Abstract

We review the formalism of the effective average action in quantum field theory which corresponds to a coarse grained free energy in statistical mechanics. The associated exact renormalization group equation and possible nonperturbative approximations for its solution are discussed. This is applied to QCD where one observes the consecutive emergence of mesonic bound states and spontaneous chiral symmetry breaking as the coarse graining scale is lowered. We finally present a study of the phenomenological importance of non–renormalizable terms in the effective linear meson model.

1 Effective average action

Quantum chromodynamics (QCD) describes qualitatively different physics at different length scales. At short distances the relevant degrees of freedom are quarks and gluons which can be treated perturbatively. At long distances we observe hadrons and an essential part of the
dynamics can be encoded in the masses and interactions of mesons. Any attempt to deal with
this situation analytically and to predict the meson properties from the short distance physics
(as functions of the strong gauge coupling $\alpha_s$ and the current quark masses $m_q$) has to bridge
the gap between two qualitatively different effective descriptions. Two basic problems have to
be mastered for an extrapolation from short distance QCD to mesonic length scales:

- The effective couplings change with scale. This does not only concern the running gauge
coupling, but also the coefficients of non–renormalizable operators as, for example, four
quark operators. Typically, these non–renormalizable terms become important in the
momentum range where $\alpha_s$ is strong and deviate substantially from their perturbative
values. Consider the four–point function which obtains after integrating out the gluons.
For heavy quarks it contains the information about the shape of the heavy quark potential
whereas for light quarks the complicated spectrum of light mesons and chiral symmetry
breaking are encoded in it. At distance scales around 1fm one expects that the effective
action resembles very little the form of the classical QCD action which is relevant at short
distances.

- Not only the couplings, but even the relevant variables or degrees of freedom are different
for long distance and short distance QCD. It seems forbiddingly difficult to describe
the low–energy scattering of two mesons in a language of quarks and gluons only. An
appropriate analytical field theoretical method should be capable of introducing field
variables for composite objects such as mesons.

A conceptually very appealing idea for our task is the block–spin action [1, 2]. It realizes
that physics with a given characteristic length scale $l$ is conveniently described by a functional
integral with an ultraviolet (UV) cutoff $\Lambda$ for the momenta. Here $\Lambda$ should be larger than
$l^{-1}$ but not necessarily by a large factor. The Wilsonian effective action $S^W_\Lambda$ replaces then the
classical action in the functional integral. It obtains by integrating out the fluctuations with
momenta $q^2 \gtrsim \Lambda^2$. An exact renormalization group equation [2]–[6] describes how $S^W_\Lambda$
changes with the UV cutoff $\Lambda$.

We will use here the somewhat different but related concept of the effective average action $\Gamma_k$ which, in the language of statistical physics, is a coarse grained free energy with coarse
graining scale $k$. The effective average action is based on the quantum field theoretical concept
of the effective action $\Gamma$ which obtains by integrating out all quantum fluctuations. The
effective action contains all information about masses, couplings, form factors and so on, since it
is the generating functional of the 1PI Green functions. The field equations derived from $\Gamma$
are exact including all quantum effects. For a field theoretical description of thermal equilibrium
this concept is easily generalized to a temperature dependent effective action which includes now
also the thermal fluctuations. In statistical physics $\Gamma$ describes the free energy as a functional
of some convenient (space dependent) order parameter, for instance the magnetization. In
particular, the behavior of $\Gamma$ for a constant order parameter (the effective potential) specifies
the equation of state. The effective average action $\Gamma_k$ is a simple generalization of the effective
action, with the distinction that only quantum fluctuations with momenta $q^2 \gtrsim k^2$ are included.
This can be achieved by introducing in the functional integral defining the partition function (or
the generating functional for the $n$–point functions) an explicit infrared cutoff $\sim k$. Typically, this IR–cutoff is quadratic in the fields and modifies the inverse propagator, for example by adding a mass–like term $\sim k^2$. The effective average action can then be defined in complete analogy to the effective action (via a Legendre transformation of the logarithm of the partition function). The mass–like term in the propagator suppresses the contributions from the small momentum modes with $q^2 < \sim k^2$ and $\Gamma_k$ accounts effectively only for the fluctuations with $q^2 \gtrsim k^2$.

Following the behavior of $\Gamma_k$ for different $k$ is like looking at the world through a microscope with variable resolution: For large $k$ one has a very precise resolution $\sim k^{-1}$ but one also studies effectively only a small volume $\sim k^d$. Taking in QCD the coarse graining scale $k$ much larger than the confinement scale guarantees that the complicated nonperturbative physics does not play a role yet. In this case, $\Gamma_k$ will look similar to the classical action, typically with a running gauge coupling evaluated at the scale $k$. (This does not hold for Green functions with much larger momenta $p^2 \gg k^2$ where the relevant IR cutoff is $p$ and the effective coupling $\alpha_s(p)$.) For lower $k$ the resolution is smeared out and the detailed information of the short distance physics can be lost. (Again, this does not concern Green functions at high momenta.) On the other hand, the “observable volume” is increased and long distance aspects such as collective phenomena become now visible. In a theory with a physical UV cutoff $\Lambda$ we may associate $\Gamma_\Lambda$ with the classical action $S$ since no fluctuations are effectively included. By definition, the effective average action equals the effective action $\Gamma$ as $k \to 0$ is equivalent to the ability to solve the quantum field theory.

For a formal description we will consider in the first two sections a model with real scalar fields $\chi^a$ (the index $a$ labeling internal degrees of freedom) in $d$ Euclidean dimensions with classical action $S[\chi]$. We define the generating functional for the connected Green functions by

$$W_k[J] = \ln \int D\chi \exp\left\{ -S_k[\chi] + \int d^d x J_a(x) \chi^a(x) \right\}$$

where we have added to the classical action an IR cutoff $\Delta_k S$

$$S_k[\chi] = S[\chi] + \Delta S_k[\chi]$$

which is quadratic in the fields and best formulated in momentum space

$$\Delta S_k[\chi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q^2) \chi^a(-q) \chi^a(q).$$

Here $J_a$ are the usual scalar sources introduced to define generating functionals and $R_k(q^2)$ denotes an appropriately chosen (see below) IR cutoff function. We require that $R_k(q^2)$ vanishes rapidly for $q^2 \gg k^2$ whereas for $q^2 \ll k^2$ it behaves as $R_k(q^2) \simeq k^2$. This means that all Fourier components of $\chi^a$ with momenta smaller than the IR cutoff $k$ acquire an effective mass $m_{\text{eff}} \simeq k$ and therefore decouple while the high momentum components of $\chi^a$ should not be affected by $R_k$. The classical fields

$$\Phi^a \equiv \langle \chi^a \rangle = \frac{\delta W_k[J]}{\delta J_a}$$

3
depend now on \( k \). In terms of \( W_k \) the effective average action is defined via a Legendre transform

\[
\Gamma_k[\Phi] = -W_k[J] + \int d^d x J_a(x) \Phi^a(x) - \Delta S_k[\Phi].
\]

(5)

In order to define a reasonable coarse grained free energy we have subtracted in (5) the infrared cutoff piece. This guarantees that the only difference between \( \Gamma_k \) and \( \Gamma \) is the effective IR cutoff in the fluctuations. Furthermore, this has the consequence that \( \Gamma_k \) does not need to be convex whereas a pure Legendre transform is always convex by definition. (The coarse grained free energy becomes convex [9] only for \( k \to 0 \).) This is very important for the description of phase transitions, in particular first order ones. One notes

\[
\lim_{k \to 0} R_k(q^2) = 0 \quad \Rightarrow \quad \lim_{k \to 0} \Gamma_k[\Phi] = \Gamma[\Phi]
\]

\[
\lim_{k \to \Lambda} R_k(q^2) = \infty \quad \Rightarrow \quad \lim_{k \to \Lambda} \Gamma_k[\Phi] = S[\Phi]
\]

(6)

for a convenient choice of \( R_k \) like

\[
R_k(q^2) = Z_k q^2 \frac{1 + e^{-q^2/k^2} - e^{-q^2/\Lambda^2}}{e^{-q^2/\Lambda^2} - e^{-q^2/k^2}}.
\]

(7)

Here \( Z_k \) denotes the wave function renormalization to be specified below and we will often use for \( R_k \) the limit \( \Lambda \to \infty \)

\[
R_k(q^2) = \frac{Z_k q^2}{e^{q^2/k^2} - 1}.
\]

(8)

We note that the property \( \Gamma_\Lambda = S \) is not essential since the short distance laws may be parameterized by \( \Gamma_\Lambda \) as well as by \( S \). In addition, for momentum scales much smaller than \( \Lambda \) universality implies that the precise form of \( \Gamma_\Lambda \) is irrelevant, up to the values of a few relevant renormalized couplings.

A few properties of the effective average action are worth mentioning:

1. All symmetries of the model that are respected by the IR cutoff \( \Delta_k S \) are automatically symmetries of \( \Gamma_k \). In particular, this concerns translation and rotation invariance and one is not plagued by many of the problems encountered by a formulation of the block–spin action on a lattice.

2. In consequence, \( \Gamma_k \) can be expanded in terms of invariants with respect to these symmetries with couplings depending on \( k \). For the example of a scalar theory one may use a derivative expansion (\( \rho = \Phi^2/2 \))

\[
\Gamma_k = \int d^d x \left\{ U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial^\mu \Phi_a \partial_\mu \Phi^a + \ldots \right\}
\]

(9)

and expand further in powers of \( \rho \)

\[
U_k(\rho) = \frac{1}{2} \Gamma_k (\rho - \rho_0(k))^2 + \frac{1}{6} \Gamma_k (\rho - \rho_0(k))^3 + \ldots
\]

\[
Z_k(\rho) = Z_k(\rho_0(k)) + Z'_k(\rho_0(k))(\rho - \rho_0(k)) + \ldots
\]

(10)

We see that \( \Gamma_k \) describes infinitely many running couplings. (\( Z_k \) in (10) can be identified with \( Z_k(\rho_0) \).)
3. There is no problem incorporating chiral fermions since a chirally invariant cutoff $R_k$ can be formulated [10].

4. Gauge theories can be formulated along similar lines [11]–[16] even though $\Delta_k S$ may not be gauge invariant. In this case the usual Ward identities receive corrections for which one can derive closed expressions [14]. These corrections vanish for $k \to 0$.

5. The high momentum modes are very effectively integrated out because of the exponential decay of $R_k$ for $q^2 \gg k^2$. Nevertheless, it is sometimes technically easier to use a cutoff without this fast decay property (e.g. $R_k \sim k^2$ or $R_k \sim k^4/q^2$). In the latter case one has to be careful with possible remnants of an incomplete integration of the short distance modes. Also our cutoff does not introduce any non–analytical behavior as would be the case for a sharp cutoff [7].

6. Despite a similar spirit and many analogies there remains also a conceptual difference to the Wilsonian effective action $S^W_\Lambda$. The Wilsonian effective action describes a set of different actions (parameterized by $\Lambda$) for one and the same theory — the $n$–point functions are independent of $\Lambda$ and have to be computed from $S^W_\Lambda$ by further functional integration. In contrast, $\Gamma_k$ describes the effective action for different theories — for any value of $k$ the effective average action is related to the generating functional of a theory with a different action $S_k = S + \Delta_k S$. The $n$–point functions depend on $k$. The Wilsonian effective action does not generate the $1PI$ Green functions [17].

7. Because of the incorporation of an infrared cutoff, $\Gamma_k$ is closely related to an effective action for averages of fields [7], where the average is taken over a volume $\sim k^d$.

