On the number of minima of a random polynomial

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Abstract

We give an upper bound in $O(d^{(n+1)/2})$ for the number of critical points of a normal random polynomial with degree at most $d$ and $n$ variables. Using the large deviation principle for the spectral value of large random matrices we obtain the bound

$$O\left(\exp\left(-\beta n^2 + \frac{n}{2} \log(d-1)\right)\right)$$

($\beta$ is a positive constant independent on $n$ and $d$) for the number of minima of such a polynomial. This proves that most normal random polynomials of fixed degree have only saddle points. Finally, we give a closed form expression for the number of maxima (resp. minima) of a random univariate polynomial, in terms of hypergeometric functions.

1 Introduction

We consider a random polynomial $f$ over the reals with $n \geq 1$ variables and degree at most $d \geq 2$. The problem is to compute, on the average, its number of critical points (the number of real roots of the system $Df(x) = 0$), and its number of local minima. Since a generic polynomial has only nondegenerate stationary points, this last number is also given by the real roots of the system $Df(x) = 0$ such that $D^2 f(x)$ is positive definite. This reduces our problem to the computation of the number of real roots of a polynomial system under certain constraints.

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Generally speaking, let $F = (F_1, \ldots, F_n)$ be a random system of real polynomial equations with $n$ variables and degree $F_i \leq d_i$. Let $N^F(U)$ denote the number of zeros of the system $F(x) = 0$ lying in the subset $U \subset \mathbb{R}^n$ and $N^F(\mathbb{R}^n) = N^F$. Little is known on the distribution of the random variable $N^F(U)$. A classical result in the case of one polynomial of one variable is given by Kac [9], [10], who gives the asymptotic value

$$E(N^F) \approx \frac{2}{\pi} \log d$$

as $d$ tends to infinity when the coefficients of $F$ are Gaussian centered independent random variables with variances equal to 1. But, when the variance of the $i$-th coefficient is equal to $(d_i)$ (Weyl’s distribution), we have (see Bogomolny-Bohias-Leboeuf [5] and also Edelman-Kostlan [6])

$$E(N^F) = \sqrt{d}.$$ 

In 1992, Shub and Smale extended this result to a real polynomial system $F$ where

$$F_i(x_1, \ldots, x_n) = \sum_{\alpha_1 + \ldots + \alpha_n \leq d_i} a_{i,\alpha} x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$

when the coefficients $a_{i,\alpha}$ are Gaussian centered independent random variables with variances equal to

$$\binom{d}{\alpha} = \frac{d_i!}{\alpha_1! \ldots \alpha_n! (d_i - \alpha_1 \ldots - \alpha_n)!}$$

(see Kostlan [11] on this distribution and its properties). Their result is

$$E(N^F) = \sqrt{d_1 \ldots d_n}$$

that is the square root of the Bézout number of the system.

A general formula for the expected value of $N^F(U)$ when the random functions $F_i$, $1 \leq i \leq n$, are stochastically independent and their law is centered and invariant under the isometries of $\mathbb{R}^n$ can be found in Azaïs-Wschebor [3]. This includes the Shub-Smale formula as a special case.

This result has also been extended by Rojas [14] to multi-homogeneous polynomial systems, and then partially by Malajovich and Rojas [12] to sparse polynomial systems.

Wschebor in [17] studies the moments of $N^F$ and Armentano-Wschebor [2] consider random systems of equations of the type $P_i(x) + X_i(x)$, $1 \leq i \leq n$, $x \in \mathbb{R}^n$, where the $P_i$’s are non-random polynomials (the signal) and the $X_i$’s are independent Gaussian random variables (the noise).

Notice a major difference between these studies and the case considered here: the $n$ equations of the system $Df(x) = 0$ are not independent!

Through this paper we denote by $P = P_{d,n}$ the space of degree at most $d$, $n$-variate polynomials with real coefficients. This space is endowed with the
inner product:
\[ \langle f, g \rangle_P = \sum_{|\alpha| \leq d} \left( \frac{d}{\alpha} \right)^{-1} f_\alpha g_\alpha \]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) is a multi-integer, \(|\alpha| = \alpha_1 + \ldots + \alpha_n\),

\[ f(x) = \sum_{|\alpha| \leq d} f_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{|\alpha| \leq d} f_\alpha x^\alpha, \]

and again
\[ \left( \frac{d}{\alpha} \right) = \frac{d!}{\alpha_1! \cdots \alpha_n!(d - |\alpha|)!}. \]

We make \( P \) a probability space in considering the probability measure

\[ \frac{1}{\sqrt{2\pi}} \lim_{P} e^{-\|f\|_P^2/2} dP = \frac{1}{\sqrt{2\pi}} \lim_{P} e^{-\|f\|_P^2/2} \prod_{|\alpha| \leq d} \left( \frac{d}{\alpha} \right)^{-1/2} df_\alpha \]

i.e. a random polynomial has here Gaussian centered independent random coefficients with variances equal to \( \left( \frac{d}{\alpha} \right) \).

Let \( S_n \) be the space of \( n \times n \) real symmetric matrices, endowed with the Frobenius inner product \( \langle R, S \rangle = \text{Trace}(R^T S) \) and its induced norm

\[ \|S\|^2 = \sum_{1 \leq i,j \leq n} S_{ij}^2. \]

The Gaussian Orthogonal Ensemble is the space \( S_n \) together with the probability measure

\[ \frac{e^{-\|S\|_F^2/2}}{(2\pi)^{n(n+1)/4}} dS = \frac{e^{-\|S\|_F^2/2}}{2^{n/2} \pi^{n(n+1)/4}} \prod_{1 \leq i \leq j \leq n} dS_{ij}. \]

Thus, the entries of a matrix in \( S_n \) are independent Gaussian random variables with mean 0 and variance 1 for a diagonal entry, and mean 0 and variance 1/2 for a non-diagonal entry.

Our first main result is the following:

**Theorem 1.** Let \( C_{d,n} \) denote the expected number of critical points of a random polynomial of degree at most \( d \) in \( n \) variables, and \( E_{d,n} \) the expected number of minima. Let \( P_n \) be the probability that a matrix in the Gaussian Orthogonal Ensemble is positive definite. Then, for every \( n \geq 2 \),

\[ C_{2,n} = 1 \text{ and } E_{2,n} = P_n, \]

and for \( d \geq 3 \)

\[ C_{d,n} \leq \sqrt{\frac{2}{d}}(d-1)^{(n+2)/2} \text{ and } E_{d,n} \leq \sqrt{\frac{2}{d}}(d-1)^{(n+2)/2} P_n. \]
When \( n = 1 \) one has
\[
C_{d,1} = 2E_{d,1} = \frac{2\sqrt{d-1}}{\pi} \int_0^\infty \frac{\sqrt{d(d-1)r^4 + 2dr^2} + 2}{(dr^2 + 1)(r^2 + 1)}dr \leq 1 + \sqrt{d - 2}.
\]
Moreover, when \( d \to \infty \),
\[
\frac{C_{d,1}}{1 + \sqrt{d - 2}} \to 1.
\]
Let \( P_n \) be the probability that a matrix in the Gaussian Orthogonal Ensemble \( GOE(n) \) is positive definite:
\[
P_n = \int_{S_+^n} e^{-\|S\|^2/2} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j) \frac{e^{-\|\lambda\|^2/2}}{(2\pi)^{n(n+1)/4}} d\lambda
\]
where \( \lambda \in \mathbb{R}^n_+ \) if and only if \( \lambda_1 > \ldots > \lambda_n > 0 \) and
\[
Vol \mathcal{O}_n = \frac{2^{n(n+3)/4}\Gamma(1/2)^{n(n+1)/2}}{\prod_{j=1}^n \Gamma((n-j+1)/2)}
\]
(see Mehta \[13\] for the description of \( P_n \) as an integral over \( \mathbb{R}^n \) and Federer \[7\] for the volume of the orthogonal group). The following values are easy to obtain
\[
P_1 = \frac{1}{2}, \quad P_2 = \frac{2 - \sqrt{2}}{4}, \quad P_3 = \pi - \frac{2\sqrt{2}}{4\pi}.
\]
\( P_3 \) was computed by Carlos Beltrán.

