Nonsymmetric extension of the Green-Osher inequality *

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Abstract In this paper we obtain the extended Green-Osher inequality when two smooth, planar strictly convex bodies are at a dilation position and show the necessary and sufficient condition for the case of equality.

Mathematics Subject Classification 2010: 52A40, 52A10
Key words: dilation position, Green-Osher’s inequality, nonsymmetric, relative Steiner polynomial

1 Introduction
We denote by \( \mathbb{R}^n \) the usual \( n \)-dimensional Euclidean space with the canonical inner product \( \langle \cdot , \cdot \rangle \). A compact convex set \( K \) in \( \mathbb{R}^n \) is called a convex body if it contains the origin and has nonempty interior. When \( n = 2 \), it is called a planar convex body. The volume of a set \( S \subseteq \mathbb{R}^n \) is denoted by \( V(S) \). The Minkowski sum of convex bodies \( K \) and \( L \), and the Minkowski scalar product of \( K \) for positive real number \( t \) are, respectively, defined by
\[
K + L = \{ x + y \mid x \in K, y \in L \}
\]
and
\[
tK = \{ tx \mid x \in K \}.
\]
For two planar convex bodies \( K \) and \( L \), the volume of the Minkowski sum \( K + tL \) gives the relative Steiner polynomial of \( K \) with respect to \( L \):
\[
V(K + tL) = V(K) + 2V(K, L)t + V(L)t^2,
\]
where \( V(K, L) \) is the mixed area of \( K \) and \( L \). Formula (1.1) is closely related to the classical isoperimetric inequality, the Brunn-Minkowski inequality and the log-Brunn-Minkowski inequality. Many proofs, sharpened forms and generalization of the isoperimetric inequality can be found in Chavel [2], Dergiades [3], Osserman [9] and Schneider [10].

Using remarkable symmetrization, Gage [4] successfully obtained an inequality for the total squared curvature for convex curves. Following his work, for a planar strictly convex body \( K \) and a symmetric,

*The author is supported by the Doctoral Scientific Research Foundation of Liaoning Province (No.20170520382) and the Fundamental Research Funds for the Central Universities (No.3132017046).
planar strictly convex body $E$, Green and Osher [8] (see also [12]) obtained a generalized formula:

$$
\frac{1}{V(E)} \int_0^{2\pi} F(\rho(\theta))h_E(\theta)(h_E(\theta) + h_E''(\theta))d\theta \geq F(-t_1) + F(-t_2),
$$

(1.2)

where $\rho(\theta)$ is the relative curvature radius of $K$ with respect to $E$, $F(x)$ is a strictly convex function on $(0, +\infty)$, $t_1$ and $t_2$ are the two roots of the relative Steiner polynomial of $K$ with respect to $E$. Inequality (1.2) plays a significant role in studying the curve shortening flow (see Gage [5, 6] and Gage-Hamilton [7]).

A natural question is whether the Green-Osher inequality holds without symmetric condition. Similar question is asked by the log-Brunn-Minkowski inequality (see Böröczky-Lutwak-Yang-Zhang [1], Xi-Leng [11] and Yang-Zhang [13]). Xi and Leng [11] gave the definition of dilation position for the first time to prove the log-Brunn-Minkowski inequality and solve the planar Dar’s conjecture.

Let $K$ and $L$ be two convex bodies. Convex bodies $K$ and $L$ are at a dilation position, if the origin $o \in K \cap L$ and

$$
r(K, L)L \subseteq K \subseteq R(K, L)L.
$$

(1.3)

Here $r(K, L)$ and $R(K, L)$ are the inradius and outradius of $K$ with respect to $L$, i.e.,

$$
r(K, L) = \max\{t > 0 \mid x + tL \subseteq K \text{ and } x \in \mathbb{R}^n\},
$$

$$
R(K, L) = \max\{t > 0 \mid x + tL \supseteq K \text{ and } x \in \mathbb{R}^n\}
$$

Noticing that there is a common center when $K$ and $L$ are at a dilation position, then the ratio of the support functions of $K$ and $L$ belongs to the range from $r(K, L)$ to $R(K, L)$, which leads to the Green-Osher inequality holds without symmetric condition. Properties of convex bodies are at a dilation position can be found in Lemma 5.1 (see also Xi-Leng [11]).

In this paper, inspired by the impressive work in [11], we obtain the main result.

