Scale invariant scalar metric fluctuations during inflation: non-perturbative formalism from a 5D vacuum

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We extend to 5D an approach of a 4D non-perturbative formalism to study scalar metric fluctuations of a 5D Riemann-flat de Sitter background metric. In contrast with the results obtained in 4D, the spectrum of cosmological scalar metric fluctuations during inflation can be scale invariant and the background inflaton field can take sub-Planckian values.

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I. INTRODUCTION

Inflation[1] is consistent with current observations of the temperature anisotropy of the Cosmic Microwave Background (CMB)[2]. Its dynamics may be categorized in two ways: the original (supercooled) inflation and inflationary models where dissipation plays an important role. Some examples of these last are new[3], stochastic inflation[4], warm[5] and fresh[6] inflation. The most popular model of supercooled inflation is chaotic inflation[7]. In this model the expansion of the universe is driven by a single scalar field \( \phi \) called inflaton. At some initial epoch, presumably the Planck scale, the scalar field is roughly homogeneous and dominates the energy density. Its initial values are random, subject to the constraint that energy density is at the Planck scale. Inflationary model solves several difficulties which arise from the standard cosmological model, such as the horizon, flatness, and monopole problems[8]. Furthermore, it provides a mechanism for the creation of primordial density fluctuations needed to explain the structure formation in the universe. Inflaton fluctuations are responsible for scalar metric fluctuations in the universe around the background Friedmann-Robertson-Walker (FRW) metric:

\[
    ds^2 = e^{2\psi} dt^2 - a^2(t) e^{-2\psi} d\vec{r}^2.
\]

This metric describe non-perturbative gravitational fluctuations on cosmological scales, in which vector and tensor perturbations of the metric can be neglected and the fluid can be considered as irrotational. Furthermore, using the continuity equation on large scales

\[
    \frac{\partial \rho}{\partial \tau} = -3H (\rho + P),
\]

where \( H = \frac{d}{d\tau} \left[ \ln \left( a(t) e^{-\psi} \right) \right] \) and \( d\tau = e^{\psi} dt \). One can show that there exists a conserved quantity in time at any order in perturbation theory

\[
    f = \ln (ae^{-\psi}) + \frac{1}{3} \int^{\rho} \frac{d\rho'}{(P' + \rho')}.
\]

Considering the invariant \( \omega \), which characterizes the equation of state \( P = \omega \rho \). The perturbation \( \delta f = -\psi + \frac{1}{3(1+\omega)} \ln (\rho/\bar{\rho}) \), is a gauge-invariant quantity representing the non-linear extension of the curvature perturbation for adiabatic fluids on uniform energy density hypersurfaces on superhorizon scales[9, 10].
In this letter we examine scalar metric fluctuations from a 5D vacuum state, which is defined on a 5D background Riemann-flat metric, using ideas of Modern Kaluza-Klein theory. This theory allows the fifth coordinate to play an important physical role. In this framework, the Induced Matter (or Space-Time-Matter) theory [11], all classical physical quantities, such as matter density and pressure, are susceptible of a geometrical interpretation. The mathematical basis of it is the Campbell’s theorem [12], which ensures an embedding of 4D general relativity with sources in a 5D theory whose field equations are apparently empty. That is, the Einstein equations $G_{\alpha\beta} = -8\pi G T_{\alpha\beta}$ (we use $c = \hbar = 1$ units), are embedded perfectly in the Ricci-flat equations $R_{AB} = 0$. Other version of 5D gravity, which is mathematically similar, is the membrane theory, in which gravity propagates freely on the 5D bulk and the interactions of particles are confined to a 4D hypersurface called “brane” [13]. Both versions of 5D general relativity are in agreement with observations.

II. SCALAR METRIC FLUCTUATIONS ON A DE SITTER SPACETIME

In this section we study the scalar metric fluctuations of a 5D spacetime background metric $(\mathcal{M}, g)$, which is Riemann-flat. Physically, the background metric here employed describes a 5D extension of the usual de Sitter spacetime. Similarly to the Induced Matter theory, here the flatness of the 5D background metric defines a 5D apparent vacuum. In the first part of the present section, we obtain the corresponding 5D Einstein field equations. In the second part, we derive the 4D induced dynamics for the scalar metric fluctuations.

A. The non-perturbative gauge-invariant scalar metric fluctuations

In order to study scalar metric fluctuations on a 5D de Sitter spacetime, we shall use the background line element [14]

$$dS_b^2 = l^2 dN^2 - l^2 e^{2N} dr^2 - dl^2,$$

where $dr^2 = \delta_{ij}dx^i dx^j$, $x^i$ are the 3D cartesian space-like dimensionless coordinates, $N$ is a dimensionless time-like coordinate and $l$ is the space-like non-compact extra coordinate, which has length units. The non-perturbative metric fluctuations of the background metric [3], are introduced in our analysis by the line element

$$dS^2 = l^2 e^{2\psi} dN^2 - l^2 e^{2(N-\psi)} dr^2 - dl^2,$$

where the metric function $\psi(N, r, l)$ describes the gauge-invariant metric fluctuations.

