Q \). However, more general PDEs may couple fields produced by all functions \( U^{(\alpha k)}_{\gamma\beta}(\lambda; x) \), \( k, \alpha, \beta, \gamma = 1, \ldots, Q \) (see Sec.2.2, item (d.3) of Proposition).

In the next section (Sec.2) we represent a generalization of the dressing method for the classical (2+1)-dimensional N-wave hierarchy. In Sec. 2.1 we discuss the case \( \dim \ker \Psi = 0 \) giving the classical (2+1)-dimensional N-wave equation

\[
B^{(2)} v_t B^{(1)} - B^{(1)} v_t B^{(2)} + [v_{y_1}, B^{(2)}] - [v_{y_2}, B^{(1)}] + [[v, B^{(1)}], [v, B^{(2)}]] = 0,
\]

(6)

where \( B^{(i)} \) are constant diagonal matrices, \( v \) is \( Q \times Q \) matrix field, \( t \) is time and \( y_i, i = 1, 2 \), are space variables. Its generalization, system of 4-dimensional 1-st order quasilinear PDEs solvable by our algorithm with \( \dim \ker \Psi = 1 \), is proposed in Sec.2.2. The whole system of nonlinear PDEs derived there may be separated into two subsystems. The first subsystem is the complete system of evolution equations for the \( Q \times Q \) matrix fields \( w^{(p)}, q^{(p)}, p = 1, 2, v \)
and $u$:

$$s^{(4;w;4)}_{\alpha\beta} \partial_t w_{\alpha\beta} + \sum_{m=1}^{Q} s^{(4;w;m)}_{\alpha\beta} \partial_m w_{\alpha\beta} - \sum_{\gamma=1}^{\infty} w^{(p)}_{\alpha\gamma} v_{\gamma\beta} T^{(4;wwv)}_{\alpha\gamma\beta} +$$ (7)

$$\sum_{\gamma,\delta=1}^{Q} q^{(p)}_{\alpha\gamma} v_{\delta\beta} T^{(4;wwv)}_{\alpha\gamma\delta} + \sum_{\gamma=1}^{Q} 2 \sum_{i_0=1}^{\infty} q^{(p)}_{\alpha\gamma} w^{(i_0)}_{\gamma\beta} T^{(4;ww;io)}_{\alpha\gamma\beta} = 0,$$

$$s^{(4;q;4)}_{\alpha} \partial_t q_{\alpha\beta} + \sum_{m=1}^{Q} s^{(4;q;m)}_{\alpha} \partial_m q_{\alpha\beta} - \sum_{\gamma=1}^{Q} w^{(p)}_{\alpha\gamma} u_{\gamma\beta} T^{(4;quw)}_{\alpha\gamma\beta} +$$ (8)

$$\sum_{\gamma,\delta=1}^{Q} q^{(p)}_{\alpha\gamma} u_{\delta\beta} T^{(4;quv)}_{\alpha\gamma\delta} + \sum_{\gamma=1}^{Q} 2 \sum_{i_0=1}^{\infty} q^{(p)}_{\alpha\gamma} q^{(i_0)}_{\gamma\beta} T^{(4;quv;io)}_{\alpha\gamma\beta} = 0,$$

$$s^{(4;v;4)}_{\alpha\beta} \partial_t v_{\alpha\beta} + \sum_{m=1}^{Q} s^{(4;v;m)}_{\alpha\beta} \partial_m v_{\alpha\beta} - \sum_{\gamma=1}^{\infty} v^{(p)}_{\alpha\gamma} v_{\gamma\beta} T^{(4;vvv)}_{\alpha\gamma\beta} +$$ (9)

$$\sum_{\gamma,\delta=1}^{\infty} u^{(p)}_{\alpha\gamma} v_{\delta\beta} T^{(4;vvv)}_{\alpha\gamma\delta} + \sum_{\gamma=1}^{Q} u^{(p)}_{\alpha\gamma} \sum_{i_0=1}^{\infty} w^{(i_0)}_{\gamma\beta} T^{(4;vwv;io)}_{\alpha\gamma\beta} = 0, \quad \alpha \neq \beta,$$

$$s^{(4;u;4)}_{\alpha} \partial_t u_{\alpha\beta} + s^{(4;u;1)}_{\alpha} \partial_1 u_{\alpha\beta} - \sum_{\gamma=1}^{\infty} v^{(p)}_{\alpha\gamma} u_{\gamma\beta} T^{(4;uwu)}_{\alpha\gamma\beta} +$$ (10)

$$\sum_{\gamma,\delta=1}^{\infty} u^{(p)}_{\alpha\gamma} u_{\delta\beta} T^{(4;uuv)}_{\alpha\gamma\delta} + \sum_{\gamma=1}^{Q} 2 \sum_{i_0=1}^{\infty} u^{(p)}_{\alpha\gamma} u^{(i_0)}_{\gamma\beta} T^{(4;uwv;io)}_{\alpha\gamma\beta} = 0,$$

The equations of the second subsystem must be viewed as a symmetry (i.e. compatible) constraints to the system (7-10):

$$s^{(3;q;3)}_{\alpha} \partial_t q^{(p)}_{\alpha\beta} + \sum_{m=1}^{Q} s^{(3;q;m)}_{\alpha} \partial_m q^{(p)}_{\alpha\beta} - \sum_{\gamma=1}^{\infty} w^{(p)}_{\alpha\gamma} u_{\gamma\beta} T^{(3;quw)}_{\alpha\gamma\beta} +$$ (11)

$$\sum_{\gamma,\delta=1}^{Q} q^{(p)}_{\alpha\gamma} u_{\delta\beta} T^{(3;quv)}_{\alpha\gamma\delta} + \sum_{\gamma=1}^{Q} 2 \sum_{i_0=1}^{\infty} q^{(p)}_{\alpha\gamma} q^{(i_0)}_{\gamma\beta} T^{(3;quv;io)}_{\alpha\gamma\beta} = 0,$$

$$\sum_{m=1}^{Q} q^{(p)}_{\alpha\gamma} \partial_m T^{(4;wwv)}_{\alpha\gamma\delta} + \sum_{\gamma=1}^{Q} 2 \sum_{i_0=1}^{\infty} q^{(p)}_{\alpha\gamma} q^{(i_0)}_{\gamma\beta} T^{(4;wwv;io)}_{\alpha\gamma\beta} = 0.$$
Although eqs.(7-10) represent the complete system for unknown fields, the dressing method provides only such solutions which satisfy the symmetry constraints (11-13). Here the constant parameters $T$ and $s$ are expressed in terms of the dressing data, see item (c) of Proposition, eqs.(52) for details. Not all of them may be arbitrary. This system has the remarkable reduction: if $u = 0$ then the eq.(9) yields the $N$-wave equation (6) while the eq.(8) yields the following linearizable system

$$\partial_{r_\alpha}q^{(p)}_{\alpha\beta} + \sum_{\gamma=1}^{Q} \sum_{i_0=1}^{2} q^{(p)}_{\alpha\gamma} q^{(i_0)}_{\gamma\beta} T^{(j;\gamma i_0)} = 0, \quad j = 3, 4,$$

where $r_\alpha$ are characteristic variables, see eq.(53) for more details. Linearization of this system will be demonstrated in Sec.3.1.

Note, that another system of PDEs having $S$- and $C$-integrable systems as particular reductions has been derived in [21]. Similar to the PDEs derived in [21], the system (7-10) consist of 4 types of nonlinear equations, which defer by dimensionalities of their linear parts. It is 2 in the eq.(10), 3 in the eqs.(8,9) and 4 in the eq.(7). However, the structure of the nonlinear PDEs is significantly different. In particular, some coefficients of the system (7-10) depend on the parameters $R^{(i_3)}_{\gamma_1}$, $\gamma_1, \delta = 1, \ldots, Q$, reflecting the fact that dim ker $\Psi = 1$ (see eq.(41)). Such parameters do not appear in [21] since dim ker $\Psi = 0$ therein.

Let us clarify which kind of initial-boundary value problem may be solved, in principle, numerically. It may be easily seen that the fields $w^{(p)}$ may be given arbitrary initial conditions (i.e. values at $t = 0$) in the whole three dimensional space $(y_1, y_2, y_3)$. The fields $q^{(p)}$ and $v$ may be given arbitrary initial conditions only on the plane (for instance, $y_3 = 0$) because of the constraints (11,12). In order to define their initial conditions in the whole space $(y_1, y_2; y_3)$ we must solve the system (11,12). Field $u$ is even more restrictive because of the constraint (13). Initial condition for $u$ may be arbitrary on the line (for instance, $y_3 = y_2 = 0$). To define initial condition in the whole space $(y_1, y_2, y_3)$ we must solve the system (13). Finally, $t$-evolution of all fields may be established solving the system (7-10). We conclude that correctly formulated initial-boundary value problem requires the following initial-boundary conditions: two arbitrary matrix functions of 3 variables $w^{(p)}|_{t=0}$, $p = 1, 2$, three arbitrary matrix functions of 2 variables $q^{(p)}|_{t=y_3=0}$, $p = 1, 2$ and $v|_{t=y_3=0}$, and single arbitrary matrix function of one variable $u|_{t=y_3=y_2=0}$. If such initial-boundary data are provided by the dressing method we would consider the system (7-10) with compatible constraints (11-13) as a completely integrable one.
However, we will show that the represented dressing algorithm may supply only single arbitrary matrix function of three variables, two arbitrary matrix functions of two variables and single arbitrary matrix function of one variable. For this reason, using our dressing algorithm, we are not able to prescribe arbitrary initial data to all fields. This fact causes us to refer to such equations as partially integrable PDEs. It will be shown by construction that they admit infinitely many commuting flows.

Remark, that another system of nonlinear evolution PDEs with symmetry constraints for some fields in the context of the dressing method has been derived in [24].

Detailed derivation of the system (7-13) is given in Appendix, Sec.5. Sec.3 describes the solution space to the 4-dimensional system (7-13). Conclusions are represented in Sec.4.

2 First order nonlinear PDEs: higher dimensional generalization of the $S$-integrable $N$-wave equation

Dressing method for the $(2+1)$-dimensional $N$-wave equation can be generalized to higher dimensions. Here we consider a variant of such generalization. For this purpose, function $\Psi(s)(\lambda, \mu; x)$ in (4) must be related with the dressing functions $\Phi^{(sm)}(\lambda; x)$ and $C^{(m)}(\mu; x)$ by the following bi-linear system of equations introducing dependence on the variables $x_i, i = 1, 2, \ldots$:

$$\partial_{x_j} \Psi^{(s)}(\lambda, \mu; x) - A^{(j)} \partial_{x_1} \Psi^{(s)}(\lambda, \mu; x) = \sum_{k=1}^{Q} \Phi^{(sk)}(\lambda; x) B^{(kj)} C^{(k)}(\mu; x),$$  

$$B^{(k2)} = I, \quad j \geq 2.$$  

Here $I$ is the identity matrix, $B^{(kj)}$ and $A^{(j)}$ are constant diagonal matrices, dressing functions $C^{(k)}(\lambda; x), k = 1, \ldots, Q,$ will be specified below. As far as the system (15) is overdetermined system of PDEs for the functions $\Psi^{(s)}$, this system must be compatible. Compatibility condition reads

$$\sum_{k=1}^{Q} \left[ \partial_{x_j} \left( \Phi^{(sk)}(\lambda; x) B^{(ki)} C^{(k)}(\mu; x) \right) - \partial_{x_i} \left( \Phi^{(sk)}(\lambda; x) B^{(kj)} C^{(k)}(\mu; x) \right) - A^{(j)} \partial_{x_1} \left( \Phi^{(sk)}(\lambda; x) B^{(ki)} C^{(k)}(\mu; x) \right) + A^{(i)} \partial_{x_1} \left( \Phi^{(sk)}(\lambda; x) B^{(kj)} C^{(k)}(\mu; x) \right) \right] = 0.$$  

Eq.(16) consists of terms represented by products of functions depending on single spectral parameter. We would like to separate PDEs involving different spectral parameters, i.e. either parameter $\lambda$ or $\mu$. This is possible due to the diagonal form of $\Phi^{(sm)}$ which provides commutativity of $A^{(i)}$ and $\Phi^{(sk)}$:

$$A^{(i)} \Phi^{(sk)}(\lambda; x) = \Phi^{(sk)}(\lambda; x) A^{(i)}, \quad \forall \ i, s, k,$$  

We put $i = 2$ in (16) without loss of generality and separate PDEs involving different spectral parameters:

$$\partial_{x_j} \Phi^{(sk)}(\lambda; x) = \partial_{x_1} \Phi^{(sk)}(\lambda; x) P^{(kj)} + \partial_{x_2} \Phi^{(sk)}(\lambda; x) B^{(kj)},$$  

$$\partial_{x_j} C^{(k)}(\mu; x) = P^{(kj)} \partial_{x_1} C^{(k)}(\mu; x) + B^{(kj)} \partial_{x_2} C^{(k)}(\mu; x)$$  

$$P^{(kj)} = A^{(j)} - A^{(2)} B^{(kj)}, \quad j > 2.$$
Eqs. (17) and (18) must be used for the derivation of nonlinear PDEs as follows. First, substitute $\Phi^{(sn)}$ from (4) into (17):

$$\sum_{n=1}^{Q} A^{(j)}\Psi^{(sn)}(\lambda, \mu; x) * U^{(nk)}(\mu; x) - \sum_{n=1}^{Q} \Psi^{(sn)}(\lambda, \mu; x) * U^{(nk)}(\mu; x)A^{(j)} = 0,$$  \(21\)

Second, substitute $\Phi^{(sk)}$ from (4) into (18) (we use (17) to result in (22)):

$$\sum_{n=1}^{Q} \left[ \partial_{x_1}\Psi^{(sn)}(\lambda, \mu; x) * U^{(nk)}(\mu; x) \right] - A^{(j)}\partial_{x_1}\Psi^{(sn)}(\lambda, \mu; x) * U^{(nk)}(\mu; x) -
$$

$$\left( \partial_{x_2}\Psi^{(sn)}(\lambda, \mu; x) * U^{(nk)}(\mu; x) - A^{(2)}\partial_{x_1}\Psi^{(sn)}(\lambda, \mu; x) * U^{(nk)}(\mu; x) \right)B^{(kj)} = 0.$$  \(22\)

Eqs. (21,22) must be reduced to homogeneous equations (see eq.(23)). For this purpose we substitute $\Psi^{(s)}_{x_j}$ from (15) into (21,22) which results in the next homogeneous equations:

$$\sum_{n=1}^{Q} \Psi^{(sn)}(\lambda, \mu; x) * L^{(ji;nk)}(\mu; x) = 0, \quad i = 1, 2,$$  \(23\)

where

$$L^{(j1;nk)}(\lambda; x) \equiv U^{(nk)}B^{(j;n)}, \quad j \geq 2,$$  \(24\)

$$L^{(j2;nk)}(\lambda; x) \equiv \partial_{x_1}U^{(nk)} + \partial_{x_2}U^{(nk)}B^{(kj;n)} - \partial_{x_2}U^{(nk)}B^{(kj)} +
$$

$$\sum_{i_1=1}^{Q} U^{(ni_1)} \left( B^{(ij_1)}v^{(ni_1;k)} - v^{(ni_1;k)}B^{(kj)} \right), \quad j \geq 3,$$  \(25\)

fields $v^{(lik)}$ are introduced by the formulae

$$v^{(lik)}(x) = C^{i}(\mu; x) * U^{(lk)}(\mu; x),$$  \(26\)

and the diagonal matrices $B^{(j;n)}$ and $B^{(kj;n)}$ are given by the formulae

$$B^{(j;n)} = A^{(j)}I - A^{(j)}, \quad B^{(kj;n)} = B^{(kj)}A^{(2)} - A^{(j)}I.$$  \(27\)

In addition, applying operators $(\partial_{x_j} - A^{(j)}\partial_{x_1})$ to the eq.(23) with $i = 1$ and $j = j_0$, one gets one more homogeneous equation:

$$\sum_{n=1}^{Q} \Psi^{(sn)}(\lambda, \mu; x) * L^{(j3;nk)}(\mu; x) = 0,$$  \(28\)

where

$$L^{(j3;nk)}(\lambda; x) \equiv \partial_{x_2}U^{(nk)} - \partial_{x_1}U^{(nk)}A^{(j)} + \sum_{i_1=1}^{Q} U^{(ni_1)}B^{(ij_1)}v^{(ni_1;k)}B^{(kj_0;n)}, \quad j \geq 2.$$  \(29\)

Comparing the eq.(20) with the eq.(27) we see that

$$P^{(kj)}_{\alpha} = -B^{(kj;\alpha)}.$$  \(30\)

Although this paper is devoted to the integral equation (4) with dimker $\Psi^{(sn)} > 0$, we consider the case dimker $\Psi^{(sn)} = 0$ in the next subsection for the sake of completeness. The case dimker $\Psi^{(sn)} = 1$ will be considered in Sec.2.2.
2.1 \( \dim \ker \hat{\Psi}^{(sn)} = 0 \). S-integrable (2+1)-dimensional N-wave equation

If \( \dim \ker \hat{\Psi}^{(sn)} = 0 \) then the homogeneous equation

\[
\sum_{n=1}^{Q} \hat{\Psi}^{(sn)} H^{(nk)} = 0
\]  

has only the trivial solution \( H^{(nk)} = 0 \), i.e.

\[
L^{(j1;nk)} := U_{\alpha \beta}^{(nk)} B^{(jkn)}_{\beta} = 0
\]  

\[
L^{(j2;nk)} := \partial_{x_j} U_{\alpha \beta}^{(nk)} + \partial_{x_1} U_{\alpha \beta}^{(nk)} B^{(kjm)}_{\beta} - \partial_{x_2} U_{\alpha \beta}^{(nk)} B^{(kj)}_{\beta} + \sum_{i,\gamma=1}^{Q} U_{\alpha \gamma}^{(ni)} v_{\gamma \beta}^{(njk)} (B^{(ij)}_{\gamma} - B^{(kj)}_{\beta}) = 0,
\]

\[
L^{(j3;nk)} := \left( \partial_{x_j} U_{\alpha \beta}^{(nk)} - \partial_{x_1} U_{\alpha \beta}^{(nk)} A_{n}^{j} + \sum_{i,\gamma=1}^{Q} U_{\alpha \gamma}^{(ni)} v_{\gamma \beta}^{(njk)} B_{\gamma}^{(ij)} \right) B^{(jo;nn)}_{\beta} = 0, \quad j \geq 2. \]  

Since \( B^{jn}_{n} = 0 \), eq.(32) tells us that \( U_{\alpha \beta}^{(nk)} = 0 \) if \( n \neq \beta \), and consequently \( v_{\alpha \beta}^{(njk)} = 0 \) if \( n \neq \beta \). Then, eq.(34) is identical to zero, while eqs.(33) with different values of \( \beta \) become decoupled:

\[
V_{t_j}(\lambda; x) - V_{t_2}(\lambda; x)B^{(j)} - V(\lambda; x)[v(x), B^{(ij)}] = 0, \quad j = 3, 4, \ldots,
\]

where, for fixed \( \beta \),

\[
V_{tk} = U_{\alpha \beta}^{(jk)}, \quad v_{tk} = v_{\beta \beta}^{(jk)}, \quad B_{k}^{(j)} = B_{\beta}^{(kj)}, \quad \alpha, i, k = 1, \ldots, Q,
\]

\[
\partial_{t_j} = \partial_{x_j} - A_{\beta}^{(j)} \partial_{x_1}, \quad j = 2, 4, \ldots.
\]

Eqs.(35) represent the linear overdetermined system for the spectral function \( V \). Compatibility condition of (35) yields the classical \( S \)-integrable (2+1)-dimensional \( N \)-wave equation:

\[
[v_{tk}, B^{(j)}] - [v_{t_j}, B^{(k)}] + B^{(j)} v_{tk} B^{(k)} - B^{(k)} v_{t_j} B^{(j)} + [[v, B^{(k)}], [v, B^{(j)}]] = 0, \quad k \neq j = 3, 4, \ldots.
\]

This example justifies the fact that our dressing algorithm with \( \dim \ker \hat{\Psi}^{(sn)} = 0 \) gives rise to the classical \( S \)-integrable models.

