Linear Differential Equations and Orthogonal Polynomials:
A Novel Approach

N. Gurappa\(^1\) *, Prasanta K. Panigrahi\(^2\) † and T. Shreecharan\(^2\) ‡.

\(^1\)Laboratoire de Physique Theorique et Modeles Statistiques, bat. 100,
Université Paris-Sud-91405, Orsay, FRANCE;

\(^2\)School of Physics, University of Hyderabad, Hyderabad,
Andhra Pradesh, 500 046 INDIA.

Abstract

A novel method, connecting the space of solutions of a linear differential equation, of arbitrary order, to the space of monomials, is used for exploring the algebraic structure of the solution space. Apart from yielding new expressions for the solutions of the known differential equations, the procedure enables one to derive various properties of the orthogonal polynomials and functions, in a unified manner. The method of generalization of the present approach to the multi-variate case is pointed out and also its connection with the well-known factorization technique. It is shown that, the generating functions and Rodriguez formulae emerge naturally in this method.

\*gurappa@ipno.in2p3.fr

†panisp@uohyd.ernet.in

‡panisprs@uohyd.ernet.in
I. INTRODUCTION

It has been recently shown, by two of the present authors, that the solution of a linear differential equation (DE), of an arbitrary order, can be mapped to the space of monomials, if the operators, relevant to a given DE, can be separated into a part containing the Euler operator \( D \equiv xd/dx \) and the constants and another one retaining the other operators. Separating a given DE into two different parts and generating a series solution, from the solution of the simpler one, by integration has been tried earlier in the literature. The advantage of the present method derives from the fact that, the Euler operator is diagonal in the space of monomials and any other differential operator or monomial \( O^d \) is characterized by a definite degree \( d \), with respect to the Euler operator i.e., \([D, O^d] = dO^d\). By a judicious use of these two results, the space of solutions of the above DE of arbitrary order, is directly connected to the space of monomials, avoiding any explicit integration, unlike the previous approaches. This technique, not only yields novel expressions for both polynomial and functional solutions of the known DEs, but can also be straightforwardly extended to a wide class of multi-variate DEs. It can be used to find solutions of equations, involving the so called Dunkl derivatives, these type of equations are being extensively studied in the current physics and mathematics literature. Furthermore, the approach leads to the diagonalization of various many-body interacting systems of the Calogero-Sutherland type.

In the present paper, we confine ourselves, to the single variable case and explore the origin of various algebraic structures in the space of solutions of the DEs and only briefly outline the procedure to generalize this approach to the many-variable cases. These algebraic structures manifest transparently here, since they appear naturally in the space of monomials and our approach connects these monomials to the solution space. It is worth mentioning that these algebras are responsible for symmetries and various degeneracies of physical problems, associated with the DE under consideration.

The paper is organized as follows. In the subsequent section, we briefly outline the
essential steps of our method and use the same to obtain novel expressions for the solutions of the well-known hypergeometric and confluent hypergeometric equations and also show its applicability to generalized hypergeometric equations. The ladder operators and the underlying algebras are then obtained in section III. We then establish the connection of this approach with the factorization technique and Supersymmetric quantum mechanics (SUSY-QM). The utility of the method, in finding various properties of the orthogonal polynomials, is then demonstrated in section IV, by finding the generating functions and the Rodriguez formulae, for a number of cases. We conclude in section V, after pointing out a number of problems, where the present method can be profitably employed.

II. MAPPING BETWEEN THE SOLUTION SPACE OF DIFFERENTIAL EQUATIONS AND MONOMIALS

In this section, we reproduce, for the sake of completeness, a recently proposed method of solving linear differential equations of arbitrary order and explicate the same, with a number of examples.

After suitable manipulations, if a DE can be cast in the form

\[ [F(D) + P(x, d/dx)] y(x) = 0 \quad , \]

where, \( D \equiv x \frac{d}{dx}, \) \( F(D) = \sum_{n=-\infty}^{\infty} a_n D^n, \) \( a_n \)'s are some parameters and \( P(x, d/dx) \) is a function of \( x, \frac{d}{dx} \) and other operators, then the solution to the DE can be written as,

\[ y(x) = C_\lambda \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^m \right\} x^\lambda \quad , \]

provided, \( F(D)x^\lambda = 0. \) Here \( C_\lambda \) is a constant. The proof is straightforward and follows by direct substitution:

\[ [F(D) + P(x, d/dx)] \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^m \right\} x^\lambda \]

\[ = F(D) \left[ 1 + \frac{1}{F(D)} P(x, d/dx) \right] \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^m \right\} x^\lambda \]
\[
F(D) \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^m x^\lambda 
+ F(D) \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^{m+1} x^\lambda 
= F(D)x^\lambda - F(D) \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^{m+1} x^\lambda 
+ F(D) \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^{m+1} x^\lambda 
= 0 .
\] (3)

Eq. (2) connects the solution of a given DE to the space of monomials. It needs to be emphasized that, the inverse of \(F(D)\) is well defined in the above expression, since it is diagonal in the space of monomials.

