THE ASCENT-PLATEAU STATISTICS ON STIRLING PERMUTATIONS

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Abstract. In this paper, several variants of the ascent-plateau statistic are introduced, including flag ascent-plateau, double ascent and descent-plateau. We first study the flag ascent-plateau statistic on Stirling permutations by using context-free grammars. We then present a unified refinement of the ascent polynomials and the ascent-plateau polynomials. In particular, by using Foata and Strehl’s group action, we prove two bistatistics over the set of Stirling permutations of order \( n \) are equidistributed.

Keywords: Stirling permutations; Context-free grammars; Ascents; Plateaus; Ascent-plateaus

1. Introduction

A Stirling permutation of order \( n \) is a permutation of the multiset \( \{1,1,2,2,\ldots,n,n\} \) such that for each \( i, 1 \leq i \leq n \), all entries between the two occurrences of \( i \) are larger than \( i \). Denote by \( Q_n \) the set of Stirling permutations of order \( n \). Let \( \sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in Q_n \). For \( 1 \leq i \leq 2n \), we say that an index \( i \) is a descent of \( \sigma \) if \( \sigma_i > \sigma_{i+1} \) or \( i = 2n \), and we say that an index \( i \) is an ascent of \( \sigma \) if \( \sigma_i < \sigma_{i+1} \) or \( i = 1 \). Hence the index \( i = 1 \) is always an ascent and \( i = 2n \) is always a descent. Moreover, a plateau of \( \sigma \) is an index \( i \) such that \( \sigma_i = \sigma_{i+1} \), where \( 1 \leq i \leq 2n - 1 \). Let \( \text{des}(\sigma) \), \( \text{asc}(\sigma) \) and \( \text{plat}(\sigma) \) be the numbers of descents, ascents and plateaus of \( \sigma \), respectively.

Stirling permutations were defined by Gessel and Stanley [8], and they proved that

\[
(1-x)^{2k+1} \sum_{n=0}^{\infty} \binom{n+k}{n} x^n = \sum_{\sigma \in Q_k} x^{\text{des}\sigma},
\]

where \( \binom{n}{k} \) is the Stirling number of the second kind, i.e., the number of ways to partition a set of \( n \) objects into \( k \) non-empty subsets. A classical result of Bóna [2] says that descents, ascents and plateaus have the same distribution over \( Q_n \), i.e.,

\[
\sum_{\sigma \in Q_n} x^{\text{des}\sigma} = \sum_{\sigma \in Q_n} x^{\text{asc}\sigma} = \sum_{\sigma \in Q_n} x^{\text{plat}\sigma}.
\]

This equidistributed result and associated multivariate polynomials have been extensively studied by Janson, Kuba, Panholzer, Haglund, Chen et al., see [3] [9] [10] [11] and references therein.

Recently, Ma and Toufik [15] introduced the definition of ascent-plateau statistic and presented a combinatorial interpretation of the \( 1/k \)-Eulerian polynomials. The purpose of this paper is to explore variants of the ascent-plateau statistic. In the following, we collect some definitions, notation and results that will be needed throughout this paper.

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Definition 1. An occurrence of an ascent-plateau of \( \sigma \in \mathcal{Q}_n \) is an index \( i \) such that \( \sigma_{i-1} < \sigma_i = \sigma_{i+1} \), where \( i \in \{2,3,\ldots,2n-1\} \). An occurrence of a left ascent-plateau is an index \( i \) such that \( \sigma_{i-1} < \sigma_i = \sigma_{i+1} \), where \( i \in \{1,2,\ldots,2n-1\} \) and \( \sigma_0 = 0 \).

Let \( \text{ap} (\sigma) \) and \( \text{lap} (\sigma) \) be the numbers of ascent-plateaus and left ascent-plateaus of \( \sigma \), respectively. For example, \( \text{ap} (442332115665) = 2 \) and \( \text{lap} (442332115665) = 3 \).

Define
\[
M_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap} (\sigma)}, \quad N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{lap} (\sigma)}.
\]

According to [15] Theorem 2, Theorem 3, we have
\[
M(x,t) = \sum_{n \geq 0} M_n(x) \frac{t^n}{n!} = \sqrt{\frac{x-1}{x-e^{2t(x-1)}},}
\]
\[
N(x,t) = \sum_{n \geq 0} N_n(x) \frac{t^n}{n!} = \sqrt{\frac{1-x}{1-xe^{2t(1-x)}.}}
\]

It should be noted that the polynomials \( M_n(x) \) and \( N_n(x) \) are also enumerative polynomials of perfect matchings. A perfect matching of \([2n]\) is a partition of \([2n]\) into \( n \) blocks of size 2. Let \( \mathcal{M}_{2n} \) be the set of perfect matchings of \([2n]\). Let \( \text{el} (M) \) (resp. \( \text{ol} (M) \)) be the number of blocks of \( M \in \mathcal{M}_{2n} \) with even (resp. odd) larger entries. According to [17], we have
\[
M_n(x) = \sum_{M \in \mathcal{M}_{2n}} x^{\text{el} (M)}, \quad N_n(x) = \sum_{M \in \mathcal{M}_{2n}} x^{\text{ol} (M)}.
\]

Let \( \#C \) denote the cardinality of a set \( C \). Let \( \mathfrak{S}_n \) denote the symmetric group of all permutations \( \pi = \pi(1)\pi(2)\ldots \pi(n) \) of \([n] \), where \([n] = \{1,2,\ldots, n\} \). A descent of \( \pi \) is an index \( i \in [n-1] \) such that \( \pi(i) > \pi(i+1) \). For \( \pi \in \mathfrak{S}_n \), let \( \text{des} (\pi) \) be the number of descents of \( \pi \). The classical Eulerian polynomials are defined by
\[
A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des} (\pi)}.
\]