2 Exact renormalization group equation

The dependence of $\Gamma_k$ on the coarse graining scale $k$ is governed by an exact renormalization group equation (ERGE) [18]

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{Tr} \left\{ \left[ \Gamma^{(2)}_k[\Phi] + R_k \right]^{-1} \partial_t R_k \right\}.$$  

(11)

Here $t = \ln(k/\Lambda)$ with some arbitrary momentum scale $\Lambda$, and the trace includes a momentum integration as well as a summation over internal indices, $\text{Tr} = \int \frac{d^dq}{(2\pi)^d} \sum_a$. The second functional derivative $\Gamma^{(2)}_k$ denotes the exact inverse propagator

$$\left[ \Gamma^{(2)}_k \right]_{ab}(q, q') = \frac{\delta^2 \Gamma_k}{\delta \Phi^a(-q) \delta \Phi^b(q')}.$$  

(12)

The flow equation (11) can be derived from (5) in a straightforward way using

$$\partial_t \Gamma_k|_{\Phi} = -\partial_t W_k|_J - \partial_t \Delta_k S[\Phi]$$
\[
\begin{align*}
&= \frac{1}{2} \text{Tr} \left\{ \partial_t R_k \left[ \langle \Phi \Phi \rangle - \langle \Phi \rangle \langle \Phi \rangle \right] \right\} \\
&= \frac{1}{2} \text{Tr} \left\{ \partial_t R_k W_k^{(2)} \right\}
\end{align*}
\]

(13)

and

\[
W^{(2)}_{k,ab}(q,q') = \frac{\delta^2 W_k}{\delta J^a(-q) \delta J^b(q')}
\]

\[
\frac{\delta^2 W_k}{\delta J_a(-q) \delta J_b(q')} \frac{\delta^2 \left( \Gamma_k + \Delta_k S \right)}{\delta \Phi_a(-q') \delta \Phi_b(q'')} = \delta_{ac} \delta_{qq''} .
\]

(14)

It has the form of a renormalization group improved one–loop expression \[\text{[7]}\]. Indeed, the one–loop formula for \(
\Gamma_k \)
reads

\[
\Gamma_k[\Phi] = S[\Phi] + \frac{1}{2} \text{Tr} \ln \left( S^{(2)}[\Phi] + R_k \right)
\]

(15)

with \(S^{(2)}\) the second functional derivative of the classical action, similar to \((12)\). (Remember that \(S^{(2)}\) is the field dependent classical inverse propagator. Its first and second derivative with respect to the fields describe the classical three– and four–point vertices, respectively.) Taking a \(t\)–derivative of eq. \((15)\) gives a one–loop flow equation very similar to \((11)\) with \(\Gamma^{(2)}_k\) replaced by \(S^{(2)}\). It may seem surprising, but it is nevertheless true, that the renormalization group improvement \(S^{(2)} \rightarrow \Gamma^{(2)}_k\) promotes the one–loop flow equation to an exact nonperturbative flow equation which includes the effects from all loops as well as all contributions which are non–analytical in the couplings like instantons, etc.! For practical computations it is actually often quite convenient to write the flow equation \((11)\) as a formal \(\partial_t\)–derivative of a renormalization group improved one–loop expression

\[
\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \partial_t \ln \left( \Gamma^{(2)}_k + R_k \right)
\]

(16)

with \(\partial_t\) acting only on \(R_k\) and not on \(\Gamma^{(2)}_k\), i.e. \(\partial_t = (\partial R_k / \partial t) (\partial / \partial R_k)\). Flow equations for \(n\)–point functions follow from appropriate functional derivatives of \((11)\) or \((16)\) with respect to the fields. For their derivation it is sufficient to evaluate the corresponding one–loop expressions (with the vertices and propagators derived from \(\Gamma_k\) and then to take a formal \(\partial_t\)–derivative. (If the one–loop expression is finite or properly regularized the \(\partial_t\)–derivative can be taken after the evaluation of the trace.) This permits the use of (one–loop) Feynman diagrams and standard perturbative techniques in many circumstances. Most importantly, it establishes a very direct connection between the solution of flow–equations and perturbation theory. If one uses on the right hand side of eq. \((11)\) a truncation for which the propagator and vertices appearing in \(\Gamma^{(2)}_k\) are replaced by the ones derived from the classical action, but with running \(k\)–dependent couplings, and then expands the result to lowest non–trivial order in the coupling constants one recovers standard renormalization group improved one–loop perturbation theory. The formal solution of the flow equation can also be employed for the development of a systematically resummed perturbation theory \[\text{[19]}\].
For a choice of the cutoff function similar to (9) the momentum integral contained in the trace on the right hand side of the flow equation is both infrared and ultraviolet finite. Infrared finiteness arises through the presence of the infrared regulator $\sim R_k$. We note that all eigenvalues of the matrix $\Gamma_k^{(2)} + R_k$ must be positive semi–definite. The proof follows from the observation that the functional $\Gamma_k + \Delta_k S$ is convex since it is obtained from $W_k$ by a Legendre transform. On the other hand, ultraviolet finiteness is related to the fast decay of $\partial_k R_k$ for $q^2 \gg k^2$. This expresses the fact that only a narrow range of fluctuations with $q^2 \approx k^2$ contributes effectively if the infrared cutoff $k$ is lowered by a small amount $\frac{1}{6}$. Since the flow equation is manifestly finite this can be used to define a regularization scheme. The “ERGE–scheme” is specified by the flow equation, the choice of $R_k$ and the “initial condition” $\Gamma_\Lambda$. This is particularly important for gauge theories where other regularizations in four dimensions and in the presence of chiral fermions are difficult to construct. We note that in contrast to previous versions of exact renormalization group equations there is no need in the present formulation to construct an ultraviolet momentum cutoff — a task known to be very difficult in non–Abelian gauge theories.

Despite the conceptual differences between the Wilsonian effective action $S_\Lambda^W$ and the effective average action $\Gamma_k$ the exact flow equations describing the $\Lambda$–dependence of $S_\Lambda^W$ and the $k$–dependence of $\Gamma_k$ are simply related. Polchinski’s continuum version of the Wilsonian flow equation (11) can be transformed into (11) by means of a Legendre transform and a suitable variable redefinition (20).

Even though intuitively simple, the replacement of the (RG–improved ) classical propagator by the full propagator turns the solution of the flow equation (11) into a difficult mathematical problem: The evolution equation is a functional differential equation. Once $\Gamma_k$ is expanded in terms of invariants (e.g. (9), (10)) this is equivalent to a coupled system of non–linear partial differential equations for infinitely many couplings. General methods for the solution of functional differential equations are not developed very far. They are mainly restricted to iterative procedures that can be applied once some small expansion parameter is identified. This covers usual perturbation theory in the case of a small coupling, the $1/N$–expansion or expansions in the dimensionality $4 – d$ or $2 – d$. It may also be extended to less familiar expansions like a derivative expansion which is related in critical three dimensional scalar theories to a small anomalous dimension $\eta$. In the absence of a clearly identified small parameter one nevertheless needs to truncate the most general form of $\Gamma_k$ in order to reduce the infinite system of coupled differential equations to a (numerically) manageable size. This truncation is crucial. It is at this level that approximations have to be made and, as for all nonperturbative analytical methods, they are often not easy to control. The challenge for nonperturbative systems like low momentum QCD is to find flow equations which (a) incorporate all the relevant dynamics such that neglected effects make only small changes, and (b) remain of manageable size. The difficulty with the first task is a reliable estimate of the error. For the second task

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1If for some other choice of $R_k$ the right hand side of the flow equation would not remain UV finite this would indicate that the high momentum modes have not yet been integrated out completely in the computation of $\Gamma_k$.

2For gauge theories $\Gamma_\Lambda$ has to obey appropriately modified Ward identities. In the context of perturbation theory a first proposal how to regularize gauge theories by use of flow equations can be found in 12 13.
the main limitation is a practical restriction for numerical solutions of differential equations to functions depending only on a small number of variables. The existence of an exact functional differential flow equation is a very useful starting point and guide for this task. At this point the precise form of the exact flow equation is quite important. Furthermore, it can be used for systematic expansions through enlargement of the truncation and for an error estimate in this way. Nevertheless, this is not all. Usually, physical insight into a model is necessary to device a useful nonperturbative truncation!

So far, two complementary approaches to nonperturbative truncations have been explored: an expansion of the effective Lagrangian in powers of derivatives \( \rho \equiv \frac{1}{2} \Phi^a \Phi^a \)

\[
\Gamma_k[\Phi] = \int d^d x \left\{ U_k(\rho) + \frac{1}{2} Z_k(\Phi) \partial_a \Phi^a \partial^\mu \Phi_a + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial^\mu \rho + O(\partial^4) \right\}
\]

(17)
or one in powers of the fields

\[
\Gamma_k[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \left( \prod_{j=0}^{n} d^d x_j \left[ \Phi(x_j) - \Phi_0 \right] \right) \Gamma_k^{(n)}(x_1, \ldots, x_n).
\]

(18)

If one chooses \( \Phi_0 \) as the \( k \)-dependent VEV of \( \Phi \), the series (18) starts effectively at \( n = 2 \). The flow equations for the 1PI \( n \)-point functions \( \Gamma_k^{(n)} \) are obtained by functional differentiation of eq. (11). The formation of mesonic bound states, which typically appear as poles in the (Minkowskian) four quark Green function, is most efficiently described by expansions like (18). This is also the form needed to compute the nonperturbative momentum dependence of the gluon propagator and the heavy quark potential \([15, 16]\). On the other hand, a parameterization of \( \Gamma_k \) as in (17) seems particularly suited for the study of phase transitions. The evolution equation for the average potential \( U_k \) follows by evaluating (17) for constant \( \Phi \). In the limit where the \( \Phi \)-dependence of \( Z_k \) is neglected and \( Y_k = 0 \) one finds \([7]\) for the \( O(N) \)-symmetric scalar model

\[
\partial_t U_k(\rho) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\partial R_k}{\partial t} \left( \frac{N-1}{Z_k q^2 + R_k + U_k} + \frac{1}{Z_k q^2 + R_k(q) + U_k' + 2\rho U_k''} \right)
\]

(19)

with \( U_k' \equiv \frac{\partial U_k}{\partial \rho} \), etc. One observes the appearance of \( \rho \)-dependent mass terms in the effective propagators of the right hand side of eq. (19). Once \( \eta_k \equiv -\partial_t \ln Z_k \) is determined \([7]\) in terms of the couplings parameterizing \( U_k \) this is a partial differential equation \([7]\) for a function \( U_k \) depending on two variables \( k \) and \( \rho \) which can be solved numerically \([22, 25]\). (The Wilson–Fisher fixed point relevant for a second order phase transition \((d = 3)\) corresponds to a scaling solution \([20, 27]\) where \( \partial_t U_k = 0 \).) A suitable truncation of a flow equation of the type (17) will play a central role in the description of chiral symmetry breaking below.

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\(^3\)Such flow equations have been discussed earlier from a somewhat different viewpoint \([4]\). They can also be interpreted as a differential form of Schwinger–Dyson equations \([21]\).

\(^4\)In the sharp cutoff limit and for \( \eta_k = 0 \) the flow equation for \( U_k \) coincides with the one corresponding to the Wilsonian effective action \([6]\). Because of non–analyticities in the kinetic terms \([6]\) an analysis of the derivative terms is difficult for a sharp cutoff.
It should be mentioned at this point that the weakest point in the ERG approach seems to be a reliable estimate of the truncation error in a nonperturbative context. This problem is common to all known analytical approaches to nonperturbative phenomena and appears often even within systematic (perturbative) expansions. One may hope that the existence of an exact flow equation could also be of some help for error estimates. An obvious possibility to test a given truncation is its enlargement to include more variables — for example, going one step higher in a derivative expansion. This is similar to computing higher orders in perturbation theory and limited by practical considerations. As an alternative, one may employ different truncations of comparable size — for instance, by using different definitions of retained couplings. A comparison of the results can give a reasonable picture of the uncertainty if the used set of truncations is wide enough. In this context we should also note the dependence of the results on the choice of the cutoff function $R_k(q)$. Of course, for $k \to 0$ the physics should not depend on a particular choice of $R_k$ and, in fact, it does not for full solutions of (11). Different choices of $R_k$ just correspond to different trajectories in the space of effective actions along which the unique IR limit $\Gamma[\Phi]$ is reached. Once approximations are used to solve the ERGE (11), however, not only the trajectory but also its end point will depend on the precise definition of the function $R_k$. This is very similar to the renormalization scheme dependence usually encountered in perturbative computations of Green functions. One may use this scheme dependence as a tool to study the robustness of a given approximation scheme.

