A MIXED PARSEVAL–PLANCHEREL FORMULA

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Abstract. In this note, a general formula is proved. It expresses the integral on the line of the product of a function $f$ and a periodic function $g$ in terms of the Fourier transform of $f$ and the Fourier coefficients of $g$. This allows the evaluation of some oscillatory integrals.

1. Introduction and notation

In [6] the following integral was described as “difficult”:

$$\int_{-\infty}^{\infty} \frac{dx}{(\cosh a + \cos x) \cosh x} \text{ for } a > 0, \quad (1)$$

it was used to test the trapezoidal rule after transforming the integral using a “sinh” transformation. Also, in [5] S. Tsipelis proposed to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\log(\cos^2 x)}{1 + e^{2|x|}} dx. \quad (2)$$

Both integrals are of the form $\int_{\mathbb{R}} f(x)g(x)dx$ where $g$ is a 2π-periodic function. The particular case, where $f$ is of the form $x \mapsto 1/(x+z)$, (for some $z \in \mathbb{C} \setminus \mathbb{R}$,) was thoroughly investigated in [3] using methods that are different from those discussed here.

In this note, we prove a general formula, that allows us to express this kind of integrals in terms of the Fourier transform of $f$ and the Fourier coefficients of $g$.

Before we proceed, let us recall some standard notation. The spaces $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, and $L^2,loc(\mathbb{R})$ are, respectively, the space of integrable functions, the space of square integrable functions, and the space of locally square integrable functions on $\mathbb{R}$. The spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are equipped with the standard norms denoted $\|\cdot\|_1$ and $\|\cdot\|_2$:

$$\|f\|_p = \left( \int_{\mathbb{R}} |f(t)|^p dt \right)^{1/p}, \quad \text{for } p = 1, 2.$$ 

We consider also $L^1(\mathbb{T})$, (resp. $L^2(\mathbb{T})$), the space of integrable, (resp. square integrable), 2π-periodic functions. The spaces $L^1(\mathbb{T})$ and $L^2(\mathbb{T})$ are equipped with the standard norms denoted $\|\cdot\|_{L^1(\mathbb{T})}$ and $\|\cdot\|_{L^2(\mathbb{T})}$ and defined as follows:

$$\|f\|_{L^p(\mathbb{T})} = \left( \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^p dt \right)^{1/p}, \quad \text{for } p = 1, 2.$$ 

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For a function \( f \in L^1(\mathbb{R}) \) we recall that its Fourier transform \( \hat{f} \) is defined by

\[
\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} \, dt, \quad \text{for } \omega \in \mathbb{R}.
\]

And for a \( 2\pi \)-periodic function \( g \in L^1(\mathbb{T}) \) we recall that the exponential Fourier coefficient \( C_n(g) \) of \( g \) is defined by

\[
C_n(g) = \frac{1}{2\pi} \int_{\mathbb{T}} g(t) e^{-int} \, dt, \quad \text{for } n \in \mathbb{Z},
\]

In section 2 we will prove our main results and in section 3 we will give some detailed examples and applications.

2. The main result

In this section we state and prove the main theorem.

**THEOREM 1.** (The mixed Parseval-Plancherel formula) Consider a function \( f \) from \( L^2,\text{loc}(\mathbb{R}) \), and a \( 2\pi \)-periodic function \( g \) from \( L^2(\mathbb{T}) \). Suppose that

\[
M(f) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} \| \mathbb{1}_{I_k} f \|_2 < +\infty,
\]

where \( \mathbb{1}_{I_k} \) is the characteristic function of the interval \( I_k = [2\pi k, 2\pi (k+1)] \). Then

\[
\int_{\mathbb{R}} f(x) g(x) \, dx = \sum_{n \in \mathbb{Z}} \hat{f}(n) \overline{C_n(g)}.
\]

where \( \hat{f} \) is the Fourier transform of \( f \), and \( \{C_n(g)\}_{n \in \mathbb{Z}} \) is the family of exponential Fourier coefficients of \( g \).