The hyperoctahedral group \( B_n \) is the group of signed permutations of the set \( \pm [n] \) such that \( \pi(-i) = -\pi(i) \) for all \( i \), where \( \pm [n] = \{\pm 1, \pm 2, \ldots, \pm n\} \). Throughout this paper, we always identify a signed permutation \( \pi = \pi(1)\ldots \pi(n) \) with the word \( \pi(0)\pi(1)\ldots \pi(n) \), where \( \pi(0) = 0 \). For each \( \pi \in B_n \), we define
\[
\text{des}_A (\pi) = \# \{i \in [n-1] : \pi(i) > \pi(i+1)\},
\]
\[
\text{des}_B (\pi) = \# \{i \in \{0,1,2,\ldots, n-1\} : \pi(i) > \pi(i+1)\}.
\]

It is clear that
\[
\sum_{\pi \in B_n} x^{\text{des}_A (\pi)} = 2^n A_n(x).
\]

Following [1], the flag descents of \( \pi \in B_n \) is defined by
\[
\text{fdes} (\pi) = \begin{cases} 2\text{des}_A (\pi) + 1, & \text{if } \pi(1) < 0; \\ 2\text{des}_A (\pi), & \text{otherwise}. \end{cases}
\]
The Eulerian polynomial of type $B$ and the flag descent polynomial are respectively defined by

$$B_n(x) = \sum_{\pi \in B_n} x^{\text{des}_B(\pi)}, \quad F_n(x) = \sum_{\pi \in B_n} x^{\text{fdes}(\pi)}.$$ 

Very recently, we studied the following combinatorial expansions (see [18, Section 4]):

$$2^n x A_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) N_{n-k}(x), \quad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) M_{n-k}(x).$$

It is now well known that $F_n(x) = (1 + x)^n A_n(x)$ (see [1, Theorem 4.4]). This paper is motivated by exploring an expansion of $F_n(x)$ in terms of some enumerative polynomials of Stirling permutations.

This paper is organized as follows. In Section 2, we present a combinatorial expansion of $F_n(x)$. In Section 3, we study a multivariate enumerative polynomials of Stirling permutations. In particular, we consider Foata and Strehl’s group action on Stirling permutations.

2. **The flag descent polynomials and flag ascent-plateau polynomials**

Context-free grammar is a powerful tool to study exponential structures (see [5, 18] for instance). In this section, we first present a grammatical description of the flag descent polynomials by using grammatical labeling introduced by Chen and Fu [5]. And then, we study the flag ascent-plateau statistics over Stirling permutations.

2.1. **Context-free grammars.**

For an alphabet $A$, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in $A$. Following [4], a context-free grammar over $A$ is a function $G : A \rightarrow \mathbb{Q}[[A]]$ that replaces a letter in $A$ by a formal function over $A$. The formal derivative $D$ is a linear operator defined with respect to a context-free grammar $G$. More precisely, the derivative $D = D_G : \mathbb{Q}[[A]] \rightarrow \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial $u$ in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that $D$ is a derivation, and for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity.

Let us now recall a result on context-free grammars.

**Proposition 2** ([13, Theorem 10]). Let $A = \{x, y, z\}$ and

$$G = \{x \rightarrow xyz, y \rightarrow yz^2, z \rightarrow y^2z\}. \quad (3)$$

For $n \geq 0$, we have

$$D^n(xy) = xy \sum_{\pi \in B_n} y^{\text{fdes}(\pi)} z^{2n-\text{fdes}(\pi)}. \quad (4)$$
Moreover,
\[
D^n(y^2) = y^2 \sum_{\pi \in B_n} y^{2\text{des}_A(\pi)} z^{2n-2\text{des}_A(\pi)},
\]
\[
D^n(yz) = yz \sum_{\pi \in B_n} y^{2\text{des}_B(\pi)} z^{2n-2\text{des}_B(\pi)},
\]
\[
D^n(y) = y \sum_{\pi \in Q_n} y^{2\text{ap}(\pi)} z^{2n-2\text{ap}(\pi)},
\]
\[
D^n(z) = z \sum_{\pi \in Q_n} y^{2\text{ap}(\pi)} z^{2n-2\text{ap}(\pi)}.
\]

The grammatical labeling is illustrated in the following proof of (1). Let \( \pi \in B_n \). As usual, denote by \( \bar{i} \) the negative element \( -i \). We define an ascent (resp. a descent) of \( \pi \) to be a position \( i \in \{0, 1, 2, \ldots, n-1\} \) such that \( \pi(i) < \pi(i+1) \) (resp. \( \pi(i) > \pi(i+1) \)). Now we give a labeling of \( \pi \in B_n \) as follows:

(L1) If \( i \in [n-1] \) is an ascent, then put a superscript label \( z \) and a subscript label \( z \) right after \( \pi(i) \);
(L2) If \( i \in [n-1] \) is a descent, then put a superscript label \( y \) and a subscript label \( y \) right after \( \pi(i) \);
(L3) If \( \pi(1) > 0 \), then put a superscript label \( z \) and a subscript label \( x \) right after \( \pi(0) \);
(L4) If \( \pi(1) < 0 \), then put a superscript label \( x \) and a subscript label \( y \) right after \( \pi(0) \);
(L5) Put a superscript label \( y \) and a subscript label \( z \) at the end of \( \pi \).

Note that the weight of \( \pi \) is given by \( w(\pi) = xy^{\text{fdes}(\pi)+1}z^{\text{fasc}(\pi)+1} \).

Let
\[
F_n(i, j) = \{ \pi \in B_n : \text{fdes} = i, \text{fasc} = j \}.
\]

When \( n = 1 \), we have \( F_1(0, 1) = \{0_1^1, 1_0^1\} \) and \( F_1(1, 0) = \{0_0^1, 1_1^1\} \). Note that \( D(xy) = xy^2 + x^2y^2 \).

Thus the sum of weights of the elements of \( B_1 \) is given by \( D(xy) \).

