Monge equation of arbitrary degree in 1 + 1 space

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Abstract

Solution of Monge equation of arbitrary degree \( (\frac{\partial^n U}{\partial x^n}) = W(\frac{\partial^n U}{\partial z^n}) \) is connected with solution of functional equation for 4 functions with 4 different arguments. Some number solutions of this equation is represented in explicit form.

1 Introduction

So called Monge-Amper equation degree \( n + 1 \) in 1 + 1 dimensional \( x, z \) space looks as [1]

\[
\frac{\partial}{\partial x} \left( \frac{\partial^n U}{\partial x^n} \right) \frac{\partial}{\partial z} \left( \frac{\partial^n U}{\partial z^n} \right) = \frac{\partial}{\partial y} \left( \frac{\partial^n U}{\partial x^n} \right) \frac{\partial}{\partial x} \left( \frac{\partial^n U}{\partial y^n} \right)
\]

The last equation may be considered as unity to zero Jacobian between \( \left( \frac{\partial^n U}{\partial x^n} \right), \frac{\partial^n U}{\partial z^n} \), which means their function dependence or

\[
\frac{\partial^n U}{\partial x^n} = W(\frac{\partial^n U}{\partial z^n})
\]

The last equation we will call as Monge equation of \( n \) degree with notation \( M_n \). For initial Monge Amper equation we use term \((M - A)_{n+1}\). Each solution of \( M_n \) is simultaneous solution of \((M - A)_{n+1}\). General solution of \( M_n \) depend on \( n \) functions of one argument and if it will be possible to find it for arbitrary \( W \) function then it will be general solution for \((M - A)_{n+1}\) depending on \( n + 1 \) arbitrary functions of one argument.

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2  Trivial partial solution of \((M - A)_{n+1}\)

If in \(M_n\) \(W(x) = x\) then solution is obvious

\[ U = \sum_{k=1}^{n} f_k(x + \lambda_k z), \quad \lambda_k^n = 1 \]

From this example it is clear that existence of the solution of \(M_n\) will determine function \(W\) for which solution is possible.

3  \(M_1\)

\[ U_x = W(U_z) \]

Solution is well known connected with classical Monge equation \(\lambda_z = \lambda \lambda_x\) which can rewrite in the form of zero Jacobian \((x + \lambda z)_x \lambda_z - (x + \lambda z)_z \lambda_x = 0\). The last condition means functional dependence \(x + \lambda x + \lambda z = F(\lambda)\) This is exactly general solution in implicit form of classical Monge equation.

4  \(\tilde{M}_2\)

The results of this section will be used below in sections \(M_3, M_n\). Equation \(\tilde{M}_2\) looks as

\[ U_{z,z} = W(U_{x,x}, U_{z,x}) \]

Really relation above is not the equation because 3 arbitrary functions in 1+1 are always functionally dependent. The talk is about some parametrization of functions involved in this relation.

In notations \(a = U_{z,z} = W(b, c)\), \(b = U_{x,z}\), \(c = U_{xx}\) equation \(\tilde{M}_2\) looks as system of equations \(a_x = W_b b_x + W_c c_x = b_z\), \(b_z = c_x\).

This is typical hydrodynamic system. (2). By transformation \(x = X(b, c), z = Z(b, c)\) we calculate derivatives

\[ b_x = \frac{Z_c}{D}, \quad c_x = -\frac{Z_b}{D}, \quad b_z = -\frac{X_c}{D}, \quad c_z = \frac{X_b}{D}, \quad D = Jacob(X, Z) \]

Linear system of equation above is resolved in terms of one function \(R, X = R_c, Z = R_b\)

\[ R_{cc} = -W_b R_{cb} + W_c R_{bb} \]
There are no known methods for solution of this equation under arbitrary function $W$. Below we propose some trick for finding solution of this equation together with function $W$. Let us try represent second order differential equation in factorize form

$$
\frac{\partial^2}{\partial c^2} + W_{c} \frac{\partial^2}{\partial b \partial c} - W_{c} \frac{\partial^2}{\partial b^2} = (\frac{\partial}{\partial c} + \nu_1 \frac{\partial}{\partial b})(\frac{\partial}{\partial c} + \nu_2 \frac{\partial}{\partial b})
$$

(1)

Comparison left and right sides lead to conclusion

$$\nu_1 + \nu_2 = W_{b}, \quad \nu_1 \nu_2 = -W_{c}, \quad \nu_2^2 + \nu_1^2 = 0, \quad \nu_1 + \nu_2 = 0 \quad (2)$$

In what follows always $\nu_2 \equiv \nu^2$, $\nu_1 \equiv \nu^1$. There are obvious three possibility in resolving the last system of equations, which will be considered on three subsections below.

4.0.1 \(\nu_2, \nu_1 = \text{Constants}\)

In this case $R = R_1(b - \nu_1 c) + R_2(b - \nu_2 c)$, $W = (\nu_1 + \nu_2)b - \nu_1 \nu_2 c + w(R_b)$. $x = R_b = \hat{R}_1 + \hat{R}_2$, $z = R_c = -\nu_1 \hat{R}_1 - \nu_2 \hat{R}_2$. Resolving two last equations $b - \nu_2 c = L_1(x + \nu_1 z)$, $b - \nu_1 c = L_2(x + \nu_2 z)$ or finally $c = \frac{L_2 - L_1}{\nu_2 - \nu_1}$, $c = \nu_2 L_2 - \nu_1 L_1$. In the same terms $W = \nu_2^2 L_2 - \nu_1^2 L_1 + w(x)$. Let us introduce notation

$$l^k_s = \frac{\nu_1^k L_2 - \nu_2^k L_1}{\nu_2 - \nu_1}, \quad L^k_{s}[1] \quad k\text{-derivatives of } L \text{ function by their own argument. It is obvious } (l^k_s)_x = l^{k+1}_s, \quad (l^k_s)_z = l^{k+1}_s.$$

