Inequalities in approximation theory involving fractional smoothness in $L_p$, $0 < p < 1$

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Abstract. In the paper, we study inequalities for the best trigonometric approximations and fractional moduli of smoothness involving the Weyl and Liouville-Gr"unwald derivatives in $L_p$, $0 < p < 1$. We extend known inequalities to the whole range of parameters of smoothness as well as obtain several new inequalities. As an application, the direct and inverse theorems of approximation theory involving the modulus of smoothness $\omega_\beta(f^{(\alpha)}, \delta)_p$, where $f^{(\alpha)}$ is a fractional derivative of the function $f$, are derived. A description of the class of functions with the optimal rate of decrease of a fractional modulus of smoothness is given.

1. Introduction

Let $\mathbb{T} \cong [0, 2\pi)$ be the torus. As usual, the space $L_p(\mathbb{T})$, $0 < p < \infty$, consists of measurable complex functions that are $2\pi$-periodic and

$$
\|f\|_p = \left( \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.
$$

Recall that if $f \in L_1(\mathbb{T})$ and $\alpha \in \mathbb{R}$, then the fractional Weyl derivative $f^{(\alpha)}$ (if $\alpha < 0$, the fractional integral) is defined by

$$
f^{(\alpha)}(x) \sim \sum_{k \in \mathbb{Z} \setminus \{0\}} (ik)^\alpha \hat{f}_k e^{ikx}, \quad (ik)^\alpha = |k|^\alpha e^{i\frac{\alpha\pi}{2} \text{sign} k},
$$

where $\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$ are the Fourier coefficients of $f$. There are several approaches to define fractional derivatives (see [34]). In this paper, together with the fractional Weyl derivative, we use the fractional derivative in the sense of $L_p$ (or the Liouville-Gr"unwald derivative). For $f \in L_p(\mathbb{T})$, $0 < p < \infty$, we define the derivative of order $\alpha > 0$ in the sense of $L_p$ as a function $g \in L_p(\mathbb{T})$ satisfying

$$
\left\| \frac{\Delta_{h\alpha} f}{h^\alpha} - g \right\|_p \to 0 \quad \text{as} \quad h \to 0.
$$

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As usual,
\[ \Delta_0^\alpha f(x) = \sum_{\nu=0}^{\infty} \frac{\alpha}{\nu} (-1)^\nu f(x - \nu \delta) \]
and \( \binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\ldots(\alpha-\nu+1)}{\nu!} \), \( \nu \geq 1 \), \( \binom{\alpha}{0} = 1 \).

Note that in the above definition if \( f \in L_p(\mathbb{T}) \), \( 1 \leq p < \infty \), and the function \( g \) exists, then \( g \) coincides with the fractional Weyl derivative \( f^{(\alpha)} \) (see [5]). Because of this, we denote \( g = f^{(\alpha)} \) for all \( 0 < p < \infty \).

Let \( T_n \) be the set of all trigonometric polynomials of order at most \( n \). The best approximation of a function \( f \) by polynomials \( T \in T_n \) is given by
\[ E_n(f)_p = \inf_{T \in T_n} \| f - T \|_p. \]
As usual, if \( \| f - T \|_p = E_n(f)_p \) and \( T \in T_n \), then \( T \) is called a polynomial of the best approximation of \( f \) in \( L_p(\mathbb{T}) \).

Recall several known inequalities for the best trigonometric approximation of a function \( f \in L_p(\mathbb{T}) \), \( 1 \leq p < \infty \), and its fractional derivatives of order \( \alpha > 0 \). We have
\begin{equation}
(1.2) \quad E_n(f)_p \leq C(\alpha) \frac{C(\alpha)}{n^\alpha} E_n(f^{(\alpha)})_p,
\end{equation}
\begin{equation}
(1.3) \quad E_n(f^{(\alpha)})_p \leq C(\alpha) \left( n^\alpha E_n(f)_p + \sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_p \right),
\end{equation}
\begin{equation}
(1.4) \quad \| f^{(\alpha)} - T_n^{(\alpha)} \|_p \leq C(\alpha) E_n(f^{(\alpha)})_p,
\end{equation}
where \( T_n \in T_n \) is such that \( \| f - T_n \|_p = E_n(f)_p \). Remark that inequality (1.2) can be found in [4] and [41, p. 95]; inequality (1.3), which is a (weak) inverse inequality to (1.2), was proved in [41, pp. 150–153] (see also [35]); inequality (1.4), which is related to the simultaneous approximation of a function and its derivatives, was derived for the case \( \alpha \in \mathbb{N} \) in [8] and for the case \( \alpha > 0 \) in [42].

Inequalities of type (1.2)–(1.4) have been also studied in the case \( 0 < p < 1 \), mainly for the derivatives of integer order. In particular, Ivanov [13] proved that if \( f \in L_p(\mathbb{T}) \), \( 0 < p < 1 \), is such that \( \sum_{\nu=1}^{\infty} \nu^{\alpha p-1} E_\nu(f)_p < \infty \) for some \( \alpha \in \mathbb{N} \), then \( f \) has the derivative \( f^{(\alpha)} \) in the sense of \( L_p \) and
\begin{equation}
(1.5) \quad E_n(f^{(\alpha)})_p \leq C(p, \alpha) \left( n^\alpha E_n(f)_p + \left( \sum_{\nu=1}^{\infty} \nu^{\alpha p-1} E_\nu(f)_p \right)^{\frac{1}{p}} \right).
\end{equation}
For \( \alpha \geq 1 \) and \( 1/2 < p < 1 \), such result was obtained by Taberski [40].

Concerning inequalities (1.2) and (1.4), it is known that in \( L_p(\mathbb{T}) \), \( 0 < p < 1 \), these inequalities are not valid in general (see [13], [23], and [7]). In particular, from [23], it follows that for every \( C > 0 \), \( B \in \mathbb{R} \), \( 0 < p < 1 \), and \( n \in \mathbb{N} \), there exists a function \( f_0 \in AC(\mathbb{T}) \) (absolutely continuous functions) such that
\begin{equation}
(1.6) \quad E_n(f_0)_p > C n^B \| f_0 \|_p.
\end{equation}

The first positive results related to inequalities (1.2) and (1.4) have been recently obtained in [21]. In particular, it is proved that if \( \alpha \in \mathbb{N} \) and a function \( f \) is such that \( f^{(\alpha-1)} \in AC(\mathbb{T}) \), then
\begin{equation}
(1.7) \quad E_n(f)_p \leq \frac{C(\alpha, p)}{n^\alpha} \left( E_n(f^{(\alpha)})_p + \left( \frac{1}{n^1 - p} \sum_{\nu=n+1}^{\infty} E_\nu(f^{(\alpha)})_p^{\frac{1}{p}} \right)^{\frac{1}{p}} \right).
\end{equation}
It is also shown the sharpness of the form of this inequality in the sense that \( \nu^{-p} E_p(f^{(\alpha)})_p \) cannot be replaced by \( \nu^{-p-\varepsilon} E_p(f^{(\alpha)})_p \) for any \( \varepsilon > 0 \).

As a rule, problems related to the smoothness of functions in \( L_p, 0 < p < 1 \), are essentially different from the corresponding ones in the spaces \( L_p, p \geq 1 \). Especially this is the case of the derivatives of fractional order. For example, the Bernstein inequality for the fractional derivatives in the case \( 0 < p < 1 \) has the following form (see [1]):

\[
\sup_{T_n \in T_n} \| T_n^{(\alpha)} \|_p \asymp \begin{cases} n^\alpha, & \alpha \in \mathbb{N} \text{ or } \alpha \notin \mathbb{N} \text{ and } \alpha > \frac{1}{p} - 1, \\ n^{\frac{\alpha}{p} - 1} \log \frac{1}{n}, & \alpha = \frac{1}{p} - 1 \notin \mathbb{N}, \\ n^{\frac{\alpha}{p} - 1}, & \alpha \notin \mathbb{N} \text{ and } \alpha < \frac{1}{p} - 1, \end{cases}
\]

where \( \asymp \) is a two-sided inequality with absolute constants independent of \( n \). On the other hand, in the classical case \( p \geq 1 \), we have \( \| T_n^{(\alpha)} \|_p \leq C(\alpha) n^\alpha \| T_n \|_p \) for any \( \alpha > 0 \) (see, e.g., [6], [34, Ch. 4, § 19]). Other interesting "pathological" properties related to the smoothness of functions in the spaces \( L_p, 0 < p < 1 \), can be found, e.g., in [10], [12], [25], [28], [38].

Now, let us consider counterparts of inequalities (1.2) and (1.3) for fractional moduli of smoothness. Recall that the fractional modulus of smoothness of order \( \alpha > 0 \) for a function \( f \in L_p(\mathbb{T}) \) is given by

\[
\omega_\alpha(f, h)_p = \sup_{|\delta| < h} \| \Delta_\delta f \|_p.
\]

For the first time, the modulus (1.8) appeared in 1970’s (see [4, p. 788] and [39]). At present, fractional moduli of smoothness are extensively studied and have several important applications to sharp inequalities of different metrics and embedding theorems (see [18], [32], [33], [35], [36], [43], see also the monograph [27] and the literature therein).

