Parametric local stability condition of a multi-converter system

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Abstract—We study local (also referred to as small-signal) stability of a network of identical DC/AC converters having a rotating degree of freedom by closing the loop with the matching control at each converter. We develop a stability theory for a class of partitioned linear systems with symmetries that has natural links to classical stability theories of interconnected systems but improves upon them. We find stability conditions descending from a particular Lyapunov function involving an oblique projection into the invariant set of synchronous steady states and enjoying insightful structural properties. Our sufficient and explicit stability conditions can be evaluated in a fully decentralized fashion, reflect a parametric dependence on the converter’s steady-state variables, and can be one-to-one generalized to other types of systems exhibiting the same behavior, such as synchronous machines. Our conditions demand for sufficient reactive power support and resistive damping. These requirements are well aligned with practitioners’ insights.

I. INTRODUCTION

The major shift in power generation from conventional synchronous machines to renewables led to the study of problems of network stability composed mainly of DC/AC converters mimicking the electro-mechanical interaction of synchronous machines with the grid. Power system stability amounts to the ability of an electric power system, for a given initial operating condition, to regain state of desired operation, after being subjected to a disturbance [2]. Synchronous machines embody mechanical systems having a rotational degree of freedom, their rotor angle [3], [4]. As a result, power system dynamics admit trigonometric nonlinearities, diffusive coupling between electrical and mechanical angles, and a rotational symmetry. Also, converters controlled to emulate synchronous machines [5], [6] endow the closed-loop system with a virtual rotating angle and thus inherit their dynamics, which proved to be a challenge for many stability analysis approaches.

In this context, different power system stability conditions have been proposed: In [7]–[9] sufficient stability conditions are obtained for a single-machine/converter case. In [10], a sufficient algebraic condition is linked to Kuramoto oscillator and connects the synchronization of power systems with network topology and system parameters. Although these conditions give qualitative insights into the sensitivities influencing stability, they usually require strong (and often unrealistic) assumptions. On the other hand, the resulting conditions are not decentralized and can only be evaluated with omniscient knowledge of system parameters and operating point [5], [10], [11]. In general, explicit stability conditions require strong assumptions (strong mechanical or DC-side damping) [1], [10], [12], whereas implicit conditions are based on semi-definite programs and thus not very insightful [3]. [13]. Additionally, the underlying models are of reduced order (sometimes only first or second order) [3], [5], [6], [10], [14]. Reduced-order systems, where one infers stability of the whole system from looking at only a subset of variables, are not a truthful representation of the full-order system if the necessary assumptions are not met [12]. Finally, some conditions are valid only in radial networks [11].

Additionally, in most works models are often assumed to be lossless implying a weak coupling between variations in active power/angle and reactive power/voltage magnitude. This is not a valid argument for many models that can be found in a vast majority of power system literature, assuming quasi-stationary steady-state conditions [15], since both rotor angle stability and voltage stability are affected by active and reactive power flows due to inevitable resistive losses. Quite the opposite, experimental and theoretical findings underscore that sufficient (virtual) resistive damping and reactive power support are necessary for the stability of a power system [16], [17].

We consider a higher-order multi-source power system model consisting of identical DC/AC converters interconnected in general topology through lossy lines with uniform inductive-resistive dynamics. The converters are controlled through matching control [8] which renders them structurally equivalent to synchronous machines. Thus, our analysis approach also extends to synchronous machines. Our model exhibits a rotational symmetry corresponding to the absence of an absolute angle. We pursue a parametric linear stability analysis at a synchronous and rotationally invariant steady state. Towards this, we develop a novel analysis approach for a class of partitioned linear systems characterized by a stable subsystem and a one-dimensional invariant subspace. We propose a class of Lyapunov functions characterized by an oblique projection into the complement of the invariant subspace, following the direction of a matrix to be chosen as solution to Lyapunov and Ricatti equations. Our approach has natural cross-links with well-established analysis concepts for interconnected systems, e.g., passive systems, though our assumptions are less restrictive, which makes our analysis applicable to the multi-source power system. For the considered model, we arrive at explicit stability conditions that depend only on local parameters and steady-state values and can thus be evaluated in a fully decentralized fashion. Unlike other works, our conditions do not require the restrictive assumption of overly strong mechanical (respectively DC-side) damping but rather sufficient reactive power support and AC-side resis-
tance, which are well aligned with practitioners' insights.

