New examples of marginally trapped surfaces
and tubes in warped spacetimes

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Abstract. By using the classical Hopf map, we construct another fibration that allows us to
obtain examples of marginally trapped tori and marginally outer trapped tubes (MOTT) which
are foliated by tori, all of them embedded in a closed Friedman-Lemaître-Robertson-Walker
4-spacetime. In addition, we show examples of MOTTs with any causal character.

1. Preliminaries
We consider a 4-dimensional spacetime $(M^4, g)$, a compact, without boundary, embedded
spacelike surface $S$ in $(M^4, g)$. We choose two normal, future-pointing lightlike vector fields
$\overrightarrow{k}$, $\overrightarrow{l}$ such that $g(\overrightarrow{l}, \overrightarrow{k}) = -1$. We call $A_{\overrightarrow{k}}$ and $A_{\overrightarrow{l}}$ the associated shape operators with $\overrightarrow{k}$
and $\overrightarrow{l}$ respectively. We recall that $S$ is called Marginally Outer Trapped Surface (MOTS for short) when one of the expansions vanish, i.e., $\text{trace}(A_{\overrightarrow{l}}) = 0$ and $\text{trace}(A_{\overrightarrow{k}}) \neq 0$ everywhere,
or viceversa. We should point out that the mean curvature vector $\vec{H}$ of a MOTS always satisfies $\|\vec{H}\| = 0$, but the converse does not hold. Actually, we are going to solve this equation in
general, and then we will add topological hypothesis to obtain MOTS.

Next, we recall that a Marginally Outer Trapped Tube (MOTT for short) is a 3-dimensional
smooth manifold $G$ which admits a foliation by surfaces $\{S_\lambda : \lambda \in \Lambda\}$ such that there is a smooth
immersion $\Phi : G \to M^4$ satisfying:

(i) each $\Phi(S_\lambda)$ ($\lambda \in \Lambda$) is a MOTS in $M^4$,
(ii) $\Phi(S_\lambda) \cap \Phi(S_\mu) = \emptyset$ for any $\lambda \neq \mu$.

Also, it is worth pointing out that the causal character of the MOTT may vary from point to
point.

Our target is to obtain new examples MOTS and MOTT in a closed Friedman-Lemaître-
Robertson-Walker 4-spacetime. We will obtain some partial results firstly, by using CMC
surfaces in $S^3$, and secondly, by using the classical Hopf map.

2. Using CMC surfaces in $S^3$
Let us consider a smooth function $f : I \subset \mathbb{R} \to (0, \infty)$, $t \in I$, and a 3-dim. Riemannian manifold
$(M^3, g_3)$. With them, we construct the Generalized-Robertson-Walker 4-spacetime $\overline{M}_4^1 = I \times M^3$
with line element $\overline{g}_4 = -dt^2 + f^2 g_3$, where $f$ is playing the rôle of the scale factor. Next, we
consider a surface $S$ and an immersion $\varphi : S \to M^3$. For a fixed $t_o \in I$, we define

$$S \xrightarrow[\varphi]{} M^3 \xrightarrow[\psi]{} M^1 \quad \implies \quad \phi := \psi \circ \varphi$$

being $\phi$ an immersion of $S$ in $M^1$ in the $t = t_o$ slice.

From the composition $\phi = \psi \circ \varphi$ it is clear that the mean curvature vector associated with $\phi$ is a linear combination of the mean curvature vectors associated with $\psi$ and $\varphi$. Thus, if $\vec{H}_\phi$ and $\vec{H}_\varphi$ stand for the mean curvature vectors associated with $\phi$ and $\varphi$, respectively, one obtains

$$\vec{H}_\phi(p) = \frac{\vec{H}_\varphi(p)}{f^2(t_o)} + \frac{f'(t_o)}{f(t_o)} \partial t\big|_{(t_o,p)}, \quad p \in S.$$  \hspace{1cm} (1)

**Theorem 1** A surface $\phi : S \to M^4$ contained in a $t_0$-slice of $(\overline{M}^4, -dt^2 + f^2g_3)$ satisfies $\|\vec{H}_\phi\| = 0$ if, and only if, $\varphi : S \to M^3$ has constant mean curvature with $\|\vec{H}_\varphi\| = |f'(t_0)|$.

**Corollary 1** There exist MOTS with arbitrary genus in closed $(M^3 = S^3)$ FLRW spacetimes.

**Proof:** In [1] and [3], the authors show the existence of compact, with arbitrary genus, embedded surfaces in the (standard) round 3-sphere with (small) constant mean curvature. Then, by considering one of such surfaces, we construct one of our embeddings in a closed FLRW 4-spacetime at a suitable time $t_0$, and we only have to resort to Theorem 1.

