Blow-up Rate Estimates for a Semilinear Heat Equation with a Gradient Term

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Abstract

We consider the pointwise estimates and the blow-up rate estimates for the zero Dirichlet problem of the semilinear heat equation with a gradient term $u_t = \Delta u - |\nabla u|^2 + e^u$, which has been considered by J. Bebernes and D. Eberly in \cite{1}.

1 Introduction

Consider the following initial-boundary value problem

$$
\begin{aligned}
\begin{cases}
    u_t = \Delta u - h(|\nabla u|) + f(u), & (x, t) \in B_R \times (0, T), \\
    u(x, t) = 0, & (x, t) \in \partial B_R \times (0, T), \\
    u(x, 0) = u_0(x), & x \in B_R,
\end{cases}
\end{aligned}
$$

(1.1)

where $f \in C^1(R)$, $h \in C^1([0, \infty))$, $f, h > 0$, $h' \geq 0$ in $(0, \infty)$, $f(0) \geq 0$, $h(0) = h'(0) = 0$,

$$
|h(\xi)| \leq O(|\xi|^2),
$$

(1.2)

$$
sh'(s) - h(s) \leq Ks^q, \quad \text{for} \quad s > 0, \quad 0 \leq K < \infty, \quad q > 1,
$$

(1.3)

$u_0 \geq 0$ is smooth, radial nonincreasing function, vanishing on $\partial B_R$, this means it satisfies the following conditions

$$
\begin{aligned}
\begin{cases}
    u(x) = u_0(|x|), & x \in B_R, \\
    u_0(x) = 0, & x \in \partial B_R, \\
    u_0(|x|) \leq 0, & x \in B_R.
\end{cases}
\end{aligned}
$$

(1.4)

Moreover, we assume that

$$
\Delta u_0 + f(u_0) - h(|\nabla u_0|) \geq 0, \quad x \in B_R.
$$

(1.5)
The special case

\[ u_t = \Delta u - |\nabla u|^q + u|u|^{p-1}, \quad p, q > 1 \]  

(1.6)

was introduced in [2] and it was studied and discussed later by many authors see for instance [5, 12]. The main issue in those works was to determine for which \( p \) and \( q \) blow-up in finite time (in the \( L^\infty \)-norm) may occur. It is well known that it occurs if and only if \( p > q \) (see [5]). Equation (1.6) in \( \mathbb{R}^n \) was considered from similar point of view, in this case blow-up in finite time is also known to occur when \( p > q \), but unbounded global solutions always exist (see [12]). For bounded domains, it has been shown in [4] for equation (1.6) with general convex domain \( \Omega \) that, the blow-up set is compact. Moreover if \( \Omega = B_{\mathbb{R}} \), then \( x = 0 \) is the only possible blow-up point and the upper pointwise rate estimate takes the following form

\[ u \leq c|x|^{-\alpha}, \quad (x, t) \in B_R \setminus \{0\} \times [0, T), \]

for any \( \alpha > 2/(p-1) \) if \( q \in (1, 2p/(p+1)) \), and for \( \alpha > q/(p-q) \) if \( q \in [2p/(p+1), p) \). We observe that \( q/(p-q) > 2/(p-1) \) for \( q > 2p/(p+1) \), therefore, the blow-up profile of solutions of equation (1.6) is similar to that of \( u_t = \Delta u + u^p \) as long as \( q < 2p/(p+1) \) (see [3]), whereas for \( q \) greater than this critical value, the gradient term induces an important effect on the profile, which becomes more singular.

On the other hand, it was proved in [3, 4, 6, 13] that the upper (lower) blow-up rate estimate in terms of the blow-up time \( T \) in the case \( q < 2p/(p+1) \) and \( u \geq 0 \), takes the following form

\[ c(T-t)^{-1/(p-1)} \leq u(x, t) \leq C(T-t)^{-1/(p-1)}. \]

J. Bebernes and D. Eberly have considered in [1] a second special case of (1.1), where \( f(s) = e^s, h(\xi) = \xi^2 \), namely

\[
\begin{align*}
  u_t &= \Delta u - |\nabla u|^2 + e^u, \quad (x, t) \in B_R \times (0, T), \\
  u(x, t) &= 0, \quad (x, t) \in \partial B_R \times (0, T), \\
  u(x, 0) &= u_0(x), \quad x \in B_R.
\end{align*}
\]

