Quantum query complexity of graph connectivity

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12th January 2022

Abstract

Harry Buhrman et al gave an \( \Omega(\sqrt{n}) \) lower bound for monotone graph properties in the adjacency matrix query model. Their proof is based on the polynomial method. However for some properties stronger lower bounds exist. We give an \( \Omega(n^{3/2}) \) bound for Graph Connectivity using Andris Ambainis’ method, and an \( O(n^{3/2} \log n) \) upper bound based on Grover’s search algorithm. In addition we study the adjacency list query model, where we have almost matching lower and upper bounds for Strong Connectivity of directed graphs.

1 Introduction

The goal of the theory of quantum complexity is to find out what the computational speedup of a quantum computer is with respect to classical computers. Today there are only a few results which give a polynomial time quantum algorithm for some problem for which no classical polynomial time solution is known. We are interested in studying the speedup for problems for which there is already an efficient classical algorithm, but which could still be slightly improved quantunly. Basic graphs problems are interesting candidates. For example Heiligman studied the problem of computing distances in graphs [12].

This paper addresses Graph Connectivity. We are given a graph \( G(V,E) \) and would like to know whether for every vertex pair \( (s,t) \) there is a path connecting them. A generalization is Strong Connectivity where we are given a directed graph \( G(V,E) \). To be strongly connected there must be a path from \( s \) to \( t \) but also from \( t \) to \( s \). Let’s assume the vertex set \( V = \{v_0,v_1,\ldots,v_{n-1}\} \). We call an undirected graph \( k \)-regular if the degree of all its vertexes is \( k \). We say that a directed graph has outdegree \( k \) if all its vertexes have outdegree \( k \). If this also true for the indegree then we call it \( k \)-regular.

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Since it is extremely difficult to prove lower bounds for time complexity, we study the query complexity of these problems; meaning the minimal number of queries to the graph in order to solve the problem. There are essentially two query models for graphs:

The **matrix model**, where the graph is given as the adjacency matrix $M \in \{0,1\}^{n \times n}$, with $M_{ij} = 1$ iff $(v_i,v_j) \in E$,

and the **list model**. In the later case we consider for this paper only directed graphs with fixed out-degree $k$. The encoding is a function $f : [n] \times [k] \to [n]$ such that $f(u,i)$ is the $i$-th neighbor of $u$ — using an arbitrary numbering from $0$ to $k - 1$ of the outgoing edges — and which satisfy the simple graph promise

$$\forall u \in [n], i, j \in [k], i \neq j : f(u,i) \neq f(u,j) \tag{1}$$

which ensures that the graph is not a multigraph, i.e. does not have multiple edges between two nodes. For undirected graphs we require an additional promise on the input, namely that $M$ is symmetric in the matrix model, and for the list model $\forall u, v \in [n]$ if $\exists i \in [k] : f(u,i) = v$ then $\exists j \in [k] : f(v,j) = u$. Note that **Connectivity** is not what is called a promise problem, as the promise is input independent.

The classical randomized query complexity in the matrix model is well studied. While it is believed to be $\Omega(n^2)$, there is no matching lower bound at the moment. The first good lower bound of $\Omega(n^{1/3})$ is by Hajnal [11], and has recently be improved to $\Omega(n^{4/3}(\log n)^{1/3})$ by Chakrabart and Khot [7] and one year later even to $\Omega(n^2/\log n)$ in a neat paper by Friedgut, Kahn and Wigderson [9]. We are not aware of any lower bound for **Strong Connectivity**.

|          | (un)directed graphs         |
|----------|----------------------------|
| matrix model                  | $\Omega(n^{3/2})$, $O(n^{3/2} \log n)$ |
| matrix model, $k$-regular     | $O(n^{3/2}/\sqrt{k})$            |

|          | directed graphs             |
|----------|----------------------------|
| list model                  | $\Omega(n\sqrt{k})$, $O(n\sqrt{k} \log n)$ |
| list model, $k$-regular     | $\Omega(n)$                  |

Table 1: Query complexity of (Strong) Connectivity.

