Two-weighted estimates for multilinear Hausdorff operators on the Morrey–Herz spaces

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Abstract
The purpose of this paper is to establish some necessary and sufficient conditions for the boundedness of the multilinear Hausdorff operators on the product of some two-weighted function spaces such as the two-weighted Morrey, Herz and Morrey–Herz spaces. Moreover, some sufficient conditions for the boundedness of multilinear Hausdorff operators on such spaces with respect to the Muckenhoupt weights are also given.

Keywords Multilinear operator · Hausdorff operator · Hardy–Cesàro operator · Two-weighted Morrey–Herz space · Muckenhoupt weights

Mathematics Subject Classification 42B35 · 46E30 · 47B38

1 Introduction
The one-dimensional Hausdorff operator is defined by
\[ \mathcal{H}_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt, \]  

(1)

where \( \Phi \) is a locally integrable function on the positive half-line. The Hausdorff operator was initiated by Hurwitz and Silverman [29] in 1917, and then was proposed by Hausdorff [27] in order to study summability of number series (see also [24] for more details). It is well known that the Hausdorff operator is one of the significant operators in harmonic analysis, and it is used to solve certain classical problems in analysis.

It is obvious that if \( \Phi(t) = \frac{\chi_{[0,1]}(t)}{t} \), then \( \mathcal{H}_\Phi \) reduces to the Hardy operator defined by

\[ \mathcal{H}(f)(x) = \frac{1}{x} \int_0^x f(t) dt, \]

which is one of the most important averaging operators in harmonic analysis. A celebrated Hardy integral inequality can be formulated as follows

\[ \|\mathcal{H}(f)\|_{L^p(\mathbb{R})} \leq \frac{p}{p-1} \|f\|_{L^p(\mathbb{R})}, \quad \text{for all } 1 < p < \infty, \]

and the constant \( \frac{p}{p-1} \) is the best possible.

It is worth pointing out that if the kernel function \( \Phi \) is taken appropriately, then the Hausdorff operator also reduces to many other classical operators in analysis such as the Cesàro operator, Hardy–Littlewood–Pólya operator, Riemann–Liouville fractional integral operator and Hardy–Littlewood average operator (see, e.g., [18, 33] and references therein).

The Hausdorff operator is extended to the high dimensional space by Brown and Móricz [4] and independently by Lerner and Liflyand [39]. To be more precise, let \( \Phi \) be a locally integrable function on \( \mathbb{R}^n \). The Hausdorff operator \( \mathcal{H}_{\Phi,A} \) associated to the kernel function \( \Phi \) is then defined by

\[ \mathcal{H}_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(A(y)x) dy, x \in \mathbb{R}^n, \]

(2)

where \( \Phi \) is a locally integrable function on \( \mathbb{R}^n \) and \( A(y) \) is an \( n \times n \) matrix satisfying \( \det A(y) \neq 0 \) for almost everywhere \( y \) in the support of \( \Phi \) and \( x \) is assumed to be the column \( n \)-vector. It should be pointed out that if we take \( \Phi(t) = |t|^n \psi(t_1) \chi_{[0,1]}(t) \) and \( A(t) = t_1 I_n \) (\( I_n \) is an identity matrix), for \( t = (t_1, t_2, \ldots, t_n) \), where \( \psi : [0, 1] \rightarrow [0, \infty) \) is a measurable function, \( \mathcal{H}_{\Phi,A} \) then reduces to the weighted Hardy–Littlewood average operator (see [26] for more details) defined by

\[ \mathcal{H}_\psi f(x) = \int_0^1 f(tx) \psi(t) dt, x \in \mathbb{R}^n. \]

(3)
Under certain conditions on $\psi$, Carton-Lebrun and Fosset [21] proved that $\mathcal{H}_\psi$ maps $L^p(\mathbb{R}^n)$ into itself for all $1 < p < \infty$. They also pointed out that the operator $\mathcal{H}_\psi$ commutes with the Hilbert transform when $n = 1$, and with certain Calderón–Zygmund singular integral operators including the Riesz transform when $n \geq 2$. A further extension of the results obtained in [21] to the Hardy space and BMO space is due to Xiao [46].

By letting $U(y) = |y|^n \psi(y) \chi_{[0,1]^n}(y)$ and $A(y) = s(y).I_n$, where $s : [0, 1] \to \mathbb{R}$ is a measurable function, Chuong and Hung [19] studied the operator $U_{\psi,s}$, so-called the generalized Hardy–Cesàro operator associated with the parameter curve $s(x,t) := s(t)x$, (see also [32]) defined by

$$U_{\psi,s}(f)(x) = \int_0^1 f(s(t)x)\psi(t)dt, \quad x \in \mathbb{R}^n.$$ 

In recent years, the theory of weighted Hardy–Littlewood average operators, Hardy–Cesàro operators and Hausdorff operators have been significantly developed into different contexts, and studied on many function spaces such as Lebesgue, Morrey, Herz, Morrey–Herz, Hardy and BMO spaces including the weighted settings. For more details, one may find in [2–5, 31, 34–39, 41, 43, 44] and references therein.

On the other hand, it is useful to observe that Coifman and Meyer [8] discovered a multilinear point of view in their study of certain singular integral operators. Thus, the research of the theory of multilinear operators is not only attracted by a pure question to generalize the theory of linear ones but also by their deep applications in harmonic analysis. In 2015, Fu et al. [23] introduced the weighted multilinear Hardy operator of the form

$$\mathcal{H}_\psi^m(f)(x) = \int_{[0,1]^n} \left( \prod_{i=1}^m f_i(y;x) \right) \psi(y)dy, \quad x \in \mathbb{R}^n,$$

where $\psi : [0, 1]^n \to [0, \infty)$ is an integrable function, $f = (f_1, \ldots, f_m), f_i, i = 1, \ldots, m,$ are complex-valued measurable functions on $\mathbb{R}^n$. They obtained the sharp bounds for $\mathcal{H}_\psi^m$ on the product of Lebesgue spaces and central Morrey spaces. As a consequence, these results are also employed to prove sharp estimates of some inequalities due to Riemann–Liouville and Weyl. Later, Hung and Ky [28] studied the weighted multilinear Hardy–Cesàro type operators, which are generalized of weighted multilinear Hardy operators, as follows

$$U_{\psi,s}^m(f)(x) = \int_{[0,1]^n} \left( \prod_{i=1}^m f_i(s_i(y)x) \right) \psi(y)dy, \quad x \in \mathbb{R}^n,$$

where $s_1, \ldots, s_m : [0, 1]^n \to \mathbb{R}$. They are also obtained the sharp bounds of weighted multilinear Hardy–Cesàro operators on the product of weighted Lebesgue spaces and central Morrey spaces.
Very recently, Chuong et al. [12] have introduced and studied a more general class of multilinear Hausdorff operators which is defined by

$$
\mathcal{H}_{\Phi, A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} f_i(A_i(y)x) \, dy, \quad x \in \mathbb{R}^n.
$$

Let us take measurable functions $s_1(y), \ldots, s_m(y) \neq 0$ almost everywhere in $\mathbb{R}^n$. Consider a special case where the matrices $A_i(y) = \text{diag}\{s_i(y), \ldots, s_i(y)\}$, for all $i = 1, \ldots, m$. Then we also study in this paper the hybrid multilinear operator of the form

$$
\mathcal{H}_{\phi, s}(f)(x) = \int_{\mathbb{R}^n} \left( \prod_{i=1}^{m} f_i(s_i(y)x) \right) \phi(y) \, dy, \quad x \in \mathbb{R}^n.
$$

Note that by letting $\phi(y) = \psi(y) \chi_{[0,1]^n}(y)$, it is clear that the operator $\mathcal{H}_{\phi, s}$ reduces to the operator $U_{\phi, s}^{m,n}$. Recently, the multilinear Hausdorff operators have been extended to study on some function spaces in the real field as well as $p$-adic numbers field. The interested reader is referred to the works [9–14, 17, 20, 23, 28] for more details.

In the recent years, there is an increasing interest in the study of the problems concerning the two-weight norm inequality for many fundamental operators in harmonic analysis, for example, such as maximal operator, the Hilbert transform, singular integral operator, Hardy operator and Hausdorff operator. For weighted Hardy inequalities, see the survey paper by Gogatishvili and Stepanov [25] and references therein. It is therefore of interest to extend the study of the two-weight norm inequalities for the multilinear Hausdorff operators. In [15], the Hausdorff operators is studied on two-weighted Herz-type Hardy spaces. In this paper, we investigate the boundedness of the multilinear Hausdorff operators on two-weighted Morrey–Herz spaces. More details, we give some necessary and sufficient conditions for the boundedness of the multilinear Hausdorff operators on the product of the two weighted Morrey, Herz and Morrey–Herz spaces. Also, in the setting of the function spaces with the Muckenhoupt weights, some sufficient conditions for the boundedness of multilinear Hausdorff operators are also discussed.

Our paper is organized as follows. In Sect. 2, we deal with necessary preliminaries on weighted Lebesgue spaces, two-weighted central Morrey spaces, two-weighted Herz spaces, two-weighted Morrey–Herz spaces and the class of the Muckenhoupt weights that we shall use in the sequel. The main theorems and their proofs in this paper are given in Sect. 3.

## 2 Preliminaries

In this section, let us present some basic facts and notations which will be used throughout this paper. By $\|T\|_{X \rightarrow Y}$, we denote the norm of $T$ between two normed vector spaces $X$ and $Y$. The letter $C$ denotes a positive constant which is independent of the main parameters, but may be different from line to line. For any $a \in \mathbb{R}^n$ and
Let \( 0 < q < \infty \) be the space of all Lebesgue measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{L^q_0(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^q \omega(x) \, dx \right)^{\frac{1}{q}} < \infty.
\]

The space \( L^q_{\text{loc}}(\mu, \mathbb{R}^n) \) is defined as the set of all measurable functions \( f \) on \( \mathbb{R}^n \) satisfying \( \int_K |f(x)|^q \omega(x) \, dx < \infty \) for any compact subset \( K \) of \( \mathbb{R}^n \). The space \( L^q_{\text{loc}}(\mu, \mathbb{R}^n \setminus \{0\}) \) is also defined in a similar way to the space \( L^q_{\text{loc}}(\mu, \mathbb{R}^n) \).

In what follows, denote \( \chi_k = \chi_{C_k}, C_k = B_k \setminus B_{k-1} \) and \( B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\} \), for all \( k \in \mathbb{Z} \). Now, we are in a position to give some definitions of the two-weighted Morrey, Herz and Morrey–Herz spaces. For further information on these spaces as well as their deep applications in analysis, the interested readers may refer to the works [1, 6, 7, 16, 20], especially, to the monograph [40] and references therein.

**Definition 1** Let \( 0 < q < \infty \) and \( 1 + \lambda q > 0, \lambda \in \mathbb{R} \). Suppose \( v, \omega \) are two weighted functions. Then the two-weighted central Morrey space is defined by

\[
\mathcal{M}^{q, \lambda}_{v, \omega}(\mathbb{R}^n) = \{f \in L^q_{\text{loc}}(\mu) : \|f\|_{\mathcal{M}^{q, \lambda}_{v, \omega}(\mathbb{R}^n)} < \infty\},
\]

where

\[
\|f\|_{\mathcal{M}^{q, \lambda}_{v, \omega}(\mathbb{R}^n)} = \sup_{R > 0} \frac{1}{v(B(0, R))^{\lambda + \frac{1}{q}}} \|f\|_{L^q_0(B(0, R))}.
\]

In particular, if \( v = \omega \), then we write \( \mathcal{M}^{q, \lambda}_{v, v}(\mathbb{R}^n) = \mathcal{M}^{q, \lambda}_{v}(\mathbb{R}^n) \) which is called the weighted central Morrey space.

**Definition 2** Let \( 0 < p < \infty, 1 < q < \infty \) and \( \alpha \in \mathbb{R} \). Let \( v \) and \( \omega \) be two weighted functions. The homogeneous two-weighted Herz space \( K^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n) \) is defined to be the set of all \( f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, \omega) \) such that

\[
\|f\|_{K^{\alpha,p,q}_{v,\omega}(\mathbb{R}^n)} = \left( \sum_{k=\infty}^{\infty} v(B_k)^{\frac{ap}{q}} \|f \chi_k\|_{L^q_0(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} < \infty.
\]
Remark that if \( v \) is a constant function, then \( \hat{K}_{v,\omega}^{p,q}(\mathbb{R}^n) := K_{p,q}^{\omega}(\omega, \mathbb{R}^n) \) is the weighted Herz space studied in [16, 40].

**Definition 3** Let \( \alpha \in \mathbb{R}, 0 < p < \infty, \beta \geq 0 \) and \( v, \omega \) be weighted functions. Then the two-weighted Morrey–Herz space \( MK_{v,\omega}^{p,\lambda,p,q}(\mathbb{R}^n) \) is defined as the space of all functions \( f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, \omega) \) such that

\[
\|f\|_{MK_{v,\omega}^{p,\lambda,p,q}(\mathbb{R}^n)} = \left( \sum_{k_0 = -\infty}^{k_0} \left( \frac{1}{v(B_{k_0})} \right) \frac{1}{\lambda(n)\|f\|_{L^p_{\omega}(\mathbb{R}^n)}} \right)^{1/p}.
\]

In particular, when \( \omega = v \), we denote \( MK_{v,\omega}^{p,\lambda,p,q}(\mathbb{R}^n) \) instead of \( MK_{v,\omega}^{p,\lambda,p,q}(\mathbb{R}^n) \). Note that if \( \lambda = 0 \), it is easy to see that \( MK_{v,\omega}^{p,0,p,q}(\mathbb{R}^n) = \hat{K}_{v,\omega}^{p,q}(\mathbb{R}^n) \). Consequently, the two-weighted Herz space is a special case of the two-weighted Morrey–Herz space. Some applications of two-weighted Morrey–Herz spaces to the Hardy–Cesàro operators may be found, for example, in the works [16, 20]. It should be pointed out that the Herz spaces are natural generalization of the Lebesgue spaces with power weight. Moreover, the Herz spaces and Morrey–Herz spaces have some important applications to the theory of singular integrals and the theory of partial differential equations.

