Around Context-Free Grammars - a Normal Form, a Representation Theorem, and a Regular Approximation

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Abstract

We introduce a normal form for context-free grammars, called Dyck normal form. This is a syntactical restriction of the Chomsky normal form, in which the two nonterminals occurring on the right-hand side of a rule are paired nonterminals. This pairwise property allows to define a homomorphism from Dyck words to words generated by a grammar in Dyck normal form. We prove that for each context-free language \( L \), there exist an integer \( K \) and a homomorphism \( \varphi \) such that \( L = \varphi(D'_K) \), where \( D'_K \subseteq D_K \), and \( D_K \) is the one-sided Dyck language over \( K \) letters. Through a transition-like diagram for a context-free grammar in Dyck normal form, we effectively build a regular language \( R \) such that \( D'_K = R \cap D_K \), which leads to the Chomsky-Schützenberger theorem. Using graphical approaches we refine \( R \) such that the Chomsky-Schützenberger theorem still holds. Based on this readjustment we sketch a transition diagram for a regular grammar that generates a regular superset approximation for the initial context-free language.

Keywords: linear languages, context-free languages, Dyck languages, Chomsky normal form, Dyck normal form, Chomsky-Schützenberger theorem, regular approximation

Introduction

A normal form for context-free grammars consists of restrictions imposed on the structure of grammar’s productions. These restrictions concern the number of terminals and nonterminals allowed on the right-hand sides of the rules, or on the manner in which terminals and nonterminals are arranged into the rules. Normal forms turned out to be useful tools in studying syntactical properties of context-free grammars, in parsing theory, structural and descriptional complexity, inference and learning theory. Various normal forms for context-free grammars have been study so far, but the most important remain the Chomsky normal form [17], Greibach normal form [12], and operator normal form [17]. For definitions, results, and surveys on normal forms the reader is referred to [5], [17], and [20]. A normal form is correct if it preserves the language generated by the original grammar. This condition is called the weak equivalence, i.e., a normal form preserves the language but may lose important syntactical or semantical properties of the original grammar. The more syntactical, semantical, or ambiguity properties a normal form preserves, the stronger it is. It is well known that the Chomsky normal form is a strong normal form.
This paper is partly devoted to a new normal form for context-free grammars, called 
\textit{Dyck normal form}. The Dyck normal form is a syntactical restriction of the Chomsky normal 
form, in which the two nonterminals occurring on the right-hand side of a rule are paired 
nonterminals, in the sense that each left (right) nonterminal of a pair has a unique right (left) 
pairwise. This pairwise property imposed on the structure of the right-hand side of each rule 
induces a nested structure on the derivation tree of each word generated by a grammar in 
Dyck normal form. More precisely, each derivation tree of a word generated by a grammar 
in Dyck normal form, that is read in the depth-first search order is a Dyck word, hence 
the name of the normal form. Furthermore, there exists always a homomorphism between 
the derivation tree of a word generated by a grammar in Chomsky normal form and its 
equivalent in Dyck normal form. In other words the Chomsky and Dyck normal forms are 
\textit{strongly equivalent}. This property, along with several other terminal rewriting conditions 
imposed to a grammar in Dyck normal form, enable us to define a homomorphism from 
Dyck words to words generated by a grammar in Dyck normal form. We have been inspired 
to develop this normal form by the general theory of Dyck words and Dyck languages, 
that turned out to play a crucial role in the description and characterization of context-free 
languages [9], [10], and [19]. The definition and several properties of grammars in Dyck 
normal form are presented in Section 1.

For each context-free grammar $G$ in Dyck normal form we define, in Section 2, the 
\textit{trace language} associated with derivations in $G$, which is the set of all derivation trees of $G$ read 
in the depth-first search order, starting from the grammar axiom. By exploiting the Dyck 
normal form, and several characterizations of Dyck languages presented in [19], we give a 
new characterization of context-free languages in terms of Dyck languages. We prove (also 
in Section 2) that for each context-free language $L$, generated by a grammar $G$ in Dyck 
normal form, there exist an integer $K$ and a homomorphism $\varphi$ such that $L = \varphi(D'_K)$, where 
$D'_K$ (a subset of the Dyck language over $K$ letters) equals, with very little exceptions, the 
trace language associated with $G$.

In Section 3 we show that the representation theorem in Section 2 emerges, through a 
\textit{transition-like diagram} for context-free grammars in Dyck normal form, to the Chomsky-
Schützenberger theorem. By improving this transition diagram, in Section 4 we refine the 
regular language provided by the Chomsky-Schützenberger theorem, while in Section 5 we 
show that the refined graphical representation of derivations in a context-free grammar in 
Dyck normal form, used in the previous sections, provides a framework for a regular grammar 
that generates a \textit{regular superset approximation} for the initial context-free language.

The method used throughout this paper is graph-constructive, in the sense that it sup-
plies a graphical interpretation of the Chomsky-Schützenberger theorem, and consequently 
it shows how to graphically build a regular language (as minimal as possible) that satisfies 
this theorem. Even if we reach the same famous Chomsky-Schützenberger theorem, the 
method used to approach it is different from the other methods known in the literature. 
In brief, the method in [17] is based on pushdown approaches, while that in [11] uses some 
kind of imaginary brackets that simulate the work of a pushdown store, when deriving a 
context-free language. The method presented in [1] uses equations on languages and al-
gebraical approaches to derive several types of Dyck language generators for context-free 
languages. In all these works, the Dyck language is somehow hidden behind the deriva-
tive structure of the context-free language (supplementary brackets are needed to derive a Dyck language generator for a context-free language). The Dyck language provided in this paper is merely found through a pairwise-renaming procedure of the nonterminals in the original context-free grammar. Hence, it lies inside the context-free grammar it describes. Each method used in the literature to prove the Chomsky-Schützenberger theorem provides its own regular language. Our aim is to find a thinner regular language that satisfies the Chomsky-Schützenberger theorem (with respect to the method hereby used) and approaching this language to build a regular superset approximation for context-free languages (likely to be as thinner as possible).

Note that the concept of a thinner (minimal) regular language, for the Chomsky-Schützenberger theorem and for the regular superset approximation is relative, in the sense that it depends on the structure of the grammar in Dyck normal form used to generate the original context-free language. In [2], [14], [15], and [16] it is proved that there is no algorithm that builds, for an arbitrary context-free language $L$, the minimal context-free grammar that generates $L$, where the minimality of a context-free grammar is considered, in principal, with respect to descriptional measures such as number of nonterminals, rules, and loops (i.e., grammatical levels [14], encountered during derivations in a context-free grammar). Consequently, there is no algorithm to build a minimal regular superset approximation for an arbitrary context-free language. Attempts to find optimal regular superset (subsets) approximations for context-free languages can be found in [4], [6], [21], and [23]. In Sections 3, 4, and 5 we also illustrate, through several examples, the manner in which the regular languages provided by the Chomsky-Schützenberger theorem and by the regular approximation can be built, with regards to the method proposed in this paper.

1 Dyck Normal Form

We assume the reader to be familiar with the basic notions of formal language theory [17]. For an alphabet $X$, $X^*$ denotes the free monoid generated by $X$. By $|x|_a$ we denote the number of occurrences of the letter $a$ in the string $x \in X^*$, while $|x|$ is the length of $x \in V^*$. We denote by $\lambda$ the empty string. If $X$ is a finite set, then $|X|$ is the cardinality of $X$.

**Definition 1.1.** A context-free grammar\(^1\) $G = (N, T, P, S)$ is said to be in Dyck normal form if it satisfies the following conditions:

1. $G$ is in Chomsky normal form,
2. if $A \to a \in P$, $A \in N$, $A \neq S$, $a \in T$, then no other rule in $P$ rewrites $A$,
3. for each $A \in N$ such that $X \to AB \in P$ ($X \to BA \in P$) there is no other rule in $P$ of the form $X' \to B'A$ ($X' \to AB'$),
4. for each rules $X \to AB$, $X' \to A'B$ ($X \to AB$, $X' \to AB'$), we have $A = A'$ ($B = B'$).

\(^1\)A context-free grammar is denoted by $G = (N, T, P, S)$, where $N$ and $T$ are finite sets of variables and terminals, respectively, $N \cap T = \emptyset$, $S \in N - T$ is the grammar axiom, and $P \subseteq N \times (N \cup T)^*$ is the finite set of productions.
Note that the reasons for which we introduce the restrictions at items 2 – 4, are the following. The condition at item 2 allows to make a partition between those nonterminals rewritten by nonterminals, and those nonterminals rewritten by terminals (with the exception of the axiom). This enables, in Section 2, to define a homomorphism from Dyck words to words generated by a grammar in Dyck normal form. Conditions at items 3 and 4 allow to split the set of nonterminals into pairwise nonterminals, and thus to introduce bracketed pairs. The next theorem proves that the Dyck normal form is correct.

**Theorem 1.2.** For each context-free grammar \( G = (N, T, P, S) \) there exists a grammar \( G' = (N', T, P', S) \) such that \( L(G) = L(G') \) where \( G' \) is in Dyck normal form.

**Proof.** Suppose that \( G \) is a context-free grammar in Chomsky normal form. Otherwise, using the algorithm described in [20], we can convert \( G \) into Chomsky normal form. To convert \( G \) from Chomsky normal form into Dyck normal form we proceed as follows.

**Step 1** We check whether \( P \) contains two (or more) rules of the form \( A \rightarrow a \), \( A \rightarrow b \), \( a \neq b \). If it does, then for each rule \( A \rightarrow b \), \( a \neq b \), a new variable \( A_b \) is introduced. We add the new rule \( A_b \rightarrow b \), and remove \( A \rightarrow b \). For each rule \( X \rightarrow AB \) \((X \rightarrow BA)\) we add the new rules \( X \rightarrow A_b B \) \((X \rightarrow B A_b)\), while for a rule of the form \( X \rightarrow AA \) we add three new rules \( X \rightarrow A_b A, X \rightarrow A A_b, X \rightarrow A_b A \), without removing the initial rules. We call this procedure an \( A_b \)-terminal substitution of \( A \). For each rule \( A \rightarrow a \), \( a \in T \), we check whether a rule of the form \( A \rightarrow B_1 B_2, B_1, B_2 \in N \), exists in \( P \). If it does, then a new nonterminal \( A_a \) is introduced and an \( A_a \)-terminal substitution of \( A \) for the rule \( A \rightarrow a \) is performed.

**Step 2** Suppose there exist two (or more) rules of the form \( X \rightarrow AB \) and \( X' \rightarrow B'A \). If we have agreed on preserving only the left occurrences of \( A \) on the right-hand sides, then according to condition 3 of Definition 1.1, we have to remove all right occurrences of \( A \). To do so we introduce a new nonterminal \( ZA \) and all right occurrences of \( A \), preceded at the left side by \( Z \), in the right-hand side of a rule, are substituted by \( ZA \). For each rule that rewrites \( A \rightarrow Y \), \( Y \in N^2 \cup T \), we add a new rule of the form \( ZA \rightarrow Y \), preserving the rule \( A \rightarrow Y \). We call this procedure an \( ZA \)-nonterminal substitution of \( A \). According to this procedure, for the rule \( X' \rightarrow B'A \), we introduce a new nonterminal \( B'A \), we add the rule \( X' \rightarrow B'B'A \), and remove the rule \( X' \rightarrow B'A \). For each rule that rewrites \( A \), of the form \( A \rightarrow Y, Y \in N^2 \cup T \), we add a new rule of the form \( B'A \rightarrow Y \), preserving the rule \( A \rightarrow Y \).

**Step 3** Finally, for each two rules \( X \rightarrow AB \), \( X' \rightarrow A'B \) \((X \rightarrow BA, X' \rightarrow BA')\) with \( A \neq A' \), a new nonterminal \( A'B \) \((B'A)\) is introduced to replace \( B \) from the second rule, and we perform an \( A'B(B'A')-\)nonterminal substitution of \( B \), i.e., we add \( X' \rightarrow A'B \), and remove \( X' \rightarrow A'B \). For each rule that rewrites \( B \), of the form \( B \rightarrow Y, Y \in N^2 \cup T \), we add a new rule \( A'B \rightarrow Y \), preserving \( B \rightarrow Y \). In the case that \( A' \) occurs on the right-hand side of another rule, such that \( A' \) matches at the right side with another nonterminal different of \( A'B \), then the procedure described above is repeated for \( A' \), too.

Note that, if one of the conditions 2, 3, and 4 in Definition 1.1, has been settled, we do not have to resolve it once again in further steps of the procedure. The new grammar \( G' \) built as described at steps 1, 2, and 3 has the set of nonterminals \( N' \) and the set of productions \( P' \)

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2This case deals with the possibility of having \( Y = B'B'A \), too.
composed of all nonterminals from $N$ and productions from $P$, plus/minus all nonterminals and productions, respectively introduced/removed according to the substitutions performed during the above steps. Next we prove that grammars $G = (N,T,P,S)$ in Chomsky normal form, and $G' = (N',T,P',S)$ in Dyck normal form, generate the same language. Consider the homomorphism $h_d : N' \cup T \rightarrow N \cup T$ defined by $h_d(x) = x$, $x \in T$, $h_d(X) = X$, for $X \in N$, and $h_d(X') = X$ for $X' \in N' - N$, $X \in N$ such that $X'$ is a (transitive) $X'$-substitution of $X$, terminal or not, in the above construction of the grammar $G'$.

To prove that $L(G') \subseteq L(G)$ we extend $h_d$ to a homomorphism from $(N' \cup T)^*$ to $(N \cup T)^*$ defined on the classical concatenation operation. It is straightforward to prove by induction, that for each $\alpha \Rightarrow^*_G \delta$ we have $h_d(\alpha) \Rightarrow^*_{G'} h_d(\delta)$. This implies that for any derivation of a word $w \in L(G')$, i.e., $S \Rightarrow^*_G w$, we have $h_d(S) \Rightarrow^*_{G'} h_d(w)$, i.e., $S \Rightarrow^*_G w$, or equivalently, $L(G') \subseteq L(G)$.

To prove that $L(G) \subseteq L(G')$ we make use of the CYK (Cocke-Younger-Kasami) algorithm as described in [20]. Let $w = a_1a_2...a_n$ be an arbitrary word in $L(G)$, and $V_{ij}$, $i \leq j$, $i,j \in \{1,...,n\}$, be the triangular matrix of size $n \times n$ built with the CYK algorithm. Since $w \in L(G)$, we have $S \in V_{1n}$. We prove that $w \in L(G')$, i.e., $S \Rightarrow^*_G \{w\}$, where $V'_{ij}$, $i \leq j$, $i,j \in \{1,...,n\}$ forms the triangular matrix obtained by applying the CYK algorithm to $w$ according to $G'$ productions.