The quantum query complexity of any monotone graph property in the matrix model was shown to be $\Omega(\sqrt{n})$ by Buhrman, Cleve, de Wolf and Zalka [5] using the polynomial method and conjectured to be $\Omega(n)$. In this note we give the lower bound $\Omega(n^{3/2})$ for **Connectivity**, a particular monotone graph property.\(^1\) We also have an almost matching

\(^1\)In a previous version of this paper, we said that Buhrman et al conjectured $\Omega(n)$ for **Connectivity**, and since their conjecture concerns arbitrary monotone graph properties in general, we gave a false impression of improving their result. We apologize.
upper bound of $O(n^{3/2} \log n)$ and for the special case when the out-degree is $k$ there is a trivial algorithm in $O(n^{3/2} \sqrt{k})$ which is of interest when $k \in o(\log^2 n)$. These bounds do also hold for strong connectivity.

For the list model we are not aware of any classical lower bound. Trivial classical upper bound is $O(nk)$. We show that the quantum query complexity of strong connectivity is $\Omega(\sqrt{n}/\sqrt{k})$ and $O(n\sqrt{k}\log n)$, which becomes better than classical for $k \in \Omega(\log^2 n)$. However our lower bound does not hold anymore for the special case of $k$-regular directed graphs, for which we only have $\Omega(n)$.

The time complexity of our algorithms are essentially the same as the query complexity. However a log $n$ factor applies in the bit model, when each vertex is encoded using log $n$ bits.

The space requirement is $O(\log n)$ qubits and $O(n \log n)$ classical bits. If we constraint the space (both classical and quantum) to $O(\log n)$ qubits then an algorithm is basically restricted to make an oblivious random walk on the graph. Quantum random walks has been subject of several papers [2, 6, 14], in particular for the st-CONNECTIVITY problem [16].

\section{The upper bounds for Connectivity}

The upper bounds gain their speedup of classical algorithm only from Grover’s search algorithm. Since it is an important tool for this paper we restate the exact results. The algorithm solves the search problem of finding an index $i$ which a given boolean table $T$ maps to 1. Let $n$ be the size of $T$ and $t$ the number of 1-entries in $T$. In this paper we will use three versions of the search algorithm.

- When the number of solutions $t$ is known in advance, Grover’s search algorithm returns a solution after $O(\sqrt{n/t})$ queries to $T$ with probability of error $O(t/n)$ [10].

- When the number of solutions is not known in advance, then an extension of Grover’s algorithm can be used, by Boyer et al which makes $O(\sqrt{n})$ queries to $T$ and either outputs a solution or claims that $T$ is all 0. This second algorithms however errs with constant probability [4].

- There is another extension of the previous algorithm which finds the smallest index $i$ such that $T[i] = 1$, in time $O(\sqrt{n})$. Again error probability is constant [8].

The main idea for the upper bound follows immediately from using Grover’s search algorithm as a blackbox in an algorithm for constructing a spanning tree.

**Theorem 1** The quantum query complexity of Connectivity is $O(n^{3/2} \log n)$ in the adjacency matrix query model and $O(n\sqrt{k}\log n)$ in the adjacency list query model.

**Proof:** We have to decide whether from vertex $v_0$, all other vertexes can be reached. For that purpose we construct a depth first spanning tree of the connected component containing $v_0$. We maintain a stack of vertexes which neighborhood still remains to be
explored. Initially it contains only $v_0$. Whenever a new vertex is found, it is marked and pushed on the stack. At every time, we search for unmarked neighbors of the top vertex. If there is no, we remove the vertex from the stack.

Algorithm 1 (Depth-first search)
Let $A$ be an empty edge set, which will contain a spanning tree.
Initially set of marked vertice $S = \{v_0\}$
and stack of vertice $T = \{v_0\}$.
While $T \neq \{\}$ do
  choose topmost $u \in T$
  use Grover’s algorithm to find a neighbor $v$ of $u$ and not in $S$
  if success add $v$ to $S, (u,v)$ to $A$ and push $v$ on $T$
  otherwise pop $u$ from $T$.
Answer ”Yes the graph is connected” if all vertice $S = V$.

To make the total success probability constant, we need each of the at most $2n$ Grover’s searches to success with probability $p$, such that $p^{2n} = 2/3$. This can be achieved using $O(\log n)$ repetitions. Therefore the cost of adding or removing a vertex from the stack is $O(\sqrt{n}\log n)$ in the matrix model and $O(\sqrt{k}\log n)$ in the list model. This gives us the required upper bound. □

For graphs with small out-degree $k \in o(\log^2 n)$, there is a trivial but better algorithm for the matrix model. Its complexity is $O(n^{3/2}\sqrt{k})$. We use the fact, that Grover’s search procedure behaves much better, when the number of solutions is known in advance. Let $m = nk$ be the number of edges in the graph.

