On the physical limit of quantum computing

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Abstract

Experimental attempts to implement quantum speedup of computations over the past 30 years have yielded a negative result, despite the lack of physical laws prohibiting such speedup. The article formulates the restriction of quantum formalism in the form of uncertainty "complexity of the system - the accuracy of its description at the quantum level", and argues in favor of its physical status. An experiment to determine the constant of this relation through Grover’s algorithm is described. This relation directly prohibits the existence of fast quantum algorithms, but allows the simulation of the dynamics of real systems on a quantum computer, the only advantage of which is the use of quantum nonlocality.

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1 Introduction

The analysis of infinitesimals is a "sacred cow" of exact natural science, despite the fact that the notion of the limit itself is a simplification of the real picture. For example, any continuous medium consists of particles of finite size, so the traditional description of such media through differential equations is already an assumption. Hypnosis of infinitesimals was overcome for the first time in 200 years after Newton by quantum mechanics, which postulated quantization of the action as a Planck constant $\hbar \approx 10^{-27}$ erg sec, as well as quantization of the energies of the stationary states of atoms.

This great physical discipline turned out to be closely connected with the science of computing - through the quantum computer project, so that the discrete ideology of computing actually began to play the role of the mathematical apparatus of the quantum theory of complex systems\(^1\). In this paper, we formulate a restriction on quantum computations in terms of complexity theory based on the almost 30-year history of quantum computer experiments, as well as an experimental method for determining the parameters of this restriction - as constants in the uncertainty relation "complexity of the system - the accuracy of its description at the quantum level". This explicitly prohibits fast quantum algorithms, but leaves open the development of a quantum computer as a device simulating the reality.

Quantum mechanics, as a universal means of accurate description of Nature has a single, but deadly drawback - algorithmic complexity. The memory of the computer simulating quantum evolutions generally grows as an exponent of the number of real particles in the system under consideration, which makes such modeling impossible for relatively simple systems like molecules from several atoms. It is this computational complexity that was the reason for the appearance of the quantum computer project, a new look at which this article presents.

At the physical level, any computation is, by its nature, quantum, and therefore the analysis of the quantum computer is an analysis of our capabilities of computer simulation of reality as a whole. At the end of the 20-th century, a tempting way to solve the problem of modeling complex systems was proposed: a quantum computer (see the pioneering work of R. Feynman [1]), and it was proved that such modeling is possible on it, and with memory growing linearly with the increase of the number of real particles in the system, however, with a quadratic time dilation compared to the real process (see the works [2]). Almost simultaneously it was discovered an amazing property of quantum computations: their ability to solve computational tasks with the option of a "black box" faster than any classical computer ([8], [4]), and was also proved that the quantum speedup holds only for a very limited range of problems ([7], [6], [7]), in particular, hopes to speed up with it the general GMSP (generic machine simulation problem - see the definition in [8]) are vain ([9]).

Experiments on the practical construction of a quantum computer, rapidly started

\(^1\)The opposite can also be true: the deep open mathematical problems of the theory of algorithms like $P = NP(?)$ can apparently be satisfactorily resolved only by involving physics.
from the early 90-ies of the last century, and continue now. They did not give confidence in the scaling of the Feynman quantum computer in its abstract-radical form, assuming the physical existence of a Hilbert space with the number of dimensions $2^n$, where $n$ is in the hundreds, whereas this hypothesis, shared and protected by most physicists, is the basis of fast quantum algorithms.

A brief history of these experiments is as follows. The work began with NMR structures (one of the last works in this field [10]), for the main reason: the life of states of nuclear spins is hours, which is very important for fragile quantum states; there were modifications - a combination of charged states of electrons with spin states (see, for example, [11]). Quantum superconducting elements (Josephson junctions and squids: [12], [13], [14]) are attractive for their mesoscopic size. Photonic gates (for example, [15], [16]) have been intensively developed in the last 10 years - here is their advantage: the proximity of the instrument park to the problems of quantum cryptography. Hybrid computations: cold ions in Paul traps ([22]), optical-mechanical systems (e.g. [17]). Topological-type computational systems, such as the one on anions ([18]) occurring in subspaces, a-priori free of decoherence, either rely on the types of particles not yet studied, or face difficulties in organizing operations, which are the greater the stronger is the protection against decoherence.

There is also a mathematical method of struggle with decoherence - quantum error correction codes (one of the pioneering works [19]). These codes start with only a few thousand qubits; their application requires a good work of quantum computer with so many qubits without correction codes, and just this barrier may fall into the critical domain of the restriction on formalism itself, which is discussed below.

It can be concluded that decoherence - spontaneous destruction of quantum states necessary for the regulated operation of a quantum computer, is not a technical problem. The traditional interpretation of decoherence as a "harmful influence of the environment" (see the concept of an open quantum system [20]) can hardly be considered as satisfactory, since this phenomenon invariably has a destructive effect on quantum computing regardless of technology. In addition, in quantum mechanics itself, the "influence of the environment" is a methodologically dark place, since the most harmful form of this influence, the so-called measurement or observation, requires an observer - an object that obeys classical laws and cannot be considered quantingly.