We consider two relations $\hat{h}_t \subseteq (N \cup T) \times (N' \cup T)$ and $\hat{h}_{-t} \subseteq N \times N'$. The first relation is defined by $\hat{h}_t(x) = x$, $x \in T$, $\hat{h}_t(S) = S$, if $S \rightarrow t$, $t \in T$, is a rule in $G$, and $\hat{h}_t(X) = X'$, if $X'$ is a (transitive) $X'$-terminal substitution of $X$, and $X \rightarrow t$ is a rule in $G$. Finally, $\hat{h}_t(X) = X$ if $X \rightarrow t \in P$, $t \in T$. The second relation is defined as $\hat{h}_{-t}(S) = S$, $\hat{h}_{-t}(X) = \{X\} \cup \{X'|X' \text{ a (transitive) } X' \text{-nonterminal substitution of } X\}$ and $\hat{h}_{-t}(X) = X$, if there is no substitution of $X$ and no rule of the form $X \rightarrow t, t \in T$, in $G$.

Note that $\hat{h}_t(X_1 \cup X_2) = \hat{h}_t(X_1) \cup \hat{h}_t(X_2)$, for $X_i \subseteq N$, $i \in \{1,2\}$, $x \in \{t,-t\}$. Using $\hat{h}_t$, each rule $X \rightarrow t$ in $P$ has a corresponding set of rules $\{X' \rightarrow t| X' \in \hat{h}_t(X), X \rightarrow t \in P\}$ in $P'$. Each rule $A \rightarrow BC$ in $P$ has a corresponding set of rules $\{A' \rightarrow B'C'| A' \in \hat{h}_{-t}(A), B' \in \hat{h}_{-t}(B) \cup \hat{h}_t(B), C' \in \hat{h}_{-t}(C) \cup \hat{h}_t(C)\}$ are pairwise nonterminals, $A \rightarrow BC \in P'$.

Consider $V'_{ii} = \hat{h}_t(V_{ii})$ and $V'_{ij} = \hat{h}_{-t}(V_{ij})$, $i < j$, $i,j \in \{1,...,n\}$. We claim that $V'_{ij}$, $i,j \in \{1,...,n\}$, $i \leq j$, defines the triangular matrix obtained by applying CYK algorithm to rules that derive $w$ in $G'$. First, observe that for $i = j$, we have $V'_{ii} = \hat{h}_t(V_{ii}) = \{A|A \rightarrow a_i \in P\}$, $i \in \{1,...,n\}$, due to the definition of $\hat{h}_t$. Now let us consider $k = j-i$, $k \in \{1,...,n-1\}$. We want to compute $V'_{ij}$, $i < j$.

By definition, we have $V_{ij} = \bigcup_{i=l}^{j-1} \{A|A \rightarrow BC, B \in V_{il}, C \in V_{l+1j}\}$, so that $V'_{ij} = \hat{h}_{-t}(\bigcup_{i=l}^{j-1} \{A|A \rightarrow BC, B \in V_{il}, C \in V_{l+1j}\}) = \bigcup_{i=l}^{j-1} \hat{h}_{-t}(\{A|A \rightarrow BC, B \in V_{il}, C \in V_{l+1j}\}) = \bigcup_{i=l}^{j-1} \{A'|A' \rightarrow B'C', A' \in \hat{h}_{-t}(A), B' \in \hat{h}_{-t}(B) \cup \hat{h}_t(B), B \in V_{il}, C' \in \hat{h}_{-t}(C) \cup \hat{h}_t(C), C \in V_{l+1j}, B' \text{ and } C' \text{ are pairwise nonterminals, } A \rightarrow BC \in P\}$. Let us explicitly develop the last union.

\footnote{There exist $X_k \in N$, such that $X'$ is an $X'$-substitution of $X_k$, $X_k$ is an $X_k$-substitution of $X_{k-1}$,..., and $X_1$ is an $X_1$-substitution of $X$. All of them substitute $X$.}

\footnote{There may exist several terminal/nonterminal substitutions for the same nonterminal $X$. This makes $\hat{h}_t/\hat{h}_{-t}$ to be a relation.}
If \( k = 1 \), then \( l \in \{i\} \). For each \( i \in \{1, \ldots, n-1\} \) we have \( V'_{i+1} = \{A'|A' \rightarrow B'C', A' \in \hat{h}_{-t}(A), B' \in \hat{h}_{-t}(B) \cup \hat{h}_{t}(B), B \in V_{ii}, C' \in \hat{h}_{-t}(C) \cup \hat{h}_{t}(C), C \in V_{i+i+1}, B' \text{ and } C' \text{ are pairwise nonterminals}, A \rightarrow BC \in P \} \). Due to the fact that \( B \in V_{ii} \) and \( C \in V_{i+i+1} \), \( B' \) is a terminal substitution of \( B \), while \( C' \) is a terminal substitution of \( C \). Therefore, we have \( B' \notin \hat{h}_{-t}(B), C' \notin \hat{h}_{-t}(C) \), so that \( B' \in \hat{h}_{t}(B) \), for all \( B \in V_{ii} \), and \( C' \in \hat{h}_{t}(C) \), for all \( C \in V_{i+i+1} \), i.e., \( B' \in \hat{h}_{t}(V_{ii}) = V'_{ii} \) and \( C' \in \hat{h}_{t}(V_{i+i+1}) = V'_{i+i+1} \). Therefore, \( V'_{ii} = \{A'|A' \rightarrow B'C', B' \in V'_{ii}, C' \in V'_{i+i+1}\} \).

If \( k \geq 2 \), then \( l \in \{i, i+1, \ldots, j-1\} \), and \( V'_{ij} = \bigcup_{i=1}^{l-1} \{A'|A' \rightarrow B'C', A' \in \hat{h}_{-t}(A), B' \in \hat{h}_{-t}(B) \cup \hat{h}_{t}(B), B \in V_{i+l}, C' \in \hat{h}_{-t}(C) \cup \hat{h}_{t}(C), C \in V_{i+i+1}, B' \text{ and } C' \text{ are pairwise nonterminals}, A \rightarrow BC \in P \} \). We now compute the first set of the above union, i.e., \( V'_i = \{A'|A' \rightarrow B'C', A' \in \hat{h}_{-t}(A), B' \in \hat{h}_{-t}(B) \cup \hat{h}_{t}(B), B \in V_{ii}, C' \in \hat{h}_{-t}(C) \cup \hat{h}_{t}(C), C \in V_{i+i+1}, B' \text{ and } C' \text{ are pairwise nonterminals}, A \rightarrow BC \in P \} \). By the same reasoning as before, the condition \( B' \in \hat{h}_{-t}(B) \cup \hat{h}_{t}(B), B \in V_{ii} \), is equivalent with \( B' \in \hat{h}_{t}(V_{ii}) = V'_{ii} \). Because \( i+1 \neq j \), \( C' \) is a nonterminal substitution of \( C \). Therefore, \( C' \notin \hat{h}_{t}(C) \), and the condition \( C' \in \hat{h}_{-t}(C) \cup \hat{h}_{t}(C), C \in V_{i+i+1} \), is equivalent with \( C' \in \hat{h}_{-t}(V_{i+i+1}) = V'_{i+i+1} \). So that \( V_i = \{A'|A' \rightarrow B'C', B' \in V'_{ii}, C' \in V'_{i+i+1}\} \). Using the same method for each \( l \in \{i+1, \ldots, j-1\} \) we have \( V'_i = \{A'|A' \rightarrow B'C', A' \in \hat{h}_{-t}(A), B' \in \hat{h}_{-t}(B) \cup \hat{h}_{t}(B), B \in V_{i+l}, C' \in \hat{h}_{-t}(C) \cup \hat{h}_{t}(C), C \in V_{i+i+1}, B' \text{ and } C' \text{ are pairwise nonterminals}, A \rightarrow BC \in P \} = \{A'|A' \rightarrow B'C', B' \in V'_{ii}, C' \in V'_{i+i+1}\} \). In conclusion, \( V'_i = \bigcup_{i=1}^{l-1} \{A'|A' \rightarrow B'C', B' \in V'_{ii}, C' \in V'_{i+i+1}\} \), for each \( i, j \in \{1, \ldots, n\} \), i.e., \( V'_{ij}, i \leq j \), contains the nonterminals of the \( n \times n \) triangular matrix computed by applying the CYK algorithm to rules that derive \( w \) in \( G' \). Because \( w \in L(G) \), we have \( S \in V_{i+n} \). That is equivalent with \( S \in V'_{i+n} = \hat{h}_{t}(V_{i+n}), \) if \( n = 1 \), and \( S \in V'_{1+n} = \hat{h}_{t}(V_{1+n}), \) if \( n > 1 \), i.e., \( w \in L(G') \).

**Corollary 1.3.** Let \( G \) be a context-free grammar in Dyck normal form. Any terminal derivation in \( G \) producing a word of length \( n, n \geq 1 \), takes \( 2n - 1 \) steps.

**Proof.** If \( G \) is a context-free grammar in Dyck normal form, then it is also in Chomsky normal form, and all properties of the latter hold. \( \square \)

**Corollary 1.4.** If \( G = (N, T, P, S) \) is a grammar in Chomsky normal form, and \( G' = (N', T, P', S) \) its equivalent in Dyck normal form, then there exists a homomorphism \( h_d: N' \cup T \rightarrow N \cup T \), such that any derivation tree of \( w \in L(G) \) is the homomorphic image of a derivation tree of the same word in \( G' \).

**Proof.** Consider the homomorphism \( h_d: N' \cup T \rightarrow N \cup T \) defined as \( h_d(A_t) = h_d(zA) = h_d(A_Z) = A, \) for each \( A_t \)-terminal or \( zA(A_Z) \)-nonterminal substitution of \( A, \) and \( h_d(t) = t, t \in T. \) The claim is a direct consequence of the way in which the new nonterminals \( A_t, zA, \) and \( A_Z \) have been chosen. \( \square \)

Note that, due to the pairwise-renaming procedure used to reach the Dyck normal form, it may appear that a context-free grammar in Dyck normal form is more ambiguous than the original grammar in Chomsky normal form. However, this is relative. The derivation trees of a certain word have the same structure in both grammars, in Chomsky normal form.
and Dyck normal form (only some “labels” of the nodes in these trees differ). The apparent ambiguity can be resolved through the homomorphism \( h_d \) considered in Corollary 1.4.

Let \( G \) be a grammar in Dyck normal form. To emphasize the pairwise brackets occurring on the right-hand side of a rule, and also to make the connection with the Dyck language, each pair \((A, B)\), such that there exists a rule of the form \( X \rightarrow AB \), is replaced by an indexed pair of brackets \([i, j]\). In each rule that rewrites \( A \) and \( B \), we replace \( A \) by \([i]\), and \( B \) by \([j]\), respectively. Next we present an example of the conversion procedure described in the proof of Theorem 1.2 along with the homomorphism considered in Corollary 1.4.

**Example 1.5.** Consider the context-free grammar in Chomsky normal form \( G = \langle \{E_0, E, E_1, E_2, T, T_1, T_2, R\}, \{+, *, a, E_0, P'\} \rangle \), where \( P' = \{E_0 \rightarrow a/TT_1/EE_1, E \rightarrow a/TT_1/EE_1, T \rightarrow a/TT_1, T_1 \rightarrow T_2R, E_1 \rightarrow E_2T, T_2 \rightarrow *, E_2 \rightarrow +, R \rightarrow a\} \).

To convert \( G \) into Dyck normal form, with respect to Definition 1.1, item 2, we first remove \( E \rightarrow a \) and \( T \rightarrow a \). Then, according to item 3, we remove the right occurrence of \( T \) from the rule \( E_1 \rightarrow E_2T \), along with other transformations that may be required after completing these procedures. Let \( E_3 \) and \( T_3 \) be two new nonterminals. We remove \( E \rightarrow a \) and \( T \rightarrow a \), and add the rules \( E_3 \rightarrow a, T_3 \rightarrow a, E_0 \rightarrow E_3E_1, E_0 \rightarrow E_3T_1, E \rightarrow E_3E_1, E \rightarrow E_3T_1, E_1 \rightarrow E_2T_3, T \rightarrow T_3T_1 \). Let \( T' \) be the new nonterminal that replaces the right occurrence of \( T \). We extend the rules \( E_1 \rightarrow E_2T', T' \rightarrow TT_1, T' \rightarrow T_3T_1 \), and remove \( E_1 \rightarrow E_2T \). We repeat the procedure with \( T_3 \) (added in the previous step), i.e., we introduce a new nonterminal \( T_4 \), remove \( E_1 \rightarrow E_2T_3 \), add \( E_1 \rightarrow E_2T_4 \) and \( T_4 \rightarrow a \).

Due to the new nonterminals \( E_3, T_3, T_4 \), item 4 does not hold. To have accomplished this condition, we introduce three new nonterminals \( E_4 \) to replace \( E_2 \) in \( E_1 \rightarrow E_2T_4 \), \( E_5 \) to replace \( E_1 \) in \( E_0 \rightarrow E_3E_1 \) and \( E \rightarrow E_3T_1 \), and \( T_4 \) to replace \( T_1 \) in \( E_0 \rightarrow E_3T_1 \) and \( E \rightarrow T_3T_1 \). We remove all the above rules and add the new rules \( E_1 \rightarrow E_4T_4, E_4 \rightarrow +, E_0 \rightarrow E_3E_5, E \rightarrow E_3, E_5 \rightarrow E_2T', E_5 \rightarrow E_4T_4, E_0 \rightarrow E_3T_5, E_5 \rightarrow E_2T_5 \), \( E \rightarrow E_3T_4, E_0 \rightarrow E_3T_5, E_5 \rightarrow E_2R \), \( E \rightarrow E_3T_4, E_0 \rightarrow E_3T_5, E_5 \rightarrow E_2R \).

The Dyck normal form of \( G' \), in bracketed notation, is \( G'' = \{ \{E_0, [1, [2, ..., [7], 1], 2, ..., 7]\}, \{+, *, a, E_0, P''\}, P'' = \{E_0 \rightarrow a/1[1]/2[3]3[1]/4[4]3[1]/4[4]3[1]/4[4]3[1]/4[4]3[1]/4[4], 2 \rightarrow [3]3[4]/[4]3[5]/[6]6, 3 \rightarrow [3]3[5]/[6]6, 4 \rightarrow [3]3[5]/[6]6, 5 \rightarrow [3]3[5]/[6]6, 6 \rightarrow [3]3[5]/[6]6, 7 \rightarrow [3]3[5]/[6]6\} \) and the homomorphism \( h_d : N'' \cup T \rightarrow N'' \cup T, h_d(E_0) = E_0, h_d([2]) = h_d([3]) = E_1, h_d([1]) = h_d([4]) = E_2, h_d([5]) = h_d([6]) = h_d([7]) = T, h_d([1]) = h_d([4]) = T_1, h_d([7]) = T_2, h_d([7]) = T, h_d([7]) = R, h_d(t) = t, for each \( t \in T \). The string \( w = a * a * a + a \) is a word in \( L(G'') = L(G) \) generated, for instance, by a leftmost derivation \( D \) in \( G'' \) as follows:

\[
D : E_0 \Rightarrow [2][2] \Rightarrow [1][1] \Rightarrow [4][1][1] \Rightarrow a[4][1][1] \Rightarrow a[7][7][1][2] \Rightarrow a * a * [7][7][1][2] \Rightarrow a * a * a \]

Applying \( h_d \) to \( D \) in \( G'' \), we obtain a derivation of \( w \) in \( G' \). If we consider \( T \) the derivation tree of \( w \) in \( G \), and \( T' \) the derivation tree of \( w \) in \( G'' \), then \( T \) is the homomorphic image of \( T' \) through \( h_d \).
2 Characterizations of Context-Free Languages by Dyck Languages

Definition 2.1. Let $G_k = (N_k, T, P_k, S)$ be a context-free grammar in Dyck normal form with $|N_k - \{S\}| = 2k$. Let $D : S \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \ldots \Rightarrow u_{2n-1} = w$, $n \geq 2$, be a leftmost derivation of $w \in L(G)$. The trace-word of $w$ associated with the derivation $D$, denoted as $t_{w,D}$, is defined as the concatenation of nonterminals consecutively rewritten in $D$, excluding the axiom. The trace-language associated with $G_k$, denoted by $L(G_k)$, is $L(G_k) = \{t_{w,D} \mid w \in L(G_k), \text{ any leftmost derivation } D \text{ of } w\}$.

