Limit resistance law for Couette-Taylor flow and for direct round pipes at very large Reynolds numbers

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Abstract. Asymptotic expansions of well-known Prandtl–von-Karman resistance law at very large Reynolds numbers were obtained. These asymptotic forms can be used for turbulent flows between rotating coaxial cylinders (Couette-Taylor flow) and in direct round pipes.

1. Introduction

Fully-developed turbulent flows of incompressible fluid between rotating cylinders and in round smooth direct pipes are of great theoretical and practical importance. Let us limit ourselves to only resistance laws for these types of flows at very large Reynolds numbers.

According to [1], experimental data for turbulent flow of incompressible fluid between rotating cylinders (external that is at rest) satisfy very well to Prandtl–von-Karman resistance law. Flow between rotating coaxial cylinders is usually called Couette-Taylor flow.

\[ M \sqrt{G} + N \sqrt{G} \log_{10} \sqrt{G} = \text{Re}, \]  

(1)

where \( \text{Re} \) - Reynolds number, \( G \) - dimensionless torque of forces, acting on internal cylinder, \( M \) and \( N \) are different functions of von Karman constants and geometry [2], [3], [4], and \( \text{Re} \) varies from \( \text{Re} = 10^5 \) up to \( \text{Re} = 10^6 \). Constants \( M = -1.83 \) and \( N = 1.56 \) were chosen from the experiment [1].

Now we shall consider fully-developed turbulent flow in smooth direct pipe. Following to [5], friction coefficient \( \lambda \) is function only of Reynolds number \( \text{Re} \), that is \( \lambda = f(\text{Re}) \),

where

\[ \lambda = \frac{dP}{dx} \frac{D}{U^2} \] and \( \text{Re} = \frac{UD}{\nu} \),

\( dP/\text{dx} \) - pressure drop for unite length, \( D \) - diameter of pipe, \( \rho \) - fluid density, \( \nu \) - viscosity, \( U \) - fluid velocity, averaged over cross section of pipe.

According to [5], experimental data satisfy very well to some kind of Prandtl–von-Karman law, beginning from \( \text{Re} = 300 \times 10^3 \) up to \( \text{Re} = 13.6 \times 10^6 \):
\[ 1/\lambda^{1/2} = 1.930 \log_{10}(\text{Re } \lambda^{1/2}) - 0.537 \]  \hspace{1cm} (2)

Despite some similarity between the laws (1) and (2), they possess different dependence on Reynolds numbers, and, therefore, have different asymptotes at very large Reynolds numbers Re.

2. Asymptotic analysis of resistance law for turbulent Taylor-Couette flow.

Algebraic equation of resistance law (1) we can write down as:

\[ aG_t + bG_t \ln G_t = \text{Re} \]  \hspace{1cm} (3)

where \( a = M = -1.83 \), \( b = N / \ln 10 = 0.6775 \), \( G_t = \sqrt{G} \) are empirical constants.

We shall obtain an asymptotic solution of this equation at \( \text{Re} \to \infty \).

We have \( G_t = \frac{\text{Re}}{b \ln \text{Re}} \) \hspace{1cm} (4)

in zero order of approximation.

To find higher orders approximation terms we shall write down unknown variable \( G_t \) as \( G_t = \frac{\text{Re}}{b \ln \text{Re}} + \delta \), (see, for example, [6]).

Inserting this expression in equation (3), we shall have nonlinear equation for unknown \( \delta \), which, after division both parts of (3) by \( \text{Re} \), become as:

\[
\frac{a}{b \ln \text{Re}} + \frac{a \delta}{\text{Re}} + \left( \frac{1}{b \ln \text{Re}} + \frac{b \delta}{\text{Re}} \right) \ln \text{Re} + \ln \left( \frac{1}{b \ln \text{Re}} + \frac{\delta}{\text{Re}} \right) = 1 \hspace{1cm} (5)
\]

We seek value of \( \delta \) as asymptotic series:

\[ \delta = \frac{\text{Re} \ln \ln \text{Re}}{\ln^2 \text{Re} } \alpha_0 + \frac{\text{Re} \ln \ln \ln \text{Re}}{\ln^3 \text{Re} } \alpha_1 + \ldots, \]

where \( \alpha_0, \alpha_1, \ldots \) are unknown coefficients. We shall limit ourselves by obtaining only the first term of the asymptotic expansion:

\[ \delta = \frac{\text{Re} \ln \ln \text{Re}}{\ln^2 \text{Re} } \alpha_0 \hspace{1cm} (6) \]

