ON THE JONES POLYNOMIAL OF $2n$-PLAT PRESENTATIONS OF KNOTS

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ABSTRACT. In this paper, a method is given to calculate the Jones polynomial of the 6-plat presentations of knots by using a representation of the braid group $B_6$ into a group of $5 \times 5$ matrices. We also can calculate the Jones polynomial of the $2n$-plat presentations of knots by generalizing the method for the 6-plat presentations of knots.

1. Introduction

In 1985, Jones [8] discovered the polynomial knot invariant $V_K(t)$ and gave a formula to calculate the polynomials of knots that are presented as closed braids. He also gave a formula to calculate the Jones polynomials of knots that are presented as closed plats. The closed plat formula is described in [3]. Birman and Kanenobu [3] generalized the formula to the polynomials of knots which are obtained by a combination of closed braid and plat. In the case of $2n$-plat, by using the skein relation of Jones polynomial, we have $2^n$ closed braids that is related to the given $2n$-plat. Then the Jones polynomial of the $2n$-plat can be obtained from the Jones polynomials of the $2^n$ closed braids.

Kauffman [9] introduced the Kauffman bracket $\langle K \rangle$ and the writhe $w(K)$ to calculate the Kauffman polynomial $X_K(a)$, which is identical to the Jones polynomial $V_K(t)$ with the change of variable $t = a^4$.

In this paper, by using the skein relation of the Kauffman bracket, we present a method to calculate the Kauffman bracket and the writhe of 6-plat presentations of knots that is obtained directly from the 6-plat presentation. Also, we indicate how it extends to $2n$-plat presentations of knots.

Let $S^2$ be a sphere smoothly embedded in $S^3$ and let $K$ be a knot transverse to $S^2$. The complement in $S^3$ of $S^2$ consists of two open balls, $B_1$ and $B_2$. We assume that $S^2$ is the $xz$-plane $\cup \{\infty\}$. Let $p$ be the projection onto the $xy$-plane from $\mathbb{R}^3$. Then the projection of $S^2 - \{\infty\}$ onto the $xy$-plane is the $x$-axis, and $B_1$ projects to the upper half plane. Similarly, $B_2$ projects to the lower half plane. The resulting diagram of $K$ is called a plat on $2n$-strings, denoted by $p_{2n}(w)$, if it is the union of a $2n$-braid $w$ and $2n$ unlinked and unknotted arcs which connect pairs of consecutive strings of the braid at the top and at the bottom endpoints and $S^2$ meets the top of the $2n$-braid. The bridge (plat) number $b(K)$ of $K$ is the smallest possible number $n$ such that there exists a plat presentation of $K$ on $2n$ strings. We remark that the braid group $B_{2n}$ is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}$ which are twistings of two
adjacent strings. For example, \( w = \sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_1^2\sigma_4^{-2}\sigma_2^{-1} \) is the word for the 6 braid in the dotted rectangle of the first diagram of Figure 1.

![Figure 1](image1.png)

Then we say that a plat presentation is *standard* if the \( 2n \)-braid \( w \) of \( p_{2n}(w) \) involves only \( \sigma_2, \sigma_3, \ldots, \sigma_{2n-1} \).

Let \( q_{2n}(w) = p_{2n}(w) \cap B_2 \) be the *plat presentation* for the rational \( n \)-tangles \( K \cap B_2 \). [See [6].]

We say that \( \underline{q_{2n}(w)} = p_{2n}(w) \) is the *plat closure* of \( q_{2n}(w) \).

The tangle diagrams with the circles in Figure 3 give the diagrams of trivial rational 2,3-tangles as in [1], [4], [7], [10]. The right side of the diagrams show the trivial rational 2,3-tangles in \( B_2 \).

We note that \( q_{2n}(w) \) is alternating if and only if \( \underline{q_{2n}(w)} \) is alternating.

A tangle \( T \) is *reduced* alternating if \( T \) is alternating and \( T \) does not have a self-crossing which can be removed by a Type I Reidemeister move. (See [1].) We say that a knot \( K \) is in *n-bridge position* if the projection of \( K \) onto the \( xy \)-plane has a plat presentation \( p_{2n}(w) \).

![Figure 2](image2.png)

Let \( \Lambda = \mathbb{Z}[a, a^{-1}] \) and \( L \) be a link.

We recall that the Kauffman bracket \( < L > \in \Lambda \) of a link \( L \) is obtained from the following three axioms (See [1].) The symbol \( < > \) indicates that the changes are made to the diagram locally, while the rest of the diagram is fixed.

The Kauffman polynomial \( X_L(a) \in \Lambda \) is defined by
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(i) \( \langle \bigcirc \rangle = 1 \)

(ii) \( \langle \bigtriangleup \bigtriangleup \rangle = a \langle 1 \bigtriangleup \rangle + a^{-1} \langle \bigtriangleup \bigtriangleup \rangle \)

(iii) \( \langle \bigtriangleup \bigtriangleup \bigtriangleup \rangle = k \langle \bigtriangleup \rangle \), where \( k = -a^2 - a^{-2} \)

\[ X_L(a) = (-a^{-3})^{w(L)} \langle L \rangle, \]

where the writhe \( w(L) \in \mathbb{Z} \) is obtained by assigning an orientation to \( L \), and taking a sum over all crossings of \( L \) of their indices \( e \), which are given by the following rule

\[ e (\bigtriangleup \bigtriangleup \bigtriangleup) = 1, \quad e (\bigtriangleup \bigtriangleup) = -1. \]

In section 2, we introduce a theorem that explains how to calculate the Kauffman brackets for 4-plat presentations of knots.

