ON A TYPE OF QUARTER-SYMMETRIC NON-RECURRENT METRIC CONNECTION ON A P-SASAKIAN MANIFOLD

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Abstract. We study a Para-Sasakian manifold admitting a type of quarter-symmetric non-recurrent-metric connection whose concircular curvature tensor satisfies certain curvature conditions.

1. Introduction

In 1977, Adati and Matsumoto [1] defined Para-Sasakian and Special Para-Sasakian manifolds which are special classes of an almost paracontact manifold introduced by Sato [10]. Para-Sasakian manifolds have been studied by De and Pathak [5], Matsumoto, Ianus and Mihai [9], Barman [2,3] and many others.

In 1924, Friedmann and Schouten [6] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\nabla$ on a Riemannian manifold $(M, g)$ is said to be a semi-symmetric connection if the torsion tensor $T$ of the connection $\nabla$ satisfies $T(X, Y) = u(Y)X - u(X)Y$, where $u$ is a 1-form and $\rho$ is a vector field defined by $u(X) = g(X, \rho)$, for all vector fields $X \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on $M$.

In 1975, Golab [7] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. A linear connection $\nabla$ on an $n$-dimensional Riemannian manifold $(M, g)$ is called a quarter-symmetric connection [7] if its torsion tensor $T$ satisfies $T(X, Y) = u(Y)\phi X - u(X)\phi Y$, where $\phi$ is a $(1,1)$ tensor field.

In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [6]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\nabla$ satisfies the condition $(\nabla_X g)(Y, Z) = 0$, for all $X, Y, Z \in \chi(M)$, then $\nabla$ is said to be a quarter-symmetric

2010 Mathematics Subject Classification: 53C15; 53C25.

Key words and phrases: para-Sasakian manifold, quarter-symmetric non-recurrent metric connection, concircular curvature tensor, $\xi$-concircularly flat, $\phi$-concircularly flat, $\eta$-Einstein manifold.

Communicated by Zoran Rakić.
metric connection and otherwise, $\nabla$ is said to be a quarter-symmetric non-metric connection. A quarter-symmetric non-metric connection $\nabla$ whose torsion tensor $T$ satisfies $T(X, Y) = u(Y)\phi X - u(X)\phi Y$ and $(\nabla_X g)(Y, Z) = 2u(X)g(\phi Y, Z) \neq 0$, is said to be the quarter-symmetric non-recurrent-metric connection.

A transformation of an $n$-dimensional Riemannian manifold $M$, which transforms every geodesic circle of $M$ into a geodesic circle, is called a concircular transformation $[8, 11]$. A concircular transformation is always a conformal transformation $[8]$. Here geodesic circle means a curve in $M$ whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [4]). An interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{W}$. It is defined in $[11, 12]$

$\tilde{W}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]$.  

From (1.1), it follows that

$\tilde{\tilde{W}}(X, Y, Z, U) = \tilde{\tilde{R}}(X, Y, Z, U) - \frac{\tilde{r}}{n(n-1)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)],$

$\tilde{\tilde{W}}(X, Y, Z, U) = g(\tilde{W}(X, Y)Z, U)\tilde{R}(X, Y, Z, U) = g(\tilde{R}(X, Y)Z, U),$

where $X, Y, Z, U \in \chi(M)$ and $\tilde{R}$ is the curvature tensor and $\tilde{r}$ is the scalar curvature with respect to the quarter-symmetric non-recurrent-metric connection respectively. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

A Riemannian manifold $M$ is locally symmetric if its curvature tensor $R$ satisfies $\nabla R = 0$. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold $M$ is said to be semi-symmetric if its curvature tensor $R$ satisfies $R(X, Y).R = 0$, where $R(X, Y)$ acts on $R$ as a derivation.

The present paper is organized as follows: Section 2 contains some prerequisites of P-Sasakian manifolds. In Section 3 we discuss a type of quarter-symmetric non-recurrent metric connection. In Section 4 we establish the relation of the curvature tensor between the Levi-Civita connection and the quarter-symmetric non-recurrent metric connection of a P-Sasakian manifold. In Section 5 we study $\xi$-concircularly flat P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent metric connection. In Section 6 we study $\phi$-concircularly flat P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent metric connection. Now, section-wise we investigate the curvature conditions $\tilde{W}.\tilde{W} = 0$ and $\tilde{R}.\tilde{W} = 0$ in a P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\nabla$ respectively. Finally, we construct an example of a
5-dimensional P-Sasakian manifold admitting the quarter-symmetric non-recurrent metric connection which verifies the results of Section 4, Section 5, and Section 6.

