Walls and Chains of Planar Skyrmions

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(Dated: February 2, 2008)

In planar (baby) Skyrme systems, there may be extended linear structures which resemble either domain walls or chains of skyrmions, depending on the choice of potential and boundary conditions. We show that systems with a single vacuum, for example with potential \( V = 1 - \phi_3 \), admit chain solutions, whereas walls are ruled out by the uniqueness of the vacuum. On the other hand, in double-vacuum systems such as \( V = \frac{1}{2}(1 - \phi_3^2) \), one has stable wall solutions, but there are no stable chains; the walls may be viewed as the primary objects in such systems, with skyrmions being made out of them.

PACS numbers: 11.27.+d, 11.10.Lm, 11.15.-q

I. INTRODUCTION

The (2+1)-dimensional Skyrme system (known as the planar Skyrme model, or baby Skyrme model) may be viewed as a simple analog of the (3+1)-dimensional Skyrme model, and is also of considerable interest in its own right. This paper is concerned with walls and chains of planar skyrmions. Walls are extended objects of codimension 1, and chains are extended objects of dimension 1; so in the planar Skyrme system, walls and chains are both 1-dimensional objects, and they are related, as we shall see below.

In addition to the usual isolated (0-dimensional) skyrmion solutions, various other periodic, extended solutions are known for both the Skyrme and the baby Skyrme models: for example, a 3-dimensional skyrmion crystal [1], a 2-dimensional skyrmion wall [2], and a planar skyrmion crystal [3]. But 1-dimensional chains of skyrmions, or of planar skyrmions, have received less attention.

Chains of topological solitons have been intensively studied in other systems: for example, chains of instantons (‘calorons’) in the O(3) sigma model [4] and in Yang-Mills theory [5], and chains of BPS monopoles [6, 7]. A chain may be viewed as a string of solitons, and the dimensionless ratio \( C = (\text{intersoliton distance})/(\text{soliton size}) \) is an important parameter. When \( C < 1 \), interesting things can happen: the solitons lose their individual identities, and ‘constituents’ may emerge. In the caloron case, the constituents are monopoles [8, 9, 10].

One fundamental difference between chains of skyrmions or baby skyrmions on the one hand, and chains of instantons or monopoles on the other hand, is as follows. Instantons and monopoles do not exert forces on each other, and so the energy-per-period of a chain is independent of the parameter \( C \). By contrast, skyrmions and baby skyrmions may attract or repel each other, depending on their relative orientation; and so we expect a chain to contract, extend, or exhibit other dynamical properties. Of particular interest are stable chains: that is, chains whose energy cannot be lowered by any periodic deformation. Chains of this type might serve as good approximations to finite chains in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

An outline of the rest of this paper is as follows. In section 2 we review the baby Skyrme model and its topology; in sections 3 and 4 we discuss planar skyrmion walls and chains respectively; and section 5 contains some concluding remarks.

II. PLANAR SKYRMIONS

A planar Skyrme (or baby Skyrme) field is a smooth function \( \phi \) from (2+1)-dimensional Minkowski space-time to the 2-sphere \( S^2 \). In what follows, we shall restrict to static fields, and think of \( \phi \) as a unit vector \( \vec{\phi} = (\phi_1, \phi_2, \phi_3) \); so \( \vec{\phi} \cdot \vec{\phi} = 1 \). Here \( x \) and \( y \) are the usual spatial coordinates \( x^j = (x^1, x^2) = (x, y) \). The (static) energy density \( \mathcal{E} \) of the field is defined by

\[
\mathcal{E}[\phi] = \mathcal{E}_2[\phi] + \mu \mathcal{E}_4[\phi] + \mu \mathcal{E}_6[\phi],
\]

(1)
where

$$E_2 = \frac{1}{2} (\partial_i \phi \cdot \partial_j \phi), \quad E_4 = \frac{1}{2} (\partial_i \phi \times \partial_2 \phi)^2, \quad E_0 = V(\phi),$$

