New Solvable Lattice Models from Conformal Field Theory

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Abstract

In this work we build the trigonometric solutions of the Yang-Baxter equation that can not be obtained from quantum groups in any direct way. The solution is obtained using the construction suggested recently from the rational conformal field theory corresponding to the WZW model on $SO(3)_{4R} = SU(2)_{4R}/Z_2$. We also discuss the full elliptic solution to the Yang-Baxter equation whose critical limit corresponds to the trigonometric solution found below.

1 Introduction

Two dimensional systems offer the intriguing possibility of exact solvability. The solutions to the Yang-Baxter equation play a central role in theory of the integrable systems. Quantum groups yield the systematic approach of the construction of the trigonometric solutions of the Yang-Baxter equation. In this work we study the trigonometric solutions of the Yang-Baxter equation that can not be obtained from quantum groups in any direct way. We also discuss the full elliptic solution to the Yang-Baxter equation whose critical limit corresponds to the trigonometric solution found below.

We use the construction suggested recently stating that solvable lattice models may be build around any rational conformal field theory together with some primary field $\phi$. In particular in this way the models found previously were rederived from conformal field theory in a systematic way.

We will deal here with the interaction round the face (IRF) lattice models.
The partition function of the lattice model is given by

\[ Z = \sum_{\text{configurations}} \prod_{\text{faces}} w\left(\begin{array}{c} a \ b \\ c \ d \end{array} \middle| u \right). \tag{1.1} \]

The model is solvable if the Boltzmann weights (BW) \( w\left(\begin{array}{c} a \ b \\ c \ d \end{array} \middle| u \right) \) obey the Yang-Baxter equation:

\[ \sum_c w\left(\begin{array}{c} b \ d \\ a \ c \end{array} \middle| u \right) w\left(\begin{array}{c} c \ d \\ a \ b \end{array} \middle| u \right) = \sum_c w\left(\begin{array}{c} a \ b \\ c \ d \end{array} \middle| u \right) w\left(\begin{array}{c} c \ d \\ a \ b \end{array} \middle| u \right). \tag{1.2} \]

The construction described in [1] states that BW are given in terms of braiding matrices of the corresponding RCFT as:

\[
\begin{aligned}
& \quad w\left(\begin{array}{c} a \ b \\ c \ d \end{array} \middle| u \right) = \sum_j \langle a, b, d | P^{(j)} | a, c, d \rangle \rho_j(u), \\
& \quad \langle a, b, d | P^{(j)} | a, c, d \rangle = \prod_{j \neq l} B_{b,c} \left[ \frac{\phi \phi_{\phi \phi}}{\phi_{\phi_{\phi \phi}}} \right] - \delta_{b,c} \lambda_j \left( -\lambda_j - \lambda_{j^{-1}} \right), 
\end{aligned} \tag{1.3} \]

Here \( j \) labels the field exchanged in the \( u \)-channel. The fields that may appear in the \( u \)-channel are determined from the product of the field \( \phi \) with itself

\[ \phi \times \phi = \sum_j N_{j,k}^l \phi_{\phi \phi} \phi_j, \]

where \( N_{j,k}^l \) are the fusion coefficients of the RCFT under consideration and \( \rho_j(u) \) are some scalar functions which will be specified below.

Projection operators \( P^{(j)} \) are found from the braiding matrix \( B_{b,c} \left[ \frac{\phi \phi_{\phi \phi}}{\phi_{\phi_{\phi \phi}}} \right] \) whose eigenvalues \( \lambda_j \) are given by

\[ \lambda_j = \left( -1 \right)^j e^{i\pi(\Delta_j - 2\Delta_\phi)}, \]

where \( \Delta, \Delta_j \) are conformal dimensions of the primary fields \( \phi, \phi_j \). Note that the Boltzmann weights vanish, unless the admissibility condition is satisfied:

\[ N_{a,\phi}^b N_{b,\phi}^c N_{c,\phi}^d N_{d,\phi}^\phi > 0. \tag{1.5} \]

