Abstract: Rabern recently proved that any graph with \( \omega \geq \frac{3}{4}(\Delta + 1) \) contains a stable set meeting all maximum cliques. We strengthen this result, proving that such a stable set exists for any graph with \( \omega > \frac{3}{4}(\Delta + 1) \). This is tight, i.e. the inequality in the statement must be strict. The proof relies on finding an independent transversal in a graph partitioned into vertex sets of unequal size. © 2010 Wiley Periodicals, Inc. J Graph Theory 67: 300–305, 2011

1. INTRODUCTION AND MOTIVATION

When coloring a graph \( G \), we often desire a stable set \( S \) meeting every maximum clique. For example, finding such a set \( S \) efficiently is the key to coloring perfect graphs in

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polynomial time [11]. Proving the existence of $S$ has also been very useful in attacking Reed’s $\omega$, $\Delta$, and $\chi$ conjecture:

**Conjecture 1** (Reed [10]). For any graph $G$, $\chi(G) \leq \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil$.

No minimum counterexample $G$ to this conjecture has such a stable set $S$. For if it did, it would contain a maximal stable set $S'$ meeting every maximum clique, and we would have $\lceil \frac{1}{2}(\Delta(G - S') + 1 + \omega(G - S')) \rceil + 1 \leq \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil$. Since $S'$ is a stable set, $\chi(G) \leq \chi(G - S') + 1$, contradicting the minimality of $G$.

Thus such a stable set $S$ is highly desirable when attacking Reed’s conjecture for a hereditary class of graphs. The proof of Reed’s conjecture for line graphs [7] exemplifies the general approach: If the maximum degree and clique number are far apart, a combination of previously known results suffices. If they are not far apart, we can use the structure of line graphs to prove the existence of a stable set $S$ meeting all maximum cliques.

Rabern [9] recently proved that if the maximum degree and clique number are close enough, we need not consider the structure of the graph class at all:

**Theorem 2** (Rabern). If a graph $G$ satisfies $\omega(G) \geq \frac{3}{4}(\Delta(G) + 1)$, then $G$ contains a stable set $S$ meeting every maximum clique.

Here we prove the best possible theorem of this type:

**Theorem 3.** If a graph $G$ satisfies $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$, then $G$ contains a stable set $S$ meeting every maximum clique.

To see that this is best possible, let $G_k$ be the graph obtained by substituting every vertex of a 5-cycle with a clique of size $k$. Then $\omega(G_k) = 2k = \frac{2}{3}(\Delta(G_k) + 1)$, and no stable set meets every maximum clique. To prove Theorem 3, we apply Rabern’s approach with a stronger final step. Rabern applies Haxell’s theorem [4], which can be stated as follows:

**Theorem 4** (Haxell). For a positive integer $k$, let $G$ be a graph with vertices partitioned into $r$ cliques of size $\geq 2k$. If every vertex has at most $k$ neighbors outside its own clique, then $G$ contains a stable set of size $r$.

To prove our theorem, we need to deal with a graph that has been partitioned into cliques of unequal size. We use the following extension of Theorem 4:

**Theorem 5.** For a positive integer $k$, let $G$ be a graph with vertices partitioned into cliques $V_1, \ldots, V_r$. If for every $i$ and every $v \in V_i$, $v$ has at most $\min\{k, |V_i| - k\}$ neighbors outside $V_i$, then $G$ contains a stable set of size $r$.

Although this is not at all obvious, it is a straightforward consequence of observations made by Aharoni, Berger, and Ziv about the proof of Theorem 4 [1].

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\(^1\omega, \Delta, \text{ and } \chi\) denote the clique number, maximum degree, and chromatic number of a graph, respectively.

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2. HITTING THE MAXIMUM CLIQUES

To prove Theorem 3, we must investigate intersections of maximum cliques. Given a graph $G$ and the set $C$ of maximum cliques in $G$, we define the *clique graph* $G(C)$ as follows. The vertices of $G(C)$ are the cliques of $C$, and two vertices of $G(C)$ are adjacent if their corresponding cliques in $G$ intersect. For a connected component $G(C_i)$ of $G(C)$, let $D_i \subseteq V(G)$ and $F_i \subseteq V(G)$ denote the union and the mutual intersection of the cliques of $C_i$, respectively, i.e. $D_i = \bigcup_{C \in C_i} C$ and $F_i = \bigcap_{C \in C_i} C$.

The proof uses three intermediate results. The first, due to Hajnal [2] (also see [9]), tells us that for each component of $G(C)$, $|D_i| + |F_i|$ is large:

**Lemma 6 (Hajnal).** Let $G$ be a graph and $C_1, \ldots, C_r$ be a collection of maximum cliques in $G$. Then

$$\left| \bigcap_{i \leq r} C_i \right| + \left| \bigcup_{i \leq r} C_i \right| \geq 2\omega(G).$$

The second is due to Kostochka [8] (proven in English in [9]). It tells us that if $\omega(G)$ is sufficiently close to $\Delta(G) + 1$, then $|F_i|$ is large:

**Lemma 7 (Kostochka).** Let $G$ be a graph with $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ and let $C$ be the set of maximum cliques in $G$. Then for each connected component $G(C_i)$ of $G(C)$,

$$\left| \bigcap_{C \in C_i} C \right| \geq 2\omega(G) - (\Delta(G) + 1).$$

The third intermediate result is Theorem 5. Combining them to prove Theorem 3 is a simple matter.

