Cohomology of Lie superalgebras
and of their generalizations

M. Scheunert
Physikalisches Institut der Universität Bonn
Nußallee 12, D–53115 Bonn, Germany

R.B. Zhang
Department of Pure Mathematics
University of Adelaide
Adelaide, Australia

Abstract

The cohomology groups of Lie superalgebras and, more generally, of \( \varepsilon \) Lie algebras, are introduced and investigated. The main emphasis is on the case where the module of coefficients is non–trivial. Two general propositions are proved, which help to calculate the cohomology groups. Several examples are included to show the peculiarities of the super case. For \( L = sl(1|2) \), the cohomology groups \( H^1(L, V) \) and \( H^2(L, V) \), with \( V \) a finite–dimensional simple graded \( L \)-module, are determined, and the result is used to show that \( H^2(L, U(L)) \) (with \( U(L) \) the enveloping algebra of \( L \)) is trivial. This implies that the superalgebra \( U(L) \) does not admit of any non–trivial formal deformations (in the sense of Gerstenhaber). Garland’s theory of universal central extensions of Lie algebras is generalized to the case of \( \varepsilon \) Lie algebras.
I. Introduction

Cohomology is an important tool in mathematics. Its range of applications contains algebra and topology as well as the theory of smooth manifolds or of holomorphic functions. The cohomology theory of Lie algebras (whose generalization is the main objective of the present paper) has its origins in the work by E. Cartan, but the foundation of the theory, as an independent topic of research, is due to Chevalley, Eilenberg [1], to Koszul [2], and to Hochschild, Serre [3]. A unifying treatment of the cohomology theory of groups, associative algebras, and Lie algebras has been given by Cartan, Eilenberg [4].

As is well–known, several classical results in Lie algebra theory have a cohomological interpretation. Thus, if $L$ is a Lie algebra, the structure of the extensions of $L$–modules (and hence the question of semi–simplicity of $L$–modules) is described by the 1–cohomology of $L$, and the structure of the Lie algebra extensions (and hence the Levi, Malcev theorem) is related to the 2–cohomology [1, 5].

In view of these remarks, it is hardly surprising that shortly after the birth of supersymmetry, a paper appeared [6] in which the most basic constructions and results of the classical theory are generalized to the case of Lie superalgebras [7, 8]. Later on, Leites and Fuks calculated the cohomology groups of the classical Lie superalgebras with trivial coefficients [9] (see also Ref. [10]), and Leites and V. Serganova used cohomological methods to determine systems of generators and relations for these algebras [11]. In addition, we should mention a letter by Retakh and Feigin [12] and the papers by Tripathy and coworkers (see Ref. [13] and the papers cited therein). Apart from this, very little seems to be known about the cohomology of Lie superalgebras, in particular, about the cohomology with non–trivial coefficients.

After the present work was put into the internet and submitted for publication, C. Gruson sent us a copy of the galley proofs of an article [14], in which she proves the finiteness of the homology of certain modules over a Lie superalgebra. She also drew our attention to a paper by J. Tanaka [15], in which the author calculates the dimensions of the homology and cohomology groups of $sl(2|1)$ with coefficients in an arbitrary finite–dimensional simple module. This has a direct bearing on part of our work, and we shall comment on it later on.

Our present interest in this topic has been stimulated by the theory of quantum algebras [16, 17]. It is known that, for any semi–simple Lie algebra $L$, the enveloping algebra $U(L)$ does not admit of any non–trivial formal deformations in the sense of Gerstenhaber [18]. We would like to know for which of the simple Lie superalgebras the same holds true. Proceeding as in Ref. [19], this amounts to showing that the second cohomology group $H^2(L, U(L))$ is trivial. Unfortunately, we are far from proving a general result of this type. In fact, all we can say at present is that this is true if $L$ is one of the algebras $osp(1|2n)$ (this is easy to see and, certainly, was already known to Kac [4]), or if $L$ is equal to $sl(1|2)$ (this will be proved in
the present work). Moreover, in the meantime we have proved that $H^2(L, U(L))$ is also trivial for the algebras $sl(m|1)$ with $m \geq 3$. This will be published elsewhere. Accordingly, our paper is going to be explorative in character.

Let us briefly describe the setup of the present article. In Sec. II we collect the basic definitions and constructions, which will be needed in the subsequent sections. Using the graded multilinear algebra as described in Ref. [20] this is just a simple transcription of the classical situation to the graded case. In doing so, we shall follow the approach of Ref. [3] (see also Ref. [1]). In addition, we comment on some peculiarities of the graded case, and we prove two propositions, which will be the main tools in our calculations. The first one shows that it is sufficient to investigate cochains and coboundaries with certain invariance properties, the second proposition gives a general criterion for the cohomology with values in a given module to be trivial.

Sec. III contains several examples. In the first example we consider the cohomology of semi–simple Lie algebras. This example is included mainly in order to show how far we can get, under the most favourable circumstances, by only using the results of Sec. II. A short look at this example reveals that the same arguments can be applied to the Lie superalgebras $osp(1|2n)$. This will be the content of the second example. The third example contains some special information on the cohomology with trivial coefficients (again well–known in the case of semi–simple Lie algebras). In the fourth example we use the second proposition of Sec. II to determine, for the general linear Lie superalgebras $gl(m|n)$, which of the finite–dimensional simple graded $gl(m|n)$–modules might have a non–trivial cohomology. This discussion also covers the case of the special linear Lie superalgebras $sl(m|n)$, provided that $m \neq n$.

In Sec. IV we generalize Garland’s theory of universal central coverings (i.e., extensions) of Lie algebras [21] to the general graded setting. As a simple example, we determine the universal covering of the simple Lie superalgebras $sl(n|n)/\text{center}$, with $n \geq 2$.

In Sec. V we consider the Lie superalgebra $L = sl(1|2)$ and determine the finite–dimensional simple graded $L$–modules $V$ for which $H^1(L, V)$ or $H^2(L, V)$ is non–trivial. The result is then used to show that $H^2(L, U(L))$ is trivial. Of course, the modules $V$ in question can be read off from Ref. [15]. Nevertheless, we think it is worthwhile to present our own calculations. First of all, this makes our paper self–contained in this respect. Secondly, since we only determine $H^1(L, V)$ and $H^2(L, V)$, our calculations are much simpler and the results much more explicit than those of Ref. [15], where $\dim H^n(L, V)$ is determined for arbitrary $n$. Finally, this part of our paper is meant to explain how one might investigate $H^2(L, U(L))$ for more general Lie superalgebras $L$.

Sec. VI contains a brief discussion of our results. The paper is closed by two appendices. In App. A we describe the analogue of the exterior (i.e., Grassmann) algebra in the general graded setting. App. B contains some special information
related to Sec. V. This will be used to show that, for $L = sl(1|2)$, the cohomology group $H^1(L, U(L))$ is non–trivial.

We close this introduction by explaining some of our conventions. The base field will always be a field $K$ of characteristic zero. We use the standard notation $\mathbb{Z}$ for the ring of integers and $\mathbb{N}$ for the set of natural numbers (according to our convention, $\mathbb{N}$ contains 0). The set of strictly positive integers will be denoted by $\mathbb{N}_*$. As explained above, in Sec. II and Sec. IV as well as in App. A we are going to consider the so–called $\varepsilon$ Lie algebras [22] (also called colour Lie algebras [23]). In these sections, $\Gamma$ is an arbitrary Abelian group and $\varepsilon$ is a commutation factor on $\Gamma$ with values in $K$. The reader who is only interested in Lie superalgebras may simply set $\Gamma = \mathbb{Z}_2$ or $\Gamma = \mathbb{Z}$ and choose $\varepsilon$ to be the standard commutation factor of supersymmetry, defined by $\varepsilon(\alpha, \beta) = (-1)^{\alpha \beta}$ for all $\alpha, \beta \in \Gamma$. The choice $\Gamma = \mathbb{Z}_2$ leads to general Lie superalgebras, for $\Gamma = \mathbb{Z}$, we obtain the consistently $\mathbb{Z}$–graded Lie superalgebras in the sense of Ref. [7]. The multiplication in an $\varepsilon$ Lie algebra (and hence in a Lie superalgebra) will be denoted by a pointed bracket $\langle , \rangle$.

II. Cohomology of $\varepsilon$ Lie algebras:
Definitions and some basic results

Let us first introduce the notation which will be used throughout the present section. Quite generally, we adopt the conventions of Ref. [20]. In the following, $\Gamma$ denotes an Abelian group, $\varepsilon : \Gamma \times \Gamma \to K$ a commutation factor on $\Gamma$ with values in $K$, $L$ an $\varepsilon$ Lie algebra [22], and $V$ a graded $L$–module. All gradations are understood to be $\Gamma$–gradations. If an element of a graded vector space is homogeneous of a certain degree, then this degree will be denoted by the lower case “Greek analogue” of the letter denoting the element itself. For example, if $A, b', g, x$ are homogeneous elements, their degrees will be denoted by $\alpha, \beta', \gamma, \xi \in \Gamma$, respectively. Only occasionally (but in all cases where there could be any doubts) the degrees will be specified explicitly.

Let $n \geq 1$ be a natural number and let

$$Lgr_n(L, V) = Lgr_n(L, \ldots, L; V)$$

(2.1)

be the graded vector space of all $n$–linear mappings of $L^n = L \times \ldots \times L$ into $V$ which can be written as a sum of homogeneous $n$–linear mappings of $L^n$ into $V$. Recall that $Lgr_n(L, V)$ is equal to the space of all $n$–linear mappings of $L^n$ into $V$ if (for example) $\Gamma$ is finite or if $L$ is finite–dimensional.

According to Ref. [20] $Lgr_n(L, V)$ is a graded $L$–module: If $A \in L$ and $g \in Lgr_n(L, V)$ are homogeneous, the action of $A$ on $g$ is defined by
\[(A_L(g))(A_1, \ldots, A_n) = A_V(g(A_1, \ldots, A_n)) \quad (2.2)\]

\[- \sum_{r=1}^n \varepsilon(\alpha + \alpha_1 + \ldots + \alpha_{r-1}) g(A_1, \ldots, (A, A_r), \ldots, A_n),\]

for all homogeneous elements \(A_1, \ldots, A_n \in L\). (Recall that the \(\varepsilon\)-commutator of two elements \(A, B \in L\) is denoted by \(\langle A, B \rangle\). Quite generally, if \(V\) is a graded \(L\)-module, the representative of an element \(A \in L\) in \(V\) will be denoted by \(A_V\). If \(x\) is an element of \(V\), we shall frequently simplify the notation and write \(A \cdot x\) in place of \(A_V(x)\).)

On the other hand, again according to Ref. [20], there is a natural representation of the symmetric group \(S_n\) in \(\text{Lgr}_n(L, V)\). To describe it, we define a map \(\varepsilon_n : S_n \times \Gamma^n \to \mathbb{K}\) by

\[\varepsilon_n(\pi^{-1}; \gamma_1, \ldots, \gamma_n) = \prod_{i<j, \pi(i) > \pi(j)} \varepsilon(\gamma_i, \gamma_j) \quad (2.4)\]

for all \(\pi \in S_n\) and \(\gamma_1, \ldots, \gamma_n \in \Gamma\). The basic property of this map is that

\[\varepsilon_n(\pi \tau; \gamma_1, \ldots, \gamma_n) = \varepsilon_n(\pi; \gamma_1, \ldots, \gamma_n) \varepsilon_n(\tau; \gamma_{\pi(1)}, \ldots, \gamma_{\pi(n)}) \quad (2.5)\]

for all \(\pi, \tau \in S_n\) and all \(\gamma_1, \ldots, \gamma_n \in \Gamma\). Then the representation \(\pi \to \tilde{S}_\pi\) of \(S_n\) in \(\text{Lgr}_n(L, V)\) is given by

\[(\tilde{S}_\pi(g))(A_1, \ldots, A_n) = \varepsilon_n(\pi; \alpha_1, \ldots, \alpha_n) g(A_{\pi(1)}, \ldots, A_{\pi(n)}) \quad (2.6)\]

for all \(\pi \in S_n\), \(g \in \text{Lgr}_n(L, V)\), and all homogeneous elements \(A_1, \ldots, A_n \in L\). It is known that \(\tilde{S}_\pi\) is an automorphism of the graded \(L\)-module \(\text{Lgr}_n(L, V)\), for all \(\pi \in S_n\).

Now let \(C^n(L, V)\) be the set of all \(\varepsilon\)-skew–symmetric elements of \(\text{Lgr}_n(L, V)\), i.e., the set of all elements \(g \in \text{Lgr}_n(L, V)\) such that

\[\tilde{S}_\pi g = \text{sgn}(\pi) g \quad (2.7)\]

for all \(\pi \in S_n\), where \(\text{sgn}(\pi)\) denotes the signum of the permutation \(\pi\). We extend this definition to the case of integers \(n \leq 0\) and set

\[C^n(L, V) = \{0\} \quad \text{if} \quad n \leq -1 \quad (2.8)\]

\[C^0(L, V) = V. \quad (2.9)\]

Occasionally, it is useful to identify \(C^0(L, V)\) with \(\text{Lgr}(\mathbb{K}, V)\). Then \(C^n(L, V)\) can be identified with \(\text{Lgr}(\text{A}_n(L, V))\), where \(\text{A}_n(L)\) is the subspace of degree \(n\) of the \(\varepsilon\)-exterior
algebra $\wedge L$ of $L$. A few comments on the $\varepsilon$–exterior algebra of a $\Gamma$–graded vector space are contained in App. A. Obviously, $C^n(L, V)$ is a graded $L$–module, for all $n$. The elements of $C^n(L, V)$ are called the $n$–cochains of $L$ with values in $V$.

In the subsequent discussion we follow Ref. [3]. First we define, for all integers $n$, a map

$$ C^n(L, V) \times L \longrightarrow C^{n-1}(L, V), $$

(2.10)
denoted by

$$(g, A) \longrightarrow g_A,$$

(2.11)

by

$$ g_A = 0 \quad \text{if} \quad n \leq 0, $$

(2.12)

$$ g_A = g(A) \quad \text{if} \quad n = 1, $$

(2.13)

$$ g_A(A_2, \ldots, A_n) = g(A, A_2, \ldots, A_n) \quad \text{if} \quad n \geq 2, $$

(2.14)

where $g \in C^n(L, V)$ and $A, A_2, \ldots, A_n \in L$. It is easy to see that this map is bilinear, homogeneous of degree zero, and $L$–invariant, i.e.,

$$ B \cdot g_A = (B \cdot g)_A + \varepsilon(\beta, \gamma) g_{(B, A)} $$

(2.15)

for all homogeneous elements $A, B \in L$ and $g \in C^n(L, V)$.

Next we define, again for all integers $n$, the linear coboundary operator

$$ \delta^n : C^n(L, V) \longrightarrow C^{n+1}(L, V) $$

(2.16)
as follows: We set

$$ \delta^n = 0 \quad \text{if} \quad n \leq -1 $$

(2.17)

and define $\delta^n$ for $n \geq 0$ inductively by

$$ (\delta^n(g))_A = \varepsilon(\gamma, \alpha) A \cdot g - \delta^{n-1}(g_A), $$

(2.18)

where $g \in C^n(L, V)$ and $A \in L$ are homogeneous. (Note that Eq. (2.18) is trivially satisfied if $n \leq -1$.) As it stands, Eq. (2.18) defines a linear map of $C^n(L, V)$ into $\text{Lgr}_{n+1}(L, V)$ which is homogeneous of degree zero, but it is easy to see by induction on $n \geq 0$ that $\delta^n g$ is $\varepsilon$–skew–symmetric and hence an element of $C^{n+1}(L, V)$. One can now show by induction on $n$ that $\delta^n$ is a homomorphism of graded $L$–modules, i.e., that

$$ A \cdot (\delta^n g) = \delta^n (A \cdot g) $$

(2.19)

for all $n$ and all $A \in L, g \in C^n(L, V)$, and then (very easily) that

$$ \delta^{n+1} \circ \delta^n = 0 $$

(2.20)
for all \( n \). Finally, one can verify inductively that the following explicit formula holds for \( n \geq 1 \):

\[
(\delta^n g)(A_0, A_1, \ldots, A_n) = \sum_{r=0}^{n} (-1)^r \varepsilon(\gamma + \alpha_0 + \ldots + \alpha_{r-1}, \alpha_r) A_r \cdot g(A_0, \ldots, \hat{A_r}, \ldots, A_n) \\
+ \sum_{r<s} (-1)^s \varepsilon(\alpha_{r+1} + \ldots + \alpha_{s-1}, \alpha_s) g(A_0, \ldots, A_{r-1}, \langle A_r, A_s \rangle, A_{r+1}, \ldots, \hat{A_s}, \ldots, A_n),
\]

where \( g \in C^n(L, V) \) and \( A_0, A_1, \ldots, A_n \in L \) are homogeneous, and where the sign \( \hat{\cdot} \) indicates that the element below it must be omitted. Empty sums (like \( \varepsilon(\alpha_0 + \ldots + \alpha_{r-1}, \alpha_r) \) for \( r = 0 \) and \( \varepsilon(\alpha_{r+1} + \ldots + \alpha_{s-1}, \alpha_s) \) for \( s = r + 1 \)) are set equal to zero. In particular, for \( n = 1 \) we have

\[
(\delta^1 g)(A_0, A_1) = \varepsilon(\gamma, \alpha_0) A_0 \cdot g(A_1) - \varepsilon(\gamma + \alpha_0, \alpha_1) A_1 \cdot g(A_0) - g(\langle A_0, A_1 \rangle),
\]

and for \( n = 2 \) we obtain

\[
(\delta^2 g)(A_0, A_1, A_2) = \varepsilon(\gamma, \alpha_0) A_0 \cdot g(A_1, A_2) - \varepsilon(\gamma + \alpha_0, \alpha_1) A_1 \cdot g(A_0, A_2) \\
+ \varepsilon(\gamma + \alpha_0 + \alpha_1, \alpha_2) A_2 \cdot g(A_0, A_1) \\
- g(\langle A_0, A_1 \rangle, A_2) + \varepsilon(\alpha_1, \alpha_2) g(\langle A_0, A_2 \rangle, A_1) + g(A_0, \langle A_1, A_2 \rangle).
\]

Occasionally, it is useful to rewrite Eq. (2.21) in the form

\[
2(\delta^n g)(A_0, A_1, \ldots, A_n) \\
= \sum_{r=0}^{n} (-1)^r \varepsilon(\gamma + \alpha_0 + \ldots + \alpha_{r-1}, \alpha_r)(A_r \cdot g)(A_0, \ldots, \hat{A_r}, \ldots, A_n) \\
+ \sum_{r=0}^{n} (-1)^r \varepsilon(\gamma + \alpha_0 + \ldots + \alpha_{r-1}, \alpha_r) A_r \cdot g(A_0, \ldots, \hat{A_r}, \ldots, A_n).
\]

**Remark 2.1.** The relations (2.18), (2.19), (2.20) are the basic properties of the cohomology operators \( \delta^n \). Of course, they can be checked directly (regarding Eq. (2.21) as the definition of \( \delta^n \)), but it is much easier to proceed as in Ref. [3], i.e., to follow the steps described above.

