Minimax Optimal Clustering of Bipartite Graphs with a Generalized Power Method

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Abstract

Clustering bipartite graphs is a fundamental task in network analysis. In the high-dimensional regime where the number of rows $n_1$ and the number of columns $n_2$ of the associated adjacency matrix are of different order, existing methods derived from the ones used for symmetric graphs can come with sub-optimal guarantees. Due to increasing number of applications for bipartite graphs in the high dimensional regime, it is of fundamental importance to design optimal algorithms for this setting. The recent work of Ndaoud et al. (2022) improves the existing upper-bound for the misclustering rate in the special case where the columns (resp. rows) can be partitioned into $L = 2$ (resp. $K = 2$) communities. Unfortunately, their algorithm cannot be extended to the more general setting where $K \neq L \geq 2$. We overcome this limitation by introducing a new algorithm based on the power method. We derive conditions for exact recovery in the general setting where $K \neq L \geq 2$, and show that it recovers the result in Ndaoud et al. (2022). We also derive a minimax lower bound on the misclustering error when $K = L$ under a symmetric version of our model, which matches the corresponding upper bound up to a factor depending on $K$.

1 Introduction

The interactions between objects of two different types can be naturally encoded as a bipartite graph where nodes correspond to objects and edges to the links between the objects of different type. One can find examples of such data in various fields, e.g., interactions between customers and products in e-commerce (Huang et al., 2007), interactions between plants and pollinators (Young et al., 2021), investors and assets networks (Squartini et al., 2017), judges vote predictions (Guimerà and Sales-Pardo, 2011) and constraint satisfaction problems (Feldman et al., 2015).

Clustering is one of the most important analysis tasks on bipartite graphs aimed at gathering nodes that have similar connectivity profiles. To this end, several methods have been proposed in the literature, e.g., convex optimization approaches (Lim et al., 2015), spectral methods (Zhou and A.Amini, 2019), modularity function maximization (Beckett, 2016), pseudo-likelihood (Zhou and Amini, 2020) and variational approaches (Keribin et al., 2015). The performance of the algorithms are generally evaluated under the Bipartite Stochastic Block Model (BiSBM), a variant of the Stochastic Block Model (SBM), where the partitions of the rows and the columns are decoupled. In particular, edges are independent Bernoulli random variables with parameters depending only on the communities of the nodes.

When the number of rows $n_1$ (corresponding to type I objects) and the number of columns $n_2$ (corresponding to type II objects) of the adjacency matrix associated to a bipartite graph are of the same order, the BiSBM behaves similarly to the SBM. However, in the high dimensional setting where $n_2 \gg n_1$, classical methods that are applicable when $n_1$ is of the same order as $n_2$ can fail. In particular, when the bipartite graph is very sparse, it becomes impossible to consistently estimate the latent partition of the columns, whereas it is still possible to estimate the latent partition of the rows. Hence, methods based on estimating the latent partitions of both rows and columns will necessarily fail in the high dimensional regime when the bipartite graph is very sparse. However, the high dimensional regime appears in many applications, e.g., hypergraphs where the number of columns corresponds to the number of hyperedges, or in e-commerce

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where the number of customers could be much smaller than the number of products (or vice-versa). Hence it is important to understand how to design statistically optimal algorithms in this regime.

Recently, the work of Ndaoud et al. (2022) improved the state-of-the-art conditions for exact recovery of the latent partition of the rows under the BiSBM when \( n_2 \gtrsim n_1 \log n_1 \). Unfortunately, their method, which can be understood as a generalized power method, uses a centering argument that only works in the special case where there are \( K = 2 \) latent communities for the rows and \( L = 2 \) latent communities for the columns. Moreover, they only consider a Symmetric BiSBM (SBiSBM) where the edge probabilities can take two values (see Section 2). It is not clear how their method can be extended to the more general setting where \( K \neq L \geq 2 \), and with more general connectivity matrices. To overcome this limitation, we propose a new algorithm – also based on the generalized power method – that can be applied to general BiSBMs, and has similar theoretical guarantees when specialized to the setting of Ndaoud et al. (2022).

### 1.1 Main contributions

Our results can be summarized as follows.

- We present a novel iterative clustering method that can be applied to general BiSBMs, unlike the one proposed by Ndaoud et al. (2022). We analyze our algorithm under the BiSBM without restrictions on \( K \) and \( L \), and derive an upper bound on the misclustering error. In particular, we show that our algorithm achieves exact recovery when the sparsity level \( p_{max} \) of the graph satisfies \( p_{max}^2 = \Omega(\log n_1 n_2) \) for fixed \( K, L \). This is the same sparsity regime obtained in Ndaoud et al. (2022) for the SBiSBM with \( K = L = 2 \). We remark that our bounds are non-asymptotic and showcase the full dependence on \( K, L \).

- We derive a minimax lower bound for the misclustering error in the special case of an SBiSBM with \( L = K = 2 \) that matches the corresponding upper bound of our algorithm, and is the first such lower bound for this problem. It completes the work of Ndaoud et al. (2022) which only shows that an oracle version of their algorithm fails to achieve exact recovery when \( p_{max}^2 \leq \frac{\log n_1 n_2}{n_1 n_2} \) where \( \epsilon \) is a small enough constant. We also show that the above analysis extends to the case \( K = L \geq 2 \), and that the ensuing lower bound matches the upper bound up to a factor depending on \( K \).

- As part of our analysis, we derive a concentration inequality for matrices with independent centered Binomial entries (see part 3 of Lemma 5) that could be of independent interest.

Our findings are complemented by numerical experiments on synthetic data.

### 1.2 Related work

A clustering strategy based on the maximum a posteriori probability (MAP) estimate for discrete weighted bipartite graphs that can encompass the BiSBM as a special case has recently been proposed by Jo and Lee (2021). However, their method requires estimating the latent partition of the columns and hence requires the sparsity level of the graph \( p_{max} \) to satisfy \( p_{max} \gtrsim \frac{\log n_1 n_2}{n_1 n_2} \). In contrast, we need a far weaker condition \( p_{max} \gg \frac{1}{\sqrt{n_1 n_2}} \) in the high dimensional regime. This highlights one of the difficulties we face in the high dimensional regime – while it is impossible to correctly estimate all the model parameters, it is still possible to exactly recover the row partition. A similar phenomenon is known for Gaussian mixture models, see Ndaoud (2018).

Our algorithm design is based on the Generalized Power Method (GPM) which has been applied successfully in various statistical learning problems in recent years. This includes, e.g., group synchronization (Boumal, 2016), joint alignment from pairwise difference (Chen and Candès, 2018), graph matching (Onaran and Villar, 2017), low rank matrix recovery (Chi et al., 2019) and SBM (Wang et al., 2021). The work of Ndaoud et al. (2022) which we extend in the present paper is also based on the GPM. However, there are significant differences between the algorithms. We do not need the centering step used in Ndaoud et al. (2022) and since we encode the community memberships in a \( n_1 \times K \) matrix (instead of using the sign), our algorithm can be applied when \( K > 2 \). Our algorithm is closer to the one proposed by Wang et al. (2021) for clustering graphs under the SBM. In contrast to Wang et al (2021), we do not add
any constraints on the columns, and instead of solving a linear assignment problem, we directly project on the extreme points of the unit simplex.

The GPM is also related to alternating optimization, a common strategy used to solve non-convex optimization problems in an iterative way. For example, EM-type algorithms [Dempster et al., 1977] have been used since decades. In general, these methods are not guaranteed to achieve a global optimum. However, a recent line of research has shown that under various statistical models, alternating optimization can actually lead to consistent estimators [Lu and Zhou, 2016; Chen et al., 2019; Gao and Zhang, 2022; Han et al., 2020; Chen and Zhang, 2021; Braun et al., 2021b]. Our proof techniques are based on the work of Braun et al. (2021b) which itself is based on a general framework developed by Gao and Zhang (2022).

1.3 Notation

We use lowercase letters (c, a, b, . . .) to denote scalars and vectors, except for universal constants that will be denoted by c₁, c₂, . . . for lower bounds, and C₁, C₂, . . . for upper bounds and some random variables. We will sometimes use the notation a_n ≲ b_n (or a_n ≳ b_n) for sequences (a_n)_{n≥1} and (b_n)_{n≥1} if there is a constant C > 0 such that a_n ≤ C b_n (resp. a_n ≥ C b_n) for all n. If the inequalities only hold for n large enough, we will use the notation a_n = O(b_n) (resp. a_n = Ω(b_n)). If a_n ≲ b_n (resp. a_n = O(b_n)) and a_n ≳ b_n (resp. a_n = Ω(b_n)), then we write a_n ∼ b_n (resp. a_n = Θ(b_n)).

Matrices will be denoted by uppercase letters. The i-th row of a matrix A will be denoted as A_i. The column j of A will be denoted by A_j, and the (i, j)-th entry by A_{i, j}. The transpose of A is denoted by A^T and A^T_j corresponds to the j-th row of A^T by convention. I_k denotes the k × k identity matrix. For matrices, we use ||.|| and ||.||_F to respectively denote the spectral norm (or Euclidean norm in case of vectors) and Frobenius norm. The number of non-zero entries of A is denoted by nnz(A). The vector of \( \mathbb{R}^n \) with all entries equal to one is denoted by \( 1_n \). When applied to a vector \( x \) of length \( K \), diag(x) will denote the diagonal \( K \times K \) matrix formed with the entries of \( x \).

2 The statistical framework

The BiSBM is defined by the following parameters.

- A set of nodes of type I, \( \mathcal{N}_1 = [n_1] \), and a set of nodes of type II, \( \mathcal{N}_2 = [n_2] \).
- A partition of \( \mathcal{N}_1 \) into \( K \) communities \( C_1, \ldots , C_K \) and a partition of \( \mathcal{N}_2 \) into \( L \) communities \( C'_1, \ldots , C'_L \).
- Membership matrices \( Z_1 \in \mathcal{M}_{n_1,K} \) and \( Z_2 \in \mathcal{M}_{n_2,L} \) where \( \mathcal{M}_{n,K} \) denotes the class of membership matrices with \( n \) nodes and \( K \) communities. Each membership matrix \( Z_1 \in \mathcal{M}_{n_1,K} \) (resp. \( Z_2 \in \mathcal{M}_{n_2,L} \)) can be associated bijectively with a partition function \( z : [n] \to [K] \) (resp. \( z' : [n] \to [L] \)) such that \( z(i) = z_i = k \) where \( k \) is the unique column index satisfying \( (Z_1)_{ik} = 1 \) (resp. \( (Z_2)_{ik} = 1 \)). To each matrix \( Z_1 \in \mathcal{M}_{n_1,K} \) we can associate a matrix \( W \) by normalizing the columns of \( Z_1 \) in the \( \ell_1 \) norm: \( W = Z_1 D^{-1} \) where \( D = Z_1^T 1_{n_1} \). This implies that \( W^T Z_1 = I_K = Z_1^T W \).
- A connectivity matrix of probabilities between communities

\[
\Pi = (\pi_{kk'})_{k \in [K], k' \in [L]} \in [0,1]^{K \times L}.
\]

Let us write \( P = (p_{ij})_{i,j \in [n]} := Z_1 \Pi (Z_2)^T \in [0,1]^{n_1 \times n_2} \). A graph \( G \) is distributed according to BiSBM\( (Z_1, Z_2, \Pi) \) if the entries of the corresponding bipartite adjacency matrix \( A \) are generated by

\[
A_{ij} \overset{\text{ind.}}{\sim} \mathcal{B}(p_{ij}), \quad i \in [n_1], j \in [n_2],
\]

where \( \mathcal{B}(p) \) denotes a Bernoulli distribution with parameter \( p \). Hence the probability that two nodes are connected depends only on their community memberships. We will frequently use the notation \( E \) for the centered noise matrix defined as \( E_{ij} = A_{ij} - p_{ij} \), and denote the maximum entry of \( P \) by \( \max_{i,j} p_{ij} \). The latter can be interpreted as the sparsity level of the graph. We make the following assumptions on the model.
Assumption A1 (Approximately balanced communities). The communities $C_1, \ldots, C_K$, (resp. $C'_1, \ldots, C'_L$) are approximately balanced, i.e., there exists a constant $\alpha \geq 1$ such that for all $k \in [K]$ and $l \in [L]$ we have

$$\frac{n_1}{\alpha K} \leq |C_k| \leq \frac{\alpha n_1}{K}$$

and

$$\frac{n_2}{\alpha L} \leq |C'_l| \leq \frac{\alpha n_2}{L}.$$ 

Assumption A2 (Full rank connectivity matrix). The smallest eigenvalue of $\Pi \Pi^\top$, denoted by $\lambda_K(\Pi \Pi^\top)$, satisfies $\lambda_K(\Pi \Pi^\top) \gtrsim \beta p_{\text{max}}^2$.

Remark 1. The following remarks are in order.

- In the analysis, we will need an upper bound on $\|\Pi\|$. Note that $\|\Pi\| \leq c(K, L)p_{\text{max}}$ is always satisfied for $c(K, L) = \sqrt{KL}$ since

$$\|\Pi\| \leq \|\Pi\|_F \leq \sqrt{KLp_{\text{max}}}.$$ 

But in some cases we might have a tighter bound, for e.g., if $\Pi$ is an (approximately) diagonal matrix.

- The spectral gap assumption $\lambda_K(\Pi \Pi^\top) \gtrsim \beta p_{\text{max}}^2$ is common in the spectral clustering literature (see for example [Lei and Rinaldo (2013)]). However, it should be possible to recover the $K$ communities even if $\Pi$ is rank deficient (for example if one row of $\Pi$ is proportional to another row of $\Pi$ and the gap between these rows is large enough). Unfortunately, except for the isotropic Gaussian mixture setting ([Löffler et al. (2024)]), it is not known how to analyze the spectral method without the spectral gap assumption.

- Instead of using the spectral gap assumption to quantify the signal, we could have used the minimal CH-divergence between the rows of $\Pi \Pi^\top$, similarly to the work of [Abbe and Sandon (2015)]. This approach is considerably more technical and would lead to a similar result under the Symmetric BiSBM (defined below).

Assumption A3 (Diagonal dominance). There exist $\beta > 0$ and $\eta \geq 1$ such that for all $k' \neq k \in [K]$, we have

$$(\Pi \Pi^\top)_{kk} - \alpha^2(\Pi \Pi^\top)_{kk'} \geq \beta p_{\text{max}}^2$$ and $$(\Pi \Pi^\top)_{kk} - \frac{1}{\alpha^2}(\Pi \Pi^\top)_{kk'} \leq \eta \beta p_{\text{max}}^2.$$ 

Remark 2. When all communities $C'_i$ have size equal to $n_2/L$, the first condition in Assumption A3 simplifies to $(\Pi \Pi^\top)_{kk} - (\Pi \Pi^\top)_{kk'} \geq \beta p_{\text{max}}^2$ and corresponds to a diagonal dominance assumption. The second condition in Assumption A3 is useful to show that $\beta$ is (up to a parameter $\eta$ that could depend on $L$ as discussed in the remark below) the parameter that measures the minimum difference between the diagonal and off-diagonal entries of $\Pi \Pi^\top$.

Symmetric BiSBM. A particular case of interest is where $L = K$ and $\Pi = (p - q)I_K + q1_K1_K^\top$, where $1 \geq p > q \geq 0$ and $q = cp$ for some constant $0 < c < 1$. This model will be referred to as the Symmetric BiSBM, denoted by SBiSBM($Z_1, Z_2, p, q$). For $K = 2$ this corresponds to the model analyzed in the work of [Ndaoud et al. (2022)]. Since we now have (for $k' \neq k$)

$$(\Pi \Pi^\top)_{kk} = p^2 + (K - 1)q^2$$ and $$(\Pi \Pi^\top)_{kk'} = 2qp + (K - 2)q^2$$

the following observations are useful to note.

- We have $\|\Pi\| \leq Kp_{\text{max}}$, so $c(K, L) = K$.
- We have $(\Pi \Pi^\top)_{kk} - (\Pi \Pi^\top)_{kk'} = (p - q)^2 \leq p_{\text{max}}^2$, and also $(p - q)^2 \geq (1 - c)^2p_{\text{max}}^2$. Hence Assumption A3 is satisfied for $\eta = \frac{1}{(1 - c)^2}$ and $\beta = (1 - c)^2$ in the equal-size community case ($\alpha = 1$). Additionally, $\lambda_K(\Pi \Pi^\top) = (1 - c)p_{\text{max}}$. 

