Langlands parameters for epipelagic representations of $GL_n$

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February 2013

Abstract. Let $F$ be a non-Archimedean local field. An irreducible cuspidal representation of $GL_n(F)$ is epipelagic if its Swan conductor equals 1. We give a full and explicit description of the Langlands parameters of such representations.

We examine an extreme class of irreducible cuspidal representations of the general linear group $G = GL_n(F)$ over a non-Archimedean local field $F$, those designated “epipelagic” by Reeder and Yu [26]. A particular arithmetic interest of such representations was first identified by Gross and Reeder in [18], who dubbed them “simple cuspidals”. The authors of [18] asked us whether anything could be said about the Langlands parameters. This paper is the response.

The context of [26] is quite general but, for the group $GL_n(F)$, an exercise in techniques going back to [1] shows that a cuspidal representation $\pi$ is epipelagic if and only if the exponential Swan conductor $Sw(\pi)$ is equal to 1. We start from that point.

1. If $n \geq 1$ is an integer, let $A_n(F)$ be the set of equivalence classes of irreducible, cuspidal (complex) representations of $G = GL_n(F)$. Let $\mathcal{S}_n(F)$ be the set of equivalence classes of irreducible, smooth, $n$-dimensional representations of the Weil group $W_F$ of some separable algebraic closure $\bar{F}/F$ of $F$. We denote the Langlands correspondence $A_n(F) \rightarrow \mathcal{S}_n(F)$ by $\pi \mapsto L\pi$. Let $\mathcal{A}_n(F)$ be the set of $\pi \in A_n(F)$ with $Sw(\pi) = 1$ and define $\mathcal{G}_n(F)$ in the same way. The Langlands correspondence preserves Swan conductors and so maps $\mathcal{A}_n(F)$ bijectively to $\mathcal{G}_n(F)$.

It is easy to describe the elements of $\mathcal{A}_n(F)$ in terms of the standard model of [14]. A representation $\pi \in A_n(F)$, not of level zero, contains a simple character $\theta$. If we choose a character $\psi_F$ of $F$ satisfying $c(\psi_F) = -1$ (see §1 for this notation), then $\theta$ is attached to a simple stratum $[a, l, 0, \alpha]$ in the matrix algebra $A = M_n(F)$. The condition $Sw(\pi) = 1$ translates into $l$ being 1, the hereditary order $a$ being minimal and the field extension $F[\alpha]/F$ being totally ramified of degree $n$. Fixing such a stratum $[a, 1, 0, \alpha]$, one may list the elements $\pi$ of $\mathcal{A}_n(F)$ to which it is attached. This yields a classification of the elements of $\mathcal{A}_n(F)$ in terms of properties of local constants. Passing across the Langlands correspondence, one gets a parallel classification of the elements of $\mathcal{G}_n(F)$.

Mathematics Subject Classification (2000). 22E50.

Key words and phrases. Epipelagic, cuspidal representation, Langlands parameter.
2. On the Galois side, that classification exhibits a lamentable opacity. We set ourselves the task of describing explicitly the representation $\sigma = l\pi$, $\pi \in 1A_n(F)$, in terms of the structure of $\pi$.

There are not many cases where such descriptions have been seriously attempted. The essentially tame case is exhaustively worked out in [7], [8], [10]. The wildly ramified case has only been considered in prime dimension [9], [20], Kutzko [24], Mœglin [25], with the aim of verifying the Langlands conjecture, as it was at the time. The dissection of the representations was carried no further than necessary for that purpose. A thorough analysis, even of a special case, seems overdue. The epipelagic representations exhibit sufficient complexity to make the exercise worthwhile and can be used to generate further families of examples.

3. Let $\sigma \in 1G_n(F)$ and $\pi \in 1A_n(F)$ satisfy $l\pi = \sigma$. Let $p$ be the residual characteristic of $F$, and write $n = ep^r$, for integers $e$, $r$ with $e$ not divisible by $p$. There is then a totally ramified extension $K/F$ of degree $e$, and a representation $\tau \in 1G_p^e(F)$, such that $\tau$ induces $\sigma$. The pair $(K/F, \tau)$ is uniquely determined up to conjugation in $W_F$. The theory of tame lifting enables us to specify $K$ in terms of $\pi$, and also the representation $\rho \in 1A_{p^r}(K)$ such that $\tau = l\rho$.

4. We therefore concentrate on the case $l\pi = \sigma \in 1G_p^e(F)$. Here, a conductor estimate shows that $\sigma$ is primitive, and so covered by Koch’s seminal work [23].

Let $\bar{\sigma}$ be the projective representation of $W_F$ determined by $\sigma$, and define a finite Galois extension $E/F$ by $W_E = \text{Ker } \bar{\sigma}$. The restriction of $\sigma$ to $W_E$ is therefore a multiple of a character $\xi_\sigma$ of $W_E$. Let $T/F$ be the maximal tamely ramified sub-extension of $E/F$. Following [23], the restriction $\sigma_T$ of $\sigma$ to $W_T$ is irreducible. The Galois group of $E/T$ is elementary abelian of order $p^{2r}$, and $\sigma_T$ is the unique irreducible representation of $W_T$ containing $\xi_\sigma$. The classification theory of [23] then rests on the fact that $\text{Gal}(E/T)$ provides a symplectic representation of $\text{Gal}(T/F)$ over the field $\mathbb{F}_p$ of $p$ elements.

There is a helpful element of structure here. If $K/F$ is a finite tame extension, let $\sigma_K$ denote the (irreducible) restriction of $\sigma$ to $W_K$. Let $D(\sigma_K)$ be the group of characters $\chi$ of $W_K$ for which $\chi \otimes \sigma_K \cong \sigma_K$. One then has $|D(\sigma_K)| \leq p^{2r}$, with equality if and only if $K$ contains $T$. Moreover, $D(\sigma_T)$ is the dual of the Galois group $\text{Gal}(E/T) = W_T/W_E$.

We can make no further progress working only with Galois representations and have to switch to the other side. If $K/F$ is a finite tame extension, we may define $\pi_K \in A_{p^r}(K)$ by the relation $l\pi_K = \sigma_K$. Using results from [4], [5] on tame lifting, we can describe $\pi_K$ explicitly, in terms of $\pi$. We have to find the least tame extension $T/F$ for which there are $p^{2r}$ characters $\phi$ of $T^\times$ satisfying $\phi | T^\times \cong \pi_T$. An elaborate calculation is required. The outcome is an explicit polynomial, with splitting field $T$. The roots of the polynomial define the non-trivial elements of $D(\sigma_T)$, as characters of $T^\times$. Via class field theory, they determine the abelian extension $E/T$.

The structure of the argument is worthy of comment. The initial steps, from the Galois theory of local fields, are fairly straightforward. We only use the early parts of [23], not going much beyond the group-theoretic arguments of Rigby [27]. The main calculations, leading to the polynomial determining the field $T$, are quite involved and use a lot of machinery from [14], but they are essentially elementary in nature. The depth lies in the facility with which we can move from side to side, via the Langlands correspondence, without loss of explicit detail. This is absolutely reliant on a substantial portion of the theory of tame lifting, as developed in [4], [6] and [12].
5. Having identified the fields $T/F$ and $E/T$, it remains to understand the “$p$-central character” $\xi_\sigma$. Starting from the description of $T$ and $D(\sigma_T)$, classical methods of local field theory and a local constant calculation yield an expression for $\xi_\sigma$ on the unit group $U^n_F$. We shall see that the datum $(\xi_\sigma|_{U^n_F}, \det \sigma)$ determines $\sigma$ up to tensoring with an unramified character of order dividing $p^r$ (6.2 below). Since $Sw(\sigma)$ is not divisible by $p$, this uncertainty is removed by a simple local constant relation.

It is possible to specify $\xi_\sigma$ completely by calculating one further special value, but the details are voluminous. Nothing much is gained, so we have omitted them.

1. Preliminaries and notation

We set down our standard notation and recall some basic facts.

1.1. Throughout, $F$ is a non-Archimedean local field with discrete valuation ring $\mathfrak{o}_F$. The maximal ideal of $\mathfrak{o}_F$ is $p_F$, the residue field is $k_F = \mathfrak{o}_F/p_F$, $q = |k_F|$, and $\nu_F : F^\times \rightarrow \mathbb{Z}$ is the normalized additive valuation. The characteristic of $k_F$ is $p$. The unit group is $U_F = \mathfrak{o}_F^\times$ and $U^n_F = 1 + p^n_F$, $k \geq 1$. Also, $\mu_F$ is the group of roots of unity in $F$, of order relatively prime to $p$.

If $a$ is a hereditary $\mathfrak{o}_F$-order in some matrix algebra $M_n(F)$ and $p_a$ is the Jacobson radical of $a$, then $U_a = a^\times$ and $U^n_a = 1 + p^n_a$, $k \geq 1$.

Let $\bar{F}/F$ be a separable algebraic closure of $F$. Let $\mathcal{W}_F$ be the Weil group of $\bar{F}/F$ and $\mathcal{P}_F$ the wild inertia subgroup of $\mathcal{W}_F$. We identify the group of smooth characters of $\mathcal{W}_F$ with that of $F^\times$ via class field theory, switching between the two viewpoints as convenient.

Let $\hat{\mathcal{W}}_F$ be the set of equivalence classes of irreducible smooth representations of $\mathcal{W}_F$. For an integer $n \geq 1$, let $\mathcal{S}_n(F)$ be the set of $\sigma \in \hat{\mathcal{W}}_F$ of dimension $n$. Let $\mathcal{A}_n(F)$ be the set of equivalence classes of irreducible cuspidal representations of $GL_n(F)$. The Langlands correspondence gives a bijection $\mathcal{A}_n(F) \rightarrow \mathcal{S}_n(F)$ which we denote by $\pi \mapsto \tau_\pi$.

1.2. Let $\psi$ be a smooth character of $F$, $\psi \neq 1$. Define $c(\psi)$ to be the greatest integer $k$ such that $p_F^{-k} \subset \text{Ker } \psi$. We shall mostly work with the case $c(\psi) = -1$, where $\psi$ is trivial on $p_F$ but not on $\mathfrak{o}_F$.

Let $n \geq 1$ be an integer, let $\pi \in \mathcal{A}_n(F)$. The Godement-Jacquet local constant $\varepsilon(\pi, s, \psi)$ of $\pi$, relative to the character $\psi$ and a complex variable $s$, takes the form (Godement-Jacquet [17], Jacquet [21])

$$\varepsilon(\pi, s, \psi) = q^{-sf(\pi, \psi)} \varepsilon(\pi, 0, \psi),$$

for an integer $f(\pi, \psi)$ such that

$$f(\pi, \psi) = a(\pi) + nc(\psi),$$

where the Artin conductor $a(\pi)$ is independent of $\psi$. Set

$$sw(\pi) = a(\pi) - n.$$

Thus $sw(\pi) = f(\pi, \psi)$ when $c(\psi) = -1$. The integer $sw(\pi)$ is the same as the Swan conductor $Sw(\pi)$ except when $n = 1$ and $\pi$ is an unramified character of $F^\times$. In that case, $sw(\pi) = -1$ and $Sw(F) = 0$.

We denote by $\mathcal{A}_n(F)$ the set of $\pi \in \mathcal{A}_n(F)$ for which $sw(\pi) = 1$. 