4.0.2 \(\nu_1 = \text{Constant}, \nu_2 = \nu_2(b - \nu_1 c)\)

$$\nu_2 = \nu_2((b - \nu_1 c)), \quad R = R_1((b - \nu_1 c)) + R_2(\int_{1}^{b - \nu_1 c} ds\frac{1}{\nu_1 - \nu_2(s)} - b)$$

$$x = R_b = R_1'b + \hat{R}_2 \frac{\nu_1}{-\nu_2 - \nu_1}, \quad z = R_c = -\nu_1 R_1' + \hat{R}_2 \frac{\nu_2 \nu_1}{-\nu_2 - \nu_1}, \quad x + \nu_1 z = \nu_1 \hat{R}_2,$$

$$x + \nu_2 z = \Delta R' \equiv \Theta(\nu_2) \quad c = C(\nu_2) - L(x + \nu_1 z), \quad b = \int dv_2 \nu_2 C_{\nu_2} - \nu_1 L(x + \nu_1 z),$$

$$a = \int dv_2 \nu_2^2 C_{\nu_2} - \nu_1^2 L(x + \nu_1 z) + \theta(z),$$

$$\Theta(\nu_2) = \nu_2 L_2 - \nu_1 L_1.$$
4.0.3 \( \nu_c^2 + \nu^1 \nu_b^2 = 0, \quad \nu_c^1 + \nu^2 \nu_b^1 = 0 \)

In the system of equations in the title of this subsection let us perform transformaton \( b = B(\nu^1 \nu^2), \quad c = C(\nu^1 \nu^2) \). Absolutely by the same technique as above we pass to the system of equations with obvious solution

\[
B_{\nu^2} - \nu^2 C_{\nu^2} = 0, \quad B_{\nu^1} - \nu^1 C_{\nu^1} = 0
\]

\[\nu = C = C^1(\nu^1) + C^2(\nu^2), \quad b = B = \int dv^1 \nu^1 C_{\nu^1} + \int dv^2 \nu^2 C_{\nu^2}
\]

\[W = \int db(\nu_1 + \nu_2) - \int dc \nu_2 + w(x) = \int dv^1 (\nu^1)^2 C_{\nu^1} + \int dv^2 (\nu^2)^2 C_{\nu^2} + w(x)
\]

From equation

\[
\left( \frac{\partial R}{\partial c} + \nu_1 \frac{\partial R}{\partial b} \right) = 0, \quad \left( \frac{\partial R}{\partial c} + \nu_2 \frac{\partial R}{\partial b} \right) = 0
\]

(both differential operators are commutative) it follows \( R = R^1(\nu^1) + R^2(\nu^2) \)

\[
\left( \frac{\partial R}{\partial c} + \nu_1 \frac{\partial R}{\partial b} \right) = \left( \frac{\partial R}{\partial c} - \left( \frac{\partial R}{\partial b} \right) \right) = 0
\]

From which follow result above. Further

\[z = R_b = R^1_{\nu^1}(\nu^1)_b + R^2_{\nu^2}(\nu^2)_b, \quad x = R_c = R^1_{\nu^1}(\nu^1)_c + R^2_{\nu^2}(\nu^2)_c
\]

or

\[x + \nu_1 z = R^1_{\nu^1}((\nu^1)_c + \nu_1(\nu^1)_b), \quad x + \nu_2 z = R^2_{\nu^2}((\nu^2)_c + \nu_2(\nu^2)_b)
\]

After differentiation expressions above for \( b, c \) functions via \( \nu \) ones we obtain system of equations for determining derivatives of \( \nu \) functions

\[
\begin{pmatrix}
C^1_{\nu^1} \nu^1_c + C^2_{\nu^1} \nu^2_c \\
\nu^1 C^1_{\nu^1} \nu^1_c + \nu^2 C^2_{\nu^1} \nu^2_c
\end{pmatrix}
= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad
\begin{pmatrix}
C^1_{\nu^2} \nu^1_b + C^2_{\nu^2} \nu^2_b \\
\nu^1 C^1_{\nu^2} \nu^1_b + \nu^2 C^2_{\nu^2} \nu^2_b
\end{pmatrix}
= \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

Result of solution

\[
C^1_{\nu^1} \nu^1_c = \frac{\nu^2}{\nu^2 - \nu^1}, \quad C^2_{\nu^1} \nu^2_c = -\frac{\nu^1}{\nu^2 - \nu^1} \quad C^1_{\nu^1} \nu^1_b = -\frac{1}{\nu^2 - \nu^1} \quad C^2_{\nu^1} \nu^2_b = \frac{1}{\nu^2 - \nu^1}
\]

Substituting these expressions in relations connected \( x, z \) variables with \( \nu \) one we obtain finally

\[
x + \nu_1 z = \frac{R^1_{\nu^1}((\nu^1)_c)}{C^1_{\nu^1}} \equiv \Theta^1(\nu^1), \quad x + \nu_2 z = \frac{R^2_{\nu^2}((\nu^2)_c)}{C^2_{\nu^2}} \equiv \Theta^2(\nu^2)
\]
5 $M_2$

This case coincide with the previous one under condition $W_b = 0$. The second order differential equation looks as

$$R_{cc} = W_c R_{bb}$$

Solution of this equation it is possible to find in a form $R = \int dke^{kb} w(k, c)$. For $w$ we pass to ordinary differential equation of the second order $w_{cc} = k^2 W_c w$. This is typical one dimensional Schrodinger equation with zero energy and potential function $k^2 W_c(c)$. All cases when it is possible to find its solution in explicit form are described in corresponding monographies. The linear equation of second order have two fundamental solution $w_1, w_2$ and general solution is their linear combination. Thus

$$W = \int dke^{kb} (f_1(k) w_1(k) + f_2(k) w_2(k))$$

Both fundamental solutions are depended from $k$ because potential energy $k^2 W_c$ depends from this parameter. Thus $(M - A)_3$ have trivial solution depending on two one dimensional functions and series solutions connected with the cases of integrability of corresponding ordinary differential equation also depending on two one dimensional functions $f_1, f_2$.

5.1 Degenerate solution

In the main text of this section was assumed no one 2 functions from 3 ones $a, b, c$ functional dependent. Let us consider opposite situation. $c = C(a) = W^{-1}(a), \ b = B(a) \ (W^{-1} \text{ inverse with respect } W \text{ function})$. From linear system of equations $a_z = b_x, \ b_z = c_x = C_a a_x$ we immediately obtain $B^2_a = C_a$ and the first equation is usual one dimensional Monge equation $a_z = C_a \frac{1}{2} a_x$ with solution in implicit form $x + C_a \frac{1}{2} z = G(a)$. Thus we have solution in degenerate case $M_2, (M - A)_3$ equations depending on 2 arbitrary functions $(W^{-1}(a), G(a))$.