2.2 \( \dim \ker \hat{\Psi}^{(sn)} = 1 \). Higher dimensional nonlinear PDEs

Let \( \dim \ker \hat{\Psi}^{(sn)} = 1 \). Then the solution space of the homogeneous equation (31) is parametrized by the arbitrary \( Q \times Q \) matrix functions \( f^{(ik)}(x) \), \( i, k = 1, \ldots, Q \):

\[
(U^{h})^{(nk)}(\lambda; x) = \sum_{i=1}^{Q} H^{(ni)}(\lambda; x) f^{(ik)}(x).
\]

This equation establishes a linear relation between any two solutions \( U^{(nk)} \), \( L^{(j1;nk)} \), \( L^{(j3;nk)} \), \( j \geq 2 \) and \( L^{(j2;nk)} \), \( j \geq 3 \) of the homogeneous integral equation (31). These linear relations represent a new overdetermined system of linear equations for the spectral functions \( U^{(nk)} \).
Details of derivation of the system of linear PDEs for the spectral function $U^{(nk)}$ as well as derivation of the associated system of nonlinear PDEs are given in Appendix, Sec. 5. Here we collect all basic results in the following Proposition.

**Proposition.** Let $\dim \ker \hat{\Psi}^{(s)} = 1$ in the integral equation (4) where the dressing functions $\Psi^{(s)}$, $\Phi^{(sk)}$ and $C^{(k)}$ are solutions of the eqs. (15,17,18,19). Then the complete system of nonlinear PDEs can be derived for the following fields $R$, where

\[ F^{nk}(\lambda; x) = G_{x_j}^{(j_1)} G_{x_1}^{(j_2)} \theta^{(j_3)} G_{x_2}^{(j_4)}(\lambda; x), \quad j > 2 \]

with arbitrary diagonal matrices $\mathcal{P}^{(j)}$, $i = 1, 2$. Replace $x_j$ by the new independent variables

\[ \partial_{x_j} = \partial_{x_j} - A_{1}^{(j)} \partial_{x_1}, \quad j \geq 2 \quad \Rightarrow \quad t = t_5, \quad y_j = t_{j+1}, \quad j = 1, 2, 3. \]

Then the complete system of nonlinear PDEs can be derived for the following fields

\[ u^{(nk)}_{\alpha\beta}, \quad q^{(nk)}_{1\alpha\beta}, \quad v^{(nk)}_{1\alpha\beta}, \quad u^{(1\gamma\delta)}_{\alpha\beta}, \quad p = 1, 2. \]

This system of PDEs is naturally separated into two subsystems. The first subsystem is the...
The second subsystem must be viewed as a system of symmetry (i.e. compatible) constraints to the system (45-48):

\[
\begin{align*}
E^{(56;1/\beta)}_{\alpha_1} & := s^{(4;w;4)}_{\alpha_\beta} \partial_t u^{(p)}_{\alpha_\beta} + \sum_{m=1}^{3} s^{(4;w;m)}_{\alpha_\beta} \partial_{y_m} u^{(p)}_{\alpha_\beta} - \sum_{\gamma=1}^{Q} u_{\alpha_\gamma} v_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(4;wuv)} + \tag{45} \\
& \quad + \sum_{\gamma, \beta=1}^{Q} q_{\alpha_\gamma}^{(p)} v_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(4;wuv)} + \sum_{\gamma=1}^{Q} \sum_{\gamma_\beta=1}^{2} q_{\alpha_\gamma}^{(p)} w_{\gamma_\beta}^{(i_\beta)} T_{\alpha_\gamma \beta}^{(4;wuv;i_\alpha)} = 0, \\
E^{(57;1/\beta)}_{\alpha} & := s^{(4;q;4)}_{\alpha} \partial_t q^{(p)}_{\alpha_\beta} + \sum_{m=1}^{2} s^{(4;q;m)}_{\alpha} \partial_{y_m} q^{(p)}_{\alpha_\beta} - \sum_{\gamma=1}^{Q} u_{\alpha_\gamma} v_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(4;qwu)} + \tag{46} \\
& \quad + \sum_{\gamma, \beta=1}^{Q} q_{\alpha_\gamma}^{(p)} u_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(4;qqu)} + \sum_{\gamma=1}^{Q} \sum_{\gamma_\beta=1}^{2} q_{\alpha_\gamma}^{(p)} w_{\gamma_\beta}^{(i_\beta)} T_{\alpha_\gamma \beta}^{(4;qqu;i_\alpha)} = 0, \\
E^{(52;1/\alpha_\beta)}_{11} & := s^{(4;v;4)}_{\alpha_\beta} \partial_t v_{\alpha_\beta} + \sum_{m=1}^{2} s^{(4;v;m)}_{\alpha_\beta} \partial_{y_m} v_{\alpha_\beta} - \sum_{\gamma=1}^{Q} v_{\alpha_\gamma} v_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(4;vu)} + \tag{47} \\
& \quad + \sum_{\gamma, \beta=1}^{Q} u_{\alpha_\gamma} v_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(4;vuv)} + \sum_{\gamma=1}^{Q} \sum_{\gamma_\beta=1}^{2} w_{\gamma_\beta}^{(i_\beta)} T_{\alpha_\gamma \beta}^{(4;vuv;i_\alpha)} = 0, \quad \alpha \neq \beta, \\
E^{(53;1/\alpha_\beta)}_{1} & := s^{(4;w;4)}_{\alpha} \partial_t u_{\alpha_\beta} + s^{(4;w;1)}_{\alpha} \partial_{y_1} u_{\alpha_\beta} - \sum_{\gamma=1}^{Q} v_{\alpha_\gamma} u_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(4;wua)} + \tag{48} \\
& \quad + \sum_{\gamma, \beta=1}^{Q} u_{\alpha_\gamma} u_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(4;wua)} + \sum_{\gamma=1}^{Q} \sum_{\gamma_\beta=1}^{2} u_{\alpha_\gamma} q_{\gamma_\beta}^{(i_\beta)} T_{\alpha_\gamma \beta}^{(4;wua;i_\alpha)} = 0,
\end{align*}
\]

The second subsystem must be viewed as a system of symmetry (i.e. compatible) constraints to the system (45-48):

\[
\begin{align*}
\tilde{E}^{(47;1/\beta)p} & := s^{(3;q;3)}_{\alpha} \partial_{y_1} q^{(p)}_{\alpha_\beta} + \sum_{m=1}^{2} s^{(3;q;m)}_{\alpha} \partial_{y_m} q^{(p)}_{\alpha_\beta} - \sum_{\gamma=1}^{Q} w_{\alpha_\gamma} u_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(3;wuv)} + \tag{49} \\
& \quad + \sum_{\gamma, \beta=1}^{Q} q_{\alpha_\gamma}^{(p)} u_{\gamma_\beta} T_{\alpha_\gamma \beta}^{(3;wua)} + \sum_{\gamma=1}^{Q} \sum_{\gamma_\beta=1}^{2} q_{\alpha_\gamma}^{(p)} q_{\gamma_\beta}^{(i_\beta)} T_{\alpha_\gamma \beta}^{(3;wua;i_\alpha)} = 0,
\end{align*}
\]
Coefficients of this system are following (see also formulae (151)):

\[ E_{11}^{(42;1\alpha\beta)} := \sum_{\gamma,\delta=1}^{Q} u_{\alpha\gamma} u_{\beta\delta} T_{\alpha\gamma\beta}^{(3;uvw)} + \sum_{\gamma=1}^{Q} u_{\alpha\gamma} v_{\beta\gamma} T_{\alpha\gamma\beta}^{(3;vwu)} + \sum_{\gamma=1}^{Q} \sum_{i_0=1}^{2} u_{\alpha\gamma}^{(i_0)} T_{\alpha\gamma\beta}^{(3;uvw;i_0)} = 0, \quad \alpha \neq \beta, \]

\[ E_{11}^{((j+1);3;1\alpha\beta)} := \sum_{\gamma,\delta=1}^{Q} u_{\alpha\gamma} u_{\beta\delta} T_{\alpha\gamma\beta}^{(j;uuu)} + \sum_{\gamma=1}^{Q} \sum_{i_0=1}^{2} u_{\alpha\gamma} q_{\gamma\beta}^{(i_0)} T_{\alpha\gamma\beta}^{(j;uuq;i_0)} = 0, \quad j = 2, 3. \]

Coefficients of this system are following (see also formulae (151)):

\[ s_{\alpha\beta}^{(j-1;w;1)} := s_{\alpha1}^{(j;w;2;\beta)} = \left| \begin{array}{ccc} B_1^{(\beta3)} & B_1^{(\beta4)} & B_1^{(\beta j)} \\ B_1^{(\beta3)} - \rho_1^{(31)} & B_1^{(\beta4)} - \rho_1^{(41)} & B_1^{(\beta j)} - \rho_1^{(j1)} \\ B_1^{(\beta3)} - \rho_1^{(32)} & B_1^{(\beta4)} - \rho_1^{(42)} & B_1^{(\beta j)} - \rho_1^{(j2)} \end{array} \right|, \quad m = 3, 4, j, \]

\[ s_{\alpha\beta}^{(j-1;w;m-1)} := s_{\alpha1}^{(j;w;m;\beta)} = \left| \begin{array}{ccc} \delta^{(m3)} & \delta^{(m4)} & \delta^{(mj)} \\ B_1^{(\beta3)} - \rho_1^{(31)} & B_1^{(\beta4)} - \rho_1^{(41)} & B_1^{(\beta j)} - \rho_1^{(j1)} \\ B_1^{(\beta3)} - \rho_1^{(32)} & B_1^{(\beta4)} - \rho_1^{(42)} & B_1^{(\beta j)} - \rho_1^{(j2)} \end{array} \right|, \quad m = 3, 4, j, \]

\[ T_{\alpha\gamma\beta}^{(j-1;uvw)} = \left| \begin{array}{ccc} B_1^{(\beta3)} & B_1^{(\beta4)} & B_1^{(\beta j)} \\ B_1^{(\beta3)} - \rho_1^{(31)} & B_1^{(\beta4)} - \rho_1^{(41)} & B_1^{(\beta j)} - \rho_1^{(j1)} \\ B_1^{(\beta3)} - \rho_1^{(32)} & B_1^{(\beta4)} - \rho_1^{(42)} & B_1^{(\beta j)} - \rho_1^{(j2)} \end{array} \right|, \]

\[ T_{\alpha\gamma\delta}^{(j-1;uqq)} = R_{\gamma1}^{(14)} T_{\alpha\delta}^{(j-1;uvw)} \]

\[ T_{\alpha\gamma}^{(j-1;wq;io)} = \left| \begin{array}{ccc} B_1^{(\beta3)} & B_1^{(\beta4)} & B_1^{(\beta j)} - \rho_1^{(ji0)} \\ B_1^{(\beta3)} - \rho_1^{(31)} & B_1^{(\beta4)} - \rho_1^{(41)} & B_1^{(\beta j)} - \rho_1^{(j1)} \\ B_1^{(\beta3)} - \rho_1^{(32)} & B_1^{(\beta4)} - \rho_1^{(42)} & B_1^{(\beta j)} - \rho_1^{(j2)} \end{array} \right|, \]

\[ s_{\alpha}^{(j-1;q;m-1)} := s_{\alpha1}^{(j;w;m;1)} = \left| \begin{array}{ccc} \delta^{(m2)} & \delta^{(m3)} & \delta^{(mj)} \\ 1 & \rho_1^{(31)} & \rho_1^{(j1)} \\ 1 & \rho_1^{(32)} & \rho_1^{(j2)} \end{array} \right|, \quad m = 2, 3, j, \]

\[ T_{\alpha\gamma}^{(j-1;qw)} = \left| \begin{array}{ccc} B_1^{(\gamma3)} & B_1^{(\gamma j)} \\ B_1^{(\gamma3)} - \rho_1^{(31)} & B_1^{(\gamma j)} - \rho_1^{(j1)} \\ B_1^{(\gamma3)} - \rho_1^{(32)} & B_1^{(\gamma j)} - \rho_1^{(j2)} \end{array} \right|, \quad T_{\alpha\gamma\delta}^{(j-1;quq)} = R_{\gamma1}^{(14)} T_{\alpha\delta}^{(j-1;qw)} , \]

\[ T_{\alpha\gamma}^{(j-1;qqq;io)} = \left| \begin{array}{ccc} 1 & \rho_1^{(3i0)} & \rho_1^{(ji0)} \\ 1 & \rho_1^{(31)} & \rho_1^{(j1)} \\ 1 & \rho_1^{(32)} & \rho_1^{(j2)} \end{array} \right|, \]
\[
s^{(j-1;v;1)}_{\alpha\beta} = s^{(j;v;2;\alpha\beta)}_1 = \begin{vmatrix}
B^{(\beta\gamma)}_1 & B^{(\beta j)}_1 \\
B^{(\alpha\gamma)}_1 & B^{(\alpha j)}_1 
\end{vmatrix},
\]
\[
s^{(j-1;v;m-1)}_{\alpha\beta} = s^{(j;v;m;\alpha\beta)}_1 = \begin{vmatrix}
\delta^{(3m)} & \delta^{(jm)} \\
B^{(\beta\gamma)}_1 - B^{(\alpha\gamma)}_1 & B^{(\beta j)}_1 - B^{(\alpha j)}_1 
\end{vmatrix}, \quad m = 3, j,
\]
\[
T^{(j-1;v;\gamma\beta)}_{\alpha\gamma\beta} = T^{(j;v;\gamma\alpha\beta)}_1 = \begin{vmatrix}
B^{(\beta\gamma)}_1 - B^{(\alpha\gamma)}_1 & B^{(\beta j)}_1 - B^{(\alpha j)}_1 \\
B^{(\beta\gamma)}_1 - B^{(\alpha\gamma)}_1 & B^{(\beta j)}_1 - B^{(\alpha j)}_1 
\end{vmatrix},
\]
\[
T^{(j-1;u;\gamma\beta)}_{\alpha\gamma\beta} = T^{(j;u;\gamma\alpha\beta)}_1 = \begin{vmatrix}
B^{(\beta\gamma)}_1 - B^{(\alpha\gamma)}_1 & B^{(\beta j)}_1 - B^{(\alpha j)}_1 \\
B^{(\beta\gamma)}_1 - B^{(\alpha\gamma)}_1 & B^{(\beta j)}_1 - B^{(\alpha j)}_1 
\end{vmatrix},
\]
\[
s^{(j-1;u;m-1)}_{\alpha} = s^{(j;u;m;\alpha)}_1 = -\begin{vmatrix}
\delta^{(2m)} & \delta^{(jm)} \\
1 & B^{(\alpha j)}_1 
\end{vmatrix}, \quad m = 2, j,
\]
\[
T^{(j-1;u;\gamma\alpha)}_{\alpha\gamma} = T^{(j;u;\gamma\alpha\gamma)}_1 = \begin{vmatrix}
B^{(\gamma j)}_1 & B^{(\alpha j)}_1 \\
1 & B^{(\gamma j)}_1 
\end{vmatrix},
\]
\[
T^{(j-1;u;q;\alpha\gamma)}_{\alpha\gamma} = T^{(j;u;\gamma\alpha\gamma)}_1 = \begin{vmatrix}
P^{(\gamma j\alpha)}_1 & B^{(\alpha j)}_1 \\
1 & B^{(\gamma j)}_1 
\end{vmatrix},
\]

where \(R^{(1)}_\gamma\), \(B^{(ij)}_1\) and \(P^{(ij)}_1\) are arbitrary constant parameters.

Dimensionality of the system (47-51) is defined by the eqs.(45) which involve derivatives with respect to 4 independent variables, while other equations involve derivatives with respect to two and three variables. Dressing algorithm supplies one arbitrary function of three independent variables (which fixes the initial datum for one of the functions \(w^{(p)}\), \(p = 1, 2\)), two arbitrary functions of two variables (fixing the initial data for \(v\) and for one of the functions \(q^{(p)}\), \(p = 1, 2\)) and single arbitrary function of one variable (fixing the initial datum for \(u\)). Namely the system (45-51) is written in Introduction, see eqs.(7-13).

