Cross Product Bialgebras

Part I

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Abstract

The subject of this article are cross product bialgebras without co-cycles. We establish a theory characterizing cross product bialgebras universally in terms of projections and injections. Especially all known types of biproduct, double cross product and bicross product bialgebras can be described by this theory. Furthermore the theory provides new families of (co-cycle free) cross product bialgebras. Besides the universal characterization we find an equivalent (co-)modular description of certain types of cross product bialgebras in terms of so-called Hopf data. With the help of Hopf data construction we recover again all known cross product bialgebras as well as new and more general types of cross product bialgebras. We are working in the general setting of braided monoidal categories which allows us to apply our results in particular to the braided category of Hopf bimodules over a Hopf algebra. Majid's double biproduct is seen to be a twisting of a certain tensor product bialgebra in this category. This resembles the case of the Drinfel'd double which can be constructed as a twist of a specific cross product.

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Introduction

In recent years various (co-cycle free) cross products with bialgebra structure had been investigated by several authors [34, 28, 22]. The different types like tensor product bialgebra, biproduct, double cross product and bicross product bialgebra are characterized each by a universal formulation in terms of specific projections and injections of the particular tensorands into the cross product. The tensorands show interrelated (co-)module structures which are compatible with these universal properties and which allow a reconstruction of the cross product. The cross products are therefore equivalently characterized by either of the two descriptions. The multiplication and the comultiplication of the different cross products have a similar form as the multiplication and the comultiplication of the tensor product bialgebra except that the tensor transposition is replaced by a more complicated morphism with particular properties. The (co-)unit is given by the canonical tensor product (co-)unit. Up to these common aspects the defining relations of the several types of cross products seem to be different. The question arises if there exists at all a
possibility to describe the different cross products as different versions of a single unifying theory which equivalently characterizes cross product bialgebras universally and in a (co-)modular manner. The present article is concerned with this question and will give an affirmative answer.

Based on the above mentioned common properties of cross products we define cross product bialgebras or bialgebra admissible tuples (BAT). We show that there are equivalent descriptions of cross product bialgebras either by certain idempotents or by coalgebra projections and algebra injections obeying specific relations. However it is not clear if a necessary and sufficient formulation by some interrelated (co-)module structures of the particular tensor factors exists as well. For these purposes we restrict the consideration to BATs where both the algebra and coalgebra structure of the tensorands is respected at least by one of the four projections or injections such that it is at the same time an algebra and a coalgebra morphism. The corresponding cross product bialgebras will be called trivalent cross product bialgebras. This is a sufficiently general class of cross product bialgebras to cover all the known cross products of \[34, 28, 22\]. Trivalent cross product bialgebras admit a universal characterization as well. On the other hand we define so-called Hopf datum. A Hopf datum is a couple of objects which are both algebras and coalgebras and which are mutual (co-)modules obeying certain compatibility relations. One can show that a certain Hopf datum structure is canonically inherited on any BAT. Conversely Hopf data induce an algebra and a coalgebra on the tensor product \(B = B_1 \otimes B_2\) which strongly resembles the definition of a cross product bialgebra. However there is a priori no compatibility of both structures rendering the Hopf datum a bialgebra. However, Hopf data obey the fundamental recursive identities \(f = \Phi(f)\) of Proposition 2.3 for both \(f = \Delta_B \circ m_B\) and \(f = (m_B \otimes m_B) \circ (\text{id}_B \otimes \Psi_{B,B} \otimes \text{id}_B) \circ (\Delta_B \otimes \Delta_B)\). This fact leads us to the definition of so-called recursive Hopf data. Recursive Hopf data turn out to be bialgebras. We also define recursive Hopf data with finite order and show that a special class of recursive Hopf data (with order \(\leq 2\), called trivalent Hopf data, are in one-to-one correspondence with trivalent cross product bialgebras. Hence the classification of cross product bialgebras either by (co-)modular or by universal properties according to \[28\] has been achieved for all trivalent cross product bialgebras in terms of trivalent Hopf data. The new classification scheme covers all the known types of cross products with bialgebra structure \[34, 28, 22\]. And for the most general trivalent Hopf datum it provides a new family of cross product bialgebras which had not yet been studied in the literature so far. At the end of Section 2 we will apply our results in particular to Radford’s 4-parameter Hopf algebra \(H_{n,q,N,\nu}\) introduced in \[29\]. It turns out that it is a biproduct bialgebra over the sub-Hopf group algebra \(kC_N\) of \(H_{n,q,N,\nu}\).

Since we are working throughout in very general types of braided categories, we can apply our results to the special case of the braided category of Hopf bimodules over a Hopf algebra (possibly in a braided category, too). We demonstrate that Majid’s double biproduct \[26\] is a bialgebra twist of a certain tensor product bialgebra in the category of Hopf bimodules. An example of double biproduct bialgebra is Lusztig’s construction of the quantum group \(U\) \[17\].

A more thorough investigation of Hopf data and cross product bialgebras in Hopf bimodule categories will be presented in a forthcoming work. Another application of our

\[ A \quad \text{special variant of a trivalent Hopf datum has been studied in} \quad 2 \quad \text{which uniformly describes biproducts and bicross products in the symmetric category of vector spaces.} \]
results shows that the braided matched pair formulation in terms of a certain pairing only works if the mutual braiding of the two objects of the pair is involutive. This confirms in some sense a similar observation in [26].

In Section 1 we give a survey of previous results, notations and conventions which we need in the following. In particular we recall outcomes of [4, 5]. We use graphical calculus for braided categories. The subsections on Hopf bimodules and twisting will be needed only in Section 3. Section 3 is devoted to the main subject of the article. We define bialgebra admissible tuples (BAT) or cross product bialgebras, trivalent cross product bialgebras, and (recursive) Hopf data (of finite order). It turns out that Hopf data with a trivial (co-)action are recursive and of order \(\leq 2\). They will be called trivalent Hopf data. We show that trivalent Hopf data and trivalent cross product bialgebras are equivalent. They generalize the known “classical” cross products which will be recovered as certain special examples. The results of Section 3 will be applied in Section 4. Using results of [4] we demonstrate that the double biproduct [26] can be obtained as a bialgebra twist from the tensor product of two Hopf bimodule bialgebras. We show that the braided version of the matched pairing [25] yields a matched pair if and only if the braiding of the two tensorands is involutive. A matched pair is a special kind of Hopf datum studied in Section 3.

