A NEW AUTOMORPHISM OF $X_0(108)$

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ABSTRACT. Let $X_0(N)$ denote the modular curve classifying elliptic curves with a cyclic $N$-isogeny. $A_0(N)$ its group of algebraic automorphisms and $B_0(N)$ the subgroup of automorphisms coming from matrices acting on the upper halfplane. In a well-known paper, Kenku and Momose showed that $A_0(N)$ and $B_0(N)$ are equal (all automorphisms come from matrix action) when $X_0(N)$ has genus $\geq 2$, except for $N = 37$ and 63.

However, there is a mistake in their analysis of the $N = 108$ case. In the style of Kenku and Momose, we show that $B_0(108)$ is of index 2 in $A_0(108)$ and construct an explicit new automorphism of order 2 on a canonical model of $X_0(108)$.

1. INTRODUCTION

For a positive integer $N$, the modular curve $X_0(N)$ parametrises elliptic curves with a cyclic $N$-isogeny. Over $\mathbb{C}$, it is isomorphic, as a Riemann surface, to the quotient of the extended upper half-plane $\Gamma \backslash \mathbb{H}$ by the subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ consisting of determinant 1, integral matrices $\begin{pmatrix}a & b \\ c & d \end{pmatrix}$ with $c \equiv 0 \mod N$, acting as $\tau \mapsto (a\tau + b)/(c\tau + d)$ (see [Shi71] or [Miy89]).

$X_0(N)$ has a natural structure of an algebraic curve over $\mathbb{Q}$, defined in [Shi71, Ch. 7] or more technically, as the generic fibre of the compactification of a modular scheme over $\mathbb{Z}$ [KM85].

The normaliser $Nm(N)$ of $\Gamma_0(N)$ in $SL_2(\mathbb{R})$ acts on the extended upper half-plane and leads to a finite subgroup $B_0(N)$ of $A_0(N) \defeq Aut_C(X_0(N))$ isomorphic to $Nm(N)/\Gamma_0(N)$. The group-theoretic structure of $B_0(N)$ is given in [Bar08]. The natural question, when the genus $g_N$ of $X_0(N)$ is $\geq 2$ and so $A_0(N)$ is finite, is whether $B_0(N)$ is all of $A_0(N)$. For $N = 37$ this was famously known not to be the case: $X_0(37)$ is of genus 2 and is thus hyperelliptic, but the only non-trivial element of $B_0(37)$ is not a hyperelliptic involution. Ogg showed that this is the only case with $A_0(N)$ larger than $B_0(N)$ when $N$ is squarefree [Ogg77].

Using deeper properties of the minimal models of $X_0(N)$ and its Jacobian $J_0(N)$ and some further techniques, Kenku and Momose extended the analysis to all $N$ (with $g_N \geq 2$) in [KM88], claiming that $A_0(N) = B_0(N)$ except for $N = 37$ and possibly $N = 63$, and that the index of $B_0(N)$ in $A_0(N)$ is 1 or 2 in the latter case. Subsequently, Elkies showed that $N = 108$ is indeed an exceptional case and gave an elegant construction of an additional automorphism [Elk90]. Kenku and Momose eliminate almost all $N$ by a combination of arguments that lead to only seven values (including 63 and 108) for which case-by-case detailed analysis is required.

For $N = 108$, a hypothetical automorphism $\gamma$ not in $B_0(N)$ is considered and is used to construct a non-trivial automorphism $\gamma$ in $B_0(N)$ with various properties. $J_0(N)$ decomposes (up to isogeny) into 10 elliptic curve factors and it is claimed that $\gamma$ must act upon a particular one $E$ as $\pm 1$. From this, further analysis leads to a contradiction. However, $E$ has $j$-invariant 0 and it isn’t clear from the construction why $\gamma$ (which has order 2 or 3) could not act as a 3rd root of unity on $E$. I tried to derive a different contradiction assuming this, but everything

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seemed consistent if \( \gamma \) was assumed to have order 3 and act on each of the 6 CM components of \( J_0(N) \) as an appropriate 3rd root of unity.

\( B_0(108) \) has order 108 by [Bar08]. Kenku and Momose show that all automorphisms of \( X_0(108) \) are defined over the field they denote by \( k'(108) \), which is the ring class field mod 6 of \( k(108) = \mathbb{Q}(\sqrt{-3}) \) : explicitly \( k'(108) \) is \( \mathbb{Q}(\sqrt{-3}, \sqrt{2}) \). The primes \( p \) splitting completely in \( k'(108) \) are 31, 43, 109, \ldots.