Before applying a new nonperturbative method to a complicated theory like QCD it should be tested for simpler models. A good criterion for the capability of the ERGE to deal with nonperturbative phenomena concerns the critical behavior in three dimensional scalar theories. In a first step the well known results of other methods for the critical exponents have been reproduced within a few percent accuracy [26]. The ability of the method to produce new results has been demonstrated by the computation of the critical equation of state for Ising and Heisenberg models [23] which has been verified by lattice simulations [28]. This has been extended to first order transitions in matrix models [25] or for the Abelian Higgs model relevant for superconductors [24]. Analytical investigations of the high temperature phase transitions in $d = 4$ scalar theories ($O(N)$-models) have correctly described the second order nature of the transition [30], in contrast to earlier attempts within high temperature perturbation theory.

For an extension of the flow equations to Abelian and non–Abelian gauge theories we refer the reader to [11]–[16]. The other necessary piece for a description of low–energy QCD, namely the transition from fundamental (quark and gluon) degrees of freedom to composite (meson) fields within the framework of the ERGE can be found in [31]. We will describe the most important aspects of this formalism for mesons below.

3 Chiral symmetry breaking in QCD

The strong interaction dynamics of quarks and gluons at short distances or high energies is successfully described by quantum chromodynamics (QCD). One of its most striking features is asymptotic freedom [32] which makes perturbative calculations reliable in the high energy regime. On the other hand, at scales around a few hundred MeV confinement sets in. As
a consequence, the low–energy degrees of freedom in strong interaction physics are mesons, baryons and glueballs rather than quarks and gluons. When constructing effective models for these IR degrees of freedom one usually relies on the symmetries of QCD as a guiding principle, since a direct derivation of such models from QCD is still missing. The most important symmetry of QCD, its local color $SU(3)$ invariance, is of not much help here, since the IR spectrum appears to be color neutral. When dealing with bound states involving heavy quarks the so called “heavy quark symmetry” may be invoked to obtain approximate symmetry relations between IR observables \cite{33}. We will rather focus here on the light scalar and pseudoscalar meson spectrum and therefore consider QCD with only the light quark flavors $u$, $d$ and $s$. To a good approximation the masses of these three flavors can be considered as small in comparison with other typical strong interaction scales. One may therefore consider the chiral limit of QCD (vanishing current quark masses) in which the classical QCD Lagrangian does not couple left– and right–handed quarks. It therefore exhibits a global chiral invariance under

$$SU_L(N) \times SU_R(N) = SU_L(N) \times SU_R(N) \times U_V(1) \times U_A(1)$$

where $N$ denotes the number of massless quarks ($N = 2$ or 3) which transform as

$$\psi_R \equiv \frac{1 - \gamma_5}{2} \psi \quad \rightarrow \quad U_R \psi_R ; \quad U_R \in U_R(N)$$

$$\psi_L \equiv \frac{1 + \gamma_5}{2} \psi \quad \rightarrow \quad U_L \psi_L ; \quad U_L \in U_L(N) . \quad (20)$$

Even for vanishing quark masses only the diagonal $SU_V(N)$ vector–like subgroup can be observed in the hadron spectrum (“eightfold way”). The symmetry $SU_L(N) \times SU_R(N)$ must therefore be spontaneously broken to $SU_V(N)$

$$SU_L(N) \times SU_R(N) \rightarrow SU_{L+R}(N) \equiv SU_V(N) . \quad (21)$$

Chiral symmetry breaking is one of the most prominent features of strong interaction dynamics and phenomenologically well established \cite{34}, even though a rigorous derivation of this phenomenon starting from first principles is still missing. In particular, the chiral symmetry breaking \cite{21} predicts for $N = 3$ the existence of eight light parity–odd (pseudo–)Goldstone bosons: $\pi^0$, $\pi^\pm$, $K^0$, $\bar{K}^0$, $K^\pm$ and $\eta$. Their comparably small masses are a consequence of the explicit chiral symmetry breaking due to small but non–vanishing current quark masses. The axial Abelian subgroup $U_A(1) = U_{L-R}(1)$ is broken in the quantum theory by an anomaly of the axial–vector current. This breaking proceeds without the occurrence of a Goldstone boson \cite{35}. Finally, the $U_V(1) = U_{L+R}(1)$ subgroup corresponds to baryon number conservation.

The light pseudoscalar and scalar mesons are thought of as color neutral quark–antiquark bound states $\Phi^{ab} \sim \bar{\psi}_L \psi_R^a$, $a, b = 1, \ldots, N$, which therefore transform under chiral rotations \cite{20} as

$$\Phi \rightarrow U_R \Phi U_L^\dagger . \quad (22)$$

Hence, the chiral symmetry breaking pattern \cite{21} is realized if the meson potential develops a VEV

$$\langle \Phi^{ab} \rangle = \sigma_0 \delta^{ab} ; \quad \sigma_0 \neq 0 . \quad (23)$$
One of the most crucial and yet unsolved problems of strong interaction dynamics is to derive an effective field theory for the mesonic degrees of freedom directly from QCD which exhibits this behavior.

4 A semi–quantitative picture

Before turning to a quantitative description of chiral symmetry breaking using flow equations it is perhaps helpful to give a brief overview of the relevant scales which appear in relation to this phenomenon and the physical degrees of freedom associated to them. Some of this will be explained in more detail in the remainder of this talk whereas other parts are rather well established features of strong interaction physics.

At scales above approximately 1.5 GeV, the relevant degrees of freedom of strong interactions are quarks and gluons and their dynamics appears to be well described by perturbative QCD. At somewhat lower energies this changes dramatically. Quark and gluon bound states form and confinement sets in. Concentrating on the physics of scalar and pseudoscalar mesons there are three important momentum scales which appear to be rather well separated:

- The compositeness scale $k_\Phi$ at which mesonic $\bar{\psi}\psi$ bound states form due to the increasing strength of the strong interaction. It will turn out to be somewhere in the range $(600 - 700)$ MeV.

- The chiral symmetry breaking scale $k_{\chi_{SB}}$ at which the chiral condensate $\langle \bar{\psi}^a \psi^a \rangle$ or $\langle \Phi^{ab} \rangle$ will assume a non–vanishing value, therefore breaking chiral symmetry according to (21). This scale is found to be around $(400 - 500)$ MeV. Below it the quarks acquire constituent masses $M_q \simeq 350$ MeV due to their Yukawa coupling to the chiral condensate (23).

- The confinement scale $\Lambda_{\text{QCD}} \simeq 200$ MeV which corresponds to the Landau pole in the perturbative evolution of the strong coupling constant $\alpha_s$. In our context, this is the scale where deviations of the effective quark propagator from its classical form and multi–quark interactions not included in the meson physics become of crucial importance.

For scales $k$ in the range $k_{\chi_{SB}} < k \ll k_\Phi$ the most relevant degrees of freedom are mesons and quarks. Typically, the dynamics in this range is dominated by the strong Yukawa coupling $h$ between quarks and mesons: $h^2(k)/(4\pi) \gg \alpha_s(k)$. One may therefore assume that the dominant QCD effects are included in the meson physics and consider a simple model of quarks and mesons only. As one evolves to scales below $k_{\chi_{SB}}$ the Yukawa coupling decreases whereas $\alpha_s$ increases. Of course, getting closer to $\Lambda_{\text{QCD}}$ it is no longer justified to neglect the QCD effects which go beyond the dynamics of effective meson degrees of freedom. On the other hand, the final IR value of the Yukawa coupling $h$ is fixed by the typical values of constituent quark masses $M_q \simeq 350$ MeV to be $h^2/(4\pi) \simeq 4.5$. One may therefore speculate that the domination

\footnote{One may assume that all other bound states are integrated out. We will comment on this issue below.}

\footnote{A formalism for the inclusion of “residual” gluon contributions to the flow equation in a situation where part of the gluon effects are already described by mesonic bound states can be found in \cite{15}.}
of the Yukawa interaction persists down to scales \( k \simeq M_q \) at which the quarks decouple from the evolution of the mesonic degrees of freedom altogether due to their mass. Of course, details of the gluonic interactions are expected to be crucial for an understanding of quark and gluon confinement. Strong interaction effects may dramatically change the momentum dependence of the quark and gluon propagators for \( k \) around \( \Lambda_{\text{QCD}} \). Yet, as long as one is only interested in the dynamics of the mesons one is led to expect that these effects are quantitatively no too important. Because of the effective decoupling of the quarks and therefore the whole colored sector the details of confinement have only little influence on the mesonic flow equations for \( k \lesssim \Lambda_{\text{QCD}} \).

Let us imagine that we integrate out the gluon degrees of freedom while keeping an effective infrared cutoff \( k_p \simeq 1.5 \text{ GeV} \) in the quark propagators. The exact flow equation to be used for this purpose obtains by keeping for \( k < k_p \) the infrared cutoff \( R \) for the quarks fixed while lowering the one for the gluons to zero.\(^7\) Subsequently, the gluons are eliminated by solving the field equations for the gluon fields as functionals of the quarks. This will result in a non–trivial momentum dependence of the quark propagator and effective non–local four and higher quark interactions. Because of the infrared cutoff the resulting effective action for the quarks resembles closely the one for heavy quarks (at least for Euclidean momenta). The dominant effect is the appearance of an effective quark potential (similar to the one for the charm quark) which describes the effective four quark interactions. For the effective quark action at \( k_p \) we only retain this four quark interaction in addition to the classical two–point function, while neglecting \( n \)–point functions involving six and more quarks. Details of the quark two–point function for \( q^2 \lesssim \Lambda_{\text{QCD}}^2 \) will not be important as long as \( k > \Lambda_{\text{QCD}} \), and similarly for the four–point function. For \( k < \Lambda_{\text{QCD}} \) the quarks decouple from the mesonic sector as discussed above. For typical momenta larger than \( \Lambda_{\text{QCD}} \) a reliable computation of the effective quark action should be possible by using in the quark–gluon flow equation relatively simple truncations \([15, 16]\).

We next have to remove the infrared cutoff for the quarks, \( k \to 0 \). This task can be carried out by means of the exact flow equation for quarks only, starting at \( k_p \) with an initial value \( \Gamma_{k_p}[\psi] \) as obtained after integrating out the gluons. For fermions the trace in eq. (11) has to be replaced by a supertrace in order to account for the minus sign related to Grassmann variables \([10]\). A first investigation in this direction \([31]\) has used a truncation with a chirally invariant four quark interaction whose most general momentum dependence was retained

\[
\Gamma_k = \int \frac{d^4p}{(2\pi)^4} \overline{\psi}_a(p) Z_{\psi,k}(p) \left[ \hat{p} \delta^{ab} + m^{ab}(p) \gamma_5 + i\tilde{m}^{ab}(p) \right] \psi_b(p) + \frac{1}{2} \int \left( \prod_{l=1}^4 \frac{d^4p_l}{(2\pi)^4} \right) (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \times \lambda_k^{(\psi)}(p_1,p_2,p_3,p_4) \left\{ \left[ \overline{\psi}_a(-p_1) \psi_b(p_2) \right] \left[ \overline{\psi}_b(p_3) \psi_a(-p_4) \right] - \left[ \overline{\psi}_b(-p_1) \gamma_5 \psi_a(p_2) \right] \left[ \overline{\psi}_a(p_3) \gamma_5 \psi_b(-p_4) \right] \right\} . \tag{24}
\]

\(^7\)One may also only partly integrate out the gluons by stopping this evolution at a scale somewhat above \( \Lambda_{\text{QCD}} \). The remaining residual gluon fluctuation can then be included later following \([14]\).
Here, $i, j$ run from one to $N_c$ which is the number of quark colors. The indices $a, b$ denote different light quark flavors and run from 1 to $N$. The matrices $m$ and $\tilde{m}$ are hermitian and $m + \tilde{m} \gamma_5$ forms therefore the most general quark mass matrix. The ansatz (24) does not correspond to the most general chirally invariant four quark interaction. It neglects similar interactions in the $\rho$-meson and pomeron channels which are also obtained from a Fierz transformation of the heavy quark potential \cite{13}. With $V(q^2)$ the heavy quark potential in a Fourier representation, the initial value at $k_p = 1.5$ GeV is given by (\(\hat{Z}_{\psi,k} = Z_{\psi,k}(p^2 = -k_p^2)\))

$$\lambda_k^{(\psi)}(p_1, p_2, p_3, p_4) \hat{Z}_{\psi,k}^{-2} = \frac{1}{2} V((p_1 - p_3)^2) = \frac{2\pi \alpha_s}{(p_1 - p_3)^2} + \frac{8\pi \lambda}{((p_1 - p_3)^2)^2}. \quad (25)$$

For simplicity, the effective heavy quark potential is approximated here by a one gluon exchange term $\sim \alpha_s(k_p)$ and a string tension $\lambda \simeq 0.18$ GeV$^2$, but one may use as well a more precise form of the potential as determined from the phenomenology of charmonium or as computed from the flow equations with gluons \cite{15, 16}. In the simplified ansatz (25) the string tension introduces a second scale in addition to $k_p$ and it becomes clear that the incorporation of gluon fluctuations is a crucial ingredient for the emergence of mesonic bound states. For a more realistic treatment of the heavy quark potential this scale is set by the running of $\alpha_s$ or $\Lambda_{\text{QCD}}$.