We illustrate the working of this method in the context of the well-known hypergeometric differential equation (HGDE), given by,
\[
\left[ x^2 \frac{d^2}{dx^2} + (\alpha + \beta + 1) x \frac{d}{dx} + \alpha\beta - x \frac{d^2}{dx^2} - \gamma \frac{d}{dx} \right] F(\alpha, \beta; \gamma; x) = 0 .
\] (4)

Collecting the part containing the powers of the Euler operator and the constants, one gets, \(F(D) = (D + \alpha)(D + \beta)\), and the condition \(F(D)x^\lambda = 0\), gives, \(\lambda = -\alpha, -\beta\). The series solution is then
\[
F(\alpha, \beta; \gamma; x) = C_{(\alpha,\beta)} \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{-1}{(D + \alpha)(D + \beta)} \left( x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right) \right]^m \right\} x^{-(\alpha,\beta)} .
\] (5)

It is easily seen that, modulo an overall normalization factor, the above solution, is a rearranged form of the well-known hypergeometric series. Making use of the following representation of \(1/(D + \beta)\),
\[
\frac{1}{(D + \beta)} = \int_0^{\infty} ds e^{-s(D+\beta)}
\] (6)

and
\[
[(D + \beta), (x \frac{d^2}{dx^2} + \gamma \frac{d}{dx})] = - \left( x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right) ,
\] (7)
one can easily show that
\[
\left[ -\frac{1}{(D+\alpha)(D+\beta)}\left(x\frac{d^2}{dx^2} + \gamma \frac{d}{dx}\right) \right]^m x^{-\beta} = \left[ \frac{1}{(D+\alpha)}\left(x\frac{d^2}{dx^2} + \gamma \frac{d}{dx}\right) \right]^m x^{-\beta} \frac{1}{m!}.
\] (8)

Choosing the normalization constant to match the conventional definition, we can write the normalized solution for the HGDE as,
\[
F(\alpha, \beta; \gamma; x) = (-1)^{-\beta} \frac{\Gamma(\alpha - \beta)\Gamma(\gamma)}{\Gamma(\gamma - \beta)\Gamma(\alpha)} \exp \left[ -\frac{1}{(D+\alpha)} \left(x\frac{d^2}{dx^2} + \gamma \frac{d}{dx}\right) \right] x^{-\beta}.
\] (9)

The exponential form is a novel expression for \(F(\alpha, \beta; \gamma; x)\), unknown in the literature, to the best of the authors’ knowledge. The choice \(\lambda = -\alpha\) will lead again to the hypergeometric series, as the series is symmetric under the exchange of \(\alpha\) and \(\beta\). Since, \(\frac{1}{(D+\alpha)}(xd^2/dx^2 + \gamma d/dx)\) lowers the degree of a given monomial by one, it can be seen that, when \(-\beta\) is an integer, the above is a polynomial solution of the HGDE.

Similarly, the normalized polynomial solution for the confluent hypergeometric (CH) differential equation,
\[
\left[ x\frac{d^2}{dx^2} + (\gamma - x) \frac{d}{dx} - \alpha \right] \Phi(\alpha; \gamma; x) = 0,
\] (10)
is
\[
\Phi(\alpha; \gamma; x) = (-1)^{-\alpha} \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} \exp \left[ -x \frac{d^2}{dx^2} - \gamma \frac{d}{dx} \right] x^{-\alpha}.
\] (11)

It is worth mentioning that, one can also modify the HGDE, by multiplying it with \(x\), which yields, \(F(D) \equiv D(D+\gamma-1)\) and \(P(x, d/dx) \equiv -x(D+\alpha)(D+\beta)\). The corresponding solution is
\[
F(\alpha, \beta; \gamma; x) = C_{(\alpha, \gamma)} \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{D(D+\gamma-1)}x(D+\alpha)(D+\beta) \right]^m \right\} x^{(0,1-\gamma)} ,
\] (12)
since \(F(D)x^\lambda = 0\) gives \(\lambda = 0, 1 - \gamma\). For \(\lambda = 0\), the series can be written in the well-known form
\[
F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!} ,
\] (13)
where,

\[ a_n = a(a+1)(a+2)\cdots(a+n-1) \quad , \quad (14) \]

is the Pochammer symbol. The corresponding exponential form is

\[ F(\alpha, \beta; \gamma; x) = C_0 \exp \left[ x \frac{(D + \alpha)(D + \beta)}{(D + \gamma)} \right] \cdot 1 \quad . \quad (15) \]

The other choice, namely \( \lambda = 1 - \gamma \), yields the other linearly independent solution. Analogous results follow for CHDE. Hence, both the polynomial and the series solutions are obtained by the present method.

The procedure for finding the solutions for CHDE and HGDE easily extends to the generalized hypergeometric cases, given by the equations of the type,

\[ [\Theta(\Theta + b_1 - 1)\cdots(\Theta + b_p - 1) - z(\Theta + a_1)\cdots(\Theta + a_p + 1)]y = 0 \quad , \quad (16) \]

where \( \Theta \equiv zd/dz \), with \( z \) being, in general, complex. The generalized hypergeometric series, is denoted by \( {}_pF_p \), where \( p+1 \) and \( p \) are the number of parameters appearing in the numerator and the denominator of the generalized HG series, respectively. Since \( \Theta \) is the Euler operator and the DE is already separated into \( F(\Theta) \) and \( P(z, d/dz) \), the solutions easily follow. We consider the case of the \( {}_3F_2 \) series below, because of its importance in the quantum theory of angular momentum.

