New $I = J$ Rules for the Baryon Vertices in $1/N_c$ Expansion

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Abstract

We apply the $1/N_c$ expansion in QCD to the baryon vertices. We find new model independent properties for the isoscalar and isovector baryon vertices from the view point of $1/N_c$ expansion in QCD. One of these results, $I = J$ rule, have been already found. The other properties, new $I = J$ rules, are the rules about the isospin and strangeness dependence for the baryon vertices.

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1 Introduction

The $1/N_c$ expansion was proposed as a non-perturbative approach to QCD by 't Hooft in 1974[1]. He has shown that the Feynman diagrams which are relevant at the leading order of the $1/N_c$ expansion are the planar diagrams without internal quark loops. Based on this $1/N_c$ expansion Witten suggested that in the large $N_c$ limit a baryon looks like a soliton[2]. From this viewpoint the Skyrme’s conjecture that baryons are solitons of the nonlinear chiral Lagrangian for the chiral fields has been revived in early 1980’s and has succeeded in describing the baryon sector from the meson sector at least semi-quantitatively[3].

The $1/N_c$ expansion method has been considered to be a qualitative method to study QCD. However recent extensive studies of the consistency conditions [4][5] show that the $1/N_c$ expansion method is useful for obtaining also quantitative results of QCD from the model independent viewpoint.

In the previous papers[6] we have calculated the $F/D$ ratios of flavor $SU(3)$ symmetry for both spin-flip and spin-nonflip baryon vertices in the non-relativistic quark model(NRQM) and the chiral soliton model(CSM) for arbitrary color degrees of freedom $N_c$. The values of $F/D$ ratios for the spin-flip and spin-nonflip baryon vertices tends to $1/3$ and $-1$, respectively in the large $N_c$ limit. The physical meaning of these values turned out to be nothing but the $I=J$ rule indicated by Mattis and collaborators [7], namely, the isovector dominance for the spin-flip baryon vertex and the isoscalar dominance for the spin-nonflip baryon vertex.

In this paper we find new and model independent properties for the isoscalar and isovector baryon vertices from the viewpoint of $1/N_c$ expansion in QCD. There new properties are the rules about isospin and strangeness dependence for the baryon vertices.

In section 2, we define the baryon states for arbitrary $N_c$ and explain the $F/D$ ratios for the baryon vertices. In section 3, we will derive the isoscalar and isovector formulas for the baryon vertices with $N_c$. In section 4, we will find new $I=J$ rules by using the $F/D$ ratios derived from the non-relativistic quark model and the chiral soliton model.
2 The $N_c$ Dependence of Baryon States and $F/D$ Ratios

In this section we investigate the $N_c$ dependence for the baryon vertices from the flavor information in the $SU(N_c)$ QCD. From now on we assume the flavor $SU(3)$ symmetry in order to make arguments easy.

In order to study the properties of baryons in the $SU(N_c)$ symmetric QCD with arbitrary $N_c$ we have to introduce the extended baryon state which is totally antisymmetric color singlet state of the $SU(N_c)$ symmetry. There are some ambiguities in extending the baryons for large $N_c$ and we have to introduce some unphysical members of the $SU(N_f)$ multiplet. Therefore we need to fix the “physical states” of baryon in the $SU(N_c)$ QCD. We will show that the following special choice of extension for the large $N_c$ baryon is appropriate to obtain the correct large $N_c$ behavior of various baryon matrix elements.

The ground state spin 1/2 baryons for arbitrary $N_c$ belong to $(k+1)(k+3)$ dimensional representation of flavor $SU(3)$ symmetry because of total symmetry in the spin-flavor states where $k = (N_c - 1)/2$ ($k = 0, 1, 2, \cdots$). This representation is specified by the Young diagram with the first row of length $k + 1$ and the second row of length $k$ and the root diagram. The physical octet baryons are located at the top region of this root diagram. The flavor wave functions are represented by the tensors with one superscript and $k$ subscripts and the physical octet baryons are given by [5].

The hypercharge extended to arbitrary $N_c$ is given by

$$Y = \frac{N_cB}{3} + S,$$

where $B$ is the baryon number and $S$ is the strangeness which reduces to $Y = B + S$ in the physical case of $N_c = 3$.

The most general flavor octet vertex is represented as a sum of two independent terms as follows:

$$< B_f | O^a | B_i > = F \, \text{tr}(\lambda^a [\bar{B}_f, B_i]) + D \, \text{tr}(\lambda^a \{ \bar{B}_f, B_i \}),$$

where $\lambda^a$ is a flavor octet matrix with $a = 1, \ldots, 8$.

For the spin-nonflip vertex $< B_f | O^a | B_i >$, this $F/D$ ratio will be denoted by $F_+/D_+$. For the spin-flip vertex $< B_f | O^{ai} | B_i >$ we have a similar expression and the $F/D$ ratio is denoted by $F_-/D_-$. 

3
The $F/D$ ratios express the spin structure of some specific model, for instance NRQM and CSM are useful to observe the large $N_c$-dependence of QCD, independently of details of the specific effective model of QCD.