1.3. Let \( \sigma \in G_n(F) \). The Langlands-Deligne local constant \( \varepsilon(\sigma, s, \psi) \) takes the form [9], Tate [30],
\[
\varepsilon(\sigma, s, \psi) = q^{-f(\sigma, \psi)}\varepsilon(\sigma, 0, \psi),
\]
with \( f(\sigma, \psi) = a(\sigma) + nc(\psi) \), the Artin conductor \( a(\sigma) \) being independent of \( \psi \). Likewise set
\[
sw(\sigma) = a(\sigma) - n, \quad n = \dim \sigma.
\]
This relates to the Swan conductor \( Sw(\sigma) \) as before. Let \( 1G_n(F) \) be the set of \( \sigma \in G_n(F) \) with \( sw(\sigma) = 1 \). The Langlands correspondence satisfies \( sw(L\pi) = sw(\pi), \pi \in A_n(F) \), so it maps \( 1A_n(F) \) bijectively to \( 1G_n(F) \).

We make frequent use of the following fact.

**Induction Formula.** Let \( E/F \) be a finite separable extension, set \( e = e(E|F), f = f(E|F) \), and let \( D_{E/F} = p_{E}^{f} \) be the relative different of \( E/F \). If \( \tau \in \hat{W}_{E} \) has dimension \( m \) and \( \sigma = \text{Ind}_{E/F} \tau \), then
\[
(1.3.1) \quad sw(\sigma) = (sw(\tau) + m(1-e+d))f.
\]

**Proof.** If \( \psi \) is a smooth character of \( F, \psi \neq 1 \), and \( \psi_{E} = \psi \circ \text{Tr}_{E/F} \), then
\[
(1.3.2) \quad c(\psi_{E}) = d + ec(\psi).
\]
According to [9] 30.4 Corollary, the quotient \( \varepsilon(\sigma, s, \psi)/\varepsilon(\tau, s, \psi_{E}) \) is independent of \( s \). The result follows from a simple computation. \( \square \)

### 2. Classification

Let \( n \geq 1 \) be an integer. In this section, we classify the elements of \( 1A_n(F) \) using the method of [14]. We translate this into a classification in terms of properties of local constants. Via the Langlands correspondence, this gives a parallel classification of the elements of \( 1G_n(F) \).

Since we work in the context of [14], we need to choose a smooth character \( \psi_{F} \) of \( F \) with \( c(\psi_{F}) = -1 \). Thus, in the language of [14], the character \( \psi_{F} \) “has level one”. Throughout, we write \( G = \text{GL}_{n}(F) \) and \( A = \text{M}_{n}(F) \).

2.1. Let \( \pi \in A_{n}(F) \). Thus \( \pi \) contains a simple character \( \theta \) in \( G \), in the sense of [14]. This simple character is uniquely determined up to \( G \)-conjugation. It is trivial if and only if \( sw(\pi) = 0 \). Assuming \( \theta \) to be non-trivial, it is attached via \( \psi_{F} \) to a simple stratum \( [a, l, 0, \beta] \) in \( A \): in the notation of [14], \( \theta \in \mathcal{C}(a, \beta, \psi_{F}) \). The stratum \( [a, l, 0, \beta] \) is m-simple, in that \( a \) is maximal among hereditary \( \mathfrak{o}_{F} \)-orders in \( A \), stable under conjugation by \( F[\beta]^{\times} \). Correspondingly, one says that \( \theta \) is m-simple.

If \( e_{a} \) denotes the \( F \)-period of the hereditary \( \mathfrak{o}_{F} \)-order \( a \), then
\[
sw(\pi) = \ln/e_{a} \geq 1,
\]
[5] 6.1 Lemma 2. Since the stratum \( [a, l, 0, \beta] \) is m-simple, the period \( e_{a} \) equals the ramification index \( e(F[\beta]|F) \) of the field extension \( F[\beta]/F \). We encapsulate these remarks:
**Lemma 1.** Let $\pi \in \mathcal{A}_n(F)$. The following conditions are equivalent:

1. $\text{sw}(\pi) = 1$;
2. $\pi$ contains an $m$-simple character $\theta \in \mathcal{C}(a, \alpha, \psi_F)$, where $[a, 1, 0, \alpha]$ is an $m$-simple stratum in $A$ such that $e_a = n$.

When these conditions are satisfied, the field extension $F[\alpha]/F$ is totally ramified of degree $n$.

We write down the various groups and the simple characters attached to the simple stratum $[a, 1, 0, \alpha]$ in the machinery of [14] Chapter 3. Let $\text{tr}_A : A \to F$ be the matrix trace and write $\psi_A = \psi_F \circ \text{tr}_A$.

**Lemma 2.** Let $[a, 1, 0, \alpha]$ be an $m$-simple stratum in $A$ with $e_a = n$.

1. The associated groups are $$H^1(\alpha, a) = J^1(\alpha, a) = U_a^1.$$ The set $\mathcal{C}(a, \alpha, \psi_F)$ of simple characters has only the one element $$\theta_\alpha : 1+x \mapsto \psi_A(\alpha x), \quad 1+x \in U_a^1.$$ The $G$-normalizer of the character $\theta_\alpha$ is $J_\alpha = F[\alpha]^xU_a^1$.

2. Let $\Psi$ be a character of $J_\alpha$ such that $\Psi|_{U_a^1} = \theta_\alpha$. The $G$-representation $$\pi_\Psi = c\text{-Ind}_\alpha^G \Psi$$ is irreducible and cuspidal. The map $\Psi \mapsto \pi_\Psi$ is a bijection between the set of characters of $J_\alpha$ extending $\theta_\alpha$ and the set of $\pi \in \mathcal{A}_n(F)$ containing $\theta_\alpha$.

**Proof.** Part (1) follows on working through the definitions in Chapter 3 of [14]. Part (2) is an instance of [14] (8.4.1) or [12] 4.1. □

In part (1) of the lemma, the character $\theta_\alpha$ determines only the coset $\alpha + a = \alpha U_a^1$. In the situation of part (2), it will be easier to say that $\pi$ contains the $m$-simple stratum $[a, 1, 0, \alpha]$ (relative to the character $\psi_F$), rather than that it contains $\theta_\alpha$.

**Proposition.**

1. If $\pi \in \mathcal{A}_n(F)$, $n > 1$, the central character $\omega_\pi$ of $\pi$ is tamely ramified.
2. Let $\omega$ be a tamely ramified character of $F^\times$ and $[a, 1, 0, \alpha]$ an $m$-simple stratum in $A$, with $e_a = n$. There exist exactly $n$ representations $\pi \in \mathcal{A}_n(F)$ containing $[a, 1, 0, \alpha]$ and having central character $\omega_\pi = \omega$.

**Proof.** Let $\pi \in \mathcal{A}_n(F)$ contain the $m$-simple stratum $[a, 1, 0, \alpha]$. We have $U_a^1 = F^\times \cap U_a^n \subset \text{Ker} \theta_\alpha$. Therefore $\omega_\pi$ is trivial on $U_a^1$, as required for (1). The group $J_\alpha$ contains $F^\times U_a^1$ with index $n$, so (2) follows from Lemma 2. □

**2.2.** If $\pi \in \mathcal{A}_n(F)$ and if $\chi$ is a character of $F^\times$, then $\chi \pi$ will denote the representation $g \mapsto \chi(\det g)\pi(g)$. In particular, $\chi \pi \in \mathcal{A}_n(F)$.

We compute Godement-Jacquet local constants. The method originates in [3], [1], but the summary in [5] 6.1 is closer to our current notation.
Lemma. If \( \pi = \text{c-Ind}_{J_n}^G \Psi_\pi \), as in 2.1 Lemma 2, then

1. \( \varepsilon(\pi, \frac{1}{2}, \psi_F) = \Psi_\pi(\alpha)^{-1} \psi_\lambda(\alpha) \) and,
2. if \( \chi \) is a tamely ramified character of \( F^\times \), then

\[
\varepsilon(\chi \pi, s, \psi_F) = \chi(\det \alpha)^{-1} \varepsilon(\pi, s, \psi_F).
\]

We may now strengthen the description of 2.1.

Proposition. For \( i = 1, 2 \), let \( \pi_i \in \mathcal{A}_n(F) \), and let \( \omega_i \) be the central character of \( \pi_i \). Let \( \pi_i \) contain the \( m \)-simple stratum \([a, 1, 0, \alpha_i] \). The representations \( \pi_1, \pi_2 \) are equivalent if and only if the following conditions are satisfied:

1. \( \det \alpha_1 \equiv \det \alpha_2 \pmod{U_1^1} \),
2. \( \omega_1 = \omega_2 \), and
3. \( \varepsilon(\pi_1, \frac{1}{2}, \psi_F) = \varepsilon(\pi_2, \frac{1}{2}, \psi_F) \).

Proof. Let \( p_a \) be the Jacobson radical of \( a \). If the \( \pi_i \) are equivalent, the \( m \)-simple characters \( \theta_{\alpha_i} \) are conjugate in \( G \) or, equivalently, the cosets \( \alpha_i U_1^1 \) are \( U_a \)-conjugate. To elucidate this condition, we use a matrix normal form calculation. We may assume that \( a \) is the ring of matrices, with entries in \( \mathfrak{O}_F \), which become upper triangular when reduced modulo \( p_F \). The ideal \( p_a \) then consists of the matrices which are strictly upper triangular modulo \( p_F \). We have \( \alpha^{-1} a = p_a \) for both values of \( i \), this being part of the definition of a simple stratum. A matrix manipulation shows that \( \alpha_i^{-1} \) is \( U_a \)-conjugate to a matrix \( a_i u_i \), where

1. \( u_i \in U_1^1 \);
2. \( (a_i)_{j,j+1} = 1 \), for \( 1 \leq j < n \);
3. \( (a_i)_{1,1} \) is a prime element \( \omega_i \) of \( F \);
4. all other entries of \( a_i \) are zero.

The cosets \( \alpha_i U_1^1 \) are then \( U_a \)-conjugate if and only if \( \omega_1 \equiv \omega_2 \pmod{U_1^1} \), and this is equivalent to condition (a) of the proposition.

We are thus reduced to the case \( \alpha_1 = \alpha_2 = \alpha \), say. Set \( E = F[\alpha] \). There are characters \( \Psi_i \) of \( J_\alpha = E^\times U_1^1 \) such that \( \Psi_i|U_1^1 = \theta_\alpha \) and \( \pi_i \cong \text{c-Ind}_{J_\alpha}^E \Psi_i \), \( i = 1, 2 \). Moreover, \( \pi_1 \cong \pi_2 \) if and only if \( \Psi_1 = \Psi_2 \). Condition (b) of the proposition is equivalent to \( \Psi_1|_{F^\times} = \Psi_2|_{F^\times} \) (2.1 Lemma 2). Using the last lemma, condition (c) of the proposition is now equivalent to \( \Psi_1(\alpha) = \Psi_2(\alpha) \).

Since \( \alpha \) generates the finite cyclic group \( J_\alpha/U_1^1 \), the result follows. \( \Box \)

Remark. Let \( \omega \) be a tamely ramified character of \( F^\times \). Taking account of 2.1 Proposition, we see that the set \( \mathcal{A}_n(F) \) has exactly \( n(q-1) \) elements \( \pi \) such that \( \omega_\pi = \omega \).

2.3. We use the Langlands correspondence to translate the classification from 2.2 Proposition in terms of representations of \( \mathcal{W}_F \).

