Solution Generating Technique for Noncommutative Orbifolds

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We propose the relationships between the noncommutative solitons and the (fractional) D-branes on the $\mathbb{C}^2/\mathbb{Z}_N$ orbifold and extend the solution generating technique for the orbifold. As applications, we determine how tachyon condensations occur in various D-\overline{D} systems on the orbifolds. The calculations give results consistent with BSFT. The extended solution generating technique enables us to calculate more general decay modes of D-\overline{D} systems.

§1. Introduction

In recent years, nonperturbative methods of string theory have revealed some mysterious dynamical structures through investigation of the Sen’s conjectures.\(^1\) conjectures are as follows. When the tachyon of an unstable systems, such as a D-\overline{D} systems, is condensed to the minimum of the tachyon potential, the brane is reduced to a lower dimensional brane or is annihilated. Whether a lower dimensional brane remains after tachyon condensation and how lower dimensional brane remains is depend on the topological charges of the gauge field on the D-brane before the condensations.

Recently noncommutative geometry has attracted grate interest as a low energy effective theory of open string theory and the theory of D-branes. The philosophy of noncommutative geometry is that an algebra of the functions on a manifold rather than a set of points exist a priori.

Grate progress was recently made in noncommutative field theory\(^2\) (for review, see e.g. Ref\(^3\) by the discovery of GMS soliton.\(^4\) It is interpreted as D-brane,\(^5\) in accordance with the calculation of tension and the observation that the fluctuation modes around the solitonic solution are identical with the spectrum of the D-brane. Owing to this development, noncommutative field theory has become one of the most powerful tool to extract information about nonperturbative aspects of string theory.

Noncommutative field theory possesses powerful techniques for analyzing nonperturbative aspects of string theory, such as the solution generating technique\(^6\) – which enables us to generate another solution from a (in most case trivial) solution – and the methods of reading off the topological charge from the tachyon configuration citeMat, HM, Wit1. In particular, noncommutative version of the field theory obtained using BSFT\(^{10,11}\) enables us to analyze tachyon condensations in D-\overline{D} systems in the language of noncommutative geometry. There are many other methods to analyze D-\overline{D} systems such as BSFT\(^{12}\) and cubic SFT\(^{13}\)-like approach\(^{14,15}\) (for review, see Ref\(^{16}\)), and some interesting results have

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Furthermore because noncommutative geometry can be defined on various types of manifolds, these methods can be applied to various backgrounds such as tori, fuzzy spheres or orbifolds. In this paper we concentrate on \( \mathbb{C}^2/\mathbb{Z}_N \) orbifolds. These orbifolds have some peculiar features for example, that they can have fractional D-branes whose R-R charges are fractional. In Ref.\cite{24} a general framework for noncommutative field theory on the orbifold was given and some simple calculations were made. Here we develop this framework for application to nontrivial configurations which Ref\cite{24} did not deal with using some additional proposals of identifications between noncommutative solitons and (fractional) D-branes and extending the solution generating technique. This enables us to calculate more general classes of D-D decays. As we see below, we can construct \( N \) types of noncommutative solitons, and we propose some rules to identify these solitons as fractional D-branes of different types. Using this identification and by extending the solution generating technique, we can calculate various new classes of decay modes of D-D systems. Calculations of the decay of D-D systems using BSFT are given in Ref.\cite{26} We carry out explicit calculations using noncommutative field theoretical methods in some simple cases. We also show that in some tachyon configurations, D-D systems decay with extra D-D pairs and the extra D-D pairs are annihilated. We calculate general two pair of D-D systems, and show that this system decays into D0-branes and D2-branes. Our method is applicable to analysis of more general systems.

We review noncommutative field theory on orbifolds \( \mathbb{C}^2/\mathbb{Z}_N \) in \S 2. In \S 3, we calculate decay modes of various D-D systems in \( \mathbb{C}^2/\mathbb{Z}_N \) whose codimensions are two and four. Finally we give some conclusions and discussions in \S 4.

\section{Noncommutative field theory on noncommutative orbifold}

\subsection{The algebra of noncommutative orbifold}

To formulate field theory on \( \mathbb{C}^2/\mathbb{Z}_N \) orbifolds, we must know about the algebra of functions on noncommutative orbifolds. To start with, we consider the noncommutative algebra of \( \mathbb{C}^2 \). We assume that the noncommutativity of the coordinates satisfy

\begin{align}
[z_1, \bar{z}_1] &= \theta, \quad [z_2, \bar{z}_2] = \theta, \\
[a_1, a_1^\dagger] &= 1, \quad [a_2, a_2^\dagger] = 1,
\end{align}

where

\[a_1 = z_1/\sqrt{\theta}, \quad a_2 = z_2/\sqrt{\theta}.\]

We set the commutator of the \( z_1 \) plane and \( z_2 \) plane to the same value for the ease of calculation. The algebra of noncommutative \( \mathbb{C}^2 \) is generated by the operators \([\text{or tensor products of two matrices}]\) acting on the tensor product of two Fock spaces. We denote its basis as \(|n_1\otimes|n_2\rangle\) and abbreviate it as \(|n_1, n_2\rangle\) where \(n_1\) and \(n_2\) are non-negative integers.