Now let us consider a non-massive test scalar field $\varphi = \varphi(x^n, l)$ defined on the 5D spacetime. The dynamics of $\varphi$ can be derived from the action

$$(5) S = \int d^4 x \sqrt{g} \left\{ \frac{1}{16\pi G} \left[ \frac{1}{2} g^{AB} \partial A \varphi \partial B \varphi \right] + (5) R \right\},$$

where $(5) R$ is the 5D Ricci scalar, $(5) g$ is the determinant of the metric which for the case of the background metric results in $(5) g = l^8 \exp(6N)$ and for the perturbed metric becomes $(5) g = l^8 \exp[6(N - 4\psi)]$. Furthermore $(5) g_0$ is a dimensionalization constant. The scalar field $\varphi$ is a purely kinetic scalar field without a potential which means that it is non-massive and therefore there are no interactions of this field with its environment.

In order to describe the scalar metric fluctuations consistently with the idea of a 5D apparent vacuum, we will require the perturbed metric to be Ricci-flat i.e. $(5) R_{AB} = 0$ for $(5) g$. This condition is given explicitly by

$$(5) R = \frac{2e^{2(\psi - N)} - 4 e^{2N} \nabla_{\alpha} \psi \nabla^{\alpha} \psi + e^{-2N} \left\{ e^{-2\psi} \left[ 15 \frac{\partial \psi}{\partial N} - 9 \left( \frac{\partial \psi}{\partial N} \right)^2 - 6 \right] - 2l^2 \left( \frac{\partial^2 \psi}{\partial t^2} - 2 \left( \frac{\partial \psi}{\partial t} \right)^2 \right) - 10l \frac{\partial \psi}{\partial t} + 6 \right\} = 0.$$

From the action $(5)$, the field equation for the test scalar field $\varphi$ on the perturbed metric reads

$$\frac{\partial^2 \varphi}{\partial N^2} + 3 \frac{\partial \varphi}{\partial N} - 4 \frac{\partial \psi}{\partial N} \frac{\partial \varphi}{\partial N} - e^{4\psi - 2N} \nabla^2 \varphi - l^2 e^{2\psi} \left[ \frac{\partial^2 \varphi}{\partial t^2} + 2 \left( \frac{2}{l} \frac{\partial \psi}{\partial t} \right) \frac{\partial \varphi}{\partial t} \right] = 0.$$
The diagonal components of the 5D field equations \( G_{AB} = 8\pi G T_{AB} \), are given by

\[
3e^{2\psi} + 6 \frac{\partial \psi}{\partial N} - 3 \left( \frac{\partial \psi}{\partial N} \right)^2 - 3 + e^{4\psi} - 2N \left[ (\nabla^2_r \psi)^2 - 2 \nabla^2_r \psi \right] + e^{2\psi} \left[ 6l^2 \left( \frac{\partial \psi}{\partial l} \right)^2 - 12l \frac{\partial \psi}{\partial l} - 3l^2 \frac{\partial^2 \psi}{\partial l^2} \right]
\]

\[
= 8\pi G e^{2(\psi - N)} \left[ \frac{1}{2} \left( \frac{\partial \varphi}{\partial N} \right)^2 + \frac{1}{2} e^{4\psi - 2N} (\nabla^2_r \varphi)^2 + \frac{l^2}{2} e^{2\psi} \left( \frac{\partial \varphi}{\partial l} \right)^2 \right],
\tag{8}
\]

\[
24 \frac{\partial \psi}{\partial N} - 15 \left( \frac{\partial \psi}{\partial N} \right)^2 + 6 \frac{\partial^2 \psi}{\partial N^2} + 9 (e^{2\psi} - 1) + e^{4\psi - 2N} (\nabla^2_r \psi)^2 + e^{2\psi} \left[ 6l^2 \left( \frac{\partial \psi}{\partial l} \right)^2 - 12l \frac{\partial \psi}{\partial l} - 3l^2 \frac{\partial^2 \psi}{\partial l^2} \right]
\]

\[
= -8\pi G e^{4\psi - 2N} \left[ -\frac{1}{2} (\nabla^2_r \varphi)^2 + \frac{3}{2} e^{-4\psi} e^{2N} \left( \frac{\partial \varphi}{\partial N} \right)^2 - \frac{3}{2} l^2 e^{-2\psi} e^{2N} \left( \frac{\partial \varphi}{\partial l} \right)^2 \right],
\tag{9}
\]