Next, we will recall some preliminaries on the theory of \( A_p \) weights which was first introduced by Benjamin Muckenhoupt [42] in the Euclidean spaces in order to characterize the weighted \( L^p \) boundedness of Hardy–Littlewood maximal functions. Let us recall that a weight is a nonnegative, locally integrable function on \( \mathbb{R}^n \).

**Definition 4** Let \( 1 < \xi < \infty \). It is said that a weight \( \omega \in A_\xi(\mathbb{R}^n) \) if there exists a constant \( C > 0 \) such that for all balls \( B \subset \mathbb{R}^n \),

\[
\left( \frac{1}{|B|} \int_B \omega(x) \, dx \right) \left( \frac{1}{|B|} \int_B \omega(x)^{-\frac{1}{\xi-1}} \, dx \right)^{\xi-1} \leq C.
\]

We say that a weight \( \omega \in A_1(\mathbb{R}^n) \) if there is a constant \( C > 0 \) such that for all balls \( B \subset \mathbb{R}^n \),

\[
\frac{1}{|B|} \int_B \omega(x) \, dx \leq C \inf_{x \in B} \omega(x).
\]

We denote

\[
A_\infty(\mathbb{R}^n) = \bigcup_{1 \leq \xi < \infty} A_\xi(\mathbb{R}^n).
\]
Let us now recall the following standard results related to the Muckenhoupt weights. For further readings on the class of the Muckenhoupt weights as well as its deep applications, see [22, 45].

**Proposition 1**

(i) \( A_{\xi}^{\frac{1}{\xi}}(\mathbb{R}^n) \subseteq A_{\eta}^{\frac{1}{\eta}}(\mathbb{R}^n) \), for \( 1 \leq \xi < \eta < \infty \).

(ii) If \( \omega \in A_{\xi}^{1/\xi}(\mathbb{R}^n) \), \( 1 < \xi < \infty \), then there is an \( \varepsilon > 0 \) such that \( \xi - \varepsilon > 1 \) and \( \omega \in A_{\xi-\varepsilon}(\mathbb{R}^n) \).

It is worth noticing that the class \( A_{\infty}^{1}(\mathbb{R}^n) \) is intimately connected with the reverse Hölder condition. More precisely, if there exist \( r > 1 \) and a fixed constant \( C \) such that

\[
\left( \frac{1}{|B|} \int_B \omega(x)^r \, dx \right)^{\frac{1}{r}} \leq C \left( \frac{1}{|B|} \int_B \omega(x) \, dx \right),
\]

for all balls \( B \subset \mathbb{R}^n \), we then say that \( \omega \) satisfies the reverse Hölder condition of order \( r \) and write \( \omega \in RH_r \). According to Theorem 19 and Corollary 21 in [30], \( \omega \in A_{\infty} \) if and only if there exists some \( r > 1 \) such that \( \omega \in RH_r \). Moreover, if \( \omega \in RH_r, r > 1 \) then \( \omega \in RH_{r+\varepsilon} \) for some \( \varepsilon > 0 \). We thus write \( r_{\omega} = \sup \{ r > 1 : \omega \in RH_r \} \) to denote the critical index of \( \omega \) for the reverse Hölder condition.

**Proposition 2** If \( \omega \in A_{\xi}^{1/\xi}(\mathbb{R}^n), 1 \leq \xi < \infty \), then for any \( f \in L_{\text{loc}}^1(\mathbb{R}^n) \) and any ball \( B \subset \mathbb{R}^n \),

\[
\frac{1}{|B|} \int_B |f(x)| \, dx \leq C \left( \frac{1}{\omega(B)} \int_B |f(x)|^{\xi} \omega(x) \, dx \right)^{\frac{1}{\xi}}.
\]

**Proposition 3** Let \( \omega \in A_{\xi} \cap RH_{\delta}, \) for \( \xi \geq 1 \) and \( \delta > 1 \). Then there exist two constants \( C_1, C_2 > 0 \) such that

\[
C_1 \left( \frac{|E|}{|B|} \right)^{\delta} \leq \frac{\omega(E)}{\omega(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{\frac{\delta}{\xi}},
\]

for any measurable subset \( E \) of a ball \( B \). In particular, for any \( \lambda > 1 \), we have

\[
\omega(B(x_0, \lambda R)) \leq C \lambda^{n\gamma} \omega(B(x_0, R)).
\]

### 3 Main results and their proofs

Before stating our main results, we introduce some notations that are often used in this section. Throughout this section, \( \beta, \gamma; \beta_1, \gamma_1, \ldots, \beta_m, \gamma_m \) are real numbers greater than \(-n\), and \( 1 \leq p, q < \infty, 1 \leq p_i, q_i < \infty \) for all \( i = 1, \ldots, m \) satisfying...
For a matrix \( A = (a_{ij})_{n \times n} \), we define the norm of \( A \) as follows

\[
\|A\| = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.
\]

It is easy to see that \( |Ax| \leq \|A\||x| \) for any vector \( x \in \mathbb{R}^n \). In particular, if \( A \) is invertible, we then have

\[
\|A\|^{-n} \leq |\det(A^{-1})| \leq \|A^{-1}\|^{-n}.
\]

The aim of the first part of the paper is to provide some sufficient and necessary conditions of the weighted functions for the boundedness of multilinear Hausdorff operators on the two-weighted Morrey, Herz, and Morrey–Herz spaces with power weights provided

\[
\rho_\Lambda := \text{ess sup}_{t \in \mathbb{R}^n, i=1,\ldots,m} \|A_i(t)\| \cdot \|A_i^{-1}(t)\| < \infty. \quad (8)
\]

As some applications, we also obtain the boundedness and bounds of the multilinear Hardy–Cesàro operators on such spaces.

Observe that if \( A(t) \) is a real orthogonal matrix for almost everywhere \( t \) in \( \mathbb{R}^n \), then \( A(t) \) satisfies the condition \((8)\). It is useful to remark that the condition \((8)\) implies \( \|A_i(t)\| \simeq \|A_i^{-1}(t)\|^{-1} \). Moreover, it is easily seen that

\[
\|A_i(t)\|^\eta \lesssim \|A_i^{-1}(t)\|^{-\eta}, \text{ for all } \eta \in \mathbb{R}, \quad (9)
\]

and

\[
|A_i(t)x|^\eta \gtrsim \|A_i^{-1}(t)\|^{-\eta}|x|^\eta, \text{ for all } \eta \in \mathbb{R}, x \in \mathbb{R}^n. \quad (10)
\]

Now, we are in a position to state and prove our first main results concerning with the boundedness of multilinear Hausdorff operators on two-weighted central Morrey spaces.

**Theorem 1** Let \( \Phi : \mathbb{R}^n \rightarrow [0, \infty) \), and \( v(x) = |x|^\beta, \omega(x) = |x|^\gamma, \nu_i(x) = |x|^\beta_i, \omega_i(x) = |x|^\gamma_i, \) for all \( i = 1,\ldots,m \), and the following conditions hold

\[
\sum_{i=1}^{m} \beta_i \gamma_i = \beta \gamma, \quad \sum_{i=1}^{m} \left( \frac{n+\beta_i}{n+\beta} \right) \lambda_i = \lambda, \text{ and } \sum_{i=1}^{m} \frac{\gamma_i}{q_i} = \frac{\gamma}{q}.
\]

Then we have that the operator \( \mathcal{H}_{\Phi, \Lambda} \) is bounded from \( \Pi_{i=1}^{m} M_{\nu_i, \omega_i}^{\eta_i, \lambda_i}(\mathbb{R}^n) \) to \( M_{\nu, \omega}^{\eta, \lambda}(\mathbb{R}^n) \) if and only if
Moreover,

$$\|H_{f,A}\|_{L^q_{\ast}(B(0,R))} \approx C_1.$$ 

**Proof** First, we prove the sufficient condition of this theorem. For $m \sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$, it is easy to see that $\prod_{i=1}^{m} \omega_i^\frac{1}{q_i} (x) = \omega(x)$. Then by the Minkowski inequality we have

$$\|H_{f,A}(f)\|_{L^q_{\ast}(B(0,R))} = \left( \int_{B(0,R)} \left| \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} f_i(A_i(y)x) \omega_i^\frac{1}{q_i} (x) dy \right|^q dx \right)^{\frac{1}{q}} \leq \int_{\mathbb{R}^n} |\Phi(y)| \left( \prod_{i=1}^{m} \|f_i(A_i(y)x)\|_{L^q_{\ast}(B(0,R))} \right) dy.$$ 

From $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$ and applying the Hölder inequality, we see at once that

$$\|H_{f,A}(f)\|_{L^q_{\ast}(B(0,R))} \leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \left( \prod_{i=1}^{m} \|f_i(A_i(y)x)\|_{L^q_{\ast}(B(0,R))} \right) dy.$$ 

Using change of variable $z = A_i(y)x$ with $z \in A_i(y)B(0,R)$, we have

$$\|H_{f,A}(f)\|_{L^q_{\ast}(B(0,R))} \leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \left( \prod_{i=1}^{m} \int_{B(0,\|A_i(y)||R)} |f_i(z)|^q |A^{-1}_i(y)z|^{q_i} |\det A^{-1}_i(y)| dz \right)^{\frac{1}{q}} dy.$$ 

By (9), for all $\gamma_i \in \mathbb{R}$, we have the following useful inequality

$$|A^{-1}_i(y)z|^{\gamma_i} \leq \max \{\|A^{-1}_i(y)\|^{\gamma_i}, \|A_i(y)\|^{-\gamma_i}\} |z|^{\gamma_i}.$$ 

Consequently, we also obtain

$$\|H_{f,A}(f)\|_{L^q_{\ast}(B(0,R))} \leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \left( \prod_{i=1}^{m} \max \{\|A^{-1}_i(y)\|^{\gamma_i}, \|A_i(y)\|^{-\gamma_i}\} |\det A^{-1}_i(y)| \right) \times \int_{B(0,\|A_i(y)||R)} |f_i(z)|^q \omega_i(z) dz \right)^{\frac{1}{q_i}} dy \leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \max \{\|A^{-1}_i(y)\|^{\gamma_i}, \|A_i(y)\|^{-\gamma_i}\} \right)^{\frac{1}{q_i}} |\det A^{-1}_i(y)|^{\frac{1}{q_i}} = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \max \{\|A^{-1}_i(y)\|^{\gamma_i}, \|A_i(y)\|^{-\gamma_i}\} \right)^{\frac{1}{q_i}} |\det A^{-1}_i(y)|^{\frac{1}{q_i}} \|f_i\|_{L^q_{\ast}(B(0,\|A_i(y)||R))} dy.$$ 

(11)
From the definition of two-weighted central Morrey space, we get

\[
\| \mathcal{H}_{\Phi,A}(f) \|_{M_{\lambda,q}^p(\mathbb{R}^n)} \leq \sup_{R > 0} \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \frac{1}{v(B(0,R))^{1 + \frac{1}{q}}} \prod_{i=1}^{m} \max \{ \| A_i^{-1}(y) \|^{\gamma_i}, \| A_i(y) \|^{-\gamma_i} \}^\frac{1}{m} \\
\times | \det A_i^{-1}(y) |^\frac{1}{m} \| f_i \|_{L_{\lambda,q}^m(B(0,\| A_i(y) \| R))} dy. 
\]

(12)

Remark that by the conditions \( \lambda + \frac{1}{q} > 0, \lambda_i + \frac{1}{q_i} > 0, \) \( i = 1, \ldots, m, \) and \( \sum_{i=1}^{m} (\beta_i + n)(\lambda_i + \frac{1}{q_i}) = (\beta + n) \left( \lambda + \frac{1}{q} \right), \) it is easy to see that

\[
v(B(0,R))^{\frac{1}{\lambda} + \frac{1}{q}} \lesssim R^{(\beta + n)(\lambda + \frac{1}{q})},
\]

and

\[
\frac{\prod_{i=1}^{m} v_i(B(0,\| A_i(y) \| R))^{(\lambda_i + \frac{1}{q_i})}}{v(B(0,R))^{\frac{1}{\lambda} + \frac{1}{q}}} \lesssim \prod_{i=1}^{m} \| A_i(y) \|^{(\beta_i + n)(\lambda_i + \frac{1}{q_i})}. 
\]

This shows that

\[
\frac{1}{v(B(0,R))^{\frac{1}{\lambda} + \frac{1}{q}}} \lesssim \prod_{i=1}^{m} \| A_i(y) \|^{(\beta_i + n)(\lambda_i + \frac{1}{q_i})}. 
\]

By the above estimations, we have

\[
\| \mathcal{H}_{\Phi,A}(f) \|_{M_{\lambda,q}^p(\mathbb{R}^n)} \leq \sup_{R > 0} \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \max \{ \| A_i^{-1}(y) \|^{\gamma_i}, \| A_i(y) \|^{-\gamma_i} \}^\frac{1}{m} | \det A_i^{-1}(y) |^\frac{1}{m} \\
\times \prod_{i=1}^{m} \| A_i(y) \|^{(\beta_i + n)(\lambda_i + \frac{1}{q_i})} \| f_i \|_{L_{\lambda,q}^m(B(0,\| A_i(y) \| R))} dy \\
\leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \max \{ \| A_i^{-1}(y) \|^{\gamma_i}, \| A_i(y) \|^{-\gamma_i} \}^\frac{1}{m} | \det A_i^{-1}(y) |^\frac{1}{m} \\
\times \| A_i(y) \|^{(\beta_i + n)(\lambda_i + \frac{1}{q_i})} dy \cdot \prod_{i=1}^{m} \| f_i \|_{L_{\lambda,q}^m(B(0,\| A_i(y) \| R))}. 
\]

By (9) and the property of invertible matrices, we get

\[
| \det A_i^{-1}(y) |^\frac{1}{m} \leq \| A_i^{-1}(y) \|^{\frac{1}{m}}, \\
\| A_i(y) \|^{(\beta_i + n)(\lambda_i + \frac{1}{q_i})} \lesssim \| A_i^{-1}(y) \|^{-(\beta_i + n)(\lambda_i + \frac{1}{q_i})}. 
\]
and
\[
\prod_{i=1}^{m} \max \{||A_i^{-1}(y)||_v, ||A_i(y)||_{\gamma_i}^{-1} \}^{\frac{1}{\gamma_i}} \lesssim \prod_{i=1}^{m} ||A_i^{-1}(y)||_{v_i}^{\frac{1}{\gamma_i}}.
\]
This implies that
\[
\prod_{i=1}^{m} \max \{||A_i^{-1}(y)||_v, ||A_i(y)||_{\gamma_i}^{-1} \}^{\frac{1}{\gamma_i}} |\det A_i^{-1}(y)|^{\frac{1}{\gamma_i}} ||A_i(y)||^{(\beta_i+n)(\lambda_i + \frac{1}{q_i})} \lesssim \prod_{i=1}^{m} ||A_i^{-1}(y)||^{-(\beta_i+n)(\lambda_i + \frac{1}{q_i})}. \tag{13}
\]
Consequently,
\[
\|\mathcal{H}_{\phi,A}\|_{\check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)} \lesssim C_1 \prod_{i=1}^{m} \|f_i\|_{\check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)}.
\]
Therefore, the operator \(\mathcal{H}_{\phi,A}\) is bounded from \(\prod_{i=1}^{m} \check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)\) to \(\check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)\).

