Rohlin flows on the Cuntz algebra $\mathcal{O}_\infty$

Ola Bratteli  
Department of Mathematics, University of Oslo  
Blindern, P.O.Box 1053, N-0316, Norway  

and  
Akitaka Kishimoto  
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan  

and  
Derek W. Robinson  
Centre for Mathematics and its Applications, Australian National University  
Canberra, ACT 0200, Australia

January, 2005

Abstract

It is shown that certain quasi-free flows on the Cuntz algebra $\mathcal{O}_\infty$ have the Rohlin property and therefore are cocycle-conjugate with each other. This, in particular, shows that any unital separable nuclear purely infinite simple C*-algebra has a Rohlin flow.

1 Introduction

We are concerned here with Rohlin flows; a flow $\alpha$ on a unital C*-algebra $A$ is said to have the Rohlin property (or to be a Rohlin flow) if for any $p \in \mathbb{R}$ there is a central sequence $(u_n)$ in $\mathcal{U}(A)$, the unitary group of $A$, such that $\max_{t \leq 1} \| \alpha_t(u_n) - e^{ipt}u_n \| \to 0$ as $n \to \infty$. A major consequence of this property can be paraphrased as any $\alpha$-cocycle is almost a coboundary. This consequence, combined with enough information on $\mathcal{U}(A)$, may lead us to a classification theory of Rohlin flows up to cocycle conjugacy. This is a goal we have in mind (see [14, 17, 18, 20, 19]).

Since the property is rather stringent, it is not easy to present a Rohlin flow in general. But we managed to give Rohlin flows on the Cuntz algebra $\mathcal{O}_n$ with $n$ finite; moreover we can identify the quasi-free flows which have the Rohlin property. In this paper we show that certain quasi-free flows on $\mathcal{O}_\infty$ have the Rohlin property. Hence it follows that any unital C*-algebra $A$ with $A \cong A \otimes \mathcal{O}_\infty$ has Rohlin flows; the class of such $A$ includes all unital separable nuclear purely infinite simple C*-algebras, due to Kirchberg.
We have left some quasi-free flows $O_\infty$ undecided whether they have the Rohlin property or not. But, as in [17], we show that all the Rohlin flows on $O_\infty$ are cocycle conjugate with each other in the class of quasi-free flows. This is true for a wider class of flows. Noting that there is a certain maximal abelian $C^*$-subalgebra $C_\infty$ of $O_\infty$ whose elements the quasi-free flows fix, we show that Rohlin flows are cocycle-conjugate in the class of flows which are $C^{1+\epsilon}$ on $C_\infty$ (see below for details and note that our terminology of quasi-free flows is restrictive).

We will now describe the contents more precisely.

For each $n = 2, 3, \ldots$ the Cuntz algebra $O_n$ is generated by $n$ isometries $s_1, s_2, \ldots, s_n$ such that $\sum_{k=1}^{\infty} s_k s_k^* = 1$. For $n = \infty$ the Cuntz algebra $O_\infty$ is generated by a sequence $(s_1, s_2, \ldots)$ of isometries such that $\sum_{k=1}^{\infty} s_k s_k^* \leq 1$ for all $n$. It is shown in [6] that $O_n$ with $n = 2, 3, \ldots$ or $n = \infty$ is a simple purely infinite nuclear $C^*$-algebra.

For a finite (resp. infinite) sequence $(p_1, p_2, \ldots, p_n)$ in $R$ we define a flow $\alpha$, called a quasi-free flow, on $O_n$ (resp. $O_\infty$) by

$$\alpha_t(s_k) = e^{ip t} s_k.$$ 

(In the case $n = \infty$, a more general flow can be induced by a unitary flow $U$ on the closed linear subspace $H$ spanned by $s_1, s_2, \ldots$, where the inner product $\langle \cdot, \cdot \rangle$ is given by $y^* x = \langle x, y \rangle 1$, $x, y \in H$, if the generator of $U$ is not diagonal. But we will exclude them from the quasi-free flows in this paper.) It is known in [11, 21] that if $p_1, p_2, \ldots$ generate $R$ as a closed subsemigroup, then the crossed product $O_n \times_\alpha R$ is simple and purely infinite (whether $n$ is finite or infinite). It is also known in [17, 19] that if $n$ is finite, the flow $\alpha$ has the Rohlin property if and only if $O_\infty \times_\alpha R$ is simple and purely infinite. For $n = \infty$, it is known in [11, 14] that if $\alpha$ has the Rohlin property then $O_\infty \times_\alpha R$ is simple and purely infinite. In this paper we shall give a partial converse to this fact:

**Theorem 1.1** Let $(p_k)$ be an infinite sequence in $R$ such that $p_1, p_2, \ldots, p_n$ generate $R$ as a closed subsemigroup for some $n$. Then the quasi-free flow $\alpha$ on $O_\infty$ defined by $\alpha_t(s_k) = e^{ip t} s_k$ has the Rohlin property.

We shall prove that each $\alpha_t$ is $\alpha$-invariantly approximately inner, i.e., for each $t \in R$ there is a sequence $(u_n)$ in $U(O_\infty)$ such that $\alpha_t(x) = \lim \text{Ad} u_n(x)$, $x \in O_\infty$ and $\max_{s \in [0,1]} \| \alpha_s(u_n) - u_n \| \rightarrow 0$. Then we would get the above theorem, by [13, 20], from the fact that $O_\infty \times_\alpha R$ is simple and purely infinite.

Let $\mathcal{E}_n$ be the $C^*$-subalgebra of $O_\infty = C^*(s_1, s_2, \ldots)$ generated by $s_1, s_2, \ldots, s_n$. Then $\mathcal{E}_n$ is left invariant under $\alpha$ and the union $\bigcup_n \mathcal{E}_n$ is dense in $O_\infty$. Hence, to prove the assertion in the previous paragraph, it suffices to show that $\alpha|\mathcal{E}_n$ is $\alpha$-invariantly approximately inner for all large $n$. Let us state formally:

**Proposition 1.2** Let $s_1, s_2, \ldots, s_n$ be isometries such that

$$\sum_{k=1}^{n} s_k s_k^* \leq 1$$


and let $\mathcal{E}_n$ be the $C^*$-algebra generated by these $s_1, \ldots, s_n$. Let $(p_1, p_2, \ldots, p_n)$ be a finite sequence in $R$ such that $p_1, \ldots, p_n$ generate $R$ as a closed subsemigroup and define a quasi-free flow $\alpha$ on $\mathcal{E}_n$ by $\alpha_t(s_k) = e^{ip_k t}s_k$. Then each $\alpha_t$ is $\alpha$-invariantly approximately inner.

To prove this we use the following facts. Let $\mathcal{J}_n$ be the ideal of $\mathcal{E}_n$ generated by $e_0^n = 1 - \sum_{k=1}^n s_k s_k^*$. Then $\mathcal{J}_n$ is isomorphic to the $C^*$-algebra $\mathcal{K}$ of compact operators (on a separable infinite-dimensional Hilbert space) and is left invariant under $\alpha$. The quotient $\mathcal{E}_n / \mathcal{J}_n$ is isomorphic to $\mathcal{O}_n$ by mapping $s_k + \mathcal{J}_n$ into $s_k$ (the latter $s_k$’s satisfy the equality $\sum_{k=1}^n s_k s_k^* = 1$ and generate $\mathcal{O}_n$). By the assumption on $(p_k)$ the induced flow $\tilde{\alpha}$ on $\mathcal{O}_n$ has the Rohlin property \cite{[19]}, from which follows that each $\tilde{\alpha}_t$ is $\alpha$-invariantly approximately inner. We will translate this property to $\alpha_t$ on $\mathcal{E}_n$ by using the fact that $\mathcal{J}_n \cong \mathcal{K}$. See Section 3 for details.

Before embarking on the proof of the above proposition, we will have to prove that if $\alpha$ is a Rohlin flow on $\mathcal{O}_n$, then each $\alpha_t$ is not only $\alpha$-invariantly approximately inner but also $\alpha$-invariantly asymptotically inner, i.e., there is a continuous map $u : [0, \infty) \to U(\mathcal{O}_n)$ such that $\alpha_t(x) = \lim_{s \to \infty} \text{Ad} u(s)(x)$ for $x \in \mathcal{O}_n$ and $\max_{t \in [0,1]} \|\alpha_t(u(s)) - u(s)\| \to 0$. This will be proved for a wider class of $C^*$-algebras (see \cite{[22]} for details). (As a matter of fact we do not know of a single example of $\alpha$ without the above property of $\alpha$-invariant asymptotic innerness if it has covariant irreducible representations; we expect that this property holds fairly in general whether it has the Rohlin property or not.)

As a corollary to the above theorem we get that any purely infinite simple separable nuclear $C^*$-algebra has a Rohlin flow; because such a $C^*$-algebra $A$ satisfies that $A \cong A \otimes \mathcal{O}_\infty$ due to Kirchberg (see \cite{[9]}) and a flow $\alpha$ on $\mathcal{O}_\infty$ induces a flow on $A$ via $\text{id} \otimes \alpha$ on $A \otimes \mathcal{O}_\infty$ which has the Rohlin property if $\alpha$ has.

Let $\mathcal{C}_\infty$ denote the $C^*$-subalgebra of $\mathcal{O}_\infty$ generated by $s_{i_1}s_{i_2} \cdots s_{i_k}s_{i_1}^* \cdots s_{i_1}^*$ with all finite sequences $(i_1, i_2, \ldots, i_k)$ in $\mathbb{N}$. Then $\mathcal{C}_\infty$ is a weakly regular maximal abelian $C^*$-subalgebra of $\mathcal{O}_\infty$ (weakly regular in the sense that $\{u \in \mathcal{P}I(\mathcal{O}_\infty) \mid uu^*, u^*u \in \mathcal{C}_\infty, u\mathcal{C}_\infty u^* = \mathcal{C}_\infty uu^*\}$ generates $\mathcal{O}_\infty$, where $\mathcal{P}I(\mathcal{O}_\infty)$ is the set of partial isometries of $\mathcal{O}_\infty$). Moreover there is a projection of norm one of $\mathcal{O}_\infty$ onto $\mathcal{C}_\infty$ and there is a character of $\mathcal{C}_\infty$ which extends uniquely to a state of $\mathcal{O}_\infty$. (When a weakly regular masa satisfies these two additional conditions, we will say that it is a weak Cartan masa.) We note that if $\alpha$ is a quasi-free flow (in our sense) then $\alpha_t$ is the identity on $\mathcal{C}_\infty$; in other words, if $\delta_\alpha$ denotes the generator of $\alpha$, then $D(\delta_\alpha) \supset \mathcal{C}_\infty$ and $\delta_\alpha|\mathcal{C}_\infty = 0$. We consider the following condition for a flow $\gamma$ on $\mathcal{O}_\infty$: $D(\delta_\gamma) \supset \mathcal{C}_\infty$ and $\sup_{x \in \mathcal{C}_\infty, \|x\| \leq 1} \| (\gamma_t - \text{id})\delta_\gamma(x) \|$ converges to zero as $t \to 0$; which we express by saying that $\gamma$ is $C^{1+\epsilon}$ on $\mathcal{C}_\infty$ below. This is obviously satisfied if $\gamma$ is $C^2$ on $\mathcal{C}_\infty$ or $D(\delta_\gamma^2) \supset \mathcal{C}_\infty$ (because then $\delta_\gamma^2|\mathcal{C}_\infty$ is bounded).

We will also show:

\textbf{Corollary 1.3} Any two Rohlin flows on $\mathcal{O}_\infty$ are cocycle conjugate with each other if they are $C^{1+\epsilon}$ on $\mathcal{C}_\infty$.

The proof consists of two parts. In the first part we show that if the flow $\gamma$ is $C^{1+\epsilon}$ on $\mathcal{C}_\infty$ then $\delta_\gamma|\mathcal{C}_\infty$ is inner, i.e., there is an $h = h^* \in \mathcal{O}_\infty$ such that $\delta_\gamma(x) = \text{ad} ih(x), \ x \in \mathcal{C}_\infty$.
Thus we can assume, by inner perturbation, that $\delta_{\gamma}|C_\infty = 0$. In the second part we show that any two Rohlin flows are cocycle-conjugate with each other if they fix each element of $C_\infty$ (see 5.11).

Acknowledgement. One of the authors (A.K.) visited at Australian National University in March, 2004 and at University of Oslo in August-September, 2004 during this collaboration. He acknowledges partial financial supports from these institutions.

2 Rohlin property

In this section we consider the class of purely infinite simple nuclear separable C$^*$-algebras satisfying the universal coefficient theorem, which is classified by Kirchberg and Phillips [9, 10] in terms of K-theory.

Let $A$ be a unital C$^*$-algebra of the above class. Let $\ell_\infty(A)$ be the C$^*$-algebra of bounded sequences in $A$ and let, for a free ultrafilter $\omega$ on $\mathbb{N}$, $c_\omega(A)$ be the ideal of $\ell_\infty(A)$ consisting of $x = (x_n)$ with $\lim_\omega \|x_n\| = 0$. If $\alpha$ is a flow on $A$, i.e., a strongly continuous one-parameter automorphism group of $A$, we can define an action of $\mathbb{R}$ on $\ell_\infty(A)$ by $t \mapsto (\alpha_t(x_n))$ for $x = (x_n)$. Let $\ell_\infty^\omega(A)$ be the maximal C$^*$-subalgebra of $\ell_\infty(A)$ on which this action is continuous; we will denote this flow by $\alpha$. We set $A^\omega = \ell_\infty(A)/c_\omega(A)$, $A^\omega_\alpha = \ell_\infty^\omega(A)/c_\omega(A)$.

We embed $A$ into $\ell_\infty^\omega(A)$ by constant sequences. Since $A \cap c_\omega(A) = \{0\}$, we regard $A$ as a C$^*$-subalgebra of $A^\omega_\alpha \subset A^\omega$.

We recall the following result [18, 20]:

**Theorem 2.1** Let $A$ be a unital separable nuclear purely infinite simple C$^*$-algebra satisfying the universal coefficient theorem and let $\alpha$ be a flow on $A$. Then the following conditions are equivalent.

1. $\alpha$ has the Rohlin property.

2. $(A' \cap A^\omega_\alpha)^\alpha$ is purely infinite and simple, $K_0((A' \cap A^\omega_\alpha)^\alpha) \cong K_0(A' \cap A^\omega)$ induced by the embedding, and Spec($\alpha|A' \cap A^\omega_\alpha$) = $\mathbb{R}$.

3. The crossed product $A \times_\alpha \mathbb{R}$ is purely infinite and simple and the dual action $\hat{\alpha}$ has the Rohlin property.

4. The crossed product $A \times_\alpha \mathbb{R}$ is purely infinite and simple and $\alpha_{t_0}$ is $\alpha$-invariantly approximately inner for every $t_0 \in \mathbb{R}$.

If the above conditions are satisfied, it also follows that $K_1((A' \cap A^\omega_\alpha)^\alpha) \cong K_1(A' \cap A^\omega)$, which is induced by the embedding.
In the last condition of the above theorem, \( \alpha_{t_0} \) (for a fixed \( t_0 \)) is \( \alpha \)-invariantly approximately inner if there is a sequence \( (u_n) \) in \( \mathcal{U}(A) \) such that \( \alpha_{t_0} = \lim \text{Ad} u_n \) and \( \max_{t \in [0,1]} \| \alpha_t(u_n) - u_n \| \to 0 \). We will strengthen this condition as follows.

**Lemma 2.2** Let \( \alpha \) be a Rohlin flow on a unital \( C^* \)-algebra \( A \) of the above class (or in particular \( \mathcal{O}_n \)). Then each \( \alpha_{t_0} \) is \( \alpha \)-invariantly asymptotically inner, i.e., there is a continuous map \( u : [0, \infty) \to \mathcal{U}(A) \) such that \( \alpha_{t_0} = \lim_{s \to \infty} \text{Ad} u(s) \) and

\[
\lim_{s \to \infty} \max_{t \in [0,1]} \| \alpha_t(u(s)) - u(s) \| = 0.
\]

**Proof.** Since \( \mathcal{K}(\alpha_{t_0}) = \mathcal{K}(\text{id}) \), \( \alpha_{t_0} \) is asymptotically inner \([25]\), i.e., there is a continuous map \( v : [0, \infty) \to \mathcal{U}(A) \) such that

\[
\alpha_{t_0} = \lim_{s \to \infty} \text{Ad} v(s).
\]

Let \( w(s, t) = v(s)\alpha_t(v(s)^*) \), \( s \in [0, \infty) \), \( t \in \mathbb{R} \). Then for each \( s \in [0, \infty) \), the map \( t \mapsto w(s, t) \) is an \( \alpha \)-cocycle, i.e., \( t \mapsto w(s, t) \) is a continuous function into \( \mathcal{U}(A) \) such that \( w(s, t_1 + t_2) = w(s, t_1)\alpha_{t_2}(w(s, t_2)) \), \( t_1, t_2 \in \mathbb{R} \). Since for each \( x \in A \),

\[
\|[w(s, t), x]\| \leq \|\text{Ad} v(s)(\alpha_{t_0 - t}(x)) - \alpha_{-t}(x)\| + \|\text{Ad} v(s)(\alpha_{-t_0}(x)) - x\|,
\]

we get, for any \( T \gg 0 \) and for any \( x \in A \), that

\[
\sup_{0 \leq t \leq T} \|[w(s, t), x]\| \to 0
\]
as \( s \to \infty \).

More specifically let \( \mathcal{F} \) be a finite subset of \( A \) and \( \epsilon > 0 \). Then there exists an \( a > 0 \) such that if \( s \geq a \), then \( \|\text{Ad} v(s)(\alpha_{t_0 - t}(x)) - \alpha_{-t}(x)\| < \epsilon/12 \) for \( x \in \mathcal{F} \) and \( t \in [0, T] \), which entails that \( \|[w(s, t), x]\| < \epsilon/11 \) for \( x \in \mathcal{F} \) and \( t \in [0, T] \).

Furthermore, for any bounded interval \( I \) of \([0, \infty)\), there is a continuous map \( z : I \times [0, T] \to \mathcal{U}(A) \) such that

\[
\begin{align*}
z(s, 0) &= 1, \\
z(s, T) &= w(s, T), \\
\|z(s, t_1) - z(s, t_2)\| &\leq (16\pi/3 + \epsilon)|t_1 - t_2|/T, \\
\|[z(s, t), x]\| &< 10\epsilon/11, \quad x \in \mathcal{F},
\end{align*}
\]

for \( s \in I \) and \( t, t_1, t_2 \in [0, T] \). (Here we used the estimate for a particular construction of \( z(s, t) \) that

\[
\|[z(s, t), x]\| < 9 \max_{0 < t \leq T} \|[w(s, t_1), x]\| + \epsilon'
\]

for any \( \epsilon' > 0 \); see \([24]\) or 2.7 of \([18]\).) By using this \( z \), we get a continuous map \( U : I \to \mathcal{U}(A) \) such that

\[
\begin{align*}
\|w(s, t) - U(s)\alpha_t(U(s)^*)\| &\leq 6\pi|t|/T + \epsilon, \\
\|U(s, x)\| &\leq \epsilon, \quad x \in \mathcal{F},
\end{align*}
\]
where we have assumed that $16\pi/3 + \epsilon < 6\pi$.

We recall how $U_T(s) = U(s)$ is defined [14]. We define a unitary $\tilde{U}_T$ in $C(\mathbf{R}/\mathbf{Z}) \otimes A$ by

$$
\tilde{U}_T(t) = w(s, Tt) \alpha_{T(t-1)}(z_T(s, Tt)^*)
$$

where $\mathbf{R}/\mathbf{Z}$ is identified with $[0, 1]/\{0, 1\}$ and $z_T(s, t) = z(s, t)$ is defined above, and we embed $C(\mathbf{R}/\mathbf{Z}) \otimes A$ into $A$ approximately by using the Rohlin property. (If $\tau$ is the flow on $C(\mathbf{R}/\mathbf{Z})$ induced by translations on $\mathbf{R}/\mathbf{Z}$, then $\|1 \otimes w(s, t) - \tilde{U}(\tau_{s/T} \otimes \alpha_t)(\tilde{U}^*)\| \leq 6\pi|t|/T$.

We find an approximate homomorphism $\phi$ of $C(\mathbf{R}/\mathbf{Z}) \otimes A$ into $A$ such that $\phi(\tau_{s/T} \otimes \alpha_t) \approx \alpha_t \phi$ and $\phi(1 \otimes x) \approx x$, $x \in A$.) Since $z_T$ is defined in terms of $w(s, t)$, $t \in [0, T]$ and other elements which almost commute with them, we may assume that $S \in [0, T] \rightarrow z_S$ is continuous; hence that $S \in [0, T] \rightarrow \tilde{U}_S \in \mathcal{U}(C(\mathbf{R}/\mathbf{Z}) \otimes A)$ is continuous. Note also that $\tilde{U}_S$ commutes with any element to the same degree as $\tilde{U}_T$ does with it. Since $\tilde{U}_0 = 1$, we may thus assume that there is a continuous path $(U_t, t \in [0, 1])$ in the space of continuous maps of $I$ into $\mathcal{U}(A)$ such that $U_0(s) = 1$, $U_1(s) = U(s)$, and $\|[U_t(s), x]\| < \epsilon$, $x \in \mathcal{F}$.

Then we set $v_1(s) = U(s)^* v(s)$ for $s \in I$, which satisfies that

$$
\|\text{Ad} v_1(s)(x) - \alpha_{t_0}(x)\| \leq 2\epsilon, \quad x \in \alpha_{-t_0}(\mathcal{F}),
$$

$$
\max_{0 \leq t \leq 1} \|\alpha_t(v_1(s)) - v_1(s)\| \leq 6\pi/T + \epsilon.
$$

Thus we have shown the following assertion: For any finite subset $\mathcal{F}$ of $A$ and $\epsilon > 0$, there exists an $a \in [0, \infty)$ such that for any compact interval $I$ of $[a, \infty)$ we find a continuous $v_I : I \times [0, 1] \rightarrow \mathcal{U}(A)$ such that

$$
\|\text{Ad} v_I(s, t)(x) - \alpha_{t_0}(x)\| < \epsilon, \quad x \in \mathcal{F}, \quad (s, t) \in I \times [0, 1],
$$

and

$$
v_I(s, 0) = v(s), \quad s \in I,
$$

$$
\max_{0 \leq t \leq 1} \|\alpha_t(v_I(s, 1)) - v_I(s, 1)\| < \epsilon, \quad s \in I,
$$

where $v : [0, \infty) \rightarrow \mathcal{U}(A)$ has been chosen so that $\alpha_{t_0} = \lim_{k \rightarrow \infty} \text{Ad} v(s)$.

Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of $A$ such that $\bigcup_n \mathcal{F}_n$ is dense in $A$ and $(\epsilon_k)$ a decreasing sequence in $(0, \infty)$ such that $\lim_k \epsilon_k = 0$. We choose an increasing sequence $(a_k)$ in $(0, \infty)$ such that if $I$ is a compact interval of $[a_k, \infty)$ then there is a continuous $v : I \times [0, 1] \rightarrow \mathcal{U}(A)$ such that the above conditions are satisfied for $\mathcal{F} = \mathcal{F}_k$ and $\epsilon = \epsilon_k$.

Let $a_0 = 0$ and $I_k = [a_k, a_{k+1})$ for $k = 0, 1, 2, \ldots$. For each $k = 1, 2, \ldots$ we choose $v_k : I_k \times [0, 1] \rightarrow \mathcal{U}(A)$ for $\mathcal{F}_k$ and $\epsilon_k$ as above and define $v_0 : I_0 \times [0, 1] \rightarrow \mathcal{U}(A)$ by $v_0(s, t) = v(s)$, $s \in I_0$. If $v_k-1(a_k) = v_k(a_k)$ for $k = 1, 2, \ldots$, we would be finished by defining a continuous function $v : [0, \infty) \rightarrow \mathcal{U}(A)$ with the desired properties in an obvious way. But note that $v_k(a_k)v_k-1(a_k)^*$ is connected to 1 by a continuous path $(w_k(s), s \in [0, 1])$ such that $w_k(0) = 1$, $w_k(1) = v_k(a_k)v_k-1(a_k)^*$, and

$$
\|[w_k(s), x]\| < 2\epsilon_k, \quad x \in \alpha_{t_0}(\mathcal{F}_{k-1})
$$
for \( s \in [0,1] \). By modifying the path \( w_k(s), s \in [0,1] \), we have to impose the condition that \( \max_{0 \leq t \leq 1} \| \alpha_t(w_k(s)) - w_k(s) \| \) is small; then the path \( s \mapsto w_k(s)v_{k-1}(s) \) connects \( v_{k-1}(a_k) \) with \( v_k(a_k) \) and has the desired property with respect to \( \alpha \). Thus it suffices to prove the following lemma by assuming that \( (\alpha_{t_0}(F_k))_k \) is sufficiently rapidly increasing and \( (\epsilon_k) \) is sufficiently rapidly decreasing. \( \square \)

**Lemma 2.3** For any finite subset \( F \) of \( A \) and \( \epsilon > 0 \) there exist a finite subset \( \mathcal{G} \) of \( A \) and \( \delta > 0 \) satisfying the following condition: If a continuous \( v : [0,1] \rightarrow \mathcal{U}(A) \) satisfies that

\[
\begin{align*}
v(0) &= 1, \\
\| [v(s), x] \| &< \delta, \ s \in [0,1], \ x \in \mathcal{G}, \\
\| \alpha_t(v(1)) - v(1) \| &< \delta, \ t \in [0,1],
\end{align*}
\]

then there exists a continuous \( u : [0,1] \rightarrow \mathcal{U}(A) \) such that

\[
\begin{align*}
u(0) &= 1, \\
u(1) &= v(1), \\
\| [u(s), x] \| &< \epsilon, \ s \in [0,1], \ x \in F, \\
\| \alpha_t(u(s)) - u(s) \| &< \epsilon, \ t \in [0,1], \ s \in [0,1].
\end{align*}
\]

**Proof.** Suppose that \( v \) satisfies that \( v(0) = 1 \), and

\[
\begin{align*}
\| [v(s), \alpha_{-t}(x)] \| &< \delta, \ x \in \mathcal{G}, \ t \in [0,1], \\
\max_{0 \leq t \leq 1} \| \alpha_t(v(1)) - v(1) \| &< \delta.
\end{align*}
\]

We define \( w(s,t) = v(s)\alpha_t(v(s)^*) \). Then \( t \mapsto w(s,t) \) is an \( \alpha \)-cocycle for each \( s \in [0,1] \) and satisfies that \( w(0,t) = 1 \), and

\[
\begin{align*}
\max_{0 \leq t \leq 1} \| [w(s,t), x] \| &< 2\delta, \\
\max_{0 \leq t \leq 1} \| w(1,t) - 1 \| &< \delta.
\end{align*}
\]

From the latter condition there are \( b,h \in A_{sa} \) such that \( b \approx 0 \), \( h \approx 0 \), and \( w(1,t) = e^{ib}z_i^{(h)}\alpha_t(e^{-ib}) \), where \( z^{(h)} \) is a differentiable \( \alpha \)-cocycle such that \( dz_i/dt|_{t=0} = ih \). By connecting the \( \alpha \)-cocycle \( t \mapsto w(1,t) \) with the trivial \( \alpha \)-cocycle \( 1 \) by the path of \( \alpha \)-cocycles \( s \mapsto (t \mapsto e^{ish}z_i^{(sh)}\alpha_t(e^{-ish})) \) and squeezing it around \( 0 \in T = R/Z \), we get an \( \alpha \)-cocycle \( W \) in \( C(T) \otimes A \) with respect to the flow \( id \otimes \alpha \) such that \( W(0,t) = 1 \) and \( W(s,t) \approx w(s,t) \). Hence it suffices to show the following lemma, because then we find a unitary \( Z \in C(T) \otimes A \) with appropriate commutativity such that \( Z(0) = 1 \) and \( W(\cdot,t) \approx Z\alpha_t(Z)^*, \) and replace \( v \) by the path \( s \mapsto Z(s)^*v(s) \) which are almost \( \alpha \)-invariant and moves from \( v(0) = 1 \) to \( v(1) \). \( \square \)
Lemma 2.4 For any finite subset $\mathcal{F}$ of $A$ and $\epsilon > 0$ there exists a finite subset $\mathcal{G}$ of $A$ and $\delta > 0$ satisfying the following condition: Let $\pi = \text{id} \otimes \alpha$ be the flow on $C(T) \otimes A$ and let $t \mapsto W_t$ be an $\pi$-cocycle such that $W_t(0) = 1$ at $0 \in T = [0, 1]/\{0, 1\}$ and

$$\max_{0 \leq t \leq 1} \|[W_t, 1 \otimes x]\| < \delta, \ x \in \mathcal{G}.$$  

Then there exists a unitary $Z$ in $C(T) \otimes A$ such that $Z(0) = 1$ and

$$\|[Z, 1 \otimes x]\| < \epsilon, \ x \in \mathcal{F},$$

$$\max_{0 \leq t \leq 1} \|W_t - Z\pi_t(Z^*)\| < \epsilon.$$  

Proof. We just sketch the proof; see [14] or the first part of the proof of [22] for details.

To meet the last condition we choose $T \in \mathbb{N}$ such that $T^{-1} < \epsilon/6\pi$. Then we impose the condition that $\max_{0 \leq t \leq 1} \|[W_t, 1 \otimes x]\| < \delta/T$ for $x \in \bigcup_{t \leq t_0} \alpha_t(\mathcal{F})$, which can be replaced by a finite subset because it is compact. Since $\max_{0 \leq t \leq T} \|[W_t, 1 \otimes x]\| < \delta$, we find a continuous path $(U_t, \ t \in [0, T])$ in $\mathcal{U}(C(T) \otimes A)$ such that $U_0 = 1, U_T = W_T, U_t(0) = 1, \|U_t - U_s\| \leq 6\pi|t_1 - t_2|/T$, and $\|[U_t, x]\| < 9\delta$. By using $W$ and $U$ and the Rohlin property for $\alpha$, we define a unitary $Z \in C(T) \otimes A$ such that $Z(0) = 1, \max_{0 \leq t \leq 1} \|W_t - Z\alpha_t(Z^*)\| < \epsilon$, and $\|[Z, 1 \otimes x]\| < 10\delta, \ x \in \mathcal{F}$. \hfill \Box

We also give the following technical results which will be used in the next section. We assume that $\alpha$ is a Rohlin flow on $A$ as before.

Lemma 2.5 For any finite subset $\mathcal{F}$ of $A$ and $\epsilon > 0$ there exists a finite subset $\mathcal{G}$ of $A$ and $\delta > 0$ satisfying the following condition: If $(u(s), \ s \in [0, 1])$ is a continuous path in $\mathcal{U}(A)$ such that

$$\|[u(s), x]\| < \delta, \ x \in \mathcal{G},$$

$$\max_{0 \leq t \leq 1} \|\alpha_t(u(s)) - u(s)\| < \delta, \ s \in [0, 1],$$

then there exists a rectifiable path $v(s), \ s \in [0, 1]$ such that $v(0) = u(0), \ v(1) = u(1), \ v(s), \ x \in \mathcal{F},$

$$\|[v(s), x]\| < \epsilon, \ x \in \mathcal{F},$$

$$\max_{0 \leq t \leq 1} \|\alpha_t(v(s)) - v(s)\| < \epsilon, \ s \in [0, 1],$$

and the length of the path $v$ is less than $17\pi/3$. If $\mathcal{F} = \emptyset$, then $\mathcal{G} = \emptyset$ is possible.

Proof. Without the conditions with respect to $\alpha$, this is shown in [24].

To define $v$ we use certain elements of $A$ which almost commute with $u(s), \ s \in [0, 1]$. They are a certain compact subset of $\mathcal{O}_\infty$, which is then embedded centrally in $A$ in [24], by using a result due to Kirchberg and Phillips. In the present case, to meet the condition of almost $\alpha$-invariance, those elements embedded in $A$ should be almost invariant under $\alpha$. For this we use the fact that $(A' \cap A_\alpha^a)$ is purely infinite and simple [18].
Explicitly we assume that those elements of \( \mathcal{O}_\infty = C^*(s_1, s_2, \ldots) \) (before the embedding into \( A \)) is in the linear subspace spanned by a finite number of monomials in \( s_1, \ldots, s_k \) and their adjoints for some \( k \). We find a finite sequence \( (T_1, \ldots, T_k) \) of isometries in \( (A' \cap A_n^\omega)\alpha \) such that \( \sum_{i=1}^k T_i T_i^* \leq 1 \). Each \( T_i \) is represented by a central sequence \( (t_i(m)) \) of isometries in \( A \) such that \( \sum_{i=1}^k t_i(m) t_i(m)^* \leq 1 \) and \( \max_{t \in [0,1]} \| \alpha_t(t_i(m)) - t_i(m) \| \to 0 \) as \( m \to \infty \). We then express those elements in terms of \( t_1(m), \ldots, t_k(m) \) in place of \( s_1, \ldots, s_k \) respectively for a sufficiently large \( m \). Thus we get the required condition involving \( \alpha \). \( \square \)

We will denote by \( \delta_\alpha \) the generator of \( \alpha \), which is a closed derivation from a dense *-subalgebra \( D(\delta_\alpha) \) into \( A \). See [5, 2, 27] for the theory of generators and derivations.

**Lemma 2.6** Let \( (u(s), \ s \in [0, \infty)) \) be a continuous path in \( \mathcal{U}(A) \) such that

\[
\max_{0 \leq t \leq 1} \| \alpha_t(u(s)) - u(s) \|
\]

converges to zero as \( s \to \infty \). Then there is a continuous path \( (v(s), \ s \in [0, \infty)) \) of unitaries such that \( v(s) \in D(\delta_\alpha) \) and \( \delta_\alpha(v(s)) \) and \( u(s) - v(s) \) converge to zero as \( s \to \infty \).

**Proof.** Let \( f \) be a non-negative \( C^\infty \)-function on \( \mathbb{R} \) of compact support such that the integral is 1. We set

\[
z(s) = b(s) \int f(b(s)t) \alpha_t(u(s))dt,
\]

where \( b : [0, \infty) \to (0, \infty) \) is a continuous decreasing function such that \( \lim_s b(s) = 0 \), \( \|z(s) - u(s)\| < 1 \), and \( \|z(s) - u(s)\| \to 0 \). Then it follows that \( z(s) \in D(\delta_\alpha) \) and

\[
\|\delta_\alpha(z(s))\| \leq b(s) \int |f'(t)|dt,
\]

which converges to zero as \( s \to \infty \). We set \( v(s) = z(s)|z(s)|^{-1} \), which satisfies the required conditions. \( \square \)

**Lemma 2.7** For a finite subset \( \mathcal{F} \) of \( A \) and \( \epsilon > 0 \) there exists a finite subset \( \mathcal{G} \) of \( A \) and \( \delta > 0 \) satisfying the following condition. Let \( u \) be a unitary in \( C[0,1] \otimes A \) such that \( u(0) = 1 \), \( u(t) \in D(\delta_\alpha) \), \( \|\delta_\alpha(u(t))\| < \delta \), and \( \|[u(t), x]\| < \delta \), \( x \in \mathcal{G} \). Then there exist an \( h_i \in D(\delta_\alpha) \cap A_{aa} \) for \( i = 1, 2, \ldots, 10 \) such that

\[
u(1) = e^{ih_1} e^{ih_2} \cdots e^{ih_{10}},
\]

\[
\|h_i\| < \pi,
\]

\[
\|\delta_\alpha(h_i)\| < \epsilon,
\]

\[
\|[h_i, x]\| < \epsilon, \ x \in \mathcal{F}.
\]

If \( \mathcal{F} = \emptyset \), then \( \mathcal{G} = \emptyset \) is possible.
Proof. We may assume, by \[2.5\] that the length of the path \(u\) is smaller than \(17\pi/3 < 18\). Then we choose \(0 < s_1 < s_2 < \cdots < s_9 < 1\) such that \(\|u(s_i) - u(s_{i-1})\| < 9/5 < 2\) for \(i = 1, 2, \ldots, 10\), where \(s_0 = 0\) and \(s_{10} = 1\). Note that
\[
u(1) = u(s_0)u(s_1)u(s_2)\cdots u(s_9)u(s_{10}).
\]
Since \(\|u(s_{i-1})u(s_i) - 1\| < 9/5\), the spectrum of \(u(s_{i-1})u(s_i)\) is contained in
\[S = \{e^{i\theta} \mid |\theta| < \theta_0\},\]
where \(\theta_0 = \pi - 2\cos^{-1}(9/10) < \pi\). Let \(\text{Arg}\) denote the function \(e^{i\theta} \mapsto \theta\) from \(S\) onto the interval \((-\theta_0, \theta_0)\) and set \(h_i = \text{Arg}(u(s_{i-1})u(s_i))\). Then we have that \(\|h_i\| < \pi\) and \(u(1) = e^{ih_1}e^{ih_2}\cdots e^{ih_{10}}\). We shall show that these \(h_i\) satisfy the other conditions for a sufficiently small \(\delta > 0\).

In general if \(v\) is a unitary with \(\text{Spec}(v) \subset S\), then \(h = \text{Arg}(v)\) can be obtained as
\[
h = \frac{1}{2\pi i} \oint_C (\log z)(z - v)^{-1}dz,
\]
where \(\log z\) is the logarithmic function on \(C \setminus (-\infty, 0]\) with values in \(\{z \mid |\Im z| < \pi\}\) and \(C\) is a simple rectifiable path surrounding \(S\) in the domain of \(\log\). We fix \(C\) and let \(r\) be the distance between \(C\) and \(S\). Since
\[
\delta_\alpha(h) = \frac{1}{2\pi i} \oint_C \log z(z - v)^{-1}\delta_\alpha(v)(z - v)^{-1}dz,
\]
we have the estimate
\[
\|\delta_\alpha(h)\| \leq (2\pi)^{-1}M|C|r^{-2}\|\delta_\alpha(v)\|,
\]
where \(M\) is the maximum of \(|\log z|\), \(z \in C\) and \(|C|\) is the length of \(C\). Similarly we have the estimate \(|[h, x]| \leq (2\pi)^{-1}M|C|r^{-2}\|[v, x]\|\) for any \(x \in A\). (See \[17\] for details.) Thus we get the conclusion. \(\Box\)

3 Proof of Proposition 1.2

We recall that \(E_n = C^*(s_1, \ldots, s_n)\), where \(s_1, \ldots, s_n\) are isometries such that \(e_n^0 = 1 - \sum_{k=1}^n s_k s_k^*\) is a non-zero projection, and that \(J_n\) is the ideal of \(E_n\) generated by \(e_n^0\). Let \(S = \{1, 2, \ldots, n\}^*\) denote the set of all finite sequences including an empty sequence, denoted by \(\emptyset\). For \(I = (i_1, i_2, \ldots, i_m) \in S\) with \(m = |I|\), we set \(s_I = s_{i_1} s_{i_2} \cdots s_{i_m}\), where \(|I|\) is the length of \(I\); if \(|I| = 0\) or \(I = \emptyset\), then \(s_I = 1\). It then follows that \(\{s_I e_n s_J^* \mid I, J \in S\}\) forms a family of matrix units and spans \(J_n\). Thus, in particular, \(J_n\) is isomorphic to the \(C^*\)-algebra \(K\) of compact operators (on an infinite-dimensional separable Hilbert space). Hence there is a unique (up to unitary equivalence) irreducible representation \(\pi_0\) of \(E_n\) such that \(\pi_0|J_n\) is non-zero or \(\pi_0(e_n^0)\) is a one-dimensional projection. We call this
representation the Fock representation and denote by $H_0$ the representation Hilbert space of $\pi_0$.

We recall that the flow $\alpha$ is defined as $\alpha_t(s_k) = e^{ip_{ik}t} s_k$ for $k = 1, 2, \ldots, n$. For $I = (i_1, \ldots, i_m) \in S$ let $p(I) = \sum_{k=1}^m p_{ik} \in \mathbb{R}$. We set $H_0 = \sum_{I \in S} p(I)\pi_0(s_I e_n s_I^*)$, which is a well-defined self-adjoint operator on $H_0$. Then it follows that $\text{Ad} e^{itH_0} \pi_0(x) = \pi_0 \alpha_t(x)$, $x \in \mathcal{E}_n$ and, by the assumption on $p_1, \ldots, p_n$, that the spectrum of $H_0$ is the whole $\mathbb{R}$. Note that $\mathcal{E}_n/\mathcal{J}_n \cong \mathcal{O}_n = C^*(\hat{s}_1, \ldots, \hat{s}_n)$, where $\hat{s}_k = s_k + \mathcal{J}_n$; we will later on denote $\hat{s}_k$ by $s_k$. Note also that $\alpha$ induces a flow on $\mathcal{O}_n$, which we will also denote by $\alpha$. We will denote by $Q$ the quotient map of $\mathcal{E}_n$ onto $\mathcal{O}_n$.

**Lemma 3.1** For any $h \in (\mathcal{O}_n)_{sa} \cap D(\delta_\alpha)$ and $\epsilon > 0$ there exists a $b \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ such that $Q(b) = h$, $\|b\| < \|h\| + \epsilon$, and $\|\delta_\alpha(b)\| < \|\delta_\alpha(h)\| + \epsilon$.

**Proof.** Since $Q\alpha_t = \alpha_t Q$ on $\mathcal{E}_n$, it follows that $(1 + \delta_\alpha)^{-1}Q = Q(1 + \delta_\alpha)^{-1}$, which implies that $Q(D(\delta_\alpha)) = D(\delta_\alpha)$. (We will use the same symbol $\delta_\alpha$ for the generator of $\alpha|\mathcal{E}_n$ and of $\alpha|\mathcal{O}_n$.)

Thus, for any $h$ as above, there is a $b \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ such that $Q(b) = h$. By $C^\infty$-functional calculus we may suppose that $\|b\| < \|h\| + \epsilon$.

Since $H_0$ is diagonal, there exists an approximate identity $(p_k)$ for $\mathcal{J}_n$ consisting of projections in $\mathcal{J}_n \cap D(\delta_\alpha)$ such that $\delta_\alpha(p_k) = 0$. Since $Q(p_k) = 0$, we may replace $b$ by $(1 - p_k)b(1 - p_k)$. We choose a $p_k$ such that $\|(1 - p_k)\delta_\alpha(b)(1 - p_k)\| < \|\delta_\alpha(h)\| + \epsilon$ (since $\|Q(\delta_\alpha(b))\| = \|\delta_\alpha(h)\| = \lim_k \|(1 - p_k)\delta_\alpha(b)(1 - p_k)\|$). Then it follows that $b_1 = (1 - p_k)b(1 - p_k)$ belongs to $(\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ and satisfies that $\|b_1\| \leq \|b\| \leq \|h\| + \epsilon$ and

$$\|\delta_\alpha(b_1)\| = \|(1 - p_k)\delta_\alpha(b)(1 - p_k)\| < \|\delta_\alpha(h)\| + \epsilon.$$ 

Thus $b_1$ satisfies the required conditions. \hfill $\Box$

**Lemma 3.2** For any finite subset $\mathcal{F}$ of $\mathcal{E}_n$ and $\epsilon > 0$ there exists a finite subset $\mathcal{G}$ of $\mathcal{O}_n$ and $\delta > 0$ satisfying the following condition: If $h \in (\mathcal{O}_n)_{sa} \cap D(\delta_\alpha)$ such that $\|h\| < \pi$, $\|\delta_\alpha(h)\| < \delta$, and $\|[h, x]\| < \delta$, $x \in \mathcal{G}$, then there is a $b \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ such that

$$Q(b) = h,$$

$$\|b\| < \pi + \epsilon,$$

$$\|\delta_\alpha(b)\| < \epsilon,$$

$$\|[b, x]\| < \delta, \quad x \in \mathcal{F},$$

$$b^*_n = 0.$$

**Proof.** Note that in the above statement we may allow $\mathcal{F}$ and $\mathcal{G}$ to be compact subsets instead of finite subsets.