while the non-diagonal ones are

\[
\frac{\partial^2 \psi}{\partial x^i \partial N} - \frac{\partial \psi}{\partial N} \frac{\partial \psi}{\partial x^i} + \frac{\partial \psi}{\partial x^i} = 4\pi G \frac{\partial \varphi}{\partial N} \frac{\partial \varphi}{\partial x^i},
\tag{11}
\]

\[
\frac{\partial^2 \psi}{\partial l \partial N} - \frac{2}{3} \frac{\partial \psi}{\partial N} \frac{\partial \psi}{\partial l} + 2 \frac{\partial \psi}{\partial l} = \frac{8}{3} \pi G \frac{\partial \varphi}{\partial N} \frac{\partial \varphi}{\partial l},
\tag{12}
\]

\[
\frac{\partial^2 \psi}{\partial x^i \partial l} - \frac{2}{3} \frac{\partial \psi}{\partial N} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial l} = \frac{8}{3} \pi G \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial l},
\tag{13}
\]

\[
\frac{2}{3} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} = -8\pi G \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j}, \quad \text{for} \quad i \neq j.
\tag{14}
\]

By using the expression \( (11) \), the linear combination \( [3 \times \text{eq}(8) + \text{eq}(9) + 2 \times \text{eq}(10)] \) yields

\[
\frac{\partial^2 \psi}{\partial N^2} + 7 \frac{\partial \psi}{\partial N} - 4 \left( \frac{\partial \psi}{\partial N} \right)^2 + 3 \left( e^{2\psi} - 1 \right) + 2 \left( e^{4\psi - 2N} (\nabla^2_r \psi)^2 - e^{4\psi - 2N} \nabla^2_r \psi + e^{2\psi} \left[ 16l^4 \left( \frac{\partial \psi}{\partial l} \right)^2 - \frac{28}{3} \frac{\partial \psi}{\partial l} - \frac{8}{3} l^2 \frac{\partial^2 \psi}{\partial l^2} \right] \right)
\]

\[
= -\frac{4\pi G}{3} \left[ \left( \frac{\partial \varphi}{\partial N} \right)^2 + e^{4\psi - 2N} (\nabla^2_r \varphi)^2 + l^2 e^{2\psi} \left( \frac{\partial \varphi}{\partial l} \right)^2 \right].
\tag{15}
\]

These expressions provide us with the 5D dynamics of the scalar metric fluctuations \( \psi(t, \vec{r}, l) \). However, our main interest, more than describing the 5D dynamics, consists in describing the 4D dynamics induced from these perturbed field equations.

**B. The induced 4D dynamics for the non-perturbative gauge-invariant scalar metric fluctuations**

The field equations obtained in the preceding section can be used to derive an induced 4D dynamics for the 4D scalar fluctuations. To go down from 5D to 4D we assume that our 5D spacetime can be foliated by the family of hypersurfaces \( \Sigma_H : l = l_0 = H^{-1}, H \) being the Hubble constant. Physically, the class of observers living on a generic \( \Sigma_H \) are those whose 5-velocity along the fifth coordinate is null: \( U^t = 0 \). Thus, after passing to a more physical coordinates on \( \Sigma_H \) through the coordinate transformation \( t = N/H, R = r/H \), the background metric \( \Sigma \) yields the background-induced metric

\[
ds^2_b = dt^2 - e^{2Ht} dR^2,
\tag{16}
\]

\[\text{null: } U^t = 0 \]
where $t$ is the cosmic time. This metric describes the de Sitter expansion of the early universe characterized by an equation of state $p_b = -\rho_b = -3H^2/(8\pi G)$, $\rho_b$ and $\rho_b$ being the background pressure and energy densities. Similarly, the perturbed metric \[ (11) \] induces on $\Sigma_H$ the metric
\[
d s^2 = e^{2\Psi} dt^2 - e^{2(Ht-\Psi)} dR^2, \tag{17}\]
where $\Psi(t, \vec{R}) \equiv \psi(t, \vec{R}, l)|_{l=H^{-1}}$ describes the scalar metric fluctuations induced on $\Sigma_H$. The 4D Ricci scalar for the perturbed metric \[ (11) \] is given by
\[
^{(4)} R = 2 \left[ 6H^2 + 9 \left( \frac{\partial \Psi}{\partial t} \right)^2 - 3 \frac{\partial^2 \Psi}{\partial t^2} - 15H \frac{\partial \Psi}{\partial t} + e^{4\Psi-2Ht} \left( \nabla^2_R \Psi - (\nabla_R \Psi)^2 \right) \right] e^{-2\Psi}. \tag{18}\]
Note that in the absence of scalar metric fluctuations we recover the value of the Ricci scalar for the background-induced metric \[ (11) \]: \[ (4) R = 12H^2. \]

