Inverse Cooperative and Non-Cooperative Dynamic Games Based on Maximum Entropy Inverse Reinforcement Learning

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Abstract—Dynamic game theory provides mathematical means for modeling the interaction between several players, where their decisions are explained by individual cost functions. The inverse problem of dynamic games, where cost functions are sought which explain observed behavior, has recently gained attention due to its potential application for identification of biological systems and the possibility of generalizing inverse optimal control results. In this paper, we extend maximum entropy inverse reinforcement learning to the N-player case in order to solve inverse dynamic games with continuous-valued state and control spaces. On this basis, we first present a method for identification of cost function parameters in a cooperative game. Afterwards, we propose an approach for identifying cost function parameters which explain the behavior of the players in a non-cooperative setting, i.e. open-loop and feedback Nash equilibrium behaviors. Furthermore, we give results on the unbiasedness of the estimation of cost function parameters for each class of inverse dynamic game. The applicability of the methods is demonstrated with simulation examples of a nonlinear and a linear-quadratic dynamic game.

Index Terms—Game theory, inverse dynamic games, inverse reinforcement learning.

I. INTRODUCTION

DY NAMIC game theory is a useful mathematical tool for describing the behavior or decision making of multiple agents or players interacting with each other. Within this framework, the decision of each player arises from the minimization of an individual cost function. Dynamic games have received considerable attention in the automatic control community and have also been successfully applied in numerous other fields including biology [1] and economics [2], [3]. Especially in the control engineering community, dynamic games have been studied in order to describe solutions under several information structures, optimality concepts and objective assumptions. Applications include driver assistance systems [4], multi-agent collision avoidance [5] and power system control [6]. In this context, several techniques for finding the optimal controls or decisions of each player based on known objectives have been thoroughly analyzed and applied.

Recent years have seen a growing interest in the inverse problem of dynamic games, where the objectives modeled by cost or utility functions of each player are sought. This problem emerges when it is not possible to model cost functions directly and it is desired to identify them based on previously observed actions of interacting players. These actions are typically assumed to correspond to an equilibrium state of the dynamic game [1], [3], [7]–[14]. In this way, a model of one or several agents in a multiagent scenario can be obtained, e.g. of a human in human-machine interaction or of a swarm of unmanned aerial vehicles. This leads to an extension of the learning by demonstration paradigm to the multiplayer case [15]. Inspired by similar approaches in the single-agent scenario, also known as inverse optimal control, inverse dynamic game methods based on necessary conditions for Nash equilibria have been proposed, e.g. for zero-sum games [7] or non-zero-sum two-player scenarios [8]. Recent results in [9]–[11] show extensions to a general N-player case.

The inverse dynamic game problem has also been examined—especially the single-player case—in the field of computer science, where various so-called inverse reinforcement learning (IRL) methods have been proposed (e.g. [16]–[18]). Contrary to inverse optimal control, their focus lies on the identification of a cost function which may not be equal to the original cost function, but is able to explain observed trajectories. In the last years, some effort has been spent to extend these techniques to a multiplayer setting. Many of these extensions consider a scenario where one global cost function is sought (e.g. of a central controller) which can describe the behavior of several agents [19], [20] or a scenario where all of the agents have the same reward [21]. These approaches can therefore be associated to cooperative dynamic game theory, where the players may have an individual objective function but can cooperate in order to improve their performance [1]. In the case of non-cooperative dynamic games, some IRL-based methods have been proposed, e.g. [12], [13]. Nevertheless, all aforementioned methods are based on a Markov Decision Process (MDP) and are limited to discrete-valued and finite control and state spaces. Literature shows few available work which considers continuous-valued control and state spaces. First extensions of IRL methods to continuous state and action spaces build upon the work of Ziebart et al. [23] who applied the principle of maximum entropy (MaxEnt) of Jaynes [24] in a single-player IRL setting. Additionally, MaxEnt IRL has been shown to have great potential for real

1 Both these state-of-the-art approaches and this paper consider cooperative yet not coalitional games, where several groups of players may build coalitions to act noncooperatively with respect to other ones, see e.g. the definitions given in [22].
applications [25–27]. Existing work in a multiplayer case includes the work of Kretzschmar et al. [28], where it is assumed that a maximum entropy distribution can describe the navigating behavior of interacting players. Furthermore, in a similar application, Ma et al. [29] considered a maximum entropy distribution for each of the players. While these papers show first interesting applications, the theoretical foundation of maximum entropy IRL in a multiplayer scenario has not been developed, especially in the case of continuous-valued state and control spaces. This is crucial to avoid the curse of dimensionality which would arise in many applications if discrete state and action spaces were assumed.

Therefore, in this paper, we extend MaxEnt IRL to a multiplayer case such that general N-player inverse dynamic games with continuous-valued and infinite state and control spaces are solved. We provide methods for identifying cost function parameters of one or several players in a dynamic game for three different solution concepts: (i) Pareto efficient solutions in cooperative games and (ii) open-loop and (iii) feedback Nash equilibrium for non-cooperative dynamic games. We show cost functions describing an open-loop and a feedback Nash player dynamic game are shown in Section III. The results present a solution of a dynamic game with cost functions \( \forall \theta \in \mathbb{R}^P \) represents the vector of player \( i \)'s individual feature weights, i.e. the cost function’s parametrization. The features \( \{ \eta_i \} \) of the cost functions \( J_i \) of each player \( i \in P \) are assumed to be continuously differentiable with respect to all of its arguments for all \( k \in \mathbb{R}, q \in \{ 1, \ldots, p_i \} \). A main element of inverse problems in optimal control and dynamic games are the observed state and control trajectories which we define in the following:

**Definition 1.** Let

\[
\mathbf{x} = \begin{bmatrix} (x^{(1)})^\top & \ldots & (x^{(k_E)})^\top \end{bmatrix}^\top \in \mathbb{R}^{nk_E}
\]

and

\[
\mathbf{u}_i = \begin{bmatrix} (u_i^{(1)})^\top & \ldots & (u_i^{(k_E)})^\top \end{bmatrix}^\top \in \mathbb{R}^{m_i k_E},
\]

for \( i \in P \), be vectors containing all values of the system state \( x^{(k)} \) and the control values \( u_i^{(k)} \) of player \( i \in P \) for all time steps \( k \in \mathbb{R} \), respectively.

A trajectory \( \zeta := \{ \mathbf{x}, \mathbf{u}_1, \ldots, \mathbf{u}_N \} \) is defined as a set containing the values of the system state \( \mathbf{x} \) and the control values \( \mathbf{u}_i \) of all players \( i \in P \), which is feasible with respect to the system dynamics given by (1).