The generator of the orbifold group acts on the coordinates as

\[z_1 \rightarrow e^{\frac{2\pi a}{N}z_1}, \quad z_2 \rightarrow e^{-\frac{2\pi a}{N}z_2}.\]

Following the prescription given in Ref.\cite{24} the algebras which represent the orbifolded
space is “covariant” \((\text{invariant})\) in the sense of Refs \(^{24}\) and \(^{25}\)) under the action of the orbifold group \((2)\). The algebra decomposes into \(N\) subalgebras, and these subalgebras are related by the automorphisms defined by \((2)\). This situation reminds us of the fact that in boundary state construction of the D-brane on the orbifolds that there are \(N\) boundary states on the noncommutative orbifolds related to each other.

With the decomposition of the algebra, the Hilbert space (Fock space in our case) are also decomposed. The components of the decomposed Hilbert spaces transform under different representations of the orbifold group \(\mathbb{Z}_N\):

\[
\mathcal{H} = \bigoplus_{\alpha=0}^{N-1} \mathcal{H}_\alpha
\]

\[
\mathcal{H}_\alpha = |n_1, n_2\rangle \quad (n_2 - n_1 = \alpha \mod N).
\]  

(3)

The subscript \(\alpha (\alpha \in \mathbb{Z}/\mathbb{Z}_N)\) indicate that corresponding component of the decomposed Hilbert space transform under the \(\alpha\)-th representation. That is, if \(v \in \mathcal{H}_\alpha\) then \(v\) transforms as \(v \rightarrow e^{2\pi i \alpha/N} v\). The decomposition of the Hilbert space clearly reflects the decomposition of the algebra. Furthermore, we see that there are parts of the algebra that are not “covariant”. These parts of the original algebra are bimodules which connect the decomposed Hilbert spaces. We refer to these bimodules \(\mathcal{H}_\alpha\) and \(\mathcal{H}_\beta\) as \(\alpha M_\beta\).

Bimodules \(\alpha M_\beta =: A_\alpha\) are “covariant” algebras describing a noncommutative orbifold and all algebras of this form are identical. \(\alpha M_\beta (\alpha \neq \beta)\) are not algebras but bimodules on which the orbifold algebra \(A_\alpha\) acts from the right and \(A_\beta\) acts from the left.

When we specify the algebras which act from the left and right, we can redefine \(\alpha M_\beta\) as \(\bigoplus_{\gamma=0}^{N-1} \alpha + \gamma M_\beta + \gamma\). This redefinition of the bimodules implies a change of the orbifold algebra. The new orbifold algebra is direct sum of old orbifold algebras: \(\bigoplus_{\gamma=0}^{N-1} A_{\alpha + \gamma}\).

Note that although direct sum components are introduced by hand, and therefore cannot act on the specified decomposed Hilbert space, these components have some meanings, when we interpret noncommutative soliton as fractional D-brane below. We do not know why this redefinition has meanings. Roughly speaking, we conjecture that this is because string theory “sees” the covering space of the orbifold by considering the twisted sector.

The explicit form of a redefined \(\alpha M_\beta\) is the tensor product of two matrices acting on the two Fock spaces whose coefficient of the element

\[
|\gamma + pN + q\rangle \otimes |\gamma + \beta - \alpha + rN + s\rangle\langle s|,
\]

where \(p,q,r,s,\gamma \in \mathbb{Z} \quad \gamma + pN + q, \gamma + \beta - \alpha + rN + s, q, s \geq 0,\)

(5)

can be nonzero, and all the other elements are zero. Note that \(\alpha M_\beta \cong \alpha + \gamma M_{\beta + \gamma}\) for any \(\gamma\).

The product of these operators is simply defined as the product of each matrix:

\[
M_1 \otimes M_2 \times N_1 \otimes N_2 := (M_1 N_1) \otimes (M_2 N_2).
\]

(6)

This structure of the product comes from the multiplication rule of commutative functions.
2.2. Noncommutative field theory and fractional D-branes

Field theory on noncommutative orbifolds is almost the same as that on the noncommutative Euclidean space. However, there are some differences, as discussed below:

1. The normalization of the trace has a factor of $1/N$, because overall volume is $1/N$ while the identity operator in $A_\alpha$ is the same form. Thus the integration over the orbifolds is written $\frac{2\pi \theta}{N^2} \text{Tr}$.

2. Because the orbifold algebra is not an algebra of all infinite size matrices but, rather $A_\alpha$, the symmetry of the theory is not $U(\infty) \otimes U(\infty)$, but instead its subgroup $G \subset U(\infty) \otimes U(\infty)$ which preserves $A_\alpha$:

$$gA_\alpha g^\dagger = A_\alpha. \quad (g \in G) \quad (7)$$

Considering the correspondence between noncommutative solitons and D-branes in the Euclidean space, it is natural to identify noncommutative solitons on an orbifold as (fractional) D-branes. In this subsection we will give the rules of identification and give some evidences for this identification.

In the following several paragraphs, we explain how to determine the type of noncommutative solitons and D-branes. Noncommutative soliton on the D4-brane wrapping on the $\mathbb{C}^2/\mathbb{Z}_N$ orbifold is characterized by the projection operators $P \in A_\alpha$, which satisfy $PP = P$. When we consider the tachyon condensation of the D-D system, the projection operator is $1 - \bar{T}T$ or $1 - \bar{T}T$, where $T$ is the tachyon field.

