WASSERSTEIN CONVERGENCE IN BAYESIAN DECONVOLUTION MODELS

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We study the renowned deconvolution problem of recovering a distribution function from independent replicates (signal) additively contaminated with random errors (noise), whose distribution is known. We investigate whether a Bayesian nonparametric approach for modelling the latent distribution of the signal can yield inferences with asymptotic frequentist validity under the $L^1$-Wasserstein metric. When the error density is ordinary smooth, we develop two inversion inequalities relating either the $L^1$ or the $L^1$-Wasserstein distance between two mixture densities (of the observations) to the $L^1$-Wasserstein distance between the corresponding distributions of the signal. This smoothing inequality improves on those in the literature. We apply this general result to a Bayesian approach based on a Dirichlet process mixture of normal distributions as a prior on the mixing distribution (or distribution of the signal), with a Laplace or Linnik noise. In particular we construct an adaptive approximation of the density of the observations by the convolution of a Laplace (or Linnik) with a well chosen mixture of normal densities and show that the posterior concentrates at the minimax rate up to a logarithmic factor. The same prior law is shown to also adapt to the Sobolev regularity level of the mixing density, thus leading to a new Bayesian estimation method, relative to the Wasserstein distance, for distributions with smooth densities.

1. Introduction. In many applied problems of econometrics, biometrics, medical statistics, image reconstruction and signal deblurring data are observed with some random error, see, e.g., [43] for relevant examples. One observes

\[ Y_i = X_i + \varepsilon_i, \quad i \leq n, \quad \varepsilon_i \overset{\text{iid}}{\sim} \mu_\varepsilon, \quad X_i \overset{\text{iid}}{\sim} \mu_X, \]  

where the noise $\varepsilon_i$ is independent of the corresponding signal of interest $X_i$, for $i = 1, \ldots, n$. Estimation of the distribution of the $X_i$’s has received a lot of attention in the literature and is known as deconvolution problem, which is a prototypical linear inverse problem. In this paper, we consider Bayesian nonparametric estimation of $\mu_X$, when the distribution of $\varepsilon$ is known. This is an extensively studied problem, from both the theoretical and methodological perspectives. There is a rich literature on frequentist estimation of both the distribution $\mu_X$ of $X$ and its density $f_X$, with ground breaking papers of the early 90’s based on Fourier inversion techniques to construct estimators of $f_X$, see [7, 18, 14, 13] or [8] using penalized contrast estimators, just to mention a few. For instance, minimax rates for estimating $f_X$ have been studied in [18, 6]. Some recent interesting results have been obtained for the estimation of the distribution of $\mu_X$ in terms of the $L^1$-Wasserstein metric. State of the art results are provided in [12], where minimax estimation rates are derived and a minimum distance estimator of $\mu_X$ is constructed, which achieves these rates.

Keywords and phrases: Adaptation, Deconvolution, Dirichlet process mixtures, Fourier transform, Inversion inequalities, Kantorovich metric, Nonparametric Bayesian inference, Mixtures of Laplace densities, Nonparametric density estimation, Posterior contraction rates, Transport distances, Wasserstein metrics.
While a wide range of frequentist estimators have been studied from a theoretical point of view, little is known about the theoretical properties of Bayesian nonparametric procedures. Contrariwise to frequentist kernel methods where the estimators are explicit, the posterior distribution of $\mu_X$ is not explicit, which renders the analysis much more difficult. A typical way to measure how well the posterior distribution recovers $\mu_X$ is to study posterior contraction rates, i.e., to determine rates $\epsilon_n = o(1)$ such that

\[
\Pi(d(\mu_X, \mu_{0X}) > \epsilon_n \mid Y^{(n)}) = o_P(1)
\]

under model (1.1), where $\mu_{0X}$ is the true mixing distribution, $\Pi(\cdot \mid Y^{(n)})$ is the posterior distribution and $d(\cdot, \cdot)$ is a loss function on probability measures.

If the approach to study posterior convergence rates developed in [23, 24] has proven to be highly successful for a wide range of models and prior distributions, it is not enough to derive sharp bounds on posterior convergence rates for $\mu_X$, since the latter is an inverse problem. To derive posterior convergence rates for $\mu_X$, one typically first obtains posterior convergence rates in the direct problem, i.e., for $\mu_Y = \mu_\varepsilon \ast \mu_X$ or its density $f_Y$, and then combine it with an inversion inequality which translates an upper bound on a distance between $\mu_Y$ and $\mu_\varepsilon$ into an upper bound on $d(\mu_Y, \mu_\varepsilon)$. These two aspects, the direct posterior concentration rate and the inversion inequality, have an interest in themselves. This approach has been used successfully in other contexts by [28, 47] for instance. In this paper we consider the first approach which we believe sheds light on the relation between various interesting metrics in these models.

In [15] the inversion inequality strategy has been used to derive posterior $L^2$-norm convergence rates for the density $f_X$ from $L^2$-norm posterior convergence rates for the density $f_Y$. However, as mentioned in [29], the distribution function of $\mu_X$ itself is also important in practice, so that considering weaker metrics, like Wasserstein metrics, is of interest. In this paper, we are interested in recovering $\mu_X$ in terms of the $L^1$-Wasserstein distance. Over the last decade there has been a growing interest in Wasserstein metrics in statistics and machine learning for both discrete and continuous distributions and we refer to [48], [12] and [45] for discussions on the use of Wasserstein metrics as loss functions for distributions.

In the case of a Laplace noise, [20] derived posterior contraction rates in Wasserstein metrics under the assumption that $X$ has bounded support and [53] extended and improved this result to the unbounded case. However, in both cases the authors obtain suboptimal rates. Recently, [57] proposed a Bayesian nonparametric method to recover $\mu_X$ under heteroscedastic noise and proved consistency of the method, but did not derive convergence rates. In a related problem, [45] has derived Wasserstein posterior contraction rates for the mixing distribution in smooth mixture models. The author obtains the rate $n^{-1/4}$ in the $L^2$-Wasserstein metric for specific distributions $\mu_{0X}$, but this rate is suboptimal for general mixing distributions $\mu_{0X}$.

The construction of Bayesian minimax optimal methods for the estimation of $\mu_X$ in Wasserstein distances remains therefore an open issue. In this paper, we bridge the gap by studying Wasserstein posterior contraction rates under model (1.1), when the noise $\varepsilon$ has ordinary smooth density $f_\varepsilon$ with regularity $\beta \geq 1/2$. Specifically, in Theorem 3.2, we derive a general inversion inequality relating either the total variation or the $L^1$-Wasserstein distance between $\mu_Y$ and $\mu_{0Y}$ to the $L^1$-Wasserstein distance between $\mu_X$ and $\mu_{0X}$. This inversion inequality is sharper than the one obtained by [20] and [53] and allows to derive nearly minimax $L^1$-Wasserstein posterior contraction rates in the case of Laplace noise, see Section 4.1. This inversion inequality is of interest in itself and can also be used to study frequentist estimators.

Another contribution of this paper is the construction of Bayesian adaptive estimation procedures for the density $f_Y$ of $\mu_Y$. Posterior convergence rates for $f_Y$ have been widely studied in the Bayesian nonparametric mixture models literature, but mostly for Gaussian mixtures,
see, e.g., [27] and [52]. When the noise follows a Laplace distribution, [20] and [53] have obtained the rate $n^{-3/8}$ (up to a log $n$-term) in the Hellinger or $L^1$-distance for estimating $f_Y$ using a Dirichlet process mixture on $\mu_X$. As noted by [20], this corresponds to the minimax estimation rate for densities belonging to Sobolev balls with smoothness $\beta = 3/2$, which is the case for $\mu_Y$, as argued in Section 3.1. Under the additional assumption that $\mu_X$ has Lebesgue density $f_X$, we prove in Theorem 4.1 that this rate can be improved to $n^{-2/5}$ and, more generally, that the rate $n^{-\beta/(2\beta+1)}$ can be obtained when the noise follows a Linnik distribution with index $1 < \beta \leq 2$, the Linnik distribution with $\beta = 2$ being the Laplace distribution. We also study the case where $f_X$ is either H"older or Sobolev $\alpha$-regular and obtain a Hellinger convergence rate for $f_Y$ of the order $O(n^{-(\alpha+\beta)/[2(\alpha+\beta)+1]})$ up to a $(\log n)$-term, see Theorem 4.3. To obtain such results, we consider a Dirichlet process mixture of Gaussian densities as a prior on $f_X$. We believe that the approximation theory developed in Section 4.2 to approximate the true density $f_{0Y}$ by $f_\varepsilon \ast f_X$, where $f_X$ is modelled as a mixture of Gaussian densities, is itself of interest.

The paper is organized as follows. In Section 2 we present the set-up and notation used throughout the paper. Section 3 contains the general posterior contraction rate theorem in terms of the $L^1$-Wasserstein distance relative to $\mu_X$ (Section 3.1) together with an inversion inequality (Section 3.2). In Section 4 we apply the general theorem to the case where the noise has a Linnik distribution with index $1 < \beta \leq 2$ and the prior on $f_X$ is a Dirichlet process mixture of Gaussian densities. The proofs of Theorems 3.1 on posterior contraction rates for $L^1$-Wasserstein deconvolution and Theorem 3.2 on the inversion inequality are presented in Section 5. Additional proofs are presented in the Supplement [49] with lemmas, equations and sections referenced with a prefix S, to differentiate them from those of the main paper.

2. Set-up and notation. We observe a sample $Y^{(n)} = (Y_1, \ldots, Y_n)$ from the model $Y_i = X_i + \varepsilon_i$ in (1.1), where the random variables $X_i$ and $\varepsilon_i$ are independent, the noise $\varepsilon_i$ has known distribution $\mu_\varepsilon$ with Lebesgue density $f_\varepsilon$, which is assumed to be ordinary smooth with parameter $\beta > 0$, i.e., its Fourier transform $\hat{f}_\varepsilon$ verifies

\begin{equation}
\label{eq:smoothness}
d_0 |t|^{-\beta} \leq |\hat{f}_\varepsilon(t)| \leq d_1 |t|^{-\beta}, \quad \text{as } t \to \infty,
\end{equation}

for constants $d_0, d_1 > 0$. Examples of ordinary smooth densities include the Laplace ($\beta = 2$) distribution, all Linnik distributions with index $\beta \in (0, 2]$ and the gamma distribution with shape parameter $\beta > 0$.

Let $\mathcal{P}$ stand for the set of all probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\mathcal{P}_0$ for the subset of Lebesgue absolutely continuous distributions on $\mathbb{R}$. Denote by $\mathcal{F}$ the class of probability measures $\mu_Y = \mu_\varepsilon \ast \mu_X$ for $\mu_X \in \mathcal{P}_0$, with density $f_X$. Since $\mu_Y$ is Lebesgue absolutely continuous, we denote by $f_Y = f_\varepsilon \ast f_X$ its density. For any $\mathcal{P}_1 \subseteq \mathcal{P}_0$, let $\mathcal{F}(\mathcal{P}_1)$ stand for the set of probability measures $\mu_Y = \mu_\varepsilon \ast \mu_X$ with $\mu_X \in \mathcal{P}_1$. We consider a prior $\Pi$ on $\mathcal{P}$ and denote by $\Pi_n(\cdot | Y^{(n)})$ the resulting posterior distribution, with

$$\Pi_n(B | Y^{(n)}) = \frac{\int_B \prod_{i=1}^n f_Y(Y_i) \Pi(d\mu_X)}{\int_{\mathcal{P}} \prod_{i=1}^n f_Y(Y_i) \Pi(d\mu_X)}, \quad B \in \mathcal{B}(\mathbb{R}).$$

Our aim is to assess the posterior concentration rate in $L^1$-Wasserstein distance for $\mu_X$, namely, to find a sequence $\epsilon_n = o(1)$ such that, if $Y^{(n)}$ is an $n$-sample from model (1.1) with true mixing distribution $\mu_{0X}$, then

$$\Pi_n(W_1(\mu_X, \mu_{0X}) \leq \epsilon_n | Y^{(n)}) \to 1$$

in $P_{0Y}$-probability, with the $L^1$-Wasserstein distance $W_1(\mu_X, \mu_{0X})$ defined as

$$W_1(\mu_X, \mu_{0X}) := \inf_{\mu\text{-couplings}} \int |X - X'| d\mu(X, X') = \int_\mathbb{R} |F_X(x) - F_{0X}(x)| dx,$$
where $\inf_{\mu\text{-couplings}}$ denotes the infimum over all couplings of $(X, X')$ having $\mu_X$ and $\mu_{0X}$ as marginal distributions, see, e.g., [60]. The last identity, in which $F_X$ and $F_{0X}$ denote the distribution functions of $\mu_X$ and $\mu_{0X}$, respectively, is valid only in dimension one.

We now introduce some notation that will be used throughout the paper. For functions $f, g \in L^1(\mathbb{R})$, let $(f * g)(\cdot) = \int_{\mathbb{R}} f(\cdot - u)g(u) \, du$ be the convolution of $f$ and $g$. We denote by $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ the density of a standard Gaussian random variable and by $\phi_\sigma(x) = \phi(x/\sigma)/\sigma$ its rescaled version. When it exists, $f^{(k)}$ denotes the $k$th derivative of $f$, for any integer $k \geq 0$, with $f^{(0)} = f$.

Let $d_{\text{H}}(f_1, f_2) := \|\sqrt{f_1} - \sqrt{f_2}\|_2$ be the Hellinger distance between densities $f_1$ and $f_2$, where $\|f\|_r$ is the $L^r$-norm of $f$, for $r \geq 1$. For probability measures $P_{0Y}$ and $P_Y$, let $\text{KL}(P_{0Y}; P_Y) := \mathbb{E}_{0Y}[\log(f_{0Y}/f_Y)(Y)]$ be the Kullback-Leibler divergence of $P_Y$ from $P_{0Y}$ and, for $\epsilon > 0$, let

$$B_{\text{KL}}(P_{0Y}; \epsilon^2) = \left\{ P_Y \in \mathcal{P} : \text{KL}(P_{0Y}; P_Y) \leq \epsilon^2, \mathbb{E}_{0Y} \left( \frac{f_{0Y}}{f_Y} \right)^2 \leq \epsilon^2 \right\}$$

be the $\epsilon$-Kullback-Leibler type neighbourhood of $P_{0Y}$.

Let $C_b(S)$ be the set of bounded, continuous real-valued functions on $S \subseteq \mathbb{R}$. For any $\alpha > 0$, let $\mathcal{F}_\alpha(S) = \{ f : f \geq 0, \|f\|_\infty = 1 \}$ and $\mathcal{C}_\alpha(L)$ be the Hölder ball with radius $L > 0$, i.e., the set of functions $f$ on $\mathbb{R}$ that are $\ell := [\alpha] - 1$ times continuously differentiable and such that the $\ell$th derivative satisfies $|f^{(\ell)}(x + \delta) - f^{(\ell)}(x)| \leq L|\delta|^\alpha$, for every $\delta, x \in \mathbb{R}$. In the above notation, let $[x] = \max \{ k \in \mathbb{Z} : k < x \}$ be the lower integer part of $x$. Similarly, we write $[x] = \max \{ k \in \mathbb{Z} : k \geq x \}$ for the upper integer part, $\lfloor x \rfloor = \max \{ k \in \mathbb{Z} : k \leq x \}$ for the integer part and $\{ x \} = x - \lfloor x \rfloor$ for its fractional part when $x \in \mathbb{R}^+ = \{ x \in \mathbb{R} : x \geq 0 \}$.

For $f \in L^1(\mathbb{R})$, let $\hat{f}(t) := \int_{\mathbb{R}} e^{itx} f(x) \, dx$, $t \in \mathbb{R}$, be its Fourier transform. For any function $f$ for which $\int |t|^{\alpha} |\hat{f}(t)| \, dt < \infty$, with $\alpha \geq 0$, define the $\alpha$th fractional derivative of $f$ as $D^\alpha f(x) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(-itx)(-it)^\alpha \hat{f}(t) \, dt$. For $\alpha = 0$, the convention $D^0 f \equiv f$ holds.

For $\epsilon > 0$, let $D(\epsilon, B, d)$ be the $\epsilon$-packing number of a set $B$ with metric $d$, that is, the maximal number of points in $B$ such that the $d$-distance between every pair is at least $\epsilon$, where $d$ can be either the Hellinger or the $L^1$-metric.

We write $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$ and $a_+ = a \vee 0$. Also $a_n \lesssim b_n$ (resp. $a_n \gtrsim b_n$) means that $a_n \leq C b_n$ (resp. $a_n \geq C b_n$) for some $C > 0$ that is universal or depends only on $P_{0Y}$ and $a_n \approx b_n$ means that both $a_n \lesssim b_n$ and $b_n \lesssim a_n$ hold.

### 3. Posterior contraction rates for $L^1$-Wasserstein deconvolution.

In this section we present a general theorem on $L^1$-Wasserstein contraction rates for the posterior measure on the mixing distribution, which is based on properties of the prior law and the true data generating process. A key tool of the proof is an inversion inequality relating the $L^1$-Wasserstein distance $W_1(\mu_X, \mu_{0X})$ between the mixing distributions with the $L^1$-norm distance $\|f_Y - f_{0Y}\|_1$ between the corresponding mixed densities. The inequality is also of interest in itself and is given in Section 3.2.

#### 3.1. A general result on $L^1$-Wasserstein posterior contraction rates.

In order to obtain $L^1$-Wasserstein posterior contraction rates for the latent distribution $\mu_X$, we make assumptions on the “true” mixing distribution $\mu_{0X}$ and the error distribution $\mu_\varepsilon$. If $\mu_\varepsilon$ has Lebesgue density $f_\varepsilon$, then its characteristic function coincides with the Fourier transform of $f_\varepsilon$, denoted by $\hat{f}_\varepsilon$. If $|f_\varepsilon(t)| \neq 0$, $t \in \mathbb{R}$, then the reciprocal of $\hat{f}_\varepsilon$,

$$r_\varepsilon(t) := \frac{1}{\hat{f}_\varepsilon(t)}, \quad t \in \mathbb{R},$$

(3.1)
is well defined. For an \( l \)-times differentiable Fourier transform \( \hat{f}_\varepsilon \), with \( l \in \{0\} \cup \mathbb{N} \), the \( l \)th derivative of \( r_\varepsilon \) is denoted by \( r^{(l)}_\varepsilon \), with \( r^{(0)}_\varepsilon \equiv r_\varepsilon \).

**Assumption 3.1.** The probability measure \( \mu_{0X} \in \mathcal{P}_0 \) has finite first moment \( \mathbb{E}_{0X}[|X|] < \infty \) and possesses Lebesgue density \( f_{0X} \) verifying either one of the following conditions:

(i) there exist \( \alpha > 0 \) and \( L_0 \in L^1(\mathbb{R}) \) such that the derivative \( f^{(\ell)}_{0X} \) of order \( \ell = \lfloor \alpha \rfloor \) exists and satisfies

\[
|f^{(\ell)}_{0X}(x + \delta) - f^{(\ell)}_{0X}(x)| \leq L_0(x)|\delta|^\alpha - \ell, \quad \text{for every } \delta, x \in \mathbb{R},
\]

or

(ii) there exists \( \alpha > 0 \) such that

\[
\int_{\mathbb{R}} |t|^\alpha |\hat{f}_{0X}(t)| \ dt < \infty, \quad D_\alpha f_{0X} \in L^1(\mathbb{R}).
\]

**Assumption 3.2.** The error distribution \( \mu_\varepsilon \in \mathcal{P}_0 \) has finite first moment \( \mathbb{E}[|\varepsilon|] < \infty \) and possesses Lebesgue density \( f_\varepsilon \) with Fourier transform \( |\hat{f}_\varepsilon(t)| \neq 0, t \in \mathbb{R} \). Furthermore, there exists \( \beta > 0 \) such that, for \( l = 0, 1, \)

\[
|r^{(l)}_\varepsilon(t)| \lesssim (1 + |t|)^{\beta-l}, \quad t \in \mathbb{R}.
\]

The discussion of Assumptions 3.1 and 3.2 is postponed to Section 3.2. To state the main theorem, whose proof is reported in Section 5, we need to introduce some definitions. Depending on whether (i) or (ii) of Assumption 3.1 holds true, we consider a kernel of order \( (\lfloor \alpha \rfloor + 1) \) or a supersmooth kernel, respectively. The kernels we consider are functions \( K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) with compactly supported Fourier transforms. More precisely,

(a) in the case of kernels of order \( \ell \), see e.g., [43], pp. 38-39, used under (i) of Assumption 3.1, we have \( \int_{\mathbb{R}} K(z) \ dz = 1 \), while \( \int_{\mathbb{R}} z^j K(z) \ dz = 0 \) for \( j = 1, \ldots, \ell \), with \( \tilde{K} \) supported on \([-1, 1]\);

(b) in the case of supersmooth kernels, used under (ii) of Assumption 3.1, we have \( K \) symmetric satisfying \( \int_{\mathbb{R}} |z| |K(z)| \ dz < \infty \), with \( \tilde{K} \) supported on \([-2, 2]\), while \( \tilde{K} \equiv 1 \) on \([-1, 1]\).

In case (b), a key property is that there exists \( A := 1 + \|K\|_1 < \infty \) such that

\[
\sup_{|t| \neq 0} \frac{|1 - \tilde{K}(t)|}{|t|^\alpha} \leq A, \quad \text{for all } \alpha > 0.
\]

We use the notation \( K_h(\cdot) := (1/h)K(\cdot/h) \) for the rescaled kernel and \( b_{FX} := F_X * K_h - F_X \) for the “bias” of the distribution function \( F_X \) of a probability measure \( \mu_X \), with the proviso that, for \( \alpha > 0 \), the kernel \( K \) is either of order \( (\lfloor \alpha \rfloor + 1) \) or supersmooth, depending on which hypothesis between (i) or (ii) holds true.

**Theorem 3.1.** Let \( \Pi_n \) be a prior distribution on \( \mathcal{P} \). Let \( \mu_{0X} \in \mathcal{P} \) have finite first moment \( \mathbb{E}_{0X}[|X|] < \infty \) and \( \mu_\varepsilon \) satisfy Assumption 3.2 for \( \beta > 0 \). Suppose that, for a positive sequence \( \varepsilon_n \to 0 \), with \( n\varepsilon_n^2 \to \infty \), constants \( c_1, c_2, c_3, c_4 > 0 \) and sets \( \mathcal{P}_n \subseteq \mathcal{P} \), we have

\[
\log D(\mathcal{P}_n, \mathcal{P}(\mathcal{P}_n), d) \leq c_1 n\varepsilon_n^2,
\]

\[
\Pi_n(\mathcal{P} \setminus \mathcal{P}_n) \leq c_3 \exp(-c_2 n\varepsilon_n^2),
\]

\[
\Pi_n(B_{KL}(P_{0Y}, \varepsilon_n^2)) \geq c_4 \exp(-c_2 n\varepsilon_n^2),
\]

where \( D(\cdot, \cdot) \) is the Wasserstein distance (or total variation distance) between two probability measures.
Then, for $\epsilon_n := (\bar{\epsilon}_n \log n)^{1/(\beta+1)}$ and a sufficiently large constant $\bar{K}$,

$$E_{\mu_X}^n [X] < \infty \text{ for } \mu_X \in \mathcal{P}_n.$$ 

If, in addition, $\mu_{0X}$ satisfies Assumption 3.1 for $\alpha > 0$ and there exist constants $C_1$, $\bar{h} > 0$ such that, for every $\mu_X \in \mathcal{P}_n$,

$$\|b_{F_X}\|_1 \leq C_1 h^{\alpha+1} \text{ for all } h \leq \bar{h},$$

then, for $\epsilon_{n,\alpha} := (\epsilon_n \log n)^{(\alpha+1)/(\alpha+(\beta+1))}$ and a constant $K_{\alpha}$ large enough,

$$E_{\mu_X}^n [W_1(\mu_X, \mu_{0X}) > K_{\alpha} \epsilon_{n,\alpha} | Y^{(n)}]] \to 0.$$ 

Application of Theorem 3.1 to specific models gives further insight into this aspect. In Section 4 we consider a Dirichlet process mixture-of-Linnik-normals prior, with Linnik error distribution of index $1 < \beta \leq 2$, and we find the rate $n^{-1/(2\beta+1)}(\log n)^\beta$ when the latent distribution $\mu_{0X}$ is only known to have a density $f_{0X}$ and the rate $n^{-(\alpha+1)/(2(\alpha+\beta)+1)}(\log n)^{\alpha}$ when a Sobolev regularity condition on $f_{0X}$ holds true. These results are expected to be valid in greater generality. If, in fact, $|f_{0X}(t)| \sim (1 + |t|)^{-\beta}$, as $|t| \to \infty$, and $f_{0X} \in \mathcal{F}_\alpha^S$, with $\alpha > 0$, then $f_{0Y} \in \mathcal{F}_{\alpha+\beta}^S$ and the minimax rate in Hellinger or $L^1$-metric for estimating densities with Sobolev regularity $(\alpha + \beta)$ is $n^{-(\alpha+\beta)/(2(\alpha+\beta)+1)}$. This would lead to an $L^1$-Wasserstein posterior convergence rate for $\mu_X$ of the order $O(n^{-(\alpha+1)/(2(\alpha+\beta)+1)})$ (up to a log $n$ term) when $\beta \geq 1$, which reduces to $O(n^{-1/(2\beta+1)})$ when $\mu_{0X}$ is only known to have a density. The latter upper bound matches the lower bound on the convergence rate for the $L^1$-Wasserstein risk obtained by [12] in Theorem 4.1, p. 243, requiring only a moment condition on the latent distribution. The following proposition extends the lower bound result on the rate of convergence for the $L^p$-Wasserstein risk, $p \geq 1$, to the case where the mixing density is either Hölder smooth or Sobolev regular.
PROPOSITION 3.1. Assume that there exists $\beta > 0$ such that, for every $l = 0, 1, 2,$
\begin{equation}
|f^{(l)}(t)| \leq d_l (1 + |t|)^{-(\beta + l)}, \quad t \in \mathbb{R},
\end{equation}
with a constant $d_l > 0.$ There, then, exists a constant $C > 0$ such that, for any estimator $\hat{\mu}_n,$
\begin{equation}
\lim_{n \to \infty} \sup_{\mu_X \in \mathcal{P}_0(M)} \mathbb{E}[W_p^{\mu}(\hat{\mu}_n, \mu_X)] > C,
\end{equation}
where $\mathcal{C}$ stands for either a Hölder $C_\alpha(L)$ or a Sobolev $F_\alpha(L)$ class of densities with $\alpha, L > 0$ and $\mathcal{P}_0(M)$ is the class of probability measures $\mu_X \in \mathcal{P}$ with uniformly bounded $p$th absolute moment $\mathbb{E}_{\mu_X} [\|X\|] \leq M$ for some $M > 0.$

The proof develops along the lines of Theorem 4.1 in [12], appealing to intermediate results in [17] and [18], and is not reported here. Condition (3.9) is stronger than condition (22) of Theorem 4.1, which requires that, for $l = 0, 1, 2,$ the $l$th derivative $|f^{(l)}(t)| \leq c(1 + |t|)^{-\beta},$ $t \in \mathbb{R}.$ For $p = 1,$ the lower bound in (3.10) matches the upper bound of Theorem 3.1 when $\beta \geq 1.$

$L^2$-minimax rates over logarithmic Sobolev classes of densities

When no assumption on the mixing distribution $\mu_{0X}$ is postulated, the $L^2$-minimax rate for estimating a convolution density $f_{0Y} = f_x * \mu_{0X},$ with
\begin{equation}
|\hat{f}_x(t)| \lesssim (1 + |t|)^{-\beta}, \quad t \in \mathbb{R},
\end{equation}
for $\beta > 1/2$ to ensure that $\hat{f}_{0Y} \in L^2(\mathbb{R}),$ is to our knowledge unknown, even if $f_{0Y}$ belongs to a Sobolev type class. Following [30] and [31], for $\gamma > 0,$ $\delta > 1$ and $w_{\gamma, \delta}(t) := (1 + |t|^2)^{\gamma/2} \log(e + |t|))^{-\delta/2},$ $t \in \mathbb{R},$ we define the logarithmic Sobolev class of densities as
\begin{equation}
\mathcal{F}^{LS}_{\gamma, \delta}(L) := \{ f : f \geq 0, \|f\|_1 = 1 \text{ and } \|w_{\gamma, \delta}\hat{f}\|_2^2 \leq L^2 \}, \quad L > 0.
\end{equation}
The Sobolev class of densities $\mathcal{F}^S_{\gamma}(L) := \{ f : f \geq 0, \|f\|_1 = 1 \text{ and } \|(1 + |\cdot|^2)^{\gamma/2}\hat{f}\|_2^2 \leq L^2 \}$ corresponds to $\mathcal{F}^{LS}_{\gamma, 0}(L).$ The following proposition assesses the order, up to a logarithmic factor, of the $L^2$-minimax rate for estimating densities in a logarithmic Sobolev class. Although the result seems to be a well-known fact, we could not find a proof for it and, for completeness, we prove it in Section B.2 of the Supplement [49].

