On the Problems with Background Independence in String Theory

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Dedicated to my teacher L. D. Faddeev
on the occasion of his 60th birthday

The problems with background independence are discussed in the example of open string theory. Based on the recent proposal by Witten I calculate the String Field Theory action in conformal perturbation theory to second order and demonstrate that the proper treatment of contact terms leads to nontrivial equations of motion. I conjecture the form of the field theory action to all orders.

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In contrast with difficult problems in realistic 4d Field Theory models, where the theory is defined and an explicit analytic solution is not yet known, String Theory isn’t yet even defined. In many cases what we have is just a number of $S$-matrices for the processes when the background is fixed by our choice of conformal field theory in the first quantized formulation for amplitudes. Satisfactory formulation of String Theory would have been a formulation where we don’t need to refer to any particular classical background and these ”classical backgrounds” are given by solutions of some equations. The latter statement is very vague, because unfortunately it is not even clear (at least to the author) what should be the right terminology to address the question. It is believed, by analogy with the second quantized description of ordinary quantum field theory, that the understanding of vacuum structure of string theory as well as the nonperturbative character, can be achieved by developing the field theory language to describe the target space theory. It might well be that the procedure that allows us to construct second quantized field theory from Feynman sum over trajectories directly applied to string theory is not the best way to approach the problem and some other new ideas should be introduced. One of the most important ingredients of any construction has to be a background independence.

In this paper I will address the question of background independent formulation of string theory in the example of open string theory recently suggested by Witten [1]. I can’t claim that at present every point is understood for the case of open string; this paper should be considered as an attempt to single out main problems and find a correct language based on this experience. The calculations and observations presented in section 2, together with final result (see below) might serve as a proper guide. This explains the title.

I’ll show that the integrals of total derivatives do not decouple inside the correlation functions that defines the String Field Theory action in the formalism of [1] due to contact term contributions. I’ll explain that these contributions have universal character and can’t be removed by change of renormalization scheme (this statement has the same origin as the one for gauge anomalies in field theory). This fact leads to slight modifications of the assumptions made in original paper [1] and also in [3]. It was shown in [3] that under the key assumption of decoupling of total derivatives, plus the requirement that BRST operator on the boundary is coupling constant independent, theory has a linear character.

1 For the discussion of the same problem in the case of closed string field theory in a different formalism see [3] and references therein.
I’ll demonstrate here that including the contribution of total derivatives one also has to properly define "BRST" operator on the boundary, which now necessarily should depend on couplings to satisfy the consistency condition. I’ll discuss the ambiguities related to this issue and will make a particular choice. In this setup I’ll calculate the field theory action in the lines of [3] using the conformal perturbation theory around some fixed point and demonstrate the existence of following relation:

\[ S = -\beta^i \frac{\partial}{\partial t^i} Z(t) + Z(t), \]  

(1)

up to the second order in coupling constant. Here \( \beta \) is the world-sheet \( \beta \)-function and \( Z \) is a partition function. I think that (1) is true to all orders, but calculations beyond second order, as usual, are very complicated. It was discovered in [4] from very general arguments that action should have the above form with first term in (1) given by some vector field. Thus, this vector field is identified with \( \beta \)-function in our approach. This identification is consistent with the statement of [4], that zeros of vector field are the classical equations of motion (CFT on world-sheet). We will give an alternative way of explaining this statement in the section 2.

For the reasons that we are dealing with interacting field theory, we are forced to loose the background independence during the calculation of the action in the perturbation theory, so this formalism doesn’t achieve the final goal; the approach is also coordinate system dependent. In fact, the latter makes it difficult to reconstruct the final answer globally, once it is computed in the perturbation theory. But, if (1) is correct to all orders, formal background independence is preserved. At the same time any approach to write down the expressions for \( \beta \) and \( Z \) should appeal to perturbation expansion.

I’ll not discuss the important issue of gauge symmetries for (1), but as it follows from (1), all symmetries of partition function are automatically the symmetries of action \( S \). I think that the general transformation properties could be written using the results of Section 2 in the lines of [4].

