Dyadic $A_1$ Weights and Equimeasurable Rearrangements of Functions

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Received: 12 February 2014 / Published online: 3 February 2015
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Abstract We prove that the non-increasing rearrangement of a dyadic $A_1$ weight $w$ with dyadic $A_1$ constant $[w]_1^T = c$ with respect to a tree $T$ of homogeneity $k$, on a non-atomic probability space, is a usual $A_1$ weight on $(0, 1]$ with $A_1$-constant $[w^*]_1$ not more than $kc - k + 1$. We prove also that the result is sharp, when one considers all such weights $w$.

Keywords Dyadic · Rearrangement · Weight

Mathematics Subject Classification 42B25

1 Introduction

The theory of Muckenhoupt weights has been proved to be an important tool in analysis. A special characteristic of these weights is their self-improving properties (see [2,3,9]). One class of special interest is $A_1(J, c)$ where $J$ is an interval on $\mathbb{R}$ and $c$ is a constant such that $c \geq 1$, which is defined as the class of all non-negative locally integrable functions $w$, defined on $J$, such that for every subinterval $I \subseteq J$ we have that

$$\frac{1}{|I|} \int_I w(y) dy \leq c \inf_{x \in I} w(x),$$

where $| \cdot |$ is the Lebesgue measure on $\mathbb{R}$.
In [1] it is proved that if \( w \in A_1(J, c) \), then \( w^* \in A_1((0, |J|], c) \), where \( w^* \) is the non-increasing rearrangement of \( w \). That is, for every \( w \in A_1(J, c) \), the inequality

\[
\frac{1}{t} \int_0^t w^*(y)dy \leq c w^*(t),
\]

is satisfied for all \( t \in (0, |J|] \). Here for a \( w : J \to \mathbb{R}^+ \), \( w^* \) is defined in the following way:

\[
w^*(t) = \inf \{ y > 0 : |\{ x \in J : w(x) > y \}| < t \}.
\]

An equivalent formulation of the non-increasing rearrangement can be given as follows

\[
w^*(t) = \sup_{e \subseteq J} \inf_{x \in e} |w(x)|, \quad \text{for any } t \in (0, |J|].
\]

It is well known that the function \( w^* \), which is defined on \( (0, |J|] \), is non-increasing, non negative and equimeasurable to \( |w| \). Inequality (1.2) is the tool (see [1]), for the determination of all \( p > 1 \) such that \( w \in RH_J^p(c') \), for some \( 1 \leq c' < +\infty \), whenever \( w \in A_1(J, c) \). Here \( RH_J^p(c') \) denotes the class of all weights \( w \) defined on \( J \) which satisfy a reverse Hölder inequality with constant \( c' \) on all the subintervals \( I \subseteq J \). One can also see related problems for estimates for the range of \( p \) in higher dimensions in [4] and [5] (for related results see also [6,10,11]).

In this paper we are interested for those weights \( w \) defined on a dyadic cube \( Q \) on \( \mathbb{R}^n \), or on the whole \( \mathbb{R}^n \), satisfying condition (1.1) for all dyadic subcubes of its domain. More precisely, a locally integrable non-negative function \( w \) is called a dyadic \( A_1 \) weight if it satisfies the condition

\[
\frac{1}{|Q|} \int_Q w(y)dy \leq c \ ess \inf_{x \in Q} w(x),
\]

for every dyadic cube \( Q \) contained in the domain of \( w \). This condition is equivalent to the inequality

\[
\mathcal{M}_d w(x) \leq c w(x),
\]

for almost all \( x \in \mathbb{R}^n \). Here \( \mathcal{M}_d \) is the dyadic maximal operator defined by

\[
\mathcal{M}_d w(x) = \sup \left\{ \frac{1}{|Q|} \int_Q w(y)dy : x \in Q, \quad Q \subset \mathbb{R}^n \text{ is a dyadic cube} \right\}.
\]

The smallest \( c \geq 1 \) for which (1.3), [equivalently (1.4)], holds is called the dyadic \( A_1 \) constant of \( w \) and is denoted by \( [w]_1^d \).
Let us now fix a weight $w$, satisfying $[w]_1^d = c$. In [7] it is proved that it belongs to $L^p$ for any $p \in [1, p_0(n, c))$, where

$$p_0(n, c) = \frac{\log(2^n)}{\log[2^n-(2^n-1)c^{-1}]}.$$ 

Moreover it satisfies a reverse Hölder inequality for all $p$ in the above range upon all dyadic cubes on $\mathbb{R}^n$. More precisely the following is true as can be seen in [7].