The system (45-51) has the following evident properties:

\[\text{d.1) There are infinitely many commuting flows to this system.}\]
\[\text{d.2) Equations of this system have differential polynomial structure, also the number of equations is rather big. So, one has 24 scalar PDEs in the simplest case \(Q = 2\).}\]
\[\text{d.3) Independent variable \(x_1\) appears only in the combinations (43), so that it does not increase dimensionality of nonlinear PDEs. This happens due to the decoupling of equations for fields with different values of the first superscript, which, in turns, is a consequence of the eq.(41). However, if one considers another relation (40) mixing fields with different values of the first superscript, then derivatives with respect to \(x_1\) will appear explicitly in nonlinear PDEs which will become 5-dimensional. But such system of nonlinear PDEs will have more complicated structure.}\]
\[\text{d.4) Dimensionality of the nonlinear PDEs is determined by the dimensionality of the linear PDE (42) and may be increased without any problem.}\]
\[\text{d.5) Reduction to the linearizable system of nonlinear PDEs. The system (45-51) admits reduction \(v = u = 0\), which corresponds to \(C^{(i)} \equiv 0\) in the dressing algorithm. In this case the}\]
system of PDEs reduces to the single eq.(46) which now reads:

\[
\sum_{\alpha} s^{(4;4)}_{\alpha} \partial_{t} q^{(p)}_{\alpha} + \sum_{m=1}^{2} s^{(4;m)}_{\alpha} \partial_{ym} q^{(p)}_{\alpha} + \sum_{\gamma=1}^{Q} \sum_{i=1}^{2} q^{(p)}_{\gamma i} T^{(4;qq;io)}_{\alpha \gamma} = 0. 
\]

Eq.(49) is a symmetry of the eq.(53). We may introduce "characteristics variables" \( \partial_{\alpha} = s^{(4;4)}_{\alpha} \partial_{t} + \sum_{m=1}^{2} s^{(4;m)}_{\alpha} \partial_{ym}, \alpha = 1, \ldots, Q \) which indicate that the dimensionality of the eq.(53) is, essentially, \( \min(3, Q) \). The eq.(53) is "linearizable" because the matrix fields \( q^{(p)} \) are algebraically expressible in terms of solutions of some linear PDE, see Sec.3.1. Eq. (53) has been written in Introduction, see eq.(14). This equation is also partially integrable in the same sense as the system (45-51) does: dressing algorithm provides arbitrary initial condition only for one of the matrix fields \( q^{(p)} \), \( p = 1, 2 \).

**Proof:** The proofs of the items (a)-(c) and (d.1, d.5) are given in Appendix, Sec.5. The items (d.2-d.4) are self-consistent. Relation between solutions of the eq. (53) and solutions of the appropriate linear PDE (item d.5) is shown in Sec.3.1.

### 3 Solutions

In this section all superscripts and subscripts take values from 1 to \( Q \) unless different is specified. Subscripts do not always denote elements of matrices. We use Greek letters for matrix indexes in order to distinguish them from others.

Solutions of the eqs. (18,19) read:

\[
\Phi^{(sm)}(\lambda; x) = \int \Phi_{0}^{(sm)}(\lambda, \mu) e^{K^{(m)}(\mu; \mathcal{X}_{1}, \mathcal{X}_{2}; x)} d\mu,
\]

\[
C^{(m)}(\lambda; x) = \int e^{K^{(m)}(\lambda; \mathcal{X}_{2}; x)} C_{0}^{(m)}(\lambda, q) dq,
\]

\[
K^{(m)}(\mathcal{X}_{1}, \mathcal{X}_{2}; x) = (\mathcal{X}_{1} x_{1} + \mathcal{X}_{2} x_{2}) + \sum_{j=3}^{5} (P^{(mj)} \mathcal{X}_{1} + B^{(mj)} \mathcal{X}_{2}) x_{j},
\]

where \( \mathcal{X} = (\mathcal{X}_{1}, \mathcal{X}_{2}) \), \( q = (q_{1}, q_{2}) \). The dressing function \( \Psi^{(s)} \) must be taken as solution of the eq.(15) with \( k = 2 \):

\[
\Psi^{(s)}(\lambda, \mu; x) = \Psi_{p}^{(s)}(\lambda, \mu; x) + \Psi_{h}^{(s)}(\lambda, \mu; x) + \Sigma^{(s)}(\lambda, \mu)
\]

\[
\Psi_{p}^{(s)}(\lambda, \mu; x) = \int \sum_{m=1}^{Q} \Phi_{0}^{(sm)}(\lambda, \mu) (\mathcal{X}_{2} + q_{2} - (\mathcal{X}_{1} + q_{1}) A^{(2)})^{-1} e^{K^{(m)}(\mu; \mathcal{X}_{1} + q_{1}, \mathcal{X}_{2} + q_{2}; x)} \times
C_{0}^{(m)}(\mu, q) dkdq,
\]

\[
\Psi_{h}^{(s)}(\lambda, \mu; x) = \int e^{\kappa \tau} e^{(\sum_{i=2}^{5} A^{(i)} x_{i} + l x_{1})} \Psi_{h_{0}}^{(s)}(\lambda, \mu, \mathcal{X}) d\mathcal{X},
\]

where \( \Psi_{p}^{(s)}(\lambda, \mu; x) \) is a particular solution of the eq.(15) and \( \Psi_{h}^{(s)}(\lambda, \mu; x) \) is a solution of homogeneous equation associated with eq.(15). In addition, we separate matrix function \( \Sigma^{(s)}(\lambda, \mu) \)
independent on $x$ for our convenience, although it might be incorporated in $\Psi_p^{(s)}(\lambda, \mu; x)$. Both $\Psi_h^{(s)}(\lambda, \mu; x)$ and $\Sigma^{(s)}(\lambda, \mu)$ have both diagonal and off-diagonal parts.

The dressing function $G$ is a solution of the eq.(42):

$$G(\lambda; x) = \int e^{K(G)(x_1, x_2; x)}G_0(\lambda, x)d\lambda,$$

$$K(G)(x_1, x_2; x) = x_1x_1 + x_2x_2 + \sum_{j=3}^{5}(P^{(j1)}x_1 + P^{(j2)}x_2)x_j.$$

It may not be diagonal, otherwise solution space will be poor.

It is quite standard to assume that the measure $d\Omega(\lambda)$ has support on an open domain $\mathcal{D}$ of the $\lambda$-space, and on a disjoint discrete set of points $D = \{b_1, \ldots, b_M\}$, $D \cap \mathcal{D} = \emptyset$. Correspondingly, we use the following notations for the dressing functions:

$$\Phi^{(sk)}(\lambda; x) = \begin{cases} 
\phi^{(sk)}(\lambda; x) = \int \phi^{(sk)}(\lambda, x)e^{K^{(k)}(x_1, x_2; x)}d\lambda, & \lambda \in D, \\
\phi^{(sk)}(x) = \int \phi^{(sk)}(x)e^{K^{(k)}(x_1, x_2; x)}d\lambda, & \lambda = b_n, \ n = 1, \ldots, M, 
\end{cases}$$

$$C^{(k)}(\lambda; x) = \begin{cases} 
c^{(k)}(\lambda; x) = \int e^{K^{(k)}(q_1, q_2; x)}c^{(k)}_0(q, \lambda)dq, & \lambda \in D, \\
c^{(k)}(x) = \int e^{K^{(k)}(q_1, q_2; x)}c^{(k)}_0(q)dq, & \lambda = b_n, \ n = 1, \ldots, M, 
\end{cases}$$

$$G(\lambda; x) = \begin{cases} 
g(\lambda; x) = \int e^{K^{(G)}(x_1, x_2; x)}g_0(\lambda, x)d\lambda, & \lambda \in D, \\
g(x) = \int e^{K^{(G)}(x_1, x_2; x)}g_0(x)d\lambda, & \lambda = b_n, \ n = 1, \ldots, M, 
\end{cases}$$

$$U^{(lk)}(\lambda; x) = \begin{cases} 
u^{(lk)}(\lambda; x), & \lambda \in D, \\
u^{(lk)}(x), & \lambda = b_n, \ n = 1, \ldots, M 
\end{cases}$$

and we choose $\Sigma^{(s)}(\lambda, \mu)$ in the next form:

$$\Sigma^{(s)}_{\alpha\beta}(\lambda, \mu) = \begin{cases} 
\sigma^{(s)}_{\alpha\beta}(\lambda, \mu) = \delta^{(s)}_{\alpha\beta}\delta(\lambda - \mu), & \lambda \in D, \ \mu \in D \\
(\sigma^{(s)}_{0m}(\mu))_{\alpha\beta} = 0, & \lambda \in D, \ \mu = b_m, \ m = 1, \ldots, M \\
(\sigma^{(s)}_{n0}(\mu))_{\alpha\beta} = 0, & \lambda = b_n, \ n = 1, \ldots, M, \ \mu \in D \\
(\sigma^{(s)}_{nm})_{\alpha\beta} = \delta^{(s)}_{\alpha\beta}\sigma_{nm}, & \lambda = b_n, \ \mu = b_m, \ n = 1, \ldots, M \\
\delta_{nm}, & n = 1, \ldots, M - 1, \ m = 1, \ldots, M \\
\sigma_{Mm}, & n = M, \ m = 1, \ldots, M 
\end{cases}$$

where $\sigma_{Mm}$ are scalar parameters. Next,

$$\Psi_p^{(s)}(\lambda, \mu; x) =$$

$$\begin{cases} 
\psi_p^{(s)}(\lambda, \mu; x) = \sum_{i=1}^{Q} \int \phi_{0}^{(si)}(\lambda, x)R^{(i)}(\lambda, q; x)c^{(i)}_0(\mu, q)d\lambda dq, & \lambda, \mu \in D \\
\psi_{p0m}(\lambda; x) = \sum_{i=1}^{Q} \int \phi_{0}^{(si)}(\lambda, x)R^{(i)}(\lambda, q; x)c^{(i)}_{m0}(q)d\lambda dq, & \lambda \in D, \ \mu = b_m \\
\psi_{pm0}(\mu; x) = \sum_{i=1}^{Q} \int \phi_{0}^{(si)}(\lambda, x)R^{(i)}(\lambda, q; x)c^{(i)}_0(\mu, q)d\lambda dq, & \lambda = b_n, \ \mu \in D \\
\psi_{pm}^{(s)}(x) = \sum_{i=1}^{Q} \int \phi_{0}^{(si)}(\lambda, x)R^{(i)}(\lambda, q; x)c^{(i)}_{m0}(q)d\lambda dq, & \lambda = b_n, \ \mu = b_m 
\end{cases}$$
where \( n, m = 1, \ldots, M \),
\[
R^{(i)}(\mathbf{x}, q; x) = \left( \mathbf{x}_2 + q_2 - (\mathbf{x}_1 + q_1)A^{(2)} \right)^{-1}e^{K^{(i)}(\mathbf{x}_1 + q_1, \mathbf{x}_2 + q_2; x)}.
\]

Remember that the functions \( \Phi^{(sm)} \) are diagonal, i.e. \( \phi^{(sm)} \) and \( \phi^{(sm)}_n \) are diagonal as well.

The function \( \Psi^{(s)}_h \) should be taken in the next form which provides the maximal possible richness of the solution space:
\[
\Psi^{(s)}_h(\lambda, \mu; x) = \begin{cases} 
\psi^{(s)}_h(\lambda, \mu; x) = 0, & \lambda, \mu \in \mathcal{D} \\
\psi^{(s)}_{h,0m}(\lambda; x) = \int e^{\mathcal{L}(\sum_{i=2}^5 A^{(i)}x_i + Ix_1)}\psi^{(s)}_{h_0,0m}(\lambda, \mathbf{x}_1)\ d\mathbf{x}_1, & \lambda \in \mathcal{D}, \ \mu = b_m, \\
\psi^{(s)}_{h,m0}(\mu; x) = 0, & \lambda = b_n, \ \mu \in \mathcal{D} \\
\psi^{(s)}_{h,nn}(x) = 0, & \lambda = b_n, \ \mu = b_m
\end{cases}
\]
where \( n, m = 1, \ldots, M \).

Then eqs. (5) reduce to the following system of equations
\[
\phi^{(\gamma k)}_\beta(\lambda; x) = \left( \int_{\mathcal{D}} \psi^{(\gamma)}_p(\lambda, \nu; x)u^{(bk)}(\nu; x)d\nu + \sum_{j=1}^M \left( \psi^{(\gamma)}_{p,0j}(\lambda; x) + \psi^{(\gamma)}_{h,0j}(\lambda; x) \right)u^{(bk)}_j(x) \right)_{\beta\beta} + u^{(bk)}(\lambda; x), \ \lambda \in \mathcal{D}
\]
\[
0 = \left( \int_{\mathcal{D}} \psi^{(\gamma)}_p(\lambda, \nu; x)u^{(ak)}(\nu; x)d\nu + \sum_{j=1}^M \left( \psi^{(\gamma)}_{p,0j}(\lambda; x) + \psi^{(\gamma)}_{h,0j}(\lambda; x) \right)u^{(ak)}_j(x) \right)_{\alpha\beta} + u^{(ak)}(\lambda; x), \ \lambda \in \mathcal{D}, \ \alpha \neq \beta
\]
\[
(\phi^{(\gamma k)}_n(x))_\beta = \left( \int_{\mathcal{D}} \psi^{(\gamma)}_{p,n0}(\nu; x)u^{(bk)}(\nu; x)d\nu + \sum_{j=1}^M \psi^{(\gamma)}_{p,nj}(x)u^{(bk)}_j(x) \right)_{\beta\beta} + \sum_{m=1}^Q \sigma_{nm}(u^{(bk)}_m(x))_{\gamma\beta}, \ n = 1, \ldots, M
\]
\[
0 = \left( \int_{\mathcal{D}} \psi^{(\gamma)}_{p,n0}(\nu; x)u^{(ak)}(\nu; x)d\nu + \sum_{j=1}^M \psi^{(\gamma)}_{p,nj}(x)u^{(ak)}_j(x) \right)_{\alpha\beta} + \sum_{m=1}^Q \sigma_{nm}(u^{(ak)}_m(x))_{\gamma\beta}, \ n = 1, \ldots, M
\]
for the unknown matrix functions \( u^{(ak)}(\lambda; x), \ \lambda \in \mathcal{D} \) and \( u^{(ak)}_j(x), \ \lambda = b_j, \ j = 1, \ldots, M \).

Following the strategy of [22], it is enough to provide single linear relation among equations (65) and single linear relation among equations (66) in order to satisfy the condition \( \text{dim ker } \hat{\Psi} = \)
In turn, this requirement is equivalent to the next two equations:

\[
\sum_{j=1}^{M} \sum_{s=1}^{Q} A_j^{(ms)} \phi_j^{(st)} = 0, \quad \sum_{j=1}^{M} A_j^{(ms)} \sigma_{jn} = 0, \tag{67}
\]

where \(A_j^{(ms)}\) are some constants. Due to the eq.(67), eqs.(65) and (66) with \(n = 1, \ldots, M-1\) are independent equations for the functions \(u_n^{(ak)}(x)\), \(n = 1, \ldots, M-1\), where the matrix functions \(u_M^{(ak)}(x)\) may be taken as arbitrary matrix functions \(f^{(ak)}(x): u_M^{(ak)}(x) = f^{(ak)}(x)\). In order to write equations for \(f^{(ak)}\) we involve the condition (41), which reads

\[
G(\lambda; x) * U^{(ak)}(\lambda; x) \equiv g(\lambda; x) * u^{(ak)}(\lambda; x) + \sum_{n=1}^{M} g_n(x) u_n^{(ak)}(x) = R^{(ak)}, \tag{68}
\]

\(\alpha, k = 1, \ldots, Q\).

Thus, the system (65,66), \(n = 1, \ldots, M-1\) and (68) represent the complete system for the functions \(u_n^{(ak)}(x)\), \(n = 1, \ldots, M\).

Once \(u^{(ak)}\) and \(u_n^{(ak)}\), \(n = 1, \ldots, M\) have been found, one constructs the matrix fields \(v^{(nik)}\) and \(w^{(nikp)}\) using the definitions (39):

\[
v^{(nik)}(x) = \int_{\mathcal{D}} c^{(i)}(\lambda; x) u^{(nk)}(\lambda; x) d\lambda + \sum_{j=1}^{M} c_j^{(i)}(x) u_j^{(nk)}(x), \tag{69}
\]

\[
w^{(nikp)}(x) = \int_{\mathcal{D}} g_{xp}(\lambda; x) u^{(nk)}(\lambda; x) d\lambda + \sum_{j=1}^{M} g_{jxp}(x) u_j^{(ak)}(x), \quad p = 1, 2,
\]

\(n, i, k = 1, \ldots, Q\).

In Secs. 3.1, 3.2, we consider two examples when solutions to the nonlinear PDEs can be constructed explicitly taking into account that the independent variables of the nonlinear equations written in Proposition are \(t, y_i, i = 1, 2, 3\), see eqs.(43).

