5-dimensional braneworld with gravitating Nambu-Goto matching conditions

Georgios Kofinas\textsuperscript{1,*} and Vasilios Zarikas\textsuperscript{2,†}

\textsuperscript{1}Research Group of Geometry, Dynamical Systems and Cosmology
Department of Information and Communication Systems Engineering
University of the Aegean, Karlovassi 83200, Samos, Greece
\textsuperscript{2}Department of Electrical Engineering, ATEI Lamias, 35100 Lamia, Greece

\textsuperscript{*}Electronic address: gkofinas@aegean.gr
\textsuperscript{†}Electronic address: vzarikas@teilam.gr

We continue the investigation of a recent proposal on alternative matching conditions for self-gravitating defects which generalize the standard matching conditions. The reasoning for this study is the need for consistency of the various codimension defects and the existence of a meaningful equation of motion at the probe limit, things that seem to lack from the standard approach. These matching conditions arise by varying the brane-bulk action with respect to the brane embedding fields (and not with respect to the bulk metric at the brane position) in a way that takes into account the gravitational back-reaction of the brane to the bulk. They always possess a Nambu-Goto probe limit and any codimension defect is seemingly consistent for any second order bulk gravity theory. Here, we consider in detail the case of a codimension-1 brane in five-dimensional Einstein gravity, derive the generic alternative junction conditions and find the $\mathbb{Z}_2$-symmetric braneworld cosmology, as well as its bulk extension. Compared to the standard brane world cosmology, the new one has an extra integration constant which accounts for the today matter and dark energy contents, therefore, there is more freedom for accommodating the observed cosmic features. One branch of the solution possesses the asymptotic linearized LFRW regime. We have constrained the parameters so that to have a recent passage from a long deceleration era to a small today acceleration epoch and we have computed the age of the universe, consistent with current data, and the time-varying dark energy equation of state. For a range of the parameters it is possible for the presented cosmology to provide a large acceleration in the high energy regime.

I. INTRODUCTION

Distributional (thin) branes in an appropriate dimensional spacetime model the dynamics of various physical systems, as it is the universe itself. A classical infinitely thin test brane with tension (probe) moving in a given background spacetime is governed at lowest order by the Nambu-Goto action \cite{1}. Variation of this action with respect to the brane embedding fields gives the Nambu-Goto equations of motion which are geometrically described by the vanishing of the trace of the extrinsic curvature, and therefore, the worldsheet swept by the brane is extremal (minimal). When the gravitational field of the defect is taken into account both the bulk metric and the brane position become dynamical. Here, we consider throughout that the bulk metric is regular (finite and continuous) at the brane position. The standard method for obtaining the equations of motion of a back-reacting brane is to consider the bulk field equations with all the localized energy-momentum tensor included and to isolate and integrate out the distributional terms. A discontinuous extrinsic curvature or a conical singularity can source such delta functions of suitable codimension. While for thin shells Israel matching conditions are well-established \cite{2}, when the support of a generic distributional stress-energy tensor is higher-codimensional, it does not make sense to consider solutions of Einstein’s equations \cite{3}, \cite{4}, \cite{5}, \cite{6} (a pure brane tension is a special situation which is consistent \cite{7}, \cite{8}).

In \cite{9}, geometric junction conditions for a codimension-2 conical defect in six-dimensional Einstein-Gauss-Bonnet theory were derived with the hope that the above inconsistency is not due to the defect construction, but due to the inability of Einstein gravity to describe complicated distributional solutions. In \cite{10} the consistency of the whole set of junction plus bulk field equations was explicitly shown for an axially symmetric codimension-2 cosmological brane in six-dimensional EGB gravity, and it is likely that the consistency will remain for non-axial symmetry. Analogously, e.g. a 5-brane in eight dimensions is again of codimension-2 and EGB theory would suffice, but for a 4-brane in eight dimensions (codimension-3) the third Lovelock density \cite{11} would need for consistency. However, e.g. a 2-brane in six dimensions is of codimension-3 and it is probably inconsistent since the spirit of the proposal is to include higher Lovelock densities to accommodate higher codimension defects and there is no higher than the second Lovelock density in six dimensions. In brief, the generalization of the proposal is that in a $D$-dimensional spacetime the
maximal \([(D-1)/2]\) Lovelock density should be included (possibly along with lower Lovelock densities) and the branes with codimensions \(\delta = 1, 2, \ldots, [(D-1)/2]\) should be consistent according to the standard treatment; for yet higher codimensions the situation is not clear and probably inconsistent. In four dimensions the absence of higher Lovelock densities does not allow the existence of generic codimension-2 or 3 defects, but even if four dimensions are not the actual spacetime dimensionality, at certain length and energy scales it has been tested that four-dimensional Einstein gravity represents effectively the spacetime to high accuracy, so a consistent four-dimensional framework would at least be desirable. Beyond the above (possible) shortcomings, there are extra difficulties with handling distributional sources inside an equation. For example, for a codimension-2 brane there are two kinds of distributions involved, \(\delta(r)/r\) and \(\delta(r)\), where \(r\) is the radial coordinate from the brane. If both distributions are used to derive two matching conditions \([12]\), then an unnatural and undesirable inconsistency for certain boundary conditions arises \([10]\). Moreover, there is the problem of the regularization of the distributional equation, since multiplying by \(r\) only one distributional term remains, while multiplying by \(r^2\) all distributions vanish. If however one considers the corresponding variational problem of brane-bulk action (variation with respect to the bulk metric at the brane position), the volume element of integration \(rdrd\theta\) vanishes the \(\delta(r)\) distribution and only one matching condition arises, consistent with bulk dynamics.

Having stated the question of consistency of “high” codimension defects in either \(D\) or 4-dimensional spacetime, we now pass to the question of the probe limit. We note that the standard equations of motion of a self-gravitating defect do not obey the natural condition of continuous deformation from the probe limit equation of motion (which is the Nambu-Goto equation). Indeed, the Israel matching conditions under vanishing of the brane energy-momentum tensor give vanishing extrinsic curvature (geodesic motion), and similarly the probe limit of a codimension-1 brane in EGB gravity \([13]\) is another equation of motion, while the codimension-2 matching condition in EGB theory \([9], [10], [14]\) is a third equation of motion. To set an analogy, the linearized equation of motion of a point particle in four dimensions \([15], [16]\) (which is not involved in our discussion since in this case the bulk metric diverges on the brane) is a correction of the geodesic equation of motion on a given background (of course, for a 0-brane the geodesic equation coincides with the Nambu-Goto) and for a two-body system the probe limit is realized when one mass is much smaller than the other. However, in an analogous case it has been shown \([17], [18]\) that a probe point mass moves on the geodesic of a background-solution of any gravitational theory, so the above variety of probe equations of motion for different gravitational theories (or also the dependence of the equation of motion on the codimension of the defect) maybe is not acceptable. Additionally, the matching conditions in EGB or Lovelock gravity \([13], [9], [14]\) are cubic or quadratic algebraic equations in the extrinsic curvature with the total brane energy-momentum tensor on their right-hand side. Switching off this brane content the probe limit arises which is a cubic or quadratic equation, therefore, in general, it possesses a multiplicity of probe solutions. This means that these theories do not predict according to the standard approach a unique equation of motion at the probe level \(^1\). In this spirit, a correct probe limit equation of motion should be linear in the extrinsic curvature and such are the geodesic or the Nambu-Goto equations.

We would like to finish the discussion on the standard approach mentioning another possible deficiency. The variation of a bulk action with respect to the bulk metric, beyond the main bulk terms gives as usual additional “garbage” \(D\)-dimensional terms. With the exception of codimension-1 case where the inclusion of the Gibbons-Hawking term on the hypersurface cancels these terms, for all higher codimension defects such terms cannot cancel whatever terms are added on the defect. The only possible thing one could imagine is to consider a “tube” around the defect (two planes for codimension-1, a tube for codimension-2, a sphere for codimension-3, etc.), convert the unpleasant terms into “tube” terms, make the cancelation on the “tube”, and take the shrink limit. The variation of the brane-bulk action in the interior of the “tube”, considering also the relevant distributional terms and integrating out around the defect, will give the brane equation of motion. Therefore, the metric variation outside the “tube” has to be independent in order to get the bulk equations of motion, and also it has to be independent on the defect to get the brane equation of motion. For the “tube” terms to cancel, some condition on the metric variation has to be assumed on the “tube” (either Dirichlet-like if generalized Gibbons-Hawking terms \([13], [19]\) are included on the “tube”, or Neumann-like). It seems that in the shrink limit these “tube” conditions will be inconsistent with the independence of the brane metric variation.

A criticism against the standard approach in the lines of the above discussion was performed in \([20]\), together with a proposal for obtaining alternative matching conditions called “gravitating Nambu-Goto matching conditions”. These arise by varying the brane-bulk action with respect to the brane position variables (embedding fields). Although the brane energy-momentum tensor is still defined by the variation of the brane action with respect to the induced metric, however, this tensor enters the new matching conditions in a different way than before. Here, the distributional terms are still present, not inside a distributional differential equation leading directly to inconsistencies at certain

---

\(^1\) this argument was mentioned to us by J. Zanelli
cases, but rather smoothed out inside an integration. In [20], it was shown in particular the consistency of the codimension-2 defect in EGB gravity according to these alternative junction conditions, while the consistency of the codimension-2 limit of Einstein gravity [21] was also discussed. Gravitating matching conditions aim to satisfy all the previous shortcomings of the standard conditions. Four-dimensional Einstein gravity seems to be consistent for any codimension brane and the same seems also true either for Einstein or any Lovelock extension for all higher spacetime dimensions $D$ (since the inclusion of the maximal Lovelock density now is not crucial). These alternative matching conditions always have the Nambu-Goto probe limit, independently of the gravitational theory considered, the dimensionality of spacetime or the codimensionality of the defect. Finally, since the proposed equation of motion for the defect is decoupled from the bulk metric variation, the “outside” problem (outside the “tube”) is well-posed with a boundary Dirichlet or Newmann type of variation on the “tube” (the central line is defined by an independent variation with respect to the embedding fields). The proposed matching conditions generalize the standard matching conditions, and so, all the solutions of the bulk equations of motion plus the conventional matching conditions are still solutions of the current system of equations.

In the present work we study the codimension-1 case in Einstein gravity. Our approach is reminiscent of the “Dirac style” variation performed in [22], however our resulting matching conditions in the general case do not coincide with the matching conditions derived in [22]. We have applied various methods in order to confirm the derived result. Our main conceptual point, in view of the above arguments of consistency of the various codimension defects and their probe limit, is that the gravitating Nambu-Goto matching conditions may be close to the correct direction for deriving realistic matching conditions. We have applied these matching conditions for a $Z_2$ cosmological brane without imposing any restriction about the bulk. The cosmology derived is different than the cosmology derived in [22] (where the bulk was assumed to be AdS$_D$) and the bulk space found here is not AdS$_D$. The set-up of the paper is as follows: In section II the method is introduced as an extension of the Nambu-Goto variation for any codimension, so that the contribution from the gravitational back-reaction is included. In section III the generic alternative junction conditions of a codimension-1 brane in five-dimensional Einstein gravity are derived and manipulated together with the remaining effective equations on the brane. Analogous equations hold for other codimension-1 branes in other spacetime dimensions, but we choose the 3-brane as it can represent our world in the braneworld scenario. In section IV we specialize to the cosmological configuration, integrate the brane system of equations and find the brane cosmology. This cosmology has richer structure compared to the cosmology derived according to the standard conditions. In section V we find the bulk extension of the brane cosmology. In section VI we investigate the cosmological equations which provide interesting and realistic cosmological evolutions and study their phenomenological implications. Finally, in section VII we conclude.

II. A BRIEF INTRODUCTION OF THE METHOD

In order to get an idea how the proposed variation with respect to the embedding fields of the brane position is performed we give in this section a brief account of the method for any codimension (for more details see [20]). However, the exact derivation of section III in 5-dimensional spacetime is independent of this section. We start with a general four-dimensional action of the form

$$s_4 = \int_{\Sigma} d^4 \chi \sqrt{|h|} L(h_{ij})$$

(2.1)

in a $D$-dimensional spacetime, where $L$ is any scalar on $\Sigma$ built up from the induced metric $h_{ij}$. The brane coordinates are $\chi^i$ ($i,j,...$ are coordinate indices on the brane) and the bulk coordinates are $x^\mu$ ($\mu, \nu, ...$ are $D$-dimensional indices). In the present paragraph the bulk metric $g_{\mu \nu}$ is fixed and non-dynamical, while the treatment of a back-reacted metric will be given in the next paragraph of this section. The embedding fields are the external (bulk) coordinates of the brane, so they are some functions $x^\mu(\chi^i)$. Let the brane be deformed to another position described by the displacement vector $\delta x^\mu(x^\nu)$ and the corresponding variation of the various quantities is denoted by $\delta_z$. The variation of the tangent vectors on the brane $x^\mu_{;i}$ is $\delta_z(x^\mu_{;i}) = (\delta x^\mu)_{;i} = \delta x^\mu_{;i}$. Since the bulk coordinates do not change, the variation of $g_{\mu \nu}$ is

$$\delta_z g_{\mu \nu} = g_{\mu \nu ; \lambda} \delta x^\lambda$$

(2.2)

and the variation of $h_{ij} = g_{\mu \nu} x^\mu_{;i} x^\nu_{;j}$ is

$$\delta_z h_{ij} = g_{\mu \nu ; \lambda} x^\mu_{;i} x^\nu_{;j} \delta x^\lambda + g_{\mu \nu} x^\mu_{;i} \delta x^\nu_{;j} + g_{\mu \nu} x^\nu_{;j} \delta x^\mu_{;i} = x^\mu_{;i} x^\nu_{;j} (g_{\mu \nu ; \lambda} \delta x^\lambda + g_{\mu \lambda} \delta x^\nu_{; \lambda} + g_{\nu \lambda} \delta x^\mu_{; \lambda}).$$

(2.3)

The variation of $s_4$ becomes

$$\delta_z s_4 = \int_{\Sigma} d^4 \chi \sqrt{|h|} \tau^{ij} \delta_z h_{ij},$$

(2.4)
where \( \tau^{ij} = \frac{\delta}{\delta x^i} + \frac{i}{2} h^{ij} \). Substituting \( \delta x h_{ij} \), integrating by parts and imposing \( \delta x^\mu \mid_{\Sigma} = 0 \), we get

\[
\delta_x s_4 = -2 \int_\Sigma d^4 x \sqrt{|h|} g_{\mu\nu} \left[ (\tau^{ij} x^\mu)_{ij} + \tau^{ij} \Gamma_{\nu\lambda}^\mu x^\nu x^\lambda \right] \delta x^\sigma
\]

\[
= -2 \int_\Sigma d^4 x \sqrt{|h|} g_{\mu\nu} \left( \tau^{ij}_{\ |j} x^\mu_{\ i} - \tau^{ij} \kappa^{\alpha}_{\ ij} n^\mu_{\ \alpha} \right) \delta x^\sigma, 
\]