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• In the unequal-size case, we need to show choices of $\beta$, $\eta$ under which Assumption [A3] holds. To this end, note that

$$(\Pi^\top)_{kk} - \alpha^2(\Pi^\top)_{kk'} = p_{\max}^2((1-c)^2 - (\alpha^2 - 1)K),$$

and

$$(\Pi^\top)_{kk} - \frac{1}{\alpha^2}(\Pi^\top)_{kk'} = p_{\max}^2((1-c)^2 - (\alpha^2 - 1)K) \leq p_{\max}^2((1-c)^2 + (\alpha^2 - 1)K).$$

Hence if, for instance, $\alpha^2 = 1 + \frac{(1-c)^2}{2K}$, we see that Assumption [A3] is satisfied for $\beta = \frac{(1-c)^2}{2}$ and $\eta = 3$.

**Remark 3.** One of the parameters $\beta$ or $\eta$ can scale with $L$ as shown in the following examples. Assume $L$ is even and $\alpha = 1$. Let us consider $p = (p_1, \ldots, p_{L/2}), q = (q_1, \ldots, q_{L/2})$ where $p_i > q_i$ for all $l \leq L/2$ and

$$\Pi = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \in [0,1]^{2 \times L}.$$ 

Then

$$L \max_{i} (p_i - q_i)^2 \geq \min_{k \neq k'} ((\Pi^\top)_{kk}) - ((\Pi^\top)_{kk'}) = \|p - q\|^2 \geq L \min_{i} (p_i - q_i)^2,$$

hence $\beta = \Theta(L)$ and $\eta = \Theta(1)$. Let us define $q' = (q_1', q_2', q_3, \ldots, q_{L/2})$ for $q_1' = 0.5q_1$ and $q_2' = 0.5q_2$. If we now consider

$$\Pi = \begin{pmatrix} p & q' \\ q' & p \end{pmatrix} \in [0,1]^{2 \times L},$$

where $q_i$ and $p_i$ are as before and $q_1' \neq q_1, q_2' \neq q_2$. Then it is easy to check that $\max_{k \neq k'} ((\Pi^\top)_{kk}) - ((\Pi^\top)_{kk'}) \leq L p_{\max}^2$ but

$$\min_{k \neq k'} ((\Pi^\top)_{kk}) - ((\Pi^\top)_{kk'}) = (\Pi^\top)_{22} - (\Pi^\top)_{23} = \frac{q_1^2 + q_2^2}{2}$$

is independent of $L$. Hence $\beta = \Theta(1)$ and $\eta = \Theta(L)$.

**Error criterion.** The misclustering rate associated to an estimated partition $\hat{z}$ is defined by

$$r(\hat{z}, z) = \frac{1}{n} \min_{\pi \in \mathcal{S}} \sum_{i \in [n]} 1_{\{\hat{z}(i) \neq \pi(z(i))\}},$$

where $\mathcal{S}$ denotes the set of permutations on $[K]$. It corresponds to the proportion of wrongly assigned nodes labels. We say that we are in the exact recovery regime if $r(\hat{Z}, Z) = 0$ with probability $1 - o(1)$ as $n$ tends to infinity. If $P(r(\hat{Z}, Z) = o(1)) = 1 - o(1)$ as $n$ tends to infinity then we are in the weak consistency or almost full recovery regime. A more complete overview of the different types of consistency and the sparsity regimes where they may occur can be found in [Abbe 2018].

**Remark 4.** As already mentioned in the introduction, we will focus in this work on the regime where $n_2 \gg n_1 \log n_1$ and $\sqrt{n_1 n_2} p_{\max} \gtrsim \sqrt{\log n_1}$. In this parameter regime there is no hope to accurately recover $Z_2$ because the columns of $A$ are too sparse. Indeed, consider the setting where $\sqrt{n_1 n_2} p_{\max} \asymp \sqrt{\log n_1}$. Then in expectation, the sum of the entries of each column is $n_1 p_{\max} \asymp \sqrt{n_1 \log n_1 n_2} \to 0$. But by analogy to the SBM, we would need a condition $n_1 p_{\max} \to \infty$ in order to recover $Z_2$. This is actually a necessary condition obtained in a related setting by [Jo and Lee 2021]. While it is impossible to estimate $Z_2$ in this sparsity regime, it is still possible to accurately estimate $Z_1$. We will focus on this problem from now onwards.

## 3 Algorithm

### 3.1 Initialization with a spectral method

We can use a spectral method on the hollowed Gram matrix $B$ to obtain a first estimate of the partition $Z_1$. This is similar to the algorithm in [Florescu and Perkins 2016]; we only use a different rounding step so the algorithm can be applied to general bipartite graphs with $K > 2$ communities, in contrast to [Florescu and Perkins 2016].
Algorithm 1 Spectral method on \( \mathcal{H}(AA^\top) \) (Spec)

**Input:** The number of communities \( K \) and the adjacency matrix \( A \).

1. Form the diagonal hollowed Gram matrix \( B := \mathcal{H}(AA^\top) \) where \( \mathcal{H}(X) = X - \text{diag}(X) \).
2. Compute the matrix \( U \in \mathbb{R}^{n_1 \times K} \) whose columns correspond to the top \( K \)-eigenvectors of \( B \).
3. Apply approximate \( (1 + \bar{\epsilon}) \) approximate \( k\)-means on the rows of \( U \) and obtain a partition \( z^{(0)} \) of \([n_1]\) into \( K \) communities.

**Output:** A partition of the nodes \( z^{(0)} \).

**Computational complexity of Spec.** The cost for computing \( B \) is \( O(n_1 \text{nnz}(A)) \) and for \( U \) is \( \mathcal{O}(n_1^2 K \log n_1) \). Applying the \( (1 + \bar{\epsilon}) \) approximate \( k\)-means has a complexity \( \mathcal{O}(2^{(K/\bar{\epsilon})} n_1 K) \), see Kumar et al. (2004). In practice, this operation is fast and the most costly operation is the computation of \( B \) which has complexity \( \mathcal{O}(n_1^2 n_2 p_{\max}) \), since one can show that \( \text{nnz}(A) = O(n_1 n_2 p_{\max}) \) with high probability.

### 3.2 Iterative refinement with GPM

Our algorithm is based on the Generalized Power Method. In contrast to the power method proposed recently by Ndaoud et al. (2022), we do not require to center the adjacency matrix \( A \) and, instead of using the sign to identify the communities, we project on \( \mathcal{H}(A) \). Consequently, our algorithm can be applied to bipartite graphs with \( K > 2 \) and \( K \neq L \) while Ndaoud et al. (2022) require \( K = L = 2 \).

In the first step we form the diagonal hollowed Gram matrix \( B \). This is natural since in the parameter regimes we are interested in, there is no hope to consistently estimate \( Z_2 \). Then, we iteratively update the current estimate of the partition \( Z_1^{(t)} \) by a row-wise projection of \( BZ_1^{(t)} \) onto the extreme points of the unit simplex of \( \mathbb{R}^K \) denoted by \( S_K \). The projection operator \( \mathcal{P} : \mathbb{R}^K \rightarrow S_K \) is formally defined by

\[
\mathcal{P}(x) := \arg \min_{y \in S_K} \|x - y\| \quad \text{for all } x \in \mathbb{R}^K.
\]

This implies that \( (Z_1^{(t+1)})_{ik} = 1 \) for the value of \( k \) that maximizes \( B_i Z_1^{(t)} \). Ties are broken arbitrarily.

Algorithm 2 Generalized Power Method (GPM)

**Input:** Number of communities \( K \), adjacency matrix \( A \), an estimate of the row partition \( \hat{Z}_1^{(0)} \) and, number of iterations \( T \).

1. Form the diagonal hollowed Gram matrix \( B := \mathcal{H}(AA^\top) \) where \( \mathcal{H}(X) = X - \text{diag}(X) \) and compute \( W^{(0)} = (Z_1^{(0)})^\top (D^{(0)})^{-1} \) where \( D^{(0)} = \text{diag}((\hat{Z}_1^{(0)})^\top 1_{n_1}) \).
2. for \( 0 \leq t \leq T \) do
   3. Update the partition \( Z_1^{(t+1)} = \mathcal{P}(BW^{(t)}) \) where \( \mathcal{P} \) is the operator in \( \Box \) applied row-wise.
   4. Compute \( D^{(t+1)} = \text{diag}((Z_1^{(t+1)})^\top 1_{n_1}) \). Then form \( W^{(t+1)} = (Z_1^{(t+1)})^\top (D^{(t+1)})^{-1} \).
5. end for

**Output:** A partition of the nodes \( Z_1^{(T+1)} \).

**Computational complexity of GPM.** As mentioned earlier, the cost for computing \( B \) is \( O(n_1 \text{nnz}(A)) \). For each \( t \), the cost of computing \( BW^{(t)} \) is \( O(n_1^2 K) \) and the cost of the projection is \( \mathcal{O}(Kn_1) \). The cost of computing \( D^{(t+1)} \) and \( W^{(t+1)} \) is \( O(Kn_1) \). So the total cost over \( T \) iterations is \( O(Tn_1^2 K) \) and doesn’t depend on \( n_2 \).

\(^1\)The log \( n_1 \) term comes from the number of iterations needed when using the power method to compute the largest (or smallest) eigenvector of a given matrix.
Remark 5. The recent work of Wang et al. (2021) proposed a power method for clustering under the SBM, but instead of using a projection on the simplex as we did, they add an additional constraint on the column on $Z$ so that each cluster has the same size. They showed that computing this projection is equivalent to solving a linear assignment problem (LAP). But the cost of solving this LAP is $O(K^2 n_1 \log n_1)$ whereas the cost of the projection on the simplex is $O(K n_1)$. Moreover, their algorithm requires to know in advance the size of each cluster and it is not straightforward to extend their theoretical guarantees to the approximately balanced community setting.

4 Spectral initialization

We show that Algorithm 1 can recover an arbitrary large proportion of community provided that the sparsity level $p_{\text{max}}$ is sufficiently large, and $n_2/n_1 \to \infty$ sufficiently fast, as $n_1 \to \infty$.

Proposition 1. Assume that $A \sim \text{BiSBM}(Z_1, Z_2, \Pi)$ and Assumptions 4.1, 4.2 are satisfied. For any $0 < \varepsilon < 1$, suppose additionally that

$$\frac{n_2}{n_1} \geq \max \left( L \log n_1, \frac{\log^2 n_2}{\log n_1} \right), \quad p_{\text{max}}^2 \geq \alpha n_2 \varepsilon n_1 n_2, \quad (4.1)$$

$$\alpha^3 \frac{KL}{\log n_1} = o(1), \quad \alpha^{12} \frac{(KL)^3}{n_1 \log n_1} = o(1) \quad \text{as } n_1 \to \infty. \quad (4.2)$$

Then the output $z^{(0)}$ of Algorithm 1 satisfies with probability at least $1 - n^{-\Omega(1)}$ the bound $r(z^{(0)}, z) \lesssim K \varepsilon^2$.

Proof. First, we will control the noise $\|B - PP^T\|$ using Lemma 3. Note that the conditions in (4.1) imply the conditions on $n_1, n_2$ and $p_{\text{max}}$ in Lemma 3. Since $H(\cdot)$ is a linear operator and $B = H(B)$, we have the decomposition

$$B = H(PP^T) + H(PE^T) + H(E^T) + H(E^{T^*}),$$

which leads to the following bound by triangle inequality

$$\|B - PP^T\| \leq \|H(PE^T) + H(E^T)\| + \|H(PP^T) - PP^T\| \leq 4 \|EZ_2\| \|\Pi^T Z_1^T\| + \|H(E^T)\| + \|H(PP^T) - PP^T\|. \quad (4.3)$$

Now let us observe that

$$\|H(PP^T) - PP^T\| = \|H(Z_1 \Pi Z_2^T Z_2 \Pi^T Z_1^T) - Z_1 \Pi Z_2^T Z_2 \Pi^T Z_1^T\| = \|\text{diag}(Z_1 \Pi Z_2^T Z_2 \Pi^T Z_1^T)\| \leq \frac{\alpha n_2}{L} (L p_{\text{max}}) \leq n_2 p_{\text{max}}^2,$$

while,

$$\|\Pi^T Z_1^T\| \leq \|Z_1\| \|\Pi\| \leq \sqrt{\frac{\alpha n_1}{K}} \|\Pi\| \leq \sqrt{\alpha n_1 L p_{\text{max}}}. \quad \text{by the upper bound in Remark 1}$$

Using these bounds along with the bounds on $\|EZ_2\|$ and $\|H(E^T)\|$ from Lemma 3 and applying them in (4.3), it holds with probability at least $1 - n^{-\Omega(1)}$ that

$$\|B - PP^T\| \leq \max(\log n_1, \sqrt{n_1 n_2 p_{\text{max}}}) + \alpha \sqrt{n_1 n_2 p_{\text{max}}^2} \leq \alpha \sqrt{n_1 n_2 p_{\text{max}}^2} \leq \alpha \sqrt{n_1 n_2 p_{\text{max}}^2}.$$

Let us denote by $\hat{U}$ (resp. $U$) to be the matrix of top-$K$ eigenvectors of $B$ (resp. $PP^T$). Denoting $\lambda_K(PP^T)$ to be the $K$th largest eigenvalue of $PP^T$, it is not difficult to verify that

$$\lambda_K(PP^T) \geq \lambda_L(Z_2) \lambda_K(Z_1) \lambda_K(\Pi)^2 \geq \frac{n_1 n_2 p_{\text{max}}^2}{\alpha^2 KL}.$$
Then by the Davis-Kahan Theorem (Yu et al., 2014), there exists an orthogonal matrix $Q \in \mathbb{R}^{K \times K}$ such that

$$
\left\| \hat{U} - UQ \right\| \lesssim \frac{\|B - PP^T\|}{\lambda_K(PP^T)} \lesssim \frac{\alpha^2 KL \log n}{n_1 n_2 p_{\max}^2} + \frac{\alpha^2 KL}{\sqrt{n_1 n_2 p_{\max}}} + \frac{\alpha^3 KL}{n_2 p_{\max}} + \frac{\alpha^3 KL}{n_1},
$$

where we bounded the max operation by the sum. Now on account of the conditions in (4.1) and (4.2), it is easily seen that $\alpha^2 KL \log n$ $n_1 n_2 p_{\max}^2$, while the remaining three terms in the RHS of (4.1) are $o(1)$ as $n_1 \to \infty$.

Finally, we conclude by using the same argument as in the proof of Theorem 3.1 in Lei and Rinaldo (2015) to show that

$$r(z, z) \lesssim \left\| \hat{U} - U\right\|^2 \lesssim K \left\| \hat{U} - UQ \right\|^2 \lesssim K \varepsilon^2.$$

\hfill \Box

\textbf{Remark 6.} As shown in Section 7, Spec has very good empirical performance. This suggests that the previous proposition is far from being optimal and doesn’t capture the true rate of convergence of this spectral method. Also note that Proposition 3 gives a meaningful bound only when $\varepsilon = O(1/\sqrt{K})$.

# 5 Analysis of GPM

Our analysis strategy is similar to the one recently considered by Braun et al. (2021b) for clustering under the contextual SBM, which in turn is based on the framework recently developed by Gao and Zhang (2022). There are however some additional technical difficulties due to the fact that there are more dependencies in the noise since the matrix $B$ is a Gram matrix.

We will assume w.l.o.g. that $\pi^*$ the permutation that best aligns $z^{(0)}$ with see (see equation (2.1)) is the identity, if not, then replace $z$ by $(\pi^*)^{-1}(z)$. Hence there is no label switching ambiguity in the community labels of $z^{(t)}$ because they are determined from $z^{(0)}$.

First we will decompose the event “after one refinement step, the node $i$ will be incorrectly clustered given the current estimation of the partition $z^{(t)}$ at time $t$” into an event independent of $i$ and events that depend on how close $z^{(t)}$ is from $z$. Then we will analyze these events separately. Finally, we will use these results to show that the error contracts at each step.

