Infra-red expansion of entanglement entropy in the Interacting Resonant Level Model

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In this paper we develop a method to describe perturbatively the entanglement entropy in a simple impurity model, the interacting resonant level model (IRLM), at low energy (i.e. in the strong coupling regime). We use integrability results for the Kondo model to describe the infra-red fixed point, conformal field theory techniques initially developed by Cardy and Calabrese and a quantization scheme that allows one to compute exactly Renyi entropies at arbitrary order in $1/T_n$ in principle, even when the system size or the temperature is finite. We show that those universal quantities at arbitrary interaction parameter in the strong coupling regime are very well approximated by the same quantities in the free fermion system in the case of attractive Coulomb interaction, whereas a strong dependence on the interaction appears in the case of repulsive interaction.

I. INTRODUCTION

Entanglement is a property allowed by quantum mechanics that describes the fact that generically, a quantum state of a system consisting of several subparts cannot be written as a product of states of the subparts. One way to characterize, at zero temperature, this fascinating property of quantum systems, is to introduce the so-called entanglement entropy. Since it was first considered, this quantity has found many applications, ranging from the study of black holes [1,2], to modern methods of simulation of quantum systems [3].

The study of quantum impurity problems from the point of view of entanglement entropy has led to many new insights and puzzles, both in [4] and out of equilibrium situations [5, 6]. In these problems, one typically considers a 1D gapless bath coupled to a localized degree of freedom, and the questions of interest concern the entanglement of the two halves separated by the impurity [7–10], the entanglement of the impurity with the bath [11, 12], or the entanglement of a region containing the impurity with the rest of the system [13]. In all those situations, one is dealing with a bipartite entanglement entropy (associated to a partition of the system in two pieces A and B) that characterizes to what extent the groundstate of the full system can be factorized into states for the subsystems A and B. Usually, the coupling of the impurity to the bath leads to a renormalization group flow, so the entanglement entropy exhibits different crossovers, and cannot be studied by the powerful methods of conformal field theory [14].

While numerical results are available for several examples of the foregoing problems, analytical results are very few 1. Several difficulties are compounded here. One is the complexity of the entanglement entropy, which usually does not behave simply under the various Bethe ansatz and integrable quantum field theory tricks. Another, maybe less immediately obvious, comes from the fact the entanglement entropy, being a $T=0$ quantity, is very hard to access perturbatively because of Infra Red (IR) divergencies - its behavior in perturbation theory is in fact reminiscent of the screening cloud in the Kondo problem [15, 16]. Like for the Kondo model, the models we are interested in also involve flow to strong coupling at low energy: no matter how small the bare impurity/bath coupling $\gamma$ is, the asymptotically low energy states (below some scale $T_B(\gamma)$ that goes to zero when $\gamma \to 0$) of the theory are qualitatively and deeply affected by the coupling. In more technical, RG terms, the interaction between the impurity and the bath is a (maybe marginally) relevant operator, of scaling dimension $D \in [0, 1]$. The RG flow is characterized by an energy scale $T_B$ (that coincides with the Kondo temperature in the case of the Kondo model), which marks the crossover between the weakly interacting regime, or ultraviolet (UV) regime, and the strong coupling regime, or infrared (IR) regime. By analogy with the usual thermodynamics quantities - such as the impurity entropy [17] - one expects the entanglement and Renyi entropies of the impurity, to bear a universal form and depend only on dimensionless parameters in the low energy (with respect to the cutoff - typically the bandwidth) limit:

$$R_N^{\text{imp}} f_N \left( \frac{v_F}{L T_B}, \frac{v_F}{L_0 T_B}, \frac{1}{\beta T_B}, \ldots \right) = S^{\text{imp}} = S - S^{\text{bulk}} = f_S \left( \frac{v_F}{L T_B}, \frac{v_F}{L_0 T_B}, \frac{1}{\beta T_B}, \ldots \right) \quad (1)$$

1 A remarkable exception is the calculation in [5] of the rate of entanglement entropy in the steady, out of equilibrium state of two wires connected by a resonant level, i.e. without interactions.
where \( L \) the length of subsystem \( A \), \( L_0 \) the size of the system, \( \beta^{-1} \) is the temperature, ..., and where by "bulk" quantities we mean those of the system with no impurity\(^2\). Scaling analysis meanwhile shows that \( T_n \propto \gamma^{1/1-D} \). While the validity of the form (1) seems well established numerically \([4, 12]\) at least in some cases, the natural attempt to calculate the universal functions perturbatively in \( \gamma \) fails: indeed, while it is easy to set up the calculation and bring it down to the evaluation of explicit integrals, all orders turn out to be infinite \([18]\), despite the presence of a natural IR cutoff - the length \( L \) of the subsystem. While these divergences simply mean that the scaling functions are not analytic in \( \gamma \), they leave us facing the practical problem of evaluating a non perturbative quantity in a particularly difficult setup involving replicas etc. The present paper and its sequels are devoted to making progress on this important question, with the ultimate goal of computing entanglement rates in out of equilibrium, interacting situations.

The impurity models we are going to consider have in common that the impurity degree of freedom is a two-state system (occupied or empty dot in the case of the IRLM, spin \( \frac{1}{2} \) for the Kondo model). At the IR fixed point, the impurity will end up being "absorbed by the bath", the physical picture in a lattice regularization of the model being that it forms a singlet with the bath site neighboring the impurity \([19]\). In a continuum model, the hybridization of the impurity with the bath should be rather thought of as occurring in a region of size \( \lambda_b = v_f / T_b \), the so-called "Kondo cloud" in the Kondo model. At zero temperature, one thus expects that the impurity has no effect on the entropies \( R_N, S \) when the partitioning of the system is done well beyond the impurity cloud (i.e. when the cloud is well inside region \( \Lambda, L \gg \lambda_B \)), resulting in the simple limits \( f_N(0, 0, 0) = 1 \) and \( f_S(0, 0, 0) = 0 \). Another simple limit at zero temperature is that of a disconnected impurity \( \gamma \to 0^+ \) or \( T \to 0 \), where one has \( f_N(\infty, x, 0) = 2^{1-N} \) and \( f_S(\infty, x, 0) = \ln 2 \). The entanglement entropy thus has the same limits in the UV and IR regime as does the boundary entropy \( s = \ln(g) \) (in this general formula \( g \) is the "universal non-integer groundstate degeneracy") introduced by Affleck and Ludwig \([20, 21]\) when varying the temperature between \( \beta^{-1} = \infty \) and \( \beta^{-1} = 0 \). In between those two simple limits lies a crossover regime where the interactions play at full and shape the crossover functions. It is probably tempting to speculate that, since they interpolate between the same values, the entanglement and boundary entropies could be the same functions, after formally replacing the length of region \( \Lambda \) by the inverse temperature. We shall see below that this is definitely not the case, and that entanglement and boundary entropies behave very differently, as a function of the interactions in particular.

The Kondo situation is illustrated on figure 1 where we have represented the spin absorbed by the bath in the low energy (long distance) ("infrared" or IR) limit, and the spin decoupled from the bath at high energy (short distance) ("ultraviolet" or UV). It is well known in these two, conformal limits, that the entanglement entropy of the region of length \( L \) with the rest of the system behaves as\(^4\)

\[
S = \frac{c}{6} \ln \frac{2L}{a} + \frac{s_1}{2} + \ln g
\]

(2)

(where \( a \) is a cutoff length scale, and \( s_1 \) a non universal constant) so \( S^{\text{imp}} \) interpolates between \( \ln 2 \) and 0.

\(^2\) Depending on the geometry, this system with no impurity may have a boundary.

\(^3\) Note that the point \( \gamma = 0 \) is singular since then the impurity density matrix is totally arbitrary, not being specified by the contact with the bath.

\(^4\) Recall also that the entanglement entropy of a region of length \( L \) sitting not at the edge but in the bulk of an infinite system takes the related form \( S = \frac{c}{6} \ln \frac{L}{a} + s_1 \).
FIG. 1: Illustration of the low energy or long distance (IR) and high energy or short distance (UV) limits of the Kondo model, after reduction to the s-wave channel. In the low energy limit, the impurity spin is swallowed by the bath, while it is decoupled at high energy (denoted by a white circle). The black arrows denote right- and left-moving fields. A and B denote the partition of the system that is used to define the bipartite entanglement.

The Kondo model (or the IRLM also considered below) being integrable, it is natural to wonder about determining the entanglement exactly using Bethe ansatz type techniques. This however remains a seemingly unreachable goal, despite progress such as in [22]. The next best option is to obtain what are usually extraordinarily good approximations using the form factors technique, directly in the continuum, scaling theory, after taking the appropriate “massless limit” of the usual massive construction [23]. This will in fact be discussed in our next paper. Meanwhile, there is one last tool we can try to put to use, following the calculation to lowest order performed in [12]: the expansion around the strong coupling fixed point. Such an expansion is bound to be better behaved as far as IR divergences are concerned, since one perturbs by irrelevant operators, whose correlation functions decay sufficiently fast at infinity to render all integrals convergent at large distance. Indeed, the lowest order calculation proposed in [12] relies on the effective Fermi Liquid theory for the vicinity of the Kondo strong coupling fixed point, and boils down, technically, to the calculation of a single integral of the one point function of the stress energy tensor (see below), a perfectly well defined procedure, leading to a result that depends only on $T_b, L$ as required. This procedure is however bound to fail in general, beyond the first order, since there are now strong UV divergences, and perturbation by an irrelevant operator does not determine a renormalizable field theory. However, in the integrable case, the existence of an infinity of conserved quantities does, in fact, provide a full control of the low energy hamiltonian: the necessarily infinite number of counter terms are all explicitly known [24], together with a well defined (analytical) regularization procedure. This means that, while we do not know how to calculate the scaling functions at small coupling (since they are not perturbative), we have, in principle, a tool to determine them perturbatively at large coupling. Of course, only a few orders are technically manageable, but this is enough to gain an understanding of the scaling functions, and answer important qualitative questions.

The paper is organized as follows: In section II, we introduce the impurity model and its representation in the ultraviolet and infrared limits. In section III, we present the method of infrared perturbation theory in the replicated theory needed to compute the entropies. We then turn to the actual calculation of the entropies in an infinite size system in section IV: our result for the entanglement entropy is summarized in Eqs. (42,43). Section V treats the case of a system of finite size $2L_0$, our results being summarized in Eqs. (58,59).

II. MODEL

A. Interacting Resonant Level Model

We start in fact not with the Kondo hamiltonian, but with the interacting resonant level model (IRLM). This is a simple impurity model that describes a tunnel junction between a localized resonant level at $x = 0$ and two baths of free, spinless electrons. We note here that the model and all the manipulations below can be generalized to an arbitrary number of baths, but this will not change qualitatively the results very much. Later, we will discuss briefly the one-wire case.

The IRLM also includes a Coulomb interaction (that is also called “excitonic” interaction) between the level and the baths. After a mode expansion in the baths and a linearization around the Fermi points, we end up with an hamiltonian $H = H_0 + H_b$, where $H_0 = \sum_{a=1,2} -i \int_{-\infty}^{\infty} dx \psi_a^\dagger \partial_x \psi_a$ (we have unfolded the fermionic fields living on the
half-infinite lines into right-moving fields on the complete line). The boundary interaction is given by

\[ H_b = \sum_{a=1,2} \gamma_a \psi_a(0) d^\dagger + \text{h.c.} \]

\[ + U \left( \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 \right) (0)(d^d d - 1/2). \]  

(3)

It is convenient to use spin-1/2 operators \( d^d = \eta d S^+ \) and \( d^d d = S_z + 1/2 \) to represent the impurity (\( \eta d \) is a Majorana fermion). The tunneling anisotropy is parametrized with \( \gamma \sqrt{2} e^{\delta/2} = \gamma_1 + i \gamma_2 \) and we consider even and odd fermions defined as a rotation of the original fermions

\[ \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]  

(4)

We take the bosonized form of this model with \( \psi_a(x) = \eta_a : e^{i \sqrt{4 \pi} \phi_a(x)} : / \sqrt{2 \pi} \), with \( a = 1, 2, \pm \). Then (\( \kappa_a = \eta_a \eta_d \))

\[ H_0 [\varphi_a] = \sum_{a=1,2} \frac{1}{2} \int_{-\infty}^{\infty} dx (\partial_x \varphi_a)^2 = H_0 [\varphi_\pm] \]  

(5)

\[ H_b = \frac{\gamma}{\sqrt{\pi}} \left[ \kappa_+ : e^{i \sqrt{4 \pi} \phi_+(0)} : S^+ + \text{h.c.} \right] \]

\[ + \frac{U_b}{\sqrt{\pi}} (\partial_x \varphi_+ + \partial_x \varphi_-) (0) S_z \]  

(6)

where \( U_b = g(U) \) is a function of the capacitive term for fermions. This function depends on the regularizing scheme chosen to bosonize the initial model [25].