(2.5)

since \( x^\mu_{\ ij} = x^\mu_{\ ij} + \Gamma^\mu_{\ \nu\lambda} x^\nu_{\ ij} + \kappa^{\alpha}_{\ ij} n^\mu_{\ \alpha} \), where \( \kappa^{\alpha}_{\ ij} = n^\alpha_{\ ij} \) are the extrinsic curvatures on the brane and \( n^\mu_{\ \alpha} (\alpha = 1, \ldots, \delta = D - 4) \) form a basis of normal vectors to the brane. The covariant differentiations \( \mid \) and \( ; \) correspond to \( h_{ij} \) and \( g_{\mu\nu} \) respectively, while \( \Gamma^\mu_{\ \nu\lambda} \) are the Christoffel symbols of \( g_{\mu\nu} \). Due to the arbitrariness of \( \delta x^\mu \) it arises

\[
\tau^{ij}_{\ |j} x^\mu_{\ i} - \tau^{ij} \kappa^{\alpha}_{\ ij} n^\mu_{\ \alpha} = 0
\]

(2.6)

and since the vectors \( x^\mu_{\ i}, n^\alpha_{\ \mu} \) are independent, two sets of equations arise

\[
\tau^{ij} = 0, \quad \tau^{ij} \kappa^{\alpha}_{\ ij} = 0 \iff \tau^{ij} \left( x^\mu_{\ ij} + \Gamma^\mu_{\ \nu\lambda} x^\nu_{\ ij} \right) = 0.
\]

(2.7)

Note that the previous equivalence of the two expressions, one with free index \( \alpha \) and the other with free index \( \mu \) is due to that the vectors \( x^\mu_{\ ij} + \Gamma^\mu_{\ \nu\lambda} x^\nu_{\ ij} \) are normal to the brane. The variation described so far is the same with the one leading to the Nambu-Goto equation of motion. Indeed for \( L = 1 \), it is \( \tau^{ij} = \frac{1}{2} h^{ij} \) and the first equation is empty, while the second becomes \( h^{ij} \kappa^{\alpha}_{\ ij} = 0 \iff \Box h^\mu + \Gamma^\mu_{\ \nu\lambda} h^{\nu\lambda} = 0 \) which is the Nambu-Goto equation of motion. Note again that the previous equivalence of the two expressions for the Nambu-Goto equation, one with free index \( \alpha \) and the other with free index \( \mu \) is due to that the vector \( \Box h^\mu + \Gamma^\mu_{\ \nu\lambda} h^{\nu\lambda} \) is normal to the brane. Similarly, the Regge-Teitelboim equation of motion [23] is a generalization where \( L \) collects the four-dimensional terms of [3, 4], i.e.

\[ L = \frac{1}{2 \kappa^2} R - \lambda + \frac{\kappa^4}{\kappa^2 \sqrt{|h|}}. \]

It is \( \tau^{ij} = \frac{1}{2} \left( \frac{\kappa^4}{\kappa^2} G^{ij} + \lambda h^{ij} - T^{ij} \right) \), so the first equation becomes the standard conservation

\[ T^{ij} \mid = 0 \]

and the second \( \left( \frac{\kappa^4}{\kappa^2} G^{ij} + \lambda h^{ij} - T^{ij} \right) K^{\alpha}_{\ ij} = 0. \)

In order to express the back-reaction of the brane onto the bulk and vice-versa, we consider a general higher-dimensional action of the form

\[
s_D = \int_M d^D x \sqrt{|g|} \mathcal{L}(g_{\mu\nu}),
\]

(2.8)

where \( \mathcal{L} \) is any scalar on \( M \) built up from the metric \( g_{\mu\nu} \), e.g. \( \mathcal{L} = \mathcal{R}(g_{\mu\nu}) \). Under an arbitrary variation of the bulk metric \( \delta g_{\mu\nu} \), the variation of \( s_D \) is \( \delta s_D = \int_M d^D x \sqrt{|g|} E^{\mu\nu} \delta g_{\mu\nu} \), where \( E^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} + \frac{\partial}{\partial \sqrt{|h|}} \), and the stationarity \( \delta s_D = 0 \) under arbitrary variations \( \delta g_{\mu\nu} \) gives the bulk field equations \( E^{\mu\nu} = 0 \). The boundary terms arising from this variation in the presence of a defect disappear by a suitable choice for the boundary condition of \( \delta g_{\mu\nu} \), usually by choosing a Dirichlet boundary condition for \( \delta g_{\mu\nu} \). However, in the presence of the defect, inside \( E^{\mu\nu} \), beyond the regular terms which obey \( E^{\mu\nu} = 0 \), in general there are also non-vanishing distributional terms making the variation \( \delta s_D \) not identically zero. The bulk action knows about the defect through these distributional terms. Since \( \text{distr } E^{\mu\nu} \propto \delta^{(\delta)} \), where \( \delta^{(\delta)} \) is the \( \delta \)-dimensional delta function with support on the defect, it is \( \text{distr } E^{\mu\nu} \delta g_{\mu\nu} \propto \delta^{(\delta)} \delta g_{\mu\nu} \mid_{\text{brane}} \), so only the variation of the bulk metric at the brane position contributes to \( \delta s_D \), as expected. More precisely, these distributional terms always appear in the parallel to the brane components and if \( \text{distr } E^{ij} = k^{ij} \delta^{(\delta)} \), the variation \( \delta s_D \) gets the form

\[
\delta s_D = \int_M d^D x \sqrt{|g|} k^{ij} \delta^{(\delta)} \delta h_{ij} = \int_\Sigma d^4 x \sqrt{|h|} k^{ij} \delta h_{ij}.
\]

(2.9)

Therefore, there is an extra variation of the bulk metric at the brane position \( \delta g_{\mu\nu} \mid_{\text{brane}} \) (which in the adapted frame coincides with the variation of the induced metric \( \delta h_{ij} \)) which is independent of the bulk metric variation and this extra variation determines the brane equation of motion. The corresponding variation of the total action at the brane position is

\[
\delta (s_D + s_4) \mid_{\text{brane}} = \int_\Sigma d^4 x \sqrt{|h|} (k^{ij} + \tau^{ij}) \delta h_{ij}.
\]

(10.20)

In particular, if all the components of the variation \( \delta h_{ij} \) are independent from each other, the stationarity of the total action at the brane position \( \delta (s_D + s_4) \mid_{\text{brane}} = 0 \) gives the standard matching conditions (or standard brane equations
of motion) \( k^{ij} + \tau^{ij} = 0 \Leftrightarrow k^{\mu \nu} + \tau^{\mu \nu} = 0 \), where \( k^{\mu \nu} = k^{ij} x^\mu_i x^\nu_j \), \( \tau^{\mu \nu} = \tau^{ij} x^\mu_i x^\nu_j \) are parallel to the brane tensors.

Under a variation \( \delta x^\mu \) of the embedding fields, the variations of \( g_{\mu \nu}, h_{ij} \) were given in the previous paragraph and the corresponding variation of the total action at the brane position will be

\[
\delta_x (s_D + s_A)_{\text{brane}} = \int_{\Sigma} d^4 \chi \sqrt{|h|} (k^{ij} + \tau^{ij}) \delta_x h_{ij},
\]

(2.11)

Following the same steps as in the previous paragraph with \( \tau^{ij} \) replaced by \( k^{ij} + \tau^{ij} \), the stationarity \( \delta_x (s_D + s_A)_{\text{brane}} = 0 \) gives, due to the arbitrariness of \( \delta x^\mu \), the brane equations of motion

\[
(k^{ij} + \tau^{ij})_{\text{ij}} = 0, \quad (k^{ij} + \tau^{ij}) K^{\alpha}_{ij} = 0 \Leftrightarrow (k^{ij} + \tau^{ij}) (x^\mu_{ij} + \Gamma^\mu_{\alpha \lambda} x^\alpha_i x^\lambda_j) = 0.
\]

(2.12)

These can be called “gravitating Nambu-Goto matching conditions” since they collect also the contribution from bulk gravity and they form a schematic summary of our proposal. Nothing ab initio assures their consistency with the bulk field equations. However, for the non-trivial case of a codimension-2 defect in six-dimensional EGB gravity the consistency has been shown in [20]. In order to describe the previous variation of the brane position there is also the equivalent passive viewpoint of a bulk coordinate change. Therefore, \( \delta x^\mu \) is invariant under coordinate transformations, the presence of the defect, i.e. of the distributional terms inside \( s_j \), make \( \delta_x s_j_{\text{brane}} \neq 0 \).

Let the tensor fields \( \phi^I(x^\rho) \) transform according to their functional variation which is the change in their functional form,

\[
\delta_x \phi^I(x) = \phi^I(x^\rho) - \phi^I(x^\rho) = \phi^I(x^\rho) \delta x^\rho - \phi^I(x^\rho) \delta x^\rho = -\mathcal{L} \delta_x \phi^I,
\]

(2.13)

i.e. transform according to the Lie derivative with generator the infinitesimal coordinate change. Therefore, \( \delta_x g_{\mu \nu} = -(g_{\mu \nu, \lambda} \delta x^\lambda + g_{\mu \lambda} \delta x^\rho + g_{\nu \lambda} \delta x^\rho, \mu \) while \( \delta_x x^\mu = 0, \delta_x h_{ij} = (\delta x g_{\mu \nu}) x^\mu_i x^\nu_j \). The stationarity under arbitrary variations \( \delta x^\mu \) of the total brane-bulk action \( \delta_x (s_D + s_A)_{\text{brane}} = \int_{\Sigma} d^4 \chi \sqrt{|h|} (k^{ij} + \tau^{ij}) \delta_x h_{ij} \) gives again the same matching conditions as before.

III. FIVE-DIMENSIONAL SETUP AND ALTERNATIVE MATCHING CONDITIONS

Let us consider the general system of five-dimensional Einstein gravity coupled to a localized 3-brane source. The domain wall \( \Sigma \) splits the spacetime \( M \) into two parts \( M_\pm \) and the two sides of \( \Sigma \) are denoted by \( \Sigma_\pm \). The unit normal vector \( n^\mu \) points inwards \( M_\pm \). The total brane-bulk action is

\[
S = \frac{1}{2 \kappa_5^2} \int_M d^5 x \sqrt{|g|} (R - 2\Lambda_5) + \frac{1}{\kappa_5^2} \int_{\Sigma} d^4 \chi \sqrt{|h|} \left( \frac{r_c}{2\kappa_5^2} R - \lambda \right) - \frac{1}{\kappa_5^2} \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} K + \int_{\Sigma} d^4 \chi L_{\text{mat}} + \int_{\Sigma} d^4 \chi L_{\text{mat}},
\]

(3.1)

where \( g_{\mu \nu} \) is the (continuous) bulk metric tensor and \( h_{\mu \nu} = g_{\mu \nu} - n_\mu n_\nu \) is the induced metric on the brane \((\mu, \nu, ... \text{are four-dimensional coordinate indices})\). The bulk coordinates are \( x^\rho \) and the brane coordinates are \( \chi^i \) (\( i, j, ... \) are coordinate indices on the brane). The symbol \( \Sigma_\pm \) in an integral means the contribution from both sides of the surface. The calligraphic quantities refer to the bulk metric, while the regular ones to the brane metric. The brane tension is \( \lambda \) (denoted also by \( V \) in the next sections concerning cosmology) and the induced-gravity term [24], if present, has a crossover length scale \( r_c = 2\kappa_5^2 m^2 = m^2/M^3 \), where \( \kappa_5^2 = M_5^4 = 2M^3 = (8\pi G_5)^{-1} \). \( L_{\text{mat}} \) are the matter Lagrangian densities of the bulk and of the brane respectively. The contribution on each side of the wall of the Gibbons-Hawking term will also be necessary here as in the standard treatment. \( K = h^{\mu \nu} K_{\mu \nu} \) is the trace of the extrinsic curvature \( K_{\mu \nu} = h^{\mu \rho} h^{\nu \sigma} K_{\rho \sigma} \) (the covariant differentiation \( ; \) corresponds to \( g_{\mu \nu} \)).

Varying (3.1) with respect to the bulk metric we get the bulk equations of motion

\[
G_{\mu \nu} = \kappa_5^2 T_{\mu \nu} - \Lambda_5 g_{\mu \nu},
\]

(3.2)

where \( G_{\mu \nu} \) is the bulk Einstein tensor and \( T_{\mu \nu} \) is a regular bulk energy-momentum tensor. We are mainly interested in a bulk with a pure cosmological constant \( \Lambda_5 = \kappa_5^2 \Lambda = \Lambda/2M^3 \), but for the present we leave a non-vanishing \( T_{\mu \nu} \). More precisely, we define the variation \( \delta g_{\mu \nu} \) of the bulk metric to vanish on the defect. In this variation, beyond the basic terms proportional to \( \delta g_{\mu \nu} \) which give (3.2), there appear, as usually, extra terms proportional to the second covariant derivatives \( (\delta g_{\mu \nu})_{;\kappa \lambda} \) which lead to a surface integral on the brane with terms proportional to \( (\delta g_{\mu \nu})_{;\kappa} \).
Adding the Gibbons-Hawking term, the normal derivatives of $\delta g_{\mu\nu}$, i.e. terms of the form $n^\kappa (\delta g_{\mu\nu})_{,\kappa}$, are canceled. The remaining boundary terms are either terms proportional to $\delta g_{\mu\nu}$ or terms containing $h^{\mu\nu}n^\lambda (\delta g_{\mu\nu})_{,\nu}$. These last terms lead (up to an irrelevant integration on $\partial \Sigma$) again to terms proportional to $\delta g_{\mu\nu}$ and more precisely, all the boundary terms together are basically of the known form $(K^{\mu\nu} - Kh^{\mu\nu})\delta g_{\mu\nu}$. Finally, considering as boundary condition for the variation of the bulk metric its vanishing on the brane (Dirichlet boundary condition for $\delta g_{\mu\nu}$), there is nothing left beyond the terms in equation (3.2). The Gibbons-Hawking term will again contribute in a while in another independent variation performed in order to obtain the brane equations of motion.