**Theorem 1.1.** Let $K, L$ be two smooth, planar strictly convex bodies and $\rho(\theta)$ the relative curvature radius of $K$ with respect to $L$. If $K$ and $L$ are at a dilation position and $F(x)$ is a strictly convex function on $(0, +\infty)$, then

$$
\frac{1}{V(L)} \int_0^{2\pi} F(\rho(\theta))h_L(\theta)(h_L(\theta) + h_L''(\theta))d\theta \geq F(-t_1) + F(-t_2),
$$

(1.4)

where $t_1$ and $t_2$ are the two roots of the relative Steiner polynomial of $K$ with respect to $L$, and the equality in (1.4) holds if and only if $K$ and $L$ are homothetic.

This paper is organized as follows. In Section 2 we give some basic facts about planar convex bodies. In Section 3 we get the extended Green-Osher inequality when two smooth, planar strictly convex bodies are at a dilation position.

## 2 Preliminaries

Let $K$ be a planar convex body. A line $l$ is called a support line of $K$ if it passes through at least one boundary point of $K$ and if the entire planar convex body $K$ lies on one side of $l$. Let $l(\theta)$ be the support line of $K$ in the direction $u(\theta) = (\cos \theta, \sin \theta)$, where $\theta$ is the oriented angle from the positive $x$-axis to the perpendicular line of $l(\theta)$. The support function of $K$ is defined by

$$
h_K(\theta) = \sup_{x \in K} \langle x, u(\theta) \rangle, \quad u(\theta) \in S^1.
$$
It is easy to see that $h_K(\theta)$ is the signed distance of the support line $l(\theta)$ of $K$ with exterior normal vector $u(\theta)$ from the origin. Clearly, $h_K$, as a function of $\theta$, is single-valued and $2\pi$-periodic.

If $h_K(\theta)$ and $h_L(\theta)$ are continuously differentiable, then

$$V(K, L) = \frac{1}{2} \int_0^{2\pi} (h_K(\theta)h_L(\theta) - h'_K(\theta)h'_L(\theta))d\theta.$$ 

Furthermore, if $h_K(\theta)$ and $h_L(\theta)$ are smooth, then

$$V(K, L) = \frac{1}{2} \int_0^{2\pi} h_K(\theta)(h_L(\theta) + h''_L(\theta))d\theta = \frac{1}{2} \int_0^{2\pi} h_L(\theta)(h_K(\theta) + h''_K(\theta))d\theta.$$ 

From the Minkowski inequality, it follows that the expression $V(K + tL) = 0$ has two negative real roots. Denote by $t_1$ and $t_2$ ($t_1 \geq t_2$) the two roots of the relative Steiner polynomial of $K$ with respect to $L$, that is,

$$t_1 = -\frac{V(K, L)}{V(L)} + \frac{\delta}{V(L)}, \quad t_2 = -\frac{V(K, L)}{V(L)} - \frac{\delta}{V(L)}, \quad \delta = \sqrt{V(K, L)^2 - V(K)V(L)}.$$

In order to prove the extended Green-Osher inequality, we have the following definition that is similar to the Definition 3.3 of [8].

**Definition 2.1** ([8]). Let $K, L$ be two smooth, planar strictly convex bodies. Consider

$$\sup \left\{ \int_I \rho(\theta)h_L(\theta)(h_L(\theta) + h''_L(\theta))d\theta \mid I \subseteq S^1, \int_I h_L(\theta)(h_L(\theta) + h''_L(\theta))d\theta = V(L) \right\}.$$ 

Let $I_1$ denote the smallest subset of $S^1$ with measure $V(L)$ and realizing the above supremum, and let $I_2$ be its complement. Then, there exists an $a \in \mathbb{R}^+$ such that

$$I_1 \subseteq \{ \theta \mid \rho(\theta) \geq a \}, \quad I_2 \subseteq \{ \theta \mid \rho(\theta) \leq a \}.$$ 

Set

$$\rho_i = \frac{1}{V(L)} \int_{I_i} \rho(\theta)h_L(\theta)(h_L(\theta) + h''_L(\theta))d\theta, \quad i = 1, 2,$$

which yield that

$$\rho_1 + \rho_2 = \frac{2V(K, L)}{V(L)} \quad \text{and} \quad \rho_1 \geq \rho_2,$$

and there is a real number $b \geq 0$ such that

$$\rho_1 = \frac{V(K, L)}{V(L)} + b \quad \text{and} \quad \rho_2 = \frac{V(K, L)}{V(L)} - b.$$

### 3 Nonsymmetric extension of the Green-Osher inequality

In order to prove the main result, we first give four lemmas, in which Lemma 3.1 shows that convex bodies are at a dilation position by appropriate translations and the location of "dilation position" (detailed proof can be found in [11 Lemma 2.1]), Lemmas 3.2 and 3.3 are used to prove inequality (1.4), and Lemma 3.4 is used to deal with its equality case.
Lemma 3.1. Let $K, L$ be two convex bodies in $\mathbb{R}^n$.

