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Fuzzy de Sitter space-times via coherent states quantization

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Abstract

A construction of the 2d and 4d fuzzy de Sitter hyperboloids is carried out by using a (vector) coherent state quantization. We get a natural discretization of the dS “time” axis based on the spectrum of Casimir operators of the respective maximal compact subgroups $SO(2)$ and $SO(4)$ of the de Sitter groups $SO_0(1,2)$ and $SO_0(1,4)$. The continuous limit at infinite spins is examined.

1 Introduction

The Madore construction of the fuzzy sphere [1] is based on the replacement of coordinate functions of the sphere by components of the angular momentum operator in a $(2j+1)$-dimensional UIR of $SU(2)$. In this way, the commutative algebra of functions on $S^2$, viewed as restrictions of smooth functions on $\mathbb{R}^3$, becomes the non-commutative algebra of $(2j+1) \times (2j+1)$-matrices, with corresponding differential calculus. The commutative limit is recovered at $j \to \infty$ while another parameter, say $\rho$, goes to zero with the constraint $j \rho = 1$ (or $R$ for a sphere of radius $R$). The aim of the present work is to achieve a similar construction for the 2d and 4d de Sitter hyperboloids. The method is based on a generalization of coherent state quantization à la Klauder-Berezin (see [2, 3] and references therein). We recall that the de Sitter space-time is the unique maximally symmetric solution of the vacuum Einstein’s equations with positive cosmological constant $\Lambda$. This constant is linked to the constant Ricci curvature $4\Lambda$ of this space-time. There exists a fundamental length $H^{-1} := \sqrt{3/(c\Lambda)}$. 

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The isometry group of the de Sitter manifold is the ten-parameter de Sitter group $SO_0(1,4)$, the latter is a deformation of the proper orthochronous Poincaré group $P^+_\uparrow$.

2 Coherent state quantization: the general framework

Let $X$ be a set equipped with the measure $\mu(dx)$ and $L^2(X,\mu)$ its associated Hilbert space of square integrable functions $f(x)$ on $X$. Among the elements of $L^2(X,\mu)$ let us select an orthonormal set $\{\phi_n(x), n = 1,2,\ldots,N\}$, $N$ being finite or infinite, which spans, by definition, a separable Hilbert subspace $\mathcal{H}$. This set is constrained to obey: $0 < \mathcal{N}(x) := \sum_n |\phi_n(x)|^2 < \infty$. Let us then consider the family of states $\{|x\rangle\}_{x \in X}$ in $\mathcal{H}$ through the following linear superposition:

$$|x\rangle := \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \phi_n(x) \langle \phi_n|.$$

(1)

This defines an injective map (which should be continuous w.r.t. some topology affected to $X$) $X \ni x \mapsto |x\rangle \in \mathcal{H}$. These coherent states are normalized and provide a resolution of the unity in $\mathcal{H}$:

$$\langle x| x \rangle = 1, \quad \int_X |x\rangle \langle x| \mathcal{N}(x) \mu(dx) = 1_{\mathcal{H}}.$$

(2)

A classical observable is a function $f(x)$ on $X$ having specific properties. Its quantization à la Berezin-Klauder-“Toeplitz” consists in associating to $f(x)$ the operator

$$A_f := \int_X f(x)|x\rangle \langle x| \mathcal{N}(x) \mu(dx).$$

(3)

For instance, the application to the sphere $X = S^2$ with normalized measure $\mu(dx) = \sin \theta \, d\theta \, d\phi/4\pi$ is carried out through the choice as orthonormal set the set of spin spherical harmonics $\sigma Y_{jm}(\hat{r})$ for fixed $\sigma$ and $j$. One obtains [4] in this way a family of inequivalent (with respect to quantization) fuzzy spheres, labeled by the the spin parameter $0 < |\sigma| \leq j, j \in \mathbb{N}_*/2$. Note that the spin is necessary in order to get a nontrivial quantization of the cartesian coordinates.
3 Application to the 2d de Sitter spacetime

De Sitter space is seen as a one-sheeted hyperboloid embedded in a three-dimensional Minkowski space:

\[ M_H = \{ x \in \mathbb{R}^3 : \eta_{\alpha\beta} x^{\alpha} x^{\beta} = (x^0)^2 - (x^1)^2 - (x^2)^2 = -H^{-2} \}. \]  

(4)

The de Sitter group is \( SO_0(1,2) \) or its double covering \( SU(1,1) \simeq SL(2,\mathbb{R}) \). Its Lie algebra is spanned by the three Killing vectors \( K_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha \) (\( K_{12} \): compact, for “space translations”, \( K_{02} \): non compact, for “time translations”, \( K_{01} \): non compact, for Lorentz boosts). These Killing vectors are represented as (essentially) self-adjoint operators in a Hilbert space of functions on \( M_H \), square integrable with respect to some invariant inner (Klein-Gordon type) product.