It is known that for any \( f \in L_p(\mathbb{T}), 1 \leq p < \infty \), and \( \alpha, \beta > 0 \), the following two inequalities are fulfilled:

\[
\omega_{\beta+\alpha}(f, \delta)_p \leq C \delta^\beta \omega_\beta(f^{(\alpha)}, \delta)_p,
\]

\[
\omega_\beta(f^{(\alpha)}, \delta)_p \leq C \left( \int_0^\delta \frac{\omega_{\beta+\alpha}(f, t)_p}{t^{\alpha+1}} dt \right)^{\frac{1}{\beta}},
\]

where \( \theta = \min(2, p) \) and the constant \( C \) is independent of \( f \) and \( \delta \). Remark that inequality (1.9) can be found, e.g., in [4]; inequality (1.10) is proved in [35] (see also [14] and [12] for \( \alpha, \beta \in \mathbb{N} \)).

It turns out that in the case \( 0 < p < 1 \), inequalities (1.9) and (1.10) have been studied only for integer parameters \( \alpha \) and \( \beta \). At that, in this case \( (\alpha, \beta \in \mathbb{N} \text{ and } 0 < p < 1) \), the analogue of (1.10) is known and its form coincides with (1.10) (see [12]). In contrast with (1.10), inequality (1.9) is not valid if \( 0 < p < 1 \). As it was mentioned in [29, p. 188], "there is no upper estimate of \( \omega_\beta(f, \delta)_p \) by \( \omega_{k-1}(f', \delta)_p \) in the case \( 0 < p < 1 \)." However, the modulus \( \omega_k(f, \delta)_p \) can be estimated from above by means of a certain integral expression related to (1.7) with the modulus \( \omega_{k-1}(f', \nu^{-1})_p \) instead of the corresponding best approximation (see [21]).

In this paper, we obtain analogous of inequalities (1.4), (1.5), and (1.7) as well as (1.9) and (1.10) for any \( f \in L_p(\mathbb{T}), 0 < p < 1 \), and any admissible parameters \( \alpha, \beta > 0 \) (see Theorems 2.1–2.7).

As an application of inequalities of type (1.5) and (1.7), we derive the direct and inverse theorems of the approximation theory involving the modulus of smoothness \( \omega_\beta(f^{(\alpha)}, \delta)_p \) (see Theorems 2.8, 2.9, and 2.10). At the same time, corresponding analogues of (1.10) and (1.7) are applied to describe the class of functions with the optimal rate of decrease of \( \omega_\beta(f, \delta)_p \) in the case \( 0 < p < 1 \) (see Theorem 2.11).


2. Main results

2.1. Inequalities for the best approximation. We start this section with the following counterpart of inequality (1.2) in the case $0 < p < 1$.

Theorem 2.1. Let $0 < p < 1$, $\alpha > 0$, and let $f$ be such that $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then for any $n \in \mathbb{N}$ we have

$$
E_n(f)_p \leq C \left( E_n(f^{(\alpha)})_p + \left( \frac{1}{n^{1-p}} \sum_{\nu=n+1}^{\infty} E_{\nu}(f^{(\alpha)})^p \right)^{\frac{1}{p}} \right),
$$

where $C$ is a constant independent of $f$ and $n$.

Remark 2.1. 1) Inequality (1.6) implies that Theorem 2.1 is not valid without the second summand in right-hand side of (2.1).

2) In Theorem 2.1, the assumption $f, f^{(\alpha)} \in L_1(\mathbb{T})$ cannot be replaced by the much weaker assumption of existence of the derivative $f^{(\alpha)}$ in the sense of $L_p(\mathbb{T})$. Indeed, let us consider the function $f(x) = \text{sign} \sin(x)$. We have that $f$ has the derivative in the sense of $L_p$, $0 < p < 1$, and $f'(x) = 0$ a.e. Thus, in the case $\alpha = 1$, the right-hand side of (2.1) is zero while $E_n(f)_p > 0$ for all $n \in \mathbb{N}$ which is impossible. Moreover, it follows from the proof of Theorem 2.1 that the convergence of the series in (2.1) implies that $f^{(\alpha)} \in L_1(\mathbb{T})$.

The next theorem gives an inverse inequality to (2.1).

Theorem 2.2. Let $f \in L_p(\mathbb{T})$, $0 < p < 1$, and let for some $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$

$$
\sum_{\nu=1}^{\infty} \nu^{\alpha p-1} E_{\nu}(f)^p < \infty.
$$

Then $f$ has the derivative $f^{(\alpha)}$ in the sense of $L_p$ and for any $n \in \mathbb{N}$

$$
\|f^{(\alpha)} - T_n^{(\alpha)}\|_p \leq C \left( n^\alpha E_n(f)_p + \left( \sum_{\nu=n+1}^{\infty} \nu^{\alpha p-1} E_{\nu}(f)^p \right)^{\frac{1}{p}} \right),
$$

where $T_n \in \mathcal{T}_n$ is such that $\|f - T_n\|_p = E_n(f)_p$ and $C$ is a constant independent of $f$ and $n$.

Remark that in the case $\alpha \in \mathbb{N}$, Theorem 2.2 was proved in [13] while the case $1/2 < p < 1$ and $\alpha \geq 1$ was considered in [40].

Under additional restrictions on the function $f$ in Theorem 2.2, it is possible to obtain an analogue of inequality (2.3) for any $\alpha > 0$.

Theorem 2.3. Let $0 < p < 1$, $\alpha > 0$, and let $f$ be such that $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then for any $n \in \mathbb{N}$ we have

$$
\|f^{(\alpha)} - T_n^{(\alpha)}\|_p \leq C \left( \sigma_{\alpha,p}(n) E_n(f)_p + \left( \sum_{\nu=n+1}^{\infty} (\sigma_{\alpha,p}(\nu))^p \nu^{1-p} E_{\nu}(f)^p \right)^{\frac{1}{p}} \right),
$$

where $T_n \in \mathcal{T}_n$ is such that $\|f - T_n\|_p = E_n(f)_p$, $\sigma_{\alpha,p}(n) = \left\{ \begin{array}{ll}
\frac{n^\alpha}{\nu^{\alpha p-1}}, & \text{if } \alpha \in \mathbb{N} \text{ or } \alpha \notin \mathbb{N} \text{ and } \alpha > \frac{1}{p} - 1, \\
\frac{1}{p} \log p(n + 1), & \alpha = \frac{1}{p} - 1 \notin \mathbb{N}, \\
\frac{n^{\frac{1}{p}-1}}{\nu^{\frac{1}{p}-1}}, & \alpha \notin \mathbb{N} \text{ and } \alpha < \frac{1}{p} - 1,
\end{array} \right.$

and $C$ is a constant independent of $f$ and $n$. 
Combining Theorems 2.1 and 2.3, we derive a positive result about the simultaneous approximation of functions and their derivatives in the spaces $L_p(\mathbb{T})$, $0 < p < 1$.

**Theorem 2.4.** Let $0 < p < 1$, $\alpha > 0$, and let $f$ be such that $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then for any $n \in \mathbb{N}$ we have

$$\|f^{(\alpha)} - T_n^{(\alpha)}\|_p \leq C\rho(n) \left( E_n(f^{(\alpha)})_p + \left( \frac{1}{n^{1-p}} \sum_{\nu=n+1}^{\infty} E_\nu(f^{(\alpha)})_p^p \right)^{\frac{1}{p}} \right),$$

where $T_n \in \mathcal{T}_n$ is such that $\|f - T_n\|_p = E_n(f)_p$,

$$\rho_{\alpha,p}(n) = \begin{cases} 1, & \alpha \in \mathbb{N} \text{ or } \alpha \notin \mathbb{N} \text{ and } \alpha > \frac{1}{p} - 1, \\ \log^p(n+1), & \alpha = \frac{1}{p} - 1 \notin \mathbb{N}, \\ n^{\frac{1}{p} - 1 - \alpha}, & \alpha \notin \mathbb{N} \text{ and } \alpha < \frac{1}{p} - 1, \end{cases}$$

and $C$ is a constant independent of $f$ and $n$.

Using Theorems 2.1, 2.2, and 2.4, we get the following equivalences.

**Corollary 2.1.** Let $0 < p < 1$, $\gamma > 1/p - 1$, $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$, and let $f$ be such that $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then the following assertions are equivalent:

(i) $E_n(f)_p = O(n^{-\alpha - \gamma})$, $n \to \infty$,

(ii) $E_n(f^{(\alpha)})_p = O(n^{-\gamma})$, $n \to \infty$,

(iii) $\|f^{(\alpha)} - T_n^{(\alpha)}\|_p = O(n^{-\gamma})$, $n \to \infty$,

where $T_n \in \mathcal{T}_n$ is such that $\|f - T_n\|_p = E_n(f)_p$.

**2.2. Inequalities for the moduli of smoothness.** In this subsection, similarly to the above considered case of the best approximation, we obtain counterparts of (1.9) and (1.10) for $0 < p < 1$.