The DE for the \( {}_3F_2 \) series

\[ [\Theta(\Theta + b_1 - 1)(\Theta + b_2 - 1) - z(\Theta + a_1)(\Theta + a_2)(\Theta + a_3)]{}_3F_2 = 0 \quad , \quad (17) \]

yields \( F(\Theta) \equiv \Theta(\Theta + b_1 - 1)(\Theta + b_2 - 1) \) and \( P(z, d/dz) \equiv -z(\Theta + a_1)(\Theta + a_2)(\Theta + a_3) \).

The solution is then

\[ {}_3F_2 = C_{(0,1-b_1,1-b_2)} \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(\Theta)} P(z, d/dz) \right]^m \right\} z^{(0,1-b_1,1-b_2)} \quad , \quad (18) \]

where \( F(\Theta)z^\lambda = 0 \), has yielded three solutions, \( \lambda = 0, 1 - b_1, 1 - b_2 \). For \( \lambda = 0 \), the above series can be expanded and rearranged in the conventional form.
\[ 3F_2 = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!} . \] (19)

Like the previous examples, an exponential form for the \( 3F_2 \) can also be written down easily.

The other roots, \( 1 - b_1 \) and \( 1 - b_2 \), give the other two linearly independent solutions. It is clear from the expression for the \( 3F_2 \) series, that it terminates, when either \( a_1, a_2 \) or \( a_3 \), is a negative integer or zero.

Multiplying the above DE with \( -1/z \), one gets,

\[ [(\Theta + a_1)(\Theta + a_2)(\Theta + a_3) - \frac{d}{dz}(\Theta + b_1 - 1)(\Theta + b_2 - 1)] 3F_2 = 0 , \] (20)

where \( F(\Theta) \) and \( P(z,d/dz) \) are now given by \( F(\Theta) = (\Theta + a_1)(\Theta + a_2)(\Theta + a_3) \) and \( P(z,d/dz) = -(d/dz)(\Theta + b_1 - 1)(\Theta + b_2 - 1) \). Therefore, the solution, in this case, turns out to be

\[ 3F_2 = C_{a_1,a_2,a_3} \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(z,d/dz) \right]^m \right\} z^{-(a_3,a_2,a_1)} , \] (21)

where \( a_1, a_2, a_3 \) have to be negative integers to yield a polynomial solution.

By now, it is clear that for a number of DEs, \( F(D)x^\lambda = 0 \), leads to the linearly independent solutions, when the solutions are nondegenerate. In cases, where this is not possible, appropriate modification of the DE, has straightforwardly yielded the linearly independent solutions. Below, we give two more examples to illustrate these points. In particular, the second example deals with the case, where \( F(D)x^\lambda = 0 \), leads to degenerate solutions. These examples will also point out the connection of the indicial equation, in the conventional series solution method, to the present one. In the standard series solution method\cite{12}, for the DE

\[ 4x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = 0 , \] (22)

the two distinct roots of the indicial equation \( c = 0,1/2 \), lead to two linearly independent solutions. Multiplication by \( x \) yields

\[ 4x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + xy = 0 , \] (23)
which implies, \( F(D) = 4x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} = 2D(2D-1) \) and \( P(x, d/dx) = x \). Hence, \( F(D)x^\lambda = 0 \) obtains \( \lambda = 0, 1/2 \), which provide the two linearly independent solutions, as is obtained in the standard approach.

In our second example,

\[
x \frac{d^2}{dx^2} + \frac{dy}{dx} + y = 0,
\]

the roots of the *indicial equation*, \( c^2 = 0 \) are degenerate. Multiplying the above DE with \( x \), to bring it to the form given by Eq. (1), one gets \( (D^2 + x)y = 0 \); hence \( F(D) = D^2 \) and \( P(x) = x \). It is clear that, \( F(D)x^\lambda = 0 \), also leads to the same degenerate case as obtained by the method of series solution. In this scenario, one has to employ the established methods for finding out the other linearly independent solution.

The procedure developed here is applicable to a wide range of functions and polynomials. Some of the well-known ones, explicitly checked by the authors are, Meijer’s G-function, Struve, Lomel, Anger, Weber, Bessel functions, Gegenbauer, Neumann’s, Jacobi, Schläfli, Whittaker and Chebyshev polynomials. It is also applicable in the periodic cases. For example, solution of DE with the following periodic potential

\[
\frac{d^2 y}{dx^2} + a \cos(x)y = 0,
\]

can be found, after multiplying Eq. (24) by \( x^2 \) and rewriting \( x^2 \frac{d^2}{dx^2} \) as \( (D - 1)D \),

\[
y(x) = \sum_{m, (n_i) = 0}^{\infty} \frac{(-a)^m}{m!} \left\{ \prod_{i=1}^{m} \frac{(-1)^{n_i}}{(2n_i)!} \right\} \left( \prod_{r=1}^{m} \frac{(2[m + \lambda/2 + 1 - r + \sum_{i=1}^{m+1-r} n_i])!}{(2m + \lambda/2 + 1 - r + \sum_{i=1}^{m+1-r} n_i)!} \right) x^{2(m + \sum_{i=1}^{m} n_i + \lambda/2)}. \tag{26}
\]

Here, \( \lambda = 0 \) or 1. In the same manner, one can write down the solutions for the Mathieu’s equation as well.

