From Hamiltonian to Lagrangian Sp(2) BRST Quantization

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Abstract

We give a formal proof of the equivalence of Hamiltonian and Lagrangian BRST quantization. This is done for a generic $Sp(2)$-symmetric theory using flat (Darboux) coordinates. A new quantum master equation is derived in a Hamiltonian setting which contains all the Hamiltonian fields and momenta of a given theory.
1 Introduction

Both the Hamiltonian BRST quantization scheme [1, 3] and the Batalin-Vilkovisky Lagrangian BRST quantization scheme [2] are well-established quantizations procedures. It is therefore of major interest to give a general recipe for the transition from a Hamiltonian BRST quantized system to the corresponding BV Lagrangian BRST quantized system. Many articles have been devoted to this problem [6]. Grigoryan, Grigoryan and Tyutin extended [4] the phase space of a generic Hamiltonian theory with some auxiliary fields and momenta, introducing antifields as sources for the BRST transformation. By integrating out the momenta, they were able to formally derive a Lagrangian quantum master equation. De Jonghe noted [5] that the extension of the phase space could be understood and organized in terms of collective fields, using the prescription of Hamiltonian quantization. In this manner the Schwinger-Dyson BRST symmetry is automatically incorporated as first pointed out as a general principle by [7]. We improve this standard BRST construction further in section 2-3. Here we show that, even in the enlarged Hamiltonian phase space without integrating over the momenta, a new quantum master equation is fulfilled. Another improvement (of a bit more technical nature) is that part of the kinetic terms ('momenta times velocity') in the effective action are in fact BRST-exact, and can be removed. It is remarkable that this little observation happens to eliminate the necessity of the usual scaling procedure, where the fields and momenta are redefined by a power of a scaling parameter, which in the end is scaled to infinity to remove (some of the kinetic) terms [3, 4, 5].

In the recent years Sp(2)-symmetric quantization schemes have been formulated both in the Hamiltonian case [10] and in the Lagrangian case [11, 12, 13]. This is the main subject of this article. One of the reasons why we treat the standard BRST case is to show that many features are in fact already present in the standard BRST case, and can be directly transfered to the Sp(2) case. In section 4-5 we perform the transition from Hamiltonian to Lagrangian picture for a general Sp(2) theory. The guiding principle is again the Schwinger-Dyson shift symmetry [8, 9]. A new quantum master equation is derived in an extended phase space, and the Sp(2) Lagrangian theory of Batalin, Lavrov and Tuytin [11] is formally extracted.

2 Standard BRST

In the case of standard BRST let us perform [5] the transition from a Hamiltonian system to the equivalent BV Lagrangian system. In order to achieve this we extend the phase space by introducing several new fields, especially those who plays the role of antifields in the Lagrangian picture. Let us consider a Hamiltonian system with fields $\phi^A$ and momenta $\pi_A$, a real Hamiltonian $H_o$ and a real BRST charge $\Omega_o$ with (right) BRST transformation $\delta_o = \{\cdot, \Omega_o\}$, where $\{\cdot, \cdot\}$ denotes the equal time Poisson superbracket. The Hamiltonian $H_o$ and the charge $\Omega_o$ fulfill

\[
\begin{align*}
\{\Omega_o(t), \Omega_o(t)\} &= 0 \\
\{H_o(t), \Omega_o(t)\} &= 0 \\
\frac{\partial}{\partial t} \Omega_o &= 0 .
\end{align*}
\]  

(1)

The last equation, which states that there is no explicit time dependence on the BRST charge $\Omega_o$, is a further requirement that we impose to make the calculations simpler. This is a rather weak

\footnote{The term standard is merely used to distinguish from the Sp(2) symmetric case, where there also is an anti-BRST symmetry.}

\footnote{The demand of Hamiltonian and BRST charge to be real replaces the requirement of hermicity in a operator formulation. In general, complex conjugation of supernumbers corresponds to Hermitian conjugation of operators. We emphasize that $\overline{zw} = \overline{w} \overline{z}$ for two supernumbers $z$, $w$. And $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial \overline{z}}$ if $w$ is a function of $z$.}
assumption which is fulfilled for most theories of interest. The compact De Witt notation is used throughout this text, i.e. summation and integration over repeated indices is implicitly understood. An exception is in the above formula where the time-dependence is explicitly mentioned in parentheses. Then there is no integration over time.

Note that \((\phi, \pi)\) runs over all fields of the given theory, including the Lagrange multipliers, ghosts, and corresponding momenta. We stress that the Hamiltonian \(H_o\) is the full BRST quantized Hamiltonian. There is a ambiguity in \(H_o\) because one could always change \(H_o\) by a BRST exact amount. If the theory has a classical Lagrangian formulation with classical action \(S_{cl}\) (which does not depend on ghosts) one could fix this ambiguity by demanding

\[
e^\frac{i}{\hbar}S_{cl} = \int D\pi e^\frac{i}{\hbar}(\pi_A \dot{\phi}^A - \int dt H_o) .
\]

(2)

Even though this looks like a rather reasonable boundary condition to impose, it is fairly complicated to study this boundary condition in a general setting. In practice it may be that one cannot perform the \(\pi\)-integrations explicitly, or those turn out to be on the form

\[
\int D\pi e^\frac{i}{\hbar}(\pi_A \dot{\phi}^A - \int dt H_o) = \text{delta functions} \cdot e^\frac{i}{\hbar} \text{action} .
\]

(3)

A possible cure for the last problem is to reorganize (fields↔momenta). In general we have not done this analysis and we give no guarantee that the boundary condition (2) can be met. Fortunately, (2) is not essential to the constructions given below other than to impose a classical limit.