Conversely, suppose that \(\mathcal{H}_{\phi,A}\) is determined as a bounded operator from the product \(\prod_{i=1}^{m} \check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)\) to \(\check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)\). Then let us choose
\[
f_i(x) = |x|^{(\beta_i+n)(\lambda_i + \frac{1}{q_i})}.
\]
It is evident that \(\|f_i\|_{\check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)} > 0\), for all \(i = 1, \ldots, m\). Now, we need to show that
\[
\|f_i\|_{\check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)} < \infty, \text{ for all } i = 1, \ldots, m.
\]
Indeed, we have
\[
\|f_i\|_{\check{M}^{q,i}_{v_i,\alpha_i}(\mathbb{R}^n)} = \sup_{R > 0} \frac{1}{V_{\gamma_i}(B(0,R))^{\frac{1}{\gamma_i}}} \left( \int_{B(0,R)} |x|^{(\beta_i+n)(\lambda_i + \frac{1}{q_i})} dx \right)^{\frac{1}{\gamma_i}}
\]
\[
= \sup_{R > 0} \frac{1}{R^{(\beta_i+n)(\lambda_i + \frac{1}{q_i})}} \left( \int_{0}^{R} \int_{S_{n-1}} |x'|^{(\beta_i+n)(\lambda_i + \frac{1}{q_i})} r^{n-1} d\sigma(x') dr \right)^{\frac{1}{\gamma_i}}
\]
\[
\simeq \sup_{R > 0} \frac{1}{R^{(\beta_i+n)(\lambda_i + \frac{1}{q_i})}} \left( \frac{1}{\gamma_i} \right)^{\frac{1}{\gamma_i}} \lesssim (\beta_i + n)(\lambda_i + \frac{1}{q_i}) < \infty.
\]
Thus, by choosing \(f_i\) and the condition (10), we conclude that...
\[ \| \mathcal{H}_{\phi, A}^m(f) \|_{M^p_{\lambda, \gamma}(\mathbb{R}^n)} \geq \int_{\mathbb{R}^n} \Phi(y) \prod_{i=1}^{m} \| A_i^{-1}(y) \|^{-\left( (\beta_i + n) \lambda_i + (\beta_i - \gamma_i) \right)} dy \]

\[ \times \sup_{R > 0} \frac{1}{V(B(0, R))^{\frac{n+1}{4}}} \left( \int_{B(0, R)} |x| \sum_{i=1}^{m} \left( (\beta_i + n) \lambda_i + (\beta_i - \gamma_i) \right)^{q+\gamma} dx \right)^{\frac{1}{q}} \]

\[ \geq C_1 \cdot \prod_{i=1}^{m} \| f_i \|_{M^p_{\lambda_i, \gamma_i}(\mathbb{R}^n)}. \]

Therefore, Theorem 1 is completely proved.

Now, we wish to give an application of Theorem 1. Let us take the matrices
\[ A_i(y) = \text{diag}[s_i(y), \ldots, s_i(y)], \text{ for all } i = 1, \ldots, m, \text{ where the measurable functions } s_1(y), \ldots, s_m(y) \neq 0 \text{ almost everywhere in } \mathbb{R}^n. \text{ It is plain that the matrices } A_i \text{ 's satisfy the condition (8). By virtue of Theorem 1, we also obtain the necessary and sufficient conditions for the boundedness of the multilinear operator } \mathcal{H}_{\phi, s} \text{ on two-weighted central Morrey spaces.} \]

**Corollary 1** Let \( \phi \) be a nonnegative function. Under the same assumptions as Theorem 1, we have that the operator \( \mathcal{H}_{\phi, s} \) is bounded from \( \prod_{i=1}^{m} \mathcal{M}^p_{\lambda_i, \gamma_i}(\mathbb{R}^n) \) to \( \hat{M}^{p, \lambda}_\gamma(\mathbb{R}^n) \) if and only if

\[ C_{1.1} = \int_{\mathbb{R}^n} \left( \prod_{i=1}^{m} |s_i(y)|^{(\beta_i + n) \lambda_i + (\beta_i - \gamma_i) \gamma_i} \right) \phi(y) dy < \infty. \]

Moreover,

\[ \| \mathcal{H}_{\phi, s} \|_{\prod_{i=1}^{m} \mathcal{M}^p_{\lambda_i, \gamma_i}(\mathbb{R}^n) \rightarrow \hat{M}^{p, \lambda}_\gamma(\mathbb{R}^n)} \simeq C_{1.1}. \]

In particular, by virtue of Corollary 1 one can claim that the weighted multilinear Hardy–Cesàro operator \( U_{\psi, s}^{m, n} \) is bounded from \( \prod_{i=1}^{m} \hat{M}^{p, \lambda}_\gamma(\mathbb{R}^n) \) to \( \hat{M}^{p, \lambda}_\gamma(\mathbb{R}^n) \) if and only if

\[ C_{1.2} = \int_{[0,1]^n} \left( \prod_{i=1}^{m} |s_i(t)|^{(\beta_i + n) \lambda_i + (\beta_i - \gamma_i) \gamma_i} \right) \psi(t) dt < \infty. \]

Moreover,

\[ \| U_{\psi, s}^{m, n} \|_{\prod_{i=1}^{m} \hat{M}^{p, \lambda}_\gamma(\mathbb{R}^n) \rightarrow \hat{M}^{p, \lambda}_\gamma(\mathbb{R}^n)} \simeq C_{1.2}. \]

Next, we will give the neccessary and sufficient conditions for the boundedness of the multilinear Hausdorff operator on two-weighted Herz spaces.

**Theorem 2** Let \( \Phi : \mathbb{R}^n \rightarrow [0, \infty) \) and \( \omega_i(x) = |x|^{\gamma_i}, v_i(x) = |x|^{\beta_i} \), for all \( i = 1, \ldots, m \), \( \omega(x) = |x|^\gamma, v(x) = |x|^\beta \), and

\[ \Box \]
\[ \sum_{i=1}^{m} \gamma_i q_i = \gamma q, \text{ and } \sum_{i=1}^{m} \left( 1 + \frac{\beta_i}{n} \right) z_i = \left( 1 + \frac{\beta}{n} \right) z. \]

Then we have that the operator \( \mathcal{H}_{\Phi,A} \) is bounded from \( \prod_{i=1}^{m} K_{\gamma_i,\epsilon_0}^{q_i,q_i}(\mathbb{R}^n) \) to \( K_{\nu,\epsilon_0}^{p,q}(\mathbb{R}^n) \) if and only if
\[
C_2 = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} \left( \|A_i^{-1}(y)\|^{1+\beta_i} \right) dy < \infty.
\]
Moreover,
\[
\|\mathcal{H}_{\Phi,A}\| \prod_{i=1}^{m} K_{\gamma_i,\epsilon_0}^{q_i,q_i}(\mathbb{R}^n) \rightarrow K_{\nu,\epsilon_0}^{p,q}(\mathbb{R}^n) \simeq C_2.
\]

**Proof** First, suppose that \( C_2 < \infty \). Using a similar argument as the inequality (11) above, we have
\[
\|\mathcal{H}_{\Phi,A}(f)\|_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} \max \{ \|A_i^{-1}(y)\|^{|\gamma_i|}, \|A_i(y)\|^{-|\gamma_i|} \} \|\det A_i^{-1}(y)\|^{\frac{\beta_i}{m}} \|f_i\|_{L^q_{\nu}(A_i(y)C_k)} \, dy,
\]
where \( A_i(y)C_k = \{ A_i(y)z | z \in C_k \} \). Because of the condition (8), there exits the greatest integer number \( \kappa^* = \kappa^*(y) \) such that
\[
\max_{i=1,\ldots,m} \{ \|A_i(y)\| \|A_i^{-1}(y)\| \} < 2^{-\kappa^*}, \quad \text{for a.e } y \in \mathbb{R}^n.
\]
Note that from the condition \( 1 \leq \|A_i(y)\| \leq \rho_A \), for a.e \( y \in \mathbb{R}^n \), \( i = 1, \ldots, m \), we have \( |\kappa^*(y)| \approx 1 \) for a.e \( y \in \mathbb{R}^n \).

Let us now fix \( i \in \{ 1, 2, \ldots, m \} \). Since \( \|A_i(y)\| \neq 0 \), there is an integer number \( \ell_i = \ell_i(y) \) such that \( 2^{\ell_i-1} < \|A_i(y)\| \leq 2^{\ell_i} \). For simplicity of notation, we write
\[
\rho_A^*(y) = \max_{i=1,\ldots,m} \{ \|A_i(y)\| \cdot \|A_i^{-1}(y)\| \}.
\]
Then, by letting \( t = A_i(y)z \), with \( z \in C_k \), we have
\[
|t| \geq \|A_i^{-1}(y)\|^{-1} |z| \geq \frac{2^{k+\ell_i-2}}{\rho_A^*(y)} > 2^{k+\ell_i-2+\kappa^*},
\]
and
\[
|t| \leq \|A_i(y)\| |z| \leq 2^{k+\ell_i}.
\]
These estimates can be used to obtain
\[
A_i(y)C_k \subset \{ z \in \mathbb{R}^n : 2^{k+\ell_i-2+\kappa^*} < |z| \leq 2^{k+\ell_i} \},
\]
which implies that
\[
\| \mathcal{H}_{\phi,A}(f) \|_{L^p_{\nu}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \max \left\{ \|A_i^{-1}(y)\|^{\gamma_i}, \|A_i(y)\|^{-\gamma_i} \right\}^{\frac{1}{n}} \\
\times \left| \det A_i^{-1}(y) \right|^{\frac{1}{n}} \left( \sum_{r=\kappa-1}^{0} \| f_i \mathcal{H}_{k+\ell+r} \|_{L^p_{\nu}(\mathbb{R}^n)} \right) \, dy. 
\]
(14)

On the other hand, by the definition of two-weighted Herz space and the Minkowski inequality, we get

\[
\| \mathcal{H}_{\phi,A}(f) \|_{L^p_{\nu}(\mathbb{R}^n)} \\
= \left( \sum_{k=-\infty}^{\infty} v(B_k)^{\alpha} \right) \left( \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \max \left\{ \|A_i^{-1}(y)\|^{\gamma_i}, \|A_i(y)\|^{-\gamma_i} \right\}^{\frac{1}{n}} \\
\times \left| \det A_i^{-1}(y) \right|^{\frac{1}{n}} \left( \sum_{r=\kappa-1}^{0} \| f_i \mathcal{H}_{k+\ell+r} \|_{L^p_{\nu}(\mathbb{R}^n)} \right) \, dy \right)^{\frac{1}{p}} \\
\leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \max \left\{ \|A_i^{-1}(y)\|^{\gamma_i}, \|A_i(y)\|^{-\gamma_i} \right\}^{\frac{1}{n}} \left| \det A_i^{-1}(y) \right|^{\frac{1}{n}} \\
\times \left( \sum_{k=-\infty}^{\infty} v(B_k)^{\alpha} \left( \prod_{i=1}^{m} \sum_{r=\kappa-1}^{0} \| f_i \mathcal{H}_{k+\ell+r} \|_{L^p_{\nu}(\mathbb{R}^n)} \right) \right)^{\frac{1}{p}} \, dy.
\]

Notice that taking the condition \( \sum_{i=1}^{m} (n + \beta_i) \alpha_i = (n + \beta) \alpha \) into account, we get \( v(B_k)^{\alpha} \simeq \prod_{i=1}^{m} v_i(B_k)^{\alpha_i} \). Applying the Hölder inequality, we have

\[
\left( \sum_{k=-\infty}^{\infty} v(B_k)^{\alpha} \left( \prod_{i=1}^{m} \sum_{r=\kappa-1}^{0} \| f_i \mathcal{H}_{k+\ell+r} \|_{L^p_{\nu}(\mathbb{R}^n)} \right) \right)^{\frac{1}{p}} \\
\simeq \left( \sum_{k=-\infty}^{\infty} \left( \prod_{i=1}^{m} v_i(B_k)^{\alpha_i} \sum_{r=\kappa-1}^{0} \| f_i \mathcal{H}_{k+\ell+r} \|_{L^p_{\nu}(\mathbb{R}^n)} \right) \right)^{\frac{1}{p}} \\
\lesssim \prod_{i=1}^{m} \left( \sum_{k=-\infty}^{\infty} v_i(B_k)^{\alpha_i} \left( \sum_{r=\kappa-1}^{0} \| f_i \mathcal{H}_{k+\ell+r} \|_{L^p_{\nu}(\mathbb{R}^n)} \right) \right)^{\frac{1}{p}}.
\]