Using the large deviation principle for the spectral value of large random matrices (see the appendix at the end of this paper) we see that the asymptotic value of \( P_n \) for large values of \( n \) satisfies
\[
\limsup_{n \to \infty} \frac{1}{n^2} \log(P_n) \leq -\alpha
\]
where \( \alpha \) is a positive constant independent on \( n \). Thus, there exist two positive constants \( \beta \) and \( \gamma \) such that, for every \( n \geq 1 \),
\[
P_n \leq \gamma e^{-\beta n^2}.
\]
This gives our second main theorem:

**Theorem 2.** There exist two positive constants \( \beta \) and \( K \) such that for every \( n \) and \( d \) the number of minima of a random polynomial satisfies
\[
E_{d,n} \leq Ke^{-\beta n^2 + \frac{d}{2} \log(d-1)}.
\]
Remark 1. This is a quite surprising result: it shows that most of random polynomials of reasonable degree have only saddle points.

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2 The space of \( n \)-variate polynomials

The inner product space \( \mathcal{P}, \langle \cdot, \cdot \rangle_{\mathcal{P}} \) has several interesting properties resumed in the following

**Lemma 1.** 1. It admits the reproducing kernel \( K(z, x) = (1 + \langle z, x \rangle)^d \):

\[
f(x) = \langle K(., x), f \rangle_{\mathcal{P}}
\]

(1)

for any \( x \in \mathbb{R}^n \) and \( f \in \mathcal{P} \).

2. It has a representation formula for the derivatives: for any integer \( k \geq 1 \) and \( x, u_1, \cdots, u_k \in \mathbb{R}^n \) we have

\[
D^k f(x)(u_1, \cdots, u_k) = \langle K_k(., x, u_1, \cdots, u_k), f \rangle_{\mathcal{P}},
\]

(2)

with

\[
K_k(z, x, u_1, \cdots, u_k) = D_x^k K(z, x)(u_1, \cdots, u_k) = d \cdots (d - k + 1) \langle z, u_1 \rangle \cdots \langle z, u_k \rangle (1 + \langle z, x \rangle)^{d-k}.
\]

3. This scalar product is orthogonally invariant:

\[
\langle f \circ U, g \circ U \rangle_{\mathcal{P}} = \langle f, g \rangle_{\mathcal{P}}
\]

(4)

for any \( f, g \in \mathcal{P} \) and the orthogonal transformation \( U \in O_n \).

**Proof.** The first two formulas are well known and easily obtained via a direct computation. For the orthogonal invariance see [4], section 12.1, or [11].

A second interest of Weyl’s distribution for polynomials is due to the following identity: let \( f(x) = x^T S x \) (here \( S \) is a symmetric \( n \times n \) matrix) be a homogeneous degree 2 polynomial, then \( \|f\|_\mathcal{P} = \|S\|. \) This is the reason why

**Proposition 1.** \( C_{2,n} = 1 \) and \( E_{2,n} = \mathcal{P}_n \).

**Proof.** Since a generic degree 2 polynomial has only one critical point we have \( C_{2,n} = 1 \). Given \( f \in \mathcal{P}_{2,n} \) we can write it

\[
f(x) = \alpha + \sum_{1 \leq i \leq n} b_i x_i + \sum_{1 \leq i \leq n} a_{ii} x_i^2 + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j.
\]

One has

\[
\|f\|_{\mathcal{P}}^2 = \alpha^2 + \frac{1}{2} \sum_{1 \leq i \leq n} b_i^2 + \sum_{1 \leq i \leq n} a_{ii}^2 + \frac{1}{2} \sum_{1 \leq i < j \leq n} a_{ij}^2
\]
so that
\[ E_{2,n} = \int_{D^2 f(0) > 0} \frac{e^{-\|f\|^2/2}}{2^{n(n+1)/4}(2\pi)^{(n+1)(n+2)/4}} \, d\omega d\alpha d\beta = \int_{D^2 f(0) > 0} e^{-\left(\sum_i a_i^2 + \frac{1}{2} \sum_{i<j} a_{ij}^2 \right)/2} \, da. \]

To compute this last integral we let \( S = \frac{1}{2} D^2 f(0); \) this gives
\[ E_{2,n} = \int_{S > 0} \frac{e^{-\|S\|^2/2}}{2^{n(n-1)/4}(2\pi)^{n(n+1)/4}} 2^{n(n-1)/2} dS = P_n. \]

\[ \square \]

### 3 An integral formulation

Let us define
\[ \text{eval}_1 : \mathcal{P} \times \mathbb{R}^n \to \mathbb{R}^n, \text{eval}_1(f, x) = Df(x). \]

The incidence variety for real critical points of a polynomial is defined by
\[ V = \{(f, x) \in \mathcal{P} \times \mathbb{R}^n : \text{eval}_1(f, x) = 0 \}. \]

The derivative of \( \text{eval}_1 \) is given by
\[ D\text{eval}_1(f, x)(\dot{f}, \dot{x}) = D\dot{f}(x) + D^2 f(x) \dot{x} \]
for any \( f, \dot{f} \in \mathcal{P} \) and \( x, \dot{x} \in \mathbb{R}^n \). Since this derivative is onto, \( V \) is a submanifold and its dimension is
\[ \dim V = \dim \mathcal{P} = \binom{n+d}{d}. \]

The tangent space at \((f, x) \in V\) is given by
\[ T_{(f,x)} V = \ker D\text{eval}_1(f, x) = \left\{ (\dot{f}, \dot{x}) \in \mathcal{P} \times \mathbb{R}^n : D\dot{f}(x) + D^2 f(x) \dot{x} = 0 \right\}. \]

The restriction \( \pi_2 : V \to \mathbb{R}^n \) of the projection \( \mathcal{P} \times \mathbb{R}^n \to \mathbb{R}^n \) is surjective and is also a regular map because for any \( (f, x) \in V \) the derivative \( D\pi_2(f, x) : T_{(f,x)} V \to \mathbb{R}^n \) is surjective. The fiber of \( \pi_2 \) above \( x \in \mathbb{R}^n \)
\[ V_x = \{(f, x) \in \mathcal{P} \times \mathbb{R}^n : \text{eval}_1(f, x) = 0\} \]
is isomorphic to a \( \dim \mathcal{P} - n \) linear space. \( V_x \) is equipped with the volume form inherited from the induced metric.