If a noncommutative soliton has the form $|k\rangle \langle k| \otimes |l\rangle \langle l|$, we call $l - k$ (modulo $N$) the type of noncommutative soliton. The reason for this classification is as follows. When $n$ noncommutative solitons of different types coincide, the gauge symmetry is not enhanced to $U(n)$ but remains $U(1)^n$. This is because the open strings connecting solitons of different types are expressed by nontrivial representations of the orbifold group and are prohibited as a gauge field. Thus these noncommutative solitons of different types should be distinguished.

Next, we define the types of D-branes. We refer to the positions of the nodes in the quiver diagrams for gauge fields on $\mathbb{C}^2/\mathbb{Z}_N$ as the “types” of the D-branes. The positions of nodes in the quiver diagrams are related to the irreducible representations of the orbifold group $\mathbb{Z}_N$. Thus we can regard the types of the D-branes as irreducible representations of the orbifold group. We note that the important

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25) It is not clear whether we can say identity operator of $A_\alpha$ has the same form as that in noncommutative Euclidean space, because when we defined $A_\alpha$ we specified the direct sum component of the Hilbert space on which $A_\alpha$ acts and the Hilbert space is not the same one as in the case of Euclidian space. However we conjecture that this procedure is correct when we analyze string theory, because of the existence of the twisted sector.

26) When we refer to the D4-brane or the D2-brane below, we are always referring to branes wrapping the orbifold and having twisted R-R 2 or 4 form charges, not transverse to the orbifold like in (27)
The quantity for types of (fractional) D-branes is the absolute value of the difference between the types of two (fractional) D-branes, because we can rotate and overturn the diagram. In boundary state language, this fact is more obvious, because the difference of the two D-branes is just a phase factor \( \exp(2\pi i \alpha/N) \), in the string theory on \( \mathbb{C}^2/\mathbb{Z}_N \). To fix these degrees of freedom, we identify the types of the D-branes by the integer \( \alpha = 0, \ldots, N-1 \), where the generator of the irreducible representation is \( \exp(2\pi i \alpha/N) \). The elements of \( A_\alpha \) are functions on a D4-brane.

We use the convention in which the type of the D4-brane whose algebra \( A_\alpha \) is \( \alpha \).

We next propose rules of identification between noncommutative solitons on the type \( \alpha \) D4-branes and D0-branes or D2-branes as follows.

- The type of a noncommutative soliton on the D4-brane of type \( \alpha \) is determined by the following rule:  

1. If the soliton has the form \( |k\rangle\langle k| \otimes I \otimes |k\rangle\langle k| \), where \( k = \beta \pmod{N} \), the type of corresponding D2-brane is \( \alpha + \beta + 1 \) (\( \alpha - \beta \)). These D2-branes extend in the \( z_2 \) (\( z_1 \)) plane.

2. If the soliton has the form \( |k\rangle\langle k| \otimes |l\rangle\langle l| \), where \( k = \beta, l = \gamma \pmod{N} \), the type of D0-brane to which the soliton corresponds is \( \alpha + \beta - \gamma + 1 \).

These rules can be verified as follows. We will concentrate on the second rule, since the first rule can be understood in the same way. Consider two solitons \( |k\rangle\langle k| \otimes |l\rangle\langle l| \) and \( |k'\rangle\langle k'| \otimes |l'\rangle\langle l'| \). An open string that connects the former soliton to the later is written as \( |k\rangle\langle k'| \otimes |l\rangle\langle l'| \) and transforms under the generator of the orbifold group \( \mathbb{Z}_N \) as \( \exp[2\pi i ((k-k')-(l-l'))/N] \). Thus the difference between the types of the two D0-branes is \( (-k+l) - (-k'+l') \). Furthermore if we consider an open string that connects a soliton on an \( \alpha \) type D4-brane to a soliton of the same type on a \( \beta \) type D4-brane the difference of the types of these D0-branes should be \( \alpha - \beta \), because the open string is an element of \( \alpha \mathcal{M}_\beta \). Taking these points into account, the rules of correspondence between types of branes and types of noncommutative solitons is determined as above, up to the relative sign of the differences. The relative sign of the differences is determined in such a way that the calculation of D-D decay gives consistent result with the result obtained using BSFT.\(^{26}\)

It is known that the D4-branes and D2-branes have the same tensions as normal D4-branes and D2-branes respectively, while the tensions of fractional D0-branes are \( 1/N \).

\(^{24}\) We can adopt another rule in which \( |k\rangle\langle k| \otimes I \) corresponds to \( \alpha + \beta \) and \( I \otimes |k\rangle\langle k| \) corresponds to \( \alpha - \beta - 1 \). This is just a difference of convention, and the physics is the same.
smaller than the bulk D0-branes. This is explained in noncommutative language as follows. A D0 brane has fractional tension because of the $1/N$ factor of the trace. Also, the energy of D4 and D2-branes must be divided by the volume of a brane that is $1/N$ times as large as that in the $\mathbb{C}^2$ case. Therefore, the $1/N$ factor is canceled and the tensions of D2 and D4-branes are the same as those of bulk D2 and D4-branes.

When all $N$ types of the fractional D0-branes are coincident, the entire system has the same tension as the bulk D0-brane. Thus we can regard it as a bulk D0-brane, and we can move it from the origin, in contrast to the situation for individual fractional D0-branes which are pinned at the origin. This is known for the commutative case. This can also be explained in the language of noncommutative field theory. A short explanation is presented at the end of §3.1.

