Linear Stability of Dilatonic Black Holes*

Panagiota Kanti

Division of Theoretical Physics, Physics Department,
University of Ioannina, Ioannina GR 451 10, Greece.

Abstract

In this talk, we recall the most important features of the Dilatonic Black Holes which arise in the framework of the Einstein-Dilaton-Gauss-Bonnet theory and which are dressed with a classical long range dilaton field in contradiction with the existing “no-hair” theorems of the Theory of General Relativity. We demonstrate linear stability of these black hole solutions under small spacetime-dependent perturbations by making use of a semi-analytic method based on the Fubini-Sturm’s theorem. As a result, the Dilatonic Black Holes constitute one of the very few examples of stable black hole solutions with non-trivial “hair” that arise in the framework of a more generalised theory of gravity.

* Talk presented at the “International Workshop on Recent Developments in High Energy Physics” organized by the Hellenic Society for the Study of High Energy Physics at Democritos NRC, Athens, Greece, April 9-11, 1998.
1 Introduction

The Dilatonic Black Holes arise in the framework of the one-loop corrected four-dimensional effective theory of the heterotic superstrings at low energies. In ref. [1], we have demonstrated numerically the existence of the Dilatonic Black Hole solutions with a regular event horizon and an asymptotically flat behaviour at infinity in the presence of higher-derivative gravitational terms such as the Gauss Bonnet (GB) curvature-squared term. According to our analytic arguments [1], it was the presence of this term which led to the by-passing of the “no-scalar-hair” theorems [2] and the existence of black hole solutions dressed with non-trivial classical dilaton hair.

The existence itself of the Dilatonic Black Holes is beyond any doubt since our results were rederived and confirmed by other research groups in subsequent works [3] [4]. But, an important question related to the nature and fate of our black hole solutions arises next : the question of stability. Although the “no-hair” theorems refer to the existence of black hole solutions and not to their stability, there is a tendency in the literature to reject some of these solutions due to their instability under spacetime perturbations. At the same time, the conception that the “no-hair” theorems are indeed valid once we have assured the stability of the black hole solutions has started to form among the scientists. For these reasons, we need also to check the stability of our solutions, that is the Dilatonic Black Hole solutions [1].

In the next paragraph, we are going to recall the most important features of the Dilatonic Black Holes while in paragraph 3 we shall present our semi-analytic method for the study of our solutions under linear spacetime-dependent perturbations [5]. The key feature of our method is the reduction of the time-dependent equations of motion of the theory to a single one-dimensional Schrödinger type differential equation. As we shall see, the absence of bound states in the spectrum of this equation corresponds to the absence of unstable modes or equivalent to the stability of our classical black hole solutions of the time-independent equations of the theory [1]. This linear stability, if it comes out to be true, will be an outstanding result since the corresponding stable black hole solutions will be unique in the framework of a generalised theory of pure gravity.

2 Dilatonic Black Holes

The action functional of the Einstein-Dilaton-Gauss-Bonnet (EDGB) theory, in the context of which the Dilatonic Black Holes have arisen, has the following form [1]

$$S = \int d^4x \sqrt{-g} \left( -\frac{R}{2} - \frac{1}{4} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{\alpha'}{8g^2} \mathcal{R}_G^2 \phi \right)$$  (1)
where
\[ R_{GB}^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \] (2)
is the well-known curvature squared Gauss-Bonnet term and \( \phi \) stands for the dilaton field. For simplicity, we have ignored all the other scalar fields of the effective superstring theory, the axions \( a \) and \( b \) and the modulus field \( \sigma \), as well as the gauge fields.

