Geodesic motion in Bogoslovsky-Finsler Spacetimes

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Abstract

We study the free motion of a massive particle moving in the background of a Finslerian deformation of a plane gravitational wave in Einstein’s General Relativity. The deformation is a curved version of a one-parameter family of Relativistic Finsler structures introduced by Bogoslovsky, which are invariant under a certain deformation of Cohen and Glashow’s Very Special Relativity group ISIM(2). The partially broken Carroll Symmetry we derive using Baldwin-Jeffery-Rosen coordinates allows us to integrate the geodesics equations. The transverse coordinates of timelike Finsler-geodesics are identical to those of the underlying plane gravitational wave for any value of the Bogoslovsky-Finsler parameter $b$. We then replace the underlying plane gravitational wave by a homogenous pp-wave solution of the Einstein-Maxwell equations. We conclude by extending the theory to the Finsler-Friedmann-Lemaître model.

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I. INTRODUCTION

Our present fundamental physical theories are based on local Lorentz invariance and hence on local isotropy. This leads naturally to the introduction of pseudo-Riemannian geometry and its associated metric tensor. It has long been known however that a "Principle of
Relativity” is compatible with anisotropy by deforming the Lorentz group by the inclusion of
dilations \[1\] (although the experiments of Hughes, and of Drever indicate that the anisotropy
must be very week \[2\]).

Currently there is also a great deal of activity exploring the astrophysics and cosmology
of alternative gravitational theories based on standard Lorentzian geometry. Laboratory
tests of local Lorentz invariance are very well developed and have reached impressive levels
of precision.

Riemann himself envisaged more general geometries. An elegant construction combing
these ideas was provided some time ago by Bogoslovsky \[3, 4\] (for more recent accounts see
\[5\]). In what is now known as Finsler geometry, the line element is a general homogeneous
function of degree one in displacements, rather than the square root of a quadratic form.

The theory proposed by Bogoslovsky, which is the main subject of interest of this pa-
per, has turned out to be relevant for attempts to accommodate a proposal of Cohen and
Glashow \[1\], accounting for weak CP violation in the standard model of particle physics, in
gravitational background \[6, 7\].

The first significant application of Finsler geometry to physics is due to Randers \[8\] who
pointed out that the world line of a particle of mass \(m\) and electric charge \(e\) extremizes the action

\[
S_0 = \int \mathcal{L}_0 d\lambda = -\int m \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} + eA_\mu dx^\mu ,
\]

where \(\lambda\) is an arbitrary parameter and \(A_\mu\) is the electro-magnetic potential. Randers applied
this idea to Kaluza-Klein theory. Further studies followed \[9–12\] ; it has also been applied
to the gravitomagnetic effects occurring in stationary spacetimes \[13\]. For more recent work
on Finsler spaces, see \[14–21\].

The aim of the present paper is to contribute to the physical applications of the Finsle-
rian generalization of General Relativity by exploring the motion of freely moving massive
particles in the background of Bogoslovsky-Finsler deformations of plane gravitational waves
and spatially flat Friedmann-Lemaître cosmologies.

II. FINSLER SPACES

In this section we shall briefly summarise some earlier results \[9–12\]. An excellent general
reference to Finsler Geometry used by these authors is \[22\] to which we refer the reader for
more details of the general theory. If $F(x^\mu, y^\nu)$ is a Finsler function \footnote{See [9, 11, 12, 22] $y^\mu$ is a four velocity and $(x^\mu, y^\nu)$ are local coordinates on $TM$, the tangent bundle of the spacetime manifold $M$.} then $F^2(x, y)$ may be written such that [9]

$$F^2(x, y) = \mathcal{F}(x, y) g_{\mu\nu} y^\mu y^\nu,$$

(II.1)

where $\mathcal{F}(x, y)$ is homogenous degree 0 in $y^\mu$. Moreover [9] if $H(x)_{\alpha_1, \alpha_2, \ldots, \alpha_N}$ is a totally symmetric tensor of rank $N$ which is covariantly constant with respect to the Levi-Civita covariant derivative of the metric $g_{\mu\nu}$, then

$$\omega = g(x)_{\mu\nu} y^\mu y^\nu / (H_{\alpha_1, \alpha_2, \ldots, \alpha_N} y^{\alpha_1} y^{\alpha_2} \ldots y^{\alpha_N})^{2/N}$$

and $\mathcal{F}(x, y) = \mathcal{F}(\omega)$

(II.2)

then the the set of Finsler geodesics of $F$ and the set of standard Riemannian geodesics of $g_{\mu\nu}$ coincide ([9] eqn. 20.)

The case of Finsler-pp waves [15, 16] is when $H$ is a covariantly constant null co-vector and $g_{\mu\nu}$ the metric of a pp-wave, a special case of which is a plane gravitational wave. This was in effect pointed by Tavakol and Van der Bergh [12] in 1986 and elaborated and extended by Roxburgh in 1991. Bogoslovsky’s original flat Finsler metric [3, 4] is a special case of their work but no mention of Bogoslovsky is made in [9, 11, 12] and so one assumes that they were unaware of it.

We next recall some basic definitions and notation used in [9, 11, 12]. Given a Finsler function $F(x, y)$ one may define the Finsler metric tensor

$$f_{\mu\nu}(x, y) = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^\mu \partial y^\nu},$$

(II.3)

which is homogeneous of degree zero in $y^\mu$. That is, $f_{\mu\nu}(x, y)$ depends only upon the direction. Differentiating the identity $F^2(x^\alpha, \lambda y^\mu) = \lambda^2 F(x^\alpha, y^\mu)$, twice with respect to $\lambda$ implies that

$$f_{\mu\nu} y^\mu y^\nu = F^2(x, y).$$

(II.4)

The Finsler line element or arc length $ds$ along a curve $\gamma$ with tangent vector $y^\mu = \frac{dx^\mu}{d\lambda}$ is given by

$$ds^2 = F^2(x^\mu, dx^\mu) = f_{\mu\nu}(x, y) dx^\mu dx^\nu,$$

(II.5)

and a Finsler geodesic is one for which $\delta \int_{\gamma} F(x, dx^\mu) = \delta \int_{\gamma} ds = 0$. The Euler-Lagrange equations are

$$\frac{d^2 x^\mu}{ds^2} + \gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0,$$

(II.6)
where
\[
\gamma_{\mu\nu\kappa} = \frac{1}{2} \left( \frac{\partial f_{\mu\nu}}{\partial x^\kappa} + \frac{\partial f_{\mu\kappa}}{\partial x^\nu} - \frac{\partial f_{\nu\kappa}}{\partial x^\mu} \right), \quad \gamma^\mu_{\nu\kappa} = f^{\mu\sigma} \gamma_{\nu\sigma\kappa}
\] (II.7)
are the analogue of Christoffel symbols of the first and second kind, respectively. In deriving the Euler-Lagrange equations one uses the fact \( g^\kappa y^\nu \frac{\partial f_{\alpha\beta}}{\partial y^\kappa} = 0 \) because \( f_{\mu\nu} \) is homogeneous of degree 0 in \( y^\mu \). Evidently under a change of parameter \( s \to \lambda = \lambda(s) \) we have \( \frac{d}{ds} = \lambda' \frac{d}{d\lambda} \), \( f_{\mu\nu} \to f_{\mu\nu} \) since \( f_{\mu\nu} \) is homogenous degree zero in velocities. Thus, as in the standard Lorentzian situation,
\[
\frac{d^2 x^\mu}{d\lambda^2} + \gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -\frac{\lambda''}{\lambda'} \frac{dx^\mu}{d\lambda}.
\] (II.8)
If \( \lambda'' = 0 \), \( \lambda \) is called an affine parameter and in what follows, unless otherwise stated, \( \lambda \) will denote an affine parameter.

In [9, 11, 12] the quantities
\[
G^\mu = \frac{1}{2} \gamma^\mu_{\nu\kappa} y^\nu y^\kappa \quad G^\mu_{\nu\kappa} = \frac{\partial^2 G^\mu}{\partial y^\nu \partial y^\kappa}
\] (II.9)
are introduced. Although in general \( G^\mu_{\nu\kappa} \neq \gamma^\mu_{\nu\kappa} \), by virtue of the homogeneity of degree zero of \( \gamma^\mu_{\nu\kappa} \) in \( y^\mu \) one has
\[
G^\mu_{\nu\kappa} y^\nu y^\kappa = \gamma^\mu_{\nu\kappa} y^\nu y^\kappa,
\] (II.10)
and therefore Euler-Lagrangian equations may be re-written as
\[
\frac{d^2 x^\mu}{d\lambda^2} + G^\mu_{\nu\kappa} \frac{dx^\nu}{d\lambda} \frac{dx^\kappa}{d\lambda} = 0.
\] (II.11)
In general \( G^\mu_{\nu\kappa} \) depends upon the direction [12] ; a Berwald-Finsler manifold is one for which \( G^\mu_{\nu\kappa} \) is independent of the direction, i.e.,
\[
G^\mu_{\nu\kappa} = G^\mu_{\nu\kappa}(x).
\] (II.12)

The motivation for the papers of [9, 11, 12] of was a classic paper of Ehlers, Pirani and Schild [23] examining the fundamental assumptions justifying the use of pseudo-Riemannian geometry adapted in Einstein’s General Relativity. Roughly speaking the idea was that

- **The Principle of Universality of free fall** endows spacetime \( \mathcal{M} \) with a projective structure, that is an equivalence class of curves, \( \gamma : \lambda \in \mathbb{R} \to \mathcal{M} \) up to reparametrisation.

- **The Principle of Einstein Causality** endows spacetime with a causal structure such that light rays are determined by some connection.
They conjectured that the only way of achieving this was that freely falling particles and null rays follow the geodesics of a pseudo-Riemannian metric, and in the case of particles that the curves carry a privileged parametrisation given by propertime with respect to the pseudo-Riemannian metric along their paths in spacet ime whether freely falling or not.

In [11, 12] Tavakol and Van den Bergh sought to show that one could as well pass to a Finsler structure provided one assumes

- \{Ai\}

\[ F^2(x, y) = e^{2\sigma(x,y)} g_{\mu\nu}(x) y^\mu y^\nu, \]  

(II.13)

where \( g_{\mu\nu}(x) \) is a Lorentzian metric and \( \sigma(x^\alpha, y^\mu) \) is homogeneous degree zero in \( y^\mu \).

This condition ensures that the conformal structures of the Finsler metric and the Lorentzian metric agree locally, in the spirit of [23]. It is pointed out in [12] that (II.13) is not equivalent to

\[ f_{\mu\nu} = e^{2\sigma(x,y)} g_{\mu\nu} \]  

(II.14)

because it this were true then \( \sigma \) would only depend upon \( x \) and hence \( f_{\mu\nu} \) and \( g_{\mu\nu} \) would be conformally related, citing [22].

