Branched Polymer Revisited

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Abstract

We show that correlation functions for branched polymers correspond to those for $\phi^3$ theory with a single mass insertion, not those for the $\phi^3$ theory themselves, as has been widely believed. In particular, the two-point function behaves as $1/p^4$, not as $1/p^2$. This behavior is consistent with the fact that the Hausdorff dimension of the branched polymer is four.

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1 Introduction

Branched polymers are the simplest generalization of the random walk and have been studied extensively. It is of great importance not only in statistical physics but also in particle physics, in particular, for understanding the critical behavior of random surface models and quantum gravity (see, for example, [1]). In the paper [2], we have studied the dynamics of the type IIB matrix model of a constructive formulation of superstring ([3] and see [4] for review). In our matrix model approach, the eigenvalues of matrices are interpreted as space-time coordinates. In these investigations, we find the system of branched polymers in a simple approximation. Although it is far from the flat four dimensional manifold, branched polymers share the same (fractal) dimensionality four with our space-time. It might be the first indication that superstring can explain the dimensionality of our space-time.

In this letter, we comment on a field theoretic description of branched polymers. It is well-known that a system of random walks is described by a free scalar field theory if there is no effect of self-avoidance. Similarly it is widely believed that the system of branched polymers becomes a scalar field theory with a three-point coupling, that is, $\phi^3$ scalar field theory (see, for example, [1]). We will, however, show that it is not so by treating the universal part of the partition function carefully. The system of branched polymers without self-avoidance can be exactly solvable by introducing the grand canonical ensemble and using the so called Schwinger-Dyson technique. In order to extract the correct large $N$ limit ($N$ is the system size) or the thermodynamic limit, we have to check that the grand canonical ensemble is dominated by the larger size system. In other words, we have to extract the universal part.

On the contrary to a naive summation which makes us conclude that the system of branched polymers is described by a $\phi^3$ scalar field theory, a careful treatment shows that we need a single mass insertion in each n-point function of the $\phi^3$ scalar field theory. Mass insertion here means a change of a propagator in each n-point function from an ordinary one, $1/p^2$, to $1/p^4$. In particular, the two-point function behaves as $1/p^4$, not $1/p^2$. Let us count the number of points which lie within distance $R$ from a fixed point in $D (> 4)$ dimensions. In the random walk, it can be estimated as $R^2$ by using the two point function. We obtain $R^4$ for branched polymers by using the $1/p^4$ type propagator. So our finding are consistent with the claim that branched polymers are four dimensional fractals. A multi-point Green function is given by a sum of graphs of the corresponding Green function for $\phi^3$ scalar field theory at tree level with a single mass insertion in each graph.

The organization of this paper is the following. First we define a canonical ensemble for a system of branched polymers (sec. 2.1) and then introduce grand canonical ensembles (sec. 2.2). We emphasize that a definition of grand canonical ensemble is not unique. In sec. 2.3, we solve Schwinger-Dyson equations for the grand canonical ensembles and obtain naive results of the correlation functions of a scalar $\phi^3$ theory. In sec. 2.4, we consider the thermodynamic limit and give the correct universal behavior of the correlation functions. Finally in section 3, we conclude and give an interpretation of our result.

2 Branched polymer dynamics
2.1 Canonical Ensemble

Branched polymers are a statistical system of \( N \) points connected by \( N - 1 \) bonds whose lengths are of order \( a_0 \). The canonical partition function is defined as

\[
Z_N = \frac{1}{N!} \sum_{G: \text{tree graph}} \int \prod_{i=1}^{N} dx_i \prod_{(ij): \text{bond of } G} f(x_i - x_j),
\]

(1)

where \( f(x) \) is a function assigned to each bond in each graph, and it damps sufficiently fast at long distance compared to the typical length scale \( a_0 \). (See, for example, figure 1.) The presence of the factor \( 1/N! \) is due to the fact that the \( N \) points are regarded identical.

We can calculate a partition function for each \( N \), by counting the number of all possible tree graphs:

\[
\begin{align*}
Z_1 & = V \\
Z_2 & = \frac{1}{2!} \hat{f}(0)V \\
Z_3 & = \frac{1}{3!} 3 \hat{f}(0)^2V \\
Z_4 & = \frac{1}{4!} 16 \hat{f}(0)^3V \\
& \vdots \\
Z_N & = \frac{1}{N!} N^{N-2} \hat{f}(0)^{N-1}V,
\end{align*}
\]

(2)-(6)

where \( \hat{f}(p) \) is a Fourier transform of \( f(x) \);

\[
\hat{f}(p) = \int dx \ e^{ipx} f(x),
\]

(7)

and \( V \) is the total volume of the system. A derivation of the general form \( (6) \) is given in the appendix.

