SEMICROSSED PRODUCTS OF THE DISK ALGEBRA

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Abstract. If $\alpha$ is the endomorphism of the disk algebra, $A(D)$, induced by composition with a finite Blaschke product $b$, then the semicrossed product $A(D) \times_\alpha \mathbb{Z}^+$ imbeds canonically, completely isometrically into $C(T) \times_\alpha \mathbb{Z}^+$. Hence in the case of a non-constant Blaschke product $b$, the $C^*$-envelope has the form $C(S_b) \times_s \mathbb{Z}$, where $(S_b, s)$ is the solenoid system for $(T, b)$. In the case where $b$ is a constant, then the $C^*$-envelope of $A(D) \times_\alpha \mathbb{Z}^+$ is strongly Morita equivalent to a crossed product of the form $C(S_e) \times_s \mathbb{Z}$, where $e: T \times \mathbb{N} \rightarrow T \times \mathbb{N}$ is a suitable map and $(S_e, s)$ is the solenoid system for $(T \times \mathbb{N}, e)$.

1. Introduction

If $\mathcal{A}$ is a unital operator algebra and $\alpha$ is a completely contractive endomorphism, the semicrossed product is an operator algebra $\mathcal{A} \times_\alpha \mathbb{Z}^+$ which encodes the covariant representations of $(\mathcal{A}, \alpha)$: namely completely contractive unital representations $\rho: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and contractions $T$ satisfying

$$\rho(a)T = T\rho(\alpha(a)) \text{ for all } a \in \mathcal{A}.$$ 

Such algebras were defined by Peters [9] when $\mathcal{A}$ is a $C^*$-algebra.

One can readily extend Peter’s definition [9] of the semicrossed product of a $C^*$-algebra by a $*$-endomorphism to unital operator algebras and unital completely contractive endomorphisms. One forms the polynomial algebra $\mathcal{P}(\mathcal{A}, t)$ of formal polynomials of the form $p = \sum_{i=0}^{n} t^i a_i$, where $a_i \in \mathcal{A}$, with multiplication determined by the covariance relation $at = \alpha(a)t$ and the norm

$$\|p\| = \sup_{(\rho, T) \text{ covariant}} \left\| \sum_{i=0}^{n} T^i \rho(a_i) \right\|.$$

2000 Mathematics Subject Classification. 47L55.
Key words and phrases. semicrossed product, crossed product, disk algebra, $C^*$-envelope.

First author partially supported by an NSERC grant.
Second author was partially supported by a grant from ECU.
This supremum is clearly dominated by $\sum_{i=0}^n \|a_i\|$; so this norm is well defined. The completion is the semicrossed product $A \times_\alpha \mathbb{Z}_+$. Since this is the supremum of operator algebra norms, it is also an operator algebra norm. By construction, for each covariant representation $(\rho, T)$, there is a unique completely contractive representation $\rho \times T$ of $A \times_\alpha \mathbb{Z}_+$ into $\mathcal{B}(\mathcal{H})$ given by

$$\rho \times T(p) = \sum_{i=0}^n T^i \rho(a_i).$$

This is the defining property of the semicrossed product.

In this note, we examine semicrossed products of the disk algebra by an endomorphism which extends to a $*$-endomorphism of $C(T)$. In the case where the endomorphism is injective, these have the form $\alpha(f) = f \circ b$ where $b$ is a non-constant Blaschke product. We show that every covariant representation of $(A(D), \alpha)$ dilates to a covariant representation of $(C(T), \alpha)$. This is readily dilated to a covariant representation $(\sigma, V)$, where $\sigma$ is a $*$-representation of $C(T)$ (so $\sigma(z)$ is unitary) and $V$ is an isometry. To go further, we use the recent work of Kakariadis and Katsoulis [6] to show that $C(T) \times_\alpha \mathbb{Z}^+$ imbeds completely isometrically into a C*-crossed product $C(S_b) \times_\alpha \mathbb{Z}$. In fact, $C_e^*(C(T) \times_\alpha \mathbb{Z}^+) = C(S_b) \times_\alpha \mathbb{Z}$ and as a consequence, we obtain that $(\rho, T)$ dilates to a covariant representation $(\tau, W)$, where $\tau$ is a $*$-representation of $C(T)$ (so $\tau(z)$ is unitary) and $W$ is a unitary.