- \{Aii\}

\[ G_{\nu\kappa}^\nu = \left\{ \begin{array}{c} \mu \\ \nu \kappa \end{array} \right\} \]  

(II.15)

where \( \left\{ \begin{array}{c} \nu \\ \nu \kappa \end{array} \right\} \) are the Christoffel symbols of the the Lorentzian metric \( g_{\mu\nu} \). This condition ensures that the projective structures of the Finsler structure \( F(x, y) \) and the Lorentzian structure \( g_{\mu\nu} \) agree locally, again in the spirit of [23].

Tavakol and Van den Bergh [12] claimed that the necessary and sufficient condition on \( \sigma(x, y) \) is

\[ \frac{\partial \sigma}{\partial x^\mu} - y^\nu \frac{\partial \sigma}{\partial y^\kappa} \left\{ \begin{array}{c} \kappa \\ \mu \nu \end{array} \right\} = 0. \]  

(II.16)

and refer to it as the metricity condition. The name originates in the theory of the so-called Cartan connection. One defines

\[ C_{\mu\nu\kappa} = \frac{1}{2} \frac{\partial f_{\mu\nu}}{\partial y^\kappa} \]  

(II.17)

\(^2\) In fact (II.13) is obviously equivalent to (II.1) which is the form used by [9] (who appears to regard it as always true, although \( g_{\mu\nu} \) is not necessarily unique.)
which is from (II.3) totally symmetric in $\mu, \nu, \kappa$. Then one defines
\[
\Gamma_{\mu\nu\kappa} = \gamma_{\mu\nu\kappa} - (C_{\sigma\kappa\nu} \frac{\partial G}{\partial y^\mu} + C_{\sigma\nu\mu} \frac{\partial G}{\partial y^\nu} - C_{\mu\sigma\nu} \frac{\partial G}{\partial y^\kappa}).
\]
(II.18)

Acting on a vector $W^\mu(x, y)$ the Cartan covariant derivative is defined by
\[
\nabla_{\text{Cartan}} W_\mu = \frac{\partial W_\mu}{\partial x_\kappa} - \gamma^\sigma_{\kappa\mu} \frac{\partial W_\sigma}{\partial y_\kappa} + \Gamma^\mu_{\nu\sigma} W_\sigma \quad \text{(II.19)}
\]
and extended to tensors of arbitrary valence in the obvious way. The Cartan connection satisfies
\[
\nabla_{\kappa} f_{\mu\nu} = 0.
\]
(II.20)

This is equivalent to
\[
\frac{\partial F^2}{\partial x_\kappa} - \frac{\partial F^2}{\partial y_\sigma} \frac{\partial G^\sigma}{\partial y_\kappa} = 0
\]
(II.21)
and ensures that the norms of vector remain constant under parallel transport along different routes. In [9] it is written as
\[
\frac{\partial F}{\partial x_\kappa} - \frac{\partial F}{\partial y_\sigma} \frac{\partial G^\sigma}{\partial y_\kappa} = 0.
\]
(II.22)

In [12] it was suggested that
\[
g_{\mu\nu} x^\mu dx^\nu = -2 du dv + \alpha(u) dx^2 + \beta(u) dy^2
\]
(II.23)
with $(x^1, x^2, z, x^4) = (x, y, u, v)$, which they call a plane wave, might lead to a solution and they find (their eqn (34)) that
\[
\sigma = \sigma(\frac{\alpha \dot{x}^2 + \beta \dot{y}^2 - 2 \dot{u} \dot{v}}{\dot{u}^2})
\]
(II.24)
and they claim that it is indeed a solution.

The treatment of [9] starts with the helpful observation that the sums, products and ratio’s of solutions are again solutions. He investigated Lorentzian metrics with covariantly constant vector fields and pointed out that pp-waves are a special case.

### III. BOGOSLOVSKY-FINSLER METRICS

Bogoslovsky’s theory [3, 4] was based on the Finsler line element such that the propertime $\tau$ along a future-directed timelike worldline $x^\mu(\tau)$ in flat Minkowski spacetime is obtained by combining the Minkowski line element with what we call here the Bogoslovsky factor,
\[
d\tau = (-\eta_{\mu\nu} dx^\mu dx^\nu)^{\frac{1-b}{2}} (-\eta_{\mu\nu} l^\mu dx^\nu)^b,
\]
(III.1)
where \( 0 \leq b < 1 \) is a dimensionless constant, \( \eta_{\mu\nu} \) is the flat Minkowski metric tensor (with mainly positive signature) and \( l^\mu \) a constant future directed null vector, i.e.

\[
\partial_\mu l^\nu = 0, \quad \eta_{\mu\nu} l^\mu l^\nu = 0, \quad l^0 > 0.
\]  

(III.2)

where \( \partial_\mu = \frac{\partial}{\partial x^\mu} \). The Bogoslovsky factor makes (III.1) homogenous of degree one – i.e., Finslerian.

The parameter \( b \) introduces spatial anisotropy which might be relevant at the early stages of the universe [24]. The constant \( b \) is very small by Hughes-Drever - type experiments [2]; Bogoslovsky argues that \( b < 10^{-10} \) [25]. For \( b = 0 \) we recover the Minkowski propertime element, cf. (I.1) with \( A_\mu = 0 \).

Bogoslovsky’s Finsler line element has an obvious generalisation: in (III.1) one replaces \( \eta_{\mu\nu} \) by a curved pseudo-Riemannian metric \( g_{\mu\nu}(x) \) and \( l^\mu \) to by a future-directed null vector such that

\[
\nabla_\mu l^\nu = 0, \quad g_{\mu\nu} l^\mu l^\nu = 0,
\]  

(III.3)

where \( \nabla_\mu \) is the Levi-Civita connection of the Lorentzian metric \( g_{\mu\nu} \). This idea has recently been explored in [15–17, 21], where such spacetimes are called “Finsler-pp waves”.

Such spacetimes are also referred to as Brinkman [26] or Bargmann [27] spacetimes. They admit Brinkmann coordinates \( X^\mu = (V,U,X^i) \) such that

\[
g_{\mu\nu} dX^\mu dX^\nu = 2dVdU + dX^i dX^i - 2H(X^i,U) dU^2,
\]  

(III.4)

where the spacetime dimension is \( d + 1 \) and \( i = 1, 2, \ldots, d - 1 \). \( H(X^i,U) \) is an arbitrary function of its arguments \(^3\). \( U, V \) are null coordinates and may be written as

\[
V = X^- = \frac{1}{\sqrt{2}}(z - X^0), \quad U = X^+ = \frac{1}{\sqrt{2}}(z + X^0).
\]  

(III.5)

We have \( l^\mu \partial_\mu = -\partial_V \) so that \( -g_{\mu\nu} l^\mu dX^\nu = dU \). Then the Finsler-pp line element is

\[
d\tau = (-g_{\mu\nu} dX^\mu dX^\nu)^{\frac{1-b}{2}} (-g_{\mu\nu} l^\mu dX^\nu)^b = (-g_{\mu\nu} dX^\mu dX^\nu)^{\frac{1-b}{2}} (dU)^b,
\]  

(III.6)

\(^3\) Our choice of sign for \( g_{UV} \) has the advantage that raising and lowering of indices entails no minus signs, merely swapping \( U \) and \( V \) and is consistent with our previous papers. It has however the consequence that if we choose a time orientation such that \( U \) increases to the future then \( V \) decreases to future. In other words \( \frac{\partial}{\partial U} \) is a future directed null vector field and \( \frac{\partial}{\partial V} \) is a past directed vector field.
where $g_{\mu \nu}$ is the pp wave metric.

Returning to the pp-waves, we recall that the metric is Ricci flat if and only if $H(X^i, U)$ is a harmonic function of the coordinates $X^i$; it may however have arbitrary dependence upon $U$. It then represents a left (i.e. in the negative $z$ direction) moving gravitational wave such that $X^i$ are transverse to the direction of motion. The wave fronts $U = \text{constant}$ are null hypersurfaces and the covariantly constant and hence Killing null vector field $\partial_V$ lies in the wave fronts.

If in addition $H(X^i, U)$ is quadratic in the transverse coordinates then we have a \textit{plane gravitational wave}. If $d = 3$, which we assume from now on, then

$$- 2H = A_+(U)(X_1^2 - X_2^2) + A_\times(U)2X_1X_2 = K_{ij}(U)X^iX^j, \quad (\text{III.7})$$

where $A_+(U)$ and $A_\times(U)$ are the amplitudes of the two plane polarisation states.

For general $A_+(U)$ and $A_\times(U)$ there is a five dimensional isometry group $G_5$ which acts multiply transitively on the three-dimensional wave fronts $U = \text{constant}$ [28, 29]. This group is a subgroup of the 6 dimensional Carroll group Carr(2) in three spacetime dimensions [30, 31] in which the $SO(2)$ subgroup is omitted [33].

The Carroll group Carr(2) may be regarded as a subgroup of the Poincaré group $ISO(3,1)$ defined by freezing out $U$-translations [31]; it acts on the null hyperplanes $U = \text{constant}$. If we label the Killing vector fields of the Poincaré group as

$$P_\mu = \frac{\partial}{\partial X^\mu}, \quad L_{\mu \nu} = X_\mu P_\nu - X_\nu P_\mu, \quad (\text{III.8})$$

then the Carroll group is generated by

$$P_- = \frac{\partial}{\partial V}, \quad P_i = \frac{\partial}{\partial X^i}, \quad (\text{III.9a})$$

$$L_{ij} = X_iP_j - X_jP_i, \quad L_{-i} = X_- \frac{\partial}{\partial X^i} - X_i \frac{\partial}{\partial X_-} = U \frac{\partial}{\partial X^i} - X_i \frac{\partial}{\partial V} \quad (\text{III.9b})$$

$i = 1, 2$ and leaves invariant each hyperplane $U = \text{const}$. The generators in (III.9a) are translations, whereas those in (III.9b) are planar rotation and boosts. Since $d = 3$ we may re-label the generators $L_{ij} = J$ and the $U$-$V$ boost

$$N_0 = L_{+-} = X_+ \partial_- - X_- \partial_+ = V \partial_V - U \partial_U, \quad (\text{III.10})$$

and find that the four generators $N_0, J, L_{-i}$ generate a group which is abstractly isomorphic to the group SIM(2), the group of similarities, that is dilations, rotations and translations...
of the Euclidean Plane $\mathbb{E}^2$. SIM(2) is the largest proper subgroup of the Lorentz group \( \text{SO}(3, 1) \). Adjoining the generators \( P_+, P_-, P_i \) gives rise to the eight generators of ISIM(2), which is a subgroup of the Poincaré group. The group ISIM(2) acts multiply-transitively on Minkowski spacetime $\mathbb{E}^{3,1}$.

