A spectral sequence for Iwasawa adjoints

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May 11, 2014

1 Introduction

This paper is a slightly edited and corrected version of a long time unpublished but several times quoted preprint from 1994. The aim of this paper was and is to give a purely algebraic tool for treating so-called (generalized) Iwasawa adjoints of some naturally occurring Iwasawa modules for \( p \)-adic Lie group extensions, by relating them to certain continuous Galois cohomology groups via a spectral sequence.

Let \( k \) be a number field, fix a prime \( p \), and let \( k_\infty \) be some Galois extension of \( k \) such that \( \mathcal{G} = \text{Gal} (k_\infty/k) \) is a \( p \)-adic Lie-group (e.g., \( \mathcal{G} \cong \mathbb{Z}_p^r \) for some \( r \geq 1 \)). Let \( S \) be a finite set of primes containing all primes above \( p \) and \( \infty \), and all primes ramified in \( k_\infty/k \), and let \( k_S \) be the maximal \( S \)-ramified extension of \( k \); by assumption, \( k_\infty \subseteq k_S \). Let \( G_S = \text{Gal} (k_S/k) \) and \( G_{\infty,S} = \text{Gal} (k_S/k_\infty) \).

Let \( A \) be a discrete \( G_S \)-module which is isomorphic to \((\mathbb{Q}_p/\mathbb{Z}_p)^r\) for some \( r \geq 1 \) as an abelian group (e.g., \( A = \mathbb{Q}_p/\mathbb{Z}_p \) with trivial action, or \( A = E[p^\infty] \), the group of \( p \)-power torsion points of an elliptic curve \( E/k \) with good reduction outside \( S \)). We are not assuming that \( G_{\infty,S} \) acts trivially.

Let \( \Lambda = \mathbb{Z}_p[[\mathcal{G}]] \) be the completed group ring. For a finitely generated \( \Lambda \)-module \( M \) we put

\[
E^i(M) = \text{Ext}^i_\Lambda (M, \Lambda).
\]
Hence \( E^0(M) = \text{Hom}_\Lambda(M, \Lambda) =: M^+ \) is just the \( \Lambda \)-dual of \( M \). This has a natural structure of a \( \Lambda \)-module, by letting \( \sigma \in G \) act via
\[
\sigma f(m) = \sigma f(\sigma^{-1}m)
\]
for \( f \in M^+, m \in M \). It is known that \( \Lambda \) is a noetherian ring (here we use that \( G \) is a \( p \)-adic Lie group), by results of Lazard \([6]\). Hence \( M^+ \) is a finitely generated \( \Lambda \)-module again (choose a projection \( \Lambda^r \twoheadrightarrow M \); then we have an injection \( M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^r \)). By standard homological algebra, the \( E^i(M) \) are finitely generated \( \Lambda \)-modules for all \( i \geqslant 0 \) which we call the (generalized) Iwasawa adjoints of \( M \). They can also be seen as some kind of homotopy invariants of \( M \), see \([5]\), and also \([7]\) V §4 and 5.

**Examples**

(a) If \( G = \mathbb{Z}_p \), then \( \Lambda = \mathbb{Z}_p[[G]] \cong \mathbb{Z}_p[[X]] \) is the classical Iwasawa algebra, and, for a \( \Lambda \)-torsion module \( M \), it is known (see \([9]\) I.2.2, or \([1]\) 1.2 and rémarque, or \([7]\) Prop. (5.5.6))) that \( E^1(M) \) is isomorphic to the classical Iwasawa adjoint, which was defined by Iwasawa (\([2]\) 1.3) as
\[
\text{ad}(M) = \lim_{\leftarrow n} (M/\alpha_n M)^\vee
\]
where \( (\alpha_n)_{n \in \mathbb{N}} \) is any sequence of elements in \( \Lambda \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( (\alpha_n) \) is prime to the support of \( M \) for every \( n \geqslant 1 \), and where
\[
N^\vee = \text{Hom}(N, \mathbb{Q}_p/\mathbb{Z}_p)
\]
is the Pontrjagin dual of a discrete or compact \( \mathbb{Z}_p \)-module \( N \). For any finitely generated \( \Lambda \)-module \( M \), \( E^1(M) \) is quasi-isomorphic to \( \text{Tor}_\Lambda(M)^\sim \), where \( \text{Tor}_\Lambda(M) \) is the \( \Lambda \)-torsion submodule of \( M \), and \( M^\sim \) is the “Iwasawa twist” of a \( \Lambda \)-module \( M \): the action of \( \gamma \in G \) is changed to the action of \( \gamma^{-1} \).