Moreover, by using the known inequality \( \left( \sum_{i=1}^{N} |a_i| \right)^p \leq N^{p-1} \sum_{i=1}^{N} |a_i|^p \) for all \( p \geq 1 \), we have, by \( p_i \geq 1 \),

\[
\left( \sum_{r=\kappa-1}^{0} \| f_i \mathcal{H}_{k+\ell+r} \|_{L^p_{\nu}(\mathbb{R}^n)} \right)^{p_i} \leq (2 - \kappa^*)^{p-1} \sum_{r=\kappa-1}^{0} \| f_i \mathcal{H}_{k+\ell+r} \|_{L^p_{\nu}(\mathbb{R}^n)}^{p-1}. \]
(16)

Consequently, we obtain
\[
\| \mathcal{H}_{\mathbf{A}}(\mathbf{f}) \|_{K^2_{\alpha_1, \alpha_2}(\mathbb{R}^n)} \\
\lesssim \int_{\mathbb{R}^n} (2 - \kappa^*)^{m-\frac{1}{2}} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} \max \left\{ \| A_i^{-1}(y) \|^{|\gamma_i|}, \| A_i(y) \|^{-|\gamma_i|} \right\} \| \det A_i^{-1}(y) \|^{|\delta|} \cdot \mathcal{H}_1 \, dy,
\]

where

\[
\mathcal{H}_1 := \prod_{i=1}^{m} \sum_{r=\kappa^* - 1}^{0} \left( \sum_{k=-\infty}^{\infty} v_i(B_k)^{\frac{\gamma_i}{p_1}} \| f_i \mathcal{Z}_{k+\ell+r} \|_{L^p_{\alpha_1}}(\mathbb{R}^n) \right)^{\frac{1}{p_1}}.
\]

Since \(2^{\ell_i - 1} < \| A_i(y) \| \leq 2^{\ell_i}\), it implies that

\[
2^{\ell_i - 1} \lesssim \| A_i(y) \|^{-1} \Rightarrow 2^{\ell_i - (1 + \frac{\delta}{2})} \lesssim \| A_i(y) \|^{- (1 + \frac{\delta}{2})}.
\]

Also, remark that \(v_i(B_k)^{\frac{\gamma_i}{p_1}} = 2^{-(\ell_i + r)(1 + \frac{\delta}{2})} \| f_i \|_{L^p_{\alpha_1}}(\mathbb{R}^n)\). Hence, it is easy to get

\[
\mathcal{H}_1 \leq \prod_{i=1}^{m} \sum_{r=\kappa^* - 1}^{0} 2^{-(\ell_i + r)(1 + \frac{\delta}{2})} \left( \sum_{k=-\infty}^{\infty} v_i(B_k)^{\frac{\gamma_i}{p_1}} \| f_i \mathcal{Z}_{k+\ell+r} \|_{L^p_{\alpha_1}}(\mathbb{R}^n) \right)^{\frac{1}{p_1}}.
\]

Thus, we obtain

\[
\| \mathcal{H}_{\mathbf{A}}(\mathbf{f}) \|_{K^2_{\alpha_1, \alpha_2}(\mathbb{R}^n)} \\
\lesssim \int_{\mathbb{R}^n} (2 - \kappa^*)^{m-\frac{1}{2}} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} \max \left\{ \| A_i^{-1}(y) \|^{|\gamma_i|}, \| A_i(y) \|^{-|\gamma_i|} \right\} \| \det A_i^{-1}(y) \|^{|\delta|} \\
\times \| A_i(y) \|^{- (1 + \frac{\delta}{2})} \left( \sum_{r=\kappa^* - 1}^{0} 2^{-(\ell_i + r)(1 + \frac{\delta}{2})} \right) dy \prod_{i=1}^{m} \| f_i \|_{K^2_{\alpha_1, \alpha_2}(\mathbb{R}^n)}.
\]

By (9), it is easy to show that

\[
\prod_{i=1}^{m} \max \left\{ \| A_i^{-1}(y) \|^{|\gamma_i|}, \| A_i(y) \|^{-|\gamma_i|} \right\} \| \det A_i^{-1}(y) \|^{|\delta|} \| A_i(y) \|^{- (1 + \frac{\delta}{2})} \\
\lesssim \prod_{i=1}^{m} \| A_i^{-1}(y) \|^{|\delta|} \| A_i^{-1}(y) \|^{|\gamma_i|} \| A_i(y) \|^{- (1 + \frac{\delta}{2})} = \prod_{i=1}^{m} \| A_i^{-1}(y) \|^{(1 + \frac{\delta}{2})} \| A_i(y) \|^{\frac{\delta}{2}}.
\]

Therefore, by \(\kappa^* = |\kappa^*(y)| \geq 1\) for a.e \(y \in \mathbb{R}^n\), we obtain

\[
\| \mathcal{H}_{\mathbf{A}}(\mathbf{f}) \|_{K^2_{\alpha_1, \alpha_2}(\mathbb{R}^n)} \lesssim C \cdot \prod_{i=1}^{m} \| f_i \|_{K^2_{\alpha_1, \alpha_2}(\mathbb{R}^n)},
\]
which means that $\mathcal{H}_{\phi, \mathcal{A}}$ is bounded from the product space $\prod_{i=1}^m \mathcal{K}^{x_i, p_i, q_i} (\mathbb{R}^n)$ to $\mathcal{K}^{x, p, q} (\mathbb{R}^n)$.

Conversely, suppose that $\mathcal{H}_{\phi, \mathcal{A}}$ is defined as a bounded operator from $\prod_{i=1}^m \mathcal{K}^{x_i, p_i, q_i} (\mathbb{R}^n)$ to $\mathcal{K}^{x, p, q} (\mathbb{R}^n)$. It is convenient to choose the functions $f_i$ as follows

$$f_i(x) = \begin{cases} 0, & |x| < \rho_{\mathcal{A}}^{-1}, \\ |x|^{-\left(1 + \frac{\mu_i}{n}\right)}x_i^{-\frac{\mu_i}{n} - \frac{1}{n} - \epsilon}, & \text{otherwise}. \end{cases}$$

It is clear that for any integer number $k$ satisfying $k < -\frac{\ln(\rho_\mathcal{A})}{\ln 2}$, then $\|f_i \mathcal{H}k\|_{L^{n'}_{\mathcal{A}}(\mathbb{R}^n)} = 0$ for all $i = 1, \ldots, m$. Otherwise, one has

$$\|f_i \mathcal{H}k\|_{L^{n'}_{\mathcal{A}}(\mathbb{R}^n)} = \left( \int_{C_k} \int_{S_{n-1}} r^{-\left(1 + \frac{\mu_i}{n}\right)} x_i^{-n-\gamma - q_i \epsilon} r^{-1} d\sigma(x') dr \right)^\frac{1}{n'} \lesssim 2^{-k \left(\epsilon + \left(1 + \frac{\mu_i}{n}\right)x_i \right)} \left( \frac{2^{q_i \left(\epsilon + \left(1 + \frac{\mu_i}{n}\right)x_i \right)} - 1}{q_i \left(\epsilon + \left(1 + \frac{\mu_i}{n}\right)x_i \right)} \right)^\frac{1}{n}.$$

Evidently, $v_i(B_k) \approx 2^{k(n+\mu_i)}$. By some simple computations, we obtain

$$\|f_i\|_{\mathcal{K}^{x_i, p_i, q_i} (\mathbb{R}^n)} \approx \left( \sum_{k \geq \theta} v_i(B_k) \right)^\frac{1}{n'} \lesssim 2^{-k \left(\epsilon + \left(1 + \frac{\mu_i}{n}\right)x_i \right)} \left( \frac{2^{q_i \left(\epsilon + \left(1 + \frac{\mu_i}{n}\right)x_i \right)} - 1}{q_i \left(\epsilon + \left(1 + \frac{\mu_i}{n}\right)x_i \right)} \right)^\frac{1}{n} \approx \left( \sum_{k \geq \theta} 2^{-k \mu_i} \right)^\frac{1}{n'} \lesssim \left( \frac{2^{(1-\theta)\mu_i}}{2^{p_i} - 1} \right)^\frac{1}{n'} \left( \frac{2^{q_i \left(\epsilon + \left(1 + \frac{\mu_i}{n}\right)x_i \right)} - 1}{q_i \left(\epsilon + \left(1 + \frac{\mu_i}{n}\right)x_i \right)} \right)^\frac{1}{n},$$

where $\theta$ is the smallest integer number such that $\theta \geq -\frac{\ln(\rho_\mathcal{A})}{\ln 2}$. Next, consider two sets as follows

$$D_x = \bigcap_{i=1}^m \{ y \in \mathbb{R}^n : |A_i(y) x| \geq \rho_{\mathcal{A}}^{-1} \},$$

and

$$E = \{ y \in \mathbb{R}^n : \|A_i(y)\| \geq \epsilon, \quad \text{for all} \quad i = 1, \ldots, m \}.$$
Indeed, let \( y \in E \). It is easy to check that \( \| A_{i}(y) \| \{ x \} \geq 1 \) for all \( x \in \mathbb{R}^{n} \setminus B(0, \varepsilon^{-1}) \). Hence, it follows from the condition (8) that

\[
|A_{i}(y)x| \geq \| A_{i}^{-1}(y) \|^{-1} |x| \geq \rho_{A}^{-1},
\]

which implies the proof of the relation (17). Thus, by (8) and (17), for any \( x \in \mathbb{R}^{n} \setminus B(0, \varepsilon^{-1}) \), we have

\[
H_{\phi_{A}}(f)(x) \geq \int_{D_{x}} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} |A_{i}(y)x|^{-(1+\frac{\beta}{n})} \frac{x_{i}-(n+1)}{\alpha} \ dy
\]

\[
\geq \int_{E} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} |A_{i}(y)x|^{-(1+\frac{\beta}{n})} \frac{x_{i}-(n+1)}{\alpha} \ dy
\]

\[
\geq \left( \int_{E} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \|A_{i}^{-1}(y)\|^{(1+\frac{\beta}{n})} x_{i}^{\frac{n+1}{n}} \ dy \right) \left( \int_{E} |x|^{-(1+\frac{\beta}{n})} \omega(x) \ dy \right)^{\frac{q}{p}}
\]

Let \( k_{0} \) be the smallest integer number such that \( 2^{k_{0}-1} \geq \varepsilon^{-1} \). We thus have

\[
\|H_{\phi_{A}}(f)\|_{L_{q}^{\omega}(\mathbb{R}^{n})} \geq \left( \int_{E} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \|A_{i}^{-1}(y)\|^{(1+\frac{\beta}{n})} x_{i}^{\frac{n+1}{n}} \ dy \right) \left( \int_{E} |x|^{-(1+\frac{\beta}{n})} \omega(x) \ dy \right)^{\frac{q}{p}}
\]

Putting together these estimates, it follows that

\[
\|H_{\phi_{A}}(f)\|_{K_{\nu}^{p,q}(\mathbb{R}^{n})} \geq \left( \sum_{k=k_{0}}^{\infty} \nu(B_{k})^{\frac{p}{q}} \left( \int_{E} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \|A_{i}^{-1}(y)\|^{(1+\frac{\beta}{n})} x_{i}^{\frac{n+1}{n}} \ dy \right) \right)^{\frac{1}{p}}
\]

\[
\times \left( \int_{C_{k}} |x|^{-(1+\frac{\beta}{n})} \omega(x) \ dy \right)^{\frac{q}{p}}
\]

\[
\geq \left( \int_{E} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \|A_{i}^{-1}(y)\|^{(1+\frac{\beta}{n})} x_{i}^{\frac{n+1}{n}} \ dy \right)
\]

\[
\times \left( \sum_{k=k_{0}}^{\infty} \nu(B_{k})^{\frac{p}{q}} \left( \int_{C_{k}} |x|^{-(1+\frac{\beta}{n})} \omega(x) \ dy \right) \right)^{\frac{q}{p}}
\]

Observe that \( \nu(B_{k}) \simeq 2^{k(n+\beta)} \), and an elementary calculation shows that
\[
\left( \int \frac{1}{C_k} \left| x \right|^{-\left(1 + \frac{\ell}{n}\right) xq - n - m a q} dx \right)^{\frac{p}{q}} \approx 2^{-kp(m + (1 + \frac{\ell}{n}) x)} \left( \frac{2^{q(m + (1 + \frac{\ell}{n}) x) - 1}{q(m + (1 + \frac{\ell}{n}) x)} \right)^{\frac{p}{q}},
\]
so
\[
\sum_{k=k_0}^{\infty} v(B_k) \frac{\alpha}{n} \left( \int \frac{1}{C_k} \left| x \right|^{-\left(1 + \frac{\ell}{n}\right) xq - n - m a q} dx \right)^{\frac{p}{q}} \approx \left( \frac{2^{k_0 m p} \left(1 \right) 2^{q(m + (1 + \frac{\ell}{n}) x) - 1}{q(m + (1 + \frac{\ell}{n}) x)} \right)^{\frac{p}{q}}.
\]

For simplicity of exposition, we put
\[
\theta^*(\varepsilon) = \frac{\left( 2^{k_0 m p} \left(1 \right) 2^{q(m + (1 + \frac{\ell}{n}) x) - 1}{q(m + (1 + \frac{\ell}{n}) x)} \right)^{\frac{p}{q}}}{}^{1 \prod_{i=1}^{m} \left( \sum_{i=1}^{\varepsilon} \frac{1}{p_i} \right) \left( \sum_{i=1}^{\varepsilon} \frac{1}{q_i} \right)}.
\]

Using \( \sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{\rho} \) and \( \sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q'} \), it is not hard to check that
\[
\lim_{\varepsilon \to 0} \varepsilon^{-m} \theta^*(\varepsilon) = c > 0.
\]

By (18) and (19), we have
\[
\| \mathcal{H}_{\Phi, A}(f) \|_{L^p_q(R^n)} = \varepsilon^{-m} \theta^*(\varepsilon) \prod_{i=1}^{m} \| f_i \|_{L^p_q(R^n)} \times \left( \int_{E} \Phi(y) \prod_{i=1}^{m} \| A_i^{-1}(y) \|_{1, \left(1 + \frac{\ell}{n}\right) xq + \frac{n + \ell}{q_i}} \prod_{i=1}^{m} \| A_i^{-1}(y) \|_{e^{m e}} dy \right).
\]

Remark that \( \| A_i(y) \| \geq \varepsilon \) for all \( y \in E \), and by (8) we have
\[
\prod_{i=1}^{m} \| A_i^{-1}(y) \|_{e^{m e}} \leq \rho^* \leq 1,
\]
for \( \varepsilon \) sufficiently small. Then, letting \( \varepsilon \to 0 \), from assuming that \( \mathcal{H}_{\Phi, A} \) is bounded from \( \prod_{i=1}^{m} K^{x, p_i, q_i}(R^n) \) to \( K^{x, p, q}(R^n) \), by the dominated convergence theorem of Lesbegue, we obtain
\[
\int_{R^n} \Phi(y) \prod_{i=1}^{m} \| A_i^{-1}(y) \|_{1, \left(1 + \frac{\ell}{n}\right) xq + \frac{n + \ell}{q_i}} dy < \infty.
\]
This ends the proof of the theorem.