The restriction \( \pi_1 : V \to \mathcal{P} \) of the projection \( \mathcal{P} \times \mathbb{R}^n \to \mathcal{P} \) is a smooth map. A given \( f \in \mathcal{P} \) is a regular value of \( \pi_1 \) when either \( f \) has no critical point or when, for any \( x \) such that \((f, x) \in V\), \( D\pi_1(f, x) : T_{(f,x)} V \to \mathcal{P} \) is surjective. This
last condition is satisfied when the second derivative $D^2f(x)$ is an isomorphism which is the generic situation:

$$\Sigma' = \{(f, x) \in V : \det D^2f(x) = 0\}$$

is a submanifold in $V$ and $\dim \Sigma' < \dim V$. Thus $\Sigma'$ and its image $\Sigma = \pi_1(\Sigma')$ have zero measure and we may ignore them. For any $(f, x) \in V \setminus \Sigma'$ and any $\dot{f} \in \mathcal{P}$ we have $D\pi_1(f, x)(\dot{f}, \dot{x}) = \dot{f}$ for $\dot{x} = -D^2f(x)^{-1}D\dot{f}(x)$ and the fiber above $f$

$$V_f = \{(f, x) \in \mathcal{P} \times \mathbb{R}^n : \text{eval}_1(f, x) = 0\}$$

consists in a finite number of points.

Given $(f, x) \in V \setminus \Sigma'$ we are in the context of the implicit function theorem that is $V$ is locally around $(f, x)$ the graph of the function

$$G = \pi_2 \circ \pi_1^{-1}$$

where $\pi_1^{-1}$ is the local inverse of $\pi_1$ such that $\pi_1^{-1}(f) = (f, x)$. Since the graph of $DG(f)$ is the tangent space $T_{(f, x)}V$ we get

$$DG(f)\dot{f} = -D^2f(x)^{-1}D\dot{f}(x) \quad (5)$$

for any $\dot{f} \in \mathcal{P}$.

Like in [4] section 13.2, theorem 3, we have the following

**Proposition 2.** Let $U$ be a measurable subset of $V$. Let us denote by $\#(f, U)$ the number of pairs $(f, x) \in U$ and by $E_U$ the expectation of $\#(f, U)$ when $f$ is taken at random:

$$E_U = \int_{\mathcal{P}} \#(f, U) \frac{e^{-\|f\|_p^2/2}}{(2\pi)^{\dim \mathcal{P}/2}} d\mathcal{P}. \quad (6)$$

With these notations, one has

$$E_U \int_{\mathbb{R}^n} dx \int_{V_f \cap U} \det(DG(f)DG(f)^*)^{-1/2} \frac{e^{-\|f\|_p^2/2}}{(2\pi)^{\dim \mathcal{P}/2}} dV_x. \quad (7)$$

**Remark 2.** In our context two sets are of particular interest: $U = V$ to compute the average number of critical points of a polynomial $C_{d,n}$, and $U = V_+$ with

$$V_+ = \{(f, x) \in \mathcal{P} \times \mathbb{R}^n : Df(x) = 0 \text{ and } D^2f(x) > 0\}$$

(here $> 0$ means positive definite) for the average number of local minima $E_{d,n}$.

We have now to compute the determinant appearing in equation (7). This is done in the following

**Proposition 3.** Under the notations above

$$\det(DG(f)DG(f)^*) = d^n(1 + \|x\|^2)^{n(d-1)-1}(1 + d\|x\|^2) \left|\det D^2f(x)\right|^{-2}. \quad (8)$$
Proof. Let us denote $D\dot{x}(x) = D_x \dot{f}$. Since $DG(f)\dot{f} = -D^2f(x)^{-1}D_x \dot{f}$ and since $D^2f(x)$ is symmetric, we get

$$DG(f)DG(f)^* = D^2f(x)^{-1}D_x D_x^* D^2f(x)^{-1}$$

so that

$$\det(DG(f)DG(f)^*) = \det(D_x D_x^*) \det(D^2f(x))^{-2}.$$  \hfill (9)

To compute $\det(D_x D_x^*)$ we use the representation formula for the derivative (equation 2) with $k = 1$. Let us denote by $e_i$, $1 \leq i \leq n$, the canonical basis in $\mathbb{R}^n$. Then, for any $\dot{f} \in P$,

$$D_x \dot{f} = \sum_i e_i \left\langle K_1(.,x,e_i),\dot{f} \right\rangle_p$$

so that, with $\dot{x} \in \mathbb{R}^n$, $\dot{x} = \sum \dot{x}_i e_i$,

$$\left\langle D_x^* \dot{x}, \dot{f} \right\rangle_p = \left\langle \dot{x}, D_x \dot{f} \right\rangle = \sum_i \dot{x}_i \left\langle K_1(.,x,e_i),\dot{f} \right\rangle_p.$$ 

Thus, we get

$$D_x^* \dot{x} = \sum_i \dot{x}_i K_1(.,x,e_i)$$

and consequently

$$D_x D_x^* \dot{x} = \sum_i e_i \left\langle K_1(.,x,e_i), \sum_j \dot{x}_j K_1(.,x,e_j) \right\rangle_p =$$

$$\sum_i e_i \frac{\partial}{\partial \dot{z}_i} \left( \sum_j \dot{x}_j d \langle \dot{z}, e_j \rangle (1 + \langle \dot{z}, x \rangle)^{d-1} \right) \bigg|_{\dot{z}=x} =$$

$$\sum_{i,j} e_i \dot{x}_j \times \left\{ \begin{array}{ll} d(d-1)x_i x_j (1 + \|x\|^2)^{d-2} & \text{if } i \neq j \\ d(d-1)x_i^2 (1 + \|x\|^2)^{d-2} + d(1 + \|x\|^2)^{d-1} & \text{if } i = j \end{array} \right.\right.$$ 

which correspond to the matrix

$$d(d-1)(1 + \|x\|^2)^{d-2}xx^T + d(1 + \|x\|^2)^{d-1}I_n.$$ 

Its eigenvectors are $x$ and any nonzero vector in the orthogonal subspace $x^\perp$.

The corresponding eigenvalues are

$$d(d-1)(1 + \|x\|^2)^{d-2} \|x\|^2 + d(1 + \|x\|^2)^{d-1} = d(1 + \|x\|^2)^{d-2}(1 + d \|x\|^2)$$

with multiplicity 1, and

$$d(1 + \|x\|^2)^{d-1}$$

with multiplicity $n - 1$ so that

$$\det D_x D_x^* = d^n (1 + \|x\|^2)^{n(d-1)-1}(1 + d \|x\|^2).$$

Our proposition combines this value and equation 9. \hfill \square
If we combine propositions 2 and 3 we obtain the following integral formulation

**Proposition 4.** Let $U$ be a measurable subset of $V$. One has

$$E_U = \int_{\mathbb{R}^n} \int_{V \cap U} dx \frac{|\det D^2 f(x)|}{d^{n/2}(1 + \|x\|^2)^{(n(d-1)-1)/2}(1 + d \|x\|^2)^{1/2}} e^{-\|\Psi(S)\|^2/2} (2\pi)^{\dim S_n/2} dS,$$

where $\alpha_n = \frac{\text{Vol} S_{n-1}}{(2\pi)^{n/2}} \frac{2^{n/2}}{\sqrt{n!}}$, $R = \sqrt{r^2 + 1}$ and $r e_T^1 = (r, 0, \ldots, 0)$.

