Angular ordering and structure functions at small $x$

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We examine the effect of using an angular ordered evolution equation for the unintegrated gluon distribution at small $x$.

1 Introduction

Angular ordering has proven to be a universal aspect of high energy processes. The physical origin of this ordering is the coherence of the soft emissions: namely the cancellation outside the ordered region of multiple soft gluons emission. The universality consists in the fact that this cancellation works both in time-like processes (e.g. $e^+e^-$ annihilation) and in space-like processes (e.g. deep inelastic scattering). Due to its universal validity this description can in principle constitute a bridge between the usual finite-energy (DGLAP) and the soft exchange (BFKL) description.

The detailed analysis of angular ordering in multi-parton emission at small Bjorken-$x$ and the related virtual corrections has been done in Ref. [3] (see also [4]), where it was shown that to leading order the initial-state gluon radiation can be formulated as a branching process in which angular ordering in both real emissions and virtual corrections is taken into account.

In the totally inclusive sum which defines the gluon density, the higher collinear singularities which come both from real and virtual contributions cancel. As a result, to leading order the small-$x$ gluon density is obtained by resumming $\ln x$ powers coming only from IR singularities, and angular ordering contributes only to subleading corrections.

The calculation of the gluon density by resummation of $\ln x$ powers without angular ordering was done 20 years ago and led to the BFKL equation. The solution of this equation, $F(x,k)$, is the unintegrated gluon density at fixed transverse momentum $k$ and is related to the small-$x$ part of the gluon structure function $F(x,Q)$ by

$$F(x,Q) = \int d^2k \ F(x,k)\theta(Q-k).$$

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In this talk, as a first step of a systematic study of multi-parton emission in DIS, the effect of angular ordering on the small-$x$ evolution of the gluon structure function is studied, with both analytical and numerical techniques.

2 Evolution equation for the gluon density

In this section we recall the basic ingredients used to build the coherent branching equation for the gluon density at small $x$. The evolution of the gluon density is thought as a multi-branching process involving only gluons, since gluons dominate the small-$x$ region (fermion exchange is subleading by a factor $\log x$). Consider the single branching process pictorially represented in fig. 1. We denote by $x_i$ and $k_i$ respectively the energy fraction and transverse momentum of the $i$-th exchanged (t-channel) gluon. The energy fraction of the $i$-th emitted gluon is $(1 - z_i)x_i - 1$, where $z_i = x_i/x_i - 1$, and $q_i$ denotes its transverse momentum. In what follows we shall use the notation $k' = |k + q|$. With each branching is associated the factor:

\[
\frac{d\mathcal{P}_i}{\pi q_i^2} \frac{\alpha_S dz_i}{z_i} \Delta(z_i, q_i, k_i) \theta\left(\frac{q_i}{z_i - 1} - q_i - 1\right) \tag{2}
\]

where the function $\Delta$

\[
\ln \Delta(z_i, q_i, k_i) = - \int_{z_i}^{1} \frac{\alpha_S dz'}{z'} \int \frac{dq'^2}{q'^2} \theta(kv - q') \theta(q' - z'q_i) \tag{3}
\]

is the form factor which resums IR singularities (small $z_i$) which appear in virtual corrections to soft exchanged gluons. This form factor corresponds, in the BFKL equation, to the gluon Regge form factor.

To see where the angular ordering is in the splitting kernel $d\mathcal{P}$ and in the form factor $\Delta$ we can write down the condition $\theta_i > \theta_{i-1}$, where $\theta_i$ is the angle of the $i$-th emitted gluon with respect to the initial incoming gluon. In terms of the transverse momenta, the constraint will read $q_i > z_i - 1 q_{i-1}$ and is exactly what
we find in (2) and (3). The so defined branching process is accurate to leading IR order and, at the inclusive level, does not require any collinear approximation (for a proof see [3]).