Suppose we get all labeled permutations in \( F_n(i, j) \) for all \( i, j, k \), where \( n \geq 1 \). Let \( \pi' \in B_{n+1} \) be obtained from \( \pi \in F_n(i, j) \) by inserting the entry \( n + 1 \) or \( n + 1 \). We distinguish the following five cases:

(c1) Let \( i \in [n-1] \) be an ascent. If we insert \( n + 1 \) (resp. \( n + 1 \)) right after \( \pi(i) \), then \( \pi' \in F_{n+1}(i + 2, j) \), and the insertion of \( n + 1 \) (resp. \( n + 1 \)) corresponds to applying the rule \( z \to y^2z \) to the superscript (resp. subscript) label \( z \) associated with \( \pi(i) \).

c2) Let \( i \in [n-1] \) be a descent. If we insert \( n + 1 \) (resp. \( n + 1 \)) right after \( \pi(i) \), then \( \pi' \in F_{n+1}(i, j + 2) \), and the insertion of \( n + 1 \) (resp. \( n + 1 \)) corresponds to applying the rule \( y \to y^2z \) to the superscript (resp. subscript) label \( y \) associated with \( \pi(i) \).

c3) If we insert \( n + 1 \) (resp. \( n + 1 \)) at the end of \( \pi \), then \( \pi' \in F_{n+1}(i + j + 2) \) (resp. \( \pi' \in F_{n+1}(i + 2, j) \)), and the insertion of \( n + 1 \) (resp. \( n + 1 \)) corresponds to applying the rule \( y \to y^2z \) (resp. \( z \to y^2z \) to the label \( y \) (resp. \( z \) at the end of \( \pi \));

c4) If \( \pi(1) > 0 \) and we insert \( n + 1 \) (resp. \( n + 1 \)) immediately before \( \pi(1) \), then \( \pi' \in F_{n+1}(i + 2, j) \) (resp. \( \pi' \in F_{n+1}(i + 1, j + 1) \)), and the insertion of \( n + 1 \) (resp. \( n + 1 \)}
corresponds to applying the rule $z \to y^2z$ (resp. $x \to xyz$) to the label $z$ (resp. $x$) right after $\pi(0)$;

(c5) If $\pi(1) < 0$ and we insert $n + 1$ (resp. $\overline{n} + 1$) immediately before $\pi(1)$, then $\pi' \in F_{n+1}(i + 1, j + 1)$ (resp. $\pi' \in F_{n+1}(i, j + 2)$), and the insertion of $n + 1$ (resp. $\overline{n} + 1$) corresponds to applying the rule $x \to xyz$ (resp. $y \to yz^2$) to the label $x$ (resp. $y$) right after $\pi(0)$.

In general, the insertion of $n + 1$ (resp. $\overline{n} + 1$) into $\pi$ corresponds to the action of the formal derivative $D$ on a superscript label (resp. subscript label). By induction, we get a grammatical proof of (3).

Example 3. For example, let $\pi = 043152$. Then $\pi$ can be generated as follows:

\begin{align*}
0^x z_1^y &\Rightarrow 0^x z_1^y z_2^y; \\
0^x z_1^y z_2^y &\Rightarrow 0^x y z_1^y z_2^y; \\
0^x y z_1^y z_2^y &\Rightarrow 0^x y^3 z_1^y z_2^y; \\
0^x y^3 z_1^y z_2^y &\Rightarrow 0^x y^3 z_1^y z_5^y z_2^y.
\end{align*}

2.2. The flag ascent-plateau statistic.

Definition 4. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in Q_n$. The number of flag ascent-plateau of $\sigma$ is defined by

$$f_{ap}(\sigma) = \begin{cases} 
2ap(\sigma) + 1, & \text{if } \sigma_1 = \sigma_2; \\
2ap(\sigma), & \text{otherwise}.
\end{cases}$$

We can now present the first main result of this paper.

Theorem 5. Let $D$ be the formal derivative with respect to the grammar (3). For $n \geq 1$, we have

$$D^n(x) = x \sum_{\sigma \in Q_n} y^{f_{ap}(\sigma)} z^{2n - f_{ap}(\sigma)}. \quad (5)$$

Therefore,

$$\sum_{\pi \in B_n} x^{\text{fdes}(\pi)} = \sum_{k=0}^{n} \binom{n}{k} \sum_{\sigma \in Q_k} x^{f_{ap}(\sigma)} \sum_{\sigma' \in Q_{n-k}} x^{2ap(\sigma)}. \quad (6)$$

Proof. We first introduce a grammatical labeling of $\sigma \in Q_n$ as follows:

(L1) If $i \in \{2, 3, \ldots, 2n - 1\}$ is an ascent-plateau, then put a superscript label $y$ immediately before $\sigma_i$ and a superscript label $y$ right after $\sigma_i$;

(L2) If $\sigma_1 = \sigma_2$, then put a superscript label $y$ immediately before $\sigma_1$ and a superscript $x$ right after $\sigma_1$;

(L3) If $\sigma_1 < \sigma_2$, then put a superscript label $x$ immediately before $\sigma_1$;

(L4) The rest of positions in $\sigma$ are labeled by a superscript label $z$.

Note that the weight of $\sigma$ is given by

$$w(\sigma) = xy^{f_{ap}(\sigma)} z^{2n - f_{ap}(\sigma)}.$$
For example, The labeling of 122331455 and 661223314554 are respectively given as follows:

\[ x\bar{1}y\bar{2}\bar{y}2\bar{y}3\bar{y}^23\bar{y}^23\bar{y}^31\bar{y}^34\bar{y}5\bar{y}^53\bar{y}^44\bar{y}5\bar{y}5\bar{y}^53\bar{y}^44\bar{y}5\bar{y}5\bar{y}^54\bar{y}. \]

We then prove by induction. Let \( S_n(i) = \{ \sigma \in Q_n : \text{fap}(\sigma) = i \} \). For \( n = 1 \), we have \( S_1(1) = \{ y1^21^2 \} \). For \( n = 2 \), the elements of \( Q_2 \) are labeled as follows:

\[ S_2(1) = \{ y\bar{2}1^21^21^2 \}, \quad S_2(2) = \{ x\bar{1}1^21^21^2 \}, \quad S_2(3) = \{ y1^21^2y \} . \]

Note that \( D(x) = xyz \) and \( D^2(x) = xyz(y^2 + yz + z^2) \). Hence the result holds for \( n = 1, 2 \).