(i) There is a translate of $L$, say $\bar{L}$, and a translate of $K$, say $\bar{K}$, so that $\bar{K}$ and $\bar{L}$ are at a dilation position.

(ii) If $K$ and $L$ are at a dilation position, then the origin $o \in \text{int}(K \cap L) \cup (\partial K \cap \partial L)$.

Lemma 3.2. Let $K, L$ be two smooth, planar strictly convex bodies. If $K$ and $L$ are at a dilation position, then the origin $o \in \text{int}(K \cap L)$ or $o$ is the point of tangency of $\partial K$ and $\partial L$ such that $K \subseteq L$ (or $L \subseteq K$).

Proof. By Lemma 3.1(ii), the origin $o \in \text{int}(K \cap L) \cup (\partial K \cap \partial L)$. If the origin $o \in \text{int}(K \cap L)$, we are done. If the origin $o \in \partial K \cap \partial L$, then $o$ must be the point of tangency of $\partial K$ and $\partial L$ such that $K \subseteq L$ (or $L \subseteq K$). Otherwise, $o$ is the point of intersection of $\partial K$ and $\partial L$, which contradicts to Lemma 3.3.

Lemma 3.3. Let $K, L$ be two smooth, planar strictly convex bodies. If $K$ and $L$ are at a dilation position, then

$$\rho_1 \geq -t_2.$$ (3.1)

Proof. From [1, Lemma 4.1] and the Minkowski inequality, it follows that

$$-t_1 \leq r(K, L) \leq R(K, L) \leq -t_2.$$

By Lemma 3.2, the origin $o \in \text{int}(K \cap L)$ or $o$ is the point of tangency of $\partial K$ and $\partial L$ such that $K \subseteq L$ (or $L \subseteq K$).

If the origin $o \in \text{int}(K \cap L)$, then $r(K, L) \leq \frac{h_K(\theta)}{\pi L(\theta)} \leq R(K, L)$, which implies

$$-\frac{\delta}{V(L)} h_L(\theta) \leq h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) \leq \frac{\delta}{V(L)} h_L(\theta), \quad \delta = \sqrt{V(K, L)^2 - V(K)V(L)} \geq 0.$$

On $I_1$, $\rho(\theta) - a \geq 0$, combining with the above inequality, it yields

$$-\left( h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) \right) (\rho(\theta) - a) \leq \frac{\delta}{V(L)} h_L(\theta)(\rho(\theta) - a).$$

By integrating this on the interval $I_1$,

$$-\frac{1}{V(L)} \int_{I_1} \left( h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) \right) (\rho(\theta) - a) (h_L(\theta) + h_L''(\theta)) d\theta \leq \frac{\delta}{V(L)} (\rho_1 - a).$$ (3.2)

Similarly, on $I_2$, we have

$$-\frac{1}{V(L)} \int_{I_2} \left( h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) \right) (\rho(\theta) - a) (h_L(\theta) + h_L''(\theta)) d\theta \leq -\frac{\delta}{V(L)} (\rho_2 - a).$$ (3.3)

It can be seen from (3.2) and (3.3) that

$$-\frac{1}{V(L)} \int_0^{2\pi} \left( h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) \right) (\rho(\theta) - a) (h_L(\theta) + h_L''(\theta)) d\theta \leq \frac{2b\delta}{V(L)}$$

and its left-hand side can be simplified to $\frac{2\delta^2}{V(L)}$, thus we have, $b \geq \frac{\delta}{V(L)} \geq 0$, that is, $\rho_1 \geq -t_2$.

If the origin $o$ is the point of tangency of $\partial K$ and $\partial L$ such that $L \subseteq K$ (the case of $K \subseteq L$ is similar), then $r(K, L) \leq \frac{h_K(\theta)}{h_L(\theta)} \leq R(K, L)$ for $\theta \in \bar{I}$ ($\bar{I}$ is a subset of $S^1$) and $h_K(\theta) = h_L(\theta) = 0$ for $\theta \in S^1 \backslash \bar{I}$. A similar discussion implies that $\rho_1 \geq -t_2$. \qed
Lemma 3.4. Let $K, L$ be two smooth, planar strictly convex bodies. If $K$ and $L$ are at a dilation position but not homothetic, then
\[ \rho_1 > -t_2. \] (3.4)

Proof. Since $K$ and $L$ are not homothetic, by [1, Lemma 4.1] and the fact that $K$ and $L$ are smooth and strictly convex,
\[ -t_1 < r(K, L) < R(K, L) < -t_2. \]
By Lemma 3.2, the origin $o \in \text{int}(K \cap L)$ or $o$ is the point of tangency of $\partial K$ and $\partial L$ such that $K \subseteq L$ (or $L \subseteq K$).