### 3.1 Solutions to the system (53)

Consider the particular case \(C^{(i)} = 0\) leading to the eq.(53). One has \(\psi^{(\gamma)}_p = 0\) and \(\psi^{(\gamma)}_{p;j} = 0\). Thus the system (63-66) reduces to the next one:

\[
\phi_{\beta}^{(\gamma k)}(\lambda; x) = \left( \sum_{j=1}^{M} \psi_{i;0j}^{(\gamma)}(\lambda; x) u_j^{(\beta k)}(x) \right)_{\beta} + u_{\gamma \beta}^{(\beta k)}(x), \quad \lambda \in \mathcal{D} \tag{70}
\]

\[
0 = \left( \sum_{j=1}^{M} \psi_{i;0j}^{(\gamma)}(\lambda; x) u_j^{(ak)}(x) \right)_{\alpha \beta} + u_{\gamma \beta}^{(ak)}(x), \quad \lambda \in \mathcal{D}, \quad \alpha \neq \beta \tag{71}
\]

\[
(\phi_n^{(\gamma k)}(x))_{\beta} = (u_n^{(\beta k)}(x))_{\gamma \beta}, \quad n = 1, \ldots, M-1 \tag{72}
\]

\[
0 = (u_n^{(ak)}(x))_{\gamma \beta}, \quad n = 1, \ldots, M-1 \tag{73}
\]

supplemented by the eq.(68) where we take \(g_n = \delta_{n,M}\) without loss of generality. Then eq.(68) gives

\[
u_M^{(ak)}(x) = R^{(ak)} - \tilde{u}^{(ak)}(x), \quad \tilde{u}^{(ak)}(x) = g(\lambda; x) * u^{(ak)}(\lambda; x). \tag{74}
\]
We need only \( u_{\gamma\beta}^{(1k)} \) with \( \beta \neq 1 \) in order to construct the fields \( w_{\gamma\beta}^{(1k)} \), \( \beta \neq 1 \). Thus, eqs.\((71,73,74)\) yield:

\[
u_{\gamma\beta}^{(1k)}(\lambda; x) = - \left( \psi_{h,0,M}^{(\gamma)}(\lambda; x) \left( R^{(1k)} - \bar{u}^{(1k)}(x) \right) \right)_{\gamma\beta}, \quad \beta \neq 1. \tag{75}
\]

The function \( \bar{u}^{(1m)}(x) \) can be found as a solution of the following linear algebraic system which appears after applying \( \sum_{\gamma=1}^{Q} \int_{D} d\lambda \ g_{\gamma\gamma}(\lambda; x) \) to \( (75) \) and replacing \( \bar{y} \) by \( \gamma \) in the result:

\[
\bar{u}_{\gamma\beta}^{(1k)} = - \left( \xi^{(0)}(\lambda)(R^{(1k)} - \bar{u}^{(1k)}) \right)_{\gamma\beta}, \quad \beta \neq 1, \tag{76}
\]

\[
\xi^{(0)}_{\alpha\beta} = \sum_{\gamma=1}^{Q} \int_{D} d\lambda \ g_{\alpha\gamma}(\lambda; x) \left( \psi_{h,0,M}^{(\gamma)}(\lambda; x) \right)_{1\beta}. \tag{77}
\]

The functions \( w_{\gamma\beta}^{(1kp)} \) can be found by definition as follows:

\[
w_{\gamma\beta}^{(1kp)}(x) \equiv \left( g_{kp}(\lambda; x) * u^{(1k)}(\lambda; x) \right)_{\gamma\beta} = - \left( \xi^{(p)}(x)(R^{(1k)} - \bar{u}^{(1k)}(x)) \right)_{\gamma\beta}, \quad \beta \neq 1, \quad p = 1, 2, \tag{79}
\]

where

\[
\xi^{(p)}_{\alpha\beta} = \sum_{\gamma=1}^{Q} \int_{D} d\lambda \left( g_{\alpha\gamma}(\lambda; x) \right)_{\gamma\beta} \left( \psi_{h,0,M}^{(\gamma)}(\lambda; x) \right)_{1\beta}. \tag{80}
\]

Now solution \( q_{\alpha\beta}^{(p)} \) of (53) can be found by the definition \((44)\). Formulae of this section suggest us to choose \( M = 1 \) without loss of generality.

We see that the fields \( w_{\gamma\beta}^{(1kp)} \) given by \((39)\) are expressed in terms of \( \xi^{(p)}, \ p = 0, 1, 2 \). By construction, the functions \( \xi^{(p)} \) are solutions of the linear PDE due to the fact that \( g(\lambda; x) \) is solution of the linear PDE (by definition) and functions \( \left( \psi_{h,0,M}^{(\gamma)}(\lambda; x) \right)_{1\beta} \) do not depend on the variables \( t, y, \ i = 1, 2, 3 \) \((43)\) by definition. Thus, the system \((53)\) is \( C^{(i)}\)-integrable.

By construction, functions \( \xi^{(p)} \) admit arbitrary dependence on 2 variables \( y, i = 1, 2 \). Since functions \( \xi^{(p)} \) with different values of \( p \) introduce the same arbitrary matrix function of 2 variables, we have \( Q^{2} \) arbitrary scalar functions of 2 variables and \( 2Q^{2} \) independent scalar fields, which are elements of the matrices \( q^{(p)}, \ p = 1, 2 \). Thus, solving, for instance, an initial-boundary value problem, only \( Q^{2} \) scalar fields may be given arbitrary initial conditions. For this reason, we consider eq.\((53)\) as a partially integrable system.

### 3.2 Degenerate kernel, \( C^{(i)} \neq 0 \).

The system of linear equations \((63-66,68)\) has a rich manifold of explicit solutions. To construct them, we choose, as usual, a degenerate kernel:

\[
c_{0}^{(i)}(q, \mu) = \sum_{j=1}^{M} c_{1j}^{(i)}(q) c_{2j}^{(i)}(\mu), \tag{81}
\]
where $c^{(i)}_{2j}(\mu)$ are diagonal matrix functions. Then

$$
\psi^{(s)}_p(\lambda, \mu; x) = \sum_{i=1}^{Q} \sum_{j=1}^{\tilde{M}} \psi^{(si)}_{pji}(\lambda; x)c^{(i)}_{2j}(\mu),
$$

(82)

$$
\psi^{(s)}_{p,n0}(\mu; x) = \sum_{i=1}^{Q} \sum_{j=1}^{\tilde{M}} \psi^{(si)}_{p,n0ij}(x)c^{(i)}_{2j}(\mu),
$$

where

$$
\psi^{(si)}_{pji}(\lambda; x) = \int \phi^{(si)}_{0}(\lambda, \nu)R^{(i)}(\nu, q; x)c^{(i)}_{ij}(q)d\nu dq,
$$

$$
\psi^{(si)}_{p,n0ij}(x) = \int \phi^{(si)}_{n0}(\nu)R^{(i)}(\nu, q; x)c^{(i)}_{ij}(q)d\nu dq
$$

In this case, equations (63-66) reduce to the following system of linear equations for the matrix functions $u^{(ak)}_n(x), \tilde{u}^{(aik)}_j(x)$:

$$(\phi^{(l\gamma k)}_n(x))_{\beta \gamma} = \left[ \sum_{j=1}^{\tilde{M}} \sum_{i=1}^{Q} \nu^{(l\gamma i)}_{nj}(x)\tilde{u}^{(bjk)}_j(x) + \sum_{j=1}^{M} \nu^{(l\gamma i)}_{nj}(x)u^{(bk)}_j(x) \right]_{\beta \gamma} +
$$

(83)

$$
0 = \left[ \sum_{j=1}^{\tilde{M}} \sum_{i=1}^{Q} \nu^{(l\gamma i)}_{nj}(x)\tilde{u}^{(aik)}_j(x) + \sum_{j=1}^{M} \nu^{(l\gamma i)}_{nj}(x)u^{(ak)}_j(x) \right]_{\alpha \beta} +
$$

(84)

$$
(\tilde{u}^{(aik)}_j(x))_{\gamma \beta}, \ n = 1, \ldots, \tilde{M}, \quad \alpha \neq \beta
$$

$$(\phi^{(\gamma k)}_n(x))_{\alpha \beta} = \left[ \sum_{j=1}^{\tilde{M}} \sum_{i=1}^{Q} \rho^{(\gamma i)}_{nj}(x)\tilde{u}^{(bjk)}_j(x) + \sum_{j=1}^{M} \rho^{(\gamma i)}_{nj}(x)u^{(bk)}_j(x) \right]_{\alpha \beta} +
$$

(85)

$$
(\tilde{u}^{(bjk)}_j(x))_{\gamma \beta}, \ n = 1, \ldots, M - 1,
$$

(86)

$$
0 = \left[ \sum_{j=1}^{\tilde{M}} \sum_{i=1}^{Q} \rho^{(\gamma i)}_{nj}(x)\tilde{u}^{(aik)}_j(x) + \sum_{j=1}^{M} \rho^{(\gamma i)}_{nj}(x)u^{(ak)}_j(x) \right]_{\alpha \beta} +
$$

$$
(\tilde{u}^{(aik)}_j(x))_{\gamma \beta}, \ n = 1, \ldots, M - 1, \quad \alpha \neq \beta,
$$

$k, \alpha, \beta, \gamma = 1, \ldots, Q, \ n = 1, \ldots, M, \ j = 1, \ldots, \tilde{M}$, where

$$
\tilde{u}^{(aik)}_j(x) = \int_{\mathcal{D}} c^{(i)}_{2j}(\lambda)u^{(ak)}(\lambda; x)d\lambda,
$$

(87)
and where the given coefficients $\nu_{n_j}^{(l_\gamma)}, \tilde{\nu}_{n_j}^{(l_\gamma)}, \rho_{n_j}^{(\gamma)}, \tilde{\rho}_{n_j}^{(\gamma)}, \phi_{n}^{(\gamma\lambda)}$ are defined in terms of the dressing functions:

$$\nu_{n_j}^{(l_\gamma)}(x) = \int_{\mathcal{D}} d\lambda \, c_{2n}^{(l)}(\lambda) \left( \psi_{p,0j}^{(\gamma)}(\lambda; x) + \psi_{h,0j}^{(\gamma)}(\lambda; x) \right), \quad (88)$$

$$\tilde{\nu}_{n_j}^{(l_\gamma)}(x) = \int_{\mathcal{D}} d\lambda \, c_{2n}^{(l)}(\lambda) \psi_{p,j}^{(\gamma)}(\lambda),$$

$$\rho_{n_j}^{(\gamma)}(x) = \psi_{p,nj}^{(\gamma)}(x),$$

$$\tilde{\rho}_{n_j}^{(\gamma)}(x) = \psi_{p,n0j}^{(\gamma)}(x),$$

$$\phi_{n}^{(\gamma\lambda)}(x) = \int_{\mathcal{D}} d\lambda \, c_{2n}^{(l)}(\lambda) \phi^{(\gamma\lambda)}(\lambda; x).$$

Eqs.(83) and (84) are obtained applying $\int_{\mathcal{D}} d\lambda \, c_{2n}^{(l)}(\lambda; x) \cdot$ to the eqs.(63) and (64) respectively from the left.

Having constructed, from (83-86), the $u_{\alpha}^{(ak)}(x), j = 1, \ldots, M - 1$ and the $\tilde{u}_{\alpha}^{(aik)}(x)$ in terms of $u_{\alpha}^{(M)}(x)$, one obtains the functions $u^{(ak)}(\lambda; x)$ via the formulae (63) and (64):

$$u_{\gamma\beta}^{(ak)}(\lambda; x) = \phi_{\beta}^{(ak)}(\lambda; x) - \left( \sum_{j=1}^{M} \sum_{i=1}^{Q} \psi_{p,j}^{(\gamma)}(\lambda; x) \tilde{u}_{j}^{(aik)}(x) \right)_{\beta\beta} + \sum_{j=1}^{M} (\psi_{p,0j}^{(\gamma)}(\lambda; x) + \psi_{h,0j}^{(\gamma)}(\lambda; x)) u_{\alpha}^{(ak)}(x), \quad \lambda \in \mathcal{D}$$

$$u_{\gamma\beta}^{(ak)}(\lambda; x) = - \left( \sum_{j=1}^{M} \sum_{i=1}^{Q} \psi_{p,j}^{(\gamma)}(\lambda; x) \tilde{u}_{j}^{(aik)}(x) \right) + \sum_{j=1}^{M} (\psi_{p,0j}^{(\gamma)}(\lambda; x) + \psi_{h,0j}^{(\gamma)}(\lambda; x)) u_{\alpha}^{(ak)}(x), \quad \lambda \in \mathcal{D}, \ \alpha \neq \beta.$$

Substituting $u^{(ak)}$ and $u_{\alpha}^{(ak)}, j = 1, \ldots, M - 1$ into (68), one obtains the expressions for $u_{\alpha}^{(ak)}$. If $g_n = \delta_{nM}$, then $u_{\alpha}^{(ak)}$ are defined by the eqs.(74). Equations for the functions $\tilde{u}^{(ak)}$ appearing in (74) can be derived applying $\sum_{\gamma=1}^{Q} \int_{\mathcal{D}} d\lambda \ g_{\gamma\gamma}$ to the eqs. (89) and replacing $\gamma$ by $\gamma$ in the result:

$$\tilde{u}_{\gamma\beta}^{(bk)}(x) = \eta_{\gamma\beta}^{(bk)}(x) - \left( \sum_{j=1}^{M} \sum_{i=1}^{Q} \xi_{j}^{(b\beta)}(x) \tilde{u}_{j}^{(bik)}(x) + \sum_{j=1}^{M} \xi_{j}^{(b\beta)}(x) u_{j}^{(bk)}(x) \right)_{\gamma\beta}, \quad (90)$$

$$\tilde{u}_{\gamma\beta}^{(ak)}(x) = - \left( \sum_{j=1}^{M} \sum_{i=1}^{Q} \xi_{j}^{(a\alpha)}(x) \tilde{u}_{j}^{(aik)}(x) + \sum_{j=1}^{M} \xi_{j}^{(a\alpha)}(x) u_{j}^{(ak)}(x) \right)_{\gamma\beta}, \ \alpha \neq \beta.$$
where

\[
\eta_{\gamma\beta}^{(\beta k)}(x) = \sum_{\gamma_1=1}^{Q} \int_{D} d\lambda \ g_{\gamma\gamma_1}(\lambda; x) \phi_{\beta}^{(\gamma_1 k)}(\lambda; x),
\]   \hspace{1cm} (91)

\[
\left(\xi_{j}^{(\alpha i)}(x)\right)_{\gamma\beta} = \sum_{\gamma_1=1}^{Q} \int_{D} d\lambda \ g_{\gamma\gamma_1}(\lambda; x) \left(\psi_{\beta}^{(\gamma_1 i)}(\lambda; x)\right)_{\alpha\beta},
\]

\[
\left(\xi_{j}^{(\alpha)}(x)\right)_{\gamma\beta} = \sum_{\gamma_1=1}^{Q} \int_{D} d\lambda \ g_{\gamma\gamma_1}(\lambda; x) \left(\psi_{\beta}^{(\gamma_1 j)}(\lambda; x) + \psi_{\alpha}^{(\gamma_1 j)}(\lambda; x)\right)_{\alpha\beta}.
\]

Now the system (83-86, 74,90) should be considered as the complete linear algebraic system for the matrix fields \(\tilde{u}_{j}^{(\gamma k)}, u_{i}^{(\gamma k)}, \tilde{u}^{(\gamma k)}\), \(j = 1, \ldots, \tilde{M}\), \(i = 1, \ldots, M\), \(k, l, \gamma = 1, \ldots, Q\) where \(u_{M}^{(\gamma k)} = f^{(\gamma k)}\). This system is solvable, in general.

At last, one constructs the fields \(v^{(n k)}, w^{(n k p)}\) using (69), and the fields \(w^{(p)}, q^{(p)}, v, u\), solutions of the nonlinear PDEs (45-51), using the formulae (44).

Simplest case corresponds to \(M = \tilde{M} = 1\). Then the system (89) yields (using (74) for \(u_{1}^{(a k)}\)):

\[
u_{\gamma\beta}^{(\beta k)}(x) = \phi_{\beta}^{(\gamma k)}(\lambda; x) - \left(\sum_{i=1}^{Q} \psi_{p}^{(\gamma i)}(\lambda; x) \tilde{u}_{1}^{(\beta i k)}(x) + (\psi_{p}^{(\gamma i)}(\lambda; x) + \psi_{h}^{(\gamma i)}(\lambda; x)) (R^{(\beta k)} - \tilde{u}^{(\beta k)}(x))\right)_{\beta\beta}, \hspace{1cm} \lambda \in D
\]   \hspace{1cm} (92)

\[
u_{\gamma\beta}^{(a k)}(x) = - \left(\sum_{i=1}^{Q} \psi_{p}^{(\gamma i)}(\lambda; x) \tilde{u}_{1}^{(\alpha i k)}(x) + (\psi_{p}^{(\gamma i)}(\lambda; x) + \psi_{h}^{(\gamma i)}(\lambda; x)) (R^{(a k)} - \tilde{u}^{(a k)}(x))\right)_{\alpha\beta}, \hspace{1cm} \lambda \in D, \hspace{0.5cm} \alpha \neq \beta.
\]

The eqs. (83,84) read (substitute (74) for \(u_{1}^{(a k)}\))

\[
(\phi_{n}^{(\gamma k)}(x))_{\beta} = \left[\sum_{i=1}^{Q} \psi_{n i}^{(\gamma i)}(x) \tilde{u}_{1}^{(\beta i k)}(x) + \nu_{n i}^{(\gamma i)}(x) (R^{(\beta k)} - \tilde{u}^{(\beta k)}(x))\right]_{\beta\beta} + (\tilde{u}_{1}^{(\beta k)}(x))_{\gamma\beta}, \hspace{1cm} (93)
\]

\[
0 = \left[\sum_{i=1}^{Q} \psi_{n i}^{(\gamma i)}(x) \tilde{u}_{1}^{(\alpha i k)}(x) + \nu_{n i}^{(\alpha i)}(x) (R^{(a k)} - \tilde{u}^{(a k)}(x))\right]_{\alpha\beta} + (\tilde{u}_{1}^{(a k)}(x))_{\gamma\beta}, \hspace{1cm} \alpha \neq \beta
\]

while the eqs. (85,86) disappear. At last, the eqs.(90) read:

\[
u_{\gamma\beta}^{(\beta k)}(x) = \eta_{\gamma\beta}^{(\beta k)}(x) - \left(\sum_{i=1}^{Q} \xi_{i}^{(\gamma i)}(x) \tilde{u}_{1}^{(\beta i m)}(x) + \xi_{i}^{(\gamma i)}(x) (R^{(\beta k)} - \tilde{u}^{(\beta k)}(x))\right)_{\gamma\beta}, \hspace{1cm} (94)
\]

\[
\hat{u}_{\gamma\beta}^{(a k)}(x) = - \left(\sum_{i=1}^{Q} \xi_{i}^{(\gamma i)}(x) \tilde{u}_{1}^{(a i k)}(x) + \xi_{i}^{(\gamma i)}(x) (R^{(a k)} - \tilde{u}^{(a k)}(x))\right)_{\gamma\beta}, \hspace{1cm} \alpha \neq \beta.
\]

The system of linear equations (93,94) may be solved for \(\tilde{u}^{(a k)}\) and \(\hat{u}^{(a k)}\). Then, the matrix functions \(u^{(a k)}\) will be explicitly given by the eqs.(92). Next, the fields \(v^{(n i k)}, w^{(n k p)}\) can be
constructed using the eqs.(69). Finally, the fields of the nonlinear system can be found by their definitions (44). For instance, the explicit solutions in the form of rational functions of exponents have been constructed. We do not represent them here since they are too cumbersome.

