Equivariant Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

Adrian Diaconu

Abstract

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of $n$-pointed stable genus $g$ curves, and let $\mathcal{M}_{g,n}$ be the moduli space of $n$-pointed smooth curves of genus $g$. In this paper, we obtain an asymptotic expansion for the characteristic of the free modular operad $\mathcal{M}_V$ generated by a stable $S$-module $V$, allowing to effectively compute $S_n$-equivariant Euler characteristics of $\overline{\mathcal{M}}_{g,n}$ in terms of $S_{n'}$-equivariant Euler characteristics of $\overline{\mathcal{M}}_{g',n'}$ with $0 \leq g' \leq g$, $\max\{0, 3 - 2g'\} \leq n' \leq 2(g-g')+n$. This answers a question posed by Getzler and Kapranov by making their integral representation of the characteristic of the modular operad $\mathcal{M}_V$ effective. To illustrate how the asymptotic expansion is used, we give formulas expressing the generating series of the $S_n$-equivariant Euler characteristics of $\overline{\mathcal{M}}_{g,n}$, for $g = 0, 1$ and $2$, in terms of the corresponding generating series associated with $\mathcal{M}_{g,n}$.

Contents

1 Introduction ........................................ 2
2 Notation and preliminaries ......................... 3
3 A semi-classical expansion ......................... 6
  3.1 The critical points .............................. 7
  3.2 The asymptotic expansion ...................... 9
4 Equivariant Euler characteristics ............... 11

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail: cad@umn.edu
1 Introduction

Let \( \mathcal{V} \) be a stable \( S \)-module, i.e., a collection of chain complexes \( \{ \mathcal{V}(g, n) \}_{g,n \ge 0} \) with an action of the symmetric group \( S_n \) on \( \mathcal{V}(g, n) \), and such that \( \mathcal{V}(g, n) = 0 \) when \( 2g + n - 2 \le 0 \), and let

\[
\text{Ch}(\mathcal{V}) = \sum_{2(g-1)+n > 0} h^{g-1} \text{ch}(\mathcal{V}(g, n))
\]

denote the characteristic of \( \mathcal{V} \), where \( \text{ch}(\mathcal{V}(g, n)) \) is the characteristic of the \( S_n \)-representation \( \mathcal{V}(g, n) \). To the stable \( S \)-module \( \mathcal{V} \), there is the associated free modular operad \( \mathcal{M}\mathcal{V} \) generated by \( \mathcal{V} \); as such, we can also take the characteristic of \( \mathcal{M}\mathcal{V} \). Throughout, we shall consider only \( S \)-modules in the category of \( \ell \)-adic Galois representations.

The aim of this paper is to make quite effective a beautiful result of Getzler and Kapranov [15, Theorem 8.13], expressing the relationship between the characteristics of \( \mathcal{V} \) and \( \mathcal{M}\mathcal{V} \) in the form

\[
\text{Ch}(\mathcal{M}\mathcal{V}) = \log(\exp(\Delta)\exp(\text{Ch}(\mathcal{V})))
\]

(1)

where \( \Delta \) is a certain analogue of the Laplacian, \( \exp(f) \) is the plethystic exponential of \( f \), and \( \log(f) \) is the inverse of the plethystic exponential; see Section 2 for precise definitions. The above equality is a consequence of how the functor \( \mathcal{M} \) on the category of stable \( S \)-modules is tailored to the way the boundary strata of \( \mathcal{M}_{g,n} \) are obtained by gluing together moduli spaces \( \mathcal{M}_{g',n'} \).

The formula (1) is a natural generalization of Wick’s theorem [5], which gives the integral formula

\[
\sum_{2(g-1)+n > 0} \mathcal{M}_{v g,n} h^{g-1} \frac{x^n}{n!} = \log \int_{\mathbb{R}} \exp \left( \sum_{2(g-1)+n > 0} v_{g,n} h^{g-1} \frac{x^n}{n!} - \frac{(x - \xi)^2}{2h} \right) \frac{dx}{\sqrt{2\pi h}}
\]

(2)

Here \( \{ v_{g,n} : 2(g-1) + n > 0 \} \) is a set of variables, and

\[
\mathcal{M}_{v g,n} := \sum_{G \in \text{Ob} \Gamma_{g,n}} \frac{1}{|\text{Aut}(G)|} \prod_{v \in \text{Vert}(G)} v_{g(v),n(v)} \quad (\text{for } 2(g-1) + n > 0)
\]

where \( \Gamma_{g,n} \) is the finite category whose objects are isomorphism classes of stable graphs of genus \( g \) with \( n \) ordered legs, and whose morphisms are the automorphisms; see [15] for details. Thus if we define

\[
b_g = b_g(\xi) = \sum_{n \ge \max\{0, 3-2g\}} \mathcal{M}_{v_{g,n}} \frac{\xi^n}{n!} \quad \text{and} \quad a_g = a_g(x) = \sum_{n \ge \max\{0, 3-2g\}} v_{g,n} \frac{x^n}{n!}
\]

then, by performing an asymptotic expansion of the integral in the right-hand side of (2), one can obtain formulas expressing the coefficients \( b_g \) in terms of \( a_g \) with \( g' \le g \).

Our main result, Theorem 3.2, provides an asymptotic expansion of the right-hand side of (1), thus answering a question posed by Getzler and Kapranov in [15], p. 113. Letting \( a_g \) and \( b_g \) denote, as above, the coefficients of \( h^{g-1} \) in \( \text{Ch}(\mathcal{V}) \) and \( \text{Ch}(\mathcal{M}\mathcal{V}) \), respectively, then as consequences of Theorem 3.2, we shall obtain formulas for \( b_g \), \( b_1 \) and \( b_2 \), in terms of \( a_g \), \( a_1 \) and \( a_2 \), see Section 4. The formulas for \( b_0 \) and \( b_1 \) (Theorem 4.1 and Theorem 4.3, respectively) are not new, see [15, Theorem 7.17] or [13, Theorem 5.9] for the calculation of \( b_0 \), and [14] or [24] for that of \( b_1 \). The proofs of these two results were merely included as examples of how the coefficients \( b_g \) are calculated.

The argument used to prove Theorem 3.2 represents the natural generalization of the method used by Bini and Harer [6, Section 3] to study the asymptotic expansion of the integral in (2). It is also possible to give an analogous interpretation of the asymptotic expansion discussed in this paper to that in [6, Proposition 3.6], as an expansion over stable graphs, to obtain explicit formulas for the coefficients \( b_g \) when \( g \ge 3 \). The general formula for \( b_g \) can then be combined with a result of Gorsky [16], where the author establishes a formula for the generating series of the numerical \( S_n \)-equivariant Euler characteristics of \( \mathcal{M}_{g,n} \), to obtain the corresponding numerical \( S_n \)-equivariant characteristics of \( \mathcal{M}_{g,n} \).
Although we shall work throughout just in the tensor symmetric abelian category $\text{Rep}_{\mathbb{Q}}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ of $\ell$-adic Galois representations, the results in this paper are valid for stable $\mathbb{S}$-modules in any symmetric monoidal category with finite colimits and additive over a field of characteristic zero.

Acknowledgements. I am grateful to Jonas Bergström and Dan Petersen for their comments and suggestions. I would also like to thank them for making available to me the formulas for the $S_n$-equivariant Euler characteristics of $\overline{M}_{2,n}$ for small values of $n$.

2 Notation and preliminaries

Symmetric functions. Let $\mathbb{S}_k$ denote the symmetric group on $k$ letters. The completed ring of symmetric functions in infinitely many variables (see [15]) is defined by

$$\Lambda = \lim_{\rightarrow} \mathbb{Z}[x_1, \ldots, x_k]^{\mathbb{S}_k}.$$

We have the standard functions

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \quad h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} \quad \text{and} \quad p_n = \sum_{i \geq 1} x_i^n$$
called the elementary symmetric functions, the complete symmetric functions and the power sums, respectively. The ring $\Lambda$ is also the completion of the graded polynomial ring $\mathbb{Z}[e_1, e_2, \ldots]$, and thus, by the well-known identities among the standard symmetric functions, we have

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots] \quad \text{and} \quad \Lambda \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots].$$

In addition, for $\lambda = (\lambda_1, \geq \cdots, \geq \lambda_n > 0)$ we have the Schur functions

$$s_\lambda = \det(h_{\lambda_i - j})_{i,j = 1}^{n}$$

which also generate $\Lambda$.

Plethysm. One defines (see [22] and [15]) the associative operation "$\circ$" on $\Lambda$, called plethysm (or composition), characterized by the following conditions:

1. $$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$$
2. $$(f_1 f_2) \circ g = (f_1 \circ g)(f_2 \circ g)$$
3. If $f = f(p_1, p_2, \ldots)$, we have $p_n \circ f = f(p_n, p_{2n}, \ldots)$.

In other words, $\Lambda$ is a $\lambda$-ring, see [20] and [11] for generalities on $\lambda$-rings; it is the complete filtered $\lambda$-ring obtained by completing the free $\lambda$-ring on one generator $\mathbb{Z}[e_1, e_2, \ldots]$. In this language $p_n \circ f = \psi_n(f)$, where $\psi_n$ denote the Adams operations.

