Coupled Stochastic Allen-Cahn equations: Existence and Uniqueness

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Abstract

In this paper, we consider a system of stochastic PDE’s. Our system consists of two components. Each component evolves according to the stochastic Allen-Cahn equation with a symmetric double well potential and with additional small space-time white noise. The two components interact with each other by an attractive linear force. We aim to give a global existence and uniqueness result in a space of continuous functions on $\mathbb{R} \times \mathbb{R}^+$. We apply methods proposed by Doering [1].

1 Introduction and main result

We investigate the following SPDE on unbounded domain $\mathbb{R} \times \mathbb{R}^+$

\begin{align*}
\partial_t m_1 &= \frac{1}{2} \partial_{xx} m_1 - V'(m_1) + \lambda(m_2 - m_1) + \dot{\alpha}_1(x, t), \quad x \in \mathbb{R}, t \geq 0 \\
\partial_t m_2 &= \frac{1}{2} \partial_{xx} m_2 - V'(m_2) + \lambda(m_1 - m_2) + \dot{\alpha}_2(x, t), \quad x \in \mathbb{R}, t \geq 0 \\
\end{align*}

(1.1)

where $V'(m) = m^3 - m$ is the derivative with respect to $m$ of the double well polynomial $V(m) = \frac{1}{4} m^4 - \frac{1}{2} m^2$ and $\lambda$ is a given positive constant. Two terms $\dot{\alpha}_1(x, t)$ and $\dot{\alpha}_2(x, t)$ are two independent space-time white noises. We refer to Walsh [4], Faris and Jona-Lasinio [2] and Da Prato and Zabczyk [3] for the SPDE’s theory applied to (1.1). The integral version of (1.1) is

\begin{align*}
m_{1,t} &= H_t m_{1,0} - \int_0^t H_{t-s} \left[ V'(m_{1,s}) + \lambda(m_{1,s} - m_{2,s}) \right] ds + Z_{1,t} \\
m_{2,t} &= H_t m_{2,0} - \int_0^t H_{t-s} \left[ V'(m_{2,s}) + \lambda(m_{2,s} - m_{1,s}) \right] ds + Z_{2,t} \\
\end{align*}

(1.2)

where $H_t$ is the kernel of the heat operator $\partial_t - \frac{1}{2} \partial_{xx}$,

\begin{equation}
Z_{i,t} = \int_0^t H_{t-s} \dot{\alpha}_{i,s} ds \quad \forall i = 1, 2
\end{equation}

(1.3)

For every $\alpha > 0$, let \( (C^\alpha(\mathbb{R}), \| \cdot \|_{C^\alpha(\mathbb{R})}) \) and \( (C^\alpha(\mathbb{R} \times \mathbb{R}^+), \| \cdot \|_{C^\alpha(\mathbb{R} \times \mathbb{R}^+)}) \) be Banach spaces of continuous functions with norm

\begin{align*}
\| f \|_{C^\alpha(\mathbb{R})} &= \sup_{x \in \mathbb{R}} e^{-\alpha |x|} |f(x)| \\
\| f \|_{C^\alpha(\mathbb{R} \times \mathbb{R}^+)} &= \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \exp \left( -\frac{\alpha^2}{2} t - \alpha |x| \right) |f(x, t)|
\end{align*}

(1.4)

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Let $\mathcal{F}_t$ be the filtration given by $\mathcal{F}_t := \sigma\{Z_{i,t}(x,s); x \in \mathbb{R}, s \leq t, i = 1, 2\}$. We now state the main theorem

**Theorem 1.1.** If $(m_{1,0}, m_{2,0}) \in (C^\alpha(\mathbb{R}))^2$ for any $\alpha > 0$ then

$\begin{align*}
m_{1,t} &= H_t m_{1,0} - \int_0^t H_{t-s} \left[ V'(m_{1,s}) + \lambda (m_{1,s} - m_{2,s}) \right] ds + Z_{1,t} \\
m_{2,t} &= H_t m_{2,0} - \int_0^t H_{t-s} \left[ V'(m_{2,s}) + \lambda (m_{2,s} - m_{1,s}) \right] ds + Z_{2,t}
\end{align*}
(1.5)$

has a unique $\mathcal{F}_t$-adapted process $(m_{1,t}, m_{2,t}) \in \left(C^\alpha(\mathbb{R} \times \mathbb{R}^+)\right)^2$ for any $\alpha > 0$.

## 2 Finite Volume Equations

Let $F_{t,i} = H_t m_{i,0} + Z_{i,t} \ \forall i = 1, 2$. Since $F_{1,t}$ and $F_{2,t}$ are not bounded on unbounded domain, so it is convenient to consider the following SPDE

$\begin{align*}
m_{1,\Lambda,t} &= -\int_0^t H_{t-s} \Lambda \left[ V'(m_{1,\Lambda,s}) + \lambda (m_{1,\Lambda,s} - m_{2,\Lambda,s}) \right] ds + \Lambda F_{1,t} \\
m_{2,\Lambda,t} &= -\int_0^t H_{t-s} \Lambda \left[ V'(m_{2,\Lambda,s}) + \lambda (m_{2,\Lambda,s} - m_{1,\Lambda,s}) \right] ds + \Lambda F_{2,t}
\end{align*}
(2.1)$

where $0 \leq \Lambda(x) \leq 1$ is a continuous function of compact support on $\mathbb{R}$.

Firstly, it is essential to establish a local existence result by using Picard iterations as in [1].

**Proposition 2.1.** There exists $T_0 > 0$ so that there is a continuous bounded solution to (2.1) on $\mathbb{R} \times [0, T_0]$

**Proof.** We define the map $P$

$$(m_1, m_2) \rightarrow \left(-H\Lambda \left[ V'(m_1) + \lambda(m_1 - m_2) \right] + \Lambda F_1, \\
- H\Lambda \left[ V'(m_2) + \lambda(m_2 - m_1) \right] + \Lambda F_2 \right)$$
(2.2)

We shall prove the map $P$ takes the closed set

$$B = \left\{ (m_1, m_2) \in C(\mathbb{R} \times [0, T])^2 : \\
\|m_i\|_\infty \leq 2 \left( \|\Lambda F_1\|_\infty + \|\Lambda F_2\|_\infty \right) \ \forall i = 1, 2 \right\}$$
(2.3)

into itself for $T$ sufficiently small. In fact, let $(m_1, m_2) \in B$ and setting $K := \|\Lambda F_1\|_\infty + \|\Lambda F_2\|_\infty$, we estimate

