Asymptotics of a \(3F_2\) hypergeometric function with four large parameters

R. B. Paris

Division of Computing and Mathematics, Abertay University, Dundee DD1 1HG, UK

Abstract

We consider the asymptotic behaviour of the generalised hypergeometric function

\[
\begin{align*}
\, _{3}F_2\left( 1, \frac{1}{2}(1 + t)k, \frac{1}{2}(1 + t)k + \frac{1}{2}; x \right), & \quad 0 < x, t \leq 1
\end{align*}
\]

as the parameter \(k \to +\infty\). Numerical results illustrating the accuracy of the resulting expansion are given.

MSC: 33C05, 34E05, 41A60

Keywords: Hypergeometric function, asymptotic expansion, large parameters

1. Introduction

The following problem arising in two-variable moment theory [2] is the determination of the asymptotic behaviour of the generalised hypergeometric function

\[
S(x; t) := \, _{3}F_2\left( 1, \frac{1}{2}(1 + t)k, \frac{1}{2}(1 + t)k + \frac{1}{2}; x \right), \quad 0 < x, t \leq 1
\]

as the parameter \(k \to +\infty\). The parametric excess\(^1\) of this function equals \(\frac{1}{2}\) so that \(S(x; t)\) converges as \(x \to 1\). We also have the simple evaluation

\[
S(1; 0) = \, _{2}F_1\left( \frac{1}{2}k, \frac{1}{2}k + \frac{1}{2}; 1 \right) = \frac{\Gamma(k + 1)\Gamma(1)}{\Gamma\left(\frac{1}{2}k + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}k + 1\right)} = 2^k
\]

by the Gauss summation formula.

An integral representation for \(S(x; t)\) involving the modified Bessel function \(K_\nu(z)\) can be obtained from [9, p. 156] in the form

\[
S(x; t) = \frac{2}{\Gamma(\nu)} \int_0^\infty \frac{u^{(\nu - 1)/2}}{K_\nu(2\sqrt{u})} \, _{1}F_2\left( \frac{1}{2}(1 + t)k + \frac{1}{2}; 1 + tk, k + 1; xu \right) du,
\]

where \(\nu = \frac{1}{2}(1 + t)k\).

In this note we investigate the large-\(k\) behaviour of \(S(x; t)\). We first employ a contour integral representation for \(S(x; t)\) that involves the Gauss hypergeometric function with two large parameters. The known asymptotics of this latter function then lead to an expansion for \(S(x; t)\) also in terms of Gauss hypergeometric functions of a different form. The asymptotic expansion of

\(^1\)The parametric excess equals the difference between the sums of the denominator and numerator parameters.
these functions has recently been considered in [8]. In the second approach we make use of the
confluence principle as discussed in [3, pp. 56-57]. Numerical results are presented to demonstrate
the accuracy of the expansion obtained.

2. An expansion for $S(x; t)$

We employ the integral representation

$$S(x; t) = \frac{\Gamma(tk + 1)}{\Gamma(ak)} \frac{1}{2\pi i} \int_0^{(1+i)} u^{ak-1}(u-1)^{-bk} \, _2F_1\left(1, ak + \frac{1}{k}; xu\right) \, du,$$  \quad (2.1)

where we define

$$a = \frac{1}{2}(1 + t), \quad b = \frac{1}{2}(1 - t), \quad c = a(1 - a) = \frac{1}{4}(1 - t^2) \quad (0 < t < 1).$$  \quad (2.2)

The validity of this representation requires $b > 0$, which implies that $t$ must satisfy $0 < t < 1$.

The integration path is a closed loop that starts at the origin, encircles $t = 1$ in the positive sense
(excluding the point $t = 1/x$) and returns to the origin. The integral (2.1) can be established from
the representation over $[0, 1]$ for $3F_2(x)$ given in [5, (16.5.2)] extended into a contour integral by
means of [5, (5.12.10)].