**Remark 3.** The measurable sets considered here: $U = V$ and $U = V_+ = \{(f, x) \in V : D^2 f(x) > 0\}$, are clearly invariant under the action of $\mathcal{O}_n$.

### 4 The inner integral

Our objective is now to compute the integral over $V_{r e_1} \cap U$ appearing in proposition 4. Let $D^2 : V_{r e_1} \to S_n$ denote the operator $f \mapsto D^2 f(r e_1)$. We would like to compute its pseudo-inverse $\Psi : S_n \to (\ker D^2)^\perp$. This means that $\Psi$ is the minimum norm right inverse of $D^2 (D^2 \circ \Psi = \text{id}_{S_n})$.

This will allow us to ‘integrate out’ $\ker D^2$:

$$\int_{V_{r e_1} \cap U} |\det D^2 f| e^{-\|\Psi(S)\|^2/2} (2\pi)^{\dim S_n/2} dS.$$

To compute $\Psi(S)$ and $|\det \Psi^* \Psi|$ we need the following lemma:

**Lemma 2.** Let us denote
• $e_i, 1 \leq i \leq n$, the canonical basis in $\mathbb{R}^n$,

• $\partial_{e_i} = K_1(z, re_1, e_i)$,

• $\partial_{e_ie_j} = K_2(z, re_1, e_i, e_j)$,

• $R = \sqrt{1 + r^2}$.

Then,

1. $\langle \partial_{e_i}, \partial_{e_i} \rangle_p d(1 + dr^2)R^{2d-4}$
2. If $i \neq 1$, then $\langle \partial_{e_i}, \partial_{e_i} \rangle_p = dR^{2d-2}$
3. If $i \neq j$, then $\langle \partial_{e_i}, \partial_{e_j} \rangle_p = 0$
4. $\langle \partial_{e_i}, \partial_{e_ie_i} \rangle_p = d(d - 1)(dr^2 + 2)rR^{2d-6}$
5. If $(i, j, k) \neq (1, 1, 1)$, then $\langle \partial_{e_i}, \partial_{e_ie_k} \rangle_p = 0$
6. $\langle \partial_{e_ie_1}, \partial_{e_ie_1} \rangle_p = d(d - 1)(d(d - 1)r^4 + 4(d - 1)r^2 + 2)R^{2d-8}$
7. If $k \neq 1$, then $\langle \partial_{e_ie_k}, \partial_{e_ie_k} \rangle_p = d(d - 1)((d - 1)r^2 + 1)R^{2d-6}$
8. If $i \neq 1$ and $k \neq 1$, then $\langle \partial_{e_ie_k}, \partial_{e_ie_k} \rangle_p = (1 + \delta_{ik})d(d - 1)R^{2d-4}$ ($\delta_{ik}$ is the Kronecker symbol),

9. If $\{i, k\} \neq \{j, l\}$, then $\langle \partial_{e_ie_k}, \partial_{ejej} \rangle_p = 0$

Proof. It is a consequence of the representation formulas given in lemma [1].

• $\langle \partial_{e_i}, \partial_{e_i} \rangle_p = \langle K_1(z, re_1, e_i), K_1(z, re_1, e_i) \rangle_p = \frac{\partial}{\partial z_i}K_1(z, re_1, e_i)|_{z=re_1} = \frac{\partial}{\partial z_i}dz_i(1+rz_1)^{d-1}|_{z=re_1} = d(1+r^2)^{d-2}(1+dr^2)$,

and similarly

• $\langle \partial_{e_i}, \partial_{e_i} \rangle_p = \frac{\partial}{\partial z_i}K_1(z, re_1, e_i)|_{z=re_1} = \frac{\partial}{\partial z_i}dz_i(1+rz_1)^{d-1}|_{z=re_1} = d(1+r^2)^{d-2}$,

• $\langle \partial_{e_i}, \partial_{e_j} \rangle_p = \frac{\partial}{\partial z_i}K_1(z, re_1, e_j)|_{z=re_1} = \frac{\partial}{\partial z_i}dz_j(1+rz_1)^{d-1}|_{z=re_1} = 0$ when $i \neq j$,

• $\langle \partial_{e_i}, \partial_{e_ie_1} \rangle_p = \frac{\partial}{\partial z_i}K_2(z, re_1, e_i, e_1)|_{z=re_1} = \frac{\partial}{\partial z_i}d(d-1)z_i^2(1+rz_1)^{d-2}|_{z=re_1} = d(d-1)r(2+dr^2)(1+r^2)^{d-3}$,

• $\langle \partial_{e_j}, \partial_{e_e_k} \rangle_p = \frac{\partial}{\partial z_i}d(d-1)z_i z_k(1+rz_1)^{d-2}|_{z=re_1} = 0$ when $(i, j, k) \neq (1, 1, 1)$,

• $\langle \partial_{e_ie_1}, \partial_{e_ie_1} \rangle_p = \frac{\partial^2}{\partial z_i^2}d(d-1)z_i^2(1+rz_1)^{d-2}|_{z=re_1} = d(d-1)(1+r^2)^{d-4}(2+4(d-1)r^2+d(d-1)r^4)$,
\[
\langle \partial_{e_1 e_k}, \partial_{e_1 e_k} \rangle_P = \frac{\partial^2}{\partial z_k^2} d(d-1)z_k(1 + rz_1)^{d-2} \big|_{z = r e_1} = d(d-1)(1 + r^2)^{d-3}(1 + (d-1)r^2),
\]
\[
\langle \partial_{e_1 e_k}, \partial_{e_1 e_k} \rangle_P = \frac{\partial^2}{\partial z_k^2} d(d-1)z_k(1 + rz_1)^{d-2} \big|_{z = r e_1} = (1 + \delta_{ik})d(d-1)(1 + r^2)^{d-2},
\]
\[
\langle \partial_{e_1 e_k}, \partial_{e_j e_i} \rangle_P = \frac{\partial^2}{\partial z_k^2} d(d-1)z_jz_k(1 + rz_1)^{d-2} \big|_{z = r e_1} = 0 \text{ when } \{i, k\} \neq \{j, l\}.
\]

Let us now evaluate \( \Psi \). Recall that

\[
V_{re_1} = \{ f \in P : Df(re_1) = 0 \}
\]

or, in other words, \( f \in V_{re_1} \) if and only if

\[
\langle f, \partial_{e_i} \rangle_P = 0, \quad 1 \leq i \leq n.
\]

Thus, by lemma \( \text{2-3} \), \( \partial_{e_i}, \quad 1 \leq i \leq n \), constitute an orthogonal basis of \( V_{re_1} \). We also have

\[
\ker D^2 = \text{Span} \{ \partial_{e_i e_j}, \quad 1 \leq i \leq j \leq n \}^\perp \cap V_{re_1}
\]

hence,

\[
(\ker D^2)^\perp = \text{Span} \{ P\partial_{e_i e_j}, \quad 1 \leq i \leq j \leq n \}
\]

where \( P \) stands for the orthogonal projection onto \( V_{re_1} \). We have seen that for \( (i, j, k) \neq (1, 1, 1) \), \( \partial_{e_i e_j} \perp \partial_{e_k} \) (lemma \( \text{2-5} \)). Hence,

\[
P\partial_{e_i e_j} \partial_{e_1 e_1} = \partial_{e_1} \frac{\langle \partial_{e_1 e_j}, \partial_{e_1} \rangle_P}{\|\partial_{e_1}\|_P^2}
\]

and for \( (i, j) \neq (1, 1) \),

\[
P\partial_{e_i e_j} = \partial_{e_i e_j}.
\]

