A New Class of Optimization Methods Based on Coefficient Conjugate Gradient

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Abstract. The coefficient conjugate serves as the foundation for many conjugate gradient methods. The quadratic model is used to derive a novel coefficient conjugate in this study. Its global convergence result might be produced under Wolfe line search circumstances. The conjugate gradient method’s performance for unconstrained optimization problems is demonstrated through numerical tests.

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1. Introduction

One of the most important iterative approaches for solving the unconstrained optimization issue is the Conjugate Gradient (CG) method, which is as follows:

\[ f(x^*) = \min_{x \in \mathbb{R}^n} f(x). \] (1)

where \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a smooth function, see [7, 11, 12]. The goal function and its gradient are all that is required of the CG techniques in each iteration, see [13]. As a result, this strategy is particularly well suited to solving optimization problems. These approaches use the following iterative formula:

\[ x_{k+1} = x_k + \alpha_k d_k. \] (2)

If \( f \) is a quadratic, the one-dimensional minimizer along the ray may be calculated analytically as:

\[ \alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k}. \] (3)
Iterative procedures are required for generic non-linear functions, in [9] has further information about this. The Wolfe requirements are frequently employed in the convergence analysis and implementation of conjugate gradient techniques to obtain the step length $\alpha_k$ satisfying:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k$$

$$d_k^T g(x_k + \alpha_k d_k) \geq \sigma d_k^T g_k$$

where $0 < \delta < \sigma < 1$. More information is available in [8, 15]. In practice, the conjugate gradient search direction for the next iteration looks like this:

$$d_{k+1} = -g_{k+1} + \beta_k d_k$$

where $\beta_k$ is a scalar. There are two well-known methods for selecting $\beta_k$:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \quad \beta_k^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k}$$

separately by the Fletcher-Reeves (FR) approach [2] and the Dai-Yan (DY) method [1]. It approaches have good convergence properties, but their numerical results aren’t as good as the others [9]. Many variations of this method have been developed throughout the years, and some are extensively utilized in practice. Take, for example, Hideaki and Yasushi [14] and Basim [3]:

$$\beta_k^{HY} = \frac{\|g_{k+1}\|^2}{2/\alpha_k (f_k - f_{k+1})}, \quad \beta_k^B = \frac{\|g_{k+1}\|^2}{(f_k - f_{k+1})/\alpha_k - g_k^T d_k/2}$$

These algorithms have a very high numerical efficiency. The quadratic model has been established to improve efficiency for unconstrained problems, in order to maximize the benefits of the original conjugate gradient approaches. A few novel optimization approaches have been presented in this regard [4–6]. Using a conjugacy condition, we provide a modified conjugate gradient technique in this paper. The sufficient descent property is satisfied by the new search direction, which is independent of the line search and the convexity assumption on the objective function. Under the conventional assumptions, the new method has global convergence for generic functions. When dealing with unconstrained optimization issues, numerical experiments show that the new approach is superior.

### 2. Deriving new Coefficient Conjugate

As everyone is aware, the following outcomes are produced when the objective function to be decreased is quadratic and accurate line searches are used:

$$d_{k+1}^T Q d_k = 0$$

where $Q$ is Hessian matrix. The conjugacy condition is what it’s called. In [13] has further information. Because the $d_{k+1} = -g_{k+1} + \beta_k d_k$ for the conjugate gradient technique, the
following is the result:
\[ \beta_k = \frac{g_{k+1}^T Q s_k}{d_k^T Q s_k} \] (10)
as a result, a coefficient conjugacy is obtained. We now utilize a quadratic model to derive the new conjugate gradient approach, which can be represented as:
\[ f_{k+1} = f_k + s_k^T g_k + \frac{1}{2} s_k^T Q(x_k) s_k \] (11)
We may calculate the first order derivative:
\[ \nabla f_{k+1} = g_k + Q(x_k) s_k \] (12)
Using (12) in (11) we get the following:
\[ s_k^T y_k + f_{k+1} - f_k = \frac{1}{2} s_k^T Q(x_k) s_k \] (13)
We may write the equation (13) by using (3), as follows:
\[ d_k^T Q(x_k) s_k = \frac{1}{2} \alpha_k (g_k^T d_k)^2 s_k^T y_k \] (14)
We generate a new formula \( \beta_k \) by plugging (14) into (10):
\[ \beta_k = \frac{g_{k+1}^T y_k}{\rho_k s_k^T y_k}, \quad \rho_k^1 = \frac{1}{2} \frac{\alpha_k (g_k^T d_k)^2}{s_k^T y_k (s_k^T y_k + (f_{k+1} - f_k))} \] (15)
Because \( f \) is a quadratic function and exact line search is used, the following is the result:
\[ \beta_k = \frac{\|g_{k+1}\|^2}{\rho_k s_k^T y_k}, \quad \rho_k^2 = \frac{1}{2} \frac{\alpha_k (g_k^T d_k)^2}{s_k^T y_k (s_k^T y_k + (f_{k+1} - f_k))} \] (16)
Using and exact line search in (13), then (16) reduces to:
\[ \rho_k^2 = \frac{1}{2} \frac{\alpha_k (g_k^T d_k)^2}{s_k^T y_k (-s_k^T g_k + (f_{k+1} - f_k))} \] (17)
and
\[ \rho_k^3 = \frac{1}{2} \frac{\alpha_k (g_k^T d_k)^2}{s_k^T y_k (\alpha_k g_k^T g_k + (f_{k+1} - f_k))} \] (18)
As a result, our proposal, the so-called BN, becomes clear. The proposed technique is provided with a new algorithm based on the aforementioned.
2.1. Algorithm