The dilaton field and the Einstein’s equations which follow from the action (1) can be written in a covariant form in the following way
\[
\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} \partial^\mu \phi] = -\frac{\alpha'}{4g^2} e^\phi R_{GB}^2
\] (3)

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{4} g_{\mu\nu} (\partial_\rho \phi)^2 - \alpha' K_{\mu\nu}
\] (4)

where
\[
K_{\mu\nu} = (g_{\mu\rho} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\rho}) \eta^{\kappa\lambda\alpha\beta} D_\gamma [\tilde{R}_{\alpha\beta} R_{\kappa\lambda}] (5)
\]

and
\[
\eta^{\mu\nu\rho\sigma} = \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} , \quad f = \frac{e^\phi}{8g^2} , \quad \tilde{R}^{\mu\nu}_{\kappa\lambda} = \eta^{\mu\nu\rho\sigma} R_{\rho\sigma\kappa\lambda} .
\] (6)

We assume that the spacetime background is spherically symmetric and it is described by the following line element
\[
ds^2 = e^{\Gamma(r)} dt^2 - e^{\Lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) .
\] (7)

If the above ansatz is to describe a black hole solution, we must demand the following behaviour for the components of the metric tensor and the scalar field near the horizon of the black hole
\[
e^{-\Lambda(r)} = \lambda_1 (r-r_h) + \lambda_2 (r-r_h)^2 + ...\]
\[
e^{\Gamma(r)} = \gamma_1 (r-r_h) + \gamma_2 (r-r_h)^2 + ...\]
\[
\phi(r) = \phi_h + \phi'_h (r-r_h) + \phi''_h (r-r_h)^2 + ...
\] (8)

Substituting the above expansions in the equations of motion, we find that the derivative of the dilaton field at the horizon satisfies the relation
\[
\phi'_h = \frac{g^2}{\alpha'} r_h e^{-\phi_h} \left( -1 \pm \sqrt{1 - \frac{6(\alpha')^2 e^{2\phi_h}}{g^4 r_h^4}} \right)
\] (9)
which results in the following constraint for the coupling function of the dilaton field with the Gauss-Bonnet term
\[
\frac{\alpha' e^{\phi_h}}{g^2} < \frac{r_h^2}{\sqrt{6}}.
\] (10)

If the coupling function does not satisfy the above constraint, the solutions of the equations of motion of the theory can no longer be described by the concept of the black hole. At the other end of the radial space, that is in the limit \(r \to \infty\), one assumes the following asymptotically flat behaviour:
\[
e^{\mathcal{A}(r)} = 1 + \frac{2M}{r} + \frac{16M^2 - D^2}{4r^2} + O\left(\frac{1}{r^3}\right)
\]
\[
e^{\Gamma(r)} = 1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right)
\]
\[
\phi(r) = \phi_\infty + \frac{D}{r} + \frac{MD}{r^2} + O\left(\frac{1}{r^3}\right)
\] (11)

The constants \(M\) and \(D\) that appear in the above expressions stand for the ADM mass and the dilaton charge of the black hole, respectively.

If we integrate numerically the equations of motion (3)-(4) by making use of the ansatz (7) and the expansions (8) and (11), we are led to a continuous one-parameter family of regular black hole solutions. The solution for the scalar field and the metric components is displayed in Figures 1 and 2. The main feature of the solution, that we are going to make use of during the stability analysis, is the monotonic, non-intersecting behaviour of the dilaton field from \(r_h\) to infinity.

![Figure 1](image1.png)  
**Figure 1:** Dilaton configurations for a family of black hole solutions, for fixed \(r_h = 1\) and various values of \(\phi_h\).  

![Figure 2](image2.png)  
**Figure 2:** Metric components \(g_{tt}\) and \(g_{rr}\) for the \(\phi_h = -1\) and \(r_h = 1\) black hole solution.
The existence itself of these black hole solutions, of the Dilatonic Black Holes, are in disagreement with the well-known “no-hair” theorems of the Theory of General Relativity. These theorems state that the only parameters that may characterize a black hole are the mass $M$, the charge $Q$ and the angular momentum $J$ and that the only long range fields that can be associated with the black hole are the gravitational and the electromagnetic ones. The Dilatonic Black Holes are clearly characterized by another long range field, that of the dilaton field, a feature which is prohibited by the Theory of General Relativity. As we mentioned before, it is the presence of the Gauss-Bonnet term that causes the by-passing of the “no-hair” theorems and the existence of the long range scalar field. On the other hand, this disagreement with the “no-hair” theorems is only partial in the sense that we have not found any new conserved quantity apart from the aforementioned ones. The dilaton charge $D$ has turned out to be a dependent on the mass $M$ quantity, thereby leading to the secondary nature of the dilaton “hair” [6].

