On Correlation and Default Clustering in Credit Markets

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Panel A: Riskless Yield Curves

Panel B: Credit Spread Curves
Single-Name Credit Risk Pricing – What do we do?

Develop general yet tractable Markovian HJM models that

- fully incorporate information on riskless and credit spread term structures
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Develop general yet tractable Markovian HJM models that:

- fully incorporate information on riskless and credit spread term structures
- allow different volatility structures for forward rates, that can be initialized to closely match empirical structures
- credit spreads and yield curves are represented by a finite set of state variables
- allow arbitrary interest rate-credit spread correlations
- permit shocks to the economy to impact riskless yield curves and credit spreads
Multi-Name Credit Risk Pricing – What do we do?

Extend single-name Markovian HJM models to

- multi-name infection-type models
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Using Kalman filter parameter estimates, we show the importance of

- interest rate-credit spread correlations
- default contagion
- the initial credit spread curve distribution
Let $P(t, T)$ be the price at date $t$ of a pure riskless discount bond that pays $1$ at date $T$:

$$P(t, T) = e^{-\int_t^T f(t,u)du},$$

where $f(t, u)$ represents the date-$t$ forward rate for the future time increment $[u, u + dt]$. 

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We assume

$$df(t, T) = \mu_f(t, T) dt + \sigma_f(t, T) dz_f(t) + c_f(t, T) dN_f(t),$$

given $f(0, T)$. $N_f(t)$ is independent Poisson process with intensity $\eta_f$. 
Dynamics of Riskless Bond Prices

Apply Ito’s lemma for jump-diffusion processes to obtain

\[
\frac{dP(t, T)}{P(t, T)} = \left( r(t) + \frac{1}{2}\sigma_p(t, T)\sigma'_p(t, T) - \int_t^T \mu_f(t, u) du \right) dt \\
- \sigma_p(t, T) dz_f(t) + \left( e^{-K_p(t, T)} - 1 \right) dN_f(t),
\]

where

\[
\sigma_p(t, T) = \int_t^T \sigma_f(t, u) du,
\]

\[
K_p(t, T) = \int_t^T \phi(t, u) du.
\]
HJM Models: Risky Debt

For firm $A$ that has not defaulted prior to date $t$, we have

$$dY_A(t) = \begin{cases} 
1 & \text{with probability } \eta_A(X_t)dt \\
0 & \text{with probability } 1 - \eta_A(X_t)dt, 
\end{cases}$$

The date-$t$ price of a bond issued by $A$ is given by

$$\Pi_A(t, T) = \frac{1}{\tau_A > t},$$

where

$$V_A(t, T) = e^{-RT_t (f(t, u) + \lambda_A(t, u)) du} = P(t, T) S_A(t, T).$$

$\lambda_A(t) = \eta_A(t) \mathbb{A}(t)$ is firm $A$’s forward credit spread, $\eta_A(t)$ is the default arrival intensity, and $\mathbb{A}(t)$ denotes LGD.
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Credit Spreads Dynamics

We assume

\[ d\lambda_A(t, T) = \mu_A(t, T) \, dt + \sigma_A(t, T) \, dz_A(t) + c_{fA}(t, T) \, dN_f(t), \quad t \leq \tau_A, \]

where

- correlation with diffusive riskless term structure:
  \[ E(dz_f(t)dz_A'(t)) = \sum_{m \times n} dt = (\rho_{ij}^A) \, dt \]
- a jump in riskless rates could transmit to shocks in the credit spreads
- \( \sigma_A(t, T) \) is predictable
- \( c_{fA}(t, T) \) is a deterministic function of time to maturity, \( T - t \)
Proposition 1: HJM Restrictions on the Drift Terms

No arbitrage implies

\[
\mu_f(t, T) = \sigma_p(t, T)\sigma'_f(t, T) - c_f(t, T)e^{-K_p(t,T)}\eta_f
\]

\[
\mu_A(t, T) = \sigma_{S_A}(t, T)\sigma'_{A}(t, T) + c_f(t, T)\Sigma^A\sigma'_{S_A}(t, T)
\]

\[
+ \sigma_p(t, T)\Sigma^A\sigma'_{A}(t, T) + g_A(t, T),
\]

where

\[
g_A(t, T) = \eta_f \left( c_f(t, T)e^{-K_p(t,T)} - (c_f(t, T) + c_{fA}(t, T))e^{-(K_p(t,T)+K_{fA}(t,T))} \right).
\]
Proposition 1: HJM Restrictions on the Drift Terms