**Remark 2.2.** We could use the \( \varepsilon \)-skew–symmetry of \( g \) to shift the argument \( \langle A_r, A_s \rangle \) in the second sum on the right hand side of Eq. (2.21) to the first place. Then this equation would take a form which is similar to the one familiar from the non–graded theory. However, this would lead to additional \( \varepsilon \) factors which we wanted to avoid.

Next let \( Z^n(L, V) \) denote the kernel of \( \delta^n \) and let \( B^n(L, V) \) denote the image of \( \delta^{n-1} \). Eq. (2.19) implies that \( Z^n(L, V) \) and \( B^n(L, V) \) are graded submodules of \( C^n(L, V) \), and according to Eq. (2.20) we have

\[
B^n(L, V) \subset Z^n(L, V).
\]
The elements of $Z^n(L,V)$ are called $n$–cocycles, the elements of $B^n(L,V)$ are the $n$–coboundaries. Because of (2.25) we can construct the so–called cohomology groups

$$H^n(L,V) = Z^n(L,V)/B^n(L,V).$$

(2.26)

Two elements of $Z^n(L,V)$ are said to be cohomologous if their residue classes modulo $B^n(L,V)$ coincide, i.e., if their difference lies in $B^n(L,V)$.

Of course, the cohomology groups are graded $L$–modules, too. Actually, their $L$–module structure is trivial. To prove this we have to show that for any cocycle $g \in Z^n(L,V)$ and any $A \in L$, the cocycle $A \cdot g$ is a coboundary, and this follows directly from Eq. (2.18).

The following consequence of the foregoing result can be used to facilitate the calculation of the cohomology groups.

**Proposition 2.1**

We use the notation introduced at the beginning of the present section. Let $L'$ be a graded subalgebra of $L$.

a) If the $L'$–module $Z^n(L,V)$ is semi–simple (which is true if the $L'$–module $C^n(L,V)$ is semi–simple), then for any cocycle $g \in Z^n(L,V)$ there exists an $L'$–invariant cocycle $g' \in Z^n(L,V)$ which is cohomologous to $g$.

b) If the $L'$–module $C^{n-1}(L,V)$ is semi–simple, then any $L'$–invariant coboundary $b \in B^n(L,V)$ is equal to $\delta^{n-1}(g)$ with an $L'$–invariant cochain $g \in C^{n-1}(L,V)$.

(Semi–simplicity may be understood in the graded sense.)

Proof

a) If the graded $L'$–module $Z^n(L,V)$ is semi–simple, there exists a graded $L'$–submodule $X^n(L,V)$ of $Z^n(L,V)$ which is complementary to $B^n(L,V)$. By definition, the canonical map

$$Z^n(L,V) \rightarrow H^n(L,V)$$

(2.27)

induces an isomorphism

$$X^n(L,V) \rightarrow H^n(L,V)$$

(2.28)

of graded $L'$–modules. Since the $L'$–module structure of $H^n(L,V)$ is trivial, the same is true for the $L'$–module structure of $X^n(L,V)$, i.e., all elements of $X^n(L,V)$ are $L'$–invariant. Thus all we have to do is to choose an element $g' \in X^n(L,V)$ such that the canonical images of $g$ and $g'$ in $H^n(L,V)$ coincide.

b) If the graded $L'$–module $C^{n-1}(L,V)$ is semi–simple, there exists a graded $L'$–submodule $Y^{n-1}(L,V)$ of $C^{n-1}(L,V)$ which is complementary to the kernel of $\delta^{n-1}$ (i.e., to $Z^{n-1}(L,V)$). Then $\delta^{n-1}$ induces an isomorphism

$$Y^{n-1}(L,V) \rightarrow B^n(L,V)$$

(2.29)

of graded $L'$–modules. This implies b) and completes the proof of the proposition.
Next we consider some constructions which can be carried out with the cohomology groups. Let \( L' \) be a second \( \varepsilon \) Lie algebra and let
\[
\omega : L' \rightarrow L
\]
be a homomorphism of \( \varepsilon \) Lie algebras. Then the action of \( L' \) on \( V \), defined by
\[
A' \cdot x = \omega(A') \cdot x
\]
for all \( A' \in L' \) and \( x \in V \), makes \( V \) into a graded \( L' \)-module, which we denote by \( V^\omega \). If \( g \in C^n(L, V) \) with some \( n \geq 1 \), then the map \( L^n \rightarrow V \), defined by
\[
(A'_1, \ldots, A'_n) \mapsto g(\omega(A'_1), \ldots, \omega(A'_n))
\]
for all \( A'_1, \ldots, A'_n \in L' \), is an element of \( C^n(L', V^\omega) \). This assignment defines a linear map
\[
C^n(L, V) \rightarrow C^n(L', V^\omega).
\]
We extend this definition to the cases \( n \leq 0 \) by defining this map to be equal to \( \text{id}_V \) if \( n = 0 \), and to zero if \( n \leq -1 \). Obviously, the map (2.33) is homogeneous of degree zero, and it is easy to see that it is compatible with the coboundary operators, in the obvious sense. In particular, it maps cocycles into cocycles and coboundaries into coboundaries, thus inducing a corresponding map for the cohomology groups:
\[
H^n(L, V) \rightarrow H^n(L', V^\omega).
\]

Similarly, let
\[
f : V \rightarrow W
\]
be a linear map of \( V \) into a second graded \( L \)-module \( W \) and suppose that \( f \) is homogeneous of degree \( \varphi \) and \( L \)-invariant. Recall that \( L \)-invariance means that
\[
f(A \cdot x) = \varepsilon(\varphi, \alpha) A \cdot f(x)
\]
for all \( x \in V \), and for all \( A \in L \) which are homogeneous of degree \( \alpha \) (stated differently, \( f \) is \( L \)-linear in the graded sense). For any integer \( n \), we define the map
\[
f^n_c : C^n(L, V) \rightarrow C^n(L, W)
\]
by
\[
f^n_c = 0 \quad \text{if} \quad n \leq -1
\]
\[
f^n_c = f \quad \text{if} \quad n = 0
\]
\[
f^n_c(g) = f \circ g \quad \text{if} \quad n \geq 1.
\]
Obviously, \( f^n_c \) is homogeneous of degree \( \varphi \), and it is easy to see that
\[
\delta^n_W \circ f^n_c = f^{n+1}_c \circ \delta^n_V,
\]
where, temporarily, we have marked the coboundary operators for cochains with values in \( V \) and \( W \) by the subscripts \( V \) and \( W \), respectively. Eq. (2.41) implies that \( f^n_c \) maps cocycles into cocycles and coboundaries into coboundaries, and hence induces a linear map
\[
f^n : H^n(L, V) \longrightarrow H^n(L, W).
\]
(2.42)

Of course, \( f^n \) is also homogeneous of degree \( \varphi \). If \( U \) is a third graded \( L \)-module and if \( f' : U \to V \) is a linear map which is homogeneous and \( L \)-invariant, then, obviously,
\[
(f \circ f')^n_c = f^n_c \circ f'^n_c \quad \text{and} \quad (f \circ f')^n = f^n \circ f'^n.
\]
(2.43)

As a first application of this construction, choose \( \sigma \in \Gamma \) and consider the graded \( L \)-module \( V^\sigma \) which is obtained from \( V \) by a shift of the gradation by \( \sigma \) (see Ref. [20]). Recall that, considered as a vector space, \( V \) and \( V^\sigma \) coincide, and that the actions of \( L \) on \( V \) and \( V^\sigma \) are also the same, but that the gradations are related by
\[
V^\sigma_\gamma = V_{\gamma + \sigma}
\]
(2.44)
for all \( \gamma \in \Gamma \). Define the linear map
\[
f : V^\sigma \longrightarrow V
\]
(2.45)
by
\[
f(x) = \varepsilon(\sigma, \xi) x
\]
(2.46)
for all \( x \in V^\sigma_\xi \), \( \xi \in \Gamma \). Then \( f \) is bijective and homogeneous of degree \( \sigma \), and it is easy to see that \( f \) is \( L \)-invariant. Consequently,
\[
f^n : H^n(L, V^\sigma) \longrightarrow H^n(L, V)
\]
(2.47)
is a bijective linear map which is homogeneous of degree \( \sigma \). Stated differently, \( f^n \) is an isomorphism of the graded vector space \( H^n(L, V^\sigma) \) onto the graded vector space \( H^n(L, V)^\sigma \).

For our second application we suppose that \( V \) is the direct sum of a family \( (V_i)_{i \in I} \) of graded submodules,
\[
V = \bigoplus_{i \in I} V_i.
\]
(2.48)
Considering the canonical injections \( V_i \to V \) and projections \( V \to V_i \), it is easy to construct a natural linear map
\[
\bigoplus_{i \in I} H^n(L, V_i) \longrightarrow H^n(L, V)
\]
(2.49)
which is injective and homogeneous of degree zero. If \( I \) is finite or if \( L \) is finite–dimensional, this map is even bijective, i.e., an isomorphism of graded vector spaces. The details are obvious and may be left to the reader.
The third application is less trivial. Consider three graded $L$–modules $U$, $V$, $W$ and a short exact sequence

$$\{0\} \rightarrow U \xrightarrow{j} V \xrightarrow{p} W \rightarrow \{0\}, \quad (2.50)$$

where the linear mappings $j$ and $p$ are homogeneous (of degrees $\iota$ and $\pi$, respectively) and $L$–invariant. Then there exists a long exact sequence

$$\ldots \rightarrow H^n(L,U) \xrightarrow{j^n} H^n(L,V) \xrightarrow{p^n} H^n(L,W) \xrightarrow{\delta^n} H^{n+1}(L,U) \rightarrow \ldots, \quad (2.51)$$

where the maps $j^n$ and $p^n$ have been defined above. The definition of the so–called connecting homomorphisms $\delta^n$ (not to be confounded with the coboundary operators $\delta^n$ of Eq. (2.16)) is standard but a little involved (see Refs. [24, 25]). Since, in the applications, this definition generally is of little use, we don’t give it here but only mention that $\delta^n$ is homogeneous of degree $-\iota - \pi$.

Next we want to construct certain product maps between the spaces of cochains. Let $V$ and $W$ be two graded $L$–modules and let $m$ and $n$ be two integers. We are going to define a bilinear map

$$C^m(L,V) \times C^n(L,W) \rightarrow C^{m+n}(L,V \otimes W), \quad (2.52)$$

denoted by

$$(g,h) \rightarrow g \circ h, \quad (2.53)$$

as follows. If $m < 0$ or $n < 0$, this map is equal to zero. Now assume that $m, n \geq 0$, and let $g \in C^m(L,V)$ and $h \in C^n(L,W)$ be homogeneous of degrees $\gamma$ and $\eta$, respectively. If $m = n = 0$, we set

$$g \circ h = g \otimes h, \quad (2.54)$$

if $m = 0, n > 0$, we set

$$(g \circ h)(A_1, \ldots, A_n) = g \otimes h(A_1, \ldots, A_n) \quad (2.55)$$

for all $A_1, \ldots, A_n \in L$, if $m > 0, n = 0$, we set

$$(g \circ h)(A_1, \ldots, A_m) = \varepsilon(\eta, \alpha_1 + \ldots + \alpha_m) g(A_1, \ldots, A_m) \otimes h \quad (2.56)$$

for all homogeneous elements $A_1, \ldots, A_m \in L$. Finally, suppose that $m, n \geq 1$. In that case, we define the $(m+n)$–linear map $g \bar{\otimes} h$ (the graded tensor product of $g$ and $h$) by

$$(g \bar{\otimes} h)(A_1, \ldots, A_m, A_{m+1}, \ldots, A_{m+n}) \quad (2.57)$$

for all homogeneous elements $A_1, \ldots, A_{m+n} \in L$ and set

$$g \circ h = \frac{1}{m! n!} \sum_{\pi \in S_{m+n}} \text{sgn}(\pi) \hat{S}_\pi(g \bar{\otimes} h), \quad (2.58)$$
Remark 2.3. If \( m, n \geq 1 \), let \( S(m,n) \) be the set of all permutations \( \pi \in S_{m+n} \) such that \( \pi \) is increasing on \( \{1, 2, \ldots, m\} \) and on \( \{m + 1, m + 2, \ldots, m + n\} \) (the appropriate set of shuffle permutations). Then Eq. (2.58) can also be written in the form

\[
g \odot h = \sum_{\pi \in S(m,n)} \text{sgn}(\pi) \tilde{S}_\pi(g \bar{\otimes} h).
\] (2.59)

Obviously, in all cases \( g \odot h \) is an element of \( C^{m+n}(L, V \otimes W) \). Moreover, it follows from the general theory of graded \( L \)-modules \[20\] that the bilinear map (2.52) is homogeneous of degree zero and \( L \)-invariant, i.e., that

\[
A \cdot (g \odot h) = (A \cdot g) \odot h + \varepsilon(\alpha, \gamma) g \odot (A \cdot h)
\] (2.60)

for all homogeneous elements \( A \in L, g \in C^m(L, V) \), and \( h \in C^n(L, W) \).

The product is also compatible with the other operations introduced above. In fact, using (2.59) it is not difficult to show that for all integers \( m, n \) and all elements \( A \in L_\alpha, g \in C^m(L, V) \), and \( h \in C^n(L, W)_\eta \)

\[
(g \odot h)_A = \varepsilon(\eta, \alpha) g_A \odot h + (-1)^m g \odot h_A.
\] (2.61)

Using this relation and induction on \( m + n \) it is easy to rederive Eq. (2.60) and then, again by induction on \( m + n \), to obtain the following generalized Leibniz rule

\[
\delta^{m+n}(g \odot h) = (\delta^m g) \odot h + (-1)^m g \odot (\delta^n h),
\] (2.62)

which holds for all integers \( m, n \) and all \( g \in C^m(L, V) \) and \( h \in C^n(L, W) \). An immediate consequence of this equation is that the product maps (2.52) induce analogous ones for the cohomology groups:

\[
H^m(L, V) \times H^n(L, W) \longrightarrow H^{m+n}(L, V \otimes W).
\] (2.63)

As a final result about the product \( \odot \) we notice that it is associative in the obvious sense: If \( U \) is a third graded \( L \)-module, we have for all integers \( \ell, m, n \)

\[
f \odot (g \odot h) = (f \odot g) \odot h,
\] (2.64)

where \( f \in C^\ell(L, U), g \in C^m(L, V) \), and \( h \in C^n(L, W) \) (of course, it is understood that \( U \odot (V \otimes W) \) and \( (U \otimes V) \otimes W \) are canonically identified). The proof can be carried out by induction on \( \ell + m + n \).

In the foregoing discussion, the product of two cocycles \( g \in C^m(L, V) \) and \( h \in C^n(L, W) \) takes its values in \( V \otimes W \). This is the generic case, to which the other possibilities can be reduced. If \( U \) is a third graded \( L \)-module and if

\[
b : V \times W \longrightarrow U
\] (2.65)
is a homogeneous $L$-invariant bilinear map, we can use the corresponding linear map
\[ \tilde{b} : V \otimes W \rightarrow U \] (2.66)
to construct the maps
\[ \tilde{b}^n : C^n(L, V \otimes W) \rightarrow C^n(L, U) \] (2.67)
which, when composed with the product maps (2.52), yield the maps
\[ C^m(L, V) \times C^n(L, W) \rightarrow C^{m+n}(L, U). \] (2.68)

Of course, these have properties analogous to those of $\odot$, in particular, they induce bilinear maps
\[ H^m(L, V) \times H^n(L, W) \rightarrow H^{m+n}(L, U) \] (2.69)
which are homogeneous of the same degree as $b$.

We close this section by showing that for certain coefficient modules $V$ the cohomology groups $H^n(L, V)$ must vanish. Let $U(L)$ be the enveloping algebra of $L$ and let $U_+(L)$ denote the ideal of $U(L)$ which is generated by $L$. We say that an element $X$ of $U(L)$ (and, by abuse of language, also the representative $X_V$ of $X$ in an $L$–module $V$) does not contain a constant term, in case $X \in U_+(L)$.

**Proposition 2.2**

Let $L$ be a finite–dimensional $\varepsilon$ Lie algebra, let $V$ be a graded $L$–module, and let $C$ be a homogeneous element of the $\varepsilon$ center of $U(L)$ (i.e., a homogeneous Casimir element of $L$). Suppose that $C$ does not contain a constant term, and that the operator $C_v$ (the representative of $C$ in $V$) is invertible. Then we have
\[ H^n(L, V) = \{0\} \text{ if } n \neq 0. \] (2.70)

**Proof**

Let $(E_i)_{i \in I}$ be a basis of $L$, with $E_i$ homogeneous of degree $\epsilon_i$. We define for all $A \in L$ and $j \in I$
\[ \langle A, E_j \rangle = \sum_{i \in I} \text{ad}_{ij}(A) \ E_i, \] (2.71)
i.e., $A \rightarrow (\text{ad}_{ij}(A))$ is the adjoint representation, written in matrix form.