We define an (unnormalized) \( m \)-point correlation function of density operators as

\[
G_N^m(x^1, \ldots, x^m)
\]

Figure 1: Form of the function \( f(x) \). It damps sufficiently fast at long distance compared to the typical length scale \( a_0 \).
\[ = \langle \rho(x^1) \cdots \rho(x^m) \rangle_N \]
\[ = \frac{1}{N!} \sum_{G, \text{tree graph}} \int \prod_{i=1}^{N} dx^i \prod_{(ij) \text{: bond of } G} f(x^i - x^j) \ \rho(x^1) \cdots \rho(x^m), \tag{8} \]

where the density operator is defined by
\[ \rho(x) = \sum_{i=1}^{N} \delta^{(d)}(x - x^i). \tag{9} \]

### 2.2 Grand canonical ensemble

We then define partition functions and m-point correlation functions in the grand canonical ensemble in the following equations:

\[ Z_{\kappa_0, l} = \sum_{N=1}^{\infty} N^l \kappa_0^N Z_N, \tag{10} \]
\[ G_{\kappa_0, l}^m(x^1, \cdots, x^m) = \sum_{N=1}^{\infty} N^l \kappa_0^N G_N^m(x^1, \cdots, x^m), \tag{11} \]

where \( \kappa_0 \) is fugacity. There exist various ways of defining grand canonical ensemble corresponding to the freedom to choose different weights for each fixed \( N \) sector. Here, we assigned an extra \( N \)-dependent factor, \( N^l \), in addition to the usual one, \( \kappa_0^N \). The criterion for a ”good” grand canonical ensemble is such that we can take the correct thermodynamic limit, or, in other words, we can correctly take the universal part in the sum (10) or (11). That is, the correlation functions in the grand canonical ensemble at the critical value of fugacity should reproduce those in the canonical ensemble for large \( N \):

\[ \lim_{N \to \infty} \frac{G_N^m}{Z_N} = \lim_{\kappa_0 \to \kappa_0,c} \frac{G_{\kappa_0, l}^m}{Z_{\kappa_0, l}}. \tag{12} \]

We illustrate the above mentioned criterion by taking partition function as an example. Since the canonical ensemble partition function (6) behaves at large \( N \) as

\[ Z_N \sim \frac{N^{-5/2}}{\sqrt{2\pi e}} N^{N-1} \hat{f}(0)^{N-1} V, \tag{13} \]

the grand canonical partition function is approximated by

\[ Z_{\kappa_0, l} \sim \frac{V}{\sqrt{2\pi \hat{f}(0)}} \sum_{N=1}^{\infty} N^{l-5/2} (\kappa/\kappa_c)^N \tag{14} \]
\[ \sim \frac{V}{\sqrt{2\pi \hat{f}(0)}} \int_{0}^{\infty} dN \ N^{l-5/2} \ e^{-N\Delta\kappa/\kappa_c}, \tag{15} \]

where

\[ \kappa = \hat{f}(0)\kappa_0, \tag{16} \]
\[ \kappa_c = e^{-1}, \tag{17} \]
\[ \Delta\kappa = \kappa_c - \kappa. \tag{18} \]
If we take \( l \) sufficiently large, the integrand in eq. (13) has a peak at \( N \sim \kappa_c/\Delta \kappa \) and we can make \( N \) large by approaching \( \kappa \) to \( \kappa_c \). On the other hand, if \( l \) is not sufficiently large, a non-universal small \( N \) behavior dominates the summation and we cannot obtain the correct answer of the large \( N \) limit by a grand canonical ensemble.

### 2.3 Schwinger-Dyson eq.

In this subsection, we recapitulate the (naive) arguments that the correlation functions for branched polymers are given by massless \( \phi^3 \) theory [1][2]. Let's consider the following correlation functions, which are suitable for a Schwinger-Dyson analysis:

\[
F^m(x_1, \ldots, x^m) = \sum_{N=1}^{\infty} \frac{\kappa^N}{(N-m)!} \sum_{G:\text{tree graph}} \int \prod_{i=1}^{N} dy^i \delta^d(y^1 - x^1) \cdots \delta^d(y^m - x^m) \\
\times \prod_{(ij):\text{bond of } G} f(y^i - y^j). \tag{19}
\]