Consider the ring $\Lambda\langle h \rangle$ of Laurent series with the topology induced by the descending filtration

$$F^n \Lambda\langle h \rangle = \left\{ \sum_{i} f_i h^i : f_i \in F^{n-2i} \Lambda \right\}.$$

The plethysm extends to $\Lambda \times \Lambda\langle h \rangle$ by keeping conditions 1., 2. above, and replacing the last condition by:

$$p_n \circ f(h, p_1, p_2, \ldots) = f(h^n, p_n, p_{2n}, \ldots)$$

1In fact, we shall only deal with virtual $\ell$-adic Galois representations (Euler characteristics); thus it will suffice to work instead with the characters of the representations at Frobenius elements.
For $f \in F^1\Lambda(h)$, put $\Psi(f) := \sum_{n \geq 1} \frac{1}{n} \psi_n(f)$, where $\psi_n(f) = p_n \circ f$, and let $\exp(f) = \exp(\Psi(f))$; we have
\[
\exp(f + g) = \exp(f) \exp(g).
\]
The mapping $\Psi : F^1\Lambda(h) \to F^1\Lambda(h)$ has an inverse $\Psi^{-1} : F^1\Lambda(h) \to F^1\Lambda(h)$ given by
\[
\Psi^{-1}(f) = \sum_{n \geq 1} \frac{\mu(n)}{n} \psi_n(f)
\]
where $\mu(n)$ is the usual Möbius function; see [23, Lemma 20]. Thus if we define $\log : 1 + F^1\Lambda(h) \to F^1\Lambda(h)$ by
\[
\log(g) = \Psi^{-1}(\log g),
\]
then $\exp$ and $\log$ are inverses of each other.

Finally, one defines ([15, Section 8]) an analogue of the Laplacian on $\Lambda(h)$ by
\[
\Delta = \sum_{n=1}^{\infty} h^n \left( \frac{n}{2} \frac{\partial^2}{\partial p_n^2} + \frac{\partial}{\partial p_n} \right).
\]
Note that $\Delta$ preserves the filtration of $\Lambda(h)$.

**The moduli spaces** $M_{g,n}$ and $\ol{M}_{g,n}$. For $g, n \in \mathbb{N}$ with $2(g-1)+n > 0$, let $\ol{M}_{g,n}$ denote the proper and smooth Deligne-Mumford stack of stable curves of arithmetic genus $g$ with $n$ ordered distinct smooth points. Let $M_{g,n} \subset \ol{M}_{g,n}$ be the open substack of irreducible and non-singular curves; both $M_{g,n}$ and $\ol{M}_{g,n}$ are defined over $\text{Spec}(\mathbb{Z})$, and the group $\Sigma_n$ acts on them by permuting the marked points on the curves. Moreover, we know that the boundary $\ol{M}_{g,n} \smallsetminus M_{g,n}$ is a normal crossings divisor.

The stack $\ol{M}_{g,n}$ admits a stratification determined by stable graphs of genus $g$ with $n$ ordered legs; see [1, Chap. XII.10].

To each stable graph $G$, there corresponds the smooth, locally closed stratum $M(G) \subset \ol{M}_{g,n}$ parametrizing curves with dual graph isomorphic to $G$. The stratum $M(G)$ is canonically isomorphic to the quotient stack
\[
\left[ \prod_{v \in \text{Vert}(G)} M_{g(v), n(v)} \right] / \text{Aut}(G).
\]
Here $\text{Vert}(G)$ denotes the set of vertices of $G$, $g(v)$ is the arithmetic genus of the component (of a stable curve) corresponding to $v$, and $n(v)$ is the valence of the vertex $v$. The automorphism group $\text{Aut}(G)$ of $G$ is the set of graph automorphisms preserving the genus function $g$ and the ordering of the legs.

For generalities on these moduli spaces, see [8], [19], [1] and [17].

**Euler characteristics.** Put $G_0 = \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})$, and for a prime $\ell$, denote by $\text{Rep}_{\mathbb{Q}_\ell}(G_0)$ the abelian category of $\ell$-adic Galois representations of $G_0$. Let $K_0(\text{Rep}_{\mathbb{Q}_\ell}(G_0))$ denote the Grothendieck ring of this category; it carries a natural $\lambda$-ring structure, cf. [27, exposé 5] or [12], the $\lambda$-operations enjoying the property:
\[
\lambda^m([V]) = [\lambda^m V] \text{ for } m \geq 0.
\]
We also recall that a semi-simple $\ell$-adic Galois representation $V$ is determined by the traces $\text{Tr}(\sigma_p | V)$ of Frobenius elements $\sigma_p$ on the primes $p$ at which $V$ is unramified (see, for instance, [7, Proposition 2.6]).

Let $M_{g,n/\mathbb{Q}}$ denote the $\mathbb{Q}$-stack corresponding to $M_{g,n}$, i.e., the generic fiber $M_{g,n/\mathbb{Q}}$ of $M_{g,n} \to \text{Spec}(\mathbb{Z})$ base changed from $\mathbb{Q}$ to $\mathbb{Q}$. The action of the symmetric group $\Sigma_n$ on $M_{g,n}$ induces an isotypic decomposition of the $\ell$-adic cohomology $H^\prime_c(M_{g,n/\mathbb{Q}}, \mathbb{Q}_\ell)$ as an $\Sigma_n$-module,
\[
H^\prime_c(M_{g,n/\mathbb{Q}}, \mathbb{Q}_\ell) \cong \bigoplus_{\lambda \vdash n} H^\prime_{c,\lambda}(M_{g,n/\mathbb{Q}}, \mathbb{Q}_\ell)
\]
where for an irreducible representation $V_{\lambda}$ of $\Sigma_n$ indexed by the partition $\lambda$ of $n$,
\[
H^\prime_{c,\lambda}(M_{g,n/\mathbb{Q}}, \mathbb{Q}_\ell) = V_{\lambda} \otimes \text{Hom}_{\Sigma_n}(V_{\lambda}, H^\prime_c(M_{g,n/\mathbb{Q}}, \mathbb{Q}_\ell)).
\]
For a partition $\lambda$ of $n$, put
\[
e_{c,\lambda}(M_{g,n/\mathbb{Q}}) = \sum_{\lambda} (-1)^{\ell} [H^\prime_{c,\lambda}(M_{g,n/\mathbb{Q}}, \mathbb{Q}_\ell)] \in K_0(\text{Rep}_{\mathbb{Q}_\ell}(G_0)).
\]
in addition, we fix throughout a finite field $F$ of characteristic different from $\ell$ and an algebraic closure of it $\overline{F}$, and define similarly the Euler characteristic $e_{\ast, \lambda}(\mathcal{M}_{g, n, \sigma})$. Let $\mathcal{V}(g, n) = H^*_{\ast}(\mathcal{M}_{g, n, \sigma}, \mathbb{Q}_\ell)$, and set

$$\chi_n(\mathcal{V}(g, n)) := \sum_{\lambda \vdash n} \frac{1}{\dim V_\lambda} e_{\ast, \lambda}(\mathcal{M}_{g, n, \sigma}) s_\lambda$$

see also [4] where this characteristic is computed for small values of $g$ and $n$.

Letting $F : \mathcal{M}_{g, n, \sigma} \to \mathcal{M}_{g, n, \sigma}$ denote the Frobenius morphism, we note that by Grothendieck’s fixed point formula [2, 3], the trace of the geometric Frobenius $F^*$ on the characteristic $\chi_n$ of the graded $S_\ast$-module $H^*_{\ast}(\mathcal{M}_{g, n, \sigma}, \mathbb{Q}_\ell)$ is given by

$$t_{g, n}(p) = t_{g, n}(p_1, p_2, \ldots) := \sum_{\lambda \vdash n} \frac{1}{\dim V_\lambda} \text{Tr}(F^*| e_{\ast, \lambda}(\mathcal{M}_{g, n, \sigma})) s_\lambda = \frac{1}{n!} \sum_{\sigma \in S_n} |M^\sigma| p_{c(\sigma)}$$

where $c(\sigma)$ denotes the cycle type of $\sigma$, and if $c(\sigma) = (1^{p(1)}, \ldots, n^{p(n)})$ then $p_{c(\sigma)} = p^{p(1)}_1 \cdots p^{p(n)}_n$. Here we have also used Frobenius’ formula

$$s_\lambda = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) p_{c(\sigma)}$$

$\chi^\lambda$ being the character of $V_\lambda$. Since $|M^\sigma|$ depends only upon the cycle type of $\sigma$, we can also write

$$t_{g, n}(p) = \sum_{\rho \vdash n} |M^\rho| p_{c(\sigma)} z_k$$

where, for convenience, we set $|M^\rho| := |M^\rho|_{g, n, \sigma}$ for $\sigma \in S_n$ with $c(\sigma) = \rho$, and

$$z_k = \prod_{i=1}^n i^{p(i)} (i)! \quad \text{if } \rho = (1^{p(1)}, \ldots, n^{p(n)}).$$

Note that, for $k \geq 1$, we have

$$\psi_k(t_{g, n}(p)) = \sum_{\rho \vdash n} |M^\rho|_{g, n, \sigma} \frac{p_{c(\sigma)} z_k}{z_k}.$$ 

Now the characteristic of the stable $S$-module $\mathcal{V} = \{ \mathcal{V}(g, n) \}$ (that is, the representation $\mathcal{V}$ of the groupoid $S = \coprod_{n \geq 0} S_n$) is defined by the formal Laurent series

$$\text{Ch}(\mathcal{V}) = \sum_{2(g-1) + n > 0} h^{2(g-1) + n} \chi_n(\mathcal{V}(g, n))$$

with the coefficients

$$\sum_{n \geq \max\{0, 3-2g\}} \chi_n(\mathcal{V}(g, n)) \in K_0(\text{Rep}_{\mathbb{Q}_\ell}(G_0))[h_1, h_2, \ldots].$$

The corresponding generating series of $t_{g, n}(p)$ will be denoted by $T(\mathcal{V})$.

