Central extensions and conformal derivations of a class of Lie conformal algebras

Yanyong Hong

Department of Mathematics, Hangzhou Normal University, Hangzhou, People’s Republic of China

ABSTRACT
A quadratic Lie conformal algebra corresponds to a Hamiltonian pair in Gel’fend and Dorfman (Hamiltonian operators and algebraic structures related to them. Funkts Anal Prilozhen. 1979;13:13–30), which plays fundamental roles in completely integrable systems. Moreover, it also corresponds to a certain compatible pair of a Lie algebra and a Novikov algebra which was called Gel’fand–Dorfman bialgebra by Xu (Quadratic conformal superalgebras. J Algebra. 2000;231:1–38). In this paper, central extensions and conformal derivations of quadratic Lie conformal algebras are studied in terms of Gel’fand–Dorfman bialgebra. It is shown that central extensions and conformal derivations of a quadratic Lie conformal algebra are related with some bilinear forms and some operators of the corresponding Gel’fand–Dorfman bialgebra, respectively.

1. Introduction
The notion of vertex algebra was formulated by Borcherds [1], which is a rigorous mathematical definition of the chiral part of a 2-dimensional quantum field theory studied intensively by physicists since the landmark paper [2]. Lie conformal algebra introduced by Kac [3,4] is an axiomatic description of the operator product expansion (or rather its Fourier transform) of chiral fields in conformal field theory. In a way, its relationship with vertex algebra is like the relationship between Lie algebra and its universal enveloping algebra. It turns out to be a valuable tool in studying vertex algebras. In addition, Lie conformal algebras have close connections to Hamiltonian formalism in the theory of nonlinear evolution equations (see the book [5] and references therein, and also [6–9] and many other papers).

Moreover, in the purely algebraic viewpoint, conformal algebras are quite intriguing subjects. One can define the conformal analogue of a variety of ‘usual’ algebras such as Lie conformal algebras, associative conformal algebras, etc. The theory of some variety of conformal algebras sheds new light on the problem of classification of infinite-dimensional algebras of the corresponding ‘classical’ variety. In particular, Lie conformal algebras give
us powerful tools for the study of infinite-dimensional Lie algebras satisfying the locality property in [10].

The topic of this paper is about Lie conformal algebra. Simple (or semisimple) Lie conformal algebras have been intensively investigated. A complete classification of finite simple (or semisimple) Lie conformal algebras were given in [11], all finite irreducible conformal modules of finite simple (or semisimple) Lie conformal algebras were classified in [12], extensions of these conformal modules were studied in [13] and cohomology groups of finite simple Lie conformal algebras with some conformal modules were characterized in [14]. Recently, some nonsimple Lie conformal algebras were introduced and their structures including central extensions, conformal derivations were studied. For example, two new nonsimple Lie conformal algebras were studied in [15] which are obtained from two formal distribution Lie algebras, i.e. Schrödinger–Virasoro Lie algebra and the extended Schrödinger–Virasoro Lie algebra. Similarly, the Lie conformal algebras of the loop Virasoro Lie algebra and a Block type Lie algebra were studied in [16,17], respectively. In fact, all these Lie conformal algebras including semisimple Lie conformal algebras are quadratic Lie conformal algebras named by Xu [18]. A quadratic Lie conformal algebra corresponds to a Hamiltonian pair in [7], which plays fundamental roles in completely integrable systems. Moreover, it also corresponds to a certain compatible pair of a Lie algebra and a Novikov algebra which was called Gel’fand–Dorfman bialgebra by Xu [18].

As we know, the description of finite Lie conformal algebras splits into two problems (see [3]):

(1) describe Lie conformal algebras that are free $\mathbb{C}[\partial]$-modules of finite rank;
(2) find central extensions of Lie conformal algebras from (1) with centre being in torsion.

Therefore, it is meaningful to study central extensions of Lie conformal algebras in the structure theory of Lie conformal algebras. Since most of known Lie conformal algebras are quadratic Lie conformal algebras, in this paper, we plan to study conformal derivations and central extensions of quadratic Lie conformal algebras by a one-dimensional centre $\mathbb{C}c\beta$ where $\partial c\beta = \beta c\beta$ in a unified form. It is shown that central extensions of some quadratic Lie conformal algebras including those corresponding to simple Novikov algebras or Novikov algebras with a left unit or a right unit by a one-dimensional centre $\mathbb{C}c\beta$ can be directly obtained from some bilinear forms of the corresponding Gel’fand–Dorfman bialgebras satisfying several equalities. Note that this method can be also applied to study central extensions of quadratic Lie conformal algebras by an abelian Lie conformal algebra $\mathbb{C}[\partial]c$ which is free of rank one. In particular, we prove that $H^2(R,\mathbb{C}c\beta) = 0$ and $H^2(R,\mathbb{C}[\partial]c) = 0$ when $\beta \neq 0$ and $R$ is the quadratic Lie conformal algebra corresponding to a Novikov algebra with a right unit. In addition, we show that conformal derivations of some quadratic Lie conformal algebras including those corresponding to Novikov algebras with a left unit or with a right unit can be obtained from some operators of the corresponding Gel’fand–Dorfman bialgebras. Moreover, it is proved that all conformal derivations of the quadratic Lie conformal algebra corresponding to a Novikov algebra with a unit are inner.

This paper is organized as follows. In Section 2, the definitions of Lie conformal algebra and quadratic Lie conformal algebra are recalled. In Section 3, central extensions of quadratic Lie conformal algebras by a one-dimensional centre $\mathbb{C}c\beta$ where $\partial c\beta = \beta c\beta$ are
studied in terms of Gel’fand–Dorfman bialgebras. It is shown that $H^2(R, \mathbb{C}c_\beta) = 0$ and $H^2(R, \mathbb{C}[\partial]c) = 0$ when $\beta \neq 0$ and $R$ is the quadratic Lie conformal algebra corresponding to a Novikov algebra with a right unit. As an application, we characterize central extensions of several specific quadratic Lie conformal algebras by $\mathbb{C}c_0$ up to an equivalence. In Section 4, we study conformal derivations of quadratic Lie conformal algebras in terms of Gel’fand–Dorfman bialgebras. Moreover, conformal derivations of several specific quadratic Lie conformal algebras are determined.

Throughout this paper, denote by $\mathbb{C}$ the field of complex numbers; $\mathbb{N}$ is the set of natural numbers, i.e. $\mathbb{N} = \{0, 1, 2, \ldots\}$; $\mathbb{Z}$ is the set of integer numbers; denote by $C^i_j$ the corresponding binomial coefficient when $i, j \in \mathbb{N}$ and $j \geq i$. All tensors over $\mathbb{C}$ are denoted by $\otimes$. Moreover, if $A$ is a vector space, the space of polynomials of $\lambda$ with coefficients in $A$ is denoted by $A[\lambda]$.

2. Preliminaries

In this section, we will recall the definitions of Lie conformal algebra and quadratic Lie conformal algebra. These facts can be referred to [3, 18].

**Definition 2.1:** A Lie conformal algebra $R$ is a $\mathbb{C}[\partial]$-module with a $\lambda$-bracket $[\cdot, \cdot]$ which defines a $\mathbb{C}$-bilinear map from $R \times R \to R[\lambda]$, satisfying

\[
\begin{align*}
[\partial a, b] &= -\lambda [a, b], & [a, \partial b] &= (\lambda + \partial)[a, b], & \text{(conformal sesquilinearity),} \\
[a, b] &= -[b, -\lambda - \partial a], & \text{(skew-symmetry),} \\
[a, [b, c]] &= [[a, b], \lambda + c] + [b, [a, c]], & \text{(Jacobi identity)}
\end{align*}
\]

for $a, b, c \in R$.

A Lie conformal algebra is said to be *finite* if it is finitely generated as a $\mathbb{C}[\partial]$-module. Otherwise, it is called *infinite*.

In addition, there is an important Lie algebra associated with a Lie conformal algebra. Assume that $R$ is a Lie conformal algebra and set $[a, b] = \sum_{n \in \mathbb{N}} \binom{\lambda}{n} a^{(n)} b$ where $a^{(n)} b \in R$ for any $n \in \mathbb{N}$. Let $\text{Coeff}(R)$ be the quotient of the vector space with basis $a_n$ ($a \in R, n \in \mathbb{Z}$) by the subspace spanned over $\mathbb{C}$ by elements:

\[
(\alpha a)_n - \alpha a_n, (a + b)_n - a_n - b_n, (\partial a)_n + na_{n-1}, \quad \text{where } a, b \in R, \alpha \in \mathbb{C}, n \in \mathbb{Z}.
\]

The operation on $\text{Coeff}(R)$ is defined as follows:

\[
[a_m, b_n] = \sum_{j \in \mathbb{N}} \binom{m}{j} (a^{(j)} b)_{m+n-j}.
\]

Then $\text{Coeff}(R)$ is a Lie algebra and it is called the *coefficient algebra* of $R$ (see [3]).

Next, let us introduce some examples of Lie conformal algebras.

**Example 2.2:** The Virasoro Lie conformal algebra Vir is the simplest nontrivial example of Lie conformal algebras. It is defined by

\[
\text{Vir} = \mathbb{C}[\partial]L, \quad [L_\lambda L] = (\partial + 2\lambda)L.
\]

$\text{Coeff(Vir)}$ is just the Witt algebra.
Example 2.3: Let \( g \) be a Lie algebra. The current Lie conformal algebra associated to \( g \) is defined by

\[
\text{Cur}_g = \mathbb{C}[\partial] \otimes g, \quad [a, b] = [a, b], \quad a, b \in g.
\]

Moreover, we can define a semi-direct sum of Vir and Cur\(_g\). The \( \mathbb{C}[\partial] \)-module Vir \( \oplus \) Cur\(_g\) can be given a Lie conformal algebra structure by

\[
[L, L] = (\partial + 2 \lambda)L, \quad [a, b] = [a, b], \quad [L, a] = (\partial + \lambda)a,
\]

\( L \) being the standard generator of Vir, \( a, b \in g \).

Then we introduce a class of special Lie conformal algebras named quadratic Lie conformal algebras (see [18]).

**Definition 2.4:** \( R \) is a quadratic Lie conformal algebra, if there exists a vector space \( V \) such that \( R = \mathbb{C}[\partial]V \) is a Lie conformal algebra as a free \( \mathbb{C}[\partial] \)-module and the \( \lambda \)-bracket is of the following form:

\[
[a, b] = \partial u + \lambda v + w,
\]

where \( a, b, u, v, w \in V \).

**Remark 2.5:** Obviously, Vir, Cur\(_g\) and Vir \( \oplus \) Cur\(_g\) where \( g \) is a Lie algebra are quadratic Lie conformal algebras. Moreover, all Lie conformal algebras introduced in [15–17] are quadratic Lie conformal algebras.

For giving an equivalent characterization of quadratic Lie conformal algebras, we first introduce the definitions of Novikov algebra and Gel'fand–Dorfman bialgebra.