The diagonal Einstein field equations induced from the 5D field equations \[ (15), (16) \] and \[ (17) \], on the hypersurface $\Sigma_H$, read
\[
3H^2 e^{2\Psi} + 6H \frac{\partial \Psi}{\partial t} - 3 \left( \frac{\partial \Psi}{\partial t} \right)^2 - 3H^2 + e^{4\Psi-2Ht} \left[ (\nabla_R \Psi)^2 - 2(\nabla^2_R \Phi) \right] + H^2 e^{2\Psi} \left[ 6l^2 \left( \frac{\partial \psi}{\partial t} \right)^2 - 12l \frac{\partial \psi}{\partial t} - 3l^2 \frac{\partial^2 \psi}{\partial l^2} \right] \bigg|_{l=H^{-1}} \bigg], \tag{19}\]
\[
24H \frac{\partial \Phi}{\partial t} - 15 \left( \frac{\partial \Phi}{\partial t} \right)^2 + 6 \frac{\partial^2 \Phi}{\partial t^2} + 9H^2 (e^{2\Psi} - 1) + e^{4\Psi-2Ht} (\nabla_R \Phi)^2 + H^2 e^{2\Psi} \left[ 6l^2 \left( \frac{\partial \varphi}{\partial t} \right)^2 - 12l \frac{\partial \varphi}{\partial t} - 3l^2 \frac{\partial^2 \varphi}{\partial l^2} \right] \bigg|_{l=H^{-1}} \bigg], \tag{20}\]
\[
e^{4\Psi-2Ht} \nabla^2_R \Psi - 15 \frac{\partial \Phi}{\partial t} + 9 \left( \frac{\partial \Phi}{\partial t} \right)^2 + 6H^2 - 3 \frac{\partial^2 \Phi}{\partial t^2} - e^{4\Psi-2Ht} (\nabla_R \Psi)^2 - 6H^2 e^{2\Psi} \left( \frac{\partial \psi}{\partial t} \right) \bigg|_{l=H^{-1}} \bigg], \tag{21}\]
where $\Phi(t, \vec{r}) \equiv \varphi(t, \vec{r}, l)|_{l=H^{-1}}$ is the massive scalar field induced on $\Sigma_H$. The non-diagonal induced 4D Einstein equations can be derived by evaluating the equations \[ (11), (12), (13) \] and \[ (14) \] on $\Sigma_H$.

The induced dynamics of $\Phi(t, \vec{r})$ [see Eq. \[ (17) \] on $\Sigma_H$, is
\[
\frac{\partial^2 \Phi}{\partial t^2} + 3H \frac{\partial \Phi}{\partial t} - 4 \frac{\partial \Phi}{\partial l} - e^{4\Psi-2Ht} \nabla^2_R \Phi - H^2 e^{2\Psi} \left[ \frac{\partial^2 \varphi}{\partial t^2} + 2 \left( \frac{\partial \varphi}{\partial t} \right) \frac{\partial \varphi}{\partial l} \right] \bigg|_{l=H^{-1}} = 0, \tag{22}\]
whereas the dynamics of the induced scalar metric fluctuations $\Psi$, is given by the expression
\[
\frac{\partial^2 \Psi}{\partial t^2} + 7H \frac{\partial \Psi}{\partial t} - 4 \left( \frac{\partial \Psi}{\partial t} \right)^2 + 3H^2 (e^{2\Psi} - 1) + \frac{2}{3} e^{4\Psi-2Ht} (\nabla_R \Psi)^2 - e^{4\Psi-2Ht} \nabla^2_R \Psi + \frac{H^2 e^{2\Psi}}{3} \left[ 16l^2 \left( \frac{\partial \psi}{\partial t} \right)^2 - 28l \frac{\partial \psi}{\partial l} \frac{\partial^2 \psi}{\partial l^2} \right] \bigg|_{l=H^{-1}} = -\frac{4\pi G}{3} \left[ \left( \frac{\partial \Phi}{\partial t} \right)^2 + e^{4\Psi-2Ht} (\nabla_R \Phi)^2 + H^2 e^{2\Psi} \left( \frac{l \partial \varphi}{\partial l} \right)^2 \right] , \tag{23}\]
which is obtained by evaluating \[ (16) \] on $\Sigma_H$. 

Performing a semiclassical approximation for the 5D scalar field from which we can derive the commutation relations. Given the quantum nature of the fields $\Phi$ and $\Psi$ it seems convenient to use the quantization procedure. To do it, we impose the commutation relations

$$[\Phi(t, \vec{R}), \Pi^0_{\Phi}(t, \vec{R})] = i\delta(3)(\vec{R} - \vec{R}'), \quad [\Psi(t, \vec{R}), \Pi^0_{\Psi}(t, \vec{R})] = i\delta(3)(\vec{R} - \vec{R}'),$$