In contrast, if $\alpha$ is induced by a constant Blaschke product, we can no longer identify $C_e^*(C(T) \times_\alpha \mathbb{Z}^+)$ up to isomorphism. In that case, $\alpha$ is evaluation at a boundary point. Even though every covariant representation of $(A(D), \alpha)$ dilates to a covariant representation of $(C(T), \alpha)$, the theory of [6] is not directly applicable since $\alpha$ is not injective. Instead, we use the process of “adding tails to C*-correspondences” [8], as modified in [3, 7] and we identify $C_e^*(C(T) \times_\alpha \mathbb{Z}^+)$ up to strong Morita equivalence as a crossed product. In Theorem 2.6 we show that $C_e^*(C(T) \times_\alpha \mathbb{Z}^+)$ is strongly Morita equivalent to a C*-algebra of the form $C(S_e) \times_s \mathbb{Z}$, where $e: T \times \mathbb{N} \to T \times \mathbb{N}$ is a suitable map and $(S_e, s)$ is the solenoid system for $(T \times \mathbb{N}, e)$.

Semi-crossed products of the the disc algebra were introduced and first studied by Buske and Peters in [1], following relevant work of Hoover, Peters and Wogen [5]. The algebras $A(D) \times_\alpha \mathbb{Z}^+$, where $\alpha$ is an arbitrary endomorphism, where classified up to algebraic endomorphism in [2]. Results associated with their C*-envelope can be found in [1, Proposition III.13] and [10, Theorem 2]. The results of the present paper subsume and extend these earlier results.
2. The Disk Algebra

The C*-envelope of the disk algebra $A(D)$ is $C(T)$, the space of continuous functions on the unit circle. Suppose that $\alpha$ is an endomorphism of $C(T)$ which leaves $A(D)$ invariant. We refer to the restriction of $\alpha$ to $A(D)$ as $\alpha$ as well. Then $b = \alpha(z) \in A(D)$; and has spectrum $\sigma_{A(D)}(b) \subset \overline{D}$ and $\sigma_{C(T)}(b) \subset \sigma_{C(T)}(z) = T$.

Thus $\|b\| = 1$ and $b(T) \subset T$. It follows that $b$ is a finite Blaschke product. Therefore $\alpha(f) = f \circ b$ for all $f \in C(T)$. When $b$ is not constant, $\alpha$ is completely isometric.

A (completely) contractive representation $\rho$ of $A(D)$ is determined by $\rho(z) = A$, which must be a contraction. The converse follows from the matrix von Neumann inequality; and shows that $\rho(f) = f(A)$ is a complete contraction. A covariant representation of $(A(D), \alpha)$ is thus determined by a pair of contractions $(A, T)$ such that $AT = Tb(A)$. The representation of $A(D) \times_\alpha \mathbb{Z}^+$ is given by

$$\rho \times T \left( \sum_{i=0}^{n} t^i f_i \right) = \sum_{i=0}^{n} T^i f_i(A),$$

which extends to a completely contractive representation of the semicrossed product by the universal property.

A contractive representation $\sigma$ of $C(T)$ is a $*$-representation, and is likewise determined by $U = \sigma(z)$, which must be unitary; and all unitary operators yield such a representation by the functional calculus. A covariant representation of $(C(T), \alpha)$ is given by a pair $(U, T)$ where $U$ is unitary and $T$ is a contraction satisfying $UT = Tb(U)$. To see this, multiply on the left by $U^*$ and on the right by $b(U)^*$ to obtain the identity

$$U^*T = Tb(U)^* = T\overline{b}(U) = T\alpha(\bar{z})(U).$$

The set of functions $\{ f \in C(T) : f(U)T = T\alpha(f)(U) \}$ is easily seen to be a norm closed algebra. Since it contains $z$ and $\bar{z}$, it is all of $C(T)$. So the covariance relation holds.