**Theorem 1** Let $w$ be a dyadic $A_1$ weight defined on $\mathbb{R}^n$ with dyadic $A_1$ constant $[w]_1^d = c$. Then

$$\frac{1}{|Q|} \int_Q (M_d w)^p \leq \frac{2^n-1}{2^n-[2^n-(2^n-1)c^{-1}]^p} \left( \frac{1}{|Q|} \int_Q w(x) dx \right)^p$$

for every dyadic cube $Q$ on $\mathbb{R}^n$ and $p$ in the range $[1, p_0(n, c))$. Additionally the above inequality is sharp for any fixed $c \geq 1$ and $p$ in the above range.

Theorem 1 now implies that the range of $p$’s mentioned above is best possible. Let now $w$ be a weight defined on a dyadic cube $Q \subset \mathbb{R}^n$ which satisfies the $A_1$ condition on all dyadic subcubes of $Q$ with constant not more than $c > 1$. Then, as it is mentioned in [7], its non-increasing rearrangement $w^*$ does not necessarily belong to $A_1((0, |Q|], c)$. As a result certain questions arise: does $w^*$ belong to $A_1((0, |Q|], c')$ for some $c' \geq c$ and is there an upper bound for these $c'$? What is the least one? These questions are answered by the following.

**Theorem 2** Let $w$ be a dyadic $A_1$ weight on $\mathbb{R}^n$ with dyadic $A_1$ constant $[w]_1^d = c$. Let $Q$ be a fixed dyadic cube on $\mathbb{R}^n$. Then if we denote by $w/Q$ the restriction of $w$ to $Q$, the following inequality

$$\frac{1}{t} \int_0^t (w/Q)^*(y) dy \leq (2^n c - 2^n + 1)(w/Q)^*(t)$$

(1.6)

is satisfied for every $t \in (0, |Q|]$. Moreover the last inequality is sharp when one considers all dyadic $A_1$ weights with $[w]_1^d = c$.

We remark that by using a standard dilation argument it suffices to prove (1.6) for $Q = [0, 1]^n$ and for all functions $w$ defined only on $[0, 1]^n$ and satisfying the $A_1$ condition for dyadic cubes contained in $[0, 1]^n$. Actually, we will work on more general non-atomic probability spaces $(X, \mu)$ equipped with a structure $T$ similar to the dyadic one (we give the precise definition in the next section).

The paper is organized as follows: In Sect. 2 we give some tools needed for the proof of Theorem 2. These are obtained from [7] and [8]. In Sect. 3 we give the proof of Theorem 2 in its general form (as Theorem 3) and mention two applications of it.
2 Preliminaries

We fix a non-atomic probability space \((X, \mu)\) and a positive integer \(k \geq 2\). We give the following

**Definition 1** A set \(T\) of measurable subsets of \(X\) will be called a tree of homogeneity \(k\) if

(i) For every \(I \in T\) there exists a subset \(C(I) \subseteq T\) containing exactly \(k\) pairwise disjoint subsets of \(I\) such that \(I = \bigcup C(I)\) and each element of \(C(I)\) has measure \((1/k)\mu(I)\).

(ii) \(T = \bigcup_{m \geq 0} T(m)\) where \(T(0) = \{X\}\) and \(T(m+1) = \bigcup_{I \in T(m)} C(I)\).

(iii) The tree \(T\) differentiates \(L^1(X, \mu)\). That is, if \(\varphi \in L^1(X, \mu)\), then for \(\mu\)-almost all \(x \in X\) and every sequence \((I_k)_{k \in \mathbb{N}}\) such that \(x \in I_k, I_k \in T\) and \(\mu(I_k) \to 0\), we have that

\[
\varphi(x) = \lim_{k \to \infty} \frac{1}{\mu(I_k)} \int_{I_k} \varphi \, d\mu.
\]

It is clear that each family \(T(m)\) consists of \(k^m\) pairwise disjoint sets, each having measure \(k^{-m}\), whose union is \(X\). Moreover, if \(I, J \in T\) and \(I \cap J\) is non-empty then \(I \subseteq J\) or \(J \subseteq I\).

