THE THEORY OF CEERS COMPUTES TRUE ARITHMETIC

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ABSTRACT. We show that the theory of the partial order of computably enumerable equivalence relations (ceers) under computable reduction is 1-equivalent to true arithmetic. We show the same result for the structure comprised of the dark ceers and the structure comprised of the light ceers. We also show the same for the structure of \(I\)-degrees in the dark, light, or complete structure. In each case, we show that there is an interpretable copy of \((\mathbb{N}, +, \cdot)\).

1. Introduction

A major theme in investigating computability theoretic reducibilities has been to measure, and when possible to characterize, the complexity of the first order theory of their degree structures. Throughout the paper we regard a degree structure as a poset, and if \(\mathcal{P}\) is a poset then the theory of \(\mathcal{P}\), denoted by \(\text{Th}(\mathcal{P})\), is the set of sentences, in the first order language of posets, that are true in \(\mathcal{P}\). This is an important task, not only because it sheds relevant information about the reducibility, but also because it generally stimulates useful techniques and constructions which are developed for this purpose: see [31] for a comprehensive discussion of the motivations and the history of this line of research in computability theory.

Typically, a reducibility is a binary relation \(\leq\) on subsets of the set \(\omega\) of natural numbers, which gives rise to a degree structure \(\mathcal{D}\) which partitions the power set \(P(\omega)\) into equivalence classes called degrees, such that (if \(\nu : P(\omega) \rightarrow \mathcal{D}\) is the surjection so that \(\nu(X)\) is the degree of \(X\), and \(\leq\) denotes the partial ordering relation on \(\mathcal{D}\)) the relations (in \(X, Y\)) \(\nu(X) = \nu(Y)\) and \(\nu(X) \leq \nu(Y)\) are arithmetical, so that one can effectively translate first order sentences regarding degrees to second order sentences of arithmetic, yielding a reduction \(\text{Th}(\mathcal{D}) \leq_1 \text{Th}^2(\mathbb{N})\), where the latter symbol denotes true second order arithmetic. In many cases, it is true that these two theories are 1-equivalent, and the challenge is to show that the reverse reduction, i.e. \(\text{Th}^2(\mathbb{N}) \leq_1 \text{Th}(\mathcal{D})\), holds as well, thus proving by the Myhill Isomorphism Theorem that the two theories are computably isomorphic, and that \(\text{Th}(\mathcal{D})\) is as complicated as it can be. The literature here is indeed rich of classical and celebrated results, starting from Simpson [32] who showed that the theory of the Turing degrees is computably isomorphic to \(\text{Th}^2(\mathbb{N})\): see also [33]; to mention two other major reducibilities, the theory of the \(m\)-degrees ([24]), and the theory of the enumeration degrees ([34]: see also [6]) are also computably isomorphic to \(\text{Th}^2(\mathbb{N})\).
When no restriction is taken on the universe of the reducibility, then one talks about the global degree structure of that reducibility. It is common however to consider local degree structures as well, by restricting attention to special countable families of degrees. This is the case for instance (just to consider some local structures of the aforementioned global structures) of the Turing degrees below the first jump, or the Turing degrees of the computably enumerable (c.e.) sets, or the m-degrees of the c.e. sets, or the enumeration degrees below the first enumeration jump. If $D$ is a local structure then typically one finds a surjection $\nu : \omega \rightarrow D$ such that the relations (in $x,y$) $\nu(x) = \nu(y)$ and $\nu(x) \leq \nu(y)$ are arithmetical, so that one can effectively translate sentences on $D$ into first order arithmetical sentences, establishing a reduction $Th(D) \leq_1 Th^1(\mathbb{N})$, where the latter set denotes true first order arithmetic $Th(\mathbb{N}, +, \times)$. To show that $Th(D)$ is as complicated as possible (i.e. computably isomorphic to $Th^1(\mathbb{N})$) it is then enough to reverse the reduction, showing in this case that $Th^1(\mathbb{N}) \leq_1 Th(D)$. For instance, this has been done for the aforementioned local structures: for the Turing degrees below the first jump, see Shore [30]; for the c.e. Turing degrees see Nies, Shore and Slaman [27]; for the c.e. m-degrees, see Nies [25]; for the enumeration degrees below the first enumeration jump, see Ganchev and Soskova [13] (see also [14]).

The above examples are about reducibilities on sets of natural numbers. We consider in this paper a reducibility on equivalence relations on $\omega$, instead of subsets of $\omega$. The reduction is defined as follows: if $R,S$ are equivalence relations on $\omega$, we say that $R$ is computably reducible (or, simply, reducible) to $S$ (notation: $R \leq S$) if there is a computable total function $f$ such that

$$(\forall x,y)[x R y \iff f(x) S f(y)].$$

As with other reducibilities, $\leq$ gives rise to an equivalence relation $\equiv$, where $R \equiv S$ if $R \leq S$ and $S \leq R$; the equivalence class of an equivalence relation $R$ under $\equiv$ will be called the degree of $R$. The first study of computable reducibility on equivalence relations on natural numbers was initiated by Ershov in the 70s, see e.g. [8, 9]. Recently, there has been a revived interest in this reducibility, motivated by the fact that is easily recognizable, see e.g. [11], as a useful and interesting effective version of Borel reducibility on equivalence relations (which is a primary target of interest in descriptive set theory, see for instance [14]), and by its high potential as a tool for measuring the computational complexity of classification problems, which are in fact equivalence relations, in computable mathematics: for instance it is shown in [10] that the isomorphism relation for various familiar classes of computable groups is $\Sigma^1_1$-complete under this reducibility.

The global structure of the degrees of equivalence relations, however, has not been extensively studied. Much more attention has been given to its local structure ceers consisting of the degrees of the c.e. equivalence relations (commonly called ceers after [15]). Indeed, ceers have played a leading role in the tale of computable reducibility: they appeared as the main characters of what are perhaps the first results about $\leq$ (although before the notion appeared in the literature), namely Miller III’s construction ([21]) of a finitely presented group whose word problem is $\Sigma^0_1$-complete with respect to $\leq$; and Miller III’s proof ([21]) that the isomorphism problem for finitely presented groups is $\Sigma^0_1$-complete with respect to $\leq$. (For other applications of $\leq$ to word problems of finitely presented groups see [25].) The first work explicitly tackling $\leq$ on ceers was done by Ershov [8], who pointed out important examples of $\Sigma^0_1$-complete ceers, showing also that their degree is join-irreducible.

In the 80s, the reducibility $\leq$ on ceers was applied to study computability theoretic properties of the relation of provable equivalence of sufficiently expressive formal systems, see [5, 20, 22, 23, 36]. Additional interest for computable reducibility on ceers comes from the study of c.e. presentations of structures, as is shown for instance in [12, 16]. It is also worth noticing that ceers have been
investigated in computability theory also not in connection with computable reducibility: for instance, Carroll [7] studied the lattice of ceers under inclusion; Nies [26] studied ceers modulo finite differences: they both showed that the first order theory of the resulting structures is computably isomorphic to \( \text{Th}^1(\mathbb{N}) \).

More explicitly oriented toward a degree-theoretic approach are the papers on ceers by Gao and Gerdes [15], Andrews et al. [1], [2], and finally Andrews and Sorbi [3]: the last paper provides a thorough investigation of the structure \( \text{Ceers} \), with emphasis on existence and non-existence of meets and joins, minimal covers, definable classes of degrees, and automorphisms.

Using the fact that the c.e. 1-degrees embed into \( \text{Ceers} \) it was shown in [1] that \( \text{Th}(\text{Ceers}) \) is undecidable, and even the \( \Pi_3 \)-fragment is undecidable. In this paper we completely characterize the complexity of \( \text{Th}(\text{Ceers}) \) by showing that it is in fact as complicated as it can be, namely computably isomorphic to \( \text{Th}^1(\mathbb{N}) \). We do this also for two suborders of \( \text{Ceers} \), called \text{Light} and \text{Dark}, introduced in [3], which are defined in the next section, and for the quotient structures obtained by quotienting the three structures \( \text{Ceers}, \text{Light} \) and \text{Dark} modulo uniform joins with finite ceers. In each case, we show that there is an interpretable copy of \( (\mathbb{N}, +, \cdot) \) in the degree structure.

2. Background material

For more information and details about unexplained computability theoretic terminology or results exploited in the paper without any reference, the reader may consult any standard textbook, see e.g. [29, 35]. In this section we review some background material concerning ceers and computable reducibility. The \( \equiv \)-degree of an equivalence relation \( R \) will be denoted by \( \text{deg}(R) \).

2.1. The classes \( I, \text{Light}, \text{Dark} \). We recall the following partition of ceers, introduced and studied in [3]. Let \( R \) be a ceer:

- \( R \) is \textit{finite} if it has only finitely many equivalence classes: if \( n \geq 1 \), it is easy to see that \( R \) has \( n \) equivalence classes if and only if \( R \equiv \text{Id}_n \), for some \( n \geq 1 \), where \( \text{Id}_n \) is equality mod \( n \);
- \( R \) is \textit{light} if there is an infinite c.e. set \( W \) (called an \textit{infinite transversal} for \( R \)) such that \( xRy \) for each pair of distinct \( x, y \in W \); it is easy to see that a ceer \( R \) is light if and only if \( \text{Id} \leq R \), where \( \text{Id} \) denotes the equivalence relation defined by equality;
- \( R \) is \textit{dark} if it is neither finite nor light.

The symbols \( I, \text{Light}, \text{Dark} \) denote the classes of finite ceers, light ceers, and dark ceers, respectively. These classes partition the ceers, and give rise to a corresponding partition of the degrees of ceers into three classes of degrees (still denoted by \( I, \text{Light}, \text{Dark} \)): \( I \) is an initial segment of \( \text{Ceers} \) having order type \( \omega \). In \( \text{Ceers} \) (in the language of posets), the degree of \( \text{Id} \), and thus each of these three classes are first order definable [3]. \( \text{Ceers}, \text{Light} \) and \text{Dark} are neither upper nor lower semilattices: in this regard, the most spectacular case is provided by dark degrees, as no pair of incomparable dark degrees has either meet or join in \( \text{Ceers} \) or in \( \text{Dark} \).

2.2. Some general facts about ceers. We describe three constructions of new ceers starting from given ceers and/or c.e. sets.
The first construction is the uniform join $R \oplus S$ which is the equivalence relation which copies $R$ on the even numbers and $S$ on the odd numbers: $x \ R \oplus S \ y$ if there exist $u, v$ such that $x = 2u, y = 2v$ and $u \ R \ v$, or $x = 2u + 1, y = 2v + 1$ and $u \ S \ v$. This operation extends in the obvious way to any countable number of equivalence relations, see Section 2.1 of [3].

The second construction is described in detail in Section 2.3 of [3]:

**Definition 2.1.** If $E$ is a ceer and $W$ is a c.e. set then $E|W$ (called the restriction of $E$ to $W$) is the ceer $x \ E|W \ y$ if and only if $h(x) \ E \ h(y)$, where $h : \omega \to W$ is any computable surjection (up to $\equiv$ the definition does not depend on the chosen $h$).

**Remark 2.2.** It is clear that $h$ provides a reduction $E|W \leq E$, which we call the inclusion of $E|W$ into $E$. If $X \leq R$ via a reduction $f$ then $X \equiv R|W$, where $W = \text{range}(f)$.

**Fact 2.3.** If $X \leq R_1 \oplus R_2$ then there are ceers $X_1 \leq R_1$ and $X_2 \leq R_2$ such that $X \equiv X_1 \oplus X_2$.

**Proof.** If $f$ is a reduction for $X \leq R_1 \oplus R_2$ and $W = \text{range}(f)$ then $X \equiv (R_1 \oplus R_2)|W \equiv R_1|V_1 \oplus R_2|V_2$, where $V_1 = \{x \mid 2x \in W\}$ and $V_2 = \{x \mid 2x + 1 \in W\}$.

The third construction is described in the following definition.

**Definition 2.4.** Let $W \subseteq \omega^2$: by $E/W$ we denote the equivalence relation generated by the set of pairs $E \cup W$. If $W$ is a singleton, say $W = \{(x, y)\}$, then we simply write $E/(x, y)$ instead of $E/(x, y)$.