**Proof of Theorem 3.** Let $C$ be the set of maximum cliques of $G$, and let the connected components of $G(C)$ be $G(C_1), \ldots, G(C_r)$. It suffices to prove the existence of a stable set $S$ in $G$ intersecting each clique $F_i$.

Lemma 7 tells us that $|F_i| > \frac{1}{3}(\Delta(G) + 1)$. Consider a vertex $v \in F_i$, noting that $v$ is universal in $G[D_i]$. By Lemma 6, we know that $|F_i| + |D_i| > \frac{1}{3}(\Delta(G) + 1)$. Therefore $\Delta(G) + 1 - |D_i| < |F_i| - \frac{1}{3}(\Delta(G) + 1)$, so $v$ has fewer than $|F_i| - \frac{1}{3}(\Delta(G) + 1)$ neighbors in $\bigcup_{j \neq i} F_i$. Furthermore, $v$ certainly has fewer than $\frac{1}{3}(\Delta(G) + 1)$ neighbors in $\bigcup_{j \neq i} F_i$.

Now let $H$ be the subgraph of $G$ induced on $\bigcup_i F_i$, and let $k = \frac{1}{3}(\Delta(G) + 1)$. Clearly the cliques $F_1, \ldots, F_r$ partition $V(H)$. A vertex $v \in F_i$ has at most $\min[k, |F_i| - k]$ neighbors outside $F_i$. Therefore by Theorem 5, $H$ contains a stable set $S$ of size $r$. This set $S$ intersects each $F_i$, and consequently it intersects every clique in $C$, proving the theorem.

It remains to prove Theorem 5. We do this in the next section.

3. INDEPENDENT TRANSVERSALS WITH LOPSIDED SETS

Suppose we are given a finite graph whose vertices are partitioned into stable sets $V_1, \ldots, V_r$. An *independent system of representatives* or *ISR* of $(V_1, \ldots, V_r)$ is a stable
set of size \( r \) in \( G \) intersecting each \( V_i \) exactly once. A partial ISR, then, is simply a stable set in \( G \) intersecting no \( V_i \) more than once. ISRs are intimately related to both the strong chromatic number [6] and list colorings [5].

A totally dominating set \( D \) is a set of vertices such that every vertex of \( G \) has a neighbor in \( D \), including the vertices of \( D \). Given \( j \subseteq [m] \), we use \( V_j \) to denote \((V_j | i \in j)\). Given \( X \subseteq V(G) \), we use \( I(X) \) to denote the set of partitions intersected by \( X \), i.e. \( I(X) = \{ i | X \cap V_i \neq \emptyset \} \). For an induced subgraph \( H \) of \( G \), we implicitly consider \( H \) to inherit the partitioning of \( G \).

To prove our lopsided existence condition for ISRs, we use a consequence of Haxell’s proof of Theorem 4 [4] pointed out (and proved explicitly) by Aharoni, Berger, and Ziv [1]. Actually, we prove a slight strengthening of their result:

**Lemma 8.** Let \( x_1 \) be a vertex in \( V_r \), and suppose \( G[V_{[r-1]}] \) has an ISR. Suppose there is no \( J \subseteq [r-1] \) and \( D \subseteq V_J \cup \{ x_1 \} \) totally dominating \( V_J \cup \{ x_1 \} \) with the following properties:

1. \( D \) is the union of disjoint stable sets \( X \) and \( Y \).
2. \( Y \) is a (not necessarily proper) partial ISR for \( V_J \). Thus \( |Y| \leq |J| \).
3. Every vertex in \( Y \) has exactly one neighbor in \( X \). Thus \( |X| \leq |Y| \).
4. \( X \) contains \( x_1 \).

Then \( G \) has an ISR containing \( x_1 \).

**Proof.** Let \( G \) be a minimum counterexample; we can assume \( G = G[V_{[r-1]} \cup \{ x_1 \}] \). Furthermore, \( r > 1 \) otherwise the lemma is trivial. Let \( R_1 \) be an ISR of \( G[V_{[r-1]}] \) chosen such that the set \( Y_1' = Y_1 = R_1 \cap N(x_1) \) has minimum size. We know that \( R_1 \) exists because \( G[V_{[r-1]}] \) has at least one ISR, and we know that \( Y_1' \) is nonempty because \( G \) does not have an ISR. Now let \( X_1 = \{ x_1 \} \) and let \( D_1 = X_1 \cup Y_1 \).

