$\mathcal{N} = 6$ superconformal gravity in three dimensions from superspace

Sergei M. Kuzenko, Joseph Novak and Gabriele Tartaglino-Mazzucchelli

School of Physics M013, The University of Western Australia
35 Stirling Highway, Crawley W.A. 6009, Australia
joseph.novak,gabriele.tartaglino-mazzucchelli@uwa.edu.au

Abstract

A unique feature of $\mathcal{N} = 6$ conformal supergravity in three dimensions is that the super Cotton tensor $W^{IJKL}$ can equivalently be viewed, via the Hodge duality, as the field strength of an Abelian vector multiplet, $W^{IJ}$. Using this observation and the conformal superspace techniques developed in arXiv:1305.3132 and arXiv:1306.1205, we construct the off-shell action for $\mathcal{N} = 6$ conformal supergravity. The complete component action is also worked out.
1 Introduction

Recently, we have constructed the superspace actions for three-dimensional (3D) conformal supergravity theories with \( \mathcal{N} < 6 \) \cite{1}, and also worked out the complete component actions. Our construction made use of the novel off-shell formulation for \( \mathcal{N} \)-extended conformal supergravity presented in \cite{2} and called 3D \( \mathcal{N} \)-extended conformal superspace. 1 Upon degauging of certain local symmetries, conformal superspace reduces to the conventional formulation for \( \mathcal{N} \)-extended conformal supergravity \cite{5,6} with the structure group \( \text{SL}(2,\mathbb{R}) \times \text{SO}(\mathcal{N}) \). However, the former formulation proves to be much more efficient for addressing certain problems such as the construction of conformal supergravity actions.

At the component level, the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) supergravity actions given in \cite{1} coincide with those constructed in the 1980s in \cite{7} and \cite{8}, respectively, using the superconformal tensor calculus. But the off-shell \( \mathcal{N} = 3, \mathcal{N} = 4 \) and \( \mathcal{N} = 5 \) supergravity actions were derived in \cite{1} for the first time. The method employed in \cite{1} broke down for \( \mathcal{N} > 5 \) due to a technical reason to be discussed below. In this note

\footnote{The 3D \( \mathcal{N} \)-extended conformal superspace \cite{2} is a generalization of the off-shell formulations for \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) conformal supergravity theories in four dimensions \cite{3,4}.}
we make use of a unique property of the $\mathcal{N} = 6$ case to construct the corresponding off-shell action from superspace. Upon reduction to components, our action coincides with that found a week ago by Nishimura and Tanii [9] who used completely different techniques.

This paper is organized as follows. In section 2 we briefly review $\mathcal{N} = 6$ conformal supergravity in superspace and then describe the unique feature of $\mathcal{N} = 6$ conformal supergravity. In section 3 we make use of this property to construct the off-shell action for $\mathcal{N} = 6$ conformal supergravity, both in superspace and in terms of the component fields. The main results and their implications are discussed in section 4.

## 2 $\mathcal{N} = 6$ conformal supergravity

We begin with a brief review of $\mathcal{N} = 6$ conformal superspace following [2]. After that we describe the unique feature of $\mathcal{N} = 6$ conformal supergravity mentioned in the introduction.

We consider a curved three-dimensional $\mathcal{N} = 6$ superspace $\mathcal{M}^{3|12}$, parametrized by local bosonic ($x^m$) and fermionic coordinates ($\theta^\mu_I$), $z^M = (x^m, \theta^\mu_I)$, where $m = 0, 1, 2$, $\mu = 1, 2$ and $I = 1, \cdots, 6$. The $\mathcal{N} = 6$ conformal superspace [2] is obtained by gauging the $\mathcal{N} = 6$ superconformal algebra $\mathfrak{osp}(6|4, \mathbb{R})$ and then imposing appropriate constraints. The covariant derivatives have the form

$$\nabla_A = E_A^M \partial_M - \omega_A^b X_b = E_A^M \partial_M - \frac{1}{2} \Omega_A^{ab} M_{ab} - \frac{1}{2} \Phi_A^{PQ} N_{PQ} - B_A \mathbb{D} - \tilde{\mathfrak{S}}_A^B K_B . \quad (2.1)$$

Here $E_A = E_A^M(z) \partial_M$ is the inverse supervielbein, $M_{ab}$ are the Lorentz generators, $N_{IJ}$ are generators of the SO(6) group, $\mathbb{D}$ is the dilatation generator and $K_A = (K_a, S_a^I)$ are the special superconformal generators. The complete set of the generators of $\mathfrak{osp}(6|4, \mathbb{R})$ consists of $X^a = (P_A, X_a)$, where $P_A = (P_a, Q_a^I)$ are the super-Poincaré generators.

The Lorentz generators obey

$$[M_{ab}, M_{cd}] = 2 \eta_{[a} M_{b]d} - 2 \eta_{[d} M_{b]c} , \quad (2.2a)$$

$$[M_{ab}, \nabla_c] = 2 \eta_{[a} \nabla_{b]} , \quad [M_{\alpha\beta}, \nabla_{\gamma}] = \varepsilon_{\gamma(\alpha \nabla_{\beta})} . \quad (2.2b)$$

As usual, we refer to $K_a$ as the special conformal generator and $S_a^I$ as the $S$-supersymmetry generator.
The SO(6) and dilatation generators obey

\[ [N_{KL}, N^{IJ}] = 2\delta_I^J [N_L] - 2\delta_I^J [N_L], \quad [N_{KL}, \nabla^I] = 2\delta_I^I [\nabla_L], \quad (2.3a) \]

\[ [\mathbb{D}, \nabla] = \nabla, \quad [\mathbb{D}, \nabla^I] = \frac{1}{2} \nabla^I. \quad (2.3b) \]