**Proof.** First, note that \( \| \mathbb{1}_{I_k} f \|_1 \leq \sqrt{2\pi} \| \mathbb{1}_{I_k} f \|_2 \) for every \( k \in \mathbb{Z} \). It follows that

\[
\int_{\mathbb{R}} |f(x)| \, dx = \sum_{k \in \mathbb{Z}} \| \mathbb{1}_{I_k} f \|_1 \leq \sqrt{2\pi} M(f) < +\infty.
\]

Thus, \( f \) belongs to \( L^1(\mathbb{R}) \), and we can consider its Fourier transform. Similarly,

\[
\int_{\mathbb{R}} |f(x)g(x)| \, dx = \sum_{k \in \mathbb{Z}} \int_{I_k} |f(x)g(x)| \, dx \leq \sqrt{2\pi} M(f) \|g\|_{L^2(\mathbb{T})} < +\infty,
\]

thus \( fg \) belongs also to \( L^1(\mathbb{R}) \).

Now, let us consider the the family \( \{f_k\}_{k \in \mathbb{Z}} \) defined by \( f_k(x) = f(x+2\pi k) \). Clearly \( \| \mathbb{1}_{m} f_k \|_2 = \| \mathbb{1}_{m+k} f \|_2 \). Thus

\[
\sum_{k \in \mathbb{Z}} \| \mathbb{1}_{m} f_k \|_2 = M(f) < +\infty
\]
and the series \( \sum_{k \in \mathbb{Z}} 1_{m}f_{k} \) is normally convergent in \( L^{2}(\mathbb{R}) \) for every \( m \in \mathbb{Z} \). This proves that the formula \( F = \sum_{k \in \mathbb{Z}} f_{k} \) defines a function \( F \) that belongs to \( L^{2,\text{loc}}(\mathbb{R}) \). Moreover, this function is clearly \( 2\pi \)-periodic, and \( \|F\|_{L^{2}(\mathbb{T})} \leq \frac{1}{\sqrt{2\pi}} M(f) \). Now, the classical Parseval’s formula, (see [2, Chap. I, §5.] or [4, Chap. 5, §3.],) implies that

\[
\frac{1}{2\pi} \int_{\mathbb{T}} F(x)\overline{g(x)} \, dx = \sum_{n \in \mathbb{Z}} C_n(F)\overline{C_n(g)}. \tag{5}
\]

Using the fact that \( \sum_{k=-n}^{n-1} 1_{I_{0}}f_{k} \) converges to \( 1_{I_{0}}F \) in \( L^{2}(\mathbb{R}) \), and that \( 1_{I_{0}}g \in L^{2}(\mathbb{R}) \), we conclude that

\[
\int_{0}^{2\pi} F(x)\overline{g(x)} \, dx = \lim_{n \to \infty} \int_{0}^{2\pi} f_{k}(x)\overline{g(x)} \, dx
= \lim_{n \to \infty} \sum_{k=-n}^{n-1} \int_{2\pi k}^{2\pi(k+1)} f(x)\overline{g(x)} \, dx
= \int_{\mathbb{R}} f(x)\overline{g(x)} \, dx \tag{6}
\]

where, for the last equality, we used the fact that \( fg \in L^{1}(\mathbb{R}) \).

Similarly,

\[
2\pi C_{n}(F) = \int_{0}^{2\pi} F(x)e^{-int} \, dx = \lim_{n \to \infty} \sum_{k=-n}^{n-1} \int_{0}^{2\pi} f_{k}(x)e^{-int} \, dx
= \lim_{n \to \infty} \sum_{k=-n}^{n-1} \int_{2\pi k}^{2\pi(k+1)} f(x)e^{-int} \, dx
= \int_{\mathbb{R}} f(x)e^{-int} \, dx = \hat{f}(n) \tag{7}
\]

where we used again the fact that \( f \in L^{1}(\mathbb{R}) \) for the last equality. Replacing (6) and (7) in (5), the desired formula follows. \( \square \)

The next corollary is straightforward.