3 Linear Stability Analysis

In this section, we are going to present our semi-analytic method for the study of our solutions under linear spacetime-dependent perturbations [5]. For the needs of our analysis, we are going to assume that both of the metric components as well as the scalar field depend not only on the radial coordinate $r$ but also on the time coordinate $t$. Moreover, according to the linear stability analysis [7] [8], we may write the three unknown functions of the problem as a sum of two parts in the following way

$$
\Gamma(r, t) = \Gamma(r) + \delta \Gamma(r, t) = \Gamma(r) + \delta \Gamma(r)e^{i\sigma t}
$$

$$
\Lambda(r, t) = \Lambda(r) + \delta \Lambda(r, t) = \Lambda(r) + \delta \Lambda(r)e^{i\sigma t}
$$

$$
\phi(r, t) = \phi(r) + \delta \phi(r, t) = \phi(r) + \delta \phi(r)e^{i\sigma t}
$$

(12)

The first part of each of the above functions is a purely radial part which corresponds to the classical Dilatonic Black Hole solutions already found in Ref. [1]. The second part is the product of a small radial part, denoted by the prefactor $\delta$, by an harmonic function of time. These small radial parts are the linear perturbations of our solutions. Note the appearance of the parameter $\sigma$ in the above expressions. It is easy to understand that for negative $\sigma^2$, that is for purely imaginary $\sigma$, even small perturbations may be led to extremely large values at late times. For this reason, we have to assure the absence of negative $\sigma^2$ in the case of our black hole solutions.

If we substitute the expressions (12) in the time-dependent equations of motion (3)-(4) and make use of the harmonic dependence of the perturbations on time $t$, we end up with a group of differential equations with respect to the radial coordinate.
After some long and tedious algebraic computation, we manage to construct a single differential equation for the dilaton perturbation $\delta \phi$ from which the other two perturbations, $\delta \Gamma$ and $\delta \Lambda$, have been eliminated. This differential equation has the following structure

$$A \delta \phi'' + 2B \delta \phi' + C \delta \phi + \sigma^2 E \delta \phi = 0$$  \hspace{1cm} (13)$$

where $A$, $B$, $C$, and $E$ are rather complicated functions of $\phi$, $\phi'$, $\Lambda$, $\Lambda'$, $\Gamma$, $\Gamma'$ and $\Gamma''$. All of these coefficients are very well behaved near infinity, that is in the limit $r \to \infty$, according to the following expressions

$$A = 1 + O \left( \frac{1}{r^5} \right) , \quad B = \frac{1}{r} + \frac{M}{r^2} + O \left( \frac{1}{r^4} \right) \hspace{1cm} (14)$$

$$C = \frac{D^2}{2r^4} + O \left( \frac{1}{r^5} \right) , \quad E = 1 + \frac{4M}{r} + \frac{4M^2}{r^2} + O \left( \frac{1}{r^4} \right) . \hspace{1cm} (15)$$

However, three of them, $B$, $C$ and $E$, take on infinite values near the horizon of the black hole, while $A$ remains finite, in the following way

$$A = \frac{2 \sqrt{x}}{1 + \sqrt{x}} + O \left( r - r_h \right) , \quad B = \frac{\sqrt{x}}{1 + \sqrt{x}} \frac{1}{(r - r_h)} + O (1) \hspace{1cm} (16)$$

$$C = \frac{2 e^{2\phi_h}}{r_h^4 (1 + \sqrt{x})} \frac{1}{(r - r_h)^2} + O \left( \frac{1}{r - r_h} \right) \hspace{1cm} (17)$$

$$E = \frac{r_h \sqrt{x}}{\gamma_1} \frac{1}{(r - r_h)^2} + O \left( \frac{1}{r - r_h} \right) . \hspace{1cm} (18)$$

where $x = 1 - 6 e^{2\phi_h} / r_h^4$.