In the NRQM and CSM we calculate the $F/D$ ratios 

\[
\left( \frac{F_+}{D_+} \right)_{SU(6)}^{NRQM} = -\frac{N_c + 1}{N_c - 3} = -1 - \frac{4}{N_c} + \frac{12}{N_c^2} + \cdots, \tag{3}
\]

\[
\left( \frac{F_+}{D_+} \right)_{SU(3)}^{CSM} = -\frac{N_c^2 + 4N_c - 1}{N_c^2 + 4N_c - 9} = -1 + \frac{0}{N_c} - \frac{8}{N_c^2} + \cdots, \tag{4}
\]

Similarly,

\[
\left( \frac{F_-}{D_-} \right)_{SU(6)}^{NRQM} = \frac{N_c + 5}{3(N_c + 1)} = \frac{1}{3} + \frac{4}{3N_c} - \frac{4}{3N_c^2} + \cdots, \tag{5}
\]

\[
\left( \frac{F_-}{D_-} \right)_{SU(3)}^{CSM} = \frac{N_c^2 + 8N_c + 27}{3(N_c^2 + 8N_c + 3)} = \frac{1}{3} + \frac{0}{N_c} + \frac{8}{N_c^2} + \cdots, \tag{6}
\]

These calculation indicate that structure of baryon vertices become the same for both the NRQM and CSM exist in the large $N_c$ limit. Furthermore in the CSM the $1/N_c$ correction does not exist.

3 The Model Independent Analysis for the Baryon Vertices in $1/N_c$ Expansion

We calculate isoscalar and isovector baryon vertices in order to derive the $I = J$ rules from the baryon wave function which we have considered in the previous section.

According to the Wigner-Eckart theorem, the matrix element of the diagonal operator $H_8$ can generally be expressed as follows

\[
\langle B_f | H_8 | B_i \rangle = -F + \frac{1}{3} D + \frac{2}{k} (F - D) K + \frac{1}{k + 2} \left( F + D - \frac{1}{k} (F - D) \right) \left( I(I + 1) - (K + \frac{1}{2})(K + \frac{3}{2}) \right), \tag{7}
\]

where $K$ is related to hypercharge or strangeness by $Y = N_c/3B + S = N_c/3B - 2K$.

We find two different structures from this formula. One is operator structure and the other is a special pattern of coefficients.
The operator structure consists of three parts. The first line of eq.(7) gives the same contribution to all states \(-F + 1/3D\) and the increasing with \(K\) contribution. The second line of eq.(7) appears only for the states located on the inner triangle of the root diagram because of the factor \(I(I+1) - (K+1/2)(K+3/2)\).

Another structure of this formula is that the combination of \(F\) and \(D\) appear in only three patterns, \(-F + 1/3D\), \(F - D\), \(F + D\). The coefficients \(-F + 1/3D\), \(F + D\) are related to the limiting value \(F_+/D_+ = -1\) and \(F_-/D_- = 1/3\), respectively.

This formula does not depend on the extrapolation while the \(N_c\) counting of the operators \(K\) and \(I(I+1) - (K+1/2)(K+3/2)\) depend on the extrapolation. From the previous argument it is natural that isospin and strangeness are \(O(N_c^0)\).

If we identify the “physical states” to the states which have the same spin and isospin, the Okubo-Gell-Mann mass relation

\[
3\Lambda + \Sigma = 2(N + \Xi),
\]

holds for the arbitrary number \(N_c\).

Next we consider isovector diagonal matrix elements. After the tedious calculation the diagonal matrix element of operator \(H_3\) can be expressed by using the Wigner 6j symbol as follows.

\[
\begin{align*}
< I, I_3, K | H_3 | I, I_3, K > & = (-1)^{3/2-I-K} \sqrt{2\dim I} \left\{ \begin{array}{ccc} 1 & I & I \\ K & 1/2 & 1/2 \end{array} \right\} \left( \begin{array}{c} I \\ I_3 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left[ F + D + \frac{2}{k}(F - D)K \right. \\
& + \frac{3}{k + 2} \left( -F + \frac{1}{3}D - \frac{1}{k}(F - D) \right) \left( I(I+1) - (K+1/2)(K+3/2) \right) \left. \right]\end{align*}
\] (9)

This isovector formula has the same operator structure are the isoscalar formula \([7]\). The common contribution is replaced by \(F + D\) instead of \(-F + 1/3D\) in the isospin nonflip vertex. The Clebsh-Goldan coefficient express as the isospin conservation.