**COROLLARY 1.** Consider a function \( f \) from \( L^{2,\text{loc}}(\mathbb{R}) \), and a \( T \)-periodic, square integrable function \( g \). Suppose that

\[
M_{T}(f) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} \|1_{[kT,(k+1)T]}f\|_{2} < +\infty, \tag{8}
\]

Then

\[
\int_{\mathbb{R}} f(x)\overline{g(x)} \, dx = \sum_{n \in \mathbb{Z}} \hat{f}\left(\frac{2\pi n}{T}\right) \overline{C_n(g)}. \tag{9}
\]

where \( \hat{f} \) is the Fourier transform of \( f \), and \( (C_{n}(g))_{n \in \mathbb{Z}} \) is the family of exponential Fourier coefficients of \( g \).
3. Examples

**Example 1.** For positive real numbers $a$ and $b$, let $g$ and $f$ be the functions defined by

$$g(x) = \frac{1}{\cosh a + \cos x}, \quad f(x) = \frac{1}{\cosh(bx)},$$

It is known [1, Chap.I, §9] that $\hat{f}(\omega) = \frac{\pi}{b} f\left(\frac{\pi}{2b}\omega\right)$. Moreover, it is easy to note that for every $k \in \mathbb{Z}$ we have $\left\| \mathbb{I}_k f \right\|_2 \leq Be^{-2\pi|k|}$ for some absolute constant $B$.

Furthermore, it is easy to check that

$$g(x) = \frac{1}{\sinh a} \sum_{n \in \mathbb{Z}} (-1)^n e^{-|n|a} e^{inx},$$

that is

$$C_n(g) = \frac{(-1)^n e^{-|n|a}}{\sinh a}, \quad \text{for } n \in \mathbb{Z}.$$  

Applying Theorem 1, we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(\cosh a + \cos x) \cosh(bx)} = \frac{\pi}{b \sinh a} + \frac{2\pi}{b \sinh a} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-na}}{\cosh(\pi n/(2b))}.$$ 

In particular, for $b = 1$, we obtain the following expression of the integral (1) as a rapidly convergent series:

$$\int_{-\infty}^{\infty} \frac{dx}{(\cosh a + \cos x) \cosh x} = \frac{\pi}{\sinh a} + \frac{2\pi}{\sinh a} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-na}}{\cosh(\pi n/2)}.$$ 

This is a simpler alternative series expansion to the one given in [6].

**Example 2.** In our second example, let $g$ and $f$ be the functions defined by

$$g(x) = \log(\cos^2 x), \quad f(x) = \frac{1}{1 + e^{2|x|}}.$$  

It is easy to note that for every $k \in \mathbb{Z}$ we have $\left\| \mathbb{I}_k f \right\|_2 \leq Be^{-2\pi|k|}$ for some constant $B$. Moreover,

$$\hat{f}(\omega) = 2 \int_{0}^{\infty} \frac{e^{-2x}}{1 + e^{-2x}} \cos(\omega x) dx$$

$$= 2 \sum_{k=1}^{\infty} (-1)^{k-1} \int_{0}^{\infty} e^{-2kx} \cos(\omega x) dx$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{4k}{4k^2 + \omega^2}.$$ 

Further, since

$$g(x) = 2 \log |1 + e^{2ix}| - 2 \log 2 = 2 \Re \log(1 + e^{2ix}) - 2 \log 2.$$
with Log being the principal branch of the logarithm, we conclude that for every \( n \in \mathbb{Z} \) we have
\[
C_{2n+1}(g) = 0, \quad \text{and} \quad C_{2n}(g) = \begin{cases} 
(-1)^{n-1}/|n| & \text{if } n \neq 0, \\
-2 \log 2 & \text{if } n = 0.
\end{cases}
\]