PROPOSITION 3.2. For $\psi_{n, \gamma} := n^{-\gamma/(2\gamma + 1)},$
\begin{equation}
\psi_{n, \gamma}^2 \lesssim \inf_{f_n} \sup_{f \in \mathcal{F}^{LS}_{\gamma, \delta}(L)} \mathbb{E}_{f}[\|\hat{f}_n - f\|_2^2] \lesssim \psi_{n, \gamma}^2 (\log n)^{\delta/(2\gamma + 1)},
\end{equation}
where the infimum is taken over all estimators $\hat{f}_n$ for densities $f$ in $\mathcal{F}^{LS}_{\gamma, \delta}(L)$ based on $n$ observations and the expectation is with respect to the $n$-fold product measure of $P_f.$

If $f_x$ is such that $\hat{f}_x$ satisfies condition (3.11), then, for $\delta > 1,$ extending the definition of $w_{\gamma, \delta}$ to $\gamma < 0,$ we have
\begin{equation}
f_{0Y} \in \mathcal{F}^{LS}_{\beta - 1/2, \delta}(L), \quad \text{with } L = \|w_{-1/2, \delta}\|_2.
\end{equation}
In fact, $\|w_{-\beta - 1/2, \delta}\hat{f}_{0Y}\|_2^2 \leq \|w_{-\beta - 1/2, \delta}\hat{f}_x\|_2^2 \leq \|w_{-1/2, \delta}\|_2^2 < \infty.$ Thus, from Proposition 3.2, the $L^2$-minimax rate over $\mathcal{F}^{LS}_{\beta - 1/2, \delta}(L),$ with $L = \|w_{-1/2, \delta}\|_2,$ is $\psi_{n, \beta - 1/2} = n^{-(\beta - 1/2)/(2\beta)}$ up to a logarithmic factor. For a Laplace error distribution ($\beta = 2$), the rate specializes to $\psi_{n, 3/2} = n^{-3/8},$ which is attained by the Bayes’ estimator $f_{nY}^B,$ defined as the posterior mean.
associated to a Dirichlet process mixture of Laplace densities, so that \( \mathbb{E}_{\mu_Y}^n[d_R^2(f_{nY}^B, f_{0Y})] = O(n^{-3/4}(\log n)) \). Since \( f_{nY}^B = f_\varepsilon \ast \mu_{nX}^B \), using the inversion inequality in (3.13), we find the rate for the (squared) \( L^1 \)-Wasserstein distance \( \mathbb{E}_{\mu_Y}^n[W_1^2(\mu_{nX}^B, \mu_{0X})] = O(n^{-3/8}(\log n)^{3/2}) = o(n^{-1/4}) \), where the \( L^1 \)-Wasserstein rate \( n^{-1/8} \) has been obtained by [20] and [53]. Yet, this rate is larger than the \( L^1 \)-Wasserstein minimax rate \( n^{-1/5} \) pointed out by [12]. It is possible that a Dirichlet process mixture-of-Laplace prior may not achieve the \( L^1 \)-Wasserstein lower bound rate \( n^{-1/5} \), although we cannot rule out that either the Hellinger rate \( n^{-3/8}(\log n)^{1/2} \) obtained by [20] in the direct problem is not sharp or that the inversion inequality is sharp only when the Hellinger or \( L^1 \)-distance between mixture densities is of the order \( O(n^{-\beta/(2\beta+1)}) \). Note that this order is attained for Laplace mixtures when the mixing distribution possesses a density, see Section 4.

3.2. Inversion inequality. In this section we establish, under general and minimal conditions, an inversion inequality relating the \( L^1 \)-distance between mixture densities to the \( L^1 \)-Wasserstein distance between the corresponding mixing distributions, when the error density has Fourier transform decaying polynomially at infinity. This inequality is crucial in the proof of Theorem 3.1. Inequalities of this type have been previously obtained by [46] for the \( L^1 \)-Wasserstein distance and by [20] and [53] for the \( L^1 \)-Wasserstein distance, but they are not as sharp as the one given in Theorem 3.2, whose proof is postponed to Section 5.2. Starting from [12], the idea is to use a suitable kernel to smooth the mixing distributions \( F_X \) and \( F_{0X} \) and then to bound the \( L^1 \)-Wasserstein distance between the smoothed versions, meanwhile controlling the bias induced by the smoothing.

**Theorem 3.2.** Let \( \mu_X, \mu_{0X} \in \mathcal{P} \) be probability measures with finite first moments \( \mathbb{E}_{\mu_X}[|X|] < \infty \) and \( \mathbb{E}_{0X}[|X|] < \infty \). Let \( \mu_\varepsilon \in \mathcal{P}_0 \) satisfy Assumption 3.2 for \( \beta > 0 \). Then, for probability measures \( \mu_Y := \mu_\varepsilon \ast \mu_X, \mu_{0Y} := \mu_\varepsilon \ast \mu_{0X}, \) with densities \( f_Y, f_{0Y} \), and a sufficiently small \( h > 0 \),

\[
W_1(\mu_X, \mu_{0X}) \lesssim h + T,
\]

where

\begin{equation}
T \lesssim W_1(\mu_Y, \mu_{0Y}) + \begin{cases} h^{-(\beta-1)/2} + |\log h|^{1+1/(\beta-1/2)} W_1(\mu_Y, \mu_{0Y}), \\ h^{-(\beta-1)} + |\log h| d(f_Y, f_{0Y}), \end{cases}
\end{equation}

for \( d \) being either the Hellinger or the \( L^1 \)-metric. If, in addition, \( \mu_X \) verifies condition (3.7) and \( \mu_{0X} \) satisfies Assumption 3.1 for \( \alpha > 0 \), then

\begin{equation}
W_1(\mu_X, \mu_{0X}) \lesssim h^{\alpha+1} + T,
\end{equation}

for \( T \) as in (3.13).

We comment on Assumptions 3.1 and 3.2. With Assumption 3.1 we are restricting attention to the set \( \mathcal{P}_0 \) of Lebesgue absolutely continuous probability measures on \( \mathbb{R} \) as mixing distributions. Assumption 3.1 is, in fact, a smoothness/regularity condition on \( f_{0X} \). It requires that either \( f_{0X} \) is locally Hölder smooth, namely, it has \( \ell \) derivatives, for \( \ell \) the largest integer strictly smaller than \( \alpha \), with the \( \ell \)th derivative being Hölder of order \( \alpha - \ell \) and integrable envelope function \( L_0 \) to bound the \( L^1 \)-norm of the bias, or \( f_{0X} \) has (global) Sobolev regularity \( \alpha \). Indeed, requiring that \( D^\alpha f_{0X} \in L^2(\mathbb{R}) \) is equivalent to impose that \( f_{0X} \in \mathcal{F}_0^S \), the difference being that \( D^\alpha f_{0X} \) is here assumed to be in \( L^1(\mathbb{R}) \), see condition (3.3). Hölder and Sobolev classes of densities are common nonparametric families of smooth functions. If \( f_{0X} \) verifies Assumption 3.1, then \( \|b_{f_{0X}}\|_1 = O(h^{\alpha+1}) \), see Lemmas C.1 and
C.2. Note that if the random density is modelled as a Gaussian mixture, $f_X = \mu_H \ast \phi_\sigma$, then
\[ \| b_{f_X} \|_1 \leq C_1 h^{a+1} \| \mu_H \ast \phi_\sigma \|_1 = C_1 h^{a+1} \] for a constant $C_1 > 0$ not depending on $\mu_H$, see Lemma G.1, so that condition (3.7) is verified.

In Assumption 3.2 it is required that $f_\varepsilon$ is everywhere non-null. This is a standard hypothesis in density deconvolution problems related to identifiability with respect to the $L^1(\mathbb{R})$-metric, which is a necessary condition for the existence of (weak) consistent density estimators of $f_{0X}$ with respect to the $L^1(\mathbb{R})$-metric, see [43], pp. 23–26. Finiteness of the first moment of $\varepsilon$ is a technical condition with a two-fold aim. First, jointly with the existence of the first moment of $\mu_{0X}$, it implies that also $\mu_{0Y}$ has finite expected value. This guarantees that $\mu_{0X}$ and $\mu_{0Y}$ have finite $L^1$-Wasserstein distances from $\mu_X$ and $\mu_Y$, respectively, which, in turn, are required to possess finite expectations. Secondly, it implies that $f_\varepsilon$ is continuously differentiable on $\mathbb{R}$ and that the derivative is $f_\varepsilon^{(1)}(t) = \int_{\mathbb{R}} \exp (stu)(u)f_\varepsilon(u)\,du$, $t \in \mathbb{R}$. Then, $r_\varepsilon^{(1)}$ exists and is well defined. Note that, for $l = 0$, condition (3.4) is equivalent to $|f_\varepsilon(t)| \gtrsim (1 + |t|)^{-\beta}$, $t \in \mathbb{R}$. The requirement is satisfied for ordinary smooth distributions covering the following examples.

- The symmetric Linnik distribution with $f_\varepsilon(t) = (1 + |t|^\beta)^{-1}$, $t \in \mathbb{R}$, for index $0 < \beta \leq 2$ and scale parameter equal to 1. The standard Laplace distribution corresponds to $\beta = 2$, see § 4.3 in [36], pp. 249–276.
- The gamma distribution with $f_\varepsilon(t) = (1 - nt)^{-\beta}$, $t \in \mathbb{R}$, for shape parameter $\beta > 0$ and scale parameter equal to 1. The standard exponential distribution corresponds to $\beta = 1$.
- An error distribution with characteristic function $f_\varepsilon$ that is the reciprocal of a polynomial, $r_\varepsilon(t) = \sum_{j=0}^m a_j t^{s_j}$, $t \in \mathbb{R}$, with $a_j \in \mathbb{C}$, $j = 0, \ldots, m$, and exponents $0 \leq s_0 < s_1 < \ldots < s_m = \beta$ for $\beta > 0$. This example extends Example 1 in [2], p. 487, wherein the $s_j$’s are taken to be non-negative integers $s_j = j$, for $j = 0, \ldots, \beta$.
- The error distribution in Example 2 of [2], p. 487, with $f_\varepsilon(u) = \gamma \left[ g_0(u - \mu) + g_0(u + \mu) \right] / 2 + (1 - \gamma) g_0(u)$, $u \in \mathbb{R}$, for a density $g_0$, constants $0 < \gamma < 1/2$ and $\mu \neq 0$, having $f_\varepsilon(t) = [(1 - \gamma) + \gamma \cos(\mu t)] g_0(t)$, $t \in \mathbb{R}$, with $|g_0(t)| \gtrsim (1 + |t|)^{-\beta}$ for $\beta > 0$.

Location and/or scale transformations of random variables with distributions as in the previous examples, as well as their convolutions as in the last example, satisfy condition (3.4), whereas the uniform, triangular and symmetric gamma distributions do not satisfy it. Differently from [12], in condition (3.4) we do not assume that $r_\varepsilon$ is at least twice continuously differentiable. Instead, as in [10], we only assume the existence of the first derivative such that $|r_\varepsilon^{(1)}(t)| \lesssim (1 + |t|)^{-\beta}$, $t \in \mathbb{R}$.

The result of Theorem 3.2 falls within the scope of inversion inequalities, which translate an $L^p$-distance, $p \geq 1$, between kernel mixtures into a proximity measure between the corresponding mixing distributions. A first inequality has been obtained by [45], Theorem 2, p. 377, for ordinary and super-smooth kernel densities in convolution mixtures, see also [32]. A refined version for the ordinary smooth case translating the Hellinger or $L^2/L^1$-distance between mixtures, with kernel density having polynomially decaying Fourier transform, into the $L^1$-Wasserstein distance between mixing distributions with finite Laplace transforms in a neighborhood of zero, has been elaborated by [20] and [53]. A lower bound on the $L^p$-Wasserstein risk, $p \geq 1$, with only a moment condition on the latent distribution, has been obtained by [12], who have also shown that it is attained by a minimum distance estimator $\hat{\mu}_n$. Their proof, however, does not rely on an inversion inequality, but on the explicit expression of $W_1(\hat{\mu}_n, \mu_{0X})$.

When $\beta \geq 1$, we show in Lemma B.1 that the Kullback-Leibler condition in (3.5) implies a posterior concentration rate for the $L^1$-Wasserstein distance $W_1(\mu_Y, \mu_{0Y})$ of the order
$O(\tilde{\epsilon}_n)$, which is equal to the order of $d(f_Y, f_{0Y})$, possibly up to a logarithmic factor. By (3.14), this in turn implies that, with posterior probability tending to one,

$$W_1(\mu_X, \mu_{0X}) \lesssim h^{\alpha+1} + \epsilon_n + h^{-(\beta-1)} + |\log h|\epsilon_n.$$  

Optimizing with respect to $h$ and neglecting logarithmic factors leads to

$$W_1(\mu_X, \mu_{0X}) \lesssim d(f_Y, f_{0Y})^{(\alpha+1)/(\alpha+\beta)}.$$  

Then, $d(f_Y, f_{0Y}) = O(n^{-(\alpha+\beta)/2(\alpha+\beta)+1})$ under the posterior distribution, which yields the minimax-optimal rate $W_1(\mu_X, \mu_{0X}) = O(n^{-(\alpha+1)/2(\alpha+\beta)+1})$, see Proposition 3.1 on the lower bound. One deficit of Theorem 3.2 is that the $d$-version of inequality (3.14) does not satisfactorily cover the case where $0 < \beta < 1$. When applied to this case, in fact, it yields the rate $n^{-(\alpha+\beta)/2(\alpha+\beta)+1}$, times a logarithmic factor, rather than $n^{-(\alpha+1)/2(\alpha+\beta)+1}$. The $W_1$-version of (3.14) would improve the rate to

$$(3.15)$$

$$n^{-(\alpha+1)/[2\alpha+(2\beta\vee 1)+1]},$$

within (at most) a log-factor, provided that $W_1(\mu_Y, \mu_{0Y}) = O(n^{-1/2})$, whose proof, however, remains elusive. The existence of an elbow effect with different rate régimes for $0 < \beta \leq 1/2$ and $\beta > 1/2$ has already been pointed out by [9] and [10] for distribution function estimation. In fact, because of the identity $W_1(\mu_X, \mu_{0X}) = \|F_X - F_{0X}\|_1 = \|F_X^{-1} - F_{0X}^{-1}\|_1$, where $F_X^{-1}$ and $F_{0X}^{-1}$ denote the left-continuous inverse or quantile functions, see, e.g., [56], pp. 64–66, the $L^1$-Wasserstein posterior convergence rate for estimating a latent probability measure coincides with the $L^1$-metric rate for estimating the corresponding distribution or quantile function. On the other hand, the sequence in (3.15) is the minimax rate for estimating single quantiles in deconvolution when the error density $f_\varepsilon$ satisfies Assumption 3.2 for $\beta > 0$ and the mixing density $f_{0X}$ is locally $\alpha$-Hölder continuous for $\alpha > 0$, see Corollary 2.9 of [10], p. 151.

A further remark concerns potential application of the $W_1$-version of (3.14) to bound the $L^1$-Wasserstein risk of a frequentist estimator for the latent distribution.

**Proposition 3.3.** Let $\tilde{\mu}_n$ be an estimator based on $n$ iid observations from a probability measure $\mu_{0Y} = \mu_{\varepsilon}\ast \mu_{0X}$, with $\mu_{0X}$ and $\mu_{\varepsilon}$ satisfying Assumptions 3.1 and 3.2, respectively. If $\tilde{\mu}_n$ is a probability measure such that $\mathbb{E}[W_1(\mu_{\varepsilon}\ast \tilde{\mu}_n, \mu_{0Y})] = O(n^{-1/2})$ up to a logarithmic factor, then, within a log-factor,

$$\mathbb{E}[W_1(\tilde{\mu}_n, \mu_{0X})] = O(n^{-(\alpha+1)/[2\alpha+(2\beta\vee 1)+1]}).$$

**4. Deconvolution by a Dirichlet mixture-of-Linnik-normals prior.** In this section we study the problem of density deconvolution for mixtures with a Linnik error distribution, the Laplace distribution being the most well known example. This family of symmetric distributions was introduced in 1953 by Yu. V. Linnik [41]. Being scale mixtures of normal distributions, see, e.g., [35], Linnik distributions can serve as the one-dimensional distributions of a special subordinated Wiener process used to model the evolution of stock prices and financial indexes. Also, generalized Linnik distributions are good candidates for modelling financial data which exhibit high kurtosis and heavy tails [44].

We use a Dirichlet process mixture-of-normals prior on the mixing density $f_X = \mu_H \ast \phi_\sigma$, so that $f_Y = f_\varepsilon \ast f_X = f_\varepsilon \ast (\mu_H \ast \phi_\sigma)$, with $\mu_H \sim \mathcal{D}_{H_0}$, for some finite, positive measure $H_0$, and $\sigma \sim \Pi_{\sigma}$. We consider the following assumptions on $H_0$ and $\Pi_{\sigma}$.

**Assumption 4.1.** The base measure $H_0$ has a continuous and positive density $h_0$ on $\mathbb{R}$ such that, for constants $b_0$, $c_0 > 0$ and $0 < \delta \leq 1$,

$$h_0(u) = c_0 \exp(-b_0 |u|^\delta), \quad u \in \mathbb{R}.$$
Assumption 4.2. The prior distribution $\Pi_\sigma$ for $\sigma$ has a continuous and positive density $\pi_\sigma$ on $(0, \infty)$ such that, for constants $D_1$, $D_2 > 0$, $s_1$, $s_2$, $t_1$, $t_2 \geq 0$ and $0 < \gamma < \infty$,

$$\sigma^{-s_1} \exp \left( -D_1 \sigma^{-\gamma} |\log \sigma|^{t_1} \right) \leq \pi_\sigma(\sigma) \leq \sigma^{-s_2} \exp \left( -D_2 \sigma^{-\gamma} |\log \sigma|^{t_2} \right)$$

for all $\sigma$ in a neighborhood of 0. Furthermore, for some $0 < \varpi < \infty$, the tail probability $\Pi_\sigma((\sigma, \infty)) \lesssim \sigma^{-\varpi}$ as $\sigma \to \infty$.

Assumption 4.1 on the base measure $H_0$ of the Dirichlet process is analogous to (4.8) in [51], p. 288, and holds true, for example, when the density $h_0$ is proportional to an exponential power distribution with shape parameter $0 < \delta \leq 1$, the Laplace distribution, which corresponds to $\delta = 1$, being the most popular case. Assumption 4.2 on the scale parameter $\sigma$ of the Gaussian kernel has become common in the literature since the articles of [59], [11] and [38], when a full-support prior for $\sigma$ is allowed. An inverse-gamma distribution $\IG(\nu, \gamma)$, with shape parameter $\nu > 0$ and scale parameter $\gamma > 0$, is an eligible prior on $\sigma$ for $s_1 = s_2 = 0$, $t_1 = t_2 = 0$ and $\gamma = 1$. The condition on the tail behaviour at $\infty$ is satisfied for $\varpi = \nu$.

Hereafter, we first study the case in which the sampling density is a location mixture of standard Linnik densities, with mixing distribution that is only known to have density with Sobolev regularity $\alpha$. In the latter case, the prior distribution on the mixing density does not depend on $\alpha$, yet leads to an adaptive posterior contraction rate. We call these two cases as non-adaptive and adaptive, respectively, and treat them separately.

4.1. Non-adaptive case. Let $\Pi$ be the prior law induced by the product measure $\mathcal{D}_{H_0} \otimes \Pi_\sigma$ on the parameter $(\mu_H, \sigma)$ of the random density $f_Y = f_\varepsilon \ast (\mu_H \ast \phi_\sigma)$, for a standard Linnik error density $f_\varepsilon$ with index $1 < \beta \leq 2$. Let $f_{0Y} = f_\varepsilon \ast f_{0X}$ be the Linnik mixture, with mixing density $f_{0X}$ satisfying the following exponential tail decay condition.

Assumption 4.3. The probability measure $\mu_{0X} \in \mathcal{D}_0$ has density $f_{0X}(x) \lesssim e^{-(1+C_0)|x|}$, $x \in \mathbb{R}$, for some constant $C_0 > 0$.

We begin by assessing posterior contraction rates in $L^1$-metric for Linnik convolution mixtures with mixing distributions having exponentially decaying tails.

Theorem 4.1. Let $Y_1, \ldots, Y_n$ be i.i.d. observations from $f_{0Y} := f_\varepsilon \ast f_{0X}$, where $f_\varepsilon$ is the density of a standard Linnik distribution with index $1 < \beta \leq 2$ and $f_{0X}$ satisfies Assumption 4.3. Let $\Pi$ be the prior law induced by $\mathcal{D}_{H_0} \otimes \Pi_\sigma$, where $H_0$ verifies Assumption 4.1 and $\Pi_\sigma$ verifies Assumption 4.2 for $\gamma = 1$. There, then, exist constants $D$ large enough and $\kappa > 0$ so that

$$\Pi_n(\mu_Y : \|f_Y - f_{0Y}\|_1 > Dn^{-\beta/(2\beta+1)}(\log n)^\kappa | Y^{(n)} ) \to 0$$

in $P_{0Y}^n$-probability.

Proof. To prove Theorem 4.1, we verify that assumption (3.5) is satisfied for $\tilde{\varepsilon}_n = n^{-\beta/(2\beta+1)}(\log n)^\tau$, with $\tau > 0$. The Kullback-Leibler condition, third equation of assumption 3.5, is verified in Lemma D.3, which is based on the construction of an approximation of $f_{0Y}$ by $f_\varepsilon \ast (\mu_H \ast \phi_\sigma)$ for a carefully chosen probability measure $\mu_H$. This construction uses the representation of Linnik densities with index $0 < \beta < 2$ as scale mixtures of Laplace densities and adapts the proof of Lemma 2 of [20], pp. 615–616, to obtain an approximation error of the order $O(\tilde{\varepsilon}_n)$. These results are detailed in Lemma D.2. The entropy and remaining mass conditions, first two equations of assumption (3.5), are a consequence of Theorem 5 of [55],
p. 631, because, for any pair of densities \( f_1 \) and \( f_2 \), we have \( \| f_\varepsilon * (f_1 - f_2) \|_1 \leq \| f_1 - f_2 \|_1 \). Finally, since, for \( Z \sim N(0, 1) \),
\[
\mathbb{E}_{\mu_X}[|X|] \leq \mathbb{E}_{\mu_H}[|U|] + \sigma \mathbb{E}[|Z|],
\]
where \( \mu_X \) has density \( \mu_H * \phi_\sigma \), with \( U \sim \mu_H \), and \( \Pi(\mathbb{E}_{\mu_H}[|U|] = \infty) = 0 \), condition (3.6) holds true.

A rate of the order \( O(n^{-\beta/(2\beta+1)}) \), up to a logarithmic factor, is achieved for estimating mixtures of Linnik densities with index \( 1 < \beta \leq 2 \), if a kernel mixture prior on the mixing density is constructed using a Gaussian kernel with an inverse-gamma bandwidth and a Dirichlet process on the mixing distribution. The result is new in Bayesian density estimation and is a preliminary step for \( L^1 \)-Wasserstein density deconvolution.

**Theorem 4.2.** Under the assumptions of Theorem 4.1, there exist constants \( K \) large enough and \( \nu > 0 \) so that
\[
\Pi_n(\mu_X : W_1(\mu_X, \mu_{0X}) > Kn^{-1/(2\beta+1)}(\log n)^\nu \mid Y^{(n)}) \rightarrow 0 \text{ in } P_{Y^n} \text{-probability.}
\]

**Proof.** Under Assumption 4.1, we have \( \int_\mathbb{R} u h_0(u) \, du < \infty \), which implies that, almost surely, the random variable \( U \), with distribution \( \mu_H \sim \mathcal{G}_{H_0} \), has finite first moment. Then, also \( Y = (U + Z) + \varepsilon \), where \( Z \sim N(0, 1) \) and \( \varepsilon \) is a standard Linnik random variable with index \( 1 < \beta \leq 2 \), has finite first moment, see, e.g., Proposition 4.3.18 in [36], p. 267. Apply Theorem 4.1 and Theorem 3.1 to conclude.