According to the standard method, the interaction of the brane with the bulk comes from the variation $\delta g_{\mu\nu}$ at the brane position of the action (3.1), which is equivalent to adding on the right-hand side of equation (3.2) the term $\kappa_5^2 T_{\mu\nu} \delta^{(1)}$, where $T_{\mu\nu} = \sqrt{|h|}/|z| \left[ T_{\mu\nu} - \lambda h_{\mu\nu} - (r_c/\kappa_5^2) G_{\mu\nu} \right]$. $T_{\mu\nu}$ is the brane energy-momentum tensor, $G_{\mu\nu}$ the brane Einstein tensor and $\delta^{(1)}$ the one-dimensional delta function with support on the defect. This approach leads to the (generalized due to $r_c$) Israel matching conditions, it has been analyzed in numerous papers and discussed in the Introduction.

Here, we discuss an alternative approach where the interaction of the brane with bulk gravity is obtained by varying the total action (3.1) with respect to $\delta x^\mu$, the embedding fields of the brane position [20], [22]. The embedding fields are some functions $x^\mu (\chi^i)$ and their variations are $\delta x^\mu (x^\nu)$. While in the standard method the variation of the bulk metric at the brane position remains arbitrary, here the corresponding variation is induced by $\delta x^\mu$ as explained in section II. It is given by

$$\delta g_{\mu\nu} = \delta x g_{\mu\nu} = g'_{\mu\nu} (x^\rho) - g_{\mu\nu} (x^\rho) = -(g_{\mu\nu,\lambda} \delta x^\lambda + g_{\mu\lambda} \delta x^{\lambda,\nu} + g_{\nu\lambda} \delta x^{\lambda,\mu}) = -\mathcal{L}_{\delta x} g_{\mu\nu}, \quad (3.3)$$

and is obviously independent from the variation leading to (3.2). The induced metric $h_{ij} = g_{\mu\nu} x^\mu, i x^\nu, j$ enters the localized terms of the action (3.1) and depends explicitly and implicitly (through $g_{\mu\nu}$) on the embedding fields. Also the bulk terms of (3.1) contribute implicitly to the brane variation under the variation of the embedding fields. The result of $\delta x^\mu$ variation gives, as we will see, as coefficient of $\delta x^\mu$ a combination of vectors parallel and normal to the brane, therefore, two sets of equations will finally arise as matching conditions instead of one. Instead of directly expressing $\delta g_{\mu\nu}$, $\delta h_{ij}$ in terms of $\delta x^\mu$, it is convenient to include the constraints in the action and vary independently (however, we will also perform the direct calculation). So, the first constraint $h_{ij} = g_{\mu\nu} x^\mu, i x^\nu, j$ implies the independent variation of $h_{ij}$. The variation $\delta x^\mu$ affects the variation of the parallel to the brane vectors $x^\mu, i$, which in turn influences the variation of the normal vector $n^\mu$. So, the additional constraints $n_\mu x^\mu, i = 0$, $g_{\mu\nu} n^\mu n^\nu = 1$ have to be added, and $\delta n_\mu$ is another independent variation. Finally, the third variation $\delta g_{\mu\nu}$ depends on $\delta x^\mu$ by (3.3). Therefore, $\delta g_{\mu\nu}$ are independent from $\delta n_\mu$, $\delta h_{ij}$, but the various $\delta g_{\mu\nu}$ components are not all independent from each other, so in the end they have to be expressed in terms of $\delta x^\mu$ which are independent. If $\lambda^i$, $\lambda^j$, $\lambda^0$ are the Lagrange multipliers corresponding to the above constraints, the constraint action added to $S$ is

$$S_c = \int_{\Sigma_{\pm}} d^4 \chi \sqrt{\kappa} \left[ \lambda^i (h_{ij} - g_{\mu\nu} x^\mu, i x^\nu, j) + \lambda^i n_\mu x^\mu, i + \lambda^0 (g_{\mu\nu} n^\mu n^\nu - 1) \right]. \quad (3.4)$$

In general, the various Lagrange multipliers are different among the two sides $\Sigma_{\pm}$. Moreover, since there are two normals $n_+^\mu, n_-^\mu$, there are two independent variations with respect to the normals at the two sides. These two variations are independent since one could consider, for example, the case where only the half space $M_+$ exists. Although the bulk metric $g_{\mu\nu}$ is continuous across the defect, the variation $\delta g_{\mu\nu}$ is different among the two sides $\Sigma_{\pm}$. Indeed, the extrinsic curvature is in general discontinuous on the brane and from equation (3.3), $\delta g_{\mu\nu}$ contains derivatives of the metric. Therefore, $\delta g_{\mu\nu}$ can be expressed in terms of quantities on either side of the defect, but not simultaneously in terms of quantities on both sides. Finally, there is the extra independent variation $\delta h_{ij}$ with respect to the brane metric $h_{ij}$.
Variation of $S + S_c$ with respect to $n_\mu, h_{ij}, g_{\mu\nu}$ at the brane gives

$$
\delta(S + S_c)_{\text{brane}} = \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left( \lambda^i x^\mu_{,i} + 2\lambda^0 n_\mu - \frac{1}{\kappa_5^2} K n_\mu \right) \delta n_\mu \\
+ \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left[ \lambda^i + \frac{1}{\kappa_5^2} \left( K^{ij} - \frac{K}{2} h^{ij} \right) \right] \delta h_{ij} + \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left[ \frac{1}{2} (T^{ij} - \lambda h^{ij}) - \frac{r_c}{2\kappa_5^2} G^{ij} \right] \delta h_{ij} \\
- \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left( \lambda^i x^\mu_{,i} x^\nu_{,j} + \lambda^0 n_\mu n^\nu \right) \delta g_{\mu\nu} \\
- \frac{1}{2\kappa_5^2} \int_M d^5 x \sqrt{|g|} \left[ G^{\mu\nu} - \kappa_5^2 T^{\mu\nu} + \Lambda g^{\mu\nu} \right] \delta g_{\mu\nu} \bigg|_{\text{brane}} - \frac{1}{2\kappa_5^2} \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} h^{\mu\nu} n^\lambda \\
\left[ (\delta g_{\mu\lambda})_{;\nu} - (\delta g_{\mu\nu})_{;\lambda} \right].
$$

(3.5)

When $r_c \neq 0$, one should add in $\text{[3.1]}$ the integral of the extrinsic curvature $k$ of $\partial \Sigma$ (if $\partial \Sigma$ is not empty) to cancel some terms from the variation $\delta R$; this, in general, does not affect the dynamics of $\Sigma_{\pm}$.

The basic root of difficulty for deriving (3.5) is the Gibbons-Hawking term $K$. Its treatment, due to the imposition of the constraints (3.4), is different from the conventional treatment of the variation $\delta g_{\mu\nu}$ described after equation (3.2). In the first line of (3.5), the $\delta n_\mu$ terms arise from the appropriate terms of (3.4) and the $K = h^{ij} K_{ij}$ term of (3.1), where the identities $K_{ij} = 0$ are used. $-n_\mu (x^j_{,i} + \Gamma^\mu_{i\nu} x^\nu_{,j})$, $-K_{ij} = x^\mu_{,ij}$, $x^\mu_{,ij}$, and $\Gamma^\mu_{i\nu}$ have been used (\Gamma^\mu_{i\nu} are the Christoffel symbols of $g_{\mu\nu}$). These identities show that $K_{ij}$ is a function of the independent variables $n_\mu, g_{\mu\nu}$, but it does not depend on the independent variable $h_{ij}$. Next, the $\delta h_{ij}$ terms in the second line of (3.5) arise again from the appropriate terms of (3.4) and all the four-dimensional terms of (3.1). Finally, the variation $\delta g_{\mu\nu}$ on the brane is more difficult. The first contribution comes from the appropriate terms of (3.4) and the derived terms are those of the third line of (3.5), where special care is needed for the variation of $g_{\mu\nu} n^\mu n^\nu$ since $n_\mu$ is kept fixed and not $n^\mu$. Second, it arises from the five-dimensional terms of (3.1) and the derived terms are those of the fourth line of (3.5). Third, it arises from the $K$ term of (3.1) and the derived terms are those of the fifth line of (3.5). In order to get these last terms, the identity $K = h^{ij} n_{(i,} x^\mu_{,j)} - n_\mu \Gamma^\mu_{i\nu} h^{i\nu\lambda}$ was used, which arises from $K_{ij} = n_\mu (x^j_{,i} + \Gamma^\mu_{i\nu} x^\nu_{,j}) - n_{(i} x^\mu_{,j)} - n_\mu \Gamma^\mu_{i\nu} x^\nu_{,i} x^\lambda_{,j}$ and $h^{\mu\nu} = h^{ij} x^\mu_{,i} x^\nu_{,j}$.

The last term of the fourth line of (3.5) cancels the last term of the fifth line, so the normal derivatives of $\delta g_{\mu\nu}$ cancel. Then, use is made of the identity $h^{\mu\nu} n^\lambda (\delta g_{\mu\lambda})_{;\nu} = (h^{\mu\nu} n^\lambda \delta g_{\mu\lambda})_{;\nu} - (K^{\mu\nu} - K n^{\mu} n^{\nu}) \delta g_{\mu\nu}$, where $\{ \}$ denotes covariant differentiation with respect to $h_{\mu\nu}$ (or $h_{ij}$), to convert the remaining $(\delta g_{\mu\lambda})_{;\nu}$ terms of (3.5) to $\delta g_{\mu\nu}$ terms (up to an irrelevant integration on $\partial \Sigma$). Finally, the quantity in curly brackets appearing in the fourth line of (3.5) vanishes since it coincides with equation (3.2) which is also valid on the brane. The variation (3.5) takes the form

$$
\delta(S + S_c)_{\text{brane}} = \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left( \lambda^i x^\mu_{,i} + 2\lambda^0 n_\mu - \frac{1}{\kappa_5^2} K n_\mu \right) \delta n_\mu \\
+ \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left[ \lambda^i + \frac{1}{\kappa_5^2} \left( K^{ij} - \frac{K}{2} h^{ij} \right) \right] \delta h_{ij} + \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left[ \frac{1}{2} (T^{ij} - \lambda h^{ij}) - \frac{r_c}{2\kappa_5^2} G^{ij} \right] \delta h_{ij} \\
- \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left( \lambda^i x^\mu_{,i} x^\nu_{,j} + \lambda^0 n_\mu n^\nu \right) \delta g_{\mu\nu} - \frac{1}{2\kappa_5^2} \int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left( K^{\mu\nu} - K n^{\mu} n^{\nu} \right) \delta g_{\mu\nu}.
$$

(3.6)

As explained above, $\delta n_\mu, \delta h_{ij}$ are independent variations, but $\delta g_{\mu\nu}$ depends on $\delta x^\mu$ which are also independent. So, $\delta(S + S_c)_{\text{brane}} = 0$ gives

$$
\lambda^i x^\mu_{,i} + (2\lambda^0 - \frac{1}{\kappa_5^2} K) n_\mu = 0
$$

(3.7)

$$
\lambda^i_j + \lambda^i_j + \frac{1}{\kappa_5^2} \left( K^{ij} - \frac{K}{2} h^{ij} \right) = \frac{1}{\kappa_5^2} \left( K^{ij} - \frac{K}{2} h^{ij} \right) + \frac{1}{2} (T^{ij} - \lambda h^{ij}) - \frac{r_c}{2\kappa_5^2} G^{ij} = 0
$$

(3.8)

$$
\int_{\Sigma_{\pm}} d^4 \chi \sqrt{|h|} \left[ \lambda^i x^\mu_{,i} x^\nu_{,j} + \lambda^0 n_\mu n^\nu \right] \delta g_{\mu\nu} = 0
$$

(3.9)

where $\delta g_{\mu\nu}$ obeys (3.3). Equation (3.7) holds separately for each side $\Sigma_{\pm}$. Since the vectors $x^\mu_{,i}, n_\mu$ are independent, equation (3.7) implies for any side separately $\lambda^i = 0$, $\lambda^0 = \frac{1}{2\kappa_5^2} K$. Equation (3.8) contains the combination $\lambda^i_j + \lambda^i_j$.
and it will be seen that the matching conditions contain the same combination, so the matching conditions will be unambiguously determined. Then, equation (3.10), with \( \lambda^{ij} \) satisfying (3.8), takes the form

\[
\int_{\Sigma_+} d^4\chi \sqrt{|h|} \left( \frac{1}{2\kappa^2_5} K_{\mu\nu} + \lambda^{ij} x^{\mu}_{,i} x^{\nu}_{,j} \right) \delta g_{\mu\nu} = 0.
\]

(3.10)

Since \( K_{\mu\nu} = K^{ij} x^{\mu}_{,i} x^{\nu}_{,j} \), equation (3.10) is written as

\[
\int_{\Sigma_+} d^4\chi \sqrt{|h|} \mu^{ij} x^{\mu}_{,i} x^{\nu}_{,j} \delta g_{\mu\nu} = 0 \quad , \quad \mu^{ij} = \frac{1}{2\kappa^2_5} K^{ij} + \lambda^{ij} .
\]

(3.11)

Contrary to the present situation, had we considered all \( \delta n_\mu, \delta h_{ij}, \delta g_{\mu\nu} \) independent, equations (3.7), (3.8) would still arise. Equation (3.11) would be written as \( \int_{\Sigma} d^4\chi \sqrt{|h|} \left( \mu^{ij} + \lambda^{ij} \right) x^{\mu}_{,i} x^{\nu}_{,j} \delta g_{\mu\nu} = 0 \) providing \( \mu^{ij} + \lambda^{ij} = 0 \). Then, using equation (3.8), the Israel matching condition \( (K_{ij} - Kh_{ij})_+ + (K_{ij} - Kh_{ij})_- = \kappa^2_5 (\lambda h_{ij} - T_{ij}) + r_c G_{ij} \) would arise.