from which we can derive the commutation relations

$$\left[\Phi(t, \vec{R}), \Phi(t, \vec{R}')\right] = i e^{2\psi} \sqrt{|(4)g_0|/|4g|} \delta(3)(\vec{R} - \vec{R}'), \quad \left[\Psi(t, \vec{R}), \Psi(t, \vec{R}')\right] = i \frac{4}{9}\pi G e^{2\psi} \sqrt{|(4)g_0|/|4g|} \delta(3)(\vec{R} - \vec{R}'),$$

with $|4g| = [\exp(3Ht - 2\Psi)/H^3]^2$ being the determinant of the effective perturbed metric (17), and $\Pi^0_{\Phi} = \Phi \exp(-2\Psi)\sqrt{|(4)g_0|/|4g|}$ and $\Pi^0_{\Psi} = [6/16\pi G](6\Psi - 5H)\exp(-2\Psi)\sqrt{|(4)g_0|/|4g|}$ the momentums conjugate to $\Phi$ and $\Psi$, respectively.

C. 5D scalar metric fluctuations in the weak field limit

In the previous sections we have derived, in a general manner, the 5D and 4D dynamical equations for the inflaton field and gauge-invariant metric fluctuations on a perturbed de Sitter spacetime, regarding that these fluctuations are non-perturbative in nature. Until this moment, we have not imposed any restriction about the amplitude of these scalar fluctuations. However, as it is well known these fluctuations are very small on cosmological scales (in particular in inflationary scenarios). Thus, a first-order approximation in the gauge scalar fluctuations of the form $e^{\pm 2\psi} \approx 1 \pm 2\psi$, will be sufficient in order to have a good description of these fluctuations during inflation, in the present formalism.

Under this weak field limit approximation, linearizing the equation (7) with respect to $\psi$, one obtains

$$\frac{\partial^2 \varphi}{\partial N^2} + 3 \frac{\partial \varphi}{\partial N} - e^{-2N} \nabla^2 \varphi - 4 \frac{\partial \psi}{\partial N} \frac{\partial \varphi}{\partial N} - l^2 \left[\frac{\partial^2 \varphi}{\partial t^2} + 4 \frac{\partial \varphi}{\partial t} \frac{\partial \psi}{\partial t} - 2 \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \varphi}{\partial t} - 2 \varphi \frac{\partial^2 \psi}{\partial t^2} + 4 \frac{\partial \varphi}{\partial t} \right] = 0. \quad (28)$$

Performing a semiclassical approximation for the 5D scalar field $\varphi$ in the form $\varphi(N, \vec{r}, l) = \varphi_b(N, l) + \delta\varphi(N, \vec{r}, l)$ (with $\varphi_b$ denoting the background part of $\varphi$ and $\delta\varphi$ denoting the quantum fluctuations of $\varphi$), Eq. (28) results in the system

$$\frac{\partial^2 \varphi_b}{\partial N^2} + 3 \frac{\partial \varphi_b}{\partial N} - l^2 \left[\frac{\partial^2 \varphi_b}{\partial t^2} + 4l \frac{\partial \varphi_b}{\partial t} \right] = 0, \quad (29)$$

$$\frac{\partial^2 \delta\varphi}{\partial N^2} + 3 \frac{\partial \delta\varphi}{\partial N} - e^{-2N} \nabla^2 \delta\varphi - l^2 \left[\frac{\partial^2 \delta\varphi}{\partial t^2} + 4 \frac{\partial \delta\varphi}{\partial t} \frac{\partial \psi}{\partial t} - 2 \varphi \frac{\partial^2 \psi}{\partial t^2} + 4 \frac{\partial \varphi}{\partial t} \right] = 0. \quad (30)$$

The expression (29) gives the dynamics of the background scalar field $\varphi(N, \vec{r}, l)$, whereas Eq. (30) describes the dynamics for the quantum fluctuations $\delta\varphi(N, \vec{r}, l)$ in terms of the scalar metric fluctuations $\psi$ and the background field $\varphi_b$. Linearizing the 5D field equations (8), (9) and (10), we obtain the independent Einstein equations for the background field $\varphi_b$, on (3) and (4) respectively:

$$\left(\frac{\partial \varphi_b}{\partial N}\right)^2 - l^2 \left(\frac{\partial \varphi_b}{\partial t}\right)^2 = 0, \quad (31)$$

$$\left(\frac{\partial \varphi_b}{\partial N}\right)^2 + l^2 \left(\frac{\partial \varphi_b}{\partial t}\right)^2 = 0. \quad (32)$$
and for the field $\delta \varphi$, we have

$$6\psi + 6\frac{\partial \psi}{\partial N} - 2e^{-2N}\nabla^2 \psi - 12\frac{\partial^2 \psi}{\partial l \partial l} = 8\pi G \left[ \frac{\partial \varphi_b}{\partial N} \frac{\partial \delta \varphi}{\partial N} + \frac{\partial^2 \varphi_b}{\partial l \partial l} \frac{\partial \delta \varphi}{\partial l} + 3\psi \left( \frac{\partial \varphi_b}{\partial N} \right)^2 + 2\psi \left( \frac{\partial \varphi_b}{\partial l} \right)^2 \right].$$  