## 5.1 Error decomposition

By definition, a node $i$ is misclustered at step $t + 1$ if there exists a $k \neq z_i \in [K]$ such that

$$B_{ik} W_{ik}^{(t)} \geq B_{ik} W_{z_i}^{(t)}.$$  \hfill (5.1)

By decomposing $B$ as

$$B = \mathcal{H}(PP^T) + \mathcal{H}(EP^T + PE^T + EE^T),$$

one can show that condition (5.1) is equivalent to

$$\hat{E}_{ik}(W_{zi} - W_{ik}) \leq -\Delta^2(z_i, k) + F_{ik}^{(t)} + G_{ik}^{(t)}$$  \hfill (5.2)

where

$$\Delta^2(z_i, k) = \hat{P}_{ik}(W_{zi} - W_{ik}),$$

$$F_{ik}^{(t)} = \langle \hat{E}_{ik}(W^{(t)} - W), e_{zi} - e_{ik} \rangle,$$

and

$$G_{ik}^{(t)} = \langle P_{ik}(W^{(t)} - W), e_{zi} - e_{ik} \rangle.$$
Here \( e_1, \ldots, e_K \) denotes the canonical basis of \( \mathbb{R}^K \). The terms \( F_{ik}^{(t)} \) and \( G_{ik}^{(t)} \) can be interpreted as error terms (due to \( W^{(i)} \neq W \)) while \( \Delta^2(z_i, k) \) corresponds to the signal. Indeed, let us denote \( Q = \Pi(Z_1^T Z_2)\Pi^T \). Then we obtain \( PP^T = Z_1QZ_1^T \) which implies
\[
\Delta^2(z_i, k) = \tilde{P}_i(W_{zi} - W_{ik}) = Q_{zi} - Q_{zik} - (PP^T)_{ii}(W_{zi} - W_{ik}) \\
= (1 - \frac{K}{n_1})Q_{zi} - Q_{zik}.
\]

We have for all \( k, k' \in [K] \) that
\[
\frac{n_2}{\alpha L}((\Pi_i^T)_{kk'}) \leq Q_{kk'} \leq \frac{\alpha n_2}{L}((\Pi_i^T)_{kk'})
\]
which implies
\[
\frac{n_2}{\alpha L}((\Pi_i^T)_{zi, zi} - \alpha^2((\Pi_i^T)_{zi, k})) \leq \Delta^2(z_i, k) \leq \frac{\alpha n_2}{L}((\Pi_i^T)_{zi, zi} - \frac{1}{\alpha^2}((\Pi_i^T)_{zi, k}))
\]
for \( n_1 \) large enough. By Assumption [A3] this implies
\[
\beta \frac{n_2 P_{\max}^2}{\alpha L} \leq \Delta^2(z_i, k) \leq \eta \beta \frac{n_2 P_{\max}^2}{L}.
\]
when \( n_1 \) is large enough.

5.2 Oracle error

We want to show that the condition (5.2) cannot occur with high probability. First we will show that this is indeed the case when we ignore the \( F \) and \( G \) error terms; the subsequent error will be referred to as the oracle error
\[
\xi(\delta) = \sum_{i=1}^{n_1} \sum_{k \in [K] \setminus z_i} \Delta^2(z_i, k) \mathbf{1}_{\Omega_{i,k}(\delta)} \text{ for } \delta \in (0, 1)
\]
where
\[
\Omega_{i,k}(\delta) = \left\{ \tilde{E}_i(W_{zi} - W_{ik}) \leq -(1 - \delta)\Delta^2(z_i, k) \right\}.
\]

Let us denote
\[
\Delta_{\text{min}} = \min_{a \neq b \in [K]} \Delta(a, b)
\]
to be the minimal separation of the parameters associated with the different communities. We will also denote by
\[
\tilde{\Delta}^2 := \frac{\beta^2}{12eL\alpha^3} \frac{n_3 n_2 P_{\max}^2}{KL},
\]
to be the approximate signal-to-noise ratio (SNR) associated with the model.

In general the rate of decay of \( \xi(\delta) \) leads to the convergence rate of iterative refinement algorithms, hence it is important to control this quantity by showing that the following condition is satisfied. Let us define
\[
\tau^{(0)} = \epsilon' \delta \min(1, \beta^2 \frac{n_1 \Delta_{\text{min}}^2}{K})
\]
for a small enough constant \( \epsilon' > 0 \). This parameter will be referred to as the minimal error required for the initial estimate of our algorithm.

Condition C1 (oracle error). Assume that there exists \( \delta \in (0, 1) \) such that
\[
\xi(\delta) \leq \frac{3}{4} \tau^{(0)}
\]
holds with probability at least \( 1 - \eta_1 \).
5.3 Contraction of the error at each step

Let
\[ l(z, z') = \sum_{i \in [n]} \Delta^2(z_i, z'_i) \mathbf{1}_{\{z_i \neq z'_i\}} \]
be a measure of the distance between two partitions \( z, z' \in [K]^n \). We want to show that \( l(z^{(t)}, z) \) decreases until reaching the oracle error. To this end, we will need to control the noise level. In particular, we are going to show that the following two conditions are satisfied.

**Condition C2 (F-error type).** Assume that
\[
\max_{\{z^{(t)}: l(z, z^{(t)}) \leq \tau^{(0)}\}} \sum_{i=1}^{n} \max_{b \in [K] \setminus z_i} \frac{(F_{ib}^{(t)})^2}{\Delta^2(z_i, b)} \leq \frac{\delta^2}{256}
\]
holds with probability at least \( 1 - \eta_2 \).

**Condition C3 (G-error type).** Assume that
\[
\max_{i \in [n]} \max_{b \in [K] \setminus z_i} \frac{|G_{ib}^{(t)}|}{\Delta^2(z_i, b)} \leq \frac{\delta}{4}
\]
holds uniformly on the event \( \{z^{(t)}: l(z, z^{(t)}) \leq \tau^{(0)}\} \) with probability at least \( 1 - \eta_3 \).

Under these conditions, we can show that the error contracts at each step.

**Theorem 1.** Assume that \( l(z^{(0)}, z) \leq \tau^{(0)} \). Additionally assume that Conditions **C1** **C2** and **C3** hold. Then with probability at least \( 1 - \sum_{i=1}^{3} \eta_i \)
\[
l(z^{(t)}, z) \leq \xi(\delta) + \frac{1}{8} l(z^{(t-1)}, z), \forall t \geq 1.
\]
(5.4)
In particular, we have for all \( t \gtrsim \log(1/\delta) \) that
\[
l(z^{(t)}, z) \leq \xi(\delta) + \tau^{(0)} (1/8)^{t-\Theta(\log(1/\delta))}.
\]

**Proof.** It is an immediate adaptation of Theorem 3.1 in [Gao and Zhang (2022)]. The last part can be derived in the same way as in Corollary 1 in [Braun et al. (2021a)].

5.4 Application to BiSBM

When applied to the BiSBM the previous theorem leads to the following result. Recall \( c(K, L) \) as defined within Remark [1].

**Theorem 2.** Assume that \( A \sim BiSBM(Z_1, Z_2, \Pi), K^2L \leq \tilde{\Delta}^2, \tilde{\Delta} \to \infty, (4.1) \) and (4.2) are satisfied. Under Assumptions [4] [5] and [6] if GPM is initialized with a \( z^{(0)} \) such that
\[
l(z, z^{(0)}) \leq \tau^{(0)} = \epsilon' \delta \min(1, \beta^2) \frac{n_1 \Delta^2_{\min}}{\alpha^2 c(K, L)^2 K}
\]
for a small enough constant \( \epsilon' > 0 \) and \( \delta = \frac{1}{4 \exp} \), then with probability at least \( 1 - n_1^{-\Omega(1)} \) we have for all \( t \gtrsim \log n_1 \)
\[
r(z^{(t)}, z) \leq \exp(- (1 - o(1)) \tilde{\Delta}^2).
\]
In particular, if \( \tilde{\Delta}^2 > \log n_1 \), we can exactly recover \( Z_1 \).

**Corollary 1.** Under the assumptions of Theorem 2 **Spec** returns an estimate \( z^{(0)} \) such that
\[
l(z, z^{(0)}) \leq \eta \beta \frac{\alpha n_1 n_2 p_{\max}^2}{L} r(z, z^{(0)}) \leq \eta \beta K^2 \epsilon \frac{n_1 \Delta^2_{\min}}{c(K, L)^2 K}
\]
and hence satisfies \( l(z, z^{(0)}) \leq \tau^{(0)} \) for \( \epsilon = O(\frac{\min(1, \beta^2)}{K c(K, L) n \sqrt{n_1}}) \).
Consequently we get
\[ \exp(-\tilde{\Delta}) \leq \alpha \beta K n_1 n_2 p_{max}^2 e^{-\tilde{\Delta}^2} \lesssim \frac{\alpha^4}{\beta} K^2 L \tilde{\Delta}^2 e^{-\tilde{\Delta}^2} = e^{-(1-o(1))\tilde{\Delta}^2} \]
by assumptions on \( \tilde{\Delta} \) and \( KL \). Then, by Markov inequality, we obtain
\[ \mathbb{P} \left( \xi(\delta) \geq \exp(\tilde{\Delta})\mathbb{E}\xi(\delta) \right) \leq \exp(-\tilde{\Delta}). \]
Consequently we get
\[ \exp(\tilde{\Delta})\mathbb{E}\xi(\delta) \leq \exp(-1-o(1))\tilde{\Delta}^2 \]
and hence with probability at least \( 1 - \exp(-\tilde{\Delta}) \)
\[ \xi(\delta) \leq \exp(-1-o(1))\tilde{\Delta}^2 \leq \frac{3}{4} \tau(0) \]
because \( \exp(-1-o(1))\tilde{\Delta}^2 = o(1) \ll \tau(0) = \Omega(1) \) for \( n_1 \) large enough. This shows that Condition [C1] is satisfied.

**F-error term.** With high probability, for all \( z^{(t)} \) such that \( l(z^{(t)}, z) \leq \tau(0) \) we have
\[ \sum_{i=1}^{n_1} \max_{b \in [K] \setminus z_i} \sum_{i=1}^{n_2} \frac{\left( I_{ik}^{(t)} \right)^2}{\Delta^2(z_i, b)l(z, z^{(t)})} \leq \sum_{i=1}^{n_1} \left\| E_i(W^{(t)} - W) \right\|_F^2 \max_{b \in [K] \setminus z_i} \frac{\| e_{z_i} - e_b \|^2}{\Delta^2(z_i, b)l(z, z^{(t)})} \]  
(by Cauchy-Schwartz)
\[ \leq \frac{2 \left\| E(W^{(t)} - W) \right\|_F^2}{\Delta_{min}^2 l(z, z^{(t)})} \]
\[ \leq \frac{2 \left\| E \right\|_F^2 \left\| W^{(t)} - W \right\|_F^2}{\Delta_{min}^2 l(z, z^{(t)})}. \]

By Lemma [2] we have w.h.p.
\[ \| H(EE^T) \| \lesssim \| EE^T - E(EE^T) \| \lesssim \max(\log n_1, \sqrt{n_1 n_2 p_{max}}) \]
and
\[ \| EP^T \| \lesssim \| EZ_2 \| \| \Pi Z_1 \| \lesssim \alpha \sqrt{n_1 n_2 p_{max}} \frac{c(K, L)p_{max}}{L} \sqrt{\frac{n_1}{K}} = \alpha \sqrt{n_1 n_2 c(K, L)p_{max}} \sqrt{\frac{n_1 p_{max}}{KL}}. \]
Since \( \|E\| \leq \|H(E^\top)\| + \|EP^\top\| \) we obtain
\[
\sum_{i=1}^n \max_{b \in \{K\} \setminus z_i} \frac{(F^{(i)}_{\text{bl}})^2}{\Delta^2(z_i, b) \|l_i(z, z^{(t)})\|} \leq \frac{\max (\log^2 n_1, n_1n_2p_{\max}^2, \alpha^2 c(K, L)^2 (KL)^{-1} n_1^2 n_2 p_{\max}^3)}{n_1^2 \Delta_{\min}^2} \alpha^3 K^3 l(z^{(t)}, z)
\]
\[
\leq \frac{\alpha^3 K^2 \Delta_{\max}^2 (\log^2 n_1, n_1n_2p_{\max}^2, \alpha^5 c(K, L)^2 (KL)^{-1} n_1^2 n_2 p_{\max}^3)}{n_1^2 \Delta_{\min}^2} K \tau^{(0)}
\]
\[
\leq \frac{\alpha^5 K^2 L^2 \Delta_{\max}^2 (\log^2 n_1, n_1n_2p_{\max}^2, \alpha^5 c(K, L)^2 (KL)^{-1} n_1^2 n_2 p_{\max}^3)}{\beta^2 n_1^2 \Delta_{\min}^2} (\delta \beta^2)
\]
\[
\leq \frac{\alpha^5 K^2 L^2 \Delta_{\max}^2 (\log^2 n_1, n_1n_2p_{\max}^2, \alpha^2 c(K, L)^2 (KL)^{-1} n_1^2 n_2 p_{\max}^3)}{n_1^2 \Delta_{\min}^2} \delta.
\]

From the assumptions of the theorem, one can verify that,
\[
\frac{\alpha^5 K^2 L^2 \Delta_{\max}^2 (\log^2 n_1, n_1n_2p_{\max}^2, \alpha^2 c(K, L)^2 (KL)^{-1} n_1^2 n_2 p_{\max}^3)}{n_1^2 \Delta_{\min}^2} = O(1), \quad \frac{\alpha^7 K^2 L^2}{n_1^2 n_2 p_{\max}^2} = o(1) \quad \text{and} \quad \frac{\alpha^2 KLc(K, L)^2}{n_1^2 n_2 p_{\max}^2} = o(1).
\]

Hence for an appropriately small choice of \( \delta \) Condition C2 is satisfied.

**G-error term.** With high probability, for all \( z^{(t)} \) such that \( l(z^{(t)}, z) \leq \tau^{(0)} \) we have
\[
\frac{|G^{(t)}_{ij}|}{\Delta^2(z_i, b)} \leq \sqrt{2} \frac{\|P_t(W^{(t)} - W)\|}{\Delta_{\min}^2} \quad \text{(by Cauchy-Schwartz)}
\]

But since \( \|P_t\| \leq \|P_t P^\top\| \leq \|\Pi\|^2 \|Z_2 Z_2^\top\| \|Z_1\| \leq c(K, L)^2 (KL)^{-1} n_1^2 n_2 p_{\max}^2 \sqrt{\frac{\max \Delta}{K}} \) we obtain
\[
\frac{|G^{(t)}_{ij}|}{\Delta^2(z_i, b)} \leq \frac{\alpha^{1.5} (K, L)^2 n_2 p_{\max}^2 \sqrt{\max \Delta} \|W^{(t)} - W\|}{L \sqrt{K} \Delta_{\min}^2}
\]
\[
\leq \frac{\alpha^3 c(K, L)^2 n_2 p_{\max}^2 \Delta_{\min}^2}{L \Delta_{\min}^2 n_1 \Delta_{\min}^2} \frac{K l(z^{(t)}, z)}{\beta n_1 \Delta_{\min}}
\]
\[
\leq \frac{\alpha^3 c(K, L)^2 K \tau^{(0)}}{\beta n_1 \Delta_{\min}^2}.
\]

Now by choosing \( \delta \) to be a suitably small constant \((< 1)\), we obtain
\[
\frac{|G^{(t)}_{ij}|}{\Delta^2(z_i, b)} \leq \frac{\delta}{4}.
\]

This shows that Condition C3 is satisfied.

**6 Minimax lower bound**

Let us denote the admissible parameters space for a SBiSBM (where \( K = L \)) by
\[
\Theta = \{ P \in [0,1]^{n_1 \times n_2} : P = Z_1 \Pi Z_2^\top \mid \Pi = q \mathbf{1}_K \mathbf{1}_K^\top + (p - q) \mathbf{1}_K, 1 > p > 0, \quad q = cp \text{ for some constant } c \in (0,1), Z_1 \in M_{n_1, K}, Z_1 \in M_{n_2, K} \text{ with } \alpha = 1 + O \left( \sqrt{\frac{\log n_1}{n_1}} \right) \}.
\]
We want to lower bound \( \inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z)) \) where \( \hat{z} \) is an estimator of \( Z_1 \). We will first present the core argument to obtain a lower bound for the special case \( K = 2 \) (Theorem 3). Then, we show that the general case (where \( K = L \geq 2 \)) can be handled by a reduction argument to the setting \( K = 2 \) (Theorem 4).

