CONTROLLABILITY ANALYSIS OF NONLINEAR FRACTIONAL ORDER DIFFERENTIAL SYSTEMS WITH STATE DELAY AND NON-INSTANTANEOUS IMPULSIVE EFFECTS

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Abstract. This manuscript prospects the controllability analysis of non-instantaneous impulsive Volterra type fractional differential systems with state delay. By enrolling an appropriate Grammian matrix with the assistance of Laplace transform, the conditions to obtain the necessary and sufficiency for the controllability of non-instantaneous impulsive Volterra-type fractional differential equations are derived using algebraic approach and Cayley-Hamilton theorem. A distinctive approach presents in the manuscript, i have taken non-instantaneous impulses into the fractional order dynamical system with state delay and studied the controllability analysis, since this not exists in the available source of literature. Inclusively, i have provided two illustrative examples with the existence of non-instantaneous impulse into the fractional dynamical system. So this demonstrates the validity and efficacy of our obtained criteria of the main section.

1. Introduction. In recent years, the analysis and applicability of mathematical model of systems with fractional differential equations in engineering and knowledge-based disciplines increases the research attention, since as it applied in the arena of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc., involves non-integer order. Hence, it acts as a wonderful hardware for interpreting the properties inherently from different background and methods. Moreover, it can also be assess as an alternative appliance to nonlinear differential equations. Furthermore, in the evolution progress of dynamic system there occurs a sudden change such as disturbance, reap, or natural disasters, and so on. This sudden change or discontinuous jumps in the process can be generally called as impulses. Therefore, the study of impulsive differential equations grow a large observation among the researchers from all over the world. Also, it motivates many authors to develop the solvability or controllability results of impulsive differential equations, see for references [14, 11, 29, 36, 37]. Basically, the impulses can be dissected into two types: One is the instantaneous impulse and the other one is non-instantaneous impulse. In general the fractional impulsive equations consists of two components:  
(i) Fractional differential equations describes the continuous part of the solution.  
(ii) Impulsive component describes the sudden change and the discontinuity of
the solution. In real-world application, the role of instantaneous impulses cannot present the dynamics of evolution processes in many fields, such as Pharmacotherapy. For this reason they need the non instantaneous impulse. In the study instantaneous impulse the duration of the changes is relatively short compared to the overall duration of the whole process, whereas it is not in the case of non-instantaneous impulse that the action of impulses starts abruptly at a fixed point and it continues up to some finite time interval. For fractional order differential equations with instantaneous impulses one can refer ([1, 2, 6, 9, 10, 13, 19, 28, 33, 34]). Moreover, for non-instantaneous impulse, one can see some fruitful references ([3, 4, 5, 7, 8, 12, 16, 21, 25, 26, 30, 31, 32, 35]). This kind of impulse is observed in lasers, and in the intravenous introduction of drugs in the bloodstream. Hernandez et al. in [15], introduced a new class of abstract differential equations where the impulses are not instantaneous, and they investigated the existence of mild and classical solutions.

On the other hand, controllability play a key role in the qualitative reaction of a dynamical system. The theory was started in the sixties of nineteenth century based on the mathematical description of the dynamical system. Generally, it can have the ability to control a dynamical system from the initial position to the desired final position with the assistance of some set of admissible controls. Hence the combination of fractional-order derivatives and as well as integrals in the theory of controllability leads to better results than the integer-order derivatives. In most of the dynamical system, the whole state of the dynamical system is not affected by the control, but it acts as a part of it, see for references [17, 18, 22, 23].

The dominant contributions of this manuscript is exposè here:

- In the existing literature, there is no work that describes the controllability analysis of non-instantaneous Volterra-type fractional differential equations with state delay.
- Here i have proved the controllability results using algebraic approach which is very effective and easier to calculate the results.
- An example is provided to illustrate the existing theory in the available source of literature.

The contour of this manuscript is exhibit here. Section 2, explores necessary definitions will be applicable to prove the theoretical results. Section 3, initiate the necessary and sufficient conditions of the controllability criteria. Next in Section 4, we contribute some examples to illustrate the effectiveness and applicability of the controllability results. Finally conclusions are drawn in Section 5.