Although this paper is devoted to the single variable case, it is worth pointing out that the generalization of this method to the many-variable case is immediate. This can be accomplished by denoting \( \bar{D} = \sum_i D_i \equiv \sum_i x_i \frac{d}{dx_i} \), where \( i = 1, 2, \cdots N \). Using the fact,
\( F(D)X^\lambda = 0 \) has solutions, in the space of monomial symmetric functions, a number of many-body equations can be solved, in a manner analogous to the single variable case.

III. ALGEBRAIC STRUCTURE OF THE SOLUTION SPACE

We now proceed to study the algebraic properties of the space of solutions. The advantages of the present approach, as compared to the previous ones, lie in the following two facts. First of all, \( a \text{ priori} \), no symmetry of the DE is assumed. Secondly, the ladder operators are straightforward to construct in the space of the monomials, which can be brought to the space of solutions, via a similarity transformation, with the aid of the exponential form of the solutions. The only criterion in the choice of the ladder operators, in the space of monomials, is that, after the similarity transformation, the resulting operators, are well-defined; these operators then yield the symmetry algebra.

Keeping in mind, the appearance of CHDE and HGDE in diverse physical systems, we explicitly work out the generators of the symmetry algebras, for these two cases, for the corresponding polynomial solutions. Application to other DEs can be carried out in a similar manner.

The solution of the CHDE,

\[
\Phi(\alpha; \gamma; x) = (-1)^{-\alpha} \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} \exp \left[ -x \frac{d^2}{dx^2} - \gamma \frac{d}{dx} \right] \cdot x^{-\alpha},
\]

leads to a polynomial solution, only if, \( \alpha \) is a negative integer (\( -n \)) or zero. The above form of the solution, immediately suggests the lowering operator to be,

\[
J_- \equiv \left( x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right),
\]

since \( J_- \) commutes with the exponential and hence can lead to a lowering operator at the level of the polynomial. \( J_- \) reduces the degree of the polynomial by one,

\[
J_- e^{-J_+} x^n = n(\gamma + n - 1) e^{-J_-} x^{n-1}.
\]

After taking into account the normalization factors, we get
\[ 
\left( x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right) \Phi(-n; \gamma; x) = -n \Phi(-n + 1; \gamma; x) . \tag{30} 
\]

Choosing \( J_+ \equiv x \), as the raising operator at the level of the monomials and introducing an identity operator, in the following manner,

\[ e^{-J_-} x^n = e^{-J_-} x e^{+J_-} e^{-J_-} x^n , \tag{31} \]

one gets,

\[ \left[ x - 2x \frac{d}{dx} - \gamma + x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right] \Phi(-n; \gamma; x) = -(n + \gamma) \Phi(-n - 1; \gamma; x) . \tag{32} \]

The similarity transformation of \( x \), the raising operator, at the level of the monomials, led to the raising operator at the level of the polynomials. It can be easily seen that, \([J_+, J_-] = -2J_0\), where \( J_0 = D + \gamma/2 \) and

\[ [J_0, J_\pm] = \pm J_\pm . \tag{33} \]

This is the well-known \( SU(1, 1) \) dynamical algebra, in the solution space of the CHDE. With an appropriate choice of the measure, the operators \( J_+ \) and \( J_- \) can be made the formal adjoints of each other. It is worth pointing out that, the simplest choice \( d/dx \) as the lowering operator for the monomials, does not lead to a convergent expression at the level of the polynomials. It should be pointed out that other forms of ladder operators can also be found.

A straightforward calculation, taking \( x \) and \( d/dx \) as the raising and lowering operators at the level of the monomials, leads to the Heisenberg algebra, \([a, a^\dagger] = 1\), for the Hermite polynomials

\[ H_n(x) = C_n \exp\left(-\frac{1}{4} \frac{d^2}{dx^2}\right) x^n . \tag{34} \]

The exponential form of the solution of HGDE suggests the simplest lowering operator to be

\[ \tilde{J}_- = \frac{1}{(D + \alpha)} \left( x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right) \equiv \tilde{T} J_- , \tag{35} \]
which lowers the degree of the monomials by one. Here \( \tilde{T} = 1/(D+\alpha) \) and \( J_- = [x(d^2/dx^2) + \gamma(d/dx)] \). It is to be noted that \( J_- \) can also act as the lowering operator at the level of the monomials; however, the latter choice will not lead to a convergent expression in the solution space.

Operating \( \tilde{J}_- \) on \( x^n \), one gets,

\[
\tilde{J}_- x^n = \frac{n(\gamma + n - 1)}{(\alpha + n - 1)} x^{n-1},
\]

(36)

which straightforwardly extends to,

\[
\frac{1}{(D+\alpha)} \left( x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right) F(\alpha, -n; \gamma; x) = -nF(\alpha; -n + 1; \gamma; x),
\]

(37)

at the level of the polynomials. The raising operator, at the level of the monomials, needs to be chosen carefully due to the presence of \( \tilde{J}_- \) in the exponential.