The free modular operad $\mathcal{M}\mathcal{V} = \{ \mathcal{M}\mathcal{V}(g, n) \}$ generated by $\mathcal{V}$ is obtained by taking $\mathcal{M}\mathcal{V}(g, n) = H^*_{\ast}(\mathcal{M}_{g, n, \sigma}, \mathbb{Q}_\ell)$; the characteristic $\text{Ch}(\mathcal{M}\mathcal{V})$ and the corresponding generating series $T(\mathcal{M}\mathcal{V})$ are defined as for $\mathcal{V}$.

The connection between the characteristics $\text{Ch}(\mathcal{V})$ and $\text{Ch}(\mathcal{M}\mathcal{V})$ is given by the following theorem of Getzler and Kapranov [15, Theorem 8.13]:

**Theorem 2.1.** — We have

$$\text{Ch}(\mathcal{M}\mathcal{V}) = \text{Log}(\exp(\Delta)\text{Exp}(\text{Ch}(\mathcal{V}))).$$

Here $\text{Exp}(-)$ and $\text{Log}(-)$ are defined as before.

**Integral representation.** Following Getzler and Kapranov, we shall now express $\text{Ch}(\mathcal{M}\mathcal{V})$ as a formal Fourier transform. The resulting formula is in complete analogy with the formula (2) in Wick’s theorem.

\[\text{The smooth Deligne-Mumford stack } \mathcal{M}_{g, n} \text{ is of finite type and has relative dimension } 3(g-1)+n \text{ over } \text{Spec}(\mathbb{Z}), \text{ thus fulfilling the conditions of [2, Theorem 3.1.2].}\]
For a partition \( \rho = (1^{\rho_1(0)}, 2^{\rho_2(0)}, \ldots) \), where \( \rho(j) = 0 \) for all but finitely many \( j \), put, as before, \( p_\rho = p_1^{\rho(1)} p_2^{\rho(2)} \ldots \). Let \( \Lambda_{\text{alg}} \) denote the space of finite linear combinations of the \( p_\rho \). On \( \text{Spec}(\Lambda_{\text{alg}} \otimes \mathbb{R}) \cong \mathbb{R}^\infty \) with coordinates \( p_1, p_2, \ldots \), let \( d\mu \) denote the formal Gaussian measure defined by

\[
d\mu = \prod_{n \text{ odd}} e^{-\frac{p^2}{2n^2h^n}} d\rho_n \prod_{n \text{ even}} e^{-\frac{p^2}{2n^2h^n} + p_{\rho_n}/nhn/2} d\rho_n = \text{Exp}(-e_{z/h}) \int_{\mathbb{R}^\infty} \prod_{n=1}^{\infty} \frac{dp_n}{e^{\epsilon_n/2n^2} \sqrt{2\pi nh^n}}
\]

where \( \epsilon_n = 0 \) or 1 according as \( n \) is odd or even. With this measure, for each monomial \( p_\rho = p_1^{\rho(1)} p_2^{\rho(2)} \ldots \), define the integral

\[
\int_{\mathbb{R}^\infty} p_\rho d\mu(p) = \prod_{n \text{ odd}} \int_{-\infty}^{\infty} p_{\rho(n)} e^{-\frac{p^2}{2n^2h^n}} dp_n \prod_{n \text{ even}} \int_{-\infty}^{\infty} p_{\rho(n)} e^{-\frac{p^2}{2n^2h^n} + p_{\rho(n)},/nhn/2} dp_n
\]

where the left-hand side is considered as a function of \( q = (q_1, q_2, \ldots) \) and \( h \).

With this definition, we have the following interpretation of the formula in Theorem 2.1, see [15, Theorem 8.18]:

**Theorem 2.2.** — We have

\[
h^{-1} h_z + \text{Ch}(M\mathbb{V}) = \log \left( \int_{\mathbb{R}^\infty} \text{Exp}(h^{-1} p, q, + \text{Ch}(\mathbb{V})) d\mu(p) \right)
\]

where the left-hand side is considered as a function of \( q = (q_1, q_2, \ldots) \) and \( h \).

For our purposes it will be more convenient to switch the roles of \( p = (p_1, p_2, \ldots) \) and \( q = (q_1, q_2, \ldots) \), and write (3) in the form

\[
\text{Exp} (\text{Ch}(M\mathbb{V})) = \int_{\mathbb{R}^\infty} e^{\mathcal{K}(p, q, h)} d'^q
\]

with \( \mathcal{K}(p, q, h) \) defined by

\[
\mathcal{K}(p, q, h) = -\sum_{m=1}^{\infty} \left( q_m - p_m - e_m^{h^m/2}/2m^{h^m} \right) \right)^2 + \Psi(\text{Ch}(\mathbb{V}))
\]

and the measure

\[
d'^q = m \prod_{n=1}^{\infty} \frac{dq_n}{\sqrt{2\pi nh^n}}
\]

Formula (4) follows easily by exponentiating (3) and applying the identities \( e_{z/h} = (q_1^2 - q_2)/2 \) and \( h_z = (p_1^2 + p_2)/2 \).

### 3 A semi-classical expansion

To express the coefficients of \( \text{Ch}(M\mathbb{V}) \) in terms of those of \( \text{Ch}(\mathbb{V}) \), both \( \text{Ch}(\mathbb{V}) \) and \( \text{Ch}(M\mathbb{V}) \) considered as formal Laurent series in \( h \), we shall study the integral (4) corresponding to the generating series \( T(\mathbb{V}) \) of \( \mathbb{A}_{g, n} \) over any finite field \( \mathbb{F} \) of characteristic different from \( \ell \). Here, we recall that the Euler characteristics \( e_{c, \lambda} \) are elements of the Grothendieck ring \( K_0(\text{Rep}_{\mathbb{F}}(G)) \).

In what follows, we shall denote by \( \text{ch}_1(\mathbb{V}) \) (resp. \( \text{ch}_2(M\mathbb{V}) \)) the coefficient of \( h^{g-1} \) in the series \( T(\mathbb{V}) \) (resp. \( T(M\mathbb{V}) \)), i.e.,

\[
\text{ch}_1(\mathbb{V})(q) = \sum_{n \geq \max(0, 3-2g)} \sum_{\rho \leq n} \left| M^{\mathbb{F}}_{g, n, \rho} \right| \frac{q_\rho}{z_\rho}
\]

and

\[
\text{ch}_2(M\mathbb{V})(p) = \sum_{n \geq \max(0, 3-2g)} \sum_{\rho \leq n} \left| M^{\mathbb{F}}_{g, n, \rho} \right| \frac{p_\rho}{z_\rho}
\]

To make use of Theorem 2.2 of Getzler and Kapranov in the form (4) to obtain formulas expressing the coefficients \( \text{ch}_2(M\mathbb{V}) \) in terms of the coefficients \( \text{ch}_1(\mathbb{V}) \) with \( g' \leq g \), we shall perform a semi-classical expansion of the corresponding integral giving \( T(h\mathbb{V}) \). The analogous expansion in the context of the usual orbifold Euler characteristics \( \chi(M_{g, n}) \) and \( \chi(M_{q, n}) \) is discussed by Bini and Harer in [6, 3.1]. Our arguments are a natural extension of theirs.
3.1 The critical points

To apply the principle of stationary phase to the integral giving $T(MV)$, we first need to determine the critical points of the function (still denoted by $K(p, q, h)$) given by

$$K(p, q, h) = -\sum_{m=1}^{\infty} \left\{ (q_m - p_m - \varepsilon_m h^{m/2})^2 / 2m h^m \right\} + \Psi(T(V)).$$

More precisely, for $m = 1, 2, \ldots$, we have to find the power series

$$q_m = \tilde{q}_m(p, h) = \sum_{s=0}^{\infty} \tilde{c}_{m,s}(p) h^s$$

satisfying the system of differential equations

$$\frac{\partial K}{\partial q_m}(p, \tilde{q}, h) = 0 \quad (\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \ldots))$$

for $m = 1, 2, \ldots$, or written explicitly,

$$\tilde{q}_m = p_m + \varepsilon_m h^{m/2} + \sum_{k=1}^{m} \frac{m}{k} \sum_{q=0}^{\infty} h^{kq + m-k} \frac{\partial \psi_k(ch_k(V))}{\partial q_m}(\tilde{q}).$$

(5)

Here $\psi_k(ch_k(V))$ is given by

$$\psi_k(ch_k(V))(\tilde{q}) = \sum_{n \geq \max(0, 3-2q)} \sum_{\rho \in \rho_F^{F'r}(q, n)} |\lambda^{\rho_F^{F'r}}_{q, n}| q_{\rho}. k_{\rho}.$$ 