3.2.1 Richness of the solution space to the system (45-51)

Let us discuss the richness of solution space, taking into account that the fields of the nonlinear equations underlined in Proposition (Sec.2.2) are expressed in terms of the scalar fields \( u^{(1k)}_{\alpha\beta} \) and \( w^{(1k;p)}_{\alpha\beta} \), \( i, k, \beta = 1, \ldots, Q, p = 1, 2 \). Using eqs.(63, 64) as the definitions of the function \( u(\lambda; x) \) and eqs.(69) for the fields \( w^{(1k;p)}_{\alpha\beta} \) and \( v^{(1k)}_{1\beta} \) we observe in the small field limit:

\[
\begin{align*}
\sum_{\gamma=1}^{Q} g_{\alpha\gamma x_p}(\lambda; x) & \sim \left( \sum_{j=1}^{M} \psi^{(\gamma)}_{h,0j}(\lambda; x) u^{(1k)}_{j}(x) \right)_{1\beta}, \quad \beta \neq 1, \\
\sum_{\gamma=1}^{Q} g_{\alpha\gamma x_p}(\lambda; x) & \sim \left( \sum_{j=1}^{M} \phi^{(\gamma)}_{1}(\lambda; x) \right), \\
\sum_{\gamma=1}^{Q} c^{(i)}_{1\gamma x_p}(\lambda; x) & \sim \left( \sum_{j=1}^{M} \phi^{(\gamma)}_{1}(\lambda; x) \right)_{1\beta}, \quad \beta \neq 1.
\end{align*}
\]  

By construction, the function \( g \) has arbitrary dependence on 2 variables \( y_1 \) and \( y_2 \). Functions \( \phi^{(\gamma)}_{1} \) and \( c^{(i)}_{1\gamma} \) have arbitrary dependence on single variable \( y_1 \). Finally, \( \psi_{h,0j} \) do not depend on \( t, y_i \).

As we have seen in Introduction, the correctly formulated initial-boundary value problem involves two arbitrary matrix functions of three variables \( y_i \), \( i = 1, 2, 3 \), for the fields \( w^{(p)} \), \( p = 1, 2 \), three arbitrary matrix functions of two variables \( y_i \), \( i = 1, 2 \), for the fields \( q^{(p)} \), \( p = 1, 2 \), and \( u \) and one arbitrary matrix function of single variable \( y_1 \) for the field \( u \) (see eqs.(44) for definitions of these fields). However, eqs.(95) show that we have single arbitrary matrix function of three variables (formula (95a) after applying \( \sum_{k,\beta=1}^{Q} B^{(2,1)}_{\beta}(\hat{R}^{-1})^{k_1}_{\beta} \)), two arbitrary matrix function of two variables (formulae (95b,c)) and one arbitrary matrix function of single variables (formula (95d) after applying \( \sum_{k,\beta=1}^{Q} B^{(2,1)}_{\beta}(\hat{R}^{-1})^{k_1}_{\beta} \)). Thus we may not supply all necessary initial data in order to completely formulate an initial-boundary value problem. For this reason, the system of PDEs underlined in Proposition is considered as a partially integrable system.

4 Conclusions

We have represented a new type of nonlinear PDEs integrable by a new version of the dressing method. This algorithm is based on two principal novelties introduced in [22]:

1. The nontrivial kernel of the integral operator used in the dressing method: \( \text{dim } \text{ker } \Psi^{(en)} = 1 \). This allows one to increase the dimensionality of nonlinear PDEs. In addition, it generates arbitrary functions of \( x \) in solution space, which allow one to introduce different
relations among the fields and, consequently, to increase variety of nonlinear PDEs treatable by the dressing method.

2. The external dressing function $G(\lambda; x)$ which allows one (a) to increase dimensionality of solution space and, similarly to the previous item, (b) to increase variety of nonlinear PDEs solvable by the dressing method.

These two novelties seemed out to be extremely promising in development of theory of integrable PDEs. This paper is devoted to the investigation of two particular problems outlined in [22]:

1. Find reductions of nonlinear PDEs derived in [22] which simplify they structure.

2. Enrich solution space of the constructed nonlinear PDEs (remember that $n$-dimensional nonlinear PDEs derived in [22] have at most $(n - 2)$ dimensional solution space, which is not enough for full integrability).

We succeeded in simplification of nonlinear PDEs. The systems (45-51) and (53) have differential polynomial form, although the number of equations is rather big. We also have enriched solution space. Thus, solution space of the constructed 4-dimensional PDEs is parametrized by one arbitrary matrix function of 3 independent variables, by two arbitrary matrix function of 2 independent variables and by one arbitrary matrix function of single independent variable. However, we need one more arbitrary matrix function of 3 independent variables and one more arbitrary matrix function of two independent variables in order to achieve full integrability. In other words, we achieve only partial integrability of the derived nonlinear PDEs.

A natural generalization of this algorithm is an increase of dimensionality of the kernel of the integral operator:

$$\dim \ker \hat{\Psi}^{(en)} > 1.$$  \hspace{1cm} (96)

The study of associated (partially) integrable equations is postponed to future investigation.

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5 Appendix: proof of Proposition, items (a-c) and (d.1, d.5)

As we have seen in Sec.2.2, the eq.(38) leads to the linear relations between any two solutions $U^{(nk)}$, $L^{(j1;nk)}$, $L^{(j3;nk)}$, $j \geq 2$ and $L^{(j2;nk)}$, $j \geq 3$ of the homogeneous integral equation (31). In other words, one has the following system of independent linear equations for the spectral functions $U^{(nk)}$:

$$L^{(j2;nk)} = \sum_{i1,\gamma=1}^Q L^{(21;ni1)}_{\alpha\gamma} F^{(j;i1;k)}_{\gamma\beta} \Rightarrow$$

$$\partial_{\gamma} U^{(nk)}_{\alpha\beta} + \partial_{i1} U^{(nk)}_{\alpha\beta} B^{(kj;ni)}_{\beta} - \partial_{i2} U^{(nk)}_{\alpha\beta} B^{(kj)}_{\beta} +$$

$$Q \sum_{i1,\gamma=1}^Q U^{(ni1)}_{\alpha\gamma} U^{(ni1,k)}_{\gamma\beta} (B^{(ij)}_{\gamma\beta} - B^{(kj)}_{\beta}) = \sum_{i1,\gamma=1}^Q U^{(ni1)}_{\alpha\gamma} B^{(2;ni)}_{\gamma\beta} F^{(j;i1;k)}_{\gamma\beta}, \quad j \geq 3,$$  \hspace{1cm} (97)
\[ L_{\alpha\beta}^{(j;ilk)} = \sum_{i_1,\gamma=1}^{Q} L_{\alpha\gamma}^{(2;il1)} F_{\gamma\beta}^{(j;ik)} \Rightarrow \] (98)

\[ \left( \partial_{x_j} U_{\alpha\beta}^{(nk)} - \partial_{x_1} U_{\alpha\beta}^{(nk)} A_n^{(j)} + \sum_{i_1,\gamma=1}^{Q} U_{\alpha\gamma}^{(ni_1)} v_{\gamma\beta}^{(ni_1k)} B_{\gamma}^{(i_1j)} \right) B_{\beta}^{(2;nn)} = \]

\[ \sum_{i_1,\gamma=1}^{Q} U_{\alpha\gamma}^{(ni_1)} B_{\gamma}^{(2;nn)} F_{\gamma\beta}^{(j;ik)} , \quad j \geq 2. \]

\[ L_{\alpha\beta}^{(j;ilk)} = \sum_{i_1,\gamma=1}^{Q} L_{\alpha\gamma}^{(2;il1)} F_{\gamma\beta}^{(j;ik)} \Rightarrow \] (99)

\[ U_{\alpha\beta}^{(nk)} B_{\beta}^{(j;nn)} = \sum_{i_1,\gamma=1}^{Q} U_{\alpha\gamma}^{(ni_1)} B_{\gamma}^{(2;nn)} F_{\gamma\beta}^{(j;ik)} \]

where \( F_{\gamma\beta}^{(j;ik)} \), \( \tilde{F}_{\gamma\beta}^{(j;ik)} \) and \( \hat{F}_{\gamma\beta}^{(j;ik)} \) are scalar functions of \( x \) to be defined. This set of equations can be taken as a basis. It is simple to check that any other linear equation for the spectral functions \( U^{(nk)} \) is a linear combination of (97-99). Remark, that if \( n = \beta \), then LHS of the eq. (98) disappears yielding algebraic relations among elements of matrix spectral functions:

\[ \sum_{i_1,\gamma=1}^{Q} U_{\alpha\gamma}^{(\beta;il1)} B_{\gamma}^{(2;\beta\beta)} F_{\gamma\beta}^{(j;ik)} = 0, \] (100)

which must be involved into consideration as compatible constraints. We consider equations (97,98) and (100) as an overdetermined system of linear equations for the spectral functions \( U^{(nk)} \) disregarding equation (99). It will be shown in the Sec.(5.1) that eq.(99) follows from the eq.(100).

Now we are going (a) to derive nonlinear equations for the fields (39) and (b) to express functions \( F(x) \), \( \tilde{F}(x) \) and \( \hat{F}(x) \) in terms of the fields (39).

Nonlinear equations for the fields \( v^{(nik)} \), \( v^{(nik;1)} \) result from the eqs.(97,98,100) after applying \( \sum_{\alpha=1}^{Q} C^{(i)}_{\alpha\alpha} \) to them and using (19) for \( C^{(i)}_{\alpha\alpha} \), \( k > 2 \) (we replace \( \tilde{\alpha} \) by \( \alpha \) in the result):

\[ \partial_{x_j} v_{\alpha\beta}^{(nik)} + \partial_{x_1} v_{\alpha\beta}^{(nik)} B_{\beta}^{(kj;nn)} - \partial_{x_2} v_{\alpha\beta}^{(nik)} B_{\beta}^{(kj)} + \]

\[ v_{\alpha\beta}^{(nik;1)} S_{\alpha\beta}^{(v;j1;nik)} + v_{\alpha\beta}^{(ik;2)} S_{\alpha\beta}^{(v;j2;ik)} + \sum_{i_1,\gamma=1}^{Q} v_{\alpha\gamma}^{(ni_1)} v_{\gamma\beta}^{(ni_1k)} \left( B_{\gamma}^{(i_1j)} - B_{\beta}^{(k)} \right) = \]

\[ \sum_{i_1,\gamma=1}^{Q} v_{\alpha\gamma}^{(ni_1)} B_{\gamma}^{(2;nn)} F_{\gamma\beta}^{(j;ik)} , \quad j \geq 3, \quad n \neq \beta, \]
\[
\left( \partial_{x_j} v_{\alpha\beta}^{(nk)} - \partial_{x_1} v_{\alpha\beta}^{(nk)} A_n \right) + \\
\sum_{i_1, \gamma=1}^Q v_{\alpha\gamma}^{(ni_1)} B_{\gamma}^{(i_1j)} B_{\beta}^{(2;n)} = \\
\sum_{i_1, \gamma=1}^Q v_{\alpha\gamma}^{(ni_1)} B_{\gamma}^{(2;n)} \tilde{F}_{\gamma\beta}^{(j;i_1k)}, \quad j \geq 2, \ n \neq \beta,
\]

where

\[
S_{\alpha\beta}^{(v;j1:nk)} = B_{\alpha}^{\beta;\alpha} - B_{\beta}^{(kj;n)} = A_n^{(j)} - A_n^{(j)} + B_{\alpha}^{(ij)} A_n^{(2)} - B_{\beta}^{(kj)} A_n^{(2)},
\]

\[
S_{\alpha\beta}^{(v;j2:ik)} = B_{\beta}^{(kj)} - B_{\alpha}^{(ij)},
\]

\[
\tilde{S}_{\alpha}^{(v;21:n)} = A_n^{(2)}, \quad \tilde{S}_{\alpha}^{(v;22:i)} = -1,
\]

\[
\tilde{S}_{\alpha}^{(v;j1:n)} = A_n^{(j)} + B_{\alpha}^{(ij;\alpha)} = A_n^{(j)} - A_n^{(j)} + B_{\alpha}^{(ij)} A_n^{(2)}, \quad \tilde{S}_{\alpha}^{(v;j2:i)} = -B_{\beta}^{(ij)}, \quad j > 2.
\]

In order to fix functions \( F^{(j;i_1k)}(x) \) we introduce the external dressing matrix function \( G(\lambda; x) \) and the associated fields

\[
u^{(nk)} = G \ast U^{(nk)}, \quad u^{(nk;p)} = G_{xp} \ast U^{(nk)}, \quad w^{(nk;ps)} = G_{xp} x_s \ast U^{(nk)}, \ldots, \quad w^{(nk;sp)} = w^{(nk;sp)}
\]

(see eq.(39) in Proposition ). Applying \( \sum_{\alpha=1}^Q G_{\alpha} \ast \) to the eqs.(97,98,100) one gets the first set of equations for these fields (we replace \( \alpha \) by \( \beta \) in the result):

\[
\left( \partial_{x_j} w_{\alpha\beta}^{(nk)} - \partial_{x_1} w_{\alpha\beta}^{(nk)} A_n \right) + \\
\sum_{i_1, \gamma=1}^Q w_{\alpha\gamma}^{(ni_1)} \tilde{B}_{\gamma}^{(i_1j)} \tilde{B}_{\beta}^{(2;n)} = \\
\sum_{i_1, \gamma=1}^Q w_{\alpha\gamma}^{(ni_1)} \tilde{B}_{\gamma}^{(2;n)} \tilde{F}_{\gamma\beta}^{(j;i_1k)}, \quad j \geq 2, \ n \neq \beta,
\]
Applying $\sum_{\alpha=1}^{Q}(G_{\alpha\alpha})_{x_p}*$ to the eqs.(97,98,100) and replacing $\tilde{\alpha}$ by $\alpha$ in the result, one gets the second set of equations for the fields (105):

$$\begin{align*}
\partial_{x_j}w^{(nk;p)}_{\alpha\beta} + \partial_{x_1}w^{(nk;p)}_{\alpha\beta} B_{\beta}^{(kj;n)} - \partial_{x_2}w^{(nk;p)}_{\alpha\beta} B_{\beta}^{(k\gamma;j)} - w^{(nk;p)}_{\alpha\beta} \\
-w^{(nk;p)}_{\alpha\beta} B_{\beta}^{(kj;n)} + w^{(nk;p)}_{\alpha\beta} B_{\beta}^{(k\gamma;j)} + \sum_{i_1,\gamma=1}^{Q} w^{(ni_1;\gamma)}_{\alpha\gamma} v^{(ni_1;\gamma)}_{\gamma\beta} (B_{\gamma}^{(i_1;j)} - B_{\beta}^{(k\gamma;j)}) = \\
\sum_{i_1,\gamma=1}^{Q} w^{(ni_1;\gamma)}_{\alpha\gamma} B_{\gamma}^{(2;n)} F_{\gamma\beta}^{(j;i_1;k)}.
\end{align*}$$

(109)

$$\begin{align*}
\left(\partial_{x_j}w^{(nk;p)}_{\alpha\beta} - \partial_{x_1}w^{(nk;p)}_{\alpha\beta} A_{n}^{(j)} - w^{(nk;p)}_{\alpha\beta} A_{n}^{(j)} + \sum_{i_1,\gamma=1}^{Q} w^{(ni_1;\gamma)}_{\alpha\gamma} v^{(ni_1;\gamma)}_{\gamma\beta} B_{\gamma}^{(i_1,j)}\right) B_{\beta}^{(2;n)} = \\
\sum_{i_1,\gamma=1}^{Q} w^{(ni_1;\gamma)}_{\alpha\gamma} B_{\gamma}^{(2;n)} F_{\gamma\beta}^{(j;i_1;k)}, \quad j \geq 2, \quad n \neq \beta.
\end{align*}$$

(110)

$$\sum_{i_1,\gamma=1}^{Q} w^{(\beta i_1;\gamma)}_{\alpha\gamma} B_{\gamma}^{(2;\beta)} F_{\gamma\beta}^{(j;i_1;k)} = 0.$$  

(111)

We refer to the eqs.(106-111) as equations for the fields $w^{(nk;p)}$ and $w^{(nk;ps)}$, also the nonlinear parts of these equations involve fields $v^{(nk)}$ as well.

The set of nonlinear equations may be continued applying $\sum_{\alpha=1}^{Q}(G_{\alpha\alpha})_{x_p}*$ to the eqs.(97,98,100), but the system of nonlinear PDEs constructed in this way may not be completed since (a) its solution space has arbitrary functions of all variables $x$, (see functions $f^{(ik)}$ in the eq.(38)) and (b) the dressing function $G$ is arbitrary, i.e. all derivatives of $G$ with respect to $x_p$, $p = 1, 2, \ldots$ are independent. This ends the proof of the item (a) of Proposition, i.e. we have derived a set of nonlinear PDEs (101-103), (106-111) for the fields (39) which does not represent a complete system of PDEs.

Now we pay attention to the completeness of the derived nonlinear system.

In order to fix arbitrary functions in the solution space we must introduce an extra largely arbitrary relation among the fields

$$\mathcal{F}^{rs}(\text{all fields}) = 0, \quad r, s = 1, \ldots, Q,$$

(112)

where $\mathcal{F}^{rs}$ are the $Q \times Q$ matrices. (See the item (b) of Proposition). For instance, in order to obtain a system of nonlinear PDEs having differential polynomial form, we choose the eq. (112) in the form (41) (see the item (c) in Proposition).

