Vacuum Rabi oscillation of an atom without rotating-wave approximation

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We have investigated vacuum Rabi oscillation of an atom coupled with single-mode cavity field exactly, and compared the results with that of J-C model. The results show that, for resonant case, there is damping Rabi oscillation for an atom, even in strong coupling regime. For small detuning and weak coupling case, the probability for the atom in excited state oscillates against time with different frequency and amplitude from that of J-C model. It exhibits the counter-rotating wave interaction could significantly effect the dynamic behavior of the atom, even under the condition in which the RWA is considered to be justified. On the other hand, the results also reveal that there is Rabi oscillation for initially unexcited atom, which is contrary to that of J-C model.

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A two-level atom interacting with a single cavity mode, described by the Jaynes-Cummings (J-C) model, which neglects counter rotating terms, is widely used in quantum optics and has the potential to constitute the basic building block of quantum computers[1, 2, 3, 4]. Making rotating-wave approximation (RWA) in Hamiltonian strongly simplifies the mathematical treatment of the problem and usually give exact solution of the approximate Hamiltonian. In spite of the simplicity of the J-C model, the dynamics have turned out to be various and complex, describing many physical phenomena, such as Rabi oscillations, collapse-revivals, squeezing, atom-field entanglement[5].

Generally, the RWA, which neglecting counter rotating, is justified for small detuning and small ratio of the atom-field coupling divided by the atomic transition frequency. In atom-field cavity systems, this ratio is typically of the order $10^{-7} \sim 10^{-6}$. Recently, cavity systems with very strong couplings have been discussed[6]. The ratio might become order of magnitudes larger in solid state systems and the counter-rotating wave terms must be considered[7]. In this paper, we investigate the influence of counter-rotating wave terms on the decay behavior of an atoms coupled with one-mode cavity, without rotating-wave approximation.

Now we restrict our attention to a two-level atom coupled with a perfect one-mode cavity field, of which the Hamiltonian is

$$H = H_a + H_f + H_{af}$$

where

$$H_a = \omega_0 \sigma_z / 2$$

$$H_f = \omega a^\dagger a$$

$$H_{af} = g(\sigma_+ + \sigma_-)(a^\dagger + a)$$

where $\omega_0$ is the atomic transition frequency between the ground state $|0\rangle$ and excited state $|1\rangle$. $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$, $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$ are pseudo-spin operators of atom. $a^\dagger$ and $a$ are creation and annihilation operators of the cavity field mode corresponding frequency $\omega$. And $g$ is the coupling constant between the transition $|1\rangle - |0\rangle$ and the field mode.

If the cavity field is initially in vacuum state, the non-perturbative reduced master equation of the atom could be derived by path integrals[8]

$$\frac{\partial}{\partial t} \rho_a = [\varepsilon_0 J_0 + \varepsilon_+ J_+ + \varepsilon_- J_-] \rho_a - g^2 [\alpha^R + f] \rho_a + [\nu_0 K_0 + \nu_+ K_+ + \nu_- K_-] \rho_a$$

Where $\varepsilon_0 = -2i (\omega_0 - g^2 \alpha^I + g^2 f^I)$, $\varepsilon_+ = g^2 (\alpha + f^*)$, $\varepsilon_- = g^2 (\alpha^* + f)$, $\nu_0 = 2g^2 (\alpha^R - f^R)$, $\nu_+ = 2g^2 \alpha^R$, $\nu_- = 2g^2 f^R$, $J_0, J_+, J_-, K_0, K_+, K_-$ are superoperators defined as

$$J_0 \rho_a = \left[ \frac{\sigma_z}{4}, \rho_a \right]$$

$$J_+ \rho_a = \sigma_+ \rho_a \sigma_+$$

$$J_- \rho_a = \sigma_- \rho_a \sigma_-$$

$$K_0 \rho_a = (\sigma_+ \sigma_- \rho_a + \rho_a \sigma_+ \sigma_- - \rho_a) / 2$$