In the supervised case, i.e., when \( Z_2 \) is known, we can use the same strategy as the one use for the degree-corrected SBM by [Gao et al. (2018)] and obtain a lower bound of the order \( e^{-(1+o(1))n_2(p-q)/2} \) corresponding to the failure probability of the optimal test associated to the following two hypothesis problem

\[
H_0 : \otimes_{i=1}^{n_2/2} B(p) \otimes_{i=n_2/2+1}^{n_2} B(q), \quad \text{vs} \quad H_1 : \otimes_{i=1}^{n_2/2} B(q) \otimes_{i=n_2/2+1}^{n_2} B(p).
\]

However when \( n_2 \gg n_1 \log n_1 \) the error associated with this hypothesis testing problem is of order \( \exp(-n_2(p-q)/2) \approx \exp(-\sqrt{n_2 \log n_2/n_1}) \) but this is far smaller than \( \exp(-n_1n_2p_{\max}^2) \), the misclustering rate obtained for our algorithm. A similar phenomena appears for high-dimensional Gaussian mixture models. Indeed, as shown in [Ndaoud (2018)], it is essential to capture the hardness of estimating the model parameters in the minimax lower bound in order to get the right rate of convergence. The argument developed for obtaining a lower bound of the minimax risk in [Ndaoud (2018)] relies heavily on the Gaussian assumption (they set a Gaussian prior on the model parameters and use the fact that the posterior distribution is also Gaussian) and cannot directly be extended to this setting.

**Theorem 3.** Suppose that \( A \sim SBiSBM(Z_1, Z_2, p, q) \) with \( K = L = 2 \), \( n_2 \gg n_1 \log n_1 \), \( n_1n_2p_{\max}^2 \to \infty \) and \( n_1n_2p_{\max}^2 = O(\log n_1) \). Then there exists a constant \( c_1 > 0 \) such that

\[
\inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z)) \geq \exp(-c_1n_1n_2p_{\max}^2)
\]

where the infimum is taken over all measurable functions \( \hat{z} \) of \( A \). Moreover, if \( n_1n_2p_{\max}^2 = \Theta(1) \), then \( \inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z)) \geq c_2 \) for some positive constant \( c_2 \).

**Remark 9.** This lower-bound shows that the rate of convergence of our estimator is optimal up to a constant factor. Indeed, for exact recovery, the minimax lower bound implies that \( n_1n_2p_{\max}^2 \gg \log n_1 \) is necessary, while for weak recovery, we need \( n_1n_2p_{\max}^2 \to \infty \). It also shows that when \( n_1n_2p_{\max}^2 = O(1) \) it is not possible to consistently estimating \( Z_1 \).

In the general case where \( K = L \geq 2 \), we obtain by a reduction argument (to the case \( K = 2 \)) the following theorem.

**Theorem 4.** Suppose that \( A \sim SBiSBM(Z_1, Z_2, p, q) \) with \( K = L \geq 2 \), \( n_2 \gg n_1 \log n_1 \), \( n_1n_2p_{\max}^2/(KL) \to \infty \) and \( n_1n_2p_{\max}^2/(KL) = O(\log n_1) \). Then there exists a constant \( c_1 > 0 \) such that

\[
\inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z)) \geq \exp(-(c_1n_1n_2p_{\max}^2/(KL))
\]

where the infimum is taken over all measurable functions \( \hat{z} \) of \( A \). Moreover, if \( n_1n_2p_{\max}^2/(KL) = \Theta(1) \), then \( \inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z)) \geq c_2/K^3 \) for some positive constant \( c_2 \).

**Remark 10.** The minimax lower bound is of order \( O(\Theta(n_2p_{\max}^2)) \) and matches the upper bound \( \exp(O(2n_2p_{\max}^2)) \) up to a \( 1/L \) factor. Extending our proof technique to the general non-symmetric case seems more challenging because the posterior distributions that appears in Step 2 (in the proof of Theorem 3) have a far more complex expression.

### 6.1 Proof of Theorem 3

The general idea of the proof is to first lower bound the minimax risk by an error occurring in a two hypothesis testing problem and then to replace this hypothesis testing problem by a simpler one. The steps are detailed below.
**Step 1.** Recall that $z, z'$ denote the partition functions associated with $Z_1$ and $Z_2$ respectively. We choose as a prior on $z$ and $z'$ a product of independent, centered Rademacher distributions. Since the marginals of $z, z'$ are sign invariant, then by using standard arguments, the results in [Gao et al. (2018)](see the proof of Theorem 2) or [Ndaooud (2018)] show that (for any given $i = 1, \ldots, n_1$)

$$\inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z)) \geq \inf_{\hat{z}_i} \mathbb{E}_{z, z'} \mathbb{E}_{A|z, z'}(\phi_i(A))$$

where $\hat{z}$ is a measurable function in $A$ and $\phi_i(A) = 1_{\hat{z}_i \neq z_i}$.

**Step 2.** We can write

$$\mathbb{E}_{z, z'} \mathbb{E}_{A|z, z'}(\phi_i(A)) = \mathbb{E}_{z - i} \mathbb{E}_{z_i} \mathbb{E}_{A|z_i, z'}(\phi_i(A)) = \mathbb{E}_{z - i} \mathbb{E}_{z_i} \mathbb{E}_{A|z_i, z'}(\phi_i(A))$$

where $z_{-i} := (z_j)_{j \neq i}$ and $\mathbb{E}_{X|Y}$ means that we integrate over the random variable $X$ conditioned on $Y$. Since $z'$ is random, the entries of $A$ are no longer independent (this is the reason why we use an uninformative prior on $z'$). Note that the quantity $\mathbb{E}_{z'} \mathbb{E}_{A|z, z'}(\phi_i(A)) = \mathbb{E}_{A|z_i}(\phi_i(A))$ only depends on $f(A|z)$ the density of $A$ conditionally on $z$. Since the columns of $A$ are independent conditionally on $z$ (because $(z'_j)_{j=1}^n$ are independent) we have that

$$f(A|z) = \prod_j f(A_{ij}|z),$$

i.e. $f(A|z)$ is the product of the densities of the columns $A_{ij}$ conditionally on $z$. Let us denote $A_{-ij} := (A_{ij'})_{i' \neq i}$ and $A_{-i} := (A_{ij})_{i \neq j}$. Now for each $j$ we can write

$$f(A_{ij}|z) = f(A_{ij}|A_{-ij}, z)f(A_{-ij}|z).$$

Since $A_{-ij}$ doesn’t depend on $z_i$ we have $f(A_{-ij}|z) = f(A_{-ij}|z_{-i})$. Assume that $z_i = 1$. Then $A_{ij} \sim \mathcal{B}(p)$ if $z'_j = 1$ or $A_{ij} \sim \mathcal{B}(q)$ if $z'_j = -1$. This in turn implies

$$\mathbb{P}(A_{ij} = 1|A_{-ij}, z) = p\mathbb{P}(z'_j = 1|A_{-ij}, z) + q\mathbb{P}(z'_j = -1|A_{-ij}, z)$$

$$= p\mathbb{P}(z'_j = 1|A_{-ij}, z_{-i}) + q\mathbb{P}(z'_j = -1|A_{-ij}, z_{-i})$$

Let us denote by $\alpha_j$ the random variable $\mathbb{P}(z'_j = 1|A_{-ij}, z_{-i})$. When $z_i = -1$, similar considerations imply

$$\mathbb{P}(A_{ij} = 1|A_{-ij}, z_{-i}) = q\alpha_j + p(1 - \alpha_j).$$

This shows that the conditional distribution

$$A_{ij}|A_{-ij}, z \sim \mathcal{B}(\alpha_j p + (1 - \alpha_j)q) \quad \text{when } z_i = 1 \quad (6.1)$$

and

$$A_{ij}|A_{-ij}, z \sim \mathcal{B}(\alpha_j q + (1 - \alpha_j)p) \quad \text{when } z_i = -1. \quad (6.2)$$

We can write $R_i$ as

$$R_i = \mathbb{E}_{z - i} \mathbb{E}_{z_i} \mathbb{E}_{A|z, z'}(\phi_i(A)) = \mathbb{E}_{z - i} \mathbb{E}_{z_i} \mathbb{E}_{A|z, z'}(\phi_i(A))$$

$$= \mathbb{E}_{z - i} \mathbb{E}_{A|z, z'}(\phi_i(A))$$

The term $R'_i$ corresponds to the risk associated with the following two hypothesis testing problem (conditionally on $(A_{ij'})_{i' \neq i}$ and $z_{-i}$)

$$H_0 : \otimes_{j=1}^{n_2} \mathcal{B}(\alpha_j p + (1 - \alpha_j)q) \quad \text{vs} \quad H_1 : \otimes_{j=1}^{n_2} \mathcal{B}(\alpha_j q + (1 - \alpha_j)p). \quad (6.3)$$
Our goal is to replace this two-hypothesis testing problem by a simpler hypothesis test associated with a smaller error. The proof strategy is the following. When $\alpha_j$ is very close to 1/2 it is not possible to statistically distinguish $(B(\alpha_j p + (1 - \alpha_j) q))$ from $(B(\alpha_j q + (1 - \alpha_j) p))$ and these factors can be dropped. When $\alpha_j$ is significantly different from 1/2, then the risk associated to the test can be lower bounded by a the risk of testing between a product of $B(p)$ vs. a product of $B(q)$. More precisely, we will show that the number of indices $j$ for which $\alpha_j$ is significantly different from 1/2 is of order $n_1 n_2 p$ and hence the risk is lower bounded by the one associated by the two hypothesis testing problem

$$H_0' : \otimes_{i=1}^{n_2} B(p) \otimes_{i=1}^{n_1} B(q) \text{ vs } H_1' : \otimes_{i=1}^{n_2} B(q) \otimes_{i=1}^{n_1} B(p).$$

It is well known that the error associated with the above testing problem is of the order $e^{-\Theta(n_2 n_1 p^2)}$ (see e.g. Gao et al. [2018]) which corresponds to the rate of convergence of our algorithm. Let us now formalize the above argument.

**Step 3.** First, let us define

$C_+ = \{i' \neq i : z_{i'} = 1\}$,  
$C_- = \{i' \neq i : z_{i'} = -1\}$,  
$\theta_j = \frac{\alpha_j}{1 - \alpha_j}$ for $j = 1 \ldots n_2$,  
$\epsilon = \Theta(p \sqrt{n_1 \log n_1}) = o(1), \quad J_b = \{j \in [n_2] : \theta_j \in [1 - \epsilon, 1 + \epsilon]\}$,  
$J_g = \{j \in [n_2] : \theta_j \notin [1 - \epsilon, 1 + \epsilon]\}$,  
$T_j \overset{ind.}{\sim} B(\alpha_j)$, for all $j \in J_g$,

and the events

$$E_1 = \left\{ |C_+| - |C_-| \in [-C \sqrt{n_1 \log n_1}, C \sqrt{n_1 \log n_1}] \right\} \text{ for some constant } C > 0,$$

$$E_2 = \left\{ \sum_j 1_{\{\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_-} A_{i'j} \neq 0\}} = \Theta(n_2 n_1 p) \right\},$$

$$E_3 = \left\{ \sum_{j \in J_b} A_{ij} = \Theta(n_2 p) \right\}.$$

We will show later that these events occur with high probability. They are useful for obtaining a lower bound on the densities $f(A_{ij} | A_{-ij}, z)$. Since we are integrating over positive functions one can write

$$R_t \geq E_z, 1_{E_{-z}} E_{A_{-ij} | z} E_z E_{A_i | z, A_{-i}} (1_{E_3} \phi_t (A)).$$

(6.4)

**Step 4.** For all $j \in J_b$, the set for which $\alpha_j \approx \frac{1}{2}$, we are going to lower bound the densities $f(A_{ij} | A_{-ij}, z)$ by $g(A_{ij})$ corresponding to the density of $B(\frac{p+q}{2})$. A simple calculation shows that

$$\theta_j = \frac{\alpha_j}{1 - \alpha_j} = \frac{\prod_{i' \neq i : z_{i'} = 1} p A_{i'j} (1 - p)^{1 - A_{i'j}} \prod_{i' \neq i : z_{i'} = -1} q A_{i'j} (1 - q)^{1 - A_{i'j}}}{\prod_{i' \neq i : z_{i'} = 1} q A_{i'j} (1 - q)^{1 - A_{i'j}} \prod_{i' \neq i : z_{i'} = -1} p A_{i'j} (1 - p)^{1 - A_{i'j}}}.$$ 

The previous expression can be rewritten as

$$\theta_j = \left( \frac{p (1 - q)}{q (1 - p)} \right)^{\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_-} A_{i'j}} \left( \frac{1 - p}{1 - q} \right)^{|C_+| - |C_-|}.$$

We also have the relation $\alpha_j = \frac{\theta_j}{1 + \theta_j}$, so $\alpha_j$ is close to 1/2 if and only if $\theta_j$ is close to 1. If $z_{-i}$ were an exactly balanced partition, i.e., $|C_+| - |C_-| = 0$, then $\theta_j = 1$ would be equivalent to $\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_-} A_{i'j} = 0$. 

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However the contribution of the term \((1-\frac{p}{1-q})^{|C_+|-|C_-|}\) is small under \(\mathcal{E}_1\). Indeed, we have \(|C_+| - |C_-| \in [-C\sqrt{n_1 \log n_1}, C\sqrt{n_1 \log n_1}]\). Note that \(\log(1-\frac{p}{1-q}) \in [-c(p-q), c(p-q)]\) by using Taylor's formula for some constant \(c > 0\). Hence, under \(\mathcal{E}_1\)

\[
\left(1 - \frac{p}{1-q}\right)^{|C_+|-|C_-|} \in \left[e^{-c\sqrt{n_1 \log n_1}}, e^{c\sqrt{n_1 \log n_1}}\right]
\]

for some constant \(c' > 0\). But we have

\[
\max(|e^{c'\sqrt{n_1 \log n_1}} - 1|, |e^{-c'\sqrt{n_1 \log n_1}} - 1|) \leq c' p \sqrt{n_1 \log n_1} := \epsilon.
\]

Therefore, it follows that \((1-\frac{p}{1-q})^{|C_+|-|C_-|} \in [1-\epsilon, 1+\epsilon]\) under \(\mathcal{E}_1\). It is easy to check \(\theta_j \in [1-\epsilon, 1+\epsilon]\) implies \(\alpha_j \in [1/2 - \epsilon', 1/2 + \epsilon']\) for \(\epsilon'\) proportional to \(\epsilon\). Since the constant involved here doesn’t matter, we won’t make a distinction between \(\epsilon\) and \(\epsilon'\).

Now recall that by (6.1) and (6.2) we have

\[
f(A_{ij} | A_{-ij}, z) = \alpha_j p^{A_{ij}} (1-p)^{1-A_{ij}} + (1-\alpha_j) q^{A_{ij}} (1-q)^{1-A_{ij}} \quad \text{when } z_i = 1
\]

and

\[
f(A_{ij} | A_{-ij}, z) = \alpha_j q^{A_{ij}} (1-q)^{1-A_{ij}} + (1-\alpha_j) p^{A_{ij}} (1-p)^{1-A_{ij}} \quad \text{when } z_i = -1.
\]

- When \(z_i = 1\), we have for all \(j \in [n_2]\) such that \(\alpha_j \in [1/2 - \epsilon, 1/2 + \epsilon]\) that

\[
\alpha_j p^{A_{ij}} (1-p)^{1-A_{ij}} + (1-\alpha_j) q^{A_{ij}} (1-q)^{1-A_{ij}} \geq (1-\epsilon) \frac{p+q}{2} 1_{A_{ij}=1} + (1-p\epsilon) \frac{1-p+1-q}{2} 1_{A_{ij}=0}
\]

and

\[
\prod_{j \in J_b} \alpha_j p^{A_{ij}} (1-p)^{1-A_{ij}} + (1-\alpha_j) q^{A_{ij}} (1-q)^{1-A_{ij}} \geq (1-\epsilon) \sum_{j \in J_b} A_{ij} (1-p\epsilon) \sum_{j \in J_b} (1-A_{ij}) \prod_{j \in J_b} p^{A_{ij}} (1-p)^{1-A_{ij}} + q^{A_{ij}} (1-q)^{1-A_{ij}}.
\]

- When \(z_i = -1\), we have for all \(j \in [n_2]\) such that \(\alpha_j \in [1/2 - \epsilon, 1/2 + \epsilon]\)

\[
\alpha_j q^{A_{ij}} (1-q)^{1-A_{ij}} + (1-\alpha_j) p^{A_{ij}} (1-p)^{1-A_{ij}} \geq (1-\epsilon) \frac{p+q}{2} 1_{A_{ij}=1} + (1-q\epsilon) \frac{1-p+1-q}{2} 1_{A_{ij}=0}
\]

and

\[
\prod_{j \in J_b} \alpha_j q^{A_{ij}} (1-q)^{1-A_{ij}} + (1-\alpha_j) p^{A_{ij}} (1-p)^{1-A_{ij}} \geq (1-\epsilon) \sum_{j \in J_b} A_{ij} (1-q\epsilon) \sum_{j \in J_b} (1-A_{ij}) \prod_{j \in J_b} p^{A_{ij}} (1-p)^{1-A_{ij}} + q^{A_{ij}} (1-q)^{1-A_{ij}}.
\]

