Linear automorphism groups of relatively free groups

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1 Introduction

Let $F = F(x_1, x_2, \ldots)$ be an absolutely free group with basis $x_1, x_2, \ldots$. Recall that a group $H$ satisfies an identity $w(x_1, \ldots, x_n) = 1$ for a word $w(x_1, \ldots, x_n) = w \in F$ if $w$ vanishes under every homomorphism $F \to H$. A variety of groups is a class of groups consisting of all groups that satisfy some set of identities.

A relatively free group $G$ is a free group in a group variety $\mathcal{V}$, i.e., $G$ belongs to $\mathcal{V}$, and $G$ is generated by a set $Y$ such that every mapping $Y \to H$, where $H \in \mathcal{V}$, extends to a homomorphism $G \to H$. The (free) rank of $G$ is the cardinality of $Y$. (See details in [N].) We denote by $F_n$, by $A_n = F_n/F_n'$, and by $M_n = F_n/F_n''$ the absolutely free group, the free abelian and the free metabelian groups of rank $n$, respectively.

The group of inner automorphisms of a group $G$ is normal in $Aut(G)$, and so the factor group $G/Z(G)$, where $Z(G)$ is the center of $G$, can be canonically identified with a normal subgroup of $Aut(G)$. Since the center of $M_n$ is trivial for $n \geq 2$ (see [N], 25.63), one can identify $M_n$ with the normal subgroup of $Aut(M_n)$ consisting of the inner automorphisms of $M_n$.

Our paper is inspired by the following result of V.P. Platonov that answers a question raised by H. Mochizuki [M].

**Theorem 1.1.** (V.P. Platonov [P]) (1) Let $\rho$ be a finite-dimensional linear representation of the automorphism group $Aut(M_n)$ over a field $k$. Then the image $\rho(M_n)$ is a virtually nilpotent group. (2) It follows that the group $Aut(M_n)$ is not linear for $n > 1$.

As usual, a group $G$ is called linear if it is isomorphic to a subgroup of $GL_m(k)$ for some field $k$ and some integer $m \geq 1$.

In Section 2, we give an alternative and shorter proof of Theorem 1.1. Then a similar approach and the utilization of some known properties of group varieties lead to the complete description of relatively free groups $G$ for which $Aut(G)$ is a linear group.

**Theorem 1.2.** Let $G$ be a relatively free but not absolutely free group. The automorphism group $Aut(G)$ is linear if and only if $G$ is a finitely generated virtually nilpotent group.

Furthermore, if the group $G$ is finitely generated but not virtually-nilpotent, then there is an automorphism $\phi$ of $G$ such that the extension $P$ of $G/Z(G)$ by $\phi$ is a non-linear subgroup of $Aut(G)$; and if $G$ is finitely generated and virtually nilpotent, then the holomorph $Hol(G)$ is linear over $Z$.

Recall that a group $G$ is virtually nilpotent if it contains a (normal) nilpotent subgroup of finite index. (The “if” part of the statement does not need the hypothesis that $G$ is relatively free.)

**Remark 1.3.** The automorphism group $Aut(F_n)$ is not linear for $n \geq 3$ (Formanek, Procesi [FP]) but the group $Aut(F_2)$ is linear (Krammer [K]).

**Remark 1.4.** Note that the formulation of Theorem 1.2 is similar to those contained in the papers of O.M. Mateiko and O.I. Tavgen` [MT] and A.A. Korobov [Ko]. Nevertheless we prove Theorem 1.2 here for the following reasons. (1) There is a mistake in both [MT] and [Ko]. Namely, the proofs essentially use the "known property" of Fitting subgroups to be fully characteristic. But this does not hold even for

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Lemma 3.1. Let \( \text{free group of rank } n \) be a free group of rank \( n > 1 \) of the variety generated by the alternating group \( \text{Alt}(5) \). (2) The formulation of the main theorems is not quite correct in both [MT] and [Ko] because it is not proved there that the virtually nilpotency of a free group of rank \( n > 1 \) in a variety implies virtually nilpotency of free groups having rank \( n > n \). (3) Our proof is simpler. (The authors of both papers [MT] and [Ko] refer, in particular, to the statement that all locally finite groups of exponent dividing \( m \) form a variety. This claim is equivalent to the restricted Burnside problem for groups of exponent \( m \), and the affirmative solution is bases on the Classification Hypothesis for finite simple groups.)

2 Free metabelian case

The following Kolchin–Mal’cev Theorem is the most–known fact on linear solvable groups.

