Poincaré invariance constraints on NRQCD and potential NRQCD

Nora Brambilla\textsuperscript{1}

\textit{Dipartimento di Fisica, Università degli Studi di Milano}
\textit{via Celoria 16, 20133 Milano, Italy}
\textit{and}

\textit{Institut für Theoretische Physik, Universität Heidelberg}
\textit{Philosophenweg 16, 69120 Heidelberg, Germany}

Dieter Gromes\textsuperscript{2}

\textit{Institut für Theoretische Physik, Universität Heidelberg}
\textit{Philosophenweg 16, 69120 Heidelberg, Germany}

Antonio Vairo\textsuperscript{3}

\textit{Theory Division CERN, 1211 Geneva 23, Switzerland}

Abstract

We discuss the constraints induced by the algebra of the Poincaré generators on non-relativistic effective field theories. In the first part we derive some relations among the matching coefficients of the HQET (and NRQCD), which have been formerly obtained by use of reparametrization invariance. In the second part we obtain new constraints on the matching coefficients of pNRQCD.
1 Introduction

For any Poincaré invariant theory the generators $H, P, J, K$ of time translations, space translations, rotations, and Lorentz transformations satisfy the Poincaré algebra:

\[
\begin{align*}
[P^i, P^j] &= 0, \\
[P^i, H] &= 0, \\
[J^i, P^j] &= i\epsilon_{ijk}P^k, \\
[J^i, H] &= 0, \\
[J^i, J^j] &= i\epsilon_{ijk}J^k, \\
[H, K^i] &= -i\delta_{ij}H, \\
[J^i, K^j] &= i\epsilon_{ijk}K^k, \\
[K^i, K^j] &= -i\epsilon_{ijk}J^k.
\end{align*}
\]

It has been pointed out, as early as in Ref. [1], that the algebra induces non trivial constraints on the form of the Hamiltonian of non-relativistic systems where Poincaré invariance is no longer explicit. Indeed, the algebra has been used in the past to constrain the form of the relativistic corrections to phenomenological potentials [2].

In this letter by Poincaré invariance we mean the explicit realization of the algebra (1)-(9). We study such realization in some of the modern non-relativistic effective field theories of QCD: Heavy Quark Effective Theory (HQET) [3], Non Relativistic QCD (NRQCD) [4] and potential NRQCD (pNRQCD) [5, 6]. One may expect that also in these cases Poincaré invariance induces non trivial constraints on the form of the interaction. More specifically, one expects to obtain some exact relations among the matching coefficients of the considered effective field theory. This study has never been done before. In the case of the HQET some exact relations between matching coefficients have been derived using a specific invariance of the theory, known as reparametrization invariance [7, 8]. We will derive in Sec. 2 some of these relations, showing in this way that reparametrization invariance is just a manifestation of the Poincaré invariance of the theory. In the case where dynamical gauge fields have been integrated out from pNRQCD, Poincaré invariance has been studied in this framework in [9]. There some relations among the potentials have been obtained that were previously known only from the transformation properties under Lorentz boost of the representation of the potentials in terms of Wilson loops [10, 11, 12]. Here we will extend that study to the situation with dynamical gluons. This is the situation of interest, for instance for weakly coupled heavy quarkonium states like the $\Upsilon(1S)$.

The letter is organized as follows. In Sec. 2 we illustrate how Poincaré invariance (in the sense specified above) works in NRQCD/HQET. In particular we construct all generators of the Poincaré transformation up to the relevant order and verify the algebra. We obtain in this way some of the constraints already known from reparametrization invariance. In Sec. 3 we apply the same machinery to pNRQCD. After constructing all generators of the Poincaré transformations up to the relevant order, we derive, by imposing the algebra, some new constraints on the matching coefficients of the theory. In Sec. 4 we summarize our results and discuss some possible future developments.
2 NRQCD

After integrating out the hard scale $m$ from QCD, one obtains NRQCD [4]. Neglecting operators that involve light-quark fields [13], the most general NRQCD Lagrangian density (up to field redefinitions) for a quark and an antiquark of mass $m$ up to order $1/m^2$ is given by (we display also the term $D^4/(8m^3)$ for further use):

$$\mathcal{L}_{\text{NRQCD}} =$$

$$\psi^\dagger \left\{ iD_0 - m + c_1 \frac{D^2}{2m} + c_2 \frac{D^4}{8m^3} + c_F g \cdot \mathbf{B} + c'_D g \frac{[\mathbf{D}, \mathbf{E}]}{8m^2} + ic_S g \frac{\sigma \cdot [\mathbf{D} \times \mathbf{E}]}{8m^2} \right\} \psi$$

$$+ \chi^\dagger \left\{ iD_0 + m - c_1 \frac{D^2}{2m} - c_2 \frac{D^4}{8m^3} - c_F g \cdot \mathbf{B} + c'_D g \frac{[\mathbf{D}, \mathbf{E}]}{8m^2} + ic_S g \frac{\sigma \cdot [\mathbf{D} \times \mathbf{E}]}{8m^2} \right\} \chi$$

$$+ \frac{d_{ss}}{m^2} \psi^\dagger \psi \chi^\dagger \chi + \frac{d_{sv}}{m^2} \psi^\dagger \sigma \psi \cdot \chi^\dagger \chi + \frac{d_{us}}{m^2} \psi^\dagger T^a \psi \chi^\dagger T^a \chi + \frac{d_{uv}}{m^2} \psi^\dagger T^a \sigma \psi \cdot \chi^\dagger T^a \sigma \chi$$

$$- \frac{1}{4} F^{a\mu} F_a^{\mu\nu} + \frac{d_3^a}{m^2} g f_{abc} F_a^{\mu} F_b^{\nu} F_\nu^{\alpha},$$

where $\psi$ is the Pauli spinor field that annihilates the fermion and $\chi$ is the Pauli spinor field that creates the antifermion, $iD^0 = i\partial^0 - gA^0$, $iD = i\nabla + g\mathbf{A}$, $[\mathbf{D}, \mathbf{E}] = \mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D}$, and $[\mathbf{D} \times \mathbf{E}] = \mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D}$, $\mathbf{E}^i = F_i^{\mu}$ and $\mathbf{B}^i = -\epsilon_{ijk} F_j^{jk}/2$, $\epsilon_{ijk}$ being the usual three-dimensional antisymmetric tensor ($(a \times b)^i = \epsilon_{ijk} a^j b^k$). We display the mass terms, that are usually removed by a field redefinition, for the further use of the canonical formalism to derive the time-translation generator. The one-loop expressions in the $\overline{\text{MS}}$ scheme for the coefficients $c_F$, $c'_D$, $c_S$ and $d_3^a$ can be found in [8] (according to the definitions of $c'_D$ and $d_3^a$ given in [14]) and for $d_{ij}$ ($i, j = s, v$) in [15].

It has been proved in [8] that in the bilinear sector the NRQCD Lagrangian is equivalent to the Lagrangian of the HQET. Therefore, at the order at which only the bilinear sector matters (and this will be the case for the rest of this section), the results that we obtain are valid both for NRQCD and HQET. The main reason for introducing here the NRQCD Lagrangian up to order $1/m^2$, is that it is needed to obtain the pNRQCD Lagrangian at $\mathcal{O}(1/m^2)$, which is the subject of Sec. 3.