$\begin{align*}
\| -H\Lambda[V'(m_1) + \lambda(m_1 - m_2)] + \Lambda F_1 \|_\infty &\leq \|V'(m_1) + \lambda(m_1 - m_2)\|_\infty \times \int_0^T \int_\mathbb{R} H_{t-s}(x, y)dyds + \|\Lambda F_1\|_\infty \\
&\leq T \left( \|m_1\|_\infty^3 + \|m_1\|_\infty + \lambda(\|m_1\|_\infty + \|m_2\|_\infty) \right) + \|\Lambda F_1\|_\infty \\
&\leq 8TK^2 + \frac{1}{8} + \frac{\lambda}{4} + \|\Lambda F_1\|_\infty \quad \text{(note that $\|m_1\|_\infty, \|m_2\|_\infty \leq 2K$)}
\end{align*}$
(2.4)
Choosing $T > 0$ such that
\[ 8T(K^2 + \frac{1}{8} + \frac{\lambda}{4}) \leq 1 \iff T \leq \frac{1}{8(K^2 + 1/8 + \lambda/4)} \] (2.5)
we derive
\[ \| -HA[V'(m_1) + \lambda(m_1 - m_2)] + \Lambda F_1 \|_\infty \leq K + \| \Lambda F_1 \|_\infty \leq 2K \] (2.6)
Since $V'$ is a polynomial, so it is Lipschitz continuous on bounded set in $C(\mathbb{R} \times [0, T])$. We claim that $P$ is a contraction mapping from $B \to B$ for $T_0$ sufficiently small. Taking $(m'_1, m'_2), (m_1, m_2) \in B$ then
\[ P(m'_1, m'_2) - P(m_1, m_2) \]
\[ = \left[ -HA\left(V'(m'_1) - V(m_1) + \lambda(m'_1 - m_1 - m_2 + m'_2)\right) \right. \]
\[ \left. - HA\left(V'(m'_2) - V(m_2) + \lambda(m'_2 - m_2 - m_1 + m'_1)\right) \right] \] (2.7)
Since
\[ V'(a) - V'(b) = (a - b)(a^2 + ab + b^2 - 1) \quad \forall a, b \in \mathbb{R} \] (2.8)
Hence, in the ball $B$
\[ \| -HA\left(V'(m'_1) - V(m_1) + \lambda(m'_1 - m_1 - m_2 + m'_2)\right) \|_\infty \]
\[ \leq \left(V'(m'_1) - V(m_1) + \lambda(m'_1 - m_1 - m_2 + m'_2)\right) \int_0^T \int_{\mathbb{R}} H_{t-s}(x, y) dy ds \]
\[ \leq T(12K^2 + 1) \| m'_1 - m_1 \|_\infty + \lambda T \left( \| m'_1 - m_1 \|_\infty + \| m'_2 - m_2 \|_\infty \right) \]
\[ \leq T(12K^2 + 1 + 2\lambda) \| (m'_1 - m_1, m'_2 - m_2) \|_\infty \] (2.9)
Thus, we have
\[ \| P(m'_1, m'_2) - P(m_1, m_2) \|_\infty \]
\[ \leq T(12K^2 + 1 + 2\lambda) \| (m'_1 - m_1, m'_2 - m_2) \|_\infty \] (2.10)
Choosing $T$ such that
\[ T(12K^2 + 1 + 2\lambda) < 1 \iff T < \frac{1}{12K^2 + 1 + 2\lambda} \] (2.11)
$P$ is a contraction mapping from $B \to B$. From (2.5) and (2.11), $T_0$ should satisfies
\[ T_0 = \min \left( \frac{1}{8(K^2 + 1/8 + \lambda/4)}, \frac{1}{2(12K^2 + 1 + 2\lambda)} \right) \] (2.12)
then $P$ has a unique fixed point by Contraction principle (Theorem ??).

To show global solution, it is sufficient to prove that
\[ \sup_{t \in [0, T^*]} \sup_{x \in \mathbb{R}} |m_{i, A, t}(x)| < \infty \quad \forall i = 1, 2 \] (2.13)
where $(m_{1, A, t}, m_{2, A, t})$ is a continuous solution on $[0, T^*)$. We recall the following useful lemma proved in Lemma 1, [1].
Lemma 2.1. If \( f(x, t) \) and \((\partial_t - \frac{1}{2} \Delta)f\) are continuous on \(\mathbb{R} \times [0, T]\) and \(f(x, 0) = 0, |f|^{2n+2} \text{ and } |\partial_x f|^{2n+2} \in L^1(\mathbb{R} \times [0, T], dxdt)\), then

\[
\int_0^T \int_{\mathbb{R}} f^{2n+1}(x, t)(\partial_t - \frac{1}{2} \Delta)f(x, t)dxdt \geq 0 \tag{2.14}
\]

Lemma 2.2. If \((m_{1, \Lambda}, m_{2, \Lambda})\) satisfies (2.1) on \(\mathbb{R} \times [0, T]\). Then \(m_{1, \Lambda}\) and \(\partial_x m_{i, \Lambda}\) vanish exponentially as \(|x| \to \infty\) for all \(t \leq T\).

Proof. Assume \(x\) is not in \(\text{supp}(\Lambda)\) and denote \(V^*(m_{1, \Lambda}) = -V'(m_{1, \Lambda}) + \lambda(m_{2, \Lambda} - m_{1, \Lambda})\). Then

\[
m_{1, \Lambda}(x, t) = \int_0^t \int_{\mathbb{R}} \Lambda(y) H_{t-s}(x, y)V^*(m_{1, \Lambda})(y, s)dyds \tag{2.15}
\]

This implies

\[
|m_{1, \Lambda}(x, t)| \leq \sup_{s \leq T, y \in \mathbb{R}} |\Lambda(y)V^*(m_{1, \Lambda}(y, s))| \int_0^t \int_{\text{supp}(\Lambda)} H_{t-s}(x, y)dyds \tag{2.16}
\]

and define \(d(x, \text{supp}(\Lambda)) := \inf \{|x - y| : y \in \text{supp}(\Lambda)\}\), then

\[
\int_0^t \int_{\text{supp}(\Lambda)} H_{t-s}(x, y)dyds \leq \int_0^t \int_{\text{supp}(\Lambda)} e^{-\frac{d^2(x, \text{supp}(\Lambda))}{2(t-s)}}dyds \tag{2.17}
\]

\[
\leq \|\text{supp}(\Lambda)\| \int_0^t e^{-\frac{d^2(x, \text{supp}(\Lambda))}{2(t-s)}}ds
\]