Out of interest, I decided to compute the automorphisms over the finite fields \( \mathbb{F}_p \) for the first few split primes using MAGMA [BCP97]. Equations for a canonical model of \( X_0(108) \) are found directly and reduced mod \( p \) using the modular form machinery. The full set of automorphisms is then returned very quickly by the built-in functions provided by Florian Hess. To my surprise, in each case the number of automorphisms was 216, twice the order of \( B_0(108) \)!

A slightly longer MAGMA computation over \( \mathbb{F}_{31} \) returned an abstract group \( G \) representing the automorphism group and it was readily verified by further MAGMA function calls that \( G \) did contain a subgroup \( H \) of index 2 with the structure of \( B_0(108) \) as described by Bars. Finally, having gleaned this information from mod 31 computer computations, it remains to construct a new \( u \) in characteristic 0. It is possible to just mechanically run the MAGMA routines again but working over \( k'(108) \) for the automorphism group computations is much slower and besides, as an important special case, it is desirable to provide a construction with some level of transparent mathematical detail rather than just the output of a generic computer program.

In the next section, I review Kenku and Momose’s analysis of the \( N = 108 \) case and show that, after removing the error, it can be adapted to prove that \( B_0(108) \) is of index 1 or 2 in \( A_0(108) \) and also give the isomorphism type of \( A_0(108) \) in the index 2 case.

In the third section, I give my construction of a new automorphism \( u \) of order 2, using detailed modular information about \( X_0(108) \) and its Jacobian. Specifically, I use the explicit action of standard generators of \( B_0(108) \) on a natural modular form basis for the differentials of \( X_0(108) \) along with the commutator relations within \( A_0(108) \) for \( u \) to find a fairly simple matrix, involving 2 undetermined parameters \( a \) and \( b \), giving the action of \( u \) on the differentials. To proceed further, I used computer computations to first determine the relations for the canonical model of \( X_0(108) \) w.r.t. the differential basis and then to solve for \( a \) and \( b \). The latter involves computing a Gröbner basis for the zero-dimensional ideal in \( a \) and \( b \) that comes from substituting the matrix for \( u \) into the canonical relations. The resulting automorphism on the canonical model is defined over \( k'(108) \) but not \( k(108) \) (the field of definition of \( B_0(108) \)), as it should be.

Subsequent to my discovery and semi-computerised construction of a new automorphism, Elkies learned of the error through Mark Watkins. Using a neat function-theoretic argument, similar in some respects to the \( N = 63 \) case, he was able to derive a particularly simple geometric model of \( X_0(108) \) as the intersection of two cubics in \( \mathbb{P}^4 \) as well as explicitly writing down all automorphisms without the need for computer computations. Elkies construction gives an independent verification of the corrected result for \( N = 108 \) and will be published by him elsewhere.

Finally, I believe that there are no other exceptional \( N \). [KM88] is a very nice paper but it does seem to contain a number of mistakes, most of which have no bearing on the final result. In particular

- The last two numbers in the statement of Lemma 1.6 should be \( 2^3 \cdot 3^2 \) and \( 2^2 \cdot 3^3 \)
- A number of expressions in the proof of Lemma 1.6 are incorrect. Especially, most of the expressions in the table for \( \mu(D, p) \) in different cases are wrong.
\begin{itemize}
\item $N = 216 = 2^3\cdot 3^3$ listed in Corollary 1.11 and treated as a special case in the rest of the paper along with the other 3 values is not actually a special case! This is clearly harmless.
\item In Lemma 2.15, $p^2 - 1$ should be $p^2 + 1$.
\item In the first part of the main theorem, 2.17, the list of $N$ for various $l \leq 11$ that cannot be eliminated by lemmas 2.14 and 2.15 contain a number of cases that can be easily eliminated by Corollary 2.11. However, there are a number of unlisted cases that cannot be eliminated by any of these results. If I have worked it out correctly, I think that these are (I’m ignoring the obvious typos in the lists here - e.g. the second $N$ listed for $l = 11$ should have $2^2$ rather than $2^3$ as a factor) $2^2 \cdot 3^2 \cdot 7$ for $l = 5$, $2^4 \cdot 11$ for $l = 3$ and $3^3 \cdot 5$ and $3^2 \cdot 19$ for $l = 2$. This gives an upper bound 26 for $X_0(N)(\text{F}_4)$ which is less than the actual values (30 and 28 respectively) for the two cases.
\end{itemize}

The special analyses for the final six cases (ignoring $N = 37$ and 63) in the proof of Theorem 2.17 all seem OK to me except when $N = 108$.

2. Automorphisms of $X_0(108)$: Generalities

We reconsider the analysis of the $X_0(108)$ case as given on pages 72 and 73 of [KM88] and show that the correct conclusion is that $B_0(108)$ is of index 1 or 2 in $A_0(108)$ rather than that $A_0(108)$ is necessarily equal to $B_0(108)$.