In view of Theorem 2, we also obtain the necessary and sufficient conditions for the boundedness of the operator \( \mathcal{H}_{\Phi, A} \) as well as multilinear Hardy–Cesàro operators on the two-weighted Herz spaces. Namely, the following is true.
Corollary 2 Let $\phi$ be a nonnegative function. Under the same assumptions as Theorem 2, we have that the operator $\mathcal{H}_{\phi,s}$ is bounded from $\prod_{i=1}^{m} K_{v_i,\omega_i}^{\lambda_i,\psi,\omega_i} (\mathbb{R}^n)$ to $\dot{K}_{v,\omega}^{\alpha,p,q} (\mathbb{R}^n)$ if and only if

$$C_{2.1} = \int_{\mathbb{R}^n} \left( \prod_{i=1}^{m} |s_i(y)|^{-\frac{n+\lambda_i}{\lambda_i q_i}} \right) \phi(y) dy < \infty.$$ 

Moreover,

$$\| \mathcal{H}_{\phi,s} \| \prod_{i=1}^{m} K_{v_i,\omega_i}^{\lambda_i,\psi,\omega_i} (\mathbb{R}^n) \rightarrow \dot{K}_{v,\omega}^{\alpha,p,q} (\mathbb{R}^n) \simeq C_{2.1}.$$ 

In particular, we have that the weighted multilinear Hardy–Cesàro operator $U_{\psi,s}^{m,n}$ is bounded from $\prod_{i=1}^{m} K_{v_i,\omega_i}^{\lambda_i,\psi,\omega_i} (\mathbb{R}^n)$ to $\dot{K}_{v,\omega}^{\alpha,p,q} (\mathbb{R}^n)$ if and only if

$$C_{2.2} = \int_{[0,1]^n} \left( \prod_{i=1}^{m} |s_i(t)|^{-\frac{n+\lambda_i}{\lambda_i q_i}} \right) \psi(t) dt < \infty.$$ 

Moreover,

$$\| U_{\psi,s}^{m,n} \| \prod_{i=1}^{m} K_{v_i,\omega_i}^{\lambda_i,\psi,\omega_i} (\mathbb{R}^n) \rightarrow \dot{K}_{v,\omega}^{\alpha,p,q} (\mathbb{R}^n) \simeq C_{2.2}.$$ 

It should be pointed out that the results of Corollary 2 are also true if the power weighted functions $v, \omega, v_i, \omega_i$, for $i = 1, \ldots, m$, are replaced by the appropriately weights of absolutely homogeneous type. The proof can be completed as in the same way as above. For further readings on the absolutely homogeneous weights, one may find in [16, 19, 20]. Thus, Corollary 2 extends the results of Theorem 3.2 in [20] to two-weighted setting.

Now, let us establish the boundedness for the multilinear Hausdorff operators on the two-weighted Morrey–Herz spaces.

Theorem 3 Let $\Phi : \mathbb{R}^n \rightarrow [0, \infty)$, and $\lambda_i > 0, \omega(x) = |x|^\gamma, v(x) = |x|^\beta$, $\omega_i(x) = |x|^\gamma_i, v_i(x) = |x|^\beta_i$ for all $i = 1, \ldots, m$, and the following conditions hold

$$\sum_{i=1}^{m} \left( 1 + \frac{\beta_i}{n} \right) \lambda_i = \left( 1 + \frac{\beta}{n} \right) \lambda, \quad \sum_{i=1}^{m} \frac{\gamma_i}{q_i} = \frac{\gamma}{q}, \quad \text{and} \quad \sum_{i=1}^{m} \left( 1 + \frac{\beta_i}{n} \right) x_i = \left( 1 + \frac{\beta}{n} \right) x.$$

Then we have that the operator $\mathcal{H}_{\Phi,A}$ is bounded from $\prod_{i=1}^{m} M K_{v_i,\omega_i}^{\lambda_i,\psi,\omega_i} (\mathbb{R}^n)$ to $MK_{v,\omega}^{\alpha,p,q} (\mathbb{R}^n)$ if and only if

$$C_{3} = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} |A_i^{-1}(y)| \left( 1 + \frac{\beta_i}{n} \right) (x_i - \lambda_i) + \frac{\gamma_i}{q_i} dy < \infty.$$ 

Moreover,
\[ \| \mathcal{H}_{\Phi, A} \|_{MK_{\alpha, \beta}^{p, q}(\mathbb{R}^n)} \leq \sup_{k_0 \in \mathbb{Z}} v(B_{k_0})^{\frac{m}{m - 1}} \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} \max \{ \| A_i^{-1}(y) \|^{\beta_i}, \| A_i(y) \|^{-\beta_i} \} \frac{1}{|y|^n} \det A_i^{-1}(y) \frac{1}{|y|^n} \times \left( \sum_{k=\infty}^{k_0} v(B_k) \prod_{i=1}^{m} \left( \sum_{r=k^* - 1}^{0} \| f_i \chi_{k+\ell, r} \|_{L_{q_0}^p(\mathbb{R}^n)} \right) \right)^{\frac{1}{n}} dy. \]

By (15) and (16), we also obtain
\[ \| \mathcal{H}_{\Phi, A}(f) \|_{MK_{\alpha, \beta}^{p, q}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} (2 - \kappa^*)^{m-\frac{m}{m}} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} \max \{ \| A_i^{-1}(y) \|^{\beta_i}, \| A_i(y) \|^{-\beta_i} \} \frac{1}{|y|^n} \det A_i^{-1}(y) \frac{1}{|y|^n} \times \prod_{i=1}^{m} \sum_{r=\kappa^* - 1}^{0} \sup_{k_0 \in \mathbb{Z}} v_i(B_{k_0}) \left( \sum_{k=\infty}^{k_0} v_i(B_k) \prod_{i=1}^{m} \left( \sum_{r=\kappa^* - 1}^{0} \| f_i \chi_{k+\ell, r} \|_{L_{q_0}^p(\mathbb{R}^n)} \right) \right)^{\frac{1}{n}} dy. \]

For simplicity, set
\[ \mathcal{H}_2 := \prod_{i=1}^{m} \sum_{r=\kappa^* - 1}^{0} \sup_{k_0 \in \mathbb{Z}} v_i(B_{k_0}) \left( \sum_{k=\infty}^{k_0} v_i(B_k) \prod_{i=1}^{m} \left( \sum_{r=\kappa^* - 1}^{0} \| f_i \chi_{k+\ell, r} \|_{L_{q_0}^p(\mathbb{R}^n)} \right) \right)^{\frac{1}{n}}. \]

Using a similar argument as \( \mathcal{H}_1 \), we also get
\[ \mathcal{H}_2 \lesssim \prod_{i=1}^{m} \| A_i(y) \|^{( \lambda_i - \lambda_0 ) \left( 1 + \frac{\alpha}{m} \right)} \sum_{r=\kappa^* - 1}^{0} 2^r \left( 1 + \frac{\alpha}{m} \right) ( \lambda_i - \lambda_0 ) \| f_i \|_{MK_{\alpha, \beta}^{p, q}(\mathbb{R}^n)}. \]

Consequently,
\[ \| \mathcal{H}_{\Phi, A}(f) \|_{MK_{\alpha, \beta}^{p, q}(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^n} (2 - \kappa^*)^{m-\frac{m}{m}} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} \max \{ \| A_i^{-1}(y) \|^{\beta_i}, \| A_i(y) \|^{-\beta_i} \} \frac{1}{|y|^n} \det A_i^{-1}(y) \frac{1}{|y|^n} \times \| A_i(y) \|^{( \lambda_i - \lambda_0 ) \left( 1 + \frac{\alpha}{m} \right)} \sum_{r=\kappa^* - 1}^{0} 2^r \left( 1 + \frac{\alpha}{m} \right) ( \lambda_i - \lambda_0 ) \| f_i \|_{MK_{\alpha, \beta}^{p, q}(\mathbb{R}^n)}. \]

Similarly, by (9) it is easily seen that
\[
\prod_{i=1}^{m} \max \{ \|A_i^{-1}(y)\|^{\frac{1}{q_i}} \|A_i(y)\|^{-\frac{1}{q_i}} \} \| \det A_i^{-1}(y) \|^{\frac{1}{2}} \|A_i(y)\| \left( (1 + \frac{\beta}{n}) (x_i - \lambda_i) + \frac{n + \gamma}{n} \right) \\
\lesssim \prod_{i=1}^{m} \|A_i^{-1}(y)\| \left( 1 + \frac{\beta}{n} \right) (x_i - \lambda_i)^{\frac{n + \gamma}{n}}.
\]

Therefore, we obtain
\[
\|\mathcal{H}_{\boldsymbol{\phi}i,\mathbf{A}}(\mathbf{f})\|_{MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)} \lesssim C_3 \cdot \prod_{i=1}^{m} \|f_i\|_{MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)}.
\]

This shows that \(\mathcal{H}_{\boldsymbol{\phi}i,\mathbf{A}}\) is bounded from \(\prod_{i=1}^{m} MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)\) to \(MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)\).

Conversely, suppose that \(\mathcal{H}_{\boldsymbol{\phi}i,\mathbf{A}}\) is determined as a bounded operator from \(\prod_{i=1}^{m} MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)\) to \(MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)\). For each \(i = 1, \ldots, m\), let us take
\[
f_i(x) = |x|^{(\lambda_i - \alpha_i)(1 + \frac{\beta}{n}) - \frac{(n + \gamma)}{\alpha}}.
\]

It is not hard to check that
\[
\|f_i\|_{L_{\alpha,\lambda}^q(\mathbb{R}^n)} \begin{cases} \ln 2, & \text{for } \lambda_i - \alpha_i = 0, \\
2^k(\lambda_i - \alpha_i)(1 + \frac{\beta}{n}) \left( 1 - 2^{\gamma} (\lambda_i - \alpha_i)(1 + \frac{\beta}{n}) \right) \frac{1}{\alpha}, & \text{otherwise.}\end{cases}
\]

Hence, we have
\[
0 < \|f_i\|_{MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)} \simeq \sup_{k \in \mathbb{Z}} 2^{-k_0 \left( 1 + \frac{\beta}{n} \right) \lambda_i} \left( \sum_{k=\infty}^{k_0} 2^k \left( 1 + \frac{\beta}{n} \right) \lambda_i \right) \frac{1}{\alpha} < \infty.
\]

By (10) and the condition \(\sum_{i=1}^{m} (n + \beta_i) \alpha_i = (n + \beta) \alpha\), we conclude
\[
|A_i(y)x| \leq \prod_{i=1}^{m} \|A_i^{-1}(y)\| \left( 1 + \frac{\beta}{n} \right) (x_i - \lambda_i)^{\frac{n + \gamma}{n}} |x|^{(\lambda_i - \alpha_i)(1 + \frac{\beta}{n}) - \frac{(n + \gamma)}{\alpha}}.
\]

This leads to that
\[
\|\mathcal{H}_{\boldsymbol{\phi}i,\mathbf{A}}(\mathbf{f})(x)\|_{MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)} \lesssim \left( \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^{m} \|A_i^{-1}(y)\| \left( 1 + \frac{\beta}{n} \right) (x_i - \lambda_i)^{\frac{n + \gamma}{n}} dy \right) \times \|x|^{(\lambda - \alpha)(1 + \frac{\beta}{n}) - \frac{(n + \gamma)}{\alpha}} \|_{MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)}.
\]

Therefore, we obtain
\[
C_3 \lesssim \|\mathcal{H}_{\boldsymbol{\phi}i,\mathbf{A}}\|_{MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n) \to MK^{2,\beta}_{(\frac{\alpha}{n})}^{x,\lambda,q}(\mathbb{R}^n)} < \infty,
\]
which finishes the proof of this theorem.\

By Theorem 3, we also have the following useful corollary.

**Corollary 3** Let \( \phi \) be a nonnegative function. Under the same assumptions as Theorem 3, we have that the operator \( \mathcal{H}_{\phi,s} \) is bounded from \( \prod_{i=1}^{m} M_{K_{V_i,y_i}}^{x_i,y_i,p_i,q_i}(\mathbb{R}^n) \) to \( M\mathbb{K}_{v,e}^{x,p,q}(\mathbb{R}^n) \) if and only if

\[
C_{3.1} = \int_{\mathbb{R}^n} \left( \prod_{i=1}^{m} |s_i(y)|^{r_i} \right) \left( 1 + \frac{\|y\|}{\eta_i} \right)^{\frac{\eta_i}{\eta}} \phi(y) dy < \infty.
\]

Moreover,

\[
\|\mathcal{H}_{\phi,s}\|_{\prod_{i=1}^{m} M_{K_{V_i,y_i}}^{x_i,y_i,p_i,q_i}(\mathbb{R}^n) \to M\mathbb{K}_{v,e}^{x,p,q}(\mathbb{R}^n)} \simeq C_{3.1}.
\]

In the rest of this paper, we will address some sufficient conditions for the boundedness of the operator \( \mathcal{H}_{\phi,A} \) on two-weighted Morrey, Herz, and Morrey–Herz spaces associated with the class of the Muckenhoupt weights.