Observed trajectories are assumed to be generated by \( p(\zeta) \) which denotes a probability density function over all feasible trajectories \( \zeta \). Following previous literature on IRL, the observations are denoted as expert trajectories \( E \subseteq \{ \mathbf{E}, \mathbf{E}_1, \ldots, \mathbf{E}_E, \mathbf{N} \} \). These are assumed to represent a solution of a dynamic game with cost functions \( J_i \) parameterized by the unknown expert feature weights \( \theta_{E,i} \), \( \forall i \in P \). In the course of this paper, the expert trajectories will correspond to either open-loop Nash, feedback Nash and Pareto efficient solutions, depending on the considered case.

A key value in IRL methods is the feature count, which we introduce in the following.

**Definition 2.** The feature count \( \mu_i(\zeta) \in \mathbb{R}^{p_i} \) of a player \( i \in P \) along a trajectory \( \zeta \) is defined as a vector containing the accumulated values of the features along that trajectory, i.e.

\[
\mu_i(\zeta) = \sum_{k=1}^{k_E} \eta_i \left( x^{(k)}, u_i^{(k)}, \ldots, u_N^{(k)} \right),
\]

with \( x^{(k)}, u_i^{(k)} \in \zeta, \forall i \in P, k \in \mathbb{R} \).

Using the feature counts \( \mu_i(\zeta) \) and (2), the costs along a trajectory \( \zeta \) for any player \( i \in P \) can be rewritten as

\[
J_i(\zeta) = -\theta_i^\top \mu_i(\zeta).
\]
corresponding to the cost functions \( J_i \), \( \forall i \in \mathbb{P} \). In order to state a relationship between observed trajectories \( \zeta_E \) and the probability distribution \( p(\zeta | \theta_{E,1:N}) \) which generated them, we make the following assumption:

**Assumption 1.** The feature count along the expert trajectory \( \zeta_E \) (denoted as \( \mu_{E,i} \) for all players \( i \in \mathbb{P} \)) represents the expectation of the feature count \( \mathbb{E}_{p(\zeta | \theta_{E,1:N})} \{ \mu_i(\zeta) \} \) based on the expert probability density function \( p(\zeta | \theta_{E,1:N}) \) which results from the feature expert weights \( \theta_{E,1}, \ldots, \theta_{E,N} \), i.e.

\[
\mathbb{E}_{p(\zeta | \theta_{E,1:N})} \{ \mu_i(\zeta) \} = \mu_{E,i}, \quad \forall i \in \mathbb{P}.
\]  

(6)

Furthermore, if \( n_e \in \mathbb{N} \) expert trajectories are given, i.e. a set of trajectories \( \mathcal{D}_N = \{ \zeta_{E,1}, \ldots, \zeta_{E,n_e} \} \), the expectation of the expert feature count of player \( i \) is given by

\[
\mathbb{E}_{p(\zeta | \theta_{E,1:N})} \{ \mu_i(\zeta) \} = \frac{1}{n_e} \sum_{i=1}^{n_e} \mu_i(\zeta_i).
\]  

(7)

With these definitions, we proceed to define the problem of inverse dynamic games with IRL.

**Problem 1.** Find feature weights \( \hat{\theta}_i \), \( \forall i \in \mathbb{P} \), such that the expected expert cost of a trajectory sampled from the probability density \( p(\zeta | \hat{\theta}_{1:N}) \) resulting from the identified feature weights \( \hat{\theta}_i \) corresponds to the cost functions \( \hat{J}_{E,i} \) for each player \( i \in \mathbb{P} \) to the expected costs of the expert trajectory sampled from the expert probability density \( p(\zeta | \theta_{E,1:N}) \), i.e.

\[
\mathbb{E}_{p(\zeta | \theta_{E,1:N})} \{ J_{E,i}(\zeta | \theta_{E,1:N}) \} = \mathbb{E}_{p(\zeta | \hat{\theta}_{1:N})} \{ \hat{J}_{E,i}(\zeta | \hat{\theta}_{1:N}) \}, \quad \forall i \in \mathbb{P}.
\]  

(8)

The requirement (8) arises from the demand of obtaining for each player a cost function that results in an individual performance as good as the observed one, where the performance is measured with respect to each player’s unknown true cost function \( J_{E,i} \).

A difficulty of inverse problems in optimal control and dynamic games consists in their inherent ill-posed nature. There are several parametrizations of the cost functions \( J_i \) which lead to trajectories with the same costs as the expert trajectories. For example, in inverse optimal control and inverse dynamic Nash games, it is well-known that multiplying any cost function with a constant \( c \in \mathbb{R}^+ \) does not affect the resulting trajectories. The next section will show how to apply the principle of maximum entropy to deal with the ill-posedness issue.

### III. Application of the Principle of Maximum Entropy to Inverse Dynamic Games

For one-player-scenarios (\( N = 1 \)) with discrete state and action spaces, Ziebart et al. [23] propose the maximum entropy IRL framework which resolves the ill-posedness in the inverse problem by applying the principle of maximum entropy introduced by Jaynes [24]. According to Jaynes, this method leads to the “least biased estimate possible on the given information”.

In this section, we transfer the maximum entropy approach to inverse dynamic games with \( N \) players.

The following lemma provides a useful result in order to solve Problem 1:

**Lemma 1.** Let the expectation of the feature count be equal for both the probability density \( p(\zeta | \theta_{1:N}) \) resulting from the identified feature weights and the probability function \( p(\zeta | \theta_{E,1:N}) \) of the expert, i.e.

\[
\mathbb{E}_{p(\zeta | \theta_{1:N})} \{ \mu_i(\zeta) \} = \mathbb{E}_{p(\zeta | \theta_{E,1:N})} \{ \mu_i(\zeta) \}
\]  

(9)

for each player \( i \in \mathbb{P} \). Then, for any expert feature weights with \( \| \theta_{E,i} \|_2 \leq \infty \), [8] is fulfilled.

**Proof:** By rewriting (8), we can state the following relations:

\[
0 \leq \left| \mathbb{E}_{p(\zeta | \theta_{1:N})} \{ J_{E,i}(\zeta | \theta_{E,1:N}) \} - \mathbb{E}_{p(\zeta | \theta_{1:N})} \{ \hat{J}_{E,i}(\zeta | \hat{\theta}_{1:N}) \} \right|
\]  

(10)

\[
= \left| \mathbb{E}_{p(\zeta | \theta_{1:N})} \{ \theta_{E,i} \mu_i(\zeta) \} - \mathbb{E}_{p(\zeta | \theta_{E,1:N})} \{ \theta_{E,i} \mu_i(\zeta) \} \right|
\]  

(11)

\[\leq \| \theta_{E,i} \|_2 \left\| \mathbb{E}_{p(\zeta | \theta_{1:N})} \{ \mu_i(\zeta) \} - \mathbb{E}_{p(\zeta | \theta_{E,1:N})} \{ \mu_i(\zeta) \} \right\|_2
\]  

(12)

Therefore, if (9) holds, then the right side of (12) is equal to zero and hence, together with the inequality in (10), this implies that (8) holds as well.