The form of (1) is quite general one could conjecture that it should be true also for the closed string theory even the analog of (1) (the corresponding space of 2d field theories together with background independent formulation) is not yet known. The expression (1) is simple, but the objects that enter, \( \beta \) and \( Z \), usually are impossible to calculate in closed

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2 The fact that the string field theory action on the classical equations of motion is given by the world-sheet bosonic partition function was previously suspected in [3] [4].
form. As a result this formalism doesn’t avoid the usual technical difficulties that are present in any other formulation of string field theory, although it is formally background independent and contains all string modes.

1. Boundary Problem and Open String Field Theory

In the beginning of this section we first will describe the construction of [1] with the emphasis of places where some assumptions are made. The idea of the construction in [1] is based on BV formalism. Let $M$ be a supermanifold which is equipped with closed, nondegenerate odd simplectic structure $\omega$ and $U(1)$ symmetry, called ghost number $U$. This means that in Darboux coordinates $\psi, \theta$ on $M$ with $\psi$ fermionic and $\theta$ bosonic $\omega = d\psi d\theta$ and $\omega$ has ghost number 1. In analogy with ordinary (bosonic) simplectic manifolds one can define the Poisson Brackets, antibracket, with

$$\{A, B\} = \frac{\partial_r A}{\partial t^k} \omega_{kj} \frac{\partial_t B}{\partial t^j}.$$ (1.1)

One can show that the following two simple facts take place:

i. If $V$ is a vector field that generates the symmetries of $\omega$, which means that $\left( d_i^V + i_V d \right) \omega = 0$, then there exists a function $S$ that

$$dS = i_V \omega.$$ (1.2)

ii. Vector field defined by (1.2) generates a symmetry of $\omega$ for any function $S$.

From the above immediately follows that the Poisson brackets of function $S$, $\{S, S\}$, defined by (1.2) is annihilated by $d$ and thus the function $\{S, S\}$ is a constant

$$d\{S, S\} = 0.$$ (1.3)

The equation

$$\{S, S\} = 0$$ (1.4)

is called the BV master equation and $S$ is the action functional if it solves the master equation. Every solution of (1.4) is automatically gauge invariant [1], [4]. ; the variation of the action under any symplectic transformation

$$\delta t^i = \{t^i, K\},$$ (1.5)
generated by Hamiltonian function $K$ (odd function), is given by $\delta S = \{S,K\}$ and for $\{S,K\} = 0$, it vanishes; trivial transformations are given with $K = \{S,\Lambda\}$.

In the quantization of gauge theories the action $S$ is given on subspace of $M$ with $U = 0, S_0$, and we have to find $S$. That is in fact what the Faddeev-Popov procedure does in the case of Gauge Theories. In the case of String Field Theory, the idea of [1] was to identify the antibracket and vector field in terms of the world-sheet theory and thus identify $S$ as the action of the corresponding target space theory. It was claimed in [1] that this can be done in the case of Open String in a background independent way. The following identifications were proposed:

$$\omega : \omega(\delta O, \delta O) = \int_{\partial \Sigma} d\sigma_1 \int_{\partial \Sigma} d\sigma_2 \langle \delta O(\sigma_1) \delta O(\sigma_2) >$$

(1.6)

$$V : \delta V O = \{Q,O\}$$

(1.7)

where $< ... >$ formally is defined through the world-sheet theory given by path integral corresponding to the 2d action

$$L_a = \int_{\Sigma} d^2 z \left( \frac{1}{8\pi} g^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j \eta_{ij} + \frac{1}{2\pi} b^{ij} D_i c_j \right) + \int_{\partial \Sigma} d\theta V(X,b,c,t)$$

(1.8)

Here, the first term is the closed string background and the second term describes an arbitrary boundary interaction, parametrised by coupling constants $t^i$ (in general there are infinite number of coupling constants), with the condition that the boundary operator $V$ has the form

$$V = b_{-1} O,$$

(1.9)

with $O$ being a general operator of ghost number 1. $Q$ is a BRST operator defined by BRST current: $Q = \int_C d\sigma J_{BRST}$ with contour $C$ approaching the boundary $\partial \Sigma$ and

$$b_{-1} = \int_C b(v), b(v) = v^i b_{ij} \epsilon_k^i dx^k$$

with $v^i$ being the Killing vector that generates the rotation of disc. This world-sheet action is also equipped with an ultraviolet cutoff $a$.