In (2.1) the integration path can be arranged so that $|zu| < 1$ everywhere on the loop. For
$k \to +\infty$, we can then employ the expansion of the Gauss hypergeometric function appearing in
(2.1). This has two large parameters with $a < 1$; from [7, Section 3.1], this function is associated
with the Laplace integral

$$\int_0^1 f(\tau)e^{\psi(\tau)} d\tau, \quad \psi(\tau) := a \log \tau + (1 - a) \log(1 - \tau), \quad f(\tau) := \frac{\tau^{-1/2}(1 - \tau)^{-1/2}}{1 - z\tau}.$$

Expansion of this integral by Laplace’s method about the saddle point $\tau = a$ yields the result [7, Eq. (3.6)]

$$\_2F_1\left(1, \frac{ak + \frac{1}{2}}{k} ; z\right) \sim \Xi(a, k) \left(1 + \frac{c_2(a, z)}{k} + \frac{3c_4(a, z)}{4k^2} + \cdots\right) \quad (a < 1),$$  \quad (2.3)

where

$$\Xi(a, k) = \frac{\Gamma(k + 1)}{\Gamma(ak + \frac{1}{2})\Gamma((1 - a)k + \frac{1}{2})} \left(\frac{k}{2\pi}\right)^{-1/2} a^{ak}(1 - a)^{(1-a)k}.$$

The coefficients $c_2(a, z)$ and $c_4(a, z)$ are defined by

$$c_2(a, z) = -\frac{1}{\psi'^\prime}\left\{F_2 - \Psi_3 F_1 + \frac{5}{12}\Psi_3^2 - \frac{1}{2}\Psi_4\right\},$$

$$c_4(a, z) = \frac{1}{(\psi'^\prime)^2}\left\{\frac{1}{6}F_4 - \frac{5}{9}\Psi_3 F_3 + \frac{5}{12}\left(\frac{7}{3}\Psi_3^2 - \Psi_4\right) F_2 - \frac{35}{36}\left(\Psi_3^3 - \Psi_3\Psi_4 + \frac{5}{12}\Psi_5\right) F_1\right\}$$

$$+ \left(\frac{35}{36}\left(\frac{7}{3}\Psi_3^2 - \frac{5}{9}\Psi_4\right) \Psi_4 + \frac{5}{12}\Psi_3\Psi_5 - \frac{1}{36}\Psi_6\right),$$  \quad (2.4)

where, for brevity, we have defined

$$\Psi_k := \frac{\psi^{(k)}(a)}{\psi''(a)} \quad (k \geq 3), \quad F_k := \frac{f^{(k)}(a)}{f(a)} \quad (k \geq 1);$$  \quad (2.5)

see, for example, [1, p. 119], [4, p. 127] or [6, p. 13]. Substitution of the above forms of $\psi(\tau)$
and $f(\tau)$ into (2.4) and (2.5) yields after some laborious algebra (carried out with the aid of
Mathematica) the coefficient values given by

$$c_2(a, z) = \frac{2cz^2}{(1 - az)^2} + \frac{(1 - 2a)z}{1 - az} - \frac{1 + 2c}{12c}$$.
where the quantity \( c \) is specified in (2.2).

Application of Stirling’s formula for the gamma function shows that

\[
\Xi(a, k) = 1 + \frac{1 + 2c}{24ck} + \frac{(1 + 2c)^2}{1152c^2k^2} + O(k^{-3}).
\]

as \( k \to +\infty \). Then with \( z = xu \) in (2.6) and use of (2.7) we obtain

\[
S(x; t) \sim \frac{\Gamma(tk + 1)\Gamma(bk)}{\Gamma(ak)} \Xi(a, k) \frac{1}{2\pi i} \int_0^{(1+)} \frac{u^{ak-1}(u-1)^{-bk}}{1-axu} \left\{ 1 + \frac{c_2(a, xu)}{2k} + \frac{3c_4(a, xu)}{4k^2} + \ldots \right\} \, du,
\]

where

\[
c_4'(a, xu) = c_4(a, xu) + \frac{(1 + 2c)}{36c} c_2(a, xu).
\]