Lemma. Let \( \sigma \in \hat{\mathcal{W}}_F \) and suppose that \( \text{sw}(\sigma) \geq 1 \). There exists \( \gamma_\sigma \in F^\times \) such that

\[
\varepsilon(\chi \otimes \sigma, s, \psi_F) = \chi(\gamma_\sigma)^{-1} \varepsilon(\sigma, s, \psi_F),
\]

for any tamely ramified character \( \chi \) of \( \mathcal{W}_F \). This property determines the coset \( \gamma_\sigma U_1^1 \) uniquely.

Proof. The lemma is a case of the main result of [16]. \( \Box \)
Proposition. Let \( \sigma \in \mathfrak{S}_n(F) \). Define \( \pi \in \mathcal{A}_n(F) \) by \( \mathcal{L}_\pi = \sigma \). If \( [a, 1, 0, \alpha] \) is an \( m \)-simple stratum contained in \( \pi \), then
\[
\gamma_{\sigma} \equiv \det \alpha \pmod{U^1_F},
\]
\[
\det \sigma = \omega_{\pi},
\]
\[
\varepsilon(\sigma, s, \psi_F) = \varepsilon(\pi, s, \psi_F).
\]
Moreover, \( \pi \) is the only element of \( \mathcal{A}_n(F) \) with these properties.

Proof. The Langlands correspondence \( \mathcal{A}_n(F) \to \mathfrak{S}_n(F) \) takes the central character to the determinant, it preserves the local constant and respects twisting with characters. The result therefore follows from the lemma and 2.2 Lemma.

Combining the proposition with 2.2 Proposition, we get:

Corollary. Let \( \sigma_1, \sigma_2 \in \mathfrak{S}_n(F) \). The representations \( \sigma_1, \sigma_2 \) are equivalent if and only if the following conditions hold:

1. \( \gamma_{\sigma_1} \equiv \gamma_{\sigma_2} \pmod{U^1_F} \),
2. \( \det \sigma_1 = \det \sigma_2 \), and
3. \( \varepsilon(\sigma_1, s, \psi_F) = \varepsilon(\sigma_2, s, \psi_F) \).

Reflection. It is not difficult to reconstruct \( \pi \in \mathcal{A}_n(F) \) from properties of local constants. In particular, the function \( \chi \mapsto \varepsilon(\chi \circ \sigma, \frac{1}{2}, \psi) \), with \( \chi \) ranging over the tame characters of \( F^\times \), reveals completely the simple character contained in \( \pi \). For \( \sigma \in \mathfrak{S}_n(F) \), the corresponding property gives the coset \( \gamma_{\sigma} U^1_F \), and that determines the simple character in \( \pi \), \( \mathcal{L}_\pi = \sigma \). According to the Ramification Theorem [6] 8.2 or [12] 6.1, it must also determine the restriction of \( \sigma \) to the wild inertia subgroup of \( W_F \). The process by which it does this is the central concern of the paper.

3. PRIMITIVITY

While the results of \( \S 2 \) classify the elements of \( \mathfrak{S}_n(F) \), they do not describe them at all effectively. We investigate further.

3.1. Any \( \sigma \in \mathfrak{S}_n(F) \) is totally ramified, in the following sense.

Lemma. Let \( \sigma \in \mathfrak{S}_n(F) \) and let \( \chi \) be an unramified character of \( W_F \). If the representations \( \chi \otimes \sigma, \sigma \) are equivalent then \( \chi = 1 \).

Proof. By 2.3 Proposition, \( \nu_F(\gamma_{\sigma}) = -1 \) in this case. If the unramified character \( \chi \) is non-trivial, then \( \varepsilon(\chi \circ \sigma, s, \psi_F) = \chi(\gamma_{\sigma})^{-1} \varepsilon(\sigma, s, \psi_F) \neq \varepsilon(\sigma, s, \psi_F) \), whence \( \chi \otimes \sigma \not\equiv \sigma \).

Equivalently, a representation \( \sigma \in \mathfrak{S}_n(F) \) restricts irreducibly to the inertia subgroup of \( W_F \).

Proposition 1. Let \( n = ep^r \), for integers \( e, r \) such that \( p \) does not divide \( e \). Let \( \sigma \in \mathfrak{S}_n(F) \).

1. There exists a totally ramified extension \( K/F \), of degree \( e \), and a representation \( \tau \in \hat{W}_K \) such that \( \sigma \cong \text{Ind}_{K/F} \tau \). This relation determines the pair \( (K/F, \tau) \) uniquely up to \( W_F \)-conjugation.
2. The representation \( \tau \) satisfies \( \text{sw}(\tau) = 1 \).
Proof. Part (1) is an instance of [6] 8.6 Proposition. Part (2) follows directly from the Induction Formula (1.3.1). □

We describe the representation $\tau$ in the manner of §2. Let $\sigma = L_\pi, \pi \in \mathcal{A}_n(F)$. As in 2.1, $\pi$ contains the simple character $\theta_\alpha \in \mathbb{C}(a, \alpha, \psi_F)$, for a simple stratum $[a, 1, 0, \alpha]$ in $A = M_n(F)$.

Let $T/F$ be the maximal tamely ramified sub-extension of $F[\alpha]/F$. Let $B \cong M_{p^r}(T)$ be the $A$-centralizer of $T$ and set $b = a \cap B$. Write $\psi_T = \psi_F \circ \text{Tr}_{T/F}$. The quadruple $[b, 1, 0, \alpha]$ is then an $m$-simple stratum in $B$ and the set $\mathbb{C}(b, \alpha, \psi_T)$ has only one element $\theta_\alpha^2$, as in 2.1 Lemma 2.

We take $(K/F, \tau)$ as in Proposition 1.

### Proposition 2.

1. The fields $T, K$ are $F$-isomorphic.
2. Let $\rho \in \mathcal{A}_{p^r}(K)$ satisfy $L_\rho = \tau$. There is an $F$-isomorphism $f : K \to T$ such that the representation $f_*\rho \in \mathcal{A}_{p^r}(T)$ contains the simple character $\theta_\alpha^2$.

Proof. Part (1) follows from the Tame Parameter Theorem of [12] 6.3 and the second from [12] 6.2. □

To save notation, we identify $T$ with $K$ via the isomorphism $f$, and let $\det_B : B^\times \to K^\times$ be the determinant map.

**Corollary.** Let $1_K$ be the trivial character of $\mathbb{W}_K$. Set $R_{K/F} = \text{Ind}_{K/F} 1_K$ and $\delta_{K/F} = \det R_{K/F}$. The representation $\tau$ satisfies:

1. $\gamma_\tau \equiv \det_B \alpha \pmod{U_K^1}$;
2. $\det \tau|_{F^\times} = \delta_{K/F}^{-1} \det_\sigma$;
3. $\varepsilon(\sigma, s, \psi_F)/\varepsilon(\tau, s, \psi_K) = (\varepsilon(R_{K/F}, s, \psi_F)/\varepsilon(1_K, s, \psi_K))^{p^r}$.

These relations determine $\tau$ uniquely.

Proof. Part (1) is implied by Proposition 2 and 2.3 Proposition. For parts (2) and (3), see for instance [9] 29.2, (29.4.1). Uniqueness follows from 2.3 Corollary. □

The values of the Langlands constant $\varepsilon(R_{K/F}, s, \psi_F)/\varepsilon(1_K, s, \psi_K)$ are tabulated in, for example, [2] 10.1. The corollary allows one to specify the representation $\rho$, such that $L_\rho = \tau$, in the manner of 2.2.

### 3.2.

The results of 3.1 reduce us to the case where $\dim \sigma$ is a power of $p$. In particular, $\sigma$ is **totally wildly ramified**, in that it remains irreducible on restriction to the wild inertia subgroup $\mathcal{P}_F$ of $\mathbb{W}_F$.

Recall that a representation $\tau \in \hat{\mathbb{W}}_F$ is **imprimitive** if there exists a finite separable extension $K/F, K \neq F$, and a representation $\rho \in \hat{\mathbb{W}}_K$ such that $\tau \cong \text{Ind}_{K/F} \rho$. Thus $\tau$ is called **primitive** if it is not imprimitive.

**Proposition.** If $\sigma \in \mathcal{A}_{p^r}(F)$, for an integer $r \geq 1$, then $\sigma$ is primitive.

Proof. Suppose, for a contradiction, that $\sigma \cong \text{Ind}_{K/F} \tau$, for a finite extension $K/F, K \neq F$, and $\tau \in \hat{\mathbb{W}}_K$. Thus $[K:F] = p^a$ and $\dim \tau = p^b$, for integers $a, b$ such that $a+b = r$. By (1.3.1),
sw(σ) = 1 is divisible by f(K|F), so K/F must be totally ramified. Since [K:F] is a power of p, the extension K/F is totally wildly ramified. If \( \mathcal{O}_{K/F} = p^{d(K|F)} \) is the different of K/F, this means d(K|F) \( \geq [K:F] = p^a \). Therefore

\[ 1 = sw(\sigma) \geq sw(\tau) + p^b(1-p^a+p^a), \]

or

\[ sw(\tau) \leq 1-p^b \leq 0. \]

The only possibility here is \( p^b = \dim \tau = 1 \) and \( \tau \) tamely ramified. The character \( \tau \) of \( K^\times \) is then of the form \( \xi \circ N_{K/F} \), for a tamely ramified character \( \xi \) of \( F^\times \). The character \( \xi \) must occur as a component of \( \text{Ind}_{K/F} \tau \), which is therefore reducible. This contradicts our hypothesis. □

4. Structure of primitive representations

Following 3.2 Proposition, a representation \( \sigma \in \text{Ind}_{Gr}^G(F), r \geq 1 \), is necessarily primitive. Primitive representations are described in [23]. We now see how \( \sigma \) fits into the scheme of [23], as a basis for the detailed analysis of the next section.

4.1. We make some remarks related to Clifford theory. For \( \tau \in \hat{W}_F \), let \( D(\tau) \) denote the group of characters \( \chi \) of \( W_F \) such that \( \chi \otimes \tau \sim \tau \).

**Lemma.** Let \( \tau \in \hat{W}_F \) have dimension \( n \). The abelian group \( D(\tau) \) has exponent dividing \( n \) and order at most \( n^2 \).

**Proof.** Let \( \chi \in D(\tau) \). The relation \( \chi \otimes \sigma \sim \sigma \) implies \( \det \sigma = \det(\chi \otimes \sigma) = \chi^n \det \sigma \), whence \( \chi^n = 1 \). Let \( \bar{\tau} \) denote the contragredient of \( \tau \), and let \( \chi \) be a character of \( W_F \). Frobenius Reciprocity yields

\[ \text{Hom}_{W_F}(\chi \otimes \tau, \tau) \cong \text{Hom}_{W_F}(\chi, \bar{\tau} \otimes \tau). \]

We deduce that \( \chi \in D(\tau) \) if and only if \( \chi \) is a component of \( \bar{\tau} \otimes \tau \). This proves the second assertion. □

**Proposition.** Let \( \tau \in \hat{W}_F \) be totally wildly ramified.

1. If \( \chi \in D(\tau) \) is tamely ramified, then \( \chi = 1 \).
2. If \( K/F \) is a finite tame extension, the canonical map

\[ D(\tau) \rightarrow D(\tau_K), \]

\[ \chi \mapsto \chi_K = \chi \circ N_{K/F}, \]

is injective. If \( K/F \) is Galois, the image of \( D(\tau) \) is the set of \( \text{Gal}(K/F) \)-fixed points in \( D(\tau_K) \).