It was suggested by Cohen and Glashow \[1\] that ISIM(2), which may be thought of as the subgroup of ISO(3, 1) leaving invariant a null direction, could explain weak CP violation while being compatible with tests of Lorentz-invariance since it would rule out spurions, that is tensor vacuum expectation values.

In \[34\] it was pointed out that Ricci flat pp-waves are strongly universal. In particular they have non-vanishing scalar invariants constructed from the Riemann tensor and as a consequence satisfy almost any set of covariant field equations. Quantum corrections to the metric vanish. Thus this property may be thought of as the analogue for the proposed curved Bogoslovsky-Finsler structures with with a Ricci flat metric $g_{\mu\nu}$ of Cohen and Glashow’s No Spurions condition.

In \[6\], an attempt was made to find a link with General Relativity in which Minkowski spacetime $\mathbb{E}^{3,1}$ may be regarded as the coset ISO(3, 1)/SO(3, 1). The only two deformations of the Poincaré group led to the two de-Sitter groups SO(4, 1) and SO(3, 2) for which translations act in a non-commutative fashion on the cosets de Sitter spacetime $dS_4 = SO(4, 1)/SO(3, 1)$ and Anti-de Sitter spacetime $AdS_4 = SO(3, 2)/SO(3, 1)$.

They therefore investigated the deformations of ISIM(2) and found that there exists a family of deformations depending upon two dimensionless parameters $a$ and $b$. However for all $a$ and $b$ the translations $P_+, P_-, P_i$ failed to commute. In general the rotation $J$ became a non-compact generator unless $a = 0$ leaving DISIM$_b(2)$ depending on a dimensionless parameter $b$. They then observed that this is precisely the symmetry of the Bogoslovsky’s Finsler metric (III.1). For a review of these ideas and their relation to much earlier work of Voigt \[35\], the reader is directed to the recent review \[36\]. For a recent discussion of Bogoslovsky-Finsler deformations in the light of ideas of Segal see \[37\].

In the recent paper \[15\], the authors have shown, among other things, that the Bogoslovsky-Finsler pp-waves enjoy the same universal properties with respect to generalisations of the Einstein equations to Finsler-Einstein equations as do those in the pseudo-Riemannian case discussed in \[34\].
IV. GEODESICS

The geodesics of a Finsler metric with Finsler function \( F(x^\mu, \dot{x}^\mu) \), where \( F \) is homogeneous of degree one in \( \dot{x}^\mu \) are extrema of

\[
I = \int F(x^\mu, \frac{dx^\mu}{d\lambda}) d\lambda. \tag{IV.1}
\]

In the case we are considering, we restrict attention to future directed timelike curves for which both \( g_{\mu\nu}l^\mu \dot{x}^\nu \) and \(-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu\) are strictly positive in order to ensure that \( F \) is real. For a particle of mass \( m \) the action resp. Lagrangian are

\[
S_b = -m \int F d\lambda. \tag{IV.2}
\]

where \( F \) is the Bogoslovsky-Finsler line element (III.6). The integral is independent of the parameter \( \lambda \). Therefore if \( p_\mu = \frac{\partial(-mF)}{\partial \dot{x}^\mu} \), then \( \mathcal{H} = p_\mu \dot{x}^\mu + mF \) is a constant of the motion. This is indeed true, but because \( F(x^\mu, \dot{x}^\mu) \) is homogeneous of degree one in \( \dot{x}^\mu \) one has \( \dot{x}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = F \) and consequently the constant vanishes identically. Standard Riemannian or Lorentzian metrics are of course a special case of this general fact.

Now we analyse the motion along geodesics in Bogoslovsky-Finsler Plane Gravitational waves adapting the discussion for the standard Einstein case given in [33, 38]. Firstly, we find it convenient to pass to Baldwin-Jeffery-Rosen (BJR) coordinates \( x^\mu = (v, u, x^i) \), defined by

\[
X^i = P_{ij} x^j, \quad U = u, \quad V = v - \frac{1}{4} \frac{da_{ij}}{du} x^i x^j, \tag{IV.3}
\]

where \( a \equiv (a_{ij}) = P^t P \), and the matrix \( P \) satisfies the matrix Sturm-Liouville equation

\[
\frac{d^2 P}{du^2} = KP, \quad P^t \frac{dP}{du} = \frac{dP^t}{du} P. \tag{IV.4}
\]

In BJR coordinates we have

\[
g_{\mu\nu} d\lambda^\mu d\lambda^\nu = 2du dv + a_{ij}(u) \dot{x}^i \dot{x}^j, \quad l^\mu \frac{\partial}{\partial x^\mu} = -\frac{\partial}{\partial v}. \tag{IV.5}
\]

Here \( \dot{x}^\mu = \frac{dx^\mu}{d\lambda} \) where \( \lambda \) is an arbitrary parameter. The Lagrangian is proportional to the Bogoslovsky-Finsler function,

\[
\mathcal{L}_b = -mF \quad \text{where} \quad F = \left( -2\dot{u}v - a_{ij}(u) \dot{x}^i \dot{x}^j \right) \frac{1}{2} (1-b) \left( \dot{u} \right)^b \tag{IV.6}
\]
For the curve to be time-like we must have $\dot{u} \dot{v} < 0$ and $\dot{u} > 0$. Since the integral (IV.1) is independent of the choice of the parameter $\lambda$, we are entitled to make the choice $\lambda = u$ and extremise

$$
\int \left( -2 \frac{dv}{du} - a_{ij}(u) \frac{d\dot{x}^i}{du} \frac{d\dot{x}^j}{du} \right)^{\frac{1}{2}} (1-b) \, du.
$$

(IV.7)

With this choice of parametrisation the integrand of (IV.7) is now no longer homogeneous in the velocities $\frac{dv}{du}$ and $\frac{dx_i}{du}$ but because $a_{ij}$ depends on the “time” $u$, there is no conserved analogue of the quantity $H$. The symmetry aspects will be further investigated in sec.V.

Before analysing the general case we recall, for later comparison, some aspects of the geodesics of a pp wave described by the square-root Lagrangian (I.1).

**Geodesics in a pp wave**

Let us thus first consider a pp wave written in BJR coordinates, whose the geodesics are described by (I.1) with $A_\mu = 0$,

$$
\mathcal{L}_0 = -m \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = -m \sqrt{-a_{ij}(u) \dot{x}^i \dot{x}^j - 2 \dot{u} \dot{v}}.
$$

(IV.8)

The canonical momenta $p_\mu = \frac{\partial \mathcal{L}_0}{\partial \dot{x}_\mu}$ are

$$
p_u = \frac{m \dot{v}}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}, \quad p_i = \frac{ma_{ij} \dot{x}^j}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}, \quad p_v = \frac{m \dot{u}}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}
$$

(IV.9)

of which $p_i$ and $p_v$ are constants of the motion since $a_{ij} = a_{ij}(u)$. For a $u$-dependent profile $p_u$ is not conserved, though. The geodesic equations of motion are

$$
\ddot{u} = \dot{u} \frac{d}{d\lambda} \ln \left( \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right),
$$

(IV.10a)

$$
\ddot{x}^i + \dot{u} a_{ij} a'_{jk} \dot{x}^k = \dot{x}^i \frac{d}{d\lambda} \ln \left( \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right),
$$

(IV.10b)

$$
\ddot{v} - \frac{1}{2} a'_{ij} \dot{x}^i \dot{x}^j = \dot{v} \frac{d}{d\lambda} \ln \left( \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right),
$$

(IV.10c)

where $a'_{ij} = \frac{da_{ij}}{du}$. Using the first equation the two remaining ones simplify \(^4\),

$$
\ddot{x}^i + \dot{u} a_{ij} a'_{jk} \dot{x}^k = \dot{x}^i \frac{\ddot{u}}{\dot{u}},
$$

(IV.11a)

$$
\ddot{v} - \frac{1}{2} a'_{ij} \dot{x}^i \dot{x}^j = \dot{v} \frac{\ddot{u}}{\dot{u}}.
$$

(IV.11b)

\(^4\) Choosing the affine parameter $\lambda = u$, the rhs would vanish.
An ingenious way to solve these equations is to use the conserved quantities of the problem. We define first the constants of the motion by setting

\[ P_i = \frac{p_i}{p_v} = \frac{a_{ij} \dot{x}^j}{\dot{u}}. \]  

(IV.12)

The resulting first order differential equation for \( x^i \) is solved at once as

\[ x^i(u) = S^{ij}(u)P_j + x^i_0, \]  

(IV.13)

where \( S \equiv S^{ij} \) is the Souriau matrix \([33]\), defined by

\[ \frac{dS(u)}{du} = a^{-1}(u). \]  

(IV.14)

\( p_v \) in (IV.9) provides us in turn with a first order equation for \( v \),

\[ \dot{v} = -\frac{1}{2} a^{ij} P_i P_j \dot{u} - \frac{1}{2} \mu^2_0 \dot{u} \quad \text{where} \quad \mu_0 = \frac{m}{p_v}. \]  

(IV.15)

This equation is then solved as

\[ v = -\frac{1}{2} P_i P_j S^{ij}(u) - \frac{1}{2} \mu^2_0 u + v_0. \]  

(IV.16)

The transverse motion (IV.13) is the same for all values of the mass \( m \), which enters only the \( v \)-motion (IV.16) by a shift which is linear in \( u \) and proportional to the mass-quotient term \( \mu_0 \) in (IV.15), familiar from [39].