**Theorem 2.1.** Let $b$ be a finite Blaschke product, and let $\alpha(f) = f \circ b$. Then $A(D) \times_\alpha \mathbb{Z}^+$ is (canonically completely isometrically isomorphic to) a subalgebra of $C(T) \times_\alpha \mathbb{Z}^+$.

**Proof.** To establish that $A(D) \times_\alpha \mathbb{Z}^+$ is completely isometric to a subalgebra of $C(T) \times_\alpha \mathbb{Z}^+$, it suffices to show that each $(A, T)$ with $AT = Tb(A)$ has a dilation to a pair $(U, S)$ with $U$ unitary and $S$ a contraction such that $US = Sb(U)$ and $P_H S^n U^m |_H = T^n A^m$ for all $n, m \in \mathbb{Z}$. 


$m, n \geq 0$. This latter condition is equivalent to $\mathcal{H}$ being semi-invariant for the algebra generated by $U$ and $S$.

The covariance relation can be restated as

$$\begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix} \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix}$$

Dilate $A$ to a unitary $U$ which leaves $\mathcal{H}$ semi-invariant. Then $\begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix}$ dilates to $\begin{bmatrix} U & 0 \\ 0 & b(U) \end{bmatrix}$. By the Sz.Nagy-Foiaş Commutant Lifting Theorem, we may dilate $\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}$ to a contraction of the form $\begin{bmatrix} \ast & S \\ \ast & \ast \end{bmatrix}$ which commutes with $\begin{bmatrix} U & 0 \\ 0 & \alpha(U) \end{bmatrix}$ and has $\mathcal{H} \oplus \mathcal{H}$ as a common semi-invariant subspace. Clearly, we may take the $\ast$ entries to all equal 0 without changing things. So $(U, S)$ satisfies the same covariance relations $US = Sb(U)$.

Therefore we have obtained a dilation to the covariance relations for $(C(\mathcal{T}), \alpha)$.

Once we have a covariance relation for $(C(\mathcal{T}), \alpha)$, we can try to dilate further. Extending $S$ to an isometry $V$ follows a well-known path. Observe that

$$b(U)S^*S = S^*US = S^*Sb(U).$$

Thus $D = (I - S^*S)^{1/2}$ commutes with $b(U)$. Write $b^{(n)}$ for the composition of $b$ with itself $n$ times, Hence we can now use the standard Schaeffer dilation of $S$ to an isometry $V$ and simultaneously dilate $U$ to $U_1$ as follows:

$$V = \begin{bmatrix} S & 0 & 0 & 0 & \ldots \\ D & 0 & 0 & 0 & \ldots \\ 0 & I & 0 & 0 & \ldots \\ 0 & 0 & I & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad U_1 = \begin{bmatrix} U & 0 & 0 & 0 & \ldots \\ 0 & b(U_1) & 0 & 0 & \ldots \\ 0 & 0 & b^{(2)}(U_1) & 0 & \ldots \\ 0 & 0 & 0 & b^{(3)}(U_1) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$ 

A simple calculation shows that $U_1V = Vb(U_1)$. So as above, $(U, V)$ satisfies the covariance relations for $(C(\mathcal{T}), \alpha)$.

We would like to make $V$ a unitary as well. This is possible in the case where $b$ is non-constant, but the explicit construction is not obvious. Instead, we use the theory of $\text{C}^*$-envelopes and maximal dilations. First we need the following.

