A DEPENDENT THEORY WITH FEW INDISCERNIBLES

ITAY KAPLAN AND SAHARON SHELAH

Abstract. We continue the work started in [KS12], and prove the following theorem: for every \( \theta \) there is a dependent theory \( T \) of size \( \theta \) such that for all \( \kappa \) and \( \delta \), \( \kappa \rightarrow (\delta)_{T,1} \) iff \( \kappa \rightarrow (\delta)_{\theta}^{\omega} \).

This means that unless there are good set theoretical reasons, there are large sets with no indiscernible sequences.

1. Introduction

In the summer of 2008, Saharon Shelah announced in a talk in Rutgers that he had proved some very important results in dependent (NIP) theories. One of these was the existence of indiscernible, an old conjuncture of his. Here is the definition:

Definition 1.1. Let \( T \) be a theory. For a cardinal \( \kappa \), \( n \leq \omega \) and an ordinal \( \delta \), \( \kappa \rightarrow (\delta)_{T,n} \) means: for every set \( A \subseteq \mathcal{C}^n \) of size \( \kappa \), there is a non-constant sequence of elements of \( A \) of length \( \delta \) which is indiscernible.

In stable theories, it is known that for any \( \lambda \) satisfying \( \lambda = \lambda^{|T|} \), \( \lambda^+ \rightarrow (\lambda^+)_{T,n} \) (proved by Shelah in [She90], and follows from local character of non-forking). The first claim of such nature was proved by Morley in [Mor65] for \( \omega \)-stable theories. Later, in [She12], Shelah proved:

Fact 1.2. If \( T \) is strongly dependent \(^1\), then for all \( \lambda \geq |T| \), \( \beth_{|T|}^+ (\lambda) \rightarrow (\lambda^+)_{T,n} \) for all \( n < \omega \).

This definition was suggested by Grossberg and Shelah and appears in [She86] pg. 208, Definition 3.1(2)], in a slightly different form\(^2\). There it is also proved that such a theorem does not hold for simple unstable theories in general. So the natural generalization is to NIP theories, and indeed it is conjectured already in [She86] pg. 209, Conjecture 3.3].

This conjuncture is connected to a result by Shelah and Cohen: in [CS09], they proved that a theory is stable iff it can be presented in some sense in a free algebra in a fix vocabulary but allowing function symbols with infinite arity. If this result could be extended to: a theory is

---

\(^1\)For more on strongly dependent theories, see Section 6.

\(^2\)The definition there is: \( \kappa \rightarrow (\delta)_{T,n} \) if and only if for each sequence of length \( \kappa \) (of \( n \)-tuples), there is an indiscernible sub-sequence of length \( \delta \). For us there is no difference because we are dealing with examples where \( \kappa \not\rightarrow (\mu)_{T,n} \). It is also not hard to see that when \( \delta \) is a cardinal these two definitions are equivalent.
dependent if it can be represented as an algebra with ordering, then this could be used to prove existence of indiscernibles.

Despite announcing it, there was a mistake in the proof for dependent theories, and here we shall see a counter-example. In the previous paper [KS12], we have shown that:

**Theorem 1.3.** There exists a countable dependent theory $T$ such that:

For any two cardinals $\mu \leq \kappa$ with no (uncountable) strongly inaccessible cardinals in $[\mu, \kappa]$,

$\kappa \not\rightarrow (\mu)_{T,1}$.

And in this paper we improve it by:

**Theorem 1.4.** For every $\theta$ there is a dependent theory $T$ of size $\theta$ such that for all $\kappa$ and $\delta$,

$\kappa \rightarrow (\delta)_{T,1}$ iff $\kappa \rightarrow (\delta)^{<\omega}_{\theta}$.

Where:

**Definition 1.5.** $\kappa \rightarrow (\delta)^{<\omega}_{\theta}$ means: for every coloring $c : [\kappa]^{<\omega} \rightarrow \theta$ there is a homogeneous sub-sequence of length $\delta$ (i.e. there exists $\langle \alpha_i \mid i < \delta \rangle \in \delta \kappa$ and $\langle c_n \mid n < \omega \rangle \in \omega \theta$ such that $c(\alpha_{i_0}, \ldots, \alpha_{i_{n-1}}) = c_n$ for every $i_0 < \ldots < i_{n-1} < \delta$).

One can see that $\kappa \rightarrow (\delta)^{<\omega}_{\theta}$ implies $\kappa \rightarrow (\delta)_{T,1}$, so this is the best result possible.

We also should note that a related result can be found in an unpublished paper in Russian by Kudajbergenov that states that for every ordinal $\alpha$ there exists a dependent theory (but it may be even strongly dependent) $T_\alpha$ such that $|T_\alpha| = |\alpha| + \aleph_0$ and $\beth_\alpha(|T_\alpha|) \not\rightarrow (\aleph_0)_{T_\alpha,1}$ and thus seem to indicate that the bound in Fact 1.2 is tight.

1.1. **The idea of the proof.** The example is a “tree of trees” with functions between the trees. More precisely, for all $\eta$ in the base tree $\omega \geq 2$ we have a unary predicate $P_\eta$ and an ordering $<_\eta$ such that $(P_\eta, <_\eta)$ is a discrete tree. In addition we will have functions $G_{\eta, \eta^+} \{i\} : P_\eta \rightarrow P_{\eta^+}(i)$ for $i = 0, 1$. The idea is to prove that if $\kappa \rightarrow (\delta)^{<\omega}_\theta$ then $\kappa \rightarrow (\delta)_{T,1}$ by induction on $\kappa$, i.e. to prove that we can find a subset of $P(\eta)$ of size $\kappa$ without an indiscernible sequence in it. For $\kappa$ regular but not strongly inaccessible or $\kappa$ singular the proof is similar to the one in [KS12]: we just push our previous examples into deeper levels.

The main case is when $\kappa$ is strongly inaccessible.

We have a coloring $c$ that witnesses that $\kappa \rightarrow (\delta)^{<\omega}_\theta$ and we build a model $M_c$ that uses it. In this model, the base tree will be $\omega$ and not $\omega \geq 2$, i.e. for each $n < \omega$ we have a predicate $P_n$ with tree-ordering $<_n$ and functions $G_n : P_n \rightarrow P_{n+1}$. In addition, $P_0 \subseteq \kappa$. On $P_n$ we will define an equivalence relation $E_n$ refining the neighboring relation ($x, y$ are neighbors if they succeed the same element) so that every class of neighbors (neighborhood) is a disjoint union of less than $\kappa$ many classes of $E_n$. We will prove that if there are indiscernibles in $P_0$, then there is some $n < \omega$...
such that in $P_n$ we get an indiscernible sequence $\langle t_i \mid i < \delta \rangle$ that looks like a fan, i.e. there is some $u$ such that $t_i \wedge t_j = u$ and $t_i$ is the successor of $u$, and in addition $t_i$ and $t_j$ are not $E_n$ equivalent for $i \neq j$.

Now embed $M_c$ in a model of our theory (i.e. now the base tree is again $\omega^2 \geq 2$), and in each neighborhood we send every $E_n$ class to an element from the model we get from the induction hypothesis (as there are less than $\kappa$ many classes, it is possible).

By induction, we get there is no indiscernible sequence in $P_0$ and finish.

1.2. Description of the paper. In Section 2 we give some preliminaries on dependent theories and trees. In Section 3 we describe the theory and prove quantifier elimination and dependence. In Section 4 we prove the theorem up to inaccessible cardinals, and in Section 5 we finish the proof. In Section 6 we explain some of the choices we made during the constructions and discuss strongly dependent theories.

2. Preliminaries

Notation.

We use standard notation. $a, b, c$ are elements, and $\bar{a}, \bar{b}, \bar{c}$ are finite or infinite tuples of elements.

$\mathcal{C}$ will be the monster model of the theory.

$S_n (A)$ is the set of complete types over $A$, and $S^q_n (A)$ is the set of all quantifier free complete types over $A$. For a finite set of formulas with a partition of variables, $\Delta (\bar{x}; \bar{y})$, $S_{\Delta (\bar{x}; \bar{y})} (A)$ is the set of all $\Delta$-types over $A$, i.e. maximal consistent subsets of

$$\{ \varphi (\bar{x}, \bar{a}), \neg \varphi (\bar{x}, \bar{a}) \mid \varphi (\bar{x}, \bar{y}) \in \Delta \& \bar{a} \in A \}.$$  

Similarly we define $\text{tp}_{\Delta (\bar{x}; \bar{y})} (\bar{b}/A)$ as the set of formulas $\varphi (\bar{x}, \bar{a})$ such that $\varphi (\bar{x}, \bar{y}) \in \Delta$ and $\mathcal{C} \models \varphi (\bar{b}, \bar{a})$.

Dependent theories.

For completeness, we give here the definitions and basic facts we need on dependent theories.

Definition 2.1. A first order theory $T$ is dependent if it does not have the independence property which means: there is no formula $\varphi (\bar{x}, \bar{y})$ and elements in $\langle \bar{a}_i, \bar{b}_s \mid i < \omega, s \subseteq \omega \rangle$ in $\mathcal{C}$ such that $\varphi (\bar{a}_i, \bar{b}_s)$ iff $i \in s$.

We shall need the following fact:

Fact 2.2. [She90 II, 4] Let $T$ be any theory. Then for all $n < \omega$, $T$ is dependent if and only if $\square_n$ if and only if $\square_1$ where for all $n < \omega$,

$$\square_n \quad \text{For every finite set of formulas } \Delta (\bar{x}, \bar{y}) \text{ with } n = \lg (x), \text{there is a polynomial } f \text{ such that for every finite set } A \subseteq M = T, |S_\Delta (A)| \leq f (|A|).$$

$^3 \bar{a} \in A$ means: $\bar{a}$ is a tuple of elements from $A$. 

Trees.
Let us remind the reader of the basic definitions and properties of trees.

\textbf{Definition 2.3.} A tree is a partially ordered set \((A, <)\) such that for all \(a \in A\), the set \(A_{<a} = \{x \mid x < a\}\) is linearly ordered.

\textbf{Definition 2.4.} We say that a tree \(A\) is well ordered if \(A_{<a}\) is well ordered for every \(a \in A\).

Assume now that \(A\) is well ordered.

- For every \(a \in A\), denote \(\text{lev} (a) = \text{otp} (A_{<a})\) — the level of \(a\) is the order type of \(A_{<a}\).
- The height of \(A\) is \(\sup \{\text{lev} (a) \mid a \in A\}\).
- \(a \in A\) is a root if it is minimal.
- \(A\) is normal when for all limit ordinals \(\delta\), and for all \(a, b \in A\) such that 1) \(\text{lev} (a) = \text{lev} (b) = \delta\), and 2) \(A_{<a} = A_{<b}, a = b\).
- If \(a < b\) then we denote by \(\text{suc} (a, b)\) the successor of \(a\) in the direction of \(b\), i.e. \(\min \{c \leq b \mid a < c\}\).
- We write \(a <_{\text{suc}} b\) if \(b = \text{suc} (a, b)\).
- We call \(A\) standard if it is well ordered, normal, and has a root.

For a standard tree \((A, <)\), define \(a \wedge b = \max \{c \mid c \leq a \& c \leq b\}\).

3. Construction of the example

\textbf{The first order theory.}

\textbf{The language:}

Let \(S\) be a standard tree, and let \(L^S\) be the language:

\[ \{P_\eta, <_\eta, \wedge_\eta, G_{\eta_1, \eta_2}, \text{suc}_\eta, \text{pre}_\eta, \text{lim}_\eta \mid \eta, \eta_1, \eta_2 \in S, \eta_1 <_{\text{suc}} \eta_2 \} \] .

Where:

- \(P_\eta\) is a unary predicate, \(<_\eta\) is a binary relation symbol, \(\wedge_\eta\) and \(\text{suc}_\eta\) are 2 place function symbols, \(G_{\eta_1, \eta_2}\), \(\text{pre}_\eta\), and \(\text{lim}_\eta\) are 1 place function symbols.

\textbf{Definition 3.1.} Let \(L'_S = \{P_\eta, <_\eta, \wedge_\eta, G_{\eta_1, \eta_2}, \text{lim}_\eta \mid \eta \in S, \eta_1 <_{\text{suc}} \eta_2 \in S\}\) (i.e. \(L_S\) without \(\text{pre}\) and \(\text{suc}\)).

\textbf{The theory:}

\textbf{Definition 3.2.} The theory \(T'_S\) says:

- \((P_\eta, <_\eta)\) is a tree.
- \(\eta_1 \neq \eta_2 \Rightarrow P_{\eta_1} \cap P_{\eta_2} = \emptyset\).
- \(\wedge_\eta\) is the meet function: \(x \wedge_\eta y = \max \{z \in P_\eta \mid z \leq_\eta x \& z \leq_\eta y\}\) for \(x, y \in P_\eta\) (so its existence is part of the theory).
• \( \text{suc}_\eta \) is the successor function — for \( x, y \in P_\eta \) with \( x \prec_\eta y \), \( \text{suc}_\eta (x, y) \) is the successor of \( x \) in the direction of \( y \) (i.e. the trees are discrete). The axioms are:
  - \( \forall x \prec_\eta y \ (x \prec_\eta \text{suc}_\eta (x, y) \leq_\eta y) \), and
  - \( \forall x \leq_\eta y \ \exists z \leq_\eta \text{suc}_\eta (x, y) [z = x \lor z = \text{suc}_\eta (x, y)] \).

• \( \lim_\eta (x) \) is the greatest limit element below \( x \). Formally,
  - \( \lim_\eta : P_\eta \rightarrow P_\eta \), \( \forall \lim_\eta (x) \leq_\eta x, \ \forall x \prec_\eta y \ (\lim_\eta (x) \leq_\eta \lim_\eta (y)) \),
  - \( \forall x \prec_\eta y \ (\lim_\eta (\text{suc}_\eta (x, y)) = \lim_\eta (x)), \ \forall x \lim_\eta (\lim_\eta (x)) = \lim_\eta (x) \).

• Define that \( x \) is a successor if \( \lim_\eta (x) \prec_\eta x \), and denote
  \[
  \text{Suc}(P_\eta) = \{ x \in P_\eta | \lim_\eta (x) \prec_\eta x \}.
  \]

• \( \text{pre}_\eta \) is the immediate predecessor function —
  \( \forall x \neq \lim_\eta (x) \ (\text{pre}_\eta (x) < x \land \text{suc}_\eta (\text{pre}_\eta (x), x) = x) \).

• (regressive) If \( \eta_1 \prec_\text{suc} \eta_2 \) then \( G_{\eta_1, \eta_2} \) satisfies: \( G_{\eta_1, \eta_2} : \text{Suc}(P_{\eta_1}) \rightarrow P_{\eta_2} \) and if \( x \prec_{\eta_1} y \), both \( x \) and \( y \) are successors, and \( \lim_\eta (x) = \lim_\eta (y) \) then \( G_{\eta_1, \eta_2} (x) = G_{\eta_1, \eta_2} (y) \).