(2)

and where $\mu$ is a positive constant. (Taking the coefficients of $E_4$ and $E_0$ to be equal fixes the length-scale.) Two commonly-studied choices for the potential $V$ are:

- $V(\phi) = (1 - \phi_3)$ (the ‘old baby Skyrme model’, see eg [11])
- $V(\phi) = \frac{1}{2} (1 - \phi_3^2)$ (the ‘new baby Skyrme model’, see eg [12, 13])

The energy densities $E_2$ and $E_4$ are invariant under global SO(3) rotations of the target space $S^2$, but the choice of potential $V$ breaks this symmetry; in the old and new baby Skyrme models, the symmetry group is broken to SO(2).

The energy of a Skyrme field $\phi$ is

$$E[\phi] = \frac{1}{4\pi} \int_M E[\phi] \, dx^1 \, dx^2.$$  

(3)

(The factor of $1/4\pi$ is for convenience.) Of particular interest are stable skyrmions, namely finite-energy Skyrme fields which are local minima of $E$. For isolated skyrmions on $\mathbb{R}^2$, finiteness of energy requires that $\phi$ tends to a constant as $x^2 + y^2 \to \infty$. It follows that $\phi$ can be extended continuously to a map from $S^2$ to $S^2$, and it therefore has a degree (winding number), called the topological charge $N \in \mathbb{Z}$. There is an integral formula for $N$, namely $N[\phi] = \int_{S^2} N[\phi] \, dx \, dy$, where

$$N[\phi] = \frac{1}{4\pi} \phi \cdot (\partial_1 \phi \times \partial_2 \phi)$$

(4)

is the topological charge density. A standard Bogomolny-type argument shows that the topology gives a lower bound on the energy, namely

$$E \geq N \left(1 + \sqrt{2} \mu \int_{S^2} \sqrt{V} \, d\omega\right),$$

(5)

where $d\omega$ denotes the standard integration measure on $S^2$.

In each of the old and the new baby Skyrme models, there exist isolated $N$-Skyrmion solutions. For $N = 1, 2$ in the old baby Skyrme model, and for all $N$ in the new baby Skyrme model, these solutions are rotationally-symmetric: a spatial O(2) rotation can be compensated by a rotation of the target 2-sphere. In fact, these solutions have the O(2)-symmetric ‘hedgehog’ form

$$\phi = (\sin f(r) \cos (\theta - \chi), \sin f(r) \sin (\theta - \chi), \cos f(r)),$$

(6)

where $(r, \theta)$ are polar coordinates on $\mathbb{R}^2$, $\chi \in [0, 2\pi)$ is a phase angle, and $f(r)$ is a function satisfying $f(0) = \pi$ and $f(r) \to 0$ as $r \to \infty$. The force between separated skyrmions depends partly on their relative phase.

In the sections that follow, we shall impose periodicity (or anti-periodicity) in $y$, so that the field $\phi$ lives on $\mathbb{R} \times S^1$ rather than $\mathbb{R}^2$. By periodic with period $\beta$ we mean that $\tilde{\phi}(x, y + \beta) = \tilde{\phi}(x, y)$; while by anti-periodic we mean that $\tilde{\phi}(x, y + \beta) = \sigma \tilde{\phi}(x, y)$, where $\sigma$ is some element of the global symmetry group SO(2) satisfying $\sigma^2 = 1$. The topological classification and Bogomolny bound extend to these cases, as we shall see below.

For some choices of potential $V$, the Bogomolny bound (5) can be saturated, and the corresponding solutions can be written down explicitly. The procedure for deriving such potentials is well-known [3, 14], and the details may be summarized as follows. It is convenient to work in terms of the stereographic projection $W$ of $\tilde{\phi}$, given by

$$W = \frac{\phi_1 + i \phi_2}{1 + \phi_3}.$$  

(7)