We use the eqs.(106,107) to express $F_{\gamma\beta}^{(j;i_1;k)}$, $\tilde{F}_{\gamma\beta}^{(j;i_1;k)}$ in terms of the fields (39). In view of
the eq.(41), the eqs.(106,107) read:

\[-w_{\alpha\beta}^{(nk;j)} - w_{\alpha\beta}^{(nk;1)}B_{\beta}^{(k;j;n)} + w_{\alpha\beta}^{(nk;2)}B_{\beta}^{(k;j)} + \sum_{i_1,\gamma=1}^{Q} R_{\alpha\gamma}^{(ni_1)}v_{\gamma\beta}^{(ni_1k)}(B_{\gamma}^{(i_1;j)} - B_{\beta}^{(k;j)}) = \tag{113}\]

\[
\sum_{i_1,\gamma=1}^{Q} R_{\alpha\gamma}^{(ni_1)}B_{\gamma}^{(2;n)}F_{\gamma\beta}^{(j;i_1k)}, \ j \geq 3,
\]

\[
\left(-w_{\alpha\beta}^{(nk;j)} + w_{\alpha\beta}^{(nk;1)}A_{n}^{(j)} + \sum_{i_1,\gamma=1}^{Q} R_{\alpha\gamma}^{(ni_1)}v_{\gamma\beta}^{(ni_1k)}B_{\gamma}^{(i_1;j)}B_{\beta}^{(2;n)} = \right.
\]

\[
\sum_{i_1,\gamma=1}^{Q} R_{\alpha\gamma}^{(ni_1)}B_{\gamma}^{(2;n)}\tilde{F}_{\gamma\beta}^{(j;i_1k)}, \ j \geq 2, \ n \neq \beta.
\]

The matrices \(R^{(ik)}\) must provide unique solvability of (113) with respect to \(F_{\gamma\beta}^{(j;i_1k)}\) and \(\tilde{F}_{\gamma\beta}^{(j;i_1k)}\), which become linear functions of the fields \(v\) and \(w\):

\[
F_{\alpha\beta}^{(j;ik)} = \sum_{i_1,i_2,\gamma=1}^{Q} S_{\alpha\gamma\beta}^{(j;i_1i_2k)}v_{\gamma\beta}^{(i_1i_2k)} + \left(\hat{R}^{-1}\right)^{(i_1i_2i_3)}(\gamma\beta)B_{\gamma}^{(i_1i_2i_3)}B_{\beta}^{(i_1i_2i_3)}, \ j \geq 3,
\]

\[
\tilde{F}_{\alpha\beta}^{(j;ik)} = \sum_{i_1,i_2,\gamma=1}^{Q} \tilde{S}_{\alpha\gamma\beta}^{(j;i_1i_2)}v_{\gamma\beta}^{(i_1i_2)} + \left(\hat{R}^{-1}\right)^{(i_1i_2i_3)}(\gamma\beta)B_{\gamma}^{(i_1i_2i_3)}B_{\beta}^{(i_1i_2i_3)}, \ j \geq 2,
\]

where

\[
S_{\alpha\gamma\beta}^{(j;i_1i_2k)} = \sum_{\gamma_1=1}^{Q} \left(\hat{R}^{-1}\right)^{(i_1i_2)}(\gamma_1\gamma)R_{\gamma_1\gamma}^{(i_1i_2)}(B_{\gamma}^{(i_1i_2)} - B_{\beta}^{(k;j)}), \tag{115}\]

\[
\tilde{S}_{\alpha\gamma\beta}^{(j;i_1i_2i_3)} = \sum_{\gamma_1=1}^{Q} \left(\hat{R}^{-1}\right)^{(i_1i_2i_3)}(\gamma_1\gamma)R_{\gamma_1\gamma}^{(i_1i_2i_3)}B_{\gamma}^{(i_1i_2i_3)}B_{\beta}^{(i_1i_2i_3)}
\]

and the operator \(\hat{R}^{-1}\) is the inverse of the operator \(R^{(ni)}B^{(2;n)}\), i.e.

\[
\sum_{n,\gamma_1=1}^{Q} \left(\hat{R}^{-1}\right)^{(ln)}(\alpha\gamma_1\gamma_1\gamma)B_{\gamma_1\gamma}^{(2;n)} = \delta^{(ln)}\delta^{(\alpha\gamma)}, \quad \text{or} \tag{116}\]

\[
\sum_{i_1,\gamma=1}^{Q} R_{\gamma\gamma_1}^{(ni_1)}B_{\gamma_1}^{(2;n)}(\hat{R}^{-1})^{(i_1i_l)}(\gamma_{1\gamma}) = \delta^{(nl)}\delta^{(\alpha\gamma)}.
\]

The eq.(100) may be transformed as follows. Substituting \(\tilde{F}^{(j;ik)}\) into (100) one gets

\[
\sum_{i_1,i_2,\gamma_1,\gamma_2=1}^{Q} U_{\alpha\gamma_1}^{(\beta i_1)}B_{\gamma_1}^{(2;\beta)}(\hat{R}^{-1})^{(i_1i_2)}(\gamma_{1\gamma_2})B_{\beta}^{(2;i_2)}E_{\gamma_2k}^{(i_2j)} = 0 \tag{117}\]
where
\[
\mathcal{E}_{\gamma_2k}^{(i_2j)} = \sum_{i_3,\gamma_3=1}^{Q} B_{\gamma_2\gamma_3}^{(i_2i_3)} B_{\gamma_3}^{(i_3j)} v_{\gamma_3}^{(i_2i_3k)} - w_{\gamma_2\beta}^{(i_2k;1)} + w_{\gamma_2\beta}^{(i_2k;1)} A_{i_2}^{(j)}, \tag{118}
\]
\[i_2 = 1, \ldots, Q, \quad j = 2, \ldots, Q + 1, \quad k, \gamma_2 = 1, \ldots, Q.\]

Assuming large arbitrariness of \(v_{\gamma_3\beta}^{(i_2i_3k)}\) and, consequently, of \(E_{\gamma_2k}^{(i_2j)}\), we conclude that coefficients ahead of \(E_{\gamma_2k}^{(i_2j)}\) are zeros:

\[
\sum_{i_1,\gamma_1=1}^{Q} U_{\alpha\gamma_1}^{(i_1i_2)} B_{\gamma_1}^{(2;\beta)} (\hat{R}^{-1})^{(i_1i_2)} B_{\beta}^{(2;ij_2)} = 0,
\tag{119}
\]
\[i_2, \alpha, \beta, \gamma_2 = 1, \ldots, Q, \quad i_2 \neq \beta.\]

We take \(i_2 \neq \beta\) in (119) since LHS of this equation is identical to zero if \(i_2 = \beta\) due to the equality \(B_{\beta}^{(2;\beta)} = 0\). Thus we have \(Q^2(Q - 1)\) equations and the same number of the elements of the spectral functions \(U_{\alpha\gamma_1}^{(i_1i_2)}\), \(\gamma_1 \neq \beta\) in (119). Remark that eqs. (119) are linearly dependent. Namely, they admit \(Q\) linear algebraic relations for each pair \((\alpha, \beta)\). To show this, one applies \(\sum_{i_2=1}^{Q} \sum_{\gamma_2=1}^{Q} R_{\gamma_2}^{(i_2k)}\), \(k = 1, \ldots, Q\), to the eq.(119). This yields identity due to the eq.(116).

One concludes that the system (119) with fixed \(\alpha\) and \(\beta\) consists of \(Q(Q - 1) - Q\) equations and consequently \(Q(Q - 1) - [Q(Q - 1) - Q] = Q\) elements of the spectral functions \(U_{\alpha\gamma_1}^{(i_1i_2)}\), \(\gamma_1 \neq \beta\), may be independent for each pair \((\alpha, \beta)\). If \(\beta \neq \alpha\), then \(U_{\alpha\gamma_1}^{(i_1i_2)}\), \(i_1 = 1, \ldots, Q\) may be taken as independent functions in the set \(U_{\alpha\gamma_1}^{(i_1i_2)}\), \(i_1, \gamma_1 = 1, \ldots, Q, \gamma_1 \neq \beta\).

Applying \(\sum_{\alpha=1}^{Q} C_{\alpha\alpha}^{(i)}\) and \(\sum_{\alpha=1}^{Q} (G_{\alpha\alpha})_{x_p}\) to the eq.(119) and replacing \(\tilde{\alpha}\) by \(\alpha\) in the result one gets the following algebraic relations among the fields \(v_{\alpha\gamma_1}^{(i_1i_2)}\) and \(w_{\alpha\gamma_1}^{(i_1i_2;p)}\):

\[
\sum_{i_1,\gamma_1=1}^{Q} v_{\alpha\gamma_1}^{(i_1i_2)} B_{\gamma_1}^{(2;\beta)} (\hat{R}^{-1})^{(i_1i_2)} B_{\beta}^{(2;ij_2)} = 0, \tag{120}
\]
\[
\sum_{i_1,\gamma_1=1}^{Q} w_{\alpha\gamma_1}^{(i_1i_2;p)} B_{\gamma_1}^{(2;\beta)} (\hat{R}^{-1})^{(i_1i_2)} B_{\beta}^{(2;ij_2)} = 0, \tag{121}
\]
\[i, i_2, \alpha, \beta, \gamma_2 = 1, \ldots, Q, \quad i_2 \neq \beta.\]

The last step which must be done to complete the system of nonlinear PDEs is introducing additional PDEs for the dressing functions \(G\) establishing relations among derivatives \(G_{x_p}\) and, consequently, among the fields \(w_{(i;j;k)}\). The simplest case leading to 4-dimensional nonlinear PDEs is following:

\[
\partial_{x_k} G = P^{(k1)} \partial_{x_1} G + P^{(k2)} \partial_{x_2} G, \quad k > 2.
\tag{122}
\]

Then

\[
w_{(i;j;k)} = P^{k1} w_{(i;j;1)} + P^{k2} w_{(i;j;2)}, \quad k > 2.
\tag{123}
\]
Not all equations of the system (101-103), (106-111) are independent. Namely, although the linear equations (97) can be written for any spectral function $U^{(nj)}$, it is not necessary to use all these equations because the eqs.(97) with $n \neq \beta$ are combinations of the eqs.(98) by construction. Thus, we need only those equations (97) which correspond to $n = \beta$, while the equations for $U^{(nj)}_{\alpha \beta}$, $n \neq \beta$, are represented by the system (98), which is simpler. Since the eq.(97) generates nonlinear equations (101), (106) and (109), one needs only those of them which correspond to $n = \beta$, while the rest equations of the nonlinear system will be generated by the eq.(98), see eqs.(102,107, 110)

Now we write down the complete system of nonlinear PDEs for the fields (39) using eqs.(120,121) and (116) (see eqs.(124-137)). Thus, eq.(101), $n = \beta$, yields:

$$
E^{(j;ik)}_{\alpha \beta} := \partial_{x_j} v^{(\beta k)}_{\alpha \beta} + \partial_{x_1} v^{(\beta k)}_{\alpha \beta} B^{(k;j;\beta)}_{\beta} - \partial_{x_2} v^{(\beta k)}_{\alpha \beta} B^{(k;j)}_{\beta} + v^{(\beta i;k)}_{\alpha \beta} S^{(\nu j;1;ik)}_{\alpha \beta} + v^{(\beta i;k)}_{\alpha \beta} S^{(\nu j;2;ik)}_{\alpha \beta} + \sum_{i=1, \gamma_1=1 \atop \gamma_1 \neq \gamma}^Q v^{(\beta i;1;ik)}_{\gamma_1} + \sum_{i=1, \gamma_1=1 \atop \gamma_1 \neq \gamma}^Q v^{(\beta i;1;ik)}_{\gamma_2} + \sum_{i=1, \gamma_1=1 \atop \gamma_1 \neq \gamma}^Q w^{(\beta i;1;ik)}_{\gamma_2} S^{(\nu j;1;ik)}_{\gamma_1, \gamma_2} = 0, \; j \geq 3,
$$

$$
S^{(w;1;j;ik)}_{\gamma_1, \gamma_2} = \mathcal{E}^{(i_1 i_3 \gamma_2)}_{\gamma_1} - \delta^{(i_1 i_3)}_{\gamma_1} \delta^{(i_1 i_2)}_{\gamma_1} S^{(w;1;j;ik)}_{\gamma_1},
$$

$$
S^{(w;2;j;ik)}_{\gamma_1, \gamma_2} = \mathcal{E}^{(i_1 i_3 \gamma_2)}_{\gamma_1} - \delta^{(i_1 i_3)}_{\gamma_1} \delta^{(i_1 i_2)}_{\gamma_1} S^{(w;2;j;ik)}_{\gamma_1},
$$

where

$$
S^{(w;1;j;ik)}_{\alpha \beta} = -B^{(j)}_{\alpha \beta} - B^{(j)}_{\alpha \beta} A^{(j)}_{\alpha \beta} + A^{(j)}_{\alpha \beta},
$$

$$
S^{(w;2;j;ik)}_{\alpha \beta} = -B^{(j)}_{\alpha \beta} + B^{(j)}_{\alpha \beta}, \; j \geq 3.
$$

Writing the expression (125) we used eq.(120) and eq.(130b) from the following list of identities:

$$
\sum_{i_1, \gamma_1, \gamma_2=1}^Q U^{(n_1 \gamma_2)}_{\alpha \gamma} B^{(2;n)}_\gamma (\hat{R}^{-1})_{\gamma_1, \gamma_2} = \sum_{i_1, \gamma_1, \gamma_2=1}^Q U^{(n_1 \gamma_2)}_{\alpha \gamma} \delta^{(i_1 \gamma_2)}_{\gamma_1}, \; \gamma_1 \neq n, \; \Rightarrow \; (130)
$$

$$
\sum_{i_1, \gamma_1, \gamma_2=1}^Q v^{(n_1 i_1 \gamma_2)}_{\alpha \gamma} B^{(2;n)}_\gamma (\hat{R}^{-1})_{\gamma_1, \gamma_2} = \sum_{i_1, \gamma_1, \gamma_2=1}^Q v^{(n_1 i_1 \gamma_2)}_{\alpha \gamma} \delta^{(i_1 \gamma_2)}_{\gamma_1},
$$

$$
\sum_{i_1, \gamma_1, \gamma_2=1}^Q w^{(n_1 i_1 \gamma_2)}_{\alpha \gamma} B^{(2;n)}_\gamma (\hat{R}^{-1})_{\gamma_1, \gamma_2} = \sum_{i_1, \gamma_1, \gamma_2=1}^Q w^{(n_1 i_1 \gamma_2)}_{\alpha \gamma} \delta^{(i_1 \gamma_2)}_{\gamma_1},
$$

$n, i, \alpha, \gamma_1 = 1, \ldots, Q, \; p = 1, 2, \; \gamma_1 \neq n.$

To prove equality (130a) one must apply $\sum_{\gamma_1, i_1=1}^Q \cdot B^{(2;n)}_\gamma (\hat{R}^{-1})_{i_1 \gamma_1} \gamma_1 \beta$ to (130) and use (119) to simplify the RHS of the resulting equation. Eqs.(130b,c) follows from the eq. (130a) after applying $\sum_{\gamma_1, i_1=1}^Q \mathcal{E}^{(i_1 \gamma_1 \gamma_2)}_{\alpha \gamma} \gamma_1 \gamma_1 \beta$ and $\sum_{\alpha, \gamma_1=1}^Q (\mathcal{E}_{\alpha \gamma})_{i \gamma_1 \gamma_2} \gamma_1 \gamma_1 \beta$, $p = 1, 2$ to the eq.(130a) and replacing $\tilde{\alpha}$ by $\alpha$ in the result.
The eq. (102) yields

\[ \tilde{E}_{\alpha\beta}^{(i;nk)} := \left( \partial_{x_i} v_{\alpha\beta}^{(nk)} - \partial_{x_1} v_{\alpha\beta}^{(nk)} A_n \right) + \left( \partial_{x_1} v_{\alpha\beta}^{(nk;1)} \tilde{S}_{\alpha\beta}^{(v;j;1;i_1i_2)} + \partial_{x_1} v_{\alpha\beta}^{(nk;2)} \tilde{S}_{\alpha\beta}^{(v;j;2;i_1i_2)} \right) B_{\beta}^{(2n)} + \right.

\left. \sum_{i_1, \gamma_1 = 1}^{Q} v_{\alpha\beta}^{(n1i_1)} \left( \sum_{i_2 = 1}^{Q} v_{\gamma_1\beta}^{(n2i_2)} \tilde{S}_{\gamma_1\beta}^{(v;j;1;i_1i_2)} \right) + \sum_{\gamma_2 = 1}^{Q} \sum_{i_0 = 1}^{2} w_{\gamma_2\beta}^{(nk;i_0)} \tilde{S}_{\gamma_1\gamma_2\beta}^{(v;i_0;j;0i_1i_2)} \right), \quad j \geq 2, \ n \neq \beta,

\[ \tilde{S}_{\gamma_1\beta}^{(v;j;1;i_1i_2)} = B_{\beta}^{(2n)} \left( E_{\gamma_1}^{(i)} - E_{\gamma_1}^{(2n)} \delta_{\gamma_1\gamma_2} \right) \tilde{S}_{\gamma_1\gamma_2\beta}^{(v;j;2;i_1i_2)}, \]

\[ \tilde{S}_{\gamma_1\gamma_2\beta}^{(v;j;1;i_1i_2)} = B_{\gamma_1}^{(2n)} \left( \tilde{R}_{\gamma_1\gamma_2}^{(i) (i_1i_2)} \right) \tilde{S}_{\gamma_2}^{(v;j;2;i_1i_2)}, \]

\[ \tilde{S}_{\gamma_1\gamma_2\beta}^{(v;j;2;i_1i_2)} = B_{\gamma_1}^{(2n)} \left( \tilde{R}_{\gamma_1\gamma_2}^{(i) (i_1i_2)} \right) \tilde{S}_{\gamma_2}^{(v;j;2;i_1i_2)}, \]

where

\[ \tilde{S}_{\alpha}^{(v;1;2);n} = A_{\alpha}^{(2)}, \quad \tilde{S}_{\alpha}^{(v;2;2);n} = -1, \]

\[ \tilde{S}_{\alpha}^{(v;1;2);n} = -\mathcal{P}_{\alpha}^{(1)} + A_{\alpha}, \quad \tilde{S}_{\alpha}^{(v;2;2);n} = -\mathcal{P}_{\alpha}^{(2)}, \quad j \geq 3. \]

Writing the expression (132) we used the eqs.(120) and (130b).

The eqs.(109) with \( n = \beta \) gives

\[ E_{\alpha\beta}^{(j;3;k;p)} := \partial_{x_j} w_{\alpha\beta}^{(3k;p)} + \partial_{x_1} w_{\alpha\beta}^{(3k;p)} B_{\beta}^{(k;j)} - \partial_{x_2} w_{\alpha\beta}^{(3k;p)} B_{\beta}^{(k;j)} + \right.

\left. \sum_{i_0 = 1}^{2} w_{\alpha\beta}^{(3k;p;i_0)} S_{\alpha\beta}^{(v;j;i_0);k} B_{\beta}^{(k;j)} + \right.