The quadratic Casimir operator has eigenvalues which determine the UIR’s:

\[ Q = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = -j(j+1)\mathbb{I} = (\rho^2 + \frac{1}{4})\mathbb{I} \]  

(5)

where \( j = -\frac{1}{2} + i\rho, \rho \in \mathbb{R}^+ \) for the principal series.

Comparing the geometric constraint (4) to the group theoretical one (5) (in the principal series) suggests the fuzzy correspondence:

\[ x^{\alpha} \mapsto \hat{x}^{\alpha} = \frac{r}{2} \epsilon^{\alpha\beta\gamma} M_{\beta\gamma}, \text{ i.e. } \hat{x}^0 = rM_{21}, \hat{x}^1 = rM_{02}, \hat{x}^2 = rM_{10}, \]  

\( r \) being a constant with length dimension. The following commutation rules are expected

\[ [\hat{x}^0, \hat{x}^1] = ir\hat{x}^2, \quad [\hat{x}^0, \hat{x}^2] = -ir\hat{x}^1, \quad [\hat{x}^1, \hat{x}^2] = ir\hat{x}^0, \]  

(6)

with \( \eta_{\alpha\beta}\hat{x}^{\alpha}\hat{x}^{\beta} = -r^2(\rho^2 + \frac{1}{4})\mathbb{I} \), and its “commutative classical limit”, \( r \to 0, \rho \to \infty, r\rho = H^{-1} \).

Let us now proceed to the CS quantization of the 2d dS hyperboloid. The “observation” set \( X \) is the hyperboloid \( M_H \). Convenient global coordinates are those of the topologically equivalent cylindrical structure: \( (\tau, \theta) \), \( \tau \in \mathbb{R}, 0 \leq \theta < 2\pi \), through the parametrization, \( x^0 = r\tau \), \( x^1 = r\tau \cos \theta - H^{-1} \sin \theta \), \( x^2 = r\tau \sin \theta + H^{-1} \cos \theta \), with the invariant measure: \( \mu(dx) = \frac{1}{2\pi} d\tau d\theta \). The functions \( \phi_m(x) \) forming the orthonormal system needed to construct coherent states are suitably weighted Fourier exponentials:

\[ \phi_m(x) = \left( \frac{\kappa}{\pi} \right)^{1/4} e^{-\frac{1}{\kappa} (\tau-m)^2} e^{im\theta}, \quad m \in \mathbb{Z}, \]  

(7)
where the parameter $\epsilon > 0$ can be arbitrarily small and represents a necessary regularization. Through the superposition (8) the coherent states read

$$ |\tau, \theta \rangle = \frac{1}{\sqrt{\mathcal{N}(\tau)}} \left( \frac{\epsilon}{\pi} \right)^{1/4} \sum_{m \in \mathbb{Z}} e^{-\frac{\epsilon}{2}(\tau-m)^2} e^{-im\theta} |m\rangle, \quad (8) $$

where $|\phi_m\rangle \simeq |m\rangle$. The normalization factor $\mathcal{N}(\tau) = \sqrt{\frac{\epsilon}{\pi}} \sum_{m \in \mathbb{Z}} e^{-\epsilon(\tau-m)^2} < \infty$ is a periodic train of normalized Gaussians and is proportional to an elliptic Theta function.

The CS quantization scheme (3) yields the quantum operator $A_f$, acting on $\mathcal{H}$ and associated to the classical observable $f(x)$. For the most basic one, associated to the coordinate $\tau$, one gets

$$ A_{\tau} = \int_X \tau |\tau, \theta \rangle \langle \tau, \theta | \mathcal{N}(\tau) \mu(dx) = \sum_{m \in \mathbb{Z}} m|m\rangle\langle m|. \quad (9) $$