2. P-Sasakian manifolds

An $n$-dimensional differentiable manifold $M$ is said to be an almost para-contact manifold $(\phi, \xi, \eta, g)$, if there exists $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric on $M$ which satisfy the conditions

\begin{align}
\phi \xi &= 0, \quad \eta(\phi X) = 0, \quad g(X, \xi) = \eta(X), \\
\phi^2(X) &= X - \eta(X)\xi, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\
(\nabla_X \eta)Y &= g(X, \phi Y) = (\nabla_Y \eta)X,
\end{align}

for any vector fields $X, Y$ on $M$.

If moreover, $(\phi, \xi, \eta, g)$ satisfy the conditions $d\eta = 0$, $\nabla_X \xi = \phi X$, and

\begin{align}
(\nabla_X \phi)Y &= -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi,
\end{align}

then $M$ is called a para-Sasakian manifold or briefly a P-Sasakian manifold.

In a P-Sasakian manifold, the following relations hold [1][10]:

\begin{align}
\eta(R(X, Y)Z) &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \\
R(\xi, X)Y &= \eta(Y)X - g(X, Y)\xi, \\
R(\xi, X)\xi &= X - \eta(X)\xi, \\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\
S(X, \xi) &= -(n-1)\eta(X), \\
S(\phi X, \phi Y) &= S(X, Y) + (n-1)\eta(X)\eta(Y),
\end{align}

where $R$ and $S$ are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3. Quarter-symmetric non-recurrent metric connection

Let $M$ be an $n$-dimensional P-Sasakian manifold with Riemannian metric $g$. If $\bar{\nabla}$ is the quarter-symmetric non-metric connection of a P-Sasakian manifold $M$, a linear connection $\nabla$ is given by

\begin{align}
\bar{\nabla}_X Y &= \nabla_X Y - \eta(X)\phi Y.
\end{align}

Using (3.1), the torsion tensor $T$ of $M$ with respect to the connection $\nabla$ is

\begin{align}
T(X, Y) &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y.
\end{align}

The connection $\bar{\nabla}$ is called a quarter-symmetric connection.

Further using (3.1), we have

\begin{align}
(\bar{\nabla}_X g)(Y, Z) &= \nabla_X g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = 2\eta(X)g(\phi Y, Z) \neq 0.
\end{align}
A relation satisfying (3.3) is said to be non-recurrent metric connection. Therefore, $\nabla$ defined by (3.1) satisfying (3.2) and (3.3) is a type of quarter-symmetric non-recurrent metric connection.

Conversely, we show that a linear connection $\bar{\nabla}$ defined on $M$ satisfying (3.2) and (3.3) is given by (3.1). Let $H$ be a tensor field of type $(1, 2)$ and

\[ (\bar{\nabla}_X Y) = \nabla_X Y + H(X, Y). \]

Then we conclude that

\[ T(X, Y) = H(X, Y) - H(Y, X). \]

Further using (3.3), it follows that

\[ (\bar{\nabla}_X g)(Y, Z) = \nabla_X g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = -g(H(X, Y), Z) - g(Y, H(X, Z)). \]

In view of (3.3) and (3.6), it yields

\[ g(H(X, Y), Z) + g(Y, H(X, Z)) = -2\eta(X)g(\phi Y, Z). \]

Also using (3.7) and (3.5), we derive that

\[ g(T(X, Y), Z) + g(T(Z, X), Y) + g(T(Z, Y), X) = 2g(H(X, Y), Z) - 2g(Z)g(X, \phi Y) + 2\eta(Y)g(\phi X, Z) + 2\eta(X)g(\phi Y, Z). \]

From the above equation, we have

\[ g(T(X, Y), Z) = \frac{1}{2}[g(T(X, Y), Z) + g(T(Z, X), Y) + g(T(Z, Y), X)] \]
\[ + \eta(Z)g(X, \phi Y) - \eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z). \]

