REFLECTION GROUPS OF THE QUADRATIC FORM

\[-px_0^2 + x_1^2 + \ldots + x_n^2\] WITH \( p \) PRIME.

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Abstract. We complete the classification of reflective quadratic forms \(-px_0^2 + x_1^2 + \ldots + x_n^2\) for \( p \) prime. We show that for \( p = 5 \), it is reflective for \( 2 \leq n \leq 8 \), for \( p = 7 \) and 11 it is reflective for \( n = 2 \) and 3, and it is not reflective for higher values of \( n \). We also show that it is non-reflective for \( n > 2 \) when \( p = 13, 17, 19, \) and 23. This completes the classification of these forms with \( p \) prime.

1. Introduction

There are only finitely many reflective arithmetic quadratic forms [?]. In [?], Nikulin lists all the strongly square-free lattices of rank 3. In [?] Allcock extended that list to all of the rank 3 ones. One of the tools used by both Nikulin and Allcock in their classifications is Vinberg’s algorithm, which is a method for producing the simple roots of a lattice given the associated quadratic form. Vinberg [?] and McLeod [?] studied quadratic forms

\[f(x) = -px_0^2 + x_1^2 + \ldots + x_n^2\] with \( p = 1, 2, 3 \). In the present paper we consider these same forms for other prime values of \( p \).

We restrict to \( p \) prime because it simplifies the implementation of Vinberg’s algorithm. We wish to know for which values of \( p \) and \( n \geq 2 \) this form is reflective. Nikulin’s list [?] contains no lattices of prime determinant larger than 23. All reflective lattices with form \( \text{(1.1)} \) and square-free determinant appear on his list. We show that with the exception of \( p = 5, 7, \) and 11, Nikulin’s list along with Vinberg’s and McLeod’s is a complete classification of these forms. We show that when \( p = 5 \), it is reflective for \( 2 \leq n \leq 8 \) and when \( p = 7 \) and 11 it is reflective for \( n = 2 \) and 3. We show that it is non-reflective in all higher dimensions, and for all \( n > 2 \) when \( p = 13, 17, 19, \) and 23.

The non-reflectivity of \( \text{(1.1)} \) when \( p = 23 \) and \( n = 3 \) presents an interesting feature. For all other values of \( p \), the first non-reflective case fails to be reflective by containing a face that passes out of hyperbolic space at a point that cannot be a vertex of the fundamental domain. In the \( p = 23 \) case, the fundamental domain is a cylinder whose faces are all hexagons with finite vertices. We show that it is infinite by showing that it has a translation symmetry of infinite order. In this case what we have is a reflection group of hyperbolic type, which Nikulin studied in [?].

We implemented our version of Vinberg’s algorithm with C++. We verified the computations by hand for \( p = 5, 7, \) and 11. For higher values of \( p \) this was impractical.
In what follows, let $V = V^{n+1}$ be an $(n + 1)$-dimensional real vector space with basis $v_0, \ldots, v_n$, and quadratic form $(1.1)$. Let $L = L^{n+1}$ be the integer lattice generated by the same basis. The group $\Theta$ of integral automorphisms is the group of symmetries of $L$ preserving the form $(1.1)$ and mapping each connected component of the set $\{x : f(x) < 0\}$ to itself. This group splits as a semidirect product

$$\Theta = \Gamma \rtimes H$$

where $\Gamma$ is generated by reflections and $H$ is a group of symmetries of an associated polyhedron in hyperbolic $n$-space [?]. We say $L$ is reflective if $H$ is a finite group.

Vinberg gives an algorithm for finding the simple roots of such a lattice in [?]. By root, we mean a primitive vector $r = \sum_{i=0}^{n} k_i v_i \in L$ of positive norm such that reflection across the mirror $r^\perp$ preserves the lattice. This is characterized by the crystallographic condition, which reduces to

$$(1.2) \quad \frac{2k_i}{(r, r)} \in \mathbb{Z} \text{ for } i > 0 \text{ and } \frac{2pk_0}{(r, r)} \in \mathbb{Z}$$

This gives a nice simplification in our case, since having prime determinant means there are fewer possible norms for roots.

Vinberg’s algorithm performs a batch search in order of increasing height, where height is given by the formula

$$(1.3) \quad \frac{k_0^2}{(r, r)}$$

and is understood to be the distance of a potential new root from the control vector $v_0$. We extend a system of simple roots $e_1, \ldots, e_n$ for the stabilizer of $v_0$, where $e_i = v_{i+1} - v_i$ for $1 \leq i < n$ and $e_n = -v_n$.