Notice that if $W$ is c.e. and $E$ is a ceer then $E/W$ is a ceer as well.

We will also make use of the following easy facts about ceers:

**Fact 2.5.** The following hold:

1. For every pair $S,T$ of ceers and $k \geq 0$, $S \equiv T$ if and only if $S \oplus \text{Id}_k \equiv T \oplus \text{Id}_k$.
2. If $f$ is a reduction from $R$ to $S$ which omits exactly $k$ $S$-classes then $R \oplus \text{Id}_k \equiv S$.
3. If $W$ is a c.e. set missing exactly $k$ equivalence classes of a ceer $R$ then $R|W \oplus \text{Id}_k \equiv R$.

**Proof.** The first item is essentially [3, Lemma 2.1]. The second item comes from Lemma 2.8 in [3]; the third item follows from the previous one and the fact that under the assumptions the inclusion reduction $R|W \leq R$ misses exactly $k$ equivalence classes.

2.3. Reductibility modulo $\mathcal{I}$. We recall the following reducibility from [3].

**Definition 2.6.** We say $R \leq_\mathcal{I} S$ if there is some finite ceer $\text{Id}_k$ so that $R \leq S \oplus \text{Id}_k$.

On equivalence relations define $R \equiv_\mathcal{I} S$ if $R \leq_\mathcal{I} S$ and $S \leq_\mathcal{I} R$. The equivalence class of $R$ under $\equiv_\mathcal{I}$ will be denoted by $\deg_\mathcal{I}(X)$ and called the $\mathcal{I}$-degree of $R$. So $\text{Ceers}/_\mathcal{I}$ is the collection of all $\deg_\mathcal{I}(R)$, where $R$ is a ceer.

**Fact 2.7.** If $X \leq_\mathcal{I} R_1 \oplus R_2$ then there are ceers $X_1 \leq R_1$ and $X_2 \leq R_2$ so that $X \equiv_\mathcal{I} X_1 \oplus X_2$.

**Proof.** There is an $n$ so that $X \leq R_1 \oplus R_2 \oplus \text{Id}_n$. By Fact 2.3, this shows that there are $X_1 \leq R_1$ and $X_2 \leq R_2$ and $F \leq \text{Id}_n$ so that $X \equiv X_1 \oplus X_2 \oplus F$. Then $X \equiv_\mathcal{I} X_1 \oplus X_2$.

We consider the six structures: $\text{Ceers}, \text{Dark}, \text{Light}, \text{Ceers}/_\mathcal{I}, \text{Dark}/_\mathcal{I}, \text{Light}/_\mathcal{I}$. For elementary differences between these classes, and for more on their structural properties, see [3]. We note:
Lemma 2.8 ([3], Obs. 9.7). In each of Ceers, Dark, and Light, the equivalence relation $\equiv_I$, and thus the partial order $\leq_I$, is definable.

Notice also:

Lemma 2.9. If $D$ is any of the six structures Ceers, Dark, Light, $\text{Dark} / \text{I}$, $\text{Light} / \text{I}$, and $\text{Ceers} / \text{I}$ then $\text{Th}(D) \leq_1 \text{Th}^1(\mathbb{N})$.

Proof. Take $\{R_z \mid z \in \omega\}$ be the indexing of ceers defined in [1], and let $\nu : \omega \rightarrow \text{Ceers}$ be given by $\nu(x) = [R_x]$, where $[R_x]$ is the degree of $R_x$ with respect to $\equiv$. Then it is easy to see that the relations $(x, y) : \nu(x) = \nu(y)$ and $\nu(x) \leq \nu(y)$ are arithmetical, so that one can effectively translate sentences on degrees into first order arithmetical sentences, getting $\text{Th}(\text{Ceers}) \leq_1 \text{Th}^1(\mathbb{N})$. Since each of the other five structures are definable in Ceers, this shows that their theories also are $\leq_1 \text{Th}^1(\mathbb{N})$. □

For each of the structures of degrees of ceers mentioned in Lemma 2.9, we show that there is a copy of $(\mathbb{N}, +, \cdot)$ which is interpreted in the structure, and thus by Lemma 2.9 the theory is 1-equivalent to true arithmetic. In view of Lemma 2.8 to yield the result we need only find an interpreted copy of $(\mathbb{N}, +, \cdot)$ in the three structures $\text{Dark} / \text{I}$, $\text{Light} / \text{I}$, and $\text{Ceers} / \text{I}$.

2.4. Self-full ceers. The following obvious facts about dark ceers hold:

Fact 2.10. The following hold:

1. If $S$ is not finite, $S \leq_I T$ and $T$ is dark then so is $S$;
2. If $R$ is dark and $S$ is either finite or dark then $R \oplus S$ is dark;
3. If $R$ is dark then so is any $R/W$: in particular, if $R, S$ are dark then so is any $(R \oplus S)/(2x, y+1)$.

Proof. Item (1) is [3, Observation 6.3]; item (2) is [3, Observation 3.2]; item (3) follows straight from the definitions. □

Definition 2.11. A ceer $R$ is self-full if $R \oplus \text{Id}_1 \neq R$. Equivalently ([3, Observation 4.2]), $R$ is self-full if whenever $\varphi : R \rightarrow R$ is a reduction, $\varphi$ is onto the classes of $R$.

The following fact collects useful properties of self-full ceers and the $\oplus \text{Id}_1$ operation:

Fact 2.12. The following hold:

1. For any ceers $R$ and $S$: If $S < R \oplus \text{Id}_1$ then $S \leq R$; and if $R < S$ then $R \oplus \text{Id}_1 \leq S$.
2. Every dark ceer is self-full.

Proof. Item (1) is [3, Lemma 4.5]; item (2) is [3, Lemma 4.6]. □

It is also useful to notice:

Observation 2.13. If $R$ is self-full and $R \equiv E$, then any reduction $\varphi : R \rightarrow E$ must be onto the classes of $E$.

Proof. Consider the pair of reductions $R \leq E \leq R$. If we were to have a reduction of $R$ into $E$ which is not onto the classes of $E$, then the composition would be a reduction of $R$ to itself which is not onto the classes of $R$, contradicting $R$ being self-full. □
3. Minimal classes in Dark$_I$

We now proceed with examining some facts about dark minimal $I$-degrees, on which we will code models of arithmetic. There are two types of minimal degrees in Dark$_I$. The first is the $I$-degree of a ceer $D$ which is a dark minimal ceer. The second is an $I$-degree which, as a class of $\equiv$-degrees, has the order type of $\mathbb{Z}$ and bounds no other non-zero $I$-degree. Indeed, suppose that $R$ is a dark ceer, and $\deg_I(R)$ is minimal. First of all notice that the ceers $\{R_k \mid k \in \omega\}$, where $R_k = R \oplus \text{Id}_k$, give rise, as $k$ strictly increases, to a strictly increasing sequence of $\equiv$-degrees within $\deg_I(R)$ (where we agree that $R_0 = R$), which comprises all $\equiv$-degrees within $\deg_I(R)$ that are greater than $\deg(R)$: this easily follows from Facts 2.10 and 2.12. On the other hand, if $S < R$ in $\deg_I(R)$ then there is $k \geq 1$ such that $R \equiv S \oplus \text{Id}_k$, and all $T$ such that $R \equiv T \oplus \text{Id}_k \equiv S$ (using Fact 2.5(1)), so for every $k \geq 1$ for which there is an $S$ as above we can choose such a (unique up to $\equiv$) $S$ and define $R_{-k} = S$. If there is no $\leq$-minimal element in $\deg_I(R)$ then all ceers in $\deg_I(R)$ which are smaller than $R$ are $\equiv$ to some $R_{-k}$, which form a chain $\cdots < R_{-k} < \cdots < R_{-1}$; thus the $I$-degree $\deg_I(R)$ consists of a $\mathbb{Z}$-chain of $\equiv$-degrees, namely the degrees of $\cdots < R_{-2} < R_{-1} < R < R_1 < R_2 < \cdots$.

Definition 3.1. A ceer in this second type of minimal $I$-degree we call a $\mathbb{Z}$-dark minimal ceer.

Example 3.2. Examples of dark minimal degrees are from Theorem 4.10 and Corollary 4.15 of [3]. As to examples of $\mathbb{Z}$-dark minimal ceers, they come from Theorem 4.11 and Corollary 4.15 of [3]: indeed, one can use the proof of Theorem 4.11, but without coding any ceer $A$, to build $\leq_I$-minimal dark ceers with finite classes. That the construction in Theorem 4.11 suffices is justified by Lemma 3.5 below.

The following fact was shown in [18], the right-to-left implication being already in [3] Lemmas 3.4 and 3.5:

Fact 3.3. $R$ is a dark minimal ceer if and only if $R$ has infinitely many classes and every c.e. set $W$ which intersects infinitely many $R$-classes intersects every $R$-class.

An easy consequence of this fact is the following:

Observation 3.4. If $R$ is a dark minimal ceer, then every pair of classes $[a]_R \neq [b]_R$ are computably inseparable. As a consequence, if $R$ is a dark minimal ceer and $R \leq_I S$ then $R \leq S$.

Proof. Suppose that $X$ is a computable set which separates $[a]_R$ and $[b]_R$. Either $X$ or the complement of $X$ must intersect infinitely many classes in $R$. But neither can intersect every class, since $[a]_R \subseteq X$ and $[b]_R \cap X = \emptyset$. This contradicts Fact 3.3. The latter claim follows from the fact that if $R$ is a dark minimal ceer then no reduction $R \leq S \oplus \text{Id}_k$ can hit $\text{Id}_k$ since no $R$-equivalence class is computable. \hfill $\square$

In the next lemma we show the corresponding fact for the $\mathbb{Z}$-dark minimal ceers.

Lemma 3.5. A ceer $R$ is a $\mathbb{Z}$-dark minimal ceer if and only if it has infinitely many computable classes and every c.e. set $W$ which intersects infinitely many $R$-classes intersects co-finitely many $R$-classes.

Proof. Suppose that $R$ is a $\mathbb{Z}$-dark minimal ceer. For every $k$, $R \equiv S \oplus \text{Id}_k$ for some $S$. By self-fullness of $R$ and Observation 2.13, the reduction of $R$ to $S \oplus \text{Id}_k$ is onto the classes of $S \oplus \text{Id}_k$, so
$R$ has at least $k$ computable classes: this is true for every $k$, so $R$ has infinitely many computable classes. Suppose $W$ is a c.e. set which intersects infinitely many $R$-classes. Since $R|W \leq R$, we see by $\mathcal{I}$-minimality of $R$ that $R|W \oplus \text{Id}_k = R$ for some $k \geq 0$, where we mean $R|W \oplus \text{Id}_0 = R|W$. Suppose now that $W$ omits $k + 1$ classes: then from the inclusion $R|W \leq R$ and sending the $k$-classes of $\text{Id}_k$ to $k$ of the classes omitted by $W$ it is possible to build a reduction $R|W \oplus \text{Id}_k \leq R$ which omits at least one class in its range. But this, together with $R \leq R|W \oplus \text{Id}_k$ contradicts Observation 2.13. It follows that $W$ hits co-finitely many classes.

For the converse: First we check that, under the assumptions, $R$ is dark. Suppose $W$ enumerated an infinite transversal. Consider $V$ any c.e. co-infinite subset of $W$. Since $W$ is a transversal, $V$ cannot contain co-finitely many classes in $R$. Yet $V$ hits infinitely many classes in $R$, contradicting the hypothesis on $R$. Thus $R$ is dark.

Since $R$ has infinitely many computable classes, then for every $k$ we can collapse any $k + 1$ of these together to find an $E$ so that $E \oplus \text{Id}_k = R$, thus $R$ is not in the $\mathcal{I}$-class of a dark minimal ceer, for choosing such an $E$ and taking $R_{-k} = E$, we must have an infinite strictly descending chain provided by the $=\text{-}$degrees of the various $R_{-k}$. To show $\mathcal{I}$-minimality of $R$, let now $X \leq \mathcal{I} R$, where $X$ is not finite. By Fact 2.7, $X \equiv_{\mathcal{I}} Y$ for some $Y \leq R$. By Remark 2.2, there is some $W$ so that $Y \equiv R|W$. Since $X$ is not finite, $Y$ must not be finite, and thus $W$ intersects infinitely many $R$-classes. Thus $W$ intersects co-finitely many $R$-classes. Thus, by Fact 2.5(3), $Y = R_m$ for some $m$. Thus, $R \leq_{\mathcal{I}} R_{-m} \leq_{\mathcal{I}} X$. Thus $R$ is $\mathcal{I}$-minimal.