The factor \( \frac{1}{(N-m)!} \) means that \( (N-m) \) points other than the fixed \( m \) points are regarded identical. We can see from eq. (11) and eq. (8) that

\[
F^m(x_1, \ldots, x^m) \simeq C^m_{\kappa_0, l=0}(x_1, \ldots, x^m) \tag{20}
\]

in the large \( N \) limit. We write a Fourier transform of \( F^m(x_1, \ldots, x^m) \) as \( \hat{F}^m(p_1, \ldots, p^{m-1}) \):

\[
(2\pi)^d \delta^d(p_1 + \cdots + p^m) \hat{F}^m(p_1, \ldots, p^{m-1}) = \int d^d x_1 \cdots d^d x^m e^{ip^1 x_1} \cdots e^{ip^m x^m} F^m(x_1, \ldots, x^m). \tag{21}
\]

Schwinger Dyson equation for 1-point function \( \hat{F}^1 \) becomes

\[
b = \kappa e^b, \tag{22}\]

where

\[
b \equiv \hat{f}(0) \hat{F}^1, \tag{23}\]

as can be seen from figure 4. Figure 4 illustrates eq. (22).

At the critical point,

\[
b_c = 1, \tag{24}\]
\[
\kappa_c = e^{-1}, \tag{25}\]

1-point function \( \partial b/\partial k \) diverges. Near this critical point, \( N \) becomes large;

\[
\Delta b \simeq \sqrt{2e \sqrt{\Delta \kappa}} \sim 1/\sqrt{N}, \tag{26}\]

where

\[
\Delta b = b_c - b, \tag{27}\]
\[
\Delta \kappa = \kappa_c - \kappa. \tag{28}\]
\[ b = \kappa e^{-b} \]

Figure 2: Schwinger-Dyson equation for one point function. A grey blob and a black point mean \( \hat{F}_1 \) and \( \kappa_0 \), respectively.

\[ \hat{F}^2(p) = \sum_{s=1}^{\infty} \hat{f}(p)^s (\hat{F}^{-1})^{s+1} \]
\[ = \frac{b^2 h(p)}{\hat{f}(0)(1 - bh(p))}, \quad (29) \]
\[ h(p) \equiv \frac{\hat{f}(p)}{\hat{f}(0)} = 1 - ca_0^2 b^2 + \cdots. \quad (31) \]

Figure 3: Schwinger-Dyson equation, \( \kappa = be^{-b} \). At the critical point, \( b_c = 1, \kappa_c = e^{-1} \).

Next, we consider the 2-point function \( \hat{F}^2(p) \). When we pick up any two points on a tree graph, we can fix the path connecting these two points. Thus, as can be seen from figure 3, 2-point function is calculated to be

\[ \hat{F}^2(p) = \sum_{s=1}^{\infty} \hat{f}(p)^s (\hat{F}^{-1})^{s+1} \]
\[ = \frac{b^2 h(p)}{\hat{f}(0)(1 - bh(p))}, \quad (29) \]
\[ h(p) \equiv \frac{\hat{f}(p)}{\hat{f}(0)} = 1 - ca_0^2 b^2 + \cdots. \quad (31) \]

Here \( c \) is a positive constant of order one. Recall that \( f(x) \) damps rapidly out of the region \( 0 < x < a_0 \), as shown in figure 1.

Near the critical point, \( b \sim b_c = 1 \), 2-point correlation function behaves as

\[ \hat{F}^2(p) \sim \frac{1}{\Delta b + ca_0^2 p^2} \quad (33) \]
$F = F^2$ is made out of 1-point function $F^1$, which is written by a gray blob.

\[
\sim \frac{1}{N^{-1/2} + ca_0b^2}.
\]

Equation (34)

Here we used eq. (26). Thus, the correlation length is $\xi = a_0 N^{1/4}$, which implies the Hausdorff dimension of branched polymer is four. If the relevant length scale is shorter than the correlation length, $\hat{F}^2(p) \sim 1/p^2$ behaves like a propagator of a massless scalar field. Let us consider the following region:

\[
a_0 \ll x \ll \xi = a_0 N^{1/4}.
\]

Equation (35)

$a_0$ gives an ultraviolet cut-off whereas $\xi$ gives an infrared cut-off length over which correlation functions damp rapidly. Note that, in this region and near the critical point, the following inequality holds:

\[
1 \ll \frac{1}{1 - bh(p)} \ll \frac{1}{1 - b}.
\]

Equation (36)

