K-THEORETIC ANALOGUE OF FACTORIAL SCHUR P- AND Q-FUNCTIONS

TAKEISHI IKEDA AND HIROSHI NARUSE

Abstract. We introduce a $K$-theoretic analogue of factorial Schur $P$- and $Q$-functions. These functions are non-homogeneous symmetric functions generalizing Ivanov’s factorial Schur $P$- and $Q$-functions. We prove various combinatorial expressions for these functions, e.g. as a ratio of Pfaffians, a sum over set-valued shifted tableaux, a sum over excited Young diagrams. As a geometric application, we show that these functions represent the Schubert classes in the $K$-theory of torus equivariant coherent sheaves on the maximal isotropic Grassmannians of symplectic and orthogonal types. This generalizes a corresponding result for the equivariant cohomology given by the authors. We also discuss a cancellation property enjoyed by these functions, which we call $K$-supersymmetric property. We prove that the $K$-theoretic $P$-functions form a (formal) basis of the ring of $K$-supersymmetric functions.

1. Introduction

In [33], Schur introduced a family of symmetric polynomials, now called Schur $Q$-functions, in order to describe the irreducible characters of projective representations of symmetric groups. It plays distinguished parts not only in representation theory but also in combinatorics and geometry (see e.g. [7], [24], [29], [34] and reference therein). In [15], Ivanov introduced a multi-parameter deformation of the $Q$-functions, which we call the factorial $P$- and $Q$-functions, and proved various combinatorial formulas for them analogous to ones for the original $Q$-functions. In this paper, we introduce a “$K$-theoretic” analogue of Ivanov’s functions, and study combinatorial properties of these functions. As an application, we show that these functions represent the structure sheaves of the Schubert varieties in the torus equivariant $K$-theory of the maximal isotropic Grassmannians of symplectic or orthogonal types.

Let $n$ be a positive integer. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a strict partition of length $r \leq n$, i.e. a sequence of positive integers such that $\lambda_1 > \cdots > \lambda_r$. We define polynomials $GP_\lambda(x_1, \ldots, x_n|b)$ and $GQ_\lambda(x_1, \ldots, x_n|b)$, which we call $K$-theoretic factorial $P$- and $Q$-functions, depending on a parameter $\beta$ and also on “equivariant parameters” $b = (b_1, b_2, \ldots)$. Although these polynomials depend on $\beta$, we do not indicate this dependence in the notation. These are non-homogeneous symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in the polynomial ring $\mathbb{Z}[\beta, b]$. There are two important specializations. First letting $\beta = 0$, our functions become the factorial $P$- and $Q$-functions as studied in [15]. Secondly, let $GP_\lambda(x_1, \ldots, x_n)$ and $GQ_\lambda(x_1, \ldots, x_n)$ denote the functions obtained by the specialization $b_i = 0$ for all $i$. We simply call these functions the $K$-theoretic $P$- and $Q$-functions. Note that even this “non-equivariant” version has not appeared in the

Date: February 22, 2012.
2010 Mathematics Subject Classification. Primary 05E05; Secondary 14M15,19L47.
T.I. was partially supported by Grant-in-Aid for Scientific Research (C) 20540053.
1For a case of special values of deformation parameter the definition is due to A. Yu. Okounkov. See remark 2.1.
literature before an announcement paper \[12\]. If we perform both specializations, then the functions become the classical \(P\)-and \(Q\)-functions.

As the name suggests, our motivation of this work comes from geometry. In our previous paper \[8\],\[10\],\[11\], the factorial \(P\)- and \(Q\)-functions of Ivanov were interpreted as the Schubert classes in equivariant cohomology of the maximal isotropic Grassmannians of symplectic or orthogonal types. These spaces are generalized flag varieties \(G/P\) of type \(C\), and \(B\) or \(D\) (see \[8\]). Our original aim is to extend this results to equivariant \(K\)-theory. Thus the primary goal of this study is to prove that the polynomials \(GP\) and \(GQ\) represent the structure sheaves of Schubert varieties in the \(K\)-theory of torus equivariant coherent sheaves on the maximal isotropic Grassmannians of symmetric and orthogonal types (Thm. \[3.3\]).

Regardless of geometric applications, the polynomials have independent interest. They will play some significant role in combinatorics, representation theory and invariant theory. Thus another purpose of this paper is to establish some fundamental properties of the polynomials, as Ivanov did in \[15\]. We obtained the following different expressions for \(GP\) and \(GQ\):

- Hall-Littlewood-type formula (Def. \[2.1\])
- Nimmo-type formula as ratio of Pfaffians (Prop. \[2.1\])
- Combinatorial formula using shifted set-valued tableaux (Thm. \[9.1\])
- Combinatorial formula using excited Young diagrams (Thm. \[9.2\]).

The coincidence of these expressions is the main result about the combinatorics of the \(K\)-theoretic factorial \(P\)-and \(Q\)-functions.

In the course of this study, a remarkable notion of a cancellation property for multi-variable polynomials arises. We call it the \(K\)-supersymmetric property. Let \(\beta\) be a formal parameter.

**Definition 1.1.** A polynomial \(f(x_1, \ldots, x_n)\) in \(\mathbb{Z}[\beta][x_1, \ldots, x_n]\) is called \(K\)-supersymmetric, if \(f(x_1, \ldots, x_n)\) is symmetric in variables \(x_1, \ldots, x_n\), and moreover, if for all \(1 \leq i < j \leq n\) the rational function

\[
 f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{j-1}, \Theta t, x_{j+1}, \ldots, x_n)
\]

does not depend on \(t\), where \(\Theta t = -t/(1 + \beta t)\).

The polynomials enjoying this property naturally form a ring, which we denote by \(GT_n\). A fundamental result about this ring is the basis theorem, which asserts that \(GP_\lambda(x_1, \ldots, x_n)\)’s form a \(\mathbb{Z}[\beta]\)-basis of the ring of \(K\)-supersymmetric polynomials (see Thm \[3.1\] for a precise statement). This generalizes work of Pragacz \[29\] in case of \(\beta = 0\).

Let us explain key ideas of our construction in more detail. For simplicity of notation, we restrict attention to case of type \(D\) (the maximal isotropic Grassmannians of even orthogonal type) in the introduction. Let \(\mathcal{R}\) be the localization of the polynomial ring \(\mathbb{Z}[\beta][b_1, b_2, \ldots]\) by the multiplicative system formed by the products of \(1 + \beta b_i\). The ring \(\mathcal{R}\) is essentially the ring of Laurent polynomials of infinitely many variables. In fact, when we specialize \(\beta = -1\), \(\mathcal{R}\) is isomorphic to \(\mathbb{Z}[e^\pm t_1, e^\pm t_2, \ldots]\) where \(b_i\) is identified with \(1 - e^{b_i}\). Let \(GT\) denote the ring of \(K\)-supersymmetric functions defined to be \(\varprojlim GT_n\). For each strict partition \(\lambda\) we can construct a certain limit of \(GP_\lambda(x_1, \ldots, x_n|b)\), an element \(GP_\lambda(x|b)\) in the ring \(\mathcal{R} \otimes_{\mathbb{Z}[\beta]} GT\). The functions \(GP_\lambda(x|b)\) when \(\lambda\) runs for the set \(SP\) of all strict partitions, form a ‘basis’ of the ring \(\mathcal{R} \otimes_{\mathbb{Z}[\beta]} GT\) over \(\mathbb{Z}[\beta]\).
Let $\Psi$ be the ring defined to be the subalgebra of $\text{Fun}(\mathcal{SP}, \mathcal{R})$ characterized by a $K$-theoretic analogue of the so-called Goresky-Kottwitz-MacPherson conditions (cf. [6], see §8.1 for the definition). For each $\mu \in \mathcal{SP}$, we define a sequence $b_\mu$ of elements in $\mathcal{R}$ (§7.1).

The localization map is defined as follows:

$$\Phi : \mathcal{R} \otimes_{\mathbb{Z}[\beta]} \mathcal{G} T \longrightarrow \Psi, \quad F(x|b) \mapsto (\mathcal{SP} \ni \mu \mapsto F(b_\mu|b) \in \mathcal{R})$$

given by substitution of $x$ to $b_\mu$ for all $\mu \in \mathcal{SP}$. The image $\{\Phi(GP_\lambda(x|b))\}_{\lambda \in \mathcal{SP}}$ is shown to be a family of Schubert classes (Def. 5.2). It gives a distinguished ‘basis’ of $\Psi$ as an $\mathcal{R}$ module (Thm. 7.1). Key facts in the proof of the identification of ‘basis’ is the divided difference equations (Thm. 8.1), and also vanishing property of $GP_\lambda(x|b)$ (Prop. 7.1). This technique of identification is also used in combinatorial arguments in §10.

The localization map can be interpreted as the restriction map to torus fixed points. Let $\mathcal{G}_n$ be the maximal isotropic Grassmannian (of even orthogonal type). An algebraic torus $T \cong (\mathbb{C}^\times)^{n+1}$ acts on $\mathcal{G}_n$ with finitely many fixed points naturally identified with the set $\mathcal{SP}(n)$ of strict partitions $\lambda$ such that $\lambda_1 \leq n$. The same set parametrize the Schubert varieties $\Omega_\lambda$ in $\mathcal{G}_n$. The Grothendieck group $K_T(\mathcal{G}_n)$ of $T$-equivariant coherent sheaves of $\mathcal{G}_n$ has an $R(T)$-algebra structure, where $R(T)$ is the representation ring of $T$; $R(T)$ is identified with $\mathbb{Z}[e^{\pm t_1}, \ldots, e^{\pm t_{n+1}}]$. As an $R(T)$-module, $K_T(\mathcal{G}_n)$ has a free basis formed by the classes $[\mathcal{O}_{\Omega_\lambda}]_T$ of the structure sheaves of the Schubert varieties. Let $i : \mathcal{G}_n^T \hookrightarrow \mathcal{G}_n$ be the inclusion map. Then we have a pullback

$$i^* : K_T(\mathcal{G}_n) \longrightarrow K_T(\mathcal{G}_n^T) \cong \text{Fun}(\mathcal{SP}(n), R(T)),$$

which is known to be injective. If we denote by $i_{\mu}^*$ the component of $i^*$ at the $T$-fixed point corresponding to $\mu \in \mathcal{SP}(n)$, Thm. 7.1 (combined with Thm. 8.3) can be restated in the following geometric form:

**Theorem 1.1.** For all $\lambda, \mu \in \mathcal{SP}(n)$ we have

$$i_{\mu}^*([\mathcal{O}_{\Omega_\lambda}]_T) = GP_\lambda(b_\mu|b),$$

where $GP_\lambda(b_\mu|b)$ is naturally considered to be an element in $R(T)$ by specialization $\beta = -1$.

Therefore, via the localization map, we have constructed canonical morphic surjective maps

$$\mathcal{R} \otimes_{\mathbb{Z}[\beta]} \mathcal{G} T \longrightarrow K_T(\mathcal{G}_n)$$

for all positive integers $n$. This map sends $GP_\lambda(x|b)$ to $[\mathcal{O}_{\Omega_\lambda}]_T$ if $\lambda \in \mathcal{SP}(n)$ and to zero otherwise (see Thm. 8.3 for a more precise statement). One may think of the ring $\mathcal{R} \otimes_{\mathbb{Z}[\beta]} \mathcal{G} T$ as an equivariant $\bar{K}$-ring of an infinite dimensional Grassmannian “$\mathcal{G}_\infty$”. This point of view will be discussed elsewhere.

### 1.1. Organization

In Section 2 we define the $K$-theoretic factorial $P$-and $Q$-polynomials $GQ_\lambda(x_1, \ldots, x_n|b)$ and $GP_\lambda(x_1, \ldots, x_n|b)$ in a Hall-Littlewood type form. These are symmetric polynomials with coefficients in $\mathbb{Z}[\beta, b]$. We show a Nimmo type formula expressing these functions as a ratio of Pfaffians (this result is not used in the rest of this paper). This section include a preliminary result on the factorial Grothendieck polynomials (25) used in the next section. Section 3 establishes some basic results for $K$-supersymmetric polynomials (and functions). Within this section, we mainly consider the case of $b = 0$. We prove the basis theorem using a factorization property of $GP_\lambda(x_1, \ldots, x_n|b)$. We also discuss the span of $GQ_\lambda(x_1, \ldots, x_n)$. In Section 4 we sets up notation for Weyl groups and root systems. In Section 5 we introduce GKM ring $\Psi$ in a combinatorial manner. We introduce a family of Schubert classes $\{\psi_\lambda\}_\lambda$ and discuss its basic properties. These
elements (if exists) gives a natural basis for $\Psi$. In Section 6 we prove the divided difference equation for $GQ_\lambda(x|b)$ and $GP_\lambda(x|b)$. In Section 7 we define the localization map which relate the ring spanned by $GP_\lambda(x|b)$ (or $GQ_\lambda(x|b)$) and the GKM ring $\Psi$. The goal of this section is to identify $\psi_\lambda$ and $GP_\lambda(x|b)$ (or $GQ_\lambda(x|b)$) via the localization map. Section 8 explains geometric meaning of our polynomials and the ring of $K$-supersymmetric functions. In Section 9 we state the results of expressing $GP_\lambda(x|b)$ and $GQ_\lambda(x|b)$ in terms of combinatorial objects; shifted set-valued tableaux and exited Young diagrams. Section 10 is devoted to the proof the results in Section 9.

1.2. Related works. For the (non-equivariant) $K$-theory of odd maximal orthogonal Grassmannians, Clifford, Thomas, and Yong [4] proved a Littlewood-Richardson rule which gives an explicit combinatorial description for the structure constants for the Schubert basis. In view of Cor. 8.1, these constants are equal to the structure constant for $GP_\lambda(x_1, \ldots, x_n)$’s. Note that [4] are based on a work of Buch and Ravikumar [3]. Our result in this paper is independent from their result.

For the Lagrangian Grassmannians, Ghorpade and Raghavan [31] gave a detailed description of Gröbner degeneration of an open affine piece of a Schubert variety around a torus fixed point. Using this result, Kreiman [21], derived a combinatorial expression for the restriction of Schubert to fixed point set of the torus action class of the equivariant $K$-theory. Note also Raghavan and Upadhyay extend [31] to orthogonal types.

It will be worth noticing that in [3], the center of the affine BMW-algebra is identified with a subalgebra of the ring of symmetric Laurent polynomials having a cancelation property quite similar to the $K$-supersymmetry.

It is a natural task to extend the $K$-theoretic factorial $P$-and $Q$-functions to family of functions relevant for the equivariant $K$-theory of the full flag varieties of types $B, C, D$. Kirillov and the second named author [16] introduced such family of functions, the double Grothendieck polynomials of type $B, C$ and $D$, by using Id-Coxeter algebra, a $K$-theoretic analogue of the nil-Coxeter algebra.

An outstanding open question is the determination of the structure constant for $GP_\lambda(x|b)$ and $GQ_\lambda(x|b)$. As for the “equivariant case”, i.e. for the case of arbitrary $b$, we do not even have a conjecture, except for very restricted cases.

In [12], we introduced a bumping procedure for the set-valued shifted tableaux. As an application we proved a Pieri type rule for $GQ_\lambda(x_1, \ldots, x_n)$, which is combinatorially equivalent to a problem of counting the $KLG$-tableau in the language of [3], where they derive the description utilizing a more geometric arguments. Bumping procedure for ‘orthogonal’ type, which is relevant for $GP_\lambda(x_1, \ldots, x_n)$, is not known now.

Acknowledgments. We thank Mark Shimozono for discussions and his continuous interest in the work. We also like to thank Leonardo Mihalcea and Alex Yong for useful comments on the draft.

2. $K$-theoretic factorial Schur $Q$- and $P$-functions

We define $K$-theoretic analogue of factorial $P$- and $Q$-functions. This generalize a form for $P$- and $Q$-functions as a particular case of the Hall-Littlewood functions with the parameter $t$ equal to $-1$. These are symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}[\beta, b]$. We give another expression for these polynomials as ratios of Pfaffians. We also give some supplementary discussion on type $A$ case, i.e. on the factorial Grothendieck polynomials defined and studied by McNamara.
2.1. The polynomials \(GP_\lambda(x_1,\ldots,x_n|b)\) and \(GP_\lambda(x_1,\ldots,x_n|b)\). Let \(\beta\) be a parameter. Define binary operators \(\oplus\) and \(\ominus\) by

\[
x \oplus y := x + y + \beta xy, \quad x \ominus y := \frac{x - y}{1 + \beta y}.
\]

We also define a deformation of \(k\)-th powers of \(x\) with parameters \(b = (b_1, b_2, \ldots)\) by

\[
[x|b]^k := (x \oplus b_1)(x \oplus b_2) \cdots (x \oplus b_k),
\]

and its variant by \([[x|b]^k := (x \ominus x)(x \ominus b_1)(x \ominus b_2) \cdots (x \ominus b_{k-1}).

Let \(\lambda = (\lambda_1, \ldots, \lambda_r)\) be a strictly decreasing sequence of positive integers. We call \(\lambda\) a strict partition, and \(r\) the length of \(\lambda\). Let \(\mathcal{SP}_n\) denote the set of all strict partitions \(\lambda\) such that the length \(r \leq n\). We set \([x|b]^\lambda = \prod_{j=1}^r [x|b]^{\lambda_j}\) and \([[x|b]^\lambda = \prod_{j=1}^r [[x|b]]^{\lambda_j}\).

**Definition 2.1.** Let \(\lambda = (\lambda_1, \ldots, \lambda_r)\) be a strict partition in \(\mathcal{SP}_n\). We define functions on \(n\) variables \(x_1,\ldots,x_n\) with parameters \(\beta\) and \(b_1, b_2, \ldots\) as follows:

\[
(2.1) \quad GP_\lambda(x_1,\ldots,x_n|b) := \frac{1}{(n-r)!} \sum_{w \in S_n} \left[ [x|b]^\lambda \prod_{i=1}^r \prod_{j=i+1}^n x_i \oplus x_j \right],
\]

\[
(2.2) \quad GQ_\lambda(x_1,\ldots,x_n|b) := \frac{1}{(n-r)!} \sum_{w \in S_n} \left[ [[x|b]^\lambda \prod_{i=1}^r \prod_{j=i+1}^n x_i \ominus x_j \right],
\]

where \(w \in S_n\) acts as a permutation on the variables \(x_1,\ldots,x_n\).

These functions are obviously symmetric in the variables \(x_1,\ldots,x_n\), and are polynomials with coefficients in \(\mathbb{Z}[\beta][b_1, b_2, \ldots, b_m]\) with \(m = \lambda_1\) for \(GP_\lambda\) and \(m = \lambda_1 - 1\) for \(GQ_\lambda\). The fact that these are polynomials follows from a standard argument. See [24] Chap. III, 1, or [13], Prop. 1.1 (a) for a proof. We call these polynomials the \(K\)-theoretic factorial \(P\)- and \(Q\)-functions.

**Remark 2.1 (Comments on terminology).** If we set \(\beta = 0\) and replace \([x|b]^k\) with the \(k\)th falling factorial \(\prod_{i=1}^k (x - i + 1)\) in the expression (2.1), the polynomial coincides with the factorial \(P\)-function, whose definition is due to A. Yu. Okounkov (see [13], [14]). Ivanov [15] developed the theory by using \((x|a)^k := \prod_{i=1}^k (x - a_i)\) with arbitrary sequence of parameters \(a = (a_i)_{i \geq 1}\). The function was first called the generalized \(Q\)(\(P\\)-) functions in [14], and then the multi-parameter Schur \(P\)-function in [15]. In [8], we simply called Ivanov’s function the factorial Schur \(Q\)-function. This terminology is consistent with a convention in type \(A\) case (e.g. [20]).

Both the functions \(GP_\lambda(x_1,\ldots,x_n|b)\) and \(GQ_\lambda(x_1,\ldots,x_n|b)\) can be expanded in the following forms:

\[
GP_\lambda(x_1,\ldots,x_n|b) = \sum_{k \geq |\lambda|} \beta^{k-|\lambda|} GP_\lambda(x_1,\ldots,x_n|b)_k,
\]

\[
GQ_\lambda(x_1,\ldots,x_n|b) = \sum_{k \geq |\lambda|} \beta^{k-|\lambda|} GQ_\lambda(x_1,\ldots,x_n|b)_k,
\]

where \(GP_\lambda(x_1,\ldots,x_n|b)_k\) and \(GQ_\lambda(x_1,\ldots,x_n|b)_k\) are symmetric polynomials in \(x_1,\ldots,x_n\) with coefficients in \(\mathbb{Z}[b] = \mathbb{Z}[b_1, b_2, \ldots]\), which are homogeneous of degree \(k\) when we set \(\deg(x_i) = \deg(b_i) = 1\). The lowest homogeneous parts are the factorial Schur functions, \(P_\lambda(x_1,\ldots,x_n|b)\) and \(Q_\lambda(x_1,\ldots,x_n|b)\) respectively. It is known that \(P_\lambda(x_1,\ldots,x_n|b)\)'s (resp. \(Q_\lambda(x_1,\ldots,x_n|b)\)'s), \(\lambda \in \mathcal{SP}_n\), are linearly independent over \(\mathbb{Z}[b]\) (cf. [15]). From
this fact, it turns out that $GP_\lambda(x_1, \ldots, x_n|b)$’s (resp. $GP_\lambda(x_1, \ldots, x_n|b)$’s), $\lambda \in \mathcal{SP}_n$, are linearly independent over the ring $\mathbb{Z}[\beta][b]$.