**Extension to general scale mixtures of Laplace densities.** The method of proof used in Theorems 4.1 and 4.2 for Linnik error densities with index \( 1 < \beta \leq 2 \) uses the representation of a Linnik density as a scale mixture of Laplace densities of the form
\[
f_\varepsilon(u) = \int_{\mathbb{R}^+} v e^{-v|u|} f_V(v; \beta) \, dv, \quad u \neq 0, \quad f_V(v; \beta) \propto \frac{v^{\beta-1}}{1 + v^{2\beta} + 2v^\beta \cos(\pi \beta/2)}, \quad v > 0.
\]
We can extend the results to more general scale mixtures of Laplace densities \( f_\varepsilon(u) = \int_{\mathbb{R}^+} v e^{-v|u|} f_V(v; \beta) \, dv, \ u \in \mathbb{R} \), with mixing density \( f_V(\cdot; \beta) \) which, for some constant \( \beta > 0 \), satisfies the following conditions:

i) the random variable \( V \) has finite expectation \( \mathbb{E}[V] = \int_{\mathbb{R}^+} v f_V(v; \beta) \, dv < \infty \) and
\[
\int_0^1 \frac{v^2}{\mathbb{E}[V 1_{\{V < v\}}]} f_V(v; \beta) \, dv < \infty,
\]
ii) for every \( t \in \mathbb{R} \),
\[
\int_{\mathbb{R}^+} \frac{v^2}{v^2 + t^2} f_V(v; \beta) \, dv \leq \frac{1}{1 + |t|^\beta},
\]
iii) there exist constants \( C_\beta > 0 \) and \( p_\beta > 2 \), possibly depending on \( \beta \), such that
\[
f_\varepsilon(u) \sim C_\beta |u|^{-p_\beta}, \quad |u| \to \infty,
\]
iv) the Fourier transform \( \hat{f}_\varepsilon \) of \( f_\varepsilon \) verifies condition (3.4) of Assumption 3.2.

Conditions ii)–iv) are the essential ones. Condition i) is a technical requirement that may possibly be due to an artifact of the proof. Granted the assumptions of Theorem 4.1 on \( f_{0X} \) and the prior \( \Pi \), conditions i)–iii) are sufficient for extending the theorem to cover error densities with heavy tails. If, in addition, condition iv) holds true, then also Theorem 4.2 goes through. These results are not separately stated here.
For another application of Theorem 4.2, consider the case where $Z$ has a monotone non-increasing density $f_{Z}$ on $\mathbb{R}^+$. Following [61], it is known that $Z$ is a scale mixture of uniforms so that $Y = \log Z = X + \varepsilon$, where $\varepsilon \sim \text{Exp}(1)$ and is independent of $X$. If the distribution of $X$ satisfies the assumptions of Theorem 4.2, then the posterior distribution of $\mu_X$ concentrates around the true $\mu_{0X}$ at rate $n^{-1/3}$, up to a $(\log n)$-term, in the $L^1$-Wasserstein metric. Writing $f_{Z}(z) = \int_{\mathbb{R}^+}(1_{[z,\infty)})/v \, dF(v)$ and $f_{0Z}(z) = \int_{\mathbb{R}^+}(1_{[z,\infty)})/v \, dF_0(v)$, we have

$$
\Pi_n(d(F, F_0) > Mn^{-1/3}(\log n)^\nu \mid Z^{(n)}) = o_p(1), \quad \text{with } d(F, F_0) := \int_{\mathbb{R}^+} \frac{|F(v) - F_0(v)|}{v} \, dv.
$$

4.2. Sobolev-regularity adaptive case. In this section we focus on the case in which the sampling density is a mixture of Laplace densities with a Sobolev regular mixing distribution. The Laplace case is worked out in detail, whereas the analogous results for Linnik mixtures are briefly mentioned to avoid duplications.

We still consider $\Pi$ the prior law induced by $\mathcal{D}_{H_0} \otimes \Pi_\sigma$ on the parameter $(\mu_H, \sigma)$ of $f_Y = f_\varepsilon \ast (\mu_H \ast \phi_\sigma)$, for a standard Laplace error density $f_\varepsilon$, and $\Pi_n(\cdot \mid Y^{(n)})$ the posterior distribution based on i.i.d. observations $Y_1, \ldots, Y_n$ drawn from $f_Y$, with $f_Y \sim \Pi$. The sampling density $f_{0Y} = f_\varepsilon \ast f_{0X}$ is assumed to be a Laplace mixture, with mixing density $f_{0X}$ satisfying the following conditions.

**Assumption 4.4.** There exists $\alpha > 0$ such that

$$
\forall b = \mp 1/2, \quad \int_{\mathbb{R}} |t|^{2\alpha} |e^b f_{0X}(t)|^2 \, dt < \infty.
$$

**Assumption 4.5.** There exist $0 < \nu \leq 1$, $L_0 \in L^1(\mathbb{R})$ and $R \geq m/\nu$, for the smallest integer $m \geq (\alpha + 2)$, such that $f_{0X}$ satisfies

$$
|f_{0X}(x + \zeta) - f_{0X}(x)| \leq L_0(x)|\zeta|^\nu, \quad \text{for every } \zeta, x \in \mathbb{R},
$$

and

$$
\int_{\mathbb{R}} e^{\nu|x|/2} f_{0X}(x) \left( \frac{L_0(x)}{f_{0X}(x)} \right)^R \, dx < \infty.
$$

Assumption 4.4 requires that, for every $b = \mp 1/2$, the function $e^b f_{0X}$ belongs to a Sobolev space of order $\alpha$, while Assumption 4.5 requires that $f_{0X}$ is locally Hölder smooth of order $\nu$, with envelope function $L_0$ satisfying the integrability condition (4.2). The model $f_Y = f_\varepsilon \ast (\mu_H \ast \phi_\sigma)$ acts as an approximation scheme for automatic posterior rate adaptation to the global regularity of $f_{0Y}$, without any knowledge of the regularity of $f_{0X}$ being used in the prior specification. We show that a rate-adaptive estimation procedure for Laplace mixtures can be obtained if the prior is properly constructed, for instance, as a mixture of Laplace-normal convolutions, with an inverse gamma bandwidth and a Dirichlet process on the mixing distribution.

**Theorem 4.3.** Let $Y_1, \ldots, Y_n$ be i.i.d. observations from $f_{0Y} := f_\varepsilon \ast f_{0X}$, where $f_\varepsilon$ is the density of a standard Laplace distribution and $f_{0X}$ satisfies Assumption 4.3, Assumption 4.4 for $\alpha > 0$ and Assumption 4.5. Let $\Pi$ be the prior law induced by $\mathcal{D}_{H_0} \otimes \Pi_\sigma$, where $H_0$ verifies Assumption 4.1 and $\Pi_\sigma$ verifies Assumption 4.2 for $\gamma = 1$. There then exist constants $D'$ large enough and $\kappa' > 0$ so that

$$
\Pi_n(\mu_Y : \|f_Y - f_{0Y}\|_1 > D'n^{-(\alpha+2)/(2\alpha+5)}(\log n)^\kappa' \mid Y^{(n)}) \to 0 \text{ in } P_{0Y}^n\text{-probability.}
$$
The key step of the proof of Theorem 4.3, which is deferred to Section 5, is a novel approximation result reported in Lemma 5.1.

By combining Theorem 4.3 with Theorem 3.1 and Lemma G.1, we obtain adaptive posterior contraction rates for $L^1$-Wasserstein density deconvolution of Laplace mixtures.

**Theorem 4.4.** Granted the assumptions of Theorem 4.3, there exist constants $K'$ large enough and $\nu' > 0$ so that

$$\Pi(\mu_Y : W_1(\mu_X, \mu_{0X}) > K'n^{-(\alpha+1)/(2\alpha+5)}(\log n)^{\nu'} | Y(n)) \to 0 \text{ in } P_{0Y}^n \text{-probability.}$$

Using the fact that every Linnik density with index $0 < \beta < 2$ admits a representation as a scale mixture of Laplace densities, together with the arguments used for proving Theorems 4.1 and 4.2, the results of Theorems 4.3 and 4.4 can be extended to mixtures of Linnik densities with index $1 < \beta < 2$ to obtain adaptive rates $n^{-(\alpha+\beta)/[2(\alpha+\beta)+1]}$ and $n^{-(\alpha+1)/[2(\alpha+\beta)+1]}$, respectively, up to logarithmic factors. Further extension to general scale mixtures of Laplace densities can be pursued using the conditions listed in Section 4.1.

**5. Proofs.**

5.1. **Proof of Theorem 3.1.** By the conditions in (3.5), Theorem 2.1 of [23], p. 503, implies that, for sufficiently large $M$,

$$\mathbb{E}_{0Y}^n[\Pi_n(\mu_X : d(f_Y, f_{0Y}) > \tilde{M}\varepsilon_n | Y(n))] \to 0$$

and, as a by-product, that $\mathbb{E}_{0Y}^n[\Pi_n(\mathcal{P}^c_n | Y(n))] \to 0$. Next, the case where Assumption 3.1 holds true is treated in details. By conditions (3.6) and (3.7), Theorem 3.2 implies that $W_1(\mu_X, \mu_{0X}) \lesssim h^{\alpha+1} + W_1(\mu_Y, \mu_{0Y}) + h^{-(\beta-1)}, d(f_Y, f_{0Y}) \log(1/h)$ with constants that are uniform over $\mathcal{P}_n$. Minimizing with respect to $h$, we get

$$W_1(\mu_X, \mu_{0X}) \lesssim W_1(\mu_Y, \mu_{0Y}) + [d(f_Y, f_{0Y}) \log(1/h)]^{(\alpha+1)/[\alpha+(\beta+1)]},$$

where $\alpha + (\beta + 1) = \alpha + 1 + (\beta - 1)$. For sufficiently large $M > 0$, defined the set $\mathcal{S}_n := \{\mu_X : W_1(\mu_Y, \mu_{0Y}) \leq M\varepsilon_n\}$, under the third listed condition in (3.5), by Lemma B.1, we have $\mathbb{E}_{0Y}^n[\Pi_n(\mathcal{S}^c_n | Y(n))] \to 0$. Define

$$\omega_n := \sup_{\mu_X \in (\mathcal{S}_n \cap \mathcal{S}_n) : d(f_Y, f_{0Y}) \leq \tilde{M}\varepsilon_n} W_1(\mu_X, \mu_{0X}),$$

we have $\omega_n \lesssim \varepsilon_n + \varepsilon_{n,\alpha} \lesssim \varepsilon_{n,\alpha}$. Reasoning as in Theorem 2.1 of [34], p. 2094, we have $\mathbb{E}_{0Y}^n[\Pi_n(\mu_X : W_1(\mu_X, \mu_{0X}) > K_\alpha\varepsilon_{n,\alpha} | Y(n))] \leq \mathbb{E}_{0Y}^n[\Pi_n(\mu_X : W_1(\mu_X, \mu_{0X}) > \omega_n | Y(n))] \leq \mathbb{E}_{0Y}^n[\Pi_n(\mu_X \in (\mathcal{S}_n \cap \mathcal{S}_n) : d(f_Y, f_{0Y}) > M\varepsilon_n | Y(n))] + \mathbb{E}_{0Y}^n[\Pi_n(\mathcal{S}_n^c | Y(n))] \to 0$ and the convergence in (3.8) follows. The case where no assumption on $\mu_{0X}$ is required, except for the first moment condition, follows similarly from the inversion inequality with $h$ in lieu of $h^{n+1}$. 

5.2. **Proof of Theorem 3.2.** Because $\mu_X$ and $\mu_{0X}$ have finite first moments, $W_1(\mu_X, \mu_{0X}) < \infty$, see, e.g., [60], p. 94. Since $\mathbb{E}[|\varepsilon|] < \infty$, also $\mu_Y$ and $\mu_{0Y}$ have finite first moments. Thus, $W_1(\mu_Y, \mu_{0Y}) < \infty$.

In what follows, we distinguish the case where no assumption, except for the first moment existence condition, is imposed on $\mu_{0X}$, from the case where Assumption 3.1 is in force.

**Case 1: No smoothness assumption on $\mu_{0X}$**

The kernel $K$ can be taken to be a symmetric probability density with finite first moment
\[ \int_{\mathbb{R}} |z| K(z) \, dz < \infty. \] For \( h > 0 \), let \( F_{K_h}(z) := \int_{-\infty}^{\infty} K_h(s) \, ds, \, z \in \mathbb{R} \), be the cumulative distribution function of the rescaled kernel \( K_h \) and let \( \mu_{K_h} \) be its probability law. Let \( Z \) be a random variable with distribution \( \mu_{K_h} \) and \( X_1 \) a random variable, independent of \( Z \), with distribution \( \mu_{0X} \). Then, \( Z + X_1 \sim \mu_{K_h} * \mu_{0X} \) and, since \( \mathbb{E}[|Z|] = h \int_{\mathbb{R}} |z| K(z) \, dz < \infty \), we have \( \mathbb{E}[|Z + X_1|] \leq \mathbb{E}[|Z|] + \mathbb{E}[|X_1|] < \infty \) so that \( W_1(\mu_{K_h} * \mu_{0X}, \mu_{0X}) < \infty \). By definition of the \( L^1 \)-Wasserstein distance, \( W_1(\mu_{K_h} * \mu_{0X}, \mu_{0X}) \leq \mathbb{E}[(Z + X_1) - X_1] = \mathbb{E}[|Z|] \leq h. \) Analogously, \( W_1(\mu_{X_1}, \mu_{K_h} * \mu_{X_1}) \leq h. \) Then,

\[
W_1(\mu_X, \mu_{0X}) \leq W_1(\mu_X, \mu_{K_h} * \mu_X) + W_1(\mu_{K_h} * \mu_X, \mu_{K_h} * \mu_{0X}) + W_1(\mu_{K_h} * \mu_{0X}, \mu_{0X})
\]

\[
(5.1) \leq h + W_1(\mu_{K_h} * \mu_X, \mu_{K_h} * \mu_{0X}).
\]

- **Case 2:** Smoothness Assumption 3.1 on \( \mu_{0X} \) holds true

If condition (i) of Assumption 3.1 is in force, then \( K \) is taken to be an \((\lfloor \alpha \rfloor + 1)\)-order kernel satisfying, in addition, \( \int_{\mathbb{R}} |z|^{\lfloor \alpha + 1 \rfloor} |K(z)| \, dz < \infty \). For every \( 0 \leq \alpha \leq 2 \), the kernel can be a density and the same reasoning as for Case 1, leading to an analogue of inequality (5.1), could be adopted. However, since, for \( \alpha > 2 \), no non-negative function can be a higher-order kernel, some adjustments are required. We therefore lay out arguments that do not rely on the fact that \( K \) is a density. If, instead, condition (ii) of Assumption 3.1 is in force, then \( K \) is taken to be a superkernel such that \( K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) is symmetric, with \( \int_{\mathbb{R}} |z| |K(z)| \, dz < \infty \) and \( \tilde{K} \equiv 1 \) on \([-1, 1]\), while \( \tilde{K} \equiv 0 \) on \([-2, 2]\).

Let \( F_X \) and \( F_{0X} \) denote the distribution functions of \( \mu_X \) and \( \mu_{0X} \), respectively. Recalling that \( W_1(\mu_X, \mu_{0X}) = \| F_X - F_{0X} \|_1 < \infty \), by the triangular inequality,

\[
W_1(\mu_X, \mu_{0X}) = \| F_X - F_{0X} \|_1 \leq \| F_{K_h} * \mu_X - F_X \|_1 + \| F_{K_h} * (\mu_X - \mu_{0X}) \|_1 + \| F_{K_h} * \mu_{0X} - F_{0X} \|_1,
\]

where all terms on the right-hand side of the last line are finite. In fact, for suitable \( \gamma \in (0, 1) \), we have \( \| F_{K_h} * \mu_X - F_{0X} \|_1 = \| F_{0X} * K_h - F_{0X} \|_1 = \| F_{0X} * K_h - F_{0X} \|_1 = \| \int_{\mathbb{R}} z \, dx \, dz = h \int_{\mathbb{R}} \| K_h(z) \| \, dx \, ds < \infty. \) By the same reasoning, also \( \| F_X - F_{K_h} * \mu_X \|_1 = \| F_X - K_h - F_X \|_1 < \infty \). Besides, by Young’s inequality \( \| f \cdot g \|_p \leq \| f \|_1 \| g \|_p \) valid for every \( f \in L^p(\mathbb{R}) \) and \( g \in L^p(\mathbb{R}), 1 \leq p \leq \infty \), we have \( \| F_{K_h} * (\mu_X - \mu_{0X}) \|_1 = \| (F_X - F_{0X}) * K_h \|_1 \leq \| F_X - F_{0X} \|_1 \| K_h \|_1 = \| K_h \|_1 = W_1(\mu_X, \mu_{0X}) < \infty. \) From condition (3.7) on \( \| b_{F_X} \|_1 \) and Assumption 3.1 on \( f_{0X} \), jointly with Lemma C.1 or Lemma C.2 on \( \| b_{F_{0X}} \|_1 \), we obtain that

\[
W_1(\mu_X, \mu_{0X}) \leq h^{\alpha + 1} + T, \quad T = W_1(\mu_{K_h} * \mu_X, \mu_{K_h} * \mu_{0X}).
\]

We show that, for sufficiently small \( h \), inequality (3.13) holds true in Cases 1 and 2. Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a symmetric, continuously differentiable function, equal to 1 on \([-1, 1]\) and to 0 outside \([-2, 2]\). For example, one such function can be defined as \( \chi(t) = e^{\exp \{-1/[1 - (|t| - 1)^2]\}} \) for \( |t| \in (1, 2) \). For a unified treatment of the cases where \( K \) is an \((\lfloor \alpha \rfloor + 1)\)-order kernel or a supersmooth kernel, we set

\[
\tilde{h} := \begin{cases} h, & \text{if } K \text{ is an } (\lfloor \alpha \rfloor + 1)\text{-order kernel}, \\ h/2, & \text{if } K \text{ is a supersmooth kernel}. \end{cases}
\]

Analogously, we set \( \tilde{\gamma} \) equal to 1 if \( K \) is an \((\lfloor \alpha \rfloor + 1)\)-order kernel and equal to 1/2 if \( K \) is a supersmooth kernel. For \( h > 0 \), define

\[
w_{1,h}(t) := \tilde{K}(ht) \chi(t) r_\varepsilon(t), \quad t \in \mathbb{R}, \quad \text{and} \quad w_{2,h}(t) := \tilde{K}(ht) [1 - \chi(t)] r_\varepsilon(t), \quad t \in \mathbb{R}.
\]

Note that \( K \in L^1(\mathbb{R}) \) implies that \( \tilde{K} \) is well-defined and \( \| \tilde{K} \|_1 := \sup_{t \in \mathbb{R}} \| \tilde{K}(t) \|_1 < \infty. \) Since \( \tilde{K}(\cdot) \in C_b([-1, 1]) \), we have \( \tilde{K} \in L^1(\mathbb{R}) \) and \( K(\cdot) = (2\pi)^{-1} \int_{\mathbb{R}} \exp(-ut^2) \tilde{K}(t) \, dt. \)

If $h < 1/2$, the function $w_{1,h}$ is equal to 0 outside $[-2, 2]$, while $w_{2,h}$ is equal to 0 on $[-1, 1]$ and outside $[-1/h, 1/h]$. Thus, $w_{j,h} \in L^1(\mathbb{R})$, for $j = 1, 2$. In fact, by inequality (3.4) with $l = 0$, we have $\|w_{1,h}\|_1 \leq \int_{|t| \leq 2} |\hat{K}(ht)| |\chi(t)|(1 + |t|)^{\beta} \, dt < \infty$ because the integrand is in $C_b([-2, 2])$. Analogously, $\|w_{2,h}\|_1 \leq \int_{1 < |t| \leq 1/h} |\tilde{K}(ht)||1 - \chi(t)|(1 + |t|)^{\beta} \, dt < \infty$. Then, the inverse Fourier transform of $w_{j,h}$,

$$z \mapsto K_{j,h}(z) := \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-itz)w_{j,h}(t) \, dt,$$

is well defined and, as a consequence of Lemma C.3 and Lemma A.1 in Section A, is in $L^1(\mathbb{R})$, for $j = 1, 2$. We can then define the mappings

$$z \mapsto F_{j,h}(z) := \int_{-\infty}^{z} K_{j,h}(s) \, ds, \quad j = 1, 2.$$

Using the decomposition $\hat{K}(ht)r_{\varepsilon}(t) = w_{1,h}(t) + w_{2,h}(t)$, $t \in \mathbb{R}$,

$$[K_h * (\mu_X - \mu_{0,X})](y) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-it\varepsilon)\hat{K}(ht)\hat{r}_{\varepsilon}(t)(\hat{f}_Y - \hat{f}_0Y)(t) \, dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-it\varepsilon)[w_{1,h}(t) + w_{2,h}(t)](\hat{f}_Y - \hat{f}_0Y)(t) \, dt$$

$$= [K_{1,h} * (\mu_Y - \mu_{0,Y})](y) + [K_{2,h} * (\mu_Y - \mu_{0,Y})](y), \quad y \in \mathbb{R},$$

and we can write $[F_{K_h} * (\mu_X - \mu_{0,X})] = [K_{1,h} * (F_Y - F_{0Y})] + [K_{2,h} * (F_Y - F_{0Y})] = [K_{1,h} * (F_Y - F_{0Y})] + [F_{2,h} * (\mu_Y - \mu_{0,Y})]$, where $F_Y$ and $F_{0Y}$ denote the distribution functions of $\mu_Y$ and $\mu_{0Y}$, respectively. Then,

$$T \leq \|K_{1,h} * (F_Y - F_{0Y})\|_1 + \|K_{2,h} * (F_Y - F_{0Y})\|_1$$

$$= \|K_{1,h}\|_1 \times W_1(\mu_Y, \mu_{0Y}) \lesssim W_1(\mu_Y, \mu_{0Y})$$

because $\|K_{1,h}\|_1 = O(1)$ in virtue of Lemma C.3.

- **Study of the term $T_1$**

  By Young’s inequality,

  $$T_1 := \|K_{1,h} * (F_Y - F_{0Y})\|_1 \leq \|K_{1,h}\|_1 \times \|F_Y - F_{0Y}\|_1$$

  $$= \|K_{1,h}\|_1 \times W_1(\mu_Y, \mu_{0Y}) \lesssim W_1(\mu_Y, \mu_{0Y})$$

- **Study of the term $T_2$**

  For every $\beta > 0$, by Young’s inequality and Lemma A.1 in Appendix A,

  $$T_2 := \|K_{2,h} * (F_Y - F_{0Y})\|_1 \leq \|K_{2,h}\|_1 \times W_1(\mu_Y, \mu_{0Y})$$

  $$\lesssim h^{-(\beta - 1/2)} \log h^{1 + 1/(\beta - 1/2)} W_1(\mu_Y, \mu_{0Y}).$$

  Analogously, for every $\beta > 0$, using Lemma A.2 in Appendix A,

  $$T_2 := \|F_{2,h} * (\mu_Y - \mu_{0Y})\|_1 \leq \|F_{2,h}\|_1 \times \|f_Y - f_{0Y}\|_1 \lesssim h^{-(\beta - 1)} \log h \|f_Y - f_{0Y}\|_1.$$

  Combining the bounds on $T_1$ and $T_2$, we obtain inequality (3.13), which, together with (5.2), proves the inversion inequality. The statement for the Hellinger distance follows from LeCam’s inequality $\|f_Y - f_{0Y}\|_1 \leq 2d_{\text{H}}(f_Y, f_{0Y})$, see [40], p. 40. The proof is thus complete. \qed
5.3. **Proof of Theorem 4.3.** To obtain a suitable approximation of \( f_{0Y} = f_{\varepsilon} \ast f_{0X} \), we need an auxiliary result concerning the approximation of a smooth function by convolutions. For \( h > 0 \), let

\[
H(x) := \frac{1}{2\pi} \hat{\tau}(x) e^{-(hx)^2/2}, \quad x \in \mathbb{R},
\]

where \(|\hat{\tau}(x)| \leq (16^2/15) e^{-\sqrt{|x|}/15}, x \in \mathbb{R}\), is the Fourier transform of \( \tau : \mathbb{R} \to [0, 1] \) defined in Theorem 25 of [5], p. 29, such that

\[
\tau(u) = \begin{cases} 
1, & \text{if } |u| < 1, \\
0, & \text{if } |u| > 17/15.
\end{cases}
\]

The function \( \tau \) is such that

\[
(5.3) \quad \text{for any } i \in \{0\} \cup \mathbb{N}, \quad |\hat{\tau}^{(i)}(x)| = O(|x|^{-\nu}) \quad \text{for large } x \in \mathbb{R} \text{ and every } \nu > 0.
\]

Given \( m \in \mathbb{N}, b = \mp 1/2, \delta, \sigma > 0 \) and a function \( f : \mathbb{R} \to \mathbb{R} \), we define the transform

\[
T_{m,b,\sigma} f := f + \sum_{k=1}^{m-1} (\mp 1)^k \sigma 2^k k! \sum_{j=0}^{2k} \binom{2k}{j} (-b)^{2k-j} [f * (e^{-b \cdot D^j H_\delta})],
\]

where \( H_\delta(\cdot) := \delta^{-1} H(\cdot/\delta) \). We have \( M_{0X}(b) < \infty \). For

\[
(5.4) \quad \bar{h}_{0,b} := \frac{e^b f_{0X}}{M_{0X}(b)}
\]

and \( \gamma := -(1 - e^{-\sigma^2/8}) \), let

\[
(5.5) \quad h_{m,b,\sigma} := \frac{1}{\gamma} \sum_{k=1}^{m-1} (\mp 1)^k \sigma 2^k k! \sum_{j=0}^{2k} \binom{2k}{j} (-b)^{2k-j} (\bar{h}_{0,b} * D^j H_\delta).
\]

Note that \([M_{0X}(b)]^{-1} f_{m,b,\sigma} f_{0X} = \bar{h}_{0,b} + \gamma h_{m,b,\sigma} \).

