On the Lattice of Conceptual Measurements

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Abstract

We present a novel approach for data set scaling based on scale-measures from formal concept analysis, i.e., continuous maps between closure systems, and derive a canonical representation. Moreover, we prove said scale-measures are lattice ordered with respect to the closure systems. This enables exploring the set of scale-measures through the use of meet and join operations. Furthermore, we show that the lattice of scale-measures is isomorphic to the lattice of sub-closure systems that arises from the original data. Finally, we provide another representation of scale-measures using propositional logic in terms of data set features. Our theoretical findings are discussed by means of examples.

Keywords: FCA, Measurements, Data Scaling, Lattice, Closure System

1. Introduction

The discovery and analysis of patterns and dependencies in the realm of data science does strongly depend on the measurement of the data. Each data set is subject to one or more scales of measure \cite{1}, i.e., maps from the data into variable of some (mathematical) space, e.g., the real line, an ordered set, etc. Beyond that, almost every data set is further scaled prior to (data)processing to meet the requirements of the employed data analysis method, such as the introduction of artificial metrics, the numerical representation of nominal features, etc. This scaling is usually accompanied by a grade of detail, which in turn is becoming more and more of a problem for data science tasks as the availability of features increases and their human explainability decreases. Often used methods to deal with this problem from the field of machine learning, such as principal component analysis, do enforce particular, possible inapt, levels of measurement, e.g., food tastes represented by real numbers, and amplify the problem for explainability.

Therefore, understanding the set of possible scaling maps, identifying its (algebraic) properties, and deriving to some extent human explainable control over it, is a pressing problem. This is especially important since found patterns and dependencies may be artifacts of some scaling map and may therefore corrupt any subsequent task, e.g., classification tasks. In the case Boolean data sets the field of formal concept analysis provides a well-formalized, yet insufficiently studied, approach for mathematically grasping the process of data scaling, called

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Figure 1: This Figure shows a Ben and Jerry’s context and its concept lattice.

scale-measure maps. These maps are continuous with respect to the closures systems that emerge from the original Boolean data set and scale, which resembles also a Boolean data set, i.e., the preimage of a closed set is closed. Equipped with this notion for data scaling we discover and characterize consistent scale-refinements and derive a theory that is able to provide new insights to data sets by comparing different scale-measures. Building up on this we prove that the set of all scale-measures bears a lattice structure and we show how to transform scale-measures using lattice operations. Moreover, we introduce an equivalent representation of scale-measures using propositional logic expressions and how they emerge naturally while scaling data.

Altogether, we present methods that are able to generate different conceptual measurements of a data set by computing meaningful features such that they are consistent with the conceptual knowledge of the original data set.

2. Scales and Measurement

2.1. Measurements and Categorical Data

Formalizing and understanding the process of measurement is, in particular in data science, an ongoing discussion. Representational Theory of Measurement (RTM) [2, 3] reflects the most recent and widely acknowledged current standpoint on this. RTM relies on homomorphisms from an (empirical) relational structure $E = (E, (R_i)_{i \in I})$ to a numerical relational structure $B = (B, (S_i)_{i \in I})$, very well explained by J. Pfanzagl [4], where $B$ is often chosen to be the real line $\mathbb{R}$ or a $n$ dimensional vector space on it. However, it might be beneficial to allow for other, more algebraic (measurement) structures [5, p. 253]. This is particularly true in cases where the empirical data does not allow for a meaningful measurement into the ratio level (cf [1]), e.g., taxonomic ranks in biology or
types of faults in software engineering. Both examples are instances of categorical data, which is classified to the nominal level with respect to S. S. Stevens [1]. If such data is also naturally equipped with a rank order relation, e.g., the Likert scale or school grades, it is situated on the ordinal level.

A mathematical framework well equipped for the nominal as well as the ordinal level is formal concept analysis (FCA) [6, 7]. In FCA we represent data in the form of formal contexts as seen in Figure 1 (top). A formal context is a triple \((G, M, I)\) with \(G\) being a finite set of object, \(M\) being a finite set of attributes and \(I \subseteq G \times M\) an incidence relation between them. With \((g, m) \in I\) means that object \(g\) has attribute \(m\). We visualize formal context using cross tables, as depicted for the running example Ben and Jerry’s in Figure 1 (top). A cross in the table indicates that an object (ice cream flavor) has an attribute (ice cream ingredient). A context \(\mathcal{S} = (H, N, J)\) is called an induced sub-context of \(\mathbb{K}\), if \(H \subseteq G, N \subseteq M\) and \(J_S = I \cap (H_S \times N)\), denoted \(\mathcal{S} \leq \mathbb{K}\). The incidence relation gives rise to two derivation operators. The first is the derivation of an attribute \(A \subseteq M\) where \(A' = \{g \in G \mid \forall m \in A : (g, m) \in I\}\). The object derivation \(B'\) for \(B \subseteq G\) is defined analogously. The consecutive application of the two derivation operators on an attribute set (object set) constitutes a closure operators, i.e., a idempotent, monotone, and extensive, map. Therefore, the pairs \((G,')\) and \((M,'')\) are closure spaces with \(': \mathcal{P}(G) \rightarrow \mathcal{P}(G)\) and \('' : \mathcal{P}(M) \rightarrow \mathcal{P}(M)\). For example, \{Dough, Vanilla\} = \{Choco, Dough, Vanilla\} in Figure 1.

A formal concept is a pair \((A, B) \in \mathcal{P}(G) \times \mathcal{P}(M)\) with \(A' = B\) and \(A = B'\), where \(A\) is called extent and \(B\) intent. We denote with \(\operatorname{Ext}(\mathbb{K})\) and \(\operatorname{Int}(\mathbb{K})\) the sets of all extents and intents, respectively. Each of these sets forms a closure system associated to the closure operator on the respective base set, i.e., the object set or the attribute set. Both closure systems are represented in the concept lattice \(\mathcal{B}(\mathbb{K}) = (\mathcal{B}(\mathbb{K}), \subseteq)\), where \(\mathcal{B}(\mathbb{K})\) denotes the set of all concepts in \(\mathbb{K}\) and for \((A, B), (C, D) \in \mathcal{B}(\mathbb{K})\) we have \((A, B) \leq (C, D) :\iff A \subseteq C\).