For this family \(T\), we define operator \(M_T\) acting on integrable functions \(\varphi : X \to \mathbb{R}\) by

\[
M_T \varphi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\varphi| \, d\mu : x \in I \in T \right\}. \quad (2.1)
\]

Furthermore we will say that a non-negative integrable function \(w\) is an \(A_1\) weight with respect to \(T\) if

\[
M_T \varphi(x) \leq C \varphi(x), \quad (2.2)
\]

for almost every \(x \in X\). The smallest constant \(C\) for which (2.2) holds will be called the \(A_1\) constant of \(w\) with respect to \(T\) and will be denoted by \([w]_{A_1}^T\). We give now the following:

**Definition 2** Every non-constant function \(w\) of the form \(w = \sum_{P \in T(m)} \lambda_P \chi_P\), for a specific \(m > 0\), and for positive \(\lambda_P\), will be called a \(T\)-step function (\(\chi_P\) denotes the characteristic function of \(P\)).

It is then clear that every \(T\)-step function is an \(A_1\) weight with respect to \(T\). Let now \(w\) be a weight as in Definition 2. Let also \([w]_1^T = c > 1\) and for any \(I \in T\) write

\[
Av_I(w) = \frac{1}{\mu(T)} \int_I w \, d\mu.
\]

Next for every \(x \in X\), let \(I_w(x)\) denote the largest element of the set \(\{I \in T : x \in I\}\) and \(M_T w(x) = Av_I(w)\) (which is non-empty since \(Av_J(w) = Av_P(w)\) for every \(P \in T(m)\) and \(J \subseteq P\)).
For any $I \in \mathcal{T}$ we define the set

$$A_I = A(w, I) = \{ x \in X : I_w(x) = I \}$$

and let $S = S_w$ denote the set of all $I \in \mathcal{T}$ such that $A_I$ is non-empty. It is clear that each such $A_I$ is a union of certain $P$ from $\mathcal{T}_{(m)}$ and moreover

$$\mathcal{M}_T(w) = \sum_{I \in S} A_{I}(w) \chi_{A_I}.$$

We also define the correspondence $I \mapsto I^*$ with respect to $S$ as follows: $I^*$ is the smallest element of $\{ J \in S_w : I \subseteq J \}$. This is defined for every $I \in S$ that is not maximal with respect to $\subseteq$.

We recall now two lemmas from [7] and for the sake of completeness we present their proof.

**Lemma 1** Let $w$ be as above. Then for all $I \in \mathcal{T}$ we have $I \in S$ if and only if $A_{v_Q}(w) < A_{v_I}(w)$ whenever $I \subseteq Q \in \mathcal{T}$, $I \neq Q$. In particular $X \in S$ and so $I \mapsto I^*$ is defined for all $I \in S$ such that $I \neq X$.

**Proof** If $I \in S$ then it is clear that the condition that is described above holds. Let now $I \in \mathcal{T}$ for which $A_{v_Q}(w) < A_{v_I}(w)$ for any $Q$ that strictly contains $I$ and belongs to the tree $\mathcal{T}$. Assume that $I \in \mathcal{T}_{(s)}$, then since

$$A_{v_J}(w) = \frac{\sum_{F \in C(J)} \mu(F) A_{v_F}(w)}{\sum_{F \in C(J)} \mu(F)}$$

we conclude that for each $J \in \mathcal{T}$ there exists $F \in C(J)$ such that $A_{v_F}(w) \leq A_{v_J}(w)$. Applying the above $m - s$ times, we get a chain $I = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_{m-s}$ such that $I_r \in \mathcal{T}_{(s-r)}$ for each $r$ and moreover $A_{v_{I_{m-s}}}(w) \leq A_{v_{I_{m-s-1}}}(w) \leq \cdots \leq A_{v_{I_1}(w)}$. Now because of the assumption on $I$ and the last mentioned inequalities, we conclude that $I_w(x) = I$ for every $x \in I_{m-s}$, therefore $I \in S$. $\square$

In what follows, we will write $y_I$ for the average $A_{v_I}(w)$, $I \in \mathcal{T}$.