From the above identifications we have the definition of string field theory action:

$$dS = \frac{1}{2} \int_{0}^{2\pi} d\theta_1 d\theta_2 < dO(\theta_1) \{Q,O\}(\theta_2) >,$$

(1.10)

\footnote{We will not worry about generality and assume that $\Sigma$ is just a disc.}
where $d = dt^i d/dt^i$ and $< ... >$ again denotes un-normalized correlation function. Witten has shown that (1.6) gives a closed form and it is invariant under (1.7). We need for future use to repeat his arguments and stress the points where some assumptions are made.

The fact that $\omega$ defined by (1.6) is closed follows from the identity:

$$0 = <b_{-1}(A_1(\theta_1)\ldots A_n(\theta_n))>$$

(1.11)

Here we use the definition of $b_{-1}$ and take two limits: first we shrink the contour $C$ to a point and get zero; second we take the limit when the contour approaches the boundary $\partial \Sigma$ and get the right hand side in (1.11). Thus, in the notation $\delta_i O = \frac{\partial}{\partial t^i} O$ we have:

$$d\omega(\delta_i O, \delta_j O, \delta_k O) = \frac{\partial}{\partial t_k} \omega(\delta_i O, \delta_j O) -$$

$$- \text{cyclic permutations} = <b_{-1} \delta_k O> \delta_i O \delta_j O - \text{cyclic perm.} = 0$$

(1.12)

and the last step follows from (1.11).

BRST invariance of (1.6) is equivalent to exactness of the right hand side in (1.10). This follows from the simple observation that because the transformation law of $\omega$ is $\omega' = \omega + \epsilon(i_V d + di_V)\omega$ and we already have shown that $d\omega = 0$, what we have to show is that $di_V\omega = 0$. We have

$$d <dO\{Q,O\}> = <(b_{-1}dO)dO\{Q,O\}> -$$

$$- <dO\{dQ,O\}> - <dO\{Q,dO\}>.$$  

(1.13)

If we use the identity (1.11) for the first term in (1.13) and the definition (1.9) we get:

$$< (b_{-1}dO)dO\{Q,O\}> - <dOb_{-1}dO\{Q,O\}> +$$

$$+ <(dO)^2[L_0,O]> - <(dO)^2[Q,V]> = 0.$$  

(1.14)

We are considering a deformation of the critical string, so we can drop all terms of the type $<\{Q,...\}>_0$, where subscript 0 means the expectation value in the unperturbed theory of some number of operators, using the argument of the contour deformation in the definition for $Q$, and thus in the last term of (1.14) we can integrate by parts in the path.

\footnote{One should note that the action defined by (1.10) differs from Zamolodchikov’s $c$-function $\tilde{c}$, but like a $c$-function, it has to have a local minimum at points where world-sheet theory is conformal invariant. Probably (1.10) could be considered as a boundary problem version of $c$-function.}
integral to obtain $+ <\{ Q, (dO)^2 \}>$; the same is true for the last term in (1.13), which leads to $+\frac{1}{2} <\{ Q, (dO)^2 \}>$. The first two terms in (1.14) are equal and contribute as $2 <(b_{-1}dO)dO\{ Q, O \}>$. Combining all the terms we get after some cancellations:

$$d <dO\{ Q, O \}> = - <dO\{ dQ, O \}> + \frac{1}{2} <(dO)^2[L_0, O] > = 0 \quad (1.15)$$

and we see that $\omega$, defined by (1.6) is BRST invariant only if the right hand side in the equation (1.15) is identically zero.

It was concluded in [1] that (1.6) is BRST invariant because of the following two assumptions:

$$\frac{\partial Q}{\partial t^i} = 0 \quad (1.16)$$

$$[L_0, O] = \int \frac{\partial}{\partial \theta} O(\theta) d\theta = 0 \quad (1.17)$$

here, in (1.17), the first identity means that $L_0$ is a generator of the rotation of circle, and the second identity assumes that total derivatives decouple inside the correlation functions.\[5\]

