Existence and Stability of Anti-Periodic Solutions for Fuzzy Cohen-Grossberg Neural Networks with Time-varying Delays on Time Scales

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\textbf{Abstract.} By applying the novel method, some sufficient conditions are established for the existence and global exponential stability of anti-periodic solutions for a kind of fuzzy Cohen-Grossberg neural networks on time scales. Moreover an example is given to illustrate our results.

1. Introduction

Over the past few years, Cohen and Grossberg neural networks (CGNNs) [1] have been extensively studied and applied in many different fields such as associative memory, signal processing and some optimization problems. In such applications, it is of prime importance to ensure that the designed neural networks are stable [2]. In practice, due to the finite speeds of the switching and transmission of signals, time delays do exist in a working network and thus should be incorporated into the model equation. The dynamical behaviors, such as, the existence and stability of equilibrium point, periodic and almost periodic solutions, for CGNNs have been investigated for the sake of theoretical interest as well as application considerations.(see for example Refs. [3-10] and the references therein).

In this paper, we would like to integrate fuzzy operations into Cohen-Grossberg neural networks. Speaking of fuzzy operations, Yang and Yang [13] first introduced fuzzy cellular neural networks (FCNNs) combining those operations with cellular neural networks. So far researchers have founded that FCNNs are useful in image processing. Some results have been reported on dynamical behaviors including the existence and stability of equilibrium, periodic solution for FCNNs [14-18].

In contrast, however, very few results are available on the existence and exponential stability of anti-periodic solutions for fuzzy Cohen-Grossberg neural networks (FCGNNs). Arising from problems in applied sciences, the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [19-24]). Moreover, both continuous and discrete systems are very important in implementing and applications. But it is troublesome to study the existence of anti-periodic solutions for discrete and continuous systems, respectively. Therefore it is meaningful to study FCGNNs on time scales, which was initiated by Hilger [25] in order to unify continuous and discrete systems.

Motivated by the above discussions, in this paper, we consider the following FCGNNs on time scales.

\begin{equation}
\dot{x}_j(t) = -a_j(x_j(t))\dot{g}_j(x_j(t)) - \sum_{j=1}^{n} c_{ij}(t)f_j(x_j(t)) - \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t - \tau_{ij}(t)))
\end{equation}

\begin{equation}
- \sum_{j=1}^{n} B_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) - I_j(t)
\end{equation}

(1)
where $t \in \mathbb{T}$, $\mathbb{T}$ is an $\omega$-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. $n$ corresponds to the number of units in the neural networks. For $i = 1, 2, \ldots, n$, $x_i(t)$ corresponds to the state of the $i$th neuron. $f_i(\cdot)$, $g_i(\cdot)$ are signal transmission functions. $\tau_j(t)$ corresponds to the transmission delay along the axon of the $j$th unit from the $i$th unit and satisfies $0 \leq \tau_j(t) \leq \tau$ ($\tau$ is a constant). $a_i(x_i(t)) > 0$ represents an amplification function at time $t$. $b_i(x_i(t))$ is an appropriately behaved function at time $t$; $c_i(t)$ represents the elements of the feedback template. $I_i(t)$ is external input to the $i$th unit. $\alpha_i(t), \beta_i(t)$ are elements of fuzzy feedback MIN template and fuzzy feedback MAX template, respectively; $\wedge$ and $\vee$ denote the fuzzy AND and fuzzy OR operation, respectively.

The main aim of this article is to establish some sufficient conditions for the existence and exponential stability of anti-periodic solutions of (1). The organization of this paper is as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, we establish sufficient conditions for the existence and exponential stability of the anti-periodic solutions of system (1). In Section 4, an example is given to demonstrate the effectiveness of our results. Conclusions are drawn in Section 5.

2. Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho: \mathbb{T} \rightarrow \mathbb{T}$ and the graininess $\mu: \mathbb{T} \rightarrow \mathbb{R}^+$ are defined, respectively, by $\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}: s < t\}$, $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense if $t \in \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t \in \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^k = \mathbb{T}\setminus\{m\}$, otherwise $\mathbb{T} = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_k = \mathbb{T}\setminus \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If $f$ is continuous at each right-dense point and each left-dense point, then $f$ is said to be a continuous function on $\mathbb{T}$.