II. MODELING AND SETUP

A. DC/AC Converter Model

Consider an averaged and balanced three-phase DC/AC converter model, as in [8] and as illustrated in Figure 1. The dynamics, formulated in a rotating frame with angle $\omega^* t$, with $\omega^*$ being the nominal frequency are

$$ C_{dc} \dot{v}_{dc} = -G_{dc} (v_{dc} - v_{dc}^*) - \frac{1}{2} m_{dq} i_{dq} + i_{dc} $$(1a)

$$ L_{dq} \dot{i}_{dq} = -Z_R i_{dq} + \frac{1}{2} m_{dq} v_{dc} - v_{dq} $$ (1b)

$$ C_i \dot{v}_i = -Z_V v_{dq} + i_{dq} - i_{net} $$ (1c)

where we defined the impedance matrices $Z_R = R + J \omega^* L \in \mathbb{R}^{2 \times 2}$, $Z_V = G + J \omega^* C \in \mathbb{R}^{2 \times 2}$ and $Z_{net} = R_{net} + J \omega^* L_{net} \in \mathbb{R}^{2 \times 2}$, and used the shorthand $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The DC circuit is represented by its capacitance $C_{dc} > 0$, conductance $G_{dc} > 0$ and a controllable current source $i_{dc} \in \mathbb{R}$. The voltage across the DC capacitor is denoted by $v_{dc} \in \mathbb{R}$. The switching pattern induced by the modulation signal, representing the principal converter control input, is given by $m_{dq} \in [-1, 1]^2$ and transforms DC- into AC signals filtered at the output of the converter via a resistor $R > 0$, an inductor $L > 0$, and a capacitor $C > 0$. The voltage at the output capacitor is denoted by $v_{dq} \in \mathbb{R}^2$, and the current flowing through the inductance is $i_{dq} \in \mathbb{R}^2$. The conductance $G > 0$ models a resistive load attached to the converter at its output. The current flowing out from the converter into the network is denoted by $i_{net} \in \mathbb{R}^2$.

The following matching control is based on the concept of matching the converter model to a synchronous machine; see Remark 1. The converter modulation signal is controlled as a sinusoid with constant magnitude $\mu \in [0, 1]$ and frequency $\gamma \in \mathbb{R}$ (relative to $\omega^*$) given by the DC voltage deviation [12],

$$ \dot{\gamma} = \eta (v_{dc} - v_{dc}^*) $$ (2a)

$$ m_{dq} = \mu \begin{bmatrix} -\sin(\gamma) \\ \cos(\gamma) \end{bmatrix} $$ (2b)

where $\gamma \in \mathbb{S}^1$ is the virtual angle, and $\eta > 0$ is a gain.

Next, we design the current source $i_{dc}$ as a feedback term represented by $i_{dc} = G_{dc} v_{dc}^*$, and a feedforward term on the DC voltage deviation

$$ i_{dc} = i_{dc}^* - K_p (v_{dc} - v_{dc}^*) - G_{dc} v_{dc}^*, $$ (3)

where $K_p, i_{dc}^* > 0$. The modulation amplitude $\mu$, feedforward current $i_{dc}^*$, the control gain $K_p$ are regarded as constants usually determined offline or in outer control loops. Finally, the overall closed-loop DC/AC converter can be written as

$$ \dot{\gamma} = \eta (v_{dc} - v_{dc}^*) $$ (4a)

$$ C_{dc} \dot{v}_{dc} = -K_p (v_{dc} - v_{dc}^*) - \mu \begin{bmatrix} -\sin(\gamma) \\ \cos(\gamma) \end{bmatrix} i_{dq} + i_{dc}^* $$ (4b)

$$ L_{dq} \dot{i}_{dq} = -Z_R i_{dq} + \frac{1}{2} m_{dq} v_{dc} - v_{dq} $$ (4c)

$$ C_i \dot{v}_i = -Z_V v_{dq} + i_{dq} - i_{net} $$ (4d)

where we used the shorthand $\dot{K}_p = G_{dc} + K_p$. The results derived below can conceptually also be applied to synchronous machines; see also Remark 2.