**Finsler Geodesics.**

Let us now consider the Bogoslovsky-Finsler Lagrangian \( L_b \) in (IV.6). Its canonical momenta are

\[ p_u = m(\dot{u})^{-b-1} \left( -2\dot{u}\dot{v} - a_{ij} \dot{x}^i \dot{x}^j \right) \left( (1 + b)\dot{\dot{u}} + ba_{ij} \dot{x}^i \dot{x}^j \right), \]  

(IV.17a)

\[ p_i = m(1 - b)(a_{ij} \dot{x}^i) \dot{u}^b \left( -2\dot{u}\dot{v} - a_{ij} \dot{x}^i \dot{x}^j \right)^{-\frac{1+b}{2}}, \]  

(IV.17b)

\[ p_v = m(1 - b) \dot{u}^{-b+1} \left( -2\dot{u}\dot{v} - a_{ij} \dot{x}^i \dot{x}^j \right)^{-\frac{1+b}{2}}. \]  

(IV.17c)

\( p_i \) and \( p_v \) are constants of the motion as before and we have the dispersion relation eqn # (18) of [6],

\[ p^2 \equiv g^{\mu \nu} p_\mu p_\nu = -m^2 (1 - b^2) \dot{u}^{2b} \left( -2\dot{u} - a_{ij} \dot{x}^i \dot{x}^j \right)^{-b}. \]  

(IV.18)

The geodesic equations,
\[(b + 1)\ddot{u} = \dot{u} \frac{d}{d\lambda} \ln \left( -2\dot{u}\dot{v} - a_{ij}\dot{x}^i\dot{x}^j \right)^{\frac{1+b}{2}}, \quad \text{(IV.19a)}\]
\[\dot{x}^i + \dot{u} a^{ij} a_{jk} \ddot{x}^k + b\frac{\ddot{u}}{\dot{u}} \dot{x}^i = \dot{x}^i \frac{d}{d\lambda} \ln \left( -2\dot{u}\dot{v} - a_{ij}\dot{x}^i\dot{x}^j \right)^{\frac{1+b}{2}}, \quad \text{(IV.19b)}\]
\[\ddot{v} + \frac{3b - 1}{2(1 + b)} a^{ij}_{ij} \ddot{x}^i \ddot{x}^j + \frac{2b}{\dot{u}(1 + b)} a_{ij}\dot{x}^i\dot{x}^j = \frac{1}{b + 1} \left( \dot{v} + \frac{2b}{1 + b} a_{ij}\dot{x}^i\dot{x}^j \right) \left( \frac{d}{d\lambda} \ln \left( -2\dot{u}\dot{v} - a_{ij}\dot{x}^i\dot{x}^j \right)^{\frac{1+b}{2}} \right), \quad \text{(IV.19c)}\]
reduce to (IV.10) when \(b = 0\).

The remarkable fact is that using (IV.19a) the two remaining equations become the same, (IV.11), as for the square root Lagrangian, (IV.8).

This not imply identical solutions, though, as seen by solving the geodesics equations along the same lines as before. Setting once again \(P_i = \frac{p_i}{p_v}\) provides us with the transverse motion,

\[x^i(\lambda) = S^{ij}(u(\lambda))P_j + x_0^i \quad \text{(IV.20)}\]

which is again (IV.13). Then from (IV.17b), we infer that

\[\dot{v} = -\frac{1}{2} \left( a^{ij} P_i P_j + \mu_b^2 \right) \dot{u}, \quad \text{where} \quad \mu_b = \left( \frac{m}{p_v} (1 - b) \right)^{-\frac{1}{1+b}}, \quad \text{(IV.21)}\]

whose integration yields,

\[v = -\frac{1}{2} S^{ij}(u) P_i P_j - \frac{1}{2} \mu_b^2 u + v_0. \quad \text{(IV.22)}\]

Let us observe that this takes the same form as for (IV.16), however with a new, \(b\)-dependent mass-quotient term, \(\mu_b\). For \(b = 0\) the latter reduces to \(\mu_0\) and the massive equation (IV.16) is recovered.

The family of pp-wave geodesics are given by (IV.13) and (IV.16) and are labelled by the constants of integration \(P_i, x_0^i, v_0\) and \(\mu_0\). The Finsler geodesics are given by (IV.20) and (IV.22) and are labelled by the constants of integration \(P_i, x_0^i, v_0\) and \(\mu_b\). It is clear that the two sets of geodesics are identical up to a \(b\)-dependent relabelling of the last constant of integration.

In the massless case \(m = 0\) (photons) that the \(b\)-dependent term drops out from (IV.22). Letting \(b \rightarrow 1\) turns off the mass-quotient term, \(\mu_b \rightarrow 0\), and all geodesics behave as if they were massless, consistently with (IV.18). See the plot 1 in sec.VII for an illustration.
Another way to see the surprising identity of the geodesics is to consider the Euler-
Lagrange equations
\[
E_\mu = \frac{\partial L_0}{\partial x^\mu} - \frac{d}{dt} \left( \frac{\partial L_0}{\partial \dot{x}^\mu} \right) = 0 \quad \text{and} \quad \tilde{E}_\mu = \frac{\partial L_b}{\partial x^\mu} - \frac{d}{dt} \left( \frac{\partial L_b}{\partial \dot{x}^\mu} \right) = 0
\] (IV.23)
of the two Lagrangians \( L_0 \) and \( L_b \) in (IV.8) and in (IV.6) respectively.

Both systems can be described by three independent equations, since the following identities hold: \( \dot{x}^\mu E_\mu \equiv 0, \dot{x}^\mu \tilde{E}_\mu \equiv 0 \). Then

1. For the first system the combinations:
\[
\frac{(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2}}{m} \left( \frac{2\dot{u}}{u} E_v + \frac{\dot{x}^i}{u} E_i \right) = 0
\] (IV.24)
\[
\frac{(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2}}{m} \left( \frac{\dot{x}^i}{u} E_v - a^{ij} E_j \right) = 0
\] (IV.25)

2. For the second system the combinations
\[
\frac{(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1+b}}{m} \left[ \frac{1}{(1-b^2)\dot{u}^{b+2}} \left( 2\dot{u} \dot{v} - ba_{ij} \dot{x}^i \dot{x}^j \right) \tilde{E}_v + \frac{\dot{x}^i}{(1-b)\dot{u}^{b+1}} \tilde{E}_i \right] = 0
\] (IV.26)
\[
\frac{(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1+b}}{m} \left[ \frac{\dot{x}^i}{(1-b)\dot{u}^{b+1}} \tilde{E}_v + \frac{a^{ij}}{(b-1)\dot{u}^b} \tilde{E}_j \right] = 0
\] (IV.27)

(where \( i, j = 1, 2 \)) yield both eqns (IV.11).

V. PARTIALLY BROKEN CARROLL SYMMETRY

Generic plane gravitational waves are invariant under the same 5 parameter group we
denote by \( G_5 \) \([28, 29]\). Expressed in BJR coordinates, \( G_5 \) is implemented as \[33],
\[
\begin{align*}
u & \to u, \quad x \to x + S(u)b + c, \quad v \to v - b \cdot x - \frac{1}{2}b \cdot S(u)b + f,
\end{align*}
\] (V.1)
where \( S \) is Souriau’s matrix (IV.14). The 2-vectors \( b \) and \( c \) and \( f \) are constants, interpreted
as boosts, and as transverse and vertical translations. These same tranformations are isometries
also for the Bogoslovsky-Finsler metric (III.6) because \( u \) is fixed, and the pp isometries
leave the pp-wave metric, and hence their powers invariant. The transformations in (V.1)
are generated by the vector fields
\[
B_i = S_{ij}(u)\partial_j - x_i\partial_v, \quad \partial_i \quad \text{and} \quad \partial_v,
\] (V.2)
respectively. The only nonvanishing Lie bracket is

$$[\partial_i, B_j] = -\delta_{ij} \partial_v.$$  \hfill (V.3)

Rotations, generated by $L_{ij} = x_i \partial_j - x_j \partial_i$, are not symmetries in general.

The restriction of a pp wave to the $u = 0$ hypersurface $C_0$ carries a Carroll structure. The “vertical” coordinate $v$ is interpreted as “Carrollian time” [30–32]. $C_0$ is left invariant by the action (V.1) and the generators then satisfy the Carroll algebra in two space dimensions with rotations omitted [33]. Then eqn. (V.1) tells us how the Carroll group is implemented on any hypersurface $u = u_0 = \text{const}$.

In the flat case, $a_{ij} = \delta_{ij}$, we have further symmetries. In particular, adding the vector fields $\partial_u, L_{ij}$ and $L_{+-} = v\partial_v - u\partial_u$ yield the Lie algebra of a 8-parameter subgroup of the Poincaré group. We now have

$$[v\partial_v - u\partial_u, \partial_v] = -\partial_v,$$  \hfill (V.4)

and so the direction of null Killing vector field $\partial_v$ is preserved. The 8-dimensional group they generate is ISIM(2). Omitting the translations $\partial_v, \partial_u, \partial_i$ gives SIM(2), the largest proper subgroup of the Lorentz group $\text{SO}(3,1)$. This is the symmetry of Cohen and Glashow’s Very Special Relativity [1].

Returning to the case of general pp-waves and its Bogoslovsky-Finsler version (III.6), we emphasise that the BJR matrix $a = (a_{ij})$ and thus the Souriau matrix $S$ depend on the pp-wave metric only, but not on the deformation parameter $b$. Therefore the isometries in (V.1) act, for (III.6), exactly as for standard plane waves.

The invariance of the Bogoslovsky-Finsler model can be confirmed with respect to the partially broken Carroll group. The infinitesimal version of (V.1) is $Y_{iso}$ in (V.2). The linear momenta in (IV.17) are readily recovered; using (V.2) for boosts we get in turn,

$$k^i = p_v x^i - S_{ij} p_j,$$  \hfill (V.5)

– just as for a gravitational wave [33]. Its conservation follows from Noether’s theorem, and can also be confirmed by a direct calculation. The dependence on $b$ is hidden in the momenta in (IV.17). The initial position $x^i_0$ in (IV.13) is the conserved value of $k^i$.

For $b = 0$ the flat Bogoslovsky-Finsler model has one more isometry, identified with the u-v- boost $N_0 = L_{+-}$ in (III.10). For $b \neq 0$ this generator is broken but not entirely lost: Let us explain how this comes about.
As said above, the (rotation-less) Carroll isometry group $G_5$ in (V.1) of the initial pp-wave remains a symmetry with identical generators for its Bogoslovsky-Finsler extension.

To see what happens to $u$-$v$ boosts we start with the Minkowski metric, $\eta_{\mu\nu}dx^\mu dx^\nu = \delta_{ij}dx^i dx^j + 2dudv$. An $u$-$v$ boost, implemented as,

$$u \rightarrow \lambda^{-1}u, \quad x^i \rightarrow x^i, \quad v \rightarrow \lambda v \tag{V.6}$$

where $\lambda = \text{const.} > 0$ is an isometry. Moreover, it’s $b$-dependent deformation of (V.6),

$$u \rightarrow \lambda^{b-1}u, \quad x^i \rightarrow \lambda^b x^i, \quad v \rightarrow \lambda^{b+1}v \tag{V.7}$$

it is readily seen to leave the Bogoslovsky-Finsler line element (III.1) invariant – although for $b \neq 0$ it is only a conformal transformation for the Minkowski metric, $\eta_{\mu\nu}dx^\mu dx^\nu \rightarrow \lambda^{2b}\eta_{\mu\nu}dx^\mu dx^\nu$, and not an isometry $^5$. We record for later use that the $b$-deformed boost (V.7) is generated by

$$N_b = (b-1)u\partial_u + (b+1)v\partial_v + bx^i\partial_i. \tag{V.8}$$

Both (V.6) and (V.7) leave the hypersurface $u = 0$ invariant, and extend the Carroll action (V.1). We note that the restriction to $C_0$ of the deformed $u$-$v$ boost (V.7) scales the Carrollian time, $v \rightarrow \lambda^{b+1}v$. Therefore it is not $\partial_v$ itself but only its direction which preserved,

$$\partial_v \rightarrow \lambda^{-1-b} \partial_v \tag{V.9}$$

the isometry (III.1) is “chronoprojective” [38, 40].