In section 3, we show the main theorem that gives us a formula to calculate the Kauffman brackets for 6-plat presentations of knots.

In section 4, we generalize the formulas given in sections 3 to the Kauffman brackets of \( 2n \)-plat presentations of knots.

Then, we give a method to calculate the writhes of of \( n \)-bridge presentations in section 5.

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2. THE KAUFFMAN BRACKETS OF THE 4-PLAT PRESENTATIONS OF KNOTS

For given 2-tangles \( T \) and \( U \), we denote by \( T + U \) the tangle sum of \( T, U \) and by \( T \ast U \) the “vertical sum” of \( T, U \) as in Figure 4.

\[ \begin{array}{ccc}
0 & \infty & 1 \\
T & U & T \ast U \\
T \ast U \\
T + U
\end{array} \]

Figure 3. the tangles 0, \( \infty \), 1 and the tangle combinations \( T + U, T \ast U \).

Goldman and Kauffman [6] define the bracket polynomial of the rational 2-tangle diagram \( T \) as \( \langle T \rangle = f^T(a) \langle 0 \rangle + g^T(a) \langle \infty \rangle \), where the coefficients \( f^T(a) \) and \( g^T(a) \) are Laurent polynomials that are obtained by starting with \( T \) and using the three axioms repeatedly until only the two trivial tangles \( T_0 \) and \( T_\infty \) in the expression given for \( T \) are left. Then we define the bracket vector of \( T \) to be the ordered pair \([f^T(a), g^T(a)]^t\), and denote it by \( \text{br}(T) \).

For example, \( \text{br}(1) = [a^{-1}, a]^t \), where 1 is the rational 2-tangle with only one positive crossing.

Eliahou-Kauffman-Thistlethwaite [5] established the following.
Proposition 2.1. For given 2-tangles $T$ and $U$, and $k = -a^2 - a^{-2}$,

$$
br(T + U) = \begin{bmatrix} f^U(a) & 0 \\ g^U(a) & f^U(a) + kg^U(a) \end{bmatrix} \text{br}(T) \text{ and, } \text{br}(T \ast U) = \begin{bmatrix} kf^U(a) + g^U(a) & f^U(a) \\ 0 & g^U(a) \end{bmatrix} \text{br}(T).
$$

So, if $U = 1$ in Proposition 2.1 then we have the following equalities.

$$\text{br}(T + 1) = M_+ \cdot \text{br}(T), \text{ and, } \text{br}(T * 1) = M_+ \cdot \text{br}(T),$$

where

$$M_+ = \begin{bmatrix} a^{-1} & 0 \\ a & -a^3 \end{bmatrix} \text{ and } M_* = \begin{bmatrix} -a^{-3} & a^{-1} \\ 0 & a \end{bmatrix}.$$

Two rational 2-tangles, $T, T'$, in $B^3$ are isotopic, denoted by $T \sim T'$, if there is an orientation-preserving self-homeomorphism $h : (B^3, T) \rightarrow (B^3, T')$ that is the identity map on the boundary.

We say that $T^{hflip}$ is the horizontal flip of the 2-tangle $T$ if $T^{hflip}$ is obtained from $T$ by a $180^\circ$-rotation around a horizontal axis on the plane of $T$, and $T^{vflip}$ is the vertical flip of the tangle $T$ if $T^{vflip}$ is obtained from $T$ by a $180^\circ$-rotation around a vertical axis, see Figure 5 for illustrations. Then we have the following lemma by Kauffman.

Lemma 2.2. (Flipping Lemma [7]) If $T$ is rational 2-tangle, then $T \sim T^{vflip}$ and $T \sim T^{hflip}$.

We note that any rational 2-tangle $T$ can be obtained from an element $u$ of the braid group $B_3$ as in the first bottom diagram of Figure 5. (Refer to [7].)

For $u = \sigma_1^{c_1} \sigma_2^{c_2} \cdots \sigma_n^{c_n}$, the reverse word of $u$, denoted by $u^r$, is defined by the word $u^r = \sigma_n^{c_n} \cdots \sigma_{k+1}^{c_1}$. Then by Lemma 2.2, we see how to get a word $u^r$ for a 4-plat presentation of a rational 2-tangle $T$ as in the bottom diagrams of Figure 5.

Suppose that $R$ is a rational 2-tangle. Let $u$ be the word for a standard 4-plat presentation of $R$. Then, by modifying the diagrams of $R + 1$ and $R \ast 1$ as in Figure 6, we see that $u^r = \sigma_3^{-1} w$ and $u^{r'} = \sigma_2 w$, where $u'$ and $u''$ are the words for 4-plat presentations of $R + 1$
and $R \ast 1$ respectively.

\[ \text{Figure 5.} \]

Now, consider the following theorem.

**Theorem 2.3.** [Conway, (See [10])] If $K$ is a 2-bridge knot, then there exists a word $w$ in $\mathbb{B}_4$ so that the plat presentation $p_4(w)$ is reduced alternating and standard and represents a knot isotopic to $K$.