Let $T'$ be a tensor field of type $(1, 2)$ given by

\[ g(T'(X, Y), Z) = g(T(Z, X), Y). \]

Adding (2.1), (3.2) and (3.9), we obtain

\[ T'(X, Y) = \eta(X)\phi Y - g(\phi X, Y)\xi. \]

From (3.8), we have by using (3.9) and (3.10)

\[ g(H(X, Y), Z) = \frac{1}{2}[g(T(X, Y), Z) + g(T(Z, X), Y) + g(T(Z, Y), X)] \]
\[ + \eta(Z)g(X, \phi Y) - \eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z) = -\eta(X)g(\phi Y, Z). \]

Now contracting $Z$ in (3.11) and using (2.1), it implies that

\[ H(X, Y) = -\eta(X)\phi Y. \]

Combining (3.3) and (3.12), it follows that $\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y$.

Now, we are in a strong position to state the following theorem:

**Theorem 3.1.** The linear connection $\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y$ is a quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ on $P$-Sasakian manifolds.
4. Curvature tensor of a P-Sasakian manifold
with respect to the quarter-symmetric non-recurrent metric connection

**Theorem 4.1.** For a P-Sasakian manifold $M$ with respect to the quarter-symmetric non-recurrent metric connection $\nabla$

(i) The curvature tensor $\tilde{R}$ is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \eta(X)g(Y,Z)\xi + \eta(Y)\tilde{S}(Y,Z)\xi - \eta(X)\eta(Z)\eta(Y),$$

(ii) The Ricci tensor $\tilde{S}$ is given by

$$\tilde{S}(Y,Z) = S(Y,Z) - g(Y,Z) + n\eta(Y)\eta(Z),$$

(iii) The scalar curvatures with respect to the Levi-Civita connection $\nabla$ and the quarter-symmetric non-recurrent metric connection $\nabla$ are equal,

(iv) $\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z$,

(v) The Ricci tensor with respect to the quarter-symmetric non-recurrent metric connection $\nabla$ is symmetric.

**Proof.** Let $M$ be a P-Sasakian manifold. A relation between the curvature tensor $\tilde{R}$ of the quarter-symmetric non-recurrent metric connection $\nabla$ and the curvature tensor $R$ of the Levi-Civita connection $\nabla$ is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z - (\nabla_X\eta)(Y)\phi Z + (\nabla_Y\eta)(X)\phi Z + \eta(X)(\nabla_Y\phi)(Z) - \eta(Y)(\nabla_X\phi)(Z).$$

Then by making use of (4.1), (4.2) and (4.3), we get

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi - \eta(X)\eta(Z)\eta(Y).$$

From (4.2), we obtain that the curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z.$$

Taking the inner product of (4.2) with $U$, it follows that

$$\tilde{R}(X,Y,Z,U) = \tilde{R}(X,Y,Z,U) - \eta(X)\eta(U)g(Y,Z) + \eta(Y)\eta(U)g(X,Z) - \eta(X)\eta(Z)g(Y,U) + \eta(Y)\eta(Z)g(X,U),$$

where $\tilde{R}(X,Y,Z,U) = g(\tilde{R}(X,Y)Z,U)$.

Now putting $X = \xi$ in (4.2) and using (2.4) and (2.5), we have

$$\tilde{R}(\xi,Y)Z = 2\eta(Y)\eta(Z)\xi - 2g(Y,Z)\xi.$$

Again putting $Z = \xi$ in (4.2) and using (2.4) and (2.5), we get

$$\tilde{R}(X,Y)\xi = 0.$$

Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of vector fields in $M$. Then by putting $X = U = e_i$ in (4.2), summing over $i$, $1 \leq i \leq n$ and using (2.4), we get

$$\tilde{S}(Y,Z) = S(Y,Z) - g(Y,Z) + n\eta(Y)\eta(Z),$$

where $\tilde{S}$ is the Ricci tensor of the quarter-symmetric non-recurrent metric connection $\nabla$. 