In this paper we will derive the solvable lattice model based on the extended current algebra of WZW model on \( SO(3)_{4R} = SU(2)_{k=4R}/Z_2 \) with the primary field \( \phi_1 \) (adjoint). Its fusion rules and torus modular \( S \) matrix were derived in [8, 10]. We quote the result here for the torus modular \( S \) matrix for \( R \) even

\[ S = \begin{pmatrix}
2S_{j,i} & S_{j,R} & S_{j,R'} \\
S_{R,i} & x & z \\
S_{R',i} & z & x
\end{pmatrix}, \quad x = \frac{S_{R,R} + 1}{2}, \quad z = \frac{S_{R,R} - 1}{2}. \tag{1.6} \]
where \( S_{a,b} = \sqrt{\frac{1}{2R+1}} \sin \frac{(2a+1)(2b+1)}{4R+2} \pi \). The entries of this matrix are restricted to the singlets of \( Z_2 \) (only integer isospins may appear) and due to the symmetry \( S_{j,i} = S_{\sigma(j),i} = S_{j,\sigma(i)} \), where \( \sigma \) is the external automorphism \( \sigma(j) = 2R - j \), the primary fields are labeled modulo \( \sigma \). The fixed point of \( \sigma \) is resolved into two fields: \( R, R' \), these fields differ by an additional quantum number but have the same conformal weights and transformation properties under \( SU(2) \). The fusion rules for \( R \) even are given by:

\[
\phi_j \times \phi_i = \sum_{p=|j-i|}^{\min(R,i+j)} m_p \phi_p, \quad i, j \neq R, R', \quad (1.7)
\]

where \( m_R = m_{R'} = 1 \) and \( m_p = 2 \) iff \( p, 2R - p \in \{ |i - j|, \ldots, i + j \} \).

\[
\phi_R \times \phi_R = \phi_{R'} \times \phi_{R'} = \sum_{p=0}^{R-2} \phi_{2p}, \quad (1.8)
\]

\[
\phi_R \times \phi_{R'} = \sum_{p=0}^{R-2} \phi_{2p+1}, \quad (1.9)
\]

\[
\phi_j \times \phi_R = \sum_{p=|j-R|}^{R-1} \phi_p + \frac{1 + (-1)^j}{2} \phi_R + \frac{1 - (-1)^j}{2} \phi_{R'}, \quad j \neq R, R' \quad (1.10)
\]

The scalar functions \( \rho^{(j)}(u) \) in this case are given by:

\[
\rho_0(u) = \frac{\sin(\lambda - u) \sin(\omega - u)}{\sin(\lambda) \sin(\omega)}, \quad \rho_1(u) = \frac{\sin(\lambda + u) \sin(\omega - u)}{\sin(\lambda) \sin(\omega)}, \quad (1.11)
\]

\[
\rho_2(u) = \frac{\sin(\lambda + u) \sin(\omega + u)}{\sin(\lambda) \sin(\omega)}, \quad (1.12)
\]

where \( \lambda, \omega \) are the crossing parameters of the model that are related to the conformal weights of the fields appearing in the operator product expansion of field \( \phi_1 \) with itself: \( \phi_1 \times \phi_1 = 1 + \phi_1 + \phi_2 \)

\[
\lambda = \frac{\pi}{2} (\Delta_1 - \Delta_0) = \frac{\pi}{2R + 2}, \quad \omega = \frac{\pi}{2} (\Delta_2 - \Delta_1) = 2\lambda. \quad (1.13)
\]

The incidence diagram of matrix \( N^b_{a,\phi_1} \) representing the admissibility condition for the lattice variables is shown at Fig.1

*For the sake of concreteness we chose to label primary fields by minimal isospin \( \min(j, 2R - j) \), so that primary fields of the theory are \( \phi_0 \) (identity primary field), \( \phi_1, \ldots, \phi_{R-1}, \phi_R, \phi_{R'} \).
Fig. 1 Admissibility graph

Here $R$ and $R'$ are two primary fields corresponding to the fixed point $[10]$.

2 Trigonometric Solution

2.1 Boltzmann Weights without Fixed Point Fields

We will start by exploiting the connection between our model and the model described at [9] which in this language corresponds to the IRF model build around $SU(2)_{k=4R}$ together with the field $\phi_1$. We will refer in the sequel to this model restricted to the singlets of $Z_2$ as the diagonal model.

