A discreteness criterion for the automorphism group of an ˜A₂-building

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Abstract

Let ∆ be a locally finite thick building of type ˜A₂. We show that, if the type-preserving automorphism group Aut(∆)⁰ of ∆ is transitive on panels of each type, then either ∆ is Bruhat-Tits or Aut(∆) is discrete. For ˜A₂-buildings which are not panel-transitive but only vertex-transitive, we give additional conditions under which the same conclusion holds.

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1 Introduction

A (locally finite) thick ˜A₂-building ∆ can be characterized as a simply connected simplicial complex of dimension 2 such that all simplicial spheres of radius 1 around vertices are isomorphic to the incidence graph of a (finite) projective plane. In this paper, ∆ will always be a locally finite thick ˜A₂-building and we will see ∆ as a simplicial complex. The simplices of dimension 2 in ∆ (i.e. triangles) are the chambers of ∆, and those of dimension 1 (i.e. edges) are the panels of ∆. Of course a vertex of ∆ is a simplex of dimension 0. The type function t: P(∆) → {1, 2, 3} associates a type to each panel of ∆, so that each chamber of ∆ has one panel of each type. Each vertex v of ∆ is then

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adjacent to panels of two different types $a$ and $b$, and we say that the type of $v$ is $\{a, b\}$. It follows that each chamber has one vertex of each type: $\{1, 2\}$, $\{2, 3\}$ and $\{3, 1\}$.

In this paper, $\text{Aut}(\Delta)$ denotes the full automorphism group of $\Delta$ (as a simplicial complex), while $\text{Aut}(\Delta)^0$ is the subgroup of $\text{Aut}(\Delta)$ consisting of the automorphisms which preserve the types. It is clear that $[\text{Aut}(\Delta) : \text{Aut}(\Delta)^0] \leq 6$, so that the locally compact group $\text{Aut}(\Delta)$ (equipped with the topology of pointwise convergence) is non-discrete if and only if $\text{Aut}(\Delta)^0$ is non-discrete.

The known sources of examples of $\tilde{A}_2$-buildings are the following:

1. Following [Wei08], we say that $\Delta$ is Bruhat-Tits if its spherical building at infinity (which is a compact projective plane) is Moufang (see §2 for the definition of a Moufang projective plane). The only (locally finite and thick) Bruhat-Tits $\tilde{A}_2$-buildings are the ones associated to $\text{PGL}(3, D)$ for $D$ a finite dimensional division algebra over a local field, see [Wei08, Chapter 28]. In particular, it follows that $\text{Aut}(\Delta)$ is non-discrete when $\Delta$ is Bruhat-Tits. A $\tilde{A}_2$-building which is not Bruhat-Tits is called exotic.

2. On can construct $\tilde{A}_2$-buildings inductively, starting from a point $O$ and gluing triangles to the ball $B(O, r)$ of radius $r$ to obtain $B(O, r + 1)$. This kind of construction is explained in [Ron86] and [BP07], where it was observed that $\tilde{A}_2$-buildings can be “really” exotic. It is hard to have any information on the automorphism group of a building constructed in that way, but it will typically be trivial.

3. $\tilde{A}_2$-buildings with lattices have been studied a lot: some of them with a panel-regular lattice (see for instance [Ess13] and [Wit16]), others with a vertex-regular lattice (see [CMSZ93a] and [CMSZ93b]), and also one with a lattice having two orbits of vertices (see [Bar00, Section 3]). For the small examples, i.e. the ones with a small enough thickness (the number of chambers adjacent to a single panel), it could be checked with a computer that the automorphism group was discrete as soon as the building was exotic. Note that there exist exotic $\tilde{A}_2$-buildings with lattices and with arbitrarily large thickness, see [BCL16, Appendix D].

4. Some exotic $\tilde{A}_2$-buildings can also be constructed from valuations on planar ternary rings, see [VM87]. The automorphism group of the $\tilde{A}_2$-buildings constructed in that way in [VM90, Section 7] is vertex-transitive and non-discrete, but it fixes a vertex at infinity, and is thus not unimodular by [CM13, Theorem M] (in particular, it cannot contain any lattice). This remark about the non-unimodularity will be important when stating Theorem B below.

The goal of this paper is to provide sufficient conditions under which an exotic $\tilde{A}_2$-building has a discrete automorphism group. Our main result is the following.

**Theorem A.** Let $\Delta$ be a locally finite thick $\tilde{A}_2$-building and suppose that $\text{Aut}(\Delta)^0$ is transitive on panels of each type. Then either:

(a) $\Delta$ is Bruhat-Tits; or

(b) $\text{Aut}(\Delta)$ is discrete.

In particular, Theorem A applies to all locally finite thick $\tilde{A}_2$-buildings with a panel-regular lattice (see (3) above). A natural question to ask is whether the panel transitivity can be weakened in this theorem, and for instance replaced by vertex transitivity. Because of the $\tilde{A}_2$-building described in [VM90, Section 7] (see (4) above), such a result cannot be true in these general terms. The next theorem however gives additional hypotheses under which a similar conclusion can be obtained.

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Theorem B. Let $\Delta$ be a locally finite thick $\tilde{A}_2$-building. Suppose that $\text{Aut}(\Delta)$ is transitive on vertices and unimodular, that $\text{Aut}(\Delta)^0$ is transitive on vertices of each type, and that $\Delta$ has thickness $p+1$ for some prime $p$. Then either:

(a) $\Delta$ is Bruhat-Tits; or
(b) $\text{Aut}(\Delta)$ is discrete.

Theorem B can in particular be applied to the locally finite thick $\tilde{A}_2$-buildings $\Delta$ with a vertex-regular lattice (see (2) above) as soon as the thickness of $\Delta$ is $p+1$ for some prime $p$ (i.e. the local projective planes in $\Delta$ have order $p$).

Theorems A and B can also be viewed as results giving weak hypotheses on $\text{Aut}(\Delta)$ under which $\Delta$ is automatically Bruhat-Tits. It was proved in [VMVS98] by Van Maldeghem and Van Steen that $\Delta$ is Bruhat-Tits as soon as $\text{Aut}(\Delta)$ is Weyl-transitive. Recall that $\text{Aut}(\Delta)$ is Weyl-transitive if for any two pairs of chambers $(c, d), (c', d')$ in $\Delta$ with equal Weyl-distances ($\delta(c, d) = \delta(c', d')$), there exists $g \in \text{Aut}(\Delta)^0$ such that $g(c) = c'$ and $g(d) = d'$. Theorem A actually shows that having $\text{Aut}(\Delta)^0$ transitive on panels of each type and non-discrete (which is strictly weaker than requiring the Weyl-transitivity) is already sufficient to have the same conclusion. Our proof of Theorem A actually uses the machinery developed by the authors in [VMVS98].

Note that the fact that Weyl-transitivity implies Bruhat-Tits was later proved to be true in any Euclidean building. Indeed, if $X$ is an Euclidean building and if $\text{Aut}(X)$ is Weyl-transitive, then $\text{Aut}(X^\infty)$ is strongly transitive on the building at infinity $X^\infty$ (i.e. transitive on pairs $(A, c)$ where $A$ is an apartment of $X^\infty$ and $c$ is a chamber of $A$), which implies that $\text{Aut}(X)$ is strongly transitive on $X$ by [CC15, Theorem 1.1] and then that $X$ is Bruhat-Tits by [CM15, Corollary E] or [Rad17, Corollary B].

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2 Preliminaries about Hjelmslev planes

This section gives the definition and first properties of Hjelmslev planes, which will be of central importance in the whole text. It is largely inspired from [VMVS98].

Given a vertex $O$ in $\Delta$ and a natural number $n \geq 1$, we define the geometry $^{n}H(O)$ as follows. The geometry $^{1}H(O)$ is just the residue of $O$, which is a projective plane. So the points of $^{1}H(O)$ are certain vertices of $\Delta$ adjacent to $O$, and similarly for the lines of $^{1}H(O)$. Now for $n \geq 1$, the points (respectively lines) of $^{n}H(O)$ are the sequences $(v_1, v_2, \ldots, v_n)$ of vertices of $\Delta$, where $v_1$ is a point (resp. a line) of $^{1}H(O)$ and $\{v_{i-1}, v_{i+1}\}$ is a pair of non-incident point and line in $^{1}H(v_i)$ (where $v_0 := O$). A point $(p_1, p_2, \ldots, p_n)$ of $^{n}H(O)$ is incident with a line $(\ell_1, \ell_2, \ldots, \ell_n)$ if all vertices $p_1, \ldots, p_n, \ell_1, \ldots, \ell_n$ are contained in a common apartment and if $p_1$ and $\ell_1$ are adjacent in $\Delta$. This geometry $^{n}H(O)$ is called a projective Hjelmslev plane of level $n$. When the vertex $O$ has no real importance, we write $^nH$ instead of $^{n}H(O)$. The point set (respectively line set) of $^nH$ is then $^nP$ (resp. $^n\mathcal{L}$), while incidence is denoted by $^n\iota$. We will also sometimes identify an element $(v_1, v_2, \ldots, v_n)$ of $^nH$ with the vertex $v_n$ of $\Delta$ (the other vertices $v_1, \ldots, v_{n-1}$ being uniquely determined by $v_n$).