1 Preliminaries

We presume reader’s knowledge of the theory of braided monoidal categories. Braided categories have been introduced in the work of Joyal and Street [14, 15]. Since then they were studied intensively by many authors. For an introduction to the theory of braided categories we recommend to have a short look into the above mentioned articles or in standard references on quantum groups and braided categories [10, 35, 16, 23]. Because of Mac Lane’s Coherence Theorem for monoidal categories [20, 21] we may restrict our consideration to strict braided categories. In our article we denote categories by caligraphic letters \(\mathcal{C}, \mathcal{D}\), etc. For a braided monoidal category \(\mathcal{C}\) the tensor product is denoted by \(\otimes\), the unit object by \(1_{\mathcal{C}}\), and the braiding by \(^C\Psi\). If it is clear from the context we omit the index ‘\(\mathcal{C}\)’ at the various symbols. Henceforth we consider braided categories which admit split idempotents [4, 18, 19]; for each idempotent \(\Pi = \Pi^2 : M \to M\) of any object \(M\) in \(\mathcal{C}\) there exists an object \(M_\Pi\) and a pair of morphisms \((i_\Pi, p_\Pi)\) such that \(p_\Pi \circ i_\Pi = \text{id}_{M_\Pi}\) and \(i_\Pi \circ p_\Pi = \Pi\). This is not a severe restriction of the categories under consideration since every braided category can be canonically embedded into a braided category which admits split idempotents [4, 18].

We use and investigate algebraic structures in braided categories. We suppose that the reader is familiar with the generalization of algebraic structures to braided categories. Essentially we are working with algebras, coalgebras, bialgebras, Hopf algebras, modules, comodules, bimodules and bicomodules in braided categories [18, 23, 27]. Structures like Hopf bimodules or crossed modules will be reviewed in the following. We use throughout the symbol \(m\) for the multiplication and \(\eta\) for the unit of an algebra, \(\Delta\) for the comultiplication and \(\varepsilon\) for the counit of a coalgebra, \(S\) for the antipode of a Hopf algebra, \(\mu\) for the (left or right) action of an algebra on a module, and \(\nu\) for the (left or right) coaction of a coalgebra on a comodule. We call a morphism \(\rho : M \otimes N \to 1_{\mathcal{C}}\) in \(\mathcal{C}\) a pairing of \(M\) and \(N\). The graphical calculus for (strict) braided monoidal categories [14, 13, 16, 31, 31, 28, 35]
will be used throughout the paper. We compose morphisms from up to down, i.e. the
domains of the morphisms are at the top and the codomains are at the bottom of the
graphics. Tensor products are represented horizontally in the corresponding order. We
present our own conventions \cite{4, 5} in Figure 1 and omit an assignment to a specific object
if there is no fear of confusion. To elucidate graphical calculus we will represent below
the bialgebra axiom of multiplicativity of the comultiplication both in the ordinary way
of composition and by graphical symbols.

\[
\Delta \circ m = (m \otimes m) \circ (\text{id} \otimes \Psi \otimes \text{id}) \circ (\Delta \otimes \Delta)
\]

\[
\chi = \begin{array}{c}
\Psi \\
\Psi^{-1}
\end{array}
\]

The results of the following subsections on Hopf bimodules and twisting will be needed
in Section \ref{3}. They are not relevant for the central part of the article presented in Section
\ref{2}.

**Hopf Bimodules**

Hopf bimodules over a bialgebra \(B\) in \(\mathcal{C}\) are \(B\)-bimodules and \(B\)-bicomodules such that
the actions are bicomodule morphisms through the diagonal coactions on tensor products
of comodules and the canonical comodule structure on \(B\) \cite{4}.

\(B\)-Hopf bimodules and bimodule-bicomodule morphisms constitute the category \(\mathbf{HC}_H^B\). For the symmetric category
of \(k\)-vector spaces Hopf bimodules have been introduced in \cite{36} under the name bicovariant
bimodules.

Suppose that \(H\) is a Hopf algebra in \(\mathcal{C}\). Then there exists a tensor bifunctor rendering \(\mathbf{HC}_H^H\),
a (braided) monoidal category \cite{4, 5}. The proper formulation of the corresponding theorem
requires two auxiliary bifunctors \(\odot\) and \(\square\) on \(\mathbf{HC}_H^H\). Two objects \(X\) and \(Y\) of \(\mathbf{HC}_H^H\) yield the
\(H\)-Hopf bimodule \(X \square Y = X \otimes Y\) with diagonal left and right actions \(\mu_{l,d}^X\) and \(\mu_{r,d}^Y\),
and with induced left and right coactions \(\nu_{l,d}^{X \square Y} = \nu_{l,1}^X \otimes \text{id}_Y\) and \(\nu_{r,d}^{X \square Y} = \text{id}_X \otimes \nu_{r,1}^Y\). The
Hopf bimodule \(X \odot Y\) is obtained by categorical dualization of the previous structures. For
Hopf bimodule morphisms \(f\) and \(g\) we define \(f \odot g = f \square g = f \otimes g\). Then the categories
\((\mathbf{HC}_H^H, \odot)\) and \((\mathbf{HC}_H^H, \square)\) are semi-monoidal, i.e. they are categories which are almost
monoidal, except that the unit object and the relations involving it are not required. For
the definition of the braiding of \(\mathbf{HC}_H^H\) we use the natural transformation \(\Theta : \odot \to \square^{op}\)
given by \( \Theta_{X,Y} := (\mu^Y \otimes \mu^X) \circ (\text{id}_H \otimes \Psi_{X,Y} \otimes \text{id}_H) \circ (\nu^X \otimes \nu^Y) : X \otimes Y \to Y \otimes X \). In the following theorem the braided monoidal structure of \( \mathcal{H} \mathcal{C}_H \) is described \([4, 5]\).

**Theorem 1.1** The category \( \mathcal{H} \mathcal{C}_H \) of Hopf bimodules over \( H \) is monoidal. The unit object is the canonical Hopf bimodule \( H \), and the tensor product \( \otimes_H \) is uniquely defined (up to monoidal equivalence) by one of the following equivalent conditions for any pair of \( H \)-Hopf bimodules \( X \) and \( Y \).

- The \( H \)-Hopf bimodule \( X \otimes_H Y \) is the tensor product over \( H \) of the underlying modules, and the canonical morphism \( \lambda^H_{X,Y} : X \otimes Y \to X \otimes_H Y \equiv X \otimes Y \) is functorial in \( H \mathcal{C}_H \), i.e. \( \lambda^H : \otimes \to \otimes_H \).
- The \( H \)-Hopf bimodule \( X \otimes_H Y \) is the cotensor product over \( H \) of the underlying comodules, and the canonical morphism \( \rho^H_{X,Y} : X \square_H Y \equiv X \otimes H Y \to X \square Y \) is functorial in \( H \mathcal{C}_H \), i.e. \( \rho^H : \square \to \square_H \).