Let $L^*$ be the coadjoint $L$–module, i.e., the graded dual of $L$ endowed with the representation contragredient to the adjoint one, and let $(E'_i)_{i \in I}$ be the basis of $L^*$ dual to $(E_i)_{i \in I}$, defined by
\[ E'_i(E_j) = \delta_{ij} \text{ for all } i, j \in I. \] (2.72)

Then $E'_i$ is homogeneous of degree $-\epsilon_i$. 


Now consider an $L$–invariant $r$–linear form $\phi$ on $L^*$ which is homogeneous of degree $\eta$. Then
\[
C(\phi) = \sum_{i_1, \ldots, i_r \in I} \phi(E'_{i_1}, \ldots, E'_{i_r}) E_{i_r} \ldots E_{i_1}
\] (2.73)
is a Casimir element of $L$ (i.e., an element of the $\varepsilon$ center of $U(L)$) which is homogeneous of degree $\eta$ (see Ref. [26]). We define the following elements $C_i(\phi); i \in I,$ of $U(L)$:
\[
C_i(\phi) = \sum_{i_2, \ldots, i_r \in I} \phi(E_{i_1}, E_{i_1}', \ldots, E_{i_r}') E_{i_r} \ldots E_{i_1}.
\] (2.74)
Obviously, we have
\[
C(\phi) = \sum_{i \in I} C_i(\phi) E_i,
\] (2.75)
moreover, $C_i(\phi)$ is homogeneous of degree $\eta - \epsilon_i$, and the $L$–invariance of $\phi$ implies that
\[
\langle A, C_i(\phi) \rangle = -\sum_{j \in I} \varepsilon(\alpha, \eta - \epsilon_j) \text{ad}_{ij}(A) C_j(\phi)
\] (2.76)
for all homogeneous elements $A \in L$ and all $i \in I$.

Now let $C$ be the Casimir element described in the proposition, and suppose that $C$ is homogeneous of degree $\eta$. Then it is known [26] that $C$ can be written as a sum of Casimir elements of the form $C(\phi)$ considered above. Consequently, there exist elements $C_i \in U(L); i \in I$, which are homogeneous of degree $\eta - \epsilon_i$ and satisfy
\[
C = \sum_{i \in I} C_i E_i
\] (2.77)
as well as
\[
\langle A, C_i \rangle = -\sum_{j \in I} \varepsilon(\alpha, \eta - \epsilon_j) \text{ad}_{ij}(A) C_j
\] (2.78)
for all homogeneous elements $A \in L$ and all $i \in I$.

After these preparations the proof of the proposition is easy. For any integer $n \geq 1$ we define a linear map
\[
d_n : C^n(L, V) \longrightarrow C^{n-1}(L, V)
\] (2.79)
by
\[
(d_n(g))(A_2, \ldots, A_n) = \sum_{i \in I} \varepsilon(\epsilon_i, \gamma) C_i \cdot g(E_i, A_2, \ldots, A_n)
\] (2.80)
for all homogeneous elements $g \in C^n(L, V)$ and all $A_2, \ldots, A_n \in L$ (with the obvious interpretation if $n = 1$). Then one can check that
\[
d_{n+1} \circ \delta^n + \delta^{n-1} \circ d_n = (C_V)^n_c,
\] (2.81)
where $C_V$ is the representative of $C$ in $V$ (see the analogous but simpler calculation in the proof of Prop. VII.5.6 in Ref. [27]). Of course, $C_V$ is homogeneous of degree
\( \eta \) and \( L \)-invariant. Thus if \( g \in C^n(L, V) \) is a cocycle, the Eqs. (2.81) and (2.41) imply that
\[
g = \delta^{n-1}((C_V^{-1})_{n-1}d_n(g)),
\]
(2.82)
hence \( g \) is a coboundary and the proposition is proved.

The case \( n = 0 \) (not covered by the proposition) can easily be dealt with: Since \( B^0(L, V) = \{0\} \), we have
\[
H^0(L, V) = Z^0(L, V) = \{x \in V \mid A \cdot x = 0 \text{ for all } A \in L\},
\]
i.e., \( H^0(L, V) \) is the space of all \( L \)-invariant elements of \( V \).

**Remark 2.4.** The definitions and constructions of the present section (but not the two propositions) can be generalized by choosing, for the basic domain of scalars, instead of the field \( \mathbb{K} \) an arbitrary associative \( \varepsilon \)-commutative graded algebra \( S \) over \( \mathbb{K} \). This follows immediately from the multilinear algebra over \( S \) as sketched in the appendix of Ref. [28]. A generalization of this type could be useful if one considers the deformations of a superalgebra over a Grassmann algebra, and the deformation parameter is allowed to be odd. (The algebra \( L \) may also act on \( S \) by \( \varepsilon \)-derivations; for the super case, see Ref. [6]).

**III. Examples**

1. **Semi–simple Lie algebras**

In our first example we consider a (finite–dimensional) semi–simple Lie algebra \( L \) and a finite–dimensional \( L \)-module \( V \), and we want to describe the cohomology groups \( H^n(L, V) \). Needless to say, this case is well–known, we consider it here in order to show how our propositions can be applied under the most favourable circumstances.

The main input is that all finite–dimensional \( L \)-modules are semi–simple (i.e., the corresponding representations are completely reducible). Because of the isomorphism (2.41) we may, therefore, assume that the \( L \)-module \( V \) is simple (of course, in practice it may be difficult to decompose an \( L \)-module into simple submodules). If \( V \) is non–trivial, it is well–known that there exists a quadratic Casimir element \( C \) of \( L \) such that \( C_V \neq 0 \). Since \( V \) is simple, it follows that \( C_V \) is invertible. Thus Prop. 2.2 implies that \( H^n(L, V) = \{0\} \) if \( n \neq 0 \), and the same is true for \( n = 0 \) since \( 0 \) is the sole \( L \)-invariant element of \( V \).

Thus we may now assume that \( V \) is the trivial \( L \)-module \( \mathbb{K} \) and that \( n \geq 0 \). According to Prop. 2.1.a every cohomology class contains an \( L \)-invariant element. Conversely, it follows from Eq. (2.21) that every \( L \)-invariant cochain (with values in \( \mathbb{K} \)) is a cocycle. Moreover, if \( g \) and \( g' \) are cohomologous elements of \( Z^n(L, \mathbb{K}) \) and if \( g \) and \( g' \) are \( L \)-invariant, then Prop. 2.1.b tells us that \( g - g' = \delta^{n-1}b \) with an \( L \)-invariant cochain \( b \in C^{n-1}(L, \mathbb{K}) \). As we just have seen, this implies that \( \delta^{n-1}b = 0 \), i.e., that \( g = g' \).
Summarizing, we have proved that $H^n(L, \mathbb{K})$ with $n \geq 1$ can be identified with the space of all $L$–invariant skew symmetric $n$–linear forms on $L$.

The cohomology groups $H^n(L, \mathbb{K})$ with $n \in \{0, 1, 2\}$ can easily be determined. Obviously, we have

$$H^0(L, \mathbb{K}) = \mathbb{K} \quad (3.1)$$

(see Eq. (2.83)). Let us next recall that the commutator algebra $[L, L]$ of $L$ is equal to $L$. In view of Eq. (2.22), this implies that $Z^1(L, \mathbb{K}) = \{0\}$ and hence

$$H^1(L, \mathbb{K}) = \{0\} \quad (3.2)$$

Moreover, it also follows from $[L, L] = L$ that every $L$–invariant bilinear form on $L$ is symmetric. This shows that

$$H^2(L, \mathbb{K}) = \{0\} \quad (3.3)$$

On the other hand, it is easy to see that

$$H^3(L, \mathbb{K}) \neq \{0\} \quad (3.4)$$

In fact, if $\phi$ is the Killing form of $L$, the assignment

$$(A, B, C) \rightarrow \phi([A, B], C) \quad (3.5)$$

(with $A, B, C \in L$) defines a non-zero, $L$–invariant, skew symmetric, trilinear form on $L$.

2. The Lie superalgebras $osp(1|2n)$

For (finite–dimensional) simple Lie superalgebras the situation is much more complicated, even for the basic classical ones. However, there is one particular class of algebras (already mentioned by Kac in Sec. 5.5.3 of Ref. [7]), namely, the $osp(1|2n)$ algebras, for which the discussion of the preceding example goes through almost verbatim. All we have to do is to interpret it in the super sense and to recall the bijection (2.47). The pertinent information on the $osp(1|2n)$ algebras is contained in Ref. [29]. As already mentioned, the cohomology groups $H^n(L, \mathbb{K})$ of the non–exceptional classical simple Lie superalgebras have been described in Ref. [3] (see also Chap. 2, §6 of Ref. [10] and Ref. [14]; in particular, the exceptional algebras are considered in the latter article).

3. Trivial coefficients

Let us return to the general assumptions (where $L$ is an arbitrary $\varepsilon$ Lie algebra), and let us consider the case where $V = \mathbb{K}$ is the trivial graded $L$–module. As noted in Example 1, Eq. (2.24) implies that every $L$–invariant cochain is a cocycle. We would like to mention a classical construction of such cocycles (which, in the case of simple Lie algebras, essentially does the whole job, see Ref. [30] for more details).
Let $m \geq 0$ be an integer and let $\phi \in \text{Lgr}_{m+1}(L, \mathbb{K})$ be an $L$–invariant $(m+1)$–linear form. We want to construct an $L$–invariant cochain out of $\phi$. An obvious ansatz is to consider the $\varepsilon$–skew–symmetrization of the $2(m+1)$–linear form on $L$ defined by

$$\phi(A_0, A_1, \ldots, A_{2m}) \rightarrow \phi(\langle A_0, A_1 \rangle, \langle A_2, A_3 \rangle, \ldots, \langle A_{2m}, A_{2m+1} \rangle)$$

(3.6)

for all $A_i \in L$. However, the $L$–invariance of $\phi$ and the $\varepsilon$ Jacobi identity imply, that the $\varepsilon$–skew–symmetrization of the form (3.6) is equal to zero. The well–known way out is to consider for $m \geq 1$ the $\varepsilon$–skew–symmetrization of the $(2m+1)$–linear form $\psi \in \text{Lgr}_{2m+1}(L, \mathbb{K})$ defined by

$$\psi(A_0, A_1, \ldots, A_{2m}) = \phi(\langle A_0, A_1 \rangle, \langle A_2, A_3 \rangle, \ldots, \langle A_{2m-2}, A_{2m-1} \rangle, A_{2m})$$

(3.7)

for all $A_i \in L$.

The situation simplifies a little if $\phi$ is $\varepsilon$–symmetric (and $L$–invariant). In that case it is sufficient to $\varepsilon$–skew–symmetrize $\psi$ in $A_1, A_2, \ldots, A_{2m-1}$, i.e., already the $(2m+1)$–linear form on $L$ defined by

$$\phi(A_0, A_1, \ldots, A_{2m}) \rightarrow \sum_{\pi \in S_{2m-1}} \text{sgn}(\pi) \varepsilon_{2m-1}(\pi; \alpha_1, \ldots, \alpha_{2m-1})$$

(3.8)

(\text{for homogeneous elements } A_i \in L) \text{ is } \varepsilon\text{–skew–symmetric and, of course, } L\text{–invariant. Note that the trilinear form (3.3) is the simplest example of the foregoing construction.}

The possibility to obtain $L$–invariant cochains from $L$–invariant $\varepsilon$–symmetric multilinear forms is welcome and interesting, since the latter forms make their appearance in quite a different context. In fact, if $L$ is finite–dimensional and admits a non–degenerate $L$–invariant bilinear form which is homogeneous of degree zero, then there is a close relationship between the Casimir elements of $L$ and the $\varepsilon$–symmetric $L$–invariant multilinear forms on $L$. For more details, we refer the reader to Ref. [26]. However, we note that we don’t know of any reason to hope that the situation might be as nice as for semi–simple Lie algebras.

4. Irreducible $\text{gl}(m|n)$–modules for which all Casimir operators are equal to zero

In this example we assume that the field $\mathbb{K}$ is algebraically closed. We want to determine all those finite–dimensional simple graded $\text{gl}(m|n)$–modules, for which all Casimir operators without a constant term are equal to zero. According to Prop. 2.2, among the finite–dimensional simple graded modules, only these might have a non–trivial cohomology. In view of the detailed knowledge of the Casimir elements of $\text{gl}(m|n)$ and of their eigenvalues in a highest weight module [31, 26, 32, 33] (see also [34, 35, 36]), it is not difficult to solve this problem.
We are going to use the notation and results of the cited references. To begin with, let us recall some of them. Let \( m \) and \( n \) be any strictly positive integers. The Lie superalgebra \( \mathfrak{gl}(m|n) \) is defined through its elementary representation in \( \mathbb{K}^{m+n} \). If \( (e_i)_{1 \leq i \leq m+n} \) is the canonical basis of \( \mathbb{K}^{m+n} \), we define the gradation of \( \mathbb{K}^{m+n} \) by demanding that the elements \( e_1, \ldots, e_m \) are even and the elements \( e_{m+1}, \ldots, e_{m+n} \) are odd. Correspondingly, we define

\[
\sigma_i = \begin{cases} 
1 & \text{if } 1 \leq i \leq m \\
-1 & \text{if } m+1 \leq i \leq m+n.
\end{cases}
\]

In Ref. [33] certain generators \( X_{ij} \) of the Lie superalgebra under consideration have been introduced. For \( \mathfrak{gl}(m|n) \), these are just the usual basic \((m+n) \times (m+n)\) matrices \( E_{ij} ; 1 \leq i, j \leq m+n \).

Now let \( s \geq 1 \) be an integer and let \( C_s \) be the element of the universal enveloping algebra \( U(\mathfrak{gl}(m|n)) \) defined by

\[
C_s = \sum_{i_1, \ldots, i_s} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{s-1}} X_{i_si_1} X_{i_1i_2} \cdots X_{i_{s-1}i_s}.
\]

Then \( C_s \) is a Casimir element of \( \mathfrak{gl}(m|n) \), and (as shown in Ref. [32]) the super center \( Z(\mathfrak{gl}(m|n)) \) of \( U(\mathfrak{gl}(m|n)) \) is generated (as an algebra) by the unit element and these elements \( C_s \). Stated differently, every Casimir element can be written as a polynomial in the \( C_s ; s \geq 1 \). (By the way, none of the \( C_s \) may be dropped. In particular, the algebra \( Z(\mathfrak{gl}(m|n)) \) is not finitely generated.)

The elements \( X_{ii} ; 1 \leq i \leq m+n \) span the standard Cartan subalgebra \( H \) of \( \mathfrak{gl}(m|n) \). Let \( \varepsilon_i ; 1 \leq i \leq m+n \) be the linear form on \( H \) which satisfies

\[
\varepsilon_i(X_{jj}) = \delta_{ij}
\]

for \( 1 \leq j \leq m+n \). We choose the usual system of positive roots of \( \mathfrak{gl}(m|n) \) with respect to \( H \), namely

\[
\Delta^+ = \{ \varepsilon_i - \varepsilon_j | 1 \leq i < j \leq m+n \}.
\]

Now let \( \Lambda \in H^* \) be any linear form on \( H \) and let \( V \) be a graded highest weight \( \mathfrak{gl}(m|n) \)-module with highest weight \( \Lambda \). If \( C \in Z(\mathfrak{gl}(m|n)) \) is any Casimir element, the corresponding Casimir operator \( C_V \) is a scalar multiple of \( \text{id}_V \),

\[
C_V = \chi_{\Lambda}(C) \text{id}_V,
\]

where \( \chi_{\Lambda} \) is the so-called central character associated to \( \Lambda \). To solve our problem, we need the eigenvalues \( \chi_{\Lambda}(C) \). According to the foregoing remarks, it is sufficient to know the \( \chi_{\Lambda}(C_s) ; s \geq 1 \). These have been investigated in Ref. [33]. To describe the pertinent result, we introduce some more notation.
Let \( 2\rho \) be the sum of the even positive roots minus the sum of the odd positive roots, i.e.,

\[
\rho = \frac{1}{2} \sum_{i<j} \sigma_i \sigma_j (\varepsilon_i - \varepsilon_j).
\] (3.14)

We set

\[
\lambda = \Lambda + \rho
\] (3.15)

and define the numbers \( r_i \) and \( \ell_i \) through

\[
\begin{align*}
  r_i &= \sigma_i \rho (X_{ii}) \\
  \ell_i &= \sigma_i \lambda (X_{ii}).
\end{align*}
\] (3.16) (3.17)

If \( C \in Z(gl(m|n)) \) is fixed, we have

\[
\chi_\Lambda(C) = P_C(\ell_1, \ldots, \ell_{m+n}),
\] (3.18)

where \( P_C \) is a polynomial in \( m+n \) indeterminates \( Y_1, \ldots, Y_{m+n} \). If \( C \) runs through all of \( Z(gl(m|n)) \), the polynomials \( P_C \) form an algebra \( T(m|n) \) (the algebra of supersymmetric polynomials in super dimension \( (m|n) \)) [31]; for a recent reference see Ref. [37], and the map of \( Z(gl(m|n)) \) onto \( T(m|n) \) defined by \( C \rightarrow P_C \) is an algebra isomorphism (the generalized Harish–Chandra isomorphism [26]). Finally, the algebra \( T(m|n) \) is generated by the unit element and the polynomials

\[
Q_s = \sum_{i=1}^{m+n} \sigma_i (Y^s_i - r^s_i)
\] (3.19)

with \( s \geq 1 \) (see the end of Sec. 2 in Ref. [33], in particular, the lines below Eq. (2.36) and Eq. (2.37)). Note that the constant term of a Casimir element \( C \) is equal to \( \chi_0(C) \). Consequently, this term is equal to zero if and only if \( P_C \) can be written as a polynomial in the \( Q_s \) without a constant term.