6 $M_3$

$$U_{xxx} = W(U_{zzz})$$
This equation as in section possible system form in notations \( a = U_{x,z}, \ b = U_{x,z}, \ c = U_{x,z}, \ d = U_{x,z}, \ d_{z} = c_{x} = W(a), \ a_{z} = b_{x}, \ b_{x} = c_{z}. \) In two dimensional all three functions are functionally depended. Thus it is possible represented \( a = f(b, c) \) The system of equations under such assumption take form

\[
\begin{align*}
    f_{b}b_{x} + f_{c}c_{x} &= b_{z}, \quad b_{x} = c_{z}, \quad c_{x} = W_{b}b_{z} + W_{c}c_{z}, \quad W = W(f)
\end{align*}
\]

After the same transformation as in section \( \tilde{M}_{2} \) we pass to two equations in partial derivatives which must be self consistent

\[
\begin{align*}
    f_{b}R_{b,c} - f_{c}R_{b,b} &= -R_{c,c}, \quad -R_{b,b} = -W_{b}R_{c,c} + W_{c}R_{c,b}, \quad W = W(f), \quad x = R_{c}, \quad z = R_{b}
\end{align*}
\]

### 6.1 \( \nu_{2}, \nu_{1} = \text{Constants} \)

In this case \( R = R_{1}(b - \nu_{1}c) + R_{2}(b - \nu_{2}c), \ W = (\nu_{1} + \nu_{2})b - \nu_{1}\nu_{2}c + w(R_{b}). \) Let \( x = R_{b} = R_{1} + R_{2}, \ z = R_{c} = -\nu_{1}R_{1} - \nu_{2}R_{2}. \) Resolving two last equations \( b - \nu_{2}c = L_{1}(x + \nu_{1}z), \ b - \nu_{1}c = L_{2}(x + \nu_{2}z) \) or finally \( c = \frac{L_{2} - L_{1}}{\nu_{2} - \nu_{1}}, \ b = \frac{\nu_{2}L_{1} - \nu_{1}L_{2}}{\nu_{2} - \nu_{1}}. \) In the same terms \( f = \frac{\nu_{2}L_{2} - \nu_{1}L_{1}}{\nu_{2} - \nu_{1}} + \sigma(x) \) For determination \( W \) we have additional equations

\[
\begin{align*}
    \nu_{1}^{-1} + \nu_{2}^{-1} &= W_{c}, \quad (\nu_{1}\nu_{2})^{-1} = -W_{b}, \quad \nu_{c}^{2} + \nu_{b}^{2} = 0, \quad \nu_{c}^{1} + \nu_{b}^{1} = 0
\end{align*}
\]

\[
W = c(\nu_{1}^{-1} + \nu_{2}^{-1}) - b(\nu_{1}\nu_{2})^{-1} + \theta(z) = \frac{\nu_{2}^{-1}L_{2} - \nu_{1}^{-1}L_{1}}{\nu_{2} - \nu_{1}} + \theta(z)
\]

Thus in notation of the section \( \tilde{M}_{2} \) we have

\[
\begin{align*}
    c = l_{0}, \quad b = l_{1}, \quad a = f = l_{2} + \theta(z), \quad W = l_{-1} + \sigma(x)
\end{align*}
\]

But \( W = W(f). \) This fact is equivalent as equality to zero Jacobian between this functions \( \text{Jacobian}(f, W) = 0 \) with arbitrary two arguments of the problem. Choosing \( x, z \) as such arguments have

\[
\begin{align*}
    f_{x}W_{z} = f_{z}W_{x}, \quad (\tilde{l}_{0}^{1} + \theta_{x})(\tilde{l}_{-1}^{1}x + \sigma_{x}) = \tilde{l}_{2}^{1}l_{0}^{1}
\end{align*}
\]

or

\[
\begin{align*}
    \sigma_{x}\theta_{z} + \sigma_{z}l_{3}^{1} + \theta_{x}l_{-1}^{1} + \frac{l_{2}l_{0}^{1}}{\nu_{2}^{-1}L_{1}} = 0
\end{align*}
\]
In what follows following notations are used

\[ \Delta = \nu_2 - \nu_1, \quad \text{[]} = (\nu_1^3 + \nu_1 \nu_2 + \nu_2^3), \quad \rho = \frac{\nu_1 + \nu_2}{\nu_1 \nu_2} \]

Equation above is functional equation connected 4 functions \( \hat{L}_2, L'_1, \sigma_x, \theta_z \) with 4 different arguments \( x + \nu_2 z, x + \nu_1 z, x, z \). This functional equation may be rewritten in many different forms. We present one of them from which we will be able to obtain some number of partial solutions

\[ \frac{\theta_z + \nu_1^2 L'_1}{\nu_1^{-1} L'_1 - \nu_2 \sigma_x} = \frac{\theta_z + \nu_2^2 \hat{L}_2}{\nu_2^{-1} \hat{L}_2 - \nu_1 \sigma_x} \]  

(6)

Both functional equations are invariant with respect to the following transformation

\[ \sigma_x \rightarrow \frac{\text{[]} \nu_2 \sigma_x}{\nu_1 \nu_2 \sigma_x}, \quad \theta_z \rightarrow \frac{\text{[]}}{\theta_z}, \quad L'_1 \rightarrow \frac{1}{\nu_1 \nu_2 L'_1}, \quad \hat{L}_2 \rightarrow \frac{1}{\nu_1 \nu_2 \hat{L}_2} \]

We would like to show that if one derivatives is constant functional equation have some explicit partial solution.