For $y: \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}$, we define the delta derivative of $y(t)$, $y^\Delta(t)$ to be the number (if exists) with the property that for given $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that $\|y(\sigma(t)) - y(s) - y^\Delta(t)[\sigma(t) - y(s)]\| < \varepsilon$ for all $s \in U$. If $y$ is continuous, then $y$ is right-dense continuous, and $y$ is delta differentiable at $t$, then $y$ is continuous at $t$. Let $y$ be right-dense continuous. If $Y^\Delta(t) = y(t)$, then we define the delta integral by $\int_a^b y(s)ds = Y(t) - Y(a)$.

**Definition 2.1** [27] If $a \in \mathbb{T}$, $\sup \mathbb{T} = \mathbb{R}$ and $f$ is rd-continuous on $[0, \infty)$, then we define the improper integral by $\int_a^b f(t)\Delta t = \lim_{b \to \infty} \int_a^b f(t)\Delta t$.

Provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

**Definition 2.2** [28] For each $t \in \mathbb{T}$, let $N$ be a neighborhood of $t$, then, for $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$. Define $D^\Delta V(t, x(t))$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \subset N$ of $t$ such that

$$\frac{V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t, s) f(t, x(t))}{\mu(t, s)} < D^\Delta V(t, x(t)) + \varepsilon.$$
for each \( s \in \mathbb{N} \), \( s > t \), where \( \mu(t,s) = \sigma(t) - s \). If \( t \) is rd and \( V(t,x(t)) \) is continuous at \( t \), this reduces to 
\[
D^rV^\lambda(t,x(t)) = \frac{V(\sigma(t),x(\sigma(t))) - V(t,x(\sigma(t)))}{\sigma(t) - t}.
\]

**Definition 2.3** [11] We say that a time scale \( \mathbb{T} \) is periodic if there exists \( p > 0 \) such that if \( t \in \mathbb{T} \), then \( t \pm t \in \mathbb{T} \). For \( \mathbb{T} \neq \mathbb{R} \), the least positive \( p \) is called the period of the time scale.

Let \( \mathbb{T} \neq \mathbb{R} \) be a periodic time scale with period \( p \). We say that the function \( f : \mathbb{T} \rightarrow \mathbb{R} \) is \( \omega \) anti-periodic if there exists a natural number \( n \) such that \( \omega = np \), \( f(t + \omega) = f(t) \) for all \( t \in \mathbb{T} \) and \( \omega \) is the least number such that \( f(t + \omega) = f(t) \). If \( \mathbb{T} = \mathbb{R} \), we say that \( f \) is \( \omega \) anti-periodic if \( \omega \) is the least positive number such that \( f(t + \omega) = f(t) \) for all \( t \in \mathbb{T} \).

A function \( r : \mathbb{T} \rightarrow \mathbb{R} \) is called regressive if \( 1 + \mu(t)r(t) \neq 0 \), for all \( t \in \mathbb{T} \).

If \( r \) is regressive function, then the generalized exponential function \( e_r \) is defined by
\[
e_r(t,s) = \exp\{\int_s^t \xi_p(r(\tau))\Delta \tau\}, s,t \in \mathbb{T}
\]

with the cylinder transformation
\[
\xi_h(z) = \begin{cases} 
\frac{\log(1+hz)}{h}, & h \neq 0 \\
z, & h = 0
\end{cases}
\]

Let \( p,q : \mathbb{T} \rightarrow \mathbb{R} \) be two regressive functions, we define
\[
p \oplus q := p + q + \mu pq; \quad p\Theta q := p \oplus (\Theta q); \quad \Theta p := \frac{p}{1+\mu p}
\]

**Lemma 2.1.** [12] Let \( p,q \) be regressive functions on \( \mathbb{T} \). Then
(i) \( e_p(t,s) = 1 \) and \( e_p(t,t) = 1 \);  
(ii) \( e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s) \);
(iii) \( e_p(t,s)e_p(s,r) = e_p(t,r) \);  
(iv) \( e_p^\lambda(\cdot,s) = pe_p(\cdot,s) \).