B. Multi-DC/AC converter model

We extend the closed-loop DC/AC converter model (3) to a network of $n$ interconnected converters. With some abuse of notation, we use the same symbols for parameters and states of the $n$-converter study.

Every closed-loop converter model is as in (1) with identical parameters, modulation signal (2), and connected to the grid through a line with resistance $R_{net} > 0$ set in series with line inductance $L_{net} > 0$. Let $B$ denote the incidence matrix of a connected grid at each phase, then $B = B \otimes I$, where $I$ denote an identity matrix of appropriate dimension. The dynamics of the current $i_{net}$ flowing through the lines are described by

$$ \dot{i}_{net} = -Z_{net} i_{net} + B^T v_{dq}, $$ (5a)

$$ \dot{v}_{dc} = -\text{diag}(\dot{K}_p) (v_{dc} - v_{dc}^*) $$ (5b)

$$ -\frac{1}{2} \text{diag}(\mu) (\gamma) i_{dq} + i_{dc}^* $$ (5c)

$$ C_i \dot{v}_i = -Z_V v_{dq} + i_{dq} - B i_{net} $$ (5d)

$$ L_{net} \dot{i}_{net} = -Z_{net} i_{net} + B^T v_{dq}, $$ (5e)

where $v_{dc} = [v_{dc,1}, \ldots, v_{dc,n}]^T \in \mathbb{R}^n$, $i_{dq} = [i_{dq,1}, \ldots, i_{dq,n}]^T \in \mathbb{R}^{2n}$, $\dot{K}_p = [\dot{K}_{p,1}, \ldots, \dot{K}_{p,n}]^T \in \mathbb{R}^n$, $i_{net} = [i_{net,1}, \ldots, i_{net,n}]^T \in \mathbb{R}^{2n}$, where $Z_R = I \otimes Z_R \in \mathbb{R}^{2n}$, $Z_V = I \otimes Z_V \in \mathbb{R}^{2n}$, and $R(\gamma) = \text{diag} \{r(\gamma_1), \ldots, r(\gamma_n)\} \in \mathbb{R}^{2n \times 2n}$. A gain example of two DC/AC converters is shown in Figure 1.

Observe that the dynamics (5) are invariant under a rigid rotation of all AC variables, i.e., under the map

$$ [\dot{\gamma}^T v_{dc}^T i_{dq}^T v_{dq}^T i_{net}^T]^T \mapsto [\dot{\gamma}^T + \theta_0 I_n v_{dc}^T R(\theta_0) i_{dq}^T R(\theta_0) v_{dq}^T R(\theta_0) i_{net}^T]^T $$ (6)

where $\theta_0 \in \mathbb{S}^1$ and $R(\theta_0)$ is a rotation matrix. The physical insight is that there is no absolute angle in a power system. We will observe this rotational symmetry also in the linearization.

C. Synchronous steady-state characterization

We are particularly interested in a synchronous steady-state (see Theorem 2 in [18]) with the following properties:

- The frequencies are synchronized at the nominal value $\omega^*$ mapped into a nominal DC voltage $v_{dc}^*$

$$ |\omega| = \{ \omega \in \mathbb{R}^n_{\ge 0} | \omega = \omega^* \}, $$

$$ |v_{dc}| = \{ v_{dc} \in \mathbb{R}^n_{\ge 0} | v_{dc} = v_{dc}^* \}. $$
\[ \begin{align*}
\dot{x} &= A(x^*) x, \\
A(x^*) &= \begin{bmatrix}
-\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\gamma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -L & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -L & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -L & 0
\end{bmatrix}, \\
J &= I \otimes J, \\
W(x^*) &= \frac{1}{2} \text{diag}(\mu) \text{diag}(J \gamma^*) = \text{diag}(w_k(x_k^*)), \\
Y(x^*) &= \frac{1}{2} \text{diag}(\mu) R \gamma^*, \\
M(x^*) &= \frac{1}{2} \text{diag}(v^*, J R \gamma^*).
\end{align*} \]

III. STABILITY OF A CLASS OF LINEAR SYSTEMS

In this section, we develop a stability theory for a general class of linear systems enjoying some of the properties featured by the system matrix \(A(x^*)\) in (7).