(b) If \( G = \mathbb{Z}_p^r \), \( r \geqslant 1 \), then the \( E^i(M) \) are the standard groups considered in local duality. By duality for the ring \( \mathbb{Z}_p[[G]] = \mathbb{Z}_p[[x_1, \ldots, x_r]] \), they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex. More precisely, \( E^i(M) \cong H^{r+1-i}_m(M)^\vee \), as recalled in \([1]\) 1.

The main result of this note is the following observation.

**Theorem 1**

There is a spectral sequence of finitely generated \( \Lambda \)-modules
\[
E_2^{p,q} = E^p(H^q(G_{\infty,S}, A)^\vee) \Rightarrow \lim_{k',m} \lim_{k} H^{p+q}(G_{S}(k'), A[p^m]) = \lim_{k'} H^{p+q}(G_{S}(k'), T_p A).
\]

Here the inverse limits runs through the finite extensions \( k'/k \) contained in \( k_\infty \), and the natural numbers \( m \), via the corestrictions and the natural maps
\[
H^n(G_S(k'), A[p^{m+1}]) \to H^n(G_S(k'), A[p^m]),
\]

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respectively. The groups

\[ H^{p+q}(G_S(k'), T_pA) = \lim_{\leftarrow m} H^{p+q}(G_S(k'), A[p^m]) \]

are the continuous cohomology groups of the Tate module \( T_pA = \lim_{\leftarrow m} A[p^m] \).

2 Some consequences

Before we give the proof of a slightly more general result (cf. Theorem 11 below), we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

\[ 0 \rightarrow E^1(H^0(G_\infty, s, A)^\vee) \xrightarrow{\inf} \lim_{k'} H^1(G_S(k'), T_pA) \rightarrow (H^1(G_\infty, s, A)^\vee)^+ \rightarrow E^2(H^0(G_\infty, s, A)^\vee) \xrightarrow{\inf} \lim_{k'} H^2(G_S(k'), T_pA). \]

To say more, we make the following assumption.

A.1 Assume that \( p > 2 \) or that \( k_\infty \) is totally imaginary.

It is well-known that this implies

A.2 \( H^r(G_\infty, s, A) = 0 = \lim_{k'} H^r(G_S(k'), T_pA) \) for all \( r > 2 \).

Corollary 2 Assume in addition that \( H^2(G_\infty, s, A) = 0 \). (This is the so-called “weak Leopoldt conjecture” for \( A \). It is stated classically for \( A = \mathbb{Q}_p/\mathbb{Z}_p \) with trivial action, and there are precise conjectures when this is expected to hold for modules \( A \) coming from algebraic geometry, cf. [4].) Then the cokernel of \( \inf^2 \) is

\[ \ker \left( E^1(H^1(G_\infty, s, A)^\vee) \rightarrow E^3(H^0(G_\infty, s, A)^\vee) \right), \]

and there are isomorphisms

\[ E^i(H^1(G_\infty, s, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^0(G_\infty, s, A)^\vee) \]

for \( i \geq 2 \).
**Proof** This comes from A.2 and the following picture of the spectral sequence

![](image)

**Corollary 3** Assume that \( H^0(G_\infty, S, A) = 0 \). Then

(a) \[
\lim_{k'} H^1(G_S(k'), T_pA) \xrightarrow{\sim} H^1(G_\infty, S, A)^+.
\]

(b) There is an exact sequence

\[
0 \rightarrow E^1(H^1(G_\infty, S, A)^+) \rightarrow \lim_{k'} H^2(G_S(k'), T_pA)
\]

\[
\rightarrow (H^2(G_\infty, S, A)^+) \rightarrow E^2(H^1(G_\infty, S, A)^+) \rightarrow 0.
\]

(c) There are isomorphisms

\[
E^n(H^2(G_\infty, S, A)^+) \xrightarrow{\sim} E^{n+2}(H^1(G_\infty, S, A)^+)
\]

for \( i \geq 1 \).

**Proof** In this case, the spectral sequence looks like

![](image)
Corollary 4 Assume that $\mathcal{G}$ is a $p$-adic Lie group of dimension 1 (equivalently: an open subgroup is $\cong \mathbb{Z}_p$). Then $E^i(\mathcal{G}) = 0$ for $i \geq 3$. Let

$$B = \text{im} \left( \text{inf}^2 : E^2(H^0(G, S, A)^\vee) \to \lim_{k'} H^2(G_S(k'), T_p A) \right)$$

Then $B$ is finite, and there is an exact sequence

$$0 \to E^1(H^1(G, S, A)^\vee) \to \lim_{k'} H^2(G_S(k'), T_p A)/B \to (H^2(G, S, A)^\vee)^+ \to E^2(H^1(G, S, A)^\vee) \to 0,$$

and

$$E^1(H^2(G, S, A)^\vee) = 0 = E^2(H^2(G, S, A)^\vee),$$

i.e., $(H^2(G, S, A)^\vee)$ is a projective $\Lambda$-module.