**Lemma 5.1.** Let \( f_{\varepsilon} \) be the standard Laplace density. Let \( f_{0X} \) be a density such that \( (e^{1/2} f_{0X}) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and which satisfies Assumption 4.4 for \( \alpha > 0 \). Then, for \( m \geq (\alpha + 2) \) and \( \sigma > 0 \) small enough,

\[
(5.6) \quad \sum_{b = \mp 1/2} \| e^b \{ f_{\varepsilon} * (T_{m,b,\sigma} f_{0X}) - f_{0X} \} \|_2^2 \lesssim \sigma^{2(\alpha + 2)}
\]

and

\[
(5.7) \quad \forall b = \mp 1/2, \quad \int_{\mathbb{R}} h_{m,b,\sigma}(x) \, dx = 1 + O(\sigma^{m-2}).
\]

**Proof.** We first obtain an equivalent expression for the \( L^2 \)-norm in (5.6). Denoting the Fourier transform operator by \( \mathcal{F} \), for any \( f \in L^1(\mathbb{R}) \), we have \( \mathcal{F}\{ f \} := \hat{f} \). Recall that, for \( f_{\varepsilon}(u) = e^{-|u|}/2, u \in \mathbb{R} \), we have \( \mathcal{F}\{ f_{\varepsilon} \} (t) = [1 - g_\delta(t)], \) where \( g_\delta(t) := [1 - \psi_\delta^2(t)] \) and \( \psi_\delta(t) := -(t + b), t \in \mathbb{R} \). Noting that \([M_{0X}(b)]^{-1} \mathcal{F}\{ e^b(T_{m,b,\sigma} f_{0X}) \} = \mathcal{F}\{ \bar{h}_{0,b} \} + \gamma \mathcal{F}\{ h_{m,b,\sigma} \}, \) where \( M_{0X}(b) < \infty \) for \( b = \mp 1/2 \) by the assumption \((e^{1/2} f_{0X}) \in L^1(\mathbb{R}) \), we get that

\[
\Delta_0 := \sum_{b = \mp 1/2} \| e^b \{ f_{\varepsilon} * (T_{m,b,\sigma} f_{0X}) - f_{0X} \} \|_2^2
\]
Decomposing \( \parallel (5.8) \parallel \) we have that

\[ |\mathcal{F}\{H\}(\delta) - |\mathcal{F}\{H\}(\delta) - \mathcal{F}\{H\}(\delta)\parallel_1 < \infty. \]

Besides, as \( 0 \leq \tau \leq 1, \)

\[ |\mathcal{F}\{H\}(\delta)| = |(\tau * \phi_h)(-\delta t)| \leq \|\phi_h(-\delta t - \cdot)\|_1 = \|\phi_{-\delta t,h}\|_1 = 1, \quad t \in \mathbb{R}. \]

Let \( Z \) be a standard normal random variable. For constants \( 0 < c_\delta, c_h < 1, \) take \( \delta := c_\delta \sigma \) and \( h := c_h |\log \sigma|^{-1/2}. \) Fix \( u_0 \) such that \( 0 < c_\delta < u_0 < 1. \) Then, for \( \omega > 0 \) and \( c_h \) such that \( (1 - u_0) \geq c_h \sqrt{2\omega}, \) we have \( |t| \leq u_0/\delta, \)

\[ |1 - \mathcal{F}\{H\}(\delta)| \leq 2 \int_{|u| \geq 1} \phi_{-\delta t,h}(u) \, du \leq 2P(|Z| \geq (1 - \delta|t|)/h) \]

(5.10)

\[ \leq 2P(|Z| \geq (1 - u_0)|\log \sigma|^{1/2}/c_h) \lesssim \omega \]

as soon as \( \sigma \) is small enough. For every \( j \in \{0\} \cup \mathbb{N}, \) we have \( \mathcal{F}\{D^j H_b\}(t) = (-it)^j \mathcal{F}\{H\}(\delta t), \)

\( t \in \mathbb{R}. \) Then, recalling that \( \psi_h(t) = -(it + b), \)

\[ \forall b = \pm 1/2, \quad \mathcal{F}\{h_{m,b,\sigma}\}(t) = \frac{1}{\gamma} \mathcal{F}\{h_{0,b}\}(t) \mathcal{F}\{H\}(\delta t) \sum_{k=1}^{m-1} \frac{(-1)^k [\sigma \psi_h(t)]^{2k}}{2k!}, \quad t \in \mathbb{R}. \]

Decomposing \( \mathcal{F}\{h_{0,b}\}(t) \) by means of \( \mathcal{F}\{H\}(\delta t) \) and \( |1 - \mathcal{F}\{H\}(\delta)|, \) the numerator of the integrand of \( \Delta_0 \) in (5.8) can be bounded above by

\[ J_0^2(t) := |e^{\sigma^2 \psi_h^2(t)/2} - 1| \mathcal{F}\{h_{0,b}\}(t) \mathcal{F}\{H\}(\delta t) + \mathcal{F}\{h_{m,b,\sigma}\}(t) \]

\[ \lesssim \mathcal{F}\{h_{0,b}\}(t) \mathcal{F}\{H\}(\delta t), \quad t \in \mathbb{R}. \]

Set \( \Delta_0 := \sum_{b=\pm 1/2} \int_{|\delta| \leq u_0} [J_0^2(t)/|\mathcal{F}\{h_{0,b}\}(t)|^2] \, dt \) and \( \Delta_0 := \sum_{b=\pm 1/2} \int_{|\delta| > u_0} [J_0^2(t)/|\mathcal{F}\{h_{0,b}\}(t)|^2] \, dt, \)

we have that \( \Delta_0 \lesssim \Delta_{01} + \Delta_{02}. \) We now prove that \( \Delta_{0j} \lesssim \sigma^{2(\alpha + 2)}, \) for \( j = 1, \) and \( \Delta_{02} \). Taking into account that \( |e^{\sigma^2 \psi_h^2(t)/2}| = e^{-\sigma^2(t^2 - \beta^2)} = e^{-\sigma^2(t^2 - 1/4)} \), for \( \omega \geq m \geq (\alpha + 2) \) and \( \sigma > 0 \) small enough, by Lemma D.1, relationships (5.9) and (5.10), we have

\[ \Delta_{01} \lesssim \sum_{b=\pm 1/2} \int_{|\delta| \leq u_0} \frac{1}{|\mathcal{F}\{h_{0,b}\}(t)|^2} (|\sigma^2(t^2 + 1/4)|^m \]

\[ + \sigma^{2\omega} \min\{4, \sigma^4(t^2 + 1/4)^2/4\}) |\mathcal{F}\{h_{0,b}\}(t)|^2 \, dt \]

\[ \lesssim \sigma^{2(\alpha + 2)} \sum_{b=\pm 1/2} \int_{|\delta| \leq u_0} (t^2 + 1)|e^{\sigma^2 \psi_h^2(t)/2} \, dt \lesssim \sigma^{2(\alpha + 2)}, \]
because $\mathcal{F}\{\tilde{h}_{0,b}\}(t) = [M_{0X}(b)]^{-1}e^{\delta b f_{0X}(t)}$, $t \in \mathbb{R}$, and $\int_{\mathbb{R}}(|t|^{2\alpha} \vee 1)|e^{\delta b f_{0X}(t)}|^2 \, dt < \infty$ by Assumption 3.1 and $\|e^{\delta b f_{0X}}\|_1 \leq \|e^{1/2 f_{0X}}\|_1 < \infty$. Analogously, for $\sigma|t| > (u_0/c_\delta) > 1,$

$$\Delta_{02} \lesssim \sum_{b = 1/2}^{\delta \|t\| > u_0} \int_{|t| > u_0} \frac{1}{\mathcal{F}\{\tilde{h}_{0,b}\}(t)^2} \left( \left| e^{\sigma^2 \psi^2(t)/2} \sum_{k=0}^{m-1} \frac{(-1)^k \sigma^k \psi(t)^2k}{2^k k!} - 1 \right|^2 
+ \left| e^{\sigma^2 \psi^2(t)/2} - 1 \right|^2 \right) \mathcal{F}\{\tilde{h}_{0,b}\}(t)^2 \, dt$$

$$\lesssim \sum_{b = 1/2}^{\delta \|t\| > u_0} \int_{|t| > u_0} \frac{1}{\mathcal{F}\{\tilde{h}_{0,b}\}(t)^2} \left( e^{-\sigma^2(t^2 - 1/4)/2} \left( \frac{1}{\sigma^4} + \min\{2, \sigma^2(t^2 + 1/4)/2\} \right)^2 \mathcal{F}\{\tilde{h}_{0,b}\}(t)^2 \, dt \right.$$

$$\lesssim \sigma^{2(\alpha+2)} \sum_{b = 1/2}^{\delta \|t\| > u_0} \int_{|t| > u_0} \frac{t^4}{\mathcal{F}\{\tilde{h}_{0,b}\}(t)^2} \left( e^{-\sigma^2t^2} (\sigma|t|)^{4m-2(\alpha+2)} + 1 \right) \mathcal{F}\{\tilde{h}_{0,b}\}(t)^2 \, dt$$

$$\lesssim \sigma^{2(\alpha+2)} \sum_{b = 1/2}^{\delta \|t\| > u_0} \int_{|t| > u_0} \left| t^{2\alpha}e^{\delta b f_{0X}(t)} \mathcal{F}\{\tilde{h}_{0,b}\}(t)^2 \, dt \lesssim \sigma^{2(\alpha+2)}.$$

We now prove relationship (5.7). Since $\mathcal{F}\{\tilde{h}_{0,b}\}(0) = 1$, $(1 - e^{-\sigma^2/8})/\gamma = -1$ and $\sigma^2/8 \leq e^{\sigma^2/8}|\gamma|$, from previous computations for the term $\Delta_{01}$ we have

$$\sigma^2/8 |\mathcal{F}\{h_{m,b,\sigma}\}(0) - 1| \leq e^{\sigma^2/8}|\gamma| |\mathcal{F}\{h_{m,b,\sigma}\}(0) + \frac{(1 - e^{-\sigma^2/8})}{\gamma}| \lesssim \mathcal{J}_0(0) \lesssim \sigma^m,$$

whence $\int_{\mathbb{R}} h_{m,b,\sigma}(x) \, dx = \mathcal{F}\{h_{m,b,\sigma}\}(0) = 1 + O(\sigma^{m-2})$. The proof is thus complete. 

**Proof of Theorem 4.3.** The entropy condition (2.8) and the small ball prior probability estimate (2.10) of Theorem 2.1 in [21], p. 1239, are satisfied for $\bar{\epsilon}_n = n^{-(\alpha+2)/(2\alpha+5)}(\log n)^{\tau'}$ and $\bar{\epsilon}_n = n^{-(\alpha+2)/(2\alpha+5)}(\log n)^{\tau'_0}$, with exponents $\tau' > \tau'_0 > 1$. Then, the posterior rate is $\epsilon_n := (\epsilon_n \vee \bar{\epsilon}_n) = \epsilon_n = n^{-(\alpha+2)/(2\alpha+5)}(\log n)^m$. For the details of the entropy and remaining mass conditions, see, e.g., Theorem 5 of [55], p. 631, while for the small-ball prior probability estimate apply Lemma E.1 together with a modified version of Lemma D.3 with $(\alpha + 2)$ in place of $\beta = 2$. 

**6. Final remarks.** In this paper we have studied, from a Bayesian perspective, the problem of deconvolution, which, as described in section 1, is of primary importance in many applications. If some optimal results have been obtained using kernel estimators, no optimal Bayesian procedure has been studied so far. One of the key results of this work is an inversion/transportation inequality relating the $L^1$-Wasserstein distance between $\mu_X$ and $\mu_{0X}$ to either the total variation or the Wasserstein distance between the mixed distributions $\mu_Y$ and $\mu_{0Y}$. This inequality is derived under mild conditions on the noise density $f_\varepsilon$ and covers noise regularity $\beta$ ranging in $[1/2, \infty)$ showing the existence of different régimes. The upper bound is expressed in terms of the total variation distance between the mixed distributions $\mu_Y$ and $\mu_{0Y}$ for $\beta \geq 1$ and in terms of the $L^1$-Wasserstein distance for $\beta \geq 1/2$. The version expressed in terms of the $L^1$-Wasserstein metric would lead to the minimax posterior contraction rate for $W_1(\mu_X, \mu_{0X})$ if a posterior convergence rate for $W_1(\mu_Y, \mu_{0Y})$ of the order $O(1/\sqrt{n})$ were preliminarily obtained. However, deriving such a rate for nonparametric mixture models is a non trivial task and we are not aware of any such results in the literature.

The inversion inequality can be used in various contexts nevertheless, in particular it can be used outside the Bayesian inference or it can be used as a first step to obtain Bernstein-von Mises type results on linear functionals of $\mu_Y$ or $\mu_X$. 


We have used the inversion inequality to derive a general result on $L^1$-Wasserstein posterior convergence rates for the mixing distribution. We have then studied the case where the error density is a Linnik distribution with true mixing density having exponentially decaying tails. In the special case of a Laplace error, we have further studied adaptive $L^1$-Wasserstein estimation of $\mu_X$ when $f_{0X}$ is Sobolev regular. Adaptation is obtained using as a prior distribution on the mixing density $f_X$ a mixture of Gaussian densities. This result is derived by constructing an approximation of $f_{0Y}$ of the form $f_\varepsilon \ast \phi_\sigma \ast f_1$ with a suitable function $f_1$. This approximation is significantly more involved than the one devised in [39], which would not lead to the correct error rate in the present context and is of interest in itself.

Acknowledgements. The authors gratefully acknowledge financial support from the Institut Henri Poincaré (IHP), Sorbonne Université (Paris), within the RIP program on “Bayesian Wasserstein deconvolution” that has taken place in 2019 at the IHP-Centre Émile Borel, where part of this work was written. Catia Scricciolo has also been partially supported by Università di Verona. She wishes to dedicate this work to her mother and sister Emilia, with deep love and immense gratitude.

The project leading to this work has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 834175).

APPENDIX A: LEMMAS A.1 AND A.2 IN THE PROOF OF THEOREM 3.2

For a unified treatment of the cases (i) and (ii) of Assumption 3.1, we set
\[ \tilde{h} := \begin{cases} h, & \text{if } K \text{ is an } \left(\lceil \alpha \rceil + 1 \right) \text{-order kernel,} \\ h/2, & \text{if } K \text{ is a supersmooth kernel.} \end{cases} \]

Analogously, $\tilde{t}$ is equal to 1 if $K$ is an $\left(\lceil \alpha \rceil + 1 \right)$-order kernel and to 1/2 if $K$ is a supersmooth kernel. Also, $K_{2,h}(\cdot) := (2\pi)^{-1} \int_{\mathbb{R}} \exp(-it\cdot)w_{2,h}(t) \, dt$ is the inverse Fourier transform of $w_{2,h}(t) := \hat{K}(ht)[1 - \chi(t)]r_\varepsilon(t)$, $t \in \mathbb{R}$, while $F_{2,h}(z) := \int_{-\infty}^{z} K_{2,h}(s) \, ds$, $z \in \mathbb{R}$, is the “distribution function” of $K_{2,h}$.

**Lemma A.1.** If $\mu_\varepsilon \in \mathcal{P}_0$ satisfies Assumption 3.2 for $\beta > 0$, then, for $h > 0$ small enough,
\begin{align*}
\|K_{2,h}\|_1 & \lesssim h^{-(\beta - 1/2)} + |\log h|^{1+1(\beta=1/2)}/2. 
\end{align*}

**Proof.** Consider the decomposition
\[ \|K_{2,h}\|_1 = \left( \int_{|z| \leq h^{3/2}} + \int_{h^{3/2} < |z| \leq 1} + \int_{|z| > 1} \right) |K_{2,h}(z)| \, dz =: K_{2,1} + K_{2,2} + K_{2,3}. \]

By condition (3.4) with $l = 0$ and any $\beta > 0$, since $\hat{K}(\tilde{t}) \in C_b([-1, 1])$, we have
\begin{align*}
K_{2,1} & := \int_{|z| \leq h^{3/2}} |K_{2,h}(z)| \, dz < 2h^{3/2} \int_{1 < |t| \leq 1/h} |\hat{K}(ht)||1 - \chi(t)||r_\varepsilon(t)|| \, dt \\
& \lesssim h^{3/2} \int_{1 < |t| \leq 1/h} |\hat{K}(ht)||t|^{\beta} \, dt \\
& \lesssim h^{-(\beta - 1/2)} \int_{1 < |t| \leq 1/h} h|h\hat{K}(ht)| \, dt \lesssim h^{-(\beta - 1/2)} \|\hat{K}\|_1.
\end{align*}
Thus, $K_2^{(1)} = O(h^{-(\beta-1/2)^+})$. To bound $K_2^{(2)}$ and $K_2^{(3)}$, we apply identity (C.3) to $K_{2,h}$. Note that
\[ z \neq 0, \quad K_{2,h}(z) = \frac{1}{2\pi(iz)} \int_{\mathbb{R}} \exp(-tz)w_{2,h}^{(1)}(t) \, dt, \]
provided that $w_{2,h}^{(1)}(t) = h\tilde{K}^{(1)}(ht)[1 - \chi(t)]r_{\varepsilon}(t) - \tilde{K}(ht)\{\chi^{(1)}(t)\varepsilon(t) - [1 - \chi(t)]r_{\varepsilon}^{(1)}(t)\}$, $t \in \mathbb{R}$, is in $L^1(\mathbb{R})$. We show that $w_{2,h}^{(1)} \in L^2(\mathbb{R})$. Then, by the same arguments, also $w_{2,h}^{(1)} \in L^1(\mathbb{R})$. We have
\[
\|w_{2,h}^{(1)}\|_2^2 \lesssim \int_{\mathbb{R}} |h\tilde{K}^{(1)}(ht)[1 - \chi(t)]r_{\varepsilon}(t)|^2 \, dt + \int_{\mathbb{R}} |\tilde{K}(ht)|^2 |\chi^{(1)}(t)r_{\varepsilon}(t) - [1 - \chi(t)]r_{\varepsilon}^{(1)}(t)|^2 \, dt
\]
\[ =: J_1^2 + J_2^2. \]

For every $\beta > 0$, since also $\tilde{K}^{(1)}(1\cdot) \in C_b([-1, 1])$, $J_1^2 \lesssim \int_{1<|t|\leq 1/h} |\tilde{K}^{(1)}(ht)|^2 |1 - \chi(t)|^2 (1 + |t|)^2 \beta \, dt \lesssim h^{-2(\beta-1/2)^+} \|\tilde{K}^{(1)}\|_2^2 \lesssim h^{-2(\beta-1/2)^+}$ so that $J_1 = O(h^{-(\beta-1/2)^+})$. For every $h \leq 1/2$,
\[
J_2^2 \lesssim \int_{|t|<2} |\tilde{K}(ht)|^2 |\chi^{(1)}(t)|^2 |r_{\varepsilon}(t)|^2 \, dt + \int_{1<|t|\leq 1/h} |\tilde{K}(ht)|^2 |1 - \chi(t)|^2 |r_{\varepsilon}^{(1)}(t)|^2 \, dt
\]
\[ \lesssim \|\chi^{(1)}r_{\varepsilon}\|_2^2 + (1 - \chi)r_{\varepsilon}^{(1)}\|_2^2, \]
where $\|\chi^{(1)}r_{\varepsilon}\|_2 = O(1)$ and $\|(1 - \chi)r_{\varepsilon}^{(1)}\|_2^2 \lesssim \int_{1<|t|\leq 1/h} |t|^{2(\beta-1)} \, dt \lesssim h^{-2(\beta-1/2)^+} |\log h|^{1(\beta=1/2)}$. So, $J_2 = O(h^{-(\beta-1/2)^+} |\log h|^{1(\beta=1/2)/2})$. It follows that $\|w_{2,h}^{(1)}\|_2 = O(h^{-(\beta-1/2)^+} |\log h|^{1(\beta=1/2)/2})$. By the same arguments, also $\|w_{2,h}^{(1)}\|_1 = O(h^{-(\beta-1/2)^+} |\log h|^{1(\beta=1/2)/2})$. Then,
\[
K_2^{(2)} := \int_{h^{1/2}<|z|\leq 1} |K_{2,h}(z)| \, dz \lesssim \left( \int_{h^{1/2}<|z|\leq 1} \frac{1}{|z|} \, dz \right) \|w_{2,h}^{(1)}\|_1
\]
\[ \lesssim h^{-2(\beta-1/2)^+} |\log h|^{1+1(\beta=1/2)/2}. \]

By the Cauchy–Schwarz inequality,
\[
K_2^{(3)} := \int_{|z|>1} |K_{2,h}(z)| \, dz = \int_{|z|>1} \frac{1}{|z|} \left| \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-tz)w_{2,h}^{(1)}(t) \, dt \right| \, dz
\]
\[ \lesssim \left( \int_{|z|>1} \frac{1}{|z|^2} \, dz \right)^{1/2} \|w_{2,h}^{(1)}\|_2 \lesssim \|w_{2,h}^{(1)}\|_2 \lesssim h^{-2(\beta-1/2)^+} |\log h|^{1(\beta=1/2)/2}. \]

Inequality (A.1) follows by combining the bounds on $K_2^{(1)}$, $K_2^{(2)}$ and $K_2^{(3)}$. \hfill \Box

The following lemma is analogous to Lemma A.1 and gives the order, in terms of the kernel bandwidth, of the $L^1$-norm of the “distribution function” $F_{2,h}$ of $K_{2,h}$.

**Lemma A.2.** If $\mu_\varepsilon \in \mathcal{P}_0$ satisfies Assumption 3.2 for $\beta > 0$, then, for $h > 0$ small enough,
\[
(A.2) \quad \|F_{2,h}\|_1 \lesssim h^{-(\beta-1)^+} |\log h|. \]
Proof. By the same arguments used for the function \(G_{2,h}\) in [12], pp. 251–252, for \(h < 1\), we have

\[
F_{2,h}(z) = \frac{1}{2\pi} \int_\mathbb{R} \exp(-itz) \frac{w_{2,h}(t)}{(-it)} \, dt, \quad z \in \mathbb{R},
\]

where \(t \mapsto [w_{2,h}(t)/t]\) is in \(L^1(\mathbb{R})\) because \(\int_{|t| \leq 1/h} [|w_{2,h}(t)|/|t|] \, dt \lesssim \|w_{2,h}\|_1 < \infty\). Consider the integral decomposition

\[
\|F_{2,h}\|_1 = \left( \int_{|z| \leq h} + \int_{h < |z| \leq 1} + \int_{|z| > 1} \right) |F_{2,h}(z)| \, dz =: F^{(1)}_2 + F^{(2)}_2 + F^{(3)}_2.
\]

By condition (3.4) with \(l = 0\) and any \(\beta > 0\), since \(\hat{K}(\cdot) \in C_b([-1, 1])\), we have

\[
F^{(1)}_2 := \int_{|z| \leq h} |F_{2,h}(z)| \, dz < 2h \int_{1 < |t| \leq 1/h} \|\hat{K}(ht)||1 - \chi(t)|r_\varepsilon(t)|/|t| \, dt
\]

\[
\lesssim h \int_{1 < |t| \leq 1/h} \|\hat{K}(ht)||1 - \chi(t)| (1 + |t|)^\beta /|t| \, dt
\]

\[
\lesssim h^{-(\beta-1)} \times \begin{cases} 
\int_{1 < |t| \leq 1/h} \frac{1}{|t|} \, dt, & \text{if } 0 < \beta < 1, \\
\|\hat{K}\|_1, & \text{if } \beta \geq 1,
\end{cases}
\]

Therefore, \(F^{(1)}_2 = O(h^{-(\beta-1)} \|\log h\|^{(1-\beta)_+})\). To bound \(F^{(2)}_2\) and \(F^{(3)}_2\), preliminarily note that, by applying identity (C.3) to \(F_{2,h}\), we have

\[
\text{for } z \neq 0, \quad F_{2,h}(z) = \frac{1}{2\pi(tz)} \int_\mathbb{R} \exp(-itz) \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{w_{2,h}(t)}{it} \right) \, dt,
\]

where

\[
\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{w_{2,h}(t)}{t} \right) = \hat{K}^{(1)}(ht)[1 - \chi(t)] \frac{r_\varepsilon(t)}{t} - \hat{K}(ht) \{ \chi^{(1)}(t) \frac{r_\varepsilon(t)}{t} - [1 - \chi(t)] \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{r_\varepsilon(t)}{t} \right) \}. 
\]

Then,

\[
I^2 := \int_\mathbb{R} \left| \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{w_{2,h}(t)}{t} \right) \right|^2 \, dt
\]

\[
\lesssim \int_\mathbb{R} \left| h \hat{K}^{(1)}(ht)[1 - \chi(t)] \frac{r_\varepsilon(t)}{t} \right|^2 \, dt
\]

\[
+ \int_\mathbb{R} \left| \hat{K}(ht) \left\{ \chi^{(1)}(t) \frac{r_\varepsilon(t)}{t} - [1 - \chi(t)] \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{r_\varepsilon(t)}{t} \right) \right\} \right|^2 \, dt
\]

\[
=: I^2_1 + I^2_2.
\]

For every \(\beta > 0\), since also \(\hat{K}^{(1)}(\cdot) \in C_b([-1, 1])\),

\[
I^2_1 \lesssim \int_{1 < |t| \leq 1/h} h^2 |\hat{K}^{(1)}(ht)|^2 |1 - \chi(t)|^2 (1 + |t|)^{2\beta} /|t|^2 \, dt \lesssim h^{-2(\beta-1)_+}.
\]
For every $h \leq 1/2$, 

$$I_2^2 := \int_{\mathbb{R}} |\hat{K}(ht)|^2 \left| \chi^{(1)}(t) \frac{r_\varepsilon(t)}{t} - |1 - \chi(t)| \frac{d}{dt} \left( \frac{r_\varepsilon(t)}{t} \right) \right|^2 dt$$

$$\lesssim \int_{1 < |t| < 2} |\hat{K}(ht)|^2 |\chi^{(1)}(t)|^2 \left| \frac{r_\varepsilon(t)}{t} \right|^2 dt$$

$$+ \int_{1 < |t| \leq 1/h} |\hat{K}(ht)|^2 |1 - \chi(t)|^2 \left| \frac{d}{dt} \left( \frac{r_\varepsilon(t)}{t} \right) \right|^2 dt$$

(A.4)

where the first integral in (A.4) is $O(1)$. Since

$$\left| \frac{d}{dt} \left( \frac{r_\varepsilon(t)}{t} \right) \right| = \left| \frac{t r_\varepsilon^{(1)}(t) - r_\varepsilon(t)}{|t|^2} \right| \leq \left( \frac{|r_\varepsilon^{(1)}(t)|}{|t|} + \frac{|r_\varepsilon(t)|}{t^2} \right) \lesssim \frac{1 + |t|^{\beta - 1}}{|t|} + \frac{1 + |t|^{\beta}}{t^2}, \quad t \in \mathbb{R},$$

the second integral in (A.4) can be bounded above as follows:

$$\int_{1 < |t| \leq 1/h} \left| \frac{d}{dt} \left( \frac{r_\varepsilon(t)}{t} \right) \right|^2 dt \lesssim \int_{1 < |t| \leq 1/h} |t|^{2(\beta - 2)} dt$$

$$\lesssim \begin{cases} 1, & \text{if } 0 < \beta < 1, \\ \int_{1 < |t| \leq 1/h} |t|^{2(\beta - 3/2)} dt, & \text{if } \beta \geq 1, \\ \frac{1}{h^{2(\beta - 1)}}, & \text{if } 0 < \beta < 1, \\ & \text{if } \beta \geq 1, \end{cases}$$

and $I_2 \lesssim h^{-(\beta - 1)_+}$. By the same arguments used to bound $I$, we also have that

$$\int_{\mathbb{R}} \left| \frac{d}{dt} \left( \frac{w_{2,h}(t)}{t} \right) \right| dt = O(h^{-(\beta - 1)_+}),$$

so that, by virtue of identity (A.3),

$$F_2^{(2)} := \int_{h < |z| \leq 1} |F_{2,h}(z)| \, dz \lesssim \left( \int_{h < |z| \leq 1} \frac{1}{|z|} \, dz \right) \left( \int_{\mathbb{R}} \left| \frac{d}{dt} \left( \frac{w_{2,h}(t)}{t} \right) \right| dt \right)$$

$$\lesssim h^{-(\beta - 1)_+} \log h.$$

Applying Cauchy–Schwarz inequality to the expression of $F_{2,h}$ in (A.3), we have

$$F_2^{(3)} := \int_{|z| > 1} |F_{2,h}(z)| \, dz \lesssim \left( \int_{|z| > 1} \frac{1}{|z|^2} \, dz \right)^{1/2} \left( \int_{\mathbb{R}} \left| \frac{d}{dt} \left( \frac{w_{2,h}(t)}{-it} \right) \right|^2 dt \right)^{1/2}$$

$$\lesssim \left( \int_{\mathbb{R}} \left| \frac{d}{dt} \left( \frac{w_{2,h}(t)}{-it} \right) \right|^2 \, dt \right)^{1/2} \lesssim h^{-(\beta - 1)_+}.$$

Inequality (A.2) follows by combining the bounds on $F_2^{(1)}$, $F_2^{(2)}$ and $F_2^{(3)}$. \[\square\]
APPENDIX B: LEMMA B.1 AND PROOF OF PROPOSITION 3.2

In this section, we provide auxiliary results associated to Section 3.