**Theorem 2.5.** Let $0 < p < 1$, $\alpha > 0$ and $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$ be such that $\alpha + \beta \in \mathbb{N} \cup (1/p - 1, \infty)$, $r \in \mathbb{N}$, and $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then for any $\delta > 0$

$$\omega_{\beta+\alpha}(f, \delta)_p \leq C\delta^\alpha \left( \omega_\beta(f^{(\alpha)}, \delta)_p + \left( \delta^{1-p} \int_0^\delta \frac{\omega_\beta(f^{(\alpha)}, t)_p}{t^{2-p}} dt \right)^{\frac{1}{p}} \right),$$

where $C$ is a constant independent of $f$ and $\delta$.

In particular, under the conditions of Theorem 2.5, one has

$$\omega_{\beta+\alpha}(f, \delta)_p \leq C\delta^{\alpha+\frac{1}{p} - 1} \left( \int_0^\delta \frac{\omega_\beta(f^{(\alpha)}, t)_p}{t^{2-p}} dt \right)^{\frac{1}{p}}.$$

A converse result is given by the following theorem.

**Theorem 2.6.** Let $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $\alpha, \beta \in \mathbb{N} \cup (1/p - 1, \infty)$. Then

$$\omega_\beta(f^{(\alpha)}, \delta)_p \leq C \left( \int_0^\delta \frac{\omega_{\beta+\alpha}(f, t)_p}{t^{p \alpha + 1}} dt \right)^{\frac{1}{p}},$$

where $C$ is some constant independent of $f$ and $\delta$. Inequality (2.7) means that if the right-hand is finite, then there exists $f^{(\alpha)}$ in the sense (1.1), $f^{(\alpha)} \in L_p(\mathbb{T})$, and (2.7) holds.

Under additional restrictions on $f$, we obtain an analogue of inequality (2.7) for any $\alpha > 0$. 
Theorem 2.7. Let $0 < p < 1$, $\alpha > 0$ and $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$ be such that $\alpha + \beta \in \mathbb{N} \cup (1/p - 1, \infty)$, and $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then for any $\delta > 0$

$$\omega_\beta(f^{(\alpha)}, \delta)_p \leq C \left( \int_0^\delta \frac{\omega_{\beta + \alpha}(f, t)_p}{t^{\alpha+p} \sigma_{\alpha,p}(\frac{1}{t})} dt \right)^{\frac{1}{p}},$$

where $\sigma_{\alpha,p}(\cdot)$ is defined in Theorem 2.3 and $C$ is some constant independent of $f$ and $\delta$.

The next corollary easily follows from Theorems 2.5 and 2.6.

Corollary 2.2. Let $0 < p < 1$, $\gamma > 1/p - 1$, $\alpha, \beta \in \mathbb{N} \cup (1/p - 1, \infty)$, and let $f$ be such that $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then the following assertions are equivalent:

(i) $\omega_{\alpha+\beta}(f, \delta)_p = O(\delta^{\alpha+\gamma})$, $\delta \to 0$,

(ii) $\omega_\beta(f^{(\alpha)}, \delta)_p = O(\delta^{\gamma})$, $\delta \to 0$.

2.3. The direct and inverse approximation theorems. Let us recall two basic inequalities in approximation theory (the so-called direct and inverse approximation theorems).

Proposition 2.1. (See [32].) Let $f \in L_p(\mathbb{T})$, $0 < p < 1$, $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$, and $n \in \mathbb{N}$. Then

$$E_n(f)_p \leq C\omega_\beta \left( f, \frac{1}{n} \right)_p,$$

(2.8)

$$\omega_\beta \left( f, \frac{1}{n} \right)_p \leq C \left( \sum_{\nu=0}^n (\nu+1)^{\beta-1} E_\nu(f)_p \right)^{\frac{1}{p}},$$

(2.9)

where $C$ is a constant independent of $n$ and $f$.

Remark that in the case $\beta \in \mathbb{N}$ inequality (2.8), which is also called the Jackson type inequality, was proved in [37] (see also [38] and [13]) and inequality (2.9) was proved in [13] (see also [40] concerning the case $\beta \geq 1$ and $1/2 \leq p < 1$).

Using Theorem 2.1, it is not difficult to obtain the following extensions of inequality (2.8) involving fractional derivatives of the function $f$.

Theorem 2.8. Let $0 < p < 1$, $\alpha > 0$, $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$, and let $f$ be such that $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then for any $n \in \mathbb{N}$ we have

$$E_n(f)_p \leq C \left( \int_0^{1/n} \frac{1}{t^{\alpha+p}} \omega_\beta(f(t), t)_p dt \right)^{\frac{1}{p}},$$

(2.10)

where $C$ is a constant independent of $f$ and $n$.

Note that in the case $1 \leq p < \infty$, inequality (2.10) holds in the following form:

$$E_n(f)_p \leq C \left( \frac{1}{n^{\alpha+p+1}} \int_0^{1/n} \omega_\beta(f(t), t)_p dt \right)^{\frac{1}{p}},$$

(2.11)

Sometimes (2.11) is called the second Jackson inequality (see, e.g., [44, p. 260]). Let us again emphasize that this inequality is not valid if $0 < p < 1$ (see (1.6)).

In the next theorem, using Theorem 2.2 and (2.9), we obtain a converse inequality to (2.10).

Theorem 2.9. Let $f \in L_p(\mathbb{T})$, $0 < p < 1$, $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$, and let for some $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$

$$\sum_{\nu=1}^{\infty} \nu^{\alpha p-1} E_\nu(f)_p < \infty.$$

(2.12)
Then $f$ has the derivative $f^{(\alpha)}$ in the sense of $L_p$ and for any $n \in \mathbb{N}$

\[(2.13) \quad \omega_{\beta}\left(f^{(\alpha)}, \frac{1}{n}\right)_p \leq C \left(\frac{1}{n^{\beta p}} \sum_{\nu=0}^{n} (\nu + 1)^{(\alpha + \beta)p - 1} E_{\beta}(f)_p^p + \sum_{\nu=n+1}^{\infty} \nu^{\beta p - 1} E_{\beta}(f)_p^p \right)^{\frac{1}{p}},\]

where $C$ is a constant independent of $f$ and $n$.

Remark that in the case $1 \leq p < \infty$, inequalities of type (2.13) can be found in [45] and [41, p. 154] (see also the general case in [35]).

Similarly to Theorem 2.3, under additional restrictions on the function $f$, we obtain the following extension of Theorem 2.9 to the case $\alpha \leq 1/p - 1$.

**Theorem 2.10.** Let $0 < p < 1$, $\alpha > 0$, $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$, and let $f$ be such that $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then

\[
\omega_{\beta}\left(f^{(\alpha)}, \frac{1}{n}\right)_p \leq C \left(\frac{1}{n^{\beta p}} \sum_{\nu=0}^{n} (\sigma_{\alpha,p}(\nu + 1))^{p} (\nu + 1)^{\beta p - 1} E_{\beta}(f)_p^p \right. \\
+ \left. \sum_{\nu=n+1}^{\infty} (\sigma_{\alpha,p}(\nu))^{p} \nu^{-1} E_{\beta}(f)_p^p \right)^{\frac{1}{p}},
\]

where $\sigma_{\alpha,p}(\cdot)$ is defined in Theorem 2.3 and $C$ is a constant independent of $f$ and $n$.

Recall that the assertions of Proposition 2.1 imply that for $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$ and $0 < \gamma < \beta$ the condition $\omega_{\beta}(f, \delta)_p = \mathcal{O}(\delta^\gamma)$ is equivalent to $E_n(f)_p = \mathcal{O}(n^{-\gamma})$. Combining Theorems 2.8 and 2.9, we obtain an analogue of this equivalence involving the fractional derivative of a function $f$.

**Corollary 2.3.** Let $0 < p < 1$, $\alpha, \beta \in \mathbb{N} \cup (1/p - 1, \infty)$, $1/p - 1 < \gamma < \beta$, and let $f$ be such that $f, f^{(\alpha)} \in L_1(\mathbb{T})$. Then the following assertions are equivalent:

(i) $E_n(f)_p = \mathcal{O}(n^{-\alpha - \gamma})$, $n \to \infty$,

(ii) $\omega_{\beta}(f^{(\alpha)}, \delta)_p = \mathcal{O}(\delta^\gamma)$, $\delta \to 0$.

2.4. On decreasing of the fractional modulus of smoothness. The following inequality plays a crucial role in the proofs of the main results of this paper:

\[(2.14) \quad \omega_{\beta}(f, \lambda \delta)_p \leq C(p, \beta)(1 + \lambda)^{\beta + \frac{1}{p} - 1} \omega_{\beta}(f, \delta)_p, \quad \lambda, \delta > 0,
\]

where $\beta \in \mathbb{N} \cup (1/p_1 - 1, \infty)$, $p_1 = \min(p, 1)$ (see [4] for the case $p \geq 1$ and [32] for the case $0 < p < 1$). Inequality (2.14) implies that the optimal rate of decrease of the modulus of smoothness $\omega_{\beta}(f, h)_p$ as $h \to 0$ is $\mathcal{O}(h^{\beta + 1/p_1 - 1})$, that is if $\omega_{\beta}(f, h)_p = o(h^{\beta + 1/p_1 - 1})$, then $f \equiv \text{const}$ (see also Proposition 5.1 in [5]). It arises a natural question about characterization of the class of functions $f \in L_p(\mathbb{T})$, $0 < p < \infty$, such that

\[
\omega_{\beta}(f, h)_p = \mathcal{O}(h^{\beta + 1/p_1 - 1}) \quad \text{as} \quad h \to 0.
\]

The first characterization of this class was derived by Hardy and Littlewood [15] in the case $\beta = 1$ and $1 \leq p < \infty$. Their result was extended to the moduli of smoothness of integral order in [2] (see also [9, Ch. 1, §9] and [46, Theorem 4.6.14]) and to the fractional moduli of smoothness in [5]. In particular, the following proposition was proved by Butzer and Westphal [5].