Remark 2.2. Here we adopt the notation in [9] for the factorial $P$- and $Q$-functions, with the identification $b_i = -t_i$. This is slightly different from the original one in [15]. See [9], §4.2 for the precise correspondence.

The specialization obtained by setting $\beta = -1$ is called the $K$-theoretic specialization, because the case is relevant when we apply these functions to $K$-theory. If we denote by $F_\lambda(x_1, \ldots, x_n|b)$ the $K$-theoretic specialization of $GP_\lambda(x_1, \ldots, x_n|b)$, then we have

$$(-\beta)^{-|\lambda|}F_\lambda(-\beta x_1, \ldots, -\beta x_n| - \beta b_1, -\beta b_2, \ldots) = GP_\lambda(x_1, \ldots, x_n|b)$$

and similarly for $GQ_\lambda$. Thus the case of arbitrary $\beta$ is recovered from the $K$-theoretic specialization. In an application to torus equivariant $K$-theory in [8] the parameters $b_i$ are identified with $1 - e^{t_i}$ where $e^{t_i}$ is a standard character of the torus.

If all the deformation parameters $b_i$ ($i \geq 1$) are specialized to zero, then the $K$-theoretic factorial $P$- and $Q$- functions are simply denoted by $GP_\lambda(x_1, \ldots, x_n)$ and $GQ_\lambda(x_1, \ldots, x_n)$. These are relevant to the non-equivariant $K$-theory of the isotropic Grassmannians.

2.2. Some useful identities. Next result will be used in the proofs of Thm. 6.1.

Lemma 2.1.

(2.3) \[ \prod_{i=1}^{m} x_i \oplus x_i \prod_{j \neq i} x_i \oplus x_j + \prod_{i=1}^{m} t \oplus x_i = 1. \]

Proof. We prove this by induction on $m$. We can write the left-hand side as

$$\frac{F(x_m)}{(x_m - t)^{m-1}}\prod_{i=1}^{m-1}(x_m - x_i)$$

where $F(x_m)$ is a polynomial in $x_m$. We claim that $\deg F(x_m) \leq m$. It is easy to see that $\deg F(x_m) \leq m + 1$. By letting $x_i$ to $\ominus x_i$ and $t$ to $\ominus t$ in (2.3) for $m - 1$ we have

$$\prod_{i=1}^{m-1} x_i \oplus x_i \prod_{j \neq i} x_i \oplus x_j + \prod_{i=1}^{m-1} t \oplus x_i = \prod_{i=1}^{m-1}(1 + \beta x_i).$$

This equation implies that the coefficient of degree $m + 1$ of $F(x_m)$ vanishes. Therefore it suffices to show the equation $F(x_m) = (x_m - t)^{m-1}\prod_{i=1}^{m-1}(x_m - x_i)$ for $m + 1$ different values of $x_m$. In fact, by letting $x_m = \ominus x_i$ ($1 \leq i \leq m - 1$) and $x_m = 0, \ominus t$ we can check the equality by using inductive hypothesis. \[\square\]

Next result will be used in the proofs of Thm. 6.1 and also Prop. 2.1. Analogous equation for the case of $\beta = 0$ can be found in [14], Prop. 2.4.

Lemma 2.2. For $k = 0, 1, 2$, we have

(2.4) \[ \sum_{i=1}^{n} (1 + \beta x_i)^k \prod_{j \neq i} x_i \ominus x_j = \begin{cases} \prod_{i=1}^{n}(1 + \beta x_i)^k - \prod_{i=1}^{n}(1 + \beta x_i) & \text{if } n \text{ is even} \\ \prod_{i=1}^{n}(1 + \beta x_i)^k & \text{if } n \text{ is odd} \end{cases} \]
Proof. Let $E_{n,k}$ denote the equation. We prove $E_{n,k}$ ($k = 0, 1, 2$) by induction on $n$. For $n = 1, 2$, the verification is straightforward. Suppose $n$ is odd and $n \geq 3$. Assume $E_{m,k}$ hold for $m < n, k = 0, 1, 2$. We write the left-hand side of $E_{n,0}$ as

\[
\frac{F(x_n)}{\prod_{i=1}^{n-1} (x_n - x_i)}
\]

where $F(x_n)$ is a polynomial in $x_n$. It is easy to see that $\deg F(x_n) \leq n$. Using $E_{n-1,1}$ we see that the coefficient of $x_n^n$ in $F(x_n)$ is zero. Hence we have $\deg F(x_n) \leq n - 1$. Now in order to prove $E_{n,0}$ we only have to show this for $n$ distinct values of $x_n$. Let us put $x_n = \ominus x_i$ ($1 \leq i \leq n - 1$). Then $E_{n,0}$ is reduced to $E_{n-1,0}$. If we put $x_n = 0$ then $E_{n,0}$ is reduced to $E_{n-1,0}$. Thus we have $E_{n,0}$. Note that $E_{n,2}$ is obtained from $E_{n,0}$ by $x_i \mapsto \ominus x_i$ ($1 \leq i \leq n$).

Next we prove

\[
\sum_{i=1}^{n} (1 + \beta x_i) \prod_{j \neq i} \frac{x_i \ominus x_j}{x_i \ominus x_j} - \prod_{i=1}^{n} (1 + \beta x_i) = 0
\]

which is equivalent to $E_{n,1}$. Write the left-hand side as in (2.5). Then by using $E_{n-1,2}$ we see that $\deg F(x_n) \leq n - 1$. Now evaluation $x_n = \ominus x_i$ ($1 \leq i \leq n - 1$) is reduced to $E_{n-2,1}$, whereas the specialization $x_n = 0$ is reduced to $E_{n-1,1}$. Thus we obtained $E_{n,1}$.

In a similar way, $E_{n+1,0}$ follows from $E_{n,1}$, $E_{n,0}$, and $E_{n+1,0}$. Then we have $E_{n+1,2}$ by switching $x_i \mapsto \ominus x_i$ ($1 \leq i \leq n$). Finally $E_{n+1,1}$ follows from $E_{n+1,2}$, $E_{n,1}$, and $E_{n-1,1}$. □

2.3. Nimmo type formula for $GQ_{\lambda}, GP_{\lambda}$. Let $A$ be a skew-symmetric matrix of even size. We denote by $\text{Pf}(A)$ the Pfaffian of $A$. There is a formula which is an analogue of Schur’s evaluation of Pfaffian (33):

Lemma 2.3.

\[
Pf_{1 \leq i < j \leq 2m} \left( \frac{x_i - x_j}{x_i \oplus x_j} \right) = \prod_{1 \leq i < j \leq 2m} \frac{x_i - x_j}{x_i \oplus x_j}.
\]

Proof. By Knuth’s theorem 17, we only have to show this for $m = 2$, which is straightforward. Indeed, (2.7) is a special case of (4.4) in [17]. □

Nimmo 28 derived an expression for $Q_{\lambda}(x_1, \ldots, x_n)$ as a ratio of Pfaffians. See also 24, Ex. III. 8.13. Next result is a generalization of Nimmo’s formula for $GQ_{\lambda}(x_1, \ldots, x_n|b)$ and $GP_{\lambda}(x_1, \ldots, x_n|b)$.

Theorem 2.1. Let $\lambda$ be a strict partition of length $r \leq n$. Let $m = r$ if $n - r$ is even, and $m = r + 1$ if $n - r$ is odd, and set $\lambda_{r+1} = 0$. Let $A_0 = A_0(x_1, \ldots, x_n)$ be a skew symmetric $n \times n$ matrix with $(i, j)$ entry $(x_i - x_j)/(x_i \oplus x_j)$, and let $B_{\lambda}$ is an $n \times m$ matrix with $(i, j)$ entry $[(x_i|b)^{\lambda_{m-j+1}}(1 + \beta x_i)^{-1}]$. Let

\[
A_{\lambda}(x_1, \ldots, x_n|b) = \begin{pmatrix} A_0 & B_{\lambda} \\ -B_{\lambda} & 0 \end{pmatrix},
\]

which is a skew symmetric matrix of $(n + m) \times (n + m)$. Put

\[
\text{Pf}_0(x_1, \ldots, x_n) = \begin{cases} \text{Pf} A_0(x_1, \ldots, x_n) & \text{if } n \text{ is even} \\ \text{Pf} A_0(x_1, \ldots, x_n, 0) & \text{if } n \text{ is odd} \end{cases}
\]
Then
\[
GQ_\lambda(x_1, \ldots, x_n|b) = \prod_{i=1}^{n}(1 + \beta x_i)^m \frac{\text{Pf} A_\lambda(x_1, \ldots, x_n|b)}{\text{Pf}_0(x_1, \ldots, x_n)}.
\]

The similar formula holds for \(GP_\lambda\) when we use \([x|b]^k\) instead of \([[x|b]]^k\).

Our proof of this theorem goes along the same line as in [28].

**Proposition 2.1** (cf. [28], (A10)). Under the same notation as in Thm. 2.1, we have
\[
\prod_{j=1}^{n}(1 + \beta x_j)^m \prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{x_i \oplus x_j} \times GQ_\lambda(x_1, \ldots, x_n|b)
= \sum_{w \in \hat{S}_{n,m}} \text{sgn}(w) \times \det \left(\left[\left([x_{w(i)}|b]\right)i^j(1 + \beta x_{w(i)}j^{i-m-1})\right] \times \prod_{m+1 \leq i < j \leq n} \frac{x_{w(i)} - x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}},
\right)
\]
where \(\hat{S}_{n,m}\) denote the set of permutations \(w\) of \(\{1, \ldots, n\}\) satisfying \(w(1) < \cdots < w(m)\), \(w(m+1) < \cdots < w(n)\). The similar formula holds for \(GP_\lambda(x_1, \ldots, x_n|b)\) when we use \([x|b]\) instead of \([[x|b]]\).

**Proof.** We can rewrite the definition of \(GQ_\lambda(x_1, \ldots, x_n|b)\) as follows:
\[
(\text{2.9}) \quad \frac{1}{(n-m)!} \sum_{w \in \hat{S}_{n,m}} \prod_{i=1}^{m} \text{sgn}(w) \times \det \left(\left[\left([x_{w(i)}|b]\right)i^j(1 + \beta x_{w(i)}j^{i-m-1})\right] \times \prod_{1 \leq i < j \leq n} \frac{x_{w(i)} \oplus x_{w(j)} \times x_{w(i)} - x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}},
\right)
\]
In fact, if \(n-r\) is even (so \(m = r\)), this is immediate by using \(x \oplus y = (x-y)(1+\beta y)\). If \(n-r\) is odd and then \(m = r+1\), we can deduce this expression by using Lemma 2.2 (the case of \(k = 0\) and the number of variables is odd). Now we write the last factor as
\[
\prod_{i=1}^{m} \prod_{j=i+1}^{n} \frac{x_{w(i)} \oplus x_{w(j)} \times x_{w(i)} - x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}} = \prod_{1 \leq i < j \leq n} \frac{x_{w(i)} \oplus x_{w(j)} \times x_{w(i)} - x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}} \times \prod_{m+1 \leq i < j \leq n} \frac{x_{w(i)} - x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}},
\]
Noting that \(\prod_{j=1}^{n}(1 + \beta x_{w(j)})^m = \prod_{j=1}^{m}(1 + \beta x_{j})^m\), which is invariant under \(S_n\), and
\[
\prod_{1 \leq i < j \leq n} \frac{x_{w(i)} \oplus x_{w(j)} \times x_{w(i)} - x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}} = \text{sgn}(w) \prod_{1 \leq i < j \leq n} \frac{x_i \oplus x_j \times x_i - x_j}{x_i \oplus x_j},
\]
we deduce that (2.9) equals
\[
(\text{2.10}) \quad \frac{1}{(n-m)!} \times \prod_{j=1}^{m}(1 + \beta x_{j})^m \prod_{1 \leq i < j \leq n} \frac{x_i \oplus x_j \times x_i - x_j \sum_{w \in \hat{S}_{n,m}} \text{sgn}(w) \times \prod_{i=1}^{m} \left(\left([x_{w(i)}|b]\right)i^j(1 + \beta x_{w(i)}j^{i-m-1})\right)}{x_{w(i)} \oplus x_{w(j)}},
\]
Now for each \(w \in \hat{S}_{n,m}\), we perform the summation over the subgroup \(G_w\) of \(S_n\) stabilizing \(\{x_{w(1)}, \ldots, x_{w(m)}\}\) and \(\{x_{w(m+1)}, \ldots, x_{w(n)}\}\). Clearly \(G_w \cong S_m \times S_{n-m}\). We write \(\sigma \in G_w\) as \(\sigma_1 \sigma_2\) correspondingly. Summation over \(\sigma_1\) creates \(\text{det} \left(\left(\left([x_{w(i)}|b]\right)i^j(1 + \beta x_{w(i)}j^{i-m-1})\right), \right)\), while \(\sigma_2\) gives a factor \((n-m)!\) since the last factor in (2.10) is anti-symmetric in \(n-m\) variables \(\{x_{w(m+1)}, \ldots, x_{w(n)}\}\). This completes the proof. □

We recall the following rule for expanding a Pfaffian of a matrix with blocks of zeros.
Lemma 2.4 (cf. [28], (A8)). Let $A$ be an $n \times n$ skew-symmetric matrix. Let $m$ be a positive integer such that $m \leq n$ and $n - m$ is even. Let $B = (b_{ij})$ be an $n \times m$ matrix. Then
\begin{equation}
\text{Pf} \left( \begin{array}{cc}
A & B \\
-B & 0
\end{array} \right) = (-1)^{m(m-1)/2} \sum_{w \in S_{n,m}} \text{sgn}(w) \det(B_{w(1),\ldots,w(m);1,\ldots,m}) \text{Pf}(A_{w(m+1),\ldots,w(n)}),
\end{equation}
where $B_{w(1),\ldots,w(m);1,\ldots,m}$ denotes an $m \times m$ matrix with $(i, j)$ entry $b_{w(i),j}$, and $A_{w(m+1),\ldots,w(n)}$ denotes the submatrix of $A$ whose indices of rows and columns are $w(m+1), \ldots, w(n)$.

Proof of Thm. 2.7 Since $n - m$ is even, we have by Lemma 2.3
\begin{equation}
\prod_{m+1 \leq i < j \leq n} \frac{x_{w(i)} - x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}} = \text{Pf} \left( \begin{array}{c}
x_{w(m+i)} - x_{w(m+j)} \\
1 \leq i < j \leq n - m
\end{array} \right).
\end{equation}
Then the right-hand side of equation of Prop. 2.1 becomes the form of (2.11). By permuting columns of $B$ we can eliminate the sign factor $(-1)^{m(m-1)/2}$ to obtain (2.9). □

2.4. Factorial Grothendieck polynomials. For a positive integer $n$, let $\mathcal{P}_n$ denote the set of partitions of length less than or equal to $n$; i.e. $\mathcal{P}_n$ is a set consisting of a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of non-negative integers such that $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Let $\lambda \in \mathcal{P}_n$. Define the following symmetric function
\begin{equation}
G_\lambda(x_1, \ldots, x_n|b) = \frac{\det([x_i|b]^{\lambda_1+n-j}(1 + \beta x_i)^{j-1})_{1 \leq i < j \leq n}}{\prod_{1 \leq i < j \leq n}(x_i - x_j)}.
\end{equation}

Remark 2.3. McNamara [25] introduced the factorial Grothendieck polynomials in terms of set-valued semistandard tableaux. It can be proved that (2.12) is identical to McNamara’s function. In fact, the argument in [10] modified to type A case suitably, is applicable to show this fact. Note that we do not use this coincidence in this paper. For our present purpose, the expression (2.12) is useful.

Given a partition $\lambda \in \mathcal{P}_n$. Define the sequence $b_\lambda = (\ominus b_{\lambda_1+n}, \ldots, \ominus b_{\lambda_n+n-i+1}, \ldots, \ominus b_{\lambda_i+1})$. We identify a partition with its Young diagram as usual (see [24]).

Proposition 2.2 (cf. [25]). Let $\lambda, \mu \in \mathcal{P}_n$. Then
\begin{equation}
G_\lambda(b_\mu|b) = \begin{cases} 
0 & \text{if } \lambda \not\subseteq \mu \\
\prod_{(i,j) \in \lambda} (b_{\lambda_{n+j} - \lambda_j} \ominus b_{\lambda_i + n - i + 1}) & \text{if } \lambda = \mu,
\end{cases}
\end{equation}
where $\lambda_j' = \#\{i \mid \lambda_i \geq j\}$.

Proof. It is convenient to write the definition of $G_\lambda(x|b)$ in the following form:
\begin{equation}
G_\lambda(x_1, \ldots, x_n|b) = \sum_{w \in S_n} w \left[ \frac{[x|b]^{\lambda+n-1}}{\prod_{1 \leq i < j \leq n}(x_i \ominus x_j)} \right],
\end{equation}
where $\rho_{n-1} = (n - 1, \ldots, 2, 1, 0)$. Then the proposition is proved by straightforward calculations (cf. [13], [26]). □

Let $\mathcal{R}$ be the localization of $\mathbb{Z}[[\beta]][b_1, b_2, \ldots]$ by the multiplicative system formed by products of $1 + \beta b_i$ ($i \geq 1$). Note that $G_\lambda(b_\mu|b)$ belongs to $\mathcal{R}$. The following lemma was proved in [25]. As our definition of $G_\lambda(x|b)$ is different from the one in [25], we give a proof here for completeness.
Lemma 2.5 (cf. [25]). $G_\lambda(x_1, \ldots, x_n|b)$ ($\lambda \in \mathcal{P}_n$) form an $\mathcal{R}$-basis of $\mathcal{R}[x_1, \ldots, x_n]^{S_n}$.

Proof. Let $m$ be a positive integer. Let $\mathcal{P}_{n,m}$ be the set of partitions $\lambda$ in $\mathcal{P}_n$ such that $\lambda_1 \leq m$. Let $L_{n,m}$ be the $\mathbb{Z}[\beta][b]$-span of $m_\mu(x)$ ($\mu \in \mathcal{P}_{n,m}$), where $m_\mu(x)$ is the monomial symmetric function in variables $x_1, \ldots, x_n$ corresponding to $\mu$ (cf. [24], I, 2). Note that the highest possible power of each $x_i$ in $G_\lambda(x_1, \ldots, x_n|b)$ is $m$ (this fact can be seen from (2.12)). Thus we can define the elements $d_{\lambda\mu}(\beta, b) \in \mathbb{Z}[\beta][b]$ by

$$
G_\lambda(x_1, \ldots, x_n|b) = \sum_{\mu \in \mathcal{P}_{n,m}} d_{\lambda\mu}(\beta, b) m_\mu(x) \quad (\lambda \in \mathcal{P}_{n,m}).
$$

We claim that $\det(d_{\lambda\mu}(\beta, b))$ is a product of factors of the form $1 + \beta b_i$ for some $i$. Once the claim is proved, we can invert the system of linear equations (2.14) over the ring $\mathcal{R} = \mathbb{Z}[\beta][b][(1 + \beta b)^{-1}(\beta \geq 1)]$ to express $m_\mu(x)$’s as $\mathcal{R}$-linear combination of $G_\lambda(x_1, \ldots, x_n|b)$’s with $\lambda \in \mathcal{P}_{n,m}$. Since each element in $\mathcal{R}[x_1, \ldots, x_n]^{S_n}$ are in $\mathcal{R} \otimes_{\mathbb{Z}[\beta][b]} L_{n,m}$ for some $m$ we have the lemma.

In order to prove the claim, we first show that $\det(d_{\lambda\mu}(0,0)) = 1$. Now we specialize $\beta$ and $b$ to 0 then $G_\lambda(x_1, \ldots, x_n|b)$ becomes the classical Schur function $s_\lambda(x_1, \ldots, x_n)$. It follows that, $d_{\lambda\mu}(0,0)$ is equal to the Kostka number (cf. [24], I, 6) if $|\lambda| = |\mu|$. Moreover, since $s_\lambda(x_1, \ldots, x_n)$ is homogeneous polynomial of degree $|\lambda|$, we have $d_{\lambda\mu}(0,0) = 0$ if $|\lambda| \neq |\mu|$. Therefore the matrix $(d_{\lambda\mu}(0,0))_{\lambda\mu}$ is lower triangular (with respect to a linear order which is a refinement of the dominance order) and all the entries of the main diagonal are 1. This implies $\det(d_{\lambda\mu}(0,0)) = 1$.

Now using Lemma 2.2, it can be proved that the only possible irreducible factors of $\det(d_{\lambda\mu}(\beta, b))$ are of the form $1 + \beta b_i$ (see [25], Lemma 4.7 and an argument after the lemma). Because we have $\det(d_{\lambda\mu}(0,0)) = 1$ we can conclude that $\det(d_{\lambda\mu}(\beta, b))$ is actually a product of $1 + \beta b_i$ for some $i$.

As a by-product of the proof of the preceding lemma, we have the following:

Corollary 2.1. $G_\lambda(x_1, \ldots, x_n)$ ($\lambda \in \mathcal{P}_n$) form a $\mathbb{Z}[\beta]$-basis of $\mathbb{Z}[\beta][x_1, \ldots, x_n]^{S_n}$.

Proof. Indeed, by letting $b_i = 0$ for all $i$, the proof of Lemma 2.5 works when we consider $\mathbb{Z}[\beta]$ instead of $\mathcal{R}$. $\square$

2.5. Factorization formula.

Proposition 2.3. For every $k \geq 1$ let $\rho_k$ denote the partition $(k, k-1, \ldots, 2, 1)$.