At this point, we note that, the existence of a canonical conjugate operator \( \tilde{J}_+ \) for \( \tilde{J}_- \), such that, \( [\tilde{J}_-, \tilde{J}_+] = 1 \), akin to the Heisenberg algebra, would imply

\[
e^{-\tilde{J}_-} \tilde{J}_+ x^n = (\tilde{J}_+ - 1) e^{\tilde{J}_-} x^n.
\]

(38)

This suggests \( \tilde{J}_+ \) can be used as a raising operator in the space of monomials, which can be written in a compact form in the space of solutions of the HGDE.

The procedure for finding the canonical conjugate is straightforward\(^3\). We briefly outline the same below and use it for finding \( \tilde{J}_+ \). Although we have used this procedure for convenience, other forms of raising operators are possible to construct. The method for obtaining them will be illustrated below. Denoting \( J_0 \equiv xd/dx + \gamma/2 \), one finds \( [J_0, \tilde{J}_-] = -\tilde{J}_- \). Starting with a function \( T(J_0) \), whose required properties will become clear in the subsequent steps, we define

\[
\tilde{J}_+ = xT(J_0)
\]

\[= J_+ T(J_0) \quad .
\]

(39)

It is easy to check that,
\[ [J_0, J_+] = J_+ \quad \text{and} \quad [J_+, J_-] = -2J_0 \quad . \] (40)

Since \(-2J_0 = g(J_0) - g(J_0 - 1)\), with \(g(J_0) = -J_0(J_0 + 1)\), the Casimir operator \(C\) commuting with all the three generators can be written as, \(C = J_-J_+ + g(J_0) = J_+J_- + g(J_0 - 1)\). Starting from \([\bar{J}_-, \bar{J}_+] = 1\), one finds

\[ \bar{T}(J_0)T(J_0)J_-J_+ - T(J_0 - 1)\bar{T}(J_0 - 1)J_+J_- = 1 \quad , \]

which leads to

\[ T(J_0) = [\bar{T}(J_0)]^{-1} \frac{J_0 + \delta}{[C - g(J_0)]} \quad . \]

Here \(\delta\) is an arbitrary constant. Simplification yields

\[ \bar{J}_+ = \frac{(J_0 + \alpha - 1)(J_0 + \delta - 1)}{(J_0 + \gamma - 1)J_0} x \quad . \]

Demanding that \([\bar{J}_-, \bar{J}_+] = 1\), holds on the lowest monomial i.e., 1, one obtains \(\delta = 1\). This leads to

\[ \bar{J}_+ = \frac{(J_0 + \alpha - 1)}{(J_0 + \gamma - 1)} J_+ \quad . \] (41)

Now the raising operator in the solution space can be obtained via a similarity transformation:

\[ e^{-\bar{J}_-} \hat{J}_+ e^{\bar{J}_-} e^{-\bar{J}_-} x^n = \frac{(\alpha + n)}{(\gamma + n)} e^{-\bar{J}_-} x^{n+1} \quad . \] (42)

The above expression after restoring the normalization factors yields,

\[ \left[ 1 - \frac{(J_0 + \alpha - 1)}{(J_0 + \gamma - 1)} x \right] F(\alpha; -n; \gamma; x) = F(\alpha; -n - 1; \gamma; x) \quad . \] (43)

As noted earlier, the raising operator obtained above is not unique, one can construct other raising operators; some examples are

\[ \tilde{J}_+ = (x + x^2 \frac{d}{dx}) \quad \text{and} \quad \hat{J}_+ = (x + x^2 \frac{d}{dx})\hat{T}(C, \tilde{J}_0) \quad . \] (44)

For the former case, explicit computation leads to
\[
\left[1 - \frac{(D + \alpha - 1)}{(D + \gamma - 1)D}(x + x^2 \frac{d}{dx})\right] F(\alpha; -n; \gamma; x) = F(\alpha; -n - 1; \gamma; x) , \quad (45)
\]

It is interesting to note that, the algebra satisfied by \(\bar{J}_+, J_-\) and \(D\) is a quadratic algebra, since

\[
[J_+, J_-] = -2(\gamma + 1/2)D - 3D^2 - \gamma , \quad (46)
\]

and

\[
[D, J_\pm] = \pm J_\pm.
\]

The algebra satisfied by \(J_+, J_-\) and \(J_0\) is the well known \(SU(1, 1)\) algebra. Here \(J_+, J_-\) can be made formal adjoints of each other with the appropriate choice of the measure. The point to note is that, both the above mentioned \(SU(1, 1)\) and the quadratically deformed algebra, have the solution space of HGDE, as their irreducible representations.

The above ladder operators can be suitably rearranged to yield the ladder operators obtained by the factorization method (FM) and SUSY-QM. These ladder operators can be used for obtaining wavefunctions of a number of quantum mechanical problems, after appropriate measures are introduced. We take the well-known example of the Laguerre DE, for the purpose of establishing the above mentioned connection:

\[
\left[x \frac{d^2}{dx^2} + (\alpha + 1 - x) \frac{d}{dx} + n\right] L_\alpha^n = 0 . \quad (47)
\]

The solution of the above DE is the familiar Laguerre polynomial,

\[
L_\alpha^n = \frac{(-1)^n}{n!} \exp \left[-x \frac{d^2}{dx^2} - (\alpha + 1) \frac{d}{dx}\right] . x^n . \quad (48)
\]