\left. \sum_{i_1, \gamma_1 = 1}^{Q} w_{\alpha\beta}^{(3k;1;i_1)} \left( \sum_{i_2 = 1}^{Q} v_{\beta\beta}^{(3k;i_2)} \tilde{S}_{\gamma_1\beta}^{(v;j;1;i_1i_2)} + \sum_{\gamma_2 = 1}^{Q} \sum_{i_0 = 1}^{2} w_{\gamma_2\beta}^{(3k;i_0)} \tilde{S}_{\gamma_1\gamma_2\beta}^{(v;i_0;j;0i_1i_2)} \right) = 0. \]

Finally, the eq.(110) reads:

\[ E_{\alpha\beta}^{(j;nk;p)} := \left( \partial_{x_j} w_{\alpha\beta}^{(nk;p)} - \partial_{x_1} w_{\alpha\beta}^{(nk;p)} A_n \right) + \sum_{i_0 = 1}^{Q} w_{\alpha\beta}^{(nk;p;i_0)} S_{\alpha\beta}^{(v;i_0;j;0i_1i_2)} \right) B_{\beta}^{(2n)} + \right.

\left. \sum_{i_1, \gamma_1 = 1}^{Q} w_{\alpha\beta}^{(n1i_1)} \left( \sum_{i_2 = 1}^{Q} v_{\beta\beta}^{(n2i_2)} \tilde{S}_{\gamma_1\beta}^{(v;j;i_1i_2)} \right) + \sum_{\gamma_2 = 1}^{Q} \sum_{i_0 = 1}^{2} w_{\gamma_2\beta}^{(nk;i_0)} \tilde{S}_{\gamma_1\gamma_2\beta}^{(v;i_0;j;0i_1i_2)} \right), \quad j \geq 2, \ n \neq \beta.

Deriving the coefficients in the nonlinear parts of the eqs.(136, 137) we used the eqs. (121) and (130c).

Now we consider such combinations of the eqs. (124), (131) which do not involve the fields \( v^{(nk;i)} \), \( j = 1, 2 \), and such combinations of the eqs. (136) and (137) which do not involve the fields \( w^{(nk;p;i_0)} \), \( p, i_0 = 1, 2 \). There are several types of such combinations depending on the relations among the coefficients ahead of the fields \( v^{(nk;i)} \) in the eqs.(124), (131) and ahead of the fields \( w^{(nk;p;i_0)} \) in the eqs.(136), (137). Remember that the parameters \( \tilde{S}_{\alpha\beta}^{(v;j;1;1;i_1i_2)}, \tilde{S}_{\alpha\beta}^{(v;j;2;1;i_1i_2)}, \tilde{S}_{\alpha\beta}^{(v;j;2;2;i_1i_2)}, \tilde{S}_{\alpha\beta}^{(v;j;1;i_1i_2)}, \tilde{S}_{\alpha\beta}^{(v;j;2;i_1i_2)} \) and \( B_{\alpha}^{(ij;k)} \) are given by the eqs.(104,129,135,27).

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1. Let \( n = \beta = \alpha, \ i = k \). Then \( S_{\alpha\alpha}^{(\nu j;1\alpha k)} = S_{\alpha\alpha}^{(\nu j;2\alpha k)} = 0 \). The eq. (124) reads:

\[
E_{\alpha\alpha}^{(j1;\alpha k)} := \partial_{\nu} v_{\alpha\alpha}^{(\alpha k)} + \partial_{j1} v_{\alpha\alpha}^{(\alpha k)} B_{\alpha}^{(kj;\alpha)} - \partial_{\nu x} v_{\alpha\alpha}^{(\alpha k)} B_{\alpha}^{(kj)} + \sum_{i_1, \gamma_1 = 1}^{Q} v_{\alpha\gamma_1}^{(\alpha i_1)} \frac{\sum_{i_2=1}^{Q} v_{\alpha\alpha}^{(\alpha i_2 k)}}{\gamma_1} + \sum_{\gamma_2=1}^{Q} \sum_{i_0=1}^{2} w_{\gamma_2,\alpha}^{(\alpha i_0 \gamma) \gamma_1}\delta_{\alpha\gamma_1}^{(\gamma i_0 i_1 \gamma))} = 0.
\]

(138)

2. Let \( \beta = \alpha \) in the eq.(124) and \( n = \alpha \) in the eq. (131). Then \( S_{\alpha\alpha}^{(\nu j;1\alpha i)} = -A_{\alpha}^{(2)} S_{\alpha\alpha}^{(\nu j;2\alpha i)}, \ \bar{S}_{\alpha\beta}^{(\nu j;1\alpha i)} = -A_{\alpha}^{(2)} \bar{S}_{\alpha\beta}^{(\nu j;2\alpha i)} \). So that the following combinations of the eqs. (124) and (131) have no fields \( v_{\alpha\alpha}^{(\nu i k ; j)} \):

\[
E_{\alpha\alpha}^{(j2;\alpha i k)} := \begin{vmatrix} E_{\alpha\alpha}^{(j2;ak)} & E_{\alpha\alpha}^{(j2;ik)} \\ E_{\alpha\alpha}^{(j3;ak)} & E_{\alpha\alpha}^{(j3;ik)} \end{vmatrix}, \ j \geq 4,
\]

(139)

\[
E_{\alpha\beta}^{(j3;\alpha i k)} := \begin{vmatrix} E_{\alpha\beta}^{(j2;ak)} & E_{\alpha\beta}^{(j2;ik)} \\ E_{\alpha\beta}^{(j3;ak)} & E_{\alpha\beta}^{(j3;ik)} \end{vmatrix}, \ j \geq 3,
\]

which have the following explicit forms:

\[
E_{\alpha\alpha}^{(j2;\alpha i k)} := \sum_{i_1 = 1}^{Q} v_{\alpha\alpha}^{(\alpha i_1 i)} T_{\alpha}^{(j\nu i \nu i_1)} + \sum_{i_1, \gamma_1 = 1}^{Q} v_{\alpha\alpha}^{(\alpha i_1 k)} \delta_{\alpha\gamma_1}^{(\alpha i_1 k)} \delta_{\gamma_1}^{(j\nu i \nu i_1)} + \sum_{i_0=1}^{2} w_{\gamma_2,\alpha}^{(\alpha i_0 \gamma) \gamma_1}\delta_{\alpha\gamma_1}^{(\gamma i_0 i_1 \gamma))} = 0, \ i \neq k, \ j \geq 4,
\]

(140)

\[
E_{\alpha\beta}^{(j3;\alpha i k)} := \sum_{i_1 = 1}^{Q} v_{\alpha\beta}^{(\alpha i_1 i)} T_{\alpha}^{(j\nu i \nu i_1)} + \sum_{i_1, \gamma_1 = 1}^{Q} v_{\alpha\beta}^{(\alpha i_1 k)} \delta_{\alpha\gamma_1}^{(\alpha i_1 k)} \delta_{\gamma_1}^{(j\nu i \nu i_1)} + \sum_{i_0=1}^{2} w_{\gamma_2,\alpha}^{(\alpha i_0 \gamma) \gamma_1}\delta_{\alpha\gamma_1}^{(\gamma i_0 i_1 \gamma))} = 0, \ j \geq 3, \ \alpha \neq \beta,
\]

(141)

where

\[
s_{\alpha}^{(j\nu i ;1i k)} = \begin{vmatrix} B_{\alpha}^{(k3;\alpha)} - B_{\alpha}^{(3;\alpha)} & B_{\alpha}^{(k3;\alpha)} - B_{\alpha}^{(i ;j)} \end{vmatrix},
\]

(142)

\[
s_{\alpha}^{(j\nu i ;2i k)} = -\begin{vmatrix} B_{\alpha}^{(k3;\alpha)} - B_{\alpha}^{(3;\alpha)} & B_{\alpha}^{(k3;\alpha)} - B_{\alpha}^{(i ;j)} \end{vmatrix},
\]

\[
s_{\alpha}^{(j\nu i ;m i k)} = \begin{vmatrix} B_{\alpha}^{(k3;\alpha)} - B_{\alpha}^{(3;\alpha)} & B_{\alpha}^{(k3;\alpha)} - B_{\alpha}^{(i ;j)} \end{vmatrix}, \ m = 3, j,
\]

30
\[ T_{\alpha}^{(j;\omega;\bar{i};k)} = \begin{vmatrix} B_{\alpha}^{(k;3)} - B_{\alpha}^{(i;3)} & B_{\alpha}^{(k;ij)} - B_{\alpha}^{(i;ij)} \end{vmatrix}, \quad \Rightarrow \quad T_{\alpha}^{(j;\omega;\bar{i};k)} = T_{\alpha}^{(j;\omega;\bar{i};k_1)} = T_{\alpha}^{(j;\omega;\bar{i};k_2)} = 0, \]

\[ T_{\alpha \gamma}^{(j;\omega;\bar{i};k_1;1)} = - \begin{vmatrix} B_{\alpha}^{(k;3)} + P_{\gamma}^{(31)} & B_{\alpha}^{(k;i,\alpha)} + P_{\gamma}^{(ij)} \end{vmatrix}, \quad \Rightarrow \quad T_{\alpha \gamma}^{(j;\omega;\bar{i};k_1;1)} = 0, \]

\[ T_{\alpha \gamma}^{(j;\omega;\bar{i};k_2;2)} = \begin{vmatrix} B_{\alpha}^{(k;3)} - B_{\alpha}^{(i;3)} & B_{\alpha}^{(k;ij)} - B_{\alpha}^{(i;ij)} \end{vmatrix}, \quad \Rightarrow \quad T_{\alpha \gamma}^{(j;\omega;\bar{i};k_2;2)} = 0, \]

\[(143)\]

\[ s_{\alpha}^{(j;\omega;\bar{i};ii)} = \begin{vmatrix} A_{\alpha}^{(2)} & A_{\alpha}^{(j)} \end{vmatrix}, \quad s_{\alpha}^{(j;\omega;\bar{i};mi)} = - \begin{vmatrix} \delta^{(2m)} & \delta^{(jm)} \end{vmatrix}, \quad m = 2, j, \]

\[ \tilde{t}_{\alpha}^{(j;\omega;\bar{i};i_1)} = \begin{vmatrix} 1 & B_{\alpha}^{(i;j)} \end{vmatrix}, \quad \Rightarrow \quad \tilde{t}_{\alpha}^{(j;\omega;\bar{i};i)} = 0 \]

\[ \tilde{t}_{\alpha \gamma}^{(j;\omega;\bar{i};i_1;1)} = - \begin{vmatrix} A_{\alpha}^{(2)} & A_{\alpha}^{(j)} - P_{\gamma}^{(ij)} \end{vmatrix}, \quad \tilde{t}_{\alpha \gamma}^{(j;\omega;\bar{i};i_2;2)} = \begin{vmatrix} 1 & P_{\gamma}^{(j;\omega;\bar{i};i_2;2)} \end{vmatrix} \]

3. As for other values of \( n, i, k, \alpha, \beta \), the following combinations of eqs.(124) and (131) have no fields \( v_{\alpha \beta}^{(nik;j)} \):

\[ E_{\alpha \beta}^{(j4;\beta;ik)} := \begin{vmatrix} E_{\alpha \beta}^{(3;\beta;ik)} & E_{\alpha \beta}^{(4;\beta;ik)} & E_{\alpha \beta}^{(j;\beta;ik)} \end{vmatrix}, \quad j \geq 5, \]

\[ E_{\alpha \beta}^{(j5;nik)} := \begin{vmatrix} \tilde{E}_{\alpha \beta}^{(2;nik)} & \tilde{E}_{\alpha \beta}^{(3;nik)} & \tilde{E}_{\alpha \beta}^{(j;nik)} \end{vmatrix}, \quad j \geq 4, \quad n \neq \beta. \]
These expressions have the following explicit forms:

\[
E_{\alpha\beta}^{(j;4;ik)} := s_{\alpha\beta}^{(j;v;m;ik)} \partial_{x_j} v_{\alpha\beta} + \sum_{m=1}^{4} s_{\alpha\beta}^{(j;v;m;ik)} \partial_{x_m} v_{\alpha\beta} - \\
\sum_{i_1, i_1 \neq k}^{Q} v_{\alpha\beta}^{(\beta_{i_1} i_1)} T_{\alpha\beta}^{(j;v;v;i_1 i_1)} + \sum_{i_1, i_2, i_2 \neq k}^{Q} v_{\alpha\beta}^{(\beta_{i_1} i_1)} v_{\alpha\gamma}^{(\beta_{i_2} i_2)} E_{\gamma}^{(i_1 i_2)} T_{\alpha\beta}^{(j;v;v;i_1 i_2)} \\
\sum_{i_1, i_1, \gamma \neq \beta}^{Q} \sum_{i_1}^{2} v_{\alpha\gamma}^{(\beta_{i_1} i_1)} w_{\gamma i_2}^{(\beta;i_0)} B_{\gamma}^{(2;\beta)} (\hat{R}^{(1)}_{\gamma i_1}) T_{\alpha\gamma}^{(j;v;v;i_1 i_0)} = 0, \quad j \geq 5
\]

\[
E_{\alpha\beta}^{(j;5;nk)} := s_{\alpha\beta}^{(j;v;m;ni)} \partial_{x_j} v_{\alpha\beta} + \sum_{m=1}^{3} s_{\alpha\beta}^{(j;v;m;ni)} \partial_{x_m} v_{\alpha\beta} - \\
\sum_{i_1}^{Q} v_{\alpha\beta}^{(n i_1)} T_{\alpha}^{(j;v;v;i_1 i_1 n)} + \sum_{i_1}^{Q} v_{\alpha\beta}^{(n i_1)} v_{\alpha\gamma}^{(n i_2)} E_{\gamma}^{(i_1 i_2 n)} T_{\alpha}^{(j;v;v;i_1 i_2 n)} + \\
\sum_{i_1, \gamma \neq \beta}^{Q} \sum_{i_1}^{2} v_{\alpha\gamma}^{(n i_1)} w_{\gamma i_2}^{(n;i_0)} B_{\gamma}^{(2;n)} (\hat{R}^{(1)}_{\gamma i_1}) T_{\alpha\gamma}^{(j;v;v;i_1 i_0)} = 0, \quad j \geq 4, \quad n \neq \beta, \quad n \neq \alpha,
\]

where

\[
\begin{align*}
    s_{\alpha\beta}^{(j;v;1;ik)} & = \begin{vmatrix}
    B_{\beta}^{(k3;\beta)} \\
    S_{\alpha\beta}^{(v;31;\beta i k)} \\
    S_{\alpha\beta}^{(v;2;2;ik)} \\
    S_{\alpha\beta}^{(v;32;ik)}
    \end{vmatrix}, \\
    s_{\alpha\beta}^{(j;v;2;ik)} & = \begin{vmatrix}
    B_{\beta}^{(k3)} \\
    S_{\alpha\beta}^{(v;31;\beta i)k} \\
    S_{\alpha\beta}^{(v;41;\beta i k)} \\
    S_{\alpha\beta}^{(v;32;ik)}
    \end{vmatrix}, \\
    s_{\alpha\beta}^{(j;v;m;ik)} & = \begin{vmatrix}
    S_{\alpha\beta}^{(v;31;\beta i k)} \\
    S_{\alpha\beta}^{(v;41;\beta i k)} \\
    S_{\alpha\beta}^{(v;2;2;ik)} \\
    S_{\alpha\beta}^{(v;32;ik)}
    \end{vmatrix}, \quad m = 2, 3, j, \quad s_{\alpha\beta}^{(j;v;m;ki)} = s_{\beta\alpha}^{(j;v;m;ki)},
\end{align*}
\]

\[
T_{\alpha\beta}^{(j;v;i_1 i_k)} = \begin{vmatrix}
    S_{\alpha\beta}^{(v;32;i_1 i k)} \\
    S_{\alpha\beta}^{(v;42;i_1 i k)} \\
    S_{\alpha\beta}^{(v;2;2;i_1 i k)} \\
    S_{\alpha\beta}^{(v;32;ik)}
    \end{vmatrix}, \quad \Rightarrow T_{\alpha\beta}^{(j;v;v;ikki)} = 0,
\]

\[
T_{\alpha\gamma\beta}^{(j;v;v;i_1 i_0)} = \begin{vmatrix}
    S_{\gamma\beta}^{(v;31;\beta i k)} \\
    S_{\gamma\beta}^{(v;41;\beta i k)} \\
    S_{\gamma\beta}^{(v;2;2;\beta i k)} \\
    S_{\gamma\beta}^{(v;32;ik)}
    \end{vmatrix},
\]

32
As we shall see in the end of this section, the eqs. (144,145) do not appear in the final complete system of nonlinear PDEs. For this reason, we leave coefficients $T$, $T_\gamma$, $S$, and $\tilde{S}$ in general form.