For $m \geq 1$, define $c_m(q)$ by

$$c_m(q) = q_m - \frac{\partial \psi_m(ch_m(V))}{\partial q_m}(q).$$

Note that $c_m = \psi_m(c_1)$. If we put $C(q) = (c_m(q))_{m \geq 1}$ and $\bar{C}(p) = (\bar{c}_{m,0}(p))_{m \geq 1}$, then equating the constant terms (i.e., setting $h = 0$) in both sides of (5), we must have

$$C(C(p)) = p.$$

Remark. In [18, Lemma (2.8)], Kisin and Lehner obtained the formula

$$|\lambda_{\rho_F^{F'r}}^{\rho_F^{F'r}}| = \prod_{\phi \leq 0} \left( \frac{s_\phi(q^{\rho_F^{F'r}})}{q^{\rho_F^{F'r}} - 1} \right) \left( \text{with } \rho = (\rho_F^{F'r}, \ldots, \rho_F^{F'r}) \text{ and } |\rho_F^{F'r}| = q \right)$$

where $s_\phi(q) = q^{\phi} + 1$, and $s_i(q) = \sum_{d|i} \mu(d) q^d$ for $i \geq 2$. Accordingly, we can write

$$ch_i(V)(p) = \left[ \prod_{n \geq 1} (1 + p_n)^{s_i(q^{\rho_F^{F'r}})} \right] - 1 = \frac{p_1^2}{2(q-1)} - \frac{p_1}{q(q-1)} - \frac{p_2}{2(q+1)}.$$ 

see also [13]. Thus we can express

$$c_1(p) = p_1 - \frac{\partial ch_1(V)}{\partial p_1}(p) = \frac{q p_1}{q - 1} - \frac{\left[ \prod_{n \geq 1} (1 + p_n) \left( \frac{1}{2} \sum_{d|n} \mu(d) q^d \right) \right] - 1}{q(q-1)}.$$ 

(6)

Since $c_1(q)$ has no constant term, and the linear part of $C(q)$ is easily seen to be invertible, it follows that $C(q)$ admits, indeed, a compositional inverse. Note that

$$\psi_m(\bar{c}_{1,0})C(p) = p_m \circ \bar{c}_{1,0}(C(p)) = p_m$$

for all $m \geq 1$ and thus $\bar{c}_{m,0} = \psi_m(\bar{c}_{1,0})$, that is, $c_1$ and $\bar{c}_{1,0}$ are plethystic inverses.
Proposition 3.1. — The system (5) has a unique solution of the form
\[ \tilde{q}_m(p, \hbar) = \tilde{c}_{m,0}(p) + O(\hbar^{(m+1-\varepsilon_m)/2}) \]
for all \( m \geq 1 \), with \( \varepsilon_m = 0 \) or 1 according as \( m \) is odd or even. The coefficients \( \tilde{c}_{m,m/2} \) of this solution are given by the formula
\[ \tilde{c}_{m,m/2} = \frac{1 + 2\psi_m \left( \frac{\partial c_{m}(V)}{\partial p_2} \right)}{1 - \psi_m \left( \frac{\partial^2 c_{m}(V)}{\partial p_1^2} \right)} \tilde{c}_{1,0} \]
for all even \( m \geq 2 \).

Proof. Splitting the right-hand side of (5) according as \( kq + m - k \) (the exponent of \( \hbar \)) is zero (i.e., \( g = 0 \) and \( k = m \)) or not, we rewrite (5) as
\[ c_m(q) - p_m = \hbar^m/2 + \sum \left( \frac{m h^{k(g-1)+m}}{k} \right) \partial_p \left( \partial c_m(V) \right) (\tilde{q}) \]
the sum being over \( k \mid m \) and \( g \geq 0 \) such that \( k(g-1) + m > 0 \). Equating the coefficients of \( h^s \), for any \( s \geq 1 \), on both sides of (7), we get recursive relations of the form
\[ \sum_{r \geq 1} \tilde{c}_{m,s} \psi_m \left( \tilde{c}_{r,m} \right) (\tilde{c}(p)) = R_m(p) \quad (\text{for all} \ m \geq 1) \]
where \( R_m(p) \) is an expression involving only coefficients \( \tilde{c}_{t,s}(p) \) \((t \geq 1)\) with \( s' < s \). Notice that the right-hand side of (7) is \( O(h^{(m+1-\varepsilon_m)/2}) \), thus the coefficient of \( h^s \) in \( c_m(q) \) must be zero when \( m - \varepsilon_m \geq 2s \), that is, \( m \geq 2s + 1 \).

We proceed now by induction on \( s \). Having the coefficients \( \tilde{c}_{m,0} = \psi_m(\tilde{c}_{1,0}) \) already determined, assume that \( \tilde{c}_{m,s'}(p) \), for all \( s' < s \) and \( m \geq 1 \), were also determined, and that, for every \( s' < s \), we have \( \tilde{c}_{m,s'} = 0 \) for all \( m \geq 2s' + 1 \). Under these assumptions, we have \( R_m(p) = 0 \) if \( m \geq 2s + 1 \). Indeed, the right-hand side of (7) is certainly \( O(h^{s+1}) \), and the coefficients in \( c_m(q) \) getting into \( R_m(p) \) are of the form \( \tilde{c}_{m,s'} \) with \( s' < s \); these are all zero, since \( m > 2s' + 1 \). Let \( \tilde{c}_{m,s'} := 0 \) for \( m \geq 2s + 1 \), hence (8) holds trivially for these values of \( m \). From (6), it is easy to see that \( \psi_m(\tilde{c}_{m,0}) (\tilde{c}(p)) \) is non-zero for all \( m \geq 1 \), implying that the linear system
\[ \sum_{r \geq 1} \tilde{c}_{m,s} \psi_m \left( \tilde{c}_{r,m} \right) (\tilde{c}(p)) = R_m(p) \quad (\text{for all} \ m \geq 1) \]
has a unique solution; thus (8) has a unique solution of the form we asserted. This completes the induction.

To compute the coefficients \( \tilde{c}_{m,m/2}(p) \), we just notice that (8) corresponding to \( s = m/2 \) is
\[ \tilde{c}_{m,m/2}(p) = \tilde{c}_{m,m/2}(p) \psi_m \left( \frac{\partial c_{m}(V)}{\partial p_2} \right) (\tilde{c}(p)) = 1 + 2\psi_m \left( \frac{\partial c_{m}(V)}{\partial p_2} \right) (\tilde{c}(p)). \]
Since \( \tilde{c}_{r,0} = \psi_r(\tilde{c}_{1,0}) \), for all \( r \geq 1 \), it follows that
\[ \tilde{c}_{m,m/2} = \frac{1 + 2\psi_m \left( \frac{\partial c_{m}(V)}{\partial p_2} \right)}{1 - \psi_m \left( \frac{\partial^2 c_{m}(V)}{\partial p_1^2} \right)} \tilde{c}_{1,0} \]
as claimed. This completes the proof.

Remark. It is easy to see that the infinite homogeneous linear system associated to (8) \((m \geq 2s + 1 \geq 3)\) has only the trivial solution if we assume that the unknowns \( \tilde{c}_{m,s}(p) \) are subject to a growing condition. Indeed, rewriting (6) as
\[ \sum_{m \geq 1} (1 + p_m) + \sum_{m \geq 1} \psi_m \left( \frac{1}{2} \right)^m q^d = -q(q-1)c_1(p) + q^2 p_1 + 1 \]
one finds easily that
\[ \psi_m \left( \frac{\partial c_{1}(V)}{\partial p_1} \right)(p) = \frac{1 + q^m c_m(p) - q^m p_m}{1 + p_m} \quad \text{and} \quad \psi_m \left( \frac{\partial c_{1}(V)}{\partial p_2} \right)(p) = s_r(q^m)(q^m(q^{m-1})c_m(p) - q^2 p_m - 1) \quad (\text{if} \ r \geq 2). \]

\[^3\]The exponent \( k(g-1) + m \) of \( \hbar \) is at least \((m + 1 - \varepsilon_m)/2\).
Letting \(|p_m| < \delta^n/q^n\) \((n \geq 1)\), for a fixed \(0 < \delta < 1\), we have that
\[
\left| \psi_m \left( \frac{\partial c_1}{\partial p_i} \right)(p) \right| < \frac{q^{n-m}}{\tau(q^n-1)(1-q^{-m})} \left(1 + q^{-m}\delta^m \right)^{-1} \cdot \prod_{n \geq 1} (1 + q^{-m}\delta^m)^{y_n(q^n)/n}
= \frac{q^{n-m}}{\tau(q^n-1)(1-q^{-m})} \cdot \frac{1 - q^{-m}\delta^m}{1 - \delta^m}.
\]
Similarly, as long as \(\delta\) is not too close to 1 (for instance, one can take \(\delta \leq 1 - \frac{3}{q^{r-1}}\)), we have the lower bound
\[
\left| \psi_m \left( \frac{\partial c_1}{\partial p_i} \right)(p) \right| > \frac{q^{n-m}}{q^m - 1} \left(1 - \frac{1 - q^{-m}\delta^m}{q^m(1-\delta^m)(1-q^{-m})} \right) > 0 \quad \text{(for } m \geq 2s + 1 \text{ and } r \geq 2)\).
\]
It follows that, for \(|p_m| < \frac{1}{\sqrt{\tau}} \left(1 - \frac{3}{q^{r-1}}\right)^n\), we have the estimate
\[
\left| \psi_m \left( \frac{\partial c_1}{\partial p_i} \right)(p) / \psi_m \left( \frac{\partial c_1}{\partial p_i} \right)(\bar{q}) \right| < \frac{q^{n-m}}{2r} \quad (m \geq 2s + 1 \text{ and } r \geq 2).
\]
Our assertion follows now – for instance, from the classical results of von Koch [21]; clearly this gives rise inductively to the solution \(\bar{q}_m(p, h)\) of (5) in the proposition.