### 2. The reflective lattices

We will treat the $p = 5$ case in detail to show how the computation works. For the other cases we include Coxeter diagrams and tables of roots in the appendix.

When $p = 5$, $(1.2)$ implies that if $r$ is a root, $(r, r) = 1, 2, 5, \text{ or } 10$, and if $(r, r) = 5 \text{ or } 10$ then $5 \nmid k_0$ and $5|k_j$ for $j \neq 0$. 

| $\frac{k_i}{(r, r)}$ | $e_i$ | $(e_i, e_i)$ | $i$ | $n$ |
|----------------------|------|-------------|-----|-----|
| $\frac{2}{2}$       | $v_0 + 2v_1 + v_2 + v_3 + v_4$ | 2   | $n + 3$ | $\geq 4$ |
| $v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7$ | 2   | $n + 4$ | $\geq 7$ |
| $\frac{4}{2}$       | $2v_0 + 5v_1$ | 5   | $n + 1$ | $\geq 2$ |
| $\frac{1}{2}$       | $v_0 + 2v_1 + v_2 + v_3$ | 1   | $n + 3$ | 3    |
| $v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6$ | 1   | $n + 4$ | 6    |
| $\frac{9}{2}$       | $3v_0 + 5v_1 + 5v_2$ | 5   | $n + 2$ | $\geq 2$ |

**Table 1.** Vectors found with Vinberg’s Algorithm when $p = 5$. The labels $i$ are chosen for convenience to later arguments rather than the order in which the algorithm finds them.
Proposition 2.1. The first several vectors that Vinberg’s algorithm produces are listed in Table 1.

Proof. The batches labeled $\frac{1}{10}, \frac{1}{5},$ and $\frac{4}{10}$ are empty because $(e, e) = 5$ or 10, and there is no way to write $(e, e) + 5k_0^2 = 15, 20$ or 30 as a sum of squares of integers all divisible by 5.

The batch labeled $\frac{1}{7}$ consists of vectors $e = \sum_{i=0}^{n} k_i v_i$ where

$$\sum_{i=1}^{n} k_i^2 = (e, e) + 5k_0^2 = 7$$

7 can be written as a sum of squares in two ways. This batch contains one vector if $n \geq 4,$ and two if $n \geq 7$. These two vectors have inner product 0, so they correspond to orthogonal wallks of the fundamental chamber.

The batch labeled $\frac{4}{5}$ consists of vectors $e = \sum_{i=0}^{n} k_i v_i$ where

$$\sum_{i=1}^{n} k_i^2 = (e, e) + 5k_0^2 = 25$$

and $5|k_i$ for all $i > 0$. The vector $2v_0 + 5v_1$ has inner product 0 and $-5$ with the vectors in the previous nonempty batch so we keep it for all $n \geq 2$.

The batch labeled $\frac{9}{10}$ is empty since $(e, e) = 10$ and $(e, e) + 5k_0^2 = 55$ cannot be written as a sum of squares of integers all divisible by 5.

The batch labeled $\frac{1}{2}$ consists of vectors $e = \sum_{i=0}^{n} k_i v_i$ where

$$\sum_{i=1}^{n} k_i^2 = (e, e) + 5k_0^2 = 6$$

There are two ways to write 6 as a sum of squares. One of these produces the vector $v_0 + 2v_1 + v_2 + v_3$. This has inner product 0 with the vector in batch $\frac{1}{7}$, so we keep it when $n \geq 3$, but it has positive inner product with the vector $v_0 + 2v_1 + v_2 + v_3 + v_4$ in batch $\frac{1}{7}$, so we throw it out when $n \geq 4$.

The other way to write 6 as a sum of squares produces the vector $v_0 + v_2 + v_3 + v_4 + v_5 + v_6$. This has inner product 0 with the vector $v_0 + 2v_1 + v_2 + v_3 + v_4$ in batch $\frac{1}{7}$ and inner product $-5$ with the vector in batch $\frac{4}{5}$, so we keep it when $n = 6$. It has positive inner product with the vector $v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7$ in batch $\frac{1}{7}$, so we throw it out when $n \geq 7$.