**Lemma 2.2** [11] Assume that \( f, g : \mathbb{T} \rightarrow \mathbb{R} \) are delta differentiable at \( t \in \mathbb{T}^+ \), then
\[
(fg)^\lambda = f^\lambda(t)g(t) + f(\sigma(t))g^\lambda(t) = f(t)g^\lambda(t) + f^\lambda(t)g(\sigma(t)).
\]

**Lemma 2.3** [12] Assume that \( p(t) \geq 0 \), for \( t > s \), then \( e_p(t,s) \geq 1 \).

**Lemma 2.4** [11] Assume that \( p \in \mathbb{R} \) is \( \omega \) periodic, then \( e_p(t+n\omega,s) = (e_p(t+\omega,s))^n \) for \( n \in \mathbb{N} \).

**Lemma 2.5** [27] Let \( f \) be continuous on \( [a,b]_\mathbb{T} \) and delta differentiable on \( [a,b]_\mathbb{T} \), then there exist \( \xi, \zeta \in [a,b]_\mathbb{T} \) such that \( f^\lambda(\xi)(b-a) \leq f(b) - f(a) \leq f^\lambda(\zeta)(b-a) \).

**Lemma 2.6** [14] Suppose \( x \) and \( y \) are two states of system (1), then we have
\[
\left| \sum_{j=1}^{n} \alpha_{ij}(t)g_{ij}(x) - \sum_{j=1}^{n} \alpha_{ij}(t)g_{ij}(y) \right| \leq \sum_{j=1}^{n} \| \alpha_{ij}(t) \| \| g_{ij}(x) - g_{ij}(y) \|
\]

and
\[
\left| \sum_{j=1}^{n} \alpha_{ij}(t)g_{ij}(x) - \sum_{j=1}^{n} \alpha_{ij}(t)g_{ij}(y) \right| \leq \sum_{j=1}^{n} \| \alpha_{ij}(t) \| \| g_{ij}(x) - g_{ij}(y) \|
\]

**Definition 2.4.** The anti-periodic solution \( x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T \) of system (1) with initial value \( \varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \cdots, \varphi_n^*(t))^T \) is said to be globally exponentially stable if there exists a positive constant \( M = M(\eta) \geq 1 \) and \( \varepsilon > 0 \) such that, for every \( \eta \in \mathbb{T} \),
\[
\| x(t) - x^* \| \leq Me^{\varepsilon t}(t,\eta)\| \varphi - x^* \|.
\]
where \[ \| \phi - x^* \| = \sup_{s \in [\tau,0]} \max_{t \in [0,\alpha]} \left| \phi_i(s) - x_i^*(s) \right| . \]

For the sake of convenience, we introduce some notations
\[ \tau = \max_{t \in [\tau,0]} \left| \tau_g(t) \right|, \quad \bar{\tau} = \max_{t \in [0,\alpha]} \left| \tau_i(t) \right|, \quad \bar{c}_{ij} = \max_{t \in [\tau,0]} \left| c_{ij}(t) \right|, \quad \bar{\sigma}_g = \max_{t \in [0,\alpha]} \left| \sigma_g(t) \right|, \quad \bar{\beta}_j = \max_{t \in [\tau,0]} \left| \beta_j(t) \right| . \]

Let \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in C(\mathbb{T}, \mathbb{R}^n) , \| x \| = \sum_{i=1}^n \max_{t \in [\tau,0]} \left| x_i(t) \right| \). The initial conditions associated with system (1) are of the form
\[ x_i(t) = \phi_i(t), t \in [-\tau,0]_\mathbb{T} , \]
where \( \phi_i(t) , i = 1,2, \ldots, n \) are continuous functions on \([ -\tau,0]_\mathbb{T} \).