Note that $\mathcal{E}_n$ is nuclear as well as $\mathcal{O}_n$ and $\mathcal{J}_n$. Let $T > 0$ be so large that $\pi/T < \epsilon/2$. Since $\mathcal{F}_t = \bigcup_{-T \leq t \leq T} \alpha_t(\mathcal{F})$ is compact, there is a $w = (w_1, \ldots, w_K) \in M_{1K}(\mathcal{E}_n)$ for some $K \in \mathbb{N}$ such that $ww^* = 1$ and

$$\|[wxw^*, a]\| \leq (\epsilon/\pi)\|x\|, \quad x \in \mathcal{E}_n,$$
for any \( a \in \mathcal{F}_1 \), where \( w x w^* = \sum_{i=1}^K w_i x w_i^* \). Note that \( x \mapsto w x w^* \) is a kind of unital averaging map of \( A \) into \( A \) (this is due to Haagerup [8]; see also [22]). We set \( \mathcal{G} = \{ Q(\alpha_t(w_i)) \mid i = 1, 2, \ldots, K, |t| \leq T \} \), which is a compact subset of \( O_n \). Let \( \delta \in (0, \epsilon/2) \) and let \( h \in (O_n)_{sa} \cap D(\delta_\alpha) \) be such that \( \|h\| < \pi, \|\delta_\alpha(h)\| < \delta \), and \( \|[h, x]\| < \delta, \ x \in \mathcal{G} \). We assume that \( \delta > 0 \) is so small that we get

\[
\|Q(\alpha_t(w))hQ(\alpha_t(w)) - h\| < \epsilon, \ t \in [-T, T].
\]

Let \( b \in (E_n)_{sa} \cap D(\delta_\alpha) \) be such that \( Q(b) = h, \|b\| < \pi \), and \( \|\delta_\alpha(b)\| < \delta \).

We define

\[
b_1 = \frac{1}{2T} \int_{-T}^T \alpha_t(w)b\alpha_t(w)^*dt.
\]

Then it follows that \( \|b_1\| < \pi \) and \( \|Q(b_1) - h\| < \epsilon \). It also follows that \( b_1 \in (E_n)_{sa} \cap D(\delta_\alpha) \) and

\[
\delta_\alpha(b_1) = (2T)^{-1}(\alpha_T(w)b\alpha_T(w)^* - \alpha_{-T}(w)b\alpha_{-T}(w)^*) + \frac{1}{2T} \int_{-T}^T \alpha_t(w)\delta_\alpha(b)\alpha_t(w)^*dt,
\]

which implies that \( \|\delta_\alpha(b_1)\| < \pi/T + \delta < \epsilon \).

Let \( a \in \mathcal{F} \). Since \( \|[\alpha_t(w)b\alpha_t(w)^*, a]\| = \|[w\alpha_{-t}(b)w^*, \alpha_{-t}(a)]\| \leq (\epsilon/\pi)\|b\| < \epsilon \) for \( t \in [-T, T] \), we get that \( \|[b_1, a]\| < \epsilon \), \( a \in \mathcal{F} \).

To meet the condition \( b_1 \in \mathcal{G} \nabla \), let \( (p_k) \) be an approximate identity for \( \mathcal{J}_n \) in \( \mathcal{J}_n \cap D(\delta_\alpha) \) such that \( p_k \geq e_n^0, \delta_\alpha(p_k) = 0 \) and \( \|[p_k, x]\| \to 0 \) for all \( x \in E_n \). We replace \( b_1 \) by \( (1 - p_k)b_1(1 - p_k) \) for a sufficiently large \( k \).

In this way we get a \( b \in (E_n)_{sa} \cap D(\delta_\alpha) \) which satisfies all the required conditions except for \( Q(b) = h \); instead of which we have that \( \|Q(b) - h\| < \epsilon \). By the previous lemma, since \( \|\delta_\alpha(Q(b) - h)\| < \epsilon + \delta \), we get a \( c \in (E_n)_{sa} \cap D(\delta_\alpha) \) such that \( Q(c) = h - Q(b), \|c\| < \epsilon \), and \( \|\delta_\alpha(c)\| < \epsilon + \delta \). We may also require that \( ce_n^0 = 0 \). Thus we can take \( b + c \) for \( b \), which satisfies the required conditions if we start with a smaller \( \epsilon \). \( \square \)

Fix \( t_0 \in \mathbb{R} \). We choose, by [22] and [26], a continuous \( u : [0, \infty) \to \mathcal{U}(O_n) \cap D(\delta_\alpha) \) such that \( \alpha_{t_0} = \lim_{s \to \infty} \text{Ad} u(s) \) and \( \lim_{s \to \infty} \delta_\alpha(u(s)) = 0 \). Since the unitary group of \( O_n \) is connected, we may suppose that \( u(0) = 1 \).

Let \( (\mathcal{F}_k) \) be an increasing sequence of finite subsets of \( E_n \) such that the union \( \bigcup_k \mathcal{F}_k \) is dense in \( E_n \) and \( (\epsilon_k) \) a decreasing sequence of positive numbers such that \( \sum_k \epsilon_k \equiv \epsilon \ll 1 \). We choose, by [22] \( \mathcal{G}_k = \mathcal{G} \) and \( \delta_k = \delta \) for \( \mathcal{F} = \mathcal{F}_k \) and \( \epsilon = \epsilon_k \). We may suppose that \( (\mathcal{G}_k) \) is increasing and \( (\delta_k) \) is decreasing to zero.

For the above continuous map \( u : [0, \infty) \to \mathcal{U}(O_n) \cap D(\delta_\alpha) \), we will choose an increasing sequence \( (s_k) \) in \( [0, \infty) \) with \( s_0 = 0 \) such that \( \|\delta_\alpha(u(s_k)^*u(s_{k+1}))\| \) is sufficiently small for \( k \geq 0 \) and \( u(s_k)^*u(s_{k+1}) \) is sufficiently central for \( k \geq 1 \). Specifically, by [27], we assume
that \(u(s_k)^* u(s_{k+1})\) has the following factorization:

\[
u(s_k)^* u(s_{k+1}) = e^{ih_{k,1}} e^{ih_{k,2}} \cdots e^{ih_{k,10}},
\]

\[
\|h_{ki}\| < \pi,
\]

\[
\|\delta_\alpha(h_{ki})\| < \delta_k,
\]

\[
\|\left[h_{ki}, x\right]\| < \delta_k, \quad x \in G_k,
\]

where \(G_0 = \emptyset\). Then by \(3.2\) we choose \(b_{ki} \in (E_n)_{sa} \cap D(\delta_\alpha)\) such that

\[
Q(b_{ki}) = h_{ki},
\]

\[
\|b_{ki}\| < \pi + \epsilon_k,
\]

\[
\|\delta_\alpha(b_{ki})\| < \epsilon_k,
\]

\[
\|\left[b_{ki}, x\right]\| < \epsilon_k, \quad x \in F_k,
\]

\[
\pi_0(b_{ki}) \Omega_0 = 0,
\]

where \(F_0 = \emptyset\) and \(\Omega_0\) is a unit vector in \(\mathcal{H}_0\) such that \(\pi_0(e_n^0) \Omega_0 = \Omega_0\).

We set \(w_0 = 1\) and lift \(u(s_k) = u(s_{k-1}) \cdot u(s_{k-1})^* u(s_k)\) for \(k \geq 1\) to a unitary in \(E_n \cap D(\delta_\alpha)\) as

\[
w_k = w_{k-1} e^{ib_{k,1}} e^{ib_{k,2}} \cdots e^{ib_{k,10}},
\]

It then follows that \(\text{Ad} w_k\) converges on \(E_n\) as \(k \to \infty\) and \(Q \circ (\lim \text{Ad} w_k) = \alpha_{t_0} \circ Q\). When we choose \(G_k\) and \(\delta_k\), we should choose them for \(F = F_k \cup \text{Ad} w_{k-1}^*(F_k)\) and \(\epsilon = \epsilon_k\), which will make sure that \(\text{Ad} w_k^*\) also converges. In this way we have \(\beta = \lim \text{Ad} w_k\) as an automorphism of \(E_n\), which satisfies that

\[
\beta \circ Q = Q \circ \alpha_{t_0}.
\]

Since \(\|\alpha_t(w_k) - w_k\|\) is dominated by

\[
\|\alpha_t(w_{k-1}) - w_{k-1}\| + \sum_{j=1}^{10} \|\alpha_t(e^{ib_{k,j}}) - e^{ib_{k,j}}\|,
\]

and since \(\|\alpha_t(e^{ib_{k,j}}) - e^{ib_{k,j}}\| \leq \|\delta_\alpha(b_{kj})\||t| \leq \epsilon_k|t|\), we get that

\[
\|\alpha_t(w_k) - w_k\| \leq \|\alpha_t(w_{k-1}) - w_{k-1}\| + 10 \epsilon_k|t|.
\]

Thus if \(|t| \leq 1\), then it follows that

\[
\|\alpha_t(w_k) - w_k\| \leq 10 \sum_{i=1}^{k} \epsilon_i < 10 \epsilon.
\]

In the Fock representation \(\pi_0\), since \(\pi_0(w_k) \Omega_0 = \Omega_0\), we have that

\[
\pi_0(w_k) \pi_0(x) \Omega_0 = \pi_0(\text{Ad} w_k(x)) \Omega_0
\]
converges strongly to \( \pi_0(\beta(x))\Omega_0 \) for any \( x \in \mathcal{E}_n \). Hence \( \pi_0(w_k) \) converges strongly to a unitary, which we will denote by \( W \). Note that \( \text{Ad} W \pi_0(x) = \pi_0(\beta(x)), \quad x \in \mathcal{E}_n \) and \( W\Omega_0 = \Omega_0 \).

As we have remarked before, the unitary flow \( U_t \equiv e^{itH_0} \) implements \( \alpha \) in \( \pi_0 \), where \( H_0 = \sum_{t \in \mathbb{S}} p(I)\pi_0(s_t e_n s_t^*) \). Since \( \|U_t\pi_0(w_k)U_t^* - \pi_0(w_k)\| < 10\epsilon \) for \( |t| \leq 1 \), we obtain that
\[
\|U_t W U_t^* - W\| \leq 10\epsilon, \quad t \in [-1, 1].
\]
We also have that
\[
\pi_0^{-1} \alpha_t \beta = \text{Ad}(W^* U_t W U_t^*) \pi_0 \alpha_t.
\]

On the other hand, \( \beta^{-1} \alpha_t \beta \) is obtained as the limit of \( \text{Ad}(w_k^* \alpha_t(w_k)) \alpha_t \), which implies that
\[
\|\alpha_t - \beta^{-1} \alpha_t \beta\| \leq 20\epsilon, \quad t \in [-1, 1].
\]
Hence there exists an \( \alpha \)-cocycle \( u \) in \( \mathcal{E}_n \) such that \( \text{Ad} u_t \alpha_t = \beta^{-1} \alpha_t \beta \) and \( \max_{t \in [-1, 1]} \|u_t - 1\| \) is at most of order of \( 400\epsilon \) (see p. 296 of [5]). Combining the observation in the previous paragraph and noting that \( \pi_0 \) is irreducible, this implies that
\[
\pi_0(u_t) = c(t) W^* U_t W U_t^*
\]
for some constant \( c(t) \in \mathbb{T} \). Then it follows by simple computation that \( c(t) = e^{ipt} \) for some \( p \in \mathbb{R} \). Thus we know that \( t \mapsto U_t W U_t^* \) is continuous in norm and that \( W^* U_t W U_t^* \in \pi_0(\mathcal{E}_n) \). Since \( \text{QAd} u_t \alpha_t = \text{Ad}(u_t) \alpha_t Q = \alpha_t Q \) on \( \mathcal{E}_n \), we also have that \( \text{Q}(u_t) \in \mathcal{C}1 \subset \mathcal{O}_n \), which implies that
\[
W^* U_t W U_t^* \in \pi_0(\mathcal{J}_n + \mathcal{C}1).
\]

Since any automorphism of \( \mathcal{E}_n \) is weakly inner in \( \pi_0 \), there is a unitary \( V \) on \( \mathcal{H}_0 \) such that \( \text{Ad} V \pi_0 = \pi_0 \beta^{-1} \alpha_t \alpha_t \). Since the vector state of \( \mathcal{E}_n \) defined through \( \Omega_0 \) is left invariant under by \( \beta^{-1} \alpha_t \alpha_t \), we may define \( V \) by \( V \pi_0(x)\Omega_0 = \pi_0(\beta^{-1} \alpha_t(x))\Omega_0 \), \( x \in \mathcal{E}_n \). Since \( Q \beta^{-1} \alpha_t \alpha_t = Q \), we have that \( [V, \pi_0(x)] \in \mathcal{K}(\mathcal{H}_0) \) for \( x \in \mathcal{E}_n \). Regarding \( V \in \mathcal{M}(\mathcal{J}_n) \), the multiplier algebra of \( \mathcal{J}_n \) which identifies with \( B(\mathcal{H}_0) \) through \( \pi_0 \), and \( \mathcal{E}_n \subset \mathcal{M}(\mathcal{J}_n) \), we have that \( \beta^{-1} \alpha_t \beta = \text{Ad}(VU_t V^* U_t^*) \alpha_t \). Since \( \beta^{-1} \alpha_t \beta = \text{Ad}(W^* U_t W U_t^*) \alpha_t \) from above and \( VU_t V^* U_t^* \Omega_0 = \Omega_0 = W^* U_t W U_t^* \Omega_0 \), we have that
\[
VU_t V^* U_t^* = W^* U_t W U_t^*, \quad t \in \mathbb{R},
\]
which implies that \( t \mapsto U_t V U_t^* \) is norm-continuous and
\[
\|U_t V U_t^* - V\| = \|U_t W U_t^* - W\|.
\]
In particular we have that
\[
\max_{t \in [0, 1]} \|U_t V U_t^* - V\| \leq 10\epsilon.
\]

Let us denote by \( \mathcal{M}(\mathcal{J}_n) \), the C*-subalgebra consisting of \( x \in \mathcal{M}(\mathcal{J}_n) \) such that \( t \mapsto \alpha_t(x) = U_t x U_t^* \) is norm-continuous. Summing up the above we have shown:
Lemma 3.3 Let \((p_1, \ldots, p_n)\) be a finite sequence in \(\mathbb{R}\) and define a quasi-free flow \(\alpha\) on \(E_n = C^*(s_1, \ldots, s_n)\) by
\[
\alpha_t(s_j) = e^{ip_j t}s_j.
\]
Suppose that \(p_1, \ldots, p_n\) generates \(\mathbb{R}\) as a closed subsemigroup. (Hence the flow \(\dot{\alpha}\) on the quotient \(O_n\) induced by \(\alpha\) has the Rohlin property and each \(\dot{\alpha}_t\) is \(\alpha\)-invariantly asymptotically inner.)

Fix \(t_0 \in \mathbb{R}\). For any \(\epsilon > 0\) there exists an automorphism \(\beta\) of \(E_n\), a sequence \((w_k)\) in \(U(E_n)\), and a unitary \(V \in M(J_n)\) such that \(t \mapsto V^*\alpha_t(V) \in J_n + C1\) is an \(\alpha\)-cocycle,
\[
Q(V) \in (O_n)',
\]
\[
Q\beta = Q\alpha_{t_0},
\]
\[
\beta^{-1}\alpha_{t_0} = \text{Ad} V,
\]
\[
\beta = \lim_{k} \text{Ad} w_k,
\]
\[
\max_{|t| \leq 1} \|\alpha_t(w_k) - w_k\| < \epsilon,
\]
\[
\max_{|t| \leq 1} \|\alpha_t(V) - V\| < \epsilon,
\]
where \(Q\) denotes the quotient map of \(M(J_n)\) onto \(M(J_n)/J_n\), which maps \(E_n\) onto \(O_n\).

To show that \(\alpha_{t_0}\) is \(\alpha\)-invariantly approximately inner, we have to approximate \(\beta^{-1}\alpha_{t_0}\) by \(\text{Ad} v\), where \(v\) is a unitary in \(J_n + 1\) which is almost \(\alpha\)-invariant. We will use the following result whose proof we will postpone to the next section.

Lemma 3.4 For any \(\epsilon > 0\) there exists a \(\delta > 0\) satisfying the following condition: Let \(V \in M(J_n)\) be a unitary such that \(Q(V) \in (O_n)'\) and
\[
\max_{|t| \leq 1} \|\alpha_t(V) - V\| < \delta.
\]
Then there exists a rectifiable path \((V_s, s \in [0, 1])\) in \(U(M(J_n)\)) such that \(V_0 = 1, V_1 = V\),
\[
\|\lambda(Q(V_s)) - Q(V_s)\| < \epsilon,
\]
\[
\sup_{s \in [0,1]} \max_{|t| \leq 1} \|\alpha_t(V_s) - V_s\| < \epsilon,
\]
where \(\lambda\) is the unital endomorphism of \(M(J_n)/J_n\) defined by
\[
\lambda(x) = \sum_{i=1}^n Q(s_i)xQ(s_i)^*.
\]

The following is a key lemma for the proof of Proposition 1.2.
Lemma 3.5 For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: If $V$ is a unitary in $M(J_n)$ such that $\beta = \text{Ad} V$ is an automorphism of $E_n$, $Q(V) \in (O_n)'$, and $\max_{|t|\leq 1} \|\alpha_t(V) - V\| < \delta$, then there is a unitary $v$ in $J_n + 1$ such that

\[
\|\beta(s_i) - vs_tv^*\| < \epsilon, \quad i = 1, 2, \ldots, n,
\]

\[
\max_{|t|\leq 1} \|\alpha_t(v) - v\| < \epsilon.
\]

Proof. We define a non-unital endomorphism $\lambda$ of $M(J_n)$ by $\lambda(x) = \sum_{i=1}^n s_i x s_i^*$. Note that $Q\lambda = \lambda Q$, where the latter $\lambda$ is the unital endomorphism defined in 3.3.

Let $\epsilon > 0$. We choose $\delta > 0$ so small that we find a continuous path $(V_s, s \in [0,1])$ in $M(J_n)$ such that $V_0 = V, V_1 = 1, \|\lambda(V_s) - V_s + J\| < \epsilon$, and $\max_{|t|\leq 1} \|\alpha_t(V_s) - V_s\| < \epsilon$.

Let $\epsilon' > 0$. We choose an increasing sequence $(\mu_k)$ in $[0, 1)$ such that $\mu_0 = 0, \lim_{k \to \infty} \mu_k = 1$, and $\|V_{\mu_k} - V_{\mu_{k+1}}\| < \epsilon'$. We will denote $V_{\mu_k}$ by $V_k$ below.

Note that $p_m = \sum_{|t|\leq m} s_t e_t^* e_t$ is an $\alpha$-invariant projection in $J_n$ for each $m \in \mathbb{N}$ and that $(p_m)$ forms an approximate identity for $J_n$.

Since $V_s s_k V_s^* = V_\alpha(V_s) s_k$ and $\|1 - V_s \lambda(V_s^*) + J\| < \epsilon$, we have that $\|V_s s_k V_s^* - s_k + J\| < \epsilon$. Since $\{V_s s_k V_s^* - s_k \mid s \in [0,1]\}$ is compact, we have a projection $p \in J_n$ such that $\|(V_s s_k V_s^* - s_k)(1 - p)\| < \epsilon$ and $\|(1 - p)(V_s s_k V_s^* - s_k)\| < \epsilon$ for $s \in [0,1]$ and $k = 1, \ldots, n$. We may suppose that $p = p_m$ for some $m$. From the convex combinations of $(p_m)$, we find an approximate unit $(e_k)$ in $J_n$ such that $e_0 = 0 \leq p \leq e_1, \alpha_t(e_k) = e_k, e_{k+1} e_k = e_k$, and

\[
\|(e_{k+1} - e_k)^{1/2} s_t\| < \epsilon' 2^{-k-1}, \quad i = 1, 2, \ldots, n,
\]

\[
\|(e_{k+1} - e_k)^{1/2} V_j\| < \epsilon', \quad j \leq k + 1,
\]

\[
\|(e_k - e_k^2)^{1/2} V_j\| < \epsilon', \quad j \leq k + 1.
\]

Since $-1 \leq \sum_{k=0}^K (e_{k+1} - e_k)^{1/2} x_k (e_{k+1} - e_k)^{-1/2} \leq 1$ for any $K$ and any $x_k = x_k^*$ with $\|x_k\| \leq 1$, we can define

\[
z = \sum_{k=0}^\infty (e_{k+1} - e_k)^{1/2} V_k (e_{k+1} - e_k)^{-1/2},
\]

which converges in the strict topology in $M(J_n)$ and has $\|z\| \leq 2$. Since $\|V_k - 1\| \to 0$, it follows that $z - 1 \in J_n$. We claim that $z$ is close to a unitary by writing

\[
zz^* = \sum_k (e_{k+1} - e_k)^{1/2} V_k (e_{k+1} - e_k) V_k^* (e_{k+1} - e_k)^{-1/2}
\]

\[
+ \sum_k (e_{k+1} - e_k)^{1/2} V_k (e_{k+1} - e_k^2)^{1/2} V_k^* (e_{k+1} - e_k^2)^{-1/2}
\]

\[
+ \sum_k (e_{k+2} - e_{k+1})^{1/2} V_{k+1} (e_{k+1} - e_{k+1}^2)^{1/2} V_{k+1}^* (e_{k+1} - e_{k+1})^{1/2},
\]

16
where we have used that \((e_{k+1} - e_k)^{1/2}(e_{j+1} - e_j)^{1/2} = 0\) if \(|k - j| > 1\) and \((e_{k+1} - e_k)^{1/2}(e_{k+2} - e_{k+1})^{1/2} = (e_{k+1} - e_{k+1}^2)^{1/2}\). By splitting out each summation into the sum over even integers and the sum over odd integers and noting that
\[
\|V_k(e_{k+1} - e_k)V_k^* - (e_{k+1} - e_k)\| < 2\epsilon'
\]
and
\[
\|V_k(e_{k+1} - e_{k+1}^2)^{1/2}V_k^* - (e_{k+1} - e_{k+1}^2)^{1/2}\| < 2\epsilon',
\]
we estimate
\[
\|zz^* - 1\| < 2(2\epsilon' + 2\epsilon' + 2\epsilon') = 12\epsilon'.
\]
Similarly we get that \(\|zz^* - 1\| < 12\epsilon'.\) Thus if \(\epsilon'\) is sufficiently small, then \(v = z|z^*z|^{-1/2}\) is a unitary in \(J_n + 1\) and satisfies that
\[
\|v - z\| \leq \|v\|\|1 - |z^*z|^{1/2}\| < \epsilon'',
\]
with \(\epsilon'' \approx 6\epsilon'.\)

Since
\[
\|s_i z^* - \sum_{k=0}^{\infty} (e_{k+1} - e_k)^{1/2} s_i V_k^* (e_{k+1} - e_k)^{1/2}\| < \epsilon',
\]
we have that
\[
zs_i z^* \approx \sum_{k} (e_{k+1} - e_k)^{1/2} (e_{k+1} - e_k)V_k s_i V_k^* (e_{k+1} - e_k)^{1/2}
\]
\[
+ \sum_{k} (e_{k+1} - e_k)^{1/2} (e_{k+1} - e_{k+1}^2)^{1/2} V_k + s_i V_k^* (e_{k+2} - e_{k+1})^2
\]
\[
+ \sum_{k} (e_{k+2} - e_{k+1})^{1/2} (e_{k+1} - e_{k+1}^2)^{1/2} V_k s_i V_k^* (e_{k+1} - e_{k+1}^2)^{1/2},
\]
with the norm difference less than \(\|z\|\epsilon' + 12\epsilon' \leq 14\epsilon'.\) By using \(\|(V_k s_i V_k^* - s_i)(1 - p)\| < \epsilon\) etc. for all terms except for the \(k = 0\) term of the first summation, we have that
\[
zs_i z^* \approx e_1^{3/2} V s_i V^* e_1^{1/2} + \sum_{k \geq 1} (e_{k+1} - e_k)^{3/2} s_i (e_{k+1} - e_k)^{1/2}
\]
\[
+ \sum_{k \geq 0} (e_{k+1} - e_k)^{1/2} (e_{k+1} - e_{k+1}^2)^{1/2} s_i (e_{k+2} - e_{k+1})^2
\]
\[
+ \sum_{k \geq 0} (e_{k+2} - e_{k+1})^{1/2} (e_{k+1} - e_{k+1}^2)^{1/2} s_i (e_{k+1} - e_k)^2,
\]
with the norm difference less than \(14\epsilon' + 6\epsilon.\) Since \(\|[(V s_i V^* e_1^{1/2})]\| < 3\epsilon'\) and \(\|[(e_{k+1} - e_k)^{1/2}, s_i]\| < \epsilon' 2^{-k-1}\), we have that
\[
zs_i z^* \approx e_1^2 V s_i V^* + (1 - e_1^2) s_i,
\]
with the norm difference less than $14\epsilon' + 6\epsilon + 3\epsilon' + 3\epsilon' = 6\epsilon + 20\epsilon'$. Since $e_1^2 V s_i V^* + (1 - e_1^2) s_i - \beta(s_i) = (1 - e_1^2)(s_i - \beta(s_i))$, it follows that $\|z s_i z^* - \beta(s_i)\| < 7\epsilon + 20\epsilon'$, which implies that $\|z s_i z^* - \beta(s_i)\| < 7\epsilon + 20\epsilon' \approx 7\epsilon + 32\epsilon'$.