We isolate the following fact which is immediate from Lemma 3.5.

**Corollary 3.6.** If $R$ is a $\mathcal{I}$-dark minimal ceer and $W$ intersects infinitely many classes, then the closure of $W$ is computable, and every class omitted from $W$ is computable.

**Proof.** The closure of $W$ along with each of the finitely many omitted classes forms a partition of $\omega$ into finitely many c.e. sets. Thus, each of these sets is computable. \qed

**Remark 3.7.** Notice that although in Fact 3.3 and Lemma 3.5 we distinguish between the cases of being dark minimal and being $\mathcal{I}$-dark minimal for a ceer lying in a dark minimal $\mathcal{I}$-degree, it is correct to say, unifying the two cases, that if $R$ lies in a dark minimal $\mathcal{I}$-degree then every c.e. set intersecting infinitely many $R$-equivalence classes intersects co-finitely many classes: if $R$ is dark minimal then “co-finitely many classes” means in fact in this case “all classes”.

**Definition 3.8.** We say that $a$ is an $\mathcal{I}$-strongly minimal cover of the pair of $\mathcal{I}$-degrees $d, e$ if $d$ and $e$ are $\mathcal{I}$-incomparable, $d, e \leq_{\mathcal{I}} a$, and the only degrees $\leq_{\mathcal{I}} a$ are $\leq_{\mathcal{I}} d$ or $\leq_{\mathcal{I}} e$.

**Lemma 3.9.** If $r_1$ and $r_2$ are distinct dark minimal $\mathcal{I}$-degrees, then $r_1 \oplus r_2$ is an $\mathcal{I}$-strongly minimal cover of the pair $r_1, r_2$.

**Proof.** Let $R_1$ and $R_2$ be in the $\mathcal{I}$-degrees $r_1, r_2$ respectively, and suppose $X \leq_{\mathcal{I}} R_1 \oplus R_2$: we want to show that either $X \leq_{\mathcal{I}} R_1$ or $X \leq_{\mathcal{I}} R_2$. We may assume that $X$ is not finite, otherwise the claim is trivial. From Fact 2.7, we have that $X \equiv_{\mathcal{I}} X_1 \oplus X_2$ where $X_1 \leq R_1$, and $X_2 \leq R_2$. By the $\mathcal{I}$-minimality of $R_1$, we have that either $R_1 \leq_{\mathcal{I}} X_1$ or $X_1$ is finite. Similarly, either $R_2 \leq_{\mathcal{I}} X_2$ or $X_2$ is finite. If both $R_1 \leq_{\mathcal{I}} X_1$ and $R_2 \leq_{\mathcal{I}} X_2$, then $R_1 \oplus R_1 \leq_{\mathcal{I}} X$ contradicting $X \leq_{\mathcal{I}} R_1 \oplus R_2$. Thus without loss of generality we may assume $X_2$ is finite. Then $X \equiv_{\mathcal{I}} X_1 \oplus X_2 \leq_{\mathcal{I}} X_1 \leq_{\mathcal{I}} R_1$. \qed

**Lemma 3.10.** If $r_1$ and $r_2$ are distinct dark minimal $\mathcal{I}$-degrees represented by $R_1$ and $R_2$, $x$ is even and $y$ is odd, then the $\mathcal{I}$-degree of $(R_1 \oplus R_2)/(x,y)$ is an $\mathcal{I}$-strongly minimal cover of the pair $r_1$ and $r_2$. 
Proof. Suppose $X \leq_I (R_1 \oplus R_2)/(x,y)$. We want to show that either $X \leq_I R_1$ or $X \leq_I R_2$. We may assume that $X$ is not finite, otherwise the claim is trivial. Take $n$ so that $X \leq (R_1 \oplus R_2)/(x,y) \oplus \text{Id}_n$. By Fact 2.3, we can write $X \equiv X_0 \oplus X_1$ where $X_0 \leq (R_1 \oplus R_2)/(x,y)$ and $X_1 \leq \text{Id}_n$. Thus $X \equiv_I X_0$, and since we are only considering $X$ up to $I$-degree, we may replace $X$ by $X_0$ and thus we may assume that $X < (R_1 \oplus R_2)/(x,y)$. Let $W = \text{range}(f)$ where $f$ reduces $X \leq (R_1 \oplus R_2)/(x,y)$. If $W \cap 2\omega$ hits infinitely many classes, then $\frac{W}{2}$ hits cofinitely many classes in $R_1$. Similarly on the odd classes.

We first rule out the possibility that $W$ hits both infinitely many even and infinitely many odd classes: if this were the case, then there would be $h, k$ such that $\frac{W}{2}$ would omit $h R_1$-classes and $\frac{W-1}{2}$ would omit $k R_2$-classes. If $f$ does not hit the $(R_1 \oplus R_2)/(x,y)$-class of $x$ then $f$ omits exactly $h + k - 1$ classes, otherwise $f$ omits exactly $h + k$ classes. Let $j$ be the number of classes (either $h + k - 1$ or $h + k$) omitted by $f$. Then by Fact 2.5(2), $X \oplus \text{Id}_j \equiv (R_1 \oplus R_2)/(x,y)$, so $X \equiv_I (R_1 \oplus R_2)/(x,y)$. This contradicts the assumption that $X < (R_1 \oplus R_2)/(x,y)$.

Therefore, we may suppose that $W$ hits only finitely many even classes, the case of $W$ hitting only finitely many odd classes being similar. Now we aim to show that $X \leq_I R_2$. Let $k$ be the number of the finitely many even classes that are hit by $f$, and choose representatives $a_0, \ldots, a_{k-1}$ for these classes; assume also that $0, 1, \ldots, k - 1$ are representatives of the $k$ equivalence classes of $\text{Id}_k$. Consider the function $g$ computed by the following procedure: given a number $x$, if $f(x)$ is even then search for the first $a_i$ such that $f(x) = (R_1 \oplus R_2)/(x,y) a_i$ and let $g(x) = a_i$; if $f(x)$ is odd, then let $g(x) = f(x)$. Using $g$ we see that if $W$ does not hit the $(R_1 \oplus R_2)/(x,y)$-equivalence class of $x$ then there is a reduction $X \leq \text{Id}_k \oplus R_2$, which, for each $i < k$, matches e.g. the $(R_1 \oplus R_2)/(x,y)$-equivalence class of $a_i$ with the $\text{Id}_k \oplus R_2$-equivalence class of $2i$; otherwise there is a reduction $X \leq (\text{Id}_k \oplus R_2)/(0,y) \equiv \text{Id}_{k-1} \oplus R_2$, where we may assume without loss of generality that the $(R_1 \oplus R_2)/(x,y)$-equivalence classes of $a_0$ and $x$ coincide.

In either case, we have $X \leq I R_2$. \hfill \Box

Lemma 3.11. If $R_1$ and $R_2$ are $I$-incomparable and are each dark minimal or $Z$-dark minimal ceers, and $[x_0]_{R_1}$ and $[y_0]_{R_2}$ are non-computable, then, letting $x = 2x_0$ and $y = 2y_0 + 1$, we have that $R_1 \oplus R_2$ and $(R_1 \oplus R_2)/(x,y)$ are $I$-incomparable ceers. (Note that the condition that $[x]_{R_1}$ and $[y]_{R_2}$ are non-computable must hold if $R_1$ and $R_2$ are each dark minimal, by Observation 3.4.)

Proof. Since any ceer $<_I$ either $R_1 \oplus R_2$ or $(R_1 \oplus R_2)/(x,y)$ must be $<_I$ either $R_1$ or $R_2$ (from the previous two lemmas), we only need to show that $R_1 \oplus R_2$ and $(R_1 \oplus R_2)/(x,y)$ are not $I$-equivalent ceers, and thus it is enough to show that $(R_1 \oplus R_2)/(x,y) \not<_I R_1 \oplus R_2$. Suppose towards a contradiction that $(R_1 \oplus R_2)/(x,y) \leq_I R_1 \oplus R_2$. Then let $n$ be so that $(R_1 \oplus R_2)/(x,y) \leq R_1 \oplus R_2 \oplus \text{Id}_n$. Consider the reduction $R_1 \leq (R_1 \oplus R_2)/(x,y) \leq R_1 \oplus R_2 \oplus \text{Id}_n$: in this reduction, let $W$ be the set of elements sent into the copy of $R_1$ in $R_1 \oplus R_2 \oplus \text{Id}_n$. If $W$ intersects only finitely many $R_1$-classes, then $R_1 \leq_I R_2$, which contradicts $R_1$ and $R_2$ being $I$-incomparable. So, $W$ intersects infinitely many $R_1$-classes, thus it intersects co-finitely many $R_1$-classes and misses only computable classes. Thus, $x_0$ is sent into $R_1$ in $R_1 \oplus R_2$. Similarly, $y_0$ is sent into $R_2$ in $R_1 \oplus R_2$ in the reduction $R_2 \leq (R_1 \oplus R_2)/(x,y) \leq R_1 \oplus R_2 \oplus \text{Id}_n$. But in $(R_1 \oplus R_2)/(x,y)$ their images are equivalent, a contradiction. \hfill \Box

Lemma 3.12. If $A$ is a $Z$-dark minimal ceer, and $E$ is an $I$-strongly minimal cover of the pair $A$, $B$ (where $A$, $B$ are $<_I$-incomparable), then either $A$ has a non-computable class or $A \oplus B \equiv_I E$. 
Definition 4.1. and E

Proof. Immediate from the preceding lemma.

We now show how to code any computable graph in Dark in the following by a -incomparable -classes is not empty. In this case we show that either at least one of them is not computable, or , and each of the remaining classes is computable by Corollary 3.6. Then let A I of first coordinates in A A,B therefore , and let Y to be the computable set of elements y such that g(y) lands in Id_k; then map 2x for x ∈ X to the element which corresponds to f(x) in the first copy of Id_k; map 2x to the element corresponding to f(x) in the copy of E if x /∈ X; map 2y + 1 for y ∈ Y to the element which corresponds to g(y) in the second copy of Id_k; map 2y + 1 to the element corresponding to g(y) in the copy of E if y /∈ Y: this gives a reduction A ⊕ B I ≺ E ⊕ Id_k ⊕ Id_k. Therefore A, B ≺ E A ≺ B I, and A ⊕ B I ≺ E: since E is an I-strongly minimal cover of the pair A, B, we have that E I ≺ A ≺ B I, and thus we may assume that E = A ⊕ B I. Now, the set W of first coordinates in ~ is a c.e. set. We first rule out the possibility that it intersects infinitely many classes in A. Suppose for a contradiction that this is the case: then W intersects co-finitely many classes, and each of the remaining classes is computable by Corollary 3.6. Then let A_0 be the ceer A restricted to this co-finite set of classes, i.e. A_0 = A|V where V is the computable union of these classes: we have that A_0 ≺ B, but A ≡ A_0 ⊕ Id_n by Fact 2.5(2), where n is the number of the missed classes, showing A ≺ B, a contradiction. Therefore, we conclude that the set of first coordinates in ~ intersects only finitely many classes in A. If it does not intersect any class then A ⊕ B = (A ⊕ B) I, showing that A ⊕ B ≺ E as desired. Otherwise this finite set of A-classes is not empty. In this case we show that either at least one of them is not computable, or A ⊕ B ≺ A ⊕ B I. To see this, suppose that each of these classes is computable. Let A_0 be the ceer comprised of A restricted to the complement of these classes. Then we see that A_0 ⊕ B ≡ E: the reduction A_0 ⊕ B ≺ E is obvious as A_0 ≺ A via the inclusion reduction, which omits the first coordinates of ~; to see that E ≺ A_0 ⊕ B map 2x to 2x if x is not A-equivalent to a first component of ~ and to 2y + 1, where (2x, 2y + 1) ∈ ~ , otherwise, and map 2y + 1 to 2y + 1. But by Fact 2.5(2) A ≡ I A_0, thus A_0 ⊕ B ≺ I A ⊕ B, giving A ⊕ B ≺ I E as desired.