For $i \leq n$, the natural morphism from $^iH$ to $^IH$ is denoted by $^i\pi$. If $P, Q \in ^nP$ satisfy $^i\pi(P) = ^i\pi(Q)$ for some $0 < i \leq n$, then we call $P$ and $Q$ $i$-neighboring. For $i = 1$ we
just say that $P$ and $Q$ are neighboring. Similarly for lines. Also, if $P \in {}^n\mathcal{P}$ and $\ell \in {}^n\mathcal{L}$ are such that $\pi(P) \not\in \pi(\ell)$ for some $0 < i \leq n$ then we say that $P$ is $i$-near $\ell$. Once again, $P$ is near $L$ when $i = 1$.

A collineation $\alpha$ of $^n\mathcal{H}$ is, as usual, a bijection from $^n\mathcal{P}$ to itself and a bijection from $^n\mathcal{L}$ to itself which preserve $^nI$. It is not hard to see that all $i$-neighboring relations are determined by the geometry of $^n\mathcal{H}$, so that every collineation $\alpha$ of $^n\mathcal{H}$ induces in $^1\mathcal{H}$ a unique collineation $\alpha^*$. When acting on elements of $^1\mathcal{H}$, $\alpha^*$ will sometimes be replaced by $\alpha$, so as to simplify the notation. For a fixed vertex $O$ in $\Delta$, the group of all collineations of $^n\mathcal{H}(O)$ which are induced from an automorphism in $\text{Aut}((\Delta)^0$ fixing $O$ will be denoted by $^n\Psi(O)$. When $\alpha \in {}^n\Psi(O)$ is induced by $g \in \text{Aut}(\Delta)^0$, it will be convenient to talk about the action of $\alpha$ (instead of $g$) on $\Delta$.

Given $P \in {}^n\mathcal{P}$ and $\ell \in {}^n\mathcal{L}$ with $P \not\in \ell$, an elation of $^n\mathcal{H}$ with axis $\ell$ and center $P$ is a collineation of $^n\mathcal{H}$ fixing all points incident with $\ell$ and fixing all lines incident with $P$. As the next lemma shows, such an elation also fixes additional points and lines.

**Lemma 2.1.** Let $\alpha$ be an elation of $^n\mathcal{H}$ ($n \geq 2$) with axis $\ell$ and center $P$. Then $\alpha$ fixes all points (respectively lines) of $^n\mathcal{H}$ that are $(n - 1)$-neighboring $P$ (resp. $\ell$).

*Proof.* See [VMVS98, Lemma 5]. □

An elation $\alpha$ of $^n\mathcal{H}$ such that $\alpha^{*-1}$ is trivial is called a $^1h$-collineation of $^n\mathcal{H}$. (All elations of $^1\mathcal{H}$ are $^1h$-collineations.) By definition, an elation $\alpha$ with axis $\ell$ and center $P$ fixes all points incident with $\ell$ and all lines incident with $P$. The following lemma states that when $\alpha$ is a $^1h$-collineation, it also fixes the points near $\ell$ and the lines near $P$.

**Lemma 2.2.** Let $\alpha$ be a $^1h$-collineation of $^n\mathcal{H}$ with axis $\ell$ and center $P$. Then $\alpha$ fixes all points (respectively lines) of $^n\mathcal{H}$ that are near $\ell$ (resp. $P$).

*Proof.* See [VMVS98, Lemma 14]. □

We then get the following result as a direct consequence.

**Lemma 2.3.** Let $\alpha$ be a $^1h$-collineation of $^n\mathcal{H}$ with axis $\ell$ and center $P$. Then for each $\ell' \in {}^n\mathcal{L}$ neighboring $\ell$ and each $P' \in {}^n\mathcal{P}$ neighboring $P$, $\alpha$ is also a $^1h$-collineation with axis $\ell'$ and center $P'$.

*Proof.* By Lemma 2.2, $\alpha$ fixes all points (respectively lines) of $^n\mathcal{H}$ that are near $\ell$ (resp. $P$). Given $P'$ neighboring $P$ and $\ell'$ neighboring $\ell$, this is equivalent to saying that $\alpha$ fixes all points (resp. lines) that are near $\ell'$ (resp. $P'$). In particular, $\alpha$ fixes all points (resp. lines) incident with $\ell'$ (resp. $P'$), which means that $\alpha$ is an elation (and thus a $^1h$-collineation) with axis $\ell'$ and center $P'$. □

The following lemma also comes from [VMVS98]. For $n = 1$, this is a particular case of the well-known result [Tit74, Theorem 4.1.1].

**Lemma 2.4.** Let $\alpha$ be a non-trivial $^1h$-collineation of $^n\mathcal{H}$ with axis $\ell$ and center $P$. Then $\alpha$ does not fix any point (respectively line) of $^n\mathcal{H}$ which is not near $\ell$ (resp. $P$).

*Proof.* See [VMVS98, Lemma 16 (iv)]. □

From this lemma we can easily deduce the more general next result.

**Lemma 2.5.** Let $\alpha$ be a non-trivial elation of $^n\mathcal{H}$ with axis $\ell$ and center $P$. Then $\alpha$ does not fix any point (respectively line) of $^n\mathcal{H}$ which is not near $\ell$ (resp. $P$). In particular, if $m \in {}^n\mathcal{L}$ is incident with $P$ but not neighboring $\ell$, then the group of all elations with axis $\ell$ and center $P$ acts freely on the points incident with $m$ but not neighboring $P$. □
Proof. Let us prove it by induction on $n$. For $n = 1$, this is equivalent to Lemma 2.4. Now assume the assertion is proved in $n^{-1}H$ and let $\alpha$ be a non-trivial elation of $nH$ with axis $\ell$ and center $P$. It is clear that $\alpha^{n-1}$ is an elation of $n^{-1}H$, with axis $n^{-1}A(\ell)$ and center $n^{-1}A(P)$. If $\alpha^{n-1}$ is trivial then $\alpha$ is a $h$-collineation of $nH$ and we can directly apply Lemma 2.4 to conclude. If on the contrary $\alpha^{n-1}$ is not trivial then it is a non-trivial elation of $n^{-1}H$ and the result follows from the induction hypothesis. \hfill \Box

We now explain what it means for $nH$ to be Moufang. First fix $P \in nP$ and $\ell \in n\ell$ with $P \not\in \ell$. Given $m \in n\ell$ incident with $P$ but not neighboring $\ell$, we say that $nH$ is $(P, \ell)$-transitive if the group of all elations with axis $\ell$ and center $P$ acts transitively on the points incident with $m$ but not neighboring $P$. In view of Lemma 2.5, this condition does not depend on the choice for $m$ and the action is then automatically simply transitive. When $nH$ is $(P, \ell)$-transitive for all $P \in nP$ and all $\ell \in n\ell$ with $P \not\in \ell$, we say that $nH$ is Moufang. For $n = 1$, this definition is of course equivalent to the definition of a Moufang projective plane.

Finally, for our future needs we give a name to some vertices of $\Delta$. Given $P \in nP(O)$ and $\ell \in n\ell(O)$ with $P \not\in \ell$ (where $O$ is a vertex of $\Delta$), the consecutive vertices of the geodesic from $P$ to $\ell$ in $\Delta$ are denoted by $P = v_0(P, \ell), v_1(P, \ell), \ldots, v_n(P, \ell) = \ell$.

## 3 Proof of Theorem A

The goal of this section is to prove Theorem A. The idea is to assume that $\text{Aut}(\Delta)^0$ is transitive on panels of each type and non-discrete, and to show that all Hjelmslev planes $nH(O)$ in $\Delta$ are Moufang. This will then imply that the building at infinity of $\Delta$ is Moufang, i.e. that $\Delta$ is Bruhat-Tits.

### 3.1 Observing non-discreteness

In this first subsection, we observe that the non-discreteness of $\text{Aut}(\Delta)$ together with its transitivity on vertices of each type implies the existence of many non-trivial $h$-collineations in $\text{Aut}(\Delta)$. We start with an easy result valid in any $\tilde{A}_2$-building $\Delta$.

**Lemma 3.1.** Let $v_0, \ldots, v_k$ be consecutive vertices of a wall of $\Delta$. Consider vertices $w_0, \ldots, w_{k-1}$ with $w_i$ adjacent to $v_i, v_{i+1}$ and $w_{i-1}$ (if $i \geq 1$) for each $i \in \{0, \ldots, k-1\}$. Similarly, consider vertices $w'_0, \ldots, w'_{k-1}$ with $w'_i$ adjacent to $v_i, v_{i+1}$ and $w'_{i-1}$ (if $i \geq 1$) for each $i \in \{0, \ldots, k-1\}$. If $g \in \text{Aut}(\Delta)^0$ fixes $v_0, \ldots, v_k$ and if $g(w_0) = w'_0$, then $g(w_i) = w'_i$ for each $i \in \{0, \ldots, k-1\}$.

**Proof.** For each $i \in \{1, \ldots, k-1\}$, the fact that $g$ fixes $v_{i-1}, v_i$ and $v_{i+1}$ clearly implies that $g$ sends $w_{i-1}$ to $w'_{i-1}$ if and only if $g$ sends $w_i$ to $w'_i$ (see Figure 1 for an illustration). The conclusion then follows immediately. \hfill \Box

![Figure 1: Illustration of Lemma 3.1.](image-url)
This enables us to show the following.