Next we recall that every finite–dimensional simple graded \( gl(m|n) \)–module is a highest weight module (this is the sole place where we use that the base field is algebraically closed). Summarizing, we conclude that our problem can be reformulated as follows:

Determine all linear forms \( \Lambda \in H^* \) which correspond to finite–dimensional simple graded \( gl(m|n) \)–modules and satisfy

\[
Q_s(\ell_1, \ldots, \ell_{m+n}) = 0
\] (3.20)

for all \( s \geq 1 \).

Let us set

\[
L_i = \Lambda(X_{ii})
\] (3.21)

for \( 1 \leq i \leq m+n \). (Note that in contrast to Eqs. (3.16), (3.17) we have not included a factor of \( \sigma_i \) on the right hand side.) Then the first condition is satisfied if and only if

\[
L_i - L_{i+1} \in \mathbb{N}
\] (3.22)
for $1 \leq i \leq m - 1$ and for $m + 1 \leq i \leq m + n - 1$.

The validity of the relations (3.20) is easily seen to be equivalent to the following condition:

*Up to the ordering, the numbers*

$$\ell_1, \ell_2, \ldots, \ell_m, r_{m+1}, r_{m+2}, \ldots, r_{m+n}$$

(3.23)

*coincide with the numbers*

$$r_1, r_2, \ldots, r_m, \ell_{m+1}, \ell_{m+2}, \ldots, \ell_{m+n}.$$  (3.24)

The subsequent discussion depends on whether $m = n$, $m > n$, or $m < n$. Using Eq. (3.22) and some special properties of the numbers $r_i$, we first conclude that the conditions (3.20) are satisfied if and only if the following holds true.

If $m = n$, we have

$$\ell_{m+i} = \ell_{m+1-i} \text{ for } 1 \leq i \leq m,$$  (3.25)

if $m > n$, there exists a number $k \in \{0, 1, \ldots, n\}$ such that

$$\ell_{m+i} = \ell_{m+1-i} \text{ for } 1 \leq i \leq k,$$  (3.26)

$$r_{j-n} = \ell_{j-k} \text{ for } n + 1 \leq j \leq m,$$  (3.27)

$$\ell_{m+i} = \ell_{n+1-i} \text{ for } k + 1 \leq i \leq n,$$  (3.28)

and if $m < n$, there exists a number $h \in \{0, 1, \ldots, m\}$ such that

$$\ell_{m+i} = \ell_{m+1-i} \text{ for } 1 \leq i \leq h,$$  (3.29)

$$\ell_{h+j} = r_{m+j} \text{ for } m + 1 \leq j \leq n,$$  (3.30)

$$\ell_{n+i} = \ell_{m+1-i} \text{ for } h + 1 \leq i \leq m.$$  (3.31)

Expressed in terms of the coefficients $L_i$, our final result reads as follows.

Suppose first that $m = n$. Choose the numbers $L_1, L_2, \ldots, L_m$ in accordance with Eq. (3.22) and define

$$L_{m+i} = -L_{m+1-i} \text{ for } 1 \leq i \leq m.$$  (3.32)

Suppose next that $m > n$. Let $k$ be any element of $\{0, 1, \ldots, n\}$. Choose the numbers $L_1, L_2, \ldots, L_m$ in accordance with Eq. (3.22) and such that

$$L_{n+1-k} = L_{n+2-k} = \ldots = L_{m-k} = n - k.$$  (3.33)

Define

$$L_{m+i} = -L_{m+1-i} \text{ for } 1 \leq i \leq k,$$  (3.34)

$$L_{m+i} = -L_{n+1-i} - (m - n) \text{ for } k + 1 \leq i \leq n.$$  (3.35)
Finally, suppose that $m < n$. Let $h$ be any element of $\{0, 1, \ldots, m\}$. Choose the numbers $L_{m+1}, L_{m+2}, \ldots, L_{m+n}$ in accordance with Eq. (3.22) and such that

$$L_{m+1+h} = L_{m+2+h} = \ldots = L_{n+h} = -(m-h). \tag{3.36}$$

Define

$$L_i = -L_{m+n+1-i} + (n-m) \quad \text{for} \quad 1 \leq i \leq m-h \tag{3.37}$$

$$L_i = -L_{2m+1-i} \quad \text{for} \quad m-h + 1 \leq i \leq m \tag{3.38}$$

(for $h = m$ resp. $h = 0$, the first resp. second of these equations should be dropped).

Then, in all three cases, our conditions are satisfied. Conversely, every solution to our conditions is of this form.

We note that the relation (3.20) with $s = 1$ is equivalent to

$$\sum_{i=1}^{m+n} L_i = 0, \tag{3.39}$$

and it is easily checked that the numbers $L_i$ specified above satisfy this condition.

Remark 3.1. Needless to say the modules we have found are atypical [35]. Actually, Eqs. (3.25) – (3.31) show that they are maximally atypical in the sense that $\Lambda$ satisfies the maximal number (equal to $\min(m, n)$) of atypicality conditions.

The foregoing results can immediately be extended to the special linear Lie superalgebras $sl(m|n)$, provided that $m \neq n$. In fact, all we have to do is to interpret them appropriately, as explained in Ref. [33]. The elements $X_{ij}$ are now given by

$$X_{ij} = E_{ij} - \frac{1}{d} \sigma_i \delta_{ij} I, \tag{3.40}$$

where we have set $d = m - n$, and where $I$ is the unit matrix. Furthermore, the linear forms $\varepsilon_i$ are given by

$$\varepsilon_i(X_{jj}) = \delta_{ij} - \frac{1}{d} \sigma_j. \tag{3.41}$$

This implies that

$$\sum_{i=1}^{m+n} X_{ii} = 0 \tag{3.42}$$

and

$$\sum_{i=1}^{m+n} \sigma_i \varepsilon_i = 0. \tag{3.43}$$

Apart from these modifications, everything goes through, and the highest weights of the modules we are looking for are given by the same formulae as above. (We
have already noted that the relation (3.39), which should be satisfied because of Eq. (3.42), indeed holds true.

IV. Central extensions of $\varepsilon$ Lie algebras

We start with the basic definitions. Let $L$ be an $\varepsilon$ Lie algebra. A central extension of $L$ by an $\varepsilon$ Lie algebra $H$ is an exact sequence of $\varepsilon$ Lie algebras,

$$\{0\} \rightarrow H \xrightarrow{\iota} E \xrightarrow{\pi} L \rightarrow \{0\}$$

(4.1)

such that $\iota(H)$ is in the $\varepsilon$ center of $E$. Thus $\iota$ and $\pi$ are homomorphisms of $\varepsilon$ Lie algebras (which implies that they are homogeneous of degree zero), $\iota$ is injective, $\pi$ is surjective, and the kernel of $\pi$ is equal to the image of $\iota$. In particular, $H$ is Abelian. Thus $H$ is nothing but a graded vector space, endowed with the trivial structure of an $\varepsilon$ Lie algebra. If

$$\{0\} \rightarrow H' \xrightarrow{\iota'} E' \xrightarrow{\pi'} L \rightarrow \{0\}$$

(4.2)

is a second central extension of $L$, then by a morphism from the central extension (4.1) to the central extension (4.2), we mean a pair $(\varphi, \psi)$ of $\varepsilon$ Lie algebra homomorphisms,

$$\varphi : E \rightarrow E' , \quad \psi : H \rightarrow H'$$

(4.3)

such that

$$\pi = \pi' \circ \varphi , \quad \varphi \circ \iota = \iota' \circ \psi ,$$

(4.4)

i.e., such that the corresponding diagram is commutative. (Obviously, $\psi$ is uniquely determined by $\varphi$.) The morphism $(\varphi, \psi)$ is said to be an isomorphism of the central extensions if $\varphi$ and $\psi$ are bijective. Actually, if one of the maps $\varphi$ and $\psi$ is bijective, then so is the other.

Consider next the case where the second central extension has the form

$$\{0\} \rightarrow H \xrightarrow{\iota'} E' \xrightarrow{\pi'} L \rightarrow \{0\}. $$

(4.5)

Then the extensions (4.1) and (4.5) are said to be equivalent if there exists a morphism $(\varphi, \psi)$ from the extension (4.1) to the extension (4.5) such that $\psi = id_H$ (which implies that $\varphi$ is bijective).

As in the Lie algebra case, the central extensions of $L$ by $H$ can be classified by means of the cohomology groups $H^2(L, H)$, where $H$ is regarded as a graded vector space, endowed with the trivial $L$–module structure. To show this, we choose a homogeneous section $\sigma$ (in the category of vector spaces) of $\pi$, i.e., a homogeneous linear map

$$\sigma : L \rightarrow E$$

(4.6)

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(necessarily of degree zero) such that
\[ \pi \circ \sigma = \text{id}_L. \]  
(4.7)

Then we have
\[ \pi(\langle \sigma(A), \sigma(B) \rangle - \sigma(\langle A, B \rangle)) = 0 \]  
(4.8)

for all \( A, B \in L \). Consequently, there exists a unique map
\[ g : L \times L \to H \]  
(4.9)

such that
\[ \iota(g(A, B)) = \langle \sigma(A), \sigma(B) \rangle - \sigma(\langle A, B \rangle) \]  
(4.10)

for all \( A, B \in L \), and it is easy to see that \( g \) is a homogeneous 2–cocycle of degree zero, i.e., \( g \in Z^2(L, H)_0 \).

Consider in addition the central extension (4.5), choose a homogeneous section \( \sigma' \) of \( \pi' \), and construct the corresponding 2–cocycle \( g' \). Then the central extensions (4.1) and (4.5) are equivalent if and only if \( g \) and \( g' \) are cohomologous.

Conversely, let \( g \) be any element of \( Z^2(L, H)_0 \). Make the graded vector space \( L \times H \) into an \( \varepsilon \) Lie algebra by defining
\[ \langle (A, X), (B, Y) \rangle = (\langle A, B \rangle, g(A, B)) \]  
(4.11)

for all \( A, B \in L \) and all \( X, Y \in H \). This \( \varepsilon \) Lie algebra will be denoted by \( L(g) \). Let
\[ \sigma : L \to L \times H, \quad \iota : H \to L \times H \]  
(4.12)

be the canonical injections and let
\[ \pi : L \times H \to L \]  
(4.13)

be the canonical projection. Then
\[ \{0\} \to H \xrightarrow{\iota} L(g) \xrightarrow{\pi} L \to \{0\} \]  
(4.14)

is a central extension of \( L \) by \( H \), \( \sigma \) is a homogeneous section of \( \pi \), and the corresponding 2–cocycle of \( L \) with values in \( H \) is just the 2–cocycle \( g \) we started with.

Summarizing, we conclude that there is a bijection between the set of equivalence classes of central extensions of \( L \) by \( H \) and the cohomology group \( H^2(L, H) \). (More generally, an analogous result holds for all those extensions (4.1) of \( L \) for which \( H \) is only assumed to be Abelian; for more details, see Ref. [5].)

In the Lie algebra case, all this is well–known, and it is not at all surprising that this material can be generalized to the present graded setting. We have included these results in order to fix our notation and as a preparation for the investigations to follow. In fact, we now are ready to generalize Garland’s theory of universal

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central extensions of Lie algebras \[21\]. As we are going to see, most of Garland’s statements and proofs can be adopted almost verbatim. In addition, we comment on the homological background of these results, and we show how universal central extensions can be constructed.

Let us agree to call an \(\varepsilon\) Lie algebra \(L\) perfect if it is equal to its own commutator algebra, i.e., if \(\langle L, L \rangle = L\). Using this notion, we call the central extension (4.1) a covering of \(L\) if \(E\) is perfect. In this case, we also call \(E\) or the pair \((E, \pi)\) a covering of \(L\), and we shall say that \(E\) (or \(\pi\) or \((E, \pi)\)) covers \(L\). Obviously, if an \(\varepsilon\) Lie algebra admits a covering, then it is perfect.

**Lemma 4.1**

If the central extension (4.1) is a covering of \(L\), then there exists at most one morphism from (4.1) to any second central extension of \(L\).

**Proof**

Let \((\varphi, \psi)\) and \((\varphi', \psi')\) be two morphisms from the central extension (4.1) to the central extension (4.2). Obviously, we have

\[
(\varphi - \varphi')(\langle A, B \rangle) = \langle \varphi(A) - \varphi'(A), \varphi(B) \rangle + \langle \varphi'(A), \varphi(B) - \varphi'(B) \rangle
\]

for all \(A, B \in E\). On the other hand, we have

\[
\pi'(\varphi(A) - \varphi'(A)) = \pi(A) - \pi(A) = 0
\]

and hence

\[
\varphi(A) - \varphi'(A) \in \iota'(H')
\]

for all \(A \in E\). Consequently, the right hand side of Eq. (4.15) vanishes, which implies that \(\varphi = \varphi'\), and hence also \(\psi = \psi'\).

**Definition 4.1**

We say that a covering of an \(\varepsilon\) Lie algebra \(L\) is universal, if for every central extension of \(L\) there is a unique morphism (in the sense of central extensions) from the covering to the central extension.

We note that the uniqueness requirement in Def. 4.1 automatically follows from Lemma 4.1. Obviously, the definition and Lemma 4.1 imply the following proposition.

**Proposition 4.2**

Any two universal coverings of an \(\varepsilon\) Lie algebra \(L\) are isomorphic as central extensions, and the two mutually inverse isomorphisms are unique.

We have already noted that if an \(\varepsilon\) Lie algebra admits a covering, then it is perfect. Conversely, we have:

**Proposition 4.3**

If the \(\varepsilon\) Lie algebra \(L\) is perfect, then \(L\) has a universal covering.
Proof

Let $\bigwedge \epsilon L$ be the $\epsilon$ exterior algebra of the graded vector space $L$ (see App. A for more information), let $\wedge$ denote the multiplication in $\bigwedge \epsilon L$, and let $W' = \bigwedge \epsilon L$ be the subspace of $\bigwedge \epsilon L$ of $\mathbb{Z}$–degree 2. Consider the subspace $I$ of $W'$ which is generated by all elements of the form

$$- \langle A, B \rangle \wedge \epsilon C + \epsilon(\beta, \gamma) \langle A, C \rangle \wedge \epsilon B + A \wedge \epsilon \langle B, C \rangle ,$$

with homogeneous elements $A, B, C \in L$. Obviously, $I$ is a graded subspace of $W'$. We set $W = W'/I$ and denote the canonical image of $A \wedge \epsilon B$ in $W$ by $f(A, B)$, for all $A, B \in L$. By definition,

$$f : L \times L \rightarrow W$$

is a 2–cocycle which is homogeneous of degree zero. Thus we have the central extension

$$\{0\} \rightarrow W \xrightarrow{\bar{\iota}} L(f) \xrightarrow{\bar{\pi}} L \rightarrow \{0\}.$$  

Consider an arbitrary central extension of $L$ by an Abelian $\epsilon$ Lie algebra $H$. We know that this extension is equivalent to the extension (4.14) with a suitable 2–cocycle $g \in Z^2(L, H)$. Since $g$ is a 2–cocycle, there exists a unique linear map

$$\psi' : W \rightarrow H$$

such that

$$\psi'(f(A, B)) = g(A, B)$$

for all $A, B \in L$, and $\psi'$ is homogeneous of degree zero. We define a map

$$\varphi' : L(f) = L \times W \rightarrow L \times H = L(g)$$

by

$$\varphi'(A, X) = (A, \psi'(X))$$

for all $A \in L$ and $X \in W$. Then $(\varphi', \psi')$ is a morphism from the central extension (4.20) to the central extension (4.14).

Now let $\hat{L} = \langle L(f), L(f) \rangle$ be the commutator algebra of $L(f)$. Since $L$ is perfect, we have $\hat{L} + W = L(f)$ and hence

$$\bar{\pi}(\hat{L}) = L$$

$$\hat{L} = \langle \hat{L} + W, \hat{L} + W \rangle = \langle \hat{L}, \hat{L} \rangle .$$

Thus $\hat{L}$ is perfect. If $\hat{H}$ is the preimage of $\hat{L}$ under $\bar{\iota}$, the central extension (4.20) induces a central extension

$$\{0\} \rightarrow \hat{H} \xrightarrow{\bar{\iota}} \hat{L} \xrightarrow{\bar{\pi}} L \rightarrow \{0\} ,$$

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which is a covering of \( L \). Furthermore, if \( \varphi \) is the restriction of \( \varphi' \) to \( \hat{L} \) and if \( \psi \) is the restriction of \( \psi' \) to \( \hat{H} \), the pair \((\varphi, \psi)\) is a morphism from the central extension (4.27) to the central extension (4.14). This shows that (4.27) is a universal covering of \( L \), and thus the proposition is proved.

In the present work we do not consider the homology of \( \varepsilon \) Lie algebras (it is fairly obvious how this should be introduced). Nevertheless, it would be rather unsatisfactory not to mention the homological interpretation of the preceding proof.

Consider the linear map

\[
\partial_2 : \bigwedge^2 \varepsilon L \longrightarrow L
\]

(4.28)
given by

\[
\partial_2(A \wedge B) = -\langle A, B \rangle,
\]

(4.29)
and the linear map

\[
\partial_3 : \bigwedge^3 \varepsilon L \longrightarrow \bigwedge^2 \varepsilon L
\]

(4.30)
specified by

\[
\partial_3(A \wedge B \wedge C) = -\langle A, B \rangle \wedge C + \varepsilon(\beta, \gamma) \langle A, C \rangle \wedge B + A \wedge \langle B, C \rangle,
\]

(4.31)
for all homogeneous elements \( A, B, C \in L \). The \( \varepsilon \) Jacobi identity is equivalent to

\[
\partial_2 \circ \partial_3 = 0.
\]

(4.32)
Let \( Z_2(L) \) be the kernel of \( \partial_2 \) and let \( B_2(L) \) be the image of \( \partial_3 \). Then Eq. (1.32) implies that \( B_2(L) \subset Z_2(L) \), and a closer look at the definition of \( \hat{H} \) in the proof of Prop. 4.3 shows that

\[
\hat{H} = Z_2(L)/B_2(L) = H_2(L)
\]

(4.33)
(where the last equation is a definition). By definition, \( C^n(L, \mathbb{K}) \) is the graded dual of \( \bigwedge^n \varepsilon L \) (see the remark below Eq. (2.9)), and the coboundary map

\[
\delta^n : C^n(L, \mathbb{K}) \longrightarrow C^{n+1}(L, \mathbb{K})
\]

(4.34)
is the graded transpose of the boundary map

\[
\partial_{n+1} : \bigwedge^{n+1} \varepsilon L \longrightarrow \bigwedge^n \varepsilon L
\]

(4.35)
(we are considering the cases \( n = 1, 2 \) only, but, in fact, this is true for all \( n \)). Consequently, the graded dual \( \hat{H}^{\text{gr}} \) of \( \hat{H} \) is canonically isomorphic to \( H^2(L, \mathbb{K}) \).