B. UV hamiltonian : anisotropic Kondo model

As in [26] one can absorb the interaction via a unitary transformation \( U_\alpha = e^{i \alpha (\varphi_+ + \varphi_-)}(0) S_z \), with \( \alpha = U_b/\sqrt{\pi} \) that relates the model to the anisotropic Kondo hamiltonian where the boson \( \phi_- \) has decoupled

\[ H = H_0 [\phi_\pm] + \frac{\gamma}{\sqrt{\pi}} \kappa_+ : e^{i \lambda \varphi_+ - \lambda} : S^+ + \text{h.c.} \]  

(7)

where \( \lambda^2 = \frac{\pi}{2} \left( U_b - \pi \right)^2 + 2 \pi \) and the bosons \( \phi_\pm \) are again a rotation of the bosons \( \varphi_\pm \)

\[ \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} \sqrt{\pi} U_b & -U_b \\ 2\pi \sqrt{\lambda^2} & 2\pi \sqrt{\lambda^2} \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \]  

(8)

the scaling dimension of the tunneling operator is \( D = \lambda^2/8 \pi \), thus for \( \lambda^2 > 8 \pi \) the tunneling is irrelevant and the model flows to its UV fixed point which consist of free right-moving bosons living on the complete line.

In the following, we will be interested only in the interesting situation where the boundary term is relevant \((D < 1)\), and deeply influences the low-energy physics of the system by driving it to a strong coupling fixed point: the model flows at low energy to the Kondo fixed point where the impurity hybridizes with the wires and we get again free bosons with different boundary conditions for \( \phi_+ \), namely \( \phi_+(0^+) = \phi_+(0^-) + \lambda/4 \), whereas we have trivial boundary conditions for \( \phi_- \), that is decoupled from the boundary. The renormalization group flow to strong coupling generates dynamically a new energy scale \( T_b \), that marks the crossover between the UV fixed point \((\gamma = 0)\) and the IR fixed point. All physical quantities are universal functions of \( 1/\beta T_b \), \( 1/L T_b \), etc. This scale \( T_b \) itself is of course non-universal, and it will depend on \( \gamma \) and \( D \). A simple scaling argument leads to the power-law dependence \( T_b \sim \gamma^{1/D} \).

C. The entanglement entropy

The quantity we are initially interested in is the entanglement between a region (region A) of size \( 5 \ 2L \) centered around the resonant level and the rest of the system (region B). Unfolding gives us two chiral wires connected by the

\[ \text{The unfolding procedure doubles the spatial size of region A.} \]
resonant level as illustrated on figure 2. After the unitary transformation, the entanglement can be decomposed into a sum of contributions from the $\phi_+$ and $\phi_-$ modes. Only the $\phi_+$ modes interact with the impurity as illustrated on figure 3. Finally, one may “refold” back the result to obtain exactly the Kondo geometry from figure 1.

![Figure 2: The IRLM after unfolding the wires to get only right movers.](image)

![Figure 3: The $\phi_+$ degrees of freedom correspond to a single chiral wire coupled to the impurity.](image)

Note that the geometry we consider does not correspond to studying the entanglement between the two wires connected by the resonant level [7]. This situation is somewhat more complicated, in particular because the problem then cannot be turned into a purely chiral one - we will discuss it elsewhere [18].

### D. Dual hamiltonian: approach of IR fixed point

Since the boson $\phi_-$ decouples from the impurity, we are left with a single boson and in the following we simplify notations and set $\phi_+ \equiv \phi$. Remarkably, as was shown by [27], due to the integrability of the anisotropic Kondo hamiltonian, one has a dual expression of the hamiltonian that is an expansion around the IR hamiltonian $H_0^{IR} = \frac{i}{2} \int_{-\infty}^{\infty} dx (\partial_x \phi)^2$ (altogether with its boundary conditions). The expansion takes the form of a series of irrelevant boundary operators

$$H = H_0^{IR} + \sum_{n \geq 0} \frac{g_{2n+2}}{T_8^{2n+1}} O_{2n+2}(0). \quad (9)$$

The operators $O_{2n+2}$ are expressed only in term of a (holomorphic) deformed energy momentum tensor $\tilde{T} = T - i \sqrt{2} \pi \alpha \partial^2 \phi$, with $T(z) = -2\pi : \partial \phi^2 : (z)$ and the coefficient $\alpha$ is related to the scaling dimension of the boundary perturbation via $\alpha = \frac{1-D}{\sqrt{D}}$. The deformed theory with energy momentum tensor $\tilde{T}(z)$ has a modified central charge $\tilde{c} = 1 - 6\alpha^2 = 1 - 6(1-D^2)$. The operators $O_{2n+2}$ are conserved quantities of the UV hamiltonian, they commute with $H$ and between one another. The first terms read:

$$O_2 = \tilde{T} \quad (10)$$
$$O_4 = (\tilde{T}^2) \quad (11)$$
$$O_6 = (O_2 O_4) + \frac{\tilde{c} + 2}{12} (O_2 \partial^2 O_2) \quad (12)$$

We also know the value of the coefficients $g_{2n+2}$, they read [27]

$$g_{2n+2} = \frac{1}{2\pi} \frac{(-1)^{n+1} (D/\pi)^n}{(n+1/2)(n+1)!} \times \left[ \frac{\Gamma \left( \frac{D}{2(1-D)} \right)}{\Gamma \left( \frac{1}{2(1-D)} \right)} \right]^{2n+1} \frac{\Gamma \left( \frac{2n+1}{2(1-D)} \right)}{\Gamma \left( \frac{2n+1}{2(1-D)} \right)}. \quad (13)$$
III. METHOD

A. Renyi and entanglement entropies

When a system is divided into two sub-systems $A$ and $B$ (more precisely when the Hilbert space can be written as $\mathcal{H}_A \otimes \mathcal{H}_B$), one can define the Renyi entropy of order $N$ associated to the partition as:

$$R_N \equiv R_N^{AB} = \text{Tr}_A (\rho_A)^N$$

where $\rho_A$ is the reduced density matrix of subsystem $A$ obtained by tracing out $B$ degrees of freedom, $\rho_A = \text{Tr}_B \rho$ and the partial traces $\text{Tr}_X() = \sum_{|\xi\rangle \in \mathcal{H}_X} \langle \xi | \cdot | \xi \rangle$ take the trace on only a subpart $X$ of the degrees of freedom. Note that Eq.(14) makes sense (i.e. the sum converges absolutely) for complex $N$ with $\text{Re}(N) > 1$ The Renyi entropy has the property $R_N^{AB} = R_N^{BA}$, hence the abreviation $R_N$ used in (14). The bipartite entanglement entropy between $A$ and $B$ is defined as

$$S = S_{AB} = \text{Tr}_A \rho_A \ln \rho_A = S_{BA}$$

One can deduce the entanglement entropy from the knowledge of the Renyi entropies by using $S = -\lim_{N \to 1} \partial_N R_N$. The replica trick is a way to compute the logarithm: if we take integer powers of the reduced density matrix, the Renyi entropy can be interpreted (up to a factor) as the partition function of a big system consisting of $N$ (coupled) copies of the original theory. Then, one analytically continues the result to arbitrary $N \in \mathbb{C}$, $\text{Re}(N) > 1$ to obtain $\mathcal{S}$ (see Ref. 30 for considerations about the non-existence of ambiguities in the continuation – essentially because $N = \infty$ is an accumulation point). Note that the replica trick

B. The replica trick

We now specify the partition $A/B$, and choose a spatial division of the system into a single finite interval and its complement, i.e. $x \in A$ if $-L < x < L$ and $x \in B$ otherwise. In the case where the initial full system has conformal symmetry, it was shown in [29, 30] that the replicated theory bears a very elegant structure and can be described by a $\mathbb{Z}_N$ orbifold theory [31, 32].

To arrive at this, Calabrese and Cardy first work with an Euclidian path integral representation of the full density matrix $\rho$ at temperature $\beta^{-1}$. If the action depends on a collection of fields: $\mathcal{S} [\{\phi(x, \tau)\}]$, a matrix element of the density matrix $\rho$ is written:

$$\langle \{\phi_2\} | \rho | \{\phi_1\} \rangle = \frac{\int_{\{\phi(x,0) = \phi_1(x)\}} \mathcal{D}\phi \: e^{\mathcal{S}[\{\phi\}]} }{\int_{\{\phi(x,0) = \phi(x,\beta)\}} \mathcal{D}\phi \: e^{\mathcal{S}[\{\phi\}]} }$$

which is represented graphically by figure 4a. Note that in our case, contrarily to [29, 30], the action $S = S_0 + S_\text{b}$ is not the free CFT action $S_0$.

A matrix element of the reduced density matrix $\rho_A$ is labelled by two spatial configurations of the field $\phi_{1(2)}^A(x \in A)$ which live only in the region $A$: $\langle \phi_2^A | \rho_A | \phi_1^A \rangle$. Since the partial trace amounts to take $\phi(x \in B,0) = \phi(x \in B,\beta)$ and to sum over the fields $\phi(x \in B)$, graphically this is represented by a gluing of the boundaries at $\tau = 0$ and $\tau = \beta$ in region $B$, leaving a surface with a cut along $x \in [-L,L]$ over the lips of which the fields $\phi_1^A(x) \in A$ live. This is shown at finite and zero temperature in figure 4b and figure 4c.

Performing the last trace over $A$ yields a compact expression for the Renyi entropies $R_N$ that can be expressed as a ratio of partition functions:

$$R_N = \frac{Z_N}{(Z_1)^N}$$

$$Z_N = \int_{R_N(\beta)} \mathcal{D}\phi \: e^{-\mathcal{S}[\phi]}.$$
FIG. 4: Graphical representation of full (figure 4a) and reduced (figures 4b and 4c) density matrix elements \( \langle \phi_1(x) | \rho_{A} | \phi_2(x) \rangle \). The fields \( \phi_1, \phi_2 \) that label the matrix element are the boundaries in the path integral formalism. In this formulation, the partial trace over \( B \) amounts to gluing together the boundaries in region \( x \in B \). The fields \( \phi \) propagate with the complete action \( S \).

(b) Reduced density matrix at finite temperature \( \beta^{-1} \).

(c) At zero temperature \( \beta = \infty \).

FIG. 5: The \( \mathcal{R}_N \) geometry. The thick line at \( x = 0 \) is the integration contours of the boundary operators we insert. In this example we have inserted two operators, coming from two different copies of the system.

Here \( Z_N \) is the partition function of the theory living on the surface \( \mathcal{R}_N(\beta) \) that is obtained by gluing cyclically the \( N \) open cuts of figures 4b and 4c, so that \( \mathcal{R}_N(\beta) \) is the \( N \)-sheeted Riemann surface \( \mathcal{R}_N \) when \( \beta = \infty \), and a more complicated surface of higher genus at finite temperature.

The \( \mathcal{R}_N \) geometry We now see the geometry in which we will have to compute correlators of boundary operators in order to get the Renyi entropies. The matrix \( \rho_{A}^N \) is a fusion of the cylinders/planes via the "lips" of the cuts, and taking the trace over region \( B \) amounts to fuse the last one with the first one. This gives the \( \mathcal{R}_N \) geometry, shown in figure 5. In the path integral formalism this means that we take \( N \) copies of the system, we have a collection of fields \( \phi_{(j)} \), \( j = 1 \ldots N \), and two inverted branching points, one at \( z = -iL \) and one at \( z = iL \), so that \( \phi_{(j)}(0^+ - ix) = \phi_{(j-1)}(0^- - ix) \) for \(-L < x < L \).

C. Infra-red perturbation

1. Inserting operators on \( \mathcal{R}_N \)

We want to make a perturbative expansion around the free action \( S_0 : S = S_0 + S_b \) with \( S_b = \int_0^\beta d\tau H_b(0, \tau) \) and we develop in the path integral (16) the exponential

\[
\exp(-S_b) = \sum_{n=0}^{\infty} \int \! d\tau_1 \ldots \int \! d\tau_n \frac{(-1)^n}{n!} \prod_{i=1}^{n} H_b(0, \tau_i). \quad (19)
\]
Graphically a matrix element of $\rho$ is represented within this expansion in figure 6.

![Matrix Element Diagram]

FIG. 6: A matrix element of the full density matrix $\rho$, here at order $n = 3$. The fields $\phi_1, \phi_2$ that label the matrix element are the boundaries in the path integral formalism. Propagation of the fields $\phi$ with the free action $S_0$ and insertion of boundary operators.

Expanding these path integrals around $H_0$, one gets an expansion in powers of $1/T_b$ of the Renyi entropy and thus of the entanglement entropy. The expansion is of the form

$$R_N = \int_{R_N} D\phi_+ D\phi_- e^{-S} \left( \int_{R_1} D\phi_+ D\phi_- e^{-S} \right)^N$$

$$= R_{N\text{ir}}^{R} \left( \frac{e^{-S_{\text{ir}}}}{e^{-S_{\text{ir}}}} \right)^{R_{N\text{ir}}} = R_{N\text{ir}}^{R} \frac{Z_{\text{imp}}}{Z^{(N)}}$$

where $R_{N\text{ir}}^{R}$ is the unperturbed Renyi entropy, and is obtained from Eq. (20) by replacing the action $S$ by $S_0 [\phi_+, \phi_-]$ (i.e. setting $H_b = 0$).