Here \(\|\text{supp}(\Lambda)\|\) is the volume of \(\text{supp}(\Lambda)\). By changing variables, \(r := \frac{1}{2(t-s)} \Rightarrow dr = 2r^2 ds\). Thus \(\forall t \leq T\), we get

\[
\int_0^t e^{-\frac{d^2(x, \text{supp}(\Lambda))}{2(t-s)}}ds \leq \int_{1/2T}^\infty e^{\frac{-r d^2(x, \text{supp}(\Lambda))}{2r^{3/2}}}dr \tag{2.18}
\]

\[
\leq CT^{3/2} e^{-\frac{d^2(x, \text{supp}(\Lambda))/2T}{d^2(x, \text{supp}(\Lambda))}}
\]

From (2.16),(2.17) and (2.18), \(m_{1, \Lambda}\) vanish exponentially as \(|x| \to \infty\). To see \(\partial_x m_{1, \Lambda}\) also vanish exponentially, we write

\[
\partial_x m_{1, \Lambda}(x, t) = \int_0^t \int_{\mathbb{R}} \Lambda(y) H_{t-s}(x, y)\frac{-(x-y)}{t-s}V^*(m_{1, \Lambda}(y, s))dyds \tag{2.19}
\]

As a result,

\[
|\partial_x m_{1, \Lambda}(x, t)| \leq \sup_{s \leq T, y \in \mathbb{R}} |\Lambda(y)V^*(m_{1, \Lambda}(y, s))| \int_0^T \int_{\text{supp}(\Lambda)} H_{t-s}(x, y)\frac{|x-y|}{t-s}dyds \tag{2.20}
\]
Moreover,
\[
\int_0^t \int_{y \in \text{supp}(\Lambda)} H_{t-s}(x, y) \frac{|x-y|}{t-s} dy ds = \int_0^t \int_{y \in \text{supp}(\Lambda)} e^{\frac{-(x-y)^2}{2(t-s)}} \frac{|x-y|}{t-s} dy ds
\]
\[
\leq \frac{1}{2} \int_0^t \int_{y \in \text{supp}(\Lambda)} e^{\frac{-(x-y)^2}{2(t-s)}} dy ds + \frac{1}{2} \int_0^t \int_{y \in \text{supp}(\Lambda)} e^{\frac{-(x-y)^2}{2(t-s)}} dx ds =: D_1 + D_2
\]

(2.21)

\[D_1 \leq C_1 \|\text{supp}(\Lambda)\| T \frac{e^{-d^2(x, \text{supp}(\Lambda))/2T}}{d^2(x, \text{supp}(\Lambda))} \]

(2.22)

\[D_2 \leq C_2 \int_0^t \int_{y \in \text{supp}(\Lambda)} e^{\frac{-4(t-s)}{(t-s)^{3/2}}} dy ds \]

(2.23)

and using the following inequality: \(xe^{-x} \leq 2e^{-x/2} \forall x \in \mathbb{R}\) we have

\[D_2 \leq C_3 \|\text{supp}(\Lambda)\| T^{1/2} \frac{e^{-d^2(x, \text{supp}(\Lambda))/4T}}{d^2(x, \text{supp}(\Lambda))} \]

(2.24)

Therefore, \(\partial_x m_{1,\Lambda}\) also vanishes exponentially as \(|x| \to \infty\). We also have analogous properties for \(m_{2,\Lambda}\). \(\square\)

**Lemma 2.3.** Let \((m_{1,\Lambda}, m_{2,\Lambda})\) be a continuous solution to (2.1) on \(\mathbb{R} \times [0, T^*]\), then for each \(\alpha > 0\) and \(p \geq 1\), \(m_{1,\Lambda}, m_{2,\Lambda} \in L^p(\mathbb{R} \times [0, T], \Lambda(x) dx dt)\), i.e., \(t\), here exists a constant \(C^* = C^*(\alpha, p, T^*)\) such that

\[\|m_{1,\Lambda}\|_{p, \Lambda}^{T^*, \Lambda} \leq C \quad \forall i = 1, 2 \]

(2.25)

Here \(\|f\|_{p, \Lambda}^{T^*, \Lambda} := \left( \int_0^T \int_{\mathbb{R}} |f|^p(x, t) \Lambda(x) dx dt \right)^{1/p}\)

**Proof.** By the Lemma 2.1 and let \(T < T^*\), we obtain

\[0 \leq \int_0^T \int_{\mathbb{R}} (m_{1,\Lambda} - \Lambda F_1)^{2n+1}(\partial_t \frac{1}{2} \partial_x)(m_{1,\Lambda} - \Lambda F_1) dx dt \]

\[= - \int_0^T \int_{\mathbb{R}} (m_{1,\Lambda} - \Lambda F_1)^{2n+1}\frac{1}{2}\left( m_{1,\Lambda}^3 m_{1,\Lambda} + \lambda (m_{1,\Lambda} - m_{2,\Lambda}) \right) dx dt \]