More explicitly, the canonical isomorphism \( \phi : H^2(L, \mathbb{K}) \rightarrow \hat{H}^{\text{gr}} \) can be described as follows. Let \( g \in Z^2(L, \mathbb{K}), \gamma \) be a cocycle which is homogeneous of degree \( \gamma \). We have to give the image of the cohomology class \( g + B^2(L, \mathbb{K}) \) in \( \hat{H}^{\text{gr}} \). The cocycle \( g \) can be regarded as an element \( g_0 \in Z^2(L, \mathbb{K}_\gamma) \) (recall Eq. (2.44)). Let

\[
\{0\} \longrightarrow \mathbb{K}\gamma \longrightarrow E \longrightarrow L \longrightarrow \{0\}
\]

(4.36)
be a central extension of $L$ which belongs to the equivalence class corresponding to $g_0 + B^2(L, \mathbb{K})$, and let $(\varphi, \psi)$ be the morphism from the universal covering (4.27) to (4.36). Then $\psi_g$ can be regarded as a linear form $\hat{H} \rightarrow \mathbb{K}$ which is homogeneous of degree $\gamma$. This form depends only on the cohomology class $g + B^2(L, \mathbb{K})$ and is the image we are looking for.

The inverse of the isomorphism $\phi$ can be described even more simply. Let $\hat{g} \in Z^2(L, \hat{H})_0$ be a cocycle such that the cohomology class $\hat{g} + B^2(L, \hat{H})$ corresponds to the equivalence class of the universal covering (4.27). If $\lambda$ is any element of $\hat{H}^{gr}$, then $\lambda \circ \hat{g}$ is an element of $Z^2(L, \mathbb{K})$ and we have

$$\phi^{-1}(\lambda) = \lambda \circ \hat{g} + B^2(L, \mathbb{K}).$$

(4.37)

If $L$ is a perfect $\varepsilon$ Lie algebra for which $H^2(L, \mathbb{K})$ is finite–dimensional (of dimension $s$, say), the preceding discussion leads to the following construction of a universal covering of $L$. Let $(g_r)_{1 \leq r \leq s}$ be a family of homogeneous 2–cochains $g_r \in Z^2(L, \mathbb{K})$, such that the cohomology classes $g_r + B^2(L, \mathbb{K})$ form a basis of $H^2(L, \mathbb{K})$, and let $\gamma_r$ denote the degree of $g_r$. Choose a graded vector space $H$ such that $H$ has a homogeneous basis $(e_r)_{1 \leq r \leq s}$, with $e_r$ homogeneous of degree $-\gamma_r$. Define the bilinear map

$$g : L \times L \longrightarrow H$$

(4.38)

by

$$g(A, B) = \sum_{r=1}^{s} g_r(A, B) e_r$$

(4.39)

for all $A, B \in L$. Then $g \in Z^2(L, H)_0$, and the corresponding central extension (4.14) is a universal covering of $L$.

**Remark 4.1.** Let

$$\{0\} \longrightarrow \hat{H} \xrightarrow{\hat{i}} \hat{L} \xrightarrow{\hat{\pi}} L \longrightarrow \{0\}$$

(4.40)

be a universal covering of the $\varepsilon$ Lie algebra $L$. Then $\hat{L}$ is a universal covering of every $\varepsilon$ Lie algebra $E$ which covers $L$. In fact, if

$$\{0\} \longrightarrow H \longrightarrow E \longrightarrow L \longrightarrow \{0\}$$

(4.41)

is a covering of $L$ and if $(\varphi, \psi)$ is a morphism from (4.40) to (4.41), then it is easy to see that

$$\{0\} \longrightarrow \ker(\varphi) \longrightarrow \hat{L} \xrightarrow{\hat{\varphi}} E \longrightarrow \{0\}$$

(4.42)

is a covering of $E$. One can then show directly that this covering is universal, but the simplest way to prove this is to invoke Thm. 4.1.

**Remark 4.2.** The universal coverings have a property which, apparently, is more general than the one required in their definition. Let

$$\{0\} \longrightarrow H \xrightarrow{i} E \xrightarrow{\pi} L \longrightarrow \{0\}$$

(4.43)
be a universal covering of an \( \varepsilon \) Lie algebra \( L \) and let

\[
\{0\} \rightarrow H' \xrightarrow{\iota'} E' \xrightarrow{\pi'} L' \rightarrow \{0\}
\]  

(4.44)

be a central extension of an \( \varepsilon \) Lie algebra \( L' \). If \( \varphi : L \rightarrow L' \) is a homomorphism (of \( \varepsilon \) Lie algebras), there exists a unique pair \((\varphi, \psi)\) of homomorphisms

\[
\varphi : E \rightarrow E' , \quad \psi : H \rightarrow H'
\]

(4.45)

such that

\[
\varrho \circ \pi = \pi' \circ \varphi , \quad \varphi \circ \iota = \iota' \circ \psi ,
\]

i.e., such that the corresponding diagram is commutative.

**Remark 4.3.** One might think to generalize the definition of a universal covering to all (not necessarily perfect) \( \varepsilon \) Lie algebras, as follows: A central extension (4.1) of an \( \varepsilon \) Lie algebra \( L \) is said to be universal if, for any central extension (4.2) of \( L \), there exists a unique morphism from (4.1) to (4.2). Actually, it can be shown that if (4.1) is universal in this sense, then \( L \) and \( E \) are perfect, and nothing beyond the universal coverings has been obtained.

**Definition 4.2**

a) An \( \varepsilon \) Lie algebra \( L \) is said to be simply connected, if for every central extension

\[
\{0\} \rightarrow H \xrightarrow{\iota} E \xrightarrow{\pi} L \rightarrow \{0\}
\]

(4.47)

of \( L \), there is a unique homomorphism of \( \varepsilon \) Lie algebras \( \sigma : L \rightarrow E \), such that \( \pi \circ \sigma = \text{id}_L \).

b) A covering

\[
\{0\} \rightarrow H \rightarrow E \rightarrow L \rightarrow \{0\}
\]

(4.48)

of an \( \varepsilon \) Lie algebra \( L \) is said to be simply connected, in case \( E \) is simply connected. Every simply connected \( \varepsilon \) Lie algebra \( L \) is perfect. In fact, choose \( H = L/\langle L, L \rangle \) and consider the trivial central extension

\[
\{0\} \rightarrow H \rightarrow L \times H \rightarrow L \rightarrow \{0\}
\]

(4.49)

of \( L \), where \( H \rightarrow L \times H \) is the canonical injection and \( L \times H \rightarrow L \) is the canonical projection. Of course, the canonical injection \( L \rightarrow L \times H \) is a homomorphism \( \sigma \) of the type described in Def. 4.2.a. But if \( \varphi : L \rightarrow L/\langle L, L \rangle \) is the canonical homomorphism, the map

\[
\sigma' : L \rightarrow L \times H
\]

(4.50)

defined by

\[
\sigma'(A) = (A, \varphi(A))
\]

(4.51)

for all \( A \in L \) is likewise. Consequently, the uniqueness of \( \sigma \) implies our claim.

**Theorem 4.1**

A covering of an \( \varepsilon \) Lie algebra \( L \) is universal if and only if it is simply connected.
Proof

Let

\[ \{0\} \to \hat{H} \overset{i}{\to} \hat{L} \overset{\hat{\pi}}{\to} L \to \{0\} \quad (4.52) \]

be a simply connected covering of \( L \). We have to show that there exists a (necessarily unique) morphism from this covering to an arbitrary central extension \([4.1]\) of \( L \).

Define

\[ E' = \{(A, B) \in E \times \hat{L} \mid \pi(A) = \hat{\pi}(B)\} \quad (4.53) \]

Obviously, \( E' \) is a graded subalgebra of the \( \varepsilon \) Lie algebra \( E \times \hat{L} \). Let \( \hat{\rho} : E' \to \hat{L} \) be the projection onto the second factor of \( E \times \hat{L} \). Then \( \hat{\rho} \) is a surjective homomorphism of \( \varepsilon \) Lie algebras, and the kernel of \( \hat{\rho} \) is equal to

\[ \ker(\hat{\rho}) = \{(\iota(X), 0) \mid X \in H\} \quad (4.54) \]

which is contained in the center of \( E' \). Since \( \hat{L} \) is simply connected, there exists a unique homomorphism of \( \varepsilon \) Lie algebras \( \sigma : \hat{L} \to E' \), such that \( \hat{\rho} \circ \sigma = id_{\hat{L}} \). Let \( \lambda : \hat{L} \to E \) denote the composition of \( \sigma \) with the projection of \( E' \) onto the first factor of \( E \times \hat{L} \). The definitions imply that

\[ \sigma(B) = (\lambda(B), B) \in E' \quad (4.55) \]

for all \( B \in \hat{L} \) and hence (by definition of \( E' \)) that \( \pi \circ \lambda = \hat{\pi} \). In particular, we have \( \pi \circ \lambda \circ i = 0 \), which shows that

\[ (\lambda \circ i)(\hat{H}) \subset \iota(H) \quad (4.56) \]

Consequently, there exists a unique homomorphism of \( \varepsilon \) Lie algebras \( \mu : \hat{H} \to H \) such that \( \lambda \circ i = \iota \circ \mu \). Then \((\lambda, \mu)\) is a morphism from the covering \([4.52]\) to the central extension \([4.1]\).

Conversely, let us assume that \([4.52]\) is a universal covering of \( L \). Consider a central extension

\[ \{0\} \to H \overset{i}{\to} E \overset{\pi}{\to} \hat{L} \to \{0\} \quad (4.57) \]

We have to show that there exists a unique homomorphism of \( \varepsilon \) Lie algebras \( \sigma : \hat{L} \to E \) such that \( \pi \circ \sigma = id_{\hat{L}} \).

If \( \sigma \) exists, the pair \((\sigma, 0)\) is a morphism from the covering

\[ \{0\} \to \{0\} \to \hat{L} \overset{id}{\to} \hat{L} \to \{0\} \quad (4.58) \]

to the central extension \([4.57]\). Since \( \hat{L} \) is perfect, Lemma 4.1 implies that \( \sigma \) is uniquely determined. Hence it remains to prove that \( \sigma \) exists.

Let \( \hat{E} = \langle E, E \rangle \) be the commutator algebra of \( E \). Since \( \hat{L} \) is perfect, we have \( \pi(\hat{E}) = \hat{L} \) and hence \( E = \hat{E} + \iota(H) \). This in turn implies that

\[ \hat{E} = \langle \hat{E} + \iota(H), \hat{E} + \iota(H) \rangle = \langle \hat{E}, \hat{E} \rangle, \quad (4.59) \]
i.e., that $\hat{E}$ is perfect. Let $\hat{H}$ be the preimage of $\hat{E}$ under $\iota$. Then the central extension (4.57) induces a central extension
\[
\{0\} \longrightarrow \hat{H} \longrightarrow \hat{E} \longrightarrow \hat{L} \longrightarrow \{0\},
\]
and the homomorphism $\sigma$ exists, if the analogous homomorphism exists for (4.60). Thus we may assume that $E$ is perfect.

Let $\nu : E \rightarrow L$ be the (surjective) homomorphism of $\varepsilon$ Lie algebras defined by $\nu = \hat{\pi} \circ \pi$. We are going to show that the kernel of $\nu$ is contained in the center of $E$. Thus suppose that $A$ is a homogeneous element of $E$ such that $\hat{\pi}(\pi(A)) = 0$. Then $\pi(A)$ is central in $\hat{L}$. Consequently, we have
\[
\pi(\langle A, B \rangle) = \langle \pi(A), \pi(B) \rangle = 0
\]
for all $B \in E$, i.e., $\langle A, B \rangle$ belongs to the kernel of $\pi$ and hence to the center of $E$. If $B$ and $B'$ are any two homogeneous elements of $E$, this implies that
\[
\langle A, \langle B, B' \rangle \rangle = \langle \langle A, B \rangle, B' \rangle + \varepsilon(\alpha, \beta) \langle B, \langle A, B' \rangle \rangle = 0,
\]
and since $E$ is perfect it follows that $A$ lies in the center of $E$.

Thus
\[
\{0\} \longrightarrow \ker(\nu) \longrightarrow E \xrightarrow{\nu} L \longrightarrow \{0\}
\]
is a central extension. Since we are assuming that (1.52) is a universal covering, there exists a morphism $(\varphi, \psi)$ from (1.52) to (1.63). We are going to show that $\varphi$ is the homomorphism $\sigma$ we are looking for. To do so we have to show that $\pi \circ \varphi = id_{\hat{L}}$. Let $\omega : \hat{L} \rightarrow L$ be the linear map defined by $\omega = \pi \circ \varphi - id_{\hat{L}}$. Since $(\varphi, \psi)$ is a morphism of central extensions, we have
\[
\hat{\pi} \circ \omega = \hat{\pi} \circ (\pi \circ \varphi) - \hat{\pi} = \pi \circ \varphi - \hat{\pi} = 0,
\]
and hence $\omega$ takes its values in the center of $\hat{L}$. This implies that
\[
\omega(\langle A, B \rangle) = \langle \omega(A), (\pi \circ \varphi)(B) \rangle + \langle A, \omega(B) \rangle = 0
\]
for all $A, B \in \hat{L}$. Since $\hat{L}$ is perfect, it follows that $\omega = 0$ and the theorem is proved.

**Example: The universal covering of** $sl(n|n)/\mathbb{K} \cdot I_{2n}$

For abbreviation, we write $L$ instead of $sl(n|n)/\mathbb{K} \cdot I_{2n}$ (here and in the following, $I_{2n}$ denotes the $2n \times 2n$ unit matrix). It is well–known [8] that $L$ inherits from $gl(n|n)$ a consistent $\mathbb{Z}$–gradation such that
\[
L = L_{-1} \oplus L_0 \oplus L_1.
\]
To begin with, we note that
\[
\{0\} \longrightarrow \mathbb{K} \xrightarrow{\iota} sl(n|n) \xrightarrow{\pi} L \longrightarrow \{0\}
\]
is a central extension of $L$, where $\iota$ is defined by

$$\iota(c) = \frac{c}{2n} I_{2n}$$

for all $c \in \mathbb{K}$, and where $\pi$ is the canonical homomorphism. Obviously, there exists a unique linear map

$$\sigma : L \longrightarrow sl(n|n)$$

such that

$$\sigma(\pi(A)) = A - \frac{1}{2n} \text{Tr}(A) I_{2n}$$

for all $A \in sl(n|n)$, and we have $\pi \circ \sigma = id_L$. It follows that the 2–cocycle $g \in Z^2(L, \mathbb{K})$ defined by Eq. (4.10) is given by

$$g(\pi(A), \pi(B)) = \text{Tr}(\langle A, B \rangle)$$

for all $A, B \in sl(n|n)$. Obviously, $g$ is $L_0$–invariant (but, of course, not $L$–invariant).

If $n = 1$, the Lie superalgebra $L$ is Abelian and hence the theory of universal coverings does not apply (see Remark 4.3). Thus from now on we shall assume that $n \geq 2$. Then $L$ is simple and hence perfect. Our main task is to determine $H^2(L, \mathbb{K})$. Since $L_0 \simeq sl(n) \times sl(n)$ is semi–simple, Prop. 2.1 implies that every cohomology class contains an $L_0$–invariant cocycle. On the other hand, there does not exist a non–zero $L_0$–invariant linear form on $L$, i.e., $C^1(L, \mathbb{K})$ does not contain a non–trivial $L_0$–invariant element. Consequently, Prop. 2.1 shows that $H^2(L, \mathbb{K})$ can be identified with the space of all $L_0$–invariant 2–cochains. Thus we are going to determine this space.

For $n \geq 3$, this is easy. Let us first recall that every $L_0$–invariant bilinear form on $L_0$ is symmetric. Using this fact, a short look at the representations of $L_0$ carried by $L_0$ and $L_{\pm 1}$ shows that there exists, up to the normalization, a unique non–zero super–skew–symmetric $L_0$–invariant bilinear form on $L$. As we already know, $g$ has these properties, moreover, $g$ is a 2–cocycle. Thus we conclude, for $n \geq 3$, that $H^2(L, \mathbb{K})$ is one–dimensional and that (4.67) is a universal covering of $L = sl(n|n)/\mathbb{K} \cdot I_{2n}$.

Due to the peculiarities of the algebra $sl(2|2)/\mathbb{K} \cdot I_4$, the case $n = 2$ is more interesting. In principle, we could proceed in this case as above. However, we prefer to use a different realization of $sl(2|2)/\mathbb{K} \cdot I_4$ which makes the special properties of this algebra manifest (and which has already been used in the description of the super derivations of this algebra [3]).