The equation (5) becomes the next expression:

\[
\frac{a}{b \ln \text{Re}} + \frac{a \ln \ln \text{Re}}{\ln^2 \text{Re} } \alpha_0 + \left( \frac{1}{b \ln \text{Re}} + \frac{b \ln \ln \text{Re}}{\ln^2 \text{Re} } \alpha_0 \right) \ln \text{Re} + \ln \left( \frac{1}{b \ln \text{Re}} + \frac{\ln \ln \text{Re}}{\ln^2 \text{Re} } \alpha_0 \right) = 1
\]

Estimation of terms in leading order gives an equation:

\[
\frac{a}{b \ln \text{Re}} + \frac{a \ln \ln \text{Re}}{\ln^2 \text{Re} } \alpha_0 = - \frac{\ln(b \ln \text{Re})}{\ln \text{Re} } + \frac{b \ln \ln \text{Re}}{\ln \text{Re} } \alpha_0 = 0 \hspace{1cm} (7)
\]

We shall obtain after further simplification the unknown value of \( \alpha_0 \):

\[ \alpha_0 = \frac{1}{b} \hspace{1cm} . \]

Asymptotic expression for function \( G_t \) at \( \text{Re} \to \infty \) will be:

[---]
\[ G_1 = \frac{\text{Re}}{b \ln \text{Re}} + \frac{\text{Re} \ln \ln \text{Re}}{b \ln^2 \text{Re}}. \]  

(8)

Therefore, an expression for dimensionless torque \( G \) will be in zero approximation as:

\[ G = \frac{\text{Re}^2}{b^2 \ln^3 \text{Re}}. \]  

(9)

and in the next approximation, taking two terms of asymptotic expansion:

\[ G = \frac{\text{Re}^2}{b^2 \ln^3 \text{Re}} + \frac{2 \text{Re}^2 \ln \ln \text{Re}}{b^2 \ln^3 \text{Re}}. \]  

(10)

A comparison with some experiments will be given further.

3. Asymptotes for resistance law of turbulent flow in round pipe

We shall rewrite relation (2) in terms of natural logarithm instead of decimal logarithm. As a result, we obtain the relation:

\[ \frac{1}{\lambda^{1/2}} = A \ln(\text{Re} \lambda^{1/2}) + B, \]  

(11)

where \( A = \frac{1.930}{\ln 10} = 0.8382, \; B = -0.537 \).

We are interested in asymptotic behavior \( \lambda \) at \( \text{Re} \to \infty \). We introduce new variables:

\( g = \lambda^{1/2}, \; p = \ln \text{Re} \).

Then relation (11) turns into the next form:

\[ \frac{1}{g} = A(p + \ln g) + B \]  

(12)

We shall look for asymptotic behavior of \( g \) at \( p \to \infty \), or \( \text{Re} \to \infty \).

We shall input new variables:

\[ f = Ap + B. \]

Then relation (12) takes a new form:

\[ \frac{1}{g} = A \ln g + f \]  

(13)

We shall investigate asymptotic behavior of \( g \) at \( f \to \infty \).

Since function \( \ln g \) varies in zero – order approximation more slowly than \( g^{-1} \), we can approximately write down:

\[ g_0 = \frac{1}{f}. \]

The next approximation we shall seek as:
\[ g = \frac{1}{f} + \delta, \text{ where } |\delta| < \frac{1}{f} \]

Inserting this expression in (13), we shall obtain algebraic equation for the variable \( \delta \):

\[ \frac{f}{1 + \delta f} = A[-\ln f + \ln(1 + \delta f)] + f \] (14)

Considering \( \delta f \) as small variable and using Taylor series for \( \ln(1 + \delta f) \),

And preserving only the first term of expansion, we can approximately write down

\[ \ln(1 + \delta f) = \delta f \].

Inserting this expression in the equation (14), we can write down an equation:

\[ f = (1 + \delta f)(-A \ln f + A\delta f + f) \]

Considering \( \delta f \) as small variable, we can obtain an expression for the variable \( \delta \):

\[ \delta = \frac{A \ln f}{f(A - A \ln f + f)} \]

Conserving leading term in the denominator at \( f \to \infty \),

We obtain an expression

\[ \delta = \frac{A \ln f}{f^2} \].