**B.1. Lemma for Theorem 3.1 on posterior contraction rates for \( L^1 \)-Wasserstein deconvolution.** We state a sufficient condition on the prior concentration rate on Kullback-Leibler type neighborhoods of the “true” probability measure for posterior \( L^1 \)-Wasserstein contraction around the true distribution to take place at least as fast. The assertion is in the same spirit of Lemma 1 in [54], pp. 123–125, providing sufficient conditions on the sampling distribution and the prior concentration rate for the posterior law to contract at a nearly \( \sqrt{n} \)-rate on Kolmogorov neighborhoods. The underlying idea is to construct tests with type I error probability controlled using a Dvoretzky-Kiefer-Wolfowitz (DKW) type inequality for the \( L^1 \)-Wasserstein metric.

**Lemma B.1.** Let \( \mu_{0Y} \in \mathcal{P}_0 \) have finite first moment \( \mathbb{E}_{0Y}[Y] < \infty \). Let \( \Pi \) be a prior law on probability measures \( \mu_Y \in \mathcal{P}_0 \), each one having finite first moment \( \mathbb{E}_{\mu_Y}[Y] < \infty \). If, for a constant \( C > 0 \) and a positive sequence \( \varepsilon_n \to 0 \) such that \( n\varepsilon_n^2 \to \infty \), we have

\[
\Pi(B_{\text{KL}}(P_{0Y}; c^2)) \geq \exp \left(-Cn\varepsilon_n^2\right),
\]

then, for \( M := \xi(1 - \theta)^{-1}(C + 1/2)^{1/2} \), with \( \theta \in (0, 1) \) and \( \xi > 1 \),

\[
\Pi_n(\mu_Y : W_1(\mu_Y, \mu_{0Y}) > M\varepsilon_n | Y^{(n)}) \to 0 \text{ in } P_{0Y}^n\text{-probability.}
\]

**Proof.** Because \( \mu_Y \) and \( \mu_{0Y} \) have finite first moments, \( W_1(\mu_Y, \mu_{0Y}) < \infty \), see, e.g., [60], p. 94. The posterior probability of \( A^c_n := \{\mu_Y : W_1(\mu_Y, \mu_{0Y}) > M\varepsilon_n\} \) is given by

\[
\Pi_n(A^c_n \mid Y^{(n)}) = \frac{\int_{A^c_n} \prod_{i=1}^n f_Y(Y_i) \Pi_n(d\mu_Y)}{\int_{\mathcal{P}_0} \prod_{i=1}^n f_Y(Y_i) \Pi_n(d\mu_Y)}.
\]

We construct (a sequence of) tests \( (\Psi_n)_{n \in \mathbb{N}} \) for the hypothesis \( H_0 : P = P_{0Y} \equiv \mu_{0Y} \) versus \( H_1 : P = P_Y \equiv \mu_Y, \mu_Y \in A^c_n \), where \( \Psi_n \equiv \Psi_n(Y^{(n)}; P_{0Y}) : \mathbb{R}^n \to \{0, 1\} \) is the indicator function of the rejection region of \( H_0 \), such that

\[
\mathbb{E}_{0Y}[\Psi_n] = o(1), \quad \sup_{\mu_Y \in A^c_n} \mathbb{E}_{\mu_Y}[1 - \Psi_n] \leq 2 \exp \left(-2(M - K)^2n\varepsilon_n^2\right)
\]

for \( n \) large enough,
with \( K := \theta M \). Define \( \Psi_n := 1_{R_n} \), with rejection region \( R_n := \{ y^{(n)} : W_1(\mu_n, \mu_{0Y}) > K\bar{c}_n \} \), where \( \mu_n \) is the empirical probability measure of the sample \( Y^{(n)} \). Since \( \mu_{0Y} \) has continuous distribution function \( F_{0Y} \) by assumption, we have that \( P_{0Y}^n(y^{(n)}) = W_1(\mu_n, \mu_{0Y}) > t) \leq 2e^{-2nt^2} \), see, e.g., [4], pp. 2304–2305, which is based on the DKW inequality [16], with the tight universal constant in [42]. Then, \( E_{0Y}^n[\Psi_n] = P_{0Y}^n(R_n) \leq 2\exp(-2K^2n\bar{c}_n^2) \).

Therefore,

\[
\tag{B.3} E_{0Y}^n[\Pi_n(A^n_c \mid Y^{(n)})] \leq 2\exp(-2K^2n\bar{c}_n^2) + E_{0Y}^n[\Pi_n(A^n_c \mid Y^{(n)})(1 - \Psi_n)].
\]

To control the second term in (B.3), defined

\[
D_n := \left\{ y^{(n)} : \prod_{i=1}^n \frac{f_Y(y_i)}{f_{0Y}(y_i)} \Pi_n(d\mu_Y) \leq \Pi(B_{KL}(P_{0Y}; \bar{c}_n^2)) \exp(-(C + 1)n\bar{c}_n^2) \right\},
\]

we consider the decomposition \( E_{0Y}^n[\Pi_n(A^n_c \mid Y^{(n)})(1 - \Psi_n)(1_{D_n} + 1_{D_n^c})] \). From Lemma 8.1 of [22], p. 524, we have that \( P_{0Y}^n(D_n) \leq (C^2n\bar{c}_n^2)^{-1} \). It follows that

\[
\tag{B.4} E_{0Y}^n[\Pi_n(A^n_c \mid Y^{(n)})(1 - \Psi_n)1_{D_n}] \leq P_{0Y}^n(D_n) \leq (C^2n\bar{c}_n^2)^{-1}.
\]

By assumption (B.1) and Fubini’s theorem,

\[
\tag{B.5} E_{0Y}^n[\Pi_n(A^n_c \mid Y^{(n)})(1 - \Psi_n)1_{D_n^c}] \leq \exp((2C + 1)n\bar{c}_n^2) \int_{A^n_c} E_{\mu_Y}^n[1 - \Psi_n] \Pi(d\mu_Y).
\]

Next, we give an exponential upper bound on \( \sup_{\mu_Y \in A^n_c} E_{\mu_Y}^n[1 - \Psi_n] \). Over the acceptance region \( R^n_c \), by the triangular inequality, for every \( \mu_Y \in A^n_c \), we have \( M\bar{c}_n < W_1(\mu_Y, \mu_{0Y}) \leq W_1(\mu_Y, \mu) + W_1(\mu, \mu_{0Y}) \leq W_1(\mu_Y, \mu_n) + K\bar{c}_n \), which implies that \( W_1(\mu_Y, \mu_n) > (M - K)\bar{c}_n \). Since \( F_Y \) is continuous, by the DKW type inequality for the \( L^1 \)-Wasserstein metric, we have

\[
\sup_{\mu_Y \in A^n_c} E_{\mu_Y}^n[1 - \Psi_n] \leq \sup_{\mu_Y \in A^n_c} P_Y^n(y^{(n)} : W_1(\mu_n, \mu_Y) > (M - K)\bar{c}_n) \leq 2\exp(-2(M - K)^2n\bar{c}_n^2).
\]

Combining the preceding inequality with (B.5), we have

\[
\tag{B.6} E_{0Y}^n[\Pi_n(A^n_c \mid Y^{(n)})(1 - \Psi_n)1_{D_n^c}] \leq \exp(-2([M - K]^2 - (C + 1/2)]n\bar{c}_n^2),
\]

where the right-hand side of (B.6) converges to zero because \( (M - K) = (1 - \theta)M > (C + 1/2)^{1/2} \). The convergence in (B.2) follows by combining the bounds in (B.3), (B.4) and (B.6).

**Remark B.1.** If condition (B.1) is replaced by

\[
\Pi(N_{KL}(P_{0Y}; \bar{c}_n^2)) \geq \exp(-Cn\bar{c}_n^2),
\]

where \( N_{KL}(P_{0Y}; \bar{c}_n^2) := \{ P_Y \in \mathcal{P}_0 : KL(P_{0Y} ; P_Y) \leq \bar{c}_n^2 \} \) is a Kullback-Leibler neighborhood of \( P_{0Y} \), then, by Lemma 6.26 of [26], p. 145, with \( P_{0Y}^n \)-probability at least equal to \( (1 - L_n^{-1}) \), for a sequence \( L_n \to \infty \) such that \( nL_n\bar{c}_n^2 \to \infty \), we have

\[
\tag{B.7} \int_{\mathcal{P}_0} \prod_{i=1}^n \frac{f_{0Y}(Y_i)}{f_Y(Y_i)} \Pi(d\mu_Y) \geq \exp(-(C + 2L_n)n\bar{c}_n^2).
\]

Following the proof of Lemma B.1 and applying the lower bound in (B.7), the convergence statement in (B.2) continues to hold true with \( M\bar{c}_n \) replaced by \( M_n\bar{c}_n \), for \( M_n := \xi(1 - \theta)^{-1}(C/2 + L_n)^{1/2} \), with \( \theta \in (0, 1) \) and \( \xi > 1 \). Taking \( L_n \) to be a slowly varying sequence,
Kullback-Leibler type neighborhoods can be replaced by Kullback-Leibler neighborhoods at the cost of an additional log-factor in the rate, which thus becomes equal to $L_n^{1/2} \varepsilon_n$. An extra log-factor is common in convergence rates of posterior distributions. It arises from the “testing-prior mass” approach, which we also adopt here. We refer the reader to [33] and [19] for a better understanding of this phenomenon.

**Remark B.2.** Lemma B.1 is stated for the case where $P_{0Y}$ is a convolution of probability measures, but the result holds true for every $P_{0Y}$ with continuous distribution function and finite first moment.

**B.2. Proof of Proposition 3.2 on $L^2$-minimax rates over logarithmic Sobolev classes.** We first show that $\psi_{n, \gamma}^2$ is a lower bound on the $L^2$-minimax risk. Let $\delta > 1$. For every $\gamma, L > 0$, the inclusion $\mathcal{F}_\gamma^S(L) \subseteq \mathcal{F}_{\gamma, \delta}^{LS}(L)$ holds true. Thus, for a constant $c > 0$, possibly depending on $\gamma$ and $L$,

$$c\psi_{n, \gamma}^2 \leq \inf_{\hat{f}_n} \sup_{f \in \mathcal{F}_{\gamma, \delta}^{LS}(L)} \mathbb{E}_f^n[\|\hat{f}_n - f\|^2] \leq \inf_{\hat{f}_n} \sup_{f \in \mathcal{F}_{\gamma, \delta}^{LS}(L)} \mathbb{E}_f^n[\|\hat{f}_n - f\|^2],$$

see Theorem 2.9 in [58], p. 107, for the left-hand side inequality. To show that $\psi_{n, \gamma}^2(\log n)^{\delta/(2\gamma + 1)}$ is an upper bound on the $L^2$-minimax risk, we restrict the class of estimators to kernel density estimators $f_K(\cdot) := n^{-1}\sum_{i=1}^n K(Y_i - \cdot),$ with a symmetric kernel $K \in L^2(\mathbb{R})$. Letting $\tilde{w}_{\gamma, \delta}(t) := |t|^{\gamma}(\log(e + |t|))^{-\delta/2}$, $t \in \mathbb{R}$, and $S_K^2 := \sup_{t \neq 0}[1 - K(t)/\tilde{w}_{\gamma, \delta}(t)]^2$, for every $f \in \mathcal{F}_{\gamma, \delta}^{LS}(L)$ we have $$\|1 - K\|_2^2 < \|w_{\gamma, \delta}\|_2^2 S_K^2 \leq L^2 S_K^2.$$ Using the decomposition (1.41) in Theorem 1.4 of [58], pp. 21–22, for the MISE of a kernel density estimator $f_K$, we have, writing $\inf_{\hat{f}_n} \sup_{f}$ for the supremum over $\mathcal{F}_{\gamma, \delta}^{LS}(L)$,

$$\inf_{\hat{f}_n} \sup_{f} \mathbb{E}_f^n[\|\hat{f}_n - f\|^2] \leq \inf_{f} \sup_{K \in \mathcal{F}_{\gamma, \delta}^{LS}(L)} \mathbb{E}_f^n[\|f_K - f\|^2]$$

$$= \inf_{f} \sup_{K} \mathbb{E}_f^n[\|1 - K\|_2^2 + \frac{1}{n} \|K\|_2^2 - \|K\hat{f}\|_2^2] \frac{2}{2\pi}$$

Since the first term is bounded by $L^2 S_K^2$, we have

$$\inf_{\hat{f}_n} \sup_{f} \mathbb{E}_f^n[\|\hat{f}_n - f\|^2] < \inf_{f} \left[ L^2 S_K^2 + \sup_{K} \frac{1}{n} \mathbb{E}_f^n(\|K\|_2^2 - \|K\hat{f}\|_2^2) \right]$$

$$\leq \inf_{c > 0} \inf_{K : |1 - K(t)| \leq c\tilde{w}_{\gamma, \delta}(t)} \left[ L^2 S_K^2 + \sup_{K} \frac{1}{n} \mathbb{E}_f^n(\|K\|_2^2 - \|K\hat{f}\|_2^2) \right]$$

$$\leq \inf_{c > 0} \left[ c^2 L^2 + \frac{2}{n} \int_{\{K : |1 - K(t)| \leq c\tilde{w}_{\gamma, \delta}(t)\}} \||\hat{K}\|_2^2 \right] \leq \inf_{c > 0} \left[ c^2 L^2 + \frac{2}{n} \int_{\mathbb{R}} [1 - c\tilde{w}_{\gamma, \delta}(t)]^2 dt \right]$$

$$= \left[ c^2\min L^2 + \frac{2}{n} \int_{\mathbb{R}} [1 - c\min \tilde{w}_{\gamma, \delta}(t)]^2 dt \right] \leq \psi_{n, \gamma}^2(\log n)^{\delta/(2\gamma + 1)},$$

where $\min_{x > 0} A x^2 + B x^{-1 - \gamma} \log^{\delta/(2\gamma)}(1/x)$ is achieved at $x_{\min} \propto n^{-\gamma/(2\gamma + 1)}(\log n)^{\delta/(2(2\gamma + 1))}$. The assertion follows by taking $c_{\min} = x_{\min}$.

**Appendix C:** Auxiliary Lemmas Used in the Proof of Theorem 3.2 on the Inversion Inequality

The following lemma assesses the order of the bias, in terms of the kernel bandwidth, of a distribution function having derivatives up to a certain order, with locally H"older continuous derivative of the highest order.
LEMMA C.1. Let $F_{0X}$ be the distribution function of $\mu_{0X} \in \mathcal{P}_0$ satisfying condition (3.2) of Assumption 3.1 for $\alpha > 0$. Let $K$ be a kernel of order $\lfloor \alpha \rfloor + 1$ satisfying
\[
\int_\mathbb{R} |z|^\alpha K(z)\,dz < \infty. \tag{C.1}
\]
Then, for every $h > 0$,
\[
\|F_{0X} * K_h - F_{0X}\|_1 = O(h^{\alpha+1}).
\]

PROOF. Recalled that $b_{F_{0X}} := (F_{0X} * K_h - F_{0X})$, write $b_{F_{0X}}(x) = \int_\mathbb{R} [F_{0X}(x - hu) - F_{0X}(x)] K(u)\,du$. Let $\ell = \lfloor \alpha \rfloor$. For any $x, u \in \mathbb{R}$ and $h > 0$, by Taylor’s expansion,
\[
F_{0X}(x - hu) = F_{0X}(x) - hu f_{0X}(x) + \ldots + \frac{(-hu)^{\ell+1}}{\ell!} \int_0^1 (1 - \tau)^\ell f_{0X}^{(\ell)}(x - \tau hu)\,d\tau.
\]
Since $K$ is a kernel of order $\ell + 1 = \lfloor \alpha \rfloor + 1$, we have
\[
b_{F_{0X}}(x) = \int_\mathbb{R} K(u) \frac{(-hu)^{\ell+1}}{\ell!} \int_0^1 (1 - \tau)^\ell \left[f_{0X}^{(\ell)}(x - \tau hu) - f_{0X}^{(\ell)}(x)\right]\,d\tau\,du.
\]
Condition (3.2) yields that
\[
\|b_{F_{0X}}\|_1 \leq \int_\mathbb{R} \left| \int_\mathbb{R} K(u) \frac{|hu|^{\ell+1}}{\ell!} \int_0^1 (1 - \tau)^\ell \left[f_{0X}^{(\ell)}(x - \tau hu) - f_{0X}^{(\ell)}(x)\right]\,d\tau\,du \right| dx
\]
\[
\leq h^{\alpha+1} \|L_0\|_1 \frac{1}{\ell!} \left( \int_\mathbb{R} |u|^{\alpha+1} |K(u)|\,du \right) \int_0^1 (1 - \tau)^\ell \tau^{\alpha-\ell} d\tau.
\]
By the assumptions $L_0 \in L^1(\mathbb{R})$ and $\int_\mathbb{R} |z|^\alpha K(z)\,dz < \infty$, we conclude that $\|b_{F_{0X}}\|_1 = O(h^{\alpha+1})$.

Analogously to Lemma C.1, the next lemma assesses the order of the bias, in terms of the kernel bandwidth, of a distribution function with density in a Sobolev type space.

LEMMA C.2. Let $F_{0X}$ be the distribution function of $\mu_{0X} \in \mathcal{P}_0$ satisfying condition (3.3) of Assumption 3.1 for $\alpha > 0$. Let $K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be symmetric, with $\int_\mathbb{R} |z||K(z)|\,dz < \infty$, and $\tilde{K} \in L^1(-1,1)$ such that $\tilde{K} \equiv 1$ on $[-1, 1]$. Then, for every $h > 0$,
\[
\|F_{0X} * K_h - F_{0X}\|_1 = O(h^{\alpha+1}).\tag{C.2}
\]

PROOF. Recalled the notation $b_{F_{0X}} := (F_{0X} * K_h - F_{0X})$, by the same arguments used for the function $G_{2,h}$ in [12], pp. 251–252, we have
\[
\|b_{F_{0X}}\|_1 = \int_\mathbb{R} \left| \frac{1}{2\pi} \int_{|t| > 1/h} \exp(-itx) \frac{1 - \tilde{K}(ht)}{(-it)} f_{0X}(t)\,dt \right| dx
\]
because $t \mapsto [1 - \tilde{K}(ht)][\hat{f}_{0X}(t)1_{[-1,1]}(ht)/t]$ is in $L^1(\mathbb{R})$ due to the first part of condition (3.3). By Young’s inequality and the second part of condition (3.3) on $D^\alpha f_{0X} \in L^1(\mathbb{R})$, we have
\[
\|b_{F_{0X}}\|_1 = \int_\mathbb{R} \left| \frac{1}{2\pi} \int_{|t| > 1/h} \exp(-itx) \frac{1 - \tilde{K}(ht)}{(-it)^{\alpha+1}} \frac{(-it)^\alpha \hat{f}_{0X}(t)}{(\alpha+1)!}\,dt \right| dx
\]
\[
\leq \|D^\alpha f_{0X}\|_1 \left( \int_{|x| \leq h} + \int_{|x| > h} \right) \left| \frac{1}{2\pi} \int_{|t| > 1/h} \exp(-itx) \frac{1 - \tilde{K}(ht)}{(-it)^{\alpha+1}}\,dt \right| dx,
\]
where \( B_1 \lesssim h \int_{|t|>1/h} [1 + |\hat{K}(ht)|] |t|^{-(\alpha+1)} dt \lesssim h^{\alpha+1} \) because \( \|\hat{K}\|_{\infty} \leq \|K\|_1 < \infty \). Therefore, \( B_1 = O(h^{\alpha+1}) \). To bound \( B_2 \), we recall that, for every \( j \in \mathbb{N} \), letting \( \hat{f}^{(j)} \) denote the \( j \)th derivative of the Fourier transform \( \hat{f} \) of a function \( f : \mathbb{R} \rightarrow \mathbb{C} \), if \( \hat{f}^{(j)} \in L^1(\mathbb{R}) \), then

\[
(C.3) \quad \text{for } z \neq 0, \quad f(z) = \frac{1}{2\pi|z|^j} \int_{\mathbb{R}} \exp(-zt) \hat{f}^{(j)}(t) \, dt.
\]

Since \( K \in L^1(\mathbb{R}) \) and \( zK(z) \in L^1(\mathbb{R}) \) jointly imply that \( \hat{K} \) is continuously differentiable with \( |\hat{K}^{(1)}(t)| \rightarrow 0 \) as \( |t| \rightarrow \infty \), so that \( \hat{K}^{(1)}(t) \in C_b(\mathbb{R}) \), defined \( \tilde{f}(t) := [1 - \hat{K}(ht)](it)^{-\alpha} \mathbf{1}_{[-1,1]}(ht) \), \( t \in \mathbb{R} \), we have \( f^{(1)}(t) = \{-i\hat{K}^{(1)}(ht)(ht)^{-\alpha} - \alpha \hat{K}(ht)(ht)^{-\alpha+1}\} \mathbf{1}_{[-1,1]}(ht) \). For \( f(\cdot) := (2\pi)^{-1} \int_{|t|>1/h} \exp(-\nu t) \hat{f}(t) \, dt \), which is well defined because \( \hat{f} \in L^1(\mathbb{R}) \), by the identity (C.3) and the Cauchy–Schwarz inequality,

\[
B_2 := \int_{|x|>h} |f(x)| \, dx = \int_{|x|>h} \left| \frac{1}{2\pi} \int_{|t|>1/h} \exp(-zt) \hat{f}^{(1)}(t) \, dt \right| \, dx \\
\lesssim \left( \int_{\mathbb{R}} \frac{1}{x^2} |f(x)|^2 \, dx \right)^{1/2} \left( \int_{|t|>1/h} |f^{(1)}(t)|^2 \, dt \right)^{1/2} \\
\lesssim h^{-1/2} h^{\alpha+3/2} \lesssim h^{\alpha+1}.
\]

Conclude that \( B_2 = O(h^{\alpha+1}) \). The assertion follows by combining the bounds on \( B_1 \) and \( B_2 \).

**Remark C.1.** Constants in (C.1) and (C.2) may depend on the kernel \( K \) and the distribution function \( F_{0X} \).

The next lemma provides the order of the \( L^1 \)-norm, in terms of the kernel bandwidth, of function arising when controlling the term \( T_1 \) in Theorem 3.2. We recall the notation. Let \( \chi : \mathbb{R} \rightarrow \mathbb{R} \) be a symmetric, continuously differentiable function, equal to 1 on \([-1,1]\) and to 0 outside \([-2,2]\). Let \( K \) be the kernel defined in Section 3.1. Recall that an \((\lfloor \alpha/2 \rfloor + 1)\)-order kernel is used when \( f_{0X} \) verifies (i) of Assumption 3.1 as in Lemma C.1, whereas a superkernel is used when \( f_{0X} \) verifies (ii) of Assumption 3.1 as in Lemma C.2. Recall that the Fourier transform \( \hat{K} \) of \( K \) has compact support. For \( h > 0 \), let \( w_{1,h}(t) := \hat{K}(ht) \chi(t) r_\varepsilon(t) \), \( t \in \mathbb{R} \), where \( r_\varepsilon \) is as defined in (3.1) and satisfies Assumption 3.2. The function \( \hat{K}_{1,h}(\cdot) := (2\pi)^{-1} \int_{\mathbb{R}} \exp(-\nu t) w_{1,h}(t) \, dt \) is the inverse Fourier transform of \( w_{1,h} \).

**Lemma C.3.** If \( \mu_\varepsilon \in \mathcal{P}_0 \) satisfies Assumption 3.2 for \( \beta > 0 \), then, for sufficiently small \( h > 0 \),

\[
\|K_{1,h}\|_1 = O(1).
\]

**Proof.** Denoted by \( w_{1,h}^{(1)} \) the derivative of \( w_{1,h} \) with respect to \( t \), we have \( \|K_{1,h}\|_1 \leq 2^{-1/2}(\|w_{1,h}\|_2^2 + \|w_{1,h}^{(1)}\|_2^2)^{1/2} \), see the proof of Theorem 4.2 in [3], pp. 1030–1031. For \( h \leq 1/2 \), by condition (3.4) with \( l = 0 \), we have \( \|w_{1,h}\|_2^2 \lesssim \int_{|t|<2} |\hat{K}(ht)|^2 |\chi(t)|^2 (1 + |t|)^{2\beta} dt \lesssim \|\chi\|_2^2 < \infty \) as \( \hat{K} \) is bounded on any compact set. Analogously, for \( w_{1,h}^{(1)}(t) = [h\hat{K}^{(1)}(ht)\chi(t) + \hat{K}(ht)\chi^{(1)}(t)] r_\varepsilon(t) + \hat{K}(ht)\chi(t)r_\varepsilon^{(1)}(t), t \in \mathbb{R} \), using condition (3.4) with
\( l = 1 \) and \( \beta > 0 \), we have
\[
\|w_{1,\Delta}^{(1)}\|_2^2 \lesssim \int_{|t| \leq 2} [h|\dot{K}^{(1)}(ht)||\chi(t)| + |\dot{K}(ht)||\chi^{(1)}(t)||^2(1 + |t|)^{2\beta} dt \\
+ \int_{|t| \leq 2} |\dot{K}(ht)|^2 |\chi(t)|^2(1 + |t|)^{2(\beta - 1)} dt
\]
\[
\lesssim \|\chi\|_2^2 + \|\chi^{(1)}\|_2^2 < \infty
\]
because also \( \dot{K}^{(1)} \) is bounded on any compact set by continuity. The assertion follows. \( \square \)

**APPENDIX D: LEMMAS FOR THEOREM 4.1 ON POSTERIOR CONTRACTION RATES FOR DIRICHLET LINNIK-NORMAL MIXTURES**

In the following lemmas we prove the existence of a compactly supported discrete mixing probability measure such that the corresponding Linnik-normal mixture has Hellinger distance of the appropriate order from a Linnik mixture and, furthermore, the prior law on Linnik-normal mixtures concentrates on Kullback-Leibler neighborhoods of the true density at optimal rate, up to a logarithmic factor.