Note that $t_{w,D}, w \in L(G)$, can also be read from the derivation tree in the depth-first search order starting with the root, but ignoring the root and the leaves. The trace-word associated with $w$ and the leftmost derivation $D$ in Example 2.5 is $t_{a*a*a+a,D} = \{E \left[ T \left[ T_1 \right] T_2 \right] R \left[ T_1 \right] T_2 \right] R \left[ E_1 \left[ E_4 \right] \right] T_3$.

Definition 2.2. A one-sided Dyck language over $k$ letters, $k \geq 1$, is a context-free language defined by the grammar $\Gamma_k = (\{S\}, T_k, P, S)$, where $T_k = \{[1], [2, \ldots, [k], 1], 2, \ldots, k\}$ and $P = \{S \rightarrow [i]S, S \rightarrow SS, S \rightarrow [i]i \mid 1 \leq i \leq k\}$.

Let $G_k = (N_k, T, P_k, S)$ be a context-free grammar in Dyck normal form. To emphasize possible relations between the structure of trace-words in $L(G_k)$ and the structure of words in the Dyck language, and also to keep control of each bracketed pair occurring on the right-hand side of each rule in $G_k$, we fix $N_k = \{S, [1], [2, \ldots, [k], 1], 2, \ldots, k\}$, and $P_k$ to be composed of rules of the forms $X \rightarrow [i]i, 1 \leq i \leq k$, and $Y \rightarrow t, X, Y \in N_k, t \in T$. From [19], we have adopted the next characterizations of $D_k, k \geq 1$, (Definition 2.3, and Lemmas 2.4 and 2.5).

Definition 2.3. For a string $w$, let $w_{i,j}$ be its substring starting at the $i^{th}$ position and ending at the $j^{th}$ position. Let $h$ be a homomorphism defined as follows:

- $h(1) = h(2) = \ldots = h(k) = [1], \quad h(1) = h(2) = \ldots = h(k) = [1]$
- $h[i_1] = h[i_2] = \ldots = h[i_{2n-1}] = w_{i,j}$, $1 \leq i \leq j \leq |w|$, where $|w|$ is the length of $w$. We say that $(i, j)$ is a matched pair of $w$, if $h(w_{i,j})$ is balanced, i.e., $h(w_{i,j})$ has an equal number of [1’s and [1’s and, in any prefix of $h(w_{i,j})$, the number of [1’s is greater than or equal to the number of ]1’s.

Lemma 2.4. A string $w \in \{[1], [1]\}^*$ is in $D_1$ if and only if it is balanced.

Consider the homomorphisms defined as follows (where $\lambda$ is the empty string)

- $h_1[1] = [1], \quad h_1[1] = [1], \quad h_1[2] = h_1[1] = \ldots = h_1[k] = h_1[k] = \lambda$
- $h_2[2] = [1], \quad h_2[2] = [1], \quad h_2[1] = h_2[1] = \ldots = h_2[k] = h_2[k] = \lambda, \ldots$
- $h_k[k] = [1], \quad h_k[k] = [1], \quad h_k[1] = h_k[1] = \ldots = h_k[k-1] = h_k[k-1] = \lambda$

Lemma 2.5. We have $w \in D_k, k \geq 2$, if and only if the following conditions hold: $i$) $(1, |w|)$ is a matched pair, and $ii)$ for all matched pairs $(i, j)$, $h_k(w_{i,j})$ are in $D_1$, where $k \geq 1$.

Definition 2.6. Let $w \in D_k, (i, j)$ is a nested pair of $w$ if $(i, j)$ is a matched pair, and either $j = i + 1$, or $(i + 1, j - 1)$ is a matched pair.
Definition 2.7. Let \( w \in D_k \) and \((i, j)\) be a matched pair of \( w \). We say that \((i, j)\) is reducible if there exists an integer \(i', i < i' < j\), such that \((i, i')\) and \((i' + 1, j)\) are matched pairs.

Let \( w \in D_k \), if \((i, j)\) is a nested pair of \( w \) then \((i, j)\) is an irreducible pair. If \((i, j)\) is a nested pair of \( w \) then \((i + 1, j - 1)\) may be a reducible pair.

Theorem 2.8. The trace-language associated with a context-free grammar, \( G = (N_k, T, P_k, S) \) in Dyck normal form, with \(|N_k| = 2k + 1\), is a subset of \( D_k \).

Proof. Let \( N_k = \{S, [1, ..., [k], ]_1, ..., ]_k\} \) be the set of nonterminals, \( w \in L(G) \), and \( D \) a leftmost derivation of \( w \). We show that any subtree of the derivation tree, read in the depth-first search order, by ignoring the root and the terminal nodes, corresponds to a matched pair in \( t_{w,D} \). In particular, \((1, |t_{w,D}|)\) will be a matched pair. Denote by \( t_{w,D[i:j]} \) the substring of \( t_{w,D} \) starting at the \( i^{th} \) position and ending at the \( j^{th} \) position of \( t_{w,D} \). We show that for all matched pairs \((i, j), h_{k'}(t_{w,D[i:j]})\) belong to \( D_1 \), \( 1 \leq k' \leq k \). We prove these claims by induction on the height of subtrees.

Basis. Certainly, any subtree of height \( n = 1 \), read in the depth-first search order, looks like \([i,]_1\), \( 1 \leq i \leq k \). Therefore, it satisfies the above conditions.

Induction step. Assume that the claim is true for all subtrees of height \( h, h < n \), and we prove it for \( h = n \). Each subtree of height \( n \) can have one of the following structures. The level 0 of the subtree is marked by a left or right bracket. This bracket will not be considered when we read the subtree. Denote by \( [m \) the left son of the root. Then the right son is labeled by \( ]_m \). They are the roots of a left and right subtree, for which at least one has the height \( n - 1 \).

Suppose that both subtrees have the height \( 1 \leq h \leq n - 1 \). By the induction hypothesis, let us further suppose that the left subtree corresponds to the matched pair \((i_1, j_1)\), and the right subtree corresponds to the matched pair \((i_r, j_r)\), \( i_r = j_l + 2 \), because the position \( j_l + 1 \) is taken by \( ]_m \). As \( h \) is a homomorphism, we have \( h(t_{w,D[i_1-1:j_r]}) = h([m t_{w,D[i_1:j_l]}] m t_{w,D[j_l+2:j_r]} \) = \( h([m]) h(t_{w,D[i_1:j_l]} h([m]) h(t_{w,D[j_l+2:j_r]}) \). Therefore, \( h(t_{w,D[i_1-1:j_r]} \) satisfies all conditions in Definition 2.3, and thus \((i_l - 1, j_r)\) that corresponds to the considered subtree of height \( n \), is a matched pair. By the induction hypothesis, \( h_{k'}(t_{w,D[i_1:j_l]} \) and \( h_{k'}(t_{w,D[i_r:j_r]} \) are in \( D_1 \), \( 1 \leq k' \leq k \). Hence, \( h_{k'}(t_{w,D[i_1-1:j_r]} = h_{k'}([m]) h_{k'}(t_{w,D[i_1:j_l]} h_{k'}([m]) h_{k'}(t_{w,D[j_l+2:j_r]} \in \{h_{k'}(t_{w,D[i_1:j_l]} h_{k'}(t_{w,D[j_l+2:j_r]}), [h_{k'}(t_{w,D[i_1:j_l]} h_{k'}([m]) h_{k'}(t_{w,D[j_l+2:j_r]}) \} belong to \( D_1 \), \( 1 \leq k' \leq k \). Note that in this case the matched pair \((i_l - 1, j_r)\) is reducible into \((i_l - 1, j_l + 1)\) and \((j_l + 2, j_r)\), where \((i_l - 1, j_l + 1)\) corresponds to the substring \( t_{w,D[i_1-1:j_l+1]} \). We refer to this structure as the left embedded subtree, i.e., \((i_l - 1, j_l + 1)\) is a nested pair.

A similar reasoning is applied for the case when one of the subtrees has the height 0. Analogously, it can be shown that the initial tree corresponds to the matched pair \((1, |t_{w,D}|)\), i.e., the first condition of Lemma 2.5 holds. So far, we have proved that each subtree of the derivation tree, and also each left embedded subtree, corresponds to a matched pair \((i, j)\) and \((i_1, j_1)\), such that \( h_{k'}(t_{w,D[i:j]} \) and \( h_{k'}([m t_{w,D[i:j]}]_m \), \( 1 \leq k' \leq k \), are in \( D_1 \).

Next we show that all matched pairs from \( t_{w,D} \) correspond only to subtrees, or left embedded subtrees, from the derivation tree. To derive a contradiction, let us suppose that there exists a matched pair \((i, j)\) in \( t_{w,D} \), that does not correspond to any subtree, or left
embedded subtree, of the derivation tree read in the depth-first search order. We show that
this leads to a contradiction.

Since \((i, j)\) does not correspond to any subtree, or left embedded subtree, there exist
two adjacent subtrees \(\theta_1\) (a left embedded subtree) and \(\theta_2\) (a right subtree) such that \((i, j)\)
is composed of two adjacent “subparts” of \(\theta_1\) and \(\theta_2\). In terms of matched pairs, if \(\theta_1\)
corresponds to the matched pair \((i_1, j_1)\) and \(\theta_2\) corresponds to the matched pair \((i_2, j_2)\),
such that \(i_2 = j_1 + 2\), then there exists a suffix \(s_{i_1-1:j_1+1}\) of \(t_{w,D_{i_1-1:j_1+1}}\), and a prefix \(p_{i_2:j_2}\)
of \(t_{w,D_{i_2:j_2}}\), such that \(t_{w,D_{i,j}} = s_{i_1-1:j_1+1} + 1p_{i_2:j_2}\). Furthermore, without loss of
generality, we assume that \((i_1, j_1)\) and \((i_2, j_2)\) are nested pairs. Otherwise, the matched pair \((i, j)\) can be
“narrowed” until \(\theta_1\) and \(\theta_2\) are characterized by two nested pairs. If \((i_1, j_1)\) is a nested pair,
then so is \((i_1 - 1, j_1 + 1)\). As \(s_{i_1-1:j_1+1}\) is a suffix of \(t_{w,D_{i_1-1:j_1+1}}\) and \((i_1 - 1, j_1 + 1)\) is a
matched pair, with respect to Definition 2.3, the number of \([1]s\) in \(h(s_{i_1-1:j_1+1})\) is greater
than or equal to the number of \([1]s\) in \(h(s_{i_1-1:j_1+1})\). On the other hand, \(s_{i_1-1:j_1+1}\) is also a
prefix of \(t_{w,D_{i,j}}\), because \((i, j)\) is a matched pair, by the induction hypothesis. Therefore, the
number of \([1]s\) in \(h(s_{i_1-1:j_1+1})\) is greater than or equal to the number of \([1]s\) in \(h(s_{i_1-1:j_1+1})\).

Hence, the only possibility for \(s_{i_1-1:j_1+1}\) to be, in the same time, a suffix for \(t_{w,D_{i_1-1:j_1+1}}\)
and a prefix for \(t_{w,D_{i,j}}\), is the equality between the number of \([1]s\) and \([1]s\) in \(h(s_{i_1-1:j_1+1})\).
This property holds if and only if \(s_{i_1-1:j_1+1}\) corresponds to a matched pair in \(t_{w,D_{i_1-1:j_1+1}}\);
\(i.e., if \(i_s\) and \(j_s\) are the start and the end positions of \(s_{i_1-1:j_1+1}\) in \(t_{w,D_{i_1-1:j_1+1}}\), then \((i_s, j_s)\)
is a matched pair. Thus, \((i_1 - 1, j_1 + 1)\) is a reducible pair into \((i_1 - 1, i_s - 1)\) and \((i_s, j_s)\),
where \(j_s = j_1 + 1\). We have reached a contradiction, \(i.e., (i_1 - 1, j_1 + 1)\) is reducible.

Therefore, the matched pairs in \(t_{w,D}\) correspond to subtrees, or left embedded subtrees,
in the derivation tree. For these matched pairs we have already proved that they satisfy
Lemma 2.5. Accordingly, \(t_{w,D} \in D_k\), and consequently the trace-language associated with
\(G\) is a subset of \(D_k\).

\textbf{Theorem 2.9.} Given a context-free grammar \(G\) there exist an integer \(K\), a homomorphism
\(\varphi\), and a subset \(D'_K\) of the Dyck language \(D_K\), such that \(L(G) = \varphi(D'_K)\).

\textbf{Proof.} Let \(G\) be a context-free grammar and \(G_k = (N_k, T, P_k, S)\) be the Dyck normal form
of \(G\), such that \(N_k = \{S, [1, ..., k, 1, ..., k]\}\). Let \(L(G_k)\) be the trace-language associated with
\(G_k\). Consider \(\{t_{k+1}, ..., t_{k+p}\}\) the ordered subset of \(T\), such that \(S \rightarrow t_{k+i} \in P, 1 \leq i \leq p\).
We define \(N_{k+p} = N_k \cup \{[t_{k+1}, ..., t_{k+p}], [t_{k+1}, ..., t_{k+p}] t_{k+i} \rightarrow \lambda, S \rightarrow t_{k+i} \in P, 1 \leq i \leq p\}\).
The new grammar \(G_{k+p} = (N_{k+p}, T, P_{k+p}, S)\) generates the same language as \(G_k\).

Let \(\varphi: (N_{k+p} - \{S\})^* \rightarrow T^*\) be the homomorphism defined by \(\varphi(N) = \lambda\), for each rule
of the form \(N \rightarrow XY\), \(N, X, Y \in N_k - \{S\}\), and \(\varphi(N) = t\), for each rule of the form \(N \rightarrow t, N \in N_k - \{S\}\), and \(t \in T\), \(\varphi([k+1]) = t_{k+i}\), and \(\varphi([k+i]) = \lambda\), for each \(1 \leq i \leq p\). Obviously,
\(L = \varphi(D'_K)\), where \(K = k + p, D'_K = L(G_k) \cup L_p, L_p = \{[t_{k+1}, ...], [t_{k+p}] t_{k+i}\}\).