Proof Quite generally, for a $p$-adic Lie group $\mathcal{G}$ of dimension $n$ one has $\text{vcd}_p(\mathcal{G}) = n$ for the virtual cohomological $p$-dimension of $\mathcal{G}$, and hence $E^i(\mathcal{G}) = 0$ for $i > n + 1$, cf. [3] Cor. 2.4. The finiteness of $E^2(M)$, for a $\Lambda$-module $M$ which is finitely generated over $\mathbb{Z}_p$ (like our module $H^0(G, S, A)^\vee$) follows from Lemma 5 below. In fact, the exact sequence $0 \to M_{\text{tor}} \to M \to M \to 0$, in which $M_{\text{tor}}$ is the torsion submodule of $M$, induces a long exact sequence

$$\ldots \to E^i(\tilde{M}) \to E^i(M) \to E^i(M_{\text{tor}}) \to \ldots$$

in which we have $E^i(M_{\text{tor}}) = 0$ for $i \neq n + 1$ and finiteness of $E^{n+1}(M_{\text{tor}})$ by Lemma 5 (b), and $E^i(\tilde{M}) = 0$ for $i \neq n$ for the torsion-free module $M$ by Lemma 5 (a). In our case we have $n = 1$ and therefore the finiteness of $E^2(M)$. The remaining claims follow from the following shape of the spectral sequence:
Lemma 5 Assume that $\mathcal{G}$ is a $p$-adic Lie group of dimension $n$ (this holds, e.g., if $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^n$), and let $M$ be a $\Lambda$-module which is finitely generated as a $\mathbb{Z}_p$-module. Then the following holds.

(a) $E^i(M) = 0$ for $i \neq n, n + 1$.

(b) If $M$ is torsion-free, then

$$E^i(M) = \begin{cases} 0 & \text{for } i \neq n \text{ or } i = n, \\ \text{Hom} \left( D, M^\vee \right) & \text{for } i = n, \end{cases}$$

where $D$ is the dualising module for $\mathcal{G}$ (which is a divisible cofinitely generated $\mathbb{Z}_p$-module, e.g., $D = \mathbb{Q}_p/\mathbb{Z}_p$ if $\mathcal{G} = \mathbb{Z}_p^n$). In particular, $E^n(M)$ is a torsion-free finitely generated $\mathbb{Z}_p$-module.

(c) If $M$ is finite, then

$$E^i(M) = \begin{cases} 0 & \text{for } i \neq n + 1 \\ \text{Hom} \left( M, D^\vee \right) & \text{for } i = n + 1. \end{cases}$$

In particular, $E^{n+1}(M)$ is a finite $\mathbb{Z}_p$-module.

Proof See [5], Cor. 2.6. For (a) note the isomorphism

$$\text{Hom}(D \otimes_{\mathbb{Z}_p} M, \mathbb{Q}_p/\mathbb{Z}_p) \cong \text{Hom}(D, \text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)) = \text{Hom}(D, M^\vee).$$

Corollary 6 Let $\mathcal{G}$ be a $p$-adic Lie group of dimension 2 (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^2$). If $G_{\infty,S}$ acts trivially on $A$, then there are exact sequences

$$0 \to \lim_{\leftarrow k'} H^1(G_S(k'), T_pA) \to (H^1(G_{\infty,S}, A)^\vee)^+ \to T_pA \xrightarrow{\text{inf}^2} \lim_{\leftarrow k'} H^2(G_S(k'), T_pA)$$

and

$$0 \to E^1(H^1(G_{\infty,S}, A)^\vee) \to \lim_{\leftarrow k'} H^2(G_S(k'), T_pA)/\text{im inf}^2 \to (H^1(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee) \to 0,$$

an isomorphism

$$E^1(H^2(G_{\infty,S}, A)^\vee) \sim E^3(H^1(G_{\infty,S}, A)^\vee),$$

and one has

$$E^2(H^2(G_{\infty,S}, A)^\vee) = 0 = E^3(H^2(G_{\infty,S}, A)^\vee).$$
**Proof** The spectral sequence looks like

![Spectral Sequence Diagram]

**Corollary 7** Let $\mathcal{G}$ be a $p$-adic Lie group of dimension 2 (So $E^i(-) = 0$ for $i \geq 4$). If $H^0(G_{\infty, S}, A)$ is finite, then

$$
\lim_{\mathclap{k'}} H^1(G_{S(k')}, T_p A) \cong (H^1(G_{\infty, S}, A)^\vee)^+ .
$$