In the flat case two more isometries, namely $u$-translations and rotations complete the algebra to one with 8-parameter. With some abuse we will refer to $G_5$ extended with $u$-translations (but with no rotations) still “Carroll” for simplicity and denote it by $G_6$. Its further extension by $u-v$ boosts will be called chrono-Carroll [38] and denoted by $G_7$.

The Lie algebra structure is most easily checked by taking the commutators of the vector fields in (III.9) and (III.10) and compare with those given in eqn (9) of ref. [6], which gives the structure constants of the deformed group DISIM$_b$(2); those of ISIM(2) are obtained by setting $b = 0$.

$^5$ Noting that $\lambda^{b-1} = (\lambda^b)^{\frac{b-1}{b}}$ shows that (V.7) has dynamical exponent $z = 1 - \frac{1}{b}$ which corresponds to the conformal Galilei algebra labeled by $z$ [42, 43].
Further insight is gained by decomposing the deformed u-v boost generator $N_b$ in (V.8) into the sum of the undeformed expression $N_0 = L_{+-}$ and a relativistic dilation, $D$,

$$N_b = v \partial_v - u \partial_u + b(u \partial_u + v \partial_v + x^i \partial_i) = N_0 + bD.$$  \hfill (V.10)

For $b \neq 0$ $N_0$ is broken and it is only the above combination of u-v boosts and dilations which is a symmetry – a situation familiar from gravitational plane waves [33, 38, 41].

It is instructive to see how this comes about. In flat Minkowski case $a_{ij} = \delta_{ij}$ and (IV.17) yield

$$p_i = (\dot{x}_i / \dot{u}) p_v \quad \text{and} \quad p_u = \frac{(1 + b) \dot{u} \dot{v} - b \dot{x}_i \dot{x}_i}{(1 - b) \dot{u}^2} p_v.$$  \hfill (V.11)

Then for $\mathcal{D} = D^\mu p_\mu$ and $\mathcal{N}_0 = N^\mu p_\mu$ we have

$$\dot{\mathcal{N}} = \frac{(2 \dot{u} \dot{v} + \dot{x}_i \dot{x}_i)}{(1 - b) \dot{u}} p_v \quad \text{and} \quad \dot{\mathcal{N}}_0 = -b \frac{(2 \dot{u} \dot{v} + \dot{x}_i \dot{x}_i)}{(1 - b) \dot{u}} p_v = -b \dot{\mathcal{N}},$$ \hfill (V.11)

so that the combination of the two expressions is conserved:

$$\dot{N}_b = 0 \quad \text{for} \quad N_b = N_0 + bD.$$ \hfill (V.12)

Now we turn to the curved case. Let us consider a conformal transformation $f$ of a pp wave with metric $g_{\mu\nu}$,

$$f_\ast g_{\mu\nu} = \Omega^2 g_{\mu\nu},$$  \hfill (V.13)

where $f_\ast$ is the pullback map. This changes the “pp factor” in (IV.5) as

$$(g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}(1-b)} \to \Omega^{1-b} (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}(1-b)}.$$ \hfill (V.14)

The change can be compensated by the “B-F factor”, though. Assuming that $f_\ast l_\mu = \Omega^a l_\mu$ for some constant $a$ yields

$$(l_\mu dx^\mu)^b (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}(1-b)} \to \Omega^{ab + 1-b} (l_\mu dx^\mu)^b (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}(1-b)}.$$ \hfill (V.15)

In the undeformed case $b = 0$ this is a conformal transformation with conformal factor $\Omega$; obtaining an isometry requires $f$ to be an isometry for the pp wave. This is consistent with our findings in the flat case for the u-v boost (V.6).

For $b \neq 0$ we have another option. If the exponent of $\Omega$ in (V.15) vanishes,

$$ab = b - 1,$$ \hfill (V.16)
then we do get an isometry again. For the $b$-deformed $u$-$v$ boost in (V.7) we have $\Omega = \lambda^b$, consistently with (V.16).

In the case of plane gravitational waves one drops the angular momentum $J$ and $\partial_U$ and then the issue is what about $N$? Our only certainty so far is that the flat-space implementation (V.8) does not work.

The symmetry of the Bogoslovsky-Finsler model is in fact of the Very Special Relativity (VSR) type, more precisely, a subgroup of the 8-parameter DISIM$_b(2)$ where $0 < b < 1$ is a deformation parameter [7, 42]. DISIM$_b(2)$ is isomorphic to the Conformal Galilei group with dynamical exponent [43]

$$z = 1 - \frac{1}{b}.$$  

(V.17)

For $b \neq 0$ u-$v$ boosts (which are isometries for the Minkowski case) are deformed to (V.8), a combination of u-$v$-boosts and relativistic dilations.

One can be puzzled if the “deformation trick” can work also for a non-trivial profile. The answer is: it might work for a particular profile. Let us consider, for example a pp wave (III.4) written in Brinkmann coordinates with the (singular) profile,

$$2H(X^i, U) = -\frac{K^0_{ij}}{U^2} X^i X^j. \quad K^0_{ij} = \text{const.}$$  

(V.18)

This wave has a 6-parameter isometry group [29, 38, 41, 44]. It is in particular invariant under a $U$ - $V$ boost, (V.6) Then we find that the deformed $U$ - $V$ boost,

$$U \rightarrow \Lambda U, \quad X \rightarrow \Lambda^{\frac{b}{b-1}} X, \quad V \rightarrow \Lambda^{\frac{b+1}{b-1}} V$$  

(V.19)

leaves the Bogoslovsky-Finsler line element

$$ds_{BF} = \left( -2dUdV - dX^2 - \frac{K^0_{ij}}{U^2} X^i X^j dU^2 \right)^{\frac{1+b}{2}} (dU)^b$$  

(V.20)

invariant. The usual $U$ - $V$ boost is recovered for $b = 0$. Writing $\lambda = \Lambda^{\frac{b}{b-1}}$ shows, moreover, that when $b \neq 0$, the dynamical exponent is $z = -1 + \frac{1}{b}$, minus that in (V.17). Note that (V.19) is, once again, a conformal transformation of the pp wave metric (III.4)-(V.18) with conformal factor $\Omega^2 = \Lambda^{\frac{2b}{2b-1}}$.

VI. PROLONGATION VECTORS AND SYMMETRIES

The connection of the aforementioned symmetries to integrals of the motion is established through Noether’s first theorem [45]: each generator of any finite dimensional Lie group of
transformations which leaves the action form 

$$\text{forme invariant up to a surface term} \ [46]$$

is associated to a conserved quantity.

Consider, for example, the most general point transformation that a dynamical system with dependent resp. independent variables $x^\mu(\lambda)$ and resp. $\lambda$ can have,

$$\Upsilon = \sigma(\lambda, x) \frac{\partial}{\partial \lambda} + Y^\mu(\lambda, x) \frac{\partial}{\partial x^\mu},$$

(VI.1)

where the coefficient $\sigma(\lambda, x)$ accounts for transformations which might involve also the parameter $\lambda$. This vector can be extended to the space of the first derivatives $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$, i.e., we can consider the first prolongation of $\Upsilon$, defined as [47, 48]

$$pr^{(1)}\Upsilon = \Upsilon + \Phi^\mu \frac{\partial}{\partial \dot{x}^\mu},$$

where

$$\Phi^\mu = \frac{dY^\mu}{d\lambda} - \dot{x}^\mu \frac{d\sigma}{d\lambda}.$$  

(VI.2)

The coefficient $\Phi^\mu$ here is to guarantee the correct transformation law for the derivatives. Given for example the generator (VI.1), up to first order in the transformation parameter (say $\epsilon$) we may write,

$$\bar{\lambda} \sim \lambda + \epsilon \sigma(\lambda, x) \quad \bar{x}^\mu \sim x^\mu + \epsilon Y^\mu(\lambda, x)$$

(VI.3)

which furthermore imply

$$\frac{d\bar{x}^\mu}{d\bar{\lambda}} \sim \frac{d(x^\mu + \epsilon Y^\mu)}{d(\lambda + \epsilon \sigma)} \sim \left( \frac{dx^\mu}{d\lambda} + \epsilon \frac{dY^\mu}{d\lambda} \right) \left( 1 - \epsilon \frac{d\sigma}{d\lambda} \right) \simeq \dot{x}^\mu + \epsilon \Phi^\mu.$$ 

(VI.4)

With the use of the extended vector $pr^{(1)}\Upsilon$, the initial requirement of Noether’s theorem written as

$$\delta(Ld\lambda) = d\Sigma$$

(VI.5)

where $\Sigma = \Sigma(\lambda, x)$ is some function can be cast in infinitesimal form as

$$pr^{(1)}\Upsilon(L) + L \frac{d\sigma}{d\lambda} = \frac{d\Sigma}{d\lambda}.$$  

(VI.6)

To an appropriate generator $\Upsilon$ and a function $\Sigma$ satisfying the above relation there corresponds a conserved quantity is

$$I = Y^\mu \frac{\partial L}{\partial \dot{x}^\mu} - \sigma(\dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} - L) - \Sigma.$$ 

(VI.7)

The geodesic system is invariant under arbitrary changes of the parameter $\lambda$, therefore the inclusion of the coefficient $\sigma$ into (VI.1) does not contribute in the conservation law. As it can
be seen using (VI.7), \( \sigma \) essentially multiplies the Hamiltonian, which is identically zero for Lagrangians which are homogeneous functions of degree one in the velocities. The coefficient \( \sigma \) plays a rôle instead in Noether’s second theorem and an identity among the Euler-Lagrange equations of motion [49]. As a result, we may restrict ourselves to consider pure space-time transformations generated by vectors \( Y = Y^\alpha(x) \partial_\alpha \). Then the first prolongation becomes

\[
pr^{(1)} Y = Y + \frac{dY^\alpha}{d\lambda} \frac{\partial}{\partial \dot{x}^\alpha} = Y^\alpha(x) \frac{\partial}{\partial x^\alpha} + \frac{\partial Y^\alpha}{\partial x^\beta} \frac{\partial}{\partial \dot{x}^\alpha} .
\]

and (VI.6)-(VI.7) reduce to

\[
pr^{(1)} Y(L) = \frac{d\Sigma}{d\lambda} , \quad I = Y^\mu \frac{\partial L}{\partial \dot{x}^\mu} - \Sigma .
\]

If for a given space-time vector \( Y \), the relation \( pr^{(1)} Y(L) = 0 \) is satisfied (as for isometries of the geodesic system), then \( \Sigma \) is just a constant and can be omitted, thus having \( \tilde{I} = I + \Sigma = Y^\mu \frac{\partial L}{\partial \dot{x}^\mu} = \text{const.} \)

To illustrate the prolongation technique, we note that for a system in the background \( g_{\mu\nu} \) (IV.5) whose Lagrangian is \( L \) the first prolongation of the isometries (V.2),

\[
Y_{\text{iso}} = (S^{ij} \beta_j + \gamma^i) \partial_i + (-\beta_i x^i + \varphi) \partial_v ,
\]

is

\[
pr^{(1)} Y_{\text{iso}}(L) = \left( Y^\alpha_{\text{iso}} \partial_\alpha + \frac{\partial Y^\alpha_{\text{iso}}}{\partial x^\beta} \dot{x}^\beta \frac{\partial}{\partial \dot{x}^\alpha} \right) (L) .
\]

If the r.h.s. is a total derivative, then we have a symmetry for the system.