Suppose we get all labeled Stirling permutations of \( S_n(i) \) for all \( n \), where \( n \geq 2 \). Let \( \sigma' \in Q_{n+1} \) be obtained from \( \sigma \in S_n(i) \) by inserting the pair \((n + 1)(n + 1)\) into \( \sigma \). We distinguish the following three cases:

1. If \( \sigma_1 = \sigma_2 \) and the pair \((n + 1)(n + 1)\) is inserted at the front of \( \sigma \), then the change of labeling is illustrated as follows:

\[ y\sigma_1^y \sigma_2 \cdots \mapsto y(n + 1)^y(n + 1)^z \sigma_1^z \sigma_2 \cdots . \]

In this case, the insertion corresponds to the rule \( y \mapsto yz^2 \) and \( \sigma' \in S_{n+1}(i) \);

2. If \( \sigma_1 < \sigma_2 \) and the pair \((n + 1)(n + 1)\) is inserted at the front of \( \sigma \), then the change of labeling is illustrated as follows:

\[ x\sigma_1 \cdots \mapsto y(n + 1)^y(n + 1)^z \sigma_1 \cdots . \]

In this case, the insertion corresponds to the rule \( x \mapsto xyz \) and \( \sigma' \in S_{n+1}(i + 1) \);

3. If \( i \) is an ascent plateau of \( \sigma \), and the pair \((n + 1)(n + 1)\) is inserted immediately before or right after \( \sigma_i \), then the change of labeling are illustrated as follows:

\[ \cdots \sigma_{i-1}^y \sigma_i \sigma_{i+1} \cdots \mapsto \cdots \sigma_{i-1}^y(n + 1)^y(n + 1)^z \sigma_i^z \sigma_{i+1} \cdots , \]

\[ \cdots \sigma_{i-1}^y \sigma_i \sigma_{i+1} \cdots \mapsto \cdots \sigma_{i-1}^y(n + 1)^y(n + 1)^z \sigma_{i+1} \cdots . \]

In this case, the insertion corresponds to the rule \( y \mapsto yz^2 \) and \( \sigma' \in S_{n+1}(i) \);

4. If the pair \((n + 1)(n + 1)\) is inserted to a position with the label \( z \), then the change of labeling are illustrated as follows:

\[ \cdots \sigma_i^z \cdots \mapsto \cdots \sigma_i^y(n + 1)^y(n + 1)^z \cdots . \]

In this case, the insertion corresponds to the rule \( z \mapsto y^2z \) and \( \sigma' \in S_{n+1}(i + 2) \).

It is routine to check that each element of \( Q_{n+1} \) can be obtained exactly once. By induction, we present a constructive proof of \([\text{5}]\). Using the Leibniz’s formula, we have \( D^y(xy) = \sum_{k=0}^{n} D^k(x)D^{n-k}(y) \). Combining \([\text{5}]\) and Proposition \([\text{2}]\) we get the desired formula \([\text{6}]\). □

Let

\[ T_n(x) = \sum_{\sigma \in Q_n} x^{\text{fap}(\sigma)} = \sum_{k \geq 1} T(n, k)x^k. \]

From the proof of \([\text{5}]\), we see that the numbers \( T(n, k) \) satisfy the recurrence relation

\[ T(n + 1, k) = kT(n, k) + T(n, k - 1) + (2n - k + 2)T(n, k - 2). \]
with the initial conditions $T(0, 0) = 1$, $T(1, 1) = 1$ and $T(1, k) = 0$ for $k \neq 1$. It should be noted that $T(n, k)$ is also the number of dual Stirling permutations of order $n$ with $k$ alternating runs (see [16]). Recall that (see [20, A008292]):

$$A(x, t) = \sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{x - 1}{x - e^{t(x-1)}}.$$  

Hence

$$F(x, t) = \frac{x - 1}{x - e^{t(x-1)}}.$$  

Let $T(x, t) = \sum_{n \geq 0} T_n(x) \frac{t^n}{n!}$. Write the formula (6) as follows:

$$F_n(x) = \sum_{k=0}^{n} \binom{n}{k} T_k(x) M_{n-k}(x^2).$$  

Thus, $F(x, t) = T(x, t) M(x^2, t)$. Combining (1), we get

$$T(x, t) = \frac{F(x, t)}{M(x^2, t)} = \frac{x - 1}{x - e^{t(x^2-1)}} \sqrt{\frac{x^2 - e^{2t(x^2-1)}}{x^2 - 1}}. \tag{7}$$

Combining (2) and (7), we have

$$T(x, t) N(x^2, t) = 1 - x + x A(x, t(1 + x)).$$

Therefore, a dual formula of (6) is given as follows:

$$\sum_{\pi \in \mathcal{B}_n} x^{\text{des} (\pi) + 1} = \sum_{k=0}^{n} \binom{n}{k} \sum_{\sigma \in \mathcal{Q}_k} x^{\text{fap} (\sigma)} \sum_{\sigma \in \mathcal{Q}_{n-k}} x^{2 \text{lap} (\sigma)},$$

for $n \geq 1$.

Let $\delta_{i,j}$ be the Kronecker delta, i.e., $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. It is not hard to verify that $T(x, t) T(-x, t) = 1$. In other words,

$$\sum_{k=0}^{n} \binom{n}{k} T_k(x) T_{n-k}(-x) = \delta_{0,n}.$$

3. Multivariate polynomials over Stirling polynomials

Let

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc} (\sigma)}.$$  

The polynomials $C_n(x)$ and $N_n(x)$ respectively satisfy the following recurrence relation

$$C_{n+1}(x) = (2n + 1)x C_n(x) + x(1 - x) C'_n(x),$$

$$N_{n+1}(x) = (2n + 1)x N_n(x) + 2x(1 - x) N'_n(x),$$

with the initial conditions $C_0(x) = N_0(x) = 1$ (see [3, 8, 14] for instance). In this section, we shall present a unified refinement of the polynomials $C_n(x)$ and $N_n(x)$.

In the sequel, we always assume that Stirling permutations are prepended by 0. That is, we identify an $n$-Stirling permutation $\sigma_1 \sigma_2 \cdots \sigma_{2n}$ with the word $\sigma_0 \sigma_1 \sigma_2 \cdots \sigma_{2n}$, where $\sigma_0 = 0$.  