**Example 3.3.** Assume that \( (M, <) \) is a standard tree. Then \( M \) has a natural structure for the language \( L(\emptyset) \) (where \( \text{lim}^M_\eta (a) \) is the greatest element below \( a \) of a limit level).

**Model completion.**

Here we prove that this theory has a model completion. As models may be infinite, we cannot use Fraïssé’s theorem.

If \( S_1, S_2 \) are standard trees, we shall treat them as structures in the language \( \{ \prec_{\text{suc}}, < \} \), so when we write \( S_1 \subseteq S_2 \), we mean that \( S_1 \) is a substructure of \( S_2 \) in this language. Sometime we write \( <, \text{suc} \) instead of \( \text{suc}_\eta, \prec_\eta \) or \( \text{suc}^M_\eta, \prec^M_\eta \) where \( M \) and \( \eta \) are clear from the context.

**Remark 3.4.**

1. \( T^\emptyset_{S_2} \) is a universal theory.
2. \( T^\emptyset_{S_2} \) has the joint embedding property (JEP).
3. If \( S_1 \subseteq S_2 \) then \( T^\emptyset_{S_1} \subseteq T^\emptyset_{S_2} \) and moreover, if \( M \models T^\emptyset_{S_2} \) is existentially closed, \( M \models L_{S_1} \) is an existentially closed model of \( T^\emptyset_{S_1} \).

Assume from now until Corollary 3.25 below that \( S \) is finite.

**Definition 3.5.**

1. Suppose \( \Sigma \) is a finite set of terms from \( L_S \). We define the following closure operators on
terms:
   a. \( \text{cl}_\wedge (\Sigma) = \Sigma \cup \{ \wedge_\eta (\Sigma^2) | \eta \in S \} = \Sigma \cup \{ t_1 \wedge_\eta t_2 | t_1, t_2 \in \Sigma \} \).
   b. \( \text{cl}_G (\Sigma) = \Sigma \cup \{ G_{\eta_1, \eta_2} (\Sigma) | \eta_1 \prec_\text{suc} \eta_2 \in S \} \).
   c. \( \text{cl}_{\lim} (\Sigma) = \Sigma \cup \{ \lim_\eta (\Sigma) | \eta \in S \} \).
(d) \( \text{cl}^0 (\Sigma) = \text{cl}_G (\text{cl}_{\lim} (\text{cl}_G (\ldots (\text{cl}_G (\text{cl}_{\lim} (\text{cl}_G (\Sigma))))))) \) where the number of compositions is the length of the longest branch in \( S \).

(e) \( \text{cl}_{\text{suc}} (\Sigma) = \bigcup \{ \text{suc}_\eta (\Sigma^2) \cup \text{pre}_\eta (\Sigma) \mid \eta \in S \} \cup \Sigma \).

(f) \( \text{cl} (\Sigma) = \text{cl}^0 (\text{cl}_{\text{suc}} (\Sigma)) \).

(2) Denote \( \text{cl}^{(0)} = \text{cl}^0 \) and for a number \( k < \omega \), \( \text{cl}^{(k)} (A) = \text{cl} (\ldots \text{cl} (\Sigma)) \) where the composition is done \( k \) times.

In the same way, if \( \bar{t} = \langle t_i \mid i < n \rangle \) is an \( n \)-tuple of terms then \( \text{cl} (\bar{t}) \) is \( \text{cl} (\{ t_i \mid i < n \}) \).

If \( S' \subseteq S \), we define \( \text{cl}_{S'} (\Sigma) \) by restricting all operations to those from \( L_{S'} \).

(3) For a model \( M \models T^r_S \), and \( \bar{a} \in M \), define \( \text{cl}^{(0)} (\bar{a}) = (\text{cl}^{(0)} (\bar{x}))^M (\bar{a}) \) where \( \bar{x} \) is a sequence of variables in the length of \( \bar{a} \). In the same way define \( \text{cl}_\lambda (\bar{a}), \text{cl}_{\lim} (\bar{a}) \), \( \text{cl}_G (\bar{a}) \), \( \text{cl}_{\text{suc}} (\bar{a}) \) and \( \text{cl}^{(k)} (\bar{a}) \). For a set \( A \subseteq M \), define \( \text{cl}^{(0)} (A) = \text{cl}^{(0)} (\bar{a}) \) where \( \bar{a} \) is an enumeration of \( A \), and in the same way the other closure operators.

Claim 3.6. For \( A \subseteq M \), \( \text{cl}^{(0)} (A) \) is closed under \( \wedge, \lim, \text{and} \ G_{\eta_1, \eta_2} \). So it is the substructure generated by \( A \) in the language \( L_S' \).

Proof. It is easy to see that \( \text{cl}_{\lim} (\text{cl}_\lambda (A)) \) is closed under \( \lim \) and \( \wedge \) for all \( \eta \in S \).

For \( n < \omega \), let \( \text{cl}^{(n)} (A) = \text{cl}_G (\text{cl}_{\lim} (\ldots (\text{cl}_G (\text{cl}_{\lim} (\text{cl}_G (A)))))) \) where there are \( n \) compositions. For \( \eta \in S \), let \( r (\eta) = r_S (\eta) = | \{ \nu \in S \mid \nu \leq \eta \} | \), so \( \text{cl}^{(0)} = \text{cl}^{0}_{\max \{ r (\eta) \mid \eta \in S \}} \). Let \( B \supseteq A \) be the closure of \( A \) under these operations in \( M \). Then \( B \) is in fact \( \text{cl}^{(0)} (A) = \bigcup \{ \text{cl}^{(0)} (A) \mid n < \omega \} \). One can prove by induction on \( r (\eta) \) that \( B \cap P_\eta = \text{cl}^{(0)} (A) \cap P_\eta \). Hence \( B = \text{cl}^{(0)} (A) \).

Claim 3.7. For every \( k < \omega \), there is a polynomial \( f_k \) such that for every finite \( A \), \( | \text{cl}^{(k)} (A) | \leq f_k (|A|) \).

Proof. Obvious.

Definition 3.8.

1. For a term \( t \), we define its successor rank as follows: if \( \text{suc} \) and \( \text{pre} \) do not appear in \( t \), then \( r_{\text{suc}} (t) = 0 \). For two terms \( t_1, t_2 \): \( r_{\text{suc}} (\text{suc}_\eta (t_1, t_2)) = \max \{ r_{\text{suc}} (t_1), r_{\text{suc}} (t_2) \} + 1 \), \( r_{\text{suc}} (\text{pre}_\eta (t_1)) = r_{\text{suc}} (t_1) + 1, r_{\text{suc}} (t_1 \wedge t_2) = \max \{ r_{\text{suc}} (t_1), r_{\text{suc}} (t_2) \}, r_{\text{suc}} (G_{\eta_1, \eta_2} (t_1)) = r_{\text{suc}} (t_1) \) and \( r_{\text{suc}} (\text{lim}_\eta (t_1)) = r_{\text{suc}} (t_1) \).

2. For a quantifier free formula \( \varphi \), let \( r_{\text{suc}} (\varphi) \) be the maximal rank of a term appearing in \( \varphi \).

3. For \( k < \omega \) and an \( n \)-tuple of variables \( \bar{x} \), denote by \( \Delta^1_k \bar{x} \) the set of all atomic formulas \( \varphi (\bar{x}) \) such that for every term \( t \) in \( \varphi, t \in \text{cl}^{(k)} (\bar{x}) \). Note that since \( \text{cl}^{(k)} (\bar{x}) \) is a finite set, so is \( \Delta^1_k \bar{x} \).

Claim 3.9. Assume that \( M \models T^r_S \) and \( \bar{a} \in M \). Then \( \text{cl}^{(k)} (\bar{a}) = \{ t^M (\bar{a}) \mid r_{\text{suc}} (t) \leq k \} \).

Proof. Easy (by induction on \( k \)).
Definition 3.10. Assume that \(M_1, M_2 \models T^y_S\) and \(a \in M_1, b \in M_2\). For \(k < \omega\), we say that \(a \equiv_k b\) if there is an isomorphism of \(L'_S\) from \(\text{cl}^{(k)}(a)\) to \(\text{cl}^{(k)}(b)\) taking \(a\) to \(b\). For \(S' \subseteq S\), denote \(\bar{a} \equiv_{S'}^S \bar{b}\) when the isomorphism is of \(L'_{S'}\) and from \(\text{cl}^{(k)}_{S'}(a)\) to \(\text{cl}^{(k)}_{S'}(b)\).

In this notation we assume that \(M_1, M_2\) are clear from the context.

Definition 3.11. For \(\bar{a} \in M \models T^y_S\) a finite tuple, \(A \subseteq M\) a finite set, and \(k < \omega\), let \(\text{tp}_k(\bar{a}/A) = \text{tp}_{\Delta^y_S(x;\bar{y})}(\bar{a}/\bar{y})\) where \(\bar{y}\) is of length \(|A|\). For \(S' \subseteq S\), \(\text{tp}_{S'}^S(\bar{a}/A)\) is defined in the same way where we reduce to \(L'_{S'}\) formulas.

Claim 3.12. Assume \(M_1, M_2 \models T^y_S\). Assume \(\bar{a} \in M_1, \bar{b} \in M_2\) and \(k < \omega\). Then \(\bar{a} \equiv_k \bar{b}\) iff \(\text{tp}_k(\bar{a}) = \text{tp}_k(\bar{b})\) iff for every quantifier free formula \(\varphi(\bar{x})\) with \(r_{\text{suc}}(\varphi) \leq k\),

\[M_1 \models \varphi(\bar{a}) \iff M_2 \models \varphi(\bar{b}).\]

Proof. Assume \(\bar{a} \equiv_k \bar{b}\) and \(f : \text{cl}^{(k)}(\bar{a}) \to \text{cl}^{(k)}(\bar{b})\) is an \(L'_S\)-isomorphism taking \(\bar{a}\) to \(\bar{b}\). It is easy to see that for every term \(t \in \text{cl}^{(k)}(\bar{x})\), \(f(t(\bar{a})) = t(\bar{b})\), and so \(\text{tp}_k(\bar{a}) = \text{tp}_k(\bar{b})\).

On the other hand, if \(\text{tp}_k(\bar{a}) = \text{tp}_k(\bar{b})\), then define \(f\) by: for every term \(t \in \text{cl}^{(k)}(\bar{x})\), \(f(t(\bar{a})) = t(\bar{b})\).

As the types are equal, \(f\) is well defined and is an isomorphism.

The last statement is obviously strongly than the second one. The converse follows by Claim 3.9 for every term \(t(\bar{x})\) with rank \(r_{\text{suc}}(t) \leq k\) there is a term \(t' \in \text{cl}^{(k)}(\bar{x})\) such that \(M_1 \models t'(\bar{a}) = t(\bar{a})\). By induction on \(k\) and \(t\), one can show that \(M_2 \models t'(\bar{b}) = t(\bar{b})\) and that suffices. \(\square\)

Similarly, we have:

Claim 3.13.

1. if \(\bar{a} \equiv_k \bar{b}\) then there is a unique isomorphism that shows it.
2. Assume \(k_2 \geq k_1\). Then \(\bar{a} \equiv_{k_2} \bar{b}\) implies \(\bar{a} \equiv_{k_1} \bar{b}\).
3. If \(\bar{a}' \equiv_k \bar{b}'\) then \(\bar{a} \equiv_k \bar{b}\).
4. If \(\bar{a} \equiv_{k+1} \bar{b}\) then \(\text{cl}(\bar{a}) \equiv_k \text{cl}(\bar{b})\).
5. Claim 3.12 and (1) – (4) are still true when we replace \(\equiv_k\) by \(\equiv_{S'}^S\) for \(S' \subseteq S\) (with the obvious adjustments).

Definition 3.14. For \(x <_n y\) from \(P_n\) in a model of \(T^y_S\), we say that the distance between \(x\) and \(y\) is \(n\) if \(y\) is the \(n\)-th successor of \(x\) or vice-versa. We say the distance is infinite if for no \(n < \omega\) the distance is \(n\). Denote \(d(x,y) = n\).

Lemma 3.15. (Quantifier elimination lemma) For all \(m_1, k < \omega\) there is \(m_2 = m_2(m_1, k, S) < \omega\) such that if:

- \(M_1, M_2 \models T^y_S\) are existentially closed.
- \(\bar{a} \in M_1\) and \(\bar{b} \in M_2\).
\[
\bullet \quad \bar{a} \equiv_{m_2} \bar{b}.
\]

Then for all \(\bar{c} \in kM_1\) there is some \(\bar{d} \in kM_2\) such that \(\bar{c} \bar{a} \equiv_{m_1} \bar{d} \bar{b}\).

**Proof.** Given \(S\), it is enough to prove the lemma for \(k = 1\), since we can define by induction \(m_2(m_1, k + 1, S) = m_2(m_2(m_1, k, S), 1, S)\).

The proof is by induction on \(|S|\). We may assume that \(m_2(m_1, k, S') > \max\{m_1, k, |S'|\}\) for \(|S'| < |S|\) (by enlarging \(m_2\) if necessary).

For \(|S| = 0\) the claim is trivial because \(T^S_0\) is just a theory of a set with no structure.

Assume \(0 < |S|\). Let \(\eta_0\) be the root of \(S\) and denote \(S_0 = \{\eta_0\}\) and \(S = \{\eta_0\} \cup \bigcup \{S_i | 1 \leq i < k\}\) where the \(S_i\)'s are the connected components of \(S\) above \(\eta_0\). Let

\[m_2 = m_2(m_1, 1, S) = \max\{2m_2(m_1, K, S_i) | 1 \leq i < k\} + 2m_1 + 1\]

where \(K\) is a natural number independent of \(m_1, k\) and \(S\) that will be described below.

Let \(P_{\eta} = \bigvee (P_\eta | \eta \in S_i)\). Suppose \(M_1, M_2, \bar{a}\) and \(\bar{b}\) are as in the lemma.

Let \(f : \text{cl}(m_2)(\bar{a}) \rightarrow \text{cl}(m_2)(\bar{b})\) be the isomorphism showing that \(\bar{a} \equiv_{m_2} \bar{b}\) and we are given \(c\).

For \(1 \leq i \leq k\), let \(A_i = \text{cl}(m_1)(\bar{a}) \cap P_{S_i}^{M_1}\) and \(B_i = \text{cl}(m_1)(\bar{b}) \cap P_{S_i}^{M_1}\).

Since \(\bar{a} \equiv_{m_2} \bar{b}\), it follows that \(\text{cl}(m_2(m_1, K, S_i))(\bar{a}) \equiv_{m_2(m_1, K, S_i)} \text{cl}(m_2(m_1, K, S_i))(\bar{b})\) and in particular \(A_i \equiv_{S_i(m_2, K, S_i)} B_i\) when we think of \(A_i\) and \(B_i\) enumerated in a way that \(f\) witnesses this.