**Definition 2.6:** A Novikov algebra \( V \) is a vector space over \( \mathbb{C} \) with a bilinear product \( \circ : V \times V \to V \) satisfying (for any \( a, b, c \in V \)):

\[
(a \circ b) \circ c - a \circ (b \circ c) = (b \circ a) \circ c - b \circ (a \circ c),
\]

\[
(a \circ b) \circ c = (a \circ c) \circ b.
\]

If for any \( a \in V \), there exists \( x \in V \) such that \( x \circ a = a \) or \( a \circ x = a \), then \( x \) is called a left unit or right unit of \((V, \circ)\). If an element in \( V \) is not only a left unit but also a right unit, then we call it a unit. If \( I \) is a subspace of a Novikov algebra \((V, \circ)\) and \( V \circ I \subseteq I, I \circ V \subseteq I \), then \( I \) is called an ideal of \((V, \circ)\). Obviously, 0 and \( V \) are ideals of \((V, \circ)\), which are called trivial. \((V, \circ)\) is called simple, if \((V, \circ)\) has only trivial ideals and \( a \circ b \neq 0 \) for some \( a, b \in V \).

**Remark 2.7:** Novikov algebra was essentially stated in [7] that it corresponds to a certain Hamiltonian operator. Such an algebraic structure also appeared in [19] from the point of view of Poisson structures of hydrodynamic type. The name ‘Novikov algebra’ was given by Osborn in [20].
Definition 2.8 (see [7] or [18]): A Gel’fand–Dorfman bialgebra \( V \) is a vector space over \( \mathbb{C} \) with two algebraic operations \([\cdot, \cdot]\) and \( \circ \) such that \((V, [\cdot, \cdot])\) forms a Lie algebra, \((V, \circ)\) forms a Novikov algebra and the following compatibility condition holds:

\[ [a \circ b, c] - [a \circ c, b] + [a, b] \circ c - [a, c] \circ b - a \circ [b, c] = 0, \]

for any \( a, b, \) and \( c \in V \). We usually denote it by \((V, \circ, [\cdot, \cdot])\).

An equivalent characterization of quadratic Lie conformal algebra is presented as follows.

Theorem 2.9 (see [18]): \( R = \mathbb{C}[\partial]V \) is a quadratic Lie conformal algebra if and only if the \( \lambda \)-bracket of \( R \) is given as follows:

\[ [a, b] = \partial(b \circ a) + [b, a] + \lambda(a * b), \quad a, b \in V, \]

and \((V, \circ, [\cdot, \cdot])\) is a Gel’fand–Dorfman bialgebra, where \( a * b = a \circ b + b \circ a \). Therefore, \( R \) is called the quadratic Lie conformal algebra corresponding to the Gel’fand–Dorfman bialgebra \((V, \circ, [\cdot, \cdot])\).

Remark 2.10: By the definition of Gel’fand–Dorfman bialgebra, any Lie algebra with the trivial Novikov algebra structure and any Novikov algebra with the trivial Lie algebra structure are Gel’fand–Dorfman bialgebras. For convenience, we usually say a quadratic Lie conformal algebra corresponds to a Novikov algebra, if the corresponding Gel’fand–Dorfman bialgebra has the trivial Lie algebra structure.

Finally, we denote the coefficient algebra of the Lie conformal algebra corresponding to \((V, \circ, [\cdot, \cdot])\) by \( \mathcal{L}(V) \).

3. Central extensions

In this section, central extensions of quadratic Lie conformal algebras by a one-dimensional centre \( \mathbb{C}c_\beta \) with \( \partial c_\beta = \beta c_\beta \) are investigated. Denote \( c_0 \) by \( c \). The method using here can also be applied to central extensions of quadratic Lie conformal algebras by an abelian Lie conformal algebra \( \mathbb{C}[\partial]c \) which is free of rank one as a \( \mathbb{C}[\partial] \)-module.

Let \( R \) be a Lie conformal algebra and \( C \) an abelian Lie conformal algebra. If there is a short exact sequence of Lie conformal algebras

\[ 0 \rightarrow C \rightarrow \hat{R} \rightarrow R \rightarrow 0, \]

and \([C, \hat{R}] = 0\), then \( \hat{R} \) is called a central extension of \( R \) by \( C \).

Next, we will focus on central extensions \( \hat{R} \) of \( R \) by a one-dimensional centre \( \mathbb{C}c_\beta \) where \( \hat{R} \) is a quadratic Lie conformal algebra. Since \( R \) is free as a \( \mathbb{C}[\partial] \)-module, we get that \( \hat{R} = R \oplus \mathbb{C}c_\beta \), and

\[ [a, b]_{\hat{R}} = [a, b]_R + \alpha_\lambda(a, b)c_\beta, \]

where \( a, b \in R \), and \( \alpha_\lambda(\cdot, \cdot) \) is a \( \mathbb{C} \)-bilinear map from \( R \times R \) to \( \mathbb{C}[\lambda] \). In order to make \( \hat{R} \) be a Lie conformal algebra, the following conditions for \( \alpha_\lambda(\cdot, \cdot) \) hold:

\[ \alpha_\lambda(\partial a, b) = -\lambda \alpha_\lambda(a, b), \quad \alpha_\lambda(a, \partial b) = (\lambda + \beta) \alpha_\lambda(a, b), \quad (2) \]
If \( n \in \mathbb{N} \), then \( \alpha \lambda(\cdot, \cdot \cdot) \) is a 2-cocycle in \( H^2(\mathbb{C}[\partial]\mathbb{C}, \mathbb{C}) \) where \( \mathbb{C} = \mathbb{C} \) is a trivial \( \mathbb{R} \)-module and \( \partial k = \beta k \) for any \( k \in \mathbb{C} \). Denote \( \mathbb{C}_0 \) by \( \mathbb{C} \). According to [14], 2-cocycles \( \alpha \lambda(\cdot, \cdot \cdot) \) and \( \alpha' \lambda(\cdot, \cdot \cdot) \) are equivalent if and only if there is a \( \mathbb{C}[\partial] \)-module homomorphism \( \varphi : \mathbb{C} \rightarrow \mathbb{C} \) such that for all \( a, b \in \mathbb{R} \), \( \alpha \lambda(a, b) = \alpha' \lambda(a, b) + \varphi([a, b]) \).

Central extensions of quadratic Lie conformal algebras by a one-dimensional centre \( \mathbb{C} \mathbb{C}_0 \) are characterized as follows.

**Theorem 3.1:** Let \( \mathbb{C} = \mathbb{C}[\partial]\mathbb{C} \) be the quadratic Lie conformal algebra corresponding to \( (\mathbb{C}, 0, [\cdot, \cdot]) \). Set \( \mathbb{R} = \mathbb{R} \oplus \mathbb{C} \mathbb{C}_0 \) be a central extension of \( \mathbb{C} \) by \( \mathbb{C} \mathbb{C}_0 \) with the following \( \lambda \)-brackets

\[
[a \cdot b] = \partial(b \circ a) + \lambda(a \ast b) + [b, a] + \alpha \lambda(a, b)\mathbb{C}_0,
\]

where \( \alpha \lambda(a, b) = \sum_{i=0}^{n} \lambda^i \alpha \lambda_i(a, b) \in \mathbb{C}[\lambda] \) for all \( a, b \in \mathbb{C} \), and \( \alpha_n(a, b) \neq 0 \) for some \( a, b \in \mathbb{C} \). Then one can get

1. Assume that \( \alpha \lambda(a, b) = \sum_{i=0}^{3} \lambda^i \alpha \lambda_i(a, b) \) for all \( a, b \in \mathbb{C} \). Then for any \( a, b, c \in \mathbb{C} \),

\[
\sum_{i=0}^{3} \lambda^i \alpha \lambda_i(a, b) = -\sum_{i=0}^{3} (-\lambda - \beta)^i \alpha \lambda_i(b, a),
\]

\[
\alpha \lambda_3(a, b, c) = \alpha \lambda_3(a \circ b, c) = \alpha \lambda_3(b \circ a, c), (7)
\]

\[
\alpha \lambda_2(a, b, c) = \beta \alpha \lambda_3(a, b, c) + \alpha \lambda_3(c, a, b) = \alpha \lambda_2(a \circ b, c) + \alpha \lambda(a, [b, c]), (8)
\]

\[
\alpha \lambda_2(a, b, c) = \alpha \lambda_2(a, [b, c], b) + \alpha \lambda_2(c, a, b) = 2\alpha \lambda_3(a \circ b, c) + 3\alpha \lambda([b, a], c), (9)
\]

\[
\alpha \lambda_1(a, b, c) = \beta \alpha \lambda_2(a, b, c) + \alpha \lambda_2(c, a, b) = \alpha \lambda_1(a \circ b, c) + \alpha \lambda([a, b], c), (10)
\]

\[
\alpha \lambda_1(a, b, c) = \alpha \lambda_1(a, b \circ c) + \alpha \lambda(a, [b, c], b) = \alpha \lambda(a \circ b, c) + 2\alpha \lambda([a, b], c), (11)
\]

\[
\alpha \lambda_0(a, b, c) = \beta \alpha \lambda_1(a, b, c) + \alpha \lambda_1(a, [b, c], b) = \alpha \lambda_0(a \circ b, c) + \alpha \lambda([a, b], c), (12)
\]

\[
\beta \alpha \lambda_0(a, b, c) = \beta \alpha \lambda_0(a \circ b, c) + \alpha \lambda([a, b], c), (13)
\]

2. If \( n > 3 \), then \( \alpha \lambda_n(a, b, c) = 0 \) for any \( a, b, c \in \mathbb{C} \).

3. \( \alpha \lambda(\cdot, \cdot \cdot) \) and \( \alpha' \lambda(\cdot, \cdot \cdot) \) are equivalent if and only if there is a linear map \( \varphi : \mathbb{C} \rightarrow \mathbb{C} \) such that

\[
\alpha \lambda(a, b) = \alpha' \lambda(a, b) + \beta \varphi(b \circ a) + \lambda \varphi(a \ast b) + \varphi([b, a]), \quad \text{for all } a, b \in \mathbb{C}. (14)
\]

**Proof:** According to (3), we get \( \sum_{i=0}^{n} \lambda^i \alpha \lambda_i(a, b) = -\sum_{i=0}^{n} (-\lambda - \beta)^i \alpha \lambda_i(b, a). \)

By (2) and (3), (4) becomes

\[
(\lambda + \beta) \alpha \lambda_3(a, b, c) + \mu \alpha \lambda_3(a, b \circ c) + \alpha \lambda(a, [b, c]) - (\mu + \beta) \alpha \lambda([a, b], c)
\]
\[ -\lambda \alpha_\mu (b, a \ast c) - \alpha_\mu (b, [c, a]) \]
\[ = (-\lambda - \mu)\alpha_{\lambda+\mu}(b \circ a, c) + \lambda \alpha_{\lambda+\mu}(a \ast b, c) + \alpha_{\lambda+\mu}([b, a], c). \]

If \( n > 3 \), according to \( \alpha_\lambda(a, b) = \sum_{i=0}^{n} \lambda^i \alpha_i(a, b) \) and \( \lambda^2 \mu^{n-1} \) and \( \lambda^{n-1} \mu^2 \), we obtain
\[
na_n(a \circ b, c) - C_n^2 \alpha_n(b \circ a, c) = 0, \quad C_n^2 \alpha_n(a \circ b, c) - n\alpha_n(b \circ a, c) = 0.
\]

Since \( n > 3 \), \( \alpha_n(a \circ b, c) = 0 \) for any \( a, b, c \in V \).