The corresponding natural morphisms \( \lambda^H \) and \( \rho^H \) obey the identity

\[
\rho^H_{X,Y} \circ \lambda^H_{X,Y} = (\mu^X_r \otimes \mu^Y_r) \circ (\text{id}_X \otimes \Psi_{H,H} \otimes \text{id}_Y) \circ (\nu^X_l \otimes \nu^Y_l) \cdot (1.1)
\]

The category \( \mathcal{H} \mathcal{C}_H \) is pre-braided through the pre-braiding \( \mathcal{H} \mathcal{C}_H \Psi_{X,Y} \) uniquely defined by the condition \( \rho^H_{Y,X} \circ \rho^H_{H,Y} \Psi_{X,Y} = \Theta_{X,Y} \equiv \Theta_{X,Y} = \Theta_{X,Y} \). It is braided if the antipode of \( H \) is an isomorphism in \( \mathcal{C} \). ■

Another concept closely related to Hopf bimodules are crossed modules \([36, 37, 3, 4]\). The connection of both notions had been studied in \([36]\) and was reformulated in \([32]\) for symmetric categories of modules over commutative rings. The general investigation for braided categories which admit split idempotents can be found in \([4]\).

A right crossed module over the Hopf algebra \( H \) is an object \( M \) in \( \mathcal{C} \) which is both right \( H \)-module and right \( H \)-comodule such that the following identity holds.

\[
\rho^H_{X,Y} \circ \lambda^H_{X,Y} = (\mu^X_r \otimes \mu^Y_r) \circ (\text{id}_X \otimes \Psi_{H,H} \otimes \text{id}_Y) \circ (\nu^X_l \otimes \nu^Y_l) \cdot (1.2)
\]

Identity (1.2) is the graphical representation of the equation

\[
(\mu_r \otimes m_H) \circ (\text{id}_M \otimes \Psi_{M,H} \otimes \text{id}_H) \circ (\nu_r \otimes \Delta_H)
= (\text{id}_M \otimes m_H) \circ (\Psi_{H,M} \otimes \text{id}_H) \circ (\text{id}_H \otimes \nu_r \otimes \mu_r) \circ (\Psi_{M,H} \otimes \text{id}_H) \circ (\text{id}_M \otimes \Delta_H).
\]

The right \( H \)-crossed modules and the corresponding module-comodule morphisms form a category which is denoted by \( \mathcal{D} \mathcal{Y}(\mathcal{C})_H \). In this notation \( \mathcal{D} \) stands for Drinfel’d who introduced the quantum double \( \mathcal{D}(H) \) of a Hopf algebra \( H \), and \( \mathcal{Y} \) stands for Yetter who identified the category of representations of \( \mathcal{D}(H) \) with the category of \( H \)-crossed
modules. Therefore crossed modules are sometimes called Drinfel’d-Yetter modules or Yetter-Drinfel’d modules. \( DY(C)^H \) is a monoidal category through the tensor product and the unit object of \( C \), and the diagonal (co)-actions for tensor products of crossed modules \([1,3]\). In the following we will outline the relation of Hopf bimodules and crossed modules \([1,3]\). If \( H \) is a Hopf algebra in the category \( C \) and \((X, \mu_r, \mu_l, \nu_r, \nu_l)\) is an \( H \)-Hopf bimodule then there exists an object \( HX \) such that \( HX \cong \mathbb{1} \otimes X \), \( HX \cong \mathbb{1} \boxtimes X \) and \( \chi \circ \lambda = \text{id}_{HX} \) where \( \chi p : \mathbb{1} \otimes X \cong X \to HX \cong \mathbb{1} \otimes X \) and \( \lambda \chi : \mathbb{1} \boxtimes X \cong HX \to X \cong \mathbb{1} \otimes X \) are the corresponding universal morphisms. The assignment \( H(\cdot) : H^{CH}_H \to DY(C)^H \), which is given through \( H(X) := (HX, \chi \circ \mu_r \circ (\chi \circ \text{id}_H), (\chi \circ \text{id}_H) \circ \nu_r \circ \lambda \chi) \) for an object \( X \), and through \( H(f) = \chi \circ \mu_r \circ f \circ \lambda \chi \) for a Hopf bimodule morphism \( f : X \to Y \), defines a functor into the category of crossed modules. Conversely a full inclusion functor \( H \times (\cdot) : DY(C)^H \to H^{CH}_H \) of the category of right \( H \)-crossed modules into the category of \( H \)-Hopf bimodules is defined by \( H \times (X) = (H \otimes X, \mu_{i,l}^{H \otimes X}, \nu_{i,l}^{H \otimes X}, \mu_{d,r}^{H \otimes X}, \nu_{d,r}^{H \otimes X}) \) for any right crossed module \( X \) and by \( H \times (f) = \text{id}_H \otimes f \) for any crossed module morphism \( f \). The action \( \mu_{i,l}^{H \otimes X} \) is the left action induced by \( H \) and \( \mu_{d,r}^{H \otimes X} \) is the diagonal action of the tensor product \( H \)-module. In the dual way the coactions \( \nu_{i,l}^{H \otimes X} \) and \( \nu_{d,r}^{H \otimes X} \) are defined. The following theorem holds \([1] \).

**Theorem 1.2** Let \( H \) be a Hopf algebra in \( C \) with isomorphic antipode. Then the categories \( DY(C)^H \) and \( H^{CH}_H \) are braided monoidal equivalent by \( DY(C)^H \xrightarrow{H(-)} H^{CH}_H \). \( \square \)

If not otherwise mentioned we subsequently assume that the antipode of the Hopf algebra \( H \) is an isomorphism in \( C \).

**Remark 1** A mirror symmetric result corresponding to Theorem 1.2 holds for left \( H \)-crossed modules and \( H \)-Hopf bimodules. Henceforth we will denote the idempotents \( x \circ \chi \circ \eta \) and \( i \chi \circ \eta \) of a Hopf bimodule \( X \) by \( x \Pi \) and \( \Pi X \) respectively. Explicitly it holds \( x \Pi = \mu_{i,l}^{X} \circ (S_H \otimes \text{id}_X) \circ \nu_{i,l}^{X} \) and \( \Pi X = \mu_{d,r}^{X} \circ (\text{id}_X \otimes S_H) \circ \nu_{d,r}^{X} \). \([1]\).