A. Separable Lyapunov Analysis for Systems with Symmetries

We consider a class of partitioned linear systems of the form
\[ \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x, \] (8)
where \(x = [x_1^T \ x_2^T]^T\) denotes the partitioned state vector, and the block matrices \(A_{11}, A_{12}, A_{21}, A_{22}\) are of appropriate dimensions.

In the following, we assume stability of the subsystem characterized by \(A_{11}\) and the existence of a symmetry, i.e., an invariant zero eigenspace:

**Assumption 1.** The block matrix \(A_{11}\) in (8) is Hurwitz.

**Assumption 2.** There exists a vector \(v = [v_1^T \ v_2^T]^T\), so that
\[ A \cdot \text{span}\{v\} = 0. \]

Assumption 2 is compatible with the nullspace of \(A(x^*)\) in (7). However, we remark that all of our analysis holds analogously if this assumption is removed; see Corollary III.3.

Our objective is to derive conditions, so that \(\text{span}\{v\}\) is asymptotically stable, hence guaranteeing that the system matrix \(A\) has all its eigenvalues in the left-half plane except for one at zero, whose eigenspace is \(\text{span}\{v\}\).

We start by defining the Lyapunov candidate as
\[ V(x) = x^T \left( P - \frac{P v v^T P}{v^T P} \right) x, \] (9)
where \(P\) is a positive definite matrix. Our Lyapunov candidate construction is based on two key observations:

- **First**, the function \(V(x)\) is a projection into the complement of \(\text{span}\{v\}\) along the columns of the matrix \(P\).
- **Second**, the positive definite matrix \(P\) is a degree of freedom that can specified later to provide sufficient and favorable (e.g., decentralized) stability conditions.

In standard Lyapunov analysis, one seeks a pair of matrices \((P, Q)\) with suitable positive (semi)-definiteness properties so that the Lyapunov equation \(P A + A^T P = -Q\) is met. In the following, we apply a helpful twist and parameterize the \(Q\)-matrix as a quadratic function \(Q(P)\) of \(P\), which renders the
Lyapunov equation to an algebraic Riccati equation. We choose the following structure for the matrix $Q(P)$,

$$Q(P) = \begin{bmatrix} Q_1 & H(P)^\top \\ H(P) & H(P)Q_1^{-1}H(P)^\top + Q_2 \end{bmatrix},$$

where $Q_1$ is a positive definite matrix, $Q_2$ is a positive semi-definite matrix with respect to $\text{span}\{v_2\}$, $P$ is block-diagonal,

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},$$

with $P_1 = P_1^\top > 0$ and $P_2 = P_2^\top > 0$, i.e., the Lyapunov function is separable, and finally $H(P) = A_{12}P_1 + P_2A_{21}$ is a shorthand.

We need to introduce a third and final assumption.

**Assumption 3.** Consider the matrix $F = A_{22} + A_{21}Q_1^{-1}P_1A_{12}$. Assume that $(F,A_{21}Q_1^{-1/2})$ is stabilizable and $(F,D)$ is detectable, where $DD^\top = A_{12}^\top P_1Q_1^{-1}P_1A_{12} + Q_2$.

Assumption 3 will guarantee suitable definiteness and decay properties of the separable Lyapunov function $\Pi$ under comparatively mild conditions, as discussed in Section III-B.