If \( n \leq 3 \), taking \( \alpha_\lambda(a, b) = \sum_{i=0}^{3} \lambda^i \alpha_i(a, b) \) into (15) and comparing the coefficients of \( \lambda^4, \lambda^2 \mu^2, \lambda^3, \lambda^2 \mu, \lambda^2, \lambda \mu, \lambda \) and \( \lambda^0 \mu^0 \), we get
\[
\alpha_3(a, c \circ b) = \alpha_3(a \circ b, c), \quad (16)
\]
\[
-\alpha_3(b, a \ast c) = -3\alpha_3(b \circ a, c) + \alpha_3(a \circ b, c), \quad (17)
\]
\[
\alpha_3(b \circ a, c) = \alpha_3(a \circ b, c), \quad (18)
\]

and (7)–(13).

By (16) and (18), (17) is reduced to
\[
\alpha_3(b, a \circ c) = \alpha_3(b \circ a, c). \quad (19)
\]

Since \( \alpha_3(a, b) = \alpha_3(b, a) \), we can obtain (19) from (7) as follows:
\[
\alpha_3(b, a \circ c) = \alpha_3(b \circ a, c) = \alpha_3(c \circ b, a) = \alpha_3(c, a \circ b) = \alpha_3(a \circ b, c) = \alpha_3(b \circ a, c).
\]

Therefore, (16)–(18) are equivalent to (7). Then it is easy to get that \( \alpha_\lambda(\cdot, \cdot) = \sum_{i=0}^{3} \lambda^i \alpha_i(\cdot, \cdot) \) is a 2-cocycle if and only if (6)–(13) hold.

Finally, by [14], \( \alpha_i(\cdot, \cdot) \) and \( \alpha'_i(\cdot, \cdot) \) are equivalent if and only if there is a \( \mathbb{C}[\hat{\beta}] \)-module homomorphism \( \varphi : R \to \mathbb{C}_\beta \) such that for all \( a, b \in V \), \( \alpha_\lambda(a, b) = \alpha'_\lambda(a, b) + \varphi(\partial(b \circ a)) \lambda(a \ast b) + \beta[a, b] \). Since \( \varphi \) is a \( \mathbb{C}[\hat{\beta}] \)-module homomorphism, \( \varphi(\partial(b \circ a)) \beta \varphi(b \circ a) \). Moreover, since \( R = \mathbb{C}[\hat{\beta}]V \) is free as a \( \mathbb{C}[\hat{\beta}] \)-module, a \( \mathbb{C}[\hat{\beta}] \)-module homomorphism \( \varphi : R \to \mathbb{C}_\beta \) can be determined by the restricted linear map \( \varphi |_V : V \to \mathbb{C} \). Then (3) can be directly obtained.

**Remark 3.2:** Note that \( \alpha_i(\cdot, \cdot) \) is a bilinear form on \( V \) for each \( 0 \leq i \leq n \). By (3) in Theorem 3.1, if there exists some nonzero \( \alpha_i(\cdot, \cdot) \) for \( i \geq 2 \), then there are nontrivial central extensions of this Lie conformal algebra up to an equivalence.

**Remark 3.3:** If we consider central extensions of quadratic Lie conformal algebras by \( \mathbb{C}[\hat{\beta}]V \) which is free of rank one as a \( \mathbb{C}[\hat{\beta}] \)-module, Theorem 3.1 also holds, when we replace \( \mathbb{C}[\hat{\beta}]V \) by \( \mathbb{C}[\lambda] \), \( \beta \) by \( \partial \), \( \alpha_\lambda(a, b) \in \mathbb{C}[\lambda] \) by \( \alpha_\lambda(a, b) \in \mathbb{C}[\lambda, \partial] \) for any \( a, b \in V, \mathbb{C} \in (3) \) by \( \mathbb{C}[\hat{\beta}] \). Note that in this case, \( \alpha_i(a, b) \in \mathbb{C}[\hat{\beta}] \) for any \( a, b \in V \) and \( i \in \{0, \ldots, n\} \).

**Corollary 3.4:** Let \( (V, \circ, [\cdot, \cdot]) \) be a finite-dimensional Gel’fand–Dorfman bialgebra with \( V = V \circ V \) and \( R = \mathbb{C} \hat{\lambda} \) the corresponding quadratic Lie conformal algebra. Suppose that \( \hat{R} = R \oplus \mathbb{C} c_\beta \) be a central extension of \( (R, [\cdot, \cdot]) \) with the \( \lambda \)-brackets defined by (5). Then for all \( a, b \in V, \alpha_\lambda(a, b) = \sum_{i=0}^{3} \lambda^i \alpha_i(a, b) \), where all \( \alpha_i(\cdot, \cdot) : V \times V \to \mathbb{C} \) are bilinear forms satisfying (6)–(13). Furthermore, \( \alpha_\lambda(\cdot, \cdot) \) and \( \alpha'_\lambda(\cdot, \cdot) \) are equivalent if and only if
for all \( a, b \in V \), \( \alpha_i(a, b) = \alpha'_i(a, b) \) for \( i = 2 \) and \( i = 3 \), and there is a linear map \( \varphi : V \to \mathbb{C} \) such that

\[
\alpha_1(a, b) = \alpha'_1(a, b) + \varphi(a \ast b),
\]
\[
\alpha_0(a, b) = \alpha'_0(a, b) + \beta \varphi(b \circ a) + \varphi([b, a]).
\]

**Proof:** According to that \( R = \mathbb{C}[\partial]V \) is a finite Lie conformal algebra, one can set \( \alpha_\lambda(a, b) = \sum_{i=0}^{n} \lambda^i \alpha_i(a, b) \) for any \( a, b \in V \) and some \( n \in \mathbb{N} \). Then this corollary can be directly obtained by Theorem 3.1. \( \blacksquare \)

**Corollary 3.5:** Let \((V, \circ)\) be a Novikov algebra with \( V = V \circ V \) and \( R = \mathbb{C}[\partial]V \) be the corresponding quadratic Lie conformal algebra. Suppose that \( \hat{R} = R \oplus \mathbb{C} \beta \) be a central extension of \((R, [-, -])\) with the \( \lambda \)-brackets defined by

\[
[a, b] = \partial(b \circ a) + \lambda(a \ast b) + \alpha_\lambda(a, b) \beta,
\]

where \( a, b \in V \) and \( \alpha_\lambda(a, b) \in \mathbb{C}[\lambda] \). Then for all \( a, b \in V \), \( \alpha_\lambda(a, b) = \sum_{i=0}^{2} \lambda^i \alpha_i(a, b) \), where all \( \alpha_i(\cdot, \cdot) : V \times V \to \mathbb{C} \) are bilinear forms satisfying (6), (7) and

\[
\alpha_2(a, c \circ b) + \beta \alpha_3(a, c \circ b) = \alpha_2(a \circ b, c),
\]
\[
\alpha_2(a, b \ast c) + \alpha_2(b \circ a, c) = 2 \alpha_2(a \circ b, c),
\]
\[
\alpha_1(a, c \circ b) + \beta \alpha_2(a, c \circ b) = \alpha_1(a \circ b, c),
\]
\[
\alpha_1(a, b \ast c) - \alpha_1(b, a \ast c) = -\alpha_1(b \circ a, c) + \alpha_1(a \circ b, c),
\]
\[
\alpha_0(a, c \circ b) + \beta \alpha_1(a, c \circ b) - \alpha_0(b, a \ast c) = \alpha_0(a \circ b, c),
\]
\[
\beta(\alpha_0(a, c \circ b) - \alpha_0(b, c \circ a)) = 0,
\]

for all \( a, b, c \in V \). In particular, when \( \beta = 0 \), (23)–(28) are equivalent to

\[
\alpha_2(a, b \circ c) + \alpha_2(b \circ a, c) = \alpha_2(a \circ b, c) = \alpha_2(a, c \circ b),
\]
\[
\alpha_1(a, c \circ b) = \alpha_1(a \circ b, c),
\]
\[
\alpha_0(c \circ a, b) - \alpha_0(c \circ b, a) = \alpha_0(a \circ b, c) - \alpha_0(a \circ c, b).
\]

Furthermore, \( \alpha_\lambda(\cdot, \cdot) \) and \( \alpha'_\lambda(\cdot, \cdot) \) are equivalent if and only if for all \( a, b \in V \), \( \alpha_i(a, b) = \alpha'_i(a, b) \) for \( i = 2, 3 \), and there is a linear map \( \varphi : V \to \mathbb{C} \) such that

\[
\alpha_1(a, b) = \alpha'_1(a, b) + \varphi(a \ast b),
\]
\[
\alpha_0(a, b) = \alpha'_0(a, b) + \beta \varphi(b \circ a).
\]

**Proof:** This corollary can be directly obtained by Theorem 3.1 and a similar discussion with that in Corollary 4.4 in [21]. \( \blacksquare \)

**Remark 3.6:** Note that Corollary 3.5 also holds when \( V \) is infinite-dimensional.

**Corollary 3.7:** If for Novikov algebra \((V, \circ)\), there is a right unit \( e \in V \), then the central extensions of quadratic Lie conformal algebra \( R = \mathbb{C}[\partial]V \) by \( \mathbb{C} \beta \) with \( \beta \neq 0 \) are trivial, i.e. \( H^2(R, \mathbb{C} \beta) = 0 \).
When \( \beta = 0 \) and \((V, \circ)\) is a Novikov algebra with a unit 1 in Corollary 3.5, then \( \alpha_2(a, b) = \alpha_0(a, b) = 0 \) for any \( a, b \in V \).

**Proof:** Obviously, if \((V, \circ)\) has a right unit \( e \) or a unit 1, \( V = V \circ V \).

When \( \beta \neq 0 \), setting \( b = e \) in (23), (25), (27) and (28), we get that \( \alpha_2(a, c) = \alpha_0(a, c) = 0 \), \( \alpha_1(a, c) = \frac{\alpha_0(a, c, d)}{\beta} \), and \( \alpha_0(a, c) = \alpha_0(e, c \circ a) \) for any \( a, c \in V \). Set \( \varphi(a) = \frac{\alpha_0(c, a)}{\beta} \) for any \( a \in V \) in Corollary 3.5. By Corollary 3.5, we can make \( \alpha_1(\cdot, \cdot) \) and \( \alpha_0(\cdot, \cdot) \) be zero. Therefore, in this case, \( H^2(R, C \varphi(\cdot)) = 0 \).

When \( \beta = 0 \) and \((V, \circ)\) is a Novikov algebra with a unit 1, letting \( b = 1 \) in (29), we can directly obtain \( \alpha_2(a, c) = 0 \) for any \( a, c \in V \). Similarly, by (6), we get \( \alpha_0(a, b) = -\alpha_0(b, a) \) for any \( a, b \in V \). Letting \( c = b = 1 \) in (31) and using \( \alpha_0(a, 1) = -\alpha_0(1, a) \), we have \( \alpha_0(a, 1) = 0 \) for any \( a \in V \). Then setting \( c = 1 \) in (31) and using \( \alpha_0(a, 1) = 0 \) for any \( a \in V \), \( \alpha_0(a, b) = 0 \) is obtained for any \( a, b \in V \). \( \blacksquare \)

**Remark 3.8:** By Remark 3.3, Corollary 3.5 also holds if we replace \( \epsilon_\beta \) by \( \mathbb{C}[\partial] \epsilon_\beta \), \( \beta \) by \( \partial \) and \( \alpha_1(a, b) \in \mathbb{C}[\lambda] \) by \( \alpha_1(a, b) \in \mathbb{C}[\lambda, \partial] \) for any \( a, b \in V \).

