On the Distribution of Witnesses in the Miller-Rabin Test

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Abstract: We show that the set of normalized Miller-Rabin witnesses becomes equidistributed in the unit interval. This will be done by exhibiting cancellation in certain exponential sums.

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1 Introduction and Notation

For convenience, the following notation will be put to use. For a set $S$, $\#S$ will denote the number of elements of $S$. The greatest common divisor of two integers $a$ and $b$ will be represented as $(a,b)$. We will denote the group of units modulo $n$ as $(\mathbb{Z}/n\mathbb{Z})^*$. A function $f(n)$ is said to be $o(g(n))$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$. Likewise, a function $f(n)$ is said to be $O(g(n))$ if $|f(n)| \leq c|g(n)|$ for some constant $c$. The function $e(x)$ is an exponential function to be defined as $e(x) = e^{2\pi ix}$.

Definition 1. Let $n$ be an odd integer and write $n - 1 = d2^s$ with $d$ odd. Then an integer $a$, $1 < a < n$, is a Miller-Rabin witness (for the compositeness of $n$) if the following conditions hold:

1. $(a,n) = 1$
2. $a^d \not\equiv 1 \mod n$
3. For all integers $j$ with $0 \leq j < s$, $a^{d2^j} \not\equiv -1 \mod n$

Given $n$, let $W(n)$ denote the set of such witnesses. There are two theorems of note proved by Miller and Rabin respectively.

Theorem [Miller] [3]

Let $n$ be odd and composite. Assuming the Generalized Riemann Hypothesis, then

$$\min W(n) = O(\log(n)^2)$$

The specific constant was later proved to be 2 by Erich Bach [1], so the least witness would be no larger than $2\log n^2$, assuming GRH.

Theorem [Rabin] [5]

Let $n$ be odd and composite, then $\#W(n)$ obeys the following bound

$$\#W(n) > \frac{3(n-1)}{4}$$

This theorem allowed Rabin to alter the deterministic version into a probabilistic version of the test, the Miller-Rabin primality test. The purpose of this paper is to study the (normalized) distribution of $W(n)$ in $\mathbb{Z}/n\mathbb{Z}$. The main result being the following.

Theorem 1. As $n \to \infty$ along odd composite numbers, then the normalized witness set becomes equidistributed in the unit interval.

By this we mean: $\forall[a,b] \subset [0,1]$

$$\frac{\# \{ \frac{W(n)}{n} \cap [a,b] \} }{\#W(n)} \to b - a \text{, as } n \to \infty$$

To illustrate this, see Figure 1. The proof is elementary, the main ingredient being reduction of certain exponential sums into Gauss sums.

2 Proof of Theorem 1

Proof. By Rabin’s bound for $\#W(n)$ and Weyl’s Criterion, it suffices to show that for fixed $k \neq 0$, $k \in \mathbb{Z}$ that

$$(*) \quad S = \sum_{w\in W(n)} e\left(\frac{kw}{n}\right) = o(n)$$
Let $W(n) = \{0, 1, 2, ..., n-1\} \setminus W(n)$ be the set of non-witnesses, and define

$$S = \sum_{w \in W(n)} e\left(\frac{kw}{n}\right)$$

so that $S + S = 0$. Then since $n \nmid k$, for $k \neq 0$, $(\star)$ is equivalent to

$$(\dagger) \quad |S| = o(n)$$

$W(n)$ can be partitioned based on its membership conditions as follows:

$W_1(n) = \{w \in W(n) | (w, n) > 1\}$.

$W_2(n) = \{w \in W(n) | w^d \equiv 1 \mod n \}$.

$W_3(n) = \{w \in W(n) | \exists j \leq s, w^{2^jd} \equiv -1 \mod n\}$.

Hence, $W(n) = W_1(n) \sqcup W_2(n) \sqcup W_3(n)$ and $(\dagger)$ follows from showing

$$S_j = \sum_{w \in W_j(n)} e\left(\frac{kw}{n}\right) = o(n) \quad j = 1, 2, 3$$

2.1 Estimation of $S_1$

Lemma 2. For each fixed $k \neq 0$, $|S_1| = O_k(1)$ as $n \to \infty$.

Proof. For $S_1$ we can represent the sum over those $w$ as

$$\sum_{(w, n) > 1} e\left(\frac{kw}{n}\right) = -\sum_{(w, n) = 1} e\left(\frac{kw}{n}\right)$$

Then upon a Möbius inversion of this sum we arrive at:

$$\sum_{(w, n) = 1} e\left(\frac{kw}{n}\right) = \sum_{s | (n, k)} s\mu\left(\frac{n}{s}\right)$$
As this sum is just a divisor sum, and that \((n, k) \leq k\), we find that for any \(\epsilon > 0\),
\[
|S_1| = \left| \sum_{s \mid (n, k)} s\mu\left(\frac{n}{s}\right) \right| \leq \left| \sum_{s \mid (n, k)} s \right| < k^{1+\epsilon}
\]
Thus \(S_1\) is of order \(O_k(1)\) as \(n \to \infty\).

\[
\square
\]

### 2.2 Cancellation Lemma

The following lemma will be applied to the estimation of \(S_2\) and \(S_3\).

**Lemma 3.** Let \(\alpha, n \in \mathbb{N}\) and let \(b\) be an element in \((\mathbb{Z}/n\mathbb{Z})^*\). Let \(W' = \{w \in (\mathbb{Z}/n\mathbb{Z})^* \mid \omega^{\alpha} \equiv b \mod n\}\). Fix \(k \neq 0 \in \mathbb{Z}\), and define the sum
\[
S' = \sum_{w \in W'} e\left(k\frac{w}{n}\right)
\]
then \(|S'| = O_k(\sqrt{n})\) as \(n \to \infty\).