If

$$
d_2^{1,1} : E^1(H^1(G_{\infty, S}, A)^\vee) \longrightarrow E^3(H^0(G_{\infty, S}, A)^\vee)
$$

is the differential of the spectral sequence in the theorem, then one has an exact sequence

$$
0 \longrightarrow \ker d_2^{1,1} \longrightarrow \lim_{\mathclap{k'}} H^2(G_{S(k')}, T_p A) \longrightarrow
$$

$$
\longrightarrow \ker(d_2^{0,2} : (H^2(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee)) \longrightarrow \coker d_2^{1,1} \longrightarrow 0 ,
$$

an isomorphism

$$
E^1(H^2(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty, S}, A)^\vee) ,
$$

and the vanishing

$$
E^2(H^2(G_{\infty, S}, A)^\vee) = 0 = E^3(H^2(G_{\infty, S}, A)^\vee) .
$$
**Proof** The spectral sequence looks like

\[
\begin{array}{c}
\bullet \bullet \bullet \\
\bullet \bullet \bullet \\
\end{array}
\]

**Remark** In the situation of Corollary 5, one has an exact sequence up to finite modules:

\[
0 \rightarrow E^1(H^1(G_\infty, s, A)^\vee) \rightarrow \lim_{k'} H^2(G_S(k'), T_pA)
\]

\[
\rightarrow (H^2(G_\infty, s, A)^\vee)^+ \rightarrow E^2(H^1(G_\infty, s, A)^\vee) \rightarrow 0.
\]

**Corollary 8** Let $G$ be a $p$-adic Lie group of dimension > 2. Then

\[
(H^1(G_\infty, s, A)^\vee)^+ \cong \lim_{k'} H^1(G_S(k'), T_pA)
\]

**Proof** The first three columns of the spectral sequence look like

\[
\begin{array}{c}
\bullet \bullet \bullet \\
\bullet \bullet \bullet \\
\end{array}
\]
3 Proof of the Main Theorem

We will now prove Theorem 1, by proving a somewhat more general result. For any profinite group $G$, let $\Lambda(G) = \mathbb{Z}_p[[G]]$ be the completed group ring over $\mathbb{Z}_p$, and let $M_G = M_{G,p}$ be the category of discrete (left) $\Lambda(G)$-modules. These are the discrete $G$-modules $A$ which are $p$-primary torsion abelian groups. For such a module $A$, its Pontrjagin dual $A^\vee = \text{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ is a compact $\Lambda(G)$-module. In fact, Pontrjagin duality gives an anti-equivalence between $M_G$ and the category $C_G = C_{G,p}$ of compact (right) $\Lambda(G)$-modules.

Let $M_G^\mathbb{N}$ be the category of inverse systems $(A_n) : \ldots \to A_3 \to A_2 \to A_1$ in $M_G$ as in [3]. Denote by $H^i_{\text{cont}}(G, (A_n))$ the continuous cohomology of such a system and recall that one has an exact sequence for each $i$

$$0 \to R^1 \lim_{n} H^{i-1}(G, A_n) \to H^i_{\text{cont}}(G, (A_n)) \to \lim_{n} H^i(G, A_n) \to 0,$$

in which the first derivative $R^1 \lim_{n}$ of the inverse limit, also noted as $\lim_{n}^1$, vanishes if the groups $H^{i-1}(G, A_n)$ are finite for all $n$ (cf. loc. cit.).

**Definition 9** For a closed subgroup $H \leq G$ and a discrete $G$-module $A$ in $M_G$ define the relative cohomology $H^m(G, H; A)$ as the value at $A$ of the $m$-th derived functor of the left exact functor (with $\text{Ab}$ being the category of abelian groups)

$$H^0(G, H; -) : M_G \to \text{Ab}$$

$$A \mapsto \lim_{U} H^0(U, A),$$

where $U$ runs through all open subgroups $U \subset G$ containing $H$, and the transition maps are the corestriction maps. For an inverse system $(A_n)$ of modules in $M_G$ define the continuous relative cohomology $H^m_{\text{cont}}(G, H; (A_n))$ as the value at $(A_n)$ of the $m$-th right derivative of the functor

$$H^0_{\text{cont}}(G, H; -) : M_G^\mathbb{N} \to \text{Ab}$$

$$(A_n) \mapsto \lim_{n} \lim_{U} H^0(U, A_n),$$

where the limit over $U$ is as before, and the limit over $n$ is induced by the transition maps $A_{n+1} \to A_n$. 