2.1 Canonical quantization

To be definite we quantize NRQCD in the $A^0 = 0$ gauge [16]. The pairs of canonical variables are $(\psi, i\psi^\dagger)$, $(\chi, i\chi^\dagger)$ and $(\mathbf{A}_{i a}, \Pi_a^i = \partial \mathcal{L}_{\text{NRQCD}} / \partial (\partial_0 A_a^i))$. The physical states $|\text{phys}\rangle$ are constrained by the Gauss law:

$$(\mathbf{D} \cdot \mathbf{\Pi})^a |\text{phys}\rangle = g(\psi^\dagger T^a \psi + \chi^\dagger T^a \chi) |\text{phys}\rangle.$$  (11)

The canonical variables satisfy the usual equal time commutation relations:

$$[\Pi_a^i(x, t), A_b^j(y, t)] = i\delta_{ij}\delta_{ab}\delta^{(3)}(x - y),$$  (12)

$$[A_a^i(x, t), A_b^j(y, t)] = [\Pi_a^i(x, t), \Pi_b^j(y, t)] = 0,$$  (13)

$$\{\psi_\alpha(x, t), \psi_\beta(y, t)\} = \{\chi_\alpha(x, t), \chi_\beta(y, t)\} = \delta_{\alpha\beta}\delta^{(3)}(x - y),$$  (14)

$$\{\psi_\alpha(x, t), \chi_\beta(y, t)\} = \{\psi_\alpha^\dagger(x, t), \psi_\beta^\dagger(y, t)\} = 0,$$  (15)

$$\{\chi_\alpha(x, t), \chi_\beta(y, t)\} = \{\chi_\alpha^\dagger(x, t), \chi_\beta^\dagger(y, t)\} = 0.$$  (16)
All other commutators are zero. In the following, in order to fulfill the Gauss-law constraint (11), we will assume the commutators to act on a space spanned by the physical states.

### 2.2 Poincaré algebra generators in NRQCD

The construction of the generators proceeds in the following way. The generators $H$, $P$ and $J$ can be derived from the symmetric energy-momentum tensor (see for instance [17]). Since translational and rotational invariance remain exact symmetries when going to the effective theory, the transformation properties of the new particle fields under these symmetries are the same as in the original theory. The derivation of the Lorentz-boost generators is more problematic, since the non-relativistic expansion has destroyed the manifest covariance under boosts. A consistent way to construct $K$ is to write down the most general expression, with some obvious restrictions already included, containing all operators consistent with its symmetries and to match it to the QCD Lorentz-boost generator, which is

$$K = -t P + \int d^3x \frac{1}{2} \left\{ x, \frac{\Pi^a \cdot \Pi^a + B^a \cdot B^a}{2} + \bar{\psi}(-i D \cdot \gamma + m) \psi \right\}. $$

This is very much the same procedure that is used in the construction of the NRQCD Lagrangian. Accordingly, new matching coefficients, typical of $K$, will appear.

Specifically for the Lagrangian (10) we obtain ($P$ and $J$ are exact, $H$ is displayed up to order $1/m^2$, including the kinetic energy up to order $1/m^3$, and $K$ is given up to order $1/m$):

$$H = \int d^3x \ h, $$

$$P = \int d^3x \ \left( \bar{\psi} (-iD) \psi + \bar{\chi} (-iD) \chi + \frac{1}{2} [\Pi^a \times , B^a] \right), $$

$$J = \int d^3x \ \left( \bar{\psi} \left( x \times (-iD) + \frac{\sigma}{2} \right) \psi + \bar{\chi} \left( x \times (-iD) + \frac{\sigma}{2} \right) \chi \right. $$

$$+ \frac{1}{2} x \times [\Pi^a \times , B^a], $$

$$K = -t P + \int d^3x \ \left\{ \frac{x}{2} h \right\} $$

$$- k^{(1)} \int d^3x \ \left( \frac{1}{2m} \bar{\psi} \frac{\sigma}{2} \times (-iD) \psi - \frac{1}{2m} \bar{\chi} \frac{\sigma}{2} \times (-iD) \chi \right), $$

where $k^{(1)}$ is a matching coefficient specific of $K$. 

3
2.3 Poincaré algebra constraints in NRQCD

Let us now consider the constraints induced by the Poincaré algebra (1)-(9) on the NRQCD generators \( H \) and \( K \). The constraint \([P^i, K^j] = -i\delta_{ij} H\) has been already used in Eq. (20). Indeed, this commutation relation forces \( K \) to have the form

\[
\int d^3x \frac{\{x, h(x, t)\}}{2} + \text{translational-invariant terms that depend on } x \text{ only through the canonical variables. From } [K^i, K^j] = -i\epsilon_{ijk} J^k \text{ at } \mathcal{O}(1/m^0) \text{ it follows that}
\]

\[
-i\epsilon_{ijk} (1 - k^{(1)}) \int d^3x \left( \frac{\sigma^k}{2} \psi + \frac{\sigma^k}{2} \chi \right) = 0 \Rightarrow k^{(1)} = 1. \tag{21}
\]

From \([H, K^i] = -iP^i \) at \( \mathcal{O}(1/m^0) \) it follows that

\[
-i (1 - c_1) \int d^3x \left( \psi^\dagger (-i \nabla^i) \psi + \chi^\dagger (-i \nabla^i) \chi \right) = 0 \Rightarrow c_1 = 1, \tag{22}
\]

and at \( \mathcal{O}(1/m) \)

\[
-i (2c_F - c_S - 1) \int d^3x \left( \frac{\sigma \times g\Pi^i}{4m} \psi - \frac{\sigma \times g\Pi^i}{4m} \chi \right) = 0 \Rightarrow 2c_F - c_S - 1 = 0. \tag{23}
\]

Finally from \([H, K^i] = -iP^i \) at \( \mathcal{O}(\nabla^2 \nabla^i/m^2) \) we obtain

\[
(1 - c_2) \int d^3x \left( \frac{\nabla^2 \nabla^i}{2m^2} \psi + \frac{\nabla^2 \nabla^i}{2m^2} \chi \right) = 0 \Rightarrow c_2 = 1. \tag{24}
\]

All other commutation relations are satisfied at the order we are working. The constraints (22), (23) and (24) were first derived in the framework of reparametrization invariance in [7, 8].