The BRST transformation rules of the phase space variables are organized as

\[
\delta_o \phi^A = i \varepsilon^A R^A(\phi, \pi) \\
\delta_o \pi_A = R_A(\phi, \pi) .
\]

(4)

The factor \(i\) is chosen such that the BRST field structure function \(R^A\) and the BRST momenta structure function \(R_A\) are real. We further assume that Hamiltonian theory is free of anomalies in the sense that the Ward identities\[^3\] \(< \delta_o G >_o = 0\) is fulfilled for every function \(G = G(\phi, \pi)\). At least this is very reasonable requirement seen from an operator formulation point of view. This leads (by a variational argument) to an equation

\[
\left( i^A \frac{\delta^r}{\delta \phi^A} R^A + \frac{\delta^r}{\delta \pi_A} R_A \right) e^\frac{i}{\hbar}(\pi_A \dot{\phi}^A - \int dt H_o) = 0 ,
\]

(5)

which can be viewed as a Hamiltonian master equation for the original theory. It is assumed that all constraints \(G_\alpha(\phi, \pi)\) are first class:

\[
\{G_\alpha(t), G_\beta(t)\} = C_{\alpha\beta}^\gamma(t)G_\gamma(t) \\
\{G_\alpha(t), H_\alpha(t)\} = V_\alpha^\beta(t)G_\beta(t) .
\]

(6)

We want to impose the Schwinger-Dyson shift symmetry \[^4\]. Therefore, the phase space is extended with shift fields \(\varphi^A_i\), shift momenta \(\pi^A_i\), shift ghosts \(C^A_i\) and shift antighosts \(P^A_i\). The index \(i = 1, 2\) is a sector index. \(i = 1\) is often called the minimal sector, while \(i = 2\) is referred to as the non-minimal sector. Thus for each original field there are four new fields:

\[
\text{Fields} \quad \Phi^A = \{\phi^A, \varphi^A_1, C^A_1\} \\
\text{Momenta} \quad \Pi^A = \{\pi^A, \pi^A_1, P^A_1\} .
\]

(7)

\[^3\]It is enough to impose the Ward identities for zero gauge fermion. In fact the principle that BRST invariant Green functions do not depend on the gauge fermion can be derived from this.
We use the convention
\[
\Phi_A = \Pi_A = (-1)^{\epsilon A} \Pi_A ,
\] (8)
so that Grassmann odd momenta is imaginary. The Grassmann parities and ghost numbers are given by
\[
\begin{align*}
\epsilon(\pi_A) &= \epsilon(\phi_A) = \epsilon(\pi_A) = \epsilon(\phi_A) \equiv \epsilon_A \\
\epsilon(P_A) &= \epsilon(C_A) = \epsilon_A + 1
\end{align*}
\] (9)
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\epsilon(P_A) &= \epsilon(C_A) = \epsilon_A + 1
\] (9)
The extended equal time Poisson superbracket reads:
\[
\{ F(t), G(t) \} = \frac{\delta^r F(t)}{\delta \Phi^A(t)} \frac{\delta^l G(t)}{\delta \Pi_A(t)} - (-1)^{\epsilon A} \frac{\delta^r F(t)}{\delta \Pi_A(t)} \frac{\delta^l G(t)}{\delta \Phi^A(t)} ,
\] (10)
or equivalently in terms of fundamental fields:
\[
\begin{align*}
\{ \Phi^A(t), \Pi_B(t) \} &= \delta^A_B \\
\{ \Phi^A(t), \phi^B(t) \} &= 0 \\
\{ \Pi_A(t), \Pi_B(t) \} &= 0 .
\] (11)
As additional shift constraints \( \chi^i_A \) we take
\[
\begin{align*}
\chi^1_A &= \pi_A + \pi^1_A \\
\chi^2_A &= \pi_A + \pi^2_A
\] (12)
For an arbitrary function \( F \) of the extended phase space variables \( (\Phi^A, \Pi_A) \) a shift operation \( \sim \) is defined as
\[
F(\phi, \pi, \pi^1, \pi^2, \cdots) \sim \tilde{F} \equiv F(\phi - \chi^1, \pi - \chi^2, 0, 0, \cdots) = F(\phi - \pi^1, -\pi^2, 0, 0, \cdots) .
\] (13)
This operation has several nice features:
\[
\begin{align*}
\{ \tilde{F}, \tilde{G} \} = \tilde{F} \tilde{G} \\
\{ F(t), G(t) \} &= \{ \tilde{F}(t), \tilde{G}(t) \} \\
\{ \tilde{F}(t), \phi(t) \} &= 0 \\
\{ \tilde{F}(t), \chi(t) \} &= 0 .
\] (14)
We now define an extended Hamiltonian \( H \) and an extended BRST charge \( \Omega \) as
\[
\begin{align*}
H(t) &= \tilde{H}_o(t) \\
\Omega(t) &= i^{\epsilon A} \chi^i_A(t) C^A_i(t) + \tilde{\Omega}_o(t) .
\] (15)
(No integration over time in the last formula.) Note that with this choice of Hamiltonian \( H \) the shift operation \( \sim \) and total time derivation commute. It is easy to check that
\[
\{ \Omega(t), \Omega(t) \} = 0
\] (15)
\[ \{H(t), \Omega(t)\} = 0, \]  