9
Lemma 10 If $G/H$ has a countable basis of neighbourhoods of identity, i.e., if there is a countable family $U_\nu$ of open subgroup, $H \leq U_\nu \leq G$, with $\bigcup U_\nu = H$, and if, in addition, $H^i(U, A_n)$ is finite for all these $U$ and all $n$, then

$$H^i_{cont}(G, H; (A_n)) = \lim\limits_{\leftarrow n} \lim\limits_{\leftarrow U} H^i(U, A_n).$$

Proof More generally, without assuming the finiteness of the groups $H^i(U, A)$, we claim that we have a Grothendieck spectral sequence for the composition of the functors $(A_n) \mapsto (H^0(U, A_n))_{U,n}$ with the functor $\lim\limits_{\leftarrow n,U} E_{p,q} = R^p \lim\limits_{\leftarrow n,U} H^q(U, A_n) \Rightarrow H^{p+q}(G, H; (A_n)).$

For this we have to show that the first functor sends injective objects to acyclics for the second functor. But if $(I_n)$ is an injective system, then all $I_n$ are injective and all morphisms $I_{n+1} \to I_n$ are split surjections, see [3] (1.1), so $H^0(U, I_{n+1}) \to H^0(U, I_n)$ is surjective for any open subgroup $U \subset G$.

On the other hand, if $I$ is an injective $G$-module, then for any pair of open subgroups $U' \subset U$ the corestriction $\text{cor} : H^0(U', I) \to H^0(U, I)$ is surjective. In fact, we may assume that $I = \text{Ind}^G_U(B)$ is an induced module for a divisible abelian group $B$. (Any such module is injective, and any discrete $G$-module can be embedded into such a module, see [10] p. 28 and 29., so that any injective is a direct factor of such a module). Moreover, since the formation of corestrictions is transitive, we may consider an open subgroup $U'' \subset U'$ which is normal in $G$. Then $\text{Ind}^G_U(B)_{U''} \cong \text{Ind}^G_{U''}(B)$ for $\overline{G} = G/U''$, and it is known that this is a cohomologically trivial $\overline{G}$-module. Therefore, letting $\overline{U} = U/U''$, we have

$$\text{Ind}^G_U(B)_{U} = \text{Ind}^G_{\overline{U}}(B)_{\overline{U}} = tr_{\overline{G}} \text{Ind}^G_{\overline{U}}(B) = \text{cor}_{U''/U} \text{Ind}^G_U(B)_{U''}$$

as claimed.

By assumption, the inverse limit over the open subgroups $U$ containing $H$ can be replaced by a cofinal set of subgroups $U_m$ with $m \in \mathbb{N}$ and $U_{m+1} \subset U_m$, and then the limit over these $U_m$ and over $n$ can be replaced by the ‘diagonal’ limit over the pairs $(U_n, n)$ for $n \in \mathbb{N}$. For such an inverse limit it is well-known that $R^p \lim_{\leftarrow n} = 0$ for $p > 1$, and that $R^1 \lim_{\leftarrow n} H^0(U_n, I_n) = 0$, since the transition maps $H^0(U_{n+1}, I_{n+1}) \to H^0(U_n, I_n)$ are surjective, as shown above (so the system trivially satisfies the Mittag-Leffler condition).

This shows the existence of the above spectral sequence. If, in addition, all $H^q(U, A_n)$ are finite, then, reasoning as above,

$$R^1 \lim_{\leftarrow n,U} H^q(U, A_n) = 0$$

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by the Mittag-Leffler property, and we get the claimed isomorphisms.

Now we come to the spectral sequence in theorem 1. Any module $A$ in $M_G$ gives rise to two inverse systems, viz., the system $(A[p^n])$, where the transition maps $A[p^{n+1}] \to A[p^n]$ are induced by multiplication with $p$ in $A$, and the system $(A/p^n)$, where the transition maps are induced by the identity of $A$. For reasons explained later, denote by $H^m_{\text{cont}}(G, H; R_{\mathbb{Z}_p}A)$ the value at $A$ of the $m$-th derived functor of the left exact functor

$$F : A \leadsto \lim_{\leftarrow n} \lim_{\leftarrow U} H^0(U, A[p^n])$$

where $U$ runs through all open subgroups $U \subset G$ containing $H$, and the transition maps are the corestriction maps and those coming from $A[p^{n+1}] \to A[p^n]$, respectively. If $H$ is a normal subgroup, then we may restrict to normal open subgroups $U \leq G$ containing $H$ in the above inverse limit, and the limit is a (left) $\Lambda(G/H)$-module in a natural way.

**Theorem 11** Let $H$ be a closed subgroup of a profinite group $G$ such that $G/H$ has a countable basis of neighbourhoods of identity (see Lemma 10), and let $A$ be a discrete $\Lambda(G)$-module.

(a) There are short exact sequences

$$0 \to H^i_{\text{cont}}(G, H; (A[p^n])) \to H^i_{\text{cont}}(G, H; R_{\mathbb{Z}_p}A) \to H^{i-1}_{\text{cont}}(G, H; (A/p^n)) \to 0.$$

If $H$ is a normal subgroup, then these are exact sequences of $\Lambda(G/H)$-modules.