**Proof.** Let \( \chi \in D(\tau) \) be tamely ramified, \( \chi \neq 1 \). The kernel of \( \chi \) is then \( W_K \), for a cyclic tame extension \( K/F \) and, by Clifford theory, \( \tau \cong \text{Ind}_{K/F} \rho \), for some \( \rho \in \hat{W}_K \). The restriction of \( \tau \) to \( W_K \) is therefore not irreducible. However, \( W_K \) contains \( P_K = P_F \) so this contradicts our hypothesis. Part (1) is proved.

In part (2), let \( \chi \in D(\tau) \). View \( \chi \) as a character of \( F^\times \) and suppose that \( \chi_K = 1 \). The field norm \( N_{K/F} \) induces a surjection \( U_K^1 \rightarrow U_F^1 \), so \( \chi \) is trivial on \( U_K^1 \). Part (1) implies \( \chi = 1 \). For
the second assertion, write $\Gamma = \text{Gal}(K/F)$. By transitivity and the first assertion, we need only treat the case where $\Gamma$ is cyclic. Let $\phi \in D(\tau_K)^\tau$. This implies $\phi = \chi_K$, for a character $\chi$ of $F^\times$. We then have $(\chi \otimes \tau)_K \cong \tau_K$, whence it follows that $\chi \otimes \tau \cong \delta \otimes \tau$, for a character $\delta$ of $F^\times$ such that $\delta_K = 1$. In particular, $\delta^{-1} \chi \in D(\tau)$ while $(\delta^{-1} \chi)_K$ agrees with $\phi$ on $U^1_K$. Therefore $(\delta^{-1} \chi)_K = \phi$, as required. □

4.2. We recall some facts about representations of Heisenberg type. Let $\tau \in \hat{W}_F$ be totally ramified of dimension $p^r$, for some $r \geq 1$. Let $\bar{\tau} : W_F \rightarrow \text{PGL}_{p^r}(\mathbb{C})$ be the associated projective representation. Thus $\text{Ker} \bar{\tau} = W_E$, for a finite Galois extension $E/F$, and the restriction of $\tau$ to $W_E$ is a multiple of a character $\xi$ of $W_E$. We call $E$ the $p$-kernel field of $\tau$ and $\xi$ the $p$-central character of $\tau$.

We are concerned with the case where $\Delta = \text{Gal}(E/F) = \text{Im} \bar{\tau} \cong (\mathbb{Z}/p\mathbb{Z})^{2r}$. The quantity $\xi(x, y) = \xi(xyx^{-1}y^{-1})$, $x, y \in W_F$, is then of order dividing $p$. We identify the group $\mu_p(\mathbb{C})$ of $p$-th roots of unity in $\mathbb{C}$ with the additive group of the field $\mathbb{F}_p$ of $p$ elements. The group $\Delta$ is a vector space over $\mathbb{F}_p$ and the identification $\mu_p(\mathbb{C}) = \mathbb{F}_p$ allows us to view the pairing $h_\tau : (x, y) \mapsto \xi(x, y)$ as a bilinear form $\Delta \times \Delta \rightarrow \mathbb{F}_p$. This form $h_\tau$ is alternating and nondegenerate. One says that $\tau$ is of Heisenberg type, since the image $\tau(W_F)$ is a Heisenberg group.

Lemma. Let $\tau \in \hat{W}_F$ be of Heisenberg type, with $p$-kernel field $E$, $p$-central character $\xi$, and dimension $p^r$.

1. If $\theta$ is an irreducible representation of $W_F$ such that $\theta|_{W_F}$ contains $\xi$, then $\theta \cong \tau$.

2. Let $\Delta'$ be a subgroup of $\Delta$, of order $p^r$, such that $h_\tau(x, y) = 0$, for all $x, y \in \Delta'$. Let $W_{E'}$ be the inverse image of $\Delta'$ in $W_F$.

   a. There exists a character $\psi$ of $W_{E'}$ such that $\psi|_{W_F} = \xi$.

   b. For any such character $\psi$, the induced representation $\text{Ind}_{E'/F} \psi$ is equivalent to $\tau$. In particular, $\text{Ind}_{E'/F} \xi_\tau$ is a sum of $p^r$ copies of $\tau$.

3. The group $D(\tau)$ has order $p^{2r}$ and consists of all characters of $F^\times$ trivial on norms from $E$. That is, $D(\tau)$ is the group $\hat{\Delta}$ of characters of $\Delta$.

Proof. This is an exercise, for which hints may be found in, for example, [2] 8.3. □

We recall the structure of the irreducible primitive representations of $W_F$. Leaving aside the trivial one-dimensional case, any such representation has dimension $p^r$, $r \geq 1$, and is totally wildly ramified.

Proposition. Let $\sigma \in \hat{W}_F$ be primitive of dimension $p^r$, $r \geq 1$. Let $E/F$ be the $p$-kernel field of $\sigma$, and let $T/F$ be the maximal tamely ramified sub-extension of $E/F$. Let $\Pi = \text{Gal}(E/T)$ and $\Gamma = \text{Gal}(T/F)$. Let $\xi_\sigma$ be the $p$-central character of $\sigma$.

1. The restriction $\sigma_T$ of $\sigma$ to $W_T$ is irreducible of Heisenberg type.

2. The alternating form $h_\sigma$ is invariant under $\Gamma$,

$$h_\sigma(\gamma u, \gamma v) = h_\sigma(u, v), \quad u, v \in \Pi, \gamma \in \Gamma.$$ 

3. The is a nondegenerate pairing $\mathbb{F}_p$-representation of $\Gamma$ afforded by $\Pi$ is anisotropic and faithful.

Proof. The proposition summarizes [23] Theorem 2.2, Theorem 4.1. □

We shall refer to the extension $T/F$ as the imprimitivity field of $\sigma$. 
4.3. It will be useful to have an external description of the imprimitivity field $T/F$ arising in 4.2.

**Proposition.** Let $\sigma \in \hat{W}_F$ be primitive of dimension $p^r$. Let $T/F$ be the imprimitivity field of $\sigma$.

1. The group $D(\sigma_T)$ has order $p^{2r}$.
2. If $L/F$ is a finite tame extension, then $|D(\sigma_L)| \leq p^{2r}$. Moreover, $|D(\sigma_L)| = p^{2r}$ if and only if there exists an $F$-embedding $T \to L$.

In particular, $T/F$ is the unique minimal tame extension for which $D(\sigma_T)$ has order $p^{2r}$.

**Proof.** Part (1) has been noted in 4.2, and the first assertion of (2) comes from 4.1 Lemma. Next, let $K/F$ be a finite, Galois, tame extension containing both $L$ and $T$. By 4.1 Proposition and part (1), the canonical map $D(\sigma_T) \to D(\sigma_K)$ is bijective while $D(\sigma_L) \to D(\sigma_K)$ is injective. The latter identifies $D(\sigma_L)$ with the group of $\text{Gal}(K/L)$-fixed points in $D(\sigma_K)$. However, if $\phi \in D(\sigma_K)$, the $\text{Gal}(K/F)$-isotropy group of $\phi$ contains $\text{Gal}(K/T)$, whence the result follows. □

5. The Langlands parameter

Let $\pi \in \Gamma_{Ap^r}(F)$, for some $r \geq 1$, and set $\sigma = \iota \pi \in \Gamma_{Ap^r}(F)$. We use the representation $\pi$ to identify the imprimitivity field $T/F$ of $\sigma$, and the group $D(\sigma_T)$ of characters $\chi$ of $W_T$ such that $\chi \otimes \sigma_T \sim \sigma_T$. As in 4.2 Lemma, the group of characters $D(\sigma_T)$ determines the p-kernel field $E/T$ of $\sigma$.

5.1. We describe $\pi$, following the outline of 2.1. We therefore choose a character $\psi_F$ of $F$ such that $c(\psi_F) = -1$. To get clean results, it will be necessary to impose the normalization

$$\psi_F(\zeta^p) = \psi_F(\zeta), \quad \zeta \in \mu_F,$$

although it will not be invoked until a late stage (5.14 below).

There is an m-simple stratum $[\alpha, 0, \alpha]$ in $A = M_{p^r}(F)$, with $e_\alpha = p^r$, such that $\pi$ contains the character

$$\theta_\alpha : 1 + x \mapsto \psi_A(\alpha x), \quad 1 + x \in U^1_a,$$

where $\psi_A = \psi_F \circ \text{tr}_A$.

**Theorem.** The imprimitivity field $T$ of $\sigma$ is the splitting field over $F$ of the polynomial

$$X^{p^{2r} - 1} - (-1)^p \det \alpha^{p^r - 1}.$$  

The non-trivial elements of $D(\sigma_T)$ are the characters $\Delta_c$ of $T^\times$ such that

$$\Delta_c = 1, \quad \Delta_c|_{\mu_T} = 1, \quad \Delta_c(\det \alpha) = 1,$$

and

$$\Delta_c(1 + y) = \psi_T(cy), \quad 1 + y \in U^1_T,$$

where $c$ ranges over the roots of the polynomial (5.1.3) and $\psi_T = \psi_F \circ \text{Tr}_{T/F}$. 


5.2. The proof of the theorem will occupy the rest of the section, but first we draw some conclusions.

The conditions (5.1.4), (5.1.5) determine the character $\Delta_c$ uniquely. The representation $\pi$ determines only the coset $(\det \alpha)U_1^F$ (2.2). Changing $\det \alpha$ within that coset changes neither $T$ (as a subfield of $\bar{F}$) nor the group $D(\sigma_T)$. Let $E/F$ be the $p$-kernel field of $\sigma$. As in 4.2 Lemma, the abelian extension $E/T$ is given by the relation

$$N_{E/T}(E^\times) = \bigcap_c \text{Ker} \Delta_c,$$

with $c$ ranging over the roots of the polynomial (5.1.3). Thus $E$ is also determined by the coset $(\det \alpha)U_1^F$.

In the other direction, the datum $(T/F, D(\sigma_T))$ does not fully determine the coset $(\det \alpha)U_1^F$.

**Corollary.** Let $\pi_i \in \mathfrak{A}_p^r(F)$ contain the simple stratum $[a, 1, 0, \alpha_i]$, $i = 1, 2$. Let $E_i/F$ be the $p$-kernel field of $\sigma_i = \mathfrak{A}_p^r$ and $T_i/F$ its imprimitivity field. The following conditions are equivalent:

1. $E_1 = E_2$;
2. $T_1 = T_2$;
3. $\det \alpha_1 \equiv \zeta \det \alpha_2 \pmod{U_1^F}$, for some $\zeta \in F^\times$ such that $\zeta^p = \zeta$.

**Proof.** Suppose first that $E_1 = E_2$. Therefore $T_1 = T_2$ and comparison of the polynomials (5.1.3) defining the two fields yields condition (3). Supposing that (3) holds, 5.1 Theorem implies $T_1 = T_2 = T$, say, and $D(\sigma_{1,T}) = D(\sigma_{2,T})$. The relation (5.2.1) implies $E_1 = E_2$, as required for (1). \(\Box\)

5.3. We start the proof of 5.1 Theorem with a preliminary estimate of the imprimitivity field $T/F$.

**Theorem.** Let $\sigma \in \mathfrak{A}_p^r(F)$, $r \geq 1$, and let $T/F$ be its imprimitivity field.

1. The field $T$ satisfies $e(T/F) = 1+p^r$ and it contains a root of unity of order $p^{2r}-1$.
2. If $\chi \in D(\sigma_T)$ and $\chi \neq 1$, then $\text{sw}(\chi) = 1$.