2.2. Scales

A fundamental problem for the analysis, the computational treatment, and the visualization of data is the high dimensionality and complex structure of modern data sets. Hence, the tasks for scaling data sets to a lower number of dimensions and decreasing their complexity has growing importance. Many unsupervised (machine learning) procedures were developed and applied, for example, multidimensional scaling [8, 9] or principal component analysis. These scaling methods use non-linear projections of data objects (points) into a lower dimensional space. While preserving the notion of object they lose the interpretability of features as well as the original algebraic object-feature relation. Therefore, the advantage of explainability when analyzing nominal or ordinal data cannot be preserved. Furthermore, most scaling approaches require the representation of the data points in a real coordinate space of some dimension, which is in turn, already a scaling for many data sets.

A more fundamental approach to scaling, in particular for nominal and ordinal data, that preserves the interpretable features can be found in FCA.

**Definition 1 (Scale-Measure (cf. Definition 91, [7]))** Let \(\mathbb{K} = (G, M, I)\) and \(\mathcal{S} = (G_S, M_S, I_S)\) be a formal contexts. The map \(\sigma : G \rightarrow G_S\) is called an \(\mathcal{S}\)-measure of \(\mathbb{K}\) into the scale \(\mathcal{S}\) iff the preimage \(\sigma^{-1}(A) := \{g \in G \mid \sigma(g) \in A\}\) of every extent \(A \in \operatorname{Ext}(\mathcal{S})\) is an extent of \(\mathbb{K}\).
This definition corresponds the notion for continuity between closure spaces $(G_1, c_1)$ and $(G_2, c_2)$, i.e., a map $f : G_1 \rightarrow G_2$ is continuous iff

$$\text{for all } A \in \mathcal{P}(G_2) \text{ we have } c_1(f^{-1}(A)) \subseteq f^{-1}(c_2(A)).$$

(1)

This property is equivalent to the requirement in Definition 1 that the preimage of closed sets is closed, more formally,

$$\text{for all } A \in \mathcal{P}(G_2) \text{ with } c_2(A) = A \text{ we have } f^{-1}(A) = c_1(f^{-1}(A)).$$

(2)

Conditions in (1) and (2) are known to be equivalent, since (1) $\Rightarrow$ (2) follows from $x \in c_1(f^{-1}(A)) \Rightarrow x \in f^{-1}(c_2(A)) \Rightarrow x \in f^{-1}(c_2(A)) \Rightarrow x \in c_1(f^{-1}(A))$.

In the following we may address by $\sigma^{-1}(\text{Ext}(S))$ the set of all extents $\sigma^{-1}(\text{Ext}(S))$ of the objects in some scale context. In this sense we understand the map $\sigma$ as an interpretation of the objects from $K$ in $S$.

The following corollary can be deduced from the continuity property above and will be used frequently throughout our work.

**Corollary 2 (Composition Scale-Measures)** Let $K$ be a formal context, $\sigma$ a $S$-measure of $K$ and $\psi$ a $T$-measure of $S$. Then is $\psi \circ \sigma$ a $T$-measure of $K$.

In Figure 2 we depict a scale-measure and its concept lattice for our running example context $Ben and Jerry's K_{BJ}$, cf. Figure 1. This scale-measure uses the same object set as the original context and maps every object to itself. The
attribute set is comprised of six elements, which may reflect the taste, instead of the original nine attributes that indicated the used ingredients. The specified scale-measure map allows for a human comprehensible interpretation of $\sigma^{-1}$, as indicated by the grey colored concepts in Figure 2 (bottom). In this figure we observe that the concept lattice of the scale-measure reflects ten out of the sixteen concepts in $\mathcal{B}(\mathcal{K}_{B3})$.

The empirical observations about the afore presented example scale-measure for some context $\mathcal{K}$ lead to the question whether scale-measures are always at least as comprehensible as the context $\mathcal{K}$ itself. A typical (objective) measure for the complexity of lattices is given by the following quantity.

**Definition 3 (Order Dimension (cf. Definition 82, [7]))** An ordered set $(P, \leq)$ has order dimension $\dim(P, \leq) = n$ iff it can be embedded in a direct product of $n$ chains and $n$ is the smallest number for which this is possible.

The order dimension of $\mathcal{B}(\mathcal{K}_{B3})$ is three whereas the concept lattice of the given scale-measure is two. Finding low dimensional scale-measures for large and complex data sets is a natural approach towards comprehensible data analysis, as demonstrated in Proposition 24. In particular, we will answer the question if the order dimension of scale-measures is bound by the order dimension of $\mathcal{B}(\mathcal{K})$.

Another notion for comparing scale-measures is provided by a natural order relation amongst scales [7, Definition 92]). We may present in the following a more general definition within the scope of scale-measures.

**Definition 4 (Scale-Measure Refinement)** Let the set of all scale-measures of a context be denoted by $\mathcal{S}(\mathcal{K}) := \{ (\sigma, S) \mid \sigma \text{ is a } S\text{-measure of } \mathcal{K} \}$. For $(\sigma, S), (\psi, T) \in \mathcal{S}(\mathcal{K})$ we say $(\sigma, S)$ is a coarser scale-measure of $\mathcal{K}$ than $(\psi, T)$, iff $\sigma^{-1}(\text{Ext}(S)) \subseteq \psi^{-1}(\text{Ext}(T))$. Analogously we then say $(\psi, T)$ is finer than $(\sigma, S)$.

We remark that the finer relation as well as coarser relation constitute (partial) order relations on the set of all scale-measure for context $\mathcal{K}$, since they are obviously reflexive, anti-symmetric, and the transitivity follow from the continuity of the composition of scale maps. Hence, we may refer to the refinement (order) using the symbol $\leq$. By computing scale-measures with coarser scale contexts with respect to the refinement order we can provide a more general conceptual view on a data set. The study of such views, e.g. the ice cream tastes in our running example presented in Figure 2, is in a similar fashion to the Online Analytical Processing tools for multidimensional databases.

Moreover, the set of all scale-measure for some formal context enables an abstract analytical structure to navigate and explore a data set with. Yet, despite the supposed usefulness of the scale-measures, there are up until now no existing methods, to the best of our knowledge, for the generation and evaluation of scale-measures, in particular with respect to data science applications.

Both tasks, the generation and the evaluation of scale-measures, will be tackled in the next section using a novel navigation approach among them.

### 3. Navigation though Conceptual Measurement

Based on the just introduced refinement order of scale-measures we provide in this section the means for efficiently browsing this structure. Given a data set,
the presented methods are able to compute arbitrary scale abstractions and the structure operations that connect them, which resembles a navigation through conceptual measurements. To lay the foundation for the navigation methods we start with analyzing the structure of all scale-measures. Thereafter we will present a thorough description of the navigation problem and its solution.