The expectation value $\langle \cdot \rangle_{R_N}$ is taken in the theory living on the Riemann surface $R_N$:

$$\langle X \rangle_{R_N} = \int_{R_N} D\phi_+ X e^{-S_{\text{ir}}[\phi_+]} / \int_{R_N} D\phi_+ e^{-S_{\text{ir}}[\phi_+]}$$

where we have factorized the odd modes since the interaction $H_b$ does not depend on $\phi_-$. Our computation is then a standard perturbative expansion, but it is carried on a complicated worldsheet. There are three factors in Eq. (21):

- The quantity $R_{N\text{ir}}^{R}$ is the Renyi entropy at the conformal point. It has been computed in [30] and we will come back to it in the next section and in the appendix A.

$$R_{N\text{ir}}^{R} = c_N \left( \frac{2L}{a_0} \right)^{\frac{c}{6}(N - \frac{3}{2})}$$

$$S^{\text{ir}} = - \lim_{N \to 1} \frac{\partial}{\partial N} (R_{N\text{ir}}^{R}) = \frac{c}{3} \ln \left( \frac{2L}{a_0} \right) + c'_1$$

with $c = 1$ the central charge and $c_N$ and $c'_1$ are non universal constants.

- The denominator in (21) is nothing but the impurity partition function of the system raised to the power $N$, where the impurity partition function is defined as the ratio of the partition function of the full model $Z = \text{Tr} e^{-\beta (H + H_b)}$ by that of the system as the IR fixed point:

$$Z^{\text{imp}} = \frac{Z}{Z^{(0)}} = \langle e^{-S_b} \rangle_{R_1}$$

Its computation involves a series of expectation values on a single-sheet geometry. For example, in the case of infinite size and zero temperature, the expectation values are taken on the complex plane and we will see shortly that the denominator is exactly equal to one.

- Similarly, the numerator can be written as a ratio of partition functions, but for the system living on a $N$-sheet geometry:

$$Z^{\text{imp}}_N = \frac{Z_N}{Z_N^{(0)}} = \langle e^{-S_b} \rangle_{R_N}$$

We will see in the next section how to compute it perturbatively.
We now turn to the description of the perturbative expansion of the Renyi entropies, or equivalently the partition function on the Riemann surface $\mathcal{R}_N$ of the full theory with boundary interaction (9).

The partition function $Z_N$ is now an infinite sum of contributions involving $k$-uple integrals (see figure 6) of $k$-points correlators on $\mathcal{R}_N$ of operators $\mathcal{O}_n, i \prod_{i=1}^k \mathcal{O}_{n_i}(z_i)$, where $z_i = \tau_i$ is a point on $\mathcal{R}_N$ at position $x = 0$. To start with, the exponential $e^{-S_0}$ is time-ordered in imaginary time.

Recall the effective IR Hamiltonian is obtained from the expression of the boundary state describing the full interacting theory in the closed-channel geometry [27], whose expression is naturally regularized by inserting the perturbing operators at points $z_j = \tau_j + ix_j$ with $0 < -x_1 < -x_2 < ... < -x_k \ll 1$. Note that this regularization of the UV divergences by point splitting is the only one consistent with integrability. Within the point splitting regularization, the $k$-uple integral is done over paths that are slightly displaced and therefore never cross.

Furthermore, recalling that the perturbing operators $\mathcal{O}_n$ are conserved quantities, one observes that we can freely exchange the order of two contours. Indeed the term that is generated when two contours $i$ and $j$ are exchanged is simply $\int dz_j \frac{1}{z_j} dz_i \mathcal{O}_{n_i}(z_i) \mathcal{O}_{n_j}(z_j) = 0$ (here ... indicates that this expression appears inside some correlators and some integrals). The vanishing of this expression resulting from the $\mathcal{O}_n$'s being commuting conserved quantities, that translates in the OPE: $\mathcal{O}_n(z) \cdot \mathcal{O}_m(\omega) \sim ... + \frac{1}{z-\omega} \partial \mathcal{O}_{n,m}(\omega) + \text{reg.}$ with $\mathcal{O}_{n,m}$ a local operator.

We are now in a position to write down the expansion of the impurity Renyi entropy $R_N^{\text{imp}}$:

$$R_N^{\text{imp}} = R_N^{\text{IR}} = \sum_n C(n) A^{(n)}$$

$$A^{(n)} = \prod_{i=1}^k \int_{(x=0)} dz_i \prod_{j \neq i} \langle \prod_{i,j} \mathcal{O}_{2n_i+2}(z_i) \rangle_{\mathcal{R}_N}$$

$$C(n) = (-1)^k T^\epsilon \sum_{N=1}^{2n+1} \prod_{i=1}^p \frac{(\lambda_{2m_i+2})^n}{q_i}$$

where the sum in (27) is taken over all integers vectors $n = (0 \leq n_1 \leq ... \leq n_k)$, and one has introduced the integers $m_i, q_i$ writing $n = (\underbrace{m_1, ..., m_1}_{q_1}, \underbrace{m_2, ..., m_2}_{q_2}, ..., \underbrace{m_p, ..., m_p}_{q_p})$ with $m_i < m_{i+1}$. For fixed $n_i$, the integral over $z_i$ is to be taken along the $N$ possible disconnected paths on $\mathcal{R}_N$ going from $-\infty$ to $+\infty$ in imaginary time, and position $x = 0$ (thick line in figure 5).

In the next section we will see how we can compute parts of the $A^{(n)}$ in the case of infinite size and $T = 0$, where the sheets of $\mathcal{R}_N$ are the complex plane.

IV. INFINITE SIZE

In this section we report results for the perturbative expansion in the simplest case of a system of infinite size and at zero temperature. We use an orbifold representation of the expectation values that appear in (21) and conformal transformations to explicitly compute the expansion up to order $\frac{1}{(LT_B)^{\epsilon}}$. We refer to appendices A and B for details about twist fields and orbifold representation.

A. Infrared fixed point

We start by briefly describing the theory at its IR fixed point (formally obtained by setting $T_B = \infty$) focusing on its equivalence with a $\mathbb{Z}_N$ orbifold theory. This method was introduced in Ref. [30] by Calabrese and Cardy to compute the first term $R_N^{(0)}$. It relies on the introduction of "twist fields" which are inserted in a path integral on a more simple geometry and which encode the twisted geometry of $\mathcal{R}_N$.

Comparing (A1) and (22), Calabrese and Cardy find that $\int_{\mathcal{R}_N} D\varphi e^{-S_0} \propto \langle \Phi(u) \bar{\Phi}(v) \rangle$, where $\Phi, \bar{\Phi}$ are the twist fields, that are inserted at points $u = -iL$ and $v = +iL$. This leads to the well-known result for the Renyi entropy of
an interval of length $2L$ with the rest of the system in a free theory

$$P_N^{\text{IR}} = c_N \left( \frac{2L}{a_0} \right)^{-\frac{c}{12}}$$

(30)

where we recall that we have chiral fields but two copies $\phi_{\pm}$, resulting in the factor $c/6$ in the exponent. When taking the limit $N \to 1$, this results in the by-now well-known expression [2, 29, 30] for the entanglement entropy at the strong coupling fixed point:

$$S^\text{IR} = \frac{c}{3} \ln \left( \frac{2L}{a_0} \right) + \text{non universal}$$

(31)

with $c = 1$ in our case.

**B. Away from the IR fixed point**

A generic term in the perturbative expansion corresponding to the insertion of $k$ boundary operators, consists in a $k$-fold integral on the surface $\mathcal{R}_N$. One can actually deform the paths of integration continuously, one by one: the integrands are $k$-points correlation functions of the $O_i$’s, therefore analytical everywhere on $(\mathcal{R}_N)^k$, except for the branching points at $z = u, v = -iL, +iL$. Bringing the contours past the singular point $z = v$ leaves behind a contribution that is simply a path $P_N$ circling anti-clockwise the singular point $z = v$ on $\mathcal{R}_N$. The other contribution, being past $z = v$, can be moved to infinity, where it vanishes due to the clustering properties of the $k$-point correlator. This is shown in figure 7 in the case $N = 3$ and one contour (if there are many contours, one obtains nested integrals around $z = v$).

![Fig. 7: Deforming one contour on $\mathcal{R}_N$, illustrated with $N = 3$.](image)

In fact, since the path of integration $P_N$ is invariant under cyclic permutations of the replicas, we are dealing here only with very special operators, namely $\mathbb{Z}_N$ invariant operators $X^{(0)}(z) = \sum_{j=1}^{N} X(z^{(j)}) = \sum_{j=1}^{N} X_j(z)$, with $j$ a sheet index. The equivalence between the theory defined on $\mathcal{R}_N$ and an orbifold theory allows to express the mean value of any such operator as:

$$\langle X^{(0)}(z) \rangle_{\mathcal{R}_N} = \frac{\langle X^{\text{orb}}(z) \Phi(u) \Phi(v) \rangle}{\langle \Phi(u) \Phi(v) \rangle}$$

(32)

where expectation values in the right hand side are taken in the orbifold theory on the plane. Hence, introducing the notation $\text{Res}[X^{\text{orb}} \cdot \Phi] = \{ X^{\text{orb}} \Phi \}_1$ for the terms in $\frac{1}{z-w}$ in the OPE $X^{\text{orb}}(z) \cdot \Phi(\omega) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-w)^n} \{ X^{\text{orb}} \Phi \}_n(\omega)$, one arrives at the point where one can represent, at the operator level, the integral over the path $P_N$:

$$\int_{\{z=0\}} dz \langle O_n(z) \rangle_{\mathcal{R}_N} = 2i\pi \left( \frac{\Phi(u) \text{Res}[O_{\text{orb}} \cdot \Phi](v)}}{\langle \Phi(u) \Phi(v) \rangle} \right)$$

(33)

This is easily iterated (the different contours are nested) with the result:

$$A^{(n_1)}(u - v) = (2i\pi)^k \left( \frac{\Phi(u) \text{Res}[\Phi](v)}}{\langle \Phi(u) \Phi(v) \rangle} \right)$$

(34)

$$\text{Res}_{\Phi} \equiv \text{Res}[O_{2n_1+2} \cdot \ldots \cdot \text{Res}[O_{2n_k+2} \cdot \Phi] \ldots]$$

(35)
Considering for a brief moment terms with only $O_2$ insertions (i.e., $n_1 = n_2 = \ldots = n_k = 0$), and using the fact that $\text{Res}[T_{\text{orb}} \cdot X] = \partial X$ for any local operator $X$, one immediately finds $A^{(0,\ldots,0)}(u - v) = \left\langle \Phi(u) \tilde{\Phi}(v) \right\rangle^{-1} (2\pi)^k \partial_b^k \left\langle \Phi(u) \tilde{\Phi}(v) \right\rangle$, so that $\sum_{k \geq 0} \left(\frac{-g_2/T_b}{k!}\right)^k A^{(0,\ldots,0)} = \left\langle \Phi(u) \tilde{\Phi}(v) \right\rangle^{-1} \left\langle \Phi(u) \tilde{\Phi}(v - \frac{2k g_2}{T_b}) \right\rangle = \left(1 + \frac{1}{LT_b}\right)^{-\frac{N}{(N-1)}}$ where one has used that $g_2 = -\frac{1}{3}$, $u = -iL$ and $v = +iL$. One can thus resum the full series for those terms consisting only on $O_2$ contributions.

More generally, since an insertion of $O_2$ amounts to a derivative with respect to the point $v$, one can resum all $O_2$ subdiagrams, $\sum_{k \geq 0} \left(\frac{-g_2/T_b}{k!}\right)^k A^{(n_1,\ldots,n_r,0,\ldots,0)}(u - v) = \left\langle \Phi(u) \tilde{\Phi}(v) \right\rangle^{-1} \left\langle \Phi(u) \tilde{\Phi}(v + 2i/T_b) \right\rangle A^{(n_1,\ldots,n_r)}(u - v - 2i/T_b)$: This allows for a partial resummation of the perturbative expansion, all $O_2$ insertions simply amounting to a shift $v \to v' = v + 2i/T_b$. We thus arrive at the final compact form:

$$R_{\text{imp}}^N = \left(1 + \frac{1}{LT_b}\right)^{-2h} \overline{\mathcal{P}}_N (u - v - \frac{2i}{T_b})$$

$$S_{\text{imp}} = S_{O_2} + \overline{\mathcal{P}} \left( u - v - \frac{2i}{T_b} \right)$$

$$S_{O_2} = \frac{1}{6} \ln \left(1 + \frac{1}{LT_b}\right)$$

where one has introduced the reduced impurity Renyi entropy free of $O_2$ insertions:

$$\overline{\mathcal{P}}_N (u - v') = \sum_{n_i > 0} C_{\{n_i\}} A^{\{n_i\}} (u - v')$$

and $\overline{\mathcal{P}} = -\lim_{N \to 1} \partial_N \overline{\mathcal{P}}_N$. This rewriting considerably reduces the number of diagrams that need to be computed to achieve a given order in the expansion: only diagrams free of $O_2$ insertions have to be considered, and at the end of the calculation, one replaces $v'$ by $v' = v + 2i/T_b = iL + 2i/T_b$.