$$8\frac{\partial \psi}{\partial N} + 2\frac{\partial^2 \psi}{\partial N^2} + 6\psi - \frac{4\frac{\partial \psi}{\partial l} + 3\frac{\partial^2 \psi}{\partial l \partial l}}{\partial l \partial l} = -8\pi G \left[ \frac{\partial \varphi_b}{\partial N} \frac{\partial \delta \varphi}{\partial N} - \frac{1}{2} \frac{\partial^2 \varphi_b}{\partial l \partial l} \frac{\partial \delta \varphi}{\partial l} + \frac{1}{2} \psi \left( \frac{\partial \varphi_b}{\partial l} \right)^2 \right].$$  

$$e^{-2N}\nabla^2 \psi - 15\frac{\partial \psi}{\partial N} - 3\frac{\partial^2 \psi}{\partial N^2} - 12\psi + 6\frac{\partial \psi}{\partial l} = 8\pi G \left[ \frac{\partial^2 \varphi_b}{\partial l \partial l} \frac{\partial \delta \varphi}{\partial l} + \frac{\partial \varphi_b}{\partial N} \frac{\partial \delta \varphi}{\partial N} + \frac{1}{2} \psi \left( \frac{\partial \varphi_b}{\partial l} \right)^2 \right].$$  

Now, linearizing the non-diagonal field equations (11), (12) and (13), we obtain the system

$$\frac{\partial^2 \psi}{\partial N^2} + \frac{\partial \psi}{\partial N} + 6\psi - e^{-2N}\nabla^2 \psi - \frac{2}{3} \left[ 14\frac{\partial \psi}{\partial l} + 4\frac{\partial^2 \psi}{\partial l \partial l} \right] = -8\pi G \left[ \frac{\partial \varphi_b}{\partial N} \frac{\partial \delta \varphi}{\partial N} + \frac{1}{2} \psi \left( \frac{\partial \varphi_b}{\partial l} \right)^2 \right].$$  

This equation describes the dynamics of the 5D metric fluctuations $\psi$ in terms of the 5D inflaton field fluctuations $\delta \varphi$, at first order in $\psi$ and $\delta \varphi$. Note that this equation also can be obtained by using a linear combination of Eqs. (33), (34) and (35).

### D. Induced 4D dynamics for metric fluctuations in the weak field limit

Inherent to the semiclassical approximation for $\varphi(N, \vec{r}, l)$, is the semiclassical approximation for the induced 4D inflaton field $\Phi(t, \vec{r})$. This means that the semiclassical approximation $\Phi(t, \vec{r}) = \Phi_b(t) + \delta \Phi(t, \vec{r})$ is also valid, where $\Phi_b(t)$ is the 4D induced background inflaton field. Evaluating the expression (29) on $\Sigma_0$, we obtain

$$\frac{\partial^2 \Phi_b}{\partial t^2} + 3H \frac{\partial \Phi_b}{\partial t} + H^2 m^2 \Phi_b = 0$$  

where we have used $[\frac{1}{2} (\frac{\partial^2 \varphi_b}{\partial t^2}) + 4l (\frac{\partial \varphi_b}{\partial t})]_{l=H^{-1}} = -m^2 \Phi_b$, $m$ being a dimensionless separation constant. In addition, according to the action (26) and the induced metric (10), the background field $\Phi_b$ satisfies the dynamical Friedmann equation

$$\left( \frac{\partial \Phi_b}{\partial t} \right)^2 + H^2 \left( \frac{\partial \Phi_b}{\partial l} \right)^2 = \frac{3H^2}{4\pi G}.$$  

A particular solution of (10), which is valid at the beginning of inflation (when slow rolling conditions are fulfilled), is given by $(\frac{\partial \Phi_b}{\partial t}) = 0$. In this case necessarily $m = 0$ and Eq. (11) leads to a constant solution for the induced field $\Phi_b$

$$\Phi_b = \frac{1}{\sqrt{12\pi G}} = \frac{M_p}{\sqrt{12\pi}}.$$  

where we have used that $[\frac{1}{2} (\frac{\partial^2 \varphi_b}{\partial t^2})]_{l=H^{-1}} = 9\Phi_b^2$. The remarkable in this result is the fact that in this formalism $\Phi_b$ remains below the Planckian mass.