**Lemma 3.2.** Let $O$ be a vertex of $\Delta$ and let $\alpha \in \Psi(O)$ ($n \geq 2$) be a non-trivial collineation such that $\alpha^{*n-1}$ is trivial. Then there exists $P \in \mathcal{P}(O)$ and $\ell \in \mathcal{L}(O)$ with $P \not\mathcal{H} \ell$ and such that $\alpha$ does not fix $v_1(P, \ell), v_2(P, \ell), \ldots, v_{n-1}(P, \ell)$.

**Proof.** For any $P \in \mathcal{P}(O)$ and $\ell \in \mathcal{L}(O)$ with $P \not\mathcal{H} \ell$ we know by Lemma 3.1 (and since $\alpha^{*n-1}$ is trivial) that either all vertices $v_1(P, \ell), \ldots, v_{n-1}(P, \ell)$ are fixed by $g$ or none of them is fixed by $g$.

We therefore proceed by contradiction, assuming that for all such $P$ and $\ell$, all the vertices $v_1(P, \ell), \ldots, v_{n-1}(P, \ell)$ are fixed by $g$. We show that, in this case, $\alpha$ is trivial (which gives the contradiction). Consider any point $P \in \mathcal{P}(O)$, and write $Q = \mathcal{P}(P)$ and $R = \mathcal{N}(P)$ (see Figure 2). The fact that $\alpha^{*n-1}$ is trivial implies that $g$ fixes all chambers having both vertices $Q$ and $R$. Now let $c$ and $d$ be two of these chambers and denote by $S$ (resp. $T$) the third vertex of $c$ (resp. $d$). Because of our assumption, $g$ fixes all chambers having vertices $Q$ and $S$ and all chambers having vertices $Q$ and $T$. This implies that $g$ is trivial on $\mathcal{H}(Q)$, hence it fixes $P$. This can be done for any choice of a point $P \in \mathcal{P}(O)$, and similarly for any choice of a line $\ell \in \mathcal{L}(O)$, so $\alpha$ is trivial. \hfill \Box

![Figure 2: Illustration of Lemma 3.2.](image)

**Proposition 3.3.** Suppose that $\text{Aut}(\Delta)$ is non-discrete and transitive on vertices of each type. Then for each vertex $O$ in $\Delta$ and each $n \geq 1$, there exists a non-trivial $\mathcal{H}$-collineation in $\Psi(O)$.

**Proof.** We fix $n \geq 1$ and show that there exists at least one non-trivial $\mathcal{H}$-collineation in $\Psi(O)$ for each vertex $O$ in $\Delta$. In view of the transitivity of $\text{Aut}(\Delta)$ on vertices of each type, it suffices to find three vertices $O_1, O_2, O_3$ of respective types $\{2,3\}, \{3,1\}$ and $\{1,2\}$ such that $\Psi(O_i)$ contains a non-trivial $\mathcal{H}$-collineation for each $i \in \{1,2,3\}$.

Fix some vertex $O_0$ in $\Delta$. The non-discreteness of $\text{Aut}(\Delta)$ implies that there exist infinitely many $N \geq 1$ such that $\Psi(O_0)$ contains a non-trivial collineation $\alpha$ with $\alpha^{*N-1}$ trivial. Consider one such $N$ with $N \geq n + 4$ and one such $\alpha \in \Psi(O_0)$. By Lemma 3.2, there exists $P \in \mathcal{P}(O_0)$ and $\ell \in \mathcal{L}(O_0)$ with $P \not\mathcal{H} \ell$ and such that none of the vertices $v_1(P, \ell), \ldots, v_{N-1}(P, \ell)$ is fixed by $\alpha$. Now write $X = \mathcal{N}(P)$ and $Y = \mathcal{N}(\ell)$ (see Figure 3). As $N-n \geq 4$, the geodesic from $X$ to $Y$ in $\Delta$ contains at least three vertices different from $X$ and $Y$. Since three consecutive vertices in such a configuration always have the three different types, there exist $O_1, O_2$, and $O_3$ with types $\{2,3\}, \{3,1\}$ and $\{1,2\}$ and strictly between $X$ and $Y$. For each $i \in \{1,2,3\}$, the action induced by $\alpha$ on $\mathcal{H}(O_i)$ is non-trivial, because $\alpha$ acts non-trivially on $v_1(P, \ell), \ldots, v_{N-1}(P, \ell)$. There remains to check that it is a $\mathcal{H}$-collineation of $\mathcal{H}(O_i)$, but this is a clear consequence of the fact that $\alpha^{*N-1}$ is trivial. \hfill \Box

The previous proposition shows the existence of a non-trivial $\mathcal{H}$-collineation in $\Psi(O)$, in some circumstances. We already know some properties of such collineations (see Lemma 2.4), but the next lemma is even more precise.
Lemma 3.4. Let $O$ be a vertex of $\Delta$ and consider $P \in {}^n\mathcal{P}(O)$ and $\ell \in {}^n\mathcal{L}(O)$ with $P \not\sim \ell$ $(n \geq 2)$. Let also $Q \in {}^n\mathcal{P}(O)$ be a point not near $\ell$ and $o \in {}^n\mathcal{L}(O)$ be a line not near $P$, such that $Q \not\sim o$.

(1) Let $\alpha \in {}^n\Psi(O)$ be a non-trivial $1^h$-collineation with axis $\ell$ and center $P$. Then $\alpha$ does not fix $v_i(Q, o)$, for any $i \in \{0, 1, \ldots, n\}$.

(2) Denote by $m \in {}^n\mathcal{L}(O)$ the line incident with $P$ and $Q$ (see Figure 4). We know by definition of an elation that $\alpha$ fixes $m$, and the fact that $\alpha^n$ is trivial implies that it also fixes $n^{1-}\pi(Q)$. Hence, from Lemma 3.1 applied to the segment from $n^{1-}\pi(m)$ to $n^{1-}\pi(Q)$, we get that $\alpha$ fixes $v_1(Q, m)$. Assertions (1) and (2) then follow thanks to another application of Lemma 3.1 to the segment with vertices $v_1(Q, m), n^{1-}\pi(Q), v_1(n^{1-}\pi(Q), n^{1-}\pi(o)), \ldots, n^{1-}\pi(o)$. (Recall, for (1), that when $\alpha$ is non-trivial it does not fix $Q$ nor $o$ by Lemma 2.4.)

Proof. Let $\alpha \in {}^n\Psi(O)$ be a $1^h$-collineation with axis $\ell$ and center $P$. Let also $m \in {}^n\mathcal{L}(O)$ be the line incident with $P$ and $Q$ (see Figure 4). We know by definition of an elation that $\alpha$ fixes $m$, and the fact that $\alpha^n$ is trivial implies that it also fixes $n^{1-}\pi(Q)$. Hence, from Lemma 3.1 applied to the segment from $n^{1-}\pi(m)$ to $n^{1-}\pi(Q)$, we get that $\alpha$ fixes $v_1(Q, m)$. Assertions (1) and (2) then follow thanks to another application of Lemma 3.1 to the segment with vertices $v_1(Q, m), n^{1-}\pi(Q), v_1(n^{1-}\pi(Q), n^{1-}\pi(o)), \ldots, n^{1-}\pi(o)$. (Recall, for (1), that when $\alpha$ is non-trivial it does not fix $Q$ nor $o$ by Lemma 2.4.)

Proof. Let $\alpha \in {}^n\Psi(O)$ be a $1^h$-collineation with axis $\ell$ and center $P$. Let also $m \in {}^n\mathcal{L}(O)$ be the line incident with $P$ and $Q$ (see Figure 4). We know by definition of an elation that $\alpha$ fixes $m$, and the fact that $\alpha^n$ is trivial implies that it also fixes $n^{1-}\pi(Q)$. Hence, from Lemma 3.1 applied to the segment from $n^{1-}\pi(m)$ to $n^{1-}\pi(Q)$, we get that $\alpha$ fixes $v_1(Q, m)$. Assertions (1) and (2) then follow thanks to another application of Lemma 3.1 to the segment with vertices $v_1(Q, m), n^{1-}\pi(Q), v_1(n^{1-}\pi(Q), n^{1-}\pi(o)), \ldots, n^{1-}\pi(o)$. (Recall, for (1), that when $\alpha$ is non-trivial it does not fix $Q$ nor $o$ by Lemma 2.4.)

Figure 3: Illustration of Proposition 3.3.

Figure 4: Illustration of Lemma 3.4.
3.2 Non-discrete and panel transitive implies chamber transitive

In this subsection, we prove that if Aut(Δ) is non-discrete and transitive on i-panels for each i ∈ {1, 2, 3}, then Aut(Δ) is even transitive on chambers. We start by the following easy lemma, valid in any projective Hjelmslev plane of level 1 (i.e. any projective plane).

**Lemma 3.5.** Let α be a non-trivial elation of 1H with axis ℓ and center P. Let m ∈ 1L be incident with P but different from ℓ. Then the permutation induced by α on the set of q points incident with m but different from P is a product of k ≥ 1 disjoint cycles of the same length c ≥ 2, where k · c = q. Moreover, k and c do not depend on m.