**Lemma 2** Let $I \in S$. Then, if $J \in S$ is such that $J^* = I$, then $y_J < y_I \leq (k - (k - 1) c^{-1}) y_I$.

**Proof** The inequality on the left follows immediately from Lemma 1. Consider now the unique $F \in \mathcal{T}$ such that $J \in C(F)$. Obviously $F \subseteq I$. It is also true that $A_{v_F}(w) \leq y_I$. Indeed $J \in S$ implies that $A_{v_Q}(w) < y_I$ whenever $I \subseteq Q$, $I \neq Q$ and so if $A_{v_F}(w) > y_I$ there would exist $F_1 \in \mathcal{T}$ such that $F \subseteq F_1 \subseteq I$ with $F_1 \neq I$ and $A_{v_{F_1}}(w) > A_{v_{I}}(w)$ whenever $F_1 \subseteq Q$, $F_1 \neq Q$. This combined with Lemma 1 implies that $F_1$ must lie in $S$, which contradicts with our hypothesis that $J^* = I$. Now note that for every $x$ that belongs to the set-theoretic difference $F \setminus J$ we have

$\square$ Springer
\[ w \uparrow^T w(x) \geq \mathcal{M}_T w(x) \geq y_I, \] hence integrating over \( F \setminus J \) and using all the above we get
\[
y_I \geq A_{v_F}(w) \geq \frac{\mu(J)}{\mu(F)} y_J + \frac{\mu(F \setminus J)}{\mu(F)} \frac{1}{[w]_1^T} y_I = \frac{y_J + (k - 1)c^{-1} y_I}{k}.
\]
This completes the proof. \( \square \)

3 Main Theorem and Proof

In this section we will prove the following.

**Theorem 3** Let \( T \) be a tree of homogeneity \( k \geq 2 \) on the probability non-atomic space \((X, \mu)\), and let \( w \) be \( A_1 \) weight with respect to \( T \) with \( A_1 \) constant \([w]_1^T = c\). Then the non-increasing rearrangement \( w^* \) of \( w \) satisfies \( \frac{1}{k} \int_0^t w^*(y)dy \leq (kc - k + 1)w^*(t) \) for every \( t \in (0, 1] \). Moreover the constant appearing in the right of the last inequality is sharp, if one considers all such weights with \( A_1 \) constant with respect to \( T \) equal to \( c \).

**Proof** First consider a positive \( T \)-step function \( w \). Fix \( t \in (0, 1] \) and consider the set
\[
E_t = \{ x \in X : \mathcal{M}_T w(x) > c w^*(t) \} = \{ \mathcal{M}_T w > c \lambda \}
\]
where \( \lambda = w^*(t) \). Then \( E_t \) is a measurable subset of \( X \). We first assume that \( \mu(E_t) > 0 \). We consider the family of all those \( I \in T \) maximal under the condition \( A_{v_I}(w) > c \lambda \), and denote it by \( (I_j)_j \). Then \( (I_j)_j \) is pairwise disjoint and \( E_t = \bigcup I_j \). Additionally for every \( j \) and \( I \in T \) such that \( I \nsubseteq I_j \) we have that \( \frac{1}{\mu(I)} \int_I w d\mu = A_{v_I}(w) \leq c \lambda \) because of the maximality of \( I_j \). In view of Lemma 1 this gives \( I_j \in S_w = S \), for every \( j \).

For every \( I_j \), consider \( I_j^* \in S \). Then by Lemma 2, \( y_{I_j} \leq [k - (k - 1)c^{-1}]y_{I_j^*} \). By the above discussion we now have \( y_{I_j^*} \leq c \lambda \). Thus we obtain as a consequence that
\[
y_{I_j} \leq [k - (k - 1)\delta]c \lambda = (kc - k + 1)\lambda, \quad \text{for every } j.
\]
This gives
\[
\int_{I_j} w d\mu \leq (kc - k + 1)\lambda \mu(I_j) \Rightarrow \int_{E_t} w d\mu \leq (kc - k + 1)\lambda \mu(E_t)
\]
\[
\Rightarrow \frac{1}{\mu(E_t)} \int_{E_t} w d\mu \leq (kc - k + 1)\lambda.
\]
Since \( \mathcal{M}_T w \leq cw \) \( \mu \)-a.e on \( X \), and \( E_t = \{ \mathcal{M}_T w > c \lambda \} \), we obviously have \( E_t \subseteq [w > \lambda] \cup H = \{ w > w^*(t) \} \cup H \), where \( H \) is suitable subset of \( X \) with \( \mu(H) = 0 \).