In our approach \( \delta g_{\mu\nu} \) has to be expressed via (3.3) in terms of quantities on either side and equation (3.11) becomes

\[
\int_{\Sigma_+ \text{ or } \Sigma_-} d^4\chi \sqrt{|h|} M^{ij} x^{\mu}_{,i} x^{\nu}_{,j} \delta g_{\mu\nu} = 0 \quad , \quad M^{ij} = \mu^{ij} + \lambda^{ij} ,
\]

(3.12)

or equivalently

\[
\int_{\Sigma_+ \text{ or } \Sigma_-} d^4\chi \sqrt{|h|} M^{ij} \left( g_{\mu\nu,\lambda} x^{\mu}_{,i} x^{\nu}_{,j} \delta x^\lambda + 2 g_{\mu\nu} x^{\mu}_{,i} x^{\nu}_{,j} \delta x^{\nu}_{,\lambda} \right) = 0 .
\]

(3.13)

After an integration of (3.13) by parts and imposing \( \delta x^\mu |_{\partial \Sigma} = 0 \), we get

\[
\int_{\Sigma_+ \text{ or } \Sigma_-} d^4\chi \sqrt{|h|} g_{\mu\sigma} \left[ M^{ij} x^{\mu}_{,j} + M^{ij} \left( \Gamma^{\mu}_{\nu\lambda} x^{\nu}_{,i} x^{\lambda}_{,j} \right) \right] \delta x^\sigma = 0 .
\]

(3.14)

and since the extrinsic curvature satisfies \(-K_{ij} n^\mu = x^{\mu}_{,ij} + \Gamma^{\mu}_{\nu\lambda} x^{\nu}_{,i} x^{\lambda}_{,j} \), equation (3.14) becomes

\[
\int_{\Sigma_+ \text{ or } \Sigma_-} d^4\chi \sqrt{|h|} g_{\mu\sigma} \left( M^{ij} x^{\mu}_{,j} - M^{ij} K_{ij} n^\mu \right) \delta x^\sigma = 0 .
\]

(3.15)

Due to the arbitrariness of \( \delta x^\mu \) it holds

\[
M^{ij} x^{\mu}_{,j} - M^{ij} K_{ij} n^\mu = 0 \quad , \quad M^{ij} x^{\mu}_{,i} - M^{ij} K_{ij} n^\mu = 0 ,
\]

(3.16)

therefore, two sorts of matching conditions arise

\[
M^{ij} K_{ij}^+ = 0 \quad , \quad M^{ij} K_{ij}^- = 0 \quad , \quad M^{ij} K_{ij}^+ = 0 .
\]

(3.17)

Substituting \( \lambda^{ij}_+ + \lambda^{ij}_- \) of \( M^{ij} \) from (3.8), we get the matching conditions of codimension-1 Einstein gravity

\[
[[K_{ij}^+ - K_+ h^{ij}] + (K_{ij}^+ - K_- h^{ij}) + \kappa^2_5 (T^{ij} - \lambda h^{ij}) - r_c G^{ij}] K_{ij}^+ = 0 \]

(3.19)

\[
[[K_{ij}^+ - K_+ h^{ij}] + (K_{ij}^+ - K_- h^{ij}) + \kappa^2_5 (T^{ij} - \lambda h^{ij}) - r_c G^{ij}] K_{ij}^- = 0 \]

(3.20)

\[
[[K_{ij}^+ - K_+ h^{ij}] + (K_{ij}^+ - K_- h^{ij}) + \kappa^2_5 T^{ij}]_{ij} = 0 .
\]

(3.21)

It is obvious that any solution of the standard matching conditions is still solution of the above matching conditions, therefore, the new space of solutions is expected to be a continuous deformation of the space of solutions of the standard theory.

If the extrinsic curvatures \( K_{ij}^+, K_{ij}^- \) are proportional to each other, i.e. \( K_{ij}^- = \eta K_{ij}^+ \), the previous matching conditions become

\[
[[1 + \eta] (K_{ij}^+ - K h^{ij}) + \kappa^2_5 (T^{ij} - \lambda h^{ij}) - r_c G^{ij}] K_{ij} = 0 \]

(3.22)

\[
[[1 + \eta] (K_{ij}^+ - K h^{ij}) + \kappa^2_5 T^{ij}]_{ij} = 0 ,
\]

(3.23)
where \( K_{ij} \equiv K^+_{ij} \).

A \( Z_2 \)-symmetric brane obeys \( K^\pm_{ij} = K^\pm_{ij} = K_{ij} \), so it corresponds to \( \eta = 1 \). The matching conditions become

\[
\begin{aligned}
[K_{ij} - Kh^{ij} + \frac{\kappa_5^2}{2} (T^{ij} - \lambda h^{ij}) - \frac{r_c G^{ij}}{2}] K_{ij} &= 0 \\
T^{ij}_{ij} &= -\frac{2}{\kappa_5^2} (K_{ij} - Kh^{ij}),
\end{aligned}
\]  

(3.24)

(3.25)

The “smooth” brane has \( K_{ij} \) continuous, so \( -K^\pm_{ij} = K^\pm_{ij} \equiv K_{ij} \) and corresponds to \( \eta = -1 \). The matching conditions are

\[
\begin{aligned}
[k_5^2 (T^{ij} - \lambda h^{ij}) - r_c G^{ij}] K_{ij} &= 0 \\
T^{ij}_{ij} &= 0.
\end{aligned}
\]  

(3.26)

(3.27)

In [22], the matching conditions derived are not the same with (3.19)-(3.21). The reason is that although equations (3.19)-(3.21) were still valid there, equation \( \int_\Sigma d^4 \sqrt{|h|} \mu_{ij} \propto x_i x_j \delta \sigma_i \delta \sigma_j \) \( \delta g_{\mu \nu} = 0 \) was considered instead of equation (3.12). Therefore, equations \( \mu_{ij} K^+_{ij} = 0, \mu_{ij} K^-_{ij} = 0, \mu_{ij} \mid_{ij} = 0, \mu_{ij} \mid_{ij} = 0 \) were derived instead of equations (3.17), (3.18). From equations \( \mu_{ij} \mid_{ij} = 0, \mu_{ij} \mid_{ij} = 0 \), the matching condition (3.18) is derived. However, since in equations \( \mu_{ij} K^+_{ij} = 0, \mu_{ij} K^-_{ij} = 0 \) the combination \( \lambda_5^+ + \lambda_5^- \) cannot appear, there arises the unnatural situation that the matching conditions contain undetermined Lagrange multipliers. This cannot be realistic since Lagrange multipliers are supplementary objects and additionally any physical matching conditions should be well-defined. However for the case \( K^- = \eta K^+ \) the matching conditions of [22] reduce to the present matching conditions.

Without the use of Lagrange multipliers, we could also proceed with the variation of (3.1) with respect to the bulk metric at the brane position and get

\[
\delta S |_{brane} = \frac{1}{2 \kappa_5^2} \int_\Sigma d^4 \sqrt{|h|} \left[ (K_+^{\mu \nu} - K_+ h^{\mu \nu}) + (K_-^{\mu \nu} - K_- h^{\mu \nu}) + \kappa_5^2 (T^{\mu \nu} - \lambda h^{\mu \nu}) - r_c G^{\mu \nu} \right] \delta g_{\mu \nu}
\]  

(3.28)

If \( \delta g_{\mu \nu} \) are independent, we take again the Israel matching conditions. If \( \delta g_{\mu \nu} \) are subject to (3.3), we take the matching conditions (3.19)-(3.21). This method is a straightforward one which convinces us about the validity of equations (3.19)-(3.21).

Let us describe in brief, before we continue, another method for deriving the matching conditions (3.19)-(3.21). This method was described in section II and has also been applied in [20] in the treatment of a codimension-2 brane, so it is applicable in all codimensions. The transverse to the brane first derivative of the metric is in general discontinuous between the two sides of the brane, and therefore, \( \delta g_{\mu \nu} \) contains a distributional piece beyond the regular one, which is \( \delta g_{\mu \nu} \equiv -\delta g_{\mu \nu} \mid (K_\mu - K h_{\mu \nu}) \mid (K_\mu - K h_{\mu \nu}) \) \( \delta (y) \). As usually done when dealing with distributional sources, the matching conditions are derived by integrating around the singular space. In [20] for a codimension-2 brane the corresponding six-dimensional distributional curly terms of (3.3) were integrated over the \( (r, \theta) \) transverse disc of radius \( \epsilon \) in the limit \( \epsilon \rightarrow 0 \). For a codimension-3 defect the appropriate integration would occur in a spherical region \( r, \theta, \phi \) of radius \( \epsilon \) in the limit \( \epsilon \rightarrow 0 \). Here, the codimension-1 brane is “sandwiched” between two “parallel” hypersurfaces \( H_\pm \) each at a distance \( \epsilon \) from the brane. The integration of the distributional curly bracket in (3.3) gives in the limit \( \epsilon \rightarrow 0 \) the first integral in the third line of (3.29). The second integral in the third line of (3.29) consists of the usual “remnant” terms of the metric variation and the volume of integration \( M \) refers to the space between the hypersurfaces \( H_\pm \). Concerning the Gibbons-Hawking term, this should not now be included in the action (3.1), since already the correct \( K^{\mu \nu} - K h^{\mu \nu} \) term is present in (3.29), so the inclusion of \( K \) would attribute a wrong factor of two. Therefore, the fifth line of (3.5) is absent in (3.29), as well as the various \( K_{ij} \) terms of (3.5) disappear in (3.29). The characteristic of the absence of the Gibbons-Hawking term in this treatment of considering the distributional terms in the action is similar to the fact that in the standard derivation of the Israel matching conditions from the distributional Einstein equations there is no Gibbons-Hawking term. Finally, the variation (3.5) gets the alternative
form

\[ \delta(S + S_c)_{\text{brane}} = \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} \left( \hat{\lambda} x^\mu,_{\mu, i} + 2 \hat{\lambda} n^\mu \right) \delta n_\mu + \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} \hat{\lambda}^{ij} \delta h_{ij} + \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} \left[ \frac{1}{2} (T^{ij} - \lambda h^{ij}) - \frac{r_c}{2\kappa_5^2} G^{ij} \right] \delta h_{ij} \]

\[ - \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} \left( \hat{\lambda}^{ij} x^\mu,_{\mu, j} + \hat{\lambda}^0 n^\mu n^\nu \right) \delta g_{\mu\nu} + \frac{1}{2\kappa_5^2} \int_M d^5 x \sqrt{|g|} g^{\lambda[\mu} g^{\nu\rho]} (\delta g_{\mu\nu})_{\lambda\lambda}, \]

where the various coefficients \( \hat{\lambda} \)'s are in general different from \( \lambda \)'s. Since the quantities which multiply \( (\delta g_{\mu\nu})_{\kappa\lambda} \) in (3.29) do not have distributional pieces and the variational fields \( \delta g_{\mu\nu} \) are considered as usually smooth functions, the corresponding integral vanishes at the shrink limit and (3.29) becomes

\[ \delta(S + S_c)_{\text{brane}} = \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} \left( \hat{\lambda} x^\mu,_{\mu, i} + 2 \hat{\lambda} n^\mu \right) \delta n_\mu + \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} \hat{\lambda}^{ij} \delta h_{ij} + \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} \left[ \frac{1}{2} (T^{ij} - \lambda h^{ij}) - \frac{r_c}{2\kappa_5^2} G^{ij} \right] \delta h_{ij} \]

\[ - \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} \left( \hat{\lambda}^{ij} x^\mu,_{\mu, j} + \hat{\lambda}^0 n^\mu n^\nu \right) \delta g_{\mu\nu} + \frac{1}{2\kappa_5^2} \int_{\Sigma_\pm} d^4 \chi \sqrt{|h|} (K^{\mu\nu} - Kh^{\mu\nu}) \delta g_{\mu\nu}, \]

(3.30)

There is another way to see why the last integral of (3.29) vanishes. This integral, being a volume integral, can be converted to a surface integral. So, it takes the form of the second integral of the fourth line in (3.3) with the only difference that the sign \( \Sigma_\pm \) of (3.3) has to be replaced by \( \Sigma_\pm \cup H_\pm \). However, the normals \( n^\lambda \) of the two hypersurfaces \( \Sigma_\pm, H_\pm \) are opposite to each other, and therefore, at the shrink limit \( \epsilon \to 0 \) the quantities \( h^{\mu\nu} n^\lambda (\delta g_{\mu\nu} \lambda - (\delta g_{\mu\nu})_{\kappa\lambda} \) from \( \Sigma_\pm, H_\pm \) cancel each other (the same also happens for \( \Sigma_+ , H_- \)). Continuing the process from equation (3.31), we arrive anew to the matching conditions (3.19)-(3.21). Note that the independent bulk metric variation \( \delta g_{\mu\nu} \) in the space outside \( H_\pm \) gives the bulk field equations (3.2) under some boundary variational condition on \( H_\pm \) (Newmann-like or if the Gibbons-Hawking term is added on \( H_\pm \) a Dirichlet one).

The general matching conditions (3.19), (3.20) do not provide the equation of motion for the defect, but only relations on the discontinuity of the extrinsic curvature. However, imposing a relation between \( K_{ij}^+, K_{ij}^- \), an equation of motion arises. E.g. for \( K_{ij}^- = \eta K_{ij}^+ \), equation (3.22) is the algebraic in the extrinsic curvature equation of motion. It is a quadratic equation in the extrinsic curvature, contrary to the Israel matching condition which is linear in the extrinsic curvature. Equation (3.22) is the generalization of the Nambu-Goto equation of motion when the self-gravitating brane interacts with bulk gravity. In the limiting case of no back-reaction, a probe brane with tension \( \lambda \) moving in a fixed background arises. Indeed, in the probe limit, all the geometric quantities \( h_{ij}, K_{ij}, G_{ij} \) get their background values when the bulk gravity coupling goes to zero (i.e. \( 1/\kappa_5^2 \to 0 \) and the extra brane sources vanish (i.e. \( T_{ij} \to 0, r_c/\kappa_5^3 \to 0 \)). Then, equation (3.22) becomes \( h^{ij} K_{ij} = 0 \) which is the Nambu-Goto equation of motion. Inversely, whenever any extra term beyond \( \lambda h^{ij} K_{ij} \) (or all terms) appears in (3.22), (3.23) and these equations are consistent with all the other bulk equations, then these matching conditions are meaningful back-reacted matching conditions. In this spirit, the “smooth” matching conditions (3.20), (3.24) without extrinsic curvature discontinuity form an unusual but interesting example. In this case, only the localized matter and four-dimensional gravity terms participate in the brane equations of motion, and although the higher-dimensional bulk terms do not have a direct imprint in these equations, there is still back-reaction since the bulk equations have also to be satisfied at the brane position. These “smooth” matching conditions correspond to the Regge-Teitelboim equations of motion (29), (20) with the crucial difference, however, that there, that there are no higher-dimensional gravity terms in the action and the bulk is prefixed (usually Minkowski). Therefore, possible difficulties discussed in (21) are irrelevant here, since they emanate from the embeddibility restrictions in the given non-dynamical bulk space, while the matching conditions here dynamically propagate in a non-trivial bulk space. “Smooth” matching conditions are also meaningful in codimension-1 standard treatment (28), without of course the \( K_{ij} \) contraction (where there is no balance of distributional terms between the two sides of the distributional equation, but the right-hand side vanishes on its own), although there, they lose their significance since there is no Nambu-Goto probe limit so that these matching conditions to signal a minimal departure from that limit.

