Optimal error estimates of the penalty finite element method for the unsteady Navier-Stokes equations with nonsmooth initial data

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Abstract

In this paper, both semidiscrete and fully discrete finite element methods are analyzed for the penalized two-dimensional unsteady Navier-Stokes equations with nonsmooth initial data. First order backward Euler method is applied for the time discretization, whereas conforming finite element method is used for the spatial discretization. Optimal $L^2$ error estimates for the semidiscrete as well as the fully discrete approximations of the velocity and of the pressure are derived for realistically assumed conditions on the data. The main ingredient in the proof is the appropriate exploitation of the inverse of the penalized Stokes operator, negative norm estimates and time weighted estimates. Two numerical examples one in 2D and one in 3D are presented whose results are conforming our theoretical findings. Finally, computational experiments on benchmark problem: one on lid driven cavity problem and other on flow around a cylinder with low viscosity are discussed.

Key Words: Navier-Stokes equations, penalty method, backward Euler method, optimal $L^2$ error estimates, uniform error estimates, benchmark computation.

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1 Introduction

Let $\Omega$ be a bounded convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial \Omega$. Now consider the following incompressible Navier-Stokes system in a space-time domain $\Omega \times (0, \infty)$

\begin{equation}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p = f \quad \text{in } \Omega, \ t > 0,
\end{equation}

with incompressibility condition

\begin{equation}
\nabla \cdot u = 0 \quad \text{on } \Omega, \ t > 0,
\end{equation}

and initial with boundary conditions

\begin{equation}
u(x, 0) = u_0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \ t \geq 0.
\end{equation}

Here, $u$ denotes the velocity vector, $p$ represents the pressure of the fluid and $\nu > 0$ is the kinematic coefficient of viscosity. Further, the forcing term $f$ and the initial velocity $u_0$ are given functions in their respective domains of definition.

The time-dependent Navier-Stokes equations (NSEs) for the incompressible flow has always been a major challenge in computational PDEs. The main difficulty in computation is that, at each time step, the velocity $u$ and the pressure $p$ are coupled together by the incompressibility condition, $\text{div } u = 0$. A common way to tackle this difficulty is to address the imposition of the incompressibility condition in an appropriate way so as to obtain a pseudo-compressible system. In this regard, methods, which come to our mind, are the penalty method, the artificial compressibility method, the pressure stabilized method, the pressure correction method, the projection method, etc. (see, [1, 3, 8, 16, 17, 20, 22, 25, 27, 32] and references, therein).

In the present paper, a completely discrete penalty finite element method to the NSEs is applied to circumvent this difficulty. It is the simplest and effective finite element implementation to handle the incompressibility. This method is often employed in order to decouple the pressure equation from the system of nonlinear algebraic equations in velocity which is obtained from finite element (or finite difference) discretizations of the Navier-Stokes equations at each time level. The basic idea of the proposed method is to

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add a pressure term to the continuity equation. The resulting penalized system in our case is to approximate the solution \((u, p)\) of (1.1)-(1.3) by \((u_c, p_c)\) satisfying

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{u_c}{\nu} + u_c \cdot \nabla u_c - \nu \Delta u_c + \frac{1}{\nu} (\nabla \cdot u_c) u_c + \nabla p_c = f(x, t) \quad \text{in } \Omega, \quad t > 0, \\
\nu \nabla \cdot u_c + \varepsilon p_c = 0 \quad \text{on } \Omega, \quad t > 0, \\
u u_c |_{t=0} = u_{00} \quad \text{in } \Omega, \quad u_c = 0 \quad \text{on } \partial \Omega, \quad t \geq 0.
\end{array} \right.
\end{aligned}
\]

Here \(\varepsilon > 0\) is the penalty parameter. One advantage of this approach is that we can eliminate pressure \(p_c\) from (1.3) to obtain an equation \(u_c\) only. We note here that it is standard to add the term \(\frac{1}{\nu} (\nabla \cdot u_c) u_c\) to the non-linear term, which was introduced by Temam [30], that ensures the dissipativity of the system (1.4).

The penalty method first appeared in the work of Courant [6] for a membrane problem, in the framework of the calculus of variations. Since then, it has been considerably developed over the time. Besides its applications to constrained variational problems and variational inequalities, the penalty method was proved to be quite successful for numerical computations in continuum fluid and solid mechanics. Its application to NSEs has been initiated in the late 1960’s by Temam [30]. Subsequently, many publications have been devoted to the study of the penalty method for the steady Stokes and NSEs, as well as for the unsteady NSEs, in continuous, semidiscrete and fully discrete cases, see, for example, [3][11][13][10][29] and references therein. Recent results on the penalty method can be categorized as follows: two-grid penalty method [2][4][15], iterative penalty method [7][14], methods based on different boundary conditions such as nonlinear slip boundary conditions [2][7][22], friction boundary conditions [23][25], slip boundary conditions [32], etc. Moreover, penalty method is used, very recently, for the stochastic 2-D incompressible NSEs [20], and for the incompressible NSEs with variable density [11].

This article deals with optimal \(L^2\)-error estimates for the velocity and for the pressure term in the semidiscrete as well as in the fully discrete case when penalty finite element method is applied to NSE (1.1)-(1.3) with nonsmooth initial data. Earlier, Shen [29] has applied the backward Euler method to discretize only in time and has derived optimal error estimates with respect to the penalty parameter and also with respect to the discretizing parameter in time. Later on, He [13] has extended it to space discretization and the following error estimate has been established for the conforming fully discrete finite element method for all \(t_n \in [0, T], T > 0\),

\[
\tau(t_n) \|u(t_n) - u_{n}^h\|_{H^1} + \left( k \sum_{m=0}^{n} \tau^2(t_m) \|p(t_m) - p_{n}^h\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C(\varepsilon + h + k),
\]

where \((u(t_n), p(t_n))\) and \((u_{n}^h, p_{n}^h)\) are the solutions of the NSEs and its fully discrete penalized system, respectively. Here, \(C\) is the positive constant, \(h\) is the mesh size, \(k\) is the time step, \(t_n = nk, 0 \leq n \leq N = T/k, \tau(t_n) = \min\{t_n, 1\}\). In both these paper, smallness condition on \(k\) is assumed like \(Ck < 1\), where \(C\) depends on \(\nu^{-1}\), Sobolev constants, etc. Subsequently, Lu and Lin [24] have discussed optimal error estimates for the semidiscrete problem under smooth initial data. But to the best of our knowledge, optimal \(L^2\) error estimates for the velocity and the pressure in the semidiscrete and fully discrete cases of the penalized unsteady NSEs with nonsmooth initial data have not been obtained in the literature although these results have been established numerically on various occasions. The purpose of this paper is to fill this gap. We extend the work of Shen [29] and He [13], and obtain the optimal error estimates for the velocity and the pressure in \(L^2\)-norm.

Our analysis is based under realistically assumed conditions on the initial data \(u_0\) in \(H^3\). We take into account the lack of regularity endured by the solutions of the Navier-Stokes system at the initial time \(t = 0\). Assuming otherwise requires the data to satisfy some local-compatibility conditions, which are not natural and difficult to verify in practice, see [18]. We note that in [29], the penalized error estimates have been obtained for \(u_0\) in \(H^3\), but the time discrete error estimates have been obtained under additional assumption of \(u_0\) in \(H^2 \cap H^1\). Also in [13], the results have been obtained for the smooth initial data \((u_0)\) in \(H^2 \cap H^1\). We take a more realistic approach and consider nonsmooth initial data, that is, \(u_0\) in \(H^3\), which poses more serious difficulties, mainly in both semidiscrete and fully discrete analyses.

The main results of this article consist of the following:

- uniform in time bounds for the semidiscrete as well as fully discrete solution are established with no additional smallness assumptions on time step \(k\).
- optimal error estimates for the finite element approximation of the penalized velocity and pressure are derived,

\[
\|u(t_n) - u_{n}^h\| + h(\|\nabla (u(t_n) - u_{n}^h(t_n))\| + \|p(t_n) - p_{n}^h(t_n)\|) \leq C h^{m+1} t^{-\frac{m}{2}},
\]

where \((u_c, p_c)\) and \((u_{n}^h, p_{n}^h)\) are the solution of (1.4) and its semidiscrete system, respectively.
- optimal error bounds for the velocity and the pressure terms, for the fully discrete penalized system, are derived which are of the form

\[
\|u(t_n) - U_n^\varepsilon\| \leq C \left( (\varepsilon + k) t^{-\frac{1}{2}} + h^{m+1} t^{-\frac{m}{2}} \right),
\]

\[
\|\nabla (u(t_n) - U_n^\varepsilon)\| + \|p(t_n) - P_n^\varepsilon\| \leq C \left( (\varepsilon + k) t^{-1} + h^{m} t^{-\frac{m}{2}} \right),
\]
where $(u, p)$ and $(U^n_v, P^n_p)$ are the solution of (L1)-(L3) and the fully discrete system of (L4), respectively.

- since constants involved in the error estimates of both semidiscrete and fully discrete schemes depend exponentially on time, using uniqueness assumption, the error estimates are shown to be valid uniformly in time.
- a couple of numerical examples are discussed to verify the theoretical findings and another couple of examples on benchmark problems with small viscosity are presented.

The remaining part of this paper is arranged as follows: Notations, assumptions and a couple of standard results are stated in the first part of the Section 2, whereas in the second part, we briefly look at the penalty method. The semidiscrete error analysis is carried out in Section 3 and in Section 4, backward Euler method is applied to the penalized system. Finally, in Section 5, some numerical examples are given which validate our theoretical findings.

## 2 Preliminaries

For our subsequent use, we denote by bold face letters, the $\mathbb{R}^2$-valued function space such as $H^1_0 = [H^1_0(\Omega)]^2$, $L^2 = [L^2(\Omega)]^2$ and $H^m = [H^m(\Omega)]^2$. We denote by $\| \cdot \|$ the usual norm of the Sobolev space $H^m$, and $(\cdot, \cdot)$ and $\| \cdot \|$ represent the inner product and norm on $L^2$ or $L^2$, respectively. The space $H^1_0$ is equipped with the norm

$$\| \nabla v \| = \left( \sum_{i,j=1}^{2} (\partial_j v_i, \partial_i v_j) \right)^{\frac{1}{2}} = \left( \sum_{i=1}^{2} (\nabla v_i, \nabla v_i) \right)^{\frac{1}{2}}.$$ 

Let $H^m/\mathbb{R}$ be the quotient space of equivalent classes of functions in $H^m$ differ by constant with norm $\| \phi \|_{H^m/\mathbb{R}} = \inf_{c \in \mathbb{R}} \| \phi + c \|_m$. For $m = 0$, it is denoted by $L^2/\mathbb{R}$. For any Banach space $X$, let $L^p(0, T; X)$ denote the space of measurable $X$-valued functions $\phi$ on $(0, T)$ such that

$$\int_0^T \| \phi(t) \|_X^p \, dt < \infty, \text{ if } 1 \leq p < \infty, \text{ and } \text{ess sup}_{0 < t < T} \| \phi(t) \|_X < \infty, \text{ if } p = \infty.$$ 

The dual space of $H^m(\Omega)$, denoted by $H^{-m}(\Omega)$, is defined as the completion of $C^\infty(\Omega)$ with respect to the norm

$$\| \phi \|_{-m} := \sup \left\{ \frac{(\phi, \psi)}{\| \psi \|_m} : \psi \in H^m(\Omega), \| \psi \|_m \neq 0 \right\}.$$ 

Throughout this paper, we make the following assumption:

**A1** For $g \in H^{m-1}$ with $m \geq 1$, let the unique pair of solution $(v, q) \in H^1_0 \times L^2/\mathbb{R}$ for the steady state Stokes problem

$$- \Delta v + \nabla q = g, \quad \nabla \cdot v = 0, \quad \text{in } \Omega, \quad v|_{\partial \Omega} = 0,$$

satisfies $(v, q) \in H^{m+1} \times H^{-m}$ and the regularity result [19]: $\| v \|_{m+1} + \| q \|_{H^{-m}} \leq C \| g \|_{m-1}$.

Assumption **A1** simply talk about the regularity of the boundary $\partial \Omega$, that is $\partial \Omega \in C^m$. We note here that **A1** implies

$$\| v \|_{m+1} \leq C \| \nabla v \|_{(m+1)/2}, \quad \forall v \in H^1_0 \cap H^{m+1} \quad \text{and} \quad \| v \|_{m-1} \leq \lambda_1^{-\frac{1}{2}} \| v \|_m, \quad \forall v \in H^1_0 \cap H^m,$$

where $\lambda_1 > 0$ to be the least eigenvalue of the Stokes operator.

We present below a couple of lemmas for subsequent use.

**Lemma 2.1** (uniform Gronwall’s Lemma [4]). Let $g, h, y$ be three locally integrable non-negative functions on the time interval $[0, \infty)$. Assume that $y$ is absolutely continuous and

$$\frac{dy}{dt} \leq gy + h, \quad \forall t \geq 0,$$

and

$$\int_t^{t+T} g(s) ds \leq \alpha_1, \quad \int_t^{t+T} h(s) ds \leq \alpha_2, \quad \int_t^{t+T} y(s) ds \leq \alpha_3, \quad \forall t \geq 0,$$

where $T, \alpha_1, \alpha_2, \alpha_3$ are positive constants. Then

$$y(t + T) \leq \left( \frac{\alpha_3}{T} + \alpha_2 \right) \exp(\alpha_1), \quad \forall t \geq 0.$$
Lemma 2.2 (discrete uniform Gronwall’s Lemma [26]). Let \( k \) and \( \{y^i, g^i, h^i\}_{i \in \mathbb{N}} \) be non-negative numbers satisfying
\[
\frac{y^i - y^{i-1}}{k} \leq g^{i-1} y^{i-1} + h^{i-1}, \quad \forall i \geq 1,
\]
and there exist \( a_1(r), a_2(r), a_3(r) \) depend on \( t_r = rk \) such that
\[
k \sum_{i=i_0}^{i_0+r} g^i \leq a_1(r), \quad k \sum_{i=i_0}^{i_0+r} h^i \leq a_2(r), \quad k \sum_{i=i_0}^{i_0+r} y^i \leq a_3(r), \quad \forall i_0 \geq 1.
\]
Then,
\[
y^n \leq \left( \frac{a_3(r)}{t_r} + a_2(r) \right) \exp(a_1(r)), \quad \forall n \geq r + 1.
\]

We are now in a position to look at the variational formulation of the penalized system (1.4). The corresponding variational formulation of penalized NSEs is to find \((u_ε(t), p_ε(t))\), \( t > 0 \) in \( H^1_0 \times L^2 \) satisfying
\[
(2.1) \quad \begin{cases} (u_{εt}, \phi) + \nu a_ε(u_ε, \phi) + \tilde{b}(u_ε, u_ε, \phi) - (p_ε, \nabla \cdot \phi) = (f, \phi), & \forall \phi \in H^1_0, \\ \nu(\nabla \cdot u_ε, \chi) + \varepsilon(p_ε, \chi) = 0, & \forall \chi \in L^2,
\end{cases}
\]
with \( u_ε(0) = u_ε^0 \). Here \( a(v, w) = (\nabla v, \nabla w) \) and
\[
\tilde{b}(v, w, \phi) = (\tilde{B}(v, w), \phi), \quad \text{where} \quad \tilde{B}(v, w) := (v \cdot \nabla)w + \frac{1}{2} \nabla (v \cdot v)w.
\]
We can easily check with the help of integration by parts that
\[
\tilde{b}(v, w, \phi) = \frac{1}{2} \{ b(v, w, \phi) - b(v, \phi, w) \}, \quad \forall v, w, \phi \in H^1_0,
\]
where \( b(v, w, \phi) = (v \cdot \nabla)w, \phi \). Hence, it follows that
\[
\tilde{b}(v, w, w) = 0, \quad \text{and} \quad \tilde{b}(v, w, \phi) = -\tilde{b}(v, \phi, w), \quad \forall v, w, \phi \in H^1_0.
\]
In order to omit the pressure term, we choose \( \chi = \nabla \cdot \phi \) in the second equation of (2.1). We then obtain
\[
(2.2) \quad (u_{εt}, \phi) + \nu a_ε(u_ε, \phi) + \tilde{b}(u_ε, u_ε, \phi) = (f, \phi), \quad \forall \phi \in H^1_0,
\]
where
\[
a_ε(v, w) = a(v, w) + \frac{1}{\varepsilon}(\nabla \cdot v, \nabla \cdot w),
\]
with \( u_ε(0) = u_ε^0 \). Setting
\[
A_εv := -\Delta v - \frac{1}{\varepsilon} \nabla (\nabla \cdot v),
\]
we rewrite the system in abstract form as
\[
(2.3) \quad u_{εt} + \nu A_ε u_ε + \tilde{B}(u_ε, u_ε) = f.
\]
The operator \( A_ε \), which is associated with the penalty method, is a self-adjoint and positive definite operator from \( H^2 \cap H^1_0 \) onto \( L^2 \). Similar to the Stokes operator, we can talk of various powers of \( A_ε \), namely, \( A_ε^r, r \in \mathbb{R} \). For details, we refer to Temam [4] and Shen [29]. It is observed in [4] that \( \|A_εv\| \) is a norm on \( H^2 \cap H^1_0 \) and is, in fact, equivalent to that of \( H^2 \), i.e.,
\[
(2.4) \quad \|A_εv\| \approx \|v\|_2,
\]
with constants depending on \( \varepsilon \). In [4], one of the inequalities of (2.5), that is, \( \|\Delta v\| \leq C_3\|A_εv\| \), for some positive constant \( C_3 \) with \( \varepsilon C_3 < 1 \) is proved to be independent of \( \varepsilon \), which is crucial for our subsequent analysis. We present below a Lemma, to support this. For a proof, we again refer to [4, pp. 6] and [29, pp. 388].