**Proof.** Let σ be the permutation induced by α on this set of q points. By Lemma 2.4, σ has no fixed point. Now it suffices to prove that two cycles in the cycle decomposition of σ always have the same length. Suppose for a contradiction that there are two cycles of different lengths c1 < c2. Then αc1 is an elation of 1H which is non-trivial (because σc1 is non-trivial) and σc1 has fixed points, which contradicts Lemma 2.4. So all k cycles in the cycle decomposition must have the same length c ≥ 2, and of course k · c = q. Note that k and c do not depend on m, otherwise we would once again get a power of α which is non-trivial but has forbidden fixed points.

**Proposition 3.6.** Suppose that Aut(Δ) is non-discrete and transitive on panels of each type. Then Aut(Δ) is chamber-transitive.

**Proof.** Assume for a contradiction that Aut(Δ) is not chamber-transitive. Then we can color the chambers of Δ in blue and red so that each color is used at least once and two chambers with different colors do not belong to the same orbit. (For instance, color one orbit of chambers in blue and all other orbits in red.) For each i ∈ {1, 2, 3}, the transitivity on i-panels implies that there exist bi and ri such that all i-panels contain bi blue chambers and ri red chambers. Note that bi ≥ 1 and ri ≥ 1, otherwise all chambers of Δ would be of the same color. In Δ, each panel has the same number of chambers, say 1 + q, so bi + ri = 1 + q for each i ∈ {1, 2, 3}.

We first claim that b1 = b2 = b3 (and r1 = r2 = r3). Indeed, take i, j ∈ {1, 2, 3} with i ≠ j and consider a vertex v of type {i, j} in Δ. The number of blue chambers adjacent to v (i.e. in the residue corresponding to v) is equal to pi · bi, where pi is the number of i-panels adjacent to v. Since the residue associated to v is a projective plane of order q, we have pi = q2 + q + 1 and the number of blue chambers adjacent to v is (q2 + q + 1) · bi. But for the same reason with j instead of i, this number is also equal to (q2 + q + 1) · bj. So bi = bj and ri = rj. In the following we therefore write b = b1 = b2 = b3 and r = r1 = r2 = r3. Recall that b + r = 1 + q.

Now consider a vertex O in Δ and a non-trivial elation α in 1Ψ(O), whose existence is ensured by Proposition 3.3. Let P ∈ 1P(O) and ℓ ∈ 1L(O) be the center and axis of the elation α. Consider m ∈ 1L(O) a line incident with P but different from ℓ. By Lemma 3.5, the permutation induced by α on the set of q points incident with m but different from P is a product of k ≥ 1 cycles of length c ≥ 2, with k · c = q. If the chamber with vertices O, P and m is blue, then this implies that b ≡ 1 (mod c) and r ≡ 0 (mod c). If it is red, then r ≡ 1 (mod c) and b ≡ 0 (mod c). But c does not depend on m, so this reasoning is valid for any choice of m. As b cannot be congruent to both 0 and 1 modulo c (because c ≥ 2), this means that all the chambers with vertices O, P and some m ≠ ℓ have the same color. We can assume that this common color is blue, so that b ≡ 1 (mod c) and r ≡ 0 (mod c). In particular we have r ≥ c ≥ 2, but this is a contradiction with the fact that there is at most one red chamber in the panel defined by O and P.

As a conclusion, such a coloring cannot exist and Aut(Δ) is chamber-transitive. □
3.3 Non-discrete and chamber transitive implies locally Moufang

The following theorem is due to Kantor [Kan87] and concerns finite projective planes with a collineation group transitive on incident point-line pairs. This result will be helpful to get a local information about Aut($\Delta$).

**Theorem 3.7** (Kantor, 1987). Let $\Pi$ be a projective plane of order $q$, and let $F$ be a collineation group of $\Pi$ transitive on incident point-line pairs. Then either

(i) $\Pi$ is Desarguesian and $F \geq \text{PSL}(3,q)$, or

(ii) $F$ is a Frobenius group of odd order $(q^2 + q + 1)(q + 1)$, and $q^2 + q + 1$ is prime.

**Proof.** See [Kan87, Theorem A].

**Corollary 3.8.** Suppose that Aut($\Delta$)$^0$ is non-discrete and chamber-transitive. Then for each vertex $O$ in $\Delta$, the projective plane $^1H(O)$ is Desarguesian and $^1\Psi(O) \geq \text{PSL}(3,q)$, where $q + 1$ is the number of chambers in each panel of $\Delta$. In particular, $^1H(O)$ is Moufang and $^1\Psi(O)$ contains all elations of $^1H(O)$.

**Proof.** For any vertex $O$ in $\Delta$, $^1H(O)$ is a projective plane of order $q$. The chamber-transitivity of $\Delta$ directly implies that $^1\Psi(O)$ is transitive on incident point-line pairs of $^1H(O)$. Hence, by Theorem 3.7, either $^1H(O)$ is Desarguesian and $^1\Psi(O) \geq \text{PSL}(3,q)$, or $|^1\Psi(O)| = (q^2 + q + 1)(q + 1)$. We only need to show that the latter is impossible. Note that there are exactly $(q^2 + q + 1)(q + 1)$ incident point-line pairs in $^1H(O)$, so the equality $|^1\Psi(O)| = (q^2 + q + 1)(q + 1)$ would imply that the action of $^1\Psi(O)$ on these point-line pairs is free. However, by Proposition 3.3, there exists a non-trivial elation in $^1\Psi(O)$. So the action is not free and the statement stands proven.

Note that, for a finite projective plane $\Pi$ (say of order $q$), being Desarguesian is equivalent to being Moufang. Also, in this case, the group generated by all elations of $\Pi$ is called the *little projective group* and is exactly $\text{PSL}(3,q)$.

3.4 Non-discrete and chamber transitive implies Bruhat-Tits

We have seen with Corollary 3.8 that $^1H(O)$ is Moufang as soon as Aut($\Delta$)$^0$ is non-discrete and chamber-transitive. Our next goal is to show that all $^nH(O)$ are Moufang under these hypotheses.

We start with the next easy corollary of Proposition 3.3.

**Lemma 3.9.** Suppose that Aut($\Delta$)$^0$ is non-discrete and chamber-transitive. Then for each vertex $O$ in $\Delta$, each $n \geq 1$ and each point $P \in ^nP(O)$ and line $\ell \in ^nL(O)$ with $P$ $^nI \ell$, there exists a non-trivial $^1h$-collineation in $^n\Psi(O)$ with axis $\ell$ and center $P$.

**Proof.** By Proposition 3.3, there exists a non-trivial $^1h$-collineation $\alpha \in ^n\Psi(O)$, say with axis $\ell' \in ^nL(O)$ and center $P' \in ^nP(O)$. Let $c$ (resp. $c'$) be the chamber of $\Delta$ with vertices $O$, $^1\pi(\ell)$ and $^1\pi(P)$ (resp. $O$, $^1\pi(\ell')$ and $^1\pi(P')$). Since Aut($\Delta$)$^0$ is chamber-transitive, there exists $g \in \text{Aut}(\Delta)^0$ such that $g(c) = c'$. Then $g^{-1}og$ is a non-trivial $^1h$-collineation, and by Lemma 2.3 it has axis $\ell$ and center $P$.

**Lemma 3.10.** In the following, $O$ is a vertex of $\Delta$, $n$ is an integer $\geq 1$, $P$ is a point in $^nP(O)$ and $\ell$ is a line in $^nL(O)$ with $P$ $^nI \ell$, $Q$ is a point in $^nP(O)$ not near $\ell$, and $m \in ^nL(O)$ is the line incident with $P$ and $Q$. 


(1) Suppose that for any $O$, $n$, $P$ and $\ell$, there exists a non-trivial $h$-collineation in $n\Psi(O)$ with axis $\ell$ and center $P$. Then for any $O$, $n$, $P$ and $\ell$ and each $1 \leq k \leq n$, there exists an elation $\alpha \in n\Psi(O)$ with axis $\ell$ and center $P$ such that $\alpha^{*k-1}$ is trivial but $\alpha^k$ is non-trivial.

(2) Suppose that for any $O$, $n$, $P$, $\ell$ and $Q$, the group of all $h$-collineations in $n\Psi(O)$ with axis $\ell$ and center $P$ acts transitively on the set of points $(n-1)$-neighboring $Q$ and incident with $m$. Then for any $O$, $n$, $P$, $\ell$ and each $1 \leq k \leq n$, the group of all elations $\alpha \in n\Psi(O)$ with axis $\ell$ and center $P$ and with $\alpha^{*k-1}$ trivial is transitive on the set of points in $n\mathcal{P}(O)$ which are $(k-1)$-neighboring $\kappa(m)$ and incident with $\kappa(m)$.

Proof. Fix $O$, $n$, $\ell$, $P$ and $1 \leq k \leq n$. Let $A$ be an appartment of $\Delta$ containing $O$, $\ell$ and $P$ (seen as vertices of $\Delta$). In $A$, we denote by $O'$ the reflection of $O$ over the line through $\ell$ and $P$ (see Figure 5). Also, $P'$ (respectively $\ell'$) is the vertex of $A$ at distance $2n + k$ from $O'$ such that $O'$ lies on the segment from $\ell$ to $P'$ (resp. from $P$ to $\ell'$).