On the other hand we have that $\|\alpha_t(z) - z\| \leq 2\sup_k \|\alpha_t(V_k) - V_k\|$, which implies that $\max_{|t| \leq 1} \|\alpha_t(v) - v\| < 2\epsilon + 2\epsilon''$. This concludes the proof. □

Proof of Proposition 1.2

This follows by combining 3.2, 3.3 and 3.5.

Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$. Then we choose $\delta > 0$ for $\epsilon$ as in 3.3. By 3.3 we have an automorphism $\beta$ of $\mathcal{E}_n$ and a unitary $V \in M(J_n)_\alpha$ such that $Q \beta = \alpha_{t_0} Q$ and $\beta^{-1} \alpha_{t_0} = \text{Ad} V, \max_{|t| \leq 1} \|\alpha_t(V) - V\| < \delta$, etc. Then by 3.5 we get a unitary $v \in J_n + 1$ such that $\|\beta^{-1} \alpha_{t_0}(s_i) - vs_i v^*\| < \epsilon$ and $\max_{|t| \leq 1} \|\alpha_t(v) - v\| < \epsilon$. Since $\beta = \lim_k \text{Ad} w_k$, we get that $\|\alpha_{t_0}(s_i) - w_k v s_i v^* w_k\| < \epsilon$ for a large $k$ and $\max_{|t| \leq 1} \|\alpha_t(w_k v) - w_k v\| < \epsilon + \delta$ from the properties imposed on $(w_k)$. Thus we can conclude that $\alpha_{t_0}$ is $\alpha$-invariantly approximately inner.

4 Proof of Lemma 3.4

We recall that $H_0$ is a self-adjoint operator on $\mathcal{H}$ with $\text{Spec}(H_0) = \mathbb{R}$ and that we let $\alpha_t = \text{Ad} e^{itH_0}$ on $M(J_n)_\alpha$, where $M(J_n)$ is identified with $B(\mathcal{H})$ and $M(J_n)_\alpha$ is the largest $C^*$-subalgebra of $M(J_n)$ on which $t \mapsto \text{Ad} e^{itH_0}$ is strongly continuous. The following can be proved by adopting the arguments in 3.10.

Lemma 4.1 For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: If $Z \in \mathcal{U}(M(J_n)_\alpha)$ satisfies that $\max_{|t| \leq 1} \|\alpha_t(Z) - Z\| < \delta$, then there exists a rectifiable path $(Z_s, s \in [0, 1])$ in $\mathcal{U}(M(J_n)_\alpha)$ such that $Z_0 = 1, Z_1 = Z$,

$$\sup_{s \in [0, 1]} \max_{|t| \leq 1} \|\alpha_t(Z_s) - Z_s\| < \epsilon,$$

and the length of $(Z_s, s \in [0, 1])$ is less than $2\pi + \epsilon$.

Proof. We choose a non-negative $C^\infty$-function on $\mathbb{R}$ such that $\hat{f}(0) = 1$ and $\text{supp} \hat{f} \subset (-\delta_0, \delta_0)$ for a small $\delta_0 > 0$ and modify $Z$ by

$$X = \int f(t) \alpha_t(Z) dt \in M(J_n)_\alpha$$

which is still close to $Z$, e.g., $\|X - Z\| < \mu$, where $\mu$ can be arbitrarily close to zero depending on $\delta$. Let $E$ be the spectral measure of $H_0$ and define, for any $k \in \mathbb{Z}$,

$$E_k = E[2k\delta_0, 2(k + 1)\delta_0) \in M(J_n)_\alpha,$$
which is a projection of infinite rank. Noting that the $\alpha$-spectrum of $X$ is contained in $(-\delta_0, \delta_0)$, we have, as in the proof of 4.1 of [16], that

$$XE_k X^* \leq E[(2k-1)\delta_0, (2k+3)\delta_0],$$

$$XE_k X^* - (XE_k X^*)^2 \leq 2\mu \cdot XE_k X^* \leq 2\mu 1,$$

$$\text{Spec}(XE_k X^*) \subset \{0\} \cup [1 - 2\mu, 1],$$

$$Y_k^+ - (Y_k^+)^2 \leq 6\mu 1,$$

$$\text{Spec}(Y_k^+) \subset [0, 6\mu'] \cup [1 - 6\mu', 1],$$

$$Y_k^- - (Y_k^-)^2 \leq 6\mu,$$

$$\text{Spec}(Y_k^-) \subset [0, 6\mu'] \cup [1 - 6\mu', 1],$$

where $\mu \approx \mu'$ and

$$Y_k^+ = E[(2k+1)\delta_0, (2k+3)\delta_0)XE_k X^*E[(2k+1)\delta_0, (2k+3)\delta_0),$$

$$Y_k^- = E[(2k-1)\delta_0, (2k+1)\delta_0)XE_k X^*E[(2k-1)\delta_0, (2k+1)\delta_0).$$

We define $F_k^\pm$ to be the spectral projection of $Y_k^\pm$ corresponding to $[1 - 6\mu', 1]$ and let $F_k$ denote the projection $F_k^+ + F_k^-$. We note that these projections are of infinite rank, because $E(I)$ is a projection of infinite rank for any non-empty open subset $I$ of $\mathbb{R}$, and that

$$\|F_k - XE_k X^*\| < 14\mu.$$

We define

$$G_k^+ = E[(2k+1)\delta_0, (2k+3)\delta_0) - F_{k+1}^-,$$

$$G_k^- = E[(2k-1)\delta_0, (2k+1)\delta_0) - F_{k-1}^+,$$

$$G_k = G_k^- + G_k^+.$$

We note that $G_k^\pm$ and $G_k$ are projections of infinite rank and that

$$\|XE_k X^* - G_k\| < 34\mu.$$

By using the facts that $\sum_k E_k = 1$ and

$$\sum_k F_{2k} + \sum_k G_{2k-1} = 1,$$

we check that

$$W = \sum_k F_{2k}XE_{2k} + \sum_k G_{2k-1}XE_{2k-1},$$

19
is close to a unitary; we denote by $V$ the unitary part of the polar decomposition of $W$. Since $\| \alpha_t(F_{2k} X E_{2k}) - F_{2k} X E_{2k} \| \leq \| \alpha_t(X) - X \| + 4\delta_0|t|$, $\| \alpha_t(G_{2k-1} X E_{2k-1}) - G_{2k-1} X E_{2k-1} \| \leq \| \alpha_t(X) - X \| + 4\delta_0|t|$, and

$$\| \alpha_t(W) - W \| \leq \sup_k \| \alpha_t(F_{2k} X E_{2k}) - F_{2k} X E_{2k} \| + \sup_k \| \alpha_t(G_{2k-1} X E_{2k-1}) - G_{2k-1} X E_{2k-1} \|,$$

we have that $W, V \in M(\mathcal{J}_n)$. We estimate that $\| Z - V \| < 10\mu''$, where $\mu'' \approx \mu^{1/2}$. We should also note that

$$V E_{2k} V^* = F_{2k} \leq E[(4k - 1)\delta_0, (4k + 3)\delta_0),
V E_{2k-1} V^* = G_{2k-1} \leq E[(4k - 3)\delta_0, (4k + 1)\delta_0).$$

It then follows that the $\alpha$-spectrum of $V$ is contained in $[-3\delta_0, 3\delta_0]$.

Since $\|VZ^* - 1\| < 10\mu''$, there is a self-adjoint $B \in M(\mathcal{J}_n)$ such that $VZ^* = e^{iB}$ and $\|B\|$ is of the order $10\mu''$. Then the path $\gamma : [0, 1] \ni s \mapsto e^{isB}Z$ goes from $Z$ to $V$ of length $\|B\|$. Since $\| \alpha_t(e^{isB}Z) - e^{isB}Z \| \leq s \| \alpha_t(B) - B \| + \| \alpha_t(Z) - Z \|$, we have that $\max_s \| \alpha_t(\gamma_s) - \gamma_s \| \rightarrow 0$ as $t \rightarrow 0$ and

$$\sup \max_{|t| \leq 1} \| \alpha_t(\gamma_s) - \gamma_s \| \leq 2\|B\| + \max_{|t| \leq 1} \| \alpha_t(Z) - Z \|,$$

which is arbitrarily small.

Then we find a rectifiable path $(\zeta_t, t \in [0, 1])$ in $\mathcal{U}(M(\mathcal{J}_n))$ such that $\zeta_t$ commutes with $E[(2k - 1)\delta_0, (2k + 1)\delta_0]$ for all $k \in \mathbb{Z}$ and

$$\text{Ad}_{\zeta_t}(F_{2k}^+) = E[(4k + 1)\delta_0, (4k + 2)\delta_0),
\text{Ad}_{\zeta_t}(F_{2k}^-) = E[4k\delta_0, (4k + 1)\delta_0),$$

and the length of $\zeta$ is at most $\pi$. Then we get that

$$\text{Ad}(\zeta_1 V) E_{2k} = E_{2k}.$$

Since $\text{Ad}\zeta_1(G_{2k+1}^-) = E[(4k + 2)\delta_0, (4k + 3)\delta_0)$ etc., we also get that

$$\text{Ad}(\zeta_1 V) E_{2k+1} = E_{2k+1}.$$

Note that the $\alpha$-spectrum of $\zeta_t$ is contained in $[-2\delta_0, 2\delta_0]$, which implies that $\zeta_t \in M(\mathcal{J}_n)$ and $\max_{|t| \leq 1} \| \alpha_t(\zeta_s) - \zeta_s \| \leq 2\delta_0$.

There is a rectifiable path $(\eta_t, t \in [0, 1])$ in $\mathcal{U}(M(\mathcal{J}_n))$ such that $\eta_t$ commutes with $E[(2k - 1)\delta_0, (2k + 1)\delta_0]$ for all $k \in \mathbb{Z}, \eta_0 = 1, \eta_1 = (\zeta_1 V)^*$, and the length of $\eta$ is at most $\pi$. Note also that the $\alpha$-spectrum of $\eta_t$ is contained in $[-2\delta_0, 2\delta_0]$, which implies that $\eta_t \in M(\mathcal{J}_n)$ and $\max_{|t| \leq 1} \| \alpha_t(\eta_s) - \eta_s \| \leq 2\delta_0$.

Combining these paths we get the conclusion.

Next we will prove another version of the above lemma; we will replace $M(\mathcal{J}_n)$ by the $C^*$-tensor product $C(\mathcal{T}) \otimes M(\mathcal{J}_n)$, with the flow $\text{id} \otimes \alpha$, which will sometimes be denoted by $\alpha$. 

20
For the preparation we present the following two lemmas, which are just concerned with $B(\mathcal{H})$, the bounded operators on an infinite-dimensional Hilbert space $\mathcal{H}$, without a flow on it. We note that the $C^*$-tensor product $C(\mathcal{T}) \otimes B(\mathcal{H})$ identifies with the norm-continuous functions on $\mathcal{T}$ into $B(\mathcal{H})$.

**Lemma 4.2** Let $E, F, P$ be projections in $C(\mathcal{T}) \otimes B(\mathcal{H})$ such that $E(s), F(s), P(s)$ are of infinite rank, $P(s) = P(0)$ for $s \in \mathcal{T} = [0, 1]/\{0, 1\}$, and $EP = 0 = FP$. Then there is a rectifiable path $(U_t, t \in [0, 1])$ in $\mathcal{U}(C(\mathcal{T}) \otimes B(\mathcal{H}))$ such that $U_0 = 1$, $\text{Ad} U_1(E) = F$, and the length of $U$ is at most $\pi$. If furthermore $E(0) = F(0)$ at $0 \in \mathcal{T}$, then the condition $U_t(0) = 1, t \in [0, 1]$ can be imposed.

Moreover if $E_k, F_k, P_k$ are such a triple of projections in $C(\mathcal{T}) \otimes B(\mathcal{H})$ for each $k \in \mathbb{N}$ such that $\mathcal{T} \ni s \mapsto \bigoplus_k E_k(s)$ and $\mathcal{T} \ni s \mapsto \bigoplus_k F_k(s)$ are continuous in $\prod_{k=1}^\infty C(\mathcal{T}) \otimes B(\mathcal{H})$, there are rectifiable paths $(U^k_t, t \in [0, 1])$ in $\mathcal{U}(C(\mathcal{T}) \otimes B(\mathcal{H}))$ as above such that $\mathcal{T} \ni s \mapsto \bigoplus_k U^k_t(s)$ is continuous and the length of $[0, 1] \ni t \mapsto \bigoplus_k U^k_t$ is at most $\pi$.

**Proof.** We have expressed the base space $\mathcal{T}$ as $[0, 1]/\{0, 1\}$. Since $t \mapsto E(t)$ is norm-continuous, there exists a norm-continuous $V : [0, 1] \mapsto \mathcal{U}(B(\mathcal{H}))$ such that $V(0) = 1$ and $\text{Ad} V(t)(E(0)) = E(t), t \in [0, 1]$.

More specifically we choose a finite number of points $t_0 = 1 < t_1 < t_2 < \cdots < t_{m-1} < t_m = 1$ such that $\|E(s) - E(t)\| < 1/2$ for $s, t \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, m$. We define, for $t \in [t_{i-1}, t_i]$,

$$Z_i^t = E(t)E(t_{i-1}) + (1 - E(t))(1 - E(t_{i-1})).$$

Since $\|(Z_i^t)^*Z_i^t - 1\| < 1/2$ etc. the polar decomposition of $Z_i^t$ gives a unitary $V_i^t = Z_i^t|Z_i^t|^{-1}$. Then we define for $t \in [t_{i-1}, t_i]$,

$$V_t = V_i^tV_{i-1}^tV_{i-2}^t\cdots V_1^t.$$

Since $\text{Ad} V_i^t(E(t_{i-1})) = E(t)$, we get that $\text{Ad} V(t)(E(0)) = E(t)$. It is obvious that $t \mapsto V_t$ is norm-continuous. But more is true.

Note that $\|V_i^t - V_s^t\| \leq 2^{1/2}\|E(t) - E(s)\| + \|Z_i^t\|^{-1} - |Z_s^t|^{-1}\|$ for $s, t \in [t_{i-1}, t_i]$.

Let $S$ be the set of self-adjoint elements $h \in B(\mathcal{H})$ such that Spec($h$) $\subset [1/2, 1]$; then $|Z_i^t|^2 = (Z_i^t)*Z_i^t \in S$. The map $S \ni h \mapsto h^{-1/2}$ is uniformly continuous in $h \in S$. Since $\|Z_i^t\|^2 - |Z_s^t|^2| \leq \|E(t) - E(s)\|$, the continuity of $t \mapsto V_t$ only depends on the continuity of $t \mapsto E(t)$ on each $[t_{i-1}, t_i]$, i.e., there is a non-decreasing continuous function $\varphi_1$ on $[0, 1]$ such that $\varphi_1(0) = 0$ and $\|V_s^t - V_t^t\| \leq \varphi_1(|E(s) - E(t)|)$ for $s, t \in [t_{i-1}, t_i]$ with all $i$. Since $[0, 1] \ni t \mapsto E(t)$ is uniformly continuous, there is a non-decreasing continuous function $\varphi_2$ on $[0, 1]$ into $[0, 1]$ such that $\varphi_2(0) = 0$ and $\|E(s) - E(t)\| \leq \varphi_2(|s - t|)$. Let $\varphi = \varphi_1 \circ \varphi_2$. Combining these estimates, we have that $\|V_s^t - V_t^t\| \leq \varphi(|s - t|)$ for $s, t \in [t_{i-1}, t_i]$.

Let $\Delta = \min_{1 \leq i \leq m}|t_i - t_{i-1}|$ and assume, by replacing $\varphi$ by a bigger one if necessary, that $\varphi(s) + \varphi(t) \leq \varphi(s + t)$ for $s, t > 0$ with $s + t \leq 1$. If $s, t \in [0, 1]$ satisfies that
0 < s − t < 2\Delta, then either \( t_{i−1} < t < t_i \) or \( t_{i−1} ≤ t < s ≤ t_i \) for some \( i \). In the former case \( \| V_s − V_t \| \leq \| V_s − V_t \| + \| V_t − V_s \| \leq \varphi_1(\| E(s) − E(t) \| ) + \varphi_1(\| E(t) − E(t) \| ) \leq \varphi(s − t) + \varphi(t_i − t) ≤ \varphi(s − t) \). With the same estimate for the latter case we have that \( \| V_s − V_t \| \leq \varphi(|s − t|) \) if \(|s − t| < 2\Delta\). By redefining \( \varphi \) on \([2\Delta, 1]\), we may suppose that \( \| V_s − V_t \| \leq \varphi(|s − t|) \) is valid for all \( s, t \in [0, 1] \).

We define a function \( F \) on \( \mathbf{T} = \{ z \in C \mid |z| = 1 \} \) by \( F(e^{i\theta}) = t \) for \( t \in (−\pi, \pi) \) and let \( H = F(V(1)) \). Since \( [V(1), E(0)] = 0 \), we have that \([H, E(0)] = 0\). We replace \( V \) by \( V' : t \mapsto V_te^{−iH}. \) Then we get that \( V'(1) = 1 = V'(0), \) \( \text{Ad}(V'(t))(E(0)) = E(t) \), and the continuity of \( t \mapsto V'_t \) depends only on the continuity of \( t \mapsto E(t) \). If we replace \( \varphi \) by \( s \mapsto \varphi(s) + \pi s \), then we have that \( \| V'(s) − V'(t) \| \leq \varphi(|s − t|) \).

In this way we construct \( V, W \in \mathcal{U}(C(\mathbf{T}) \otimes B((1 − P(0))\mathcal{H})) \), which satisfies that \( V(1 \otimes E(0))V^* = E, V(0) = 1, W(1 \otimes F(0))W^* = F, \) and \( W(0) = 1 \). Moreover we may assume that there is a non-decreasing continuous function \( \varphi \) on \([0, 1]\) such that \( \varphi(0) = 0 \) and

\[
\| V(s) − V(t) \| \leq \varphi(|s − t|), \quad \| W(s) − W(t) \| \leq \varphi(|s − t|).
\]

Note that \( \varphi \) depends only on the choice of \( t_i \)'s and the continuity of \( s \mapsto E(s) \) and of \( s \mapsto F(s) \).

If \( E(0) = F(0) \), the unitary \( Z \equiv WV^* \) in \( C(\mathbf{T}) \otimes B((1 − P(0))\mathcal{H}) \) satisfies that \( \text{Ad}(Z)(E) = F \) and \( Z(0) = 1 − P(0) \). Let \( Y \in B(\mathcal{H}) \) be such that \( Y^*Y = P(0) \) and \( YY^* = 1 − P(0) \) and let

\[
X_t = \cos(\pi t/2) + (Y − Y^*) \sin(\pi t/2), \quad t \in [0, 1].
\]

Then \( X_t \) is a unitary on \( \mathcal{H} \). The path \( U : t \mapsto (Z + P)(1 \otimes X_t)(Z^* + P)(1 \otimes X_t^*) \) in \( \mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H})) \) satisfies that \( U_0 = 1, U_1 = Z + Y^*Z^*Y, U_t(0) = 1, \) and

\[
\| U_{t_1} − U_{t_2} \| ≤ 2\| X_{t_1} − X_{t_2} \| ≤ \pi |t_1 − t_2|.
\]

Moreover \( U_t \) satisfies that

\[
\| U_t(s_1) − U_t(s_2) \| ≤ 2\varphi(|s_1 − s_2|).
\]

Since \( \text{Ad}U_1(E) = F \), this concludes the proof for the case \( E(0) = F(0) \).