### 6.1.1 \( \sigma_x = A = \text{Constant} \)

In this case let us introduce in the last equation notation \( u^{-1}(x + \nu_1 z) = \nu_1^{-1} L'_1 - \nu_2 A, \quad u^{-1}(x + \nu_2 z) = \nu_2^{-1} \hat{L}_2 - nu_1 A \) and determine \( \theta_z \) in this terms. Result is the following one

\[ \theta_z = \frac{\nu_2^3 - \nu_1^3}{u - v} + \frac{\nu_2^3 \nu_1 A u - \nu_2 \nu_1^3 A v}{u - v} \]

Derivative with respect to \( x \) the last expression equal to zero and \( u_x = u', v_x = \dot{v} \) The final result

\[ \frac{u'}{u + (A)^{-1}} = \frac{\dot{v}}{(v + (A)^{-1})} = k, \quad \ln(v + (\bar{A})^{-1}) = k(x + \nu_2 z) + d_2 \equiv kD_2, \quad \bar{A} \equiv \rho A \]

\[ v^{-1} = (e^{kD_2} - (\bar{A})^{-1})^{-1} = \nu_2^{-1} \hat{L}_2 - \nu_1 A, \quad u^{-1} = (e^{kD_1} - 1)^{-1} = \nu_1^{-1} L'_1 - \nu_2 A, \]

\[ \theta_z = A \nu_2 \nu_1 \frac{\nu_2^2 e^{kD_1} - \nu_1^2 e^{kD_2}}{e^{kD_1} - e^{kD_2}} \]
Trivial integration of the last expressions lead to
\[ L_2 = \frac{\nu_2 \bar{A}}{k} \ln(1 - \bar{A}e^{-kD}) + \nu_2 \nu_1 AD_2, \quad L_1 = \frac{\nu_1 \bar{A}}{k} \ln(1 - \bar{A}e^{D_1}) + \nu_2 \nu_1 AD_2 \]
\[ \theta = -A \frac{\nu_2^3 - \nu_1^3}{k \Delta} \ln(e^{-kD_2} - e^{-kD_1}) - A \nu_2 \nu_1 [(\nu_2 + \nu_1)x + \lceil z \rceil] \quad \sigma = Ax \]
\[ W = l_0 + \sigma = A \frac{\nu_1 D_2 - \nu_2 D_1}{\Delta} + \frac{\bar{A}}{k} \ln \frac{1 - (\bar{A})^{-1}e^{-kD_2}}{1 - \bar{A}e^{-kD_1}} + Ax = \frac{\bar{A}}{k} \ln \frac{1 - (\bar{A})^{-1}e^{-kD_2}}{1 - (\bar{A})^{-1}e^{-kD_1}} \]
\[ f = a = l_0 + \theta = A \frac{\nu_2^2 D_2 - \nu_1^2 D_1}{\Delta} + \nu_1 \frac{\bar{A}}{\Delta k} \ln(1 - \bar{A}e^{-kD_2}) - \]
\[ \nu_1^3 \frac{\bar{A}}{\Delta k} \ln(1 - \bar{A}e^{-kD_1}) - A \frac{\nu_2^3 - \nu_1^3}{k \Delta} \ln(e^{-kD_2} - e^{-kD_1}) - A \nu_2 \nu_1 [(\nu_2 + \nu_1)x + \lceil z \rceil] = \]
\[ \nu_2^3 \frac{\bar{A}}{\Delta k} \ln \left( \frac{1 - \bar{A}e^{-kD_2}}{e^{-kD_2} - e^{-kD_1}} \right) - \nu_1^3 \frac{\bar{A}}{\Delta k} \ln \left( \frac{1 - \bar{A}e^{-kD_1}}{e^{-kD_2} - e^{-kD_1}} \right) \]
or
\[ \frac{\Delta k}{A} f = (\nu_2^3 - \nu_1^3) \ln(e^{W_D} - 1) + \nu_1^3 \frac{W_k}{A} \]
(7)

6.1.2 \( L_1 = D = \text{Constant} \)

In this case let us rewrite the main equation in equivalent form
\[ \frac{\nu_2 L_2 - \nu_1 \sigma_x L'_1}{\nu_1 L'_1 - \nu_2 \sigma_x} = \frac{\theta_x + \nu_2 \bar{L}_2}{\theta_x + \nu_1 \bar{L}'_1} \]
(8)

and introduce notations
\[ u^{-1}u^{-1}(x) = \nu_1^{-1}D - \nu_2 \sigma_x, \quad v^{-1}(z) = \theta_x + \nu_1^2 D. \]
In this notations the last equation resolved with respect to \( \bar{L}_2 \) looks as
\[ \bar{L}_2 = \frac{1 - \nu_1}{\nu_2} \frac{\nu_2^2 u - \nu_1^2 v}{\nu_2^2 u - \nu_1^2 v} \]

After action on the this expression by operator \( H = \frac{\partial}{\partial z} - \nu_2 \frac{\partial}{\partial x} \) keeping in mind that \( HL_2 = 0, \quad Hu = -\nu_2 u_x, \quad Hv = v_z \) we pass to equation for \( u, v \) functions with obvious solution
\( \bar{D} = \bar{D}(\nu_1, \nu_2) \)
\[ \frac{u_x}{\nu_2 \bar{D}^{-1} + u} + \nu_2 \frac{v_x}{\nu_1 \bar{D}^{-1} + v} = 0, \quad \nu_2 \bar{D}^{-1} + v = e^{-k\nu_2 z}, \quad \nu_2 \bar{D}^{-1} + u = e^{kx} \]
Substituting these expressions into equation for $\dot{L}_2$ and using definitions of $u, v$ functions we obtain explicit expressions for all derivations

$$\dot{L}_2 = D + D(\nu_2^2 - \nu_1^2) \frac{e^{-kD_2}}{\nu_2^{-1} - \nu_2^2 e^{-kD_2}}, \quad \sigma_x = \nu_1^{-1} \nu_2^{-1} D - \nu_2^{-1} \frac{1}{e^{kx} - \nu_2^2 D^{-1}}$$

$$L'_1 = D, \quad \theta_z = -\nu_2^2 D + \frac{1}{e^{-\nu_2 k z} - \nu_2^{-1} D^{-1}}$$

Trivial integration leads to explicit expressions of $L_2, L_1, \theta, \sigma$ functions

$$L_2 = \frac{\nu_2^2}{\nu_2^2} D x + \nu_2 D z + D \frac{\nu_2^2 - \nu_2^2}{k \nu_2^2} \ln(\nu_2^{-1} e^{kx} - \nu_2^2 e^{-k\nu_2 z}), \quad L_1 = D(x + \nu_1 z)$$

$$\theta = -D \nu_2 + D \nu_2 z - D k \ln(e^{-k\nu_2 z} - \nu_2 D^{-1}), \quad \sigma = \nu_2^{-1} \nu_1^{-1} \frac{D x - \nu_2^{-3} \nu_1^{-1}}{e^{kx} - \nu_2^2 D^{-1}} k \ln(e^{k x} - \nu_2^2 D^{-1})$$