**Lemma 1** represents the principle of matching feature expectations for all players. This principle was introduced in [16] and used as a basis for numerous single-player IRL methods.

Since Problem 1 implies (8), by the results of Lemma 1 and using Assumption 1 we require

\[
\mathbb{E}_{p(\zeta | \theta_{1:N})} \{ \mu_i(\zeta) \} = \mu_{E,i},
\]  

(13)

for each player \( i \in \mathbb{P} \).

Our aim is to find a probability density function \( p(\zeta | \theta_{1:N}) \) which represents the probability of trajectories \( \zeta \) as a function of the feature weight parameters \( \theta_1, \ldots, \theta_N \), yet considering (13) as only a-priori knowledge. Moreover, for a density function,

\[
\int_{\zeta} p(\zeta | \theta_{1:N}) d\zeta = 1
\]  

(14)

must apply.

Since the conditions (13) and (14) do not lead to a unique solution for the probability density function, the principle of maximum entropy is applied. For a continuous density function the differential entropy is given by [31] Section 6.1

\[
h(p(\zeta | \theta_{1:N})) = -\int_{\zeta} p(\zeta | \theta_{1:N}) \ln(p(\zeta | \theta_{1:N})) d\zeta.
\]  

(15)

We aim to determine a probability density function \( p(\zeta | \theta_{1:N}) \) that only takes the information of (13) and (14) into consideration. To achieve this, the differential entropy (15) is maximized with the requirements (13) and (14) as
optimization constraints. The density function which leads to maximum entropy [31] Section 12.1 in dynamic games is presented in the following lemma.

**Lemma 2.** The maximum entropy probability density under the constraints defined by (13) and (14) is given by

\[
p(\zeta|\theta_{1:N}) = \frac{\exp \left( \sum_{i=1}^{N} \theta_i^T \mu_i(\zeta) \right)}{\int_{\mathcal{W}} \exp \left( \sum_{i=1}^{N} \theta_i^T \mu_i(\zeta) \right) d\zeta}.
\]

**Proof:** To maximize the differential entropy (15) under the constraints given by (13) and (14), we introduce Lagrange multipliers \(\lambda \in \mathbb{R}\) and \(\theta_i \in \mathbb{R}^p_i\), \(\forall i \in \mathbb{P}\), and set up the objective function

\[
\Lambda(p(\zeta|\theta_{1:N}), \lambda, \theta_1, \ldots, \theta_N) = - \int_{\mathcal{W}} \ln \left( p \left( \zeta \big| \theta_{1:N} \right) \right) p \left( \zeta \big| \theta_{1:N} \right) d\zeta + \ldots
\]

\[
+ \lambda \left( \int_{\mathcal{W}} p \left( \zeta \big| \theta_{1:N} \right) d\zeta - 1 \right) + \ldots
\]

\[
+ \theta_1^T \left( \int_{\mathcal{W}} p \left( \zeta \big| \theta_{1:N} \right) \mu_1 \left( \zeta \right) d\zeta - \mu_{E,1} \right) + \ldots
\]

\[
\vdots
\]

\[
+ \theta_N^T \left( \int_{\mathcal{W}} p \left( \zeta \big| \theta_{1:N} \right) \mu_N \left( \zeta \right) d\zeta - \mu_{E,N} \right).
\]

(16)

In this way, the expression

\[
\frac{\partial \Lambda}{\partial p(\zeta|\theta_{1:N})} = - \int_{\mathcal{W}} \ln \left( p \left( \zeta \big| \theta_{1:N} \right) \right) p \left( \zeta \big| \theta_{1:N} \right) d\zeta - \ldots
\]

\[
+ \lambda \int_{\mathcal{W}} 1 d\zeta + \ldots
\]

\[
+ \theta_1^T \int_{\mathcal{W}} \mu_1 \left( \zeta \right) d\zeta + \ldots + \theta_N^T \int_{\mathcal{W}} \mu_N \left( \zeta \right) d\zeta
\]

\[
= \int_{\mathcal{W}} \ln \left( p \left( \zeta \big| \theta_{1:N} \right) \right) - 1 + \lambda + \sum_{i=1}^{N} \theta_i^T \mu_i \left( \zeta \right) d\zeta
\]

\[
\frac{\partial^2 \Lambda}{\partial p(\zeta|\theta_{1:N})^2} = - \int_{\mathcal{W}} \frac{1}{p(\zeta|\theta_{1:N})} \frac{d\zeta}{d\zeta} < 0
\]

(17)

Due to \(p(\zeta|\theta_{1:N}) \geq 0\).

In order to solve an inverse dynamic game based on the derived probability function, i.e. obtain the cost function parameters \(\theta_i\), the knowledge of which solution concept lies at hand becomes necessary. We first consider cooperative games, where players can make an agreement which they strictly adhere to in order to obtain better results. Afterwards, we consider the non-cooperative case, where each player acts greedily in order to unilaterally minimize its own cost function.

**IV. INVERSE COOPERATIVE DYNAMIC GAMES**

In this section, we show a method to identify cost function parameters out of a solution of the dynamic game in the sense of Pareto. Furthermore, we prove the unbiasedness of the estimation.

**A. Preliminaries**

We restrict ourselves to Pareto efficient solutions which can be described by a global cost function given by the sum of weighted player cost functions. Several global cost functions are possible depending on the selected weighting parameters to build the sum. One particular global cost function is given by the sum of uniformly weighted player cost functions defined as follows.

**Definition 3.** The uniformly weighted sum of all player cost functions is given by

\[
J_\Sigma = \sum_{i=1}^{N} J_i = \sum_{i=1}^{N} -\theta_i^T \mu_i =: -\theta_\Sigma^T \mu_\Sigma
\]

(23)

with

\[
\theta_\Sigma = \left[ \theta_1^T \ldots \theta_N^T \right]^T,
\]

(24a)

and

\[
\mu_\Sigma = \left[ \mu_1^T \ldots \mu_N^T \right]^T.
\]

(24b)

We further introduce the following assumption.