In the case \( E(0) \neq F(0) \) we choose a unitary \( T \) on \((1 − P(0))\mathcal{H}\) such that \( TE(0)T^* = F(0) \). Then we can proceed as above with \( Z = W(1 \otimes T)V^* \).

Suppose that \( E_k, F_k, P_k \) are given as in the statement. Let \( E_\infty = \bigoplus_k E_k \) and \( F_\infty = \bigoplus_k F_k \) in \( \Pi_\infty C(\mathbf{T}) \otimes B(\mathcal{H}) \). Then we choose a finite number of points \( t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1 \) such that \( \| E_\infty(s) − E_\infty(t) \| < 1/2 \) and \( \| F_\infty(s) − F_\infty(t) \| < 1/2 \) for \( s, t \in [t_{i−1}, t_i) \) with all \( i \). By using these points in \([0, 1]\) we construct a path \( U_k : [0, 1] \mapsto \mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H})) \) as above. Then there is a non-decreasing continuous function \( \varphi \) on \([0, 1]\) such that \( \varphi(0) = 0 \) and \( \| U_k(s_1) − U_k(s_2) \| ≤ 2\varphi(|s_1 − s_2|) \). Note that we can define \( \varphi \) based on the functions \( s \mapsto E_\infty(s) \) and \( s \mapsto F_\infty(s) \), i.e., \( \varphi \) is independent of \( k \). Hence it follows that \( \mathbf{T} \ni s \mapsto U_\infty(s) \equiv \bigoplus_k U_k(s) \) is continuous for each \( t \in [0, 1] \). Since
\[ \|U_{t_1}^k - U_{t_2}^k\| \leq \pi |t_1 - t_2|, \] we also get that \[ \|U_{t_1}^\infty - U_{t_2}^\infty\| \leq \pi |t_1 - t_2|. \] This concludes the proof. \[ \square \]

**Lemma 4.3** Let \( U \in \mathcal{U}(C(\mathbb{T}) \otimes B(\mathcal{H})) \). If there is a projection \( E \in B(\mathcal{H}) \) such that \( E \) and \( 1 - E \) are of infinite rank and \( [U, 1 \otimes E] = 0 \), then there is a rectifiable path \( (V_t, \ t \in [0, 1]) \) in \( \mathcal{U}(C(\mathbb{T}) \otimes B(\mathcal{H})) \) such that \( V_0 = 1, V_1 = U, \|V_t(s_1) - V_t(s_2)\| \leq 6\|U(s_1) - U(s_2)\|, \) and the length of \( V \) is at most \( 3\pi \).

**Proof.** Let \( Y \in B(\mathcal{H}) \) be a partial isometry such that \( Y^*Y = E \) and \( YY^* = 1 - E \) and let \( X_t = \cos(\pi t/2) + (Y - Y^*) \sin(\pi t/2) \) as in the proof of the previous lemma. Then the map \( [0, 1] \ni t \mapsto (U(1 \otimes E) + 1 \otimes (1 - E))(\otimes X_t)(U(1 \otimes (1 - E)) + 1 \otimes E)(1 \otimes X_t^*) \) moves from \( U \) into \( U_1 \equiv U(1 \otimes E)(\otimes Y)U(1 \otimes Y^*) + 1 \otimes (1 - E) \). Note that the length of this path is at most \( \pi \).

Let \( W = U(1 \otimes E)(\otimes Y^*)U(1 \otimes Y) \), which is a unitary in \( C(\mathbb{T}) \otimes B(E\mathcal{H}) \). Since \( 1 - E \) is of infinite rank, letting \( \mathcal{H}_k = E\mathcal{H} \) for all \( k \in \mathbb{Z} \), we identify \( \mathcal{H} \) with \( \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k \) and \( E\mathcal{H} \) with \( \mathcal{H}_0 \). Thus we regard \( U_1(s) \) as

\[ \cdots \oplus 1 \oplus 1 \oplus W(s) \oplus 1 \oplus \cdots, \]

where \( W(s) \) is on \( \mathcal{H}_0 \). Let \( W_s = \bigoplus_{k \leq 0} 1 \oplus \bigoplus_{k \geq 1} W \in C(\mathbb{T}) \otimes B(\bigoplus_k \mathcal{H}_k) \). Let \( H \) be a self-adjoint operator on \( \bigoplus_k \mathcal{H}_k \) such that \( \text{Spec}(H) = [-\pi, \pi] \), and \( e^{itH} \) induces the shift to right. Then the map \( [0, 1] \ni t \mapsto U_1W_r(1 \otimes e^{-itH})W_r^*(1 \otimes e^{itH}) \) moves from \( U_1 \) into 1. The length of this path is at most \( 2\pi \).

Combining these two paths we get the desired one \( (V_t, \ t \in [0, 1]) \). Since \( \|W(s_1) - W(s_2)\| \leq 2\|U(s_1) - U(s_2)\| \), we get \( \|V_t(s_1) - V_t(s_2)\| \leq 6\|U(s_1) - U(s_2)\| \) for any \( t \in [0, 1] \). This concludes the proof. \[ \square \]

**Lemma 4.4** For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) satisfying the following condition: If \( Z \in \mathcal{U}(C(\mathbb{T}) \otimes M(\mathcal{J}_n)_\alpha) \) satisfies that \( \max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(Z) - Z\| < \delta \), then there exists a rectifiable path \( (Z_s, \ s \in [0, 1]) \) in \( \mathcal{U}(C(\mathbb{T}) \otimes M(\mathcal{J}_n)_\alpha) \) such that \( Z_0 = 1, Z_1 = Z, \)

\[ \max_{s \in [0, 1]} \max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(Z_s) - Z_s\| < \epsilon, \]

and the length of \( (Z_s, \ s \in [0, 1]) \) is less than \( 4\pi + \epsilon \). Furthermore if \( Z(0) = 1 \) at \( 0 \in \mathbb{T} \), then the path is chosen to satisfy that \( Z_s(0) = 1 \) for \( s \in [0, 1] \).

**Proof.** We will prove this result for \( C(\mathbb{T}) \otimes M(\mathcal{J}_n)_\alpha \) mostly following the proof of Lemma 4.1 for \( M(\mathcal{J}_n)_\alpha \), by using Lemmas 4.2 and 4.3 where the counterparts for \( M(\mathcal{J}_n)_\alpha \) are trivial. We will indicate below how to use 4.2 and 4.3 in the proof of Lemma 4.1.

We will apply 4.2 when we construct the unitary path \( \zeta \) in the proof of 4.1.
More explicitly we define $F_k^\pm$ as in the proof of \ref{lem:1} which entails that $F_k^\pm \in C(T) \otimes M(J_n)_\alpha$ and

$$F_k^+ \leq 1 \otimes E[(2k + 1)\delta_0, (2k + 3)\delta_0],$$

$$F_{2k}^- \leq 1 \otimes E[(2k - 1)\delta_0, (2k + 1)\delta_0].$$

With $\mathcal{H} = E[(2k - 1)\delta_0, (2k + 1)\delta_0]\mathcal{H}_0$, we then construct a unitary path $\zeta^k$ in each $C(T) \otimes B(\mathcal{H})$, where $B(\mathcal{H})$ equals

$$E[(2k - 1)\delta_0, (2k + 1)\delta_0]M(J_n)_\alpha E[(2k - 1)\delta_0, (2k + 1)\delta_0).$$

For $\mathcal{H} = E[(4k + 1)\delta_0, (4k + 3)\delta_0]\mathcal{H}_0$, we have to find a path $\zeta^{2k+1}$ in $U(C(T) \otimes B(\mathcal{H}))$ such that $\zeta_0^{2k+1} = 1$,

$$\text{Ad} \zeta_1^{2k+1}(F_{2k}^+) = 1 \otimes E[(4k + 1)\delta_0, (4k + 2)\delta_0],$$

and

$$\|\zeta_1^{2k+1} - \zeta_2^{2k+1}\| \leq \pi|t_1 - t_2|.$$}

We can apply \ref{lem:2} because $1 \otimes E[(4k + 3)\delta_0 - \delta', (4k + 3)\delta_0)$ is of infinite rank and orthogonal to $F_{2k}^+$ and $1 \otimes E[(4k + 1)\delta_0, (4k + 2)\delta_0)$ for a small $\delta' > 0$.

For $\mathcal{H} = E[(4k - 1)\delta_0, (4k + 1)\delta_0]\mathcal{H}_0$, we find a path $\zeta^k$ in $U(C(T) \otimes B(\mathcal{H}))$ such that $\zeta_0^{2k} = 1$, $\text{Ad} \zeta^k(F_{2k}^+) = 1 \otimes E[4k\delta_0, (4k + 1)\delta_0)$, and $\|\zeta_1^{2k} - \zeta_2^{2k}\| \leq \pi|t_1 - t_2|.$

Note that $T \ni s \mapsto Y_k(s)$ is equi-continuous in $k \in \mathbb{Z}$; see the definition of $Y_k^+$ in the proof of \ref{lem:1}. Since the spectrum of $F_k^+$ is contained in $[0, 6\mu'] \cup [1 - 6\mu', 1]$ and $F_k^+$ is the spectral projection of $Y_k^+$ corresponding to $[1 - 6\mu', 1]$, it follows that $T \ni s \mapsto F_k^+(s)$ is equi-continuous in $k$. Similarly we get that $T \ni s \mapsto F_k^-(s)$ is equi-continuous in $k$. By \ref{lem:2} we can choose these paths $(\zeta^k)$ such that $T \ni s \mapsto \bigoplus \zeta^k_t(s)$ is continuous for each $t \in [0, 1]$, which implies that $\zeta_t \equiv \sum_k \zeta_t^k \in C(T) \otimes M(J_n)$. Since $\zeta_t$ commutes with $E[(2k - 1)\delta_0, (2k + 1)\delta_0)$, it follows that $\zeta_t \in C(T) \otimes M(J_n)_\alpha$ and the $\alpha$-spectrum of $\zeta_t$ is contained in $[-2\delta_0, 0]$. Note that $(\zeta_t, t \in [0, 1])$ has length of at most $\pi$.

When we construct the path $\eta$, we will use Lemma \ref{lem:3}.

More explicitly $\zeta_1 V$ commutes with $1 \otimes E_k$. We apply Lemma \ref{lem:3} and find a path $\eta^k$ in $U(C(T) \otimes B((E_{2k} + E_{2k+1})\mathcal{H}_0))$ which connects $\zeta_1 V(E_{2k} + E_{2k+1})$ with $E_{2k} + E_{2k+1}$ for each $k$. Since $\|\eta^k_1(s_1) - \eta^k_2(s_2)\| \leq 6\|\zeta_1 V(s_1) - \zeta_2 V(s_2)\|$, we have that $T \ni s \mapsto \eta_t(s) \equiv \sum_k \eta^k_t(s)$ is continuous, i.e., $\eta_t \in C(T) \otimes M(J_n)$. Note that the length of $(\eta_t, t \in [0, 1])$ is at most $3\pi$ and that the $\alpha$-spectrum of $\eta_t$ is contained in $[-4\delta_0, 4\delta_0]$. In this way we get the conclusion.

We recall that the flow $\alpha$ on $\mathcal{E}_n(\subset M(J_n)_\alpha)$ induces a flow on the quotient $\mathcal{O}_n(\subset M(J_n)_\alpha/J_n)$, which we will also denote by $\alpha$. By using $Q(s_i) \in \mathcal{O}_n$ we define a unital endomorphism $\lambda$ of $\mathcal{O}_n$ by $\lambda(x) = \sum_{i=1}^n Q(s_i)xQ(s_i)^*$. We know that $\lambda$ has the Rohlin property (as a version for a single endomorphism or automorphism); see \cite{26, 12, 14}. But we have more:
Lemma 4.5 For any $N \in \mathbb{N}$ and $\epsilon > 0$, there are $n^N$ projections $e_i$ in $\mathcal{O}_n$ for $i = 0, 1, \ldots, n^N - 1$ such that $e_i \in D(\delta_\alpha)$,

$$\sum_{i=0}^{2^N-1} e_i = 1,$$

$$\|\delta_\alpha(e_i)\| < \epsilon,$$

$$\max_i \|\lambda(e_i) - e_{i+1}\| < \epsilon,$$

with $F_{n^N} = F_0$.

Proof. Define an action $\gamma$ of $\mathbb{T}$ on $\mathcal{O}_n$ by $\gamma_z(s_k) = z ks_k$ for $z = (z_1, z_2, \ldots, z_n) \in \mathbb{T}^n$. We embed $\mathbf{T}$ into $\mathbb{T}^n$ by $z \mapsto (z, z, \ldots, z)$. Then the fixed point algebra of $\mathcal{O}_n$ under $\gamma|\mathbf{T}$ is the closed linear span of $Q(sIs_j)$, $|I| = |J|$, where $I, J \in \{1, 2, \ldots, n\}^*$ and is isomorphic to the UHF algebra of type $n^\infty$. We will denote it by $\text{UHF}_n$. Then the restriction of $\lambda$ onto $\text{UHF}_n$ is the one-sided shift and is known to have the Rohlin property.

Note that $\mathcal{O}_n^\gamma \subset \text{UHF}_n$, $\mathcal{O}_n^\alpha \subset \mathcal{O}_n^{\alpha_n}$, and by [19] that $\lambda|\mathcal{O}_n^\alpha$ has the so-called one-cocycle property. (Since $\mathcal{O}_n^\gamma$ has $n$ distinct characters, $\lambda|\mathcal{O}_n^\alpha$ cannot have the Rohlin property but has an approximate Rohlin property.) In particular for any $k \in \mathbb{N}$ we have mutually orthogonal projections $f^k_i$, $i = 0, \ldots, M - 1$ in $\mathcal{O}_n^\gamma$, with $M = n^N$, such that

$$\|\lambda(f^k_i) - f^k_{i+1}\| < 1/k$$

with $f^k_M = f_0$ and $[f^k_0] = [1]$ in $K_0(\mathcal{O}_n^\gamma) = \mathbb{Z}/(n - 1)\mathbb{Z}$. Let $e^k = \sum_{i=0}^{M-1} f^k_i \in \mathcal{O}_n^\gamma$, which is an $\alpha$-invariant projection. Since $\|\lambda(e^k) - e^k\| < M/k$, the sequence $(e^k)$ forms a central sequence in $A = \mathcal{O}_n$. Hence $(e^k)$ defines a projection $E$ in $(A' \cap A^\omega)^\alpha$, where $\omega$ is an ultrafilter on $\mathbb{N}$ and $(A' \cap A^\omega)^\alpha$ is the $\alpha$-fixed point algebra of the central sequence algebra divided by the ideal vanishing at $\omega$. Since $[1] = [e^k]$ in $K_0(A) = \mathbb{Z}/(n - 1)\mathbb{Z}$, we have that $[1] = [E]$ in $K_0(A' \cap A^\omega)$ [20]. (This is not entirely trivial; we use the fact that $K_1(A) = 0$ for $A = \mathcal{O}_n$.)

By [18] [20] we have that $K_0((A \cap A^\omega)^\alpha) = K_0(A' \cap A^\omega)$. Since $1, E \in (A' \cap A^\omega)^\alpha$, we have a $W \in (A' \cap A^\omega)^\alpha$ such that $WW^* = E$ and $W^*W = 1$. Suppose that $(w_k)$ represent $W$; then it follows that $\lim_\omega \max_{|t| \leq 1} \|\alpha_t(w_k) - w_k\| = 0$, $\lim_\omega \|w_k w_k^* - e^k\| = 0$, and $\lim_\omega \|w_k^* w_k - 1\| = 0$. Thus we can make an isometry $w$ in $A = \mathcal{O}_n$ from $w_k$ for some $k$ with $\epsilon/3 > 1/k$ such that $\|\lambda(w) - w\| < \epsilon/3$, $w \in D(\delta_\alpha)$, $\|\delta_\alpha(w)\| < \epsilon/2$, $ww^* = e^k$, and $w^*w = 1$. We can take the projections $e_i = w^* f^k_i w$, $i = 0, 1, \ldots, M - 1$ because then $\sum_i e_i = 1$,

$$\|\lambda(e_i) - e_{i+1}\| < \epsilon$$

and so on. \qed

Lemma 4.6 For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: Let $H_\sigma$ be a self-adjoint operator on a Hilbert space $\mathcal{H}_\sigma$ for $\sigma = 1, 2$ such that $\dim(\mathcal{H}_1) = \ldots$
dim(\mathcal{H}_2) < \infty. If there is a unitary operator \( W \) from \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \) such that

\[
\max_{|t| \leq 1} \|e^{itH_2} W e^{-itH_1} - W\| < \delta,
\]

then it follows that \(|\lambda_i^{(1)} - \lambda_i^{(2)}| < \epsilon\) for all \( i \), where \((\lambda_i^{(\sigma)})\) is the increasing sequence of eigenvalues of \( H_\sigma \) (each repeated as often as its multiplicity) for \( \sigma = 1, 2 \).

**Proof.** Let \( f \) be a non-negative integrable continuous function on \( \mathbb{R} \) such that \( \hat{f}(0) = 1 \) and \( \text{supp}(\hat{f}) \subset [-\epsilon, \epsilon] \), where \( \hat{f}(p) = \int f(t)e^{-ipt}dt \). We choose a sufficiently small \( \delta > 0 \) so that \( W_f = \int f(t)e^{itH_2} W e^{-itH_1}dt \) is close to \( W \) and so is invertible for \( W \) satisfying the condition in the lemma. Then it follows that for any finite subset \( \mathcal{F} \) of the eigenvalues of \( H_1 \) the number of eigenvalues \( \lambda \) of \( H_2 \) with \(|\lambda - \mu| < \epsilon\) for some \( \mu \in \mathcal{F} \) is greater than or equal to the number of elements of \( \mathcal{F} \). (Because if \( L \) is the linear subspace spanned by eigenvectors for \( H_1 \) whose eigenvalues are in \( \mathcal{F} \), then the linear subspace \( W_f L \) is contained in the linear subspace \( L' \) spanned by eigenvectors for \( H_2 \) whose eigenvalues are in \( \{\lambda \mid \exists \mu \in \mathcal{F} \mid |\lambda - \mu| < \epsilon\} \). Since \( W_f \) is invertible, the dimension of \( L' \) must be no less than the dimension of \( L \).) This implies, by the matching theorem, that there is a bijection \( \phi \) from the eigenvalues of \( H_1 \) to those of \( H_2 \) such that \(|\phi(\lambda) - \lambda| < \epsilon\) for any eigenvalue \( \lambda \) of \( H_1 \). This implies the above conclusion. \( \square \)

**Lemma 4.7** For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) satisfying the following condition: If \( v \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha/\mathcal{J}_n \) is a unitary such that \( v(0) = 1 \) at \( 0 \in \mathbf{T} = [0, 1]/\{0, 1\} \) and \( \max_{|t| \leq 1} \| (\text{id} \otimes \alpha_t)(v) - v \| < \delta \), then there is a unitary \( Z \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha \) such that \( Q(Z) = v \), \( Z(0) = 1 \), and \( \max_{|t| \leq 1} \| (\text{id} \otimes \alpha_t)(Z) - Z \| < \epsilon \).

**Proof.** Regarding \( v \) as a continuous function from \([0, 1]\) into \( \mathcal{U}(M(\mathcal{J}_n)_\alpha/\mathcal{J}_n) \), we can find a continuous function \( V \) from \([0, 1]\) into \( \mathcal{U}(M(\mathcal{J}_n)_\alpha) \) such that \( V(0) = 1 \) and \( Q(V(s)) = v(s) \) for \( s \in [0, 1] \). Since \( V(1) \in 1 + \mathcal{J}_n \) and \( \mathcal{U}(1 + \mathcal{J}_n) \) is connected, we may suppose that \( V(1) = 1 \), i.e., we have a unitary \( V \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha \) such that \( V(0) = 1 \) and \( Q(V) = v \). If \( \max_{|t| \leq 1} \| (\text{id} \otimes \alpha_t)(v) - v \| < \delta \), then we have that \( \max_{|t| \leq 1} \| \alpha_t(V(s)) - V(s) + \mathcal{J}_n \| < \delta \) for all \( s \in \mathbf{T} \). Thus we may start with such a unitary \( V \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha \) instead of \( v \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha/\mathcal{J}_n \).

Let \( (p_k) \) be an increasing sequence of projections in \( \mathcal{J}_n \) such that \( xp_k \to x \) for any \( x \in \mathcal{J}_n \) and \( \alpha_t(p_k) = p_k \) for all \( k \). (Note that \( \alpha_t = \text{Ad} e^{itH_0} \) on \( M(\mathcal{J}_n)_\alpha \) and \( H_0 \) is diagonal.)

Since \( \mathbf{T} \times [-1, 1] \ni (s, t) \mapsto \alpha_t(V(s)) - V(s) \in M(\mathcal{J}_n)_\alpha \) is norm-continuous, the range \( \{\alpha_t(V(s)) - V(s) \mid s \in \mathbf{T}, \ t \in [-1, 1]\} \) is compact. Suppose that \( \| \alpha_t(V(s)) - V(s) + \mathcal{J}_n \| < \delta \) for all \( (s, t) \in \mathbf{T} \times [-1, 1] \) for a sufficiently small \( \delta > 0 \). Then there is a \( p = p_k \) such that

\[
\| (\alpha_t(V(s)) - V(s))(1 - p) \| < \delta
\]

for all \( (s, t) \in \mathbf{T} \times [-1, 1] \).
Let $P(s) = V(s)pV(s)^*$ for $s \in T$. Then $s \mapsto P(s)$ defines a projection $P$ in $C(T) \otimes J_n$, such that
\[
\max_{|t| \leq 1} \|(id \otimes \alpha_t)(P) - P\| < 2\delta.
\]
Hence, for any $\epsilon' > 0$ by choosing a sufficiently small $\delta > 0$, we may suppose that there is a unitary $U \in C(T) \otimes (J_n + C1)$ and a self-adjoint $b \in C(T) \otimes J_n$ such that $\|U - 1\| < \epsilon'$, $\|b\| < \epsilon'$, and $(id \otimes \delta_\alpha + ad ib)(UPU^*) = 0$. We may suppose that $U(0) = 1$ and $b(0) = 0$ since $P(0) = p$ is $\alpha$-invariant. Let $P' = UPU^* = UV(1 \otimes p)V^*U^*$, which implies that $P'(0) = p$. Since $H_0 + b(s)$ commutes with $P'(s)$, we can compute $\text{Spec}((H_0 + b(s))P'(s))$. Let $E_0 = \min_{s \in T} \text{Spec}((H_0 + b(s))P'(s))$ and let $E_1 = \max_{s \in T} \text{max} \text{Spec}((H_0 + b(s))P'(s))$.