We divide into cases:

**Case 1.** \(c \not\in P_{\eta_i}^{M_i}\) for every \(\eta \in S\).

Here finding \(d\) is easy.

**Case 2.** \(c \in P_{S_i}^{M_i}\) for some \(1 \leq i \leq k\).

\(A_i \equiv_{S_i(m_2, K, S_i)} B_i\), so by the induction hypothesis (and by Remark 3.4 (3)) we can find \(d \in M_2\) and expand \(f \upharpoonright \text{cl}(m_1)(A_i)\) to \(f' : \text{cl}(m_1)(cA_i) \rightarrow \text{cl}(m_1)(dB_i)\). It follows that

\[f \upharpoonright \text{cl}(m_1)(\bar{a}) \cup f' \upharpoonright \text{cl}(m_1)(c\bar{a})\]

is an isomorphism from \(\text{cl}(m_1)(c\bar{a})\) to \(\text{cl}(m_1)(d\bar{b})\) that shows that \(c\bar{a} \equiv_{m_1} d\bar{b}\) (note that \(P_{S_i}^{M_i} \cap \text{cl}(m_1)(\bar{ac}) = A_j\) for \(j \neq i\) and that if \(x \in \text{cl}(m_1)(\bar{ac}) \cap P_{S_i}^{M_1}\) then \(x \in \text{cl}(m_1)(cA_i)\), and so the domain is indeed \(\text{cl}(m_1)(c\bar{a})\)).

**Case 3.** \(c \in P_{\eta_0}\).

Let \(A_0 = \text{cl}(0)(\bar{a}) \cap P_{\eta_0}^{M_1}\), \(B_0 = \text{cl}(0)(\bar{b}) \cap P_{\eta_0}^{M_2}\) and \(\eta_i = \min(S_i)\) for \(1 \leq i \leq k\).

Note that \(\text{cl}(A_0) \cap P_{\eta_0}^{M_1} = \text{cl}_{\text{auc}}(A_0)\) and the same is true for \(\text{cl}(0)(A_0c)\). Let \(F = \text{cl}(0)(A_0c) \cap P_{\eta_0}^{M_1}\). For notational simplicity, let \(c \equiv_{\eta_0} c\). Let \(\lim_{\eta_0} A_0 = \lim_{\eta_0} c\).

The main idea is:
Say that an element of $c\mathit{l}^{(m_1)}(A_0c) \cap P_{\eta_0} = c\mathit{l}^{(m_1)}_{\text{suc}}(F)$ is new if it is a successor and not in $c\mathit{l}^{(m_1)}(A_0)$. If $e$ is a new element, and it is in $c\mathit{l}^{(r)}_{\text{suc}}(F)$ for $1 < r$, then as for $1 \leq i \leq k$, the function $G_{\eta_0, \eta_i}$ is regressive, there is some $e' \in c\mathit{l}^{(r)}_{\text{suc}}(F)$ such that $G_{\eta_0, \eta_i}(e) = G_{\eta_0, \eta_i}(e')$. By the case study below, there are at most $K$ new elements in $c\mathit{l}^{(r)}_{\text{suc}}(F)$. Enumerate them by $\langle \epsilon_i \vert l < K \rangle$ and denote their images by $\epsilon_i = G_{\eta_0, \eta_i}(\epsilon_i)$. Fix $1 \leq i \leq k$. By assumption, $A_i = M_{\eta_2, m_1, N, S_i} B_i$ (and $f$ witnesses this), so by the induction hypothesis there is $d_i^l \in M_2$ for $l < K$ such that $\langle \epsilon_i \vert l < K \rangle A_i = S_{m_1, i}$, $\langle d_i \vert l < K \rangle B_i$ as witnessed by some isomorphism $g_i$ extending $f \upharpoonright A_i$. By choice of $m_2$ it follows (see below, in the division to subcases) that we can find a model of $T_{\eta_0}^\Psi$: $M_3' \supseteq P_{\eta_0}^{M_3}$ and some $d' \in M_3'$ such that $f \upharpoonright A_0$ can be extended to some $f'$ showing that $A_0c = M_{\eta_1} B_0 d'$ (we may assume that $M_{\eta_1}'$ is the minimal structure generated by $P_{\eta_0} M_{\eta_1}$ and $d'$). Let $M_3$ be a model of $T_{\eta_0}^\Psi$ satisfying $P_{\eta_0}^{M_3} = P_{\eta_0}^{M_3'}, M_3 \supseteq M_2$ and $G_{\eta_0, \eta_i}(f'(\epsilon_i)) = d_i^l$ (for $l < K$ and $1 \leq i \leq k$). It exists — all we need to check is that $G_{\eta_0, \eta_i}$ is well defined, but this is clear because $A_0c = M_{\eta_1} = M_{\eta_1} B_0 d'$ and $\langle d_i^l \vert l < K \rangle A_i = M_{\eta_1} \langle d_i \vert l < K \rangle B_i$.

Now it is easy to see that $$f \upharpoonright c\mathit{l}^{(m_1)}(\bar{a}) \cup f' \cup \bigcup \{ g_i \upharpoonright c\mathit{l}^{(m_1)}(\bar{ac}) \vert 1 \leq i < k \}$$ is the isomorphism extending $f \upharpoonright c\mathit{l}^{(m_1)}(\bar{a})$. So $\bar{c} \bar{a} \equiv M_{\eta_1} \bar{d}' \bar{b}$, i.e. $\text{tp}_{M_{\eta_1}}(\bar{c} \bar{a}) = \text{tp}_{M_{\eta_1}}(\bar{d}' \bar{b})$, and if $\Psi$ is the conjunction of all formulas appearing in $\text{tp}_{M_{\eta_1}}(\bar{c} \bar{a})$ then $M_3 \models \exists x \Psi(\bar{d} \bar{b})$.

As $M_3$ is existentially closed there is some $d \in M_2$ such that $\Psi(\bar{d} \bar{b})$, i.e. $\bar{c} \bar{a} \equiv M_{\eta_1} \bar{d} \bar{b}$.

Now we shall analyze the different subcases and show that there really is a bound $K$ for the number of new elements in $c\mathit{l}^{(r)}_{\text{suc}}(A_0c)$ and that we can find $d'$ as above.

Case i. $c \in A_0$: obvious — there are no new elements.

Case ii. $c$ is in a branch of $A_0$, i.e. there is $c < y \in A_0$ and assume $y$ is minimal in this sense. We again divide into cases:

Case a. There is no $x \in A_0$ below $c$. The new elements from $c\mathit{l}^{(r)}_{\text{suc}}(F)$ are contained in $D = \{ c, \text{pre}(c) \cup \text{suc}(c, y) \cup \text{suc}(\text{lim}(c), c) \}$. Now it is easy to see that $d'$ and $M_3'$ exist (note that here $y = \text{lim}(y)$).

Case b. There is some $x \in A_0$ such that $x < c$. Assume $x$ is maximal in this sense. New elements again come from $D$.

If $\text{lim}(x) < \text{lim}(y)$ then necessarily $\text{lim}(x) \leq x < c < \text{lim}(y) = y$ and finding $d'$ and $M_3'$ is easy.

If, on the other hand, $\text{lim}(x) = \text{lim}(y)$, we must make sure that the distance between $f(x)$ and $f(y)$ is big enough, so that we can place $d'$ in the right spot between them. In $c\mathit{l}^{(m_1)}_{\text{suc}}(F) \setminus c\mathit{l}^{(m_1)}_{\text{suc}}(A_0)$
we may add $m_1$ successors to $c$ in the direction of $y$ and $m_1$ predecessors. This is why we chose $m_2 \geq 2m_1 + 1$.

**Case iii.** $c$ starts a new branch in $A_0$, i.e. there is no $y \in A_0$ such that $c < y$. In this case, let $c' = \{ \max (c \land b) \mid b \in A_0 \}$. Note that if there is an element in $\text{cl}_\lambda (A_0c) \setminus A_0c$, it must be $c'$. Adding $c'$ falls under Subcase 3.12 (if it is indeed new), so the new elements will be those which come from $c'$ as before, and perhaps

$$\{ c, \text{pre}(c), \text{suc}(c), \text{suc}(\lim (c), c) \}.$$ 

By the previous case, we can find $M_3'' \supseteq P_{M_3}^{M_1}$ and $d'' \in M_3''$ such that $c'A_0 \equiv_{m_1} d'B_0$, and then we can find $M_3' \supseteq M_3''$ and $d' \in M_3'$ such that $cA_0 \equiv_{m_1} d'B_0$ and we are done.

**Claim 3.16.** For every formula $\varphi (\bar{x})$ (with free variables) there is a quantifier free formula $\psi (\bar{x})$ such that for every existentially closed model $M \models T_5^v$, we have $M \models \psi \equiv \varphi$.

**Proof.** It is enough to check formulas of the form $\exists y \varphi (y, \bar{x})$ where $\varphi$ is quantifier free and $\lg (\bar{x}) = n > 0$. Let $k = r_{\text{succ}} (\varphi)$. Let $m = m_2 (k, 1, S)$ from Lemma 3.15. By Claim 3.12 if $M_1, M_2 \models T_5^v$ are existentially closed and $a \in M_1, b \in M_2$ are of length $n$ and $a \equiv_m b$, then $M_1 \models \exists y \varphi (y, a)$ iff $M_2 \models \exists y \varphi (y, b)$.

Assume $|\Delta_m^2| = N$ and let $\{ \varphi_i \mid i < N \}$ be an enumeration of $\Delta_m^2$. For every $\eta : N \rightarrow 2$, let $\varphi_m^\eta (\bar{x}) = \bigwedge_{i < N} \varphi_i^{\eta(i)} (\bar{x})$ (where $\varphi^0 = \neg \varphi$ and $\varphi^1 = \varphi$).

Let

$$R = \{ \eta : N \rightarrow 2 \mid \exists \text{ e.c. } M \models T_5^v \& \exists \bar{c} \in M \left( M \models \varphi_m^\eta (\bar{c}) \land \exists y \varphi (y, \bar{c}) \right) \}.$$ 

Let $\psi (\bar{x}) = \bigvee_{\eta \in R} \varphi_m^\eta (\bar{x})$. By Claim 3.12 it follows that $\psi$ is the desired formula. \qed

**Corollary 3.17.** If $M_1$ and $M_2$ are two existentially closed models of $T_5^v$ then $M_1 \equiv M_2$ and their theory eliminates quantifiers.

**Proof.** Assume first that $M_1 \subseteq M_2$, then $M_1 \prec M_2$: for formulas with free variables it follows directly from the previous claim, and for a sentence $\varphi$ we consider the formula $\varphi \land (x = x)$.

Now the corollary follows from the fact that the theory is universal (so every model can be extended to an existentially closed one) and has JEP. \qed

**Definition 3.18.** Let $T_5$ be the theory of all existentially closed models of $T_5^v$. This is the model completion of $T_5^v$. 
Dependence.

Now let us prove that $T_S$ is dependent.

**Definition 3.19.** Assume $A \subseteq M \models T_S$ is a finite set and $k < \omega$.

(1) We say that $a, b \in M$ are $k$-isomorphic over $A$, denoted by $a \equiv_{A,k} b$ iff $aA \equiv_k bA$ (i.e. there is an isomorphism $f : cl_{L_S}(aA) \to cl_{L_S}(bA)$ of $L_S$ taking $a$ to $b$ and fixing $A$ point-wise).

(2) Similarly for tuples from $M^{<\omega}$.

(3) Similarly for $S' \subseteq S$, and we shall write $a \equiv_{S',A,k} b$ for $aA \equiv_{S',k} bA$.

**Claim 3.20.** Assume $M \models T_S^\eta$, $k < \omega$, $A \subseteq M$ is finite and $\bar{a}, \bar{b} \in M$. Then $\bar{a} \equiv_{A,k} \bar{b}$ iff $tp_k(\bar{a}/A) = tp_k(\bar{b}/A)$ iff for every quantifier free formula over $A$, $\varphi(\bar{x})$ such that $r_{\text{suc}}(\varphi) \leq k$, $M \models \varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$. The analogous claim is true for every $S' \subseteq S$ when we reduce to $L_{S'}$ formulas.

**Proof.** Follows from the definitions and from Claim 3.12. □

**Claim 3.21.** Assume $|S| = 1$ and $k < \omega$. Then there is a polynomial $f$ such that for every finite set $A \subseteq M$, $|\{M/ \equiv_{A,k}\}| \leq f(|A|)$.

**Proof.** As $|S| = 1$, we can forget the index $\eta$ and write $<, \lim$, etc. instead of $<_\eta, \lim_\eta$, etc.

Given $a < b \in M$, the $k$-distance between them is defined by

$$d_k(a, b) = \min\{d(a, b), 2k + 1\}.$$  

Assume $a \in M$ and $A \subseteq M$ is finite.

Let $B = cl^0(A)$, and enumerate $B$ by $\{b_i | i < l\}$. Recall that $l \leq f_0(|A|)$ (see Claim 3.7), hence it is polynomially bounded.

The following data is enough to determine the $k$-isomorphism type of $a$ over $A$.

(1) Is $a \in P$ or not?

(a) If not, then it is enough to know whether $a \in B$, and if yes, for which $i < l$ is it that $a = b_i$.

If yes:

(2) What is the $k$-distance between $a$ and $\lim(a)$?

(3) Is there some $i < l$ such that $a \leq b_i$?

If yes:

(a) Is $a = b_i$? if not, what is the minimal element such that this happens (recall that $B$ is closed under $\wedge$, so if there is some $i < l$ such that $a < b_i$, there must be a minimal one). Call it $b_i$.

(b) Is $\lim(a) = \lim(b_i)$?

If yes:

(i) What is the $k$-distance between $a$ and $b_i$?

(ii) What is the maximal $b_j$ such that $b_j \leq a$?
(iii) What is the $k$-distance between $a$ and $b_j$?
If not (i.e. $\lim(a) \neq \lim(b_i)$):
(i) Is there some $b_j$ such that $b_j \leq a$?
If yes:
(A) What is the maximal such $b_j$?
(B) Is it true that $\lim(a) = \lim(b_j)$? If yes what is the $k$-distance between $a$ and $b_j$?
If not (there is no $b_i$):
(a) Let $a' = \max\{a \land b_i \mid i < l\}$.
(b) Give the same information for $a'$ (of course $a'$ is smaller than or equal to some $b_i$).

There is a polynomial bound on the number of $k$-isomorphism types for $a'$ (by the analysis above).
(c) Is it true that $\lim(a) = \lim(a')$?
(i) If yes, what is the $k$-distance between $a$ and $a'$?