Let us denote by $F_{ijkl}^{p}(f_{ijkl}^{p})$ correspondingly the conformal blocks of extended (unextended) theory, the corresponding braiding matrices are denoted by $B$ and $C$. Below we assume for the moment that none of the fields is equal to the fixed point field $\phi_R$

\[
F_{ijkl}^{p} = \sum_{\sigma} f_{ijkl}^{\sigma}, \quad (2.1)
\]

\[
F_{ijkl}^{p} = \sum_{p'} B_{p,p'} \left[ \phi_j \phi_k \phi_l \right] F_{ijkl}^{p'}, \quad (2.2)
\]

\[
f_{ijkl}^{p} = \sum_{p'} C_{p,p'} \left[ \phi_j \phi_k \phi_l \right] f_{ijkl}^{p'}, \quad (2.3)
\]
so using only definitions

\[
\sum_{p'} B_{p,p'} \left[ \phi_j \phi_k \right] f_{ijkl}^{p'jk} = \sum_{\sigma} \sum_{p'} C_{p,p'} \left[ \sigma \left( jk \right) \right] f_{ijkl}^{\sigma(jkl)},
\]  

(2.4)

where in the previous equation by \( \sigma(i, j, k, l) \) we mean some collection of Dynkin labels \( \sigma(i), \sigma'(j), \sigma''(k), \sigma'''(l) \) and \( \sigma \in \mathbb{Z}_2 \). Of course some of the conformal blocks \( f_{ijkl}^{\sigma(jkl)} \) in the RHS of Eq.(2.4) may vanish. Note that Eqs.(2.1-2.4) imply the relation of the form \( \sum_j b_j f_j(z) = \sum_j c_j f_j(z) \), where \( f_j \) are independent functions and \( c_j, b_j \) are some coefficients. From this follows the equality of the coefficients, namely we have

\[
B_{p,p'} \left[ \phi_j \phi_k \right] = C_{p,p'} \left[ jk \right], \quad p, p', i, j, k, l \neq R. \quad (2.5)
\]

It means that the Boltzmann weights which do not contain fixed points fields are equal to the corresponding Boltzmann weights of the model described at [9].

### 2.2 Boltzmann Weights Involving Fixed Points

First we calculate the braiding matrix \( B_{p,q} \left[ \phi_1 \phi_1 \phi_R \phi_R' \right] \). Note that correlator \( \langle \phi_R \phi_1 \phi_1 \phi_R' \rangle \) receives contribution in the s-channel only from the field \( \phi_{R-1} \), so that corresponding braiding matrix \( B_{p,q} \left[ \phi_1 \phi_1 \phi_R \phi_R' \right] \) is fixed by monodromy invariance

\[
B_{p,q} \left[ \phi_1 \phi_1 \phi_R \phi_R' \right] = e^{2\pi i(\Delta_R - \Delta_{R-1})}.
\]

Now we calculate the braiding matrix \( B_{p,q} \left[ \phi_1 \phi_1 \phi_R \phi_R' \right] \). From the fusion rules we know that the relevant space of conformal blocks is two dimensional: \( p, q = \phi_{R-1}, \phi_R, \phi_{R'} \), so that it is enough to find only one entry of the braiding matrix and the others will be fixed from the invariance under monodromy. Using pentagon identity

\[
B_{p_1,q_1} \left[ \phi_{j_1} \phi_{j_2} \phi_{j_3} \right] B_{p_2,q_2} \left[ \phi_{j_1} \phi_{j_4} \phi_{j_5} \right] e^{i\pi(\Delta_{p_1} - \Delta_{q_2} - \Delta_{j_1})} = \\
= \sum_s e^{i\pi \Delta_s} B_{p_2,s} \left[ \phi_{j_2} \phi_{j_3} \phi_{j_4} \phi_{j_5} \right] B_{p_1,q_1} \left[ \phi_{j_1} \phi_{j_4} \phi_{j_5} \right] B_{s,q_1} \left[ \phi_{j_2} \phi_{j_3} \phi_{j_4} \right],
\]  

(2.6)

For R-even let us set

\[
\begin{align*}
  j_2 &= p_2 = j_4 = j_5 = \phi_1, \\
  j_1 &= j_3 = \phi_R, \\
  p_1 &= q_1 = \phi_R', \\
  q_2 &= \phi_{R-1}.
\end{align*}
\]

†This form of pentagon identity is obtained from the conventional using compatibility between braiding and fusing.
After some algebra we have:

\[
B_{\phi_R, \phi_R'} \left[ \phi_1 \phi_1 \right] = B_{\phi_R, \phi_{R-1}} \left[ \phi_1 \phi_1 \right] B_{\phi_{R-1}, \phi_R'} \left[ \phi_1 \phi_1 \right], \tag{2.9}
\]