Let $A_2 = M_s$ and $A_3 = M_s^{-1}$.

Then we can derive the following theorem which shows how to calculate the Jones polynomials of 4-plat presentations of knots.

**Theorem 2.4.** Suppose that $q_4(w)$ is a plat presentation of a rational 2-tangle $T$ which is reduced alternating and standard so that $w = \sigma_3^{-\epsilon_1} \sigma_2^{\epsilon_2} \cdots \sigma_2^{-\epsilon_{2n-1}} \sigma_2^{\epsilon_{2n}}$ for some positive integers $\epsilon_i$ ($2 \leq i \leq 2n$) and non-negative integer $\epsilon_1$. Let $A = A_3^{-\epsilon_1} A_2^{\epsilon_2} \cdots A_3^{-\epsilon_{2n-1}} A_2^{\epsilon_{2n}}$. Then,

\[ < T > = f^T(a) < T_0 > + g^T(a) < T_\infty >, \]

where $f^T(a)$ and $g^T(a)$ are given by $br(T) = [f^T(a), g^T(a)]^t = A[0, 1]^t$ and $< K > = f^T(a) - (a^2 + a^{-2})g^T(a)$ for $K$, where $K$ is represented by the plat presentation $q_4(w)$.

Therefore, $X_K = (-a^{-3})^{w(K)}(f^T(a) - (a^2 + a^{-2})g^T(a))$, where $w(K)$ is the writhe of the knot $K$.

**Proof.** Let $l = |\epsilon_1| + |\epsilon_2| + \cdots + |\epsilon_{2n}|$.

We will show this theorem by using induction on $l$.

Suppose that $w = \sigma_2$. Then $T = \infty \ast 1$.

Therefore, by Proposition 2.1., $br(T) = M_s[0, 1]^t = A_2[0, 1]^t$.

Now, we assume that $br(T) = [f^T(a), g^T(a)]^t = A[0, 1]^t$ if $T$ is a reduced alternating standard rational 2-tangle with the plat presentation $q_4(w)$, where $w = \sigma_3^{-\epsilon_1} \sigma_2^{\epsilon_2} \cdots \sigma_2^{-\epsilon_{2n}}$ for some positive integers $\epsilon_i$ ($2 \leq i \leq 2n$) and non-negative integer $\epsilon_1$, and $l = |\epsilon_1| + |\epsilon_2| + \cdots + |\epsilon_{2n}| = k$.

Now, consider a reduced alternating standard rational 2-tangle $T'$ with the plat presentation $q_4(w')$, where $w' = \sigma_3^{-\epsilon'_1} \sigma_2^{\epsilon'_2} \cdots \sigma_2^{\epsilon'_{2m}}$ for some positive integers $\epsilon_i$ ($2 \leq i \leq 2m$) and non-negative integer $\epsilon'_1$, and $l = |\epsilon'_1| + |\epsilon'_2| + \cdots + |\epsilon'_{2m}| = k + 1$. 
If \( \epsilon_i' \geq 1 \) then we set \( w'' = \sigma_3^{-(\epsilon_i'-1)} \sigma_2^i \cdots \sigma_2^{\epsilon_{2m}} \). Then \( w' = \sigma_3^{-1} w'' \).

Let \( T'' \) be the reduced alternating standard rational 2-tangle with the plat presentation \( q_4(w'') \).

Let \( A' = A_3^{-\epsilon_i'} A_2^{\epsilon_2'} \cdots A_2^{\epsilon_{2m}} \) and \( A'' = A_3^{-(\epsilon_i'-1)} A_2^{\epsilon_2'-1} \cdots A_2^{\epsilon_{2m}} \).

Since \( |\epsilon_1' - 1| + |\epsilon_2'| + \cdots + |\epsilon_{2m}'| = k \), we note that \( br(T'') = [f^{T''}(a), g^{T''}(a)]^t = A''[0, 1]^t \) by assumption. We note that \( T' = T'' + 1 \). So, by Proposition 2.1., \( br(T') = M_+ \cdot br(T'') = A_3^{-1}(A''[0, 1]^t) = (A_3^{-1}A'')[0, 1]^t = A'[0, 1]^t \).

If \( \epsilon_1 = 0 \) then we set \( w'' = \sigma_2^{\epsilon_2'-1} \cdots \sigma_2^{\epsilon_{2m}} \). Then \( w' = \sigma_2 w'' \).

Let \( T'' \) be the reduced alternating standard rational 2-tangle with the plat presentation \( q_4(w'') \).
Let \( A' = A_2^{\epsilon_2'} \cdots A_2^{\epsilon_{2m}} \) and \( A'' = A_2^{\epsilon_2'-1} \cdots A_2^{\epsilon_{2m}} \).

Since \( |\epsilon_2' - 1| + \cdots + |\epsilon_{2m}'| = k \), we note that \( br(T'') = [f^{T''}(a), g^{T''}(a)]^t = A''[0, 1]^t \) by assumption.
We note that \( T' = T'' \ast 1 \). So, by Proposition 2.1., \( br(T') = M_+ \cdot br(T'') = A_2(A''[0, 1]^t) = (A_2A'')[0, 1]^t = A'[0, 1]^t \).