(1) For positive integer $n$ we have

$$
GP_{\rho_{n-1}}(x_1, \ldots, x_n|b) = GP_{\rho_{n-1}}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i \oplus x_j),
$$

$$
GQ_{\rho_{n}}(x_1, \ldots, x_n|b) = GQ_{\rho_{n-1}}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i \oplus x_j).
$$

(2) For positive integer $n$ and $\lambda \in \mathcal{P}_n$, we have

$$
GP_{\rho_{n-1}+\lambda}(x_1, \ldots, x_n|b) = GP_{\rho_{n-1}}(x_1, \ldots, x_n)G_\lambda(x_1, \ldots, x_n|b),
$$

$$
GQ_{\rho_{n}+\lambda}(x_1, \ldots, x_n) = GQ_{\rho_{n}}(x_1, \ldots, x_n)G_\lambda(x_1, \ldots, x_n|b).
$$
Theorem 3.1. The corresponding term vanishes because the claim that each summand corresponding to \( w \) in independent from \( F \) basis of \( \mathbb{P} \) \( \text{GP} \) \( \prod \) Let \( K \) \( \text{version} \) \( \text{property (Def. 1.1)} \) form a subring in \( \mathbb{Z} \) \( K \text{3.1} \). \( \text{Proposition 3.1} \). \( \text{GP} \) \( \text{GP} \) \( \text{version} \) \( \text{K-supersymmetric polynomials} \). The polynomials with the K-supersymmetric property (Def. 1.1) form a subring in \( \mathbb{Z}[\beta][x_1, \ldots, x_n] \). We denote it by \( \text{GT}_n \).

Proposition 3.1. \( \text{GP}_\lambda(x_1, \ldots, x_n) \) and \( \text{GQ}_\lambda(x_1, \ldots, x_n) \) are in \( \text{GT}_n \).

This is a consequence of the next Lemma which was proved in [13] for the case of \( \beta = 0 \).

Lemma 3.1. Let \( r \leq n \), and \( f(x_1, \ldots, x_r) \) be a polynomial in \( \mathbb{Z}[\beta][x_1, \ldots, x_r] \). Then the following functions are K-supersymmetric polynomial

\[
R_n(x_1, \ldots, x_n) = \sum_{w \in S_n} f(x_{w(1)}, \ldots, x_{w(r)}) \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_{w(i)} \oplus x_{w(j)}}{x_{w(i)} \ominus x_{w(j)}}.
\]

If moreover \( f \) is divisible by \( x_1 \cdots x_r \) then \( \tilde{R}_n(x_1, \ldots, x_n) = \frac{1}{(n-r)!} R_n(x_1, \ldots, x_n) \) has stability. i.e. \( \tilde{R}_{n+1}(x_1, \ldots, x_n, 0) = \tilde{R}_n(x_1, \ldots, x_n) \).

Proof. Let \( x_i = t, x_j = \ominus t \), for arbitrary integers \( i, j \) such that \( 1 \leq i < j \leq n \). We claim that each summand corresponding to \( w \in S_n \) does not depend on \( t \). To see this, set \( F_n(x_1, \ldots, x_n) = \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i \oplus x_j}{x_i \ominus x_j} \). In fact, if one of \( i, j \) is in \( \{w(1), \ldots, w(r)\} \) then \( F_n(x_{w(1)}, \ldots, x_{w(n)}) \) vanishes. If \( i, j \in \{w(r+1), \ldots, w(n)\} \), then by using identity

\[
\frac{x \oplus t \times \ominus t}{x \ominus t \times \oplus t} = 1
\]

we see that \( F_n(x_{w(1)}, \ldots, x_{w(n)}) \) does not depend on \( t \). Also \( f(x_{w(1)}, \ldots, x_{w(r)}) \) is obviously independent from \( t \). Hence the claim follows.

Next we prove the second assertion. Let \( x_{n+1} = 0 \). If \( n+1 \) is in \( \{w(1), \ldots, w(r)\} \), then the corresponding term vanishes because \( f(x_{w(1)}, \ldots, x_{w(r)}) = 0 \) by the assumption. If \( n+1 \in \{w(r+1), \ldots, w(n)\} \), then \( F_{n+1}(x_{w(1)}, \ldots, x_{w(n+1)}) = F_n(x_{w(1)}, \ldots, x_{w(n)}) \) where \( w' \) is a permutation obtained by multiplying the transposition \( (w(n+1), n+1) \) from the left to \( w \). This implies the desired result. \( \square \)

Theorem 3.1 (Basis theorem). The polynomials \( \text{GP}_\lambda(x_1, \ldots, x_n) (\lambda \in \mathcal{S} \mathcal{P}_n) \) form a \( \mathbb{Z}[\beta] \) basis of \( \text{GT}_n \).
3.2. Proof of Basis Theorem. We use the same strategy of proof as in [29], with the aid of the following lemma.

Lemma 3.2. Let $A$ be any unique factorization domain, $\beta, x, y$ be independent variables over $A$. If $f(\beta, x, y) \in A[\beta, x, y]$ vanishes when we made the substitution $x = \varnothing y$, then $f(\beta, x, y)$ is divisible by $x \oplus y$.

Proof. We can prove this lemma by applying the division algorithm for $f(\beta, x, y)$ as a polynomial in $x$. □

Proof of Thm 3.1 We first consider the case when $n$ is even integer. We use induction on $n$. Let $n = 2$ and let $f(x_1, x_2)$ be a $K$–supersymmetric polynomial. We may assume the constant term of $f$ is zero so that $f(0, 0) = 0$. By $K$–supersymmetry we have $f(t, \oplus t) = f(0, 0) = 0$. Then by Lemma 3.2, $f(x_1, x_2)$ is divisible by $(x_1 \oplus x_2)$. Thus $f$ can be written as $f(x_1, x_2) = (x_1 \oplus x_2)g(x_1, x_2)$. Since $g(x_1, x_2)$ is a symmetric polynomial we can write $g(x_1, x_2) = \sum_{\lambda \in \mathcal{P}_2} c_{\lambda} G_{\lambda}(x_1, x_2)$, $c_{\lambda} \in \mathbb{Z}[\beta]$. By the factorization formula, we have

$$f(x_1, x_2) = \sum_{\lambda \in \mathcal{P}_2} c_{\lambda} G_{\lambda + \rho_2}(x_1, x_2).$$

Thus $f(x_1, x_2)$ is a $\mathbb{Z}[\beta]$–linear combination of $G_{\lambda}(x_1, x_2)$, $\lambda \in \mathcal{S}\mathcal{P}_2$.

If $n \geq 4$. We proceed as follows. $f(x_1, \ldots, x_{n-2}, 0, 0)$ is $\beta$–supersymmetric. By induction, $f' = f(x_1, \ldots, x_{n-2}, 0, 0)$ is a linear combination of $G_{\lambda}(x_1, \ldots, x_{n-2})$’s where $\lambda \in \mathcal{S}\mathcal{P}_{n-2}$. Let

$$f' = \sum_{\lambda} c_{\lambda} G_{\lambda}(x_1, \ldots, x_{n-2}), \quad c_{\lambda} \in \mathbb{Z}[\beta].$$

Consider the polynomial $g(x) = \sum_{\lambda} c_{\lambda} G_{\lambda}(x_1, \ldots, x_n)$. Let $h(x) = f(x) - g(x)$. We have $h(x_1, \ldots, x_{n-2}, t, \oplus t) = h(x_1, \ldots, x_{n-2}, 0, 0) = 0$. This implies that $x_{n-1} \oplus x_n$ divides $h(x)$. Since $h(x)$ is symmetric, $h(x)$ is a multiple of $G_{\rho_n-1}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i \oplus x_j)$. Thus

$$f = g + G_{\rho_n-1}(x_1, \ldots, x_n)s(x)$$

where $s(x) \in \mathbb{Z}[\beta][x_1, \ldots, x_n]_{\mathcal{S}_n}$. Write by Cor. 2.1

$$s(x) = \sum_{\lambda \in \mathcal{P}(n)} c'_{\lambda} G_{\lambda}(x_1, \ldots, x_n), \quad c'_{\lambda} \in \mathbb{Z}[\beta].$$

Hence we have

$$(3.1) \quad f(x) = \sum_{\lambda \in \mathcal{S}\mathcal{P}_{n-2}} c_{\lambda} G_{\lambda}(x_1, \ldots, x_n) + \sum_{\lambda \in \mathcal{P}_n} c'_{\lambda} G_{\rho_n-1} + \lambda(x_1, \ldots, x_n).$$

Note that if $\lambda \in \mathcal{P}_n$ then $\rho_n-1 + \lambda \in \mathcal{S}\mathcal{P}_n$. This completes the proof.

Next consider when $n$ is odd. Let $n = 1$. This case is obvious since $G_{\Gamma_1} = \mathbb{Z}[\beta][x_1]$ by definition and $G_{\rho_n}(x_1) = x_1^n$. If $n \geq 3$ the proof is the same as the even case. □

3.3. Characterization of subring spanned by $G_{\lambda}(x)$. Let $G_{\Gamma_n,+}$ be the set consisting of $F \in G_{\Gamma_n}$ such that $F(t, x_2, \ldots, x_n) - F(0, x_2, \ldots, x_n)$ is divisible by $t \oplus t$. Obviously $G_{\Gamma_n,+}$ is a subring of $G_{\Gamma_n}$.

Proposition 3.2 (Basis theorem for $G_{\Gamma_n,+}$). $G_{\Gamma_n,+} = \bigoplus_{\lambda \in \mathcal{S}\mathcal{P}_n} \mathbb{Z}[\beta]G_{\lambda}(x_1, \ldots, x_n)$. 
Proof. We know that \( GQ_\lambda(x_1, \ldots, x_n) \) is \( K \)-supersymmetric. By definition, one sees that \( GQ_\lambda(t, x_2, \ldots, x_n) - GQ_\lambda(0, x_2, \ldots, x_n) \) is divisible by \( t \). So \( GQ_\lambda(x_1, \ldots, x_n) \) is an element of \( G\Gamma_{n+} \). Let \( F \in G\Gamma_{n+} \). We will show that \( F \) is a \( \mathbb{Z}[\beta] \)-linear combination of \( GQ_\lambda(x_1, \ldots, x_n) \) \( (\lambda \in SP_n) \) by induction on \( n \). We can proceed in the same way as in the proof for Thm. 3.1 with using \( GQ_\lambda(x_1, \ldots, x_n) \) instead of \( GP_\lambda(x_1, \ldots, x_n) \). □

3.4. Inverse limit. As with the other types of symmetric functions, our polynomials have the following stability property.

Proposition 3.3. Let \( \lambda \) be a strict partition of length \( r \) in \( SP_n \). Then

1. \( GQ_\lambda(x_1, \ldots, x_{n-1}, 0) = GQ_\lambda(x_1, \ldots, x_{n-1}) \),
2. \( GP_\lambda(x_1, \ldots, x_{n-1}, 0) = GP_\lambda(x_1, \ldots, x_{n-1}) \),

where the right-hand side are zero if \( r = n \).

Proof. We can apply Lemma 3.1 □

Remark 3.1. Note that \( K \)-supersymmetric implies

\[ GP_\lambda(x_1, \ldots, x_{n-2}, 0, 0|b) = GP_\lambda(x_1, \ldots, x_{n-2}|b) \]

However, the property \( GP_\lambda(x_1, \ldots, x_{n-1}, 0|b) = GP_\lambda(x_1, \ldots, x_{n-1}|b) \) does not hold in general; for example, we have \( GP_1(x_1, x_2) = x_1 \oplus x_2 \), whereas \( GP_1(x_1) = x_1 \oplus b_1 \). In contrast to this fact, we have \( GQ_\lambda(x_1, \ldots, x_{n-1}, 0|b) = GQ_\lambda(x_1, \ldots, x_{n-1}|b) \), which holds since \([|x|]|^k\) is divisible by \( x_1 \ldots x_r \), whereas \([|x|]|^k\) is not (see Lemma 3.1).

Let \( \varphi_{n+1} : G\Gamma_{n+1} \to G\Gamma_n \) be the morphism of the \( \mathbb{Z}[\beta] \)-algebras given by the specialization \( x_{n+1} = 0 \). Then \( \{G\Gamma_n, \varphi_n\} \) form an inverse system. Let \( G\Gamma \) denote the inverse limit \( \varprojlim G\Gamma_n \). We call this the ring of \( K \)-supersymmetric functions. Then, by the stability property of \( GP_\lambda(x_1, \ldots, x_n) \) (Prop. 3.3) we have \( GP_\lambda(x) := \varprojlim GP_\lambda(x_1, \ldots, x_n) \in G\Gamma \). Similarly we define \( G\Gamma_+ = \varprojlim G\Gamma_{n+} \), which is a subring of \( G\Gamma \). Then \( GQ_\lambda(x) := \varprojlim GQ_\lambda(x_1, \ldots, x_n) \in G\Gamma_+ \).

Remark 3.2. Both the functions \( GP_\lambda(x) \) and \( GQ_\lambda(x) \) can be expressed in the following form:

\[ F_{|\lambda|}(x) + \beta F_{|\lambda|+1}(x) + \beta^2 F_{|\lambda|+2}(x) + \cdots, \quad F_k(x) \in \Lambda^k \]

where \( \Lambda^k \) \((k \geq 0)\) is the space of homogeneous symmetric functions of degree \( k \) (see [21]). The initial terms \( F_{|\lambda|}(x) \) are \( P_\lambda(x) \) and \( Q_\lambda(x) \) respectively. In particular we see that the functions \( GP_\lambda(x) \) (resp. \( GP_\lambda(x) \)) \( \lambda \in SP \), are linearly independent over \( \mathbb{Z}[\beta] \).

Remark 3.3. From combinatorial results proved in [20] one can see that all the homogeneous parts \( F_k(x) \) are actually non-zero. Moreover, each \( F_k(x) \) is a non-negative linear combination of monomial symmetric functions \( m_\mu(x) \) such that \(|\mu| = k\).

Here is the ‘basis’ theorem for \( G\Gamma \).

Proposition 3.4. Any \( f(x) \in G\Gamma \) can be expressed uniquely as a possibly infinite \( \mathbb{Z}[\beta] \)-linear combination of \( GP_\lambda \)’s,

\[ f(x) = \sum_{\lambda \in SP} c_\lambda \cdot GP_\lambda(x), \quad c_\lambda \in \mathbb{Z}[\beta], \]

such that for all positive integer \( n \) the set \( \{\lambda \in SP_n \mid c_\lambda \neq 0\} \) is finite.
Proof. This is a direct consequence of Thm. 3.1. □

We can introduce $c_{\lambda \mu}' \in \mathbb{Z}[\beta]$ by the following expansion with possibly infinitely many terms:

$$GP_\lambda(x) \cdot GP_\mu(x) = \sum_{\nu} c_{\lambda \mu}' GP_\nu(x).$$

We expect that the right-hand side is actually a finite sum. For type $A$ case, the corresponding statement is true as a consequence of the explicit description of LR-coefficients given there ([2], Cor. 5.5).

**Conjecture 3.1.** $\bigoplus_{\lambda \in SP} \mathbb{Z}[\beta]GP_\lambda(x)$ is a subring of $GT$.

**Remark 3.4.** Our result (Thm. 8.1) shows that the constants $c_{\lambda \mu}'$, when specialized to $\beta = -1$, are equal to $K$-theory Littlewood-Richardson coefficients of the maximal orthogonal Grassmannians. Clifford, Thomas and Yong [4] has given a combinatorial interpretation for the LR-coefficients. We do not know the description in [4] implies Conjecture 3.1.

The algebra $GT$ has a natural decreasing filtration $GT = F^0 \supset F^1 \supset \cdots \supset F^k \supset \cdots$, defined by

$$F^k = F^k GT = \left\{ \sum_{\lambda \in SP} c_{\lambda} \cdot GP_\lambda(x) \in GT \mid c_{\lambda} = 0 \text{ if } |\lambda| < k \right\}.$$ 

By proposition 3.4 and a consideration of degree in $x_i$, one sees that $F^k$ is actually an ideal and $F^k, F^{k+1}$ isomorphic to $\mathbb{Z}[\beta] \otimes \mathbb{Z} \Gamma$, where $\Gamma$ denote the ring of Schur $P$-functions, the $\mathbb{Z}$-span of the $P_\lambda(x) (\lambda \in SP)$. This fact follows from Remark 3.2.

We have similar result for $GT_+$. This is a consequence of Prop. 3.2.

**Proposition 3.5.** Any $f(x) \in GT_+$ can be expressed uniquely as a possibly infinite $\mathbb{Z}[\beta]$-linear combination of $GQ_\lambda$’s,

$$f(x) = \sum_{\lambda \in SP} c_{\lambda} \cdot GQ_\lambda(x), \quad c_{\lambda} \in \mathbb{Z}[\beta],$$

such that for all positive integer $n$ the set $\{ \lambda \in SP_n \mid c_{\lambda} \neq 0 \}$ is finite.

**Conjecture 3.2.** $\bigoplus_{\lambda \in SP} \mathbb{Z}[\beta]GQ_\lambda(x)$ is a subring of $GT_+$.

**Remark 3.5.** The Pieri type rule in [12] implies that $GQ_\lambda(x)GQ_k(x)$ is a finite $\mathbb{Z}[\beta]$ linear combination of $GQ_\mu(x)$’s.

4. Preliminary on Weyl groups and root systems

4.1. Weyl groups. Let $X = B, C, D$ and $W = W(X_\infty)$ be the Weyl group of type $X_\infty$. This is a Coxeter group whose generators $\{s_i\}_{i \in I}$ are indexed by $I = \{0, 1, 2, \ldots\}$ for $B_\infty$, $C_\infty$ and $I = \{1, 1, 2, \ldots\}$ for $D_\infty$. If $W$ is type $C_\infty (B_\infty)$ then the defining relations are $s_i^2 = 1 (i = 0, 1, 2 \ldots)$ and

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (i \geq 1),$$

$$s_i s_j = s_j s_i \quad (i, j \geq 1, |i - j| \geq 2),$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \quad s_0 s_i = s_i s_0 \quad (i \geq 1).$$
If $W$ is type $D_\infty$ then the defining relations are $s_i^2 = 1$ ($i = \hat{1}, 1, 2 \ldots$), (4.1), (4.2), and (4.4) 
\begin{align*}
  s_1s_2s_1 &= s_2s_1s_2, \\
  s_1s_j &= sj (j \neq 2).
\end{align*}

The corresponding graphs are given as follows:

\[
C_\infty (B_\infty) \quad \begin{array}{cccccccc}
  s_0 & s_1 & s_2 & \cdots & s_{n-1} & s_n \\
  s_1 & s_2 & \cdots & s_{n-1} & s_n & s_0 \\
\end{array} \quad D_\infty \quad \begin{array}{cccccccc}
  s_0 & s_1 & s_2 & \cdots & s_{n-1} & s_n \\
  s_1 & s_2 & \cdots & s_{n-1} & s_n & s_0 \\
\end{array}
\]

Let $I_n = \{0, 1, \ldots, n - 1\}$ for $X = B, C$ and $\{1, 1, 2, \ldots, n - 1\}$ for $X = D$. Let $W(X_n)$ denote the subgroups of $W(X_\infty)$ generated by $s_i$ ($i \in I_n$).

### 4.2. Weyl groups and signed permutations

Let $\mathbb{N}$ be the set of positive integers $\{1, 2, \ldots\}$. Denote by $\overline{\mathbb{N}}$ a ‘negative’ copy $\overline{\mathbb{N}} = \{\bar{1}, \bar{2}, \ldots\}$ of $\mathbb{N}$. A signed permutation $w$ of $\mathbb{N}$, by definition, is an bijection on the set $\mathbb{N} \cup \overline{\mathbb{N}}$ such that $w(i) = w(\bar{i})$ for all $i \in \mathbb{N}$ and $w(i) = i$ for $i > n$ for some $n \geq 1$. Denote by $S_\infty$ the group of all signed permutations. We often denote $w \in S_\infty$ by one-line form $w(1)w(2) \cdots w(n) \cdots$. Note that we only have to specify $w(i)$ for positive $i$. For example $w = 52134 \cdots$ is a signed permutation, where dots mean the part that $w(i) = i$ holds.

Define the signed permutations $s_i$ ($i \geq 0$) by 
\begin{align*}
  s_0(1) &= \bar{1}, \\
  s_0(i) &= i (i \in \mathbb{N}, i \neq 1), \\
  s_i(i) &= i + 1, \\
  s_i(i + 1) &= i, \\
  s_i(j) &= j (j \in \mathbb{N}, j \neq i).
\end{align*}

Then (4.1), (4.2), and (4.3) hold, and $W(C_\infty) = W(B_\infty)$ can be identified with $S_\infty$. Let $s_1 = s_0s_1s_0$. Explicitly, we have 
\begin{align*}
  s_1(1) &= \bar{2}, \\
  s_1(2) &= \bar{1}, \\
  s_1(i) &= i (i \in \mathbb{N}, i \neq 1, 2).
\end{align*}

Then $W(D_\infty)$ can be identified with the subgroup of $S_\infty$ generated by $s_1$, $s_i$ ($i \geq 1$), which we denote this subgroup by $S_{\infty,+}$.