For the matrices in the RHS we may use the expression which was found for arbitrary 2 \times 2 braiding matrix. Other entries are calculated from the monodromy invariance

\[
\sum_{p'} e^{2\pi i (\Delta_R + \Delta_{p'})} B_{p', p} \left[ \phi_1 \phi_1 \right] B_{p', q} \left[ \phi_1 \phi_1 \right] = \delta_{p, q} e^{4\pi i \Delta_R}, \tag{2.10}
\]

After some algebra we have:

\[
B_{p, q} \left[ \phi_1 \phi_1 \right] = \frac{1}{2R^2} \left( \frac{1}{q} \frac{[2R + 2][2R - 2]}{\sqrt{1 - \frac{2R + 2}{2R - 2}}} \right) e^{2\pi i \Delta_R}, \tag{2.11}
\]

where \(z \equiv \frac{q^2 - 1}{2q}, \quad q = e^{\frac{\pi i \Delta_R}{R}}.\)

Now we will turn to the calculation of the braiding matrix \(B_{p, q} \left[ \phi_1 \phi_1 \right],\) it is 4 \times 4 matrix (the relevant space of conformal blocks is 4 dimensional \(p, q = \phi_{R-2}, \phi_{R-1}, \phi_R, \phi_{R'}.\)) Using twice pentagon identity we have:

\[
B_{\phi_R, \phi_R} \left[ \phi_1 \phi_1 \right] B_{\phi_1, \phi_{R-1}} \left[ \phi_R \phi_1 \right] e^{2\pi i (\Delta_R - \Delta_{R-1})} = \]

\[
= B_{\phi_1, \phi_{R-1}} \left[ \phi_{R-1} \phi_1 \right] B_{\phi_{R}, \phi_R'} \left[ \phi_{R-1} \phi_1 \right] B_{\phi_{R-1}, \phi_{R'}} \left[ \phi_{R-1} \phi_1 \right], \tag{2.12}
\]

and similarly

\[
B_{\phi_{R'}, \phi_{R'}} \left[ \phi_1 \phi_1 \right] B_{\phi_1, \phi_{R}} \left[ \phi_R \phi_1 \right] e^{2\pi i (\Delta_R - \Delta_{R-1})} = \]

\[
= B_{\phi_1, \phi_{R}} \left[ \phi_{R-1} \phi_1 \right] B_{\phi_{R}, \phi_R'} \left[ \phi_{R-1} \phi_1 \right] B_{\phi_{R-1}, \phi_{R'}} \left[ \phi_{R-1} \phi_1 \right]. \tag{2.13}
\]

From these equations one may show:

\[\text{We did not use this expression directly, because exponent sum rule in this case is not obeyed.}\]
\[ B_{R,R} \left[ \phi_1 \phi_1 \right] = \pm B_{R,R'} \left[ \phi_1 \phi_1 \right]. \]  

(2.14)

Now using monodromy invariance

\[ \sum_{R'} e^{2\pi i (\Delta_p + \Delta_{p'})} B_{p,p'} \left[ \phi_1 \phi_1 \right] B_{p',q} \left[ \phi_1 \phi_1 \right] = \delta_{p,q} e^{4\pi i \Delta R}, \]  

(2.15)

we find

\[ B_{R,R} \left[ \phi_1 \phi_1 \right] = -B_{R,R'} \left[ \phi_1 \phi_1 \right] = \frac{1}{2} e^{2\pi i (\Delta_R - \Delta_{R'})}, \]  

(2.16)

Using Eqs. (1.3-1.4, 1.11-1.13) one may find the Boltzmann weights of the model. Let us consider for example the \( R-1 \bigcap R-1 \), the only field exchanged in the u-channel is \( \phi_1 \) so that the corresponding BW is given by:

\[ R-1 \bigcap R-1 = \rho_1(u), \]  

(2.17)

after some straightforward calculation the rest of the Boltzmann Weights are found to be:

\[ R-1 \bigcup R-1 = R-1 \bigcup R-1 = 2\rho_0(u)P_{R-1,R-1}^{(0)} + 2\rho_2(u)P_{R-1,R-1}^{(2)}, \]  

(2.18)

\[ R-1 \bigcup R = R-1 = \sqrt{2}\rho_0(u)P_{R,R-1}^{(0)} + \sqrt{2}\rho_2(u)P_{R,R-1}^{(2)}, \]  