Now, assume that \( q_4(w) \) be a 4-plat presentation of a rational 2-tangle \( T \) which is reduced alternating and standard so that \( w = \sigma_3^{-\epsilon_1} \sigma_2^i \cdots \sigma_3^{-\epsilon_{2n-1}} \sigma_2^i \) for some negative integers \( \epsilon_i \) \((2 \leq i \leq 2n)\) and non-positive integer \( \epsilon_1 \).

We can calculate the Kauffman bracket of \( q_4((w^{-1})^r) \) by using the previous theorem.

We note that \( q_4(w) \) is the mirror image of \( q_4((w^{-1})^r) \) which is obtained by interchanging the over and under crossings.

So, we switch \( a \) and \( a^{-1} \) to calculate the Kauffman bracket of the 4-plat presentation \( q_4(w) \) of the rational 2-tangle \( T \).

3. The Kauffman Brackets of the 6-plat Presentations of Knots

Now, suppose that \( K \) is in 3-bridge position. Then we have a plat presentation \( q_6(w) \) for the rational 3-tangle \( K \cap B_2 \). Then, let \( w = \sigma_{k_1}^{\epsilon_1} \sigma_{k_2}^{\epsilon_2} \cdots \sigma_{k_{n-1}}^{\epsilon_{2n-1}} \sigma_{k_{n}}^{\epsilon_{2n}} \) for some non-zero integers \( \epsilon_i \) \((1 \leq i \leq n)\), where \( k_i \in \{1, 2, 3, 4, 5\} \).

H. Cabrera-Ibarra [4] defined the bracket polynomial of the rational 3-tangle \( T \) as \( < T > = f_1^T(a) < 0_1 \rangle + f_2^T(a) < 0_2 \rangle + f_3^T(a) < 0_3 \rangle + f_4^T(a) < 0_4 \rangle + f_5^T < 0_5 \rangle \), where \( f_i^T(a) \) are polynomials in \( a \) and \( a^{-1} \) that are obtained by starting with \( T \) and using the three axioms repeatedly until only the five trivial tangles \( < 0_j \rangle \) in the expression given for \( T \) are left. (See Figure 6 and 7.)
Let $\mathcal{A} = < 0_1 >, \mathcal{B} = < 0_2 >, \mathcal{C} = < 0_3 >, \mathcal{D} = < 0_4 >$ and $\mathcal{E} = < 0_5 >$.

Let $B_1^{\pm 1} = \begin{bmatrix} a^{\pm 1} & 0 & 0 & 0 & 0 \\ 0 & a^{\pm 1} & 0 & 0 & 0 \\ 0 & a^{\pm 1} & -a^{\mp 3} & a^{\mp 1} & 0 \\ 0 & 0 & 0 & a^{\pm 1} & 0 \\ a^{\mp 1} & 0 & 0 & 0 & -a^{\mp 3} \end{bmatrix}$, $B_2^{\pm 1} = \begin{bmatrix} -a^{\mp 3} & 0 & 0 & a^{\mp 1} & a^{\mp 1} \\ 0 & -a^{\mp 3} & a^{\mp 1} & 0 & 0 \\ 0 & 0 & a^{\pm 1} & 0 & 0 \\ 0 & 0 & 0 & a^{\pm 1} & 0 \\ 0 & 0 & 0 & 0 & a^{\pm 1} \end{bmatrix}$,

$B_3^{\pm 1} = \begin{bmatrix} a^{\pm 1} & 0 & 0 & 0 & 0 \\ 0 & a^{\pm 1} & 0 & 0 & 0 \\ 0 & a^{\pm 1} & -a^{\mp 3} & 0 & a^{\mp 1} \\ a^{\mp 1} & 0 & 0 & -a^{\mp 3} & 0 \\ 0 & 0 & 0 & 0 & a^{\pm 1} \end{bmatrix}$, $B_4^{\pm 1} = \begin{bmatrix} -a^{\mp 3} & a^{\mp 1} & 0 & a^{\mp 1} & 0 \\ 0 & a^{\pm 1} & 0 & 0 & 0 \\ 0 & 0 & a^{\pm 1} & 0 & 0 \\ 0 & 0 & 0 & a^{\pm 1} & 0 \\ 0 & 0 & a^{\pm 1} & 0 & -a^{\mp 3} \end{bmatrix}$,

$B_5^{\pm 1} = \begin{bmatrix} a^{\pm 1} & 0 & 0 & 0 & 0 \\ a^{\mp 1} & -a^{\mp 3} & 0 & 0 & 0 \\ 0 & 0 & -a^{\mp 3} & a^{\pm 1} & a^{\mp 1} \\ 0 & 0 & a^{\pm 1} & 0 & 0 \\ 0 & 0 & 0 & 0 & a^{\pm 1} \end{bmatrix}$.

Let $B = B_{k_1}^{e_1}B_{k_2}^{e_2}\cdots B_{k_{n-1}}^{e_{n-1}}B_{k_n}^{e_n}$.

Then we have the following theorem to calculate the Kauffman polynomial of $K$.

**Theorem 3.1.** Suppose that $q_6(w)$ is a plat presentation for a rational 3-tangle $T$ and $w = \sigma_{k_1}^{e_1}\sigma_{k_2}^{e_2}\cdots \sigma_{k_{n-1}}^{e_{n-1}}\sigma_{k_n}^{e_n}$ for some non-zero integers $e_i$ ($1 \leq i \leq n$), where $k_i \in \{1, 2, 3, 4, 5\}$.