3.1. A grammatical labeling of Stirling permutations.

**Definition 6.** Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in Q_n$. For $1 \leq i \leq 2n$, a double ascent of $\sigma$ is an index $i$ such that $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, a descent-plateau of $\sigma$ is an index $i$ such that $\sigma_{i-1} > \sigma_i = \sigma_{i+1}$.

Let $\text{dasc}(\sigma)$ and $\text{dp}(\sigma)$ denote the numbers of double ascents and descent-plateaus of $\sigma$, respectively. For example, $\text{dasc}(244332115665) = 2$ and $\text{dp}(244332115665) = 2$. It is clear that

$$\text{asc}(\sigma) = \text{lap}(\sigma) + \text{dasc}(\sigma), \text{plat}(\sigma) = \text{lap}(\sigma) + \text{dp}(\sigma).$$

Define

$$P_n(x, y, z) = \sum_{\sigma \in Q_n} x^{\text{lap}(\sigma)} y^{\text{dasc}(\sigma)} z^{\text{dp}(\sigma)} = \sum_{i,j,k} P_n(i, j, k) x^i y^j z^k,$$

where $1 \leq i \leq n, 0 \leq j \leq n - 1, 0 \leq k \leq n - 1$. In particular,

$$P_n(x, x, 1) = P_n(x, 1, x) = C_n(x), \quad P_n(x, 1, 1) = N_n(x).$$

The first few of the polynomials $P_n(x, y, z)$ are given as follows:

$$P_1(x, y, z) = x,$$

$$P_2(x, y, z) = xy + xz + x^2,$$

$$P_3(x, y, z) = x(y^2 + z^2) + 4x^2(y + z) + 2xyz + 2x^2 + x^3.$$

Now we present the second main result of this paper.

**Theorem 7.** Let $A = \{x, y, z, p, q\}$ and

$$G = \{x \rightarrow xzq, y \rightarrow yzp, z \rightarrow xyz, p \rightarrow xyz, q \rightarrow xyz\}.$$  

Then

$$D^n(z) = z \sum_{i,j,k} P_n(i, j, k)(xy)^i q^j p^k z^{2n-2i-j-k},$$

where $1 \leq i \leq n, 0 \leq j \leq n - 1, 0 \leq k \leq n - 1$ and $2i + j + k \leq 2n$. Set $P_n = P_n(x, y, z)$. Then the polynomials $P_n(x, y, z)$ satisfy the recurrence relation

$$P_{n+1} = (2n + 1)x P_n + (xy + xz - 2x^2) \frac{\partial}{\partial x} P_n + x(1 - y) \frac{\partial}{\partial y} P_n + x(1 - z) \frac{\partial}{\partial z} P_n,$$

with the initial condition $P_0(x, y, z) = 1$.

**Proof.** Now we give a labeling of $\sigma \in Q_n$ as follows:

(L1) If $i$ is a left ascent-plateau, then put a superscript label $y$ immediately before $\sigma_i$ and a superscript label $x$ right after $\sigma_i$;

(L2) If $i$ is a double ascent, then put a superscript label $q$ immediately before $\sigma_i$;

(L3) If $i$ is a descent-plateau, then put a superscript label $p$ right after $\sigma_i$;

(L4) The rest positions in $\sigma$ are labeled by a superscript label $z$. 

The weight of $\sigma$ is defined by

$$w(\sigma) = z(xy)^{\text{lap}(\sigma)}q^{\text{dasc}(\sigma)}p^{\text{dp}(\sigma)}z^{2n-2\text{lap}(\sigma)-\text{dasc}(\sigma)-\text{dp}(\sigma)}.$$  

For example, the labeling of $552442998813316776$ is as follows:

$$y^5x^5y^4x^4y^3g^2y^2g^2y^2x^3y^3x^2y^2x^2y^2x^2y.$$  

We proceed by induction on $n$. Note that $Q_1 = \{y^1x^1z\}$ and $Q_2 = \{y^1x^1y^2z^2, y^1y^2x^2z^1, y^2x^2z^1x\}$. Thus the weight of $y^1x^1z$ is given by $D(z)$ and the sum of weights of elements in $Q_2$ is given by $D^2(z)$, since $D(z) = x y z$ and $D^2(x) = z(x y q z + x y p z + x^2 y^2)$.

Assume that the result holds for $n = m - 1$, where $m \geq 3$. Let $\sigma$ be an element counted by $P_{m-1}(i, j, k)$, and let $\sigma'$ be an element of $Q_m$ obtained by inserting the pair $mm$ into $\sigma$. We distinguish the following five cases:

1. If the pair $mm$ is inserted at a position with label $x$, then the change of labeling is illustrated as follows:

   $$\cdots \sigma_i^y \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^y m^x m^z \sigma_{i+1} \cdots.$$  

   In this case, the insertion corresponds to the rule $x \mapsto x q$ and produces $i$ permutations in $Q_m$ with $i$ left ascent-plateaus, $j + 1$ double ascents and $k$ descent-plateaus;

2. If the pair $mm$ is inserted at a position with label $y$, then the change of labeling is illustrated as follows:

   $$\cdots \sigma_i^y \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^y m^x m^z \sigma_i^p \sigma_{i+1} \cdots.$$  

   In this case, the insertion corresponds to the rule $y \mapsto y p$ and produces $i$ permutations in $Q_m$ with $i$ left ascent-plateaus, $j$ double ascents and $k + 1$ descent-plateaus;

3. If the pair $mm$ is inserted at a position with label $z$, then the change of labeling is illustrated as follows:

   $$\cdots \sigma_i^z \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^y m^x m^z \sigma_{i+1} \cdots.$$  

   In this case, the insertion corresponds to the rule $z \mapsto x y z$ and produces $2m-2-2i-j-k$ permutations in $Q_m$ with $i + 1$ left ascent-plateaus, $j$ double ascents and $k$ descent-plateaus;

4. If the pair $mm$ is inserted at a position with label $q$, then the change of labeling is illustrated as follows:

   $$\cdots \sigma_i^q \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^y m^x m^z \sigma_{i+1} \cdots.$$  

   In this case, the insertion corresponds to the rule $q \mapsto x y z$ and produces $j$ permutations in $Q_m$ with $i + 1$ left ascent-plateaus, $j - 1$ double ascents and $k$ descent-plateaus;

5. If the pair $mm$ is inserted at a position with label $p$, then the change of labeling is illustrated as follows:

   $$\cdots \sigma_i^p \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^y m^x m^z \sigma_{i+1} \cdots.$$
In this case, the insertion corresponds to the rule $p \mapsto xyz$ and produces $k$ permutations in $Q_m$ with $i+1$ left ascent-plateaus, $j$ double ascents and $k-1$ descent-plateaus.