In the sequel, grammar \(G_{k+p}\) is called the extended grammar of \(G_k\). \(G_k\) has an extended
grammar if and only if \(G_k\) (or \(G\)) has rules of the form \(S \rightarrow t, t \in T \cup \{\lambda\}\). If \(G_k\) does not
have an extended grammar then \(D'_K = D_k = L(G_k)\).
3 On the Chomsky-Schützenberger Theorem

Let $G_k = (N_k, T, P_k, S)$ be an arbitrary context-free grammar in Dyck normal form, with $N_k = \{S, [1, \ldots, k], [1, \ldots, k]\}$, and $\varphi: (N_k - \{S\})^* \to T^*$ the restriction of the homomorphism $\varphi$ in the proof of Theorem 2.9. We divide $N_k$ into three main sets $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ as follows:

1. $[i]$ and $]_i$ belong to $N^{(1)}$ if and only if $\varphi([i]) = t$ and $\varphi(]_i) = t'$, $t, t' \in T$,
2. $[i]$ and $]_i$ belong to $N^{(2)}$ if and only if $\varphi([i]) = t$ and $\varphi(]_i) = \lambda$, or vice versa $\varphi([i]) = \lambda$ and $\varphi(]_i) = t$, $t \in T$,
3. $[i, ]_i \in N^{(3)}$ if and only $\varphi([i]) = \lambda$ and $\varphi(]_i) = \lambda$.

Certainly, $N_k - \{S\} = N^{(1)} \cup N^{(2)} \cup N^{(3)}$ and $N^{(1)} \cap N^{(2)} \cap N^{(3)} = \emptyset$. $N^{(2)}$ is further divided into $N^{(2)}_l$ and $N^{(2)}_r$, where $N^{(2)}_l$ contains those pairs $[i, ]_i \in N^{(2)}$ such that $\varphi([i]) \neq \lambda$, while $N^{(2)}_r$ contains those pairs $[i, ]_i \in N^{(2)}$ such that $\varphi([i]) \neq \lambda$.

Definition 3.1. A grammar $G_k$ is in linear-Dyck normal form if $G_k$ is in Dyck normal form and $N^{(3)} = \emptyset$.

Theorem 3.2. For each linear grammar $G$, there exists a grammar $G_k$ in linear-Dyck normal form such that $L(G) = L(G_k)$, and vice versa.

Proof. Each linear grammar $G$, in standard form, is composed of rules of the forms $X \to \lambda$, $X \to t$, $X \to t_1 Y$, $X \to Y t_2$, $X \to t_1 Y t_2$, $t, t_1, t_2 \in T$, $X, Y \in N$. Transforming $G$ into Chomsky normal form, and then into the Dyck normal form, we obtain a grammar $G_k$ in linear-Dyck normal form. Since the standard form for linear languages, Chomsky normal form, and Dyck normal form are weakly equivalent we obtain $L(G) = L(G_k)$. The converse statement is trivial.

Next we consider more closely the structures of the derivation trees associated with words generated by linear and context-free grammars in linear-Dyck normal form and Dyck normal form, respectively. We are interested on the structure of the trace-words associated with words generated by these grammars.

Let $G_k = (N_k, T, P_k, S)$ be an arbitrary (linear) context-free grammar in (linear-)Dyck normal form, and $L(G_k)$ the language generated by this grammar. Let $w \in L(G_k)$, $D$ a leftmost derivation of $w$, and $t_w, D$ the trace-word of $w$ associated with $D$. From the structure of the derivation tree, read in the depth-first search order, it is easy to observe that each bracket $[i]$, such that $[i,]_i \in N^{(1)}$, is immediately followed, in $t_w, D$ by its pairwise $]_i$. The same property holds for those pairs $[i, ]_i \in N^{(2)}_l$. If $[i, ]_i \in N^{(2)}_r \cup N^{(3)}$ then the pair $[i,]_i$ should embed a left subtree, i.e., the case of the left embedded subtree in the proof of Theorem 2.8. In this case the bracket $[i]$ may have a left, long distance, placement from its pairwise $]_i$, in $t_w, D$.

Suppose that $G_k$ is a linear grammar in linear-Dyck normal form, i.e., $N^{(3)} = \emptyset$, such that $N^{(2)}_l \neq \emptyset$ and $N^{(2)}_r \neq \emptyset$. Each word $w = a_1 a_2 \ldots a_n \in L(G_k)$, of an arbitrary length $n$, has the property that there exists an index $n_t$, $1 \leq n_t \leq n - 1$, and a unique pair

\[\text{To emphasize which of the brackets in the pair } ([i,]_i) \text{ produces a terminal, we also use the notation } [i,]_i \text{ if and only if } [i,]_i \in N^{(2)}_l, [i,]_i \text{ if and only if } [i,]_i \in N^{(2)}_r, \text{ and } [i,]_i \text{ if and only if } [i,]_i \in N^{(1)}.\]
\[ \text{[[1, 1]] in N(1), such that } [j] \rightarrow a_{nt} \text{ and } [j] \rightarrow a_{nt+1}. \] Using the homomorphism \( \varphi \) in Theorem 2.9, we have \( \varphi([j]) = a_{nt} \) and \( \varphi([j]) = a_{nt+1} \). For the position \( nt \) already “marked”, there is no other position in \( w \) with the above property. We call \([j] \) \( j \) the core segment of the trace-word \( t_{w,D} \). Trace-words of words generated by context-free grammars in Dyck normal form have more than one core segment. Each core segment induces in a trace-word (both for linear and context-free languages) a symmetrical distribution of right brackets in \( N^{(2)}_l \cup N^{(3)}_r \) (always placed at the right side of the core segment) according to left brackets in \( N^{(2)}_l \cup N^{(3)}_r \) (always placed at the left side of the respective core). The structure of the trace-word of a word \( w \in L(G_k) \), for a grammar \( G_k \) in linear-Dyck normal form, is depicted in (1), where by vertical lines we emphasize the image through the homomorphism \( \varphi \) of each bracket occurring in \( t_{w,D} \).

\[
t_{w,D} = \begin{array}{cccccccc}
\lambda & \ldots & \lambda & a_1 & \lambda & \ldots & \lambda & a_2 & \ldots & a_{nt-1} & \lambda & \ldots & \lambda \\
\lambda & \ldots & \lambda & \ldots & \lambda & \ldots & \lambda & \ldots & \lambda & \ldots & \lambda & \ldots & \lambda \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
\end{array}
\]

Next our aim is to find a connection between Theorem 2.9 and the Chomsky-Schützenberger theorem. More precisely we want to compute, from the structure of trace-words, the regular and the Dyck languages yielded by the Chomsky-Schützenberger theorem. Therefore, we build a transition-like diagram for context-free grammars in Dyck normal form. First we build some directed graphs as follows.

**Construction 3.3.** Let \( G_k = (N_k, T, P_k, S) \) be an arbitrary context-free grammar in Dyck normal form. A dependency graph of \( G_k \) is a directed graph \( G^X = (V_X, E_X) \), \( X \in \{[j][l, j] \in N^{(3)}_r \} \cup \{S\} \), in which vertices are labeled with variables in \( \tilde{N}_k \cup \{X\} \), \( \tilde{N}_k \) \( = \{[i][l, i] \in N^{(1)}_r \cup N^{(2)}_l \cup N^{(3)}_r \} \cup \{[j][l, j] \in N^{(1)}_r \} \) and the set of edges is built as follows. For each rule \( X \rightarrow [i] \in P_k \), \( [i][j] \in N^{(2)}_l \), \( G^X \) contains a directed edge from \( X \) to \( [j] \), for each rule \( X \rightarrow [i] \in P_k \), \( [i][l, i] \in N^{(1)}_r \cup N^{(2)}_r \cup N^{(3)}_r \), \( G^X \) contains a directed edge from \( X \) to \( [i] \). There exists an edge in \( G^X \) from a vertex labeled by \([i][j, i] \in N^{(2)}_l \cup N^{(3)}_l \), to a vertex labeled by \([j]/k, [j][l, j] \in N^{(2)}_l \), \([k], k \in N^{(1)}_r \cup N^{(2)}_l \cup N^{(3)}_r \), if there exists a rule in \( P_k \) of the form \([i][j]/l \rightarrow [j]/k, k \). There exists an edge in \( G^X \) from a vertex labeled by \([i], [i][l, i] \in N^{(2)}_l \), to a vertex labeled by \([j]/k, [j][l, j] \in N^{(2)}_l \), \([k], k \in N^{(1)}_r \cup N^{(2)}_l \cup N^{(3)}_r \), if there exists a rule in \( P_k \) of the form \([i][j]/l \rightarrow [j]/k, k \). The vertex labeled by \( X \) is called the initial vertex of \( G^X \). Any vertex labeled by a left bracket in \( N^{(1)}_r \) is a final vertex.

Let \( G^X \) be a dependency graph of \( G_k \). Consider the set of all possible paths in \( G^X \) starting from the initial vertex to a final vertex. Such a path is called terminal path. A loop or cycle in a graph is a path from \( v \) to \( v \) composed of distinct vertices. If from \( v \) to \( v \) there is no other vertex, then the loop is a self-loop. The cycle rank of a graph is a measure...
of the loop complexity formally defined\footnote{In brief, the rank of a cycle $C$ is 1 if there exists $v \in C$ such that $C - v$ is not a cycle. Recursively, the rank of a cycle $C$ is $k$ if there exists $v \in C$ such that $C - v$ contain a cycle of rank $k - 1$ and all the other cycles in $C - v$ have the rank at most $k - 1$.} and studied in \cite{3} and \cite{7}. In \cite{7} it is proved that from each two vertices $u$ and $v$ belonging to a digraph of cycle rank $k$, there exists a regular expression of star-height at most $k$ that describes the set of paths from $u$ to $v$. On the other hand, the cycle rank of a digraph with $n$ vertices is upper bounded by $n \log n$ \cite{13}.

Hence any regular expression obtained from a digraph with $n$ vertices has the star-height at most $n \log n$. Consequently, the (infinite) set of paths from an initial vertex to a final vertex in $G^X$, can be divided into a finite number of classes of terminal paths. Paths belonging to the same class are characterized by the same regular expression, in terms of $\ast$ and $+$ Kleene operations, of star-height at most $|V_X| \log |V_X|$ (which is finite related to the lengths of strings in $L(G_k)$).

Denote by $\mathcal{R}^X_n$ the set of all regular expressions over $\tilde{N}_k \cup \{X\}$ that can be read in $G^X$, starting from the initial vertex $X$ and ending in the final vertex $\lambda$. The cardinality of $\mathcal{R}^X_n$ is finite. Define the homomorphism $h_G: \tilde{N}_k \cup \{X\} \rightarrow \{i, i, i \in N_r^{(2)} \cup N^{(3)}\} \cup \{\lambda\}$ such that $h_G(i[i, i, i \in N_r^{(2)}) \cup N^{(3)}\}$, $h_G(X) = h_G(\lambda) = h_G(\lambda) = \lambda$, for any $[i, i, i \in N^{(1)}$ and $[i, i, i \in N_r^{(2)}$. For any element $r.e_i \in \mathcal{R}^X_n$ we build a new regular expression $r.e_i = h_G^{-1}(r.e_i)$, where $h_G$ is the mirror image of $h_G$. Consider $r.e_i \in \mathcal{R}^X_n \cup \mathcal{R}^X_n$. For a certain $X$ and $\lambda$, denote by $\mathcal{R}.e^X_n$ the set of all regular expressions $r.e_i$ obtained as above.

Furthermore, $\mathcal{R}.e^X_n = \bigcup_{i, j} \mathcal{R}.e^X_n$ and $\mathcal{R}.e = \mathcal{R}.e^S \cup \bigcup_{i, j} \mathcal{R}.e^X_n$.\footnote{Informally, this is the (maximal) power of a nested $\ast$-loop occurring in the description of a regular expression. For the formal definition the reader is referred to \cite{17} and \cite{15} (see also Definition 4.1, Section 4).}

\textbf{Construction 3.4.} Let $G_k = (N_k, T, P_k, S)$ be a context-free grammar in Dyck normal form and $\{G^X | X \in \{i, j, j, j \in N^{(3)} \cup \{S\}\}\}$ the set of dependency graphs of $G_k$. The extended dependency graph of $G_k$, denoted by $G_e = (V_e, E_e)$, is a directed graph for which $\forall e = \tilde{N}_k \cup \{S\} \cup \{i, i, i \in N_r^{(2)} \cup N^{(3)}\}$, $S$ is the initial vertex of $G_e$ and $E_e$ is built as follows:

1. - $S[i, j] -$ there exists an edge in $G_e$ from the vertex labeled by $S$ to a vertex labeled by $[i$ (from $S$ to $[j], [i, i, i \in N^{(1)} \cup N_r^{(2)} \cup N^{(3)} ([j], j \in N_r^{(2)}), if there exists a regular expression in $\mathcal{R}.e^S$ with a prefix of the form $S[i, j$, respectively).

2. - $[i, j] -$ there exists an edge in $G_e$ from a vertex labeled by $[i$ to a vertex labeled by $[j$, $[i, j, j \in N_r^{(2)}$ if there exists a regular expression in $\mathcal{R}.e$ having a substring of the form $[i, j$ (if $i = j$ then $[i, i$ forms a self-loop in $G_e$).

3. - $[i, j] (or $[i, j]$) - there exists an edge in $G_e$ from a vertex labeled by $[i$ to a vertex labeled by $[j$ (or vice versa from $[j$ to $[i]$) such that $[i, i, i \in N_r^{(2)}$ and $[j], j \in N_r^{(2)} \cup N^{(3)}$, if there exists a regular expression in $\mathcal{R}.e$ having a substring of the form $[i, j$, respectively).

4. - $[i, j] -$ there exists an edge in $G_e$ from a vertex labeled by $[i$ to a vertex labeled by $[j$, $[i, i, i, j \in N_r^{(2)} \cup N^{(3)}$, if there exists a regular expression in $\mathcal{R}.e$ having a substring of the form $[i, j$.
form $[i]_j$ (if $i = j$ then $[i]_i$ forms a self-loop in $G_e$).

5. - $[i]_j$ - there exists an edge in $G_e$ from a vertex labeled by $[i]$ to a vertex labeled by $[j]$, if there exists a regular expression in $R$ with a substring of the form $[i]_j$ (or $[i]_j$ respectively), if there exists a regular expression in $R.e$ with a substring of the form $[i]_j$ (or $[i]_j$ respectively).

6. - $[j]_i$ - there exists an edge in $G_e$ from a vertex labeled by $[j]$ to a vertex labeled by $[i]$, if there exists a regular expression in $R.e$ with a substring of the form $[j]_i$ (or $[j]_i$ respectively).