and that the constraints \((\chi^i_A, \tilde{G}_\alpha)\) form a first class algebra. Therefore, the Schwinger-Dyson shift symmetry is fully integrated in the enlarged phase space. Note that the only genuine collective shift field is \(\varphi^A_{i=1}\). The new BRST transformation rules with \(\delta = \{, \Omega\}\) are:

\[ \begin{align*}
\delta \phi^A_i & = i^A C^A_i \\
\delta \varphi^A_1 & = i^A C^A_1 - i^A \tilde{R}^A \\
\delta \varphi^A_2 & = i^A C^A_2 \\
\delta C^A_i & = 0 \\
\delta \chi^i_A & = 0 \\
\delta \pi^A_1 & = -\tilde{R}^A \\
\delta \pi^A_2 & = 0.
\end{align*} \]  

The extended effective action \(S_{\text{eff}}\) takes the form:

\[ S_{\text{eff}} = \Pi_A \dot{\phi}^A + \int dt \left(-H + \delta (\psi_s + \psi_{gf})\right), \]  

where it is convenient to extract a certain 'shift' fermion \(\psi_s\) from the total gauge fermion \(\psi_s + \psi_{gf}\).

\[ \begin{align*}
\psi_s(t) & = -(-i)^2 \left( \mathcal{P}^j_1(t) \varphi^A_j(t) + g^j_i \mathcal{P}^j_1(t) \varphi^A_j(t) \right) \\
g^j_i & \equiv g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\end{align*} \]  

\(g\) is a constant of dimension \((\text{time})^{-1}\). By a straightforward calculation, the extended effective action \(S_{\text{eff}}\) can be rewritten in a more useful manner

\[ S_{\text{eff}} = \hat{S} + S_\delta + \int dt \, \delta \psi_{gf}. \]  

Here we have defined an action \(\hat{S}\)

\[ \hat{S} \equiv \pi_A \dot{\phi}^A - \int dt \, \tilde{H}_o + \mathcal{P}^1_A \dot{\tilde{R}}^A + \phi^* \tilde{R}^A, \]  

and an auxiliary 'delta function term' \(S_\delta = S_\delta^1 + S_\delta^2:\)

\[ \begin{align*}
-S_\delta & = g^j_i \chi^j_A \varphi^A_j + g^j_i \mathcal{P}^j_1 \varphi^A_j = -S_\delta^1 - S_\delta^2 \\
-S_\delta^1 & \equiv g \left( \chi^j_A \varphi^A_j + \mathcal{P}^1_A \varphi^A_j \right) = (-i)^2 g \delta (\mathcal{P}^1_A \varphi^A_j) \\
-S_\delta^2 & \equiv g \chi^j_A \varphi^A_j + \phi^*_A \varphi^A_j.
\end{align*} \]  

We have identified antifields \(\phi^*_A\):

\[ \phi^*_A \equiv g \mathcal{P}^2_A, \]  

as the generators of the field BRST symmetry \(\tilde{R}^A\) in the (extended) quantum action \(\hat{S}\).

If we now consider an arbitrary function \(G = G(\phi, \pi)\) and gauge fixing fermion \(\psi_{gf} = \psi_{gf}(\phi, \pi)\) of the original phase space variables \((\varphi^A, \pi_A)\), the corresponding Green function in the extended phase space equals the original Green function

\[ < G > = \frac{\int D\Phi D\Pi \, G \, e^{\hat{S}(S + S_\delta + \int dt \, \delta \psi_{gf})}}{\int D\Phi D\Pi \, e^{\hat{S}(S + S_\delta + \int dt \, \delta \psi_{gf})}} = \frac{\int D\phi D\pi \, G \, e^{\hat{S}(\pi_A \dot{\phi}^A + \int dt (-H_o + \delta \psi_{gf}))}}{\int D\phi D\pi \, e^{\hat{S}(\pi_A \dot{\phi}^A + \int dt (-H_o + \delta \psi_{gf}))}} = < G >_o. \]  

(24)
Therefore the original theory is included in the extended theory. This justifies the raison d’être of the whole ‘shift’ construction. The proof of (24) goes as follows: First of all, it is convenient to change the integration variables in path integrals over the extended phase space from \((\Phi, \Pi) = (\cdots, \pi, \cdots, \pi^4, \cdots)\) to \((\cdots, \pi, \cdots, \chi^1, \cdots)\). (The dots indicate unchanged variables.) The Jacobian of this change of variables is 1, due to (12). Now integration over \(\chi^2, \varphi_2, C_2\) and \(C_1\), produces delta functions in \(\varphi_1, \chi^1, P^1\) and \(\phi^* \equiv gP^2\) respectively, due to the \(S_\delta\) term. This proves equation (24).