Notation:

$X_0(108)$, $J = J_0(108)$, $A_0(108)$, $B_0(108)$, $k(108)$ and $k'(108)$ are as described in the introduction. $\sigma$ denotes one of the two generators of $G(k'(108)/k(108))$.

$w_n$ will denote the Atkin-Lehner involution on $X_0(108)$ for $n|108$, $(n, 108/n) = 1$ (see, eg, [Miy89] or [Bar08]). Explicitly, we could take matrix representatives mod $\mathbb{R}^*\Gamma_0(108)$ for the actions of $w_4, w_{27}$ and $w_{108}$ on the extended upper half-plane as

$$w_4 = \begin{pmatrix} 28 & 1 \\ 108 & 4 \end{pmatrix} \quad w_{27} = \begin{pmatrix} 27 & -7 \\ 108 & -27 \end{pmatrix} \quad w_{108} = \begin{pmatrix} 0 & -1 \\ 108 & 0 \end{pmatrix}$$

$w_1$ is trivial. Up to scalars, the matrix for $w_{27}$ is an involution and the matrix for $w_{108}$ is the product of those for $w_{27}$ and $w_4$.

For $v|6$, $S_v$ will denote the element of $B_0(108)$ represented by the matrix $\begin{pmatrix} 1 & (1/v) \\ v & 1 \end{pmatrix}$.

$B_0(108)$ is generated by $w_4, w_{27}, S_2$ and $S_3$. Its group structure is described fully later in the next section. We note here that the first three generators have order 2 and the last has order 3. The subgroup $S$ of $B_0(108)$ commuting with $\langle w_4, w_{27} \rangle$ is

$$\langle w_4 \rangle \times \langle w_{27} \rangle \times \langle \tau_3 \rangle$$

where $\tau_3$ is the element of order 3 in the centre of $B_0(108)$ defined by

$$\tau_3 := S_3 w_{27} S_3 w_{27}$$

($S_3$ and $w_{27} S_3 w_{27}$ commute).

The argument on page 73 of [KM88] considers a hypothetical automorphism $u$ in $A_0(108)$ not lying in $B_0(108)$. It is shown that $u$ is defined over $k'(108)$ but not over $k(108)$ and the non-trivial automorphism $\gamma$ is defined as $u^\sigma u^{-1}$. Note that all cusps are defined over $k(108)$,
that $B_0(108)$ acts transitively on the cusps and that, by their Corollary 2.3, any automorphism is determined by its images of $\infty$ and any other cusp. This shows, in particular, that all elements of $B_0(108)$ are defined over $k(108)$. $\gamma$ is shown to lie in $B_0(108)$.

Let $f_{36}$, $f_{36}$ and $f_{108}$ denote the primitive cusp forms associated to the unique isogeny classes of elliptic curves with conductors 27, 36 and 108 respectively. These curves all have complex multiplication by orders of $\gamma$. Kenku and Momose consider the decomposition up to isogeny of $J$ into the product $J_H \times J_{C_1} \times J_{C_2}$ where $J_H$ is the part without CM, $J_{C_1}$ is associated to the eigenforms \{f_{36}(z), f_{36}(3z), f_{108}(z)\} and $J_{C_2}$ is associated to the eigenforms \{f_{27}(z), f_{27}(2z), f_{27}(4z)\}. They show that $\gamma$ acts trivially on the $J_H$ factor and that its order $d$ and the genus $g_Y$ of the quotient $X_0(108)/\langle \gamma \rangle$ satisfies (i) $d = 2$, $g_Y = 4, 5$ or (ii) $d = 3$, $g_Y = 4$. It is also shown that $\gamma$ commutes with $w_4$ and $w_{27}$ and so lies in $S$.

$E$ is the new elliptic curve factor of $J_{C_1}$ corresponding to $f_{108}$. The error comes with the line “Then $\gamma$ acts on $E$ under $\pm(1)$”. This eliminates case (ii) above and leads to a contradiction on the existence of $u$. However, there is the possibility that

(*) $\gamma$ acts on (the optimal quotient isogeny class of) $E$ by a non-trivial 3rd root of unity and case (ii) occurs.

We see in the next section that this actually can occur when we explicitly construct such a $u$. To have order 3 and lie in $S$, $\gamma$ must equal $\tau_3$ or $\tau_3^{-1}$. That (*) holds for such $\gamma$ follows from the determination of the action of the generators of $B_0(108)$ on a nice basis for the cusps of the congruence, given in the construction. This shows that $\tau_3$ fixes the non-CM forms defining the $J_H$ factor and multiplies each of the six CM Hecke eigenforms given above by some non-trivial 3rd root of unity as required.