The special conformal generators \( K_A \) transform under Lorentz and SO(6) transformations as

\[ [M_{ab}, K_c] = 2\eta_{[a} K_{b]} , \quad [M_{\alpha\beta}, S^I_\gamma] = \varepsilon_{\gamma(\alpha} S^I_{\beta)}, \quad (2.4a) \]
\[ [N_{KL}, S^I_\alpha] = 2\delta_I^I [K_{aL}], \quad (2.4b) \]

while under dilatations as

\[ [\mathbb{D}, K_a] = -K_a , \quad [\mathbb{D}, S^I_\alpha] = -\frac{1}{2} S^I_\alpha. \quad (2.5) \]

Among themselves, the generators \( K_A \) obey the algebra

\[ \{ S^I_\alpha, S^J_\beta \} = 2i\delta^{IJ} (\gamma^c)_{\alpha\beta} K_c. \quad (2.6) \]

Finally, the algebra of \( K_A \) with \( \nabla_A \) is given by

\[ [K_a, \nabla_A] = 2\eta_{ab} \mathbb{D} + 2M_{ab}, \quad (2.7a) \]
\[ [K_a, \nabla^I_A] = -i(\gamma_A)_{\alpha}^{\beta} S^I_{\beta}, \quad (2.7b) \]
\[ [S^I_\alpha, \nabla_A] = i(\gamma_A)_{\alpha}^{\beta} \nabla^I_{\beta}, \quad (2.7c) \]
\[ \{ S^I_\alpha, \nabla^J \} = 2\varepsilon_{\alpha\beta} \delta^{IJ} \mathbb{D} - 2\delta^{IJ} M_{\alpha\beta} - 2\varepsilon_{\alpha\beta} N^{IJ}. \quad (2.7d) \]

The remaining (anti-)commutators are not essential here and may be found in [2].

Under the supergravity gauge group, the covariant derivatives transform as

\[ \delta_g \nabla_A = [\mathcal{K}, \nabla_A], \quad (2.8) \]

where \( \mathcal{K} \) denotes the first-order differential operator

\[ \mathcal{K} = \xi^C \nabla_C + \frac{1}{2} \Lambda^{ab} M_{ab} + \frac{1}{2} \Lambda^{IJ} N_{IJ} + \Lambda_\mathbb{D} \mathbb{D} + \Lambda^K K_A. \quad (2.9) \]

Covariant (or tensor) superfields transform as

\[ \delta_g T = \mathcal{K} T. \quad (2.10) \]

The covariant derivatives obey the (anti-)commutation relations of the form

\[ [\nabla_A, \nabla_B] = -T_{AB}^C \nabla_C - R_{AB}^C X_a, \]
\[ = -T_{AB}^C \nabla_C - \frac{1}{2} R(M)_{AB}^{cd} M_{cd} - \frac{1}{2} R(N)_{AB}^{PQ} N_{PQ} \]
\[ - R(\mathbb{D})_{AB} \mathbb{D} - R(S)_{AB}^{K} S^K - R(K)_{AB}^c K_c, \quad (2.11) \]
where $T$ is the torsion, and $R(M), R(N), R(D), R(K)$ are the curvatures.

The algebra of covariant derivatives corresponding to $\mathcal{N} = 6$ conformal supergravity is

\begin{align}
\{\nabla^I, \nabla^J\} &= 2i\delta^I_J \nabla_{\alpha\beta} + i\varepsilon_{\alpha\beta} W^{IJKL} N_{KL} - \frac{i}{3} \varepsilon_{\alpha\beta}(\nabla_K W^{IJKL}) S_{\gamma L} \\
&\quad + \frac{1}{24} \varepsilon_{\alpha\beta}(\gamma^\gamma)(\nabla_\gamma \nabla_\delta W^{IJKL}) K_c , \tag{2.12a}
\\
[\nabla_a, \nabla_b] &= \frac{1}{6} (\gamma_a)_{\beta\gamma}(\nabla_\gamma W^{JPQK}) N_{PQ} \\
&\quad - \frac{1}{24} (\gamma_a)_{\beta\gamma}(\nabla_\gamma L \nabla_\delta W^{JKLP}) S_{\delta K} \\
&\quad - \frac{i}{240} (\gamma_a)_{\beta\gamma}(\gamma^\delta)_{\beta\rho}(\nabla_K \nabla_L \nabla_\rho W^{JKLP}) K_c , \tag{2.12b}
\\
[\nabla_a, \nabla_b] &= \frac{1}{48} \varepsilon_{abc}(\gamma^d)_{\alpha\beta} \left( i(\nabla^\alpha \nabla^\beta W^{PQIJ}) N_{PQ} \\
&\quad + \frac{1}{3} (\nabla_I \nabla_J W^{LJK}) S_{\gamma L} \\
&\quad + \frac{1}{60} (\gamma^d)_{\gamma\delta}(\nabla_I \nabla^\delta \nabla_K \nabla_L W^{LJKL}) K_d \right) , \tag{2.12c}
\end{align}

where $W^{IJKL} = W^{[IJKL]}$ is the super Cotton tensor, which is a completely antisymmetric primary superfield of dimension 1,

\begin{equation}
S^P_{\alpha} W^{IJKL} = 0 , \quad D W^{IJKL} = W^{IJKL} . \tag{2.13}
\end{equation}

It satisfies the Bianchi identity

\begin{equation}
\nabla^I W^{JKLP} = \nabla^I W^{JKLP} - \frac{4}{3} \nabla_\alpha W^{QJKL} \delta^P_{\alpha} . \tag{2.14}
\end{equation}

The $\mathcal{N} = 6$ case is special because it has the important property that the Hodge dual of the Cotton tensor

\begin{equation}
W^{IJ} := \frac{1}{4!} \varepsilon^{IJLPQ} W_{KLPQ} \tag{2.15}
\end{equation}

satisfies the Bianchi identity for the field strength of an Abelian $\mathcal{N} = 6$ vector multiplet\footnotemark[3]

\begin{equation}
\nabla^I W^{[JK]} = \nabla^I W^{[JK]} - \frac{2}{5} \delta^I_{[J} \nabla_\alpha W^{L]K} . \tag{2.16}
\end{equation}

Therefore, associated with the $\mathcal{N} = 6$ Weyl multiplet is a uniquely defined Abelian $\mathcal{N} = 6$ vector multiplet.