Applying (VI.10) first to the pp wave Lagrangian

\[
L_{pp} = \dot{u} \dot{v} + \frac{1}{2} a_{ij} \dot{x}^i \dot{x}^j
\]

confirms that \( Y_{\text{iso}} \) is an symmetry for the pp wave.

Next, for the Bogoslovsky-Finsler Lagrangian \( \mathcal{L}_b \) in (IV.6) we find that the r.h.s. of (VI.10) vanishes,

\[
pr^{(1)} Y_{\text{iso}}(\mathcal{L}_b) = \left( m(1 - b)(-a_{ij} \dot{x}^i \dot{x}^j - 2 \dot{u} \dot{v})^{-\frac{1+b}{2}} u^b \right) \quad pr^{(1)} Y_{\text{iso}}(L_{pp}) = 0 ,
\]

proving that the Carroll group [with broken rotations] generates symmetries also for the Bogoslovsky-Finsler metric. The conserved quantities listed in sec.V are recovered using (VI.9).
Turning now to $u$-$v$ boosts we check first that for the flat Minkowski metric the prolongation of the deformed boost $N_b$ in (V.8) vanishes,

$$\text{pr}^{(1)} N_b(\mathcal{L}_0) = 0 \quad \text{(VI.13)}$$

and thus generates the constant of the motion $\mathcal{N}_b$ in (V.12).

However the same calculation carried out in the curved background $g_{\mu\nu}$ (IV.5) yields instead

$$\text{pr}^{(1)} N_b(\mathcal{L}_b) = m u \, i^b(b - 1)^2 \left( \frac{da_{ij}}{du} \dot{x}^i \dot{x}^j \right) \left( -a_{ij} \dot{x}^i \dot{x}^j - 2 \dot{u} \dot{v} \right)^{-\frac{1}{2}}. \quad \text{(VI.14)}$$

Consistently with what we said before, this vanishes for the flat metric $\eta_{\mu\nu}$. However it is manifestly not a total derivative in general whenever $a = (a_{ij})$ is not a constant matrix.

**VII. AN EINSTEIN-MAXWELL EXAMPLE**

In this section we treat the motion in the Bogoslovsky-Finsler deformation of a pp-wave which is not Ricci flat. It is

$$ds^2 = (dX^1)^2 + (dX^2)^2 + 2dUdV - \frac{\omega^2}{4} \left( (X^1)^2 + (X^2)^2 \right) dU^2. \quad \text{(VII.1)}$$

From p. 385 eqn (24.5) of ref. [29] one learns that it belongs to a class first considered by Baldwin and Jeffery [50]. It is a conformally flat and is an Einstein-Maxwell solution with a covariantly constant null Maxwell field. From the Bargmann point of view this metric describes an isotropic harmonic oscillator in the plane with frequency $\omega$ [27]. The kinematic group arising from the null reduction is the Newton-Hooke group [51]. Because the metric (VII.1) is of the form (III.4) with

$$-2H = K_{ij} X^i X^j \quad \text{(VII.2)}$$

where $K_{ij}$ is non-degenerate and independent of $U$ its is also a Cahen-Wallach symmetric space [52–55]. Following the procedure outlined in sec.IV, the metric (VII.1) can presented in the BJR form. We put $a = P^t P$ where

$$P = \begin{bmatrix} \left( 1 - \sin(\omega u) \right)^{1/2} \cos \phi & -\left( 1 + \sin(\omega u) \right)^{1/2} \sin \phi \\ \left( 1 - \sin(\omega u) \right)^{1/2} \sin \phi & \left( 1 + \sin(\omega u) \right)^{1/2} \cos \phi \end{bmatrix} \quad \text{(VII.3)}$$
is a solution of the Sturm-Liouville equation (IV.4) with diagonal profile \( K = -\frac{\omega^2}{4} \text{Id} \). Then using eqn (IV.3) we end up with
\[
\begin{align*}
ds^2 &= (1 - \sin(\omega u)) dx^2 + (1 + \sin(\omega u)) dy^2 + 2 dudv \tag{VII.4}
\end{align*}
\]
which has \( a = P^T P = \text{diag}(1-\sin(\omega u), 1+\sin(\omega u)) \). On p. 386 of [29] this result is ascribed to Brdička [56]. Eqn (VII.4) shows that the u-v boost symmetry is manifestly broken.

The Souriau matrix is found by integrating the inverse of \((a_{ij})\), cf. (IV.14),
\[
S(u) = \frac{1}{3} \begin{pmatrix}
\tan \left( \frac{\omega u}{2} + \frac{\pi}{4} \right) + C_1 & 0 \\
0 & \tan \left( \frac{\omega u}{2} - \frac{\pi}{4} \right) + C_2
\end{pmatrix}, \tag{VII.5}
\]
where \( C_{1,2} \) are integration constants. Choosing \( u_0 = 0 \) yields \( C_1 = -1 \) and \( C_2 = 1 \). The trajectories (IV.13)-(IV.22) for different values of \( b \) are depicted in Fig.1.

We just mention that the profile of the metric (VII.1) is \( U \)-independent and therefore \( U \)-translation, \( U \rightarrow U + \epsilon \) is an additional isometry. This carries over trivially to its Finslerized line element (III.6), since both the pp-wave metric and the “Bogoslovsky-Finsler factor” are invariant.

VIII. BOGOSLOVSKY-FINSLER-FRIEDMANN-LEMAÎTRE MODEL

In this section we shall describe a simple extension of Bogoslovsky’s theory to take into account the expansion of the universe. For some previous work see [57–60]. In contrast to our work, these authors consider only Finsler metrics which share the isotropy and spatial homogeneity of Friedmann-Lemaître models. This necessarily excludes the use of a null vector field.

A. The \( \Lambda \)CDM model

The simplest standard model consistent with current observational data is the spatially flat Friedmann-Lemaître model with metric
\[
g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2 dx^2 \tag{VIII.1}
\]
where \( a = a(t) \) and \( x = (x, y, z) \). The “scale factor” \( a(t) \) is determined by the Einstein equations once the matter content has been specified. The favoured \( \Lambda \)CDM models has
\[
a(t) = \sinh^2 \left( \frac{\sqrt{3\Lambda}}{4} t \right), \tag{VIII.2}
\]
FIG. 1: Consistently with (IV.13), the Bogoslovsky-Finsler geodesics project to the same curve in 2D transverse space for all values of the parameter $b$ while their $v$ coordinates differ, according to (IV.22), in a $b$-dependent term, which is linear in retarded time, $u$. Experiments indicate that the anisotropy and hence $b$ is very small. When $b \to 1$ the trajectory approaches to the massless one (in heavy black), consistently with (IV.22).

which enjoys the remarkable property that the jerk equals one,

$$j = a^2 \left( \frac{da}{dt} \right)^2 - 3 \frac{d^3a}{dt^3} = 1.$$ 

See [61] for details and original references.

Here we shall leave the precise form of $a(t)$ unspecified. The coordinate $t$ is called cosmic time. The spatial coordinate $x$ is usually said to be comoving since the world lines of the cosmic fluid have constant $x$. Two events simultaneous with respect to constant time, i.e. with $x_1^\mu = (0, x_1)$ and $x_2^\mu = (0, x_2)$ have a time-dependent proper separation $a(t) (x_1 - x_2)$.

**B. The choice of null vector field**

The vector field

$$l^\mu \frac{\partial}{\partial x^\mu} = g(t) \frac{1}{\sqrt{2}} \left( a \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right)$$

where $g(t)$ is a non-vanishing arbitrary function is past directed and null but is neither covariantly constant nor Killing, as it can be checked by a tedious calculation. The associated
one form is
\[ l_\mu dx^\mu = -a^2 g \frac{1}{\sqrt{2}} \left( \frac{1}{a} dt + dz \right). \]  

(VIII.5)

If \( \dot{x}^\mu = \frac{dx^\mu}{dx} \) then
\[ \mathcal{L} = -m(a^2 g)^b \left( \frac{1}{\sqrt{2}} \left( a(t)^{-1} \dot{t} + \dot{\mathbf{x}} \right) \right)^b (\dot{t}^2 - a^2 \dot{\mathbf{x}}^2)^{1/2(1-b)} \]  

is a possible Bogoslovsky-Finsler type Lagrangian for a particle of mass \( m \). It admits three commuting symmetries generated by \( \frac{\partial}{\partial x} \) and hence three conserved momenta \( p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \).

If \( b = 0 \) then (VIII.6) is the standard action for a freely moving particle in a flat isotropic Friedmann-Lemaître universe.