Now according to Eqs. (37) to (39), we obtain that on the hypersurface $\Sigma_H$ the scalar fluctuations $\Psi$ satisfy

$$\frac{\partial^2 \Psi}{\partial t^2} + 7H \frac{\partial \Psi}{\partial t} - e^{-2Ht} \nabla^2 \Psi + (7H^2 - \lambda^2) \Psi = 0,$$  

where $\lambda$ is the slow rolling parameter.
where we have used \((8\pi G)H^2[2(d^2\psi/dl^2) + 4l(d\psi/dl)]|_{l=H^{-1}} = \lambda^2\Psi\), with the separation constant \(\lambda\) having mass units. Equation (36), evaluated on \(\Sigma_H\), gives the condition \((\partial \Psi/\partial t) = -H \Psi\). Thus, by making use of this condition, Eq. (39) can be written as

\[
\frac{\partial^2 \Psi}{\partial t^2} + 3H \frac{\partial \Psi}{\partial t} - e^{-2Ht} \nabla^2_R \Psi + (3H^2 - \lambda^2) \Psi = 0. \tag{44}
\]

Following a canonical quantization process, the field \(\Psi\) can be written as a Fourier expansion

\[
\Psi(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[ a_k e^{i\vec{k} \cdot \vec{R}} \xi_k(t) + a_k^\dagger e^{-i\vec{k} \cdot \vec{R}} \xi_k^*(t) \right], \tag{45}
\]

where the annihilation and creation operators \(a_k\) and \(a_k^\dagger\) satisfy the usual commutation algebra

\[
[a_k, a_{k'}] = \delta^{(3)}(\vec{k} - \vec{k'}), \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0. \tag{46}
\]

Using the commutation relation (27) and the Fourier expansion (45), we obtain the normalization condition for the modes \(\xi_k(t)\):

\[
\xi_k(t)\xi_k^*(t) - \xi_k^*(t)\xi_k(t) = i \frac{4\pi G}{9} \left(\frac{a_0}{a}\right)^3, \tag{47}
\]

where \(a(t) = \exp(\partial t)\) is the scale factor in (16) and \(a_0 = \exp(\partial t_0)\), \(t_0\) being the cosmic time at the end of inflation. Inserting (45) into (14), we obtain that the modes \(\xi_k(t)\) satisfy

\[
\frac{d^2 \xi_k}{dt^2} + 3H \frac{d \xi_k}{dt} + \left[ k^2 e^{-2Ht} + 3H^2 - \lambda^2 \right] \xi_k = 0. \tag{48}
\]

By means of Eq. (47) and choosing the Bunch-Davies vacuum condition, the normalized solution of (48) is

\[
\xi_k(t) = \frac{\pi}{\sqrt{3V}} \frac{G}{H} \left(\frac{a_0}{a}\right)^{3/2} \mathcal{H}_\nu^2[x(t)], \tag{49}
\]

where \(\mathcal{H}_\nu^2[x(t)]\) is the second-kind Hankel function, \(\nu = [1/(2H)]\sqrt{4\lambda^2 - 3H^2}\) and \(x(t) = (k/H)e^{-Ht}\). The parameter \(\nu\) is well defined for values of \(\lambda\) that satisfy \(\lambda^2 > (3/4)H^2\). For \(\lambda = \pm \sqrt{3}H\) the parameter \(\nu\) results in the value \(\nu = 3/2\), which, as it will be shown in further calculations, is the value corresponding to a scale-invariant spectrum for the mean squared scalar fluctuations of the metric \(\langle \Psi^2 \rangle\). The mean-squared fluctuations for \(\Psi\) on the IR sector (on super Hubble scales), are

\[
\langle \Psi^2 \rangle = \frac{1}{2\pi^2} \int_{\xi_0(k_0(t))}^{\infty} dk \ k^2 \xi_k \xi_k^* = \frac{2^{2\nu-1}}{9\pi^2} a_0^3 G H^2 c_0^3 \Gamma^2(\nu) \left[ \frac{\sqrt{\lambda^2 - (3/4)H^2}}{H} \right]^{3-2\nu}, \tag{50}
\]

where \(\langle \rangle\) is denoting expectation value, \(c_0 \simeq 10^{-3}\) is a dimensionless constant and \(k_0(t) = a\sqrt{\lambda^2 - (3/4)H^2}\). Notice that to obtain the last term of (50), we have employed the asymptotic expansion \(\mathcal{H}_\nu^2 \simeq (i/\pi)\Gamma(\nu)(x/2)^{-\nu}\) valid for \(x \ll 1\). It is easy to show that the energy density fluctuations associated with cosmological \((k \ll k_0)\) metric fluctuations \(\Psi\), are in our case given by

\[
\frac{\delta \rho}{\rho} \simeq 2 < \Psi^2 >^{1/2} \simeq 2 \Psi. \tag{51}
\]