Lemma 3.13. If A is a Z-dark minimal ceer and B is any ceer so that the pair A,B has two I-incomparable I-strongly minimal covers, then A has a non-computable class.

Proof. Immediate from the preceding lemma.

4. Coding graphs into the partial order Dark/I using parameters

In the following by a graph we mean a structure G = ⟨V, E⟩ where V is a nonempty set of vertices, and E, called the edge relation, is just an irreflexive and symmetric binary relation on V.

Definition 4.1. Given any dark degree c, we describe a graph G_c as follows:

- (vertices) the vertex set V of the graph are the I-minimal degrees ≺ c.
- (edges) For each pair d, e ∈ V, we put an edge between d and e if and only if there are two incomparable I-degrees a, b ≺ I c which are both I-strongly minimal covers of the pair d, e.

We now show how to code any computable graph in Dark/I.
Theorem 4.1. If $G = (V, E)$ is a computable graph, then there exists a dark degree $c$ so that $G_c$ is isomorphic to $G$ along with a set of isolated vertices.

Proof. Fix a computable presentation of $G$. We also fix a uniform c.e. sequence of pairwise $\leq_I$-incomparable dark minimal ceers $\{R_i \mid i \in \omega\}$; for this, just observe that by the proof of [3, Theorem 3.3] from any finite set $R_0, \ldots, R_n$ of dark minimal ceers we can uniformly find a dark minimal ceer $R_{n+1}$ so that $R_{n+1} \nleq R$ where $R = \bigoplus_{i \in n} R_i$, and thus $R_{n+1} \nleq R$ by Observation 3.4, at the same time, for each $i$, $R_i \nleq_I R_{n+1}$ otherwise (again by Observation 3.4) $R_i \nleq R$, by minimality of $R_{n+1}$. If $R, E$ are ceers we say that the $n$-th column of $E$ codes $R$ (or $R$ is copied in the $n$-th column of $E$) if for every $x, y$, if $x R y$ if and only if $\langle n, x \rangle E \langle n, y \rangle$.

Requirements and strategies. We construct the ceer $C$ (with $I$-degree $c$) with the following requirements:

- $\text{Code}_i$: Some column of $C$ codes $R_i$.
- $\text{Dark}_j$: If $W_j$ is infinite, then there are distinct $x, y \in W_j$ so that $x C y$.
- $\text{Edge}_{j,i}$: If there is an edge between $i$ and $j$ in $G$, then some column of $C$ codes $(R_i \oplus R_j)_{/0,1}$.

The priority order of the requirements is

$\text{Code}_0 \prec \text{Edge}_0 \prec \text{Dark}_0 \prec \cdots \prec \text{Code}_j \prec \text{Edge}_j \prec \text{Dark}_j \prec \cdots$

A requirement $\text{Edge}_{j,i}$ such that the graph $G$ has an edge between $i$ and $j$ will be called binding. As we consider only symmetric irreflexive graphs, we assume that if $(i)_0 = (i)_1$ then $\text{Edge}_i$ is not binding, and $\text{Edge}_{j,i}$ is not binding if $\langle i, j \rangle < \langle j, i \rangle$.

We outline the strategies to meet the requirements.

For the sake of the $\text{Code}_i$-requirement, we act by picking a new column and determining that this column will copy $R_i$. It restrains this entire column.

For the sake of the $\text{Dark}_j$-requirement, while the requirement is not satisfied (it becomes permanently satisfied when distinct numbers $x, y$ appear such that $x, y \in W_j$ and $x C y$) we simply wait for $W_j$ to enumerate two distinct numbers $x, y$ which are not in columns restrained by higher priority requirements, then we collapse to a single class the entire columns of $x$ and $y$.

We will use the following notations: $\omega[i]$ denotes the $i$-th column of $\omega$; $\omega[i,j] = \bigcup_{i \leq r \leq j} \omega[r]$, and $\omega[i,j] = \bigcup_{i \leq r < j} \omega[r]$. It will follow from the construction that $\text{Dark}_j$ works with a parameter $\epsilon_j$ so that the columns restrained by higher priority requirements will be the columns $\omega[n]$ for $n \leq \epsilon_j$; eventually $\epsilon_j$ stabilizes in the limit, and the interval $[0, \epsilon_j]$ is eventually partitioned into subintervals, each one being either a singleton $\{i\}$ so that in the $i$-th column codes a dark ceer (with $R_0$ coded in the 0-th column); or a subinterval $[a, b]$ so that all columns $\omega[i]$ with $i \in [a, b]$ are collapsed to a single (clearly decidable) class. Being thus computably bijective with a uniform join of dark ceers and copies of $\text{Id}_1$ and being infinite, this finite set of columns can be viewed as a dark ceer $R$ (see Fact 2.10). If an infinite $W_j$ is contained in this finite set of columns, we need not do anything, as in this case $W_j$ is not a transversal of $C$: otherwise (as shown in the verification) from $W_j$ one could find an infinite c.e. set which is a transversal of $R$, which is impossible by darkness of $R$.

In the following we will distinguish between coding columns in which we code dark ceers, and column
blocks, comprised of finitely many consecutive columns of \( \omega \) all collapsed to a single \( C \)-equivalence class which is decidable.

If Dark, acts by collapsing, it re-initializes all lower priority requirements, it leaves untouched all coding columns and column blocks in the restrained interval \([0, \epsilon_j]\), and collapses to a single class all other columns up to the biggest column so far used in the construction: these newly collapsed columns contain also the witnesses \( x, y \) which are thus collapsed and Dark \( j \) is permanently satisfied.

For the sake of the Edge \( (i, j) \)-requirement, we act exactly as in the other Coding requirements Code\( i \). The only distinction is that the existence of any one of these Edge-requirements (i.e. whether or not it is binding) is determined by the computable graph \( G \).

It follows that in the end \( C \) will consist of single coding columns (used for coding ceers of the form \( R_i \) or \((R_i \oplus R_j)/(0,1)\)) and column blocks comprised of finitely many consecutive columns of \( \omega \) all collapsed to a single \( C \)-equivalence class.

We first observe that the only dark minimal ceers \( \leq_T C \) are the \( R_i \)'s that we began with. To see this, first of all notice that by Observation 3.4, if \( E \) is dark minimal and is \( \leq_T C \) then it is also \( \leq C \), and must reduce to a single coding column: in fact (again by Observation 3.4) no class can be mapped to a column block as no \( E \)-class is decidable, and no two distinct \( E \)-classes can be mapped to distinct coding columns by recursive inseparability of \( E \). So \( E \leq R_i \), or \( E \leq (R_i \oplus R_j)/(0,1) \) for some \( i, j \); in the former case \( E \) is equivalent to \( R_i \); in the latter case, Lemma 3.10 shows that \( E \) must be equivalent to either \( R_i \) or \( R_j \).

**Adequacy of the requirements.** We now suppose that \( C \) has been constructed satisfying all the requirements and we verify that \( G \in \cong \) (where \( \cong \) denotes isomorphism) modulo a set of isolated vertices, which are \( I \)-degrees of \( Z \)-dark minimal ceers with only computable classes.

We first observe that the only \( Z \)-dark minimal ceers \( \leq_T C \) have only computable classes. Suppose towards a contradiction that \( A \) is a \( Z \)-dark minimal ceer with a non-computable class \([m]_A\), and \( f : A \leq_T C \) is an \( I \)-reduction, say \( f \) is a reduction \( f : A \leq C \oplus \Id_r \), for some \( r \). Let \( f(m) = 2 \cdot \langle n, x \rangle \), where clearly the \( n \)-th column is a coding column, say coding the ceer \( X \), by undecidability of \([m]_A\). Let \( W \) be the computable set of \( y \) so that \( f(y) \) does not land in the \( n \)-th column of \( C \). By Corollary 3.6 \( W \) cannot hit infinitely many classes, as otherwise \([m]_A\), which is omitted by \( W \), should be computable. Thus we must have that \( W \) intersects only finitely many classes. Thus the reduction \( f \) lands in the \( n \)-th column of \( C \) plus a finite collections of single equivalence classes of \( C \oplus \Id_r \). Thus \( A \leq X \oplus \Id_s \) for some \( s \), so \( A \leq_T X \). Now, \( X \) is either a dark minimal ceer or one of \((R_i \oplus R_j)/(0,1)\). Since a dark minimal ceer cannot bound a \( Z \)-dark minimal ceer, the former case is impossible. In the latter case, we have that \( A \equiv_T (R_i \oplus R_j)/(0,1) \) or \( A \leq_T R_i \) or \( A \leq_T R_j \). In any case \( A \) is not a \( Z \)-dark minimal ceer, which is a contradiction.

Thus, the universe of \( G \in \cong \) is comprised of the dark minimal ceers \( R_i \) along with perhaps some \( Z \)-dark minimal ceers which have all computable classes. By Lemma 3.13, these are isolated points in \( G \).

If there is an edge between \( i \) and \( j \) in \( G \), then we have \( R_i \oplus R_j \) and \((R_i \oplus R_j)/(0,1)\) being both \( \leq C \). By Lemmas 3.9 and 3.10 we have an edge between \( R_i \) and \( R_j \) in \( G \).

Now, suppose that there is no edge between \( i \) and \( j \) in \( G \). Then we do not place any columns in \( C \) of the form of \((R_i \oplus R_j)/(0,1)\). Suppose \( X \) is an \( I \)-strongly minimal cover below \( C \) of the pair \( R_i, R_j \). Consider the composition reductions \( R_i \leq X \leq_T C \) and \( R_j \leq X \leq_T C \): by Observation 3.4 the first reduction in each of the two chains is just \( \leq \), and by computable inseparability these
reductions reduce to single coding columns of \( C \). Each coding column is either some \( R_k \), or has the form \((R_k \oplus R_l)/(0,1)\). In the latter case, by Lemma 3.10 only \( R_k \) and \( R_l \) reduce to that column. Thus, the two coding columns in which \( R_i \) and \( R_j \) are reducing to \( C \) are different columns. Thus we see \( R_i \oplus R_j \leq X \), giving \( R_i \oplus R_j \equiv_{\mathbb{I}} X \) as \( X \) is an \( \mathbb{I} \)-strongly minimal cover of the pair \( R_i, R_j \). Thus we can only have one \( \mathbb{I} \)-strongly minimal cover of the pair \( R_i, R_j \) below \( C \), and there is no edge between \( R_i \) and \( R_j \) in \( G_c \).

**The construction.** In the formal construction we make use of several parameters: \( \gamma_i(s) \), if defined, denotes the column in which at \( s \) we code \( R_i \); \( \epsilon_{(i,j)}(s) \), if defined and Edge\(c_{(i,j)} \) is binding, denotes the column in which at \( s \) we code \((R_i \oplus R_j)/(0,1)\); the parameter \( r(s) \) denotes the least number \( n \) so that the corresponding column is still fresh i.e. no parameter \( \gamma_i(t) \) or \( \epsilon_i(t) \) for \( t \leq s \) was defined and \( \geq n \). At each stage \( s \) there will always be a unique number \( i \) such that we define for the first time, or redefine, \( \gamma_i(s) \). This will determine also the definition of \( \epsilon_i(s) \) as

\[
\epsilon_i(s) = \begin{cases} 
\gamma_i(s) + 1, & \text{if Edge}_i \text{ is binding,} \\
\gamma_i(s), & \text{otherwise.}
\end{cases}
\]

This means that we plan to code \( R_i \) in the \( \gamma_i(s) \)-column; and if there is an edge in the graph from \((i)_0\) to \((i)_1\) then we plan to code \((R_i \oplus R_i)/(0,1)\) in the next column; otherwise Edge\(c_i \) is not binding and does not need to be coded in any column. A requirement Dark\(j \) is satisfied at \( s \), if \( W_j \) has already enumerated a pair of distinct numbers \( x, y \) which \( C \) has already collapsed. Finally, at stage \( s \) we define a ceer \( C_s \), so that \( C_0 \subseteq C_1 \subseteq \cdots \), and the sequence \( \{C_s \mid s \in \omega \} \) is c.e., so that \( C = \bigcup_s C_s \) is our desired final ceer.