So for any automorphism $u$, $u^\sigma u^{-1}$ is trivial or equal to $\tau_3$ or $\tau_3^{-1}$. As Kenku and Momose show that all automorphisms defined over $k(108)$ are in $B_0(108)$, this implies that $B_0(108)$ is of index at most three in $A_0(108)$.

However, this can be improved by considering the action on the reduction mod 31 of $X_0(108)$ and arguing as Kenku and Momose do to show that $\gamma$ is defined over $k(108)$. All automorphisms are defined over $F_{31}$ as 31 splits in $k'(108)$. If $u$ and $v$ are two automorphisms not in $B_0(108)$, then exactly the same argument near the top of page 73 applied to $u$ and $u^\sigma$ can be applied to $u$ and $v$ to show that $vu^{-1}$ lies in $B_0(108)$. This shows that $B_0(108)$ is of index at most 2 in $A_0(108)$. Note that the sentence on page 73 starting “Applying lemma 2.16 to $p = 7 \ldots$ ” should contain $p = 31$ rather than $p = 7$ and there should be a comment that lemma 2.16 is being applied here with any pair of cusps replacing 0 and $\infty$, which is permissible as the same proof works. So, replacing $\sigma$ by $\tau_3$ if necessary, we have that (remembering that $\tau_3$ is in the centre of $B_0(108)$)

($+$) $B_0(108)$ is of index 1 or 2 in $A_0(108)$ and any automorphism $u \notin B_0(108)$ satisfies $u^\sigma u^{-1} = \tau_3$.

Now, we assume that $A_0(108)$ is bigger than $B_0(108)$ and show that its group structure can then be determined from the above information and the abstract group structure of $B_0(108)$.

We denote a cyclic group of order $n$ by $C_n$. [Bar08] gives the structure of $B_0(108)$. Abstractly, it is the direct product $D_6 \times (C_3 \times C_2)$ where the first factor is the dihedral group of order 6 and the second is the order 18 wreath product (the semidirect product of $C_3 \times C_2$ by $C_2$, the generator of $C_2$ swapping the two $C_3$ factors). The $D_6$ factor is generated by $S_2$ and $w_4$, which both have order 2. The wreath product is generated by $S_3$ (order 3) and $w_{27}$ (order 2), so that $S_3$ and $w_{27}, S_3 w_{27}$ are two commuting elements of order 3 generating the order 9 subgroup. The centre of $B_0(108)$ is of order 3, generated by $\tau_3 = S_3 w_{27}, S_3 w_{27}$. 
The automorphism group of $B_0(108)$ is easy to determine from the decomposition of as $D_6 \times D_6 \times C_3$. The outer automorphism group is $C_2 \times C_2$. This can also be easily checked in MAGMA, for example.

Now if $u$ is not in $B_0(108)$, $u^2$ is in $B_0(108)$ and so is fixed by $\sigma$. Then, $(+) \text{ above shows that } u\tau_3^{-1} = \tau_3^{-1}$ so that $u$ acts by conjugation on $B_0(108)$ (which is normal in $A_0(108)$ having index 2) as an outer automorphism, since $\tau_3$ is central in $B_0(108)$. Also, we can assume $u$ has 2-power order and, as the kernel of the map of $B_0(108)$ elements. Another element would leads to a $u$ the outer automorphism group $H$ of $B_0(108)$. $H$ has 3 non-trivial elements giving extensions of $B_0(108)$ of degree 2. But the condition that $u$ doesn’t centralise $\tau_3$ excludes one of these elements. Another element would leads to a $u$ of order 2 commuting with $\langle w_4, w_{27} \rangle$ and $S_2$. From the explicit action of $S_2$ on weight 2 forms (see next section) we see that $u$ would have to preserve the $(w_{27} - 1)C_{12} (= E)$ and $(w_4 - 1)C_{12}$ elliptic curve factors of the Jacobian, so act as $\pm 1$ on $E$. $\gamma$ would then act trivially on $E$ and the argument of Kenku and Momose would properly lead to a contradiction. Thus, there is only one possibility for $u$ in $H$ and one possible group structure for $A_0(108)$. Explicitly, we find

**Lemma 2.1.** If $A_0(108)$ is larger than $B_0(108)$, then it contains $B_0(108)$ as a subgroup of index 2 and is generated by $B_0(108)$ and an element $u$ of order 2 that acts on $B_0(108)$ by conjugation as follows:

$$uw_4u = w_{27} \quad uw_{27}u = w_4$$

$$uS_2u = S_3w_{27}S_3^{-1} = S_3^{-1}\tau_3^{-1}w_{27} \quad uS_3u = S_2w_4\tau_3$$

For an appropriate choice of $\sigma$, $u^\sigma u^{-1} = \tau_3$.