### 4.3. Strict partitions and Grassmannian elements

Let $\mathcal{SP}$ denote the set of all strict partitions, i.e. $\mathcal{SP} = \bigcup_{n \geq 0} \mathcal{SP}_n$. For each $w \in S_\infty$, define $\lambda(w) = \overline{\mathbb{N}} \cap w(\mathbb{N})$. For example if $w = 52134 \cdots$ then $\lambda(w) = \{5, 3, 1\}$. Such $\lambda(w)$ can be considered as a strict partition; for example the last $\lambda(w)$ corresponds to $(5, 3, 1) \in \mathcal{SP}$. Thus we have a surjective map $\lambda : S_\infty \to \mathcal{SP} ; w \mapsto \lambda(w)$. This map gives a bijection $S_\infty / S_\infty \cong \mathcal{SP}$, where $S_\infty$ is the subgroup of $S_\infty$ consisting of ‘ordinary’ permutations, i.e. those $w \in S_\infty$ such that $w(\mathbb{N}) \subset \mathbb{N}$.

An element $w \in S_\infty$ is called Grassmannian, if $w(1) < w(2) < \cdots$, where the elements are ordered as $\cdots < 2 < 1 < 1 < 2 < \cdots$. For example $w = 53124 \cdots$ is Grassmannian. Let $S_\infty^0$ denote the set of Grassmanian elements in $S_\infty$. Note that $S_\infty^0$ is equal to the set of elements $w \in S_\infty$ such that $\ell(ws_i) > \ell(w)$ for all $i$ other than $i = 0$, where $\ell$ is the length function of $W(B_\infty) = W(C_\infty) \cong S_\infty$.

Next fact is well-known. See for example [1], [10] (note that our convention here is different from there).

**Proposition 4.1.** The set $S_\infty^0$ forms a set of coset representatives for $S_\infty / S_\infty$. Let $\lambda$ denote the resulting bijection $S_\infty^0 \cong \mathcal{SP}$. Then for $w, v \in S_\infty$ we have $|\lambda(w)| = \ell(w)$, and $\lambda(w) \subset \lambda(v) \iff w \leq v$ (Bruhat-Chevalley order).
Next consider type $D$ case, i.e. $\overline{S}_{\infty,+}$. The group $\overline{S}_{\infty,+}$ consists of the signed permutations such that the cardinality of $\lambda(w)$ is even. The image $\lambda(\overline{S}_{\infty,+}) \subset SP$ is a subset of $\lambda$ having even length. This image can be also identified with $SP$ by "removing all the diagonal boxes" as illustrated by the following:

![Diagram of diagonal boxes]

Let $\lambda_+ : \overline{S}_{\infty,+} \rightarrow SP$ denote the obtained map. For example if $w = \overline{643125} \cdots$ then $\lambda_+(w) = (5,3,2) \in SP$. Thus $\lambda_+$ gives a bijection $\overline{S}_{\infty,+}/S_\infty \cong SP$. The set of Grassmannian elements in $\overline{S}_{\infty,+}$ is denoted by $\overline{S}_{\infty,+}^1$. As in the previous case of type $B,C$, the set $\overline{S}_{\infty,+}^1$ is equal to the set of elements $w \in \overline{S}_{\infty,+}$ such that $\ell(ws_i) > \ell(w)$ for all $i \in I - \{1\}$.

The corresponding result (also well-known) for type $D$ is the following:

**Proposition 4.2.** The set $\overline{S}_{\infty,+}^1$ forms a set of coset representatives for $\overline{S}_{\infty,+}/S_\infty$. Let $\lambda_+$ denote the resulting bijection $\overline{S}_{\infty,+}^1 \cong SP$. Then for $w,v \in \overline{S}_{\infty,+}^1$ we have $|\lambda_+(w)| = \ell(w)$, and $\lambda(w) \subset \lambda(v) \iff w \preceq v$ (Bruhat-Chevalley order).

Note that the length function $\ell$ and Bruhat order are those of type $D_\infty$.

Through the bijections given in Prop. 4.1 and Prop. 4.2 Weyl group acts naturally on $SP$. We will describe the action explicitly. Define $c(\alpha) \in I$ of $\alpha \in \mathbb{D}(\lambda)$ to be $j-i$ for type $B$ and $C$ case. For type $D$ case, we define $c(i,j) = j - i + 1$ ($i < j$) and $c(i,i) = 1$ of $i$ is even and $c(i,i) = 1$ of $i$ is odd. For each type, a strict partition $\lambda$ is $i$-removable if there is a box $\alpha \in \mathbb{D}(\lambda)$ with $c(\alpha) = i$ such that $\mathbb{D}(\lambda) - \{\alpha\}$ is a shifted diagram. Then we denote the diagram $\mathbb{D}(\lambda) - \{\alpha\}$ by $\lambda^{(i)}$. Conversely $\lambda$ is $i$-addable if there is $\mu$ such that $\mu$ is $i$-removable and $\mu^{(i)} = \lambda$.

**Proposition 4.3.** Let $\lambda \in SP$. Then
- $s_i \lambda < \lambda \iff \lambda$ is $i$-removable,
- $s_i \lambda > \lambda \iff \lambda$ is $i$-addable,
- $s_i \lambda = \lambda \iff \lambda$ is neither $i$-removable nor $i$-addable.

Furthermore, if $s_i \lambda < \lambda$, then $s_i \lambda = \lambda^{(i)}$, and if $s_i \lambda > \lambda$, then $(s_i \lambda)^{(i)} = \lambda$.

**Example 4.1.** If $\lambda = (3,1)$ then for type $B,C$ case we have the following examples:

$$
\begin{align*}
\lambda &= \begin{array}{ccc}
1 & 2 & 0 \\
0 & & \\
\end{array} \\
s_0 \lambda &= \begin{array}{ccc}
0 & 1 & 2 \\
1 & & \\
\end{array} \\
s_1 \lambda &= \begin{array}{ccc}
0 & 1 & 2 \\
0 & & \\
\end{array} \\
s_2 \lambda &= \begin{array}{ccc}
0 & & \\
0 & & \\
\end{array} \\
s_3 \lambda &= \begin{array}{ccc}
0 & 1 & 2 \\
0 & & \\
\end{array}
\end{align*}
$$

where we filled $c(\alpha)$ in each box $\alpha$. For the same $\lambda$ in type $D$ case, we have

$$
\begin{align*}
\lambda &= \begin{array}{ccc}
1 & 2 & 3 \\
1 & & \\
\end{array} \\
s_1 \lambda &= \begin{array}{ccc}
1 & 2 & 3 \\
1 & & \\
\end{array} \\
s_1 \lambda &= \begin{array}{ccc}
1 & 2 & 3 \\
1 & & \\
\end{array} \\
s_2 \lambda &= \begin{array}{ccc}
1 & 2 & 3 \\
1 & & \\
\end{array} \\
s_3 \lambda &= \begin{array}{ccc}
1 & 2 & 3 \\
1 & & \\
\end{array}
\end{align*}
$$

etc.

### 4.4. Grassmannian elements in the finite rank case.

Let $SP(n)$ denote the set of strict partitions such that $\lambda \subset \rho_n$. Consider a subgroup of $W(X_n)$ generated by $s_1, \ldots, s_{n-1}$, which is isomorphic to $S_n$. If we denote it simply by $S_n$, we have the following natural bijection:

$$SP(n) \cong W(C_n)/S_n = W(B_n)/S_n \cong W(D_{n+1})/S_{n+1}.$$
Indeed, when we identify \( W(C_\infty) \) with \( \mathbb{F}_\infty \), a set of coset representatives for \( W(C_\infty)/S_n \) is given by \( W(C_n) \cap \mathbb{F}_\infty \), which is naturally identified with \( S^P(n) \). Similarly, \( W(D_{n+1})/S_{n+1} \) can be identified with \( W(D_{n+1}) \cap \mathbb{F}_\infty, \) isomorphic to \( S^P(n) \).

4.5. **The ring \( \mathcal{R} \) and \( W \) action.** So far we have worked over the ring of coefficients \( \mathbb{Z}[\beta][b_1, b_2, \ldots] \). Now we need to work over a larger coefficient ring:

\[
\mathcal{R} := \mathbb{Z}[\beta][b_1, b_2, \ldots, b_1, b_2, \ldots]/\langle b_i \pm b_j | i \geq 1 \rangle
\]

This is essentially the ring of Laurent polynomials in infinite variables with coefficient in \( \mathbb{Z}[\beta] \). One see that \( \mathcal{R} \) is isomorphic to \( \mathbb{Z}[\beta][b_1, b_2, \ldots]|(1 + \beta b_i)^{-1}(i \geq 1) \) (see [2.1] and also to \( \mathbb{Z}[\beta][b_1, b_2, \ldots]|(1 + \beta b_1)^{-1}(i \geq 1) \). \( \mathcal{R} \) has a natural action of the Weyl group defined by \( w(b_i) = b_{w(i)}, \ w(b_1) = b_{\overline{w(i)}} \) \((w \in W, i = 1, 2, \ldots) \) with \( \beta \) fixed.

4.6. **Roots.** We fix notation about root systems. Let \( L \) denote the free \( \mathbb{Z} \)-module with basis \( \{t_i\}_{i \geq 1} \). The **positive roots** \( \Delta^+ \subset L \) (set \( \Delta^- := -\Delta^+ \) the **negative roots**) are defined by

\[
\text{Type } B_\infty : \quad \Delta^+ = \{t_i | i \geq 1\} \cup \{t_j \pm t_i | j > i \geq 1\},
\]
\[
\text{Type } C_\infty : \quad \Delta^+ = \{2t_i | i \geq 1\} \cup \{t_j \pm t_i | j > i \geq 1\},
\]
\[
\text{Type } D_\infty : \quad \Delta^+ = \{t_j \pm t_i | j > i \geq 1\}.
\]

For a positive integer \( n \), define \( \Delta^+_n := \Delta^+ \cap L_n \) where \( L_n := \bigoplus_{i=1}^n \mathbb{Z} t_i \). These are the corresponding positive root system of finite rank \( n \) of type \( B_n, C_n, \) and \( D_n \).

The following elements of \( \Delta^+ \) are called the **simple roots**:

\[
\text{Type } B_\infty : \quad \alpha_0 = t_1, \quad \alpha_i = t_{i+1} - t_i \quad (i \geq 1),
\]
\[
\text{Type } C_\infty : \quad \alpha_0 = 2t_1, \quad \alpha_i = t_{i+1} - t_i \quad (i \geq 1),
\]
\[
\text{Type } D_\infty : \quad \alpha_1 = t_1 + t_2, \quad \alpha_i = t_{i+1} - t_i \quad (i \geq 1).
\]

We can define a map \( e : L = \bigoplus_{i=1}^\infty \mathbb{Z} t_i \rightarrow \mathcal{R} \) satisfying \( e(t_i) = b_i, e(-t_i) = b_i, e(\alpha + \gamma) = e(\alpha) \oplus e(\gamma), e(\alpha - \gamma) = e(\alpha) \ominus e(\gamma) \) \((\alpha, \gamma \in L)\). Note that the map \( e \) is compatible with the natural action of \( W \) on \( L \) and the action on \( \mathcal{R} \) defined in [4.5]. For simple roots \( \alpha_i \), the explicit form of \( e(\alpha_i) \) written in terms of \( b_i \)'s are given by

\[
\text{Type } B_\infty : \quad e(\alpha_0) = b_1, \ e(\alpha_i) = b_{i+1} \ominus b_i \quad (i \geq 1),
\]
\[
\text{Type } C_\infty : \quad e(\alpha_0) = b_1 \oplus b_1, \ e(\alpha_i) = b_{i+1} \ominus b_i \quad (i \geq 1),
\]
\[
\text{Type } D_\infty : \quad e(\alpha_1) = b_1 \oplus b_2, \ e(\alpha_i) = b_{i+1} \ominus b_i \quad (i \geq 1).
\]

5. **GKM ring and its Schubert basis**

In this section we introduce a ring \( \Psi \), which is defined by the \( K \)-theoretic GKM condition. A notion of “Schubert classes” is defined in a combinatorial way.

5.1. **GKM ring \( \Psi \).** Let \( \text{Fun}(S^P, \mathcal{R}) \) denote set of all maps from \( S^P \) to \( \mathcal{R} \). \( \text{Fun}(S^P, \mathcal{R}) \) is naturally an \( \mathcal{R} \)-algebra; the \( \mathcal{R} \)-module structure is given by diagonal multiplication, and multiplication is defined in point wise manner. For each \( \alpha \in \Delta^+ \) we have \( \alpha = w(\alpha_i) \) for some \( i \in I \) and \( w \in W \). Then let \( s_\alpha = ws_w^{-1} \).

**Definition 5.1** (GKM ring). Let \( \Psi \) be the subring of \( \text{Fun}(S^P, \mathcal{R}) \) defined by the following condition:

\[
\psi(s_\alpha \mu) - \psi(\mu) \in e(\alpha) \cdot \mathcal{R} \quad \text{for all } \mu \in S^P \text{ and } \alpha \in \Delta^+.
\]

This condition and the associated geometry is discussed in [8.3]
5.2. **Divided difference operators on $\Psi$.** We now define the divided difference operators $\pi_i$ ($i \in I$) on $\Psi$ by the following formula:

$$
(\pi_i \psi)(\mu) = \frac{s_i(\psi(s_i\mu)) - (1 + \beta e(\alpha_i))\psi(\mu)}{e(\alpha_i)} \quad (\psi \in \Psi).
$$

This can be rewritten as follows:

$$
(\pi_i \psi)(\mu) = \frac{\psi(\mu) - (1 + \beta e(-\alpha_i))s_i(\psi(s_i\mu))}{e(-\alpha_i)}.
$$

If $\psi \in \Psi$ then the family $\{ (\pi_i \psi)(\mu) \}_\mu$ is actually gives an element of $\Psi$. This fact can be shown by a similar argument in [19] (the first Lemma in Appendix).

The operators $\pi_i$ satisfy following relation: If $W$ is type $C_{\infty}$ ($B_{\infty}$) then $\pi_i^2 = -\beta \pi_i$ ($i = 0, 1, 2 \ldots$) and

$$
\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \quad (i \geq 1),
$$

$$
\pi_i \pi_j = \pi_j \pi_i \quad (i, j \geq 1, |i - j| \geq 2),
$$

$$
\pi_0 \pi_1 \pi_0 \pi_1 = \pi_1 \pi_0 \pi_1 \pi_0, \quad \pi_0 \pi_i = \pi_i \pi_0 \quad (i \geq 1).
$$

If $W$ is type $D_{\infty}$ then $\pi_i^2 = -\beta \pi_i$ ($i = 1, 2 \ldots$), (5.3), (5.4), and

$$
\pi_1 \pi_2 \pi_1 = \pi_2 \pi_1 \pi_2, \quad \pi_1 \pi_j = \pi_j \pi_1 \quad (j \neq 2).
$$

**Remark 5.1.** The operators $\pi_i$ are ‘left’ divided difference operators. These operators are not appear explicitly in [20], where ‘right’ divided difference operators are used. Unfortunately we cannot find a literature that clarify a geometric origin of ‘left’ one. However, it turns out that the left divided difference operators are very useful especially for ‘parabolic’ situations. See [18] and [19] for some discussions and applications of the left divided difference operators in equivariant cohomology.

5.3. **Schubert classes.**

**Definition 5.2.** A set of elements $\{ \psi_\lambda \mid \lambda \in \mathcal{SP} \} \subset \Psi$ is called a family of Schubert classes if the following conditions are satisfied:

$$
\pi_i \psi_\lambda = \begin{cases} 
\psi_{\lambda(i)} & \text{if } s_i \lambda < \lambda \\
-\beta \psi_\lambda & \text{if } s_i \lambda \geq \lambda,
\end{cases}
$$

(5.7)

$$
\psi_\lambda(\emptyset) = \delta_{\lambda, \emptyset} \quad (Kronecker’s \ delta).
$$

(5.8)

A family of Schubert classes exists. This fact can be proved by a geometric argument in [18]. We have a proof of the existence as a consequence of Thm [7.4].

**Lemma 5.1.** Let $\{ \psi_\lambda \}_\lambda$ be a family of Schubert classes. Suppose $\mu$ be a strict partition such that $\mu \notin \emptyset$. Let $i \in I$ be such that $s_i \mu < \mu$. Then

$$
\psi_\lambda(\mu) = \begin{cases} 
(1 + \beta e(-\alpha_i))s_i(\psi_\lambda(s_i \mu)) + e(-\alpha_i) s_i(\psi_{\lambda, \lambda}(s_i \mu)) & \text{if } s_i \lambda < \lambda \\
s_i(\psi_\lambda(s_i \mu)) & \text{if } s_i \lambda \geq \lambda.
\end{cases}
$$

(5.9)

**Proof.** This is a direct consequence of the divided difference equation. $\square$

**Remark 5.2.** Recurrence equation (5.9) is called ‘left-hand’ recurrence in [23], Remark 2.3.

**Proposition 5.1.** A family of Schubert classes, if exists, is unique.
Proof. Let $\{\psi_\lambda\}_{\lambda \in \mathcal{SP}}$ be a family of Schubert classes. By definition, we have $\psi_\lambda(\emptyset) = \delta_{\lambda,0}$. Suppose $\mu \neq \emptyset$. Then there is $i \in I$ such that $s_i \mu < \mu$. By the recurrence equation (5.2), $\{\psi_\lambda(\mu) \mid \lambda \in \mathcal{SP}\}$ is uniquely determined from $\{\psi_\lambda(s_i \mu) \mid \lambda \in \mathcal{SP}\}$. Hence by using induction on $|\mu|$, we conclude the uniqueness. \hfill $\square$

Proposition 5.2. Let $\{\psi_\lambda\}_\lambda$ be a family of Schubert classes.

1. If $\lambda \not\subseteq \mu$ then $\psi_\lambda(\mu) = 0$.
2. We have $\psi_\lambda(\lambda) = \prod_{\alpha \in \text{Inv}(\lambda)} e(-\alpha)$, where $\text{Inv}(\lambda) = \{\alpha \in \Delta^+ \mid s_\alpha \lambda < \lambda\}$.

Proof. (1) We prove by induction on $|\mu|$. For $\mu = \emptyset$, the vanishing property hold by the initial condition. Let $\mu \neq \emptyset$. There exists $i$ such that $s_i \mu < \mu$. Then we have $\lambda \not\subseteq s_i \mu$. So by inductive hypothesis we have $\psi_\lambda(s_i \mu) = 0$. Thus if $s_i \lambda \geq \lambda$ then from (5.2) we have $\psi_\lambda(\mu) = 0$. Next suppose $s_i \lambda < \lambda$. Then since both $\lambda$ and $\mu$ are $i$-removable, we have $s_i \lambda \not\subseteq s_i \mu$. By inductive hypothesis, we have $\psi_\lambda(s_i \mu) = \psi_{s_i \lambda}(s_i \mu) = 0$. Hence from (5.2) we have $\psi_\lambda(\mu) = 0$.

(2) There exists $i$ such that $s_i \lambda < \lambda$. By the recurrence equation (5.2) together with $\psi_\lambda(s_i \lambda) = 0$ as a consequence of (1), we have $\psi_\lambda(\lambda) = e(-\alpha) s_i \psi_{s_i \lambda}(s_i \lambda)$. By this equation $\psi_\lambda(\lambda)$ are determined inductively. We see that $\prod_{\alpha \in \text{Inv}(\lambda)} e(-\alpha)$ satisfies this equation since we have $\text{Inv}(\lambda) = s_i \text{Inv}(s_i \lambda) \cup \{\alpha_i\}$. \hfill $\square$

Proposition 5.3. Let $\{\psi_\lambda\}_\lambda$ be a family of Schubert classes. Any element in $\Psi$ can be uniquely expressed as a possibly infinite $\mathcal{R}$-linear combination of $\psi_\lambda$ ($\lambda \in \mathcal{SP}$).

Proof. Let $\psi \in \Psi$. Define $\text{Supp}(\psi) = \{\mu \in \mathcal{SP} \mid \psi(\mu) \neq 0\}$. Let $\lambda \in \text{Supp}(\psi)$ be a minimum element in the ordering of strict partitions by inclusion. From the GKM condition and Prop. 5.2 we see that $\psi(\lambda)$ is divisible by $e(-\alpha)$ for all $\alpha \in \text{Inv}(\lambda)$. We see that the elements $\{e(-\alpha) \mid \alpha \in \text{Inv}(\lambda)\}$ are relatively prime, and hence $\psi(\lambda)$ is divisible by their product $\psi_\lambda(\lambda) = \prod_{\alpha \in \text{Inv}(\lambda)} e(-\alpha)$. Let $\psi' = \psi - \psi_\lambda(\lambda) \cdot \psi_\lambda$. By the minimality of $\lambda$ in $\text{Supp}(\psi)$ and Prop. 5.2 (1), $\text{Supp}(\psi') \subsetneq \text{Supp}(\psi)$. By repeating this, we may write $\psi$ as a possibly infinite $\mathcal{R}$-linear combination of $\psi_\lambda$'s. \hfill $\square$

6. Divided difference equation for $GP_\lambda(x|b)$ and $GQ_\lambda(x|b)$

In this section, we prove a divided difference equation for $GP_\lambda(x|b)$ and $GQ_\lambda(x|b)$.

6.1. Ring $\mathcal{G}_\mathcal{R}^X$ and inverse limit of the $K$-theoretic factorial $P$- and $Q$-functions.

Let $X$ be $B, C$ or $D$. In order to state results for different types in a parallel way, we use the following convention: We denote by $\mathcal{G}_\mathcal{R}^X$ the following: $\mathcal{G}_\mathcal{R}^C = \mathcal{R} \otimes_{\mathbb{Z}[\beta]} \mathcal{G}_+$, $\mathcal{G}_\mathcal{R}^C = \mathcal{G}_\mathcal{R}^D = \mathcal{R} \otimes_{\mathbb{Z}[\beta]} \mathcal{G}_T$. We often suppress $X$ when there is no fear of confusion.