Comment: the second correlator in (1.15), the one with total derivative inside the integral, generically is not zero and might receive the contribution from the boundary of moduli space (position of operators on the circle are moduli). Thus, we have to treat such terms and include their contribution. Or, if we want to set up such a scheme during the evaluation of correlation functions in (1.10), when (1.16) is satisfied once inside the correlator, we have to make sure that our regularization scheme leads to decoupling of total derivatives in the second term in (1.15). The latter is a nontrivial statement and in the next section we are going to address this question in detail. The identity in (1.15) should be considered as the requirement for operator $Q$; so, $Q$, when the contour approaches the boundary $\partial \Sigma$ should depend on the couplings according to equation (1.15) and this leads to consistency condition on the construction. From the point of view of conformal perturbation theory the above requirement means that we have to use the parallel transport of $Q$, consistent with (1.15) when we move away from the critical point $t_{CFT}$. It happens that only the $PSL(2, R)$ subalgebra of Virasoro algebra is relevant, thus what one needs is to deform this subalgebra by including the contributions of boundary term. In the

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5 I would like to thank K. K. Li and Erik Verlinde for discussions on importance of (1.16) in [1] (see also [3] and [4]) and E. Witten, who insisted that the whole perturbation theory should be used for proper definition of BRST commutator in (1.10).
next section we will evaluate the right hand side in (1.10) and formulate this consistency condition in more clear terms for the case when the boundary interaction doesn’t mixes ghosts and matter.

At the end of this section as an illustration I would like to discuss a known example of perturbation of a conformal field theory (closed string) by dimension one operator, where the decoupling of total derivatives doesn’t takes place and the obstruction is a $\beta$-function \[10\]. Similar calculations will be performed in the next section for open strings.

Consider some CFT perturbed by a dimension one operator $V_i t^i$. \[6\] We will denote the correlation functions in the perturbed theory by $<< ... >>$ and those in the unperturbed theory by $< ... >$; so, the partition function for can be written as $<< 1 >>$, or $< exp(i \int V) >$. One can calculate the trace of stress-tensor in the perturbed theory in the following way. We start from the expectation value of the holomorphic part of stress tensor $<< T(z, \bar{z}) >>$ and use the operator expansion algebra

$$
T(z, \bar{z})V_i(w) = \frac{1}{(z-w)^2} + \frac{1}{z-w} \frac{\partial}{\partial w} V(\omega) + ... = \frac{\partial}{\partial w} \left( \frac{1}{z-w} V + ... \right)
$$

(1.18)

(here we used the fact that $V$ has dimension one in the CFT that we are perturbing; we are speaking about closed string) to deduce that expectation value of the stress tensor is given by total derivatives, integrated over the points where the operator is inserted:

$$
<< T(z, \bar{z}) >> = \sum_i < \int d^2w_i \frac{\partial}{\partial w_i} \left( \frac{1}{z-w_i} V(w_i) + ... \right) \sum_n \frac{1}{n(n-1)!} (\int V)^{n-1} >.
$$

(1.19)

If we claim that the contribution of the total derivative in the right hand side is zero, we will conclude that the expectation values of stress tensor is zero; but the latter is wrong – we know that the following relation holds:

$$
\frac{\partial}{\partial \bar{z}} << T(z, \bar{z}) >> = \frac{\partial}{\partial z} << trT(z, \bar{z}) >> = \beta^i \frac{\partial}{\partial z} << V_i(z, \bar{z}) >>
$$

(1.20)

Thus the obstruction for decoupling is the $\beta$-function, so we have to calculate the contribution of contact terms. We have to use the operator expansion algebra for $V_i$ to proceed further:

\[6\] I would like to thank A. Polyakov for pointing out to me the following example.
\[ V_i(w_1, \bar{w}_1)V_j(w_2, \bar{w}_2) = \frac{g_{ij}}{|w_1 - w_2|^4} + \frac{C_{ij}^k}{|w_1 - w_2|^2}V_k(w_2) + \ldots \] (1.21)

Now we see that because of the poles in (1.19) and (1.21), there is nonzero boundary contribution in the integral over \( w_i \) in (1.19). For this we substitute (1.21) in (1.19). After integration over \( w_i \) in (1.19) we are left with the boundary contour integral, with small contour surrounding each point \( w_j \); denoting \( w_i = w_j + \rho e^{i\theta} \) with small \( \rho \), we get:

\[
<< T(z, \bar{z}) >> = \sum_{ij} t^i t^j \int d^2 w_j \int d\theta \rho e^{-i\theta} \frac{1}{z - w_j - \rho e^{i\theta}} \left( \frac{g_{ij}}{\rho^4} + \frac{C_{ij}^k}{\rho^2}V_k(w_j) + \ldots \right) \sum_n \frac{1}{(n-2)!} (\int V)^{n-2} >.
\] (1.22)