The sensitivity to decoherence is a special property of the states of quantum systems, which does not coincide with the degree of their entanglement. Thousands of atoms may be in entangled states (e.g., acoustic vibrations of atoms in a diamond - see [21], [22]), but these states do not provide fast algorithms. The states that are required for quantum speedup are surprisingly sensitive to decoherence.

These arguments, along with the apparent change that quantum computing brings to the theory of algorithmic complexity, suggest a deep internal connection between the algorithmic description of nature and quantum mechanics. In particular, they raise the suspicion that the algorithmic description of real evolutions, with all its limitations, can be an independent form of physical laws.
In quantum mechanics, there is already a thesis that the form of description of reality is a property of reality itself. This revolutionary thesis was confirmed by the fundamental Bohr - Heisenberg uncertainty relation: \( \delta x \delta p = \hbar \), according to which the better the momentum can be known, the worse the coordinate could be known and vice versa, as well as the similar energy-time relation.

If we assume that there is a law prohibiting fast quantum algorithms, its simplest form is the uncertainty relation "complexity of the system under consideration - accuracy of its quantum description". We argue in favor of such a relation, and show how one can experimentally determine a constant in this relation analogous to the Planck constant \( \hbar \). This experiment is based on the well-known Grover algorithm for solving the NP-complete search problem ([4]), and it can actually be carried out on the available prototypes of bounded quantum computers if its number of basic states does not exceed this constant.

2 Feynman quantum computer and quantum computing

Complex systems are associated with the concept of information. Its syntax is the number of bits; it is given by the Holevo bound for the throughput of quantum communication channel ([23]). Semantics, in the simplest case, is a Boolean function \( f \) from \( n \) variables whose values are zero or one. Semantic analysis, therefore, is the search for the roots of the equation

\[
  f(x) = 1. \tag{2.1}
\]

Just this computational task can be sped up on a hypothetical quantum computer if it existed - this is the quantum Grover algorithm ([4]). Speedup of the classical search in solving this problem requires quantum entanglement, that is, the use of states \( |\Psi\rangle \), not representable as a tensor product of states of the components of the composite system: \( |\Psi\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle \), since without entanglement the quantum computer will be equivalent to the classical one.

2.1 General scheme of quantum computer

A quantum computer is a composite device consisting of two parts: a classical one, which controls the work, and a quantum-working one, which performs the calculation itself. A state of the quantum part is a superposition

\[
  |\Psi\rangle = \sum_{j \in J} \lambda_j |j\rangle \tag{2.2}
\]

of the basic states \( |j\rangle \), each of which we identify with a string of zeros and ones - values of the corresponding qubits. The state of the classical part of the computer
indicates what action should be performed on the quantum part at a given time. For the gate, Feynman model, this can be the operation NOT: $|x\rangle \rightarrow |x \oplus 1\rangle$ over a certain qubit\(^2\), or the operation CNOT: $|x, y\rangle \rightarrow |x, x \oplus y\rangle$ over two defined qubits. For an equivalent, adiabatic model (see, for example, [24]), the action can be, for example, the establishment of a certain potential $V(0)$ supplied to the contacts at the initial time, and the law of its slow change $V(t)$ together with a finite time value $t_{fin}$. In any case, the classical part uniquely defines a certain control action on the quantum part, which causes the corresponding unitary operation on the qubit system of the quantum part of the computer.

If we set a certain sequence of states of the classical part of the computer, called a quantum algorithm, then the implementation of this algorithm on any physical realization of the quantum part will result in a sequence of unitary operations on the state of the quantum part: $|\psi_0\rangle \rightarrow |\psi_1\rangle \rightarrow ... \rightarrow |\psi_t\rangle$, called quantum computation. Its number of steps $t$ is also determined by the classical part of the computer in advance. At the end, the final state $|\psi_t\rangle$ is measured, and the resulting basic state is the result of the computation.

If there is a simple procedure to encode the classical state $x$ (a set of bits) to the initial state $|\psi_0\rangle = |\psi_0(x)\rangle$ of a quantum computer, and the result of the computation starting with such a state for all $x$ with a certain probability $p_0 \approx 1$ coincides with the value of some function $F(x) = y$, then this quantum algorithm computes the function $f$ with the probability $p_0$. The value $t = t(x)$ is then called the complexity of the computation on this word $x$, and the function $s(n) = max_{|x|\leq n} t(n)$ is called the complexity of the quantum algorithm, where the maximum time is taken for all Boolean words of length not greater than $n$. This computational scheme is called absolute, since it does not use any external information device.

If we use an external device that implements the unitary operator $A$ during the computation, we obtain a quantum computation with the quantum oracle $A$. At certain moments, the algorithm applies operator $A$ to a certain set of qubits instead of usual quantum gates. Then the computation continues normally until the next time when again it will be necessary to apply $A$, etc. In this case, the complexity refers not to the number $t$ of all the operations, but only the number of oracle queries - applications of $A$. The oracle is usually implemented by a much more complex device than the computer itself; it may be of biological origin, for example, it may be a request to a human, etc.

Quantum computing is an information procedure of a special kind. It can, of course, be modeled on a classic supercomputer, but it will require memory of at least $2^n$, where $n$ is the number of all qubits in the quantum part. If $n < 40$, this simulation will be, at least in principle, possible with the use of modern supercomputers; at $n > 200$ this will be impossible because the current state vector is too large; the impossibility of a more clever classical algorithm follows from the existence of so-called fast quantum algorithms.

