Common Fixed Point Theorems for Multivalued Generalized $F$-Suzuki-Contraction Mappings in Complete Strong $b$–Metric Spaces

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Abstract

This paper introduces a new version of multivalued generalized $F$-Suzuki-Contraction mapping and then establish some new common fixed point theorems for these new multivalued generalized $F$-Suzuki-Contraction Mappings in complete strong $b$–Metric Spaces.

Keywords: Common Fixed Point Problem, Multivalued Generalized $F$-Suzuki-Contraction Mapping, Complete Strong $b$–metric Space.

1. Introduction

Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to \mathbb{R}^+_0$ is said to be a $b$-metric if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair $(X, d)$ is called a $b$-metric space with constant $s$. A strong $b$–metric is a semimetric space $(X, d)$ if there exists $s \geq 1$ for which $d$ satisfies the following triangular inequality.

$$d(x, y) \leq d(x, z) + sd(z, y), \text{ for each } x, y, z \in X. \quad (1)$$

In 1922, a mathematician Banach [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle, which states that every self-mapping $T$ defined on a complete metric space $(X, d)$ satisfying

$$\forall x, y \in X, d(Tx, Ty) \leq \lambda d(x, y), \text{ where } \lambda \in (0, 1)$$

has a unique fixed point and for every $x_0 \in X$ a sequence $\{T_n x_0\}_{n=1}^\infty$ converges to the fixed point. Subsequently, in 1962, Edelstein [2] proved the following version of the Banach contraction principle. Let $(X, d)$ be a compact metric space and let $T : X \to X$ be a self-mapping. Assume that for all $x, y \in X$ with $x \neq y$,

$$d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y).$$

Then $T$ has a unique fixed point in $X$. In 2012, Wardowski [3] introduced a new type of contractions called F-contraction and proved a new fixed point theorem concerning F-contractions.

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$
where $F : R^+ \rightarrow R$ is a mapping satisfying the following conditions:

F1 $F$ is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y$, $F(x) < F(y)$;
F2 For each sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \to \infty} a_n = 0$ if and only if $\lim_{n \to \infty} F(a_n) = -\infty$;
F3 There exists $k \in (0, 1)$ such that $\lim_{a \to 0^+} a^k F(a) = 0$.

We denote by $\xi$, the set of all functions satisfying the conditions (F1) – (F3). Wardowski [3] then stated a modified version of the Banach contraction principle as follows. Let $(X, d)$ be a complete metric space and let $T : X \rightarrow X$ be an F-contraction. Then $T$ has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $(T_n x)_{n=1}^{\infty}$ converges to $x^*$. In 2014, Hossein, P. and Poom, K. [15] defined the F-Suzuki contraction as follows and gave another version of theorem. Let $(X, d)$ be a metric space. A mapping $T : X \rightarrow X$ is said to be an F-Suzuki-contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $T(x) \neq T(y)$

$$d(x, Tx) < d(x, y) \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : R^+ \rightarrow R$ is a mapping satisfying the following conditions:

F1 $F$ is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y$, $F(x) < F(y)$;
F2 For each sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \to \infty} a_n = 0$ if and only if $\lim_{n \to \infty} F(a_n) = -\infty$;
F3 $F$ is continuous on $(0, \infty)$.

We denote by $\xi$, the set of all functions satisfying the conditions (F1) – (F3).

Let $T$ be a self-mapping of a complete metric space $X$ into itself. Suppose $T \in \xi$ and there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Then $T$ has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $(T_n x_0)_{n=1}^{\infty}$ converges to $x^*$.

Following this direction of research (see examples, [4, 5, 6, 7, 8, 9, 10, 16, 17]), in this paper, fixed point results of Piri and Kumam [11], Ahmad et al. [9], Suzuki [18] and Suzuki [19] are extended by introducing common fixed point problem for multivalued generalized F-Suzuki-contraction mappings in strong b-metric spaces.

**Definition 1.1. (Hardy and Rogers [14])**

(1) There exist non-negative constants $a$, satisfying $\sum_{i=1}^{7} a_i < 1$ such that, for each $x, y \in X$, $d(f(x), f(y)) < a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x)) + a_6 d(x, f(x)) + a_7 d(y, f(y))$.

(2) There exist monotonically decreasing functions $a_i(t) : (0, \infty) \rightarrow [0, 1)$ satisfying $\sum_{i=1}^{7} a_i(t) < 1$ such that, for each $x, y \in X$, $d(f(x), f(y)) < a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x)) + a_6 d(x, f(x)) + a_7 d(y, f(y))$.