**Lemma 2.1.** Every linear solvable group has a subgroup \( H \) of finite index such that the derived subgroup \( H' \) is nilpotent. \( \square \)

Let \( \phi \) be an automorphism of the free abelian group \( A_n \) and \( B_\phi = \langle A_n, \phi \rangle \) the extension of \( A_n \) by the automorphism \( \phi \). Assume that no root of the characteristic polynomial of \( \phi \) is an \( k \)-th root of 1 for any integer \( k > 0 \). The following property of the group \( B_\phi \) is folklore.

**Lemma 2.2.** Let \( C \) be a subgroup of finite index in \( B_\phi \). Then the derived subgroup \( C' \) has finite index in \( A_n \).

**Proof.** Since \( C \) is of finite index in \( B_\phi \), it must contain \( \phi^k \) and \( mA_n \) for some positive integers \( m \) and \( k \). (We use the additive notation for \( A_n \) in this proof.) Therefore \( C' \) contains the subgroup \( \{ m\phi^k(a) - ma | a \in A_n \} \.

Proving by contradiction, assume that the index \( [A_n : \langle m\phi^k(a) - id | a \in A_n \}] \) is infinite. Then the image of \( mA_n \) under the mapping \( \phi^k - id \) is of rank \( < n \). Hence 0 is an eigenvalue of \( \phi^k - id \), and so \( \lambda^k = 1 \) for a characteristic root \( \lambda \) of \( \phi \); a contradiction. \( \square \)

**Proof of Theorem 1.1** There is nothing to prove if \( n = 1 \). For every \( n \geq 2 \), there exists an automorphism \( \phi \) of \( A_n \) whose characteristic roots are not roots of 1. (One can easily find such automorphisms for \( n = 2,3 \) and note for \( n > 3 \) that \( A_n \) is a direct sum of subgroups isomorphic to \( A_2 \) and \( A_3 \).) We keep the same notation \( \phi \) for a lifting of \( \phi \) to \( \text{Aut}(M_n) \). (Recall that \( A_n \cong M_n/M_n' \), and the induced homomorphism \( \text{Aut}(M_n) \to \text{Aut}(A_n) \) is surjective by \([N], 41.21\).

Let \( P = \langle M_n, \phi \rangle \) be the extension of \( M_n \) by the automorphism \( \phi \). This \( P \) is a subgroup of \( \text{Aut}(M_n) \) since \( \langle \phi \rangle \cap M_n = \{ 1 \} \); and \( P \) is solvable because the factor-group \( P/M_n' \) is cyclic. By Lemma 2.1 there is a normal subgroup \( T \) of \( A_n \) in \( P \) such that the subgroup \( \rho(T') \) is nilpotent.

The canonical image \( C = TM_n'_{M_n'} \) of \( T \) in \( B_\phi = P/M_n' \) has finite index, and therefore \( C' \) is of finite index in \( A_n = M_n/M_n' \) by Lemma 2.2. Hence the inverse image \( D = T'M_n'_{M_n'} \) of \( d \) is of finite index in \( M_n \).

The normal subgroup \( \rho(M_n') \) of \( \rho(M_n) \) is abelian since \( M_n \) is metabelian. Thus, \( \rho(D) \) is nilpotent being a product of two nilpotent normal in \( \rho(P) \) subgroups \( \rho(T') \) and \( \rho(M_n') \). Since \( (M_n : D) < \infty \), the theorem is proved. \( \square \)

3 Few lemmas on varieties of groups

The **product** \( \mathcal{UV} \) of two group varieties contains all the groups \( G \) having a normal subgroup \( N \) such that \( N \in \mathcal{U} \) and \( G/N \in \mathcal{V} \); \( \mathcal{UV} \) is also a group variety (\([N], 21.12\).

**Lemma 3.1.** \([SH]\). Let \( L \) be a free group of rank \( n \geq 2 \) in a product of varieties \( \mathcal{UV} \). Assume that the free group of rank \( n \) in the variety \( \mathcal{V} \) is infinite and \( \mathcal{U} \) contains a non-trivial group. Then the center of \( L \) is trivial. \( \square \)

We denote by \( \mathcal{A} \) (by \( \mathcal{A}_k \)) the variety of all abelian groups (of all abelian groups of exponent dividing \( k \)), and denote by \( M_{k,n} \) the free group of rank \( n \) in the variety \( \mathcal{M}_k = \mathcal{A}_k \mathcal{A} \).

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Lemma 3.2. If $n, k \geq 2$, then the group $M_{k,n}$ is not virtually nilpotent.