The next lemma provides an upper bound on the remainder term (or truncation error) associated with the \( (r - 1) \)th order Taylor polynomial about zero of the complex exponential function, see, e.g., Lemma 10.1.5 in [1], pp. 320–321.

**Lemma D.1.** \( \) For every \( r \in \mathbb{N} \), we have
\[
\left| \exp(tx) - \sum_{k=0}^{r-1} \frac{(tx)^k}{k!} \right| \leq \min \left\{ \left| x \right|^r, \frac{2\left| x \right|^{r-1}}{(r-1)!} \right\}, \quad x \in \mathbb{R}.
\]

For later use, we recall that the bilateral Laplace transform of a function \( f : \mathbb{R} \to \mathbb{C} \) is defined as \( B\{f\}(s) := \int_{\mathbb{R}} \exp(-sx)f(x)\,dx \) for all \( s \in \mathbb{C} \) such that \( \int_{\mathbb{R}} |\exp(-sx)f(x)|\,dx = \int_{\mathbb{R}} \exp(-\text{Re}(s)x)|f(x)|\,dx < \infty \), where \( \text{Re}(s) \) denotes the real part of \( s \). With abuse of notation, for a probability measure \( \mu \) on \( \mathbb{R} \), we define \( B\{\mu\}(s) := \int_{\mathbb{R}} \exp(-sx)\mu(dx), \ s \in \mathbb{C} \). For all \( t \in \mathbb{R} \) such that \( \int_{\mathbb{R}} \exp(tx)\mu(dx) < \infty \), the mapping \( t \mapsto M_\mu(t) := \int_{\mathbb{R}} \exp(tx)\mu(dx) \) is the moment generating function of \( \mu \) and \( M_\mu(t) = B\{\mu\}(-t), \ t \in \mathbb{R} \).

In the following lemma, we prove the existence of a compactly supported discrete mixing probability measure, with a sufficiently small number of support points, such that the corresponding Linnik-normal mixture has Hellinger distance of the order \( O(\sigma^3) \) from the sampling density \( f_{0Y} \).

**Lemma D.2.** \( \) Let \( f_\varepsilon \) be a standard Linnik density with index \( 0 < \beta < 2 \). Let \( \mu_{0X} \in \mathcal{P}_0 \) be a probability measure supported on \([-a, a]\), with density \( f_{0X} \) such that \((e^{-1/2}f_{0X}) \in L^2(\mathbb{R})\). For \( \sigma > 0 \) small enough, there exists a discrete probability measure \( \mu_H \) on \([-a, a]\), with at most \( N = O((a/\sigma)^{1/2}) \) support points, such that, for \( f_Y := f_\varepsilon \ast (\mu_H \ast \phi_\sigma) \) and \( f_{0Y} := f_\varepsilon \ast f_{0X} \),
\[
d_H(f_Y, f_{0Y}) \lesssim \delta_0^{-1/2} e^{a_0^2/2} \sigma^3,
\]
as soon as \( P_{0X}(|X| \leq a_0) \geq \delta_0 \) for some \( 0 < a_0 < a \) and \( 0 < \delta_0 < 1 \).
**Proof.** We begin by analysing the case of a Laplace error distribution, which corresponds to \( \beta = 2 \). We then use the fact that a Linnik density with index \( 0 < \beta < 2 \) is a scale mixture of Laplace densities, see [37], to complete the proof.

- **Laplace error distribution** (\( \beta = 2 \)): \( f_\varepsilon(\cdot) = e^{-|\cdot|/2} \)

  For \( a_0 \) and \( \delta_0 \) as in the statement, we have

  \[
  f_{0Y}(y) \geq \int_{|x| \leq a_0} f_\varepsilon(y - x) f_{0X}(x) \, dx \geq \frac{1}{2} e^{- \langle y \rangle + \sigma_0^2} P_{0X}(|X| \leq a_0) \geq \frac{\sigma_0}{2} e^{- \langle y \rangle + \sigma_0^2}, \quad y \in \mathbb{R}.
  \]

  Define

  \[
  (D.1) \quad U(y) := e^{-y^2/2} + e^{y^2/2}.
  \]

  By the inequality \( e^{\langle y \rangle^2/2} \leq U(y) \) valid for every \( y \in \mathbb{R} \), we have

  \[
  d_H^2(f_y, f_{0Y}) \leq 2\sigma_0^{-1} e^{\sigma_0^2} \int_\mathbb{R} [e^{\langle y \rangle^2/2}(f_y - f_{0Y})(y)]^2 \, dy \leq 2\sigma_0^{-1} e^{\sigma_0^2} \| g_Y - g_{0Y} \|^2_2,
  \]

  where \( g_Y := U f_y \) and \( g_{0Y} := U f_{0Y} \). For \( b = \mp 1/2 \), noting that \( e^b f_{0Y} = (e^b f_{0X}) \ast \phi_\beta \), where \( e^b f_{0X} \in L^1(\mathbb{R}) \) for compactly supported \( f_{0X} \) and \( e^b f_{\varepsilon} \in L^p(\mathbb{R}) \) for every \( 1 \leq p \leq \infty \), we have \( \| e^b f_{0Y} \|_p \leq \| e^b f_{\varepsilon} \|_p \), \( \| e^b f_{0X} \|_1 < \infty \). Analogously, \( e^b f_y = (e^b f_{\varepsilon} \ast \phi_\beta) \ast (e^b \mu_H) \), where \( \mu_H(b) < \infty \) for compactly supported \( \mu_H \) and \( e^b f_{\varepsilon} \ast \phi_\beta \in L^p(\mathbb{R}) \) for every \( 1 \leq p \leq \infty \), we have \( \| e^b f_Y \|_p \leq \| e^b f_{\varepsilon} \ast \phi_\beta \|_p \times \| \mu_H \|_p \). Also, \( \| g_{0Y} \|_2^2 = (2\pi)^{-1} \| g_{0Y} \|_2^2 \) and \( \| g_Y \|_2^2 = (2\pi)^{-1} \| g_Y \|_2^2 \).

  Consequently, \( g_{0Y} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and the corresponding Fourier transforms

  \[
  \hat{g}_Y(t) := \int_\mathbb{R} e^{it \varepsilon} g_Y(y) \, dy \quad \text{and} \quad \hat{g}_{0Y}(t) := \int_\mathbb{R} e^{it \varepsilon} g_{0Y}(y) \, dy, \quad t \in \mathbb{R},
  \]

  are well defined. Also, \( \| \hat{g}_{0Y} \|_2^2 = (2\pi)^{-1} \| \hat{g}_{0Y} \|_2^2 \) and \( \| \hat{g}_Y \|_2^2 = (2\pi)^{-1} \| \hat{g}_Y \|_2^2 \). For \( \psi(t) := -(it + b) \), let \( \varrho(t) := |1 - \psi_0^2(t)| \), \( t \in \mathbb{R} \). Note that \( \varrho_{-1/2}(t) = \varrho_{1/2}(t) \) and \( \varrho_{-1/2}(t)^2 = \varrho_{1/2}(t)^2 = (t^4 + 5t^2/2 + 9/16) \).

  Noting that

  \[
  B\{f_\varepsilon(\cdot - x)\}(\psi(t)) = \frac{e^{-\psi(t)x}}{\varrho(t)} \quad t, x \in \mathbb{R},
  \]

  we have

  \[
  r(t; x) := \int_\mathbb{R} e^{it \varepsilon} U(y) f_\varepsilon(y - x) \, dy = \sum_{b = \mp 1/2} B\{f_\varepsilon(\cdot - x)\}(\psi(t)) = \sum_{b = \mp 1/2} \frac{e^{-\psi(t)x}}{\varrho(t)} \quad t, x \in \mathbb{R}.
  \]

  Then, \( \hat{g}_{0Y}(t) = \int_{|x| \leq \sigma} r(t; x) f_{0X}(x) \, dx = \sum_{b = \mp 1/2} B\{f_{0X}\}(\psi(t))/\varrho(t), \quad t \in \mathbb{R} \).

  We derive the expression of \( \hat{g}_Y \). Since

  \[
  (D.2) \quad B\{\phi_\beta(\cdot - u)\}(\psi(t)) = \exp(-\psi(t)u + \sigma^2 \psi_0^2(t)/2), \quad t, u \in \mathbb{R},
  \]

  we have

  \[
  \hat{g}_Y(t) = \int_{|u| \leq \sigma} \left( \int_{\mathbb{R}} r(t; x) \phi_\beta(x - u) \, dx \right) \mu_H(\,du) = \sum_{b = \mp 1/2} \frac{e^{\sigma^2 \psi_0^2(t)/2}}{\varrho(t)} B\{\mu_H\}(\psi(t)), \quad t \in \mathbb{R}.
  \]

  For ease of notation, we introduce the integrals

  \[
  I_b := \int_{\mathbb{R}} \frac{1}{\varrho(t)} \left| e^{\sigma^2 \psi_0^2(t)/2} B\{\mu_H\}(\psi(t)) - B\{f_{0X}\}(\psi(t)) \right|^2 \, dt, \quad b = \mp 1/2.
  \]
By Plancherel’s theorem and the triangular inequality, 
\[2\pi \|g_Y - \bar{g}_Y\|_2^2 = \|g_Y - \bar{g}_Y\|_2^2 \leq 2(I_{-1/2} + I_{1/2}).\]
Both terms \(I_{-1/2}\) and \(I_{1/2}\) can be controlled using the same arguments, we therefore consider a unified treatment for \(I_b\). For \(M > 0\), we have
\[
I_b \leq \left( \int_{|t| \leq M} + \int_{|t| > M} \right) \left| \frac{e^{\sigma^2 \psi_0^2(t)/2}}{\varrho_0(t)^2} \right| \left( B\{ \mu_H \} - B\{ f_{0X} \} \right)(\psi_b(t)) \, dt \\
+ \int_{\mathbb{R}} \frac{1}{\varrho_0(t)^2} \left| e^{\sigma^2 \psi_b^2(t)/2} - 1 \right|^2 \left( B\{ f_{0X} \} \right)(\psi_b(t)) \, dt =: \sum_{k=1}^{3} I_b^{(k)}.
\]

**Study of the term \(I_b^{(1)}\)**

The term \(I_b^{(1)}\) can be bounded similarly to \(I_1\) in Lemma 2 of [20], p. 616. Preliminarily note that, for \(\sigma < 1/|b| = 2\), we have \(|e^{\sigma^2 \psi_0^2(t)/2} - \sigma^2(t^2 + b^2)/2| < 1\). Let \(\mu_H\) be a discrete probability measure on \([-a, a]\) satisfying the constraints
\[
\int u^j \mu_H(du) = \int u^j f_{0X}(u) \, du, \quad j = 0, \ldots, J - 1, \\
\int e^{bu} \mu_H(du) = \int e^{bu} f_{0X}(u) \, du, \quad b = \pm 1/2,
\]
where \(J = \lceil \eta eaM \rceil\) for some \(\eta > 1\). Note that the last constraints can be written as \(M_{\mu_H}(b) = M_{0X}(b), b = \pm 1/2\). Using Lemma D.1 with \(r = J\), by the inequality \(J! \geq (J/e)^J\), we have
\[
I_b^{(1)} \leq \int_{|t| \leq M} \frac{1}{\varrho_0(t)^2} \left| \int \left[ e^{\psi_b(t)u} - \sum_{j=0}^{J-1} \frac{[\psi_b(t)u]^j}{j!} \right] (\mu_H - \mu_{0X})(du) \right|^2 \, dt \\
\leq \left[ M_{\mu_H}(b) + M_{0X}(b) \right]^2 \frac{1}{(J!)^2} \int_{|t| \leq M} \frac{(a|t|)^{2J}}{\varrho_0(t)^2} \, dt \\
\leq a^{2J} \frac{2J}{(J!)^2} \int_0^M t^{2(J-2)} \, dt \\
\leq a^{2J} \frac{2J}{(J!)^2} \times \frac{M^{2J-3}}{2J-3} \leq M^{-4} \frac{a^{2J}}{(J!)^2} \times \frac{M^{2J+1}}{2J-3} \leq M^{-4} \left( \frac{e a M}{J} \right)^{2J+1} \leq M^{-4}.
\]

**Study of the term \(I_b^{(2)}\)**

Choosing \(M\) so that \((\sigma M)^2 \geq |\log \sigma|\), equivalently, \(M \geq \sigma^{-1} |\log \sigma|^{1/2}\), and using the facts that \(|e^{\sigma^2 \psi_0^2(t)/2}| = O(e^{-\sigma^2 t^2})\) and \(|B\{ \mu_H \}(\psi_b(t))| \leq M_{\mu_H}(b) = M_{0X}(b)\), we have
\[
I_b^{(2)} \leq M_{0X}(b) e^{-\sigma M^2} \int_{|t| > M} \frac{1}{t^4} \, dt \leq e^{-\sigma M^2} M^{-3} \leq \sigma M^{-3} \leq \sigma^4.
\]

**Study of the term \(I_b^{(3)}\)**

Noting that
\[
B\{ f_{0X} \}(\psi_b(t)) = (e^{b f_{0X}})(t), \quad t \in \mathbb{R},
\]
and, by Lemma D.1,
\[
|e^{\sigma^2 \psi_b(t)/2} - 1| \leq \min\{2, \sigma^2(t^2 + b^2)/2\} \leq \sigma^2(t^2 + 1/4)/2,
\]
we have

\[ I_b^{(3)} \leq \frac{\sigma^4}{2} \int_{\mathbb{R}} \frac{(t^4 + b^4)}{|g_b(t)|^2} |B\{ f_{0X}\}(\psi_b(t))|^2 \, dt \leq \sigma^4 \int_{\mathbb{R}} |(e^{b f_{0X}}(t))|^2 \, dt \leq \sigma^4, \]

where, by Plancherel’s theorem, \((2\pi)^{-1} \| e^{b f_{0X}} \|^2 = \| e^{b f_{0X}} \|^2 < \infty\) by the assumption that \(e^{b f_{0X}} \in L^2(\mathbb{R})\).

The existence of a discrete probability measure \(\mu_H\) supported on \([-a, a]\), with at most \(2 (J + 1) \alpha (aM) \geq (a/\sigma) |\log \sigma|^{1/2}\) support points, is guaranteed by Lemma A.1 of [21], p. 1260. Combining the bounds on \(I_b^{(k)}\), for \(k = 1, 2, 3\), we conclude that \(\| g_Y - g_{0Y} \|_2^2 \leq \sigma^4\).

- **Linnik error distribution with index \(0 < \beta < 2\)**

Every Linnik density \(f_\varepsilon\) with index \(0 < \beta < 2\) admits a representation as a scale mixture of Laplace densities \(f_1/v(\cdot) := ve^{-v| \cdot |}/2, v > 0,\)

\[ f_\varepsilon(u) = \int_0^\infty f_1/v(u)f_V(v; \beta) \, dv, \quad u \neq 0, \]

with mixing density

\[
(D.6) \quad f_V(v; \beta) := \left(\frac{2}{\pi} \sin \frac{\pi \beta}{2}\right) \frac{v^{\beta - 1}}{1 + v^{2\beta} + 2v^\beta \cos(\pi \beta/2)}, \quad v > 0.
\]

Let \(V\) be a random variable with the density in (D.6). Note that \(\mathbb{E}[V] = \int_0^\infty v f(v; \beta) \, dv < \infty\) for \(1 < \beta < 2\) and infinite for \(0 < \beta \leq 1\), in which case \(f_\varepsilon\) has an infinite peak at \(u = 0\).

Writing \((f_V - f_{0Y})(\cdot) = [f_\varepsilon \ast (\mu_X - \mu_{0X})](\cdot) = \int_0^\infty [f_1/v \ast (\mu_X - \mu_{0X})](\cdot) f_V(v; \beta) \, dv,\)

we have

\[
d_1^2(f_Y, f_{0Y}) \leq 2 \int_{\mathbb{R}} \left\{ \left( \int_0^1 + \int_1^\infty \right) [f_1/v \ast (\mu_X - \mu_{0X})](y) \frac{f_V(v; \beta)}{f_{0Y}(y)} \, dv \right\}^2 \, dy
\]

\[
\leq 2 \int_{\mathbb{R}} \left\{ \left( \int_0^1 + \int_1^\infty \right) [f_1/v \ast (\mu_X - \mu_{0X})](y) \frac{f_V(v; \beta)}{f_{0Y}(y)} \, dv \right\}^2 \, dy
\]

\[
+ 2 \int_{\mathbb{R}} \left\{ \int_1^\infty [f_1/v \ast (\mu_X - \mu_{0X})](y) \frac{f_V(v; \beta)}{f_{0Y}(y)} \, dv \right\}^2 \, dy =: I_1 + I_2.
\]

**Study of the term \(I_1\)**

For \(z > 0\), we define the function \(z \mapsto E(z) := \mathbb{E}[V 1_{\{V < z\}}] = \int_0^z v f(v; \beta) \, dv.\) Let \(v > 0\) be fixed and \(\alpha_0, \delta_0\) as in the statement. For every \(y \in \mathbb{R},\)

\[
(D.7) \quad f_{0Y}(y) \geq P_{0X}(\|X\| \leq \alpha_0) \int_0^y \frac{v}{2} e^{-u(|y| + \alpha_0)} f_V(u; \beta) \, du \geq \frac{\delta_0}{2} e^{-v(|y| + \alpha_0)} E(v).
\]
By the Cauchy–Schwarz inequality,
\[
\frac{I_1}{2} \leq 2e^{\epsilon_0-\Delta_0} \int_\mathbb{R} \left[ \int_0^1 e^{y/2} \left( \int_{|x| \leq a} e^{-v|y-x|}(\mu_X - \mu_{0X})(x) \right) \frac{v f_V(v; \beta)}{\sqrt{E(v)}} \right]^2 \frac{dv}{dy} d\gamma
\]
\[
= 2e^{\epsilon_0-\Delta_0} \int_0^1 \int_\mathbb{R} e^{y/2} \left( \int_{|x| \leq a} e^{-v|y-x|}(\mu_X - \mu_{0X})(x) \right)^2 \frac{v^2 f_V(v; \beta)}{E(v)} \frac{dv}{dy} d\gamma
\]
\[
\leq \int_0^1 \mathcal{I}(v) \frac{v^2 f_V(v; \beta)}{E(v)} dv.
\]
Since the function \( v \mapsto \mathcal{I}(v) \) is continuous, by the mean value theorem, there exists \( \bar{v} \in (0, 1) \) such that \( \int_0^1 \mathcal{I}(v) [v^2 f_V(v; \beta)/E(v)] dv = \mathcal{I}(\bar{v}) \int_0^1 [v^2 f_V(v; \beta)/E(v)] dv \leq \mathcal{I}(\bar{v}) \) because, for every \( 0 < \gamma < 1 \),
\[
E(v) \geq \int_{(\gamma v)^a} z^{1/\beta} \frac{1}{1+z^2+2z \cos(\pi \beta/2)} \frac{dz}{dz} \int_{(\gamma v)^a} 1 \frac{(1+z^2)^{1/2} d\gamma}{d\gamma} \int_{(\gamma v)^a} 1 \frac{(1+z^2)^{1/2} d\gamma}{d\gamma}
\]
and, consequently, \( \int_0^1 [v^2 f_V(v; \beta)/E(v)] dv \leq \int_0^1 \{1/[1 + v^{2\beta} + 2v^\beta \cos(\pi \beta/2)]\} dv < \infty \). The integral \( \mathcal{I}(\bar{v}) \) can be bounded above by \( \sigma^4 \) using the same arguments as for the standard Laplace error distribution, with \( \bar{v}/2 \) playing the role of \( b \). Thus, \( I_1 \lesssim \sigma^4 \lesssim \sigma^{2\beta} \).

**Study of the term \( I_2 \)**

Taking \( v = 1 \) in (D.7), we have \( f_{0Y}(y) \geq \delta_0 e^{-(|y|+\epsilon_0)} E(1)/2, y \in \mathbb{R} \). For
\[
\tilde{f}_\epsilon(u) := \int_1^\infty f_{1/u}(u) f_V(v; \beta) dv, \quad u \neq 0,
\]
let \( \tilde{g}_{0Y} := U(\tilde{f}_\epsilon \ast \mu_{0X}) \) and \( \tilde{g}_Y := U(\tilde{f}_\epsilon \ast \mu_X) \). By the same arguments laid down for the standard Laplace error distribution case, we have \( \tilde{g}_Y, \tilde{g}_{0Y} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and the corresponding Fourier transforms \( \tilde{g}_Y(t) := \int_\mathbb{R} e^{itv} \tilde{g}_Y(y) dy \) and \( \tilde{g}_{0Y}(y) := \int_\mathbb{R} e^{itv} \tilde{g}_{0Y}(y) dy, t \in \mathbb{R} \), are well defined. We have
\[
\frac{I_2}{2} \leq 2e^{\epsilon_0-\Delta_0} E(1) \int_\mathbb{R} \left\{ e^{y/2} \int_1^\infty [f_{1/u} \ast (\mu_X - \mu_{0X})](y) f_V(v; \beta) dv \right\}^2 \frac{dy}{dy} d\gamma
\]
\[
= 2e^{\epsilon_0-\Delta_0} E(1) \int_\mathbb{R} \{e^{y/2} [\tilde{f}_\epsilon \ast (\mu_X - \mu_{0X})](y)\}^2 \frac{dy}{dy} d\gamma
\]
\[
\leq 2e^{\epsilon_0-\Delta_0} E(1) \left\| \tilde{g}_Y - \tilde{g}_{0Y} \right\|^2 \geq \frac{e^{\epsilon_0-\Delta_0}}{\pi \delta_0 E(1)} \left\| \tilde{g}_Y - \tilde{g}_{0Y} \right\|^2.
\]
We derive the expressions of \( \tilde{g}_{0Y} \) and \( \tilde{g}_Y \):
\[
\tilde{g}_{0Y}(t) = \int_{|x| \leq a} \left( \int_1^\infty f_{1/u}(u) f_V(v; \beta) dv \right) \frac{v f_{0X}(v; \beta)}{v^2 - \psi^2_b(t)} \frac{dy}{dy} d\gamma \int_\mathbb{R} f_{0X}(x) dx
\]
\[
= \sum_{b=-\pi/2}^{\pi/2} \left( \int_1^\infty f_{1/u}(u) f_V(v; \beta) dv \right) \mathcal{B} \{ f_{0X} \} (\psi_b(t)), \quad t \in \mathbb{R},
\]
and, using the expression of $B\{\phi_{\sigma}(\cdot - u)\}(\psi_b(t))$ in (D.2),

$$\hat{g}_Y(t) = \sum_{b=\mp 1/2} \left( \int_{b}^{\infty} \frac{v f_{V}(v; \beta)}{v^2 - \psi_b^2(t)} dv \right) e^{\sigma^2 \psi_b^2(t)/2} B\{\mu_H\}(\psi_b(t)), \quad t \in \mathbb{R}. $$

Thus, $\|\hat{g}_Y - \hat{g}_{0Y}\|_2^2 \leq 2(\tilde{I}_{-1/2} + \tilde{I}_{1/2})$, where, for $b = \mp 1/2$,

$$\tilde{I}_b := \int_{\mathbb{R}} \left| \int_{1}^{\infty} \frac{v f_{V}(v; \beta)}{v^2 - \psi_b^2(t)} dv \right|^2 \left| e^{\sigma^2 \psi_b^2(t)/2} B\{\mu_H\}(\psi_b(t)) - B\{f_{0X}\}(\psi_b(t)) \right|^2 dt. $$

By identity (2) in [37], p. 63, (with their $\beta = 2$),

$$\left| \int_{1}^{\infty} \frac{v f_{V}(v; \beta)}{v^2 - \psi_b^2(t)} dv \right| \leq \int_{1}^{\infty} \frac{v f_{V}(v; \beta)}{v^2 - \psi_b^2(t)} dv \lesssim \int_{0}^{\infty} \frac{v^2 f_{V}(v; \beta)}{v^2 + t^2} dv = \frac{1}{1 + |t|^\beta}, \quad t \in \mathbb{R}. $$

For $M > 0$, we have

$$\tilde{I}_b \leq \left( \int_{|t| \leq M} + \int_{|t| > M} \right) \frac{e^{\sigma^2 \psi_b^2(t)/2}}{(1 + |t|^\beta)^2} \left| (B\{\mu_H\} - B\{f_{0X}\})(\psi_b(t)) \right|^2 dt

+ \left( \int_{|t| \leq 1/\sigma} + \int_{|t| > 1/\sigma} \right) \frac{1}{(1 + |t|^\beta)^2} \left| e^{\sigma^2 \psi_b^2(t)/2} - 1 \right|^2 \left| B\{f_{0X}\}(\psi_b(t)) \right|^2 dt

= \sum_{k=1}^{4} \tilde{I}_b^{(k)}.$$

Using the same construction of $\mu_H$ on $[-a, a]$ as for the standard Laplace error distribution case, with at most $2(J + 1) \times (aM) \gtrsim (a/\sigma) |\log \sigma|^{1/2}$, for $M \geq \sigma^{-1} |\log \sigma|^{1/2}$, support points such that the constraints in (D.3) hold true. Let $\sigma < 1/|b|$. By the same arguments laid down for $I_b^{(1)}$ and $I_b^{(2)}$ of the standard Laplace error distribution case,

$$\tilde{I}_b^{(1)} \lesssim \frac{a^{2J}}{(J!)^2} \int_{0}^{M} t^{2(J-\beta)} dt \lesssim M^{-2\beta} \frac{a^{2J}}{(J!)^2} \times \frac{M^{2J+1}}{2J - 2\beta + 1} \lesssim M^{-2\beta} \lesssim \sigma^{-2\beta}$$

and

$$\tilde{I}_b^{(2)} \lesssim \int_{|t| > M} \frac{e^{-(\sigma t)^2}}{(1 + |t|^\beta)^2} dt \lesssim M^{-2\beta} \int_{|t| > M} e^{-(\sigma t)^2} dt \lesssim \sigma^{-2\beta}. $$

Recalling the identity in (D.4) and the inequality in (D.5), since $e^{b_{1}f_{0X}} \in L^2(\mathbb{R})$ by assumption,

$$\tilde{I}_b^{(3)} \lesssim \sigma^4 \int_{|t| \leq 1/\sigma} \frac{(t^4 + b^4)}{(1 + |t|^\beta)^2} \left| B\{f_{0X}\}(\psi_b(t)) \right|^2 dt

\lesssim \sigma^4 \int_{|t| \leq 1/\sigma} |t|^{2(2-\beta)} \left| B\{f_{0X}\}(\psi_b(t)) \right|^2 dt \lesssim \sigma^{2\beta} \int_{\mathbb{R}} |(e^{b_{1}f_{0X}})(t)|^2 dt \lesssim \sigma^{2\beta}. $$

Also,

$$\tilde{I}_b^{(4)} \lesssim \int_{|t| > 1/\sigma} \frac{1}{|t|^{2\beta}} \left| B\{f_{0X}\}(\psi_b(t)) \right|^2 dt \lesssim \sigma^{2\beta} \int_{\mathbb{R}} |(e^{b_{1}f_{0X}})(t)|^2 dt \lesssim \sigma^{2\beta}. $$

Combining the bounds on $I_b^{(k)}$, for $k = 1, \ldots, 4$ and $b = \mp 1/2$, we have $I_2 \lesssim \|\hat{g}_Y - \hat{g}_{0Y}\|_2^2 \lesssim \sum_{b=\mp 1/2} \tilde{I}_b \lesssim \sum_{b=\mp 1/2} \sum_{k=1}^{4} \tilde{I}_b^{(k)} \lesssim \sigma^{2\beta}.$
Conclude that $d_{\mathcal{H}}^2(f_Y, f_{0Y}) \lesssim (I_1 + I_2) \lesssim \delta_0^{-1} e^{a_0 \sigma^{2/3}}$, which completes the proof. \hfill \Box

The next lemma gives sufficient conditions on the prior law $\mathcal{D}_{H_0} \otimes \Pi_\sigma$ so that the induced probability measure on Linnik-normal mixtures $f_Y = f_\varepsilon \ast (\mu_H \ast \phi_\sigma)$ concentrates on Kullback-Leibler neighborhoods of a Linnik mixture $f_{0Y} = f_\varepsilon \ast f_{0X}$, with index $1 < \beta \leq 2$ and mixing density $f_{0X}$ having exponentially decaying tails, at a rate of the order $O(n^{-\beta/(2\beta+1)} (\log n) \tau)$ for a suitable $\tau > 0$.