**Proof.** The proof takes until the end of the next sub-section. In this one, we find an upper bound for $e(T/F)$. Set $\Pi = \text{Gal}(E/T)$ and $\Gamma = \text{Gal}(T/F)$.

**Proposition.** The $\mathbb{F}_p\Gamma$-representation afforded by $\Pi$ is irreducible.

**Proof.** Suppose otherwise. From [23] Theorem 12.2 we deduce the existence of $\sigma_1, \sigma_2 \in \hat{W}_F$ such that $\dim \sigma_i = p^{r_i}$, with $r_2 \geq r_1 \geq 1$, and $\sigma \cong \sigma_1 \otimes \sigma_2$. Let $\pi = l\sigma$, $\pi_i = l\sigma_i$, and let $\varepsilon(\pi_1 \times \pi_2, s, \psi)$ be the local constant of the pair $(\pi_1, \pi_2)$, in the sense of Jacquet, Piatetskii-Shapiro and Shahidi [22], Shahidi [29]. This takes the form

$$\varepsilon(\pi_1 \times \pi_2, s, \psi) = q^{-sp} \langle \mathfrak{F}(\pi_1, \pi_2) + c(\psi) \rangle \varepsilon(\pi_1 \times \pi_2, 0, \psi),$$

for a certain rational number $\mathfrak{F}(\pi_1, \pi_2)$ worked out in [6], [13]. We have

$$\varepsilon(\sigma, s, \psi) = \varepsilon(\pi_1 \times \pi_2, s, \psi),$$
so \( \text{sw}(\sigma) = p^r \overline{\mathfrak{f}}(\hat{\pi}_1, \pi_2) - p^r \). In present circumstances, \( \hat{\pi}_1 \) cannot be of the form \( \chi \pi_2 \) for an unramified character \( \chi \) of \( F^\times \), since that would imply \( \sigma_1 \otimes \sigma_2 \) reducible. So, from [6] 2.1 Proposition and Corollary, we obtain
\[
\overline{\mathfrak{f}}(\hat{\pi}_1, \pi_2) > 1 + \frac{c}{p^{2r_1}},
\]
for a certain integer \( c \geq 1 \). In particular, \( \text{sw}(\sigma) > p^{r_2-r_1}c \geq 1 \). That is, \( \text{sw}(\sigma) > 1 \), contrary to hypothesis. \( \square \)

**Corollary.** The ramification index \( e(T|F) \) divides \( 1+p^r \).

*Proof.* Let \( K/F \) be the maximal unramified sub-extension of \( T/F \). The representation \( \sigma_K = \sigma|_W \) satisfies the same hypotheses as \( \sigma \): it is irreducible, \( \dim \sigma_K = p^r \) and \( \text{sw}(\sigma_K) = 1 \). It is therefore primitive and the proposition applies to it unchanged. In other words, we may assume that the Galois tame extension \( T/F \) is *totally ramified*. The cyclic group \( \Gamma = \text{Gal}(T/F) \) thus admits an irreducible, faithful, anisotropic, symplectic \( \mathbb{F}_p \)-representation of dimension \( 2r \). By [10] 3.3, \( |\Gamma| \) divides \( 1+p^r \), as required. \( \square \)

*Remark.* The bound \( e(T|F) \leq 1+p^r \) applies to any primitive representation of \( W_F \) of dimension \( p^r \), irrespective of the value of the Swan conductor [19]. We have used this more demanding technique for its wider interest.

**5.4.** We continue the proof of 5.3 Theorem.

**Lemma.** Let \( K/F \) be a finite tame extension, and set \( e_K = e(K|F) \). If there exists a non-trivial character \( \chi \) of \( W_K \) such that \( \chi \otimes \sigma_K \cong \sigma_K \), then \( e_K \geq 1+p^r \).

*Proof.* The representation \( \sigma_K \) satisfies \( \text{sw}(\sigma_K) = e_K \). If \( \chi \) is a character of \( W_K \) with \( \text{sw}(\chi) = s \geq 0 \), then \( \text{sw}(\chi \otimes \sigma_K) \leq \max\{e_K, p^r s\} \), with equality if \( e_K \neq p^r s \) [19] 3.6. A relation \( \chi \otimes \sigma_K \cong \sigma_K \) implies first that \( s \geq 1 \) (cf. 4.3) and second that \( p^r s \leq e_K \). Since \( p \) does not divide \( e_K \), we get \( e_K > p^r \), as required. \( \square \)

Applying the lemma to the case \( K = T \) and recalling 5.3 Corollary, we get \( e_T = e(T|F) = 1+p^r \). Also, if \( \chi \in D(\sigma_T) \) is non-trivial and has Swan conductor \( s \), the argument of the preceding proof gives \( sp^r < 1+p^r \), whence \( s = 1 \).

Finally, since \( T/F \) is Galois with ramification index \( 1+p^r \), surely \( T \) contains a primitive \( 1+p^r \)-th root of unity. That is, if \( \sqrt[p^r]{t} = p^d \), say, then \( 1+p^r \) divides \( p^d-1 \). An elementary argument shows that \( t \) is divisible by \( 2r \), whence follows the only remaining assertion of 5.3 Theorem. \( \square \)

**5.5.** We start the proof of 5.1 Theorem. Let \( K/F \) be a finite Galois extension with \( e(K|F) = 1+p^r \): in particular, \( K/F \) is tamely ramified. The representation \( \sigma_K \) is irreducible, so we define \( \pi_K \in \mathcal{A}_p(K) \) by \( l\pi_K = \sigma_K \). A character \( \chi \) of \( K^\times \) then lies in \( D(\sigma_K) \) if and only if \( \chi \pi_K \cong \pi_K \).

We therefore approach 5.1 Theorem via the representation \( \pi_K \). Following 4.3 Proposition, we have to find the least extension \( K/F \) for which the equation \( \chi \pi_K \cong \pi_K \) has \( p^{2r} \) solutions \( \chi \).

We describe \( \pi_K \) in terms of simple strata and simple characters. The field extension \( P = F[\alpha]/F \) is totally wildly ramified, of degree \( p^r \). We set \( B = \mathcal{M}_{p^r}(K) = K \otimes_F A \). The \( K \)-algebra \( K \otimes_F P \) is a field. We denote it \( KP \) and identify it with a subfield of \( B \). Let \( b \) be the unique
We collect some technical results. In the context of part (2)(a) of the proposition, Remark. Let $\phi$ be a character of $K^\times$, $\phi \neq 1$, such that $\phi \pi_K \cong \pi_K$. Viewing $\phi$ as a character of $W_K$ via class field theory, we get $\phi \otimes \sigma_K \cong \sigma_K$. We argue as in 5.4 to conclude that $\text{sw}(\phi) = 1$. Consequently, there is an element $c \in K$, with $\nu_K(c) = -1$, such that $\phi(1+y) = \psi_K(cy)$, $y \in \mathfrak{p}_K$.

**Proposition.** Let $c \in K$, $\psi_K(c) = -1$.

1. The quadruple $[b, 1+p', 0, \alpha+c]$ is an $m$-simple stratum in $B$ and

\[ H^1(\alpha+c, b) = H^1(\alpha, b). \]

2. Let $\phi$ be a character of $K^\times$ such that $\phi(1+y) = \psi_K(cy)$, $y \in \mathfrak{p}_K$. Let $\Phi$ denote the character $\phi \circ \det_B |_{H^1(\alpha, b)}$.

   (a) The character $\Phi \phi^K_\alpha$ lies in $\mathcal{C}(b, \alpha+c, \psi_K)$ and is contained in $\phi \pi_K$.

   (b) If $\phi \pi_K \cong \pi_K$, the simple characters $\phi^K_\alpha$, $\Phi \phi^K_\alpha$ are conjugate in $\text{GL}_{p'}(K)$.

   (c) If the simple characters $\phi^K_\alpha$, $\Phi \phi^K_\alpha$ are conjugate in $\text{GL}_{p'}(K)$, there is a unique character $\phi'$ of $K^\times$ that agrees with $\phi$ on $U^1_K$ and such that $\phi' \pi_K \cong \pi_K$.

**Proof.** Assertion (1) is immediate from the definitions. Part (2)(a) is an instance of [15] Appendix. Part (2)(b) is given by [14] 8.4 (or [11] Corollary 1 for this formulation). In part (2)(c), the totally ramified representations $\pi_K$, $\phi \pi_K$ contain the same $m$-simple character and so $\phi \pi_K \cong \chi \pi_K$, for a tamely ramified character $\chi$ of $K^\times$. The result therefore holds with $\phi' = \chi^{-1} \phi$. The uniqueness of $\phi'$ is given by 4.1 Proposition.

**Remark.** In the context of part (2)(a) of the proposition, $\mathcal{C}(b, \alpha+c, \psi_K) = \Phi \mathcal{C}(b, \alpha, \psi_K)$.

5.6. We collect some technical results.
Lemma 1. If $d(KP|K)$ is the differential exponent of $KP/K$, then $d(KP|K) \geq 2p^r$.

Proof. The extension $P/F$ is totally wildly ramified, so $d(P|F) \geq [P:F] = p^r$. The transitivity property of the different gives us first

$$d(KP|F) = d(KP|P) + (1+p^r)d(P|F) \geq p^r + (1+p^r)p^r = 2p^r + p^{2r},$$

and second $d(KP|F) = d(KP|K) + p^{2r}$. It follows that $d(KP|K) \geq 2p^r$, as required. □

Let $s_K : B \to KP$ be a tame corestriction on $B$, relative to $KP/K$ (see [14] (1.3.3) for the definition).

Lemma 2. For any integer $t$, we have $s_K(p^{t}KP) \subset p^{1+p^r+t}$. 

Proof. This follows from Lemma 1 and [14] (1.3.8). □

5.8. We take $c \in K$ with $\nu_K(c) = -1$. We compare the m-simple strata $[b,1+p^r,0,\alpha]$, $[b,1+p^r,0,\alpha+c]$.

Proposition. There exists $x \in q$ such that

$$(5.8.1) (1+x)^{-1} \alpha(1+x) \equiv \alpha+c \pmod b.$$ 

Proof. Let $\mathbb{A} : B \to B$ be the map $x \mapsto \alpha x \alpha^{-1} - x$. This fits into a long exact sequence [14] 1.4

$$\cdots \to B \xrightarrow{\mathbb{A}} B \xrightarrow{s_K} B \xrightarrow{\mathbb{A}} B \xrightarrow{\mathbb{A}} \cdots.$$ 

Set $\delta = c^{-1}\alpha$: thus $\delta^{-1}$ is a prime element of $KP$ and $\delta^{-1}b = q$. Multiplying the congruence (5.8.1) on the left by $c^{-1}(1+x)$, we see it is equivalent to

$$(5.8.2) \mathbb{A}(x) - x\delta^{-1} \equiv \delta^{-1} \pmod{q^{p^r}}.$$ 

Let $i,j$ be integers, with $1 \leq j \leq p^r$. Since $p_Kb = q^{p^r}$, the quotient $V_{i,j} = q^i/q^{i+j}$ is a $k_K$-vector space. Each of the maps $s_K$, $\mathbb{A}$ induces a $k_K$-endomorphism of $V_{i,j}$, for which we use the same notation. Since $\alpha$ is minimal over $K$, we have an infinite exact sequence [14] (1.4.7), (1.4.15),

$$(5.8.3) \cdots \to V_{i,j} \xrightarrow{\mathbb{A}} V_{i,j} \xrightarrow{s_K} V_{i,j} \xrightarrow{\mathbb{A}} V_{i,j} \to \cdots.$$ 

The subspace $s_K(V_{i,j})$ is the natural image of $p^{i}_{KP}/p^{i+j}_{KP}$ in $V_{i,j}$.