7. - $[i]_j$ - there exists an edge in $G_e$ from a vertex labeled by $[i]$ to a vertex labeled by $[j]$, if there exists a regular expression in $R.e$ having a substring of the form $[i]_j$ (if $i = j$ then $[i]_i$ is a self-loop).

8. - $[i]_j$ - there exists an edge in $G_e$ from a vertex labeled by $[i]$, to a vertex labeled by $[j]$, if there exists a regular expression in $R.e$ having a substring of the form $[i]_j$, and a regular expression in $R.e^k$ that ends in $[i]$ (if $i = j$ then $[i]_i$ is a self-loop).

9. - $[i]_j$ - there exists an edge in $G_e$ from a vertex labeled by $[i]$, to a vertex labeled by $[j]$, if there exists a regular expression in $R.e$ having a substring of the form $[i]_j$, and a regular expression in $R.e^k$ that ends in $[i]$.

10. - A vertex labeled by $[i]_j$, $[i]_j \in N^{(1)}$, is a final vertex in $G_e$ if either $i$, $ii.$, or $iii.$ holds:

   i. there exists a regular expression in $R.e$ with a substring of the form $[i]_j$, and a regular expression in $R.e^S$ that ends in $[i]_j$.

   ii. there exists an edge in $G_e$ from a vertex labeled by $[i]$, to a vertex labeled by $[j]$, if there exists a regular expression in $R.e^S$ that ends in $[j]$.

   iii. there exists a regular expression in $R.e^S$ that ends in $[i]_j$, and a regular expression in $R.e^{k_1}$ that ends in $[i]_j$.

11. - A vertex labeled by $[i]_j$, $[i]_j \in N^{(2)}$, is a final vertex in $G_e$ if either $i$, $ii.$, or $iii.$ holds:

   i. there exists a regular expression in $R.e^S$ that ends in $[i]_j$.
ii. there exists \([k, k] \in N^{(3)}\), such that there exist a regular expression in \(R.e^{S}\) that ends in \([k, k]\), and a regular expression in \(R.e^{k}\) that ends in \([i, i]\).

iii. there exists \([k, k] \in N^{(3)}\) such that there exist a regular expression in \(R.e^{S}\) that ends in \([k, k]\), and \([k_1, k_1, \ldots, k_m, k_m] \in N^{(3)}\) such that there exist a regular expression in \(R.e^{k}\) ending in \([k_1]\), a regular expression in \(R.e^{k_1}\) ending in \([k_2]\), and so on, until a regular expression in \(R.e^{k_{m-1}}\) ending in \([k_m]\) and a regular expression in \(R.e^{k_m}\) ending in \([i, i]\) are reached.

Denote by \(R_e\) the set of all regular expressions obtained by reading all paths in \(G_e\) from the initial vertex \(S\) to all final vertices (i.e., all terminal paths). We have

**Theorem 3.5. (Chomsky-Schützenberger theorem)** For each context-free language \(L\) there exist an integer \(K\), a regular set \(R\), and a homomorphism \(h\), such that \(L = h(D_K \cap R)\). Furthermore, if \(G\) is the context-free grammar that generates \(L\), \(G_k\) the Dyck normal form of \(G\), and \(G_k\) has no extended grammar, then \(K = k\) and \(D_K \cap R = L(G_k)\). Otherwise, there exists \(p > 0\) such that \(K = k + p\), and \(D_K \cap R = D'_K\), where \(D'_K\) is the subset of \(D_K\) computed as in Theorem 2.9.

**Proof.** Let \(G_k = (N_k, T, P_k, S)\) be the Dyck normal form of \(G\) such that \(L = L(G)\). Suppose that \(G_k\) does not have an extended grammar. Let \(h_k: N_k \cup \{[i, i], [k, k] \in N^{(2)} \cup N^{(3)}\} \cup \{S\} \to \{[i, i], [k, k] \in N^{(2)} \cup N^{(3)}\} \cup \{\lambda\}\) be the homomorphism defined by \(h_k(S) = \lambda\), \(h_k([i, i]) = [i, i]\) for \([i, i] \in N^{(2)} \cup N^{(3)}\), \(h_k([k, k]) = [k, k]\) for \([k, k] \in N^{(2)}\), and \(h_k([i, i]) = [i, i]\) for \([i, i] \in N^{(1)}\). Then \(R = h_k(R_e)\) is a regular language such that \(D_k \cap R = L(G_k)\).

To prove the last equality, notice that each terminal path in a dependency graph \(G^X\) (Construction 3.3) provides a string equal to a substring (or a prefix if \(X = S\)) of a trace-word in \(L(G_k)\) (in which left brackets in \(N^{(2)}\) are omitted) generated (in the leftmost derivation order) from the derivation time when \(X\) is rewritten, up to the moment when the very first left bracket of a pair in \(N^{(1)}\) is rewritten. This string corresponds to a regular expression \(r.e^{(l, X)}\) in \(R_e^X\), which is extended with another regular expression \(r.e^{(r, X)}\) that is the “mirror image” of left brackets in \(N^{(2)} \cup N^{(3)}\) occurring in \(r.e^{(l, X)}\). If left brackets in \(N^{(2)} \cup N^{(3)}\) are enrolled in a star-height, then their homomorphic image (through \(h^c\)) in \(r.e^{(r, X)}\) is another star-height. The “mirror image” of consecutive left brackets in \(N^{(2)}\) (with respect to their relative core) is a segment composed of consecutive right brackets in \(N^{(2)}\). The “mirror image” of consecutive left brackets in \(N^{(3)}\) is “broken” by the interpolation of a regular expression \(r.e^{(l, X)}\) in \(R.e^{(r, X)}\), \([j, j] \in N^{(3)}\). The number of \(r.e^{(l, X)}\) insertions matches the number of left brackets \([j, j] \in N^{(3)}\) placed at the left side of the relative core (this is assured by the intersection with \(D_k\)). In fact, the extended dependency graph of \(G_k\) has been conceived such that it reproduces, on regular expressions in \(R_e\), the structure of trace-words in \(L(G_k)\). The main problem is the “star-height synchronizations” for brackets in \(N^{(2)} \cup N^{(3)}\), i.e., the number of left-brackets occurring in a loop placed at the left-side of a core segment \([i, i]\), to be equal to the number of their pairwise right-brackets occurring in the corresponding “mirror” loop placed at the right-side of its relative core, \([i, i]\), \([i, i] \in N^{(1)}\). This is controlled
by the intersection of \( h_k(R_e) \) with \( D_k \), leading to \( L(G_k) \). In few words, the proof is by
the construction described in Construction 3.4. Another problem that occurs is that the
construction of \( G_e \) allows to concatenate \( r.e.\mathcal{L}(X) \in R_{\mathcal{L}(X)} \) to its right pairwise \( r.e.\mathcal{L}(X') \) as well
as to another regular expression \( r.e.\mathcal{L}(X') \) (which by construction it is also concatenated to
its left pairwise \( r.e.\mathcal{L}(X') \)) where \( X \) and \( X' \) are not necessarily distinct. This does not change
the intersection with the Dyck language, but enlarges the regular language \( R = h_k(R_e) \)
with useless\(^9\) words.

If \( G_k \) has an extended grammar \( G_{k+p} = (N_{k+p}, T, P_{k+p}, S) \), built as in the proof of
Theorem 2.9, then \( \mathcal{R}_e \) is augmented with \( \nabla_e = \{S[t_{k+1}, \ldots, S[t_{k+p}] \} \) and \( h_k \) is extended to
\( h_K : \tilde{N}_k \cup \{S\} \cup \{[[i], j], i \in N_r^2 \cup N^{(3)} \} \cup \{[t_{k+1}, \ldots, [t_{k+p}] \rightarrow ([i], j), [i], j \in N_r^2 \cup N^{(3)} \} \cup
\{[[i], i], i \in N_r^2 \cup N^{(1)} \} \cup \{[t_{k+1}], \ldots, [t_{k+p}], j \cup \{\lambda\}, \) where \( h_k(x) = h_k(x), \) \( x \not\in
\{[t_{k+1}, \ldots, [t_{k+p}] \}, h_k([t_{k+p}]) = ([t_{k+p}]), 1 \leq j \leq p, K = k + p. \) \( L(G_k) \) is augmented with
\( L_p = \{[t_{k+1}, t_{k+1}, \ldots, [t_{k+p}] \} \) and \( D_K = h_k(R_e \cup \nabla_e) \cap D_K = L(G_k) \cup L_p. \)

The homomorphism \( h \) is equal to \( \varphi \) in Theorem 2.9, i.e., \( \varphi : (N_{k+p} - \{S\})^* \rightarrow T^*, \)
\( \varphi(N) = \lambda, \) for each rule of the form \( N \rightarrow XY, N, X, Y \in N_k, \) and \( \varphi(N) = t, \) for each
rule of the form \( N \rightarrow t, N \in N_k - \{S\}, t \in T, \varphi ([k+i]) = t_{k+i}, \) and \( \varphi ([k+i]) = \lambda, \) for each
\( 1 \leq i \leq p. \)

Note that, for the case of linear languages there is only one dependency graph \( G^S \).
The regular language in the Chomsky-Schützenberger theorem can be built without the
use of the extended dependency graph. It suffices to consider only the regular expressions
in \( \mathcal{R}_e \cap \nabla_e \cup h_k(\mathcal{R}_e \cap \nabla_e) \), where \( K = k + p, G_K, \) \( \nabla_e \) and \( \varphi \) are defined as in Theorems 2.9 and 3.5.
If \( G_k \) has no extended grammar then \( L(G_k) = \varphi(D_k \cap h_k(\mathcal{R}_e \cap \nabla_e)). \) However, a graphical
representation may be considered an interesting common framework for both, linear and
context-free languages. Below we illustrate the manner in which the regular language in the
Chomsky-Schützenberger theorem can be computed for linear (Examples 3.6) and context-
free (Example 3.7) languages.

**Example 3.6.** Consider the linear context-free grammar \( G = (\{S, [1], [7], \ldots, [7\}_1, \), \( \{a, b, c, d\}, \)
\( S, P) \) in linear-Dyck normal form, with \( P = \{S \rightarrow [1]_1, [1]_1 \rightarrow [2]_2, [2]_2 \rightarrow [3]_3, [3]_3 \rightarrow [2]_2, [4]_4 \rightarrow
[5]_5, [5]_5 \rightarrow [6]_6, [6]_6 \rightarrow [4]_4 [5]_5, [1]_1 \rightarrow [2]_2, [2]_2 \rightarrow b, [3]_3 \rightarrow b, c, [4]_4 \rightarrow d, [5]_5 \rightarrow a, [6]_6 \rightarrow a\). \)

The dependency graph \( G^S \) and extended dependency graph \( G_e \) of \( G \) are depicted in Figure
1a and 1b, respectively. There exists only one regular expression readable from \( G^S \), i.e.,
\( r.e.\mathcal{L}(S) = S([1][2]_3^+ [4]_4 [5]_5^+ [7]_7^+). \) Hence, \( r.e.\mathcal{L}(S) = S([1][2][3]^+ [4]_4 [5]_5 [6]_6^+ [7]_7^+ [7]_7^+ [5]_5 [3]_3^+). \)

The regular language provided by the Chomsky-Schützenberger theorem is
\( R = ([1]_1 [2]_2 [3]_3^+ [4]_4 [5]_5 [6]_6^+ [7]_7^+ [5]_5 [3]_3^+)^+. \)

Therefore, \( D_T^2 = D_T \cap R \) \( = \{([1]_1 [2]_2 [3]_3^+ [4]_4 [5]_5 [6]_6^+ [7]_7^+ [5]_5 [3]_3^+)^m, m \geq 1\} \) \( = L(G_k) \) and \( L(G) = \varphi(D_T^2) = \{(abb)^m aa (d(cb)^n)^m, m \geq 1\} \) \( (G \) contains no rule of the
form \( S \rightarrow t, t \in T). \)

\(^9\)In Section 4 we show how these unnecessary concatenations can be avoided, through a refinement
procedure of the regular language in the Chomsky-Schützenberger theorem.
Example 3.7. Consider the context-free grammar $G = (\{S, [\ldots, [\ldots], \ldots]\}, \{a, b, c\}, S, P)$ in Dyck normal form with $P = \{S \rightarrow [1], [1] \rightarrow [5]/[1], 1 \rightarrow [6], [2] \rightarrow [6]/[7], [3] \rightarrow [7], [5] \rightarrow \}$.

The sets of regular expressions and extended regular expressions obtained by reading $G^S$ (Figure 2.a) are $\mathcal{R}^{S}_{[1]} = \{S\}^{[1]}$ and $\mathcal{R}.e^S = \mathcal{R}.e^S_{[1]} = \{S\}^{[1]}$, respectively.

The regular expressions and extended regular expressions readable from $G^t$ (Figure 2.b) are $\mathcal{R}^{t}_{[1]} = \{1[6[3]7]^{+}[4]\}^+$ and $\mathcal{R}.e^t = \{1[6[3]7]^{+}[4][5]^{+}\}^+$, respectively. The regular expressions and extended regular expressions obtained by reading $G^t$ (Figure 2.c) are $\mathcal{R}^{t}_{[1]} = \{0[2[6[3]7]^{+}[4][5]^{+}]_{[1]}\}$ and $\mathcal{R}.e^t = \mathcal{R}.e^t_{[1]} = \{0[2[6[3]7]^{+}[4][5]^{+}]_{[1]}\}$, respectively.

The extended dependency graph of $G$ is sketched in Figure 2.d. Edges in black, are built from the regular expressions in $\mathcal{R}^{X}_{[1]}$, $X \in \{S, [\ldots], [\ldots]\}$. Orange edges emphasize symmetrical structures, built with respect to the structure of trace-words in $L(G)$. Some of them (e.g., $[1]_{[2]}$ and $[3]_{[2]}$) connect regular expressions in $\mathcal{R}^c$ between them with respect to the structure of trace-words in $L(G)$ (see Construction 3.4, item 8). The edge $[1]_{[2]}$ is added because there exists at least one regular expression in $\mathcal{R}^c$ that contains $[1]_{[1]}$ (e.g. $S[1][5][5][5][7]$), a regular expression in $\mathcal{R}.e^t_{[1]}$ that ends in $[6]_{[0]}$ (e.g. $1[6[3]7]^{+}[4][5]^{+}_{[0]}$) and a regular expression in $\mathcal{R}.e^t_{[1]}$ that ends in $[7]_{[0]}$ (see Construction 3.4, item 8.i). The + self-loop $[7]_{[7]}$ is due to the existence of a regular expression that contains $[0]_{[5]}$ (e.g. $0[2[6[3]7]^{+}[4][5]^{+}_{[0]}$) and a regular expression in $\mathcal{R}.e^t_{[1]}$ that ends in $[2]_{[0]}$ (e.g. $0[2[6[3]7]^{+}[4][5]^{+}_{[0]}$ or $0[2[6][3]7]^{+}[4][5]^{+}_{[0]}$).