The off-diagonal matrix element of \(H_3\) is given by

\[
\begin{align*}
< I - 1, I_3, K | H_3 | I, I_3, K > & = < I, I_3, K | H_3 | I - 1, I_3, K > \\
& = (-1)^{5/2-I-K} \sqrt{2\dim I} \left\{ \begin{array}{ccc} 1 & I & I - 1 \\ K & 1/2 & 1/2 \end{array} \right\} \left( \begin{array}{c} I \\ I_3 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left[ I - 1 \right) \\
& \times \sqrt{\frac{k - 2K + 1}{k + 2}} \left( F + D - \frac{1}{k}(F - D) \right) \end{align*}
\] (10)
This result is consistent with the result given by Dashen, Jenkins and Manohar\cite{5}

\[
< I', I'_3, J', J'_3; K | A^{1a} | I, I_3, J, J_3; K > = N_c g(K) (-1)^{2j'+j-K} \sqrt{\text{dim}I \text{dim}J} \times \left\{ \begin{array}{ccc}
1 & I & I' \\
K & J & J'
\end{array} \right\} \left( \begin{array}{cc}
I & 1 \\
I_3 & a
\end{array} \right) \left( \begin{array}{cc}
J & 1 \\
J_3 & i
\end{array} \right) \right) \quad (11)
\]

The Coleman-Glashow mass relation

\[
\Sigma^+ - \Sigma^- = p - n + \Xi^0 - \Xi^- \quad (12)
\]

is obtained by assuming the same extrapolation with isoscalar part only for \( N_c = 3 \) contrary to the case of Okubo-Gell-Mann mass relation.

### 4 New \( I = J \) Rules for the Baryon Vertices in \( 1/N_c \) Expansion

In the previous section we have constructed the general isoscalar and isovector baryon vertices in terms of the Wigner-Eckart theorem. However we were not able to decide the \( N_c \) dependence only by use of information on flavor. In order to decide the \( N_c \) dependence completely, we need to know the values of \( F/D \) ratios. The \( F/D \) ratios is related to the spin dependence of the system.

If the spin-nonflip part of \( H_8 \) transforms as \((SU(3)_f, SU(2)_s) = (8, 1)\), then \( F_+/D_+ = -1 + O(1/N_c) \). With this \( F_+/D_+ \) ratio for the mass formula for the baryons, the mass difference of baryons is given by

\[
H(I = J = 0) = N_c a + bK + c \frac{1}{N_c} \left( I(I + 1) - (K + \frac{1}{2})(K + \frac{3}{2}) \right), \quad (13)
\]

where \( a, b \) and \( c \) are \( O(N_c^0) \).

The isovector formula eq.\((9)\) is

\[
H^a(I = 1, J = 0) = \left[ d' + e' K + f' \left( I(I + 1) - (K + \frac{1}{2})(K + \frac{3}{2}) \right) \right] I^a, \quad (14)
\]

We find that the terms which depend on the isospin and the strangeness are suppressed to the order \( O(1/N_c) \). A similar formula is also derived in Ref.\cite{5}

On the other hand if the spin-flip part of \( H_8 \) transforms as \((SU(3)_f, SU(2)_s) = (8, 3)\), then \( F_-/D_- = 1/3 + O(1/N_c) \). We can apply this formula to the isoscalar part
of the magnetic moment. Then we obtain

\[ H^i(I = 0, J = 1) = \left[a' + b'K + c' \left( I(I + 1) - (K + \frac{1}{2})(K + \frac{3}{2}) \right) \right] J^i \quad (15) \]

where \( a', b' \) and \( c' \) are \( O(N^0_c) \).

The isovector formula which we can apply to the isovector part of the magnetic moment is

\[ H^{ia}(I = J = 1) = \left[N_c d + eK + \frac{f}{N_c} \left( I(I + 1) - (K + \frac{1}{2})(K + \frac{3}{2}) \right) \right] X_0^{ia} \quad (16) \]

From these results we find that there are contributions with two different properties.

One is the old \( I = J \) rule found by Mattis and collaborators \([7]\) and the other is the new \( I = J \) rule. The old \( I = J \) rule is concerned with the leading order in \( 1/N_c \) expansion and new \( I = J \) rule is related to the \( N_c \) dependent part of isospin and strangeness.

If spin and isospin of the octet operator are equal, then its baryon vertices are \( O(N_c) \) from (13) and (14). But if spin and isospin are different, then these are terms of order \( O(N^0_c) \) from (14) and (15). This property is the old \( I = J \) rule.

On the other hand we recognize new properties from these results. It’s properties are the isospin and strangeness dependence of baryon vertices. The isospin and strangeness are suppressed by the \( 1/N_c \) expansion if we use (13) or (14), but in (14) and (15) the isospin and strangeness dependence survives even if we take the limit \( N_c \to \infty \). These properties belong to the new \( I = J \) rules.

The isospin and strangeness dependence are not necessarily to be suppressed at the lowest order in \( 1/N_c \) expansion \([\ref{13}] [\ref{14}] [\ref{10}]\).

### 5 Summary

We have found the new \( I = J \) rules for the baryon vertices in \( 1/N_c \) expansion. The assumption which we have used to derive the new \( I = J \) rules are three, first is quark confinement, second is \( SU(3) \) flavor symmetry and the last is that spin and isospin of baryon states in the \( SU(N_c) \) QCD are \( O(N^0_c) \). We guess that the second assumption is not essential from the consideration of consistency condition approach \([\ref{13}]\).

In our discussion there exists some unsatisfactory point. One problem is that we have used the special models in order to calculate the \( F/D \) ratios. By virtue of this
argument we cannot understand $I = J$ rules as easy as the OZI rule. The second problem is that we assume $SU(3)$ flavor symmetry. We do not need this assumption to derive the $I = J$ rules.

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