**Lemma 5** The scale-measure equivalence is an equivalence relation on the set of scale-measures.

**Proof.** Let \((\sigma,S),(\psi,T)\) with \((\sigma,S)\sim (\psi,T)\) and \(\sigma,\psi\) are surjective maps. Then \(\sigma^{-1}\circ\psi\) is an order isomorphism from \((\text{Ext}(S),\subseteq)\) to \((\text{Ext}(T),\subseteq)\).

**Lemma 6** Let \((\sigma,S),(\psi,T)\in \mathcal{G}(K)\) with \((\sigma,S)\sim (\psi,T)\) and \(\sigma,\psi\) are surjective maps. Then \(\sigma^{-1}\circ\psi\) is an order isomorphism from \((\text{Ext}(S),\subseteq)\) to \((\text{Ext}(T),\subseteq)\).

**Proof.** From [7, Proposition 118] we have that \(\sigma^{-1}\) is an injective \(\land\)-preserving order embedding of \((\text{Ext}(S),\subseteq)\) into \((\text{Ext}(K),\subseteq)\) and thereby a bijective \(\land\)-preserving order embedding into \((\sigma^{-1}(\text{Ext}(S)),\subseteq)\). The analogue holds for \(\psi^{-1}\) from \(\text{Ext}(T)\) into \(\text{Ext}(T)\). Due to \((\sigma,S)\sim (\psi,T)\) we know that \(\sigma^{-1}(\text{Ext}(S)) = \psi^{-1}(\text{Ext}(T))\), which results in \(\sigma^{-1}\) being a bijective \(\land\)-preserving order embedding into \(\text{Ext}(T)\). Hence, when restricting \(\sigma^{-1}\circ\psi : \mathcal{P}(G_\gamma) \rightarrow \mathcal{P}(G_\pi)\) to the respective extent set we obtain a bijective map. The fact that all formal contexts are finite (throughout this work) and the monotonicity of the lifts of \(\sigma^{-1}\) and \(\psi\) to their respective power sets imply the required order preserving property follow.

We may stress that the required surjectivity is not constraining the application of scale-measures, since any object \(g\) of a scale-context having an empty preimage may just be removed from the scale-context without consequences to the analysis.

The just discussed equivalence relation together with the refinement order allows to cope with the set of all scale-measures \(\mathcal{G}(K)\) in a meaningful way.

**Definition 7 (Scale-Hierarchy)** Given a formal context \(K\) and its set of all scale-measures \(\mathcal{G}(K)\), we call \(\mathcal{G}(K) := (\mathcal{G}(K)/\sim,\subseteq)\) the scale-hierarchy of \(K\).

The order structure thus given represents all possible means of scaling a (contextual) data set. Yet, it seems hardly comprehensible or even applicable in that form. Therefore the goal for the rest of this section is to achieve a characterization of said structure in terms of closure systems.

**Lemma 8** Let \(G\) be a set and \(A\subseteq \mathcal{P}(G)\) be a closure system. Furthermore, let \(K_A = (G,A,\in)\) be a formal context using the element relation as incidence. Then the set of extents \(\text{Ext}(K_A)\) is equal to the closure system \(A\).

**Proof.** For any set \(D \subseteq G\) and \(A \in A\) we find \((*)\) \(D \subseteq A \implies A \in D'\). Since \(A\) is a closure system and \(D'' = \bigcap D'\) we see that \(D'' \in A\), hence, \(\text{Ext}(K_A) \subseteq A\). Conversely, for \(A \in A\) we can draw from \((*)\) that \(A'' = A\), thus \(A \in \text{Ext}(K_A)\).
We want to further motivate the constructed formal context $K_A$ and its particular utility with respect to scale-measures for some context $K$. Since both contexts have the same set of objects, we may study the use of the identity map $\text{id} : G \to G, g \mapsto g$ as scale-measure map.

**Lemma 9 (Canonical Construction)** For a context $K$ and any $S$-measure $\sigma$ is id a $K_{\sigma^{-1}(\text{Ext}(S))}$-measure of $K$, i.e., $(\text{id}, K_{\sigma^{-1}(\text{Ext}(S))}) \in \mathcal{S}(K)$.

**Proof.** Lemma 8 gives that $\text{Ext}(K_{\sigma^{-1}(\text{Ext}(S))})$ is equal to $\sigma^{-1}(\text{Ext}(S))$. Since $(\sigma, S) \in \mathcal{S}(K)$, i.e., $(\sigma, S)$ is a scale-measure of $K$, we see that the preimage $\sigma^{-1}(\text{Ext}(S)) \subseteq \text{Ext}(K)$, and thus $\text{id}^{-1}(\text{Ext}(K_{\sigma^{-1}(\text{Ext}(S))})) \subseteq \text{Ext}(K)$. □

Using the canonical construction of a scale-measure, as given above, we can facilitate the understanding of the scale-hierarchy $\mathcal{S}(K)$.

**Proposition 10 (Canonical Representation)** Let $K = (G, M, I)$ be a formal context with scale-measure $(S, \sigma) \in \mathcal{S}(K)$, then $(\sigma, S) \sim (\text{id}, K_{\sigma^{-1}(\text{Ext}(S))})$.

**Proof.** Lemma 9 states that id is a $K_{\sigma^{-1}(\text{Ext}(S))}$-measure of $K$. Furthermore, from Lemma 8 we know that the extent set of $K_{\sigma^{-1}(\text{Ext}(S))}$ is $\sigma^{-1}(\text{Ext}(S))$, as required by Definition 4. □

Equipped with this proposition we are now able to compare sets of scale-measures for a given formal context $K$ solely based on their respective attribute sets in the canonical representation. Furthermore, since these representation sets are sub-closure systems of $\text{Ext}(K)$, by Definition 1, we may reformulate the problem for navigating scale-measures using sub-closure systems and their relations. For this we want to nourish the understanding of the correspondence of scale-measures and sub-closure systems in the following.