We first prove (1). By hypothesis, there exists a non-trivial $h$-collineation $\beta \in 2n+k\Psi(O')$ with axis $\ell'$ and center $P'$. We now consider the element $\alpha \in n\Psi(O)$ induced by $\beta$. The fact that $\beta^{*2n+k-1}$ is trivial implies that $\alpha^{*k-1}$ is trivial. Also, it is clear from Lemma 3.4 (1) applied to $\beta$ that $\alpha^k$ is non-trivial. There remains to show that $\alpha$ is an elation of $\mathcal{H}(O)$ with axis $\ell$ and center $P$, i.e. that $\alpha$ fixes all points incident with $\ell$ and all lines incident with $P$. This is actually also a consequence from the fact that $\beta^{*2n+k-1}$ is trivial. Indeed, all points incident with $\ell$ (and all lines incident with $P$) in $\mathcal{H}(O)$ correspond to vertices of $\Delta$ which are contained in $2n\mathcal{P}(O)$ (more precisely, contained in the convex hull of the vertices of $\Delta$ associated to $2n\mathcal{P}(O)$ and $2n\mathcal{L}(O)$).

![Figure 5: Illustration of Lemma 3.10.](image-url)
The reasoning is the same for (2). Take $Q \in \mathcal{P}(O)$ a point not near $\ell$ and denote by $m \in \mathcal{L}(O)$ the line incident with $P$ and $Q$. Here also, we see $Q$ and $m$ as vertices of $\Delta$ and we can even assume that they belong to $A$. Let $Q'$ be the vertex of $A$ at distance $2n + k$ from $O'$, in the direction of $P$ and $m$. If $m' \in 2n+k\mathcal{H}(O')$, then the hypothesis states that the group of all $h$-collineations in $2n+k\Psi(O')$ with axis $\ell'$ and center $P'$ acts transitively on the set of points $(2n+k-1)$-neighboring $Q'$ and incident with $m'$. Using Lemma 3.4 (2) and as for (1), we obtain that the group of all elations $\alpha \in \mathfrak{g}(O)$ with axis $\ell$ and center $P$ with $\alpha^*k^{-1}$ trivial is transitive on the set of points in $k\mathcal{P}(O)$ which are $(k-1)$-neighboring $k\pi(Q)$ and incident with $k\pi(m)$.

**Proposition 3.11.** Suppose that $\text{Aut}(\Delta)^0$ is non-discrete and chamber-transitive. Let $O$ be a vertex in $\Delta$, let $n \geq 1$ and consider a point $P \in \mathcal{P}(O)$ and a line $\ell \in \mathcal{L}(O)$ with $\ell P \in \mathfrak{I}(O)$. Let $Q \in \mathcal{P}(O)$ be a point not near $\ell$ and denote by $m \in \mathcal{L}(O)$ the line incident with $P$ and $Q$. Then the group of all $h$-collineations in $\Psi(O)$ with axis $\ell$ and center $P$ acts transitively on the set of points $(n-1)$-neighboring $Q$ and incident with $m$.

**Proof.** We proceed by induction on $n$. We introduce the three following assertions, all depending on $n \geq 1$ (actually $n \geq 2$ for $(C_n)$). Remark that $(A_n)$ is exactly what we need to prove.

$(A_n)$ Let $O$ be a vertex in $\Delta$. Let $P \in \mathcal{P}(O)$ and $\ell \in \mathcal{L}(O)$ be such that $\ell P \in \mathfrak{I}(O)$, let $Q \in \mathcal{P}(O)$ be a point not near $\ell$ and denote by $m$ the line incident with $P$ and $Q$. The group of all $h$-collineations in $\Psi(O)$ with axis $\ell$ and center $P$ acts transitively on the set of points $(n-1)$-neighboring $Q$ and incident with $m$.

$(B_n)$ Let $(i,j,k)$ be a permutation of $(1,2,3)$ and let $f$ be the word $ijikijikj\ldots$ of length $2n$. For any gallery $(c_0,c_1,\ldots,c_{2n})$ of type $f$ in $\Delta$ and for any two chambers $d$ and $d'$ adjacent to both $c_0$ and $c_1$, there exists an automorphism of $\Delta$ fixing $c_0,c_1,\ldots,c_{2n}$ and sending $d$ to $d'$.

$(C_n)$ Let $O$ be a vertex in $\Delta$. Let $P \in \mathcal{P}(O)$ and $\ell \in \mathcal{L}(O)$ be such that $\ell P \in \mathfrak{I}(O)$, let $Q \in \mathcal{P}(O)$ be a point near $\ell$ but not neighboring $P$, and let $m,o \in \mathcal{L}(O)$ be two lines near $Q$ but not neighboring $\ell$. There exist a point $P' \in \mathcal{P}(O)$ $(n-1)$-neighboring $P$, a line $\ell' \in \mathcal{L}(O)$ neighboring $\ell$ (with $P' \in \mathfrak{I}(\ell')$) and an elation in $\Psi(O)$ with axis $\ell'$ and center $P'$ sending $\ell \pi(m)$ to $\ell \pi(o)$.

First note that $(A_1)$ and $(B_1)$ are true because of Corollary 3.8. We now show three different relations between these assertions. For each one, we suppose $n \geq 2$.

**Claim 1.** $(B_{n-1}) + (C_n) \Rightarrow (A_n)$.

**Proof of the claim:** Let $O$, $P$, $\ell$, $Q$ and $m$ be as in $(A_n)$. Let also $R \in \mathcal{P}(O)$ be a point $(n-1)$-neighboring $Q$ and incident with $m$ (see Figure 6). We want to prove that there exists some $h$-collineation in $\Psi(O)$ with axis $\ell$ and center $P$, sending $Q$ to $R$.

By Lemma 3.9, there exists a non-trivial $h$-collineation $\alpha \in \Psi(O)$ with axis $\ell$ and center $P$. By Lemma 2.4, $\alpha$ sends $Q$ to some $S \neq R$. We know from $(B_{n-1})$ that there exists $\beta \in \text{Aut}(\Delta)^0$ fixing $Q$, $O$ and $\pi(m)$ and sending $S$ to $R$. Then $\beta \alpha \beta^{-1}$ sends $Q$ to $R$ (as desired) and is a $h$-collineation with axis $\ell'$ and center $P'$, with $\pi(P') = \pi(P)$ and $\pi(Q) = \pi(Q)$. Now there are two different cases:

- If $\pi(\ell') = \pi(\ell)$, then also $\pi(P') = \pi(P)$, and hence $\beta \alpha \beta^{-1}$ is a $h$-collineation with axis $\ell$ and center $P$ in view of Lemma 2.3.
• If $1\pi(\ell') \neq 1\pi(\ell)$, then denote by $T \in \mathcal{P}(O)$ the point incident with $\ell$ and $\ell'$ and by $o \in \mathcal{L}(O)$ the line incident with $Q$ and $T$. By $(C_n)$, there exist a point $Q' \in \mathcal{P}(O)$ $(n-1)$-neighboring $Q$, a line $o' \in \mathcal{L}(O)$ neighboring $o$ (with $Q'$ $n$-I $o'$) and an elation $\gamma \in \mathcal{U}(O)$ with axis $o'$ and center $Q'$ sending $1\pi(\ell')$ to $1\pi(\ell)$. Note that $\gamma$ fixes $Q$ and $R$ because of Lemma 2.1. Therefore, $\gamma(\beta \alpha \beta^{-1})\gamma^{-1}$ is a $h$-collineation with axis $\ell$ and center $P$, and it still sends $Q$ to $R$. ■

Claim 2. $(B_{n-1}) + (A_n) \Rightarrow (B_n)$.

Proof of the claim: Let $(i, j, k)$, $w$, $(c_0, c_1, \ldots, c_{2n})$, $d$ and $d'$ be as in $(B_n)$. We must find an automorphism of $\Delta$ fixing $c_0, \ldots, c_{2n-2}$ and sending $d$ to $d'$.

By $(B_{n-1})$, we already have some $g \in \text{Aut}(\Delta)^0$ fixing $c_0, \ldots, c_{2n-2}$ and sending $d$ to $d'$. Denote by $c'_{2n-1}$ the image of $c_{2n-1}$ by $g$. Now taking $O$, $P$, $\ell$ and $Q$ as in Figure 7a, we can apply $(A_n)$ to get an element $h \in \text{Aut}(\Delta)^0$ fixing $c_0, \ldots, c_{2n-2}$ as well as $d$ and $d'$ and sending $c'_{2n-1}$ to $c_{2n-1}$. So $hg$ sends $d$ to $d'$ and fixes $c_0, \ldots, c_{2n-1}$. Now we can use the same method one step further: if $c'_{2n}$ denotes the image of $c_{2n}$ by $hg$, then we
can find thanks to \((A_n)\) (see Figure 7b) an element \(h'\) fixing \(c_0, \ldots, c_{2n-1}\), \(d\) and \(d'\) and sending \(c'_{2n}\) to \(c_{2n}\). The element \(h'g\) then fixes \(c_0, \ldots, c_{2n}\) and sends \(d\) to \(d'\).

\[\] **Claim 3.** \((B_{n-1}) \Rightarrow (C_n)\).