We now employ the result [5, (15.6.2)]

\[
\frac{1}{2\pi i} \int_0^{(1+)} \frac{u^{\beta-1}(u-1)^{\gamma-\beta-1}}{(1-uz)^\alpha} \, du = \frac{\Gamma(\beta)}{\Gamma(\gamma)\Gamma(1+\beta-\gamma)} {}_2F_1(\alpha; \beta; \gamma; z)
\]

when \( \Re(\beta) > 0 \) and \( \gamma - \beta \neq 1, 2, 3, \ldots \), to find that, for \( m = 0, 1, 2, \ldots \),

\[
\frac{\Gamma(tk + 1)\Gamma(bk)}{\Gamma(ak)} \frac{1}{2\pi i} \int_0^{(1+)} \frac{u^{ak+m-1}(u-1)^{-bk}}{(1-axu)^{m+1}} \, du = A_m \mathcal{F}_m,
\]

where

\[
A_m := \frac{(ak)_m}{(tk + 1)_m}, \quad \mathcal{F}_m := {}_2F_1\left( \frac{m + 1, ak + m}{tk + m + 1}; ax \right).
\]

The terms involving \((1 - axu)^{-1}\) in (2.8) combine to yield

\[
\left\{ \Xi(a, k) - \frac{1 + 2c}{24ck} - \frac{(1 + 2c)^2}{1152c^2k^2} \right\} \mathcal{F}_0 = \mathcal{F}_0\{1 + O(k^{-3})\} \quad (k \to +\infty)
\]

by (2.7).

Evaluation of the remaining terms then produces the following expansion

\[
S(x; t) \sim \mathcal{F}_0 + \frac{1}{k} \left\{ \frac{1}{2}(1 - 2a)x A_1 \mathcal{F}_1 + a(1 - a)x^2 A_2 \mathcal{F}_2 \right\}
\]

\[
+ \frac{1}{k^2} \left\{ \frac{1}{2} (2a - 1)xA_1 \mathcal{F}_1 + \frac{1}{4} (3 - 20c)x^2 A_2 \mathcal{F}_2 + \frac{7}{2}(1 - 2a)c x^3 A_3 \mathcal{F}_3 + 3c^2 x^4 A_4 \mathcal{F}_4 \right\} + \ldots
\]

as \( k \to +\infty \).

### 3. An alternative approach

We give an alternative derivation of the expansion (2.10) based on the confluence principle described in [3, pp. 56–57]. This approach is valid for \( 0 < t \leq 1 \), so that \( a \leq 1 \). Proceeding formally, we have the series representation

\[
S(x; t) = \sum_{r \geq 0} \frac{(1_r (ak)_r (a + \frac{1}{2})_r x^r}{(tk + 1)_r (k + 1)_r r!} = \sum_{r \geq 0} \frac{(1_r (ak)_r (ax)^r }{(tk + 1)_r r!} P_r,
\]
where
\[ P_r = \frac{(ak + \frac{1}{2})r}{(k+1)r} = a^{-r} \frac{1 + \frac{1}{2ak}(1 + \frac{3}{2ak}) \cdots (1 + \frac{3r-1}{2ak})}{(1 + \frac{1}{k})(1 + \frac{2}{k}) \cdots (1 + \frac{r}{k})} \]

It follows that, for \( k \to +\infty \),
\[
\log P_r = \frac{1}{2ak} \sum_{n=1}^{r} (2n-1) - \frac{1}{8a^2k^2} \sum_{n=1}^{r} (2n-1)^2 - \frac{1}{k} \sum_{n=1}^{r} n + \frac{1}{2k^2} \sum_{n=1}^{r} n^2 + O(k^{-3})
\]
\[
= \frac{r^2}{2ak} - \frac{r(4r^2 - 1)}{24a^2k^2} + \frac{r(r+1)}{2k} + O(k^{-3})
\]
\[
= \frac{r^2(1-a) - ar}{2ak} + \frac{1}{24a^2k^2} \left\{ (2a^2+1)r + 6a^2r^2 - 4(1-a)r^3 \right\} + O(k^{-3}),
\]

which upon exponentiation then yields
\[
P_r = 1 - ar - \frac{r^2(1-a)}{2ak} + \frac{1}{24a^2k^2} \left\{ (2a^2+1)r + 9a^2r^2 - 2(1-a)(5a+2)r^3 \right\} + 3(1-a)^2r^4 + O(k^{-3}).
\]

This expansion assumes that \( r^2/k \ll 1 \). But the summation index \( r \in [0, \infty) \); if the terms in the sum (3.1) are negligible for \( r \gg r_0 \), then if \( k \) is such that \( r_0^2/k \ll 1 \) we can formally neglect the tail \( r \gg r_0 \). substitute the expansion for \( P_r \) into (3.1) and evaluate the resulting series term by term.