\(^2\)Here and further it is assumed that on the superposition of the basic states any operator extends by linearity.
2.2 Fast quantum computations and its control function

If some quantum algorithm of complexity $s_{\text{quant}}(n)$ computes a function $y = f(x)$, which minimal classical complexity $s_{\text{class}}(n) > s_{\text{quant}}(n)$ exceeds the quantum one, then there is a case of quantum speedup. Suppose that the classical computation uses the oracle $F$, and the quantum one uses its quantum counterpart $F_q : |x, y\rangle \rightarrow |x, y \oplus F(x)\rangle$. This counterpart implements quantum parallelism: it is actually able to extract from the function $F$ more information than the classical oracle, because the classical oracle calls the function $F$ only on one value of its argument, whereas the quantum counterpart calls it on all values $x$, on each - with the corresponding amplitude in the quantum superposition.

A fast quantum computation leads to the goal by concentrating the amplitudes $\lambda_j$ of all base states $|j\rangle$ in (2.2) on the target state $|x_t\rangle$. This process can be made much faster than the classical computation of $x_t$ provided that the set of $J$ from which the amplitude is collected has an exponentially large size that is not achievable for any classical computer.

We illustrate the amplitude pumping by the example of Grover’s algorithm (11) for solving the equation (2.1). Here the action of oracle $f_q : |x, y\rangle \rightarrow |x, y \oplus f(x)\rangle$ on the state $|x\rangle|0\rangle$, where $|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ gives the same state if $x$ is not the solution of (2.1), and changes the sign if $x$ is the solution of (2.1). Quantum oracle thus acts as the mirror reflection $I_{x_t}$ relative to the subspace orthogonal to the solution $|x_t\rangle$ of (2.1) (we here assume that the solution is unique).

It is not difficult to build the ancillary operator $I_{\tilde{O}}$, reflecting the space relative to the subspace orthogonal to the state $|\tilde{O}\rangle = \sum_{j=0}^{N-1} |j\rangle$, where $N = 2^n$. Having these two reflection we build the operator $G = -I_{\tilde{O}}I_{x_t}$, which on the two dimensional real space spanned by vectors $|x_t\rangle$ and $|\tilde{O}\rangle$ acts as the turn to the angle $2 \arcsin \frac{1}{\sqrt{N}}$, and thus, applying this operator $[\pi \sqrt{N}/4]$ times, starting from $|\tilde{O}\rangle$, we obtain $|x_t\rangle$ with very high probability. It is readily seen that the complexity of this algorithm has the order of square root of the classical complexity $N$: $t_{\text{quant}} = O(\sqrt{t_{\text{class}}})$. This is quantum amplitude amplification.

Quantum speedup of classical computations is a rare phenomenon. It should be considered, first of all, as a criterion for building a real quantum computer, and distinction it from all its possible ersatz. In Copenhagen quantum mechanics there is no prohibition on its existence (see [25]), and therefore its construction is a radical test of quantum theory in the area where it has never been tested - in the field of complex systems. Here, the only form of verification will be the implementation of a fast quantum algorithm (for example, Grover algorithm), and if this does not work, it is necessary to introduce a restriction on operations in the Hilbert space of states in quantum theory itself.

\[3\] Square brackets here denote the integer part of the number.
2.3 Quantum modeling

The initial task of the quantum computer was to simulate the quantum dynamics of a real multiparticle system. We briefly present a quantum algorithm proposed in \cite{2} that simulates the evolution of the real state vector of a system of \( n \) particles in the qubit approximation on a quantum computer with memory \( O(n) \) in the time \( O(t^2) \), where \( t \) is the natural number of steps of real physical time. Classical space and time can be divided into a finite number of small areas so that in their qubit representation, the wave function simulation will converge to the real wave function of the system as the partition value tends to zero. This assumption is usually accepted in the theory of continuous media, where it is called the condition of continuity.

Unitary dynamics for the Hamiltonian \( H \) has the form \( U_t = \exp(-\frac{i}{\hbar}Ht) \). If we present the Hamiltonian as the sum of kinetic and potential energy, for one one-dimensional particle we will have \( H = E_{\text{kin}} + V, \ E_{\text{kin}} = \frac{p^2}{2m}, \ p = \frac{i}{\hbar}\nabla \). For small \( dt \) we can separately model parts of the unitary evolution operator for kinetic and potential energy: \( U_{\text{kin}} = \exp(-\frac{i}{\hbar}E_{\text{kin}}dt) \) and \( U_{\text{pot}} = \exp(-\frac{i}{\hbar}Vdt) \). Indeed, the potential energy operator, as it is easy to see, is diagonal, and therefore with a simple form \( V \) it can be represented as a result of the quantum algorithm, which is built on the expression for the potential \( V \). And the kinetic energy operator is reduced to the diagonal form by the transition to the impulse basis of the state space, which is carried out by the Fourier transform. P. Shore in \cite{3} constructed a quantum algorithm that implements the discrete Fourier transform, applying which we can just as easily model the kinetic energy operator and its exponent.