\footnotetext[3]{The $\mathcal{N}$-extended vector multiplet in conformal superspace is described in [2].}
As a result we can use $W^{IJ}$ to define a closed two-form $F = \frac{1}{2} E^B \wedge E^A F_{AB}$, $dF = 0$, with components

$$F^{IJ}_{\alpha \beta} = -2i \varepsilon_{\alpha \beta} W^{IJ},$$

$$F^I_{a \alpha} = \frac{1}{5} (\gamma_a)_{\alpha \beta} \nabla_{\beta J} W^{IJ},$$

$$F_{a b} = -\frac{i}{120} \varepsilon_{a b c} (\gamma^c)_{\alpha \beta} [\nabla^K_{\alpha}, \nabla^L_{\beta}] W_{KL}.$$ (2.17a)

(2.17b)

(2.17c)

It is the field strength of the vector multiplet. We associate with the field strength $F$ a gauge one-form $A$,

$$F = dA.$$ (2.18)

It is the existence of this gauge one-form which distinguishes the $\mathcal{N} = 6$ case from the $\mathcal{N} < 6$ cases.

3 Conformal supergravity action

In this section, we start by recalling the method to construct the $\mathcal{N} < 6$ conformal supergravity actions employed in [1]. After that, we present a generalization of the method that is suitable in the $\mathcal{N} = 6$ case.

3.1 Construction of $\mathcal{N} < 6$ conformal supergravity actions

The idea of the method employed in [1] is to look for two solutions, $\Sigma_{\text{CS}}$ and $\Sigma_{R}$, to the superform equation

$$d\Sigma = \langle R^2 \rangle,$$ (3.1)

where

$$\langle R^2 \rangle := \bar{R}^b \wedge R^a \Gamma_{ab}, \quad d\langle R^2 \rangle = 0$$ (3.2)

and $\Gamma_{ab}$ denotes a properly normalized Cartan-Killing metric of the $\mathcal{N}$-extended superconformal algebra; the explicit form of $\langle R^2 \rangle$ is

$$\langle R^2 \rangle = -R(N)^{IJ} \wedge R(N)_{IJ}.$$ (3.3)

Following the notation of [1], we define $\bar{R}^a := (R(P)^A, R^a)$ where $R(P)^A := T^A - \hat{T}^A$ and $\hat{T}^A$ is the flat superspace torsion.
One solution is always the Chern-Simons form $\Sigma_{CS}$ which exists for any $N$. Its explicit form is found to be

$$\Sigma_{CS} = -\hat{R}^a \wedge \Omega_a - \frac{1}{6} \Omega^c \wedge \Omega^b \wedge \Omega^a \varepsilon_{abc} - 4i E^a \wedge \bar{F}^{\alpha I} \wedge \bar{F}^{\beta}_I \bar{\gamma}_a \alpha \beta - \hat{R}^{IJ} \wedge \Phi_{IJ}$$

$$+ \frac{1}{3} \Phi^{IJ} \wedge \Phi^K_I \wedge \Phi_{KJ} + 2E^a \wedge \bar{F}_a \wedge B - 2E_I^a \wedge \bar{F}_a \wedge B + \text{exact form} \ , \quad (3.4)$$

where

$$\hat{R}^{ab} := d\Omega^{ab} + \Omega^c \wedge \Omega^a \wedge \Omega^b$$

$$\hat{R}^{IJ} := d\Phi^{IJ} + \Phi^K_I \wedge \Phi^K_J$$

(3.5)

correspond to the Riemann and SO($N$) curvature tensors. The other solution is the so-called curvature induced form $\Sigma_R$ such that its components are constructed in terms of the super Cotton tensor and its covariant derivatives. It turns out that $\Sigma_R$ exists for $N < 6$ (in the cases $N = 1$ and $N = 2$, $\Sigma_R$ vanishes), see [1] for more details.

The difference

$$\mathcal{J} = \Sigma_{CS} - \Sigma_R$$

(3.6)

is a closed three-form which may be used to construct a locally supersymmetric action

$$S = \int_{\mathcal{M}^3} \mathcal{J} = \int d^3x \, e^{*} \mathcal{J} |_{\theta = 0} \ , \quad * \mathcal{J} = \frac{1}{3!} \varepsilon^{mnp} \mathcal{J}_{mnp} . \quad (3.7)$$

This action principle is not applicable for $N \geq 6$ since $\Sigma_R$ does not exist. However, in the $N = 6$ case there is a way out and that is to make use of the closed two-form (2.17) constructed in terms of the vector multiplet field strength $W^{IJ}$, eq. (2.15).

### 3.2 Modified $N = 6$ curvature induced form

In the $N = 6$ case, we can modify the superform equation (3.3) to

$$d\Sigma = -R(N)^{IJ} \wedge R(N)_{IJ} - A F \wedge F \ , \quad (3.8)$$

where $A$ is some constant we will determine. Here $F$ is the closed two-form (2.17). The Chern-Simons solution is now modified to

$$\Sigma_{CS} = -\hat{R}^a \wedge \Omega_a - \frac{1}{6} \Omega^c \wedge \Omega^b \wedge \Omega^a \varepsilon_{abc} - 4i E^a \wedge \bar{F}^{\alpha I} \wedge \bar{F}^{\beta}_I \bar{\gamma}_a \alpha \beta - \hat{R}^{IJ} \wedge \Phi_{IJ}$$

$$+ \frac{1}{3} \Phi^{IJ} \wedge \Phi^K_I \wedge \Phi_{KJ} + 2E^a \wedge \bar{F}_a \wedge B - 2E_I^a \wedge \bar{F}_a \wedge B - A F \wedge A$$

$$+ \text{exact form} . \quad (3.9)$$
We may now attempt to find a covariant solution to eq. (3.8) which will also be called the curvature induced form and denoted $\Sigma_R$. We make the ansatz for the lowest components

$$\Sigma^{IJ\gamma}_{\alpha\beta} = 0, \quad \Sigma^{JK}_{\alpha\gamma} = i(\gamma_\alpha)_{\beta\gamma}(A\delta^{JK}W^{PQ}W_{PQ} + BW^{JP}W^K_P),$$

(3.10)

with $A$ and $B$ some constants to be determined, and turn to analyzing the superform equation (3.8) by increasing dimension.