Let $V$ be a two–dimensional vector space, let $\psi$ be a non–degenerate skew–symmetric bilinear form on $V$, and let $sp(\psi)$ (equal to $sl(V)$) be the Lie algebra of all linear maps of $V$ into itself leaving the form $\psi$ invariant. Define the bilinear map

$$P : V \times V \longrightarrow sp(\psi)$$

for all $c \in \mathbb{K}$, and where $\pi$ is the canonical homomorphism. Obviously, there exists a unique linear map

$$\sigma : L \longrightarrow sl(n|n)$$

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for all $A, B \in sl(n|n)$. Obviously, $g$ is $L_0$–invariant (but, of course, not $L$–invariant).
by

\[ P(x, y) z = \psi(x, z) y + \psi(y, z) x \] (4.73)

for all \( x, y, z \in V \). We set

\[ L_0 = sp(\psi) \times sp(\psi) \quad \text{and} \quad L_1 = V \otimes V \otimes V \] (4.74)

and regard \( L_1 \) as an \( L_0 \)-module, where the first (resp. second) factor of \( L_0 \) acts in the natural way on the first (resp. second) tensorial factor of \( L_1 \). For any \( Q \in sp(\psi) \), let \( Q_1 \) and \( Q_2 \) be the elements of \( L_0 \) defined by

\[ Q_1 = (Q, 0) \quad \text{and} \quad Q_2 = (0, Q). \] (4.75)

Using these data, the \( \mathbb{Z}_2 \)-graded vector space

\[ L = L_0 \oplus L_1 \] (4.76)

becomes a Lie superalgebra, if the bracket of two elements of \( L_1 \) is chosen such that

\[ \langle x \otimes y \otimes z, x' \otimes y' \otimes z' \rangle = c P_1(x, x')\psi(y, y')\psi(z, z') - c \psi(x, x')P_2(y, y')\psi(z, z') \] (4.77)

for all \( x, y, \ldots, z' \in V \), where \( c \in \mathbb{K} \) is an arbitrary non-zero constant.

This Lie superalgebra is isomorphic to \( \text{sl}(2|2)/\mathbb{K} \cdot I_4 \). More precisely, one can easily show the following:

Let \( (v_+, v_-) \) be any basis of \( V \). If we define

\[ L_0 = L_0, \quad L_{\pm 1} = V \otimes V \otimes \mathbb{K} v_{\pm}, \] (4.78)

then

\[ L = L_{-1} \oplus L_0 \oplus L_1 \] (4.79)

is a consistent \( \mathbb{Z} \)-gradation of \( L \), and \( L \) and \( \text{sl}(2|2)/\mathbb{K} \cdot I_4 \) are isomorphic as \( \mathbb{Z} \)-graded Lie superalgebras.

The arbitrariness in the choice of \( v_{\pm} \) reflects the peculiarity of \( \text{sl}(2|2)/\mathbb{K} \cdot I_4 \) we have been alluding to: This algebra admits a large group of outer automorphisms. In fact, let \( s \) be an arbitrary element of the group \( Sp(\psi) \) (equal to \( SL(V) \)) and let \( \tilde{s} \) be the linear map of \( L \) into itself, which induces the identity map on \( L_0 \), and on \( L_1 \) is given by

\[ \tilde{s}(x \otimes y \otimes z) = x \otimes y \otimes s(z) \] (4.80)

for all \( x, y, z \in V \). Then \( \tilde{s} \) is an automorphism of the Lie superalgebra \( L \).

It is easy to determine the \( L_0 \)-invariant super–skew–symmetric bilinear forms on \( L \). Let \( g \) be such a form. For the same reasons as above, \( g(A, B) \) has to vanish if at least one of the two homogeneous elements \( A, B \in L \) is even. On the other hand, the restriction of \( g \) to \( L_1 \times L_1 \) is given by

\[ g(x \otimes y \otimes z, x' \otimes y' \otimes z') = \psi(x, x')\psi(y, y')\varphi(z, z') \] (4.81)
for all \(x, y, \ldots, z' \in V\), where \(\varphi\) is an arbitrary symmetric bilinear form on \(V\). Conversely, it is not difficult to check that every bilinear form \(g\) with these properties is, in fact, an \(L_0\)-invariant 2-cocycle.

Obviously, the map assigning to any symmetric bilinear form \(\varphi\) on \(V\) the \(L_0\)-invariant 2-cocycle specified above is a vector space isomorphism of the spaces under consideration. Consequently, we have

\[
\dim H^2(sl(2|2)/\mathbb{K} \cdot I_4, \mathbb{K}) = 3. \tag{4.82}
\]

The universal covering of \(L\) can now be constructed along the lines specified around Eq. (4.39). Actually, this is not really necessary, since this algebra is known: It is obtained if one sets out to construct the Serre presentation of \(sl(2|2)\) and forgets about the so-called supplementary relations. More precisely, the universal covering of \(sl(2|2)/\mathbb{K} \cdot I_4\) is the Lie superalgebra \(s(2, 2)\) considered in Ref. [38]. In fact, there is a natural homomorphism of the Lie superalgebra \(s(2, 2)\) onto the Lie superalgebra \(sl(2|2)/\mathbb{K} \cdot I_4\) whose kernel is equal to the three-dimensional center of \(s(2, 2)\), moreover, it is known that \(s(2, 2)\) is perfect. Since the universal covering of \(sl(2|2)/\mathbb{K} \cdot I_4\) must cover \(s(2, 2)\) as well (see Remark 4.1), it follows that this universal covering must be isomorphic to \(s(2, 2)\). Actually, it is not difficult to construct this isomorphism explicitly. According to Remark 4.1, \(s(2, 2)\) is also a universal covering of \(sl(2|2)\). (Note that this is at variance with Thm. 2.6.1 of Ref. [10], but it is consistent with Ref. [9]. On the other hand, in the latter reference, the definition of the two basic 2-cocycles of \(sl(2|2)\) is not correct, and it is easy to see that, contrary to what is stated, we have \(H^3(L, \mathbb{K}) \neq \{0\}\) for both \(L = sl(2|2)\) and \(L = sl(2|2)/\mathbb{K} \cdot I_4\). We think that these discrepancies and mistakes are due to some trivial slips or misprints.)

V. Cohomology of the Lie superalgebra \(sl(1|2)\)

Throughout the present section we shall assume that the base field \(\mathbb{K}\) is algebraically closed. Let \(U(sl(1|2))\) denote the enveloping algebra of \(sl(1|2)\). Our main goal is to show that

\[
H^2(sl(1|2), U(sl(1|2))) = \{0\}. \tag{5.1}
\]

In order to prove this, we shall first determine \(H^1(sl(1|2), V)\) and \(H^2(sl(1|2), V)\), for all finite-dimensional simple graded \(sl(1|2)\)-modules \(V\). Actually, we only need to know those modules \(V\) for which these cohomology groups are non-trivial. As explained in the introduction, this information could be extracted from Ref. [15], but we think it is worthwhile to present our own calculations.

As an abbreviation, we write \(L\) instead of \(sl(1|2)\). We are going to use the notation and results of Ref. [39]. Let us first recall some of the most relevant facts. The basis of \(L\) we are going to use consists of four even elements \(Q_\pm, Q_3, B\) and
four odd elements $V_\pm$ and $W_\pm$. The elements $Q_\pm$, $Q_3$ span an $sl(2)$ subalgebra of $L_0$, they are normalized such that

$$\langle Q_+, Q_- \rangle = 2Q_3 \, .$$

(5.2)

The one-dimensional subspace $\mathbb{K} \cdot B$ is the center of $L_0$, and the two-dimensional subspace

$$H = \mathbb{K} \cdot B + \mathbb{K} \cdot Q_3$$

(5.3)

is a Cartan subalgebra of $L$. Thus any linear form $\lambda$ on $H$ (in particular, any weight of an $L$-module) can and will be identified with the pair $(b, q)$ of values which $\lambda$ takes on $B$ and $Q_3$, respectively. For example, $V_\pm$ and $W_\pm$ are root vectors corresponding to the roots $(\frac{1}{2}, \pm \frac{1}{2})$ and $(\frac{1}{2}, \pm \frac{1}{2})$, respectively. They are normalized such that $(V_\pm)$ and $(W_\pm)$ are standard $sl(2)$-doublets and satisfy

$$\langle V_\pm, W_\pm \rangle = \pm Q_\pm \, .$$

(5.4)

$$\langle V_\pm, W_\mp \rangle = -Q_3 \pm B \, .$$

(5.5)

The reader who prefers a matrix realization may set

$$Q_+ = E_{23} \, , \quad Q_- = E_{32} \, , \quad Q_3 = \frac{1}{2}(E_{22} - E_{33}) \, , \quad B = -\frac{1}{2}(2E_{11} + E_{22} + E_{33})$$

(5.6)

$$V_+ = E_{21} \, , \quad V_- = E_{31} \, , \quad W_+ = E_{13} \, , \quad W_- = -E_{12} \, ,$$

(5.7)

where the $E_{ij}$ are the usual basic $3 \times 3$ matrices. (The elements are chosen such that the $\mathbb{Z}$-gradation of $sl(1|2)$ as specified in Ref. [8] and the $\mathbb{Z}$-gradation given by Eq. (5.11) coincide.)

For later use we note that the linear map

$$\omega : L \rightarrow L$$

defined by

$$\omega(Q_\pm) = Q_\pm \, , \quad \omega(Q_3) = Q_3 \, , \quad \omega(B) = -B$$

(5.9)

$$\omega(V_\pm) = W_\pm \, , \quad \omega(W_\pm) = V_\pm$$

(5.10)

is an automorphism of the Lie superalgebra $L$.

The simple roots of $L$ with respect to $H$ are chosen such that $V_+$ and $W_+$ are the associated root vectors. Thus both of the simple roots are odd, i.e., we do not use a so-called distinguished basis of the root system. For $sl(1|2)$, this choice has certain advantages, for example, it is compatible with the automorphism $\omega$.

For any weight $(b, q) ; b, q \in \mathbb{K}$, and any $\gamma \in \mathbb{Z}_2$, there exists (up to isomorphism) a unique simple graded highest weight $L$-module $V(b, q, \gamma)$ such that the highest weight vectors (which are proportional to each other) have weight $(b, q)$ and $\mathbb{Z}_2$-degree $\gamma$. Of course, $V(b, q, \gamma)$ and $V(b, q, \gamma + 1)$ can be transformed into each other by a shift of the $\mathbb{Z}_2$-gradation (see Eq. (2.44)). The module $V(b, q, \gamma)$ is finite-dimensional if and only if $b = q = 0$ or $q \in \frac{1}{2}\mathbb{N}_+$, and any finite-dimensional simple
graduated $L$–module is of this type. Moreover, it is atypical if and only if $b = \pm q$, and its dimension is equal to $8q$ (resp. $4q + 1$) in the typical (resp. atypical) case.

The Casimir elements of $L$ are known, and the eigenvalues of the Casimir operators in a graded highest weight module have been calculated in terms of a generating function (see also Example 4 of Sec. III). In particular, the image of the generalized Harish–Chandra isomorphism is known. It can be regarded as an algebra of polynomials in $b, q$ and, as such, consists of all polynomials of the form $a + (b^2 - q^2) P$, where $a \in K$ is a constant and $P$ is a polynomial in $b$ and $(b^2 - q^2)$. This implies that, for a graded highest weight module with highest weight $(b, q)$, all Casimir operators without a constant term are equal to zero, if and only if $b = \pm q$.

Next we recall that $L$ has a consistent $Z$–gradation, which is fixed by the requirement that the elements $Q_{\pm}, Q_3,$ and $B$ are homogeneous of degree 0, the elements $V_{\pm}$ are homogeneous of degree 1, and the elements $W_{\pm}$ are homogeneous of degree $-1$. Thus we can apply our general theory both in the $Z_2$–graded and in the $Z$–graded sense. Actually, this essentially amounts to a matter of formulation (at least in the present article). This is because, for all $L$–modules $V$ for which a $Z$–gradation will be of interest, this gradation is simply given by the action of $B$, as follows:

$$V_r = \{ x \in V \mid 2B \cdot x = r x \}$$

for all $r \in Z$. In particular, this is true for $L$ itself and for the enveloping algebra $U(L)$ (both endowed with the adjoint action of $L$).

As a final preparatory remark we notice that there is a natural embedding of the Lie superalgebra $osp(1|2)$ into $sl(1|2)$. In fact, if we define

$$U_\pm = \frac{1}{2}(V_\pm + W_\pm),$$

the elements $Q_\pm, Q_3, U_\pm$ satisfy the commutation relations of $osp(1|2)$ as specified in Ref. [39]. In particular, we have

$$\langle U_\pm, U_\pm \rangle = \pm \frac{1}{2} Q_\pm \quad (5.13)$$

$$\langle U_\pm, U_+ \rangle = - \frac{1}{2} Q_3. \quad (5.14)$$

The $Z_2$–graded subalgebra of $L$ spanned by these elements will be denoted by $G$, by construction, it is isomorphic to $osp(1|2)$.

We recall that the finite–dimensional graded $G$–modules are semi–simple. The finite–dimensional simple graded $G$–modules can be labelled by a pair $(q, \gamma)$, with $q \in \frac{1}{2}N$ and $\gamma \in Z_2$: By definition, a highest weight vector $v$ of the corresponding module is homogeneous of degree $\gamma$ and satisfies $Q_3 \cdot v = q v$. We denote this module (which is fixed up to isomorphism) by $U(q, \gamma)$, its dimension is equal to $4q + 1$.

Of course, every graded $L$–module can also be regarded as a graded $G$–module. For example, choose $b \in K, q \in \frac{1}{2}N$, $\gamma \in Z_2$, and regard $V(b, q, \gamma)$ as a $G$–module.
If \( b \neq \pm q \), this module decomposes into the direct sum of two simple graded \( G \)-modules, isomorphic to \( U(q, \gamma) \) and \( U(q - \frac{1}{2}, \gamma + \bar{1}) \), respectively. On the other hand, the \( G \)-module \( V(\pm q, q, \gamma) \) is simple and isomorphic to \( U(q, \gamma) \). (Obviously, the latter statement remains true for \( q = 0 \).)

In particular, consider the adjoint \( L \)-module \( L \). Regarded as a \( G \)-module, it decomposes into two \( G \)-submodules: One of them is \( G \) itself, the other is spanned by \( X_\pm \) and \( B \), with

\[
X_\pm = \frac{1}{2}(V_\pm - W_\pm). \tag{5.15}
\]

The elements \( X_\pm \) have been chosen such that \( X_\pm \) and \( B \) form a basis of \( U(\frac{1}{2}, 1) \) as specified in Ref. \([39]\). We remark that \( Q_{\pm}, Q_3, iX_\pm \) also satisfy the commutation relations of \( osp(1|2) \).

After these preliminaries we are ready to tackle our main problem: We want to determine the cohomology groups \( H^n(L, V) \) and \( H^2(L, V) \), with \( V \) a finite-dimensional simple graded \( L \)-module. In view of the preceding remarks, we deduce from Prop. 2.2 that \( H^n(L, V) = \{0\} \) for all \( n \) if \( V \) is typical. Consequently, we only have to consider the cases where \( V = V(\pm q, q, \gamma) \), with \( q \in \frac{1}{2}\mathbb{N} \) and \( \gamma \in \mathbb{Z}_2 \). Actually, if \( g \) is the representation afforded by \( V(q, q, \gamma) \), then \( g \circ \omega \) is (equivalent to) the representation afforded by \( V(-q, q, \gamma) \) (where the automorphism \( \omega \) has been defined by the Eqs. (5.8) – (5.10)). Recalling the isomorphism (2.34) we may assume, therefore, that \( V = V(q, q, \gamma) \). On the other hand, the bijection (2.47) shows that it is sufficient to consider only one of the two choices for \( \gamma \). According to Eq. (5.11), we shall choose \( \gamma = 2\bar{q} \) (where \( \bar{r} = r + 2\mathbb{Z} \in \mathbb{Z}_2 \) denotes the residue class of the integer \( r \mod 2 \)), and assume that \( V(q, q, 2\bar{q}) \) is endowed with the consistent \( \mathbb{Z} \)-gradation defined by Eq. (5.11). To simplify the notation, we write \( V(q) \) instead of \( V(q, q, 2\bar{q}) \). Then we have to determine the cohomology groups \( H^n(L, V(q)) \), with \( n \in \{1, 2\} \).

The case \( n = 1 \) is simple. We present it in some detail to explain our approach. According to Prop. 2.1, any 1–cocycle of \( L \) with values in \( V(q) \) is cohomologous to a \( G \)-invariant 1–cocycle. Regarded as graded \( G \)-modules, \( L \) is isomorphic to \( U(1, \bar{0}) \oplus U(\frac{1}{2}, \bar{1}) \), and \( V(q) \) is isomorphic to \( U(q, 2\bar{q}) \). But a non–zero \( G \)-invariant linear map of a \( G \)-module \( U(p, 2\bar{p}) \) into a \( G \)-module \( U(q, 2\bar{q}) \) (with \( p, q \in \frac{1}{2}\mathbb{N} \)) exists if and only if \( p = q \), and such a map must be even. Consequently, a non–zero \( G \)-invariant linear map \( g : L \to V(q) \) exists if and only if \( q \in \{\frac{1}{2}, 1\} \). Moreover, if this condition is satisfied, a map of the type in question is even and uniquely determined up to a scalar factor.

On the other hand, let \( V \) be any graded \( L \)-module, and let \( g : L \to V \) be a \( G \)-invariant 1–cocycle which is homogeneous of degree 0. Then the \( G \)-invariance and the cocycle condition imply that, for all homogeneous elements \( X \in G \) and \( Y \in L \),

\[
X \cdot g(Y) = g(\langle X, Y \rangle) = X \cdot g(Y) - \varepsilon(\xi, \eta) Y \cdot g(X) \tag{5.16}
\]

and hence that

\[
Y \cdot g(X) = 0. \tag{5.17}
\]
In particular, we conclude that \( g((X, X')) = 0 \) for all \( X, X' \in G \), which implies that
\[
g(X) = 0 \tag{5.18}
\]
for all \( X \in G \).