That is enough. Since the number of possible answers for every question is polynomially bounded (in $k$ and $l$), there is such a polynomial. □

Claim 3.22. For every $S$ the following are equivalent:

(1) $T_S$ is dependent.
(2) For $k < \omega$ there is a polynomial $f$ such that for every model $M \models T$ and finite $A \subseteq M$,
\[ |\{M/\models A,k\}| \leq f(|A|). \]
(3) For all $k,n < \omega$, there is a polynomial $f$ such that for every model $M \models T$ and for every finite $A \subseteq M$,
\[ |\{M^n/\models A,k\}| \leq f(|A|). \]

Proof. The claim follows from Fact 2.2.

Let $A \subseteq M \models T_S$ be finite. Then $\bar{a} \equiv_{A,k} \bar{b}$ i.f. $tp_k(\bar{a}/A) = tp_k(\bar{b}/A)$ (by Claim 3.20). So (2) is equivalent to the claim that there is a polynomial bound on the size of $S_{\Delta_k^{xy}(\bar{x};\bar{y})}(A)$ (where $\lg(\bar{y}) = |A|$) in terms of $|A|$ when $\lg(\bar{x}) = 1$, and (3) for $\lg(\bar{x}) = n$. Hence (1) obviously implies (2) and (3).

Let $\Delta$ be a finite set of formulas in $\bar{x}\bar{y}$. By quantifier elimination, we may assume that $\Delta$ is quantifier free. Let $k = \max\{r_{\text{suc}}(\varphi) \mid \varphi \in \Delta\}$ and $m = |S_{\Delta(\bar{x};\bar{y})}(A)|$. Let $\{\bar{c}_i \mid i < m\}$ be a set of tuples satisfying all the different types in $S_{\Delta(\bar{x};\bar{y})}(A)$ in some model of $T_S$. If $i \neq j$ then $tp_k(\bar{c}_i/A) \neq tp_k(\bar{c}_j/A)$ (by Claim 3.20), so $m \leq |\{M^{\lg(\bar{x})}/\equiv_{A,k}\}|$, and hence (2) and (3) imply (1) by Fact 2.2 □

Corollary 3.23. $T_S$ is dependent.
Proof. The proof is by induction on $|S|$. For $|S| = 1$ it follows from Claim 3.21 and Claim 3.22 (and for $|S| = 0$ it is obvious).

Assume $1 < |S|$, and we shall show (2) from Claim 3.21. Assume we are given $k < \omega$.

Let $\eta_0$ be the root of $S$ and let $S = \{ \eta_0 \} \cup \bigcup \{ S_i | 1 \leq i < m \}$ where the $S_i$’s are the connected components of $S$ above $\eta_0$. Denote $S_0 = \{ \eta_0 \}$, and $P_{S_i} = \bigvee (P_{\eta} | \eta \in S_i)$. Let $\eta_i = \min (S_i)$.

Assume $A \subseteq M \models T_x$ is finite and $a \in M$. Let $B = cl^{(0)} (A)$, $A_0 = B \cap P_{\eta_0}$ and for every $1 \leq i < m$ let $A_i = cl^{(k)} (A) \cap P_{S_i}^M$.

As in the proof of Claim 3.21, the data below suffice to determine the $k$-isomorphism type of $a$ over $A$. In every stage there is a polynomial bound (in terms of $|A|$) on the number of possible answers.

- Is it true that $a \in B$? If yes, then determine which element (after choosing an enumeration of $B$ as in Claim 3.21).
- Is there some $\eta \in S$ such that $a \in P_{\eta}^M$? If not, we are done. If yes which is it?
- If $\eta \in S_i$ for some $1 \leq i \leq m$, then what is $tp_{S_i}^k (a/A_i)$? Since $|A_i|$ is polynomially bounded in terms of $|A|$, the number of types is polynomially bounded.

This information is enough in this case, since if $tp_{S_i}^k (a/A_i) = tp_{S_i}^k (b/A_i)$, then $a \equiv_{A_i,k} b$ (by Claim 3.20), so there is an isomorphism $f' : cl^{(k)} (A_i a) \to cl^{(k)} (A_i b)$ taking $a$ to $b$ and fixing $A_i$. Define $f : cl^{(k)} (Aa) \to cl^{(k)} (Ab)$ by

$$f' \upharpoonright (cl^{(k)} (Aa) \cap P_{S_i}^M) \cup (id \upharpoonright cl^{(k)} (A)) .$$

This is an isomorphism.

- If $\eta_0 = \eta_0$, then the following data will suffice:
  - $tp_{S_i}^k (a/A_0)$.
  - An enumeration $\langle t_l (x) | l < K \rangle$ (perhaps with repetitions) of terms of successor rank $\leq 1$ over $A_0$ such that $\{ t_l (a) | l < K \}$ lists all the new elements in $cl_{suc} (F)$ where $F = cl^{(0)} (A_0 a)$ (exactly as in the proof of Lemma 3.15 and see below).
  - $tp_{S_i}^k (\{ a_l^i | l < K \} / A_i)$ where $a_l^i = G_{\eta_0, \eta_i} (t_l (a))$ for $l < K$. By induction and by Claim 3.22, there is a polynomially bound in $|A_i|$ of the number of possible types (and $|A_i|$ itself is polynomially bounded in $|A|$).

Explantion:

As in the proof of Lemma 3.15, we say that an element of $cl^{(k)} (A_0 a) \cap P_{\eta_0}^M = cl^{(k)}_{suc} (F)$ is new if it is a successor and not in $cl^{(k)} (A_0) \cap P_{\eta_0}^M$. If $e$ is a new element in $cl^{(k)}_{suc} (F)$ for $1 < i$, then, by the fact that $G_{\eta_0, \eta_i}$ is regressive, there is $e' \in cl_{suc} (F)$ such that $G_{\eta_0, \eta_i} (e) = G_{\eta_0, \eta_i} (e')$. By that proof, there are at most $K$ new elements in $cl_{suc} (F)$ for some constant $K$.
Assume \( b \in M \) gives the same information listed above. Then \( b \equiv_{A_0, k} a \) and let \( f_0 : \text{cl}^{(k)}(aA_0) \to \text{cl}^{(k)}(bA_0) \) be a witness of that. We also know that \( f_0(t_l(a)) = t_l(b) \) and that there are isomorphisms \( f_i : \text{cl}^{(k)}(\langle a_i^l \mid l < K \rangle A_i) \to \text{cl}^{(k)}(\langle b_i^l \mid l < K \rangle A_i) \) fixing \( A_i \) (where \( b_i^l = G_{\eta_0, \eta_i}(t_l(b)) \)). Then
\[
\text{id} \upharpoonright \text{cl}^{(k)}(A) \cup f_0 \cup \bigcup_{1 \leq i < m} \left( f_i \upharpoonright \text{cl}^{(k)}(Aa) \cap P^M_{S_i} \right)
\]
is the required isomorphism.

The following remark implies that the theory \( T_S \) is not just dependent but strongly dependent and more. We shall come back to it in Section 6.

**Remark 3.24.** By closely inspecting the proof, we see that the polynomial we get as a bound for the number of \( \Delta \)-types over a finite set \( A \) in terms of \( |A| \) has the property that its degree is independent of \( \Delta \) (but depends on \( S \) and the number of variables). More formally, the degree of the polynomial \( f_{n,k} \) that bounds the number of \( \equiv_{A,k} \) classes in \( M^n \) in terms of \( |A| \) does not depend on \( k \). To see that, note that this is true for \( f_{1,k} \) and \( |S| \leq 1 \) (by the proof of Claim 3.21), and continue by induction on \( |S| \) (by the proof of Corollary 3.23) and \( n \). Note that by the quantifier elimination lemma (Lemma 3.15), one can choose \( f_{n+1,k}(x) = f_{1,k'}(x) \cdot f_{n,k}(x + 1) \) for some \( k' \geq k \).

Now we stop assuming that \( S \) is finite.

**Corollary 3.25.** If \( M \models T^\forall S \) then since
\[
\text{Th}(M) = \bigcup \{ \text{Th}(M \upharpoonright L_{S_0}) \mid S_0 \subseteq S \& |S_0| < \mathbb{N}_0 \},
\]
by Remark 3.4, Corollary 3.17 is true in the case where \( S \) is infinite. So \( T_S \) is well defined in this case as well and it is in fact \( \bigcup \{ T_{S_0} \mid S_0 \subseteq S \& |S_0| < \mathbb{N}_0 \} \). It eliminates quantifiers and is dependent.

**Adding Constants.**

We want to find an example from every cardinality, and so we add constants to the language. For a cardinal \( \theta \), the theory \( T^\forall_S \) will be \( T_S \) augmented with the quantifier free diagram of a model of \( T^\forall_S \) of cardinality \( \theta \). Formally:

**Definition 3.26.** For a cardinal \( \theta \), let \( L^\forall_S = L_S \cup \{ c_{\eta,i} \mid i < \theta, \eta \in S \} \) where \( \{ c_{\eta,i} \} \) are new constants. Let \( T^\forall_S \) be the theory \( T^\forall_S \) with the axioms stating that for all \( \eta, \eta_1, \eta_2 \in S \) and \( i, j, i', j' < \theta \) such that \( \eta_1 <_{\text{suc}} \eta_2 \),

\[
\bullet \ c_{\eta,i} \in P_{\eta}.
\]
• $i \neq j \Rightarrow c_{n,i} \neq c_{n,j}$,

• $i \neq j, i' \neq j' \Rightarrow c_{n,i} \land \eta c_{n,j} = c_{n,i'} \land \eta c_{n,j'}$,

• $\eta_1 \prec \text{suc} \eta_2 \Rightarrow G_{\eta_1,\eta_2} (c_{n,i}) = c_{\eta_2,i}$,

• $\lim_\eta (c_{n,i} \land c_{n,j}) = c_{n,i} \land c_{n,j}$ and

• $\text{suc} (c_{n,i} \land c_{n,j}, c_{n,i}) = c_{n,i}$.

**Corollary 3.27.** $T_S^{\theta,\theta}$ has a model completion — $T_S^0$ — that is complete, dependent and has quantifier elimination. This means that $T_S^{\theta,\theta}$ has JEP and the amalgamation property (AP).

Moreover, given any model $M \models T_S^{\theta}$, there is a model $M' \models T_S^{\theta,\theta}$ satisfying $M' \models L_S \supseteq M$.

**Proof.** Let $T_S^0 = T_S \cup T_S^{\theta,\theta}$. Since we only added constants, $T_S^0$ is dependent and has quantifier elimination. It is complete since the axioms specify exactly what is the quantifier free diagram of the structure generated by the constants.

The fact that $T_S^{\theta,\theta}$ has JEP and AP now follows from the fact that any model of it can be extended to an existentially closed one.

The “moreover” follows from JEP for $T_S^{\theta}$. □

4. THE MAIN THEOREM

From now on set $S = \omega > 2$. Here is the main theorem:

**Theorem 4.1.** For every cardinal $\theta$, $T_S^0$ satisfies: for every cardinal $\kappa$ and limit ordinal $\delta$, $\kappa \rightarrow \langle \delta \rangle_{T,1}$ iff $\kappa \rightarrow \langle \delta \rangle_{\theta}^{\leq \omega}$.

Let us get rid of the easy direction first:

**Proposition 4.2.** If $\kappa \rightarrow \langle \delta \rangle_{\theta}^{\leq \omega}$ then for every theory $T$ of cardinality $|T| \leq \theta$, $\kappa \rightarrow \langle \delta \rangle_{T,1}$.

**Proof.** Let $\langle a_i \mid i < \kappa \rangle$ be a sequence of elements in a model $M \models T$. Define $c : [\kappa]^{<\omega} \rightarrow L(T) \cup \{0\}$ as follows:

Given an increasing sequence $\eta \in \kappa^{<\omega}$, if $\lg (\eta)$ is odd, then $c(\eta) = 0$. If not, assume it is $2k$ and that $\eta = \langle a_i \mid i < 2k \rangle$. If $a_{a_0} \ldots a_{a_{2k-1}} = a_{a_0} \ldots a_{a_{2k-1}}$, then $c(\eta) = 0$. If not there is a formula $\varphi (x_0, \ldots, x_{k-1})$ such that $M \models \varphi (a_{a_0}, \ldots, a_{a_{2k-1}}) \land \neg \varphi (a_{a_k}, \ldots, a_{a_{2k-1}})$, so choose such a $\varphi$ and define $c(\eta) = \varphi$. By assumption there is a sub-sequence $\langle a_i \mid i < \delta \rangle$ on which $c$ is homogeneous.

Without loss, assume that $a_i = i$ for $i < \delta$.

It follows that $\langle a_i \mid i < \delta \rangle$ is an indiscernible sequence:

Assume not, and that $i_0 < i_1 < \ldots < i_{2k-1} < \delta$ and $a_{i_0} \ldots a_{i_{k-1}} \neq a_{i_k} \ldots a_{i_{2k-1}}$.

Then there is a formula $\varphi$ such that $c (\langle i_0, \ldots, i_{3k-1} \rangle) = c (\langle i_0, \ldots, i_{2k-1} \rangle) = \varphi$, meaning that

$$M \models \varphi (a_{i_0}, \ldots, a_{i_{k-1}}) \land \neg \varphi (a_{i_k}, \ldots, a_{i_{2k-1}})$$
and

\[ M \models \varphi (a_{i_k}, \ldots, a_{i_{2k-1}}) \land \neg \varphi (a_{i_{2k}}, \ldots, a_{i_{3k-1}}) \]

— a contradiction.

Let \( j < \delta \) be greater than \( k \) and \( i_k \). Then \( a_{i_0} \ldots a_{i_{k-1}} \equiv a_j \ldots a_{j+k-1} \equiv a_0 \ldots a_{k-1} \) and we are done. \( \square \)

Before starting with the difficult direction let us introduce some notation:

**Notation 4.3.** Assume \( M \models T_S^\nu \) and \( x, y \in P^M_\eta \).

1. When we say indiscernible, we shall always mean indiscernible for quantifier free formulas.
2. We say that \( x \equiv 0 \pmod{\omega} \) when \( x = \lim(x) \). For \( n < \omega \), we say that \( x \equiv n + 1 \pmod{\omega} \) where \( x \neq \lim_{\eta}(x) \) and \( \text{pre}_{\eta}(x) \equiv n \pmod{\omega} \). Note that for a fixed \( n \), the set \( \{x \mid x \equiv n \pmod{\omega} \} \) is quantifier free definable. In addition, if the trees are well ordered as in Example 3.3, then for every \( x \) there is some \( n < \omega \) such that \( x \equiv n \pmod{\omega} \) (where \( n \) is the unique number satisfying \( \text{lev}(x) = \alpha + n \) for a limit ordinal \( \alpha \)).
3. Say that \( x \equiv y \pmod{\omega} \) if there is \( n < \omega \) such that \( x \equiv n \pmod{\omega} \) and \( y \equiv n \pmod{\omega} \).