(3) For each $x, y \in X$, $x \neq y$, $d(f(x), f(y)) < \min\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}$.

**Lemma 1.1.** [13] From definition 1.1, (1) $\implies$ (2) $\implies$ (3).

Denote by $CB(X)$, the collection of all nonempty closed and bounded subsets of $X$ and let $H$ be the Hausdorff metric with respect to the metric $d$; that is,

$$H(A, B) = \max_{a \in A} \{\sup d(a, B), \sup_{b \in B} d(a, B)\}$$

for all $A, B \in CB(X)$, where $d(a, B) = \inf_{b \in B} d(a, b)$ is the distance from the point $a$ to the subset $B$.

**2. Main Results**

**Definition 2.1.** Let $\mathcal{G}$ be the family of all functions $F : R^+ \rightarrow R$ such that:

(F1) $F$ is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y$, $F(x) < F(y)$;
(F2) $\forall a_n \in (0, \infty)$, $\lim_{n \to \infty} a_n = 0$ if and only if $\lim_{n \to \infty} F(a_n) = -\infty$;
(F3) $F$ is continuous on $(0, \infty)$.

**Definition 2.2.** Let $\Psi$ be the family of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\psi(t)$ is continuous and $\psi(t) = 0$ if $t = 0$.

**Definition 2.3.** Let $(X, d)$ be a strong $b$-metric space. Mappings $T, S : X \rightarrow CB(X)$ are said to be multivalued generalized F-Suzuki-Contraction on $(X, d)$ if there exists $F \in \mathcal{G}$ and $\psi \in \Psi$ such that, $\forall x, y \in X, x \neq y$,

$$\frac{1}{1+s}d(x, Tx) < d(x, y) and \frac{1}{1+s}d(y, Sy) < d(y, STx) \implies \psi(N_\phi(x, y)) + F(s^2H(Tx, Ty)) \leq F(N_\phi(x, y)) in which$$

$$N_\phi(x, y) = \phi_1(d(x, y))(d(x, y)) + \phi_2(d(x, y))(d(y, STx)) + \phi_3(d(x, y))\left(\frac{d(y, Tx)}{2s}\right) + \phi_4(d(x, y))\left(\frac{d(x, STx) + H(Tx, Ty)}{2s}\right) + \phi_5(d(x, y))(H(STx, Ty) + H(Tx, Ty)) + \phi_6(d(x, y)))(H(STx, Ty) + d(Tx, x)) + \phi_7(d(x, y))(d(Tx, y) + d(y, Sy)).$$

for which $\phi : R^+ \rightarrow [0, 1)$, with $\sum_{i=1}^{7} \phi_i(d(x, y)) < 1$, is monotonically decreasing function.

Considering the definition $STx := \{y \in CB(X) : \forall y \in Tx\}$, we have the following result.

**Theorem 2.1.** Let $(X, d)$ be a complete strong $b$-metric space and let $T, S : X \rightarrow CB(X)$ be multivalued generalized F-Suzuki-contraction mappings. Then $T$ and $S$ has a common
fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}^\infty_n$ and $\{S^n x\}^\infty_n$ converge to $x^*$.

**Proof** Let $x_0 = x \in X$. Let $x_{n+1} \in T x_n$ and $x_{n+2} \in S x_{n+1} \forall n \in N$. If there exists $n \in N$ such that $d(x_n, T x_n) = d(x_{n+1}, S x_{n+1}) = 0$ then $x_{n+1} = x_{n+2} = x$ becomes a fixed point of $T$ and $S$, respectively, therefore the proof is complete. Now, suppose that $d(x_n, T x_n) > 0$ and $d(x_{n+1}, S x_{n+1}) > 0 \forall n \in N$ then the proof will be divided into two steps.

**Step one.** We show that $\{x_n\}^\infty_n$ is a Cauchy sequence. Let

$$d(x_n, T x_n) > 0 \text{ and } d(x_{n+1}, S x_{n+1}) > 0 \forall n \in N. \quad (3)$$

therefore, we have that

$$\frac{1}{s+1}d(x_n, T x_n) < d(x_n, T x_n) \text{ and } \frac{1}{s+1}d(x_{n+1}, S x_{n+1}) < d(x_{n+1}, S x_{n+1}) \forall n \in N. \quad (4)$$