If we expand the denominator in (1.22) in the powers of \( \rho \) and integrate over \( \theta \), we see that only second term contributes, with final answer:

\[
<< T(z, \bar{z}) >> = C_{ij} t^i t^j \int d^2 w \frac{1}{(z - w)^2} << V_k(w) >> + \ldots
\] (1.23)

Thus, we obtain the desired formula (1.20) in the second order for \( \beta \)-function.

2. Modifications and Perturbative Calculation

In this section we consider the situation when ghosts and matter are decoupled. This means that the boundary interaction \( V \) is a functional purely of the matter coordinates \( X \) and the operators \( O \) are just \( O = cV \). The following calculations were largely already presented in [3], but it was assumed that (1.16) and (1.17) are valid assumptions. As a result the calculations have captured only the linear part of \( \beta \) in (1) and the conclusion was that the theory has a linear equation of motion. Below I will modify the construction by using the parallel transport of \( Q \) for to satisfy the consistency condition (1.15). This procedure is generally ambiguous and I will suggest the possible criteria. I would like to note that there has to be a relation of this procedure with the issues discussed in [11].

On the world-sheet we are dealing with interacting field theory and thus only the way to perform the calculations is to use the perturbation theory. This means that in any calculation we will loose background independence (formally), but the goal is to get final expressions in the invariant terms that don’t appeal to a particular background. Thus, we will consider the perturbation theory near a fixed point \( t_0 \), which corresponds to some
conformal field theory with stress tensor $T$ and corresponding BRST operator $Q$. When
the contour $C$ approaches the boundary of the disc the operator

$$Q = \int d\theta c(\theta)[T_m + \frac{1}{2}T_{gh}]. \quad (2.1)$$

is ambiguous, or it is better to say, it needs to be defined. We have to understand what
the correct $Q$ is and make sure that the consistency condition is satisfied.

We could write the left hand side of equation (1.10) in the following form:

$$dS = \frac{1}{2} \int_0^{2\pi} d\theta_1 d\theta_2 \langle c(\theta_1) dV(\theta_1) c(\theta_2) \rangle + \frac{1}{2} \int_0^{2\pi} d\theta_1 d\theta_2 d\theta_3 \langle c(\theta_1) dV(\theta_1) c(\theta_2) c(\theta_3) [T_m(\theta_2), V(\theta_3)] \rangle. \quad (2.2)$$

The ghost correlation functions in (2.2) is easy to evaluate and it is given by a standard
formula, because we consider the case when ghosts and matter are decoupled. For the
general 3-point function we have:

$$\langle c(\theta_1) c(\theta_2) c(\theta_3) \rangle = 2(\sin(\theta_1 - \theta_2) - \sin(\theta_1 - \theta_3) + \sin(\theta_2 - \theta_3)). \quad (2.3)$$

Let us treat (2.2) term by term. The expression for the first term is simple:

$$\frac{1}{2} \int_0^{2\pi} d\theta_1 d\theta_2 (2 \cos(\theta_1 - \theta_2) - 2) \langle c(\theta_1) dV(\theta_1) V(\theta_2) \rangle = \int d\theta_1 d\theta_2 \cos(\theta_1 - \theta_2) \langle c(\theta_1) dV(\theta_1) V(\theta_2) \rangle - d \langle c \rangle + d \langle c \rangle. \quad (2.4)$$

If we use the notation $L_n = \int_0^{2\pi} d\theta e^{in\theta} T_m(\theta)$, the second term in (2.2) can be written as
a combination of three expressions:

$$\frac{1}{2} \left[ i(- \langle [L_{-1}, d\theta_2 V(\theta_2)] \int d\theta_1 e^{i\theta_1} dV(\theta_1) \rangle - c.c.) + 
+ i(< [L_{-1}, \int d\theta_2 e^{i\theta_2} V(\theta_2)] \int d\theta_1 dV(\theta_1) \rangle - c.c) - 
- 2 \int d\theta_1 d\theta_2 \sin(\theta_1 - \theta_2) \langle c \rangle [L_0, V(\theta_2)] \rangle. \quad (2.5)$$