Lemma 2.3. There exists a constant \( c_0 > 0 \) such that, for \( \varepsilon > 0 \) sufficiently small, the following estimates hold:
\[
\|v\|_r \leq c_0\|A_ε^{r/2}v\|, \quad \forall v \in H^r \cap H^1_0, \quad 1 \leq r \leq m + 1,
\]
\[
\|A_ε^{-1}v\| \leq c_0\|v\|_{-2}, \quad \forall v \in H^{-2}.
\]

Based on the second result, we have obtained another estimate, independent of \( \varepsilon \), which we prove in the next Lemma.

Lemma 2.4. There exists a constant \( c_0 > 0 \) such that, for \( \varepsilon > 0 \) sufficiently small, the following holds true:
\[
\|A_ε^{r/2}v\| \leq c_0\|v\|_{-1}, \quad \forall v \in H^{-1}.
\]
Proof. With \( w \in H_0^1 \), we use Lemma 2.3 to find that

\[
(A_\varepsilon^{-\frac{1}{2}} v, w) = (v, A_\varepsilon^{-\frac{1}{2}} w) \leq \|v\|_{-1} \|\nabla (A_\varepsilon^{-\frac{1}{2}} w)\| \leq c_0 \|v\|_{-1} \|w\|.
\]

and

\[
\|A_\varepsilon^{-\frac{1}{2}} v\| = \sup_{0 \neq w \in L^2} \frac{(A_\varepsilon^{-\frac{1}{2}} v, w)}{\|w\|} \leq c_0 \|v\|_{-1}.
\]

This completes the proof. Alternative way is to consider the following problem: Let \( w \) be a solution of

\[
A_\varepsilon w = v, \quad w|_{\partial \Omega} = 0.
\]

Clearly \( \|A_\varepsilon w\| = \|v\| \) and

\[
\|A_\varepsilon^\frac{1}{2} w\|^2 = (A_\varepsilon w, w) = (v, w) = (A_\varepsilon^{-\frac{1}{2}} v, A_\varepsilon^\frac{1}{2} w) \leq \|A_\varepsilon^{-\frac{1}{2}} v\| \|A_\varepsilon^\frac{1}{2} w\|
\]

and therefore

\[
(2.6) \quad \|A_\varepsilon^\frac{1}{2} w\| \leq \|A_\varepsilon^{-\frac{1}{2}} v\|.
\]

Now using (2.6) and Lemma 2.3 we note that

\[
\|A_\varepsilon^{-\frac{1}{2}} v\|^2 = (A_\varepsilon^{-1} v, v) = (A_\varepsilon^{-1} v, A_\varepsilon w) = (v, w) \leq \|v\|_{-1} \|w\|_1
\]

\[
\leq c_0 \|v\|_{-1} \|A_\varepsilon^\frac{1}{2} w\| \leq c_0 \|v\|_{-1} \|A_\varepsilon^{-\frac{1}{2}} v\|.
\]

This completes the rest of the proof. \( \square \)

We now consider the following assumptions on the given data for the penalized NSEs for our subsequent analysis.

(A2). The initial velocity \( u_{0,0} \) and the external force \( f \) satisfy for positive constant \( M_0 \), and for \( T \) with \( 0 < T < \infty \) and for some integer \( m \geq 1 \) and \( 0 \leq l \leq m \)

\[
u_{0,0} \in H_0^1(\Omega), \quad D_0^l f \in L^\infty(0, T; H^{m-1}) \quad \text{with} \quad \|A_\varepsilon^l u_{0,0}\| \leq M_0, \quad \sup_{0 < t < T} \left\{ \|D_0^l f(t)\|_{m-1} \right\} \leq M_0.
\]

Throughout, we shall use \( C \) as a generic constant depending on the data: \( \Omega, \nu_{0,0}, f, \nu, c_0 \) and \( T \), but not on mesh parameter \( h \) and \( k \). Below, we take a quick glance at the \( a \) priori estimates of the penalized problem.

**Lemma 2.5.** Assume (A1) and (A2) hold true and \( 0 < \alpha < \nu \lambda_1 / 2c_0^2 \). Then, there exists constant \( C > 0 \), independent of \( \varepsilon \), such that

\[
\|u_\varepsilon(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|A_\varepsilon^{\frac{1}{2}} u_\varepsilon(s)\|^2 \, ds \leq C,
\]

\[
\tau^m(t) \|A_\varepsilon^{\frac{m+1}{2}} u_\varepsilon(t)\|^2 + e^{-2\alpha t} \int_0^t \sigma^m(s) \|A_\varepsilon^{\frac{m+2}{2}} u_\varepsilon(s)\|^2 \, ds \leq C,
\]

\[
\tau^{m+1}(t) \|A_\varepsilon^{\frac{m+1}{2}} u_\varepsilon(t)\|^2 + e^{-2\alpha t} \int_0^t \sigma^{m+1}(s) \|A_\varepsilon^{\frac{m+1}{2}} u_\varepsilon(s)\|^2 \, ds \leq C,
\]

hold for \( m \leq 4 \), where \( \tau(t) = \min\{t, 1\} \) and \( \sigma^m(t) = \tau^m(t) e^{2\alpha t} \).

The proof goes in a similar way as that of the proofs given in [3] Lemma 2.1] and [19] Proposition 3.2. We give a sketch in the appendix.

For the sake of completeness, we present below the optimal penalty error estimate.

**Theorem 2.1.** Under the assumption of Lemma 2.5, the following hold:

\[
\sqrt{\tau(t)} \|(u - u_\varepsilon(t))\| + \tau(t) \|
abla (u - u_\varepsilon(t))\| \leq \left( \int_0^t \tau^2(s) \|(p - p_\varepsilon(s))\|^2 \, ds \right)^{\frac{1}{2}} \leq K(t) \epsilon,
\]

and

\[
\tau(t) \|(p - p_\varepsilon(t))\| \leq K(t) \epsilon,
\]

where \( \tau(t) = \min\{t, 1\} \) and \( K(t) = Ce^{Ct} \), \( C \) is the positive constant. Under the uniqueness condition:

\[
(2.7) \quad \frac{2N}{\nu^2} \|f\|_{L^\infty(0, \infty; H^{-1}(\Omega))} < 1 \quad \text{and} \quad N = \sup_{u, v, w} \frac{|b(u, v, w)|}{\|\nabla u\| \|\nabla v\| \|\nabla w\|},
\]

the above estimates are uniform in time, that is, the constant \( K(t) \) becomes \( C \).
Proof. The proof of the first estimate is available in [29, Theorem 4.1]. Using this, we can easily prove the pressure estimate in $L^\infty(L^2)$. For the uniform estimates, we sketch a proof below. We note that in [29], the error has been split into two:

$$
u \xi = (u - v) + (v - u_e) = \xi + \eta,$$

where $v$ is the solution of the linear penalized problem [29 (4.1)-(4.2)]:

$$
u_t - \nu \Delta v + \nabla \gamma = f - B(u, u),$$

$$\nu \nabla \cdot v + \varepsilon \eta = 0, \quad v(0) = u_0,$$

where $u$ is the solution of NSEs [1.1]-[1.3] and $B(u, u) = (u \cdot \nabla)u$. And the estimates of $\xi$ are uniformly in time, see [29, Lemma 4.1]. However, the estimates of $\eta$ grow exponentially in time (see, [29, Theorem 4.1]) due to the use of the Gronwall’s lemma. This can be avoided under the assumption of uniqueness condition (2.7). We first note down the equation in $\eta$, see [29, (4.7)].

$$(2.8) \quad \eta_t + \nu A \eta + \tilde{b}(u_e, \xi + \eta) + \tilde{B}(\xi + \eta, u) = 0.$$ 

Take the inner product of (2.8) with $\eta$ to obtain

$$(2.9) \quad \frac{1}{2} \frac{d}{dt}(\|\eta\|^2) + \nu \|A^\frac{1}{2} \eta\|^2 + \tilde{b}(u_e, \xi + \eta, \eta) + \tilde{B}(\xi, \eta, u, \eta) = 0.$$ 

The nonlinear terms can be bounded by using the uniqueness condition (2.7) and the bounds (2.4) as

$$\tilde{b}(u_e, \xi + \eta, \eta) + \tilde{b}(\xi + \eta, u, \eta) = \tilde{b}(u_e, \xi, \eta) + \tilde{b}(\xi, \eta, u, \eta) + \tilde{b}(\eta, u, \eta) \leq C(\|\Delta u_e\| + \|\Delta u\|)\|\xi\|\|\nabla \eta\| + N\|\nabla u\|\|\nabla \eta\|^2$$

$$\leq C(\|\Delta u_e\|^2 + \|\Delta u\|^2)\|\xi\|^2 + \frac{\nu}{2}\|A^\frac{1}{2} \eta\|^2 + N\|\nabla u\|\|\nabla \eta\|^2.$$ 

Use the above estimate in (2.9). Then, multiply by $e^{2\alpha t}$ and integrate with respect to time to arrive at

$$(2.10) \quad e^{2\alpha t}\|\eta(t)\|^2 + \nu \int_0^t e^{2\alpha s}\|A^\frac{1}{2} \eta(s)\|^2 ds \leq \|\eta(0)\|^2 + 2\alpha \int_0^t e^{2\alpha s}\|\eta(s)\| ds + N \int_0^t e^{2\alpha s}\|\nabla u(s)\|\|\nabla \eta(s)\|^2 ds$$

$$+ C \int_0^t e^{2\alpha s}(\|\Delta u_e(s)\|^2 + \|\Delta u(s)\|^2)\|\xi(s)\|^2 ds.$$ 

The last term on the right hand side of (2.10) can be written as

$$(2.11) \quad \int_0^t e^{2\alpha s}(\|\Delta u_e(s)\|^2 + \|\Delta u(s)\|^2)\|\xi(s)\|^2 ds \leq \|\xi(t)\|_{L^\infty(L^2)} \int_0^t e^{2\alpha s}(\|\Delta u_e(s)\|^2 + \|\Delta u(s)\|^2) ds$$

$$\leq Ce^{2\alpha t}\|\xi(t)\|^2_{L^\infty(L^2)}.$$ 

Now, we rewrite the last term on the left hand side of (2.10) using $\|A^\frac{1}{2} \eta\|^2 = \|\nabla \eta\|^2 + \frac{1}{\nu} \|\nabla \cdot \eta\|^2$. Then use (2.12) and multiply the above estimate by $e^{-2\alpha t}$ to obtain

$$\|\eta(t)\|^2 + e^{-2\alpha t}(\nu - 2N\|\nabla u\|)\|\nabla \eta(t)\|^2 ds + \frac{\nu}{\varepsilon} e^{-2\alpha t}(\nu - 2\|\nabla \eta\|)\|\nabla \eta(t)\|^2 ds$$

$$\leq e^{-2\alpha t}\|\eta(0)\|^2 + 2\alpha e^{-2\alpha t} \int_0^t e^{2\alpha s}\|\eta(s)\|^2 ds + C\|\xi(t)\|^2_{L^\infty(L^2)}.$$ 

Take limit on both sides as $t \to \infty$. Using $\lim_{t \to \infty}\|\nabla u\| \leq \frac{1}{\nu}\|f\|_{L^\infty(0,\infty;H^{-1}(\Omega))}$, it follows that

$$\left(\nu - \frac{2N}{\nu}\|f\|_{L^\infty(0,\infty;H^{-1}(\Omega))}\right)\lim_{t \to \infty}\|\nabla \eta(t)\|^2 \leq C\lim_{t \to \infty}\|\xi(t)\|^2_{L^\infty(L^2)}.$$ 

Since $2N\nu^{-2}\|f\|_{L^\infty(0,\infty;H^{-1}(\Omega))} < 1$, from (2.7), we conclude the following

$$\lim_{t \to \infty}\|\eta(t)\| \leq C\lim_{t \to \infty}\|\nabla \eta(t)\| \leq C\lim_{t \to \infty}\|\xi(t)\|_{L^\infty(L^2)}.$$ 

Combining the estimate of $\xi$ from [29, Lemma 4.1], with the above estimate, we finally obtain

$$\lim_{t \to \infty}\|u - u_e(t)\| \leq C\varepsilon t^{-\frac{1}{2}}.$$ 

Using the above estimate, we can easily find the estimates of $\|\nabla(u - u_e(t))\|$ and $\|(p - p_e(t))\|$, which completes the rest of the proof.
3 Galerkin Finite Element Method

This section deals with the finite element Galerkin approximations to the penalized problem (1.4) or (2.4) and some a priori bounds for the semidiscrete problem.

Let $\mathcal{T}_h = \{ K \}$ be a shape regular triangulation of the polygonal domain $\Omega$ into closed subset $K$, triangles or quadrilaterals with $h = \max_K h_K$, where $h_K$ is the diameter of $K$.

Let $H_h$ and $L_h$ be two families of finite element spaces of $H^1_0$ and $L^2/\mathbb{R}$, respectively, approximating the velocity vector and the pressure. It is assumed that the spaces $H_h$ and $L_h$ comprise of piecewise polynomial of degree at most $m$ and $m-1$ ($m \geq 2$), respectively. Assume that the following approximation properties are satisfied for the spaces $H_h$ and $L_h$:

**(B1)** For each $w \in H^0_0 \cap H^{m+1}$ and $q \in H^m/\mathbb{R}$ with $m \geq 1$, there exist approximations $i_h w \in H_h$ and $j_h q \in L_h$ such that

$$
\| w - i_h w \| + h \| \nabla (w - i_h w) \| \leq C h^{j+1} \| w \|_{j+1}, \quad \| q - j_h q \| \leq C h^j \| q \|_j, \quad 0 \leq j \leq m.
$$

Further, we assume that inverse inequality holds for $w_h$ in $H_h$:

$$
\| \nabla w_h \| \leq C h^{-1} \| w_h \|.
$$

We consider now the discrete analogue of the weak formulations (2.1) and (2.3): Find $(u_{ch}, p_{ch})$ in $H_h \times L_h$ satisfying

**(3.1)**

$$
(u_{ch}, \phi_h) + \nu a(u_{ch}, \phi_h) + b(u_{ch}, u_{ch}, \phi_h) - (p_{ch}, \nabla \cdot \phi_h) = (f, \phi_h), \quad \forall \phi_h \in H_h,
$$

where $\nu = \nu \chi_h \in L_h$ is a self-adjoint and positive definite operator. With additional assumptions (B1) and (B2), the following properties hold true for $H_h$.

**(B2)** For every $w \in H^0_0 \cap H^{m+1}$, there exists an approximation $r_h w \in H_h$ such that

$$
\| w - r_h w \| + h \| \nabla (w - r_h w) \| \leq C h^{j+1} \| w \|_{j+1}, \quad 0 \leq j \leq m.
$$

Based on it, $L^2$-projection $P_h$, defined as $P_h : L^2 \rightarrow H_h$, can be derived, satisfying the following properties (1.9):

$$
\| \phi - P_h \phi \| + h \| \nabla (\phi - P_h \phi) \| \leq C h^{j+1} \| \phi \|_{j+1}, \quad \forall \phi \in H^0_0(\Omega) \cap H^m(\Omega).
$$

Now we define the discrete operator $\Delta_h : H_h \rightarrow H_h$ through the bilinear form $a(\cdot, \cdot)$ as

$$
a(v_h, \phi_h) = (-\Delta_h v_h, \phi_h), \quad \forall v_h, \phi_h \in H_h.
$$

Next we define the discrete analogue $A_{ch} : H_h \rightarrow H_h$ of $A_c$ satisfying

**(3.3)**

$$
(A_{ch} v_h, \phi_h) = a_c(v_h, \phi_h) = a(v_h, \phi_h) + \frac{1}{\varepsilon} (\nabla \cdot v_h, \nabla \cdot \phi_h), \quad \forall v_h, \phi_h \in H_h.
$$

Note that, $A_{ch}$ is a self-adjoint and positive definite operator. With additional assumptions (B1) and (B2), the operator $A_{ch}$ mimics the estimates presented in the Lemmas 2.3 and 2.4.

**Lemma 3.1.** There exists a constant $c_0 > 0$ such that for $\varepsilon > 0$ sufficiently small, the following estimates hold:

$$
\begin{align*}
\| \Delta_h v_h \| & \leq c_0 \| A_{ch} v_h \|, \quad \forall v_h \in H_h, \\
\| \nabla v_h \| & \leq c_0 \| A_{ch} v_h \|, \quad \forall v_h \in H_h, \\
\| A_{ch}^r v_h \| & \leq c_0 \| v_h \|_{r}, \quad \forall v_h \in H_h, \quad r \in \{1, 2\}.
\end{align*}
$$
Proof. Let $v_h \in H_h$ and $A_{ch} \cdot v_h = g$. Take $q_h = -\frac{1}{2} \nabla \cdot v_h$, then (3.3) can be written as

\[(\nabla v_h, \nabla \phi_h) - (q_h, \nabla \cdot \phi_h) = (g, \phi_h), \quad \forall \phi_h \in H_h, \quad (\nabla \cdot v_h, \psi_h) + \varepsilon (q_h, \psi_h) = 0, \quad \forall \psi_h \in L_h.\]

From regularity estimate, one can find that (see [3] (1.20))

\[\|\Delta_h v_h\| + \|\nabla q_h\| \leq c_0 \|g\| + \varepsilon_0 \|\nabla q_h\|.\]

Now choose $\varepsilon$ sufficiently small such that $c_0 \varepsilon < 1$, then we conclude the first result. For the second result, we choose $\phi_h = v_h$ in (3.3) and arrive at

\[\|A_{ch}^{\frac{1}{2}} v_h\|^2 = \|\nabla v_h\|^2 + \frac{1}{\varepsilon} \|\nabla \cdot v_h\|^2 \geq \|\nabla v_h\|^2.\]

For the third one, let $w_h$ be the solution of $A_{ch}^{\frac{1}{2}} w_h = A_{ch}^{\frac{1}{2}} v_h$.

\[\|A_{ch}^{\frac{1}{2}} v_h\|^2 = (A_{ch}^{\frac{1}{2}} w_h, A_{ch}^{\frac{1}{2}} v_h) = (w_h, v_h) \leq \|w_h\|_{r} \|v_h\|_{-r} \leq c_0 \|A_{ch}^{\frac{1}{2}} w_h\| \|v_h\|_{-r} \leq c_0 \|A_{ch}^{\frac{1}{2}} v_h\| \|v_h\|_{-r}.\]

Cancelling one $\|A_{ch}^{\frac{1}{2}} v_h\|$ from both sides completes the rest of the proof.