Putting \( Y = Z = e_i \) in (4.6), summing over \( i, 1 \leq i \leq n \) and using (2.1), we have
\[
(4.7) \quad \bar{r} = r,
\]
where \( r \) and \( \bar{r} \) are the scalar curvatures with respect to the Levi-Civita connection \( \nabla \) and the quarter-symmetric non-recurrent metric connection \( \bar{\nabla} \) respectively. □

5. \( \xi \)-concircularly flat in P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent-metric connection

**Definition 5.1.** A P-Sasakian manifold is said to be \( \xi \)-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection \( \bar{\nabla} \) if
\[
W(X, Y) = 0,
\]
where \( X, Y \in \chi(M) \).

**Theorem 5.1.** An \( n \)-dimensional P-Sasakian manifold is \( \xi \)-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection \( \bar{\nabla} \) if the scalar curvature vanishes with respect to the Levi-Civita connection.

**Proof.** Putting (4.2) and (4.7) in (1.1), we get
\[
W(X, Y)Z = R(X, Y)Z - \eta(X)g(Y, Z)\xi + \eta(Y)g(X, Z)\xi - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y].
\]
From (5.1), it yields
\[
\mathbb{W}(X, Y)Z = -\mathbb{W}(Y, X)Z.
\]
Also putting \( X = \xi \) in (5.1) and using (2.1) and (2.3), it follows that
\[
\mathbb{W}(\xi, Y)Z = \frac{r}{n(n-1)}\eta(Z)Y - \left[2 + \frac{r}{n(n-1)}\right]g(Y, Z)\xi + 2\eta(Y)\eta(Z)\xi.
\]
Again putting \( Z = \xi \) in (5.1) and using (2.1) and (2.6), we obtain
\[
\mathbb{W}(X, Y)\xi = \frac{r}{n(n-1)}[\eta(X)Y - \eta(Y)X].
\]
If \( r = 0 \), then \( \mathbb{W}(X, Y)\xi = 0 \). □

6. \( \phi \)-concircularly flat P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent metric connection

**Definition 6.1.** A P-Sasakian manifold is said to be \( \phi \)-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection \( \nabla \) if
\[
\mathbb{W}(\phi X, \phi Y, \phi Z, \phi U) = 0,
\]
where \( X, Y, Z, U \in \chi(M) \).

**Definition 6.2.** A P-Sasakian manifold is said to be an \( \eta \)-Einstein manifold if its Ricci tensor \( S \) of the Levi-Civita connection is of the form
\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]
where \( a \) and \( b \) are smooth functions on the manifold.
Theorem 6.1. If a P-Sasakian manifold is $\phi$-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$, then the manifold is an $\eta$-Einstein manifold.

Proof. Putting $X = \phi X, Y = \phi Y, Z = \phi Z$ and $U = \phi U$ in (1.2), we obtain
\begin{equation}
\bar{\mathcal{W}}(\phi X, \phi Y, \phi Z, \phi U) = \bar{\mathcal{R}}(\phi X, \phi Y, \phi Z, \phi U) - \frac{r}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)].
\end{equation}
Using (2.1) and (4.3) in (6.2), we have
\begin{equation}
\bar{\mathcal{W}}(\phi X, \phi Y, \phi Z, \phi U) = \bar{\mathcal{R}}(\phi X, \phi Y, \phi Z, \phi U) - \frac{r}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)].
\end{equation}
Since the P-Sasakian manifold is $\phi$-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection. Now Combining (4.7), (6.1) and (6.3), it follows that
\begin{equation}
\bar{\mathcal{R}}(\phi X, \phi Y, \phi Z, \phi U) = \frac{r}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)].
\end{equation}
Let \{e_1, \ldots, e_{n-1}, \xi\} be a local orthonormal basis of vector fields in $M$; then \{\phi e_1, \ldots, \phi e_{n-1}, \xi\} is also a local orthonormal basis. Putting $X = U = e_i$ in (6.1) and summing over $i = 1$ to $n - 1$, we get
\begin{equation}
S(\phi Y, \phi Z) = \frac{(n-2)r}{n(n-1)}g(\phi Y, \phi Z).
\end{equation}
By making use of (2.7) and (2.2) in (6.5), we have
\begin{equation}
S(Y, Z) = \frac{(n-2)r}{n(n-1)}g(Y, Z) - \frac{n(n-1)^2 + (n-2)r}{n(n-1)}\eta(Y)\eta(Z).
\end{equation}
This result shows that the manifold is an $\eta$-Einstein manifold, where $a = \frac{(n-2)r}{n(n-1)}$, $b = \frac{n(n-1)^2 + (n-2)r}{n(n-1)}$. $\square$