(2.26)
Then, by expanding \((m_{1,\Lambda} - \Lambda F_1)^{2n+1}\),
\[
0 \leq -\int_0^T \int_{\mathbb{R}} \Lambda(x) \left[ \sum_{m=0}^{2n+1} \binom{2n+1}{m} m_{1,\Lambda}^{2n+1-m} (-\Lambda F_1)^m \right] \\
\times \left[ m_{1,\Lambda}^2 - m_{1,\Lambda} + \lambda(m_{1,\Lambda} - m_{2,\Lambda}) \right] \, dx \, dt
\] (2.28)
or
\[
\int_0^T \int_{\mathbb{R}} m_{1,\Lambda}^{2n+4} \Lambda(x) \, dx \, dt \\
\leq \int_0^T \int_{\mathbb{R}} \left[ \sum_{m=1}^{2n+1} \binom{2n+1}{m} |m_{1,\Lambda}|^{2n+4-m} |\Lambda F_1|^m + |1 - \lambda|m_{1,\Lambda}^2 \right. \\
\left. + \lambda |m_{1,\Lambda}|^{2n+1}|m_{2,\Lambda}| \right] \\
\left. + \sum_{m=1}^{2n+1} |1 - \lambda| \binom{2n+1}{m} |m_{1,\Lambda}|^{2n+2-m} |\Lambda F_1|^m \right. \\
\left. + \lambda \sum_{m=1}^{2n+1} \binom{2n+1}{m} |m_{1,\Lambda}|^{2n+1-m} |m_{2,\Lambda}| |\Lambda F_1|^m \left[ \Lambda(x) \right] \, dx \, dt
\] (2.29)
Using Young’s inequality,
\[
\frac{1}{4} \left( \|m_{1,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{2n+4} \\
\leq \sum_{m=1}^{2n+1} \binom{2n+1}{m} \left( \|\Lambda F_1\|^{T,\Lambda}_{2n+4} \right)^{m} \left( \|m_{1,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{2n+4-m} \\
+ |1 - \lambda| \left( \|1\|^{T,\Lambda}_{2n+4} \right)^{2} \left( \|m_{1,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{2n+2} \\
+ \lambda \left( \|1\|^{T,\Lambda}_{2n+4} \right)^{2n+1} \left( \|m_{2,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{m} \\
+ |1 - \lambda| \sum_{m=1}^{2n+1} \binom{2n+1}{m} \left( \|m_{1,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{2n+2-m} \left( \|\Lambda F_1\|^{T,\Lambda}_{p(m,n)} \right)^{m} \\
+ \lambda \sum_{m=1}^{2n+1} \binom{2n+1}{m} \left( \|m_{1,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{2n+1-m} \left( \|m_{2,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{m} \\
\left( \|\Lambda F_1\|^{T,\Lambda}_{p(m,n)} \right)^{m} \tag{2.30}
\]
here \(p(m,n) := \frac{(2n+4)m}{2 + m}\). Thus,
\[
\left( \|m_{1,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{2n+4} \leq P_1 \left( \|m_{1,\Lambda}\|^{T,\Lambda}_{2n+4}, \|m_{2,\Lambda}\|^{T,\Lambda}_{2n+4} \right) \tag{2.31}
\]
In a similar way to \(m_{2,\Lambda}\), we get
\[
\left( \|m_{2,\Lambda}\|^{T,\Lambda}_{2n+4} \right)^{2n+4} \leq P_2 \left( \|m_{1,\Lambda}\|^{T,\Lambda}_{2n+4}, \|m_{2,\Lambda}\|^{T,\Lambda}_{2n+4} \right) \tag{2.32}
\]
where \(P_1(x,y)\) and \(P_2(x,y)\) is a polynomial of \(x, y\) with total degree \(2n + 3\). By letting \(T \to T^*\) and note that coefficients of the polynomial \(P_1, P_2\) are all bounded as \(T \to T^*\). Thus there exists \(C\) such that
\[
\|m_{i,\Lambda}\|^{T^*,\Lambda}_{2n+4} \leq C \quad \forall i = 1, 2 \tag{2.33}
\]
Lemma 2.4. The kernel $H_t(x, y) \in L^p(\mathbb{R} \times [0, T])$ for any $T < \infty$ and $p < 3$.

Proof. We have

$$\int_0^T \int_{\mathbb{R}} H_t^p(x, y)dyds = \int_0^T \int_{\mathbb{R}} \frac{-p(x - y)^2}{2t} dydt$$

(2.34)

and by changing variables, set $z := \sqrt{p(y - x)} \Rightarrow dz = \frac{\sqrt{p}}{\sqrt{2t}} dy$. Thus

$$\int_0^T \int_{\mathbb{R}} e^{-\frac{z^2}{2t}} \frac{1}{t^{p/2 - 1/2}} dzdt = C \int_0^T \frac{1}{t^{p/2 - 1/2}} dt = Ct^{3/2 - p/2} \bigg|_{t=0}^T < \infty \Leftrightarrow p < 3.$$  (2.35)

Lemma 2.5. Let $(m_{1,\Lambda}, m_{2,\Lambda})$ be a continuous solution to (2.1) on $\mathbb{R} \times [0, T^\ast)$, then it is bounded.

Proof. Using Holder’s inequality, we get

$$\left\|\chi_{[0, T^\ast]}(m_{1,\Lambda})\right\|_{L^\infty} \leq \left\|\chi_{[0, T^\ast]}H\right\|_2 \left\|\chi_{[0, T^\ast]}\Lambda(V'(m_{1,\Lambda}) + \lambda(m_{1,\Lambda} - m_{2,\Lambda}))\right\|_2 + \left\|\chi_{[0, T^\ast]}\Lambda F_1\right\|_{L^\infty}$$

(2.36)

Moreover, by $\Lambda \leq 1$, we get

$$\left\|\chi_{[0, T^\ast]}\Lambda(V'(m_{1,\Lambda}) + \lambda(m_{1,\Lambda} - m_{2,\Lambda}))\right\|_2 \leq \left\|V'(m_{1,\Lambda}) + \lambda(m_{1,\Lambda} - m_{2,\Lambda})\right\|_{2^{T^\ast,\Lambda}}.$$  (2.37)

and implies the result.

Lemma 2.6. There is a continuous solution to (2.1) on $\mathbb{R} \times [0, \infty)$.

Proof. This follows from Lemma 2.5.

3 Infinite Volume Equations

This section aims to show global existence and uniqueness (1.2) by letting $\Lambda(x) \to 1$ in (2.1). We now introduce several measures on unbounded domain $\mathbb{R} \times \mathbb{R}^+$

$$\mu(dt, dx) = e^{-\alpha^2t/2 - \alpha|x|} dxdt$$

$$\mu_T(dt, dx) = \chi_{[0, T]}\mu(dt, dx)$$

$$\mu_\Lambda(dt, dx) = \Lambda(x)\mu(dt, dx)$$

$$\mu_{T,\Lambda}(dt, dx) = \chi_{[0, T]}\Lambda\mu(dt, dx)$$

(3.1)
Moreover, we have analogous version for measure $\mu_{T, \Lambda}$ as shown in Lemma 7, [1].

**Lemma 3.1.** If $f(x, t)$ and $(\partial_t - \frac{1}{2}\Delta)f$ are continuous on $\mathbb{R} \times [0, T]$ and $f(x, 0) = 0$, $|f|^{2n+2}$ and $|\nabla f|^{2n+2} \in L^1(\mathbb{R} \times [0, T], d\mu)$, then

$$
\int_{\mathbb{R}^n} f^{2n+1}(x, t)(\partial_t - \frac{1}{2}\Delta)f(x, t)d\mu_{T, \Lambda}(x, t) \geq 0
$$

(3.2)

Let us recall that the operator $f \to Hf$ is a bounded map from $L^p(\mu) \to L^p(\mu)$ for any $p > 1$. This fact was proved in Lemma 9, [1].