3 pNRQCD

The pNRQCD Lagrangian for a heavy quark-antiquark system is obtained from NRQCD by integrating out the soft degrees of freedom associated with the scale of the relative momentum of the two heavy quarks in the bound state [5, 6]. The name pNRQCD has been used in the literature to identify effective field theories with different degrees of freedom. Here we call pNRQCD the effective field theory that can be obtained from NRQCD by perturbative matching and contains, as degrees of freedom, the quark-antiquark field (that can be split into a colour singlet \( S = S_{1c}/\sqrt{N_c} \) and a colour octet \( O = O^a T^a/\sqrt{T_F} \) component) and (ultrasoft) gluons. The fields \( S \) and \( O^a \) are functions of \((X, t)\) and \( x \), where \( X = (x_1 + x_2)/2 \) is the centre-of-mass coordinate and \( x = x_1 - x_2 \) the relative coordinate, with \( x_1, x_2 \) the coordinates of the quark (antiquark). The coordinate \( x \) plays the role of a continuous parameter, which specifies different fields. All the gauge fields have been multipole expanded around the centre-of-mass and depend on \((X, t): F^{\mu\nu a} = F^{\mu\nu a}(X, t)\) and \( iD_\mu O = i\partial_\mu O - g[A_\mu(X, t), O] \). The terms in the pNRQCD Lagrangian are organized by powers in the \( 1/m \) and \( x \) expansions. We will indicate a
generic term of order $x^3/m^4$ as $h_{\varphi \phi}^{(i,j)}$, where $\varphi, \phi \in \{S, O\}$. Notice that the derivative $\nabla_x$ counts like $x^{-1}$ and, therefore, contributes with a negative power to the second index of $h_{\varphi \phi}^{(i,j)}$.

We aim at verifying the Poincaré algebra at order $x/m^0$ and $x^0/m$. For this purpose we need the pNRQCD Lagrangian at order $x^2/m^0$ (obtained in [5, 18]), as well as at order $x^0/m$, $(x/m) P_X$ and $(x^0/m^2) P_X$ (derived in this work). We include all local operators of the correct transformation properties and the appropriate dimensions, moreover we assume that, apart from the kinetic energy, momentum and spin operators are suppressed by a factor $1/m$ each. To the above order the pNRQCD Lagrangian density is given by (as in NRQCD, we display the mass terms, that could be removed by a field redefinition, for the further use of the canonical formalism to derive the time-translation generator):

$$\mathcal{L}_{\text{pNRQCD}} = \int d^3x \text{Tr} \left\{ S^\dagger (i\partial_0 - 2m - h_S) S + O^\dagger (iD_0 - 2m - h_O) O - \left[ \left( S^\dagger h_{SO} O + \text{H.c.} \right) + \text{C.C.} \right] - \left[ O^\dagger h_{OS} O + \text{C.C.} \right] - \left[ O^\dagger h_{OO} O + \text{C.C.} \right] \right\} - \frac{1}{4} F^a_{\mu \nu} F^{a, \mu \nu},$$

(25)

$$h_{\varphi} = \left\{ \frac{c_{\varphi}^{(1, -2)}}{2m} \left( \frac{P_x^2}{4m} + V^{(0)}(x) \right) + \frac{V^{(1)}(x)}{m} + \frac{V^{(2)}(x)}{m^2} \right\}$$

(26)

$$h_{\varphi \phi} = h_{\varphi \phi}^{(0,1)} + h_{\varphi \phi}^{(0,2)} + h_{\varphi \phi}^{(1,0)}(P_X) + h_{\varphi \phi}^{(2,0)}(P_X),$$

with

(28)

$$h_{\varphi \phi}^{(0,1)} = -\frac{V^{(0,1)}_{\varphi \phi}(x)}{2} x \cdot gE,$$

(29)

$$h_{\varphi \phi}^{(0,2)} = -\frac{V^{(0,2)}_{\varphi \phi}(x)}{8} x^i x^j (D^i gE^j) - \frac{V^{(0,2)}_{\varphi \phi}(x)}{8} x^2 (D \cdot gE),$$

(30)

$$h_{\varphi \phi}^{(1,0)} = \frac{1}{8m} V^{(1,0)}_{\varphi \phi}(x) \left\{ p_x, x \times gB \right\} - \frac{c_{\varphi}}{2m} V^{(1,0)}_{\varphi \phi}(x) \sigma^{(1)} \cdot gB$$

$$- \frac{1}{2m} \frac{V^{(1,0)}_{\varphi \phi}(x)}{x^2} (x \cdot \sigma^{(1)}) (x \cdot gB) - \frac{1}{m} \frac{V^{(1,0)}_{\varphi \phi}(x)}{2} x \cdot gE,$$

(31)

$$h_{\varphi \phi}^{(1,1)}(P_X) = \frac{1}{8m} V^{(1,0)}_{\varphi \phi}(x) \left\{ P_{X^i}, x \times gB \right\},$$

(32)

$$h_{\varphi \phi}^{(2,0)}(P_X) = \frac{c_{\varphi}}{16m^2} V^{(2,0)}_{\varphi \phi}(x) \sigma^{(1)} \cdot \left[ P_X, x \times gE \right].$$
\[ + \frac{1}{16m^2} \frac{V_{\phi d}^{(2,0)}(x)}{x^2} (x \cdot \sigma^{(1)}) \{ P_X, (gE \times x) \} \]
\[ + \frac{1}{16m^2} \frac{V_{\phi d'}^{(2,0)}(x)}{x^2} \{(x \cdot gE), P_X \cdot (x \times \sigma^{(1)}) \} \]
\[ + \frac{1}{16m^2} \frac{V_{\phi d''}^{(2,0)}(x)}{x^2} \{(x \cdot P_X), \sigma^{(1)} \cdot (x \times gE) \} \]
\[ + \frac{1}{16m^2} \{(p_x \cdot P_x), V_{\phi d''}^{(2,0)}(x)(x \cdot gE) \} \]
\[ + \frac{1}{16m^2} \{(p_x p_x') V_{\phi d''}^{(2,0)}(x) x^j gE^j \} \]
\[ + \frac{1}{16m^2} \{(p_x p_x'), V_{\phi d''}^{(2,0)}(x) x^j gE^j \} \]
\[ + \frac{1}{16m^2} \{ p_x p_x', V_{\phi d''}^{(2,0)}(x) x^i x^j (x \cdot gE) \} \]
\[ + \frac{1}{8m^2} \frac{V_{\phi d'}^{(2,0)}(x)}{x} \{ P_X, x \times gB \}, \] (33)

\[ O^{\dagger} h^{A}_{OO} O h^{B}_{OO} = O^{\dagger} h^{A}_{OO}(1) O h^{B}_{OO}(1) + O^{\dagger} h^{A}_{OO}(2) O h^{B}_{OO}(2) (P_x), \] with \[ O^{\dagger} h^{A}_{OO}(1) O h^{B}_{OO}(1) = - \frac{1}{2m} V_{\phi d}^{(1,0)}(x) O^{\dagger}(x \cdot \sigma^{(1)}) \cdot O gB \]
\[ - \frac{1}{2m} \frac{V_{\phi d}^{(1,0)}(x)}{x^2} O^{\dagger}(x \cdot \sigma^{(1)}) O (x \cdot gB), \] (35)
\[ O^{\dagger} h^{A}_{OO}(2) O h^{B}_{OO}(2) (P_x) = \frac{1}{16m^2} \frac{V_{\phi d}^{(2,0)}(x)}{x^2} \{ O^{\dagger}(x \cdot \sigma^{(1)}) P_X O, (gE \times x) \} \]
\[ + \frac{1}{16m^2} \frac{V_{\phi d'}^{(2,0)}(x)}{x^2} \{ O^{\dagger}(x \cdot \sigma^{(1)}) P_X O, (gE \times x) \} \]
\[ + \frac{1}{16m^2} \frac{V_{\phi d''}^{(2,0)}(x)}{x^2} \{ O^{\dagger}(x \cdot \sigma^{(1)}) P_X O, (x \cdot gE) \} \]
\[ + \frac{1}{16m^2} \frac{V_{\phi d''}^{(2,0)}(x)}{x^2} \{ O^{\dagger}(x \cdot \sigma^{(1)}) P_X O, (x \cdot gE) \}. \] (36)