Step 1. Given $x_1 \in \mathbb{R}^n$. Set $k = 1$ and $d_1 = -g_1$.

Step 2. Let the step size $\alpha_k$ satisfying the (4) and (5).

Step 3. Let $x_{k+1} = x_k + \alpha_k d_k$. If $\|g_{k+1}\| \leq 10^{-6}$, then stop.

Step 4. Update $\beta_k$ by the (16), then $d_{k+1}$ by (6).

Step 5. Set $k = k + 1$ and go to Step2.

The descent condition exposes a useful characteristic for the new formula conjugate gradient technique. Additional properties are highly crucial in the convergence proved in this investigation. The preceding material is summarized in the following theorem.

**Theorem 1.** If we use the new way to create $\{x_k\}$ and $\{d_k\}$, we get:

$$d_{k+1}^T g_{k+1} < 0,$$

and

$$d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k$$

(19)

**Proof.** It goes without saying that if $d_k = -g_k$ then $d_k^T g_k < 0$. Assume that $d_k^T g_k < 0$ for any $k$. From (6) and (17), it is simple to deduce:

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \beta_k d_k^T g_{k+1}$$

$$= -\beta_k \left[ \rho_k s_k^T y_k \right] + \beta_k d_k^T g_{k+1}$$

$$= \beta_k \left[ d_k^T g_{k+1} - \rho_k s_k^T y_k \right]$$

(20)

We may deduce the following from equations (15) and (20):

$$d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k$$

(21)

It is obvious that $d_k^T g_k < 0$, thus we get:

$$d_{k+1}^T g_{k+1} < 0$$

(22)

The proof is finished. Other methods, can be proven in the same way.

3. Convergence Analysis

We look at how new algorithms are converged. On the objective function, the following assumptions are made.

- $D = \{x|f(x) \leq f(x_0)\}$ is a bounded level set.
- In some neighborhood $D$ that contains $L_0$, the gradient is Lipschitz continuous; that is, $L$ exists such that:

$$\|g(\nu) - g(\omega)\| \leq L \|\nu - \omega\| \forall \nu, \omega \in D$$

(23)

For further information, read [1, 10] and Zoutendijk [16] achieved the following significant finding.
Lemma 1. Allow all assumptions to be true. Consider any iteration algorithm that uses the Wolfe line search to get \( \alpha_k \). Then:

\[
\sum_{k=1}^{\infty} \left( g_k^T d_k \right)^2 < \infty
\]  

(24)

Theorem 2. Assume that all of your assumptions are correct. If (19) is satisfied by formula \( \beta_k \), we have:

\[
\lim_{k \to \infty} \inf \| g_k \| = 0
\]  

(25)

Proof. Assume that (25) isn’t true by induction. (6) is rewritten as \( d_{k+1} + g_{k+1} = \beta_k d_k \), and upon squaring both sides, we obtain:

\[
\| d_{k+1} \|^2 + \| g_{k+1} \|^2 + 2d_{k+1}^T g_{k+1} = (\beta_k)^2 \cdot \| d_k \|^2
\]  

(26)

Applying (21), yields:

\[
\| d_{k+1} \|^2 = \frac{(d_{k+1}^T g_{k+1})^2}{(d_k^T g_k)^2} \cdot \| d_k \|^2 - 2d_{k+1}^T g_{k+1} - \| g_{k+1} \|^2
\]  

(27)