The evolution equation for the function $\lambda_k^{(\psi)}$ can be derived from the fermionic version of eq. (11) and the truncation (24). Since $\lambda_k^{(\psi)}$ depends on six independent momentum invariants it is a partial differential equation for a function depending on seven variables and has to be solved numerically \cite{31}. The ansatz (25) corresponds to the $t$-channel exchange of a colored gluonic state and it is by far not clear that the evolution of $\lambda_k^{(\psi)}$ will lead at lower scales to a momentum dependence representing the exchange of colorless mesonic bound states. Yet, at the compositeness scale

$$k_\Phi \simeq 630 \text{ MeV} \quad (26)$$

one finds an approximate factorization

$$\lambda_k^{(\psi)}(p_1, p_2, p_3, p_4) = g(p_1, p_2) \tilde{G}(s) g(p_3, p_4) + \ldots \quad (27)$$

which indicates the formation of mesonic bound states. Here $g(p_1, p_2)$ denotes the amputated Bethe–Salpeter wave function and $\tilde{G}(s)$ is the mesonic bound state propagator displaying a pole–like structure in the $s$-channel if it is continued to negative $s = (p_1 + p_2)^2$. The dots indicate the part of $\lambda_k^{(\psi)}$ which does not factorize and which will be neglected in the following. In the limit where the momentum dependence of $g$ and $\tilde{G}$ is neglected we recover the four quark interaction of the Nambu–Jona-Lasinio model \cite{34, 38}. It is therefore not surprising that our description of the dynamics for $k < k_\Phi$ will parallel certain aspects of the investigations of this model, even though we are not bound to the approximations used typically in such studies (large–$N_c$ expansion, perturbative renormalization group, etc.).

It is clear that for scales $k \lesssim k_\Phi$ a description of strong interaction physics in terms of quark fields alone would be rather inefficient. Finding physically reasonable truncations of the effective

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Our chiral conventions \cite{10} where the hermitean part of the mass matrix is multiplied by $\gamma_5$ may be somewhat unusual but they are quite convenient for Euclidean calculations.
average action should be much easier once composite fields for the mesons are introduced. The exact renormalization group equation can indeed be supplemented by an exact formalism for the introduction of composite field variables or, more generally, a change of variables [31]. For our purpose, this amounts in practice to inserting at the scale $k_\Phi$ the identities 9

$$1 = \text{const} \int \mathcal{D}\sigma_A$$
$$\times \exp \left\{ -\frac{1}{2} \text{tr} \left( \sigma_A^\dagger - K_A^\dagger \tilde{G} - m_A^\dagger - \mathcal{O}^\dagger \tilde{G} \right) \frac{1}{\tilde{G}} \left( \sigma_A - \tilde{G} K_A - m_A \right) \right\}$$

$$1 = \text{const} \int \mathcal{D}\sigma_H$$
$$\times \exp \left\{ -\frac{1}{2} \text{tr} \left( \sigma_H^\dagger - K_H^\dagger \tilde{G} - m_H^\dagger - \mathcal{O}_H^{(5)^\dagger} \tilde{G} \right) \frac{1}{\tilde{G}} \left( \sigma_H - \tilde{G} K_H - m_H \right) \right\}$$

(28)

into the functional integral which formally defines the quark effective average action. Here $K_{A,H}$ are sources for the collective fields $\sigma_{A,H}$ which correspond in turn to the anti-hermitian and hermitian parts 10 of the meson field $\Phi$. They are associated to the fermion bilinear operators $\mathcal{O}^{[5]}[\psi], \mathcal{O}^{[5]}[\psi]$ whose Fourier components read

$$\mathcal{O}^a_{\bar{b}}(q) = -i \int \frac{d^4p}{(2\pi)^4} g(-p, p + q) \overline{\psi}_a(p) \psi_{\bar{b}}(p + q)$$
$$\mathcal{O}^{(5)}_{\bar{a}b}(q) = - \int \frac{d^4p}{(2\pi)^4} g(-p, p + q) \overline{\psi}_a(p) \gamma_5 \psi_{\bar{b}}(p + q) .$$

(29)

The choice of $g(-p, p + q)$ as the bound state wave function renormalization and of $\tilde{G}(q)$ as its propagator guarantees that the four–quark interaction contained in (28) cancels the dominant factorizing part of the QCD–induced non–local four–quark interaction (24), (27). In addition, one may choose

$$m_{H,ab}^T = m_{ab}(0) g^{-1}(0, 0) Z_{\psi,k\Phi}(0)$$
$$m_{A,ab}^T = \tilde{m}_{ab}(0) g^{-1}(0, 0) Z_{\psi,k\Phi}(0)$$

(30)

such that the explicit quark mass term cancels out for $q = 0$. The remaining quark bilinear, which is proportional to $m(q) - m(0) Z_{\psi,k\Phi}(0) g(-q, q)/[Z_{\psi,k\Phi}(q) g(0, 0)]$ and breaks chiral symmetry, vanishes for zero momentum and will be neglected in the following. Without loss of generality we can take $m$ real and diagonal and $\tilde{m} = 0$. In consequence, we have replaced at the scale $k_\Phi$ the effective quark action (24) with (27) by an effective quark meson action given by

$$\hat{\Gamma}_k = \Gamma_k - \frac{1}{2} \int d^4x \text{tr} \left( \Phi^\dagger J + J^\dagger \Phi \right)$$
$$\Gamma_k = \int d^4x U_k(\Phi, \Phi^\dagger)$$

(31)

9We use here the shorthand notation $A^i GB \equiv \int \frac{d^4q}{(2\pi)^4} A^i_{\psi}(q) G^{ab}(q) B_{\bar{b}}(q)$.
10The fields $\sigma_{A,H}$ associated to $K_{A,H}$ by a Legendre transformation obey $\sigma_A^T = -\frac{i}{2}(\Phi - \Phi^\dagger), \sigma_H^T = \frac{i}{2}(\Phi + \Phi^\dagger)$. 

14
\[
+ \int \frac{d^4q}{(2\pi)^4} \left\{ Z_{\Phi,k}(q) q^2 \text{tr} \left[ \Phi^\dagger(q) \Phi(q) \right] + Z_{\psi,k}(q) \bar{\psi}_a(q) \gamma^\mu q_\mu \psi^a(q) \right\} \\
+ \int \frac{d^4p}{(2\pi)^4} \tilde{h}_k(-q, q - p) \bar{\psi}^a(q) \left( \frac{1 + \gamma_5}{2} \Phi_{ab}(p) - \frac{1 - \gamma_5}{2} \Phi_{ab}(p) \right) \psi^b(q - p) .
\]

At the scale \( k_\Phi \) the inverse scalar propagator is related to \( \tilde{G}(q) \) in eq. (27) by
\[
\tilde{G}^{-1}(q^2) = 2m^2(k_\Phi) + 2Z_{\Phi,k}(q)q^2 .
\]

This fixes the term in \( U_{k_\Phi} \) which is quadratic in \( \Phi \) to be positive, \( U_{k_\Phi} = m^2 \text{tr} \Phi^\dagger \Phi + \ldots \). The higher order terms in \( U_{k_\Phi} \) cannot be determined in the approximation (24) since they correspond to terms involving six or more quark fields. The initial value of the Yukawa coupling corresponds to the “quark wave function in the meson” in eq. (27), i.e.
\[
\tilde{h}_{k_\Phi}(-q, q - p) = g(-q, q - p)
\]
which can be normalized with \( \tilde{h}_{k_\Phi}(0, 0) = g(0, 0) = 1 \). We observe that the explicit chiral symmetry breaking from non–vanishing current quark masses appears now in the form of a meson source term with
\[
j = 2m^2Z_{\psi,k}(0)g^{-1}(0, 0) (m_{ab} + i\tilde{m}_{ab}) = 2Z_{\psi,k}m^2 \text{diag}(m_u, m_d, m_s) .
\]

This induces a non–vanishing \( \langle \Phi \rangle \) and an effective quark mass \( M_q \) through the Yukawa coupling. Spontaneous chiral symmetry breaking can be described in this language by a non–vanishing \( \langle \Phi \rangle \) in the limit \( j \to 0 \). We note that the current quark mass \( m_q \) and the constituent quark mass \( M_q \sim \tilde{h} \langle \Phi \rangle \) are identical only for a quadratic potential \( \sim \text{tr} \Phi^\dagger \Phi \). Because of spontaneous chiral symmetry breaking the constituent quark mass \( M_q \) can differ from zero even for \( m_q = 0 \). One may check that by solving the field equation for \( \Phi \) as a functional of \( \bar{\psi}, \psi \) (with \( U_k = m^2 \text{tr} \Phi^\dagger \Phi \)) one recovers from (31) the effective quark action (24). Note that \( \Gamma_k \) in (31) is chirally invariant. The flow equation for \( \Gamma_k \) therefore respects chiral symmetry which leads to a considerable simplification.

At the scale \( k_\Phi \) the propagator \( \tilde{G} \) and the wave function \( g(-q, q - p) \) should be optimized for a most complete elimination of terms quartic in the quark fields. In the present context we will, however, neglect the momentum dependence of \( Z_{\psi,k}, Z_{\Phi,k} \) and \( \tilde{h}_k \). The mass term \( m^2 \) was found in [31] for the simple truncation (24) with \( Z_\psi = 1, m = \tilde{m} = 0 \) to be \( \overline{m}(k_\Phi) \approx 120 \text{ MeV} \).

In view of the possible large truncation errors we will take this only as an order of magnitude estimate. Below we will consider the range \( \overline{m}(k_\Phi) = (45 - 120) \text{ MeV} \) for which chiral symmetry breaking can be obtained in a two flavor model. Furthermore, we will assume, as usually done in large–\( N_c \) computations within the NJL–model, that \( Z_\Phi(k_\Phi) \equiv Z_{\Phi,k}(q = 0) \ll 1 \). Moreover, the quark wave function renormalization \( Z_\psi(k) \equiv Z_{\psi,k}(q = 0) \) is set to one at the scale \( k_\Phi \) for convenience. For \( k < k_\Phi \) we will therefore study an effective action for quarks and mesons in the truncation
\[
\Gamma_k = \int d^4x \left\{ Z_\psi \bar{\psi}_a i\gamma^\mu \psi^a + Z_\Phi \text{tr} \left[ \partial_\mu \Phi^\dagger \partial^\mu \Phi \right] + U_k(\Phi, \Phi^\dagger) \right\}
\]
\[
+ \overline{h} \bar{\psi} \left( \frac{1 + \gamma_5}{2} \Phi_{ab} - \frac{1 - \gamma_5}{2} (\Phi^\dagger)_{ab} \right) \psi^b \right\} .
\]
with compositeness conditions

\[
U_{k\Phi}(\Phi, \Phi^\dagger) = m^2(k\Phi) \text{tr} \Phi^\dagger \Phi - \frac{1}{2} \overline{m}(k\Phi) \left( \text{det} \Phi + \text{det} \Phi^\dagger \right)
+ \frac{1}{2} X_1 \left( \text{tr} \Phi^\dagger \Phi \right)^2 + \frac{N - 1}{4} X_2 \text{tr} \left( \Phi^\dagger \Phi - \frac{1}{N} \text{tr} \Phi^\dagger \Phi \right)^2 + \ldots
\]

\[
\overline{m}^2(k\Phi) \equiv \frac{1}{2G(0)} \simeq (45 \text{ MeV})^2 - (120 \text{ MeV})^2
\]

\[
\overline{h}(k\Phi) = Z_\psi(k\Phi) = 1
\]

\[
Z_\Phi(k\Phi) \ll 1.
\]

As a consequence, the initial value of the renormalized Yukawa coupling which is given by \( h(k\Phi) = \overline{h}(k\Phi) Z_\psi^{-1}(k\Phi) Z_\Phi^{-1/2}(k\Phi) \) is large! Note that we have included in the potential an explicit \( U_A(1) \) breaking term \( \sim \overline{\nu} \) which mimics the effect of the chiral anomaly of QCD to leading order in an expansion of the effective potential in powers of \( \Phi \). Because of the infrared stability discussed in section 7 the precise form of the potential, i.e. the values of the quartic couplings \( \lambda_i \) and so on, will turn out to be unimportant.