Before we proceed further, we look at some standard estimates of the nonlinear term $b(\cdot, \cdot, \cdot)$ (see [3][9]), which use Hölder’s inequality and the following discrete Ladyzhenskaya’s inequality [31], for all $\phi_h \in H_h$:

\[\|\phi_h\|_{L^4} \leq C \|\phi_h\|^{\frac{1}{4}} \|\nabla \phi_h\|^\frac{3}{4}, \quad \|\nabla \phi_h\|_{L^4} \leq C \|\phi_h\|^{\frac{1}{2}} \|\Delta_h \phi_h\|^{\frac{1}{2}}, \quad \|\phi_h\|_{L^{\infty}} \leq C \|\phi_h\|^{\frac{1}{8}} \|\Delta_h \phi_h\|^{\frac{7}{8}}.\]

**Lemma 3.2.** Suppose the conditions (A1), (B1) and (B2) are satisfied. Then there exists a constant $C > 0$ such that for all $v_h, w_h, \phi_h \in H_h$, the trilinear form $b(\cdot, \cdot, \cdot)$ satisfies the following properties:

\[\tilde{b}(v_h, w_h, \phi_h) = C \left\{\begin{array}{ll}
\|v_h\|^{\frac{1}{2}} \|\Delta_h v_h\|^{\frac{1}{2}} \|\nabla w_h\| \|\phi_h\|,
\|v_h\|^{\frac{1}{2}} \|\Delta_h v_h\|^{\frac{1}{2}} \|\nabla w_h\| \|\phi_h\|,
\|v_h\| \|\nabla w_h\| \|\phi_h\| \|\Delta_h \phi_h\|^{\frac{1}{2}},
\|v_h\| \|\nabla w_h\| \|\phi_h\| \|\Delta_h \phi_h\|^{\frac{1}{2}},
\|v_h\| \|\nabla w_h\| \|\phi_h\| \|\Delta_h \phi_h\|^{\frac{1}{2}},
\|v_h\| \|\nabla w_h\| \|\phi_h\| \|\Delta_h \phi_h\|^{\frac{1}{2}}.
\end{array}\right.\]

We next look at the a priori estimates of the discrete penalized solution $u_{ch}$. Similar to the continuous case (like in Lemma 2.5), these estimates can be easily verified.

**Lemma 3.3.** Apart from the assumptions of the Lemma 2.5, we assume that (B1) and (B2) hold. Then, there exists constant $C$ independent of $\varepsilon$ and $h$ such that for $u_{ch}(0) = P_h u_0$

\[\|u_{ch}(t)\|^2 + e^{-2\alpha t} \int_0^t \|A_{ch}^{\frac{1}{2}} u_{ch}(s)\|^2 ds \leq C, \quad \tau(t) \int_0^t e^{2\alpha s} \|A_{ch}^{\frac{1}{2}} u_{ch}(s)\|^2 ds \leq C, \quad r \in \{0, 1\}, \quad e^{-2\alpha t} \int_0^t \tau(s) e^{2\alpha s} \|A_{ch}^{\frac{1}{2}} u_{ch,rr}(s)\|^2 ds \leq C, \quad r \in \{0, 1, 2\},
\]

hold, where $\tau(t) = \min\{t, 1\}$.

Proof. Choose $\phi_h = u_{ch}$ in (5.2) and use the Cauchy-Schwarz inequality and the Poincaré inequality with Lemma 5.1 ($\|u_{ch}\|^2 \leq \frac{1}{\lambda_1} \|\nabla u_{ch}\|^2 \leq \frac{\lambda_1}{\varepsilon^2} \|A_{ch}^{\frac{1}{2}} u_{ch}\|^2$) to find that

\[(3.4) \quad \frac{d}{dt} \|u_{ch}\|^2 + \nu \|A_{ch}^{\frac{1}{2}} u_{ch}\|^2 \leq \frac{c_0^2}{\nu \lambda_1} \|f\|^2.\]

Note that the non-linear term vanishes due to (2.2). Now multiply by $e^{2\alpha t}$ and integrate from 0 to $t$ to obtain

\[(3.5) \quad e^{2\alpha t} \|u_{ch}(t)\|^2 + (\nu - \frac{2\varepsilon^2 \alpha}{\lambda_1}) \int_0^t e^{2\alpha s} \|A_{ch}^{\frac{1}{2}} u_{ch}(s)\|^2 ds \leq \|P_h u_0\|^2 + \frac{c_0^2(e^{2\alpha t} - 1)}{2\nu \lambda_1} \|f\|^2.\]
With $0 < \alpha < \frac{\nu T}{2}$, we have $(\nu - \frac{2\alpha^2}{\nu}) > 0$. Multiply throughout by $e^{-2\alpha t}$ to conclude the first proof. Now, we integrate (3.4) with respect to time from $t$ to $t + T$ for any $T > 0$, we have

$$\int_{t}^{t+T} \left( \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}(s)\|^2 + \nu \|A_{\epsilon h} u_{\epsilon h}(s)\|^2 \right) ds \leq \|u_{\epsilon h}(t)\|^2 + \frac{\nu T}{\nu - \alpha} \|f\|_{L_{\infty}}^2.$$ 

(3.6)

For the second estimate, choose $\phi_h = A_{\epsilon h}^{r+1} u_{\epsilon h}$, $r \in \{0, 1\}$ in (3.2). When $r = 0$, we find that

$$\frac{1}{2} \frac{d}{dt} \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^2 + \nu \|A_{\epsilon h} u_{\epsilon h}\|^2 = (f, A_{\epsilon h} u_{\epsilon h}) - \hat{b}(u_{\epsilon h}, u_{\epsilon h}, A_{\epsilon h} u_{\epsilon h}).$$

(3.7)

We use Lemma 3.2 with Lemma 3.1, the Young’s inequalities to bound the nonlinear term as

$$\hat{b}(u_{\epsilon h}, u_{\epsilon h}, A_{\epsilon h} u_{\epsilon h}) \leq C\|u_{\epsilon h}\| \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\| \|A_{\epsilon h} u_{\epsilon h}\| \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^2 \leq C\|u_{\epsilon h}\|^2 \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^4 \leq C\left(\|u_{\epsilon h}\| \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\| + \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|\right)^4 + \frac{\nu}{4} \|A_{\epsilon h} u_{\epsilon h}\|^2.$$

(3.8)

Substitute the above estimate in (3.7) to find that

$$\frac{d}{dt} (\|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^2 + \nu \|A_{\epsilon h} u_{\epsilon h}\|^2) \leq C\left(\|u_{\epsilon h}\| \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^2 \right) \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^2 + \frac{2\nu}{4} \|f\|^2.$$

(3.9)

We now apply the uniform Gronwall’s lemma (Lemma 2.1) in (3.9) and use (3.5) and (3.6) to conclude that $\|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}(t + T)\|^2$ is uniformly bounded with respect to $t$ for all $t \geq 0$, that is, $\|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}(t)\|^2$ is uniformly bounded on $[T, \infty)$. Precisely

$$\|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}(t)\|^2 \leq C, \quad \forall t \geq T.$$ 

(3.10)

For $0 \leq t \leq T$, we use the classical Gronwall’s lemma (29) in (3.9) and obtain

$$\|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}(t)\|^2 \leq C, \quad \forall 0 \leq t \leq T.$$ 

(3.11)

Finally, multiply (3.9) by $e^{2\alpha t}$ and integrate with respect to time from $0$ to $t$ and use the estimates (3.5), (3.10) and (3.11) to complete the second proof when $r = 0$. For $r = 1$, we need some intermediate estimate. First we take $\phi_h = e^{2\alpha t} u_{\epsilon h}$ with $\tilde{u}_{\epsilon h} = e^{\alpha t} u_{\epsilon h}$ in (3.2) to obtain

$$\frac{\nu}{2} \frac{d}{dt} \|A_{\epsilon h}^{\frac{1}{2}} \tilde{u}_{\epsilon h}\|^2 + \|\tilde{u}_{\epsilon h}\|^2 = \alpha \nu \|A_{\epsilon h}^{\frac{1}{2}} \tilde{u}_{\epsilon h}\|^2 + (\tilde{f}, \tilde{u}_{\epsilon h}) - e^{2\alpha t} \hat{b}(u_{\epsilon h}, u_{\epsilon h}, u_{\epsilon h}).$$

(3.12)

We can estimate the nonlinear term on the right hand side of (3.12) using Lemma 3.2 and integrate both sides with respect to time to find that

$$\nu \|A_{\epsilon h}^{\frac{1}{2}} \tilde{u}_{\epsilon h}\|^2 + \int_0^t \|\tilde{u}_{\epsilon h}\|^2 ds \leq C \left[ \int_0^t \left( \|A_{\epsilon h}^{\frac{1}{2}} \tilde{u}_{\epsilon h}\|^2 + \|\tilde{f}\|^2 + \|A_{\epsilon h}^{\frac{1}{2}} \tilde{u}_{\epsilon h}\|^2 \right) ds \right].$$

Now a use of (3.5) and (3.10) lead us to the intermediate estimate.

$$\|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}(t)\|^2 \leq e^{-2\alpha t} \int_0^t e^{2\alpha s} \|u_{\epsilon h}(s)\|^2 ds \leq C.$$ 

(3.13)

We now differentiate (3.2) with respect to time and deduce that

$$\frac{d}{dt}(\sigma(t) ||u_{\epsilon h}||^2) + \nu \sigma(t) \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^2 \leq Ce^{2\alpha t} \|u_{\epsilon h}\|^2 + C \sigma(t) \left( \|\tilde{f}\|^2 + \|u_{\epsilon h}\|^2 \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^2 \right).$$

Integrate with respect to time and use (3.13), (3.10) and (3.11) to obtain

$$\sigma(t) \|u_{\epsilon h}(t)\|^2 + \nu e^{-2\alpha t} \int_0^t \sigma(s) \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}(s)\|^2 ds \leq C.$$ 

(3.15)

Now we are in position to complete the proof of the second estimate when $r = 1$. Set $\phi_h = A_{\epsilon h} u_{\epsilon h}$ in (3.2) and rewrite it and use (3.8) and the Cauchy-Schwarz inequality to arrive at

$$\nu \|A_{\epsilon h} u_{\epsilon h}\|^2 = (f, A_{\epsilon h} u_{\epsilon h}) - (u_{\epsilon h}, A_{\epsilon h} u_{\epsilon h}) - \hat{b}(u_{\epsilon h}, u_{\epsilon h}, A_{\epsilon h} u_{\epsilon h}) \leq C(\|f\|^2 + \|A_{\epsilon h} u_{\epsilon h}\|^2 + \|u_{\epsilon h}\|^2 \|A_{\epsilon h}^{\frac{1}{2}} u_{\epsilon h}\|^4) + \frac{\nu}{2} \|A_{\epsilon h} u_{\epsilon h}\|^2.$$ 

(3.14)
Multiply by $\tau(t)$ and use (5.5), (8.10), (5.11) and (8.15) to complete the second proof.

For the third estimate, choose $\phi_h = e^{2\alpha t}A_{\varepsilon h}^{-2}u_{\varepsilon h}$ in (5.11). Then, we use (2.23) and Lemma 3.2 with Lemma 3.1 and the Cauchy-Schwarz inequality to bound the terms on right hand side as

$$
|\langle f, A_{\varepsilon h}^{-2}u_{\varepsilon h} \rangle - \tilde{b}(u_{\varepsilon h}, u_{\varepsilon h}, A_{\varepsilon h}^{-2}u_{\varepsilon h})| \\
\leq |\langle f, A_{\varepsilon h}^{-2}u_{\varepsilon h} \rangle| + |\tilde{b}(u_{\varepsilon h}, u_{\varepsilon h}, A_{\varepsilon h}^{-2}u_{\varepsilon h})| + |\tilde{b}(u_{\varepsilon h}, A_{\varepsilon h}^{-2}u_{\varepsilon h}, u_{\varepsilon h})| \\
\leq C(\|f\|^2 + \|u_{\varepsilon h}\|^2 \|A_{\varepsilon h}^{-2}u_{\varepsilon h}\|^2) + \frac{1}{2}\|A_{\varepsilon h}^{-1}u_{\varepsilon h}\|^2.
$$

Then we arrive at

$$
e^{2\alpha t}\|A_{\varepsilon h}^{-1}u_{\varepsilon h}\|^2 + \frac{\nu}{2 \ell} \left( e^{2\alpha t}\|A_{\varepsilon h}^{-1}u_{\varepsilon h}\|^2 \right) \leq \nu e^{-\alpha t}\|A_{\varepsilon h}^{-1}u_{\varepsilon h}\|^2 + Ce^{2\alpha t}\left(\|u_{\varepsilon h}\|^2 \|A_{\varepsilon h}^{-2}u_{\varepsilon h}\|^2 + \|f_{\varepsilon}\|^2 \right).
$$

Integrate both sides with respect to time from 0 to $t$. Now, a use of the estimates obtained above results in the case $r = 0$. For $r = 1$ and $r = 2$, we take $\phi_h = \sigma(t)A_{\varepsilon h}^{-1}u_{\varepsilon h}$ and $\phi_h = \sigma^2(t)u_{\varepsilon h}$ with $\sigma^2(t) = (r)^2(t)e^{2\alpha t}$, respectively, and do similar analysis as above. This completes the rest of the proof. \qed

4 Error Estimates for the Semidiscrete Problem

This section deals with the error analysis of the finite element Galerkin approximation for the penalized system (2.23). Here, the goal is to provide optimal $L^\infty(L^2)$ error estimates for the velocity and the pressure when $u_0 \in H^1_0$. The main result of this section is as follows.

**Theorem 4.1.** Let the conditions (A1), (A2), (B1) and (B2) be satisfied. Further, let the discrete initial velocity $u_{\varepsilon h}(0) = P_h u_0$, where $u_0 \in H^1_0(\Omega)$. Then, there exists a positive constant $C$ such that for $t > 0$

$$
\|u(t) - u_{\varepsilon h}(t)\| + h(\|\nabla(u - u_{\varepsilon h}(t))\| + \|p - p_{\varepsilon h}(t)\|) \leq K(t)h^{m+1}e^{-\frac{t}{2}}.
$$

where, $K(t) = Ce^{Ct}$. Under the additional condition (2.7), the estimates are uniformly in time, that is $K(t) = C$.

The proof of the theorem is realized via a numbers of lemmas. However before we visit the lemmas, we need to develop some preliminary tools, with which we begin our discussion. Let $f \in L^2$. We define the linear inverse operators $A_{\varepsilon}^{-1}: L^2 \to H^1_0$ and $A_{\varepsilon h}^{-1}: H_h \to H_h$ satisfying

$$
\begin{align*}
a_{\varepsilon}(A_{\varepsilon}^{-1}f, \phi) &= (\nabla A_{\varepsilon}^{-1}f, \nabla \phi) + \frac{1}{\varepsilon}(\nabla \cdot A_{\varepsilon}^{-1}f, \nabla \cdot \phi) = (f, \phi), \quad \forall \phi \in H^1_0 \\
a_{\varepsilon}(A_{\varepsilon h}^{-1}P_h f, \phi_h) &= (\nabla A_{\varepsilon h}^{-1}P_h f, \nabla \phi_h) + \frac{1}{\varepsilon}(\nabla \cdot A_{\varepsilon h}^{-1}P_h f, \nabla \cdot \phi_h) = (P_h f, \phi_h), \quad \forall \phi_h \in H_h.
\end{align*}
$$

Following the work of Heywood and Rannacher [13], we have the following result.

**Proposition 4.1.** The map $A_{\varepsilon h}^{-1}P_h A_{\varepsilon}: H^1_0 \cap H^2 \to H_h$ satisfies the following estimate:

$$
\|v - A_{\varepsilon h}^{-1}P_h A_{\varepsilon}v\| + h\|\nabla(v - A_{\varepsilon h}^{-1}P_h A_{\varepsilon}v)\| \leq Ch^{m+1}\|A_{\varepsilon h}^{-1}v\|.
$$

We next consider the penalized steady Stokes problem: For a given function $g \in L^2$, let $v \in H^1_0 \cap H^2$, $q \in H^1/R$ be the unique solution of

$$
a(v, \phi) - (q, \nabla \cdot \phi) = (g, \phi), \quad \forall \phi \in H^1_0, \\
(\nabla \cdot v, \chi) + \varepsilon(q, \chi) = 0, \quad \forall \chi \in L^2.
$$

Getting rid of $q$, we obtain

$$
a(v, \phi) + \frac{1}{\varepsilon}(\nabla \cdot v, \nabla \cdot \phi) = (g, \phi), \quad \forall \phi \in H^1_0.
$$

The finite element approximation $v_h \in H_h$ of $v$ satisfies the following equation

$$
a(v_h, \phi_h) + \frac{1}{\varepsilon}(\nabla \cdot v_h, \nabla \cdot \phi_h) = (g, \phi_h), \quad \forall \phi_h \in H_h.
$$

**Lemma 4.1.** Let the conditions (A1), (A2), (B1) and (B2) be satisfied. Then, there exists a positive constant $C$ such that

$$
\|v - v_h\| + h\|\nabla(v - v_h)\| \leq Ch^{m+1}\|A_{\varepsilon h}^{-1}v\|.
$$
Proof. The main idea of the proof is adapted from the paper of Heywood and Rannacher [18]. From (4.1) and (4.2), we have the following error equation

\[(4.3)\]
\[a(v - v_h, \phi_h) + \frac{1}{\varepsilon}(\nabla \cdot (v - v_h), \nabla \cdot \phi_h) = 0, \quad \forall \phi_h \in H_h.\]

We choose \(\phi_h = i_h v - v_h \in H_h\) in the above equation to obtain

\[\|\nabla (v - v_h)\|^2 + \frac{1}{\varepsilon}\|\nabla \cdot (v - v_h)\|^2 = a(v - v_h, v - i_h v) + \frac{1}{\varepsilon}(\nabla \cdot (v - v_h), \nabla \cdot (v - i_h v)).\]

A use of the Cauchy-Schwarz inequality with (B1) and Lemma 2.3 yields

\[(4.4)\]
\[\|\nabla (v - v_h)\|^2 + \frac{1}{\varepsilon}\|\nabla \cdot (v - v_h)\|^2 \leq C(\|\nabla (v - i_h v)\|^2 + \frac{1}{\varepsilon}\|\nabla \cdot (v - i_h v)\|^2)\]
\[\leq Ch^{2m}(\|v\|_{m+1}^2 + \frac{1}{\varepsilon}\|\nabla \cdot v\|_{m}^2)\]
\[\leq Ch^{2m}\|A_{\varepsilon} \frac{m+1}{2} v\|^2.\]

To obtain the \(L^2\) estimate, we need to consider a duality problem

\[(4.5)\]
\[A_{\varepsilon} w = v - v_h, \quad w|_{\Omega} = 0,\]

satisfying

\[(4.6)\]
\[\|A_{\varepsilon} w\| \leq C\|v - v_h\|.