**Lemma 3.2.** The operator $H$ with heat kernel $H_t(x, y)$ is a bounded map from $L^p(d\mu) \to L^p(\mu)$ for any $p > 1$.

*Proof.* Let $g \in L^p(\mu)$ and set $f := Hg$. We prove that $f$ is also in $L^p(\mu)$ and $\|H\| < \infty$. We have

$$
f(x, t) = \int_0^\infty \int_{\mathbb{R}} \chi_{[0, \infty]}(t-s)H_{t-s}(x, y)\chi_{[0, \infty]}(s)g(y, s)dyds
$$

(3.3)

Multiplying both sides $e^{-\alpha|x|/p-\alpha^2t/2p}$ and note that $e^{-\alpha|x|} \leq e^{\alpha|x-y|e^{-|y|}}$, we derive

$$
e^{-\alpha|x|/p-\alpha^2t/2p}f(x, t)
\leq \int_0^\infty \int_{\mathbb{R}} \left( \chi_{[0, \infty]}(t-s)H_{t-s}(x, y)e^{\alpha|x-y|/p-\alpha^2(t-s)/2p} \right.
\left. \times \chi_{[0, \infty]}(s)g(y, s)e^{-\alpha|y|/p-\alpha^2s/2p} \right)dyds
$$

(3.4)

The RHS is a convolution of $g$ and $H_t$ and it is suggested to apply Young’s inequality

$$
\|f\|_{L^p(\mu)} \leq \|g\|_{L^p(\mu)} \int_0^\infty \int_{\mathbb{R}} H_t(x, y)e^{\alpha|x-y|/p-\alpha^2t/2p}dydt
$$

(3.5)

Moreover,

$$
\|H\|_{L^p(\mu)\to L^p(\mu)} \leq \int_0^\infty \int_{\mathbb{R}} H_t(x, y)e^{\alpha|x-y|/p-\alpha^2t/2p}dydt
\leq \int_0^\infty e^{-\alpha^2t/2p} \int_{\mathbb{R}} \frac{e^{-(x-y)^2/2t+\alpha|x-y|/p}}{\sqrt{2\pi t}}dydt
\leq \int_0^\infty e^{-\alpha^2t/2p(1-1/p)}dt \int_{\mathbb{R}} e^{-\frac{\sqrt{\alpha}}{\sqrt{2}p}z}dz < \infty \iff p > 1.
$$

(3.6)

The following important property was given in Lemma 12, [1]

**Lemma 3.3.** The operator $H$ with heat kernel $H_t(x, y)$ is a bounded map from $L^p(d\mu) \to C^0(\mathbb{R}^+ \times \mathbb{R})$ for any $p > 3/2$. 

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Proof. Let \( g \in L^p(\mu) \) and set \( f := Hg \). We prove that \( f \) is also in \( L^p(\mu) \) and \( \|H\| < \infty \). We have

\[
f(x, t) = \int_0^\infty \int_\mathbb{R} \chi_{[0, \infty]}(t-s)H_{t-s}(x, y)\chi_{[0, \infty]}(s)g(y, s)dyds \quad (3.7)
\]

Multiplying both sides \( e^{-\alpha|x|}e^{-\alpha^2 t/2} \) and note that \( e^{-\alpha|x|} \leq e^{\alpha|x-y|}e^{-|y|} \), we derive

\[
e^{-\alpha|x|}e^{-\alpha^2 t/2}|f(x, t)| \leq \int_0^\infty \int_\mathbb{R} \left( \chi_{[0, \infty]}(t-s)H_{t-s}(x, y)e^{\alpha|x-y|}-\alpha^2(t-s)/2 \right) \times \chi_{[0, \infty]}(s)g(y, s)e^{-\alpha|y|-\alpha^2 s/2} dyds \quad (3.8)
\]

The RHS is a convolution of \( g \) and \( H_t \) and it is suggested to apply Young’s inequality

\[
\|f\|_{C^\alpha(\mathbb{R} \times \mathbb{R}^+)} \leq \|g\|_{L^p(\mu)} \int_0^\infty \int_\mathbb{R} H_t^q(x, y)e^{\alpha|x-y|-\alpha q^2 t/2} dydt \quad (3.9)
\]

Moreover,

\[
\|H\|_{L^p(\mu) \rightarrow C^\alpha(\mathbb{R} \times \mathbb{R}^+)} \leq \int_0^\infty \int_\mathbb{R} H_t^q(x, y)e^{\alpha^2 t/2} \sqrt{2\pi t} dydt
\]

\[
\leq \int_0^\infty \int_\mathbb{R} \frac{e^{-q(x-y)^2/2t+\alpha^2(x-y)}-q^2 t/2} \sqrt{2\pi t} dydt
\]

\[
\leq C \int_0^\infty t^{(1-q)/2} dt \int_\mathbb{R} e^{-\left(\frac{\sqrt{q}}{\sqrt{2t}}(y-x)-\frac{\alpha\sqrt{q}}{2}\right)^2} dy < \infty
\]

\[
\Leftrightarrow q < 3 \Leftrightarrow p > 3/2. \quad (3.10)
\]

Lemma 3.4. Let \((m_{1,\Lambda}, m_{2,\Lambda})\) be a continuous solution to (2.1) on \( \mathbb{R} \times \mathbb{R}^+ \), then for each \( \alpha > 0 \) and \( p \geq 1 \), there exists a constant \( C = C(\alpha, p) \) independent of \( \Lambda \) such that

\[
\|m_{i,\Lambda}\|_{L^p(\mu)} \leq C \quad \forall i = 1, 2 \quad (3.11)
\]

Proof. By the Lemma 3.1 and take arbitrarily \( T > 0 \), we obtain

\[
0 \leq \int_{\mathbb{R}^+} \int_\mathbb{R} (m_{1,\Lambda} - \Lambda F_1)^{2n+1} (\partial_t - \frac{1}{2} \partial_{xx}) (m_{1,\Lambda} - \Lambda F_1) d\mu_T(x, t)
\]

\[
= -\int_0^T \int_\mathbb{R} (m_{1,\Lambda} - \Lambda F_1)^{2n+1} \Lambda \left( m_{1,\Lambda}^3 - m_{1,\Lambda} + \lambda (m_{1,\Lambda} - m_{2,\Lambda}) \right) d\mu(x, t) \quad (3.12)
\]