Then $< T > = f_1^T(a)\mathcal{A} + f_2^T(a)\mathcal{B} + f_3^T(a)\mathcal{C} + f_4^T(a)\mathcal{D} + f_5^T(a)\mathcal{E}$, where $f_i^T(a)$ are given by

$[f_1^T(a) f_2^T(a) f_3^T(a) f_4^T(a) f_5^T(a)]^r = B[0 \ 0 \ 1 \ 0 \ 0]^r$, and $B = B_{k_1}^{e_1}B_{k_2}^{e_2}\cdots B_{k_{n-1}}^{e_{n-1}}B_{k_n}^{e_n}$. (i.e., the third column of $B$)
Also, \( < K >= f^T_1(a) + k(f^T_2(a) + f^T_4(a) + f^T_5(a)) + k^2 f^T_3(a) \), where \( k = -a^2 - a^{-2} \) and \( K \) is the knot which is represented by the plat presentation \( q_0(w) \).

Therefore, \( X_K = (-a^{-3})^{w(\hat{K})}(f^T_1(a) + k(f^T_2(a) + f^T_4(a) + f^T_5(a)) + k^2 f^T_3(a)) \).

\textbf{Proof.} Suppose that \( K \) is a 3-bridge link. Then, we have a link \( K' \) which is isotopic to \( K \) and the projection onto the \( xy \)-plane has a plat presentation \( p_0(w) \). Then we define \( q_0(w) \) that is the plat presentation of the tangle \( T = K' \cap B_2 \) as in the first bottom diagram of Figure 6.

Suppose that \( < T >= f^T_1(a)A + f^T_2(a)B + f^T_3(a)C + f^T_4(a)D + f^T_5(a)E \) for some polynomials \( f^T_1(a) \).

Let \( T_{\sigma_j^\pm 1} \) be the new rational 3-tangle in \( B_2 \) which is obtained from \( T \) by adding \( \sigma_j^\pm 1 \) for \( 1 \leq j \leq 5 \) as in the bottom diagrams of Figure 6.

Suppose that \( < T_{\sigma_j^\pm 1} >= f^T_{1\sigma_j^\pm 1}(a)A + f^T_{2\sigma_j^\pm 1}(a)B + f^T_{3\sigma_j^\pm 1}(a)C + f^T_{4\sigma_j^\pm 1}(a)D + f^T_{5\sigma_j^\pm 1}(a)E \) for some polynomials \( f^T_{1\sigma_j^\pm 1}(a) \).

For convenience, let \( f_i(a) = f^T_i(a) \) and \( f'_i(a) = f^T_{i\sigma_j^\pm 1}(a) \).

Then, we get \( < T_{\sigma_j} >= a < T > + a^{-1} < T' > \). We note that \( < T'> >= f_1(a)E + f_2(a)C + kf_3(a)C + f_4(a)C + kf_5(a)E \) as in Figure 7, where \( k = -(a^2 + a^{-2}) \).

![Figure 7](image-url)

Therefore, \( < T_{\sigma_j} >= a < T > + a^{-1} < T' > = a(f_1(a)A + f_2(a)B + f_3(a)C + f_4(a)D + f_5(a)E) + a^{-1}(f_1(a)E + f_2(a)C + kf_3(a)C + f_4(a)C + kf_5(a)E) = af_1(a)A + af_2(a)B + (af_3(a) + \ldots) \).
\[ a^{-1}(f_2(a) + kf_3(a) + f_4(a))C + af_4(a)D + (af_5(a) + a^{-1}(f_1(a) + kf_5(a)))E. \]

So, we have \( f'_1(a) = af_1(a), f'_2(a) = af_2(a), f'_3(a) = af_3(a) + a^{-1}(f_2(a) + kf_3(a)), \)
\( f'_4(a) = af_4(a) \) and \( f'_5(a) = af_5(a) + a^{-1}(f_1(a) + kf_5(a)). \)

Similarly, By Figure 8, we have \( \langle T_{\sigma^{-1}} \rangle = a < T' > + a^{-1} < T >. \)

Therefore, \( \langle T_{\sigma^{-1}} \rangle = a < T' > + a^{-1} < T > = a(f_1(a)E + f_2(a)C + kf_3(a)C + f_4(a)) + kC + f_5(a)E) + a^{-1}(f_1(a)A + f_2(a)B + f_3(a)C + f_4(a)D + f_5(a)E) = a^{-1}f_1(a)A + a^{-1}f_2(a)B + a^{-1}f_3(a)C + (a^{-1}f_4(a)D + (a^{-1}f_5(a) + a(f_5(a) + kC + f_5(a))))E. \)

So, we have \( f'_1(a) = a^{-1}f_1(a), f'_2(a) = a^{-1}f_2(a), f'_3(a) = a^{-1}f_3(a) + a(f_2(a) + kC + f_3(a)) \),
\( f'_4(a) = a^{-1}f_4(a) \) and \( f'_5(a) = a^{-1}f_5(a) + a(f_1(a) + kC + f_5(a)) \).