One can simplify (2.3) first noting that:
\[ \langle \int d\theta_1 e^{i\theta_1} dV(\theta_1)[L_{-1}, \int d\theta_2 V(\theta_2)] \rangle = \langle \int d\theta e^{i\theta} dV(\theta)[L_{-1}, \exp(iL^{int})] \rangle, \quad (2.6) \]

\[ \langle \int d\theta_1 dV(\theta_1)[L_{-1}, \int d\theta_2 e^{i\theta_2} V(\theta_2)] \rangle = \langle d(\exp(iL^{int}))[L_{-1}, \int d\theta e^{i\theta} V(\theta)] \rangle. \quad (2.7) \]

We need two more identities:

\[ \langle d(\exp(iL^{int}))[L_{-1}, \int d\theta e^{i\theta} V(\theta)] \rangle = d \langle [L_{-1}, \int d\theta e^{i\theta} V(\theta)] \rangle - \langle [L_{-1}, \int d\theta e^{i\theta} V(\theta)] \rangle - \langle [dL_{-1}, \int d\theta e^{i\theta} V(\theta)] \rangle, \quad (2.8) \]

and finally for the second term in (2.8)

\[ \langle [L_{-1}, \int d\theta e^{i\theta} dV(\theta)] \rangle = - \langle [L_{-1}, \exp(iL^{int})] \int d\theta e^{i\theta} dV(\theta) \rangle + \langle [L_{-1}, \exp(iL^{int})] \int d\theta e^{i\theta} dV(\theta) \rangle. \quad (2.9) \]

Here we used the notation where \( L^{int} \) is the boundary interaction term, and \( dL_{-1} = dt^i \frac{\partial}{\partial \theta^i} L_{-1} \), assuming that \( L_{-1} \) might depend on the couplings.

Now we see that the difference between (2.6) and (2.7), which enters in (2.5) is given by:

\[ \langle \int d\theta_1 e^{i\theta_1} dV(\theta_1)[L_{-1}, \int d\theta_2 V(\theta_2)] \rangle - \langle \int d\theta_1 dV(\theta_1)[L_{-1}, \int d\theta_2 e^{i\theta_2} V(\theta_2)] \rangle = \langle d(\exp(iL^{int}))[L_{-1}, \int d\theta e^{i\theta} V(\theta)] \rangle - \langle [dL_{-1}, \int d\theta e^{i\theta} V(\theta)] \rangle. \quad (2.10) \]

The last term in (2.10) could be dropped because it is an expectation value of operator \([L_{-1}, ...]\) in a critical theory, thus we can integrate by parts in path integral and this term is identically zero. So, if we combine (2.2), (2.4) and (2.10) we get:
\[ dS = d^2\theta (\llbracket [L_{-1}, \int d\theta e^{i\theta} V(\theta)] \rrbracket \llbracket c.c. \rrbracket + dZ - d <\int d\theta V(\theta)\rrbracket + X, \] 

where we denote by one form \( X \) the following expression:

\[ X = \int d\theta_1 d\theta_2 \frac{\partial}{\partial \theta_2} \sin(\theta_1 - \theta_2) \llbracket dV(\theta_1) V(\theta_2) \rrbracket \llbracket c.c. \rrbracket - \int d\theta_1 d\theta_2 \sin(\theta_1 - \theta_2) \llbracket dV(\theta_1) [L_0 - \frac{\partial}{\partial \theta_2} V(\theta_2)] \rrbracket \llbracket c.c. \rrbracket - i \frac{1}{2} (\llbracket [dL_{-1}, \int d\theta e^{i\theta} V(\theta)] \rrbracket \llbracket c.c. \rrbracket). \] 

(2.12)