$$K_+ \rho_a = \sigma_+ \rho_a \sigma_-$$

$$K_- \rho_a = \sigma_- \rho_a \sigma_+$$

and

$$\alpha = \frac{1 - \exp(-i \Delta t)}{i \Delta}$$

$$f = \frac{\exp(i \delta t) - 1}{i \delta}$$

where $\Delta = \omega + \omega_0$, $\delta = \omega_0 - \omega$. $\alpha^R, \alpha^I, \alpha^*$ and $f^R, f^I, f^*$ are real part, image part and conjugate of $\alpha$ and of $f$, respectively. $\alpha$ comes from the counter-rotating wave interaction and $f$ comes from the rotating-wave interaction.
Using algebraic approach, the formal solution of Eq. (5) is obtained \[2, 3\]

\[
\rho_a(t) = e^{-\Gamma_k t} \tilde{T} e^{i J_m' dt} (e^{\omega t_0 + \epsilon^+ J_+ + \epsilon^- J_-}) \times \tilde{T} e^{-i J_m' dt} \rho_a(0) 
\]

(8)

where \( \Gamma_k = g^2(\tilde{\alpha}^R + F^R) \) and

\[
\tilde{\alpha} = \int_0^t \alpha dt = \frac{1 - \exp(-i\Delta t) - i\Delta t}{\Delta^2} \equiv \tilde{\alpha}^R + i\tilde{\alpha}^I 
\]

\[
F = \int_0^t f dt = \frac{1 + i\delta t - \exp(i\delta t)}{\delta^2} \equiv F^R + iF^I 
\]

(9)

where \( \tilde{\alpha}^R, \tilde{\alpha}^I, F^R, F^I \) are real part, image part and conjugate of \( \tilde{\alpha} \) and of \( F \), respectively.

Using the decomposition of SU(2) operator, the time-ordered exponential operators could be disentangled \[2\].

The exact solution of master equation Eq. (5) is obtained

\[
\rho_a(t) = e^{-\Gamma_k t} \tilde{\rho}(t) 
\]

(10)

\[
\tilde{\rho}(t) = \left( \begin{array}{cc} l \rho_{11}^a(0) + np_{10}^a(0) & x \rho_{10}^a(0) + y \rho_{01}^a(0) \\ q \rho_{01}^a(0) + q' \rho_{10}^a(0) & n \rho_{00}^a(0) + p \rho_{11}^a(0) \end{array} \right) 
\]

(11)

\[
l = e^{k_0/2} + e^{-k_0/2}k_+k_-, \quad m = e^{-k_0/2}k_+ \quad \text{(12)} 
\]

\[
n = e^{-k_0/2}, \quad p = e^{-k_0/2}k_- \quad \text{(13)} 
\]

\[
q = e^{-j_0/2}, \quad r = e^{-j_0/2}j_- \quad \text{(14)} 
\]

\[
x = e^{j_0/2} + e^{-j_0/2}j_+j_-, \quad y = e^{-j_0/2}j_+ \quad \text{(15)} 
\]

where \( j_+, j_0, j_- \) and \( k_+, k_0, k_- \) satisfy the following differential equation \[2\]

\[
\dot{X}_+ = \mu_+ - \mu X_+^2 + \mu_0 X_+ \quad \text{(16)} 
\]

\[
\dot{X}_0 = \mu_0 - 2\mu X_+ \quad \text{(17)} 
\]

\[
\dot{X}_- = \mu_\pm \exp(X_0) \quad \text{(18)} 
\]

\( \mu = \varepsilon \) for \( X = j \) and \( \mu = \nu \) for \( X = k \). Generally, the Riccati equation could not be solved analytically. Next, we will investigate it numerically.

When the atom is initially in excited state \( \rho_{11}^a(0) = 1 \), the probability \( P_e \) for the atom in excited state at time \( t \) is \( P_e = l \). The counterpart result for Jaynes-Cummings model is

\[
P_{JC} = 1 - \left[ \frac{2g \sin(\Omega t/2)}{\Omega} \right]^2 
\]

(19)

where \( \Omega = \sqrt{\delta^2 + 4g^2} \).

When the atom is initially in ground state \( \rho_{00}^a(0) = 1 \), the probability \( P_e \) for the atom in excited state at time \( t \) is \( P_e = m \). The counterpart result for Jaynes-Cummings model is \( P_{JC} = 0 \).

First, we investigate the case for initially excited atom. And we refer to our exact solution as exact model in the following discussion.