Now using the Trotter formula

\[
\exp \left( -\frac{i}{\hbar} (E_{\text{kin}} + V)t \right) \approx \left( \exp\left( -\frac{i}{\hbar} E_{\text{kin}} dt \right) \exp\left( -\frac{i}{\hbar} V dt \right) \right)^{t/dt}
\]

true to an accuracy of \( dt^2 \), we obtain that the sequential application of our quantum algorithms for kinetic and potential parts gives the required model of the real evolution. Such a simulation will have a quadratic slow down compared to real physical time\footnote{Despite that the time here is considered in the steps of evolution, thus being a natural number, the slowdown is quite real, physical.}, but the memory required for the case of \( n \) particles in this method will grow as \( n \), not as \( \exp(n) \) like in the case of classical modeling. Thus, the problem of modeling real evolutions on a quantum computer also has no restrictions in the framework of Copenhagen mechanics. Moreover, this result suggests that quantum computing is an equally powerful representation of any quantum evolution in general.

The principal difference between Grover and Zalka-Wiesner algorithms is that the verification of the correct operation of the first algorithm is rigidly connected with the exponentially large dimension of the state space: in the case of any restriction of this space, it simply will not work. While Zalka-Wiesner algorithm can work in the case of some, a-priori unknown limitation of the state space, since this restriction...
will be applied not only to the algorithm itself, but also to the real evolution modeled by it, and therefore the coincidence of the simulation results with the experiment (with the necessary statistical processing arising from the probabilistic nature of the modeled wave function) may well occur.

We thus have to distinguish Feynman quantum computer designed to simulate reality, which allows the same limitations on Hilbert space of states as the actual system from a mathematical abstraction - a radical quantum computer, which computation takes place in the abstract Hilbert space that allows quantum speedup of classical computations.

Success in building Feynman computer for the modeling is a real and technologically achievable goal. While the implementation of fast quantum algorithms, in light of the history of experiments in this field, is unlikely. Therefore, experiments on fast quantum algorithms should be interpreted as a search for the limits of applicability of the standard apparatus of Hilbert spaces of states and as a test of possible restrictions on these spaces. Below we show the most natural of these restrictions, in the form of uncertainty "complexity of the quantum system state - the accuracy of its description".

3 Constructive collapse of the wave function of many-particle systems

We accept the possibilities of modeling real evolutions on a quantum computer, keeping in mind only its only advantage over the classical one, established in confirmed physical experiments: quantum nonlocality. Such an abstract quantum computer simulating reality will have states $|j\rangle$ in the superposition

$$|\Psi\rangle = \sum_{j \in J} \lambda_j |j\rangle,$$

(3.1)

which may include spatially distant qubits, but the set $J$ must have the power available to classical memory, that is, not growing exponentially with the number of real particles in the system under consideration.

This requirement follows not from the obvious dramatic difficulties of the physical implementations of fast quantum algorithms, but from the probabilistic interpretation of the amplitudes $\lambda_j$: $|\lambda_j|^2$ is the probability to find the ensemble in a state $|j\rangle$ in the result of the measurement of its state (3.1). The restriction on $J$ is dictated simply by the possibility to collect the statistics sufficient to verify this state in the process of quantum tomography.

Standard quantum theory allows any arbitrarily small grinding of amplitudes $\lambda_j$ that is again impossible to verify otherwise than to build a fast quantum algorithm - where the (infinitely) small amplitudes are summed in enormous quantities, giving the observed result. Thus, the verification of the standard theory for complex
systems lies only through fast quantum algorithms. And if they persistently fail to build, we must consider an alternative.

The alternative is to quantize the amplitude, according to the logic of constructive mathematics (see Appendix 1). Let there be a quantum of amplitude $\epsilon$, such that if the amplitude becomes smaller than it, it simply turns to zero, and the amplitudes of all other states increase proportionally to preserve the normalization of state. If we make the usual measurement of the wave function $|\Psi\rangle$, it will be equivalent to the above reduction of small amplitudes (see Appendix 2).

So, if Hilbert formalism yields the state (3.1), where the summation extends to an exponentially large set $J$ of various basic states $|j\rangle$ of the system, then our algorithmic approach involves truncating this series to a limited sum, which is obtained if we drop all terms with coefficients $\lambda_j$, which modules are less than some pre-known threshold value $\epsilon$. Such sum should contain no more than $1/\epsilon^2$ terms. Let $N$ be the number of basic states for one particle. Then we can assume that $\epsilon = \frac{1}{\sqrt{N}}$. Thus, for the state (3.1), we have $J = \{0, 1, ..., N - 1\}$, $N = 2^n$ and some $\lambda_j$ can be zero.

Given a state $|\Psi\rangle$ we call its reduction the elimination of all terms in the state record whose module of amplitude are less than $\epsilon$, and we consider physically acceptable only those states that are obtained as a result of the reduction. We call the constant $\epsilon$ the amplitude quantum.

4 On the size of the amplitude quantum

The natural question is: what is the size of the amplitude quantum has no exact answer, just as the question of the limit of the accuracy of the particle coordinate measurement. On the one hand, it is known that for very simple systems, for example, for an electron in a hydrogen atom, $\epsilon \approx 0$ that is, this quantum is negligible and does not play any role: the wave functions of stationary states are determined very accurately. Here a huge number of equally "prepared" hydrogen atoms, which can be measured in parallel plays a role, so that the collected statistics will perfectly match the analytically found solution.