We have defined \( P_X = -iD_X \) and \( p_x = -i\nabla_x = (p_1 - p_2)/2 \); when acting on a singlet field \( P_X \) reduces to \( -i\nabla_x = p_1 + p_2, p_j \) being \( -i\nabla_{x_j} \). The abbreviation C.C. stands for charge conjugation, H.C. for Hermitian conjugation. For the potential \( V(2) \) we have used a notation close to the traditional one of [14]; we indicate momentum- and spin-independent potentials with \( V_r \), momentum-dependent potentials with \( V_{p^2} \) and \( V_{L^2} \), spin-orbit potentials with \( V_{L_S} \), spin-spin potentials with \( V_{S_2} \) and spin-tensor potentials with \( V_{S_{12}} \). The letters \( a, b, \ldots \) that appear in some of the matching coefficients are used to label the different kinds of operators. For \( h^{(1,1)} \) and \( h^{(2,0)} \) only the \( P_X \)-dependent terms are displayed.

Under color trace all displayed terms of the type \( h_{SS}^{(ij)} \) vanish. Due to charge conjugation the terms proportional to \( V_{SOa}^{(0,2)} \), \( V_{SOa}^{(2,0)} \), \( V_{SOa}^{(1,0)} \), \( V_{SOc'}^{(2,0)} \), \( V_{SOc'}^{(2,0)} \), \( V_{SOc''}^{(2,0)} \), \( V_{SOc''}^{(2,0)} \) and \( V_{SOd}^{(2,0)} \) vanish. The matching coefficients \( c_{\phi^{(-2)}}^{(1,0)}, c_{\phi^{(1,1)}}^{(1,0)}, c_{\phi^{(0,0)}}^{(1,0)} \), \( V_{OOa}^{(0,1)} \), \( V_{OOa}^{(0,1)} \), \( V_{\phi \phi}^{(1,1)} \), \( V_{\phi \phi}^{(1,1)} \) and \( V_{\phi \phi}^{(1,0)} \) are
equal to 1 at tree level. All other matching coefficients are zero at $\mathcal{O}(\alpha_s^0)$. The tree-level matching has been performed by multipole expanding the NRQCD Lagrangian (10) and projecting on singlet and octet two-particles states.

### 3.1 Canonical quantization

The canonical variables and their conjugates are $(S, iS^\dagger)$, $(O_a, iO_a^\dagger)$, and $(A_i, \Pi^i_a = \partial L_{\text{NRQCD}}/\partial (\partial_0 A^a_i))$. The physical states $|\text{phys}\rangle$ are constrained to satisfy the Gauss law:

$$ (\mathbf{D} \cdot \Pi)^a|\text{phys}\rangle = \int d^3x\, \text{Tr} \left\{ \mathbf{O}^\dagger [g \Gamma^a, \mathbf{O}] \right\} |\text{phys}\rangle. $$

The canonical commutation relations are

$$ [\Pi^i_a(X, t), A^j_b(Y, t)] = i\delta_{ab}\delta_{ij}\delta^{(3)}(X - Y), $$

$$ [S(x, X, t), S^\dagger(y, Y, t)] = \delta^{(3)}(x - y)\delta^{(3)}(X - Y), $$

$$ [O_a(x, X, t), O^\dagger_b(y, Y, t)] = \delta_{ab}\delta^{(3)}(x - y)\delta^{(3)}(X - Y), $$

all the other commutators are zero. As in the case of NRQCD, in order to fulfil the Gauss-law constraint (37), we will assume the Poincaré algebra commutators to act on a space spanned by the physical states.

### 3.2 Poincaré algebra generators in pNRQCD

As in the case of NRQCD, since translational and rotational invariance are exact symmetries of the effective theory, the generators $H$, $\mathbf{P}$ and $\mathbf{J}$ can be derived from the symmetric energy-momentum tensor. The pNRQCD Lorentz-boost generators $\mathbf{K}$ can be derived by writing down the most general expression, with some obvious restrictions already included, containing all operators consistent with its symmetries and by matching it to the NRQCD Lorentz-boost generator (20).

In our case we obtain at order $x^2/m^0$, $x^0/m$, $(x/m)\mathbf{P}_X$ and $(x^0/m^2)\mathbf{P}_X$ (and dropping terms involving 4 matter fields that show up at $\mathcal{O}(x^2/m^0)$, but shall not affect our results):

$$ h \equiv \frac{\Pi^{a2} + B^{a2}}{2} + \int d^3x\, \text{Tr} \left\{ S^\dagger(2m + h_S)S + O^\dagger(2m + h_O)O 
+ \left[ (S^\dagger h_{SO}O + \text{H.C.}) + \text{C.C.} \right] + \left[ O^\dagger h_{OO}O + \text{C.C.} \right] + \left[ O^\dagger h^A h^B_{OO} + \text{C.C.} \right] \right\} |\mathbf{E} = \Pi\rangle, $$

$$ H = \int d^3X\, h. $$

The generators $\mathbf{P}$ and $\mathbf{J}$ are exactly known:

$$ P = \int d^3X \, \int d^3x\, \text{Tr} \left\{ S^\dagger \mathbf{P}_X S + O^\dagger \mathbf{P}_X O \right\} + \frac{1}{2} \int d^3X \, [\Pi^a \times, B^a], $$

$$ J = \int d^3X \, \int d^3x\, \text{Tr} \left\{ S^\dagger \left( \mathbf{X} \times \mathbf{P}_X + \mathbf{x} \times \mathbf{p}_x + \frac{\sigma^{(1)} + \sigma^{(2)}}{2} \right) S 
+ O^\dagger \left( \mathbf{X} \times \mathbf{P}_X + \mathbf{x} \times \mathbf{p}_x + \frac{\sigma^{(1)} + \sigma^{(2)}}{2} \right) O \right\} + \frac{1}{2} \int d^3X \, \mathbf{X} \times [\Pi^a \times, B^a]. $$
The Lorentz-boost generators at order \(x^2/m^0, x^0/m\) and \((x/m)P_X\) are given by:

\[
K = -tP + \int d^3X \frac{1}{2} \{X, \hbar\} + \int d^3X \int d^3x \text{Tr} \left\{ \left[S^i k_{SS} S + \text{C.C.} \right]
+ \left[S^i k_{SO} O + \text{H.C.} \right] + \text{C.C.} \right\},
\]