In order to find the raising operator for \(L_\alpha^n\), analogous to the ones obtained by FM and SUSY-QM, we start with

\[
\left[x + x \frac{d}{dx} - n\right] x^n = x^{n+1} ,
\]

as the raising operator at the level of the monomials. Following the method employed for CHDE, we get
This can be cast in the form

$$L_n^\alpha = \prod_{k=1}^{n} \left[ x \frac{d}{dx} + \alpha + k - x \right] 1$$ \hspace{1cm} (50)$$

or

$$L_n^\alpha = A^\dagger(\alpha+1)A^\dagger(\alpha+2)\cdots A^\dagger(\alpha+n)1$$

where $A^\dagger(\alpha+n) \equiv \left[ x \frac{d}{dx} + \alpha + n - x \right]$. These shifted operators have found application in the construction of coherent states. This form of the polynomial, modulo normalizations, matches with the ones obtained from FM and SUSY-QM. Using the raising operator for CHDE and HGDE, one can also obtain similar expressions for other polynomials.

It should be emphasized that FM and SUSY-QM, have taken recourse to a special property of the equations under study called shape invariance for arriving at these results. Furthermore it was necessary to introduce a host of intermediate systems. In contrast the present technique does not presume any such special properties of the DE.

Since we have already provided the raising operator, for the sake of completeness, we derive the lowering operator for $L_n^\alpha$ in a convenient form. Starting from,

$$\left[ x \frac{d}{dx} - n + x \frac{d^2}{dx^2} + (\alpha + 1) \frac{d}{dx} \right] x^n = n(n + \alpha)x^{n-1}$$

a suitable similarity transformation yields

$$\left[ x \frac{d}{dx} - n \right] L_n^\alpha = -(n + \alpha)L_{n-1}^\alpha \hspace{1cm} (51)$$

Eq. (49) and Eq. (51) are the standard recurrence relations for the Laguerre polynomials. We can also derive those operators which change the values of $\alpha$. The following steps lead to,

$$\frac{d}{dx} L_n^\alpha(x) = \frac{(-1)^n}{n!} \frac{d}{dx} \exp \left[ -x \frac{d^2}{dx^2} - (\alpha + 1) \frac{d}{dx} \right] x^n \hspace{1cm} (52)$$
\[ = (\frac{-1}{n!})^n \exp \left[ -\frac{d}{dx} x \frac{d}{dx} - (\alpha + 1) \right] \frac{d}{dx} x^n \]

\[ = -L_{n-1}^{\alpha+1}(x) \quad . \quad (53) \]

Similarly for HG series we get,

\[ \frac{d}{dx} F(\alpha, \beta; \gamma; x) = -\beta(-1)^{n-\beta} \frac{\Gamma(\alpha - \beta) \Gamma(\gamma)}{\Gamma(\gamma - \beta) \Gamma(\alpha)} \times \]

\[ \times \exp \left[ \frac{-1}{(D + \alpha + 1)} \left( x \frac{d^2}{dx^2} + (\gamma + 1) \frac{d}{dx} \right) \right] . x^{-\beta-1} \quad (54) \]

\[ = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x) \quad (55) \]

**IV. APPLICATION TO CLASSICAL ORTHOGONAL POLYNOMIALS**

**A. Rodriguez Formula**

In this section, we elaborate on the applicability of the approach developed here for finding other properties of the solution space. We start with the Rodriguez formula (RF) and show how these can be obtained, with the examples of Laguerre and Hermite polynomials.

**RF for Laguerre polynomials**

For simplicity, we consider the expression of \( L_0^n \),

\[ L_0^n(x) = \frac{(-1)^n}{n!} \exp \left( -x \frac{d^2}{dx^2} - \frac{d}{dx} \right) x^n \quad . \quad (56) \]

Defining \( B \equiv xd^2/dx^2 + d/dx \) and introducing an identity operator, we get

\[ L_0^n(x) = \frac{(-1)^n}{n!} e^{-B} x^n e^B e^{-B} 1 \quad . \]

The above expression simplifies to

\[ L_0^n(x) = \frac{(-1)^n}{n!} \left[ x - 2x \frac{d}{dx} - 1 + x \frac{d^2}{dx^2} + \frac{d}{dx} \right]^n e^{-B} 1 \quad , \quad (57) \]

which can be written in the form
Further simplification yields,

$$L^0_n(x) = \frac{(-1)^n}{n!} e^x e^{-x} \left[ x - 2x \frac{d}{dx} - 1 + x \frac{d^2}{dx^2} + \frac{d}{dx} \right]^n e^x e^{-x} 1 .$$

and hence

$$L^0_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n} \left( e^{-x} x^n \right) .$$

This is the well-known Rodriguez formula for the Laguerre polynomials.