We find a projection $q \in J_n$ such that $q \leq 1 - p$, $[H_0, q] = 0$, $\text{Spec}(H_0q) \subset [E_0, E_1]$, $\text{rank}(q) \gg \text{rank}(p)$, and the eigenvalues of $H_0q$ are almost equally distributed in $[E_0, E_1]$. By the last condition we should have, in particular, the following: If $(\lambda_k)_{k=1}^{\text{rank}(q)}$ is the increasing sequence of eigenvalues of $H_0q$ (each of which is repeated as its multiplicity indicates), we get $|\lambda_k - \lambda_\ell| < \epsilon$ for any $k$ and $\ell$ with $|k - \ell| \leq \text{rank}(p)$.

Let $Q = V(1 \otimes q)V^*$, which satisfies that $Q(0) = q$ and
\[
\max_{|t| \leq 1} \|(id \otimes \alpha_t)(Q) - Q\| < 2\delta.
\]

Since we can choose the projection $q$ so that $\|(1 \otimes q)P'\| \approx 0$ and $\|QP'\| \approx 0$ as precisely as we like, we may suppose that the above $U \in C(T) \otimes (J_n + C1)$ and $b \in C(T) \otimes J_n$, are chosen to satisfy that $(1 \otimes (\delta_\alpha + ad ib))(UQU^*) = 0$ in addition, by keeping the original relation $P' = UPU^*$. Note, for each $s \in T$, that $U(s)V(s)q$ is a partial isometry from $qH_0$ onto $U(s)V(s)qH_0$ and that
\[
\max_{|t| \leq 1} \|e^{it(H_0+b(s))}U(s)V(s)qe^{-itH_0} - U(s)V(s)q\| < \delta + 3\epsilon',
\]
where we have use that $\|b\| < \epsilon'$, $\|U - 1\| < \epsilon'$, and
\[
\|e^{itH_0}V(s)e^{-itH_0}q - V(s)q\| < \delta, \quad t \in [-1, 1].
\]

By the previous lemma, if $(\lambda_i)$ is the increasing sequence of eigenvalues of $H_0q = H_00q$ and $(\mu_i)$ is the increasing sequence of eigenvalues of $(H_0+b(s))q_s$ with $q_s = U(s)V(s)qV(s)^*U(s)^*$, it follows, by choosing sufficiently small $\delta$ and $\epsilon'$, that $|\lambda_i - \mu_i| < \epsilon$ for all $i$.

If $S_r$ denotes the set of eigenvalues of $H_0r = rH_0$ with $r$ a projection, then $S_{p+q} = S_p \cup S_q$ (which is a disjoint union since we count the multiplicity). We have assumed that the rank of $q$ is much larger than the rank of $p$; hence the cardinality of $S_q$ is much larger than that of $S_p$. We have also assumed that $S_q$ overwhelms $S_p$; if we align $S_{p+q}$ in the increasing order, the difference of a pair of values which differ by the rank of $p$ in this order is at most $\epsilon$.

Note that $S_{p+q}(s) = S_p(s) \cup S_q(s)$ for each $s \in T$, where $S_r(s)$ is the set of eigenvalues of $(H_0+b(s))U(s)V(s)rV(s)^*U(s)^*$. Since $S_q(s)$ is close to $S_q = S_q(0)$ (as $\|b(s)\| < \epsilon$).
\( \epsilon' \ll \epsilon \) and \( S_q \) overwhelms \( S_p(s) \), we can conclude the following: If \( (\lambda_i) \) is the increasing sequence of eigenvalues of \( H_0(p + q) \) and \( (\lambda_i^{(s)}) \) is the increasing sequence of eigenvalues of \((H_0 + b(s))U(s)V(s)(p + q)V(s)^*U(s)^*\), then \(|\lambda_i - \lambda_i^{(s)}| < 2\epsilon \) for all \( i \).

There are continuous functions \((f_i)\) on \( \mathcal{T} \) such that

\[
 f_i(s) \in \text{Spec}((1 \otimes H_0 + b(s))\text{Ad}(U(s)V(s))(p + q))
\]

for all \( s \in \mathcal{T}, f_i \leq f_{i+1}, \) and \((f_i(s)) = (\lambda_i^{(s)})\) as sequences. By perturbing \( 1 \otimes H_0 + b \) restricted to \( U(V(1 \otimes (p + q))V^*U^*) \) slightly, we may suppose that there are projections \( p_i \in C(\mathcal{T}) \otimes \mathcal{J}_n \) such that \((1 \otimes H_0 + b)(UV(1 \otimes (p + q))V^*U^*) = \sum_i f_i p_i \) (see [II]). By a further perturbation up to \( \epsilon \), we may suppose that \( f_i(s) = \lambda_i, s \in \mathcal{T} \) for all \( i \). By using this, we can construct a partial isometry \( W \in C(\mathcal{T}) \otimes \mathcal{J}_n \) such that \( W^*W = 1 \otimes (p + q), WW^* = UV(1 \otimes (p + q))V^*U^* \) and \( e^{it(1 \otimes H_0 + b)}W e^{-itH_0} = W \) for all \( t \), where \( b \) has now the estimate \(|b| < 2\epsilon \). Since \( U(0)V(0)(p + q)V(0)^*U(0)^* = p + q \), we have that \( W(0) = p + q \). Then it follows that \( U^*W \) is a partial isometry from \( 1 \otimes (p + q) \) to \( P + Q = V(1 \otimes (p + q))V^* \) with \((U^*W)(0) = p + q \) and that

\[
 \max_{|t| \leq 1} \| e^{it(1 \otimes H_0)}U^*W e^{-it(1 \otimes H_0)} - U^*W \| \leq 2\| U - 1 \| + 2\| b \|.
\]

Thus we can set \( Z = V(1 - 1 \otimes (p + q)) + U^*W \), which is the desired unitary in \( C(\mathcal{T}) \otimes M(\mathcal{J}_n)_\alpha \).

**Lemma 4.8** If \( e \) is a projection in \( M(\mathcal{J}_n)_\alpha/\mathcal{J}_n \cap D(\delta_\alpha) \), there is a projection \( E \in M(\mathcal{J}_n)_\alpha \cap D(\delta_\alpha) \) such that \( Q(E) = e \).

**Proof.** From the proof of Lemma 3.1 we can choose \( B \in M(\mathcal{J}_n)_\alpha \cap D(\delta_\alpha) \) such that \( Q(B) = e \) and \( B^* = B \). Since \( Q(B) \) is a projection, the essential spectrum of \( B \) is \( \{0, 1\} \) (except for the trivial cases), i.e., \( \text{Spec}(B) \) has a gap between 0 and 1. We can define a projection \( E \) from \( B \) by a \( C^\infty \)-functional calculus such that \( E \in D(\delta_\alpha) \) and \( Q(E) = e \). \( \square \)

When \( e_0 \in M(\mathcal{J}_n)_\alpha/\mathcal{J}_n \cap D(\delta_\alpha) \) is a projection such that \( \| \delta_\alpha(e_0) \| \) is small, we set \( h = -i(\delta_\alpha(e_0))^00_0 - e_0 \delta_\alpha(e_0)) \in M(\mathcal{J}_n)_\alpha/\mathcal{J}_n \). There is an \( H \in M(\mathcal{J}_n)_\alpha \) such that \( H^* = H \), \( Q(H) = h \), and \( \| H \| = \| h \| = \| \delta_\alpha(e_0) \| \). If \( E_0 \in M(\mathcal{J}_n)_\alpha \cap D(\delta_\alpha) \) is a projection such that \( Q(E_0) = e_0 \) in the above lemma, then we have that \( [H_0 - H, E_0] \in \mathcal{J}_n \). Hence, setting \( K = -i(\delta_\alpha - \text{ad} \ H)(E_0)E_0 - E_0(\delta_\alpha - \text{ad} \ H)(E_0) \in \mathcal{J}_n \), we have that \( [H_0 - H - K, E_0] = 0 \). Thus we can talk about the essential spectrum of \((H_0 - H - K)E_0 \), which is independent of particular choice of \( H \) and \( E_0 \) and may be obtained as follows.

We define a homomorphism \( \phi \) of \( C_0(\mathcal{R}) \) into \( e_0(M(\mathcal{J}_n)_\alpha/\mathcal{J}_n)e_0 \) by

\[
 \phi(f) = Q(f(H_0 - H - K)E_0), \quad f \in C_0(\mathcal{R}).
\]

Then the kernel of \( \phi \) is identified with \( \{ f \in C_0(\mathcal{R}) \mid f|S = 0 \} \) for some closed subset \( S \) of \( \mathcal{R} \). The essential spectrum of \((H_0 - H - K)E_0 \) is equal to \( S \).
Lemma 4.9 For any canonical unitary group implementing $\alpha$. More precisely, then there is a rectifiable path $w_s$, $s \in [0,1]$ of partial unitaries in $C(T) \otimes M(J_n) \alpha / J_n$ such that $w_s^* w_s = 1 \otimes e = w_s^* w_s$, $w_0 = 1 \otimes e$, $w_1 = u$, $w_s(0) = e$, $\max_{|s| \leq 1} \|\alpha(t) - w_s\| < \epsilon$, and the length of $(w_s)$ is less than $4\pi + \epsilon$.

Proof. We will decide $\delta > 0$ later and suppose that $u$ is given as above.

We have that $\|\delta_{\alpha}(e)\| < \delta$. Let $h = -i(\delta_{\alpha}(e) e - e \delta_{\alpha}(e)) \in M(J_n) \alpha / J_n$, which satisfies that $\delta_{\alpha}(e) = \text{ad } h(e)$ and has norm less than $\delta$. We find a self-adjoint $H \in \mathcal{E}_n \subset M(J_n) \alpha$ such that $Q(H) = h$ and $\|H\| = \|h\|$.

Let $E \in \mathcal{E}_n \cap D(\delta_{\alpha})$ be a projection such that $Q(E) = e$. Note that $(\delta_{\alpha} - \text{ad } iH)(E) \in J_n$. Hence there is a self-adjoint $K \in J_n$ such that
\[
(\delta_{\alpha} - \text{ad } i(H + K))(E) = 0.
\]

Since $u(0) = e$, we can find a continuous path $(U(s), s \in [0,1])$ in $\mathcal{U}(EM(J_n) \alpha E)$ such that $U(0) = E$ and $Q(U(s)) = u(s)$, where $T$ is regarded as the quotient space $[0,1]/\{0,1\}$. Since $Q(U(1)) = e$ or $U(1) \in E(J_n+1) E$ and the unitary group of $E(J_n+1) E$ is connected, we may suppose that $U(1) = E$, i.e., we have a lifting $u$ to the unitary $U$ in $C(T) \otimes EM(J_n) \alpha$ such that $U(0) = E$. Let $\beta$ be the flow on $M(J_n) \alpha$ generated by $\delta_{\alpha} - \text{ad } iH - \text{ad } iK$, which leaves $E$ invariant, i.e., $\beta_t$ is implemented by $e^{it(H_0 - H - K)}$. Since $Q(\beta_t(U(s)) - U(s)) = \alpha(-h)(u(s)) - u(s)$, we have that
\[
\max_{|t| \leq 1} \|\beta_t(U(s)) - U(s) + J_n\| < 2\|h\| + \delta < 3\delta,
\]
where $\alpha(-h)$ is the flow on $M(J_n) \alpha$ generated by $\delta_{\alpha} - \text{ad } iH$. We have remarked that the essential spectrum of $(H_0 - H - K) E$ is $R$, where we may suppose that $(H_0 - H - K) E$ is diagonal by changing $K$ if necessary. We will apply Lemmas 4.7 and 4.11 below for the flow $\beta$ on $M(EJ_n E) \beta$ in place of $\alpha \beta M(J_n) \alpha$, where $EJ_n E$ is the compact operators on $EH_0$ as $J_n$ is the compact operators on $H_0$ and $M(EJ_n E) \beta = EM(J_n) \beta E = EM(J_n) \alpha E$. This is possible because the properties we needed for $\alpha_t = \text{Ad } e^{itH_0}$ on $M(J_n) \alpha$ in these lemmas are that $\text{Spec}(H_0) = R$ and that $H_0$ is diagonal; the same properties hold for $\beta_t = \text{Ad } e^{it(H_0 - H - K)}$ on $M(EJ_n E) \beta$ as asserted above.
Lemma 4.10 For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: If $V \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ is a unitary such that $V(0) = 1$ at $0 \in \mathbf{T}$ and

$$\max_{|t| \leq 1} \left\| (\text{id} \otimes \alpha_t)(V) - V \right\| < \delta,$$

then there is a unitary $Z \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ such that $Z(0) = 1$, $\|Q(V) - Q(Z)\lambda(Q(Z))^*\| < \epsilon$, and

$$\max_{|t| \leq 1} \left\| (\text{id} \otimes \alpha_t)(Z) - Z \right\| < \epsilon.$$

Proof. Let $N \in \mathbb{N}$ and $M = n^N$. We assume that $M^{-1} < \epsilon$. By Lemma 4.5 there are projections $e_i, i = 0, \ldots, M - 1$ in $\mathcal{O}_n \cap D(\delta_0) \subset M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$ such that $\sum_i e_i = 1$, $\|\delta_0(e_i)\| < \epsilon$, and $\|\lambda(e_i) - e_i\| < \epsilon'$ for an arbitrarily small $\epsilon' > 0$.

For each $s \in \mathbf{T}$ there is an isomorphism $\phi_s$ of $\mathcal{O}_n = C^*(Q(s_1), \ldots, Q(s_n))$ onto $C^*(Q(V(s)s_1), \ldots, Q(V(s)s_n)) \subset M(\mathcal{J}_n)_\alpha$. Note that $s \mapsto \phi_s$ is continuous in the sense that $s \mapsto \phi_s(x)$ is continuous for each $x \in \mathcal{O}_n$. Since $\phi_s\lambda|\mathcal{O}_n = \text{Ad} Q(V(s))\lambda \phi_s|\mathcal{O}_n$, the projections $f_i(s) = \phi_s(e_i)$ satisfy that $\sum_i f_i(s) = 1$ and $\|\text{Ad} Q(V(s))\lambda(f_i(s)) - f_{i+1}(s)\| < \epsilon'$. Since $\alpha_t(V(s)) \approx V(s)$ or $\phi_s\alpha_t|\mathcal{O}_n \cong \alpha_t\phi_s|\mathcal{O}_n$ depending on $\delta$, we can also ascertain $\max_{|t| \leq 1} \|\alpha_t(f_i(s)) - f_i(s)\| < \epsilon'$, by choosing $\delta > 0$ sufficiently small. Note that $f_i : s \mapsto f_i(s)$ defines a projection in $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$.

We assert that there is a partial isometry $z \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$ such that $z^*z = 1 \otimes e_0$, $zz^* = f_0$, and $\max_{|t| \leq 1} \left\| (\text{id} \otimes \alpha_t)z - z \right\|$ is small.

Let $h = -i(\delta_0(e_0)e_0 - e_0\delta_0(e_0)) \in (\mathcal{O}_n)_{sa}$. Then we have that $(\delta_0 - \text{ad } ih)(e_0) = 0$.

From the proof of Lemma 4.3 $e_0$ is defined as $w^*p_0w$, where $p_0$ is a projection in $\mathcal{O}_n^\alpha \subset \mathcal{U}HF_n$, and $w$ is an isometry in $\mathcal{O}_n$ with $ww^* \in \mathcal{O}_n^\alpha$ and $ww^* \geq p_0$. We may suppose that $p_0$ and $ww^*$ are in the linear span of $Q(s_is_j^*)$, $|I| = |J| = L$ for a large $L \in \mathbb{N}$. Since $\phi_s(s_is_j^*) = Q(V(s))_Ls_is_j^*Q(V(s))^*_L$ for $I, J \in \{1, 2, \ldots, n\}^*$ with $|I| = |J| \leq L$, it follows that

$$f_0(s) = \phi_s(e_0) = \phi_s(w^*)Q(V(s))_Lp_0Q(V(s))^*_L\phi_s(w),$$

where $Q(V(s))^*_L$ is defined inductively by $Q(V(s))_0 = 1$ and

$$Q(V(s))^*_L = Q(V(s))\lambda(Q(V(s)))_{L-1}.$$
Thus we can set
\[ z(s) = \phi_s(w^*)Q(V(s))Lw e_0, \quad s \in T, \]
which has the required property as a partial isometry from \( 1 \otimes e_0 \) onto \( f_0 \) if we choose \( \delta \) sufficiently small. (For example, \( z(s)^*z(s) = e_0 w^* Q(V(s))^\ast \phi_s(w w^*) Q(V(s)) L w e_0 = e_0 w^* \cdot w w^* \cdot w e_0 = e_0. \)) Since \( V(0) = 1 \) and \( \phi_0 = \text{id} \), we have that \( z(0) = e_0. \) (Given \( \epsilon > 0 \) we choose \( N \in \mathbb{N} \) and the projections \( e_i \) for \( i = 0, 1, \ldots, n^N - 1 \); in particular the above \( w, p_0, \) and \( L \). Then making \( \delta > 0 \) sufficiently small depending on \( w \) and \( L \), we get that \( \phi_s(w^*) \) and \( Q(V)L \) are almost \( \alpha \)-invariant.)

Now we proceed as follows. We choose unitaries \( \zeta, \eta \) in \( C(T) \otimes M(J_n)/J_n \) such that \( \zeta(s) = \zeta(0) = \eta(0), \zeta \approx 1, \operatorname{Ad} \zeta \lambda (1 \otimes e_1) = 1 \otimes e_{i+1}, \eta \approx 1, \) and \( \operatorname{Ad} (\eta Q(V)) \lambda (f_i) = f_{i+1}, \) where \( \lambda \) denotes the endomorphism \( \text{id} \otimes \lambda \) on \( C(T) \otimes M(J_n)/J_n \). We define a map \( \mu \) on \( C(T) \otimes M(J_n)/J_n \) by \( \mu = L_{\eta Q(V)} R_{\epsilon} - \lambda \), where \( L_x \) (resp. \( R_x \)) is the left (resp. right) multiplication of \( x \). Note that \( \mu|C(T) \otimes 1 = \text{id}, \mu(z) = \eta Q(V) \lambda(z) \zeta^* \) is a partial isometry from \( 1 \otimes e_1 \) to \( f_1 \), etc. We let
\[ y = z^* \mu^M(z) \approx z^* Q(V) \lambda(Q(V)) \cdots \lambda^{M-1}(Q(V)) \lambda^M(z) \]
which is a unitary in \( C(T) \otimes e_0(M(J_n)/J_n) e_0 \) such that \( (\text{id} \otimes \alpha_t)(y) \approx y \) for \( t \in [-1, 1] \) depending on \( \delta > 0 \). Since \( V(0) = 1 \) and \( z(0) = e_0 \), we get that \( y(0) = e_0 \).

By assuming \( \delta > 0 \) sufficiently small we can now invoke \( 4.9 \). i.e., we find a rectifiable path \((y_s, s \in [0, 1])\) in \( C(T) \otimes M(J_n)/J_n \) such that \( y_s y_s^* = 1 \otimes e_0 = y_s y_s^* \), \( y_0 = 1 \otimes e_0, \) \( y_1 = y, y_s(0) = e_0, \) \( \| \alpha_t(y_s) - y_s \| \approx 0 \) for \( t \in [-1, 1] \), and the length of \((y_s)\) is less than \( 5\pi \).

We are now in a familiar situation; we define a unitary \( u \in C(T) \otimes M(J_n)/J_n \) by
\[ u = \sum_{k=0}^{M-1} \mu^k(z y_k), \]
where \((y_k)\) is a sequence of unitaries in \( C(T) \otimes e_0(M(J_n)/J_n) e_0 \) chosen from the above path \((y_s)\) as follows: \( y_0 = y, \) \( \| y_k - y_{k+1} \| < 5\pi/M, y_k(0) = e_0, \) and \( y_{M-1} = e_0. \) Note that \( \mu^M(z) = z y \). Then \( u \) satisfies that \( \| \mu(u) - u \| < 5\pi/M, \) i.e., \( Q(V) \lambda(u) \approx u \). It also follows that \( u(0) = 1 \) and \( (\text{id} \otimes \alpha_t)(u) \approx u \) for \( t \in [-1, 1] \). Then use \( 4.7 \) to lift \( u \) to a unitary \( Z \) in \( C(T) \otimes M(J_n)/J_n \) such that \( Q(Z) \approx u, Z(0) = 1, \) and \( (\text{id} \otimes \alpha_t)Z \approx Z \) for \( t \in [-1, 1] \), which satisfies the required conditions.