We now employ the result valid for convergent series
\[
\sum_{r \geq 0} b_r r^m \chi^r = \Theta^m \sum_{r \geq 0} b_r \chi^r, \quad \Theta := \frac{d}{d\chi} \quad (m = 0, 1, 2, \ldots),
\]

where we identify the coefficients \( b_r \) with \( (ak)_r/(tk+1)_r \). Then substitution of the above expansion for \( P_r \) into (3.1) leads to
\[
S(x; t) \sim \mathcal{F}_0 + \frac{1}{2ak} \left\{ -a\Theta \mathcal{F}_0 + (1-a)\Theta^2 \mathcal{F}_0 \right\}
\]
\[
+ \frac{1}{24a^2k^2} \left\{ (2a^2+1)\Theta \mathcal{F}_0 + 9a^2\Theta^2 \mathcal{F}_0 - 2(1-a)(2+5a)\Theta^3 \mathcal{F}_0 + 3(1-a)^2 \Theta^4 \mathcal{F}_0 \right\} + \ldots,
\]

where
\[
\mathcal{F}_0 = \sum_{r \geq 0} \frac{(1)_r (ak)_r}{(tk+1)_r} \frac{\chi^r}{r!} = {}_2F_1 \left( \frac{1, ak}{tk+1}; \chi \right), \quad \chi := ax.
\]

Since
\[
\Theta \mathcal{F}_0 = \chi \mathcal{F}_0', \quad \Theta^2 \mathcal{F}_0 = \chi^2 \mathcal{F}_0'' + \chi \mathcal{F}_0', \quad \Theta^3 \mathcal{F}_0 = \chi^3 \mathcal{F}_0''' + 3\chi \mathcal{F}_0'' + \chi \mathcal{F}_0',
\]
\[
\Theta^4 \mathcal{F}_0 = \chi^4 \mathcal{F}_0^{(iv)} + 6\chi^3 \mathcal{F}_0''' + 7\chi^2 \mathcal{F}_0'' + \chi \mathcal{F}_0',
\]
we find
\[
S(x; t) \sim \mathcal{F}_0 + \frac{1}{2ak} \left\{ (1-2a)\chi \mathcal{F}_0' + (1-a)\chi^2 \mathcal{F}_0'' \right\}
\]
\[
+ \frac{1}{24a^2k^2} \left\{ 12a(2a-1)\chi \mathcal{F}_0 + 3(3-20c)\chi^2 \mathcal{F}_0'' + 14(1-a)(1-2a)\chi^3 \mathcal{F}_0''' + 3(1-a)^2 \chi^4 \mathcal{F}_0^{(iv)} \right\} + \ldots.
\]

From [5, (15.5.2)], the derivatives of \( \mathcal{F}_0 \) are given by
\[
\frac{d^n}{d\chi^m} \mathcal{F}_0 = \frac{d^n}{d\chi^m} {}_2F_1 \left( \frac{1, ak}{tk+1}; \chi \right) = m! A_m \mathcal{F}_m,
\]

where \( A_m \) and \( \mathcal{F}_m \) are defined in (2.9). This enables the expansion of \( S(x; t) \) to be written in the form
\[
S(x; t) \sim \mathcal{F}_0 + \frac{1}{k} \left\{ \frac{1}{2} (1-2a)x A_1 \mathcal{F}_1 + a(1-a)x^2 A_2 \mathcal{F}_2 \right\}
\]
\[
+ \frac{1}{k^2} \left\{ \frac{1}{2} (2a-1)x A_1 \mathcal{F}_1 + \frac{1}{4} (3-20c)x^2 A_2 \mathcal{F}_2 + \frac{7}{2} (1-2a)cx^3 A_3 \mathcal{F}_3 + 3c^2 x^4 A_4 \mathcal{F}_4 \right\} + \ldots, \quad (3.2)
\]
where we recall that $c = a(1 - a)$. This expansion agrees with that in (2.10) obtained from
the contour integral approach. We remark that its derivation has allowed us to consider $a \leq 1$
$(0 < t \leq 1)$ whereas the integral (2.1) requires $b > 0$ $(0 < t < 1)$.