By Definition 2.3, we get

$$F(H(T x_n, S x_{n+1})) \leq F(N(x, x_{n+1})) - \psi(N(x, x_{n+1})).$$

Since that

$$N(x, x_{n+1}) = \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1}))+ \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) + \phi_3(d(x_n, x_{n+1}))(2d(x_{n+1}, x_{n+2})) + \phi_4(d(x_n, x_{n+1}))(2s d(x_{n+1}, x_{n+2})) + \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \leq \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1}))+ \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) + \phi_3(d(x_n, x_{n+1}))(2d(x_{n+1}, x_{n+2})) + \phi_4(d(x_n, x_{n+1}))(2s d(x_{n+1}, x_{n+2})) + \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \leq \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1}))+ \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) + \phi_3(d(x_n, x_{n+1}))(2d(x_{n+1}, x_{n+2})) + \phi_4(d(x_n, x_{n+1}))(2s d(x_{n+1}, x_{n+2})) + \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) = [\phi_1(d(x_n, x_{n+1}))+ \phi_2(d(x_n, x_{n+1}))+ \phi_4(d(x_n, x_{n+1}))+ \phi_5(d(x_n, x_{n+1}))+ \phi_6(d(x_n, x_{n+1}))+ \phi_7(d(x_n, x_{n+1}))) \leq d(x_n, x_{n-1}) + s \epsilon. \quad (11)$$

By (11) and (9), we have that

$$\epsilon \leq \limsup_{n \to \infty} d(x_n, x_{n+1}) < s \epsilon. \quad (12)$$

then by (5) and definition 2.3, we get

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) + \phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})). \quad (6)$$

On contrary, if $d(x_{n+1}, x_{n+2}) > d(x_n, x_{n+1})$, then

$$\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) < d(x_{n+1}, x_{n+2})$$

and therefore (6) becomes

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \psi(d(x_{n+1}, x_{n+2})).$$

But, from (3) and the fact that $\psi(d(x_{n+1}, x_{n+2})) > 0$, this is a contradiction. Thus, we conclude that

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \psi(d(x_{n+1}, x_{n+2})) < F(d(x_n, x_{n+1})). \quad (7)$$

By (7) and Definition 2.1(F1), we have that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \forall n \in N. \quad (8)$$

Therefore $\{d(x_{n+1}, x_{n+2})\}$ is a nonnegative decreasing sequence of real numbers. Thus there exists $\gamma \geq 0$ such that $\lim_{n \to \infty} d(x_n, x_{n+1}) = \gamma$. From (7) as $n \to \infty$, we have that

$$F(\gamma) \leq F(\gamma) - \psi(\gamma).$$

This implies that $\psi(\gamma) = 0$ and thus $\gamma = 0$. Consequently we arrive at

$$\lim_{n \to \infty} d(x_n, T x_n) = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \quad (9)$$

Now, we claim that $\{x_n\}^\infty_n$ is a Cauchy sequence. On contrary, we assume that there exists $\epsilon > 0$ and $n, m \in N$ such that, for all $n \geq n_\epsilon$ and $n_\epsilon < n < m$,

$$d(x_n, x_m) \geq \epsilon \text{ and } d(x_{n-1}, x_m) < \epsilon. \quad (10)$$

It implies that

$$\epsilon \leq d(x_n, x_m) \leq d(x_n, x_{n-1}) + s \epsilon < d(x_n, x_{n-1}) + s \epsilon. \quad (11)$$

By (11) and (9), we have that

$$\epsilon \leq \limsup_{n \to \infty} \phi_n(x_n, x_m) < s \epsilon. \quad (12)$$
By triangle inequality, we have that
$$
\epsilon \leq d(x_n, x_m) \leq d(x_n, x_{m+1}) + sd(x_{m+1}, x_m).
$$  
(13)

By (9), (10), (12) and (13), we have that
$$
\epsilon \leq \limsup_{n \to \infty} d(x_n, x_{m+1}) < s \epsilon.
$$  
(14)

Similarly, we have that
$$
\epsilon \leq d(x_n, x_m) \leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_m)
\leq sd(x_n, x_m) + (s^2 + 1)d(x_n, x_{n+1}).
$$  
(15)

By (9), (10), (12) and (15), we have that
$$
\epsilon \leq \limsup_{n \to \infty} d(x_n, x_{n+1}) < s \epsilon.
$$  
(16)

Observe that
$$
d(x_n, x_{m+1}) \leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_m)
\leq d(x_n, x_{n+1}) + s[d(x_{n+1}, x_m) + sd(x_m, x_{m+1})]
\leq d(x_n, x_{n+1}) + s[d(x_n, x_{n+1}) + sd(x_n, x_{m})]
+ sd(x_{m+1}, x_m)].
$$  
(17)