One could suppose that the amplitude quantum and the number of particles in the system under consideration are inversely proportional or something like that, but it would also not be true. A set of interacting with each other harmonic oscillators can be arbitrarily large, but the wave function of such a system can be determined very accurately, since its basic states can be overridden by a quasi-partial representation, so that the resulting quasi-particles will not interact with each other, and therefore the accuracy of the wave function of the entire system will coincide with the accuracy of its definition for a single harmonic oscillator, that is, can be arbitrarily large.

\[5\text{Similar arguments are given in [26], but we get the Born rule from the general algorithmic concept without additional assumptions.}\]
Let $M$ be the set of qubits of the considered system consisting of $n$ qubits. We call the state $|\Psi\rangle$ composite with respect to $M$ (M-composite) if there is such a partition of $M = M_1 \cup M_2$ into two disjoint sets and the states $|\Psi_1\rangle$, $|\Psi_2\rangle$ on these sets, such that $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$. Not composite states are called $M$-solid. The complexity of the state $|\Psi\rangle$ on the set $M$ is the maximum of the natural numbers $s$, such that there is a subset $M_1 \subseteq M$ and the states $|\Psi_1\rangle$, $|\Psi_2\rangle$ on $M_1$ and $M - M_1$ respectively, such that $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$, $M_1$ contains $s$ elements and $|\Psi_1\rangle$ is $M_1$-solid. This state $|\Psi_1\rangle$ is called the quantum kernel of the state $|\Psi\rangle$, and the corresponding set $M_1$ is the kernel carrier.

In other words, complexity is the maximal number of qubits in an entangled tensor divisor of a given state; this divisor will be the quantum kernel. There can be several kernels, since the maximal number $s$ from the definition can correspond to different sets of $M_1$ qubits. Naturally, this definition may depend on very small amplitudes, so that the complex state may be very close to simple. However, if we consider only state, which amplitudes $\lambda_j$ have the form

$$\lambda_j = (k_j + il_j)\epsilon,$$

where $k_j$, $l_j$ are natural numbers, this proximity will be limited to $\epsilon$. From the following it will be clear that it is impossible to direct $\epsilon$ to zero for complex systems, and therefore the complexity is determined in this way correctly. We denote the complexity of the state $|\Psi\rangle$ by $C(|\Psi\rangle)$.

The representation of the amplitude quantum in the form (4.1), in addition to their simplicity, has another justification - the possibility of introducing micro-causality for an important class of states (see Appendix 3). It is also interesting that this class does not contain the states that should appear in quantum speedup of classical computations.

Let $\tau \in S_n$ be a permutation of the basis vectors of the state space corresponding to the set of qubits $M$. Then the state $\tau |\Psi\rangle$ is called a quasiparticle representation of the state $|\Psi\rangle$. For example, for a set of $n$ harmonic oscillators their basic state is $(q_1, q_2, ..., q_n)$ - the values of coordinates, and the Fourier transform over this sequence of the form $Q_k = \alpha \sum_j q_j e^{-\beta i k j}$ means the transition to the description of the same system of oscillators not through their coordinates $q_j$, but through phonons - quasiparticles with new coordinates $Q_k$.

Another example: generalized GZH - state of the form $\frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle)$ in which all $n$ qubits are entangled, however, it can be transformed to a state in which only 2 qubits are entangled by successive operations of $CNOT$, which are permutations of the basic vectors of space.

The absolute complexity $A(\Psi)$ of the state $|\Psi\rangle$ is the minimum of the complexities of all its quasi-partial representations. Formally:

$\text{representation of amplitudes in discrete form \ref{eq:4.1} requires an appropriate choice of classical basic states; due to the smallness of }\epsilon \text{ this does not lead to any revision of the experimentally confirmed part of the quantum theory, but only affects the scaling of the quantum computer.}$
\[ A(\Psi) = \min_{\tau \in S_N} C(\tau|\Psi) \].

(4.2)

Absolute complexity is the number of qubits required to represent the quantum kernel of a given state. The state space in which the kernel lives has therefore the dimension \(2^{A(\Psi)}\).

We assume that for the state (3.1) the amplitudes of its components always have the form (4.1) for some \(\epsilon > 0\). Then the uncertainty relation of the "amplitude-complexity" type has the form

\[ 2^{A(\Psi)}/\epsilon^2 \leq Q \]

(4.3)

for some universal constant \(Q\), which does not depend on the state \(|\Psi\rangle\). This relation can be rewritten as

\[ A(\Psi) + m(\Psi) \leq \log_2 Q \]

(4.4)

where \(m(\Psi)\) is the number of qubits required for the representation of amplitudes of the state \(|\Psi\rangle\).

The sense of the constant \(Q\): this is the maximal number of classical components of the superposition in the solid state of any physical ensemble. For example, if an ensemble consists of one elementary particle, and its state is its spatial coordinates, then it can be "smeared" in space by no more than \(Q\) points, and if it is distributed exactly by \(Q\) points, it is in them in equal probability.