No arbitrage implies

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\begin{align*}
\mu_f(t, T) &= \sigma_p(t, T)\sigma_f'(t, T) - c_f(t, T)e^{-K_p(t, T)}\eta_f \\
\mu_A(t, T) &= \sigma_{S_A}(t, T)\sigma_A'(t, T) + \sigma_f(t, T)\Sigma^A_s\sigma_A'(t, T) \\
&\quad + \sigma_p(t, T)\Sigma^A_s\sigma_A'(t, T) + g_A(t, T),
\end{align*}
\]

where

\[
g_A(t, T) = \eta_f \left( c_f(t, T)e^{-K_p(t, T)} - (c_f(t, T) + c_{fA}(t, T))e^{-(K_p(t, T) + K_{fA}(t, T))} \right).
\]

Problem with HJM models:

- In general, the dynamics are not Markovian in a small number of state variables
- To overcome this issue, we curtail the volatility structures
**Volatilities** are given by

\[
\sigma_{f_i}(t, T) = h_{f_i}(t) e^{-\kappa_{f_i}(T-t)},
\]
\[
\sigma_{A_j}(t, T) = h_{A_j}(t) e^{-\kappa_{A_j}(T-t)}.
\]

where \( h_{f_j}(t) \) and \( h_{A_j}(t) \) are predictable functions.
Markovian HJM: Volatilities and Jump-Impact Factors

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- Example: \(h_f(t) = \min\left( |\tilde{h}_f(t)|, \bar{h}_f \right)\), where \(\bar{h}_f\) is a large yet finite constant, and

  \[
  \tilde{h}_f(t) = \sigma_f r(t)
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  \[
  \tilde{h}_f(t) = \sigma_f r(t)
  \]
  
- **Jump-impact factors** are of the form
  \[
  c_{f}(t, T) = c_f e^{-\gamma_f(T-t)}, \\
  c_{fA}(t, T) = c_{fA} e^{-\gamma_{fA}(T-t)}
  \]
Proposition 2: Markovian Models for Riskless Debt

Under volatility restrictions, we get exponential affine riskless bond prices:

\[
P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( - \sum_{i=1}^{2} \sum_{j=1}^{m} H_{ij}(t, T) \psi_{ij}(t) - H_{3}(t, T) \psi_{3}(t) + H_{J}(t, T) \right),
\]

where

\[
H_{1j}(t, T) = \frac{1}{\kappa f_j^2} \left( 1 - e^{-\kappa f_j(T-t)} \right), \quad \text{for } j = 1, \ldots, m
\]

\[
H_{2j}(t, T) = -\frac{1}{(2\kappa f_j^2)} \left( 1 - e^{-2\kappa f_j(T-t)} \right), \quad \text{for } j = 1, \ldots, m
\]

\[
H_{3}(t, T) = \frac{c_f}{\gamma f} \left( 1 - e^{-\gamma f(T-t)} \right),
\]

\[
H_{J}(t, T) = \eta_f e^{-\frac{c_f}{\gamma f}} \int_{t}^{T} \left( e^{\frac{c_f}{\gamma f} e^{-\gamma f(u-t)}} - e^{\frac{c_f}{\gamma f} e^{-\gamma f u}} \right) du.
\]

The dynamics of the state variables are

\[
d\psi_{1j}(t) = (h_{lj}^2(t) - \kappa f_j \psi_{1j}(t)) dt + \kappa f_j h_{lj}(t) d\mathcal{Z}_{lj}(t)
\]

\[
d\psi_{2j}(t) = (h_{lj}^2(t) - 2\kappa f_j \psi_{2j}(t)) dt
\]

\[
d\psi_{3}(t) = -\gamma f \psi_{3}(t) dt + d\mathcal{N}_{f}(t).
\]
Proposition 2: Markovian Models for Risky Debt

Under volatility restrictions, and assuming RMV, the risky bond price at \( t \) is
\[
\Pi_A(t, T) = P(t, T)S_A(t, T)1_{\tau_A > t},
\]
where \( S_A(t, T) \) is exponential affine:
\[
S_A(t, T) = \frac{S_A(0, T)}{S_A(0, t)} \exp[-A_0(t, T) - \sum_{j=1}^{n} (K_{0,j}(t, T)\xi_{0,j} - K_{1,j}(t, T)\xi_{1,j})
+ \sum_{i=1}^{m} \sum_{j=1}^{n} (K_{2,ij}(t, T)\xi_{2,ij} - K_{3,ij}(t, T)\xi_{3,ij} - K_{4,ij}(t, T)\xi_{4,ij})
- K_{5}(t, T)\xi_{5}(t)].
\]