### 3. Construction of a new automorphism

The notation introduced at the start of the last section is still in force.

**Conventions:**

If $u$ is an automorphism of $X_0(108)$, then we also think of it as an automorphism of $J$ by the “Albanese” action: a degree zero divisor $\sum a_iP_i \mapsto \sum a_iu(P_i)$. If $X_0(108)$ is embedded in $J$ in the usual way by $i : P \mapsto (P) - (\infty)$ then the actions are compatible up to translation by $(u(\infty)) - (\infty)$. As global differentials on $J$ are translation invariant, this means that the pullback action $u^*$ on global differentials of $J$ or $X_0(108)$ is the same if we identify global differentials of $J$ and $X_0(108)$ by the pullback $i^*$.

When we identify the weight 2 cusp form $f(z)$ of $\Gamma_0(108)$ with the complex differential $(1/2\pi i)f(z)dz$ on $X_0(108)$, if $u \in B_0(108)$ then $u^*f$ is $f|_{2u}$ in the notation of Sec. 2.1 [Miy89], identifying $u$ with the 2x2 matrix representing it. As we only deal with weight 2 forms we omit the subscript 2.

If we say that $u$ is represented by matrix $M \text{ w.r.t. a basis } f_1, \ldots, f_n$ of cusp forms/differentials, we mean that $u^*f_i = \sum_j M_{ji}f_j$. So if $u$ and $v$ are represented by $M$ and $N$, then $uv$ is represented by $MN$.

$J_0(108)$ decomposes up to isogeny into a product of 10 elliptic curves defined over $\mathbb{Q}$, as partially described in [KM88]. We work with a natural basis for the weight 2 cusp forms of level 108 coming from multiples of the primitive forms $f_{27}$, $f_{36}$ and $f_{108}$ which generate the CM part as in the last section, and the two primitive level 54 forms $f_{54}^{(1)}$, $f_{54}^{(2)}$ and their multiples by 2 which generate a 4-dimensional non-CM complement. All the forms have rational $q$-expansions. We give the initial $q$-expansions of the primitive forms, isogeny classes of elliptic curves over $\mathbb{Q}$
that they correspond to and the eigenvalues for the Atkin-Lehner involutions of the base level (which we refer to as $W_n$ to differentiate from the $w_n$ involutions for level 108).

**Conductor 27**

\[ f_{27} = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + O(q^{25}) \]

\[ W_{27} = -1 \]

\[ E_{27} : y^2 + y = x^3 \approx y^2 = x^3 + 16 \]

$X_0(27)$ is of genus 1 and this is a well-known case (see [Lig75]). $f_{27}$ is $\{\eta(3z)\eta(9z)\}^2$ where $\eta$ is the Dedekind eta function (§4.4 [Miy89]).

**Conductor 36**

\[ f_{36} = q - 4q^7 + 2q^{13} + 8q^{19} + O(q^{25}) \]

\[ W_9 = 1, W_4 = -1 \]

\[ E_{36} : y^2 = x^3 + 1 \]

$X_0(36)$ is of genus 1 and this is a well-known case (see [Lig75]). $f_{36}$ is $\eta(6z)^4$.

**Conductor 108**

\[ f_{108} = q + 5q^7 - 7q^{13} - q^{19} + O(q^{25}) \]

\[ w_{27} = 1, w_4 = -1 \]

\[ E_{108} : y^2 = x^3 + 4 \]

$f_{108}$ and the action of the Atkin-Lehner operators again come from Tables 3 and 5 of [BK75] or from a modular form computer package such as William Stein’s. It is easy to check that the CM elliptic curve $E_{108}$ has conductor 108 with $f_{108}$ as its associated modular form (which is of the type described in Thm 4.8.2 of [Miy89] with $K = k(108)$).

**Conductor 54**

\[ f_{54}^{(1)} = q - q^2 + q^4 + 3q^5 - q^7 - q^8 + 3q^{10} + 3q^{11} + O(q^{12}) \]

\[ w_{27} = -1, w_2 = 1 \]

\[ f_{54}^{(2)} = q + q^2 + q^4 - 3q^5 - q^7 + q^8 - 3q^{10} + 3q^{11} + O(q^{12}) \]

\[ w_{27} = 1, w_2 = -1 \]

\[ E_{54}^{(1)} : y^2 + xy = x^3 - x^2 + 12x + 8 \quad E_{54}^{(2)} : y^2 + xy + y = x^3 - x^2 + x - 1 \]

The $f_{54}^{(i)}$ and the action of the Atkin-Lehner operators again come from Tables 3 and 5 of [BK75] or from a modular form computer package. It is easy to check that the elliptic curves given have
conductor 54 and have the respective $f_{54}^{(i)}$ as their associated modular forms. In fact the $E$s are quadratic twists of each other by $-3$ and the two $f_{54}^{(i)}$ are twists by the quadratic character of $k(108)$.