We have refrained here for simplicity from considering four quark operators with vector and pseudo–vector spin structure. Their inclusion is straightforward and would lead to vector and pseudo–vector mesons in the effective action (35). We will concentrate first on two flavors and consider only the two limiting cases \( \overline{\nu} = 0 \) and \( \overline{\nu} \to \infty \). We also omit first the explicit quark masses and study the chiral limit \( j = 0 \). (The more realistic three flavor situation with a finite but non–vanishing \( \overline{\nu} \) and non–vanishing quark masses is depicted in section 8.) Because of the positive mass term \( m^2(k\Phi) \) one has at the scale \( k\Phi \) a vanishing expectation value \( \langle \Phi \rangle = 0 \) (for \( j = 0 \)). There is no spontaneous chiral symmetry breaking at the compositeness scale. This means that the mesonic bound states at \( k\Phi \) and somewhat below are not directly connected to chiral symmetry breaking.

The question remains how chiral symmetry is broken. We will try to answer this question by following the evolution of the effective potential \( U_k \) from \( k\Phi \) to lower scales using the exact renormalization group method outlined in section 2 with the compositeness conditions (36) defining the initial values. In this context it is important that the formalism for composite fields also induces an infrared cutoff in the meson propagator. The flow equations are therefore exactly of the form (11) (except for the supertrace), with quarks and mesons treated on an equal footing. In fact, one would expect that the large renormalized Yukawa coupling will rapidly drive the scalar mass term to negative values as the IR cutoff \( k \) is lowered. This will then finally lead to a potential minimum away from the origin at some scale \( k_{\chi_{SB}} < k\Phi \) such that \( \langle \Phi \rangle \neq 0 \). The ultimate goal of such a procedure, besides from establishing the onset of chiral symmetry breaking, would be to extract phenomenological quantities, like \( f_\pi \) or meson masses, which can be computed in a straightforward manner from \( \Gamma_k \) in the IR limit \( k \to 0 \).

At first sight, a reliable computation of \( \Gamma_{k \to 0} \) seems a very difficult task. Without a truncation \( \Gamma_k \) is described by an infinite number of parameters (couplings, wave function renormalizations, etc.) as can be seen if \( \Gamma_k \) is expanded in powers of fields and derivatives. For instance,
the pseudoscalar and scalar meson masses are obtained as the poles of the exact propagator, \( \lim_{k \to 0} \Gamma^{(2)}_k(q)|_{\Phi = \langle \Phi \rangle} \), which receives formally contributions from terms in \( \Gamma_k \) with arbitrarily high powers of derivatives and the expectation value \( \sigma_0 \). Realistic nonperturbative truncations of \( \Gamma_k \) which reduce the problem to a manageable size are crucial. The remainder of this work will be devoted to demonstrate that there may be a twofold solution to this problem:

- Due to an IR fixed point structure of the flow equations in the symmetric regime, i.e. for \( k_{\chi SB} < k < k_{\Phi} \), the values of many parameters of \( \Gamma_k \) for \( k \to 0 \) will be approximately independent of their initial values at the compositeness scale \( k_{\Phi} \). For large enough \( h(k_{\Phi}) \) only a few relevant parameters (\( m^2(k_{\Phi}), \sigma(k_{\Phi}) \)) need to be computed accurately from QCD. They can alternatively be determined from phenomenology.

- Physical observables like meson masses, decay constants, etc., can be expanded in powers of the quark masses. This is similar to the way it is usually done in chiral perturbation theory \[34\]. To a given finite order of this expansion only a finite number of terms of a simultaneous expansion of \( \Gamma_k \) in powers of derivatives and \( \Phi \) are required if the expansion point is chosen properly.

In combination, these two results open the possibility for a perhaps unexpected degree of predictive power within the linear meson model.

We wish to stress, though, that a perturbative treatment of the model at hand, e.g., using perturbative RG techniques, cannot be expected to yield reliable results. The renormalized Yukawa coupling is expected to be large at the scale \( k_{\Phi} \) and even the IR value of \( h \) is still relatively big

\[
h(k = 0) = \frac{2 M_q}{f_{\pi}} \simeq 7.5
\]

and \( h \) increases with increasing \( k \). The dynamics of the linear meson model is therefore clearly nonperturbative for all scales \( k \leq k_{\Phi} \).

5 Flow equations for the linear meson model

We will next turn to the ERGE analysis of the linear meson model which was introduced in the last section. To be specific, we will try to attack the problem at hand by truncating \( \Gamma_k \) in such a way that it contains all perturbatively relevant and marginal operators, i.e. those with canonical dimensions \( d_c \leq 4 \) in four space–time dimensions and ignore the evolution and effects coming from operators with \( d_c > 4 \). The effective potential \( U_k \) is a function of only four \( SU_L(N) \times SU_R(N) \) invariants\(^{12} \) for \( N = 3 \):

\[
\rho = \text{tr} \Phi^\dagger \Phi
\]

\(^{12}\)The invariant \( \tau_3 \) is only independent for \( N \geq 3 \). For \( N = 2 \) it can be eliminated by a suitable combination of \( \tau_2 \) and \( \rho \). The additional \( U_A(1) \) breaking invariant \( \omega = i(\det \Phi - \det \Phi^\dagger) \) is \( CP \) violating and may therefore appear only quadratically in \( U_k \). It is straightforward to see that \( \omega^2 \) is expressible in terms of the invariants\(^{13} \).
\[\tau_2 = \frac{N}{N-1} \text{tr} \left( \Phi^\dagger \Phi - \frac{1}{N} \rho \right)^2\]
\[\tau_3 = \text{tr} \left( \Phi^\dagger \Phi - \frac{1}{N} \rho \right)^3\]
\[\xi = \det \Phi + \det \Phi^\dagger.\]  \hfill (38)

We will expand \(U_k\) as a function of these invariants around its minimum in the absence of explicit \(SU(3)\) breaking due to current quark masses, i.e. \(\rho = \rho_0 \equiv N \bar{\sigma}_0^2\), \(\xi = \xi_0 = 2 \bar{\sigma}_0^N\) and \(\tau_2 = \tau_3 = 0\) where
\[\bar{\sigma}_0 \equiv \frac{1}{N} \text{tr} \langle \Phi \rangle.\]  \hfill (39)

Restricting \(U_k\) to operators of canonical dimension \(d_c \leq 4\) therefore this yields in the chirally symmetric regime, i.e., for \(k_{\chi_{\text{SB}}} \leq k \leq k_{\phi}\) where \(\bar{\sigma}_0 = 0\)
\[U_k = \bar{m}^2 \rho + \frac{1}{2} \lambda_1 \rho^2 + \frac{N-1}{4} \lambda_2 \tau_2 - \frac{1}{2} \nu \xi\]  \hfill (40)

whereas in the SSB regime for \(k \leq k_{\chi_{\text{SB}}}\) we have
\[U_k = \frac{1}{2} \lambda_1(k) \left[ \rho - N \sigma_0^2(k) \right]^2 + \frac{N-1}{4} \lambda_2(k) \tau_2 + \frac{1}{2} \nu(k) \left[ \sigma_0^{N-2}(k) \rho - \xi \right].\]  \hfill (41)

Before continuing to the actual beta functions for the couplings or parameters still kept in \(\Gamma_k\) it is worthwhile to pause here and emphasize that naively (perturbatively) irrelevant operators can by no means always be neglected. The most prominent example for this is QCD itself. It is the very assumption of our treatment of chiral symmetry breaking (substantiated by the results of [31]) that the momentum dependence of the coupling constants of some six–dimensional quark operators \(\langle \bar{q}q \rangle^2\) develop poles in the \(s\)–channel indicating the formation of mesonic bound states. On the other hand, it is quite natural to assume that \(\Phi^6\) or \(\Phi^8\) operators are not really necessary to understand the properties of the potential in a neighborhood around its minimum. Yet, truncating higher dimensional operators does not imply the assumption that the corresponding coupling constants are small. In fact, this could only be expected as long as the relevant and marginal couplings are small as well. What is required, though, is that their influence on the evolution of those couplings kept in the truncation, for instance, the set of equations (43) below, is small. In this context it is perhaps also interesting to note that the truncation (43) includes the known one–loop beta functions of a small coupling expansion as well as the leading order result of the large–\(N_c\) expansion of the \(U_L(N) \times U_R(N)\) model [39]. This should provide at least some minimal control over this truncation, even though we hope that our results are significantly more accurate.

Inserting the truncation (43), (40), (41) into (11) reduces this functional differential equation for infinitely many variables to a finite set of ordinary differential equations. This yields, in particular, the beta functions for the couplings \(\lambda_1, \lambda_2, \nu\) and \(\bar{m}^2\) or \(\bar{\sigma}_0\). Details of the calculation can be found in [40]. We will refrain here from presenting the full set of flow equations but rather illustrate the main results with a few examples. Defining dimensionless renormalized
mass, VEV and coupling constants

\[ \epsilon(k) = k^{-2}m^2(k) = \overline{m}^2(k)Z^{-1}_\phi(k)k^{-2} \]
\[ \kappa(k) = k^{2-d}N\sigma_R^2 = Z\Phi k^{2-d}N\sigma_0^2 \]
\[ h^2(k) = \overline{h}^2(k)Z^{-1}_\phi(k)Z^{2-1}_\psi(k) \]
\[ \lambda_i(k) = \overline{\lambda}_i Z^{-2}_\phi(k); \quad i = 1, 2 \]
\[ \nu(k) = \overline{\nu}(k)Z^{-\frac{N}{2}}_\phi(k)k^{N-4}. \]  