We now take inner product on both sides of (4.3) by \(v - v_h\) and choose \(\phi_h = i_h w\) in (4.3) to obtain

\[\|v - v_h\|^2 = a(v - v_h, w - i_h w) + \frac{1}{\varepsilon}(\nabla \cdot (v - v_h), \nabla \cdot (w - i_h w)).\]

A use of the Cauchy-Schwarz inequality with the Young’s inequality with \(\delta > 0\), approximation property (B1), Lemma 3.3 and (4.6) shows

\[(4.7)\]
\[\|v - v_h\|^2 \leq C h^{2m+2}\|A_{\varepsilon} \frac{m+1}{2} v\|^2 + \delta(\|v\|_{m+1}^2 + \frac{1}{\varepsilon}\|\nabla \cdot v\|_{m}^2).

Now, a choice of \(C\delta = \frac{1}{2}\) completes the rest of the proof. \(\square\)

### 4.1 Error Estimates for the Velocity

In order to obtain the semidiscrete error estimates for the velocity, we denote \(e_\varepsilon = u_\varepsilon - u_{\varepsilon,h}\) and subtract (4.2) from (2.3) to find the equation of the semi-discrete error \(e_\varepsilon\):

\[(4.8)\]
\[(e_{\varepsilon,t}, \phi_h) + \nu e_\varepsilon (e_\varepsilon, \phi_h) = \tilde{b}(u_{\varepsilon,h}, u_{\varepsilon,h}, \phi_h) - \tilde{b}(u_{\varepsilon,h}, u_{\varepsilon,h}, \phi_h), \quad \forall \phi_h \in H_h.\]

By introducing an intermediate solution \(\tilde{v}_{\varepsilon,h}\) which is a finite element Galerkin approximation to a linearized penalized NSEs, that is, \(\tilde{v}_{\varepsilon,h}\) satisfies

\[(4.9)\]
\[(\tilde{v}_{\varepsilon,h,t}, \phi_h) + \nu \tilde{a}_{\varepsilon}(\tilde{v}_{\varepsilon,h}, \phi_h) = (f, \phi_h) - \tilde{b}(u_{\varepsilon,h}, u_{\varepsilon,h}, \phi_h) \quad \forall \phi_h \in H_h;\]

we split \(e_\varepsilon\) as

\[e_\varepsilon := u_\varepsilon - u_{\varepsilon,h} = (u_\varepsilon - \tilde{v}_{\varepsilon,h}) + (\tilde{v}_{\varepsilon,h} - u_{\varepsilon,h}) = e_\varepsilon - \xi_h + \eta_h.\]

Note that \(\xi_h\) is the error committed by approximating a linearized penalized NSEs and \(\eta_h\) represents the error due to the presence of non-linearity in the equation. Below, we derive some estimates of \(\xi_h\). Subtracting (4.8) from (2.3), the equation in \(\xi_h\) is written as

\[(4.10)\]
\[(\xi_{h,t}, \phi_h) + \nu \tilde{a}_{\varepsilon}(\xi_h, \phi_h) = 0, \quad \phi_h \in H_h.\]

**Lemma 4.2.** Let the assumptions of Lemma 3.3 hold and \(\tilde{v}_{\varepsilon,h}(t) \in H_h\) be a solution of (4.8) with initial condition \(\tilde{v}_{\varepsilon,h}(0) = P_h u_{\varepsilon,0}\) and \(u_\varepsilon\) be a weak solution of (2.4) with initial condition \(u_{\varepsilon,0} \in H_0\). Then, \(\xi_h\) satisfies

\[e^{-2\alpha t} \int_0^t \sigma^{m-1}(s) \|\xi_h(s)\|^2 \, ds \leq Ch^{2m+2}, \quad m \geq 1,\]

where, \(\sigma^m(t) = \tau^m(t)e^{2\alpha t}\) and \(\tau(t) = \min\{1, t\}.\]
Proof. We rewrite the equation (2.3) and (1.3) as
\[ A^{-1}_c u_{ct} + \nu u_c = A^{-1}_c (f - \tilde{B}(u_c,u_c)), \]
and
\[ A^{-1}_c \tilde{v}_{cht} + \nu \tilde{v}_c = A^{-1}_c P_h (f - \tilde{B}(u_c,u_c)). \]
In the view of above two equations, along with (2.3) we have
\[ A^{-1}_c P_h \xi_{ht} + \nu \xi_h = A^{-1}_c (P_h u_{ct} - \tilde{v}_{cht}) + \nu (u_c - \tilde{v}_c) \]
\[ = (A^{-1}_c P_h - A^{-1}_c) u_{ct} + (A^{-1}_c u_{ct} + \nu u_c) - (A^{-1}_c \tilde{v}_{cht} + \nu \tilde{v}_c) \]
\[ = (A^{-1}_c P_h - A^{-1}_c) u_{ct} + (A^{-1}_c u_{ct} + \nu u_c) - A^{-1}_c P_h (f - \tilde{B}(u_c,u_c)) \]
\[ = (A^{-1}_c P_h - A^{-1}_c) u_{ct} + (A^{-1}_e - A^{-1}_c P_h)(u_c - \nu u_c). \]
(4.10)
Taking inner product with \( \xi_h \) in above equation to arrive at
\[ \frac{1}{2} \frac{d}{dt} \| A^{1/2}_c P_h \xi_h \|^2 + \nu \| \xi_h \|^2 = -\nu (\langle u_c - A^{-1}_c P_h u_c, \xi_h \rangle). \]
A use of the Cauchy-Schwarz inequality, Proposition 4.1 in (4.11) gives
\[ \frac{d}{dt} \| A^{1/2}_c P_h \xi_h \|^2 + \nu \| \xi_h \|^2 \leq C h^{2m+2} \| A^{m/2}_c u_c \|^2. \]
We now multiply both sides of (4.12) by \( \sigma^{m-1}(t) \) and use the fact \( \frac{d}{dt} \sigma^{m-1}(t) \leq (m - 1) \sigma^{m-2}(t) + 2 \alpha \sigma^{m-1}(t) \) to obtain
\[ \frac{d}{dt} (\sigma^{m-1}(t) \| A^{1/2}_c P_h \xi_h(t) \|^2) - (m - 1) \sigma^{m-2}(t) \| A^{1/2}_c P_h \xi_h \|^2 - 2 \alpha \sigma^{m-1}(t) \| A^{1/2}_c P_h \xi_h \|^2 + \nu \sigma^{m-1}(t) \| \xi_h \|^2 \leq C h^{2m+2} \sigma^{m-1}(t) \| A^{m/2}_c u_c \|^2. \]
The third term on the left hand side can be combined with the forth term using the fact \( \| A^{1/2}_c P_h \xi_h \|^2 \leq \| \xi_h \|^2 \) and we finally integrate both sides from 0 to \( t \) and use \( \| A^{1/2}_c P_h \xi_h(0) \| = 0 \), we deduce that
\[ \sigma^{m-1}(t) \| A^{1/2}_c P_h \xi_h(t) \|^2 + \left( \nu - \frac{2 \alpha \sigma}{\lambda_1} \right) \int_0^t \sigma^{m-1}(s) \| \xi_h(s) \|^2 \, ds \leq C h^{2m+2} \int_0^t \sigma^{m-1}(s) \| A^{m/2}_c u_c(s) \|^2 \, ds. \]
(4.13)
With \( 0 < \alpha < \frac{\nu \sigma}{2 \lambda_1} \), we have \( \nu - \frac{2 \alpha \sigma}{\lambda_1} > 0 \). Now we prove the assertion of Lemma 4.2 in sequence of steps. First we take \( m = 1 \), then the last term on the right hand side of (4.13) vanishes and we obtain
\[ \int_0^t e^{2\alpha s} \| \xi_h(s) \|^2 \, ds \leq C h^4 \int_0^t e^{2\alpha s} \| A^{1/2}_c u_c(s) \|^2 \, ds. \]
For \( m = 2 \), we have from (4.13)
\[ \int_0^t \sigma(s) \| \xi_h(s) \|^2 \, ds \leq C h^6 \int_0^t \sigma(s) \| A^{1/2}_c u_c(s) \|^2 \, ds + \int_0^t e^{2\alpha s} \| A^{1/2}_c P_h \xi_h(s) \|^2 \, ds. \]
To find the estimate for the last term on the right hand side of (4.14), we take inner product with \( A^{-1}_c P_h \xi_h \) in (4.10) and argue as same as (4.14) - (4.13), we find that
\[ \int_0^t e^{2\alpha s} \| A^{1/2}_c P_h \xi_h(s) \|^2 \, ds \leq C h^6 \int_0^t e^{2\alpha s} \| A^{1/2}_c u_c(s) \|^2 \, ds. \]
For \( m > 2 \), we can follow the same technique, which completes the rest of the proof. \( \square \)
For optimal estimate of \( \xi_h \) in \( L^\infty(L^2) \), we consider a projection \( S_h^c : H_0^1 \rightarrow H_h \) satisfy
\[ a_c(u_c - S_h^c u_c, \phi_h) = 0, \quad \forall \phi_h \in H_h, \]
for some fixed \( \varepsilon > 0 \). We note that the above system, similar to \[ 18 \] (4.52)], has a positive definite operator \( A_{ch} \). Therefore, we can establish the well-posedness of the system (4.15) similar to \[ 18 \].
We now write \( \xi_h = (u_c - S_h^c u_c) + (S_h^c u_c - \tilde{v}_{ch}) =: \zeta + \theta \).
We are interested in the estimates of \( \| \xi_h \|, \| \nabla \xi_h \| \), as this is the first step towards obtaining the optimal estimate of \( \xi_h \). We present the following Lemma.
Lemma 4.3. Suppose the assumptions of Lemma\textsuperscript{3.3} are satisfied. Then, there exists a positive constant \( C \) such that
\[
\|\zeta(t)\| + h\|\nabla \zeta(t)\| \leq Ch^{m+1}\|A^{\frac{m+1}{2}}\psi_e(t)\|.
\]
Moreover, the following estimate holds:
\[
\|\zeta_t(t)\| + h\|\nabla \zeta_t(t)\| \leq Ch^{m+1}\|A^{\frac{m+1}{2}}\psi_t(t)\|.
\]

Proof. Note that, \( \zeta = u_e - S_h^1u_e \), then the first result directly comes from Lemma\textsuperscript{4.1} replacing \( v \) by \( u_e \) and \( v_h \) by \( S_h^1u_e \). For the second estimate, we differentiate (4.15) with respect to the temporal variable \( t \) and do similar set of analysis as Lemma\textsuperscript{4.3}. This completes the estimates.

Recall that, we split \( \xi_h \) as follows: \( \xi_h = \zeta + \theta \). Armed with the estimates of \( \zeta \) and \( \zeta_t \), we now pursue the estimates of \( \theta \) in order to find the optimal error estimates of \( \xi_h \) in \( L^\infty(L^2) \) and \( L^\infty(H^1) \)-norms. From (4.9) and (4.15), the equation in \( \theta \) turns out to be
\[
(\theta_t, \phi_h) + \nu a_h(\theta, \phi_h) = -(\zeta_t, \phi_h), \quad \forall \phi_h \in H_h.
\]

Lemma 4.4. Under the assumptions of Lemma\textsuperscript{3.3} there is a positive constant \( C \) such that \( \xi_h \) satisfies the following estimate for some finite time \( t > 0 \),
\[
\|\xi_h(t)\| + h\|\nabla \xi_h(t)\| \leq Ch^{m+1}\|A^{\frac{m+1}{2}}\xi_h(t)\|.
\]

Proof. Selecting \( \phi_h = \sigma^m(t)\theta \) with \( \sigma^m(t) = e^{2\alpha t}\sigma^m(t) \) in (4.10), it now follows that
\[
\frac{1}{2} \frac{d}{dt}(\sigma^m(t)\|\theta\|^2) + \nu \sigma^m(t)\|A^{\frac{1}{2}}\theta\|^2 = -\sigma^m(t)(\zeta, \theta) + \frac{1}{2}\sigma^m(t)\|\theta\|^2.
\]
A use of the Cauchy-Schwarz inequality with an appropriate use of the Young’s inequality with \( \sigma^m(t) \leq (2m+1)\sigma^{m+1}(t) \) in (4.17) yields
\[
\frac{1}{2} \frac{d}{dt}(\sigma^m(t)\|\theta\|^2) + \nu \sigma^m(t)\|A^{\frac{1}{2}}\theta\|^2 \leq \frac{1}{2}\sigma^m(t+1)\|\xi_t\|^2 + \frac{1}{2}(2\alpha + 1)\sigma^{m+1}(t)\|\theta\|^2.
\]
On integrating (4.18) with respect to time from 0 to \( t \) and write \( \theta = \xi_h - \zeta \) to find that
\[
\sigma^m(t)\|\theta(t)\|^2 + \nu \int_0^t \sigma^m(s)\|A^{\frac{1}{2}}\theta(s)\|^2 ds \leq C \int_0^t \sigma^m+1(s)\|\zeta(s)\|^2 ds + C \int_0^t \sigma^{m+1}(s)\|\theta(s)\|^2 ds
\]
\[
\leq C \int_0^t \sigma^m(s)\|\zeta(s)\|^2 ds + C \int_0^t \sigma^m(s)(\|\xi_h(s)\|^2 + \|\zeta(s)\|^2) ds.
\]
Use the Lemmas\textsuperscript{4.2} and \textsuperscript{4.3} to conclude
\[
\sigma^m(t)\|\theta(t)\|^2 + \int_0^t \sigma^m(s)\|A^{\frac{1}{2}}\theta(s)\|^2 ds \leq Ch^{2m+2}\left(\int_0^t \sigma^{m+1}(s)\|A^{\frac{m+1}{2}}\xi_h(s)\|^2 ds + \int_0^t \sigma^{m-1}(s)\|A^{\frac{m+1}{2}}u_e(s)\|^2 ds\right).
\]
Finally, a use of Lemma\textsuperscript{4.3} with Lemma\textsuperscript{2.3} triangle inequality and the inverse hypothesis completes the rest of the proof.

Recall that
\[
e_e := u_e - u_{eh} = (u_e - v_{eh}) + (v_{eh} - u_{eh}) = \xi_h + \eta_h.
\]
Since all the required estimates of \( \xi_h \) are obtained, it is now enough to estimate \( \eta_h \).

Lemma 4.5. Suppose the assumptions of Lemma\textsuperscript{3.3} hold. Let \( u_{eh}(t) \) be a solution of (3.2) with initial condition \( u_{eh}(0) = P_hu_{e0} \). Then, there exists a positive constant \( C \) such that the following holds:
\[
\int_0^t \sigma^{m-1}(s)\|e_e(s)\|^2 ds \leq K(t)h^{2m+2}\int_0^t \sigma^{m-1}(s)\|A^{\frac{m+1}{2}}u_e(s)\|^2 ds, \quad m \geq 1,
\]
where, \( K(t) = Ce^{Ct} \).
Proof. In view of the Lemma 4.2 we need to prove only the estimate of \( \eta_h \). From (3.2) and (4.8), the equation in \( \eta_h \) becomes

\[
(\eta_{ht}, \phi_h) + \nu a_z(\eta_h, \phi_h) = \tilde{b}(u_{ch}, u_{ch}, \phi_h) - \tilde{b}(u_z, u_z, \phi_h), \quad \phi_h \in H_h.
\]

(4.19)

Choose \( \phi_h = A^{-1}_{ch} \eta_h \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2 + \nu \| \eta_h \|^2 = \Lambda_h(A^{-1}_{ch} \eta_h),
\]

where

\[
\Lambda_h(\phi_h) = \tilde{b}(u_{ch}, u_{ch}, \phi_h) - \tilde{b}(u_z, u_z, \phi_h) = -\tilde{b}(e_z, u_{ch}, \phi_h) - \tilde{b}(u_z, e_z, \phi_h).
\]

Use Lemma 3.2 with Lemma 3.1 and write \( e_z = \xi_h + \eta_h \), we easily estimate \( \Lambda_h \) as similar to [9] (4.9)-(4.14) as

\[
|\Lambda_h(A^{-1}_{ch} \eta_h)| \leq \rho \| \eta_h \|^2 + C(\rho) \left( \| \nabla u_{ch} \|^2 + \| u_{ch} \| \| \nabla u_{ch} \| + \| u_z \| \| \nabla u_z \| \right) \| \xi_h \|^2
\]

\[
+ C(\rho) \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2 \left( \| \nabla u_{ch} \|^4 + \| u_{ch} \|^2 \| \nabla u_{ch} \|^2 + \| u_z \|^2 \| \nabla u_z \|^2 \right).
\]

(4.21)

With \( \rho = \nu/2 \), we obtain using (4.20) and Lemma 3.3

\[
\frac{d}{dt} \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2 + \nu \| \eta_h \|^2 \leq C(\nu) \left( \| \xi_h \|^2 + \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2 \right).
\]

(4.22)

Now, multiply by \( e^{2\alpha t} \) and apply the Gronwall’s Lemma [29] to conclude

\[
e^{2\alpha t} \| A^{-\frac{1}{2}}_{ch} \eta_h(t) \|^2 + \nu \int_0^t e^{2\alpha \tau} \| \eta_h(s) \|^2 d\tau \leq Ce^{\alpha t} \int_0^t e^{2\alpha \tau} \| \xi_h(s) \|^2 d\tau.
\]

An application of the triangular inequality with Lemma 4.2 concludes the proof for \( m = 1 \). For \( m > 1 \), multiply (4.22) by \( \sigma^{-m-1}(t) \) and using the fact \( \frac{d}{dt} \sigma^{-m-1}(t) \leq C \sigma^{-m-2}(t) \) and (4.21) with \( \rho = \nu/2 \), we obtain

\[
\frac{d}{dt} (\sigma^{-m-1}(t) \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2) + \nu \sigma^{-m-1}(t) \| \eta_h \|^2 \leq C \sigma^{-m-2}(t) \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2 + C \sigma^{-m-1}(t) \| \xi_h \|^2 + \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2.
\]

Finally, arguing in exactly the same way as in the Lemma 4.2 and applying the Gronwall’s Lemma [29], the triangular inequality with Lemma 4.2 we complete the rest of the proof.