In addition, we use the complex coordinate $z = x + iy$. Then the expressions for the topological charge density (4) and the energy density (2) become

$$\mathcal{N} = \frac{1}{\pi} \frac{|\partial_z W|^2 - |\partial_{\bar{z}} W|^2}{(|W|^2 + 1)^2}$$

$$\mathcal{E}_2 = 4 \left(\frac{|\partial_z W|^2 + |\partial_{\bar{z}} W|^2}{(|W|^2 + 1)^2}\right)$$

$$\mathcal{E}_4 = 8\pi^2 \mathcal{N}^2.$$
The existence of the Bogomolny bound \([5]\) follows from the identities
\[
E_2 = 4 \pi N + \frac{8|\partial_z W|^2}{(|W|^2 + 1)^2},
\]
\[
E_4 + E_0 = \left(2 \pi \sqrt{2} N - \sqrt{V}\right)^2 + 4 \pi \sqrt{2} N \sqrt{V},
\]
and the bound is saturated provided that
\[
\partial_z W = 0 \quad (8)
\]
\[
V(W) = \frac{8|\partial_z W|^4}{(|W|^2 + 1)^4}. \quad (9)
\]
From (8) we see that \(W\) has to be a holomorphic (or rather meromorphic) function of \(z\) — this corresponds to \(W\) being a solution of the \(\mu = 0\) limit of the system, namely the \(O(3)\) sigma model. If we have such a \(W(z)\), and we can express \(\partial_z W\) in terms of \(W\), then (9) gives the relevant potential \(V\). Examples of walls and chains of this type will be constructed below.

### III. SKYRMION WALLS

In this section, we consider fields which are periodic in \(y\) (with period \(\beta\)), and which satisfy the boundary condition \(\phi_3 \to \pm 1\) as \(x \to \pm \infty\). In order to get finite energy, the potential \(V\) has to vanish for \(\phi_3 = \pm 1\); the standard example is therefore the system \(V = \frac{1}{2}(1 - \phi_3^2)\). The topological classification is straightforward, and may be described as follows. The field \(\tilde{\phi}\) is defined on the cylinder \(\mathbb{R} \times S^1\), but the boundary condition means that the ‘ends’ at \(x = \pm \infty\) can be regarded as single points, and therefore \(\tilde{\phi}\) represents a map from \(S^2\) to \(S^2\). So it has a degree \(K \in \mathbb{Z}\). Thus the topological (or magnetic) charge of a segment of wall of length \(N\beta\) equals \(NK\); note that the examples below all have \(K = 1\). Planar skyrmion walls have been studied previously: for example, the authors of [12] investigated the interaction between a skyrmion and an uncharged (\(K = 0\)) wall.

The energy per period \(E'\), and the integer \(K\), are given by
\[
E'[\phi] = \frac{1}{4 \pi} \int_0^\beta \int_{-\infty}^{\infty} \mathcal{E}[\phi] \, dx^1 \, dx^2, \quad (10)
\]
\[
K[\phi] = \int_0^\beta \int_{-\infty}^{\infty} \mathcal{N}[\phi] \, dx^1 \, dx^2, \quad (11)
\]
where \(\mathcal{E}\) and \(\mathcal{N}\) are defined by the same expressions as before. Furthermore, the energy \(E'\) is bounded below as in \([5]\).

One can construct exact solutions by using the procedure described at the end of the previous section. For simplicity, set \(\beta = 2 \pi\). The simplest function \(W(z)\) satisfying the boundary conditions is \(W = \exp(-z)\); substituting this into \([9]\), and converting from \(W\) to \(\phi\), gives the potential
\[
V = \frac{1}{2}(1 - \phi_3^2)^2. \quad (12)
\]
For this potential, the Bogomolny bound is \(E' \geq 1 + \frac{2}{3} \mu\); the field \(W = \exp(-z)\) is a \(K = 1\) wall solution which saturates this lower bound. In particular, the wall has a preferred length-per-charge (or period), which in this case is \(2\pi\). (Multiplying \(V\) by a constant has the effect of changing \(\beta\).) Note that the size of an individual soliton is of order \(1\), so \(\beta\) is essentially the dimensionless ratio \(C\) mentioned in the Introduction.