The BRST symmetry in the extended phase space is called Schwinger-Dyson BRST symmetry, because the Schwinger-Dyson equations

\[
\int \mathcal{D}\phi \left( \frac{\hbar}{i} \frac{\delta^r G}{\delta \phi^A} + G(\phi) \frac{\delta^r S_{cl}}{\delta \phi^A} \right) e^{i(\xi + \delta \psi_{gf}(\phi))} = 0
\]  

(25)

are fulfilled whenever \(\frac{\hbar}{i} \frac{\delta^r G}{\delta \phi^A} + G(\phi) \frac{\delta^r S_{cl}}{\delta \phi^A}\) is BRST closed, if the boundary condition (2) is satisfied. The proof is composed of several steps. First, we show (in two different ways) that

\[
< \frac{\hbar}{i} \frac{\delta^r G}{\delta \phi^A} + G(\phi) \frac{\delta^r S_{cl}}{\delta \phi^A} > = 0
\]

(26)

In the above manipulations (26) we have assumed that the gauge fermion term \(\delta \psi_{gf}\) is independent of \(\phi, \varphi_1\) and \(C_1\), and we have performed partial integrations in \(\varphi_1\) and \(C_1\). Now it is clear, from arguments similar to (24), that

\[
< \frac{\hbar}{i} \frac{\delta^r G}{\delta \phi^A} + G(\phi) \frac{\delta^r S_{cl}}{\delta \phi^A} > = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\Pi \ h \delta \psi_{gf} e^{i(\delta^r (\tilde{S} + \delta \psi_{gf}))} e^{i(\delta^r (\tilde{S} + \delta \psi_{gf}))} = 0.
\]

(27)

Finally, one integrates out the original momenta \(\pi_A\) to obtain the Schwinger-Dyson equations (25), by applying boundary condition (2). \(S_{cl}\) does not depend on the original gauge ghost so one needs to chose an appropriate gauge fixing term \(\delta \psi_{gf}\) in order for the gauge ghost integrations in the Schwinger-Dyson equations (25) to become meaningful. This is allowed because it does not change the correlators. Here, \(\delta_{cl}\) denotes the original BRST symmetry in which momenta are integrated out.

3 An Extended Master Equation in the Standard BRST Case

Let us now prove a quantum master equation in the extended phase space. The principle is to impose Ward identities \(< \delta G > = 0\) for every function \(G = G(\Phi, \Pi)\) generated by the Schwinger-Dyson BRST symmetry in the extended phase space. Consider a BRST variation of an arbitrary
We have introduced the odd Laplacian function $G = G(\Phi, \Pi)$:

$$
\delta G = \frac{\delta' G}{\delta \phi^A} \epsilon^A c_1^A + \frac{\delta' G}{\delta \pi^A} \chi \tilde{R}_A + \frac{\delta' G}{\delta \varphi^1} \epsilon^A \left( c_1^A - \frac{\delta \hat{S}}{\delta \phi^A} \right) + \frac{\delta' G}{\delta \varphi^2} \epsilon^A c_2^A + \frac{\delta' G}{\delta \pi^A} (-i)^\epsilon A \chi^A_i .
$$

We have made use of the fact that $\phi^A_i$ only appears linearly in the action $\hat{S}$ as the generator for $\tilde{R}^A$. Evaluating the BRST variation inside a path integral with gauge fixing term $\psi_{gf} = 0$ and using Ward identities, we find, after performing partial integration in the $G$-derivatives:

$$
0 = -Z < \delta G > = \int D\Phi D\Pi \left( (-i)^\epsilon^A \frac{\delta^l}{\delta \phi^A} c_1^A + \frac{\delta^l}{\delta \pi^A} \right) \tilde{R}_A + \left( (-i)^\epsilon^A \frac{\delta^l}{\delta \phi^A} c_1^A + \frac{\delta^l}{\delta \pi^A} \right) \tilde{R}_A + \left( (-i)^{\epsilon^A + g} \frac{\delta^l}{\delta \phi^A} c_1^A + \frac{\delta^l}{\delta \pi^A} \right) \tilde{R}_A - i^\epsilon^A \chi^A \frac{\delta^l}{\delta \pi^A} \tilde{R}_A - i^{\epsilon^A + g} \chi^A_i \frac{\delta^l}{\delta \pi^A} \tilde{R}_A - i^\epsilon^A \chi^A \frac{\delta^l}{\delta \pi^A} \tilde{R}_A - i^{\epsilon^A + g} \chi^A_i \frac{\delta^l}{\delta \pi^A} \tilde{R}_A - i^{\epsilon^A + g} \chi^A_i \frac{\delta^l}{\delta \pi^A} \tilde{R}_A .
$$

Applying the fundamental lemma in calculus of variations, we get a quantum master equation in the extended phase space:

$$
\left( \frac{\hbar}{i} \Delta + \frac{\delta^l}{\delta \pi^A} \chi \right) \tilde{R}_A - i^{\epsilon^A} \chi^A \frac{\delta^l}{\delta \pi^A} \tilde{R}_A = 0 .
$$