2. Preliminaries and solution representation. This section introduces some preliminary concepts like definitions, lemmas and properties that assist us in proving the main controllability criterion. Also, we have presented the solution representation for the constructed model.

Definition 2.1. [27] The Caputo fractional derivative of order \( \alpha > 0, n - 1 < \alpha < n \) is defined as

\[
(C^\alpha D^\alpha_0+ f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,
\]

where the function \( f(t) \) has absolutely continuous derivatives up to order \( n - 1 \).
Definition 2.2. [27] The Mittag-Leffler function in two parameters is defined as
\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \text{for } \alpha, \beta > 0 \]
and \( z \in \mathbb{C} \), \( \mathbb{C} \) denotes the complex plane.

Definition 2.3. [27] The Laplace transform of a function \( f(t) \) is defined as
\[ F(s) = L(f(t)) = \int_0^{+\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}, \]
where \( f(t) \) is n-dimensional vector valued function.

Lemma 2.4.[27] Let \( \mathbb{C} \) be complex plane, for any \( \alpha > 0, \beta > 0 \) and \( A \in \mathbb{C}^{n \times n} \) then
\[ L[ t^{\beta-1} E_{\alpha,\beta}(At^\alpha)] = s^{\alpha-\beta} (s^\alpha I - A)^{-1}, \quad \Re(s) > \| A \|^\frac{1}{\beta} \]
holds, where \( \Re(s) \) denotes the real part of the complex numbers and \( I \) represents the identity matrix. In particular, for \( \beta = 1 \),
\[ E_{\alpha,1}(\lambda z^\alpha) = E_{\alpha}(\lambda z^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{k\alpha}}{\Gamma(ak + 1)}, \quad \lambda, z \in \mathbb{C} \]
have the interesting property that
\[ C D^\alpha E_{\alpha}(\lambda t^\alpha) = \lambda E_{\alpha}(\lambda t^\alpha) \]
and \( LE_{\alpha}(\pm at^\alpha)(s) = s^{\alpha-1} \).

Lemma 2.5. ([13]) Let \( h : Q \to \mathbb{R} \) be a continuous function. A function \( y \in C(Q, \mathbb{R}) \) is a solution of the fractional integral equation
\[ y(t) = y_a - \frac{1}{\Gamma(\alpha)} \int_0^a (a-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \]
if and only if \( x \) is a solution of the following fractional Cauchy problem
\[ C D^\alpha y(t) = h(t), \quad t \in J \]
\[ y(a) = y_a, \quad a > 0. \]