Equation (27) is divided into \( (d_{k+1}^T g_{k+1})^2 \), we get:

\[
\frac{\| d_{k+1} \|^2}{(d_{k+1}^T g_{k+1})^2} = \frac{\| d_k \|^2}{(d_k^T g_k)^2} - \frac{\| g_{k+1} \|^2}{(d_{k+1}^T g_{k+1})^2} - \frac{2}{d_{k+1}^T g_{k+1}}
\]  

(28)

The equation becomes, by completing the square:

\[
\frac{\| d_{k+1} \|^2}{(d_{k+1}^T g_{k+1})^2} \leq \frac{\| d_k \|^2}{(d_k^T g_k)^2} - \frac{\| g_{k+1} \|^2}{(d_{k+1}^T g_{k+1})^2} + \frac{1}{\| g_{k+1} \|^2}
\]  

(29)

Hence,

\[
\frac{\| d_{k+1} \|^2}{(d_{k+1}^T g_{k+1})^2} \leq \sum_{i=1}^{k+1} \frac{1}{\| g_i \|^2}
\]  

(30)

Suppose that there exists \( c_1 > 0 \) such that \( \| g_k \| \geq c_1 \) for all \( k \in n \). Then:

\[
\frac{\| d_{k+1} \|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k + 1}{c_1^2}
\]  

(31)

Assume and note in Equation (31) that:

\[
\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\| d_k \|^2} = \infty
\]  

(32)
Based on Lemma 1, we get $\lim_{k \to \infty} \inf \|g_k\| = 0$ holds. Other methods, can be proven in the same way.

4. Numerical Results

The outcomes of mathematical experiments are presented in this section. The FR-Algorithm was contrasted with the innovative conjugate gradient algorithm. Fortran was used to create both algorithms. Section 1. contains the exam questions. We investigated numerical experiments for 15 extended unconstrained optimization problems with the number of variables for each test function. We use the inequality as a termination condition. Table 1 contains the findings of the Wolfe condition test, with the following definitions for each column: The terminology used to describe the issue comprises problem, problem size, dim, number of iterations (NI), and number of function evaluations (NF). Table 1 shows how many challenges these strategies have addressed in terms of iterations and function evaluations.

Problem numbers indicate for: 1. is the Trigonometric, 2. is the Extended Rosenbrock, 3. is the Hager, 4. is the Extended Tridiagonal 1, 5. is the Generalized Tridiagonal 2, 6. is the Extended PSC1, 7. is the Extended Tridiagonal 2, 8. is EDENSCH (CUTE), 9. is the STAIRCASE S1, 10. is the DENSCHNA (CUTE), 11. is the DENSCHNC (CUTE), 12. is the Extended White & Holst, 13. is the Extended Block-Diagonal BD2, 14. is the Generalized quartic GQ2, 15. is the Extended Beale.
Table 1: The numerical results of the FR and New methods