The proof uses the following construction:

\((\ast)\) Assume \( S' \subseteq S \) is such that \( \nu \in S' \Rightarrow \nu \upharpoonright k \in S' \) for every \( k \leq \lg(\nu) \). Assume \( N \models T_S^\nu,\theta \) and that for every \( \nu \in S' \), if \( \nu^\prec x \notin S' \) for \( x \in \{0, 1\} \), we have a model \( M^\nu_\eta \models T_S^\nu,\theta \).

We may assume all models are disjoint. We build a model \( M \models T_S^\nu,\theta \) such that \( M \upharpoonright L^\theta_{S'} \supseteq N \) and: for every \( \nu \in S' \) and \( \varepsilon \in \{0, 1\} \) such that \( \nu^\prec x \notin S' \) and for every \( \eta \in S \), \( P^M_\nu(x) \upharpoonright \eta = P^M_\nu(x) \upharpoonright \eta \). In general, for every symbol \( R^\eta_\nu \) from \( L^\theta_{S'} \), let \( R^M_\nu \upharpoonright (x) \upharpoonright \eta = R^M_\nu \upharpoonright (x) \upharpoonright \eta \). For instance, \( c^M_\nu(x) \upharpoonright \eta \) and \( c^M_\nu(x) \upharpoonright \eta \), \( G^M_\nu(x) \upharpoonright \eta \) and \( G^M_\nu(x) \upharpoonright \eta \).

The last thing that remains to be defined is \( G^M_{\nu^\prec \langle x \rangle} \). After we have defined it, \( M \) is a model. Moreover, for every tuple \( \bar{a} \in M^\nu_\eta \) and for every quantifier free formula \( \varphi \) over \( L^\theta_{S'} \), there is a formula \( \varphi' \) generated by concatenating \( \nu^\prec \langle x \rangle \) to every symbol appearing in \( \varphi \) such that \( M^\nu_\eta \models \varphi(\bar{a}) \) iff \( M \models \varphi'(\bar{a}) \). In particular, if \( I \subseteq M^\nu_\eta \) is an indiscernible sequence in \( M \), it is also such in \( M^\nu_\eta \).

**Proof.** (of the main theorem) We shall prove the following: for every cardinal \( \kappa \) and limit ordinal \( \delta \) such that \( \kappa \rightarrow (\delta)^{<\omega}_\theta \), there is a model \( M \models T_S^\nu,\theta \) and a set \( A \subseteq P^M_\theta \) of size \( |A| \geq \kappa \) with no non-constant indiscernible sequence in \( \delta \). That will suffice.

The proof is by induction on \( \kappa \). Note that if \( \kappa \rightarrow (\delta)^{<\omega}_\theta \) then also \( \lambda \rightarrow (\delta)^{<\omega}_\theta \) for \( \lambda < \kappa \). The case analysis for some of the cases is very similar to the one done in [KS12], but we repeat it for completeness.

Divide into cases:
Case 1. \( \kappa \leq \theta \). Let \( M \models T_{S}^{\theta, \theta} \) be any model and \( A = \left\{ c_{M}^{\lambda, i} \mid i < \theta \right\} \).

Case 2. \( \kappa \) is singular. Assume that \( \kappa = \bigcup \left\{ \lambda_{i} \mid i < \sigma \right\} \) where \( \sigma < \kappa \) and \( \lambda_{i} < \kappa \) for all \( i < \sigma \).

Assume that \( N_{0}, A_{0} \) are the model and set given by the induction hypothesis for \( \sigma \).

For all \( i < \sigma \), let \( M_i, B_i \) be the models and substructures guaranteed by the induction hypothesis for \( \lambda_i \). Let \( N_1 \) be a model of \( T_{S}^{\theta, \theta} \) containing \( M_i \) as substructures for all \( i < \sigma \) (it exists by JEP) and \( A_1 = \bigcup \left\{ B_i \mid i < \sigma \right\} \).

Assume that \( \left\{ a_i \mid i < \sigma \right\} \subseteq A_0 \) and that \( \left\{ b_j \mid \bigcup \left\{ \lambda_{i} \mid l < i \right\} \right\} \subseteq JEP \) are enumerations witnessing that \( |A_0| \geq \sigma, |B_i| \geq \lambda_{i} \setminus \bigcup \left\{ \lambda_{i} \mid l < i \right\} \).

Define \( M' \models T_{\langle \rangle}^{\theta} \) as follows: \( P_{\langle \rangle}^{M'} = \kappa, \beta_{\langle \rangle} = \varepsilon, \) and this is a model as in Example 3.3.

Let \( N \models T_{\langle \rangle}^{\theta} \) be a model such that \( N \models L_{\langle \rangle} \supseteq M' \). Use (\(*\)) with \( M_{\langle \rangle}^{0} = N_0 \) and \( M_{\langle \rangle}^{1} = N_1 \), and define the functions \( G_{\langle \rangle}, (0) \) and \( G_{\langle \rangle}, (1) \) as follows: For a limit \( \alpha < \kappa \) and \( 0 < n < \omega \), define \( G_{\langle \rangle, (0)}^{\alpha} (\alpha + n) = a_{\min (j < \sigma) \alpha < \lambda_{j}} \) and \( G_{\langle \rangle, (1)}^{\alpha} (\alpha + n) = b_{\alpha} \).

Let \( A = \kappa = P_{\langle \rangle}^{M'} \). Assume that \( \langle \beta_{i} \mid i < \delta \rangle \) is an indiscernible sequence contained in \( A \).

Obviously it cannot be that \( \beta_{1} < \beta_{0} \). Assume that \( \beta_{0} < \beta_{1} \). There are limit ordinals \( \alpha_i \) and natural number \( n_i \) such that \( \beta_{i} = \alpha_i + n_i \), i.e., \( \beta_{i} \equiv n_i \) (mod \( \omega \)). By indiscernibility, \( n_i \) is constant, and denote it by \( n \). So \( \langle \beta_{i} \mid i < \delta \rangle \) is an indiscernible sequence of successor ordinals.

\( \langle G_{\langle \rangle, (0)}^{\beta_{2i}} (\beta_{2i} + 1) \mid i < \delta \rangle \) must be constant by the choice of \( A_0 \), and assume it is \( a_{n_0} \) for \( i_0 < \sigma \). It follows that \( \alpha_2i \in \lambda_{n_0} \setminus \bigcup \left\{ \lambda_{i} \mid l < i_0 \right\} \). This means that \( G_{\langle \rangle, (1)}^{\beta_{2i}} (\beta_{2i} + 1) = b_{\alpha_2i} \subseteq B_{n_0} \) for all \( i < \sigma \), and so \( \alpha_2i \) must be constant. This means that \( \langle \beta_{2i} \mid i < \delta \rangle \) is constant — a contradiction.

Case 3. \( \kappa \) is regular but not strongly inaccessible. Then there is some \( \lambda < \kappa \) such that \( 2^{\lambda} \geq \kappa \).

Let \( M_0 \models T_{S}^{\theta, \theta} \) and \( A_0 \subseteq P_{\langle \rangle}^{M_0} \) be from the induction hypothesis for \( \lambda \). Assume that \( A_0 \supseteq \left\{ a_i \mid i \leq \lambda \right\} \) where \( a_i \neq a_j \) for \( i \neq j \).

Let \( M' \models T_{\langle \rangle}^{\theta} \) be such that \( P_{\langle \rangle}^{M'} = \lambda, 2 \) ordered by first segment. We give it a structure of \( T_{\langle \rangle}^{\theta} \) as in Example 3.3.

Let \( N \models T_{\langle \rangle}^{\theta} \) be any model such that \( N \models L_{\langle \rangle} \supseteq M' \). We use the construction (\(*\)) to build a model \( M \models T_{S}^{\theta} \) using \( N \) and \( M_{\langle \rangle}^{0} = M_{\langle \rangle}^{1} = M_0 \): we need to define the functions \( G_{\langle \rangle, (0)}^{\lambda} \) and \( G_{\langle \rangle, (1)}^{\lambda} \):

For \( f \in P_{\langle \rangle}^{\theta} \) such that \( \lg (f) = \alpha + n \) for some limit \( \alpha \) and \( n < \omega \), define \( G_{\langle \rangle, (0)}^{\lambda} (f) = a_{\alpha} \). There are no further limitations on the functions \( G_{\langle \rangle, (0)}^{\lambda} \) and \( G_{\langle \rangle, (1)}^{\lambda} \) as long as they are regressive.

Let \( A = \lambda, 2 = P_{\langle \rangle}^{M'} \). Assume for contradiction that \( \langle f_i \mid i < \delta \rangle \) is a non-constant indiscernible sequence contained in \( A \).
It cannot be that $f_1 < f_0$, because by indiscernibility, we would have an infinite decreasing sequence.

It cannot be that $f_0 < f_1$: In that case, $\langle f_i \mid i < \delta \rangle$ is increasing. For all $i < \delta$, let $g_i = \text{succ} (f_{2i}, f_{2i+1})$. The sequence $\langle g_i \mid i < \delta \rangle$ is an indiscernible sequence contained in $\text{Suc} (P_{\delta})$ and so $g_i \equiv n \mod \omega$ for some constant $n < \omega$. Hence $\langle \lg (g_i) - n \mid i < \delta \rangle$ is increasing and $\langle G_{\delta} (j) (g_i) = a_{\lg (g_i) - n} \mid i < \delta \rangle$ is a non-constant indiscernible sequence contained in $A_0$ — a contradiction.

Denote $h_i = f_0 \land f_{i+1}$ for $i < \delta$. This is an indiscernible sequence, and by the same arguments, it cannot decrease or increase. But since $h_i < f_0$, it follows that $h_i$ is constant.

Assume that $f_0 \land f_1 < f_1 \land f_2$, then $f_1 \land f_2 < f_2 \land f_3$ and so $f_{2i} \land f_{2i+1} < f_{2(i+1)} \land f_{2(i+1)+1}$ for all $i < \delta$, and again — $\langle f_{2i} \land f_{2i+1} \rangle$ is an increasing indiscernible sequence — we reach a contradiction.

Similarly, it cannot be that $f_0 \land f_1 > f_1 \land f_2$. As both sides are smaller or equal to $f_1$, it must be that

$$f_0 \land f_2 = f_0 \land f_1 = f_1 \land f_2.$$ 

But this is a contradiction (because if $\alpha = \lg (f_0 \land f_1)$ then $|\{f_0 (\alpha), f_1 (\alpha), f_2 (\alpha)\}| = 3$, but the range of the functions is $\{0, 1\}$).

**Case 4.** $\kappa$ is strongly inaccessible. With this case we shall deal in the next section.

\[\square\]

### 5. The inaccessible case

**Assumption 5.1.** Assume from now on that $\theta < \kappa$ is strongly inaccessible and that $\kappa \nrightarrow (\delta)^{< \omega}$.  

This means that for all $\lambda < \kappa$, $\lambda \nrightarrow (\delta)^{< \omega}$, so we may use the induction hypothesis in the proof.

First let us define an abstract class of structures and analyze indiscernible sequences in it. Then we shall apply the analysis to a specific structure. This structure will be used when we return to the proof.

Again, throughout this section, indiscernible means “indiscernible for quantifier free formulas”.

**Analysis of indiscernibles in $T$.**

We introduce a class of models of $T^\theta_\omega$, and then we shall analyze indiscernible sequences in them.

**Definition 5.2.** Let $T$ be the class of models $M \models T^\theta_\omega$ that satisfy

1. For all $n < \omega$, $(P_n, <)$ is a standard tree, and we give it structure as in Example 5.3
2. For $t \in P_n$, $\text{lev} (G_{n,n+1} (t)) \leq \text{lev} (t)$. As here $S = \omega$, we denote $G_{n,n+1}$ by $G_n$.
3. $G_n : \text{Suc} (P_n) \to \text{Suc} (P_n)$ (i.e. we demand that the image is also a successor).
(4) If \( \langle s_i | i < \delta \rangle \) is an increasing sequence in \( \text{Suc}(P_n) \) such that \( s_i \equiv s_j \pmod{\omega} \) for all \( i < j < \delta \) then \( i < j \Rightarrow G_n(s_i) \neq G_n(s_j) \).

**Notation 5.3.** For \( M \in \mathcal{T} \),

1. We say that \( s, t \in P_n \) are neighbors, denoted by \( tE_{nb}s \) when \( \{ x | x < t \} = \{ x | x < s \} \). This is an equivalence relation. As \( P_n \) is a normal tree, for \( t \) of a limit level its \( E_{nb} \)-class is \( \{ t \} \).
2. For all \( M \in \mathcal{T} \), let \( \text{Suc}(M) = \bigcup \{ \text{Suc}(P_n^M) | n < \omega \} \).

We need the following generalization of indiscernible sequences for \( \mathcal{T} \):

**Definition 5.4.** A sequence \( \bar{s} = \langle s_i | i < \delta \rangle \) is called *Nearly Indiscernible* (in short NI) if

1. There is \( n < \omega \) such that for all \( \delta^* \leq \delta \) every sub-sequence \( \langle s_{i_j} | j < \delta^* \rangle \) such that \( i_j + n \leq i_{j+1} \) is indiscernible with the same quantifier free type. We call this property “the sub-sequence property”.
2. For \( i, j < \delta \) and \( k < \omega \), \( tp_{qf}(s_i, \ldots, s_i+k) = tp_{qf}(s_j, \ldots, s_j+k) \). We call this property “sequential homogeneity”.

**Definition 5.5.** A sequence \( \bar{s} = \langle s_i | i < \delta \rangle \) is called *Hereditarily Nearly Indiscernible* (in short HNI) if:

For every term \( \sigma(x_0, \ldots, x_{n-1}) \), the sequence \( \bar{t} = \langle t_i | i < \delta \rangle \) defined by \( t_i = \sigma(s_i, \ldots, s_{i+n-1}) \) is NI.

**Example 5.6.** If \( \bar{s} = \langle s_i | i < \delta \rangle \) is indiscernible, then it is HNI.

**Proof.** If \( t_i = \sigma(s_i, \ldots, s_{i+n-1}) \), then any sub-sequence of \( \bar{t} \) where the distance between two consecutive elements is at least \( n \) is an indiscernible sequence (with a constant quantifier free type).

Note that for a quantifier free formula \( \varphi \),

\[
\varphi(t_i, \ldots, t_{i+k}) = \varphi(\sigma(s_i, \ldots, s_{i+n-1}), \ldots, \sigma(s_{i+k}, \ldots, s_{i+n+k-1})).
\]

Let \( i, j < \delta \). As \( tp_{qf}(s_i, \ldots, s_{i+n+k-1}) = tp_{qf}(s_j, \ldots, s_{j+n+k-1}) \), it follows that

\[
 tp_{qf}(t_i, \ldots, t_{i+k}) = tp_{qf}(t_j, \ldots, t_{j+k}).
\]

\( \square \)

**Notation 5.7.**

1. \( \bar{s} \) and \( \bar{t} \) will denote \( \delta \)-sequences, e.g. \( \bar{s} = \langle s_i | i < \delta \rangle \).
2. If there is some \( n < \omega \) such that \( \bar{s} \) is contained in \( P_n^M \), then we write \( < \) instead of \( <_n \) etc.