As a result, the differential equation (13) is not well defined at one point only of the radial space, that is at the event horizon $r_h$. In order to remove this singularity, it is necessary to introduce a new radial coordinate, the so called “tortoise” coordinate $r^*$, which is related to the old one in the following way

$$\frac{d r^*}{d r} = e^{-\left( \Gamma - \Lambda \right)/2} . \hspace{1cm} (19)$$

The introduction of the new radial coordinate leads to the extension of the radial space $[r_h, \infty)$ over the entire real axis $(-\infty, \infty)$ in such a way that the relative slope of any two curves in Fig. 1 remains unchanged. Nevertheless, the introduction of the new coordinate is powerful enough to render all of the coefficients in the differential equation (13) finite over the entire radial space. Now, the perturbed equation for the dilaton field takes the form

$$A \frac{d^2 \delta \phi}{d r^{*2}} + 2B \frac{d \delta \phi}{d r^*} + (C + \sigma^2 E) \delta \phi = 0 \hspace{1cm} (20)$$
where
\[ A = A, \quad B = B e^{(\Gamma - \Lambda)/2} - \frac{A d(\Gamma - \Lambda)}{4 \, dr^*}, \quad C = e^{\Gamma - \Lambda} C, \quad E = e^{\Gamma - \Lambda} E \quad (21) \]

For reasons that will become clear later, we have to make another step in order to eliminate the term in equation (20) which is proportional to \( \delta \phi' \). This step involves the use of the auxiliary function \( F \) defined as
\[ F = \exp \left( \int_{-\infty}^{r^*} \frac{B}{A} \, dr^* \right) \quad (22) \]
and the definition of a new function \( u \) as the product of the auxiliary function by the dilaton perturbation, \( u = F \delta \phi \). Then, the differential equation for \( \delta \phi \) takes the form
\[ p^2 u + \left[ \frac{C}{A} + \sigma^2 \frac{E}{A} - \frac{B^2}{A^2} - p_* \left( \frac{B}{A} \right) \right] u = 0 \quad (23) \]
where
\[ p_* = \frac{d}{dr^*} \quad (24) \]

In this form, equation (23) is an ordinary Schrödinger equation with well defined coefficients over the whole radial space. At the same time, this differential equation is an eigenvalue problem where different values of the parameter \( \sigma^2 \) correspond to different eigenvalues. It is easy to understand that, since the number of negative values of \( \sigma^2 \) is equal to the number of the unstable modes of our black hole solutions, the absence of states with negative \( \sigma^2 \) in the above eigenvalue problem is equivalent to the linear stability of the Dilatonic Black Holes.

In order to demonstrate the absence of the aforementioned states from the spectrum of the eigenvalue problem (23), we are going to make use of the asymptotic behaviour of the eigenfunctions \( u \) at both limits of the radial space, \( r \to r_h \) and \( r \to \infty \), and of the Fubini-Sturm’s theorem [9]. We are going to concentrate our attention on two different kinds of eigenfunctions, namely on \( u_0 \) which corresponds to zero eigenvalue \( \sigma^2 = 0 \) and on \( u_\sigma \) which corresponds to negative eigenvalues \( \sigma^2 < 0 \). We start by noting that near the horizon of the black hole, that is in the limit \( r \to r_h \) or equivalently \( r^* \to -\infty \), the differential equation (23) assumes the form
\[ p^2_0 u_\sigma + k^2 u_\sigma = 0 \quad (25) \]
where
\[ k^2 \equiv \frac{2 \gamma_1 e^{2\phi_h}}{(1 + \sqrt{1 - 6e^{2\phi_h}}) \sqrt{1 - 6e^{2\phi_h}}} + \sigma^2 = k_0^2 + \sigma^2 \quad (26) \]
and where we have made use of the definitions (21) and the asymptotic expansions (8). It can be easily seen that, while for \( \sigma^2 = 0 \) the corresponding eigenfunction \( u_0 \) takes on a constant, non-zero value at the event horizon, for \( -k_0^2 < \sigma^2 < 0 \) the only
acceptable value of \( u_\sigma \), in order to ensure the finiteness of \( \delta \phi \), is: \( u_\sigma \sim e^{k|r^*|} \to 0 \). Here, we have to add that the eigenfunctions with \( \sigma^2 < -k_0^2 < 0 \) are absent from the spectrum of the states since they have been shown to violate the continuous, non-degenerate nature of the unbound states [3].