The batch labeled $\frac{9}{10}$ is empty since $(e, e) = 10$ and $(e, e) + 5k_0^2 = 90$ cannot be written as a sum of squares of integers all divisible by 5.

The batch labeled $\frac{9}{10}$ consists of vectors $e = \sum_{i=0}^{n} k_i v_i$ where

$$\sum_{i=1}^{n} k_i^2 = (e, e) + 5k_0^2 = 50$$

and $5|k_i$ for all $i > 0$. There is exactly one vector satisfying this. $3v_0 + 5v_1 + 5v_2$ has non-positive inner product with all the vectors in previous batches, so we keep it for all $n \geq 2$.

□

Proposition 2.2. The diagrams in figure 1 all describe acute angled polyhedra of finite volume.
Figure 1. Coxeter diagrams for the hyperbolic reflection groups associated to the form \(-5x_0^2 + x_1^2 + \ldots + x_n^2\), and their maximal affine subdiagram types.

To prove this we use Vinberg’s criterion for finite volume, which is Proposition 1 in [?]. We restate the criterion here for convenience.

**Proposition 2.3.** The necessary and sufficient condition for the polyhedron \(P\) with gram matrix \(S\) to have finite volume is that if \(G_S\) is a critical principal submatrix of \(G\),
(a) if $G_S$ is positive semi-definite, then there is a superset $T$ of $S$ such that $G_T$

is positive semi-definite of rank $n - 1$.

(b) if $G_S$ is indefinite and nondegenerate, then $K_S = \{0\}$.

Here, $G_S$ is the principal submatrix of $G$ corresponding to some subset $S$ of the
index set $I$ of the roots. Notice that if $G_S$ is critical it must be either positive
semi-definite or indefinite and nondegenerate. The polyhedral angle $K = \{x \in V : (x, e_i) \leq 0 \text{ for all } i\}$ is the continuation of the polyhedron $P$. $K_S = \{x \in K : (x, e_i) = 0 \text{ for all } i \in S\}$ is the subset of $K$ that is fixed by reflections in the roots
whose indices are in $S$.

If conditions (a) and (b) of the finite volume criterion are met, then any facet of the
polyhedral angle $K$ that passes out of the cone $\mathcal{C} = \{x \in V : f(x) < 0\}$
intersects the boundary of $\mathcal{C}$ in a line corresponding to a vertex at infinity of $P$.

In the proof we refer also to the list of the affine Coxeter diagrams which can also be found in Table 2 of [7].

Proof. The only cocompact hyperbolic diagrams occurring as subdiagrams of the
graphs in figure I are dotted line edges, and the only maximal affine subdiagrams
have rank $n - 1$.

Listed below each diagram in Figure I the types of maximal affine subdiagrams
that diagram has. Every affine subdiagram that appears can be extended to one of these, which shows that condition (a) of the finite volume criterion is satisfied for
all of the diagrams in Figure I.

It remains to show that condition (b) of the finite volume criterion is met. There are 4 different dotted line components which appear. We will show that each satisfies condition (b).

A sufficient condition for a dotted line subgraph to satisfy condition (b) is that there must exist a subset $T$ of the vertex set such that the diagram with vertex set
$T$ is a spherical subdiagram of rank at least $n - 1$, and there are no edges between
the vertices in $S$ and the vertices in $T$ (this is a corollary of Proposition 2 in [7]).

The two dotted line subgraphs that appear only when $n = 6, 7, 8$ can be shown
to satisfy condition (b) using this fact. These are the two dotted line subgraphs
that include the vertex $n + 4$, and by symmetry we need to only go through the argument for one of them. Let $S = \{n+1, n+4\}$, and let $T = \{2, 3, \ldots, n-1, n+3\}$. When $n = 6$, $T$ has type $D_5$. When $n = 7$, $T$ has type $E_6$. When $n = 8$, $T$ has type $E_7$. In each case this is a spherical diagram of rank $n - 1$ with no edges joining any
vertex in $T$ to any vertex in $S$.