Let \( x_i(t) \in C(\mathbb{T}, \mathbb{R}) , x_i(t) \) is said to be \( \theta \) periodic, if \( x_i(t+\omega) = -x_i(t) \) for all \( t \in \mathbb{T} , \omega > 0 \) is a constant. Denote \( \mathbb{R}^+ = [0, +\infty). \) Throughout this paper, we make the following assumptions

(A1) \( c_{ij}, \alpha_j, \beta_j, I_j \in C(\mathbb{T}, \mathbb{R}) , \quad c_{ij}(t+\omega) = c_{ij}(t) , \alpha_j(t+\omega) = \alpha_j(t) , \beta_j(t+\omega) = \beta_j(t) , \)

(A2) \( a_i \in C(\mathbb{R}, \mathbb{R}^+) , a_i(-u) = a(u) , \) and there exist positive constants \( \bar{a}_i, a_i \) such that
\[ 0 < a_i \leq a_i(u) \leq \bar{a}_i , \quad i = 1,2, \ldots, n \]

(A3) \( b_j \in C(\mathbb{R}, \mathbb{R}) , \quad b_j(-u) = -b_j(u) \) and there exist positive constants \( \delta_j \) such that
\[ \delta_j |u| \leq \text{sign}(u)b_j(u) \quad \text{for all} \quad u \in \mathbb{R} , \quad i = 1,2, \ldots, n \]

(A4) \( f_j, g_j \in C(\mathbb{R}) , \quad f_j(-u) = -f_j(u) , \quad g_j(-u) = -g_j(u) , \quad f_j(0) = g_j(0) = 0 , \) and there exist \( \mu_j, \nu_j (j = 1,2, \ldots, n) \) such that
\[ |f_j(u) - f_j(v)| \leq \mu_j |u-v| , \quad |g_j(u) - g_j(v)| \leq \nu_j |u-v| . \]
for all \( u,v \in \mathbb{R} , \quad j = 1,2, \ldots, n \).

3. Main results

In this section, we will prove our main results of this paper.

Lemma 3.1. Under condition (A1)-(A4), and suppose further the following condition hold
(A5) there exists a constant \( \gamma > 0 \) such that
\[ -a_i \delta_i + \bar{a}_i \sum_{j=1}^n \left( \bar{c}_{ij} \mu_j + (\bar{\alpha}_{ij} + \bar{\beta}_{ij}) \nu_j \right) < -\gamma < 0 . \]

Suppose that \( \bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))^T \) is a solution of (1) with initial condition
\[ \bar{x}_i(s) = \bar{\phi}_i(s) , \quad | \bar{x}_i(s) | < \frac{\Gamma}{\gamma} , \quad s \in [-\tau,0], i = 1,2, \ldots, n , \]
where \( \Gamma = \max_{i \in [1,2, \ldots, n]} \{ \bar{a}_i \bar{I}_j \} . \) Then
\[ | \bar{x}_i(t) | < \frac{\Gamma}{\gamma}, t \in \mathbb{T}^+, \quad i = 1,2, \ldots, n . \]

Proof. For any given initial condition, assumption (A4) guarantees the existence and unique of \( x(t) \) , the solution to (1) in \([ -\tau, +\infty) \). By way of contradiction, assume that (4) does not hold. Then, there exist \( i \in \{1,2, \ldots, n\} , \) and \( t_0 \in \mathbb{T}^+ \) such that
\[ | \bar{x}_i(t_0)| = \frac{\Gamma}{\gamma} , \quad | \bar{x}_i(t)| < \frac{\Gamma}{\gamma}, \quad t \in [-\tau,t_0)_\mathbb{T} , \quad and \quad | \bar{x}_j(t)| < \frac{\Gamma}{\gamma}, \quad t \in [-\tau,t_0)_\mathbb{T} , \quad j \neq i , \quad j = 1,2, \ldots, n . \]