**Assumption 2.** The cost functions \(J_i\) are convex for all \(i \in \mathbb{P}\).
Remark 1. We note that
\[
\arg\min_{\gamma} J_{\Sigma}(\gamma) = \arg\min_{\gamma} \sum_{i=1}^{N} \frac{1}{N} J_{i}(\gamma), \tag{25}
\]
where $\gamma := \{ \gamma_1, \ldots, \gamma_N \}$, holds since multiplying any cost function $J_{\Sigma}$ with a constant factor $c \in \mathbb{R}^+$ (here $1/N$) does not alter the solution of the optimization problem. Therefore, under Assumption 3 the minimizer of $J_{\Sigma}$ describes a Pareto efficient solution of a cooperative game [32 Theorem 6.4].

B. Identification Method and Unbiasedness of the Estimation

We aim to find cost function parameters which explain observed trajectories corresponding to a cooperative game with equally weighted cost functions as in Definition 3. Therefore, our identification results of this section are based on the following assumption:

Assumption 3. The observed expert trajectories $\zeta_E$ constitute an optimal solution of a cooperative dynamic game with system dynamics $f$ and $N$ uniformly weighted cost functions $J_i$ fulfilling Assumption 2 with unknown parameters $\theta^{(CG)}_i$.

The identification method is based on the maximum likelihood of the observed trajectories under the density (16) with maximum entropy. Before introducing the method, we use (23) and (24) to rewrite (16) as
\[
p(\zeta|\theta_{\Sigma}) = \frac{\exp(\theta_{\Sigma}^{T} \mu_{\Sigma}(\zeta))}{\int_{\zeta} \exp(\theta_{\Sigma}^{T} \mu_{\Sigma}(\zeta)) d\zeta} = \frac{\exp(-J_{\Sigma}(\zeta))}{\int_{\zeta} \exp(-J_{\Sigma}(\zeta)) d\zeta}. \tag{26}
\]

The following theorem represents our main result concerning the identification of cost functions in an inverse cooperative dynamic game.

Theorem 1. Let $n_i$ expert trajectories $D_{\Sigma} = \{ \zeta_{E_1}, \ldots, \zeta_{E_{n_i}} \}$ fulfilling Assumption 3 be available. Then, the maximum likelihood estimator of expert trajectories, i.e.
\[
\hat{\theta}_{\Sigma} = \arg\max_{\theta_{\Sigma}} \mathcal{L}(D_{\Sigma}|\theta_{\Sigma}) := \arg\max_{\theta_{\Sigma}} \ln \prod_{i=1}^{n_i} p(\zeta_{E_i}|\theta_{\Sigma}) \tag{27}
\]
where $p(\zeta_{E_i}|\theta_{\Sigma})$ is obtained by evaluating (26) with $\zeta_{E_l}, l \in \{1, \ldots, n_i\}$, leads to a probability density function for which the trajectories yield in expectation the same accumulated costs for all players as the trajectories corresponding to the expert probability density, i.e.
\[
\mathbb{E}_{p(\zeta|\theta_{E})} \{ J_{\Sigma}(\zeta|\theta_{E,\Sigma}) \} = \mathbb{E}_{p(\zeta|\theta_{E})} \{ J_{\Sigma}(\zeta|\theta_{E,\Sigma}) \}. \tag{28}
\]

Proof: Maximization of the log-likelihood function (27) leads to
\[
0 = \frac{1}{\theta_{\Sigma}} \sum_{i=1}^{n_i} \ln p(\zeta_{E_i}|\theta_{\Sigma}) \bigg|_{\theta_{\Sigma}=\hat{\theta}_{\Sigma}}. \tag{29}
\]

Using the probability density function resulting from Lemma 2 this can be rewritten as
\[
0 = \frac{1}{\theta_{\Sigma}} \sum_{i=1}^{n_i} \ln \left( \frac{\exp(\theta_{\Sigma}^{T} \mu_{\Sigma}(\zeta_{E_i}))}{\int_{\zeta} \exp(\theta_{\Sigma}^{T} \mu_{\Sigma}(\zeta)) d\zeta} \right) \bigg|_{\theta_{\Sigma}=\hat{\theta}_{\Sigma}} \tag{30}
\]
\[
= \sum_{i=1}^{n_i} \frac{1}{\theta_{\Sigma}} \left( \frac{\exp(\theta_{\Sigma}^{T} \mu_{\Sigma}(\zeta_{E_i}))}{\int_{\zeta} \exp(\theta_{\Sigma}^{T} \mu_{\Sigma}(\zeta)) d\zeta} \right) \bigg|_{\theta_{\Sigma}=\hat{\theta}_{\Sigma}} \tag{31}
\]
\[
= \sum_{i=1}^{n_i} \left( \frac{\exp(-J_{\Sigma}(\zeta_{E_i}))}{\int_{\zeta} \exp(-J_{\Sigma}(\zeta)) d\zeta} \right) \bigg|_{\theta_{\Sigma}=\hat{\theta}_{\Sigma}} \tag{32}
\]

We note that the integrals in the numerator and the denominator in (32) are independent of each other. Therefore, (32) can be rewritten as
\[
0 = \sum_{i=1}^{n_i} \left( \frac{-\int_{\zeta} \exp(-J_{\Sigma}(\zeta)) d\zeta}{\int_{\zeta} \exp(-J_{\Sigma}(\zeta)) d\zeta} \right) \bigg|_{\theta_{\Sigma}=\hat{\theta}_{\Sigma}} \tag{33}
\]

Using (26), we obtain
\[
0 = \sum_{i=1}^{n_i} \left( -\int_{\zeta} p(\zeta|\theta_{E}) \mu_{\Sigma}(\zeta) d\zeta + \mu_{\Sigma}(\zeta_{E_i}) \right) \bigg|_{\theta_{E}=\theta_{\Sigma}} \tag{34}
\]

From (34) and by Assumption 1 we obtain
\[
n_{\Sigma} \mathbb{E}_{p(\zeta|\theta_{E})} \{ \mu_{\Sigma}(\zeta) \} = \sum_{i=1}^{n_i} \mu_{\Sigma}(\zeta_{E_i}) \tag{35}
\]

The expectations of the feature count $\mu_{\Sigma}$ are equal for both probability density functions. We apply now the results of Lemma 1 to state that (35) leads to (28).

The results of Theorem 1 imply that the expectation of the global costs (under the expert’s parameters) produced by trajectories generated by both expert and estimated probability density functions are equal. Note that this result is weaker than the one required in [8] as it considers only the overall costs.
Nevertheless, for a cooperative game, it is enough to describe observed trajectories completely.

**Remark 2.** Solving the optimization problem \( (27) \) demands the possibility of evaluating the likelihood function \( L \{ D \Sigma \{ \theta_S \} \} \) and therefore the probability density function \( (26) \) at the trajectories \( \zeta_E \). The denominator in \( (26) \) includes an integral over all trajectories \( \zeta \) which are feasible with respect to the system dynamics and an initial state. Calculating this integral is intractable given the continuous-valued control and action spaces. Therefore, approximations are usually sought. In this paper, we apply the approach introduced in \([30]\).