The remaining two dotted line subgraphs appear for all $n$, and checking them
requires a more explicit calculation. The first of these has vertex set $S = \{1, n+1\}$. The two associated roots are

$$e_1 = -v_1 + v_2 \quad \text{and} \quad e_{n+1} = 2v_0 + 5v_1$$

so a vector $v \in K_S$ (fixed by reflections with respect to both $e_1$ and $e_{n+1}$) has the form

$$v = av_0 + 2a(v_1 + v_2) + \sum_{i=3}^{n} k_i v_i$$

Since $K_S \subseteq K$, $v \in K_S$ must satisfy $(v, e_i) \leq 0$ for all $e_i$. In particular, this holds
for $i = 1, \ldots, n$ so we have $2a \geq k_3 \geq \ldots \geq k_n \geq 0$, so $a \geq 0$. We also have that
\[(v, e_n + 2) = -15a + 20a \leq 0, \text{ so } a \leq 0. \] We conclude that \(a = 0\), and therefore \(K_S = \{0\}\).

By symmetry, we need not repeat the argument for \(S = \{2, n + 2\}\). 

For \(p = 7\) and 11, (1.1) is reflective for \(n = 2\) and 3. For \(p = 13, 17, 19, \) and 23, (1.1) is reflective for \(n = 2\). Tables of vectors found with Vinberg’s algorithm are in the appendix. For \(p = 7\) and 11, the Coxeter diagrams are also in the appendix. Since the only reflective case for other values of \(p\) is \(n = 2\), we describe the fundamental polygon by its norm/angle sequence and its symmetries instead of by a Coxeter diagram. A norm/angle sequence is a symbol that describes a 2-dimensional hyperbolic polygon. It has the form

\[a_1n_1a_2n_2 \ldots a_kn_k\]

where \(a_i\) is the norm of the outward pointing root orthogonal to the \(i\)-th side, and \(\frac{\pi}{n_i}\) is the angle between the \(i\)-th side and the \((i + 1)\)-st side. Cyclic permutations of such a symbol describe the same polygon if we take \(a_{k+1} = a_1\). If a polygon has a rotation symmetry, we write

\[(a_1n_1a_2n_2 \ldots a_kn_k)^m\]

to indicate that the whole symbol can be obtained by concatenating \(m\) copies of the partial symbol. These polygon symmetries need not preserve the form (1.1), but in all of our cases they do.

3. Non-reflective lattices in higher dimensions

The following proposition ensures that once we show that (1.1) is non-reflective for some \(n\), we don’t need to check any higher dimensions.

**Proposition 3.1.** If (1.1) is non-reflective for some \(n \geq 2\), then is also non-reflective in all higher dimensions.

**Proof.** The proof is by induction. Consider the root \(e_{n+1} = -v_{n+n}\) in \(L^{n+1}\). Its mirror contains a face of the fundamental polyhedron \(P\). Because \(e_{n+1}^2 = 1\), all the faces of \(P\) incident with \(e_{n+1}\) meet it in an angle of \(\frac{\pi}{n+1}\). Thus for each such face \(F\), there is a root \(r\) of \(L^{n+1}\) that is orthogonal to \(e_{n+1}\) whose mirror intersects \(e_{n+1}\) in the same line as \(F\). The reflections in these roots preserve \(e_{n+1} \cap L^{n+1}\), so they are roots of the lattice \(e_{n+1} \cap L^{n+1}\). The lattice \(e_{n+1} \cap L^{n+1}\) is just the complement to \(v_{n+1}\), which is to say the lattice generated by \(v_0, \ldots, v_n\). So it is exactly \(L^n\), with quadratic form (1.1). By assumption this lattice is non-reflective, so it has infinitely many simple roots. Thus \(L^{n+1}\) also has infinitely many simple roots. Thus by induction (1.1) is non-reflective in all higher dimensions. \[\square\]

**Proposition 3.2.** When \(p = 5\) and \(n \geq 9\), (1.1) is non-reflective.

**Proof.** By the previous theorem, we only need to show that it is non-reflective when \(n = 9\). Vinberg’s algorithm finds the same first 4 vectors as when \(n = 8\). The Coxeter diagram obtained by adding these four roots is shown in Figure 2.

The diagram has a subdiagram of type \(\tilde{D}_7\) labeled in Figure 2 by the numbers \(3, 4, 5, 6, 7, 8, 12, 13\). This affine subdiagram corresponds to a primitive vector.
e \in V$ of norm 0, obtained by taking a certain positive linear combination of the corresponding roots:
\[ e = e_3 + 2e_4 + 2e_5 + 2e_6 + 2e_7 + e_8 + e_{12} + e_{13} = 2v_0 + 3v_1 + 2v_2 + v_3 + \ldots + v_9 \]
The faces of $K = \{x \in V : (x, e_i) \leq 0\}$ corresponding to roots making up this copy of $\tilde{D}_7$ pass through $e$, so for $P$ to have finite volume, $e$ must correspond to a vertex of $P$ at infinity.