2.3. The action of the D-brane and D-\(\bar{D}\) systems

Since we will consider the gauge theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_N$, we need operator representations of the covariant derivatives. These are given as

$$C_{1,2} = a^{1,2} + i\theta^{1/2}A_{1,2}, \quad \bar{C}_{1,2} = a_{1,2} - i\theta^{1/2}\bar{A}_{1,2}. \quad (8)$$

$A_{1,2}$ and $\bar{A}_{1,2}$ are gauge fields on the orbifold. Because $A_{1,2}$ and $\bar{A}_{1,2}$ must be invariant under the action of orbifold group, they are elements of modules, $A_1, \bar{A}_2 \in \mathcal{M}_\alpha$ and $A_2, \bar{A}_1 \in \mathcal{M}_\alpha$. The reason is that the exp($\mp 2i\pi/N$) factor resulting from the action of orbifold group must be canceled by the contribution from the rotation of the vector $A_\mu \rightarrow R(g)_\mu^\nu A_\nu$, where $A_\mu[\mu = 1, 2, 3, 4]$ is the gauge field in the $x_\mu$ basis and $R(g)_\mu^\nu$ is the matrix representing the rotation by the generator of the orbifold group $\mathbb{Z}_N$. Thus these gauge fields are invariant under the action of the orbifold group.

The action of gauge theory on the noncommutative worldvolume of a D4-brane that fills the orbifold is

$$T_4 \left( \frac{2\pi\alpha'}{\theta} \right)^2 \int \frac{dx^4}{N} \left( -\frac{1}{4} F_{\mu\nu}^2 \right)$$

$$= T_4 \left( \frac{2\pi\alpha'}{\theta} \right)^2 \frac{(2\pi\theta)^2}{N} \text{Tr} \left( -\frac{\theta^{-2}}{2} \left( \left([C_1, \bar{C}_1] + I\right)^2 + \left([C_2, \bar{C}_2] + I\right)^2 - [C_1, C_2][\bar{C}_1, \bar{C}_2] - [C_1, \bar{C}_2][C_2, \bar{C}_1] \right) \right), \quad (9)$$

where $T_p$ is the tension of the Dp-brane.

We need the action of the field theory on the noncommutative orbifold for D4-D4
systems. This action is given as
\[
\mathcal{L} = T^4 \left( \frac{2\pi\alpha'}{\theta} \right) ^2 \frac{(2\pi\theta)^2}{N} \text{Tr}(f(T, \bar{T})(\text{kinetic terms of gauge fields}) \\
+ \frac{i}{4} \theta^{-1} g(T, \bar{T}) \sum_{i=1,2} ((C_i^+ T - \bar{T} C_i^-)(\bar{C}_i^+ T - \bar{T} \bar{C}_i^-) + (C_i^- T - T C_i^+)(\bar{C}_i^- T - \bar{T} \bar{C}_i^+)) \\
+ V(T\bar{T} - I) + V(\bar{T}T - I) + (\text{higher derivatives}) ),
\]
(10)
where $C_i^\pm$ and $C_i^{\mp}$ are covariant derivatives on the D-brane (superscript $+$) and anti-D-brane (superscript $-$). The tachyon field $T$ is a bifundamental field, and $T \in _\alpha \mathcal{M}_\beta$ if the D4-brane is of type $\alpha$ and the anti-D4-brane is of type $\beta$. It couples to the gauge field on both the D-brane and anti-D-brane via covariant derivatives. The gauge field on the D-brane acts from the right, and the gauge field on the $\bar{D}$-brane acts from the left. For $\bar{T}$, the actions of gauge fields are reversed. The potential $V(T\bar{T} - I) + V(\bar{T}T - I)$ is minimal at $|T\bar{T}| = 1$ and takes the value 1 at $|T\bar{T}| = 0$. This is identical to the action obtained in the BSFT description.\(^{11)}\)

The transformation law of tachyon fields under the generator $g$ of the orbifold group is given by
\[
T \in _\alpha \mathcal{M}_\beta \rightarrow e^{\frac{2\pi i}{N}(\beta - \alpha)} T.
\]
(11)

§3. Tachyon condensation on noncommutative orbifolds

3.1. Solution generating technique in orbifold theories

We now give a brief explanation of the solution generating technique. To begin with, we work on two dimensional Euclidean space. When there exists gauge symmetry, the action possesses $U(\infty)$ symmetry, $O \rightarrow gO\bar{g}$ ($g \in U(\infty)$). This $U(\infty)$ symmetry enables us to employ a technique called “solution generating technique”. To use this technique, we must use a partial isometry $U$. A partial isometry is an infinite-dimensional matrix that satisfies
\[
\bar{U}U = 1, \quad U\bar{U} \neq 1.
\]
(12)
Matrices that satisfy this relation do exist. For example, the shift operator,
\[
S = \sum_{k=0}^{\infty} |k + 1\rangle \langle k|,
\]
(13)
is partial isometry, and it satisfies the following relation:
\[
\bar{S}S = I, \quad SS = I - P_1,
\]
(14)
where $P_n$ is the projection operator $\sum_{k=0}^{n-1} |k\rangle \langle k|$. Using this isometry, we transform fields that are solution of the equation of motion as
\[
O \rightarrow UO\bar{U}.
\]
(15)
Because
\[ \frac{\delta S}{\delta \mathcal{O}} \to U \frac{\delta S}{\delta \mathcal{O}} \hat{U}, \]
the transformed fields also satisfy the equation of motion. Consequently, using the isometry we can construct another solution from a given solution.