Since on \(\mathcal{E}_3\), \(\sum_{j \in J_b} A_{ij} = \Theta(n_2p)\) we obtain that

\[
(1-\epsilon) \sum_{j \in J_b} A_{ij} \geq 1 - n_2\epsilon p = 1 - o(1)
\]

and similarly

\[
(1-p\epsilon) \sum_{j \in J_b} (1-A_{ij}) \geq 1 - o(1) \quad \text{and} \quad (1-q\epsilon) \sum_{j \in J_b} (1-A_{ij}) \geq 1 - o(1).
\]

This implies the lower bound

\[
\mathbf{1}_{\mathcal{E}_3} \prod_{j \in J_b} f(A_{ij} | A_{-ij}, z) \geq (1 - o(1)) \mathbf{1}_{\mathcal{E}_3} \prod_{j \in J_b} g(A_{ij}).
\]
Hence, by Lemma 4 in Gao et al. (2018) the risk is lower bounded by

\[ \mathbb{E}_{z_i} \mathbb{E}_{A_i \mid z_i, A_{-i}} (1 \mathbb{E}_z \phi_i(A)) \]

\[ \geq (1 - o(1)) \int_{z_i} \int_{A_i} 1 \mathbb{E}_z \phi_i(A) \prod_{j \in J_b} g(A_{ij}) \prod_{j \in J_g} f(A_{ij} | A_{-ij}, z) dA_i d\mathbb{P}(z_i) \]

\[ \geq \int_{z_i} \int_{(A_{ij})_{j \in J_b}} \prod_{j \in J_b} g(A_{ij}) \left( \phi_i(A) \prod_{j \in J_g} f(A_{ij} | A_{-ij}, z) d(A_{ij})_{j \in J_g} \right) d(A_{ij})_{j \in J_b} d\mathbb{P}(z_i) \]

\[ - \hat{\mathbb{P}}_{A_i \mid A_{-i}, -z}(E^c \mathbb{E}_z) \]

\[ \geq \int_{z_i} \int_{(A_{ij})_{j \in J_b}} \prod_{j \in J_b} g(A_{ij}) \left( \int_{z_i} \int_{(A_{ij})_{j \in J_g}} \phi_i(A) \prod_{j \in J_g} f(A_{ij} | A_{-ij}, z) d(A_{ij})_{j \in J_g} \right) d(A_{ij})_{j \in J_b} d\mathbb{P}(z_i) \]

\[ - \hat{\mathbb{P}}_{A_i \mid A_{-i}, -z}(E^c \mathbb{E}_z) \] (since \( J_b \) and \( J_g \) are disjoint)

\[ \geq \int_{(A_{ij})_{j \in J_b}} \prod_{j \in J_b} g(A_{ij}) \left( \int_{z_i} \int_{(A_{ij})_{j \in J_g}} \phi_i(A) \prod_{j \in J_g} f(A_{ij} | A_{-ij}, z) d(A_{ij})_{j \in J_g} \right) d(A_{ij})_{j \in J_b} d\mathbb{P}(z_i) \]

\[ - \hat{\mathbb{P}}_{A_i \mid A_{-i}, -z}(E^c \mathbb{E}_z) \] (since \( g(A_{ij}) \) is independent from \( z_i \))

where \( \hat{\mathbb{P}}_{A_i \mid A_{-i}, -z}(E^c \mathbb{E}_z) \) is the conditional distribution corresponding to the density product

\[ \prod_{j \in J_b} g(A_{ij}) \prod_{j \in J_g} f(A_{ij} | A_{-ij}, z) d\mathbb{P}(z_i). \]

**Step 5.** \( R''_i \) corresponds to risk associated to the testing problem (6.3) where the product is restricted to the set \( J_g \). One can lower bound this risk as follows.

By definition of \( T_j \), we have when \( z_i = 1 \) by equation (6.1)

\[ f(A_{ij} | A_{-ij}, z) = \mathbb{E}_{T_j} (T_j p_{ij} (1 - p)^{1 - A_{ij}} + (1 - T_j) q^{A_{ij}} (1 - q)^{1 - A_{ij}}) = \mathbb{E}_{T_j} (f_{T_j}(A_{ij})). \]

A similar result holds when \( z_i = -1 \). Since \( T_j \) is independent of \( z_i \) (it only depends on \( A_{-ij} \) and \( z_{-i} \) through \( \alpha_{ij} \)) we can write

\[ R''_i = \mathbb{E}_T \mathbb{E}_{z_i} \int_{(A_{ij})_{j \in J_g}} \phi_i(A) \prod_{j \in J_g} f_{T_j}(A_{ij}) d(A_{ij})_{j \in J_g} = \mathbb{E}_T (R''_i) \]

where \( T = (T_j)_{j \in J_g} \). But now \( R''_i \) is the risk associated with the following two hypothesis testing problem

\[ H'_0'' : \otimes_{T_j=1} \mathcal{B}(p) \otimes_{T_j=0} \mathcal{B}(q), \text{ vs} \]

\[ H'_1'' : \otimes_{T_j=1} \mathcal{B}(q) \otimes_{T_j=0} \mathcal{B}(p). \]

The error associated with this test is lower bounded by the error associated with the test

\[ H''_0 : \otimes_{j \in J_g} \mathcal{B}(p) \otimes_{j \in J_g} \mathcal{B}(q), \text{ vs} \]

\[ H''_1 : \otimes_{j \in J_g} \mathcal{B}(q) \otimes_{j \in J_g} \mathcal{B}(p). \]

since adding more information can only decreases the error (more formally, the lower bound follows from the fact that we multiply by density functions inferior to one). Under \( E_2 \cap \mathcal{E}_1 \) we have

\[ |J_g| = \Theta(n_1 n_2 p) \quad \text{and} \quad |J_b| = (1 - o(1)) n_2. \]  \hspace{1cm} (6.5)

Hence, by Lemma 4 in Gao et al. (2018) the risk is lower bounded by \( R''_i \geq e^{-\Theta(n_1 n_2 p^2)}. \)
Remark 11. Note that the point of the above construction is to reduce our problem to the case where $z_i$ can take only two values, namely $k$ or $k'$. Then we display only the column labels which are different from $k$ and $k'$ — those belonging to $\{k, k'\}$ are masked by the symbol ‘∗’. Indeed, the columns with labels different from $k, k'$ do not provide any information for distinguishing $z_i = k$ from $z_i = k'$.

Conclusion. It remains to integrate over all the events we conditioned on. We have so far shown that

$$R_i \geq E_{z_i} 1_{\tilde{z}_i} E_{A_{z_i}} 1_{z_i} E_{z_i} 1_{z_i} \tilde{p} A_{z_i} E_{A_{z_i}} 1_{z_i} 1_{z_i} \tilde{p} A_{z_i} (E_3')$$

$$\geq e^{-\Theta(n_1n_2p^2)} E_{z_i} 1_{\tilde{z}_i} E_{A_{z_i}} 1_{z_i} (E_2') - E_{z_i} 1_{z_i} E_{A_{z_i}} 1_{z_i} E_{A_{z_i}} 1_{z_i} E_{A_{z_i}} 1_{z_i} 1_{z_i} \tilde{p} A_{z_i} (E_3').$$  (6.6)

Now these probabilities can be controlled with the following lemma proved in the appendix, see Section 3.3.

Lemma 1. We have

1. $\mathbb{P}_{z_i}(\tilde{E}_i) \geq 1 - n_1^{-\Omega(1)}$;
2. for all realizations $z_i$, $\mathbb{P}_{A_{z_i}} 1_{z_i} (\tilde{E}_2') \geq 1 - e^{-\Theta(n_1n_2p)}$;
3. for all $z_i \in \tilde{E}_1$ and $A_{z_i} \in \tilde{E}_2$, $\tilde{p} A_{z_i} 1_{z_i} (\tilde{E}_3') \geq 1 - e^{-\Theta(n_2p)}$.

Lemma 1 and (6.6) directly imply $R_i \geq e^{-\Theta(n_1n_2p^2)}$ since $e^{-\Theta(n_1n_2p^2)} = o(e^{-\Theta(n_1n_2p^2)})$ under the assumption on $p$.

When $n_1n_2p^2_{max} = \Theta(1)$, we can use the same proof except that now the final two-hypothesis testing problem we reduced to has a non vanishing risk (see e.g. Zhang and Zhou (2016)).

6.2 Proof of Theorem 4

The main idea of the proof is to reduce the problem to the case $K = L = 2$, after which we can proceed along the lines of the proof of Theorem 3. The steps are outlined below.

Step 1. Instead of considering independent vectors with marginals given by independent Rademacher random variables, we will use $z, z'$ that are random vectors with independent marginals following the uniform law on $[K]$. The argument from Gao et al. (2018) used in the proof of Theorem 3 leads to the following lower bound (for any given $i = 1, \ldots, n_1$)

$$\inf_{\tilde{z}} \sup_{\theta \in \Theta} E(r(\tilde{z}, z)) \geq \frac{1}{K^3} \inf_{\tilde{z}} E_{z, z'} E_{A|z, z'}(\phi_i(A)) = \frac{1}{K^3} \inf_{\tilde{z}} E_{z} E_{A|z}(\phi_i(A)).$$

Step 2. Let us fix $k \neq k' \in [K]$, we then have

$$\phi_i(A) = 1_{\tilde{z}_i \neq z_i} \geq \tilde{\phi}_i(A) := 1_{\tilde{z}_i = k, z_i = k'} + 1_{\tilde{z}_i = k', z_i = k}.$$  

So if $z_i \notin \{k, k'\}$ then $\tilde{\phi}_i(A) = 0$ whatever the choice of the estimator $\tilde{z}_i$. Denoting $S$ to be the event $\{z_i \in \{k, k'\}\}$, we then have

$$E_{z_i} E_{A|z}(\phi_i(A)) \geq E_{z_i} E_{A|z}(\tilde{\phi}_i(A))$$

$$= \mathbb{P}(S) E_{z_i} E_{A|z}(\tilde{\phi}_i(A)) | S$$

$$= \frac{2}{K} E_{z_i} E_{A|z}(\tilde{\phi}_i(A)) | S.$$  

Now let us define $z'' = h(z')$ as follows. If $z_j' \in [K] \setminus \{k, k'\}$ then $z_j'' = 0$. Otherwise $z_j'' = k'$ where “*” is any symbol that does not belong to $[K]$. In other words, $z''$ only keeps the information about the column labels that are different from $k$ and $k'$.

Remark 11. Note that the point of the above construction is to reduce our problem to the case where $K = L = 2$. First, we reduce to the case where $z_i$ can take only two values, namely $k$ or $k'$. Then we display only the column labels which are different from $k$ and $k'$ — those belonging to $\{k, k'\}$ are masked by the symbol ‘∗’. Indeed, the columns with labels different from $k, k'$ do not provide any information for distinguishing $z_i = k$ from $z_i = k'$.
Since \( z'' \) is independent of \( z \), we obtain

\[
\mathbb{E}_{z_i} \left( \mathbb{E}_{A|z}(\hat{\phi}_t(A))|S \right) = \mathbb{E}_{z_i} \left( \mathbb{E}_{A|z, z''}(\hat{\phi}_t(A))|S \right) = \mathbb{E}_{z''} \mathbb{E}_{z_i} \left( \mathbb{E}_{A|z, z''}(\hat{\phi}_t(A))|S \right).
\]

Conditionally on \( z, S, z'' \), note that the columns of \( A \) are independent since \( (z'_j)_{j=1}^{n_2} \) are independent, hence

\[
f(A|z, S, z'') = \prod_j f(A_{j}|z, z'_j),
\]
i.e. \( f(A|z, S, z'') \) is the product of the densities of the columns \( A_{j} \) conditionally on \( z, S, \) and \( z'' \). For each \( j \) we can write

\[
f(A_{ij}|z_{-ij}, S, z''_{ij}) = f(A_{ij}|A_{-ij}, z, z''_{ij}) f(A_{-ij}|A_{-ij}, z, z''_{ij}) = f(A_{ij}|A_{-ij}, z, z''_{ij}) f(A_{-ij}|z_{-i}, z''_{ij})
\]
and \( f(A_{-ij}|z_{-i}, z''_{ij}) \) doesn’t depend on \( z_i \). We have the following scenarios depending on the value of \( z_i \).

1. Assume that \( z_i = k \). Then \( A_{ij} \sim \mathcal{B}(p) \) if \( z'_j = k \) or \( A_{ij} \sim \mathcal{B}(q) \) if \( z'_j \neq k \).
   - If \( z''_j = * \), we have
     \[
     P(A_{ij} = 1|A_{-ij}, z, S, z''_{ij} = *) = pP(z'_j = k|A_{-ij}, z, S, z''_{ij} = *) + qP(z'_j = k'|A_{-ij}, z, S, z''_{ij} = *) = pP(z'_j = k|A_{-ij}, z, S, z''_{ij} = *) + qP(z'_j = k'|A_{-ij}, z, S, z''_{ij} = *).
     \]
   - If \( z''_j \neq * \), then
     \[
     P(A_{ij} = 1|A_{-ij}, z, S, z''_{ij}) = q.
     \]

2. Assume that \( z_i = k' \).
   - If \( z''_j = * \), similar considerations as before show that
     \[
     P(A_{ij} = 1|A_{-ij}, z, S, z''_{ij} = *) = q\alpha_j + p(1 - \alpha_j).
     \]
   - If \( z''_j \neq * \), then
     \[
     P(A_{ij} = 1|A_{-ij}, z, S, z''_{ij}) = q.
     \]

Denote \( J = \{ j \in [n_2] : z'_j = * \} \). Then by a similar argument as in the proof of Theorem \( \text{[B]} \) we can show that we have obtained a lower bound corresponding to the risk between the two-hypothesis testing problem (conditionally on \( A_{i'}_{j'} \neq i, z_{-j} \) and \( z'' \))

\[
H_0 : \otimes_{j \in J} \mathcal{B}(\alpha_j p + (1 - \alpha_j) q) \otimes_{j \notin J} \mathcal{B}(q) \text{ vs } H_1 : \otimes_{j \in J} \mathcal{B}(\alpha_j q + (1 - \alpha_j) p) \otimes_{j \notin J} \mathcal{B}(q).
\]

This is equivalent to

\[
H_0 : \otimes_{j \in J} \mathcal{B}(\alpha_j p + (1 - \alpha_j) q) \text{ vs } H_1 : \otimes_{j \in J} \mathcal{B}(\alpha_j q + (1 - \alpha_j) p).
\]

By using a conditioning argument on an event that occurs with high probability as in the proof of Theorem \( \text{[B]} \) one can assume that \( |J| = (2 + o(1))n_2/K \). We have reduced to the problem of testing whether \( \hat{z}_i = k \) or \( k' \) with the knowledge of all the values of \( z' \) that are different from \( k \) and \( k' \). Then, to obtain the stated result it remains to apply the proof of Theorem \( \text{[B]} \) from Step 3 with

\[
\begin{align*}
C_+ &= \{ i' \neq i : z_{i'} = k \}, \\
C_- &= \{ i' \neq i : z_{i'} = k' \}, \\
\theta_j &= \frac{\alpha_j}{1 - \alpha_j} \text{ for } j = 1 \ldots n_2, \\
\epsilon &= \Theta(p\sqrt{n_1 \log n_1}) = o(1), \\
J_0 &= \{ j \in J : \theta_j \in [1 - \epsilon, 1 + \epsilon] \}, \\
J_g &= \{ j \in J : \theta_j \notin [1 - \epsilon, 1 + \epsilon] \}, \\
T_j &\overset{\text{ind}}{\sim} \mathcal{B}(\alpha_j), \text{ for all } j \in J_g.
\end{align*}
\]

\(^{2}\)By definition the distribution of \( A|z, z'' = * \) is the same as \( A|z, z'_j \in \{ k, k' \} \) and the distribution of \( A|z, z'' = l \) is the same as \( A|z, z'_j = l \notin \{ k, k' \} \).
and the events
\[ E_1 = \left\{ |C_+| - |C_-| \in [-C \sqrt{n_1/K \log(n_1/K)}, C \sqrt{n_1/K \log(n_1/K)}] \right\} \] for some constant \( C > 0 \),
\[ E_2 = \left\{ \sum_j 1_{\{\sum_{i' \in C_+} A_{i,j} - \sum_{i' \in C_-} A_{i,j} \neq 0\}} = \Theta(n_2 n_1 p/K^2) \right\}, \]
\[ E_3 = \left\{ \sum_{j \in J_k} A_{i,j} = \Theta(n_2 p/K) \right\}. \]

Steps 4 and 5. It is easy to check that
\[ \theta_j = \frac{\alpha_j}{1 - \alpha_j} = \frac{\prod_{i' \neq i: z_{i'} = k} B_{i',j} (1 - p)^{1 - A_{i',j}} \prod_{i' \neq i: z_{i'} = \hat{k}} q_{i',j} (1 - q)^{1 - A_{i',j}}}{\prod_{i' \neq i: z_{i'} = k} p_{i',j} (1 - p)^{1 - A_{i',j}}} \]
By using the same argument as in the proof of Theorem 8, the error associated with the test (6.8) can be lower bounded by
\[ H_0 : \otimes_j \in J_k \mathcal{B}(p) \text{ vs } H_1 : \otimes_j \in J_k \mathcal{B}(q). \quad (6.9) \]

We can conclude by using a similar conditioning argument so that \( |J_k| = \Theta(n_1 n_2^2) \) and the risk of the test (6.9) is lower bounded by \( e^{-\Theta(n_1 n_2^2 p)} \). The stated result is obtained by using the same steps as in Theorem 8 and multiplying by \( 2/K \) (see Step 2). This factor could possibly be removed by lower-bounding \( \phi_i(A) \) by the sum over all \( k' \neq k \) of functions of the form \( \phi_i(A) \). But when \( \frac{n_1 n_2^2 p}{K^2} \to \infty \), the factor \( 2/K \) can be absorbed by the \( \Theta(n_1 n_2^2) \). This improvement is only interesting when \( \frac{n_1 n_2^2 p}{K^2} = \Theta(1) \).