According to Corollary 3.7, if Novikov algebra \((V, \circ)\) has a right unit, then the central extensions of quadratic Lie conformal algebra \( R = \mathbb{C}[\partial] V \) by an abelian Lie conformal algebra \( \mathbb{C}[\partial] \epsilon \) are trivial, i.e. \( H^2(R, \mathbb{C}[\partial] \epsilon) = 0 \).

**Proposition 3.9:** Let \((V, \circ, [\cdot, \cdot])\) be a Gel’fand–Dorfman bialgebra and \( \alpha_i(\cdot, \cdot) \) (\( i = 0, 1, 2, 3 \)) be bilinear forms on \( V \) satisfying (6)–(13) for any \( a, b, c \in V \) with \( \beta = 0 \). In addition, let \( \pi : \mathcal{L}(V) \times \mathcal{L}(V) \rightarrow \mathbb{C} \) be the bilinear form on \( \mathcal{L}(V) \) defined by

\[
\pi(a_m, b_n) = \alpha_0(a, b) \delta_{m+n+1,0} + m \alpha_1(a, b) \delta_{m+n,0} + m(m-1) \alpha_2(a, b) \delta_{m+n-1,0} + m(m-1)(m-2) \alpha_3(a, b) \delta_{m+n-2,0},
\]

for \( a, b \in V, m, n \in \mathbb{Z} \). Then \( \pi \) is a 2-cocycle of Lie algebra \( \mathcal{L}(V) \).

**Proof:** Let \( R = \mathbb{C}[\partial] V \) be the quadratic Lie conformal algebra corresponding to a Gel’fand–Dorfman bialgebra \((V, \circ, [\cdot, \cdot])\). Denote \( \epsilon_0 \) by \( \epsilon \). Assume that \( \hat{R} \) is the central extension defined by (5). It is easy to see that the coefficient algebra of \( \hat{R} \) is \( \text{Coeff} \hat{R} = \mathcal{L}(V) \oplus \mathbb{C} c_{-1} \) with the following Lie bracket: \([a_m, c_{-1}] = 0\) and

\[
[a_m, b_n] = [a, b]_{m+n} + m(a \circ b)_{m+n-1} - (b \circ a)_{m+n-1} + (\alpha_0(a, b))_{m+n+1,0} \\
+ m \alpha_1(a, b) \delta_{m+n,0} + m(m-1) \alpha_2(a, b) \delta_{m+n-1,0} \\
+ m(m-1)(m-2) \alpha_3(a, b) \delta_{m+n-2,0} c_{-1},
\]

for any \( a, b \in V, m, n \in \mathbb{Z} \). Obviously, \( \text{Coeff} \hat{R} \) is a central extension of \( \mathcal{L}(V) \) by a one-dimensional centre \( \mathbb{C} c_{-1} \). Therefore, by (34), \( \pi \) is a 2-cocycle of \( \mathcal{L}(V) \). \( \blacksquare \)

**Corollary 3.10:** Let \((V, \circ)\) be a Novikov algebra and \( \alpha_i(\cdot, \cdot) \) (\( i = 0, 1, 2, 3 \)) be bilinear forms on \( V \) satisfying (6), (7) and (29)–(31) for any \( a, b, c \in V \) with \( \beta = 0 \). Moreover, let \( \pi_i : \mathcal{L}(V) \times \mathcal{L}(V) \rightarrow \mathbb{C} \) be bilinear forms on \( \mathcal{L}(V) \) defined by

\[
\pi_i(a \otimes t^m, b \otimes t^n) = \alpha_0(a, b) \delta_{m+n+1,0},
\]

(35)
\[ \pi_1(a \otimes t^m, b \otimes t^n) = m\alpha_1(a, b)\delta_{m+n,0}, \]
\[ \pi_2(a \otimes t^m, b \otimes t^n) = m(m-1)\alpha_2(a, b)\delta_{m+n-1,0}, \]
\[ \pi_3(a \otimes t^m, b \otimes t^n) = m(m-1)(m-2)\alpha_3(a, b)\delta_{m+n-2,0}, \]

for \( a, b \in V, m, n \in \mathbb{Z} \). Then \( \pi_0, \pi_1, \pi_2, \pi_3 \) are 2-cocycles of Lie algebra \( L(V) \).

**Proof:** According to that \( \alpha_0(\cdot, \cdot), \alpha_1(\cdot, \cdot), \alpha_2(\cdot, \cdot) \) and \( \alpha_3(\cdot, \cdot) \) do not depend on each other, this corollary can be directly obtained from Proposition 3.9.

\[ \blacksquare \]

**Remark 3.11:** This corollary can also be referred to Proposition 2.5 in [22].

By Corollary 3.7, when \( \beta \neq 0, H^2(R, \mathbb{C}c_\beta) = 0 \) for any quadratic Lie conformal corresponding to a Novikov algebra \( (V, \circ) \) with the right unit. In the following, we mainly focus on the case when \( \beta = 0 \), i.e. we use Theorem 3.1 to determine central extensions of several specific Lie conformal algebras by a one-dimensional centre \( \mathbb{C}c \) up to an equivalence.

**Example 3.12:** Let \( R(\alpha, \beta) = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]W \) be a Lie conformal algebra with the \( \lambda \)-bracket given as follows:
\[ [L, L] = (\partial + 2\lambda)L, \quad [L, W] = (\partial + \alpha \lambda + \beta)W, \quad [W, L] = 0, \]
\[ [L, W] = (\partial + \alpha \lambda + \beta)W, \quad [W, L] = 0, \]

for some \( \alpha, \beta \in \mathbb{C} \). In fact, it is the semi-direct sum of \( \text{Vir} \) and the conformal module \( \mathbb{C}[\partial]W \) of \( \text{Vir} \) with the action \( L(W) = (\partial + \alpha \lambda + \beta)W \). Moreover, it also appeared in [23].

Next, we begin to study central extensions of \( R(\alpha, \beta) \) by a one-dimensional centre \( \mathbb{C}c \) up to an equivalence.

Obviously, \( R(\alpha, \beta) \) is the quadratic Lie conformal algebra corresponding to a 2-dimensional Gel’fand–Dorfman bialgebra \( V = \mathbb{C}L \oplus \mathbb{C}W \) with the Novikov operation ‘\( \circ \)’ and Lie bracket defined as follows:
\[ L \circ L = L, \quad L \circ W = (\alpha - 1)W, \quad W \circ L = W, \quad W \circ W = 0, \]
\[ [L, L] = 0, \quad [W, L] = \beta W, \quad [W, W] = 0. \]

It is easy to see that \( (V, \circ, [\cdot, \cdot]) \) satisfies the assumption in Corollary 3.4. Therefore, according to Corollary 3.4, (6)–(13) and by some simple computations, we can obtain

1. If \( \alpha \neq 2, \alpha \neq 1, \alpha \neq 0, \) or \( \alpha = 1, \beta \neq 0 \) or \( \alpha = 2, \beta \neq 0 \),
\[ \alpha_1(L, L) = A, \quad \alpha_1(L, W) = B, \quad \alpha_1(W, L) = C, \quad \alpha_0(L, W) = \frac{\beta}{\alpha} C, \]
\[ \alpha_3(L, W) = \alpha_3(W, W) = \alpha_2(L, W) = \alpha_2(W, L) = \alpha_2(W, W) = 0, \]
\[ \alpha_1(W, W) = \alpha_0(W, W) = 0, \]

for any \( A, B, C \in \mathbb{C} \).

2. If \( \alpha = 0 \) and \( \beta = 0 \),
\[ \alpha_3(L, L) = A, \quad \alpha_1(L, L) = B, \quad \alpha_1(L, W) = C, \quad \alpha_0(L, W) = D, \]
\[ \alpha_3(L, W) = \alpha_3(W, W) = \alpha_2(L, W) = \alpha_2(L, L) = \alpha_2(W, W) = 0, \]
\[ \alpha_1(L, W) = \alpha_0(W, W) = \alpha_0(L, L) = 0, \]

for any \( A, B, C, D \in \mathbb{C} \).

(3) If \( \alpha = 0 \) and \( \beta \neq 0 \),
\[ \alpha_3(L, L) = A, \quad \alpha_1(L, L) = B, \quad \alpha_0(L, W) = C, \]
\[ \alpha_3(L, W) = \alpha_3(W, W) = \alpha_2(L, W) = \alpha_2(L, L) = \alpha_2(W, W) = 0, \]
\[ \alpha_1(L, W) = \alpha_1(W, W) = \alpha_0(W, W) = \alpha_0(L, L) = 0, \]

for any \( A, B, C \in \mathbb{C} \).

(4) If \( \alpha = 1, \beta = 0 \),
\[ \alpha_3(L, L) = A, \quad \alpha_2(L, W) = B, \quad \alpha_1(L, L) = C, \quad \alpha_1(L, W) = D, \]
\[ \alpha_1(W, W) = E, \]
\[ \alpha_3(L, W) = \alpha_3(W, W) = \alpha_2(W, W) = \alpha_2(L, L) = 0, \]
\[ \alpha_0(W, W) = \alpha_0(L, L) = \alpha_0(L, W) = 0, \]

for any \( A, B, C, D, E \in \mathbb{C} \).

(5) If \( \alpha = 2, \beta = 0 \),
\[ \alpha_3(L, L) = A, \quad \alpha_3(L, W) = B, \quad \alpha_1(L, L) = C, \quad \alpha_1(L, W) = D, \]
\[ \alpha_1(W, W) = 0, \]
\[ \alpha_3(W, W) = \alpha_2(L, W) = \alpha_2(W, W) = \alpha_2(L, L) = 0, \]
\[ \alpha_0(W, W) = \alpha_0(L, W) = \alpha_0(L, L) = 0, \]

for any \( A, B, C, D \in \mathbb{C} \).

Therefore, if \( \alpha \neq 2, \alpha \neq 1, \alpha \neq 0 \), or \( \alpha = 1, \beta \neq 0 \) or \( \alpha = 2, \beta \neq 0 \), by choosing the linear map \( \varphi : V \to \mathbb{C} \) in Corollary 3.4 defined by \( \varphi(L) = \frac{B}{2} + \lambda \) and \( \varphi(W) = \frac{C}{\alpha} \), we can make \( \alpha_1(\cdot, \cdot) \) and \( \alpha_0(\cdot, \cdot) \) be zero up to an equivalence. Therefore, by Corollary 3.4, all equivalence classes of central extensions of \( R(\alpha, \beta) \) by a one-dimensional centre \( \mathbb{C}t \) are \( R(\alpha, \beta)(A) \) with the \( \lambda \)-brackets as follows:

\[ [L_\lambda L] = (\partial + 2\lambda)L + A\lambda^3c, \]
\[ [L_\lambda W] = (\partial + \alpha\lambda + \beta)W, \quad [W_\lambda W] = 0, \]

for all \( A \in \mathbb{C} \). Note that if \( A_1 \neq A_2 \), then \( \hat{R}(\alpha, \beta)(A_1) \) is not equivalent to \( \hat{R}(\alpha, \beta)(A_2) \).