C. Hubble Friction

A notable feature of the free motion of a massive particle moving in a Friedmann-Lemaître universe is Hubble friction. The conserved momenta are
\[ p = ma^2 \frac{dx}{d\tau} \text{ where } \quad d\tau = \sqrt{1 - (a \frac{dx}{dt})^2} \ dt. \]  

(VIII.7)

Here \( d\tau \) is the increment of proper time along the world line of a particle. The four-velocity of the particle with respect to the local inertial reference frame \( \frac{\partial}{\partial t}, \frac{\partial}{\partial \mathbf{x}} \) is
\[ u = a(t) \frac{dx}{d\tau} \quad \text{whence } \quad u = \frac{p}{ma(t)}. \]  

(VIII.8)

One may also define a velocity \( v \) measured in units of cosmic time \( t \), \( v = a(t) \frac{dx}{dt} \), so that
\[ d\tau = \sqrt{1 - v^2} \ dt, \quad u = \frac{v}{\sqrt{1 - v^2}} \quad \text{and} \quad v = \frac{p}{m \sqrt{a^2 + p^2/m^2}}. \]  

(VIII.9)

Hence
\[ \frac{dx}{dt} = \frac{p}{a \sqrt{m^2 a^2 + p^2}}. \]  

(VIII.10)

Thus in an expanding phase in which \( a(t) \) increases with time both \( v \) and \( u \) decrease with time. However, as a consequence of isotropy, their directions remain constant. The fact that we have three conserved momenta and the constraint
\[ (\frac{dt}{d\tau})^2 = 1 + a^2 (\frac{dx}{d\tau})^2 \]  

(VIII.11)

implies that the system of geodesics is completely integrable. In fact
\[ dx = \frac{p}{ma^2 \sqrt{1 + (p/m)^2}} \quad \text{and} \quad d\tau = \frac{dt}{\sqrt{1 + (p/m)^2}}. \]  

(VIII.12)
D. Conformal Flatness

Before proceeding further we recall that the Friedmann-Lemaître metric (VIII.1) is conformally flat, as becomes clear if we define conformal time $\eta$ by

$$\eta(t) = \int_t^t \frac{dt}{a(t)} \quad \text{(VIII.13)}$$

where the lower limit is left unspecified for the time being. In terms of cosmic time, the Friedmann-Lemaître metric (VIII.1) becomes

$$g_{\mu\nu}dx^\mu dx^\nu = a^2\left\{-d\eta^2 + dz + dx^i dx^i\right\} = a^2\left\{2dudv + dx^i dx^i\right\} \quad \text{(VIII.14)}$$

where $a^2$ is regarded as a function of conformal time $\eta$ and we introduced the light-cone coordinates

$$u = \frac{z + \eta}{\sqrt{2}}, \quad v = \frac{z - \eta}{\sqrt{2}}. \quad \text{(VIII.15)}$$

From (VIII.5) we learn that $l_\mu dx^\mu = -a^2gdu$ whence our Bogoslovsky-Finsler-ized line element is

$$ds = a^{1+b}g^b\left(-2dudv - dx^i dx^i\right)^{\frac{1}{2}(1-b)}(du)^b. \quad \text{(VIII.16)}$$

This Lagrangian would yield Bogoslovsky’s original flat spacetime model provided we choose $g(t)$ such that

$$f = a^{1+b}g^b = 1. \quad \text{(VIII.17)}$$

The only freedom with this model would be to introduce an arbitrary factor

$$\mathcal{L}_f = f(\eta)\left(-2\dot{u}\dot{v} - \dot{x}^i \dot{x}^i\right)^{\frac{1}{2}(1-b)}(\dot{u})^b, \quad \text{(VIII.18)}$$

which amounts to saying that the mass depends upon cosmic time.

This situation is the same as in the ordinary spatially flat Friedmann-Lemaître cosmology for which $b = 0$. We can either say the universe is expanding, but our rulers, constructed from massive particles all of whose masses are constant in cosmic time $t$, or that the universe is time independent but the rulers all change with the same time dependence. In that case the phenomenon of Hubble friction would be ascribed not to the expansion of the universe but to that masses are getting heavier.
E. Redshifting

If we adopt (VIII.18) then light rays move along straight lines in \((\eta, x)\) coordinates. Emitters and Observers (e.g. Galaxies and Astronomers) are usually held to be at rest in these coordinates.

Suppose the Observer is at the origin at \((\eta_0, 0, 0, 0)\) and receives light rays from a galaxy at \((\eta_e, x_e, y_e, z_e)\) so that the duration of emission in conformal time is \(d\eta_e\) and the duration of the corresponding observation is \(d\eta_0\), then

\[
d\eta_e = d\eta_0.
\] (VIII.19)

Then the emitted and observed propertimes are \(d\tau_e = f(\eta_e)d\eta_e\), \(d\tau_0 = f(\eta_0)d\eta_0\) and so the redshift is

\[
1 + z = d\tau_0/d\tau_e = f(\eta_0)/f(\eta_e).
\] (VIII.20)

Thus if the universe is expanding, that is if \(f' > 0\), then the signal received is redshifted and contrariwise if the universe is contracting, that is \(f' < 0\).

Note that under these assumptions, the emitted light from all galaxies at the same conformal time will be redshifted in the same way. That is: the redshift should be isotropic.

F. A possible choice for \(f(\eta)\)

As mentioned earlier, our observed universe is well described by a scale factor \(a(t)\) given by (VIII.2). Applying the Einstein equations to the Friedmann–Lemaître metric (VIII.1) one finds that it is supported a pressure-free fluid (some of it visible and some of it not — so-called dark matter) and a positive cosmological constant term \(\Lambda\) often called “dark energy”. Near the “Big Bang” i.e. for small \(t\), \(a(t) \propto t^{2/3}\) because the \(\Lambda\) term is negligible. This is the Einsten-de Sitter model. At late times \(a(t) \propto \exp \sqrt{\Lambda/3} t\) which exhibits cosmic acceleration. This is de Sitter spacetime.

From (VIII.13) choosing (VIII.2) and setting \(ag = 1\) in (VIII.16) we have

\[
\eta(t) = \int_0^t \sinh^{-\frac{2}{3}}\left(\frac{\sqrt{3\Lambda}}{4} \tilde{t}\right) d\tilde{t}
\] (VIII.21a)

\[
f(\eta) = a(t) = \sinh^{\frac{2}{3}}\left(\frac{\sqrt{3\Lambda}}{4} t\right),
\] (VIII.21b)
FIG. 2: The conformal factor (VIII.21b) of the Friedmann-Lemaître model (VIII.1), expressed as function of the conformal time, $\eta$, obtained by numerical integration of (VIII.21).

This step depends only on the scale factor $a$ in (VIII.1) and does not involve the deformation parameter $b$. See Fig.2. It is worth noting that conformal time as a function of cosmic time is bounded from above – as it happens for de Sitter space, to which our spacetime tends when $t \to \infty$.

IX. BOGOSLOVSKY-FINSLER-FRIEDMANN-LEMAÎTRE GEODESICS

Written in coordinates $(\eta, x, y, z)$, the Lagrangian (VIII.18) is

$$\mathcal{L}_f = -mf(\eta)(\dot{\eta} + \dot{z})b(\dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{\frac{1}{2}(1-b)},$$

providing us with the momenta

$$p_x = m(1-b)f(\eta)\dot{x}(\dot{\eta} + \dot{z})b(\dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{-\frac{1}{2}(1+b)} \tag{IX.2a}$$

$$p_y = m(1-b)f(\eta)\dot{y}(\dot{\eta} + \dot{z})b(\dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{-\frac{1}{2}(1+b)} \tag{IX.2b}$$

$$p_z = m(1-b)f(\eta)\dot{z}(\dot{\eta} + \dot{z})b(\dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{-\frac{1}{2}(1+b)} - \frac{b}{\sqrt{2}}m f(\eta)(\dot{\eta} + \dot{z})^{1+b}(\dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{\frac{1}{2}(1-b)} \tag{IX.2c}$$

$$p_{\eta} = -mf(\eta)(\dot{\eta} + \dot{z})b^{-1}(\dot{\eta}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{-\frac{1}{2}(1+b)}$$

$$\left((1-b)\eta(\dot{\eta} + \dot{z}) + \frac{b}{\sqrt{2}}(\dot{\eta}^2 - \dot{x}^2 + \dot{y}^2 - \dot{z}^2)\right) \tag{IX.2d}$$

29
Evidently the three momenta \( p_x, p_y, p_z \) are conserved. Moreover, since
\[
\frac{p_y}{p_x} = \frac{dy}{dx} \tag{IX.3}
\]
the projection of the geodesics onto the transverse \( x - y \) plane are straight lines. Choosing the proper time as parameter, \( \lambda = \tau \), one has the constraint
\[
f(\eta)(\frac{\dot{\eta} + \dot{z}}{\sqrt{2}})^b(\dot{\eta}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{\frac{1}{2}(1-b)} = 1 \tag{IX.4}
\]
which we may re-write in terms of conformal time, \( \eta \), as an equation for \( \tau \) as
\[
\tau' = f(\eta)(\frac{1+\dot{z}'}{\sqrt{2}})(1 - (\dot{x}')^2 - (\dot{y}')^2 - (\dot{z}')^2)^{\frac{1}{2}(1-b)} . \tag{IX.5}
\]
where \((\dot{x}', \dot{y}', \dot{z}') = (\frac{dx}{d\eta}, \frac{dy}{d\eta}, \frac{dz}{d\eta})\).

If \( f' = 0 \) then the \( p_\mu / f(\eta) \) are independent of \( \eta \) leaving us with the same straight line motion at constant velocity as for the flat Bogoslovsky spacetime.

As we have seen above, if even if \( f' \neq 0 \), the projection of the motion on the \( x - y \) plane are straight lines although not with constant speed with respect to the conformal time \( \eta \). The speeds of the projections on to the \( x - z \) and \( y - z \) planes are also not at constant \( \eta \)-speed but in addition they are not straight lines either. Over conformal \( \eta \) times short compared with \( \frac{f}{f'} \) they are approximately straight lines with slopes given by \( \frac{p_x}{mf(\eta)} \) but over longer the speeds and directions change reflecting precisely the effects of Hubble Friction.