### C. Results

We now turn to the actual evaluation of the reduced partition function $\overline{Z}_N (u - v)$. According to Eq. (35), one way would be to compute nested residues of the perturbing operators $O_4, O_6, \ldots$. The situation is not as simple as for $O_2$.

For instance if we want to compute in the orbifold theory an expectation value such as $\sum_{\mathcal{S}} \langle \langle T^2(z^{(i)}) \rangle \rangle_{\mathcal{R}_N} = \sum_i \langle (T^2_i(z)) \rangle_{\mathcal{R}_N}$, the orbifold version of $\sum_i \langle T^2_i \rangle$ is not $(T^2_{\text{orb}})$ and in particular we have a priori no clue of the OPE of the former with twist fields.

It is nevertheless possible to compute such an OPE (see appendix C for the explicit form of $\text{Res}[O_4^{\text{orb}} \cdot \tilde{\Phi}]$), but it notably involves to know in advance the correlators we are at the end interested in. Contrarily to the effect of $O_2$ insertions, there is (probably) no simple differential operator representing higher operators $O_n$ at the level of correlation functions, presumably forbidding a possible resummation of the Renyi entropy in the same spirit as could be done for $O_2$ contributions.

The contributions $A^{\{n_i > 0\}}$ can nevertheless be evaluated in a straightforward way: any such contribution can be obtained from the knowledge of the $n$-point function of the operator $O_2$ on $\mathcal{R}_N$, by then performing the relevant normal order operations on the surface $\mathcal{R}_N$ (this is the route we have chosen, see for instance appendix B1b). Another equivalent way would be to derive directly the form of the orbifold operators on $\mathcal{R}_N$. For instance, the expression of the operator $O_4^{\text{orb}}(z) \equiv \sum_i O_4(z^{(i)})$ can be deduced from the transformation of the $\Lambda = (T^2) - \frac{\Lambda}{36} \partial^2 T$ quasi-primary operator, that reads

$$\Lambda_{\mathcal{R}_N} (z) = f'(z)^4 \Lambda_{\mathcal{C}} (\omega) + \frac{22 + 5c}{5c} \cdot \frac{c(f, z)}{12} \left(2f'(z)^2 T_{\mathcal{C}} (\omega) + \frac{c(f, z)}{12} \right)$$

upon the conformal mapping $\omega = f(z), z \in \mathcal{R}_N, \omega \in \mathcal{C}$ that uniformizes the surface $\mathcal{R}_N$ into the complex plane.

We now turn to the results up to order 7 in $1/(LT_b)$. The partial resummation (37) of $O_2$ insertions leads to a significant simplification of diagrams that need to be computed; from 17 diagrams $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, $A^{(0,0)}$, $A^{(0,1)}$, $A^{(1,1)}$, $A^{(0,0,0)}$, $A^{(0,0,1)}$, $A^{(1,0,0)}$, $A^{(0,0,0,0)}$, $A^{(0,0,0,0,0)}$, $A^{(0,0,0,0,0,0)}$.
to only 3 reduced diagrams (i.e. diagrams free of $O_2$ insertions) $A^{(1)}$, $A^{(2)}$ and $A^{(1,1)}$ at order 7. After tedious calculations, we find:

$$\bar{\Pi}_N(u-v) = -\frac{g_4}{T^6} A^{(1)}(u-v) - \frac{g_6}{T^6} A^{(2)}(u-v) + \frac{g_4^2}{2T^6} A^{(1,1)}(u-v) + \mathcal{O}((LT_b)^{-8})$$

$$A^{(1)}(u-v) = \frac{-i\pi(N^2 - 1)^2}{24DN^3(u-v)^3} (4-D)(1-4D)$$

$$A^{(2)}(u-v) = \frac{-i\pi(N^2 - 1)^2}{5760D^2N^6(u-v)^6} \left[ 24(2-D)(1-2D)((1-D)^2(334N^2 - 246) - D(64N^2 - 31)) + DN(-18D(1123N^2 - 667) + (1-D)^2(7N^411930N^2 - 5697)) \right]$$

$$A^{(1,1)}(u-v) = -\frac{(N^2 - 1)^2}{N^6(u-v)^6} \left[ \frac{(1-4D)^2(4-D)^2}{2880D^2} \left( 5(N^2 - 1)^3 + \frac{144}{13}N(-36 + 79(N^2 - 1)) \right) + 9N(20 - 29N^2 + 13N^4) \frac{3521D^2 - 1576D(1 + D^2) + 216(1 + D^2)}{1820D^2} \right]$$

Due to the prefactors $(N^2 - 1)^2$ in all but one of the contributions to $\bar{\Pi}_N$, the reduced entanglement entropy bears only one contribution up to order 7, that can be expressed as a function of parameter $\alpha = \frac{(1-D)}{\sqrt{D}}$:

$$\mathfrak{S}(u' - v') = \frac{18}{35} \frac{(\pi g_4)^2}{(u' - v')^6 T_b^6} (4\alpha^4 - 8\alpha^2 + 9) + \mathcal{O}((LT_b)^{-8})$$

where $u', v'$ are the generic points where the twist fields are inserted in order to compute the quantities $A^{(n_i)}$ with $n_i > 0$. Now, taking $u' = -iL, v' = iL + 2i/T_b$ in this expression and keeping only terms that are $\mathcal{O}((LT_b)^{-7})$ leads to the expression for the full entanglement entropy:

$$S = S^{\text{IR}} + S^{\text{imp}}$$

$$S^{\text{imp}} = \frac{1}{6} \ln \left( 1 + \frac{1}{LT_b} \right) - \frac{18}{35} \frac{(\pi g_4)^2}{(2LT_b)^6} \left( 1 - \frac{6}{LT_b} \right) (4\alpha^4 - 8\alpha^2 + 9) + \mathcal{O}((LT_b)^{-8})$$

with $S^{\text{IR}}$ the entanglement entropy at the IR fixed point given by (31), and we recall the $D$–dependence of the coefficients $g_4$ and $\alpha$:

$$g_4 = \frac{D}{6\pi^2} \left( \frac{\Gamma(D/2(1-D))}{\Gamma(1/2(1-D))} \right)^3 \frac{\Gamma(3/2(1-D))}{\Gamma(3D/2(1-D))}, \quad \alpha = \frac{(1-D)}{\sqrt{D}}.$$
FIG. 8: Impurity entanglement entropy: first correction to the contribution $\frac{1}{2} \ln(1 + 1/LT_b)$ stemming from $O_2$, displayed here at fixed $LT_b = 1$ as a function of the interactions. $D = \frac{1}{2}$ corresponds to the free point, where this correction is minimal (see inset). A significant enhancement of this negative correction is visible when reducing $D$.

system at finite temperature $\beta^{-1}$: $s = (1 - \beta \partial_\beta) \ln Z^{\text{imp}}$ with $Z^{\text{imp}} = \frac{Z^{(0)}(\beta)}{Z^{(0)}(\beta)}$ and $Z^{(0)}$ the partition function of the system without impurity. One immediately gets $\ln Z^{\text{imp}} = \ln \langle e^{-S_b} \rangle_\beta$ where the expectation value is taken at temperature $\beta^{-1}$, i.e. on a cylinder of circumference $\beta$. Because the perturbing operators $O_n$ are commuting conserved quantities, in the perturbative expansion of this quantity around the strong coupling fixed point, the contours can be deformed and moved one by one to infinity along the cylinder, resulting in $\langle e^{-S_b} \rangle_\beta = e^{-\langle S_b \rangle_\beta}$ and finally $\ln Z^{\text{imp}} = -\langle S_b \rangle_\beta = -\beta \sum_n g_{2n+2} T_b^{-2n+1} \langle O_{2n+2} \rangle_\beta$. The boundary entropy thus has an expansion in odd powers of $T/T_b$. Evaluating the one-point functions $\langle O_{2n+2} \rangle_\beta$ in the cylinder geometry with circumference $\beta$, one arrives at the expansion

$$s = -\pi g_2 \frac{1}{3} \left( \frac{\pi}{\beta T_b} \right)^3 - \pi g_4 \frac{9 - 4 \alpha^2}{15} \left( \frac{\pi}{\beta T_b} \right)^3 - \pi g_6 \frac{96 \alpha^4 - 340 \alpha^2 + 425}{252} \left( \frac{\pi}{\beta T_b} \right)^5 + O \left( \frac{1}{\beta T_b} \right)^7. \quad (45)$$

We note that a dependence in the interaction parameter $D$ appears at order 3 already, making the boundary entropy more sensitive to interactions than the entanglement entropy.

V. FINITE SIZE

We now want to compute perturbatively the expression of the entanglement entropy around the IR fixed point but in the case when the size of the system is finite, of length $L_0$ in the original geometry or $2L_0$ in the unfolded geometry. In this case, the replica trick leads to expressions of Renyi entropy similar to Eq. (20), but instead of considering a surface $\mathcal{R}_N$ made with $N$ infinite planes with branching cuts, one gets a surface $\mathcal{C}_N$ made of $N$ cylinders periodic along the $x$ direction with the same branching cuts between them.

FIG. 9: The reduced density matrix is now a cylinder, periodic in space, with a cut along the $x$ axis. The $\mathcal{C}_N$ geometry now consists in $N$ cylinders connected along the cuts. The partial trace over subsystem $B$ results in the gluing of two semi-infinite cylinders along the dashed line, at $\tau = 0$. The impurity is inserted at $x = 0$ (not represented).
The orbifold theory is now living on a cylinder and one has to uniformize it to the complex plane. The integration contour corresponding to the impurity insertion is then radial and goes from $\omega = 0$ to $\omega = \infty$, passing between the twist and anti-twist fields. The dashed line corresponds to the partial trace over subsystem $B$.

A. Infrared fixed point and twist fields

The first contribution to the entanglement entropy at finite size in the strong coupling limit $LT_b \to \infty$ is, as before, the entanglement entropy at the IR fixed point. This quantity, that was computed by Calabrese and Cardy [30], can be obtained as the two-point function of the twist and anti-twist operator at finite size; since those operators are primary operators, by using the conformal mapping that sends the cylinder onto the plane:

$$z \in \mathbb{C} \rightarrow \omega = e^{\frac{2\pi}{a_0}z} \in \mathbb{C}$$

one easily computes $\langle \Phi(u)\bar{\Phi}(v) \rangle_{L_0}$, yielding:

$$S_{L_0}^{\text{ir}} = \frac{c}{3} \ln \left( \frac{L_0}{\pi a_0} \sin \frac{\pi L}{L_0} \right) + c'_1.$$  \hspace{1cm} (47)

Here $a_0$ is the inverse of the ultraviolet cut-off (e.g. in a lattice regularization, it coincides with the lattice constant), and $c'_1$ is again a non universal constant.

The finite size infrared fixed point is now our reference point around which one wishes to expand the Renyi and entanglement entropies. The orbifold representation allows one to compute the expansion even when the surface is $\mathbb{C}N$. Indeed, since this representation gives the correlators of the original $N$-sheeted geometry in terms of correlators of the orbifold theory in the geometry of a lone sheet, one only has to perform the change of variables from the cylinder to the complex plane (46).

However, there is a slight difference with before. When considering the surface $\mathcal{R}_N$, one can move the integration contours freely in the orbifold representation and only get a contribution from residues around one of the $u,v$ points, and this fact greatly simplifies the computations. In the finite size case, we first have to flatten the cylinder and change the integration contour accordingly (see Fig. 10). In the new geometry the integration contour on the plane is not a circle anymore, it is instead not closed leading to more complicated integrals. It results that the insertion of the operator $O_2$ has no simple action and we cannot derive an equality such as Eq. (37).

B. IR divergences

At finite size, one also has to be careful to take into account the expansion stemming from the impurity partition function $Z_{\text{imp}} = \langle e^{-S_b} \rangle_{\mathcal{R}_1}$ (the denominator in Eq. (20)), which is now not equal to 1 because of the non trivial expectation value of $O_2$, and of higher order perturbing operators, on a cylinder.

This additional complication is easily understood by just considering the first order perturbing operator $O_2$: having put the system on a cylinder, the curvature induces a non-vanishing local energy density, $\langle O_2 \rangle = \langle T \rangle = e_0 = -\frac{c\pi^2}{6L_b^2}$. It results that the impurity partition function is now formally infinite: introducing a (small) finite temperature $\beta^{-1}$ for a moment to regularize those divergences, one has $Z_{\text{imp}} = \lim_{\beta \to \infty} e^{-e_0 \frac{L}{T_b}}$, so that each term of the expansion of $Z_{\text{imp}}$ in inverse powers of $T_b$ is formally infinite at vanishing temperature.