**Remark 3.** The result \( \hat{\theta}_S \) of \( (27) \) contains the cost function parameters of all players in one single vector according to \( (24) \). Assuming that the number of features \( p_i \) is known for every player \( i \in \mathbb{P} \), an individual parameter set \( \theta_i \) can be determined by means of \( (24a) \) out of \( \hat{\theta}_S \). This is done by using the relation

\[
\hat{\theta}_i = \theta_S^* (l^*_i : l^e_i),
\]

with

\[
l^*_i = 1 + \sum_{a=1}^i p_{a-1} \quad \text{and} \quad l^e_i = \sum_{a=1}^i p_a,
\]

with \( p_0 = 0 \) and where \( \hat{\theta}_S^* (l^*_i : l^e_i) \in \mathbb{R}^{l^*_i - l^e_i + 1} \) denotes a vector that contains the entries \( l^*_i \) to \( l^e_i \) of the vector \( \hat{\theta}_S \).

**Remark 4.** The presented method is capable of identifying cost function parameters which explain trajectories corresponding to an optimal solution based on uniformly weighted player cost functions, which is one of the Pareto efficient solutions belonging to the Pareto frontier. Pareto efficient solutions can be obtained by minimizing the sum of cost functions of all players which are nevertheless not necessarily equally weighted (see \([22] \text{ Definition 6.1}) \). However, the presented method is also capable of describing the trajectories in this scenario.

V. IDENTIFICATION IN NON-COOPERATIVE NASH GAMES

We now consider non-cooperative dynamic games, where all players act greedily and no agreements between the players exist. In a non-cooperative dynamic game, the players minimize their individual cost function \( J_i \) and may potentially find themselves in a Nash equilibrium defined as follows:

**Definition 4.** An \( N \)-tuple of control sequences \( (\mathbf{u}_1^*, \ldots, \mathbf{u}_N^*) \) constitutes a Nash equilibrium if, and only if, the inequality

\[
J_i (\mathbf{u}_1^*, \ldots, \mathbf{u}_i^*, \mathbf{u}_{i+1}^*, \ldots, \mathbf{u}_N^*; \theta_i^*) \leq J_i (\mathbf{u}_1^*, \ldots, \mathbf{u}_i^*, \mathbf{u}_{i+1}^*, \ldots, \mathbf{u}_N^*; \theta_i^*)
\]

is satisfied for all players \( i \in \mathbb{P} \), where \( \mathbf{u}_i^* \) denotes the control sequence of all players except player \( i \) (cf. \([33] \text{ p. 266}) \).

The Nash equilibrium constitutes a stable solution of a non-cooperative dynamic game since \([33] \) implies that no incentive exists for each player to deviate from their Nash equilibrium strategy.

In a dynamic game, the control decision of the players depend on the information each one of them has access to. Following the notation in \([33]\), this can be represented as

\[
\mathbf{u}_i^{(k)} = \gamma_i^{(k)} (\xi_i^{(k)}), \quad i \in \mathbb{P}, k \in \mathbb{K},
\]

where \( \gamma_i^{(k)} \) defines the strategy of player \( i \) at time step \( k \) based on the set \( \xi_i^{(k)} \) representing the information of player \( i \) available at time step \( k \). We focus in this paper on the following information patterns

- open-loop (OL) information pattern, i.e. \( \xi_i^{(k)} = \{ x^{(1)} \} \)
- memory-less perfect state (MPS) information pattern, i.e. \( \xi_i^{(k)} = \{ x^{(1)}, x^{(k)} \} \).

Depending on whether the control values arisen from the minimization of a cost function with an OL or an MPS information pattern, the control trajectories \( \mathbf{u}_i \) correspond to an open-loop or feedback Nash equilibrium, respectively\([2]\). We seek to identify cost function parameters which explain Nash equilibrium trajectories for both information patterns. In the following subsections, we give solutions for both types of inverse dynamic games by applying the maximum entropy principle.

A. Inverse Open-Loop Dynamic Games

As we consider inverse open-loop dynamic games, the results in this subsection are based on the following assumption:

**Assumption 4.** Observed expert trajectories \( \zeta_E \) constitute an open-loop Nash equilibrium solution to the game consisting of the system dynamics \( (1) \) and the \( N \) cost functions of the form \( (2) \) with unknown parameters \( \theta_i^* \) for all players \( i \in \mathbb{P} \).

Similar to Section IV we seek a suitable probability function \( p(\zeta) \) for the estimation of cost function parameters.

The non-cooperative character of the game implies that each player only considers his own cost function. By \([33] \text{ Theorem 6.1}] \) we discern that obtaining the open-loop Nash equilibrium involves the solution of a set of differential equations which includes derivatives of the system dynamics and the features with respect to the system state \( x^{(k)} \) and player \( i \)’s controls \( \mathbf{u}_i^{(k)} \). Since the other players’ controls do not depend on either of these, they do not have any influence on player \( i \)’s actions.

Therefore, we define the probability function

\[
p(\zeta | \theta_i) = \frac{\exp (-J_i (\zeta))}{\int_\zeta \exp (-J_i (\zeta')) d\zeta'}
\]

which represents the probability of a particular trajectory from the point of view of player \( i \). This density implies that the probability of a particular trajectory is inversely proportional to

\[2\]For a feedback Nash equilibrium, a further restriction needs to be added to \([38] \text{ (see [33] Definition 6.2}] \). This definition was considered in the simulation results in the next section but was omitted here for simplicity.
the costs generated by player \( i \)'s own individual cost function \( J_i \) defined by player \( i \)'s cost function parameter set \( \theta_i \).

This simplifies the probability density function \( p(\zeta \mid \theta_1:N) \) in such a way that \( N \) probability density functions \( p(\zeta \mid \theta_i) \) which depend on each player’s cost function parameters \( \theta_i \) are considered instead of one single probability density function which depends on all parameters.

Before presenting our main result for inverse non-cooperative dynamic games, we first introduce the following assumption which is analogous to Assumption 1.

**Assumption 5.** The mean of the feature count of the \( n_t \) expert trajectories fulfilling Assumption 4 corresponds to the positions corresponding to the elements of \( \zeta \) representing features which were not in \( \eta_i \) previously.