Below we apply the solution generating technique to $\mathbb{C}^2/\mathbb{Z}_N$ orbifolds. But, a straightforward application of the solution generating technique is not useful for following reasons. Because the symmetry of the action is not $U(\infty) \otimes U(\infty)$ but $G$, we should use a partial isometry related to $G$, which is complicated. Furthermore, when the types of the D-brane and anti-D-brane are different, we cannot take a trivial configuration as the start point of the solution generating.

To make more use of the solution generating technique, we can extend the technique to use the subset of the isometries related to $U(\infty) \otimes U(\infty)$ which satisfies the condition given below;

\[ |n_1, n_2\rangle \mapsto |n_1', n_2'\rangle, \]
\[ (n_2' - n_1') - (n_2 - n_1) = \gamma \pmod{N} \quad \forall n_1, n_2. \]

We call $\gamma$ in the above equation “shifting”. Note that if $U$ is a partial isometry whose “shifting” is $\gamma$, then
\[ \hat{U} U = I \otimes I, \quad U \hat{U} = I \otimes I - \hat{P}_{\gamma+pN} \otimes I, \quad (p \in \mathbb{Z}) \]

or
\[ \hat{U} U = I \otimes I, \quad U \hat{U} = I \otimes I - P_{-\gamma+pN} \otimes I. \]

The reason for including the condition given in 17 and 18 is as follows. Consider the solution generating technique on the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ which is constructed from the partial isometry whose “shifting” is $\gamma$. It takes the operator $\mathcal{O} \in \mathcal{A}_\alpha$ to $U \hat{O} U \in \mathcal{A}_{\alpha+\gamma}$. The set $\mathcal{A}_{\alpha+\gamma}$ is isomorphic to $\mathcal{A}_\alpha$, and the actions of the operators in $\mathcal{A}_\alpha$ and those of the operators in $\mathcal{A}_{\alpha+\gamma}$ have the same form. Thus the same arguments given in the case of the Euclidean spaces regarding the solution generating technique can be made here too. The nontrivial point concerns covariant derivatives. By the isometry that satisfies the above condition, the covariant derivative transforms $a_{\pm 1} \mathcal{M}_\alpha$ to $a_{\pm 1, \gamma} \mathcal{M}_{\alpha+\gamma} \cong a_{\pm 1} \mathcal{M}_\alpha$. Because covariant derivatives take arbitrary values in $a_{\pm 1, \gamma} \mathcal{M}_{\alpha+\gamma} \cong a_{\pm 1} \mathcal{M}_\alpha$, they are transformed into covariant derivatives of the brane of type $\alpha + \gamma$. Consequently, the solution obtained with the solution generating technique is the solution of the $\mathcal{A}_{\alpha+\gamma}$ system.

We use the partial isometry composed of the shift operators:
\[ S_1 = S \otimes I - \sum_{k=0}^{\infty} |k+1\rangle \langle k| \otimes I, \]
\[ S_2 = I \otimes S = I \otimes \sum_{k=0}^{\infty} |k+1\rangle \langle k|, p \]
(21)
These shift operators satisfy the condition given above. The “shifting” of $S_1$ and $S_2$ are $-1$ and $1$, respectively.

When we use the solution generating technique in the D-$\bar{D}$ system, we can act with different partial isometries on the D-brane and anti-D-brane to generate new solutions. That is, from a given solution, the transformation
\[
\mathcal{O}^+ \rightarrow U \mathcal{O}^+ \bar{U}, \\
\mathcal{O}^- \rightarrow V \mathcal{O}^- \bar{V}, \\
T \rightarrow VT \bar{U}
\] (22)
generates another solution. For example, consider a solution of the D-$\bar{D}$ system with the tachyon operator $T$ in $\alpha \mathcal{M}_\beta$. Consider the case that we act shift operator $S_2$ on the D-brane of type $\alpha$ but keep the anti-D-brane of type $\beta$ unchanged. Then we get another gauge configuration of the D-brane and $T \in \alpha + 1 \mathcal{M}_\beta$. This is a solution of the $(\alpha + 1)\bar{\beta}$ system, because the action of the $\alpha\bar{\beta}$ system and the $(\alpha + 1)\bar{\beta}$ system are the same and the logic of solution generating uses only a superficial form of the action.

As the starting point of the solution generating technique, we consider the most trivial gauge configuration of the $\alpha\bar{\alpha}$ system,
\[
C_1^+ = a_1^+, \bar{C}_1^+ = a_1, \\
C_2^+ = a_2^+, \bar{C}_2^+ = a_2, \\
C_1^- = a_1^-, \bar{C}_1^- = a_1, \\
C_2^- = a_2^-, \bar{C}_2^- = a_2, \\
T = I \otimes I.
\] (23)
This configuration is obviously a solution of the action of the D-$\bar{D}$ system (10), which represents the vacuum. We can generate many solutions from this solution using the extended solution generating technique.