### 3.2 The asymptotic expansion

To obtain the asymptotic expansion of \(T(MV)\), we first expand \(K(p, q, h)\) as a power series centered at the unique solution \(\bar{q} = (\bar{q}_1, \bar{q}_2, \ldots)\) of (5). Thus, recalling that
\[
K(p, q, h) = -\sum_{m=1}^{\infty} \left\{ (q_m - p_m - \varepsilon_m h^{m/2})^2 / 2m h^m \right\} + \Psi(T(V))
\]
we can express
\[
K(p, q, h) = K(p, \bar{q}, h) - \sum_{m=1}^{\infty} \frac{t_m^2}{2m h^m} + \sum_{|\alpha| \geq 2} \left( (h_m)^{\alpha} \Psi(T(V))(q, h) \right) \frac{t_m^\alpha}{\alpha!} \quad \text{(with } t_m = q_m - \bar{q}_m \text{ and } t = q - \bar{q})
\]
We set
\[
L(p, q, h) = L(p, t, h) := \sum_{|\alpha| \geq 2} \left( (h_m)^{\alpha} \Psi(T(V))(q, h) \right) \frac{t_m^\alpha}{\alpha!} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{\partial^2 \Psi(T(V))(q, h)}{\partial q_m^2} t_m^2
\]
so that
\[
K(p, q, h) = K(p, \bar{q}, h) - \sum_{m=1}^{\infty} \left\{ (1 - m h_m \partial^2 \Psi(T(V))(q, h)) \left( 1 - \left( \frac{t_m}{2m h^m} \right)^2 \right) \right\} + L(p, q, h).
\]

**Theorem 3.2.** — For coordinates \(t_1, t_2, \ldots\) in \(\text{Spec}(\Lambda_{\text{alg}} \otimes \mathbb{R}) \cong \mathbb{R}^\infty\), set
\[
i_m := \left( 1 - m h_m \partial^2 \Psi(T(V))(q, h) \right)^{1/2} \sqrt{2m h^m} t_m \quad \text{and} \quad \tilde{i} = (\tilde{i}_1, \tilde{i}_2, \ldots).
\]
If \(d\nu(t)\) denotes the formal Gaussian measure
\[
d\nu(t) = \prod_{m=1}^{\infty} e^{-t_m^2 / \sqrt{\pi}} \] we have
\[
\Psi(T(MV)) = K(p, q, h) - \frac{1}{2} \sum_{m=1}^{\infty} \log \left( 1 - m h_m \partial^2 \Psi(T(V))(q, h) \right) + \log \left( \int_{\mathbb{R}^\infty} e^{L(p, t, h)} d\nu(t) \right) \quad \text{(9)}
\]
Moreover, as \(h \to 0^+\), we have the first-order asymptotic expansion
\[
\log \left( \int_{\mathbb{R}^\infty} e^{L(p, t, h)} d\nu(t) \right) = \left( \frac{\partial^2 \chi_0}{\partial q_1 \partial q_0} (q) \right)^2 + \frac{5}{24} \left( \frac{\partial^2 \chi_0}{\partial q_1^2} (q) \right)^2 + \frac{1}{8} \left( 1 - \frac{\partial^2 \chi_0}{\partial q_1^2} (q) \right)^2 \right) h + O(h^2)
\]
where \(\chi_0(q) = \chi_0(V)(q) = \sum_{n \geq 2} \sum_{\mu \geq n} [M^F_{0, n, \mu}] \frac{y_n}{\bar{q}_n} \).
Proof. By Theorem 2.1, and Theorem 2.2 in the form (4), we have

\[ \text{Exp}(T(MV)) = \exp(\Delta) \text{Exp}(T(V)) = \int_{\mathbb{R}^m} e^{K(p, q, h)} d^m q \]

\[ = e^{K(p, q, h)} \int_{\mathbb{R}^m} e^{L(p, t, h)} \exp \left[ - \sum_{m=1}^{\infty} \left( 1 - m h^m \frac{2}{\partial q_m^2} \Psi(T(V)) (q, h) \right) \frac{t_m^2}{2m h^m} \right] dt. \]

Substituting \( t_m \) by \( (1 - m h^m \frac{2}{\partial q_m^2} \Psi(T(V)) (q, h))^{-1/2} \sqrt{2 m h^m} t_m \) for all \( m \geq 1 \), and then taking the logarithm, we find that

\[ \Psi(T(MV)) = K(p, \bar{q}, h) - \frac{1}{2} \sum_{m=1}^{\infty} \log \left( 1 - m h^m \frac{2}{\partial q_m^2} \Psi(T(V)) (q, h) \right) + \log \left( \int_{\mathbb{R}^m} e^{L(p, t, h)} dt \right). \]

This is the formula for \( \Psi(T(MV)) \) stated in our theorem.

Now let \( \alpha = (\alpha_1, \ldots, \alpha_s) \) be a multi-index with \( |\alpha| \geq 2 \), and put \( \bar{\alpha} := (\alpha^1, \ldots, \alpha^s) \). It is clear that when computing the partial derivative \( \partial^\alpha \Psi(T(V)) (q, h) \), the only terms

\[ h_k^{k(s-1)} \frac{\partial^{\alpha_k} \Psi(T(V)) (q, h)}{\partial q_k^{\alpha_k}} \]

in \( \Psi(T(V)) \) contributing nontrivially are among those corresponding to \( k \)'s dividing \( \bar{\alpha} \) (i.e., \( k \) divides every part of \( \bar{\alpha} \)). Using this, one finds easily that

\[ L(p, \bar{t}, h) = \frac{1}{2h^2} \left( \frac{\partial^2 (q_2 \circ \text{ch}_0(V)) (q)}{\partial q_2 \partial q_4} (q) \bar{t}_2 \bar{t}_4 + \frac{\partial^2 (q_2 \circ \text{ch}_0(V)) (q)}{\partial q_2^2} (q) \bar{t}_2^2 \right) + \frac{1}{h} \left( \frac{\partial^2 \text{ch}_0(V)}{\partial q_1 \partial q_3} (q) \bar{t}_1 \bar{t}_3 + \frac{\partial^2 \text{ch}_0(V)}{\partial q_1^2} (q) \bar{t}_1^2 \right) \]

\[ + \frac{1}{h} \left( \frac{\partial^2 \text{ch}_0(V)}{\partial q_2 \partial q_3} (q) \bar{t}_2 \bar{t}_3 + \frac{\partial^2 \text{ch}_0(V)}{\partial q_2^2} (q) \bar{t}_2^2 \right) + \frac{1}{2} \left( \frac{\partial^2 \text{ch}_0(V)}{\partial q_1 \partial q_3} (q) \bar{t}_1 \bar{t}_3 + \frac{\partial^2 \text{ch}_0(V)}{\partial q_1\partial q_3} (q) \bar{t}_1^2 \right) \] \[ + O(h^{3/2}); \]

it is also easy to see that

\[ \left( 1 - m h^m \frac{2}{\partial q_m^2} \Psi(T(V)) (q, h) \right)^{-1/2} \]

\[ = \left( 1 - \frac{\partial^2 \text{ch}_0(V)}{\partial q_m^2} (q) \right)^{1/2} \left( 1 + \frac{\delta_{m,1} \frac{\partial^2 \text{ch}_0(V)}{\partial q_1^2} (q) + 2 \delta_{m,2} \frac{\partial^2 \text{ch}_0(V)}{\partial q_2^2} (q) h}{1 - \frac{\partial^2 \text{ch}_0(V)}{\partial q_m^2} (q)} \right)^{1/2} + O(h^2). \]

Here \( \delta_{m,j} \) is the Kronecker delta. Replacing \( \bar{t}_1, \ldots, \bar{t}_3, \bar{t}_4 \) by their defining expressions and applying (10), we obtain an asymptotic expansion of \( L(p, \bar{t}, h) \) of the form

\[ L(p, \bar{t}, h) = (A_{1,2} + B_{1,2} h) \sqrt{h} t_1 t_2 + (A_{3,4} + B_{3,4} h) \sqrt{h} t_1^2 t_2 + (A_{3,4} + B_{3,4} h) t_1 t_2 \]

\[ + A_{1,2} t_1 t_3 + A_{3,4} h t_1^2 t_3 + A_{3,4} h t_3^2 + O(h^{3/2}) \]

where, for instance,

\[ A_{1,2} = 2 \sqrt{2} \frac{\frac{\partial^2 \text{ch}_0(V)}{\partial q_2 \partial q_4} (q)}{\sqrt{1 - \frac{\partial^2 \text{ch}_0(V)}{\partial q_2^2} (q)}}. \]

\[ A_{3,4} = \sqrt{2} \frac{\frac{\partial^2 \text{ch}_0(V)}{\partial q_1 \partial q_3} (q)}{\sqrt{1 - \frac{\partial^2 \text{ch}_0(V)}{\partial q_1^2} (q)}}. \]

Finally, by applying the familiar Gaussian integral identity

\[ \int_{\mathbb{R}} t^k e^{-t^2} dt \sqrt{2\pi} = \begin{cases} \frac{2^{-k/2} (k - 1)!}{\pi} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \]
one finds easily that
\[
\log \left( \int_0^\infty e^{L(p,t,h)} \, dt \right) = \left( \frac{1}{8} A_2^2 + \frac{15}{16} A_3^2 + \frac{3}{4} A_4 \right) h + O(h^2)
\]
and the asymptotic expansion stated in the theorem follows. This completes the proof.

We note that by expressing the exponential \( e^{L(p,t,h)} \) as a power series and then integrating, one can write the full asymptotic expansion of \( \Psi(T(\mathcal{M})\mathcal{V}) \). One should also be able to interpret this asymptotic expansion, as in [6, Proposition 3.6], as an expansion over stable graphs. We shall not pursue this further since the calculations are quite cumbersome. Instead, we shall just use Theorem 3.2 to obtain formulas for the generating series \( \text{ch}_n(\mathcal{M}\mathcal{V}) \) when \( g \leq 2 \).