Therefore, in this case, \( \dim H^2(R(\alpha, \beta), \mathbb{C}c) = 1 \).

If \( \alpha = 0 \), and \( \beta = 0 \), by choosing the linear map \( \varphi : V \to \mathbb{C} \) in Corollary 3.4 defined by \( \varphi(L) = \frac{B}{2} \) and \( \varphi(W) = 0 \), we can make \( \alpha_1(L, L) \) be zero up to an equivalence. Therefore, by Corollary 3.4, all equivalence classes of central extensions of \( R(0, 0) \) by a one-dimensional centre \( \mathbb{C}c \) are \( R(0, 0)(A, C, D) \) with the \( \lambda \)-brackets as follows:

\[ [L_\lambda L] = (\partial + 2\lambda)L + A\lambda^3c, \]
\[ [L_\lambda W] = \partial W + (C\lambda + D)c, \quad [W_\lambda W] = 0, \]

for all \( A, C, D \in \mathbb{C} \). Note that if \((A_1, C_1, D_1) \neq (A_2, C_2, D_2)\), then \( \tilde{R}(0, 0)(A_1, C_1, D_1) \) is not equivalent to \( R(0, 0)(A_2, C_2, D_2) \). Therefore, \( \dim H^2(R(0, 0), \mathbb{C}c) = 3 \).

If \( \alpha = 0 \) and \( \beta \neq 0 \), by choosing the linear map \( \varphi : V \to \mathbb{C} \) in Corollary 3.4 defined by \( \varphi(L) = \frac{B}{2} \) and \( \varphi(W) = \frac{C}{\beta} \), we can make \( \alpha_1(\cdot, \cdot) \) and \( \alpha_0(\cdot, \cdot) \) be zero up to an equivalence. Therefore, by Corollary 3.4, all equivalence classes of central extensions of \( R(0, \beta) \) by a one-dimensional centre \( \mathbb{C}c \) are \( R(0, \beta)(A) \) with the \( \lambda \)-brackets as follows:

\[ [L_\lambda L] = (\partial + 2\lambda)L + A\lambda^3c, \]
\[ [L_\lambda W] = (\partial + \beta)W, \quad [W_\lambda W] = 0, \]

for all \( A \in \mathbb{C} \). Note that if \( A_1 \neq A_2 \), then \( \tilde{R}(0, \beta)(A_1) \) is not equivalent to \( \tilde{R}(0, \beta)(A_2) \). Therefore, in this case, \( \dim H^2(R(0, \beta), \mathbb{C}c) = 1 \).

If \( \alpha = 1 \) and \( \beta = 0 \), by choosing the linear map \( \varphi : V \to \mathbb{C} \) in Corollary 3.4 defined by \( \varphi(L) = \frac{C}{2} \) and \( \varphi(W) = D \), we can make \( \alpha_1(L, L) \) and \( \alpha_1(L, W) \) be zero up to an equivalence. Therefore, by Corollary 3.4, all equivalence classes of central extensions of \( R(1, 0) \) by a one-dimensional centre \( \mathbb{C}c \) are \( R(1, 0)(A, B, E) \) with the \( \lambda \)-brackets as follows:

\[ [L_\lambda L] = (\partial + 2\lambda)L + A\lambda^3c, \]
\[ [L_\lambda W] = (\partial + \lambda)W + B\lambda^2c, \quad [W_\lambda W] = E\lambda c, \]

for all \( A, B, E \in \mathbb{C} \). Note that if \((A_1, B_1, E_1) \neq (A_2, B_2, E_2)\), then \( \tilde{R}(1, 0)(A_1, B_1, E_1) \) is not equivalent to \( \tilde{R}(1, 0)(A_2, B_2, E_2) \). Therefore, \( \dim H^2(R(1, 0), \mathbb{C}c) = 3 \).

If \( \alpha = 2 \) and \( \beta = 0 \), by choosing the linear map \( \varphi : V \to \mathbb{C} \) in Corollary 3.4 defined by \( \varphi(L) = \frac{C}{2} \) and \( \varphi(W) = \frac{D}{2} \), we can make \( \alpha_1(L, L) \) and \( \alpha_1(L, W) \) be zero up to an equivalence. Therefore, by Corollary 3.4, all equivalence classes of central extensions of \( R(2, 0) \) by a one-dimensional centre \( \mathbb{C}c \) are \( R(2, 0)(A, B) \) with the \( \lambda \)-brackets as follows:

\[ [L_\lambda L] = (\partial + 2\lambda)L + A\lambda^3c, \]
\[ [L_\lambda W] = (\partial + 2\lambda)W + B\lambda^3c, \quad [W_\lambda W] = 0, \]

for all \( A, B \in \mathbb{C} \). Note that if \((A_1, B_1) \neq (A_2, B_2)\), then \( \tilde{R}(2, 0)(A_1, B_1) \) is not equivalent to \( \tilde{R}(2, 0)(A_2, B_2) \). Therefore, \( \dim H^2(R(2, 0), \mathbb{C}c) = 2 \).

**Example 3.13:** The loop Virasoro Lie conformal algebra \( \mathcal{L}W \) introduced in [17] is a Lie conformal algebra with the \( \mathbb{C}[\partial]-basis \) \( \{L_i | i \in \mathbb{Z} \} \) and the \( \lambda \)-bracket given as follows:

\[ [L_i L_j] = (-\partial - 2\lambda)L_{i+j}. \quad (42) \]

Its coefficient algebra is just the loop Virasoro Lie algebra. In [17], conformal derivations, rank one conformal modules and \( \mathbb{Z} \)-graded free intermediate series modules of \( \mathcal{L}W \) were determined. But, central extensions of \( \mathcal{L}W \) were not studied. Next, we will give a characterization of central extensions of \( \mathcal{L}W \) by a one-dimensional centre \( \mathbb{C}c \) up to an equivalence.
Obviously, $\mathcal{LW}$ is the quadratic Lie conformal algebra corresponding to an infinite-dimensional Novikov algebra $V = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}L_i$ with the operation ‘$\circ$’ as follows:

$$L_i \circ L_j = -L_{i+j}, \quad \text{for any } i, j \in \mathbb{Z}. \quad (43)$$

According to Corollary 3.5, (6), (7), (29)–(31) and by some simple computations, we can obtain

$$\alpha_3(L_i, L_{j+k}) = \alpha_3(L_{i+k}, L_j), \quad \alpha_3(L_i, L_j) = \alpha_3(L_j, L_i), \quad (44)$$

$$\alpha_2(L_i, L_j) = 0, \quad (45)$$

$$\alpha_1(L_i, L_{j+k}) = \alpha_1(L_{i+j}, L_k), \quad \alpha_1(L_i, L_j) = \alpha_1(L_j, L_i), \quad (46)$$

$$2\alpha_0(L_{i+k}, L_j) = \alpha_0(L_{i+j}, L_k) + \alpha_0(L_{j+k}, L_i), \quad \alpha_0(L_i, L_j) = -\alpha_0(L_j, L_i), \quad (47)$$

for any $i, j, k \in \mathbb{Z}$. By (44), we get $\alpha_3(L_i, L_{j+k}) = \alpha_3(L_{i+j+k}, L_0) = \alpha_3(L_0, L_{i+j+k})$. Therefore, we set

$$\alpha_3(L_0, L_{i+j+k}) = f(i + j + k), \quad (48)$$

for some complex function $f$. Thus, $\alpha_3(L_i, L_{j+k}) = f(i + j + k)$. Letting $k = 0$, we obtain $\alpha_3(L_i, L_j) = f(i + j)$. Then (44) holds for any $i, j, k \in \mathbb{Z}$. Similarly, $\alpha_1(L_i, L_j) = g(i + j)$ for some complex function $g$. Set $j = 0$ in (47). We have $2\alpha_0(L_{i+k}, L_0) = \alpha_0(L_i, L_k) + \alpha_0(L_k, L_i) = 0$ for any $i, k \in \mathbb{Z}$. Therefore, $\alpha_0(L_i, L_0) = \alpha_0(L_0, L_i) = 0$. Set $k = 0$ in (47). we get

$$2\alpha_0(L_i, L_j) = \alpha_0(L_{i+j}, L_0) + \alpha_0(L_j, L_i) = \alpha_0(L_j, L_i) = -\alpha_0(L_i, L_j).$$

Thus, $\alpha_0(L_i, L_j) = 0$.

Therefore, by choosing the linear map $\varphi : V \to \mathbb{C}$ in Corollary 3.5 defined by $\varphi(L_i) = -\frac{g(i)}{2}$ for all $i \in \mathbb{Z}$, we can make $\alpha_1(\cdot, \cdot)$ be zero up to an equivalence. Therefore, by Corollary 3.5, all equivalence classes of central extensions of $\mathcal{LW}$ by a one-dimensional centre $\mathbb{C}c$ are $\widehat{\mathcal{LW}}(f) = \mathcal{LW} \oplus \mathbb{C}c$ with the $\lambda$-brackets as follows:

$$[L_i, L_j] = (-\partial - 2\lambda)L_{i+j} + f(i + j)\lambda^3c, \quad (49)$$

for all complex functions $f$. Note that if $f_1 \neq f_2$, then $\widehat{\mathcal{LW}}(f_1)$ is not equivalent to $\widehat{\mathcal{LW}}(f_2)$. Therefore, $\dim H^2(\mathcal{LW}, \mathbb{C}c) = \infty$.

4. Conformal derivations

In this section, we will investigate conformal derivations of quadratic Lie conformal algebras.

**Definition 4.1:** A conformal linear map between $\mathbb{C}[/]$-modules $U$ and $V$ is a linear map $\phi_\lambda : U \to V[\lambda]$ such that

$$\phi_\lambda(\partial u) = (\partial + \lambda)\phi_\lambda u, \quad \text{for all } u \in U. \quad (50)$$

We will abuse the notation by writing $\phi : U \to V$ any time it is clear from the context that $\phi$ is conformal linear.
Definition 4.2: Let $R$ be a Lie conformal algebra. A conformal linear map $d: R \rightarrow R$ is called a conformal derivation of $R$ if
\[
d_{\lambda}[a, b] = [d_{\lambda}a, b] + [a, d_{\lambda}b], \quad a, b \in R. \tag{51}
\]

The space of all conformal derivations of $R$ is denoted by $\text{CDer}(R)$. For any $a \in R$, there is a natural conformal derivation $\text{ad} a : R \rightarrow R$ such that
\[
(\text{ad} a)_{\lambda}b = [a, b], \quad b \in R.
\]
All conformal derivations of this kind are called inner. The space of all inner conformal derivations is denoted by $\text{CInn}(R)$.

Next, we present a characterization of conformal derivations of quadratic Lie conformal algebras.