7 Numerical experiments

In this section, we empirically compare the performance of our algorithm (GPM) with the spectral algorithm (Spec) and the algorithm introduced by [Ndaoud et al. 2022] (referred to as HL).

Case \( K = L = 2 \). In this setting, we generate a SBiSBM with parameters \( n_1 = 500, n_2 = [C n_1 \log(n_1)] \) (where \( C \geq 1 \) is a constant), \( p \in (0, 1) \), and \( q = cp \) where \( c > 0 \) is a constant. The accuracy of the clustering is measured by the Normalized Mutual Information (NMI); it is equal to one when the partitions match exactly and is zero when they are independent. The results are averaged over 20 Monte-Carlo runs. For the experiment presented in Figure 1A, we fixed \( C = 10 \); for the experiment in Figure 1B, we fixed \( C = 3 \) and \( c = 0.5 \). For the experiment presented in Figure 1C, we fixed \( p = 0.01 \) and \( c = 0.5 \). We observe that HL and GPM have similar performance in all the aforementioned experiments. Thus, there is no gain in using the specialized method HL instead of the general algorithm GPM. The spectral method Spec has only slightly worst performance than the iterative methods HL and GPM. In particular, when approaching the threshold for exact recovery, the performance gap disappears. This suggests that Spec also reaches the threshold for exact recovery. It would be interesting to obtain stronger theoretical guarantees to explain the good performance of Spec.

Case \( K = L \neq 2 \). We fix \( n_1 = 1000, n_2 = 10000, p = 0.05 \) and \( c = 0.5 \) and vary \( K \) from 2 to 10. As can be seen from Figure 1D, the performance of Spec decreases faster than GPM when \( K \) increases.

For these experiments, we assumed that the number of communities \( K \) was known. In practice, one can try to estimate the number of clusters by using the simple and popular elbow method which chooses a \( K \) that maximizes the spectral gap \( \lambda_k(B) - \lambda_{k+1}(B) \) (recall \( B \) from Algorithm 1). From a more theoretical point of view, it would be interesting to see if the selection method proposed by [Jin et al. 2021], which also comes with optimality guarantees, can be extended to the BiSBM setting.
Figure 1: Relative performance of Spec, HL and GPM for varying parameters

(a) Relative performance of Spec, HL and GPM for varying $c$.

(b) Relative performance of Spec, HL and GPM for varying $p$.

(c) Relative performance of Spec, HL and GPM for varying $C$.

(d) Relative performance of Spec and GPM for varying $K$. 

Figure 1: Relative performance of Spec, HL and GPM for varying parameters
8 Conclusion and future work

In this work, we proposed an algorithm based on the GPM to cluster the rows of bipartite graphs in the high-dimensional regime where \(n_2 \gg n_1 \log n_1\). We analyzed our algorithm under a relatively general BiSBM, that incorporates as a special case the model with \(K = L = 2\) studied recently by Ndaoud et al. (2022). When specialized to the SBiSBM with \(K = L = 2\) communities, our rate of convergence matches the one obtained by Ndaoud et al. (2022). We also extend the aforementioned work in another direction by showing that the derived rate of convergence of the GPM for the SiSBM (with \(K = L = 2\)) is minimax optimal.

We only considered binary bipartite graphs in this work, but our algorithm could possibly be extended to the weighted case. Some heterogeneity amongst blocks could also be introduced, similar to the Degree-Corrected SBM. Another interesting direction would be to improve the theoretical guarantees obtained for the spectral method as we observed that it has very good performance in practice. A natural way to do that would be to adapt the entrywise eigenvector techniques developed by Abbe et al. (2020) to the BiSBM setting.

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Supplementary Material

A Useful inequalities

Lemma 2. For all $z, z' \in [K]^{n_1}$ we have

$$h(z, z') \leq \frac{l(z, z')}{\Delta_{\min}^2}$$

where $h(z, z') = \sum_{i \in [n_1]} 1_{z_i \neq z'_i}$ is the Hamming distance between $z$ and $z'$.

Proof. For all $z, z'$ we have

$$\sum_{i \in [n_1]} 1_{z_i \neq z'_i} \leq \sum_{i \in [n_1]} \frac{\Delta(z_i, z'_i)^2}{\Delta_{\min}^2} 1_{z_i \neq z'_i} = \frac{l(z, z')}{\Delta_{\min}^2}.$$  \hfill $\square$

Lemma 3. Assume that for some $\alpha \geq 1$ we have

$$\frac{n_1}{\alpha K} \leq |C_k| \leq \frac{\alpha n_1}{K}$$

for all $k \in [K]$.

Let $C_k^{(t)} = \{i \in [n_1] : z^{(t)}_i = k\}$ be the nodes associated with the community $k$ at some iteration $t \geq 0$, with $z^{(t)} \in [K]^{n_1}$. If $l(z, z^{(t)}) \leq n_1 \Delta_{\min}^2 / (2\alpha K)$ then for all $k \in [K]$,

$$\frac{n_1}{2\alpha K} \leq |C_k^{(t)}| \leq \frac{2\alpha n_1}{K}.$$  \hfill $\square$

Proof. This is a standard result, we recall the proof for completeness. Since for all $k \in [K]$ we have $n_1 / (\alpha K) \leq |C_k| \leq \alpha n_1 / K$,

$$1 \geq \sum_{i \in C_k} 1 \geq \sum_{i \in C_k \cap C_k^{(t)}} 1 \geq \sum_{i \in [n_1]} 1 - \sum_{i \in [n_1]} 1_{z_i \neq z_i^{(t)}}$$

$$\geq \frac{n_1}{\alpha K} - h(z, z^{(t)})$$

$$\geq \frac{n_1}{\alpha K} - \frac{l(z, z^{(t)})}{\Delta_{\min}^2}$$

$$\geq \frac{\alpha n_1}{2K}$$

by assumption. The other inequality can be proved in a similar way. \hfill $\square$

Lemma 4. Assume that $A \sim BiSBM(Z_1, Z_2, \Pi)$ with approximately balanced communities. Then for all $t \geq 0$ and $z^{(t)}$ such that $l(z^{(t)}, z) \leq n_1 \Delta_{\min}^2 / (2\alpha K)$, the following holds.

1. $\max_{k \in [K]} \|W_k^{(t)} - W_k\| \lesssim \frac{(\alpha K)^{-1/2}}{n_1^{1/2}} l(z^{(t)}, z)$,

2. $\|Z_1^T W^{(t)}\| \lesssim 1$.

Proof. The proof is an adaptation of Lemma 4 in Han et al. (2020), see also Lemma 14 in Braun et al. (2021b). Since we consider a slightly different setting, we outline the details for completeness.

Let $n_{\min} = \min_k |C_k|$. First observe that $Z_1$ is rank $K$ and $\lambda_K(Z) = \sqrt{n_{\min}}$ so that

$$\|W_k^{(t)} - W_k\| \leq \|W^{(t)} - W\| \leq n_{\min}^{1/2} \|I - Z_1^T W^{(t)}\|_F.$$
For any $k \in [K]$, denote $\delta_k = 1 - (Z_i^T W^{(t)})_{kk}$. Since for all $k, k' \in [K]$

$$
(Z_i^T W^{(t)})_{kk'} = \frac{\sum_{t \in C_k} 1_{\hat{z}^{(t)} = k'}}{|C^{(t)}_{k'}|},
$$
we have

$$
0 \leq \delta_k \leq 1, \quad \sum_{k' \in [K]\setminus k} (Z_i^T W^{(t)})_{k'k} = \delta_k.
$$

Therefore,

$$
\|Z_i^T W^{(t)} - I\|_F = \sqrt{\sum_{k \in [K]} \left( \sum_{k' \in [K]\setminus k} (Z_i^T W^{(t)})_{k'k}^2 \right)}
$$

$$
\leq \sqrt{2 \sum_{k \in [K]} \left( \sum_{k' \in [K]\setminus k} (Z_i^T W^{(t)})_{k'k}^2 \right)}
$$

$$
= \sqrt{2 \sum_{k \in [K]} \sum_{i \in C_k} \sum_{\hat{z}^{(t)} \neq k} \frac{1_{\hat{z}^{(t)} = k}}{|C^{(t)}_{k'}|}}
$$

$$
\leq \sqrt{2 \max(|C^{(t)}_{k'}|)^{-1} \sum_{z \in [n]} 1_{\hat{z}^{(t)} \neq z}}
$$

$$
\lesssim \frac{K \alpha}{n_1} H(z, z^{(t)}) \lesssim \frac{K \alpha}{n_1 \Delta_{\min}}.
$$

(A.1)

For the second inequality, observe that since $Z_i^T W = I_K$ we have

$$
\|Z_i^T W^{(t)}\| \leq 1 + \|Z_i^T (W^{(t)} - W)\|
$$

$$
= 1 + \|I - Z_i^T W^{(t)}\|
$$

$$
\lesssim 1 + \alpha K \frac{I(z^{(t)}, z)}{n_1 \Delta_{\min}^2}
$$

(by Equation (A.1))

$$
\lesssim 1. \quad \text{(by assumption on } I(z^{(t)}, z))
$$

\[\square\]

## B Concentration inequalities

### B.1 General results

**Lemma 5.** Assume that $A \sim BiSBM(Z_1, Z_2, \Pi)$, and recall $E = A - E(A)$. For $n_1$ large enough, the following holds true.

1. $\mathbb{P}(|E| \lesssim \sqrt{n_2 p_{\max}}) \geq 1 - n_1^{-\Omega(1)}$ if $n_2 p_{\max} \gtrsim \log n_2$ and $n_2 \geq n_1$.

2. $\mathbb{P}(|EE^T - \mathbb{E}(EE^T)| \lesssim \max(\log n_1, \sqrt{n_1 n_2 p_{\max}})) \geq 1 - n_1^{-\Omega(1)}$. Moreover, this event implies $\|\mathcal{H}(EE^T)\| \lesssim \max(\log n_1, \sqrt{n_1 n_2 p_{\max}})$.

3. $\mathbb{P}(|EZ Z| \lesssim \sqrt{n_1 n_2 p_{\max}/L}) \geq 1 - n_1^{-\Omega(1)}$ if $\alpha n_2 p_{\max} \gtrsim L \log n_1$ and $\sqrt{n_1 n_2 p_{\max}} \gtrsim \log n_1$. 
Proof. 1. The first inequality follows from classical proof techniques. We first convert $E$ into a square symmetric matrix as follows

$$\tilde{E} = \begin{pmatrix} 0 & E^\top \\ E & 0 \end{pmatrix}.$$ 

It is easy to verify that $\|\tilde{E}\| = \|E\|$. Since $\tilde{E}$ is a square symmetric matrix with independent entries, we can use the result of Remark 3.13 in [Bandeira and van Handel (2016)] and obtain with probability at least $1 - n_1^{-\Omega(1)}$ that

$$\|\tilde{E}\| \lesssim \sqrt{n_2p_{\max}}$$

since $n_2p_{\max} \gtrsim \log(n_1 + n_2)$. This condition is ensured since $n_2 \geq n_1$ and $n_2p_{\max} \gtrsim \log n_2$.

2. The second inequality follows from the recent work of Cai et al. (2022). First observe that $c \equiv c(n_1)$, $n_2$, and $\varphi$ are all related through $L \geq \varphi(n_1)$. Hence it is easy to see that

$$c \equiv c(n_1) \leq \varphi(n_1).$$

It is however more difficult to get a high probability bound from this last inequality since we can no longer use Talagrand’s inequality as in [Bandeira and van Handel (2016)]. However, we can use the moments to obtain a tail bound as in Theorem 5 in Cai et al. (2022). This theorem is stated for matrices with Gaussian entries, but if instead of Lemma 1 we use Lemma 9 of Cai et al. (2022), we obtain a similar result for bounded sub-Gaussian entries. Since the variance parameters $\sigma_R$ and $\sigma_C$ that appear in the statement of Theorem 5 in Cai et al. (2022) satisfy $\sigma_R^2 \leq n_1p_{\max}$ and $\sigma_C^2 \leq n_2p_{\max}$ we obtain the result.

3. The last inequality can be obtained by using the proof techniques in Bandeira and van Handel (2016) as follows. In order to extend the concentration result from a matrix $Y$ with independent standard Gaussian entries to a matrix $X$ with symmetric sub-Gaussian entries, the key is to upper bound all the moments of $X_{ij}$ by moment of $Y_{ij}$. This can be done by using the boundedness of $Z_{ij}$ as in Corollary 3.2, or the sub-Gaussian norm of $X_{ij}$ as in Corollary 3.3 of Bandeira and van Handel (2016). But in our case, none of these bounds gives a good result. However, the proof of Theorem 1.1 (and its extensions) only requires control of the moments of the order $\log n_1$. For a Binomial r.v. $X$ with parameters $\alpha n_2/L$ and $p_{\max}$, we have, according to Theorem 1 in [Ahl (2022)], for all $c \in \mathbb{N}^*$,

$$\mathbb{E}(X^c) \leq (\alpha n_2p_{\max}/L)^c e^{c^2/(2\alpha n_2p_{\max}/L)}.$$ 

Let $X'$ be an independent copy of $X$. Since $\alpha n_2p_{\max}/L \gtrsim \log n_1$ by assumption, $e^{c^2/(2\alpha n_2p_{\max}/L)} \leq e^{\gamma c}$ for $c \asymp \log(n_1)$ and an absolute constant $\gamma > 0$. Hence

$$\mathbb{E}((X - X')^c) \leq \sum_{i \leq c} \binom{c}{i} \mathbb{E}(X^i) \mathbb{E}(X'^{-i}) \leq 2^c (\alpha n_2p_{\max}/L)^c e^{\gamma c}.$$ 

Observe that $(2c)^c \lesssim \mathbb{E}Y_{ij}^{2c} = O((2c)^c)$ for every $c$, so that for all even $c$ we have

$$\mathbb{E} \left( \frac{L(X - X')}{2c^{\gamma} \alpha n_2p_{\max}} \right)^c \lesssim \mathbb{E}Y_{ij}^{2c}.$$ 

We can now use the same argument as in Corollary 3.2 of [Bandeira and van Handel (2016)] to conclude that the matrix $M$ with independent entries generated with the same law as $X - X'$ satisfies with probability at least $1 - O(n_1^{-\Omega(1)})$ (since $\sqrt{\alpha n_1n_2p_{\max}/L} \gtrsim \log n_1$ by assumption)

$$\|M\| \leq \sqrt{\alpha n_1n_2p_{\max}/L}.$$
When the random variables are only centered, we can use the symmetrization argument of Corollary 3.3 to finally obtain
\[ \|EZ_i^\top\| \lesssim \sqrt{n_1 n_2 P_{\text{max}}/L}, \]
with probability at least \( 1 - O(n_1^{\Omega(1)}) \).

**Remark 12.** The second concentration inequality of the above lemma slightly improves Proposition 1 and Theorem 4 in [Ndaoud et al. (2022)]. The third concentration could be of independent interest and be applied for example in multilayer network analysis where matrices with independent Binomial entries arise naturally as the sum of the adjacency matrices of the layers, see e.g. [Paul and Chen (2020); Braun et al. (2021a)].