Stage 0. Let \( \gamma_0(0) = 0 \); since \( 0 = \langle 0, 0 \rangle \), Edge\(c_0 \) is not binding, thus \( \epsilon_0(0) = 0 \) too, and \( r(0) = 1 \). All other parameters are undefined. (The construction will ensure that \( \gamma_0 \) and \( \epsilon_0 \) will never be initialized.) Let \( C_0 \) be the cer generated by the c.e. set of pairs \( X \) where

\[
X = \{\gamma_0(0)\} \times R_0 = \{0\} \times R_0
\]

Here and below we use the notation: For \( n \in \omega \) and \( E \) a cer, \( \{n\} \times E = \{\langle n, x \rangle, \langle n, y \rangle \mid (x, y) \in E\} \).

Stage \( s + 1 \). Let us say that Dark\(j \) requires attention at \( s + 1 \) if \( j \leq s \), \( \epsilon_j = \epsilon_j(s) \) is defined, Dark\(j \) is not as yet satisfied at the end of stage \( s \), and there are now distinct numbers \( x, y \in W_j \) at \( s + 1 \), with \( x, y \in \omega^{\epsilon_j+1,r(s)} \).

1. If some Dark\(j \) requires attention, then pick the least such \( j \); redefine \( \gamma_{j+1}(s + 1) = r(s) \); this determines also \( \epsilon_{j+1}(s + 1) \) (equal to \( r(s) + 1 \) or \( r(s) \) according to whether Edge\(_{j+1} \) is binding or not), and set to be undefined all \( \gamma_i \) and \( \epsilon_i \) relative to requirements with priority less than Edge\(_{j+1} \); let \( C_{s+1} \) be the cer generated by the c.e. set of pairs \( C_s \cup X \) where, for the newly defined \( \gamma_{j+1}, \epsilon_{j+1} \),

\[
X = \begin{cases} 
(\omega^{\epsilon_{j+1},r(s)})^2 \cup (\{\gamma_{j+1}\} \times R_{j+1}) \cup (\{\epsilon_{j+1}\} \times (R_{(j+1)} \oplus R_{(j+1)}))(0,1), & \text{if Edge}_{j+1} \text{ is binding,} \\
(\omega^{\epsilon_{j+1},r(s)})^2 \cup (\{\gamma_{j+1}\} \times R_{j+1}), & \text{otherwise,}
\end{cases}
\]

(Notice that the pairs in \( \omega^{\epsilon_{j+1},r(s)} \) all \( C \)-collapse.) Declare Dark\(j \) satisfied (it will never become unsatisfied again), as Dark\(j \) has \( C \)-collapsed two distinct numbers of \( W_j \). Notice that Dark\(j \) injures all lower priority Code- and Edge-requirements: the injured highest
priority Code-requirement (namely, Code\textsubscript{j+1}) starts anew on the fresh column \(r(s)\), and, if binding, Edge\textsubscript{j+1} starts anew on the next column.

(2) If no Dark\textsubscript{j} requires attention then let \(i\) be the least number such that \(\gamma_i(s)\) is undefined. Define \(\gamma_i(s + 1) = r(s)\); this determines \(\epsilon_i(s + 1)\), and \(r(s + 1)\) too. Define \(C_{s+1}\) be the c.e. set of pairs \(C_s \cup X\) where, for the newly defined \(\gamma_i = \gamma_i(s + 1)\) and \(\epsilon_i = \epsilon_i(s + 1)\),

\[
X = \begin{cases} 
C_s \cup ((\gamma_i) \times R_i) \cup ((\epsilon_i) \times (R_{(i)_0} \oplus R_{(i)_1}))/(0,1), & \text{if Edge}_i \text{ is binding,} \\
C_s \cup ((\gamma_i) \times R_i), & \text{otherwise.}
\end{cases}
\]

**Verification.** We first observe that for every \(i\), \(\gamma_i = \lim_s \gamma_i(s)\) and \(\epsilon_i = \lim_s \epsilon_i(s)\) exist: by the way we define \(\epsilon_i\) we need in fact only show that \(\lim_s \gamma_i(s)\) exists, as \(\epsilon_i = \gamma_i + 1\) if Edge\textsubscript{(i)\textsubscript{0}(i)\textsubscript{1}} is binding, and \(\epsilon_i = \gamma_i\) otherwise. This is easily seen by induction. The claim is trivial if \(i = 0\) as for every \(s\), \(\gamma_0(s) = 0\).

Suppose now that the claim is true of every \(j \leq i\), and let \(s_0\) be the least stage such that for every \(s \geq s_0\), no such \(\gamma_j\) changes at \(s\). Consider the requirement Dark\textsubscript{j}. If it has already been satisfied by stage \(s_0\), or it will never act, then for every \(s \geq s_0\) \(\gamma_{j+1}(s) = \epsilon_j + 1\), which immediately determines also the final value of \(\epsilon_{j+1}\); on the other hand, if at some least stage \(s_1 \geq s_0\) Dark\textsubscript{j} acts, it becomes satisfied, by collapsing to a single class the column block \(\omega^{(\epsilon_j+1,r(s_1))}\); then \(\gamma_{j+1}(s) = r(s_1)\) for every \(s \geq s_1\), and this determines also the final value of \(\epsilon_{j+1}\) as well.

Finally we prove that \(C\) is dark, by showing that each Dark\textsubscript{i} is satisfied. Suppose that \(W_i\) is infinite and let \(s_0\) be a stage such that \(\gamma_i\) and \(\epsilon_i\) never change after \(s_0\). Thus the columns \(\omega^{[j]}\) with \(j \leq \epsilon_i\) are partitioned in coding columns, and column blocks, and the intersection of \(C\) with these columns will never change after \(s_0\). Call \(E\) this intersection: clearly there is a computable bijection \(f\) of \(\omega^{[0,\epsilon_i]}\) onto \(\omega\) under which \(E\) is translated into a c.e. set \(W = f(W_i)\). If \(W_i\) were an infinite transversal of \(C\) and did not contain infinitely many elements in the complement of \(\omega^{[0,\epsilon_i]}\), then \(W\) minus a finite set would be an infinite transversal of \(E\)', which would contradict the darkness of \(E\). Thus if \(W_i\) is an infinite transversal of \(C\), then there are distinct \(x, y \in W_i\) such that \(x, y \notin \omega^{[0,\epsilon_i]}\). But then, unless already \(x C y\), at some \(s \geq s_0\) Dark\textsubscript{i} would require attention for the sake of some such pair \(x, y\), and thus Dark\textsubscript{i} would collapse such a pair \(x, y\). In any case we conclude that \(W_i\) is not a transversal of \(C\).

\[\square\]

**Remark 4.2.** Notice that instead of choosing 0, 1 when adding \((R_i \oplus R_j)/(0,1)\) to \(C\) we could have chosen any pair \((x, y)\) with \(x\) even and \(y\) odd, and added \((R_i \oplus R_j)/(x, y)\). The argument in the previous proof relies on the fact that we can apply Lemma 3.11 which works as long as the equivalence classes of \(x\) and \(y\) are not computable: but \(R_i\) and \(R_j\) are dark minimal and thus by Observation 3.4 all equivalence classes are non-computable.

**Remark 4.3.** In the proof of Theorem 4.1 we start with a uniform c.e. sequence of \(\leq_{\text{I}}\)-incomparable dark minimal ceers \(\{R_i \mid i \in \omega\}\), as we are tacitly assuming that we need to code a graph with infinitely many vertices. If we need to code a finite graph, we can simply code a finite sequence of incomparable dark minimal ceers \(R_i\) along with ceers \((R_i \oplus R_j)/(0,1)\) to code edges between \(i\) and \(j\), and we needn’t even have any Dark\textsubscript{i} requirements. In this case the I-degree c is the I-degree of the uniform join of finitely many dark ceers, and thus it is automatically dark.
5. DEFINING \((\mathbb{N}, +, \times)\) IN THE PARTIAL ORDER \(\text{Dark}/\mathcal{I}\) WITHOUT PARAMETERS

We are now ready to show how to give a definition of \((\mathbb{N}, +, \times)\) without parameters.

**Definition 5.1.** Let \(\mathcal{P} = (P, \leq)\) be a poset, and \(n \geq 1\). An relation \(R \subseteq P^n\) is said to be **definable in** \(\mathcal{P}\) if there is a first order formula \(\varphi(\vec{x})\) in the language of posets (with \(\vec{x}\) an \(n\)-tuple of variables, and all free variables of \(\varphi\) are in \(\vec{x}\)) such that, for every \(\vec{a} \in P^n\),

\[
R(\vec{a}) \iff \mathcal{P} \models \varphi(\vec{a}).
\]

**Corollary 5.2.** There are definable relations \(V(x, c), E(x, y, c), NI(x, c)\) on \(\text{Dark}/\mathcal{I}\) such that if \(c\) is a dark \(\mathcal{I}\)-degree then

- \(G_c = \{x \mid V(x, c)\}\),
- \(E(x, y, c)\) if and only if \(x, y \in G_c\) and there is an edge between \(x\) and \(y\),
- \(\{x \mid NI(x, c)\}\) is the set on non-isolated vertices of \(G_c\).

**Proof.** Immediate as the definitions of vertices and edges in Definition 4.1 are given in terms of minimality and existence of strong minimal covers for pairs, which are first order properties in the language of posets.

**Remark 5.3.** From the previous corollary, we see how to effectively translate any sentence \(\sigma\) in the language of graph theory (just the binary edge relation) into a formula \(\hat{\sigma}(w)\) of posets with free variable \(w\) such that for every graph \(G\) there is a dark \(\mathcal{I}\)-degree \(c\) for which, for every such \(\sigma\),

\[
G \models \sigma \iff \text{Dark}/\mathcal{I} \models \hat{\sigma}(c).
\]

We will refer to the following result:

**Theorem 5.1.** There is a computable graph \(G\) without isolated vertices in which \(\text{Th}(\mathbb{N}, +, \times)\) is first order definable, that is there are first-order formulas \(U, \varphi_+, \varphi_\times\) in the language of graphs defining respectively the subset which is the universe of the copy of \(\mathbb{N}\) and the operations +, \(\times\) in \(G\).

**Proof.** See item 1(c)14 of the list of def-complete structures of [19], or see [17, Theorem 5.5.1].

**Remark 5.4.** In view of Theorem 5.1 in coding \(G\) in \(\text{Dark}/\mathcal{I}\) we only need the subset of the vertices of \(G_c\) which is comprised of the non-isolated vertices (all of them being dark minimal). Since by Corollary 5.2 this subgraph is recognizable in a first order way from parameter \(c\), henceforth we shall use the symbol \(G_c\) to denote this subgraph, so that \(G_c\) is henceforth understood to be without isolated vertices and \(G \cong G_c\).

Fix a graph \(G\) as in Lemma 5.1. So there are first-order formulas \(U, \varphi_+, \varphi_\times\) and a mapping \(\sigma \mapsto \sigma^o\) from arithmetical formulas to formulas in the language of graphs so that (where \(\equiv\) denotes syntactic equality) \((+(x, y, z))^\circ := \varphi_+(x, y, z), (x(x, y, z))^\circ := \varphi_\times(x, y, z)\), the mapping \(\circ\) commutes (modulo \(\equiv\)) with propositional connectives, \((\forall x)^\circ := (\forall x)(U(x) \to \sigma^o), (\exists x)^\circ := (\exists x)(U(x) \land \sigma^o)\), and finally for every sentence \(\sigma, \mathbb{N} \models \sigma\) if and only if \(G \models \sigma^o\).

**Remark 5.5.** Using Corollary 5.2, we see that the following binary relation \(U^c(x)\), and quaternary relations \(\varphi^+_c(x, y, z), \varphi^-_c(x, y, z)\) (corresponding to \(U(x), \varphi_+(x, y, z), \varphi_\times(x, y, z)\) mentioned above) are definable in \(\text{Dark}/\mathcal{I}\):

- \(U^c(x)\) if and only if \(G_c \models U(x)\) (henceforth let \(U^c = \{x \mid U^c(x)\}\)),
- \(\varphi^+_c(x, y, z)\) if and only if \(G_c \models \varphi_+(x, y, z)\),
Now that we have constructed $Z$, since there are only finitely many direct summands in $f$, and each of these are dark, $f$ is dark: see Remark 4.3. Let $f$ denote the $\mathcal{I}$-degree of $f$.