**Proposition 11** For a formal context $K$ and the set of all sub-closure systems $\mathcal{C}(K) \subseteq \mathcal{P}(\text{Ext}(K))$ together with the inclusion order, the following map is an order isomorphism:

$$i : \mathcal{C}(K) \to \mathcal{S}(K)_{/\sim}, \quad A \mapsto i(A) := (\text{id}, K_A)$$

**Proof.** Let $A, B \subseteq \text{Ext}(K)$ be two system sets on $G$. Then the images of $A$ respectively $B$ under $i$ are a scale-measures of $K$, according to Lemma 9, with extents $A$ and $B$, respectively. Since $A \neq B \iff \text{Ext}(K_A) \neq \text{Ext}(K_B)$ are different and therefore $(\text{id}, K_A) \neq (\text{id}, K_B)$, thus, $i$ is an injective map. For the surjectivity of $i$ let $[(\sigma, S)] \in \mathcal{S}(K)_{/\sim}$, then $(\text{id}, K_{\sigma^{-1}(\text{Ext}(S))}) \sim (\sigma, S)$, i.e., an equivalent representation having extents $\sigma^{-1}(\text{Ext}(S)) \subseteq \text{Ext}(K)$ and $i(\sigma^{-1}(\text{Ext}(S))) = (\text{id}, K_{\sigma^{-1}(\text{Ext}(S))})$. Finally, for $A \subseteq B$ we find that $i(A) \subseteq i(B)$, since $\text{Ext}(K_A) \subseteq \text{Ext}(K_B)$, as required. □

This order isomorphism allows us to analyze the structure of the scale-hierarchy by studying the related closure systems. For instance, the problem of computing $\mathcal{S}(K)$, i.e., the size of the scale-hierarchy. In the case of the boolean context $K_{\mathcal{P}(G)}$, this problem equivalent to the question for the number of Moore families, i.e., the number of closure systems on $G$. This number grows tremendously in $|G|$ and is known up to $|G| = 7$, for which it is known [10, 11, 12] to be $14 087 648 235 707 352 472$. In the general case the size of the scale-hierarchy is equal to the size of the order ideal $\downarrow \text{Ext}(K)$ in $\mathcal{C}(K_{\mathcal{P}(G)})$.

The fact that the set of all closure systems on $G$ is again a closure system [13], which is lattice ordered by set inclusion, allows for the following statement.
Corollary 12 (Scale-hierarchy Order) For a formal context $K$, the scale-hierarchy $S(K)$ is lattice ordered.

We depicted this lattice order relation in the form of abstract visualizations in Figure 3. In the bottom (right) we see the most simple scale which has only one attribute, $G$. The top (right) element in this figure is then the scale which has all extents of $K$. On the left we see the lattice ordered set of all closure systems on a set $G$, in which we find the embedding of the hierarchy of scales.

Proposition 13 Let $(\sigma, S), (\psi, T) \in S(K)$ and let $\wedge, \vee$ be the natural lattice operations in $S(K)$, (induced by the lattice order relation). We then find that:

\begin{align*}
\text{Meet} : (\sigma, S) \wedge (\psi, T) &= (\text{id}, K_{\sigma^{-1}(\text{Ext}(S)) \cap \psi^{-1}(\text{Ext}(T))}), \\
\text{Join} : (\sigma, S) \vee (\psi, T) &= (\text{id}, K_{\{A \cap B \mid A \in \sigma^{-1}(\text{Ext}(S)), B \in \psi^{-1}(\text{Ext}(T))\}}).
\end{align*}

Proof. 1. For the preimages $i^{-1}(\sigma, S), i^{-1}(\psi, T)$ (Proposition 11) we can compute their meet [13], which yields

$$i^{-1}(\sigma, S) \wedge i^{-1}(\psi, T) = \sigma^{-1}(\text{Ext}(S)) \cap \psi^{-1}(\text{Ext}(T)).$$

2. The join [13] of the scale-measure preimages under $i$ (Proposition 11) is equal to $\{A \cap B \mid A \in \sigma^{-1}(\text{Ext}(S)), B \in \psi^{-1}(\text{Ext}(T))\}$, which results in the required expression by applying the order isomorphism $i$. \qed

3.1. Propositional Navigation through Scale-Measures

Although the canonical representation of scale-measures is complete up to equivalence Proposition 10, this representation eludes human explanation to some degree. In particular the use of the extentional structure of $K$ as attributes provides insight to the scale-hierarchy itself, however, not to the data, i.e., the objects, attributes, and their relation. A formulation of scales using attributes from $K$, and their combinations, seems more natural and more comprehensible. For this, we employ an approach as used in [14]. In their work the authors used a logic on the context’s attributes to introduce new attributes. The advantage is that the so newly introduced attributes have a real-world semantic in terms of the measured properties. In this work we use propositional logic, which leads to the following problem description.
Problem 14 (Navigation Problem) For a formal context $\mathbb{K}$, a scale-measure $(\sigma, S) \in \mathcal{S}(\mathbb{K})$ and $M \subseteq \mathcal{L}(M, \{\land, \lor, \neg\})$, compute an equivalent scale-measure $(\psi, T) \in \mathcal{S}(\mathbb{K})$, i.e., $(\sigma, S) \sim (\psi, T)$, where $(h, m) \in I_T \iff \psi^{-1}(h)^I \models m$.

The attributes of $T$ are logical expression build from the attributes of $\mathbb{K}$, and are thus interpretable in terms of the measurements by the attributes $M$ from $\mathbb{K}$. For example, we can express the Choco taste attribute of our running example (Figure 2) as the disjunction of the ingredients Choco Ice or Choco Pieces, i.e., Choco=$\sigma$ Choco Ice$\lor$ Choco Pieces. For any scale-measure $(\sigma, S)$, such an equivalent scale-measure, as searched for in Problem 14, is not necessarily unique, and the problem statement does not favor any of the possible solutions.

To understand the semantics of the logical operations, we first investigate their contextual derivations. For $\phi \in \mathcal{L}(M, \{\land, \lor, \neg\})$ we let $\varphi(\phi)$ be the set of all propositional variables in the expression $\phi$. We require from $\phi \in \mathcal{L}(M, \{\land, \lor, \neg\})$ that $|\varphi(\phi)| > 0$.

**Lemma 15 (Logical Derivations)** Let $\mathbb{K} = (G, M, I)$ be a formal context, $\phi, \psi \in \mathcal{L}(M, \{\land\})$, $\phi \lor \in \mathcal{L}(M, \{\lor\})$, $\psi_\land \in \mathcal{L}(M, \{\land\})$, with scale contexts $(G, \{\phi\}, I_\phi)$ having the incidence $(g, \phi) \in I_\phi \iff g^1 \models \phi$ for $\phi \in \{\phi_\land, \phi_\lor, \phi_\land\}$. Then we find

i) \( |\varphi(\phi_\land)\} = \varphi(\phi_\land)^I, \)

ii) \( |\varphi(\phi_\lor)\} = \cup_{m \in \varphi(\phi_\lor)} \{m\}^I, \)

iii) \( |\varphi_\land(\phi)\} = G \setminus \{n\}^I \) with $\phi_\land = \neg n$ for $n \in M$.