Staying with this potential \([12]\), one can envisage taking a segment of the wall of length \(2\pi N\) in the \(y\)-direction, and joining the ends together to form an \(N\)-skyrmion in \(\mathbb{R}^2\). So one might expect \(N\)-skyrmions in this system to resemble rings of radius \(N\), and with energy
\[
E = N \left(1 + \frac{2}{3} \mu + \frac{c}{N^2}\right),
\]
where \(c\) is a positive constant (the \(c/N^2\) term representing a curvature contribution). A numerical simulation, assuming the hedgehog form \([7]\) and finding the profile function \(f(r)\) which minimizes \(E\), reveals that this is indeed
what happens. In particular, the energy $E/N$ of $N$-skyrmions in the range $3 \leq N \leq 20$ was computed for $\mu = 1$, and the values are plotted in Figure 1. These values are well-fitted by the curve $E/N = B + 0.29/N^2$, where $B = 5/3$ is the Bogomolny bound.

Similar (although slightly less explicit) features occur for other ‘double-vacuum’ potentials, such as $V = \frac{1}{2}(1 - \phi_3^2)$. In this case, one again has a wall solution with a preferred length-per-charge $\beta$, which now depends on the parameter $\mu$; below, we shall derive and plot the expression for $\beta = \beta(\mu)$. Isolated $N$-skyrmions in this system have previously been studied \cite{13}: they are rings with radii which grow linearly with $N$, and $E/N$ is a decreasing function of $N$, similarly to the previous example.

To calculate $\beta(\mu)$, we assume homogeneity in the $y$-direction, by taking the field to have the form

$$\vec{\phi} = (\sin(f) \cos(\nu y), \sin(f) \sin(\nu y), \cos(f)). \quad (13)$$

Here $\nu = 2\pi/\beta$, and $f = f(x)$ is a function satisfying the boundary conditions $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to \pi$ as $x \to \infty$. Note that this wall has $K = 1$ (one unit of charge per period), and the energy per period is

$$E' = \frac{\beta}{4\pi} \int_{-\infty}^{\infty} \mathcal{E} \, dx,$$

where

$$\mathcal{E} = \frac{1}{2}(f')^2 + \frac{1}{2}(\nu^2 + \mu \sin^2 f + \frac{\mu \nu^2}{2} (f')^2 \sin^2 f.$$ 

This functional $E'$ satisfies a Bogomolnyi-type inequality, namely

$$\frac{4\pi}{\beta} E' = \int_{-\infty}^{\infty} \frac{1}{2} \left( \sqrt{1 + \mu \nu^2 \sin^2 f (f')^2} - \sqrt{\mu + \nu^2 \sin^2 f} \right)^2 \, dx$$

$$+ \sqrt{1 + \mu \nu^2 \sin^2 f} \sqrt{\mu + \nu^2 \sin f} \int_{-\infty}^{\infty} f' \, dx$$

$$\geq \int_{0}^{\pi} \sqrt{1 + \mu \nu^2 \sin^2 f} \sqrt{\mu + \nu^2 \sin f} \, df, \quad (14)$$
with equality if and only if the Bogomolny equation

\[ f' = \frac{\sqrt{\mu + \nu^2} \sin f}{\sqrt{1 + \mu \nu^2 \sin^2 f}}, \]  

(15)

is satisfied. This equation can be integrated to give

\[ x = x_0 + \frac{1}{\sqrt{\mu + \nu^2}} \left( \sqrt{\mu \nu} \arctan \sqrt{\mu \nu z} + \frac{1}{2} \ln \frac{1 + z}{1 - z} \right), \]

\[ z = \frac{-\cos f}{\sqrt{1 + \mu \nu^2 \sin^2 f}}, \]

and this implicitly determines the profile function \( f(x) \). But without knowing \( f(x) \) explicitly, one can use (14, 15) to obtain an expression for the minimal energy: it is

\[ E' = \frac{1}{2 \nu} \sqrt{\mu + \nu^2} \left[ \left( \sqrt{\mu \nu} + \frac{1}{\sqrt{\mu \nu}} \right) \arcsin \sqrt{\frac{\mu \nu^2}{1 + \mu \nu^2} + 1} \right], \]