Angular ordering provides a lower bound on transverse momenta so that no collinear cutoff is needed other than virtuality for the first incoming gluon. On the other hand, in order to deduce a recurrence relation for the inclusive distribution, one has to introduce a “maximum available angle” $\theta$ in the form of a transverse momentum $p$. If $x_n$ and $k_n$ are the kinematical variables of the last exchanged gluon (nearest to the hard vertex) the constraint $\theta_n < \theta$ becomes $z_n q_n < p \simeq x_n E \theta$ where $E$ is the incoming parton energy.

The distribution for emitting $n$ initial state gluons is defined as

$$A^{(n)}(x, k, p) = \int \prod_{i=1}^{n} dP \, \theta(p - z_n q_n) \, \delta(k^2 - k_n^2) \, \delta(x - x_n) \quad (4)$$

so that the fully inclusive gluon density becomes

$$A(x, k, p) = \sum_{n=0}^{\infty} A^{(n)}(x, k, p). \quad (5)$$

This satisfies the so called CCFM equation [3]:

$$A(x, k, p) = A^{(0)}(x, k, p) +$$

$$\int \frac{d^2q}{\pi q^2} \int_{x}^{1} \frac{dx}{z} \Delta(z, q, k) \theta(p - zq) A(x, k', q) \quad (6)$$

where the inhomogeneous term $A^{(0)}(x, k, p)$ is the distribution for no gluon emission.

This equation can be partially diagonalised by introducing the $\omega$-representation

$$A_\omega(k, p) = \int_{0}^{1} dx \, x^\omega A(x, k, p). \quad (7)$$

so that (3) becomes

$$A_\omega(k, p) = A_\omega^{(0)}(k, p) +$$

$$\int \frac{d^2q}{\pi q^2} \int_{0}^{1} \frac{dx}{z} \Delta(z, q, k) \theta(z - q) A_\omega(k', q). \quad (8)$$

No further diagonalisation in transverse momentum is possible since the kernel depends both on the total momentum $k$ and on $q$ and $p$.

In order to explicitly see the collinear safety of the fully inclusive gluon density $A(x, k, p)$ it is possible to rewrite the evolution equation in an “inclusive” form:

$$A_\omega(k, p) = A_\omega^{(0)}(k, p) +$$

$$\frac{\hat{s}}{\omega} \int \frac{d^2q}{\pi q^2} \left[ A_\omega(k', q) - \theta(k - q) A_\omega(k, \tilde{q}) \right] \quad (9)$$

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where the inhomogeneous term is

\[ \tilde{A}^{(0)}(k, p) = A^{(0)}(k, p) + \frac{\bar{\alpha}_S}{\omega} \int \frac{d^2 q}{q^2} A^{(0)}(k, \tilde{q}), \]  

(10)

with \( \tilde{q} = \min(q, p) \). Forgetting the extra factor \( \delta_{\omega}(k, p) \), which is collinear safe, one sees explicitly that (9) is finite in the \( q \to 0 \) collinear limit.

### 2.1 The BFKL limit.

For \( p \to \infty \) the term \( \delta_{\omega}(k, p) \) vanishes and the gluon density \( A^{(0)}(k, p) \) becomes independent of \( p \). In this limit the equations (9) becomes

\[ F_{\omega}(k) = \tilde{F}_{\omega}(k) + \frac{\bar{\alpha}_S}{\omega} \int \frac{d^2 q}{\pi q^2} [F_{\omega}(k') - \theta(k - q) F_{\omega}(k)] \]  

(11)

which is the BFKL equation for the gluon density that possesses the well known solution

\[ F_{\omega}(k) = \int_{\frac{1}{2} - i \infty}^{\frac{1}{2} + i \infty} \frac{d \gamma}{2 \pi i} \left( \frac{k^2}{\tilde{k}_0^2} \right)^{-\gamma} \frac{\omega f_0(\omega, \gamma)}{\omega - \bar{\alpha}_S \chi(\gamma)} \]  