This operations can be expressed by the following.

\[
\begin{bmatrix}
  a^\pm 1 & 0 & 0 & 0 & 0 \\
  0 & a^\pm 1 & 0 & 0 & 0 \\
  0 & a^\mp 1 + a^\mp 1 k & a^\mp 1 & 0 \\
  0 & 0 & a^\mp 1 & 0 \\
  a^\mp 1 & 0 & 0 & a^\pm 1 + a^\mp 1 k \\
\end{bmatrix}
\begin{bmatrix}
  f_1(a) \\
  f_2(a) \\
  f_3(a) \\
  f_4(a) \\
  f_5(a) \\
\end{bmatrix}
= 
\begin{bmatrix}
  f'_1(a) \\
  f'_2(a) \\
  f'_3(a) \\
  f'_4(a) \\
  f'_5(a) \\
\end{bmatrix}
\]

Similarly, we have four more operations from \( \sigma_2^{\pm 1}, \sigma_3^{\pm 1}, \sigma_4^{\pm 1} \) and \( \sigma_5^{\pm 1} \) as follows.

\[
\begin{bmatrix}
  a^\pm 1 + a^\mp 1 k & 0 & 0 & a^\mp 1 & a^\mp 1 \\
  0 & a^\pm 1 + a^\mp 1 k & a^\mp 1 & 0 & 0 \\
  0 & 0 & a^\pm 1 & 0 & 0 \\
  0 & 0 & 0 & a^\pm 1 & 0 \\
  0 & 0 & 0 & 0 & a^\pm 1 \\
\end{bmatrix}
\begin{bmatrix}
  f_1(a) \\
  f_2(a) \\
  f_3(a) \\
  f_4(a) \\
  f_5(a) \\
\end{bmatrix}
= 
\begin{bmatrix}
  f'_1(a) \\
  f'_2(a) \\
  f'_3(a) \\
  f'_4(a) \\
  f'_5(a) \\
\end{bmatrix}
\]
Recall that $T$ is expressed by $\sigma_{k_1}^{\epsilon_1} \sigma_{k_2}^{\epsilon_2} \cdots \sigma_{k_{n-1}}^{\epsilon_{n-1}} \sigma_{k_n}^{\epsilon_n}$ for some non-zero integers $\epsilon_i$ ($1 \leq i \leq n$), where $k_i \in \{1, 2, 3, 4, 5\}$.

Then, we know that the generators $\sigma_i$ correspond to $B_i$.

So, given $B = B_{k_1}^{\epsilon_1} B_{k_2}^{\epsilon_2} \cdots B_{k_{n-1}}^{\epsilon_{n-1}} B_{k_n}^{\epsilon_n}$, we have

$$< T > = f_1(a) < T_1 > A + f_2(a)B + f_3(a)C + f_4(a)D + f_5(a)E$$

for $[f_1(a) f_2(a) f_3(a) f_4(a) f_5(a)] = B[0 0 1 0 0]^t$ since $< T_{0_3} > = 0.A+0.B+1.C+0.D+0.E$.

From $< T >$, we have $< K >= f_1(a) + k f_2(a) + f_4(a) + f_5(a)) + k^2 f_3(a)$ since $\overline{T_{0_3}}$ is the unknot, $\overline{T_{0_i}}$ for $i \in \{2, 4, 5\}$ are disjoint unions of two unknots and $\overline{T_{0_3}}$ is disjoint union of three unknots as in Figure 8.

Therefore, $X_K = (-a^{-3})^{w(R)}(f_1(a) + k(f_2(a) + f_4(a) + f_5(a)) + k^2 f_3(a))$.

We remark that the matrices $B_1^{\pm 1}, B_2^{\pm 1}, B_3^{\pm 1}, B_4^{\pm 1}$ and $B_5^{\pm 1}$ satisfy the braid group relations.

4. THE KAUFFMAN BRACKETS OF 2n-PLAT PRESENTATION KNOTS

We define the bracket polynomial of the rational $n$-tangle $T$ as $< T > = f_1^T(a) < 0_1 > + f_2^T(a) < 0_2 > + \cdots + f_m^T(a) < 0_m >$, where $f_i^T(a)$ are Laurent polynomials that are obtained by starting with $T$ and using the three axioms repeatedly until only the $m$ trivial tangles $< 0_j >$ in the expression given for $T$ are left.
So, we remark that the number $m$ of trivial rational $n$-tangles $0_i$ needs to be calculated.

To do this, let $\psi(0) = 1$. Then we define the map $\psi : 2\mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ so that $\psi(2k) = \sum_{i=1}^{k} \psi(2i - 2) \cdot \psi(2k - 2i)$.

**Lemma 4.1.** The number of trivial rational $n$-tangles is $\psi(2n)$.

**Proof.** Let $0_i$ be a trivial rational $n$-tangle. Then we note that the string with the endpoint 1 has the other endpoint at $2k + 1$ for some positive integer $k$.

If $k = 1$ then we calculate the number of trivial $n$-tangles by considering $2n - 2$ endpoints and it is $\phi(0) \cdot \psi(2n - 2)$.

If $k = 2$ then we have a nested string inside of the string with the endpoint 1 and we need to calculate the possible case for the rest of strings. Then it is $\psi(2) \cdot \psi(2n - 4)$.

By considering the all subcases with respect to $k$, we calculate the number of trivial rational $n$-tangle which is $\sum_{i=1}^{n} \psi(2i - 2) \cdot \psi(2n - 2i)$.