**Proposition III.1.** Under Assumptions 1, 2, and 3 the matrix $P$ in (11) is unique and positive definite.

**Proof.** By calculating $PA + A^\top P = -Q(P)$, where $A$ is as in (8), $P$ is as in (11), and $Q(P)$ is as in (10), we obtain

$$\begin{bmatrix} P_1A_{11} + A_{11}^\top P_1 & H(P)^\top \\ H(P) & P_2A_{22} + A_{22}^\top P_2 \end{bmatrix} = \begin{bmatrix} Q_1 & H(P)^\top \\ H(P) & H(P)Q_1^{-1}H(P)^\top + Q_2 \end{bmatrix},$$

the block-diagonal terms of which are

1. $P_1A_{11} + A_{11}^\top P_1 = -Q_1$,
2. $P_2A_{22} + A_{22}^\top P_2 = -H(P)Q_1^{-1}H(P)^\top - Q_2$.

Since $A_{11}$ is Hurwitz, there is a unique and positive definite matrix $P_1$ solving (1). Moreover, specification (2) is equivalent to solving for $P_2$ in the following algebraic Riccati equation:

$$P_2A_{21}Q_1^{-1}A_{12}P_2 + P_2F + F^\top P_2 + A_{12}P_1Q_1^{-1}P_1A_{12} + Q_2 = 0.$$ 

Under Assumption 3 there is a solution to (2) with $P_2$ unique and positive definite.

**Lemma III.2.** Under Assumptions 1, 2, and 3 the matrix $Q(P)$ in (10) is positive semi-definite. Additionally, $\text{ker}(A) = \text{span}\{v\}$.

**Proof.** First, observe that the matrix $Q(P)$ in (10) is symmetric, and the upper left block $Q_1 > 0$ is positive definite. By using the Schur complement and positive semi-definiteness of $Q_2$, we obtain that $Q(P)$ is positive semi-definite. Second, by virtue of $v^\top Q(P)v = v^\top (PA + A^\top P)v = 0$ due to Assumption 2 it follows that $\text{span}\{v\} \subseteq \text{ker}(Q(P))$.

Third, consider a general vector $s = [s_1 \ s_2]^\top$, so that $Q(P)s = 0$, we obtain the algebraic equations,

$Q_1s_1 + H(P)^\top s_2 = 0$, $H(P)s_1 + (H(P)Q_1^{-1}H(P)^\top + Q_2)s_2 = 0$. One deduces that $Q_2s_2 = 0$ and thus $s_2 \in \text{span}\{v_2\}$. The latter implies $s_1 = -Q_1^{-1}H(P)^\top \text{span}\{v_2\}$ because $Q(P)\text{span}\{v\} = 0$. Thus, it follows that $s \in \text{span}\{v_1 \ s_2\} = \text{span}\{v\}$, and we deduce that $\text{ker}(Q(P)) = \text{span}\{v\}$. Fourth and finally, for the sake of contradiction, take a vector $\bar{v} \notin \text{span}\{v\}$, so that $\bar{v} \in \text{ker}(A) \Rightarrow \bar{v}^\top (A^\top P + PA)\bar{v} = 0 \Rightarrow \bar{v}^\top Q(P)\bar{v} = 0 \Rightarrow \bar{v} \in \text{ker}(Q(P))$. This is a contradiction to $\text{ker}(Q(P)) = \text{span}\{v\}$. Hence, we conclude that $\text{ker}(A) = \text{ker}(Q(P)) = \text{span}\{v\}$.

Next, we provide the main result of this section.

**Theorem III.3.** Consider system (8). Under Assumptions 1, 2 and 3 $\text{span}\{v\}$ is an asymptotically stable subspace of $A$.

**Proof.** Consider the function $V(x)$ in (8). The matrix $P$ in (11) is positive definite by Proposition III.1. Take $y = P^{1/2}x$ and $w = P^{1/2}v$, the function $V(x)$ can be rewritten as $V(y) = y^\top \left(I - \frac{w^\top w}{w^\top w}\right)y = y^\top \Pi_w y$. The matrix $\Pi_w = I - \frac{w^\top w}{w^\top w}$ is a projector into the orthogonal complement of span($w$), and is hence positive semi-definite with one-dimensional nullspace corresponding to $P^{1/2}\text{span}\{v\}$. It follows that the function $V(x)$ is positive semi-definite with respect to span($v$). By means of $Av = Q(P)v = 0$ in $v^\top PA = v^\top (Q - A^\top P) = 0$, we obtain $V(x) = -x^\top Q(P)x$. By Lemma III.2 it holds that $V(x)$ is negative definite with respect to span($v$). We apply Lyapunov’s method to conclude that span($v$) is an asymptotically stable subspace.