7. P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ satisfying $\bar{\mathcal{W}}, \bar{\mathcal{W}} = 0$

Theorem 7.1. If a P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ satisfies $\bar{\mathcal{W}}, \bar{\mathcal{W}} = 0$, then the manifold is an $\eta$-Einstein manifold.

Proof. We suppose that the manifold under consideration is concircular semi-symmetric with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$, that is, \((\bar{\mathcal{W}}(X, Y)/\bar{\mathcal{W}}(U, V))Z = 0\). Then we have
\begin{equation}
(\bar{\mathcal{W}}(X, Y)/\bar{\mathcal{W}}(U, V))Z = \bar{\mathcal{W}}(\bar{\mathcal{W}}(X, Y)U, V)Z - \bar{\mathcal{W}}(U, \bar{\mathcal{W}}(X, Y)V)Z - \bar{\mathcal{W}}(U, V)\bar{\mathcal{W}}(X, Y)Z = 0.
\end{equation}
Putting $X = \xi$ in (7.1), it follows that

\begin{equation}
(7.2) \quad (\mathbb{W}(\xi, Y))\mathbb{W}(U, V)Z - (\mathbb{W}(\xi, Y)U, V)Z - \mathbb{W}(U, \mathbb{W}(\xi, Y)V)Z - \mathbb{W}(U, V)\mathbb{W}(\xi, Y)Z = 0.
\end{equation}

In view of (5.2), (5.3) and (7.2), it yields

\begin{equation}
(7.3) \quad (\mathbb{W}(\xi, Y)\mathbb{W})(U, V)Z - \frac{r}{n(n-1)}\eta(U)\mathbb{W}(Y, V)Z + \left[2 + \frac{r}{n(n-1)}\right]g(Y, U)\mathbb{W}(\xi, V)Z - 2\eta(Y)\eta(U)\mathbb{W}(\xi, Z)Y - 2\eta(Y)\mathbb{W}(\xi, U)Z - \frac{r}{n(n-1)}\eta(Z)\mathbb{W}(U, V)Y + \left[2 + \frac{r}{n(n-1)}\right]g(Y, Z) - 2\eta(Y)\eta(Z)\left[\frac{r}{n(n-1)}\eta(U)V - \eta(V)U\right] = 0.
\end{equation}

And also putting $X = \xi$ in (5.3) and using (2.1), it implies that

\begin{equation}
(7.4) \quad \mathbb{W}(\xi, Y)\xi = \frac{r}{n(n-1)}[Y - \eta(Y)]\xi.
\end{equation}

Again putting $U = \xi$ in (7.3) and using (6.1), (5.2) and (7.3), we get

\begin{equation}
(7.5) \quad \frac{r}{n(n-1)}\eta(Z)\mathbb{W}(\xi, Y)V - \left[2 + \frac{r}{n(n-1)}\right]g(V, Z)\mathbb{W}(\xi, Y)\xi + 2\eta(V)\eta(Z)\mathbb{W}(\xi, Y)\xi - \frac{r}{n(n-1)}\left[R(Y, V)Z - \eta(Y)g(V, Z)\xi + \eta(V)g(Y, Z)\xi - \eta(Y)\eta(Z)V + \eta(V)\eta(Z)Y - \frac{r}{n(n-1)}\left[g(Y, Z)Y - g(Y, V)\xi\right] + \left[\frac{r}{n(n-1)}\right]^2\eta(Y)\eta(Z)\xi - \left[2 + \frac{r}{n(n-1)}\right]\left[\frac{r}{n(n-1)}\right]\eta(Y)\eta(Z)\xi - \left[2 + \frac{r}{n(n-1)}\right]g(Y, V)\eta(\xi)\xi - \left[\frac{2r}{n(n-1)}\right]\eta(V)\eta(Z)Y + \left[2 + \frac{r}{n(n-1)}\right]\left[\frac{r}{n(n-1)}\right]g(Y, V)\eta(\xi)\xi - \left[\frac{2r}{n(n-1)}\right]^2\eta(Z)\eta(V)Y + \left[2 + \frac{r}{n(n-1)}\right]\left[\frac{r}{n(n-1)}\right]g(Y, Z)V = 0.
\end{equation}