**Proof.** i) For $g \in G$ if $g I_\phi, \phi_\land$ then $\{g\}^I \models \phi_\land$ and therefore $\varphi(\phi_\land) \subseteq \{g\}^I$. Hence $g \in \varphi(\phi_\land)^I$. In case $(g, \phi_\land) \notin I_\phi$, it holds that $\varphi(\phi_\land) \nsubseteq \{g\}^I$ and thereby $g \notin \varphi(\phi_\land)^I$. ii) For $g \in G$ if $g I_\phi, \phi_\lor$ we have $\{g\}^I \models \phi_\lor$. Hence, $\exists m \in \varphi(\phi_\lor)$ with $g \in m^I$ and therefore $g$ is in the union. If $(g, \phi_\lor) \notin I_\phi$, there does not exist such a $m \in \varphi(\phi_\lor)$ and $g \notin \cup_{m \in \varphi(\phi_\lor)} m^I$. iii) For any $n \in M$ we have $\phi_\land = \neg n$. Hence, for $g \in G$ if $g I_\phi, \phi_\land$ we find $g \notin \{n\}^I$. Conversely, if $(g, \phi_\land) \notin I_\phi$ it follows that $g \in \{n\}^I$. □

Naturally, the results from the lemma above generalize to scale contexts with more than one logical expression in the set of attributes. How this is done is demonstrated in Section 3.2. Moreover, more complex formulas, i.e., $\phi \in \mathcal{L}(M, \{\land, \lor, \neg\})$, can be recursively deconstructed and then treated with Lemma 15. In particular, with respect to unsupervised machine learning, we may mention the connection to the task of clustering attributes, as studied by Kwuida et al. [15].

**Proposition 16 (Logical Scale-Measure)** Let $\mathbb{K}$ be a formal context and let $\phi \in \mathcal{L}(M, \{\land, \lor, \neg\})$, then $id_G$ is a $(G, \{\phi\}, I_\phi)$-measure of $\mathbb{K}$ iff $\{\phi\}^I \in \text{Ext}(\mathbb{K})$.

**Proof.** Since $|\{\phi\}| = 1$ we find that $(G, \{\phi\}, I_\phi)$ has at least one and most two possible extents, $\{\phi\}^I \in \text{Ext}(\mathbb{K})$. If the map $id_G$ is a scale-measure of $\mathbb{K}$, then $id^{-1}_G(\{\phi\}^I) = \{\phi\}^I \in \text{Ext}(\mathbb{K})$. Conversely, if $\{\phi\}^I \in \text{Ext}(\mathbb{K})$ so is $id^{-1}_G(\{\phi\}^I)$, hence, $id_G$ is a $(G, \{\phi\}, I_\phi)$-measure of $\mathbb{K}$. □

This result raises the question for which formulas $\phi$ is $id_G$ a $(G, \{\phi\}, I_\phi)$-measure of $\mathbb{K}$. Counter examples for which $id_G$ is not a $(G, \{\phi_\lor\}, I_\phi)$- or $(G, \{\phi_\land\}, I_\phi)$-measure of a $\mathbb{K}$ are depicted in Figure 4.
Corollary 17 (Conjunctive Logical Scale-Measures) Let $\mathbb{K} = (G, M, I)$ be a formal context and $\phi \in \mathcal{L}(M, \{\land\})$, then $(\text{id}_G, (G, \{\phi\}, I_{\phi^\land})) \in \mathcal{S}(\mathbb{K})$.

Proof. According to Lemma 15 $(\phi \land^I = \text{Var}(\phi)^I$, hence, by Proposition 16 $(\text{id}_G, (G, \{\phi\}, I_{\phi})) \in \mathcal{S}(\mathbb{K})$.

3.2. Context Apposition for Scale Construction

To build more complex scale-measures we employ the apposition operator of contexts and transfer it to the realm of scale-measures. We remind the reader that the apposition of two contexts $K_1$, $K_2$ with $G_1 = G_2$ and $M_1 \cap M_2 = \emptyset$ is defined as $K_1 | K_2 := (G_1 \cup M_2, I_1 \cup I_2)$. The set of extents of $K_1 | K_2$ is known to be the set of all pairwise extents of $K_1$ and $K_2$. In the case of $M_1 \cap M_2 \neq \emptyset$ the apposition is defined alike by coloring the attribute sets.

Definition 18 (Apposition of Scale-Measures) Let $(\sigma, S)$, $(\psi, T)$ be scale-measures of $\mathbb{K}$. Then the apposition of scale-measures $(\sigma, S) | (\psi, T)$ is:

$$(\sigma, S) | (\psi, T) := \begin{cases} (\sigma, S | T) & \text{if } G_S = G_T, \sigma = \psi \\ (\sigma, S) \lor (\psi, T) & \text{else} \end{cases}$$

Note that also in the case of $G_S = G_T, \sigma = \psi$ is the scale-measure apposition is as well a join up to equivalence in the scale-hierarchy, cf. Proposition 13.

Proposition 19 (Apposition Scale-Measure) Let $G_S, (\sigma, S), (\psi, T)$ be two scale-measures of $\mathbb{K}$. Then $(\sigma, S) | (\psi, T) \in \mathcal{S}(\mathbb{K})$.

Proof. 1. In the first case we know that set of extents $\text{Ext}(S | T)$ contains all intersections $A \cap B$ for $A \in \text{Ext}(S)$ and $B \in \text{Ext}(T)$ [7]. Furthermore, we know that we can represent $\sigma^{-1}(A \cap B) = \sigma^{-1}(A) \cap \sigma^{-1}(B) = (A \cap B) \cap \psi^{-1}(B)$. Since $\sigma^{-1}(\text{Ext}(S)), \psi^{-1}(\text{Ext}(T)) \subseteq \text{Ext}(\mathbb{K})$, we can infer that the intersection $\sigma^{-1}(A) \cap \psi^{-1}(B) \in \text{Ext}(\mathbb{K})$. 2. The second case follows from Proposition 13. □

The apposition operator combines two scale-measures, and therefore two views, on a data context to a new single one. We may note that the special case of $(\sigma, S) = (\text{id}_G, \mathbb{K})$ was already discussed by Ganter and Wille [7].
**Proposition 20** Let $K = (G, M, I)$ and $S = (G_S, M_S, I_S)$ be two formal contexts and $\sigma : G \rightarrow G_S$, then TFAE:

i) $\sigma$ is a $S$-measure of $K$

ii) $\sigma$ is a $(G_S, \{n\}, I_S \cap \{G_S \times \{\{n\}\})$-measure of $K$ for all $n \in M_S$

**Proof.** (i) \(\Rightarrow\) (ii) : Assume $\hat{n} \in M_S$ s.t. $\sigma$ is not a $(G_S, \{\hat{n}\}, I_S \cap \{G_S \times \{\{\hat{n}\}\})$-measure of $K$. Then the only non-trivial extent $\{\hat{n}\}^J$ has a preimage $\sigma^{-1}(\{\hat{n}\}^J) \not\in \text{Ext}(K)$. Since $\{\hat{n}\}^J \in \text{Ext}(S)$ we can conclude that $\sigma$ is not a $S$-measure of $K$.