Now I would like to require that under proper definition of renormalization scheme and the generators of \( SL(2, R) \) subalgebra, the object \( X \) is identically zero. One should note that the consistency condition (1.15) requires that \( X \) is just an exact form, thus there is an ambiguity in the definition of \( Q \) and equivalently the symmetries of (1.6). Moreover it is not guaranteed that in the deformed theory the stress tensor, that enters in the definition of \( Q \), is deformed accordingly to this requirement. Below I will give arguments that it is indeed the case and that the vanishing of \( X \) is a natural choice. They couldn’t be considered as a rigorous proof to all orders in \( t \); they are just arguments and most likely they can lead to such a proof. Before I turn to this very important question let us evaluate the first term in (2.12), to be sure that the total derivative doesn’t make it zero. We have in the lowest order in \( t \):

\[ \int d\theta_1 d\theta_2 \frac{\partial}{\partial \theta_2} \sin(\theta_1 - \theta_2) \llbracket dV(\theta_1) V(\theta_2) \rrbracket \llbracket c.c. \rrbracket = 2dt^i t^j C_{ij}^k \int d\theta_1 \frac{\sin a}{\sin \frac{1}{2} a} \frac{\Delta_i + \Delta_j - \Delta_k}{\Delta_i + \Delta_j - \Delta_k} \llbracket dV(\theta_1) \rrbracket \llbracket c.c. \rrbracket + \ldots = \] 

(2.13)

with \( C_{ij}^k(a) = c_{ij}^k(a/2)^{\Delta_k - \Delta_i - \Delta_j}. \) Here we had used the operator expansion algebra for \( V_i \),

\[ V_i(\theta_1)V_j(\theta_2) = \frac{c_{ij}^k}{\sin \frac{1}{2} (\theta_1 - \theta_2) \Delta_i + \Delta_j - \Delta_k} V_k(\theta_1) + \ldots, \] 

(2.14)

with \( \Delta_i + \Delta_j - \Delta_k = 1 \),

(2.15)
the "resonance term", is convergent in the limit when we remove the cutoff, and others
diverge. These divergent terms can be removed by redefinition of couplings or the same,
by a subtraction procedure (see below), while the constant term is universal and can’t be
removed. Also, there should be a higher order correction in couplings in (2.13).

We see that, like in the example for closed string at the end of the previous section,
total derivative in $X$ doesn’t decouple and is proportional to $\beta$-function coefficient $C^k_{ij}$;
thus (1.17) fails, so does (1.16).

Until now we had avoided the question about the transformation properties of bound-
ary operator. To proceed further it is necessary to know the action of $L_{-1}, L_0$ and $L_1$ on
$V$’s. What we need to cancel the contribution of (2.13) in $X$ is:

$$\frac{\partial}{\partial t^i} \delta_{\epsilon} V_j = 4 C^k_{ij} V_k \partial \epsilon + ...$$

with $\epsilon = e^{-i\theta}, 1, e^{i\theta}$.

This leads to a suggestion (that has to be verified) that the proper deformation of the
action of $SL(2, R)$ algebra is given by:

$$\delta_{\epsilon} V_i = \epsilon \partial V_i + \gamma^j_i(t) \partial \epsilon V_j,$$  (2.17)

and $\gamma$ is the matrix of anomalous dimensions, which for operators $V$ are simply given by

$$\gamma^j_i(t) = \delta^j_i - \frac{\partial \beta^j_i}{\partial t^i} = \delta^j_i \Delta_i + 4 c^j_i t^k + ...,$$  (2.18)

and $\beta^j_i$ is the $\beta$-function for operator $V_j$. This deformation is a very natural one.

In fact what we need is (2.17) in the first nontrivial order in couplings to compare with (2.13),
because the latter we are able to calculate only up to this order. Here are the arguments
in support of (2.17): consider the automorphisms of disc:

$$z' = \alpha \frac{z - z_0}{1 - z_0 z},$$  (2.19)

or infinitesimally

$$z' = \alpha (z - z_0 + z_0 z^2)$$  (2.20)

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7 There are similarities between (2.17) and the boundary problem analog of anomaly equation
(1.20), recently derived by Zamolodchikov [12], see also [13]. I would like to thank A. Zamolod-
chikov for sharing his insight on the problem of boundary deformations with me.
with $|z_0| < 1$ and $|\alpha| = 1$. The requirement, that corresponding $PSL(2,R)$ subalgebra of Virasoro algebra is not broken in perturbed theory, fixes the transformation law (2.17). In the case of closed string, the transformation properties under constant shift is standard, and under dilatation is controlled by the Callan-Symanzik equation, that leads to change of classical dimension to anomalous dimension [8]; so these two elements are universal, and the third one is fixed by the requirement, mentioned above. The open string version is given by (2.17) and leads to (2.16). We see from this expression that they are compatible in the lowest order in couplings and guarantee that $X$ vanishes. What is needed to complete the proof is that one has to show the compatibility of (2.17), in particular regularization scheme (note that higher order terms in $\beta$-function are scheme dependent), with the vanishing of $X$ in (2.12) and (2.13) to all orders.