(2.19)

\[ R-1 \bigcap R = R-1 \bigcap R = \rho_0(u)P_{R,R}^{(0)} + \rho_1(u)P_{R,R}^{(1)} + \rho_2(u)P_{R,R}^{(2)}, \quad b \neq R, R'. \]  

(2.20)
\[
R^{-1} R = \rho_0(u) P_{R,R}^{(0)} + \frac{\rho_1(u)}{2} P_{R,R}^{(1)} + \frac{\rho_2(u)}{2} (P_{R,R}^{(2)} + 1), \quad (2.21)
\]

\[
R'^{-1} R' = \rho_0(u) P_{R,R}^{(0)} + \frac{\rho_1(u)}{2} P_{R,R}^{(1)} + \frac{\rho_2(u)}{2} (P_{R,R}^{(2)} - 1), \quad (2.22)
\]

\[
R^{-1} R = R'^{-1} R' = 0, \quad (2.23)
\]

\[
R^{-1} R = R'^{-1} R' = \rho_0(u) P_{R,R}^{(0)} + \rho_1(u) P_{R,R}^{(1)} + \rho_2(u) P_{R,R}^{(2)}, \quad (2.24)
\]

where \( P_{b,c}^{(j)} \equiv \langle a, b, d | P^{(j)} | a, c, d \rangle \) are projectors from \([9]\) (their explicit expression are summarized in the appendix) and scalar functions \( \rho_j(u) \) are given by:

\[
\rho_0(u) = \frac{\sin(\lambda - u) \sin(\omega - u)}{\sin(\lambda) \sin(\omega)}, \quad \rho_1(u) = \frac{\sin(\lambda + u) \sin(\omega - u)}{\sin(\lambda) \sin(\omega)}, \quad \rho_2(u) = \frac{\sin(\lambda + u) \sin(\omega + u)}{\sin(\lambda) \sin(\omega)}, \quad \lambda = \frac{1}{2}, \quad \omega = \frac{\pi}{4R + 2}, \quad (2.25)
\]

Note that the Boltzmann weights obey the following crossing relation \([1]\):

\[
w \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right) = \sqrt{S_{b,0} S_{0,0}} \cdot \frac{S_{a,0} S_{c,0}}{S_{a,0} S_{c,0}} \cdot w \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right) \cdot -\lambda - u, \quad (2.27)
\]

with the torus modular matrix \( S \) for the extended theory Eq.(1.6), where "0" designates the identity primary field.

### 3 Thermalized Boltzmann Weights

In this section we will present the full off critical solution. The Boltzmann weights are now parameterized by the elliptic theta functions \( \Theta_1(u, p) \) which is defined by:

\[
\Theta_1(u, p) = 2p^4 \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos(2u) + p^{4n}) (1 - p^{2n}) \equiv [u], \quad (3.1)
\]
where \( p \) labels the distance from criticality. In the limit \( p \to 0 \) the trigonometric solution of the previous section is recovered.

The related diagonal model was found at \( \text{[3]} \). The Boltzmann weights that do not contain fixed point fields are remained unchanged:

\[
\begin{align*}
j + 1 \left\langle j + 1 \right. & = \frac{\lambda + u}{\lambda} \frac{\omega + u}{\omega}, \\
\left. j + 1 \right\rangle & = \frac{\lambda + u}{\lambda} \frac{(j + 1)\omega - u}{(j + 1)\omega}, \\
\left\langle j \right. & = \frac{\lambda + u}{\lambda} \frac{j\omega + u}{j\omega}, \\
\left. j \right\rangle & = \frac{\lambda + u}{\lambda} \frac{\sqrt{(j + 2)\omega} \sqrt{j\omega}}{(j + 1)\omega}, \\
j - 1 \left\langle j - 1 \right. & = \frac{u}{\lambda} \frac{\lambda + u - \omega}{\lambda} \frac{\sqrt{(j + \frac{3}{2})\omega} \sqrt{(j - \frac{1}{2})\omega}}{(j + \frac{1}{2})\omega}, \\
\left. j - 1 \right\rangle & = \frac{\lambda - u}{\lambda} \frac{(2j + 1)\omega - u}{(2j + 1)\omega} + \frac{u}{\lambda} \frac{(2j + \frac{3}{2})\omega - u}{(2j + \frac{3}{2})\omega} \frac{[j\omega]}{(j + 1)\omega}, \\
j - 1 \left\langle j \right. & = \frac{\lambda + u}{\lambda} \frac{2j\omega + u}{2j\omega} - \frac{u}{\lambda} \frac{2j\omega + \lambda + u}{2j\omega} \frac{[j - \frac{1}{2}]\omega}{(j + \frac{1}{2})\omega}.
\end{align*}
\]