These divergences will also appear in the $N$–sheeted geometry $\mathcal{R}_N$, when computing $Z_{\text{imp}}^N$. The impurity partition functions $Z_{\text{imp}}^N$ and $Z_{\text{imp}}^N$ cannot be defined on their own and the quantities $A^{(n_i)}$ defined in (28) as constituents of $R_N$ are now ill-defined.
In fact these divergences compensate term by term in the expansion in $1/L_B$ of $R_N/R_N^\text{imp}$: the ratio between the impurity partition functions is perfectly well defined yielding a finite answer for the Renyi and entanglement entropies. One way to see this is to take the logarithm of the Renyi entropy:

$$\ln R_N - \ln R_N^\text{imp} = \ln \langle e^{-S_B}\rangle_{C_N} - N \ln \langle e^{-S_B}\rangle_{C_1} = \langle e^{-S_B}\rangle_{C_N}^c - 1 - N(\langle e^{-S_B}\rangle_{C_1}^c - 1)$$

(48)

where we used the well-known relationship between the generating functionals of connected and disconnected expectation values of operators. The connected expectation value of a product of operators $\mathcal{O}_{n_1}(z_i)$ is defined recursively as

$$\langle \mathcal{O}_{n_1}(z) \rangle_{C_N}^c = \langle \mathcal{O}_{n_1}(z) \rangle_{C_N}$$

(49)

and

$$\langle \mathcal{O}_{n_1}(z_1) \cdots \mathcal{O}_{n_k}(z_k) \rangle_{C_N} = \langle \mathcal{O}_{n_1}(z_1) \cdots \mathcal{O}_{n_k}(z_k) \rangle_{C_N}^c + \sum_{\text{part.}} \langle \prod_{i \in a} \mathcal{O}_{n_i}(z_i) \rangle_{C_N}^c \langle \prod_{j \in \beta} \mathcal{O}_{n_j}(z_j) \rangle_{C_N}^c \cdots$$

(50)

where $\alpha, \beta, \ldots$ are non-trivial subdivisions of a partition of $[1, k]$.

This writing allows to regroup the terms of the expansion in $1/T_B$ in well defined quantities. Indeed, these connected (or non factorizable from the point of view of the integration variables) contributions of $\langle \mathcal{O}_{n_1}^{\text{orb}}(z_1) \cdots \Phi(u)\Phi(v)\rangle_{C}/\langle \Phi(u)\Phi(v)\rangle_{C}$ can be written as the sum of two terms.

The first term depends on $u, v$ has to vanish when $u \to v$ (or $u, v \to \mp iL_0$, cutting open the cylinders) and encodes the local influence of the insertion of twist fields at $u, v$. This first contribution does not produce infinities when sending all the $z_i \to \infty$ (while keeping $u$ and $v$ finite) because of the clustering properties of the connected correlator between the $\mathcal{O}_{n_i}$ and $\Phi, \Phi$.

The second contribution which does not depend on the points $u, v$ (from $\langle \mathcal{O}_{n_1}^{\text{orb}}(z_1) \cdots \rangle_{C}$) can be shown to be divergent when all the points $z_i$ are sent to infinity. In this limit, the fields inserted at points $z_i$ do not feel any longer the cuts on the cylinders, and the correlator is asymptotically that of fields living on $N$ disconnected cylinders, yielding a factor $N$, the number of copies, for the result of the integral.

In a nutshell, it can be shown that $\langle (-S_B)^k \rangle_{C_N} = (u - v)F_1(u, v) + N \times \text{diverging part}$, where the diverging part is independent of $N$ and is exactly compensated by $N\langle (-S_B)^k \rangle_{C_1}$. This shows that the expansion in $\ln R_N - \ln R_N^\text{imp}$ are well defined.

As a result, when computing the Renyi entropy $R_N$ at finite size, one only has to take into account the connected correlators in $\mathcal{Z}_N^\text{imp}$, and do not care about the diverging parts that are in fact all canceled out by the denominator $\mathcal{Z}_N^\text{imp}$ at every order in the development in $1/T_B$.

C. Twist fields at finite size

We can now easily generalize the calculation of the first order terms to the finite size case. Under the conformal mapping (46), the singular points $u, v$ of the are mapped onto $\tilde{v} = e^{i\pi L/L_0}$ and $\tilde{u} = e^{-i\pi L/L_0}$, so that one has for instance:

$$\langle T(z^{(i)}) \rangle_{C_N} = \frac{\langle T_{\text{orb}}(z)\Phi(u)\Phi(v)\rangle_{C}}{\langle \Phi(u)\Phi(v)\rangle_{C}}$$

(51)

$$= \frac{\langle \left(\frac{dz}{d\omega}\right)^{-2} T_{\text{orb}}(\omega)\Phi(\tilde{u})\Phi(\tilde{v})\rangle_{C}}{\langle \Phi(\tilde{u})\Phi(\tilde{v})\rangle_{C}} - \frac{\epsilon}{12} \left(\frac{dz}{d\omega}\right)^{-2} \{z; \omega\}$$

The last term yields a diverging contribution that is exactly canceled out by a term coming from $(\mathcal{Z}_N^\text{imp})^N$, the denominator of Eq. (20), so that we get:

$$A_{L_0}^{(1)} = \frac{\pi}{T_B} \int_0^\infty d\omega \left[ \frac{h\omega(\tilde{u} - \tilde{v})^2}{(\omega - \tilde{u})^2(\omega - \tilde{v})^2} - \frac{N}{24\omega} \right] = -\frac{2\pi h}{L} f_1(L/L_0) + \text{div}.$$  

(52)

with the function $f_1(a) = a(1 + \pi(1 - a)\cot \pi a)$. Similarly one can obtain the higher order contributions involving insertions of the perturbing operator $\mathcal{O}_2$ (see Appendix B 2).
The next step consists in computing correlators of higher order perturbing operators \( \mathcal{O}_{n>2} \) on the surface \( \mathcal{C}_N \). One route would be to express the \( \mathbb{Z}_N \)-neutral combinations corresponding to those operators, namely \( \mathcal{O}_{n}^{\text{orb}} = \sum_{j=1}^{N} \mathcal{O}_n \left( z^{(j)} \right) \), in terms of orbifold (quasi-) primary operators. We have chosen another equivalent route, namely a quantization method that allows to directly compute correlators of more complicated operators like \( \mathcal{O}_4 \).

**D. Direct quantization**

A direct way to compute the correlators \( A^{(n_i)} \) at finite size is to look directly at the structure of the Hilbert space in the twisted sector and to consider a quantized version of the fields. This is a generalization of the quantization of fields with Neveu-Schwarz or Ramond conditions. The field living on \( N \) sheets is in fact a \( N \)-valuated field and we shall quantized it this way.

1. **Monodromy relations**

To be general, in the replica trick we consider a CFT on a Riemann surface with \( N \) sheets. Each sheet is described by the complex coordinate \( z = \tau - ix \). We consider a real free scalar field \( \varphi(z, \bar{z}) \) living on the Riemann surface, its value on a given sheet \( j = 1, \ldots, N \) is denoted by \( \varphi^{(j)}(z) \), so that we have in fact \( N \) copies of a (chiral) CFT.

These copies are not independent: we consider a branch point at \( z = 0 \), such that \( \varphi^{(j)}(0^+ - ix) = \varphi^{(j-1)}(0^- - ix) \), for all \( x > 0 \). In other words, when one turns anticlockwise around the point \( z = 0 \), one decreases the index of the sheet by one: \( \varphi^{(k)}(ze^{2i\pi}) = e^{2i\pi k/N} \varphi^{(k)}(z) \), \(^{53}\) making afterwards the conformal transformation \(^{D5}\) will allow one to consider the branch cut that we are interested in. It is more convenient to consider the Fourier-conjugate fields, in order to diagonalize the monodromy relations. We thus introduce the fields \( \varphi^{(k)}(z, \bar{z}) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2i\pi jk/N} \varphi^{(j)}(z, \bar{z}) \), which verify the relations

\[
\varphi^{(k)}(ze^{2i\pi}) = e^{2i\pi k/N} \varphi^{(k)}(z),
\]

and since the holomorphic energy-momentum tensor is given by \( T(z) = -2\pi \sum_i : \partial \varphi^{(i)} \partial \varphi^{(i)} : (z) \), we also have

\[
T(z) = -2\pi \sum_k : \partial \varphi^{(-k)} \partial \varphi^{(k)} : (z) = \sum_k T^{(k)}(z).
\]

Note that \( \varphi^{(-k)} = \varphi^{(N-k)} = \varphi^{(k)*} \): the index is to be understood modulo \( N \). \(^{6}\)

![FIG. 11: Fields living on a \( N \)-sheeted Riemann surface. The cut is along the \( x \) axis, between \( u = 0, v = \infty \).](image)

---

\(^{6}\) If \( N \) is even, we now have a free theory with two real bosonic fields \( (\varphi_0, \varphi_{N/2}) \) and the associated energy-momentum tensor is \( T^{(0)}, T^{(N/2)} \) and \( (N-2)/2 \) complex bosonic fields (the associated energy-momentum tensors are the \( T^{(k)} + T^{(-k)} = -4\pi (\partial \varphi^{(k)} \partial \varphi^{(-k)}) \)). One the other hand, if \( N \) is odd, we have only one real field \( \varphi_0 \) and \( (N-1)/2 \) complex fields.
2. Mode decomposition

A canonical quantization on a cylinder altogether with imposing the boundary conditions (54), leads to the mode decomposition

\[ \varphi^{(k)}(z) = \frac{i}{\sqrt{4\pi}} \sum_m \frac{1}{m} a_m^{(k)} z^{-m} \quad m \in (\mathbb{Z} - k/N) \]  

(56)

with the commutation relations

\[ [a_m^{(k)}, a_n^{(k)}] = m \delta_{k+k} \delta_{m+n} \]  

(57)

Acting on the twisted vacuum we have \( a_m^{(k)} |\sigma\rangle = 0 \) if \( m > 0 \). Here \( |\sigma\rangle = \bigotimes_{k=0}^{N-1} |\sigma^{(k)}\rangle \) is the twisted vacuum and the \( |\sigma^{(k)}\rangle \) are the highest-weight vectors of the representation of the algebra (57); \( k \) is the index of the monodromy sector.

The twist field \( \Phi(z) \) is the primary field associated with \( |\sigma\rangle \), and it can be written as a product of the twist fields associated with \( |\sigma^{(k)}\rangle \) : \( \Phi(z) = \prod_k \Phi^{(k)}(z) \). The prime on the product simply denotes that we count \( k \) only for the independent fields. We shall see in the next paragraph that the conformal dimension of \( \Phi_k \) is \( h_k = (2 \frac{1}{16\pi}) \), the factor 2 depending on whether \( k \) is the index of a real or complex field. In the twisted sector the vacuum expectation values are abbreviated as \( \langle X \rangle \equiv \langle \sigma | X | \sigma \rangle \).

Within this canonical quantization scheme, one can build the perturbing operators \( \mathcal{O}_n \) and evaluated arbitrary correlators thereof by using Wick’s theorem for the bosons \( \varphi^{(k)} \). Finite size is eventually taken into account by means of the conformal mapping (40). We report on the details of the actual computations in Appendix D and directly turn to the end result for the entanglement entropy.

E. Results

Finally let us summarize the results for the entanglement entropy of a segment of width 2L in a system at finite size 2L_0, up to order \( \mathcal{O}(T_n^{-4}) \). The net expression for the entanglement entropy is:

\[ S = S^\text{in}_{L_0} + S^\text{imp} \]  

(58)

\[ S^\text{imp} = \frac{1}{6LT_n} f_1(L/L_0) - \frac{1}{12(LT_n)^2} f_2(L/L_0) \]

\[ + \left( \frac{1}{2LT_n} \right)^3 \left( \frac{4}{9} f_3(L/L_0) + \frac{(2\pi)^3 g_4}{120} (9 - 4\alpha^2)(L/L_0)^2 f_1(L/L_0) \right) \]  

(59)

where \( S^\text{in}_{L_0} \) is the Calabrese-Cardy entanglement entropy (47) in the IR limit \( LT_n \rightarrow \infty \), and the impurity entanglement entropy is expressed in terms of the functions (we also give their asymptotics when \( a = L/L_0 \rightarrow 0 \)):

\[ f_1(a) = a(1 + \pi(1 - a) \cot a \pi) \sim 1 - \pi^2 a^2/3 + \pi^2 a^3/3 + \mathcal{O}(a^4) \]  

(60)

\[ f_2(a) = f_1(a)^2 + a^2(1 - a)^2 \pi^2 \sim 1 + \pi^2 a^2/3 - 4\pi^2 a^3/3 + \mathcal{O}(a^4) \]  

(61)

\[ f_3(a) = f_1(a)f_2(a) + 2a^3(1 - a)^2 \pi^2 \sim 1 + \pi^2 a^3 + \mathcal{O}(a^4) \]  

(62)

We can see that the interactions (which are hidden in the coefficients \( \alpha = \frac{1 - D}{\sqrt{D}} \) and \( g_4 \)) do not affect the entanglement entropy before the third order in \( 1/LT_n \): the effect of the interactions at strong coupling is thus a subtle one, just as in the infinite size case. We again note that this conclusion is invalidated at strong repulsive Coulomb interaction in the one-wire IRLM, where \( D \) can be made as small as wished: in this limit, the term proportional to \( g_4 \) scales as \( 1/D^2 \) and thus eventually becomes dominant at sufficiently strong \( U > 0 \).