**Proof of the claim:** Let \(O, P, \ell, Q, m\) and \(o\) be as in \((C_n)\). We must find an elation in \(^n\Psi(O)\) sending \(1\pi(m)\) to \(1\pi(o)\), with axis \(\ell'\) and center \(P'\) where \(\ell'\) is neighboring \(\ell\) and \(P'\) is \((n-1)\)-neighboring \(P\). By Lemmas 3.9 and 3.10 (1), there exists an elation \(\alpha \in ^n\Psi(O)\) with axis \(\ell\) and center \(P\) and such that \(\alpha^{*1}\) is non-trivial. In view of Lemma 2.4 (applied in \(^1H(O)\)), if \(p\) denotes the image of \(m\) by \(\alpha\), then \(1\pi(p) \neq 1\pi(m)\). By \((B_{n-1})\), there exists \(g \in \text{Aut}(\Delta)^0\) fixing \(n-1\pi(P), 1\pi(\ell)\) and \(1\pi(m)\) and sending \(1\pi(p)\) to \(1\pi(o)\) (see Figure 8). Then \(gog^{-1}\) is an elation with axis \(g(\ell)\) and center \(g(P)\) which sends \(1\pi(m)\) to \(1\pi(o)\). Since \(g(\ell)\) is neighboring \(\ell\) and \(g(P)\) is \((n-1)\)-neighboring \(P\), we are done.

Claims 1 and 3 together imply that \((B_{n-1}) \Rightarrow (A_n)\) (*), so that Claim 2 then reads as \((B_{n-1}) \Rightarrow (B_n)\). From \((B_1)\) we therefore get \((B_n)\) for all \(n \geq 1\), and hence \((A_n)\) is true for all \(n \geq 2\) by (*). (Remember that \((A_1)\) was already true.)

As a consequence of Proposition 3.11, we get the key result of this section.

**Corollary 3.12.** Suppose that \(\text{Aut}(\Delta)^0\) is non-discrete and chamber-transitive. For each vertex \(O\) in \(\Delta\) and each \(n \geq 1\), \(^n\mathcal{H}(O)\) is Moufang.

**Proof.** Consider \(P \in ^n\mathcal{P}(O)\) and \(\ell \in ^n\mathcal{L}(O)\) with \(P \neq P\ell\). We need to show that \(^n\mathcal{H}(O)\) is \((P, \ell)\)-transitive. Let \(m \in ^n\mathcal{L}(O)\) be incident with \(P\) but not neighboring \(\ell\) and let \(Q, R \in ^n\mathcal{P}(O)\) be incident with \(m\) but not neighboring \(P\). We must find an elation of \(^n\mathcal{H}(O)\) with axis \(\ell\) and center \(P\) sending \(k\pi(Q)\) to \(k\pi(R)\). For \(k = 0\) we can take the identity (because \(0\pi(Q) = 0\pi(R) = O\) by convention). Now consider \(1 \leq k \leq n\) and assume that this is true for \(k-1\). Thus there is an elation \(\alpha\) with axis \(\ell\) and center \(P\) such that \(\alpha^{(k-1)\pi(Q)} = k\pi(R)\). Denote by \(Q'\) the image of \(Q\) by \(\alpha\). Then \(Q'\) is \((k-1)\)-neighboring \(R\) and incident with \(m\), and it suffices to find an elation with axis \(\ell\) and center \(P\) sending \(k\pi(Q')\) to \(k\pi(R)\) in order to conclude. This is exactly what Proposition 3.11 and Lemma 3.10 (2) together imply.

We can now directly prove Theorem A.

![Figure 8: Illustration of Claim 3.](image-url)
Proof of Theorem A. We suppose that Aut(Δ) is non-discrete. We have by Proposition 3.6 that Aut(Δ) is chamber-transitive, and then by Corollary 3.12 that \( nH(O) \) is Moufang for each vertex \( O \) in \( Δ \) and each \( n \geq 1 \). We now show that \( Δ^∞ \) is Moufang, i.e. that \( Δ \) is Bruhat-Tits.

Consider \( ℓ^∞ \) and \( m^∞ \) two lines in the projective plane \( Δ^∞ \), and denote by \( P^∞ \) the point of \( Δ^∞ \) incident to \( ℓ^∞ \) and \( m^∞ \). We want to show that \( Δ^∞ \) is \( (P^∞, ℓ^∞) \)-transitive, i.e. that the group of all elations with axis \( ℓ^∞ \) and center \( P^∞ \) acts transitively on the points incident with \( m^∞ \) but different from \( P^∞ \). Consider \( Q^∞ \) and \( R^∞ \) two points incident with \( m^∞ \), different from \( P^∞ \). Let \( A^∞ \) be some apartment in \( Δ^∞ \) containing \( ℓ^∞ \), \( P^∞ \), \( m^∞ \) and \( Q^∞ \). There exists an apartement \( A \) in \( Δ \) whose apartement at infinity is \( A^∞ \). Now choose a vertex \( O \) in \( A \) so that the two rays from \( O \) to \( P^∞ \) and \( R^∞ \) respectively represent distinct points in \( 1H(O) \). For each \( n \geq 1 \), \( nH(O) \) is Moufang so there exists an elation \( α(n) \) in \( nH(O) \), with axis \( nπ(ℓ^∞) \) and center \( nπ(P^∞) \), sending \( nπ(Q^∞) \) to \( nπ(R^∞) \) (where \( nπ(x^∞) \) is the point or line of \( nH(O) \) represented by the ray from \( O \) to \( x^∞ \)). Notice that the sequence \( (α(n)) \) is contained in the stabilizer of \( O \), which is a compact subgroup of Aut(\( Δ \)). It then suffices to consider \( α \in \text{Aut}(Δ) \) any accumulation point of the sequence \( (α(n)) \) in Aut(\( Δ \)). In Aut(\( Δ^∞ \)), \( α \) is an elation with axis \( ℓ^∞ \) and center \( P^∞ \), sending \( Q^∞ \) to \( R^∞ \). □

4 Proof of Theorem B

The goal of this section is to prove Theorem B. Once again we will suppose that Aut(\( Δ \)) is non-discrete and, under the hypotheses of Theorem B, we will prove that Aut(\( Δ^0 \)) must be transitive on panels of each type. The conclusion will then follow from Theorem A.

4.1 About the collineation group of a finite projective plane

We begin with several lemmas concerning finite projective planes. They will become useful later in the section. The first lemma is classical.

Lemma 4.1. Let \( Π \) be a finite projective plane and let \( F \) be a collineation group of \( Π \). Then \( F \) is transitive on points of \( Π \) if and only if \( F \) is transitive on lines of \( Π \).

Proof. It is actually true that, for any collineation group \( F \) of a finite projective plane \( Π \), \( F \) has as many point orbits as line orbits, see [HP73, Theorem 13.4]. □

The following lemma is also classical but, because of the lack in finding a suitable reference, we give its proof here.

Lemma 4.2. Let \( Π \) be a finite projective plane of prime order and let \( F \) be a collineation group of \( Π \). Suppose that \( F \) contains a non-trivial elation. Then either \( F \) is transitive on points of \( Π \) or \( F \) fixes a point or a line of \( Π \).

Proof. We color the points of \( Π \) according to the orbits of \( F \). Let us suppose that \( F \) is not transitive on points of \( Π \), i.e. that there are at least 2 colors. Let us denote by \( P \) and \( ℓ \) the center and axis of a non-trivial elation \( α \) in \( F \). By Lemma 3.5 and since \( Π \) has prime order, for each line \( o \) incident to \( P \) and different from \( ℓ \), the elation \( α \) is transitive on points incident to \( o \) and different from \( P \). Thus, for each such \( o \), all points incident to \( o \) and different from \( P \) have the same color (\( * \)). Now let us distinguish several cases:

- If \( P \) has a color that no other point has, then \( P \) is fixed by \( F \).
• Otherwise, and if the only points with the same color as $P$ are incident to $\ell$, then $\ell$ is fixed by $F$.

• Now assume that there exists a point $P'$ not incident to $\ell$ but with the same color as $P$. This means that there exists $\beta \in F$ with $\beta(P) = P'$. Denote by $m$ the line through $P$ and $P'$, and write $\ell' = \beta(\ell)$. Note that, in view of $(\ast)$, for each line $o'$ incident to $P'$ and different from $\ell'$, all points incident to $o'$ and different from $P'$ have the same color $(\ast\ast)$. See Figure 9 for an illustration.

  - If $\ell' = m$, we deduce from $(\ast)$ and $(\ast\ast)$ that all points have the same color, which is a contradiction.
  - If $\ell' \neq m$, then we obtain from $(\ast)$ and $(\ast\ast)$ that all points incident to $m$ have the same color, say $c_1$, and that all points not incident to $m$ but different from $Q = \ell \cap \ell'$ have the same color, say $c_2$. We write $c_3$ for the color of $Q$. If $c_3 \neq c_1, c_2$, then $Q$ is the only point with color $c_3$ so it is fixed by $F$. If $c_3 = c_2$, then $c_1 \neq c_2$ (because there are at least two colors), and $m$ is fixed by $F$. Finally, if $c_3 = c_1$, then $c_1 \neq c_2$ and there should exist $\gamma \in F$ with $\gamma(P) = Q$. But this gives a contradiction with the coloring.