**Lemma 4.6.** Under the assumption of Lemma 4.3, the following holds:

\[
\| \eta_h(t) \| + h \| \nabla \eta_h(t) \| \leq K(t)h^{m+1}t^{-\frac{m}{2}}.
\]

**Proof.** Choose \( \phi_h = \sigma^m(t) \eta_h \) in (4.19) to obtain

\[
\frac{1}{2} \frac{d}{dt} (\sigma^m(t) \| \eta_h \|^2) + \nu \sigma^m(t) \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2 = \frac{1}{2} \sigma^m(t) \| \eta_h \|^2 + \sigma^m(t) \Lambda_h(\eta_h).
\]

We proceed as [9] (4.16)-(4.20) and integrate to find that

\[
\sigma^m(t) \| \eta_h \|^2 + \nu \sigma^m(t) \int_0^t \sigma^m(s) \| A^{-\frac{1}{2}}_{ch} \eta_h(s) \|^2 ds \leq \frac{1}{2} \sigma^m(t) \| \eta_h \|^2 + \sigma^m(t) \int_0^t \sigma^{-m-1}(s) \| \eta_h(s) \|^2 ds
\]

\[
+ C \int_0^t \tau(s) \left( \| \nabla u_z(s) \| \| \Delta u_z(s) \| + \| \nabla u_{ch}(s) \| \| \Delta u_{ch}(s) \| \right) \sigma^{-m-1}(s) \| \eta_h(s) \|^2 ds.
\]

Apply Lemmas 2.3, 3.3 and 4.5 and then multiply the resulting inequality by \( e^{-2\alpha t} \) to arrive at

\[
\tau^m(t) \| \eta_h \|^2 + \nu e^{-2\alpha t} \int_0^t \sigma^m(s) \| A^{-\frac{1}{2}}_{ch} \eta_h \|^2 ds \leq C h^{2m+2}.
\]

Since \( \eta_h \in H_h \), we use inverse hypothesis to obtain an estimate for \( \| \nabla \eta_h \| \), which completes the rest of the proof. \( \square \)
4.2 Error Estimates for the Pressure

Subtract the second equation of (2.1) from the second equation of (3.1) and obtain

\[(p_c - p_{ch}, \chi_h) = \frac{\nu}{\varepsilon} (\nabla \cdot \mathbf{e}_v, \chi_h).\]

Choose \(\chi_h = p_{ch} - j_h p_c = e_p - (p_c - j_h p_c)\) with \(e_p = p_c - p_{ch}\) in (4.23) to find that

\[\|e_p\|^2 = (e_p, p_c - j_h p_c) + \frac{\nu}{\varepsilon} (\nabla \cdot e_v, e_p) - \frac{\nu}{\varepsilon} (\nabla \cdot e_v, p_c - j_h p_c)\]

\[
\leq C h^{2m} \|p_c - p_{ch}\|^2 + \frac{C}{\varepsilon^2} \|\nabla \cdot e_v\|^2 + \frac{1}{2} \|e_p\|^2.
\]

If we use the bound \(\|\nabla \cdot \phi\| \leq C \|\nabla \phi\|\), then we observe that the error bound depends on \(1/\varepsilon\). Replacing \(\mathbf{v}\) by \(\mathbf{u}_v + S_h^l \mathbf{u}_v\) in (4.24), we can say \(\|\nabla \cdot (\mathbf{u}_v - S_h^l \mathbf{u}_v)\| \leq C \sqrt{h^{m-1} t} \varepsilon\). Since \(e_v = (u_v - S_h^l u_v) + (S_h^l u_v - u_{ch})\), hence, \(\frac{1}{\varepsilon} \|\nabla \cdot e_v\|\) will depend on \(1/\varepsilon\), and so will \(e_p\). It is clear that the error bound for the pressure always depends on \(1/\varepsilon\) or \(1/\sqrt{\varepsilon}\) if we find it directly using velocity error. But, if we choose the finite element spaces \(\mathbf{H}_h\) and \(L_h\) in such a way that satisfy the discrete inf-sup condition \((B2')\), then we can find the \(\varepsilon\)-uniform pressure error estimate as given below.

First, we split \(e_p\) as

\[\|e_p\| = \|p_c - p_{ch}\| \leq \|p_c - j_h p_c\| + \|j_h p_c - p_{ch}\|.
\]

From (B2'), we observe that

\[
\|j_h p_c - p_{ch}\| \leq C \sup_{\phi \in \mathbf{H}_h \setminus \{0\}} \left\{ \frac{|(j_h p_c - p_{ch}, \nabla \cdot \phi_h)|}{\|\nabla \phi_h\|} \right\}
\]

\[
\leq C \left( \|j_h p_c - p_{ch}\| + \sup_{\phi \in \mathbf{H}_h \setminus \{0\}} \left\{ \frac{|(p_{ch}, \nabla \cdot \phi_h)|}{\|\nabla \phi_h\|} \right\} \right).
\]

The first term on right hand side of (4.25) can be estimated by using (B1) and for the second term, we subtract (3.1) from (2.1) to obtain

\[(e_p, \nabla \cdot \phi_h) = (e_v, \phi_h) + \nu a(e_v, \phi_h) - \Lambda_h(\phi) \quad \forall \phi_h \in \mathbf{H}_h.
\]

A use of Lemma 3.2 shows

\[(\Lambda_h(\phi_h)) \leq C (\|\nabla \mathbf{u}_v\| + \|\nabla \mathbf{u}_{ch}\|) \|\nabla e_v\| \|\nabla \phi_h\|.
\]

Now, apply the Cauchy-Schwarz inequality (4.26) and use (4.27) to arrive at

\[(e_p, \nabla \cdot \phi_h) \leq \left[ C \|e_v\|_{-1; h} + C \|\nabla e_v\| \right] \|\nabla \phi_h\|,
\]

where,

\[
\|e_v\|_{-1; h} = \sup \left\{ \frac{\epsilon \cdot \mathbf{e}_h}{\|\nabla \phi_h\|} : \phi_h \in \mathbf{H}_h, \phi_h \neq 0 \right\}.
\]

Since all the estimate on right hand side in (4.28) are known except \(\|e_v\|_{-1; h}\). Also \(\|e_v\|_{-1; h} \leq \|e_v\|_{-1}\), so now we derive \(\|e_v\|_{-1}\).

Lemma 4.7. The error \(e_v\) satisfies for \(0 < t < T\)

\[\|e_v\|_{-1} \leq K(t) \left( h^m (\|u_v\|_{m-1} + \|\nabla e_v\|) \right).
\]

Proof. For any \(\phi \in H_0^1\), use the orthogonal projection \(P_h : L^2 \rightarrow \mathbf{H}_h\), we obtain using (1.7) with \(\phi_h = P_h \phi\)

\[(e_v, \phi_h) = (e_v, \phi_h - P_h \phi) + (e_v, P_h \phi)
\]

\[
= (e_v, \phi - P_h \phi) - \nu a(e_v, P_h \phi) - \Lambda_h(P_h \phi).
\]

Using approximation property of \(P_h\), we find that

\[(e_v, \phi - P_h \phi) = (u_v - P_h u_v, \phi - P_h \phi) \leq Ch^m \|u_v\|_{m-1} \|\nabla \phi\|.
\]

Also, using Lemma 3.2 with boundedness of \(u_v\) and \(u_{ch}\) to bound

\[
\Lambda_h(P_h \phi) \leq C (\|\nabla u_v\| + \|\nabla u_{ch}\|) \|\nabla e_v\| \|\nabla \phi\| \leq C \|\nabla e_v\| \|\nabla \phi\|.
\]
Now substitute (4.31)-(4.32) in (4.30) to obtain
\[
(e_{zt}, \phi) \leq C\left(h^{m}(\|u_{zt}\|_{m-1} + \|\nabla e_{z}\|)\|\nabla \phi\|\right).
\]
and therefore,
\[
\|e_{zt}\|_{1} \leq \sup \left\{ \frac{e_{zt}, \phi}{\|\nabla \phi\|} : \phi \in H_{0}^{1}, \phi \neq 0 \right\} \leq C\left(h^{m}(\|u_{zt}\|_{m-1} + \|\nabla e_{z}\|)\right).
\]
This completes the rest of the proof.

The use of (4.25)-(4.28) and Lemma 4.7 and with the help of Lemma 2.5 and Theorem 4.1 yields the following.

**Lemma 4.8.** Under the hypothesis of the Lemma 4.2 there exists a positive constant $C$ define as Lemma 4.2 such that, for all $t > 0$, it holds:
\[
\|p_{e} - p_{ch}\| \leq K(t)h^{m}t^{-\frac{m}{2}}.
\]

**Proof of the Theorem 4.1.** Combining the Lemmas 1.4, 1.6 and 1.8 we complete the proof the theorem 4.1

**Remark 4.1.** It is noted that, there is no restriction on choosing finite element spaces for the velocity error bounds. But for the $\varepsilon$-uniform pressure error bound we have to choose a proper finite element spaces which satisfy the discrete inf-sup condition. In [17], it is seen that the improper choice of finite element spaces like $(P1 - P0)$ makes the error bounds dependent of $1/\sqrt{\varepsilon}$ in the context of the steady state Navier-Stokes equations.

### 4.3 Uniform Estimates

We note here that the error estimates obtained in Theorem 4.1 are exponentially dependent on time. In order to find the uniform estimates of $e_{z}$, we need to obtain the uniform estimates of $\eta_{h}$ only using (4.24), since the estimates of $\xi_{h}$ are uniform in time (see, Lemma 4.4).

**Theorem 4.2.** Under the assumptions of Theorem 4.1 and the uniqueness condition (2.7), there exists a positive constant $C$ such that
\[
\|u_{e} - \eta_{h}\|_{t} + h(\|\nabla(u_{e} - \eta_{h})\|_{t}) + \|(p_{e} - p_{ch})\|_{t}) \leq C^{m+1}t^{-\frac{m}{2}},
\]

**Proof.** Choose $\phi_{h} = \eta_{h}$ in (4.19) to obtain
\[
\frac{1}{2}\frac{d}{dt}(\|\eta_{h}\|^{2}) + \nu\|A_{ch}^{\frac{1}{2}}\eta_{h}\|^{2} = \Lambda_{h}(\eta_{h}).
\]

We use Lemma 3.3 and 3.4 and the uniqueness condition (2.7) with Lemma 2.5 to bound $\Lambda_{h}(\eta_{h})$ as
\[
\|\Lambda_{h}(\eta_{h})\| = b(\xi_{h}, u_{ch}, \eta_{h}) + b(u_{e}, \xi_{h}, \eta_{h}) + b(\eta_{h}, u_{ch}, \eta_{h}) \leq C(\|\nabla u_{e}\|^{\frac{2}{3}} + \|\nabla \xi_{h}\|^{\frac{2}{3}} + \|\nabla u_{ch}\|^{\frac{2}{3}} + \|\nabla \eta_{h}\|^{\frac{2}{3}})(\|\nabla \eta_{h}\| + N\|\nabla u_{ch}\|\|\nabla \eta_{h}\|^{2}) \leq C^{m+1}t^{-\frac{m}{2}} + \|\nabla \eta_{h}\|^{2}.
\]

Use the above estimate in (4.33) and multiply by $e^{2\alpha t}$ then integrate with respect to time to arrive at
\[
e^{2\alpha t}\|\eta_{h}(t)\|^{2} + 2\int_{0}^{t}(\nu - N\|\nabla u_{e}\|)e^{2\alpha s}\|\nabla \eta_{h}(s)\|^{2}ds + \frac{2\nu}{\varepsilon}\int_{0}^{t}e^{2\alpha s}\|\nabla \cdot \eta_{h}(s)\|^{2}ds
\]
\[
\leq \|\eta_{h}(0)\|^{2} + 2\alpha \int_{0}^{t}e^{2\alpha s}\|\eta_{h}(s)\|^{2}ds + C^{m+1}t^{-\frac{m}{2}}\int_{0}^{t}e_{ch}^{2\alpha s}\|\nabla \eta_{h}(s)\|ds.
\]

Now, multiply by $e^{-2\alpha t}$ and take $t \to \infty$ and use $\lim_{t \to \infty}\|\nabla u_{e}\| \leq \frac{1}{N\|\nabla u_{e}\|} \leq \frac{1}{\nu}\|\nabla u_{e}\|_{L^{\infty}(0,\infty;H^{-1}(\Omega))}$ and $\lim_{t \to \infty}t^{-\frac{m}{2}}(t) = 1$ to find that
\[
(\nu - N\|\nabla u_{e}\|_{L^{\infty}(0,\infty;H^{-1}(\Omega))})\lim_{t \to \infty}\|\nabla \eta_{h}(t)\| \leq C^{m+1}\lim_{t \to \infty}\|\nabla \eta_{h}(t)\|.
\]

From (2.7), we have $N\nu^{2}\|\nabla u_{e}\|_{L^{\infty}(0,\infty;H^{-1}(\Omega))} < 1$. Then, we conclude that
\[
\lim_{t \to \infty}\|\nabla \eta_{h}(t)\| \leq \lim_{t \to \infty}\|\nabla \eta_{h}(t)\| \leq C^{m+1}.
\]

With this and Lemma 3.4 we finally obtain
\[
\lim_{t \to \infty}\|e_{z}(t)\| \leq C^{m+1}t^{-\frac{m}{2}}.
\]

Using the above estimate, we easily derive the estimates of $\|\nabla e_{z}(t)\|$ and $\|(p_{e} - p_{ch})(t)\|$, which completes the rest of the proof. \qed
where, \( U^n \) is the penalized semidiscrete NSEs (3.1): Find \( U^n \) such that for \( 1 \leq n \leq N \):

\[
\frac{\partial}{\partial t} U^n + \nu A^n U^n + \nu A^n U^n + \nu A^n U^n = 0, \quad \forall \chi_h \in H_h, \quad n \geq 0,
\]

Note that, the nonlinear term \( \tilde{b}(U^n, U^n, \phi_h) \) depends on the given data.

Proof. For \( n = 1 \), we substitute \( \phi_h = U^n \) in (5.2) and use \( \frac{1}{2} \partial_t \| U^n \|^2 \) and the Poincaré inequality and the Cauchy-Schwarz inequality with (5.3) to obtain

\[
\frac{1}{2} \partial_t \| U^n \|^2 + \frac{3}{4} \| A^n_{ch} U^n \|^2 \leq \frac{\kappa}{\nu \lambda_1} \| f \|^2.
\]

Note that, the nonlinear term \( \tilde{b}(U^n, U^n, U^n) = 0 \). Hence multiply (5.5) by \( k e^{2\sigma_1 n} \) and sum over \( i = 1 \) to \( n \) and use the fact

\[
k \sum_{i=1}^{n} e^{2\sigma_1} \frac{e^{2\sigma_1} k}{k} \| U^n \|^2 = e^{2\sigma_1} \| U^n \|^2 - \| U^n \|^2 - k \sum_{i=1}^{n} \left( \frac{e^{2\sigma_1}}{k} - 1 \right) e^{2\sigma_1} \| U^n \|^2 \geq e^{2\sigma_1} \| U^n \|^2 - \| U^n \|^2 - k \sum_{i=1}^{n} \left( \frac{e^{2\sigma_1}}{k} - 1 \right) e^{2\sigma_1} \| A^n_{ch} U^n \|^2,
\]

to conclude that

\[
e^{2\sigma_1} \| U^n \|^2 + \frac{3}{4} \| A^n_{ch} U^n \|^2 \leq \| U^n \|^2 + \frac{2c_0^2 \| f \|^2}{\nu \lambda_1} \leq \| U^n \|^2 + \frac{2c_0^2 \| f \|^2}{\nu \lambda_1} \]

where \( \| f \|_\infty = \| f \|_{L^\infty} \). With \( 0 < \alpha < \min \{ \alpha_0, \frac{\nu \lambda_1}{2c_0^2} \} \), which guarantees that \( \nu \geq c_0 \left( \frac{e^{2\sigma_1} k}{k \lambda_1} \right) \). Now, we multiply both sides by \( e^{-2\alpha_1 n} \) to conclude (5.3).