We have introduced the odd Laplacian

$$
\Delta \equiv (-i)^\epsilon^A \frac{\delta^l}{\delta \phi^A} \frac{\delta^l}{\delta \phi^A} .
$$

Alternatively, one could derive the quantum master equation from Batalin-Fradkin-Vilkovisky Theorem, which states that the partition function $Z_{\psi_{gf}}$ is independent of the gauge fixing fermion $\psi_{gf}$. Choosing an arbitrary infinitesimal variation $G = d\psi_{gf}$ around $\psi_{gf} = 0$, the equation

$$
0 = Z_{d\psi_{gf}} - Z_0 = \int D\Phi D\Pi \epsilon^+(\hat{S} + S^I) \delta d\psi_{gf} = < \delta G > ,
$$

would lead to the same proof and exactly the same conclusions.

It is easy to formally extract the BV Lagrangian theory. First, use to manipulate the exponent $S$ to $\hat{S} + S^I$ in the master equation:

$$
\left( \frac{\hbar}{i} \Delta + \frac{\delta^l}{\delta \pi^A} \chi \right) \tilde{R}_A - i^{\epsilon^A} \chi^A \frac{\delta^l}{\delta \pi^A} \tilde{R}_A = 0 .
$$

*An infinitesimal variation is denoted with a 'hard d'.
Now, let $\hat{W}$ be defined as

$$e^{\hat{W}} \equiv \int D\pi^1 D\chi^1 D\varphi^1 e^{\hat{\phi} (\hat{S} + S^1_1)} .$$

(34)

Defining the Lagrangian quantum action $W = W[\phi - \varphi_1, \phi^*]$

$$e^{\hat{W}} \equiv \int D\pi^1 D\chi^1 D\varphi^1 D\pi^1 e^{\hat{\phi} (\hat{S} + S^1_1)} ,$$

(35)

and integrating over $\pi, \varphi_2, \chi^1, C_2$ and $P^1$ in (33), one formally regains the Lagrangian quantum master equation:

$$\Delta e^{\hat{W}} = 0 .$$

(36)

Performing the integrations in (34) by means of (21,22)

$$\hat{W} = \pi_A \delta^A_0 - \int dt \left. \tilde{H}_o \right|_{\chi = 0} + \phi^*_A \tilde{R}^A \left|_{\chi = 0} ,$$

(37)

one obtains that the Lagrangian quantum action becomes

$$e^{\hat{W} = \int D\pi^1 D\chi^1 D\varphi^1 \left( \pi_A \delta^A_0 - \int dt \left. \tilde{H}_o \right|_{\chi = 0} + \phi^*_A \tilde{R}^A \right) .$$

(38)

Note that the action $W$ is not necessarily linear in the antifields $\phi^*$ due to the $\pi$ integration (if $R^A$ depend on the $\pi$‘s). Imposing the boundary condition (3) yields

$$W[\phi - \varphi_1, \phi^* = 0] = S_{cl} [\phi - \varphi_1] .$$

(39)

4 $Sp(2)$ Symmetric Case

Let us now turn to the $Sp(2)$ symmetric case. The shift construction is almost the same although we have to extend it with twice as many shift fields. We focus on the new features of the $Sp(2)$ symmetric case, and leave out the details that are carried over unchanged from the standard BRST case. Consider a Hamiltonian system with fields $\phi^A$ and momenta $\pi_A$, Hamiltonian $H_o$ and two BRST charges $\Omega^a_o, a = 1, 2$ with corresponding (right) BRST transformations $\delta^a_o = \{ \cdot, \Omega^a_o \}$. $a = 1$ denotes the standard BRST and $a = 2$ is the anti BRST. The Hamiltonian $H_o$ and the charge $\Omega^a_o$ fulfill

$$\{ \Omega^a_o(t), \Omega^b_o(t) \} = 0$$

$$\{ H_o(t), \Omega^a_o(t) \} = 0$$

$$\frac{\partial}{\partial t} \Omega^a_o = 0 .$$

(40)

We assume that the BRST transformation rules of the phase space variables are organized as

$$\delta^a_o \phi^A = \epsilon^A \Omega^a_o \phi^A$$

$$\frac{1}{2} \epsilon_{ab} \delta^b_o \phi^A = \epsilon^A \Omega^a_o \phi^A$$

$$\delta^a_o \pi_A = \Omega^a_o \pi_A .$$

(41)

The sign convention of the $\epsilon$-tensor is $\epsilon^{12} = \epsilon_{21} = +1$. 