Now contracting $Y$ in (7.5) and (2.1), we have

\begin{equation}
(7.6) \quad S(V, Z) = (n - 2)\eta(Z)\eta(V) + (3 - 2n)g(Z, V).
\end{equation}

Form (7.6), we can write $S(V, Z) = ag(V, Z) + bh\eta(V)\eta(Z)$, where $a = (3 - 2n)$, $b = (n - 2)$. This result shows that the manifold is an $\eta$-Einstein manifold.  \qed
8. P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\nabla$ satisfying $\tilde{R}(\xi, Y).\mathbb{W} = 0$

Theorem 8.1. An $n$-dimensional P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\nabla$ satisfies $\tilde{R}(\xi, Y).\mathbb{W} = 0$ if the scalar curvature tensor of the manifold with respect to the Levi-Civita connection vanishes.

Proof. We obtain a necessary condition for a P-Sasakian manifold to satisfy the derivation condition $(\tilde{R}(\xi, Y).\mathbb{W})(U, V)Z = 0$, where $\tilde{R}(\xi, Y).\mathbb{W}$ denotes $\tilde{R}(\xi, Y)$ acting on $\mathbb{W}$ as a derivation. Then we have

$$(\tilde{R}(\xi, Y))\mathbb{W}(U, V)Z - \mathbb{W}(\tilde{R}(\xi, Y)U, V)Z - \mathbb{W}(U, \tilde{R}(\xi, Y)V)Z - \mathbb{W}(U, V)\tilde{R}(\xi, Y)Z = 0.$$  \hfill (8.1)

Combining (4.4) and (8.1), we get

$$(\tilde{R}(\xi, Y))\mathbb{W}(U, V)Z - 2[\eta(U)\eta(Y) - g(Y, U)]\mathbb{W}(\xi, V)Z - 2[\eta(V)\eta(Y) - g(Y, V)]\mathbb{W}(U, \xi)Z - 2[\eta(Z)\eta(Y) - g(Y, Z)]\mathbb{W}(U, V)\xi = 0.$$  \hfill (8.2)

Putting $U = \xi$ in (8.2), we obtain

$$(\tilde{R}(\xi, Y))\mathbb{W}(\xi, V)Z - 2[\eta(V)\eta(Y) - g(Y, V)]\mathbb{W}(\xi, \xi)Z - 2[\eta(Z)\eta(Y) - g(Y, Z)]\mathbb{W}(\xi, V)\xi = 0.$$  \hfill (8.3)

Now putting $Y = \xi$ in (8.2) and using (2.1), we get

$$\mathbb{W}(\xi, \xi)Z = 0.$$  \hfill (8.4)

Using (5.2), (8.4) and (7.4) in (8.3), it follows that

$$(\tilde{R}(\xi, Y))\mathbb{W}(\xi, \xi)Z - 2[\eta(Y)\eta(Z)]\mathbb{W}(\xi, \xi)Z - 2[\eta(Z)\eta(Y)]\mathbb{W}(\xi, \xi)V - 2[\eta(Y)\eta(Z)]\mathbb{W}(\xi, V)\xi = 0.$$  \hfill (8.5)

Again putting $Y = Z = \xi$ in (8.5) and using (2.1), we have

$$\frac{2r}{n(n-1)}[2\eta(V)\xi - V] = 0.$$  

Either $r = 0$, or, $2\eta(V)\xi - V = 0$, which is not possible. Therefore, $r = 0$. \hfill \square

Theorem 8.2. An $n$-dimensional P-Sasakian manifold is semi-symmetric with respect to the quarter-symmetric non-recurrent metric connection $\nabla$ if the scalar curvature tensor of the manifold with respect to the Levi-Civita connection vanishes.