Consider the non-instantaneous Volterra-type fractional differential equation
\[ C D^\alpha y(t) = H y(t) + J u(t) + f(t, y(t)) + \int_0^t v(t, s, y(s)) ds, \quad t \in (t_m, sm), \]
\[ m = 0, 1, 2, ..., N \]
\[ \Delta y(t) = I_m(t, y(s_{m-1} - 0)), \quad \text{for } t \in (s_{m-1}, t_m), \quad m = 1, 2, ..., N, \]
\[ y(t) = \rho(t), \quad t \in [-\tau, t_m] \]
(1)
where \( C D^\alpha y(t) \) represents the \( \alpha \) th order of Caputo’s fractional derivative of \( y(t) \), \( 0 < \alpha < 1 \), \( H \) and \( J \) are known constant matrices and satisfy \( H \in \mathbb{R}^{n \times n}, J \in \mathbb{R}^{n \times m}, f : (\mathbb{R}_+) \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous, \( v : (\mathbb{R}_+) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous, \( \rho \in C([-\tau, t_m], \mathbb{R}^n) \) is the initial state function, where \( C([-\tau, t_m], \mathbb{R}^n) \) represents the space of all continuous functions mapping the interval \([-\tau, t_m]\) into \( \mathbb{R}^n \).
Lemma 2.6. A function \( y \) is a solution of the fractional integral equation
\[
y(t) = \begin{cases} 
\rho(0) + \int_0^t (t-s)^{a-1} E_{\alpha,\alpha}(H(t-s)\alpha) [H\rho(0) + Ju(s) + f(s, y(s))] 
+ \int_0^s v(s, k, y(k)) dk \text{ds}, & t \in (0, s_0], 
\end{cases}
\]
\[
y(t) = I_1(t_1, y(s_0, 0)) + \int_0^s (t-s)^{a-1} E_{\alpha,\alpha}(H(t-s)\alpha) [H\rho(0) + Ju(s) + f(s, y(s))] 
+ \int_0^s v(s, k, y(k)) dk \text{ds} - \int_0^{t_1} (t-s)^{a-1} E_{\alpha,\alpha}(H(t_m-s)\alpha) 
\times [H\rho(0) + Ju(s) + f(s, y(s))] + \int_0^s v(s, k, y(k)) dk \text{ds}, & t \in (t_1, s_1], 
\end{cases}
\]
\[
y(t) = I_2(t_2, y(s_1, 0)) + \int_0^s (t-s)^{a-1} E_{\alpha,\alpha}(H(t-s)\alpha) [H\rho(0) + Ju(s) + f(s, y(s))] 
+ \int_0^s v(s, k, y(k)) dk \text{ds} - \int_0^{t_2} (t-s)^{a-1} E_{\alpha,\alpha}(H(t_m-s)\alpha) [H\rho(0) + Ju(s)] 
+ f(s, y(s)) + \int_0^s v(s, k, y(k)) dk \text{ds}, & t \in (t_2, s_2], 
\end{cases}
\]
\[
y(t) = I_m(t_m, y(s_{m-1}, 0)) + \int_0^s (t-s)^{a-1} E_{\alpha,\alpha}(H(t-s)\alpha) [H\rho(0) + Ju(s) + f(s, y(s))] 
+ \int_0^s v(s, k, y(k)) dk \text{ds} - \int_0^{t_m} (t-s)^{a-1} E_{\alpha,\alpha}(H(t_m-s)\alpha) [H\rho(0) + Ju(s)] 
+ f(s, y(s)) + \int_0^s v(s, k, y(k)) dk \text{ds}, & t \in (t_m, s_m], \ m = 1, 2, ..., N, 
\end{cases}
\]
\[
y(t) = I_m(t, y(s_{m-1}, 0)), & t \in (s_{m-1}, t_m], \quad (6) 
\]
\[
y(t) = \rho(t), & t \in [-\tau, t_m], \quad (7) 
\]
if and only if \( y \) is a solution of the problem \((1)\).

Proof. Assume that \( y \) satisfies problem \((1)\). When \( t \in [0, s_0] \), we consider
\[
^{C}D^\alpha y(t) = H y(t) + J u(t) + f(t, y(t)) + \int_0^t v(t, s, y(s)) ds, \quad t \in (0, s_0] 
\]
with \( y(0) = \rho(0) \).

By employing Laplace transform to equation \((8)\), we have
\[
\begin{align*}
 s^\alpha L[y(t)] - s^{\alpha-1}[y(0)] &= H L[y(t)] + L[J u(t) + f(t, y(t)) + \int_0^t v(t, s, y(s)) ds] \\
 L[y(t)] &= (s^\alpha I - H)^{-1}s^{\alpha-1}[y(0)] + (s^\alpha I - H)^{-1}[J u(t) + f(t, y(t))] \\
 &\quad + \int_0^t v(t, s, y(s)) ds.
\end{align*}
\]
From the observation of Definition 2.3 and Lemma 2.4, Eqn. (8) is equivalent to

\[ L[y(t)] = (s^\alpha I - H)^{-1} s^\alpha L[y(0)] + (s^\alpha I - H)^{-1} L[J u(t) + f(t, y(t))] + \int_0^t v(t, s, y(s)) ds \]

\[ = L[y(0)] + (s^\alpha I - H)^{-1} L[Hy(0) + Ju(t) + f(t, y(t))] + \int_0^t v(t, s, y(s)) ds \]

\[ = L[y(0)] + L[t \alpha^{-1} E_{\alpha,\alpha}(H^\alpha)] L[Hy(0) + Ju(t) + f(t, y(t))] + \int_0^t v(t, s, y(s)) ds. \]

The above equation gives the form after employing the convolution

\[ L[y(t)] = L[y(0)] + L[\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[H(t-s)^\alpha][H \rho(0) + Ju(t) + f(t, y(t))] \]

\[ + \int_0^t v(t,s,y(s))] ds. \]