We conclude with a third lemma about finite projective planes of prime order which can be applied in some really precise situation.

**Lemma 4.3.** Let $\Pi$ be a finite projective plane of prime order and let $F$ be a collineation group of $\Pi$. Suppose that $F$ contains a non-trivial elation and that $F$ fixes exactly one point $Q$ and one line $m$, with $Q$ not incident to $m$. Then $F$ is transitive on points incident to $m$ and transitive on points not incident to $m$ but different from $Q$.

**Proof.** Let $\alpha$ be a non-trivial elation in $F$, say with center $P$ and axis $\ell$. From Lemma 2.4, we deduce that $Q$ is incident to $\ell$ (and different from $P$) and that $m$ is incident to $P$ (and different from $\ell$), see Figure 10. We color the points of $\Pi$ according to the orbits of $F$. By Lemma 3.5 and since $\Pi$ has prime order, for each line $o$ incident to $P$ and different from $\ell$, all points incident to $o$ and different from $P$ have the same color $(\ast)$. Now by hypothesis, $P$ is not fixed by $F$. Thus there exists $\beta \in F$ with $\beta(P) = P' \neq P$. Moreover, $P'$ is incident to $m$ since $m$ is fixed by $F$ (and hence by $\beta$). For similar reasons, we have $\beta(\ell) = \ell'$ where $\ell'$ is the line incident to $P'$ and $Q$. So by $(\ast)$, we get that for each line $o'$ incident to $P'$ and different from $\ell'$, all points incident to $o'$ and different from $P'$ have the same color $(\ast\ast)$. From $(\ast)$ and $(\ast\ast)$ we deduce that there are exactly three colors: one for $Q$, one for the points incident to $m$ and one for all other points. 

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4.2 From vertex transitivity to panel transitivity

We start this subsection with two easy lemmas.

Lemma 4.4. Suppose that $\text{Aut}(\Delta)^0$ is transitive on vertices of each type but not transitive on panels of each type. Then for each vertex $O$ in $\Delta$, $\Psi(O)$ is not transitive on $P(O)$ (resp. $L(O)$).

Proof. Suppose for a contradiction that $\Psi(O)$ is transitive on $P(O)$ for some vertex $O$ in $\Delta$, say of type $\{1, 2\}$. By Lemma 4.1, $\Psi(O)$ is also transitive on $L(O)$. Since $\text{Aut}(\Delta)^0$ is transitive on vertices of each type, this implies that $\text{Aut}(\Delta)^0$ is transitive on panels of type 1 and of type 2 of $\Delta$. Now if we consider a vertex $O'$ of type $\{1, 3\}$, then we know that the stabilizer of $O'$ in $\text{Aut}(\Delta)^0$ is transitive on panels of type 1 adjacent to $O'$. By Lemma 4.1, it is also transitive on panels of type 3 adjacent to $O'$. It follows that $\text{Aut}(\Delta)^0$ is also transitive on panels of type 3, which contradicts the hypothesis.

Lemma 4.5. Suppose that $\text{Aut}(\Delta)$ is transitive on vertices and unimodular. If $v$ and $w$ are two vertices in $\Delta$ such that the stabilizer $\text{Aut}(\Delta)^0(v)$ of $v$ in $\text{Aut}(\Delta)^0$ fixes $w$, then $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(w)$.

Proof. We have $\text{Aut}(\Delta)^0(v) \subseteq \text{Aut}(\Delta)^0(w)$ by hypothesis. Moreover, there exists $g \in \text{Aut}(\Delta)$ such that $g(v) = w$. Since $\text{Aut}(\Delta)$ is unimodular, the Haar measure $\mu$ of $\text{Aut}(\Delta)$ satisfies $\mu(\text{Aut}(\Delta)^0(v)) = \mu(g \text{Aut}(\Delta)^0(v)g^{-1}) = \mu(\text{Aut}(\Delta)^0(w))$. This implies that $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(w)$.

The main result of this section is then the following.

Proposition 4.6. Suppose that $\text{Aut}(\Delta)$ is transitive on vertices, non-discrete and unimodular, that $\text{Aut}(\Delta)^0$ is transitive on vertices of each type, and that $\Delta$ has thickness $p + 1$ for some prime $p$. Then $\text{Aut}(\Delta)^0$ is transitive on panels of each type.

Proof. Let us assume for a contradiction that $\text{Aut}(\Delta)^0$ is not transitive on panels of each type. By Lemma 4.4, this implies that $\Psi(v)$ is not transitive on $P(v)$ (and on $L(v)$) for each vertex $v$ in $\Delta$. In view of Lemmas 4.2 and 4.5, for each such $v$ there exists $w$ adjacent to $v$ in $\Delta$ such that $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(w)$. From now on, we color in red all panels (i.e. edges) $[v, w]$ in $\Delta$ such that $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(w)$. We have just seen that each vertex is adjacent to at least one red edge.

Claim 1. Let $v, w, x, y$ be vertices in $\Delta$, placed as shown below.

(i) If $[v, w]$ and $[v, x]$ are red, then $[w, x]$ is red.
(ii) If $[v, w]$ and $[v, y]$ are red, then $[v, x]$ is red.

Proof of the claim: The claim follows easily from the definition of a red edge:

(i) Having $[v, w]$ and $[v, x]$ red means that $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(w)$ and $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(x)$, so $\text{Aut}(\Delta)^0(w) = \text{Aut}(\Delta)^0(x)$ and $[w, x]$ is red.

(ii) Having $[v, w]$ and $[v, y]$ red means that $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(w)$ and $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(y)$. In particular, this implies that $\text{Aut}(\Delta)^0(v)$ fixes $x$. By Lemma 4.5, this gives us $\text{Aut}(\Delta)^0(v) = \text{Aut}(\Delta)^0(x)$ so that $[v, x]$ is red. ■

Claim 2. Let $v$ be a vertex in $\Delta$ and let $\alpha$ be a non-trivial elation of $1^H(v)$, with axis $\ell$ and center $P$. Then all vertices $w$ adjacent to $v$ with $[v, w]$ red are incident to $P$ or $\ell$.

Proof of the claim: This follows from Lemma 2.4. ■

Claim 3. For each vertex $v$ in $\Delta$, there exist two vertices $w, x$ adjacent to $v$ and opposite in $1^H(v)$ such that $[v, w]$ and $[v, x]$ are red.

Proof of the claim: By Lemmas 4.2 and 4.5, there is at least one red edge adjacent to any vertex. Since $\text{Aut}(\Delta)$ is transitive on vertices, each vertex is adjacent to the same number of red edges. This number cannot be exactly one, because then there would be an issue with the types of the red panels (because $\text{Aut}(\Delta)^0$ is transitive on vertices of each type). So each vertex is adjacent to at least two red edges.

We want to show that, for each vertex $v$, there exists $w, x$ adjacent to $v$ and opposite in $1^H(v)$ such that $[v, w]$ and $[v, x]$ are red. If this situation occurs at one vertex $v$, then it occurs at any vertex $v$ in view of the vertex-transitivity. So we assume for a contradiction that this situation does not appear anywhere.

First assume that, for some vertex $v$, there exist two vertices $w, y$ adjacent to $v$, with the same type and such that $[v, w]$ and $[v, y]$ are red. Then the edge $[v, x]$ between $w$ and $y$ must also be red, as well as $[w, x]$ and $[x, y]$ (by Claim 1). But there must also be two red edges of the same type adjacent to $w$. In all cases, we find (via Claim 1)
two opposite red edges adjacent to a same vertex. So two such red edges \([v, w]\) and \([v, x]\) cannot exist, and the only remaining possibility is to have, for each vertex \(v\) in \(\Delta\), exactly two red edges adjacent to \(v\), of different types and incident in \(\mathcal{H}(v)\) (\(\ast\)).

We now show that this situation is impossible. Let us consider some non-trivial \(1_h\)-collineation \(\alpha\) in \(\mathcal{H}(v)\), which exists by Proposition 3.3. Denote by \(P\) and \(\ell\) its center and axis. Let \(w, x\) be two vertices adjacent to \(v\) in \(\Delta\), placed as in Figure 11. Now for each vertex \(y\) adjacent to both \(w\) and \(x\) but different from \(v\), \(\alpha\) induces an elation of \(\mathcal{H}(y)\) with axis \(x\) and center \(w\). Observing (\(\ast\)) and Claim 2 at \(y\), we deduce that at least one of the edges \([y, w]\) and \([y, x]\) is red. This observation is true for any choice of \(y\). If \(p \geq 3\), there are at least three such vertices \(y\) and we get two red edges \([w, y]\) and \([w, y']\) (or \([x, y]\) and \([x, y']\)) with \(y\) and \(y'\) of the same type, which contradicts (\(\ast\)). In the particular case where \(p = 2\), we can also get a contradiction. First, if we denote by \(y\) and \(y'\) the two vertices adjacent to \(w\) and \(x\) and different from \(v\), then the only way to not have a contradiction is to have \([w, y]\) and \([x, y]\) red or \([w, y']\) and \([x, y]\) red. Now consider \(x'\) a vertex adjacent to \(v\) and \(w\), different from \(x\) and not adjacent to \(P\).