This is a good place to give an explanation of how an individual noncommutative soliton is pinned at the origin, while a collection of $N$ branes of $N$ different types can move from the origin. The solution generating technique by shift operators can be generalized as
\[
C_i \rightarrow S^n C_i \bar{S}^n + X_i,
\] (24)
where each $X_i$ is an $n \times n$ matrix.\(^{28}\) The new degrees of freedoms represented by these $X_i$s must be in the same representation of the orbifold group as $A_i$. The action for $X_\mu (\mu = 1 \cdots 4)$ is $([X_\mu, X_\nu])^2$, and this action is minimized when all $X_i$ commute. It is known that the space of such $X_i$s are one point for $n < N$ and $\mathbb{C}^2/\mathbb{Z}_N$ for $n = N.\(^{25, 27}\)$ Because the moduli spaces of noncommutative solitons are determined by the $X_i,^{21}$ only when the solitons of all $N$ types coincide they can move from the origin.

3.2. From $D_4$-$\bar{D}_4$ to $D2$

There are many ways to obtain system with an $\alpha$ type D4-brane and a $\beta$ (assume $\beta \geq \alpha$) type D4-brane using the solution generating technique. Here we give some examples.
3.2.1. Simple $\alpha$-$\bar{\beta}$ system

The most simple $\alpha - \bar{\beta}$ systems can be obtained by solution generating technique from $(\beta - \gamma)$-$[\beta - \gamma]$ system, where $\gamma$ satisfying $0 \leq \gamma \leq \beta - \alpha$ is an integer. If we act with the shift operator $S_1^{\beta-a-\gamma}$ on the D-brane and $S_2^\gamma$ on the $\bar{D}$-brane, we obtain $\alpha$-$\bar{\beta}$ systems for which

$$T = S_1^{\beta-a-\gamma} \otimes S_2^\gamma \in \alpha \mathcal{M}_\beta,$$
$$C_1^+ = S_1^{\beta-a-\gamma} a_1^+ \tilde{S}_1^{\beta-a-\gamma}, \tilde{C}_1^+ = S_1^{\beta-a-\gamma} a_1 \tilde{S}_1^{\beta-a-\gamma},$$
$$C_2^- = S_2^\gamma a_2 \tilde{S}_2^\gamma, \tilde{C}_2^- = S_2^\gamma a_2 \tilde{S}_2^\gamma,$$
$$\bar{T}T = (I - P_{\beta-a-\gamma}) \otimes I,$$
$$T\bar{T} = I \otimes (I - P_{\gamma}).$$

Following the identification rules given in §2.2, we get $\alpha + 1, \ldots, \beta - \gamma$ type D2-branes that extend in the $z_2$ plane and $\beta - \gamma + 1, \ldots, \beta$ type D2-branes extend in the $z_1$ plane. In any case, $\beta - \alpha$ D2-branes are created. The calculation in the $\gamma = 0$ case has been carried out in the context of BSFT\textsuperscript{26} and we have the same number of D2-branes, while the types of these D2-branes are the same.

The gauge field on the type $\alpha$ D4-brane is given by

$$C_1^+ = S_1^{\beta-a-\gamma} a_1^+ \tilde{S}_1^{\beta-a-\gamma}, \quad \tilde{C}_1^+ = S_1^{\beta-a-\gamma} a_1 \tilde{S}_1^{\beta-a-\gamma}. $$

The curvature on the $\alpha$ brane is

$$-i\theta^{-1}([C_1, C_1] + I) = -i\theta^{-1} P_{\beta-a-\gamma} \otimes I. $$

Integrating over the $z_1$ plane, we have the winding number $(\beta - \alpha - \gamma)/N$. Similarly we have winding number $\gamma/N$ around the $z_2$ plane on the $\bar{\beta}$ brane. This means that the $\alpha$ brane has D2-brane charge$(\beta - \alpha - \gamma)$ and $\bar{\beta}$ brane has D2-brane charge $\gamma$.

We can verify the above result by calculating the tension. The energy of the system can be calculated by the action for the D4-brane (10). The kinetic terms for the gauge fields vanish because the coefficient $f(T, \bar{T})$ vanishes. The kinetic terms for tachyon field also vanish, because these terms are zero before the solution generating. Thus it is sufficient to examine the potential term;

$$\mathcal{L} = T_4 \left( \frac{2\pi \alpha'}{\theta} \right)^2 \left( \frac{2\pi \theta^2}{N} \right) \text{Tr}(V(1 - T\bar{T}) + V(1 - \bar{T}T)) = T_2 \frac{2\pi \alpha'}{\theta} V_2(\beta - \alpha), $$

where $V_2$ is volume of $z_1$ or $z_2$ plane. Thus the tension is $T_2(2\pi \alpha'/\theta)(\beta - \alpha)$, which is the correct value for the $\beta - \alpha$ D2-branes.