**Proposition 2.2.** Let $f \in L_p(\mathbb{T})$, $1 \leq p < \infty$, and $\beta > 0$. Then $\omega_{\beta}(f, h)_p = \mathcal{O}(h^\beta)$ if and only if $f$ can be corrected on a set of measure zero to be a function $g$ such that $g^{(\beta)} \in L_p(\mathbb{T})$ for $1 < p < \infty$ and $g^{(\beta - 1)} \in BV(\mathbb{T})$ (functions of bounded variation on $\mathbb{T}$) for $p = 1$. 
In the spaces $L_p(\mathbb{T})$, $0 < p < 1$, the class of functions with the optimal rate of decrease of the modulus of smoothness has different nature. Indeed, it is easy to see that for any step function $f$ one has $\omega_1(f, h)_p = O(h^{1/p})$. A complete description of such functions was obtained by Krotov [25].

**Proposition 2.3.** (See [25].) Let $f \in L_p(\mathbb{T})$, $0 < p < 1$. Then $\omega_1(f, h)_p = O(h^{1/p})$ if and only if $f$ can be corrected on a set of measure zero to be a function $g$ such that $g(x) = d_0 + \sum_{x_k < x} d_k$, where $\sum_k |d_k|^p < \infty$ and $\{x_k\}$ is a set of pairwise distinct point from $[0, 2\pi)$.

See also in [3] and [16] analogues of Proposition 2.3 with the moduli of smoothness of arbitrary integer order.

It is worth mentioning the following unusual property of the modulus of continuity: if $f \in AC(\mathbb{T})$ (absolutely continuous functions on $\mathbb{T}$) and

$$\omega_1(f, h)_p = o(h) \quad \text{as} \quad h \to 0$$

for some $0 < p < 1$, then $f \equiv \text{const} \text{ a.e. on } \mathbb{T}$ (see Lemma 1.5 in [38]).

Using Theorem 2.6, it is easy to extend this property to the moduli of smoothness of fractional order. In particular, we have the following result:

**Proposition 2.4.** Let $0 < p < 1$, $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$, $f^{(\beta-1)} \in AC(\mathbb{T})$, and

$$\omega_\beta(f, \delta)_p = o(\delta^\beta), \quad \delta \to 0,$$

then $f \equiv \text{const} \text{ a.e. on } \mathbb{T}$.

In the next theorem, we generalize Proposition 2.2 to the case $0 < p < 1$ and Proposition 2.3 to the fractional moduli of smoothness of arbitrary order $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$.

**Theorem 2.11.** Let $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $\beta \in \mathbb{N} \cup (1/p - 1, \infty)$. Then the following assertions are equivalent:

1. $\omega_\beta(f, h)_p = O(h^{\beta+1/p-1})$ as $h \to 0$,
2. $f \in L_1(\mathbb{T})$ and it can be corrected on a set of measure zero to be a function $g$ such that $g^{(\beta-1)}(x) = d_0 + \sum_{x_k < x} d_k$, where $\sum_k |d_k|^p < \infty$ and $\{x_k\}$ is a set of pairwise distinct point from $[0, 2\pi)$.

### 3. Auxiliary results

**3.1. Properties of the fractional moduli of smoothness.** First of all, we note that in the case $0 < p < 1$, considering the fractional derivatives in the sense of $L_p$ and the corresponding moduli of smoothness, we restrict ourselves to the parameter $\alpha$ belonging to the set $\mathbb{N} \cup (1/p - 1, \infty)$. This restriction is natural since for $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$ we always have

$$\|\Delta_\beta^\alpha f\|_p^p \leq \sum_{\nu=0}^\infty \left(\frac{\alpha}{\nu}\right)\|f\|_p^p \leq C(\alpha, p)\|f\|_p^p. \quad (3.1)$$

The last inequality follows from the fact that $\binom{\alpha}{\nu} = O(\nu^{-\alpha-1})$ as $\nu \to \infty$ (see, e.g., [34, Ch. 1, §1]).

Let us recall two basic properties of the fractional moduli of smoothness. For $f \in L_p(\mathbb{T})$, $0 < p \leq \infty$, and $\alpha \in (1/p_1 - 1, \infty) \cup \mathbb{N}$, we have

$$\omega_\alpha(f + g, \delta)^p_\rho \leq \omega_\alpha(f, \delta)^p_\rho + \omega_\alpha(g, \delta)^p_\rho, \quad \delta > 0, \quad (3.2)$$

$$\omega_\alpha(f, \delta)_p \leq C(p, \alpha)\|f\|_p, \quad \delta > 0, \quad (3.3)$$

where $p_1 = \min(p, 1)$. Inequality (3.2) is obvious while inequality (3.3) can be derived from (3.1).
It is well known (see [4]) that if $1 \leq p \leq \infty$, the modulus of smoothness is equivalent to the $K$-functional given by

$$K_\alpha(f, \delta)_p = \inf_{g^{(\alpha)} \in L_p(\mathbb{T})} \left( \|f - g\|_p + \delta^\alpha \|g^{(\alpha)}\|_p \right),$$

that is,

$$\omega_\alpha(f, \delta)_p \asymp K_\alpha(f, \delta)_p, \quad \delta > 0.$$ 

This equivalence fails for $0 < p < 1$ since $K_\alpha(f, \delta)_p \equiv 0$ (see [10]). A suitable substitute for the $K$-functional for $p < 1$ is the realization concept given by

$$\mathcal{R}_\alpha(f, \delta)_p = \inf_{T \in \mathcal{T}_{\alpha/\delta}} \left( \|f - T\|_p + \delta^\alpha \|T^{(\alpha)}\|_p \right).$$

Let us recall some properties of the realization $\mathcal{R}_\alpha(f, \delta)_p$.

**Lemma 3.1.** Let $f \in L_p(\mathbb{T})$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N} \cup (1/p_1 - 1, \infty)$. Then

$$\mathcal{R}_\alpha(f, \delta)_p \asymp \omega_\alpha(f, \delta)_p, \quad \delta > 0,$$

where $\asymp$ is a two-sided inequality with absolute constants independent of $f$ and $\delta$.

Remark that in the case $\alpha \in \mathbb{N}$, Lemma 3.1 was proved in [10]; the case $\alpha > 1/p_1 - 1$ was considered in [18] and [35].

The next lemma gives an analogue of inequality (2.14) for the realizations of $K$-functional.

**Lemma 3.2.** (See [31, Theorem 4.22], [32]). Let $f \in L_p(\mathbb{T})$, $0 < p \leq \infty$, and $\alpha > 0$. Then

$$\mathcal{R}_\alpha(f, \lambda \delta)_p \leq C(1 + \lambda)^{\alpha + \frac{1}{p_1} - 1} \mathcal{R}_\alpha(f, \delta)_p, \quad \lambda, \delta > 0,$$

where $C$ is a constant, which depends only on $p$ and $\alpha$.

Note that in above inequality in contrast with (2.14), we do not assume that $\alpha > 1/p - 1$ in the case $0 < p < 1$.

### 3.2. Inequalities for trigonometric polynomials.

We need the following three important results for trigonometric polynomials in $L_p$. The first one is the Nikolskii–Stechkin type inequality (see [10] for the case $\alpha \in \mathbb{N}$ and [17] for the case $\alpha > 0$).

**Lemma 3.3.** Let $0 < p < 1$, $n \in \mathbb{N}$, $0 < h \leq \pi/n$, and $\alpha > 0$. Then for any trigonometric polynomial $T_n \in \mathcal{T}_n$, we have

$$h^\alpha \|T_n^{(\alpha)}\|_p \asymp \|\Delta_h^\alpha T_n\|_p,$$

where $\asymp$ is a two-sided inequality with absolute constants independent of $T_n$ and $h$. Moreover, if $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$ and $T_n$ is a polynomial of the best approximation of $f \in L_p(\mathbb{T})$, then

$$\|\Delta_h^\alpha T_n\|_p \leq C \omega_\alpha \left( f, \frac{1}{n} \right)_p,$$

where $C$ is a constant independent of $T_n$, $h$, and $f$.

The second result is the well-known Nikolskii inequality of different metrics (see, e.g., [26, p. 133] and [9, Ch. 4, § 2]).

**Lemma 3.4.** Let $0 < p < q \leq \infty$. Then for any $T_n \in \mathcal{T}_n$, $n \in \mathbb{N}$, one has

$$\|T_n\|_q \leq C_n^{\frac{1}{p} - \frac{1}{q}} \|T_n\|_p,$$

where $C$ is a constant independent of $T_n$.