Then with the same argument as above we get two vertices \(t\) and \(t'\) adjacent to \(w\) and \(x'\) and such that \([w, t]\) and \([x, t']\) are red. This gives a contradiction with (\(\ast\)) at \(w\): the two edges \([w, y]\) and \([w, t]\) are red but \(y\) and \(t\) have the same type.

\[\boxed{\text{Claim 4.} \text{ For each vertex } v \text{ in } \Delta, \text{ there are exactly two red edges adjacent to } v, \text{ and they are opposite in } \mathcal{H}(v).}\]

\[\text{Proof of the claim: For each vertex } v \text{ in } \Delta, \text{ we have two red edges adjacent to } v \text{ and opposite in } \mathcal{H}(v), \text{ by Claim 3. Now assume for a contradiction that some (and hence any) vertex is adjacent to a third red edge.} \]

For some vertex \(v\), we consider some non-trivial \(1_h\)-collineation \(\alpha\) in \(\mathcal{H}(v)\), with axis \(\ell\) and center \(P\). Let \(w, x\) be two vertices adjacent to \(v\) in \(\Delta\), placed as in Figure 12. Given a vertex \(y\) adjacent to both \(w\) and \(x\) but different from \(v\), \(\alpha\) induces an elation of \(\mathcal{H}(y)\) with axis \(x\) and center \(w\). Applying Claims 2 and 3 at \(y\), we obtain two red edges \([y, s]\) and \([y, t]\), with \(s\) adjacent to \(w\) and \(t\) adjacent to \(x\), see Figure 12. We assumed that there is a third red edge \([y, r]\) adjacent to \(y\). By Claim 2, \(r\) must be adjacent to \(w\) or \(x\). Via Claim 1, this implies that all edges \([y, w]\), \([y, x]\), \([s, w]\), \([w, x]\) and \([x, t]\) are red. Now we can do the same reasoning with another vertex \(y\) adjacent to \(w\) and \(x\) but different from \(v\). This gives us two vertices \(s'\) and \(t'\) with \([y', s]\), \([y', t]\), \([y', w]\), \([y', x]\), \([w, t]\) and \([x, s]\).
[s', w] and [x, t'] red. In particular, we get that the three edges [w, s], [w, x] and [w, s'] are red, with s, x and s' having the same type. In view of Claim 2, since there exists a non-trivial elation of \(1^H(w)\), these three edges should be incident to a common edge. This is not the case, so we have our contradiction.

Claim 5. For each vertex \(v\) in \(\Delta\), there is a red bi-infinite geodesic through \(v\).

Proof of the claim: This follows directly from Claim 4, with an easy induction.

Claim 6. Let \(v, w, x, y, z\) be vertices in \(\Delta\) placed as shown below. If \([v, w]\) and \([v, x]\) are red, then \([y, z]\) is red.

Proof of the claim: Consider some non-trivial \(1^h\)-collineation \(\alpha\) of \(2^H(v)\) given by Proposition 3.3 and denote by \(P\) and \(\ell\) its center and axis. Assume without loss of generality that the vertex \(1^\pi(P)\) has the same type as \(x\) (resp. \(w\)). Recall from Claims 4 and 5 that there is a red bi-infinite geodesic through \(w, v\) and \(x\). We deduce that \(w\) cannot be opposite to \(1^\pi(P)\) in \(1^H(v)\), because then \(\alpha\) would fix a line not near \(P\), contradicting Lemma 2.4. So \(w\) must be adjacent to \(1^\pi(P)\). In the same way, we deduce that \(x\) must be adjacent to \(1^\pi(\ell)\). Moreover, since \(\text{Aut}(\Delta)^0(v)\) fixes \(w\) and \(x\) and is transitive on points adjacent to \(v\) and \(w\) (by Lemma 4.3), we can assume without loss of generality that \(y\) and \(z\) are different from \(1^\pi(P)\) and \(1^\pi(\ell)\), as in Figure 13.

We now prove that \([y, z]\) is red. We already know by the previous claims that there is a (unique) vertex \(s\) adjacent to \(y\) and with the same type as \(z\) such that \([y, s]\) is red. Our goal is to show that \(s = z\). First observe that \(s\) cannot be opposite to \(v\) in \(1^H(v)\) (as \(s_1\) in Figure 13). Indeed, if this was the case, then it would mean that \(\alpha\) fixes \(s\), a point of \(2^H(v)\) not near \(P\). This is impossible by Lemma 2.4. So \(s\) is adjacent to \(v\).

We now prove that \([y, z]\) is red. We already know by the previous claims that there is a (unique) vertex \(s\) adjacent to \(y\) and with the same type as \(z\) such that \([y, s]\) is red. Our goal is to show that \(s = z\). First observe that \(s\) cannot be opposite to \(v\) in \(1^H(v)\) (as \(s_1\) in Figure 13). Indeed, if this was the case, then it would mean that \(\alpha\) fixes \(s\), a point of \(2^H(v)\) not near \(P\). This is impossible by Lemma 2.4. So \(s\) is adjacent to \(v\).

Of course we cannot have \(s = w\) since \([w, v]\) and \([w, y]\) cannot be both red. In order to show that \(s = z\), there remains to show that \(s\) is adjacent to \(x\). We proceed by contradiction, assuming that \(s\) is not adjacent to \(x\) (as \(s_2\) in Figure 13). We thus have a red edge \([y, s]\) with \(y\) and \(s\) adjacent to \(v\), \(y\) adjacent to \(w\) but \(s\) not adjacent to \(x\).

In the case where \(p \geq 3\), the contradiction will come from Lemma 4.3. Indeed, if we denote by \(Y\) the set of vertices adjacent to \(v\) and \(w\), and by \(S\) the set of vertices with the same type as \(s\), adjacent to \(v\) and not adjacent to \(x\), then Lemma 4.3 tells us that \(\text{Aut}(\Delta)^0(v)\) is transitive on \(Y\) and on \(S\). But \(|Y| = p + 1\) and \(|S| = p^2 - 1\), so if \(p \geq 3\)

![Figure 13: Illustration of Claim 6.](image-url)
then having a red edge \([y, s]\) from a vertex in \(Y\) to a vertex in \(S\) implies that each vertex in \(Y\) has more than one red edge going to a vertex in \(S\). This is impossible, as \(s\) is the only vertex of that type with \([y, s]\) red.

Let us now consider the last case \(p = 2\). We continue our proof by contradiction, assuming that \(s \neq z\). This time we have \(|Y| = 3 = |S|\), and each vertex in \(Y\) is adjacent to a unique vertex in \(S\). This gives us three red edges. If we do the same reasoning around \(z\) instead of \(y\), then we denote by \(Z\) the set of vertices adjacent to \(v\) and \(x\), by \(S'\) the set of vertices with the same type as \(y\), adjacent to \(v\) and not adjacent to \(w\), and we get three other red edges, each one connecting a vertex of \(Z\) and a vertex of \(S'\). In total, we got six red edges connecting neighbors of \(v\). Now since \(\text{Aut}(\Delta)\) is transitive on vertices, this whole situation around \(v\) also occurs around \(w\). If we denote by \(a\) the vertex adjacent to \(w\) such that \([w, a]\) is red (with \(a \neq v\)), this means that \([y, b]\) is red, where \(b\) is the unique vertex adjacent to \(w\) and \(y\), different from \(v\) and not adjacent to \(a\) (see Figure 14). But then, around \(y\), we have \([y, b]\) and \([y, s]\) red, while \([w, v]\) is also red. This situation is different from the one around \(v\), so we get our contradiction. ■

**Figure 14: Illustration of Claim 6.**

We now find a new contradiction. This will show that our hypotheses were wrong since the beginning, i.e. that \(\text{Aut}(\Delta)^0\) must be transitive on panels of each type.

Fix a vertex \(v\) in \(\Delta\) and consider a non-trivial \(1h\)-collineation \(\alpha\) of \(1H(v)\) given by Proposition 3.3, say with axis \(\ell\) and center \(P\). We choose a vertex \(w\) adjacent to \(v\) and \(1\pi(P)\) but different from \(1\pi(\ell)\) and a vertex \(x\) adjacent to \(w\) and \(v\) but different from \(1\pi(P)\), as shown in Figure 15. The \(1h\)-collineation \(\alpha\) induces a non-trivial elation of \(1H(x)\) with axis \(v\) and center \(w\). By Claim 2, this implies that the two red edges adjacent to \(x\) (given by Claim 4) are incident to \(w\) and \(v\) in \(1H(x)\). Hence, we conclude via Claim 6 that \([v, w]\) is also red. However, this reasoning could be done for any choice of \(w\). So if \(w'\) is another vertex adjacent to \(v\) and \(1\pi(P)\) but different from \(1\pi(\ell)\), then we also get that \([v, w']\) is red. This gives a contradiction with Claim 4.

Theorem B now follows immediately.

**Proof of Theorem B.** This follows from Proposition 4.6 and Theorem A. ■

**Figure 15: Illustration of Proposition 4.6.**
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