(44)

\[
k_{\phi\phi}^{(0,2)} = k_{\phi\phi}^{(0,2)} + k_{\phi\phi}^{(1,-1)} + k_{\phi\phi}^{(1,0)} + k_{\phi\phi}^{(1,1)} (P_X),
\]

(45)

\[
k_{\phi\phi}^{(0,2)} = -\frac{1}{8} k_{\phi\phi}^{(0,2)} (x) x^i x^j \sigma^{(1)}_{ij} \cdot P_x,
\]

(46)

\[
k_{\phi\phi}^{(1,0)} = \frac{1}{8} \left\{ k_{\phi\phi}^{(1,0)} (x), \left( \sigma^{(1)} \times P_x \right) \right\}
+ \frac{1}{8} \left\{ k_{\phi\phi}^{(1,0)} (x), P_x \right\}
+ \frac{1}{8} \left\{ k_{\phi\phi}^{(1,0)} (x), P_x P_x \right\}
+ \frac{1}{8} \left\{ k_{\phi\phi}^{(1,0)} (x), P_x P_x P_x \right\}
- \frac{1}{8} k_{\phi\phi}^{(1,0)} (x) \left( \sigma^{(1)} \times P_x \right)^i
- \frac{1}{8} k_{\phi\phi}^{(1,0)} (x) \left( \sigma^{(1)} \cdot P_x \right)
- \frac{1}{8} k_{\phi\phi}^{(1,0)} (x) \left( \sigma^{(1)} \cdot P_x \right)
+ \frac{1}{8} k_{\phi\phi}^{(1,0)} (x) \left( \sigma^{(1)} \cdot P_x \right),
\]

(47)

\[
k_{\phi\phi}^{(1,1)} (P_X) = 0,
\]

(48)

where \(k_{\phi\phi}^{(i,j)}\) are matching coefficients specific of \(K\). For \(k_{\phi\phi}^{(1,1)}\) only the \(P_X\)-dependent terms are displayed.

The terms of \(K\) proportional to \(k_{SS}^{(0,2)}, k_{SO}^{(1,0)}, \) and \(k_{SO}^{(1,0)}\) vanish under color trace. Due to charge conjugation, also the term proportional to \(k_{SO}^{(0,2)}\) vanishes. The matching coefficients \(k_{SOa}^{(0,2)}, k_{SOa}^{(1,-1)}, k_{SOa}^{(1,0)}\) and \(k_{SOa}^{(1,0)}\) are equal to 1 at tree level. All other matching coefficients, which are specific of \(K\), are zero at \(O(a_s^0)\). Also in the case of \(K\) the tree-level matching has been performed by multipole expanding the NRQCD Lorentz-boost generators (20) and projecting on singlet and octet two-particles states. We note that
loop corrections can, in principle, be calculated as they are for the matching coefficients of the pNRQCD Lagrangian.

3.3 Poincaré algebra constraints in pNRQCD

Let us now consider the constraints induced by the Poincaré algebra (1)-(9) on the pNRQCD generators \( H \) and \( K \).

As in the NRQCD case, the constraint \([P^i, K^j] = -i\delta_{ij}H\) has been already used in writing Eq. (44). Indeed, it forces \( K \) to have the form

\[
\int d^3X \{X, h(X, t)\}/2 + \text{translational-invariant terms that depend on } X \text{ only through the canonical variables.}
\]

From \([K^i, K^j] = -i\epsilon_{ijk}J^k\) at \(O(x^0/m^0)\) it follows that

\[
k^{(1,0)}_{SSa} - k^{(1,0)}_{SSa'} = k^{(1,0)}_{OOa} - k^{(1,0)}_{OOa'} = 1,
\]

\[
k^{(1,0)}_{SSc} = k^{(1,0)}_{OOc} = 1,
\]

\[
k^{(1,0)}_{SSd} = k^{(1,0)}_{OOd} = 0,
\]

\[
k^{(1,0)}_{SSd'} + k^{(1,0)}_{SSd''} = k^{(1,0)}_{OOd''} + k^{(1,0)}_{OOd'} = 0,
\]

\[
c^{(1,0)}_S = c^{(1,0)}_O = 1.
\]

The constraint (54) follows also from \([H, K^i] = -iP^i\) at \(O(x^0/m^0)\).

From \([K^i, K^j] = -i\epsilon_{ijk}J^k\) and \([H, K^i] = -iP^i\) at \(O(x/m^0)\) it follows that

\[
V^{(0,1)}_{SO} = V^{(1,1)}_{SO},
\]

\[
V^{(0,1)}_{OO} = V^{(1,1)}_{OO}.
\]

From \([H, K^i] = -iP^i\) at \(O(x^0/m)\), we obtain:

\[
V^{(1,0)}_{SO} = V^{(2,0)}_{SO},
\]

\[
V^{(1,0)}_{OO} = V^{(2,0)}_{OO},
\]

\[
V_{LS Sa} + \left(k^{(1,-1)}_{SSa} + k^{(1,-1)}_{SSb'}\right) \frac{V^{(0)}_S}{2x} = 0,
\]

\[
V_{LS Oa} + \left(k^{(1,-1)}_{OOa} + k^{(1,-1)}_{OOb'}\right) \frac{V^{(0)}_O}{2x} = 0,
\]

\[
V_{L^2 Sa} + \left(k^{(1,0)}_{SSa'} + k^{(1,0)}_{SSb'} + k^{(1,0)}_{SSb}\right) \frac{xV^{(0)}_S}{2} = 0,
\]

\[
V_{L^2 Oa} + \left(k^{(1,0)}_{OOa'} + k^{(1,0)}_{OOa''} + k^{(1,0)}_{OOb}\right) \frac{xV^{(0)}_O}{2} = 0,
\]

\[
V_{p^2 Sa} + V_{L^2 Sa} + \frac{V^{(0)}_S}{2} - k^{(1,0)}_{SSa''} \frac{xV^{(0)}_S}{2} = 0,
\]
\[ V_{p^2Oa} + V_{L^2Oa} + \frac{V^{(0)}}{2} - k_{Oa}^{(1,0)} \frac{xV^{(0)\prime}}{2} = 0, \]  
(64)

\[ 2c_F V_{SOb}^{(1,0)} - c_s V_{SOa}^{(2,0)} - k_{SSa}^{(1,1)} V_{SO}^{(0,1)} = 0, \]  
(65)

\[ 2c_F V_{OOa}^{(1,0)} - c_s V_{OOa}^{(2,0)} - k_{OOa}^{(1,0)} \frac{V_{OO}^{(0,1)}}{2} - k_{OOa}^{(1,0)} = 0, \]  
(66)

\[ 2V_{O\otimes Ob}^{(1,0)} - V_{O\otimes Oa}^{(2,0)} - \frac{k_{OOa}^{(1,1)} V_{OO}^{(0,1)}}{2} + k_{OOc}^{(1,0)} = 0, \]  
(67)

\[ k_{SSa}^{(1,1)} - k_{Oa}^{(1,1)} = 0, \]  
(68)