At the lowest dimension we find that we must set $A = -2$ and

$$\Sigma^{JK}_{\alpha\beta} = 8i(\gamma_\alpha)_{\beta\gamma}(W^{JP}W^K_P - \frac{1}{4}\delta^{JK}W^{PQ}W_{PQ}).$$

(3.11a)

The higher dimension components are found to be

$$\Sigma_{ab\gamma} = -2\varepsilon_{abc}(\gamma^c)_{\gamma\gamma}\left(\nabla^a_{[K}W^{PQ]}W_{PQ} - \frac{2}{5}\nabla^a_{P}W_{QP}W^{QK}\right),$$

(3.11b)

$$\Sigma_{abc} = i\varepsilon_{abc}\left(\frac{1}{5}\nabla^a_{[I}\nabla_{\gamma}W^{JK]}W_{IJ} + \frac{1}{3}\nabla^a[g_{IJK}]\nabla_{\gamma}I W_{JK} - \frac{2}{25}\nabla^a_{I}W^{K}I\nabla_{J}W_{KJ}
- \frac{i}{3}\varepsilon^{IKLPQ}W_{IJ}W_{KL}W_{PQ}\right),$$

(3.11c)

To derive these results, we have made use of the following consequences of the Bianchi identity (2.16):

$$\nabla^a_{[I}\nabla_{\gamma}W^{JK]}W_{IJ} = \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^a_{[I}\nabla_{\gamma}W^{JK]} - 5i\nabla^a_{\alpha\beta}W_{IJ} + \frac{1}{6}\delta^{IJ}\nabla_{(\alpha\beta}W_{L)K}^{KL},$$

(3.12a)

$$\nabla^a_{P}\nabla_{\gamma}W^{IJ} = \frac{2}{5}\nabla^a[g_{IJK}]\nabla_{\gamma}W_{JK} - 4iW^{IJKL}W_{KL},$$

(3.12b)

$$\nabla^a_{[I}\nabla_{\beta}W^{KL]} = \nabla_{(I}W^{\alpha\beta}W_{KL}^{\alpha}) + \frac{3}{10}\varepsilon_{\alpha\beta}\delta^{[I}\nabla_{\gamma}W_{\gamma]P}W^{LP} + 3i\delta^{[I}\nabla_{\alpha\beta}W^{KL]} + 6i\varepsilon_{\alpha\beta}W^{P[JKW}_{L]P}.$$ 

(3.12c)

Now, the closed three-form $3 = \Sigma_{CS} - \Sigma_R$ generates a locally supersymmetric action according to the rule (3.7).

### 3.3 The component action

The complete component analysis of the $N$-extended Weyl multiplet was given in [1]. Here we specialize to the $N = 6$ case where the auxiliary fields coming from the

5When referring to components of the curvature induced form we denote $\Sigma_R$ by $\Sigma$ to avoid awkward notation.
super Cotton tensor are defined as:

\[ w_{IJKL} := W_{IJKL} \equiv \frac{1}{2} \varepsilon_{IJKLPQ} w^{PQ}, \quad (3.13a) \]

\[ w_{\alpha}^{IJK} := -\frac{i}{6} \nabla_{\alpha L} W_{IJKL} \equiv \frac{1}{3!} \varepsilon^{IJKLPQ} \tilde{w}_{\alpha LPQ}, \quad (3.13b) \]

\[ y^{IJKL} := i \nabla_{\gamma} \nabla_{\gamma p} W_{IJKLP} \equiv \frac{1}{2} \varepsilon^{IJKLPQ} y_{PQ}, \quad (3.13c) \]

\[ X_{\alpha}^{I_1 \ldots I_5} := i \nabla_{\gamma} W_{I_1 I_2 \ldots I_5} \equiv \varepsilon^{I_1 \ldots I_5 J} X_{\alpha J}. \quad (3.13d) \]

These definitions agree with [10, 11]. There is also an additional component field

\[ X_{\alpha}^{I_1 \ldots I_6} := i \nabla_{(\alpha} \nabla_{\beta)} W_{I_1 I_2 \ldots I_6} = -\varepsilon^{I_1 \ldots I_6} F_{\alpha \beta} \].

However, this field turns out to be a composite object as it is the component U(1) field strength \( F_{ab} \) up to contributions involving the gravitino:

\[ F_{ab} = F_{ab} + i \psi_{[ab} (\gamma_{b]} \beta^\gamma) X^{\gamma J} - \frac{1}{2} i \psi_{[ab} \psi_{b]} \alpha \gamma w^{IJ}, \quad \mathcal{F}_{ab} := 2 e_a^m e_b^n \partial_{[m} A_{n]}. \quad (3.14) \]