Thus all we have to do is to determine \( H^1(L, V(\frac{1}{2})) \). To construct a non–zero \( G \)–module homomorphism of \( L \) into \( V(\frac{1}{2}) \) we introduce a suitable basis in \( V(\frac{1}{2}) \). According to Ref. [39] there exists a basis \((e_+, e_0, e_-)\) of \( V(\frac{1}{2}) \) such that \( e_\pm \) are odd and \( e_0 \) is even, such that \( e_\pm \) and \( e_0 \) are weight vectors corresponding to the weights \((\frac{1}{2}, \pm\frac{1}{2})\) and \((1, 0)\), respectively, and such that
\[
V_\pm \cdot e_\pm = 0 \quad V_\pm \cdot e_\mp = \mp e_0 \tag{5.19}
\]
\[
W_\pm \cdot e_\pm = W_\pm \cdot e_\mp = 0 \tag{5.20}
\]
\[
V_\pm \cdot e_0 = 0 \quad W_\pm \cdot e_0 = -e_\pm \tag{5.21}
\]
(Obviously, this basis is uniquely fixed up to a non–zero overall factor.) The action of \( Q_\pm \) on the basis vectors can be derived from the commutation relations and the formulae above, it is such that \((e_\pm)\) is a standard \( sl(2) \)–doublet and \( e_0 \) an \( sl(2) \)–singlet. It follows that \( e_\pm \) and \( e_0 \) also form a basis of the \( G \)–module \( V(\frac{1}{2}) \cong U(\frac{1}{2}, \bar{1}) \) as described in Ref. [39]. Consequently, the linear map
\[
g : L \longrightarrow V(\frac{1}{2}) \tag{5.22}
\]
defined by
\[
g(X) = 0 \quad \text{if} \quad X \in G \tag{5.23}
\]
\[
g(B) = e_0 \tag{5.24}
\]
\[
g(V_\pm - W_\pm) = 2e_\pm \tag{5.25}
\]
is a homomorphism of graded \( G \)–modules. Since
\[
g(V_\pm + W_\pm) = g(2U_\pm) = 0 \tag{5.26}
\]
it follows that
\[
g(V_\pm) = -g(W_\pm) = e_\pm. \tag{5.27}
\]
It is now easy to check that \( g \) is a 1–cocycle. Moreover, since \( V(\frac{1}{2}) \) does not contain a non–zero \( G \)–invariant element, \( g \) is not a coboundary. Summarizing, we have shown that
\[
H^1(L, V(q)) = \{0\} \quad \text{if} \quad q \neq \frac{1}{2} \tag{5.28}
\]
\[
\dim H^1(L, V(\frac{1}{2})) = 1. \tag{5.29}
\]
The fact that \( V(\frac{1}{2}) \) is (essentially) the sole finite–dimensional simple graded \( L \)–module \( V \) for which \( H^1(L, V) \) is non–trivial has already been mentioned in the Additional remarks at the end of Ref. [4].
The cocycle $g$ can be replaced by a simpler one. Let us regard $L$ as a consistently $\mathbb{Z}$–graded Lie superalgebra and $V(\frac{1}{2})$ as a $\mathbb{Z}$–graded $L$–module (recall that the $\mathbb{Z}$–gradations are chosen according to Eq. (5.11)). Then $g$ can be decomposed into its $\mathbb{Z}$–homogeneous components,

$$g = g_0 + g_2,$$

(5.30)

where $g_r, r \in \{0, 2\}$, is homogeneous of degree $r$. Explicitly, the non–vanishing values of $g_0$ and $g_2$ on the basis vectors $Q_{\pm}, Q_3, B, V_{\pm}, W_{\pm}$ are given by

$$g_0(V_{\pm}) = e_{\pm}$$

(5.31)

$$g_2(B) = e_0, \quad g_2(W_{\pm}) = -e_{\pm}.$$  

(5.32)

Both $g_0$ and $g_2$ are 1–cocycles. Hence exactly one of them is not a coboundary. Obviously, $g_0$ cannot be a coboundary, for $V(\frac{1}{2})$ does not contain a non–zero $\mathbb{Z}$–homogeneous element of degree 0. Consequently, $g_2$ must be a coboundary. In fact, we have

$$g_2(X) = X \cdot e_0$$

(5.33)

for all $X \in L$.

The 1–cocycle $g_0$ can be used to construct higher order cocycles, as follows. Choose any integer $n \geq 1$. Using the product of cocycles defined in Eq. (2.58) and the associativity of this product (see Eq. (2.64)), we can construct the $n$–cocycle $g_0^{\otimes n} = g_0 \otimes \ldots \otimes g_0$ (n factors $g_0$) with values in $V(\frac{1}{2})^{\otimes n}$, which is given by

$$g_0^{\otimes n}(X_1, \ldots, X_n) = \sum_{\pi \in S_n} \text{sgn}(\pi) \varepsilon_n(\pi; \xi_1, \ldots, \xi_n) g_0(X_{\pi(1)}) \otimes \ldots \otimes g_0(X_{\pi(n)})$$

(5.34)

for all homogeneous elements $X_i \in L$. Obviously, $g_0^{\otimes n}$ is non–zero and takes its values in the graded $L$–submodule $W(n)$ of super–skew–symmetric tensors in $V(\frac{1}{2})^{\otimes n}$.

Let $f_n : L^n \to W(n)$ denote the $n$–cocycle defined by $g_0^{\otimes n}$. Of course, $f_n$ is $\mathbb{Z}$–homogeneous of degree zero. Since the degree of a non–zero $\mathbb{Z}$–homogeneous element of $W(n)$ is at least equal to $n$, there does not exist a non–zero $(n - 1)$–linear map of $L^{n-1}$ into $W(n)$, which is $\mathbb{Z}$–homogeneous of degree zero. Consequently, $f_n$ is not a coboundary. On the other hand, it is easy to see that $W(n)$, even when regarded as a $\mathbb{Z}$–graded $L$–module, is isomorphic to $V(\frac{n}{2})$. Thus we have proved that

$$H^n(L, V(\frac{1}{2})) \neq \{0\} \quad \text{for all} \quad n \geq 1.$$  

(5.35)

Remark 5.1. Our results show that Prop. 2.2 exactly describes the finite–dimensional simple graded $L$–modules with a non–trivial cohomology, and the same holds for the $osp(1|2n)$ algebras. It would be interesting to know whether this is also true for other simple Lie superalgebras.

Let us next determine the cohomology groups $H^2(L, V(q))$. Since we are going to proceed as for $H^1(L, V(q))$, it should suffice to give the main intermediate steps. For
any graded $L$–module $V$, the graded $L$–module $C^2(L,V)$ is canonically isomorphic to $\text{Lgr}(L \wedge L, V)$ (see App. A). Thus we have to determine the structure of $L \wedge L$. It is known that this module is isomorphic to the graded submodule of super–skew–symmetric tensors in $L \otimes L$. This submodule is semi–simple (whereas $L \otimes L$ is not), and its simple submodules can be determined, with the result that

$$L \wedge L \simeq V(\frac{1}{2}, \frac{3}{2}, \bar{1}) \oplus V(-\frac{1}{2}, \frac{3}{2}, \bar{1}) \oplus V(0, 1, \bar{0}).$$

This implies that, regarded as a graded $G$–module, $L \wedge L$ is isomorphic to $2U(\frac{3}{2}, \bar{1}) \oplus 3U(1, \bar{0}) \oplus U(\frac{1}{2}, \bar{1})$.

Recalling that the graded $G$–module $V(q)$ is isomorphic to $U(q, \frac{2q}{2})$, we conclude as before that $H^2(L, V(q))$ is trivial unless $q \in \{\frac{1}{2}, 1, \frac{3}{2}\}$.

Let us consider the case $q = \frac{1}{2}$. Visibly, there exists, up to a scalar factor, a unique non–zero $G$–invariant linear map $\bar{f}$ of $L \wedge L$ into $V(\frac{1}{2})$, and this map is even and vanishes on the submodule $W$ isomorphic to

$$W \simeq V(\frac{1}{2}, \frac{3}{2}, \bar{1}) \oplus V(-\frac{1}{2}, \frac{3}{2}, \bar{1}).$$

Let $f \in C^2(L, V(\frac{1}{2}))$ be the associated bilinear map. We are going to show that $f$ is not a cocycle. According to Prop. 2.1, this will imply that

$$H^2(L, V(\frac{1}{2})) = \{0\}.$$  \hfill (5.39)

In fact, the product map $\langle , \rangle$ of $L$ induces an $L$–module homomorphism of $L \wedge L$ onto $L$, whose kernel is equal to $W$. Consequently, there exists a unique homomorphism of graded $G$–modules

$$g' : L \longrightarrow V(\frac{1}{2})$$

such that

$$f(X, Y) = g'(\langle X, Y \rangle)$$

for all $X, Y \in L$. But we know that there exists, up to a scalar factor, a unique homomorphism of graded $G$–modules of $L$ into $V(\frac{1}{2})$, namely, the map $g$ defined by the Eqs. \ref{eq:5.22} – \ref{eq:5.23}. Thus, by multiplying $f$ by a suitable scalar factor, we may assume that

$$f(X, Y) = g(\langle X, Y \rangle)$$

for all $X, Y \in L$. If $f$ would be a 2–cocycle, its $\mathbb{Z}$–homogeneous components $f_r$ given by

$$f_r(X, Y) = g_r(\langle X, Y \rangle), \quad r \in \{0, 2\}$$

would be 2–cocycles as well. However, using Eq. \ref{eq:5.33}, it is easy to see that $f_2$ is not a cocycle.

To proceed, we remark that the condition $q \in \{\frac{1}{2}, 1, \frac{3}{2}\}$ derived above can be improved by regarding $L \wedge L$ and $V(q)$ as $L_0$–modules. The same type of argument
as before then shows that \( H^2(L, V(q)) \) is trivial unless \( q \in \{0, \frac{1}{2}, 1\} \), moreover, there exists, up to a scalar factor, a unique \( \mathcal{L}_0 \)-invariant linear map of \( L \wedge L \) into \( V(1) \).

Since (again by Prop. 2.1) any 2–cocycle on \( L \) with values in \( V(1) \) is cohomologous to an \( \mathcal{L}_0 \)-invariant cocycle, we conclude from Eq. (5.32) that \( H^2(L, V(1)) \) is one–dimensional. Summarizing, we have shown that

\[
H^2(L, V(q)) = \begin{cases} 
\{0\} & \text{if } q \neq 1 \quad (5.44) \\
\dim H^2(L, V(1)) = 1 & \quad (5.45)
\end{cases}
\]

As an application of the preceding result, let us now prove that \( H^2(L, U(L)) \) is trivial. The following lemma contains the pertinent information on the \( L \)-module \( U(L) \).

**Lemma 5.1**

Any finite–dimensional atypical simple graded subquotient of the adjoint \( L \)-module \( U(L) \) is isomorphic to one of the three graded \( L \)-modules \( V(0, 0, \bar{0}) \) and \( V(\pm \frac{1}{2}, \frac{1}{2}, \bar{1}) \).

**Proof**

Consider two graded submodules \( U' \subset U \) of \( U(L) \) such that \( U/U' \) is a finite–dimensional atypical simple graded \( L \)-module. Then \( U/U' \) is isomorphic to one of the modules \( V(\pm q, q, \gamma) \), with \( q \in \mathbb{N} \) and \( \gamma \in \mathbb{Z}_2 \), and we have to show that \( q \in \{0, \frac{1}{2}\} \) and \( \gamma = \frac{2q}{q} \).

Let \( \text{ad} \) denote the adjoint representation of \( L \) in \( U(L) \). Then \( \text{ad}B \) is diagonalizable, and the sole eigenvalues of this operator are 0, \( \pm \frac{1}{2} \), and \( \pm 1 \). Since, for \( q \geq \frac{1}{2} \), the module \( V(\pm q, q, \gamma) \) contains non–zero weight vectors with weight \( (\pm q \pm \frac{1}{2}, q - \frac{1}{2}) \), the cases \( q \geq 1 \) are not possible. On the other hand, an eigenvector of \( \text{ad}B \) is odd if and only if the corresponding eigenvalue is equal to \( \pm \frac{1}{2} \), and it is even otherwise. This implies that \( \gamma = \frac{2q}{q} \) and proves the lemma.

**Remark 5.2.** As shown in App. B, the adjoint \( L \)-module \( U(L) \) really has graded subquotients isomorphic to \( V(0, 0, \bar{0}) \) and \( V(\pm \frac{1}{2}, \frac{1}{2}, \bar{1}) \).

**Remark 5.3.** A decomposition of the adjoint \( L \)-module \( U(L) \) into indecomposable submodules has been constructed in a recent paper by Benamor [10]. (We are grateful to the referee for drawing our attention to this article.) This decomposition is rather complicated, thus we think it would not be adequate to invoke that reference to prove the simple lemma above.

It is now easy to see that \( H^2(L, U(L)) = \{0\} \). The image of any 2–cochain of \( L \) with values in \( U(L) \) is contained in a finite–dimensional graded \( L \)-submodule of \( U(L) \) (use the canonical filtration of \( U(L) \)). Thus it is sufficient to show that \( H^2(L, V) \) is trivial, for any finite–dimensional graded \( L \)-submodule \( V \) of \( U(L) \). To prove this we choose an increasing sequence of graded \( L \)-submodules of \( U(L) \),

\[
\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = V, \quad (5.46)
\]
such that \( V_r/V_{r-1} \) is a simple graded \( L \)-module, for \( 1 \leq r \leq n \). (Up to the opposite ordering, (5.46) is a Jordan–Hölder sequence of the graded \( L \)-module \( V \).) We show by induction on \( r \) that

\[
H^2(L, V_r) = \{0\} .
\]  

(5.47)

The case \( r = 0 \) is trivial. Now suppose that Eq. (5.47) is true for some \( r \in \{0, 1, \ldots, n-1\} \). We have a short exact sequence of graded \( L \)-modules:

\[
\{0\} \longrightarrow V_r \longrightarrow V_{r+1} \longrightarrow V_{r+1}/V_r \longrightarrow \{0\} .
\]  

(5.48)

Consider the following part of the exact cohomology sequence (2.51) associated to (5.48):

\[
\ldots \longrightarrow H^2(L, V_r) \longrightarrow H^2(L, V_{r+1}) \longrightarrow H^2(L, V_{r+1}/V_r) \longrightarrow \ldots .
\]  

(5.49)

By assumption, \( H^2(L, V_r) \) is trivial, and Lemma 5.1, combined with Eq. (5.44), shows that \( H^2(L, V_{r+1}/V_r) \) is trivial as well. Thus, \( H^2(L, V_{r+1}) \) must be trivial.

Summarizing, we have shown that

\[
H^2(L, U(L)) = \{0\} ,
\]  

(5.50)

i.e., we have proved Eq. (5.44). On the other hand, we shall see in App. B that

\[
H^1(L, U(L)) \neq \{0\} ,
\]  

(5.51)

in contrast to what is known for semi–simple Lie algebras.

VI. Discussion

In the present work we have taken some exploratory steps towards a better understanding of the cohomology of Lie superalgebras and their generalizations. It is hardly surprising that the basic definitions and formal techniques known from the cohomology of Lie algebras can immediately be generalized to the general graded setting. In particular, this applies to Garland’s theory of universal central extensions of Lie algebras. The picture changes if we try to actually calculate the cohomology groups of Lie superalgebras, especially, when the coefficients are non–trivial. Our main tools in this project were two simple propositions proved in Sec. II.

As for many questions from the theory of Lie superalgebras, the \( osp(1|2n) \) algebras behave very much like simple Lie algebras. On the other hand, already for \( sl(1|2) \) the situation changes drastically. For any integer \( n \geq 0 \), there exists a finite–dimensional simple graded \( sl(1|2) \)-module \( V \) such that \( H^n(sl(1|2), V) \) is non–trivial. Moreover, in App. B we are going to show that

\[
H^1(sl(1|2), U(sl(1|2))) \neq \{0\} .
\]  

(6.1)
Thus it is remarkable that, nevertheless,

$$H^2(sl(1|2), U(sl(1|2))) = \{0\}.$$  \hspace{1cm} (6.2)

Actually, one of the main reasons for our present investigation was to try to prove the analogous result for as many simple Lie superalgebras $L$ as possible, from which it follows that the associative superalgebra $U(L)$ does not admit of non–trivial formal deformations in the sense of Gerstenhaber [18]. To see this, all we have to do is to transcribe the corresponding discussion of Ref. [19] to the present setting. Then Eq. (5.50) implies that the Hochschild 2–cohomology (in the super sense) of the associative superalgebra $U(L)$ with values in $U(L)$ (considered as a graded $U(L)$–bimodule) vanishes. In this connection, we recall that there is an isomorphism relating the cohomology of a Lie algebra $L$ with values in $U(L)$ to the Hochschild cohomology of the associative algebra $U(L)$ (see Theorem 5.1 of chapter XIII in Ref. [4]). It would be interesting to know whether an analogous isomorphism exists in the general graded setting.

The methods used in our investigation are completely elementary, and already for the 3–cohomology of $sl(1|2)$ the calculations become quite extensive. On the other hand, in the meantime we have shown that $H^2(L, U(L)) = \{0\}$ for $L = sl(m|1)$, $m \geq 3$, but up to now we were not able to prove or disprove this for $L = sl(3|2)$. Thus it might turn out that some more profound techniques must be used both in the study of higher order cohomology groups and in the case of more complicated Lie superalgebras.

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Appendix

A The $\varepsilon$–exterior algebra of a graded $L$–module

We use the notation introduced at the beginning of Sec. II. Let $V$ be a graded $L$–module and let

$$T(V) = \bigoplus_{n \in \mathbb{Z}} T_n(V)$$  \hspace{1cm} (A.1)
be the tensor algebra of $V$. It is known that $T(V)$ is a $\mathbb{Z} \times \Gamma$–graded associative algebra with a unit element and that

\[
T_n(V) = \begin{cases} 
\{0\} & \text{if } n \leq -1 \\
\mathbb{K} & \text{if } n = 0 \\
V \otimes \ldots \otimes V & \text{if } n \geq 1 .
\end{cases}
\] (A.2)

Moreover, $T(V)$ is a graded $L$–module, and $L$ acts on $T(V)$ by $\varepsilon$–derivations. Note also that each $T_n(V)$ is a graded $L$–submodule of $T(V)$.