\(^3\)We use here the fact that the model corresponding to the vector representation of \( B_1 \) is equivalent to the symmetric tensor of degree 2 \( A_1 \) model.
The Boltzmann weights containing fixed point fields are given by:

\[
\begin{align*}
R' \bigg\lfloor R = R' \bigg\rfloor R^{-1} = \frac{\lambda - u}{\lambda} (j + \frac{1}{2}) \omega - u \bigg\lfloor \frac{u}{\lambda} (j + 1) \omega - u \bigg\rfloor \bigg\lfloor \frac{j \omega}{\lambda} (j + \frac{1}{2}) \omega \bigg\rfloor + \frac{u}{\lambda} (j + 1) \omega \bigg\lfloor \frac{j \omega}{\lambda} (j + \frac{1}{2}) \omega \bigg\rfloor,
\end{align*}
\]

(3.9)

The Yang-Baxter equation may be proved easily if one notes the following relations between the Boltzmann Weights of our model and the corresponding model of [3]. Indeed we have:

\[
\begin{align*}
R' \bigg\lfloor R = R' \bigg\rfloor R^{-1} &= \frac{\lambda - u}{\lambda} (2R - 1) \omega - u \bigg\lfloor \frac{u}{\lambda} (2R - 1) \omega - u \bigg\rfloor \bigg\lfloor \frac{R - 1}{\lambda} \omega \bigg\rfloor + \frac{1}{2} \lambda + u \bigg\lfloor u \omega + u \bigg\rfloor
\end{align*}
\]

(3.10)

\[
\begin{align*}
R \bigg\lfloor R' = R \bigg\rfloor R^{-1} &= \frac{\lambda - u}{\lambda} (2R - 1) \omega - u \bigg\lfloor \frac{u}{\lambda} (2R - 1) \omega - u \bigg\rfloor \bigg\lfloor \frac{R - 1}{\lambda} \omega \bigg\rfloor + \frac{1}{2} \lambda + u \bigg\lfloor u \omega + u \bigg\rfloor
\end{align*}
\]

(3.11)

\[
\begin{align*}
R^{-1} \bigg\lfloor R^{-1} = R^{-1} \bigg\rfloor R^{-1} &= \frac{\lambda + u}{\lambda} \omega - u
\end{align*}
\]

(3.12)

\[
\begin{align*}
R' \bigg\lfloor R = R' \bigg\rfloor R^{-1} &= \frac{\lambda - u}{\lambda} (R + \frac{1}{2}) \omega - u \bigg\lfloor \frac{u}{\lambda} (R + 1) \omega - u \bigg\rfloor \bigg\lfloor \frac{R + 1}{\lambda} \omega \bigg\rfloor + \frac{1}{2} \lambda + u \bigg\lfloor u \omega + u \bigg\rfloor
\end{align*}
\]

(3.13)

The Yang-Baxter equation may be proved easily if one notes the following relations between the Boltzmann Weights of our model and the corresponding model of [3]. Indeed we have:

\[
\begin{align*}
R^{-1} \bigg\lfloor R^{-1} = R^{-1} \bigg\rfloor R^{-1} &= R^{-1} \bigg\lfloor R - 1 \bigg\rfloor R^{-1} + R^{-1} \bigg\lfloor R + 1 \bigg\rfloor R^{-1},
\end{align*}
\]

(3.14)

\[
\begin{align*}
R^{-1} \bigg\lfloor R^{-1} = R^{-1} \bigg\rfloor R^{-1} &= R^{-1} \bigg\lfloor R + 1 \bigg\rfloor R^{-1} + R^{-1} \bigg\lfloor R - 1 \bigg\rfloor R^{-1},
\end{align*}
\]

(3.15)
\[ R^{R-1} R' = \frac{1}{2} (R \bigtriangleup R - R \bigtriangleup R), \quad (3.16) \]

\[ R' \bigtriangleup R' = \frac{1}{2} (R \bigtriangleup R + R \bigtriangleup R), \quad (3.17) \]

\[ R' \bigtriangleup j = \frac{1}{\sqrt{2}} R \bigtriangleup j, \quad j \neq R, R' \quad (3.18) \]

\[ R \bigtriangleup R-1 = \sqrt{2} R \bigtriangleup R-1, \quad (3.19) \]

\[ R' \bigtriangleup R' = R \bigtriangleup R, \quad (3.20) \]

where the Boltzmann Weights in the RHS of the Eqs. (3.14-3.20) are that of the corresponding diagonal model.