As the action is invariant with respect to the gauge transformations (2.8) up to a total derivative, it follows that the dependence on \( b_m \) must drop out. Equivalently, we can simply adopt the \( K \)-gauge \( b_m = 0 \). Using the action (3.7) and the Chern-Simons form (3.9), we find the Chern-Simons contribution to be

\[ S_{CS} = \frac{1}{4} \int d^3 x \varepsilon^{abc} \left( \omega_a f^g R_{bcfg} - \frac{1}{2} \omega_a^g \omega_b^h \omega_c^f - \frac{i}{2} \tilde{\psi}_{bc} (\gamma_d) \alpha^\beta (\gamma_a) \beta^\epsilon \varepsilon^{def} \tilde{\psi}_{ef} \gamma \right. \]

\[ - 2 R_{ab}^{IJ} V_{cIJ} - \frac{4}{3} V_a^{IJK} V_{bIK} + 4 \mathcal{F}_{ab} A_c) , \quad (3.15) \]

where the component curvatures \( R_{ab}^{cd} \) and \( R_{ab}^{IJ} \) are defined as

\[ R_{ab}^{cd} := 2 e_a^m e_b^n \partial_{[m} \omega_{n]}^{cd} - 2 \omega_a^c \omega_b^d f^{ef}, \quad (3.16a) \]

\[ R_{ab}^{IJ} := 2 e_a^m e_b^n \partial_{[m} \omega_{n]}^{IJ} - 2 V_a^{IJK} V_{bJK}. \quad (3.16b) \]

Using the formula

\[ \frac{1}{3!} \varepsilon^{mnp} \Sigma_{mnp} = \frac{1}{3!} \varepsilon^{mnp} E_p^C E_n^B E_m^A \Sigma_{ABC} \]

\[ = \frac{1}{3!} \varepsilon^{abc} (\Sigma_{abc} + \frac{3}{2} \tilde{\psi}_{aI} \psi_{bI} \alpha \Sigma^{IJ} + \frac{3}{4} \tilde{\psi}_{bJ} \psi_{aI} \Sigma^{IJ}) \]

\[ + \frac{1}{8} \tilde{\psi}_{cK} \psi_{bI} \psi_{aI} \Sigma^{IJ} \]

(3.17)

for the component projection of a three-form along with the explicit expressions for the components of \( \Sigma_{ABC} \), we find

\[ \frac{1}{3!} \varepsilon^{mnp} \Sigma_{mnp} = y^{IJ} w_{IJ} + \frac{4i}{3} \tilde{\omega}^{\alpha IJK} \tilde{w}_{\alpha IJK} - 2 i X^K X^K + \frac{2}{3} \varepsilon^{IJKLPQ} w_{IJL} w_{KPQ} \]

\[ + 2 i \tilde{\psi}_{aI} (\gamma_{a})^\alpha \beta (\tilde{w}_{\beta}^{IJK} w_{JK} + X_{\beta J} w^{IJ}) \]

\[ - i \varepsilon^{abc} (\gamma_a)^\alpha \beta \psi_{aI} \alpha \beta^I (w_{IK} w_{KL} - \frac{1}{4} \delta^{IJ} w_{KL} w_{KL}), \quad (3.18) \]
where we have used the relations

\[ \tilde{w}_\alpha^{IJK} = -\frac{i}{2} \nabla_a^{[I} W^{JK]} |, \quad (3.19a) \]

\[ y^{IJ} = -\frac{i}{5} \nabla^\gamma [\nabla_\gamma W^{J} | P - \frac{1}{2} \varepsilon^{IJKLPQ} W_{KLW_{PQ}} |, \quad (3.19b) \]

\[ X^I_a = -\frac{i}{5} \nabla_a W^{IJ} |. \quad (3.19c) \]

Combining this result with the Chern-Simons contribution gives the full action

\[ S = \frac{1}{4} \int d^3 x \varepsilon \left\{ \varepsilon^{abc} (\omega_a^{fg} R_{bcfg} - \frac{2}{3} \omega_a^{fg} \omega_{bg} \omega_{ch} f - \frac{i}{2} \Psi_{bcf}^{\alpha_d} (\gamma_d)^{\alpha} (\gamma_a)^{\beta} \gamma_d \varepsilon^{def} \Psi^{ef}_I \right. \]

\[ - 2 R^{IJ}_{ab} V_{cIJ} - \frac{4}{3} V^{IJ}_{ab} K V_{cKJ} + 4 F_{ab} A_c \}

\[ - 4 y^{IJ} w_{IJ} - \frac{16i}{3} \tilde{w}_a^{IJK} \tilde{w}_\alpha^{IJK} + 8i X^K \gamma^K - \frac{8}{3} \varepsilon^{IJKLPQ} w_{IJ} w_{KLw_{PQ}} \]

\[ - 8i \psi_a^{\alpha} (\gamma_d)^{\alpha} (\tilde{w}_d^{IJK} w_{JK} + X_{\betaJ} w^{IJ}) \]

\[ + 4 i \varepsilon^{abc} (\gamma_d)^{\alpha} \psi_b^{\alpha} \psi_d^{\beta} (w^{IK} w^K_{J} - \frac{1}{4} \delta^{IJ} w^{KLW_{KL}}) \right\}. \quad (3.20) \]

Our choice of normalization for the auxiliary fields allows a simple truncation to lower values of \( \mathcal{N} \). From the above action one can truncate the auxiliary fields to \( \mathcal{N} = 5 \) by taking (with \( I, J, K = 1, 2, 3, 4, 5 \))

\[ w_{IJ} \rightarrow 0, \quad \tilde{w}_\alpha^{IJK} \rightarrow 0, \quad X^I_a \rightarrow 0, \quad y^{IJ} \rightarrow 0, \quad \left\{ \right. \]

\[ w^I \rightarrow w^I, \quad \tilde{w}_\alpha^{I6} \rightarrow \tilde{w}_\alpha^{I6}, \quad y^I \rightarrow y^I. \quad (3.21) \]

For the gauge fields one must switch off the U(1) gauge field \( A_b \rightarrow 0 \), while truncation is obvious for the other gauge fields. The \( \mathcal{N} < 5 \) cases can be obtained via the truncation procedure given in [1].