The geodesics are conveniently studied by switching to conformal time, \( \eta \). Introducing
\[
\dot{u} = \frac{\dot{\eta} + \dot{z}}{\sqrt{2}} \quad \text{and} \quad \dot{w} = \dot{\eta}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 \tag{IX.6}
\]
in place of \( \dot{\eta} \) and \( \dot{z} \), the equations (IX.2a)-(IX.2b)-(IX.2c) become
\[
(m(b - 1)f(\eta)\dot{x}\dot{u}^b + px\dot{w}^{\frac{b+1}{2}} = 0 \tag{IX.7a}
\]
\[
(m(b - 1)f(\eta)\dot{y}\dot{u}^b + py\dot{w}^{\frac{b+1}{2}} = 0 \tag{IX.7b}
\]
\[
\frac{mf(\eta)}{2\sqrt{2}}\dot{w}^{b-1}\dot{w}^{\frac{1}{2}(b-1)}((2(b - 1)\dot{u}^2 + (b + 1)\dot{w} - (b - 1)(\dot{x}^2 + \dot{y}^2)) + p_z = 0. \tag{IX.7c}
\]

The first two eqns imply identical evolution,
\[
\dot{x} = \frac{\dot{u} - b\dot{w}^{\frac{b+1}{2}}}{m(1 - b)f(\eta)}p_x, \quad \dot{y} = \frac{\dot{u} - b\dot{w}^{\frac{b+1}{2}}}{m(1 - b)f(\eta)}p_y \tag{IX.8}
\]
which confirms once again that the transverse projection is a straight line, owing to \( \dot{x}/\dot{y} = p_x/p_y = \text{const.} \) With their help (IX.7c) becomes

\[
-\sqrt{2}(b-1)m^2 f(\eta)^2 \dot{u}^{2b} (2(b-1)\dot{u}^2 + (b+1)\dot{w}) - 4(b-1)m p_z f(\eta) \dot{u}^{b+1} w^{\frac{b+1}{2}} + \sqrt{2} (p_x^2 + p_y^2) \dot{w}^{b+1} = 0.
\]  

(IX.9)

By reparametrizing \( w(\lambda) \) as

\[
w(\lambda) = \int \sigma(\lambda)^2 \dot{u}(\lambda)^2 d\lambda,
\]

(IX.10)

where \( \sigma(\lambda) \) is a new function that we introduce, equation (IX.9) reduces from a differential one to an algebraic

\[
-\sqrt{2}(b-1)m^2 f(\eta)^2 ( (b+1)\sigma(\lambda)^2 + 2(b-1) ) - 4(b-1)m f(\eta) p_z \sigma(\lambda)^{b+1} + \sqrt{2} (p_x^2 + p_y^2) \sigma(\lambda)^{2(b+1)} = 0.
\]

(IX.11)

For \( b = 0 \) this is simply quadratic in \( \sigma(\lambda) \), but for \( b \neq 0 \) it is not trivial to solve it for \( \sigma \), cf. Fig. 3a. However as it is quadratic in \( f(\eta) \), the inverse problem (which amounts to choosing \( \sigma(\lambda) \) to find the corresponding \( f(\lambda) \)) still works. The functions \( \sigma(\eta) \) and \( w(\eta) \) are plotted in Fig.3. Using (IX.6), (IX.10) and (IX.8) we get

![Graphs of \( \sigma(\eta) \) and \( w(\eta) \) for different values of \( b \).](image)

FIG. 3: (a) \( \sigma(\eta) \) and (b) \( w(\eta) \) in (IX.10) plotted for \( b = 0 \) and for \( b = 0.5 \).
\[ \dot{\eta} = \frac{\dot{u}}{2\sqrt{2}} \left( 2 + \sigma^2 + \frac{(p_x^2 + p_y^2) \sigma^{2(b+1)}}{(b-1)^2 m^2 f(\eta)^2} \right) \]  
(IX.12a)

\[ \dot{x} = \dot{u} \frac{\sigma^{b+1}}{(1-b)m f(\eta)} p_x \]  
(IX.12b)

\[ \dot{y} = \dot{u} \frac{\sigma^{b+1}}{(1-b)m f(\eta)} p_y \]  
(IX.12c)

\[ \dot{z} = \frac{\dot{u}}{2\sqrt{2}} \left( 2 - \sigma^2 - \frac{(p_x^2 + p_y^2) \sigma^{2(b+1)}}{(b-1)^2 m^2 f(\eta)^2} \right) \]  
(IX.12d)

together with the algebraic constraint between \( \sigma(\lambda) \) and \( f(\eta(\lambda)) \) in (IX.11).

The joint system can be shown to satisfies the Euler-Lagrange equations.

The \( u(\lambda) \) that remains unspecified in (IX.12) and is disappeared from (IX.11) serves as a gauge parameter (by seeing the ratios of derivatives that are being formed in (IX.12)), which we can simply set \( u(\lambda) = \lambda \). So in this time-gauge \( \eta(\lambda) + z(\lambda) = \sqrt{2}\lambda \), which is compatible with the (IX.12a) and (IX.12d) as seen above. We thus have, in the “conformal-time gauge”

\[ \frac{dx}{d\eta} = \pm \frac{2\sqrt{2}(1-b)m \sigma^{b+1} f(\eta)}{(1-b)^2 m^2 (\sigma^2 + 2) f(\eta)^2 + (p_x^2 + p_y^2) \sigma^{2(b+1)}} p_x \]  
(IX.13a)

\[ \frac{dy}{d\eta} = \pm \frac{2\sqrt{2}(1-b)m \sigma^{b+1} f(\eta)}{(1-b)^2 m^2 (\sigma^2 + 2) f(\eta)^2 + (p_x^2 + p_y^2) \sigma^{2(b+1)}} p_y \]  
(IX.13b)

\[ \frac{dz}{d\eta} = \pm \frac{2 - \sigma^2 - \frac{(p_x^2 + p_y^2) \sigma^{2(b+1)}}{(1-b)^2 m^2 f(\eta)^2}}{2 + \sigma^2 + \frac{(p_x^2 + p_y^2) \sigma^{2(b+1)}}{(1-b)^2 m^2 f(\eta)^2}} \]  
(IX.13c)

**The Friedmann-Lemaître case \( b = 0 \)**

If \( b = 0 \), the algebraic relation (IX.11) is quadratic and can be solved for \( \sigma \),

\[ \sigma(\eta) = \pm \frac{\sqrt{2} m f(\eta)}{\sqrt{m^2 f(\eta)^2 + p^2}} p_z, \]  
(IX.14)

shown by the blue line in Fig. 3a for the upper sign \( ^6 \). In terms of conformal time \( \eta \)

\[ \frac{dx}{d\eta} = \pm \frac{p}{\sqrt{m^2 f(\eta)^2 + p^2}}. \]  
(IX.15)

\( ^6 \) Choosing the lower sign would amount to an overall sign change when that of \( p_z \) is also reversed.
to be compared with (VIII.10). The consistency with the eqns in sec.VIII C follows from
\[
\begin{align*}
\frac{d\tau}{m f^2 d\eta} &= \frac{d\eta}{\sqrt{m^2 f^2 + p^2}}, \\
\frac{dx}{d\eta} &= \frac{dp}{\sqrt{m^2 f^2 + p^2}} \Rightarrow \frac{dx}{d\tau} = \frac{1}{m f^2} p.
\end{align*}
\] (IX.16)

We couldn’t get analytical expressions, however using the numerically calculated values of \(f(\eta)\) (see Fig.2) allow us to plot \(x(\eta)\) by solving (IX.15), as shown in Fig.4 7.

FIG. 4: For \(b = 0\) all trajectories follow straight lines and have identical evolution. For \(b = 0.5\) \(z(\eta)\) become different from the transverse trajectories \((x(\eta), y(\eta))\) consistently with (IX.13), as shown in Fig.5.

FIG. 5: For \(b > 0\) the motion in the \(x - z\) plane is not more along a straight line (as it is for \(b = 0\)). The Hubble friction slows down the \(z\) motion for \(b = 0\) but not when \(b > 0\).

7 The two signs in (IX.14) can be compensated by \(p \rightarrow -p\) implying an overall sign change. In what follows the upper sign will be choosen.
The $b = 0$ case nicely illustrates *Hubble Friction* : all trajectories slow down and ultimately come to rest. For $b \neq 0$ it seems that this happens only for the transverse motion but not for the motion in the $z$-direction. The slowing down in the transverse case is plausible from equations (IX.12b) - (IX.12c).

**FIG. 6:** For the Friedmann-Lemaître model for $b = 0$ the 3D trajectory is a straight line. For the Bogoslovsky-Finsler modification $b = 0.5$, however, while the projection to the $x - y$ plane is still along a straight line, the $z$-component becomes curved, consistently with Fig.5.

**X. CONCLUSION**

In this paper, motivated by work by Bogoslovsky [3–5, 7], and by that of Tavakol and Van den Bergh, and Roxburgh [9, 11, 12], and by Cohen and Glashow [1], and more recently by others [14–21], we have studied the free motion of a massive particle moving in a one parameter family of Finslerian deformations of a plane gravitational wave. By free motion we mean that it extremises the proper time along its timelike world line. Finslerian proper time is measured by replacing the usual square-root integrand

\[ \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \]

by

\[ (-g_{\mu\nu}l^\mu \frac{dx^\nu}{d\lambda})^b (-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda})^{\frac{1}{2}(1-b)}, \]
where $l^\mu$ is a null vector field and $b$ is a dimensionless constant.

In earlier work we have shown that because of the five dimensional isometry group of plane gravitional waves, the motion of the usual timelike geodesics is completely integrable. In the present paper we have shown that the five dimensional partially broken Carroll symmetry group $G_5$ remains a symmetry of our Finslerian line element provided we choose the null vector field $l^\mu$ to be the covariantly constant null vector of the underlying gravitational wave. As a consequence we find that that not only is the free motion completely integrable but it differs only in that the “vertical” coordinate $v$ involves in turn a $b$-dependent term, which is linear in the retarded time coordinate $u$. The motion in the transverse directions is unchanged. The situation is analogous to what happens for massive vs. massless geodesics in a pp wave [39].

The symmetry of the Bogoslovsky-Finsler model is in fact of the Very Special Relativity (VRS) type; in the Minkowski case it is the 8-parameter DISIM$_b(2)$ [7, 37, 42]. The clue is to deform an $u$-$v$ boost $N_0$ to $N_b$ in as (V.8). The trick works for certain non-trivial profiles, as for the $U^{-2}$ discussed at the end of sec. V.

We have also examined the free motion of a Finslerian deformation of a homogeneous pp wave which is an Einstein-Maxell solution. The resulting spacetime is a Cahen-Wallach symmetric space [52] and arises in a wide variety of physical applications and whose null reduction in the fashion of Eisenhart, and Duval et al [27] is a simple harmonic oscillator with a Newton-Hooke type symmetry. Here again, the free motion is qualitatively independent of the deformation parameter $b$.

We have also studied a simple anisotropic cosmological model based on that of Friedmann and Lemaître with vanishing spatial curvature. Because the latter is conformally flat, the motion of massive particles is equivalent to motion in flat Bogoslovsky spacetime except that all masses become time dependent with identical time dependence.

Although our present universe shows little sign of anisotropy of the sort that arises in Bogoslovsky-Finsler metrics, that may not have been true earlier in the history of the universe since the absence of anistropy now is usually ascribed to a rapid phase of inflation during which the scale factor of the universe increased by a factor of perhaps 60 e-folds. It is of interest therefore to study geodesics in Bogoslovsky-Finsler deformations of Friedmann-Lemaître metrics.
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