At the other end of the radial space, in the limit \( r \to \infty \), it can be easily seen from the expressions (14)-(15) that the differential equation (23) takes the simple form

\[
p_\sigma^2 u_\sigma + \sigma^2 u_\sigma = 0 .
\]

In the same way, the only acceptable behaviour of the eigenfunction \( u \) for \( \sigma^2 < 0 \), in the limit \( r \to \infty \), is: \( u_\sigma \sim e^{-|\sigma|r^*} \to 0 \) while for \( \sigma^2 = 0 \), the solution is \( u_0 = c_1 r + c_2 \). Note that only the eigenfunctions \( u_\sigma \) that correspond to negative values of the parameter \( \sigma^2 \) vanish at both limits of the radial space. For this reason, only these eigenfunctions correspond to physical perturbations of our black hole solutions [4].

Now, we arrive at the final step of our stability analysis. This final step involves the use of the Fubini-Sturm’s theorem of ordinary differential equations [3]. According to the theorem, we consider the following two differential equations:

\[
\begin{align*}
    u''_1 + (q_1 - p_1^2 - p_1') u_1 &= 0 \\
    u''_2 + (q_2 - p_2^2 - p_2') u_2 &= 0
\end{align*}
\]

If the coefficients of the above equations satisfy the following relation

\[
p_2^2 + p_1^2 = q_2 \leq p_1' + p_1^2 - q_1
\]

throughout the interval \([a, b] \), then, between any two consecutive zeroes of function \( u_1 \), in the interval \([a, b] \), there is at least one zero of function \( u_2 \).

If we apply the above theorem to the spectrum of bound states of a Schrödinger type eigenvalue problem, we obtain the well-known “node rule”, according to which, if we arrange all the bound states in an increasing order of eigenvalues, the \( n \)-th eigenfunction has \( n-1 \) nodes [10]. This means that the ground state in the spectrum of the bound states, denoted by \( u_b \), has no zeroes throughout the interval \([a, b] \).

Finally, we consider the case where \( u_1 = u_b \) with \( \sigma_b^2 \) being the most negative eigenvalue of the system and \( u_2 = u_0 \) with \( \sigma_0^2 = 0 \). We multiply equation (25) by \( u_2 \) and equation (23) by \( u_1 \), subtract the two equations and integrate the result over the entire domain of \( r^* \). Then, we obtain:

\[
(u_\sigma p_u u_0 - u_0 p_u u_\sigma) \bigg|_{-\infty}^{\infty} = (\sigma_0^2 - \sigma_b^2) \int_{-\infty}^{\infty} \frac{E}{A} u_b u_0 dr^*
\]

By making use of the asymptotic behaviour of the eigenfunctions \( u_0 \) and \( u_b \) at both limits of the radial space, a behaviour which was discussed previously, it can
be easily seen that the left hand side of the above equation is zero. The same
must also hold for the right hand side of this equation. Note that the integral that
appears on the right hand side contains the product of three functions, $E/A$, $u_b$ and
$u_0$. The ratio of the coefficients $E$ and $A$, after the introduction of the “tortoise”
coordinate, is a well defined function over the entire radial space as it is shown
in Figure 3. Moreover, it is a positive definite function without any zeroes throughout
the radial domain. On the other hand, according to the “node rule”, the function $u_b$
is the ground state in the spectrum of bound states of the eigenvalue problem and,
as such, it has no zeroes as well. The function $u_0$ demands some more attention
since it corresponds to a vanishing value of the parameter $\sigma$. In this case, the time
dependence in the expressions (12) of the linear perturbations disappears and the
small radial parts can be absorbed into the large radial parts which describe the
classical Dilatonic Black Hole solutions. As a result, the eigenfunction $u_0$ can be
easily constructed out of the difference of any two of the curves in Figure 1. The
monotonic, non-intersecting behaviour of these curves ensures the absence of zeroes
in the expression of $u_0$. Since each one of the three functions that appear inside
the integral has a definite sign, their product will have a definite sign as well and
the integral can never vanish. Then, the only consistent case is the degenerate one
$\sigma_b^2 = \sigma_0^2 = 0$, which means that, actually, the ground state of the spectrum is the
one with the zero eigenvalue and that all of the bound states with negative values
of the parameter $\sigma^2$ are absent. Since the number of negative values of $\sigma^2$ is equal
to the number of the unstable modes of our solutions, the absence of bound states
implies the linear stability of the Dilatonic Black Holes.