Now let $J(V, \varepsilon)$ denote the two–sided ideal of $T(V)$, generated by the elements of the form

\[ x \otimes y + \varepsilon(\xi, \eta) y \otimes x , \] (A.5)

where $x$ and $y$ are homogeneous elements of $V$. Then the quotient algebra

\[ \bigwedge \varepsilon V = T(V)/J(V, \varepsilon) \] (A.6)

is called the $\varepsilon$–exterior (or $\varepsilon$–Grassmann) algebra of $V$. The multiplication in $\bigwedge V$ will be denoted by $\wedge \varepsilon$.

Since $J(V, \varepsilon)$ is a $\mathbb{Z} \times \Gamma$–graded ideal, the algebra $\bigwedge \varepsilon V$ inherits from $T(V)$ a canonical $\mathbb{Z} \times \Gamma$–gradation. In particular, we have

\[ \bigwedge \varepsilon V = \bigoplus_{n \in \mathbb{Z}} \bigwedge^n \varepsilon V , \] (A.7)

where $\bigwedge^n \varepsilon V$ is the canonical image of $T_n(V)$ in $\bigwedge \varepsilon V$. It follows that

\[
\bigwedge^n \varepsilon V = \begin{cases} 
\{0\} & \text{if } n \leq -1 \\
\mathbb{K} & \text{if } n = 0 \\
V & \text{if } n \geq 1 .
\end{cases}
\] (A.8)

For $n \geq 2$ we also use the notation

\[ \bigwedge^n \varepsilon V = V \bigwedge \ldots \bigwedge V \ (n \text{ factors}) . \] (A.11)

On the other hand, $J(V, \varepsilon)$ is a $\Gamma$–graded $L$–submodule of $T(V)$, and hence $\bigwedge \varepsilon V$ inherits from $T(V)$ the structure of a $\Gamma$–graded $L$–module. Then $L$ acts on $\bigwedge \varepsilon V$ by $\varepsilon$–derivations, and each $\bigwedge^n \varepsilon V$ is a $\Gamma$–graded $L$–submodule of $\bigwedge \varepsilon V$. 

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The theory of $\varepsilon$–exterior algebras can easily be reduced to the theory of $\varepsilon'$–symmetric algebras [20], with $\varepsilon'$ a commutation factor on a suitable Abelian group $\Gamma'$. In fact, set $\Gamma' = \mathbb{Z} \times \Gamma$ and define the commutation factor $\varepsilon'$ on $\Gamma'$ by

$$\varepsilon'((r, \alpha), (s, \beta)) = (-1)^{rs} \varepsilon(\alpha, \beta)$$

(A.12)

for all $r, s \in \mathbb{Z}$ and all $\alpha, \beta \in \Gamma$. Regard $V$ as a $\Gamma'$–graded vector space (denoted by $V'$), such that, for all $(r, \gamma) \in \Gamma'$,

$$V_{(1, \gamma)} = V_{\gamma} \quad \text{(A.13)}$$

$$V_{(r, \gamma)} = \{0\} \quad \text{if } r \neq 1 \quad \text{(A.14)}$$

On the other hand, regard $L$ as a $\Gamma'$–graded $\varepsilon'$ Lie algebra (denoted by $L'$), such that, for all $(r, \gamma) \in \Gamma'$,

$$L'_{(0, \gamma)} = L_{\gamma} \quad \text{(A.15)}$$

$$L'_{(r, \gamma)} = \{0\} \quad \text{if } r \neq 0 \quad \text{(A.16)}$$

while leaving the product map unchanged. Obviously, $V'$ and $T(V)$ have a natural structure of a $\Gamma'$–graded $L'$–module. A similar remark applies to the $\varepsilon'$–symmetric algebra $S(V', \varepsilon')$ of $V'$.

A priori, $T(V')$ is a $\mathbb{Z} \times \Gamma' = \mathbb{Z} \times \mathbb{Z} \times \Gamma$–graded vector space, where the $\mathbb{Z}$–gradation corresponding to the first factor is the canonical $\mathbb{Z}$–gradation of the tensor algebra $T(V')$, and where the one corresponding to the second factor stems from the $\mathbb{Z}$–gradation of $V'$. Obviously, these two $\mathbb{Z}$–gradations coincide. Thus we may simply forget about one of them and regard $T(V')$ as a $\Gamma'$–graded algebra and a $\Gamma'$–graded $L'$–module. A similar remark applies to the $\varepsilon'$–symmetric algebra $S(V', \varepsilon')$ of $V'$.

With these conventions, $T(V)$ and $T(V')$ coincide, and $\bigwedge V$ is equal to $S(V', \varepsilon')$ (where these statements may be interpreted both in the sense of $\Gamma'$–graded algebras and of $\Gamma'$–graded $L'$–modules).

The foregoing observation implies that for any result about $\varepsilon'$–symmetric algebras there is an analogous one for $\varepsilon$–exterior algebras. In particular, from Sec. 12 of Ref. [20] we draw the following conclusions.

Let $n \geq 1$ be an integer and let

$$\omega_n : V^n \longrightarrow \bigwedge^n_v V \quad \text{(A.17)}$$

be the canonical mapping defined by

$$\omega_n(x_1, \ldots, x_n) = x_1 \bigwedge \ldots \bigwedge x_n \quad \text{(A.18)}$$

for all $x_i \in V$. Then $\omega_n$ is an $\varepsilon$–skew–symmetric $n$–linear map, which is homogeneous of degree zero and $L$–invariant, and the pair $(\bigwedge_v^n V, \omega_n)$ has the following universal
property:
For any vector space $W$ and any $\varepsilon$–skew–symmetric $n$–linear map
\[ g : V^n \longrightarrow W, \]
there exists a unique linear map
\[ \hat{g} : \bigwedge \varepsilon V \longrightarrow W \]
such that
\[ g = \hat{g} \circ \omega_n. \]
Now suppose in addition that $W$ is a graded $L$–module, and let $\text{Lgr}^a_n(V,W;\varepsilon)$ denote the graded $L$–submodule of $\text{Lgr}_n(V;\ldots,V;W)$ consisting of all elements which are $\varepsilon$–skew–symmetric. Then the assignment $g \rightarrow \hat{g}$ described above induces a graded $L$–module isomorphism of $\text{Lgr}^a_n(V,W;\varepsilon)$ onto $\text{Lgr}(\bigwedge \varepsilon V,W)$.

Next, let $T^a_n(V,\varepsilon)$ denote the graded $L$–submodule of $T_n(V)$ consisting of all $\varepsilon$–skew–symmetric tensors in $T_n(V)$. The $\varepsilon$–skew–symmetrization
\[ t \longrightarrow \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi) S_\pi(t) \]
defines a graded $L$–module homomorphism of $T_n(V)$ onto $T^a_n(V,\varepsilon)$ (recall that the mappings $S_\pi$ have been defined in Ref. [20]). Using the universal property of $(\bigwedge \varepsilon V,\omega_n)$, it follows that there exists a unique surjective graded $L$–module homomorphism
\[ \chi_n : \bigwedge \varepsilon V \longrightarrow T^a_n(V,\varepsilon) \]
such that
\[ \chi_n(x_1 \varepsilon \ldots \varepsilon x_n) = \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi) S_\pi(x_1 \otimes \ldots \otimes x_n) \]
for all $x_i \in V$, and it is easy to see that $\chi_n$ is injective, and hence an isomorphism of graded $L$–modules.

On the other hand, from the Poincaré, Birkhoff, Witt theorem [22] we deduce that to any homogeneous basis of $V$ there corresponds, in the obvious way, a homogeneous basis of $\bigwedge \varepsilon V$.

Finally, if $V$ and $W$ are two graded $L$–modules, then $\bigwedge \varepsilon (V \oplus W)$ (regarded as a $\mathbb{Z} \times \Gamma$–graded algebra and as a $\Gamma$–graded $L$–module) is canonically isomorphic to $(\bigwedge \varepsilon V) \bar{\otimes}_{\varepsilon'} (\bigwedge \varepsilon W)$, where $\bar{\otimes}_{\varepsilon'}$ denotes the graded tensor product of $\mathbb{Z} \times \Gamma$–graded algebras with respect to $\varepsilon'$. This is a special case of an analogous result for $\varepsilon$ Lie algebras: If $G$ and $L$ are two $\varepsilon$ Lie algebras, the (universal) enveloping algebra $U(G \times L)$ of the direct product $G \times L$ of $G$ and $L$ is canonically isomorphic to the graded tensor product $U(G) \bar{\otimes}_\varepsilon U(L)$ of the $\Gamma$–graded algebras $U(G)$ and $U(L)$ (with
respect to $\varepsilon$). The proof follows directly from the universal property of enveloping algebras.

In particular, let $U = U_0 \oplus U_1$ be a $\mathbb{Z}_2$–graded vector space. Applying the preceding result to $V = U_0$ and $W = U_1$ (regarded as $\mathbb{Z}_2$–graded vector spaces), we conclude that the super–exterior algebra $\wedge \varepsilon U$ is canonically isomorphic to $(\wedge U_0) \otimes S(U_1)$, where $\wedge U_0$ (resp. $S(U_1)$) is the usual exterior (resp. symmetric) algebra of the vector space $U_0$ (resp. $U_1$), and where $\otimes$ denotes the graded tensor product of $\mathbb{Z}$–graded algebras with respect to the commutation factor $(r, s) \mapsto (-1)^{rs}$. In particular, we have the vector space isomorphism

$$\bigwedge^n \varepsilon U \simeq \bigoplus_{r=0}^{n} (\wedge^n U_0) \otimes S_{n-r}(U_1). \quad (A.25)$$

This enables us to make contact with the definition of the cohomology of Lie superalgebras given in Ref. [10]. First, we notice that we have chosen a different overall sign for the coboundary operator (which does not matter). Moreover, there is another discrepancy, which (in our language) amounts to dropping $\gamma$ in the first sum on the right hand side of Eq. (2.21). This is serious, for it spoils the $L$–invariance of $\delta^n$ (but not the $\varepsilon$–skew-symmetry of $\delta^n g$ and not the validity of the fundamental equation (2.20)). We suspect that this is a misprint.

## B The tensorial square of the adjoint $sl(1|2)$–module

We keep the notation of Sec. V and write $L$ instead of $sl(1|2)$. Our goal is to investigate the graded $L$–module $L \otimes L$. Of course, $L \otimes L$ decomposes into the direct sum of the graded $L$–submodule $T^a_2(L, \varepsilon)$ of super–symmetric tensors with the graded $L$–submodule $T^a_2(L, \varepsilon)$ of super–skew–symmetric tensors,

$$L \otimes L = T^a_2(L, \varepsilon) \oplus T^a_2(L, \varepsilon). \quad (B.1)$$

Moreover, it is known that $T^a_2(L, \varepsilon)$ (resp. $T^a_2(L, \varepsilon)$) is isomorphic to $S_2(L, \varepsilon)$ (resp. to $\wedge^2 L$) (see Ref. [20] and App. A).

The decomposition of $T^a_2(L, \varepsilon)$ into the direct sum of simple graded submodules has already been used in Sec. V, we have

$$T^a_2(L, \varepsilon) \simeq V(\frac{1}{2}, \frac{3}{2}, \bar{1}) \oplus V(-\frac{1}{2}, \frac{3}{2}, \bar{1}) \oplus V(0, 1, \bar{0}). \quad (B.2)$$

Thus all we have to do is to investigate $T^a_2(L, \varepsilon)$. This $L$–module is not semi–simple. In fact, we find that

$$T^a_2(L, \varepsilon) \simeq V(0, 2, \bar{0}) \oplus V(0, 1, \bar{0}) \oplus V_8, \quad (B.3)$$

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where \( V_8 \) is an 8–dimensional indecomposable graded \( L \)–module, which can be described as follows. It has a basis consisting of four even weight vectors \( t, s, v, w \) and four odd weight vectors \( v_{\pm}, w_{\pm} \), corresponding to the weights \((0, 0)\), \((0, 0)\), \((1, 0)\), \((-1, 0)\) and \((1/2, \pm 1/2)\), \((-1/2, \pm 1/2)\), respectively. The action of \( L \) on \( V_8 \) is fixed by the equations

\[
\begin{align*}
V_\pm \cdot t &= v_\pm, \\
W_\pm \cdot t &= w_\pm \\
V_\pm \cdot v^\mp &= \pm v, \\
W_\pm \cdot w^\mp &= \pm w,
\end{align*}
\]

while the remaining images of the basis vectors under the action of \( V_\pm \) or \( W_\pm \) are equal to zero. We note that these equations fix the normalization of the basis vectors up to an overall factor. The even vectors \( t, s, v, w \) are \( sl(2) \)–singlets, while \( (v_\pm) \) and \( (w_\pm) \) are standard \( sl(2) \)–doublets. Obviously, \( t \) generates the \( L \)–module \( V_8 \) (but, of course, it is not a highest weight vector). Needless to say, \( t \) can be replaced by any vector of the form \( t + \lambda s \), with \( \lambda \in \mathbb{K} \).

We do not want to specify the eight basis vectors inside \( T^*_2(L, \varepsilon) \). Suffice it to note that \( t \) can be chosen such that

\[
t = Q_+ \otimes Q_- + Q_- \otimes Q_+ + 2Q_3 \otimes Q_3 + 2B \otimes B.
\]

**Remark B.1.** The decomposition (B.3) of \( T^*_2(L, \varepsilon) \) has been found independently of Ref. [40]. As already noted, in this article a decomposition of \( U(L) \) into indecomposable \( L \)–submodules has been determined, and (B.3) is just the first non–trivial part of it. The module \( V_8 \) has been known since the early days of the theory of Lie superalgebras. In fact, it was proved in Sec. 3.B of Ref. [39] that, for both \( \gamma \in \mathbb{Z}_2 \), we have

\[
V(0, \frac{1}{2}, \gamma) \otimes V(0, \frac{1}{2}, \gamma) \simeq V(0, 1, \bar{0}) \oplus V_8,
\]

thus showing that the tensor product of two (typical) simple graded \( L \)–modules need not be semi–simple.

The graded \( L \)–module \( V_8 \) has the following non–trivial graded submodules:

\[
\begin{align*}
V_1 &= \mathbb{K} s \\
V_4 &= \mathbb{K} v_+ \oplus \mathbb{K} v_- \oplus \mathbb{K} v \oplus \mathbb{K} s \\
\bar{V}_4 &= \mathbb{K} w_+ \oplus \mathbb{K} w_- \oplus \mathbb{K} w \oplus \mathbb{K} s \\
V_7 &= V_4 + \bar{V}_4.
\end{align*}
\]

Obviously, \( V_1 \) is the submodule consisting of the \( L \)–invariant elements of \( V_8 \). The modules \( V_4 \) and \( \bar{V}_4 \) are indecomposable, and we have

\[
V_7/\bar{V}_4 \simeq V_4/V_1 \simeq V(\frac{1}{2}, \frac{1}{2}, \bar{1})
\]
\[ V_7/V_4 \simeq V_4/V_1 \simeq V\left(-\frac{1}{2}, \frac{1}{2}, \bar{1}\right). \]  
(B.16)

Finally, \( V_8/V_7 \) is a trivial one-dimensional \( L \)-module.

Let us next calculate \( \dim H^1(L, V_8) \). Using the information obtained in Sec. V, this is easily done by means of the long cohomology sequence \((2.51)\), applied to various short exact sequences of graded \( L \)-modules. In fact, considering the sequence

\[ \{0\} \rightarrow V_1 \rightarrow V_4 \rightarrow V_4/V_1 \rightarrow \{0\}, \]  
(B.17)

and the analogous one with \( V_4 \) replaced by \( V_4 \), we obtain

\[ \dim H^1(L, V_4) = \dim H^1(L, \bar{V}_4) = 1 \]  
(B.18)

\[ H^2(L, V_4) = H^2(L, \bar{V}_4) = \{0\}. \]  
(B.19)

Considering next the sequence

\[ \{0\} \rightarrow V_4 \rightarrow V_7 \rightarrow V_7/V_4 \rightarrow \{0\}, \]  
(B.20)

we conclude that

\[ \dim H^1(L, V_7) = 2. \]  
(B.21)

Finally, from the cohomology sequence corresponding to

\[ \{0\} \rightarrow V_7 \rightarrow V_8 \rightarrow V_8/V_7 \rightarrow \{0\}, \]  
(B.22)

and from Eq. \((2.83)\) we deduce that

\[ \dim H^1(L, V_8) = 1. \]  
(B.23)

The corresponding 1-cocycles can easily be determined. In fact, define two linear maps

\[ g : L \rightarrow V_4, \quad \bar{g} : L \rightarrow \bar{V}_4 \]  
(B.24)

by

\[ g(V \pm) = v \pm, \quad \bar{g}(W \pm) = w \pm \]  
(B.25)

\[ -g(B) = \bar{g}(B) = s, \]  
(B.26)

with \( g(X) = \bar{g}(Y) = 0 \) for all other elements \( X, Y \) of our standard basis of \( L \).

Then \( g \) and \( \bar{g} \) are 1-cocycles with values in \( V_4 \) and \( \bar{V}_4 \), respectively, and they are not coboundaries. If we regard \( g \) and \( \bar{g} \) as 1-cocycles with values in \( V_7 \), their cohomology classes in \( H^1(L, V_7) \) are linearly independent. Finally, if we regard \( g \) and \( \bar{g} \) as 1-cocycles with values in \( V_8 \), they are still not coboundaries, but \( g \) and \( -\bar{g} \) are cohomologous, with

\[ g + \bar{g} = \delta^0 t. \]  
(B.27)

The foregoing results have an immediate bearing on the cohomology of \( U(L) \). In fact, it is known \([20]\) that \( U(L) \), regarded as a graded \( L \)-module (and also as a
graded coalgebra) is canonically isomorphic to $S(L, \varepsilon)$. Consequently, there exists a graded $L$–submodule of $U(L)$, which is a direct summand of $U(L)$ and which is isomorphic to $V_8$. (Note that the simple graded subquotients of $V_8$ are exactly those which according to Lemma 5.1 are allowed.) Using the isomorphism (2.49), we conclude from Eq. (B.23) that

$$H^1(L, U(L)) \neq \{0\}.$$  

(B.28)

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