4. Numerical results

In this section we expand the quantities $A_m$ appearing in (2.10) in inverse powers of the large
parameter $k$ to obtain a modified expansion for $S(x; t)$.

For $k \to +\infty$ (with $t$ bounded away from zero) we have

$$A_1 = \frac{a}{t} \left( 1 - \frac{1}{kt} + O(k^{-2}) \right), \quad A_2 = \frac{a^2}{t^2} \left( 1 - \frac{\alpha}{4k} + O(k^{-2}) \right)$$

with

$$\alpha = 4 \left( \frac{3}{t} - \frac{1}{a} \right) = \frac{4(1 + a)}{at},$$

and generally $A_m = (a/t)^m \{ 1 + O(k^{-1}) \}$. Then, upon expanding the quantities $A_m$ $(1 \leq m \leq 4)$
and noting that $2a - 1 = t$, we obtain from (2.10) or (3.2) the expansion in the following modified
form

$$S(x; t) \sim \mathcal{F}_0 - \frac{1}{k^2} \left\{ \frac{1}{2} tk X \mathcal{F}_1 - cX^2 \mathcal{F}_2 \right\}$$

$$+ \frac{1}{k^2} \left\{ ax \mathcal{F}_1 + \frac{1}{4} [3 - (20 + \alpha)c] X^2 \mathcal{F}_2 - \frac{7}{2} ctk X^3 \mathcal{F}_3 + 3c^2 X^4 \mathcal{F}_4 \right\} + \ldots, \quad (4.1)$$

where we have put $X := ax/t$. This expansion holds for $k \to +\infty$, provided $t$ is bounded away
from zero such that $kt \to +\infty$. In the case $t = 1$ (so that $a = 1, c = 0$), (4.1) reduces to

$$S(x; 1) \sim \mathcal{F}_0 - \frac{X \mathcal{F}_1}{2k} + \frac{1}{k^2} \left\{ X \mathcal{F}_1 + \frac{3}{4} X^2 \mathcal{F}_2 \right\} + \ldots \quad (k \to +\infty). \quad (4.2)$$

The leading term of the above expansions is given by

$$\mathcal{F}_0 = \mathcal{F}_1 \left( \frac{1, ak}{tk + 1}; ax \right).$$

Since the parametric excess associated with this function equals $-bk < 0$ the series converges if
$ax < 1$. Thus $\mathcal{F}_0$ converges for $x \leq 1$ if $t < 1$ and for $x < 1$ if $t = 1$. In Table 1 we present values of
the absolute relative error in the computation of $S(x; t)$ using the expansions (4.1) and (4.2) when
truncated at the term $O(k^{-M})$, $0 \leq M \leq 2$.

Table 1: Values of the absolute relative error in the computation of $S(x; t)$ from (4.1) and (4.2) as a function of
the truncation index $M$ and different values of $k, x$ and $t.$

| $M$ | $k = 100$ | $x = 0.50, t = 0.75$ | $x = 0.50, t = 1$ | $k = 200$ | $x = 0.50, t = 0.50$ | $k = 300$ | $x = 0.50, t = 0.50$ |
|-----|----------|---------------------|------------------|----------|---------------------|----------|------------------|
| 0   | $5.723 \times 10^{-3}$ | $9.481 \times 10^{-3}$ | $1.357 \times 10^{-1}$ | $9.638 \times 10^{-4}$ | $6.171 \times 10^{-4}$ | $1.286 \times 10^{-4}$ |
| 1   | $9.925 \times 10^{-5}$ | $3.288 \times 10^{-4}$ | $2.380 \times 10^{-4}$ | $4.238 \times 10^{-6}$ | $3.15 \times 10^{-4}$ | $6.018 \times 10^{-5}$ |
| 2   | $1.223 \times 10^{-6}$ | $1.315 \times 10^{-5}$ | $5.973 \times 10^{-6}$ | $1.825 \times 10^{-6}$ | $1.285 \times 10^{-6}$ | $1.285 \times 10^{-6}$ |