The regular language provided by the Chomsky-Schützenberger theorem is the homomorphic image, through $h_k$ (defined in Theorem 3.5), of all regular expressions associated with all paths in the extended dependency graph in Figure 2.d, reachable from the initial vertex $S$ to the final vertex labeled by $[1]_{[2]}$, i.e., terminal paths.

The interpretation that emerges from the graphical method described in this paper is that the regular language in the Chomsky-Schützenberger theorem intersected with a (certain) Dyck language lists all derivation trees (read in the depth-first search order) associated with words in a context-free grammar, in Dyck normal form or in Chomsky normal form.
Figure 2: a. - d. The dependency graphs of the context-free grammar $G$ in Example 3.7. e. The extended dependency graph of $G$. In all graphs, vertices colored in red are initial vertices, while vertices colored in blue are final vertices. Edges colored in orange, in d. emphasize symmetrical structures obtained by linking the dependency graphs between them.

(since these derivation trees are equal, up to an homomorphism). The intersection forms (with very little exceptions) the trace-language associated with the respective context-free grammar.

In the next section we refine the extended dependency graph $G_e$ to provide a thinner regular language in the Chomsky-Schützenberger theorem with respect to the structure of the context-free grammar in Dyck normal form obtained through the algorithm described in the proof of Theorem 1.2. Based on this readjustment in Section 5 we sketch a transition diagram for a finite automaton and a regular grammar that generates a regular superset approximation for the initial context-free language.

4 Further Refinements of the Regular Language in the Chomsky-Schützenberger Theorem

One of the main disadvantages of considering $*$-height regular expressions in building the extended dependency graph associated with a context-free grammar in Dyck normal form is that some $*$-loops composed of right brackets in $N_r(2) \cup N(3)$ may not be symmetrically arranged according to their corresponding left brackets in $N_r(2) \cup N(3)$, if we consider their corresponding core segment as a symmetrical center. This is due to the possibility of having “$\lambda$-loops”. This deficiency does not affect the intersection with a Dyck language, but it has the disadvantage of enlarging considerably the regular language in the Chomsky-Schützenberger theorem. This can be avoided by considering only loops described in terms of $+$ Kleene closure.

Another disfunction of the extended dependency graph built through Construction 3.4 is the concatenation of a regular expression $r.e_{i_1}^{l,X}$ with another regular expression $r.e_{i_2}^{r,X'}$,
Let $\Sigma$ be a finite alphabet. The plus-height $h(r)$ of a regular expression $r$ is defined recursively as follows: i. $h(\lambda) = h(\emptyset) = h(a) = 0$ for $a \in \Sigma$, ii. $h(r_1 \cup r_2) = h(r_1) + h(r_2) = \max\{h(r_1), h(r_2)\}$, and $h(r^+) = \max(h(r_1), h(r_2))$, and $h(r^+) = h(r) + 1$.

Note that for any star-height regular expression it is possible to build a digraph, with an initial vertex $v_i$ and a final vertex $v_f$, such that all paths in this digraph, from $v_i$ to $v_f$, to provide the respective regular expression (which can be done in a similar manner as in Construction 3.4). However, if the regular expression is described in terms of plus-height then this statement may not be true (due to the repetition of some symbols). To force this statement be true, also for plus-height regular expressions, each repetition of a bracket is marked by a distinct symbol (e.g., $]_6[(_7[3]^+)\]_7$ becomes $]_6[(_7[3]^+)\]_7^\prime$), and then, for the new plus-height regular expression obtained in this way, we build a digraph with the above property. In order to recover the initial plus-height regular expression from the associated digraph, a homomorphism that maps all the marked brackets (by distinct symbols) into the initial one must be applied. Each time it is required, we refer to such a vertex as a $h$-marked vertex. Therefore, due to the technical transformations described above and the symmetrical considerations used in the construction of a trace language, we may assume to work only with plus-height regular expressions.

Let $G_k = (N_k, T, P_k, S)$ be an arbitrary context-free grammar in Dyck normal form, and $\mathcal{G}X$ the dependency graph of $G_k$ (see Construction 3.3). Denote by $\mathcal{P}^X_{l,i}$ the set of all plus-height regular expressions over $\tilde{N}_k \cup \{X\}$ that can be read in $\mathcal{G}X$, starting from the initial vertex $X$ and ending in the final vertex $[l]$. The cardinality of $\mathcal{P}^X_{l,i}$ is finite. Now, we consider the same homomorphism, as defined for the case of the set $\mathcal{R}^X_{l,i}$, i.e., $h_\mathcal{G} : \tilde{N}_k \cup \{X\} \to \{[i] | [i] \in N^2_r \cup N^3_r \} \cup \{\lambda\}$ such that $h_\mathcal{G}(i) = i$ for any pair $[i], [i] \in N^2_r \cup N^3_r$, $h_\mathcal{G}(X) = h_\mathcal{G}([i]) = h_\mathcal{G}([i]) = \lambda$, for any $[i], [i] \in N^1_r$ and $[i], [i] \in N^2_r$. For any element $r.e^{(l,X)}_{l,i} \in \mathcal{P}^X_{l,i}$ we build a new plus-height regular expression $r.e^{(l,X)}_{l,i} = h_\mathcal{G}(r.e^{(l,X)}_{l,i})$, where $h_\mathcal{G}$ is the mirror image of $h_\mathcal{G}$. Consider $r.e^X_{l,i} = r.e^{(l,X)}_{l,i}r.e^{(r,X)}_{l,i}$. For a certain $X$ and $[l]$, denote by $\mathcal{P}.e^X_{l,i}$ the set of all (plus-height) regular expressions $r.e^X_{l,i}$ obtained as above. Furthermore, $\mathcal{P}.e^X = \bigcup_{[l], [l] \in N^1_r} \mathcal{P}.e^X_{l,i}$, and $\mathcal{P}.e = \mathcal{P}.e^S \cup (\bigcup_{[l], [l] \in N^1_r} \mathcal{P}.e^X_{l,i})$.
Note that linear languages do not need an extended dependency graph. The set of all regular expressions $\mathcal{P}.e^S$ suffices to build a regular language in the Chomsky-Schützenberger theorem (see Theorem 3.5) that cannot be further adjusted by using the graphical method proposed in this section. Furthermore $|\mathcal{R}.e^S| \leq |\mathcal{P}.e^S|$. Equality takes place only for the case when each regular expression in $\mathcal{R}.e^S$ is a plus-height regular expression (see Example 3.6). For the case of context-free languages the plus-height regular expressions in $\mathcal{P}.e$ must be linked with each other in such a way it approximates, as much as possible, the trace-language associated with the respective context-free language.

In order to find an optimal connection of the regular expressions in $\mathcal{P}.e$ we consider the following labeling procedure of elements in $\mathcal{P}.e$. Denote by $c_0$ the cardinality of $\mathcal{P}.e^S$, i.e., $|\mathcal{P}.e^S| = c_0$, and by $c_j$ the cardinality of $\mathcal{P}.e^j$, where $[j], j \in N^3$. Each regular expression $r \in \mathcal{P}.e^S$ is labeled by a unique $q$, $1 \leq q \leq c_0$, and each regular expression $r \in \mathcal{P}.e^j$, is labeled by a unique $q$, such that $\sum_{r=0}^{i-1} e_r + 1 \leq q \leq \sum_{r=0}^{i} e_r$, $1 \leq i \leq s$, and $s = |\{[j], j \in N^3\}|$. Denote by $r^q$ the labeled version of $r$. To preserve symmetric structures that characterize trace-words of context-free languages, then when we link regular expressions in $\mathcal{P}.e$ between them, each bracket in a regular expression $r^q$ is upper labeled by $q$. Exception makes the first bracket occurring in $r^q$ (which is a bracket in $\{[j], j \in N^3\}$). Now, a refined extended digraph can be built similar to that described in Construction 3.4.

To have a better picture of how the labeled regular expressions must be linked to each other, and where further relabeling procedures may be required (to obtain a better approximation of the trace-language), we first build for each plus-height regular expression $r^q \in \mathcal{P}.e^j$, $[j], j \in N^3$, a digraph and then we connect all digraphs between them. Denote by $G^q,j$ the digraph associated with $r^q \in \mathcal{P}.e^j$, such that $[j]$ is the initial vertex and the final vertex is the last bracket occurring in $r^q$. Each digraph $G^q,j$ read from the initial vertex $[j]$ to the final vertex provides the regular expression $r^q$. Hence, any digraph $G^q,j$ has vertices labeled by brackets of the forms $\{[j], j \in N^1 \cup N^2 \cup N^3\} \cup \{[j], j \in N^1 \cup N^2 \cup N^3\}$, $c_0 \leq q \leq \sum_{r=0}^{s} e_r$, with the exception of the initial vertex $[j]$, $[j], j \in N^3$. Some of vertices in $G^q,j$, besides the $q$-index, may also be $h$-marked, in order to prevent repetitions of the same vertex which may occur in a plus-height regular expression. As the construction of the dependency graph does not depend on $h$-markers, unless it is necessary, we avoid $h$-marked notations in further explanations when building this digraph.

The adjacent vertex $Y$ to $[j]$ in $G^q,j$, is called sibling. Any edge of the form $[i], k \in N^3$, is called dummy edge, while $[i], k \in N^3$, is a dummy vertex. An edge that is not a dummy edge is called stable edge. Denote by $G^j$ the set of all digraphs $G^q,j$, i.e., their initial vertex is $[j]$. Any digraph $G^q,j$ has only one bracket $[j], k \in N^1$, which stands for a core segment in a trace-word. Right brackets $[j], j \in N^1 \cup N^3$, must be symmetrically arranged according to their left pairwise $[j], j \in N^1 \cup N^3$, that occur at the left side of $[j]$. A dummy vertex labeled by $[i], k \in N^3$, allows the connection with any digraph in $G^j$. A digraph in $G^j$ with a final vertex labeled by a bracket $[i], k \in N^1$, or by a bracket $[i], k \in N^3$, is called terminal, because the vertex $[i]$ or $[i]$, respectively, does not allow more connections.

Next we describe the procedure that builds a refined extended digraph with the property
that reading this digraph (in which each loop is a plus-loop) from the initial vertex (which is \( S \)) to all its final vertices, we obtain those (plus-height) regular expressions that form a regular language that provides the best approximation of the corresponding trace-language.

**Step 1.** First we build a digraph \( G.e^S \) that describes all (plus-height) regular expressions in \( P.e^S \). This can be done by connecting all digraphs in \( G^S \) to \( S \). Since each bracket labeling a vertex in \( G^q,S \), \( 1 \leq q \leq c_0 \), is uniquely labeled by \( q \), and there exists a finite number of brackets, \( G.e^S \) is correct (in the sense that it is finite and any vertex occurs only one time). The initial vertex of \( G.e^S \) is \( S \). If a graph in \( G^S \) has a final vertex labeled by a bracket \([_i,j]_i \in N^{(1)} \) or by a bracket \([_i,j]_i \in N^{(2)}_r \), then this is also a final vertex in \( G.e^S \).

If \( G_k \) is a grammar in linear-Dyck normal form then \( G.e^S \), built in this way, suffices to build the regular language in the Chomsky-Schützenberger theorem. The set of all paths from \( S \) to each final vertex to which we apply the homomorphism \( h_k \), defined in the proof of Theorem 3.5, yields a regular language \( R_m \) that cannot be further adjusted, such that the Chomsky-Schützenberger theorem still holds. Therefore, we call the \( R_m \) language, as minimal with respect to the grammar \( G_k \) and the Chomsky-Schützenberger theorem, i.e., the equality \( \varphi(D_k \cap R_m) = \varphi(L(G_k)) \) still holds, where \( \varphi \) is the homomorphism defined in the proof of Theorem 2.9.

**Step 2.** For each vertex \([_i,j]_i \in G^V \), such that \([_i,j]_i \in N^{(3)} \), we connect all digraphs in \( G^V \) to \( G.e^S \). This can be done by adding to \( G.e^S \) a new edge \([_i,j]_i \) for each sibling \( Y \) of \([_i,j]_i \) (in \( G^V \)). If \( Z \) is the adjacent vertex of \([_i,j]_i \) (in the former version of \( G.e^S \)), i.e., \([_i,j]_i \) \( Z \) is a dummy edge, then we remove in \( G.e^S \) the edge \([_i,j]_i \), while in \( G^V \) (connected to \( G.e^S \) through \([_i,j]_i \)) we remove the vertex \([_i,j]_i \) and consequently, the edge \([_i,j]_i \) \( Y \). For the moment, all the other edges in \( G^V \) are preserved in \( G.e^S \), too. Besides, if \( V \) is the final vertex of \( G^V \), then a new edge \( V \) \( Z \) is added to \( G.e^S \). If \( V \in \{[_{i,k}]_{i,k} \in N^{(1)} \} \cup \{[_{i,l}]_{i,l} \in N^{(2)}_r \} \), i.e., \( G^V \) is a terminal digraph then the edge \( V \) \( Z \) is a glue edge, i.e., it is a stable edge that makes the connection of \( G^V \) into \( G.e^S \) (or more precisely the connection of \( G^V \) to \( G.e^S \) digraph in which it has been inserted). Otherwise, \( V \) \( Z \) is a dummy edge, which will be removed at a further connection with a digraph in \( G^V \). Since for the case of linear languages generated by a grammar in linear-Dyck normal form, \( G.e^S \) does not contain any dummy vertex, the construction of \( G.e^S \) is completed at Step 1.

A vertex in \( G.e^S \) labeled by a bracket \([_i,j]_i \), \([_i,j]_i \in N^{(3)} \), that has no adjacent vertex, i.e., the out degree of the vertex labeled by \([_i,j]_i \) is 0, is called pop vertex. When connecting a digraph \( G^V \) to \( G.e^S \), through a pop vertex, if \( G^V \) is a terminal digraph, then the final vertex of \( G^V \) becomes a final vertex of \( G.e^S \). If \( G^V \) is not a terminal digraph, then the final vertex of \( G^V \) becomes a pop vertex for \( G.e^S \).

If there exist more then one vertex labeled by an upper indexed bracket \([_i,j]_i \), \([_i,j]_i \in N^{(3)} \), then, if \( G^{V \}\_i \) has been already added to \( G.e^S \) there is no need to add another “copy” of \( G^V \). It is enough to connect \([_i,j]_i \) to the digraph existing in \( G.e^S \), i.e., to add a new edge \([_i,j]_i \) \( Y \), where \( Y \) is a sibling of \([_i,j]_i \) in \( G^{V \}_\_i \). This observation holds for any element in \( G^V \).