Now, there exists a Lebesgue measurable set \( E_t^* \subseteq (0, 1] \) such that \( |E_t^*| = \mu(E_t) =: t_1 \), and such that \( \int_{E_t^*} w^*(y)dy = \int_{E_t} w d\mu \). By the equimeasurability of
we choose the set $E_t^*$ so that $E_t^* \subseteq \{w^* > w^*(t)\} \subseteq (0, t)$. As an immediate consequence we get that $t_1 \leq t$.

Since $T$ differentiates $L^1(X, \mu)$, we have that $\mu$-almost every element of the set $\{w > c\lambda\} \subseteq X$ belongs to $E_t$. Since $\mu(E_t) > 0$, we also have that $\mu(\{w > c\lambda\}) > 0$. Let now $t_2$ be such that

$$w^*(t) > \lambda c \text{ for every } t \in (0, t_2) \text{ and } w^*(t) \leq c\lambda, \text{ for every } t \in (t_2, 1).$$

By the definition of $E_t^*$ we have that $E_t^* = (0, t_2) \cup A_t$, where $A_t$ is a Lebesgue measurable subset of $(t_2, t)$ and $|A_t| = t_1 - t_2$ (Of course $t_2 = |(0, t_2)| = |\{w^* > \lambda c\}| = \mu(\{w > \lambda c\}) \leq \mu(\{M_T w > \lambda c\}) = \mu(E_t) = t_1$).

We will now prove the following inequality

$$\frac{1}{\mu(E_t)} \int_{E_t} w^* d\mu \geq \frac{1}{t} \int_0^t w^*(y) dy. \quad (3.2)$$

This inequality is equivalent to

$$\frac{1}{t_1} \int_{E_t^*} w^*(y) dy \geq \frac{1}{t} \int_0^t w^*(y) dy \iff \int_0^{t_2} w^*(y) dy + t \int_{A_t} w^*(y) dy \geq\quad (3.3)$$

$$t_1 \int_0^{t_2} w^*(y) dy + t_1 \int_{t_2}^t w^*(y) dy \iff (t - t_1) \int_0^{t_2} w^*(y) dy + t \int_{A_t} w^*(y) dy \geq$$

$$\geq t_1 \int_{t_2}^t w^*(y) dy. \quad (3.4)$$

We define $\Gamma_t = (t_2, t) \setminus A_t$. Then (3.3) becomes

$$\int_0^{t_2} w^*(y) dy + (t - t_1) \int_{A_t} w^*(y) dy \geq t_1 \int_{\Gamma_t} w^*(y) dy \iff (t - t_1) \int_{E_t^*} w^*(y) dy \geq t_1 \int_{\Gamma_t} w^*(y) dy. \quad (3.4)$$

Moreover,

$$\int_{E_t^*} w^*(y) dy = \int_{E_t} w d\mu > \mu(E_t) \cdot c\lambda = c\lambda \cdot t_1,$$

since $E_t$ is the pairwise disjoint union of $(I_j)_j$. Thus if we prove that

$$\int_{\Gamma_t} w^*(y) dy \leq c\lambda(t - t_1), \quad (3.5)$$
we complete the proof of (3.2). But (3.5) is obvious since \( w^*(y) \leq c\lambda \) on \((t_2, t)\), \( \Gamma_i \subseteq (t_2, t) \) and

\[
|\Gamma_i| = |(t_2, t) \setminus A_i| = (t - t_2) - |A_i| = t - t_1.
\]

We thus have proved that for every \( w, T \)-step function, and \( t \) such that \( \mu(\{M_T w > c \cdot w^*(t)\}) > 0 \), the following inequality is true:

\[
\frac{1}{t} \int_0^t w^*(y) dy \leq (kc - k + 1) w^*(t).
\] (3.6)

If \( t \) is such that \( \mu(\{M_T w > cw^*(t)\}) = 0 \), then obviously \( M_T w(x) \leq cw^*(t) \), for \( \mu \)-almost every \( x \in X \), so since \( T \) differentiates \( L^1(X, \mu) \) we get that \( w(y) \leq cw^*(t) \) for almost all \( y \in X \). This obviously gives (3.6), since \( c \leq kc - k + 1 \).

Next if \( w \) is a general \( A_1 \) weight with respect to \( T \), then an approximation argument by \( T \)-step \( A_1 \) weights gives the result for \( w \). More precisely one can easily see that if \( w \) is a \( A_1 \) weight with respect to \( T \), with \( A_1 \) constant \( [w]_T^1 = c \), then there exists an increasing sequence \((w_n)_n\) of \( T \)-step functions such that \( w_n \leq w \) and \([w_n]_T^1 = c_n \leq c\) with the additional properties \( w_n \to w \), \( \mu \) a.e. and \( c_n \to c \) as \( n \to \infty \). Thus in order to finish the proof of Theorem 3 we just need to prove the sharpness of the result.