If the origin $o \in \text{int}(K \cap L)$, then
\[-\frac{\delta}{V(L)} h_L(\theta) < h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) < \frac{\delta}{V(L)} h_L(\theta), \quad \delta = \sqrt{V(K, L)^2 - V(K)V(L)} > 0.\]
For $I_1$ and $I_2$, $\rho(\theta) \equiv a$ holds on at most one interval, unless $K$ and $L$ are homothetic. Without loss of generality, assume that $\rho(\theta) > a$ on a subinterval $I_1'$ of $I_1$. On $I_1'$, $\rho(\theta) > a$ and
\[-\left( h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) \right) (\rho(\theta) - a) < \frac{\delta}{V(L)} h_L(\theta)(\rho(\theta) - a).\]
Integrating this expression over the interval $I_1$ yields
\[-\frac{1}{V(L)} \int_{I_1} \left( h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) \right) (\rho(\theta) - a) (h_L(\theta) + h''_L(\theta)) d\theta < \frac{\delta}{V(L)} (\rho_1 - a),\]
which, together with (3.3), gives
\[-\frac{1}{V(L)} \int_0^{2\pi} \left( h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) \right) (\rho(\theta) - a) (h_L(\theta) + h''_L(\theta)) d\theta < \frac{2b\delta}{V(L)}.\]
By a similar argument as in Lemma 3.3, $b > \frac{\delta}{V(L)} > 0$, which implies that $\rho_1 > -t_2$.

If the origin $o$ is the point of tangency of $\partial K$ and $\partial L$ such that $L \subseteq K$ (the case of $K \subseteq L$ is similar), then
\[-\frac{\delta}{V(L)} h_L(\theta) < h_K(\theta) - \frac{V(K, L)}{V(L)} h_L(\theta) < \frac{\delta}{V(L)} h_L(\theta),\]
for $\theta \in \hat{I}$. Similar with the case that the origin $o \in \text{int}(K \cap L)$, one can get $\rho_1 > -t_2$. \hfill \Box

Now, we give the proof of Theorem 1.1

Proof of Theorem 1.1 By Jensen’s inequality on $I_i$, $i = 1, 2$, one has
\[ \frac{1}{V(L)} \int_{I_i} F(\rho(\theta)) h_L(\theta) (h_L(\theta) + h''_L(\theta)) d\theta \geq F(\rho_i). \]
Then
\[ \frac{1}{V(L)} \int_0^{2\pi} F(\rho(\theta)) h_L(\theta) (h_L(\theta) + h''_L(\theta)) d\theta \geq F(\rho_1) + F(\rho_2), \] (3.5)
where $\rho_1 = \frac{V(K, L)}{V(L)} + b$, $\rho_2 = \frac{V(K, L)}{V(L)} - b$ and $b \geq 0$. Again from (3.1), it follows that $b \geq \frac{\delta}{V(L)} \geq 0$ and $\delta = \sqrt{V(K, L)^2 - V(K)V(L)}$. Since function $F(x)$ is strict convexity,
\[ F(\rho_1) + F(\rho_2) = F \left( \frac{V(K, L)}{V(L)} + b \right) + F \left( \frac{V(K, L)}{V(L)} - b \right). \]
\[
\geq F \left( \frac{V(K, L)}{V(L)} + \frac{\delta}{V(L)} \right) + F \left( \frac{V(K, L)}{V(L)} - \frac{\delta}{V(L)} \right) = F(-t_1) + F(-t_2),
\]

which together with (3.5) yields inequality (1.4).

On one hand, if \( K \) and \( L \) are homothetic, then \(-t_1 = -t_2 = \rho(\theta)\), it is clear that the equality holds in (1.4). On the other hand, in order to prove that \( K \) and \( L \) are homothetic when the equality holds in (1.4), it is enough to show that inequality (1.4) is strict when \( K \) and \( L \) are not homothetic. If \( K \) and \( L \) are not homothetic, then \( \delta = \sqrt{V(K, L)^2 - V(K)V(L)} > 0 \), and by (3.4), one has \( b > \frac{\delta}{V(L)} > 0 \). It follows from the strict convexity of function \( F(x) \) that (3.6) is strict, which together with (3.5) implies that (1.4) is strict.

\begin{remark}
If \( \mathbb{R}^2 \) is equipped with a suitable Minkowski metric such that \( \partial L \) becomes the isoperimetrix of the Minkowski plane, then (1.4) turns into an inequality in Minkowski geometry (see [12], Remark 3.6).
\end{remark}

\section*{Acknowledgements}

I am grateful to the anonymous referee for his or her careful reading of the original manuscript of this paper and giving us many invaluable comments. I would also like to thank Professor Shengliang Pan for posing this problem to me.

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