**Lemma 2.2.** Let $b$ be a finite Blaschke product, and let $\alpha(f) = f \circ b$. Then

$$\text{C}^*_e(A(\mathcal{D}) \times_{\alpha} \mathbb{Z}^+) \simeq \text{C}^*_e(C(\mathcal{T}) \times_{\alpha} \mathbb{Z}^+).$$
Proof. The previous Theorem identifies $A(D) \times_\alpha \mathbb{Z}^+$ completely isometrically as a subalgebra of $C(T) \times_\alpha \mathbb{Z}^+$. The C*-envelope $C$ of $C(T) \times_\alpha \mathbb{Z}^+$ is a Cuntz-Pimsner algebra containing a copy of $C(T)$ which is invariant under gauge actions. Now $C$ is a C*-cover of $C(T) \times_\alpha \mathbb{Z}^+$, so it is easy to see that it is also a C*-cover of $A(D) \times_\alpha \mathbb{Z}^+$. Since $A(D) \times_\alpha \mathbb{Z}^+$ is invariant under the same gauge actions, its Shilov ideal $J \subseteq C$ will be invariant by these actions as well. If $J \neq 0$ then by gauge invariance $J \cap C(T) \neq 0$. Since the quotient map $A(D) \rightarrow C(T)/J \cap C(T)$ is completely isometric, we obtain a contradiction. Hence $J = 0$ and the conclusion follows.

We now recall some of the theory of semicrossed products of C*-algebras. When $A$ is a C*-algebra, the completely isometric endomorphisms are the faithful $\ast$-endomorphisms. In this case, Peters shows [9, Prop.I.8] that there is a unique C*-algebra $B$, a $\ast$-automorphism $\beta$ of $B$ and an injection $j$ of $A$ into $B$ so that $\beta \circ j = j \alpha$ and $B$ is the closure of $\bigcup_{n \geq 0} \beta^{-n}(j(A))$. It follows [9, Prop.II.4] that $A \times_\alpha \mathbb{Z}^+$ is completely isometrically isomorphic to the subalgebra of the crossed product algebra $B \times_\beta \mathbb{Z}$ generated as a non-self-adjoint algebra by an isomorphic copy $j(A)$ of $A$ and the unitary $u$ implementing $\beta$ in the crossed product. Actually, Kakariadis and the second author [6, Thm.2.5] show that $B \times_\beta \mathbb{Z}$ is the C*-envelope of $A \times_\alpha \mathbb{Z}^+$.

In the case where $A = C(X)$ is commutative and $\alpha$ is induced by an injective self-map of $X$, the pair $(B, \beta)$ has an alternative description.

Definition 2.3. Let $X$ be a Hausdorff space and $\varphi$ a surjective self-map of $X$. We define the solenoid system of $(X, \varphi)$ to be the pair $(S_\varphi, s)$, where

$$S_\varphi = \{(x_n)_{n \geq 1} : x_n = \varphi(x_{n+1}), x_n \in X, n \geq 1\}$$

equipped with the relative topology inherited from the product topology on $\prod_{i=1}^{\infty} X_i$, $X_i = X$, $i = 1, 2, \ldots$, and $s$ is the backward shift on $S_\varphi$.

It is easy to see that in the case where $A = C(X)$ and $\alpha$ is induced by an injective self-map $\varphi$ of $X$, the pair $(B, \beta)$ for $(A, \alpha)$ described above, is conjugate to the solenoid system $(S_\varphi, s)$. Therefore, we obtain

Corollary 2.4. Let $b$ be a non-constant finite Blaschke product, and let $\alpha(f) = f \circ b$ on $C(T)$. Then

$$C^*_e(A(D) \times_\alpha \mathbb{Z}^+) = C^*_e(C(S_b) \times_s \mathbb{Z})$$

where $(S_b, s)$ is the solenoid system of $(T, b)$. 
It is worth restating this theorem as a dilation result.

**Corollary 2.5.** Let \( \alpha \) be an endomorphism of \( A(D) \) induced by a non-constant finite Blaschke product and let \( A, T \in \mathcal{B}(H) \) be contractions satisfying \( AT = T\alpha(A) \). Then there exist unitary operators \( U \) and \( W \) on a Hilbert space \( K \supset H \) which simultaneously dilate \( A \) and \( T \), in the sense that \( P_H W^m U^n |_H = T^m A^n \) for all \( m, n \geq 0 \), so that \( UW = W\alpha(U) \).