Let us consider as an example the following equation:

\[ \sum_p w \left( \begin{array}{c} R \hline p \\ R-2 \hline R-1 \end{array} \right) \times w \left( \begin{array}{c} R \hline R-1 \hline p \end{array} \right) \times w \left( \begin{array}{c} p \hline R \hline R-2 \end{array} \right) = \]

\[ = \sum_p w \left( \begin{array}{c} R \hline R-1 \\ p \hline R-1 \end{array} \right) \times w \left( \begin{array}{c} R-1 \hline R' \hline p \end{array} \right) \times w \left( \begin{array}{c} p \hline R-1 \hline R \end{array} \right), \quad (3.21) \]

Using Eqs. (3.14-3.20) one may show that this equation is equivalent up to a factor $\frac{1}{\sqrt{2}}$ to the difference of the two following Yang-Baxter equations which should hold as was proved in [3]:

\[ \sum_p w \left( \begin{array}{c} R \hline p \\ R-2 \hline R-1 \end{array} \right) \times w \left( \begin{array}{c} R \hline R-1 \hline p \end{array} \right) \times w \left( \begin{array}{c} p \hline R \hline R-2 \end{array} \right) = \]

\[ = \sum_p w \left( \begin{array}{c} R \hline R-1 \\ p \hline R-1 \end{array} \right) \times w \left( \begin{array}{c} R-1 \hline R \hline p \end{array} \right) \times w \left( \begin{array}{c} R-1 \hline p \hline R \end{array} \right), \quad (3.22) \]
\[
\sum_p \sum_{p'} \sum_{v} \sum_{u} \left( \begin{array}{c|c}
R+p & u \\
R-2 & R-1 \\
\end{array} \right) w \left( \begin{array}{c|c}
R+1 & v \\
R & p \\
\end{array} \right) w \left( \begin{array}{c|c}
p & R \\
R-1 & R-2 \\
\end{array} \right) \left( \begin{array}{c|c}
u + v & \\
R & R \\
\end{array} \right) =
\sum_p \sum_{p'} \sum_{v} \sum_{u} \left( \begin{array}{c|c}
R+1 & u + v \\
p & R-1 \\
\end{array} \right) w \left( \begin{array}{c|c}
R & p \\
R-1 & R \\
\end{array} \right) w \left( \begin{array}{c|c}
u & v \\
R & R \\
\end{array} \right),
\] (3.23)

Therefore Eq. (3.21) is obeyed. The rest of the equations may be proved similarly using the relations (3.14-3.20).

It is straightforward to show that the BW of our model enjoy the following properties:

**Initial condition:**

\[w \left( \begin{array}{c|c}
a & b \\
c & d \\
\end{array} \right) \left| 0 \right> = \delta_{bd},\] (3.24)

**Reflection symmetry:**

\[w \left( \begin{array}{c|c}
a & b \\
c & d \\
\end{array} \right) \left| u \right> = w \left( \begin{array}{c|c}
a & b \\
c & d \\
\end{array} \right) \left| -u \right>,\] (3.25)

**Rotational symmetry:**

\[w \left( \begin{array}{c|c}
a & b \\
c & d \\
\end{array} \right) \left| u \right> = \sqrt{G_d G_b G_c G_a} w \left( \begin{array}{c|c}
d & a \\
b & c \\
\end{array} \right) \left| -\lambda - u \right>, \quad G_j \equiv (2 - \delta_{j,R} - \delta_{j,R'})(j + \frac{1}{2} \omega),\] (3.26)