Equation (3.21) is the second matching condition and expresses a non-conservation equation of the brane energy-momentum tensor, where the energy exchange between the brane and the bulk is due to the variability along the brane of the extrinsic geometry. Actually, equation (3.21) arises also in the conventional treatment by differentiation of the standard matching conditions. In the following, equation (3.21) will be written in a more convenient form (also present in the standard approach), from where it will be seen that the non-conservation of energy is only due to a possible non-vanishing brane-bulk energy-momentum exchange.

Having finished with the brane equations of motion arising from the distributional parts, we pass to the bulk equations of motion. These bulk equations are also defined limitingly on the brane, and therefore, additional equations
have to be satisfied at the brane position beyond the matching conditions. In the Gaussian-normal coordinate system the spacetime metric takes the form
\[ ds_5^2 = dy^2 + g_{ij}(\chi, y)d\chi^i d\chi^j, \] (3.31)
where the braneworld metric \( h_{ij}(\chi) = g_{ij}(\chi, 0) \) is assumed to be regular everywhere with the possible exception of isolated singular points. From appendix A the 55 component of the bulk equations (3.2) at the brane position gets the form (the index 5 refers to the extra dimension \( y \))
\[ K_{ij} K^{ij} - K^2 + R = 2\Lambda_5 - 2\kappa_5^2 T_{55}, \] (3.32)
which holds separately on each side of the brane. Then, the algebraic matching conditions (3.19), (3.20) are written equivalently in the following form
\[
\begin{align*}
&\left[ (K_{ij}^+ - K_{-h}^{ij}) + \kappa_5^2(T_{ij}^{+} - \lambda h^{ij}) - r_e G_{ij}^{+} \right] K_{ij}^+ = \left( R + 2\kappa_5^2 T_{55} - 2\Lambda_5 \right), \\
&\left[ (K_{ij}^- - K_{+h}^{ij}) + \kappa_5^2(T_{ij}^{-} - \lambda h^{ij}) - r_e G_{ij}^{-} \right] K_{ij}^- = \left( R + 2\kappa_5^2 T_{55} - 2\Lambda_5 \right).
\end{align*}
\] (3.33)
(3.34)
Accordingly, the matching condition (3.22) for \( K_{ij}^- = \eta K_{ij}^+ = \eta K_{ij} \) is written equivalently in the following simpler form linear in the extrinsic curvature
\[ \left[ \kappa_5^2(T_{ij}^{\eta} - \lambda h^{ij}) - r_e G_{ij}^{\eta} \right] K_{ij} = (1 + \eta) \left( R + 2\kappa_5^2 T_{55} - 2\Lambda_5 \right), \] (3.35)
where \( T_{55} \equiv T_{55}^{\eta} \), while in particular the \( Z_2 \)-equation (3.24) gets the form
\[ \left[ \kappa_5^2(T_{ij}^{\eta} - \lambda h^{ij}) - r_e G_{ij}^{\eta} \right] K_{ij} = 2\left( R + 2\kappa_5^2 T_{55} - 2\Lambda_5 \right). \] (3.36)
Similarly, from appendix A the \( i5 \) component of the bulk equations (3.2) at the brane position gets the form
\[ K_{ij}^{\eta} - K_{ij} h^{ij} = \kappa_5^2 T^{\eta 5}, \] (3.37)
which holds separately on each side of the brane. Then, the non-conservation equation (3.21) gets a simpler form
\[ T_{ij}^{\eta} = -(T_{i 5}^{\eta} + T_{j 5}^{\eta}), \] (3.38)
which expresses the non-conservation of the brane energy-momentum due to the flux of energy-momentum from the two sides of the brane. Accordingly, equation (3.22) is written as
\[ T_{ij}^{\eta} = -(1 + \eta) T^{\eta 5}, \] (3.39)
where \( T^{\eta 5} \equiv T_{i 5}^{\eta} \), while in particular the \( Z_2 \)-equation (3.25) gets the form
\[ T_{ij}^{\eta} = -2 T^{\eta 5}. \] (3.40)
Therefore, for the case that the extrinsic curvatures on the two sides of the brane are linearly related, the system of equations which have to be satisfied on the brane consists of equations (3.22) (or equivalently (3.33)), (3.24) (or equivalently (3.39)), (3.32) and (3.37). What remain to be satisfied on the brane are the \( ij \) bulk equations which however contain beyond \( h_{ij}, K_{ij} \) also \( \partial_y K_{ij}, \) so they are decoupled from the other equations. The previous system of equations for \( K_{ij} \) forms a set of 6 algebraic-differential equations for the 10 independent components of \( K_{ij} \), therefore, the brane-bulk system is always consistent. In the next sections, we are going to study the full system of brane-bulk equations for the case of cosmology, therefore the brane evolution and the dynamical bulk space will be found. After the brane geometry is determined, the brane data will be used as initial data to determine the evolution of the bulk geometry. Concerning this point, the main difference with [22] is that there, although the brane evolution is also back-reacted, however, the bulk was assumed from the beginning to be the AdS space.

**IV. CODIMENSION-1 BRANE COSMOLOGY WITH Z_2-SYMMETRY**

We focus on the \( Z_2 \)-symmetric case and consider that the only bulk energy-momentum content consists of a pure cosmological constant \( \Lambda \). We rewrite the matching conditions (3.24), (3.25)
\[
\begin{align*}
&\left[ K_{ij}^{\eta} - K_{h}^{ij} + \frac{1}{4M^3}(T_{ij}^{\eta} - V h^{ij}) - \frac{m^2}{2M^3} G_{ij}^{\eta} \right] K_{ij} = 0, \\
&T_{ij}^{\eta} = -4M^3 \left( K_{ij}^{\eta} - K_{h}^{ij} \right). 
\end{align*}
\] (4.1)
(4.2)
or the equivalent equations \((3.36), (3.40)\)

\[ (T^{ij} - V h^{ij} - 2m^2 G^{ij}) K_{ij} = 4(M^3 R - \Lambda) \]  
\[ T^{ij}_{|j} = 0 . \]  

In order to search for cosmological solutions we consider the corresponding form for the bulk metric in the Gaussian-normal coordinates

\[ ds_5^2 = dy^2 - n^2(t, y) dt^2 + a^2(t, y) \gamma^{ij}(\chi^i) d\chi^i d\chi^j , \]  

where \(\gamma^{ij}\) is a maximally symmetric 3-dimensional metric \((i, j, \ldots = 1, 2, 3)\) characterized by its spatial curvature \(k = -1, 0, 1\). The energy-momentum tensor on the brane (beyond that of the brane tension \(V > 0\)) is assumed to be the one of a perfect cosmic fluid with energy density \(\rho\) and pressure \(p\).

It is convenient to define the quantities

\[ A = \frac{a'}{a}, \quad N = \frac{n'}{n} \]  
\[ X = H^2 + \frac{k}{a^2}, \quad Y = \frac{H^n}{2nH} + X, \quad H = \frac{\dot{a}}{na}, \]

where a prime denotes \(\partial/\partial y\) and a dot denotes \(\partial/\partial t\). The cosmic scale factor, lapse function and Hubble parameter arise as the restrictions on the brane of the functions \(a(t, y), n(t, y)\) and \(H(t, y)\) respectively. Other quantities also have their corresponding values when restricted on the brane, and since all the following equations in this section will refer to the brane position, we will use the same symbols for the restricted quantities without confusion.

The 05 bulk equation \((3.37)\) at the position of the brane becomes

\[ \dot{A} + nH(A - N) = 0, \]  

while the 55 bulk equation \((3.32)\) becomes

\[ A(A + N) - (X + Y) + \frac{\Lambda}{6M^3} = 0 . \]

Equation \((4.3)\) is linear in \(A, N\), and the system \((4.3), (4.9)\) could be algebraically solved for \(A, N\) in terms of \(X, Y\) and replaced into equation \((4.8)\). However, the arising equation would contain \(\ddot{H}\) and the difficulty for integrating such an equation would increase considerably. Hopefully, we can do better because \(A\) can be integrated from equations \((4.8), (4.9)\). Indeed, eliminating \(N\) between equations \((4.8), (4.9)\) we obtain the equation

\[ (A^2)^{\dot{}} + 4nHA^2 - 2nH(X + Y - \frac{\Lambda}{6M^3}) = 0 , \]  

and since \(X + Y = (Xa^4)^{\dot{}}/(2na^4)\), equation \((4.10)\) becomes a total derivative

\[ \left( A^2a^4 - Xa^4 + \frac{\Lambda}{12M^3}a^4 \right)^{\dot{}} = 0 . \]  

The integration of \((4.11)\) gives the equation

\[ A^2 - X + \frac{\Lambda}{12M^3} + \frac{C}{a^4} = 0 \]

\((C\) is integration constant), with two branches for \(A\)

\[ A = \pm \sqrt{X - \frac{C}{a^4} - \frac{\Lambda}{12M^3}} . \]  

Since \(G^0_0 = -3X, G^i_j = -(X + 2Y)\delta^i_j\), the matching condition \((4.3)\) becomes

\[ 3[\rho - V + 2m^2(X + 2Y)]A - (\rho + V - 6m^2X)N = 24M^3(X + Y) - 4\Lambda , \]  

\((4.14)\)
from which the quantity $N$ could also be found. Combining equations (4.9), (4.14) to eliminate $N$, we obtain the following algebraic equation for $A$

$$
\left(\rho + 3p - 2V + 12m^2 Y\right)A^2 - 4\left[6M^3(X + Y) - \Lambda\right]A - \left(\rho + V - 6m^2 X\right)\left(X + Y - \frac{\Lambda}{6M^3}\right) = 0. 
$$

(4.15)

Substituting $A$ from (4.13) in (4.15), we obtain the final Raychaudhuri equation for the brane cosmology

$$
\left(\frac{\rho + 3p - 2V}{4M^3} + \frac{3m^2}{M^3} Y\right)\left(X - \frac{C}{a^4} - \frac{\Lambda}{12M^3}\right) = \left(X + Y - \frac{\Lambda}{6M^3}\right)\left(\rho + V - 6m^2 X\right) = \frac{3m^2}{2M^3} X \pm 6\sqrt{X - \frac{C}{a^4} - \frac{\Lambda}{12M^3}}. 
$$

(4.16)

This equation contains one integration constant, and therefore is distinct from the corresponding equation of [22], where there is no integration constant present. In the following, we are going to ignore the four-dimensional Einstein term [24] and set $m = 0$, because in this case we were able to integrate equation (4.16). However, before doing so, we can study the limiting case where the 5-dimensional gravity vanishes and only the 4-dimensional one is present (this corresponds to the Regge-Teitelboim equation of motion [23], but with the difference that now the bulk is not prefixed). Therefore, setting $M = 0$ with $m \neq 0$, $\Lambda \neq 0$ in equation (4.16) we get

$$
\frac{\dot{H}}{n} + 2H^2 + \frac{k}{a^2} = \frac{\rho - 3p + 4V}{12m^2}. 
$$

(4.17)

For a single component perfect fluid with $p = w\rho$, its conservation equation (4.4) takes the standard form

$$
\dot{\rho} + 3nH(\rho + p) = 0 
$$

(4.18)

and (4.17) is integrated to

$$
H^2 + \frac{k}{a^2} = \frac{\rho}{6m^2} + \frac{\rho}{6m^2} - \frac{C_1}{a^4}, 
$$

(4.19)

where $C_1$ is integration constant. This is the standard FRW solution with cosmological constant, but with an extra dark radiation term.

From now on, we set $m = 0$ and equation (4.16) is written as

$$
\frac{\dot{H}}{n} + 2H^2 + \frac{k}{a^2} - \frac{\Lambda}{6M^3} = \frac{\rho + 3p - 2V}{4M^3} \left(H^2 + \frac{k}{a^2} - \frac{C}{a^4} - \frac{\Lambda}{12M^3}\right) \left[\frac{\rho + V}{4M^3} \pm 6\sqrt{H^2 + \frac{k}{a^2} - \frac{C}{a^4} - \frac{\Lambda}{12M^3}}\right]^{-1}. 
$$

(4.20)

It is seen from (4.20) that for $C = k = \rho = p = 0$, the lower branch contains as solution the Minkowski brane under the assumption of the Randall-Sundrum fine-tuning $\Lambda + V^2/(12M^3) = 0$ [20]. We will not assume this condition in our analysis, so in the absence of matter our cosmology may have a de-Sitter vacuum. It is assumed that the quantity inside the square root in equation (4.20) is positive.

Equation (4.20), although pretty complicated, it can be integrated. We consider a single component perfect fluid with $p = w\rho$ and its conservation equation (4.4) takes the standard form (4.18). We can show that for $\Lambda \neq 0$ the variable

$$
\Xi = \frac{1}{2} \ln \left[\frac{12M^3}{|\Lambda|}\left(H^2 + \frac{k}{a^2} - \frac{C}{a^4} - \frac{\Lambda}{12M^3}\right)\right] 
$$

(4.21)

obeys the differential equation

$$
\frac{d\Xi}{d\rho} = \frac{1}{3(1+w)(\rho - V)} \left[2 - \frac{(1+3w)\rho - 3(1+w)V}{\rho \pm 6e^{\Xi}}\right], 
$$

(4.22)

where

$$
\dot{\rho} = \sqrt{\frac{12M^3}{|\Lambda|}} \frac{\rho + V}{4M^3}, \quad \dot{V} = \frac{V}{\rho_*}, \quad \rho_* = 4M^3 \sqrt{\frac{|\Lambda|}{12M^3}}. 
$$

(4.23)

Note that the Randall-Sundrum fine-tuning corresponds to the value $V = 3$. Finally, changing to the variable

$$
\Phi = (\dot{\rho} \pm 6e^\Xi)^2, 
$$

(4.24)
we get, after some cancelations, a linear differential equation
\[ \frac{d\Phi}{d\bar{\rho}} - \frac{4}{3(1+w)(\bar{\rho} - \bar{V})}\Phi = 2\bar{\rho} \frac{1+3w}{3(1+w)} \bar{\rho} - 3(1+w) \bar{V}, \]
with general solution
\[ \Phi = \left( \frac{\rho}{\rho_*} + \bar{V} \right)^2 + c \left( \frac{\rho}{\rho_*} \right)^{\frac{w+1}{1-w}}, \]
where \( c \) is integration constant. For \( c < 0 \), due to that \( \Phi > 0 \), there are some restrictions on the allowed values of \( \rho \).