**Proof.** Every covariant representation \( (A, T) \) of \( (A(D), \alpha) \) dilates to a covariant representation \( (U_1, V) \) of \( (C(T), \alpha) \). This in turn dilates to a maximal dilation \( \tau \) of \( C(T) \times \alpha \mathbb{Z}^+ \), in the sense of Dritschel and McCullough [4]. The maximal dilations extend to \( * \)-representations of the \( C^* \)-envelope. Then \( A \) is dilated to \( \tau(j(z)) = U \) is unitary and \( T \) dilates to the unitary \( W \) which implements the automorphism \( \beta \) on \( \mathfrak{B} \), and restricts to the action of \( \alpha \) on \( C(T) \).

The situation changes when we move to non-injective endomorphisms \( \alpha \) of \( A(D) \). Indeed, let \( \lambda \in T \) and consider the endomorphism \( \alpha_\lambda \) of \( A(D) \) induced by evaluation on \( \lambda \), i.e., \( \alpha_\lambda(f)(z) = f(\lambda) \), \( \forall z \in D \). (Thus \( \alpha_\lambda \) is the endomorphism of \( A(D) \) corresponding to a constant Blaschke product.) If two contractions \( A, T \) satisfy \( AT = T\alpha_\lambda(A) = \lambda T \), then the existence of unitary operators \( U, W \), dilating \( A \) and \( T \) respectively, implies that \( A = \lambda I \). It is easy to construct a pair \( A, T \) satisfying \( AT = \lambda T \) and yet \( A \neq \lambda I \). This shows that the analogue Corollary 2.5 fails for \( \alpha = \alpha_\lambda \) and therefore one does not expect \( C^*_e(A(D) \times \alpha \mathbb{Z}^+) \) to be isomorphic to the crossed product of a commutative \( C^* \)-algebra, at least under canonical identifications. However as we have seen, a weakening of Corollary 2.5 is valid for \( \alpha = \alpha_\lambda \) if one allows \( W \) to be an isometry instead of a unitary operator. In addition, we can identify \( C^*_e(A(D) \times \alpha \mathbb{Z}^+) \) as being strongly Morita equivalent to a crossed product \( C^* \)-algebra. Indeed, if

\[
e: T \times \mathbb{N} \to T \times \mathbb{N}
\]

is defined as

\[
e(z, n) = \begin{cases} (1, 1) & \text{if } n = 1 \\ (z, n - 1) & \text{otherwise,} \end{cases}
\]

then

**Theorem 2.6.** Let \( \alpha = \alpha_\lambda \) be an endomorphism of \( A(D) \) induced by evaluation at a point \( \lambda \in T \). Then \( C^*_e(A(D) \times \alpha \mathbb{Z}^+) \) is strongly Morita equivalent to \( C(S_e) \times_s \mathbb{Z} \), where \( e: T \times \mathbb{N} \to T \times \mathbb{N} \) is defined above and \( (S_e, s) \) is the solenoid system of \( (T \times \mathbb{N}, e) \).
Proof. In light of Lemma 2.2 it suffices to identify the C*-envelope of \( C(T) \times_\alpha Z^+ \). As \( \alpha \) is no longer an injective endomorphism of \( C(T) \), we invoke the process of adding tails to C*-correspondences [8], as modified in [3, 7].

Indeed, [7, Example 4.3] implies that the C*-envelope of the tensor algebra associated with the dynamical system \( (C(T), \alpha) \) is strongly Morita equivalent to the Cuntz-Pimsner algebra associated with the injective dynamical system \( (T \times \mathbb{N}, e) \) defined above. Therefore by invoking the solenoid system of \( (T \times \mathbb{N}, e) \), the conclusion follows from the discussion following Lemma 2.2.

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