\[ 2V_{SOc}^{(1,0)} - V_{SOb}^{(2,0)} - k_{SSb}^{(1,1)} V_{SO}^{(0,1)} = 0, \]  
(69)

\[ 2V_{OOc}^{(1,0)} - V_{OOb}^{(2,0)} - \frac{k_{OOb}^{(1,1)} V_{OO}^{(0,1)}}{2} - \frac{k_{OOc}^{(1,0)}}{2} = 0, \]  
(70)

\[ 2V_{O\otimes Oa}^{(1,0)} - V_{O\otimes Ob}^{(2,0)} - \frac{k_{OOa}^{(1,1)} V_{OO}^{(0,1)}}{2} + \frac{k_{OOc}^{(1,0)}}{2} = 0, \]  
(71)

\[ k_{SSb}^{(1,1)} - k_{OOa}^{(1,1)} = 0, \]  
(72)

\[ V_{SOb}^{(2,0)} + k_{SSa}^{(1,1)} x V_{SO}^{(0,1)\prime} + k_{SSb}^{(1,1)} x V_{SO}^{(0,1)\prime} = 0, \]  
(73)

\[ V_{OOb}^{(2,0)} + \frac{k_{OOa}^{(1,1)} x V_{OO}^{(0,1)\prime}}{2} + \frac{k_{OOb}^{(1,1)} x V_{OO}^{(0,1)\prime}}{2} + \frac{k_{OOc}^{(1,0)}}{2} = 0, \]  
(74)

\[ V_{O\otimes Ob}^{(2,0)} + \frac{k_{OOa}^{(1,1)} x V_{OO}^{(0,1)\prime}}{2} + \frac{k_{OOb}^{(1,1)} x V_{OO}^{(0,1)\prime}}{2} - \frac{k_{OOc}^{(1,0)}}{2} = 0, \]  
(75)

\[ k_{SSb}^{(1,1)} - k_{OOa}^{(1,1)} = 0, \]  
(76)

\[ V_{SOa}^{(1,0)} + k_{SSa}^{(1,1)} V_{SO}^{(0,1)} = 0, \]  
(77)

\[ V_{OOa}^{(2,0)} + \frac{k_{OOa}^{(1,1)} V_{OO}^{(0,1)}}{2} + \frac{k_{OOc}^{(1,0)}}{2} = 0, \]  
(78)

\[ V_{O\otimes Ob}^{(2,0)} + \frac{k_{OOa}^{(1,1)} V_{OO}^{(0,1)}}{2} - \frac{k_{OOc}^{(1,0)}}{2} = 0, \]  
(79)

\[ k_{SSb}^{(1,1)} - k_{OOa}^{(1,1)} = 0, \]  
(80)

\[ V_{OOa}^{(1,0)} - c_O^{(1,2)} k_{OOa}^{(0,2)} + 2 k_{OOa}^{(1,0)} + V_{OOc}^{(2,0)} = 0, \]  
(81)

\[ V_{OOa}^{(1,0)} + c_O^{(1,2)} k_{OOa}^{(0,2)} - 2 k_{OOa}^{(1,0)} - V_{OOc}^{(2,0)} = 0, \]  
(82)

\[ c_O^{(1,2)} \left( x^2 k_{OOa}^{(0,2)\prime} \right) - 2 k_{OOa}^{(1,0)} - V_{OOc}^{(2,0)} = 0, \]  
(83)

\[ c_O^{(1,2)} x k_{OOa}^{(0,2)\prime} - 2 k_{OOa}^{(1,0)} - V_{OOd}^{(2,0)} = 0, \]  
(84)

where \( f' \equiv df/dx \). More precisely, as in the case of NRQCD discussed in Sec. 2.3, what we obtain are the above combinations of matching coefficients multiplying some
pNRQCD operators. For simplicity, here we have not displayed the expressions involving the operators. However, they may turn out to be important in considering the QED case of the above equalities. So, for instance, the expressions appearing in Eqs. (66) and (67) contribute to the same operator in the QED case. Therefore, the two equalities reduce to their sum, which is given by Eq. (65). Same considerations apply to the couples of equations (70)-(71), (74)-(75) and (78)-(79). Moreover, the combinations of matching coefficients appearing in Eqs. (81)-(84) multiply operators that vanish in the QED case. Therefore, these constraints have no QED analogue. For what concerns the equalities (55)-(64) in the QED case, they become pairwise identical.

3.4 Unitary transformations

The generators of the Poincaré algebra are defined up to unitary transformations. We can use this freedom in order to change or reduce our basis of operators. (For a similar use of unitary transformations in a slightly different context we refer to [19].) In particular, we can look for a basis where some of the matching coefficients of $K$ are fixed. Consider the unitary transformation:

$$U = \exp \left( i \int d^3x \int d^3x \, \text{Tr} \left\{ \left[ S^\dagger \left( \frac{P_X \cdot (k^{(1,-1)}_{SS} + k^{(1,0)}_{SS})}{2m} \right) S + \text{C.C.} \right] + \left[ O^\dagger \left( \frac{P_X \cdot (k^{(1,0)}_{OO} + k^{(1,0)}_{OO})}{2m} \right) O + \text{C.C.} \right] \right\} \right),$$

(85)

where the operators $\tilde{k}_{\varphi_\phi}^{(i,j)}$ are equal to the operators $k_{\varphi_\phi}^{(i,j)}$ but with arbitrary coefficients $\tilde{k}_{\varphi_\phi}^{(i,j)}$. The transformed Lorentz-boost generators $U K U^\dagger$ have the same structure as the original ones up to order $x^2/m^0$, $x^0/m$ and $(x/m) P_X$, but with the matching coefficients shifted as follows:

$$
\begin{align*}
  k_{\varphi_\phi a}^{(1,-1)} &\rightarrow k_{\varphi_\phi a}^{(1,-1)} + \tilde{k}_{\varphi_\phi a}^{(1,-1)}, \\
  k_{\varphi_\phi b}^{(1,-1)} &\rightarrow k_{\varphi_\phi b}^{(1,-1)} + \tilde{k}_{\varphi_\phi b}^{(1,-1)}, \\
  k_{\varphi_\phi b'}^{(1,-1)} &\rightarrow k_{\varphi_\phi b'}^{(1,-1)} + \tilde{k}_{\varphi_\phi b'}^{(1,-1)}, \\
  k_{\varphi_\phi b''}^{(1,-1)} &\rightarrow k_{\varphi_\phi b''}^{(1,-1)} + \tilde{k}_{\varphi_\phi b''}^{(1,-1)}, \\
  k_{\varphi_\phi a}^{(1,0)} &\rightarrow k_{\varphi_\phi a}^{(1,0)} + \tilde{k}_{\varphi_\phi a}^{(1,0)} + \tilde{k}_{\varphi_\phi a''}, \\
  k_{\varphi_\phi a''}^{(1,0)} &\rightarrow k_{\varphi_\phi a''}^{(1,0)} + \tilde{k}_{\varphi_\phi a''}^{(1,0)} + \tilde{k}_{\varphi_\phi a}, \\
  k_{\varphi_\phi a''''}^{(1,0)} &\rightarrow k_{\varphi_\phi a''''}^{(1,0)} + 2 \tilde{k}_{\varphi_\phi a''''}, \\
  k_{\varphi_\phi b}^{(1,0)} &\rightarrow k_{\varphi_\phi b}^{(1,0)} + 2 \tilde{k}_{\varphi_\phi b}, \\
  k_{\varphi_\phi b'}^{(1,0)} &\rightarrow k_{\varphi_\phi b'}^{(1,0)} + \tilde{k}_{\varphi_\phi b'}^{(1,0)} - \tilde{k}_{\varphi_\phi b''}, \\
  k_{\varphi_\phi b''}^{(1,0)} &\rightarrow k_{\varphi_\phi b''}^{(1,0)} - \tilde{k}_{\varphi_\phi b''}^{(1,0)} + \tilde{k}_{\varphi_\phi b''}, \\
  k_{\varphi_\phi b'''}^{(1,0)} &\rightarrow k_{\varphi_\phi b'''}^{(1,0)} - \tilde{k}_{\varphi_\phi b'''}^{(1,0)} + \tilde{k}_{\varphi_\phi b''''}, \\
  k_{\varphi_\phi b''''}^{(1,0)} &\rightarrow k_{\varphi_\phi b''''}^{(1,0)} - \tilde{k}_{\varphi_\phi b''''}^{(1,0)} + \tilde{k}_{\varphi_\phi b'''}.
\end{align*}
$$