**Two inversion relations:**

\[\sum_g w \left( \begin{array}{c|c}
a & g \\
c & d \\
\end{array} \right) \left| u \right> w \left( \begin{array}{c|c}
a & b \\
c & g \\
\end{array} \right) \left| -u \right> = \delta_{bd} \frac{[\lambda + u][\lambda - u][\omega + u][\omega - u]}{[\lambda]^2 [\omega]^2}.\] (3.27)

\[\sum_g w \left( \begin{array}{c|c}
a & b \\
g & d \\
\end{array} \right) \left| \lambda - u \right> w \left( \begin{array}{c|c}
a & b \\
g & d \\
\end{array} \right) \left| \lambda + u \right> = \delta_{bd} \frac{[\lambda + u][\lambda - u][\omega + u][\omega - u]}{[\lambda]^2 [\omega]^2},\] (3.28)

where

\[\overline{w} \left( \begin{array}{c|c}
a & b \\
c & d \\
\end{array} \right) \left| \lambda - u \right> \overline{w} \left( \begin{array}{c|c}
a & b \\
c & d \\
\end{array} \right) \left| \lambda + u \right> = \delta_{bd} \frac{[\lambda + u][\lambda - u][\omega + u][\omega - u]}{[\lambda]^2 [\omega]^2}.\] (3.29)
4 Discussion

We built the solvable lattice model starting from the rational conformal field theory given by WZW model on \( SO(3)_{4R} = SU(2)_{k=4R}/Z_2 \). Note that although the RCFT we started with was defined for \( R \) even, the solution of the Yang-Baxter equation so obtained is also valid for \( R \) odd.

We note that the result obtained here seems to be closely related to that of [1], where by the orbifold procedure in the context of the lattice models the direct application of the relations similar to Eqs.(3.14-3.20) was implied. In this way many of the known ADE lattice models [12] were found to be related in a simple manner [1].

The construction applied here [1] admits natural generalization for the higher representations (for the fused graphs) as well as the generalization to the higher rank, for example for the models based on \( SU(n)_{k=nR}/Z_m \) (\( m \) divisor of \( n \)) extended current algebra [10].

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APPENDIX

Here we list the explicit expressions for the projectors \( P^{(j)} \) from [9]:

\[
P^{(2)} = 1 - P^{(0)} - P^{(1)}, \quad P^{(i)} P^{(j)} = \delta_{ij} P^{(j)}. \tag{A1}
\]

The matrix elements of \( P^{(0)} \) and \( P^{(1)} \) \( \langle a, b, d | P^{(j)} | a, c, d \rangle \) are given by:

\[
\langle a, b, d | P^{(0)} | a, c, d \rangle = 0, \quad a \neq d, \tag{A2}
\]

\[
\langle j, b, j | P^{(0)} | j, c, j \rangle = \frac{1}{3 [2j + 1]} \begin{pmatrix}
[2j - 1] & \ast & \ast \\
\sqrt{[2j - 1][2j + 1]} & [2j + 1] & \ast \\
\sqrt{[2j - 1][2j + 3]} & \sqrt{[2j + 1][2j + 3]} & [2j + 3]
\end{pmatrix}, \quad b, c = j - 1, j, j + 1 \tag{A3}
\]

\( P^{(1)} \):
\[
\langle j, b, j+1 | P^{(1)} | j, c, j+1 \rangle = \frac{[2]}{[4][2j+2]} \left( \frac{[2j]}{\sqrt{[2j][2j+4]}} \right)^* \left( \frac{[2j+4]}{[2j]} \right)^*, \quad b, c = j, j+1,
\]

(A4)

\[
\langle j, b, j-1 | P^{(1)} | j, c, j-1 \rangle = \frac{[2]}{[4][2j]} \left( -\frac{[2j+2]}{\sqrt{[2j+2][2j-2]}} \right)^* \left( \frac{[2j-2]}{[2j]} \right)^*, \quad b, c = j, j-1,
\]

(A5)

\[
\langle j, b, j | P^{(1)} | j, c, j \rangle = \frac{[2]}{[4]} \left( \frac{1 - \frac{[2]}{[2j][2j+1]}}{\sqrt{[2j+1][2j-1]}} \right)^* \left( \frac{2^j q^{2j+1} + 2^j q^{4j-2}}{[2j][2j+2]} \right)^* \left( \frac{2^j q^{2j+1} + 2^j q^{4j-2}}{[2j][2j+2]} \right)^* \left( 1 - \frac{[2]}{[2j+2][2j+4]} \right)^*, \quad b, c = j-1, j, j+1.
\]

(A6)

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