For (5.2), we substitute \( \phi_h = A^n_{ch} U^n \) in (5.2) to obtain

\[
\frac{1}{2} \partial_t \| A^n_{ch} U^n \|^2 + \nu \| A^n_{ch} U^n \|^2 \leq (f^i, A^n_{ch} U^n) - \tilde{b}(U^n, U^n, A^n_{ch} U^n).
\]
Now a use of the Cauchy-Schwarz inequality and the Young’s inequality yields
\[
|\langle f', A_{zh} U_{\varepsilon}^i \rangle - \tilde{b}(U_{\varepsilon}^i, U_{\varepsilon}^i, A_{zh} U_{\varepsilon}^i)\rangle | \leq \|f'\| \|A_{zh} U_{\varepsilon}^i\| + C \|U_{\varepsilon}^i\|^2 \|A_{zh} U_{\varepsilon}^i\|^2 \|A_{zh} U_{\varepsilon}^i\|^3/2
\]
(5.7)
\[
\leq C \|f'\|^2 + \|U_{\varepsilon}^i\|^2 \|A_{zh} U_{\varepsilon}^i\|^4 + \frac{\nu}{2} \|A_{zh} U_{\varepsilon}^i\|^2.
\]
Insert (5.7) in (5.6) to obtain
\[
\partial_t \|A_{zh} U_{\varepsilon}^i\|^2 + \nu \|A_{zh} U_{\varepsilon}^i\|^2 \leq C (\|f'\|^2 + \|U_{\varepsilon}^i\|^2 \|A_{zh} U_{\varepsilon}^i\|^4).
\]
(5.8)
Now, choose \(\phi_h = \partial_t U_{\varepsilon}^i\) in (5.2) with \(n = i\) and use the fact \(\partial_t \phi_h, \phi_h^n = \frac{1}{k} \partial_t \phi_h^n + \frac{1}{2} ||\partial_t \phi_h^n||^2\) to obtain
\[
\|\partial_t U_{\varepsilon}^i\|^2 + \frac{\nu}{2} \|\partial_t U_{\varepsilon}^i\|^2 + \frac{k\nu}{2} \|\partial_t A_{zh} U_{\varepsilon}^i\|^2 \leq (f', \partial_t U_{\varepsilon}^i) - \tilde{b}(U_{\varepsilon}^i, U_{\varepsilon}^i, \partial_t U_{\varepsilon}^i).
\]
(5.9)
Using (2.2), the last term on the right hand side of (5.9) can be written as
\[
\|\tilde{b}(U_{\varepsilon}^i, U_{\varepsilon}^i, \partial_t U_{\varepsilon}^i)\| \leq \frac{1}{k} \|\tilde{b}(U_{\varepsilon}^i, U_{\varepsilon}^i, U_{\varepsilon}^i - U_{\varepsilon}^{i-1})\| = \frac{1}{k} \|\tilde{b}(U_{\varepsilon}^i, U_{\varepsilon}^i, U_{\varepsilon}^{i-1})\| = \frac{1}{k} \|\tilde{b}(U_{\varepsilon}^i, U_{\varepsilon}^i, U_{\varepsilon}^{i-1}, U_{\varepsilon})\|.
\]
A use of Lemma 3.2 with 3.1, the Cauchy-Schwarz inequality and the Young’s inequality yields
\[
\frac{1}{k} \|\tilde{b}(U_{\varepsilon}^i, U_{\varepsilon}^{i-1}, U_{\varepsilon})\| \leq \frac{C}{k} \|U_{\varepsilon}^i\|^2 \|\nabla U_{\varepsilon}^{i+1}\|^2 \|\nabla U_{\varepsilon}^{i-1}\|^2 \|\nabla U_{\varepsilon}^i\|^2 \|\nabla U_{\varepsilon}^i\|^2
\]
\[
\leq \frac{C}{k} \|U_{\varepsilon}^i\| \|A_{zh} U_{\varepsilon}^i\| \|A_{zh} U_{\varepsilon}^{i-1}\|
\]
\[
\leq \frac{\nu}{4k} \|A_{zh} U_{\varepsilon}^i\|^2 + \frac{C}{k\nu} \|U_{\varepsilon}^i\|^2 \|A_{zh} U_{\varepsilon}^i\|^2.
\]
(5.10)
We use the inequality (5.10) with \(\|U_{\varepsilon}^i\| \leq C\) in (5.9) and use the fact \(\partial_t \|A_{zh} U_{\varepsilon}^i\|^2 = \frac{1}{2} (\|A_{zh} U_{\varepsilon}^i\|^2 - \|A_{zh} U_{\varepsilon}^{i-1}\|^2)\), then we obtain
\[
\|\partial_t U_{\varepsilon}^i\|^2 + \frac{\nu}{4k} \|A_{zh} U_{\varepsilon}^i\|^2 + \frac{k\nu}{4} \|\partial_t A_{zh} U_{\varepsilon}^i\|^2 \leq \frac{C}{k\nu} \|f'\|^2 + \frac{C}{k} \|A_{zh} U_{\varepsilon}^{i-1}\|^2.
\]
Now, drop the first and the last terms on the left hand side and multiply the resulting equation by \(4k/\nu\) to arrive at
\[
\|A_{zh} U_{\varepsilon}^i\|^2 \leq C \|A_{zh} U_{\varepsilon}^{i-1}\|^2 + C \|f'\|^2.
\]
(5.11)
A use of (5.11) in the last term on the right hand side of (5.8) leads to
\[
\partial_t \|A_{zh} U_{\varepsilon}^i\|^2 + \nu \|A_{zh} U_{\varepsilon}^i\|^2 \leq g^{-1} \|A_{zh} U_{\varepsilon}^{i-1}\|^2 + h^{i-1},
\]
where, \(g^{-1} = C \|U_{\varepsilon}^i\|^2 \|A_{zh} U_{\varepsilon}^i\|^2\) and \(h^{i-1} = C (1 + \|U_{\varepsilon}^i\|^2 \|A_{zh} U_{\varepsilon}^i\|^2) \|f'\|^2\). We now use the discrete uniform Gronwall’s lemma (Lemma 2.2) to derive
\[
\|A_{zh} U_{\varepsilon}^i\|^2 \leq C, \quad \forall n \geq N_1 + 1,
\]
(5.13)
where \(N_1 \leq N\). For \(1 \leq n \leq N_1\), we use the classical discrete Gronwall’s lemma \([19,20]\) to obtain
\[
\|A_{zh} U_{\varepsilon}^i\|^2 \leq C, \quad \text{for } 1 \leq n \leq N_1.
\]
(5.14)
We multiply (5.12) by \(e^{2\alpha t_n}\) and sum from \(i = 1\) to \(n\) and use (5.3), (5.13) and (5.14). Finally, we multiply the resulting equation by \(e^{-2\alpha t_n}\) to conclude (5.3), when \(r = 1\).
For \(r = 2\), we need a couple of intermediate estimates. For this, we first use Lemma 3.2 with Lemma 3.1 and the Cauchy-Schwarz inequality to bound the terms on the right hand side of (5.9) as
\[
|\langle f', \partial_t U_{\varepsilon}^i \rangle - \tilde{b}(U_{\varepsilon}^i, U_{\varepsilon}^i, \partial_t U_{\varepsilon}^i)\rangle | \leq C \|f'\|^2 + \|A_{zh} U_{\varepsilon}^i\|^2 \|A_{zh} U_{\varepsilon}^i\|^2 + \frac{1}{2} \|\partial_t U_{\varepsilon}^i\|^2.
\]
(5.15)
After using (5.15) in (5.9), we multiply the resulting equation by \(ke^{2\alpha t_i}\) and take sum from \(i = 1\) to \(n\). Then a use of (5.3) with \(r = 1\) gives
\[
\nu \|A_{zh} U_{\varepsilon}^i\|^2 + e^{-2\alpha t_n} \sum_{i=1}^{n} e^{2\alpha t_i} \|\partial_t U_{\varepsilon}^i\|^2 \leq C.
\]
(5.16)
For the second intermediate estimate, we consider (5.2) for \( n = i \) and for \( n = i - 1 \). Then subtract them and divide by \( k \) to obtain

\[
\frac{1}{2} \partial_t \| \partial_t U^i_t \|^2 + 2 \nu A^i_{1t} \partial_1 U^i_t \|^2 \leq \frac{1}{k} \| \partial_t U^i_t \|^2 + C \| \partial_t U^i_t \|^2 + \| \partial_1 U^i_t \|^2 + \| \partial_\epsilon U^i_t \|^2 + \frac{1}{2} \| A^i_{1t} \|^2.
\]

(5.17)

Now, choose \( \phi_h = \partial_1 U^i_t \) in (5.17) and use Lemma 3.2 with Lemma 3.1 and the Cauchy-Schwarz inequality to find

\[
\frac{1}{2} \partial_t \| \partial_t U^i_t \|^2 + \nu \| A^i_{1t} \|^2 \leq \frac{1}{k} \| \partial_t U^i_t \|^2 + C \| \partial_t U^i_t \|^2 + \| \partial_1 U^i_t \|^2 + \| \partial_\epsilon U^i_t \|^2 + \frac{1}{2} \| A^i_{1t} \|^2.
\]

(5.18)

Multiply (5.18) by \( k \epsilon \epsilon^2 \), and sum from \( i = 1 \) to \( n \) and use (5.16) to deduce

\[
\tau_n \| \partial_t U^n_t \|^2 \leq C.
\]

(5.19)

Now, choose \( \phi_h = A_{ch} U^n_t \) in (5.2) and use (5.7) to rewrite it as

\[
\nu \| A_{ch} U^n_t \|^2 = (f^n, A_{ch} U^n_t) - \partial_t (U^n_t, U^n_t, A_{ch} U^n_t) - (\partial_t U^n_t, A_{ch} U^n_t)
\]

\[
\leq C(\| f^n \| + \| U^n_t \| + \| A_{ch} U^n_t \|^2 + \| \partial_t U^n_t \|) \| A_{ch} U^n_t \|.
\]

Multiply by \( \tau_n \) and use (5.3), (5.13) and (5.19) to conclude (5.4), when \( r = 2 \), which completes the proof.

\[\square\]

**Remark 5.1.** In Lemma 5.1 such a choice of \( \alpha_0 > 0 \) is possible by choosing \( \alpha_0 < \frac{\log(1 + e^{-20\epsilon})}{e} \). Note that for large \( k \), \( \alpha_0 \) is small but as \( k \to 0 \), \( \frac{\log(1 + e^{-20\epsilon})}{e} \to \frac{\nu \lambda_1}{20 \epsilon} \). Thus, with \( 0 < \alpha < \min\{\alpha_0, \frac{\nu \lambda_1}{20 \epsilon}\} \), the result in Lemma 5.1 is valid.

### 5.1 Fully Discrete Error Estimate

Define \( u_{ch}(t_n) = u^n_{ch} \) and \( e^n_t = U^n_t - u^n_{ch} \). Consider (5.2) at \( t = t_n \) and subtract from (5.2) to arrive at

\[
(\partial_t e^n_i, \phi_h) + \nu a_{\epsilon} e^n_i, \phi_h) = R^n_h(\phi_h) + A^n_h(\phi_h),
\]

where

\[
R^n_h(\phi_h) = (u^n_{ch}, \phi_h) - (\partial_t u^n_{ch}, \phi_h) = (u^n_{ch}, \phi_h) - \frac{1}{k} \int_{t_{n-1}}^{t_n} (u_{chs}, \phi_h) \, ds
\]

(5.21)

\[
A^n_h(\phi_h) = -\partial_t (u^n_{ch}, \phi_h) - \partial_\epsilon (U^n_t, U^n_t, \phi_h) = -\partial_t (u^n_{ch}, \phi_h) - \partial_\epsilon (U^n_t, e^n_t, \phi_h),
\]

(5.22)

The optimal error estimates for velocity and pressure for time discretization only, have already been proved in [29] when the initial data \( u_{00} \in H^2 \cap H^1 \) and under the assumption of sufficiently small \( k \) so that \( Ck < 1 \). In our case, \( u_{00} \in H^2_0 \) and without smallness assumption on \( k \), we have proved the following optimal error estimates.

**Lemma 5.2.** Assume that (A1),(A2), (B1) and (B2) hold true. Then, under the assumption of Lemma 5.1 there exists some positive constant \( C \), that depends on the given data, the following holds

\[
\| A_{ch}^{1/2} e^n_t \|^2 + ke^{-20\epsilon} \sum_{j=1}^{n} e^{20\epsilon} \| A_{ch}^{(j+1)/2} e^n_t \|^2 \leq K_n k^{1-j}, \quad j = -1, 0, 1,
\]

where \( K_n = C e^{C_1 n} \).

**Proof.** For \( j = 0 \), take \( n = i \) and \( \phi_h = e^n_i \) in (5.20) to arrive at

\[
(\partial_t e^n_i)^2 + 2 \nu \| A_{ch}^{1/2} e^n_i \|^2 \leq 2R_h^i(e^n_i) + 2A_h^i(e^n_i).
\]

(5.23)

Then multiply (5.23) by \( ke^{20\epsilon} \) and take sum from \( i = 1 \) to \( n \) and use the fact

\[
k \sum_{i=1}^{n} e^{20\epsilon} \| \partial_t e^n_i \|^2 \geq e^{20\epsilon} \| e^n_i \|^2 - \sum_{i=1}^{n} e^{20\epsilon} (e^{20\epsilon} - 1) \| e^n_i \|^2
\]

(5.24)

\[
\geq e^{20\epsilon} \| e^n_i \|^2 - C(\epsilon_{\text{opt}})^2 \sum_{i=1}^{n} e^{20\epsilon} \| A_{ch}^{1/2} e^n_i \|^2.
\]
to find that

\[(5.25) \quad e^{2\alpha t_n} \| \epsilon_t^i \|^2 + \left( 2 \nu - c_0^2 \left( \frac{e^{2\alpha k}}{k\lambda_1} - 1 \right) \right) k \sum_{i=1}^n e^{2\alpha t_i} \| A_{ch}^h \epsilon_t^i \|^2 \leq 2k \sum_{i=1}^n e^{2\alpha t_i} \left( R_h^i \left( \epsilon_t^i \right) + A_h^i \left( \epsilon_t^i \right) \right). \]

A use of the Cauchy-Schwarz’s inequality with \((5.21)\) and \(t - t_{i-1} \leq t_i, t \in [t_{i-1}, t_i] \) yields

\[
k \sum_{i=1}^n e^{2\alpha t_i} R_h^i \left( \epsilon_t^i \right) \leq k \sum_{i=1}^n e^{2\alpha t_i} \left( \frac{1}{k} \int_{t_{i-1}}^{t_i} \left( s - t_{i-1} \right) \| A_{ch}^{-\frac{1}{2}} u_{chss} \| \, ds \right) \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|
\leq \frac{1}{\nu k} \sum_{i=1}^n \left( \int_{t_{i-1}}^{t_i} \left( s - t_{i-1} \right) \, ds \right) \left( \int_{t_{i-1}}^{t_i} e^{2\alpha t_i} \left( s - t_{i-1} \right) \| A_{ch}^{-\frac{1}{2}} u_{chss} \|^2 \, ds \right) + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2
\leq Ck \int_0^{t_i} \sigma(s) \| A_{ch}^{-\frac{1}{2}} u_{chss} \|^2 \, ds + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2.
\]

From Lemma 3.1 and 3.2 we find that

\[
k \sum_{i=1}^n e^{2\alpha t_i} A_h^i \left( \epsilon_t^i \right) \leq Ck \sum_{i=1}^n e^{2\alpha t_i} \| \nabla u_t^i \|^2 \| \epsilon_t^i \|^2 + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2
\leq Ck \sum_{i=1}^n e^{2\alpha t_i} \| \epsilon_t^i \|^2 + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2
\leq Cke^{2\alpha t_n} \| \epsilon_t^i \|^2 + Ck \sum_{i=1}^{n-1} e^{2\alpha t_i} \| \epsilon_t^i \|^2 + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2.
\]

Inserting \((5.26)-(5.27)\) in \((5.26)\) with the boundedness of \(\| \epsilon_t^i \|^2 \leq \| u_t^i \|^2 + \| U_t^i \|^2 \leq C\) and Lemma 3.3 we conclude that

\[
e^{2\alpha t_n} \| \epsilon_t^i \|^2 + \left( \nu - c_0^2 \left( \frac{e^{2\alpha k}}{k\lambda_1} - 1 \right) \right) k \sum_{i=1}^n e^{2\alpha t_i} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2 \leq Cke^{2\alpha t_n} + Ck \sum_{i=1}^{n-1} e^{2\alpha t_i} \| \epsilon_t^i \|^2.
\]

With \(0 \leq \alpha < \min\{\alpha_0, \frac{\nu A}{2\alpha} \}\), we have \(\nu - c_0^2 \left( \frac{e^{2\alpha k}}{k\lambda_1} - 1 \right) > 0\). Finally, a use of the discrete Gronwall’s lemma [19,20] completes the rest of proof for the case \(j = 0\). For the case \(j = -1\), take \(n = i\) and \(\Phi_h = A_{ch}^{\frac{1}{2}} \epsilon_t^i \) in \((5.21)\) and multiply the resulting equation by \(ke^{2\alpha t_i}\) and take sum from \(i = 1\) to \(n\) and use the similar fact \((5.21)\) to obtain

\[
(5.28) \quad e^{2\alpha t_n} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2 + \left( 2 \nu - c_0^2 \left( \frac{e^{2\alpha k}}{k\lambda_1} - 1 \right) \right) k \sum_{i=1}^n e^{2\alpha t_i} \| \epsilon_t^i \|^2 \leq 2k \sum_{i=1}^n e^{2\alpha t_i} \left( R_h^i \left( A_{ch}^{\frac{1}{2}} \epsilon_t^i \right) + A_h^i \left( A_{ch}^{-\frac{1}{2}} \epsilon_t^i \right) \right).
\]

A use of the Cauchy-Schwarz’s inequality with \((5.21)\) yields

\[
k \sum_{i=1}^n e^{2\alpha t_i} R_h^i \left( A_{ch}^{\frac{1}{2}} \epsilon_t^i \right) \leq k \sum_{i=1}^n e^{2\alpha t_i} \left( \frac{1}{k} \int_{t_{i-1}}^{t_i} \left( s - t_{i-1} \right) \| A_{ch}^{-\frac{1}{2}} u_{chss} \| \, ds \right) \| \epsilon_t^i \|
\leq Ck \sum_{i=1}^n \left( \int_{t_{i-1}}^{t_i} \| A_{ch}^{-\frac{1}{2}} u_{chss} \|^2 \, ds \right) + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| \epsilon_t^i \|^2
\leq Ck \int_0^{t_i} e^{2\alpha s} \| A_{ch}^{-\frac{1}{2}} u_{chss} \|^2 \, ds + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| \epsilon_t^i \|^2.
\]

From Lemma 3.1 and 3.2 it follows that

\[
k \sum_{i=1}^n e^{2\alpha t_i} A_h^i \left( A_{ch}^{-\frac{1}{2}} \epsilon_t^i \right) \leq Ck \sum_{i=1}^n e^{2\alpha t_i} \left( \| \nabla u_t^i \|^2 + \| \nabla U_t^i \|^2 \right) \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2 + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| \epsilon_t^i \|^2
\leq Ck \sum_{i=1}^n e^{2\alpha t_i} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2 + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| \epsilon_t^i \|^2
\leq Cke^{2\alpha t_n} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2 + Ck \sum_{i=1}^{n-1} e^{2\alpha t_i} \| A_{ch}^{\frac{1}{2}} \epsilon_t^i \|^2 + \frac{\nu}{4} k \sum_{i=1}^n e^{2\alpha t_i} \| \epsilon_t^i \|^2.
\]
Proof. Take $\sigma$ where (5.34) and take sum from $i = 1$ to $n$ to arrive at

$$k \sum_{i=1}^{n} \sigma_i \partial_t \|e_i^*\|^2 \geq \sigma_n \|e_n^*\|^2 - k \sum_{i=1}^{n-1} e^{2\alpha t_i} \|e_i^*\|^2,$$

so we only give a sketch of the proof.