(86)-(95)
We have, now, the freedom to choose the coefficients $\tilde{k}^{(i,j)}$. A convenient choice is that one that fixes the above matching coefficients to their tree level values:

$$
k_{\varphi\phi}^{(1,-1)}, k_{\varphi\phi}^{(1,0)} \to 1, \quad k_{\varphi\phi}^{(1,-1)}, k_{\varphi\phi}^{(1,1)}, k_{\varphi\phi}^{(1,0)}, k_{\varphi\phi}^{(1,0)}, k_{\varphi\phi}^{(1,0)}, k_{\varphi\phi}^{(1,0)} \to 0. \quad (96)
$$

The unitary transformation leaves invariant the generators $P$ and $J$, while the changes in $U H U^\dagger$ may be reabsorbed in a redefinition of the matching coefficients.

Finally, we note that other unitary transformations are possible and some of them may, in principle, further reduce the number of operators in $K$ or $H$. We will not examine this issue here, which, however, deserves further studies.

### 3.5 Discussion

We comment now on the obtained relations. Equation (54) fixes the centre-of-mass kinetic energy to be equal to $P^2 X/4m$. It may come as a surprise that, at least at the order at which we are working here, Poincaré invariance does not fix the coefficient of the kinetic energy $p^2 x/m$ of the quarks in the centre-of-mass frame. However, one should consider that the coordinate $x$ is no more a dynamical variable of the theory. Nevertheless, we may argue that, because no other momentum-dependent operator than the kinetic energy of NRQCD, $-\bar{\psi} \nabla^2/(2m) \psi + \bar{\chi} \nabla^2/(2m) \chi$, may contribute to the kinetic energy of pNRQCD, the coefficients $c_S^{(1,0)}$ and $c_S^{(1,-2)}$ have to be equal. It follows then that also $c_S^{(1,-2)} = 1$ (analogously for $c_O^{(1,-2)}$).

From Eqs. (59) and (60) we obtain:

$$
\frac{V_{LS} S_a}{V_S^{(0)y}} = \frac{V_{LS} O_a}{V_O^{(0)y}} = -\frac{1}{2x}, \quad (97)
$$

where the last equality holds in the basis of operators discussed in Sec. 3.4. Eq. (97) in the singlet sector is the relation between the spin-orbit potentials and the static potential first derived in [10] by boosting the potentials expressed in terms of Wilson loops. It was also obtained in [9]. In all these previous cases the Lorentz-boost generators were written in the basis of Eq. (96). The extension to the octet sector is new.

From Eqs. (61)-(64) we obtain:

$$
V_{L^2} S_a + \frac{x V_S^{(0)y}}{2} = V_{L^2} O_a + \frac{x V_O^{(0)y}}{2} = 0, \quad (98)
$$

$$
V_{p^2} S_a + V_{L^2} S_a + \frac{V_S^{(0)}}{2} = V_{p^2} O_a + V_{L^2} S_a + \frac{V_O^{(0)}}{2} = 0. \quad (99)
$$

These equations hold in the basis of Sec. 3.4. In the singlet sector they are the relations between the momentum-dependent potentials first derived in [11] by boosting the potentials expressed in terms of Wilson loops. They were also obtained in [9]. The extension to the octet sector is new.

Equations (55)-(58) constrain the fields to enter in the singlet-octet and octet-octet sectors of the Lagrangian just in the combination

$$
x \cdot \left( gE + \frac{1}{2} \left\{ \frac{P X}{2m} \times, gB \right\} \right), \quad (100)
$$
i.e. like in the Lorentz force. The coefficients \( V_{SO}^{(0,1)} \) and \( V_{OO}^{(0,1)} \) were called \( V_A \) and \( V_B \) respectively in the previous literature [20, 6]. These are the coefficients associated to the next-to-leading order terms of the multipole expansion of the static pNRQCD Lagrangian. They play an important role in the running of the static potentials [20, 21] and in several observables [18]. Eqs. (65)-(80) mix them with the matching coefficients of operators appearing at order \( 1/m \) and \( 1/m^2 \). Eqs. (73)-(75) involve also the derivatives of \( V_{SO}^{(0,1)} \) and \( V_{OO}^{(0,1)} \). Eqs. (65) and (66) contain combinations of matching coefficients inherited from NRQCD. Somehow these relations reflect at the level of the pNRQCD potentials the relation (23) among the NRQCD matching coefficients.

The four sets of Eqs. (65)-(68), (69)-(72), (73)-(76) and (77)-(80) can be combined in order to give the following four equations that involve only potentials appearing in the Lagrangian:

\[
\begin{align*}
2c_FV_{SO}^{(1,0)} - c_sV_{SO}^{(2,0)} &= \frac{2}{V_{SO}^{(0,1)}} \left( c_FV_{OOb}^{(1,0)} + V_{O\otimes Ob}^{(1,0)} \right) - \left( c_sV_{OOa}^{(2,0)} + V_{O\otimes Oa}^{(2,0)} \right) = 1, \\
2V_{SOc}^{(1,0)} - V_{SOv}^{(2,0)} &= \frac{2}{V_{SO}^{(0,1)}} \left( V_{OOc}^{(1,0)} + V_{O\otimes Oc}^{(1,0)} \right) - \left( V_{OOb}^{(2,0)} + V_{O\otimes Ob}^{(2,0)} \right) = 0, \\
-V_{SOv'}^{(2,0)} + 2c_FV_{SOb}^{(1,0)} - c_sV_{SOa}^{(2,0)} &= \left( xV_{SO}^{(0,1)} \right)' - V_{OOv'}^{(2,0)} - V_{O\otimes Ov'}^{(2,0)} + 2 \left( c_FV_{OOb}^{(1,0)} + V_{O\otimes Ob}^{(1,0)} \right) - \left( c_sV_{OOa}^{(2,0)} + V_{O\otimes Oa}^{(2,0)} \right) = 1, \\
\frac{V_{SOv''}^{(2,0)}}{V_{SO}^{(0,1)}} &= \frac{V_{OOv''}^{(2,0)} + V_{O\otimes Ov''}^{(2,0)}}{V_{OO}^{(0,1)}} = 0,
\end{align*}
\]

where the last equalities hold in the basis of Sec. (3.4).