This observation leads to the following useful lemma.

**Lemma 1.3** Suppose that \( X \) and \( Y \) are \( H \)-Hopf bimodules and \( f, g : X \odot Y \to X \boxtimes Y \) are Hopf bimodule morphisms. Then the identity

\[
H^{CH}_H \Psi_{X,Y} \circ \lambda_{X,Y}^H \circ (x \Pi \otimes \Pi Y) = \lambda_{Y,X}^H \circ \Psi_{X,Y} \circ (x \Pi \otimes \Pi Y)
\]

(1.3)

holds. The identity

\[
(\chi \Pi \otimes \Pi Y) \circ f \circ (\chi \Pi \otimes \Pi Y) = (\chi \Pi \otimes \Pi Y) \circ g \circ (\chi \Pi \otimes \Pi Y)
\]

(1.4)

implies \( f = g \).

**Proof.** The composition of both sides of (1.3) with the monomorphism \( \rho_{Y,X} \) obviously leads to the identity \( \Theta_{Y,X} \circ (\chi \Pi \otimes \Pi Y) = (\mu_{i,l}^{X} \otimes \mu_{i,l}^{Y}) \circ (\text{id}_X \otimes \Psi_H \otimes \text{id}_Y) \circ (\nu_{i,l}^{X} \otimes \nu_{i,l}^{Y}) \circ (\chi \Pi \otimes \Pi Y) \).
Π_Y) which in turn coincides with \( \Psi_{Y,X} \circ (\chi \Pi \otimes \Pi_Y) \). To prove the second statement of the lemma observe that any Hopf bimodule morphism \( f : X \otimes Y \rightarrow X \square Y \) can be expressed in terms of \( f' = f \circ (\chi \Pi \otimes \Pi_Y) \) and subsequently in terms of \( f'' = (\chi \Pi \otimes \Pi_Y) \circ f \circ (\chi \Pi \otimes \Pi_Y) \) in the following way

\[
\begin{align*}
     f &= \begin{tikzpicture}[baseline=(current bounding box.center), thick, scale=0.5]
        \node (X) at (0,0) {$X \otimes Y$};
        \node (Y) at (2,0) {$Y$};
        \node (X1) at (0,-2) {$X$};
        \node (Y1) at (2,-2) {$Y$};
        \draw[->] (X) to node[auto] {$\chi \Pi$} (Y);
        \draw[->] (X) to node[auto,swap] {$\Pi_Y$} (Y);
    \end{tikzpicture} \\
    &= \begin{tikzpicture}[baseline=(current bounding box.center), thick, scale=0.5]
        \node (X) at (0,0) {$X \otimes Y$};
        \node (Y) at (2,0) {$Y$};
        \node (X1) at (0,-2) {$X$};
        \node (Y1) at (2,-2) {$Y$};
        \draw[->] (X) to node[auto] {$\chi \Pi$} (Y);
        \draw[->] (X) to node[auto,swap] {$\Pi_Y$} (Y);
    \end{tikzpicture} \\
    &= \begin{tikzpicture}[baseline=(current bounding box.center), thick, scale=0.5]
        \node (X) at (0,0) {$X \otimes Y$};
        \node (Y) at (2,0) {$Y$};
        \node (X1) at (0,-2) {$X$};
        \node (Y1) at (2,-2) {$Y$};
        \draw[->] (X) to node[auto] {$\chi \Pi$} (Y);
        \draw[->] (X) to node[auto,swap] {$\Pi_Y$} (Y);
    \end{tikzpicture}
\end{align*}
\]

where \( \nu_{ad,r}^X, \nu_{ad,l}^Y \) are the (braided) adjoint coactions \([3, 4]\). The second identity of (1.5) is derived from the module properties of \( f \). In a similar way one obtains the third identity of (1.5).

Relation (1.3) is the braided counterpart of Woronowicz’s definition of braiding of Hopf bimodules (see \([36]\)).

Finally we recall the first part of the canonical transformation procedure of bialgebras in \( H \otimes H \) into bialgebras in \( C \) \([4]\) which we need in the following.

**Proposition 1.4** Let \( H \) be a Hopf algebra in \( C \). A bialgebra \( \overline{B} = (B, m_B, \eta_B, \Delta_B, \epsilon_B) \) in \( H \otimes H \) can be turned into a bialgebra \( B = (B, m_B, \eta_B, \Delta_B, \epsilon_B) \) in \( C \) where the structure morphisms are given by

\[
    m_B = m_B \circ \chi_B^H, \quad \eta_B = \eta_B \circ \chi_H, \quad \Delta_B = \rho_B^H \circ \Delta_B, \quad \epsilon_B = \epsilon_H \circ \bar{\epsilon}_B. \quad (1.6)
\]

If \( \overline{B} = (B, m_B, \eta_B, \Delta_B, \epsilon_B, S_B) \) is Hopf algebra in \( H \otimes H \) then \( B = (B, m_B, \eta_B, \Delta_B, \epsilon_B, S_B) \) is Hopf algebra in \( C \) with antipode \( S_B \) given by \( S_B = S_B \circ S_B/H = S_B/H \circ S_B/H \) where \( S_B/H = \mu \circ (id_H \otimes \mu_r) \circ (S_H \otimes id_B \otimes S_B) \circ (id_H \otimes \psi_r) \circ \psi_l \).

**Twisting**

In this subsection we present the twisting construction for bialgebras in a braided category \( C \). We proceed along the lines of \([12, 24]\).

Let \( C \) be a coalgebra and \( \chi : C \rightarrow \mathbb{I}_C \) be a morphism into the unit object. Henceforth we will use the following notations

\[
    \chi.f := (\chi \otimes f) \circ \Delta, \quad f.\chi := (f \otimes \chi) \circ \Delta
\]

for any morphism \( f : C \rightarrow B \) in \( C \).

**Definition 1.5** If \( B \) is a bialgebra in \( C \) and \( \chi : B \otimes B \rightarrow \mathbb{I}_C \) is a morphism obeying the identities

\[
    \chi \circ (id_B \otimes \chi, m) = \chi \circ (\chi, m \otimes id_B), \quad (1.8)
\]

\[
    \chi \circ (\eta \otimes id_B) = \varepsilon = \chi \circ (id_B \otimes \eta) \quad (1.9)
\]
then \(\chi\) is called a 2-cocycle of the bialgebra \(B\). If \(\chi\) is a convolution invertible 2-cocycle, then the twist \(m^\chi_B\) of the multiplication \(m_B\) is defined by \(m^\chi_B := \chi \cdot m_B \cdot \chi^-\). If \(B\) is a Hopf algebra then the twist \(S^\chi_B\) of the antipode \(S\) is given by \(S^\chi_B := u \cdot S_B \cdot u^-\) where \(u = \chi \circ (\text{id}_B \otimes S_B) \circ \Delta_B\).