(A) For resonant and weak coupling case, Fig. 1 reveals that the probability \( P_e \), for J-C model, oscillates against time, while that periodically decays and revives with damping amplitude for exact model. Eq. (10) and (19) show that there is an attenuation factor \( \exp(-g^2t^2/2) \) for exact model when \( \omega_0 t \gg 1 \), while \( P_{JC} = 1 - \sin^2(gt) \) for J-C model.

(B) For small detuning and weak coupling case, Fig. 2 shows that \( P_e \) oscillates against time for the two model except for different amplitude and different oscillating frequency. For exact model, the result shows that the oscillating frequency is \( \delta \) for exact model, while that is \( \Omega \) for J-C model.

From Fig. 3, we could find, with the decreasing of the ratio of coupling strength to the atom transmission frequency, the difference between the J-C model and the exact model becomes smaller. And the result also reveals that the frequency for atom decay and recover is only
FIG. 3: $P_e$ as a function of $\delta t$ for initially excited atom with $\omega_0 = 50g$ and $\delta = 0.1\omega_0$. dotted line for J-C model, solid line for exact model.

FIG. 4: $P_e$ as a function of $gt$ for initially excited atom with $\omega_0 = 10g$. solid line for $\delta = 0.2\omega_0$ and dotted line for $\delta = 0.6\omega_0$.

FIG. 5: $P_e$ as a function of $gt$ for initially unexcited atom with $\omega_0 = 20g$ and $\delta = 0.5\omega_0$.

dependent on the detuning $\delta$.

(C) Here, the decay behavior of an atom in strong coupling regime is discussed. For the detuning case, Fig. 4 shows that $P_e$ will periodically decrease and recover, accompanying with small amplitude rapid oscillation. As the detuning value increases, the oscillation amplitude decreases, while the oscillation frequency increases.

Form Eq. (5) and Eq. (6), we could find that the contribution of energy-conserving process, corresponding to rotating-wave terms $\sigma_- a^\dagger$ and $\sigma_+ a$ in Hamiltonian, varies with frequency of $\omega_0 - \omega$. And that of energy-non-conserving process, corresponding to counter-rotating wave terms $\sigma_+ a^\dagger$ and $\sigma_- a$ in Hamiltonian, varies with frequency of $\omega_0 + \omega$. The attenuation factor in Eq. (10) comes from the destructive superposition of the different frequency contribution associated with rotating-wave terms and counter-rotating wave terms. With the increasing of coupling strength, the contribution of virtual processes will increase and result in the decay rate obviously modified.

Then, we focus on the case for initially unexcited atom. For J-C model, the probability $P_e$ is always equal to zero, which indicates no Rabi oscillation for an atom. However, the results for exact model is in opposition to that.

For detuning case, Fig. 5 shows that probability $P_e$ periodically increases and decays. It indicates that there is Rabi oscillation for initially unexcited atom coupled with vacuum single-mode field, which is contrary to that of J-C model. From another point of view, this result theoretically testifies the existence of vacuum energy fluctuation. The numeric result also exhibits that, as the coupling strength enhances, the Rabi oscillation amplitude will increases.

From above discussion, we find that $P_e$ will oscillate with damping amplitude for resonant case and there is Rabi oscillation for detuning case. That is because spontaneous emission is inhibited if there is detuning between atomic frequency and cavity mode, and enhanced if the cavity is resonant. [10, 11]

In summary, we have investigated vacuum Rabi oscillation of an atom coupled with single-mode cavity field exactly, and compared the results with that of J-C model. Firstly, the results show that, for resonant case, there is damping Rabi oscillation for an atom, even in strong coupling regime, while there is Rabi oscillation for an atom in J-C model. Secondly, for small detuning and weak coupling case, the probability for the atom in excited state will oscillate against time with different frequency and amplitude from that of J-C model. Thirdly, the results also reveal that the Rabi oscillation frequency is only dependent on the detuning value, while that is dependent on both the coupling strength and the detuning value for J-C model. Fourthly, there is Rabi oscillation for initially unexcited atom coupled with vacuum single-mode field, which is contrary to that of J-C model.

On the whole, it exhibits that there is a significant effect on the dynamic behavior of the atom due to the
counter-rotating wave interaction, even under the condition in which the RWA is considered to be justified. The vacuum energy fluctuation and counter-rotating wave interaction could invoke many different behaviors from that of J-C model.

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