Our result for the entanglement entropy has the expected property that \( S^\text{imp} \) vanishes order by order in the limit \( L \rightarrow L_0 \): in this limit, region B shrinks to a point (and this limit is not singular since region B does not contain the impurity) so that all Renyi entropies evaluate to 1.

VI. CONCLUSIONS

The entanglement entropy is an elusive quantity, which in many respects differs from usual local observables. In 1+1 dimensional many-body systems at their critical point described by a CFT, it has been shown that the replica
method allows for a computation of the Renyi and the entanglement entropies. In this paper, we have shown how to combine the replica method with infrared perturbation theory to compute the bipartite entanglement entropy of a segment of length $2L$ with the rest of the system in the case of a CFT perturbed at its boundary by the interaction with an impurity, namely the Interacting Resonant Level Model, or the one-channel anisotropic $s = 1/2$ Kondo model. Our approach allows for a determination of the Renyi and entanglement entropies in a systematic expansion around the strong coupling fixed point that corresponds to the limit $L \gg T_b$, where $T_b$ is the boundary energy scale generated by the interaction with the impurity. We have also investigated the case of a finite size system, and shown that a similar expansion can be carried out in this case.

Our results indicate that generically, the effect of the interactions on the entanglement entropy is weak: the first contribution to the entanglement entropy that depends on the interactions scale as $1/(L^2 T_b^2)$ at infinite size, and as $1/(L^2 T_b^2)$ at finite size. On the other hand, we emphasize that the effect of interactions is strongly enhanced in the case of strong coulombic repulsion $U > 0$ in the one-wire IRLM.

Note that our method could also be extended to more complicated geometries, like finite temperature and finite size, the only (but significant regarding the difficulty in computing the correlators!) difference being that the orbifold theory should be put on a torus.

At the end of such a technical paper, it is of course natural to ask how the perturbative expansion compares with say a numerical evaluation of the quantity $S^\text{imp}$ over the whole flow from UV to IR, and how, in particular, the UV limit - which is non perturbative in $\gamma$ - is approached. This will be discussed elsewhere [18]. Similarly, it is clear that in most circumstances, the quantity of physical interest will not be $S^\text{imp}$ itself, but rather the entanglement between the two wires connected by the resonant level. This will involve more technicalities, and will also be discussed elsewhere.

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Appendix A: Twist fields and orbifold representation

We consider a CFT of central charge $c$ (later one will apply this to the case $c = 1$) with fields living on the surface $\mathcal{R}_N$ with branching conditions between the $N$ sheets of the surface. As argued in Refs. [31, 33], this is equivalent to considering a theory leaving on the complex plane but with twisted vacuums $|\text{orb}\rangle = \Phi(u)|0\rangle$ and $|\text{orb}\rangle = |0\rangle\bar{\Phi}(v)$ and central charge $c = Nc$. The equivalence as the following meaning: given a local operator $X(z^{(j)})$ in the original theory defined on $\mathcal{R}_N$ (here $z$ belongs to the punctured complex plane $\mathbb{C}\setminus\{u,v\}$ and $j = 1...N$ is a sheet index), one can form linear combinations $X^{(k)}(z) = \sum_j e^{2i\pi j k/N} X(z^{(j)})$ that carry a well defined charge under the $\mathbb{Z}_N$ cyclic exchange of the sheets. The equivalence states that to each field $X^{(k)}$ there corresponds a field $X^{(k)}_{\text{orb}}$ in the twisted theory such that the correlators of $X^{(k)}$ fields on $\mathcal{R}_N$ coincide with those of $X^{(k)}_{\text{orb}}$ according to:

$$\langle ...X^{(k)}(z)... \rangle_{\mathcal{R}_N} = \frac{\langle ...X^{(k)}_{\text{orb}}(z)...\Phi(u)\bar{\Phi}(v)\rangle}{\langle \Phi(u)\bar{\Phi}(v) \rangle} \quad \text{(A1)}$$

and in the following we will denote $X^{(k)} \equiv X^{(k)}_{\text{orb}}$ this equivalence between operators.

The twist (and anti-twist) fields $\Phi$ (and $\bar{\Phi}$), are primary fields of conformal dimension $h = \frac{j}{N} (N^2 - 1)$. (Note that in this paper the CFT one considers are chiral ones as the result of our unfolding of the original problem, so that $\hat{h}$ is not defined). $|0\rangle$ is the vacuum of the chiral theory living on a single sheet. The two point function on the plane of these twist fields reads:

$$\langle \Phi(u)\bar{\Phi}(v) \rangle = 1/(u - v)^{2h} \quad \text{(A2)}$$

Appendix B: Energy momentum tensor in the orbifold theory

In this appendix one derives some identities for the operator product expansion of the energy-momentum tensor and its "orbifold powers", by using the equivalence between operators in the untwisted sector of the orbifold theory and $\mathbb{Z}_N$-neutral combinations of operators in the original theory living on the Riemann surface $\mathcal{R}_N$. 

...
1. The orbifold energy-momentum tensor at infinite size

We start by considering the \( \mathbb{Z}_N \)-neutral combination \( T^{(0)}(z) = \sum_{j=1}^N T_{\mathcal{R}_N}(z^{(j)}) = \sum_{j=1}^N T_j(z) \) of the stress energy tensor (here \( j \) labels the sheets of the Riemann surface \( \mathcal{R}_N \)), and show, following Ref. 30, that it coincides with the orbifold energy-momentum tensor \( T_{\text{orb}} \) (and we will write \( T_{\text{orb}} = \sum_j T_j \)) in the sense that

\[
\langle X \rangle_{\mathcal{R}_N} = \frac{\langle X_{\text{orb}} \Phi(u)\Phi(v) \rangle}{\langle \Phi(u)\Phi(v) \rangle} \quad (\text{B1})
\]

where \( X = T^{(0)}(z_1) \ldots T^{(0)}(z_k) \) is a collection of energy-momentum tensors inserted at arbitrary points on \( \mathbb{C} \setminus \{ u, v \} \). The left hand side of Eq. (B1) (first route) can be computed by using the uniformizing map that sends \( \mathcal{R}_N \) onto the punctured complex plane:

\[
z \rightarrow \xi = \frac{z-u}{z-v} \rightarrow \omega = \xi^{1/N}.
\]

On the other hand (second route), the right-hand side of Eq. (B1) can be computed using a Ward identity involving the energy-momentum tensor \( T_{\text{orb}} \).

\[a. \quad \text{One point correlator}\]

Let us follow the first route to compute the one-point function of \( T^{(0)} \) on \( \mathcal{R}_N \). Under the uniformizing transformation (B2), the energy-momentum tensor transforms as:

\[
T_{\mathcal{R}_N}(z) = (\partial_z \omega)^2 \left[ T_\mathbb{C}(\omega) + \frac{\kappa}{\omega^2} \right] \quad \kappa = \frac{c}{24} (N^2 - 1)
\]

so that

\[
\langle T_{\mathcal{R}_N}(z) \rangle = \kappa \left( \frac{\partial_z \omega}{\omega} \right)^2 = \frac{\kappa}{|N(v-u)|^2} \frac{(1-\xi)^4}{\xi^4}
\]

(B4)

where one used \( \partial_z \omega = \frac{\omega^2}{N(v-u)} \frac{(1-\xi)^2}{\xi} \). Since this one-point function doesn’t depend on the Riemann subsheet where it is evaluated, one thus gets the final result by just multiplying by \( N \):

\[
\langle T^{(0)}(z) \rangle_{\mathcal{R}_N} = \frac{\kappa (1-\xi)^4}{N|v-u|^2 \xi^2} = \frac{\kappa}{N} \frac{(u-v)^2}{(z-u)^2(z-v)^2}.
\]

(B5)

According to the second route, one also has: \( \langle T^{(0)}(z) \rangle_{\mathcal{R}_N} = \frac{T_{\text{orb}}(\Phi(u)\Phi(v))}{\Phi(u)\Phi(v)} \) with \( T_{\text{orb}} \) the energy-momentum tensor of the orbifold theory with central charge \( \hat{c} = cN \), and \( \langle \Phi(u)\Phi(v) \rangle = (u-v)^{-2h} \) where \( h \) is the conformal dimension of the twist fields. As noted in Ref. 30, one can indeed recover the result (B5) by using the fundamental OPE:

\[
T_{\text{orb}}(z) \cdot \Phi(u) = \frac{h\Phi(u)}{(z-u)^2} + \frac{\partial_u \Phi(u)}{z-u} + \text{reg.}
\]

(B6)

that leads to the following Ward identity:

\[
\langle T_{\text{orb}}(z)\Phi(u)\Phi(v) \rangle = \left[ \frac{h}{(z-u)^2} + \frac{h}{(z-v)^2} + \frac{\partial_z}{z-u} + \frac{\partial_z}{z-v} \right] \langle \Phi(u)\Phi(v) \rangle = \frac{h(u-v)^2 - 2h}{(z-u)^2(z-v)^2}
\]

(B7)

where the conformal dimension of the twist \( h \) is identified as:

\[
h = \frac{\kappa}{N} = \frac{c}{24} \left( N - \frac{1}{N} \right).
\]

(B8)
Note that we can now easily compute the simplest contribution $A^{(0)} = \int_{x=0} dz \left\langle \bar{T}(z) \right\rangle_{R_N}$. First one notices that in this expression, $\bar{T}$ can be replaced by $T$ since the term $\bar{T} - T \propto \partial Q_{R_N}$ is a total derivative and immediately

$$A^{(0)} = \int_{x=0} dz \frac{h}{N} \frac{(u-v)^2}{(z-u)^2(z-v)^2} = \frac{4i\pi h}{u-v} \tag{B9}$$

where one recalls that $u = -iL$ and $v = iL$, and where a factor of $N$ in the last expression comes from the $N$ disconnected part of the path of integration. An easy way to compute the integral in (B9) consists in deforming the contour of integration in the half plane $\text{Im}(z) > 0$ and pick up a residue at $z = v$.

\[ b. \text{ Higher order correlators} \]

One can explicitly check the equivalence of the two routes when considering higher order correlators. Let us for example compute the two-point function of $T_{\text{orb}}$; one gets

$$\sum_{j_1, j_2} \langle T_{j_1}(z_1) T_{j_2}(z_2) \rangle_{R_N} = \sum_{j_1, j_2} \frac{(1 - \xi_1)^4 (1 - \xi_2)^4}{(N(u-v))^4 (\xi_1 \xi_2)^2} \left[ \kappa^2 + \frac{c}{2} \frac{(\omega_{1}^{(j_1)} \omega_{2}^{(j_2)})^2}{(\omega_{1}^{(j_1)} - \omega_{2}^{(j_2)} \xi_{1})^2}\right] \tag{B10}$$

where one has written $\omega^{(j)} = e^{2i\pi j/N} \omega^0$, and one has introduced the function

$$f_{p,q}(x) = x^q \sum_{j_1, j_2} \frac{e^{2i\pi j_1 j_2}}{(1 - xe^{2i\pi j_1 j_2})^{p+q}} = \delta(q) + x^{-p} \binom{-p+q-1}{p+q-1} \partial_y^{p+q-1} \left[ \frac{y^{q-1}}{y^N - 1} \right] \bigg|_{y = x^{-1}}, \tag{B12}$$

The last equality shows that $f_{p,q}$ is in fact an entire function of $x^N$; it can be derived by using an integral representation $f_{p,q}(x) = \frac{1}{2\pi i} \int_{C_{1\pm}} \frac{dy}{y^N - 1} \frac{x^{y}}{(1-x^{y})^{p+q}}$, where $C_{1\pm}$ circles the unit circle, clockwise at radius $1 - 0^+$ and counter-clockwise at radius $1 + 0^+$. Moving the contour one gets Eq. (B12), so that $f_{2,2}(x) = N^3 \left( \frac{e^{x^n}}{(1-x^N)^2} \right) + \frac{x^N N(N-1)}{6} \left( \frac{e^{x^n}}{(1-x^N)^2} \right)$. Note that, as it should, the final result (B11) does not depend any longer on the sheet the points $z_1$ and $z_2$ are chosen in. Putting everything together, one gets:

$$\sum_{j_1, j_2} \langle T_{j_1}(z_1) T_{j_2}(z_2) \rangle_{R_N} = h^2 D^2 + \frac{2hD}{(z_{12}^2)^2} + \frac{cN}{2(z_{12}^2)^4} \quad \text{with} \quad D = \frac{(v-u)^2}{(z_1-z)(z_2-z)(z_2-u)^2} \tag{B13}$$

This is again consistent with the Ward identity:

$$\left\langle T_{\text{orb}}(z_1) T_{\text{orb}}(z_2) \Phi(u) \Phi(v) \right\rangle = \frac{\hat{c}}{2(z_{12})^4} \left\langle \Phi(u) \Phi(v) \right\rangle + \left[ \frac{2}{(z_{12})^8} + \partial_{z_1} \frac{h}{(z-v)^2} + \partial_{z_2} \frac{h}{(z-u)^2} + \partial_{z_2} \frac{h}{(z-v)^2} \right] \left\langle T_{\text{orb}}(z_2) \Phi(u) \Phi(v) \right\rangle \tag{B14}$$

\[ 2. \text{ Computations at finite size} \]