(42)

one finds, e.g., for the spontaneous symmetry breaking (SSB) regime and \( \overline{\nu} = 0 \)

\[ \frac{\partial \kappa}{\partial t} = -(2 + \eta_\Phi)\kappa + \frac{1}{16\pi^2} \left\{ N^2l_1^4(0) + 3l_1^4(2\lambda_1\kappa) \right. \]
\[ + (N^2 - 1) \left[ 1 + \frac{\lambda_2}{\lambda_1} \right] l_1^4(\lambda_2\kappa) - 4N_c \frac{h^2}{\lambda_1} l_1^4(F^4) \left( \frac{1}{N} h^2\kappa \right) \}
\[ \frac{\partial \lambda_1}{\partial t} = 2\eta_\phi \lambda_1 + \frac{1}{16\pi^2} \left\{ N^2\lambda_2^2 l_1^4(0) + 9\lambda_2^2 l_1^4(2\lambda_1\kappa) \right. \]
\[ + (N^2 - 1) [\lambda_1 + \lambda_2^2] l_2^4(2\lambda_2\kappa) - 4\frac{N_c}{N} h^4 l_2^4(F^4) \left( \frac{1}{N} h^2\kappa \right) \}\n
(43)

\[ \frac{\partial \lambda_2}{\partial t} = 2\eta_\phi \lambda_2 + \frac{1}{16\pi^2} \left\{ \frac{N^2}{4} \lambda_2^2 l_1^4(0) + \frac{9}{4}(N^2 - 4) \lambda_2^2 l_1^4(2\lambda_1\kappa) \right. \]
\[ - \frac{1}{2} N^2 \lambda_2^2 l_1^4(0, \lambda_2\kappa) + 3[\lambda_2 + 4\lambda_1] l_2^4(2\lambda_1\kappa, \lambda_2\kappa) \]
\[ - 8\frac{N_c}{N} h^4 l_2^4(F^4) \left( \frac{1}{N} h^2\kappa \right) \}\n
\[ \frac{\partial h^2}{\partial t} = \left[ d - 4 + 2\eta_\psi + \eta_\phi \right] h^2 + 4N_c \psi_\chi h^4 \left\{ N^2 l_1^4(FB)_d \left( \frac{1}{N} \kappa h^2, \epsilon; \eta_\psi, \eta_\phi \right) \right. \]
\[ - (N^2 - 1) l_1^4(FB)_d \left( \frac{1}{N} \kappa h^2, \epsilon + \kappa \lambda_2; \eta_\psi, \eta_\phi \right) - l_1^4(FB)_d \left( \frac{1}{N} \kappa h^2, \epsilon + 2\kappa \lambda_1; \eta_\psi, \eta_\phi \right) \}\]

Here \( \eta_\Phi = -\partial_t \ln Z_\Phi, \eta_\psi = -\partial_t \ln Z_\psi \) are the meson and quark anomalous dimensions, respectively \[40\]. The symbols \( l_n^4, l_{n1,n2}^4 \) and \( l_n^{(F)} \) denote mass threshold functions. A typical example is

\[ l_n^4(w) = 8n\pi^2 k^{2n-4} \int \frac{d^4q}{(2\pi)^4} \partial_t (Z\Phi^{-1} R_k(q^2)) \]
\[ \times \left| P(q^2) + k^2 w \right|^{n+1} \]  

(44)

with \( P(q^2) = q^2 + Z\Phi^{-1} R_k(q^2) \). These functions decrease monotonically with their arguments \( w \) and decay \( \sim w^{-(n+1)} \) for \( w \gg 1 \). Since the arguments \( w \) are generally the (dimensionless) squared masses of the model, the main effect of the threshold functions is to cut off quantum fluctuations of particles with masses \( M^2 \gg k^2 \). Once the scale \( k \) is changed below a certain mass threshold, the corresponding particle no longer contributes to the evolution of the couplings and decouples smoothly.
Within our truncation the beta functions for the dimensionless couplings look almost the same as in one-loop perturbation theory. There are, however, two major new ingredients which are crucial for our approach: First, there is a new equation for the running of the mass term in the symmetric regime or for the running of the potential minimum in the regime with spontaneous symmetry breaking. This equation is related to the quadratic divergence of the mass term in perturbation theory and does not appear in the Callan–Symanzik or Coleman–Weinberg treatment of the renormalization group. Obviously, this equation is the key for a study of the onset of spontaneous chiral symmetry breaking as is lowered from to zero. Second, the most important nonperturbative ingredient in the flow equations for the dimensionless Yukawa and scalar couplings is the appearance of effective mass threshold functions which account for the decoupling of modes with masses larger than . Their form is different for the symmetric regime (massless fermions, massive scalars) or the regime with spontaneous symmetry breaking (massive fermions, massless Goldstone bosons). Without the inclusion of the threshold effects the running of the couplings would never stop and no sensible limit could be obtained due to unphysical infrared divergences. The threshold functions are not arbitrary but have to be computed carefully. The mass terms appearing in these functions involve the dimensionless couplings. Expanding the threshold functions in powers of the mass terms (or the dimensionless couplings) makes their non–perturbative content immediately visible. (It is the presence of threshold functions which explains why one–loop type formulae could be used for a necessarily nonperturbative computation of critical exponents in three–dimensional scalar theories.)

6 The chiral anomaly and the $O(4)$–model

We have seen how the mass threshold functions in the flow equations describe the decoupling of heavy modes from the evolution of $\Gamma_k$ as the IR cutoff $k$ is lowered. In the chiral limit with two massless quark flavors ($N = 2$) the pions are the massless Goldstone bosons whereas all other mesons have masses larger than $k_\Phi$ and are therefore practically decoupled already at the scale $k_\Phi$. The effect of the physical pion mass of $m_\pi \simeq 140$ MeV, or equivalently of the two small but non–vanishing current quark masses, can then easily be mimicked by stopping the flow of $\Gamma_k$ at $k = m_\pi$ by hand. This situation changes significantly once the strange quark is included. Now the $\eta$ and the four $K$ mesons appear as additional massless Goldstone modes in the spectrum. They would artificially drive the running of $\Gamma_k$ at scales $m_\pi \lesssim k \lesssim 500$ MeV where they should already be decoupled because of their physical masses. It is therefore advisable to focus on the two flavor case $N = 2$ as long as the chiral limit of vanishing current quark masses is considered.

It is straightforward to obtain an estimate of the (renormalized) coupling $\nu$ parameterizing the explicit $U_A(1)$ breaking due to the chiral anomaly. From (41) we find

$$m_{\eta'}^2 = \frac{N}{2} \nu \sigma_0 N^{-2} \simeq 1 \text{ GeV}$$

which translates for $N = 2$ into

$$\nu(k \to 0) \simeq 1 \text{ GeV}.$$
This suggests that $\nu \to \infty$ can be considered as a realistic limit. An important simplification occurs for $N = 2$ and $\nu \to \infty$, related to the fact that for $N = 2$ the chiral group $SU_L(2) \times SU_R(2)$ is (locally) isomorphic to $O(4)$. Thus, the complex $(2, 2)$ representation $\Phi$ of $SU_L(2) \times SU_R(2)$ may be decomposed into two vector representations, $(\sigma, \pi^k)$ and $(\eta', a^k)$ of $O(4)$:

$$\Phi = \frac{1}{2} (\sigma - i\eta') + \frac{1}{2} (a^k + i\pi^k) \tau_k .$$

(47)

For $\nu \to \infty$ the masses of the $\eta'$ and the $a^k$ diverge and these particles decouple. We are then left with the original $O(4)$ symmetric linear $\sigma$–model of Gell–Mann and Levy [43] coupled to quarks. The flow equations of this model have been derived previously [10, 7] for the truncation of the effective action used here. For mere comparison we also consider the opposite limit $\nu \to 0$. Here the $\eta'$ meson becomes an additional Goldstone boson in the chiral limit which suffers from the same problem as the $K$ and the $\eta$ in the case $N = 3$. Hence, we may compare the results for two different approximate limits of the effects of the chiral anomaly:

- the $O(4)$ model corresponding to $N = 2$ and $\nu \to \infty$
- the $U_L(2) \times U_R(2)$ model corresponding to $N = 2$ and $\nu = 0$.

For the reasons given above we expect the first situation to be closer to reality. In this case we may imagine that the fluctuations of kaons, $\eta$ $\eta'$ and the scalar mesons (as well as vector and pseudovector mesons) have been integrated out in order to obtain the initial values of $\Gamma_{k_{SB}}$ — in close analogy to the integration of the gluons for the effective quark action $\Gamma_{k_{q}}[\bar{\psi}]$ discussed in section [4]. We will keep the initial values of the couplings $m^2(k, \Phi), \lambda_1(k, \Phi)$ and $Z(\Phi)$ as free parameters. Our results should be quantitatively accurate to the extent to which the local polynomial truncation is a good approximation.

7 Infrared stability

Eqs. (13) and the corresponding set of flow equations for the symmetric regime constitute a coupled system of ordinary differential equations which can be integrated numerically. Similar equations obtain for the $O(4)$ model where $N = 2$ and the coupling $\lambda_2$ is absent. The most important result is that chiral symmetry breaking indeed occurs for a wide range of initial values of the parameters including the presumably realistic case of large renormalized Yukawa coupling and a bare mass $\overline{m}(k, \Phi)$ of order 100 MeV. A typical evolution of the renormalized mass $m(k)$ is plotted in figure [4]. Driven by the strong Yukawa coupling, $m$ decreases rapidly and goes through zero at a scale $k_{\chi_{SB}}$ not far below $k_{SB}$. Here the system enters the SSB regime and a non–vanishing (renormalized) VEV $\sigma_R$ for the meson field $\Phi$ develops. The evolution of $\sigma_R(k)$ turns out to be reasonably stable already before scales $k \approx m_{\pi}$ where the evolution is stopped. We take this result as an indication that our truncation of the effective action $\Gamma_k$ leads at least qualitatively to a satisfactory description of chiral symmetry breaking. The reason for the relative stability of the IR behavior of the VEV (and all other couplings) is that the quarks acquire a constituent mass $M_q = \hbar \sigma_R \simeq 350$ MeV in the SSB regime. As a consequence
they decouple once $k$ becomes smaller than $M_q$ and the evolution is then dominantly driven by the massless Goldstone bosons. This is also important in view of potential confinement effects expected to become important for the quarks for $k$ around $\Lambda_{QCD} \approx 200\text{MeV}$. Since confinement is not explicitly included in our truncation of $\Gamma_k$, one might be worried that such effects could spoil our results completely. Yet, as discussed in some more detail in section 4, only the colored quarks should feel confinement and they are no longer important for the evolution of the meson couplings for $k$ around 200 MeV. One might therefore hope that a precise treatment of confinement is not crucial for this approach to chiral symmetry breaking.

Most importantly, one finds that the system of flow equations exhibits an IR fixed point in the symmetric phase. As already pointed out one expects $Z_\Phi$ to be rather small at the compositeness scale $k_\Phi$. In turn, one may assume that, at least for the initial range of running in the symmetric regime the mass parameter $\epsilon \sim Z_\Phi^{-1}$ is large. This means, in particular, that all threshold functions with arguments $\sim \epsilon$ may be neglected in this regime. As a consequence, the flow equations simplify considerably. We find, for instance, for the $U_L(2) \times U_R(2)$ model

$$
\begin{align}
\partial_t \bar{\epsilon} &\equiv \partial_t \frac{\epsilon}{\hbar^2} \simeq -2\bar{\epsilon} + \frac{N_c}{4\pi^2} \\
\partial_t \bar{\lambda}_1 &\equiv \partial_t \frac{\lambda_1}{\hbar^2} \simeq \frac{N_c}{4\pi^2} \hbar^2 \left[ \frac{1}{2} \bar{\lambda}_1 - \frac{1}{N} \right] \\
\partial_t \bar{\lambda}_2 &\equiv \partial_t \frac{\lambda_2}{\hbar^2} \simeq \frac{N_c}{4\pi^2} \hbar^2 \left[ \frac{1}{2} \bar{\lambda}_2 - \frac{2}{N} \right]
\end{align}
$$

(48)
\[ \partial_t h^2 \simeq \frac{N_c}{8\pi^2} h^4 \]
\[ \eta_\Phi \simeq \frac{N_c}{8\pi^2} h^2 \]
\[ \eta_\psi \simeq 0. \]

This system possesses an attractive IR fixed point for the quartic scalar self interactions

\[ \tilde{\lambda}_1 = \frac{1}{2} \tilde{\lambda}_2 = \frac{2}{N}. \]  

(49)

Furthermore it is exactly soluble \[40\]. Because of the strong Yukawa coupling the quartic couplings \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \) generally approach their fixed point values rapidly, long before the system enters the broken phase (\( \epsilon \to 0 \)) and the approximation of large \( \epsilon \) breaks down. In addition, for large \( h^2(k_\Phi) \) the value of the Yukawa coupling at the scale \( k_s \) where \( \epsilon \) vanishes (or becomes small) only depends on the initial value

\[ \tilde{\epsilon}_0 \equiv \frac{\epsilon(k_\Phi)}{h^2(k_\Phi)} = \frac{m^2(k_\Phi)}{k_\Phi^2}. \]

(50)