Taking into account that the experiments allow us to establish such effects as the Lamb shift of energy levels, which is about \(10^{-4}\) from the main energy gap, or even less, we can suppose that the constant \(Q\), most likely, should be not less than ten thousands; if it is not more than \(10^{10}\), it could be found in the experiment (see below).

It can be seen from this uncertainty relation that as the absolute complexity of the state increases, the representation of the amplitudes of its components must become more and more coarse. This fully corresponds to the constructive approach to the description of the quantum system dynamics: this description should fit into the memory of a classical computer. Therefore, the accuracy of the description of quantum coherence should decrease with the growth of the quantum kernel. If the dynamics leads to the violation of this inequality, there is a collapse of the state in which it is automatically reduced to the size dictated by the inequality (4.3).

The hypothetical relation (4.4) has a form similar to the fundamental Bohr-Heisenberg uncertainty relation. The hypothesis of amplitude quantum may also mean that Planck constant, which is the main constant of the fundamental uncertainty relation, has a computational nature, and the possibilities of simultaneous measurement of the coordinate and momentum are limited by the power of an abstract computer simulating real dynamics. This point of view is quite uncommonly for physics, but it is no less extravagant than the miracles in the quantum comput-
ing, which follows from the standard unlimited formalism of Hilber spaces. Not to mention, unlike these miracles, it is perfectly consistent with experiments.

5 Experimental finding of the constructive constant $Q$

The constructive dimensionless constant $Q$ has a physical nature and is therefore subject to experimental search. To do this, it is necessary to distribute the amplitude over a very large number of classical states of an ensemble, so that the constructive reduction would give a gross deviation from the coherent dynamics, which could be recorded in the experiment. If one deals with well-studied physical ensembles, it is very difficult to do so, since they are amenable to study by standard means precisely because their absolute complexity is small and therefore the accuracy of amplitude determination - the value of $m$ from (4.3) can be large.

The most reliable way is to implement Grover’s algorithm for as many $n$ qubits as possible. We have to reproduce the computation by this algorithm to find the root of the equation

$$f(x) = 1,$$  \hspace{1cm} (5.1)

where $f$ is a Boolean function of $n$ variables such that the solution is unique. We suppose that the number of qubits $n$ exceeds the barrier $\log_2 Q$.

Grover’s algorithm is a successive application of the operator $G = I_0 I_{x_t}$ to the state $|\tilde{0}\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle$, where $I_a$ is the reflection of space with respect to the hyperplane orthogonal to the vector $|a\rangle$, and $x_t$ is the only solution of the equation $f(x) = 1$. We show that the constructive reduction of the current state of the computer at the first step will destroy the state, which was considered as the result in quantum theory.

The absolute complexity of the initial state $|\tilde{0}\rangle$ is one, since this state is a tensor $n$-th degree of the state $|0\rangle + |1\rangle \sqrt{2}$. The implementation of the $I_{x_t}$ operator will give the state of the form $|\Psi'\rangle = \frac{1}{\sqrt{N}} \left( \sum_{j \neq x_t} |j\rangle - |x_t\rangle \right)$, where all the amplitudes of the components are less than $\epsilon$ and equal in modulus. The absolute complexity of the state $|\Psi'\rangle$ is $n$. Indeed, since at any permutation of the basis vectors $\tau$ the state $x_t$ will go to exactly one basic state $\tau|x_t\rangle$, any qubit in the state $\tau|\Psi'\rangle$ will be entangled with all the others, that is $C(\tau|\Psi'\rangle) = n$ for all $\tau \in S_n$.

Thus, due to (4.3), the state $|\Psi'\rangle$ should be reduced by elimination of small amplitudes. But since there all amplitudes are equally small, the smallest amplitudes will be cancelled, the choice of which will be based on inaccuracy of gates inside $G$. This will lead to a radical change of state compared to what the (unlimited) theory predicts. We see that the implementation of the Grover algorithm can be used for
determining the value of the constant $Q$: the logarithm of this value coincides with the maximum number of qubits for which the Grover algorithm can work.

Consider the implementation of the Grover algorithm in more detail. Let $x_t$ be the only solution to the equation (5.1). The state of the quantum computer at the step with the number $t$ in the computation is as follows

$$|\Psi(t)\rangle \approx \sin(\tilde{t})|x_t\rangle + \cos(\tilde{t})|X\rangle \quad (5.2)$$

where $\tilde{t} = \frac{t}{2\sqrt{N}}$ with high accuracy, $|X\rangle = \frac{1}{\sqrt{N}} \sum_{j \neq x_t} |j\rangle$. It follows that for $N \ll Q$, the implementation of Grover’s algorithm on a quantum computer will go exactly as quantum theory predicts. But when $N$ approaches the barrier $Q$, the following will occur. The current state of the computer $|\Psi(t)\rangle$ with high accuracy will coincide with the desired $|x_t\rangle$ already at the first step. A further increase in the number of qubits $n = \log_2 N$ will give the opposite result: the state $|x_t\rangle$ will appear when measuring $|\Psi(t)\rangle$ at any $t$ with vanishingly low probability. Indeed, in the first step we will have the state $|\Psi\rangle = \frac{1}{\sqrt{N}} (-|x_t\rangle + \sum_{j \neq x_t} |j\rangle)$, where all amplitudes are less than $\epsilon$, so that the reduction result will depend on the inaccuracy of the gates themselves and will not be associated with $x_t$.