(b) Let $H'$ be a normal subgroup of $G$, with $H' \subset H$. There is a spectral sequence

$$E_{2}^{p,q} = H^p_{\text{cont}}(G/H', H/H'; R_{\mathbb{Z}_p}H^q(H', A)) \Rightarrow H^{p+q}_{\text{cont}}(G, H; R_{\mathbb{Z}_p}A).$$

If $H$ is a normal subgroup, too, this is a spectral sequence of $\Lambda(G/H)$-modules.

(c) If $H$ is a normal subgroup of $G$, then for every discrete $\Lambda(G)$-module $A$ one has canonical isomorphisms of $\Lambda(G/H)$-modules

$$H^m(G, H; R_{\mathbb{Z}_p}A) \cong \text{Ext}^m_{\Lambda(G)}(A^\vee, \Lambda(G/H))$$

for all $m \geq 0$, where $\Lambda(G/H)$ is regarded as a $\Lambda(G)$-module via the ring homomorphism $\Lambda(G) \to \Lambda(G/H)$. More precisely, the $\delta$-functor

$$M_G \to \text{Mod}_{\Lambda(G/H)} \ , \ A \leadsto (H^m(G, H; R_{\mathbb{Z}_p}A) | m \geq 0)$$

is canonically isomorphic to the $\delta$-functor

$$M_G \to \text{Mod}_{\Lambda(G/H)} \ , \ A \leadsto (\text{Ext}^m_{\Lambda(G)}(A^\vee, \Lambda(G/H)) | m \geq 0).$$
Here and in the following, the Ext-groups $\text{Ext}_{\Lambda(G)}(-,-)$ are taken in the category $C_G$ of compact $\Lambda(G)$-modules. We note that these Ext-groups are $\Lambda(G)$-modules, but not necessarily compact.

(d) In particular, let $H$ be a normal subgroup of $G$, and let $\mathcal{G} = G/H$. If $A$ is a discrete $\Lambda(G)$-module, then one has a spectral sequence of $\Lambda(\mathcal{G})$-modules

$$E_2^{p,q} = \text{Ext}_{\Lambda(\mathcal{G})}^p(H^q(H, A)^\vee, \Lambda(\mathcal{G})) \Rightarrow H_{\text{cont}}^{p+q}(G, H; RT_pA) = \text{Ext}_{\Lambda(G)}^{p+q}(A^\vee, \Lambda(\mathcal{G})).$$

Before we give the proof of Theorem 11, we note that it implies Theorem 1. In fact, we apply Theorem 11 to $G = G_S$ and $H = G_{\infty,S}$. If $A$ is a $G_S$-module of cofinite type as in Theorem 1, then $A/p^n = 0$ and $A[p^n]$ is finite, for all $n$. Moreover, $H^i(U, B)$ is known to be finite for all open subgroups $U \leq G_S$ and all finite $U$-modules $B$. By (a) and Lemma 10 we deduce

$$H_{\text{cont}}^m(G_S, G_{\infty,S}; RT_pA) = \lim_{\leftarrow n,U} H^m(U, A[p^n]) = \lim_{\leftarrow n,k'} H^m(G_S(k'), A[p^n]),$$

where $k'$ runs through all finite subextensions of $k_{\infty}/k$. Moreover, one has canonical isomorphisms

$$\lim_{\leftarrow n} H^m(U, A[p^n]) \cong H^m(U, T_pA)$$

where the latter group is continuous cochain group cohomology, cf. [Ja 1]. By applying Theorem 11 (d) we thus get the desired spectral sequence. Finally, $H^m(H, A)^\vee$ is a finitely generated $\Lambda(\mathcal{G})$-module for all $m \geq 0$, so that the initial terms of the spectral sequence are finitely generated $\Lambda(\mathcal{G})$-modules as well, and so are the limit terms. In fact, let $N$ be the kernel of the homomorphism $G_S \rightarrow \text{Aut}(A)$ given by the action of $G_S$ on $A$, and let $H' = H \cap N$. Then $G/H'$ is a $p$-adic analytic Lie group, since $G/H$ and $G/N$ are. It is well-known that $H^m(H', \mathbb{Q}_p/\mathbb{Z}_p)$ is a cofinitely generated discrete $\Lambda(G/H')$-module for all $m \geq 0$; hence the same is true for $H^m(H', A) \cong H^m(H', \mathbb{Q}_p/\mathbb{Z}_p) \otimes T_pA$. The claim then follows from the Hochschild-Serre spectral sequence $H^p(H/H', H^q(H', A)) \Rightarrow H^{p+q}(H, A)$.