Hence, the system is approximately independent in the IR upon the initial values of \( \lambda_1, \lambda_2 \) and \( h^2 \), the only “relevant” parameter being \( \tilde{\epsilon}_0 \). In other words, the effective action loses almost all its “memory” in the far IR of where in the UV it came from. This feature of the flow equations leads to a perhaps surprising degree of predictive power. In addition, also the dependence of \( f_\pi = 2\sigma R(k \to 0) \) on \( \tilde{\epsilon}_0 \) is not very strong for a large range in \( \tilde{\epsilon}_0 \), as shown in figure \( \text{ figure } 2 \). The relevant parameter \( \tilde{\epsilon}_0 \) can be fixed by using the constituent quark mass \( M_q \equiv (h\sigma R)(k = 0) \simeq 350 \text{ MeV} \) as a phenomenological input. One obtains for the \( O(4) \) model

\[ \tilde{\epsilon}_0 \simeq 0.02. \]

(51)

The resulting value for the decay constant is

\[ f_\pi = (91 - 100) \text{ MeV} \]

(52)

for \( h^2(k_\Phi) = 10^4 - 300 \). It is striking that this comes close to the real value \( f_\pi = 92.4 \text{ MeV} \) but we expect that the uncertainty in the determination of the compositeness scale \( k_\Phi \) and the truncation errors exceed the influence of the variation of \( h^2(k_\Phi) \). We have furthermore used this result for an estimate of the chiral condensate:

\[ \langle \bar{\psi} \psi \rangle \equiv -\tilde{\epsilon}_0 f_\pi k_\Phi^2 Z_\Phi^{-1/2}(k = 0)A \simeq -(195 \text{ MeV})^3 \]

(53)

where the factor \( A \simeq 1.7 \) accounts for the change of the normalization scale of \( \langle \bar{\psi} \psi \rangle \) from \( k_\Phi \) to the commonly used value 1 GeV. Our value is in reasonable agreement with results from

\[ \text{Once quark masses and a proper treatment of the chiral anomaly are included for } N = 3 \text{ one expects that } m_q \text{ and } \nu \text{ are additional relevant parameter. Their values may be fixed by using the masses of } \pi, K \text{ and } \eta' \text{ as phenomenological input.} \]
Figure 2: The pion decay constant $f_\pi$ as a function of $\tilde{\epsilon}_0$ for $k_\Phi = 630$ MeV, $\lambda_1(k_\Phi) = \lambda_2(k_\Phi) = 0$ and $h^2(k_\Phi) = 300$ (solid line) as well as $h^2(k_\Phi) = 10^4$ (dashed line).

This result is non-trivial since not only $k_\Phi$ and $f_\pi$ enter but also $\tilde{\epsilon}_0$ and the IR value $Z_\Phi(0)$. Integrating (48) for $\eta_\Phi$ one finds

$$Z_\Phi(k) = Z_\Phi(k_\Phi) + N_c \frac{8\pi^2}{8\pi^2} \ln \frac{k_\Phi}{k}.$$  

Thus $Z_\Phi(k)$ will indeed be practically independent of its initial value $Z_\Phi(k_\Phi)$ already after some running as long as $Z_\Phi(k_\Phi)$ is small compared to 0.01.

The alert reader may have noticed that the beta functions (48) correspond exactly to those obtained in the one–quark–loop approximation or, in other words, to the leading order in the large–$N_c$ expansion for the Nambu–Jona-Lasinio model [39]. The fixed point (49) is then nothing but the large–$N_c$ boundary condition on the evolution of $\lambda_1$ and $\lambda_2$ in this model. Yet, we wish to stress that nowhere we have made the assumption that $N_c$ is a large number. On the contrary, the physical value $N_c = 3$ suggests that the large–$N_c$ expansion should a priori only be trusted on a quantitatively rather crude level. The reason why we expect (48) to nevertheless give rather reliable results is based on the fact that for small $Z_\Phi(k_\Phi)$ all (renormalized) meson masses are much larger than the scale $k$ for the initial part of the running. This implies that the mesons are effectively decoupled and their contribution to the beta functions is negligible leading to the one–quark–loop approximation. Yet, already after some relatively short period of running the renormalized meson masses approach zero and our approximation of neglecting mesonic threshold functions breaks down. Hence, the one–quark–loop approximation is reasonable only for scales close to $k_\Phi$ but is bound to fail around $k_{\chi,SB}$ and in the SSB regime. What is important
in our context is not the numerical value of the partial fixed points \( \lambda_1(k_0) \) but rather their mere existence and the presence of a large coupling \( h^2 \) driving the \( \lambda_1 \) fast towards them. This is enough for the IR values of all couplings to become almost independent of the initial values \( \lambda_1(k_0) \). Similar features of IR stability are expected if the truncation is enlarged, for instance, to a more general form of the effective potential \( U_{k_0} \).

8 Phenomenology of the linear meson model for three flavors

In the preceding sections we have presented an investigation of the linear meson model subject to some simplifications and approximations:

- We have restricted ourselves to the case of two light quark flavors, \( N = 2 \), whereas it is known from chiral perturbation theory \(^{[34]}\) that the strange quark can also be considered as light.

- Explicit chiral symmetry breaking due to current quark masses has been neglected. For \( N = 2 \) its main effect is to give the pions a non–vanishing mass which is easily mimicked by stopping the RG evolution around \( k = m_\pi \) by hand. For \( N = 3 \) this would be rather inaccurate, since the kaons and the \( \eta \) have masses quite different from the pions and should therefore decouple much earlier. This necessitates an explicit consideration of the source term \( \sim j \).

- We have treated the effects of the chiral anomaly only for two limiting cases: \( \nu = 0 \) and \( \nu \to \infty \). In reality \( \nu \) is, however, large but finite.

- The IR–cut–off effective action \( \Gamma_k \) was truncated in such a way that only operators of canonical dimension \( d_c \leq 4 \) were kept.

It is, of course, necessary to investigate the effects of these simplifications systematically. It is straightforward to relax the first three points, i.e. to include three massive quark flavors and allow for a non–vanishing and finite \( \nu \). A systematic study of the truncation errors is more involved. Even though it is possible to include additional terms in \( \Gamma_k \) it would be desirable to have a guiding principle at hand, which would allow to determine the invariants which one should keep in a truncation of \( \Gamma_k \) and those which can be neglected. Clearly, such a principle will depend on the kind of physical questions one poses. In the following we will apply three different criteria for this purpose:

1. Phenomenological observables like meson masses or decay constants can be expanded systematically in powers of (current) quark masses in the linear meson model in a similar way as it is usually done in chiral perturbation theory. This can be used to determine the operators of \( \Gamma_k \) which are necessary to compute an observable to a given order in \( m_q \).
2. We have assumed so far that all other low–energy degrees of freedom of strong inter-
action physics like vector and axial–vector mesons have been integrated out. Carrying
out this procedure explicitly in some cases induces relatively large values for several
non–renormalizable interactions in the linear meson model. They have to be taken into
account, at least for comparison with phenomenology.

3. Even though the range of evolution from \( k_{\Phi} \) to \( k = m_\pi \) is not too large \( (m_\pi/k_{\Phi} \approx 0.2) \),
it is likely that higher dimensional operators are dimensionally suppressed at \( k = m_\pi \) as
long as their anomalous dimension is not very large. This is related to the “triviality”
of the linear quark meson system, i.e. the existence of the interaction free “Gaussian”
fixed point. We will therefore assume that of two operators contributing to a physical
observable at the same order in the quark mass expansion, the lower dimensional one
dominaates, provided the other one does not receive large corrections from other degrees
of freedom assumed to be integrated out. This can also be understood as an expansion
in powers of the chiral condensate \( \sim \sigma_0 \).

We will sketch in the following how these criteria can be applied to the linear meson model.
The details of this procedure can be found in [45].

In order to perform a quark mass expansion of meson observables it is useful to decompose
\( \Phi \) with respect to the vector–like \( SU_V(3) \) symmetry

\[
\Phi = \sigma_0 + \frac{1}{\sqrt{2}} \left( i\Phi_p + \frac{i}{\sqrt{3}} \chi_p + \Phi_s + \frac{1}{\sqrt{3}} \chi_s \right). \tag{55}
\]

For \( N = 3 \) the hermitean traceless \( 3 \times 3 \) matrices \( \Phi_p \) and \( \Phi_s \) represent (up to mixing effects) the
pseudoscalar and scalar octets, respectively. The real fields \( \chi_p \) and \( \chi_s \) are associated with the
parity odd and even singlets corresponding to the \( \eta' \) meson and the \( \sigma \)–resonance, respectively.
The constant \( \sigma_0 \) is chosen as the VEV of \( \Phi \) in the limit of equal quark masses \( m_q = (m_u + m_d + m_s)/3 \). Differences between the strange, down and up, quark masses imply

\[
\langle \Phi_s \rangle = \frac{1}{\sqrt{2}Z_h} \left( w \lambda_3 - \sqrt{3} v \lambda_8 \right). \tag{56}
\]

Here \( \lambda_3 \) and \( \lambda_8 \) denote the two diagonal Gell–Mann matrices. The parameters \( w \) and \( v \) corre-
respond to the explicit isospin and \( SU_V(3) \) violation in the pseudoscalar meson decay constants

\[
\left( \frac{Z_m}{Z_h} \right)^{\frac{1}{2}} w = \overline{f}_{K^\pm} - \overline{f}_{K^0}
\]

\[
\left( \frac{Z_m}{Z_h} \right)^{\frac{1}{2}} v = \frac{1}{3} \left( \overline{f}_{K^\pm} + \overline{f}_{K^0} - 2\overline{f}_\pi \right). \tag{57}
\]

Here \( Z_m \) and \( Z_h \) denote the wave functions renormalization constants of \( \Phi_p \) and \( \Phi_s \), respectively. The \( \overline{f}_i \) are related to the physical decay constants \( f_i \) by additional wave function renormalization
effects which will be discussed below.
It is convenient to expand the effective potential around $\Phi = \sigma_0$ or $\rho = 3\sigma_0^2$, $\xi = 2\sigma_0^3$ and $\tau_2 = \tau_3 = 0$:

$$U(\rho, \tau_2, \tau_3, \xi) = \frac{m_g^2}{2}(\rho - \rho_0) - \frac{1}{2}\sigma[\xi - \xi_0 - \sigma_0(\rho - \rho_0)]$$
$$+ \left\{ \begin{array}{l}
\frac{1}{2}\lambda_1(\rho - \rho_0)^2 + \frac{1}{2}\lambda_2\tau_2 + \frac{1}{2}\lambda_3\tau_3 \\
\frac{1}{2}\beta_1(\rho - \rho_0)(\xi - \xi_0) + \frac{1}{2}\beta_2(\rho - \rho_0)\tau_2 \\
\frac{1}{2}\beta_3(\xi - \xi_0)\tau_2 + \frac{1}{2}\beta_4(\xi - \xi_0)^2 + \ldots .
\end{array} \right. \tag{58}$$

The bare meson mass squared matrix is obtained from the second derivatives of $U$ with respect to the components of the fields $\Phi_s$, $\Phi_p$, $\chi_s$ and $\chi_p$ evaluated at the minimum of $U_k = \frac{1}{2} \text{tr} J(\Phi + \Phi^\dagger)$. Decomposing the chiral invariants $\rho$, $\tau_2$, $\tau_3$ and $\xi$ with respect to the $SU_V(3)$ multiplets $\Phi_s$, $\Phi_p$, $\chi_s$ and $\chi_p$ and taking into account that the only non–vanishing VEV is $\langle \Phi_s \rangle \sim O(m_q)$ it is straightforward to see that the potential (58) contains all terms which are necessary and sufficient for a computation of the bare pseudoscalar meson masses to second order and the scalar octet and pseudoscalar singlet masses to linear order in the quark masses $m_q$.