**B. Contextualizing and Relations to Existing Results**

In what follows, we consider a few special cases and put Theorem III.3 as well as Assumptions 1, 2 and 3 into context.

First, note that our approach is general and can be applied to matrices that do not have a nullspace as in Assumption 2.

In this case, the Lyapunov function (13) is simply chosen as $V(x) = x^\top Px$ with $P$ being positive definite and separable as in (11), and $Q_1$ is positive definite, and $Q_2$ in (10) is chosen to be positive definite. In this case our analysis applies analogously.

**Corollary III.4.** Consider system (8). Under Assumptions 1, 2 and 3 the system is asymptotically stable.

Second, we put Assumption 3 into context and compare it to other system classes admitting separable Lyapunov functions. Towards this it will be helpful to consider system (8) as an interconnected closed-loop system, as illustrated in Figure 2. For simplicity of presentation, in the remainder of this section, we consider only the case of Corollary III.4 that is, we discard the existence of a nullspace as in Assumption 2.

We will also consider only the sufficient condition that $F = A_{22} + A_{21}Q_1^{-1}P_1A_{12}$ is Hurwitz which implies Assumption 3.
Small-gain interpretation: Let us first provide a qualitative insight into Assumption 1. As in Figure 2, we define the stable (due to Assumption 1) subsystem \( \Sigma_1 : x_1 = A_{11} x_1 + u_1 \) with state \( x_1 \), input \( u_1 = A_{12} x_2 \), and output \( y_1 = A_{21} x_1 \). Recall that if one chooses \( Q_1 = 2 \gamma' P_1 \) with \( \gamma > 0 \), then \( P_1 \) measures the exponential decay of \( x_1 \). Now if \( A_{22} \) is Hurwitz as well, then the condition that \( F = A_{22} + A_{21} Q_1^{-1} P_1 A_{12} = A_{22} + A_{21} A_{12} / (2 \gamma') \) is Hurwitz can be understood as a small gain condition: \( A_{11} \) and \( A_{12} \) have to be sufficiently stable so that the interconnection through \( A_{12} \) and \( A_{21} \) remains stable.

Passive systems: The sufficient conditions that \( A_{11} \) and \( F \) must be Hurwitz include the case of passive systems, which are a well-known class of systems admitting separable Lyapunov functions. To see this consider again Figure 2 but with a different partitioning. Consider the stable (due to Assumption 1) subsystem \( \Sigma_1 : x_1 = A_{11} x_1 + A_{12} u_1 \) with state \( x_1 \), input \( u_1 = -x_2 \), and output \( y_1 = A_{21} x_1 \). Consider also the subsystem \( \Sigma_2 : x_2 = A_{22} x_2 + u_2 \) with state \( x_2 \), input \( u_2 = x_1 \), and output \( y_2 = x_2 \). Assume that both \( \Sigma_1 \) and \( \Sigma_2 \) are strictly passive. Hence, their negative feedback connection is known to be asymptotically stable. Under these conditions it can be shown that \( A_{11} \) and \( F \) must be Hurwitz. To see this note that, by the Kalman-Yakubovich-Popov (KYP) Lemma, \( \Sigma_1 \) is strictly passive with input \( u_1 \) and output \( y_1 \) if and only if there are positive definite matrices \( P_1 \) and \( Q_1 \) so that \( P_1 A_{11} + A_{11}^T P_1 = -Q_1 \) (which equals Assumption 1) and \(-P_1 A_{12} = A_{21}^T\). Likewise, by the KYP Lemma, \( \Sigma_2 \) is strictly passive with input \( u_2 \) and output \( y_2 \) if and only if the associated storage function is \( x^T P_2 x = x^T x \), that is, \( A_{22} \) is negative definite. In this case, \( F = A_{22} + A_{21} Q_1^{-1} P_1 A_{12} = A_{22} - A_{21} Q_1^{-1} A_{12} \) is negative definite (Hurwitz), and Assumptions 1 and 3 are satisfied.