4 Equivariant Euler characteristics

In this section we shall apply Theorem 3.2 to express \( \text{ch}_n(\mathcal{M}\mathcal{V}) \), when \( g = 0, 1 \) and \( 2 \), in terms of \( \text{ch}_n(\mathcal{V}) \), \( g' \leq g \). The formulas for \( \text{ch}_n(\mathcal{M}\mathcal{V}) \) and \( \text{ch}_0(\mathcal{M}\mathcal{V}) \) are not new; they were first obtained by Getzler and Kapranov [15] when \( g = 0 \), and by Getzler [14] when \( g = 1 \).

From now on we adopt the following notation. Put \( a_x := \text{ch}_x(\mathcal{V}) \), \( b_x := \text{ch}_x(\mathcal{M}\mathcal{V}) \), and for \( g, \alpha_1, \ldots, \alpha_s \in \mathbb{N} \), write
\[
a_y^{(\alpha_1, \ldots, \alpha_s)} := \frac{\partial^{\alpha_1 + \cdots + \alpha_s} \text{ch}_y(\mathcal{V})}{\partial q_1^{\alpha_1} \cdots \partial q_s^{\alpha_s}} \quad \text{and} \quad b_y^{(\alpha_1, \ldots, \alpha_s)} := \frac{\partial^{\alpha_1 + \cdots + \alpha_s} \text{ch}_y(\mathcal{M}\mathcal{V})}{\partial q_1^{\alpha_1} \cdots \partial q_s^{\alpha_s}}.
\]

In what follows, we shall need the notion of the Legendre transform for symmetric functions, see [15, Theorem 7.15]. To define this notion, let \( \mathbb{Q}[x] \) denote the set of all formal power series \( f \in \mathbb{Q}[x] \) of the form
\[
f(x) = \sum_{s=2}^{\infty} a_s x^s \quad (\text{with } a_2 \neq 0).
\]
Let \( \text{rk} : \Lambda \to \mathbb{Q}[x] \) be the homomorphism defined by \( h_n \mapsto x^n/n! \), and denote by \( \Lambda_* \) the set of symmetric functions \( f \in \Lambda \) such that \( \text{rk}(f) \in \mathbb{Q}[x]_* \); note that \( \text{rk}(f) \) can also be obtained from \( \text{rk}(f_1, f_2, \ldots) \) by setting: \( f_1 = x \), and \( f_n = 0 \) if \( n \geq 2 \). If \( f \in \Lambda_* \), there is a unique element \( g = \mathcal{L} f \in \Lambda_* \), called the Legendre transform of \( f \), determined by the formula
\[
g = \frac{\partial f}{\partial p_1} + f = p_1 \frac{\partial f}{\partial p_1}.
\]
With the above notation and terminology, we have (see [15, Theorem 7.17] or [13, Theorem 5.9]):

**Theorem 4.1.** — Put \( f = e_2 - a_2 \) and \( g = h_2 + b_2 \). Then \( \mathcal{L} f = g \).

**Proof.** By the definition of the Legendre transform and the identities \( e_2 = (p_1^2/p_2) = 2 \) and \( h_2 = (p_1^2/p_2) = 2 \), we have to show that
\[
\left( \frac{p_1^2 + p_2}{2} + b_0 \right) \circ (p_1 - a_2) = \frac{p_1^2 + p_2}{2} - p_1 a_2 + a_2.
\]
To see this, we first observe that, by (10), the only piece in the right-hand side of (9) contributing negative powers of \( h \) to \( \Psi(T(\mathcal{M})\mathcal{V}) \) is \( \mathcal{K}(p, q, h) \). From the definition of \( \mathcal{K}(p, q, h) \) it is clear that
\[
\mathcal{K}(p, q, h) = \sum_{m=1}^{\infty} \left\{ -\left( \bar{q}_m - p_m - \varepsilon_m h^{m/2} \right)^2 + 2\psi_m(a_2)(\bar{q}) \right\} / 2m h^m + O(1)
\]
with \( \psi_m(a_2)(\bar{q}) \) given explicitly by
\[
\psi_m(a_2)(\bar{q}) = \sum_{n \geq 3} \sum_{\rho + \eta = n} |\mathcal{M}_{\rho, \eta}^m|^2 \frac{\bar{q}_{m,n}}{2^\rho}.
\]
Thus we have:
\[
\Psi(T(\mathcal{M})\mathcal{V}) = \sum_{m=1}^{\infty} \left\{ -\left( \bar{q}_m - p_m - \varepsilon_m h^{m/2} \right)^2 + 2\psi_m(a_2)(\bar{q}) \right\} / 2m h^m + O(1).
\]
Now, since
\[
\bar{q}_r(p, h) = \bar{e}_{r,0}(p) + O(h^{(r+1-\varepsilon_r)/2}) \quad (\text{for all } r \geq 1)
\]
it follows that for every partition \( \rho = (1^{\rho(1)}, \ldots, n^{\rho(n)}) \), we have

\[
q_{m, \rho}(p, h)h^{-m} = \left( \bar{c}_{m, o}(p)h^{-m} + \rho(1)h^{(1-m-\epsilon_m)/2}\bar{c}_{m, o}(p)U_m(p, h) \right) \prod_{l=2}^{n} \bar{c}_{m, o}(p) + O(1)
\]

where \( U_m(p, h) = (q_m(p, h) - \bar{c}_{m, o}(p))h^{-(m+1-\epsilon_m)/2} \). Thus we can replace \( \bar{q}_{m, \rho}h^{-m} \), for every \( m \), by

\[
h^{-m} \prod_{l=1}^{n} \bar{c}_{m, o}(p) + U_m(p, h)\bar{c}_{m, o}^{-1}(1) \prod_{l=2}^{n} \bar{c}_{m, o}(p).
\]

This immediately yields

\[
h^{-m} \psi_m(a_o)(q) = h^{-m} \psi_m(a_o)(\bar{C}(p)) + h^{(1-m-\epsilon_m)/2}U_m(p, h)(\bar{c}_{m, o}(p) - p_m) + O(1).
\]

Similarly,

\[
-\left( \bar{q}_{m} - p_m - \epsilon_m h^{m/2} \right)/2h^m = -\left( \bar{c}_{m, o}(p) - p_m \right)^2/2h^m + \epsilon_m(\bar{c}_{m, o}(p) - p_m)h^{m/2}
\]

\[
- h^{(1-m-\epsilon_m)/2}U'_m(p, h)(\bar{c}_{m, o}(p) - p_m) + O(1).
\]

Putting these calculations together, we find that

\[
\Psi(T[M]) = -\frac{1}{2}\psi_\left( \frac{(\bar{c}_{1, o}(p) - p_1)^2}{h} \right) + \sum_{m \text{ even}} \frac{1}{m} \psi_m(\bar{c}_{1, o}(p) - p_1) \sqrt{h} - \psi_2(\bar{c}_{1, o})(p) + O(1).
\]

To this asymptotic expansion, we apply the operation \( \Psi^{-1}(-) = \sum_{d \geq 1} \frac{\mu(d)}{d} \psi_d(-) \). Using the identity

\[
\sum_{d \geq 1} \frac{\mu(d)}{d} \psi_d(\bar{c}_{1, o}) = \begin{cases} 1 & \text{if } N = 2 \\ 0 & \text{if } N \neq 2 \end{cases}
\]

and then equating the coefficients of \( h^{-1} \) in the resulting asymptotic expansion, one obtains the formula:

\[
b_{a_0} = -\frac{1}{2}(\bar{c}_{1, o} - p_1)^2 - \frac{1}{2}p_2 + \frac{1}{2} \psi_2(\bar{c}_{1, o}) + (a_o \circ \bar{c}_{1, o}). \tag{11}
\]

The theorem follows now by applying \( \circ c_1 \) (i.e., \( (p_1 - a_{b_0}^{(1)}) \)) on the right of (11), and by recalling that \( \bar{c}_{1, o} \circ c_1 = p_1 \).

**Corollary 4.2.** — The symmetric functions

\[ p_1 = a_{b_0}^{(1)} \text{ and } p_1 + b_{a_0}^{(1)} \]

are plethystic inverses. Thus \( \bar{c}_{1, o} = p_1 + b_{a_0}^{(1)} \).

**Proof.** Let \( f \) and \( g \) be as in Theorem 4.1. Then

\[
\frac{\partial f}{\partial p_1} = c_1 = p_1 - a_{b_0}^{(1)} \text{ and } \frac{\partial g}{\partial p_1} = p_1 + b_{a_0}^{(1)}.
\]

Since \( Lf = g \), the first assertion follows at once from [15, Theorem 7.15 (c)]. The second assertion is an immediate consequence of the fact that \( c_1 \) and \( \bar{c}_{1, o} \) are plethystic inverses.

The following theorem is the main result of [14]; for a direct combinatorial proof, the reader may consult [24].