### B.2 Control of the oracle error term

**Lemma 6.** Assume that the assumptions of Theorem 4 hold. Recall that
\[ \Omega_{i,k}(\delta) = \left\{ \tilde{E}_i(W_{zi} - W_{z}) \leq -(1 - \delta)\Delta^2(z_i, k) \right\}. \]
We have for all \( 0 < \delta \leq \frac{1}{4np} \), \( k \neq z_i \in [K] \) and \( i \in [n_1] \)
\[ \mathbb{P}(\Omega_{i,k}(\delta)) \leq e^{-\frac{\delta^2}{12n^2 L^2}} \leq e^{-\Delta^2}. \]

**Proof.** The event \( \Omega_{i,k}(\delta) \) holds if and only if
\[ B_{i}(W_{zi} - W_{zi}) \geq -\delta \Delta^2(z_i, k). \]
But we can decompose this quantity as
\[ B_{i}(W_{zi} - W_{zi}) = \sum_{j \neq i} B_{ij}(W_{jk} - W_{jzi}) \]
\[ = |C_k|^{-1} \sum_{j \in C_k \setminus \{i\}} (A_{i,j}, A_{j};) - |C_i|^{-1} \sum_{j \in C_i \setminus \{i\}} (A_{i,j}, A_{j};) \]
\[ = (A_{i,k} - \tilde{A}_k, \tilde{A}_z) \]
where \( \tilde{A}_k = \frac{1}{|C_k|} \sum_{j \in C_k} A_j \) and \( \tilde{A}_z = \frac{1}{|C_z|} \sum_{j \in C_z \setminus \{i\}} A_j \) are independent random variables. Since the index \( i \) doesn’t appear in the second sum, \( \tilde{A}_z \) is independent from \( A_i \). By definition the entries of \( \tilde{A}_k \) and \( \tilde{A}_z \) are independent normalized binomial random variables whose parameters vary depending on the community associated with the entry.

We are now going to bound the moment generating function of \( M := (A_i, \tilde{A}_k - \tilde{A}_z) \), conditionally on \( A_i \). Observe that \( M \) is a sum of independent random variables. Recall that if \( X \sim B(p) \) then \( \mathbb{E}(e^{tX}) = (e^t p + 1 - p). \)
Hence, for \( t > 0 \), conditionally on \( A_{ij} \), each summand has a m.g.f equal to
\[
\log \mathbb{E}(e^{tA_{ij}(\tilde{A}_k - \tilde{A}_z)} | A_{ij}) = |C_k| \log(e^{t\tilde{A}_k} \Pi_{z'j} + 1 - \Pi_{z'j}) + (|C_z| - 1) \log(e^{-t\tilde{A}_z} \Pi_{z'j} + 1 - \Pi_{z'j})
\]
\[ \leq |C_k| \Pi_{z'j} (e^{t\tilde{A}_k} - 1) + (|C_z| - 1) \Pi_{z'j} (e^{-t\tilde{A}_z} - 1) \]
by using the fact that \( \log(1 + x) \leq x \) for all \( x > -1 \).

Fix \( t = t^* = \epsilon \frac{\log K}{\alpha K} \) for a parameter \( \epsilon \in (0, 1) \) that will be fixed later. We have by Taylor Lagrange formula for all \( t \leq \frac{\alpha}{\alpha K} \)
\[ e^{\frac{t}{|C_k|}} - 1 \leq \frac{t}{|C_k|} + \frac{e}{2} \left( \frac{t}{|C_k|} \right)^2 \]
and
\[ e^{\frac{t}{|C_z|}} - 1 \leq -\frac{t}{|C_z|} + \frac{e}{2} \left( \frac{t}{|C_z|} \right)^2. \]
By using these upper bounds we get
\[
\log \mathbb{E}(e^{t^* A_{ij}}(A_{ij} - \hat{A}_{ij}) | A_{ij}) \leq \left( \prod_{k: z_k' = z_k} \left| \frac{C_{z_k} | - 1}{|C_{z_k}|} \Pi_{z_k} z_k' \right| \right) t^* A_{ij} + \frac{e}{2} (t^*)^2 A_{ij} \left( \frac{\prod_{k: z_k' = z_k} \Pi_{z_k} z_k' + \Pi_{z_k} z_k'}{|C_{z_k}|} \right).
\]
Hence by using independence we obtain
\[
\log \mathbb{E}(e^{t^* M} | A_{ij}) \leq \sum_{j \in \{1, \ldots, n\}} \left( \prod_{k: z_k' = z_k} \left| \frac{C_{z_k} | - 1}{|C_{z_k}|} \Pi_{z_k} z_k' \right| t_{ti} A_{ij} + \frac{e}{2} (t^*)^2 \left( \frac{\prod_{k: z_k' = z_k} \Pi_{z_k} z_k' + \Pi_{z_k} z_k'}{|C_{z_k}|} \right) \right) A_{ij}.
\]
Using Markov inequality and the fact that \(\Delta^2(z, l) \leq \eta \beta n_2 p_{max}^2 / L\) by (5.3) leads to
\[
\mathbb{E}(P(\Omega_{i,k}(\delta) | A_{ij})) \leq e^{\frac{\delta}{n_2 p_{max}}} \Delta^2(z, l) \mathbb{E}(\mathbb{E}(e^{t^* M} | A_{ij})) \leq e^{\frac{\delta}{n_2 p_{max}}} \Pi_j \mathbb{E}(\mathbb{E}(e^{t^* A_{ij}})).
\]
But since \(A_{ij}\) is a Bernoulli random variable with parameter \(\Pi_{z_i} z_j\) we have
\[
\mathbb{E}(e^{t_{ij} A_{ij}}) = (e^{t_{ij}} - 1) \Pi_{z_i} z_j + 1.
\]
By our choice of \(t^*\), \(t_{ij} = O(n_2 p_{max}) = o(1)\) so that
\[
\prod_j (\Pi_{z_i} z_j + 1) \leq e^{\sum_j (\Pi_{z_i} z_j - 1)}.
\]
Here we use the fact that for \(x_1, \ldots, x_n > -1\) we have \(\prod_{i \in [n]} (1 + x_i) \leq e^{\sum_{i \in [n]} x_i}\). But again, by using Taylor Lagrange formula, we have \(e^{t_{ij}} - 1 = t_{ij} + O(t_{ij}^2)\). Consequently
\[
\prod_j \mathbb{E}(e^{t_{ij} A_{ij}}) \leq e^{\sum_j (t_{ij} + o(t_{ij})) \Pi_{z_i} z_j}.
\]
We can write \(\sum_j t_{ij} \Pi_{z_i} z_j\) as
\[
t^* \sum_{l \in [L]} |C_l| \left( \Pi_{z_i} \Pi_{kl} - \Pi_{z_i}^2 \left( \frac{|C_{z_i} | - 1}{|C_{z_i}|} \right) \right) + \frac{e (t^*)^2}{2} \sum_{l \in [L]} |C_l| \left( \Pi_{z_i} \Pi_{kl} \left( \frac{\Pi_{z_k} z_k'}{|C_k|} + \Pi_{z_k} z_k' \left( \frac{|C_{z_k} | - 1}{|C_{z_k}|} \right) \right) \right).
\]
By Assumption \(A1\) and \(A3\) we have
\[-A_1 \geq \frac{n_2}{\alpha L} (\beta p_{max}^2 + o(p_{max}^2))\]
and
\[-A_2 \leq \frac{\alpha K}{n_1} n_2 p_{max}^2.\]
Let \(\epsilon = \frac{\theta}{\alpha L}\) (recall that \(t^* = \epsilon \frac{n_2}{\alpha L}\)). By this choice of \(\epsilon\) we have
\[
\epsilon \ln \frac{\epsilon n_1}{2 \alpha K} A_2 \leq \frac{|A_1|}{2}.
\]
Consequently we have for all \(\delta \leq 1/(4 \eta \alpha)\)
\[
\mathbb{P}(\Omega_{i,k}(\delta)) \leq e^{- \frac{\delta}{n_2 p_{max}} \frac{\epsilon n_1 p_{max}^2}{2 \alpha K} \Delta^2(z, k) \leq e^{- \frac{n_2 p_{max}^2}{4 \alpha L} \frac{\epsilon n_1 p_{max}^2}{2 \alpha K} \Delta^2(z, k) \leq e^{- \frac{\epsilon^2 n_1 p_{max}^2}{4 \alpha L} \Delta^2(z, k)}.
\]
\(\square\)
B.3 Proof of Lemma 1

The first inequality is a direct consequence of Hoeffding concentration inequality. For the second inequality, observe that
\[
P_{A^{-i}|z^{-i}}(E_2) = \mathbb{E}_{z'} P_{A^{-i}|z^{-i},z'}(E_2)
\]
and
\[
P_{A^{-i}|z^{-i},z'}(\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_+} A_{i'j} = 0) \geq P(A_{i'j} = 0 \text{ for all } i') \geq (1 - p)^n = 1 - n_1 p.
\]

To obtain an upper bound on this probability, we can use Paley-Zigmund inequality as follows. Let us denote
\[
Z = |\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_+} A_{i'j}|.
\]
We then have
\[
P_{A^{-i}|z^{-i},z'}(Z = 0) = 1 - P_{A^{-i}|z^{-i},z'}(Z > 0) \leq 1 - P_{A^{-i}|z^{-i},z'}(Z > \theta E Z)
\]
\[
\leq 1 - (1 - \theta)^2 \frac{(E_{A^{-i}|z^{-i},z'}(Z))^2}{E_{A^{-i}|z^{-i},z'}(Z^2)}
\]
for $\theta \in (0, 1)$ by Paley-Zigmund inequality. It is easy to check that
\[
E_{A^{-i}|z^{-i},z'}Z \geq n_1 (p - q) \text{ and } E_{A^{-i}|z^{-i},z'}(Z^2) \asymp n_1 p.
\]
Consequently, $P_{A^{-i}|z^{-i},z'}(Z = 0) = 1 - \Theta(n_1 p)$. By using independence over $j$ and Chernoff’s multiplicative bound we obtain that
\[
P_{A^{-i}|z^{-i},z'} \left( \sum_j 1_{\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_-} A_{i'j} \neq 0} = \Theta(n_2 n_1 p) \right) \geq 1 - e^{-cn_2 n_1 p}
\]
for any realization of $z^{-i}$ and $z'$. Hence the stated bound follows.

Finally it remains to control the probability of $E_3$. Under $\tilde{P}_{A_{-i}|A_{-i},z}$, for each $j \in J_b$, $A_{ij}$ are independent and distributed as a mixture of Bernoulli of parameters $p$ and $q$. The size of the set $J_b$ is also controlled by the assumption $A_{-i} \in E_2$ (see equation (6.5)). Since under $\tilde{P}_{A_{-i}|A_{-i},z}$ for all $j \in J_b$, $A_{ij}$ are independent r.v. with density $g$, we obtain by stochastic domination (we can replace a mixture of Bernoulli with parameters $p$ and $q$ by a Bernoulli of parameter $p$) and Chernoff’s bound that $\tilde{P}_{A_{-i}|A_{-i},z}(E_3) \geq 1 - e^{-\Theta(n_2 p)}$. 

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1 The statistical framework

The Bipartite SBM (BiSBM) is defined by the following parameters.

* A set of nodes of type I, $\mathcal{N}_1 = [n_1]$, and a set of nodes of type II, $\mathcal{N}_2 = [n_1]$.
* A partition of $\mathcal{N}_1$ into $K$ communities $C_1, \ldots, C_K$ of equal sizes $n_1/K$ and a partition of $\mathcal{N}_2$ into $L$ communities $C'_1, \ldots, C'_L$ of equal sizes $n_2/L$.
* Membership matrices $Z_1 \in \mathcal{M}_{n_1,K}$ and $Z_2 \in \mathcal{M}_{n_2,L}$ where $\mathcal{M}_{n,K}$ denotes the class of membership matrices with $n$ nodes and $K$ communities. Each membership matrix $Z_1 \in \mathcal{M}_{n_1,K}$ can be associated bijectively with a function $z : [n] \rightarrow [K]$ such that $z(i) = z_i = k$ where $k$ is the unique column index satisfying $(Z_1)_{ik} = 1$. To each matrix $Z \in \mathcal{M}_{n_1,K}$ we can associate a matrix $W$ by normalizing the columns of $Z_1$ in the $\ell_1$ norm: $W = ZD^{-1}$ where $D = \text{diag}(n_1/K, \ldots, n_1/K) \in \mathbb{R}^{K \times K}$. This implies that $W^\top Z = I_K = Z^\top W$.
* A connectivity matrix of probabilities between communities

$$
\Pi = (\pi_{kk'})_{k \in [K], k' \in [L]} \in [0,1]^{K \times L}.
$$

Let us write $P = (p_{ij})_{i,j\in[n]} := Z_1 \Pi (Z_2)^\top \in [0,1]^{n_1 \times n_2}$. A graph $G$ is distributed according to $\text{BiSBM}(Z_1, Z_2, \Pi)$ if the entries of the corresponding bipartite adjacency matrix $A$ are generated by $A_{ij} \overset{\text{ind.}}{\sim} B(p_{ij})$, $1 \leq i, j \leq n$,

where $B(p)$ denotes a Bernoulli distribution with parameter $p$. Hence the probability that two nodes are connected depends only on their community memberships. We will frequently use the notation $E$ for the centered noise matrix defined as $E_{ij} = A_{ij} - p_{ij}$, and denote the maximum entry of $P$ by $p_{\max} = \max_{i,j} p_{ij}$. The latter can be interpreted as the sparsity level of the graph.

**Assumption A1.** Throughout this work we will assume that $n_2 \gtrsim n_1 \log n_1$.

For the analysis, we will consider a particular case where $L = K$ and $\Pi = (p - q)1_K1_K^\top + qI_K$ where $1 > p > q > 0$. This model will be referred as the Symmetric BiSBM (SBiSBM).

The **misclustering rate** associated to an estimated partition $\hat{z}$ is defined by

$$
r(\hat{z}, z) = \frac{1}{n} \min_{\pi \in \mathcal{G}} \sum_{i \in [n]} 1_{\{\hat{z}(i) \neq \pi(z(i))\}},
$$

where $\mathcal{G}$ denotes the set of permutations on $[K]$. It corresponds to the proportion of wrongly assigned nodes labels. We say that we are in the **exact recovery** regime if $r(\hat{Z}, Z) = 0$ with probability $1 - o(1)$ as $n$ tends to infinity. If $\mathbb{P}(r(\hat{Z}, Z) = o(1)) = 1 - o(1)$ as $n$ tends to infinity then we are is the **weak consistency** or **almost full recovery regime**. A more complete overview of the different types of consistency and the sparsity regimes where they occur can be found in ?.

2 Minimax lower bound

Let us denote by

$$
\Theta = \{P \in [0,1]^{n_1 \times n_2} : P = Z_1 \Pi Z_2^\top \text{ where } \Pi = (p - q)1_K1_K^\top + qI_K
$$

and $Z_1 \in \mathcal{M}_{n_1,K}^\prime$, $Z_2 = \mathcal{M}_{n_2,K}^\prime$} the admissible parameters space ($\mathcal{M}_{n_1,K}^\prime$ corresponds to set of approximated partition with $K$ communities).

We want to lower bound $\inf_\mathcal{Z} \sup_{\mathcal{P} \in \Theta} \mathbb{E}(r(\hat{z}, z))$ where $\hat{z}$ is an estimator of $Z_1$. For simplicity, we will assume that $K = 2$, but we believe that the same argument can be extended to general $K$. In the supervised case, ie. when $Z_2$ is known, we can use the same strategy as the one use for the degree-corrected SBM by ? and
obtain a lower bound of the order $e^{-(1+o(1))n_2(p-q)/2}$ corresponding to the failing probability of the optimal test associated to the following two hypothesis problem

$$
H_0 : G \sim \otimes_{i=1}^{n_2/2} \mathcal{B}(p) \otimes_{i=n_2/2+1}^{n_2} \mathcal{B}(q), \text{ vs } \ H_1 : G \sim \otimes_{i=1}^{n_2/2} \mathcal{B}(q) \otimes_{i=n_2/2+1}^{n_2} \mathcal{B}(p).
$$

However when $n_2 \gg n_1 \log n_1$ this rate of convergence $n_2(p-q)/2 \approx \sqrt{n_2 \log n_1 \over n_1}$ is faster than $n_1 n_2 p_{\text{max}}^2 = \log n_1$. A similar phenomena appears for high-dimensional Gaussian mixture. Indeed, as shown in ?, it is essential to capture the hardness of estimating the model parameters in the minimax lower bound in order to get the right rate of convergence. The argument developed for obtaining a lower bound of the minimax risk in ? relies heavily on the Gaussian assumption (they set a Gaussian prior on the model parameters and use the fact that the posterior distribution is also Gaussian) and cannot directly be extended to this setting. Our result are summarised in the following theorem.