For each $\lambda \in \mathcal{SP}_n$, we define $GX_\lambda^{(n)}(x|b)$ by

$$
\begin{align*}

(6.1) \quad GB_\lambda^{(n)}(x|b) &= GP_\lambda(x_1, \ldots, x_n|0, b_1, b_2, \ldots), \\

(6.2) \quad GC_\lambda^{(n)}(x|b) &= GQ_\lambda(x_1, \ldots, x_n|b_1, b_2, \ldots), \\

(6.3) \quad GD_\lambda^{(n)}(x|b) &= \begin{cases} 
GP_\lambda(x_1, \ldots, x_n|b_1, b_2, \ldots) & \text{if $n$ is even} \\
GP_\lambda(x_1, \ldots, x_n, 0|b_1, b_2, \ldots) & \text{if $n$ is odd} 
\end{cases}
\end{align*}
$$

We can define $GX_\lambda(x|b) = \lim_{\lambda \to \lambda^0} GX_\lambda^{(n)}(x|b) \in \mathcal{G}_\mathcal{R}^X$. 

Proposition 6.1. Any \( f(x) \in \text{GT}_R^X \) can be expressed uniquely as a possibly infinite \( \mathcal{R} \)-linear combination of \( GX_{\lambda}(x|b) \)'s,
\[
(6.4) \quad f(x) = \sum_{\lambda \in \mathcal{S}P} c_\lambda \cdot GX_{\lambda}(x|b), \quad c_\lambda \in \mathcal{R},
\]
such that for all positive integer \( n \) the set \( \{ \lambda \in \mathcal{S}P_n \mid c_\lambda \neq 0 \} \) is finite.

Proof. Using Lem. 2.5 in place of Cor. 2.4 we can show this by the same argument as in \[3.2\] (cf. Prop. 3.4 and Prop. 3.5). \( \square \)

6.2. Action of \( W \) on the \( K \)-supersymmetric algebras. We define an action of \( W(X_\infty) \) on \( \text{GT}_R^X \). Let \( s_i \) \( (i \geq 1) \) act on \( \mathcal{R} \otimes \mathbb{Z}[\beta] \) \( \mathcal{G} \) by
\[
(6.5) \quad (s_0 \phi)(x_1, x_2, \ldots) = \phi(b_1, x_1, x_2, \ldots),
\]
which is a well-defined element in \( \mathcal{R} \otimes \mathbb{Z}[\beta] \) \( \mathcal{G} \). In general, define
\[
(6.6) \quad s_0 \left( \sum \alpha c_\alpha \phi_\alpha \right) = \sum \alpha s_0(c_\alpha) \cdot s_0(\phi_\alpha) \quad (c_\alpha \in \mathcal{R}, \phi_\alpha \in \mathcal{G}).
\]
We have \( s_0^2 = \text{id} \) on \( \mathcal{R} \otimes \mathbb{Z}[\beta] \) \( \mathcal{G} \) by virtue of the \( K \)-supersymmetric property (cf. \[9\]). Define also \( s_1 = s_0s_1s_0 \).

Proposition 6.2. The operators \( s_i \) \( i \in I \) give an action of \( W(X_\infty) \) on \( \text{GT}_R^X \).

Proof. The proof is straightforward (cf. \[9\], Prop. 7.2). \( \square \)

6.3. Divided difference operators \( \pi_i \) on \( \text{GT}_R^X \). The divided difference operator \( \pi_i \) on \( \text{GT}_R^X \) is defined by
\[
(6.7) \quad \pi_iF = \frac{s_iF - (1 + \beta e(\alpha_i))F}{e(\alpha_i)} \quad \text{for all } F \in \text{GT}_R^X.
\]
One can check directly that \( s_iF - F \) is divisible by \( e(\alpha_i) \) so the right-hand side is an element in \( \text{GT}_R^X \).

Remark 6.1. Using equation \( (1 + \beta e(\alpha_i))(1 + \beta e(-\alpha_i)) = 1 \) we can rewrite \[6.7\] as
\[
\pi_iF = \frac{F - (1 + \beta e(-\alpha_i))s_iF}{e(-\alpha_i)} \quad \text{for all } F \in \text{GT}_R^X.
\]

Theorem 6.1. We have
\[
\pi_iGX_{\lambda}(x|b) = \begin{cases} 
GX_{\lambda(\iota)}(x|b) & \text{if } s_i \lambda < \lambda \\
-\beta GX_{\lambda}(x|b) & \text{if } s_i \lambda \geq \lambda.
\end{cases}
\]

Proof. We first prove type \( C \) case. Let \( i \geq 1 \). We will work over finite \( n \) variables \( x_1, \ldots, x_n \). Since the operator \( \pi_i \) acts only non-trivially on \( b_i \) and \( b_{i+1} \), we only have to show
\[
\pi_i[[x|b]]^\lambda = \begin{cases} 
[[x|b]]^{\lambda(\iota)} & \text{if } s_i \lambda < \lambda \\
-\beta[[x|b]]^\lambda & \text{if } s_i \lambda \geq \lambda.
\end{cases}
\]
If $s_i \lambda \geq \lambda$ then one sees that $[[x|b]]^\lambda$ is symmetric with respect to $b_i$ and $b_{i+1}$. It follows that $\pi_i GQ_\lambda(x|b) = -\beta GQ_\lambda(x|b)$. If $s_i \lambda < \lambda$ then $\lambda_k = i + 1$ for some $k$. Then we note that $\prod_{j<k}[[x|b]]^\lambda$ is symmetric with respect to $b_i$ and $b_{i+1}$, and $\prod_{j>k}[[x|b]]^\lambda$ does not depend on $b_i$ and $b_{i+1}$. So we only have to calculate $[[x|b]]^\lambda$. It is easy to check the equation

$$x_k \oplus b_{i+1} - (1 + \beta b_{i+1} \ominus b_i)(x_k \oplus b_i) = b_{i+1} \ominus b_i,$$

which is equivalent to $\pi_i(x_k \oplus b_i) = 1$. So we have $\pi_i([[x|b]]^{i+1}) = [[x|b]]^i$.

Next we consider the operator $\pi_0$. Define $S_{n,r} = \{w \in S_n \mid w(r+1) < \cdots < w(n)\}$. Then the definition of $GQ_\lambda(x_1, \ldots, x_n, x_{n+1} \mid b_1, b_2, \ldots)$ reads

$$(6.8) \quad \sum_{w \in S_{n+1,r}} \prod_{i=1}^r (x_{w(i)} \oplus x_{w(i)}) x_{w(i) \ominus b_1}(x_{w(i)} \oplus b_2) \cdots \prod_{i=1}^{r-1} x_{w(i) \oplus b_{\lambda-1} \ominus b_1} \times \prod_{j=i+1}^{r+1} \prod_{i=1}^{n+1} x_{w(i)} \oplus x_{w(j)} x_{w(i) \ominus x_{w(j)}}.$$

Suppose $s_0 \lambda < \lambda$ then we have $\lambda_j \geq 2$ for $1 \leq j \leq r - 1$, $\lambda_r = 1$. We will prove

$$GQ_\lambda(x_1, \ldots, x_n, b_1 \ominus b_1, b_2, \ldots) - (1 + \beta(b_1 \oplus b_1)) GQ_\lambda(x_1, \ldots, x_n b_1, b_2, \ldots)$$

$$b_1 \oplus b_1$$

$$(6.9) = GQ_{\lambda(0)}(x_1, \ldots, x_n | b_1, b_2, \ldots).$$

If $w(k) = n + 1$ for some $k$ such that $1 \leq k \leq r - 1$ then the corresponding term in (6.8) vanishes when we substitute $x_{n+1} = b_1$. Define $S_{n+1,r}^{\lambda}$ to be the set of elements $w \in S_{n+1,r}$ such that $w(r) = n + 1$. Then the corresponding part of (6.8), after substituting $x_{n+1} = b_1$, becomes

$$(6.10) \quad (b_1 \oplus b_1) \sum_{w \in S_{n+1,r}^{\lambda}} \prod_{j=r+1}^{n+1} b_1 \oplus x_{w(j)} x_{w(i) \ominus b_1} x_{w(i) \oplus b_2} \cdots \prod_{i=1}^{r-1} x_{w(i) \oplus b_{\lambda-1} \ominus b_1} \times \prod_{j=i+1}^{r+1} \prod_{i=1}^{n+1} x_{w(i)} \oplus x_{w(j)} x_{w(i) \ominus x_{w(j)}},$$

where we used an obvious equation

$$(6.11) \quad (x_{w(i)} \ominus b_1) \left( \frac{x_{w(i)} \oplus b_1}{x_{w(i) \ominus b_1}} \right) = x_{w(i)} \oplus b_1.$$

Define $w' \in S_n$ by $w'(i) = w(i)$ ($1 \leq i \leq r - 1$), $w'(i) = w(i + 1)$ ($r \leq i \leq n$). Then $w' \in S_{n,r}$ and this correspondence gives a bijection $S_{n+1,r}^{\lambda} \rightarrow S_{n,r-1}$. Then (6.10) is written as follows:

$$(6.12) \quad (b_1 \oplus b_1) \sum_{w' \in S_{n,r-1}} \prod_{j=r}^{n} b_1 \oplus x_{w'(j)} x_{w'(i) \ominus b_1} x_{w'(i) \oplus x_{w'(j)}} \times \prod_{i=1}^{r-1} x_{w'(i) \oplus b_{\lambda-1} \ominus b_1} \times \prod_{j=i+1}^{r+1} \prod_{i=1}^{n+1} x_{w'(i)} \oplus x_{w'(j)} x_{w'(i) \ominus x_{w'(j)}}.$$

Define $S_{n+1,r}^{\mu}$ to be the set of elements $w \in S_{n+1,r}$ such that $w(n + 1) = n + 1$. We calculate the corresponding part in (6.8), after substitution $x_{n+1} = b_1$. This, using (6.11) again, is equal to

$$(6.13) \quad \sum_{w \in S_{n+1,r}^{\mu}} \frac{x_{w(r)} \oplus b_1}{x_{w(r) \oplus b_1}} \times \prod_{i=1}^{r} x_{w(i) \ominus b_1} x_{w(i) \oplus b_1} \times \prod_{j=i+1}^{r+1} \prod_{i=1}^{n+1} x_{w(i)} \oplus x_{w(j)} x_{w(i) \ominus x_{w(j)}}.$$

Let $F_\lambda(x)$ denote the last function. The natural embedding $S_n \subset S_{n+1}$ given by $w(n + 1) = n + 1$ gives a bijection $S_{n,r} \cong S_{n+1,r}^{\mu}$. Using the identity

$$(6.14) \quad \frac{x_{w(r)} \oplus b_1}{x_{w(r) \oplus b_1}} - (1 + \beta(b_1 \oplus b_1)) = \frac{b_1 \oplus b_1}{x_{w(r) \ominus b_1}}.$$
we have

\[ F_\lambda(x) - (1 + \beta(b_1 \oplus b_1))GQ_\lambda(x_1, \ldots, x_n | b_1, b_2, \ldots) \]

\[ = \sum_{w \in S_{n,r}} \frac{b_1 \oplus b_1}{x_{w(r)} \oplus b_1} \prod_{i=1}^{r} \left( \frac{x_{w(i)} \oplus x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}} \right) \times \left( \prod_{i=1}^{n} \prod_{j=i+1}^{n} \frac{x_{w(i)} \oplus x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}} \right) \]

\[ \times \prod_{i=1}^{r-1} \left[ \prod_{i=1}^{n} \prod_{j=i+1}^{r-1} \frac{x_{w(i)} \oplus x_{w(j)}}{x_{w(i)} \oplus x_{w(j)}} \right]. \]

(6.15)

Now using (6.12), (6.15), and (2.3) in Lemma 2.1, with \( m = n - r + 1, t = b_1 \), we reduce (6.9).

Next suppose \( s_0 \lambda \geq \lambda \), which is equivalent to \( \lambda_r \geq 2 \), where \( r \) is the length of \( \lambda \). We show

\[ GQ_\lambda(x_1, \ldots, x_n, b_1 | \oplus b_1, b_2, \ldots) = GQ_\lambda(x_1, \ldots, x_n | b_1, b_2, \ldots). \]

In (6.8), those \( w \in S_{n+1} \) with \( 1 \leq w(n+1) \leq r \) will vanish after the specialization \( x_{n+1} = b_1 \). Using (6.11), we see that the remaining terms, i.e. those elements satisfying \( w(n+1) = n + 1 \), coincide with the summands of \( GQ_\lambda(x_1, \ldots, x_n | b_1, b_2, \ldots) \) in the definition.

Next we consider the case of type B. Calculations for equations for \( \pi_i (i \geq 1) \) are almost the same as for the case of type C. Indeed, we now use \( x \prod_{i=1}^{k-1} (x \oplus b_i) \) instead of \( [[x|b]]^k \).

For the equation with respect to the operator \( \pi_0 \), if \( s_0 \lambda \leq \lambda \), we use the identity

\[ \frac{x_{w(r)} \oplus b_1}{x_{w(r)} \oplus b_1} - (1 + \beta b_1) = \frac{b_1(2 + \beta x_{w(r)})}{x_{w(r)} \oplus b_1} \]

(6.16)

instead of (6.14). Noting that \( x_{w(r)}(2 + \beta x_{w(r)}) = x_{w(r)} \oplus x_{w(r)} \) the rest is quite similar using the same equation (2.1) in the same way. The case of \( s_0 \lambda \geq \lambda \) is the same as for type C.

Finally we consider the case of type D. We also use finite \( n \) variables \( x_1, \ldots, x_n \), but note that \( n \) should be even here. Calculations for equations for \( \pi_i (i \geq 2) \) are quite similar to the case of type C. For the equation with respect to the operator \( \pi_1 \), we need to calculate the function

\[ GP_\lambda(x_1, \ldots, x_n, b_1, b_2 | \oplus b_1, \ominus b_2, b_3, \ldots). \]

The case \( s_1 \lambda \geq \lambda \) is easy. Indeed (6.17) is equal to \( GP_\lambda(x_1, \ldots, x_n | b_1, b_2, b_3, \ldots) \). For the case \( s_1 \lambda \leq \lambda \), we can deduce

\[ GP_\lambda(x_1, \ldots, x_n, b_1, b_2 | \oplus b_1, \ominus b_2, b_3, \ldots) - (1 + \beta (b_1 \oplus b_2))GP_\lambda(x_1, \ldots, x_n | b_1, b_2, b_3, \ldots) \]

\[ = GP_{\lambda^{\hat{\lambda}}}(x_1, \ldots, x_n | b_1, b_2, b_3, \ldots) \]

by using the following identities

\[ \frac{x_{w(r)} \oplus b_2}{x_{w(r)} \oplus b_1} - (1 + \beta (b_1 \oplus b_2)) = \frac{b_1 \oplus b_2}{x_{w(r)} \oplus b_1}, \]

(6.18)

\[ \prod_{i=1}^{m} \frac{u_i \oplus t}{u_i \ominus t} \prod_{j \neq i} \frac{u_i \oplus u_j}{u_i \ominus u_j} + \prod_{i=1}^{m} \frac{t \oplus u_i}{t \ominus u_i} = 1, \]

(6.19)
where the latter holds for even integer \( m \). The last equation is exactly (2.2) for \( k = 0 \) and \( n \) odd. Equation for \( \pi_1 \)

\[ \square \]

7. Localization map

In this section we define localization map \( \Phi \). This map gives an injective \( \mathcal{R} \)-algebra homomorphism from the ring of \( K \)-supersymmetric functions into \( \Psi \). The image of \( GX_\lambda(x|b) \)'s under the map \( \Phi \) is shown to be a (unique) family of Schubert classes.

7.1. Vanishing property. Let \( v \) be the Grassmannian element in \( W \) corresponding to a strict partition \( \mu \), i.e. \( \lambda(v) = \mu \) (\( \lambda_+(v) = \mu \) for type \( D \) case) in the notation of §4.3.

Define a sequence \( b_\mu \) of elements in \( \mathcal{R} \) by

\[
(b_\mu)_i = \begin{cases} 
b_{v(i)} & \text{if } v(i) \text{ is negative} \\
0 & \text{otherwise.} \end{cases}
\]

Explicitly the sequences \( b_\mu \) for types \( B, C \) are as follows:

\[
b_\mu = (\ominus b_{\mu_1}, \ldots, \ominus b_{\mu_r}, 0, \ldots),
\]

where \( r \) is the length of \( \mu \), and for type \( D \):

\[
b_\mu = \begin{cases} 
(\ominus b_{\mu_1+1}, \ldots, \ominus b_{\mu_r+1}, 0, \ldots) & \text{if } r \text{ is even} \\
(\ominus b_{\mu_1+1}, \ldots, \ominus b_{\mu_r+1}, \ominus b_1, 0, \ldots) & \text{if } r \text{ is odd.}
\end{cases}
\]

Remark 7.1. We defined \( b_\mu \) for \( \mu \in \check{P}_n \) in §2.3 (type A case). Notify that the above definition is for types \( B, C, \) and \( D \).

**Proposition 7.1** (Vanishing property). Let \( \lambda, \mu \) be strict partitions. Then \( GX_\lambda(b_\mu|b) = 0 \) unless \( \lambda \subset \mu \) and \( GX_\lambda(b_\mu|b) = \prod_{\alpha \in \text{Inv}(\lambda)} e(-\alpha) \).

**Proof.** We only show this for \( GP_\lambda(x|b) \) and the case when the length \( r \) of \( \mu \) is even. Other cases are similar. Using the stability property, \( GP_\lambda(b_\mu|b) \) can be evaluated as \( GP_\lambda(\ominus b_{\mu_1+1}, \ldots, \ominus b_{\mu_r+1}|b) \).

Suppose \( \lambda \not\subset \mu \). So we have \( \mu_j < \lambda_j \) for some \( 1 \leq j \leq r \). Then we easily see that the polynomial \( \prod_{i=1}^{r} x_{w(i)}^{\lambda_i} \) vanishes for any \( w \in S_r \) when we specialize \( x_i \) to \( \ominus b_{\mu_i+1} \) (\( 1 \leq i \leq r \)). Note that the denominator of the factor

\[
\prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_{w(i)} \oplus x_{w(j)}}{x_{w(i)} \ominus x_{w(j)}}
\]

does not vanish identically under the specialization. Thus we have \( GP_\lambda(b_\mu|b) = 0 \).

Next we consider the case \( \lambda = \mu \). We see that the terms other than the one comes from \( w = e \) are vanish, by the similar reason to the previous case. The term corresponding to \( w = e \) is evaluated by using the following equation:

\[
(\ominus b_{\lambda_1+1} \oplus b_1) \cdots (\ominus b_{\lambda_1+1} \oplus b_\lambda) \prod_{k=\lambda_{i+1}}^{r} \frac{\ominus b_{\lambda_{i+1}} \ominus b_{\lambda_k+1}}{\ominus b_{\lambda_{i+1}} \oplus b_{\lambda_k+1}}
\]

\[
= \prod_{k \in \{1, \ldots, n\} \setminus \cup_{i=\lambda_{i+1}}^{r} (\lambda_{i+1})} (\ominus b_{\lambda_{i+1}} \ominus b_k) \prod_{k=\lambda_{i+1}}^{r} (\ominus b_{\lambda_{i+1}} \ominus b_{\lambda_k+1}),
\]

which is a simple consequence of cancellation. This last expression is readily seen to be \( \prod_{\alpha \in \text{Inv}(\lambda)} e(-\alpha) \). \( \square \)
7.2. Algebraic localization map $\Phi$. Let $\mu$ be a strict partition. Define an $\mathcal{R}$-algebra homomorphism $\Phi_\mu : \Gamma \mathcal{R}_\mathcal{R} \to \mathcal{R}$ by “substitution” $x = b_\mu$. This is well-defined. In fact, we know that arbitrary element $F(x)$ in $\Gamma \mathcal{R}$ can be written as a possibly infinite $\mathcal{R}$-linear combination $F(x) = \sum_{\lambda \in \mathcal{SP}} c_\lambda \cdot GX_\lambda(x|b)$ of $GX_\lambda$’s. Since $GX_\lambda(b_\mu|b)$ can be non-zero for only finitely many $\lambda$’s such that $\lambda \subset \mu$, $F(b_\mu) = \sum_{\lambda \in \mathcal{SP}} c_\lambda \cdot GX_\lambda(b_\mu|b)$ is a well-defined element in $\mathcal{R}$.

**Definition 7.1** (Localization map). Define the homomorphism of $\mathcal{R}$-algebras $$\Phi : \Gamma \mathcal{R}_\mathcal{R} \to \text{Fun}(\mathcal{SP}, \mathcal{R}) \quad f \mapsto (v \mapsto \Phi_\mu(f)).$$

$\Phi$ is called the localization map for $\Gamma \mathcal{R}_\mathcal{R}$.

We often call $\Phi$ the algebraic localization map in order to form a contrast to geometric one. See $\mathbb{K}$ for geometric background of the map $\Phi$.

**Proposition 7.2.** $\text{Im}(\Phi) \subset \Psi$.

This is a consequence of the following more explicit statement.