**Theorem 4.3.** — We have

\[
b_{a_1} = \left( a_1 - \frac{1}{2} \sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log \left( 1 - \psi_m(a_{b_0}^{(1)}) \right) + \frac{a_{b_0}^{(0, 1)}(a_{b_0}^{(0, 1)} + 1) + \frac{1}{2} \psi_2(a_{b_0}^{(1)})}{1 - \psi_2(a_{b_0}^{(1)})} \right) \circ (p_1 + b_{a_0}^{(1)})
\]

where \( \varphi(m) \) is Euler’s totient function.
Proof. The proof is similar to that of Theorem 4.1. By Theorem 3.2 and the definition of \( K(p, q, h) \), we have that

\[
\Psi(T(MV)) = \Psi(a_1)(q) + \sum_{m=1}^{\infty} \left\{ \left( \bar{q}_m - p_m - \epsilon_m h^{m/2} \right)^2 + 2\psi_m(a_0)(q) \right\} / 2mh^m
\]

\[-\frac{1}{2} \sum_{m=1}^{\infty} \log \left( 1 - mh^m \frac{\partial^2 \Psi(T(V))}{\partial q_m^2} (q, h) \right) + O(h). \]

By Corollary 4.2, the constant term of \( \Psi(a_1)(\bar{q}) \) is:

\[
\Psi(a_1)(q(p, 0)) = \Psi(a_1)(\bar{C}(p)) = (\Psi(a_1) \circ \bar{e}_{1,0})(p) = \Psi(a_1) \circ (p_1 + b_0^{(1)}(p)).
\]

By applying the linear operation \( \Psi^{-1} \) to this, we obtain the first contribution to \( \mathbf{b}_1 \), stated in the theorem.

The second contribution corresponds to

\[
-\frac{1}{2} \sum_{m=1}^{\infty} \log \left( 1 - mh^m \frac{\partial^2 \Psi(T(V))}{\partial q_m^2} (q, h) \right)_{h=0}. \quad (12)
\]

From (10) and Corollary 4.2, we have

\[
\log \left( 1 - mh^m \frac{\partial^2 \Psi(T(V))}{\partial q_m^2} (q, h) \right)_{h=0} = \log \left( 1 - \psi_m(a_0^{(2)}) \right) \circ (p_1 + b_0^{(1)}(p))
\]

for all \( m \geq 1 \). Thus the contribution (12) is given by

\[
-\frac{1}{2} \left( \sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log \left( 1 - \psi_m(a_0^{(2)}) \right) \right) \circ (p_1 + b_0^{(1)}(p)).
\]

As before, we now apply the operation \( \Psi^{-1}(-) = \sum_{r=1}^{\varphi(m)} \frac{\mu(r)}{r} \psi_r(-) \). Summing first over \( m = kr \), and using the well-known Möbius inversion formula

\[
\sum_{r|m} \frac{\mu(r)}{r} = \frac{\varphi(m)}{m}
\]

we find that the contribution to \( \mathbf{b}_2 \), corresponding to (12) is

\[
-\frac{1}{2} \left( \sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log \left( 1 - \psi_m(a_0^{(2)}) \right) \right) \circ (p_1 + b_0^{(1)}(p)).
\]

It remains to compute the constant term of

\[
\sum_{m=1}^{\infty} \left\{ \left( \bar{q}_m - p_m - \epsilon_m h^{m/2} \right)^2 + 2\psi_m(a_0)(q) \right\} / 2mh^m.
\]

Just as in the proof of Theorem 4.1, for every \( m \geq 1 \), the constant term of the summand is

\[
\frac{1}{m} U_{2m}(p, 0) \psi_m(a_0^{(0, 1)})(\bar{C}(p)) + \frac{\epsilon_m}{m} \left\{ \frac{1}{2} \left( -1 + \psi_m(a_0^{(2)}) \right) \bar{C}(p) \right\} U_{2m}(p, 0) + U_m(p, 0) - \frac{1}{2};
\]

by Proposition 3.1, we know that

\[
U_m(\cdot, 0) = \bar{e}_{m, m/2} = \frac{1 + 2\psi_m(a_0^{(0, 1)})}{1 - \psi_m(a_0^{(2)})} \circ \bar{e}_{1,0} \quad \text{(when \( m \) is even)}.
\]

Applying the operation \( \Psi^{-1} \), it follows that the final contribution to \( \mathbf{b}_2 \) is

\[
\frac{(a_0^{(0, 1)})^2 + a_0^{(0, 1)} + \frac{1}{3} \psi_2(a_0^{(2)})}{1 - \psi_2(a_0^{(2)})} \circ \bar{e}_{1,0}.
\]

The formula of \( \mathbf{b}_2 \), stated in the theorem follows now from Corollary 4.2. □
Theorem 4.4.

To state the analogous result when \( g = 2 \), let us introduce some notation. For \( m \geq 2 \) even, write \( \bar{c}_{m/m/2} = v_m \circ \bar{c}_{1,0} \), where, by Proposition 3.1,

\[
v_m = \frac{1 + 2\psi_m(a^{(1,1)}_0) + \frac{a^{(1,1)}_0}{1 - \psi_m(a^{(2)}_0)} + a^{(1)}_1}{1 - a^{(1)}_0}.
\]

Define

\[
\begin{align*}
\mathbf{w}_1 &= \frac{a^{(1,1)}_0 (1 + 2a^{(1,1)}_0)}{(1 - a^{(2)}_0)} \frac{1}{1 - \psi_2(a^{(2)}_0)} + \frac{a^{(1)}_1}{1 - a^{(1)}_0} + a^{(2)}_0 \frac{a^{(2)}_0}{1 - \psi_2(a^{(2)}_0)}, \\
\mathbf{w}_1 &= \psi_2(a^{(1,1)}_1) + 2a^{(1,1)}_0 + 2a^{(1)}_0 w_1 + \frac{2a^{(2,1)}_0 (1 + 2a^{(1,1)}_0)}{(1 - \psi_2(a^{(2)}_0))^2} + \frac{1}{2} \left( 1 + 2a^{(1,1)}_0 \right)^2 \psi_2(a^{(1,1)}_0) \\
\mathbf{w}_2 &= \frac{1}{1 - \psi_2(a^{(2)}_0)} + \frac{3a^{(0,0,1)}_0}{1 - \psi_3(a^{(2)}_0)}, \\
\mathbf{w}_4 &= \frac{2 (1 + 2a^{(1,1)}_0) \psi_2(a^{(1,1)}_0)}{(1 - \psi_2(a^{(2)}_0))(1 - \psi_4(a^{(2)}_0))} + \frac{4a^{(0,0,1,1)}_0}{1 - \psi_3(a^{(2)}_0)}, \quad \mathbf{w}_6 = \frac{3\psi_2(a^{(0,0,1)}_0)}{1 - \psi_4(a^{(2)}_0)}
\end{align*}
\]

and \( w_m = 0 \) for \( m \geq 5 \), \( m \neq 6 \). By solving for the coefficients \( \bar{c}_{m,(m+1+\varepsilon_m)/2}(p) \), we see that

\[
\bar{c}_{m,(m+1+\varepsilon_m)/2} = \mathbf{w}_m \circ \bar{c}_{1,0}.
\]

With these quantities we now define

\[
\mathbf{B}_2 := \underbrace{a_2 + w_1 a^{(1)}_1 + v_2 a^{(1,1)}_0 + \frac{1}{2} \psi_2 (a^{(1,1)}_0)}_{g = 2} + \underbrace{v_2 w_m a^{(1,1)}_0 + \frac{1}{2} \psi_2 a^{(2,1)}_0 + w_0 a^{(0,0,1,1)}_0 + v_2 a^{(0,0,0,1)}_0}_{g = 0 \text{ and } m \neq 1} + \sum_{m = \text{even}} \frac{v_m w_m \psi_m(a^{(2)}_0)}{m} + \frac{w_m (1 - v_m)}{m} + \sum_{m = \text{odd}} \frac{w_{2m} \psi_m(a^{(1,1)}_0) + \frac{1}{2} w_2^2 \psi_m(a^{(2)}_0)}{m} + \frac{w_m (1 - v_m)}{m} - \sqrt{\frac{(q_m - p_m)^{m/2}}{2m^{m/2}}} \sum_{m = \text{odd}} \frac{w_{2m} \psi_m(a^{(1,1)}_0) + \frac{1}{2} w_2^2 \psi_m(a^{(2)}_0)}{2m} - \frac{w_m}{2m} - \sum_{m = \text{even}} \frac{w_{2m} \psi_m(a^{(1,1)}_0) + \frac{1}{2} w_2^2 \psi_m(a^{(2)}_0)}{2m} - \frac{w_m}{2m}
\]

\[
\begin{align*}
+ \frac{1}{2} \left( \frac{w_{2m} \psi_m(a^{(1,1)}_0) + \frac{1}{2} w_2^2 \psi_m(a^{(2)}_0)}{2m} \right)^2 & \quad \frac{\left( a^{(1,1)}_0 \right)^2}{(1 - a^{(2)}_0)(1 - \psi_2(a^{(2)}_0))} + \frac{5}{8} \frac{\left( a^{(0)}_0 \right)^2}{(1 - \psi_2(a^{(2)}_0))} + \frac{1}{8} \frac{\left( a^{(0,0,1)}_0 \right)^2}{(1 - \psi_2(a^{(2)}_0))} - \frac{1}{2} \sum_{m = 1} \log \left( 1 - m h^m \frac{\partial^2 \Psi(T(V))}{\partial q_m^2} (q, h) \right) \\
& \quad \log \left( \int_{\mathbb{R}^m} e^{\varepsilon (p, k, h)} \, d\nu(t) \right)
\end{align*}
\]

\( (13) \)