**Proof of Lemma 3.4**

Let \( \delta > 0 \), which will be decided during the proof. Suppose that we are given a unitary \( V \in M(J_n) \) such that \( Q(V) \in (O_n)' \) and \( \max_{|t| \leq 1} \| \alpha_t(V) - V \| < \delta. \) Then by applying Lemma 4.1 there is a rectifiable path \((V_s, s \in [0, 1])\) in \( U(M(J_n)) \) such that \( V_0 = 1, V_1 = V, \) and \( \epsilon \equiv \max_{s \in [0, 1]} |t| \leq 1 \| \alpha_t(V_s) - V_s \| \) is very small depending on \( \delta. \)
Since $Q(V\lambda(V)^*) = 1$, the map $s \in [0,1] \mapsto Q(V_s)\lambda(Q(V_s)^*)$ defines a unitary $w \in C(T) \otimes M(J_n) / J_n$ such that $w(0) = 1$ and $\max_{|t| \leq 1} \|(id \otimes \alpha_t)(w) - w\| \leq 2\varepsilon$ for $t \in [-1, 1]$. By \[17\] we obtain a unitary $W \in C(T) \otimes M(J_n) / J_n$ such that $Q(W) = w$, $W(0) = 1$, and $\max_{|t| \leq 1} \|(id \otimes \alpha_t)(W) - W\| \equiv \varepsilon'$ is small depending on $\varepsilon$. Then by \[4.10\] we obtain a unitary $Z \in C(T) \otimes M(J_n) / J_n$ such that $Z(0) = 1$, $w = Q(W) \approx Q(Z)^*\lambda(Q(Z))$, and $\max_{|t| \leq 1} \|(id \otimes \alpha_t)(Z) - Z\|$ is small depending on $\varepsilon'$. Set $V_s = Z(s)V_s$, $s \in [0,1]$. Then $V_0 = 1$, $V_1 = V$, $\lambda(Q(V_s)) \approx Q(V_s)$, and $\max_{s \in [0,1]} \max_{|t| \leq 1} \|\alpha_t(V_s) - V_s\| \approx 0$. Thus we find the desired path $(V_s, s \in [0,1])$ connecting 1 with $V$ in $U(M(J_n) / J_n)$. This concludes the proof.

5 Cocycle conjugacy

When $A$ is a unital C*-algebra, we denote by $U(A)$ the unitary group of $A$. When $C$ is a C*-subalgebra of $A$ such that $A \cap C^* = C$, we call $C$ a masa of $A$. When $C$ is a masa of $A$ and $u \in U(A)$, $u$ is said to normalize $C$ if $uCu^* = C$. We denote by $N(C)$ the set of those $u \in U(A)$ normalizing $C$. Note that $N(C) \supset U(C)$ and $N(C)$ is a closed subgroup of $U(A)$. When $N(C)$ generates $A$ as a C*-algebra (or equivalently the closed linear span of $N(C)$ is $A$), $C$ is called a regular masa of $A$.

When a regular masa $C$ of $A$ satisfies the conditions that there is a norm-one projection of $A$ onto $C$ and that there is a character of $C$ which uniquely extends to a state of $A$, we call $C$ a Cartan masa of $A$.

When $\alpha$ is a flow on a C*-algebra $A$, we denote by $\delta_\alpha$ its generator and by $D(\delta_\alpha)$ the domain of $\delta_\alpha$, which is a dense *-subalgebra of $A$.

**Theorem 5.1** Let $A$ be a unital simple C*-algebra and let $C$ be a Cartan masa of $A$. Let $\alpha$ be a flow on $A$ such that $D(\delta_\alpha) \supset C$. Then the following conditions are equivalent:

1. $\sup\{\|\alpha_t\delta_\alpha(x) - \delta_\alpha(x)\| \mid x \in C, \|x\| \leq 1\} \to 0$ as $t \to 0$.

2. $\delta_\alpha|C$ is inner, i.e., there is an $h = h^* \in A$ such that $\delta_\alpha(x) = \text{ad} \, ih(x)$, $x \in C$.

**Remark 5.2** The first condition of the above theorem follows if $D(\delta_\alpha^2) \supset C$. This is because then $\delta_\alpha^2|C$ is bounded (see, e.g., [2]) and

$$\|\alpha_t\delta_\alpha(x) - \delta_\alpha(x)\| \leq \|\delta_\alpha^2|C|||t||\|x\|$$

for $x \in C$.

In the above theorem it is obvious that (2)$\Rightarrow$(1). Because, by using $h$ in (2), we have the estimate that

$$\|\alpha_t\delta_\alpha(x) - \delta_\alpha(x)\| \leq 2(\|\alpha_t(h) - h\| \|x\| + \|h\| \|\delta_\alpha|C|||t||), \quad x \in C.$$  

Now we assume (1). To derive (2) we first show the following lemmas, the first of which is proved in a more general context.
Lemma 5.3 Let $\alpha$ be a flow on a $C^*$-algebra $A$ and let $C$ be an abelian $C^*$-subalgebra of $A$ such that $C \subseteq D(\delta_\alpha)$. Then for each character $\omega$ of $C$ there is a state $\varphi$ of $A$ such that $\|\varphi_\alpha - \varphi\| \to 0$ as $t \to 0$ and $\varphi|C = \omega$.

Proof. Let $\omega$ be a character of $C$. Then there is a decreasing net $(x_\nu)$ in $\{x \in C \mid \|x\| \leq 1\}$ such that $\omega(x_\nu) = 1$ and $(x_\nu)$ converges to the minimal projection $p \in C^{**}$ with $\omega(p) = 1$. We regard $C^{**} \subseteq A^{**}$ and set $c = \|\delta_\alpha|C\|$. Since $\|\alpha_t(z_\nu) - z_\nu\| \leq c|t|$, we get that $\|\alpha_t^*(p) - p\| \leq c|t|$, i.e., $t \mapsto \alpha_t^*(p)$ is norm-continuous. Let $B$ be the $C^*$-subalgebra of $A^{**}$ generated by $A$ and $\alpha_t^*(p)$, $t \in \mathbb{R}$. Then $B$ is left invariant under $\alpha^{**}$ and the restriction $\beta = \alpha^{**}|B$ is a flow, i.e., is strongly continuous. There is a $\beta$-cocycle $u$ such that $\text{Ad} u\beta_t(p) = p$. Hence there is a $\text{Ad} u\beta$-invariant pure state $\hat{\varphi}$ of $B$ such that $\hat{\varphi}(p) = 1$.

We let $\varphi = \hat{\varphi}|A$. Since $\alpha = \beta|A = \alpha^{**}|A$, we have that $\|\varphi_\alpha - \varphi\| \leq \|\hat{\varphi}_\beta - \hat{\varphi}\|$. Since $\|\hat{\varphi}_\beta - \hat{\varphi}\| \leq \|(\hat{\varphi} - \hat{\varphi}\text{Ad} u_t)\hat{\beta}_t\| \leq 2\|u_t - 1\|$, we have that $\|\varphi_\alpha - \varphi\| \leq 2\|u_t - 1\|$. Since $\varphi|C = \omega$, this concludes the proof. \qed

Lemma 5.4 Let $\omega$ be a character of $C$ such that $\omega$ uniquely extends to a state $\varphi$ of $A$, which is necessarily a pure state. Then $\pi_\varphi(C)^\prime\prime$ is a completely atomic masa of $\pi_\varphi(A)^\prime\prime = B(\mathcal{H}_\omega)$, i.e., there is a family $\{p_i\}$ of one-dimensional projections on $\mathcal{H}_\omega$ such that $\pi_\varphi(C)^\prime\prime$ is generated by $p_i$’s. If $\Phi$ is a norm one projection of $A$ onto $C$, then $\pi_\varphi \Phi(x) = \sum_i p_i \pi_\varphi(x) p_i$ for $x \in A$.

Proof. The support projection $p$ of $\omega$ in $C^{**}$ is minimal in $A^{**}$. Hence $\pi_\varphi^{**}(p) \equiv p$ is a one-dimensional projection belonging to $\pi_\varphi(C)^\prime\prime$. Since $C$ is regular, the supremum of $\pi_\varphi(u)p\pi_\varphi(u)^* \subseteq \pi_\varphi(C)^\prime\prime$ over $u \in \mathcal{N}(C)$ should commute with $\pi_\varphi(A)^\prime\prime$; hence equals 1. Thus we conclude that there is a family $\{p_i\}$ of one-dimensional projections in $\pi_\varphi(C)^\prime\prime$ such that $\sum_i p_i = 1$.

Let $x \in A$. If $z \in C$ satisfies that $0 \leq z \leq 1$ and $\pi_\varphi(z)\Omega_\varphi = \Omega_\varphi$, then

$$\pi_\varphi \Phi(x)\Omega_\varphi = \pi_\varphi \Phi(x)\pi_\varphi(z^2)\Omega_\varphi = \pi_\varphi \Phi(zxz)\Omega_\varphi.$$ 

Since $zxz \approx \varphi(x)z^2$ for a suitably chosen $z$, we get that

$$\pi_\varphi \Phi(x)\Omega_\varphi = \varphi(x)\Omega_\varphi = p\pi_\varphi(x)p\Omega_\varphi$$

for $x \in A$, where $p$ is the projection onto $C\Omega_\varphi$. Let $u \in \mathcal{N}(C)$. Since $x \in C \mapsto \langle \pi_\varphi(x)\pi_\varphi(u)^*\pi_\varphi(u), \pi_\varphi(u)\Omega_\varphi \rangle = \omega(u^* xu) = \Omega_\varphi$ is also a character, we apply the same argument and get that if $u \in \mathcal{N}(C)$,

$$\pi_\varphi \Phi(x)\pi_\varphi(u)\Omega_\varphi = \varphi(u^* xu)\pi_\varphi(u)\Omega_\varphi = p_u \pi_\varphi(x) p_u \pi_\varphi(u) \Omega_\varphi = (\sum_i p_i \pi_\varphi(x) p_i) \pi_\varphi(u) \Omega_\varphi$$

for $x \in A$, where $p_u = \pi_\varphi(u)p\pi_\varphi(u)^* \subseteq \pi_\varphi(C)^\prime\prime$ is the projection onto $C\pi_\varphi(u)\Omega_\varphi$ and $\{p_i\}$ is chosen in the first paragraph. This concludes the proof. \qed

33
Lemma 5.5 Under the condition (1) of the theorem, it follows that $\mathcal{N}(C) \subset D(\delta_\alpha)$.

Proof. Let $\omega$ be a character of $C$ which uniquely extends to a state, say $\varphi$, of $A$. Note that $\varphi$ is a pure state of $A$. From 5.3 it follows that $\|\varphi\alpha_t - \varphi\| \to 0$ as $t \to 0$. Hence there is a unitary flow $U$ on $\mathcal{H}_\varphi$ such that $\text{Ad} U_t \pi_\varphi(x) = \pi_\varphi(\alpha_t(x))$, $x \in A$. We will let $H$ be the self-adjoint generator of $U$; $U_t = e^{itH}$.

Let $h$ be an invariant mean of the function on the abelian group $\mathcal{U}(C)$ defined by $u \mapsto -i\pi_\varphi(\delta_\alpha(u)u*) \in B(\mathcal{H}_\varphi)$. Then it follows that $h^* = h \in B(\mathcal{H}_\varphi)$ and $\text{ad} ih(\pi_\varphi(x)) = \pi_\varphi(\delta_\alpha(x))$, $x \in C$. We will show that $t \mapsto \text{Ad} U_t(h)$ is norm-continuous.

Let $c = \|\delta_\alpha|C\|$. We have that

$$\|\alpha_t(\delta_\alpha(u)u*) - \delta_\alpha(u)u*\| \leq \|\delta_\alpha(\alpha_t(u) - u)\| + c^2|t|$$

for $u \in \mathcal{U}(C)$. Since $\text{Ad} U_t(ih) - ih$ is the invariant mean of $u \mapsto \pi_\varphi(\alpha_t(\delta_\alpha(u)u*) - \delta_\alpha(u)u*)$, it follows that

$$\|\text{Ad} U_t(h) - h\| \leq \sup_{u \in \mathcal{U}(C)} \|\delta_\alpha(\alpha_t(u) - u)\| + c^2|t|.$$

Thus we get that $t \mapsto \text{Ad} U_t(h)$ is norm-continuous.

Let $B$ be the $C^*$-algebra generated by $\pi_\varphi(A)$ and $\text{Ad} U_t(h)$, $t \in \mathbb{R}$. Then $\text{Ad} U_t$ leaves $B$ invariant and $x \mapsto \text{Ad} U_t(x)$ is norm-continuous for $x \in B$. Hence $t \mapsto \text{Ad} U_t|B$ defines a flow on $B$, which we denote by $\beta$. Since $h \in B$, $\text{Ad} e^{it(H-h)}$ leaves $B$ invariant and defines a perturbed flow $\beta^{(-h)}$, which fixes each element of $\pi_\varphi(C)$, i.e., $e^{it(H-h)} \in \pi_\varphi(C)'$.

Let $u \in \mathcal{N}(C)$ and define

$$W_t = \pi_\varphi(u)\beta^{(-h)}_t(\pi_\varphi(u*))$$

$t \in \mathbb{R}$.

Then $W_t$ is a unitary in $B$ and $t \mapsto W_t$ is a $\beta^{(-h)}$-cocycle. For $z \in \mathcal{U}(C)$ we have that $[\pi_\varphi(z), W_t] = 0$, which implies that $W_t \in B \cap \pi_\varphi(C)' \subset \pi_\varphi(C)'$. Since $\beta^{(-h)}_t|B \cap \pi_\varphi(C)'$ is the identity map, we get that $t \mapsto W_t$ is a norm-continuous unitary flow, i.e., there is a $k = k^* \in B \cap \pi_\varphi(C)'$ such that $W_t = e^{ikt}$. Thus it follows that $\pi_\varphi(u)$ is in the domain of the generator of $\beta^{(-h)}$ or $\beta$. Since $\beta_t \pi_\varphi(u) = \pi_\varphi(\alpha_t(u))$ and $\pi_\varphi$ is isometric, we get that $u \in D(\delta_\alpha)$. This completes the proof of the inclusion $\mathcal{N}(C) \subset D(\delta_\alpha)$. □

Continued from the above proof we get that for $u \in \mathcal{N}(C)$,

$$\pi_\varphi(u\delta_\alpha(u*)) - \pi_\varphi(u)\text{ad} ih(\pi_\varphi(u*)) = ik,$$

where $k = k^* \in \pi_\varphi(C)'$.

Let $\Phi$ be the norm-one projection of $B(\mathcal{H}_\varphi)$ onto $\pi_\varphi(C)'$. For $z \in C$ we have that

$$\pi_\varphi(\Phi \delta_\alpha(z)) = \overline{\Phi} \pi_\varphi \delta_\alpha(z) = [i\Phi(h), \pi_\varphi(z)] = 0.$$

Since $h$ is the invariant mean of $-i\pi_\varphi(\delta_\alpha(z)z^*)$, $z \in \mathcal{U}(C)$ and

$$\overline{\Phi}(-i\pi_\varphi(\delta_\alpha(z)z^*)) = -i\pi_\varphi(\Phi(\delta_\alpha(z))z^*) = 0,$$
we obtain that $\Phi(h) = 0$. Using that $\Phi(\pi_\varphi(u)h\pi_\varphi(u^*)) = \pi_\varphi(u)\Phi(h)\pi_\varphi(u^*) = 0$ for $u \in N(C)$, we get, from the equation in the previous paragraph,

$$\pi_\varphi(\Phi(u\delta_\alpha(u^*))) = ik,$$

i.e., $k \in \pi_\varphi(C)$. Hence we get that

$$\text{ad}\ ih(\pi_\varphi(u^*)) = \pi_\varphi(\delta_\alpha(u^*)) - \pi_\varphi(u^*)ik \in \pi_\varphi(A).$$

Since this is the case for all $u \in N(C)$, we can conclude that $\text{ad}\ ih$ defines a bounded derivation on $\pi_\varphi(A)$. Since $\pi_\varphi(A)$ is unital and simple and $\pi_\varphi(A)^{\prime\prime} = B(\mathcal{H}_\varphi)$, this implies that $h \in \pi_\varphi(A)$ (because any derivation of a unital simple $\mathcal{C}^*$-algebra is inner). Since $\text{ad}\ ih\ \pi_\varphi(z) = \pi_\varphi(\delta_\alpha(z))$, $z \in C$, this concludes the proof of Theorem 5.1.

In the above theorem, the regularity of the masa $C$ in that strong sense is not really needed. We call a masa $C$ of $A$ weakly regular if

$$\{u \in \mathcal{P}\mathcal{I}(A) \mid uu^*, u^*u \in C, u Cu^* = Cuu^*\}$$

generates $A$, where $\mathcal{P}\mathcal{I}(A)$ is the set of partial isometries of $A$. If a weakly regular masa $C$ satisfies the conditions that there is a norm-one projection of $A$ onto $C$ and that there is a character of $C$ which uniquely extends to a state of $A$, then we call $C$ a weak Cartan masa. We can get the following from the above proof straightforwardly.

**Corollary 5.6** Let $A$ be a unital simple $\mathcal{C}^*$-algebra and let $C$ be a weak Cartan masa of $A$.

Let $\alpha$ be a flow on $A$ such that $D(\delta_\alpha) \supset C$. Then the following conditions are equivalent.

1. $\sup\{\|\alpha_t\delta_\alpha(x) - \delta_\alpha(x)\| \mid x \in C, \|x\| \leq 1\} \to 0$ as $t \to 0$.

2. $\delta_\alpha$ is inner, i.e., there is an $h = h^* \in A$ such that $\delta_\alpha(x) = \text{ad}\ ih(x)$, $x \in C$.

**Remark 5.7** Let $A$ be a unital simple AF $\mathcal{C}^*$-algebra. Let $C$ be a masa of $A$. We call $C$ a canonical AF masa if there is an increasing sequence $(A_n)$ of finite-dimensional $\mathcal{C}^*$-subalgebras of $A$ such that the union is dense in $A$ and $C$ is generated by $C \cap A_n \cap A_n'$, $n = 1, 2, \ldots$ with $A_0 = 0$. If $C$ is a canonical AF masa of $A$, then the pair $(A, C)$ satisfies the assumptions of Theorem 5.1. Moreover if $\alpha$ is a flow on $A$ such that $D(\delta_\alpha) \supset C$, then $\delta_\alpha|C$ is inner [16]. Note also that a canonical AF masa of $A$ is unique up to the transform by automorphisms.

**Proposition 5.8** Let $B$ be a stable AF $\mathcal{C}^*$-algebra, $C$ a canonical AF masa, and $\gamma$ an automorphism of $B$ such that $\gamma(C) = C$ and $\gamma$ fixes no non-trivial ideals of $B$. Suppose that $\gamma_*(g)$ is strictly smaller than $g$ for any $g \in K_0(B)_+$, i.e., if $\phi$ is a positive homomorphism of $K_0(B)$ into $\mathbb{R}$ such that $\phi(g) > 0$, then $\phi(g) > \phi\gamma_*(g)$. Let $e$ be a non-zero projection in $(C)$ such that $\gamma(e) \leq e$. Let $A = e(B \times, \mathbb{Z})e$ and regard $C = Ce$ as a $\mathcal{C}^*$-subalgebra of $A$. Then it follows that
1. A is a unital simple $C^*$-algebra.

2. There exists a norm-one projection of $A$ onto $C$.

3. There exists a character of $C$ which uniquely extends to a state of $A$.

If $B$ is simple in addition, $C$ is a regular masa of $A$.

Proof. Since the Connes spectrum of $\gamma$ is full (otherwise some power of $\gamma$ would be universally weakly inner on some non-zero ideal of $B$), it is well-known that $B \times \gamma \mathbb{Z}$ is simple; thus (1) follows.

By using the dual action of $T$ on $B \times \gamma \mathbb{Z}$, we get a norm-one projection of $A = e(B \times \gamma \mathbb{Z})e$ onto $eBe$. We also have a norm-one projection of $eBe$ onto $C = Ce$. Composing them we get the desired norm-one projection from $A$ onto $C$. Thus (2) follows.

To prove (3) we assert that there is a decreasing sequence $(p_k)$ of non-zero projections in $C$ such that $\|p_k u \gamma^n(p_k)\| \to 0$ for all $u \in \mathcal{N}(C) \cap eBe$ for all $n \in \mathbb{N}$. Then we take a character $\omega$ of $C$ such that $\omega(p_k) = 1$ for all $k$. If $\varphi$ is an extension of $\omega$ to a state of $A$, then we get that $\varphi(xU^n) = \lim_k \varphi(p_k x \gamma^n(p_k) U^n) = 0$ for all $x \in eBe$ and all $n = 1, 2, \ldots$, where $U$ is the canonical unitary multiplier of $B \times \gamma \mathbb{Z}$ and we have used that $C$ is a regular masa of $eBe$. This implies that $\varphi$ is uniquely determined by $\omega$.

To prove the above assertion on $(p_k)$, let $(u_m, n_m)_m$ be a dense sequence in $(\mathcal{N}(C) \cap eBe) \times \mathbb{N}$. Suppose that we have chosen a non-zero projection $p \in C$ such that $pu_m \gamma^{n_m}(p) = 0$ for $m < \ell$. Then we have to find a non-zero projection $p' \in C$ such that $p' \leq p$ and $p' u_{\ell} \gamma^{n_{\ell}}(p') = 0$. Since $q = Ad u_{\ell} \gamma^{n_{\ell}}(p)$ is a projection in $C$ whose equivalence class is strictly smaller than $[p]$, we get that $p(1 - q) \neq 0$; thus we may set $p' = p(1 - q)$. We can get the desire sequence of projections by repeating this procedure.

Recall that we have set $\mathcal{N}(C) = \{ u \in \mathcal{U}(A) \mid uCu^* = C \}$. To prove the last statement we have to show that $\mathcal{N}(C)$ generates $A$. Note that $\lim_{k \to \infty} \phi([\gamma^k(e)]) = 0$ for any positive homomorphism $\phi$ of $K_0(B)$ into $\mathbb{R}$ (otherwise $\lim_k \phi \gamma^k$ would define a non-zero $\gamma$-invariant positive homomorphism $\psi$ of $K_0(B)$ into $\mathbb{R}$ such that $\psi([e]) > 0$). We choose $k > 1$ such that $[\gamma^{k-1}(e)] + [\gamma^k(e)] \leq [e]$ and then choose $v, w \in \mathcal{N}(C) \cap eBe$ such that

$$\gamma^k(e) \perp Ad v \gamma^{k-1}(e), \quad \gamma^{k-1}(e) \perp Ad w \gamma^k(e),$$

and

$$\gamma^k(e) + Ad v \gamma^{k-1}(e) = \gamma^{k-1}(e) + Ad w \gamma^k(e) \equiv f.$$ 

Then it follows that

$$S^k S^{k-1} + v S^{k-1} S^k w^* + e - f \in \mathcal{N}(C),$$

where $S = Ue$. Hence by multiplying $\gamma^{k-1}(e) \in C$ from the right, we get that $S^k S^{k-1} = S \gamma^{k-1}(e)$ belongs to the $C^*$-algebra $C^*(\mathcal{N}(C))$ generated by $\mathcal{N}(C)$. For $u \in \mathcal{N}(C) \cap eBe$ we have that $\gamma(u) + e - \gamma(e) \in \mathcal{N}(C)$ and that $\gamma(u) S \gamma^{k-1}(e) u^* = S u \gamma^{k-1}(e) u^*$ belongs to...
bounded perturbation to $\delta^*$ then generated by $C$ $\beta$ invariant under $\beta$. We may suppose that $D_{e,\gamma}A$ Cartan masa of $k$ $n$. We recall that the above automorphism $\gamma$ of $B$ may have the Rohlin property (see [II]). In this case we can show:

**Lemma 5.9** Let $A$ be a unital nuclear simple purely infinite $C^*$-algebra and $C = Ce$ be a Cartan masa of $A$ as in Proposition 5.8. Suppose also that the automorphism $\gamma$ of $B$ has the Rohlin property.