(12)

where \( k_0 \) and the function \( f_0 \) are fixed by the inhomogeneous term and \( \chi(\gamma) \) is the BFKL characteristic function \( \chi(\gamma) = 2 \psi(1) - \psi(\gamma) - \psi(1 - \gamma) \). For a general initial condition the asymptotic behaviour of \( F_{\omega}(k) \) for \( k \gg k_0 \) and for \( k \ll k_0 \) is given by solving the characteristic equation \( \omega = \bar{\alpha}_S \chi(\gamma) \) in the regions \( 0 < \gamma < \frac{1}{2} \) and \( \frac{1}{2} < \gamma < 1 \) respectively. Using the saddle point approximation around the leading singularity of \( \gamma(\alpha_S/\omega) \) in the \( \omega \)-plane, which is at \( \gamma_c = \gamma(\alpha_S/\omega_c) = 1/2 \), one obtains the asymptotic behaviour of \( F(x, k) \) at small \( x \) which reads:

\[ x F(x, k) \sim x^{-\omega_c} \left( \frac{k^2}{\tilde{k}_0^2} \right)^{\gamma_c} \]  

(13)

where \( \omega_c = \bar{\alpha}_S \chi(\frac{1}{2}) = 4 \bar{\alpha}_S \ln 2 \).

### 2.2 Properties of the gluon distribution.

Consider a solution to (8) of the form:

\[ A^\omega(k, p) = 1 \left( \frac{k^2}{\tilde{k}_0^2} \right)^{\tilde{\gamma}} G \left( \frac{p}{k} \right), \]  

(14)

with \( \tilde{\gamma} \) a function of \( \alpha_S \) and \( \omega \). This form allows a direct comparison with the BFKL asymptotic behaviour (see (13)). With \( 0 < \tilde{\gamma} < 1 \), from (8) one finds the equation:

\[ p \partial_p G(p/k) = \bar{\alpha}_S \int_p \frac{d^2 q}{\pi q^2} \left( \frac{p}{q} \right)^{\omega} \Delta(\frac{p}{q}, q, k) G(\frac{q}{k^2}) \left( \frac{k^2}{\tilde{k}_0^2} \right)^{\tilde{\gamma} - 1} \]  

(15)
where we have chosen the boundary condition \( G(\infty) = 1 \). The function \( \tilde{\gamma} \) must satisfy the “characteristic” equation:

\[
1 = \frac{\bar{\alpha}}{\omega} \tilde{\chi}(\tilde{\gamma}, \alpha_S)
\]

\[
\tilde{\chi} = \int \frac{d^2 q}{2\pi q^2} \left\{ \left( \frac{q^2}{k^2} \right)^{\tilde{\gamma} - 1} G\left( \frac{k}{q} \right) - \theta(k - q) G\left( \frac{k}{q} \right) \right\}.
\]

obtained from (9) in the \( p \to \infty \) limit.

Notice that when \( \bar{\alpha}_S \to 0 \) the function \( G \) reduces to a constant and this implies that \( \tilde{\chi} \) becomes the BFKL characteristic function. Since \( 1 - G \) is of order \( \bar{\alpha}_S \) and doesn’t have the \( 1/\omega \) enhancement factor, we can conclude that the effect of angular ordering on the asymptotic behaviour is of subleading nature.

Moreover it’s possible to show \([5]\) that in the limit \( \gamma \to 0 \) the difference \( \chi(\gamma) - \tilde{\chi}(\gamma, \alpha_S) \) tends to a constant (in \( \alpha_S \)) implying a behaviour of the form \( \alpha_S^3/\omega^2 \).

As we shall see from the numerical analysis, the characteristic function \( \tilde{\chi}(\tilde{\gamma}, \alpha_S) \) decreases with \( \tilde{\gamma} \), reaches a minimum at \( \tilde{\gamma}_c < 1 \) (for reasonable \( \alpha_S \)), and then rises again. As in the BFKL case, we shall denote by \( \tilde{\omega}_c \) the leading singularity in \( \omega \) which corresponds to the minimum of the characteristic function at \( \tilde{\gamma} = \tilde{\gamma}_c \).