By (17), we have that
$$
\frac{\epsilon}{s} \leq \limsup_{n \to \infty} d(x_{n+1}, x_{m+1}) < s^2 \epsilon.
$$  
(18)

By (9) and (10), we select $n_\epsilon > 0 \in N$ such that
$$
\frac{1}{s+1} d(x_n, T x_n) < \frac{1}{s+1} \epsilon < \epsilon \leq d(x_n, x_m) \forall n \geq n(\epsilon)
\implies \frac{1}{s+1} d(x_n, T x_n) < \frac{1}{s+1} \epsilon < d(x_n, x_m)
\forall n \geq n(\epsilon)
$$
and
$$
\frac{1}{s+1} d(x_{n+1}, S x_{n+1}) < \frac{1}{s+1} \epsilon < \epsilon \leq d(x_{n+1}, x_{m+1}) \forall n \geq n_\epsilon
\implies \frac{1}{s+1} d(x_{n+1}, S x_{n+1}) < \frac{1}{s+1} \epsilon
\leq d(x_{n+1}, x_{m+1}) \forall n \geq n_\epsilon
$$

It follows that from Definition 2.3, we have, for every $n \geq n_\epsilon$
$$
F(H(x_{n+1}, x_{m+1})) \leq F(N_\delta(x_n, x_m)) - \psi(N_\delta(x_n, x_m)).
$$  
(19)

Since that
$$
d(x_n, x_m) \leq N_\delta(x_n, x_m)
= \phi_1(d(x_n, x_m))(d(x_n, x_m)) + \phi_2(d(x_n, x_m))(d(x_n, x_{m+1}))
+ \phi_3(d(x_n, x_m)) \left( \frac{d(x_{n+1}, x_m) + d(x_n, x_{n+1})}{2s} \right)
+ \phi_4(d(x_n, x_m)) \left( \frac{d(x_{n+2}, x_n) + d(x_{n+2}, x_{n+1})}{2s} \right)
+ \phi_5(d(x_n, x_m))(d(x_{n+2}, x_{m+1}) + d(x_{n+2}, x_{n+1})).
$$

Similarly, we have that
$$
n_\epsilon \leq \limsup_{n \to \infty} N_\delta(x_n, x_m) < s \epsilon.
$$  
(21)

and therefore
$$
\epsilon \leq \liminf_{n \to \infty} N_\delta(x_n, x_m) < s^3 \epsilon.
$$  
(22)

By (19), (21) and (22), we have that
$$
F(s^3 \epsilon) = F(s^4 \epsilon) \leq F(s^4 \limsup_{n \to \infty} d(x_{n+1}, x_{m+1}))
\leq F(\limsup_{n \to \infty} N_\delta(x_n, x_m)) - \psi(\limsup_{n \to \infty} N_\delta(x_n, x_m))
\leq F(s^3 \epsilon) - \psi(\epsilon).
$$  
(23)

By (23) and the fact that $\epsilon > 0$, this is a contradiction. Hence \{x_n\} is a Cauchy sequence in X. By completeness of (X, d), \{x_{n_{r=1}}^\infty\} and \{x_{n_{r+1}}^\infty\} converge to some point $x^* \in X$, that is,
$$
\lim_{n \to \infty} d(x_n, x^*) = 0 \text{ and } \lim_{n \to \infty} d(x_{n+1}, x^*) = 0.
$$  
(24)

There exists increasing sequences \{n_k\}, \{n_1 + k\} \in N such that $x_{n_k} \in T x^*$ and $x_{n_1 + k} \in S x^*$ for all $k \in N$. Since $Tx^*$ and $Sx^*$ are closed and
$$
\lim_{n \to \infty} d(x_{n_k}, x^*) = 0 \text{ and } \lim_{n \to \infty} d(x_{n_1 + k}, x^*) = 0,
$$

we get $x^* \in Tx^*$ and $x^* \in Sx^*$.

Step two. We show that $x^*$ is a common fixed point of $T$ and $S$. It suffices to show that
$$
\frac{1}{1+s} d(x_n, T x_n) < d(x_n, x^*) \text{ and } \frac{1}{1+s} d(x_{n+1}, S x_{n+1}) < d(x_{n+1}, x^*),
$$
for every $n \in N$.

(25)
implies
\[ F(d(T x^*, x^*)) \leq F(N_\phi(x^*, T x^*)) - \psi(N_\phi(x^*, T x^*)) \]
and
\[ F(d(S x^*, x^*)) \leq F(N_\phi(S x^*, x^*)) - \psi(N_\phi(S x^*, x^*)�)
respectively.