Furthermore, let the extended parameter vector \( \vec{\theta} \) be defined such that

\[
J_i(\zeta) = \vec{\theta}_i^\top \mu(\zeta) = \vec{\theta}_i^\top \hat{\mu}(\zeta), \quad \forall i \in \mathbb{P}. \tag{42}
\]

Note that, for \( \eta \) to hold, \( \vec{\theta}_i \) has to include zeros in the positions corresponding to the elements of \( \eta \), i.e., \( \theta_i \) includes all features \( \eta_i \) of \( \eta_i \) all players such that no feature is included more than once. The extended feature count \( \hat{\mu}(\zeta) \) is defined analogously according to Definition 2.

Furthermore, let the extended parameter vector \( \vec{\theta} \) be defined such that

\[
J_i(\zeta) = \vec{\theta}_i^\top \mu(\zeta) = \vec{\theta}_i^\top \hat{\mu}(\zeta), \quad \forall i \in \mathbb{P}. \tag{43}
\]

**Proof:**

Using the the extended parameter vector, \( \{ \theta_i \} \) can be rewritten as

\[
E_{p(\zeta \mid \theta_i)} \{ J_j(\zeta \mid \theta_i^*) \} = E_{p(\zeta \mid \hat{\theta}_i)} \{ J_j(\zeta \mid \theta_i^*) \} \tag{45}
\]

for all \( j \in \mathbb{P} \). The maximization of the log-likelihood function (43) implies

\[
0 = \frac{\partial}{\partial \theta_i} \sum_{l=1}^{n_s} \ln \left( \frac{\exp (\vec{\theta}_i^\top \hat{\mu}(\zeta_j))}{\int \exp (\vec{\theta}_i^\top \hat{\mu}(\zeta)) d\zeta} \right) \bigg|_{\hat{\theta}_i = \hat{\theta}_i}, \tag{46}
\]

where we used (42). Continuing analogously to the proof of Theorem 1 and using (41) with (42), we obtain

\[
E_{p(\zeta \mid \theta_i)} \{ \hat{\mu}(\zeta) \} = \frac{1}{n_t} \sum_{l=1}^{n_t} \hat{\mu}(\zeta_j) = E_{p(\zeta \mid \theta_i)} \{ \hat{\mu}(\zeta) \}. \tag{47}
\]

Thus, the expectations of the feature count \( \hat{\mu} \) are equal for both probability density functions \( p(\zeta \mid \theta_i) \) and \( p(\zeta \mid \theta_i^*) \). We apply now the results of Lemma 1 and conclude that (47) leads to (45) and (44).

The results of Theorem 2 guarantee that the costs of the estimated and the expert cost functions are the same for all players, thus ensuring the fulfillment of (8) and solving Problem 1 for open-loop dynamic games.

**Remark 5.** The evaluation of the likelihood function \( L(D \mid \theta_j) \) and the probability density function \( p(\zeta \mid \theta_j) \) at the trajectories \( \zeta_j \) is done analogously to the previous section (see Remark 2). The same holds for the results of the next subsection.

### B. Inverse Feedback Nash Dynamic Games

In this section, we give solutions for inverse dynamic games with the feedback Nash equilibrium as a solution concept. Therefore, we consider the MPS information structure given by

\[
u^{(k)}_i = \gamma_i(x^{(k)}). \tag{48}
\]

The results of this subsection are based on the following assumptions:

**Assumption 6.** Observed expert trajectories \( \zeta_E \) constitute a feedback Nash equilibrium solution to the game consisting of the system dynamics (1) and the \( N \) cost functions of the form (2) with unknown parameters \( \theta_i^* \) for all players \( i \in \mathbb{P} \).

**Assumption 7.** The players’ Nash equilibrium feedback control laws \( \gamma_i^{(k)} \) are known or can be estimated from the expert trajectories \( \zeta_E \).

**Remark 6.** Assumption 7 is not restrictive for linear-quadratic dynamic games, since the Nash equilibrium controls are given by

\[
u^{(k)}_i = \gamma^{(k)}_i(x^{(k)}) = K_i^* x^{(k)} \tag{49}\]

According to [33], p. 278, the feedback Nash equilibrium solution under the MPS information pattern solely depends on \( x^{(k)} \) at the time step \( k \). The dependency on \( x^{(k)} \) is given only for \( k = 1 \).

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3According to [33], p. 278, the feedback Nash equilibrium solution under the MPS information pattern solely depends on \( x^{(k)} \) at the time step \( k \). The dependency on \( x^{(k)} \) is given only for \( k = 1 \).
with $\mathbf{K}^*_i \in \mathbb{R}^{m_i \times n}$ [32, Section 8.3]. The estimation of $\mathbf{K}^*_i$ can easily be performed by means of a least-squares approach, see e.g. [17]. In the case of general nonlinear feedback Nash equilibria, not only the estimation of the control law is non-trivial, but also the calculation of the equilibria themselves which implies the solution of coupled partial differential equations.

If Assumption 7 holds, the control laws of the players $j \in \mathcal{P}$, $j \neq i$ can be used to rewrite the system dynamics (1) from player $i$’s perspective as

$$x^{(k+1)} = f^{(k)} \left( x^{(k)}, \gamma^*_i \left( x^{(k)} \right), \ldots, u_i^{(k)}, \ldots, \gamma^*_N \left( x^{(k)} \right) \right)$$

$$=: f^*_i \left( x^{(k)}, u_i^{(k)} \right). \quad (50)$$

In this way, it is possible for player $i$ to represent the system dynamics as a function of the system state $x$ and his own control variable $u_i$. The effect of the other players’ controls are considered due to the MPS information structure and the implied knowledge of the system state in every step. Analogously, the features $\eta_i$ of player $i$’s cost function can be rewritten as a function of the state $x$ and the control variables $u_i$:

$$\eta_i = \eta_i \left( x^{(k)}, u_i^{(k)}, \ldots, u_N^{(k)} \right)$$

$$= \eta_i \left( x^{(k)}, \gamma^*_i \left( x^{(k)} \right), \ldots, u_i^{(k)}, \ldots, \gamma^*_N \left( x^{(k)} \right) \right) \quad (51)$$

Based on the representation (50) of the system dynamics from player $i$’s perspective and the rewritten features (51), we state the following theorem which describes the method for an unbiased estimation of cost function parameters in an inverse feedback Nash dynamic game.