By induction, we see that grammar (1) generates all of the permutations in $Q_m$.

Combining the above five cases, we see that

$$P_{n+1}(i,j,k) = iP_n(i,j-1,k) + iP_n(i,j,k-1) + (j+1)P_n(i-1,j+1,k) + (k+1)P_n(i-1,j,k+1) + (2n+3-2i-j-k)P_n(i-1,j,k).$$

Multiplying both sides of the above recurrence relation by $x^iy^jz^k$ for all $i, j, k$, we get (10) \( \Box \)

3.2. **Equidistributed statistics.**

Let $i \in \{2n\}$ and let $\sigma = \sigma_1 \sigma_2 \ldots \sigma_{2n} \in Q_n$. We define the action $\varphi_i$ as follows:

- If $i$ is a double ascent, then $\varphi_i(\sigma)$ is obtained by moving $\sigma_i$ to the right of the second $\sigma_i$, which forms a new plateau $\sigma_i \sigma_i$;
- If $i$ is a descent-plateau, then $\varphi_i(\sigma)$ is obtained by moving $\sigma_i$ to the right of $\sigma_k$, where $k = \max\{j \in \{0, 1, 2, \ldots, i-1\} : \sigma_j < \sigma_i\}$.

For instance, if $\sigma = 2447887332115665$, then

$$\varphi_1(\sigma) = 44478873322115665, \quad \varphi_4(\sigma) = 2448877332115665,$$

and $\varphi_9(\varphi_1(\sigma)) = \varphi_6(\varphi_4(\sigma)) = \sigma$. In recent years, the Foata and Strehl’s group action has been extensively studied (see [3] [12] for instance). We define the Foata-Strehl action on Stirling permutations by

$$\varphi'_i(\sigma) = \begin{cases} \varphi_i(\sigma), & \text{if } i \text{ is a double ascent or descent-plateau;} \\ \sigma, & \text{otherwise.} \end{cases}$$

It is clear that the $\varphi'_i$'s are involutions and that they commute. Hence, for any subset $S \subseteq \{2n\}$, we may define the function $\varphi'_S : Q_n \mapsto Q_n$ by $\varphi'_S(\sigma) = \prod_{i \in S} \varphi'_i(\sigma)$. Hence the group $\mathbb{Z}_2^{2n}$ acts on $Q_n$ via the function $\varphi'_S$, where $S \subseteq \{2n\}$.

The third main result of this paper is given as follows, which is implied by (10).

**Theorem 8.** For any $n \geq 1$, we have

$$P_n(x, y, z) = P_n(x, z, y). \tag{11}$$

Furthermore,

$$\sum_{\sigma \in Q_n} x^{\text{Dasc} (\sigma)} y^{\text{asc} (\sigma)} = \sum_{\sigma \in Q_n} x^{\text{Dasc} (\sigma)} y^{\text{plat} (\sigma)}. \tag{12}$$

**Proof.** For any $\sigma \in Q_n$, we define

$$\text{Dasc} (\sigma) = \{i \in \{2n\} : \sigma_{i-1} < \sigma_i < \sigma_{i+1}\},$$

$$\text{DP} (\sigma) = \{i \in \{2n\} : \sigma_{i-1} > \sigma_i = \sigma_{i+1}\},$$

$$\text{LAP} (\sigma) = \{i \in \{2n\} : \sigma_{i-1} < \sigma_i = \sigma_{i+1}\}.$$

Let $S = S(\sigma) = \text{Dasc} (\sigma) \cup \text{DP} (\sigma)$. Note that

$$\text{Dasc} (\varphi'_S(\sigma)) = \text{DP} (\sigma),\quad \text{DP} (\varphi'_S(\sigma)) = \text{Dasc} (\sigma) \text{ and } \text{LAP} (\varphi'_S(\sigma)) = \text{LAP} (\sigma).$$
Therefore,
\[ P_n(x, y, z) = \sum_{\sigma \in Q_n} x^{\text{lap}(\sigma)} y^{\text{dasc}(\sigma)} z^{\text{dp}(\sigma)} \]
\[ = \sum_{\sigma' \in Q_n} x^{\text{lap}(\varphi'_S(\sigma))} y^{\text{dasc}(\varphi'_S(\sigma))} z^{\text{dp}(\varphi'_S(\sigma))} \]
\[ = \sum_{\sigma \in Q_n} x^{\text{lap}(\sigma)} z^{\text{dasc}(\sigma)} y^{\text{dp}(\sigma)} \]
\[ = P_n(x, z, y). \]

Combining (8) and (11), we see that \( P_n(xy, y, 1) = P_n(xy, 1, y) \). This completes the proof. □

**Theorem 9.** For \( n \geq 1 \), we have
\[ \sum_{\sigma \in Q_n} x^{\text{lap}(\sigma)} y^{\text{dasc}(\sigma)} z^{\text{dp}(\sigma)} = \sum_{1 \leq i \leq n, 0 \leq j \leq n-1} \gamma_{n,i,j} x^i(y+z)^j, \]
where
\[ \gamma_{n,i,j} = \#\{ \sigma \in Q_n : \text{lap}(\sigma) = i, \text{dasc}(\sigma) = j, \text{dp}(\sigma) = 0 \}. \]

**Proof.** Define
\[ \text{NDP}_{n,i,j} = \{ \sigma \in Q_n : \text{lap}(\sigma) = i, \text{dasc}(\sigma) = j, \text{dp}(\sigma) = 0 \}. \]

For any \( \sigma \in \text{NDP}_{n,i,j} \), let
\[ [\sigma] = \{ \varphi'_S(\sigma) | S \subseteq \text{Dasc}(\sigma) \}. \]