Applying the inverse Laplace transform, we obtain

\[ y(t) = \rho(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[H(t-s)^\alpha][H \rho(0) + Ju(s) + f(s, y(s))] \]

\[ + \int_0^s v(s,k, y(k)) dk] ds, \quad t \in [0, s_0]. \]

when \( t \in (s_0, t_1], \ y(t) = I_1(t, y(s_0 - 0)). \) Further, when \( t \in (t_1, s_1], \) we consider

\[ C D^\alpha y(t) = H y(t) + Ju(t) + f(t, y(t)) + \int_0^t v(t,s,y(s)) ds, t \in (0, s_0] \]

with \( y(t_1) = I_1(t_1, y(s_0 - 0)). \)

By Lemma 2.5 and using the above procedure, we have

\[ y(t) = I_1(t_1, y(s_0 - 0)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[H(t-s)^\alpha][H \rho(0) + Ju(s) + v(s, y(s))] \]

\[ + \int_0^s v(s,k, y(k)) dk] ds + \int_0^{t_1} (t-s)^{\alpha-1} E_{\alpha,\alpha}[H(t_1-s)^\alpha][H \rho(0) + Ju(s)] \]

\[ + f(s, y(s)) + \int_0^s v(s,k, y(k)) dk] ds \]

when \( t \in (s_1, t_2], \ y(t) = I_2(t, y(s_1 - 0)). \) When \( t \in (t_2, s_2], \) we consider

\[ C D^\alpha y(t) = H y(t) + Ju(t) + f(t, y(t)) + \int_0^t v(t,s,y(s)) ds, \quad t \in (0, s_0] \]

with \( y(t_2) = I_2(t_2, y(s_1 - 0)). \)

By Lemma 2.5 and using the above procedure, we have

\[ y(t) = I_2(t_2, y(s_1 - 0)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[H(t-s)^\alpha][H \rho(0) + Ju(s) + f(s, y(s))] \]

\[ + \int_0^s v(s,k, y(k)) dk] ds + \int_0^{t_2} (t-s)^{\alpha-1} E_{\alpha,\alpha}[H(t_2-s)^\alpha][H \rho(0) + Ju(s)] \]

\[ + f(s, y(s)) + \int_0^s v(s,k, y(k)) dk] ds \]
Proceeding like this, in general, when \( t \in (s_{m-1}, t_m] \), we consider
\[
C^D y(t) = H y(t) + Ju(t) + f(t, y(t)) + \int_0^t v(t, s, y(s))ds, \quad t \in (0, s_0]
\]
with \( y(t_m) = I_m(t_m, y(s_{m-1} - 0)) \).

By Lemma 2.5 and using the above procedure, we have
\[
y(t) = I_k(t_k, y(s_{m-1} - 0)) + \int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}[H(t - s)^\alpha][H \rho(0) + Ju(s) + f(s, y(s))]
+ \int_0^s v(s, k, y(k))dk]ds + \int_0^{t_m} (t - s)^{\alpha - 1} E_{\alpha, \alpha}[H(t_m - s)^\alpha][H \rho(0) + Ju(s)
+ v(s, y(s)) + \int_0^s v(s, k, y(k))dk]ds.
\]

If \( t \in [-\tau, t_m] \) then
\[
y(t) = \rho(t), \quad t \in [-\tau, t_m]
\]
is obviously holds. Conversely, by proceeding the steps the proof is similar. Thus, the proof of the Lemma 2.6 is completed. \( \square \)

3. Controllability criteria. This module, explores the main theorems of our system.

Definition 3.1. System (1) is called controllable on \([0, w](w \in (0, T))\); for any initial function \( \rho \in C([-\tau, t_m], \mathbb{R}^n) \) and any state \( y_w \in \mathbb{R}^n \), there exists a control input \( u(t) \in C_p([0, w], \mathbb{R}^n) \), such that the corresponding solution of (1) satisfies \( y(w) = y_w \).