As we have seen in the section V, in order to compute the correlators on $C_N$, with the equivalence with the orbifold theory, one has to make another change of variables from the plane to the cylinder $z = \frac{2iL}{\text{Im}(\omega)}$ or $\omega$, that change the integration contour and thus forbid to move it and wrap it around the twist fields, as we did in the infinite size case. For instance

$$\sum_{i=1}^{N} \left\langle T(z^{(i)}) \right\rangle_{C_N} \left\langle \Phi(u) \Phi(v) \right\rangle_{C_N} = \frac{\left( \hat{c} \frac{dz}{d\omega} \right)^2 \left\langle T_{\text{orb}}(\omega) \Phi(u) \Phi(v) \right\rangle_{C_N}}{\left\langle \Phi(u) \Phi(v) \right\rangle_{C_N}} \tag{B15}$$

$$= \frac{\left( \frac{dz}{d\omega} \right)^2 T_{\text{orb}}(\omega) \Phi(u) \Phi(v)}{\left\langle \Phi(u) \Phi(v) \right\rangle_{C_N}} + \frac{\hat{c}}{12} \left( \frac{dz}{d\omega} \right)^2 \left\{ z; \omega \right\}$$
And without too much pain, one can manage to compute the very first terms of the contributions to the Renyi entropy, up to order \( O(T_0^{-2}) \) (remember that \( \partial \phi (-k) = \partial \phi (k^*) \) and \( \bar{v} = e^{i \pi L / L_0} = \bar{u}^* \))

\[
A^{(0)}_{L_0} = \frac{\pi}{L_0} \int_0^{\infty} d\omega \left[ \frac{h \omega (u - \bar{v})^2}{|\omega - \bar{v}|^4} - N \frac{24 \omega}{24 \omega} \right] = -2 \frac{\pi h}{L} f_1(L / L_0) - \int_0^{\infty} d\omega \frac{\pi N}{24 L_0 \omega} \text{div. (B16)}
\]

\[
A^{(0,0)}_{L_0} = \left( A^{(0)}_{L_0} \right)^2 + \int_0^{\infty} d\omega_1 d\omega_2 \left[ \frac{h \pi^2 \omega_1 \omega_2}{L_0^2 (\omega_1 - \omega_2)^2} \frac{(u - \bar{v})^2}{|\omega_1 - \bar{v}|^2 |\omega_2 - \bar{v}|^2} \frac{N \pi^2 \omega_1 \omega_2}{24 L_0 (\omega_1 - \omega_2)^2} \right] \text{div. (B17)}
\]

where

\[
f_1(a) = a (1 + \pi (1 - a) \cot \pi a) \quad \text{(B18)}
\]

\[
f_2(a) = f_1(a)^2 + a^2 (1 - a)^2 \pi^2 \quad \text{(B19)}
\]

Note that these integrals are well defined, even in the presence of the \((\omega_i - \omega_j)^{-k}\) terms, because the integration points are ”avoiding each other”. This is a consequence of our point-splitting regularization scheme.

As discussed in the main text, the diverging terms in the integrals stemming from ”intricated” integrands are proportional to \( N \) only and are cancelled out by terms stemming from the denominator of Eq. (20).

Appendix C: The composite field \( \mathcal{O}_4 \)

In this appendix we establish the operator product expansions of the first composite field \( \mathcal{O}_4^{\text{orb}} \) in the neutral sector of the orbifold theory with the twist operator \( \Phi \). The orbifold form of the perturbing operator is:

\[
\mathcal{O}_4^{\text{orb}}(z) = \sum_{j=1}^{N} \overline{T}_j \overline{T}_j (z) \quad ; \quad \overline{T}_j = T_j + \alpha \partial Q_j \quad \text{(C1)}
\]

where \( z \) is a coordinate on the complex plane (punctured at points where the twist fields is inserted), and on the right hand side the index \( j \) indicates the sheet of \( \mathcal{R}_N \). The parenthesis in the second line of (C1) indicates normal ordering \( \langle A(z)B(z) \rangle = \frac{1}{2\pi i} \oint \frac{dz}{z} A(z + x)B(z) \) that is performed on \( \mathcal{R}_N \).

Introducing the Virasoro generators’ action on a local field \( O(z) \) via \( L_{-k}(z) O(z) = \frac{1}{2\pi i} \oint \frac{dx}{x - z} (x - z)^{-k} T_{\text{orb}}(x) O(z) \) [34], one parametrizes the OPE of \( \mathcal{O}_4^{\text{orb}} \) and \( \Phi \) as follows (only the descendants of the twist field can appear):

\[
\mathcal{O}_4^{\text{orb}}(z) \cdot \Phi (0) = \left[ \frac{\tilde{a}_4}{\bar{z}} + \frac{\tilde{a}_4}{\bar{z}} \right] + \left[ \frac{\tilde{a}_4}{\bar{z}} + \frac{\tilde{a}_4}{\bar{z}} \right] \Phi (0) \quad \text{(C2)}
\]

We will determine the coefficients \( \tilde{a}_i, \tilde{b}_i, \tilde{c}_i \) (see Eqs.(C14)) following the simplest route: one starts by computing correlators \( G_{X\mathcal{O}} = \langle X(z_i) \mathcal{O}(z) \rangle_{\mathcal{R}_N} \) where \( X(z_i) \) is an insertion of a collection of stress energy tensors \( T_{\text{orb}} \) at points \( z_i \), and one then expands this expression when \( z \to u \). On the other hand, one has \( G_{X\mathcal{O}} = \langle \Phi(u) \Phi(v) \rangle^{-1} \langle X(z_i) \mathcal{O}(z) \Phi(u) \Phi(v) \rangle \) by identification this fixes constraints on the OPE \( \mathcal{O}(z) \cdot \Phi(u) \) that are sufficient to determine the \( a \) priori unknown coefficients in Eq. (C2).

The operator (C1) can be split in 3 pieces,

\[
\mathcal{O}_4^{\text{orb}} = (T^2)_{\text{orb}} + \alpha \left( \{ \partial Q, T \} \right)_{\text{orb}} + \alpha^2 \left( \partial Q \partial Q \right)_{\text{orb}}
\]

Let’s start with \( (T^2)_{\text{orb}}(z) \equiv \sum_j \langle T_j T_j \rangle (z) = \frac{1}{2} \sum_{z_1, z_2} \sum_j T_j (z_1) T_j (z_2) \). Note that the summation implies that \( (T^2)_{\text{orb}} \neq (T_{\text{orb}})^2 \). Starting from the correlator (B10), inserting \( \delta (j_1 - j_2) \) in the sum, and performing the normal order one finds:

\[
\sum_j \langle T_j T_j \rangle (z)_{\mathcal{R}_N} = \frac{1}{N^3 (u - v)^4} \left[ \kappa^2 (1 - \xi)^2 + \frac{\kappa}{60} P_1 (\xi) \right] \quad \text{(C3)}
\]
with \( P_1(X) = (1 - X)^2(29N^2 - 11) + 90N^2(1 + X)^2 \)

Parametrizing the OPE of \((T^2)_{\text{orb}}\) with the twist field as:

\[
(T^2)_{\text{orb}}(z) \cdot \Phi(0) = \left[ \frac{a_1 L_{-1}^1}{z^4} + \frac{a_2 L_{-1}^1}{z^2 + b_2 L_{-2}^2} + \frac{a_4 L_{-1}^1 + b_1 L_{-1}^1 L_{-2} + c_1 L_{-3}^3}{z} \right] \Phi(0),
\]

one can then expand the correlator \( \langle (T^2)_{\text{orb}}(z) \Phi(u) \widetilde{\Phi}(v) \rangle \) when \( z \) approaches one of the twist fields, say \( \widetilde{\Phi}(v) \), and identify the two expansions obtained via the OPE (C4) and via (C3). The first expansion involves correlators:

\[
\langle \Phi(u) L_{-n} \widetilde{\Phi}(v) \rangle = \frac{1}{24\pi} \int_u \frac{dz}{(z-v)^{n+1}} \langle T_{\text{orb}}(z) \Phi(u) \widetilde{\Phi}(v) \rangle = \frac{(n+1)h}{(u-v)^{n+2h}} \quad (n \geq 0)
\]

\[
\langle \Phi(u) L_{-1} L_{-2} \widetilde{\Phi}(v) \rangle = \frac{6h(h+1)}{(u-v)^{3+2h}}
\]

(C5)

Now expanding (C3) when \( z \to u \), one gets the following identifications:

\[
a_4 = \frac{1}{5cN} [h(22 + 5c) + 9cN] \\
a_3 = \frac{1}{5cN} [2h(22 + 5c) + 3cN] \\
2(2h + 1)a_2 + 3b_2 = \frac{2h}{cN}(22 + 5c) \\
(2h + 1)(2h + 2)a_1 + (3h + 1)b_1 + 2c_1 = \frac{2h}{cN}(22 + 5c)
\]

(C6)

In order to remove the ambiguity in the determination of the unknown coefficients \( a_i, b_i, c_i \), one needs to consider higher order correlator with more insertion of \( T(0) \): 

\[
\sum_{j,k} (T_j(z_1)(T_k(z_2)))_{\mathcal{R}_N} = \frac{(1 - \xi_1)^4(1 - \xi_2)^8}{N^6(u-v)^6 \xi_1^4 \xi_2^4} \sum_{j,k} \left[ \kappa^3 + \frac{\kappa c}{1440(1 - \xi_1^2)^2} P_2(\xi_2) \right.
\]

\[
\left. + \frac{c}{12(1 - \xi_2^2)^2} \sum_{p=0}^{2} \frac{\omega_1^2 \omega_2^{4-p} Q_p(\xi_2)}{\omega_2^{4-p}} \right]
\]

\[
= \frac{(1 - \xi_1)^4(1 - \xi_2)^8}{N^6(u-v)^6 \xi_1^4 \xi_2^4} \sum_{j,k} \left[ \kappa^3 + \frac{\kappa c P_2(\xi_2)}{1440(1 - \xi_2^2)^2} \right. \\
\left. + \frac{\kappa c f_{2,2}(\xi_2)}{1440(1 - \xi_2^2)^2} \frac{\omega_0^0}{\omega_1^0} \left( \frac{\omega_2^0}{\omega_0^0} \right) \right]
\]

(C7)

with

\[
P_2(X) = (-22 - 100N^2 + 122N^4)X + (11 - 130N^2 + 119N^4)(1 + X^2) \\
Q_0(X) = 36(1 - X)^2 \\
Q_1(X) = -36(1 - X)[(N + 1)X + N - 1] \\
Q_2(X) = 5(1 - X)^2 + 18N(X^2 - 1) + N^2(13X^2 + 10X + 13)
\]

(C8)

This fixes the coefficients \( a_2, b_2 \). A last constraint is needed: we compute the 3 points correlator \( \sum_{j,k,l} \langle T_j(z_1)(T_k(z_2))(T_l(T)) \rangle_{\mathcal{R}_N} \) (its expression is not particularly enlightening so we do not display it) and we
take the limit $z_3 \to v$. Finally we find:

\[
\begin{align*}
a_1 &= \frac{22 + 5c}{5\tilde{c}} \left(-\epsilon^2 + 23\epsilon h - 22h^2\right) \\
b_1 &= \frac{22 + 5c}{5\tilde{c}} \left(2(\epsilon - 19\epsilon h + 3\epsilon^2 h + 54\epsilon^2 h^2 - 21\epsilon h^2 + 2\epsilon^2 h^2 - 90h^3 + 22\epsilon h^3 + 48h^4)\right) \\
c_1 &= \frac{22 + 5c}{5\tilde{c}} \left(2(\epsilon - 12h + \epsilon h + 3h^2)\right) \\
a_2 &= \frac{22 + 5c}{5\tilde{c}} \left(-\epsilon + 10h + 2\epsilon h + 16h^2\right) \\
b_2 &= \frac{22 + 5c}{5\tilde{c}} \left(2h(\epsilon - 22h + 2\epsilon h + 16h^2)\right) \\
a_3 &= \frac{3 + 2h}{5} + \frac{22 + 5c}{5\tilde{c}} \\
a_4 &= \frac{9}{5} + \frac{22 + 5c}{5\tilde{c}}
\end{align*}
\] (C9)

So far, the computation was totally general and applies to an arbitrary CFT put on a $\mathbb{Z}_N$. In our case, one deals with a free boson $Q \propto i\partial \phi_+$, so that one should put $c = 1$ and $\tilde{c} = N$ in Eqs. (C9).

We are not done so far: there is a second piece to $O_4^{\text{orb}}$, namely terms that depend on the deformation parameter $\alpha$. On the plane, this deformation parameter simply amounts to a change of the central charge: $\hat{T}$ is an energy-momentum tensor of a $c = 1 - 6\alpha^2$ CFT. However, it would be wrong to identify $O_4^{\text{orb}}$ with $(\hat{T}^2)_{\text{orb}}$, the orbifold version of the energy-momentum tensor of the deformed theory. Instead, what one should do is put the $c = 1$ theory on the orbifold, and determine the orbifold version of the operators $\alpha^2 \sum_j (\partial Q_j \partial Q_j)$ and $\alpha \sum_j (\{\partial Q_j, T_j\})$.