| $M$ | $k = 200$ | $x = 0.50, t = 0.75$ | $x = 0.50, t = 1$ | $k = 300$ | $x = 0.50, t = 0.50$ | $k = 400$ | $x = 0.50, t = 0.50$ |
|-----|----------|---------------------|------------------|----------|---------------------|----------|------------------|
| 0   | $2.919 \times 10^{-3}$ | $4.866 \times 10^{-3}$ | $1.073 \times 10^{-1}$ | $7.293 \times 10^{-4}$ | $1.150 \times 10^{-3}$ | $5.973 \times 10^{-5}$ |
| 1   | $2.458 \times 10^{-5}$ | $8.476 \times 10^{-5}$ | $6.018 \times 10^{-5}$ | $1.285 \times 10^{-6}$ | $1.285 \times 10^{-6}$ | $1.285 \times 10^{-6}$ |
| 2   | $1.897 \times 10^{-7}$ | $1.710 \times 10^{-6}$ | $1.285 \times 10^{-6}$ | $1.285 \times 10^{-6}$ | $1.285 \times 10^{-6}$ | $1.285 \times 10^{-6}$ |
If we write $\epsilon = a/t$, $\lambda = tk$ and $\chi = ax$, we have when $0 < t < 1$
\[
\mathcal{F}_0 = 2F_1 \left( 1, \epsilon \lambda ; 1 + \lambda ; \chi \right), \quad \epsilon > 1.
\] (4.3)

The expansion of the Gauss hypergeometric function $2F_1(1, \alpha + \epsilon \lambda; 1 + \lambda ; \chi)$ for $\epsilon > 0$ and $\lambda \to +\infty$ has been considered in [7], and more recently when $\epsilon > 1$ in [8]. The analysis of this function when $\epsilon > 1$ is complicated by the presence in its Laplace-type integral representation (where a multiplicative factor is omitted)
\[
\frac{1}{2\pi i} \int_0^{(1+)} \frac{x^{\alpha-1} (\tau - 1)^{\beta - a - 1}}{1 - \chi x} e^{-x(\tau - 1) - x \log \tau} d\tau,
\]
where $\phi(\tau) = (\epsilon - 1) \log(\tau - 1) - x \log \tau$
of a saddle point at $\tau = e$ and a simple pole at $\tau = 1/\chi$. These points coalesce when $\epsilon \chi = 1$; that is, since $\epsilon \chi = a^2 x/t$, when
\[
x = x^* = \frac{4t}{(1 + t)^2}.
\]

From [7, Eq. (3.13)], we have as $\lambda \to +\infty$ (when $\epsilon \chi < 1$)
\[
\mathcal{F}_0 \sim \frac{G(\lambda)}{\sqrt{2\pi}} \left( \frac{\epsilon}{\epsilon - 1} \right)^{-1/2} \frac{e^{-x\phi(\epsilon)}}{1 - \epsilon \chi} \sum_{k=0}^{\infty} \frac{c_{2k} \Gamma(k + \frac{1}{2})}{\lambda^{k+\frac{1}{2}} \Gamma(\frac{1}{2})} \left( \epsilon > 1, x < x^* \right),
\] (4.4)

where
\[
G(\lambda) := \frac{\Gamma(1 + \lambda) \Gamma((\epsilon - 1) \lambda)}{\Gamma(\epsilon \lambda)} \sim (2\pi \lambda)^{1/2} \left( \frac{\epsilon}{\epsilon - 1} \right)^{1/2} e^{x\phi(\epsilon)} \quad (\lambda \to +\infty)
\]
by application of Stirling’s formula. The coefficient $c_0 = 1$ with $c_2, c_4$ given in [7, Eqs. (3.4), (3.5)]. This shows that the leading large-$\lambda$ behaviour of $\mathcal{F}_0$ when $x < x^*$ is given by $(1 - \epsilon \chi)^{-1}$.