Proof. From the definition of concircular curvature tensor, it follows that $\tilde{R}(\xi, Y).\mathbb{W} = \tilde{R}(\xi, Y).\tilde{R}$. \hfill \square
9. Example

In this section, we construct an example on P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\nabla$ which verifies the results of Section 4 Section 5 and Section 6.

We consider the 5-dimensional manifold $\{(x, y, z, u, v) \in \mathbb{R}^5\}$, where $(x, y, z, u, v)$ are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial u}, \quad e_5 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v},$$

which are linearly independent at each point of $M$.

Let $g$ be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j; \quad i, j = 1, 2, 3, 4, 5. \end{cases}$$

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_5)$, for any $Z \in \mathfrak{X}(M)$. Let $\phi$ be the $(1, 1)$-tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = e_3, \quad \phi(e_4) = e_4, \quad \phi(e_5) = 0.$$

Using the linearity of $\phi$ and $g$, we have

$$\eta(e_5) = 1, \quad \phi^2 Z = Z - \eta(Z)e_5, \quad g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any vector fields $Z, U \in \mathfrak{X}(M)$. Thus for $e_5 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines an almost paracontact metric structure on $M$. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_1, e_4] = 0, \quad [e_1, e_5] = e_1,$$

$$[e_2, e_3] = [e_2, e_4] = 0, \quad [e_2, e_5] = e_2,$$

$$[e_3, e_4] = 0, \quad [e_3, e_5] = e_3, \quad [e_4, e_5] = e_4.$$

The Levi-Civita connection $\nabla$ of the metric tensor $g$ is given by Koszul’s formula

$$2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y)$$

$$- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul’s formula, we get

$$\nabla_{e_1} e_1 = -e_5, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_5, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = 0, \quad \nabla_{e_3} e_5 = e_3,$$

$$\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = -e_5, \quad \nabla_{e_4} e_5 = e_4,$$

$$\nabla_{e_5} e_1 = 0, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0.$$

In view of the above relations, we see that

$$\nabla_X \xi = \phi X, \quad (\nabla_X \phi) Y = -g(X, Y) \xi - \eta(Y) X + 2\eta(X) \eta(Y) \xi,$$

for all $e_5 = \xi$. 


Therefore the manifold is a P-Sasakian manifold with the structure \((\phi, \xi, \eta, g)\).

Using (3.1) in the above equations, we obtain
\[
\begin{align*}
\bar{\nabla}e_1e_1 &= -e_5, \quad \bar{\nabla}e_1e_2 = 0, \quad \bar{\nabla}e_1e_3 = 0, \quad \bar{\nabla}e_1e_4 = 0, \quad \bar{\nabla}e_1e_5 = 2e_1, \\
\bar{\nabla}e_2e_1 &= 0, \quad \bar{\nabla}e_2e_2 = -e_5, \quad \bar{\nabla}e_2e_3 = 0, \quad \bar{\nabla}e_2e_4 = 0, \quad \bar{\nabla}e_2e_5 = 2e_2, \\
\bar{\nabla}e_3e_1 &= 0, \quad \bar{\nabla}e_3e_2 = 0, \quad \bar{\nabla}e_3e_3 = -e_5, \quad \bar{\nabla}e_3e_4 = 0, \quad \bar{\nabla}e_3e_5 = 2e_3, \\
\bar{\nabla}e_4e_1 &= 0, \quad \bar{\nabla}e_4e_2 = 0, \quad \bar{\nabla}e_4e_3 = 0, \quad \bar{\nabla}e_4e_4 = -e_5, \quad \bar{\nabla}e_4e_5 = 2e_4, \\
\bar{\nabla}e_5e_1 &= -e_1, \quad \bar{\nabla}e_5e_2 = -e_2, \quad \bar{\nabla}e_5e_3 = -e_3, \quad \bar{\nabla}e_5e_4 = -e_4, \quad \bar{\nabla}e_5e_5 = e_5.
\end{align*}
\]