**Remark 2** Under the condition of Definition 1.5 the first identity in (1.9) holds if and only if the second one is valid.

In analogy to [12, 24] the following proposition can be verified in the braided case because nowhere in the proof the involutivity \(\Psi^2 = \text{id}\) is needed. Therefore we will only sketch how to prove the proposition.

**Definition and Proposition 1.6** Let \(B\) be bialgebra (Hopf algebra) and \(\chi : B \otimes B \to \mathbb{I}_C\) be an invertible 2-cocycle. Then \(B_{\chi} := (B, m^\chi_B, \eta_B, \Delta_B, \varepsilon_B, (S^\chi_B))\) with the twisted multiplication \(m^\chi_B\) (and twisted antipode \(S^\chi_B\)) is again bialgebra (Hopf algebra). \(B_{\chi}\) is called the twisted bialgebra (Hopf algebra) of \(B\) obtained by the twist \(\chi\).

**Proof.** At first we will demonstrate that the bialgebra axiom \(L := \Delta \circ m = (m \otimes m) \circ (\text{id} \otimes \Psi \otimes \text{id}) \circ (\Delta \otimes \Delta) := R\) for \(B\) is equivalent to the bialgebra axiom \(L_{\chi} = R_{\chi}\) for \(B_{\chi}\). This follows from the identities \((L_{\chi}) \cdot \chi = \chi \cdot L\) and \((R_{\chi}) \cdot \chi = \chi \cdot R\). Secondly, using the previous fact we show that the associativity \(A_{\chi} := m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m) =: A^r\) of \(B\) is equivalent to the associativity of \(B_{\chi}\), denoted by \(A^r_{\chi} = A^r_{\chi} \). This is proved with the help of the identities \((A^r_{\chi}) \cdot (\chi \circ (\text{id}_A \otimes \chi \cdot m)) = (\chi \circ (\chi \cdot m \otimes \text{id}_A)) \cdot A^r\). \(\Box\)

2 **Cross Product Bialgebras and Hopf Data**

Section 2 is the central part of the article. We define cross product bialgebras or bialgebra admissible tuples (BAT) and Hopf data. We consider certain specializations of these definitions, which we call trivalent cross product bialgebras and recursive Hopf data respectively. Trivalent cross product bialgebras form a sufficiently general class to cover the cross product bialgebras of \([12, 28, 22]\). Additionally there arise new explicit examples of trivalent cross product bialgebras. All of them will be completely classified in terms of recursive Hopf data. Therefore an equivalent description either through interrelated (co-)module structures or through universal projector decompositions is found.

**Cross Product Bialgebras**

Suppose now there are two objects \(B_1\) and \(B_2\) in \(\mathcal{C}\), and morphisms \(\varphi_{1,2} : B_1 \otimes B_2 \to B_2 \otimes B_1\) and \(\varphi_{2,1} : B_2 \otimes B_1 \to B_1 \otimes B_2\).

**Definition 2.1** We call \(((B_1, m_1, \eta_1, \Delta_1, \varepsilon_1), (B_2, m_2, \eta_2, \Delta_2, \varepsilon_2), \varphi_{1,2}, \varphi_{2,1})\) a bialgebra admissible tuple (BAT) in the category \(\mathcal{C}\) if \((B_j, m_j, \eta_j)\) is an algebra and \((B_j, \Delta_j, \varepsilon_j)\) is
a coalgebra for $j \in \{1, 2\}$, such that $\varepsilon_j \circ \eta_j = \text{id}_{\mathcal{C}}$, and the object $B_1 \otimes B_2$ is a bialgebra through

\[
\begin{align*}
m_\times &= (m_1 \otimes m_2) \circ (\text{id}_{B_1} \otimes \varphi_{2,1} \otimes \text{id}_{B_2}) , \\
\Delta_\times &= (\text{id}_{B_1} \otimes \varphi_{1,2} \otimes \text{id}_{B_2}) \circ (\Delta_1 \otimes \Delta_2) ,
\end{align*}
\]

(2.1)

This bialgebra will be called the cross product bialgebra associated to the bialgebra admissible tuple $(B_1, B_2, \varphi_{1,2}, \varphi_{2,1})$ and is denoted by $B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2$.

One observes that the definition of a cross product bialgebra differs from the usual definition of a canonical tensor product bialgebra (in a symmetric category) only through the substitution of the tensor transposition by the morphisms $\varphi_{1,2}$ and $\varphi_{2,1}$.

Of course the known cross products with bialgebra structure are cross product bialgebras in the sense of Definition 2.1 if the structures of the objects $B_1$ and $B_2$ and of the morphisms $\varphi_{1,2}$ and $\varphi_{2,1}$ are chosen correctly.

The following proposition allows us to express cross product bialgebras in terms of idempotents or projections and injections.

**Proposition 2.2** Let $A$ be a bialgebra in $\mathcal{C}$, then the following statements are equivalent.

1. $A$ is bialgebra isomorphic to a cross product bialgebra $B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2$.

2. There are idempotents $\Pi_1, \Pi_2 \in \text{End}(A)$ such that

\[
\begin{align*}
m_A \circ (\Pi_j \otimes \Pi_j) &= \Pi_j \circ m_A \circ (\Pi_j \otimes \Pi_j) , \\
\Pi_j \circ \eta_A &= \eta_A , \\
(\Pi_j \otimes \Pi_j) \circ \Delta_A &= (\Pi_j \otimes \Pi_j) \circ \Delta_A \circ \Pi_j , \\
\epsilon_A \circ \Pi_j &= \epsilon_A
\end{align*}
\]

for $j \in \{1, 2\}$, and the sequence $A \otimes A \xrightarrow{\text{id}_A \otimes \Pi_2} A \xrightarrow{(\Pi_1 \otimes \Pi_2) \circ \Delta_A} A \otimes A$ is a splitting of the idempotent $\Pi_1 \otimes \Pi_2$ of $A \otimes A$.