Let $M = e^\perp \cap L$, and $\overline{M} = M/\langle e \rangle$. Lemma 3.1 in [?] says that $e$ corresponds to a vertex of $P$ at infinity if and only if the positive definite lattice $\overline{M}$ is spanned by roots. $\overline{M}$ contains a primitive $D_7$ sublattice spanned by the $\overline{e}_3, \ldots, \overline{e}_8, \overline{e}_{12}$, where the bar denotes the image in the quotient by $\langle e \rangle$. The orthogonal complement to this sublattice in $\overline{M}$ is generated by the image in the quotient of a primitive vector $f_1 = v_0 - 5v_2$ of norm 20. This vector is not a root since its norm is not 1, 2, 5, or 10. The sum of the $D_7$ with its complement has index 4 in $\overline{M}$ with cyclic quotient, so the glue vector $f_2 = v_2 - 2v_3$ is needed to generate all of $\overline{M}$. The vector $f_2$ has norm 5, but is also not a root since the coefficients of $v_i$ for $i > 0$ are not all divisible by 5. Since $\overline{M}$ is not generated by roots, $e$ does not correspond to a vertex of $P$ at $\infty$, and $P$ does not have finite volume. \[\Box\]

For $p = 7, 11, 13, 17$, and 19, the proof is essentially the same. We find a primitive vector of norm zero corresponding to a rank $n - 2$ affine subdiagram, and show that its orthogonal sublattice is not generated by roots. The tables listing all these vectors are in the appendix. For $p = 23$, $n = 3$, Vinberg’s algorithm will never produce an affine subdiagram of rank 2, so we need to do something slightly different. In this case, the fundamental polygon is an infinite cylinder in hyperbolic 3-space with hexagonal faces. We prove that it is infinite by showing it has an infinite order automorphism.

**Proposition 3.3.** The form \((1.1)\) is nonreflective when $p = 23$ and $n = 3$

**Proof.** The first 14 vectors found by Vinberg’s algorithm are listed in Table 3. Figure 10 in appendix A shows an unwrapped picture of a piece of the surface of this polyhedron. Let $p$ be the primitive lattice vector in the positive light cone.
in \langle e_1, e_7, e_{10} \rangle^\perp and let \( q \) be a primitive lattice vector in the positive light cone in \langle e_{14}, e_{11}, e_3 \rangle^\perp. These two corners are marked with a • in the figure. We computed these to be

\[ p = 45v_0 + 138v_1 + 138v_2 + 92v_3 \quad \text{and} \quad q = 91v_0 + 414v_1 + 138v_2 \]

These vectors have the same norm, namely \(-23\). Thus if we write down the matrix for the linear transformation taking the ordered basis \( \{ e_6, e_{13}, e_9, p \} \) to the ordered basis \( \{ e_{12}, e_8, e_5, q \} \), we get a matrix of integers:

\[
\begin{pmatrix}
-495 & -152 & -12 & 2484 \\
-440 & -135 & -12 & 2208 \\
-348 & -108 & -9 & 1748 \\
-156 & -48 & -4 & 7
\end{pmatrix}
\]

Checking shows that it preserves the form (1.1), so it is a lattice isometry. It also has infinite order, meaning \( P \) has infinitely many faces. □

In addition to the translation symmetry used in the proof, this polyhedron also has a glide reflection.