From the definition (4.24) we can find that
\[ \bar{\rho} = \frac{24M^3}{\rho_*} \sqrt{H^2 + \frac{k}{a^2} - \frac{C}{a^4} - \frac{\Lambda}{12M^3}} = \epsilon \sqrt{\Phi}. \]

In this equation the sign index \( \epsilon \) is \(+1\) or \( -1 \) has been used to denote a new different bifurcation from the previous \( \pm \) branches. It is seen from (4.27) that the sign \( \epsilon = -1 \) is only consistent with the lower \( \pm \) branch, while the sign \( \epsilon = +1 \) is consistent with both \( \pm \) branches. The distinction, however, introduced by the sign index \( \epsilon \) will be lost in the following expressions for the expansion rate and the acceleration parameter and only the sign \( \epsilon \) will distinguish the two branches of solutions.

The expansion rate of the new cosmology for \( \Lambda \neq 0 \) arises by squaring equation (4.27) and is given by
\[ H^2 + \frac{k}{a^2} - \frac{C}{a^4} = \left( \frac{\rho_*}{24M^3} \right)^2 \left\{ \left( \frac{\rho}{\rho_*} + \bar{V} - \epsilon \sqrt{\left( \frac{\rho}{\rho_*} + \bar{V} \right)^2 + c \left( \frac{\rho}{\rho_*} \right)^{\frac{w+1}{1-w}}} \right)^2 + 36 \text{sgn}(\Lambda) \right\}, \]
where \( \text{sgn}(\Lambda) \) is the sign of \( \Lambda \). The positivity of the quantity inside the square root of equation (4.20) is now always assured. The solution (4.28) contains two integrations constants. The first constant \( C \) is associated with the usual dark radiation term reflecting the non-vanishing bulk Weyl tensor. The second constant \( c \) is the new feature that does not appear in the cosmology of the standard matching conditions and signals new characteristics in the cosmic evolution. Setting \( c = 0 \) in the branch \( \epsilon = -1 \) we obtain the cosmology of the standard matching conditions. Of course, there is always the extra integration constant from the integration of the conservation equation (4.18) for \( \rho \) which is adjusted by the today matter content. We will see that \( c \) will be accounted for this today matter content (or equivalently the today dark energy content). The solution also contains three free parameters \( M, \Lambda, V \) or \( M, \rho_*, \bar{V} \).

For \( \Lambda = 0 \) the variable
\[ \tilde{\Xi} = \frac{1}{2} \ln \left( \frac{16M^6}{V^2} \left( H^2 + k \frac{V}{a^2} \right) \right) \]
obeyes the differential equation
\[ \frac{d\tilde{\Xi}}{d\bar{\rho}} = \frac{1}{3(1+w)(\bar{\rho} - 1)} \left[ 2 - \frac{(1+3w)(\bar{\rho} - 3(1+w))}{\bar{\rho} + 6\epsilon \tilde{\Xi}} \right], \]
where
\[ \bar{\rho} = \frac{\rho}{V} + 1. \]

Defining
\[ \bar{\Phi} = (\bar{\rho} + 6\epsilon \tilde{\Xi})^2, \]
we get the equation
\[ \frac{d\bar{\Phi}}{d\bar{\rho}} = \frac{4}{3(1+w)(\bar{\rho} - 1)} \bar{\Phi} = 2\bar{\rho} \frac{(1+3w)(\bar{\rho} - 3(1+w))}{3(1+w)(\bar{\rho} - 1)}, \]
with general solution
\[ \bar{\Phi} = \left( \frac{\rho}{V} + 1 \right)^2 + \bar{c} \left( \frac{\rho}{V} \right)^{\frac{w+1}{1-w}}, \]
where \( \bar{c} \) is integration constant. Finally, the expansion rate for \( \Lambda = 0 \) is
\[ H^2 + k \frac{V}{a^2} = \left( \frac{V}{24M^3} \right)^2 \left[ \left( \frac{\rho}{V} + 1 - \epsilon \sqrt{\left( \frac{\rho}{V} + 1 \right)^2 + \bar{c} \left( \frac{\rho}{V} \right)^{\frac{w+1}{1-w}}} \right)^2 \right]. \]
V. BULK SOLUTION

Equations (4.8)–(4.11) are not only valid on the brane, but also in the bulk. The 00 bulk equation (5.2) is written as

\[ A' + 2A^2 - X + \frac{\Lambda}{6M^3} = 0. \]  

(5.1)

From equation (4.9) we get \( H' = -HA, \) \( X' = -2XA \) and using (5.1) we obtain

\[ \left( A^2a^4 - Xa^4 + \frac{\Lambda}{12M^3}a^4 \right)' = 0. \]  

(5.2)

From (4.11), (5.2), we obtain the integral (4.12), but now valid everywhere in the bulk, with \( C \) integration constant. Finally, the spatial \( ij \) bulk equations (3.2) consist of only one equation

\[ 2A' + N' + 3A^2 + N^2 + 2AN - X - 2Y + \frac{\Lambda}{2M^3} = 0. \]  

(5.3)

From (4.8) we get \( \frac{1}{M}H' = H^2(2A - N) - AY, Y' = -(A + N)Y. \) Then, differentiating (4.9) with respect to \( y \) and using (5.1) we get an equation for \( N' \)

\[ AN' = (2A + N)\left( 2A^2 - X + \frac{\Lambda}{6M^3} \right) - 2AX - (A + N)Y. \]  

(5.4)

Substituting equations (5.1), (5.4) into (5.3) and using (4.9), equation (5.3) becomes identically satisfied.

Equation (4.8) is also written as

\[ \frac{\dot{a}(t, y)}{a(t)} = 0, \]  

(5.5)

which is integrated to

\[ \dot{a} = f(t) n, \]  

(5.6)

where \( f(t) \) is an arbitrary function of time. Adopting the cosmic time on the brane \( n(t, 0) = 1 \) we have \( f(t) = \dot{a}_0, \) where we adopt temporarily in this section the notation that \( a_0 \) denotes the brane scale factor. It is

\[ n(t, y) = \frac{\dot{a}(t, y)}{a_0(t)}. \]  

(5.7)

Using (5.7), equation (5.2) becomes

\[ (a^2)'' + \frac{\Lambda}{3M^3}a^2 = g(t), \]  

(5.8)

where \( g(t) = 2(\dot{a}_0^2 + k). \) The solution of (5.8) is

\[ a^2(t, y) = \begin{cases} A_1 \cosh (\mu y) + A_2 \sinh (\mu y) - 2\mu^{-2}a_0^2 X, & \Lambda < 0 \\ B_1 \cos (\mu y) + B_2 \sin (\mu y) + 2\mu^{-2}a_0^2 X, & \Lambda > 0 \end{cases}, \quad \mu = \sqrt{\frac{|\Lambda|}{3M^3}}, \]  

(5.9)

where \( X \) is given by the brane expansion rate (4.28), (4.35) and \( A_1, A_2, B_1, B_2, D_1, D_2 \) are functions of time \( t. \) This solution corresponds to the right hand side of the brane. Due to the \( Z_2 \) symmetry, the solution on the left hand side arises by the substitution \( A_2 \rightarrow -A_2, B_2 \rightarrow -B_2, D_2 \rightarrow -D_2. \) Solution (5.9) gives for \( y = 0 \) the coefficients \( A_1 = a_0^2(1 + 2\mu^{-2}X), B_1 = a_0^2(1 - 2\mu^{-2}X), D_1 = a_0. \) The remaining coefficients are found by differentiating (5.9) with respect to \( y \) at the brane position giving \( A_2 = 2\mu^{-1}a_0^2 A, D_2 = 2a_0^2 A, \) where the brane value \( A \) is given by (4.13) in terms of \( X. \) Finally, we write the bulk solution as

\[ \frac{a^2(t, y)}{a^2(t, 0)} = \begin{cases} (1 + 2\mu^{-2}X) \cosh (\mu y) + 2\mu^{-1}A \sinh (\mu y) - 2\mu^{-2}X, & \Lambda < 0 \\ (1 - 2\mu^{-2}X) \cos (\mu y) + 2\mu^{-1}A \sin (\mu y) + 2\mu^{-2}X, & \Lambda > 0 \end{cases}, \quad \mu = \sqrt{\frac{|\Lambda|}{3M^3}}, \]  

(5.10)

where \( X(t) \) and \( A(t) \) are given by (4.28), (4.35) and (4.13) respectively. We note that the cosmological solution (4.28) we have derived is quite different than the solution found in [22], where the domain wall was assumed to move in a 5-dimensional AdS space. This mismatch shows that the bulk space described by the solution (5.10) is not AdS, but reduces to AdS for the particular choice of the integration constant \( c = 0. \)
VI. INVESTIGATION OF THE COSMOLOGY

We will focus in the following in the case of a negative bulk cosmological constant, \( \Lambda < 0 \), which involves the situation with \( \Lambda \) being a small deformation from the Randall-Sundrum value. The scale factor for the branch \( \epsilon = -1 \) with \( V < 3 \) is bounded from above, and the same happens for the branch \( \epsilon = +1 \) with any value of \( V \). However, the branch \( \epsilon = -1 \) with \( V \geq 3 \) possesses the late-times asymptotic linearized regime with a positive effective cosmological constant

\[
H^2 + \frac{k}{a^2} - \frac{C}{a^4} \approx 2 \gamma \rho + \frac{\Lambda_{\text{eff}}}{3},
\]

where
\[
\gamma = \frac{4\pi G_N}{3} = \frac{V}{144M^5} \tag{6.2}
\]
\[
\Lambda_{\text{eff}} = 3 \left( \frac{\rho_0}{4M^3} \right)^2 \left( \frac{V^2}{9} - 1 \right) = \frac{1}{4M^3} \left( \Lambda + \frac{V^2}{12M^3} \right). \tag{6.3}
\]

The identification of Newton’s constant \( G_N \) with parameters of the theory reduces the number of free parameters from three to two.

At early times, the dominant behaviour for the branch \( \epsilon = -1 \) is \( H^2 + \frac{k}{a^2} - \frac{C}{a^4} \approx (\frac{\rho}{12M})^2 \), while for \( \epsilon = +1 \) it is \( H^2 + \frac{k}{a^2} - \frac{C}{a^4} \approx \text{constant} \) (which may help to inflation). In the temporal gauge choice of the brane \( n(t,0) = 1 \) (cosmic time on the brane), the acceleration-deceleration eras during the cosmic evolution are studied by evaluating the quantity \( \ddot{a}/a = \dot{H} + H^2 \). From equations (4.20), (4.27) we obtain

\[
\frac{\ddot{a}}{a} = \left( \frac{\rho + 3\rho - 2V}{\epsilon \rho \sqrt{\Phi}} - 1 \right) \left( \frac{\rho + V - \epsilon \rho \sqrt{\Phi}}{24M^3} \right)^2 + \frac{\Lambda}{12M^3} - \frac{C}{a^4}, \tag{6.4}
\]

where \( \Phi \) is given by (4.26). Note that for \( \epsilon = -1 \), the late-times behaviour \( \rho \to 0 \) of equation (6.4) is \( \ddot{a}/a \to (\rho_0/4M^3)^2 (V^2/9 - 1) \), which is positive for \( V > 3 \).

It is convenient for the investigation of the cosmological behaviour, instead of using the parameter \( \rho \), to express the expansion rate and the acceleration in terms of the parameter

\[
y = \frac{\rho}{\rho_0} = (1 + z)^{3(1+w)}, \tag{6.5}
\]

where a subscript \( o \) characterizes present values. Today epoch corresponds to \( y = 1 \) and past to \( y > 1 \). The redshift is denoted as usually by \( z \). Then, equations (4.28), (4.34) become

\[
H^2 + \frac{k}{a^2} - \frac{C}{a^4} = \left( \frac{\rho_0 V}{24M^3} \right)^2 \left[ \left( 1 + \frac{x_o}{V^2 - \epsilon \sqrt{\varphi}} \right)^2 - \frac{36}{V^2} \right], \tag{6.6}
\]
\[
\frac{\ddot{a}}{a} = \left( \frac{\rho_0 V}{24M^3} \right)^2 \left\{ \left[ \frac{(1+3w)(1 + \frac{x_o}{\epsilon \sqrt{\varphi}}) - 3(1+w)}{\epsilon \sqrt{\varphi}} - 1 \right] \left( 1 + \frac{x_o}{V^2 - \epsilon \sqrt{\varphi}} \right)^2 - \frac{36}{V^2} \right\} - \frac{C}{a^4}, \tag{6.7}
\]

where
\[
\varphi = \frac{\Phi}{V^2} = \left( 1 + \frac{x_o}{V} \right)^2 + \frac{\epsilon L_o^{4/3}}{V^2} \left( \frac{y}{x_o} \right)^{4(1+w)}, \quad x_o = \frac{\rho_o}{\rho_0}. \tag{6.8}
\]

For \( c < 0 \), due to that \( \varphi > 0 \), there are some restrictions on the allowed values of \( y \).

The age of the universe \( t_o \) can also be expressed in terms of the parameter \( y \). Considering that the main contribution in \( t_o \) comes from the dust era, we have the approximation

\[
t_o \approx \frac{1}{3} \int_{1}^{\infty} \frac{dy}{yH(y)}. \tag{6.9}
\]

We will restrict our attention to the study of the more reasonable branch \( \epsilon = -1 \) which possesses the late-times linearized LFRW regime. In the following we provide a more detailed analysis of the phenomenological consequences of this cosmology. The branch \( \epsilon = +1 \) is not necessarily precluded, but along with a proper dark energy/vacuum content [31], may meet recent observations of the early epoch or of late times.
A. Branch $\epsilon = -1$

Equation (6.1), beyond the linear term, possesses a series of extra terms which become significant away from the asymptotic regime. These extra terms consist the dark energy contribution, while the linear term consists the matter contribution observed today. Therefore, the existence of the linear term in the full expansion assures for $V \geq 3$ the relation (6.2) between the parameters of the theory and Newton’s constant. This relation reduces the number of free parameters by one and makes our analytical estimates relatively easier. In the following analysis we will consider this case $V \geq 3$.