**RF for the Hermite polynomials**

The normalized solution of the Hermite DE is given by,

$$H_n(x) = 2^n \exp \left( - \frac{1}{4} \frac{d^2}{dx^2} \right) x^n .$$

Denoting $A \equiv \frac{1}{4} \frac{d^2}{dx^2}$ and writing

$$H_n(x) = 2^n e^{-A} x^n e^A e^{-A} 1 ,$$

we obtain

$$H_n(x) = \left[ 2x - \frac{d}{dx} \right]^n e^{-A} 1 .$$

Introducing an identity operator in the form,

$$H_n(x) = e^{x^2} e^{-x^2} \left[ 2x - \frac{d}{dx} \right]^n e^{x^2} e^{-x^2} e^{-A} 1$$

one obtains the RF as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^2}{dx^2} e^{-x^2} .$$

Eq. (57) and Eq. (62) reveal that simplification of the raising operators by introducing appropriate identity operators led to the RF. Using the raising operators obtained in the previous section, one can easily extend these results to other polynomials.
One can also start from the RF and obtain Eq.(60). Explicitly,

\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \]

\[ = (-1)^n \left[ \frac{d}{dx} - 2x \right]^n \]

\[ = 2^n e^{-A} e^A \left[ x - \frac{1}{2} \frac{d}{dx} \right]^n e^{-A} e^A 1 \]  \hspace{1cm} (65) 

The above expression, by suitable manipulations, leads to

\[ H_n(x) = 2^n \exp\left( -\frac{1}{4} \frac{d^2}{dx^2} \right) x^n . \]

This shows the procedure to obtain the exponential form of the solutions, starting from the known RF for a given solution.

**B. Generating Functions**

Below, we outline the method of getting the generating functions (GF) taking Laguerre and Chebyshev as examples. The advantage of having an exponential form for the solution comes out naturally through these examples.

**GF for the Laguerre Polynomial.**

Defining the generating function as

\[ g(x, t) = \sum_{n=0}^{\infty} L_0^n(x)t^n \]  \hspace{1cm} (66) 

and substituting the expression for \( L_0^n(x) \) in the above equation, we get,

\[ g(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{-B} x^n t^n \]

\[ = e^{-B} e^{-xt} \]

\[ = \left[ 1 - B + \frac{B^2}{2!} - \frac{B^3}{3!} + \cdots \right] e^{-xt} . \]  \hspace{1cm} (67) 

Action \( B \) on \( e^{-xt} \) is easy to compute:

\[ g(x, t) = e^{-xt} \left[ 1 - (xt^2 - t) + \frac{1}{2!}(x^2 t^4 - 4xt^3 + 2t^2) - \cdots \right] \]  \hspace{1cm} (68)
This series can be summed and written in the compact form

\[ g(x, t) = \frac{\exp[-xt/(1-t)]}{(1-t)}, \]

\(g(x, t)\) is the well-known GF for \(L_0^\alpha(x)\). The procedure outlined above can be straightforwardly extended to include, \(\alpha \neq 0\) cases also.

**Chebyshev type II**

We consider type II Chebyshev polynomial for illustration. The solution of Chebyshev type II DE can be written as

\[ U_n(x) = 2^n \exp \left[ -\frac{1}{2(D+n+2)} \frac{d^2}{dx^2} \right] x^n, \quad (69) \]

and its GF is defined as,

\[ g(x, t) = \sum_{n=0}^{\infty} U_n(x) t^n. \quad (70) \]

Substituting the expression for \(U_n(x)\) in the above equation, we obtain,

\[ g(x, t) = \sum_{n=0}^{\infty} 2^n \exp \left[ -\frac{1}{2(D+n+2)} \frac{d^2}{dx^2} \right] x^n t^n, \]

\[ = 1 + 2xt + (4x^2 - 1)t^2 + \cdots, \quad (71) \]

which can be summed to yield the GF as

\[ g(x, t) = \frac{1}{1 - 2xt + t^2}. \]

This procedure for obtaining GF can be generalized to other orthogonal polynomials. It is clear that the exponential form of the solutions enables one to find the GFs straightforwardly.

Generalized GFs find application in the construction of coherent and squeezed states\(^{18, 26}\). It is worth mentioning that the exponential form of the Hermite polynomials has been connected with the Gauss transform\(^{27}\). Our results provide analytic expressions for the Gauss transform, with appropriate parameter value, of various polynomials. Construction of generalized coherent states, associated with these polynomials is currently under progress and will be reported elsewhere.
V. CONCLUSIONS.

In conclusion, the solution of a wide class of liner differential equations, which can be cast into a form, where a function of the Euler operator and constants separates from the rest, can be written in a closed form, which makes the algebraic properties of the solution space transparent. Explicit examples dealing with confluent hypergeometric, hypergeometric and generalized hypergeometric equations were analyzed, where, not only novel expressions connecting the solution space, with the space of the monomials were written down, but also, utilized for unravelling the dynamical symmetries underlying the solution spaces.

The fact that, a priori no assumption was made about the symmetry of the equation understudy, makes this approach attractive. Although, we have analyzed here well-known examples, an exhaustive study reveals that the present approach extends to a host of other functions and polynomials. Some of these are, Meijer’s G-Function, Struve, Lomel, Anger, Weber, Bessel functions, Gegenbauer, Neumann’s, Jacobi, Schläfli, Whittaker, Chebyshev polynomials. These functions and polynomials manifest in diverse branches of physics and mathematics. The novel expressions for the solutions presented here will help in unravelling various properties of these functions and polynomials.

Here, we have only briefly mentioned about the multivariate cases dealing with correlated systems. The solution space of these equations have rich symmetry and they are connected with random matrices which find application in diverse areas. Hence, a deeper analysis of the the problem is also warranted. Furthermore, the connection of this approach with the Gauss transform can be used for constructing generalized coherent states. It is worth mentioning that, the exponential form of the Hermite polynomial has already found application in the construction of coherent and squeezed states.