4 Discussion

Our component action for \( \mathcal{N} = 6 \) conformal supergravity agrees with that derived recently in [9], where alternative techniques were used involving the consistent truncation of the off-shell multiplet of \( \mathcal{N} = 8 \) conformal supergravity [12]. It also correctly reduces to the action for \( \mathcal{N} = 5 \) conformal supergravity [1] via the truncation procedure (3.21). Eliminating the auxiliary fields is equivalent to removing the last three lines in (3.20). The resulting on-shell action for \( \mathcal{N} = 6 \) conformal supergravity does not agree with that obtained in [13, 14] by gauging the \( \mathcal{N} = 6 \) superconformal
algebra in \(x\)-space (the action given in \[13,14\] does not contain the U(1) Chern-Simons term). Instead it coincides with the action given in \[15\].

In conclusion, we comment on the structure of a supercurrent multiplet associated with a superconformal matter theory coupled to \(\mathcal{N}\)-extended conformal supergravity (see also \[2\]). In general, the supergravity-matter system is described by an action of the form

\[
S = \frac{1}{\kappa} S_{\text{CSG}} + S_{\text{matter}}, \quad (4.1)
\]

where \(S_{\text{CSG}}\) denotes the conformal supergravity action and \(S_{\text{matter}}\) the matter action. The conformal supergravity equation is

\[
\frac{1}{\kappa} W + T = 0. \quad (4.2)
\]

Here \(W\) is the \(\mathcal{N}\)-extended super Cotton tensor (with its indices suppressed) and \(T\) the matter supercurrent multiplet. The supercurrent has the same algebraic type as \(W\) and obeys the same differential constraints \(W\) is subject to. For any \(\mathcal{N}\), the super Cotton tensor (and also the supercurrent) is a conformal primary superfield,

\[
S^I_{\alpha} W = 0, \quad \Box W = \Delta W W, \quad (4.3)
\]

with \(\Delta W\) the dimension of \(W\). We now recall the structure of \(W\) for various values of \(\mathcal{N}\) following \[2\].

The \(\mathcal{N} = 1\) super Cotton tensor \[16\] is a completely symmetric spinor \(W_{\alpha\beta\gamma}\) of dimension 5/2. It obeys the conformally invariant constraint

\[
\nabla^\alpha W_{\alpha\beta\gamma} = 0. \quad (4.4)
\]

In the \(\mathcal{N} = 2\) case, the super Cotton tensor \[17,18\] is a completely symmetric spinor \(W_{\alpha\beta}\) of dimension 2. The corresponding conformally invariant constraint is

\[
\nabla^\alpha_I W_{\alpha\beta} = 0. \quad (4.5)
\]

In the \(\mathcal{N} = 3\) case, the super Cotton tensor is a symmetric spinor \(W_{\alpha}\) of dimension 3/2. It obeys the conformally invariant constraint

\[
\nabla^\alpha_I W_{\alpha} = 0. \quad (4.6)
\]

For \(\mathcal{N} > 5\), the super Cotton tensor \[5,6\] is a completely antisymmetric tensor \(W^{IJKL}\) of dimension 1. It obeys the conformally invariant constraint

\[
\nabla^I W^{JKLP} = \nabla^I_{\alpha} W^{JKLP} - \frac{4}{\mathcal{N} - 3} \nabla_{\alpha} Q W^{[JKL} \delta P] W^{I}. \quad (4.7)
\]
In the $\mathcal{N} = 4$ case, the super Cotton tensor is equivalently described by a scalar primary dimension-1 superfield $W^{IJKL} := \varepsilon^{IJKL}W$. The corresponding conformally invariant constraint is

$$\nabla^\alpha I \nabla^\alpha J W = \frac{1}{4} \delta^{IJ} \nabla^\alpha P \nabla^\alpha _P W.$$  \hfill (4.8)

As we have demonstrated, the specific feature of the $\mathcal{N} = 6$ case is that the super Cotton tensor is equivalent to the U(1) vector multiplet field strength (2.15). Therefore, the $\mathcal{N} = 6$ supercurrent $T^{IJ}$ has the same multiplet structure. This agrees with the Nishimura-Tanii analysis [9] of the supercurrent of the ABJM model [19] coupled to conformal supergravity.

**Acknowledgements:**

We are grateful to Daniel Butter for collaboration at the early stage of this project. The work of SMK and JN was supported in part by the Australian Research Council, project No. DP1096372. The work of GT-M and JN was supported in part by the Australian Research Council’s Discovery Early Career Award (DECRA), project No. DE120101498.

**A The supersymmetry transformations**

In this appendix we present the complete $Q$- and $S$-supersymmetry transformations for the component fields of the Weyl multiplet for $\mathcal{N} = 6$. The component action (3.20) is manifestly supersymmetric by virtue of our superspace construction. We refer the reader to [1] for details on the component projection rules and the precise definition of the gauge component fields.

The $Q$-supersymmetry transformations of the connections are

$$\delta_Q(\xi)e_m^a = i\psi_m^{\alpha I}(\gamma^a)^{\alpha}_{\beta} \xi_{I\beta},$$  \hfill (A.1a)

$$\delta_Q(\xi)\psi_m^{\alpha I} = 2D_m^{\alpha I} = 2\partial_m^{\alpha I} + \omega_m^{\alpha} \beta^{\alpha I} + b_m^{\alpha I} - 2V_m^{I} j^{\alpha J},$$  \hfill (A.1b)

$$\delta_Q(\xi)b_m = -\phi_m^{\alpha J} \xi_{\alpha J},$$  \hfill (A.1c)

$$\delta_Q(\xi)V_m^{IJ} = 2\phi_m^{\alpha I} [\xi_{\alpha}]^{J} - \frac{i}{2} \varepsilon^{IJKLPQ} \psi_{mK}^{\alpha} \xi_{\alpha L} w_{PQ} - \frac{i}{3} \varepsilon^{IJKLPQ} \xi_{J}^{\alpha} \bar{w}_{\beta LPQ},$$  \hfill (A.1d)

$$\delta_Q(\xi)\omega_m^{ab} = -\varepsilon^{abc} \phi_m^{\alpha I} (\gamma_c)^{\alpha}_{\beta} \xi_{I\beta},$$  \hfill (A.1e)