| \( \frac{e_i}{(e_i, e_i)} \) | \( e_i \) | \( (e_i, e_i) \) | \( i \) |
|---|---|---|---|
| \( \frac{1}{2} \) | \( v_0 + 4v_1 + 3v_2 \) | 2 | 4 |
| \( \frac{1}{2} \) | \( v_0 + 5v_1 \) | 2 | 5 |
| \( \frac{1}{3} \) | \( v_0 + 4v_1 + 2v_2 + 2v_3 \) | 1 | 5 |
| \( \frac{1}{4} \) | \( 2v_0 + 7v_1 + 6v_2 + 3v_3 \) | 2 | 6 |
| \( \frac{1}{4} \) | \( 2v_0 + 9v_1 + 3v_2 + 2v_3 \) | 2 | 7 |
| \( \frac{9}{2} \) | \( 3v_0 + 9v_1 + 8v_2 + 8v_3 \) | 2 | 7 |
| \( \frac{15}{2} \) | \( 4v_0 + 12v_1 + 12v_2 + 9v_3 \) | 1 | 8 |
| \( \frac{19}{2} \) | \( 6v_0 + 27v_1 + 10v_2 + v_3 \) | 2 | 9 |
| \( \frac{19}{2} \) | \( 7v_0 + 32v_1 + 10v_2 + 2v_3 \) | 1 | 10 |
| \( \frac{100}{2} \) | \( 10v_0 + 33v_1 + 27v_2 + 22v_3 \) | 2 | 11 |
| \( \frac{144}{2} \) | \( 12v_0 + 55v_1 + 17v_2 \) | 2 | 12 |
| \( \frac{225}{2} \) | \( 15v_0 + 48v_1 + 43v_2 + 32v_3 \) | 2 | 13 |
| \( \frac{400}{2} \) | \( 20v_0 + 92v_1 + 27v_2 + 3v_3 \) | 2 | 14 |

| Table 2. Vectors found by Vinberg’s Algorithm with \( p = 23 \). |
Appendix A. Tables and diagrams for other values of $p$

Acknowledgment. Thanks to Daniel Allcock for posing the question.

| $e_i \in \mathbb{S}_{(e_i,e_i)}$ | $e_i$ | $(e_i,e_i)$ | $i$ | $n$ |
|---|---|---|---|---|
| $\frac{1}{2}$ | $v_0 + 3v_1$ | 2 | $n+1$ | $\geq 2$ |
| | $v_0 + 2v_1 + 2v_2 + v_3$ | 2 | $n+2$ | $\geq 3$ |
| $\frac{1}{3}$ | $v_0 + 2v_1 + 2v_2$ | 1 | $n+2$ | 2 |

Table 3. Vectors found with Vinberg’s Algorithm for $p = 7$

Figure 3. $p = 7$, reflective with $n = 2$ and 3

Figure 4. The first non-reflective case with $p = 7$ is $n = 4$

Affine subdiagram: $A_2$
Norm 0 vector: $v_0 + 2v_1 + v_2 + v_3 + v_4$
Complement: $2v_0 + 7v_1$
Complement norm: 21
Order 3 glue: $v_1 - 2v_2$
### Table 4. Vectors found with Vinberg’s Algorithm for $p = 11$

| $\frac{k^2}{(v_i, v_i)}$ | $v_i$ | $(v_i, e_i)$ | $i$ | $n$ |
|-------------------------|-------|--------------|-----|-----|
| $\frac{22}{22}$        | $3v_0 + 11v_1$ | $22$ | $n+1$ | $\geq 2$ |
| $\frac{1}{2}$           | $v_0 + 3v_1 + 2v_2$ | $2$ | $n+2$ | $\geq 2$ |
|                          | $v_0 + 2v_1 + 2v_2 + 2v_3 + v_4$ | $2$ | $n+3$ | $\geq 4$ |
| $\frac{1}{2}$           | $v_0 + 2v_2 + 2v_3$ | $1$ | $n+3$ | $3$ |
|                          | $v_0 + 3v_1 + v_2 + v_3 + v_4$ | $1$ | $n+5$ | $\geq 4$ |
| $\frac{64}{22}$        | $8v_0 + 22v_1 + 11v_2 + 11v_3$ | $22$ | $n+4$ | $\geq 3$ |

**Figure 5.** $p = 11$, reflective with $n = 2$ and $3$

**Figure 6.** The first non-reflective case with $p = 11$ is $n = 4$

- **Affine subdiagram:** $\tilde{A}_1^2$
- **Norm 0 vector:** $v_0 + 3v_1 + v_2 + v_3$
- **Complement:** $5v_0 + 11v_1 + 11v_2 + 11v_3$
- **Complement norm:** 88
- **Order 4 glue:** $v_1 - 3v_2$
| $e_i$ | $e_i$ | $(e_i, e_i)$ | $i$ | $n$ |
|---|---|---|---|---|
| $v_0 + 3v_1 + 2v_2 + v_3$ | 1 | 4 | 3 |
| $\frac{25}{13}v_0 + 13v_1 + 13v_2$ | 13 | 3 | 2 |
| $\frac{4}{1}v_0 + 7v_1 + 2v_2$ | 1 | 4 | 2 |
| $\frac{64}{13}v_0 + 26v_1 + 13v_2$ | 13 | 5 | 2 |
| $\frac{324}{13}v_0 + 65v_1$ | 13 | 6 | 2 |
| $\frac{144}{2}v_0 + 43v_1 + 5v_2$ | 2 | 7 | 2 |
| $\frac{2209}{2}v_0 + 169v_1 + 13v_2$ | 13 | 8 | 2 |