We note that this method of determining the $Q$ with the approximation $n$ to $\log_2 Q$, possibly, will work best for the adiabatic form of quantum computing, as in gate version will be the implementation of the operators $I_{\tilde{0}}, I_{x_t}$ may distort the picture.

This behavior of the state vector in the implementation of the Grover algorithm directly follows from the hypothesis of the existence of the amplitude quantum $\epsilon$ and the uncertainty relation (4.4). This behavior for sufficiently small $Q$, e.g. for $Q \leq 10^6$ can be detected using quantum tomography due to the specific forms of the states (5.2). Confirmation of this behavior will be a strong argument in favor of the uncertainty relation (4.4) if the found value of $Q$ does not depend on the quantum gate technology.

6 Conclusion

Numerous experiments on quantum computers, conducted since the late 80-ies of the last century, give reason to consider a quantum computer as an object of fundamental scientific analysis, a problem, not a technical project. A natural limitation of quantum formalism in the field of complex systems was proposed, in which decoherence is actually a consequence of the limitation of computing resources of an abstract computer that simulates the real evolution. The restriction is formulated in the form of the uncertainty relation "complexity of the system - the accuracy of its quantum description”, in which the accuracy of determining the amplitude of the components of the quantum state is in competition with the size of the quantum kernel of the system under consideration. This restriction does not affect the
achievements of quantum theory, but makes fast quantum algorithms fundamentally impossible.

The constant of this uncertainty relation can be determined in the course of the Grover algorithm implementation on a limited quantum memory of several tens of qubits. Such experiments get a whole new meaning.

The constructive reduction of the system state vector arising from this uncertainty relation naturally leads to the appearance of the classical state after a short period of quantum evolution, which is fully consistent with quantum mechanics.

The computational capabilities of quantum computers are not scalable, like the classical ones, and the speed of quantum algorithms turns out to be limited by this uncertainty relation. The only fundamental difference between quantum and classical computers is the possibility of using quantum nonlocality.

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Appendix 1. Constructive mathematical analysis

It is possible to build all the mathematics on the concept of the algorithm; this was partly done by A.A. Markov - Junior, see [27, 28]. In constructive mathematics, real numbers are treated as algorithms that produce their successive rational approximations. At the same time, a constructive function is also an algorithm that produces an approximation of the function value by a given approximation of the argument. Constructive analysis differs from the classical one in that the constructive real function is always continuous, that is, constructivism excludes constructions of the type of the Heaviside function. This makes constructivism a more convenient
mathematical apparatus in comparison with conventional mathematics, where problems arise such as the non-normalized nature of the eigenfunction of the momentum operator.

The main thing in constructivism: the fundamental role of the grain of space-time resolution, while the transition to the limit is carried out only after all physical constructions, and not immediately, as in classical analysis. Analytical differentiation and integration is convenient only for simple systems for which the finite grain size of the resolution is secondary, for example, it was so for the search of stationary states of model Hamiltonian that occupied physicists of the first three quarters of the 20-th century.

But already in an attempt to avoid ultraviolet divergences in quantum electrodynamics, it was found that the main constants of the traditional quantum theory - the charge and the mass of the electron, depend on the grain of resolution, which led to the concept of renormalization ([29]). There will be no logical contradictions only if we fix the grain of the resolution of the main quantities: space and time; and only after performing all standard physical actions, for example, calculations of the amplitudes of stationary states, we can move to the limit at \( dx \to 0 \).

Appendix 2. Derivation of the Born rule from amplitude quantization

The reduction procedure, which involves cancellation of small amplitudes, leads to the Born rule for quantum probability (see [30]). We reduce the calculation of the probability of obtaining a certain basic state \( A \) when measuring the quantum state \( \Psi \) to the application of the classical rule \( p(A) = \frac{N_{\text{suc}}}{N_{\text{tot}}} \) where \( N_{\text{suc}} \) is the number of favorable outcomes (that is, those elementary outcomes at which the event \( A \) happens), \( N_{\text{tot}} \) is the total number of elementary outcomes. We call elementary outcomes those states of the extended system (measured system + measuring device), which module of amplitude in this quantum state equals the amplitude quantum \( \epsilon \). The set of elementary outcomes will thus depend on the quantum state of the extended system.

Let \( |\Psi_j\rangle \) denote the basic states of the measured system, \( |\Phi_j\rangle \) the basic states of the measuring device (which can be, for example, the observer’s eye), the contact of these two objects will give the state

\[
|\Psi_{\text{cont}}\rangle = \sum_j \lambda_j |\Psi_j\rangle \bigotimes |\Phi_j\rangle
\]

Now, since the measuring device is very massive in comparison with the measured object, when trying to describe its quantum states, we will have to divide components of \( |\Psi_{\text{cont}}\rangle \) by the sum of \( l_j \) basis states (it is necessary to take into account, for example, the states of all nuclei, electrons, etc., included in the measuring device). That is, even if at the contact we had a state \( |\Phi_j\rangle \), as a result of evolution, a state of the form \( |\Phi_j'\rangle = \sum_{k=1}^{l_j} \mu_{j,k} |\phi_{j,k}\rangle \) will arise very quickly, where over time \( l_j \)
will grow rapidly, until the decreasing amplitudes reach the size of the amplitude quantum $\epsilon$, after which those of them that are less than this threshold will disappear. Therefore, all amplitude modules $|\mu_{j,k}|$ should be considered approximately as equal. If we substitute the expression for $|\Phi'_j\rangle$ instead of $|\Phi_j\rangle$ in the expression for $|\Psi_{cont}\rangle$, the amplitudes of states $\phi_{j,k}$ will be equal to $\frac{\lambda_j}{\sqrt{l_j}}$ by virtue of the unitarity of quantum evolution.