The third result is the Bernstein type inequality involving the Weyl fractional derivative (see [1]).
Lemma 3.5. Let $0 < p < 1$. Then
\[
\sup_{T_n \in \mathcal{T}_n, \|T_n\|_p \leq 1} \|T_n^{(\alpha)}\|_p \asymp \begin{cases} 
\frac{n^\alpha}{|\alpha|}, & \alpha \in \mathbb{Z}_+ \text{ or } \alpha \notin \mathbb{Z}_+ \text{ and } \alpha > \frac{1}{p} - 1, \\
\frac{n^{\frac{1}{p}-1}}{\log n}, & \alpha = \frac{1}{p} - 1 \notin \mathbb{Z}_+, \\
\frac{n^{\frac{1}{p}-1}}{\log n}, & \alpha \notin \mathbb{Z}_+ \text{ and } \alpha < \frac{1}{p} - 1,
\end{cases}
\]
where $\asymp$ is a two-sided inequality with absolute constants independent of $n$.

3.3. Approximation of a function and its derivatives. In the spaces $L_p$ with $p \geq 1$, the following fact is well-known: if a sequence of functions $\{\varphi_n\}_{n=1}^\infty \subset L_p$ is such that $\varphi_n^{(r-1)} \in AC$, $n \in \mathbb{N}$, and for some $f, g \in L_p$ one has
\[
\|f - \varphi_n\|_{L_p} + \|g - \varphi_n^{(r)}\|_{L_p} \to 0 \quad \text{as} \quad n \to \infty,
\]
then (in the sense of distribution) $g = f^{(r)}$ (see [26, Ch. 4]).

In the case $0 < p < 1$, this result does not hold in general. In particular, for $f_0(x) = x$ there exists a sequence of functions $\varphi_n \in AC[0,1]$, $n \in \mathbb{N}$, such that $\varphi_n \to f_0$ as $n \to \infty$ in $L_p[0,1]$, but $\|\varphi_n^{(r)}\|_{L_p[0,1]} \to 0$ as $n \to \infty$ (see [12]). This is an undesirable property of the spaces $L_p$, $0 < p < 1$. However, as it is shown in Lemma 3.8 below, under certain additional restrictions on $f$ and $\varphi_n$, this feature can be fixed (see also [12], in which the case of the derivatives of integer order was considered).

To prove the main result of this subsection (see Lemma 3.8), we need the following two lemmas. As usual, the Fourier transform of a function $f \in L_1(\mathbb{R})$ is denoted by
\[
\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ixy}dx.
\]

Lemma 3.6. (See [46, 4.1.1].) Let $0 < p \leq 1$, a function $\phi \in C(\mathbb{R})$ have a compact support, and $\hat{\phi} \in L_p(\mathbb{R})$. Then
\[
\sup_{h > 0} h^{1-\frac{1}{r}} \|\Phi_h\|_{L_p(\mathbb{T})} = \sqrt{2\pi} \|\hat{\phi}\|_{L_p(\mathbb{R})},
\]
where
\[
\Phi_h(x) = \sum_{k=-\infty}^{\infty} \hat{\phi}(hk) e^{ikx}.
\]

In the case $p = 1$, the next lemma can be found in [20]; the general case see in [19].

Lemma 3.7. Let $0 < p \leq 1$, $1 < q < \infty$, $1 < r < \infty$, $s > 1/p - 1 + 1/r$, $s \in \mathbb{N}$, let a function $f$ be such that $f \in C(\mathbb{R}) \cap L_1(\mathbb{R})$, $\lim_{|x| \to \infty} f(x) = 0$, and $\hat{f} \in L_1(\mathbb{R})$. Suppose also that $f \in L_q(\mathbb{R})$, $f^{(s)} \in L_r(\mathbb{R})$, and
\[
\frac{1 - \theta}{q} + \frac{\theta}{r} > \frac{1}{2}, \quad \theta = \frac{1}{s} \left( \frac{1}{p} - \frac{1}{2} \right).
\]
Then
\[
\|\hat{f}\|_{L_p(\mathbb{R})} \leq C \|f\|_{L_q(\mathbb{R})} \|f^{(s)}\|_{L_r(\mathbb{R})},
\]
where $C$ is a constant independent of $f$.

Now, we are ready to formulate and prove a key result for obtaining Theorems 2.2 and 2.6.

Lemma 3.8. Let $f \in L_p(\mathbb{T})$, $0 < p < 1$, $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$, and $T_n \in \mathcal{T}_n$, $n \in \mathbb{N}$, be such that
\[
\|f - T_n\|_p = o\left( \frac{1}{n^\alpha} \right) \quad \text{and} \quad \|g - T_n^{(\alpha)}\|_p = o(1) \quad \text{as} \quad n \to \infty.
\]
Then $f^{(\alpha)} = g$, i.e. $g$ satisfies (1.1).
Proof. For any sufficiently small $\varepsilon > 0$, we choose $n_0 = n_0(\varepsilon)$ such that for any $n \geq n_0$ one has

\begin{equation}
\|f - T_n\|_p \leq \frac{\varepsilon}{n^\alpha} \quad \text{and} \quad \|g - T_n^{(\alpha)}\|_p \leq \varepsilon.
\end{equation}

Let $h$ be such that $\varepsilon^n - 1 \leq h \leq 2\varepsilon^n - 1$, where $0 < \lambda < \alpha^{-1}$. We have

\begin{equation}
\left\| \frac{\Delta^\alpha_h f - g}{h^\alpha} \right\|_p \leq \left\| \frac{\Delta^\alpha_h (f - T_n)}{h^\alpha} \right\|_p
\end{equation}

\begin{equation}
+ \left\| \frac{\Delta^\alpha_h T_n}{h^\alpha} - T_n^{(\alpha)} \right\|_p + \|g - T_n^{(\alpha)}\|_p
= J_1 + J_2 + J_3.
\end{equation}

Using (3.3) and (3.4), we get

\begin{equation}
J_1 = \left\| \frac{\Delta^\alpha_h (f - T_n)}{h^\alpha} \right\|_p \leq C h^{-\alpha p} \|f - T_n\|_p \leq C \varepsilon^{(1 - \lambda \alpha)p},
\end{equation}

\begin{equation}
J_3 \leq \varepsilon^p.
\end{equation}

Let us consider $J_2$. Set

\begin{equation}
T_{n, h, \alpha}(t) = \frac{\Delta^\alpha_h T_n(t)}{h^\alpha} - T_n^{(\alpha)}(t).
\end{equation}

It is easy to see (here and throughout we use the principal branch of the logarithm) that

\begin{equation}
T_{n, h, \alpha}(t) = \sum_{k=-n}^{n} (ik)^\alpha \left( \frac{1 - e^{-ikt}}{ikt} \right)^\alpha c_k e^{ikt},
\end{equation}

where \(\{c_k\}_{k=-n}^{n}\) are the coefficients of $T_n$. We also have the following equality

\begin{equation}
T_{n, h, \alpha}(t) = (K_{h, \alpha} * T_n^{(\alpha)})(t),
\end{equation}

where

\begin{equation}
K_{h, \alpha}(t) = \sum_{k \in \mathbb{Z}} \eta_{\alpha, \varepsilon}(hk) e^{ikt}, \quad \eta_{\alpha, \varepsilon}(x) = \left( \frac{1 - e^{-ix}}{ix} \right)^\alpha v \left( \frac{x}{2\varepsilon^\lambda} \right),
\end{equation}

and the function $v$ is such that $|v(x)| \leq 1$, $v \in C^\infty(\mathbb{R})$, $v(x) = 1$ for $|x| \leq 1$ and $v(x) = 0$ for $|x| \geq 2$.

Note that $K_{h, \alpha}(x) T_n^{(\alpha)}(t - x)$ is a trigonometric polynomial of order at most $4n$ in variable $x$. Thus, using Lemma 3.4, we obtain

\begin{equation}
|T_{n, h, \alpha}(t)|^p \leq \left( \frac{1}{2\pi} \int_T |K_{h, \alpha}(x) T_n^{(\alpha)}(t - x)| dx \right)^p
\leq C n^{1-p} \int_T |K_{h, \alpha}(x) T_n^{(\alpha)}(t - x)|^p dx.
\end{equation}

Integrating the above inequality by $t$ and applying Fubini’s theorem, we get

\begin{equation}
\|T_{n, h, \alpha}\|_p \leq C n^{1-p} \|K_{h, \alpha}\|_p T_n^{(\alpha)}\|_p.
\end{equation}

Now, let us consider the function $\eta_{\alpha, \varepsilon}$. Noting that for sufficiently small $x$

\begin{equation}
\frac{1}{2} \leq \left| \frac{e^{-ix} - 1}{ix} \right| \leq 1,
\end{equation}