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\[
\begin{align*}
    d\xi_{0,j}(t) &= (h_{A_j}^2(t) - \kappa_{A_j}\xi_{0,j}(t)) \, dt + \kappa_{A_j}h_{A_j}(t) \, dZ_{A_j}(t) \\
    d\xi_{1,j}(t) &= (h_{A_j}^2(t) - 2\kappa_{A_j}\xi_{1,j}(t)) \, dt \\
    d\xi_{2,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - (\kappa_{A_j} + \kappa_{f_i})\xi_{2,ij}(t)) \, dt \\
    d\xi_{3,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - \kappa_{f_i}\xi_{3,ij}(t)) \, dt \\
    d\xi_{4,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - \kappa_{A_j}\xi_{4,ij}(t)) \, dt \\
    d\xi_{5}(t) &= -\gamma_{fA}\xi_{5}(t) + dN_f(t).
\end{align*}
\]
Stochastic Drivers vs State Variables

- Assume forward rates are driven by \( m \) **stochastic drivers**, and credit spreads by \( n \)
  - Computational burden is limited to that of \((m + n)\) -dim affine models with jumps
Stochastic Drivers vs State Variables

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  - Computational burden is limited to that of $(m + n)$ -dim affine models with jumps

- Number of state variables: $3mn + 2(m + n + 1)$
Stochastic Drivers vs State Variables

- Assume forward rates are driven by \( m \) stochastic drivers, and credit spreads by \( n \)
  - Computational burden is limited to that of \((m + n)\) -dim affine models with jumps

- Number of state variables: \( 3mn + 2(m + n + 1) \)

- Number of state variables can sometimes be reduced:
  - \( m = n = 1 \): 8
  - \( m = n = 1 \), no jumps and constant \( h(\cdot) \) functions: 2
  - \( m = n = 1 \), no jumps and no correlations between interest rates and credit spreads: 4
What’s the Big Deal?

Consider a HJM model with $m = n = 1$, and no jumps. Assume a 30-year time horizon.
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Standard HJM:

- Forward rates of $30 \times 12 = 360$ monthly interest rates and credit spreads need to be tracked.
- As such, the model is Markovian in 720 state variables.
- If the time partitions are refined to weeks, the number of state variables increases to 2,880.
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Markovian HJM:
- A maximum of 8 state variables need to be maintained, no matter what the partition.
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- Proposition 2 can be generalized to enable humped volatility structures even when \(m = n = 1\).
Are the Volatility Restrictions Severe?

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- Proposition 2 can be generalized to enable humped volatility structures even when $m = n = 1$.

- In fact, we are able to establish arbitrary shapes:

$$\sigma_f(t, T) = h_f(t) \sum_{j=1}^{k} a_j e^{-\kappa_j(T-t)}, \quad k > 1.$$
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The drift terms of the path statistics offset spot rate volatilities in a manner that allows bond yields to be affine in the states, even though the state variables themselves do not have to be affine processes.

As a result, the family of models we have established are very rich in structure, yet are easy to implement. In that sense, our analysis complements Duffie-Kan '96.
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Relationship with Duffie-Kan Affine Models

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Empirical Evidence: Using Kalman Filter

1yr Treasury yield

5yr Treasury yield

AMR: 1yr credit spread

AMR: 5yr credit spread

Lennar: 1yr credit spread

Lennar: 5yr credit spread
Importance of Interest Rate-Credit Spread Correlations

Bond options

- K=1.05
- K=1
- K=0.95

percent

\( \rho^A \)

\(-1\) \(-0.5\) \(0\) \(0.5\) \(1\)
The model already allows market-wide events to affect the riskless yield curve and credit spread curves.
Systemic Credit Shocks and Default Clustering

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- We now allow a primary firm’s default to impact the credit spread of surviving (secondary) firms.
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- We now allow a primary firm’s default to impact the credit spread of surviving (secondary) firms.

Examples: Secondary firm

- could carry significant debt of the primary firm,
- may sell much of its goods to a primary firm,
- may be in competition with the primary firm.
Credit Spread Dynamics: Secondary Firms

We assume

\[ d\lambda_B(t, T) = \mu_B(t, T)dt + \sigma_B(t, T)dz_B(t) + c_{fB}(t, T)dN_f(t) \]

\[ + \sum_{i=1}^{m_B} c_{AiB}(1 - Y_{Ai}(t))dY_{Ai}(t), \forall t \leq \tau_B, \]

where

- correlation with diffusive riskless term structure:
  \[ E(dz_f(t)dz'_B(t)) = \Sigma^B_{m\times n} dt = \left(\rho^B_{ij}\right) dt \]