$$\delta_Q(\xi)\phi_m^{\alpha I} = -2\xi^{\beta J} (\gamma_\beta)^{\alpha}_{\gamma} j_{m}^{\beta} + \frac{1}{3} \varepsilon^{IJKLPQ} \psi_{mJ}^{\alpha} \xi_{\beta} \bar{w}_{\alpha LPQ} + i\xi^{\beta J} (\gamma_\beta)^{\alpha}_{\gamma} R(N)^{\alpha J},$$  \hfill (A.1f)
\[ \delta_Q(\xi)\bar{f}_m^a = \frac{i}{2} \bar{\psi}_m^a \xi_{\alpha J} R(N)^{a J} + \xi^\alpha (\gamma_m)_{\alpha \beta} R(S)^{a \beta} , \] (A.1g)
\[ \delta_Q(\xi)A_m = -i \xi^\alpha (\gamma_m)_{\alpha \beta} X^K_{\beta} - i \psi_{m J} \xi_{\beta K} w^{JK} . \] (A.1h)

Here we have made use of the covariant derivative
\[ \mathcal{D}_a = e_a^m \mathcal{D}_m = e_a^m (\partial_m - \frac{1}{2} \omega_m^{bc} M_{bc} - \frac{1}{2} V_{mIJ} N_{IJ} - b_m \mathcal{D}) \] (A.2)
and the following results for the projection of the SO(6) and S-supersymmetry curvatures
\[ R(N)_{ab}^{IJ} = R_{ab}^{IJ} + 2 \psi_{[a}^{\alpha[I} \phi_{b]}^{J]} + \frac{i}{3} \epsilon^{IJKLPQ} \psi_{[a K}^{\beta} (\gamma_{b]}^{\gamma})^\beta \bar{\psi}_{\gamma \lambda \rho} \] (A.3a)
\[ R(S)^{\alpha I} = e_a^m e_b^n \mathcal{D}_m \phi_{n I} + i \psi_{[a I}^{\alpha} \psi_{b]}^{J]} (\gamma_{c]}^{\gamma})^\gamma \alpha R(N)^{c \ J] \] (A.3b)

The curvature tensors \( R(N)^{eIJ} \) and \( R(S)^{eI} \) in eqs. (A.11), (A.1g) and (A.3b) are the Hodge-duals of \( R(N)_{ab}^{IJ} \) and \( R(S)_{ab}^{I} \), respectively. The S-supersymmetry transformations of the connections are
\[ \delta_S(\eta)e_m^a = 0 , \] (A.4a)
\[ \delta_S(\eta)\psi_m^{\alpha I} = -2i \eta^{\beta I} (\gamma_m)_J^\beta \] (A.4b)
\[ \delta_S(\eta)\psi^{I} = 0 , \] (A.4c)
\[ \delta_S(\eta)V_m^{IJ} = 2 \psi_{m [a I}^{\alpha} \psi_{b]}^{J]} , \] (A.4d)
\[ \delta_S(\eta)\omega_m^{ab} = - \epsilon^{abc} \psi_m^{a I} (\gamma_c)_J^\beta \psi_{b]}^{J] , \] (A.4e)
\[ \delta_S(\eta)\phi_m^{\alpha I} = 2 \mathcal{D}_m \eta^{\alpha I} = 2 \omega_m^{a I} + \omega_m^{a \beta} \eta_{b]}^{\beta I} - b_m \eta^{\alpha I} - 2 V_{mJ}^{I} \eta^{a J} , \] (A.4f)
\[ \delta_S(\eta)\bar{f}_m^a = i \phi_m^{a I} (\gamma_c)_J^\beta \psi_{b]}^{J] , \] (A.4g)
\[ \delta_S(\eta)A_m = 0 . \] (A.4h)

It should be pointed out that all the auxiliary fields defined in (3.13), as well as the Cottino \( w_{\alpha \beta \gamma}^{I} \) and Cotton \( w_{\alpha \beta \gamma}^{\delta} \) tensors \( \Pi \), are ordinary primary fields annihilated by the special conformal generator \( K_a \).

Along with the transformation laws of the gauge fields of the Weyl multiplet it is necessary to have the transformation rules of the auxiliary fields given by eqs. (3.13) and (3.19). Their \( Q- \) and \( S- \)supersymmetry transformations are
\[ \delta_Q(\xi)w_{IJ}^{a} = 2i \epsilon^{a}_{\alpha} \bar{w}_{K}^{I} X_{\alpha J} , \] (A.5a)
\[ \delta_Q(\xi)w_{I}^{a} = 0 , \] (A.5b)
\[ \delta_Q(\xi)X_{I}^{a} = 0 . \] (A.5c)

---

\( ^6 \)Given a two-form \( F_{ab} \), its Hodge-dual is \( F^c = \frac{1}{2} \epsilon^{cab} F_{ab} \).
\[ \delta_S(\eta) w^{IJ} = 0 , \]  
\[ \delta_Q \bar{\psi}_a IJK = \frac{1}{4} (\gamma^a)_{\alpha\beta} \xi_{\alpha\beta}^p \varepsilon^{PJKST} R(N)_{aST} + \frac{3}{2} (\gamma^a)_{\alpha\beta} \xi_{\alpha\beta}^p [I \tilde{\nabla}_a w^{JK}] - \frac{3}{4} \xi^{|I} y^{JK} , \]  
(A.5b)  
\[ \delta_S \bar{w}_a IJK = 3i \eta_a [I w^{JK}] , \]  
(A.5c)  
\[ \delta_Q X^I_{\alpha} = \xi^I F_{\alpha} | - \frac{1}{2} \xi_K \nabla_{\gamma \beta} w^{KIJ} + \frac{1}{4} \varepsilon^{IJKLPQ} \xi_{aJ} w_{KL} w_{PQ} , \]  
(A.5d)  
\[ \delta_S X^I_{\alpha} = 2i \eta_K [I w^{JK}] , \]  
(A.5e)  
\[ \delta_Q y^{IJ} = -4i \xi^I [I \tilde{\nabla}_{\gamma \beta} X^{\gamma \beta J}] + 4i \xi_K \tilde{\nabla}_{\gamma \beta} \bar{w}^{KIJ} + \frac{4i}{3} \xi_K \bar{w}^M_{\gamma \beta} \varepsilon_{MNP} \varepsilon_{MNP} K^{I|Q} w^{J|Q} - \frac{4i}{3} \bar{w}^M_{\gamma \beta} w_{QS} \varepsilon_{MNPQS} [I \xi_{\gamma J}] , \]  
(A.5f)  
\[ \delta_S y^{IJ} = -4 \eta \delta^{[I} X^{J]} - 4 \eta_L \bar{w}_{\delta} L^{IJ} , \]  
(A.5g)  