**Lemma D.3.** Let $f_{0Y} := f_\varepsilon \ast f_{0X}$, where $f_\varepsilon$ is the density of a standard Linnik distribution with index $1 < \beta \leq 2$ and $f_{0X}$ satisfies Assumption 4.3. Let the model be $f_Y := f_\varepsilon \ast (\phi_\sigma \ast \mu_H)$, with $\mu_H \in \mathcal{P}$. If the base measure $H_0$ of the Dirichlet process prior $\mathcal{D}_{H_0}$ for $\mu_H$ satisfies Assumption 4.1 and the prior $\Pi_\sigma$ for $\sigma$ satisfies Assumption 4.2 with $0 < \gamma < 1$, then $(\mathcal{D}_{H_0} \otimes \Pi_\sigma)(N_{\mathcal{KL}}(P_{0Y}; \tilde{\phi}_n^2)) \gtrsim \exp(-C n^2)$ for $\tilde{\epsilon}_n = n^{-\beta/(2\beta+1)} (\log n)^{1/2+\beta(t_v/3)/2(\beta+1)}$.

**Proof.** We show that, for some constant $C > 0$, the prior probability of a Kullback-Leibler neighborhood of $P_{0Y}$ of radius $\epsilon_n^2$ is at least $\exp(-C n^2 \epsilon_n^2)$. We apply Lemma B2 of [55], pp. 638–639, to relate $N_{\mathcal{KL}}(P_{0Y}; \tilde{\phi}_n^2)$ to a Hellinger ball of appropriate radius. By Assumption 4.3, there exists $C_0 > 0$ such that $\mu_{0X}([-a, a]) \leq e^{-(1+C_0)a}$ for a large enough. Set $a_\eta := a_0 \log(1/\eta)$, with $a_0 \geq 2/(1+C_0)$ and $\eta > 0$ small enough, we have $\mu_{0X}([-a_\eta, a_\eta]) \leq \eta^2$. Then, Lemma A.3 of [27], p. 1261, shows that the $L^1$-distance between $f_0Y$ and $f_0^\ast := f_\varepsilon \ast f_{0X}$, where $f_{0X}$ is the density of the renormalized restriction of $\mu_{0X}$ to $[-a_\eta, a_\eta]$, denoted by $\mu_{0X}^\ast$, is bounded above by $2\eta^2$. From $d_{\mathcal{H}}^2(f_{0Y}, f_{0Y}^\ast) \leq \|f_{0Y} - f_{0Y}^\ast\|_1 \leq 2\eta^2$, we have $d_{\mathcal{H}}(f_{0Y}, f_{0Y}^\ast) \lesssim \eta$. Lemma D.2 applied to $\mu_{0X}^\ast$ (which plays the role of $\mu_{0X}$ in the statement) shows that, for $\sigma > 0$ small enough, there exists a discrete probability measure $\mu_H^\ast$, supported on $[-a_\eta, a_\eta]$, with at most $N = O((a_\eta/\sigma) \log \sigma)^{1/2}$ support points, such that $f_Y^\ast := f_\varepsilon \ast (\mu_H^\ast \ast \phi_\sigma)$ satisfies

$$d_{\mathcal{H}}(f_Y^\ast, f_{0Y}^\ast) \lesssim \sigma^\beta.$$

An analogue of Corollary B1 in [55], p. 16, shows that $\mu_H^\ast = \sum_{j=1}^{N} p_j \delta_{u_j}$ has support points inside $[-a_\eta, a_\eta]$, with at least $\sigma^{1+2\beta}$-separation between every pair of points $u_i \neq u_j$, and that $d_{\mathcal{H}}(f_Y^\ast, f_{0Y}^\ast) \lesssim \sigma^\beta$. Consider disjoint intervals $U_j$ centred at $u_1, \ldots, u_N$ with length $\sigma^{1+2\beta}$ each. Extend $\{U_1, \ldots, U_N\}$ to a partition $\{U_1, \ldots, U_K\}$ of $[-a_\eta, a_\eta]$ such that each $U_j$, for $j = N + 1, \ldots, K$, has length at most $\sigma$. Further extend this to a partition $U_1, \ldots, U_M$ of $\mathbb{R}$ such that, for some constant $a_1 > 0$, we have $a_1 \sigma \leq H_0(U_j) \leq 1$ for all $j = 1, \ldots, M$. The whole process can be done with a total number $M$ of intervals of the same order as $N$. Define $p_j = 0$ for $j = N + 1, \ldots, M$. Let $\mathcal{P}_\sigma$ be the set of probability measures $\mu_H \in \mathcal{P}$ with

$$\sum_{j=1}^{K} |\mu_H(U_j) - p_j| \leq 2\sigma^{2\beta+1} \quad \text{and} \quad \min_{1 \leq j \leq K} \mu_H(U_j) \geq \sigma^{2(2\beta+1)/2}.$$

Note that $\sigma^{2\beta+1} K < 1$. By Lemma 5 in [25], p. 711, or Lemma B1 in [55], p. 16, with $V_0 := \bigcup \{U_j \mid N < U_j \}$ and $V_j \equiv U_j$, $j = 1, \ldots, N$, for any $\mu_H \in \mathcal{P}_\sigma$ we have $d_{\mathcal{H}}^2(f_Y, f_{0Y}) \leq \|f_Y - f_{0Y}^\ast\|_1 \lesssim \sigma^{2\beta}$. Then, for $\eta = O(\sigma^2)$, we have $d_{\mathcal{H}}^2(f_Y, f_{0Y}) \lesssim d_{\mathcal{H}}^2(f_Y, f_{0Y}^\ast) + d_{\mathcal{H}}^2(f_{0Y}^\ast, f_{0Y}) \lesssim \sigma^{2\beta}$. To apply Lemma B2 of [55], pp. 16–17, we study the quotient $(f_Y/f_{0Y})$. Let $\mu_H \in \mathcal{P}_\sigma$.

- Linnik error distribution with index $1 < \beta < 2$
For every $1 < \beta < 2$, we have $\|f_{0Y}\|_\infty \leq \mathbb{E}[V] < \infty$. Let $b_\sigma := \sigma^{-2\beta/(\beta-1)}$. Since $a_\eta < b_\sigma$ and $f_\varepsilon(u) \sim \Gamma(1+\beta)\sin(\pi\beta/2)|u|^{-(1+\beta)/\pi}$, as $|u| \to \infty$, see the asymptotic expansion (4.3.39) in [37], p. 262, for $|y| < b_\sigma$ with $\sigma$ small enough,

$$\frac{f_y}{f_{0Y}}(y) \geq \int_{|x| \leq a_\eta} f_\varepsilon(y-x) \int_{|x-u| \leq \sigma} \phi_\sigma(x-u)\mu_H(du)\,dx$$

$$\geq \frac{a_\eta}{\sigma} f_\varepsilon(2b_\sigma)\mu_H(U_{J(x)}) \geq a_\eta b_\sigma^{-2\beta} \sigma^{2\beta+1},$$

where $J(x)$ denotes the index $j \in \{1, \ldots, K\}$ for which $U_j \ni x$, because the interval $U_{J(x)}$ with length at most $\sigma$ is a subset of an interval of radius $\sigma$ centred at $x$. Analogously, for $|y| \geq b_\sigma$,

$$\frac{f_y}{f_{0Y}}(y) \geq \int_{|x| \leq a_\eta} f_\varepsilon(y-x) \int_{|x-u| \leq \sigma} \phi_\sigma(x-u)\mu_H(du)\,dx$$

$$\geq \frac{a_\eta}{\sigma} e^{-|y|+a_\eta} \mu_H(U_{J(x)}) \geq a_\eta e^{-2|y|} \sigma^{2\beta+1}.$$  

For $\lambda = a_\eta b_\sigma^{-1} \sigma^{2\beta+1}$, we have $\log(1/\lambda) \leq \log(1/\sigma)$. Since $\{y: (f_Y/\sigma_{0Y})(y) \leq \lambda\} \subseteq \{y: |y| \geq b_\sigma\}$,

$$P_{0Y}\left(\log(\frac{f_{0Y}}{f_Y})\right) \leq \frac{f_{0Y}}{f_Y}(y) f_{0Y}(y) \,dy$$

$$\leq \log(1/\sigma) \int_{|y| \geq b_\sigma} |y| f_{0Y}(y) \,dy,$$

where, by Assumption 4.3 on $f_{0X}$ that guarantees that $\int_{\mathbb{R}} e^{|x|} f_{0X}(x) \,dx < \infty$ and by Proposition 4.3.13 in [36], p. 262, with $n = 1$,

$$\int_{|y| \geq b_\sigma} |y| f_{0Y}(y) \,dy$$

$$\leq \int_{|y| \geq b_\sigma} |y| \int_{\mathbb{R}} \left( e^{|x|} \int_0^1 e^{-|y-x|} + \int_1^\infty e^{-|y-x|} \right) v_{f_Y}(v; \beta) \,dv \right) f_{0X}(x)\,dx \,dy$$

$$\leq \left( \int_{\mathbb{R}} e^{|x|} f_{0X}(x) \,dx \right) \int_{|y| \geq b_\sigma} |y| f_\varepsilon(y) \,dy \leq \int_{|y| \geq b_\sigma} |y|^{-\beta} \,dy \leq b_\sigma^{-2(\beta-1)} \leq \sigma^{2\beta}.$$  

It follows that $P_{0Y}\left(\log(\frac{f_{0Y}}{f_Y})\right) \leq \sigma^{2\beta}$.

**Laplace error distribution ($\beta = 2$)**

Since $\|f_{0Y}\|_\infty \leq 1/2$, for $|y| < a_\eta$,

$$\frac{f_y}{f_{0Y}}(y) \geq \int_{|x| \leq a_\eta} f_\varepsilon(y-x) \int_{|x-u| \leq \sigma} \phi_\sigma(x-u)\mu_H(du)\,dx$$

$$\geq \frac{a_\eta}{\sigma} e^{-2a_\eta} \mu_H(U_{J(x)}) \geq a_\eta e^{-2a_\eta} \sigma^{4\beta+1},$$

while, for $|y| \geq a_\eta$,

$$\frac{f_y}{f_{0Y}}(y) \geq \int_{|x| \leq a_\eta} f_\varepsilon(y-x) \int_{|u| \leq a_\eta} \phi_\sigma(x-u)\mu_H(du)\,dx \geq \frac{a_\eta}{\sigma} e^{-|y|} e^{-a_\eta e^{-2(a_\eta/\sigma)^2}},$$

where $\mu_H([-a_\eta, a_\eta]) \geq 1 - 2\sigma^{2\beta+1}$ because of the first condition in (D.8). For $\lambda = a_\eta e^{-2a_\eta} \sigma^{4\beta+1}$, we have $\log(1/\lambda) \leq \log(1/\sigma)$. Since $\{y: (f_Y/\sigma_{0Y})(y) \leq \lambda\} \subseteq \{y: |y| \geq b_\sigma\}$,
Thus, for every \(1 < \beta \leq 2\), we have \(P_{0Y}(\log (f_{0Y}/f_Y) \leq \lambda) \leq \sigma^{2\beta}\). Lemma B2 of [55], pp. 16–17, implies that \(P_{0Y}(\log (f_{0Y}/f_Y))\) is bounded above by \(\sigma^{2\beta} \log \sigma\). By Lemma 10 of [25], p. 714, we have \(D_{H_0}(\mathcal{S}) \geq \exp (-c_1 K \log (1/\sigma)) \geq \exp (-c_2 (n\sigma/\sigma) \log^{3/2} (1/\sigma))\) for constants \(c_1, c_2 > 0\) that depend on \(H_0(\mathbb{R})\) and \(a_1\). Given \(\sigma > 0\), define \(\mathcal{S} := \{\sigma' : \sigma (1 + \sigma^\delta) - 1 \leq \sigma' \leq \sigma\}\) for a constant \(0 < d \leq s_1 - 1\). Then, \(\Pi(\mathcal{S}) \geq \exp (-D_1 \sigma^{-\gamma} \log^4 (1/\sigma))\).

Replace \(\sigma\) at every occurrence with \(\sigma' \in \mathcal{S}\). For \(\xi := \sigma^{\beta} \log^{1/2} (1/\sigma)\), noting that \(\log (1/\sigma) \leq \log (1/\xi)\), since \(\gamma \leq 1\) we have

\[
(\mathcal{D}_{H_0} \otimes \Pi(\{\mathcal{S} = \mathcal{S}^2\})) \geq \exp (-c_3 (a_{\eta}/\sigma) \log^3 (1/\sigma)) \exp (-D_1 \sigma^{-\gamma} \log^4 (1/\sigma))
\]

\[
\geq \exp (-c_4 \xi^{-1/\beta} \log^{1/2} (1/\sigma))
\]

Replacing \(\xi\) with \(\tilde{\xi}_{n} = n^{-\beta/(2\beta + 1)} (\log n)^{1/2 + \beta (1/2\beta + 1)}\), for a suitable constant \(C > 0\) we have \((\mathcal{D}_{H_0} \otimes \Pi(\{\mathcal{S} = \mathcal{S}^2\})) \geq \exp (-C n \tilde{\xi}_n^2)\) and the proof is complete. \(\square\)

**APPENDIX E: LEMMAS FOR THEOREM 4.3 ON ADAPTIVE POSTERIOR CONTRACTION RATES FOR DIRICHLET LAPLACE-NORMAL MIXTURES**

We introduce some more notation. For \(h = o(1)\), let \(\delta = o(h)\). For \(m \in \mathbb{N}\), \(b = \mp 1/2\) and \(\sigma = o(1)\), we define the set

\[
A_{b,\sigma} := \{x \in \mathbb{R} : \gamma h_{m,b,\sigma}(x) > -\tilde{h}_{0,b}(x)/2\},
\]

with \(\tilde{h}_{0,b}\) and \(h_{m,b,\sigma}\) as in (5.4) and (5.5), respectively, as well as the function

\[
g_{b,\sigma} := M_{0X}(b)e^{-b} \gamma h_{m,b,\sigma} 1_{A_{b,\sigma}} - \frac{1}{2} f_{0X} 1_{A_{b,\sigma}}.
\]

In the following lemma, we prove the existence of a compactly supported discrete mixing probability measure, with a sufficiently small number of support points, such that the corresponding Laplace-normal mixture has Hellinger distance of the order \(O(\sigma^{\alpha+2})\) from the sampling density \(f_{0Y}\) having an \(\alpha\)-Sobolev regular mixing density \(f_{0X}\) with exponentially decaying tails.

**LEMMA E.1.** Let \(f_{\xi}\) be the standard Laplace density. Let \(f_{0X}\) be a density satisfying Assumption 4.3, Assumption 4.4 for \(\alpha > 0\) and Assumption 4.5. For \(\sigma > 0\) small enough, there exist a constant \(A_0 > 0\) and a discrete probability measure on \([-a_\sigma, a_\sigma]\), with \(a_\sigma :=

A_0|\log \sigma|$, having at most $N = O((a_\sigma/\sigma)|\log \sigma|^{1/2})$ support points, such that, for $f_Y := f_\varepsilon \ast (\mu_H \ast \phi_\sigma)$ and $f_{0Y} := f_\varepsilon \ast f_{0X}$.

$$d_H(f_Y, f_{0Y}) \lesssim \delta_0^{-1/2} e^{a_\sigma/2\sigma^2}$$
as soon as $P_{0X}(|X| \leq a_\sigma) \geq \delta_0$ for some $0 < a_\sigma < a_\sigma$ and $0 < \delta_0 < 1$.

**Proof.** Reasoning as in Lemma D.2, for $a_\sigma$, $\delta_0$ as in the statement, $d^2_{H}(f_Y, f_{0Y}) \leq 2\delta_0^{-1} e^{a_\sigma}||g_Y - g_{0Y}||_2^2$, where $g_Y := U f_Y$ and $g_{0Y} := U f_{0Y}$, with $U$ defined in (D.1). Note that $(e^{1/2} f_{0X}) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by Assumption 4.3. Also, $g_Y, g_{0Y} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ so that, not only are the corresponding Fourier transforms $\hat{g}_Y, \hat{g}_{0Y}$ well defined, but $||g_Y||^2_2 = (2\pi)^{-1}||\hat{g}_Y||^2_2$ and $||g_{0Y}||^2_2 = (2\pi)^{-1}||\hat{g}_{0Y}||^2_2$. In order to bound $||g_Y - g_{0Y}||^2_2$, some definitions and preliminary facts are exposed. For $T \geq \lceil (\alpha + 2)/\vartheta \rceil$, with $\vartheta$ suitably chosen later on, we define the set $E_\sigma := \{x \in \mathbb{R} : f_{0X}(x) > \sigma^T\}$. The tail condition on $f_{0X}$ of Assumption 4.3 implies that $E_\sigma \subset \{|x| \leq A_0|\log \sigma|\}$ for some $A_0 > 0$. Note that $A_0$ can be chosen arbitrarily large by choosing $T$ large enough because $A_0$ is proportional to $T/(1 + C_0)$. Set $B_0 := E_{0X}|\sigma^{\frac{2d}{\vartheta}}(X) < \infty$, then

(E.3) $P_{0X}(E_\sigma^{\vartheta}) \leq B_0 \sigma^{\vartheta T} \lesssim \sigma^{\alpha + 2}$

by definition of $T$. Introduced the densities

$$\bar{h}_{b,\sigma} := \frac{f_{0X} + g_{b,\sigma}}{||f_{0X} + g_{b,\sigma}||_1}, \quad \text{and} \quad \bar{h}_{b,\sigma} 1_{E_\sigma} \frac{1}{||\bar{h}_{b,\sigma} 1_{E_\sigma}||_1},$$

where $g_{b,\sigma}$ is as defined in (E.2), we consider the decomposition

$$||g_Y - g_{0Y}||_2^2 \leq \sum_{b = \pm 1/2} ||e^{b} \{f_\varepsilon \ast [f_{0X} - \phi_\sigma \ast (T_{m, b, \sigma} f_{0X})]\}||_2^2$$

$$+ \sum_{b = \pm 1/2} ||e^{b} \{f_\varepsilon \ast \phi_\sigma \ast (T_{m, b, \sigma} f_{0X}) - (f_{0X} + g_{b,\sigma})\}||_2^2$$

$$+ \sum_{b = \pm 1/2} ||e^{b} \{f_\varepsilon \ast \phi_\sigma \ast (f_{0X} + g_{b,\sigma}) - \bar{h}_{b,\sigma}\}||_2^2$$

$$+ \sum_{b = \pm 1/2} ||e^{b} \{f_\varepsilon \ast \phi_\sigma \ast (\bar{h}_{b,\sigma} - (\bar{h}_{b,\sigma} 1_{E_\sigma} ||\bar{h}_{b,\sigma} 1_{E_\sigma}||_1))\}||_2^2$$

$$+ \sum_{b = \pm 1/2} ||e^{b} \{f_\varepsilon \ast \phi_\sigma \ast (\bar{h}_{b,\sigma} 1_{E_\sigma} ||\bar{h}_{b,\sigma} 1_{E_\sigma}||_1) - \mu_H\}||_2^2$$

$$=: \sum_{r = 1}^5 V_r.$$  

We show that each term $V_1, \ldots, V_5$ is of the order $O(\sigma^{2(\alpha + 2)})$. First by inequality (5.6) of Lemma 5.1, we have $V_1 \lesssim \sigma^{2(\alpha + 2)}$.

**Study of the term $V_2$**

We recall that $g_b(t) := [1 - \psi_b^2(t)]$, with $\psi_b(t) := -(t + b), t \in \mathbb{R}$, and that

$$h_{m, b, \sigma} := \frac{1}{\gamma} \sum_{k=1}^{m-1} \frac{(-1)^k \bar{a}^{2k}}{2^{2k} k!} \sum_{j=0}^{2k} \binom{2k}{j} (-b)^{2k-j} (\bar{h}_{0,b} \ast D^j H_b).$$
We write
\[
\int_{\mathbb{R}} [e^{s\psi(t)^2/2} F^2 - e^{s\psi(t)^2/2} F^2]^2 dt
\]
as
\[
\int_{\mathbb{R}} [e^{s\psi(t)^2/2} F^2 - e^{s\psi(t)^2/2} F^2]^2 dt
\]
so that, using the definition of \( g_{b,\sigma} \) in (E.2),
\[
V_2 = \sum_{b=\pm 1/2} \left\| e^{b} \left\{ f_{\varepsilon} * \phi_{\sigma} * \left[ M_{0X}(b) e^{-b} \gamma h_{m,b,\sigma} 1_{A_{b,\sigma}^c} - \frac{1}{2} f_{0X} 1_{A_{b,\sigma}^c} \right] \right\} \right\|^2_2
\]
\[
\lesssim \sum_{b=\pm 1/2} \int_{\mathbb{R}} \left| \frac{e^{s\psi(t)^2/2} F}{\| g_0(t) \|^2} \right|^2 \left| \mathcal{F} \{ [2\gamma h_{m,b,\sigma} - \bar{h}_{0,b}] 1_{A_{b,\sigma}^c} \} (t) \right|^2 dt
\]
\[
\lesssim \sum_{b=\pm 1/2} (\| \gamma h_{m,b,\sigma} 1_{A_{b,\sigma}^c} \|_1 + \| \bar{h}_{0,b} 1_{A_{b,\sigma}^c} \|_1)^2,
\]
where we have used the facts that
\[
\| \mathcal{F} \{ [2\gamma h_{m,b,\sigma} - \bar{h}_{0,b}] 1_{A_{b,\sigma}^c} \} \|_{\infty} \leq \| \gamma h_{m,b,\sigma} 1_{A_{b,\sigma}^c} \|_1 + \| \bar{h}_{0,b} 1_{A_{b,\sigma}^c} \|_1
\]
and
\[
\frac{|e^{s\psi(t)^2/2} F|^2}{\| g_0(t) \|^2} \lesssim \frac{1}{1 + t^4}.
\]
Finally, using the inequalities in (F.4) of Lemma F.3, we obtain that
\[
V_2 \lesssim \sigma^{2eR} \lesssim \sigma^{2(\alpha+2)}.
\]

Study of the term \( V_3 \)
By the inequalities in (F.4) and (F.5) of Lemma F.3, noting that \( \| \mathcal{F} \{ \bar{h}_{0,b} \} \|_\infty \leq 1 \), we have
\[
V_3 = \sum_{b=\pm 1/2} \left( 1 - \frac{1}{\| f_{0X} + g_{b,\sigma} \|_1} \right)^2 \left\| e^{b} \left\{ f_{\varepsilon} * \phi_{\sigma} * (f_{0X} + g_{b,\sigma}) \right\} \right\|^2_2
\]
\[
\lesssim \sigma^{2eR} \sum_{b=\pm 1/2} \int_{\mathbb{R}} \left[ \frac{e^{s\psi(t)^2/2} F}{\| g_0(t) \|^2} \right]^2 \left\| \mathcal{F} \{ \bar{h}_{0,b} \} (t) \right\|^2 + \left\| \mathcal{F} \{ y_{m,b,\sigma} \} (t) \right\|^2 dt.
\]
Recalling that \( |\mathcal{F} \{ D^j H_{b} \} (t) | \leq |t|^j |\mathcal{F} \{ H \} (\delta t) | \leq |t|^j \) for \( j = 0, \ldots, 2k \), then
\[
\int_{\mathbb{R}} \left[ \frac{e^{s\psi(t)^2/2} F}{\| g_0(t) \|^2} \right]^2 \left\| \mathcal{F} \{ y_{m,b,\sigma} \} (t) \right\|^2 dt \leq \sum_{k=1}^{m-1} \frac{e^{s^2/4}}{2^k k!} \int_{\mathbb{R}} \left[ \frac{\sigma^2 (t^2 + 1/4)}{e^{s^2/2} \| g_0(t) \|^2} \right]^2 \left\| \mathcal{F} \{ \bar{h}_{0,b} \} (t) \right\|^2 dt
\]
\[
\lesssim 2^{eR} \lesssim \sigma^{2eR} \lesssim \sigma^{2(\alpha+2)}.
\]
which in turns implies that
\[
V_3 \lesssim \sigma^{2eR} \lesssim \sigma^{2(\alpha+2)}.
\]
Study of the term $V_4$