**Proof.** Let \(b\) be the inverse of \(b \in (\mathbb{Z}/n\mathbb{Z})^*\) and consider \(\frac{1}{\phi(n)} \sum_{\chi} \chi(b\omega^\alpha)\), a sum over Dirichlet characters modulo \(n\), note that
\[
\frac{1}{\phi(n)} \sum_{\chi} \chi(b\omega^\alpha) = \begin{cases} 
1 & w \in W' \\
0 & \text{otherwise}
\end{cases}
\]
Insert this into the sum and interchange the order of summation to obtain
\[
S' = \left| \sum_{\omega^\alpha \equiv b \mod n} \sum_{w \mod n} e\left(k\frac{w}{n}\right) \frac{1}{\phi(n)} \sum_{\chi} \chi(b\omega^\alpha) \right| = 
\]
\[
\left| \frac{1}{\phi(n)} \sum_{\chi} \chi(b) \sum_{w \mod n} e\left(k\frac{w}{n}\right) \chi^\alpha(w) \right| \leq \frac{1}{\phi(n)} \sum_{\chi} \left| \sum_{w \mod n} \chi^\alpha(w)e\left(k\frac{w}{n}\right) \right|
\]
For each \(\chi, \chi^\alpha\) could be the trivial character mod \(n\) or not. If it is the trivial character then the inside sum breaks down into a Ramanujan sum, and is estimated as in Lemma 2. If it is nontrivial, note that \(\sum_{w \mod n} \chi^\alpha(w)e\left(k\frac{w}{n}\right)\) is a type of Gauss sum, and it is a known fact [2] that for primitive characters
\[
\left| \sum_{w \mod n} \chi^\alpha(w)e\left(k\frac{w}{n}\right) \right| \leq \sqrt{n}
\]
If \(\chi^\alpha\) is imprimitive with conductor \(q\), writing \(n = ql\), there are two cases to be handled. If \(l \nmid k\), then the sum is zero. If \(l \mid k\) then
\[
\left| \sum_{w \mod n} \chi^\alpha(w)e\left(k\frac{w}{n}\right) \right| \leq l\sqrt{q}
\]
We have that \(\chi^\alpha\) is induced by a character \(\chi_1\) which is primitive modulo \(q\). Upon writing \(w = qj + r\) with \(r \mod q\) and \(j \mod l\) we have that
\[
\sum_{w \mod n} \chi^\alpha(w)e\left(k\frac{w}{n}\right) = \sum_{j \mod l} \sum_{r \mod q} \chi_1(qj + r)e\left(k\frac{qj + r}{ql}\right)
\]
Denote the inside sum by

\[ S'' = \sum_{r \mod q} \chi_1(qj + r)e(k\frac{qj + r}{ql}) \]

To handle the first case, that \( l \nmid k \), multiply \( S'' \) by \( e\left(\frac{k}{l}\right) \) and note that \( e\left(\frac{k}{l}\right)S'' = S'' = 0 \). For the second case, we have \( k = k'l \), and

\[ S'' = \sum_{r \mod q} \chi_1(qj + r)e\left(k\frac{r}{l}\right) = \sum_{r \mod q} \chi_1(r)e\left(k'\frac{r}{q}\right) \]

Now \( S'' \) can be handled as above (as \( \chi_1 \) is primitive modulo \( q \)), and it is summed exactly \( l \) times.

As we have \( l \mid k \) and \( q \mid n \), we have \( l \leq k \) and \( q \leq n \), so \( \sum_{w \mod n} \chi^o(w)e(k\frac{w}{n}) = O_k(\sqrt{n}) \).

It follows that \( |S''| = O_k(\sqrt{n}) \) as \( n \to \infty \).

\[ \square \]

### 2.3 Estimation of \( S_2 \) and \( S_3 \)

**Lemma 4.** For each fixed \( k \neq 0 \), \( |S_2| = O_k(\sqrt{n}) \) as \( n \to \infty \).

**Proof.** \( S_2 \) is a sum over the set \( \overline{W}_2(n) = \{ w \in \overline{W}(n) | w^d \equiv 1 \mod n \} \). As \( \overline{W}_2(n) \) is of the type \( W' \), then we can apply Lemma 3. \[ \square \]

**Lemma 5.** For each fixed \( k \neq 0 \), \( |S_3| = O_k(\sqrt{n} \log n) \) as \( n \to \infty \).

**Proof.** \( S_3 \) is the sum over the set \( \overline{W}_3(n) = \{ w \in \overline{W}(n) | \exists j < s, w^2j^d \equiv -1 \mod n \} \). We can write \( S_3 \) as:

\[ S_3 = \sum_{j=0}^{s-1} \sum_{w^{2jd} \equiv -1 \mod n} e\left(k\frac{w}{n}\right) \]

Applying the cancellation lemma to the innermost sum and then bounding the outside sum by \( \log n \) yields that \( |S_3| = O_k(\sqrt{n} \log n) \) as \( n \to \infty \).

\[ \square \]

### 2.4 Proof of Theorem 1

As previously stated, the statement (⋆) is equivalent to

\[ |S| = \left| \sum_{w \in \overline{W}(n)} e\left(k\frac{w}{n}\right) \right| \leq |S_1| + |S_2| + |S_3| = o(n) \]

(with each sum shown individually to be \( o(n) \) in Lemmas 2, 4, and 5 respectively). The main result then follows from Weyl’s criterion.

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