**Theorem 3.** Let a set of expert trajectories $\mathcal{D} = \{ \xi_{E_1}, \ldots, \xi_{E_n} \}$ be given such that Assumptions 5 and 6 are fulfilled. Furthermore, let Assumption 7 hold such that the feedback Nash control laws $\gamma^*_i$ are given for all $i \in \mathcal{P}$. Then, the maximum likelihood estimator with respect to the observed expert trajectories, i.e.

$$\hat{\theta}_i = \arg \max_{\theta_i} \ln \mathcal{L} \left( \mathcal{D} | \theta_i \right) = \arg \max_{\theta_i} \sum_{l=1}^{n_l} \ln \left( p \left( \xi_{E_l} | \theta_i \right) \right), \quad (52)$$

where $p \left( \xi_{E_l} | \theta_i \right)$ is obtained by evaluating (40) with $\xi_{E_l}$, $l \in \{1, \ldots, n_l\}$ and with respect to the system dynamics (50), leads to parameters $\hat{\theta}_i$ such that

$$E_{p(\xi|\theta_i)} \left\{ J_i \left( \xi | \theta_i \right) \right\} = E_{p(\xi|\hat{\theta}_i)} \left\{ J_i \left( \xi | \theta_i \right) \right\} \quad (53)$$

holds for all $i, j \in \mathcal{P}$ (cf. Theorem 2).

**Proof:** The cost function $J_i$ can be rewritten using the modified features (51). Afterwards, the proof is the same as in Theorem 2. \hfill \blacksquare

**VI. SIMULATION EXAMPLES**

In this section, we present simulations to illustrate the identification of cost functions in a cooperative game and in a noncooperative game with open-loop and feedback Nash equilibria. The first example is a nonlinear dynamic game with a ball-on-beam system as the dynamic system to be controlled. In the second example, we linearize the system dynamics of the ball-on-beam system to obtain a linear-quadratic (LQ) dynamic game such that it is possible to calculate Nash equilibria in order to verify the inverse feedback Nash dynamic game solutions. After showing the results, we end this section with a discussion.

**A. Nonlinear Dynamic Game**

This first example is used for demonstrating the performance and comparing the results of two inverse dynamic game methods, namely the approach presented in Section III for cooperative games with Pareto efficient solutions and the method presented in Section IV-A for open-loop Nash equilibria. The considered system is the well-known ball-on-beam system which is typically used for testing nonlinear controllers (e.g. [34]). Contrary to the standard ball-on-beam system, this system is controlled by two players simultaneously with the goal of balancing a ball in the middle of the beam.

The considered ball-on-beam system is depicted in Fig. 1. Here, $\alpha_x$ denotes the angle of the beam towards the horizontal. In addition, $(s_X, s_Y)$ and $(s_x, s_y)$ represent the ball positions in earth-fixed and beam-fixed coordinate systems, respectively, both centered at the beam’s center of rotation. Both players are allowed to interact with the system by applying a torque $u_i(t) = M_i(t)$, $i \in \{1, 2\}$, with respect to the beam’s rotational axis. Let the system state be defined as $x(t) = [s_X(t) \ s_Y(t) \ \alpha_x(t) \ \dot{\alpha}_x(t)]^\top$. Then, the system dynamics are described by the nonlinear differential equation

$$\dot{x} = \begin{bmatrix} x_2 \\ -m_g x_1 x_2 x_4 + m_i x_1 x_2 \cos(x_3) + x_1 + u_2 \\ -2m_b x_1 x_2 x_4 + m_1 g x_1 \cos(x_3) + x_1 + u_2 \\ -\frac{m_b x_2^2 + \Theta_p}{\Theta_p} \end{bmatrix}, \quad (54)$$

where $g$ denotes gravity, $\Theta_p$ is the inertia of the beam and $r_b$, $m_b$ and $\Theta_b$ are the radius, mass and inertia of the ball, respectively [35]. We dropped the time-dependency here for brevity. The model parameter values are given in Table 1.

Each player acts based on an individual cost function of the form (2), where the feature vector is given by

$$\eta_i = [x_1^2 \ x_2^2 \ x_3^2 \ x_4^2 \ u_i^2]^\top, \quad \forall i \in \{1, 2\}. \quad (55)$$

![Ball on beam system](image_url)
This feature vector describes both player’s individual preferences to zero the ball’s displacement from the center of the beam, its velocity, the beam’s angle and angular velocity. Furthermore, it represents the desire to keep their individual torque small. In the following, units are neglected as all quantities are given in SI units and angles in rad. To model the players’ behavior by means of cost functions, let the expert ground truth parameters be given by $\theta_{E,1} = [20 \ 1 \ 1 \ 1 \ 2]$ and $\theta_{E,2} = [1 \ 1 \ 10 \ 1 \ 1]$. In this way, the first player focuses on bringing the ball to the center of the beam whereas the second player mainly focuses on bringing the beam to a horizontal position (see state definition).

1) Cooperative Game Solution: We first assume that the players are able to act upon a binding agreement to reduce the overall costs of all players. Therefore, the ground truth parameters were used to determine optimal trajectories by solving an optimal control problem with the global cost function resulting from (23), leading to

$$J_S = \theta_E^\top \eta_S = [21 \ 2 \ 11 \ 2 \ 3] \eta_t. \quad (56)$$

This was done by applying Pontryagin’s minimum principle and solving the resulting two-point boundary value problem. Using the initial state $x^{(1)} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \end{bmatrix}^\top$, we obtain the assumed expert trajectories $\zeta_{E,CG}$ of the cooperative game (CG) solution.

2) Non-Cooperative Open-Loop Game Solution: In a next step, we assume a scenario where the players act greedily based on their individual cost function. For this case, we also determine the open-loop Nash equilibrium trajectories by applying Pontryagin’s minimum principle and then solving the resulting two-point boundary value problem. We observe that the open-loop Nash equilibrium exists and is unique since the conditions of [36, Lemma 4.2] are fulfilled. Using the same initial state $x^{(1)}$ as before, we obtain $\zeta_{E,NOLN}$ corresponding to the (nonlinear) open-loop Nash equilibrium (NOLN).

3) Inverse Dynamic Game Solutions: In order to solve the inverse dynamic game problems corresponding to each aforementioned solution concept, the system was discretized using a sampling time $\Delta T = 0.02 \text{ s}$. The optimization problem (27) was solved with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method and the approach in Remark 3 was applied to obtain the estimations $\hat{\theta}_i^{(CG)}$. Similarly, (43) was solved to obtain the estimated (nonlinear) open-loop Nash (NOLN) parameters $\hat{\theta}_i^{(NOLN)}$. All resulting parameters are given in Table I. These were used to generate estimated trajectories $\hat{\zeta}_{E,CG}$ and $\hat{\zeta}_{E,NOLN}$. All four sets of trajectories are depicted in Fig 2. Error measures will be given in Subsection VI-C.

### B. Linear-Quadratic Dynamic Game

We now consider an LQ dynamic game to evaluate the presented identification method for feedback Nash equilibria.