3. There exist objects $B_1$ and $B_2$ in $\mathcal{C}$ which are at the same time algebras and coalgebras, and algebra morphisms $i_j$, coalgebra morphisms $p_j$, $B_j \xrightarrow{i_j} A \xrightarrow{p_j} B_j$, such that $p_j \circ i_j = \text{id}_{B_j}$ for $j \in \{1, 2\}$, and the morphisms $m_A \circ (i_1 \otimes i_2) : B_1 \otimes B_2 \rightarrow A$ and $(p_1 \otimes p_2) \circ \Delta_A : A \rightarrow B_1 \otimes B_2$ are inverse to each other.

**Proof.** “(2) $\Rightarrow$ (3)”: Since $\Pi_j$ for $j \in \{1, 2\}$ are idempotents there are morphisms $i_j : B_j \rightarrow A$ and $p_j : A \rightarrow B_j$ which split $\Pi_j$. We define

\[
\begin{align*}
m_j &:= p_j \circ m_A \circ (i_j \otimes i_j) , \\
\eta_j &:= p_j \circ \eta_A , \\
\Delta_j &:= (p_j \otimes p_j) \circ \Delta_A \circ i_j , \\
\epsilon_j &:= \epsilon_A \circ i_j
\end{align*}
\]

for $j \in \{1, 2\}$. One immediately verifies that $(B_j, m_j, \eta_j)$ are algebras and $(B_j, \Delta_j, \epsilon_j)$ are coalgebras, and $i_j$ are algebra morphisms, $p_j$ are coalgebra morphisms for $j \in \{1, 2\}$. Because $\Pi_j$ are idempotents it follows $m_A \circ (i_1 \otimes i_2) \circ (p_1 \otimes p_2) \circ \Delta_A = m_A \circ (\Pi_1 \otimes \Pi_2) \circ \Delta_A$ which completes the proof.


\[\Pi_2 \circ (\Pi_1 \otimes \Pi_2) \circ \Delta_A = \text{id}_A\] where the last equation follows by assumption. Similarly \((p_1 \otimes p_2) \circ \Delta_A \circ \text{id}_A \circ (i_1 \otimes i_2) = \text{id}_{B_1 \otimes B_2}\) is proven.

“(3) \Rightarrow (2)”: For \(j \in \{1, 2\}\) we consider the idempotents \(\Pi_j = i_j \circ p_j\). Statement (2) is then proven easily with the help of the assumed properties of \(i_j\) and \(p_j\).

“(1) \Rightarrow (3)”: Let \(\phi : B_1 \varphi_{1,2} \otimes \varphi_{2,1} B_2 \to A\) be the isomorphism of bialgebras. Then in particular

\[\varphi_{2,1} \circ (\eta_2 \otimes \text{id}_{B_1}) = \text{id}_{B_1} \otimes \eta_2\]
\[\varphi_{2,1} \circ (\text{id}_{B_2} \otimes \eta_1) = \eta_1 \otimes \text{id}_{B_2}\]  \hspace{1cm} (2.2)

and dually analogous for \(\varphi_{1,2}\) and \(\varepsilon_1, \varepsilon_2\). We define the morphisms

\[i_1 := \phi \circ (\text{id}_{B_1} \otimes \eta_2), \quad p_1 := (\text{id}_{B_1} \otimes \varepsilon_2) \circ \phi^{-1},\]
\[i_2 := \phi \circ (\eta_1 \otimes \text{id}_{B_2}), \quad p_2 := (\varepsilon_1 \otimes \text{id}_{B_2}) \circ \phi^{-1}.\]  \hspace{1cm} (2.3)

Using (2.3) one verifies without problems that \(i_j\) are algebra morphisms. In a dual manner it is proven that \(p_j\) are coalgebra morphisms for \(j \in \{1, 2\}\). Since \(\varepsilon_1 \otimes \eta_1 = \text{id}_1 = \varepsilon_2 \otimes \eta_2\) it follows \(p_1 \circ i_1 = \text{id}_{B_j}, j \in \{1, 2\}\). Because \(\phi\) is bialgebra isomorphism it holds \(m_A \circ (i_1 \otimes i_2) = m_A \circ (\phi \otimes \phi) \circ (\text{id}_{B_1} \otimes \eta_2 \otimes \eta_1 \otimes \text{id}_{B_2}) = \phi \circ m_{B_1 \varphi_{1,2} \otimes \varphi_{2,1}} B_2 \circ (\text{id}_{B_1} \otimes \eta_2 \otimes \eta_1 \otimes \text{id}_{B_2}) = \phi\) where (2.2) has been used in the third equation. Dually one obtains \((p_1 \otimes p_2) \circ \Delta_A = \phi^{-1}\).

“(3) \Rightarrow (1)”: By assumption the isomorphism \(\phi := m_A \circ (i_1 \otimes i_2)\) induces a bialgebra structure on \(B := B_1 \otimes B_2\) through

\[m_B = \phi^{-1} \circ m_A \circ (\phi \otimes \phi), \quad \Delta_B = (\phi^{-1} \otimes \phi^{-1}) \circ \Delta_A \circ \phi,\]
\[\eta_B = \phi^{-1} \circ \eta_A, \quad \varepsilon_B = \varepsilon_A \circ \phi.\]  \hspace{1cm} (2.4)

We have to show that \(B\) with the structure (2.4) is a cross product bialgebra. At first we prove that the (co-)units of \(B\) are given by the tensor products of the particular (co-)units of \(B_1\) and \(B_2\). It holds \(\eta_B = (p_1 \otimes p_2) \circ \Delta_A \circ \eta_A = p_1 \circ \eta_A \otimes p_2 \otimes \eta_A\) because \(A\) is a bialgebra. Furthermore \(\eta_A = i_1 \circ \eta_1 = i_2 \circ \eta_2\) is satisfied since \(i_1\) and \(i_2\) are algebra morphisms. Combining these two equations then yields \(\eta_B = \eta_1 \otimes \eta_2\) since \(p_j \circ i_j = \text{id}_{B_j}\) for \(j \in \{1, 2\}\) by assumption. Dually \(\varepsilon_B = \varepsilon_1 \otimes \varepsilon_2\) can be proven. Now we are going to prove that \(B\) has the structure of a cross product bialgebra if we use the morphisms \(\varphi_{1,2} = (\varepsilon_1 \otimes \text{id}_{B_2} \otimes \text{id}_{B_1} \otimes \varepsilon_2) \circ \Delta_B\) and \(\varphi_{2,1} = m_B \circ (\eta_1 \otimes \text{id}_{B_2} \otimes \text{id}_{B_1} \otimes \eta_2)\). Thereto we need the auxiliary relations

\[(\text{id}_{B_1} \otimes m_2) \circ (\phi^{-1} \otimes p_2) = \phi^{-1} \circ m_A \circ (\text{id}_A \otimes \Pi_2),\]
\[(m_1 \otimes \text{id}_{B_1}) \circ (p_1 \otimes \phi^{-1}) = \phi^{-1} \circ m_A \circ (\Pi_1 \otimes \text{id}_A)\]  \hspace{1cm} (2.5)

which can be proven easily because \(\phi\) is a bialgebra isomorphism and \(i_1\) and \(i_2\) are algebra morphisms. Using (2.3) we show that \(m_B\) is indeed a multiplication of the form (2.1).