Thus, the first term in equation (2.11) defines the string field theory action

$$S = \frac{i}{2} \langle \langle L_{-1}, \int d\theta e^{i\theta} V(\theta) \rangle \rangle - \text{c.c.} - \langle \langle \int d\theta V(\theta) \rangle \rangle + \langle \langle 1 \rangle \rangle; \quad (2.21)$$

and we know the transformation of boundary operator with respect to $SL(2,R)$ from $X = 0$ or (2.17). We had derived this expression in the second order for conformal perturbation theory. At present it is difficult to make any rigorous statement to all orders, but I think that (2.21) should be correct up to all orders. My believe is based on important check which is provided by the expression (2.9); if one calculates the left and right hand sides of this identity using (2.17), or (2.16), he will find out that these transformation laws are consistent with (2.9). For direct proof we need to calculate $X$ in all orders and there are two problems involved: first, it is difficult to perform the calculation beyond second order; second, we have to calculate the parallel transport of Virasoro generators and derive (2.17) in the perturbation theory (or its modification in higher orders) and make sure that $X$ vanishes to all orders. In fact, there are ambiguities related to both calculations, caused by renormalization scheme dependence beyond second order.

Our final expression (2.21) can be written in the form announced in the introduction:

$$S = -\beta^i \frac{\partial}{\partial t^i} Z(t) + Z(t). \quad (2.22)$$

If we remember that structure constants in (2.13) were cutoff dependent and this dependence we removed by a subtraction, we might have kept it up to (2.22). The reason is...
that as it follows from Poincare-Dulac Theorem about vector fields, every vector field can be linearized by appropriate redefinition of coordinates up to the resonant terms, and the resonant condition is related to the zero modes of linear part $\alpha_1, \ldots, \alpha_n$. The $N$-th order term can not be removed by this redefinition if and only if there exists the integer relation of the form:

$$\alpha_s = \sum_{i=1}^{N} m_i \alpha_i$$

(2.23)

with $(m_1, \ldots)$ integers and $m_k \geq 0, \sum m_k \geq 2$ and (2.23) is called the resonance relation. The linear term for $\beta^i$ is given by $(1 - \Delta_i) t^i$, thus the resonance condition in the second order is the one we had written before (2.15): 

$$1 - \Delta_k = 1 - \Delta_i + 1 - \Delta_j.$$ 

They correspond to finite terms in (2.13) and can’t be removed by coordinate transformation. This in fact proves that the non-vanishing of total derivative term in $S$ is universal.

The expression (2.22) shows that for exactly marginal deformations of base point (which is a particular CFT) action $S$ is the same as partition function and this confirms the statement of [3]. Obviously, assuming that $\beta = 0$ ($\Delta_i = 1$ and $c^k_{ij} = 0$ in the perturbation theory) from the beginning and going through our calculations again we get $dS = 0$. Because any attempt to calculate string field theory action in the present approach should use the perturbation theory we can’t make any statement about global properties of action. Also, it is difficult to check above statement about equation of motion directly from final expression (2.22) without going to world-sheet and using the identities described in this Section; it would be very interesting to find such procedure. Last important comment related to these questions: we couldn’t compare two actions (note that dependance on $t$’s enter through $\beta$) if they are calculated in perturbation theory around different points in space of $t$’s. For this we will need the natural parallel transport in the space of theories that we unfortunately don’t have; we only know the deformed relations for $SL(2, R)$ generators in perturbation theory. Thus, the result is not truly background independent even it looks so formally.

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8 The relevance of Poincare-Dulac theorem to $\beta$-function related issues was stressed many times by Zamolodchikov, see [14].
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