The procedure described at Step 2 is repeated for each new dummy or pop vertex added

\(^{10}\)As \( G.e^S \) is finite, there cannot exist in \( G.e^S \) two right brackets \([_i,j]_i \), \([_i,j]_i \in N^{(3)} \), upper indexed by the same value.
to \( \mathcal{G}.e^S \). For each transformation performed on \( \mathcal{G}.e^S \), we maintain the same notation \( \mathcal{G}.e^S \) for the new obtained digraph. The construction of \( \mathcal{G}.e^S \) ends up then when each vertex \( [j]_k \), \( [j], j \in \mathbb{N}^3 \), has been connected to a digraph in \( \mathcal{G}_{ji}^l \), i.e., no dummy and pop vertices exist in \( \mathcal{G}.e^S \). The only permissible contexts under which a bracket \( [j]_k \), \( [j], j \in \mathbb{N}^3 \), may occur, in the final version of \( \mathcal{G}.e^S \), are of the forms \( [1]_k \), \( [n], n \in \mathbb{N} \), \( [2]_l \), \( [2], l \in \mathbb{N}^3 \), where \( [i, i] \in \mathbb{N}^2 \), \( [i, i] \in \mathbb{N}^1 \), \( [k, j] \in \mathbb{N}^1 \cup \mathbb{N}_r^2 \cup \mathbb{N}^3 \), \( [i, i] \in \mathbb{N}^2 \).

There are several refinements that can be done on \( \mathcal{G}.e^S \) such that the resulted regular language better approximates the trace language associated with the considered context-free grammar. Two peculiar situations may occur when adding digraphs to \( \mathcal{G}.e^S \): 

First, suppose that during the construction of \( \mathcal{G}.e^S \), \( [k, j, j] \in \mathbb{N}^3 \), we reach a terminal digraph with a finite vertex \( [j]_k \), \( [k, j] \in \mathbb{N}^3 \), such that \( [j]_k \) is linked to \( Z \), forming thus a stable (glue) edge \( [j]_k Z \). Denote by \( \varphi = [j]_k [j]_k Z \) the path (in \( \mathcal{G}.e^S \)) from \( [j] \) to \( [j]_k Z \), obtained at this stage. If the vertex that precedes \( [j]_k \) in \( \varphi \) is \( [j]_k \), \( [j], j \in \mathbb{N}^3 \), i.e., \( \varphi = [j]_k [j]_k Z \), then connecting \( [j]_k \) (\( [j]_k \) is a dummy edge), through its siblings, to digraphs in \( \mathcal{G}_{ji}^l \) another edge \( [j]_k [j]_k \) preceded by \( [j]_k [j]_k \), is added to \( \mathcal{G}.e^S \), i.e., \( \varphi \) becomes \( \varphi = [j]_k [j]_k Z \). Since \( [j]_k [j]_k \) is a dummy edge, the vertex \( [j]_k \) must be again connected to digraphs in \( \mathcal{G}_{ji}^l \), and so on, until \( [j]_k \) is connected to a terminal digraph \( \mathcal{G}_{ji}^l \in \mathcal{G}_{ji}^l \), \( \bar{q} \neq q' \), that has a final vertex labeled by a bracket \( [n]_k \), \( [k, k] \in \mathbb{N}^3 \) \( (m \text{ and } k \text{ not necessarily distinct}) \), or by a bracket \( [m, [m, [m] \in \mathbb{N}^3 \) such that \( [m]_k \) is not preceded by a bracket of the form \( [j]_k \), \( [j], j \in \mathbb{N}^3 \). Then \( \varphi \) is either of the form \( [j]_k [m]_k [m]_k + Z \) or of the form \( [j]_k [m]_k [m]_k + Z \), respectively. On the other hand, since \( \mathcal{G}_{ji}^l \in \mathcal{G}_{ji}^l \) the digraph \( \mathcal{G}_{ji}^l \) can be added to \( \mathcal{G}.e^S \), through \( [j]_k \), from the very first beginning, avoiding thus the plus-loop \( ([j]_k) \), i.e., there should exist in \( \mathcal{G}.e^S \) a new path \( \varphi' = [j]_k [m]_k [m]_k Z \) or \( \varphi' = [j]_k [m]_k [m]_k Z \) (where \( \varphi' \) is a path in \( \mathcal{G}_{ji}^l \)). This allows two other new paths to be created in \( \mathcal{G}.e^S \), i.e., \( \varphi'' = [j]_k [n]_k [m]_k Z \) (or \( \varphi'' = [j]_k [n]_k [m]_k Z \) and \( \varphi'' = [j]_k [n]_k [m]_k Z \) (or \( \varphi'' = [j]_k [n]_k [m]_k Z \), which are of no use in approximating the trace language (hence in building the regular language in the Chomsky-Schützenberger theorem). Paths \( \varphi \) and \( \varphi' \) \( \varphi'' \) do not affect the intersection with the Dyck language but they enlarge the regular language with useless words.

In order to avoid the paths \( \varphi \) and \( \varphi' \) (or \( \varphi'' \)) the terminal digraph \( \mathcal{G}_{ji}^l \) receives a new label \( \bar{q} \), besides of label \( q \) (which is maintained to allow \( \varphi \) to be produced). To allow the shorter path \( \varphi' \) to be created, instead of \( \mathcal{G}_{ji}^l \) the terminal digraph \( \mathcal{G}_{ji}^l \) is connected to \( \mathcal{G}.e^S \) through the dummy vertex \( [j]_k \). Hence \( \varphi' \) becomes \( \varphi' = [j]_k [m]_k Z \) (or \( \varphi' = [j]_k [m]_k Z \), while \( \varphi \) remains \( [j]_k [m]_k Z \) (or \( [j]_k [m]_k Z \)), respectively). This relabeling procedure is used for any case similar to that described above encountered during the computation of \( \mathcal{G}.e^S \). As there may exist a finite number of plus-loops in \( \mathcal{G}.e^S \), there will be a finite number of

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Footnotes:

11For instance, \( [j]_k \) may also be a dummy vertex and \( [j]_k Z \) a dummy edge.

12The plus-height of a regular expression obtained from any digraph in \( \mathcal{G}_{ji}^l \) is finite related to the length of the strings in \( L(G_k) \).
"relabeled" digraphs (not necessarily terminal). A loop (not necessarily a self-loop) may be reached through different paths that must be "renamed" (if we want to avoid that loop).

$T_2$. Another situation that requires a relabeling procedure may occur when connecting a digraph to $G.e^S$ through a pop vertex. Suppose that $\mathcal{I}^q_j$, $[i,j]_j \in N(3)$, is a pop vertex, and the digraph $G^{(2)}_q$ that must be added to $G.e^S$ has already connected through a dummy vertex labeled by $\mathcal{I}^q_j$ (i.e., $G^{(2)}_q$ has been already inserted in $G.e^S$). According to the procedure described at Step 2 the vertex $\mathcal{I}^q_j$ is linked to the sibling of $\mathcal{I}^q_j$ in $G^{(2)}_q$ already existing in $G.e^S$. Since the connection of $G^{(2)}_q$ to $G.e^S$ has been done through a dummy vertex, the final vertex in $G^{(2)}_q$ cannot be neither a final vertex in $G.e^S$ (if $G^{(2)}_q$ is a terminal digraph) nor a pop vertex.

To forbid a pop vertex $\mathcal{I}^q_j$ to overlap with a dummy vertex $\mathcal{I}^q_j$, each of the digraphs connected to $G.e^S$ through a pop vertex, is renamed by a new label. Denote by $G^{(2)}_q$ the labeled version of $G^{(2)}_q$. Then connections through pop vertices will be done by using only digraphs in $G^{(2)}_q$. However, any dummy vertex $\mathcal{I}^q_j$, that is not a pop vertex, obtained by connecting digraphs in $G^{(2)}_q$ to $G.e^S$ should be connected to the original digraphs in $G^{(2)}_q$, unless a relabeling procedure described at $T_1$ is required.

Denote by $\bar{N}_k = \{(i, i) \in N(1) \cup N_r(2) \cup N(3) \} \cup \{(i, j) \in N_r(2) \cup N(1) \} \cup \{(i, j) \in N_r(2) \cup N(1) \}$, the set of vertices composing $G.e^S$, in which some brackets may be $h$-marked (by distinct $h$-markers). To reach the regular language in the Chomsky-Schützenberger theorem we denote by $R_G$ the set of all regular expressions obtained by reading $G.e^S$ from the initial vertex $S$ to any final vertex. First, suppose that $G_k$ does not have an extended grammar. We have $K = k$ and $D_k = L(G_k)$. Consider the homomorphism $h_k : \bar{N}_k \cup \{S\} \rightarrow \{(i, i) | (i, i) \in N_r(2) \cup N(3) \} \cup \{(i, j) | (i, j) \in N_r(2) \cup N(1) \} \cup \{\lambda\}$, defined by $h_k(S) = \lambda$, $h_k([i, i]) = [i, h_k([i, i])] = [i]$, for any $[i, i] \in N_r(2)$, $h_k([i, j]) = [i, j]$ for any $[i, j] \in N_r(2)$, $h_k([i, i]) = [i]$, for any $[i, i] \in N(1)$. Then $R_m = h_k(R_G)$ is a regular language with $D_k \cap R_m = L(G_k)$. Furthermore, $R_m$ is a stronger refinement of $R$, such that the Chomsky-Schützenberger theorem still holds. This is because when building regular expressions in $\mathcal{P}e$, each $r.e^{(r, X)}_{[i]}$ is linked only to its right pairwise $r.e^{(r, X)}_{[i]i}$ (due to plus-height considerations and labeling procedures). In this way all plus-loops in $r.e^{(r, X)}_{[i]}$ are correctly mirrored (through $h^r_k$) into its correct pairwise $r.e^{(r, X)}_{[i]}$. The case of $\lambda$-loops is taken by the relabeling procedure described at $T_1$. This is also applicable each time we want to fork a path in $G.e^S$ in order to avoid useless loops on that path. The relabeling procedure $T_2$ allows to leave $G.e^S$ without re-loading another useless path. That is why the regular language $R_m$ built this way is a tighter approximation of $L(G_k)$. A finer language than $R_m$ can be found by searching for a more efficient grammar in Dyck normal form, with respect to the number of rules and nonterminals.

If $G_k$ has an extended grammar $G_{k+p} = (N_{k+p}, T, P_{k+p}, S)$ (built as in the proof of Theorem 2.9) then $R_G$ is augmented with $\nabla e = \{S[\mathcal{I}_{k+1}], \ldots, S[\mathcal{I}_{k+p}]\}$ and $h_k$ is extended to $h_K$, $h_K : \bar{N}_k \cup \{S\} \cup \{[\mathcal{I}_{k+1}], \ldots, [\mathcal{I}_{k+p}] \} \rightarrow \{(i, i) | (i, i) \in N_r(2) \cup N(3) \} \cup \{(i, j) | (i, j) \in N_r(2) \cup N(1) \} \cup \{[\mathcal{I}_{k+1}], [\mathcal{I}_{k+1}], \ldots, [\mathcal{I}_{k+p}] \} \cup \{\lambda\}$, $h_K(x) = h_k(x)$, $x \notin \{[\mathcal{I}_{k+1}], \ldots, [\mathcal{I}_{k+p}]\}$, and $h_K([\mathcal{I}_{k+j}]) = [\mathcal{I}_{k+j}]_{[\mathcal{I}_{k+j}], 1 - j \leq p}$, $K = k + p$. 23
Figure 3: a.- e. Graphs associated with regular expressions in $\mathcal{P}.e$ (Example 4.2). Initial vertices are colored in red, final vertices in blue, while purple vertices mark a core segment. $\bar{4}$ is a marked vertex to allow the plus-loop. $G_1^2$, $G_2^1$, $G_3^6$, $G_4^3$, $G_5^5$, $G_6^7$ are the graphs associated with regular expressions in $\mathcal{P}.e$ (Example 4.2).

**Example 4.2.** Consider the context-free grammar in Example 3.7 with the dependency graphs sketched in Figure 3. The set $\mathcal{P}.e$ of labeled plus-loop regular expressions built from the dependency graphs is composed of

$$S([1]^{1}+[1][11][1]'^+[1]_{1}^2)$$

(with the associated digraph $G_{1}^{1}$, Fig. 3.a),

$$S([1]^{2}+[1][11][1]'^+[1]_{1}^2)$$

(with $G_{1}^{2}$, Fig. 3.b),

$$S([1]^{3}+[1][11][1]'^+[1]_{1}^2)$$

(with $G_{1}^{3}$, Fig. 3.c),

$$S([4]^{1}+[4][11][1]'^+[1]_{1}^2)$$

(with $G_{1}^{4}$, Fig. 3.d),

$$S([5]^{1}+[5][11][1]'^+[1]_{1}^2)$$

(with $G_{1}^{5}$, Fig. 3.e).

The extended dependency graph built with respect to the refinement procedure is sketched in Figure 4. The terminal digraphs $G_{6}^{6}$ and $G_{6}^{7}$ are introduced with respect to the relabeling procedure $I_1$, in order to prevent the loop yielded by the "iterated" digraph $G_{1}^{3}$ to occur between $G_{1}^{2}$ and $G_{1}^{6}$ (or $G_{1}^{7}$). It also forbids the self-loop $[1]^{1}$ to be linked to $G_{6}^{6}$ (or to $G_{7}^{7}$), then when the digraph $G_{1}^{3}$ is not added to the corresponding path. Due to the self-loop $[1]^{1}$, in which $[1]$ is a pop vertex, we did not apply the relabeling procedure described at $I_2$ (applying it leads to the same result).

5 A Regular Superset Approximation for Context-Free Languages

A regular language $R$ may be considered a superset approximation for a context-free language $L$, if $L \subseteq R$. A good approximation for $L$ is that for which the set $R - L$ is as small as possible. There are considerable methods to find a regular approximation for a context-free language. The most significant consist in building, through several transformations applied to the original pushdown automaton (or context-free grammar), the most appropriate finite automaton (regular grammar) recognizing (generating) a regular superset approximation of the original context-free language. How accurate the approximation is, depends on the transformations applied to the considered devices. However, the perfect regular superset (or subset) approximation for an arbitrary context-free language cannot be built. For surveys on approximation methods and their practical applications in computational linguistics (es-
Figure 4: The refined dependency graph of the context-free grammar in Examples 3.7 and 4.2. $S$ is the initial vertex, vertices colored in green are final vertices, vertices colored in blue are dummy vertices, vertices colored in purple mark a core segment. Orange edges emphasize symetrical structures built with respect to the structure of the trace language. Green edges are glue edges.

especially in parsing theory) the reader is referred to [21] and [22]. Methods to measure the accuracy of a regular approximation can be found in [4], [8], and [23].

In the sequel we propose a new approximation technique that emerges from the Chomsky-Schützenberger theorem. In brief, the method consists in transforming the original context-free grammar into a context-free grammar in Dyck normal form. For this grammar we build the refined extended dependency graph $G_\epsilon$. From $G_\epsilon$ we depict a state diagram $A_\epsilon$ for a finite automaton and a regular grammar $G_r = (N_r, T, P_r, S)$ that generates a regular (superset) approximation for $L(G_k)$ (which is nothing else than the image through $\varphi$ of the language $R_m$ built in Section 4).