(16)

and this is minimized with respect to variations in \( \nu \) if

\[ \left( \sqrt{\mu \nu} - \frac{1}{\sqrt{\mu \nu}} - \frac{2 \sqrt{\mu}}{\nu^3} \right) \arcsin \sqrt{\frac{\mu \nu^2}{1 + \mu \nu^2} + 1} = 0. \]

(17)

Solving (17) for \( \nu \) then gives the preferred period \( \beta = 2\pi/\nu \) as a function of \( \mu \). This function \( \beta(\mu) \) is plotted in Figure 2, for the range \( 0 \leq \mu \leq 4 \), from which one sees that the dependence on \( \mu \) is not very strong. Note that \( \beta/\pi \to 2^{3/4} \approx 1.68 \) as \( \mu \to \infty \).

![Graph showing the relationship between \( \mu \) and \( \beta/\pi \) for the preferred length-per-charge of a wall in the system \( V = \frac{1}{2}(1 - \phi_3^2) \).](image)

FIG. 2: Preferred length-per-charge \( \beta \) of a wall in the system \( V = \frac{1}{2}(1 - \phi_3^2) \).
IV. SKYRMION CHAINS

In this section, we consider fields which are anti-periodic in $y$, and which satisfy the boundary condition $\phi_3 \to 1$ as $x \to \pm \infty$. We call static solutions satisfying these boundary conditions Skyrmion chains, and we shall investigate such chains in the two systems $V = \frac{1}{2}(1 - \phi_3^2)$ and $V = 1 - \phi_3$.

The energy $E'$ and magnetic charge $K$ per period are given by the same expressions as before. It is not immediately obvious that $K$ has to be an integer (since the field is anti-periodic rather than periodic), but the following argument shows that it does (i.e. $K \in \mathbb{Z}$). Suppose that $\vec{\phi}(x,y)$ satisfies the anti-periodicity condition $(\phi_1(x,y + \beta), \phi_2(x,y + \beta), \phi_3(x,y + \beta)) = (-\phi_1(x,y), -\phi_2(x,y), \phi_3(x,y))$.

We will construct a new field $\vec{\psi}(x,y)$ such that $K[\psi] = K[\phi]$, and such that $\vec{\psi}$ is strictly periodic (with period $\beta$). But then as described in the previous section, $K[\psi]$ is the degree of $\psi$, and so is necessarily an integer; therefore $K[\phi]$ has to be an integer as well. The new field is defined as $\psi(x,y) = R(y)\phi(x,y)$, where

$$R(y) = \begin{pmatrix} \cos(\pi y/\beta) & -\sin(\pi y/\beta) & 0 \\ \sin(\pi y/\beta) & \cos(\pi y/\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

The charge density of $\psi$ is

$$N[R\phi] = \frac{1}{4\pi} (R\phi) \cdot (R\partial_1 \phi) \times (R\partial_2 \phi + \partial_2 R\phi)$$

$$= N[\phi] + \frac{1}{4\pi} \phi \cdot \partial_1 \phi \times [R^{-1}(\partial_2 R)\phi]$$

$$= N[\phi] - \frac{1}{4\beta} \partial_1 \phi_3.$$  

And integrating this gives $K[R\phi] = K[\phi]$, as claimed, since $\lim_{x \to \infty} \phi_3 = \lim_{x \to -\infty} \phi_3$.