Lemma.

(1) The endomorphism $\mathbb{A}$ of $V_{i,j}$ is nilpotent such that $\mathbb{A}^{p^r} = 0$ and

$$\mathbb{A}^{p^r-1}(V_{i,j}) = s_K(V_{i,j}) = p^{i}_{KP}/p^{i+j}_{KP} \neq 0.$$ 

(2) There is a unit $u \in U_{KP}$ such that

$$\mathbb{A}^{p^r-1}(v) = us_K(v), \quad v \in V_{i,j}.$$
We apply the lemma to the case \( V = V_{i,p^r-1} = q/q^{p^r} \). We define another \( k_K \)-endomorphism \( \mathbb{B}_c \) of \( V \) by
\[
\mathbb{B}_c(v) = \mathbb{A}(v) - v\delta^{-1}, \quad v \in V.
\]
To prove the proposition, we have to show that the equation \( \mathbb{B}_c(x) = \delta^{-1} \) has a solution \( x = x_c \) in \( V \). To do this, we abbreviate \( m = p^r - 2 \) and define
\[
X_c(z) = \mathbb{A}^m(z) + \mathbb{A}^{m-1}(z)\delta^{-1} + \mathbb{A}^{m-2}(z)\delta^{-2} + \ldots + \mathbb{A}(z)\delta^{1-m} + z\delta^{-m}, \quad z \in V,
\]
so that
\[
\mathbb{B}_c(X_c(z)) = \mathbb{A}^{m+1}(z) - z\delta^{-(m+1)} = \mathbb{A}^{m+1}(z)
\]
in \( V \). By the lemma, we may choose \( z \in V \) such that \( \mathbb{A}^{m+1}(z) = \mathbb{A}^{p^r-1}(z) = \delta^{-1} \), and then \( x = X_c(z) \) provides the desired solution to the congruence (5.8.2). \( \square \)

5.9. We gather the threads for the next phase of the proof of 5.1 Theorem.

Let \( c \in K \), \( \nu_K(c) = -1 \) and let \( x = x_c \in q \) satisfy
\[
(1+x)^{-1}\alpha(1+x) \equiv \alpha+c \pmod{b},
\]
as in 5.8 Proposition. The formula \( \psi_{K,c} : 1+t \mapsto \psi_K(ct) \) defines a character of \( U^1_K \). This gives a character
\[
\chi_c : h \mapsto \psi_{K,c}(\det_B h)
\]
of \( H^1(\alpha, b) \). As in 5.6 Remark, \( \mathcal{C}(b, \alpha+c, \psi_K) = \mathcal{C}(b, \alpha, \psi_K) \chi_c \).

**Lemma.** The element \( 1+x \) normalizes \( H^1(\alpha, b) = H^1(\alpha+c, b) \) and conjugation by \( 1+x \) gives a bijection
\[
\mathcal{C}(b, \alpha, \psi_K) \longrightarrow \mathcal{C}(b, \alpha+c, \psi_K),
\]
\[
\vartheta \mapsto \vartheta^{1+x}.
\]

There exists \( \vartheta_c \in \mathcal{C}(b, \alpha, \psi_K) \) such that \( (\theta^K_c)^{1+x} = \vartheta_c \chi_c \).

**Proof.** The equality of \( H^1 \)-groups has been noted in 5.6 Proposition, whence the other assertions follow. \( \square \)
The next step in the proof of 5.1 Theorem is to find all elements \( c \) for which \( \vartheta_c = \theta^K_\alpha \), that is,

\[
(\theta^K_\alpha)^{1+x_c} = \theta^K_\alpha \chi_c.
\]

For such an element \( c \), 5.6 Proposition gives a character \( \Delta_c \) of \( K^\times \), extending \( \psi_{K,c} \), such that \( \Delta_c \pi_K \cong \pi_K \).

5.10. We have to compute the quantity

\[
\theta^K_\alpha ((1+x)h(1+x)^{-1}), \quad h \in H^1(\alpha, b) = U^1_K p U^1 b U^{1+[1+p^r/2]},
\]

where \( x = x_c \). The definition of \( x_c \) yields

\[
(5.10.1) \quad \theta^K_\alpha ((1+x)h(1+x)^{-1}) = \theta^K_\alpha (h) \chi_c (h), \quad h \in U^1 b U^{1+[1+p^r/2]}.
\]

It is therefore enough to consider elements \( h = 1+y \in U^1_K p \).

**Proposition.** If \( y \in p_K p \) and \( x = x_c \), then \( (\theta^K_\alpha)^{1+x}(1+y) = \psi_B (-cxy) \).

**Proof.** We need a preliminary calculation.

**Lemma.** If \( y \in p_K p \), then

\[
(1+x)(1+y)(1+x)^{-1} \equiv 1+v \pmod{U^1 b U^{1+p^r}},
\]

where \( v \in p_K p \) and \( v \equiv y \pmod{U^2_K p} \).

**Proof.** We write \( [x,y] = xy - yx \), so that

\[
1+t = (1+x)(1+y)(1+x)^{-1} = 1 + y + [x,y](1+x)^{-1}.
\]

Surely \( t \equiv y \pmod{q^2} \), so the result reduces to showing \( at - ta \in \alpha q^{1+p^r} = b \). We use two forms of the defining relation for \( x \), namely

\[
\alpha x - xa \equiv c(1+x) \pmod{b},
\]

\[
(1+x)^{-1} \alpha \equiv (\alpha+c)(1+x)^{-1} \pmod{b}.
\]

Expanding,

\[
at - ta = \alpha [x,y](1+x)^{-1} - [x,y](1+x)^{-1} \alpha
\]

\[
= (\alpha [x,y] - [x,y](\alpha+c))(1+x)^{-1}
\]

\[
\equiv \alpha [x,y] - [x,y](\alpha+c) \pmod{b}.
\]

We recall that \( c \) is central and that \( y \) commutes with \( \alpha \). Expanding the commutators, the defining relation yields \( at - ta \equiv 0 \pmod{b} \), as required. \( \square \)

We use the lemma to write

\[
(1+x)(1+y)(1+x)^{-1} = 1+v+h, \quad h \in q^{1+p^r}, \quad v \in p_K p,
\]
with \( v \equiv y \pmod{p_{KP}^2} \). Thus
\[
(\theta^K)_{\alpha}^{1+x}(1+y) = \theta^K_{\alpha}(1+y) \psi_B(\alpha h).
\]
We have \( \theta^K_{\alpha}(1+y) = 1 = \psi_B(\alpha y) \) by (5.5.2), 5.7 Lemma 2, respectively, so
\[
(\theta^K)_{\alpha}^{1+x}(1+y) = \psi_B(\alpha(v+h)), \quad \text{and} \quad v+h = y + [x,y](1+x)^{-1}.
\]
We have to compute
\[
\psi_B(\alpha[x,y](1+x)^{-1}) = \psi_B\left(\sum_{i \geq 0} (-1)^i \alpha(xy-yx)x^i\right).
\]
We expand the inner sum using the symmetry properties of the trace and the defining relation for \( x \) in the form
\[
[\alpha, x] \equiv c(1+x) \pmod{b}.
\]
The term \( i = 0 \) contributes \( \psi_B(\alpha(xy-yx)) = 1 \), since \( y \) commutes with \( \alpha \). The general term \( i \geq 1 \) gives
\[
\psi_B((-1)^i \alpha(xy-yx)x^i) = \psi_B((-1)^i x^i[\alpha, x]y) = \psi_B((-1)^i c(x^i + x^{i+1})).
\]
In all, we get
\[
\psi_B(\alpha[x,y](1+x)^{-1}) = \psi_B(-cxy),
\]
or \( (\theta^K)_{\alpha}^{1+x}(1+y) = \psi_B(-cxy) \), as required. \( \Box \)

**5.11.** We examine the character \( \eta_c : 1+y \mapsto \psi_B(-cxy) \) of \( U_{1,KP}^1 \) arising in 5.10 Proposition. We use the map \( \mathcal{A} \) of 5.8 and let \( s^K_{\alpha} \) be a tame corestriction on \( B \), relative to \( KP/K \), such that
\[
s^K_{\alpha}(t) \equiv \mathcal{A}^{p^{r-1}}(t) \pmod{q^{p^r}}, \quad t \in q,
\]
(cf. 5.8 Lemma (2)).

**Lemma.** There is a unique character \( \epsilon \) of \( KP \) such that \( \epsilon(\epsilon) = -1 \) and
\[
(5.11.1) \quad \psi_B(b) = \epsilon(s^K_{\alpha}(b)), \quad b \in B.
\]
The character \( \eta_c \) then satisfies
\[
(5.11.2) \quad \eta_c(1+y) = \epsilon(-\alpha^{1-p^r}c^{p^r}y), \quad y \in \mathfrak{p}_{KP}.
\]
Consequently, \( \eta_c \) is trivial on \( U_{1,KP}^1 \).

**Proof.** The first assertion is given by [14] (1.3.7).

We return to the construction of \( x_c \) in 5.8. As there, we put \( \delta = c^{-1} \alpha \). We choose \( z \in q \) such that \( \mathcal{A}^{p^{r-1}}(z) \equiv \delta^{-1} \pmod{q^{p^r}} \); this yields \( x_c = X_c(z) \). The choice of \( s^K_{\alpha} \) then gives
\[
s^K_{\alpha}(cx_c \equiv c\delta^{1-p^r} \pmod{b},
\]
while \( c\delta^{1-p^r} = \alpha^{1-p^r}c^{p^r} \). The lemma now follows from the relation (5.11.1). \( \Box \)
5.12. We compare the characters \( \eta_c, \chi_c \) on \( U^1_{KP} \). Both are trivial on \( U^2_{KP} \) and, for \( y \in \mathfrak{p}_{KP} \),
\[
\chi_c(1+y) = \psi_{K,c}(N_{KP/K}(1+y)).
\]
We may take \( y = \zeta \alpha^{-1} \), for some \( \zeta \in \mu_{KP} = \mu_K \).

**Lemma.** If \( y = \zeta \alpha^{-1}, \zeta \in \mu_K \), then
\[
N_{KP/K}(1+y) \equiv 1 + \zeta^{p^r} \phi^\gamma \det \alpha^{-1} \pmod{U^2_K}.
\]

*Proof.* Nothing is changed if we replace \( \alpha \) by \( \alpha u, u \in U^1_a \). We may therefore assume (as in the proof of 2.2 Proposition) that \( \alpha^{-p^r} \) is a prime element of \( F \). Therefore \( (\alpha^{-1})^p \) is a prime element of \( K \). We may choose a matrix representation so that \( \alpha^{-1} \) is a monomial matrix as in 2.2. We may re-write the identity to be proved as
\[
\det(1+y) \equiv 1 + \det(\zeta \alpha^{-1}) \pmod{U^2_K}.
\]
The lemma is then given by an elementary calculation. \( \square \)

It follows that
\[
\chi_c(1+y) = \psi_K(\zeta^{p^r} c^{1+p^r} \det \alpha^{-1}), \quad y = \zeta \alpha^{-1}.
\]
In these terms, (5.11.2) says
\[
\eta_c(1+y) = \epsilon((-1)^{p^r} c^{1+p^r} \zeta \det \alpha^{-1}), \quad y = \zeta \alpha^{-1}.
\]

5.13. We relate the characters \( \psi_K, \epsilon \).

**Lemma.** The characters \( \epsilon, \psi_K \) satisfy \( \epsilon(\zeta) = \psi_K(\zeta) \), for all \( \zeta \in \mu_K \).