Let $\alpha$ be a Rohlin flow on $A$ and let $\beta$ be a flow on $A$ such that $\beta | C = \text{id}$. Then $\beta$ is an approximate cocycle perturbation of $\alpha$.

*Proof. Let $(B_n)$ be an increasing sequence of finite-dimensional $C^*$-subalgebras of $B$ such that $e, \gamma(e) \in B_1$, the central support of $\gamma(e)$ in $eB_1e$ is $e$, $\gamma^\pm_1(B_n) \subset B_{n+1}$, and $C$ is generated by $C \cap B_n \cap B_n'$, $n = 1, 2, \ldots$, where $B_0 = 0$. Let $u \in N(C) \cap eB_ne$. We have then $\beta_t(u) = e^{itk}u$ for some $k = k^* \in C$. Since $N(C) \cap eB_ne$ generates $eB_ne$, we get that $\beta$ fixes $eBe$ and that $D(\delta_\beta)$ contains the union of $eB_ne$, $n = 1, 2, \ldots$.

For each $n$ there exists a $z_n = z^*_n \in C$ such that $\delta_\beta | eB_ne = \text{ad}z_n | eB_ne$. We make a bounded perturbation to $\delta_\beta$ and pass to a subsequence of $(B_n)$ so that $z_n \in C \cap eB_{n+1}e$, i.e., the $C^*$-subalgebra $D_n$ of $eB_{n+1}e$ generated by $eB_ne$ and $e(B_{n+1} \cap B_n)e \cap C$ is left invariant under $\beta$. Note that $eB_ne \subset D_n \subset eB_{n+1}e$.

The domain $D(\delta_\alpha)$ may not contain $\bigcup_n eB_ne$, but we find a $u \in U(A)$ such that $D(\delta_\alpha) \supset ueB_neu^*$ for all $n$. Thus by replacing $\alpha_t$ by a cocycle perturbation $Ad(u^*\alpha_t(u))\alpha_t$, we may suppose that $D(\delta_\alpha) \supset \bigcup_n eB_ne = \bigcup_n D_n$.

There exists $y_n = y^*_n \in A$ such that $\delta_\alpha | D_n = \text{ad}y_n | D_n$. Then it follows that $(\delta_\alpha + \text{ad}(z_n - y_n)) | D_n = \delta_\beta | D_n$, which implies that $\alpha_t^{(z_n - y_n)}(x) = \beta_t(x)$ for $x \in D_n$. Thus, for any $n$, by a bounded perturbation on $\alpha$ we can always assume that $\alpha = \beta$ on $D_n$.

Let $S = Ue$ and define $w_t = S^*\beta_t(S)$, which is an $\beta$-cocycle. We know that $w_t \in C$ and hence that there is a $k = k^* \in C$ such that $w_t = e^{itk}$. We approximate $k$ by $k_n = k^*_n \in Ce \cap eB_ne = Ce \cap D_n$ such that $\|k - k_n\| \to 0$ as $n \to \infty$.

Let $T$ be a large constant and $n$ be a large integer such that $1/T \approx 0$ and $T\|k - k_n\| \approx 0$. Let $N$ be also a large integer. We suppose that $\alpha_t(x) = \beta_t(x)$, $x \in D_{n+N}$.

Let $v_t = S^*\alpha_t(S)$, which is a unitary since $\alpha_t(SS^*) = SS^*$ and forms a $\alpha$-cocycle. Let $x \in Ce \cap eB_{n+N}e = Ce \cap D_{n+N-1}$. Then since $SS^* \in C \cap B_{n+N+1}e \subset D_{n+N}e$, we have that $x_{\alpha} = S^*SS^*\alpha_t(S) = S^*\alpha_t(SS^*) = v_tx$, which implies, in particular, that $e^{-itk_n}v_t = v_te^{-itk_n}$ (as $k_n \in Ce \cap D_{n-1}$). Since $\alpha_t(e^{-itk_n}) = e^{-itk_n}$, it follows that $t \mapsto e^{-itk_n}v_t$ is an $\alpha$-cocycle.

Let $x \in D_{n+N-1}$. Then since $S\beta_{-t}(x)S^* = S\alpha_{-t}(x)S^* \in D_{n+N}$ we have that $x_{\beta_{-t}(S^*)\alpha_t(S)} = \beta_t(S^*\beta_{-t}(x)S^*)\alpha_t(S) = \beta_t(S^*)\alpha_t(S)x$, i.e., $e^{-itk_n}v_t \in A \cap D'_{n+N-1}$. Hence,
for $x \in D_{n+N-1}$ and $t \in [0,T]$, we get that $\|e^{-itk_n}v_t, x\| = \|e^{-itk_n}(e^{itk}x)\| \leq 2T\|k - k_n\||x||$.

Since $\alpha$ has the Rohlin property and $t \mapsto e^{-itk_n}v_t$ is an $\alpha$-cocycle, we obtain a unitary $w$ in $U(A)$ such that $\|w\alpha_t(w^*) - e^{-itk_n}v_t\|$ is of the order of $|t|/T$. Specifically, let us assume that $\|w\alpha_t(w^*) - e^{-itk_n}v_t\| \leq \epsilon$ for $t \in [-1, 1]$ with $\epsilon \approx 1/T$. To define such a unitary $w$ we use the path $(e^{-itk_n}v_t)_{0 \leq t \leq T}$ which connects 1 with $e^{-itk_n}v_T$, along which any element of the unit ball of $D_{n+N-1}$ almost commutes up to the order of $T\|k - k_n\|$, which insures that $\|w, x\|$ for $x \in D_{n+N-1}$ with $\epsilon'$ of the order of $T\|k - k_n\|$. Since $\|w, e^{-itk_n}\| \leq \epsilon'$, we get that

$$\|\alpha_t(Sw) - Sw e^{itk_n}\| = \|Sw_\alpha_t(w)e^{-itk_n}w^* - S\|$$

is less than $\epsilon + \epsilon'$ for $t \in [-1, 1]$.

From the construction of $w$, we should also note that $w$ is connected to 1 by a path which almost commutes with $D_{n+N-1}$. We may just as well assume that $w \in A \cap D'_{n+N-1}$ with $\max_{|t| \leq 1}\|\alpha_t(Sw) - Sw e^{itk_n}\| \leq \epsilon + 3\epsilon'$.

Since $\gamma$ has the Rohlin property and satisfies that $\gamma^\pm(D_k) \subset D_{k+1}$, we can choose a unitary $u \in A \cap (eB_2)e'$ such that $\|w - u\lambda(u^*)\|$ is of order of $1/N$ and $u$ belongs to $A \cap D_n$, where $\lambda$ is a unital homomorphism of $A \cap (eB_2)e'$ into $A \cap (eB_1)e'$ such that $\lambda(x)\gamma(e) = SxS^*$, $x \in A \cap (eB_2)e'$. (To construct $u$ we use a path of unitaries which connects 1 with $w\lambda(w)\lambda^2(w)\cdots\lambda^M(w)$ with $M \approx N$ and lies in the commutant of $D_n$. This is why we get $u$ from $A \cap D'_{n}$ while $w \in A \cap D_{n+N-1}$; see, e.g., [7].) Since $\lambda(u)S\lambda(u^*) = Su\lambda(u^*) \approx Sw$, $\alpha_t(\lambda(u)S\lambda(u^*)) \approx Sw e^{itk_n}$, and $[e^{itk_n}, \lambda(u)] = 0$, we have for $v = \lambda(u)$ that

$$\text{Ad } v^*\alpha_t\text{Ad }v(S) \approx v^*Sw e^{itk_n}v = Su^*we^{itk_n}u \approx S e^{itk_n} \approx \beta_t(S).$$

Moreover we have that for $x \in D_{n-1}$

$$\text{Ad } v^*\alpha_t\text{Ad }v(x) = \alpha_t(x) = \beta_t(x).$$

This concludes the proof. \hfill $\Box$

When $\alpha$ is a flow on $A$, $\delta_\alpha$ is the generator of $\alpha$, and $C$ is a $C^*$-subalgebra of $A$, we say that $\alpha$ is $C^{1+\epsilon}$ if $D(\delta_\alpha) \supset C$ and

$$\sup_{x \in C, \|x\| \leq 1} \| (\alpha_t - \text{id})\delta_\alpha(x) \|$$

converges to 0 as $t \to 0$.

We recall the following result from [17]: If $A$ is a unital separable nuclear purely infinite simple $C^*$-algebra, and if each of two Rohlin flows on $A$ is an approximate cocycle perturbation of the other, then they are cocycle conjugate with each other. By 3.1 and 3.9, we obtain:
Corollary 5.10 Let $A$ be a unital separable nuclear simple purely infinite $C^*$-algebra and $C = C e$ be a Cartan masa of $A$ as in Proposition 5.8. Suppose also that $\gamma$ has the Rohlin property.

Let $\alpha$ and $\beta$ be Rohlin flows on $A$ such that both $\alpha$ and $\beta$ are $C^{1+\epsilon}$ on $C$. Then $\alpha$ and $\beta$ are cocycle-conjugate.

Proof. By 5.1 we may suppose that $\alpha|C = \text{id} = \beta|C$ by inner perturbation. Then by 5.9 $\alpha$ is a cocycle perturbation of $\beta$ and vice versa. By using the result of 17 quoted above, we get the conclusion. □

Let $K$ denote the $C^*$-algebra of compact operators on an infinite-dimensional separable Hilbert space and let $K = K + C 1$. We fix a minimal projection $p$ in $K$. For a bounded interval $I$ of $\mathbb{Z}$ we define $\mathcal{K}(I) = \bigotimes_I K$ and $\mathcal{K}(I) = \bigotimes_I K \subset \mathcal{K}(I)$. When $I = [a, b]$, we embed $\mathcal{K}([a, b])$ into $\mathcal{K}([a, b + 1])$ by $x \mapsto x \otimes 1$ and $\mathcal{K}([a, b])$ into $\mathcal{K}([a - 1, b])$ by $x \mapsto p \otimes x$. Let $\hat{B}$ denote the inductive limit of the system $(\mathcal{K}(I))_I$ with these embeddings and let $B$ be the $C^*$-subalgebra of $\hat{B}$ generated by $(\mathcal{K}(I))_I$. Let $\sigma$ denote the automorphism of $\hat{B}$ induced by the shift on $\mathbb{Z}$ to right; in particular $1 \in \mathcal{K}([0])$ maps into $p \in \mathcal{K}([0])$. We will denote by $e \in B$ the projection corresponding to $1 \in \mathcal{K}([0])$ or equivalently to $p \in \mathcal{K}([0, -1])$.

We note that $\sigma$ leaves $B$ invariant and $e \in B$.

Note that $B$ is an AF algebra. It is known $[3]$ that the crossed product $B \times_{\sigma} \mathbb{Z}$ is isomorphic to $\mathcal{K} \otimes \mathcal{O}_\infty$ and that $e(B \times_{\sigma} \mathbb{Z}) e \cong \mathcal{O}_\infty$.

Let $(e_{ij})_{i,j=1}^\infty$ be a family of matrix units in $\mathcal{K}$ such that $e_{1,1} = p$ and $\mathcal{K}$ is the closed linear span of $(e_{ij})$. Let $C$ be the abelian $C^*$-subalgebra of $\mathcal{K}$ generated by those $e_{i,j}$'s and $1$. It follows that $C$ is a regular masa of $\mathcal{K}$.

Let $C_\infty$ be the $C^*$-subalgebra generated by $\mathcal{K}(I) \cap \bigotimes_I C$ with all bounded intervals $I$. It follows that $e \in C_\infty$ and that $C_\infty$ is a regular masa of $B$ and is left invariant under $\sigma$. Let $U$ denote the unitary multiplier of $B \times_{\sigma} \mathbb{Z}$ which implements $\sigma$. We let $S = U e \in e(B \times_{\sigma} \mathbb{Z}) e$, which is an isometry. For each $x \in \mathcal{K}([0])$ we denote by the same symbol $x$ the corresponding element in $B$.

Under the isomorphism $e(B \times_{\sigma} \mathbb{Z}) e \cong \mathcal{O}_\infty$ which sends $e_{k1} S$ onto $s_k$, $e_{C_\infty}$ is the abelian $C^*$-subalgebra $C_\infty$ generated by $s_i s_j^*$, $I \in \mathbb{N}^*$, where $\mathbb{N}^*$ is the set of finite sequences in $\mathbb{N}$. It is immediate that $C_\infty$ is a weak Cartan masa of $\mathcal{O}_\infty$.

Hence Corollary 5.6 is applicable to the pair $A = \mathcal{O}_\infty$ and $C_\infty = e_{C_\infty}$. Since $B$ is not simple and $\sigma$ does not have the Rohlin property, we cannot apply Lemma 5.9 to this pair $(\mathcal{O}_\infty, C_\infty)$. (We have used that $B$ is simple at the beginning of the proof, but this is required to define the unital partial endomorphism $\lambda$ of $e B e$ out of $\gamma$, whose Rohlin property we would need.) But we can use instead the fact that any unital endomorphism of $\mathcal{O}_\infty$ is approximately inner $[23]$, in the proof of 5.9.

More precisely we define a unital endomorphism $\phi$ of $\mathcal{O}_\infty$ by $\phi(s_k) = s_k w$, or $\phi(e_{k1} S) = e_{k1} S w$, for $k = 1, 2, \ldots$, where $w$ is the unitary described in the proof of 5.9. In particular $w \in \mathcal{O}_\infty \cap D_{n+1, n-1}$. Note that $\phi(e_{ij}) = \phi(s_i s_j^*) = s_i w w^* s_j^* = s_i s_j^* = e_{ij}$. For $e_{i1, j1} \otimes e_{i2, j2} \in \mathcal{O}_\infty \cap D_{n+1, n-1}$.
If $e_{i_2,j_2}$ commutes with $w$, we have that

$$\phi(e_{i_1,j_1} \otimes e_{i_2,j_2}) = \phi(s_{i_1}s_{j_2}^*s_{j_1}^*) = s_{i_1}we_{i_2,j_2}w^*s_{j_1}^* = e_{i_1,j_1} \otimes e_{i_2,j_2}.$$

In this way, we can show that if $e_k = \sum_{i=1}^k e_{ii}$ is a projection in $K$ such that $w$ is in the commutant of $D(k) = Ce + e_kKe_k + e_kKe_k \otimes e_kKe_k + \cdots + e_kKe_k \otimes e_kKe_k \otimes \cdots \otimes e_kKe_k$ ($k+1$ terms), then $\phi|D(k) = id$. Hence we can find a unitary $v \in \mathcal{O}_\infty$ such that $vSv^* \approx Sw$ and $vuv^* \approx x$ for $x \in D(k)$. Since $\bigcup_k D(k)$ is dense in $eBe$, this leads us to the same conclusion of Corollary 5.9.

Thus we have proved Corollary 5.10 just as Corollary 5.10. Since there are quite a few Rohlin flows on $\mathcal{O}_\infty$, this result is certainly non-void.

We can define a Cartan masa $\mathcal{C}_n$ of $\mathcal{O}_n$ in the same way as $\mathcal{C}_\infty$. Since Corollary 5.10 is applicable to the pair $(\mathcal{C}_n, \mathcal{C}_n)$, let us state:

**Corollary 5.11** Let $n = 2, 3, \ldots$ or $n = \infty$ and let $\mathcal{C}_n \subset \mathcal{O}_n$ be as above. Then there are Rohlin flows which are $C^{1+\epsilon}$ on $\mathcal{C}_n$ and any two Rohlin flows of this type are cocycle conjugate with each other.

Let $\lambda \in (0, 1)$ and let $G_\lambda$ be the subgroup of $\mathbb{R}$ generated by $\lambda^n$, $n \in \mathbb{Z}$. Then $G_\lambda$, as an ordered subgroup of $\mathbb{R}$, is a simple dimension group and there is a stable simple AF algebra $B_\lambda$ such that $K_0(B_\lambda) \cong G_\lambda$. Let $\gamma$ be an automorphism of $B_\lambda$ such that $\gamma$ induces the multiplication by $\lambda$ on $K_0(B_\lambda) = G_\lambda$. We may suppose that $\gamma$ leaves a canonical AF masa $C$ of $B_\lambda$ invariant. By [7] $\gamma$ has the Rohlin property and by [26] the crossed product $B_\lambda \times_\gamma \mathbb{Z}$ is purely infinite and simple. (We can get more examples of this kind from [4].)

If $\{f \in \mathbb{Z}[t] \mid f(\lambda) = 0\} = p(t)\mathbb{Z}[t]$ for some non-zero $p(t) \in \mathbb{Z}[t]$, then $A_\lambda \equiv B_\lambda \times_\gamma \mathbb{Z}$ is isomorphic to $\mathcal{O}_n \otimes K$ where $n = |p(1)| + 1$; otherwise $A_\lambda$ is isomorphic to $\mathcal{O}_\infty \otimes K$. By cutting off $A_\lambda$ by a projection $e \in C$ with $[e]$ a generator of $K_0(A_\lambda)$, we get a Cartan masa $\mathcal{C} = Ce$ in $eA_\lambda e$ which is isomorphic to $\mathcal{O}_n$ with $n$ depending on $\lambda$ as above. Thus there are many ways to construct a Cartan masa of $\mathcal{O}_n$ as in Corollary 5.9, but we do not know whether we can get a new Cartan masa (in case $n < \infty$), which is not obtained as an image of the above $\mathcal{C}_n$ by an automorphism, and if we can, whether we have a Rohlin flow which is trivial on this Cartan masa.

**References**

[1] B. Blackadar, O. Bratteli, G.A. Elliott, and A. Kumjian, Reduction of real rank in inductive limits of $C^*$-algebras, Math. Ann. 292 (1992), 111–126.

[2] O. Bratteli, Derivations, dissipations and group actions on $C^*$-algebras, Lecture Notes in Math. 1229 (1986), Springer.

[3] O. Bratteli, D.E. Evans, G.A. Elliott, and A. Kishimoto, Homotopy of a pair of approximately commuting unitaries in a simple purely infinite unital $C^*$-algebra, J. Funct. Anal.160 (1998), 466–523.
[4] O. Bratteli and A. Kishimoto, Trace scaling automorphisms of certain stable AF algebras, II, Quarterly J. Math. Oxford 51 (2000), 131-154.

[5] O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, I, Springer, 1979.

[6] J. Cuntz, Simple C*-algebras generated by isometries, Commun. Math. Phys. 57 (1977), 173–185.

[7] D.E. Evans and A. Kishimoto, Trace scaling automorphisms of certain stable AF algebras, Hokkaido Math. J. 26 (1997), 211–224.

[8] U. Haagerup, All nuclear C*-algebras are amenable, Invent. Math. 74 (1983), 305–319.

[9] E. Kirchberg and N.C. Phillips, Embedding of exact C*-algebras in the Cuntz algebra $O_2$, J. reine angew. Math. 525 (2000), 17–53.

[10] E. Kirchberg and N.C. Phillips, Embedding of continuous fields of C*-algebras in the Cuntz algebra $O_2$, J. reine angew. Math. 525 (2000), 55–94.

[11] A. Kishimoto, Simple crossed products of C*-algebras by locally compact abelian groups, Yokohama Math. J. 28 (1980), 69–85.

[12] A. Kishimoto, The Rohlin property for automorphisms of UHF algebras, J. reine angew. Math. 465 (1995), 183–196.

[13] A. Kishimoto, The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras, J. Funct. Anal. 140 (1996), 100–123.

[14] A. Kishimoto, A Rohlin property for one-parameter automorphism groups, Commun. Math. Phys. 179 (1996), 599-622.

[15] A. Kishimoto, Locally representable one-parameter automorphism groups of AF algebras and KMS states, Rep. Math. Phys. 45 (2000), 333–356.

[16] A. Kishimoto, Examples of one-parameter automorphism groups of UHF algebras, Commun. Math. Phys. 216 (2001), 395–408.

[17] A. Kishimoto, Rohlin flows on the Cuntz algebra $O_2$, Internat. J. Math. 13 (2002), 1065–1094.

[18] A. Kishimoto, Rohlin property for flows, Contemporary Math. 335 (2003), 195–207.

[19] A. Kishimoto, The one-cocycle property for shifts, to appear in Ergod. Th. & Dynam. Sys.

[20] A. Kishimoto, Central sequence algebras of a purely infinite simple C*-algebra, Canad. J. Math. 56 (2004), 1237–1258.

[21] A. Kishimoto and A. Kumijian, Crossed products of Cuntz algebras by quasi-free automorphisms, Fields Inst. Commun. 13 (1997), 173–192.
[22] A. Kishimoto, N. Ozawa, and S. Sakai, Homogeneity of the pure state space of a separable C*-algebra, Canad. Math. Bull. 46 (2003), 365–372.

[23] H. Lin and N.C. Phillips, Approximate unitary equivalence of homomorphisms from $O_\infty$, J. reine angew. Math. 464 (1995), 173–186.

[24] H. Nakamura, Aperiodic automorphisms of nuclear purely infinite simple C*-algebras, Ergod. Th. & Dynam. Sys. 20 (2000), 1749–1765.

[25] N.C. Phillips, A classification theorem for nuclear purely infinite simple C*-algebras, Doc. Math. 5 (2000), 49–114 (electronic).

[26] M. Rørdam, Classification of certain infinite simple C*-algebras, J. Funct. Anal. 131 (1995), 415–458.

[27] S. Sakai, Operator Algebras in Dynamical Systems, Cambridge Univ. Press, 1991.