7
$\mathcal{R}^A$ is often referred to as the commutator of the BRST and anti-BRST transformation. A Hamiltonian counterpart to a master equation is fulfilled in the original theory

$$
\left(\epsilon^A \frac{\delta\gamma}{\delta \phi^A} \mathcal{R}^{Aa} + \frac{\delta\gamma}{\delta \pi_A} \mathcal{R}_A^a\right) e^{i \tilde{\Phi}(\pi_A \phi^A - \int dt H_o)} = 0 . \tag{42}
$$

We now impose the Schwinger-Dyson BRST symmetry and extend the phase space with shift fields $\varphi_i^A$, $\pi_i^A$, shift momenta $\pi_i^A$, $\varphi_A^i$ shift ghost/antighost fields $C_i^{Aa}$, and shift ghost/antighost momenta $\mathcal{P}_{Aa}$. Altogether, we have a ninefold increase in the number of fields:

$$
\text{Fields } \Phi^A = \left\{ \phi^A, \varphi_i^A, \pi_i^A, C_i^{Aa} \right\},
$$

$$
\text{Momenta } \Pi_A = \left\{ \pi_A, \pi_i^A, \varphi_A^i, \mathcal{P}_{Aa} \right\} . \tag{43}
$$

Note that $\pi_i^A$ are fields, and $\varphi_A^i$ momenta. The Grassmann parity, ghost number and new ghost number $\tilde{\Omega}$ are given by

$$
\begin{align*}
\epsilon(\pi_i^A) &= \epsilon(\varphi_i^A) = \epsilon(\pi_i^A) = \epsilon(\varphi_i^A) = \epsilon(\pi_A) = \epsilon(\phi^A) \equiv \epsilon_A \\
\epsilon(\pi_{Aa}) &= \epsilon(\pi_i^A) = \epsilon(\phi^A) = \epsilon(\phi^A) = \epsilon(\phi^A) \\
\text{gh}(\pi_i^A) &= \text{gh}(\varphi_i^A) = \text{gh}(\pi_i^A) = \text{gh}(\phi^A) \equiv \text{gh}_A \\
\text{gh}(\pi_A^i) &= \text{gh}(\varphi_i^A) = \text{gh}(\pi_A^i) = \text{gh}(\pi_A^i) = \text{gh}(\pi_A^i) \\
\text{gh}(C_i^{Aa}) &= \text{gh}_A - (-1)^a = -\text{gh}(\mathcal{P}_{Aa}) \tag{44}
\end{align*}
$$

The shift constraints $\chi_A^i$ are

$$
\begin{align*}
\chi_A^1 &= \pi_A + \pi_1^A \\
\chi_A^2 &= \pi_A^2 \\
\chi_A^3 &= \pi_A^3 \\
\chi_A^4 &= \pi_A^4 . \tag{45}
\end{align*}
$$

The shift operation $\sim$ is unchanged compared to the standard BRST case, although it now operates on a larger space.

$$
F(\phi, \pi, \varphi_1, \pi_1, \varphi_2, \pi_2, \ldots) \sim \tilde{F} \equiv F(\phi - \varphi_1, \pi - \chi^1_1, 0, 0, \ldots) = F(\phi - \varphi_1, -\pi^1, 0, 0, \ldots) . \tag{46}
$$

We now define an extended Hamiltonian $H$ and extended BRST charges $\Omega^a$ as

$$
\begin{align*}
H(t) &= \tilde{H}_o(t) \\
\Omega^a(t) &= i^A \chi^i_A(t) C_i^{Aa}(t) + (-i)^{a+1} \epsilon_c^{ab} \chi^{A}_{i}(t) \mathcal{P}_{A}^{b}(t) + \tilde{\Omega}_o(t) . \tag{47}
\end{align*}
$$

(No integration over time in the last formula.) It is easy to check that

$$
\begin{align*}
\left\{ \Omega^a(t), \Omega^b(t) \right\} &= 0 \\
\left\{ H(t), \Omega^a(t) \right\} &= 0 , \tag{48}
\end{align*}
$$

8
and that the constraints \((\chi_A^i, \chi^A_i, \tilde{G}_\alpha)\) form a first class algebra. The BRST transformation rules 
\[\delta^\alpha = \{\cdot, \Omega^\alpha\} \]
are:
\[\begin{align*}
\delta^\alpha \phi^A &= i\epsilon_A C^{Aa}_1 \\
\delta^\alpha \varphi^i_A &= i\epsilon_A C^{Aa}_1 - i\epsilon_A \tilde{R}^{Aa} \\
\delta^\alpha \chi^A_i &= i\epsilon_A C^{Aa}_2 \\
\delta^\alpha \varphi^i_A &= i\epsilon_A \varphi^i_A + \epsilon^{ab} \chi^A_i \\
\delta^\alpha \chi^A_i &= 0 \\
\delta^\alpha \pi^A_i &= 0.
\end{align*}\]

The action \(S_{\text{eff}}\) takes the form
\[\begin{align*}
S_{\text{eff}} &= \Pi_A \dot{\Phi}^A + \int dt \left( -H + \frac{1}{2} \epsilon_{ab} \delta^a \delta^b (\chi_s + \chi_{gf}) \right) \\
&= \hat{S} + S_\delta + \int dt \frac{1}{2} \epsilon_{ab} \delta^a \delta^b \chi_{gf}.
\end{align*}\]

We have extracted a 'shift' boson \(\chi_s\) from the total gauge boson \(\chi_s + \chi_{gf}\):
\[\chi_s(t) \equiv i \varphi^i_A(t) \varphi^i_A(t) + ig \varphi^i_A(t) \varphi^i_A(t) .\]