**Theorem 4.4.** — With the above notation, we have \( \mathbf{b}_2 = \mathbf{B}_2 \circ (p_1 + b^{(1)}_0) \).

**Proof.** The assertion follows, as before, from Theorem 3.2 by computing the coefficient of \( h \) in the right-hand side of (9). Each contribution to this coefficient corresponds to a piece of \( \mathbf{B}_2 \), as indicated in (13). For instance, let us verify the contribution to \( \mathbf{b}_2 \) coming from

\[
- \frac{1}{2} \sum_{m = 1} \log \left( 1 - m h^m \frac{\partial^2 \Psi(T(V))}{\partial q_m^2} (q, h) \right)
\]

\( \varepsilon \)The operation \( \Psi^{-1} \) does not have any effect in this case.
By (10), it suffices to compute the coefficient of $h$ in
\[ -\frac{1}{2} \sum_{m=1}^{\infty} \log \left( 1 - \psi_m(a_0^{(2)}(q)) \right) + \frac{a_0^{(2)}(q)}{1 - a_0^{(2)}(q)} \cdot \frac{h}{2} + \frac{a_0^{(2)}(q)}{1 - \psi_2(a_0^{(2)}(q))} \cdot h. \] (14)

The logarithmic terms do not contribute to the coefficient when $m \geq 3$. If $m = 1$ we have
\[ a_0^{(2)}(q) = (a_0^{(2)} \circ \bar{c}_{1,0})(p) + h \left( (w_1 a_0^{(3)} + v_2 a_0^{(2,1)}) \circ \bar{c}_{1,0} \right)(p) + O(h^2) \]
and
\[ \psi_2(a_0^{(2)}(q)) = (\psi_2(a_0^{(2)}) \circ \bar{c}_{1,0})(p) + h \left( \psi_2(a_0^{(3)}) \circ \bar{c}_{1,0} \right)(p) + O(h^2) \]
if $m = 2$. Thus the coefficient of $h$ in (14) is indeed
\[ \left( \frac{1}{2} \left( \frac{w_1 a_0^{(3)} + v_2 a_0^{(2,1)}}{1 - a_0^{(2)}} + \frac{v_2 \psi_2(a_0^{(2)})}{1 - \psi_2(a_0^{(2)})} + \frac{a_0^{(2)}}{1 - a_0^{(2)}} \right) \right) \circ \bar{c}_{1,0} \]
(evaluated at $p$); we recall that $\bar{c}_{1,0} = p_1 + b_0^{(3)}$.

All the other contributions are computed similarly. \qed

We note that using the above results, the generating series $b_2$ can be effectively computed. Indeed, since $a_0 = ch_0(V)$ is known (see the remark preceding Proposition 3.1), it suffices to compute the generating series $a_1$ and $a_2$. An expression in closed form for $a_1$ can be read off directly from Getzler’s formula [12, Eq. (5.5)] for the generating series of the $S_n$-equivariant Serre characteristic of $M_{1,n}$. To compute $a_2$, let $A_2$ denote the moduli space of principally polarized abelian surfaces. Via the Torelli map, we can view $M_2$ as an open substack of $A_2$, and set $A_{1,1} = A_2 \setminus M_2$. For $\lambda = (\lambda_1, \lambda_2) \geq 0$, we have (see [10]) natural $\ell$-adic smooth étale sheaves $\mathcal{V}_\lambda$ on $A_2 \otimes \mathbb{Z}[1/\ell]$ corresponding to irreducible algebraic representations of $GSp_4(\mathbb{Q})$. From the results of [12] (or rather their $\ell$-adic realization), we know that computing the $S_n$-equivariant Euler characteristic $ch_n(V(2, n))$ amounts to the same as computing the Euler characteristics
\[ e_*(\mathcal{X}, \mathcal{V}_\lambda) = \sum_i (-1)^i [H^i(\mathcal{X}_x, \mathcal{V}_\lambda)] \]
for $\mathcal{X} = A_2$ and $x = A_{1,1}$, and all $\lambda$ with $\lambda_1 + \lambda_2 \leq n$. A formula for $e_*(A_2, \mathcal{V}_\lambda)$ was conjectured by Faber and van der Geer in [9]. Their conjecture was proved by Weissauer [28] when $\lambda$ is regular, that is, $\lambda_1 > \lambda_2 > 0$, and by Petersen [26] for arbitrary $\lambda$. The characteristic $e_*(A_{1,1}, \mathcal{V}_\lambda)$ can be computed, for instance, using the branching formula in [25, Section 3]. Thus taking the trace of Frobenius one obtains a formula expressing $a_2$ in terms of traces of Hecke operators on spaces of elliptic and genus 2 vector-valued Siegel modular forms.

References

[1] E. Arbarello, M. Cornalba, and P.A. Griffiths: Geometry of algebraic curves. Volume II. With a contribution by Joseph Daniel Harris. Grund. Math. Wiss. 268, Springer, Heidelberg, 2011, xxx+963.
[2] K.A. Behrend: The Lefschetz trace formula for algebraic stacks. Invent. Math. 112 (1993), no. 1, 127–149.
[3] K.A. Behrend: Derived l-adic categories for algebraic stacks. Mem. Amer. Math. Soc. 163 (2003), no. 774, viii+93.
[4] J. Bergström and O. Tommasi: The rational cohomology of $\overline{M}_4$. Math. Ann. 338 (2007), no. 1, 207–239.
[5] D. Bessis, C. Itzykson, and J.B. Zuber: Quantum field theory techniques in graphical enumeration. Adv. Appl. Math. 1 (1980) 109–157.
[6] G. Bini and J. Harer: Euler characteristics of moduli spaces of curves. (JEMS) 13 (2011), no. 2, 487–512.
[7] H. Darmon, F. Diamond, and R. Taylor: Fermat’s last theorem. In: Elliptic Curves, Modular Forms & Fermat’s Last Theorem (Hong Kong, 1993), pp. 2–140, Int. Press, Cambridge, MA, 1997.
[8] P. Deligne and D. Mumford: The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.
[9] C. Faber and G. van der Geer: Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, I. II. C. R. Math. Acad. Sci. Paris 338 (2004), no. 5, 381–384 and no. 6, 467–470.
[10] G. Faltings and C.-L. Chai: *Degeneration of abelian varieties*. With an appendix by David Mumford. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 22, Springer-Verlag, Berlin, 1990, xii+316.

[11] W. Fulton and S. Lang: *Riemann-Roch algebra*. Grundlehren der Mathematischen Wissenschaften 277, Springer-Verlag, New York, 1985, x+203.

[12] E. Getzler: *Resolving mixed Hodge modules on configuration spaces*. Duke Math. J. 96 (1999), no. 1, 175–203.

[13] E. Getzler: *Operads and moduli spaces of genus 0 Riemann surfaces*. In: The moduli space of curves (Texel Island, 1994), Progr. Math. 129, pp. 199–230, Birkhäuser Boston, 1995.

[14] E. Getzler: *The semi-classical approximation for modular operads*. Comm. Math. Phys. 194 (1998), no. 2, 481–492.

[15] E. Getzler and M.M. Kapranov: *Modular operads*. Compositio Math. 110 (1998), no. 1, 65–126.

[16] E. Gorsky: *The equivariant Euler characteristic of moduli spaces of curves*. Adv. Math. 250 (2014), 588–595.

[17] J. Harris and I. Morrison: *Moduli of curves*. Graduate Texts in Mathematics 187, Springer-Verlag, New York, 1998, xiv+366.

[18] M. Kisin and G.I. Lehrer: *Equivariant Poincaré polynomials and counting points over finite fields*. J. Algebra 247 (2002), no. 2, 435–451.

[19] F.F. Knudsen: *The projectivity of the moduli space of stable curves. II. The stacks Mg,n*. Math. Scand. 52 (1983), no. 2, 161–199.

[20] D. Knutson: *λ-rings and the representation theory of the symmetric group*. Lecture Notes in Mathematics, Vol. 308, Springer-Verlag, Berlin-New York, 1973, iv+203.

[21] H. von Koch: *On regular and irregular solutions of some infinite systems of linear equations*. In: Proceedings of the fifth International Congress of Mathematicians, Vol.I (Cambridge,22-28 August 1912), pp.352–365, Cambridge University Press, 1913. Available at: http://www.mathunion.org/ICM/ICM1912.1/ICM1912.1.ocr.pdf

[22] I.G. Macdonald: *Symmetric functions and Hall polynomials*. With contribution by A.V. Zelevinsky and a foreword by Richard Stanley. Oxford Classic Texts in the Physical Sciences, Second Edition, Clarendon Press, Oxford University Press, New York, 2015, xii+475.

[23] S. Mozgovoy: *A computational criterion for the Kac conjecture*. J. Algebra 318 (2007), no. 2, 669–679.

[24] D. Petersen: *A remark on Getzler’s semi-classical approximation*. In: Geometry and arithmetic, EMS Ser. Congr. Rep., pp. 309–316, Eur. Math. Soc., Zürich, 2012.

[25] D. Petersen: *Cohomology of local systems on loci of d-elliptic abelian surfaces*. Michigan Math. J. 62 (2013), no. 4, 705–720.

[26] D. Petersen: *Cohomology of local systems on the moduli of principally polarized abelian surfaces*. Pacific J. Math. 275 (2015), no. 1, 39–61.

[27] SGA 6: *Théorie des intersections et théorème de Riemann-Roch*. Lecture Notes in Math. 225, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967, dirigé par P. Berthelot, A. Grothendieck et L. Illusie (avec la collaboration de D. Ferrand, J.P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J.P. Serre), Springer-Verlag, Berlin-New York, 1971, xii+700.

[28] R. Weissauer: *The trace of Hecke operators on the space of classical holomorphic Siegel modular forms of genus two*. Available at arXiv:0909.1744.