Using Theorem 1, we obtain
\[
\int_{-\infty}^{\infty} \frac{\log(\cos^2 x)}{1 + e^{2|x|}} \, dx = \sum_{n \in \mathbb{Z}} \hat{f}(2n) \overline{C_{2n}(g)}
\]
\[
= -2 \log 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} + 2 \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+n}}{(k^2 + n^2)n} \right)
\]
\[
= -2 \log^2 2 + 2J
\]
with
\[
J = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+n}}{(k^2 + n^2)n} \right)
\]
(10)

Now, this double series is not absolutely convergent, so we must be careful. First, exchanging the roles of \( k \) and \( n \) we have
\[
J = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{(-1)^{k+n}n}{k(n^2 + k^2)} \right)
\]
(11)

Now, using the properties of convergent alternating series we have
\[
\sum_{n=1}^{\infty} \frac{(-1)^{k+n}n}{k(n^2 + k^2)} = \sum_{n=1}^{q-1} \frac{(-1)^{k+n}n}{k(n^2 + k^2)} + R_q(k),
\]
with
\[
R_q(k) = \frac{(-1)^k}{k} \sum_{n=q}^{\infty} \frac{(-1)^n n}{n^2 + k^2} \quad \text{and} \quad |R_q(k)| \leq \frac{1}{k} \cdot \frac{q}{k^2 + q^2}
\]

Thus
\[
J = \sum_{n=1}^{q-1} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+n}n}{k(n^2 + k^2)} \right) + \varepsilon_q
\]
(12)
with \( \varepsilon_q = \sum_{k=1}^{\infty} R_q(k) \) and
\[
\varepsilon_q \leq \sum_{k=1}^{\infty} \frac{q}{k(k^2 + q^2)}.
\]

Now, since
\[
\frac{q}{k(k^2 + q^2)} \leq \frac{1}{2k^2} \quad \text{for every } q,
\]
- the series \( \sum_{k=1}^{\infty} \frac{1}{2k^2} \) is convergent,
• and \( \lim_{q \to \infty} \frac{q}{k^2 + q^2} = 0 \) for every \( k \),

we conclude that \( \lim_{q \to \infty} \epsilon_q = 0 \). So, letting \( q \) tend to \( +\infty \) in (12) we get

\[
J = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+n} n}{k(n^2 + k^2)} \right)
\]

Taking the sum of the two expressions (11) and (13) of \( J \) we obtain

\[
2J = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+n} (n + k)}{n^2 + k^2} \right) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+n}}{nk} \right) = (-\log 2)^2 = \log^2 2.
\]

Replacing back in (10) we obtain

\[
\int_{-\infty}^{\infty} \frac{\log(\cos^2 x)}{1 + e^{2|x|}} \, dx = -\log^2 2.
\]

**Example 3.** For positive real numbers \( a \) and \( b \), let \( g \) and \( f \) be the functions defined by

\[
g(x) = \frac{1}{\cosh a - \cos x}, \quad f(x) = e^{-x^2/(4b)},
\]

It is known [1, Chap.I, §4] that \( \hat{f}(\omega) = 2\sqrt{\pi b} f(2b\omega) \). Moreover,

\[
C_n(g) = \frac{e^{-|n|a}}{\sinh a}, \quad \text{for } n \in \mathbb{Z}.
\]

Hence

\[
\int_{\mathbb{R}} \frac{e^{-x^2/(4b)}}{\cosh a - \cos x} \, dx = \frac{2\sqrt{\pi} b}{\sinh a} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-an - bn^2} \right)
\]

In particular, for \( b = a \) we get

\[
\int_{\mathbb{R}} \frac{e^{-x^2/(4a)}}{\cosh a - \cos x} \, dx = \frac{2\sqrt{\pi} a}{\sinh a} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-an(n+1)} \right) = \frac{2\sqrt{\pi} a}{\sinh a} \left( e^{a/4} \vartheta_2(0, e^{-a}) - 1 \right),
\]

where \( \vartheta_2(u, q) \) is one of the well-known Jacobi Theta functions [7, Chap. XXI].

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