**Definition 5.8.** Assume \( M \in \mathcal{T} \).
Remark 5.9. If $\bar{s} \in \text{ai}(M)$, then $s_i \wedge s_{i+n} = s_i \wedge s_{i+1}$ for all $2 \leq n < \omega$ and $i < \delta$ (prove by induction on $n$).

Proposition 5.10. $\text{HNind}(M) = \text{HNind}_{\text{ai}}(M) \cup \text{HNind}_f(M)$.

Proof. Assume that $\bar{s} \in \text{HNind}(M)$. Since $\bar{s}$ is NI, there is some $n < \omega$ that witnesses the subsequence property. As for $i < j < k$, $s_i \wedge s_j$ is comparable with $s_j \wedge s_k$, by Ramsey, there is an infinite subset $A \subseteq \omega$ that satisfies one of the following possibilities:

1. For all $i < j < k \in A$, $s_i \wedge s_j = s_j \wedge s_k$, or
2. For all $i < j < k \in A$, $s_i \wedge s_j < s_j \wedge s_k$.

(note that it cannot be that $s_j \wedge s_k < s_i \wedge s_j$ because the trees are well ordered).

Assume (1) is true.

It follows that if $i < j < k < l \in A$ then $s_i \wedge s_j = s_j \wedge s_k = s_k \wedge s_l$. If $n \leq j - i, k - j, l - k$, then by the choice of $n$, the same is true for all $i < j < k < l < \delta$ where the distances are at least $n$. Moreover, given $i < j, k < l$ such that $n \leq j - i$ and $n \leq l - k$, then $s_i \wedge s_j = s_{\text{max}(j,l)+2n} \wedge s_{\text{max}(j,l)+2n}$, and the same is true for $s_k \wedge s_l$. It follows that $s_j \wedge s_k = s_k \wedge s_l$.

Choose some $0 < i < n$.

Assume for contradiction that $s_0 \wedge s_i < s_i \wedge s_{2i}$, then by sequential homogeneity $\langle s_{i\alpha} \mid \alpha < \delta \rangle \in \text{ai}(M)$. In this case, by Remark 5.9, $s_0 \wedge s_i < s_i \wedge s_{2i} = s_i \wedge s_{ni+i}$. But $s_0 \wedge s_{ni+i} = s_i \wedge s_{ni+i}$, and so on the one hand $s_0 \wedge s_i < s_0 \wedge s_{ni+i}$, and on the other $s_0 \wedge s_{ni+i} \leq s_i$ — together it’s a contradiction.

It cannot be that $s_0 \wedge s_i > s_i \wedge s_{2i}$ since the trees are well ordered.
So (again by the sequential homogeneity) it must be that \( s_0 \land s_i = s_i \land s_{2i} = \ldots = s_n \land s_{ni+i} \).

So necessarily \( s_0 \land s_i \leq s_0 \land s_{ni} \), but in addition \( s_0 \land s_{ni} = s_0 \land s_{ni+i} \) (since the distance is at least \( n \)) and so \( s_0 \land s_i = s_{ni} \land s_{ni+i} \geq s_0 \land s_{ni} \), and hence \( s_0 \land s_i = s_0 \land s_{ni} = s_0 \land s_n \).

It follows that \( s_0 \land s_i = s_{i0+i} = s_{i0+n} = s_0 \land s_n \) for every \( i_0 < \delta \). This is true for all \( i \) such that \( i_0 + i < \delta \) and so \( s_i \land s_j = s_0 \land s_n \) for all \( i < j < \delta \). So in this case \( \bar{s} \in \text{HNInd}_{f} (M) \).

Assume (2) is true. Assume that \( i < j < k \) and \( s_i \land s_j < s_j \land s_k \), it follows from the sub-sequence property that \( \langle s_{n\alpha} \mid \alpha < \delta \rangle \in \text{ai} (M) \) and that \( (s_0, s_{n+1}, s_{3n}, s_{4n}, \ldots) \in \text{ai} (M) \). In particular, by Remark 5.9, \( s_0 \land s_n = s_0 \land s_{3n} = s_0 \land s_{n+1} \).

If \( s_0 \land s_1 < s_1 \land s_2 \), then \( \bar{s} \in \text{HNInd}_{a} (M) \) by sequential homogeneity and we are done, so assume this is not the case.

It cannot be that \( s_0 \land s_1 > s_1 \land s_2 \) (because the trees are well ordered).

Assume for contradiction that \( s_0 \land s_1 = s_1 \land s_2 \). By sequential homogeneity it follows that \( s_0 \land s_1 = s_n \land s_{n+1} \). We also know that \( s_0 \land s_n = s_0 \land s_{n+1} \), and together we have \( s_0 \land s_1 = s_0 \land s_{n+1} \), and again by sequential homogeneity, \( s_n \land s_{2n+1} = s_n \land s_{n+1} \), and so \( s_n \land s_{2n+1} = s_0 \land s_n \) — a contradiction (because the distances are at least \( n \)).

**Definition 5.11.** Define the function \( H : \text{HNInd}_{a} (M) \rightarrow \text{HNInd} (M) \) as follows: given \( \bar{s} \in \text{HNInd}_{a} (M) \), let \( H (\bar{s}) = \bar{t} \) where \( t_i = G (\text{su}c (s_i \land s_{i+1}), s_{i+1})). \)

**Remark 5.12.** \( H \) is well defined: if \( \bar{s} \in \text{HNInd}_{a} (M) \) then \( H (\bar{s}) \) is in \( \text{HNInd} (M) \). This is because \( \bar{t} = H (\bar{s}) \) is not constant — by clause (1) of Definition 5.2 (it is applicable: the sequence \( (s_i \land s_{i+1} \mid i < \delta) \) is NI and increasing, so there is some \( n < \omega \) such that \( s_i \land s_{i+1} \equiv n \) (mod \( \omega \)) for all \( i < \delta \), and hence \( (\text{su}c (s_i \land s_{i+1} \mid i < \delta) \) is increasing).

**Corollary 5.13.** Let \( \bar{s} \in \text{HNInd}_{a} (M) \). If for no \( n < \omega \), \( H^{(n)} (\bar{s}) \in \text{HNInd}_{f} (M) \), then for all \( n < \omega , H^{(n)} (\bar{s}) \in \text{HNInd}_{a} (M) \). Moreover, in this case there exists some \( K < \omega \) such that for all \( n \geq K \), if \( \bar{t} = H^{(n)} (\bar{s}) \) then \( \text{su}c (s_i \land s_{i+1}), t_i = t_i \).

**Proof.** By Proposition 5.10, it follows by induction on \( n < \omega \) that \( H^{(n)} (\bar{s}) \in \text{HNInd}_{a} (M) \) and so \( H^{(n+1)} (\bar{s}) \) is well defined.

For \( n < \omega \), let \( \bar{s}_{n} = H^{(n)} (\bar{s}) \), and let us enumerate this sequence by \( \bar{s}_{n} = (s_{n,i} \mid i < \delta) \).

\( \text{lev} (\text{su}c (s_{n,0} \land s_{n,1})) < \text{lev} (s_{n,0}) \) because \( \text{lev} (s_{n,0}) \) is a successor ordinal (by clause (3) of Definition 5.2) while \( \text{lev} (\text{su}c (x)) \) is a limit for all \( x \in M \).

So \( \text{lev} (\text{su}c (s_{n,0} \land s_{n,1}), s_{n,1}) \leq \text{lev} (s_{n,0}) \), and so by clause (2) of Definition 5.2

\[ (\text{lev} (s_{n,0}) \mid n < \omega) \]

is a \( \leq \)-decreasing sequence.

Hence there is some \( K < \omega \) and some \( \alpha \) such that \( \text{lev} (s_{n,0}) = \alpha \) for all \( K \leq n \). Assume without loss of generality that \( K = 0 \).
Let \( n < \omega \). We know that
\[
\text{lev}(s_{n+1,0}) \leq \text{lev}(\text{suc}(\lim (s_{n,0} \land s_{n,1}), s_{n,1})) = \text{lev}(\text{suc}(\lim (s_{n,0} \land s_{n,1}), s_{n,0})) \leq \text{lev}(s_{n,0})
\]

But the left side and the right side are equal and \( \text{suc}(\lim (s_{n,0} \land s_{n,1}), s_{n,0}) \leq s_{n,0} \), so
\[
\text{suc}(\lim (s_{n,0} \land s_{n,1}), s_{n,0}) = s_{n,0}.
\]

By sequential homogeneity, \( \text{suc}(\lim (s_{n,i} \land s_{n,i+1}), s_{n,i}) = s_{n,i} \) for all \( i < \delta \) as desired. \( \square \)

**Constructing a model in \( \mathcal{T} \).**

By Assumption 5.1, we have a coloring \( c : [\kappa]^\omega \to \theta \) that witnesses the fact that \( \kappa \twoheadrightarrow (\delta)^{<\omega}_\theta \). Fix \( c \), and also a pairing function (a bijection) \( \text{pr} : \theta \times \theta \to \theta \) and projections \( \pi_1, \pi_2 : \theta \to \theta \) (defined by the equations \( \pi_1(\text{pr}(\alpha, \beta)) = \alpha \) and \( \pi_2(\text{pr}(\alpha, \beta)) = \beta \)). For us, \( 0 \) is considered to be a limit ordinal, and for an ordinal \( \alpha \), let \( \text{Lim}(\alpha) = \{ \beta < \alpha | \beta \text{ is a limit} \} \).

**Definition 5.14.** \( P = P_{\theta, \kappa} \) is the set of triples \( p = (d, M, E) = (d_p, M_p, E_p) \) where:

1. \( M \) is a model of \( T_{\emptyset}^\kappa \) as in Example 3.3 and \( M = P^M_\emptyset \) (i.e. \( M \) is a standard tree). Some notation:
   (a) We write \( <_p \) instead of \( <^M \) etc., or omit \( p \) when it is clear.
   (b) Let \( \text{Suc}_{\lim}(M) \) be the set of all \( t \in \text{Suc}(M) \) such that \( \text{lev}(t) - 1 \) is a limit.
2. \( E \) is an equivalence relation refining \( E^{\text{nb}} \). Moreover, for levels that are not \( \alpha + 1 \) for limit \( \alpha \) it equals \( E^{\text{nb}} \). By normality \( E \) is equality on limit elements, so it is interesting only on \( \text{Suc}_{\lim}(M) \).
3. For every \( E^{\text{nb}} \) equivalence class \( C, |C/E| < \kappa \).
4. \( d \) is a function from \( \{ \eta \in ^{<\omega}_{\text{Suc}_{\lim}(M)} | \eta(0) < \ldots < \eta(\lg(\eta) - 1) \} \) to \( \theta \).
5. We say that \( p \) is hard if there is no increasing sequence of elements \( s \) of length \( \delta \) from \( \text{Suc}_{\lim}(M) \) such that:
   
   For all \( n < \omega \) there is \( c_n \) such that for every \( i_0 < \ldots < i_{n-1} < \delta, d(s_{i_0}, \ldots, s_{i_{n-1}}) = c_n \).

**Example 5.15.** \( \kappa \) is a standard tree. Let \( p_\kappa = (c \upharpoonright \text{Suc}_{\lim}(\kappa), \kappa, =) \in P \). Then \( p_\kappa \) is hard.

**Definition 5.16.** Let \( p = (d, M, E) \in P, x \) a variable and \( A \subseteq \text{Suc}_{\lim}(M) \) a linearly ordered set.

1. Say that \( p \) is a \( d \)-type over \( A \) if \( p \) is a consistent set of equations of the form
   \[
   d(a_0, \ldots, a_{n-1}, x) = \varepsilon \text{ where } n < \omega, \varepsilon < \theta \text{ and } a_0 < \ldots < a_{n-1} \in A.
   \]
2. Consistency here means that \( p \) does not contain a subset of the form
   \[
   \{d(a_0, \ldots, a_{n-1}, x) = \varepsilon, d(a_0, \ldots, a_{n-1}, x) = \varepsilon'\}
   \]
   for \( \varepsilon \neq \varepsilon' \).
Definition 5.17. For $p \in \mathbf{P}$, define $\mathbf{q} = \mathbf{q}(p) = (M_{\mathbf{q}}, d_{\mathbf{q}}, E_{\mathbf{q}}) = (M, d, E) \in \mathbf{P}$ by:

- $M$ is the set of pairs $a = (\Gamma, \eta) = (\Gamma_n, \eta_n)$ such that:
  1. There is $\alpha < \kappa$ such that $\eta : \alpha \rightarrow \text{Suc}_{\text{lim}}(M_p)$ and $\Gamma : \text{Lim}(\alpha) \rightarrow R(M_p)$. Denote $\lg (\Gamma, \eta) = \lg (\eta) = \alpha$. If $\alpha$ is a successor ordinal, let $l_{(\Gamma, \eta)} = \eta(\alpha - 1) \in M_p$.
  2. For $\beta < \alpha$ limit, $\Gamma(\beta) \in R(\{\eta(\beta') | \beta' \leq \beta\})$.
  3. If $0 < \alpha$ then $\eta(0) \models \Gamma(0) \nvdash \emptyset$.
  4. For $\beta' < \beta < \alpha$, $\eta(\beta') < \eta(\beta)$ (\(\eta\) is increasing in $M_p$).
  5. If $\beta' < \beta < \alpha$ are limit ordinals then $\Gamma(\beta') \subseteq \Gamma(\beta)$.
  6. If $\beta' < \beta < \alpha$ and $\beta'$ is a limit ordinal then $\eta(\beta) \models \Gamma(\beta')$.
  7. For $\beta < \alpha$, there is no $t < \eta(\beta)$ that satisfies
     a. $t \in \text{Suc}_{\text{lim}}(M_p)$,
     b. $\eta(\beta') < t$ for all $\beta' < \beta$,
     c. $t \models \Gamma(0) \nvdash \emptyset$, and
     d. $t \models \Gamma(\beta')$ for all limit $\beta' < \beta$.
  8. The order on $T$ is $(\Gamma, \eta) < (\Gamma', \eta')$ iff $\Gamma < \Gamma'$ and $\eta < \eta'$ (where $<$ means first segment).

It follows that for $a = (\Gamma, \eta)$, $\text{lev}(a) = \lg (a)$.

- $d$ is defined as follows: suppose $a_0 < \ldots < a_{n-1} \in \text{Suc}_{\text{lim}}(M)$ and $a_i = (\Gamma_i, \eta_i)$.