- correlation with firm A’s diffusive term:
  \[ E(dz_A(t)dz'_B(t)) = \Sigma^{AB}_{n\times n} dt = \left(\rho^{AB}_{ij}\right) dt \]

- volatility structures are curtailed
Proposition 3: Pricing Risky Debt of Secondary Firms

The price of a risky bond issued by a secondary firm is given by
\[ \Pi_B(t, T) = V_B(t, T)1_{\tau_B > t}, \]
where \( V_B(t, T) = P(t, T)S_B(t, T) \) and
\[
S_B(t, T) = \frac{S_B(0, T)}{S_B(0, t)} e^{-B_0(t, T) - \sum_{j=1}^{n} (K_{0,j}(t, T)\xi_{0,j} - K_{1,j}(t, T)\xi_{1,j})}
\times e^{\sum_{i=1}^{m} \sum_{j=1}^{n} (K_{2,ij}(t, T)\xi_{2,ij} - K_{3,ij}(t, T)\xi_{3,ij} - K_{4,ij}(t, T)\xi_{4,ij})}
\times e^{-K_{5}B(t, T)\xi_{5}(t)}
\times e^{\sum_{i=1}^{m_B} \left(1 - e^{-c_{A_iB}(T-t)}\right) U_{A_iB}(t) - c_{A_iB}(T-t) Y_{A_i}(t)}.
\]

Here, \( B_0(t, T) = \int_{t}^{T} \int_{0}^{t} g_B(v, u) dv du \). The \( K^B \) coefficients and \( \xi^B \) state variables are defined as in Proposition 2, and
\[
U_{A_iB}(t) = \int_{0}^{t\wedge \tau_{A_i}} \eta_{A_i}(u)e^{-c_{A_iB}(t-u)} du, \quad \text{for } i = 1, \ldots, m_B.
\]
Importance of Default Contagion: Counterparty Risk in Insurance Contracts

\[ \rho^A = \rho^B \]

\[ \rho^{AB} \]

\[ c^{AB} \]

\[ \eta_f \]
Importance of Default Contagion: CDS Index Tranches

10–15

15–25

25–35
Generating Default Clustering

- \( c_{AB} = 0 \)
- \( c_{AB} = 0.05 \)
- \( c_{AB} = 0.1 \)
# Importance of the Initial Credit Spread Curve Distr.

| Distribution of initial credit spread curves | Tranche spreads |
|---------------------------------------------|-----------------|
| $\lambda(0, t) = 0.05$                      | 3451 1450 111   |
| $\lambda(0, 0) \sim \text{Uniform}(0.025, 0.075)$ and $\lambda(0, t) = \lambda(0, 0)$ | 3523 1528 128  |
| $\lambda(0, 0) \sim \text{Uniform}(0, 0.1)$ and $\lambda(0, t) = \lambda(0, 0)$ | 3528 1528 121  |
| $\lambda(0, t)$ incr from $\lambda(0, t) = 0.0125$ to $\lambda(0, t) = 0.0875$ | 2698 1340 107  |
| $\lambda(0, t)$ incr from $\lambda(0, t) = 0.025$ to $\lambda(0, t) = 0.075$ | 2883 1366 108  |
| $\lambda(0, t)$ incr from $\lambda(0, t) = 0.0375$ to $\lambda(0, t) = 0.0625$ | 3130 1405 110  |
| $\lambda(0, t) = 0.05$                      | 3451 1450 111   |
| $\lambda(0, t)$ decr from $\lambda(0, t) = 0.0625$ to $\lambda(0, t) = 0.0375$ | 3857 1514 115  |
| $\lambda(0, t)$ decr from $\lambda(0, t) = 0.075$ to $\lambda(0, t) = 0.025$ | 4345 1601 120  |
| $\lambda(0, t)$ decr from $\lambda(0, t) = 0.0875$ to $\lambda(0, t) = 0.0125$ | 4947 1772 141  |
Summary

- Develop a family of models for pricing interest and credit derivatives on single and multiple names
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- Fairly easy to implement

- The variance structures need not be affine
- The number of state variables is decoupled from the number of stochastic drivers
- Allow flexible specification of correlations between interest rates and credit spreads
- Permit default clustering through a variety of channels (diffusive correlations, jumps, contagion effects)
Summary

- Develop a family of models for pricing interest and credit derivatives on single and multiple names
- Fairly easy to implement
- Models have exponentially affine representations for riskless and risky bond prices
  - Yet the variance structures need not be affine
  - The number of state variables is decoupled from the number of stochastic drivers
  - Allow flexible specification of correlations between interest rates and credit spreads
  - Permit default clustering through a variety of channels (diffusive correlations, jumps, contagion effects)