With help of these formulae and definitions above we have

$$W = \ell_0^1 + \sigma = \nu_2^{-3} D \frac{\nu_2 + \nu_1}{k} \ln(\nu_2^{-1} e^{kx} - \nu_2^2 e^{-k\nu_2 z} - \nu_2^2 D^{-1})$$

$$a = f = \ell_0^1 + \theta = D \frac{\nu_2 + \nu_1}{k} \ln(\nu_2^{-1} e^{kx} - \nu_2^2 e^{-k\nu_2 z} - \nu_2^2 D^{-1} - \nu_2^2)$$

or functions $W, f$ are functionally dependent

$$\nu_2^{-1} e^{-k\nu_2^2 W} - \nu_2^2 e^{-k f} = 1 \quad (9)$$

6.1.3 \( \theta_z = E = \text{Constant} \)

After introduction new notation $\frac{1}{u}(x + \nu_1 z) = E + \nu_1 L'_1, \quad \frac{1}{v}(x + \nu_2 z) = E + \nu_2 \dot{L}_2$ and resolving the main equation (??) with respect $\theta \sigma_x$ we have

$$\sigma_x(\nu_1 v - \nu_2 u) = \nu_2^{-3} - \nu_1^{-3} + E(\nu_1^{-3} u - \nu_2^{-3} v)$$

After calculation derivative with respect to $z$ argument and trivial manipulations as in previous sub subsections we pass to equations for $u, v$ functions with obvious solution

$$\frac{\nu_1 u'}{u + \frac{\nu_1}{E \rho}} = k = \frac{\nu_2 v'}{v + \frac{\nu_2}{E \rho}}, \quad u = e^{\frac{k}{\nu_1} D_1} - \frac{\nu_1}{E \rho}, \quad v = e^{\frac{k}{\nu_2} D_2} - \frac{\nu_2}{E \rho}$$
From definitions of \( u, v \) functions above \( \sigma_x \) via these functions we obtain

\[
\nu_2^2 \hat{L}_2 = -E + \frac{1}{e^{\nu_2 D_2} - \frac{\nu_2}{E^\rho}}, \quad \nu_1^2 \hat{L}_1 = -E + \frac{1}{e^{\nu_1 D_1} - \frac{\nu_1}{E^\rho}}, \quad \sigma_x = E \frac{\nu_1^{-3} e^{\frac{\nu_1}{E} D_1} - \nu_1^{-3} e^{\frac{\nu_1}{E} D_2}}{\nu_1 e^{\nu_2 D_2} - \nu_2 e^{\nu_1 D_1}}
\]

Result of integration

\[
\nu_2^2 L_2 = -ED_2 + \frac{\rho}{k} \ln(1 - \frac{\nu_2}{E^\rho} e^{-\frac{\nu_2}{E} D_2}), \quad \nu_1^2 L_1 = -ED_1 + \frac{\rho}{k} \ln(1 - \frac{\nu_1}{E^\rho} e^{-\frac{\nu_1}{E} D_1})
\]

\[
\sigma = -\nu_1^{-3} \nu_2^{-1} - \frac{\nu_1 + \nu_2}{\nu_2^3 \nu_1^3} D_2 - \frac{\nu_1 + \nu_2}{k \nu_2^3 \nu_1^3} \ln(\nu_1 e^{\frac{\nu_1}{E} D_1} - \nu_2 e^{\frac{\nu_2}{E} D_2})
\]

\[
f = \frac{\rho}{k \Delta} \ln \left(\frac{1 - \frac{\nu_2}{E^\rho} e^{-\frac{\nu_2}{E} D_2}}{1 - \frac{\nu_1}{E^\rho} e^{-\frac{\nu_1}{E} D_1}}\right), \quad W = \frac{\rho}{k \Delta} \left[\nu_2^{-3} \ln(1 - \frac{\nu_2}{E^\rho} e^{-\frac{\nu_2}{E} D_2}) - \nu_1^{-3} \ln(1 - \frac{\nu_1}{E^\rho} e^{-\frac{\nu_1}{E} D_1}) - \nu_2^{-3} \ln(\nu_1 e^{-\frac{\nu_1}{E} D_1} - \nu_2 e^{-\frac{\nu_2}{E} D_2})\right].
\]

From the last two relations above we obtain

\[-\frac{\rho}{k \Delta} W = \nu_2^{-3} \ln(1 - e^{f \frac{\nu_2}{k \Delta}}) - \nu_1^{-3} \ln(e^{-f \frac{\nu_1}{k \Delta}} - 1)\]

### 6.2 \( \nu_1 = \text{Constant}, \nu_2 = \nu_2(b - \nu_1 c) \)

\[
\nu_2 = \nu_2((b - \nu_1 c)), \quad R = R_1((b - \nu_1 c)) + R_2(f^{(b - \nu_1 c) \frac{1}{v_1 - v_2(s)}} - b)
\]

\[
x = R_b = R'_1 + R''_1 \frac{\nu_1}{-\nu_2 - \nu_1}, \quad z = R_c = -\nu_1 R'_1 + R''_1 \frac{\nu_2 \nu_1}{-\nu_2 - \nu_1}, \quad x + \nu_1 z = \nu_1 R'_2,
\]

\[
x + \nu_2 z = \Delta R' \equiv \Theta(\nu_2) \quad c = C(\nu_2) - L(x + \nu_1 z), \quad b = \int \nu_2 \nu_2^1 C_{\nu_2} - \nu_1 L(x + \nu_1 z),
\]

\[
a = \int d\nu_2 \nu_2^2 C_{\nu_2} - \nu_2^2 L(x + \nu_1 z) + \theta(z), \quad W = \int d\nu_2 \nu_2^{-1} C_{\nu_2}^{-1} L(x + \nu_1 z) + \sigma(x)
\]

and only one equation remains

\[
W = W(a), \quad \sigma_x \theta_z + \theta_z (\nu_2^{-1} C_{\nu_2}(\nu_2)_x - \nu_1^{-1} L') + \sigma_x (\nu_2^{-1} C_{\nu_2}(\nu_2)_x - \nu_1^{-1} L') = \frac{\Delta^2}{\nu_1 \nu_2} (\nu_2)_x C_{\nu_2} L'.
\]

also functional equation of the same kind as above and below ones.
6.2.1 $\theta_z = E = \text{constant}$