\(g^i\) is defined as in equation (19). In the second line of (50), we have introduced an action \(\hat{S}\) and a 'delta function' part \(S_\delta = S_\delta^1 + S_\delta^2:\)
\[\begin{align*}
\hat{S} &\equiv \pi_A \dot{\phi}^A - \int dt \tilde{H}_o + \varphi^i_A \tilde{\chi}^A + \tilde{\phi}_A \tilde{R}^A + \varphi^A_1 \tilde{R}^A + \phi^A_1 \tilde{R}^A \\
- S_\delta &\equiv g \varphi^i_A \varphi^i_A + g \varphi^i_A \varphi^A_i + g \varphi^A_i \varphi^A_1 \\
- S_\delta^1 &\equiv g \left( \chi^A \varphi^A + \varphi_A \chi^A + \varphi^A_1 \chi^A \right) = -\frac{i}{2} \epsilon_{ab} \delta^a \delta^b \left( \varphi^A_1 \varphi^A_2 \right) \\
- S_\delta^2 &\equiv g \left( \varphi^A_1 \varphi^A_2 + \varphi^A_2 \varphi^A_1 \right) + \phi^A_1 \chi^A \varphi^A_1 .
\end{align*}\]

Here we have identified \(\tilde{\phi}_A\) and antifields \(\phi^A_{\alpha}\)
\[\begin{align*}
\tilde{\phi}_A &\equiv g \varphi^2_A \\
\phi^A_{\alpha} &\equiv g \varphi^2_A
\end{align*}\]
as the generators of \(\mathcal{R}^{A}\) and \(\mathcal{R}^{Aa}\) respectively.

If we now take an arbitrary function \(G = G(\phi, \pi)\) and a gauge fixing boson \(\chi_{gf} = \chi_{gf}(\phi, \pi)\) of the original phase space variables \((\phi^A, \pi_A)\), the corresponding Green function in the extended phase space equals the original Green function, because integration over \(\chi_{gf}^2, \varphi_2, \chi_2, \chi_1, C_2\) and \(C_1\), produces delta functions in \(\varphi_1, \chi_1, \chi^A, \phi \equiv g \varphi^2, \varphi^A_1\) and \(\phi^2 \equiv g \varphi^2\) respectively, due to the \(S_\delta\) term.
\[< G > = \frac{\int \mathcal{D}\Phi \mathcal{D}\Pi \mathcal{D}\pi \mathcal{D}\chi_{gf} e^{\pi \left( \hat{S} + S_\delta \right) + \int dt \frac{1}{2} \epsilon_{ab} \delta^a \delta^b \chi_{gf} \right)}}{\int \mathcal{D}\Phi \mathcal{D}\Pi \mathcal{D}\pi \mathcal{D}\chi_{gf} e^{\pi \left( \hat{S} + S_\delta \right) + \int dt \frac{1}{2} \epsilon_{ab} \delta^a \delta^b \chi_{gf}}}} = < G > .\]

Again, we see that all physical relevant quantities of the original Hamiltonian theory can be reproduced in the extended Hamiltonian theory.
5 An Extended Master Equation in the \(Sp(2)\)-symmetric Case

Consider a BRST variation of an arbitrary function \(G = G(\Phi, \Pi)\):

\[
\delta^a G = \frac{\delta' G}{\delta \phi^A} \bar{v}^A C^A_1 + \frac{\delta' G}{\delta \pi_A} \bar{R}^a_A + \frac{\delta G}{\delta \phi^A} C^A_1 \left( C^A_1 \frac{\delta \hat{S}}{\delta \phi^A} \right) + \frac{\delta G}{\delta \phi^A} C^A_2 \\
+ \frac{\delta G}{\delta C_i} (-i)^{a+1} e^{ab} \chi^A_i + \frac{\delta G}{\delta \phi^A} e^{ab} P_{ab} + \frac{\delta G}{\delta P_{ab}} (-i)^{a+1} \chi^A_i. \tag{55}
\]

We have made use of the fact that \(\phi^A_a\) only appears linearly in the action \(\hat{S}\) as the generator for \(\mathcal{R}^{Aa}\). Evaluating the BRST variation inside a path integral with gauge fixing term \(\chi_{gf} = 0\) and using Ward identities, we find, after performing partial integration in the \(G\)-derivatives:

\[
0 = -Z < \delta^a G > \\
= \int \mathcal{D}[\pi, \phi] G \left( (-i)^{a} \frac{\delta' \hat{S}}{\delta \phi^A} C^A_1 + \frac{\delta \hat{S}}{\delta \pi_A} \bar{R}^a_A \right) \\
+ (-i)^{a} \frac{\delta \hat{S}}{\delta \phi^A} \left( C^A_1 - \frac{\delta \hat{S}}{\delta \phi^A} \bar{R}^a_A \right) \\
+ (-i)^{a+1} e^{ab} \frac{\delta \hat{S}}{\delta P_{ab}} \chi^A_i + (-i)^{a+1} e^{ab} P_{ab} \chi^A_i \\
+ (-i)^{a+1} e^{ab} \frac{\delta \hat{S}}{\delta P_{ab}} \chi^A_i \chi^A_i. \tag{56}
\]