**Proof of Theorem 11 (a):** We can write $F$ as the composition of the two left exact functors

$$T_p : M_G \rightarrow M_G^{[1]}$$

and

$$H^0_{\text{cont}}(G, H; -) : M_G^{[1]} \rightarrow Ab$$

and

$$\lim_{\leftarrow n} \lim_{\leftarrow U} H^0(U, A_n),$$

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where the limit over $U$ runs through all open (normal) subgroups of $G$ containing $H$, with the corestrictions as transition maps. With the arguments in the proof of lemma 10, we can deduce that $T_p$ maps injectives to $H^0_{\text{cont}}(G, H; -)$-acyclics. In fact, we may assume injective $G$-modules given as induced modules $I = \text{Ind}_G^B(B[n])$ with a divisible abelian group $B$. Then each module $I[p^n] = \text{Ind}_G^B(B[n])$ is induced, hence acyclic for the functor $H^0(U, -)$, and since $I$ is divisible, we have exact sequences

$$0 \to I[p^n] \to I[p^{r+1}] \to I[p^r] \to 0.$$  

Therefore the transition maps $H^0(U, I[p^{r+1}]) \to H^0(U, I[p^r])$ are surjective, and as in the proof of Lemma 10 we conclude that the corestrictions $\text{cor} : H^0(U', I[p^n]) \to H^0(U, I[p^n])$ are surjective for open subgroups $U' \subset U$ of $G$. Therefore the system $I[p^n]$ is acyclic for the $\lim\leftarrow_{U,n}$, noting that

$$\lim\leftarrow_{U,n} H^0(U, I[p^n]) = \lim\leftarrow_{n} H^0(U_n, I[p^n])$$

for a cofinal family $(U_n)$ of subgroups between $H$ and $G$.

Therefore we get a spectral sequence

$$E^{p,q}_2 = H^p_{\text{cont}}(G, H; R^qT_pA) \Rightarrow H^{p+q}_{\text{cont}}(G, H; RT_pA).$$

From the snake lemma one immediately gets

$$R^qT_pA = \begin{cases} (A/p^mA) & q = 1 \\ 0 & q > 1 \end{cases}$$

(note that the described functor $A \sim (A/p^nA)$ is effacable, since $A$ embeds into an injective, hence divisible $G$-module. Hence we get a short exact sequences

$$0 \to H^n_{\text{cont}}(G, H; T_pA) \to H^n_{\text{cont}}(G, H; RT_pA) \to H^{n-1}_{\text{cont}}(G, H; R^1T_pA) \to 0.$$  

This shows (a) and also explains the notation for $R^nF$. In fact, $H^n_{\text{cont}}(G, H; R T_pA)$ is the hypercohomology with respect to $H^0_{\text{cont}}(G, H; -)$ of a complex $RT_pA$ in $M_G$ computing the $R^1T_pA$.

(b): If $H$ is a normal subgroup, we can regard the functor $F$ as a functor from $M_G$ to the category $\text{Mod}_{\Lambda(G/H)}$ of $\Lambda(G/H)$-modules. On the other hand, we can also write $F$ as the composition of the left exact functors

$$H^0(H, -) : M_G \to M_{G/H}, \quad A \sim A^H.$$
and

\[ \tilde{F} : \text{Mod}_{\Lambda(G/H)} \rightarrow \text{Mod}_{\Lambda(G/H)}, B \mapsto \lim_{\leftarrow n} \lim_{\leftarrow U/H} H^0(U/H, B[p^n]) = H^0(G/H, \{1\}; RT_p B). \]

(Note that \(U/H\) runs through all open (normal) subgroups of \(G/H\).) This immediately gives the spectral sequence in (b).

(c): We claim that the functor \(F\) is isomorphic to the functor \(M_{G/H} \rightarrow \text{Mod}_{\Lambda(G/H)} B \mapsto \text{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)).\)

In fact, writing \(\text{Hom}_{\Lambda(G)}(-, -)\) for the homomorphism groups of compact \(\Lambda(G)\)-modules, we have (cf. [Ja 3] p. 179)

\[
\text{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)) = \lim_{\leftarrow U} \text{Hom}_{\Lambda(G)}(B^\vee, Z_p[G/U])
\]

\[
= \lim_{\leftarrow n} \lim_{\leftarrow U} \text{Hom}_{\text{cont}}(H^0(U, B)^\vee, Z/p^n Z)
\]

\[
= \lim_{\leftarrow n} \lim_{\leftarrow U} \text{Hom}_{\text{cont}}(H^0(U, B[p^n])^\vee, Z/p^n Z)
\]

\[
= \lim_{\leftarrow n} \lim_{\leftarrow U} H^0(U, B[p^n]),
\]

where \(U\) runs through all open subgroups of \(G\) containing \(H\), and hence

\[
\text{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)) = H^0(G, H; RT_p B).
\]

Since taking Pontrjagin duals is an exact functor \(M_G \rightarrow C_G\) taking injectives to projectives, the derived functors of the functor \(B \mapsto \text{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H))\) are the functors \(B \mapsto \text{Ext}^m_{\Lambda(G)}(B^\vee, \Lambda(G/H))\), and we get (c). Finally, by applying (b) for \(H' = H\) and (c) for \(H = \{1\}\) we get (d).