Taking into account that $\|f_{0X} + g_{b,\sigma}\|_1 \geq 1$, see inequalities (F.5) of Lemma F.3, we have

$$V_4 \lesssim \sum_{b = \mp 1/2} \left( \|\bar{h}_{b,\sigma}1_{E_\sigma}\|^2_1 \times \|e^b \{f_\varepsilon \ast \phi_\sigma \ast (\bar{h}_{b,\sigma}1_{E_\sigma} / \|\bar{h}_{b,\sigma}1_{E_\sigma}\|_1)\}\|_2^2 \right. \left. + \|e^b \{f_\varepsilon \ast \phi_\sigma \ast (\bar{h}_{b,\sigma}1_{E_\sigma})\}\|_2^2 \right)$$

$$\lesssim \sum_{b = \mp 1/2} \| (f_{0X} + g_{b,\sigma})1_{E_\sigma}\|^2_1 \times \int_R \frac{|e^{\sigma^2 \psi_\sigma(t)^2/2}|^2}{|g_b(t)|^2} |\mathcal{F}\{e^b \bar{h}_{b,\sigma}1_{E_\sigma} / \|\bar{h}_{b,\sigma}1_{E_\sigma}\|_1\}(t)|^2 \, dt$$

$$+ \sum_{b = \mp 1/2} \int_R \frac{|e^{\sigma^2 \psi_\sigma(t)^2/2}|^2}{|g_b(t)|^2} |\mathcal{F}\{e^b \bar{h}_{b,\sigma}1_{E_\sigma}\}(t)|^2 \, dt.$$\hspace{1cm}$$\text{Note that}$$

$$\|(f_{0X} + g_{b,\sigma})1_{E_\sigma}\|_1 \leq \frac{3}{2} P_0X(E_\sigma^c) + M_0X(b) \int_{A_{b,\sigma} \cap E_\sigma^c} e^{-bx} |\gamma h_{m,b,\sigma}(x)| \, dx$$

$$\leq \frac{3B_0}{2} \sigma^{\alpha T} + M_0X(b) \int_{A_{b,\sigma} \cap E_\sigma^c} e^{-bx} |\gamma h_{m,b,\sigma}(x)| \, dx,$$

where, as hereafter shown,

(E.4) \hspace{1cm} \int_{A_{b,\sigma} \cap E_\sigma^c} e^{-bx} |\gamma h_{m,b,\sigma}(x)| \, dx \lesssim \sigma^{\alpha + 2},$$

and

(E.5) \hspace{1cm} \|\mathcal{F}\{e^b \bar{h}_{b,\sigma}1_{E_\sigma}\}\|_{\infty} \lesssim \sigma^{\alpha + 2}.
To prove inequality (E.4), note that, by Lemma F.2 and inequality (E.3), for every $j = 0, \ldots, 2k$, since $R > 2$, by Hölder’s inequality,
\[
\int_{A_{b,\sigma} \cap E^\sigma_\alpha} \left| \int_{\mathbb{R}} e^{-bu} f_{0X}(x - u) D^j H_\delta(u) \, du \right| \, dx \\
\leq \int_{A_{b,\sigma} \cap E^\sigma_\alpha} \left| \int_{\mathbb{R}} [e^{-bu} f_{0X}(x - u) - f_{0X}(x)] D^j H_\delta(u) \, du \right| \, dx \\
+ \int_{A_{b,\sigma} \cap E^\sigma_\alpha} \int_{\mathbb{R}} |D^j H_\delta(u)| \, du \, dx \\
\lesssim C_j \delta^{-j+\nu} \int_{A_{b,\sigma} \cap E^\sigma_\alpha} [L_0(x) + f_{0X}(x)] \, dx + C_0, j \int_{A_{b,\sigma} \cap E^\sigma_\alpha} f_{0X}(x) \, dx \\
\lesssim C_j \delta^{-j+\nu} \int_{A_{b,\sigma} \cap E^\sigma_\alpha} \left( \frac{L_0(x)}{f_{0X}(x)} \right)^{1/R} f_{0X}^{1/R}(x) f_{0X}^{1-1/R}(x) \, dx + [C_0, j + C_j \delta^{-j+\nu}] P_{0X}(E^\sigma_\alpha) \\
\lesssim \delta^{-j+\nu} \left( \int_{A_{b,\sigma} \cap E^\sigma_\alpha} f_{0X}(x) \left( \frac{L_0(x)}{f_{0X}(x)} \right)^{R} \, dx \right)^{1/R} P_{0X}(E^\sigma_\alpha)^{1-1/R} + \sigma^{\vartheta T-j+\nu} \\
\lesssim \delta^{-j+\nu} \left( \int_{\mathbb{R}} f_{0X}(x) \left( \frac{L_0(x)}{f_{0X}(x)} + 1 \right)^{R} \, dx \right)^{1/R} [P_{0X}(E^\sigma_\alpha)]^{1-1/R} + \sigma^{\vartheta T-j+\nu} \\
\lesssim \delta^{-j+\nu} \left( \int_{\mathbb{R}} f_{0X}(x) \left( \frac{L_0(x)}{f_{0X}(x)} + 1 \right)^{R} \, dx \right)^{1/R} \sigma^{\vartheta T(1-1/R)} + \sigma^{\vartheta T-j+\nu}.
\]

Consequently,
\[
\int_{A_{b,\sigma} \cap E^\sigma_\alpha} e^{-bx} |\gamma h_{m,b,\sigma}(x)| \, dx \lesssim \sigma^{\nu + \vartheta T(1-1/R)} + \sigma^{\vartheta T} \lesssim \sigma^{\alpha+2}
\]

by choosing $[(\alpha + 2)/T] \leq \vartheta < [1 \wedge (v R/T)]$. Thus $\|f_{0X} + g_{b,\sigma}1_{E^\sigma_\alpha}\|_1 \lesssim \sigma^{\alpha+2}$. Inequality (E.5) essentially follows from the tail condition on $f_{0X}$ of Assumption 4.3 and the definition of the set $E^\sigma_\alpha$.

**Study of the term $V_5$**

Recalling that $B$ stands for the bilateral transform operator, we have
\[
V_5 = \sum_{b = \pm 1/2} \int_{\mathbb{R}} \frac{e^{\sigma^2 \psi_b(t)^2 / 2}}{\| \theta_b(t) \|^2} \| B\{\bar{h}_{b,\sigma}1_{E_\sigma} / \| \bar{h}_{b,\sigma}1_{E_\sigma} \|_1 \} - B\{\mu_H\}(\psi_b(t)) \|^2 \, dt.
\]

For $M > 0$, split the integral domain into $|t| \leq M$ and $|t| > M$ and let the corresponding terms be denoted by $V_5^{(1)}$ and $V_5^{(2)}$. Let $\mu_H$ be a discrete probability measure on $E_\sigma$ satisfying the constraints
\[
\int_{E_\sigma} w_j \mu_H(du) = \int_{E_\sigma} w_j \bar{h}_{b,\sigma}(u) \| \bar{h}_{b,\sigma}1_{E_\sigma} \|_1 \, du, \quad \text{for } j = 1, \ldots, J - 1,
\]

with $J = [\eta e a M]$ for some $\eta > 1$, together with
\[
\int_{E_\sigma} e^{bu} \mu_H(du) = \int_{E_\sigma} e^{bu} \bar{h}_{b,\sigma}(u) \| \bar{h}_{b,\sigma}1_{E_\sigma} \|_1 \, du,
\]

(E.6)
where the integral on the right-hand side of (E.6) is finite because \( \int_{E_\sigma} e^{b_1 \bar{h}_{b,\sigma}(u)} \, du \leq \int_{\mathbb{R}} e^{b_1 \bar{h}_{b,\sigma}(u)} \, du \lesssim M_0 X(b) [1 + \int_{\mathbb{R}} \gamma h_{m,b,\sigma}(u) \, du] = M_0 X(b) [1 + \gamma [1 + O(\sigma^{-m-2})]] \) by relationship (5.7) of Lemma 5.1. Thus,

\[
\int_{E_\sigma} e^{b_1 \bar{h}_{b,\sigma}(u)} \, du = O(1).
\]

Also, by the lower bound inequality in (F.5) of Lemma F.3 and the previously proven fact that \( \| (f_0X - g_{b,\sigma}) 1_{E_\sigma} \|_1 \lesssim \sigma^{\alpha+2} \), we have \( \| h_{b,\sigma} 1_{E_\sigma} \|_1 = 1 - \| \bar{h}_{b,\sigma} 1_{E_\sigma} \|_1 \geq 1 - \| (f_0X - g_{b,\sigma}) 1_{E_\sigma} \|_1 \geq 1 - \sigma^{\alpha+2} \). Therefore,

\[
\| B \{ h_{b,\sigma} 1_{E_\sigma} / \| \bar{h}_{b,\sigma} 1_{E_\sigma} \|_1 \} (\psi_b) \|_\infty \leq (\| \bar{h}_{b,\sigma} 1_{E_\sigma} \|_1)^{-1} \| e^{b_1 \bar{h}_{b,\sigma} 1_{E_\sigma}} \|_1 \lesssim \int_{E_\sigma} e^{b_1 \bar{h}_{b,\sigma}(u)} \, du.
\]

Then, using Lemma D.1 with \( r = J \), by the inequality \( J! \geq (J/e)^J \), we have

\[
V_{5}^{(1)} := \sum_{b=\mp 1/2} \left[ \int_{|t| \leq M} \frac{|e^{\alpha^2 \psi_b(t)^2 / 2}|^2}{|g_b(t)|^2} \right] \| B \{ \bar{h}_{b,\sigma} 1_{E_\sigma} / \| \bar{h}_{b,\sigma} 1_{E_\sigma} \|_1 \} - B \{ M_H \} (\psi_b(t)) \|^2 \, dt 
\leq \frac{a^{2J}}{(J!)^2} \int_0^M t^{2(J-2)} \, dt
\leq M^{-2(\alpha+2)} \times \frac{a^{2J}}{(J!)^2} \frac{M^{2J+\alpha+1}}{2J - 3} \lesssim M^{-2(\alpha+2)} \left( \frac{e^M}{J} \right)^{2J+1} M^{2\alpha} \lesssim M^{-2(\alpha+2)}
\]

because \( (e^M/J)^{2J+1} M^{2\alpha} < e^{-2(\log M + 1)\alpha} M^{2\alpha} < 1 \). Choosing \( M \) so that \( (\sigma M)^2 \geq (2\alpha + 1) |\log \sigma| \), equivalently, \( M \geq \sigma^{-1} [(2\alpha + 1) |\log \sigma|]^{1/2} \), and taking into account that \( |e^{\sigma^2 \psi_b^2(t)^2 / 2}| = O(e^{-\sigma^2 t^2}) \), we have

\[
V_{5}^{(2)} := \sum_{b=\mp 1/2} \left[ \int_{|t| > M} \frac{|e^{\alpha^2 \psi_b(t)^2 / 2}|^2}{|g_b(t)|^2} \right] \| B \{ \bar{h}_{b,\sigma} 1_{E_\sigma} / \| \bar{h}_{b,\sigma} 1_{E_\sigma} \|_1 \} - B \{ M_H \} (\psi_b(t)) \|^2 \, dt
\leq e^{-(\sigma M)^2} \int_{|t| > M} t^{-4} \, dt \lesssim e^{-(\sigma M)^2} M^{-3} \lesssim \sigma^{2\alpha+1} M^{-3} \lesssim \sigma^{2(\alpha+2)}.
\]

Therefore, \( V_5 \lesssim \sigma^{2(\alpha+2)} \). Finally, \( \| g_Y - g_0 Y \|_2^2 \lesssim \sum_{r=1}^5 V_r \lesssim \sigma^{2(\alpha+2)} \) and the assertion follows.

APPENDIX F: TECHNICAL LEMMAS

**Lemma F.1.** For \( r \geq 0, a \in \mathbb{R} \) and \( j \in \{0\} \cup \mathbb{N} \) fixed, there exists a constant \( C_{r,j} < \infty \) such that, for \( h = o(1) \) and \( \delta = o(h) \),

(F.1) \[
\int_{\mathbb{R}} |x|^r e^{a \delta x} |D^j H(x)| \, dx \leq C_{r,j}.
\]

**Proof.** Recalling that \( H(x) = (2\pi)^{-1/2} \hat{\tau}(x) e^{-(hx)^2/2}, x \in \mathbb{R} \), we have

\[
D^j H(x) = \frac{1}{2\pi} \sum_{i=0}^j \binom{j}{i} \hat{\tau}^{(i)}(x) D^{j-i} e^{-(hx)^2/2}, \quad x \in \mathbb{R},
\]

where \( D^{j-i} e^{-(hx)^2/2} \) is a linear combination of terms of the form \( e^{-(hx)^2/2} (-1)^{j_1} h^{j_2} x^{j_3} \), where \( 0 \leq j_1, j_2, j_3 \leq (j - i) \). Note that \( e^{a \delta x} e^{-(hx)^2/2} = e^{-hx(a\delta/h)^2/2} (e^{a \delta x/h^2})^{j-i} \lesssim (e^{a \delta x/h^2})^{j-i} \).
\(e^{(a\delta/h)^2/2}, \text{ where } e^{(a\delta/h)^2/2} = 1 + o(1) \text{ because } (\delta/h) = o(1).\) Then, by condition (5.3), for \(\nu > (r+j+1)\) and \(0 \leq j_1, j_2, j_3 \leq (j-i),\)

\[
|x|^r e^{a\delta z} |\hat{f}^{(i)}(x)| e^{-(h_0)^2/2} |x|^{j_1} \leq |x|^{r+j_1} |\hat{f}^{(i)}(x)|,
\]

where the function on the right-hand side is integrable. The assertion follows. \(\square\)

**Lemma F.2.** Suppose that \(f_{0X}\) satisfies the local Hölder condition (4.1) of Assumption 4.5 for \(0 < \nu \leq 1\) and \(L_0 \in L^1(\mathbb{R}).\) For every \(b \in \mathbb{R}\) and \(j \in \{0\} \cup \mathbb{N},\) if \(h = o(1)\) and \(\delta = o(h),\) then

\[
(\text{F.2}) \quad \left| \int_{\mathbb{R}} [e^{-bu} f_{0X}(x-u) - f_{0X}(x)]D^j H_0(u) \, du \right| \leq C_j \|D^\nu [L_0(x) + f_{0X}(x)], x \in \mathbb{R},
\]

where \(C_j := \max \{3C_{v,j}, C_{1,j} \} > 0,\) with \(C_{v,j}\) as in (F.1).

**Proof.** For every \(x \in \mathbb{R},\) by the local Hölder condition (4.1) of Assumption 4.5 and Lemma D.1,

\[
\delta^j \left| \int_{\mathbb{R}} [e^{-bu} f_{0X}(x-u) - f_{0X}(x)]D^j H_0(u) \, du \right| = \left| \int_{\mathbb{R}} [e^{-b\delta z} f_{0X}(x - \delta z) - f_{0X}(x)]D^j H(z) \, dz \right| \\
\leq \int_{\mathbb{R}} |e^{-b\delta z} - 1||f_{0X}(x - \delta z) - f_{0X}(x)||D^j H(z)| \, dz \\
+ f_{0X}(x) \int_{\mathbb{R}} |e^{-b\delta z} - 1||D^j H(z)| \, dz \\
+ \int_{\mathbb{R}} |f_{0X}(x - \delta z) - f_{0X}(x)||D^j H(z)| \, dz \\
\leq 3 \int_{\mathbb{R}} |f_{0X}(x - \delta z) - f_{0X}(x)||D^j H(z)| \, dz + f_{0X}(x) \int_{\mathbb{R}} |e^{-b\delta z} - 1||D^j H(z)| \, dz \\
\leq 3\delta^\nu L_0(x) \int_{\mathbb{R}} |z|^{\nu} ||D^j H(z)| \, dz + b\delta f_{0X}(x) \int_{\mathbb{R}} |z| ||D^j H(z)| \, dz \\
\leq 3\delta^\nu C_{v,j} L_0(x) + b\delta C_{1,j} f_{0X}(x) < C_j \delta^\nu [L_0(x) + f_{0X}(x)].
\]

Inequality (F.2) follows. \(\square\)

**Lemma F.3.** For \(h = o(1),\) let \(\delta = o(h).\) For \(m \in \mathbb{N}, b = \mp 1/2\) and \(\sigma = o(1),\) let the set \(A_{b,\sigma}\) be defined as in (E.1). Under Assumptions 4.3 and 4.5 on \(f_{0X},\) the latter with \(0 < \nu \leq 1,\)

\(L_0 \in L^1(\mathbb{R})\) and any \(R > 0,\) there exists a constant \(C_m > 0,\) depending on \(m\) and \(\nu,\) such that, for \(\sigma\) small enough,

\[
(\text{F.3}) \quad \forall b = \mp 1/2, \quad A_{b,\sigma}^C \subseteq B_{\sigma},
\]

with \(B_{\sigma} := \{x : |L_0(x) + f_{0X}(x)| > \tilde{C}_m^{-1} \sigma^{-\nu} f_{0X}(x)\}.\) Furthermore, there exist constants \(C_R, S_R > 0,\) depending on \(m, \nu,\) and \(R,\) so that

\[
(\text{F.4}) \quad \|h_{0,b} 1_{A_{b,\sigma}}\|_1 < C_R \sigma^{\nu R}, \quad \|h_{m,b,\sigma} 1_{A_{b,\sigma}}\|_1 < 2C_R \sigma^{\nu R}
\]

and the function \(f_{0X} + g_{b,\sigma},\) with \(g_{b,\sigma}\) as defined in (E.2), which is non-negative, has

\[
(\text{F.5}) \quad 1 \leq \|f_{0X} + g_{b,\sigma}\|_1 \leq 1 + S_R \sigma^{\nu R}.
\]
where \(0 < \varepsilon < 1\) and \(\tilde{h}_{0,b}(x)\) is small enough, by Lemma F.2,

\[
\bigg|\sum_{k=1}^{m-1} \frac{(-1)^k \sigma^{2k}}{2^k k!} \sum_{j=0}^{2k} \binom{2k}{j} (-b)^{2k-j} \int_{\mathbb{R}} \tilde{h}_{0,b}(x-u) D^j H_\delta(u) \, du\bigg|
\]

\[
\leq \frac{1}{M_0X(b)} \sum_{k=1}^{m-1} \sigma^{2k} \sum_{j=0}^{2k} \binom{2k}{j} |b|^{2k-j} e^{bx} \int_{\mathbb{R}} |e^{-bu} f_0X(x-u) - f_0X(x)| D^j H_\delta(u) \, du
\]

\[
< \frac{1}{M_0X(b)} \left( \sum_{k=1}^{m-1} \frac{1}{(2\varepsilon \sigma)^{k!}} \max_{0 \leq j \leq 2k} C_j \right) \sigma^{2k} e^{bx} [L_0(x) + f_0X(x)]
\]

where \(0 < \tilde{C}_m < \infty\). Then, for \(\tilde{C}_m := 4\tilde{C}_m\), we have \(A_{\tilde{h}_{0,b}} \subseteq B_{\sigma}\).

We now prove the inequalities in (F.4). Concerning the first one,

\[
\int_{A_{\tilde{h}_{0,b}}(\sigma)} \tilde{h}_{0,b}(x) \, dx < \sigma^{vR} \tilde{C}_m \frac{R}{M_0X(b)} \int_{B_{\sigma}} e^{bx} f_0X(x) \left( \frac{L_0(x)}{f_0X(x)} + 1 \right)^R \, dx \leq C_R \sigma^{vR},
\]

where

\[
C_R = \sigma^{vR} \tilde{C}_m \frac{R}{M_0X(-1/2) \wedge M_0X(1/2)} \int_{\mathbb{R}} e^{|x|/2} f_0X(x) \left( \frac{L_0(x)}{f_0X(x)} + 1 \right)^R \, dx > \infty
\]
because of condition (4.2) and Assumption 4.3. Concerning the second inequality, from previous computations,
\[
\int_{A_{b,\sigma}} |(\tilde{h}_{0,b} * D^j \beta)(x)| \, dx \leq \delta^{-j} \int_{A_{b,\sigma}} \int_{\mathbb{R}} |\tilde{h}_{0,b}(x - \delta u) - \tilde{h}_{0,b}(x)| |D^j \beta(u)| \, du + C_{j,0} C_R \sigma^{v_R}
\]
which implies that \( \| \gamma h_{m,b,\sigma} \mathbf{1}_{A_{b,\sigma}} \|_1 \leq 2 C_R \sigma^{v_R} \).

To prove the last part of the lemma, we begin by noting that
\[
f_{0X} + g_{b,\sigma} = [f_{0X} + M_0X(b)e^{-bx}\gamma h_{m,b,\sigma}] \mathbf{1}_{A_{b,\sigma}} + \frac{1}{2} f_{0X} \mathbf{1}_{A_{b,\sigma}} > \frac{1}{2} f_{0X} \geq 0
\]
and
\[
M_0X(b) \int_{\mathbb{R}} e^{-bx}\gamma h_{m,b,\sigma}(x) \, dx = 0.
\]
In fact, since \( \int_{\mathbb{R}} e^{-bx} H(x) \, dx < \infty \) because \( (\delta/h) = o(1) \), we have
\[
M_0X(b) \int_{\mathbb{R}} e^{-bx}\gamma h_{m,b,\sigma}(x) \, dx = \sum_{k=1}^{m-1} \frac{(-1)^k \sigma^{2k}}{2^k k!} \sum_{j=0}^{2k} \binom{2k}{j} (-b)^{2k-j} \int_{\mathbb{R}} e^{-bu} D^j \beta(u) \, du
\]
\[
= \sum_{k=1}^{m-1} \frac{(-1)^k \sigma^{2k}}{2^k k!} \sum_{j=0}^{2k} \binom{2k}{j} (-b)^{2k-j} b^j \int_{\mathbb{R}} e^{-bx} H(x) \, dx = 0.
\]
Then, since \( g_{b,\sigma} = M_0X(b)e^{-bx}\gamma h_{m,b,\sigma} - [M_0X(b)e^{-bx}\gamma h_{m,b,\sigma} + (f_{0X}/2)] \mathbf{1}_{A_{b,\sigma}} \), we have
\[
\int_{\mathbb{R}} (f_{0X} + g_{b,\sigma})(x) \, dx = 1 + \int_{\mathbb{R}} M_0X(b)e^{-bx}\gamma h_{m,b,\sigma}(x) \, dx
\]
\[
= 1 - \int_{A_{b,\sigma}} \left[ M_0X(b)e^{-bx}\gamma h_{m,b,\sigma}(x) + \frac{1}{2} f_{0X}(x) \right] \, dx \geq 1.
\]
On the other side, using Lemma F.2 and reasoning as in the first part of the present lemma,
\[
\int_{\mathbb{R}} (f_{0X} + g_{b,\sigma})(x) \, dx = 1 - \int_{A_{b,\sigma}} \left[ M_0X(b)e^{-bx}\gamma h_{m,b,\sigma}(x) + \frac{1}{2} f_{0X}(x) \right] \, dx
\]
\[
\leq 1 + \int_{A_{b,\sigma}} \left[ M_0X(b)e^{-bx}|\gamma h_{m,b,\sigma}(x)| + \frac{1}{2} f_{0X}(x) \right] \, dx
\]
\[
\leq 1 + 2[M_0X(1/2) \vee M_0X(-1/2)] C_R \sigma^{v_R}.
\]
Conclude that \( 1 \leq \| f_{0X} + g_{b,\sigma} \| \leq 1 + S_R \sigma^{v_R} \). The proof is thus complete. \( \square \)
**Remark F.1.** Although in condition (4.1) of Assumption 4.5 the constant $R$ is such that $R \geq m/v$, with the smallest integer $m \geq (\alpha + 2)$, in Lemma F.3, indeed, $R$ can be any positive real.

**APPENDIX G: LEMMA FOR THEOREM G.1 ON ADAPTIVE POSTERIOR CONTRACTION RATES FOR $L^1$-WASSERSTEIN DECONVOLUTION OF LAPLACE MIXTURES**

The following lemma assesses the order of the bias of the distribution function corresponding to a Gaussian mixture, where the mixing distribution is any probability measure on the real line and the scale parameter is equal to the kernel bandwidth times a logarithmic factor. It shows that condition (3.7) of Theorem 3.2 is verified for a universal constant $C_1$.

**Lemma G.1.** Let $F_X$ be the distribution function of $\mu_X = \mu_H * \phi_\sigma$, with $\mu_H \in \mathcal{P}$ and $\sigma > 0$. Let $K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be symmetric, with $\int \|z||K(z)||dz < \infty$, and $\hat{K} \in L^1(\mathbb{R})$ such that $\hat{K} \equiv 1$ on $[-1, 1]$. Given $h > 0$, for $\sigma = O(\sqrt{2}h \log h^{2\alpha+1}/|t|/2)$, we have

\[
\|F_X * K_h - F_X\|_1 = O(h^{\alpha+1}).
\]

**Proof.** Let $b_{F_X} := (F_X * K_h - F_X)$. Defined $\hat{f}(t) := [1 - \hat{K}(ht)][\hat{\phi}(\sigma/\sqrt{2})1_{[-1,1]}(ht)/t]$, $t \in \mathbb{R}$, since $t \rightarrow [\hat{\mu}_H(t)\hat{\phi}(\sigma/\sqrt{2})][\hat{f}(t)$ is in $L^1(\mathbb{R})$, arguing as for $G_{2,h}$ in [12], pp. 251–252, we have

\[
\|b_{F_X}\|_1 = \int_\mathbb{R} \left\{ \frac{1}{2\pi} \int_{|t|>1/h} \exp(-itx)\hat{\mu}_H(t)\hat{\phi}(\sigma/\sqrt{2})\hat{f}(t) \right\} dt dx
\]

\[
= \|\mu_H * \phi_{\sigma/\sqrt{2}} * f\|_1 \leq \|\mu_H * \phi_{\sigma/\sqrt{2}}\|_1 \times \|f\|_1 \leq (\|\hat{f}\|_2 + \|\hat{f}(1)\|_2)^{1/2}/\sqrt{2},
\]

see, e.g., [3], p. 1031, for the last inequality. Since $\int_1^\infty \hat{\phi}(\sigma t) dt \leq (h/\sigma^2)\hat{\phi}(\sigma/h)$, we have $\|\hat{f}\|_2 \leq \hat{\phi}(\sigma/h) \int_{|t|>1/h} [1 + (\hat{K}(ht))^2]t^{-2} dt \leq h^{2(\alpha+1)}$ because $\|\hat{K}\|_\infty \leq \|K\|_1 < \infty$. Write

\[
\hat{f}(1)(t) = -\left\{ h\hat{K}(1)(ht) + \left( \frac{1}{t} + \frac{\sigma^2}{2} \right) \hat{\phi}(\sigma/\sqrt{2}) \right\} 1_{[-1,1]}(ht), \quad t \in \mathbb{R}.
\]

Since $K \in L^1(\mathbb{R})$ and $zK(z) \in L^1(\mathbb{R})$ jointly imply that $\hat{K}$ is continuously differentiable with $|\hat{K}(1)(t)| \rightarrow 0$ as $|t| \rightarrow \infty$, so that $\hat{K}(1) \in C_b(\mathbb{R})$, we have $\|\hat{f}(1)\|_2 \leq \int_{|t|>1/h} (h^2\hat{K}(1)(ht))^2 + (h^4 + \sigma^4)[1 + (\hat{K}(ht))^2] \hat{\phi}(\sigma t) dt \leq h^{2(\alpha+1)}$. Hence, $\|\hat{f}(1)\|_2 \leq h^{2(\alpha+1)}$. The assertion follows.

**Remark G.1.** Due to the exponentially decaying tails of the Gaussian density and a suitable choice of the scale parameter $\sigma$ as a multiple of the kernel bandwidth $h$, times a logarithmic factor, a different argument than that used in Lemma C.2 is used to bound the bias of $F_X$.

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