(1.7)

The semilinear equation in (1.7) can be viewed as the limiting case of the critical splitting as \( p \to \infty \) in the equation (1.6). It has been proved that, the solution of the above problem with \( u_0 \) satisfies (1.4) may blow up in finite time and the only possible blow-up point is \( x = 0 \). Moreover, if we consider the problem in any general bounded domain \( \Omega \) such that \( \partial \Omega \) is analytic, then the blow-up set is a compact set. On the other hand, they proved that, if \( x_0 \) is a blow-up point for problem (1.7) with the finite blow-up time \( T \); then

\[
\lim_{t \to T^-} [u(x_0, t) + m \log(T - t)] = k,
\]
for some \( m \in \mathbb{Z^+} \) and for some \( k \in \mathbb{R} \). The analysis therein is based on the observation that the transformation \( v = 1 - e^{-u} \) changes the first equation in problem (1.7) into the linear equation \( v_t = \Delta v + 1 \), moreover, \( x_0 \) is a blow-up point for (1.7) with blow-up time \( T \) if and only if \( v(x_0, T) = 1 \).

In this paper we consider problem (1.7) with (1.4), our aim is to derive the upper pointwise estimate for the classical solutions of this problem and to find a formula for the upper (lower) blow-up rate estimate.

### 2 Preliminaries

The local existence and uniqueness of classical solutions to problem (1.1), (1.4) is well known by [7, 9]. Moreover, the gradient function \( \nabla u \) is bounded as long as the solution \( u \) is bounded due to (1.2) (see [11]).

The following lemma shows some properties of the classical solutions of problem (1.1) with (1.4). We may denote for simplicity \( u(r, t) = u(x, t) \).

**Lemma 2.1.** Let \( u \) be a classical solution to the problem classical solution of problem (1.1) with (1.4). Then

(i) \( u > 0 \) and it is radial nonincreasing in \( B_R \times (0, T) \). Moreover if \( u_0 \neq 0 \), then \( u_r < 0 \) in \( (0, R] \times (0, T) \).

(ii) \( u_t \geq 0 \) in \( \overline{B_R} \times [0, T) \).

Depending on Lemma 2.1 the problem (1.1) with (1.4) can be rewritten as follows

\[
\begin{align*}
  u_t &= u_{rrr} + \frac{n-1}{r} u_r - h(-u_r) + f(u), \quad (r, t) \in (0, R) \times (0, T), \\
  u_r(0, t) &= 0, \quad u(R, t) = 0, \quad t \in [0, T), \\
  u(r, 0) &= u_0(r), \\
  u_r(r, t) &= 0, \quad (r, t) \in (0, R] \times (0, T). 
\end{align*}
\]  

(2.1)

### 3 Pointwise Estimate

In order to derive a formula to the pointwise estimate for problem (2.1), we need first to recall the following theorem, which has been proved in [4].

**Theorem 3.1.** Assume that, there exist two functions \( F \in C^2([0, \infty)) \) and \( c_\varepsilon \in \mathcal{C}^2([0, R]), \varepsilon > 0 \), such that

\[
c_\varepsilon(0) = 0, c_\varepsilon' \geq 0, \quad F > 0, F', F'' \geq 0, \quad \text{in} \quad (0, \infty), \tag{3.1}
\]

\[
f'F - fF' - 2c_\varepsilon F' F + c_\varepsilon^2 F'' F^2 - 2q^{-1} Kc_\varepsilon^2 F^q F' + AF \geq 0, \quad u > 0, 0 < r < R, \tag{3.2}
\]
where
\[ A = \frac{c_\varepsilon'}{c_\varepsilon} + \frac{n - 1}{r} c_\varepsilon' - \frac{n - 1}{r^2}, \]
\[ \frac{c_\varepsilon(r)}{r} \to 0 \text{ uniformly on } [0, R] \text{ as } \varepsilon \to 0, \text{ and} \]
\[ G(s) = \int_s^\infty \frac{du}{F(u)} < \infty, \quad s > 0. \]

Let \( u \) be a blow-up solution to problem (2.1), where \( u_0 \) satisfies
\[ u_0 r \leq -\delta, \quad r \in (0, R], \quad \delta > 0. \tag{3.3} \]

Suppose that, \( T \) is the blow-up time. Then the point \( r = 0 \) is the only blow-up point, and there is \( \varepsilon_1 > 0 \) such that
\[ u(r, t) \leq G^{-1}\left(\int_0^r c_\varepsilon(z)dz\right), \quad (r, t) \in (0, R] \times (0, T). \tag{3.4} \]

We are ready now to drive a formula to the pointwise estimate for the blow-up solutions of problem (1.7) with (1.4).