**Lemma 7.1.** Let $t_{ij} = s_{t_{ij}} - t_i, s_{ij} = s_{t_{ij}} + t_i (j > i \geq 1), s_{ii} = s_{t_{ii}} = s_{2t_i}$. If $F(x)$ is a $K$-supersymmetric function in $\Gamma \mathcal{R}$ then, for any $\mu \in \mathcal{SP}$, we have

\begin{equation}
F(b_{t_{ij}}) - F(b_\mu) \in \langle b_j \oplus b_i \rangle \quad \text{for } j > i \geq 1, \quad (7.1)
\end{equation}

\begin{equation}
F(b_{s_{ij}}) - F(b_\mu) \in \langle b_j \oplus b_i \rangle \quad \text{for } j > i \geq 1, \quad (7.2)
\end{equation}

\begin{equation}
F(b_{s_{ii}}) - F(b_\mu) \in \langle b_i \rangle \quad \text{for } i \geq 1. \quad (7.3)
\end{equation}

If $F(x)$ is in $\Gamma_\mathcal{R}_+$ then we have

\begin{equation}
F(b_{s_{ii}}) - F(b_\mu) \in \langle b_i \oplus b_i \rangle \quad \text{for } i \geq 1. \quad (7.4)
\end{equation}

**Proof.** There are the following possibilities: (i) $i, j \in \lambda$, (ii) $i, j \notin \lambda$, $i \in \lambda$ and $j \notin \lambda$, (iv) $i \notin \lambda$ and $j \in \lambda$. Actions of $t_{ij}$ and $s_{ij}$ are given as follows:

| $i,j \in \lambda$ | $i,j \notin \lambda$ | $i \in \lambda, j \notin \lambda$ | $i \notin \lambda, j \in \lambda$ |
|------------------|-------------------|------------------|------------------|
| $t_{ij} \cdot \lambda$ | $\lambda$ | $(\lambda \setminus \{i\}) \cup \{j\}$ | $(\lambda \setminus \{j\}) \cup \{i\}$ |
| $s_{ij} \cdot \lambda$ | $\lambda \setminus \{i,j\}$ | $\lambda \cup \{i,j\}$ | $\lambda$ |

We may concentrate on two variables $x_i, x_j$. The check for (7.1) is easy in view of the fact that $F$ is symmetric (note that $(e(-\alpha)) = (e(\alpha))$). To show that (7.2), it suffices to consider the cases (i) and (ii) (the cases (iii) and (iv) are obvious). Let $G = F(\oplus b_i, \oplus b_j) - F(0, 0)$. Then $G$ as a polynomial of $b_i$ vanishes at $\oplus b_j$, because $G$ is $K$-supersymmetric. This implies that $G$ is divisible by $b_i \oplus b_i$ (cf. Lemma 3.2). Hence we have (7.2).

Next we show (7.3) and (7.4). It is obvious that $F(\oplus b_i) - F(0)$ is divisible by $\oplus b_i$. The divisibility for $F(\oplus b_i) - F(0)$ by $b_i \oplus b_i$ is the very condition that $F$ is a member of $\Gamma_\mathcal{R}_+$. This conclude the proof. \qed

The following is the main result of this paper.

**Theorem 7.1.** $\{\Phi(GX_\lambda(x|b))\}_{\lambda \in \mathcal{SP}}$ is a family of Schubert classes.

**Proof.** By the uniqueness of Schubert classes (Prop. 5.1), it suffices to check the defining property. We know that $\Phi(GX_\lambda(x|b)) \in \Psi$ by Prop. 7.2. Note that $\Phi \circ \pi_i = \pi_i \circ \Phi$ holds (cf. 2, Prop. 7.4.). By this fact and Thm. 6.1 the divided difference equation (5.7) is satisfied. The initial condition (5.8) is satisfied because $GX_\lambda(x|b)$ vanishes at $x = b_\emptyset = (0, 0, \ldots)$ for $\lambda \neq \emptyset$, and we have $GX_\emptyset(x|b) = 1$. \qed
Corollary 7.1. $\Phi$ is injective.

Remark 7.2. One can prove the injectivity of $\Phi$ in the same way as [9], Lemma 6.5.

8. Equivariant $K$-theory of the maximal isotropic Grassmannians

In this section, we show an application of the $K$-theoretic factorial $P$- and $Q$-functions to the equivariant $K$-theory of the maximal isotropic Grassmannians.

8.1. Maximal isotropic Grassmannians. Let $n$ be a positive integer. Suppose $G$ is one of the groups $SO(2n + 1, \mathbb{C}), Sp(2n, \mathbb{C})$, or $SO(2n + 2, \mathbb{C})$. Let $T$ be a maximal torus of $G$, and let $B$ be a Borel subgroup containing $T$. Then we have the corresponding root system, which we identify it with the one defined in §4.6 for types $B_n, C_n, D_{n+1}$ respectively. Let $P$ be the standard maximal parabolic subgroups corresponding to the simple root $\alpha_0$ for $B_n$ and $C_n$, and $\alpha_1$ for $D_{n+1}$. Let $G_n$ denote the homogeneous variety $G/P$. Then $G_n$ is $OG(n, 2n + 1), LG(n), OG(n + 1, 2n + 2)$ respectively for type $B_n, C_n$, and $D_{n+1}$, where $LG(n)$ denote the Lagrangian Grassmannian, and $OG(n, 2n + 1), OG(n + 1, 2n + 2)$ the maximal odd and even orthogonal Grassmannians. It is known that $OG(n, 2n + 1)$ and $OG(n + 1, 2n + 2)$ are isomorphic as algebraic varieties, however, note that here we are considering actions of different tori.

We denote by $G_n^T$ the set of $T$-fixed points in $G_n$. We have a natural bijection $G_n^T \cong SP(n)$, where $SP(n)$ is the set of strict partitions $\lambda$ such that $\lambda \subset \rho_n$. Let $e_\lambda \in G_n^T$ denote the $T$-fixed point corresponding to $\lambda \in SP(n)$. Let $B_-$ be the opposite Borel subgroup, so that we have $B \cap B_- = T$. The Schubert varieties $\Omega_\lambda$ in $G_n$ is defined to be the closure of the orbit $B_- e_\lambda \subset G/P$. Then the co-dimension of $\Omega_\lambda$ is given by $|\lambda|$. Note that $e_\mu \in \Omega_\lambda$ if and only of $\lambda \subset \mu$.

Let $K_T(G_n)$ denote the Grothendieck group of the abelian category of $T$-equivariant coherent sheaves on $G_n$. It is known that $K_T(G_n)$ has a natural ring structure. Since $\Omega_\lambda$ is $T$-stable, its structure sheaf $O_{\Omega_\lambda}$ defines a class $[O_{\Omega_\lambda}]_T$ in $K_T(G_n)$. $K_T(G_n)$ has a natural $R(T)$-algebra structure, where $R(T)$ is the representation ring of $T$, which is also identified with $K_T(pt)$. As an $R(T)$-module $K_T(G_n)$ is free and has basis consisting of the Schubert structure sheaves $[O_{\Omega_\lambda}]_T$ ($\lambda \in SP(n)$).

8.2. GKM ring. Let $\Delta(n)^+$ denote the set of positive roots associated to the pair $(G, B)$. Then $\Delta(n)^+$ is $\Delta^+_B$ for types $B, C$, and $\Delta^+_n$ for type $D$ (see §1.6 for the notation $\Delta^+_B$). We realize $R(T)$ as the ring of Laurent polynomials $R(T) = \mathbb{Z}[e^{\pm a_1}, \ldots, e^{\pm a_n}]$ for types $B, C$ and $R(T) = \mathbb{Z}[e^{\pm a_1}, \ldots, e^{\pm a_n}]$ for type $D$. Let us denote by $Fun(SP(n), R(T))$ the set of all functions from $SP(n)$ to $R(T)$. This is naturally an $R(T)$-algebra by pointwise multiplication.

Definition 8.1. Let $\Psi_n$ be the $R(T)$-subalgebra of $Fun(SP(n), R(T))$ defined as follows:

a map $\psi : SP(n) \to R(T)$ is in $\Psi_n$ if and only if

$$\psi(s_\alpha \mu) - \psi(\mu) \in (1 - e^\alpha)R(T)$$

for all $\mu \in SP(n)$, $\alpha \in \Delta(n)^+$.

Letting $\beta = -1$ we consider $R(T)$ as a subalgebra of $\mathbb{Z} \otimes_{\mathbb{Z}[\beta]} \mathcal{R} = \mathbb{Z}[e^{\pm a_1}, e^{\pm a_2}, \ldots]$. If $\lambda, \mu$ are strict partitions in $SP(n)$, we see that $\psi_\lambda(\mu) \in R(T)$. Thus we have $\psi_\lambda(\mu) := \psi_\lambda|_{SP(n)} \in Fun(SP(n), R(T))$, which is obviously an element in $\Psi_n$. Thus we have the following:

Proposition 8.1. $\Psi_n = \bigoplus_{\lambda \in SP(n)} R(T)\psi_\lambda(n)$. 

8.3. $K_T(G_n)$ and its Schubert basis. Note that $K_T(G_n^T) \cong \prod_{e_\mu \in G_n^T} K_T(e_\mu)$ is naturally identified with $\text{Fun}(SP(n), R(T))$, since we have $G_n^T \cong SP(n)$.

**Theorem 8.1** ([20]). Let $i$ be the inclusion map $G_n^T \hookrightarrow G_n$. The induced map

$$i^*: K_T(G_n) \longrightarrow K_T(G_n^T) \cong \text{Fun}(SP(n), R(T))$$

is injective, and its image is equal to the $R(T)$-subalgebra $\Psi_n$.

**Proof.** For the proof, see [20], Thm. 3.13 and Corollary 3.20.

**Theorem 8.2** ([20]). The image $i^*[O_{\Omega_w}]_T$, $\lambda \in SP(n)$, is equal to $\psi_\lambda(n) \in \Psi_n$, where $\psi_\lambda(n)$ is the restriction of $\psi_\lambda$ to $SP(n)$.

An $R(T)$-basis $\{\tau_w\}_{w \in W}$ for $K_T(G/B)$ was constructed in [20]. The class $\tau_w$ is closely related to $[O_{\Omega_w}]_T$ but not-exactly coincides with it, where $\Omega_w$ is the Schubert variety in $G/B$ corresponding to $w \in W$. For the precise comparison with the class $[O_{\Omega_w}]_T$, see e.g. [23], [30]. Now we need the corresponding result for parabolic case, i.e. for $K_T(G/P)$. The reader can consult [23] for the result.

For $X = B, C, D$, we denote $G_T^B_n = G_T^D_n = G_T_n$ and $G_T^C_n = G_T_{n,+}$. We consider $R(T)$-algebras $R(T) \otimes_{\mathbb{Z}[\beta]} G_T^X_n$, with the specialization $\beta = -1$. For any strict partition $\lambda$ in $SP_n$, let $GX_\lambda(n)(x|1-e^t)$ be functions in $R(T) \otimes_{\mathbb{Z}[\beta]} G_T^X_n$, defined as follows:

$$
\begin{cases}
GB_\lambda(n)(x|1-e^{t_1}, \ldots, 1-e^{t_n}, 0, \ldots) & \text{if } X = B, \\
GQ_\lambda(n)(x|1-e^{t_1}, \ldots, 1-e^{t_n}, 0, \ldots) & \text{if } X = C, \\
GP_\lambda(n)(x|1-e^{t_1}, \ldots, 1-e^{t_n}, 1-e^{t_{n+1}}, 0, \ldots) & \text{if } X = D.
\end{cases}
$$

See [6, 7] for notation $GX_\lambda(n)(x|b)$. Then $\{GX_\lambda(n)(x|1-e^t) \mid \lambda \in SP_n\}$ form an $R(T)$ basis of $R(T) \otimes_{\mathbb{Z}[\beta]} G_T^X_n$.

**Theorem 8.3.** There exists a surjective homomorphism of $R(T)$-algebras

$$\pi_n: R(T) \otimes_{\mathbb{Z}[\beta]} G_T^X_n \longrightarrow K_T(G_n),$$

which sends $GX_\lambda(n)(x|1-e^t)$ to $[O_{\Omega_\lambda}]_T$ if $\lambda \subset \rho_n$ and to 0 if $\lambda \not\subset \rho_n$.

**Proof.** The algebraic localization map $\Phi$ naturally induces a homomorphism of $R(T)$-algebras $\Phi_n: R(T) \otimes_{\mathbb{Z}[\beta]} G_T^X_n \to \Psi_n$, given by $\Phi_n(F) = (\Phi_{F}(F))_{\mu \in SP(n)}$. By Thm. [7] we know that $\Phi_n$ maps $GX_\lambda(n)(x|1-e^t)$ to $\psi_\lambda(n)$ if $\lambda \subset \rho_n$ and to 0 if $\lambda \not\subset \rho_n$. Then there is a unique $R(T)$-algebra homomorphism $\pi_n$ making the following diagram commutative:

$$
\begin{array}{ccc}
R(T) \otimes_{\mathbb{Z}[\beta]} G_T^X_n & \xrightarrow{\pi_n} & K_T(G_n) \\
\downarrow{\Phi_n} & & \downarrow{i^*} \\
\Psi_n & \xrightarrow{i^*} & \Psi_n
\end{array}
$$

Since $i^*$ is an isomorphism (Thm. [8.1]), $\pi_n$ is determined by the above diagram, and has the desired property. \hfill \Box

Now we derive the non-equivariant analogue of Thm. [8.3]. We first recall that two cases $B_n$ and $D_{n+1}$ coincide in non-equivariant setting.

Let $\varepsilon: R(T) \to \mathbb{Z}$ be the ring homomorphism given by the evaluation at the identity element of $T$; it is given by $e^{t_i} \mapsto 1 \ (1 \leq i \leq r)$ or equivalently $b_i \mapsto 0$. Let $GB_\lambda(n)(x) = GD_\lambda(n)(x) = GP_\lambda(x_1, \ldots, x_n)$ and $GC_\lambda(n)(x) = GQ_\lambda(n)(x_1, \ldots, x_n)$. 
Corollary 8.1. There exists a surjective homomorphism of rings

\[ \pi_n : G_T^X \rightarrow K(G_n), \]

which sends \( GX^{(n)}_\lambda(x) \) to \([\mathcal{O}_R\lambda]\) if \( \lambda \subset \rho_n \) and to 0 if \( \lambda \not\subset \rho_n \).

Proof. This follows from Thm. 8.3 since the canonical map

(8.1) \[ \mathbb{Z} \otimes_{R(T)} K_T(G_n) \rightarrow K(G_n), \]

is an isomorphism of rings, where \( \mathbb{Z} \) is considered as an \( R(T) \)-algebra by \( \varepsilon \) (see [20], Proposition 3.25). \( \Box \)

9. Combinatorial expressions

In this section we present two combinatorial expressions for \( GQ_\lambda(x|b) \) and \( GP_\lambda(x|b) \) (Thm. 9.1 and Thm. 9.2). The proof will be given in §10.

9.1. Shifted set-valued tableaux. For each strict partition \( \lambda \), let \( \mathbb{D}(\lambda) \) be the shifted diagram of \( \lambda \), which is defined as \( \mathbb{D}(\lambda) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i \leq j \leq \lambda_i + i - 1, 1 \leq i \leq \ell(\lambda)\} \).

Let \( \mathcal{A} \) denote the ordered alphabet

\[ 1' < 1 < 2' < 2 < \cdots < n' < n. \]

Let \( \mathcal{X} \) denote the set of all non-empty subsets of \( \mathcal{A} \).

A set-valued shifted tableau \( T \) of shape \( \lambda \) is an assignment \( T : \mathbb{D}(\lambda) \rightarrow \mathcal{X} \) such that

(1) \( \max T(i, j) \leq \min T(i, j + 1), \) \( \max T(i, j) \leq \min T(i, j + 1), \)

(2) Each \( a (= 1, 2, \ldots, n) \) appears at most once in each columns,

(3) Each \( a' (= 1', 2', \ldots, n') \) appears at most once in each rows.

We denote by \( \mathcal{T}(\lambda) \) the set of all set-valued shifted tableaux of shape \( \lambda \). Define \( \mathcal{T}'(\lambda) \) to be the subset of \( \mathcal{T}(\lambda) \) consisting of \( T \in \mathcal{T}(\lambda) \) satisfying the condition that for each \( i \) (1 \( \leq i \leq n \)), \( T(i, i) \) contains only 1, 2, 3, 4, \ldots if \( i \) is odd, and 1', 2', 3', 4', \ldots if \( i \) is even. Define \( \mathcal{T}''(\lambda) \) to be the subset of \( \mathcal{T}(\lambda) \) consisting of \( T \in \mathcal{T}(\lambda) \) satisfying the condition that for each \( i \) (1 \( \leq i \leq n \)), \( T(i, i) \) contains no primed symbols \( \{1', 2', \ldots, n'\} \).

Example 9.1. The following are examples of set-valued shifted tableaux of shape \( \lambda = (4, 3, 1) \):

\[
\begin{align*}
T_1 &= \begin{array}{cccc}
1' & 12 & 23 & 41 \\
2' & 4' & 6' & \\
6' & & &
\end{array} \\
T_2 &= \begin{array}{cccc}
1 & 2 & 2 & 2 \\
23 & 3 & 3 & 4 \\
& & &
\end{array} \\
T_3 &= \begin{array}{cccc}
1 & 2 & 23 & 34 \\
2 & 4' & 6' & \\
& & &
\end{array}
\end{align*}
\]

Of these three tableaux in \( \mathcal{T}(\lambda) \), only \( T_2 \) is in \( \mathcal{T}'(\lambda) \), and only \( T_3 \) is in \( \mathcal{T}''(\lambda) \).

For each integers \( i, j \) such that 1 \( \leq i \leq n \), \( i \leq j \) and \( a \in \mathcal{A} \), we define

\[ w_I(i, j; a) = \begin{cases} 
 x_a \oplus b_{j-i} & \text{if } a \in \{1, 2, \ldots, n\} \\
 x_{[a]} \oplus b_{j-i} & \text{if } a \in \{1', 2', \ldots, n'\},
\end{cases} \]

where \( b_0 = 0 \) and

\[ w_{II}(i, j; a) = \begin{cases} 
 x_a \oplus b_{j-i+1} & \text{if } a \in \{1, 2, \ldots, n\} \\
 x_{[a]} \oplus b_{j-i+1} & \text{if } a \in \{1', 2', \ldots, n'\}.
\end{cases} \]
For $T \in \mathcal{T}(\lambda)$ we define
\begin{equation}
(x|b)_T^T = \prod_{(i,j) \in D(\lambda), a \in T(i,j)} w_I(i,j;a), \quad (x|b)_H^T = \prod_{(i,j) \in D(\lambda), a \in T(i,j)} w_H(i,j;a).
\end{equation}

Recall that $G\lambda_x^{(n)}(x|b)$ is defined in [56.1]

**Theorem 9.1** (Combinatorial formula in terms of set-valued tableaux). Let $\lambda$ be a strict partition of length $\leq n$. Then we have
\begin{align*}
(9.2) & \quad \text{Type B : } GB_x^{(n)}(x|b) = \sum_{T \in T^n(\lambda)} \beta_{|T| - |\lambda|}(x|b)_T^T \\
(9.3) & \quad \text{Type C : } GC_x^{(n)}(x|b) = \sum_{T \in T(\lambda)} \beta_{|T| - |\lambda|}(x|b)_T^T \\
(9.4) & \quad \text{Type D : } GD_x^{(n)}(x|b) = \sum_{T \in T(\lambda)} \beta_{|T| - |\lambda|}(x|b)_H^T
\end{align*}

For example, let $T_1$ be the tableau in Example 9.1, then the corresponding summand on the right-hand side of (9.3) equals:
\[x_1(x_1 \oplus b_1)x_2(x_2 \oplus b_1)(x_2 \oplus b_2)(x_3 \oplus b_2)(x_3 \oplus b_3)(x_4 \oplus b_1)(x_4 \oplus b_3)x_6(x_6 \oplus b_2).\]

9.2. Excited Young diagrams. Let $\mathcal{D}_n$ denote the subset of $\mathbb{Z} \times \mathbb{Z}$ consisting of $(i, j)$ satisfying $1 \leq i \leq n, \ i \leq j$. We call an element $(i,j)$ in $\mathcal{D}_n$ a box. Let $D$ be a finite set of boxes, i.e. a finite subset of $\mathcal{D}_n$. Suppose $(i, j) \in D$ and $(i + 1, j), (i + 1, j + 1), (i, j + 1) \notin D$ (if $i = j$, the condition $(i + 1, j) \notin D$ is vacuous). Then it is said that $D' = (D \setminus (i, j)) \cup (i + 1, j + 1)$ is obtained from $D$ by an elementary excitation, and we denote $D \to D'$. Let $\mathcal{E}^I_n(\lambda)$ denote the set of all subsets $D$ of $\mathcal{D}_n$ obtained from $D_0 = \mathbb{D}(\lambda)$ by a sequence of successive elementary excitations
\[D_0 \to D_1 \to \cdots \to D_r = D.\]

Let $D \in \mathcal{E}^I_n(\lambda)$. We denote by $B^I(D)$ the set of boxes $(i, j)$ satisfying the following conditions:
\begin{enumerate}
\item $(i,j) \notin D,$
\item there is a positive integer $k$ such that $(i + k, j + k) \in D,$
\item $(i + s, j + s) \notin D$ for $1 \leq s < k,$
\item $(i + s, j + s + 1) \notin D$ for $0 \leq s < k,$
\item $(i + s - 1, j + s) \notin D$ for $0 \leq s < k$ (this condition is ignored if $i = j$).
\end{enumerate}

**Example 9.2.** Let $n = 4$. The following are example of elements of $\mathcal{E}^I_4(\lambda)$ for $\lambda = (4, 2)$.

The boxes with symbol $\times$ are the elements of $B^I(D)$.