Every contribution in the expansion rate (4.28) can be parameterized by normalized variables

$$\Omega_m + \Omega_{DE} + \Omega_k + \Omega_C = 1,$$

(6.10)

where the matter contribution is

$$\Omega_m = \frac{2\gamma \rho}{H^2},$$

(6.11)

the dark energy component is

$$\Omega_{DE} = \left(\frac{\rho_s}{24M^3}\right)^2 \frac{1}{H^2} \left\{ \frac{\rho}{\rho_s} + \tilde{V} + \sqrt{\left(\frac{\rho}{\rho_s} + \tilde{V}\right)^2 + c \left(\frac{\rho}{\rho_s}\right)^{\frac{4}{1+\omega}} } \right\}^2 - 36 - 8\tilde{V} \frac{\rho}{\rho_s},$$

(6.12)

while the topology contribution and the dark radiation portion are

$$\Omega_k = -\frac{k}{a^2 H^2} , \quad \Omega_C = \frac{C}{a^4 H^2}.$$

(6.13)

In order for the energy density of dark radiation $\rho_C = \frac{c}{2\gamma^a}$ not to violate the nucleosynthesis constraints it should be $-1.23 \leq \frac{\rho_C}{\rho_r} \leq 0.11$ [32], where $\rho_r$ is the energy density of the radiation component. Since $\rho_r$ is insignificant today, $\rho_C$ is also insignificant today. Therefore, the term $C/a^4$ can be neglected (at least for a relatively recent epoch).

Note also that

$$\frac{x_o}{\sqrt{V}} = \frac{\Omega_{m,o}}{2} \left(\frac{H_o}{12M^3 \gamma}\right)^2 \equiv \eta \approx 2 \times 10^{-30} [M(\text{TeV})]^{-6}.$$

(6.14)

Equation (6.13) becomes

$$\frac{1}{(6M^3 \gamma)^2} \left( H^2 + \frac{k}{a^2} - \frac{C}{a^4} \right) = (1 + \sqrt{\varphi + \eta y})^2 - \frac{36}{V^2},$$

(6.15)

where

$$\varphi = (1 + \eta y)^2 + \frac{c x^{4/3}_o}{V^2} \left(\frac{V}{x^{1/3}_o}\right)^{\frac{4}{1+\omega}}.$$

(6.16)

From equation (6.12) it is seen that the integration constant $c$ can be adjusted to get the measured dark energy component and this is the main difference of the current cosmological model compared to the cosmology of the standard matching conditions. Combining the present values of equations (6.11), (6.12) we get the equation

$$\tilde{V} + x_o + \sqrt{(\tilde{V} + x_o)^2 + c x^{4/3}_o} = 4(9 + 2\omega_o \tilde{V} x_o), \quad \omega_o = 1 + \frac{\Omega_{DE,o}}{\Omega_{m,o}},$$

(6.17)

which can be solved for $c$ in terms of the parameters

$$c x^{4/3}_o = 4 \sqrt{9 + 2\omega_o \tilde{V} x_o} \left[ \sqrt{9 + 2\omega_o \tilde{V} x_o} - (\tilde{V} + x_o) \right].$$

(6.18)

From equations (6.17), (6.18) it arises that the parameters have to satisfy the constraint $(\tilde{V} + x_o)^2 < 4(9 + 2\omega_o \tilde{V} x_o)$, or equivalently due to (6.14), $\tilde{V} < 6 + 6(4\omega_o - 1)\eta$. Therefore, $x_o \lesssim 6\eta$, i.e. $x_o \lesssim 10^{-29} [M(\text{TeV})]^{-6}$. Additionally, since $\tilde{V} x_o \lesssim 36\eta$, the quantity $c x^{4/3}_o$ in (6.18) can be approximated by

$$\frac{1}{12} c x^{4/3}_o \approx 3 - \tilde{V} + \left(\frac{\omega_o}{3}\tilde{V} - 1\right)\tilde{V} \eta,$$

(6.19)
and set into (6.16). Therefore, \( c \) is accounted for the today matter content \( \Omega_{m,o} \) and the today dark energy content \( \Omega_{DE,o} \) (which are approximately known).

We are now in position to capture the basic cosmological characteristics. **Dust era** \((w = 0)\): Equation (6.14), neglecting the dark radiation, becomes

\[
\frac{1}{(6M^3\gamma)^2} \frac{\ddot{a}}{a} = \frac{1}{\sqrt{\gamma}} (2 - \sqrt{\varphi} - \eta y) \left( 1 + \sqrt{\varphi} + \eta y \right)^2 - \frac{36}{V^2},
\]

(6.20)

where

\[
\varphi = (1 + \eta y)^2 + \frac{c_{x/o}^4/3}{V^2} y^{4/3}.
\]

(6.21)

We finally observe that equations (6.15), (6.20) contain only the two free parameters \( \tilde{V} \), \( M \) and the flatness value \( \Omega_{m,o} \).

To continue our analysis we discern the following possible cases:
- \( \tilde{V} = 6 + v \), where \( |v| \ll O(\sqrt{\eta}) \). This case means that \( \tilde{V} \) is between the values 6 and 6, but not extremely close to 3 or 6. Then, \( c_{x/o}^4/3 \approx -12(\tilde{V} - 3) \), \( \varphi_o \approx \left( \frac{b}{\sqrt{\gamma}} - 1 \right)^2 \). Therefore, \( \frac{1}{(6M^3\gamma)^2} \frac{\ddot{a}}{a} \approx \frac{144}{2} \frac{\tilde{V} - 3}{V^2} > 0 \). For \( \eta y < 1 \), i.e. \( z < 10^7(M(\text{TeV}))^{3/2} \), it is \( \varphi \approx 1 - 12V^2 \sim 1 - 12V^2(1 + z)^4 \) and the requirement \( \varphi > 0 \) means \( z < \left( \frac{V^2}{12} \right)^{1/4} - 1 \). Therefore, except if \( \tilde{V} \) is very close to the value 3, the values of \( z \) are restricted close to 0. Furthermore, \( \frac{1}{(6M^3\gamma)^2} \frac{\ddot{a}}{a} \approx \frac{2}{\sqrt{\gamma}} - \varphi - 1 + 4(1 - \frac{\eta}{\varphi}) \), which is permanently positive giving always acceleration. So, these values of \( \tilde{V} \) are not particularly interesting for the dust era of the universe.
- \( \tilde{V} = 6 + v \), where \( \frac{\tilde{V}}{V^2} \) is extremely close to the value 6 from above or from below. It is \( c_{x/o}^4/3 \approx -3 - v + 6(\omega_o - 1) \eta \), \( \varphi_o \approx 4\omega_o \eta + \frac{y^2}{\sqrt{\gamma}} \). Therefore, except if \( \omega_o \) do not have particular significance.
- \( \tilde{V} = 3 + v \), where \( v = \frac{3}{4} \vartheta \varphi_o \), \( \frac{1}{2} < \vartheta < 1 - \frac{2}{\varphi_o} \), \( \varphi_o \approx 2(\omega_o - 1) \). It is indeed \( 1 - \frac{2}{\varphi_o} > \frac{1}{2} \) if \( \Omega_{DE,o} > 2\Omega_{m,o} \). This case corresponds to an extreme fine-tuning close to the Randall-Sundrum value \( \tilde{V} = 3 \), which, however, possesses interesting phenomenological implications. For the typical values \( \Omega_{m,o} = 0.3 \), \( \Omega_{DE,o} = 0.7 \), we get \( 1/2 < \vartheta \approx 0.82 \). It is \( c_{x/o}^4/3 \approx 9 [1 - (1 - \vartheta) \omega_o - 2 \eta \varphi] \). Therefore, we get acceleration today. For \( y < 1 \), i.e. \( z < 10^7(M(\text{TeV}))^{3/2} \), it is \( \varphi \approx 1 + \eta y + 2(1 - \vartheta) \omega_o \eta > 0 \), therefore we get acceleration today.

Comparing equation (6.22) with the usual FRW equation \( 2\dot{H} + 3H^2 = -8\pi G_N(p + p_{DE}) - \frac{k}{a^2} \), we can define an
effective dark energy pressure $p_{DE}$ ($C/\alpha^4$ is ignored)

$$\frac{1}{6M^6\gamma}p_{DE} = \frac{1}{\sqrt{\varphi}}(\sqrt{\varphi} + 2\eta y - 4)(\sqrt{\varphi} + \eta y + 1)^2 + \frac{108}{\sqrt{2}}v$$,  

(6.23)

while from (6.12) and the usual FRW equation $H^2 + \frac{\dot{a}}{a} = \frac{8\pi G}{3}(\rho + \rho_{DE})$, we get the effective dark energy density $\rho_{DE}$

$$\frac{1}{18M^6\gamma}\rho_{DE} = (\sqrt{\varphi} + \eta y + 1)^2 - 8\eta y - \frac{36}{\sqrt{2}}v.$$  

(6.24)

Therefore, in the scenario at hand the dark energy equation of state $w_{DE} = \frac{p_{DE}}{\rho_{DE}}$ is given by

$$w_{DE} = \frac{1}{3\sqrt{\varphi}}\left(\frac{\sqrt{\varphi} + 2\eta y - 4}{\sqrt{\varphi} + \eta y + 1} \right)^2 - 8\eta y - \frac{36}{\sqrt{2}}v.$$  

(6.25)

For the fine-tuned case $\tilde{V} = 3 + v$ and $z << 10^7[M(\text{TeV})]^3/2$, we obtain the expression

$$w_{DE}(z) \approx \frac{1}{3}[(1 - \vartheta)\varphi_0 - 2](1 + z)^2 - 3\varphi_0.$$  

(6.26)

Then, the today value is $w_{DE,\vartheta} \approx \frac{1}{3} - \frac{4}{3}\vartheta(1 - \frac{2}{\varphi_0})^{-1} > -1$. For $\vartheta$ tending to the maximum value $1 - \frac{2}{\varphi_0}$, it is $w_{DE,\vartheta} \rightarrow -1$. E.g. for $\vartheta = 0.8$, $\Omega_{m,o} = 0.3$, it is $w_{DE,\vartheta} \approx -0.96$. The function $w_{DE}(z)$ for $\vartheta = 0.8$ is shown in Fig. 2 for various values of $\Omega_{m,o}$. In the far future where $z \rightarrow -1$, the function $w_{DE}(z)$ always approaches the value $-1$ which corresponds to the cosmological constant, as expected from equation (6.11). Note that the effective dark energy density and pressure are written as $\rho_{DE} \approx 36M^6\gamma\eta\{((1 - \vartheta)\varphi_0 - 2)y + 3\varphi_0\}$, $p_{DE} \approx 12M^6\gamma\eta\{((1 - \vartheta)\varphi_0 - 2)y^2/3 - 3\varphi_0\}$, and therefore, these components satisfy the weak and dominant energy conditions, while the strong energy condition is violated for $y^{1/3} < \varphi_0/((1 - \vartheta)\varphi_0 - 2)$.

Let us discuss in brief the situation of the standard matching conditions. Equations (6.15), (6.20) are replaced by

$$\frac{1}{(6M^6\gamma)^2}\left(H^2 + \frac{\dot{a}}{a} - \frac{C}{a^2}\right) = 4(1 + \eta y)^2 - 36v, \quad \frac{1}{(6M^6\gamma)^2}\frac{\ddot{a}}{a} = 4\left(1 - \frac{\vartheta}{v^2}\right) - 4\eta y(1 + 2\eta y)$$ which still contain the two parameters $\tilde{V}, M$ and $\Omega_{m,o}$. However, to incorporate the flatness values $\Omega_{m,o}, \Omega_{DE,o}$ in these standard equations, the relation $\frac{\Omega_{DE,o}}{\Omega_{m,o}} = \frac{1}{2\eta}(1 - \frac{\vartheta}{v^2} + \eta^2)$ must be satisfied, which means that one of the two parameters is constrained by $\tilde{V} \approx 3 + 3\frac{\Omega_{DE,o}}{\Omega_{m,o}}\eta$, and finally only one parameter remains free. This value of $\tilde{V}$ means that the standard matching conditions correspond to the limiting value $\vartheta = 1 - \frac{2}{\varphi_0}$. If $\eta y << 1$, we find for the standard equations

$$\frac{1}{(6M^6\gamma)^2}\left(H^2 + \frac{\dot{a}}{a} - \frac{C}{a^2}\right) \approx 8\left(\frac{\Omega_{DE,o}}{\Omega_{m,o}} + y\right)\eta, \quad \frac{1}{(6M^6\gamma)^2}\frac{\ddot{a}}{a} \approx 4\left(2\frac{\Omega_{DE,o}}{\Omega_{m,o}} - y\right)\eta.$$ These mean that the characteristic quadratic energy density term is insignificant at least for a recent era and this cosmology coincides with the $\Lambda CDM$ one in a recent era. Indeed, today it provides acceleration with a passage from deceleration to acceleration at $z_p \approx 0.67$ for $\Omega_{m,o} = 0.3$. 

![FIG. 1: Dimensionless age parameter $H_{,\vartheta}$ as a function of $\vartheta$ for some selected values of $\Omega_{m,o}$. The dotted line corresponds to $\Omega_{m,o} = 0.1$, the continuous to 0.3 and the dashed to 0.4](image1)

![FIG. 2: Dark energy equation of state $w_{DE}$ as a function of $z$ for $\vartheta = 0.8$ for some selected values of $\Omega_{m,o}$. The dotted line corresponds to $\Omega_{m,o} = 0.1$, the continuous to 0.3 and the dashed to 0.32](image2)
while the age of the universe (for \( k = C = 0, \Omega_{m,o} = 0.3 \)) is found to be \( t_o \approx \frac{1}{3H_o\sqrt{\Omega_{m,o}}} \int\left(\frac{\Omega_{DE}}{\Omega_{m,o}} + y\right)^{-1/2} \approx 13.5 \text{Gyr} \). From equation (6.25) setting \( \sqrt{\varphi} = 1 + \eta y \), or from equation (6.20) setting the above value of \( \vartheta \), it is found that \( w_{DE} = -1 \), i.e. the exact cosmological constant. In this standard picture there is no essential free parameter left due to the constraint from the observed Hubble constant \( H_o (\Omega_{m,o} \) is still as always free, but approximately known). As a result, the proposed cosmology, compared to the standard one, possesses more freedom in accommodating the observed characteristics of the universe, e.g. age of the universe, recent passage from deceleration to acceleration, time variability of the dark energy equation of state, etc.