Only with the aim to demonstrate self consistence of this equation let us consider the simplest case $L' = 0$ and simplest solution $\nu^2 = -\frac{x}{z}, \Theta = 0, \nu_x^2 = -\frac{1}{z}, \nu_z^2 = \frac{x}{z}$. Functional equation reduced to

$$\frac{1}{C_{\nu_2}(\nu_2)} + \frac{1}{x\sigma_x} + \frac{1}{z^4\theta_z} x^3 = 0,$$

$$\frac{1}{\nu_2}(\frac{1}{C_{\nu_2}(\nu_2)})\nu_2 + \frac{1}{z^4\theta_z} z^4 = 0$$

The second equation arises after differentiation the first one with respect to $z$. Solution of this system is the following one

$$C_{\nu_2} = k \nu_2^3 + \alpha, \quad \theta = -\frac{1}{3k} \ln(kz^{-3} + \beta), \quad \sigma = \frac{1}{3\alpha} \ln(\alpha x^{-3} + k)$$

After simple calculations we obtain functional dependence $W, a = f$ functions in a form $e^{3\alpha W} + \frac{2}{k} e^{-3k f} = \frac{2}{k}$.

6.3 $\nu^2_c + \nu^1_b = 0, \quad \nu^1_c + \nu^2_c = 0$

We present only finally formulae of the corresponding subsubsection of section 4.

$$c = C^1_{\nu_1} + C^2_{\nu_2}, \quad b = \int dv^1 \nu^1 c^1_{\nu_1} + \int dv^2 \nu^2 c^2_{\nu_2}$$

$$f = \int [(\nu_1 + \nu_2) dc - \nu_1 \nu_2 db] + \theta(z) = \int dv_1 \nu^1_1 C^1_{\nu_1} + \int dv_2 \nu^2_2 C^2_{\nu_2} + \theta, \quad (\nu^1 = \nu_1)$$

$$W = \int [(\nu^{-1}_1 + \nu^{-1}_2) db - (\nu_1 \nu_2)^{-1} dc] + \sigma(x) = \int dv_1 \nu^{-1}_1 C^1_{\nu_1} + \int dv_2 \nu^{-1}_2 C^2_{\nu_2} + \sigma$$

Substituting these expressions in relations connected $x, z$ variables with ($\nu$) ones we obtain finally

$$z + \nu_1 x \Theta^1(\nu^1), \quad z + \nu_2 x = \Theta^2(\nu^2), \quad z = \frac{\Theta^2 - \Theta^1}{\Delta}, \quad = \frac{\nu_1 \Theta^2 - \nu_2 \Theta^1}{\Delta}$$

Thus only one problem which remains to resolve is functionally dependence of $f, W$ functions or $f_x W_x = f_x W_z$. Taking into account equations defined $\nu$ functions we have

$$\nu_z^1 = \frac{1}{\Theta^1_{\nu_1} - z}, \quad \nu_z^2 = \frac{1}{\Theta^1_{\nu_2} - z}, \quad \nu_x^2 = \frac{1}{\Theta^2_{\nu_1} - z}, \quad \nu_x^2 = \frac{1}{\Theta^2_{\nu_2} - z}$$
\[ \Delta z \sigma_x + (\nu_1^2 C_{\nu_1}^1 \frac{\Delta}{q_1} + \nu_2^2 C_{\nu_2}^2 \frac{\Delta}{q_2}) \sigma_x + (\nu_1^{-1} C_{\nu_1}^1 \frac{\Delta}{q_1} + \nu_2^{-1} C_{\nu_2}^2 \frac{\Delta}{q_2}) \theta_z = \]

\[ \left( \frac{\nu_1^2}{\nu_1} - \frac{\nu_2^2}{\nu_1} \right) \frac{\Delta}{q_1} C_{\nu_1}^1 C_{\nu_2}^2, \quad q_1 = \Theta_{\nu_1}^1 - z, \quad q_1 = \Theta_{\nu_2}^2 - z \]  

(10)

which can be rewritten in equivalent form

\[ \frac{\nu_2^{-1} C_{\nu_2}^2 \frac{\Delta}{q_2} - \nu_1 \sigma_x}{-\nu_1^{-1} C_{\nu_1}^1 \frac{\Delta}{q_1} - \nu_2 \sigma_x} = \frac{\theta_z + \nu_2^2 C_{\nu_2}^2 \frac{\Delta}{q_2}}{\theta_z - \nu_1^2 C_{\nu_1}^1 \frac{\Delta}{q_1}} \]

an thus (10) and last above is functional equation connected 4 functions \( C^1, C^2, \theta, \sigma \) with corresponding arguments \( (\nu^1, \nu^2, z, x) \).

Of course we have no idea how to find general solution of (10) and we present below one of its partial solution to prove its self consistence.

### 6.3.1 \( \theta_z = E = \text{Constant} \)

Absolutely by the same way as in previous subsubsection we assume \( \theta_z = A = \text{Const} \) and choose \( \Theta \) functions in special form \( \Theta_1^1 = \nu_1^2, \ \Theta_2^1 = \nu_2^2 \) for which \( z = \nu_1 + \nu_2, \ x = -\nu_1 \nu_2, \ q_2 = \Delta, \ q_1 = -\Delta \). Under such restrictions we resolve equation (10) with respect to \( \sigma \) function in a form

\[ \sigma_x = \frac{\nu_1^{-1} C_{\nu_1}^1 (E + \nu_2^2 C_{\nu_2}^2) - \nu_2^{-1} C_{\nu_2}^2 (E + \nu_1^2 C_{\nu_1}^1)}{\nu_2 (E + \nu_2^2 C_{\nu_2}^2) - \nu_1 (E + \nu_1^2 C_{\nu_1}^1)} = -x^{-1} \frac{P - \bar{P}}{Q - \bar{Q}} \]  