We claim that the following algorithm has the required complexity.

Algorithm 2 (Learning the adjacency matrix)
Empty adjacency list $L$.
Matrix $M' \in \{0,1\}^{n \times n}$ initially all zero.
For $t = m$ downto 1
  Quantum search $i, j$ s.t. $M_{ij} = 1$ and $M'_{ij} = 0$ until a solution is found
  (using the fact that there are $t$ solutions in a search space of size $n^2$)
  add $(i,j)$ to $L$ and set $M'_{ij} = 1$
Run classical spanning tree algorithm using adjacency list $L$.

The expected number of queries to $M$ is $O(\sqrt{n^2/m} + \sqrt{n^2/(m-1)} + \ldots + \sqrt{n^2}) = O(n \sum_{x=1}^{m} x^{-1/2})$. We bound the sum by

$$\sum_{x=1}^{m} x^{-1/2} \leq \int_{x=1}^{m} x^{-1/2} = \left[2x^{1/2}\right]_{1}^{m} \in O(\sqrt{m}).$$

Ronald de Wolf informed us that only $\sqrt{\log n}$ repetitions are necessary, since in [5] it is shown that $\sqrt{n \log(1/\epsilon)}$ queries are necessary to obtain an error probability $\epsilon$ in Grover’s search algorithm.
So in expected time $O(n\sqrt{m})$ we know the complete graph, and can compute classically the solution. Stopping the overall algorithm at twice the expected time gives an algorithm with success probability at least $1/2$ and worst case running time $O(n\sqrt{m})$.

3 The upper bound for Strong Connectivity

A directed graph is strongly connected if for a fixed vertex $v_0$, (1) all other vertexes can be reached by $v_0$, and (2) $v_0$ can be reached by all other vertexes. The algorithms we gave in the previous section verifies condition (1), and in the matrix model, condition (2) can easily be verified by applying the same algorithm on the transposed adjacency matrix, i.e. taking all edges in reverse order.

For the adjacency list model however more work is needed.

**Theorem 2** The quantum query complexity of Strong Connectivity for directed graphs with out-degree $k$ in the list model is $O(n\sqrt{k\log n})$.

**Proof:** In a first stage we use algorithm 1 to construct a directed spanning tree $A \subseteq E$ rooted in $v_0$. Assume vertexes to be named according to the order algorithm 1 marked them, so $v_0$ was the first to be marked, $v_1$, the second, and so on.

Then in a second stage we search for every vertex $v_i \in V$, the neighbor $v_j$ with smallest index. The result is a set of backward edges $B \subseteq E$. This search can be done with $O(n\sqrt{k\log n})$ queries using Dürr and Høyer’s variant of Grover’s algorithm [8]. We claim that the graph $G(V,E)$ is strongly connected iff its subgraph $G'(V,A \cup B)$ is strongly connected, which would conclude the proof.

Clearly if $G'$ is strongly connected then so is $G$ since $A \cup B \subseteq E$. Therefore to show the converse assume $G$ strongly connected. For a proof by contradiction let $v_i$ be the vertex with smallest index, which is not connected to $v_0$ in $G'$. However by assumption there is a path in $G$ from $v_i$ to $v_0$. Let $(v_l, v_l')$ be its first edge with $l \geq i$ and $l' < i$. We will use the following property of depth first search.

**Lemma 1** Let $v_l$ and $v_{l'}$ be two vertexes in the graph $G$ with $l < l'$. If there is a path from $v_l$ to $v_{l'}$ in $G(V,E)$ then $v_{l'}$ is in the subtree of $G''(V,A)$ with root $v_l$.

Therefore we can replace in the original path the portion from $v_i$ to $v_l$ by a path only using edges from $A$. Let $v_{l''}$ be the neighbor of $v_l$ with smallest index. Clearly $l'' \leq l' < i$. By the choice of $v_i$, there exists a path from $v_{l''}$ to $v_0$ in $G'$. Together this gives a path from $v_i$ to $v_0$ in $G'$ contradicting the assumption and therefore concluding the proof. $\square$

4 The lower bounds

We will first give a simple lower bound for Connectivity (and Strong Connectivity) in the list model, by a reduction from Parity. As we recently found out, this reduction
has first been used by Henzinger and Fredman for the on-line connectivity problem \[13\]. We show later how to improve this construction.