On contrary, suppose there exists \( m \in \mathbb{N} \) such that
\[
\frac{1}{1 + s} d(x_m, Tx_m) \geq d(x_m, x^*) \quad \text{or} \quad \frac{1}{1 + s} d(x_{m+1}, S x_{m+1}) \geq d(x_{m+1}, x^*). \tag{26}
\]

By (26), we have that
\[
(s + 1)d(x_m, x^*) \leq d(x_m, Tx_m) \leq d(x_m, x^*) + sd(T x_m, x^*)
or
\[
(s+1)d(x_{m+1}, x^*) \leq d(x_{m+1}, S x_{m+1}) \leq d(x_{m+1}, x^*) + sd(S x_{m+1}, x^*),
\]
and therefore
\[
d(x_m, x^*) \leq d(T x_m, x^*) = d(x_{m+1}, x^*) \quad \text{and} \quad d(x_{m+1}, x^*) \leq d(S x_{m+1}, x^*) = d(x_{m+2}, x^*). \tag{27}
\]

By (8), (26) and (27), this is a contradiction. Hence, (25) holds, and therefore
\[
F(d(x_{n+1}, x^*)) = F(H(T x_n, S x^*)) \leq F(N_\phi(x_n, x^*)) - \psi(N_\phi(x_n, x^*)), \tag{28}
\]
and
\[
F(d(x_{n+2}, x^*)) = F(H(S x_{n+1}, T x^*)) \leq F(N_\phi(x_{n+1}, x^*)) - \psi(N_\phi(x_{n+1}, x^*)�). \tag{29}
\]

Since that
\[
d(x^*, T x^*) \leq N_\phi(x_n, x^*)
= \phi_1(d(x_n, x^*))(d(x_n, x^*) + \phi_2(d(x_n, x^*) + (d(x_{n+2}, x^*)))
+ \phi_3(d(x_n, x^*)) \left( \frac{d(x_{n+1}, x^*) + d(x_{n+2}, S x^*)}{2s} \right)
+ \phi_4(d(x_n, x^*)) \left( \frac{d(x_n, S x^*) + d(S x^*, x_{n+2})}{2s} \right)
+ \phi_5(d(x_n, x^*) + (d(x_{n+2}, S x^*) + d(x_{n+1}, S x^*) \quad \text{and} \quad + \phi_6(d(x_n, x^*)) \left( d(x_n, x_{n+1}) + d(x_{n+2}, T x^*) \right)
+ \phi_7(d(x_n, x^*) + (d(T x^*, x_n+1)) \leq \max\{d(x_n, x^*), d(x_{n+2}, x^*)
\frac{d(x_{n+1}, x^*) + d(x_{n+2}, S x^*)}{2s}
\frac{d(x_n, S x^*) + sd(S x^*, x_{n+2}) + d(S x^*, x_{n+2})}{2s}
\frac{d(S x^*, x_{n+2}) + d(x_{n+1}, S x^*) + d(x_n, x_{n+1}) + d(x_{n+2}, T x^*)}{2s},
\]
and
\[
d(x^*, S x^*) \leq N_\phi(x_{n+1}, x^*)
= \phi_1(d(x_{n+1}, x^*)) + \phi_2(d(x_{n+1}, x^*) + (d(x_{n+2}, x^*) + \phi_3(d(x_{n+1}, x^*)) \left( \frac{d(x_{n+2}, x^*) + d(x_{n+1}, x^*)}{2s} \right)
+ \phi_4(d(x_{n+1}, x^*)) \left( \frac{d(x_{n+2}, S x^*) + d(S x^*, x_{n+2})}{2s} \right)
+ \phi_5(d(x_{n+1}, x^*) + (d(x_{n+2}, S x^*) + d(x_{n+2}, S x^*) \quad \text{and} \quad + \phi_6(d(x_{n+1}, x^*)) \left( d(x_{n+1}, x_{n+2}) + d(x_{n+3}, S x^*) \right)
+ \phi_7(d(x_{n+1}, x^*) + (d(S x^*, x^*) + d(x^*, x_{n+2})) \leq \max\{d(x_{n+1}, x^*), d(x_{n+3}, S x^*)
\frac{d(x_{n+2}, x^*) + d(x_{n+1}, x^*) + d(x_{n+2}, x^*)}{2s}
\frac{d(x_{n+3}, S x^*) + d(x_{n+2}, S x^*) + d(x_{n+1}, x_{n+2}) + d(x_{n+3}, S x^*)}{2s}
\frac{d(x_{n+2}, S x^*) + d(x_{n+1}, x_{n+2}) + d(x_{n+3}, S x^*)}{2s},
\]