IV. LOCAL STABILITY OF THE MULTI-DC/AC CONVERTER

This section aims at characterizing the local stability behavior of the nonlinear DC/AC converter model in (5). Ultimately, we would like to find an answer to the question: given the dynamics in (5), under which conditions is the synchronous steady state locally asymptotically stable? For this, we apply the theory developed in the previous section to give nuts and bolts on how to derive sufficient parametric conditions for the closed-loop multi-converter model (5). We consider the linearized model (7) and identify the following matrices

\[
A_{11} = \begin{bmatrix} 0 & \frac{\eta}{2} M(x) \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & M(x) \end{bmatrix}, \\
A_{21} = \begin{bmatrix} L^{-1} M(x) & L^{-1} Y(x) \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & \frac{\eta}{2} \end{bmatrix},
\]

Define the Lyapunov function \( V(x) \) as in (9) with \( v = v(x) \). Hence, \( V(x) \) is positive semi-definite with respect to span\{\( v(x) \)\}. Next, we fix the matrix \( Q(P) \) given by (10), where we set \( Q_1 = I \), \( Q_2 = I - \frac{\gamma' P_1}{2} \), and search for the corresponding matrix \( P \), so that \( PA(x') + A^T(x')P = -Q(P) \). Analogous to (11), we choose the block diagonal matrix \( P \) as

\[
P = \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},
\]

where \( P_{11}, P_{12} \), and \( P_{22} \) are chosen to be block diagonal. Notice that the chosen structure of \( P \) and the zeros in the off-diagonals in \( P \) originates from the physical intuition of the tight coupling between the angle of the converter and its corresponding DC voltage (proportional to the AC frequency), as enabled by the matching control (2). The same type of coupling comes into play in synchronous machines between the rotor angle and its frequency, due to the presence of the electrical power in the swing equation (2). The 4 × 4 matrix \( P_2 \) is dense with off-diagonals coupling at each phase the inductance current of one converter to the other and vice versa, reflecting the physical resistive and inductive line in dq-frame.

In the sequel, we show that this structure allows for sufficient and fully decentralized stability conditions.

Assumption 4 (Algebraic Synchronization Condition). Assume the following condition is satisfied,

\[
16 Q_{x,k} > \mu L W \gamma^2 \frac{v_d^2}{R},
\]

where \( Q_{x,k} = w_k(x) v_d \).

Next, we provide the main result of this Section.

Theorem IV.1 (Local Asymptotic Stability). Consider the closed-loop multi-converter model described by (5). Under Assumption 4, span\{\( v(x) \)\} is locally asymptotically stable for any trajectory of (5).

Proof. Since \( v(x) \in \ker(A) \), Assumption 2 is satisfied. If Assumption 4 is true, then \( w_k(x) > 0 \), the submatrix \( A_{11} \) is Hurwitz, and Assumption 1 is valid. Next, we verify Assumption 3. First, the matrix \( P_1 \) can be identified from specification 3 with \( Q_1 = I \) by

\[
P_{11} = \frac{1}{\gamma'} \begin{bmatrix} \text{diag} \{ \hat{K}_P \} & W(x)^{-1} \\ \frac{1}{2} \end{bmatrix} - \frac{1}{2} \text{diag} \{ \hat{K}_P \} (I + \eta C_d W(x)^{-1}) \]

\[
P_{12} = \frac{1}{2} \begin{bmatrix} W(x)^{-1} C_d \\ 0 \end{bmatrix}
\]

\[
P_{22} = \frac{1}{2} \text{diag} \{ \hat{K}_P \} (I + \eta C_d W(x)^{-1})
\]