  Let $t_i = l_{a_i} = \eta_i(\lg (a_i) - 1)$ and $p = \Gamma_{n-1}(\lg (a_{n-1}) - 1)$. Let $\varepsilon \in \theta$ be the unique color such that $d(t_0, \ldots, t_{n-1}, x) = \varepsilon \in p$. Then

$$d(a_0, \ldots, a_{n-1}) = \text{pr}(\varepsilon, c(\text{lev}(a_0), \ldots, \text{lev}(a_{n-1}))).$$

- $E$ is defined as follows: $(\Gamma_1, \eta_1) E (\Gamma_3, \eta_2)$ iff
  1. $\lg (\eta_1) = \lg (\eta_2)$, so equals to some $\alpha < \kappa$, say that $p$ is complete if for every increasing sequence $\langle a_0, \ldots, a_{n-1} \rangle$ from $A$ there is such an equation in $p$.
  4. If $B \subseteq A$ then for a $d$-type $p$ over $A$, let

$$p \upharpoonright B = \{ d(a_0, \ldots, a_{n-1}, x) = \varepsilon \in p \mid a_0, \ldots, a_{n-1} \in B \}.$$
- \( \eta_1 \upharpoonright \beta = \eta_2 \upharpoonright \beta, \Gamma_1 \upharpoonright \beta = \Gamma_2 \upharpoonright \beta \) for all \( \beta < \alpha \) (so they are \( E^{nb} \)-equivalent),
- \( \Gamma_1 (0) \upharpoonright \emptyset = \Gamma_2 (0) \upharpoonright \emptyset \), and
- If \( \alpha = \beta + n \) for \( \beta \in \text{Lim} (\alpha) \) and \( n < \omega \) then for all \( \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \beta, \)
\[
d (\eta_1 (\alpha_0), \ldots, \eta_1 (\alpha_{k-1}), \eta_1 (\beta), x) = \varepsilon \in \Gamma_1 (\beta) \iff \nond\ld (\eta_2 (\alpha_0), \ldots, \eta_2 (\alpha_{k-1}), \eta_2 (\beta), x) = \varepsilon \in \Gamma_2 (\beta)
\]

Note that it follows that if \( 1 < n \), and \( (\Gamma_1, \eta_1) E^{nb} (\Gamma_2, \eta_2) \), then \( \Gamma_1 (\beta) = \Gamma_2 (\beta) \) and 
\( \eta_1 (\beta) = \eta_2 (\beta), \) so they are \( E \)-equivalent.

In the next claims we assume that \( p \in P \) and \( q = q (p) \).

**Remark 5.18.** \( \text{lev} (a) = \text{lg} (a) \) for all \( a \in M_q \) and \( a E^{nb} b \) iff \( \text{lev} (a) = \text{lev} (b) \) and \( a \upharpoonright \alpha = b \upharpoonright \alpha \) for all \( \alpha < \text{lev} (a) \).

**Claim 5.19.** \( q \in P_{\theta, \kappa} \) and moreover it is hard.

**Proof.** The fact that \( M_q \) is a standard tree is trivial. Also, \( E \) refines \( E^{nb} \) by definition.

We must show that the number of \( E \)-classes inside a given \( E^{nb} \)-class is bounded.

Given a (partial) \( d \)-type \( p \) over \( M_p \) and \( t \in M_p \), let \( p' \) be the set of equations we get by replacing all appearances of \( t \) by a special letter \( * \).

Assume that \( A \) is an \( E^{nb} \)-class contained in \( \text{Suc}_{lim} (M_q) \), and that for every \( a \in A \), \( \text{lev} (a) = \alpha + 1 \) where \( \alpha \) is limit. Assume \( a \in A \) and let \( B = \{ * \} \cup \text{im} (\eta_a) \setminus \{ l_a \} \) (since \( A \) is an \( E^{nb} \)-class, this set does not depend on the choice of \( a \)). Consider the map \( \varepsilon \) defined by \( a \mapsto \Gamma_a (\alpha)^{l_a} \). Then, \( a, b \in A \) are \( E \) equivalent iff \( \varepsilon (a) = \varepsilon (b) \). Therefore this map induces an injective map from \( A/E \) to this set of types. The size of this set is at most \( 2^{|B|+\theta+\aleph_0} \). But \( |B| = |\alpha| < \kappa \), and \( \theta < \kappa \) by assumption, so \( |A/E| < \kappa \) (as \( \kappa \) is a strong limit).

\( q \) is hard: if \( \bar{s} = \langle s_i | i < \delta \rangle \) is a counterexample then \( \langle \text{lev} (s_i) | i < \delta \rangle \) contradicts the choice of \( c \).

**Proposition 5.20.**

1. For all \( a \in \text{Suc} (M_q) \), \( \text{lev}_{M_q} (a) \leq \text{lev}_{M_p} (l_a) \).
2. Assume \( t \in \text{Suc}_{lim} (M_p) \). Then there is some \( a = (\Gamma, \eta) \in M_q \) such that \( l_a = t \).

**Proof.**
1. Let \( \text{lev}_{M_q} (a) = \alpha. \) Then \( \langle \text{pre} (\eta_a (\beta)) | \beta < \alpha \rangle \) is an increasing sequence below \( l_a \), hence \( \alpha \leq \text{lev}_{M_p} (l_a) \).
2. We try to build \( (\Gamma_\alpha, \eta_\alpha) \) by induction on \( \alpha < \kappa \) so that \( (\Gamma_\alpha, \eta_\alpha) \in M_q \); \( \text{lg} (\eta_\alpha) = \alpha; \) it is an increasing sequence in \( q; \eta_\alpha (\beta) < t \) for \( \beta < \alpha \) and if \( \beta \) is a limit then \( \Gamma_\alpha (\beta) = \text{dtp} (t/\{ \eta_\alpha (\beta') | \beta' \leq \beta \}) \).

We must get stuck in a successor stage, i.e. there is some \( \beta \) such that we cannot continue the construction for \( \alpha = \beta + 1 \). In fact, we will get stuck at the latest in stage \( \text{lev}_{M_p} (t) \) by (I).
Define \( \eta = \eta_\beta \cup \{ (\beta, t) \} \), \( \Gamma = \Gamma_\beta \) unless \( \beta \) is a limit in which case let \( \Gamma (\beta) \) be any complete type in \( x \) over \( \{ \eta (\beta') | \beta' \leq \beta \} \) containing \( \bigcup \{ \Gamma (\beta') | \beta' \in \text{Lim} (\beta) \} \cup \{ d (x) = d_p (t) \} \).

By construction, \( (\Gamma, \eta) \in M_q \).

Now we build a model in \( \mathcal{T} \) using \( P \):

**Definition 5.21.**

1. Define \( p_0 = p_c \) (see Example \ref{ex:5.14}), and for \( n < \omega \), let \( p_{n+1} = q (p_n) \).
2. Define \( P_n = M_p \), \( d_n = d_{p_n} \) and \( E_n = E_{p_n} \).
3. Let \( M_\epsilon = \bigcup_{n < \omega} P_n \) (we assume that the \( P_n \)'s are mutually disjoint). So \( P_n^{M_\epsilon} = P_n \).
4. \( M_\epsilon \models T_\omega \) when we interpret the relations in the language as they are induced from each \( P_n \) and in addition:
5. Define \( G_n^{M_\epsilon} : \text{Suc} (P_n) \rightarrow \text{Suc} (P_{n+1}) \) as follows: let \( a \in \text{Suc} (P_n) \) and \( a' = \text{Suc} (\text{lim} (a)) \).

By the claim above, there is an element \( (\Gamma, \eta)_a \in \text{Suc} (P_{n+1}) \) such that \( l_{(\Gamma, \eta)_a} = a' \). Choose such an element for each \( a \), and define \( G_n^{M_\epsilon} (a) = (\Gamma, \eta)_a \).

**Corollary 5.22.** \( M_\epsilon \in \mathcal{T} \).

**Proof.** All the demands of Definition \ref{def:5.22} are easy. For instance, clause \( \text{[2]} \) follows from Proposition \ref{prop:5.20}. Clause \( \text{[4]} \) follows from the fact that if \( \langle s_i | i < \delta \rangle \) is an increasing sequence in \( P_n \) such that \( s_i \equiv s_j \) (mod \( \omega \)) then \( \text{Suc} (\langle s_i, s_j | i < \delta \rangle) \) is increasing, so \( l_{G_n(s_i)} \neq l_{G_n(s_j)} \) for \( i \neq j \).

**Notation 5.23.** Again, we do not write the index \( p_n \) when it is clear from the context (for instance we write \( d (s_0, \ldots, s_k) \) instead of \( d_{p_n} (s_0, \ldots, s_k) \)).

**Lemma 5.24.** Assume that \( \bar{s} \in \text{HNind}_{\alpha_1} (M_\epsilon) \) and \( \bar{t} = H (\bar{s}) \in \text{HNind}_{\alpha_1} (M_\epsilon) \) (see Definition \ref{def:5.17}) satisfy that for all \( i < \delta \):

- \( \text{Suc} (\langle s_i \land s_{i+1}, s_i \rangle) = s_i \), and
- \( \text{Suc} (\langle t_i \land t_{i+1}, t_i \rangle) = t_i \).

**Claim 5.25.** Then, if \( u_i = \text{Suc} (\langle s_i \land s_{i+1}, s_{i+1} \rangle) \) and \( v_i = \text{Suc} (\langle t_i \land t_{i+1}, t_{i+1} \rangle) \) for \( i < \delta \), then:

1. \( \langle d(u_i) | 1 \leq i < \delta \rangle \) is constant.
2. \( d(u_{i_0}, \ldots, u_{i_n}) = \pi_1 (d(v_{i_0}, \ldots, v_{i_{n-1}})) \) for \( 1 \leq i_0 < \ldots < i_n < \delta \) (recall that \( \pi_1 \) is defined by \( \pi_1 (\text{pr} (i, j)) = i \)).

**Proof.** (1) By definition, \( t_i = G (u_i) \). Denote \( t_i = (\Gamma_i, \eta_i) \). As \( \langle t_i \land t_{i+1} | i < \delta \rangle \) is an increasing sequence (because \( \bar{t} \in \text{HNind}_{\alpha_1} (M_\epsilon) \)), \( 0 < \text{lev} (t_1 \land t_2) \). Let \( p = G(t_1 \land t_2) \). Then \( p = G_0 (\emptyset) \mid \emptyset \) for all \( 1 \leq i \) (it may be that \( t_1 \land t_0 = \emptyset \) and in this case we have no information on \( t_0 \)). Assume
Lemma 5.27. If \( d(x) = \varepsilon \) for some \( \varepsilon < \theta \). Then, by Definition 5.17, clauses (3) and (6), \( d(\eta_i(\beta)) = \varepsilon \) for all \( 1 \leq i < \delta \) and \( \beta < \lg(\eta_i) \). As \( u_i = l_t \), we are done.

(2) Denote \( v_i = (\Gamma'_i, \eta'_i) \). By our assumptions on \( \bar{t}, t_iE_{\bar{t}}v_i \) hence if \( \bar{t} \) is increasing then \( v = \bar{t} \).

Assume that it is not increasing. Then \( t_i \land t_{i+1} < t_i \) so \( \lim (t_i \land t_{i+1}) = t_i \land t_{i+1} \). Let \( \alpha_i = \beta_i + 1 = \lg(t_i) = \lg(\eta'_i) \), then \( \beta_i \) is a limit ordinal and \( t_i \upharpoonright \beta_i = v_i \upharpoonright \beta_i \). By the same argument as before, for \( 1 \leq i, \Gamma'_i(0) \upharpoonright 0 = \Gamma_i(0) \upharpoonright 0 = p \) and \( \Gamma'_i \upharpoonright \beta_i = \Gamma_i \upharpoonright \beta_i \).

Note that for \( 1 \leq i, l_{t_i} \) and \( l_{v_i} \) are both below \( u_{i+1} = l_{t_{i+1}} \) (as \( v_i \leq t_{i+1} \) and \( l_{t_i} = u_i < u_{i+1} \)), that they both satisfy \( p \) and that they both satisfy the equations in \( \Gamma_i(\beta) \) for each limit \( \beta < \beta_i \), so if for instance \( l_{t_i} < l_{v_i} \), we will have a contradiction to Definition 5.17 clause (7).

By choice of \( \bar{v} \) and the assumptions on \( \bar{t} \), \( v \) is increasing so \( d \) is defined on finite subsets of it. Assume \( 1 \leq i_0 < \ldots < i_n < \delta \). Then for every \( \sigma < \theta \), by the choice of \( d \) in Definition 5.17

\[
\begin{align*}
\Box \quad &\pi_1(d(v_{i_0}, \ldots, v_{i_n-1})) = \sigma \text{ iff} \\
\Box \quad &d(l_{v_{i_0}}, \ldots, l_{v_{i_n-1}}, x) = \sigma \in \Gamma'_{i_n-1}(\beta_{i_n-1}) \text{ iff} \\
\Box \quad &d(l_{v_{i_0}}, \ldots, l_{v_{i_n-1}}, x) = \sigma \in \Gamma'_{i_n}(\beta_{i_n-1}) \text{ (because } \Gamma'_{i_n} \upharpoonright \alpha_{i_n-1} = \Gamma'_{i_n-1} \upharpoonright \alpha_{i_n-1}) \text{ iff} \\
\Box \quad &d(l_{v_{i_0}}, \ldots, l_{v_{i_n-1}}, l_{v_{i_n}}) = \sigma \text{ (this follows from clause (3) of Definition 5.17) iff} \\
\Box \quad &d(u_{i_0}, \ldots, u_{i_n}) = \sigma \text{ (because } l_{v_i} = u_i).
\end{align*}
\]

\[\square\]

Corollary 5.26. If \( s \in \text{HNind}_{\text{al}}(M_c) \) then there must be some \( n < \omega \) such that \( H^{(n)}(s) \in \text{HNind}_{f}(M_c) \) (see Definition 5.17).

Proof. If not, by Corollary 5.13 for all \( n < \omega \), \( H^{(n)}(s) \in \text{HNind}_{\text{al}}(M_c) \). Moreover, there exists some \( K < \omega \) such that for all \( K \leq n \), if \( \bar{t} = H^{(n)}(s) \) then \( \text{su}(\lim (t_i \land t_{i+1}), t_i) = t_i \). Without loss, \( K = 0 \) (i.e. this is true also for \( s \)).

Claim. If \( s \) is such a sequence then \( d(u_{i_0}, \ldots, u_{i_n-1}) \) is constant for all \( 1 \leq i_0 < \ldots < i_{n-1} < \delta \) where \( u_i = \text{su}(\text{lim} (s_i \land s_{i+1}), s_{i+1}) \) for \( i < \delta \).