Now, we can easily obtain the non-zero components of the curvature tensors
\[
\begin{align*}
R(e_1, e_2)e_1 &= e_2, \quad R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_1 = e_3, \quad R(e_1, e_3)e_3 = -e_1, \\
R(e_1, e_4)e_1 &= e_4, \quad R(e_1, e_4)e_2 = -e_1, \quad R(e_1, e_5)e_1 = e_5, \quad R(e_1, e_5)e_5 = -e_1, \\
R(e_2, e_3)e_2 &= e_3, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_2, e_4)e_2 = e_4, \quad R(e_2, e_4)e_4 = -e_2, \\
R(e_2, e_5)e_2 &= e_5, \quad R(e_2, e_5)e_3 = -e_2, \quad R(e_3, e_4)e_3 = e_4, \quad R(e_3, e_4)e_4 = -e_3, \\
R(e_3, e_5)e_3 &= e_5, \quad R(e_3, e_5)e_4 = -e_3, \quad R(e_4, e_5)e_4 = e_5, \quad R(e_4, e_5)e_5 = -e_4.
\end{align*}
\]

and
\[
\begin{align*}
\bar{R}(e_1, e_2)e_1 &= e_2, \quad \bar{R}(e_1, e_2)e_2 = -e_1, \quad \bar{R}(e_1, e_3)e_1 = e_3, \quad \bar{R}(e_1, e_3)e_3 = -e_1, \\
\bar{R}(e_1, e_4)e_1 &= e_4, \quad \bar{R}(e_1, e_4)e_2 = -e_1, \quad \bar{R}(e_1, e_5)e_1 = 2e_5, \quad \bar{R}(e_1, e_5)e_5 = 0, \\
\bar{R}(e_2, e_3)e_2 &= e_3, \quad \bar{R}(e_2, e_3)e_3 = -e_2, \quad \bar{R}(e_2, e_4)e_2 = e_4, \quad \bar{R}(e_2, e_4)e_4 = -e_2, \\
\bar{R}(e_2, e_5)e_2 &= 2e_5, \quad \bar{R}(e_2, e_5)e_3 = -e_2, \quad \bar{R}(e_3, e_4)e_3 = e_4, \quad \bar{R}(e_3, e_4)e_4 = -e_3, \\
\bar{R}(e_3, e_5)e_3 &= 2e_5, \quad \bar{R}(e_3, e_5)e_4 = 0, \quad \bar{R}(e_4, e_5)e_4 = 2e_5, \quad \bar{R}(e_4, e_5)e_5 = 0.
\end{align*}
\]

With the help of the above results we find the Ricci tensors
\[
\begin{align*}
S(e_1, e_1) &= S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = S(e_5, e_5) = -4, \\
S(e_1, e_1) &= S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = -5, \quad S(e_5, e_5) = 0.
\end{align*}
\]

Also, it follows that the scalar curvature tensors with respect to the Levi-Civita connection and the quarter-symmetric non-recurrent metric connection \(\bar{\nabla}\) are \(r = -20\) and \(\bar{r} = -20\) respectively. Thus the manifold under consideration satisfies equations (15, 16) and (17) of Section 3.

Let \(X, \ Y, \ Z\) and \(U\) be any four vector fields given by
\[
\begin{align*}
X &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5, \quad Y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5, \\
Z &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5, \quad W = d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5.
\end{align*}
\]
where \(a_i, b_i, c_i, d_i\), for all \(i = 1, 2, 3, 4, 5\) are all non-zero real numbers.

Using the above curvature tensors and the scalar curvatures of the quarter-symmetric non-recurrent metric connection \(\bar{\nabla}\), we obtain
\[
\bar{\nabla}(X, Y)\xi = [(a_5b_1 - a_1b_5)e_1 + (a_5b_2 - a_2b_5)e_2 + (a_5b_3 - a_3b_5)e_3 + (a_5b_4 - a_4b_5)e_4].
\]
Hence P-Sasakian manifolds will be $\xi$-concircularly flat with respect to the quarter-symmetric non-recurrent metric connections $\bar{\nabla}$ if

$$a_1b_2c_2d_1 + a_2b_1c_1d_2 = \frac{4}{3}$$

which verifies the result of Section 5.

From the above relations, we see that $\mathbb{W}(\phi X, \phi Y, \phi Z, \phi U) = 0$.

The above arguments tell us that the 5-dimensional P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent metric connections $\bar{\nabla}$ under consideration agree with the Section 6.

Acknowledgement. The author is thankful to the referee for his/ her valuable comments towards the improvement of my paper.

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