Proof. The wreath product $W = \mathbb{Z}_k \wr \mathbb{Z}$ of a cyclic group of order $k$ and an infinite cyclic group is 2-generated, and it belongs to the variety $M_k$. Therefore $W$ is a homomorphic image of the group $M_{k,n}$. Since $W$ is not virtually nilpotent, $M_{k,n}$ is not virtually nilpotent too.

A variety $\mathcal{V}$ is called solvable if all the groups of $\mathcal{V}$ are solvable.

Lemma 3.3. Let $\mathcal{S}$ be a solvable variety of groups. Then either $\mathcal{S}$ contains as a subvariety the product $\mathcal{M}_p = \mathcal{A}_p \mathcal{A}$ for some prime $p$ or every finitely generated group in $\mathcal{S}$ is virtually nilpotent. □

Further we call a variety $\mathcal{V}$ proper if it does not contain all groups, or equivalently, the absolutely free group $F_2$ does not belong to $\mathcal{V}$. It is easy to see that a product of two proper varieties is proper. (See also [N] .) The minimal variety containing a group $Q$ is denoted by $\text{var} Q$. Given a group $G$ and a variety $\mathcal{V}$, the verbal subgroup $\text{var} V(G)$ corresponding to $\mathcal{V}$ is the smallest normal subgroup $N$ of $G$ such that $G/N \in \mathcal{V}$.

4 Proof of Theorem 1.2

By Auslander – Baumslag’s theorem [AB], the holomorph of any finitely generated (virtually) nilpotent group $G$ is linear over $\mathbb{Z}$. Thus, it remains to consider a non-virtually-nilpotent free group $G$ of rank $n \geq 2$ in a proper variety $\mathcal{V}$ and construct the required automorphism $\phi$. (A non-trivial relatively free group of infinite rank admits the automorphisms from an infinite symmetric group that is not linear.)

Now the quotient $H = G/Z(G)$ is non-virtually-nilpotent normal subgroup of $\text{Aut}(G)$. We may assume that $H$ is a linear group since otherwise there is nothing to prove.

Since both $G$ and $H$ satisfy a non-trivial identity and $H$ is linear, the group $H$ is virtually solvable by Platonov’s theorem [P1], i.e., the solvable radical $R$ of $H$ is of finite index in $H$. Therefore $R$ is a finitely generated but not virtually nilpotent solvable group.

By Lemma 3.3 there is a prime $p$ such that $\mathcal{M}_p \subseteq \text{var} R \subseteq \text{var} H \subseteq \text{var} G$.

Therefore there are canonical epimorphisms $G \to M_{p,n}$ and $G \to A_n$. The kernels are the verbal subgroups of $G$ corresponding to the varieties $\mathcal{M}_p$ and $\mathcal{A}$, respectively. The latest kernel is just the derived subgroup $G'$, and we denote by $M_p(G)$ the former one.

The center of $M_{p,n}$ is trivial by Lemma 3.1 and so $Z(G)$ is contained in $M_p(G) \subseteq G'$. Consequently, we have isomorphisms $G/M_p(G) \cong H/M_p(H) \cong M_{p,n}$ and $G/G' \cong H/H' \cong A_n$.

Now, as in the proof of Theorem 1.1 we introduce an automorphism $\phi$ of $A_n$ whose action has no characteristic roots equal to any root of 1. As there (by [N], 41.21) one can lift $\phi$ to $\text{Aut}(G)$ and also to $\text{Aut}(H)$ since the center $Z(G)$ is a characteristic subgroup of $G$. Denote by $P = \langle H, \phi \rangle$ the extension of $H$ by the automorphism $\phi$. It is a subgroup of $\text{Aut}(G)$ as in the proof of Theorem 1.1.

Proving by contradiction, assume that $P$ is a linear group. Since $P \in \mathcal{V} \mathcal{A}$, the group $P$ satisfies a non-trivial identity, and by [P1], $P$ must have a solvable normal subgroup of finite index. By Lemma 2.1 $P$ contains a normal subgroup $T$ of finite index with nilpotent derived subgroup $T'$. Applying Lemma 2.2 to the image of $T$ in $B_\phi = P/H'$, we have $(H : (T'H')) < \infty$.

The quotient $H'/M_p(H)$ is an abelian normal subgroup of $H/M_p(H) \cong M_{p,n}$. Since $T'$ is nilpotent and normal in $P$, the image of the subgroup $T'/H'$ under the canonical epimorphism $H \to M_{p,n}$ is nilpotent too. But this image is of finite index in $M_{p,n}$ because $(H : (T'H')) < \infty$. This contradicts the statement of Lemma 3.2 and so the theorem is proved. □

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References

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