Let the assumption of Lemma 5.2 be satisfied. Then, for some positive constant $C,$ that depends on $T,$ there holds

$$\tau_n \|e_n^*\|^2 + k e^{-2\alpha t_n} \sum_{i=1}^{n} \sigma_i \|A_{ch}^{-1} e_i^*\|^2 \leq K_n k^2,$$

where $\sigma_i = \tau_i e^{2\alpha t_i}$ and $\tau_i = \tau(t_i) = \min\{1, t_i\}.$

Proof. Take $n = i$ and $\phi_h = \sigma_i e_i^*$ in (5.20) to arrive at (5.31)

$$\sigma_i \partial_t \|e_i^*\|^2 + 2\nu \sigma_i \|A_{ch}^{-1} e_i^*\|^2 \leq 2 R_i^h(\sigma_i e_i^*) + 2 \Lambda_i^h(\sigma_i e_i^*).$$

Now multiply (5.31) by $k$ and take sum from $i = 1$ to $n$ and use the fact

$$k \sum_{i=1}^{n} \sigma_i \partial_t \|e_i^*\|^2 \geq \sigma_n \|e_n^*\|^2 - k \sum_{i=1}^{n-1} e^{2\alpha t_i} \|e_i^*\|^2,$$

to obtain

$$\sigma_n \|e_n^*\|^2 + 2\nu k \sum_{i=1}^{n} \sigma_i \|A_{ch}^{-1} e_i^*\|^2 \leq k \sum_{i=1}^{n} e^{2\alpha t_i} \|e_i^*\|^2 + 2k \sum_{i=1}^{n} \sigma_i (R_i^h(e_i^*) + \Lambda_i^h(e_i^*)).$$

A use of the Cauchy-Schwarz inequality with the Young’s inequality, (5.21) and Lemma 2.4 yields

$$k \sum_{i=1}^{n} \sigma_i R_i^h(e_i^*) \leq \frac{1}{k} \sum_{i=1}^{n} \sigma_i \left( \int_{t_{i-1}}^{t_i} (s - t_{i-1}) \|A_{ch}^{-1} u_{ch,h}^s\| ds \right) \|A_{ch}^{-1} e_i^*\|$$

$$\leq C k \sum_{i=1}^{n} \left( \int_{t_{i-1}}^{t_i} ds \right) \left( \int_{t_{i-1}}^{t_i} \sigma_i(s) \|A_{ch}^{-1} u_{ch,h}^s\|^2 ds \right) + \frac{\nu}{4} k \sum_{i=1}^{n} \sigma_i \|A_{ch}^{-1} e_i^*\|^2$$

$$\leq C k \int_0^{t_n} \sigma(s) \|A_{ch}^{-1} u_{ch,h}^s\|^2 ds + \frac{\nu}{4} k \sum_{i=1}^{n} \sigma_i \|A_{ch}^{-1} e_i^*\|^2.$$

From the Lemmas 5.2 and 5.3, we deduce that

$$k \sum_{i=1}^{n} \sigma_i \Lambda_i^h(e_i^*) \leq \frac{1}{k} \sum_{i=1}^{n} \sigma_i (\|\Delta_h u_{i,h}^s\|^2 + \|\Delta_h U_i^s\|^2) \|e_i^*\|^2 + \frac{\nu}{4} k \sum_{i=1}^{n} \sigma_i \|\nabla e_i^*\|^2$$

$$\leq C k \sum_{i=1}^{n} e^{2\alpha t_i} \|e_i^*\|^2 + \frac{\nu}{4} k \sum_{i=1}^{n} \sigma_i \|A_{ch}^{-1} e_i^*\|^2.$$

Using (5.33)–(5.34) in (5.32), we conclude the rest of the result.

Lemma 5.4. Let the assumption of Lemma 5.2 be satisfied. Then, for some positive constant $C,$ that depends on $T,$ there holds

$$\tau_n^2 \|A_{ch}^{-1} e_t^*\|^2 + k e^{-2\alpha t_n} \sum_{i=1}^{n} \sigma_i^2 \|A_{ch} e_i^*\|^2 \leq K_n k^2,$$

where $\sigma_i^2 = \tau_i^2 e^{2\alpha t_i}.$

Proof. The proof is quite similar to the proof of the previous lemma. So we only give a sketch of the proof. Choose $\phi_h = \sigma_i^2 A_{ch} e_i^*$ with $n = i$ in (5.20) to arrive at

$$\sigma_i^2 \partial_t \|A_{ch}^{-1} e_i^*\|^2 + 2\nu \sigma_i^2 \|A_{ch} e_i^*\|^2 \leq 2 R_i^h(\sigma_i^2 A_{ch} e_i^*) + 2 \Lambda_i^h(\sigma_i^2 A_{ch} e_i^*).$$
Multiplying by \( k \) and summing from \( i = 1 \) to \( n \), we obtain

\[
\sigma_n^2 \| A_n^e e_n^1 \|^2 + 2nk \sum_{i=1}^{n} \sigma_i^2 \| A_{ch} e_i^1 \|^2 \leq k \sum_{i=1}^{n-1} \sigma_i \| A_n^e e_i^1 \|^2 + 2k \sum_{i=1}^{n} \sigma_i^2 (R_h^e(A_{ch} e_i^1) + \Lambda_h(A_{ch} e_i^1)).
\]

We apply the Cauchy-Schwarz inequality and the Young’s inequality with (5.21) and Lemma 2.4 to bound

\[
k \sum_{i=1}^{n} \sigma_i^2 R_h^e(A_{ch} e_i^1) \leq k \sum_{i=1}^{n} \sigma_i^2 \left( \frac{1}{k} \int_{t_{i-1}}^{t_i} (s-t_{i-1}) \| u_{chss} \| \, ds \right) \| A_{ch} e_i^1 \| \\
\leq Ck \sum_{i=1}^{n} \left( \int_{t_{i-1}}^{t_i} ds \right) \left( \int_{t_{i-1}}^{t_i} \sigma_i^2(s) \| u_{chss} \|^2 ds \right) + \frac{\nu}{4} k \sum_{i=1}^{n} \sigma_i^2 \| A_{ch} e_i^1 \|^2 \\
\leq Ck^2 \int_0^{t_n} \sigma^2(s) \| u_{chss} \|^2 ds + \frac{\nu}{4} k \sum_{i=1}^{n} \sigma_i^2 \| A_{ch} e_i^1 \|^2.
\]

(5.36)

With the help of the Lemmas 3.2, 3.1 and 5.2 we can bound the nonlinear terms as

\[
k \sum_{i=1}^{n} \sigma_i^2 \Delta_h(A_{ch} e_i^1) \leq k \sum_{i=1}^{n} \tau_i \left( \| \Delta_h u_{ch} \|^2 + \| \Delta_h U^e_1 \|^2 \right) \sigma_i \| A_n^e e_i^1 \|^2 + \nu \frac{k}{4} \sum_{i=1}^{n} \sigma_i^2 \| A_{ch} e_i^1 \|^2
\]

(5.37)

Using (5.36)-(5.37) in (5.35) with the Lemmas 5.2 and 5.4 we conclude the rest of the proof.

We now also derive the error estimate for the pressure term. In fact, similar to the semidiscrete pressure error estimate, we can easily prove that \( \tau_n \| \partial_t e_n^p \| \leq K_n k \). Now using this and the available estimates for \( e_n^p \), we can easily prove the following lemma:

**Lemma 5.5.** Let the assumption of Lemma 5.2 be satisfied. Then, for some positive constant \( C \), that depends on \( T \), there holds

\[
\tau_n \| P_n^e - p_{ch}(t_n) \| \leq K_n k.
\]

**Proof.** From (3.1) and (5.1), we can write the pressure error equation as

\[
(P_n^e - p_{ch}, \nabla \cdot \phi_h) = (\partial_t e_n^p, \phi_h) + \nu a(e_n^p, \phi_h) - R_h^e(\phi_h) - \Lambda_h(\phi_h),
\]

where \( R_h^e \) and \( \Lambda_h \) are defined by (5.21) and (5.22), respectively. A use of Lemma 3.2 gives

\[
(P_n^e - p_{ch}, \nabla \cdot \phi_h) = \left( \| \partial_t e_n^p \|_{-1,h} + \nu \| \nabla e_n^p \| + C(\| \nabla u_{ch} \| + \| \nabla U^e_1 \|) \| \nabla e_n^p \| \right) \\
+ \frac{1}{k} \int_{t_{n-1}}^{t_n} (t-t_{n-1}) \| u_{chss} \|_{-1} \, ds \| \nabla \phi_h \|,
\]

where \( \| \cdot \|_{-1,h} \) is defined in (4.29) and clearly \( \| \cdot \|_{-1,h} \leq \| \cdot \|_{-1} \leq C \| \cdot \| \). Finally, we use the Lemmas 3.3, 5.1 and 5.4 to complete the rest of the proof.

**Lemma 5.6.** Let the assumption of Lemma 5.2 be satisfied. Then,

\[
\sqrt{\tau_n} \| e_n^p \| + \tau_n \| \nabla e_n^p \| + \tau_n \| P_n^e - p_{ch}(t_n) \| \leq C k,
\]

where \( C \) depends exponentially on time. Under the uniqueness condition (5.7), the above estimate (5.38) holds uniformly in time.

**Proof.** A use of Lemma 5.3, 5.4 and 5.5 with triangular inequality shows (5.38). Since the constant \( C \) depends exponentially in time due to the use of Lemma 5.2 which is not uniform in time. For improving the estimates of Lemma 5.2 using (5.25), we multiply (5.24) by \( ke^{2\alpha t} \) and take sum from \( i = i_0 + 1 \) to \( n \) and use (5.24) to obtain

\[
e^{2\alpha t} \| e_n^p \|^2 + \left( 2\nu - \frac{\sigma_n^2}{\lambda_1} - \frac{e^{2\alpha t} - 1}{k} \right) k \sum_{i=i_0+1}^{n} e^{2\alpha t i} \| A_n^e e_i^1 \|^2 \\
\leq e^{2\alpha t_0} \| e_n^p \|^2 + 2k \sum_{i=i_0+1}^{n} e^{2\alpha t i} (R_h^e(e_i^p) + \Lambda_h(e_i^p)).
\]

(5.39)
From (5.40), it follows that

\[ 2k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} R_{i}^{(0)}(e_i^0) \leq Ck e^{2\alpha t_n} + \frac{\nu}{2} k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} \| A_{ch}^1 e_i^1 \|^2. \]

Now, we bound the nonlinear terms using (2.7) as

\[ |A_{ch}^1(e_i^0)| \leq N \| \nabla u_i \| \| \nabla e_i \|^2. \]

By (5.39), we can easily derive that \( \lim_{t \to \infty} \| \nabla u_i \| \leq \nu^{-1} \| f \|_{L^\infty(0,T;H^1)} \) implies

\[ 2k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} A_{ch}^1(e_i^0) \leq 2N \nu^{-1} \| f \|_{L^\infty(0,T;H^1)} k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} \| \nabla e_i \|^2. \]

Inserting (5.40)-(5.42) in (5.39), we conclude that

\[ e^{2\alpha t_n} \| e_n^0 \|^2 + \left( \nu - c_0^2 \frac{(2\alpha k - 1)}{k\lambda_1} \right) k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} \| A_{ch}^1 e_i^1 \|^2 + \frac{\nu}{2} \sum_{i=i_0+1}^{n} e^{2\alpha t_i} \| \nabla e_i \|^2 \leq C k e^{2\alpha t_n} + e^{2\alpha t_n} \| e_n^0 \|^2. \]

With \( 0 < \alpha < \min\{\alpha_0, \frac{\lambda_1}{2c_0} \} \), we have \( \nu - c_0^2 \frac{(2\alpha k - 1)}{k\lambda_1} > 0 \) and from (2.7), \( \nu - 2N \nu^{-1} \| f \|_{L^\infty(0,T;H^1)} \geq 0 \). Then, we obtain

\[ \| e_n^0 \|^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \| A_{ch}^1 e_i^1 \|^2 \leq C k. \]

Next, we choose \( \phi_i = A_{ch}^{-1} e_i^0 \) in (5.20) with \( n = i \) and multiply the resulting equation by \( k e^{2\alpha t_i} \), and take sum from \( i = i_0 + 1 \) to \( n \). Then, arguing similar set of analysis as above one can obtain

\[ \| A_{ch}^{-1} e_n^0 \|^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \| e_i^1 \|^2 \leq C k^2. \]

Finally, a use of (5.43) and (5.44) instead of Lemma 5.2 in the proof of Lemma 5.3, 5.4 and 5.5 complete the rest of proof.

Finally, combining Theorem 2.1, 3.1 and Lemma 5.6 we conclude the following theorem:

**Theorem 5.1.** Assume that (A1), (A2), (B1) and (B2) hold true. Then, for some positive constant \( C \), that depends on \( T \), there holds:

\[ \| u(t_n) - U_n^0 \| \leq K_n \left( \epsilon + k \right) t^{-\frac{1}{2}} + h^{m+1} t^{-\frac{3}{2}}, \]

\[ \| \nabla (u(t_n) - U_n^0) \| \leq K_n \left( \epsilon + k \right) t^{-1} + h^m t^{-\frac{3}{2}}, \]

\[ \| p(t_n) - P_n^0 \| \leq K_n \left( \epsilon + k \right) t^{-1} + h^m t^{-\frac{3}{2}}, \]

where the positive constant \( K_n = C e^{C t_n} \) depends exponentially on time. The estimates are uniform in time under the uniqueness condition (2.7), that is, the constant \( K_n \) becomes \( C \).

**Remark 5.2.** Although we have discussed conforming finite element spaces on this article, but all the results remain valid for \( (P_1^{NC} - P_0) \) nonconforming elements. Therefore, the present analysis improves upon the results of Lu and Lin [24] in the sense that optimal estimates in \( L^2 \)-norm are obtained when initial data are in \( H_0^1 \). To this effect, a numerical experiment is presented in Section 6.

## 6 Numerical Experiments

In this section, we present some numerical experiments that verify the results of previous section, mainly verify the order of convergence of the error estimates.

We consider the NSEs in the domain \( \Omega = [0, 1] \times [0, 1] \) subject to homogeneous Dirichlet boundary conditions. We approximate the equation using \( P_2-P_1, P_3-P_2 \) and \( P_4^{NC}-P_0 \) elements over a triangulation of \( \Omega \). The domain is partitioned into triangles with size \( h = 2^{-i}, i = 1, 2, \ldots, 6 \). To verify the theoretical result, we consider the following examples.
Example 6.1. For the experiment in 2D, we take the forcing term \( f(x, y, t) \) such that the solution of the problem to be
\[
\begin{align*}
  u_1(x, y, t) &= 2e^t x^2(x - 1)^2 y(y - 1)(2y - 1), \\
  u_2(x, y, t) &= -2e^t x(x - 1)(2x - 1)y^2(y - 1)^2, \\
  p(x, y, t) &= 2e^t x - y.
\end{align*}
\]

| h   | \(|u(t_0) - U^n|_L^2\) | Rate | \(|u(t_0) - U^n|_H^1\) | Rate | \(|p(t_0) - P^n|_L^2\) | Rate |
|-----|---------------------|------|---------------------|------|---------------------|------|
| 1/2 | 3.30633896e-03      |      | 2.96918336e-02      |      | 3.29192462e-02      |      |
| 1/4 | 5.11077157e-04      | 2.6936 | 8.36078857e-03      | 1.8284 | 6.96272136e-03      | 2.2412 |
| 1/8 | 5.2570055e-05       | 3.2826 | 1.99271639e-03      | 2.0689 | 9.29102388e-04      | 2.9057 |
| 1/16| 6.32085098e-06      | 3.0546 | 5.32350596e-04      | 1.9042 | 2.78763334e-04      | 1.7368 |
| 1/32| 7.91350016e-07      | 2.9977 | 1.35504728e-04      | 1.9740 | 7.93302459e-05      | 1.8131 |
| 1/64| 9.89942025e-08      | 2.9989 | 3.39036941e-05      | 1.9988 | 2.01629829e-05      | 1.9762 |

Table 1: Errors and convergence rates with \( \nu = 1, k = \varepsilon = \mathcal{O}(h^3) \) at time \( t = 1 \) using \( P_2 \) - \( P_1 \) element

| h   | \(|u(t_0) - U^n|_L^2\) | Rate | \(|u(t_0) - U^n|_H^1\) | Rate | \(|p(t_0) - P^n|_L^2\) | Rate |
|-----|---------------------|------|---------------------|------|---------------------|------|
| 1/2 | 8.72519439e-04      |      | 9.64515596e-03      |      | 2.42266708e-02      |      |
| 1/4 | 7.86457181e-05      | 3.4717 | 2.53372020e-03      | 1.9285 | 3.77601493e-03      | 2.6816 |
| 1/8 | 5.40305188e-06      | 3.8635 | 3.35107253e-04      | 2.9156 | 4.04997117e-04      | 3.2209 |
| 1/16| 3.65282701e-07      | 3.8866 | 4.43890795e-05      | 2.9476 | 3.73502090e-05      | 4.3427 |
| 1/32| 2.40020778e-08      | 3.9278 | 5.55651150e-06      | 2.9667 | 3.34112900e-06      | 4.8427 |

Table 2: Errors and convergence rates with \( \nu = 1, k = \varepsilon = \mathcal{O}(h^4) \) at time \( t = 1 \) using \( P_3 \) - \( P_2 \) element

In Tables 1 and 2 we present the numerical errors and convergence rates obtained on successive meshes with the backward Euler scheme, applied to the system (1.1)-(1.3) using \( P_m - P_{m-1} \) elements for \( m = 2, 3 \), respectively. The numerical analysis shows that the rate of convergence are \( \mathcal{O}(h^{m+1}) \) in \( L^2 \)-norm and \( \mathcal{O}(h^m) \) in \( H^1 \)-norm for the velocity and \( \mathcal{O}(h^m) \) in \( L^2 \)-norm for the pressure with the choice of \( k = \varepsilon = \mathcal{O}(h^{m+1}) \) at the final time level, that is, when \( t = 1 \) and \( \nu = 1 \). These results support the optimal convergence rates obtained in Theorem 5.1. In Table 3, we present the numerical results for \( P_1^{NC} - P_0 \) element. It is observed in Table 3 that the rate of convergence in \( L^2 \)-norm and \( H^1 \)-norm of the velocity is 2 and 1, respectively. Moreover it is linear in pressure in \( L^2 \)-norm.