The feasibility of specification 2 with the positive semi-definite matrix \( Q_2 = I - \frac{\gamma' P_1}{2} \) is given by

\[
p_2 E E^T + p_2 F + F^T p_2 + s S + Q_2 = 0
\]

where \( F = A_{22} + E P_1 S^T \), \( E = \begin{bmatrix} L^{-1} W(x) & L^{-1} Y(x) \end{bmatrix} \), and \( S = \begin{bmatrix} -C_d Y(x) & -C_d Y(x) \end{bmatrix} \). If \( F \) is Hurwitz, then \( (F, S) \) is stabilizable and \( (F, D) \) is detectable, where \( DD^T = SS^T + Q_2 \).
Next, we find sufficient and fully decentralized conditions, for which $F$ satisfies the Lyapunov equation $FP + PF = -QF$. We choose $P_F$ and $Q_F$ to be block-diagonal matrices

$$P_F = \begin{bmatrix}
0 & 0 \\
0 & C_r
\end{bmatrix}, 
Q_F = \begin{bmatrix}
\Gamma & 0 \\
0 & 2G_1 & 0 \\
0 & 0 & 2k_w^r I
\end{bmatrix}$$

with $\Gamma = 2RI + C_{dc}^{-1}(M(x^r)P_3Y(x^r)\top + Y(x^r)P_2M(x^r)\top + 2C_{dc}^{-1}(Y(x^r)P_2Y(x^r)\top))$ being itself block-diagonal. Aside from $\Gamma$, all diagonal blocks of $P_F$ and $Q_F$ are positive definite. We evaluate the block-diagonal matrix $P_F$ for positive definiteness by exploring its two-by-two block diagonals, where trace and determinant of each block are positive under AssumptionByName. By applying TheoremByName, we deduce that span{$v(x^r)$} is locally asymptotically stable for any trajectory of (7).

Remark 2 (Evaluation, interpretation, and satisfaction). Condition [16] can be evaluated in a fully decentralized fashion and depends on the converter’s resistance $R$, modulation amplitude $\mu$, nominal DC voltage $v_{dc}^*$, and reactive power output $Q^r$. Condition [13] requires sufficient reactive power support and resistive damping, which are well-known stability conditions [16]. In fact, the sufficient resistive damping is often enforced by virtual impedance control. A simple implementation thereof is based on the modulation signal

$$m' = m + 2k_m i_{dq}/v_{dc},$$

where $m$ is a vector whose components are as in [5], $k_m > 0$ is a control gain, and the DC voltage is assumed nonzero. The virtual impedance control [13] adds $k_m I$ to the resistive part of $Z_R$ in the model [5c], but it also affects the DC dynamics [50]. To counter-act the latter, the DC current source is controlled as $i_{dc} = i_{dc} + k_m i_{dq}/v_{dc}$, with $i_{dc}$ as in (3). A robust alternative to the feed-forward term $k_m i_{dq}/v_{dc}$ in practice is PI or high-gain control of $i_{dc}$.

In a more general setting with heterogeneous converter and RL lines parameters, an analogous condition can be obtained. The analogous multi-machine stability condition; see Remark [7] is

$$4Q^r > \frac{L_m^2 a^2 \omega^2}{R},$$

where $Q^r$ is the reactive power output neglecting stator losses, $L_m$ is the mutual inductance, and $i_f$ is the constant rotor excitation current. This condition sets an upper bound on the back electromotive force dependent on the reactive power and the stator resistance [8]. Observe that the mechanical damping (equivalent to $\dot{K}_{dc}$ [8]) has no effect on local stability contrary to many other stability conditions [7]–[10].

V. CONCLUSIONS

We derived sufficient and fully decentralized conditions for local asymptotic stability of a synchronous steady state of a power system composed of identical DC/AC converters in closed-loop with the matching control. We explored techniques from Lyapunov theory to analyze the small-signal trajectories of the system, where the basic idea is to resort to oblique projection, while exploiting the different degrees of freedom of the proposed Lyapunov function. We establish a stability theory that encompasses a class of linear systems inspired by our problem statement and that coalesce into well-known concepts for stability of interconnected systems. Since solutions of local stability problems should not be at the expense of other system aspects, future venues for our work include the study of conservativeness of our derived stability condition.

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