**Definition 3.1.** $\delta_n$ is the operator on modular forms given by the matrix $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$, so that if $f(z)$ is a weight 2 form, $(f|\delta_n)(z)$ is the form $nf(nz)$.

**Definition 3.2.** $e_1, \ldots, e_{10}$ are the basis for the weight 2 cusp forms of $\Gamma_0(108)$ defined as follows:

\[ e_1 = f_{54}^{(2)} - f_{54}^{(2)}|\delta_2 \quad e_2 = f_{54}^{(2)} + f_{54}^{(2)}|\delta_2 \quad e_3 = f_{54}^{(1)} - f_{54}^{(1)}|\delta_2 \quad e_4 = f_{54}^{(1)} + f_{54}^{(1)}|\delta_2 \]

\[ e_5 = f_{27} + f_{27}|\delta_4 \quad e_6 = f_{27}|\delta_2 \quad e_7 = f_{36} + f_{36}|\delta_3 \]

\[ e_8 = f_{108} \quad e_9 = f_{27} - f_{27}|\delta_4 \quad e_{10} = f_{36} - f_{36}|\delta_3 \]

The standard decomposition into new and old forms (Section 4.6, [Miy89]) shows that $e_1, \ldots, e_{10}$ form a basis for the weight 2 cusp forms of $\Gamma_0(108)$.

Identifying these cusp forms with differential forms on $X_0(108)$ and $J, V := \langle e_1, \ldots, e_4 \rangle$ is the subspace corresponding to differentials of $J_H$ and $W := \langle e_5, \ldots, e_{10} \rangle$ the subspace corresponding to $J_{C_1} + J_{C_2}$. All endomorphisms of $J$ preserve these subspaces.

**Lemma 3.3.** With respect to the basis $e_1, \ldots, e_4$ of $V$, $w_4, w_{27}, S_2$ and $S_3$ act by the following matrices ($\zeta := \exp(2\pi i/3)$, $\sqrt{-3} := \zeta - \zeta^{-1}$)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & \sqrt{-3} & 0 \\
0 & -1 & 0 & \sqrt{-3} \\
\sqrt{-3} & 0 & -1 & 0 \\
0 & \sqrt{-3} & 0 & -1
\end{pmatrix}
\]

With respect to the basis $e_5, \ldots, e_{10}$ of $W, w_4, w_{27}, S_2$ and $S_3$ act by the following matrices

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\zeta & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & -(1/2)\zeta^{-1} & 0 & 0 & -(1 - \zeta)/2 \\
0 & 0 & \zeta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta & 0 \\
0 & 0 & -(1 - \zeta)/2 & 0 & 0 & -(1/2)\zeta^{-1}
\end{pmatrix}
\]
Proof. The proof is a straightforward computation using relations between Atkin-Lehner involutions and the $\delta_i$ and congruence conditions on the exponents of the non-zero terms of the $q$-expansions to find the $S_2$ and $S_3$ actions.

For $S_2$: All forms $f|\delta_i$ where $i$ is 2 or 4 are clearly fixed by $S_2$. Generally, considering $q$-expansions, if a form $f$ is an eigenvalue of the Hecke operator $T_2$ at its even base level with eigenvalue $e$, then we see that $f|S_2 = -ef|\delta_2$. Note that $f_{36}|\delta_3$ is still an eigenvector of $T_2$ with eigenvalue 0. This leaves only $f_{27}$ to consider. As it is killed by $T_2$, the definition of the $T_2$ action quickly leads to $f_{27}|S_2 = -(f_{27} + f_{27}|\delta_4)$.

For $S_3$: $f_{27}, f_{36}$ and $f_{108}$ all have $q$-expansions where all non-zero terms $a_nq^n$ have $n = 1$ mod 3. This follows from the fact that they are eigenvalues of all Hecke operators and that $a_p = 0$ if $p = 2$ mod 3, $p > 2$, as the associated elliptic curves have supersingular reduction at these primes so $p|a_p$ and $|a_p| < 2\sqrt{p}$. $a_2$ and $a_3$ are clearly also zero. As $f_{54}^{(1)}$ and $f_{54}^{(2)}$ are twists by the $\mathbb{Z}_2$ quadratic character and are killed by the $T_3$ operator, $f_{54}^{(1)} + f_{54}^{(2)} = \sum_{n=1(3)} a_nq^n$ and $f_{54}^{(1)} - f_{54}^{(2)} = \sum_{n=2(3)} b_nq^n$. From these facts, the action of $S_3$ on the basis follows easily.