Let us now show that

\[
\Psi(S) = \sum_{1 \leq i \leq j \leq n} S_{ij} \frac{P\partial_{e_i e_j}}{\|P\partial_{e_i e_j}\|_P^2}.
\]

Since this expression is clearly in \( (\ker D^2)^\perp \) it suffices to prove that \( D^2 \circ \Psi(S) = S \) for any \( S \in S_n \) i.e.

\[
D^2 \Psi(S)(r e_1)(e_k, e_l) = S_{kl}
\]

or, using lemma \( \text{1} \) that

\[
\left\langle \partial_{e_k e_l}, \sum_{1 \leq i \leq j \leq n} S_{ij} \frac{P\partial_{e_i e_j}}{\|P\partial_{e_i e_j}\|_P^2} \right\rangle_P = S_{kl}.
\]

This last equality holds because \( P\partial_{e_i e_j}, \quad 1 \leq i \leq j \leq n \), constitute an orthogonal basis of \( (\ker D^2)^\perp \).
It is important to have in mind that $\Psi$ is not an isometry, we have

$$
\|\Psi(S)\|_P^2 = \sum_{1 \leq i \leq j \leq n} \frac{S_{ij}^2}{\|P\partial_{e_i}e_j\|_P^2}
$$

We introduce now the functions

$$
A(d, r) = \sqrt{d(d-1)r^4 + 2dr^2 + 2} \quad \text{and} \quad B(d, r) = \sqrt{(d-1)r^2 + 1},
$$

where again $R = \sqrt{1 + r^2}$.

**Lemma 3.** Let $i \leq j$. Then,

$$
\|P\partial_{e_i}e_j\|_P^2 d(d-1)R^{2d-4} \times \begin{cases} A(d, r)^2 & \text{if } i = 1 \text{ and } j = 1 \\
B(d, r)^2 & \text{if } i = 1 \text{ and } j \neq 1 \\
(1 + \delta_{ij}) & \text{if } i \neq 1 \text{ and } j \neq 1
\end{cases}
$$

with $\delta_{ij} = 1$ when $i = j$ and 0 otherwise.

Let us now compute $\det \Psi^*\Psi$. For any $f = \sum_{1 \leq i \leq j \leq n} f_{ij} P\partial_{e_i}e_j \in (\ker D)^\perp$ and for any $S \in S_n$ we have

$$
\langle \Psi^*(f), S \rangle = \langle f, \Psi(S) \rangle = \sum_{1 \leq i \leq j \leq n} f_{ij} S_{ij}
$$

Therefore, we have always for any $T \in S_n$:

$$
\langle \Psi^*\Psi(T), S \rangle = \sum_{1 \leq i \leq j \leq n} \frac{T_{ij}S_{ij}}{\|P\partial_{e_i}e_j\|_P^2}
$$

We write the matrix of the operator $\Psi^*\Psi$ with respect to the orthonormal basis of $S$ given by $e_1e_1^T, \ldots, e_ne_n^T$ and then, for $i < j$, $\frac{1}{\sqrt{2}} (e_ie_j^T + e_je_i^T))$:

$$
\Psi^*\Psi = \begin{bmatrix}
\frac{1}{\|P\partial_{e_1}e_1\|^2} & \cdots & \\
& \ddots & \\
& \cdots & \frac{1}{\|P\partial_{e_n}e_n\|^2}
\end{bmatrix}
$$

Using lemma\[2\] we obtain:
Lemma 4.

\[(\det \Psi^*\Psi)^{1/2} = 2^{\frac{n(n+1)}{4}} (d(d-1)R^{2d-4})^{-\frac{n(n+1)}{4}} A(d, r)^{-1} B(d, r)^{-(n-1)}.\]

At this point

**Proposition 6.** Under the conditions above,

\[
E_U \frac{\alpha_n}{d^n/2} \int_0^\infty \frac{(\det \Psi^*\Psi)^{\frac{1}{2}} r^{n-1} dr}{(dr^2 + 1)^{1/2} R^{(d-1)n-1}} \int_{D^{2(U\cap V_{rel})}} \frac{|\det S|}{(2\pi)^{dim S_n/2}} e^{-\|\Psi(S)\|^2_{\ell^n/2}} dS_n.
\]

In particular,

\[
C_{d,n} \frac{\alpha_n}{d^n/2} \int_0^\infty \frac{(\det \Psi^*\Psi)^{\frac{1}{2}} r^{n-1} dr}{(dr^2 + 1)^{1/2} R^{(d-1)n-1}} \int_{S_n^+} \frac{|\det S|}{(2\pi)^{dim S_n/2}} e^{-\|\Psi(S)\|^2_{\ell^n/2}} dS_n,
\]

and

\[
E_{d,n} \frac{\alpha_n}{d^n/2} \int_0^\infty \frac{(\det \Psi^*\Psi)^{\frac{1}{2}} r^{n-1} dr}{(dr^2 + 1)^{1/2} R^{(d-1)n-1}} \int_{S_n^+} \frac{|\det S|}{(2\pi)^{dim S_n/2}} e^{-\|\Psi(S)\|^2_{\ell^n/2}} dS_n,
\]

where \(S_n^+\) denotes the set of positive definite matrices. When \(n = 1\),

\[
C_{d,1} = 2E_{d,1} = \frac{2\sqrt{d-1}}{\pi} \int_0^\infty \frac{\sqrt{d(d-1)r^4 + 2dr^2 + 2}}{(dr^2 + 1)(r^2 + 1)} dr.
\]

**Proof.** The three first formulas are obtained in combining proposition 6 equation 11 and lemma 4. For the case \(n = 1\) we obtain

\[
E_{d,1} = \frac{2\sqrt{d-1}}{\pi} \int_0^\infty \frac{dr}{A(dr^2 + 1)^{1/2} R^{2d-4}} \int_0^\infty \frac{s e^{-\frac{2s^2 - 4s}{d(d-1)R^{2d-4} + 4s} + s}}{\sqrt{2\pi}} ds =
\]

\[
\frac{\sqrt{d-1}}{\pi} \int_0^\infty \frac{\sqrt{d(d-1)r^4 + 2dr^2 + 2}}{(dr^2 + 1)(r^2 + 1)} dr.
\]

The identity \(C_{d,1} = 2E_{d,1}\) is easy. \(\square\)

5 Some integral lemmas

The term \(e^{-\|\Psi(S)\|^2_{\ell^n/2}}\) in the inner integrals of Proposition 6 can be simplified through additional changes of coordinates. We reparametrize the spaces \(S_n\) and \(S_n^+\) though a stretching \(S \mapsto T = \Delta^{-1} S \Delta^{-1}\).