For any \( \sigma' \in [\sigma] \), suppose that \( \sigma' = \varphi'_S(\sigma) \) for some \( S \subseteq \text{Dasc}(\sigma) \). Then
\[ \text{lap}(\sigma') = \text{lap}(\sigma), \text{dasc}(\sigma') = \text{dasc}(\sigma) - |S| \text{ and dp}(\sigma') = |S|. \]

Moreover, \( \{[\sigma] | \sigma \in \text{NDP}_{n,i,j} \} \) form a partition of \( Q_n \). Hence,
\[ \sum_{\sigma \in Q_n} x^{\text{lap}(\sigma)} y^{\text{dasc}(\sigma)} z^{\text{dp}(\sigma)} = \sum_{\sigma \in \text{NDP}_{n,i,j}} \sum_{\sigma' \in [\sigma]} x^{\text{lap}(\sigma')} y^{\text{dasc}(\sigma')} z^{\text{dp}(\sigma')} \]
\[ = \sum_{\sigma \in \text{NDP}_{n,i,j}} \sum_{S \subseteq \text{Dasc}(\sigma)} x^{\text{lap}(\varphi'_S(\sigma))} y^{\text{dasc}(\varphi'_S(\sigma))} z^{\text{dp}(\varphi'_S(\sigma))} \]
\[ = \sum_{\sigma \in \text{NDP}_{n,i,j}} \sum_{S \subseteq \text{Dasc}(\sigma)} x^{\text{lap}(\sigma)} y^{\text{dasc}(\sigma) - |S|} z^{|S|} \]
\[ = \sum_{\sigma \in \text{NDP}_{n,i,j}} x^{\text{lap}(\sigma)} \sum_{S \subseteq \text{Dasc}(\sigma)} y^{\text{dasc}(\sigma) - |S|} z^{|S|} \]
\[ = \sum_{\sigma \in \text{NDP}_{n,i,j}} x^{\text{lap}(\sigma)} (y+z)^{|S|} z^{|S|} \]
\[ = \sum_{i,j} \gamma_{n,i,j} x^i(y+z)^j. \]

□
Taking \( y = z = 1 \) in Theorem 9 we have
\[
N_n(x) = \sum_{\sigma \in Q_n} x^{\text{lap} (\sigma)} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-1} 2^j \gamma_{n,i,j} \right) x^i.
\]

Let \( N_n(x) = \sum_{k=1}^{n} N(n,k)x^k \). According to [14, Eq. (24)],
\[
N_n(x) = \sum_{k=1}^{n} 2^{n-2k} \cdot \binom{2k}{k} \frac{n}{k} \binom{n}{j} x^k (1-x)^{n-k}.
\]

Thus, for \( n \geq 1 \), we have
\[
\sum_{j=0}^{n-1} 2^j \gamma_{n,i,j} = \sum_{j=1}^{i} (-1)^{i-j} 2^{n-2j} \cdot \binom{2j}{j} \frac{n-j}{j} \binom{j}{j} \binom{n}{j}.
\]

**Theorem 10.** Let \( A = \{u,v,w\} \) and \( G = \{u \rightarrow uv, v \rightarrow 2uw, w \rightarrow uw\} \). Then
\[
D^n(w) = \sum_{1 \leq i \leq n, 0 \leq j \leq n-1} \gamma_{n,i,j} u^i v^j w^{2n+1-2i-j}.
\]

Furthermore, the numbers \( \gamma_{n,i,j} \) satisfy the recurrence relation
\[
\gamma_{n+1,i,j} = i \gamma_{n,i,j-1} + 2(j+1) \gamma_{n,i,j+1} + (2n+3-2i-j) \gamma_{n,i-1,j},
\]
with the initial conditions \( \gamma_{1,1,0} = 1 \) and \( \gamma_{1,i,j} = 0 \) for \( i > 1 \) and \( j \geq 0 \).

**Proof.** From the grammar [9], we see that
\[
D(xy) = xyz(p+q),
\]
\[
D(p+q) = 2xyz,
\]
\[
D(z) = xyz.
\]

Set \( u = xy, v = p+q \) and \( w = z \). Then \( D(u) = uvw, D(v) = 2uw \) and \( D(w) = uw \). Combining Theorem 4 and Theorem 9, we get [13]. Since \( D^{n+1}(w) = D(D^n(w)) \), we obtain [14].

\[
D^{n+1}(w) = D \left( \sum_{i,j} \gamma_{n,i,j} u^i v^j w^{2n+1-2i-j} \right)
\]
\[
= \sum_{i,j} i \gamma_{n,i,j} u^{i+1} v^{j+1} w^{2n+2-2i-j} + 2 \sum_{i,j} j \gamma_{n,i,j} u^{i+1} v^{j-1} w^{2n+2-2i-j} +
\]
\[
\sum_{i,j} (2n+1-2i-j) \gamma_{n,i,j} u^{i+1} v^{j} w^{2n+1-2i-j}.
\]

Equating the coefficients of \( u^i v^j w^{2n+1-2i-j} \) on both sides of the above equation, we obtain [14]. \( \square \)

Let \( G_n(x,y) = \sum_{i,j} \gamma_{n,i,j} x^i y^j \). Multiplying both sides of the recurrence relation [14] by \( x^i y^j \) for all \( i,j \), we get that
\[
G_{n+1}(x,y) = (2n+1)xG_n(x,y) + (xy-2x^2) \frac{\partial}{\partial x} G_n(x,y) + (2x-xy) \frac{\partial}{\partial y} G_n(x,y).
\]

(15)
The first few of the polynomials $G_n(x, y)$ are given as follows:

\[ G_0(x, y) = 1, \quad G_1(x, y) = x, \quad G_2(x, y) = xy + x^2, \quad G_3(x, y) = xy^2 + 4x^2y + 2x^2 + x^3. \]

3.3. Connection with Eulerian numbers.

Recall that the Eulerian numbers are defined by

\[ \langle n \rangle = \#\{\pi \in S_n : \des(\pi) = k\}. \]

The numbers $\langle n \rangle$ satisfy the recurrence relation

\[ \langle n + 1 \rangle = (k + 1)\langle n \rangle + (n + 1 - k)\langle n - 1 \rangle, \]

with the initial conditions $\langle 1 \rangle = 1$ and $\langle n \rangle = 0$ for $k \geq 1$.