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we derive for any $s = 0, 1, \ldots$ the following estimates
\[
|\eta_{\alpha,\varepsilon}^{(s)}(x)| = \left| \sum_{\nu=0}^{s} \left( \frac{e^{-ix} - 1}{ix} \right)^{\alpha} \left( v(x) \right)^{(s-\nu)} \right|
\leq \frac{1}{\varepsilon^{s \lambda}} \sum_{\nu=0}^{s} C_{\nu, s, \alpha} |v^{(s-\nu)}(x/2\varepsilon^\lambda)|.
\]
Thus, it is easy to see that for any $1 < q < \infty$, $1 < r < \infty$, and $s \in \mathbb{N}$, we have
\[
\|\eta_{\alpha,\varepsilon}\|_{L_q(\mathbb{R})} \leq C \varepsilon^{\frac{\lambda}{q}} \quad \text{and} \quad \|\eta_{\alpha,\varepsilon}^{(s)}\|_{L_r(\mathbb{R})} \leq C \varepsilon^{\frac{\lambda}{r}-\lambda s}.
\]
Thus, by Lemmas 3.6 and 3.7, we derive
\[
\tag{3.10}
n\varepsilon^{-\lambda} K_{h,\alpha} \leq C \varepsilon^{\frac{\lambda}{p}} \|K_{h,\alpha}\|_p \leq C \varepsilon^{\frac{\lambda}{p}-1} \|\eta_{\alpha,\varepsilon}\|_{L_p(\mathbb{R})} \leq C \varepsilon^{\lambda(1-\theta)\frac{q}{2}+\theta\frac{r}{2}} = C \varepsilon^\gamma.
\]
It is obvious that we can choose $q$ and $r$ such that $\gamma = \lambda((1 - \theta)/q + \theta/r - 1/2) > 0$. Then, using (3.9) and (3.10), we get $\|T_{n,h,\alpha}\|_p \leq C \varepsilon^\gamma \|T_n^{(\alpha)}\|_p$. From this inequality, taking into account (3.4) and (3.8), we obtain
\[
\tag{3.11}
J_2 \leq C \varepsilon^\gamma (\varepsilon^p + \|g\|_p^p).
\]
Finally, combining inequalities (3.5)–(3.7) and (3.11), we derive
\[
\left\| \frac{\Delta_{h}^\alpha f}{h^\alpha} - g \right\|_p \leq C(\varepsilon^{1-\lambda \alpha} + \varepsilon^\gamma \|g\|_p + \varepsilon).
\]
The last inequality implies that $f^{(\alpha)} = g$ in the sense (1.1). \qed

The next proposition shows that conditions of Lemma 3.8 are sharp.

**Proposition 3.1.** Let $0 < p < 1$ and $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$. Then there exists $f_\alpha \in L_1(\mathbb{T})$ and a sequence of polynomials $T_{n,\alpha} \in \mathcal{T}_n$, $n \in \mathbb{N}$, such that $f_\alpha^{(\alpha)}(x) \equiv \text{const} \neq 0$ a.e. on $[0, \pi)$ and
\[
\|f_\alpha - T_{n,\alpha}\|_p = O\left( \frac{1}{n^\alpha} \right), \quad \text{but} \quad \|T_{n,\alpha}^{(\alpha)}\|_p \to 0 \quad \text{as} \quad n \to \infty,
\]
where $\asymp$ is a two-sided inequality with positive constants independent of $n$.

**Proof.** We will use some ideas from [22]. Let $r \in \mathbb{N}$. Set
\[
f_r(x) = \begin{cases} x^r, & x \in [0, \pi), \\ (2\pi - x)^r, & x \in [\pi, 2\pi], \end{cases}
\]
and
\[
g_{n,r}(x) = \begin{cases} \frac{k}{n} x^{r-1}, & \frac{k}{n} \leq x < \frac{k+1}{n} - \frac{1}{n^{r+1}}, \\ \frac{k}{n} x^{r-1} + x^{r-1} \left( x - \frac{k+1}{n} + \frac{1}{n^{r+1}} \right), & \frac{k+1}{n} - \frac{1}{n^{r+1}} \leq x < \frac{k+1}{n}, \end{cases}
\]
for $k = 0, 1, \ldots, n - 1$, $g_{n,r}(x) = 1 - g_{n,r}(x-1)$ for $1 < x \leq 2$, and
\[
\varphi_{n,r}(x) = \pi g_{n,r} \left( \frac{x}{\pi} \right) \quad \text{for} \quad x \in [0, 2\pi).
\]
We need the following inequalities
\[
\tag{3.12}
\omega_r(\varphi_{n,r}, n^{-1})_q \leq C n^{-r} \|\varphi_{n,r}^{(r)}\|_q \leq C n^{-r} n^{-\frac{r}{q}+1}, \quad 0 < q < \infty.
\]
The first inequality can be found in [24], the second one can be verified by simple calculation. It is also easy to see that

$$\|f_r - \varphi_{n,r}\|_p = O(n^{-r}) \tag{3.13}$$

Let $T_{n,r} \in T_n$ be a polynomial of the best approximation of $\varphi_{n,r}$ in $L_p$. Using (2.8), (3.13), and (3.12), we obtain

$$\|f_r - T_{n,r}\|_p \leq C(\|f_r - \varphi_{n,r}\|_p + \|\varphi_{n,r} - T_{n,r}\|_p) \leq C(n^{-r} + \omega_r(\varphi_{n,r}, n^{-1})_p) \leq Cn^{-r}. \tag{3.14}$$

At the same time, by Lemma 3.3 and (3.12), one has

$$\|T_{n,r}^{(r)}\|_p \leq Cn^r \omega_r(\varphi_{n,r}, n^{-1})_p \leq Cn^{1-\frac{1}{r}}. \tag{3.15}$$

Thus, we have proved the proposition in the case $\alpha = r \in \mathbb{N}$.

Now let $\alpha \notin \mathbb{N}$. Choose $r \in \mathbb{N}$ such that $r > \alpha$ and denote $f_{\alpha} = f_r^{(r-\alpha)}$ and $T_{n,\alpha} = T_{n,r}^{(r-\alpha)}$. Note that if $f \in L_p(\mathbb{T})$, $\gamma > \beta > 1/p - 1$, and $T_n \in T_n$, $n \in \mathbb{N}$, are such that

$$\|f - T_n\|_p = O(n^{-\gamma}) \quad \text{as} \quad n \to \infty, \tag{3.16}$$

then $f$ has the derivative $f^{(\beta)}$ in the sense of $L_p$ and

$$\|f^{(\beta)} - T_{n,\alpha}\|_p = O(n^{-(\gamma - \beta)}) \quad \text{as} \quad n \to \infty. \tag{3.17}$$

This can be verified repeating the proof of Theorem 2.2 presented below. Thus, using (3.14) and taking into account (3.16) and (3.17), we obtain

$$\|f_{\alpha} - T_{n,\alpha}\|_p = \|f_r^{(r-\alpha)} - T_{n,r}^{(r-\alpha)}\|_p \leq Cn^{-\alpha}. \tag{3.18}$$

At the same time, by (3.15), we get

$$\|T_{n,\alpha}\|_p = \|T_{n,r}^{(r)}\|_p \leq Cn^{1-\frac{1}{r}}. \tag{3.19}$$

The last two inequalities prove the proposition. \hfill \Box

### 4. Proofs of the main results

**Proof of Theorem 2.1.** It is clear that we can assume that

$$\sum_{\nu=1}^{\infty} \nu^{-p} E_{\nu}(f^{(\alpha)})_p < \infty. \tag{4.1}$$

Let $U_n \in T_n$ and $T_n \in T_n$, $n \in \mathbb{N}$, be such that

$$\|f^{(\alpha)} - U_n\|_p = E_n(f^{(\alpha)})_p \tag{4.2}$$

and

$$T_n^{(\alpha)}(x) = U_n(x) - \frac{1}{2\pi} \int_0^{2\pi} U_n(x) dx. \tag{4.3}$$

Choosing $m \in \mathbb{N}$ such that $2m-2 \leq n < 2m-1$, we derive

$$E_n(f)_p \leq E_n(T_{2m})_p + E_n(f - T_{2m})_p. \tag{4.4}$$

Let us estimate $E_n(T_{2m})_p$. Set

$$\tau_u(x) = \tau_u,2m,n(x) = \Delta_u^1(T_{2m}(x) - T_n(x)), \quad u > 0.$$
Applying (2.8) with $\beta = \alpha + r$, $r > 1/p$, and Lemma 3.1, we obtain
\begin{equation}
E_n(T_{2m})_p = E_n(T_{2m} - T_n)_p \leq C \omega_{\alpha + r}(T_{2m} - T_n, n^{-1})_p
\end{equation}

\begin{equation}
= C \sup_{0 < h \leq n^{-1}} \|\Delta_h^{a + r - 1} \tau_h\|_p \leq C \sup_{0 < h \leq n^{-1}} \|\Delta_h^{a + r - 1} \tau_u\|_p
\end{equation}

\begin{equation}
\leq C \sup_{u > 0} \omega_{\alpha + r - 1}(\tau_u, n^{-1})_p \leq C \sup_{u > 0} \mathcal{R}_{\alpha + r - 1}(\tau_u, n^{-1})_p.
\end{equation}

Next, let $V_n \in T_n$, $n \in \mathbb{N}$, be such that
\begin{equation}
\|\tau_u - V_n\|_p + n^{-\alpha} \|V_n^{(n)}\|_p \leq 2\mathcal{R}_\alpha(\tau_u, n^{-1})_p.
\end{equation}