Theorem 4.3: Let $R = \mathbb{C}[\partial]V$ be the quadratic Lie conformal algebra corresponding to a Gel'fand–Dorfman bialgebra $(V, \circ, [\cdot, \cdot])$ and $d$ be a conformal derivation of $R$. Assume that
\[
d_{\lambda}(a) = \sum_{i=0}^{n} \partial^{i}d_{\lambda}^{i}(a), \quad d_{\lambda}^{i}(a) \in V[\lambda], \quad a \in V, \quad i \in \{0, 1, \ldots, n\}, \tag{52}
\]
and there exists some $a \in V$ such that $d_{\lambda}^{i}(a) \neq 0$. Then we obtain (for any $a, b \in V$).

1. Assume that $d_{\lambda}^{i}(a) = \sum_{i=0}^{n} \partial^{i}d_{\lambda}^{i}(a)$ for all $a \in V$. Then for any $a, b \in V$,
\[
d_{\lambda}^{3}(b \circ a) = d_{\lambda}^{3}(a \circ b) = d_{\lambda}^{3}(b) \circ a, \tag{53}
\]
\[
b \circ d_{\lambda}^{3}(a) = a \circ d_{\lambda}^{3}(b), \tag{54}
\]
\[
2d_{\lambda}^{3}(b) \circ a + a \circ d_{\lambda}^{3}(b) = 0, \tag{55}
\]
\[
\lambda d_{\lambda}^{3}(b \circ a) + d_{\lambda}^{3}(b \circ a) + d_{\lambda}^{3}[b, a] = [d_{\lambda}^{3}(b), a] + d_{\lambda}^{3}(b) \circ a, \tag{56}
\]
\[
d_{\lambda}^{3}(a \ast b) = 2d_{\lambda}^{3}(b) \circ a + d_{\lambda}^{3}(b) \ast a + 3[d_{\lambda}^{3}(b), a], \tag{57}
\]
\[
-3\lambda b \circ d_{\lambda}^{3}(a) + b \circ d_{\lambda}^{3}(a) + d_{\lambda}^{3}(b) \circ a + 2d_{\lambda}^{3}(b) \ast a + 3[d_{\lambda}^{3}(b), a] = 0, \tag{58}
\]
\[
-4\lambda d_{\lambda}^{3}(a) \ast b + d_{\lambda}^{3}(a) \ast b - [b, d_{\lambda}^{3}(a)] + d_{\lambda}^{3}(b) \ast a + [d_{\lambda}^{3}(b), a] = 0, \tag{59}
\]
\[
\lambda d_{\lambda}^{3}(b \circ a) + d_{\lambda}^{3}(b \circ a) + d_{\lambda}^{3}[b, a] = d_{\lambda}^{3}(b) \circ a + [d_{\lambda}^{3}(b), a], \tag{60}
\]
\[
d_{\lambda}^{3}(a \ast b) = \sum_{i=1}^{3} (-1)^{i}C_{i}^{1}\lambda^{i-1}b \circ d_{\lambda}^{i}(a) + d_{\lambda}^{i}(b) \circ a + d_{\lambda}^{i}(b) \ast a + 2[d_{\lambda}^{i}(b), a], \tag{61}
\]
\[
\sum_{i=1}^{3} (-1)^{i}C_{i}^{2}\lambda^{i-2}b \circ d_{\lambda}^{i}(a) + d_{\lambda}^{i}(b) \circ a + d_{\lambda}^{i}(b) \ast a + [d_{\lambda}^{i}(b), a] = 0, \tag{62}
\]
\[
\lambda d_{\lambda}^{i}(b \circ a) + d_{\lambda}^{i}(b \circ a) + d_{\lambda}^{i}[b, a] = \sum_{i=0}^{3} (-1)^{i}\lambda^{i}b \circ d_{\lambda}^{i}(a) + d_{\lambda}^{i}(b) \circ a + [d_{\lambda}^{i}(b), a], \tag{63}
\]
If $n > 3$, comparing the coefficients of $\mu^{n-1}\alpha^2$, $\mu^2\alpha^{n-1}$ in (66) and (67), we obtain the following equalities:

$$nd_{\lambda}^n(b) \circ a + C_n^2 d_{\lambda}^n(b) \ast a = 0, \quad C_n^2 d_{\lambda}^n(b) \circ a + nd_{\lambda}^n(b) \ast a = 0. \tag{68}$$

Since $n > 3$, by (68), we get $d_{\lambda}^n(b) \circ a = d_{\lambda}^n(b) \ast a = 0$ for any $a, b \in V$. Since $d_{\lambda}^n(b) \ast a = d_{\lambda}^n(b) \circ a + a \circ d_{\lambda}^n(b)$, we also have $a \circ d_{\lambda}^n(b) = 0$. 

(2) If $n > 3$, then $d_{\lambda}^n(b) \circ a = a \circ d_{\lambda}^n(b) = 0$.

**Proof:** Since $d$ is a conformal derivation of $R$, by (51) and (52), we obtain

$$d_{\lambda}[a_{\mu} b] = d_{\lambda}(\alpha(b \circ a) + \mu(a \ast b) + [b, a])$$

$$= (\lambda + \alpha) d_{\lambda}(b \circ a) + \mu d_{\lambda}(a \ast b) + d_{\lambda}[b, a],$$

$$= (\lambda + \alpha) \sum_{i=0}^{n-1} \alpha^i d_{\lambda}^i(b \circ a) + \mu \sum_{i=0}^{n-1} \alpha^i d_{\lambda}^i(a \ast b) + \sum_{i=0}^{n-1} \alpha^i d_{\lambda}^i[b, a] \tag{66}$$

and

$$[(d_{\lambda} a)_{\lambda + \mu} b] + [a_{\mu}(d_{\lambda} b)]$$

$$= \left[ \left( \sum_{i=0}^{n-1} \alpha^i d_{\lambda}^i(a) \right)_{\lambda + \mu} b \right] + \left[ a_{\mu} \left( \sum_{i=0}^{n-1} \alpha^i d_{\lambda}^i(b) \right) \right]$$

$$= \sum_{i=0}^{n} (-\lambda - \mu)^i [d_{\lambda}^i(a)_{\lambda + \mu} b] + \sum_{i=0}^{n} (\lambda + \alpha)^i [a_{\mu} d_{\lambda}^i(b)]$$

$$= \sum_{i=0}^{n} (-\lambda - \mu)^i (\alpha(b \circ d_{\lambda}^i(a)) + (\lambda + \mu) d_{\lambda}^i(a) \ast b + [b, d_{\lambda}^i(a)])$$

$$+ \sum_{i=0}^{n} (\mu + \alpha)^i (\alpha(d_{\lambda}^i(b) \circ a) + \mu d_{\lambda}^i(b) \ast a + [d_{\lambda}^i(b), a]). \tag{67}$$

If $n > 3$, comparing the coefficients of $\mu^{n-1}\alpha^2$, $\mu^2\alpha^{n-1}$ in (66) and (67), we obtain the following equalities:

$$nd_{\lambda}^n(b) \circ a + C_n^2 d_{\lambda}^n(b) \ast a = 0, \quad C_n^2 d_{\lambda}^n(b) \circ a + nd_{\lambda}^n(b) \ast a = 0. \tag{68}$$
If $n \leq 3$, assume
\[
\mu \partial^d \lambda \frac{\partial^3}{\partial^3} \sum_{i=0}^{3} \partial^i d^i \lambda (a), \quad \text{for any} \ a \in V.
\tag{69}
\]
Taking (69) into (66) and (67) and by comparing the coefficients of $\partial^4, \partial^3 \mu, \partial^2 \mu^2, \partial \mu^3, \mu^4$, we can get
\[
\begin{align*}
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (b \circ a) &= d^3 \lambda (b) \circ a, \\
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (a * b) &= 3d^3 \lambda (b) \circ a + d^3 \lambda (b) * a, \\
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (b \circ a) &= d^3 \lambda (b) * a = 0, \\
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (b) \circ a &= 0, \\
- b \circ d^3 \lambda (a) + d^3 \lambda (b) \circ a + 3d^3 \lambda (b) * a &= 0, \\
- d^3 \lambda (a) * b + d^3 \lambda (b) * a &= 0.
\end{align*}
\tag{70} \tag{71} \tag{72} \tag{73} \tag{74}
\]
It is easy to check that (53)–(55) are equivalent to (70)–(74). Similarly, by comparing the coefficients of $\partial^3, \partial^2 \partial, \mu^2 \partial, \mu^3, \partial^2, \mu \partial, \mu^2, \partial \mu$ and $\mu \partial ^0 \partial$, we can immediately obtain the equalities (66)–(65).

By now, the proof is finished. $\blacksquare$

**Remark 4.4:** By the definition of quadratic Lie conformal algebra, if there exists some nonzero $d^i$ in Theorem 4.3 for $i \geq 2$, then there are noninner conformal derivations of this Lie conformal algebra.

**Corollary 4.5:** Let $R = \mathbb{C} [\mathfrak{g}] V$ be the quadratic Lie conformal algebra corresponding to a finite-dimensional Gel’fand–Dorfman bialgebra $(V, \circ, [\cdot, \cdot])$. Then we have

1. If $(V, \circ)$ is a simple Novikov algebra, then any conformal derivation $d$ of $R$ is of the form $d \lambda \frac{\partial^3}{\partial^3} (a) = \sum_{i=0}^{3} \partial^i d^i \lambda (a)$ for any $a \in V$, where $d^i \lambda (a) \in V[\lambda]$ for $i \in \{0, 1, 2, 3\}$ satisfy (53)–(65).

2. If for $(V, \circ)$, there exists $x \in V$ such that $b \circ x = kb$ for any $b \in V$ and some fixed $k \in \mathbb{C} \setminus \{0\}$, then any conformal derivation $d$ of $R$ is of the form $d \lambda \frac{\partial^3}{\partial^3} (a) = d_0 \lambda \frac{\partial^3}{\partial^3} (a) + \partial d^1 \lambda (a)$ for any $a \in V$, where $d_0 \lambda \frac{\partial^3}{\partial^3} (a), d^1 \lambda \frac{\partial^3}{\partial^3} (a) \in V[\lambda]$ and they satisfy
\[
\begin{align*}
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (b \circ a) &= d^1 \lambda (b) \circ a, \\
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (a * b) &= d^1 \lambda (b) * a, \\
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (b \circ a) &= d^1 \lambda (b) * a = 0, \\
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (b) \circ a &= 0, \\
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (b) * a &= 0, \\
\mu \partial^3 \lambda \frac{\partial^3}{\partial^3} (b) &\in V[\lambda] \text{ and they satisfy}
\end{align*}
\tag{75} \tag{76} \tag{77} \tag{78}
\]
for any $a, b \in V$. Moreover, if $(V, \circ)$ also satisfies that there exists some $y \in V, y \circ V = V$, then any conformal derivation $d$ of $R$ is of the form $d = D + \tilde{d}^0$ where $D$ is an inner conformal derivation and $\tilde{d}^0$ satisfies
\[
\tilde{d}^0 \lambda \frac{\partial^3}{\partial^3} (b \circ a) = b \circ \tilde{d}^0 \lambda \frac{\partial^3}{\partial^3} (a) + \tilde{d}^0 \lambda \frac{\partial^3}{\partial^3} (b) \circ a,
\tag{79}
\]

\[ \lambda \overline{d}_\lambda^0(b \circ a) + \overline{d}_\lambda^0[b, a] = \lambda \overline{d}_\lambda^0(a) + b + [b, \overline{d}_\lambda^0(a)] + [\overline{d}_\lambda^0(b), a], \] (80)

for any \( a, b \in V \).