(11)

where \( P(\nu_1) = \frac{C_{\nu_1}^1}{\nu_1 (E + \nu_1^2 C_{\nu_1}^1)}, \ \bar{P}(\nu_2) = \frac{C_{\nu_2}^2}{\nu_2 (E + \nu_2^2 C_{\nu_2}^2)}, \ Q = \frac{1}{\nu_1 (E + \nu_1^2 C_{\nu_1}^1)}, \ \bar{Q} = \frac{1}{\nu_2 (E + \nu_2^2 C_{\nu_2}^2)}, \ \frac{1}{\nu_1 (E + \nu_1^2 C_{\nu_1}^1)} \nu_1^2 P + E \bar{Q} = \frac{1}{\nu_1}, \ \nu_2^2 \bar{P} + EQ = \frac{1}{\nu_2} \). Let us choose \( Q = a \nu_1 + E^{-1} \nu_1^{-1}, \ P = -a D \nu_1^{-1}, \ \bar{Q} = a \nu_2 + E^{-1} \nu_2^{-1}, \ \bar{P} = -a D \nu_2^{-1} \). Under such choice we have

\[ x \sigma_x = \frac{E}{x + (aD)^{-1}}, \ C_{\nu_1}^1 = -\frac{E}{\nu_1^2 + (aD)^{-1}}, \ C_{\nu_1}^1 = -\frac{E}{\nu_2^2 + (aD)^{-1}} \]

\( \theta_z = E \)

From main equation it follows that it is invariant with respect to multiplication all unknown functions \( \theta, \sigma, C^2, C^1 \) on common factor. By this reason we will omit common factor \( E \) in solution above and denote \( (aD)^{-1} = g \). Now in connection with beginning of section 6 we calculate

\[ W = -\int d\nu_1 \frac{1}{(\nu_1^2 + g)\nu_1} - \int d\nu_2 \frac{1}{(\nu_2^2 + g)\nu_2} + \sigma = \frac{1}{2g} \ln \left( \frac{(\nu_2^2 + g)(\nu_1^2 + g)}{\nu_2 \nu_1^2} \right) + \]
\[
\frac{1}{g} \ln \frac{x}{x+g} = \frac{1}{2g} \ln (1 + g \frac{z^2}{(x+g)^2}), \quad z = \nu_2 + \nu_1, \quad x = -\nu_2 \nu_1
\]

Thus in this case solution of \( M_3 \) exists if function \( W \) as function of its argument \( f \) looks as

\[
e^{2gW} = \frac{1}{\cosh^2 (f \sqrt{-g})}
\]

There are other solution if assume from the beginning that \( E = 0 \). Then equation (12) takes the form

\[
\sigma_x = \frac{\nu_2^3 - \nu_3^3}{\nu_2^3 C_{\nu_2}^2 C_{\nu_1}^1} - \frac{1}{a} (\nu_1^3 + \alpha_1)(\nu_2^3 + \alpha_2) = -x -1 \frac{\nu_1^3 - \nu_2^3}{Q - \bar{Q}}
\]

where \( Q = \frac{1}{\nu_1^3 c_{\nu_1}^1}, \quad \bar{Q} = \frac{1}{\nu_1^3 c_{\nu_1}^1}. \) Resolving of (12)

\[
Q = a \nu_1^3 + b + c \nu_1^{-3}, \quad \bar{Q} = Q = a \nu_2^3 + b + c \nu_2^{-3}, \quad \sigma_x = -\frac{1}{x(a x^3 + c)}
\]

\[
\sigma = \frac{1}{3c} \ln (a + c x^{-3}), \quad C_{\nu_1}^1 = \frac{1}{a} (\nu_1^3 + \alpha_1)(\nu_2^3 + \alpha_2), \quad C_{\nu_2}^2 = \frac{1}{a} (\nu_2^3 + \alpha_1)(\nu_2^3 + \alpha_2)
\]

where \( \alpha_1 + \alpha_2 = \frac{b}{a}, \quad \alpha_1 \alpha_2 = \frac{c}{a} \).

\[
f = \int \frac{d \nu_1}{a(\nu_1^3 + \alpha_1)(\nu_2^3 + \alpha_2)} + (\nu_1 \rightarrow \nu_2) = \frac{1}{3a(\alpha_2 - \alpha_1)} \ln \frac{(\nu_3^3 + \alpha_1)(\nu_3^3 + \alpha_1)}{(\nu_1^3 + \alpha_1)(\nu_2^3 + \alpha_2)}
\]

\[
W = \int \frac{d \nu_1}{a(\nu_1^3 + \alpha_1)(\nu_2^3 + \alpha_2)} + (\nu_1 \rightarrow \nu_2) + \Delta = \frac{1}{3a(\alpha_2 - \alpha_1)} (\ln \nu_1^3 \nu_2^3) -
\]

\[
\frac{\alpha_2}{\alpha_2 - \alpha_1} \ln Q_1 + \frac{\alpha_1}{\alpha_2 - \alpha_1} \ln Q_2 + \frac{1}{3c} \ln \frac{\alpha_2 Q_1 - \alpha_1 Q_2}{(\alpha_1 - \alpha_2)x^3} =
\]

\[
\frac{\alpha_2}{3c(\alpha_2 - \alpha_1)} \ln \frac{\alpha_2 - \alpha_1 Q_2}{Q_2} - \frac{\alpha_1}{3c(\alpha_2 - \alpha_1)} \ln \frac{\alpha_2 Q_2 - \alpha_1}{(\alpha_2 - \alpha_1)}
\]

And finally dependence of \( W \) from \( f \) under which solution of \( M_3 \) exists in this case is the following one

\[
3c(\alpha_2 - \alpha_1)W = \alpha_2 \ln \frac{\alpha_2 - \alpha_1 e^{-3a(\alpha_2 - \alpha_1)f}}{(\alpha_1 - \alpha_2)} - \alpha_1 \ln \frac{\alpha_2 e^{3a(\alpha_2 - \alpha_1)f} - \alpha_1}{(\alpha_1 - \alpha_2)} \quad (13)
\]
The results of previous sections may be generalized on the case of $M_n$ equations with arbitrary $n$.