Theorem 3.2. System (1) is controllable on \([0, w] \) if and only if the Grammian matrix
\[
W_j[0, w] = \int_0^w (w - s)^{-\alpha} \left[ E_{\alpha, \alpha}(H(w - s)^\alpha)JJ^*[E_{\alpha, \alpha}(H^*(w - s)^\alpha)] \right] ds \quad (9)
\]
is nonsingular for some \( w \in (0, T) \), where \( E_{\alpha, \alpha}(\cdot) \) is the Mittag-Leffler function and * denotes the matrix transpose.

Proof. Assume that \( W_j[0, w] \) is non singular, then \( W_j^{-1}[0, w] \) is well defined. Let
\[
u(t) = [E_{\alpha, \alpha}(H^*(w - t)^\alpha)]W_j^{-1}[0, w][y_w - \rho(0) - \int_0^w (w - \theta)^{-\alpha} E_{\alpha, \alpha}[H(w - \theta)^\alpha]
\times \left[ H \rho(0) + f(\theta, y(\theta)) + \int_0^\theta v(\theta, k, x(k))d\theta \right]. \quad (10)
\]
when \( w \in [0, s_0] \). Put \( t = w \) in (2) and Substituting in (10), we get
\[
y(w) = \rho(0) + \int_0^w (w - s)^{-\alpha} E_{\alpha, \alpha}[H(w - s)^\alpha][H \rho(0) + f(s, y(s)) + \int_0^s v(s, k, x(k))dk
+ JJ^*[E_{\alpha, \alpha}(H^*(w - s)^\alpha)]W_j^{-1}[0, w][y_w - \rho(0) - \int_0^w (w - \theta)^{-\alpha} E_{\alpha, \alpha}[H(w - \theta)^\alpha]
\times \left[ H \rho(0) + f(\theta, y(\theta)) + \int_0^\theta v(\theta, k, x(k))d\theta \right]. \quad (11)
\]
Hence system (1) is controllable on \([0, w]\), where \(w \in [0, s_0]\). When \(w \in (t_1, s_1]\), Let

\[
u(t) = J^*[E_{\alpha,\alpha}(H^*(w - t)^\alpha)]W_j^{-1}[0, w][y_w + y(t_1)] - I_1(t_1, y(s_0 - 0)) - \int_0^w (w - \theta)^{\alpha - 1}E_{\alpha,\alpha}[H(w - \theta)^\alpha][H\rho(0) + f(\theta, y(\theta)) + \int_0^\theta v(\theta, k, y(k))dk]d\theta.
\]

Put \(t = w\) in (3) and Substituting in (11), we get

\[
y(w) = I_1(t_1, y(s_1 - 0)) + \int_0^w (w - s)^{\alpha - 1}E_{\alpha,\alpha}[H(w - s)^\alpha][H\rho(0) + f(s, y(s)) + \int_0^s v(s, k, y(k))dk]ds
\]

\begin{align*}
&+ \int_0^s v(s, k, y(k))dk + JJ^*[E_{\alpha,\alpha}(H^*(w - s)^\alpha)]W_j^{-1}[0, w][y_w + y(t_1)] - I_1(t_1, y(s_1 - 0)) - \int_0^w (w - \theta)^{\alpha - 1}E_{\alpha,\alpha}[H(w - \theta)^\alpha][H\rho(0) + f(\theta, y(\theta)) + \int_0^\theta v(\theta, k, y(k))dk]d\theta
\end{align*}

\[
= I_1(t_1, y(s_1 - 0)) + \int_0^w (w - s)^{\alpha - 1}E_{\alpha,\alpha}[H(w - s)^\alpha][H\rho(0) + f(s, y(s)) + \int_0^s v(s, k, y(k))dk]ds
\]

\[
+ \int_0^s v(s, k, y(k))dk + [y_w + y(t_1) - I_1(t_1, y(s_1 - 0)) - \int_0^w (w - \theta)^{\alpha - 1}E_{\alpha,\alpha}[H(w - \theta)^\alpha][H\rho(0) + f(\theta, y(\theta)) + \int_0^\theta v(\theta, k, y(k))dk]d\theta]]ds
\]

\[
= y_w.
\]