Therefore, the number of trivial rational $n$-tangles is $\psi(2n)$. \hfill \Box

Recall $\mathcal{T}$ that is the tangle closure of the tangle $T$ to have the knot with the $2n$-plat presentation.

Now, we have a corollary to calculate the Kauffman polynomial of $n$-plat presentation as follows.

**Corollary 4.2.** There exist $(4n - 2) \psi(2n) \times \psi(2n)$ matrices to calculate the coefficients $f_{1}^T(a), \ldots, f_{\psi(2n)}^T(a)$ of the Kauffman bracket for a rational $n$-tangle $T$ with a $2n$-plat presentation $q_{2n}(w)$. Moreover, we calculate the Kauffman bracket of $\overline{q_{2n}(w)}$ and the Kauffman polynomial of $\overline{q_{2n}(w)}$ from this.

**Proof.** This is the generalization of Theorem 2.1. \hfill \Box

5. A WAY TO CALCULATE THE WRITHE OF A $n$-BRIDGE KNOT ($n \geq 2$)

First, assume that the projection onto the $xy$-plane of a $n$-bridge knot $K$ has a plat presentation $p_{2n}(w)$ with $w = \sigma_{k_1}^{\epsilon_1} \sigma_{k_2}^{\epsilon_2} \cdots \sigma_{k_m}^{\epsilon_{m-1}} \sigma_{k_m}^{\epsilon_m}$ for some non-zero integers $\epsilon_i$ (1 \leq i \leq m), where $k_i \in \{1, 2, \ldots, 2n - 1\}$ and $k_j \neq k_{j+1}$ for 1 \leq j \leq m - 1.

Then we have the plat presentation $q_{2n}(w)$ of the tangle $T = K \cap B_2$ so that $\overline{q_{2n}(w)} = p_{2n}(w)$.

Let $\mathcal{P}(\sigma_{i}^{\pm 1})$ be the $2n \times 2n$ matrix which is obtained by interchanging the $i$th and $i + 1$st rows of $I$. Then $\mathcal{P}$ extends to a homomorphism from $\mathbb{B}_{2n}$ to $GL_n(\mathbb{Z})$. 
Lemma 5.1. $w(u) = [1, 2, \cdots, 2n]P(w)$.

Proof. This is proven by induction on $l = |\epsilon_1| + |\epsilon_2| + \cdots + |\epsilon_m|$.

Let $R$ be the $2n \times 2n$ matrix which is obtained by interchanging the $2i + 1$st and $2i + 2$nd rows of $I$ for all $i$ such that $0 \leq i \leq n - 1$.

Recall that $w^r$ is the reverse word of $w$.

Let $[o_j(1), o_j(2), \cdots, o_j(2n)] = [1, 2, \cdots, 2n](P(w)RP(w^r)R)^{j-1}P(w)RP(w^r)$ for $1 \leq j \leq n$.

Also, let $[o_j'(1), o_j'(2), \cdots, o_j'(2n)] = [1, 2, \cdots, 2n](P(w)RP(w^r)R)^{j}$ for $0 \leq j \leq n$.

We note that $[o_0'(1), o_0'(2), \cdots, o_0'(2n)] = u$.

Consider the case that $j = 1$. In order to get $[1, 2, \cdots, 2n]P(w)$, we follow the strings of the braid down while preserving the numbers of the strings. Then $[1, 2, \cdots, 2n]P(w)R$ is obtained by following along the trivial arcs $\gamma_i$ while preserving the numbers of the strings. Then we follow the strings of the braid up while preserving the numbers of the strings to get $[1, 2, \cdots, 2n]P(w)RP(w^r)$ which is $[o_1(1), o_1(2), \cdots, o_1(2n)]$. After this, we follow along the trivial arcs $\delta_i$ while preserving the numbers of the strings to get $[1, 2, \cdots, 2n]P(w)RP(w^r)R$ which is $[o_1'(1), o_1'(2), \cdots, o_1'(2n)]$.

Generally speaking, from the $[o_j'(1), o_j'(2), \cdots, o_j'(2n)]$ we follow the strings of the braid down and follow along the trivial arcs $\gamma_i$ and follow the strings of the braid up to get $[o_j+1(1), o_j+1(2), \cdots, o_j+1(2n)]$ while preserving the numbers of the strings. Then, we get $[o_j+1(1), o_j+1(2), \cdots, o_j+1(2n)]$ by following along the trivial arcs $\delta_i$ while preserving the numbers of the strings.

We note that $o_0'(i) = i$, $o_1'(i), \cdots o_{n-1}'(i)$ are distinct points. Otherwise, $K$ is a link, not a knot. Also, we know that $o_j'(i) = o_{n+j}'(i)$.

Similarly, $o_1(i), o_2(i), \cdots o_n(i)$ are distinct points and $o_j(i) = o_{n+j}(i)$.

Also, we note that for a trivial arc $\delta_k$ there exists a unique $i$ ($1 \leq i \leq n$) so that either $o_i(1) = 2k - 1$ or $o_i(1) = 2k$.