**Radiation era** \((w = 1/3)\): Equation (6.7) becomes

\[
\frac{1}{(6M^3\gamma)^2} \frac{\ddot{a}}{a} = \frac{1}{\sqrt{\varphi}} (2 - \sqrt{\varphi} - 2\eta y) (1 + \sqrt{\varphi} + \eta y)^2 - \frac{36}{V^2} - \frac{1}{(6M^3\gamma)^2} \frac{C}{a^4},
\]

where the constant \( C \) now cannot be ignored at early times, and

\[
\varphi = (1 + \eta y)^2 + \frac{c_\varphi^{4/3}}{\eta^{1/3}V^{7/3}} y.
\]

We will discuss the fine-tuned case \( \dot{V} = 3 + v \), where it is important to note that the same parameters which gave the late-times evolution will be used also for the early-times behaviour. So, the value of \( c_\varphi \) found at late times will be used also here. This way, we will have a unified, all-times cosmology. For \( \eta y^{3/2} \gg 1 \), i.e. \( z \gg 10^3 [M(\text{TeV})] \), it is \( \varphi \approx (\eta y)^2 + \frac{1}{\sqrt{\varphi}} ((1 - \theta)\varphi - 2\eta y)^2 \). More precisely, for \( 10^5 [M(\text{TeV})] < z < 10^{10} [M(\text{TeV})]^2 \) it is \( \varphi \approx \frac{1}{\sqrt{\varphi}} ((1 - \theta)\varphi - 2\eta y)^2 \) and \( \frac{\ddot{a}}{a} \approx -6M^3\gamma \varphi - \frac{\ddot{a}}{a} \approx -\frac{1}{\sqrt{\varphi}} (6M^3\gamma)^2 ((1 - \theta)\varphi - 2\eta y)^2 z^4 - C z^4 \); for \( z \sim 10^{10} [M(\text{TeV})]^2 \) both terms are significant in (6.28) and \( \frac{\ddot{a}}{a} \approx -6M^3\gamma \varphi - \frac{\ddot{a}}{a} \approx -6M^3\gamma \varphi z^4 - C z^4 \). This last case is the most interesting one since it can encapsulate an inflationary accelerating period, where for temperature of inflation \( T_{inf} \approx 1 \text{TeV} \) it is \( z_{inf} \approx \frac{V}{\sqrt{\varphi}} \approx 10^{16} \) (these refer to the end of inflation), while for \( T_{inf} \approx M_P \) it is \( z_{inf} \approx 10^{29} \). It is obvious that initially the dominant power in the acceleration expression is \( z^4 \), which has negative sign, therefore the universe starts with deceleration. However, if \( C < 0 \), the universe necessarily enters an accelerating phase during the evolution. To make a more precise estimate, if \( z_{inf} \) denotes the entrance into the accelerating phase, it is \( |C| \approx 12(6M^3\gamma)^2 \varphi z_{inf} \approx 10^{-120} [M(\text{TeV})]^{-6} \varphi z_{inf} \). If the accelerated phase is to be interpreted as the inflationary era, the redshift \( z_{ent} \) should be several orders of magnitude larger than \( z_{inf} \). Since the maximum acceleration occurs at redshift \( z_{max} \) it is \( z_{ent} \approx 2 - 1/4 \), the value of this maximum acceleration is \( \frac{\ddot{a}}{a} \) and its ratio to the today acceleration is \( \frac{\ddot{a}}{a} = \frac{z_{max} - \frac{\ddot{a}}{a}}{z_{ent} - \frac{\ddot{a}}{a}} \), which means that the early acceleration is huge compared to the today tiny value. Although this is interesting, there is a caveat. Since \( H_o \approx 10^{-90} [\text{TeV}]^2 \), for \( z_{ent} \gg 10^{10} [M(\text{TeV})]^2 \), it arises \( |C| > H_o \). On the other hand, from equation (6.10) it is \( H^2 + \frac{\ddot{a}}{a} = \frac{3C}{8\pi G z^4} (\rho + \rho_{DE} + \rho_{c}) \), where the energy density of dark radiation is \( \rho_{c} = \frac{3C}{8\pi G z^4} \). In order not to violate the nucleosynthesis bounds it should be \( -1.23 < \frac{\ddot{a}}{a} \). However, this is not necessarily the case, as the energy density of the radiation component. Since \( \rho_{c} = \frac{3C}{8\pi G z^4} \), we get \( -1.23 < \frac{\ddot{a}}{a} \). Therefore, there is a contradiction. Although a sufficient negative value of \( C \) leads to a large early-times acceleration, in order for this phase to be interpreted as inflation the value of \( C \) should violate the nucleosynthesis constraints.

Other values of \( \dot{V} \), different than \( \dot{V} = 3 + v \), may also be interesting for providing a geometric origin inflationary period in the early cosmic era, but this investigation is beyond the scope of the present study. For example, assuming a varying tension in the early universe, it might be possible to unify a large value of \( \dot{V} \) in the inflationary period with the above fine-tuned value of \( \dot{V} \) which provides a solution to the dark energy problem. The physical motivation behind this dependence of the brane tension can be ideas inspired by the temperature dependence of the fluid membrane tension, cosmological phase transitions that modify brane tension, brane-bulk energy exchange, or particle creation on the brane. A hybrid type of inflation could be used with the varying tension being the “field” that produces the exponential expansion while an extra scalar field such as the one from the braneworld scenario. While an equation of the general form bulk gravity tensor is some smooth matter content or some matter

**VII. CONCLUSIONS**

In this paper, we continue the investigation of a recent proposal [20] on alternative matching conditions for self-gravitating branes. The bulk metric is assumed to be regular at the brane position, as e.g. happens in the braneworld scenario. While an equation of the general form bulk gravity tensor some smooth matter content or some matter
content of a “thick” brane is certainly correct, we claim that it cannot be correct in the shrink limit of distributional branes. A different treatment of the delta function characterizing the defect is needed for extracting its equation of motion. If this is so, the Israel matching conditions, as well as their generalizations where the bulk gravity tensor instead of Einstein is replaced by Lovelock extensions and the branes have differing codimensions, may be physically inadequate.

Our reasoning is based on two points: First, the incapability of the conventional matching conditions to accept the Nambu-Goto probe limit. Even the geodesic limit of the Israel matching conditions is not an acceptable probe limit since being the geodesic equation a kinematical fact it should be preserved independent of the gravitational theory or the codimension of the defect, which however is not the case for these matching conditions. Furthermore, even the non-geodesic probe limit of the standard equations of motion for various codimension defects in Lovelock gravity theories is not accepted, since this consists of higher order algebraic equations in the extrinsic curvature, therefore, a multiplicity of probe solutions arise instead of a unique equation of motion at the probe level. Second, in the \( D \)-dimensional spacetime we live (maybe \( D = 4 \)), classical defects of any possible codimension could in principle be constructed, and therefore, they should be compatible. The standard matching conditions fail to accept codimension-2 and 3 defects for \( D = 4 \) (which represents effectively the spacetime at certain length and energy scales) and most probably fail to accept high enough codimensional defects for any \( D \) since there is no corresponding high enough Lovelock density to support them.

According to our proposal the problem is not the distributional character of the defects, neither the gravitational theory used, but the equations of motion of the defects. The proposed matching conditions (“gravitating Nambu-Goto matching conditions”) may be close to the correct direction of finding realistic matching conditions since they always have the Nambu-Goto probe limit (independently of the gravity theory, the dimensionality of spacetime or codimensionality of the brane), and moreover, with these matching conditions, defects of any codimension seem to be consistent for any (second order) gravity theory. These alternative matching conditions arise by promoting the embedding fields of the defect to the fundamental entities. Instead of varying the brane-bulk action with respect to the bulk metric at the brane position and derive the standard matching conditions, we vary with respect to the brane embedding fields in a way that takes into account the gravitational back-reaction of the brane to the bulk. The proposed matching conditions generalize the standard matching conditions, and so, all the solutions of the bulk equations of motion plus the conventional matching conditions are still solutions of the current system of equations. Therefore, only interesting extensions are expected by using the proposed matching conditions.

In the present work we have considered in detail the case of a 3-brane in five-dimensional Einstein gravity and derived the generic alternative matching conditions. Of course, same or similar results are true for other codimension-1 defects in other spacetime dimensions. Since a 3-brane can represent our world in the braneworld scenario, we have investigated the cosmological equations and found the general solution for the cosmic evolution, as well as its bulk extension. One branch of the solution, that we have investigated further, possesses the asymptotic linearized LFRW regime. Compared to the conventional 5-dimensional braneworld cosmology, here, the Friedmann equation is much more complicated and has an extra integration constant.

Both in the standard and the alternative cosmologies, there are three parameters in the action: the higher dimensional mass scale, the bulk cosmological constant and the brane tension. Since both cosmologies possess the linearized asymptotic regime, the satisfaction of Newton’s constant constrains the parameters from three to two. The main difference between the two cosmologies is the satisfaction of the (approximately known) today flatness parameters. For the standard braneworld cosmology (which coincides with \( \Lambda \)CDM at least recently) this requirement is achieved in charge of one of these two parameters, and finally, only one parameter remains free (actually no essential free parameter remains due to the normalization to the present Hubble constant). On the contrary, in the proposed cosmology, the existence of the extra integration constant accounts for the today matter and dark energy contents, and finally, the two parameters remain free (again only one free parameter is essential due to the today Hubble value, and this parameter is denoted in the paper by \( \tilde{V} \) or \( \vartheta \)). Therefore, the new cosmology possesses an extra freedom for accommodating better the observed characteristics of the universe. We have found that for values of \( \tilde{V} \) extremely close to the Randall-Sundrum fine-tuning there is a small today acceleration with a recent passage from the long deceleration era to the present epoch. We have estimated the age of the universe which is consistent with current data, and calculated the time variability of the dark energy equation of state. For the same values of \( \tilde{V} \) a unified cosmology is defined for all times which possesses in the radiation regime a large acceleration (however, this cannot easily be interpreted as inflation since the nucleosynthesis bounds are violated). In general, depending on the parameters, a variety of behaviours can be exhibited which need further investigation and the model should be confronted against real data.

**Acknowledgements** We wish to thank E. Saridakis and J. Zanelli for useful discussions.
Appendix A: Geometric Components

For the metric \( \text{eq} \), the extrinsic curvature tensor defined everywhere in the bulk is \( K_{ij} (\chi, y) = \frac{1}{2} g_{ij}'(\chi, y) \) (a prime denotes \( \partial / \partial y \)). In the text we denote the two side value of \( K_{ij} \) at the brane position by \( K_{ij}^+ (\chi) \equiv K_{ij} (\chi, 0^+) \), \( K_{ij}^- (\chi) \equiv K_{ij} (\chi, 0^-) \). The non-vanishing components of the necessary geometric quantities are

\[
\Gamma^i_{ij} = -K_{ij} \quad \Gamma^j_{is} = K^j_i \quad \Gamma^i_{jk} = \frac{1}{2} g^{is} (g_{ij,k} + g_{ik,j} - g_{jk,i})
\]

where \( K^i_j = g^{ik} K_{kj} \)

\[
R_{i55} = -K'_{ij} + K_{ik} K^k_j \quad R_{5i5k} = K_{ijk} - K_{ikj} \quad R_{ik5\ell} = R_{ijkl} + K_{ik} K_{jk} - K_{ik} K_{jk}
\]

where \( | \) denotes the covariant derivative with respect to the metric \( g_{ij} \)

\[
R_{55} = -K' - K_{ij} K^{ij} \quad R_{i5} = K_{i5} - K_{ij} \quad R_{ij} = R_{ij} - K_{ij} + 2K_{ik} K^k_j - K K_{ij}
\]

where \( K = K^i_j \)

\[
R = R - 2K' - K_{ij} K^{ij} - K'^2
\]

\[
G_{55} = \frac{1}{2} K^2 - \frac{1}{2} K_{ij} K^{ij} - \frac{1}{2} R \quad G_{i5} = K_{i5} - K_{ij} \quad G_{ij} = G_{ij} - K_{ij} + 2K_{ik} K^k_j - K K_{ij} + \left( K' + \frac{1}{2} K_{ik} K^{k\ell} + \frac{1}{2} K'^2 \right) g_{ij}
\]
[25] R. Capovilla and J. Guven, Phys. Rev. D 57 (1998) 5158 [math-ph/9804002].
[26] B. Carter, Int. J. Theor. Phys. 40 (2001) 2099; A. Battye and B. Carter, Phys. Lett. B 509 (2001) 331.
[27] S. Deser, F.A.E. Pirani and D.C. Robinson, Phys. Rev. D 14 (1976) 3301.
[28] G. Kofinas, JHEP 0108 (2001) 034 [hep-th/0108013].
[29] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 [hep-ph/9905221]; L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690 [hep-th/9906064].
[30] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477 (2000) 285 [hep-th/9910219].
[31] S. Kachru and R. Kallosh and A.D. Linde and S.P. Trivedi, Phys. Rev. D 68 (2003) 046005 [hep-th/0301240]; V. Sahni and A.A. Starobinsky, Int. J. Mod. Phys. D 9 (2000) 373 [astro-ph/9904308]; S.M. Carroll, Living Rev. Rel. 4 (2001) 1 [astro-ph/0004075]; P.J.E. Peebles and B. Ratra, Rev. Mod. Phys. 75 (2003) 559 [astro-ph/0207347]; T. Padmanabhan, Phys. Rept. 380 (2003) 235 [astro-ph/0212290]; L. Amendola and S. Tsujikawa, Dark energy: Theory and Observations, Cambridge University Press (2010).
[32] K. Ichiki, M. Yahiro, T. Kajino, M. Orito and G.J. Mathews, Phys. Rev. D 66 (2002) 043521 [astro-ph/0203272].
[33] A.G. Riess et al., Astron. J. 116 (1998) 1009 [astro-ph/9805201]; S. Perlmutter et al., Astrophys. J. 517 (1999) 565 [astro-ph/9812133].
[34] D.N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148 (2003) 175 [astro-ph/0302209]; P.A.R. Ade et al. [Planck Collaboration], astro-ph/1303.5076.
[35] D.I. Eisenstein et al. [SDSS Collaboration], Astrophys. J. 633 (2005) 560 [astro-ph/0501171].
[36] L.A. Gergely, Phys. Rev. D 78 (2008) 084006 [gr-qc/0806.3857]; M.C.B. Abdalla, J.M. Hoff da Silva and R. da Rocha, Phys. Rev. D 80 (2009) 046003 [hep-th/0907.1321]; K.C. Wong, K.S. Cheng and T. Harko, Eur. Phys. J. C 68 (2010) 241 [gr-qc/1005.3101].