By directly computing the upper left derivative of \( x_i(t) \), Combining with (A1)-(A5), we can obtain that
0 ≤ D^T |\ddot{x}(t_0)|^2 ≤ \text{sign}(\ddot{x}(t_0))\{-a_i(\dot{x}(t_0))[b_j(\ddot{x}(t_0)) - \sum_{j=1}^n c_{ij}(t)f_j(\ddot{x}(t_0))]

- \wedge_{j=1}^n \alpha_g(t)g_j(\ddot{x}(t_0) - \tau_g(t_0)) - \vee_{j=1}^n \beta_g(t)g_j(\ddot{x}(t_0) - \tau_g(t_0)) - I_j(t_0)]

≤-a_i(\ddot{x}(t_0))\text{sign}(\ddot{x}(t_0))b_j(\ddot{x}(t_0)) + |a_i(\ddot{x}(t_0))|\sum_{j=1}^n c_{ij}(t)f_j(\ddot{x}(t_0))

+|a_i(\ddot{x}(t_0))||\sum_{j=1}^n c_{ij}(t)f_j(\ddot{x}(t_0)) - \sum_{j=1}^n c_{ij}(t)f_j(0)|

≤-a_i(\ddot{x}(t_0))\text{sign}(\ddot{x}(t_0))b_j(\ddot{x}(t_0)) + |a_i(\ddot{x}(t_0))|\sum_{j=1}^n c_{ij}(t)f_j(0)\Gamma

≤-\gamma + a_iI_i ≤ 0.

(A6) There exists a positive constant \( \rho_i \) such that for all \( u,v \in R \)

\[ |a_i(u) - a_i(v)| ≤ \rho_i |u - v|, i = 1,2,\ldots,n. \]

(A7) There exists a positive constant \( \xi \) such that for all \( u,v \in R \),

\[ |a_i(u)b_i(u) - a_i(v)b_i(v)| |u - v| ≥ 0, \quad |a_i(u)b_i(u) - a_i(v)b_i(v)| ≥ \xi, \quad |u - v|, i = 1,2,\ldots,n. \]

Lemma 3.2 Let (A1)-(A7) hold, and let \( x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T \) be the solution of (1) with initial conditions (3), suppose that

\[ (A8) \quad -\xi_i + \rho_i \sum_{j=1}^n (c_{ij} + \alpha_{ij} + \beta_{ij})|\ddot{x}| + \rho_i I_i + a_i \sum_{j=1}^n (c_{ij} + \alpha_{ij} + \beta_{ij})|\dot{x}| < 0, \quad i = 1,2,\ldots,n. \]

Then \( x^*(t) \) is exponentially stable.

Proof. The proof is similar to Lemma 3.2 of [22]. So we omit it.

Theorem 3.1. Suppose (A1)-(A8) hold, then (1) has an anti periodic solution \( x^*(t) \) which is globally exponentially stable.

Proof. Let \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \) be a solution of (1) with initial condition

\[ x_i(s) = \varphi_i(s), \quad \rho_i(s) < \frac{\Gamma}{\gamma}, s \in [\tau,0], i = 1,2,\ldots,n. \]

By Lemma 3.1, the solution \( x(t) \) is bounded and

\[ |x_i(t)| < \frac{\Gamma}{\gamma}, t \in [\tau,\infty], i = 1,2,\ldots,n. \]

From (1), (A1)-(A4) and Lemma 2.1, we have

\[ |(-1)^{k+1}x_i(t+(k+1)\omega)|^\frac{1}{k+1} = (-1)^{k+1}x_i(t+(k+1)\omega) \]