**Theorem 11.** For $n \geq 1$ and $0 \leq k \leq n - 1$, we have

\[ \gamma_{n,n-k,k} = \langle n \rangle. \]

**Proof.** Set $a(n, k) = \gamma_{n,n-k,k}$. Then $a(n, k - 1) = \gamma_{n,n-k+1,k-1}$. Using (14), it is easy to verify that

\[ \gamma_{n,i,j} = 0 \quad \text{for } i + j > n. \]

Hence $\gamma_{n,n-k,k+1} = 0$. Therefore, the numbers $a(n, k)$ satisfy the recurrence relation

\[ a(n + 1, k) = (k + 1)a(n, k) + (n + 1 - k)a(n, k - 1). \]

Since the numbers $a(n, k)$ and $\langle n \rangle$ satisfy the same recurrence relation and initial conditions, so they agree. This completes the proof. \qed

A bijective proof of Theorem 11

**Proof.** Let $\sigma \in Q_n$. Note that every element of $[n]$ appears exactly two times in $\sigma$. Let $\alpha(\sigma)$ be the permutation of $S_n$ obtained from $\sigma$ by deleting all of the first $i$ from left to right, where $i \in [n]$. Then $\alpha$ is a map from $Q_n$ to $S_n$. For example, $\alpha(344355661221) = 435621$. Let

\[ D_n = \{\sigma \in Q_n : \text{lap}(\sigma) = i, \text{dasc}(\sigma) = n - i, \text{dp}(\sigma) = 0\}. \]

Let $x$ be a given element of $[n]$. For any $\sigma \in Q_n$, we define the action $\beta_x$ on $Q_n$ as follows:

- Read $\sigma$ from left to right and let $i$ be the first index such that $\sigma_i = x$;
- Move $\sigma_i$ to the right of $\sigma_k$, where $k = \max\{j \in \{0, 1, 2, \ldots, i - 1\} : \sigma_j < \sigma_i\}$, where $\sigma_0 = 0$.

For example, if $\sigma = 3443578876652211$, then

\[ \beta_1(\sigma) = 13443578876652211, \quad \beta_2(\sigma) = 23443578876652211, \quad \beta_6(\sigma) = 3443567887652211. \]

It is clear that $\beta_x(\beta_y(\sigma)) = \beta_y(\beta_x(\sigma))$ for any $x, y \in [n]$. For any $S \subseteq [n]$, let $\beta_S : Q_n \mapsto Q_n$ be a function defined by

\[ \beta_S(\sigma) = \prod_{x \in S} \beta_x(\sigma). \]
It is easy to verify that 
\[ \beta_{[n]}(\sigma) \in D_n, \quad \alpha(\sigma) = \alpha(\beta_{[n]}(\sigma)), \quad \beta_{[n]}(\sigma) = \sigma \text{ if } \sigma \in D_n. \]

Let \( \alpha|_{D_n} \) denote the restriction of the map \( \alpha \) on the set \( D_n \). Then \( \alpha|_{D_n} \) is a map from \( D_n \) to \( S_n \). Let \( \pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n \). The inverse \( \alpha|_{D_n}^{-1} \) is defined as follows:

- let \( \sigma = \sigma_1\sigma_2\ldots\sigma_{2n} \) be the Stirling permutation such that \( \sigma_{2i-1} = \sigma_{2i} = \pi(i) \) for each \( i = 1, 2, \ldots, n \);
- let \( S(\pi) = \{ \pi_i : \pi_{i-1} > \pi_i, 2 \leq i \leq n \} \);
- let \( \alpha|_{D_n}^{-1}(\pi) = \beta_{S(\pi)}(\sigma) \).

Note that 
\[ \text{lap}(\alpha|_{D_n}^{-1}(\pi)) + \text{dasc}(\alpha|_{D_n}^{-1}(\pi)) = n \text{ and } \text{dasc}(\alpha|_{D_n}^{-1}(\pi)) = \text{des}(\pi). \]

Then \( \alpha|_{D_n} \) is a bijection from \( D_n \) to \( S_n \). This completes the proof. \( \square \)

**Example 12.** The bijection between \( S_3 \) and \( D_3 \) is demonstrated as follows:

- \( 123 \leftrightarrow 112333 \quad (S = \emptyset) \leftrightarrow \beta_S(112333) = 112333; \)
- \( 132 \leftrightarrow 113322 \quad (S = \{2\}) \leftrightarrow \beta_S(113322) = 112332; \)
- \( 213 \leftrightarrow 221133 \quad (S = \{1\}) \leftrightarrow \beta_S(221133) = 122133; \)
- \( 231 \leftrightarrow 223311 \quad (S = \{1\}) \leftrightarrow \beta_S(223311) = 122331; \)
- \( 312 \leftrightarrow 331122 \quad (S = \{1\}) \leftrightarrow \beta_S(331122) = 133122; \)
- \( 321 \leftrightarrow 332211 \quad (S = \{1, 2\}) \leftrightarrow \beta_S(332211) = 123321. \)

4. **Concluding remarks**

In this paper, we introduce several variants of the ascent-plateau statistic on Stirling permutations. Recall that Park \[19\] studied the \((p, q)\)-analogue of the descent polynomials of Stirling permutations:

\[ C_n(x, p, q) = \sum_{\sigma \in Q_n} x^{\text{des}(\sigma)} p^{\text{inv}(\sigma)} q^{\text{maj}(\sigma)}. \]

It would be interesting to study the relationship between \( C_n(x, p, q) \) and the following polynomials:

\[ \sum_{\sigma \in Q_n} x^{\text{ap}(\sigma)} y^{\text{lap}(\sigma)} p^{\text{inv}(\sigma)} q^{\text{maj}(\sigma)}. \]

In \[6\], Egge introduced the definition of Legendre-Stirling permutation, which shares similar properties with Stirling permutation. One may study the ascent-plateau statistic on Legendre-Stirling permutations.
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