Fix \( k \geq 2 \). We suppose that we are given a tree \( T \) of homogeneity \( k \), and consider \( T_2(2) \).

\[
T_2(2) = \{P_1, \ldots, P_k, P_{k+1}, \ldots, P_{2k}, \ldots, P_{k^2-k+1}, \ldots, P_{k^2}\}
\]

where

\[
T_1(1) = \bigcup_{i=1}^{k} P_i, \bigcup_{i=k+1}^{k^2} P_i, \ldots, \bigcup_{i=k^2-k+1}^{k^2} P_i = \{I_1, I_2, \ldots, I_k\}.
\]

We have that \( \mu(P_i) = \frac{1}{k^2} \), for all \( i \).

Suppose \( \delta > 0 \) is such that \( \delta < \frac{1}{k^2} \), and consider a set \( A_\delta \) of measure \( \mu(A_\delta) = \delta \) such that \( A_\delta \subseteq P_1 \) \(((X, \mu) \text{ is non-atomic})\). Let \( c \geq 1 \) and \( \alpha, \epsilon > 0 \) be such that \( \epsilon < \alpha \) and \( kc - k + 1 = \frac{a}{\epsilon} \). Let \( \varphi = \varphi_\delta \) be the function defined as follows:

\[
\varphi/A_\delta := \alpha, \quad \varphi/I_1 \setminus A_\delta := \epsilon, \quad \varphi/P_{k+1} := \alpha, \quad \varphi/(I_2 \setminus P_{k+1}) := \epsilon, \quad \varphi/P_{2k+1} := \alpha, \quad \varphi/(I_3 \setminus P_{2k+1}) := \epsilon, \quad \ldots \quad \varphi/P_{k^2-k+1} := \alpha, \quad \varphi/(I_k \setminus P_{k^2-k+1}) := \epsilon.
\]

It is easy to see that \( \varphi = \varphi_\delta \) is a \( A_1 \) weight, with \( A_1 \) constant

\[
c_\delta = [\varphi]_1^T = \frac{A_{\varphi \times I_2}(\varphi)}{\epsilon} = \frac{k}{\epsilon} \int_{I_2} \varphi \, d\mu = \frac{k}{\epsilon} \left[ \alpha \cdot \frac{1}{k^2} + \left( \frac{1}{k} - \frac{1}{k^2} \right) \epsilon \right].
\]
Then $c_\delta = c$, for every $\delta$. Moreover, $\varphi_\delta^*(1/k) = \epsilon$, so $\varphi_\delta^*(1/k)(kc - k + 1) = \alpha$, while $k \int_0^{1/k} \varphi_\delta^*(y) dy$ tends to $\alpha$, as $\delta \to 1/k^2$.

This completes the proof of Theorem 3. $\square$

Theorem 1 of Sect. 1 is an immediate Corollary of Theorem 3. Also, the following are consequences of Theorem 3.

**Corollary 1** Let $w$ be an $A_1$ weight with respect to the tree $T$ of homogeneity $(k \geq 2)$ on $(X, \mu)$ with $[w]_T = c$. Consider $((0, 1], |\cdot|)$ equipped with the usual $k$-adic tree $T_k$, where $|\cdot|$ is the Lebesgue measure on $(0, 1]$. Then $[w^*]_{T_k} \leq kc - k + 1$ and this result is sharp.

**Corollary 2** Let $w$ be $A_1$-weight on $\mathbb{R}^n$ as described in Sect. 1 Then $w^* : (0, +\infty) \to \mathbb{R}^+$ has the following property:

$$\frac{1}{t} \int_0^t w^*(y) dy \leq (kc - k + 1)w^*(t), \text{ for every } t \in (0, +\infty),$$

and the inequality is sharp.

**Proof** We expand $\mathbb{R}^n$ as a union of an increasing sequence $(Q_j)_j$ of dyadic cubes, and use Theorem 3 for each of these. $\square$

**Acknowledgments** The author would like to thank professor A. Melas for suggesting the problem in this paper. This research has been co-financed by the European Union and Greek national funds through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF), Aristeia Code: MAXBELLMAN 2760, Research code: 70/3/11913.

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