Consider $\sum_j (\partial Q_j \partial Q_j)$: the same game can be played with it, its OPE reading:

\[
(\partial Q \partial Q)_{\text{orb}}(z) \cdot \Phi(0) = \left[\frac{\tilde{a}_4 L_0}{z^4} + \frac{\tilde{a}_3 L_{-1}}{z^3} + \frac{\tilde{a}_2 L_{-2}}{z^2} + \frac{\tilde{a}_1 L_{-3} + \tilde{b}_2 L_{-2} + \tilde{c}_1 L_{-1} + \epsilon_1 L_{-2}}{z}\right] \Phi(0)
\]

(C10)

Proceeding in a similar way (we do not reproduce the steps, that are identical), we obtain the coefficients: $(\tilde{a}_i, \tilde{b}_i, \tilde{c}_i) = -\frac{1}{9}(a_i, b_i, c_i)$, except for $a_3 = -\frac{1}{9}a_3 + \tilde{a}_3$ and $a_4 = -\frac{1}{9}a_4 + 2$. Comparing this with the OPE:

\[
\partial^2 T_{\text{orb}}(z) \cdot \Phi(0) = \frac{6h \Phi(0)}{z^4} + \frac{2L_{-1} \Phi(0)}{z^3},
\]

we find the remarkable relation:

\[
\sum_j \left[(\partial Q_j \partial Q_j) + \frac{4}{9}(T_j T_j)\right] = \frac{1}{3} \partial^2 \sum_j T_j
\]

(C12)

so that keeping only terms that are even in $\alpha$ (terms linear in $\alpha$ are easily showed to be total derivatives, $\{T, \partial Q\}_{\mathcal{R}_N} = \frac{2}{9} \partial Q(T Q)_{\mathcal{R}_N}$, and in general are odd under $Q \to -Q$, and they don’t contribute to expectation values):

\[
O_4^{\text{orb}} \bigg|_{\alpha-\text{even}} = \left(1 - \frac{4\alpha^2}{9}\right) (T^2)_{\text{orb}} + \frac{\alpha^2}{3} \partial^2 T_{\text{orb}}
\]

(C13)

Restricting again to terms that are even in $\alpha$, one gets:

\[
\begin{align*}
\{\tilde{a}_i, \tilde{b}_i, \tilde{c}_i\}_{i \leq 2} &= \{\lambda a_i, \lambda b_i, \lambda c_i\} \\
\tilde{a}_3 &= \lambda a_3 + 2\alpha^2/3 \\
\tilde{a}_4 &= \lambda a_4 + 2\alpha^2
\end{align*}
\] (C14)

with $\lambda = 1 - 4\alpha^2/9$, and $\{a_i, b_i, c_i\}$ are given by Eq. (C9) with $c = 1$.

**Appendix D: Computations with a direct quantization**

We hereunder report some details useful in order to understand how to explicitly compute correlators with the mode decomposition.
1. Energy density on the plane

With the mode decomposition it is easy to show that
\[ \langle \varphi'(k) \partial \varphi(-k)(\xi') \rangle = \sum_{m>0} \frac{1}{4\pi x'} \left( \frac{\xi'}{x} \right)^m, \]
so that we find
\[ g(k)(x',x) = \langle \partial \varphi(k)(x) \partial \varphi(-k)(x') \rangle \]
\[ = -\frac{1}{4\pi(x-x')^2} \left[ (1 - \frac{k}{N}) + \frac{x' k}{N} \right] \]
we then have the expression for the \( k \)-th energy-momentum tensor
\[ \langle T^{(k)}(\xi) \rangle = -\frac{k(N-k)}{4N^2\xi^2} \]
and summing up this expression over \( k \) to get the total energy-momentum tensor, we have
\[ \langle T(\xi) \rangle = \frac{c(N^2-1)}{24N^2} \frac{1}{\xi^2}. \]

2. Cut of finite size on the plane and on the cylinder

If we are interested in a cut between two points \( u, v \) on the \( z \)-plane, all we have to do is make a conformal transformation that sends \( v \to 0 \) and \( u \to \infty \). Namely
\[ \xi = \frac{z-v}{z-u} \]
since \( \partial \varphi(k) \) is a primary field, it is easy to compute with the correlator (D2)
\[ \langle T^{(k)}(z) \rangle^{(u,v)} = \left( \frac{d\xi}{dz} \right)^2 \langle T^{(k)}(\xi) \rangle^{(0,\infty)} = \frac{k(1-k)(u-v)^2}{4N^2(z-u)^2(z-v)^2} \]
and summing up over \( k \), we find the expectation value of Eq. (B17)
\[ \langle T(z) \rangle^{(u,v)} = \frac{c(N^2-1)}{24N^2} \frac{(u-v)^2}{(z-u)^2(z-v)^2} \]
we see in the next section that we can find directly this result using the energy-momentum tensor’s transformation identity for \( T^{(k)} \) since the Schwartzian of the transformation (D5) is equal to zero.

Now we want to compute correlations on \( C_N \) : cylinders of size \( 2L_0 \) linked by a finite cut between \( u = -iL \) and \( v = iL \). To come back to the usual situation where the fields acquire a phase when turning around \( z = 0 \), we have to make the following transformations : first explode the cylinder on the plane, where \( u = e^{-i\pi L/L_0}, v = e^{i\pi L/L_0} \) and then transform \( v \to 0 \) and \( u \to \infty \), namely it is given by
\[ \xi = \frac{e^{\pi z/L_0} - e^{+i\pi L/L_0}}{e^{\pi z/L_0} - e^{-i\pi L/L_0}} \]
where \( z \in \mathcal{C} \).
3. Schwartzian

It is interesting to check explicitly that even in this twisted geometry, at finite size, and with this mode decomposition one finds that the schwartzian appears when computing

$$\langle T^{(k)}(z) \rangle^{(u,v)}_{cyl.} = \text{lim}_{\epsilon \to 0} -2\pi \left[ \langle \partial \varphi^{(k)}(z + \epsilon) \partial \varphi^{(-k)}(z) \rangle^{(u,v)}_{cyl.} + \frac{1}{\epsilon^2} \right]$$

$$= -2\pi \left[ \frac{\partial}{\partial z} \langle \partial \varphi^{(k)}(z + \epsilon) \partial \varphi^{(-k)}(z) \rangle^{(0,\infty)}_{pl.} + \frac{1}{\epsilon^2} \right]$$

$$= \frac{\pi^2 k (1 - k) e^{2\pi z/L_0} (u - v)^2}{4N^2 L_0^2 (e^{\pi z/L_0} - u)^2 (e^{\pi z/L_0} - v)^2} - \frac{c\pi^2}{24L_0^2}$$

(D9)

so that we will now directly use the well-known identity: $$T'(z) = \left( \frac{d}{dz} \right)^2 \langle T(\xi) + \frac{1}{\pi} \langle \xi; z \rangle \rangle$$, even for the $$T^{(k)}$$ energy-momentum tensors.

4. Example with ($$\tilde{T}^2$$)

Where the quantization scheme becomes really useful is when one is interested in computing expectation values of operators that have no simple expression in their orbifold version, for instance ($$\tilde{T}^2$$).

All we have to do is to use the mode decomposition 56 of the basic fields $$\varphi^{(i)}$$ and use the Wick theorem to compute explicitly

$$\int_{z=0} d\xi \langle (\tilde{T}^2)(z) \rangle_{c_N} = \sum_j \int_{z=0} d\xi \langle (T^2_j)(z) \rangle_{cyl.} + \alpha^2 \sum_j \int_{z=0} d\xi \langle [\partial Q_j \partial Q_j](z) \rangle_{c_N}$$

(D10)

where only the non vanishing expectation values have been written. The integrand can be expressed with the fields $$\partial \varphi^{(k)}$$ as

$$\sum_j \langle (T^2_j)(z) \rangle_{c_N} = \frac{1}{2\pi N} \sum_{k,k'} \int_0 d\omega \frac{d\epsilon}{\epsilon} \left[ \langle T^{(k)}(z + \epsilon) \rangle \langle T^{(k')}(z) \rangle \right.$$

$$+ 2(-2\pi)^2 \frac{\langle \partial \varphi^{(k)}(z + \epsilon) \partial \varphi^{(-k)}(z) \rangle \langle \partial \varphi^{(k')}(z) \partial \varphi^{(-k')}(z + \epsilon) \rangle}{B} \right]$$

(D11)

**Term A** The term A expressed in the $$\omega$$-plane (change of variables 46) is given by (we denote $$a = L/L_0$$) is

$$A = \int_0^\infty d\omega \left[ \frac{h^2 \pi^3 (\bar{\omega} - \bar{\nu})^4 \omega^3}{L_0^3 N^3 |\omega - \bar{\nu}|^8} - \frac{h^3 \pi (\bar{\omega} - \bar{\nu})^2 \omega^2}{12L_0^3 N |\omega - \bar{\nu}|^4} + \frac{N^3}{576L_0^3 |\omega|^4} \right]$$

$$= \frac{h^2 \pi^3 a^2}{3NL^3} \left( 15 \frac{f_1(a)}{\sin^2 a \pi} - 6 f_1(a) - 5a \right) + \frac{h^3 \pi^3}{6L^3 a^2} f_1(a) + \text{div.}$$

(D12)

**Term B** The term B gives

$$B = \int_0^\infty d\omega \left[ \frac{22h^2 \pi^3 (\bar{\omega} - \bar{\nu})^4 \omega^3}{5NL_0^3 |\omega - \bar{\nu}|^8} + \frac{h^3 \pi (\bar{\omega} - \bar{\nu})^2 \omega^2}{30L_0^3 |\omega - \bar{\nu}|^4} \left( \frac{25}{|\omega - \bar{\nu}|^4} + \frac{90 \omega (\bar{\nu} + \bar{\omega})}{|\omega - \bar{\nu}|^6} + \frac{54 (\bar{\nu} - \bar{\omega})^2 \omega^2}{|\omega - \bar{\nu}|^8} \right) \right] + \frac{11N^3}{1440L_0^3 |\omega|^4}$$

$$B = \frac{22h^2 \pi^3 a^2}{15NL^3} \left( 15 \frac{f_1(a)}{\sin^2 a \pi} - 6 f_1(a) - 5a \right) + \frac{11h^3 \pi^3}{15L^3 a^2} f_1(a) + \text{div.}$$

(D13)
Term C  And finally, the term C gives

$$C = \int_0^\infty d\omega \left[ -\frac{12h^2\pi^3(\bar{u} - \bar{v})^4\omega^3}{5NL^3|\omega - \bar{u}|^8} + \frac{h\pi^3(\bar{u} - \bar{v})^2\omega}{5L^3} \left( \frac{5}{|\omega - \bar{u}|^4} + \frac{10\omega(\bar{u} + \bar{v})}{|\omega - \bar{u}|^6} + \frac{6(\bar{u} - \bar{v})^2\omega^2}{|\omega - \bar{u}|^8} \right) - \frac{N\pi^3}{240L^3\omega} \right]$$

$$= -\frac{4h^2\pi^3a^2}{5NL^3} \left( 15 \frac{f_1(a)}{\sin^2(a\pi)} - 6f_1(a) - 5a \right) - \frac{2h\pi^3}{5L^3} a^2 f_1(a) + \text{div.} \quad (D14)$$

The IR divergences that come when integrating at large distance from the insertion of the twist operators will actually not depend on the twisted geometry. Hence these divergences will only get multiplied by a factor \(N\), so that they will be compensated at every orders. This is verified for the diverging terms stemming from A and B which are indeed linear in \(N\), so that the expansion in power series of \(O(1/LT_b)\) of the exponential \(\exp(-\int_{R_N} dz A_{div}^{(1)})\) will be exactly compensated from terms of the denominator \(\exp(-N\int_{R_1} dz A_{div}^{(1)})\) in the expression of \(R_N\).

Combining the results one finally gets \((a = L_0/L)\)

$$A_{L_0}^{(1)} = \frac{(9 - 4a^2)}{5} \left[ \frac{3h^2\pi}{NL^3} f_4(a) + \frac{h\pi^3a^2f_1(a)}{2L^3} \right] \quad (D15)$$

where

$$f_4(a) = \frac{\pi^2a^2}{15} \left( 15f_1(a)/\sin^2(a\pi) - 6f_1(a) - 5a \right) \quad (D16)$$

$$= 1 - 2\pi^2a^2/5 + \pi^4a^4/15 + O(a^6) \quad (D17)$$

It is easy to check the result known at infinite size for \(A^{(1)}\) (from Eq. (40)):

$$A^{(1)} = \frac{3(9 - 4a^2)\hbar^2\pi}{5NL^3} = -\frac{3\pi\hbar^2(4 - D)(1 - 4D)}{DNL^3} \quad (D18)$$

Note that now, \(A_{L_0}^{(1)}\) gives non-vanishing corrections to the entropy (terms proportional to \(h\)). These corrections are due to finite size effects, they are proportional to \(a^2\) and depend on the interactions (\(a^2\) term).

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