Without loss of generality, give the clockwise orientation to the trivial arc $\delta_1$ in $B_1$ with $\partial \delta_1 = \{1, 2\}$ from 1 to 2 along $\delta_1$. So, the initial point of $\delta_1$ is 1 and the terminal point of $\delta_1$ is 2 for the given orientation. Then, the orientations of the other trivial arcs $\delta_2, \cdots, \delta_n$ in
Lemma 5.2. The trivial arc $\delta_k$ has the same clockwise orientation as $\delta_1$ if $k = o_i(1)/2$ for some $i \ (1 \leq i \leq n)$. The trivial arc $\delta_k$ has the opposite orientation (counterclockwise) as $\delta_1$ if $k = (o_i(1) + 1)/2$ for some $i \ (1 \leq i \leq n)$.

Proof. If $k = o_i(1)/2$ for some $i$ then the endpoints of $\delta_k$ are $o_i(1) - 1$ and $o_i(1)$. Also, the orientation of $\delta_k$ is from $o_i(1) - 1$ to $o_i(1)$. Therefore, the $\delta_k$ has the same orientation as $\delta_1$.

If $k = (o_i(1) + 1)/2$ for some $i$ then the endpoints of $\delta_k$ are $o_i(1)$ and $o_i(1) + 1$. Also, the orientation of $\delta_k$ is from $o_i(1) + 1$ to $o_i(1)$. Therefore, the $\delta_k$ has the opposite orientation as $\delta_1$. \qed

Recall the ordered sequence of numbers $u = [1, 2, \ldots, 2n]$. Now, we define a new sequence of numbers $r = [r(1), r(2), \ldots, r(2n)]$ as follows. For the orientation given above, we replace the original numbers of $u = [1, 2, \ldots, 2n]$ for the initial points of $\delta_i$ by 1 and we replace the original numbers for the terminal points of $\delta_i$ by 2.

Now, let $r_0 = r$.

Let $r_i = [r_i(1), r_i(2), r_i(3), \ldots, r_i(2n)] = rP(\sigma_{k_1}^{e_1} \sigma_{k_2}^{e_2} \cdots \sigma_{k_i}^{e_i})$ for $1 \leq i \leq m$.

Let $v(i) = \begin{cases} 0 & \text{if } r_{i-1}(k_i) = r_{i-1}(k_i + 1) \\ 1 & \text{if } r_{i-1}(k_i) \neq r_{i-1}(k_i + 1) \end{cases}$

Then we calculate the writhe of $K$ as follows.

Theorem 5.3. $w(K) = \sum_{i=1}^{m} (-1)^{v(i)} e_i$.

Proof. For the $2n$ strings of the braid $w$, we assign the number $r(k)$ to each string with the upper endpoint $k$ for $1 \leq k \leq 2n$.

Without loss of generality, we give the orientation (clockwise) to $\delta_1$ from 1 to 2 along $\delta_1$. Then the orientation at 1 is up and the orientation at 2 is down. Then we know that the orientation at $j$ is up if $r(j) = 1$ and it is down if $r(j) = 2$.

Fix a value $i \ (1 \leq i \leq m)$.

Case 1: Suppose that $r_{i-1}(k_i) = r_{i-1}(k_i + 1)$.

Then the two strings for the $(\sum_{j=1}^{i-1} |e_j| + 1)$-th crossing have the same orientation since $r_{i-1}(k_i) = r_{i-1}(k_i + 1)$, i.e., the numbers of the two strings for $(\sum_{j=1}^{i-1} |e_j| + 1)$-th crossing are the same. If $r_{i-1}(k_i) = r_{i-1}(k_i + 1) = 1$ then the orientations are up and if $r_{i-1}(k_i) = r_{i-1}(k_i + 1) = 2$ then the orientations are down.

Then we note that the index $e$ of $(\sum_{j=1}^{i-1} |e_j| + 1)$-th crossing is +1 if the crossing is positive and the index $e$ of $(\sum_{j=1}^{i-1} |e_j| + 1)$-th crossing is −1 if the crossing is negative.
We note that all the crossings in $\sigma_{k_i}^t$ have the same index.

Therefore, the contribution of $\sigma_{k_i}^t$ to the writhe is $\epsilon_i$

Since $v(i) = 0$, we check that $(-1)^{v(i)}\epsilon_i = (-1)^0\epsilon_i = \epsilon_i$ is the contribution of $\sigma_{k_i}^t$ to the writhe.

Case 2: Suppose that $r_{i-1}(k_i) \neq r_{i-1}(k_i + 1)$.

Then we know that the orientations of the two strings for $(\sum_{j=1}^{i-1} |\epsilon_j| + 1)$-th crossing are either up and down or down and up since $r_{i-1}(k_i) \neq r_{i-1}(k_i + 1)$.

So, we check that the index $e$ of $(\sum_{j=1}^{i-1} |\epsilon_j| + 1)$-th crossing is $-1$ if the crossing is positive and the index $e$ of $(\sum_{j=1}^{i-1} |\epsilon_j| + 1)$-th crossing is $+1$ if the crossing is negative.

Therefore, the contribution of $\sigma_{k_i}^t$ to the writhe is $-\epsilon_i$

Since $v(i) = 1$, we check that $(-1)^{v(i)}\epsilon_i = (-1)\epsilon_i = -\epsilon_i$ is the contribution of $\sigma_{k_i}^t$ to the writhe.

By adding all the indices of $\sigma_{k_i}^t$, we have the given formula for the writhe. 

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