where we have defined

\[ \tilde{\nabla}_a w^{IJ} := D_a w^{IJ} - i \psi_{a \gamma} \bar{w}^K_{\gamma IJ} - i \psi^{[I} w_{\gamma J]} , \]  
(A.6a)  
\[ \tilde{\nabla}_a X^{\beta J} := D_a X^{\beta J} - \frac{1}{2} \psi_{a \gamma} F^{\beta \gamma} | + \frac{1}{4} \psi_{a L} y^{LJ} + \frac{1}{2} (\gamma^b)_{\beta \gamma} \psi_{a L} \tilde{\nabla}_b w^{LJ} + \frac{1}{8} (\gamma^a)_{\alpha\beta} \varepsilon^{JMKLPQ} \psi_{a K} \bar{w}_L w_{PQ} + i (\gamma^a)_{\alpha\beta} \phi_{a L} w^{JL} , \]  
(A.6b)  
\[ \tilde{\nabla}_a \bar{w}^{IJK} := D_a \bar{w}^{IJK} - \frac{1}{8} (\gamma^b)_{\beta \gamma} \varepsilon^{JMKLPQ} \psi_{a \gamma L} R(N)_{bPQ} | + \frac{3}{8} \psi_{a \beta} [I y^{JK}] + \frac{3}{4} (\gamma^b)_{\beta \delta} \psi_{a L} \bar{w}^{IJ} \delta_{L} - \frac{3i}{2} \phi_{a \beta} [I w^{JK}] . \]  
(A.6c)  

References

[1] D. Butter, S. M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, “Conformal supergravity in three dimensions: Off-shell actions,” JHEP 1310, 073 (2013) [arXiv:1306.1205 [hep-th]].

[2] D. Butter, S. M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, “Conformal supergravity in three dimensions: New off-shell formulation,” JHEP 1309, 072 (2013) [arXiv:1305.3132 [hep-th]].

[3] D. Butter, “N=1 Conformal superspace in four dimensions,” Annals Phys. 325, 1026 (2010) [arXiv:0906.4399 [hep-th]].

[4] D. Butter, “N=2 Conformal superspace in four dimensions,” JHEP 1110, 030 (2011) [arXiv:1103.5914 [hep-th]].

[5] P. S. Howe, J. M. Izquierdo, G. Papadopoulos and P. K. Townsend, “New supergravities with central charges and Killing spinors in 2+1 dimensions,” Nucl. Phys. B 467, 183 (1996) [arXiv:hep-th/9505032].

[6] S. M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, “Off-shell supergravity-matter couplings in three dimensions,” JHEP 1103, 120 (2011) [arXiv:1101.4013 [hep-th]].
[7] P. van Nieuwenhuizen, “D = 3 conformal supergravity and Chern-Simons terms,” Phys. Rev. D 32, 872 (1985).

[8] M. Roček and P. van Nieuwenhuizen, “N ≥ 2 supersymmetric Chern-Simons terms as d = 3 extended conformal supergravity,” Class. Quant. Grav. 3, 43 (1986).

[9] M. Nishimura and Y. Tanii, “N=6 conformal supergravity in three dimensions,” arXiv:1308.3960 [hep-th].

[10] J. Greitz and P. S. Howe, “Maximal supergravity in three dimensions: supergeometry and differential forms,” JHEP 1107, 071 (2011) [arXiv:1103.2730 [hep-th]].

[11] U. Gran, J. Greitz, P. Howe and B. E. W. Nilsson, “Topologically gauged superconformal Chern-Simons matter theories,” JHEP 1212, 046 (2012) [arXiv:1204.2521 [hep-th]].

[12] M. Nishimura and Y. Tanii, “Coupling of the BLG theory to a conformal supergravity background,” JHEP 1301, 120 (2013) [arXiv:1206.5388 [hep-th]].

[13] U. Lindström and M. Roček, “Superconformal gravity in three dimensions as a gauge theory,” Phys. Rev. Lett. 62, 2905 (1989).

[14] H. Nishino and S. J. Gates Jr., “Chern-Simons theories with supersymmetries in three dimensions,” Int. J. Mod. Phys. A 8, 3371 (1993).

[15] X. Chu and B. E. W. Nilsson, “Three-dimensional topologically gauged N=6 ABJM type theories,” JHEP 1006, 057 (2010) [arXiv:0906.1655 [hep-th]].

[16] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, “Conformal supergravities as Chern-Simons theories revisited,” JHEP 1303, 113 (2013) [arXiv:1212.6852 [hep-th]].

[17] B. M. Zupnik and D. G. Pak, “Superfield formulation of the simplest three-dimensional gauge theories and conformal supergravities,” Theor. Math. Phys. 77 (1988) 1070 [Teor. Mat. Fiz. 77 (1988) 97].

[18] S. M. Kuzenko, “Prepotentials for N=2 conformal supergravity in three dimensions,” JHEP 1212, 021 (2012) [arXiv:1209.3894 [hep-th]].

[19] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP 0810, 091 (2008) [arXiv:0806.1218 [hep-th]].