As mentioned in Section V-B, the approach relies on the estimation of the control law $\gamma_i$ which is possible e.g. for linear state feedback as in (49). We assume again that the cost functions $J_i$, $i \in \{1, 2\}$ have a form according to (2) and a feature vector given by (55). Therefore, the cost function is quadratic. In order to obtain linear system dynamics, we

---

**TABLE I**

PARAMETERS OF THE BALL-ON-BEAM SYSTEM USED FOR SIMULATION

| $m_b$ | $r_b$ | $g_c$ | $\Theta_b$ | $\Theta_p$ |
|-------|-------|-------|-----------|-----------|
| 0.02 kg | 25 mm | 9.81 m/s² | 5\times 10^{-6} kg m² | 0.667 kg m² |

**TABLE II**

IDENTIFIED COST FUNCTION PARAMETERS IN THE NONLINEAR DYNAMIC GAME

| $E$ | NOLN | CG |
|-----|------|----|
| 20  | 19.697 | 10.116 |
| 1   | 0.915  | 1.207  |
| 1   | 2.350  | 0.61   |
| 1   | 0.643  | 1.317  |
| 2   | 2.000  | 2.000  |

---

**Fig. 2.** State and control trajectories used for identification (cooperative game and open-loop Nash solution) and corresponding model trajectories generated from identified parameters.
Finally, the resulting optimization problem \((43)\) was also resolved with the BFGS method for both the open-loop and feedback matrix \(K\). For the open-loop Nash equilibrium trajectories, we previously estimated the densities analogously \[32, \text{Theorem 7.13}\]. Both theorems allow for noise measurements of the states and controls. The noisy signals are given by \(\tilde{x}_i(t) = \pi_i(t) + \varepsilon_i, \forall l \in \{1, ..., n\}\) and \(\tilde{u}_{i,k}(t) = \pi_i,k(t) + \varepsilon_i,k, \forall k \in \{1, ..., m_i\}, \forall i \in \mathbb{P}\). Here, \(\pi_i(t)\) and \(\tilde{u}_i(t)\), \(i \in \mathbb{P}\), stand for the observed Pareto efficient, open-loop or feedback Nash equilibrium trajectories, depending on the considered case. The Gaussian noise was chosen such that all signals have a particular signal-to-noise ratio (SNR). We used different SNR levels for the evaluation.

In order to analyze the performance of the methods, we define the error measure for state trajectories

\[
e^x = \max \left\{ e_1^x, \ldots, e_n^x \right\},
\]

and for the control trajectories

\[
e^u = \max \left\{ e_1^u, \ldots, e_N^u \right\},
\]

\[
e_i^u = \max \left\{ e_1^u, \ldots, e_i^u \right\}, \quad \forall i \in \{1, \ldots, N\},
\]

\[
e_i^u = \max \left\{ e_1^u, \ldots, e_i^u \right\}, \quad \forall l \in \{1, \ldots, m_i\},
\]

which represent the normalized maximum absolute error of the states and the controls over all time steps, respectively. The results are given in Table \[IV\]. We denote with \(\text{SNR} = \infty\) the case in which no noise was added to all signals.

### C. Identification with Noisy Measurements

Now we evaluate the performance of the identification methods in the presence of a signal disturbance. We added white Gaussian noise to all trajectories to simulate noisy measurements of the states and controls. The noisy signals are given by \(\tilde{x}_i(t) = \pi_i(t) + \varepsilon_i, \forall l \in \{1, ..., n\}\) and \(\tilde{u}_{i,k}(t) = \pi_i,k(t) + \varepsilon_i,k, \forall k \in \{1, ..., m_i\}, \forall i \in \mathbb{P}\). Here, \(\pi_i(t)\) and \(\tilde{u}_i(t)\), \(i \in \mathbb{P}\), stand for the observed Pareto efficient, open-loop or feedback Nash equilibrium trajectories, depending on the considered case. The Gaussian noise was chosen such that all signals have a particular signal-to-noise ratio (SNR). We used different SNR levels for the evaluation.

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### D. Discussion

We observe that the parameters for the open-loop and feedback Nash dynamic games were mostly correctly identified. Some parameter deviations can be recognized, e.g. the third entry of \(\theta_1\) in the nonlinear open-loop case. Similarly, in the feedback Nash case, the same parameter was identified as a negative parameter which denotes a minor reward of the deviations of the beam from the horizontal position. Nevertheless, the ratios between parameters are well approximated which explains the correct approximation of the trajectories shown in Fig. \[2\] and Fig. \[3\]. In the case of the identified cooperative game parameters, we observe that the individual parameters deviate considerably from the ground truth. However, the corresponding equally weighted sum resembles the ground truth global cost function in \[50\]. Additionally, the trajectories are still well approximated, indicating that the identified parameters also belong to the same Pareto frontier.

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that trajectories which originated from a Nash equilibrium cannot be explained by cost functions which were estimated using a method which assumes that a cooperative game has taken place.

VII. CONCLUSION

In this paper, we developed methods for inverse cooperative and non-cooperative dynamic games based on maximum entropy inverse reinforcement learning. Unbiasedness results are proposed which guarantee the performance of the approaches. The suitability of the methods was demonstrated using examples of nonlinear and linear-quadratic dynamic games. Furthermore, the simulations give further evidence that the solution concept plays a fundamental role in the use of inverse dynamic game methods. For applying identification of cost functions, it is essential to analyze which solution concept (cooperative, non-cooperative open-loop Nash and feedback Nash) is more suitable for a given scenario. Future work will include the analysis of these methods with experimental data.

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The maximum entropy principle does not favour any player and therefore, the individual parameters are almost equal for all common features (all except $u_k^2$).

In the case where only noisy measurements are available, we observe that the identification of the cooperative game solution described by the global cost function is robust to measurement noise. The other methods’ results deteriorate for signals with an SNR of less than 20dB. The presented approaches seek for parameters $\theta_i$ such that the feature count $\langle \gamma \rangle$ of the expert trajectory is equalized. The feature count of the noisy measurements can differ considerably with respect to the original observations. Therefore, the results suggest that a low-pass filtering of the observations is necessary for good identification results. Furthermore, this effect grows with an increasing number of maximum likelihood estimations, as it is necessary in the inverse Nash games. In addition, it is important to stress that the results of each identification rely on the knowledge of which solution concept is currently at hand. The simulations show that the resulting trajectories of all three solution concepts differ considerably, especially in the control trajectories. Therefore, we conjecture

Fig. 3. State and control trajectories used for identification (open-loop and feedback Nash solution) and corresponding model trajectories generated from identified parameters.

\[ \begin{align*}
\phi & \in [0, 2\pi] \\
M_1 & \in \mathbb{R} \\
M_2 & \in \mathbb{R}
\end{align*} \]

\[ t = k\Delta T \text{ in s} \]
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