By (24) and (30), we have that
\[
\lim_{n \to \infty} N_\phi(x_n, x^*) = d(T x^*, x^*). \tag{31}
\]

By (24) and (31), we have that
\[
\lim_{n \to \infty} N_\phi(x_{n+1}, x^*) = d(S x^*, x^*). \tag{32}
\]

By (28) and (29) and by the continuity of \( F \) and \( \psi \), we have that
\[
F(d(x^*, T x^*)) \leq F(N_\phi(x^*, T x^*)) - \psi(N_\phi(x^*, T x^*)) \tag{33}
\]
and
\[
F(d(x^*, S x^*)) \leq F(N_\phi(x^*, S x^*)) - \psi(N_\phi(x^*, S x^*). \tag{34}
\]

Hence, since \( T x^* \) and \( S x^* \) are closed then we have \( x^* \in T x^* \) and \( x^* \in S x^* \), that is, \( x^* \) is a fixed point of \( T \) and \( S \).

In Theorem 2.1, when \( T = S = U \), then we have the following result.

**Corollary 2.1.1.** Let \( (X, d) \) be a complete strong \( b \)-metric space and let \( U : X \to CB(X) \) be a multivalued generalized \( F \)-Suzuki-Contractation mapping. Then \( U \) has a fixed point \( x^* \in X \) and for every \( x \in X \) the sequence \( \{U^n x\}_{n=1}^{\infty} \) converges to \( x^* \).

In Corollary 2.1.1, when \( U \) is a single-valued then we have another new result as follows.

**Corollary 2.1.2.** Let \( (X, d) \) be a complete strong \( b \)-metric space and let \( U : X \to X \) be a single-valued generalized \( F \)-Suzuki-Contractation mapping. Then \( U \) has a fixed point \( x^* \in X \) and for every \( x \in X \) the sequence \( \{U^n x\}_{n=1}^{\infty} \) converges to \( x^* \).

In Theorem 2.1, when \( T \) and \( S \) are two single-valued then the
following result holds.

**Corollary 2.1.3.** Let \((X, d)\) be a complete strong \(b\)-metric space and let \(T, S : X \to X\) be two single-valued generalized \(F\)-Suzuki-Contraction mappings. Then \(T\) and \(S\) have a common fixed point \(x^* \in X\) and for every \(x \in X\) the sequence \(\{T^n x\}_{n=1}^\infty\) and \(\{S^n x\}_{n=1}^\infty\) converge to \(x^*\).

In Theorem 2.1, when \((X, d)\) is a complete \(b\)-metric space then the following new result holds.

**Corollary 2.1.4.** Let \((X, d)\) be a complete \(b\)-metric space and let \(T, S : X \to X\) be two single-valued generalized \(F\)-Suzuki-Contraction mappings. Then \(T\) and \(S\) have a common fixed point \(x^* \in X\) and for every \(x \in X\) the sequence \(\{T^n x\}_{n=1}^\infty\) and \(\{S^n x\}_{n=1}^\infty\) converge to \(x^*\).

In corollary 2.1.4, when \(T = S = U\), then we have the following result.

**Corollary 2.1.5.** Let \((X, d)\) be a complete \(b\)-metric space and let \(U : X \to CB(X)\) be a multivalued generalized \(F\)-Suzuki-Contraction mapping. Then \(U\) has a fixed point \(x^* \in X\) and for every \(x \in X\) the sequence \(\{U^n x\}_{n=1}^\infty\) converges to \(x^*\).

**Corollary 2.1.6.** Let \((X, d)\) be a complete strong \(b\)-metric space and let \(U : X \to CB(X)\) be a multivalued generalized \(F\)-Suzuki-Contraction mapping such that there exists \(F \in \mathcal{U}\) and \(\psi \in \Psi\), \(\forall x, y \in X, x \neq y, \frac{1}{s+1} d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^2 d(Ux, Uy)) \leq F(N(x, y))\) in which

\[
N(x, y) = \max\{d(x, y), d(y, U^2 x), \\
\frac{(d(y, Ux)) + (d(x, Uy))}{2s}, \\
\frac{(d(x, Uy)) + (d(U^2 x, Uy))}{2s}, \\
\frac{d(U^2 x, Uy)}{2s} + \frac{d(U^2 x, Uy)}{2s}, \\
\frac{d(U^2 x, Uy)}{2s} + \frac{d(U^2 x, Uy)}{2s}, \\
\frac{d(Ux, y)}{2s} + \frac{d(Ux, y)}{2s}\}.
\]

Then \(U\) has a fixed point \(x^* \in X\) and for every \(x \in X\) the sequence \(\{U^n x\}_{n=1}^\infty\) converges to \(x^*\).