Then, by expanding \( m_{1,\Lambda} - \Lambda F_1)^{2n+1},

\[
0 \leq -\int_0^T \int_\mathbb{R} \Lambda(x) \left[ \sum_{m=0}^{2n+1} \frac{(2n+1)}{m} m_{1,\Lambda}^{2n+1-m} (-\Lambda F_1)^m \right] \times \left[ m_{1,\Lambda}^3 - m_{1,\Lambda} + \lambda (m_{1,\Lambda} - m_{2,\Lambda}) \right] d\mu(x, t) \quad (3.13)
\]
Take the highest order term above to the left side to derive the estimate

\[
\int_{0}^{T} \int_{\mathbb{R}} m_{1,\Lambda}^{2n+4} \Lambda(x) d\mu(x, t) \\
\leq \int_{0}^{T} \int_{\mathbb{R}} \left( \sum_{m=1}^{2n+1} \binom{2n+1}{m} m_{1,\Lambda}^{2n+4-m} |\Lambda F_1|^m \right) \\
+ |1 - \lambda|^{m_{1,\Lambda}^{2n+2} + \lambda m_{1,\Lambda}^{2n+1} |m_{2,\Lambda}|} \\
+ \sum_{m=1}^{2n+1} |1 - \lambda|^{2n+1} m_{1,\Lambda}^{2n+2-m} |\Lambda F_1|^m \\
+ \lambda \sum_{m=1}^{2n+1} \binom{2n+1}{m} m_{1,\Lambda}^{2n+1-m} |m_{2,\Lambda}| |\Lambda F_1|^m \right| \Lambda(x) d\mu(x, t)
\]

and using Young’s inequality,

\[
\frac{1}{4} \left( \left\| m_{1,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)^{2n+4} \\
\leq \sum_{m=1}^{2n+1} \binom{2n+1}{m} \left( \left\| \Lambda F_1 \right\|_{L^{2n+4}(\mu_{T,A})} \right)^{m} \left( \left\| m_{1,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)^{2n+4-m} \\
+ |1 - \lambda|^{2n+2} \left( \left\| m_{1,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)^{2n+1} \left( \left\| m_{2,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)^{2n+2-m} \left( \left\| \Lambda F_1 \right\|_{L^{p(m,n)}(\mu_{T,A})} \right)^{m} \\
+ \lambda \sum_{m=1}^{2n+1} \binom{2n+1}{m} \left( \left\| m_{1,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)^{2n+1-m} \times \\
\left( \left\| m_{2,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right) \left( \left\| \Lambda F_1 \right\|_{L^{p(m,n)}(\mu_{T,A})} \right)^{m}
\]

(3.15)

here \( p(m, n) := \frac{(2n + 4)m}{2 + m} \). Thus,

\[
\left( \left\| m_{1,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)^{2n+4} \leq \mathcal{T}_1 \left( \left\| m_{1,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} , \left\| m_{2,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)
\]

(3.16)

where \( P(x, y) \) is a polynomial of \( x, y \) with total degree \( 2n + 3 \).

In a similar way to \( m_{2,\Lambda} \), we get

\[
\left( \left\| m_{2,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)^{2n+4} \leq \mathcal{T}_2 \left( \left\| m_{2,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} , \left\| m_{1,\Lambda} \right\|_{L^{2n+4}(\mu_{T,A})} \right)
\]

(3.17)

note that coefficients of the polynomial \( \mathcal{T}_1, \mathcal{T}_2 \) are all bounded as \( T \to \infty \) and \( \Lambda \to 1 \)

\[
\left\| \Lambda F_i \right\|_{L^{p}(\mu_{T,A})} \leq \left\| F_i \right\|_{L^{p}(\mu)} < \infty , \quad \left\| 1 \right\|_{L^{p}(\mu_{T,A})} \leq \left\| 1 \right\|_{L^{p}(\mu)} < \infty \]

(3.18)
Thus there exists $C'$ independent of $T$ and $\Lambda$ such that
\[
\|m_{i,\Lambda}\|_{L^{2n+4}(\mu_{T,\Lambda})} \leq C' \quad \forall i = 1, 2 \quad (3.19)
\]
So there is $C = C(\alpha, p) > 0$
\[
\|m_{i,\Lambda}\|_{L^p(\mu_{T,\Lambda})} \leq C \quad \forall i = 1, 2 \quad (3.20)
\]
Since $\Lambda \leq 1$ and (3.11) and letting $T \to \infty$
\[
\|\Lambda m_{i,\Lambda}\|_{L^p(\mu)} \leq \lim_{T \to \infty} \|m_{i,\Lambda}\|_{L^p(\mu_{T,\Lambda})} < \infty \quad (3.21)
\]
Moreover, the heat operator $H$ is bounded from $L^p(\mu)$ to $L^p(\mu)$ (Lemma 3.2).
Thus, from
\[
\|m_{i,\Lambda}\|_{L^p(\mu)} \leq \|H\| \left\| \Lambda \left( V'(m_{i,\Lambda}) + \lambda(m_{i,\Lambda} - m_{i+1,\Lambda}) \right) \right\|_{L^p(\mu)} + \|F_i\|_{L^p(\mu)}
\]
here $\|H\| = \|H\|_{L^p(\mu) \to L^p(\mu)}$. This is combined with (3.21), the lemma follows.
\[
\square
\]

**Lemma 3.5.** For $\beta$ large enough and for all $i = 1, 2$, $e^{-\beta t}\Lambda m_{i,\Lambda}$ and $e^{-\beta t}\Lambda m_{i,\Lambda}$ are Cauchy in $L^p(\mu)$ as $\Lambda \to 1$.