Let us note that the proof of theorem 11 gives the following \(Z/p^n\)-analogue (by omitting the inverse limits over \(n\)). For a profinite group \(G\) let \(\Lambda_n(G) = \Lambda(G)/p^n = Z/p^n[[G]]\) be the completed group ring over \(Z/p^n\).

**Theorem 12** Let \(H\) and \(H'\) be normal subgroups of a profinite group \(G\), with \(H' \subset H\), and let \(A\) be a discrete \(\Lambda_n(G)\)-module.

(a) There is a spectral sequence of \(\Lambda_n(G/H)\)-modules

\[ E_2^{p,q} = H^p(G/H', H/H'; H^q(H', A)) \Rightarrow H^{p+q}(G, H; A). \]

(b) On the category of discrete \(\Lambda_n(G)\)-modules the \(\delta\)-functor \(A \mapsto (H^m(G, H; A) | m \geq 0)\) with values in the category of \(\Lambda_n(G/H)\)-modules is canonically isomorphic to the \(\delta\)-functor \(A \mapsto (\text{Ext}^m_{\Lambda_n(G)}(A^\vee, \Lambda_n(G/H)) | m \geq 0),\)
where the Ext-groups are taken in the category of compact $\Lambda_n(G)$-modules.

(c) In particular, if $G = G/H$, and $A$ is a discrete $\Lambda_n(G)$-module, then one has a spectral sequence of $\Lambda_n(G)$-modules

$$E_{2}^{p,q} = \text{Ext}^{p}_{\Lambda_n(G)}(H^q(H, A)^\vee, \Lambda_n(G)) \Rightarrow H^{p+q}(G, A) = \text{Ext}^{p+q}_{\Lambda_n(G)}(A^\vee, \Lambda(G)).$$

**Corollary 13** With the notations as for Theorem 1, let $A$ be a finite $\Lambda_n(G_S)$-module, and $\Lambda_n = \Lambda(G)$. Then there is a spectral sequence of finitely generated $\Lambda_n$-modules

$$E_{2}^{p,q} = \text{Ext}^{p}_{\Lambda_n(S)}(H^q(G_\infty, A)^\vee, \Lambda_n) \Rightarrow \lim_{\leftarrow \ k'} H^{p+q}(G_S(k'), A) = \text{Ext}^{p+q}_{\Lambda_n(G_S)}(A^\vee, \Lambda_n),$$

where $k'$ runs through the finite subextensions $k'/k$ of $k_\infty/k$.

On the other hand, Theorem 1 also has the following counterpart for finite modules.

**Theorem 14** With notations as for Theorem 1, let $A$ be a finite $p$-primary $G_S$-module, of exponent $p^n$. Then there is a spectral sequence

$$E_{2}^{p,q} = \text{Ext}^{p}_{\Lambda}(H^q(G_\infty, S, A)^\vee, \Lambda) \Rightarrow \lim_{\leftarrow \ k'} H^{p+q}(G_S(k'), A) = \text{Ext}^{p+q-1}_{\Lambda_n(G_S)}(A^\vee, \Lambda_n),$$

where, in the inverse limit, $k'$ runs through the finite extension $k'$ of $k$ inside $k_\infty$ and the transition maps are the corestrictions.

**Proof** As in the proof of Theorem 1, Theorem 11 (d) applies to $G = G_S$ and $H = G_\infty, S$. But now the inverse system $(A[p^n])$ is Mittag-Leffler-zero in the sense of [3]: if the exponent of $A$ is $p^d$, then the transition maps $A[p^{n+d}] \to A[p^n]$ are zero. This implies that $H^m_{\text{cont}}(G_S, (A[p^n])) = 0$ for all $m \geq 0$, cf. [Ja 1]. On the other hand it is clear that the system $(A/p^n)$ is essentially constant ($A/p^n = A$ for $n \geq d$). From Theorem 11 (a) and Lemma 10 we immediately get

$$H^m_{\text{cont}}(G_S, G_\infty, S; RT_p A) \cong H^{m-1}_{\text{cont}}(G_S, G_\infty, S; (A/p^n)) \cong \lim_{\leftarrow \ k'} H^{p+q-1}(G_S(k'), A),$$

and hence the claim, by applying Theorem 12 (b) in addition.
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