(3) If for \((V, \circ)\), there exists \( x \in V \) such that \( x \circ b = kb \) for any \( b \in V \) and some fixed \( k \in \mathbb{C} \setminus \{0\} \), then any conformal derivation \( d \) of \( R \) is of the form \( d = D + \overline{d}^0 \) where \( D \) is an inner derivation and \( \overline{d}^0 \) satisfies (79) and (80).

**Proof:** Since \( R = \mathbb{C}[\partial]V \) is a finitely generated and free \( \mathbb{C}[\partial] \)-module, we can assume that \( d_\lambda(a) = \sum_{i=0}^\infty \partial^i d_\lambda^i(a) \) for any \( a \in V \) and some non-negative integer \( n \). If \( n > 3 \), by Theorem 4.3 and the condition that \( V \) is a simple Novikov algebra or there exists \( x \in V \) such that \( x \circ y = ky \) or \( y \circ x = ky \) for any \( y \in V \) and \( k \in \mathbb{C} \setminus \{0\} \), we can get \( d_\lambda^0(b) = 0 \) for any \( b \in V \). Therefore, for any \( a \in V \), we can assume that \( d_\lambda(a) = \sum_{i=0}^\infty \partial^i d_\lambda^i(a) \) by Theorem 4.3. (1) can be directly obtained.

Next, we prove (2). According to the above discussion, we get \( d_\lambda(a) = \sum_{i=0}^3 \partial^i d_\lambda^i(a) \) for any \( a \in V \). By Theorem 4.3, (53)–(65) hold. Setting \( a = b = x \) in (60), we get \( k\lambda \overline{d}_\lambda^3(x) = [d_\lambda^2(x), x] \). Since \( k \neq 0 \), we can obtain \( d_\lambda^2(x) = 0 \). Letting \( a = b = x \) in (57) we can get \([d_\lambda^3(x), x] = 0 \). Then it is easy to get that \( x \circ d_\lambda^3(x) = 0 \) from (58). Therefore, setting \( a = b = x \) in (59), one can obtain \( d_\lambda^3(x) \circ x = k\overline{d}_\lambda^3(x) = 0 \). Consequently, \( d_\lambda^3(x) = 0 \). Letting \( a = x \) in (54), we get \( x \circ d_\lambda^1(b) = 0 \) for any \( b \in V \). Setting \( a = x \) in (55), one can directly obtain \( d_\lambda^3(b) = 0 \) for any \( b \in V \). Therefore, we get \( d_\lambda(a) = \sum_{i=0}^2 \partial^i d_\lambda^i(a) \) for any \( a \in V \). Then by Theorem 4.3, we get

\[
\begin{align*}
d_\lambda^2(b \circ a) &= d_\lambda^2(b) \circ a, \quad (81) \\
d_\lambda^2(a \circ b) &= 2d_\lambda^2(b) \circ a + d_\lambda^2(b) \circ a, \quad (82) \\
b \circ d_\lambda^2(a) + d_\lambda^2(b) \circ a + 2d_\lambda^2(b) \circ a &= 0, \quad (83) \\
d_\lambda^2(a \circ b) + d_\lambda^2(b) \circ a &= 0. \quad (84)
\end{align*}
\]

Letting \( a = x \) in (84), we can have \( d_\lambda^2(b) \circ x = 0 \). Then according to (83) with \( a = x \), \( kd_\lambda^3(b) = 0 \). Therefore, \( d_\lambda^2(b) = 0 \) for any \( b \in V \). Therefore, \( d_\lambda(a) = d_\lambda^0(a) + \partial d_\lambda^1(a) \) for any \( a \in V \). By Theorem 4.3, we get that \( d^0 \) and \( d^1 \) satisfy (75)–(78). Set \( d_\lambda^1(y) = \sum_{i=0}^m \lambda^i y_i \) where \( y_i \in V \) for \( i \in \{0, 1, 2, \ldots, m\} \). If \( y \circ V = V \), then there exist some \( b_i \) such that \( y \circ b_i = y_i \) for all \( i \). It is easy to see that \( (d - \text{ad}(\sum_{i=0}^m (-\partial^i b_i)))y) \in V[\lambda] \). Therefore, we can assume that \( d_\lambda^1(y) = 0 \). Then letting \( b = y \) in (75) and by \( y \circ V = V \), we can get \( d^1 = 0 \). Therefore, \( d = D + \overline{d}^0 \) where \( D \) is an inner conformal derivation. Obviously, \( \overline{d}^0 \) is a conformal derivation and \( \overline{d}_\lambda^0(a) \in V[\lambda] \) for any \( a \in V \). Therefore, by (77) and (78), \( \overline{d}^0 \) satisfies (79) and (80).

Finally, we prove (3). Similarly, we can set \( d_\lambda(a) = \sum_{i=0}^3 \partial^i d_\lambda^i(a) \) for any \( a \in V \). Letting \( a = x \) in (53) and (55), we get \( d_\lambda^3(b) \circ x = k d_\lambda^3(b) \) and \( 2d_\lambda^3(b) \circ x + k d_\lambda^3(b) = 0 \). Since \( k \neq 0 \), we get \( d_\lambda^3(b) = 0 \) for any \( b \in V \). By (81), from (82), we can get

\[
d_\lambda^2(a \circ b) = 2d_\lambda^2(b) \circ a + a \circ d_\lambda^2(b). \quad (85)
\]

Setting \( a = x \) in (85), we get \( d_\lambda^2(b) \circ x = 0 \). Then letting \( a = x \) in (83), we obtain

\[
b \circ d_\lambda^2(x) + 2kd_\lambda^2(b) = 0, \quad \text{for any } b \in V. \quad (86)
\]
Letting \( b = x \) in (86), we get \( 3kd^2_k(x) = 0 \). Therefore, \( d^2_k(x) = 0 \). Then by (86), we get \( d^2_k(b) = 0 \) for any \( b \in V \). Therefore, \( d_k(a) = d^0_k(a) + \partial d^1_k(a) \) for any \( a \in V \). Then \( d^0 \) and \( d^1 \) satisfy (75)–(78). Since \( x \circ V = V \), with a similar discussion as that in the proof of (2), we may assume that \( d^1 = 0 \). Therefore, \( d = D + \tilde{d}^0 \), where \( D \) is an inner conformal derivation and \( \tilde{d}^0 \) satisfies (79) and (80).

By now, the proof is finished.

**Corollary 4.6:** Let \( R = \mathbb{C}[\partial]V \) be the quadratic Lie conformal algebra corresponding to a Novikov algebra \( (V, \circ) \). Then we have

1. If \( V \) is a simple Novikov algebra, then any conformal derivation \( d \) of \( R \) must be of the following form: \( d_k(a) = \sum_{i=0}^3 \delta^i d^i_k(a) \) for any \( a \in V \), where \( d^i_k(a) \in V[\lambda] \) for \( i \in \{0, 1, 2, 3\} \) satisfy (53)–(65) with \([\cdot, \cdot]\) trivial.
2. If for \( (V, \circ) \), there exists \( x \in V \) such that \( b \circ x = kb \) for any \( b \in V \) and some fixed \( k \in \mathbb{C} \setminus \{0\} \), then any conformal derivation \( d \) of \( R \) is of the form \( d_k(a) = d^0_k(a) + \partial d^1_k(a) \) for any \( a \in V \), where \( d^0_k(a), d^1_k(a) \in V[\lambda] \) and they satisfy (75), (76) and

\[
\begin{align*}
\tilde{d}^0_k(b \circ a) + \lambda \tilde{d}^1_k(b \circ a) &= b \circ \tilde{d}^0_k(a) - \lambda b \circ \tilde{d}^1_k(a) + d^0_k(b) \circ a, \\
\tilde{d}^0_k(b \circ a) &= d^0_k(a) \ast b - \lambda d^1_k(a) \ast b,
\end{align*}
\]

for any \( a, b \in V \). Moreover, if \( (V, \circ) \) also satisfies that there exists some \( y \in V, y \circ V = V \), then any conformal derivation \( d \) of \( R \) is of the form \( d = D + \tilde{d}^0 \) where \( D \) is an inner conformal derivation and \( \tilde{d}^0 \) satisfies

\[
\begin{align*}
\tilde{d}^0_k(b \circ a) &= b \circ \tilde{d}^0_k(a) + \tilde{d}^0_k(b) \circ a, \\
\tilde{d}^0_k(b) \circ a &= \tilde{d}^0_k(a) \circ b,
\end{align*}
\]

for all \( a, b \in V \).

3. If for \( (V, \circ) \), there exists \( x \in V \) such that \( x \circ b = kb \) for any \( b \in V \) and some fixed \( k \in \mathbb{C} \setminus \{0\} \), then any conformal derivation \( d \) of \( R \) is of the form \( d = D + \tilde{d}^0 \) where \( D \) is an inner conformal derivation and \( \tilde{d}^0 \) satisfies (89) and (90). Moreover, if the element \( x \) also satisfies that for any \( b \in V \) and the same number \( k, b \circ x = kb \), then \( \text{CDer}(R) = \text{CInn}(R) \).

**Proof:** For any \( a \in V \), set \( d_k(a) = \sum_{i=0}^{n} \delta^i d^i_k(a), \) where \( d^i_k(a) \in V[\lambda] \) for \( i \in \{0, 1, \ldots, n_a\} \), and \( n_a \) is a non-negative integer depending on \( a \). With the same process as in Theorem 4.3, (51) becomes

\[
(\lambda + \partial) d_k(b \circ a) + \mu d_k(a \ast b) = [(d_k(a)\lambda + \mu) + [a_\mu(d_k(b))].
\]

For fixed \( a, b \), there are four elements of \( V \) under the actions of \( d_k \) in (91). Therefore, we may assume the degrees of \( \partial \) in \( d_k(b \circ a), d_k(a \ast b), d_k a \) and \( d_k b \) in (91) are smaller than some non-negative integer. So, we set \( d_k(b \circ a) = \sum_{i=0}^{n} \delta^i d^i_k(b \circ a), \) and \( d_k(b) = \sum_{i=0}^{n} \delta^i d^i_k(b) \). Obviously, \( n \) depends on \( a \) and \( b \).

Taking them into (91), we get

\[
(\lambda + \partial) \sum_{i=0}^{n} \delta^i d^i_k(b \circ a) + \mu \sum_{i=0}^{n} \delta^i d^i_k(a \ast b)
\]
\[
\sum_{i=0}^{n} (-\lambda - \mu)^i (\partial (b \circ d_{\lambda}^i (a)) + (\lambda + \mu) d_{\lambda}^i (a) * b) \\
+ \sum_{i=0}^{n} (\mu + \partial)^i (\partial (d_{\lambda}^i (b) \circ a) + \mu d_{\lambda}^i (b) * a).
\] (92)

If \( n > 3 \), by comparing the coefficients of \( \mu^{n-1}\partial^2 \) and \( \mu^2 \partial^{n-1} \) in (92), we get

\[
nd_{\lambda}^m (b) \circ a + C_n^2 d_{\lambda}^n (b) * a = 0, \quad C_n^2 d_{\lambda}^n (b) \circ a + nd_{\lambda}^n (b) * a = 0.
\]

Therefore, \( d_{\lambda}^n (b) \circ a = 0 \) and \( d_{\lambda}^m (b) * a = 0 \). Repeating this process, we can get \( d_{\lambda}^m (b) \circ a = d_{\lambda}^m (b) * a = 0 \) for all \( n \geq m > 3 \).