3.2.2. Additional D2-$\bar{D}$2 pairs creation

In the $\alpha$-$\bar{\beta}$ system, there is another type of the tachyon configuration that enables us to obtain the D2-$\bar{D}$2 system. Consider the case that we act with the shift operator $S_1^\gamma$ on the D-brane and with $S_2^{\beta-a+\gamma}$ on the $\bar{D}$-brane for the $(\alpha - \gamma)$$\overline{(\alpha - \gamma)}$ system, where $\gamma$ is
a positive integer. We then obtain \( \alpha - \bar{\beta} \) system, where

\[
T = I \otimes S_2^{\beta-\alpha+\gamma} \bar{S}_2^\gamma \in \mathcal{M}_\beta,
\]

\[
C_2^+ = S_2^\gamma a_2^+ \bar{S}_2^\gamma, \quad \bar{C}_2^+ = S_2^\gamma a_\beta \bar{S}_2^\gamma,
\]

\[
C_2^- = S_2^- \beta - \alpha + \gamma a_2^- \bar{S}_2^{\beta-\alpha+\gamma}, \quad \bar{C}_2 = S_2^\beta a_2 \bar{S}_2^{\beta-\alpha+\gamma},
\]

\[
TT = I \otimes (I - P_{\beta-\alpha+\gamma}),
\]

\[
\bar{T} \bar{T} = I \otimes (I - P_{\gamma}).
\]

We have \( \alpha - \gamma + 1, \ldots, \beta \) type \( \bar{D}2 \)-branes and \( \alpha - \gamma + 1, \ldots, \alpha \) type \( D2 \)-branes. Since \( D2 \)-branes and \( \bar{D}2 \)-branes of same type annihilate, this system will be pair annihilate, yielding \( \alpha + 1, \ldots, \beta \) type \( D2 \)-branes. We note that this is consistent with the claim that the topological charge is determined by the asymptotic behavior of the tachyon field. Upper-left part of the matrix represents the behavior near the origin and the (infinitely) lower-right part represents the asymptotic behavior far from the origin. As is easily seen from the explicit form of \( T \), asymptotic behavior does not depend on the value of \( \gamma \) but, rather is determined by the value of \( \beta - \alpha \):

\[
T = S_2^\gamma \bar{S}_2^{\beta-\alpha+\gamma} = I \otimes \gamma \begin{pmatrix} 0 & \beta - \alpha + \gamma \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 1 & 0 & 0 & \ddots \\ \vdots & \ddots & 0 & 1 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 & \ddots \end{pmatrix}.
\]

Thus the total R-R charge does not depend on \( \gamma \).

3.3. \textit{From D2-}\( \bar{D}2 \) to D0

This case is almost the same as that in the previous subsection, but here we must work only in the \( z_1 \) plane or the \( z_2 \) plane. Thus we must restrict the operators which appear in the calculation to the form \( I \otimes M \) or \( M \otimes I \). From the \( \alpha \) type \( D2 \)-brane and \( \beta \) type \( \bar{D}2 \)-brane we get \( \alpha + 1, \ldots, \beta \) type D0-branes.

3.4. \textit{From D4-}\( \bar{D}4 \) to D0

In the commutative case, to see the decay mode by which the D4-\( \bar{D}4 \) system decays into D0 system, we must consider two pairs of D4-D4. Therefore we will modify the operator representation of the start point of the solution generating technique by tensoring the CP-factor, \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). In this way, we can calculate two general pairs of D4-\( \bar{D}4 \) systems.
3.4.1. ABS-like construction

To consider the codimension four case, we will begin with the relatively simple configuration of the \((\alpha, \beta + \delta)-(\alpha + \delta, \bar{\beta})\) system. The reason for considering this configuration is that this configuration allows the ABS(Atiyah-Bott-Shapiro)-like construction\(^8\),\(^9\),\(^26\) where only D0-branes are created as we will see below. The tachyon field is written

\[
T \propto (\bar{T}^2 \ T_1 \ T_2),
\]

where \(\gamma_1\) and \(\gamma_2\) are any integer and negative power of the \(S_i\) means \(S_i^{-\alpha} = \tilde{S}_i^{\alpha}\). The proportionality here means that \(T\) must be normalized by the square root of \(\bar{T} T\) from the right. As the solutions of the equations \(\bar{T} T T = \bar{T}\) and \(T \bar{T} T = T\), we found solutions \((\gamma_1, \gamma_2) = (\alpha - \beta, 0), (0, -\delta)\). In this case, we have

\[
\bar{T} T = \begin{pmatrix} I \otimes I & 0 \\ 0 & I \otimes I \end{pmatrix}, \quad T \bar{T} = \begin{pmatrix} I \otimes I & 0 \\ 0 & I \otimes I - P_{\beta - \alpha} \otimes P_\delta \end{pmatrix},
\]

for \((\gamma_1, \gamma_2) = (\alpha - \beta, 0)\), and

\[
\bar{T} T = \begin{pmatrix} I \otimes I & 0 \\ 0 & I \otimes I \end{pmatrix}, \quad T \bar{T} = \begin{pmatrix} I \otimes I - P_\delta \otimes P_{\beta - \alpha} & 0 \\ 0 & I \otimes I \end{pmatrix},
\]

for \((\gamma_1, \gamma_2) = (\alpha - \beta, 0)\). These operators are apparently projection operators.

The solution \((\gamma_1, \gamma_2) = (\alpha - \beta, 0)\) gives a \(\delta(\beta - \alpha)\) fractional \(\bar{D}0\)-brane on the \(\bar{\alpha} + \delta\) brane and the types of the \(\bar{D}0\)-branes are

\[
\alpha + \delta + 1, \ldots, \beta + \delta
\]

\[
\vdots \quad \ddots \quad \vdots
\]

\[
\alpha + 2, \ldots, \beta + 1
\]

(34)

Because only \(\bar{D}0\)-branes exist, \(1/2\) supersymmetry is preserved after the tachyon condensations. The solution \((\gamma_1, \gamma_2) = (0, -\delta)\) gives the same result. This result gives same number and types of D0 branes as the result of Ref 26). The solution \((\gamma_1, \gamma_2) = (0, -\delta)\) gives \(\delta(\beta - \alpha)\) fractional \(\bar{D}0\)-branes on the \(\bar{\beta}\) brane and the type of the \(\bar{D}0\)-brane is the same as that in the case \((\gamma_1, \gamma_2) = (\alpha - \beta, 0)\).