The stretching coefficients are \(\Delta_i = (2d(d-1)R^{2d-4})^{-1/4}\) for \(i \geq 2\), \(\Delta_1 = B(d, r)\Delta_2\) and, \(\Delta = \text{Diag}(\Delta_1, \Delta_2, \ldots, \Delta_n)\). We obtain

\[
\|\Psi(S)\|_{\ell^n/2} = \frac{1}{d(d-1)R^{2d-4}} \left( \frac{S_{11}^2}{A^2} + \sum_{j=2}^n \frac{S_{jj}^2}{B_j^2} + \sum_{1 \leq i \leq j \leq n} \frac{1}{1 + \delta_{ij}} S_{ij}^2 \right)
\]
\[ \|\Delta^{-1} S \Delta^{-1}\|^2 = \frac{1}{d(d-1)R^{2d-4}} \left( \frac{S_{11}^2}{2B^4} + \sum_{j=2}^{n} \frac{S_{1j}^2}{B^2} + \sum_{1 < i, j \leq n} \frac{1}{1 + \delta_{ij}} S_{ij}^2 \right) \]

so that

\[ \|\Psi(S)\|^2_P = \|\Delta^{-1} S \Delta^{-1}\|^2 + \left( \frac{1}{A^2} - \frac{1}{2B^4} \right) \frac{S_{11}^2}{d(d-1)R^{2d-4}}. \]

Let us define \( T = \Delta^{-1} S \Delta^{-1} \). We get

\[ \|\Psi(S)\|^2_P = \|T\|^2 + \left( \frac{2B^4}{A^2} - 1 \right) T_{11}^2 \]

so that, via this change of variable,

\[ \int_{D^2(U)} \frac{|\det S|}{\sqrt{2\pi} \lim S_n} e^{-\|\Psi(S)\|^2/2} dS = \left( \prod_{i=1}^{n} \Delta_i \right)^{n+3} \int_{D^2(U \cap V_r)} \frac{|\det T|}{\sqrt{2\pi} \lim S_n} e^{-\frac{1}{2} \left( \|T\|^2 + \left( \frac{2B^4}{A(d,r)^2} - 1 \right) T_{11}^2 \right)} dT. \]

If \( U \subset V \), we define the auxiliary quantity

\[ C_U(d, r, n) = \int_{D^2(U \cap V_r)} \frac{|\det T|}{\sqrt{2\pi} \lim S_n} e^{-\frac{1}{2} \left( \|T\|^2 + \left( \frac{2B^4}{A(d,r)^2} - 1 \right) T_{11}^2 \right)} dT. \]

There are two cases of interest corresponding to \( U = V \) for the average of critical points and \( U = V_+ \) for the average number of local minima. The corresponding functions are denoted \( C_V(d, r, n) \) and \( C_{V_+}(d, r, n) \). Using proposition 6 we get (the proof is easy and left to the reader)

**Proposition 7.**

\[ E_U = \frac{2\sqrt{2}(d-1)^{n/2}}{\Gamma(n/2)} \int_{0}^{\infty} \frac{r^{n-1}}{R^2 \sqrt{d(d-1)r^4 + 2dr^2 + 2Rn-1}} C_U(d, r, n) dr \]

Moreover

\[ C_V(d, r, n) = \int_{S_n} \frac{|\det T|}{\sqrt{2\pi} \lim S_n} e^{-\frac{1}{2} \left( \|T\|^2 + \left( \frac{2B^4}{A(d,r)^2} - 1 \right) T_{11}^2 \right)} dT \]

and

\[ C_{V_+}(d, r, n) = \int_{S_n^+} \frac{|\det T|}{\sqrt{2\pi} \lim S_n} \frac{d}{\sqrt{2\pi} \lim S_n} e^{-\frac{1}{2} \left( \|T\|^2 + \left( \frac{2B^4}{A(d,r)^2} - 1 \right) T_{11}^2 \right)} dT. \]
6 Prove of Theorem 1

To prove our main theorem we use both the proposition 7 and the case $d = 2$ already investigated in the proposition 1. We have

$$1 = C_{2,n} = \frac{2\sqrt{2}}{\Gamma(n/2)} \int_0^\infty \frac{r^{n-1} C_V(2, r, n)}{\sqrt{2}} dr$$

and

$$C_V(2, r, n) = \int_{S_n} \frac{|\det T|}{(2\pi)^{n(n+1)/4}} e^{-\frac{1}{2}(\|T\|^2 + 2r^2 T_n^2)} dT.$$ 

Lemma 5. The quantity $\Lambda(d, r) = \frac{2 B(d, r)^2}{A(d, r)^2}$ satisfies, for all $r > 0$ and $d \geq 2$, the scaling law:

$$\Lambda(2, r\sqrt{d-1}) \leq \Lambda(d, r) \leq \Lambda(2, \frac{\sqrt{5}}{2} r\sqrt{d-1})$$

Proof. We write

$$\Lambda(d, r) = 2(d-1)r^2 + \frac{d-2}{d} \frac{(d-1)r^4}{(d-1)r^4 + 2r^2 + \frac{2}{d}}.$$

The lower bound is now obvious. The upper bound is obtained as follows:

$$\Lambda(d, r) = 2(d-1)r^2 + \frac{d-2}{d} \frac{(d-1)r^4}{(d-1)r^4 + 2r^2 + \frac{2}{d}} \leq \Lambda(d, r) = 2(d-1)r^2 + \frac{d-2}{2d} (d-1)r^2 \leq \frac{5}{4} \Lambda(2, r\sqrt{d-1}) = \Lambda(2, \frac{\sqrt{5}}{2} r\sqrt{d-1}).$$

It follows from Lemma 5 that

$$C_V(d, r, n) = \int_{S_n} \frac{|\det T|}{\sqrt{2\pi} \dim S_n} e^{-\frac{1}{2}(\|T\|^2 + \Lambda(d, r) T_n^2)} \leq \int_{S_n} \frac{|\det T|}{\sqrt{2\pi} \dim S_n} e^{-\frac{1}{2}(\|T\|^2 + \Lambda(2, r\sqrt{d-1}) T_n^2)} = C_V(2, r\sqrt{d-1}, n)$$

and similarly $C_{V+}(d, r, n) \leq C_{V+}(2, r\sqrt{d-1}, n)$. Now we have:

$$C_{d,n} = \frac{2\sqrt{2}(d-1)^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{((d-1)r^2 + 1)^2}{R^2 \sqrt{d(d-1)r^4 + 2dr^2 + 2 R^{n-1} C_V(d, r, n)} dr$$

$$\leq \frac{2\sqrt{2}(d-1)^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{((d-1)r^2 + 1)^2}{R^2 \sqrt{d(d-1)r^4 + 2dr^2 + 2 R^{n-1} C_V(2, r\sqrt{d-1}, n)} dr.$$
We set $s = r\sqrt{d-1}$ and $S = \sqrt{d-1 + s^2}$ to obtain:

$$C_{d,n} \leq \frac{2\sqrt{2} (d-1)^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{(d-1)s^2}{S^2 \sqrt{\frac{d}{d-1}s^4 + 2 \frac{d}{d-1}s^2 + 2}} \frac{s^{n-1}}{S^{n-1}} C_V(2, s, n) \frac{1}{\sqrt{d-1}} \, ds.$$ 

Since

$$\frac{(d-1)s^2}{S^2 \sqrt{\frac{d}{d-1}s^4 + 2 \frac{d}{d-1}s^2 + 2}} \leq \sqrt{\frac{d-1}{d}}$$

and $C_{2,n} = 1$ we obtain

$$C_{d,n} \leq \frac{(d-1)^{(n+2)/2}}{\sqrt{d}}$$

and the same argument holds for $E_{d,n}$.