**Theorem 4** Let \( 1 \leq q^*, \xi, \eta < \infty, -\frac{1}{q_i} < \lambda_i < 0, \) for all \( i = 1, \ldots, m, \) and \( \omega \in A_\xi, v \in A_\eta \) with the finite critical index \( r_\omega, r_v \) for the reverse Hölder condition such that \( \omega(B(0,R)) \leq v(B(0,R)) \) for all \( R > 0. \) Assume that \( q > q^* \xi r_\omega ', \delta_1 \in (1, r_\omega), \delta_2 \in (1, r_v), \) and

\[
\lambda^* = \lambda_1 + \cdots + \lambda_m,
\]

\[
C_4 = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^m} \prod_{i=1}^{m} \left| \det A_i^{-1}(y) \right| |A_i(y)|^{\frac{\delta_1}{\delta} A_i(y)} dy < \infty,
\]

where

\[
A_i(y) = \prod_{i=1}^{m} \left( \left\| A_i(y) \right\|^{\lambda_i + \frac{1}{q_i}} \frac{\eta_i}{\eta_i} Z_{\{y \in \mathbb{R}^n : \|A_i(y)\| \leq 1\}} + \left\| A_i(y) \right\|^{\eta_i} \frac{\lambda_i + \frac{1}{q_i}}{\eta_i} Z_{\{y \in \mathbb{R}^n : \|A_i(y)\| > 1\}} \right)
\]

\[
\times \prod_{i=1}^{m} \left( \left\| A_i(y) \right\|^{\lambda_i + \frac{1}{q_i}} \frac{\eta_i}{\eta_i} Z_{\{y \in \mathbb{R}^n : \|A_i(y)\| \leq 1\}} + \left\| A_i(y) \right\|^{\eta_i} \frac{\lambda_i + \frac{1}{q_i}}{\eta_i} Z_{\{y \in \mathbb{R}^n : \|A_i(y)\| > 1\}} \right).
\]

Then \( \mathcal{H}_{\phi,A} \) is bounded from \( \prod_{i=1}^{m} M_{K_{V_i,y_i}}^{x_i,y_i,p_i,q_i}(\mathbb{R}^n) \) to \( M\mathbb{K}_{v,e}^{x^*,p^*,q^*}(\mathbb{R}^n). \)

**Proof** By the Minkowski inequality, we have

\[
\|\mathcal{H}_{\phi,A}(f)\|_{L_{v,e}^{x^*,p^*,q^*}(B(0,R))} \leq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^m} \left( \int_{B(0,R)} \prod_{i=1}^{m} |f_i(A_i(y),x)|^{q_i} \omega(x) dx \right)^{\frac{1}{q}} dy.
\]
From the condition \( q > q^* \xi r_\infty \), there exists \( r \in (1, r_\infty) \) such that \( \frac{q}{r} = q^* \xi r' \). Applying the reverse Hölder property and the Hölder inequality with \( \frac{q}{r} = \frac{q}{q_1} + \cdots + \frac{q}{q_m} \), we infer

\[
\left( \int_{B(0,R)} \prod_{i=1}^m |f_i(A_i(y)x)|^{q_i} \omega(x) dx \right)^{\frac{1}{q'}} \leq \left( \int_{B(0,R)} \prod_{i=1}^m |f_i(A_i(y)x)|^{\frac{q}{q_i}} dx \right)^{\frac{q_i}{q}} \left( \int_{B(0,R)} \omega(x) r dx \right)^{\frac{1}{r'}} \leq \left( \int_{B(0,R)} \prod_{i=1}^m |f_i(A_i(y)x)|^{\frac{q}{q_i}} dx \right)^{\frac{q_i}{q}} |B(0,R)|^{-\frac{q}{q'} \omega(B(0,R))^{\frac{1}{q'}}} \leq \prod_{i=1}^m \left( \int_{B(0,R)} |f_i(A_i(y)x)|^{\frac{q_i}{q_i}} dx \right)^{\frac{q_i}{q}} |B(0,R)|^{-\frac{q_i}{q'}} |B(0,R)|^{\frac{1}{q'}}.
\]

By the change of variables \( z = A_i(y)x \), we have

\[
\| \mathcal{H}_{\Phi,A}(f) \|_{L^{q'_*}_{\infty}(B(0,R))} \leq \omega(B(0,R))^{\frac{1}{q'_*}} |B(0,R)|^{-\frac{q}{q'}} \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|} \prod_{i=1}^m |\det A_i^{-1}(y)|^{\frac{q_i}{q_i}} \| f_i \|_{L^{q_i}_{\infty}(B(0,\|A_i(y)\|\|R\|))}^{\frac{q_i}{q}} dy.
\]

In view of Proposition 2, one has

\[
\| f_i \|_{L^{q_i}_{\infty}(B(0,\|A_i(y)\|\|R\|))} \leq |B(0, \|A_i(y)\|\|R\|)|^{-\frac{q_i}{q_i}} \omega(B(0, \|A_i(y)\|\|R\|)) \left( \int_{B(0,\|A_i(y)\|\|R\|)} |f_i|^{q} \omega(x) dx \right)^{\frac{1}{q}} = |B(0, \|A_i(y)\|\|R\|)|^{-\frac{1}{q}} |f_i|_{L^{q}_{\infty}(B(0,\|A_i(y)\|\|R\|))}.
\]

Therefore, we obtain

\[
\| \mathcal{H}_{\Phi,A}(f) \|_{L^{q'_*}_{\infty}(B(0,R))} \leq \omega(B(0,R))^{\frac{1}{q'_*}} |B(0,R)|^{-\frac{q}{q'}} \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|} \prod_{i=1}^m |\det A_i^{-1}(y)|^{\frac{q_i}{q_i}} |B(0, \|A_i(y)\|\|R\|)|^{\frac{1}{q_i}} dy \times \omega(B(0, \|A_i(y)\|\|R\|))^{-\frac{1}{q}} |f_i|_{L^{q}_{\infty}(B(0,\|A_i(y)\|\|R\|))} \left( \int_{B(0,\|A_i(y)\|\|R\|)} |f_i|^{q} \omega(x) dx \right)^{\frac{1}{q}} \leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|} \prod_{i=1}^m \left( |\det A_i^{-1}(y)|^{\frac{q_i}{q_i}} |A_i(y)|^{\frac{q_i}{q_i}} \right)^{\frac{1}{q_i}} \omega(B(0, \|A_i(y)\|\|R\|))^{\frac{1}{q'}} \times \left( \omega(B(0,R))^{\frac{1}{q'_*}} |B(0,R)|^{-\frac{q}{q'}} \prod_{i=1}^m |f_i|_{L^{q}_{\infty}(B(0,\|A_i(y)\|\|R\|))} \right) dy.
\]

\( \text{Birkhäuser} \)
Consequently,

\[
\| \mathcal{H}_\phi A(f) \|_{M^\alpha (\mathbb{R}^n)} = \sup_{R > 0} \frac{1}{v(B(0, R))^\lambda^+} \| \mathcal{H}_\phi A(f) \|_{L^\infty_y(B(0, R))}
\]

\[
\leq \sup_{R > 0} \frac{1}{v(B(0, R))^\lambda^+} \int_{\mathbb{R}^n} \frac{|\phi(y)|}{|y|^m} \prod_{i=1}^m \left( |\det A_i^{-1}(y)|^\frac{\lambda_i}{\gamma_i} |A_i(y)|^\frac{\gamma_i}{\lambda_i} \frac{1}{\omega(B(0, |A_i(y)| |R|)\gamma_i^\lambda)} \right)
\times \left( \frac{\omega(B(0, R))^{\gamma_i}}{v(B(0, R))^{\lambda_i + \frac{\lambda_i}{\gamma_i}}} \prod_{i=1}^m \frac{v(B(0, |A_i(y)| |R|)}{\omega(B(0, |A_i(y)| |R|)\gamma_i^\lambda)} \right) dy
\]

\[
\leq \sup_{R > 0} \int_{\mathbb{R}^n} \frac{|\phi(y)|}{|y|^m} \prod_{i=1}^m |\det A_i^{-1}(y)|^\frac{\lambda_i}{\gamma_i} |A_i(y)|^\frac{\gamma_i}{\lambda_i} A_i(R) \prod_{i=1}^m \| f_i \|_{M^\alpha (\mathbb{R}^n)} dy,
\]

where

\[
A_i(R) = \left( \frac{\omega(B(0, R))^{\gamma_i}}{v(B(0, R))^{\lambda_i + \frac{\lambda_i}{\gamma_i}}} \right) \prod_{i=1}^m \frac{v(B(0, |A_i(y)| |R|)}{\omega(B(0, |A_i(y)| |R|)\gamma_i^\lambda)} \right).
\]

By \( \frac{1}{q^i} > \frac{1}{q} \), then the condition \( \omega(B(0, R)) \leq v(B(0, R)) \) implies that

\[
\left( \frac{\omega(B(0, R))^{\gamma_i}}{v(B(0, R))^{\lambda_i + \frac{\lambda_i}{\gamma_i}}} \right) \leq \left( \frac{\omega(B(0, R))^{\gamma_i}}{v(B(0, R))^{\lambda_i + \frac{\lambda_i}{\gamma_i}}} \right)^{\frac{1}{\gamma_i}}.
\]

Hence, by the conditions \( \lambda^+ = \sum_{i=1}^m \lambda_i \frac{1}{q_i} = \sum_{i=1}^m \frac{1}{q_i} \), we have

\[
A_i(R) \leq \prod_{i=1}^m \left( \frac{v(B(0, |A_i(y)| |R|)}{\omega(B(0, |A_i(y)| |R|)} \right)^{\frac{1}{\gamma_i}} \prod_{i=1}^m \left( \frac{v(B(0, |A_i(y)| |R|)}{\omega(B(0, |A_i(y)| |R|)} \right)^{\lambda_i} \prod_{i=1}^m \left( \frac{v(B(0, |A_i(y)| |R|)}{\omega(B(0, |A_i(y)| |R|)} \right) \right)
\]

\[
= \prod_{i=1}^m \left( \frac{v(B(0, |A_i(y)| |R|)}{\omega(B(0, |A_i(y)| |R|)} \right)^{\lambda_i + \frac{1}{\gamma_i}} \prod_{i=1}^m \left( \frac{v(B(0, |A_i(y)| |R|)}{\omega(B(0, |A_i(y)| |R|)} \right)^{\lambda_i} \right)^{\frac{1}{\gamma_i}}.
\]

Now, using the assumptions \( \omega \in A_{\frac{\lambda}{\gamma}}, \nu \in A_{\eta}, \delta_1 \in (1, r, \omega), \delta_2 \in (1, r, \nu), \frac{1}{q_i} + \lambda_i > 0, \) and by Proposition 3, we consider the following two cases.

Case 1: \( 0 < |A_i(y)| \leq 1 \). Then we have

\[
\left( \frac{v(B(0, |A_i(y)| |R|)}{\omega(B(0, R))} \right)^{\lambda_i + \frac{1}{\gamma_i}} \leq \left( \frac{|B(0, |A_i(y)| |R|)}{|B(0, R)|} \right)^{\lambda_i + \frac{1}{\gamma_i}} = |A_i(y)|^{\nu} \frac{\lambda_i + \frac{1}{\gamma_i}}{\delta_2},
\]

\[
\left( \frac{\omega(B(0, R))}{\omega(B(0, |A_i(y)| |R|)} \right)^{\frac{1}{\gamma_i}} \leq \left( \frac{|B(0, R)|}{|B(0, |A_i(y)| |R|)} \right)^{\frac{1}{\gamma_i}} = |A_i(y)|^{-\frac{\lambda_i}{\gamma_i}}.
\]

Case 2: \( |A_i(y)| > 1 \). We also get
Thus, we obtain
\[
\mathcal{A}_i(R) \leq \prod_{i=1}^{m} \left( \|A_i(y)\|^n \left( \frac{\lambda_i+\frac{1}{q_i}}{\lambda_i} \right)^{\frac{n-1}{q_i}} \chi_{\mathbb{R}^n: \|A_i(y)\| \leq 1} + \|A_i(y)\|^{m} \left( \frac{\lambda_i+\frac{1}{q_i}}{\lambda_i} \right)^{\frac{n-1}{q_i}} \chi_{\mathbb{R}^n: \|A_i(y)\| > 1} \right) \times \prod_{i=1}^{m} \left( \|A_i(y)\|^{\frac{\eta}{q_i}} \chi_{\mathbb{R}^n: \|A_i(y)\| \geq 1} + \|A_i(y)\|^{\frac{\eta}{q_i}} \chi_{\mathbb{R}^n: \|A_i(y)\| \leq 1} \right).
\]

Thus, we obtain
\[
\|\mathcal{H}_{\Phi, A}(f)\|_{M^{\mu, \lambda^*}_{\nu,\eta} (\mathbb{R}^n)} \leq \left( \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^m} \prod_{i=1}^{m} |\det A_i^{-1}(y)|^{\frac{\xi_i}{q_i}} \|A_i(y)\|^\eta A_i(y) \, dy \right)^{\frac{1}{n}} \prod_{i=1}^{m} \|f_i\|_{M^{\mu_i, \lambda_i^*}_{\nu_i, \eta_i} (\mathbb{R}^n)}.
\]

Therefore, Theorem 4 is completely proved. \(\square\)

Finally, we also obtain some sufficient conditions for the boundedness of \(\mathcal{H}_{\Phi, A}\) on two-weighted Morrey–Herz spaces associated with the Muckenhoupt weights.