Figure 3: The coefficient $E/A$ for the $r_h = 1$ and $\phi_h = -1$ black hole solution.
The definite sign of this coefficient is obvious. Also shown is the coefficient $E/A$
before the introduction of the “tortoise” coordinate.
4 Conclusions

By the use of a semi-analytic method, we have demonstrated the linear stability of the Dilatonic Black Holes, which arise in the framework of the Einstein-Dilaton-Gauss-Bonnet theory, under small spacetime-dependent perturbations. As a result, our black hole solutions have no reason to be rejected, even if we make use of the argument of stability, and the concept of the validity of the “no-hair” theorem among the stable solutions has proven to be false. Our result is extremely important since it renders our black hole solutions linearly stable and makes the long range dilaton field, or equivalently the “dilaton hair”, one of the very few examples of stable, although of “secondary type”, hair that bypasses the “no-hair” theorems of the Theory of General Relativity.

The stability itself of the Dilatonic Black Holes under spacetime-dependent perturbations is very important since the lifetime of these black hole solutions becomes substantially longer. As a result, Dilatonic Black Holes, that may have been formed in the past, are rather possible to have survived until the present epoch. In this case, the possibility of detecting them is not negligible at all and this might provide a test for the predictions of superstring theory at low energies.

As a final remark, we would like to note that the Schwarzschild Black Hole, which arises in the framework of the Theory of General Relativity, is also linearly stable. This means that our black hole solutions and the Schwarzschild Black Hole share the same kind of stability. Moreover, the Dilatonic Black Holes arise in the framework of the Einstein-Dilaton-Gauss-Bonnet theory which is a pure gravity theory—the dilaton field simply plays the role of the connecting link between the scalar curvature $R$ and the Gauss-Bonnet term. Taking into account the above as well as the fact that these two black hole solutions are characterized by the same parameter at infinity, the mass $M$, we are led to the conclusion that the Dilatonic Black Holes can be considered as the most direct generalisation of the Schwarzschild Black Hole in the framework of the effective superstring theory.

References

[1] P. Kanti, N.E. Mavromatos, J. Rizos, K. Tamvakis and E. Winstanley, Phys. Rev. D54 (1996), 5049.

[2] J. Bekenstein, Phys. Rev. D5 (1972), 1239; Phys. Rev. D51 (1995), 6608; A. Mayo and J. Bekenstein, Phys. Rev. D54 (1996), 5059.

[3] S.O. Alexeev and M.V. Pomazanov, Phys. Rev. D56 (1996), 2110.

[4] T. Torii, H. Yajima and K. Maeda, Phys. Rev. D55 (1996), 739.
[5] P. Kanti, N.E. Mavromatos, J. Rizos, K. Tamvakis and E. Winstanley, Phys. Rev. D57 (to appear in the 15th May 1998 issue), hep-th/9703192.

[6] S. Coleman, J. Preskill and F. Wilczek, Nucl. Phys. B378 (1992), 175.

[7] N. Straumann and Z.H. Zhou, Phys. Lett. B237 (1990), 353; Phys. Lett. B243 (1990), 33;
   M. Heusler, S. Droz, and N. Straumann, Phys. Lett. B285 (1992), 21.

[8] P. Boschung, O. Brodbeck, F. Moser, N. Straumann and M.S. Volkov, Phys. Rev. D50 (1994), 3842;
   M.S. Volkov and D.V. Gal’tsov, Phys. Lett. B341 (1995), 279;
   G. Lavrelashvili and D. Maison, Phys. Lett. B343 (1995), 214.
   M.S. Volkov, O. Brodbeck, G. Lavrelashvili, and N. Straumann, Phys. Lett. B349 (1995), 438;
   E. Winstanley and N.E. Mavromatos, Phys. Lett. B352 (1995), 242;
   O. Brodbeck and N. Straumann, Phys. Lett. B324 (1994); J. Math. Phys. 37 (1996), 1414;

[9] G. Birkhoff and G-C. Rota, *Ordinary Differential Equations*, (Wiley 1989).

[10] See for instance, A. Messiah, *Quantum Mechanics*, Vol. I (North-Holland Publishing Co., Amsterdam 1970).