By the discussion above, for any \( a, b \in V \), we get \( d_{\lambda}^m (b) \circ a = a \circ d_{\lambda}^m (b) = 0 \) for all \( m > 3 \). By the condition that \( V \) is a simple Novikov algebra, or there exists \( x \in V \) such that \( x \circ y = ky \) or \( y \circ x = ky \) for any \( y \in V \) and \( k \in \mathbb{C}\setminus\{0\} \), we get \( d_{\lambda}^m (b) = 0 \) for any \( b \in V \) and \( m > 3 \). Therefore, we can assume that \( d_{\lambda} (a) = \sum_{i=0}^{3} \partial^i d_{\lambda}^i (a) \), for any \( a \in V \). Then (1), (2) and the first part of (3) can be directly obtained from Corollary 4.5.

Finally, we prove the second part of (3). By the first part of (3), we only need to determine \( \tilde{d}^0 \). For computing \( \tilde{d}^0 \), we only need to compute the operator \( T : V \rightarrow V \) such that

\[
T(b \circ a) = b \circ T(a) + T(b) \circ a,
\] (93)

\[
T(b) \circ a = T(a) \circ b,
\] (94)

for any \( a, b \in V \). Letting \( a = b = x \) in (93), we can directly obtain that \( T(x) = 0 \). Setting \( a = x \) in (94), we get that \( T(b) = 0 \) for any \( b \in V \). Therefore, \( \tilde{d}^0 = 0 \). Thus, all conformal derivations of \( R \) are inner.

**Remark 4.7:** Note that this corollary also holds when \( V \) is infinite-dimensional. Moreover, by (3) in Corollary 4.6, all conformal derivations of the quadratic Lie conformal algebra corresponding to the Novikov algebra with a unit are inner.

**Remark 4.8:** It should be pointed out that when \( \mathfrak{g} \) is a finite-dimensional Lie algebra, every conformal derivation \( d \) of \( \text{Curg} \) is of the form \( d_{\lambda} (a) = p(\lambda) (\partial + \lambda) a + \tilde{d}_a (a) \), where \( \tilde{d} \) is an inner conformal derivation and \( p(\lambda) \in \mathbb{C}[\lambda] \). This characterization can be referred to [11].

Finally, we will use the above results to study conformal derivations of some specific Lie conformal algebras.

**Example 4.9:** It is known that \( \mathfrak{vir} \) is the quadratic Lie conformal algebra corresponding to \( (V = \mathbb{C} L, \circ) \) where \( L \circ L = L \). Obviously, \( L \) is a unit of \( V \). By (3) in Corollary 4.6, \( \text{CDer(Vir)} = \text{Clinn(Vir)} \). This result can also be found in [11].

**Example 4.10:** The corresponding Novikov algebra of loop Virasoro conformal algebra \( \mathcal{L}V = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\delta] L_i \) introduced in Example 3.13 is \( V = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} L_i \) with the product \( L_i \circ L_j = -L_{i+j} \) for any \( i, j \in \mathbb{Z} \). Since \( L_0 \circ L_j = -L_j \) and \( L_j \circ L_0 = -L_j \) for any \( j \in \mathbb{Z} \), by (3) in Corollary 4.6, \( \text{CDer} (\mathcal{L}V) = \text{Clinn} (\mathcal{L}V) \). This is a result in [17].
**Example 4.11:** Let $R(\alpha, \beta)$ be the Lie conformal algebra given in Example 3.12. Next, let us consider conformal derivations of $R(\alpha, \beta)$.

Since $L \circ L = L$, $W \circ L = W$, by Corollary 4.5, we get $n \leq 1$. Therefore, by Corollary 4.5, we only need to determine $d^0$ and $d^1$ satisfying (75)–(78). According to that $L \circ L = L$ and $L \circ W = (\alpha - 1)W$, by (2) in Corollary 4.5, we can discuss it in two cases, i.e. $\alpha \neq 1$ and $\alpha = 1$.

When $\alpha \neq 1$, we have $L \circ R(\alpha, \beta) = R(\alpha, \beta)$. Then by (2) in Corollary 4.5, we only need to determine $\tilde{d}^0$ satisfying (79) and (80). By (79), we first study the operator $T : V \to V$ satisfying $T(b \circ a) = b \circ T(a) + T(b) \circ a$. It is easy to check that $T$ is of the form: $T(L) = 0$ and $T(W) = c_2 W$ for some $c_2 \in \mathbb{C}$. Therefore, $\tilde{d}^0_\chi(L) = 0$ and $\tilde{d}^0_\chi(W) = a(\lambda)W$ for some $a(\lambda) \in \mathbb{C}[\lambda]$. Replacing $a, b$ by $L, W$ in (80), we can easily obtain $a(\lambda) = 0$. Therefore, $\tilde{d}^0_\chi(L) = 0$, $\tilde{d}^0_\chi(W) = 0$ and (80) holds for other cases. So, in this case, $\text{CDer}(R(\alpha, \beta)) = \text{Clinn}(R(\alpha, \beta))$.

Finally, we consider the case when $\alpha = 1$. First, we consider $d^1$. For obtaining $d^1$, by (75) and (76), we only need to study the operator $T : V \to V$ satisfying the following equalities:

$$T(b \circ a) = T(b) \circ a,$$
$$T(a) \ast b = T(b) \ast a.$$  

By a simple computation, $T$ is of the following form: $T(L) = a_1 L + a_2 W$ and $T(W) = a_1 W$ for any $a_1, a_2 \in \mathbb{C}$. Therefore, $d^1$ is of the form as follows: $d^1_\chi(L) = A(\lambda) L + B(\lambda) W$, $d^1_\chi(W) = A(\lambda) W$ for any $A(\lambda), B(\lambda) \in \mathbb{C}[\lambda]$. Let $D = \text{ad}(A(-\delta)L)$. Then it is easy to check that $D = D^0 + \delta D^1$ where $D^0(\alpha) \in \mathbb{C}[V]$ for any $i \in \{0, 1\}$. Then, we first consider the operator $T : V \to V$ satisfying $T(b \circ a) = b \circ T(a) + T(b) \circ a$. It is easy to check that $T$ is of the form: $T(L) = d_2 W$ and $T(W) = e_2 W$ for some $d_2, e_2 \in \mathbb{C}$. Therefore, $D^0_\chi(L) = F(\lambda) W$ and $D^0_\chi(W) = G(\lambda) W$ for some $F(\lambda), G(\lambda) \in \mathbb{C}[\lambda]$. Replacing $a, b$ by $L, W$ in (80), we can easily obtain $G(\lambda) = 0$. Therefore, $D^0_\chi(L) = F(\lambda) W$, $D^0_\chi(W) = 0$ and (80) holds for other cases. If $F(\lambda) = \sum_{i=0}^m a_i (\lambda - \beta)^i$, then let $\gamma(\lambda) = \sum_{i=0}^{m-1} a_{i+1} (\lambda - \beta)^i$. Then, $D^0 = \text{ad}(\gamma(-\delta)L) + Q$, where $Q_\chi(L) = a_0 W$, $Q_\chi(W) = 0$. Thus, $\text{CDer}(R(1, \beta)) = \text{Clinn}(R(1, \beta)) \oplus M$, where $M$ is the vector space spanned by $Q$, where $Q_\chi(L) = W$, $Q_\chi(W) = 0$.

Therefore, by the discussion above, $\text{CDer}(R(\alpha, \beta)) = \text{Clinn}(R(\alpha, \beta)) \oplus M$, where $M$ is the vector space spanned by $Q$, where $Q_\chi(L) = \delta_{\alpha, 1} W$, $Q_\chi(W) = 0$.

**Example 4.12:** Let $R = \bigoplus_{i \geq -1} \mathbb{C}[\partial] L_i$ be a Lie conformal algebra with the following $\lambda$-bracket:

$$[L_i, L_j] = ((i + 1)\partial + (i + j + 2)\lambda)L_{i+j}, \quad \text{for } i, j \geq -1.$$  

It is the graded algebra of general conformal algebra $gc_1$ (see [24]).

Obviously, it is the quadratic Lie conformal algebra corresponding to a Novikov algebra $(V = \bigoplus_{i \geq -1} \mathbb{C} L_i, \circ)$ where

$$L_i \circ L_j = (j + 1)L_{i+j}, \quad \text{for } i, j \geq -1.$$  

(98)
Since \( L_i \circ L_0 = L_i \) for all \( i \geq -1 \), and \( L_{-1} \circ V = V \), by (2) in Corollary 4.6, we only need to determine \( \tilde{d}^0 \) satisfying (89) and (90). In fact, by (89) and (90), it can be changed into find all operations \( T: V \rightarrow V \) satisfying

\[
T(a \circ b) = T(a) \circ b + a \circ T(b), \quad T(a) \circ b = T(b) \circ a. \tag{99}
\]

Let \( T: V \rightarrow V \) be an operator satisfying (99). Define \( T_i(L_j) = \pi_{i+j} T(L_j) \) where in general \( \pi_i \) is the natural projection from \( V \) onto \( L_i \). Then \( T_i \) is an operator satisfying (99) and \( T = \sum_{i \geq -1} T_i \) in the sense that for any \( x \in V \) only finitely many \( T_i(x) \neq 0 \). Therefore, set \( T_c(L_i) = f(i)L_{i+c} \). Replacing \( T \) by \( T_c \), \( a \) by \( L_i \) and \( b \) by \( L_j \) in (99), and comparing the coefficients of \( L_{i+j+c} \), we obtain

\[
(j + 1)f(i + j) = f(i)(j + 1) + f(j)(j + c + 1), \tag{100}
\]

\[
f(i)(j + 1) = f(j)(i + 1), \quad \text{for } i, j \geq -1. \tag{101}
\]

By (101), we can get \( f(i) = A(i + 1) \) for all \( i \) and some \( A \in \mathbb{C} \). Letting \( i = 0 \) in (100), we immediately get \( c = -1 \) or \( f(j) = 0 \) for all \( j \). Therefore, \( T_c = 0 \), if \( c \neq -1 \) and \( T_{-1} = A(i + 1)L_{i-1} \). Thus, \( T(L_i) = A(i + 1)L_{i-1} \) for some \( A \in \mathbb{C} \). So, \( d^0_i(L_i) = a(\lambda)(i + 1)L_{i-1} \) for any \( a(\lambda) \in \mathbb{C}[\lambda] \). If \( a(\lambda) = \sum_{i=0}^m a_i \lambda^i \), set \( b(\lambda) = \sum_{i=0}^{m-1} a_i \lambda^i \). Then, \( d = \text{ad}(b(-\partial)L_{-1}) + Q \), where \( Q_i(L_i) = a_0(i + 1)L_{i-1} \) for any \( i \geq -1 \).

Therefore, by the discussion above, \( \text{CDer}(R) = M \oplus \text{Clmn}(R) \), where \( M \) is the vector space spanned by \( Q \), where \( Q_i(L_i) = (i + 1)L_{i-1} \) for any \( i \geq -1 \).

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