Finally, we consider correlation functions of $m > 2$. As in the case of the two-point function, $m$ points are fixed on each tree graph. We can uniquely fix the path connecting them on each graph. Therefore, an $m$-point function $\hat{F}^m$ is represented as a summation over all tree diagrams with $m$ fixed points in which $\hat{F}^2$ appear as propagators. For example,

\[
\hat{F}^3(p, q) = (b/\hat{f}(0))^{-2} \hat{F}^2(p) \hat{F}^2(q) \hat{F}^2(p + q) + (b/\hat{f}(0))^{-1} [\hat{F}^2(p) \hat{F}^2(q) + \hat{F}^2(p) \hat{F}^2(p + q) + \hat{F}^2(q) \hat{F}^2(p + q)],
\]

Equation (37)

as we can see from figure 4. However, because of the inequality (36), the diagrams with the maximum number of propagators dominate. In the case of 3-point function, the first term

\[
\hat{F}^3 = \hat{F}^2 + \hat{F}^2 + \hat{F}^2 + \hat{F}^2
\]

\[
\ldots = \hat{F}^2 \hat{F}^2
\]

Figure 5: 3-point function $\hat{F}^3$ is a summation of tree diagrams whose propagators are $\hat{F}^2$. 

Figure 4: 2-point function $\hat{F}^2$ is made out of 1-point function $\hat{F}^1$, which is written by a gray blob.
in eq. (37) dominates. In general, we obtain a naive result for \( m \)-point correlation functions;

\[
\hat{F}^m \sim \text{correlation functions of massless } \phi^3 \text{ theory at tree level.} \tag{38}
\]

In the next subsection, we will see that these naive results do not correspond to the correct thermodynamic results.

### 2.4 Correlation functions in thermodynamic limit

As we mentioned in eq. (20), \( \hat{G}_{\kappa_0, l=0}^m \) is equal to \( F^m \) in the large \( N \) limit:

\[
\hat{G}_{\kappa_0, l=0}^m(p^1, \cdots, p^{m-1}) \simeq \hat{F}^m(p^1, \cdots, p^{m-1}). \tag{39}
\]

Then, let us consider the \( m \)-point correlation functions with \( l \geq 1 \). From the definition (11), they can be obtained by applying \( l \)-th derivative to the \( l = 0 \) case:

\[
\hat{G}_{\kappa_0, l}^m(p^1, \cdots, p^{m-1}) = (\kappa_0 \frac{\partial}{\partial \kappa_0})^l \hat{G}_{\kappa_0, l=0}^m(p^1, \cdots, p^{m-1}) \tag{40}
\]

\[
= (\frac{b}{1-b} \frac{\partial}{\partial b})^l \hat{G}_{\kappa_0, l=0}^m(p^1, \cdots, p^{m-1}) \tag{41}
\]

2-point function with \( l = 1 \) is given by

\[
\hat{G}_{\kappa_0, l=1}^2(p) \simeq \frac{b}{1-b} \frac{\partial}{\partial b} \left( \frac{b^2 h(p)}{1-bh(p)} \right) \tag{42}
\]

\[
\simeq \frac{1}{(1-b)(1-bh(p))^2} \tag{43}
\]

\[
\simeq \frac{1}{(1-b)p^4} \tag{44}
\]

Here, we have used the inequality (36) since we are interested in the correlation functions in the physical region (35) near the critical point. This behavior is different from that of \( \hat{G}_{\kappa_0, l=0}^2 \sim 1/p^2 \). Similarly for \( l > 1 \), the behavior of 2-point function becomes

\[
\hat{G}_{\kappa_0, l \geq 1}^2(p) \sim \frac{1}{(1-b)^{2l-1} (1-bh(p))^2} \tag{45}
\]

\[
\sim \frac{1}{(1-b)^{2l-1} p^4} \tag{46}
\]

and coincides with the \( l = 1 \) result. Due to the inequality (36), the derivative \( \frac{\partial}{\partial b} \) is dominated to act on \( \frac{1}{1-b} \), not on \( \frac{1}{1-bh(p)} \). Thus, their \( p \)-dependences are all the same. This is the correct thermodynamic limit. That is, we should consider a grand canonical correlation function with \( l \geq 1 \), otherwise a non-universal small \( N \) behavior affects the summation and we cannot obtain the universal result. Note that the behavior of \( \hat{G}_{\kappa_0, l \geq 1}^2(p) \sim 1/p^4 \) is consistent with the fact that the Hausdorff dimension of the branched polymer is four. A naive argument expected from the figure is that the effect of branching can be absorbed by renormalizing the mass. If so, the propagator behaves as that of random walks. We discuss in section 3 why this argument is not correct.
Similarly, 3-point functions become
\begin{align}
\hat{G}_{\kappa_0, l=0}^3(p, q) & \sim g(p)g(q)g(p + q), \\
\hat{G}_{\kappa_0, l\geq 1}^3(p, q) & \sim g(p)g(q)g(p + q) + g(p)g(q)g(p + q) + g(p)g(q)g(p + q)',
\end{align}
where
\begin{align}
g(p) &= \frac{1}{1 - bh(p)} \sim \frac{1}{p^2}, \\
g(p)' &= \frac{1}{(1 - bh(p))^2} \sim \frac{1}{p^4}.
\end{align}
Note that only one propagator in a graph is replaced by $g'(p)$, since the derivative $\frac{\partial}{\partial b}$ is dominated to act on the factor $\frac{1}{1 - b}$, as in the case of 2-point functions. Therefore, for $m$-point correlation functions,
\begin{align}
\hat{G}_{\kappa_0, l=0}^m & \sim \text{correlation functions for } \phi^3 \text{ theory at tree level} \\
\hat{G}_{\kappa_0, l\geq 1}^m & \sim \text{correlation functions for } \phi^3 \text{ theory at tree level with a mass insertion}
\end{align}
and the correct correlation function in the thermodynamic limit should be given by eq. (52), not by eq. (51).