The gauge field on the \(\bar{D}4\)-brane in this case is

\[
C^-_{1,2} \propto T \begin{pmatrix} a^{1,2} & 0 \\ 0 & a^{1,2} \end{pmatrix} \bar{T},
\]

(35)
The gauge field on the D4-brane is trivial. The curvature on the \( \bar{D}4 \)-branes is

\[
-i\theta^{-1}([C_i^-, \bar{C}^-_i] + I) = \begin{cases} 
-i\theta^{-1}\left(P_{\beta-\alpha} \otimes P_\delta \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) & \text{for } (\gamma_1, \gamma_2) = (\alpha - \beta, 0), \\
-i\theta^{-1}\left(0 \otimes P_\delta \otimes P_{\beta-\alpha}\right) & \text{for } (\gamma_1, \gamma_2) = (0, -\delta).
\end{cases}
\]

The winding number is \( \delta(\beta - \alpha)/N \). This means \( \bar{D}4 \)-branes have a \( \delta(\beta - \alpha) \) D0-brane charge.

### 3.4.2. Decay of two general pairs of D4-\( \bar{D}4 \)

We can extend the above calculation to the \( (\alpha, \beta + \delta') - (\alpha + \delta, \bar{\beta}) \) system. This is the general two pairs of D-\( \bar{D} \) system. Without loss of generality, we can \( \delta' \geq \delta \). We also choose the tachyon field as

\[
T \sim \begin{pmatrix} T_2 & T'_1 \\ T_1 & -T'_2 \end{pmatrix},
\]

\[
T_1 = S^{\beta-\alpha} \otimes I,
\]

\[
T'_1 = S^{\beta-\alpha} \otimes \bar{S}^{\delta'-\delta},
\]

\[
T_2 = I \otimes S^\delta,
\]

\[
T'_2 = I \otimes \bar{S}^{\delta'}.
\]

This system decays into anti-D0-branes of type described by (34) and D2-branes extending in the \( z_1 \) plane of type \( \delta + 1, \ldots, \delta' \). This is consistent with the result of the charge calculation using a boundary state. Furthermore, the existence of the D2-brane breaks 1/2 supersymmetry.

The gauge field is the same as that in the case of §3.4.1 for anti-D4-branes and

\[
C = \begin{pmatrix} a^1 \\ 0 \\ S_2^{\delta'-\delta} a \bar{S}_2^{\delta'-\delta} \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} a \\ 0 \\ S_2^{\delta'-\delta} a \bar{S}_2^{\delta'-\delta} \end{pmatrix}
\]

for D4-branes. The winding number of the gauge field around the \( z_2 \) plane on D4-branes is \( (\delta' - \delta)/N \). This means that a \( \delta' - \delta \) D2-brane charge exists.

### §4. Conclusion and discussion

In this paper, we have summarized the construction of the noncommutative field theory on the noncommutative orbifold \( \mathbb{C}^2/\mathbb{Z}_N \) following the prescription of the Ref.\(^{24}\) and proposed identification rules of the noncommutative solitons as fractional branes. Those rules determine the correspondence between the types of the noncommutative solitons and these of the fractional branes. We showed that the noncommutative solitons on the orbifolds have the same nature – such as fractional tension and pinning – as those on (fractional) D-brane.
The solution generating technique was extended to give more solutions in the orbifold theory. Combined with our identification rules, this method provide powerful tools to analyze nonperturbative aspects of string theory on orbifolds.

As examples, by calculating the various decay modes of D-ĀD systems on noncommutative orbifolds, we obtained results consistent with the result calculated using BSFT. Furthermore, using the extended solution generating technique, we were able to calculate the result in the case of more general decay modes of D-ĀD systems on noncommutative orbifolds. In particular, we derived the decay mode of two D4-ĀD4 pairs, which decay into D0-branes and D2-branes. The extension of this calculation to any pairs of D4-ĀD4 is straightforward.

As further problems, we must confirm the identification given in §2.2 more rigorously. We determined some signs in the formula that determines the type of D0-brane and D2-brane so that the calculation in §3 give results consistent with the result obtained using BSFT. But there should be the reason for these signs.

Furthermore we need a more detailed derivation showing why the redefinition of the algebra in §2.1 has some physical meaning. The properties of the redefined algebra have important roles in the correspondence between the noncommutative solitons and D-branes. As we stated in §2.1, we believe that this is because string theory “sees” the covering space of the orbifold by including the twisted sectors.

Furthermore, although we concentrated on $\mathbb{C}^2/\mathbb{Z}_N$, we can calculate non-abelian orbifolds such as $\mathbb{C}^2/\Gamma$, where $\Gamma$ is $D_n$ or $E_{6,7,8}$ or higher dimensional orbifold. These calculations will provide a good check of our identifications given in §2.2.

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