**Theorem 5** Let \(1 \leq q^*, \xi, \eta < \infty, \lambda_i < 0, \lambda_i \geq 0, \text{ for all } i = 1, \ldots, m, \text{ and } \omega \in A_\xi, \nu \in A_\eta \) with the finite critical index \(r_\omega, r_\nu \) for the reverse Hölder condition such that \(\omega(B_k) \lesssim \nu(B_k) \), for every \(k \in \mathbb{Z} \). Assume that \(q > \max\{mq^*, q^* \xi r_\omega^*\} \), \(\delta_1 \in (1, r_\omega) \), \(\delta_2 \in (1, r_\nu) \) and \(\lambda^*, \lambda^* \) are two real numbers such that
\[
\lambda^* = \lambda_1 + \cdots + \lambda_m, \quad \text{and} \quad \frac{1}{m} \left( \frac{q^*}{n} + \frac{1}{q^*} \right) = \frac{\lambda_i}{n} + \frac{1}{q_i}, \quad i = 1, \ldots, m.
\]

If \(\frac{\xi}{n} + \frac{1}{q^*} \leq 0 \) and
\[
\mathcal{C}_{5.1} = \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^m} |\det A_i^{-1}(y)|^{\frac{\mu_i}{q_i}} \|A_i(y)\|^\eta B_i(y) \, dy \right)^{\frac{1}{n}} < \infty,
\]
where
\[
B_i(y) = \|A_i(y)\|^{\frac{\lambda_i+\frac{1}{q_i}}{\lambda_i} + (\lambda_i^* - \lambda_i) \frac{\lambda_i - \delta_1^*}{\nu_i} - \xi \lambda_i} \chi_{\mathbb{R}^n: \|A_i(y)\| < 1} + \|A_i(y)\|^{\frac{\lambda_i+\frac{1}{q_i}}{\lambda_i} + \eta (\lambda_i^* - \lambda_i) + \delta_2 \frac{\lambda_i - \lambda_i^*}{\nu_i}} \chi_{\mathbb{R}^n: \|A_i(y)\| \geq 1},
\]
\(\xi\) Birkhäuser
or \( \frac{z'}{n} + \frac{1}{q'} > 0 \) and

\[
C_{5,2} = \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} |\det A_i^{-1}(y)| \frac{m_i}{n} \|A_i(y)\| \frac{m_i}{n} B_2(y) dy \right)^{\frac{1}{n}} < \infty,
\]

where

\[
B_2(y) = \|A_i(y)\|^{-\frac{1}{n} + \frac{1}{q'} \eta(z' - m_i)} \chi_{\{y \in \mathbb{R}^n : \|A_i(y)\| < 1\}} + \|A_i(y)\|^{-\frac{1}{n} + \frac{1}{q'} \eta(z' - m_i)} \chi_{\{y \in \mathbb{R}^n : \|A_i(y)\| \geq 1\}},
\]

then we have \( \mathcal{H}_{\Phi,A} \) is bounded from \( \prod_{i=1}^{m} M_{K_{\lambda_i,\lambda_i}^{q',q'}}(\mathbb{R}^n) \) to \( \prod_{i=1}^{m} M_{K_{\lambda_i,\lambda_i}^{q',q'}}(\mathbb{R}^n) \).

**Proof** By the same arguments as Theorem 4, we also have

\[
\|\mathcal{H}_{\Phi,A}(f)\chi_k\|_{L_{\lambda}^{q'}(\mathbb{R}^n)} \leq \left( \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \left( \int_{C_k} \prod_{i=1}^{m} |f_i(A_i(y)x)|^{q'} \omega(x) dx \right)^{\frac{1}{n}} dy \right)^{\frac{1}{n}}.
\]

and

\[
\left( \int_{C_k} \prod_{i=1}^{m} |f_i(A_i(y)x)|^{q'} \omega(x) dx \right)^{\frac{1}{n}} \lesssim \left( \int_{C_k} \prod_{i=1}^{m} |f_i(A_i(y)x)|^{q} dx \right)^{\frac{1}{n}} |B_k|^{-\frac{1}{n} q \omega(B_k)}.
\]

Combining the Hölder inequality again with variable transformation \( z = A_i(y)x \), we have

\[
\|\mathcal{H}_{\Phi,A}(f)\chi_k\|_{L_{\lambda}^{q'}(\mathbb{R}^n)} \lesssim |B_k|^{-\frac{1}{n} q \omega(B_k)} \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} |\det A_i^{-1}(y)|^{\frac{1}{n}} \|f_i\|_{L_{B(0, |A_i(y)|^{2k})}^{q'}(\mathbb{R}^n)} dy.
\]

It follows from Proposition 2 that

\[
\|f_i\|_{L_{B(0, |A_i(y)|^{2k})}^{q'}(\mathbb{R}^n)} \lesssim |B(0, |A_i(y)|^{2k})|^{\frac{1}{n-q}} \omega(B(0, |A_i(y)|^{2k}))^{-\frac{1}{n-q}} \|f_i\|_{L_{B(0, |A_i(y)|^{2k})}^{q'}(\mathbb{R}^n)}.
\]

Hence, we obtain

\[
\|\mathcal{H}_{\Phi,A}(f)\chi_k\|_{L_{\lambda}^{q'}(\mathbb{R}^n)} \lesssim |B_k|^{-\frac{1}{n} q \omega(B_k)} \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \left( \prod_{i=1}^{m} |\det A_i^{-1}(y)|^{\frac{1}{n}} B(0, |A_i(y)|^{2k})^{\frac{1}{n-q}} \right)^{\frac{1}{n}} \times \omega(B(0, |A_i(y)|^{2k}))^{-\frac{1}{n-q}} \|f_i\|_{L_{B(0, |A_i(y)|^{2k})}^{q'}(\mathbb{R}^n)} dy
\]

\[
\lesssim \left( \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^{m} \left( |\det A_i^{-1}(y)|^{\frac{1}{n}} |A_i(y)|^{\frac{1}{n-q}} \omega(B_k)^{\frac{1}{n-q}} \omega(B(0, |A_i(y)|^{2k}))^{-\frac{1}{n-q}} \right) \|f_i\|_{L_{B(0, |A_i(y)|^{2k})}^{q'}(\mathbb{R}^n)} dy \right)^{\frac{1}{n}} (21)
\]
Thus, by $\lambda^* = \lambda_1 + \cdots + \lambda_m$, the Minkowski and the Hölder inequalities, we have

$$
\|H_{\Phi,A}(f)\|_{MK^{s_1,\epsilon_1,p_1,q_1}(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^m} \prod_{i=1}^m \left( |\det A_i^{-1}(y)|^{\frac{1}{m}} \|A_i(y)\|_2^{\frac{m}{m}} \right) \times
$$

$$
\times \sup_{k_0 \in \mathbb{Z}} v(B_{k_0})^{-\frac{1}{m}} \left( \sum_{k = -\infty}^{k_0} \prod_{i=1}^m \frac{v(B_k)^{\frac{1}{m}}}{\omega(B_k)^{\frac{1}{m}}} \left| \frac{\omega(B_k)}{\omega(B(0, ||A_i(y)||2^k))} \right|^{\frac{1}{m}} \|f_i\|_{L_{C_1}^p(B(0, ||A_i(y)||2^k))} \right)^{\frac{1}{m}}
$$

$$
\lesssim \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^m} \prod_{i=1}^m \left( |\det A_i^{-1}(y)|^{\frac{1}{m}} \|A_i(y)\|_2^{\frac{m}{m}} \right) \times
$$

$$
\times \prod_{i=1}^m \sup_{k_0 \in \mathbb{Z}} v(B_{k_0})^{-\frac{1}{m}} \left( \sum_{k = -\infty}^{k_0} \frac{v(B_k)^{\frac{1}{m}}}{\omega(B_k)^{\frac{1}{m}}} \left| \frac{\omega(B_k)}{\omega(B(0, ||A_i(y)||2^k))} \right|^{\frac{1}{m}} \|f_i\|_{L_{C_1}^p(B(0, ||A_i(y)||2^k))} \right)^{\frac{1}{m}}.
$$

Consequently,

$$
\|H_{\Phi,A}(f)\|_{MK^{s_1,\epsilon_1,p_1,q_1}(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^m} \prod_{i=1}^m \left( |\det A_i^{-1}(y)|^{\frac{1}{m}} \|A_i(y)\|_2^{\frac{m}{m}} C_1(y) \right) dy,
$$

where

$$
C_1(y) := \sup_{k_0 \in \mathbb{Z}} v(B_{k_0})^{-\frac{1}{m}} \left( \sum_{k = -\infty}^{k_0} \left( \frac{v(B_k)^{\frac{1}{m}}}{\omega(B_k)^{\frac{1}{m}}} \left| \frac{\omega(B_k)}{\omega(B(0, ||A_i(y)||2^k))} \right|^{\frac{1}{m}} \|f_i\|_{L_{C_1}^p(B(0, ||A_i(y)||2^k))} \right)^{\frac{1}{m}}. \right)
$$

It follows readily from the Hölder inequality for $\frac{1}{m} + \cdots + \frac{1}{m} = 1$ that

$$
\|H_{\Phi,A}(f)\|_{MK^{s_1,\epsilon_1,p_1,q_1}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \left( \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^m} |\det A_i^{-1}(y)|^{\frac{m}{m}} \|A_i(y)\|_2^{\frac{m}{m}} C_1(y) \right)^{\frac{1}{m}}
$$

$$
=: \prod_{i=1}^m E_{1,i}^{\frac{1}{m}}.
$$

Fix $i \in \{1, \ldots, m\}$. Since $\|A_i(y)\| \neq 0$, there is $j = j(i,y)$ such that $2^{j-1} \leq \|A_i(y)\| < 2^j$. Thus, $B_{k+j-1} \subseteq B(0, ||A_i(y)||2^k) \subseteq B_{k+j}$. It implies that $\omega(B_{k+j-1}) \leq \omega(B(0, ||A_i(y)||2^k))$ and $B(0, ||A_i(y)||2^k) \subseteq B_{k+j} \subseteq B_{k+i,\epsilon}$. Combining these and using the inequality $(\sum |a_i|\theta \leq \sum |a_i|^\theta$ for all $0 < \theta \leq 1$, we have
\[ C_{I}(y) \leq \sup_{k_0 \in \mathbb{Z}} v(B_{k_0})^{-\frac{j_i}{\beta}} \left( \sum_{k=-\infty}^{k_0} \left( \frac{v(B_k)^{\frac{z_i}{m}} \omega(B_k)^{\frac{1}{m'}}}{\omega(B(0, ||A_i(y)||^{2k}))^{\frac{1}{n}}} ||f_i||_{L^{\infty}_y(B_{k+i})} \right)^{p_i} \right)^{\frac{1}{p_i}} \]

\[ \leq \sup_{k_0 \in \mathbb{Z}} v(B_{k_0})^{-\frac{j_i}{\beta}} \left( \sum_{k=-\infty}^{k_0} \left( \frac{v(B_k)^{\frac{z_i}{m}} \omega(B_k)^{\frac{1}{m'}}}{\omega(B(0, ||A_i(y)||^{2k}))^{\frac{1}{n}}} \right)^{\frac{1}{p_i}} \times \left( \sum_{\ell=-\infty}^{j} \left( \frac{v(B_{k+j})}{v(B_{k+j+\ell})} \right)^{\frac{z_i}{p_i}} ||f_i||_{L^{\infty}_y}(c_{k+i}) \right)^{p_i} \right) \]

\[ \leq \sup_{k_0 \in \mathbb{Z}} v(B_{k_0})^{-\frac{j_i}{\beta}} \left( \sum_{k=-\infty}^{k_0} \left( D_k^j \sum_{\ell=-\infty}^{j} \left( \frac{v(B_{k+j})}{v(B_{k+j+\ell})} \right)^{\frac{z_i}{p_i}} ||f_i||_{L^{\infty}_y}(c_{k+i}) \right)^{p_i} \right) \]

where

\[ D_k^j := \frac{v(B_{k+j})^{\frac{z_i}{m}} \omega(B_{k+j})^{\frac{1}{m'}}}{\omega(B_{k+j+1})^{\frac{1}{n}} v(B_{k+j})^{\frac{1}{n}}} \]

By virtue of Proposition 3, \( \ell \leq j \) and \( \alpha_i < 0 \), we have

\[ \left( \frac{v(B_{k+j})}{v(B_{k+j+\ell})} \right)^{\frac{z_i}{p_i}} \leq \left( \frac{B_{k+j}}{B_{k+j+\ell}} \right)^{\frac{z_i}{p_i}} \leq 2^{(j-\ell)z_i}. \]

Observe that

\[ D_k^j = \frac{v(B_{k+j})^{\frac{z_i}{m}} \omega(B_{k+j})^{\frac{1}{m'}}}{\omega(B_{k+j+1})^{\frac{1}{n}} v(B_{k+j})^{\frac{1}{n}}} = \left( \frac{\omega(B_{k+j+1})}{\omega(B_{k+j})} \right)^{\frac{1}{n}} \frac{v(B_{k+j})^{\frac{z_i}{m}} \omega(B_{k+j})^{\frac{1}{m'}}}{\omega(B_{k+j+1})^{\frac{1}{n}} v(B_{k+j})^{\frac{1}{n}}} \]

\[ \leq \left( \frac{B_{k+j}}{B_{k+j+1}} \right)^{\frac{z_i}{p_i}} \frac{v(B_{k+j})^{\frac{z_i}{m}} \omega(B_{k+j})^{\frac{1}{m'}}}{\omega(B_{k+j})^{\frac{1}{n}} v(B_{k+j})^{\frac{1}{n}}} \]

\[ = \frac{\omega(B_{k+j})^{\frac{1}{m'}}}{\omega(B_{k+j})^{\frac{1}{n}}} \frac{v(B_{k+j})^{\frac{z_i}{m}} \omega(B_{k+j})^{\frac{1}{m'}}}{\omega(B_{k+j})^{\frac{1}{n}} v(B_{k+j})^{\frac{1}{n}}} \]

\[ = \left( \frac{\omega(B_{k+j})}{\omega(B_{k+j})} \right)^{\frac{1}{m'}} \frac{v(B_{k+j})^{\frac{z_i}{m}} \omega(B_{k+j})^{\frac{1}{m'}}}{\omega(B_{k+j})^{\frac{1}{n}} v(B_{k+j})^{\frac{1}{n}}} \]

By the conditions \( \frac{z_i}{m} + \frac{1}{m'} = \frac{z_i}{n} + \frac{1}{q_i} \) and \( q > mq^* \), it is obvious that \( \frac{1}{m'} > \frac{1}{q} > \frac{1}{q_i} \)

and \( \frac{z_i}{m} < \frac{z_i}{n} \) for all \( i = 1, \ldots, m \). Thus, the condition \( \omega(B_{k+j}) \leq v(B_{k+j}) \), for every \( k \in \mathbb{Z} \), implies that

\[ \left( \frac{\omega(B_{k+j})}{v(B_{k+j})} \right)^{\frac{z_i}{n}} \leq \left( \frac{\omega(B_{k+j})}{v(B_{k+j})} \right)^{\frac{z_i}{m}} \]

Therefore, we obtain
Now, let us consider two cases as follows.