Thus system (1) is controllable on \([0, w], w \in (t_1, s_1]\). For \(w \in (t_m, s_m]\), \(m = 1, 2, ..., N\), we take the control function as

\[
u(t) = J^*[E_{\alpha,\alpha}(H^*(w - t)^\alpha)]W_j^{-1}[0, w][y_w + y(t_m)] - I_m(t_m, y(s_{m-1} - 0)) - \int_0^w (w - \theta)^{\alpha - 1}E_{\alpha,\alpha}[H(w - \theta)^\alpha][H\rho(0) + f(\theta, y(\theta)) + \int_0^\theta v(\theta, k, y(k))dk]d\theta.
\]

Put \(t = w\) in (5) and Substituting (12), then by the similar procedure, we get

\[
y(w) = y_w. Hence system (1) is controllable on \([0, w]\).
\]

Conversely, Suppose that \(W_j[0, w]\) is singular, without loss of generality for \(w \in (t_m, s_m], m = 1, 2, ..., N\), there exists a non zero vector \(z_0\) such that \(z_0^*W_j[0, w]z_0 = 0\). That is

\[
\int_0^w z_0^*(w - s)^{\alpha - 1}[E_{\alpha,\alpha}(H(w - s)^\alpha)]JJ^*[E_{\alpha,\alpha}(H^*(w - s)^\alpha)]z_0 ds = 0.
\]
Then
\[ z_0^* E_{\alpha,\alpha}[H(w-s)^\alpha] J = 0 \] (13)
on \( s \in [0, w] \), Since system (1) is controllable, there exists control inputs \( u_1(t) \) and \( u_2(t) \) such that
\[
y(w) = I_m(t_m, y(s_{m-1} - 0)) + \int_0^w (w-s)^{\alpha-1} E_{\alpha,\alpha}[H(w-s)^\alpha][H\rho(0) + f(s, y(s))
+ \int_0^s v(s, k, y(k)) dk + J u_1(s)] ds = 0 \] (14)
\[ z_0 = I_m(t_m, y(s_{m-1} - 0)) + \int_0^w (w-s)^{\alpha-1} E_{\alpha,\alpha}[H(w-s)^\alpha][H\rho(0) + f(s, y(s))
+ \int_0^s v(s, k, y(k)) dk + J u_2(s)] ds \] (15)
Combining (14) and (15) yields
\[ z_0 - \int_0^w (w-s)^{\alpha-1} E_{\alpha,\alpha}(H(w-s)^\alpha) J[u_2(s) - u_1(s)] ds = 0. \] (16)
Multiplying \( z_0^* \) on both sides of (16), we get
\[ z_0^* z_0 - \int_0^w (w-s)^{\alpha-1} E_{\alpha,\alpha}(H(w-s)^\alpha) J[u_2(s) - u_1(s)] ds = 0. \]
By (13) \( z_0^* E_{\alpha,\alpha}[H(w-s)^\alpha] J = 0 \), we have \( z_0^* z_0 = 0 \).
Thus \( z_0 = 0 \). This is a contradiction. Hence the proof. Theorem 3.2 presents
a geometric type criterion. By the algebraic transform and computation, we can
obtain an algebraic criterion which is similar to the famous Kalman’s rank condition
[18].
**Theorem 3.3.** System (1) is controllable on \([0, w]\) if and only if \( \text{rank}[J \mid HJ \mid H^2J \ldots \mid H^{n-1}J] = n \).
**Proof.** By Cayley-Hamilton theorem, \( t^{\alpha-1} E_{\alpha,\alpha}(H t^\alpha) \) can be represented as
\[
t^{\alpha-1} E_{\alpha,\alpha}(H t^\alpha) = \sum_{i=0}^{\infty} \frac{t^{i \alpha + \alpha - 1}}{\Gamma(i \alpha + \alpha)} H^i
= \sum_{i=0}^{n-1} G_i(t) H^i \] (17)
For \( w \in [0, s_0] \),
\[
y(w) = \rho(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(H(t-s)^\alpha)[H\rho(0) + J u(s) + f(s, y(s))
+ \int_0^s v(s, k, y(k)) dk] ds,
= \rho(0) + \sum_{i=0}^{n-1} \int_0^w G_i(w-s) H^i[H\rho(0) + J u(s) + f(s, y(s))
+ \int_0^s v(s, k, y(k)) dk + J u(s)] ds \] (18)
Let 
\[ \psi = \rho(0) + \sum_{i=0}^{n-1} \int_0^w G_i(w-s)H^i[H\rho(0)] + Ju(s) + f(s,y(s)) + \int_0^s v(s,k,y(k))dk ds \] (19)