A. Saturating the Bogomolny Bound

The Bogomolny bound \cite{14} holds for skyrmion chains; as was done previously for walls, one may write down simple configurations $W(z)$ which saturate the bound for potentials given by \cite{14}. One possible choice, satisfying the chain boundary conditions, is

$$W(z) = c \text{sech}(\pi z/\beta),$$  

where $c$ is a positive constant. The potential defined by this field and \cite{14} is

$$V = \frac{1}{2} \left( \frac{\pi}{\beta c} \right)^4 (1 - \phi_3)^2 \left[ \left( (c^2 + 1)\phi_3 + (c^2 - 1) \right)^2 + 4c^2\phi_3^2 \right].$$  

This potential $V$ has three vacua, at

$$\vec{\phi} = (0,0,1), \left( \frac{\pm 2c}{1 + c^2}, 0, \frac{1 - c^2}{1 + c^2} \right).$$  

In the limit where $\beta \to \infty$, with $c \to 0$ so that $\beta c$ remains constant, we get the potential $V = (1 - \phi_3)^4$. This was to be expected, since it is exactly this potential which admits a ‘saturating’ 1-skyrmion solution on the plane \cite{14}.

B. Chains in the old baby Skyrme model

In this subsection, we consider chains in the old baby Skyrme model, in other words with potential $V = 1 - \phi_3$. We begin with an analytic approximation describing a chain of well-separated skyrmions (this corresponds to $\beta \gg 1$), and then we describe some numerical results.
The $\beta \gg 1$ approximation is an adaptation of the corresponding one for widely-separated skyrmions in $\mathbb{R}^2$, used in [11]. It enables one to compute the energy of a collection of skyrmions in terms of their separations and relative phases. Let $W^{(1)}, W^{(2)}$ be two skyrmions [6], with phases $\chi^{(1)}, \chi^{(2)}$, expressed in stereographic coordinates [7]. Suppose that these two skyrmions are given a separation $D$, and consider the field $W = W^{(1)} + W^{(2)}$ representing their superposition. In the limit where $D \to \infty$, the energy of $W$ is twice the energy $E_1$ of a single skyrmion. One can calculate the leading contribution to the difference

$$I_2 := E[W^{(1)} + W^{(2)}] - 2E_1,$$

in the limit where $D \to \infty$. The result [11] is that

$$I_2 \approx \frac{p^2 \mu^2}{4\pi^2} \cos(\chi^{(2)} - \chi^{(1)}) K_0(\sqrt{D}),$$

for large $D$, where $K_0$ is the modified Bessel function of order zero, and where $p$ is a constant related to the decay of the profile function $f(r)$ by

$$f(r) \approx \frac{p\mu}{2\pi} K_1(\sqrt{p}r).$$

This constant $p$ depends on $\mu$, and has to be determined numerically: for example, when $\mu^2 = 0.1$, one finds $p = 24.16$. This approximation is called the dipole approximation, because the asymptotic formula for $I_2$ matches the interaction energy of a pair of orthogonal scalar dipoles. Notice that the sign of $I_2$ depends on the relative orientation $\chi^{(2)} - \chi^{(1)}$; it follows that a pair of aligned skyrmions repel, while a pair of skyrmions with relative phase $\pi$ attract.

This approximation can be adapted to the case of an infinite chain of skyrmions. We shall do the computation both for in-phase skyrmions (so that the resulting chain is periodic), and for anti-periodic chains where each pair of neighbouring skyrmions has a relative phase of $\pi$. For each $j \in \mathbb{Z}$, let $W^{(j)}$ be a skyrmion located at $(x,y) = (0,\beta j)$, with phase $\chi = 0$. Then the fields

$$W^\pm := \sum_{j=-\infty}^{\infty} (\pm 1)^j W^{(j)}$$

are skyrmion chains, with relative phase 0 or $\pi$ according to whether one chooses the upper or the lower sign. Let $I_{\pm} = E'[W^\pm] - E_1$ be the interaction energy per period of the chain. One can show analytically that

$$I_{\pm} \approx \sum_{j=1}^{\infty} (\pm 1)^j \frac{p^2 \mu^2}{4\pi^2} K_0(\sqrt{p}j\beta)$$

(21)

to leading order, in the limit where $\beta \to \infty$. In fact, in this limit, only the first term in the series [21] is significant, and so this approximation is really the same as [20]. As before, $I_+(\beta)$ is a positive decreasing function, and $I_-(\beta)$ is a negative increasing function (for large $\beta$). So, a chain of well-separated skyrmions will extend if the skyrmions are aligned, and contract if they are anti-aligned.