**Case 1:** \( \frac{z^*}{n} + \frac{1}{q^*} \leq 0 \).

For \( j \geq 1 \), we get

\[
\left( \frac{\omega(B_k)}{\omega(B_{k+j})} \right)^{\frac{1}{3n^{\frac{1}{3}}}} \left( \frac{v(B_k)}{v(B_{k+j})} \right)^{\frac{z^*}{3n}} \leq \frac{|B_k|}{|B_{k+j}|} \left( \frac{|B_{k+j}|}{|B_k|} \right)^{\frac{x^{\frac{1}{4} - 1}}{n}} = 2^{-jnx^{\frac{1}{2} - 1}}.
\]

For \( j \leq 0 \), we have

\[
\left( \frac{\omega(B_k)}{\omega(B_{k+j})} \right)^{\frac{1}{3n^{\frac{1}{3}}}} \left( \frac{v(B_k)}{v(B_{k+j})} \right)^{\frac{z^*}{3n}} \leq \frac{|B_k|}{|B_{k+j}|} \left( \frac{|B_{k+j}|}{|B_k|} \right)^{\frac{x^{\frac{1}{4} - 1}}{n}} = 2^{-jnx^{\frac{1}{2} - 1}}.
\]

So, we obtain

\[
\mathcal{D}_k^j \lesssim \begin{cases} 2^{\frac{1}{2}(\frac{nx^{\frac{1}{4} - 1} - jx^{\frac{1}{4} - 1} - jx^{\frac{1}{2} - 1} + jx^{\frac{1}{2} - 1}}{q^*})}, & j \leq 0 \\ 2^{\frac{1}{2}(\frac{-jx^{\frac{1}{2} - 1} + jx^{\frac{1}{4} - 1} + jx^{\frac{1}{4} - 1}}{q^*})}, & j \geq 1. \end{cases}
\]

**Case 2:** \( \frac{z^*}{n} + \frac{1}{q^*} > 0 \).

Similarly to Case 1, for \( j \geq 1 \), we also have

\[
\left( \frac{\omega(B_k)}{\omega(B_{k+j})} \right)^{\frac{1}{3n^{\frac{1}{3}}}} \left( \frac{v(B_k)}{v(B_{k+j})} \right)^{\frac{z^*}{3n}} \leq \frac{|B_k|}{|B_{k+j}|} \left( \frac{|B_{k+j}|}{|B_k|} \right)^{\frac{x^{\frac{1}{4} - 1}}{n}} = 2^{-jnx^{\frac{1}{2} - 1}}.
\]

For \( j \leq 0 \), we get
\[
\left( \frac{\omega(B_k)}{\omega(B_{k+j})} \right)^{\frac{1}{m^*} + \frac{s}{m}} \lesssim \left( \frac{|B_k|}{|B_{k+j}|} \right)^{\frac{s}{m} + \frac{1}{m^*} + \frac{s}{m}} = 2^{-j \xi n \left( \frac{1}{m^*} + \frac{s}{m} \right)}
\]
\[
\left( \frac{\omega(B_{k+j})}{\omega(B_k)} \right)^{\frac{s}{m}} \lesssim \left( \frac{|B_{k+j}|}{|B_k|} \right)^{\frac{s}{m} + \frac{1}{m^*} + \frac{s}{m}} = 2^{j \xi^2 \frac{s}{m}}
\]
\[
\left( \frac{v(B_k)}{v(B_{k+j})} \right)^{\frac{s}{m}} \lesssim \left( \frac{|B_k|}{|B_{k+j}|} \right)^{\frac{s^2 + 1}{m^*} + \frac{s}{m}} = 2^{-j \frac{s^2 + 1}{m^*} \frac{s}{m}}.
\]
Consequently,
\[
\mathcal{D}_{k,j} \lesssim \begin{cases} 
2^{\frac{j}{n} \left( \frac{\mu_1}{m_1} - j\xi x^* \frac{s^2}{m_2} \right)}, & j \leq 0 \\
2^{\frac{j}{n} \left( \frac{\mu_1}{m_1} - j\eta x^* \frac{s^2}{m_2} \right)}, & j \geq 1.
\end{cases}
\]

Now, let us estimate \( \mathcal{E}_{1j} \) for the case 1. We have
\[
\mathcal{E}_{1j} \lesssim \int_{\{y : ||A(y)|| < 1\}} \frac{||\Phi(y)||}{|y|^n} |\det A_j^{-1}(y)|^{\frac{m_i}{n}} |A_i(y)|^{\frac{m_i}{n}} \times 
\times \left( \sup_{k_0 \in \mathbb{Z}} v(B_{k_0}) \right)^{-\frac{n}{m}} \sum_{k = -\infty}^{k_0} \left( 2^{\frac{j}{n} \left( \frac{\mu_1}{m_1} - j\xi x^* \frac{s^2}{m_2} \right)} \right) dy
\times \int_{\{y : ||A(y)|| \geq 1\}} \frac{||\Phi(y)||}{|y|^n} |\det A_j^{-1}(y)|^{\frac{m_i}{n}} |A_i(y)|^{\frac{m_i}{n}} \times 
\times \left( \sup_{k_0 \in \mathbb{Z}} v(B_{k_0}) \right)^{-\frac{n}{m}} \sum_{k = -\infty}^{k_0} \left( 2^{\frac{j}{n} \left( \frac{\mu_1}{m_1} - j\eta x^* \frac{s^2}{m_2} \right)} \right) dy
\lesssim \int_{\{y : ||A(y)|| < 1\}} \frac{||\Phi(y)||}{|y|^n} |\det A_j^{-1}(y)|^{\frac{m_i}{n}} |A_i(y)|^{\frac{m_i}{n}} |A_i(y)| \times 
\times \left( \sup_{k_0 \in \mathbb{Z}} v(B_{k_0}) \right)^{-\frac{n}{m}} \sum_{k = -\infty}^{k_0} \left( \int_{-\infty}^{j} 2^{\frac{j}{n} \left( \frac{\mu_1}{m_1} - j\xi x^* \frac{s^2}{m_2} \right)} |f_i|_{L_{m_1}^n(C_{k+i})} \right) \frac{d\eta}{n} dy + 
+ \int_{\{y : ||A(y)|| \geq 1\}} \frac{||\Phi(y)||}{|y|^n} |\det A_j^{-1}(y)|^{\frac{m_i}{n}} |A_i(y)|^{\frac{m_i}{n}} |A_i(y)| \times 
\times \left( \sup_{k_0 \in \mathbb{Z}} v(B_{k_0}) \right)^{-\frac{n}{m}} \sum_{k = -\infty}^{k_0} \left( \int_{-\infty}^{j} 2^{\frac{j}{n} \left( \frac{\mu_1}{m_1} - j\eta x^* \frac{s^2}{m_2} \right)} |f_i|_{L_{m_1}^n(C_{k+i})} \right) \frac{d\eta}{n} dy.
\]
Notice that \( \|A_i(y)\|^{-1} \simeq 2^{-j} \) and \( \sum_{\ell=-\infty}^{j} 2^{(j-\ell)}\alpha_{\ell}^{-1-n} = \frac{1}{1-2^{\frac{j-n}{2}}} \), for all \( \alpha_{\ell} < 0 \), and

\[
v(B_{k_0})^{-\frac{j}{2}} = \left( \frac{v(B_{k_0})}{v(B_{k_0})} \right)^{\frac{j}{2}} v(B_{k_0})^{-\frac{j}{2}} \leq \left( \frac{v(B_{k_0+j})}{v(B_{k_0})} \right)^{\frac{j}{2}} v(B_{k_0+j})^{-\frac{j}{2}},
\]

for all \( \ell \leq j \). It is easy to show that

\[
\left( \frac{v(B_{k_0+j})}{v(B_{k_0})} \right)^{\frac{j}{2}} \lesssim \begin{cases} 2^{j\alpha_{j-1}n-1} & j \leq 0 \\ 2^j & j \geq 1 \end{cases}
\]

Then by the Minkowski inequality for \( p_i \geq 1 \) we deduce

\[
\mathcal{E}_{1i} \lesssim \int_{\{y: \|A_i(y)\| < 1\}} \frac{|\Phi(y)|}{|y|^n} \left| \det A_i^{-1}(y) \right| \|A_i(y)\|^{\frac{m_i}{q_i}} \left( \frac{\xi_i + \eta_i}{\xi_i} \right)^{\frac{m_i}{q_i}} \left( \frac{\xi_i - \eta_i}{\xi_i} \right)^{1-n} dy +
\]

\[
\int_{\{y: \|A_i(y)\| \geq 1\}} \frac{|\Phi(y)|}{|y|^n} \left| \det A_i^{-1}(y) \right| \|A_i(y)\|^{\frac{m_i}{q_i}} \left( \frac{\xi_i + \eta_i}{\xi_i} \right)^{\frac{m_i}{q_i}} \left( \frac{\xi_i - \eta_i}{\xi_i} \right)^{1-n} dy
\]

Finally, we obtain

\[
\| \mathcal{H}_{A,i}(f) \|_{MK_{1,0}^{\alpha,\beta,\gamma,\delta,\rho,\sigma}(\mathbb{R}^n)} \lesssim \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \left| \det A_i^{-1}(y) \right| \|A_i(y)\|^{\frac{m_i}{q_i}} \|A_i\|^{\frac{m_0}{q_0}} B_{1;i}(y) dy \right)^{\frac{1}{m}} \prod_{i=1}^{m} \|f_i\|_{MK_{1,0}^{\alpha,\beta,\gamma,\delta,\rho,\sigma}(\mathbb{R}^n)}
\]

Arguing as in case 1, we also get the desired results for case 2 under the condition \( \frac{\alpha_{j-1}n}{n} + \frac{1}{q} > 0 \). More precisely, the following is true.
\[
\| \mathcal{H}_{\phi, A}(f) \|_{M_{2q}^{2,q} (\mathbb{R}^n)} \\
\lesssim \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{|\phi(y)|}{|y|^n} |\det A_i^{-1}(y)|^{\frac{m_i}{\tilde{q}}} |A_i(y)|^{\frac{n_0}{\tilde{q}}} B_{2q}(y) \, dy \right)^{\frac{1}{m}} \prod_{i=1}^{m} \| f_i \|_{M_{2q}^{2,q}(\mathbb{R}^n)} \\
\lesssim C_{5.2} \prod_{i=1}^{m} \| f_i \|_{M_{2q}^{2,q}(\mathbb{R}^n)}.
\]

Therefore, the proof of the theorem is completed. \(\square\)

In particular, in the case \(\omega = \nu\), we also obtain some sufficient conditions for the boundedness of \(\mathcal{H}_{\phi, A}\) on the weighted Morrey–Herz spaces which are actually better than ones of Theorem 5. The proof is similar to one of Theorem 5, and it is omitted.

More precisely, we have the following theorem.

**Theorem 6** Let \(1 \leq q^*, \zeta < \infty, \alpha_i < 0, \lambda_i \geq 0, \text{ for all } i = 1, \ldots, m, \text{ and } \omega \in A_\xi\) with the finite reverse Hölder critical index \(r_\omega\). Assume that \(q > q^* \zeta r_\omega, \delta \in (1, r_\omega)\) and \(x^*, \lambda^*\) are two real numbers satisfying

\[
\lambda^* = \lambda_1 + \cdots + \lambda_m \quad \text{and} \quad \frac{x^*}{n} = \frac{1}{q^*} = \sum_{i=1}^{m} \frac{\alpha_i}{n} + \frac{1}{q}.
\]

If \(\frac{\alpha_i}{n} + \frac{1}{q_i} \leq 0, \text{ for all } i = 1, \ldots, m, \text{ and}

\[
C_{6.1} = \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{|\phi(y)|}{|y|^n} |\det A_i^{-1}(y)|^{\frac{m_i}{\tilde{q}}} |A_i(y)|^{\frac{n_0}{\tilde{q}}} \psi_{1i}(y) \, dy \right)^{\frac{1}{m}} < \infty,
\]

where

\[
\psi_{1i}(y) = \|A_i(y)\|^{m\left(\lambda_i - \frac{n}{2}(1 + \frac{1}{q_i})\right)} L_{\{y \in \mathbb{R}^n : |A_i(y)| < 1\}} + \|A_i(y)\|^{m\left(\lambda_i - \frac{n}{2}(1 + \frac{1}{q_i})\right)} L_{\{y \in \mathbb{R}^n : |A_i(y)| \geq 1\}},
\]

or \(\frac{\alpha_i}{n} + \frac{1}{q_i} > 0, \text{ for all } i = 1, \ldots, m, \text{ and}

\[
C_{6.2} = \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{|\phi(y)|}{|y|^n} |\det A_i^{-1}(y)|^{\frac{m_i}{\tilde{q}}} |A_i(y)|^{\frac{n_0}{\tilde{q}}} \psi_{2i}(y) \, dy \right)^{\frac{1}{m}} < \infty,
\]

where

\[
\psi_{2i}(y) = \|A_i(y)\|^{m\left(\lambda_i - \frac{n}{2}(1 + \frac{1}{q_i})\right)} L_{\{y \in \mathbb{R}^n : |A_i(y)| < 1\}} + \|A_i(y)\|^{m\left(\lambda_i - \frac{n}{2}(1 + \frac{1}{q_i})\right)} L_{\{y \in \mathbb{R}^n : |A_i(y)| \geq 1\}},
\]

then we have \(\mathcal{H}_{\phi, A}\) is bounded from \(\prod_{i=1}^{m} M_{K_{p_i,q_i}}(\omega, \mathbb{R}^n)\) to \(M_{K_{p^*,q^*}}(\omega, \mathbb{R}^n)\).
As consequences of Theorems 5 and 6, by letting $\lambda_1 = \cdots = \lambda_m = 0$, we also obtain the sufficient conditions for the boundedness of the multilinear Hausdorff operators on two-weighted Herz spaces with the $A_\xi$ weights.

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