\[(m_1 \otimes m_2) \circ (\text{id}_{B_1} \otimes \varphi_{2,1} \otimes \text{id}_{B_2}) = (m_1 \otimes \text{id}_{B_2}) \circ (p_1 \otimes \phi^{-1}) \circ (\text{id}_A \otimes m_A) \circ (i_1 \otimes i_2 \otimes \phi)\]
\[= \phi^{-1} \circ m_A \circ (\phi \otimes \phi)\]
\[= m_B.\]  \hspace{1cm} (2.6)
In the first equation of (2.4) two times (2.3) has been used. In the second equality we applied $p_1 \circ i_1 = \text{id}_{B_1}$ and again (2.3). The third equation of (2.4) holds by definition. Similarly it can be shown by dualization that the comultiplication is that of a cross product bialgebra, i.e. $\Delta_B = (\text{id}_{B_1} \otimes \varphi_{1,2} \otimes \text{id}_{B_2}) \circ (\Delta_1 \otimes \Delta_2)$. Hence $(B_1, B_2, \varphi_{1,2}, \varphi_{2,1})$ is a BAT

Since identities (2.2) and their dual analogues for $\varphi_{1,2}$ and $\varepsilon_1, \varepsilon_2$ hold for cross product bialgebras, we immediately derive

**Corollary 2.3** Let $B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2$ be a cross product bialgebra. Then $\eta_1 \otimes \text{id}_{B_2}$ and $\text{id}_{B_1} \otimes \eta_2$ are algebra morphisms, and $\varepsilon_1 \otimes \text{id}_{B_2}$, $\text{id}_{B_1} \otimes \varepsilon_2$ are coalgebra morphisms.

It is not clear if the very general definition of a cross product bialgebra is in one-to-one correspondence with a description in terms of pairs of (co-)algebras with certain interrelated compatible (co-)module structures. Hence a classification of cross product bialgebras in the sense of (2.3) may not succeed at this general level. But for reasons of classification and reconstruction this aspect is important. The known cross products with bialgebra structure in [34, 28, 22] admit such a description. For later use we will therefore define trivalent cross product bialgebras as follows.

**Definition 2.4** A cross product bialgebra $B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2$ is called trivalent if at least one of the morphisms $\eta_1 \otimes \text{id}_{B_2}$, $\text{id}_{B_1} \otimes \eta_2$, $\varepsilon_1 \otimes \text{id}_{B_2}$, $\text{id}_{B_1} \otimes \varepsilon_2$ is both an algebra and a coalgebra morphism. In a slight abuse of notation we denote the corresponding bialgebra by $B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2$ without indication of the specific algebra-coalgebra morphism.

In particular all cross products in [34, 28, 22] are trivalent. Up to now we investigated universality of cross product bialgebras. In the following subsection we study cross product bialgebras from a (co-)modular point of view.

**Hopf Data**

**Definition 2.5** A Hopf datum $(B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)$ in $\mathcal{C}$ consists of two objects $B_1$ and $B_2$ which are both counital algebras and unital coalgebras, and $(B_1, \mu_1)$ is left $B_2$-module, $(B_1, \nu_1)$ is left $B_2$-comodule, $(B_2, \mu_r)$ is right $B_1$-module, and $(B_1, \nu_r)$ is right $B_1$-comodule obeying the identities

\[
\begin{align*}
\mu_r \circ (\eta_2 \otimes \text{id}_1) &= \eta_2 \circ \varepsilon_1 = (\text{id}_2 \otimes \varepsilon_1) \circ \nu_1, \\
\mu_1 \circ (\text{id}_2 \otimes \eta_1) &= \eta_1 \circ \varepsilon_2 = (\varepsilon_2 \otimes \text{id}_1) \circ \mu_r,
\end{align*}
\]

\[
\begin{align*}
\nu_r \circ \eta_2 &= \eta_2 \otimes \eta_1 = \varepsilon_1 \circ \mu_1, \\
\nu_r \circ \eta_2 &= \eta_2 \otimes \eta_1 = \varepsilon_1 \circ \mu_1,
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Diagram:}
\end{array}
\end{align*}
\]

**Algebra-coalgebra compatibility,**
Module-comodule compatibility,

\[
\begin{align*}
\mathcal{P} = & \quad \mathcal{P} \\
\mathcal{Q} = & \quad \mathcal{Q}
\end{align*}
\]

Module-algebra compatibility,

\[
\begin{align*}
\mathcal{H} = & \quad \mathcal{H} \\
\mathcal{L} = & \quad \mathcal{L}
\end{align*}
\]

Comodule-coalgebra compatibility,

\[
\begin{align*}
\mathcal{H} = & \quad \mathcal{H} \\
\mathcal{L} = & \quad \mathcal{L}
\end{align*}
\]

Module-coalgebra compatibility,

\[
\begin{align*}
\mathcal{H} = & \quad \mathcal{H} \\
\mathcal{L} = & \quad \mathcal{L}
\end{align*}
\]

Comodule-algebra compatibility.

At first sight the defining relations of Hopf data seem to be rather complicated and impenetrable. However all the compatibility identities only relate the different (co-)algebra and (co-)module structures. Besides there are two remarkable symmetries of the definition of Hopf data. The first one is the usual categorical duality in conjunction with the transformation “\(m \leftrightarrow \Delta\)”, “\(\eta \leftrightarrow \varepsilon\)”, “\(\mu_l \leftrightarrow \nu_l\)” and “\(\mu_r \leftrightarrow \nu_r\)”. The second one is a kind of mirror symmetry with respect to a vertical axis of the defining equations considered as graphics in three dimensional space, followed by the transformation of the indices “\(1 \leftrightarrow 2\)” and “\(l \leftrightarrow r\)”. This observation considerably simplifies subsequent considerations and calculations. In a first step we recover canonical (co-)algebra structures of Hopf datum.