**Theorem 3.2.** Let \( u \) be a blow-up solution to problem (1.7), assume that \( u_0 \) satisfies (1.4) and (3.3). Then the upper pointwise estimate takes the following form
\[ u(x, t) \leq \frac{1}{2\alpha} \left| \log C - m \log(r) \right|, \quad (r, t) \in (0, R] \times (0, T), \]
where \( \alpha \in (0, 1/2], C > 0, m > 2. \)

**Proof.** Let \( c_\varepsilon = \varepsilon r^{1+\delta} \), where \( \delta \in (0, \infty). \)

It is clear that \( c_\varepsilon \) satisfies the assumptions (3.1) in Theorem 3.1, so that (3.2) becomes
\[
\begin{align*}
 f' F - f F' - 2\varepsilon (1 + \delta) r^{1+\delta} F' + \varepsilon^2 r^{2+2\delta} F'' F^2 \\
 - 2^{q-1} K \varepsilon^q r^{q+\delta q} F^q F' + \frac{\delta(n + \delta)}{r^2} F \geq 0, \quad u > 0, \quad 0 < r < R. \tag{3.5}
\end{align*}
\]

For the semilinear equation in (1.7) it is clear that \( K \geq 1, q = 2. \) To make use of Theorem 3.1 for problem (1.7), assume that
\[ F(u) = e^{2\alpha u}, \quad \alpha \in (0, 1/2]. \]

It is clear that \( F \) satisfies all the assumptions (3.1) in Theorem 3.1. With this choice of \( F \) the inequality (3.5) takes the form
\[
\begin{align*}
 (1 - 2\alpha) e^{(1+2\alpha)u} + 4\varepsilon^2 r^{2(1+\delta)} e^{6\alpha u} + \frac{\delta(n + \delta)}{r^2} e^{2\alpha u} \geq \\
 4\alpha \varepsilon (1 + \delta) r^{1+\delta} e^{4\alpha u} + 4\varepsilon^2 r^{2(1+\delta)} e^{6\alpha u}, \quad u \geq 0, \quad 0 < r \leq R
\end{align*}
\]
provided $\alpha \leq \frac{1}{2 + 2\epsilon R(1 + \delta)}$.

Define the function $G$ as in Theorem 3.1 as follows

$$G(s) = \int_s^\infty \frac{du}{e^{2\alpha u}} = \frac{1}{2\alpha e^{\alpha s}}, \quad s > 0.$$  

Clearly,

$$G^{-1}(s) = -\frac{1}{2\alpha} \log(2\alpha s), \quad s > 0.$$  

Thus (3.4) becomes

$$u(r, t) \leq \frac{1}{2\alpha} [\log C - m \log(r)], \quad (r, t) \in (0, R] \times (0, T),$$

where $C = \frac{2 + \delta}{2\alpha}, \quad m = 2 + \delta$.

**Theorem 3.2** shows that, with choosing $\alpha = 1/2$, the upper point-wise estimate for problem (1.7) is the same as that for $u_t = \Delta u + e^u$, which has been considered in [8]. Therefore, the gradient term in problem (1.7) has no effect on the pointwise estimate.

## 4 Blow-up Rate Estimate

Since under the assumptions of Theorem 3.2, $r = 0$ is the only blow-up point for the problem (1.7), therefore, in order to estimate the blow-up solution it suffices to estimate only $u(0, t)$. The next theorem, which has been proved in [4], considers the upper blow-up rate estimate for the general problem (1.1).