Accordingly, we can obtain expression for the function \( g \):

\[ g = \frac{1}{f} + \frac{A \ln f}{f^2} \].

Omitting the evident transforms, we obtain two leading terms of asymptotic expansion of function \( \lambda^2 \) at \( \text{Re} \to \infty \).

\[ \lambda^2 = \frac{1}{A \ln \text{Re}} + \frac{\ln \ln \text{Re}}{A \ln^2 \text{Re}} \]

Leading term of asymptotic expansion is

\[ \lambda = \frac{1}{A^2 \ln^2 \text{Re}} \] (15)

Conserving two leading terms of asymptotic expansion, we obtain final expression for variable \( \lambda \):

\[ \lambda = \frac{1}{A^2 \ln^2 \text{Re}} + \frac{2 \ln \ln \text{Re}}{A^2 \ln^3 \text{Re}} \] (16)

Calculations using asymptotic expansions and numeric solutions of algebraic equations for resistance laws (1) and (2) by Newton’s method will be given in the next section.
4. Results and discussion
Calculation results of the resistance parameters for the turbulent Couette-Taylor flow under turbulent flow in the round pipe are given in table 1 and table 2 respectively.

Table 1. Variable $G$ dependence on Reynolds number $Re$. $G_{arym}$ - calculation the same variable using the expression (9), $G_{2aryn}$ - calculation of $G$, following expression (10), $G_{Pr-K}$ - numeric solution of algebraic equation (1) for variable $G$.

| $Re$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|------|------|------|------|------|
| $G_{arym}$ | $1.6437 \times 10^8$ | $1.1414 \times 10^{10}$ | $8.3860 \times 10^{11}$ | $6.4205 \times 10^{13}$ |
| $G_{2aryn}$ | $2.3417 \times 10^8$ | $1.5754 \times 10^{10}$ | $1.1279 \times 10^{12}$ | $8.4515 \times 10^{13}$ |
| $G_{Pr-K}$ | $4.175 \times 10^8$ | $2.5318 \times 10^{10}$ | $1.6838 \times 10^{12}$ | $1.1943 \times 10^{14}$ |

Table 2. Variable $\lambda$ dependence on Reynolds number $Re$. $\lambda_{arym}$ - calculation, using expression (15), $\lambda_{2aryn}$ - calculation, using expression (16), $\lambda_{Pr-K}$ - numeric solution of algebraic equation (2) for variable $G$.

| $Re$ | $3 \times 10^5$ | $1.36 \times 10^7$ |
|------|----------------|-------------------|
| $\lambda_{arym}$ | 0.0089 | 0.0053 |
| $\lambda_{2aryn}$ | 0.0125 | 0.0071 |
| $\lambda_{Pr-K}$ | 0.0146 | 0.0080 |

Calculations, given in table 1 show, that accuracy of using asymptotic expression (10) for dimensionless torque $G$ for turbulent Couette-Taylor flow is 62 percentages in comparison with expression (1) at Reynolds number $Re = 10^6$, 67 percentages at $Re = 10^7$, 71 percentages at $Re = 10^8$. We should note, that expression (9) was originally obtain in [7]. Resistance law, kind of (1), for independent rotations as inner and as well outer cylinders was obtain in [8]. Leading term of asymptotic expansion of dimensionless torque $G$ at tending Taylor number to infinity ($\overline{Ta} \rightarrow \infty$), similar to expression (9), was obtain in [8] also.

Calculations of the variable $\lambda$ for turbulent flow in round pipe, given in Table 2, show, that accuracy of asymptotic expression (16) is about 89 percentages at Reynolds number $Re = 1.36 \times 10^7$. This asymptotic analysis can be extended to similar resistance laws, obtained in [9].

We made conclusion, that these expressions may be used in various applications, for instance, in hydraulics in considering range of Reynolds numbers.

Prandtl-von-Karman resistance law is also used for the approximation of skin friction in turbulent pipe flow [10]. The expression (15) coincides with Filonenko’s formula [11]. Solution of equation (11) can be written down with help of Lambert W-function [12], [13].

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