Proof. (of claim) Prove by induction on \( n \) using Lemma 5.24 \[\square\]

But this claim contradicts the fact that for all \( k < \omega \), \( p_k \) is hard. \[\square\]

Lemma 5.27. If \( s \in \text{HNind}_{\text{al}}(M_c) \) and \( \bar{t} = H(s) \in \text{HNind}_{f}(M_c) \) then \( \neg (v_iEv_j) \) for \( i < j < \delta \) where \( v_i = \text{su}(\text{lim} (t_{i+1} \land t_i), t_i) \).

Proof. Let \( t = t_0 \land t_1 \), so \( t = t_i \land t_j \) for all \( i < j < \delta \). Let \( u_i = \text{su}(t, t_i) \). As \( t_i \neq t_j \) for \( i < j < \delta \), \( u_i \neq u_j \). In addition

\[
l_{u_i} \leq l_{t_i} = \text{su}(\text{lim} (s_i \land s_{i+1}), s_{i+1}) \leq s_{i+1} \land s_{i+2}
\]
and \( \langle s_i \wedge s_{i+1} \mid i < \delta \rangle \) is increasing so \( l_{u_i} \) and \( l_{u_j} \) are comparable.

First assume that \( \alpha = \text{lev}(t) > 0 \). Then \( \Gamma_\ell(0) = \Gamma_{\ell+1}(0) \) for \( i < \delta \). For all \( i < j < \delta \), \( l_{u_i} = \Gamma_{u_i}(0) \) if \( l_{u_i} \), \( l_{u_j} \) is greater than \( \eta_{u_i}(\beta) = \eta_{u_j}(\beta) \) for all \( \beta < \alpha \) and \( l_{u_j} = \Gamma_{u_j}(\beta) = \Gamma_{\ell}(\beta) \) for all limit \( \beta < \alpha \). So by Definition \( 5.17 \) clause \( 3 \), \( l_{u_i} = l_{u_j} \), so \( \eta_{u_i} = \eta_{u_j} \) for all \( i < j < \delta \).

But since \( u_i \neq u_j \), it necessarily follows that \( \Gamma_{u_i} \neq \Gamma_{u_j} \). If \( \alpha = \beta + 1 \) for some \( \beta \), then by choice of \( q \), \( \Gamma_{u_i} = \Gamma_{u_j} \) (because \( \Gamma \) was defined only for limit ordinals). So necessarily \( \alpha \) is a limit, and it follows that \( \lim(t) = t \) so \( v_i = u_i \). Now it is clear that \( \Gamma_{v_i}(\alpha) \neq \Gamma_{v_j}(\alpha) \) and by definition of \( E \), \( \neg(v_iEv_j) \) for all \( i < j < \delta \).

If \( \alpha = 0 \), then as before \( v_i = u_i \) (because \( \lim(t) = t \)). We cannot use the same argument (because \( \Gamma_\ell(0) \) is not defined), so we take care of each pair \( i < j < \delta \) separately. If \( \Gamma_{v_i}(0) \) then the argument above will work and \( \neg(v_iEv_j) \). If \( \Gamma_{v_i}(0) \) then \( \neg(v_iEv_j) \) follows directly from the definition.

Finally we have

**Corollary 5.28.** If \( \bar{s} \in \text{HNInd}_\text{f}(M_c) \), then there is some \( \bar{v} \in \text{HNInd}_\text{f}(M_c) \) such that \( v_i = \text{suc}(\text{lim}(v_i), v_i) P_{i<j} v_j \) but \( \neg(v_iEv_j) \) for \( i < j < \delta \).

**Proof.** By Corollary \( 5.26 \) there is some minimal \( n < \omega \) such that \( \bar{t} = H^{(n+1)}(\bar{s}) \in \text{HNInd}_\text{f}(M_c) \). Let \( v_i = \text{suc}(\text{lim}(t_{i+1} \wedge t_i), t_i) \) for \( i < \delta \). By Lemma \( 5.27 \) we have that \( v_iP_{i<j} v_j \) but \( \neg(v_iEv_j) \) for \( i < j < \delta \) (in particular \( v_i \neq v_j \)). So necessarily \( t = t_i \wedge t_j \) is a limit and \( v_i = \text{suc}(t, v_i) \). \( \square \)

**Corollary 5.29.** If there is some \( \bar{s} \in \text{ind}(M_c) \) and \( s_i \in P_0 \) for all \( i < \delta \), then there is some \( \bar{v} \in \text{ind}_\text{f}(M_c) \) such that \( v_i \in \text{Suc}_\text{lim}(M_c) \), \( v_iP_{i<j} v_j \) but \( \neg(v_iEv_j) \) for \( i < j < \delta \).

**Proof.** Since \( P_0 = \kappa \), any sequence \( \bar{s} \) in \( \text{ind}(M_c) \) in \( P_0 \) must be increasing. So by the last corollary there is some \( \bar{v} \in \text{HNInd}_\text{f}(M_c) \) like here. But then there is some \( n < \omega \) such that \( \langle v_{ni} \mid i < \delta \rangle \) is indiscernible. \( \square \)

**End of the proof of Theorem 4.1**

Assume that \( M_\lambda, A_\lambda \) are the model and sets guaranteed by the induction hypothesis for \( \lambda < \kappa \). We may assume they are disjoint. Let \( N \) be a model of \( T^{x, \theta}_S \) containing \( M_\lambda \) for \( \lambda < \kappa \) (\( N \) exists by JEP), and let \( A = \bigcup \{ A_\lambda \mid \lambda < \kappa \} \subseteq N \). Let \( N_c \models T^{x, \theta}_{S'} \) be a model such that \( N_c \models L_S \supseteq M_c \). Let \( S' = \omega^\ast 1 \) (finite sequences of zeros). We may think of \( N_c \) as a model of \( T^{x, \theta}_{S'} \). Denote \( 0_n = \langle 0, \ldots, 0 \rangle \) where \( \text{lg}(0_n) = n \).

We use the construction \( (\ast) \) and \( S' \) to build a model \( M \) of \( T^{x, \theta}_S \):

- For all \( n < \omega \), let \( M^{\text{b}_n}_0 = N \).
- Define \( G^{M}_{n_0, n_0} \) as follows:
  - Recall that \( P^M_{n_0} \supseteq P^M_n = M_{n_0} \). Assume that \( B \subseteq \text{Suc}_\text{lim}(P^M_n) \) is an \( E_{\text{lb}} \) class. By definition, \( |B/E_{\text{lb}}| < \kappa \).
Choose some enumeration of the classes \( \{ c_i \mid i < |B/E_{p_n}| \} \), and an enumeration \( A_{|B/E_{p_n}|} \supseteq \{ a_i \mid i < |B/E_{p_n}| \} \) of pairwise distinct elements. Now, \( G_{0_n,0_n,\cdot}(1) \) maps every class \( c_i \) (i.e. every element in \( c_i \)) to \( a_i \). Define \( G_{0_n,0_n,\cdot}(1) \) on other elements in \( P_{M_n} \) by regressiveness. Outside of \( P_{M_n} \), define \( G_{0_n,0_n,\cdot}(1) \) arbitrarily as long as it is regressive.

Let \( A = \text{Suc}_\lim \left( P_{M_n} \right) \), i.e. \( A = \text{Suc}_\lim (\kappa) \). Assume for contradiction that \( A \) contains a \( \delta \)-indiscernible sequence.

By Corollary 5.29 there is \( n < \omega \) and an indiscernible sequence \( \bar{v} \) in \( \text{Suc}_\lim \left( P_{0_n} \right) \) such that for \( i < j < \delta \), \( v_i E^{ab} v_j \) but \( \neg (v_i E v_j) \). So \( \left( G_{0_n,0_n,\cdot}(1) (v_i) \mid i < \delta \right) \) is a non-constant indiscernible sequence in \( A_{|[v_0/E^{ab}]/E_{p_n}|} \) — a contradiction.

6. Further comments

Strongly dependent theories.

Strongly dependent theories are a sub-class of dependent theories. They were introduced and discussed in [She12, She09] as a possible solution to the equation dependent / x = stable / superstable. There, the author also introduced other classes of smaller sub-classes, namely strongly\(^l\) dependent for \( l = 2, 3, 4 \). Groups which are strongly\(^2\) dependent are discussed in [KST1]. The theories of the reals and the \( p \)-adics are both strongly dependent, but neither is strongly\(^2\) dependent.

As we said in the introduction, in [She12] it is proved that \( \sum_{|T|} (\lambda) \rightarrow (\lambda^+)_{T,n} \) for strongly dependent \( T \) and \( n < \omega \).

In [KST12] we show that in \( \mathbb{R} \) there is a similar phenomenon to what we have here, but for \( \omega \)-tuples: there are sets from all cardinalities with no indiscernible sequence of \( \omega \)-tuples (up to the first strongly inaccessible cardinal). This explains why the theorem mentioned was only proved for \( n < \omega \).

The example we described here is not strongly dependent, but it can be modified a bit so that it will be, and then give a similar theorem for strongly dependent theories (or even strongly\(^2\) dependent), but for \( \omega \)-tuples.

Here are the definitions:

**Definition 6.1.** A theory \( T \) is said to be **not strongly dependent** if there exists a sequence of formulas \( \langle \varphi_i (x, y_i) \rangle \) (where \( x, y_i \) are tuples of variables), an array \( \langle a_{i,j} \mid i, j < \omega \rangle \) (\( \lg (a_{i,j}) = \lg (y_i) \)) and tuples \( \langle b_{\eta} \mid \eta : \omega \rightarrow \omega \rangle \) (\( \lg (b_{\eta}) = \lg (x) \)) such that \( \models \varphi_i (b_{\eta}, a_{i,j}) \iff \eta (i) = j \).

**Notation 6.2.** We call an array of elements (or tuples) \( \langle a_{i,j} \mid i, j < \omega \rangle \) an **indiscernible array** over \( A \) if for \( i_0 < \omega \), the \( i_0 \)-row \( \langle a_{i_0,j} \mid j < \omega \rangle \) is indiscernible over the rest of the sequence \( \{ \langle a_{i,j} \mid i \neq i_0, i, j < \omega \} \} \) and \( A \), i.e. when the rows are mutually indiscernible.
Definition 6.3. A theory $T$ is said to be not strongly dependent if there exists a sequence of formulas $\langle \varphi_i(x, y_i, z_i) \mid i < \omega \rangle$, an array $\langle a_{i,j} \mid i, j < \omega \rangle$ and $b_k \in \{ a_{i,j} \mid i < k, j < \omega \}$ such that

- The array $\langle a_{i,j} \mid i, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{ \varphi_i(x, a_{i,0}, b_i) \land \neg \varphi_i(x, a_{i,1}, b_i) \mid i < \omega \}$ is consistent.

Theorem 6.4. For every $\theta$ there is a strongly dependent theory $T$ of size $\theta$ such that for all $\kappa$ and $\delta$, $\kappa \rightarrow (\delta)_{\omega}$ iff $\kappa \rightarrow (\delta)_{\theta}^{<\omega}$.

Proof. First note that the easy direction works the same way as before.

For $n < \omega$ let $T_n$ be the theory defined in Definition 3.26 for $S_n = n \geq 2$. Let $T$ be the theory $\sum_{n<\omega} T_n$: the language is $\{ Q_n \mid n < \omega \} \cup \{ R_n \mid R \in L_{S_n}^\theta \}$ where $Q_n$ are unary predicates, and the theory says that they are mutually disjoint and that each $Q_n$ is a model of $T_n$. It is easy to see that this theory is complete and has quantifier elimination. Denote $S = \omega \geq 2$ as before. If $M$ is a model of $T_n$, then $M$ naturally induces a model $N$ of $T$ (where $Q_n^N = (M \times \{ n \}) \upharpoonright L_{S_n}^\theta$). For all $a \in M$, let $f_a \in \prod_{n<\omega} Q_n^N$ be defined by $f_a (n) = (a, n)$ for $n < \omega$. Now, if $A \subseteq P^M_\emptyset$ is any set with no $\delta$-indiscernible sequence then the set $\{ f_a \mid a \in A \}$ is a sequence of $\omega$-tuples with no indiscernible sequence of length $\delta$.

By Remark 3.24 it follows that each $T_n$ is strongly dependent, and so also $T$. \hfill \Box

Comments about the proof.

We would like to explain some of the choices made through the proof.

1. Why did we use discrete trees and not dense ones (as in [KSI2])?

   This is because indiscernible sequences could be in a diagonal comb situation (i.e. in $\text{ind}_a$), and we needed the ability to make an element be a successor to the meet (otherwise we have no control on its coloring). Trial and error has shown that adding the function “successor to the meet” instead of just successor causes the lost of amalgamation, so we needed the successor function. The predecessor function is not necessary (in existentially closed models, if $x > \lim \eta(x)$, $x$ has a predecessor), but there is no price to adding it, and it simplify the theory a bit.

2. Why did we need the regressiveness in the definition of $T_S$?

   Because otherwise there is no quantifier elimination: for $a <_{\eta_2} b$ we can ask whether

   $$\exists x (a <_{\eta_1} x <_{\eta_1} b \land G_{\eta_1, \eta_2}(x) \neq G_{\eta_1, \eta_2}(b)) .$$

   This formula is not equivalent to a quantifier free formula without regressiveness, because for every $m$ we can find $a, b$ and $a', b'$ such that $ab \equiv_m a'b'$ but one of the pairs satisfies the formula and the other does not.

3. Why in the definition of $Q$ we demanded that the image of $\eta$ is in $\text{Suc}_{\lim}$ and that $\Gamma$ is relevant only in limit levels?
Had we given $\Gamma$ the freedom to give values in every ordinal, then the “fan” (i.e., the sequence in $\text{ind}_f$) which we got in the end (in Lemma 5.27) might not have been in a successor to a limit level, so we would have no freedom in applying $G$ on it. As $\Gamma$ is relevant only for limit levels, the coloring was defined only on sequence in $\text{Suc}_{\text{lim}}$, so we needed $\eta$ to give elements from there.

References

[CS09] Moran Cohen and Saharon Shelah. Stable theories and representation over sets, 2009. arXiv:0906.3050.

[KS11] Itay Kaplan and Saharon Shelah. Chain conditions in dependent groups. Proceedings of the Logic Colloquium 2011, 2011. accepted, Number 993 in Shelah Archive. arXiv:1112.0897.

[KS12] Itay Kaplan and Saharon Shelah. Examples in dependent theories. submitted, 2012. Number 946 in Shelah archive, arXiv:1009.5420.

[Mor65] Michael Morley. Categoricity in power. Transaction of the American Mathematical Society, 114:514–538, 1965.

[She86] Saharon Shelah. Around classification theory of models, volume 1182 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986.

[She90] Saharon Shelah. Classification theory and the number of nonisomorphic models, volume 92 of Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, second edition, 1990.

[She09] Saharon Shelah. Dependent first order theories, continued. Israel J. Math., 173:1–60, 2009.

[She12] Saharon Shelah. Strongly dependent theories. Israel Journal of Mathematics, 2012. accepted.