**Proof.** Rewrite (2.1) into PDE
\[
(\partial_t - \frac{1}{2}\partial_{xx})(m_{1,\Lambda} - \Lambda F_1) = -\Lambda \left( V'(m_{1,\Lambda}) + \lambda(m_{1,\Lambda} - m_{2,\Lambda}) \right) \quad (3.23)
\]
it follows that for any $\beta > 0$
\[
(\partial_t - \frac{1}{2}\partial_{xx}) \left( e^{-\beta t}(m_{1,\Lambda} - \Lambda F_1) \right)
= -e^{-\beta t} \Lambda \left( V'(m_{1,\Lambda}) + \beta m_{1,\Lambda} + \lambda(m_{1,\Lambda} - m_{2,\Lambda}) \right) - \beta e^{-\beta t}(1 - \Lambda)m_{1,\Lambda}
+ \beta e^{-\beta t} \Lambda F_1 \quad (3.24)
\]
Thus
\[
(\partial_t - \frac{1}{2}\partial_{xx}) \left( e^{-\beta t}(m_{1,\Lambda'} - m_{1,\Lambda}) - e^{\beta t}(\Lambda' - \Lambda)F_1 \right)
= -e^{-\beta t} \left\{ \Lambda' \left( V'(m_{1,\Lambda'}) + \beta m_{1,\Lambda'} + \lambda(m_{1,\Lambda'} - m_{2,\Lambda'}) \right) \right. \\
- \Lambda \left( V'(m_{1,\Lambda}) + \beta m_{1,\Lambda} + \lambda(m_{1,\Lambda} - m_{2,\Lambda}) \right) \left. \right\} \\
- \beta e^{-\beta t}(1 - \Lambda)m_{1,\Lambda'} - (1 - \Lambda)m_{1,\Lambda}
+ \beta e^{-\beta t}(\Lambda' - \Lambda)F_1 \quad (3.25)
\]
Using the Lemma 3.1, we get

\[
0 \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left( e^{-\beta t} (m_{1,A} - m_{1,A'}) - e^{-\beta t} (\Lambda' - \Lambda) F_1 \right)^{2n+1} \times \\
\left( \partial_t - \frac{1}{2} \Delta \right) \left( e^{-\beta t} (m_{1,A'} - m_{1,A}) - e^{-\beta t} (\Lambda' - \Lambda) F_1 \right) d\mu
\]

\[
= - \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} \left( (m_{1,A'} - m_{1,A}) - (\Lambda' - \Lambda) F_1 \right)^{2n+1} \times \\
\left[ \Lambda' \left( V'(m_{1,A'}) + \beta m_{1,A'} - \lambda (m_{2,A'} - m_{1,A'}) \right) \\
- \Lambda \left( V'(m_{1,A}) + \beta m_{1,A} - \lambda (m_{2,A} - m_{1,A}) \right) \right] d\mu
\]

\[
= - \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} \left( \Lambda' m_{1,A'} - \Lambda m_{1,A} \right)^{2n+1} \times \\
\left[ V'(\Lambda m_{1,A'}) + \beta \Lambda m_{1,A'} - \lambda (\Lambda m_{2,A'} - \Lambda m_{1,A'}) \\
- \left( V'(\Lambda m_{1,A}) + \beta \Lambda m_{1,A} - \lambda (\Lambda m_{2,A} - \Lambda m_{1,A}) \right) \right] d\mu + R_1
\]

\[
= - \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} \left( \Lambda' m_{1,A'} - \Lambda m_{1,A} \right)^{2n+1} (A_1 + A_2) d\mu + R_1
\]

where

\[
A_1 = V'(\Lambda m_{1,A'}) + \beta \Lambda m_{1,A'} - V'(\Lambda m_{1,A}) - \beta \Lambda m_{1,A} \\
A_2 = -\lambda \left( \Lambda m_{2,A'} - \Lambda m_{2,A} - (\Lambda m_{1,A'} - \Lambda m_{1,A}) \right)
\]

and \( R_1 \) consists of terms vanishing as \( \Lambda, \Lambda' \to 1 \). Setting

For \( \beta \) large enough, \( V' + \beta I d \) is a monopolynomial, so there exists \( k > 0 \) such that

\[
(V'(x) + \beta x - V'(y) - \beta y)(x - y) \geq k(x - y)^4 \quad \forall x, y \in \mathbb{R}
\]

(3.28)

Setting \( X_1 = \Lambda m_{1,A'} - \Lambda m_{1,A} \) and \( X_2 = \Lambda m_{2,A'} - \Lambda m_{2,A} \)

\[
k \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} X_1^{2n+4} d\mu + \lambda \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} X_1^{2n+1} (X_1 - X_2) d\mu \leq R_1
\]

(3.29)

We also get similar expression for \( X_2 \)

\[
k \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} X_2^{2n+4} d\mu + \lambda \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} X_2^{2n+1} (X_2 - X_1) d\mu \leq R_2
\]

(3.30)

Taking the sum of (3.29) and (3.30)

\[
k \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} X_1^{2n+4} + X_2^{2n+4} d\mu
\]

\[
+ \lambda \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} X_1^{2n+1} - X_2^{2n+1} (X_1 - X_2) d\mu \leq R_1 + R_2
\]

(3.31)

It is easy to see that the second term of LHS is non-negative and can be neglected

\[
k \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-2(2n+2)\beta t} X_1^{2n+4} + X_2^{2n+4} d\mu \leq R_1 + R_2
\]

(3.32)
Since \(e^{-(2n+2)\beta t} \geq e^{-(2n+4)\beta t}\), we derive
\[
\|e^{-\beta t}(\Lambda'm_{i,N} - \Lambda m_{i,A})\|_{L^{2n+4}(\mu)} \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t} x^{2n+4} d\mu \leq \frac{R_1 + R_2}{k} \to 0
\]  
(3.33)
as \(\Lambda, \Lambda' \to 1\), which implies the statement.

With help of above lemma, we are ready to establish global existence and uniqueness in space \(C^\alpha(\mathbb{R} \times \mathbb{R}^+)\).

### 3.0.1 Proof of Theorem 1.1

**Proof.** (Existence) By Lemma 3.5, there is a \(\beta > 0\) such that
\[
e^{-\beta t}\Lambda m_{i,A} \to e^{-\beta t} m_i \quad \text{in } L^p(d\mu)
\]  
(3.34)
but \(e^{-\beta t}\Lambda m_{i,A} = e^{-\beta t}m_{i,A} + e^{-\beta t}(1 - \Lambda)m_{i,A}\) and this follows
\[
e^{-\beta t}m_{i,A} \to e^{-\beta t}m_i \quad \text{in } L^p(d\mu) \quad \text{as } \Lambda \to 1
\]  
(3.35)
If \(\beta < \alpha^2/2\), we choose \(q, r \geq 1\) satisfies \(\frac{1}{p} = \frac{1}{q} + \frac{1}{r}, 1 < q < \frac{\alpha^2}{2\beta}\) and applying Young’s inequality
\[
\|m_{i,A} - m_i\|_{L^p(\mu)} = \left\|e^{\beta \xi}(e^{-\beta t}m_{i,A} - e^{-\beta t}m_{i,A})\right\|_{L^p(\mu)} \leq \left\|e^{\beta \xi}\right\|_{L^r(\mu)} \left\|e^{-\beta t}m_{i,A} - e^{-\beta t}m_i\right\|_{L^q(\mu)}
\]  
(3.36)
Since \(1 < q < \frac{\alpha^2}{2\beta}\), this implies \(\|e^{\beta \xi}\|_{L^r(\mu)} < \infty\) and from (3.35), we obtain as \(\Lambda \to 1\)
\[
m_{i,A} \to m_i \quad \text{in } L^p(\mu) \quad \forall i = 1, 2
\]  
(3.37)
Therefore, (1.2) has a solution \((m_1, m_2)\) in \((L^p(\mu))^2\). Moreover, the operator \(H\) is a bounded map from \(L^p\) to \(C^\alpha(\mathbb{R} \times \mathbb{R}^+)\) (Lemma 3.3), (1.2) has also a continuous solution in \(\left(C^\alpha(\mathbb{R} \times \mathbb{R}^+)\right)^2\) with \(\alpha^2/2 > \beta\).