\[ = (-1)^{k+1}[a_i(x_i(t+(k+1)\omega))|b_j(x_i(t+(k+1)\omega))| - \sum_{j=1}^n c_{ij}(t+(k+1)\omega)f_j(x_i(t+(k+1)\omega))] \]
\[
\begin{align*}
- \sum_{j=1}^{n} \alpha_j(t+(k+1)) g_j(x_j(t+(k+1)) \omega - \tau_j(t+(k+1))) \\
- \sum_{j=1}^{n} \beta_j(t+(k+1)) g_j(x_j(t+(k+1)) \omega - \tau_j(t+(k+1))) - I_j(t+(k+1)) \\
= -a_i((-1)^k x_i(t+(k+1)) \omega - \tau_i(t+(k+1))) \\
- \sum_{j=1}^{n} c_{ij}(t) f_j((-1)^k x_i(t+(k+1)) \omega - \tau_i(t+(k+1))) \\
- \sum_{j=1}^{n} \beta_j(t) g_j(x_j(t+(k+1)) \omega - \tau_j(t+(k+1))) - I_j(t+(k+1)) \\
\end{align*}
\]

Therefore, for any natural number \(k, (-1)^k x_i(t+(k+1)) \omega - \tau_i(t+(k+1))\) is the solution of (1). Then, from Lemma 3.2, there exists a constant \(M > 1\), such that

\[
|(-1)^k x_i(t+(k+1)) \omega - \tau_i(t+(k+1))| \leq M e_{\infty}(t+k \omega, \eta) \sup_{s \in [-\tau_i(t), t]} \max \{x_i(s+\omega) \omega - x_i(s)\} 
\]

\[
\leq 2M e_{\infty}(t+k \omega, \eta) \frac{\gamma}{\zeta} 
\]

where \(t+k \omega \in T^+, \ i=1,2, \ldots, n\). Thus, for any natural \(p\), we have

\[
(-1)^p x_i(t+(p+1) \omega) = x_i(t) + \frac{p}{\gamma} \sup_{s \in [-\tau_i(t), t]} \max \{x_i(s+\omega) \omega - x_i(s)\} 
\]

Then

\[
|(-1)^p x_i(t+(p+1) \omega)| \leq x_i(t) + \frac{p}{\gamma} \sup_{s \in [-\tau_i(t), t]} \max \{x_i(s+\omega) \omega - x_i(s)\} 
\]

In view of (6), we can choose a sufficiently large constant \(N > 0\) and a positive constant \(\zeta\) such that

\[
|(-1)^k x_i(t+(k+1) \omega) \omega - \tau_i(t+(k+1) \omega)| \leq \zeta e_{\infty}(t+k \omega, \eta) \frac{\gamma}{\zeta}, k > N. 
\]

On any compact set of \(R\). Together with (7) and (8), it follows that \((-1)^p x_i(t+p \omega)\) uniformly converges to a continuous function \(x^*(t)\) on any compact set of \(R\).

Now, we need to prove that \(x^*(t)\) is an anti periodic solution of (1). First \(x^*(t)\) is \(\omega\) anti-periodic, since

\[
x^*_i(t+\omega) = \lim_{p \to \infty} (-1)^p x_i(t+p \omega) = -\lim_{p \to \infty} (-1)^p x_i(t+p \omega) = -x^*_i(t) 
\]

Next, we show that \(x^*(t)\) is a solution of (1). Noting (1) and (5), we obtain that \((-1)^p x_i(t+(p+1) \omega)\) uniformly converges to continuous function on any compact set of \(R\). letting \(p \to \infty\), we have

\[
(x^*_i(t)) = -a_i(x^*_i(t))[b_i(x^*_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(x^*_j(t)) - \sum_{j=1}^{n} \alpha_j(t) g_j(x^*_j(t+\tau_j(t))) \\
- \sum_{j=1}^{n} \beta_j(t) g_j(x_j(t+\tau_j(t)) - I_j(t)) 
\]

That is \(x^*(t)\) is a solution of (1). It follows from Lemma 3.2 that \(x^*(t)\) is exponentially stable.

This completes the proof of Theorem 3.1.

4. An illustrative example

In this section, we will give an example to illustrate the feasibility and effectiveness of our results obtained in Section 3.