The expansion (4.4) breaks down as $x \to x^*$. From [8, Theorem 1] we have the uniform expansion (corresponding to $\alpha = 0$, $\beta = 1$)
\[
\mathcal{F}_0 \sim \frac{G(\lambda)}{2} \left\{ \chi e^{-x\phi(1/\chi)} \text{erfc} (\pm \lambda \frac{1}{\epsilon} p) + \frac{e^{-x\phi(\epsilon)}}{\epsilon \chi} \sum_{k=0}^{\infty} d_{2k} \frac{\Gamma(k + \frac{1}{2})}{\lambda^{k+\frac{1}{2}}} \right\},
\] (4.5)

where $\text{erfc}$ denotes the complementary error function and $p = (\phi(\epsilon) - \phi(1/\chi))/1/2 \geq 0$. A closed-form expression for the coefficients $d_{2k}$ is given in [8, Eq. (2.17)] and
\[
d_0 = \left( \frac{2(\epsilon - 1)}{\epsilon} \right)^{1/2} \frac{\chi}{1 - \epsilon \chi} + \frac{\chi}{p}.
\] (4.6)

The upper signs in (4.5) and (4.6) apply when $\epsilon \chi < 1$ ($x < x^*$) and the lower signs when $\epsilon \chi > 1$ ($x^* < x < 1$). These coefficients possess a removable singularity when the saddle and pole coincide, corresponding to $\epsilon \chi = 1$ ($x = x^*$) and $p = 0$; see [8, Section 3] for the expansion at coalescence and an explicit representation of the first three coefficients. In Table 2 we show values of the absolute relative error in the computation of $\mathcal{F}_0$ from (4.5) when $t = 1/3$, $k = 150$ (so that $x^* = 0.75$), corresponding to $\epsilon = 2$, $\chi = 2x/3$ and $\lambda = 50$, as a function of the truncation index $M$. The coefficients $d_{2k}$ for $k = 1, 2$ are taken from [8].

Table 2: Values of the absolute relative error in the computation of $\mathcal{F}_0$ from (4.5) as a function of the truncation index $M$ when $t = 1/3$ and $k = 150$.

| $M$ | $x = 0.45$ | $x = 0.72$ | $x = 0.78$ | $x = 0.90$ | $x = 1$ |
|-----|------------|------------|------------|------------|--------|
| 0   | 6.025 × 10^{-5} | 1.535 × 10^{-6} | 7.122 × 10^{-7} | 3.455 × 10^{-7} | 1.270 × 10^{-8} |
| 1   | 6.803 × 10^{-7} | 1.664 × 10^{-8} | 8.762 × 10^{-9} | 4.168 × 10^{-9} | 1.474 × 10^{-10} |
| 2   | 2.403 × 10^{-9} | 4.600 × 10^{-11} | 2.422 × 10^{-11} | 1.224 × 10^{-11} | 4.835 × 10^{-13} |

The expansion of the functions $\mathcal{F}_m$ with $m \geq 1$ can be obtained from that of $2F_1(1, \alpha + \epsilon \lambda; 1 + \lambda ; \chi)$, with $\alpha = m$, $\beta = m + 1$, by means of a recurrence relation; see [8, Section 4] for details.
References

[1] R.B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*, Academic Press, London 1973.

[2] J. Geronimo, Private communication (2018).

[3] Y.L. Luke, *The Special Functions and Their Approximations*, Vol.1, Academic Press, New York, 1969.

[4] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press, New York 1974. Reprinted A.K. Peters, Massachusetts 1997.

[5] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge 2010.

[6] R.B. Paris, *Hadamard Expansions and Hyperasymptotic Evaluation: An Extension of the Method of Steepest Descents*, Cambridge University Press, Cambridge 2011.

[7] R.B. Paris Asymptotics of the Gauss hypergeometric function with large parameters, I., J. Classical Anal. 2 (2013) 183–203.

[8] R.B. Paris Asymptotics of the Gauss hypergeometric function with large parameters, IV.: A uniform expansion. [arXiv: 1809.08794].

[9] L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge 1966.