For the remainder of this subsection, we restrict to the anti-periodic case, and in particular pose the question: is there a chain with a preferred length-per-charge? In other words: which period $\beta$ (if any) will minimize the energy $E'$? Anti-periodic chains in this system were studied in [15] by using an ansatz for $\tilde{\phi}$ which, in effect, imposed homogeneity in the $y$-direction. Then there is a Bogomolny-type argument which yields a formula for $E'$, and one can minimize $E'$ with respect to $\beta$. The result is plotted in Figure 3 (dashed curve), for the system with $\mu = 1$. However, the actual solution, for each $\beta$, is not of this homogeneous type, and the actual energy is lower than that of a homogeneous field. This minimal energy has to be determined numerically, and the result of such a computation is also presented in Figure 3 (solid curve). The conclusion, therefore, is that a chain of anti-aligned skyrmions in the old baby Skyrme system will contract until its period $\beta$ reaches a value which minimizes $E'$; and for $\mu = 1$ this preferred period is $\beta \approx 0.76\pi$, as one sees from Figure 3. Figure 4 depicts the chain at $\beta = 0.76\pi$, with plots of the three components $\phi_j$ as well as the energy density $E'$.

The numerical procedure used in deriving Figures 3–5 is as follows. The $xy$-space is replaced by a rectangular grid, with a transformation of $x$ to ensure that $x \to \pm \infty$ is included. The expression for the energy $E'$ is discretized with finite differences, and is then minimized using a conjugate-gradient method. Richardson extrapolation is used to reduce the error in $E'$ to significantly less than 1%.
FIG. 3: Energy-per-period of a chain in the system $V = 1 - \phi_3$, as a function of the period $\beta$. The solid curve gives the minimal energy; the dashed curve gives the energy of a chain which is $y$-homogeneous.

C. Chains in the new baby Skyrme model

In this subsection, we consider chains in the new baby Skyrme model; that is, with potential $V = \frac{1}{2}(1 - \phi_3^2)$. The dipole approximation described above works equally well in this case, and the same result is obtained: namely that a chain of anti-aligned skyrmions will contract. As before, we pose the question of whether there is a preferred period $\beta$ corresponding to a minimal-energy chain.

A new feature in this case is that the system also admits wall solutions. One may construct a field with chain boundary conditions by taking it to consist of a parallel wall-antiwall pair. From this it is clear that there is now a possible decay mode for chains: namely that a chain could split into a wall and an antiwall which separate, and move in the negative $x$-direction and the positive $x$-direction respectively. Whether or not the wall and antiwall actually do move apart depends on the force between them, and this force can be studied analytically, as follows.

Let $\phi^{(1)}$ be a domain wall (13) with $\nu^{(1)} = \pi/\beta$, with profile function $f^{(1)}$ given by (16), and located at $x = -D/2$ for some real number $D > 0$. Let $\phi^{(2)}$ be a second domain wall given by (13) with $\nu^{(2)} = -\pi/\beta$, with profile function $f^{(2)}(x) = f^{(1)}(-x)$. So this second wall is an antiwall, and it is located at $x = D/2$. Let $W^{(1)}$ and $W^{(2)}$ be the stereographic projections of these, as in (7). Finally, define a superposition $W$ by

$$\frac{1}{W} = \frac{1}{W^{(1)}} + \frac{1}{W^{(2)}}.$$ 

The field $W$ satisfies the boundary conditions for anti-periodic chain with period $\beta$, and the charge per unit period is 1. This field resembles a parallel wall-antiwall pair, separated by a distance $D$, with $\phi_3 = 1$ between the walls and $\phi_3 = 1$ on either side. We want the interaction energy

$$I_w := E[W] - E[W^{(1)}] - E[W^{(2)}]$$

of the superposition, which is a function of $\beta$ and $\mu$.