| P. No. | n   | FR algorithm |      |      |       |      |      |      |      |      |       |      |      |      |      |      |      |
|--------|-----|--------------|------|------|-------|------|------|------|------|------|-------|------|------|------|------|------|------|
|        |     | NI           | NF   | NI   | NF    | NI   | NF   | NI   | NF   | NI   | NF    | NI   | NF   | NI   | NF   | NI   | NF   |
| 1      | 100 | 19           | 35   | 18   | 33    | 18   | 33   | 18   | 34   | 18   | 33    | 18   | 33   | 18   | 34   |
|        | 1000| 38           | 65   | 32   | 57    | 34   | 62   | 38   | 65   | 38   | 65    | 38   | 65   | 38   | 65   |
| 2      | 100 | 47           | 93   | 40   | 80    | 40   | 82   | 42   | 90   | 42   | 90    | 42   | 90   | 42   | 90   |
|        | 1000| 78           | 131  | 37   | 78    | 34   | 74   | 39   | 85   | 39   | 85    | 39   | 85   | 39   | 85   |
| 3      | 100 | 61           | 1024 | 47   | 665   | 25   | 43   | 28   | 46   | 28   | 46    | 28   | 46   | 28   | 46   |
|        | 1000| Fail         | Fail | Fail | Fail  | Fail | Fail | Fail | Fail | Fail | Fail  | Fail | Fail | Fail | Fail |
| 4      | 100 | 32           | 64   | 10   | 21    | 10   | 21   | 13   | 27   | 13   | 27    | 13   | 27   | 13   | 27   |
|        | 1000| 77           | 129  | 16   | 31    | 16   | 31   | 15   | 31   | 15   | 31    | 15   | 31   | 15   | 31   |
| 5      | 100 | 37           | 67   | 40   | 63    | 40   | 63   | 44   | 70   | 44   | 70    | 44   | 70   | 44   | 70   |
|        | 1000| 73           | 115  | 65   | 102   | 67   | 107  | 61   | 98   | 61   | 98    | 61   | 98   | 61   | 98   |
| 6      | 100 | 15           | 31   | 8    | 17    | 8    | 17   | 8    | 17   | 8    | 17    | 8    | 17   | 8    | 17   |
|        | 1000| 8            | 17   | 7    | 15    | 7    | 15   | 7    | 15   | 7    | 15    | 7    | 15   | 7    | 15   |
| 7      | 100 | 40           | 65   | 35   | 56    | 37   | 58   | 36   | 56   | 36   | 56    | 36   | 56   | 36   | 56   |
|        | 1000| 43           | 68   | 49   | 382   | 39   | 59   | 42   | 65   | 42   | 65    | 42   | 65   | 42   | 65   |
| 8      | 100 | 69           | 1202 | 45   | 637   | 38   | 364  | 26   | 48   | 26   | 48    | 26   | 48   | 26   | 48   |
|        | 1000| 98           | 1967 | 65   | 1279  | 43   | 440  | 33   | 257  | 33   | 257   | 33   | 257  | 33   | 257  |
| 9      | 100 | 671          | 1066 | 441  | 689   | 480  | 763  | 612  | 982  | 612  | 982   | 612  | 982  | 612  | 982  |
|        | 1000| Fail         | Fail | Fail | Fail  | Fail | Fail | Fail | Fail | Fail | Fail  | Fail | Fail | Fail | Fail |
| 10     | 100 | 20           | 33   | 10   | 19    | 10   | 19   | 10   | 19   | 10   | 19    | 10   | 19   | 10   | 19   |
|        | 1000| 19           | 35   | 9    | 18    | 9    | 18   | 9    | 18   | 9    | 18    | 9    | 18   | 9    | 18   |
| 11     | 100 | 49           | 80   | 15   | 28    | 14   | 26   | 15   | 26   | 15   | 26    | 15   | 26   | 15   | 26   |
|        | 1000| 129          | 166  | 13   | 26    | 13   | 26   | 13   | 26   | 13   | 26    | 13   | 26   | 13   | 26   |
| 12     | 100 | 43           | 88   | 38   | 86    | 33   | 75   | 37   | 79   | 37   | 79    | 37   | 79   | 37   | 79   |
|        | 1000| 46           | 92   | 34   | 77    | 35   | 76   | 40   | 89   | 40   | 89    | 40   | 89   | 40   | 89   |
| 13     | 100 | 122          | 156  | 12   | 23    | 12   | 23   | 12   | 23   | 12   | 23    | 12   | 23   | 12   | 23   |
|        | 1000| 130          | 166  | 12   | 23    | 12   | 23   | 12   | 23   | 12   | 23    | 12   | 23   | 12   | 23   |
| 14     | 100 | 112          | 147  | 34   | 57    | 37   | 61   | 34   | 58   | 34   | 58    | 34   | 58   | 34   | 58   |
|        | 1000| 110          | 145  | 38   | 58    | 40   | 60   | 35   | 55   | 35   | 55    | 35   | 55   | 35   | 55   |
| 15     | 100 | 32           | 52   | 12   | 24    | 12   | 24   | 16   | 32   | 16   | 32    | 16   | 32   | 16   | 32   |
|        | 1000| 22           | 42   | 12   | 24    | 12   | 24   | 19   | 38   | 19   | 38    | 19   | 38   | 19   | 38   |
| Total  |     | 2240         | 7341 | 1194 | 4668  | 1175 | 2687 | 1314 | 2472 |       |       |       |       |       |       |

The novel procedure saves NI and NF over time when compared to the conventional Fletcher and Reeves (FR) methodology, notably for our set of test problems, as demonstrated in Table 2. Table 1 compares the new methods to the Fletcher and Reeves [2] convex optimization strategy.
Table 2: Ratio of algorithm New cost to HS cost

|         | FR algorithm | BN1 (16) | BN2 (17) | BN3 (18) |
|---------|--------------|----------|----------|----------|
| NI      | 100 %        | 53.30 %  | 52.45 %  | 58.66 %  |
| NF      | 100 %        | 63.58 %  | 36.60 %  | 33.67 %  |

5. Final Thoughts

In this study, we provide a novel, globally convergent, functional nonlinear CG method, and fulfills the descent property under certain assumptions, and is based on the strictly convex quadratic function represented by (10) According to computer studies, the unique kinds given in this study are successful.

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