Then, by the definition of the realization $\mathcal{R}_\alpha$, using Lemmas 3.5 and 3.2, inequalities (4.4) and (3.3), and taking into account that
\begin{equation}
\mathcal{R}_{\alpha + r - 1}(\tau_u, n^{-1})_p \leq \|\tau_u - V_n\|_p + n^{-(\alpha + r - 1)} \|V_n(\alpha + r - 1)^{n}\|_p
\end{equation}

\begin{equation}
\leq \|\tau_u - V_n\|_p + Cn^{-\alpha} \|V_n^{(n)}\|_p \leq C\mathcal{R}_\alpha(\tau_u, n^{-1})_p
\end{equation}

\begin{equation}
\leq C\mathcal{R}_\alpha(\tau_u, 2^{-m})_p \leq C2^{-ma} \|\tau_u\|_p
\end{equation}

\begin{equation}
= C2^{-ma} \|\Delta_1^{n}(T_{2m}^{(n)} - T_n^{(n)})\|_p = C2^{-ma} \|\Delta_1^{n}(U_{2m} - U_n)\|_p
\end{equation}

\begin{equation}
\leq Cn^{-\alpha} \|U_{2m} - U_n\|_p \leq Cn^{-\alpha} E_n(f^{(n)})_p.
\end{equation}

Combining (4.3) and (4.5), we derive
\begin{equation}
E_n(T_{2m})_p \leq Cn^{-\alpha} E_n(f^{(n)})_p.
\end{equation}

Now, let us consider the second term in the right-hand side of (4.2). For any $N > m$, we have
\begin{equation}
E_n(f - T_{2m})_p^p \leq \sum_{m=m}^{N-1} E_n(T_{2m+1} - T_{2m})_p^p + E_n(f - T_{2N})_p^p.
\end{equation}

Using Hölder’s inequality and (1.2), we obtain
\begin{equation}
E_n(f - T_{2N})_p \leq CE_n(f - T_{2N})_1 \leq Cn^{-\alpha} E_n(f^{(n)} - T_{2^N}^{(n)})_1
\end{equation}

\begin{equation}
= Cn^{-\alpha} E_n(f^{(n)} - U_{2N})_1 \leq Cn^{-\alpha} \|f^{(n)} - U_{2N}\|_1.
\end{equation}

Let us show that $\|f^{(n)} - U_{2N}\|_1 \to 0$ as $N \to \infty$. By Lemma 3.4, we have
\begin{equation}
\sum_{\mu=1}^{\infty} ||U_{2\mu+1} - U_{2\mu}||_1^p \leq C \sum_{\mu=1}^{\infty} 2^{(1-p)\mu} ||U_{2\mu+1} - U_{2\mu}||_p^p
\end{equation}

\begin{equation}
\leq C \sum_{\mu=1}^{\infty} 2^{(1-p)\mu} E_{2\mu}(f^{(n)})_p^p \leq C \sum_{\nu=1}^{\infty} \nu^{-p} E_{2\nu}(f^{(n)})_p^p.
\end{equation}

In view of (4.1), this implies that there exists $g \in L_1(\mathbb{T})$ such that $U_{2^\mu} \to g$ as $\mu \to \infty$ in $L_1(\mathbb{T})$. By the definition of $U_n$, we know that $U_{2^\mu} \to f^{(n)}$ as $\mu \to \infty$ in $L_p(\mathbb{T})$. Therefore, $g = f^{(n)}$ a.e. on $\mathbb{T}$ and
\begin{equation}
U_{2^\mu} \to f^{(n)} \text{ as } \mu \to \infty \text{ in } L_1(\mathbb{T}).
\end{equation}

Thus, combining (4.7) and (4.8) and taking into account (4.9), we get
\begin{equation}
E_n(f - T_{2m})_p^p \leq \sum_{\mu=m}^{\infty} E_n(T_{2\mu+1} - T_{2\mu})_p^p.
\end{equation}
Now, applying the same arguments as in (4.3) and (4.5) to the function $\tau_u(x) = \tau_{u,2^\nu+1,2^\nu}(x) = \Delta_{u}^{1}(T_{2^{\nu+1}}(x) - T_{2^\nu}(x))$, we derive

$$E_n(T_{2^{\nu+1}} - T_{2^\nu})_p \leq C\omega_{\alpha+r}(T_{2^{\nu+1}} - T_{2^\nu}, n^{-1})_p \leq C\sup_{u>0} R_\alpha(\tau_u, n^{-1})_p$$

(4.11)

$$\leq C(2^{\mu+1}n^{-1})^{\alpha+\frac{1}{p}-1}\sup_{u>0} R_\alpha(\tau_u, 2^{-\mu-1})_p$$

$$\leq Cn^{-\alpha-\frac{1}{p}-1}\mu\sup_{u>0} \parallel \tau_u^{(\alpha)} \parallel_p.$$  

Note that in the third inequality, we use Lemma 3.2 and take into account that $n < 2^{m-1} \leq 2^{\mu+1}$. Next, by (3.3)

$$\parallel \tau_u^{(\alpha)} \parallel_p = \parallel \Delta_{u}^{1}(T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}) \parallel_p = \parallel \Delta_{u}^{1}(U_{2^{\nu+1}} - U_{2^\nu}) \parallel_p$$

(4.12)

$$\leq C\parallel U_{2^{\nu+1}} - U_{2^\nu} \parallel_p \leq CE_{2^\nu}(f^{(\alpha)})_p.$$  

Combining (4.10)–(4.12), we obtain

$$E_n(f - T_{2^m})_p \leq Cn^{-\alpha p} \sum_{\mu=m}^\infty 2^{(1-p)\mu} E_{2^\nu}(f^{(\alpha)})_p$$

(4.13)

$$\leq Cn^{-\alpha p} \sum_{\nu=n+1}^\infty \nu^{-p} E_{2^\nu}(f^{(\alpha)})_p.$$  

Finally, combining (4.2), (4.6), and (4.13), we get (2.1). \qed

**Proof of Theorem 2.2.** Let $N \in \mathbb{N}$ be such that $2^{N-1} \leq n < 2^N$. Assuming for a moment that $f^{(\alpha)}$ exists, we get

$$\parallel f^{(\alpha)} - T_n^{(\alpha)} \parallel_p = \parallel \Delta_{n}^{1}(T_{2^N}^{(\alpha)} - T_{2^\nu}^{(\alpha)}) \parallel_p = \parallel \Delta_{n}^{1}(U_{2^N} - U_{2^\nu}) \parallel_p$$

(4.14)

$$\leq C\parallel U_{2^N} - U_{2^\nu} \parallel_p \leq CE_{2^\nu}(f^{(\alpha)})_p.$$  

By Lemma 3.5, we obtain

$$\parallel T_{2^\nu}^{(\alpha)} - T_{2^\nu}^{(\alpha)} \parallel_p \leq C2^{\alpha Np} \parallel T_{2^N} - T_{2^\nu} \parallel_p \leq Cn^{\alpha p} E_n(f)_p$$

(4.15)

and

$$\sum_{\nu=N}^\infty \parallel T_{2^{\nu+1}}^{(\alpha)} - T_{2^{\nu}}^{(\alpha)} \parallel_p \leq C \sum_{\nu=N}^\infty 2^{\alpha \nu p} \parallel T_{2^{\nu+1}} - T_{2^{\nu}} \parallel_p$$

(4.16)

$$\leq C \sum_{\nu=N}^\infty 2^{\alpha \nu p} E_{2^{\nu}}(f)_p.$$  

Thus, by the completeness of $L_p(\mathbb{T})$ and condition (2.2), there exists a function $g \in L_p(\mathbb{T})$ such that

$$\parallel g - T_{2^N}^{(\alpha)} \parallel_p = \lim_{l\to\infty} \parallel T_{2^l}^{(\alpha)} - T_{2^N}^{(\alpha)} \parallel_p \leq C \left( \sum_{\nu=N}^\infty 2^{\alpha \nu p} E_{2^{\nu}}(f)_p \right)^{\frac{1}{p}}.$$  

(4.17)

In (4.17), we use the equality $T_{2^l} - T_{2^N} = \sum_{\nu=N}^{l-1} (T_{2^{\nu+1}} - T_{2^\nu})$ and (4.16). It is also easy to see that

$$\parallel f - T_{2^N} \parallel_p \leq C2^{-\alpha N} \left( 2^{N\alpha} E_{2^N}(f)_p \right) = o(2^{-\alpha N}) \quad \text{as} \quad N \to \infty.$$  

(4.18)

Therefore, by Lemma 3.8, (4.18), and (4.17), we obtain that $g = f^{(\alpha)}$. Finally, combining (4.14), (4.15), and (4.17), we get (2.3). \qed
Proof of Theorem 2.3. The proof is similar to the proof of Theorem 2.2 using Lemma 3.5 for all $\alpha > 0$.  

Proof of Theorem 2.4. We prove the theorem only in the case $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$. The other cases for $\alpha$ can be obtained repeating the arguments presented below.

Using (2.1), we obtain

$$\sum_{\nu=1}^{\infty} \nu^{\alpha p - 1} E_\nu(f)_p^p \leq C \sum_{\nu=1}^{\infty} \left( \nu^{\alpha p - 1} E_\nu(f^{(\alpha)}(f))_p^p + \nu^{p-2} \sum_{\mu=\nu+1}^{\infty} \mu^{-p} E_\mu(f^{(\alpha)}(f))_p^p \right)$$

(4.19)

$$\leq C \sum_{\nu=1}^{\infty} \nu^{p-1} \nu^{-p} E_\nu(f^{(\alpha)}(f))_p^p + C \left( \sum_{\nu=1}^{\infty} \nu^{p-2} \right) \sum_{\mu=\nu+1}^{\infty} \mu^{-p} E_\mu(f^{(\alpha)}(f))_p^p$$

$$\leq C n^{p-1} \sum_{\nu=1}^{\infty} \nu^{-p} E_\nu(f^{(\alpha)}(f))_p^p.$$ 

Therefore, combining (2.4), (2.1), and (4.19), we get (2.5).