In a similar way one can see that modifications of the minimal kinetic term for $\Phi$ beyond the approximation (35) are necessary for a determination of the physical meson masses to the desired order in $m_q$. There are various two–derivative terms all contributing to the same order in the quark mass expansion to, e.g., meson masses. As argued above we will, however, assume that the lowest dimensional ones dominate. The ones relevant here are given by

$$L_{\text{kin}} = Z_\Phi \text{Tr} \partial^\mu \Phi^\dagger \partial^\mu \Phi$$
$$+ \frac{1}{2}U_\epsilon \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \left( \Phi_{a_1 b_1} \partial^\mu \Phi_{a_2 b_2} \partial^\mu \Phi_{a_3 b_3} + \Phi_{a_1 b_1}^\dagger \partial^\mu \Phi_{a_2 b_2} \partial^\mu \Phi_{a_3 b_3}^\dagger \right)$$
$$- \frac{1}{8}X^- \left\{ \begin{array}{l}
\text{Tr} \left( \Phi^\dagger \partial_\mu \Phi - \partial_\mu \Phi^\dagger \Phi \right) \left( \Phi^\dagger \partial^\mu \Phi - \partial^\mu \Phi^\dagger \Phi \right) \\
+ \text{Tr} \left( \Phi \partial_\mu \Phi^\dagger - \partial_\mu \Phi \Phi^\dagger \right) \left( \Phi \partial^\mu \Phi^\dagger - \partial^\mu \Phi \Phi^\dagger \right) \} \\
- \frac{1}{8}X^+ \left\{ \begin{array}{l}
\text{Tr} \left( \Phi^\dagger \partial_\mu \Phi + \partial_\mu \Phi^\dagger \Phi \right) \left( \Phi^\dagger \partial^\mu \Phi + \partial^\mu \Phi^\dagger \Phi \right) \\
+ \text{Tr} \left( \Phi \partial_\mu \Phi^\dagger + \partial_\mu \Phi \Phi^\dagger \right) \left( \Phi \partial^\mu \Phi^\dagger + \partial^\mu \Phi \Phi^\dagger \right) \} + \ldots .
\end{array} \right. \tag{59}$$

These modifications of the kinetic term induce different wave function renormalization constants $Z_h$, $Z_m$, $Z_s$ and $Z_p$ for the four multiplets $\Phi_s$, $\Phi_p$, $\chi_s$ and $\chi_p$, respectively, e.g.

$$Z_m = Z_\Phi + U_\Phi \sigma_0 + X^- \sigma_0^2$$
$$Z_p = Z_\Phi - 2U_\Phi \sigma_0 + X^- \sigma_0^2$$
$$Z_h = Z_\Phi - U_\Phi \sigma_0 - X^+ \sigma_0.$$

\(^{14}\)If one counts the effects of the chiral anomaly as being of $O(m_q)$, eq. (58) is also sufficient for the bare pseudoscalar singlet or (up to mixing) the $\eta$' meson mass to quadratic order in $m_q$. 27
Furthermore, they lead to a split in the wave function renormalizations for the individual members of each multiplet:

\[
\begin{align*}
Z_\pi &= 1 - \omega_m v \\
Z_{K^\pm} &= 1 + \frac{1}{2} \omega_m (v + w) \\
Z_{K^0} &= 1 + \frac{1}{2} \omega_m (v - w) \\
Z_8 &= 1 + \omega_m v
\end{align*}
\] (61)

where

\[
\omega_m = \left( X_{\Phi} \Phi_0 - U_\Phi \right) Z_h^{-1/2} Z_m^{-1}
\] (62)

The physical meson masses and decay constants are then given by

\[
\begin{align*}
M^2_\pi &= \hat{M}^2_\pi Z^{-1}_\pi Z^{-1}_m \\
f_\pi &= \bar{f}_\pi Z_1^{1/2} \ 	ext{etc.}
\end{align*}
\] (63)

where the \( \hat{M}^2_i \) are the bare meson mass terms given by the eigenvalues of the matrix of second derivatives of the potential \( U \). For the pseudoscalar octet the relations (63) with (61) contain all contributions to the meson masses to second order and the decay constants to linear order in the quark mass expansion.

In addition, the non–minimal kinetic terms (59) result in a correction relevant for the \( \eta - \eta' \) mixing. This is related to a quark mass dependence of kinetic terms and off–diagonal kinetic terms arising from

\[
\Delta L = \frac{1}{\sqrt{2}} \omega_m Z_m Z_h^{1/2} \text{tr} \Phi_s \partial^\mu \Phi_p \partial_\mu \Phi_p + \omega_{pm} (Z_p Z_m Z_h)^{1/2} \partial_\mu \chi_p \text{tr} \Phi_s \partial^\mu \Phi_p
\] (64)

with

\[
\omega_{pm} = \frac{1}{\sqrt{6}} \left( 2X_{\Phi} \Phi_0 + U_\Phi \right) (Z_p Z_h Z_m)^{-1/2}
\] (65)

The parameter \( \omega_{pm} \) induces a momentum dependence in the \( \eta - \eta' \) mixing angle. Consequently, the mixing angle will be different for the on–shell decays \( \eta \to 2\gamma \) and \( \eta' \to 2\gamma \) through which it is usually determined experimentally. This qualitative result is, in fact, in agreement with observation. For a quantitative comparison with experiment it is, however, more convenient to compute the measured ratios

\[
\frac{f_\eta}{f_\pi} = \left[ \frac{M^3_\eta \Gamma(\eta \to 2\gamma)}{M^3_\pi \Gamma(\pi \to 2\gamma)} \right]^{1/2} \\
\frac{f'_\eta}{f_\pi} = \left[ \frac{M^3_\eta' \Gamma(\eta' \to 2\gamma)}{M^3_\pi \Gamma(\pi \to 2\gamma)} \right]^{1/2}
\] (66)

from (58) and (59). The results, as well as \( M_\eta \) and \( M_{\eta'} \), are now functions of \( U_\Phi \) and \( X_{\Phi} \) or \( Z_p/Z_m \) and \( \omega_m \). Using the observed values for \( M_\pi, K_{K^\pm}, M_{K^0}, M_{\eta'}, f_\pi \) and \( f_{K^\pm} \) as input we have plotted \( M_\eta \) including all corrections of second order in the quark mass expansion as a function of \( \omega_m v \) for various values of \( Z_p/Z_m \) in figure 3. Figure 4 shows the ratios (66) of
Figure 3: The plot shows $M_\eta$ as a function of $\omega_m v$ for various values of $Z_p/Z_m$. The solid line corresponds to $Z_p/Z_m = 1$ and the difference in $Z_p/Z_m$ between two adjacent lines is 0.1. The horizontal dotted line indicates the experimental value $M_\eta \simeq 547.5$ MeV.

decay constants as functions of $\omega_m v$ for the same values of $Z_p/Z_m$. The agreement with the experimental values is satisfactory for $Z_p/Z_m \simeq 0.9$ and $\omega_m v \simeq -0.19$. In addition, it is clear from figure 4 that a phenomenologically acceptable value of $f_\eta/f_\pi$ can only be obtained once the non–minimal kinetic terms in (59) are taken into account. We furthermore note that the deviation of $Z_p/Z_m$ from one is rather small. This is compatible with the assumption that the chiral anomaly which induces a non–vanishing $U_\phi$ is an effect which is comparable in size to corrections linear in the quark masses.

Interestingly, the non–minimal kinetic term $\sim X_\Phi^-$ in (59) which is responsible for the relatively large value of $\omega_m v$ receives a large contribution from the exchange of axial–vector mesons. More precisely, the longitudinal component of $\partial_\mu \rho_A^\mu$, with $\rho_A$ the axial–vector meson nonet, represents a $0^{++}$ state which therefore mixes with the $\eta$ and $\eta'$ mesons — a mechanism usually referred to as “partial Higgs effect”. A rough estimate of the contribution to $X_\Phi$ resulting from axial–vector meson exchange yields a value of $\omega_m^{(0)} v \simeq -0.16$. This is already quite close to the value $\omega_m v \simeq -0.19$ suggested by figures 3 and 4. One is therefore tempted to even assume that all kinetic terms (including those of $\mathcal{O}(\partial^4)$) beyond the minimal one $\sim Z_\phi$ are dominated by the contributions from axial–vector meson exchange. This “leading mixing approximation” leads, in fact, to a satisfactory description of the $\eta$–$\eta'$ system [15].

Analogously one may apply the quark mass expansion to the scalar sector of the linear meson model. As an interesting result one obtains a Gell–Mann–Okubo type relation for the scalar octet

$$M_{\sigma_n}^2 = \frac{1}{3} \left( 4M_{K^*}^2 - M_{\rho_0}^2 \right)$$

(67)
Figure 4: The plots show the ratios $f_\eta/f_\pi$ and $f_\eta'/f_\pi$ as functions of $\omega_m v$ for various values of $Z_p/Z_m$. The solid lines correspond to $Z_p/Z_m = 1$ and the difference in $Z_p/Z_m$ between two adjacent lines is 0.1. The experimentally allowed windows (1$\sigma$) for both quantities are bounded by the horizontal solid lines.
where $M_{\sigma_{q}}$ denotes the mass of the scalar partner of the $\eta$ meson. This relation can be seen to be exact to linear order in the quark mass expansion. Using $M_{K^*} = 1430$ MeV and $M_{a_0} = 983$ MeV as input one obtains

$$M_{\sigma_{q}} = 1550 \text{ MeV}.$$  \hfill (68)

Yet, it can be seen that the relation (67) is subject to large corrections of second order in the quark mass expansion induced by mixing of the charged scalar state with the longitudinal component $\partial_{\mu} A_{\mu}$ of the vector meson nonet.

Finally, it should be mentioned that the linear meson model as outlined so far also leads to satisfactory results for the ratios of light quark masses \[46\]. In particular, one finds

$$\frac{m_u + m_s}{m_u + m_d} = \frac{M_{K^\pm} f_{K^\pm}}{M_{\pi^\pm} f_{\pi}} \left( \frac{Z_{K^\pm}}{Z_{\pi^\pm}} \right)^{1/2},$$

$$\frac{m_u + m_s}{m_d + m_s} = \frac{M_{K^0} f_{K^0}}{M_{K^0} f_{K^0}} \left( \frac{Z_{K^0}}{Z_{\pi^\pm}} \right)^{1/2}. \hfill (69)$$

Inserting the $Z_i$ defined in (61) with $\omega_{mv}$ as determined from the $\eta - \eta'$ sector this leads to values for the quark mass ratios which are in excellent agreement with the ones obtained in chiral perturbation theory. We take this as strong support for our treatment of the linear meson model and, in particular, for the inclusion of the non–renormalizable terms in (58) and (59). Taking these terms into account appears to be crucial for obtaining results for meson masses and decay constants as well as current quark ratios in agreement with experiment or well established results from other methods. These terms are the main difference between our phenomenological treatment of the linear meson model and that of previous works \[47\].

From our phenomenological study we conclude that at least the couplings $m_g^2 = \text{tr} j/(2N\sigma_0)$, $\tau$, $\lambda_1$ and $\lambda_2$ should be included in the truncation (58) of the potential. Also the difference between $Z_h$ and $Z_m$ (which turns out to be $Z_h/Z_m \approx 0.4$ for $k \to 0$) needs to be incorporated. This is the minimal setting for which the meson masses have qualitatively the correct values for $k \to 0$. It seems likely that a reasonable estimate of $\sigma_0$ and $Z_m$ and therefore the flavor average of the decay constants, $(2f_{K^*} + f_{\pi^*})/3 \approx 2\sigma_0 Z_1^{-1/2}$, becomes already possible with this type of truncation. For a more detailed comparison with experiment the coupling $X_{\sigma}$ should also be taken into account. A computation of $\lambda_3$ would be helpful for an understanding of the mass splitting in the scalar $(0^{++})$ sector.

We conclude that it would be very desirable to gain additional information on the parameters of the linear meson model contained in (58) and (59). This would allow to establish relations between different scalar and pseudoscalar meson observables beyond those based on chiral symmetry. In particular, in the scalar sector which is not very well understood this would be quite interesting. The application of the exact RG to the effective action (58), (59), in fact, offers this prospect due to the infrared fixed point structure established in section 7. If it is indeed realized as indicated by the simplified analysis for two flavors outlined in this work, it should allow for a determination of many of the couplings and wave function renormalization constants introduced in (58) and (59). Work in this direction is in progress.
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