Combining (18) with (19), we get
\[ y(w) = \sum_{i=0}^{n-1} H^iJ \int_0^w G_i(w-s)u(s)ds \] (20)
\[ y(w) = [J \mid HJ \mid H^2J \ldots \mid H^{n-1}J] \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}, \] (21)

where \( d_i = \int_0^w G_i(w-s)u(s)ds, i = 0, 1, 2, \ldots, n - 1 \). For \( y(w) \in \mathbb{R}^n \), the if and only if condition to have a control input \( u(t) \) satisfying (20) is that \( \text{rank}[J \mid HJ \mid H^2J \ldots \mid H^{n-1}J] = n \). For \( w \in (t_m, s_m] \), \( m = 1, 2, \ldots, N \),

\[ y(w) = I_m(t_m, y(s_m-1, 0)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[H(t-s)^{\alpha}][H\rho(0) + Ju(s) + f(s,y(s))] ds \]
\[ + \int_0^s v(s,k,y(k))dk ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[H(t-m-s)^{\alpha}][H\rho(0) + Ju(s)] + f(s,y(s)) ds \] (22)
\[ = I_m(t_m, y(s_m-1, 0)) + \sum_{i=0}^{n-1} \int_0^w G_i(w-s)H^i[H(t-s)^{\alpha}][H\rho(0) + Ju(s)] + f(s,y(s)) ds \]
\[ + \int_0^s v(s,k,y(k))dk ds - \sum_{i=0}^{n-1} \int_0^w G_i(w-s)H^i[H(t-m-s)^{\alpha}][H\rho(0) + Ju(s)] + f(s,y(s)) ds \]
\[ + f(s,y(s)) + \int_0^s v(s,k,y(k))dk ds, \] (23)

Combining (22) with (23), we get
\[ y(w) - \psi = I_m(t_m, y(s_{k-1}, 0)) + \sum_{i=0}^{n-1} H^iJ \int_0^w G_i(w-s)u(s)ds \]
\[ = [J \mid HJ \mid H^2J \ldots \mid H^{n-1}J] \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}, \] (24)
Note that, for \( y(w) \in \mathbb{R}^n \), the if and only if condition to have a control input \( u(t) \) satisfying (23) is that \( \text{rank}[J \mid H J \mid H^2 J \ldots \mid H^{n-1} J] = n \). Thus the proof completed. \( \square \)

**Remark 3.6.** In [5], Agarwal et al. investigated the stability of solutions to impulsive Caputo fractional differential equations, furthermore, they established the Monotone iterative technique for the initial value problem for differential equations with non-instantaneous impulses in [7]. Recently, Zhou et al. in [1] developed the existence results for differential equations with fractional order and impulses. More recently in [6] Agarwal et al. presented a survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations. With the motivation from the above works, in this paper, we proposed a new controllability criteria for non-instantaneous impulsive Volterra-type fractional dynamical systems and it shows the novel over the works mentioned above.

### 4. Examples

This module describes the efficacy of the controllability criterion.

**Example 4.1.** Consider the controllability of linear fractional differential systems with state delay and impulses as follows:

\[
C D^\frac{1}{2} y(t) = H y(t) + \frac{e^{-t}|y(t)|}{(19 + e^t)(1 + |y(t)|)} + \int_0^9 \frac{(t-s)^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} ds + J u(t), t \in [0, 9] - \{1, 2, 3, 4, 5, 6, 7, 8\}
\]

\[
\Delta y(t_i) = \frac{1}{2} y(t_i^-), t_i = i, \quad i = 1, 2, 3, 4, 5, 6, 7, 8
\]

\[
y(t) = e^t, \quad t \in [-\frac{1}{3}, 0],
\]

Our aim is to prove that system (25) is controllable on \([0, 9]\) by using Theorem 3.1.