Proof of Theorem 2.5. Let $n \in \mathbb{N}$ be such that $1/(n+1) < \delta \leq 1/n$ and let $T_n \in \mathcal{T}_n$ be polynomials of the best approximation of $f$ in $L_p(\mathbb{T})$. By (3.2), we get

$$\omega_{\alpha+\beta}(f, \delta)_p^p \leq \omega_{\alpha+\beta}(f, 1/n)_p^p$$

(4.20)

$$\leq \omega_{\alpha+\beta}(f - T_n, 1/n)_p^p + \omega_{\alpha+\beta}(T_n, 1/n)_p^p = M_1 + M_2.$$ 

Using Lemma 3.3, (3.2), and (3.3), we obtain

$$M_2 \leq C n^{-\alpha p} \omega_{\beta}(T_n^{(\alpha)}, 1/n)_p^p$$

(4.21)

$$\leq C n^{-\alpha p} \left( \| f^{(\alpha)}(f) - T_n^{(\alpha)} \|_p^p + \omega_{\beta}(f^{(\alpha)}, 1/n)_p^p \right).$$ 

Next, by Theorem 2.4 and the Jackson–type inequality (2.8), we have

$$\| f^{(\alpha)}(f) - T_n^{(\alpha)} \|_p^p \leq C \left( \omega_{\nu}(f^{(\alpha)}, 1/n)_p^p + n^{p-1} \sum_{\nu=1}^{\infty} \nu^{-p} \omega_{\nu}(f^{(\alpha)}, 1/n)_p^p \right)$$

(4.22)

$$\leq C n^{p-1} \int_0^{1/n} \frac{\omega_{\nu}(f^{(\alpha)}, t)_p^p}{t^{2-p}} dt.$$ 

At the same time, by (3.3), Theorem 2.1, and (2.8), we derive

$$M_1 \leq C \| f - T_n \|_p^p$$

(4.23)

$$\leq C n^{-\alpha p} \left( \omega_{\nu}(f^{(\alpha)}, 1/n)_p^p + n^{p-1} \sum_{\nu=1}^{\infty} \nu^{-p} \omega_{\nu}(f^{(\alpha)}, 1/n)_p^p \right)$$

$$\leq C n^{p-1-\alpha p} \int_0^{1/n} \frac{\omega_{\nu}(f^{(\alpha)}, t)_p^p}{t^{2-p}} dt.$$ 

Thus, combining (4.20)–(4.23) and taking into account (2.14) and $1/(n+1) < \delta \leq 1/n$, we get (2.6).

Proof of Theorem 2.6. The proof is similar to the proof of Theorem 2.5 combining Theorem 2.2, Lemma 3.3, and the Jackson inequality (2.8).
Proof of Theorem 2.7. The proof is similar to the proof of Theorems 2.5 and 2.6. We only note that we use Theorem 2.3 instead of Theorem 2.2.

Proof of Theorem 2.8. The proof easily follows from inequality (2.8) and Theorem 2.1.

Proof of Theorem 2.9. In view of (2.12) and Theorem 2.2, the function $f^{(\alpha)}$ in the sense of $L_p$. Using inequalities (2.3) and (2.9), we obtain

$$
\omega_\beta(f^{(\alpha)}, n^{-1})_p \leq C n^{-\beta p} \sum_{\nu=0}^{n} (\nu + 1)^{\beta p - 1} E_\nu(f^{(\alpha)})^p
$$

(4.26)

$$
\leq C n^{-\beta p} \sum_{\nu=0}^{\infty} (\nu + 1)^{\beta p - 1} \left( (\nu + 1)^{\alpha p} E_\nu(f)^p + \sum_{\mu=\nu+1}^{\infty} \mu^{\alpha p - 1} E_\mu(f)^p \right)
$$

(4.25)

$$
= C n^{-\beta p} \left( \sum_{\nu=0}^{n} (\nu + 1)^{\alpha p} E_{\nu+1}(f)^p + \sum_{\nu=0}^{n} (\nu + 1)^{\beta p - 1} \left( \sum_{\mu=\nu}^{\infty} + \sum_{\mu=\nu+1}^{\infty} \right) (\mu + 1)^{\alpha p - 1} E_{\mu+1}(f)^p \right).
$$

Further, we have

$$
\sum_{\nu=0}^{n} (\nu + 1)^{\beta p - 1} \sum_{\mu=\nu}^{n} (\mu + 1)^{\alpha p - 1} E_{\mu+1}(f)^p
$$

(4.25)

$$
= \sum_{\mu=0}^{n} (\mu + 1)^{\alpha p - 1} E_{\mu+1}(f)^p \sum_{\nu=0}^{n} (\nu + 1)^{\beta p - 1}
$$

$$
\leq C n^{\beta p} \sum_{\mu=0}^{n} (\mu + 1)^{\alpha p} E_{\mu+1}(f)^p.
$$

At the same time, we derive

$$
n^{-\beta p} \sum_{\nu=0}^{n} (\nu + 1)^{\beta p - 1} \sum_{\mu=\nu+1}^{\infty} (\mu + 1)^{\alpha p - 1} E_{\mu+1}(f)^p \leq C \sum_{\mu=\nu+1}^{\infty} \mu^{\alpha p - 1} E_\mu(f)^p.
$$

(4.26)

Finally, combining (4.24)–(4.26), we get (2.13).

Proof of Theorem 2.10. The proof is similar to the proof of Theorem 2.9 by using inequality (2.4) instead of (2.3).

Proof of Theorem 2.11. First, we show that (i) implies (ii). By Ulyanov's type inequality (see, e.g., [11]) and inequality (3.3), for any $r > \beta + 1/p - 1$, $r \in \mathbb{N}$, we have

$$
\|f\|_p^p \leq C \left( \int_0^1 \left( \frac{\omega_r(f,t)^p}{t^{1/p-1}} \right) \frac{dt}{t} + \|f\|_p^p \right)
$$

$$
\leq C \left( \int_0^1 \left( \frac{\omega_r(f,t)^p}{t^{1/p-1}} \right) \frac{dt}{t} + \|f\|_p^p \right) \leq C \left( \int_0^1 t^{\beta p - 1} dt + \|f\|_p^p \right) < \infty,
$$

that is $f \in L_1(\mathbb{T})$. Next, using Theorem 2.6 in the case $\beta \geq 1$ and Theorem 2.5 in the case $\beta < 1$, we get

$$
\omega_1(f^{(\beta-1)}, \delta)_p \leq C \left( \int_0^\delta \frac{\omega_\beta(f,t)^p}{t^{(\beta-1)p+1}} dt \right)^\frac{1}{p} = \mathcal{O}(\delta^p), \quad \beta \geq 1,
$$
and
\[
\omega_1(I_{1-\beta} f, \delta)_p \leq C \delta^{\frac{1}{p} - \beta} \left( \int_0^\delta \frac{\omega_{\beta}(f, t)_p^p}{t^{2-p}} dt \right)^{\frac{1}{p}} = O(\delta^{\frac{1}{p}}), \quad \beta < 1,
\]
where, for the clarity, we use the notation
\[
I_\alpha f = f^{(-\alpha)}
\]
to denote the fractional integral of order \(\alpha > 0\). It only remains to apply Proposition 2.3.

Now, let us prove that (ii) implies (i). Let
\[
f^{(\beta-1)}(x) = d_0 + \sum_{x_k < x} d_k = d_0 + \sum_{k=1}^\infty d_k h_{x_k}(x),
\]
where
\[
h_\eta(x) = \begin{cases} 1, & x > \eta, \\ 0, & x \leq \eta. \end{cases}
\]
Then, we have \(f(x) = d_0' + \sum_{k=1}^\infty d_k I_{1-\beta} h_{x_k}(x)\). Using (3.2), we get
\[
\omega_{\beta}(f, \delta)_p^p \leq \sum_{k=1}^\infty |d_k|^p \omega_{\beta}(I_{1-\beta} h_{x_k}, \delta)_p^p.
\]
(4.27)

Next, for any \(r \in \mathbb{N}\) and \(\eta \in \mathbb{R}\), we have
\[
\omega_r(I_{r-1} h_\eta, \delta)_p \leq C(p, r) \delta^{r+\frac{1}{p}-1}
\]
(see, e.g., [30] or [9, p. 359]). Choose \(\alpha > 1/p - 1\) such that \(\beta + \alpha = r \in \mathbb{N}\). Then, applying Theorem 2.6 and (4.28), we obtain
\[
\omega_{\beta}(I_{r-1} h_\eta, \delta)_p \leq C \left( \int_0^\delta \frac{\omega_{\alpha+\beta}(I_\alpha I_{1-\beta} h_\eta, t)_p^p}{t^{\alpha p+1}} dt \right)^{\frac{1}{p}}
\]
(4.29)

Finally, combining (4.27) and (4.29), we prove the theorem. \(\square\)

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