We must produce a reduction procedure which consists simply in dropping states $\phi_{j,k}$ which module of amplitude is small. Since the period of time during which the described partition into a huge number of terms occurs is negligible, it will actually mean that we divide each term in the expression for $|\Psi_{cont}\rangle$ by $l_j$ new terms so that all the resulting amplitudes are close to the amplitude quantum and are approximately the same, since this makes all states equal before the reduction and makes it possible to apply the classical urn scheme. But then the number $l_j$ of summands with the first factor $|\Psi_j\rangle$, which is the number of favorable outcomes, will be proportional to $|\lambda_j|^2$, and if only one summand survives in the reduction, we get exactly Born rule for quantum probability. Of course, zeroing small amplitudes will automatically lead to the growth of other amplitudes, since the total probability must be single.

Thus, the probability space depends on the choice of the wave function $|\Psi\rangle$. We consider actually conditional probabilities to get one or another outcome when measuring, provided that the system is in the state $|\Psi\rangle$.

The given explanation of Born’s rule of quantum probability is based only on our definition of the reduction of the algorithmic description of the quantum state as the rejection of small amplitudes and the corresponding pumping of the rest. This procedure is critically dependent on the size of the amplitude quantum $\epsilon$.

The constructive interpretation of Born’s rule in the form of dropping small amplitudes does not affect the quantum theory of simple systems, but puts a limit to the possible scaling of the quantum computer: the dimension of the Hilbert state space of its quantum part should be no more than $1/\epsilon^2$.

Appendix 3. Amplitude quantum and collective micro-causality

The analysis of the "dark place" of quantum mechanics: the unnatural separation of the observer and the observed system is inseparable from the analysis of its fundamentally non-deterministic nature. Disagreement with this feature of science Einstein expressed in the words: "God does not play dice". There is no concept of the wave function of a single particle. When this is said, assuming the freedom of language, it refers to the wave function of a huge number of equally prepared particles, because only under the condition of the existence of this homogeneous mass of independent particles, it makes sense to talk about the wave function.

The probabilistic basis of quantum theory stems from the fact that in an arbitrary system, any portion of the amplitude of its components on any arbitrarily small time interval $dt$ is inevitably divided into even smaller portions, each of which corresponds to a certain finite state, and almost any such possible finite state gets something,
unless the corresponding transition is prohibited. This is the essence of the matrix principle - the core of quantum mechanics.

However, for some important class of states, it is possible to introduce the so-called micro-causality in the sense that there will be a hidden parameter, the knowledge of which will uniquely determine the evolution of the system in a small period of time. These are so-called connected states characterized by the following property: any two basic components of such states can be translated into one another by a permutation of variables that commutes with a Hamiltonian - see [31].

Namely, let $H$ be the Hamiltonian of a quantum system consisting of atoms of the same type, and $G_H$ be the group of permutations on atoms commuting with $H$. The state

$$|\Psi\rangle = \sum_{j\in J} \lambda_j |j\rangle$$

is called connected with respect to Hamiltonian $H$, if for any basic states $|j\rangle$, $|j'\rangle$, such that $\lambda_j$ and $\lambda_{j'}$ are nonzero there exists $\tau \in G_H$, such that $\tau |j\rangle = |j'\rangle$.

The components of connected states are thus homogeneous with respect to permutations of variables that do not change the Hamiltonian.

Micro-causality for connected states is the following. Let for the connected state $|\Psi\rangle$ each nonzero amplitude be decomposed into the sum of small terms equal in modulus:

$$\lambda_j = \langle j|\Psi \rangle \approx \text{sign}_{\text{re}} \left( \sum_{M_j} \varepsilon + \varepsilon + \ldots + \varepsilon \right) + \text{sign}_{\text{im}} i \left( \sum_{N_j} \varepsilon + \varepsilon + \ldots + \varepsilon \right)$$

thus, for each occurrence of $\varepsilon$ in any of the amplitudes $\lambda_j$, it will be uniquely determined in what particular state $|j'\rangle$ this quantum will pass under the action of the unitary evolution $U = \exp(\frac{i}{\hbar} H dt)$ in a small time interval $dt$ (see the work [31]).

The occurrence of the symbol "$\varepsilon$" in the amplitude $\lambda_j$ will thus play the role of a hidden parameter of the quantum theory of complex systems in the case of connected states. This result cannot be generalized to the disconnected states.

The possibility of micro-causality for connected states suggests that the quantization of the amplitude as a sum of identical portions can play an important role in the dynamics of complex systems, which should be considered as quantum.