3. Examples of MOTT in closed FLRW foliated by tori with different causality

We are interested in constructing some MOTT in closed FLRW spacetimes foliated by tori with different causality. Let $\mathbb{C}$ be the complex numbers, with $i = \sqrt{-1}$, $|z|$ the modulus of $z \in \mathbb{C}$, $\overline{z}$ its complex conjugate. We consider the round 3-sphere $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, with standard metric $g_3$. Now, we recall the embedded torus with constant mean curvature $C_u$ in $S^3$ given by

$$C_u := \{(z_1, z_2) \in S^3 \subset \mathbb{C}^2 : |z_1| = \cos(u), \ |z_2| = \sin(u)\},$$

$u \in (0, \pi/2)$, with mean curvature $\|\vec{H}_u\| := |2 \cot(2u)|$. We define an auxiliary function

$$h : I \to (0, \pi/2), \quad h(t) = \frac{1}{2} \arccot\left(\frac{f'(t)}{2}\right) = \frac{\pi}{4} - \frac{1}{2} \arctan\left(\frac{f'(t)}{2}\right),$$

and now, the embedding

$$\phi : I \times S^1 \times S^1 \to I \times S^3,$$

$$\phi(t, e^{i\theta}, e^{i\nu}) = \left(\frac{f(t)}{2} \cos(h(t)), \ \frac{f(t)}{2} \sin(h(t))\right).$$

For each $t \in I$, the map $\phi(t, -, -) : S^1 \times S^1 \to \overline{M}^4$ is an embedding of a torus in the $t$-slice, with constant mean curvature $\|\vec{H}_u\| = |2 \cot(2u)\big|_{u=h(t)} = |f'(t)|$. By Theorem 1, each torus is a MOTS, and therefore, $\phi$ is a MOTT. The induced line element on $I \times S^1 \times S^1$ is

$$\phi^*g_4 = \frac{g_4(\phi_t, \phi_t)}{dt^2 + (f(t) \cos(h(t)))^2 d\theta^2 + (f(t) \sin(h(t)))^2 d\nu^2}.$$ 

Clearly, the causal character depends only on $\phi_t$, so we are going to study it by defining the function

$$z(t) := \frac{g_4(\phi_t, \phi_t)}{f(t) f''(t)}.$$  \hspace{1cm} (2)

Next, we exhibit four different examples, showing that any causal character can actually happen.
(i) Given $a, b > 0$ such that $a^2 = 4 + b^2$, define the function $f : I = \mathbb{R} \to (0, \infty)$, $f(t) = a \cosh(t) + b \sinh(t)$. Then, $z(t) \equiv 0$. Therefore, $\phi_t$ is everywhere lightlike.

(ii) Define the function $f : (-1, 1) \to (0, \infty)$, $f(t) = \frac{2}{1 - t^2}$. By simple computations, we obtain $z(t) \geq 3$, for any $t \in (-1, 1)$, and therefore $\phi_t$ is always spacelike.

(iii) Take real constants $c_1, c_2 > 0$. Then, the function $f : \mathbb{R} \to (0, \infty)$, $f(t) = \frac{4 + c_1^2}{4c_2} t^2 + c_1 t + c_2$ is well-defined. A simple computation shows $z(t) \equiv -3/4$. This implies that $\phi_t$ is everywhere timelike.

(iv) Given the function $f : \mathbb{R} \to (0, \infty)$, $f(t) = 3 + \cos(2t)$. A straightforward computation gives

\[
z(t) := -1 + \left(\frac{f(t)f''(t)}{4 + f'(t)^2}\right)^2 = -1 + \frac{4 \cos^2(2t)(3 + \cos(2t))^2}{(3 - \cos(4t))^2}.
\]

Finally, it is easy to check $z(0) = 15$ and $z(\pi/4) = -1$. In this case, the causal character changes with time.

4. Using the Hopf map

We regard now the standard 2-round sphere of radius $1/2$ as $S^2(1/2) = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1/4\}$, with standard line element $g_2$. The classical Hopf map is

\[
\pi : S^3 \to S^2(1/2), \quad \pi(z, w) = \left(z\bar{w}, \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2\right).
\]

This map has many marvellous properties. Among them, we recall the following:

(i) $\pi$ is a Riemannian submersion.
(ii) For each $(z, a) \in S^2(1/2)$, then $\pi^{-1}\{(z, a)\}$ is a closed geodesic in $S^3$.