(ii) \(\Rightarrow\) (i) : From Proposition 19 follows $\big|_{n \in M_S} \sigma \cdot (G_S, \{n\}, I_S \cap \{G_S \times \{n\}))$ is again a scale-measure. Furthermore, by Definition 18 we know that $S = \big|_{n \in M_S} (G_S, \{n\}, I_S \cap \{G_S \times \{n\}))$. □

**Corollary 21 (Deciding the Scale-measure Problem)** Given a formal context $(G, M, I)$ and scale-context $S := (G_S, M_S, I_S)$ and a map $\sigma : G \rightarrow G_S$, deciding if $(\sigma, S)$ is a scale-measure of $K$ is in $P$. More specifically, to answer this question does require $O(|M_S| \cdot |G_S| \cdot |G| \cdot |M|)$.

We may not that this result is favorable since the naive solution would be to compute $\text{Ext}(S)$, which is potentially exponential in the size of $S$, and checking all its elements in $K$ for their closure, which consumes $O(|G| \cdot |M|)$ for all $A \in \text{Ext}(S)$. Moreover, if the formal context $K$ is fixed as well as $G_S$, the computational cost for deciding the scale-measure problem grows linearly in $|M_S|$. Altogether, this enables a feasible navigation in the scale-hierarchy.

**Corollary 22 (Attribute Projection)** Let $K = (G, M, I)$ be a formal context, $M_S \subseteq M$, and $I_S := I \cap (G \times M_S)$, then $\sigma = \text{id}_G$ is a $(G, M_S, I_S)$-measure of $K$.

**Proof.** The map $\text{id}_G$ is a $K$-measure of $K$, hence $\text{id}_G$ is a $(G, \{n\}, I \cap (G \times \{n\}))$-measure of $K$ for every $n \in M$, and in particular $n \in M_S$, by Proposition 20, leading to $(\text{id}_G, (G, M_S, I_S))$ being a scale-measure of $K$, cf. Proposition 19. □

Due to duality one may also investigate an object projection based on the just presented attribute projection. However, an investigation of dualities in the realm of scale-measures is deemed future work. Combining our results on scale-measure apposition (Proposition 19) with the logical attributes (Proposition 16) we are now tackle the navigation problem as stated in Problem 14.

When we look at this problem again, we find that in its generality it does not always permit a solution. For example, consider the well-known Boolean formal context $B_n := ([n], [n], \not\equiv)$, a standard scale context, where $[n] := \{1, \ldots, n\}$ and $n \geq 2$. This context allows a scale-measure into the standard nominal scale $N_n := ([n], [n], =)$, the map $\text{id}_{B_n}$. Restricted to any disjunctive combination of attributes, i.e., $M_T \subseteq \mathcal{L}(M, \{\lor\})$, the afore mentioned scale-measure does not have an equivalent logical scale-measure $(\psi, T := ([n], M_T, I_T))$. This is due to the fact that 1. in nominal contexts there is for every object $g$ there is an attribute $m$, such that $m' = g$, also $|m'| = 1$, 2. all attribute derivations in Boolean context $B_n$ are of cardinality $n - 1$, 3. the derivation of a disjunctive
formula (over \([n]\)) is the union of the elemental attribute derivations (Lemma 15). Hence, the derivation of a disjunctive formula is at least of cardinality \(n - 1\) in \(T\) and therefore there must not exist an \(m \in M_T\) such that \(\left|\{m\}^{T}\right| = 1\), and therefore \(\text{Ext}(N) \neq \text{Ext}(T)\).

Despite this result, we may also report positive answers for particular instances of Problem 14 that use conjunctive formulas for \(M_T\).

**Proposition 23 (Conjunctive Normalform of Scale-measures)** Let \(\mathcal{K}\) be a context, \((\sigma, \mathcal{S}) \in \mathcal{S}(\mathcal{K})\). Then the scale-measure \((\psi, T) \in \mathcal{S}(\mathcal{K})\) given by

\[
\psi = \text{id}_G\quad\text{and}\quad T = \big|_{A \in \sigma^{-1}(\text{Ext}(\mathcal{S}))} (G, \{\phi = \land A^T\}, I_\phi)
\]

is equivalent to \((\sigma, \mathcal{S})\) and is called conjunctive normalform of \((\sigma, \mathcal{S})\).

**Proof.** We know that every formal context \((G, \{\phi = \land A^T\}, I_\phi)\) together with \(\text{id}_G\) is a scale-measure (Corollary 17). Moreover, every apposition of scale-measures (for some formal context \(\mathcal{K}\)) is again a scale-measure (Proposition 19). Hence, the resulting \((\psi, T)\) is a scale-measure of \(\mathcal{K}\).

It remains to be shown that \(\sigma^{-1}(\text{Ext}(\mathcal{S})) = \text{id}_G(\text{Ext}(T))\). Scale-measure equivalence holds if \((\psi, T)\) reflects the same set of extents in \(\text{Ext}(\mathcal{K})\) as \((\sigma, \mathcal{S})\), thus if Each \((G, \{\phi = \land A^T\}, I_\phi)\) has the extent set \(\{G, (\land A^T)^{I_\phi}\}\). In this set we find that \((\land A^T)^{I_\phi} = A\) by Lemma 15. Due to the apposition property the resulting context has the intersections of all subsets of \(\sigma^{-1}(\text{Ext}(\mathcal{S}))\) as extents. This set is closed under intersection. Therefore, \(\sigma^{-1}(\text{Ext}(\mathcal{S})) = \text{id}_G(\text{Ext}(T))\). \(\square\)

The conjunctive normalform \((\psi, T)\) of a scale-measure \((\sigma, \mathcal{S})\) may constitute a more human-accessible representation of the same scaling information. To demonstrate this in a more practical manner we applied our method to the well-known Zoo data set by R.S. Forsyth, which we obtained from the UCI repository [16]. For this we computed a canonical scale-measure (Lemma 9), for which we computed an equivalent scale-measure (Figure A.6) according to Proposition 23. In the presented example we see that the intent of animal taxons emerge naturally, which are indicated using red colored names in Figure A.6, (instead of extents as used by the canonical representation).