Now, employing the equations (30) and (38) and incorporating the solution (42), we obtain that the 4D quantum fluctuations for the inflaton field are determined by the equation

\[
\frac{\partial^2 \delta \Phi}{\partial t^2} + 3H \frac{\partial \delta \Phi}{\partial t} - e^{-2Ht} \nabla^2_R \delta \Phi + (6H^2 - M^2) \delta \Phi = 0, \tag{52}
\]

where we have used that \([2^2(\partial^2 \delta \varphi / \partial l^2) + 4l(\partial \delta \varphi / \partial l)]|_{l=H^{-1}} = (M^2 / H^2) \delta \Phi\), with \(M\) being a separation constant with mass units. Performing the Fourier expansion of \(\delta \Phi\) in the form

\[
\delta \Phi(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3K \left[ a_K e^{i\vec{K} \cdot \vec{R}} \eta_K(t) + a_K^\dagger e^{-i\vec{K} \cdot \vec{R}} \eta_K^*(t) \right], \tag{53}
\]
where the annihilation and creation operators \(a_K\) and \(a_K^\dagger\) satisfy the commutation algebra given by (16). Inserting (53) into (22) leads to

\[
\frac{\partial^2 \eta_K}{\partial t^2} + 3H \frac{\partial \eta_K}{\partial N} + (K^2 e^{-2Ht} + 6H^2 - M^2) \eta_K = 0.
\] (54)

In this case, due to the commutation relations (27), the normalization condition reads

\[
\dot{\eta}_K \eta_K - \dot{\eta}_K \eta_K^* = i \left( \frac{a_0}{a} \right)^3.
\] (55)

Assuming the vacuum Bunch-Davies condition and using (55), we find that the normalized solution of (54) can be written as

\[
\eta_K(t) = \frac{i}{2} \sqrt{\frac{\pi}{H}} \left( \frac{a_0}{a} \right)^{3/2} \mathcal{H}_\mu^{(2)}[y(t)],
\] (56)

where \(\mu = (1/(2H)) \sqrt{4M^2 - 15H^2}\) and \(y(t) = (K/H)e^{-Ht}\). The parameter \(\mu\) is well defined for \(M^2 > (15/4)H^2\) and it is always positive in this region.

Once we have calculated the normalized modes (56), we are in position to obtain the mean-squared fluctuations for the 4D induced inflaton field: \(\langle \delta \Phi^2 \rangle\). Following a similar procedure to that for calculating \(\langle \Psi^2 \rangle\), we obtain that on super Hubble scales the mean-squared fluctuations \(\langle \delta \Phi^2 \rangle\) are given by

\[
\langle \delta \Phi^2 \rangle = \frac{1}{2\pi^2} \int_{0}^{t_1} K_1(t) dK K^2 \eta_K \eta_K^* = \frac{2^{-3/2\mu} \Gamma^2(\mu)}{\pi^3} \frac{\sqrt{M^2 - (15/4)H^2}}{3 - 2\mu} \left[ \frac{\sqrt{M^2 - (15/4)H^2} - 2\mu}{H} \right]^{3-2\mu} a_0^2 H^2,
\] (57)

where \(K_1(t) = a \sqrt{M^2 - (15/4)H^2}\) and we have also used the asymptotic expansion \(\mathcal{H}_\mu^{(2)} \simeq (i/\pi)\Gamma(\mu)(y/2)^{-\mu}\) valid for \(y \ll 1\) to obtain the last term of (50). From the expression (57) it can be easily seen that the power spectrum related to the inflaton quantum fluctuations \(P_{\delta \Phi}(K) \sim [K/(\alpha H)]^{3-2\mu}\) is scale invariant when \(M = \sqrt{6}H\). Furthermore, the spectral index given by \(n = 4 - 2\mu = 4 - \sqrt{3 + (4\alpha^2 - 18)}\), with \(M = \alpha H\), becomes \(n \simeq 1\) when \(\alpha^2 \simeq 9/2\). In other words, the spectrum becomes nearly scale invariant around this value of the dimensionless parameter \(\alpha\).

### III. FINAL REMARKS

In this letter we have studied the dynamics of scalar metric fluctuations from a 5D vacuum state, which is defined on a 5D background Riemann-flat metric, using ideas of Modern Kaluza-Klein theory. We passed from a 5D Riemann-flat metric to an effective 4D metric which describes the de Sitter expansion. The dimensional reduction has been achieved by taking the foliation \(l = H^{-1}\) on the metric (3), \(H\) being the constant Hubble parameter during inflation. From the point of view of a relativistic observer, this implies that the penta-velocity is null: \(U^i = 0\).

The most remarkable result here obtained is that, as one would expect, the spectrum of the squared \(\Psi\)-fluctuations \(\langle \Psi^2 \rangle\) on cosmological scales can be scale invariant, in contrast with the results obtained using a standard 4D formalism\[10, 16\], where they have a \(k^2\)-spectrum when we take a longitudinal gauge. Of course, our results are valid only on cosmological scales, when vector and tensor perturbations can be neglected. Another interesting result here obtained is that the background value for the inflaton field at the beginning of inflation remains below the Planckian mass: \(\Phi_b = \frac{M}{\sqrt{2\pi}}\) [see Eq. (12)].

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