- $\varphi^c_\mathcal{I}(x, y, z)$ if and only if $G_c \models \varphi(x, y, z)$.

For every $c \in \text{Dark}_\mathcal{I}$ we can regard the triple $(U^c, \varphi^c_+, \varphi^c_\mathcal{I})$ as a structure for the arithmetical language $+\times$. From these, we also have definable relations $\varphi^c_\mathcal{I}(x, y), \leq^c (x, y), 0^c(x)$ in $\text{Dark}_\mathcal{I}$, corresponding to the formulas defining in $G$ the successor operation, the natural ordering on $\mathbb{N}$, and the number 0, respectively.

**Definition 5.6.** A $c \in \text{Dark}_\mathcal{I}$ is a good code if, in $G_c$, $(U^c, \varphi^c_+, \varphi^c_\mathcal{I})$ gives a model of Robinson’s system $Q$.

**Corollary 5.7.** The set of good codes is definable in the $\text{Dark}_\mathcal{I}$ degrees.

**Proof.** This immediately follows from the fact that $Q$ is finitely axiomatizable. \hfill $\square$

**Remark 5.8.** If $c$ is a good code then for the sake of simplicity without loss of generality we may assume that $\varphi^c_\mathcal{I}, \varphi^c_{+}, \varphi^c_\mathcal{I}$ are in fact operations in $U^c$, and $0^c$ is a distinguished element of $U^c$.

**Definition 5.9.** For any pair of dark minimal $\mathcal{I}$-degrees $(x, y)$, we say that a graph of the form $x \not\in a \not\in b \not\in d \not\in y$ and $a \not\in b \not\in c \not\in a$ (where $a, b, c, d$ are pairwise distinct, and distinct from $x, y$) is a graph-label for the pair $(x, y)$. See Figure 1.

![Figure 1. A graph-label for the pair (x, y)](image_url)

**Definition 5.10.** Given a set $F = \{(x_i, y_i) \mid i < n\}$ of pairs of dark minimal $\mathcal{I}$-degrees such that all the $x_i$’s are pairwise distinct and all the $y_i$’s are pairwise distinct, we say that an $f \in \text{Dark}_\mathcal{I}$ is a name for $F$ if $G_f$ is comprised of the union of graph-labels for the pairs $(x_i, y_i)$, where the various quadruples $a, b, c, d$ are distinct for each pair, and not appearing in $F$.

**Lemma 5.11.** If $F$ is a finite set of pairs of ceers, each of which is either dark minimal or is $\mathcal{I}$-dark minimal with a non-computable class, then there is a dark $\mathcal{I}$-degree which is a name for the set of $\mathcal{I}$-degrees corresponding to the pairs of ceers in $F$.

**Proof.** Fix a pair $(X, Y)$ in $F$. Without loss of generality, we assume that $[0]_X$ and $[0]_Y$ are non-computable. Let $A, B, C, D$ be distinct dark minimal ceers with $\mathcal{I}$-degrees not mentioned in $F$, and targeted only for this pair. Then we construct

$$Z_{(X,Y)} = X \oplus A \oplus D \oplus Y \oplus B \oplus C \oplus (X \oplus A)/(0,1) \oplus ((A \oplus D)/(0,1)) \oplus ((D \oplus Y)/(0,1)) \oplus ((A \oplus B)/(0,1)) \oplus ((B \oplus C)/(0,1)) \oplus (A \oplus C)/(0,1)).$$

Now that we have constructed $Z_{(X,Y)}$ for each pair $(X, Y) \in F$, define

$$f = \bigoplus_{(X,Y) \in F} Z_{(X,Y)}.$$
First we check that we have no unwanted vertices, i.e. the only minimal $I$-degrees below $f$ are equal to the $I$-degrees of the ceers $X, Y, A, B, C, D$ that we placed there: If $R$ has minimal $I$-degree and $R \leq_I f$, then $R \leq f \oplus \text{Id}_n$ for some $n$ and by Fact 2.7 $R \equiv_I \oplus \text{Id}_j E_j$ where each $E_j$ is $\leq$ one of the summands in the definition of $f$, i.e. $E_j$ is either $\leq_I$ some dark minimal ceer or $E_j$ is $\leq_I (S \oplus T)/(0,1)$ where $S$ and $T$ are in dark minimal $I$-degrees. Thus by Lemma 3.10, each $E_j$ is either finite or its $I$-degree is $\geq_I$ a dark minimal $I$-degree which is $I$-equivalent to the $I$-degree of one of the summands. Thus since $R$ is not finite, the $I$-degree of one of the $E_j$ is $\geq_I$ one of the dark minimal $I$-degree of one of the summands, and since $R$ has minimal $I$-degree, $R$ is $I$-equivalent to one of the summands.

Next we check that $f$ codes exactly the edges we intended. Since the equivalence class of 0 is non-computable in all of the ceers $X, A, D, Y, B, C$ that we consider, if we place columns for $X, A,$ and $(X \oplus A)/(0,1)$, we have ensured that the $I$-degrees of $X$ and $A$ have two $I$-incomparable $I$-strongly minimal covers below $f$. Similarly for the pairs $(A, D), (D, Y), (A, B), (B, C),$ and $(A, C)$. Thus by Lemmas 3.9 and 3.10 $f$ successfully codes every edge that we intended. Suppose now the $I$-degree of $X$ is an $I$-strongly minimal cover below $f$ of the $I$-degrees of the pair $R_1$ and $R_2$, which are minimal $I$-degrees below $f$ between which we did not explicitly code an edge. We may assume that $R_1$ and $R_2$ are among the summands we used to create $f$, in particular, the equivalence class of 0 is non-computable. Then consider the pair of reductions $R_1 \leq f \oplus \text{Id}_n$ and $R_2 \leq f \oplus \text{Id}_n$, which we get by composing the reductions $R_1 \leq X \oplus \text{Id}_k \leq f \oplus \text{Id}_n$ and $R_2 \leq X \oplus \text{Id}_k \leq f \oplus \text{Id}_n$, respectively. Since the $R_1$-equivalence class of 0 is not computable, its image under the reduction $R_1 \leq f \oplus \text{Id}_n$ must be in some column of $f$ (not in $\text{Id}_n$). Let $W$ be the set of elements whose image is not in the same column under the reduction $R_1 \leq f \oplus \text{Id}_n$. Since $[0]_R_1$ is not computable and it is not intersected by $W$, we must have that $W$ contains only finitely many $R_1$-classes. Thus $R_1 \leq_I$ this one column of $f$. Similarly for $R_2$. Since we did not explicitly code an edge between $R_1$ and $R_2$, the columns of $f$ which are $\geq_I R_1$ and $\geq_I R_2$ are not the same column, so we see that $R_1 \oplus R_2 \leq_I X$. Thus, since the $I$-degree of $X$ is assumed to be an $I$-strongly minimal cover of the $I$-degrees of the pair $R_1, R_2$, we have $X \equiv_I R_1 \oplus R_2$. Thus there can only be one $I$-strongly minimal cover $\leq_I f$ of any pair of minimal $I$-degrees aside from the pairs where we intended to place an edge. Thus there are no unwanted edges.

Therefore the $I$-degree $f$ of $f$ is the desired name for $F$. \hfill $\square$

**Lemma 5.12.** There are relations $\varepsilon^f(x, y), x \in \text{domain}^f, x \in \text{range}^f$ definable in $\text{Dark}_{/I}$ such that, if $f$ is a name for a set $F$ of pairs as in Lemma 5.11 then $F = \{(x, y) \mid \varepsilon^f(x, y)\}$, $\{x \mid \text{domain}^f(x)\} = \text{domain}(F)$ and $\{x \mid \text{range}^f(x)\} = \text{range}(F)$.

**Proof.** Immediate by the previous remarks on definability, and the fact that in a graph-label for the pair $(x, y)$, $x$ and $y$ are the only two vertices in exactly one edge, and we can distinguish in a first order way $x$ as the first component of the ordered pair since the vertex adjacent to it has three adjacent vertices, whereas the vertex adjacent to $y$ has only two adjacent vertices. \hfill $\square$

In the following let us write $\text{domain}^f = \{x \mid \text{domain}^f(x)\}$ and $\text{range}^f = \{x \mid \text{range}^f(x)\}$.

For $I$-degrees $a, b, c$ define $[a, b]^{U^c} = \{x \in U^c \mid a \leq^c x \leq^c b\}$.

This is clearly a ternary relation in $a, b, c$, which is definable in $\text{Dark}_{/I}$.

**Definition 5.13.** On pairs of dark $I$-degrees we define the equivalence relation $(c, d) \sim (c', d')$ if the two pairs coincide, or $c$ and $c'$ are good codes, $d \in U^c$ and $d' \in U^c$, and there exists a name
Lemma 5.16. \( f \) for a set of pairs \( F \) which is an order-preserving bijection between \([0^e, d]^U^c\) and \([0^e', d']^U^c\), i.e. there exists \( f \) such that \([0^e, d]^U^c = \text{domain}^f\), \([0^e', d']^U^c = \text{range}^f\) and, for all \((x, y), (x', y')\) such that \(\in^f (x, y)\) and \(\in^f (x', y')\) we have that \(x \leq^c x'\) if and only if \(y \leq^c y'\).

Lemma 5.14. The relation \( \sim \) is definable in the collection of dark \( I \)-degrees.

Proof. Use Remark 5.5 and Lemma 5.12. \( \square \)

Finally, we define:

Definition 5.15. Let \( N \) be the set of \( \sim \)-equivalence classes of pairs \((c, d)\) of dark \( I \)-degrees so that for every good code \( c' \), there exists a \( d' \) so that \((c, d) \sim (c', d')\).

Lemma 5.16. \( N \) is definable in the dark \( I \)-degrees.

Proof. By Corollary 5.7 and Lemma 5.14. \( \square \)

Lemma 5.17. Let \( c \) be a good code so that \( (U^c, \varphi^c_+, \varphi^c_\times) \simeq (\mathbb{N}, +, \times) \). Then \( N = \{ [(c, d)]_\sim \mid d \in U^c \} \) (where \([ (c, d) ]_\sim \) denotes the equivalence class of the pair \((c, d)\) under \( \sim \)).

Proof. Every model of Robinson’s \( Q \) has a standard part isomorphic to \((\mathbb{N}, +, \cdot)\). Note that if \( R \) is in a degree in \( U^c \) for any good \( c' \), then \( R \) must have a non-computable class by Lemma 3.13. By Lemma 5.11 this shows that \( \{ [(c, d)]_\sim \mid d \in U^c \} \subseteq N \). For the converse, if \( \{(c', d')\}_\sim \in N \), then by definition of \( N \), there must be some \( d \in U^c \) so that \((c', d') \sim (c, d)\). Thus \( N \subseteq \{ [(c, d)]_\sim \mid d \in U^c \} \).

This allows us to define, without parameters, + and \( \times \) on \( N \). For instance,

Definition 5.18. Let \( c \) be a good code so that \( (U^c, \varphi^c_+, \varphi^c_\times) \simeq (\mathbb{N}, +, \times) \). We define \([ (c, d) ]_\sim + [ (c', d') ]_\sim = [ (c, \varphi^c_+(d, d')) ]_\sim \), and similarly for the other operation.

Lemma 5.19. The definition of + and \( \times \) on \( N \) does not depend on the choice of \( c \) and is definable (without parameters) in the partial order of \( \text{Dark}_{I} \)-degrees.

Proof. The first claim is immediate from the definitions. We can define addition by saying that \([ (c, d) ]_\sim + [ (c', d') ]_\sim = [ (c'', d'') ]_\sim\) if and only if there exists a good code \( \hat{c} \) and \( e, e', e'' \in U^c \) so that \((c, d) \sim (\hat{c}, e), (c', d') \sim (\hat{c}, e'), (e'', d'') \sim (\hat{c}, e'')\), and that \( \varphi^c_+(e, e') = e'' \).

It is immediate that \((N, +, \times)\) is isomorphic to \((\mathbb{N}, +, \times)\), and thus we have proved:

Theorem 5.2. There is an interpretation of \((\mathbb{N}, +, \times)\) in the partial order \( \text{Dark}_{I} \). Thus \( \text{Th}(\text{Dark}_{I}) \) is computably isomorphic to \( \text{Th}^1(\mathbb{N}) \).