### 3.3. Order Dimension of Scale-measures

An important property of formal contexts, and therefore of scale-measures, is the order dimension (Definition 3). We already motivated their investigation with respect to our running example, specifically the decrease of dimension (Figure 2). The substantiate formally our experimental finding we investigate the correspondence between order dimension and scale-hierarchies. For this we employ the Ferrers dimension of contexts, which is equal to their order dimension [7, Theorem 46]. A Ferrers relation is a binary relation \(F \subseteq G \times M\) such that for \((g, m), (h, n) \in F\) it holds that \((g, n) \notin F \Rightarrow (h, m) \in F\). The Ferrers dimension of the formal context \(\mathcal{K}\) is equal to the minimum number of ferrers relations \(F_t \subseteq G \times M, t \in T\) such that \(I = \bigcap_{t \in T} F_t\).

**Proposition 24** For a context \(\mathcal{K}\) and scale-measures \((\sigma, \mathcal{S}), (\psi, T) \in \mathcal{S}(\mathcal{K})\) with \((\sigma, \mathcal{S}) \leq (\psi, T)\), where \(\sigma\) and \(\psi\) are surjective, it holds that \(\dim(\mathcal{S}) \leq \dim(T)\).
We decided for another food related data set, since we assume that this knowledge from multiple categories, such as vegetables, meat, or fish dishes. The incidence is in the following denoted by \( K \) as attributes, and the resulting context edge domain is easily to grasp. Specifically, the data set is comprised of 56 dishes as objects and 37 spices as their attributes, and the resulting context is then equal to the incidence of \( K_{\sigma^1}(\text{Ext}(S)) \). Thus, as required, \( \dim(K_{\sigma^1}(\text{Ext}(S))) \leq \dim(K_{\sigma^1}(\text{Ext}(T))) \).

Building up on this result we can provide an upper bound for the dimension of apposition of scale-measures for some formal context \( K \).

**Proposition 25** For a context \( K \) and scale-measures \( (\sigma, S), (\psi, T) \in \mathfrak{S}(K) \) with \( (\sigma, S) \mid (\psi, T) = (\delta, \emptyset) \). Then order dim. of \( \emptyset \) is bound by \( \dim(\emptyset) \leq \dim(S) + \dim(T) \).

**Proof.** Without loss of generality we consider for all scale-measures their canonical representation, only. Let \( F_T \) be a Ferrers set of the formal context \( T \) such that \( \bigcap_{t \in T} F_t = I_T \) and similarly \( \bigcap_{s \in S} F_s = I_S \). For any Ferrers relation \( F \) of \( S \) it follows that \( F \cup (G \times M_T) \) is a Ferrers relation of \( S \mid T \). Hence, the intersection of \( \bigcap_{s \in S} F_s \cup (G \times M_T) \) and \( \bigcap_{t \in T} F_t \cup (G \times M_S) \) is a Ferrers set and is equal to \( I_{S \mid T} \). Since this construction does neither change the cardinality of index set \( T \) nor the index set \( S \), the required inequality follows.

4. Implications for Data Set Scaling

We revisit the running example \( K_{\text{BJ}} \) (Figure 1) and want to outline a semi-automatically procedure to obtain a human-meaningful scale-measure from it, as depicted in Figure 2, based on the insights from Section 3. In this example, we derive new attributes \( M_T \subseteq L(M, \{\land, \lor, \neg\}) \) from the original attribute set \( M \) of \( K_{\text{BJ}} \) using background knowledge. This process results in

\[
M_T = \{\text{Choco} = \text{Choco Ice} \lor \text{Choco Pieces}, \quad \text{Caramel} = \text{Caramel Ice} \lor \text{Caramel}, \quad \text{Peanut} = \text{Peanut Ice} \lor \text{Peanut Butter}, \quad \text{Brownie, Dough, Vanilla}\}
\]

Such propositional features can be bear various meanings, in our example we interpret \( M_T \) as taste attributes (as opposed to ingredients). Another possible set \( M_T \) could represent ingredient mixtures (\( \land \)) to generate a recipe view on the presented ice creams. From \( M_T \) we can now derive semi-automatically a scale-measure (Propositions 19 and 20) if it exists (Corollary 21).

4.1. Scaling of Larger Data Set

To demonstrate the benefits of the scale-measure navigation on a larger data set, we evaluate our method on a data set that related spices to dishes [17, 18]. We decided for another food related data set, since we assume that this knowledge domain is easily to grasp. Specifically, the data set is comprised of 56 dishes as objects and 37 spices as their attributes, and the resulting context is in the following denoted by \( K_{\text{Spices}} \). The dishes in the data set are picked from multiple categories, such as vegetables, meat, or fish dishes. The incidence \( I_{K_{\text{Spices}}} \) indicates that a spice \( m \) is necessary to cook a dish \( g \). The concept
lattice of $K_{\text{Spices}}$ has 421 concepts and is therefore too large for a meaningful human comprehension. Thus, using scale-measures through our methods, we are able to generate two small-scaled views of readable size. Both scales, as depicted in Figure 5, measure the dishes in terms of spice mixtures $M_T \subseteq L(M, \{\land\})$. For the conjunction of spices we transformed intent sets $B \in \text{Int}(K_{\text{Spices}})$ to propositional formulas $\land_{m \in B} m$. However, in order to retrieve a small scale context we decided for using intents with high support, only, i.e., $B'/G$ is high with respect to some selection criterion. We employed two different selection criteria: A) high support in all dishes; B) high support in meat dishes. Afterwards we derive semi-automatically two scale-measures (Propositions 19 and 20). Both scale-measures include five spices mixtures. The concept lattice for the scale context of A) is depicted in Figure 5 (bottom), and for B) in Figure 5 (top). We named all selected intent sets to make them more easily addressable. Both scales can be used to identify similar flavored dishes, e.g., a menu such as deer in combination with red cabbage, which share the bay leaf mix. Based on the scale-measures one might be interested to further navigate in the scale-hierarchy by adding additional spice mixtures (Proposition 19), or employing other selection criterion, which result in different views on the data set $K_{\text{Spices}}$, e.g., vegetarian.

Finally, we may point out that in contrast to feature compression techniques, such as LSA (which use linear combinations of attributes), the scale-measure attributes are directly interpretable by the semantics of propositional logics on the original data set attributes.