We now construct an infinite sequence of partial ISRs \( Y_1 \subset Y_2 \subset \cdots \), which contradicts the fact that \( G \) is finite. Let \( i > 1 \), and suppose we have sets \( \{ R_j, Y_j, X_j \} | 1 \leq j < i \) such that:

- \( X_j \) is a stable set consisting of distinct vertices \( \{ x_1, \ldots, x_j \} \). For \( j > 1 \), \( x_j \) is a vertex in \( G[V_{[r-1]}] \) with no neighbor in \( X_{j-1} \cup Y_{j-1} \).
- \( R_j \) is an ISR of \( G[V_{[r-1]}] \) such that for every \( 1 \leq \ell < j \), \( R_j \cap N(x_\ell) = Y_\ell \). Subject to that, \( R_j \) is chosen so that \( Y'_j = R_j \cap N(x_j) \) is minimum. For \( 1 \leq j < i \), \( Y'_j \) is nonempty.
- \( Y_j = \bigcup_{i=1}^{j} Y'_j \).

To find \( x_i \), \( Y'_i \), and \( R_i \), we proceed as follows:

1. Let \( x_i \) be any vertex in \( G[V_{[r-1]}] \) with no neighbor in \( X_{i-1} \cup Y_{i-1} \). We know that \( x_i \) exists; otherwise the set \( D_{i-1} = X_{i-1} \cup Y_{i-1} \) would be a total dominating set for \( G[V_{[r-1]}] \), contradicting the fact that \( G \) is a counterexample.
2. Let \( R_i \) be an ISR of \( G[V_{[r-1]}] \) chosen so that for all \( 1 \leq j < i \), \( R_i \cap N(x_j) = Y'_j \). Subject to that, choose \( R_i \) so that \( Y'_i = R_i \cap N(x_i) \) is minimum. We know that \( R_i \) exists because \( R_{i-1} \) is a possible candidate for the ISR.
3. It remains to show that \( Y'_i \) is nonempty, i.e. that \( Y_i \neq Y_{i-1} \). Suppose \( Y'_i = \emptyset \). We will show that this contradicts our choice of \( R_i \) for the unique \( j < i \) such that \( x_i \in V_I(Y'_j) \).

Let \( y \) be the unique vertex in \( R_i \cap V_I(x_i) \). Construct \( R'_j \) from \( R_i \) by removing \( y \) and
inserting $x_i$. Now for every $\ell$ such that $1 \leq \ell < j$, $R'_j \cap N(x_\ell) = Y'_\ell = R_j \cap N(x_\ell)$. For $j$, $R'_j \cap N(x_j) = (R_j \cap N(x_j)) \setminus \{y\}$, a contradiction. Thus, $Y'_j$ is nonempty.

4. Set $X_i = X_{i-1} \cup \{x_i\}$ and $Y_i = Y_{i-1} \cup Y'_i$.

This choice of $X_i$, $R_i$, and $Y_i$ sets up the conditions so that we can repeat our argument indefinitely for increasing $i$, a contradiction since $G$ is finite.

The lemma easily implies Theorem 3.5 in [1], and allows us to prove a strengthening of Theorem 5:

**Theorem 9.** Let $k$ be a positive integer and let $G$ be a graph partitioned into stable sets $(V_1, ..., V_r)$. If for each $i \in [r]$, each vertex in $V_i$ has degree at most $\min\{k, |V_i| - k\}$, then for any vertex $v$, $G$ has an ISR.

**Proof.** Suppose $G$ is a minimum counterexample for a given value of $k$. Clearly we can assume that each $V_i$ has size greater than $k$, and that $G[V_J]$ has an ISR for all $J \subset [r]$. Take $v$ such that $G$ does not have an ISR containing $v$; we can assume $v \in V_r$. By Lemma 8, there is some $J \subseteq [r-1]$ and a set $D \subseteq V_J \cup \{v\}$ totally dominating $V_J \cup \{v\}$ such that (i) $D$ is the union of disjoint stable sets $X$ and $Y$, (ii) $Y$ is a partial ISR of $V_J$, (iii) $|X| \leq |Y| \leq |J|$, and (iv) $v \in X$.

Since $D$ totally dominates $V_J \cup \{v\}$, the sum of degrees of vertices in $D$ must be greater than the number of vertices in $V_J$. That is, $\sum_{v \in D} d(v) > \sum_{i \in J} |V_i|$. Clearly $\sum_{v \in X} d(v) \leq k \cdot |J|$ and $\sum_{v \in Y} d(v) \leq \sum_{i \in J} (|V_i| - k)$, so $\sum_{v \in D} d(v) \leq \sum_{i \in J} |V_i|$. Thus $D$ cannot totally dominate $V_J \cup \{v\}$, giving us the contradiction that proves the theorem.

This extends Haxell’s theorem by bounding the difference between the degree of a vertex and the size of its partition. One might hope that bounding the ratio of these by $\frac{1}{2}$ is enough, but it is not: Given $V_1$ of size four and $V_2, ..., V_5$ of size two, in which each vertex of $V_1$ dominates one of the smaller sets, there exists no ISR [3]. Lemma 8 cannot imply such a result because in the totally dominating set $D = X \cup Y$, we have no control over the average degree of a vertex in $X$—it may be $k$. So while we know that the average degree of a vertex in $Y$ behaves nicely with respect to the average partition size, the same is not necessarily true of $X$. Thus Theorem 9 gives a lopsided existence condition that is not only a useful consequence of Lemma 8, but also a natural one.

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