Proof. We have just shown that \( \text{Th}^1(\mathbb{N}) \leq_1 \text{Th}(\text{Dark}_{I}) \). On the other hand, \( \text{Th}(\text{Dark}_{I}) \leq_1 \text{Th}^1(\mathbb{N}) \) by Lemma 2.9. \( \square \)

Corollary 5.20. \( \text{Th}(\text{Dark}) \) is computably isomorphic to \( \text{Th}^1(\mathbb{N}) \).

Proof. \( I \)-equivalence is definable in \( \text{Dark} \) by Lemma 2.8 thus we have \( \text{Th}^1(\mathbb{N}) \leq_1 \text{Th}(\text{Dark}_{I}) \leq_1 \text{Th}(\text{Dark}) \leq_1 \text{Th}^1(\mathbb{N}) \). \( \square \)

Corollary 5.21. \( \text{Th}(\text{Ceers}) \) is computably isomorphic to \( \text{Th}^1(\mathbb{N}) \).
Proof. The dark ceers are definable in the structure of ceers [3 Corollary 8.1], thus we have
\[
\text{Th}^1(N) \leq_1 \text{Th}(\text{Dark}_\mathcal{I}) \leq_1 \text{Th}(\text{Dark}) \leq_1 \text{Th}(\text{Ceers}) \leq_1 \text{Th}^1(N)
\]
\[\square\]

Finally,

**Theorem 5.3.** The theory of the partial order of \(\text{Ceers}_\mathcal{I}\) is 1-equivalent to the theory of true arithmetic.

*Proof.* For any \(\mathcal{I}\)-degree \(c\), we use the same definition for graphs \(G_c\) (i.e. \(U^c\) is comprised of all \(\mathcal{I}\)-minimal ceers, and edges are witnessed by two \(\mathcal{I}\)-strong minimal covers below \(c\)), relations \(U^c, \varphi^c_+, \varphi^c_-\), and the notion of good code (which were defined from the graph). We have already constructed a \(c\) so that \((U^c, \varphi^c_+, \varphi^c_-) \simeq (\mathbb{N}, +, \times)\).

We define a new equivalence relation \(\sim\) on pairs \((c, d)\). The reason we need a new notion is that our Lemma [5.11] does not let us build a name for \(F\) if \(\text{Id}\) is among the ceers in \(F\). The \(\mathcal{I}\)-minimal degrees in the structure of \(\text{Ceers}_\mathcal{I}\) are exactly the dark \(\mathcal{I}\)-minimal degrees along with one more: \(\text{Id}\), the \(\mathcal{I}\)-degree of the identity relation \(\text{Id}\).

Given any function \(F\) from \(U^c\) to \(U^{c'}\), we define another function \(F'\) as follows: \((d, d') \in F'\) if and only if either

- \(d = 0^c\) and \(d' = 0^{c'}\)
- there are \(e\) and \(e'\) so that \(d = S^c(e)\), \(d' = S^{c'}(e')\), and \(F(e) = e'\).

We define \((c, d) \sim (c', d')\) if there exists a name \(f\) for a function \(F\) so that \(G := F \cup F' \cup F''\) is a function from \(U^c\) to \(U^{c'}\), \([0^c, d]^{U^c} \subseteq \text{domain}(G), [0^{c'}, d']^{U^{c'}} \subseteq \text{range}(G)\), and \(G\) is order-preserving.

We similarly define \(N\) to be the set of \(\sim\)-classes of pairs of \(\mathcal{I}\)-degrees \((c, d)\) so that for every good code \(c'\), there exists a \(d'\) so that \((c, d) \sim (c', d')\).

**Lemma 5.22.** Let \(c\) be a good code so that \((U^c, \varphi^c_+, \varphi^c_-) \simeq (\mathbb{N}, +, \times)\). Then \(N = \{[(c, d)]_\sim | d \in U^c\}\).

*Proof.* Every model of Robinson’s \(Q\) has standard part isomorphic to \((\mathbb{N}, +, \times)\). Note that if \(c'\) is a good code, then at most one \(\mathcal{I}\)-degree in \(U^{c'}\) can be equivalent to \(\text{Id}\) and the others must contain dark \(\mathcal{I}\)-minimal ceers which have a non-computable class. Given \(d\) in \(U^c\), and \(c'\) any good code, let \(d'\) be so that \([0^c, d]^{U^c} \cong [0^{c'}, d']^{U^{c'}}\). Let \(F\) be the set of pairs \((e, e')\) in this isomorphism for which neither \(e\) nor \(e'\) contain the ceer \(\text{Id}\). Then there is a name for \(F\) by Lemma [5.11]. Since we have only removed at most two elements from the isomorphism (one if \(\text{Id}\) is in the domain of the isomorphism and one if it is in the range), \(F \cup F' \cup F''\) will fill in the (at most 2) gaps and witnesses that \((c, d) \sim (c', d')\). Thus \(\{[(c, d)]_\sim | d \in U^c\} \subseteq N\). For the converse, if \(\{[(c', d')]_\sim \in N\), then by definition of \(N\), there must be some \(d \in U^c\) so that \((c', d') \sim (c, d)\). Thus \(N \subseteq \{[(c, d)]_\sim | d \in U^c\}\). \[\square\]

Once again, we can define \(+\) and \(\times\) on \(N\) to form our interpreted copy of \((\mathbb{N}, +, \times)\). \[\square\]

6. **Coding graphs into the partial order \(\text{Light}_\mathcal{I}\) using parameters**

We recall that the symbol \(\text{Id}\) denotes the \(\mathcal{I}\)-degree of \(\text{Id}\).
**Definition 6.1.** We say that a light $\mathcal{I}$-degree $e$ is *light minimal* if it is $\succ_\mathcal{I} \text{Id}$ and $(\text{Id}, e)_\mathcal{I}$ is empty, where $(\text{Id}, e)_\mathcal{I}$ is the interval of (light) $\mathcal{I}$-degrees $x$ such that $\text{Id} \prec_\mathcal{I} x \prec_\mathcal{I} e$.

Note that the property of being light minimal is definable in the partial order $\text{Light}_{/\mathcal{I}}$.

**Lemma 6.2.** The map $\iota : X \mapsto X \oplus \text{Id}$ induces an embedding of $\text{Dark}_{/\mathcal{I}}$ into $\text{Light}_{/\mathcal{I}}$.

**Proof.** For two dark ceers $X, Y$ we have

$$X \oplus \text{Id} \preceq_\mathcal{I} Y \oplus \text{Id} \iff X \preceq Y \oplus \text{Id},$$

the last equivalence coming from that fact that $Y \oplus \text{Id} \oplus \text{Id} \equiv Y \oplus \text{Id}$ and thus if $X \leq Y \oplus \text{Id}$ then $X \oplus \text{Id} \leq Y \oplus \text{Id}$. On the other hand, $X \leq Y \oplus \text{Id}$ if and only if $X \leq Y \oplus \text{Id}_k$ for some $k$, because the darkness of $X$ guarantees that the reduction cannot be infinite into the Id part. But since $X \leq Y \oplus \text{Id}_k$, for some $k$, is just the definition of $X \leq_\mathcal{I} Y$, we see $X \leq_\mathcal{I} Y \iff X \oplus \text{Id} \leq_\mathcal{I} Y \oplus \text{Id}$. □

**Lemma 6.3.** If $X$ is dark, $Y$ is light, and $Y \leq X \oplus \text{Id}$, then $Y \equiv \text{Id}$ or there is a dark ceer $Z \leq X$ so that $Y \equiv \text{Id}$.

**Proof.** Assume $X, Y$ as in the statement of the lemma. Since $Y \leq X \oplus \text{Id}$, by Fact 2.3 we have $Y \equiv Z \oplus Y_0$ where $Z \leq X$ (and thus $Z$ is either finite or dark) and $Y_0 \leq \text{Id}$. If $Y_0$ has only finitely many classes, then $Y \equiv Z \oplus \text{Id}_k$, but then $Z$ is dark and so is its uniform join with a finite ceer, which is impossible since $Y$ is light. It follows that $Y_0 \equiv \text{Id}$ and thus if $Z$ is finite then $Y \equiv \text{Id}$, otherwise $Y \equiv Z \oplus \text{Id}$ for some dark $Z$. □

Thus, the ceers of the form $X \oplus \text{Id}$ with $X$ dark form an initial segment in the light ceers $> \text{Id}$.

**Theorem 6.4.** Let $C$ be dark and let $c$ denote its $\mathcal{I}$-degree. Then $\text{Dark}_{/\mathcal{I}}(\leq_\mathcal{I} c) \simeq (\text{Id}, \iota(c))_{/\mathcal{I}}$.

**Proof.** Immediate from Lemma 6.2 (which shows that $\iota$ is an order-theoretic embedding) and Lemma 6.3 which gives onto-ness. □

**Definition 6.5.** For any $\text{Light}_{/\mathcal{I}}$ degree $c$, we associate a graph $H_c$ as follows:

- (vertices) the *vertices* are the light minimal degrees $\preceq_\mathcal{I} c$;
- (edges) we place an *edge* between vertices $d$ and $e$ if and only if there are two incomparable $\mathcal{I}$-degrees $a, b \preceq_\mathcal{I} c$ which are an $\mathcal{I}$-strongly minimal cover of the pair $d$ and $e$, i.e. $d, e \preceq_\mathcal{I} a, b$ and the only light $\mathcal{I}$-degrees $\prec_\mathcal{I} a$ or $\prec_\mathcal{I} b$ are $d, e$ and $\text{Id}$.

**Lemma 6.6.** For any dark ceer $c$, $H_{\iota(c)} \simeq G_c$.

**Proof.** This is immediate from Theorem 6.4 as the properties of being $\mathcal{I}$-minimal and of being a $\mathcal{I}$-strongly minimal cover are order-theoretic. □

**Corollary 6.7.** For any computable graph $G$, there is a light $\mathcal{I}$-degree $c$ so that $H_c$ is isomorphic to $G$ along with a set of isolated vertices.

**Proof.** By Theorem 4.1 and Lemma 6.6. □
7. Defining \((\mathbb{N}, +, \times)\) in the partial order \(\text{Light}_I\) without parameters

Following what we did for dark degrees in Definition 5.6 (and using the notations therein exploited), we define a light \(I\)-degree \(c\) to be good if in \(H_c\), the triple \((U^c, \varphi^c_+, \varphi^c_-)\) gives a model of Robinson’s \(Q\). Our goal is to define \(\mathcal{N}\) as in Section 5 but we have to use a different coding for finite functions. The reason is that we may have good codes which are \(\text{Light}_I\)-degrees which are not in the image of \(i\), thus we cannot use the \(i\)-image of the construction for names in \(\text{Dark}_I\).

Throughout the section an \(I\)-strongly minimal cover of an \(I\)-degree \(x\) means an \(I\)-degree \(y >_I x\) such that the interval \([\text{Id}, y)_I\) is exactly the interval \([\text{Id}, x)_I\). Since we will consider only \(I\)-strongly minimal covers of light \(I\)-degrees, they will be light as well.

**Definition 7.1.** Let \(F = \{\{a_i, b_i\} \mid i \in S\}\) be a set of pairs of light minimal \(I\)-degrees so that the \(a_i\)'s and the \(b_i\)'s are distinct (including \(a_i \neq b_j\) for any \(i, j \in S\)). We say that a light \(I\)-degree \(f\) is a name for \(F\) if the only light minimal \(I\)-degrees below \(f\) are \(\{a_i, b_i\} \mid i \in S\}\), and the \(\{a_i, b_i\}\)'s are the only pairs \(\{c, d\}\) of light minimal \(I\)-degrees less than \(f\) for which there is an \(x \prec_I f\) so the only light minimal \(I\)-degrees less than \(x\) are \(c\) and \(d\), and \(x\) has an \(I\)-strongly minimal cover \(y\), which in turn has an \(I\)-strongly minimal cover \(z\) which is \(\leq_I f\).

**Lemma 7.2.** Let \(F = \{\{a_i, b_i\} \mid i < n\}\) be a finite set of pairs of light minimal \(I\)-degrees, so that the \(a_i\)'s and the \(b_i\)'s are distinct. Then there is a name for \(F\).