Let $G_k = (N_k, T, P_k, S)$ be an arbitrary context-free grammar in Dyck normal form, and $G_\epsilon^S = (V_\epsilon, E_\epsilon)$ the extended dependency graph of $G_k$. Recall that $V_\epsilon = \{[[i],i] \in N^{(1)} \cup N_r^{(2)} \cup N^{(3)} \} \cup \{[j],[j],j] \in N_l^{(2)} \cup N_r^{(2)} \cup N^{(3)} \} \cup \{S\}$ in which some of the vertices may be $h$-marked, in order to prevent repetition of the same bracket when building the digraph associated with a plus-height regular expression. In brief, the state diagram $A_\epsilon$ can be built by skipping in $G_\epsilon^S$ all left brackets in $N_l^{(2)}$ and all brackets in $N^{(3)}$, and labeling the edges with the symbol produced by left or right bracket in $N^{(2)} \cup N^{(1)}$. This reasoning is applied no matter whether the vertex in $V_\epsilon$ is $h$-marked or not. Therefore, we avoid $h$-marker specifications when building $A_\epsilon$, unless this is strictly necessary. Denote by $s_f$ the accepting state of $A_\epsilon$. The start state of $A_\epsilon$ is $s_S$, where $S$ is the axiom of $G_k$. We proceed as follows:

1. There exists an edge in $A_\epsilon$ from $s_S$ to $s_f$, labeled by $a$, where $[i],i] \in N_l^{(2)}$ and
\[ i \rightarrow a \in P_k \], if either \( S_{j}^{q} \in E_e \), or there exists a path in \( \mathcal{G}.e^{S} \) from \( S \) to \( j \) that contains no vertex labeled by \( \mathcal{A} \), \( [j, j] \in N_{i}^{(2)} \), or by \( [\mathcal{A}, k, k] \in N^{(1)} \). We fix \( S \rightarrow a[l] \in P_r \).

2. There exists an edge in \( \mathcal{A} \) from \( s_{l} \) to \( s_{r} \), labeled by \( a \), and an edge from \( s_{q} \) to \( s_{q} \) labeled by \( b \), where \( [l, l] \in N_{i}^{(1)} \), \( l \rightarrow a \), and \( l \rightarrow b \in P_k \), if either \( S_{l}^{q} \in E_e \), or there exists a path in \( \mathcal{G}.e^{S} \) from \( l \) to \( q \) that contains no vertex labeled by \( \mathcal{A} \), \( [j, j] \in N_{i}^{(2)} \), or by \( [\mathcal{A}, k, k] \in N^{(1)} \). We fix \( S \rightarrow a[l] \rightarrow b[q] \in P_r \).

3. There exists an edge in \( \mathcal{A} \) from \( s_{l} \) to \( s_{r} \), labeled by \( a \), \( [i, j] \in N_{i}^{(2)} \), or by \( [\mathcal{A}, k, k] \in N^{(1)} \). If \( i = j \) or \( i \neq j \), there exists a path in \( \mathcal{G}.e^{S} \) from \( l \) to \( q \) that contains no vertex labeled by \( \mathcal{A} \), \( [j, j] \in N_{i}^{(2)} \). We fix \( S \rightarrow a[l] \rightarrow b[q] \in P_r \).

4. There exists an edge in \( \mathcal{A} \) from \( s_{l} \) to \( s_{r} \), labeled by \( a \), and an edge from \( s_{q} \) to \( s_{q} \) labeled by \( b \), where \( [i, i] \in N_{i}^{(1)} \), \( [j, j] \in N_{i}^{(2)} \), \( l \rightarrow a \), and \( l \rightarrow b \in P_k \), if either \( S_{l}^{q} \in E_e \), or there exists a path in \( \mathcal{G}.e^{S} \) from \( l \) to \( q \) that contains no vertex labeled by \( \mathcal{A} \), \( [j, j] \in N_{i}^{(2)} \). We fix \( S \rightarrow a[l] \rightarrow b[q] \in P_r \).

5. There exists an edge in \( \mathcal{A} \) from \( s_{l} \) to \( s_{r} \), labeled by \( a \), \( [i, i] \in N_{i}^{(1)} \), \( [j, j] \in N_{i}^{(2)} \), and \( l \rightarrow a \in P_k \), if \( S_{l}^{q} \in E_e \). If \( i = j \) and \( q = q' \), then \( S_{l} \) is a self-loop in \( \mathcal{A} \) (because \( S_{l}^{q} \) is a self-loop in \( \mathcal{G}.e^{S} \)). We fix \( S \rightarrow a[l] \in P_r \). Note that, it is also possible to have \( i \neq j \) and \( q = q' \) (with \( i \neq j \), case in which \( S_{l}^{q} \) is a glue edge in \( \mathcal{G}.e^{S} \)).

6. There exists an edge in \( \mathcal{A} \) from \( s_{l} \) to \( s_{r} \), labeled by \( a \), \( [i, i] \in N_{i}^{(1)} \), \( [j, j] \in N_{i}^{(2)} \), and \( l \rightarrow a \in P_k \), if there exists a path in \( \mathcal{G}.e^{S} \) from \( l \) to \( q \) that contains no vertex labeled by \( \mathcal{A} \), \( [j, j] \in N_{i}^{(2)} \), or by \( [\mathcal{A}, k, k] \in N^{(1)} \). We fix \( S \rightarrow a[l] \in P_r \). Note that, \( q \) may be equal to \( q' \).

7. There exists an edge \( s_{l} s_{q} s_{q} \), labeled by \( a \), and an edge \( s_{l} s_{q} s_{q} \), labeled by \( b \), where \( [i, i] \in N_{i}^{(1)} \), \( [j, j] \in N_{i}^{(2)} \), \( l \rightarrow a \), and \( l \rightarrow b \in P_k \), if there exists a path in \( \mathcal{G}.e^{S} \) from \( l \) to \( q \) that contains no vertex labeled by \( \mathcal{A} \), \( [j, j] \in N_{i}^{(2)} \), or by \( [\mathcal{A}, k, k] \in N^{(1)} \). We fix \( S \rightarrow a[l] \rightarrow b[q] \in P_r \).

8. There exists an edge in \( \mathcal{A} \) from \( s_{l} \) to \( s_{r} \), labeled by \( a \), \( [i, i] \in N_{i}^{(1)} \), \( [j, j] \in N_{i}^{(2)} \), and \( l \rightarrow a \in P_k \), if \( S_{l}^{q} \in E_e \). We fix \( S \rightarrow a[l] \in P_r \). Note that, it is possible to have \( q = q' \) (in the last case \( S_{l}^{q} \) is a glue edge in \( \mathcal{G}.e^{S} \)).

9. There exists an edge \( s_{l} s_{q} s_{q} \), labeled by \( a \), and an edge \( s_{l} s_{q} s_{q} \), labeled by \( b \), where \( [i, i] \in N_{i}^{(1)} \), \( [j, j] \in N_{i}^{(2)} \), \( l \rightarrow a \), and \( l \rightarrow b \in P_k \), if there exists a path in \( \mathcal{G}.e^{S} \) from \( l \) to \( q \).

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13This case deals also with the situation when \( S_{l}^{q} \), \( [i, i] \in N_{i}^{(2)} \), occurs in a loop in \( \mathcal{G}.e^{S} \) composed of only left brackets in \( N_{i}^{(2)} \), or \( N^{(3)} \), excepting \( S_{r} \). A loop composed of only left brackets in \( N_{i}^{(2)} \), or \( N^{(3)} \) is ignored when building \( \mathcal{A} \).

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that contains no vertex labeled by $\lambda$, $[k, i]_k \in N_i^{(2)} \cup N_i^{(2)}$, or by $[i], [s, i]_i \in N_i^{(1)}$. We fix $\gamma_i^q \rightarrow a\gamma_j^q, \gamma_j^q \rightarrow b\gamma_j^t \in P_r$. Note that, $\gamma_i^q$ may be equal to $\gamma_j^q$, i.e., $i = j$ and $q = q'$, i.e., the case of a loop in $\gamma_i^q$.

10. - For any final vertex labeled by $\lambda$, $[s, i]_s \in N_r^{(2)}$, or by $[s, i]_s, [r, i]_r \in N_r^{(1)}$, in $G_{eS}$, we add in $A_e$ a new edge $s_i^r s_f$, or $s_i^r s_f$, respectively. In both cases, this is labeled by $\lambda$. We set in $P_r$ a rule of the form $\gamma_i^q \rightarrow \lambda$ or $\gamma_i^q \rightarrow \lambda$, respectively.

The new grammar $G_r = (N_r, T, P_r, S)$, in which the set of rules $P_r$ is built as above, and $N_r = \{[g, i]_g \mid [i, i]_i \in N_i^{(2)}\} \cup \{[i], [s, i]_i \mid [i, i]_i \in N_i^{(1)}\}$ is a regular grammar generating a regular superset approximation for $L(G_k)$. Recall that, some of the brackets in $N_r$ may also be $h$-marked (by distinct symbols). It is easy to observe that $L(G_r) = \varphi(R_m)$, where $\varphi$ is the homomorphism in the proof of Theorem 2.9.

Note that since the regular language in the Chomsky-Schützenberger theorem is an approximation of the trace-language, $R_m$ depends on the considered context-free grammar in Dyck normal form. Hence, the refinement of the regular approximation depicted in this section is considered with respect to the structure of the grammar $G_k$ in Dyck normal form, where by the structure we mean the number of rules and nonterminals composing $G_k$. As for $L = L(G_k)$ there exist infinitely many grammars generating it, setting these grammars in Dyck normal form other trace languages can be drawn, and consequently other regular languages, of type $R_m$, can be built. The best approximation for $L$ is the regular language with fewer words that are not in $L$.

Denote by $G_L$ the infinite set of grammars in Dyck normal form generating $L$, by $R_m$ the set of all regular languages obtained from the refined extended dependency graphs associated with grammars in $G_L$, and by $A_L = \{\varphi(R_m) \mid R_m \in R_m\}$ the set of all superset regular approximations of $L$. It is easy to observe that $A_L$, with the inclusion relation on sets, is a partially ordered subset of context-free languages. $A_L$ has an infimum equal to the context-free language it approximates, but it does not have the least element. Indeed, as proved in [2], [14], [15], and [16], there is no algorithm to build for a certain context-free language $L$, the simplest context-free grammar that generates $L$. Hence, there is no possibility to identify the simplest context-free grammar in Dyck normal form that generates $L$. Therefore, there is no algorithm to build the minimal superset approximation for $L$. Where by the simplest grammar we refer to a grammar with a minimal number of nonterminals, rules, or loops (grammatical levels encountered during derivations). Consequently, $A_L$ does not have the least element.

It would be interesting to further study how the (refined) extended dependency graphs, associated with grammars in Dyck normal form generating a certain context-free language $L$, vary depending on the structure of these grammars, and what makes the structure of the regular language $R_m$ (hence the regular superset approximation) simpler. In other words, to find a hierarchy on $A_L$, depending on the structure of the grammars in Dyck normal form that generate $L$. These may also provide an appropriate measure to compare languages in

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\[\text{For instance, how does it look the extended dependency graph associated with a nonself-embedding grammar in Dyck normal form, and which is the corresponding regular superset approximation. Note that, a context-free -nonself-embedding grammar always generates a regular language (since the language is finite).}\]
Figure 5: The transition diagram $A_e$ built from $G.e^S$ in Example 4.2. Each bracket $[i, (S, s)]$ in $A_e$ corresponds to the state $s_i$ ($ss, s_1$) (see Example 5.1 b.). $S$ is the initial vertex, vertices colored in green lead to the final state.

$A_L$. On the other hand, for an ambiguous grammar $G_k$, there exist several paths (hence regular expressions) in the refined extended dependency graph, which “approximate” the same word in $L(G_k)$. Apparently, finding an unambiguous grammar for $L(G_k)$ may refine the language $R_m$. The main disadvantage is that, again in general, there is no algorithm to solve this problem. Moreover, even if it is possible to find an unambiguous grammar for $L(G_k)$, it is doubtful that the corresponding regular language $R_m$ is finer than the others. In [14] it is also proved that the cost of the “simplicity” is the ambiguity. In other words, finding an unambiguous grammar for $L = L(G_k)$ may lead to the increase in size (e.g. number of nonterminals, rules, levels, etc.) of the respective grammar. Which again, may enlarge $R_m$ with useless words. Therefore, a challenging matter that deserves further attention is whether the unambiguity is more powerful than the “simplicity” in determining a more refined regular superset approximation for a certain context-free language (with respect to the method proposed in this paper).

In [4] it is proved that optimal (minimal) superset approximations exist for several kind of context-free languages, but no specification is provided of how the existing minimal approximation can be built starting from the context-free language it approximates. It would be challenging to further investigate whether there exist subsets of context-free languages for which it would be possible to build a minimal superset approximation (by using the graphical method herein proposed).

**Example 5.1. a.** The regular grammar that generates the regular superset approximation of the linear language in Example 3.6 is $G_r = ([S_r, 1, 3, 5, 6, 5, 4, 6, 6], \{a, b, c, d\}, S, P_r)$, where $P_r = \{S \to a, a \to b, b \to c, c \to d, d \to \lambda\}$. The language generated by $G_r$ is $L(G_r) = \{(abb)^m a a (d (cb)^n) p | n, m, p \geq 0\}$.

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15Note that, since there is only one dependency graph that yields only one plus-height regular expression there is no need of the labeling procedure described in Section 4.
\[1\} = (ab)^+ \text{aa}(db)^+ = h(R).\] The transition diagram associated with the finite automaton that accepts \(L(G_r)\) is sketched in Figure 1.c.

b. The regular grammar that generates the regular superset approximation of the context-free language in Examples 3.7 and 4.2 is \(G_r = \{\langle S, \{a, b, c, S, P_r\rangle, \langle a, b, c \rangle, \{a, b, c\}, S, P_r\rangle, \text{ where } P_r = \{S \rightarrow c_{14}^{[t]} \rightarrow c_{14}^{[u]} \rightarrow c_{14}^{[t]}, |S| = 14, \text{ and } |P_r| = 4\}, \text{ the transition diagram associated with the finite automaton that accepts } L(G_r)\} \) is sketched in Figure 5.

6 Conclusions

In this paper we have introduced a normal form for context-free grammars, called Dyck normal form. Based on this normal form and on graphical approaches we gave an alternative proof of the Chomsky-Schützenberger theorem. From a transition-like diagram for a context-free grammar in Dyck normal form we built a transition diagram for a finite automaton and a regular grammar for a regular superset approximation of the original context-free language. A challenging problem for further investigations may be to further refine this superset approximation depending on the type of the grammar (e.g. nonself-embedding or unambiguous) or on the size of the grammar (e.g. number of nonterminals, rules, etc.) generating a certain context-free language.

The method used throughout this paper is graphically constructive, and it shows that i. derivational structures in context-free grammars can be better described through nested systems of parenthesis (Dyck languages), and ii. the Chomsky-Schützenberger theorem may render a good and efficient approximation for context-free languages. Furthermore, the method provides a graphical framework to handle derivations and descriptional structures in context-free grammars, which may be useful in further complexity investigations of context-free languages.

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