As a consistency check, the following relation between an $(m+1)$-point function and an $m$-point function must hold:
\begin{equation}
\hat{G}_{\kappa_0}^{m+1}(p^1, \ldots, p^{m-1}, p^m = 0) = \hat{G}_{\kappa_0}^m(p^1, \ldots, p^{m-1}).
\end{equation}
It actually holds because in the L.H.S. of eq. (53), the special class of diagrams dominate in which the $m$-th end point is attached to the propagator $g'(p^m)$. It is due to the inequality $g(p = 0) \ll g(p = 0)'$. Then, it is equal to the R.H.S. of eq. (53).

## 3 Conclusion and discussion

In this letter we have shown that the correlation functions for branched polymers are given by those for $\phi^3$ theory at tree level with a single mass insertion if we correctly take the thermodynamic limit. It is not given by those for $\phi^3$ theory at tree level themselves, as has been widely believed.

Our result can be interpreted as follows. Since the 1-point function behaves as
\begin{equation}
G_1^1_N \sim N^{-3/2}(e \hat{f}(0))^N
\end{equation}
at large $N$ (see eq. (13)), we obtain a relation;
\begin{equation}
G_1^1_N \gg \int dN' G_1^1_{N'} G_1^1_{(N-N')}. 
\end{equation}
This relation reminds us of the situation in the two-dimensional quantum gravity (see, for example, [3]). Two dimensional quantum gravity is known to describe a fractal space-time.
It consists of numerous tiny baby universes and a single mother universe. In our case, this analogy means that if we divide any tree graph into two by cutting a bond, we find only finite points in one of them and the most of the points belong to the other. In the case of \( m \)-point functions, there is a single mother universe on a blob in the path connecting these \( m \) points. Let’s consider the case of the two-point function as a simplest example. Naively the figure [1] implies that the propagator behaves as that of the simpler random walk if each blob is on an equal footing. However one of the blobs in figure [2] becomes the mother universe consisting of infinitely many points and it is entirely different from the other blobs. Therefore we have to divide the path into two parts by cutting out the mother universe. Each part can be considered as a propagator of random walks whose weights are dressed by blobs. This is the reason why we have obtained the propagator behaving as \( 1/p^4 \) instead of \( 1/p^2 \). In other words, the mother universe (the blob with infinitely many points) corresponds to a mass insertion and the other blobs renormalize the correlation length \( \xi \) of random walks from \( a_0 N^{1/2} \) to \( a_0 N^{1/4} \). We can also apply a similar argument for the higher-point functions. One of the blobs becomes the mother universe, which corresponds to the single mass insertion in the \( \phi^3 \) scalar field theory.

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Appendix

In this appendix, we derive the canonical-ensemble partition function (6) from the Schwinger-Dyson equation (22). Let us solve \( b \) as a form of expansion in \( \kappa \). Each coefficient is calculated to be

\[
\frac{1}{2\pi i} \int_{\kappa=0} db \frac{b}{\kappa^{N+1}} = \frac{1}{2\pi i} \int_{b=0} db (1 - b) e^{-b} \frac{b}{(be^{-b})^{N+1}} = \frac{N^{N-1}}{N!}.
\]

Hence,

\[
b = \sum_{N=1}^{\infty} \frac{N^{N-1}}{N!} \kappa^N.
\]

From the definitions of (1), (19) and (23), \( b \) is expanded as

\[
b = \hat{f}(0) \sum_{N=1}^{\infty} \frac{N}{V} Z_N \kappa_0^N.
\]

Comparing these two expansions, we get the result of eq. (3).
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