Let us take

\[
\alpha = \frac{1}{2}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 3 \\ 2 \end{pmatrix},
\]

\[
f(t, y(t)) = \frac{e^{-t}|y(t)|}{(19 + e^t)(1 + |y(t)|)}
\]

By computation, we have

\[
JJ^* = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}
\]

\[
E_{\frac{1}{2}, \frac{1}{2}}(A(9-s)^{\frac{1}{2}}) = \sum_{p=0}^1 \frac{H^p(9-s)^{\frac{p}{2}}}{\Gamma(\frac{p+1}{2})} = \begin{pmatrix} \frac{1}{\sqrt{\pi}} & (9-s)^{\frac{1}{2}} \\ 0 & 1/\sqrt{\pi} \end{pmatrix}
\]

\[
E_{\frac{1}{2}, \frac{1}{2}}(H^*(9-s)^{\frac{1}{2}}) = \sum_{p=0}^1 \frac{H^p(9-s)^{\frac{p}{2}}}{\Gamma(\frac{p+1}{2})} = \begin{pmatrix} 0 & (9-s)^{\frac{1}{2}} \\ \frac{1}{\sqrt{\pi}} & 1/\sqrt{\pi} \end{pmatrix}
\]

Put \( w = 9 \) in (12) and combining (12) with (26)-(28), we get

\[
W_J[0, 9] = \int_0^9 (9-s)^{-\frac{1}{2}}[E_{\frac{1}{2}, \frac{1}{2}}(A(9-s)^{\frac{1}{2}})] JJ^*[E_{\frac{1}{2}, \frac{1}{2}}(H^*(9-s)^{\frac{1}{2}})] ds
\]
\begin{align*}
\mathcal{W}_{J}[0, 9] & \text{ is nonsingular. Hence by Theorem 3.2, system (25) is controllable on } [0, 9]. \text{ Note: We can select any values between the selected intervals to satisfy the given example.}

\text{Example 4.2. Consider the controllability of linear fractional differential systems with state delay and impulses as follows:}

\begin{align*}
C D^\frac{1}{3}\ y(t) &= H x(t) + \frac{e^{-t} |y(t)|}{(19 + e^t)(1 + |y(t)|)} + \int_0^t (t - s)^{\frac{1}{3} - 1} ds \\
&+ J u(t), t \in [0, \frac{7}{2} \pi] - \{\pi, \frac{3}{2} \pi, 2 \pi, \frac{5}{2} \pi, 3 \pi\} \\
\Delta y(t_i) &= \frac{1}{3} y(t_i^-), \quad t_i = i \pi, \quad i = 1, 2, 3, 4, 5, 6 \\
y(t) &= \sin t, \quad t \in [-\frac{\pi}{5}, 0]
\end{align*}

\text{Our aim is to prove that system (25) is controllable on } [0, (\frac{7}{2}) \pi] \text{ by using Theorem 3.2. Let us take}

\begin{align*}
\alpha &= \frac{1}{3}, H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -6 & -5 \end{pmatrix}, \quad J = \begin{pmatrix} 2 & -1 \\ 0 & 1 \\ -2 & 2 \end{pmatrix}.
\end{align*}

\begin{align*}
f(t, y(t)) &= -\frac{e^{-t} |y(t)|}{(9 + e^t)(1 + |y(t)|)}
\end{align*}

Then

\begin{align*}
\text{rank}[J \ |H| \ |H^2|] = \text{rank} \begin{pmatrix} 2 & -1 & 2 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 2 & * & * & * \end{pmatrix} = 3
\end{align*}

Hence, by Theorem 3.3, system (30) is controllable on } [0, (\frac{7}{2}) \pi].

5. Conclusions. In this paper the controllability analysis of non-instantaneous impulsive Volterra-type fractional order dynamical systems with state delay have been investigated. A set of if and only if conditions of controllability criteria for such systems has been established based on the algebraic approach. Two numerical examples are presented to illustrate the effectiveness and applicability of the results obtained. Still now controllability criteria for non-instantaneous impulsive fractional dynamical systems are obtained by using semigroup theory. Generally speaking, both the Riemann-Liouville and the Caputo fractional operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. Motivated from the work of Jiang et al. in [17], I am ready to develop the controllability of singular systems with delay and non-instantaneous impulses, this will appear in the near future.

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