**Theorem 1.** Under the assumption $K = L = 2$, $n_1 n_2 p_{\text{max}}^2 \rightarrow \infty$, $n_2 p_{\text{max}}^2 = O(\log n_1)$ and $\sqrt{n_1 \log n_1} n_2 p_{\text{max}}^2 \rightarrow 0$, we have

$$
\inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z)) \geq \exp(-cn_1 n_2 p_{\text{max}}^2).
$$

If $n_1 n_2 p_{\text{max}}^2 = O(1)$, then $\inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z) / n) \geq c'$ for some positive constant $c'$.

**Remark 1.** This lower-bound shows that the rate of convergence of our estimator is optimal up to a constant factor. Indeed, for exact recovery the minimax lower bound implies that $n_1 n_2 p_{\text{max}}^2 \approx \log n_1$, and for weak recovery, we need $n_1 n_2 p_{\text{max}}^2 \rightarrow \infty$.

**Remark 2.** For simplicity, we only write the proof for $K = L = 2$ but it can be extended in a straightforward way when $K = L > 2$ by reducing the problem to test between two fixed community like in ?.

**Proof.** The general idea of the proof is to first lower bound the minimax risk by an error occurring in a two hypothesis testing problem and then to replace this hypothesis testing problem by a simpler one.

**Step 1.** To simplify notations, we will denote by $z$ the partition function associated with $Z_1$ instead of $z_1$. Let $\pi = \pi_1 \times \pi_2$ a prior on $(z, z_2)$ such that its marginal on $z$ is sign invariant. By using standard arguments (see ? or ?) one can show that

$$
\inf_{\hat{z}} \sup_{\theta \in \Theta} \mathbb{E}(r(\hat{z}, z)) \geq \frac{2}{n_1} \sum_{i=1}^{n_1/2} \inf_{\hat{z}_i} \mathbb{E}_{\pi}(1_{\hat{z}_i \neq z_i} | z_2, z)
$$

where $\hat{z}_i$ is a measurable function in $A$. Without lost of generality, we can always assume that $\hat{z}_i \in [-1, 1]$.

**Step 2.** Without lost of generality, we can assume that $n_2$ is odd (else just delete one column: it won’t increase the error). We choose as a prior on $z_2$ a product of independent centered Rademacher and the same for $z$. We can write

$$
\mathbb{E}_{\pi}(1_{\hat{z}_i \neq z_i} | z_2, z) = \mathbb{E}_{\pi}(1_{\hat{z}_i \neq z_i}, z_2, \mathbb{E}(1_{\hat{z}_i \neq z_i} | z_2, z))
$$

where $z_{-i} := (z_j)_{j \neq i}$ and $\mathbb{E}_{\theta}$ means that we integrate only over the random variable $\theta$. Because $z_2$ is random, the entries of $A$ are no longer independent (this is the reason why we use an uninformative prior on $z_2$). The quantity $\mathbb{E}_{\pi}(1_{\hat{z}_i \neq z_i} | z_2, z)$ only depends on $f(A | z_2)$ the density of $A$ conditionally on $z$. Since the columns of $A$ are independent (because $(z_{2j})_{j=1}^{n_2}$ are independent) it correspond to $\prod_j f(A_j | z)$ the product of the density of the columns $A_{2j}$ conditionally on $z$. For each $j$ we can write

$$
f(A_{2j} | z) = f(A_{2j} | (A_{2j}', v' \neq i, z) f((A_{2j}')_{v' \neq i} | z).
$$

Since $(A_{2j}')_{v' \neq i}$ doesn’t depend on $z_2$, we have $f(A_{2j} | (A_{2j}', v' \neq i, z) = f(A_{2j} | (A_{2j}')_{v' \neq i} | z_{-i})$. Assume that $z_{-i} = 1$. Then $A_{2j}$ is sampled from a Bernoulli with parameter $p$ if $z_{2j} = 1$ or a Bernoulli with parameter $q$ if $z_{2j} = -1$. Hence

$$
P(A_{2j} = 1 | (A_{2j}')_{v' \neq i}, z) = p P(z_{2j} = 1 | (A_{2j}')_{v' \neq i}, z) + q P(z_{2j} = -1 | (A_{2j}')_{v' \neq i}, z).
$$
Let us denote by $\alpha_j$ the random variable $\mathbb{P}(z_{2j} = 1 | (A_{i'})_{i' \neq i}, z)$. Hence we have shown that $\mathbb{E}_{z_{2j}} \mathbb{E}_z (1_{z_{2j} \neq 1} | z, z_2)$ is lower bounded by the error $\inf_\phi \mathbb{P}_{H_0, \phi} + \mathbb{P}_{H_1 (1 - \phi)}$ associated with the following two hypothesis testing problem (conditionally on $(A_{i'})_{i' \neq i}$ and $z_{-i}$):

\begin{align*}
H_0 : G &\sim \otimes_{i=1}^{n_2} (\alpha_j B(p) + (1 - \alpha_j) B(q)), \quad \text{vs} \quad H_1 : G \sim \otimes_{i=1}^{n_2} (\alpha_j B(q) + (1 - \alpha_j) B(p)).
\end{align*}

Unfortunately, for this testing problem it is difficult to obtain a simplified expression of the optimal test given by Neyman Pearson lemma. Our goal is to replace this complex two hypothesis testing problem by a simpler hypothesis problem associated with a smaller error. Here we sketch the proof strategy. When $\alpha_j$ is very close to $1/2$ it is not possible to statistically distinguish $(\alpha_j B(p) + (1 - \alpha_j) B(q))$ from $(\alpha_j B(q) + (1 - \alpha_j) B(p))$ and these factors can be dropped. When $\alpha_j > 1/2$ we can replace the testing problem $(\alpha_j B(p) + (1 - \alpha_j) B(q))$ vs $(\alpha_j B(q) + (1 - \alpha_j) B(p))$ by the simpler testing problem $B(p)$ vs $B(q)$ and do a similar simplification when $\alpha_j < 1/2$. It remains to control the number of indexes for which $\alpha_j > 1/2$ and $\alpha_j < 1/2$. It can be shown that they both have the same magnitude $n_2 n_1 p$. By consequence we obtained

$$
\inf_\phi \mathbb{P}_{H_0, \phi} + \mathbb{P}_{H_1 (1 - \phi)} \geq c(\inf_\phi \mathbb{P}_{H'_0, \phi} + \mathbb{P}_{H'_1 (1 - \phi)})
$$

where

\begin{align*}
H'_0 : &G \sim \otimes_{i=1}^{n_2 n_1 p} B(p), \quad \text{vs} \quad H'_1 : G \sim \otimes_{i=1}^{n_2 n_1 p} B(q).
\end{align*}

It is well known that the error associated with this testing problem is of order $e^{-cn_2 n_1 p^2}$ and that corresponds to the rate of convergence of our algorithm. Now we are going to formalize the above argument.

**Step 3.** A simple calculation show that

$$
\frac{\alpha_j}{1 - \alpha_j} = \frac{\prod_{i' : z_{i'} = 1} p A_{i'j} (1 - p)^{1 - A_{i'j}} \prod_{i' : z_{i'} = -1} q A_{i'j} (1 - q)^{1 - A_{i'j}}}{\prod_{i' : z_{i'} = 1} q A_{i'j} (1 - q)^{1 - A_{i'j}} \prod_{i' : z_{i'} = -1} p A_{i'j} (1 - p)^{1 - A_{i'j}}} := \theta_j.
$$

Let us denote by $C_+$ (resp $C_-$) the set $\{i' : z_{i'} = +1\}$ (resp. $\{i' : z_{i'} = -1\}$ ). The previous expression can be rewritten as

$$
\theta_j = \left( \frac{p(1 - q)}{q(1 - p)} \right)^{\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_-} A_{i'j}} \left( \frac{1 - p}{1 - q} \right)^{|C_+| - |C_-|}.
$$

We also have the relation $\alpha_j = \frac{\theta_j}{1 + \theta_j}$, so $\alpha_j$ is close to $1/2$ if and only if $\theta_j$ is close to $1$. If $z_{-i}$ were an exactly balanced partition, i.e. $|C_+| - |C_-| = 0$, $\theta_j = 1$ would be equivalent to $\sum_{i' \in C_+} A_{i'j} = \sum_{i' \in C_-} A_{i'j} = 0$. It is unfortunately not the case with our choice of prior for $z_{-i}$. However the contribution of the $\left( \frac{1 - p}{1 - q} \right)^{|C_+| - |C_-|}$ is small because it is very close to $1$. Indeed, we have $|C_+| - |C_-| \in [-C \sqrt{n_1 \log n_1}, C \sqrt{n_1 \log n_1}]$ with probability at least $1 - n_1^{-c}$ for some positive constant $c$ by Hoeffding’s inequality. We will condition on this event that is independent from $z_i$ and $A$ (it only depends on $z_{-i}$). Moreover $\log \left( \frac{1 - p}{1 - q} \right) \in [-c(p - q), c(p - q)]$ by using Taylor formula. Hence

$$
\left( \frac{1 - p}{1 - q} \right)^{|C_+| - |C_-|} \in [e^{-c' \sqrt{n_1 \log n_1 p}}, e^{c' \sqrt{n_1 \log n_1 p}}]
$$

for some positive constant $c' > 0$. But we have

$$
\max\{|e^{c' \sqrt{n_1 \log n_1 p}} - 1|, |e^{-c' \sqrt{n_1 \log n_1 p}} - 1|\} \leq c' \sqrt{n_1 \log n_1 p} := \epsilon.
$$

On the other hand when $\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_-} A_{i'j} \neq 0$ the quantity $\left( \frac{p(1 - q)}{q(1 - p)} \right)^{\sum_{i' \in C_+} A_{i'j} - \sum_{i' \in C_-} A_{i'j}}$ will be at least $p/q > 1$ or at most $q/p < 1$. This implies that the information provided by the size of the partition will only matter in the decision process when $\sum_{i' \in C_+} A_{i'j} = \sum_{i' \in C_-} A_{i'j}$. But since it provides very few information, it will be almost equivalent to toss a coin.
**Step 4.** We want to control the size of the set \( j \in [n_2] \) for which \( \theta_j \in [1 - \epsilon, 1 + \epsilon] \). It can only happens for the indexes \( j \) such that \( \sum_{i' \in \mathbb{C}_+} A_{i'j} - \sum_{i' \in \mathbb{C}_-} A_{i'j} = 0 \). But this last event occurs with probability at least \((1 - p)^{n_1 - 1} \approx 1 - n_1 p\) (the probability that all the \( A_{i'j} \) are simultaneously null). But by Chernoff’s bound we have that

\[
\sum_j 1(\sum_{i' \in \mathbb{C}_+} A_{i'j} - \sum_{i' \in \mathbb{C}_-} A_{i'j} \neq 0) \lesssim n_2 n_1 p
\]

with probability at least \( 1 - e^{-cn_2 n_1 p} \). Notice that this event is independent on \( A_k \) and \( z_i \). Let us define

\[
J_b := \{ j \in [n_2] : \theta_j \in [1 - \epsilon, 1 + \epsilon] \}
\]

and

\[
J_g := \{ j \in [n_2] : \theta_j \notin [1 - \epsilon, 1 + \epsilon] \}.
\]

Then w.h.p by the previous concentration inequality we have

\[
|J_b| = (1 - o(1)) n_2, \quad |J_g| \lesssim n_2 n_1 p.
\]

**Step 5.** It is easy to check \( \theta_j \in [1 - \epsilon, 1 + \epsilon] \) implies \( \alpha_j \in [1/2 - \epsilon', 1/2 + \epsilon'] \) for \( \epsilon' \) proportional to \( \epsilon \). Since the constant involved here doesn’t matter, we won’t make a distinction between \( \epsilon \) and \( \epsilon' \).

For all \( j \in [n_2] \) such that \( \alpha_j \in [1/2 - \epsilon, 1/2 + \epsilon] \) we can check that

\[
\alpha_j p^{A_{ij}} (1 - p)^{1 - A_{ij}} + (1 - \alpha_j) q^{A_{ij}} (1 - q)^{1 - A_{ij}} \geq (1 - \epsilon) p + q \cdot 1_{A_{ij} = 1} + (1 - p) \cdot 1_{A_{ij} = 0}.
\]

In particular

\[
\prod_{j \in J_b} \alpha_j p^{A_{ij}} (1 - p)^{1 - A_{ij}} + (1 - \alpha_j) q^{A_{ij}} (1 - q)^{1 - A_{ij}} \geq (1 - \epsilon) \sum_{j \in J_b} \alpha_j (1 - p) \cdot \sum_{j \in J_b} (1 - A_{ij}) \prod_{j \in J_b} p^{A_{ij}} (1 - p)^{1 - A_{ij}} + q^{A_{ij}} (1 - q)^{1 - A_{ij}}.
\]

But with probability at least \( 1 - e^{-c \epsilon n_2 p} \) uniformly for all choices for \( z \) and \( z_2 \), \( \sum_{j \in J_b} A_{ij} \approx n_2 p \) so \( (1 - \epsilon) \sum_{j \in J_b} A_{ij} \approx 1 - \sqrt{n_1 n_2 p^2} \). We can obtain a similar bound for \( (1 - p) \sum_{j \in J_b} (1 - A_{ij}) \approx 1 - \sqrt{n_1 n_2 p^2} \).

Since by assumption \( \sqrt{n_1 \log n_1 n_2 p^2} \to 0 \), \( (1 - \epsilon) \sum_{j \in J_b} A_{ij} (1 - p) \sum_{j \in J_b} (1 - A_{ij}) > c \) where \( c > 0 \) is a constant. This show that for all \( j \in J_b \) we can replace all the factors \( (\alpha_j B(p) + (1 - \alpha_j) B(q)) \) by \( (1/2 B(p) + 1/2 B(q)) \) in the hypothesis testing problem and obtain a proportional error.

**Step 6.** It remains to simplify the testing problem for \( j \in J_b \). Let \( T_j \sim B(\alpha_j) \) be independent Bernoulli with parameter \( \alpha_j \). Then

\[
\mathbb{E}_{T_j} (p^{A_{ij}} (1 - p)^{A_{ij}} T_j + q^{A_{ij}} (1 - q)^{A_{ij}} (1 - T_j)) = p^{A_{ij}} (1 - p)^{A_{ij}} \alpha_j + q^{A_{ij}} (1 - q)^{A_{ij}} (1 - \alpha_j).
\]

But \( T_j \) is independent from \( z_i \), so we can interchange the integration over \( T_j \) and \( z_i \). It leads to a testing problem

\[
H''_0 : G \sim \otimes_{T_j = 1} B(p) \otimes_{T_j = 0} B(q), \quad \text{vs}
\]

\[
H''_1 : G \sim \otimes_{T_j = 1} B(q) \otimes_{T_j = 0} B(p).
\]

Again, we need to control the size of \( |j : T_j = 1| \) and \( |j : T_j = 0| \). By concentration inequality \( |j : T_j = 1| \approx \sum_{j \in J_b} \alpha_j \) and \( |j : T_j = 0| \approx \sum_{j \in J_b} (1 - \alpha_j) \). But it is easy to see that \( \mathbb{P}(\alpha_j > 1/2) = \mathbb{P}(\alpha_j < 1/2) \) (because of the choice of the prior on \( z_2 \)) so the are approximately as many indexes \( j \) such that \( \alpha_j > 1/2 \) than indexes such that \( \alpha_j < 1/2 \). It implies that all the sums are of order \( n_2 n_1 p \).

**Conclusion.** We have lower bounded the minimax risk by the error occurring in the two hypothesis testing

\[
H''_0 : G \sim \otimes_{j = 1}^{m} B(p) \otimes_{j = m + 1}^{2m} B(q), \quad \text{vs}
\]

\[
H''_1 : G \sim \otimes_{j = 1}^{m} B(q) \otimes_{j = m + 1}^{2m} B(p),
\]

where \( m = n_2 n_1 p_{max} \). It gives the good rate of convergence \( e^{-c n_1 n_2 (p - q)^2} \) by the results in Lemma 4 in 7. \( \square \)