**Proof** from Lemma 1.1, since (2) \(\Rightarrow\) (32) then by the corollary 2.1.1 the result follows immediately.

**Corollary 2.1.7.** Let \((X, d)\) be a complete strong \(b\)-metric space and let \(U : X \to X\) be a single-valued generalized \(F\)-Suzuki-Contraction mapping such that there exists \(F \in \mathcal{U}\) and \(\psi \in \Psi\), \(\forall x, y \in X, x \neq y, \frac{1}{s+1} d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^2 d(Ux, Uy)) \leq F(N(x, y))\) in which

\[
N(x, y) = \max\{d(x, y), d(y, U^2 x), \\
\frac{(d(y, Ux)) + (d(x, Uy))}{2s}, \\
\frac{(d(x, Uy)) + (d(U^2 x, Uy))}{2s}, \\
\frac{d(U^2 x, Uy)}{2s} + \frac{d(U^2 x, Uy)}{2s}, \\
\frac{d(U^2 x, Uy)}{2s} + \frac{d(U^2 x, Uy)}{2s}, \\
\frac{d(Ux, y)}{2s} + \frac{d(Ux, y)}{2s}\}.
\]

Then \(U\) has a fixed point \(x^* \in X\) and for every \(x \in X\) the sequence \(\{U^n x\}_{n=1}^\infty\) converges to \(x^*\).

**Proof** from Lemma 1.1, since (2) \(\Rightarrow\) (33) then by the corollary 2.1.2 the result holds.

**Corollary 2.1.8.** Let \((X, d)\) be a complete strong \(b\)-metric space and let \(T, S : X \to X\) be two single-valued generalized \(F\)-Suzuki-Contraction mappings such that there exists \(F \in \mathcal{U}\) and \(\psi \in \Psi\), \(\forall x, y \in X, x \neq y, \frac{1}{s+1} d(x, T x) < d(x, y)\) and \(\frac{1}{s+1} d(y, S x) < d(y, ST x) \Rightarrow \psi(N(x, y)) + F(s^2 H(T x, S y)) \leq F(N(x, y))\) in which

\[
N(x, y) = \max\{d(x, y), d(x, ST x), \\
\frac{(d(y, T x)) + (d(x, S y))}{2s}, \\
\frac{(d(x, S y)) + (d(y, ST x))}{2s}, \\
\frac{d(ST x, S y)}{2s} + \frac{d(ST x, S y)}{2s}, \\
\frac{d(ST x, S y)}{2s} + \frac{d(ST x, S y)}{2s}, \\
\frac{d(y, S T x)}{2s} + \frac{d(y, S T x)}{2s}\}.
\]

Then \(T\) and \(S\) have a common fixed point \(x^* \in X\) and for every \(x \in X\) the sequence \(\{T^n x\}_{n=1}^\infty\) and \(\{S^n x\}_{n=1}^\infty\) converge to \(x^*\).

**Proof** from Lemma 1.1, since (2) \(\Rightarrow\) (34) then by the corollary 2.1.4 the result holds.

**Corollary 2.1.9.** Let \((X, d)\) be a complete \(b\)-metric space and let \(U : X \to CB(X)\) be a multivalued generalized \(F\)-Suzuki-Contraction mapping such that there exists \(F \in \mathcal{U}\) and \(\psi \in \Psi\), \(\forall x, y \in X, x \neq y, \frac{1}{s+1} d(x, U x) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^2 d(U x, U y)) \leq F(N(x, y))\) in which

\[
N(x, y) = \max\{d(x, y), d(x, U^2 x), \\
\frac{(d(y, Ux)) + (d(x, Uy))}{2s}, \\
\frac{(d(x, Uy)) + (d(U^2 x, Uy))}{2s}, \\
\frac{d(U^2 x, Uy)}{2s} + \frac{d(U^2 x, Uy)}{2s}, \\
\frac{d(U^2 x, Uy)}{2s} + \frac{d(U^2 x, Uy)}{2s}, \\
\frac{d(Ux, y)}{2s} + \frac{d(Ux, y)}{2s}\}.
\]

Then \(U\) has a fixed point \(x^* \in X\) and for every \(x \in X\) the sequence \(\{U^n x\}_{n=1}^\infty\) converges to \(x^*\).