Applying the fundamental lemma in calculus of variation, we get a quantum master equation in the extended phase space:

\[
\left( \frac{\hbar}{i} \Delta^a + V^a + \frac{\delta \hat{S}}{\delta \pi_A} \chi \bar{R}^a_A - (-i)^{a+1} e^{ab} P_{ab} \delta \phi^A \frac{\delta \hat{S}}{\delta \phi^A} + (-i)^{a+1} e^{ab} \frac{\delta \hat{S}}{\delta P_{ab}} \chi^A_i \chi^A_i \right) = 0. \tag{57}
\]

We have introduced the odd Laplacian \(\Delta^a\) and vector field \(V^a\):

\[
\Delta^a \equiv (-i)^{a} \frac{\delta \hat{S}}{\delta \phi^A} \frac{\delta \hat{S}}{\delta \phi^A} \\
V^a \equiv (-i)^{a+1} e^{ab} \frac{\delta \hat{S}}{\delta P_{ab}} \chi^A_i \frac{\delta \hat{S}}{\delta P_{ab}} \chi^A_i. \tag{58}
\]

Let us derive the Lagrangian theory of Batalin, Lavrov and Tyutin \cite{11}. First, use (52) to manipulate the exponent \(\hat{S}\) to \(\hat{S} + S^1_3\) in the master equation (57):

\[
\left( \frac{\hbar}{i} \Delta^a + V^a + \frac{\delta \hat{S}}{\delta \pi_A} \chi \bar{R}^a_A - (-i)^{a+1} e^{ab} P_{ab} \delta \phi^A \frac{\delta \hat{S}}{\delta \phi^A} + (-i)^{a+1} e^{ab} \frac{\delta \hat{S}}{\delta P_{ab}} \chi^A_i \chi^A_i \right) e^{\frac{i}{\hbar} (\hat{S} + S^1_3)} = 0. \tag{59}
\]
Now, let $\hat{W}$ be defined as
\[
e^{i\hat{h}\hat{W}} \equiv \int D\chi_2 D\varphi_1^1 D\varphi_1^2 D\chi_1^1 D\varphi_2 e^{i\hat{S} + S_1^3}.
\] (60)

The Lagrangian quantum action $W = W[\phi - \varphi_1, \bar{\phi}, \phi^*]$ is defined as
\[
e^{i\hat{h}W} \equiv \int D\pi e^{i\hat{h}W} = \int D\chi_2 D\varphi_1^1 D\varphi_1^2 D\chi_1^1 D\varphi_2 D\pi e^{i\hat{S} + S_1^3}.
\] (61)

The Lagrangian quantum master equation is formally reproduced by integrating over $\pi$, $\varphi_2$, $\chi_1$, $C_2$, $P^1$, $\varphi^1$ and $\chi_2$ in (59):
\[
\left(\frac{\hbar}{i} \Delta^a + V^a\right) e^{i\hat{h}W} = 0.
\] (62)

The integrations in (60) is easily done with the help of (52)
\[
\hat{W} = \pi_A \hat{\phi}^A - \int dt \bar{H}_a|_{\chi = 0} + \bar{\phi}_A \bar{R}^A|_{\chi = 0} + \phi^*_A \bar{R}^A|_{\chi = 0},
\] (63)

Therefore, the Lagrangian quantum action is
\[
e^{i\hat{h}W} \equiv \int D\pi e^{i\hat{h}W} \left(\pi_A \hat{\phi}^A - \int dt \bar{H}_a|_{\chi = 0} + \bar{\phi}_A \bar{R}^A|_{\chi = 0} + \phi^*_A \bar{R}^A|_{\chi = 0}\right).
\] (64)

The classical action $S_{cl}$ is regained for $\phi^*_A = 0 = \bar{\phi}_A$ if one impose the boundary condition (2).

## 6 Conclusion

We have shown in great detail how to introduce the Lagrangian antifields and quantum action in a generic Hamiltonian theory assuming no second class constraint and using canonical flat (Darboux) coordinates. This has been done both in the standard BRST case and in case of $Sp(2)$ symmetry by introducing collective shift fields. As a spin-off we have derived a new quantum master equation (30, 57) in an extended phase space. It is pleasant that the Lagrangian and Hamiltonian pictures merge together in this extended phase space.

It is amazing how smooth the $Sp(2)$ discussion extends the standard BRST case. This is a priori not at all obvious. The major difference is that in order to have a manifest $Sp(2)$ BRST exact expression, it has to be inside both the BRST transformations $\delta^1$ and $\delta^2$. The magic is the shift boson (52) quadratic in the fundamental fields. Each boson term leads to 4 terms in the action, so if one wants to modify the action by 1 term one has to alter with 3 other terms as well, because modifications should be $Sp(2)$ BRST exact.

The discussion of boundary conditions has some loose ends. A more complete treatment would require a more explicit analysis of the original theory. One should bear in mind that the classical action is often not unique. In this context, it would be desirable to have a procedure enforcing manifest Lorentz-covariance for the Lagrangian quantum action. Another problem is how to avoid the assumption of flat coordinates, i.e. to have a covariant formulation.

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