*Proof.* We continue in the situation of the proof of 5.12 Lemma. For \( \zeta \in \mu_K \), we let \( \zeta_0 \) be the matrix with \( \zeta \) in the \((1,1)\)-place and with all other entries zero. The matrix \( K^{p^r-1}(\zeta_0) \) is a scalar matrix whose last entry is \( \zeta \), whence the lemma follows. \( \square \)

5.14. Let \( \zeta_c \in \mu_K \) satisfy \( \zeta_c = c^{1+p^r} \det \alpha^{-1} \pmod{U^1_{KP}} \). Combining (5.12.1), (5.12.2), we have to solve the equation
\[
\psi_K((-1)^{1+p^r} \zeta_c) = \psi_K(\zeta_c \zeta^{p^r}), \quad \zeta \in \mu_K,
\]
for \( c \in K, v_K(c) = -1 \). We have \( \psi_K(\gamma^{p^r}) = \psi_K(\gamma) \), for \( \gamma \in \mu_K \) (5.5.1). We must therefore solve
\[
(-1)^{p^r} \zeta_c^{p^r} = \zeta_c, \text{ that is, } c^{1-p^r} = (-1)^p. \]
This translates back to
\[
c^{1-p^r} \equiv (-1)^p \det \alpha^{1-p^r} \pmod{U^1_K}.
\]
This congruence admits \( p^{2r} - 1 \) solutions in \( K^\times /U^1_K \) if and only if the field \( K \) contains the splitting field of the polynomial (5.1.3).

Let \( K \) be this splitting field and let \( c \in K \) be a root of the polynomial (5.1.3). Let \( \phi_c \) be a character of \( K^\times \) such that \( \phi_c(1+t) = \psi_K(ct), t \in \mathfrak{p}_K \). The representations \( \pi_K, \phi_c \pi_K \) both contain the simple character \( \theta^\alpha_{K} \). As in 5.6 Proposition, there is a unique character \( \Delta_c \) of \( K^\times \), agreeing with \( \phi_c \) on \( U^1_K \), such that \( \Delta_c \pi_K \cong \pi_K \). That is, \( \Delta_c \in D(\pi_K) \). Surely \( K \) is minimal for the property \( |D(\sigma_K)| = p^{2r} \), so \( K = T \), as desired.

This character \( \Delta_c \) satisfies (5.1.5) by construction. By (5.1.5), \( \Delta_c^p \) is tamely ramified and so trivial, by 4.1 Proposition. The second property in (5.1.4) expresses the fact that \( \pi_K, \Delta_c \pi_K \) have the same central character, while the third follows from the relation \( \epsilon(\Delta_c \pi_K, s, \psi_T) = \epsilon(\pi_K, s, \psi_T) \).

We have completed the proof of 5.1 Theorem. \( \square \)
6. THE P-CENTRAL CHARACTER

As in §5, $\pi \in 1^r_1(F)$ contains the simple stratum $[a,1,0,\alpha]$, relative to the character $\psi_F$ of (5.1.1). From 5.1 Theorem, the primitive representation $\sigma = L^r_\pi$ has $p$-kernel field $E/F$ and imprimitivity field $T/F$. We investigate the p-central character $\xi$ of $\sigma$ to complete our examination of the representation $\sigma$.

6.1. Let $F'/F$ be the maximal unramified sub-extension of $T/F$. Thus $T/F'$ is cyclic and totally ramified of degree $1+p^r$. We set

$$\Pi = \text{Gal}(E/T) \cong (\mathbb{Z}/p \mathbb{Z})^{2r},$$
$$\Gamma = \text{Gal}(T/F') \cong \mathbb{Z}/(1+p^r)\mathbb{Z},$$
$$\Theta = \text{Gal}(E/F').$$

The extension $E/T$ is totally wildly ramified, so there exists $\beta_E \in E^\times$ such that $N_{E/T}(\beta_E) \equiv \det \alpha \pmod{U^1_1}$. This condition determines the coset $\beta_EU^1_1$ uniquely. We write $\psi_E = \psi_F \circ \text{Tr}_{E/F}$.

**Theorem.** The character $\xi$ has the following properties:

1. $\xi$ is fixed under conjugation by $\Theta$,
2. $\text{sw}(\xi) = 1+p^r$, and
3. if $\beta_E \in E$ satisfies $N_{E/T}(\beta_E) \equiv \det \alpha \pmod{U^1_1}$, then
   $$\xi(1+x) = \psi_E(\beta_E^p x), \quad 1+x \in U^{1+p^r}_1.$$

The properties (1)–(3) determine the character $\xi|_{U^1_1}$ uniquely.

We give a more complete formula for $\xi|_{U^1_1}$ in (6.7.2) below. We note the following refinement of 5.2 Corollary.

**Corollary.** Let $\pi_i \in 1^r_1(F)$ contain the m-simple stratum $[a,1,0,\alpha]$, $i = 1,2$. Suppose that the representations $\sigma_i = L^r_\pi_i$ have $p$-kernel field $E$. If $\xi_i$ is the p-central character of $\sigma_i$, the following conditions are equivalent.

1. $\xi_1|_{U^1_1} = \xi_2|_{U^1_1}$;
2. $\xi_1|_{U^{1+p^r}_1} = \xi_2|_{U^{1+p^r}_1}$;
3. $\det \alpha_1 \equiv \det \alpha_2 \pmod{U^1_1}$.

The proofs occupy the rest of the section.

6.2. Before starting, we show how 6.1 Theorem completes our picture of the representation $\sigma$. We use the viewpoint of [12] §1. According to the Ramification Theorem of local class field theory, the Artin Reciprocity map $\mathcal{W}_E \to E^\times$ maps the wild inertia subgroup $\mathfrak{P}_E$ onto $U^1_1$. From this point of view, the restricted character $\xi|_{U^1_1}$, determined in 6.1 Theorem, gives a character $\zeta$ of $\mathfrak{P}_E$. (If we think of $\xi$ as a character of $\mathcal{W}_E$, then $\zeta = \xi|_{\mathfrak{P}_E}$.)

**Corollary.**

1. The representation $\rho = \sigma|_{\mathfrak{P}_E}$ is irreducible. It is the unique irreducible representation of $\mathfrak{P}_E$ containing $\zeta$. 

(2) The representation $\sigma$ is the unique element of $\text{Ind}^{W_F}_{\mathbb{I}}(F)$ with the following properties:

(a) $\sigma|_{\mathbb{I}} \cong \rho_\sigma$,
(b) $\det \sigma = \omega_\pi$, and
(c) $\varepsilon(\sigma, \frac{1}{2}, \psi_F) = \varepsilon(\pi, \frac{1}{2}, \psi_F)$.

Proof. The representation $\rho_\sigma$ is surely irreducible and contains $\zeta_\sigma$. The relation $\text{Ind}^{W_F}_{\mathbb{I}}(F)$ of 4.2 Lemma implies $\text{Ind}^{W_F}_{\mathbb{I}}(F) = \rho_\sigma$, whence (1) follows.

That $\sigma$ has the listed properties follows from the definition of $\rho_\sigma$ and 2.3 Proposition. If $\sigma'$ is some other representation of $W_F$ extending $\rho_\sigma$, 1.3 Proposition of [12] asserts that $\sigma' \cong \chi \otimes \sigma$, for a unique tamely ramified character $\chi$ of $W_F$. If $\det \sigma' = \chi^{p^r}$, then $\chi^{p^r} = 1$ and consequently $\chi$ is unramified. So, by 2.3 Lemma and Proposition,

$\varepsilon(\sigma', \frac{1}{2}, \psi_F) = \chi(\det \alpha)^{-1} \varepsilon(\sigma, \frac{1}{2}, \psi_F)$.

Since $v_F(\det \alpha) = -1$, a relation $\varepsilon(\sigma', \frac{1}{2}, \psi_F) = \varepsilon(\sigma, \frac{1}{2}, \psi_F)$ implies $\chi = 1$. □

Remark. The field extension $E/F$ and the representation $\rho_\sigma$ of $\mathbb{F}$ are determined entirely by the coset $(\det \alpha)U_F^r$ or, equivalently, by the endo-class $\Theta_\alpha$ of the simple character $\theta_\alpha$ occurring in $\pi$ (cf. 2.1). Indeed, $\Theta_\alpha$ is the endo-class corresponding to $\rho_\sigma$ via the Ramification Theorem of [12] 6.1.

6.3. We use the notation set up at start of 6.1.

A subgroup $\Xi$ of $\Pi$ is $\xi_\sigma$-Lagrangian if $[\Xi : \sigma] = p^r$ and the commutator pairing $(x, y) \mapsto \xi_\sigma[x, y]$ is null on $\Xi$. Equivalently, $\Xi$ is the image in $\Pi$ of a maximal abelian subgroup $\Xi$ of $\sigma(W_F)$. It follows that $\xi_\sigma$ extends to a character of $\Xi$ or, in terms of fields, $\xi_\sigma$ factors through the norm map $N_{E/K} : E^\times \to K^\times$ where $K = E^{\Xi}$, cf. 4.2 Lemma. We identify the dual $\hat{\Xi}$ of the abelian group $\Xi$ with the group of characters of $K^\times$ vanishing on norms from $E$.

Proposition. Let $\Xi$ be an $\xi_\sigma$-Lagrangian subgroup of $\Pi$ and set $K = E^{\Xi}$.

(1) The fields $T \subset K \subset E$ satisfy

$$d(E|T) = 2p^{2r} - 2,$$
$$d(E|K) = d(K|T) = 2p^r - 2.$$

(2) If $\phi \in \hat{\Xi}$, $\phi \neq 1$, then $\text{sw}(\phi) = 1$.

(3) The character $\xi_\sigma$ satisfies $\text{sw}(\xi_\sigma) = 1 + p^r$.

(4) If $\chi$ is a character of $K^\times$, such that $\xi_\sigma = \chi \circ N_{E/K}$, then $\text{sw}(\chi) = 2$.

Proof. Each of the extensions $E/K$, $K/T$ is totally ramified and $\Xi \cong (\mathbb{Z}/p\mathbb{Z})^r \cong \text{Gal}(K/T)$. As in 5.1, every non-trivial $\phi \in D(\sigma_T) = \hat{\Pi}$ satisfies $\text{sw}(\phi) = 1$, so the Artin conductor $a(\phi)$ is 2. By the conductor-discriminant formula [28] VI §3 Corollaire 2,

$$d(E|T) = \sum_{\phi \in D(\sigma_T)} a(\phi) = 2p^{2r} - 2.$$

Let $\Psi = \text{Gal}(K/T)$. The dual $\hat{\Psi}$ of $\Psi$ is a subgroup of $D(\sigma_T) = \hat{\Pi}$ of order $p^r$. The relation $d(K|T) = 2p^r - 2$ follows as before. Multiplicativity of the different in towers yields the final part of (1). Part (2) now follows from the conductor-discriminant formula.