Table 5. Vectors found with Vinberg’s Algorithm for $p = 13$

Norm/angle sequence: $(2_11_213_\infty 13_2)^2$
Symmetry: order 2 rotation preserves $[1, 1]$

Figure 7. The first non-reflective case with $p = 13$ is $n = 3$

Affine subdiagram: $\tilde{A}_1$
Norm 0 vector: $v_0 + 3v_1 + 2v_2$
Complement: $3v_0 + 13v_1$
Complement norm: 52
Order 2 glue: $2v_1 - 3v_2$
Table 6. Vectors found with Vinberg’s Algorithm for $p = 17$

| $e_i$ | $(e_i, e_i)$ | $i$ | $n$ |
|-------|--------------|-----|-----|
| $\frac{2}{1}v_0 + 3v_1 + 3v_2 + v_3$ | 2 | 4 | 3 |
| $\frac{16}{17}4v_0 + 17v_1$ | 17 | 3 | 2 |
| $\frac{1}{1}v_0 + 3v_1 + 3v_2$ | 1 | 4 | 2 |
| $\frac{16}{2}4v_0 + 15v_1 + 7v_2$ | 2 | 5 | 2 |
| $\frac{169}{17}13v_0 + 51v_1 + 17v_2$ | 17 | 6 | 2 |
| $\frac{576}{31}24v_0 + 85v_1 + 51v_2$ | 34 | 7 | 2 |

Norm/angle sequence: $2_41_217_\infty17_22_34_21_2$

**Figure 8.** The first non-reflective case with $p = 17$ is $n = 3$

Affine subdiagram: $\tilde{A}_1$

Norm $0$ vector: $v_0 + 3v_1 + 2v_2 + 2v_3$

Complement: $3v_0 + 17v_1$

Complement norm: 136

Order 4 glue: $2v_1 - 3v_2$

Table 7. Vectors found with Vinberg’s Algorithm for $p = 19$

| $e_i$ | $(e_i, e_i)$ | $i$ | $n$ |
|-------|--------------|-----|-----|
| $\frac{2}{1}v_0 + 4v_1 + 2v_2 + v_3$ | 2 | 4 | 3 |
| $\frac{36}{38}6v_0 + 19v_1 + 19v_2$ | 38 | 3 | 2 |
| $\frac{1}{1}v_0 + 4v_1 + 2v_2$ | 1 | 4 | 2 |
| $\frac{169}{38}13v_0 + 57v_1$ | 38 | 5 | 2 |
| $\frac{9}{7}3v_0 + 13v_1 + 2v_2$ | 2 | 6 | 2 |

Norm/angle sequence: $(2_41_238_2)^2$

Symmetry: order 2 rotation preserves $[11]$
Figure 9. The first non-reflective case with $p = 19$ is $n = 3$
Affine subdiagram: $\tilde{A}_1$
Norm 0 vector: $v_0 + 3v_1 + 3v_2 + v_3$
Complement: $3v_0 + 19v_1$
Complement norm: 190
Glue: none

| $\frac{k_0}{(e_i,e_i)}$ | $e_i$ | $(e_i,e_i)$ | $i$ |
|-------------------------|-------|-------------|-----|
| $\frac{1}{2}$           | $v_0 + 4v_1 + 3v_2$ | 2   | 3   |
|                         | $v_0 + 5v_1$         | 2   | 4   |
| $\frac{36}{1}$          | $6v_0 + 27v_1 + 10v_2$ | 1   | 5   |
| $\frac{144}{2}$         | $12v_0 + 55v_1 + 17v_2$ | 2   | 6   |

Table 8. Vectors found with Vinberg’s algorithm for $p = 23$, $n = 2$.

Norm/angle sequence: $(2_41_24_3)^2$
Symmetry: order 2 rotation preserves $[111]$
Figure 10. The fundamental polyhedron for $p = 23, n = 3$. The open faces on the top match up with the ones with the same labels on the bottom to make a cylinder.