5. Related Work

Measurement is an important field of study in many (scientific) disciplines that involve the collection and analysis of data. According to Stevens [1] there are four feature categories that can be measured, i.e. nominal, ordinal, interval and ratio features. Although there are multiple extensions and re-categorizations of the original four categories, e.g., most recently Chrisman introduced ten [19], for the purpose of our work the original four suffice. Each of these categories describe which operations are supported per feature category. In the realm of formal concept analysis we work often with nominal and ordinal features, supporting value comparisons by = and <, >. Hence grades of detail/membership cannot be expressed. A framework to describe and analyze the measurement for Boolean data sets has been introduced in [20] and [21], called scale-measures. It characterizes the measurement based on object clusters that are formed according to common feature (attribute) value combinations. An accompanied notion of dependency has been studied [22], which led to attribute selection based measurements of boolean data. The formalism includes a notion of consistency enabling the determination of different views and abstractions, called scales, to the data set. This approach is comparable to OLAP [23] for databases, but on a conceptual level. Similar to the feature dependency study is an approach for selecting relevant attributes in contexts based on a mix of lattice structural features and entropy maximization [24]. All discussed abstractions reduce the complexity of the data, making it easier to understand by humans.

Despite the in this work demonstrated expressiveness of the scale-measure framework, it is so far insufficiently studied in the literature. In particular algorithmical and practical calculation approaches are missing. Comparable and
Figure 5: In this figure, we display the concept lattices of two scale contexts for which the identity map is a scale-measures of the spices context. The attributes of the scales are spice mixtures generated by propositional logic. By Other we identify all objects in the top concept for better readability.
popular machine learning approaches, such as feature compressed techniques, e.g., Latent Semantic Analysis [25, 26], have the disadvantage that the newly compressed features are not interpretable by means of the original data and are not guaranteed to be consistent with said original data. The methods presented in this paper do not have these disadvantages, as they are based on meaningful and interpretable features with respect to the original features using propositional expressions. In particular preserving consistency, as we did, is not a given, which was explicitly investigated in the realm scaling many-valued formal contexts [14] and implicitly studied for generalized attributes [15].

Earlier approaches to use scale contexts for complexity reduction in data used constructs such as \((G_N \subseteq P(N), N, \supseteq)\) for a formal context \(K = (G, M, I)\) with \(N \subseteq M\) and the restriction that at least all intents of \(K\) restricted to \(N\) are also intent in the scale [27]. Hence, the size of the scale context concept lattice depends directly on the size of the concept lattice of \(K\). This is particularly infeasible if the number of intents is exponential, leading to incomprehensible scale lattices. This is in contrast to the notion of scale-measures, which cover at most the extents of the original context, and can thereby display selected and interesting object dependencies of scalable size.

6. Conclusion

Our work has broadened the understanding of the data scaling process and has paved the way for the development of novel scaling algorithms, in particular for Boolean data, which we summarize under the term Exploring Conceptual Measurements. We build our framework on the notion of scale-measures, which themselves are interpretations of formal contexts. By studying and extending the theory on scale-measures, we found that the set of all possible measurements for a formal context is lattice ordered, up to equivalence. Thus, this set is navigable using the lattice’s meet and join operations. Furthermore, we found that the problem of deciding whether for a given formal context \(K\) and a tuple \((\sigma, S)\) the latter represents a scale-measure for the former is \(\text{PTIME}\) with respect to the respective object and attribute set sizes. All this and the following is based on our main result that for a given formal context \(K\) the set of all scale-measures and the set of all sub-closure systems of \(\mathfrak{B}(K)\) are isomorphic.

To ensure our goal for human comprehensible scaling we derived a propositional logic scaling of formal contexts by transferring and extending results from conceptual scaling [14]. With this approach, we are able to introduce new features that lead to interpretable scale features in terms of a logical formula and with respect to the original data set attributes. Moreover, these features are suitable to create any possible scale measurement of the data. Finally, we found that the order dimension decreases monotonously when scale-measures are coarsened, hinting the principal improved readability of scale-measures in contrast to the original data set. We have substantiated our theoretical results with three exemplary data analyses. In particular we demonstrated that employing propositional logic on the attribute set enables us to express and apply meaningful scale features, which improved the human readability in a natural manner. All methods used throughout this work are published with the open source software \texttt{conexp-clj}\[28], a research tool for Formal Concept Analysis.

We identified three different research directions for future work, which together may lead to an efficient and comprehensible data scaling framework.
First of all, the development of meaningful criteria for ranking or valuing scale-measures is necessary. Although our results enable an efficient navigation in the lattice of scale-measures, it cannot provide a promising direction, except from decreasing the order dimension. Secondly, efficient algorithms for computing an initial, well ranked/rated scale-measure and the subsequent navigation are required. Even though we showed a bound for the computational run time complexity, we assume that this can still be improved. Thirdly, a natural approach for decreasing the computational cost of navigating conceptual measurements would be to employ a set of minimal closure generators instead of the closure system. We speculate that our results hold in this case. Yet, it is an open questions if procedures, such as TITANIC [29], can be adapted to efficiently navigate the scale-hierarchy of a formal context.

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Appendix A. Example

![Concept lattice of a scale-measure of the zoo data set with twenty-seven of the original 4579 concepts. Contained objects are animals and attributes are characteristics. Newly introduced logical attributes are a characterization of animal taxons. The objects girl, frog were omitted. We grouped: OtherFishes = \{seahorse, sole, herring, piranha, pike, chub, haddock, stingray, carp, bass, dogfish, catfish, tuna\}, OtherMammals = \{reindeer, aardvark, polecat, wolf, mole, vole, hare, boar, cavy, antelope, goat, puma, mongoose, pony, bear, pussycat, lynx, elephant, calf, mink, opossum, leopard, buffalo, lion, giraffe, cheetah, oryx, deer, hamster, raccoon\}, OtherBirds = \{gull, parakeet, crow, skua, swan, hawk, sparrow, lark, wren, dove, vulture, penguin, duck, flamingo, pheasant, rhea, ostrich, skimmer, chicken, kiwi\}.

Figure A.6: Concept lattice of a scale-measure of the zoo data set with twenty-seven of the original 4579 concepts. Contained objects are animals and attributes are characteristics. Newly introduced logical attributes are a characterization of animal taxons. The objects girl, frog were omitted. We grouped: OtherFishes = \{seahorse, sole, herring, piranha, pike, chub, haddock, stingray, carp, bass, dogfish, catfish, tuna\}, OtherMammals = \{reindeer, aardvark, polecat, wolf, mole, vole, hare, boar, cavy, antelope, goat, puma, mongoose, pony, bear, pussycat, lynx, elephant, calf, mink, opossum, leopard, buffalo, lion, giraffe, cheetah, oryx, deer, hamster, raccoon\}, OtherBirds = \{gull, parakeet, crow, skua, swan, hawk, sparrow, lark, wren, dove, vulture, penguin, duck, flamingo, pheasant, rhea, ostrich, skimmer, chicken, kiwi\}.