In part (3), we recall that $\text{Ind}_{E/T}(\xi_\sigma = p^r \sigma_T$. Since $\text{sw}(\sigma_T) = 1 + p^r$, the result is given by part (1) and (1.3.1). Finally, $\text{Ind}_{K/T}(\chi = \sigma_T$ (4.2 Lemma) and part (4) follows similarly. □
6.4. We take $\Xi$ and $K = E^\Xi$ as before.

**Proposition.** The norm map $N_{E/K}$ induces an isomorphism

$$U_E^{1+p^r}/U_E^{2+p^r} \cong U_K^2/U_K^3,$$

and an exact sequence

$$1 \rightarrow \Xi \rightarrow U_E^1/U_E^2 \xrightarrow{N_{E/K}} U_K^1/U_K^2.$$

**Proof.** We consider the ramification subgroups $\Xi_i, i \in \mathbb{Z}, i \geq 0,$ of $\Xi = \text{Gal}(E/K)$ in the lower numbering. We recall from [28] IV §2 Proposition 4 that

$$2p^r - 2 = d(E|K) = \sum_{i \geq 0} (|\Xi_i| - 1).$$

Here we have $\Xi = \Xi_0 = \Xi_1$ and $|\Xi| = p^r$. The only conclusion is that $\Xi_2$ is trivial. The derivative of the Herbrand function $\varphi_{E/K}$ is therefore given by

$$\varphi'_{E/K}(x) = \begin{cases} 1, & 0 < x < 1, \\ p^{-r}, & 1 < x, \end{cases}$$

so the inverse Herbrand function $\psi_{E/K}$ satisfies

$$\psi'_{E/K}(x) = \begin{cases} 1, & 0 < x < 1, \\ p^r, & 1 < x. \end{cases}$$

Now we use [28] V Proposition 9. Abbreviating $\Psi = \Psi_{E/K}$, we have an exact sequence

$$(6.4.1) \quad 1 \rightarrow \Xi \Psi(i)/\Xi \Psi(i)+1 \rightarrow U_E^{\Psi(i)}/U_E^{\Psi(i)+1} \xrightarrow{N_{E/K}} U_K^i/U_K^{i+1},$$

for any integer $i \geq 1$.

Since $\Psi(2) = 1+p^r$, the norm $N_{E/K}$ induces an isomorphism $U_E^{1+p^r}/U_E^{2+p^r} \cong U_K^2/U_K^3$. Since $d(E|K) = 2p^r - 2$ (6.3), the trace $\text{Tr}_{E/K}$ induces an isomorphism $p_E^{1+p^r}/p_E^{2+p^r} \rightarrow p_K^2/p_K^3$. On the other hand, for $x \in p_E^{1+p^r}$,

$$N_{E/K}(1+x) = 1 + \text{Tr}_{E/K}(x) + R,$$

where the remainder term $R$ lies in $p_E^{2+2p^r} \cap K \subset p_K^3$.

In the second assertion, the exact sequence is the case $i = 1$ of (6.4.1). □

6.5. Again let $\Xi$ be an $\xi_\sigma$-Lagrangian subgroup of $\Pi$ and $K = E^\Xi$.

**Proposition.** Let $\chi$ be a character of $K^\times$ such that $\xi_\sigma = \chi \circ N_{E/K}$. Let $\psi_K = \psi_F \circ \text{Tr}_{K/F}$ and let $\beta_K \in K^\times$. The following conditions are equivalent:

1. $\chi(1+y) = \psi_K(\beta_K y), y \in p_K^2$;
2. $N_{K/F} \beta_K \equiv \det \alpha \pmod{U_F^1}$.
Proposition. If \( \psi_T = \psi_F \circ \text{Tr}_{T/F} \), then \( c(\psi_T) = -1 \) since \( T/F \) is tame. It follows from 6.3 Proposition and (1.3.1) that
\[
c(\psi_K) = d(K|T) + e(K|T)c(\psi_T) = p^r-2.
\]
Let \( \beta_K \in K^\times \) satisfy (1). The coset \( \beta_KU_K^\times \) is thereby uniquely determined, and \( v_K(\beta_K) = -(1+p^r) \).

Let \( X_1(T) \) denote the group of tamely ramified characters of \( T^\times \). Let \( \epsilon \in X_1(T) \) and set \( \epsilon_K = \epsilon \circ N_{K/T} \). The map \( \epsilon \mapsto \epsilon_K \) is an isomorphism \( X_1(T) \to X_1(K) \), since \( K/T \) is totally wildly ramified. The induction relation
\[
\text{Ind}_{K/T} \epsilon_K \otimes \chi = \epsilon \otimes \sigma_T,
\]
implies the local constant relation
\[
\frac{\varepsilon(\epsilon \otimes \sigma_T, s, \psi_T)}{\varepsilon(\sigma_T, s, \psi_T)} = \frac{\varepsilon(\epsilon_K \otimes \chi, s, \psi_K)}{\varepsilon(\chi, s, \psi_K)}.
\]

By 5.5 Proposition and a calculation parallel to 2.2 Lemma, the element \( \gamma_{\sigma_T} \) of 2.3 Lemma is \( \det(\alpha) \) (modulo \( U_1^1 \)). It follows that \( \epsilon(\det(\alpha)) = \epsilon_K(\beta_K) \) for all \( \epsilon \in X_1(T) \), and therefore \( \det(\alpha) = N_{K/T}(\beta_K) \) (mod \( U_1^1 \)).

Conversely, let \( \beta_K^\prime \in K^\times \) satisfy \( N_{K/T} \beta_K^\prime \equiv \det(\alpha) \) (mod \( U_1^1 \)). It follows that \( \beta_K^\prime \equiv \beta_K \) (mod \( U_1^1 \)), so \( \psi_K(\beta_K^\prime) = \psi_K(\beta_K) = \chi(1+y), y \in p_1^1 \).

We re-formulate the proposition to avoid the choice of a Lagrangian.

Corollary. There exists \( \beta_E \in E^\times \) such that \( N_{E/T} \beta_E \in (\det(\alpha))U_1^1 \). For any such element \( \beta_E \), we have
\[
\xi_\sigma(1+z) = \psi_E(\beta_E^p z), \quad z \in p_1^{1+p^r},
\]
where \( \psi_E = \psi_F \circ \text{Tr}_{E/F} \).

Proof. The first assertion is immediate. If \( \beta_K = N_{E/K} \beta_E \), then \( \det(\alpha) \equiv N_{K/T} \beta_K \) (mod \( U_1^1 \)) while \( \beta_K \equiv \beta_E^p \) (mod \( U_1^1 \)). By 6.4 Proposition,
\[
N_{E/K}(1+z) \equiv 1 + \text{Tr}_{E/K}(z) \pmod{p_1^3}, \quad z \in p_1^{1+p^r},
\]
so \( \xi_\sigma(1+z) = \psi_K(\beta_K^p \text{Tr}_{E/K}(z)) = \psi_E(\beta_K z) = \psi_E(\beta_E^p z) \), as required. \( \square \)

6.6. By definition, the character \( \xi_\sigma \) is fixed under the action of \( \Theta = \text{Gal}(E/F') \) and, by 6.3 Proposition, \( \text{sw}(\xi_\sigma) = 1+p^r \). We examine some implications of these properties.

The soluble group \( \Theta \) has order \( p^{2r}(1+p^r) \). The Hall Subgroup Theorem gives a subgroup \( \Delta \) of \( \Theta \) of order \( 1+p^r \), unique up to conjugation in \( \Theta \). If we set \( H = E^\Delta \), the extension \( E/H \) is totally tamely ramified of degree \( 1+p^r \) and so \( \Delta \) is cyclic.

Proposition. Let \( \Delta \) be a subgroup of \( \Theta \) of order \( 1+p^r \), and let \( H = E^\Delta \).

(1) Let \( \tau \) be a character of \( E^\times \) fixed by \( \Delta \). There exists a character \( \tau^H \) of \( H^\times \) such that \( \tau = \tau^H \circ N_{E/H} \). Moreover, \( \text{sw}(\tau) = (1+p^r)\text{sw}(\tau^H) \).
(2) The norm map \( N_{E/H} \) induces an isomorphism

\[
U_{1}^{1+p^r}/U_{2}^{1+p^r} \rightarrow U_{1}/U_{2},
\]

\[
1+x \mapsto 1 + \text{Tr}_{E/H}(x).
\]

(3) Let \( \tau_1, \tau_2 \) be \( \Delta \)-fixed characters of \( E^\times \) satisfying \( \text{sw}(\tau_1) = \text{sw}(\tau_2) = 1+p^r \). If \( \tau_1, \tau_2 \) agree on \( U_{1}^{1+p^r} \), then \( \tau_2 = \phi \tau_1 \), for a tamely ramified character \( \phi \) of \( E^\times \).

Proof. Parts (1) and (2) are standard properties of cyclic, tamely ramified extensions. Part (3) follows from (1) and (2). □

6.7. We interpolate an alternative description of the character \( \xi_{\sigma}|_{U_E^1} \). Taking \( \Delta \) and \( H = E^\Delta \) as in 6.6, choose \( \beta_H \in H \) so that

\[
(6.7.1) \quad N_{H/F'} \beta_H \equiv \text{det} \alpha \pmod{U_{1}^{1}}.
\]

The relation (6.7.1) determines \( \beta_H \) modulo \( U_{1}^{1} \), that is, modulo \( U_{1}^{1+p^r} \).

Let \( \psi_H = \psi_F \circ \text{Tr}_{H/F} \), and define a character \( \xi^H \) of \( U_{1}/U_{2}^r \) by

\[
\xi^H(1+x) = \psi_H(\beta^r_H x), \quad x \in \mathfrak{p}_H.
\]

Thus \( \xi^H \circ N_{E/H} \) is a character of \( U_E^1 \), fixed by \( \Delta \) and agreeing with \( \xi_{\sigma} \) on \( U_{1}^{1+p^r} \). That is,

\[
(6.7.2) \quad \xi_{\sigma}(u) = \xi^H(N_{E/H}(u)), \quad u \in U_{1}^{1},
\]

by 6.6 Proposition and 6.5 Corollary.

6.8. We prove 6.1 Theorem. Property (1) is a direct consequence of the definition of \( \xi_{\sigma} \), while (2) is given by 6.3 Proposition. We have just established (3) in 6.5 Corollary. The uniqueness assertion is given by 6.6 Proposition. □

We prove 6.1 Corollary. The equivalence of (1) and (2) is given by 6.5 Proposition, and that of (2) and (3) by 6.5 Corollary and 5.2 Corollary. □

Remark. Theorem 6.1 gives \( \xi_{\sigma} \) on \( U_E^1 \) while \( \xi_{\sigma}^p = \text{det} \sigma|_{W_E} = \omega_{\pi} \circ N_{E/F} \). Together, these relations determine \( \xi_{\sigma} \) up to an unramified factor of order dividing \( p^r \). One cannot isolate this factor via the standard technique of a local constant calculation. The character \( \psi_E = \psi_F \circ \text{Tr}_{E/F} \) has \( c(\psi_E) = p^{2r-2} \) (by (1.3.2) and 6.3 Proposition). So, if \( \chi \) is unramified and \( \chi^{p^r} = 1 \), then \( \varepsilon(\chi \xi_{\sigma}, s, \psi_E) = \varepsilon(\xi_{\sigma}, s, \psi_E) \).

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