**Theorem 4.1.** Let $u$ be a blow-up solution to problem (1.1), where $u_0 \in C^2(B_R)$ and satisfies (1.4), (1.5). Assume that $T$ is the blow-up time and $x = 0$ is the only possible blow-up point. If there exist a function, $F \in C^2([0, \infty))$ such that $F > 0$ and $F', F'' \geq 0$ in $(0, \infty)$, moreover,

$$f'F - F'f + F''|\nabla u|^2 - F'[h'(|\nabla u|)|\nabla u| - h(|\nabla u|)] \geq 0, \quad \text{in } B_R \times (0, T),$$

then the upper blow rate estimate takes the form

$$u(0, t) \leq G^{-1}(\delta(T - t)), \quad t \in (\tau, T),$$

where $\delta, \tau > 0, \quad G(s) = \int_s^\infty \frac{du}{F(u)}$.

For problem (1.7), if one could choose a suitable function $F$ that satisfies the conditions, which have stated in Theorem 4.1, then the upper blow-up rate estimate for this problem would be held.
Theorem 4.2. Let \( u \) be a blow-up solution to problem (1.7), where \( u_0 \in C^2(\bar{B}_R) \) and satisfies (1.4), (3.3) and the monotonicity assumption
\[
\Delta u_0 + e^{u_0} - |\nabla u_0|^2 \geq 0, \quad x \in B_R,
\]
suppose that \( T \) is the blow-up time. Then there exist \( C > 0 \) such that the upper blow-up rate estimate takes the following form
\[
u(0, t) \leq \frac{1}{\alpha} \left[ \log C - \log(T - t) \right], \quad 0 < t < T, \quad \alpha \in (0, 1].
\]

Proof. Let
\[
F(u) = e^{\alpha u}, \quad \alpha \in (0, 1].
\]
It is clear that the inequality (4.1) becomes
\[
(1 - \alpha)e^{(1+\alpha)u} + \alpha^2 e^{\alpha u} |\nabla u|^2 - \alpha e^{\alpha u} |\nabla u|^2 \geq 0,
\]
which holds for any \( \alpha \in (0, 1] \).

Set
\[
G(s) = \int_s^\infty \frac{du}{e^{\alpha u}} = \frac{1}{\alpha e^{\alpha s}}, \quad s > 0.
\]
Clearly,
\[
G^{-1}(s) = -\frac{1}{\alpha} \log(\alpha s), \quad s > 0.
\]
From Theorem 4.1 there is \( \delta > 0 \) such that
\[
u(0, t) \leq \frac{1}{\alpha} \left[ \log(\frac{1}{\alpha \delta}) - \log(T - t) \right], \quad \tau < t < T.
\]
Therefore, there exist a positive constant, \( C \) such that
\[
u(0, t) \leq \frac{1}{\alpha} \left[ \log C - \log(T - t) \right], \quad 0 < t < T.
\]

\[ \square \]

Next, we consider the lower blow-up rate for problem (1.7), which is much easier than the upper bound.

Theorem 4.3. Let \( u \) be a blow-up solution to problem (1.7), where \( u_0 \) satisfies (1.4) and (3.3). Suppose that \( T \) is the blow-up time. Then there exist \( c > 0 \) such that the lower blow-up rate estimate takes the following form
\[
\log c - \log(T - t) \leq u(0, t), \quad 0 < t < T.
\]
Proof. Define

\[ U(t) = u(0,t), \quad t \in [0,T). \]

Since \(u\) attains its maximum at \(x = 0\),

\[ \Delta U(t) \leq 0, \quad 0 \leq t < T. \]

From the semilinear equation in (1.7) and above, it follows that

\[ U_t(t) \leq e^{U(t)} \leq \lambda e^{U(t)}, \quad 0 < t < T, \quad (4.2) \]

for \(\lambda \geq 1\). Integrate (4.2) from \(t\) to \(T\), we obtain

\[ \frac{1}{\lambda(T-t)} \leq e^{u(0,t)}, \quad 0 < t < T. \]

It follows that

\[ \log c - \log(T-t) \leq u(0,t), \quad 0 < t < T, \]

where \(c = 1/\lambda\). \(\square\)

Remark 4.4. Theorem 4.3 (Theorem 4.2, where \(\alpha = 1\)) show that, the lower (upper) blow-up rate estimate for problem (1.7) is the same as for \(u_t = \Delta u + e^u\), which has been considered in [8], therefore, we conclude that, the gradient term in problem (1.7) has no effect on the blow-up rate estimate.

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