3 Numerical results.

Fig. 2 shows the results for \( \tilde{\chi} \) as a function of \( \tilde{\gamma} \) for various \( \alpha_S \). The difference \( \delta\chi = \chi - \tilde{\chi} \) is positive and increases with \( \tilde{\gamma} \) and with \( \alpha_S \). Moreover we find \( \delta\chi \sim \tilde{\gamma} \) for \( \tilde{\gamma} \to 0 \) (\( \bar{\alpha}_S \) small and fixed) and \( \delta\chi \sim \bar{\alpha}_S \) for \( \bar{\alpha}_S \to 0 \) (\( \tilde{\gamma} \) small and fixed) as expected from analytical studies. The function \( \tilde{\chi} \) decreases faster than \( \chi \) for increasing \( \tilde{\gamma} \): the minimum of the characteristic function gets shifted to the right and is lower. This produces a milder growth of the structure function at small \( x \) and moderate \( k \). In Fig. 3a and 3b we plot as a function of \( \alpha_S \) the values \( \tilde{\chi}_c \) and \( \tilde{\gamma}_c \) of the minimum of \( \tilde{\chi} \) and its position \( \tilde{\gamma}_c \). As expected the differences compared to the BFKL values \( \chi_c = 4 \ln 2 \) and \( \gamma_c = \frac{1}{2} \) are of order \( \bar{\alpha}_S \).

In Fig. 3c we show the second derivative, \( \tilde{\chi}''_c \), of the characteristic function at its minimum. The diffusion in \( \ln k \) is inversely proportional to the square root of \( \tilde{\chi}''_c \). One can see therefore that the inclusion of angular ordering significantly reduces the diffusion compared to the BFKL case.

The loss of symmetry under \( \gamma \to 1 - \gamma \), which clearly shows up in Fig. 2, relates to the loss of symmetry between small and large scales: while in BFKL the regions of small and large momenta are equally important, the angular ordering favours instead the region of larger \( k \). However, at each intermediate

\[\text{This is strictly true only for the solution in the saddle-point approximation, nevertheless this quantity remain a good indicator due to the mild asymptotic behaviour of the } G \text{ function.}\]
branching, the region of vanishing momentum is still reachable for $x \to 0$, so that the evolution still contains non-perturbative components.

The subleading nature of the angular ordering inclusion in the evolution equation that we have seen so far is no longer guaranteed if one considers, instead of the gluon density, more exclusive quantities. This happens because the cancellations between real emissions and virtual corrections which, in the angular ordering equation, reconstruct at leading level the BFKL solution no longer work for a modified kernel, such the one used for associated distributions.

The topic requires further analysis but for now let us show some preliminary plots. In Fig. 4a and Fig. 4b we show the “rung multiplicity” and the transverse momentum flow calculated for given kinematical variables. Notice that even if for the chosen values of $x$ and $k_t$ the gluon distribution calculated in the two approaches are practically the same, the distribution shapes and normalisation are quite different for the BFKL case compared to the CCFM equation.

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**References**
Figure 3: (a) The value of the minimum of the characteristic function \( \tilde{\chi}_c \), as a function of \( \alpha_S \). (b) The position of the minimum of the characteristic function \( \tilde{\gamma}_c \), as a function of \( \alpha_S \). (c) The second derivative of the characteristic function \( \tilde{\chi}''_c \) at its minimum, as a function of \( \alpha_S \).

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Figure 4: (a) Distribution of number of emission with $q > q_0 = 1$GeV, for DGLAP, CCFM and BFKL evolution to $x = 5 \times 10^{-5}$, $k = 5$ GeV, $\alpha_S = 0.2$. (b) Transverse momentum flow in the hadronic centre of mass frame as a function of the rapidity $\eta^*$ for evolution to $x = 2 \cdot 10^{-4}$, $k = 3$ GeV, $\alpha_S = 0.2$ (the proton direction is to the left).

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