Example 6.2. For the experiment in 3D, we take the forcing term \( f(x, y, z, t) \) such that the solution of the problem to be
\[
\begin{align*}
  u_1(x, y, z, t) &= \pi e^t \sin(\pi x)\cos(\pi y)\sin(\pi z) - \cos(\pi z)\sin(\pi y), \\
  u_2(x, y, z, t) &= \pi e^t \sin(\pi y)\cos(\pi z)\sin(\pi x) - \cos(\pi x)\sin(\pi z), \\
  u_3(x, y, z, t) &= \pi e^t \sin(\pi z)\cos(\pi y)\sin(\pi x) - \cos(\pi y)\sin(\pi x), \\
  p(x, y, z, t) &= e^t (\sin(\pi x)\sin(\pi y)\sin(\pi z) - 8.0/\pi^3)).
\end{align*}
\]

We also present the numerical errors and convergence rates for Example 6.2 using \( P_m - P_{m-1} \) elements for \( m = 2, 3 \), in Tables 4 and 5 respectively. From the Tables 4 and 5 we observe that the rate of convergence are \( \mathcal{O}(h^{m+1}) \) in \( L^2 \)-norm and \( \mathcal{O}(h^m) \) in \( H^1 \)-norm for the velocity and \( \mathcal{O}(h^m) \) in \( L^2 \)-norm for the pressure with the choice of \( k = \varepsilon = \mathcal{O}(h^{m+1}) \) at the final time level, that is, when \( t = 0.1 \) and \( \nu = 1 \). These results support the optimal convergence rates obtained in Theorem 5.1.

The next two examples are related to two-dimensional Benchmark problems.
Table 4: Errors and convergence rates with $\nu = 1, k = \varepsilon = O(h^3)$ at time $t = 0.1$ using $P_2 - P_1$ element

| $h$  | $\|u(t_n) - U^n\|_{L^2}$ Rate | $\|u(t_n) - U^n\|_{H^1}$ Rate | $\|p(t_n) - P^n\|_{L^2}$ Rate |
|------|-------------------------------|-------------------------------|-------------------------------|
| 1/2  | 4.39800806e-02  | 4.97101276e-01  | 3.84531508e-01  |
| 1/4  | 4.80285161e-03  | 3.1949  | 9.96770050e-02  | 2.3182  |
| 1/8  | 4.80285161e-03  | 3.1949  | 9.96770050e-02  | 2.3182  |
| 1/16 | 5.02080312e-05  | 3.0254  | 4.61136630e-02  | 2.5844  |

Table 5: Errors and convergence rates with $\nu = 1, k = \varepsilon = O(h^4)$ at time $t = 0.1$ using $P_3 - P_2$ element

| $h$  | $\|u(t_n) - U^n\|_{L^2}$ Rate | $\|u(t_n) - U^n\|_{H^1}$ Rate | $\|p(t_n) - P^n\|_{L^2}$ Rate |
|------|-------------------------------|-------------------------------|-------------------------------|
| 1/2  | 1.45489050e-02  | 3.13468857e-01  | 1.52818314e-01  |
| 1/4  | 8.52910943e-04  | 4.0923  | 3.87617913e-02  | 3.0156  |
| 1/8  | 5.90735824e-05  | 3.8518  | 5.43288281e-03  | 2.8348  |
| 1/16 | 3.78214933e-06  | 3.9652  | 7.03179542e-04  | 2.9497  |

Example 6.3. In this example, we consider a benchmark problem related to a two-dimensional lid driven cavity flow on a unit square with zero body force. Also, no slip boundary condition are considered everywhere except the non zero velocity $u = (1, 0)^T$ on upper boundary, see Figure 1.

For numerical experiments, we have chosen lines $(0.5, y)$ and $(x, 0.5)$ and we plot the velocity profile with respect to these two lines. In Figure 2, we present the comparison between velocity obtained by penalty method and velocity obtained by Ghia et. al. [10] of NSEs for final time $t = 75$, for $\nu = 10^{-2}, 10^{-3}$ and $t = 150$, for $\nu = 10^{-4}$, respectively, with the choice of time step $k = 0.01$. From the graphs, it is observed that the velocity profiles coincide with those of Ghia’s results very well for a large time and that for $\nu$ small.

Figure 1: Domain $\Omega$ for lid-driven cavity flow.

Figure 2: Velocity components for Example 6.3.
Example 6.4. Now, we also consider a well-known benchmark problem related to the two-dimensional flow around a cylinder with zero body forces [24]. The domain $\Omega$ is the channel of size $[0, 2.2] \times [0, 0.41]$ with a circle of diameter 0.1 located at $(0.2, 0.2)$ as shown in Figure 3. The whole boundary is divided in four parts; inflow boundary $\Gamma_{in} := \{x = 0\}$, outflow boundary $\Gamma_{out} := \{x = 2.2\}$, the remaining two wall $\Gamma_{wall} := \{y = 0, y = 0.41\}$ and the boundary of the circle $\Gamma_{cyl} := \{(x-0.2)^2 + (y-0.2)^2 = 0.0025\}$. We consider the no-slip boundary at $\Gamma_{wall}$ and $\Gamma_{cyl}$ and the inflow and outflow velocity is given by $u(0, y) = u(2.2, y) = (\frac{6}{0.14\pi} \sin(\frac{\pi y}{2}))(0.41 - y), 0 \leq y \leq 0.41$.

\[
\begin{array}{ccc}
(0, 0.41) & \Gamma_{wall} & (2.2, 0.41) \\
\downarrow & \Gamma_{cyl} & \\
(0, 0) & \Gamma_{wall} & (2.2, 0)
\end{array}
\]

Figure 3: Domain $\Omega$ for flow past cylinder.

First, we approximate the velocity and pressure by Taylor-Hood element $P_2 - P_1$, and the domain is discretized with mesh size $h = 1/64$. For this test, we choose $\nu = 10^{-3}$, $k = 10^{-3}$ and the time interval $[0, 8]$. It is known that a vortex sheet develops at the cylinder’s bottom around $t = 4$. In fact, in Figures 4 and 5, we observe this phenomenon, where the velocity field and stream function have been described for different $t = 5$ and $t = 6$, and the vortices are still visible at time $T = 8$.

We also plot the evolution of the drag coefficient ($c_d(t)$) at the cylinder, lift coefficient ($c_l(t)$) at the cylinder, and differences in the pressure ($\Delta p(t)$) between the front and the back of the cylinder in Figure 6 for $P_2 - P_1$ and $P_3 - P_2$ elements. In addition, we mark the maximum value of the drag coefficient, the lift coefficient, and the final value of the pressure difference. We calculate all these parameters using the formula given in [21].

In all cases, computation were done in FreeFem++ [12].

7 Conclusion

This paper deals with a penalty finite element method along with the backward Euler method for the incompressible NSEs. With the appropriate use of the inverse of the penalized Stokes operator and the negative norm estimates, optimal error estimates for the velocity and the pressure terms in both the semi-discrete and fully-discrete schemes are derived. Analysis has been carried out for non-smooth initial data, that is, the initial velocity $u_0 \in H^1_0$. This demands time weighted estimates; the proofs are now more technical and more involved than those for the smooth case. These optimal estimates are derived under the assumption that the penalized parameter $\varepsilon$ is small. In the numerical part also, optimal convergence rates have been shown for small $\varepsilon$. Moreover, results of computational experiments on two benchmark problems show that the proposed method works well for low viscosity and their results compare well with exiting results from the literature. Although, computational results on one 3D example are encouraging, but this is with out theoretical justification and this will form a part of our future endeavour.

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Figure 4: Velocity field for Example 6.4 for $T = 4, 5, 6, 7, 8$.

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Figure 5: Stream function for Example 6.4 for $T = 4, 5, 6, 7, 8$.

Figure 6: Drag coefficient, lift coefficient, and pressure difference for Example 6.4.

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Lemma 2.3 \[ \| \alpha \| \leq \frac{\| \nu \lambda \|}{\lambda} \]  

Proof.

Proof of the Lemma 2.5:

Note that the non-linear term vanishes due to (2.2). Now multiply by \( \varepsilon \) obtain

\[ \int_0^T \| \alpha \| \nu \lambda \| \| \| \leq \frac{\| \|}{\| \|} \]

We integrate (7.1) with respect to time from \( t \) to \( T \) for any \( T > 0 \), we have

\[ \| \alpha \| \leq \frac{\| \nu \lambda \|}{\lambda} \]

For the second estimate, choose \( \phi = A_{\mu}^{m+1} \) in (2.3). When \( m = 0 \), we find that

\[ \frac{1}{2} \int_0^T \| A_{\mu}^{m+1} \| \| \| + \| A_{\mu}^{m+1} \| \| \| = 1 \]
A use of Ladyzhenskaya’s inequality \( \| \phi \|_{L^4} \leq C \| \phi \|^2 \| \nabla \phi \|^{\frac{1}{2}} \), and \( \| \nabla \phi \|_{L^4} \leq C \| \nabla \phi \|^2 \| \Delta \phi \|^{\frac{1}{2}} \) with Lemma 2.3, the Young’s inequalities, we bound the nonlinear term as
\[
\hat{b}(u_t, u, A_t u) \leq \| u \|_{L^4} \| \nabla u \|_{L^4} \| A_t u \|
\leq C \| u \|^2 \| A_t^\frac{1}{2} u \| \| A_t u \|^{3/2}
\leq C \| u \|^2 \| A_t^\frac{1}{2} u \|^4 + \frac{\nu}{4} \| A_t u \|^2.
\]
(7.5)

Substitute the above estimate in (7.4) to find that
\[
dt \left( \| A_t^\frac{1}{2} u(t) \|^2 \right) + \nu \| A_t u(t) \|^2 \leq C \left( \| u \|^2 \| A_t^\frac{1}{2} u \|^2 \right) \| A_t^\frac{1}{2} u \|^2 + \frac{2}{\nu} \| f \|^2.
\]
(7.6)

We now apply uniform Gronwall’s Lemma (Lemma 2.1) in (7.6) and use (7.2) and (7.3) to conclude that \( \| A_t^\frac{1}{2} u(t) \|^2 \) is uniformly bounded with respect to \( t \) on [0, T]. Precisely
\[
\| A_t^\frac{1}{2} u(t) \|^2 \leq C, \quad \forall t \geq T.
\]
(7.7)

For \( 0 \leq t \leq T \), we use the classical Gronwall’s lemma \([19, 26]\) in (7.6) and obtain
\[
\| A_t^\frac{1}{2} u(t) \|^2 \leq C, \quad \forall 0 \leq t \leq T.
\]
(7.8)

Finally, multiply (7.6) by \( e^{2\alpha t} \) and integrate with respect to time from 0 to \( t \) and use the estimates (7.2), (7.7) and (7.8) to complete the second proof when \( r = 0 \). For \( r = 1 \), we need some intermediate estimate. First we take \( \phi = e^{2\alpha t} u_{2t} \) with \( u_{2t} = e^{\alpha t} u \), in (2.3) to obtain
\[
\frac{\nu}{2} d_{dt} \| A_t^\frac{1}{2} u_{2t} \|^2 + \| u_{2t} \|^2 = \nu \| A_t^\frac{1}{2} u \|^2 + \left( \bar{f}, u_{2t} \right) - e^{2\alpha t} \hat{b}(u_t, u, A_t u).
\]
(7.9)

We can estimate the nonlinear term on the right hand side of (7.9) similar to (7.5) and integrate both sides with respect to time to find that
\[
\nu \| A_t^\frac{1}{2} u_{2t} \|^2 + \int_0^t \| u_{2t} \|^2 \, ds \leq C \left[ \int_0^t \left( \| A_t^\frac{1}{2} u_{2t} \|^2 + \| \bar{f} \|^2 + \| A_t^\frac{1}{2} u_{2t} \|^2 \| A_t u \|^2 \right) \, ds \right].
\]

Now a use of (7.2) and (7.3) lead us to the intermediate estimate.
\[
\| A_t^\frac{1}{2} u(t) \|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \| u_{2s}(s) \|^2 \, ds \leq C.
\]
(7.10)

We now differentiate (2.3) with respect to time and deduce that
\[
(u_{2t}, \phi) + \nu u_{2t} \phi = (f, \phi) - \hat{b}(u_t, u, \phi) - \hat{b}(u, u, \phi), \quad \forall \phi \in H_0^1.
\]
(7.11)

Take \( \phi = \sigma(t) u_{2t} \) in (7.11) and use Lemma 3.1, the Cauchy-Schwarz inequality to reach at
\[
\frac{d}{dt} (\sigma(t) \| u_{2t} \|^2) + \nu \sigma(t) \| A_t^\frac{1}{2} u_{2t} \|^2 \leq C e^{2\alpha t} \| u_{2t} \|^2 + C \sigma(t) \left( \| f \|^2 + \| u_{2t} \|^2 \| A_t^\frac{1}{2} u \|^2 \right).
\]
Integrate with respect to time and use (7.10), (7.7) and (7.8) to obtain
\[
\tau(t) \| u_{2t}(t) \|^2 + \nu e^{-2\alpha t} \int_0^t \sigma(s) \| A_t^\frac{1}{2} u_{2s}(s) \|^2 \, ds \leq C.
\]
(7.12)

Now we are in position to complete the proof of the second estimate when \( m = 1 \). For this, we set \( \phi = A_t u \) in (2.3) and rewrite it and use (7.5) and the Cauchy-Schwarz inequality to arrive at
\[
\nu \| A_t u \|^2 = (f, A_t u) - (A_{tt}, A_t u) - \hat{b}(u_t, u, A_t u) \leq C \left( \| f \|^2 + \| u_{tt} \|^2 + \| u_t \|^2 \| A_t^\frac{1}{2} u \|^4 \right) + \frac{\nu}{2} \| A_t u \|^2.
\]
Multiply by \( \tau(t) \) and use (7.2), (7.7), (7.8) and (7.12) to complete the second proof.

Proof of the well-posedness of the discrete solution of problem (5.2):
Proof. We can rewrite (5.2) as

\[\begin{align*}
(U^n, \phi_h) + \nu k a_c(U^n, \phi_h) + k \tilde{b}(U^n, U^n, \phi_h) &= k(f^n, \phi_h) + (U^{n-1}, \phi_h), \quad \forall \phi_h \in H_h.
\end{align*}\]

Consider a function \(F : H_h \to H_h\) such that

\[\begin{align*}
(F(\mathbf{v}), \phi_h) = (\mathbf{v}, \phi_h) + \nu k a_c(\mathbf{v}, \phi_h) + k \tilde{b}(\mathbf{v}, \mathbf{v}, \phi_h) - k(f^n, \phi_h) - (U^n, \phi_h), \quad \forall \phi_h \in H_h.
\end{align*}\]

Clearly, \(F\) is continuous. Then, a use of pointcaré inequality and inverse hypothesis yields

\[\begin{align*}
(F(U^n), U^n) &= (U^n, U^n) + \nu k a_c(U^n, U^n) + k \tilde{b}(U^n, U^n) - k(f^n, U^n) - (U^{n-1}, U^n) \\
&\geq \|U^n\|^2 + \nu k \|A^{1/2}_h U^n\|^2 - k \|f^n\| \|U^n\| - \|U^{n-1}\| \|U^n\| \\
&\geq \left(1 + \frac{vk\lambda_1}{c^2}\right) \|U^n\| - \left(k \|f^n\| + \|U^{n-1}\|\right) \|U^n\|.
\end{align*}\]

Now, choose \(\mathbf{U}^n \in H_h\) such that

\[U^n = \frac{2(k \|f^n\| + \|U^{n-1}\|)}{\left(1 + \frac{vk\lambda_1}{c^2}\right)} = \alpha_1.
\]

If either \(\|f^n\| \neq 0\) or \(\|U^{n-1}\| \neq 0\), then \(\alpha_1 > 0\), which implies that there exists \(\mathbf{U}^n \in H_h\) such that \(\|\mathbf{U}^n\| \leq \alpha_1\) and \(F(\mathbf{U}^n) = 0\). \(\square\)