Now, we extend it to a new submersion as follows:

\[
\varpi : (-I \times_f S^3, -dt^2 + f^2 g_3) \to (-I \times_f S^2(1/2), -dt^2 + f^2 g_2), \quad \varpi(t, p) = (t, \pi(p)).
\]

Next, we consider a curve $\alpha : J \to -I \times_f S^2(1/2)$, and its pullback $\varpi^\ast(\alpha) = J \times S^1 \to -I \times_f S^3$. We point out that the geometric elements of $\alpha$ determine the properties of the mean curvature vector of $\varpi^\ast(\alpha)$. For instance, if $\alpha$ is embedded and open/closed, then $\varpi^\ast(\alpha)$ is an embedded cylinder/torus in $-I \times_f S^3$. In addition, these surfaces may not be contained in a single $t$-slice.

To make computations, we ask $\alpha$ to be a unit spacelike Frenet curve. Next, let $\beta$ be a horizontal lift of $\alpha$, i. e., $\varpi \circ \beta = \alpha$ and $\beta'$ is orthogonal to $\ker(\varpi)_\ast$ (i. e., $\beta'$ is horizontal.) Also, for each $e^{i\theta} \in S^1$, the map

\[
\Gamma_{\theta} : -I \times_f S^3 \to -I \times_f S^3, \quad \Gamma_{\theta}(t, (z, w)) = \left(t, (e^{i\theta} z, e^{i\theta} w)\right)
\]

is an isometry, so we can construct a parametrization of the surface $\varpi^\ast(\alpha)$ by

\[
\phi : \varpi^\ast(\alpha) = J \times S^1 \to -I \times_f S^3, \quad \phi(s, \theta) = \Gamma_{\theta}(\beta(s)).
\]

If $g_3 = -dt^2 + f^2 g_2$ is the line element, since $\alpha$ is a unit Frenet curve, we can compute its Frenet apparatus $\{T = \dot{\alpha}, N, B\}$ and $\kappa, \tau$, with Frenet equations

\[
\nabla_T T = \epsilon_2 \kappa N, \quad \nabla_T N = \kappa T + \epsilon_3 \tau B, \quad \nabla_T B = -\epsilon_2 \tau N,
\]
where $\epsilon_2 = \overline{g}_3(N,N)$, $\epsilon_3 = \overline{g}_3(B,B)$, $\epsilon_2 = -\epsilon_3 = \pm 1$, and \{\(T, N, B\)\} is a positive basis along $\alpha$.

As a summary, we have the following commutative diagram:

$$
\begin{array}{cc}
\pi^*(\alpha) = J \times S^1 & \phi \\
\beta \downarrow & \downarrow \pi \\
J & \phi^{-1}(\alpha) \\
\end{array}
$$

Let $\tilde{N}$ and $\tilde{B}$ be horizontal lifts of $N$ and $B$, resp., along $\beta$.

**Lemma 1** The mean curvature vector of $\phi$ is given by

$$
\tilde{H}_\phi = \frac{\epsilon_2}{2} \left( \kappa + \frac{f'}{f} \overline{g}_3(\partial_t, N) \right) (\overline{\Gamma}_\theta)_* \tilde{N} + \frac{\epsilon_3}{2} \left( \frac{f'}{f} \overline{g}_3(\partial_t, B) \right) (\overline{\Gamma}_\theta)_* \tilde{B}
$$

**Proposition 1** The mean curvature vector $\tilde{H}_\phi$ satisfies $\|\tilde{H}_\phi\| = 0$ if, and only if,

$$
\left( \kappa + \frac{f'}{f} \overline{g}_3(\partial_t, N) \right)^2 - \left( \frac{f'}{f} \overline{g}_3(\partial_t, B) \right)^2 = 0.
$$

5. Final remarks

In [2], we obtained an open embedded surface with null mean curvature vector, and crossing two regions, one expanding and one collapsing. As a result, we pose the following open problem: to find an explicit MOTS in the 4-dim closed FLRW spacetime, which is not contained in any $t$-slice, from a closed curve in the toy model $-I \times f S^2(1/2)$.

6. Conclusions

- There exist MOTS in closed FLRW 4-spacetimes embedded in $t_0$-slices with arbitrary topology.
- This leads to MOTT in closed FLRW 4-spacetimes with any causal character. Our examples are foliated by tori, but other topologies are also possible.
- From a curve in a (toy model) closed FLRW 3-spacetime $\gamma: J \to (-I \times f S^2(1/2), -dt^2 + f^2 g_2)$, it is possible to construct embedded cylinders and tori in the closed FRLW 4-spacetime $(-I \times f S^3, -dt^2 + f^2 g_3)$ with some control of the mean curvature vector.
- Problem: to construct such a tori which is also a MOTS.

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