Example 4.1 Consider the following fuzzy Cohen-Grossberg neural networks with time-varying delays
\[ x_i^2(t) = -a_i(x_i(t))[b_i(x_i(t)) - \sum_{j=1}^{2} c_{ij}(t)f_j(x_j(t)) - \alpha_{ij}(t)g_j(x_j(t) - \tau_j(t))] \]
\[ -\sqrt{\sum_{j=1}^{2} \beta_{ij}(t)g_j(x_j(t) - \tau_j(t))} - I_i(t), \quad t \in T^+, \]  
\[ \text{where} \]
\[ a_i(u) = 5 + \cos u, b_i(u) = 5 - \cos u, f_j(u) = g_j(u) = \sin u, \quad \tau_j(t) = 0.5 \sin^2(4\pi t) \]
\[ (i, j = 1, 2). \]

\[ (c_{ij})_{2 \times 2} = \begin{bmatrix} 0.09 \sin(4\pi t) & 0.07 \cos(4\pi t) \\ 0.06 \cos(4\pi t) & 0.08 \sin(4\pi t) \end{bmatrix}, \quad (\alpha_{ij})_{2 \times 2} = \begin{bmatrix} 0.16 \cos(4\pi t) & 0.14 \sin(4\pi t) \\ 0.12 \sin(4\pi t) & 0.08 \cos(4\pi t) \end{bmatrix}, \]
\[ (\beta_{ij})_{2 \times 2} = \begin{bmatrix} 0.05 \sin(4\pi t) & 0.04 \cos(4\pi t) \\ 0.06 \cos(4\pi t) & 0.06 \sin(4\pi t) \end{bmatrix}, \quad (I_i)_{2 \times 1} = \begin{bmatrix} 0.2 \sin(4\pi t) \\ 0.1 \cos(4\pi t) \end{bmatrix}. \]

then, we have
\[ \alpha_1 = \alpha_2 = 6, c_{11} = c_{22} = 0.09, c_{21} = 0.08, c_{12} = 0.07, \quad \alpha_1 = 0.16, \alpha_2 = 0.08, \]
\[ \alpha_{11} = 0.14, \alpha_{21} = 0.12, \beta_1 = 0.05, \beta_2 = 0.06, \beta_{11} = 0.04, \beta_{21} = 0.06, \quad I_1 = 0.2, I_2 = 0.1. \]
\[ \rho_1 = \rho_2 = 1, \delta_1 = 1, \xi_1 = \xi_2 = 4, \Gamma = 2, \mu_j = \nu_j = 1, (j = 1, 2) \]

Take \( \gamma = 0.6 \), By simple computation, we have
\[ -a_1\delta_1 + a_1\sum_{j=1}^{2} (c_{1j} + \alpha_{1j} + \beta_{1j}) \nu_j = -0.7 < 0, \]  
\[ -a_2\delta_2 + a_2\sum_{j=1}^{2} (c_{2j} + \alpha_{2j} + \beta_{2j}) \nu_j = -1.24 < 0, \]
\[ -\xi_1 + \rho_1\sum_{j=1}^{2} (c_{1j} + \alpha_{1j} + \beta_{1j}) \Gamma + \rho_1 I_1 + a_1\sum_{j=1}^{2} (c_{2j} + \alpha_{1j} + \beta_{1j}) \nu_j = -1.2 < 0, \]
\[ -\xi_2 + \rho_2\sum_{j=1}^{2} (c_{2j} + \alpha_{2j} + \beta_{2j}) \Gamma + \rho_2 I_2 + a_2\sum_{j=1}^{2} (c_{2j} + \alpha_{2j} + \beta_{2j}) \nu_j = -2.06 < 0. \]

Now, we can see that conditions (A1)-(A8) hold. By Theorem 3.1, system (9) has a
\[ \frac{1}{4} - \text{anti-periodic solution which is exponentially stable}. \]

5. Conclusion

In this paper, we have studied the existence, exponential stability of the anti-periodic solution for fuzzy Cohen-Grossberg neural networks with time delays. Some sufficient conditions set up here are easily verified and these conditions are correlated with parameters of the system (1). The obtained criteria can be applied to design globally exponential stable of anti-periodic fuzzy Cohen-Grossberg neural networks.

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