Next we consider the case of type D. In the sequel, we assume $n$ is even, whenever we consider type D case. In order to treat type D case, we define $\mathcal{E}^H_n(\lambda)$, which is a subset of $\mathcal{E}^I_n(\lambda)$ consisting of EYD whose way of excitation of each diagonal box $(i,i)$ is restricted to be even steps. More precisely, elementary excitation of $(i,i) \in D$ is defined as follows: if $(i,i) \in D$ and $(i,i+1), (i+1,i+1), (i+1,i+2), (i+2,i+2) \notin D$ then
Let \( D' = (D \setminus (i, i)) \cup (i + 2, i + 2) \) is an excitation of type \( II \). For off-diagonal boxes the definition of excitation is the same. For example the left diagram of Example 9.2 is in \( \mathcal{E}_n^H(\lambda) \), but the right one is not.

Let \( D \in \mathcal{E}_n^H(\lambda) \). We define the subset \( B^H(D) \) of \( B^I(D) \) by restricting \( k \) to be even in the definition of \( B^I(D) \).

**Example 9.3.** In the following figure, the boxes with symbol \( \times \) are the elements of \( B^H(D) \).

\[
\begin{array}{cccccccc}
\times & & & & & & & \\
& \times & & & & & & \\
& & \times & & & & & \\
& & & \times & & & & \\
& & & & \times & & & \\
& & & & & \times & & \\
& & & & & & \times & \\
& & & & & & & \\
\end{array}
\]

Define for \( (i, j) \in \mathcal{D}_n \), the following weight of types \( B, C, \) and \( D \):

\[
\begin{align*}
\text{wt}^B(i, j) &= \begin{cases} 
x_i & \text{if } j \leq n \\
x_i \oplus b_{j-n} & \text{if } j > n
\end{cases}, \\
\text{wt}^C(i, j) &= \begin{cases} 
x_i \oplus x_j & \text{if } j \leq n \\
x_i \oplus b_{j-n} & \text{if } j > n
\end{cases}.
\end{align*}
\]

For type \( D \), note that we assumed \( n \) is even. We define

\[
\begin{align*}
\text{wt}^D(i, j) &= \begin{cases} 
x_i \oplus x_{j+1} & \text{if } j \leq n - 1 \\
x_i \oplus b_{j-n+1} & \text{if } j \geq n
\end{cases}.
\end{align*}
\]

**Example 9.4.** Let \( n = 4 \). The weight function \( \text{wt}^C \) on \( \mathcal{D}_4 \) is given as follows:

\[
\begin{array}{cccccccc}
x_1 \oplus x_1 & x_1 \oplus x_2 & x_1 \oplus x_3 & x_1 \oplus x_4 & x_1 \oplus b_1 & x_1 \oplus b_2 & x_1 \oplus b_3 & \cdots \\
x_2 \oplus x_2 & x_2 \oplus x_3 & x_2 \oplus x_4 & x_2 \oplus b_1 & x_2 \oplus b_2 & x_2 \oplus b_3 & \cdots \\
x_3 \oplus x_3 & x_3 \oplus x_4 & x_3 \oplus b_1 & x_3 \oplus b_2 & x_3 \oplus b_3 & \cdots \\
x_4 \oplus x_4 & x_4 \oplus b_1 & x_4 \oplus b_2 & x_4 \oplus b_3 & \cdots
\end{array}
\]

while \( \text{wt}^D \) is as follows:

\[
\begin{array}{cccccccc}
x_1 \oplus x_2 & x_1 \oplus x_3 & x_1 \oplus x_4 & x_1 \oplus b_1 & x_1 \oplus b_2 & x_1 \oplus b_3 & x_1 \oplus b_4 & \cdots \\
x_2 \oplus x_3 & x_2 \oplus x_4 & x_2 \oplus b_1 & x_2 \oplus b_2 & x_2 \oplus b_3 & x_2 \oplus b_4 & \cdots \\
x_3 \oplus x_4 & x_3 \oplus b_1 & x_3 \oplus b_2 & x_3 \oplus b_3 & x_3 \oplus b_4 & \cdots \\
x_4 \oplus b_1 & x_4 \oplus b_2 & x_4 \oplus b_3 & x_4 \oplus b_4 & \cdots
\end{array}
\]

We denote \( \mathcal{E}_n^B(\lambda) = \mathcal{E}_n^C(\lambda) = \mathcal{E}_n^I(\lambda) \) and \( \mathcal{E}_n^D(\lambda) = \mathcal{E}_n^H(\lambda) \). We also denote \( B^B(D) = B^C(D) = B^I(D) \) and \( B^D(D) = B^H(D) \). Define

\[
\text{wt}^X(D) = \prod_{(i, j) \in D} \text{wt}^X(i, j) \prod_{(i', j') \in B^X(D)} (1 + \beta \text{wt}^X(i', j')).
\]

Recall the definition (5.3) of \( GX^{(n)}(x|b) \).

**Theorem 9.2.** Let \( \lambda \) be a strict partition of length \( \leq n \). Then we have

\[
GX^{(n)}_\lambda(x|b) = \sum_{D \in \mathcal{E}_n^X(\lambda)} \text{wt}^X(D)
\]
Consider the right diagram $D$ in Example 9.2. The corresponding summands of the right-hand side of (9.6) are equal to the following:

\[ wt^B(D) = x_1(x_1 + x_2)(x_2 + x_4)(1 + \beta(x_1 + x_3))(x_3 + b_4)(1 + \beta(x_2 + b_1)) \]
\[ x_4(1 + \beta(x_2 + x_2))(1 + \beta(x_3 + x_3))(x_4 + b_1), \]
\[ wt^C(D) = (x_1 + x_1)(x_1 + x_2)(x_2 + x_4)(1 + \beta(x_1 + x_3))(x_3 + b_2)(1 + \beta(x_2 + b_1)) \]
\[ (x_4 + x_4)(1 + \beta(x_2 + x_2))(1 + \beta(x_3 + x_3))(x_4 + b_1), \]
\[ wt^{\mathcal{D}}(D) = (x_1 + x_2)(x_1 + x_3)(x_2 + b_1)(1 + \beta(x_1 + x_4))(x_3 + b_3)(1 + \beta(x_2 + b_2)) \]
\[ (x_4 + b_1)(1 + \beta(x_2 + x_3))(x_4 + b_2). \]

Note that $B_{\Pi}(D)$ is given in Example 9.3.

**Corollary 9.1.** The coefficients of $G_X^{(A)}(x \mid b)$ as a polynomial in $x_1, \ldots, x_n$ are in $\mathbb{N}[\beta, b_1, b_2, \ldots]$.

9.3. **Type A case.** There is an analogous formula for $G_\lambda(x_1, \ldots, x_n \mid b)$. Define $\mathcal{D}_n^A = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq n, j \geq 0\}$. Define for $(i, j) \in \mathcal{D}_n^A$, the following weight of type $A$:

\[ wt^A(i, j) = x_i \oplus b_j. \]

For a partition $\lambda \in \mathcal{P}_n$ the set of excited Young diagrams $\mathcal{E}_n^A(\lambda)$ is defined in similar way (see [10]). For $D \in \mathcal{E}_n^A(\lambda)$ the set $B_{\Pi}(D)$ is defined in the same way as in [9.2].

**Proposition 9.1** (EYD formula for Type A). Let $\lambda$ be a partition in $\mathcal{P}_n$. Then

\[ G_{\lambda}(x_1, \ldots, x_n \mid b) = \sum_{D \in \mathcal{E}_n^A(\lambda)} \prod_{(i, j) \in D} wt^A(i, j) \prod_{(i', j') \in B^A(D)} (1 + \beta wt^A(i', j')). \]

Proof of this proposition can be given in the same line as for Thm. 9.2. The arguments are left to the reader because these are quite similar and easier than types $B, C$, and $D$. Furthermore, we can show $G_{\lambda}(x_1, \ldots, x_n \mid b)$ is equal to a combinatorial expression in terms of set-valued tableaux given by McNamara (cf. Remark 2.3).

10. **Proofs of Thm. 9.1 and Thm. 9.2**

In this section we give proofs of Thm. 9.1 and Thm. 9.2. To make them more comprehensible, we consider type C case. The modifications for other cases are left to the reader.

10.1. **Locally equivalent weight functions.** Let $S$ be a subset of $\mathcal{D}_n$. By a **weight function** on $S$ we mean a map $w$ from $S$ to a commutative ring $R$. Suppose a weight function $w$ on $\mathcal{D}_n$ is given. For a strict partition $\lambda \in \mathcal{SP}_n$, we denote

\[ E_{\lambda}(w) = \sum_{D \in \mathcal{E}_n(\lambda)} w(D), \quad w(D) := \prod_{c \in D} w(c) \prod_{c' \in B(D)} (1 + \beta w(c')). \]

Two weight functions $w$ and $w'$ on $\mathcal{D}_n$ are **equivalent** if the equality $E_{\lambda}(w) = E_{\lambda}(w')$ holds for all $\lambda \in \mathcal{SP}_n$. Two weight functions $w_1$ and $w_2$ on $S$ are **locally equivalent** on $S$, if for any weight function $w$ on $\mathcal{D}_n \setminus S$, the following two weight functions $w \lor w_1$ and $w \lor w_2$ on $\mathcal{D}_n$ are equivalent:

\[ (w \lor w_i)(\alpha) = \begin{cases} w(\alpha) & \text{if } \alpha \in \mathcal{D}_n \setminus S \\ w_i(\alpha) & \text{if } \alpha \in S \end{cases} \quad (i = 1, 2). \]
Example 10.1. We will show that the following two weight functions on $D_2$ are equivalent for arbitrary $x, y, z$, etc. In other words, the two weight functions on $S$ are locally equivalent, where $S$ is the set of the gray boxes.

| $x$ | $a \oplus t$ | $a$ | $y$ | $\cdots$ | $x$ | $a$ | $a \oplus t$ | $y$ | $\cdots$ |
|-----|---------------|-----|-----|----------|-----|-----|---------------|-----|----------|
| 0   | 0             | $z$ | $\cdots$ |          | 0   | $t$ | $z$ | $\cdots$ |

Let $w, w'$ denote the above weight functions. If $\lambda = \boxed{\text{gray boxes}}$ then the corresponding EYD sums are

$$E_\lambda(w) = x(a \oplus t)a + x(a \oplus t)z(1 + \beta a)$$
$$E_\lambda(w') = xa(a \oplus t) + xaz(1 + \beta(a \oplus t)) + xt(1 + \beta a)z$$

We can easily check the equality $E_\lambda(w) = E_\lambda(w')$. One sees that the only remaining $\lambda$ which we have to check is $\lambda = \boxed{\text{gray boxes}}$. Then we have the following two EYD sums which are equal:

$$E_\lambda(w) = x(a \oplus t), \quad E_\lambda(w') = xa + xt(1 + \beta a).$$

Next lemma is a generalization of this example.

**Lemma 10.1.** The following weight functions on $S$ are locally equivalent:

| $a_n \oplus t$ | $a_n$ |
|----------------|-------|
| $\vdots$      | $\vdots$ |
| $a_2 \oplus t$ | $a_2$ |
| $a_1 \oplus t$ | $a_1$ |
| 0              | $a_0$ |

| $a_n$ | $a_n \oplus t$ |
|-------|-----------------|
| $\vdots$ | $\vdots$ |
| $a_2$ | $a_2 \oplus t$ |
| $a_1$ | $a_1 \oplus t$ |
| 0    | $a_0 \oplus t$ |

where $S$ is the subset of $D_{n+1}$ consisting of two columns ($i$-th and $(i+1)$-th) with $i \geq n+1$.

**Lemma 10.2.** The following weight functions on $S$ are locally equivalent:

| $b_{i-1} \oplus t$ | $b_{i-1}$ |
|---------------------|-----------|
| $\vdots$            | $\vdots$  |
| $b_1 \oplus t$      | $b_1$     |
| $c \oplus t \oplus t$ | $c \oplus t$ |
| $c \oplus t$        | $a_1 \oplus t$ |
| $a_2 \oplus t$      | $a_3 \oplus t$ |
| $c$                 | $a_1$     |
| $a_2$               | $a_3$     |

| $b_{i-1}$ | $b_{i-1} \oplus t$ |
|-----------|---------------------|
| $\vdots$  | $\vdots$            |
| $b_1$     | $b_1 \oplus t$     |
| $c \oplus t$ | $a_1$ |
| $a_2$     | $a_3$               |
| $c \oplus t \oplus t$ | $a_1 \oplus t$ |
| $a_2 \oplus t$ | $a_3 \oplus t$ |

where $S$ is the subset of $D_n$ consisting of two columns and rows ($i$-th and $(i+1)$-th) with $i < n$. 
Lemma 10.1 and Lemma 10.2 can be proved by elementary but quite tedious calculations. An alternative proof using the Yang-Baxter equation is available (the details will appear in \[27\]).

10.2. Outline of the proof of Thm 9.2. For $\lambda \in S^P_n$, define

$$E_\lambda(x_1, \ldots, x_n|b) = \sum_{D \in E_n(\lambda)} wt(D).$$

**Proposition 10.1.** $E_\lambda(x_1, \ldots, x_n|b)$ is a $K$-supersymmetric polynomial in $x_1, \ldots, x_n$.

**Proof.** Let $i$ be $1 \leq i \leq n - 1$. We apply Lemma 10.2 to $i$-th and $(i+1)$-th columns and rows with $t = x_i \oplus x_{i+1}$. It turns out that $E_\lambda(x_1, \ldots, x_n|b)$ is invariant under the exchange of $x_i$ and $x_{i+1}$. Hence $E_\lambda(x_1, \ldots, x_n|b)$ is symmetric.

Next we show the cancellation property of Def. 1.1. Let $x_1 = t, x_2 = \ominus t$. Then weight function is given as follows:

| $t \oplus t$ | $t \ominus t$ | $t \oplus x_3$ | $t \oplus x_4$ | $t \oplus b_1$ | $t \oplus b_2$ | $t \oplus b_3$ | $t \ominus b_1$ | $t \ominus b_2$ | $t \ominus b_3$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $t \ominus t$ | $t \oplus x_3$ | $t \ominus x_4$ | $t \oplus b_1$ | $t \ominus b_2$ | $t \ominus b_3$ | $x_3 \oplus x_3$ | $x_3 \ominus x_4$ | $x_3 \oplus b_1$ | $x_3 \ominus b_2$ | $x_3 \ominus b_3$ |
| $t \ominus x_3$ | $t \ominus x_4$ | $x_3 \oplus b_1$ | $x_3 \ominus b_2$ | $x_3 \ominus b_3$ | $x_4 \ominus x_4$ | $x_4 \oplus b_1$ | $x_4 \ominus b_2$ | $x_4 \ominus b_3$ |                      |                      |

This is obviously equivalent to

| $0$ | $0$ | $t \oplus x_3$ | $t \oplus x_4$ | $t \oplus b_1$ | $t \oplus b_2$ | $t \oplus b_3$ | $t \ominus b_1$ | $t \ominus b_2$ | $t \ominus b_3$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $0$ | $t \oplus x_3$ | $t \ominus x_4$ | $t \oplus b_1$ | $t \ominus b_2$ | $t \ominus b_3$ | $x_3 \oplus x_3$ | $x_3 \ominus x_4$ | $x_3 \oplus b_1$ | $x_3 \ominus b_2$ | $x_3 \ominus b_3$ |
| $x_3 \oplus x_3$ | $x_3 \ominus x_4$ | $x_3 \oplus b_1$ | $x_3 \ominus b_2$ | $x_3 \ominus b_3$ | $x_4 \ominus x_4$ | $x_4 \oplus b_1$ | $x_4 \ominus b_2$ | $x_4 \ominus b_3$ |                      |                      |

Then the sum of weights over $D_n$ reduces to the sum over the subset of $D_n$ consisting of the boxes $(i, j) \in D_n$ such that $i \geq 3$. Since this subset is identified with $D_{n-2}$ of type $C_{n-2}$ weights with respect to $x_3, \ldots, x_n$, and hence the sum is equal to $E_\lambda(0, 0, x_3, \ldots, x_n|b)$.

This implies $E_\lambda(x_1, \ldots, x_n|b)$ is $K$-supersymmetric. $\Box$

**Proposition 10.2** (Stability). $E_\lambda(x_1, \ldots, x_n, 0|b) = E_\lambda(x_1, \ldots, x_n|b)$.

**Proof.** To make the stability property clearer, it is useful to arrange the weight in the following way by using symmetry.

$$
\begin{array}{cccccc}
  x_3 \oplus x_3 & x_3 \oplus x_2 & x_3 \oplus x_1 & x_3 \oplus b_1 & x_3 \oplus b_2 & x_3 \oplus b_3 \\
  x_2 \oplus x_2 & x_2 \oplus x_1 & x_2 \oplus b_1 & x_2 \oplus b_2 & x_2 \oplus b_3 & x_2 \oplus b_4 \\
  x_1 \oplus x_1 & x_1 \oplus b_1 & x_1 \oplus b_2 & x_1 \oplus b_3 & x_1 \oplus b_4 \\
\end{array}
$$

Set $x_{n+1} = 0$. Then the weight at the box $(1, 1)$ is zero. So if $(1, 1) \in D$ for $D \in E_{n+1}(\lambda)$, then $wt(D) = 0$. The set consisting of $D \in E_{n+1}(\lambda)$ such that $(1, 1) \notin D$ is naturally in bijection with $E_n(\lambda)$. This bijection yields the equation. $\Box$

Let us denote by $E_\lambda(x|b)$ the inverse limit $\lim_{\leftarrow} E^{(n)}_\lambda(x_1, \ldots, x_n|b)$ (cf. Prop. 10.2). By Prop. 1.1 we have the following:
Lemma 10.3. \( E_{\lambda}(x|b) \in GT^C_R \).

In order to prove Thm. 9.2, we need the following lemma:

Lemma 10.4. \( \pi_i E_{\lambda}(x|b) = \begin{cases} E_{\lambda(i)}(x|b) & \text{if } s_i \lambda < \lambda \\ \beta E_{\lambda}(x|b) & \text{if } s_i \lambda \geq \lambda \end{cases} \)

With this lemma at hand, the proof for Thm. 9.2 is completed as follows.

**Proof of Thm. 9.2.** Lemma 10.3 enable us to send \( E_{\lambda}(x|b) \) by \( \Phi \) into \( \Psi \). By the same argument of proof of Thm. 7.1, Lemma 10.4 implies that \( \{ \Phi(E_{\lambda}(x|b)) \} \) is a family of Schubert classes. Since we know \( \{ \Phi(GQ_{\lambda}(x|b)) \} \) is a family of Schubert classes (Thm. 10.1), the injectivity of \( \Phi \) and the uniqueness of a family of Schubert classes implies \( E_{\lambda}(x|b) = GQ_{\lambda}(x|b) \). \( \square \)

10.3. **Proof of Lemma 10.4.** Let \( i \geq 1 \). We calculate \( s_i E_{\lambda}(x|b) \), which is equal to the weight sum over \( E_n(\lambda) \) of the weights obtained from the original one by replacing \((n+i)\)-th and \((n+i+1)\)-th columns with the following left diagram:

\[
\begin{array}{cccc}
x_1 + b & x_1 + b_i & x_1 + b_{i+1} \\
x_2 + b & x_2 + b_i & x_2 + b_{i+1} \\
\vdots & \vdots & \vdots \\
x_n + b & x_n + b_i & x_n + b_{i+1} \\
0 & 0 & 0 \\
\end{array} \rightarrow \begin{array}{cccc}
x_1 + b_i & x_1 + b_{i+1} \\
x_2 + b_i & x_2 + b_{i+1} \\
\vdots & \vdots & \vdots \\
x_n + b_i & x_n + b_{i+1} \\
0 & b_{i+1} + b_i \\
\end{array}
\]

We consider each \( D \in E_n(\lambda) \) as a subset in \( D_{n+1} \), on which the values of weight on \((n+1)\)-th row are all zero. Applying Lemma 10.1, we can change the weight with the right one without changing the weight sum in total. There is an additional box at the bottom of \((i+n)\)-th column, whose weight is \( e(\alpha_i) = b_{i+1} \oplus b_i \). For example if \( n = 3, i = 1 \) then the two weight functions looks as follows:

\[
\begin{array}{cccccccc}
x_3 \oplus x_3 & x_3 \oplus x_2 & x_3 \oplus x_1 & x_3 \oplus b_2 & x_3 \oplus b_1 & x_3 \oplus b_3 & x_3 \oplus b_4 \\
x_2 \oplus x_2 & x_2 \oplus x_1 & x_2 \oplus b_2 & x_2 \oplus b_1 & x_2 \oplus b_3 & x_2 \oplus b_4 \\
x_1 \oplus x_1 & x_1 \oplus b_2 & x_1 \oplus b_1 & x_1 \oplus b_3 & x_1 \oplus b_4 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
x_3 \oplus x_3 & x_3 \oplus x_2 & x_3 \oplus x_1 & x_3 \oplus b_2 & x_3 \oplus b_1 & x_3 \oplus b_3 & x_3 \oplus b_4 \\
x_2 \oplus x_2 & x_2 \oplus x_1 & x_2 \oplus b_2 & x_2 \oplus b_1 & x_2 \oplus b_3 & x_2 \oplus b_4 \\
x_1 \oplus x_1 & x_1 \oplus b_2 & x_1 \oplus b_1 & x_1 \oplus b_3 & x_1 \oplus b_4 \\
0 & 0 & b_2 \oplus b_1 & 0 & 0 \\
\end{array}
\]

We denote the first one by \( s_i(\wt) \) (function obtained from \( \wt \) by exchanging \( b_i \) and \( b_{i+1} \)) and the second one by \( \wt_{+i} \). Thus we have the following:

**Lemma 10.5.** For \( i \geq 1 \) we have \( s_i E_{\lambda}(x_1, \ldots, x_n|b) = \sum_{D \in E_{n+1}(\lambda)} \wt_{+i}(D) \).
Proof of Lemma 10.4 for \( i \geq 1 \). Suppose \( \lambda \) is \( i \)-removable. We denote by \( \mathcal{E}_{n+1}^i(\lambda) \) the set of \( D \in \mathcal{E}_{n+1}(\lambda) \) such that \( wt_{+i}(D) \neq 0 \). We have a map \( r_i : \mathcal{E}_{n+1}^i(\lambda) \to \mathcal{E}_n(\lambda^{(i)}) \) defined by removing the box of content \( i \) having the largest row index. One readily see that \( r_i \) is surjective. Fix \( D_0 \in \mathcal{E}_n(\lambda^{(i)}) \). It suffices to show the following equation:

\[
(10.1) \quad \sum_{D \in \mathcal{E}_{n+1}^i(\lambda), \ r_i(D) = D_0} wt_{+i}(D) - (1 + \beta e(\alpha_i)) \sum_{D \in \mathcal{E}_n(\lambda), \ r_i(D) = D_0} wt(D) = e(\alpha_i) wt(D_0)
\]

Note that each term in the left-hand side of (10.1) is clearly a multiple of \( wt(D_0) \). The differences comes from the contributions of the box to be removed by \( r_i \). An obvious equation

\[
(10.2) \quad \sum_{i=1}^{r} a_i \prod_{j=1}^{i-1} (1 + \beta a_j) - (1 + \beta a_r) \sum_{i=1}^{r-1} a_i \prod_{j=1}^{i} (1 + \beta a_j) = a_r.
\]

depicted below gives (10.1).