Let us introduce notations $a_k = (\frac{\partial^n U}{\partial z^n} - k \frac{\partial z}{\partial z} + k = 0, 1, ... n - 1$, then $M_n$ equation $a^n = W(a^0)$ may be represented in a form

$$a^k_x = a^{k+1}_x, \quad a^{n-1}_x = a^n_z = W(a^0)_z = W_0 a^0$$

Now let us use parametrization by one of 3 possibilities of previous sections. For instance $\nu_1, \nu_2 = constants$ and functions $l^g_k$ are defined by the same way as above. There are not difficult to check that functions $a_k$ defined as (all notations see in section $M_2$)

$$a^0 = l^0_{n-1} + \theta(z), \quad a^1 = l^0_{n-2}, \quad a^2 = l^0_{n-3}, ... \quad W = l^0_{-1} + \sigma(x)$$

satisfy all equations above except of condition of functionally dependence functions $W + \sigma_x$ and $a^0 + \theta(z)$

$$(l^1_n + \nu_2)(l^1_{n-1} + \nu_2) = l^1_{n-1} l^1_0, \quad \sigma_x \theta_z + \sigma_x l^1_n + \theta_z l^1_{n-1} + \frac{\nu_1}{\nu_1 \nu_2} \theta_x l^1_{n-1} = 0 \quad (14)$$

where $\nu_n = \frac{\nu_n^2 - \nu_{n-1}^2}{\nu_2 - \nu_1}$. We emphasize that among solution of last equation the trivial one is exists. Really let $\theta = \sigma = 0$ and $W = a^0$ then last condition means that (we assume also that $L_1 = 0$) $(\frac{1}{\nu_2} \theta_x l^1_2 = 0$ and this is exactly trivial solution of section 2. It is understand that solution in the considerable case $M_n$ equation may be obtained from case $M_3$ by very simple manipulations. The example below clarify situation.

7.0.2 $\theta_z = E = Constant$

Equation (??) in form resolved with respect to $\sigma_x$ in notations

$$\tilde{R}(D_2) = \frac{\theta_x}{\nu_2(\nu^2_2 - 1 L_2 + E)}, \quad \tilde{Q}(D_2) = \frac{1}{\nu_2(\nu^2_2 - 1 L_2 + E)}, \quad \nu_2 - 1 \tilde{R} + EQ = \frac{1}{\nu_2}$$

$$\tilde{R}(D_1) = \frac{\theta_x}{\nu_2(\nu^2_2 - 1 L_1 + E)}, \quad \tilde{Q}(D_1) = \frac{1}{\nu_1(\nu^2_1 - 1 L_1 + E)}, \quad \nu_1 - 1 \tilde{R} + EQ = \frac{1}{\nu_1}$$

looks as

$$\sigma_x \nu_1 \nu_2 = -\frac{\tilde{R} - R}{Q - Q}$$

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In the present paper was proposed and used the following construction.

Substituting $Q, Q$ via $R, R$ we obtain equations for determining $R, R$ functions

\[
\frac{1}{\nu_1} \left( \frac{1}{R'} + \nu_1 \nu_2 \right) \mid_{n-1} \frac{R}{R'} = \frac{1}{\nu_2} \left( \frac{1}{R} + \nu_1 \nu_2 \right) \mid_{n-1} \frac{\bar{R}}{R} = \frac{1}{k} = \text{Constant}
\]

with solution

\[
\bar{R} = -\frac{1}{\nu_1 \nu_2 \mid_{n-1}} + c e^{k \nu_2 \mid_{n-1} D_1}, \quad \bar{R} = -\frac{1}{\nu_1 \nu_2 \mid_{n-1}} + c e^{k \nu_1 \mid_{n-1} D_2}
\]

\[
EQ = \frac{\bar{R}}{\nu_1 \nu_2 \mid_{n-1}} - \nu_1^{n-1} c e^{k \nu_2 \mid_{n-1} D_1}, \quad EQ = \frac{\bar{R}}{\nu_1 \nu_2 \mid_{n-1}} - \nu_2^{n-1} c e^{k \nu_1 \mid_{n-1} D_2}
\]

Substituting these results in equation for $\sigma x \nu_1 \nu_2$

\[
\sigma x \nu_1 \nu_2 = E \frac{c e^{k \nu_1 \mid_{n-1} D_2} - c e^{k \nu_2 \mid_{n-1} D_1}}{\nu_2^{n-1} c e^{k \nu_1 \mid_{n-1} D_2} - \nu_1^{n-1} c e^{k \nu_2 \mid_{n-1} D_1}}
\]

and all other calculations as in the case of $M_3$ equation. Explicit expressions for $Q, \bar{Q}$ define $L_2, L_1'$. After integration these expressions we obtain explicit form $\sigma L_2, L_1, \theta = Ez$ functions some partial solution of $M_n$ equation and explicit form of $W$ function under which this solution exists.

### 8 Conclusion remarks

In the present paper was proposed and used the following construction. Infinite dimensional chain of of equations $a_x^k = a_x^{k+1}$ was realized on the space of 2 one dimensional functions in 3 version $\nu_1 \nu_2 L(x + \nu_1 z), \int d\nu_2 \nu_1^k C_{v_2}, \nu_1 = \text{Constant}, x + \nu_2 z = \Theta(\nu_2), \int d\nu_1 \nu_1^k C_{v_1}, \int d\nu_2 \nu_2^k C_{v_2}, x + \nu_1 z = \Theta(\nu_1), x + \nu_2 z = \Theta(\nu_2)$. After this this chain was interrupted by additional condition $a^{n+1} + \sigma(x) = W(a^0 + \theta(z)$ which lead exactly to solution of $M_n$.

In this way we come to situation when question of integrability is connected with functional equation of special form. In this connection we would like to notice that functional equations arises before in the theory of integrable system for finding general solution in implicit form [?]. (3 – 4) What
class of solutions of $M_n$ it is possible to find by this method we don’t know. What connection have all this with group theoretical approach is unknown for us at the present moment also. In general the very interesting problem may be formulated as the follows it is necessary to enumerate all functions $W$ with choice of which equation $M_n$ has integrable solution. Part of this solutions in partial cases we have presented in this paper in explicit form. What domain of mathematic responsible for this is the most interesting question for further investigation.

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