**Proof** from Lemma 1.1, since (2) \(\Rightarrow\) (35) then by the corollary 2.1.5 the result holds.

**3. Example**

Let \(X = [0, 1]\), \(T, S : [0, 1] \to CB([0, 1])\) be defined by \(T x = [0, \frac{x}{2}]\) and \(S y = [0, \frac{y}{2}]\) such that \(S T x = [0, \frac{x}{2}]\) for all \(x \in [0, 1]\). Let \(d\) be the usual metric on \(X\). Taking \(T= \frac{1}{10}\) and let \(x < y\), then \(\forall x, y \in [0, 1] \ d(x, y) > 0\) and \(d(y, S T x) = \frac{|y - \frac{x}{2}|}{2} > \frac{|y - \frac{x}{2}|}{2}\). Thus, \(F\) has the above properties, let \(\phi_1(d(x, y)) = \phi_2(d(x, y)) = \phi_3(d(x, y)) = \frac{1}{s+1}\) and \(\phi_4(d(x, y)) = \phi_5(d(x, y)) = \phi_6(d(x, y)) = \phi_7(d(x, y)) = \frac{1}{10s+1}\). Therefore, we have that

\[
F(H(Tx, S y)) = \ln (H(Tx, S y)) + H(Tx, S y)
\]

\[
= \frac{1}{10} \left( \frac{y - x}{2} \right) + \frac{1}{10} \left( \frac{y}{2} - \frac{x}{4} \right)
\]

\[
\leq \frac{1}{10} \left( \frac{y - x}{4} + \frac{x - y}{4} \right)
\]

\[
= \frac{1}{10} \left( \frac{|y - \frac{x}{2}|}{2} + \frac{|x - \frac{y}{2}|}{2} \right)
\]
\[ \frac{1}{10} \left( \left| y - \frac{x}{4} \right| + \left| x - \frac{y}{2} \right| \right) + \frac{1}{10} \left( \left| y - \frac{x}{4} \right| + \left| x - \frac{x}{8} + \frac{x}{2} - \frac{y}{2} \right| \right) \\
= \frac{1}{10} \left( \left| y - \frac{x}{4} \right| + \left| x - \frac{y}{2} \right| \right) + \frac{1}{10} \left( \left| \frac{x}{8} - \frac{y}{2} \right| + \left| x - \frac{x}{8} \right| \right) \\
+ \frac{1}{10} \left( \left| \frac{y}{2} - \frac{x}{8} \right| \leq \frac{1}{10} \left( \left| y - \frac{x}{2} \right| + \left| x - \frac{x}{4} \right| \right) \right) \\
= \frac{1}{5} \left( \left| y - \frac{x}{4} \right| + \left| x - \frac{y}{2} \right| \right) + \frac{1}{5} \left( \left| \frac{x}{8} - \frac{y}{2} \right| + \left| x - \frac{x}{8} \right| \right) \\
+ \frac{1}{10} \left( \left( \frac{y}{2} - \frac{x}{8} \right) + \left| x - \frac{x}{4} \right| \right) + \frac{1}{10} \left( \left| y - \frac{x}{2} \right| + \left| x - \frac{x}{4} \right| \right) \\
+ \frac{1}{10} \left( \left( y - x \right) + \left| \frac{y}{8} - \frac{x}{2} \right| + \left| \frac{y}{4} - \frac{x}{2} \right| \right) \]

\[ = \phi_1(d(x,y))(d(x,y)) + \phi_2(d(x,y))(d(y,STx)) + \phi_3(d(x,y))(d(y,STx)) + \phi_4(d(x,y))(d(STx,Sy)) + \phi_5(d(x,y))(d(STx,STx)) + \phi_6(d(x,y))(d(STx,Sy) + d(Tx,x)) + \phi_7(d(x,y))(d(Tx,y)) + d(y,SY) - \psi(N_\phi(x,y)). \]

4. Conclusion

Fixed point results of Piri and Kumam [11], Ahmad et al. [9], Suzuki [18] and Suzuki [19] are extended by introducing common fixed point problem for multivalued generalized F-Suzuki-contraction mappings in strong b-metric spaces. In specific, Corollary 2.1.1 and corollary 2.1.2 generalize and extend the work of Ahmad et al. [9] and Kumam and Hossein [5], respectively.

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