**Lemma 2** **Strong Connectivity** needs $\Omega(n)$ queries in the list model.

*Proof:* We will use a straightforward reduction from *Parity*.

Let $x \in \{0,1\}^p$ be an instance to the parity problem. We construct a permutation $f$ on $V = \{v_0, \ldots, v_{2^p-1}\}$ which has exactly 1 or 2 cycles depending on the parity of $x$. For any $i \in [p]$ and the bit $b = x_i$ we define $f(v_{2i}) = v_{2i+1}+b$ and $f(v_{2i+1}) = v_{2i+3}+b$ where addition is modulo $2p$. See figure 1. The graph defined by $f$ has 2 levels and $p$ columns, each corresponding to a bit of $x$. A directed walk starting at vertex $v_0$ will go from left to right, changing level whenever the corresponding bit in $x$ is 1. So when $x$ is even the walk returns to $v_0$ while having explored only half of the graph, otherwise it returns to $v_1$ connecting from there again to $v_0$ by $p$ more steps. Since the query complexity of *Parity* is $\Omega(n)$ — see for example \[5\] — this concludes the proof. □

The following proofs will all rely on Andris Ambainis’ technique for proving lower bounds, which we restate here.

**Theorem 3 (Ambainis [3, theorem 6])** Let $L \subseteq \{0,1\}^*$ be a decision problem. Let $X \subseteq L$ be a set of positive instances and $Y \subseteq \bar{L}$ a set of negative instances. Let $R \subseteq X \times Y$ be a relation between instances of same size. Let be the values $m, m', l_{x,i}$ and $l'_{y,i}$ with $x, y \in \{0,1\}^n$ and $x \in X$, $y \in Y$, $i \in [n]$ such that

- for every $x \in X$ there are at least $m$ different $y \in Y$ in relation with $x$,
- for every $y \in Y$ there are at least $m'$ different $x \in X$ in relation with $y$,
- for every $x \in X$ and $i \in [n]$ there are at most $l_{x,i}$ different $y \in Y$ in relation with $x$ which differ from $x$ at entry $i$,
- for every $y \in Y$ and $i \in [n]$ there are at most $l'_{y,i}$ different $x \in X$ in relation with $y$ which differ from $y$ at entry $i$.

Then the quantum query complexity of $L$ is $\Omega(\sqrt{mm'\max_{x,y,i} l_{x,i}l'_{y,i}})$ with the maximum taken over $xRy$. 

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*Figure 1: A standard reduction from parity*
Now we show how to improve our previous proof by changing slightly the construction.

**Theorem 4** STRONG CONNECTIVITY for directed graphs with out-degree \( k \) needs \( \Omega(n\sqrt{k}) \) queries in the list model.

**Proof:** We use a similar construction as for Lemma\(^2\) but now for every vertex the \( k-1 \) additional edges are redirected back to an origin. We would like to connect them back to a fixed vertex \( u_0 \), but this would generate multiple edges and we want the proof work for simple graphs. Therefore we connect them back to a \( k \)-clique which then is connected to \( u_0 \). See figure 2. Let be the vertex set \( V = \{v_0, \ldots, v_{2p-1}, u_0, \ldots, u_{k-1}\} \) for some integer \( p \). In the list model, the edges are defined by a function \( f : V \times [k] \to V \). We will consider only functions with the following restrictions:

- For every \( i \in [k] \) we have \( f(u_i, 0) = v_0 \) and for \( j \in \{1, \ldots, k-1\} \) \( f(u_i, j) = u_{i+j} \), where addition is modulo \( k \).

- For every \( i \in [p] \) there exist \( j_0, j_1 \in [k] \) and a bit \( b \) such that \( f(v_{2i}, j_0) = v_{2i+2+2b} \) and \( f(v_{2i+1}, j_1) = v_{2i+3-b} \), where addition is modulo \( 2p \) this time. We call these edges the forward edges. The backward edges are for all \( j \in [k] \) \( f(v_{2i}, j) = u_j \) whenever \( j \neq j_0 \) and \( f(v_{2i+1}, j) = u_j \) whenever \( j \neq j_1 \).

![Figure 2: A strongly connected graph](image)

Now all the nodes are connected to the \( k \)-clique, the clique is connected to \( v_0 \), and the graph is strongly connected if and only if there is a path from \( v_0 \) to \( v_1 \).