Eqs. (81)-(84) are typical for the non-Abelian structure of QCD and have been commented above. Here, we add that, summing Eq. (81) and (82), we obtain:

\[
V_{OOa}^{(1,0)} = 1 + \frac{V_{OOc}^{(2,0)} - V_{OOc'}^{(2,0)}}{2}.
\]

## 4 Conclusions

In this work we have shown how the implementation of the Poincaré algebra provides a feasible and useful tool to constrain the dynamics of non-relativistic effective field theories of QCD. The method was used in the past, at a quantum-mechanical level, to constrain the relativistic dynamics of some phenomenological potentials. This is its first use in an effective field theory context.

In Sec. 2 we have applied the method to the NRQCD/HQET Lagrangian up to order \( 1/m \) and \( (\nabla^2 \nabla^i)/m^2 \) in the commutators. In a simple way we have obtained some of the relations, Eqs. (21)-(24), derived formerly from reparametrization invariance. This shows that reparametrization invariance is, indeed, one way in which the Poincaré
invariance of QCD manifests itself in the HQET (for a similar observation in a different context see [22]). An obvious extension of this work would be to study the constraints induced by Poincaré invariance at higher order in $1/m$. This study would be of interest, because at some point the four-fermion operators of NRQCD will start to play a role and new relations, specific of NRQCD, will show up. To our knowledge, relativistic invariance of NRQCD, in the sector where it does not reduce to the HQET, has never been explored. We believe that the framework used here is suitable also for this exploration.

In Sec. 3 we have calculated the constraints induced in pNRQCD by the Poincaré invariance of QCD up to order $x/m^0$ and $x^0/m$. We have obtained a set of new constraints, listed in Eqs. (50)-(84). These constraints involve, in general, mixings of matching coefficients appearing in the pNRQCD Lagrangian and in the Lorentz-boost generators. There are two kinds of information that we can extract from them. First we can combine them in order to obtain relations that involve only the potentials appearing in the pNRQCD Lagrangian. These equations are (54)-(58), (97)-(99) and (101)-(105). More relations of this kind are expected in going to higher orders. In some specific bases of operators the relations become particularly simple. In the present letter we have presented a unitary transformation that fixes the matching coefficients specific of $K$ at their tree-level value. An exhaustive analysis was beyond the purposes of the present letter but surely deserves further investigations. A second information, provided by the constraints, is the form of the Lorentz-boost generators expressed in terms of the potentials appearing in the Lagrangian of pNRQCD. Once $K$ is known, the singlet and octet quark-antiquark fields transform under infinitesimal Lorentz boosts with velocity $v$ in accordance to $\delta S = i[S, v \cdot K]$ and $\delta O^a = i[O^a, v \cdot K]$.

Finally, we would like to mention that the present approach is quite general. Therefore, it may be also suited to derive exact relations among the matching coefficients of other effective field theories of QCD, where the manifest covariance under boosts has been destroyed by an expansion in some small momenta. An example is the soft-collinear effective theory [23], where also reparametrization invariance has been discussed [24].

Acknowledgements. The authors thank Heinz Rothe and Klaus Rothe for valuable discussions on the quantization procedure and Joan Soto for reading the manuscript and for useful comments and suggestions. A.V. was supported during this work by the European Community through the Marie-Curie fellowship HPMF-CT-2000-00733. N.B. gratefully acknowledge the support of the Alexander von Humboldt Foundation.
References

[1] P.A.M. Dirac, Rev. of Mod. Phys. 21, 392 (1949).

[2] L.L. Foldy, Phys. Rev. 122, 275 (1961); R.A. Krajck and L.L. Foldy, Phys. Rev. D10, 1777 (1974); K.J. Sebastian and D. Yun, Phys. Rev. D19, 2509 (1979).

[3] N. Isgur and M.B. Wise, Phys. Lett. B232, 113 (1989); B237, 527 (1990).

[4] W.E. Caswell and G.P. Lepage, Phys. Lett. B167, 437 (1986); G.T. Bodwin, E. Braaten and G.P. Lepage, Phys. Rev. D51, 1125 (1995); Erratum, ibid. D55, 5853 (1997).

[5] A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. 64, 428 (1998).

[6] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B566, 275 (2000).

[7] M. Luke and A.V. Manohar, Phys. Lett. B286, 348 (1992).

[8] A.V. Manohar, Phys. Rev. D56, 230 (1997).

[9] N. Brambilla, D. Gromes and A. Vairo, Phys. Rev. D64, 076010 (2001).

[10] D. Gromes, Z. Phys. C26, 401 (1984).

[11] A. Barchielli, N. Brambilla and G. Prosperi, Nuovo Cimento 103A, 59 (1990).

[12] Y. Chen, Y. Kuang and R.J. Oakes, Phys. Rev. D52, 264 (1995); N. Brambilla and A. Vairo, Nucl. Phys. Proc. Suppl. 74, 201 (1999).

[13] C. Bauer and A.V. Manohar, Phys. Rev. D57, 337 (1998).

[14] A. Pineda and A. Vairo, Phys. Rev. D63, 054007 (2001).

[15] A. Pineda and J. Soto, Phys. Rev. D58, 114011 (1998).

[16] A.A. Slavnov and L.D. Faddeev, Gauge Fields: Introduction to quantum theory (Benjamin-Cummings, Menlo Park, 1980); B. Hatfield, Quantum Field Theory of Point Particles and Strings (Addison-Wesley, Boston, 1992); R. Jackiw, in Current Algebra and Anomalies, eds. S.B. Treiman, R. Jackiw, B. Zumino and E. Witten (World Scientific, Singapore, 1985).

[17] S. Weinberg, The quantum theory of fields I (Cambridge University Press, Cambridge, 1995); F. Belinfante, Physica 6, 887 (1939).

[18] N. Brambilla, D. Eiras, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D67, 034018 (2003).

[19] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D63, 014023 (2001).

[20] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D60, 091502 (1999).
[21] A. Pineda and J. Soto, Phys. Lett. B495, 323 (2000).

[22] D. Eiras and J. Soto, Phys. Rev. D61, 114027 (2000).

[23] C.W. Bauer, S. Fleming, D. Pirjol and I.W. Stewart, Phys. Rev. D63, 114020 (2001); M. Beneke, A.P. Chapovsky, M. Diehl and T. Feldmann, Nucl. Phys. B643, 431 (2002).

[24] A.V. Manohar, T. Mehen, D. Pirjol and I.W. Stewart, Phys. Lett. B539, 59 (2002).