If \( \lambda \) is not \( i \)-removable, one easily sees that \( s_i E_{\lambda}(x_1, \ldots, x_n|b) = E_{\lambda}(x_1, \ldots, x_n|b) \). Then we have \( \pi_i E_{\lambda}(x|b) = -\beta E_{\lambda}(x|b) \). □

In order to prove Lemma 10.4 for \( i = 0 \), we need the following Lemma. Let \( wt_{+0} \) denote the weight function on \( \mathcal{D}_{n+1} \) defined similarly as \( wt_{+i} (i \geq 1) \).

Lemma 10.6. \( E_{\lambda}(x_1, \ldots, x_n, b_1 \uplus b_1, b_2, b_3, \ldots) = \sum_{D \in \mathcal{E}_{n+1}(\lambda)} wt_{+0}(D) \).

If \( n = 2 \) then \( E_{\lambda}(x_1, x_2, b_1 \uplus b_1, b_2, b_3, \ldots) \) is given by the following table.

| \(*\) | \(*\) |
|---|---|
| \(x_2 \uplus x_2\) | \(x_2 \uplus x_1\) | \(x_2 \uplus b_1\) | \(x_2 \uplus b_2\) | \(x_2 \uplus b_3\) | \(x_2 \uplus b_4\) |
| \(x_1 \uplus x_2\) | \(x_1 \uplus b_1\) | \(x_1 \uplus b_2\) | \(x_1 \uplus b_3\) | \(x_1 \uplus b_4\) |
| \(b_1 \uplus b_1\) | \(b_1 \uplus b_2\) | \(b_1 \uplus b_3\) | \(b_1 \uplus b_4\) |

Note that \( b_1 \uplus b_1 = 0 \). By applying Lemma 10.1 to two columns indicated by \(*\), we can deform this as follows:

| \(*\) | \(*\) |
|---|---|
| \(x_2 \uplus x_2\) | \(x_2 \uplus x_1\) | \(x_2 \uplus b_1\) | \(x_2 \uplus b_2\) | \(x_2 \uplus b_3\) | \(x_2 \uplus b_4\) |
| \(x_1 \uplus x_1\) | \(x_1 \uplus b_1\) | \(x_1 \uplus b_2\) | \(x_1 \uplus b_3\) | \(x_1 \uplus b_4\) |
| \(b_1 \uplus b_1\) | 0 | 0 | \(b_1 \uplus b_3\) | \(b_1 \uplus b_4\) |

Repeat this process sufficiently many times we have

| \(*\) | \(*\) |
|---|---|
| \(x_2 \uplus x_2\) | \(x_2 \uplus x_1\) | \(x_2 \uplus b_1\) | \(x_2 \uplus b_2\) | \(x_2 \uplus b_3\) | \(x_2 \uplus b_4\) |
| \(x_1 \uplus x_1\) | \(x_1 \uplus b_1\) | \(x_1 \uplus b_2\) | \(x_1 \uplus b_3\) | \(x_1 \uplus b_4\) |
| \(b_1 \uplus b_1\) | 0 | 0 | 0 | 0 |

The proof of Lemma 10.4 for \( i = 0 \) is completed.
10.4. **Exited Young diagrams and Set-valued Tableaux.** Let $E_\lambda(x|b)$ and $T_\lambda(x|b)$ be the functions in the right-hand of (9.6) and (9.3) respectively. In this section we prove the following:

**Proposition 10.3.** $E_\lambda(x|b) = T_\lambda(x|b)$.

The main idea of proof is to use equivalence of weight functions so that we have “separation of variables”. For example, let $n = 2$, and consider the weight function $wt^C$ defined by (9.5). One readily see that the following weight function is equivalent to the original one.

\[
\begin{array}{cccccccccccc}
  x_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_1 & x_1 \oplus b_1 & x_1 \oplus b_2 & x_1 \oplus b_3 & x_1 \oplus b_4 & \cdots \\
  0 & 0 & 0 & 0 & 0 & \cdots \\
  x_2 & 0 & 0 & 0 & \cdots \\
  x_2 & x_2 \oplus b_1 & x_2 \oplus b_2 & \cdots 
\end{array}
\]

This can be further deformed to the following equivalent one:

\[
\begin{array}{cccccccccccc}
  x_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_1 & x_1 \oplus b_1 & x_1 \oplus b_2 & x_1 \oplus b_3 & x_1 \oplus b_4 & \cdots \\
  0 & x_2 \oplus b_1 & 0 & 0 & 0 & \cdots \\
  x_2 & 0 & 0 & 0 & \cdots \\
  x_2 & x_2 \oplus b_1 & x_2 \oplus b_2 & \cdots 
\end{array}
\]

We repeat this process. For example, if $n = 3$, the following weight function is equivalent to the original one.

\[
\begin{array}{cccccccccccccccccccc}
  x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_1 & x_1 \oplus b_1 & x_1 \oplus b_2 & x_2 \oplus x_1 & x_1 \oplus b_3 & x_1 \oplus b_4 & x_1 \oplus b_5 & x_1 \oplus b_6 & x_1 \oplus b_7 & x_1 \oplus b_8 & x_1 \oplus b_9 & \cdots \\
  0 & 0 & x_2 \oplus b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & x_2 \oplus b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_2 & x_2 \oplus b_1 & x_2 \oplus b_2 & x_2 \oplus x_1 & x_2 \oplus b_3 & x_2 \oplus b_4 & x_2 \oplus b_5 & \cdots \\
  0 & 0 & x_3 \oplus b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & x_3 \oplus b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_3 & x_3 \oplus b_1 & x_3 \oplus b_2 & \cdots 
\end{array}
\]

If we only consider a strict partition $\lambda$ such that $\lambda_1 \leq 3$, we can use the following weight of “separated” form:

\[
\begin{array}{cccccccccccccccccccc}
  x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_1 & x_1 \oplus b_1 & x_1 \oplus b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & 0 & x_2 \oplus b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & x_2 \oplus b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_2 & x_2 \oplus b_1 & x_2 \oplus b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & 0 & x_3 \oplus b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & x_3 \oplus b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  x_3 & x_3 \oplus b_1 & x_3 \oplus b_2 & \cdots 
\end{array}
\]

In general, we consider a weight function of the above separated form. Precisely, for positive integer $k$, we define the weight function $wt_{sep}$ on $D_{nk+n-k+1}$ as follows: $wt_{sep}(i, j) = 0$ unless $j - i \leq k$ and $(i \equiv 2 \pmod{k+1})$ or $(j \equiv 1 \pmod{k+1})$. On $j$ th column with $j = (k+1)s - k$ $1 \leq s \leq n$, $wt_{sep}(i, j) = x_s \oplus b_{j-i}$ $(j - k + 1 \leq i \leq j)$. On $i$ th row with $i = (k+1)t - k + 1$ $(1 \leq t \leq n)$, $wt_{sep}(i, j) = x_t \oplus b_{j-i}$ $(i \leq j \leq i + k - 1)$.
Let \( \lambda \) be a strict partition. Suppose \( k \) is large enough so that \( k \geq \lambda_1 \), and let \( w_{\text{sep}} \) be the weight function defined the preceding paragraph. Let \( \mathcal{E}_{\text{sep}}(\lambda) \) denote the set of excited Young diagrams \( D \) such that \( w_{\text{sep}}(D) \neq 0 \). Then

\[
E_\lambda(x|b) = \sum_{D \in \mathcal{E}_{\text{sep}}(\lambda)} w_{\text{sep}}(D).
\]

We denote by \( \mathcal{T}_0(\lambda) \) the set of ordinary (non set-valued) tableaux of shape \( \lambda \).

**Lemma 10.7.** We have a bijection \( \mathcal{E}_{\text{sep}}(\lambda) \rightarrow \mathcal{T}_0(\lambda) \).

*Proof.* We use a term horizontal (resp. vertical) line to call a rows (resp. columns) having nonzero weights in \( w_{\text{sep}} \). We put label \( i \) to the horizontal line having weights \( x_i \oplus b_{j-i} \), and \( j' \) to the vertical line having weight \( x_j \ominus b_{j-i} \). Let \( D \in \mathcal{E}_{\text{sep}}(\lambda) \). Since each box in \( D \) is on some line, horizontal or vertical, we associate its label. Then we write the label in the original position of the box in \( \mathbb{D}(\lambda) \). This gives a tableau of shape \( \lambda \). The fact that the resulting tableau is semistandard is readily seen from the configuration of the lines. Conversely, let \( T \in \mathcal{T}_0(\lambda) \). Any box \( \alpha \) in \( \mathbb{D}(\lambda) \) has the right (excited) position, say \( \alpha' \), on the line having label \( T(\alpha) \). Semistandardness insures that the set \( \{ \alpha' \} \) is an excited Young diagram in \( \mathcal{E}_{\text{sep}}(\lambda) \). \( \square \)

**Example 10.2.** Tableau \( T_0 = \begin{array}{ccc}
1 & 2 & 3 \\
1' & 2' & 3' \\
\end{array} \) corresponds to the following EYD:

![Diagram](image)

Here ■'s are the elements in \( D \) whereas □'s are the elements in \( B(D) \).

Let \( T \) in \( \mathcal{T}(\lambda) \). The map \( \mathbb{D}(\lambda) \rightarrow \mathcal{A} \) defined by \( (i, j) \mapsto \max(T(i, j)) \) gives an element in \( \mathcal{T}_0(\lambda) \). Denote the resulting element by \( \max(T) \). We denote \( (x|b)^T \) in (9.1) simply by \( (x|b)^T \).

**Lemma 10.8.** Let \( T_0 \in \mathcal{T}_0(\lambda) \) be correspond to \( D \in \mathcal{E}_{\text{sep}}(\lambda) \) by the bijection of Lemma 10.7. Then

\[
(10.3) \quad w_{\text{sep}}(D) = \sum_{T \in \mathcal{T}(\lambda), \max(T)=T_0} \beta^{|T|-|\lambda|}(x|b)^T.
\]
If this lemma is true, the proof of Prop. 10.3 completes as follows:

\[
E_\lambda(x|b) = \sum_{D\in E_{\text{sep}}(\lambda)} \text{wt}_{\text{sep}}(D) \\
= \sum_{T_0 \in T_0(\lambda)} \left( \sum_{T \in T(\lambda), \max(T)=T_0} \beta^{[T]-[\lambda]}(x|b)^T \right) \\
= \sum_{T \in T(\lambda)} \beta^{[T]-[\lambda]}(x|b)^T = T_\lambda(x|b),
\]

where the second equality is the consequence of bijection in Lemma 10.7 and equation 10.3.

Example 10.3. In order to see that equation 10.3 holds, it is convenient to use weights \(x_j \ominus b_{j-i}\) on the vertical line instead of \(x_j \ominus b_{j-i}\). This device makes clear that each term in the right-hand side of 10.3 corresponds naturally to a term in suitably expanded form of \(\text{wt}_{\text{sep}}(D)\). If \(D\) is the EYD in Example 10.2 then \(\text{wt}_{\text{sep}}(D)\) is equal to the product of the weight in the following boxes:

\[
\begin{array}{cccc}
(1 + \beta x_1)x_1 & (1 + \beta(x_1 + b_1))(1 + \beta(x_2 + b_1))(x_2 + b_1) & (1 + \beta(x_2 + b_2))(x_3 + b_2)
\end{array}
\]

First note that the factor \((x_1(x_2 + b_1))(x_3 + b_2)x_2\), when we ignore primes, is equal to \((x|b)^{T_0}\). Expanding the remaining factors of the form \(1 + \beta \text{wt}_{\text{sep}}(\alpha)\), which comes from \(B(D)\), the weight \(\text{wt}_{\text{sep}}(D)\) can be expressed as a sum over tableaux

\[
\begin{array}{cccc}
a_1 & b_2 & c_3 \\
\end{array}
\]

where \(a, b, c\) are subsets of \(A\) such that \(a \subset \{1'\}, \ b \subset \{1, 2'\}, \ c \subset \{2\}\). Note that these tableaux are exactly \(T\)'s in \(T(\lambda)\) such that \(\max(T)=T_0\). One sees that such \(T\) gives a summand \(\beta^{[T]-[\lambda]}(x|b)^T\). For example, if \(a = \{1'\}, \ b = \{2'\}, \ c = \emptyset\) then for the corresponding \(T\), we have the term

\[
\beta^2 \cdot x_1 x_1 (x_2 + b_1)(x_2 + b_1)(x_3 + b_2)x_3'.
\]

We then ignore ‘primes’ to have the right weight \(\beta^2(x|b)^T\). Thus we have 10.3.

The argument in Example 10.3 can be generalized to the following:

Proof of Lemma 10.3. For each \(c = (i, j) \in D(\lambda)\), let \(R(c; T_0)\) denote the set of letters \(a\) in \(A\) such that joining \(a\) into \(T_0\) at \(c\) gives a semistandard set-valued tableau. Now suppose, for all \(c \in D(\lambda)\), subsets \(S_c\) in \(R(c; T_0)\) are given. Then by joining \(S_c\) into \(T_0\) at \(c\) for all \(c \in D(\lambda)\), we have a tableau \(T \in T(\lambda)\) such that \(\max(T)=T_0\). Conversely, all such tableaux are obviously given in this way. This correspondence gives

\[
(10.4) \quad (x|b)^{T_0} \sum_{(S_c)\in D(\lambda)} \prod_{c \in D(\lambda)} \beta^{[S_c]} \prod_{a \in S_c} w(c; a) = \sum_{T \in T(\lambda), \max(T)=T_0} \beta^{[T]-[T_0]}(x|b)^T,
\]

where \(S_c\) runs for all the subsets in \(R(c; T_0)\), and \(w(c; a)\) is defined in 9.1. Note that \((x|b)^{T_0} = \prod_{c \in D} \text{wt}_{\text{sep}}(c)\). Finally as illustrated in Example 10.3 we have

\[
\sum_{(S_c)\in D(\lambda)} \prod_{c \in D(\lambda)} \beta^{[S_c]} \prod_{a \in S_c} w(c; a) = \prod_{c \in B(D)} (1 + \beta \text{wt}_{\text{sep}}(c)).
\]
This together with (10.4) leads to (10.3). □

References

[1] S. Billey, V. Lakshmibai, Singular loci of Schubert varieties, Progr. Math. 182 (2000), Birkhäuser, Boston.

[2] A. S. Buch, A Littlewood-Richardson rule for the $K$-theory of Grassmannians, Acta. Math. 189 (2002) 37–78.

[3] A. S. Buch, V. Ravikumar, Pieri rules for the $K$-theory of cominuscule Grassmannians, preprint, arXiv:1005.2603 to appear in J. Reine Angew. Math.

[4] E. Clifford, H. Thomas, A. Yong, K-theoretic Schubert calculus for $OG(2, 2n+1)$ and jeu de taquin for shifted increasing tableaux, arXiv:1002.1664v2.

[5] Z. Daugherty, A. Ram, R. Virk, Affine and degenerate affine BMW algebras: The center, arXiv:1105.4207.

[6] M. Goresky, R. Kottwitz, R. MacPherson, Equivariant cohomology, Koszul duality, and the localization theorem, Invent. Math. 131 (1998) 25–83.

[7] P. N. Hoffman, J. F. Humphreys, Projective representations of the symmetric groups. Q-functions and shifted tableaux Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

[8] T. Ikeda, Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian, Adv. Math. 215 (2007), 1-23.

[9] T. Ikeda, L. Mihalcea, H. Naruse, Double Schubert polynomials for the classical groups, Adv. Math. 226 (2011) 840-886.

[10] T. Ikeda, H. Naruse, Excited Young diagrams and equivariant Schubert calculus. Trans. Amer. Math. Soc. 361 (2009), no. 10, 5193–5221.

[11] T. Ikeda, H. Naruse, Double Schubert polynomials of classical type and Excited Young diagrams, Kökuryokou Bessatsu B11 (2009),

[12] T. Ikeda, H. Naruse, Y. Numata, Bumping algorithm for set-valued shifted tableaux (extended abstract for FPSAC 2011, Reykjavik), Discrete Math. Theor. Comput. Sci. (online) Proceeding volume for fpsac 2011, 527–538.

[13] V. N. Ivanov, The dimension of skew shifted Young diagrams, and projective characters of the infinite symmetric group. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 240 (1997), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 2, 115–135, 292–293; translation in J. Math. Sci. (New York) 96 (1999), no. 5, 3517–3530.

[14] V. N. Ivanov, Combinatorial formulas for factorial Schur $Q$-functions, J. Math. Sci. vol. 107, No. 5, 2001, 4195–4211.

[15] V. N. Ivanov, Interpolation analogues of Schur $Q$-functions. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 307 (2004), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 10, 99–119, 281–282; translation in J. Math. Sci. (N. Y.) 131 (2005), no. 2, 5495–5507.

[16] A. N. Kirillov, H. Naruse, Construction of double Grothendieck polynomials of classical type using Id-Coxeter algebras, preprint.

[17] D. Knuth, Overlapping pfaffians, Electronic Journal of combinatorics, 3 (2)1996, 1–13.

[18] A. Knutson, A Schubert calculus recurrence from the noncomplex W-action on G/B, arXiv:0306304.

[19] A. Knutson, T. Tao, Puzzles and (equivariant) cohomology of Grassmannians. Duke Math. J. 119 (2003), no. 2, 221–260.

[20] B. Kostant, S. Kumar, $T$-equivariant $K$-theory of generalized flag varieties. J. Differential Geom. 32 (1990), no. 2, 549–603.

[21] V. Kreiman, Schubert classes in the $K$-theory and equivariant cohomology of the Lagrangian Grassmannian, arXiv:0602245.

[22] S. Kumar, Kac-Moody groups, their flag varieties and representation theory, Progress in Math. 204, Birkhäuser, Boston, 2002.

[23] T. Lam, A. Schilling, M. Shimozono, $K$-theory Schubert calculus of the affine Grassmannian. Compos. Math. 146 (2010), no. 4, 811–852.

[24] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, Oxford Univ. Press, Oxford 1995.

[25] P. J. McNamara, Factorial Grothendieck polynomials, Elec. J. Combin. 13 (2006), #R71.
[26] A. Molev, B. E. A. Sagan, A Littlewood-Richardson rule for factorial Schur functions. Trans. Amer. Math. Soc. 351 (1999), no. 11, 4429–4443.

[27] H. Naruse, Excited Young diagram and Yang-Baxter equation, in preparation.

[28] J. J. C. Nimmo, Hall-Littlewood symmetric functions and the BKP equation. J. Phys. A 23 (1990), no. 5, 751–760.

[29] P. Pagacz, Algebro-geometric applications of Schur $S$- and $Q$-polynomials. Topics in invariant theory (Paris, 1989/1990), 130–191, Lecture Notes in Math., 1478, Springer, Berlin, 1991.

[30] W. Graham, S. Kumar, On positivity in $T$-equivariant $K$-theory of flag varieties. Int. Math. Res. Not. IMRN 2008, Art. ID rnn 093, 43 pp.

[31] S. R. Ghorpade, K. N. Raghavan, Hilbert functions of points on Schubert varieties in the symplectic Grassmannian. Trans. Amer. Math. Soc. 358 (2006), no. 12, 5401–5423.

[32] K. N. Raghavan, K. S. Upadhyay, Hilbert functions of points on Schubert varieties in orthogonal Grassmannians. J. Algebraic Combin. 31 (2010), no. 3, 355–409.

[33] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, J. Reine Angew. Math. 139 (1911) 155–250.

[34] J. Stembridge, Shifted tableaux and the projective representations of symmetric groups, Adv. Math. 74, (1989) 87–134.

Department of Applied Mathematics, Okayama University of Science, Okayama 700-0005, JAPAN
email address: ike@xmath.ous.ac.jp

Graduate School of Education, Okayama University, Okayama 700-8530, JAPAN
email address: rdcv1654@cc.okayama-u.ac.jp