For $w_4$: Simple matrix computations show that $f|\delta_2|w_4 = f|W_2$ and $f|w_4 = (f|W_2)|\delta_2$ for a level 54 form $f$. Similarly, $f|w_4 = f|\delta_4$, $(f|\delta_2)|w_4 = f|\delta_2$ and $(f|\delta_4)|w_4 = f$ for level 27 forms; $f|w_4 = f|W_4$ and $(f|\delta_3)|w_4 = (f|W_4)|\delta_3$ for level 36 forms. The full $w_4$ action follows.

For $w_{27}$: Again, simple matrix computations show that $f|w_{27} = f|W_{27}$ and $(f|\delta_2)|w_{27} = (f|W_{27})|\delta_2$ for level 54 forms; $f|w_{27} = f|W_{27}$ and $(f|\delta_4)|w_{27} = (f|W_{27})|\delta_4$ (i = 2 or 4) for level 27 forms; $f|w_{27} = (f|W_9)|\delta_3$ and $(f|\delta_3)|w_{27} = (f|W_9)$ for level 36 forms. The full $w_{27}$ action follows.

Note: From the above lemma, we see that $\tau_3$ acts trivially on $V$ and multiplies each $e_i$ for $i \geq 5$ by $\zeta$ or $\zeta^{-1}$ as asserted in the last section.

Using the commutator conditions for a new automorphism $u$ of order 2 as described in Lemma 2.1, it is now easy to show that $u$ acts on weight two forms by a matrix $\pm M$ (w.r.t. the $e_i$ basis) with $M$ of the form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & z^{-1} & 0 \\
0 & z & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 & 0 & -za & 0 & 0 & 0 \\
0 & 0 & 0 & b & 0 & 0 \\
(za)^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & b^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & a^{-1} & 0 & 0
\end{pmatrix}
\]

with $a, b \in \mathbb{C}^*$ and $z = \sqrt{-3}$ as defined in Lemma 3.3.

We can now complete the construction of $u$ on a canonical model of $X_0(108)$ with detailed computations that can be carried out using a suitable computer algebra system. The author performed these with MAGMA. There are 3 steps.

1. Compute a basis $R$ for the degree 2 canonical relations for $X_0(108)$ embedded into $\mathbb{P}^9$ via the differential basis corresponding to the forms $e_i$. These relations generate the full ideal defining $X_0(108)$ in $\mathbb{P}^9$. 

(2) Substitute the automorphism of $P^9$ given by $M$ into $R$ treating $a$ and $b$ as indeterminates. Clear powers of $a$ and $b$ from denominators. The condition that each new degree 2 form must lie in the span of $R$ gives a number of polynomial relations on $a$ and $b$ that generate a zero dimensional ideal $I_{a,b}$ of $k(108)[a,b]$.

(3) Compute a lex Gröbner basis of $I_{a,b}$. From this, we can read off all solutions for $a$ and $b$ such that $M$ gives an automorphism of $X_0(108)$.

For the first step we need to find a basis for the linear relations between the 55 weight 4 cusp forms $e_i e_j$, $1 \leq i \leq j \leq 10$. Considering it as a regular differential of degree 2 (see Section 2.3 [Miy89] - note that there are no elliptic points here), a weight 4 form for $\Gamma_0(108)$ that vanishes to order at least 2 at each cusp is zero iff it has a $q$-expansion $\sum_{n \geq 2} a_n q^n$ with $a_n = 0$, $\forall n \leq 38$. So the computation reduces to finding a basis for the kernel of a $55 \times 37$ matrix with integer entries. In practice, it is good to work to a higher $q$-expansion precision than 38 and we actually did the computation with the expansions up to $q^{150}$. This still only took a fraction of a second in MAGMA. Applying an LLL-reduction to get a nice basis for the relations, the result is that the canonical model for $X_0(108)$ in $P^9$ with coordinates $x_i$ is defined by the ideal generated by the following 28 degree 2 polynomials:

\[
\begin{align*}
&x_3 x_4 + x_6 x_9 - x_5 x_{10}, \quad x_1 x_2 - x_6 x_9 - x_5 x_{10}, \quad x_2 x_6 - x_3 x_7 + x_1 x_{10}, \quad x_4 x_5 - x_1 x_8 + x_6 x_{10} \\
&x_1 x_8 - x_3 x_9 + x_6 x_{10}, \quad x_4 x_6 + x_1 x_7 - x_3 x_{10}, \quad x_3 x_7 - x_2 x_8 x_9 + x_2 x_{10}, \quad x_3 x_7 - x_2 x_8 x_9 + x_2 x_{10} \\
&x_2 x_5 - 2 x_3 x_8 + x_1 x_9, \quad x_2 x_5 - 2 x_3 x_8 + x_1 x_9, \quad x_2 x_4 + x_7 x_9 - x_8 x_{10}, \quad x_1 x_7 - 2 x_5 x_9 + x_3 x_{10} \\
&x_2 x_3 - x_5 x_7 - 2 x_6 x_8, \quad x_2 x_3 + x_1 x_4 - 2 x_5 x_7, \quad x_7^2 - x_2 x_8 x_9 + x_2 x_{10}, \quad x_7^2 - x_2 x_8 x_9 + x_2 x_{10} \\
&3 x_1 x_5 - 2 x_4 x_8 - x_2 x_9, \quad 3 x_2^2 - x_7^2 \rightarrow x_{10}, \quad 3 x_1 x_3 - x_7 x_9 - 2 x_8 x_{10}, \quad 3 x_1 x_6 + x_4 x_7 - x_2 x_{10} \\
&3 x_3 x_6 - x_2 x_7 + x_4 x_{10}, \quad 3 x_3 x_5 - x_7^2 - x_2 x_8 - x_{10}^2, \quad 3 x_7^2 - x_4^2 - 2 x_9 x_{10}, \quad x_2^2 - 3 x_3^2 + 2 x_9 x_{10} \\
&x_2 x_7 - 4 x_8^2 + 2 x_9 x_{10}, \quad x_2 x_7 - 4 x_8^2 + 2 x_9 x_{10}, \quad x_4 x_8 + x_2 x_9 - 2 x_7 x_{10}, \quad 3 x_2^2 - x_2 x_7 - x_3^2 - 2 x_4 x_{10}
\end{align*}
\]

We are using the fact that $X_0(108)$ is not hyperelliptic [Ogg74]. This follows from the above anyway, since there would be 36 canonical quadric relations if it were. However, it needs to be checked that $X_0(108)$ is not trigonal (having a degree 3 rational function) when there would be independent degree 3 relations. For this, it is only necessary, for example, to verify that the ideal defined by the above polynomials has the right Hilbert series. This was easily verified in MAGMA which uses a standard Gröbner based algorithm [BS92].

For the second step, we work over $K(a,b)$, $K = Q(z)$, apply the substitution $x_i \mapsto \sum_{1 \leq j \leq 10} M_{ij} x_j$ to the above polynomials, and the rest is straightforward linear algebra. Applying the Gröbner basis algorithm, we find that $I_{a,b}$ is generated as an ideal by the two polynomials

\[b - za^2, \quad a^3 + (1/2)\]

which gives 3 possibilities for $u$ with $a$ any cube root of $-1/2$. We remark that if $u$ is one of these automorphisms then the other two are $u \tau_3$ and $u^{-1} \tau_3$ as expected. We also check that $u^2 u^{-1} = \tau_3$ if $a = e^{x(2\pi i/3)}$.

**Theorem 3.4.** $B_0(108)$ is of index two in $A_0(108)$, which has the structure described in Lemma 2.7. $u$ is given explicitly on the canonical model of $X_0(108)$ with the above defining equations by

\[x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_8 : x_9 : x_{10} \mapsto (x_1 : x_3 : (1/2)^{x_2} x_4 : (c/z)^{x_7} : (c^2/z)^{x_8} : (z/c)^{x_5} : (z/c^2)^{x_6} : -c x_{10} : -(1/c)x_9)\]

where $c^3 = 2$ and $z = \sqrt{-3}$ with $\Im(z) > 0$. 

Remarks:

(1) The action of $u$ on differentials is given by $\pm M$ where $M$ is the matrix on page 8. $M$ has 1 (resp. $-1$) as an eigenvalue of multiplicity 6 (resp. 4). Let $Y = X_0(108)/(u)$. If the action was by $M$, then the genus of $Y$, $g_Y$, would be 6. The Hurwitz formula would then give a value of $-2$ for the number of fixed points of $u$. Thus $u$ acts on differentials by $-M$, $g_Y = 4$, and $u$ has two fixed points on $X_0(108)$.

(2) On our canonical model of $X_0(108)$, the cusp $\infty$ is given by the point $(1 : 1 : 1 : 1 : 0 : 1 : 1 : 1 : 1)$ and the generators of $B_0(108)$ act via the matrices given in Lemma [3,3]. It is then easy to compute all of the cuspidal points as the cusps form a single orbit under $B_0(108)$. It is then easily verified that $u(\{\text{cusps}\}) \cap \{\text{cusps}\} = \emptyset$.

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