Let \( X \) be the set of functions which define a strongly connected graph, and \( Y \) the set of functions which do not. Function \( f \in X \) is in relation with \( g \in Y \) if there are numbers \( i \in [p], j_0, j_1, h_0, h_1 \in [k] \) with \( j_0 \neq h_0, j_1 \neq h_1 \) such that the only places where \( f \) and \( g \) differ are

\[
\begin{align*}
g(v_{2i}, h_0) &= f(v_{2i+1}, j_1) & g(v_{2i+1}, h_1) &= f(v_{2i}, j_0) \\
g(v_{2i}, j_0) &= u_{j_0} & g(v_{2i+1}, j_1) &= u_{j_1} \\
f(v_{2i}, h_0) &= u_{h_0} & f(v_{2i+1}, h_1) &= u_{h_1}
\end{align*}
\]
Informally \( f \) and \( g \) are in relation if there is a level, where the forward edges are exchanged between a parallel and crossing configuration and in addition the edge labels are changed.

Then \( m = m' = O(nk^2) \), \( p \in O(n) \) for the number of levels and \((k-1)^2\) for the number of possible forward edge labels. We also have \( l_{f,v,j} = k - 1 \) if \( f(v,j) \in \{u_0, \ldots, u_{k-1}\} \) and \( l_{f,v,j} = (k-1)^2 \) otherwise. The value \( l'_{g,v,j} \) is the same. Since only one of \( f(v,j) \), \( g(v,j) \) can be in \( \{u_0, \ldots, u_{k-1}\} \) we have \( l_{\text{max}} = O(k^3) \) and the lower bound follows. \( \square \)

For the matrix model, there is a much simpler lower bound which works even for undirected graphs.

**Theorem 5** Connectivity needs \( \Omega(n^{3/2}) \) queries in the matrix model.

**Proof:** We use Ambainis’ method for the following special problem. You are given a symmetric matrix \( M \in \{0,1\}^{n \times n} \) with the promise that it is the adjacency matrix of a graph with exactly one or two cycles, and have to find out which is the case.

Let \( X \) be the set of all adjacency matrices of a unique cycle, and \( Y \) the set of all adjacency matrices with exactly two cycles each of length between \( n/3 \) and \( 2n/3 \). We define the relation \( R \subseteq X \times Y \) as \( M R M' \) if there exist \( a, b, c, d \in [n] \) such that the only difference between \( M \) and \( M' \) is that \((a,b), (c,d) \) are edges in \( M \) but not in \( M' \) and \((a,c), (b,d) \) are edges on \( M' \) but not in \( M \). See figure 3. The definition of \( Y \) implies that in \( M \) the distance from \( a \) to \( c \) is between \( n/3 \) and \( 2n/3 \).

![Figure 3: Illustration of the relation](image)

Then \( m = O(n^2) \) since there are \( n-1 \) choices for the first edge and \( n/3 \) choices for the second edge. Also \( m' = O(n^2) \) since from each cycle one edge must be picked, and cycle length is at least \( n/3 \).

We have \( l_{M,(i,j)} = 4 \) if \( M_{i,j} = 0 \) since in \( M' \) we have the additional edge \((i,j)\) and the endpoints of the second edge must be neighbors of \( i \) and \( j \) respectively. Moreover \( l_{M,(i,j)} = O(n) \) if \( M_{i,j} = 1 \) since then \((i,j)\) is one of the edges to be removed and there remains \( n/3 \) choices for the second edge.

The values \( l'_{M',(i,j)} \) are similar, so in the product one factor will always be constant while the other is linear giving \( l_{M,(i,j)}l'_{M',(i,j)} = O(n) \) and the theorem follows. \( \square \)

**5 Conclusion**

It remains to close the gap between lower and upper bound for regular directed graphs, for which we have only the bounds \( \Omega(n) \) and \( O(n^{3/2} \log n) \). Our upper bounds relied only on
Grover’s algorithm. One might hope for some cases that quantum random walks provide better upper bounds. On the other side one might obtain better lower bounds with the polynomial method, which has recently be smartly used by Aaronson [1] and Shi [15] to show lower bounds for properties of function graphs.

Acknowledgments

We are grateful to Miklos Santha for helpful discussions, to Katalin Friedl for helpful comments on Algorithm 2 to Oded Regev for helpful corrections and to Ronald de Wolf who made us aware of the problem of error propagation in repeated use of Grover’s search.

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