Now $I_w \to 0$ as $\beta \to \infty$, and we have calculated the leading contribution to $I_w$, in this limit. The calculation is similar to that for two isolated skyrmions in [11], but is more complicated in practice, for the following reason. In the case of two isolated skyrmions, the leading contribution is found using an elegant argument based on the Euler-Lagrange equations; in the case of two walls, however, the analogous term vanishes, and so we must work to higher
order. We find that

\[ I_w \approx \frac{4 \mu \exp(4\sqrt{\mu \nu} \arctan(\sqrt{\mu \nu})}{\nu(1 + \mu \nu^2)^2} \]
\[ \times (2\nu^4 + 2\mu \nu^2 - 1)D \exp(-2D\sqrt{\mu + \nu^2}) \]

provided \( D \gg 1 \), where we have written \( \nu = \pi / \beta \). So a pair of well-separated domain walls will attract when \( \beta^4 - 2\mu \pi^2 \beta^2 - 2\pi^4 > 0 \) and repel otherwise. Therefore the approximate prediction is that a chain is stable with respect to separation into two walls provided that its period \( \beta \) is greater than the critical value

\[ \beta_c = \pi \sqrt{\mu + \sqrt{\mu^2 + 2}}. \] (22)

But for \( \beta < \beta_c \), chains will decay into separating wall-antiwall pairs.

Recall from section 3 that each wall in a well-separated wall-antiwall pair has a preferred period \( \beta \); it is obtained by solving (17) for \( \nu \), and then setting \( \beta = \pi / \nu \). (We take \( \beta \) to have half the value it had in section 3, so that the charge-per-period of the wall-antiwall system is \( K = 1 \). As a consequence, the chain is anti-periodic rather than periodic.) From Figure 2 we see that for \( \mu = 1 \), the preferred period is \( \beta = 0.89\pi \). By contrast, (22) gives \( \beta_c = 1.65\pi \). So if we begin with a wall-antiwall pair with large \( \beta \), then each wall in the pair will continue to contract until the period reaches \( \beta_c \), whereupon they will begin to repel each other.

A numerical simulation for the case \( \mu = 1 \) reveals that this is indeed what happens, with the critical value of the period being only 10% below that predicted by (22). This result is illustrated in Figure 5, which plots the energy \( E' \) of a chain, numerically-minimized for a range of (fixed) values of \( \beta \). The dashed line is the energy \( E = 2.151 \) of an isolated skyrmion in this system, and so one sees that for \( \beta \geq 4\pi \) each skyrmion in the chain hardly notices its neighbours. Energy can be lowered by reducing the period (the chain tends to contract); but once the period reaches \( \beta \approx 1.49\pi \), the separating-wall instability sets in. So for chains in this system there is no preferred length-per-charge.
V. CONCLUSIONS

In single-vacuum systems such as $V = 1 - \phi_3^3$, one has chains with a preferred length-per-charge. They are stable against periodic perturbations; their stability against more general perturbations has not been investigated here. In double-vacuum systems such as $V = \frac{1}{2}(1 - \phi_3^3)$, one has stable wall solutions, but there are no stable chains. The walls are, in some sense, the primary objects in such systems — skyrmions are made out of them (although one can, in certain cases, also have crystalline chunks [3]). The same is true one dimension higher up, namely for the Skyrme model. Skyrme walls were discussed in [2], and skyrmions are hollow shells constructed out of this wall material [16]. Introducing a potential can change things: for example, potentials of $1 - \phi_3^3$ type [17], or of more general type [18].

It would be interesting to extend the investigations described here to other, related, systems. For example, in 2+1 dimensions, the baby Skyrme model is related to abelian Higgs models which admit local and semi-local vortices [19]; what are the properties of chains and walls in this general family of systems, and their $\mathbb{CP}^n$ generalizations? Going to 3+1 dimensions, do stable chains exist in the Skyrme model, and how does the choice of potential affect such structures?

Acknowledgments

This work was supported by a studentship, and research grants “Strongly Coupled Phenomena” and “Classical Lattice Field Theory”, from the UK Science and Technology Facilities Council.

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