4. As for the eqs. (136) and (137), the following combinations should be considered:

\[
E^{(j\beta;k\nu)}_{\alpha\beta} := \begin{vmatrix}
E^{(3;\beta;k\nu)}_{\alpha\beta}
\tilde{S}^{(w;1;3,k)}_{\alpha\beta}
\tilde{S}^{(w;2;3,k)}_{\alpha\beta}
\tilde{S}^{(w;2;3,k)}_{\alpha\beta}
\end{vmatrix}, \quad j \geq 5, \quad p = 1,2
\]

\[
E^{(j;\gamma;k\nu)}_{\alpha\beta} := \begin{vmatrix}
E^{(4;\beta;k\nu)}_{\alpha\beta}
\tilde{S}^{(w;1;4;k)}_{\alpha\beta}
\tilde{S}^{(w;2;4;k)}_{\alpha\beta}
\tilde{S}^{(w;2;4;k)}_{\alpha\beta}
\end{vmatrix}, \quad j \geq 4, \quad p = 1,2, \quad n \neq \beta.
\]
These expressions have the following explicit forms:

\[
E_{\alpha\beta}^{(j6;\beta;k)p} := s_{\alpha\beta}^{(j;w;1;k)} \partial_{x_j} w_{\alpha\beta}^{(\beta;k;p)} + \sum_{m=1}^{4} s_{\alpha\beta}^{(j;w;1;m;k)} \partial_{x_m} w_{\alpha\beta}^{(\beta;k;p)} - \sum_{i_1=1}^{Q} w_{\alpha\beta}^{(\beta_1;i_1;p)} u_{\beta\beta}^{(\beta_2;i_2;k)} T_{\alpha\beta}^{(j;w;1;i_1;k)} + \sum_{i_1=1}^{Q} w_{\alpha\gamma}^{(\beta_1;i_1;p)} v_{\beta\beta}^{(\beta_2;i_2;k)} \mathcal{C}_{\gamma}^{(i_1;i_2;\beta)} T_{\alpha\beta}^{(j;w;1;i_1;k)} + \sum_{i_1=1}^{Q} \sum_{i_2=1}^{2} w_{\alpha\gamma_1}^{(\beta_1;i_1;p)} w_{\gamma_2\beta}^{(\beta_2;i_2;0)} \mathcal{B}_{\gamma_1}^{(\gamma_1;i;\beta)} (\hat{R}^{-1})_{\gamma_1\gamma_2}^{(i_1;i_2;\beta)} T_{\alpha\beta}^{(j;w;1;i_1;i_2;\gamma_1;i_2;\gamma_2;\beta)} = 0, \quad j \geq 5,
\]

\[
E_{\alpha\beta}^{(j7;\beta;k)p} := \tilde{s}_{\alpha}^{(j;w;2;k)} \partial_{x_j} w_{\alpha\beta}^{(n;k;p)} + \sum_{m=1}^{3} \tilde{s}_{\alpha}^{(j;w;2;m;k)} \partial_{x_m} w_{\alpha\beta}^{(n;k;p)} - \sum_{i_1=1}^{Q} \frac{\delta_{\alpha\gamma}}{\gamma} w_{\alpha\gamma_1}^{(\beta_1;i_1;p)} w_{\gamma_2\beta}^{(\beta_2;i_2;0)} \mathcal{B}_{\gamma_1}^{(\gamma_1;i;\beta)} (\hat{R}^{-1})_{\gamma_1\gamma_2}^{(i_1;i_2;\beta)} T_{\alpha\beta}^{(j;w;2;i_1;i_2;\gamma_1;i_2;\gamma_2;\beta)} = 0, \quad j \geq 4, \quad n \neq \beta,
\]

where

\[
s_{\alpha\beta}^{(j;w;1;k)} = - \begin{bmatrix}
B_{\beta}^{(k;3;\beta)} & B_{\beta}^{(k;4;\beta)} & B_{\beta}^{(k;j;\beta)} & B_{\beta}^{(k;k;\beta)} & B_{\beta}^{(k;3;\beta)} + \mathcal{P}_{\alpha}^{(31)} & B_{\beta}^{(k;4;\beta)} + \mathcal{P}_{\alpha}^{(41)} & B_{\beta}^{(k;j;\beta)} + \mathcal{P}_{\alpha}^{(j1)} & B_{\beta}^{(k;k;\beta)} + \mathcal{P}_{\alpha}^{(j2)} \\
B_{\beta}^{(k;3;\beta)} & B_{\beta}^{(k;4;\beta)} & B_{\beta}^{(k;j;\beta)} & B_{\beta}^{(k;k;\beta)} & B_{\beta}^{(k;3;\beta)} + \mathcal{P}_{\alpha}^{(32)} & B_{\beta}^{(k;4;\beta)} - \mathcal{P}_{\alpha}^{(42)} & B_{\beta}^{(k;j;\beta)} - \mathcal{P}_{\alpha}^{(j2)} & B_{\beta}^{(k;k;\beta)} - \mathcal{P}_{\alpha}^{(j2)} \\
B_{\beta}^{(k;3;\beta)} & B_{\beta}^{(k;4;\beta)} & B_{\beta}^{(k;j;\beta)} & B_{\beta}^{(k;k;\beta)} & B_{\beta}^{(k;3;\beta)} + \mathcal{P}_{\alpha}^{(31)} & B_{\beta}^{(k;4;\beta)} + \mathcal{P}_{\alpha}^{(41)} & B_{\beta}^{(k;j;\beta)} + \mathcal{P}_{\alpha}^{(j1)} & B_{\beta}^{(k;k;\beta)} + \mathcal{P}_{\alpha}^{(j2)} \\
B_{\beta}^{(k;3;\beta)} & B_{\beta}^{(k;4;\beta)} & B_{\beta}^{(k;j;\beta)} & B_{\beta}^{(k;k;\beta)} & B_{\beta}^{(k;3;\beta)} + \mathcal{P}_{\alpha}^{(32)} & B_{\beta}^{(k;4;\beta)} - \mathcal{P}_{\alpha}^{(42)} & B_{\beta}^{(k;j;\beta)} - \mathcal{P}_{\alpha}^{(j2)} & B_{\beta}^{(k;k;\beta)} - \mathcal{P}_{\alpha}^{(j2)} \\
\end{bmatrix}, \quad m = 3, 4, j,
\]

\[
s_{\alpha\beta}^{(j;w;2;k)} = - \begin{bmatrix}
\delta^{(m3)} & \delta^{(m4)} & \delta^{(m5)} & \delta^{(m6)} & \delta^{(m7)} & \delta^{(m8)} & \delta^{(m9)} & \delta^{(m10)} \\
B_{\beta}^{(k;3;\beta)} + \mathcal{P}_{\alpha}^{(31)} & B_{\beta}^{(k;4;\beta)} + \mathcal{P}_{\alpha}^{(41)} & B_{\beta}^{(k;j;\beta)} + \mathcal{P}_{\alpha}^{(j1)} & B_{\beta}^{(k;k;\beta)} + \mathcal{P}_{\alpha}^{(j2)} & \delta^{(m3)} & \delta^{(m4)} & \delta^{(m5)} & \delta^{(m6)} \\
B_{\beta}^{(k;3;\beta)} + \mathcal{P}_{\alpha}^{(32)} & B_{\beta}^{(k;4;\beta)} + \mathcal{P}_{\alpha}^{(42)} & B_{\beta}^{(k;j;\beta)} + \mathcal{P}_{\alpha}^{(j2)} & B_{\beta}^{(k;k;\beta)} + \mathcal{P}_{\alpha}^{(j2)} & \delta^{(m3)} & \delta^{(m4)} & \delta^{(m5)} & \delta^{(m6)} \\
\end{bmatrix}, \quad m = 3, 4, j,
\]

\[
T_{\alpha\beta}^{(j;w;1;1;k)} = - \begin{bmatrix}
B_{\beta}^{(k;3;\beta)} - B_{\beta}^{(i;3;\beta)} & B_{\beta}^{(k;4;\beta)} - B_{\beta}^{(i;4;\beta)} & B_{\beta}^{(k;j;\beta)} - B_{\beta}^{(i;j;\beta)} & B_{\beta}^{(k;k;\beta)} - B_{\beta}^{(i;i;\beta)} \\
B_{\beta}^{(k;3;\beta)} + \mathcal{P}_{\alpha}^{(31)} & B_{\beta}^{(k;4;\beta)} + \mathcal{P}_{\alpha}^{(41)} & B_{\beta}^{(k;j;\beta)} + \mathcal{P}_{\alpha}^{(j1)} & B_{\beta}^{(k;k;\beta)} + \mathcal{P}_{\alpha}^{(j2)} \\
B_{\beta}^{(k;3;\beta)} - \mathcal{P}_{\alpha}^{(32)} & B_{\beta}^{(k;4;\beta)} - \mathcal{P}_{\alpha}^{(42)} & B_{\beta}^{(k;j;\beta)} - \mathcal{P}_{\alpha}^{(j2)} & B_{\beta}^{(k;k;\beta)} - \mathcal{P}_{\alpha}^{(j2)} \\
\end{bmatrix}, \quad T_{\alpha\beta}^{(j;w;1;1;k)} = 0,
\]

34
It is remarkable that the system of nonlinear PDEs (138, 140, 141, 144, 145, 146, 147) has the complete subsystem of nonlinear PDEs. Let us show this. First of all, we remark, that the fields \( v_{\alpha\beta}^{(nk)} \) appear only in the eq.(138). Moreover, the fields \( v_{\alpha\beta}^{(nk)} \), \( n \neq \alpha \) do not appear in the eqs.(140, 141, 146, 147). Thus namely these equations compose the complete system of nonlinear PDEs. Finally, we observe that, for our choice of the first superscript in the fields, the parameters \( A_1^{(j)} \), \( j > 2 \), appear in the combinations \( A_1^{(j) - P_{\alpha}^{(j1)}} \). For this reason we may put

\[
T_{\alpha\gamma\beta}^{(j;ww;k;1)} = \begin{vmatrix}
B_\beta^{(k3;\beta)} + P_\gamma^{(31)} & B_\beta^{(k4;\beta)} + P_\alpha^{(41)} & B_\beta^{(k5;\beta)} + P_\beta^{(j1)} \\
B_\beta^{(k3;\beta)} + P_\alpha^{(31)} & B_\beta^{(k4;\beta)} + P_\gamma^{(41)} & B_\beta^{(k5;\beta)} + P_\alpha^{(j1)} \\
B_\beta^{(k3;\beta)} - P_{\alpha}^{(32)} & B_\beta^{(k4;\beta)} - P_{\alpha}^{(42)} & B_\beta^{(k5;\beta)} - P_{\alpha}^{(j2)}
\end{vmatrix},\Rightarrow T_{\alpha\alpha\beta}^{(j;ww;k;1)} = 0,
\]

\[
T_{\alpha\gamma\beta}^{(j;ww;k;2)} = -\begin{vmatrix}
B_\beta^{(k3;\beta)} - P_{\alpha}^{(32)} & B_\beta^{(k4;\beta)} - P_{\alpha}^{(42)} & B_\beta^{(k5;\beta)} - P_{\alpha}^{(j2)} \\
B_\beta^{(k3;\beta)} + P_\alpha^{(31)} & B_\beta^{(k4;\beta)} + P_\alpha^{(41)} & B_\beta^{(k5;\beta)} + P_\alpha^{(j1)} \\
B_\beta^{(k3;\beta)} - P_{\alpha}^{(32)} & B_\beta^{(k4;\beta)} - P_{\alpha}^{(42)} & B_\beta^{(k5;\beta)} - P_{\alpha}^{(j2)}
\end{vmatrix},\Rightarrow T_{\alpha\alpha\beta}^{(j;ww;k;2)} = 0.
\]

\[
\begin{align*}
S_{\alpha}^{(j;w;1;n)} &= \begin{vmatrix}
A_n^{(2)} & A_n^{(3)} & A_n^{(j)} \\
A_n^{(2)} & A_n^{(3)} - P_{\alpha}^{(31)} & A_n^{(j)} - P_{\alpha}^{(j1)} \\
1 & P_{\alpha}^{(32)} & P_{\alpha}^{(j2)}
\end{vmatrix},
\end{align*}
\]

\[
\begin{align*}
S_{\alpha}^{(j;w;m;n)} &= -\begin{vmatrix}
\delta^{(m2)} & \delta^{(m3)} & \delta^{(mj)} \\
A_n^{(2)} & A_n^{(3)} - P_{\alpha}^{(31)} & A_n^{(j)} - P_{\alpha}^{(j1)} \\
1 & P_{\alpha}^{(32)} & P_{\alpha}^{(j2)}
\end{vmatrix},\quad m = 2, 3, j,
\end{align*}
\]

\[
\begin{align*}
\tilde{T}_{\alpha}^{(j;ww;i;1;n)} &= \begin{vmatrix}
1 & B_{n}^{(1;i;3)} & B_{n}^{(1;j)} \\
A_n^{(2)} & A_n^{(3)} - P_{\alpha}^{(31)} & A_n^{(j)} - P_{\alpha}^{(j1)} \\
1 & P_{\alpha}^{(32)} & P_{\alpha}^{(j2)}
\end{vmatrix},
\end{align*}
\]

\[
\begin{align*}
\tilde{T}_{\alpha\gamma}^{(j;ww;n;1)} &= -\begin{vmatrix}
A_n^{(2)} & A_n^{(3)} - P_{\gamma}^{(31)} & A_n^{(j)} - P_{\gamma}^{(j1)} \\
A_n^{(2)} & A_n^{(3)} - P_{\alpha}^{(31)} & A_n^{(j)} - P_{\alpha}^{(j1)} \\
1 & P_{\alpha}^{(32)} & P_{\alpha}^{(j2)}
\end{vmatrix},\Rightarrow \tilde{T}_{\alpha\alpha}^{(j;ww;n;1)} = 0,
\end{align*}
\]

\[
\begin{align*}
\tilde{T}_{\alpha\gamma}^{(j;ww;n;2)} &= \begin{vmatrix}
1 & P_{\gamma}^{(32)} & P_{\gamma}^{(j2)} \\
A_n^{(2)} & A_n^{(3)} - P_{\alpha}^{(31)} & A_n^{(j)} - P_{\alpha}^{(j1)} \\
1 & P_{\alpha}^{(32)} & P_{\alpha}^{(j2)}
\end{vmatrix},\Rightarrow \tilde{T}_{\alpha\alpha}^{(j;ww;n;2)} = 0.
\end{align*}
\]
\( A_i^{(j)} = 0, j > 2 \) without loss of generality. To simplify formulae, we assume that the matrix \( A^{(2)} \) satisfies the following two requirements:

\begin{align*}
1. A_1^{(2)} &= -1, \\
2. A_i^{(2)} &\neq A_j^{(2)}, \ i \neq j, \ i, j = 1, \ldots, Q.
\end{align*}

All in all, applying \( \sum_{\beta,k=1}^Q B^{(2;1)}_{\beta} (R^{-1})^{(k1)}_{\beta} \) to the eq. (141) with \( \alpha = 1 \) and to the eq. (147) with \( n = 1 \), putting \( \alpha = 1 \) into the eq. (140) and \( \beta = 1 \) into the eq. (146) we end up with the system of nonlinear PDEs (45-51) for the fields (44) (we write the symbolic set of equations in the same order as they are written in the system (45-51)):

\begin{align*}
E_{a1}^{(56;1\beta p)}, \quad E_{a1}^{(57;1\beta p)} &= \sum_{\gamma,k=1}^Q E_{a1}^{(57;1\beta p)} B^{(2;1)}_{\gamma} (R^{-1})^{(k1)}_{\gamma}, \\
E_{11}^{(52;1\beta p)}, \quad E_{11}^{(53;1\beta p)} &= \sum_{\gamma,k=1}^Q E_{11}^{(53;1\beta p)} B^{(2;1)}_{\gamma} (R^{-1})^{(k1)}_{\gamma}, \\
\tilde{E}_{a1}^{(47;1\beta p)} &= \sum_{\gamma,k=1}^Q E_{a1}^{(47;1\beta p)} B^{(2;1)}_{\gamma} (R^{-1})^{(k1)}_{\gamma}, \\
E_{11}^{(42;1\alpha p)}, \quad E_{11}^{(43;1\alpha p)} &= \sum_{\beta,k=1}^Q E_{11}^{(43;1\alpha p)} B^{(2;1)}_{\gamma} (R^{-1})^{(k1)}_{\gamma}, \ j = 3, 4,
\end{align*}

where we have redefined constant coefficients in order to improve the structure of PDEs:

\begin{align*}
s_{\alpha\beta}^{(j-1;w;m-1)} &= s_{\alpha1}^{(j-1;w;m-1)}, \ m = 2, 3, 4, j, \ T_{a\gamma\beta}^{(j-1;uw;\gamma)} = T_{a1}\gamma\beta, \\
T_{a\gamma\beta}^{(j-1;uw;\gamma)} &= T_{a\gamma1}^{(j-1;uw;\gamma)}, \ j \geq 5, \\
s_{\alpha}^{(j-1;w;m-1)} &= s_{\alpha1}^{(j-1;w;m-1)}, \ m = 2, 3, j, \ T_{a\gamma1}^{(j-1;wu;\gamma)} = T_{a\gamma1}^{(j-1;wu;\gamma)}, \\
T_{a\gamma1}^{(j-1;wu;\gamma)} &= T_{a\gamma1}^{(j-1;wu;\gamma)}, \ j \geq 4, \\
s_{\alpha}^{(j-1;w;m-1)} &= s_{1}^{(j-1;w;m;\alpha)} \beta, \ m = 2, 3, j, \ T_{a\gamma\beta}^{(j-1;wu;\gamma)} = T_{a\gamma\beta}^{(j-1;wu;\gamma)}, \\
T_{a\gamma\beta}^{(j-1;wu;\gamma)} &= T_{a\gamma1}^{(j-1;wu;\gamma)}, \ j \geq 4, \\
s_{\alpha}^{(j-1;w;m-1)} &= s_{1}^{(j-1;w;m;\alpha)} \beta, \ m = 2, j, \ T_{a\gamma1}^{(j-1;wu;\gamma)} = T_{a\gamma1}^{(j-1;wu;\gamma)}, \\
T_{a\gamma1}^{(j-1;wu;\gamma)} &= T_{a\gamma1}^{(j-1;wu;\gamma)}, \ j \geq 3 .
\end{align*}

This ends the proof of the item (c) of Proposition. The eq. (53) of the item (d.5) follows from the eq. (46) if one puts \( v_{nk} = u_{nk} = 0 \). Existence of the infinitely many commuting flows, as indicated in the item (d.1) of Proposition, is associated with the arbitrary parameter \( j \) in the system (140, 141, 146, 147). This remark ends the proof of Proposition.

### 5.1 Remark on the eq.(99)

We will show, that the linear equation (99) is automatically satisfied due to the eq.(119). In fact, applying \( \sum_{a=1}^{Q} G_{\tilde{a}a} \) to the eq.(99) and replacing \( \tilde{a} \) by \( a \) in the result, one gets the next
expression for $\hat{F}$:

$$\hat{F}_{\gamma\beta}^{(j;i_1k)} = \sum_{n=1}^{Q} \sum_{\alpha=1}^{Q} (\hat{R}^{-1})_{\gamma\alpha}^{(i_1n)} R_{\alpha\beta}^{(nk)} B_{\beta}^{(j;n)}.$$  \hfill (152)

The eq.(99) gets the next form:

$$U_{\alpha\beta}^{(nk)} = \sum_{i,\gamma,\gamma_1=1}^{Q} U_{\alpha\gamma}^{(ni)} B_{\gamma}^{(2;n)} (\hat{R}^{-1})_{\gamma\gamma_1}^{(m)} R_{\gamma\beta}^{(nk)}, \ n \neq \beta$$  \hfill (153)

in view of the eq.(119), where we have canceled $B_{\beta}^{(j;n)}$ assuming that $n \neq \beta$. Eq. (153) is the identity. To show this, we apply $\sum_{\beta,k=1}^{Q} B_{\beta}^{(2;n)} (\hat{R}^{-1})_{\beta\delta}^{(kl)}$ to the eq.(153), use the eq. (116) to simplify the RHS and use the eq.(119) to simplify the LHS of the resulting equation.

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