### 7 The Riemann surface

We rewrite the case $n = 1$ (proposition 6) for convenience as:

$$E_{d,1} = \frac{(d-1)\sqrt{d}}{2\pi} \int_{\mathbb{R}} g(z)dz$$

with

$$g(z) = \sqrt{\frac{z^4 + \frac{2}{d-1}z^2 + \frac{2}{d(d-1)}}{(1+z^2)(1+dz^2)}}$$

At this point we encounter a classical situation: we want to compute a line integral of a function $g(z)$, which is a two-branched meromorphic function of $\mathbb{C}$. In order to apply the residue theorem, we need first to replace $g$ by a regular meromorphic function, defined in the relevant Riemann surface $R$. The branching points of the Riemann surface are the roots of the polynomial inside the square root. If we set

$$\zeta = \sqrt{-1 + i \sqrt{1 - \frac{2}{d}}}$$

with the branch of the external square root in such a way that $\zeta$ belongs to the positive quadrant, we can now factorize

$$z^4 + \frac{2}{d-1}z^2 + \frac{2}{d(d-1)}(z - \zeta)(z - \bar{\zeta})(z + \zeta)(z + \bar{\zeta}).$$

It follows that the Riemann Surface $R$ is a twofold cover of $\mathbb{C}$ with branch points $\zeta$, $-\zeta$, $-\zeta$, $\zeta$. 

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Let $\gamma$ be the arc of circle (centered in the origin) joining $-\bar{\zeta}$ to $\zeta$ crossing the positive imaginary axis. Notice that it crosses the segment $[i/\sqrt{d}, i]$. Let $\mathcal{D}$ denote the upper half plane with $\gamma$ removed.

Then, the positive branch of $\sqrt{z^4 + \frac{2}{d-1} z^2 + \frac{2}{d(d-1)}}$ on $\mathbb{R}$ extends to a unique branch on $\mathcal{D}$. The square root is real and positive on $[0, i|\zeta|]$ and real and negative on $[i|\zeta|, i\infty)$.

The residue theorem is now:

$$\int_{\mathcal{R}} g(z)dz - 2 \int_{\gamma} g(z)dz 2\pi i \text{Res}_{z=i/\sqrt{d}} g(z) + 2\pi i \text{Res}_{z=i\infty} g(z)$$

Residues are respectively $-\frac{i}{2(d-1)\sqrt{d}}$ and $\frac{i\sqrt{d-2}}{2(d-1)\sqrt{d}}$. Therefore,

$$E_{d,1} = \frac{1}{2} + \frac{\sqrt{d-2}}{2} + \frac{(d-1)\sqrt{d}}{\pi} \int_{\gamma} g(z)dz$$

(We mean the integral of the branch that is positive on $i|\zeta|$).

Now, in order to integrate $g(z)$, we introduce a linear fractional transformation mapping the real line onto the circle containing $\gamma$. Namely,

$$\Psi(w) = \frac{Aw + B}{Cw + D}$$
with \( A = |\zeta| \), \( B = i|\zeta| \), \( C = i \), \( D = 1 \). For the record, \( AD - BC = 2|\zeta| \)

Let \( s = \frac{\text{Re}(\zeta)}{|\zeta|+\text{Im}(\zeta)} \). Define also \( s_1 = \frac{1-|\zeta|}{1+|\zeta|} \), \( s_2 = s_1^{-1} \), \( s_3 = \frac{1-|\zeta|\sqrt{d}}{1+|\zeta|\sqrt{d}} \) and \( s_4 = s_3^{-1} \). We have the following mapping table for \( \Psi \):

| \( w \) | \( \Psi(w) \) | \( w \) | \( \Psi(w) \) |
|-------|-------|-------|-------|
| -1    | -|\zeta|    | \( i s_1 \) | \( i \) |
| 0     | \( i|\zeta| \) | \( i s_2 \) | -i |
| 1     | \( |\zeta| \) | \( i s_3 \) | \( i/\sqrt{d} \) |
| -s^{-1}| -|\zeta|    | \( i s_4 \) | -i/\sqrt{d} |
| -s    | -|\zeta|    | s_5 |
| s     | \( \zeta \) | s_6 |

Changing coordinates,

\[
\int_{\gamma} g(z) dz = 2c(d) \text{Re} \int_{[0,s]} \frac{\sqrt{(w^2 - s^2)(w^2 - s^{-2})}}{\Pi_{k=1}^{4}(w - is_k)} dw
\]

with

\[
c(d) = \frac{(AD - BC)\sqrt{A^4 + \frac{2}{A-1}A^2C^2 + \frac{2}{A(d-1)}C^4}}{(A^2 + C^2)(dA^2 + C^2)} \in O(d^{-3/2})
\]

(More precisely: \( \lim d^{3/2}C(d) = -2^{7/4}\sqrt{\frac{2-\sqrt{2}}{\sqrt{2}+1}} \approx -6.2151 \).)

At this point, good practice seems to be:

1. Multiply numerator and denominator by the conjugate of the denominator, in order to obtain a real polynomial in the denominator.
2. Multiply numerator and denominator by the square root.
3. Expand in partial fractions.
4. Put into Legendre normal form.

5. Write down the integral in terms of elliptic functions \( K \) and II.

We expand the integrand in partial fractions:

\[
\int_{\gamma} g(z) dz = 2c(d) \int_{[0,s]} \frac{1}{\sqrt{(w^2 - s^2)(w^2 - s^{-2})}} \left( 1 + \sum_{k=1}^{4} \text{Re} \frac{R_k}{w - is_k} \right) dw
\]

\[
= 2c(d) \int_{[0,s]} \frac{1}{\sqrt{(w^2 - s^2)(w^2 - s^{-2})}} \left( 1 + \sum_{k=1}^{4} s_k^{-2} \text{Re} \frac{R_k(w + is_k)}{1 + w^2s_k^{-2}} \right) dw
\]

\[
= 2c(d) \int_{[0,s]} \frac{1}{\sqrt{(w^2 - s^2)(w^2 - s^{-2})}} \left( 1 + \sum_{k=1}^{4} s_k^{-1} \text{Re} \frac{R_ki}{1 + w^2s_k^{-2}} \right) dw
\]
(the last step uses the fact that all residues $R_k$ are pure imaginary). Residues are given in Table 1. We use formula [1] [17.4.45] to compute the parameter $m = s^4$. Then we set $\sin \alpha = s^2$ above, and also $w = s \sin \theta$ to obtain the Legendre normal form:

$$
\int g(z)dz = 2c(d)s \int_0^{\pi/2} \frac{1}{\sqrt{1 - \sin^2 \alpha \sin^2 \theta}} \left( 1 + \sum_{k=1}^{4} \frac{R_k s_k i}{1 - n_k \sin^2 \theta} \right) d\theta
$$

This is a combination of one complete elliptic integral of the first kind and 4 complete elliptic integrals of the third kind. The arguments $n_k = -s^2 s_k^{-2}$ of the integrals of the third kind are given in Table 1.