This operator reads in angular position representation (Fourier series):

$$ A_{\tau} = -\frac{i}{\epsilon} \frac{\partial}{\partial \theta}, $$

and is easily identified as the compact representative $M_{12}$ of the Killing vector $K_{12}$ in the principal series UIR. Thus, the “time” component $x^0$ is naturally quantized, with spectrum $r \mathbb{Z}$ through $x^0 \mapsto \tilde{x}^0 = -rM_{12}$. For the two other ambient coordinates one gets:

$$ \tilde{x}^1 = \frac{re^{-\frac{\epsilon}{2}}}{-2} \sum_{m \in \mathbb{Z}} \{p_m|m+1\rangle\langle m+h.c\}, \quad \tilde{x}^2 = \frac{re^{-\frac{\epsilon}{2}}}{-2i} \sum_{m \in \mathbb{Z}} \{p_m|m+1\rangle\langle m-h.c\}, $$

with $p_m = (m + \frac{1}{2} + ip)$. Commutation rules are those of $so(1, 2)$, that is those of (8) with a local modification to $[\tilde{x}^1, \tilde{x}^2] = -ire^{-\frac{\epsilon}{2}}\tilde{x}^0$. The commutative limit at $r \to 0$ is apparent. It is proved that the same holds for higher degree polynomials in the ambient space coordinates.

### 4 Application to the 4d de Sitter spacetime

The extension of the method to the 4d-de Sitter geometry and kinematics involves the universal covering of $SO_0(1, 4)$, namely, the symplectic $Sp(2, 2)$ group, needed for half-integer spins. In a given UIR of the latter, the ten Killing vectors are represented as (essentially) self-adjoint operators in Hilbert space of (spinor-)tensor valued functions on the de Sitter space-time $M_H$, square integrable with respect to some invariant inner (Klein-Gordon type) product: $K_{\alpha\beta} \to L_{\alpha\beta}$. There are now two Casimir operators whose eigenvalues determine the UIR’s:

$$ Q^{(1)} = -\frac{1}{2}L_{\alpha\beta}L^{\alpha\beta}, \quad Q^{(2)} = -W_\alpha W^\alpha, \quad W^\alpha := -\frac{1}{8}L_{\alpha\beta\gamma\delta}L^{\beta\gamma}L^{\delta\epsilon}. $$

4
Similarly to the 2-dimensional case, the principal series is involved in the construction of the fuzzy de Sitter space-time. Indeed, by comparing both constraints, the geometric one: 

\[ \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2} \]

and the group theoretical one, involving the quartic Casimir (in the principal series with spin \( s > 0 \)):

\[ Q^{(2)} = -W^\alpha W_\alpha = (\nu^2 + \frac{1}{4}) s(s + 1) \]

suggests the correspondence \( x^\alpha \rightarrow \xi^\alpha = rW^\alpha \), and the "commutative classical limit": \( r \rightarrow 0, \nu \rightarrow \infty, rs\sqrt{\nu^2 + \frac{1}{4}} = H^{-1} \).

For the CS quantization of the 4d-dS hyperboloid, suitable global coordinates are those of the topologically equivalent \( \mathbb{R} \times S^3 \) structure:

\( (\tau, \xi), \tau \in \mathbb{R}, \xi \in S^3, \) through the following parametrization, \( x^0 = r\tau, x = (x^1, x^2, x^3, x^4)^t = r\tau \xi + H^{-1} \xi^\perp, \) where \( \xi^\perp \in S^3 \) and \( \xi \cdot \xi^\perp = 0, \) with the invariant measure: \( \mu(dx) = d\tau \mu(d\xi) \). We now consider the spectrum \( \{ \tau_i \mid i \in \mathbb{Z} \} \) of the compact "dS fuzzy time" operator \( rW_0 \) in the Hilbert space \( L^2_{C^2s+1}(S^3) \) which carries the principal series UIR \( U_{s,\nu}, s > 0 \). This spectrum is discrete. Let us denote by \( \{ Z_J(\xi) \} \), where \( J \) represents a set of indices including in some way the index \( i \), an orthonormal basis of \( L^2_{C^2s+1}(S^3) \) made up with the eigenvectors of \( W^0 \). The functions \( \phi_J(x) \), forming the orthonormal system needed to construct coherent states, are suitably weighted Fourier exponentials:

\[ \phi_J(x) = \left( \frac{\epsilon}{\pi} \right)^{1/4} e^{-\frac{\epsilon}{4}(\tau - \tau_i)^2} Z_J(\xi), \]  

where \( \epsilon > 0 \) can be arbitrarily small. The resulting vector coherent states read as

\[ |\tau, \xi\rangle = \frac{1}{\sqrt{N(\tau, \xi)}} \left( \frac{\epsilon}{\pi} \right)^{1/4} \sum_J e^{-\frac{\epsilon}{4}(\tau - \tau_i)^2} \overline{Z_J(\xi)} |J\rangle, \]

with normalization factor

\[ N(x) \equiv N(\tau, \xi) = \sqrt{\frac{\epsilon}{\pi}} \sum_J e^{-\epsilon(\tau - \tau_i)^2} \overline{Z_J^*(\xi)} Z_J(\xi) < \infty. \]

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