**Proof.** In this proof we use that in \(\text{Ceers}_I\) every non-universal element has infinitely many distinct self-full strong minimal covers, see [3, Theorem 7.9]. For each pair \(\{a_i, b_i\} \in F\), we let \(c_i\) be a self-full \(I\)-strongly minimal cover of \(a_i \circ b_i\). Let \(d_i\) be a self-full \(I\)-strongly minimal cover of \(c_i\). Note that we choose these to be \(I\)-strongly minimal covers in \(\text{Ceers}_I\), not just in \(\text{Light}_I\). Let \(f = \bigoplus_i d_i\). First we check that for each pair \(\{a_i, b_i\}\) in \(F\), there is an \(x_i \prec_I f\) (take \(x_i = a_i \circ b_i\)) so that the only light minimal \(I\)-degrees less than \(x_i\) are \(a_i\) and \(b_i\), and \(x_i\) has an \(I\)-strongly minimal cover \(y_i\) (take \(y_i = c_i\)) which has an \(I\)-strongly minimal cover \(z_i \leq_I f\) (take \(z_i = d_i\)). To see that the only light minimal \(I\)-degrees \(\leq_I x_i\) are \(a_i\) and \(b_i\), assume that \(A_i \in a_i, B_i \in b_i, \text{ and } X_i = A_i \circ B_i\); if \(U \leq_I X_i\) has light minimal \(I\)-degree then by Fact 2.7 there exist \(U_0, U_1\) such that \(U \equiv_I U_0 \circ U_1\) with \(U_0 \leq A_i\) and \(U_1 \leq B_i\). By light minimality of \(A_i, B_i\) it follows that either \(U_0\) is finite, dark, \(U_0 \equiv_I A_i, \text{ or } U_0 \equiv_I \text{Id}\), and similarly \(U_1\) is finite, dark, \(U_1 \equiv_I B_i, \text{ or } U_1 \equiv_I \text{Id}\). If \(U_0 \equiv_I A_i\), then we see that \(A_i \leq_I U\), so by light minimality of \(U\) and \(A_i\), we have that \(U \equiv_I A_i\). Similarly if \(U_1 \equiv_I B_i\) then \(U \equiv_I B_i\). Thus we can assume neither of these cases holds. By lightness of \(U\), it follows at least one of \(U_0\) or \(U_1\) is light, so without loss of generality, we suppose \(U_0\) is light, i.e. \(U_0 \equiv_I \text{Id}\). If \(U_1\) is finite or \(\equiv_I \text{Id}\), then \(U \equiv_I \text{Id}\), contradicting \(U\) being of light minimal \(I\)-degree. Thus \(U_1\) must be dark \(\leq B_i\). So, \(U \equiv_I \text{Id} \oplus D\) for some dark \(D \leq B_i\). But since every dark ceer \(D\) has a join with \(\text{Id}, \text{ namely } D \oplus \text{Id} \) [3, Obs 5.1], it follows that \(U \equiv_I D \oplus \text{Id} \leq B_i\). Again, by light minimality of \(U\) and \(B_i\), we see \(U \equiv_I B_i\).

It remains to show that no other pair \(\leq_I f\) has such a triple \(x, y, z\). We begin with an easy observation about the \(I\)-degrees \(\leq d_i\). Together with \(A_i \in a_i, B_i \in b_i\), fix also representatives \(C_i \in c_i, D_i \in d_i\), and \(f \in f\). We also use the notation \(0_I\) to denote the least \(I\)-degree, which is the \(\equiv_I\)-class of the ceer \(\text{Id}\) with only one equivalence class, and is comprised exactly of all finite ceers. We observe also that for each \(a_i\) there can be at most one dark-\(I\) degree \(r_i \leq_I a_i\): to see this assume that \(r, s\) are dark \(I\)-degrees below \(a_i\), with representatives \(R \in r\) and \(S \in s\): as before, since \(R \oplus \text{Id}\) and \(S \oplus \text{Id}\) are joins, it is easy to see that \(r \oplus \text{Id}\) and \(s \oplus \text{Id}\) are joins of the two \(I\)-degrees, and thus they are both \(\leq_I a_i\), but \(a_i\) is light minimal, then we have \(r \oplus \text{Id} = s \oplus \text{Id} = a_i\). By
darkness of \( r \) we have that \( r \preceq_I s \oplus \text{Id} \) implies \( r \preceq_I s \); similarly, \( s \preceq_I r \). Therefore the two dark \( I \)-degrees coincide. If there is a (unique) dark \( I \)-degree \( r_i \preceq_I a_i \) then \( a_i = r_i \oplus \text{Id} \); in the following if there is no dark degree below \( a_i \) then we let \( r_i = a_i \). Similar considerations and notations hold for each \( b_i \): in particular, if there is a (unique) dark degree \( I \)-degree \( \leq b_i \), we call it \( s_i \), and if there is no dark \( s_i \preceq_I b_i \), then we let \( s_i = b_i \).

**Claim 7.3.** If \( w \preceq_I d_i \) is an \( I \)-degree, then \( w \in \{0_I, \text{Id}, a_i, b_i, r_i, s_i, r_i \oplus b_i, a_i \oplus s_i, a_i \oplus \text{Id}, b_i \oplus \text{Id}, a_i \oplus b_i, c_i, d_i \} \).

*Proof.* If \( W \preceq_I D_i \), then \( W \equiv_I D_i \) or \( W \equiv_I C_i \) or \( W \preceq_I A_i \oplus B_i \) since \( d_i \) is an \( I \)-strongly minimal cover of \( c_i \) which is an \( I \)-strongly minimal cover of \( a_i \oplus b_i \). If \( W \preceq_I A_i \oplus B_i \), then by Fact 2.7, \( W \equiv_I W_0 \oplus W_1 \) where \( W_0 \leq A_i \) and \( W_1 \leq B_i \). Since each \( a_i \) and \( b_i \) is light minimal, the only possibilities for the \( I \)-degree of \( W_0 \) are \( a_i \), \( r_i \), \( \text{Id} \), or \( 0_I \), and the only possibilities for the \( I \)-degree of \( W_1 \) are \( b_i \), \( s_i \), \( \text{Id} \), or \( 0_I \). The possible direct sums of these are easily seen to be the possibilities listed in the claim. \( \square \)

Suppose that the \( I \)-degree \( x \) of an \( X \preceq_I f \) bounds only \( a_i \) and \( a_j \), with \( i \neq j \) (it is no different if we consider \( a_i \) and \( b_j \) or \( b_i \) and \( b_j \)). Since \( X \preceq_I f \equiv_I \bigoplus D_i \), we see by Fact 2.7 that \( X \equiv_I \bigoplus X_r \) where each \( X_r \leq D_r \). Thus each \( X_r \) is \( I \)-equivalent to one of the \( I \)-degrees in the list: \( 0_I, \text{Id}, a_i, b_i, r_i, s_i, r_i \oplus b_i, a_i \oplus s_i, a_i \oplus \text{Id}, b_i \oplus \text{Id}, a_i \oplus b_i, c_i, d_i \). Since \( x \) only bounds \( a_i \) and \( a_j \), for the \( I \)-degrees of the various \( X_r \) we must rule out the possibilities that any such degree bounds in the above list a light minimal \( I \)-degree not in \( \{a_i, a_j\} \); finally we can remove from the list the \( I \)-degrees of \( a_i \oplus s_i \), and \( a_j \oplus s_j \); note for example that \( a_i \oplus s_i \geq_I \text{Id} \oplus s_i = b_i \), so this one cannot be the \( I \)-degree of \( X_r \), unless \( b_i \in \{a_i, a_j\} \). Then we conclude that the \( I \)-degree of any of the \( X_r \)'s is equal to one of the \( I \)-degrees in the following list: \( 0_I, \text{Id}, a_i, r_i, b_i, s_i, a_i \oplus \text{Id}, a_j \oplus b_i, a_i \oplus b_i, c_i, d_i \), for some \( k \).

We write \( a_i \) as either \( r_i \) or \( r_i \oplus \text{Id} \), depending on which is true. Similarly for \( a_j \). It then follows that the only possibility for \( x \) is to be of the form \( r_i \oplus r_j (\oplus r_k) (\oplus s_k) (\oplus \text{Id}) \) where parentheses are meant to symbolize that we may or may not be joining with these degrees. But we observe that any degree \( u \geq_I a_i \) and \( \geq_I r_k \) for any \( k \), is also \( \geq_I a_k \), as \( u \) is light (hence \( \text{Id} \leq_I u \)) and thus \( \geq_I a_k \), as in this case \( a_i \) is the join of \( \text{Id} \) and \( r_k \). Since \( x \) bounds only \( a_i \) and \( a_j \), this is impossible if \( k \notin \{i, j\} \). Thus, we are left with the only possibilities for \( x \) being \( r_i \oplus r_j (\oplus \text{Id}) \). Now, \( x \) bounded \( a_i \) and \( a_j \) and no other light minimal \( I \)-degrees, and \( y \) was an \( I \)-strongly minimal cover of \( x \), then it would also bound no other light minimal degrees, and \( z \) being an \( I \)-strongly minimal cover of \( y \) would also bound no other light minimal \( I \)-degrees: on the other hand, the same argument as before applied to \( y \) and \( z \) would show that also \( y \) and \( z \) may only have the form \( r_i \oplus r_j (\oplus \text{Id}) \). But there are at most two such \( I \)-degrees, contradicting the fact that \( x, y, z \) are three distinct \( I \)-degrees.

Thus \( f \) is a name for \( F \).

**Definition 7.4.** Given two good codes \( c, c' \) for graphs in \( \text{Light}_I \) (thus models of Robinson’s \( Q \)), we say that a pair of names \((f, g)\) is a *label* for a partial function \( H : H_c \rightarrow H_{c'} \) if \( f \) is a name for a set \( F \) and \( g \) is a name for a set \( G \) and whenever \( a \preceq_I c \) is light minimal, then there is a light minimal \( I \)-degree \( b \) so that \( \{a, b\} \) is a pair in \( F \) and \( b \preceq_I c, c' \), and \( \{b, H(a)\} \) is a pair in \( G \).

**Lemma 7.5.** If \( H \) is a finite function between \( H_c \) and \( H_{c'} \), with \( c, c' \) light \( I \)-degrees, there exists a pair \((f, g)\) which is a label for this function.
Proof. We need to use light minimal degrees which are not below $c \oplus c'$ to interpolate for the function. Such degrees exist, because there are infinitely many dark minimal degrees avoiding any lower cone \textsuperscript{[3]} Theorem 3.3. By Lemma \textsuperscript{[6.2]}, we just use the $\iota$ image of these: if $C,C'$ are representatives of $c,c'$ respectively, and $D \in \mathcal{I}_\mathcal{L}$ is dark minimal, then $\iota(D) \in \mathcal{I}_\mathcal{L}$ $C \oplus C'$ because $D \in \mathcal{I}_\mathcal{L}$ $C \oplus C'$, by Observation \textsuperscript{[3.4]}

The existence of the names for the needed sets of pairs is then given by Lemma \textsuperscript{[7.2]}.

**Definition 7.6.** On $\mathcal{I}$-degrees we define the equivalence relation $(c,d) \sim (c',d')$ if the two pairs coincide, or $c$ and $c'$ are good codes, $d \in U^c$ and $d' \in U^{c'}$ and there exists a pair of names $(f,g)$ which is a label for a function $H$ which is an order-preserving bijection between $[0^c,d]^{U^c}$ and $[0^{c'},d']^{U^{c'}}$, where the various symbols $U^c$, $U^{c'}$, $0^c$ and $0^{c'}$ have the same meanings as in Section 5.

This is what is needed to again define $\mathbb{N}$ exactly as in the dark case, as explained in Section \textsuperscript{[5]}

Thus we have shown that there is a copy of $(\mathbb{N},+\times)$ definable in the structure $\text{Light}_{\mathcal{I}}$, proving the following theorem:

**Theorem 7.1.** The first order theory of $\text{Light}_{\mathcal{I}}$ is computably isomorphic to true first order arithmetic.

**Corollary 7.7.** The first order theory of the Light is computably isomorphic to true first order arithmetic.

**Proof.** $\mathcal{I}$-equivalence on light $\mathcal{I}$-degrees is definable in the light degrees, by Lemma \textsuperscript{[2.8]}

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