Apart from finding exact solutions, the present approach can also be utilized for finding approximate solutions to differential equations. Some of these works are currently under progress and we hope to report the findings in the near future.

Acknowledgements
We acknowledge useful discussions with Profs. N. Mukunda, V. Srinivasan, R. Jagannathan, S. Chaturvedi and R. Sridhar. T.S. thanks UGC (India) for providing financial support through the JRF scheme.
REFERENCES

1. N. Gurappa and P.K. Panigrahi, [hep-th/9908127],
   N. Gurappa, P. K. Panigrahi, T. Shreecharan and S. Sree Ranjani, in *Frontiers of Fundamental Physics 4*, Eds. B. G. Sidharth and M. V. Altaisky, Kluwer, Dordrecht, 2001.

2. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer, Dordrecht, 1994.

3. N. Gurappa and P. K. Panigrahi, *Phys. Rev. B* 62, 1943 (2000).

4. C. F. Dunkl, *Amer. Math. Soc.* 311, 167 (1989).

5. T. H. Koorwinder, in *Special Functions and Differential Equations*, Eds. K. S. Rao, R. Jagannathan, G. V. Berghe and J. V. Jeugt, Allied Publishers, New Delhi, 1997 and references therein.

6. N. Gurappa and P. K. Panigrahi, *Phys. Rev. B*, R2490 (1999).

7. F. Calogero, *J. Math. Phys.* 12, 419 (1971);
   B. Sutherland, *ibid* 12, 246 (1971); 12, 251(1971).

8. Y. Alhassid, F. Gürsey and F. Iachello, *Phys. Rev. Lett.* 50, 873 (1983);
   Y. Alhassid, F. Gürsey and F. Iachello, *Ann. Phys.* 148, 346 (1983);
   A. O. Barut, A. Inomata and R. Wilson, *J. Phys. A* 20, 4075 (1987); 20, 4083 (1987);
   A. Gangopadhyaya, J. V. Mallow and U. Sukhatme, *Phys. Rev A* 58, 4287 (1998);
   S. Chaturvedi, R. Dutt, A. Gangopadhyaya, P.K. Panigrahi, C. Rasinariu and U. Sukhatme, *Phys. Lett. A* 248, 2 (1998);
   W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press, 1968.

9. I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, Series and products*, Academic Press, 1965.

10. L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, (1966).
11 K. Srinivasa Rao and V. Rajeswari, \textit{Quantum Theory of Angular Momentum}, Narosa Publishing House, 1993.

12 P. M. Morse and H. Feshbach, \textit{Methods of Theoretical Physics}, \textbf{Vol 1}, McGraw-Hill, New-York, 1953.

13 I. G. Macdonald, \textit{Symmetric Functions and Hall Polynomials}, 2nd edition, Oxford: Clarendon press, 1995.

14 P. Shanta, S. Chaturvedi, V. Srinivasan, G. S. Agarwal and C. L. Mehta, \textit{Phys. Rev. Lett.} \textbf{72}, 1447 (1994);

N. Gurappa, P. K. Panigrahi and V. Srinivasan, \textit{Mod. Phys. Lett. A} \textbf{13}, 339 (1998).

15 M. Rocek, \textit{Phys.Lett. B} \textbf{255(4)}, 554 (1991).

16 E. K. Sklyannin, \textit{Funct. Anal. Appl.} \textbf{16}, 263 (1982).

17 V. Sunilkumar, B.A. Bambah, P.K. Panigrahi and V. Srinivasan, \textit{J. Opt. B} \textbf{2}, 126 (2000).

18 F. M. Fernandez, \textit{Phys. Lett. A} \textbf{237}, 189 (1998), and references therein.

19 E. Schrödinger, \textit{Proc. Roy. Irish. Acad.} \textbf{46A}, 9 (1940) and \textbf{46A}, 183 (1941).

20 L. Infeld and T.E. Hull, \textit{Rev. Mod. Phys.} \textbf{23}, 21 (1951).

21 J. W. Dabrowska, A. Khare and U. Sukhatme \textit{J. Phys. A} \textbf{21}, L195 (1988).

22 F. Cooper, A. Khare and U. Sukhatme \textit{Phys. Rep.} \textbf{251}, 268 (1995) and references therein.

23 M. A. Jafarizadeh and H. Fakhri, \textit{Phys. Lett. A} \textbf{230}, 164 (1997).

24 L. Gendenshtein, \textit{Zh. Eksp. Teor. Fiz. Pis. Red.} \textbf{38}, 299 (1983).

25 B. Molnar and M. G. Benedict, Phys. Rev. \textbf{A 60}, R1737, (1999)

26 M. M. Nieto and D. R. Truax, \textit{Phys. Lett. A} \textbf{208}, 8 (1995).

27 M. M. Nieto and D. R. Truax, \textit{Phys. Lett. A} \textbf{237}, 192 (1998).
28 B. D. Simons, P. A. Lee and B. L. Altshuler, *Phys. Rev. Lett.* **72**, 64 (1994) and references therein.

29 P. K. Panigrahi and R. Atre under preparation.