Therefore,

$$
\int g(z)dz = 2c(d) \left( K(m) + \sum_{k=1}^{4} R_k s_k i \Pi(n_k; m) \right)
$$

where $K$ and $\Pi$ denote the complete elliptic integrals of the first and third kind, respectively.

ASYMPTOTICS: $s \to \sqrt{2} - 1$, so $m \to 0.029437251$, $\alpha \to 0.172425997 \text{rad} \simeq 9^\circ 52' 45.42''$. Also, $s_1, s_2 \to 1$ and $s_3 = s_4^{-1} = (1 - \sqrt{2})/(1 + \sqrt{2})$.

EXPERIMENTAL DATA: The hypergeometric functions were evaluated using Romberg iteration. Coefficients and residues obtained symbolically and then numerically. Digits are not guaranteed to be all significative.
Remark 4. Mark Rybowicz [15] provided the following alternative formula for $C_{d,1} = 2E_{d,1}$:

$$C_{d,1} = -\frac{4d(u - 2)}{\sqrt{u(u - 1)(u - d)}} K(v) + \frac{u + 1}{\sqrt{u(u - 1)}} \Pi\left(-\frac{(u - 1)^2}{4u}, v\right)$$

where

$$u = \sqrt{\frac{2d}{d - 1}} \quad \text{and} \quad v = \frac{\sqrt{2} - u}{2}$$

His formula agrees with ours up to six decimal places.

8 Appendix: Asymptotics for $P_n$

Let us recall briefly the large deviation principle for large random matrices. A good reference is Guionnet [8] where we have taken most of the following.

Let $X$ be a real $n \times n$ symmetric matrix. Its entries are independent Gaussian random variables with mean 0 and variances $2/n$ for a diagonal entry and $1/n$ for an off-diagonal one. Thus $\sqrt{n}X$ is in $GOE(n)$.

Let $\mathcal{P}(\mathbb{R})$ denotes the set of probability measures on $\mathbb{R}$ equipped with the weak topology. For any $\lambda \in \mathbb{R}^n$ let $\delta(\lambda) = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$ be the probability measure on $\mathbb{R}$ defined by

$$\delta(\lambda)(A) = \frac{1}{n} \# \{i : \lambda_i \in A\}.$$
We also need the function $I : P(\mathbb{R}) \to [0, \infty]$ defined by

$$I(\mu) = \frac{1}{4} \int x^2 d\mu(x) - \frac{1}{2} \int \int \log(|x - y|)d\mu(x)d\mu(y) - \frac{3}{8}$$

and the probability measure $Q^n$ on $\mathbb{R}^n$ with density

$$Q^n = \frac{1}{Z^n} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j| \exp \left( -\frac{n}{2} \sum_{i=1}^{n} \lambda_i^2 \right) d\lambda$$

with respect to the Lebesgue measure on $\mathbb{R}^n$ ($Z^n$ is the normalisation constant). Then, according to Guionnet [8] Theorem 3.1, for every closed set $A \subset P(\mathbb{R})$ one has

$$\limsup_{n \to \infty} \frac{1}{n^2} \log Q^n (\{ \lambda \in \mathbb{R}^n : \delta(\lambda) \in A \}) \leq - \inf_{\mu \in A} I(\mu).$$

To obtain an estimate on $P_n$ we take

$$A = \{ \mu \in P(\mathbb{R}) : \mu([0, \infty]) = 1 \}$$

so that, when $\lambda$ is the vector of eigenvalues of the matrix $X$, we have $\mu \in A$ if and only if $X$ is semi-positive definite or (almost surely) if and only if $X$ is positive definite. Since $Q^n$ is the joint law of the eigenvalues of $X$ we get

$$\limsup_{n \to \infty} \frac{1}{n^2} \log \text{Prob} \{ X \in S_n : X \text{ is pd} \} \leq - \inf_{\mu \in A} I(\mu)$$

where the probability is taken in $\sqrt{2/n} \text{GOE}(n)$. Since the set of positive definite matrices is invariant under scaling it is not too difficult to see that

$$P_n = \text{Prob} \{ X \in S_n : X \text{ is pd} \}$$

so that

$$\limsup_{n \to \infty} \frac{1}{n^2} \log(P_n) \leq - \inf_{\mu \in A} I(\mu) = -\alpha.$$

It remains to explain why $\alpha$ is a positive number. The map $I$ is $\geq 0$, each sub-level set $\{ \mu : I(\mu) \leq M \}$ is compact and, it achieves its minimum value at a unique probability measure on $\mathbb{R}$ described as Wigner’s semicircular law

$$\frac{1}{2\pi} \sqrt{4 - x^2} dx$$

with support at $[-2, 2]$. Since this measure is clearly not in our set $A$ we get $\alpha > 0$ and we are done.

References

[1] Abramowitz M. and I. Stegun Editors, *Handbook of Mathematical functions*. Dover Pub. Inc., New York, 1964.
[2] Armentano D., and M. Wschebor, Random systems of polynomial equations. The expected root number under smooth analysis. Preprint.

[3] Azaïs J.-M., and M. Wschebor, On the roots of a random system of equations. The theorem of Shub and Smale and some extensions. Fondations of Computational Mathematics (2005) 125-144.

[4] Blum, L., F. Cucker, M. Shub, and S. Smale, Complexity and Real Computation, Springer, 1998.

[5] Bogomolny E., O. Bohias, and P. Leboeuf, Distribution of roots of random polynomials. Phys. Rev. Letters, 68 (1992) 2726-2729.

[6] Edelman A., and E. Kostlan, How many zeros of a random polynomial are real? Bulletin of the AMS, 32 (1995) 1-37 and 33 (1996) 325.

[7] Federer H., Geometric Measure Theory, Springer, 1969.

[8] Guionnet A., Large deviations and stochastic calculus for large random matrices. Probability Surveys (2004).

[9] Kac M., On the average number of real roots of a random algebraic equation. Bull. Am. Math. Soc. 49 (1943) 314-320 and 938.

[10] Kac M., On the average number of real roots of a random algebraic equation (II). Proc. London Math. Soc. 50 (1949) 390-408.

[11] Kostlan E., On the expected number of real roots of a system of random polynomial equations. In: Foundations of Computational Mathematics, Hong Kong 2002, 149-188. World Sci. Pub., 2002.

[12] Malajovich G. and M. Rojas, High probability analysis of the condition number of sparse polynomial systems, Theoretical Computer Science 315 (2004) 525-555.

[13] Mehta M., Random matrices, Acad. Press, 1991.

[14] Rojas M., On the Average Number of Real Roots of Certain Random Sparse Polynomial Systems. In: The Mathematics of Numerical Analysis. Edited by: James Renegar, Michael Shub, and Steve Smale. Lectures in Applied Mathematics. 32 (1996).

[15] Rybowicz, M., personnel communication.

[16] Shub, M., and S. Smale, Complexity of Bézout’s Theorem II: Volumes and Probabilities in: Computational Algebraic Geometry, F. Eyssette and A. Galligo eds., em Progress in Mathematics, vol. 109, Birkhäuser, 1993, 267-285.

[17] Wschebor M., On the Kostlan-Shub-Smale model for random Polynomials Systems : Variance of the Number of Roots. Journal of Complexity, (2005) 773-789.