To show (1.2) has a solution in \(\left(C^\alpha(\mathbb{R} \times \mathbb{R}^+)\right)^2\) for every \(\alpha > 0\). Using the Lemma 3.4, there exists \((m'_1, m'_2)\) and a weakly convergence sequence such that
\[
m_{i,A_n} \to m'_i \quad \text{in } L^p(d\mu) \quad \forall i = 1, 2
\]  
(3.38)
This is combined with (3.35), we get \(m'_i = m_i\) a.e and hence \(m_i \in L^p(d\mu)\) and \(H\) is closed by the Lemma 3.3. Furthermore, the operator \(H\) is a bounded map from \(L^p(\mu)\) to \(C^\alpha(\mathbb{R} \times \mathbb{R}^+)\)(Lemma 3.3),the system has a continuous solution \((m_1, m_2)\) in \(\left(C^\alpha(\mathbb{R} \times \mathbb{R}^+)\right)^2\).

(Uniqueness) Let \((m_1, m_2)\) and \((m'_1, m'_2)\) be two solutions. Multiplying by \(e^{-\beta t}\) for a large enough \(\beta\) and use the same computation in the Lemma 3.5, we get \(e^{-\beta t}m_i = e^{-\beta t}m'_i \quad \forall i = 1, 2\). Hence \(m_i = m'_i \quad \forall i = 1, 2\). Indeed, we have
\[
(\partial_t - \frac{1}{2}\partial_{xx})(m_i - F_1) = -V'(m_i) + \lambda(m_2 - m_1)
\]  
(3.39)
Then
\((\partial_t - \frac{1}{2} \partial_{xx}) \left( e^{-\beta t}(m_1 - F_1) \right) = -e^{-\beta t} \left( V'(m_1) + \beta m_1 + \lambda (m_1 - m_2) \right) + \beta e^{-\beta t} F_1 \) \hspace{1cm} (3.40)

As a result,

\[
(\partial_t - \frac{1}{2} \partial_{xx}) \left( e^{-\beta t}(m'_1 - m'_1) \right) = -e^{-\beta t} \left\{ \left( V'(m'_1) + \beta m'_1 + \lambda (m'_1 - m'_2) \right) \right\} \]

(3.41)

Applying the Lemma 3.1,

\[
0 \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left( e^{-\beta t}(m'_1 - m_1) \right)^{2n+1} (\partial_t - \frac{1}{2} \partial_{xx}) \left( e^{-\beta t}(m'_1 - m_1) \right) d\mu
\]

\[
= -\int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}(m'_1 - m_1)^{2n+1} \left( V'(m'_1) + \beta m'_1 - V'(m_1) - \beta m_1 \right) d\mu
\]

\[
- \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}(m'_1 - m_1)^{2n+1} \left( m'_1 - m'_1 - (m'_2 - m_2) \right) d\mu =: -\overline{A}_1 - \overline{A}_2
\]

(3.42)

where

\[
\overline{A}_1 := \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}(m'_1 - m_1)^{2n+1} \left( V'(m'_1) + \beta m'_1 - V'(m_1) - \beta m_1 \right) d\mu
\]

\[
\overline{A}_2 := \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}(m'_1 - m_1)^{2n+1} \left( m'_1 - m'_1 - (m'_2 - m_2) \right) d\mu
\]

Setting \(\overline{x}_1 := m'_1 - m_1\) and \(\overline{x}_2 := m'_2 - m_2\) and recalling

\[
(V'(x) + \beta x - V'(y) - \beta y)(x - y) \geq k(x - y)^4 \quad \forall x, y \in \mathbb{R}
\]

(3.44)

We get

\[
\overline{A}_1 \geq \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}(m'_1 - m_1)^{2n+4} d\mu \geq \left\| e^{-\beta t}(m'_1 - m_1) \right\|^{2n+4}_{L^{2n+4}(\mu)}
\]

(3.45)

and

\[
\overline{A}_2 = \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}X_1^{2n+1}(X_1 - X_2)d\mu
\]

(3.46)

and from \(\overline{A}_1 + \overline{A}_2 \leq 0\), we derive

\[
\left\| e^{-\beta t}(m'_1 - m_1) \right\|^{2n+4}_{L^{2n+4}(\mu)} + \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}X_1^{2n+1}(X_1 - X_2)d\mu \leq 0
\]

(3.47)

Analogously,

\[
\left\| e^{-\beta t}(m'_2 - m_2) \right\|^{2n+4}_{L^{2n+4}(\mu)} + \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}X_2^{2n+1}(X_2 - X_1)d\mu \leq 0
\]

(3.48)

Taking the sum of (3.47) and (3.48), we have

\[
\left\| e^{-\beta t}(m'_1 - m_1) \right\|^{2n+4}_{L^{2n+4}(\mu)} + \left\| e^{-\beta t}(m'_2 - m_2) \right\|^{2n+4}_{L^{2n+4}(\mu)} + \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-(2n+2)\beta t}(X_1^{2n+1} - X_2^{2n+1})(X_1 - X_2)d\mu \leq 0
\]

(3.49)
Since the third term on LHS is non-negative, it is easy to have
\[ \| e^{-\beta t} (m'_i - m_i) \|_{L^{2n+4}(\mu)} = 0 \quad \forall i = 1, 2 \] (3.50)
As a result,
\[ m'_i = m_i \quad \forall i = 1, 2 \] (3.51)

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