**Proposition 2.6** Let \((B_1, B_2, \mu_l, \nu_l, \mu_r, \nu_r)\) be a Hopf datum. We define

\[
\begin{align*}
\phi_{1,2} = & \quad \mathcal{H} \\
\phi_{2,1} = & \quad \mathcal{L}
\end{align*}
\]  

(2.7)
Then \( B = B_1 \otimes B_2 \) is both an algebra and a coalgebra through the structure morphisms

\[
\begin{align*}
  m_B &= (m_1 \otimes m_2) \circ (\text{id}_{B_1} \otimes \phi_{2,1} \otimes \text{id}_{B_2}), \\
  \Delta_B &= (\text{id}_{B_1} \otimes \phi_{1,2} \otimes \text{id}_{B_2}) \circ (\Delta_1 \otimes \Delta_2), \\
  \eta_B &= \eta_1 \otimes \eta_2, \\
  \varepsilon_B &= \varepsilon_1 \otimes \varepsilon_2.
\end{align*}
\]  

(2.8)

**Proof.** It is a well known fact that the necessary and sufficient conditions for \((B, m_B, \eta_B)\) being an algebra are given by the following equations.

\[
\begin{align*}
  \phi_{2,1} \circ (m_2 \otimes \text{id}_{B_1}) &= (\text{id}_{B_1} \otimes m_2) \circ (\phi_{2,1} \otimes \text{id}_{B_2}) \circ (\text{id}_{B_2} \otimes \phi_{2,1}) \\
  \phi_{2,1} \circ (\text{id}_{B_2} \otimes m_1) &= (m_1 \otimes \text{id}_{B_2}) \circ (\text{id}_{B_1} \otimes \phi_{2,1}) \circ (\phi_{2,1} \otimes \text{id}_{B_1}) \\
  \phi_{2,1} \circ (\eta_2 \otimes \text{id}_{B_1}) &= \text{id}_{B_1} \otimes \eta_2 \\
  \phi_{2,1} \circ (\text{id}_{B_2} \otimes \eta_1) &= \eta_1 \otimes \text{id}_{B_2}.
\end{align*}
\]  

(2.9)

The verification of the third and fourth equation of (2.9) can be done straightforwardly using (2.7) and the defining relations of a Hopf datum. The second equation of (2.9) will be proven graphically.

\[
\begin{align*}
  B_2 & \downarrow \quad B_1 \quad B_1 \quad B_2 \downarrow \\
  B_1 \downarrow & \quad B_1 \downarrow \quad B_1 \downarrow \quad B_2 \downarrow \\
  B_1 \downarrow & \quad B_1 \downarrow \quad B_1 \downarrow \quad B_2 \downarrow
\end{align*}
\]

(2.10)

The first identity of (2.10) uses the algebra-coalgebra compatibility of Definition 2.3. In the second equation the module-algebra compatibility is used. The third equality holds because \( B_1 \) is a left \( B_2 \)-module, and the fourth identity is true because of the module-coalgebra compatibility. This proves the second equation of (2.9). The first identity of (2.9) can be verified similarly. Hence \((B, m_B, \eta_B)\) is an algebra. It will be proven dually that \((B, \Delta_B, \varepsilon_B)\) is coalgebra.

Under the conditions of Proposition 2.6 one proves that \( \eta_1 \otimes \text{id}_{B_2}, \text{id}_{B_1} \otimes \eta_2 \) are algebra morphisms, and \( \varepsilon_1 \otimes \text{id}_{B_2}, \text{id}_{B_1} \otimes \varepsilon_2 \) are coalgebra morphisms. The result of Proposition 2.6 strongly resembles the definition of a BAT. However there is a priori no compatibility of the algebra and the coalgebra structure of \( B = B_1 \otimes B_2 \) rendering \( B \) a cross product bialgebra. On the other hand the following proposition is easily proved.

**Proposition 2.7** A BAT \((B_1, B_2, \varphi_{1,2}, \varphi_{2,1})\) yields a Hopf datum \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) through

\[
\begin{align*}
  \mu_1 &= (\text{id}_{B_1} \otimes \varepsilon_2) \circ \varphi_{2,1}, \\
  \mu_r &= (\varepsilon_1 \otimes \text{id}_{B_2}) \circ \varphi_{2,1} \\
  \nu_1 &= \varphi_{1,2} \circ (\text{id}_{B_1} \otimes \eta_2), \\
  \nu_r &= \varphi_{1,2} \circ (\eta_1 \otimes \text{id}_{B_2}).
\end{align*}
\]  

(2.11)

Conversely according to eqs. (2.7) in Proposition 2.6 the resulting Hopf datum can be transformed into the cross product bialgebra \( B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2 \) because it holds \( \phi_{1,2} = \varphi_{1,2} \) and \( \phi_{2,1} = \varphi_{2,1} \).
Definition 2.8 Let \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) be a Hopf datum in \(\mathcal{C}\) and \(f \in \text{End}_C(B_1 \otimes B_2 \otimes B_1 \otimes B_2)\) be any endomorphism in \(\mathcal{C}\). Then the mapping \(\Phi : \text{End}_C(B_1 \otimes B_2 \otimes B_1 \otimes B_2) \rightarrow \text{End}_C(B_1 \otimes B_2 \otimes B_1 \otimes B_2)\) is given by

\[
\Phi(f) = \begin{array}{c}
\begin{array}{ccc}
B_1 & \rightarrow & B_2 \\
B_1 & \rightarrow & B_2 \\
B_1 & \rightarrow & B_2 \\
\end{array}
\end{array}
\]

In the following proposition the fundamental recursive relation for Hopf data will be derived.

Proposition 2.9 Let \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) be a Hopf datum in \(\mathcal{C}\) and consider the endomorphisms \(f_1 := \Delta_B \circ m_B\) and \(f_2 := (m_B \otimes m_B) \circ (\Delta_B \otimes \Delta_B)\), where \(m_B\) and \(\Delta_B\) are defined according to (2.8). Then it holds \(f_1 = \Phi(f_1)\) and \(f_2 = \Phi(f_2)\).
Proof. For \( f_1 = \Delta_B \circ m_B \) we obtain the result through the following (graphical) identities.

\[
B_1 B_2 B_1 B_2 \quad = \quad \quad \quad \quad = \quad \Delta \circ m
\]

The first identity of (2.14) uses the algebra-coalgebra compatibilities of Definition 2.5. In the second identity of (2.14) we used the module-coalgebra and the comodule-algebra compatibilities. (Co-)associativity is applied to derive the third equation of (2.14).

For \( f_2 = (\operatorname{id}_B \otimes \Psi_B \otimes \operatorname{id}_B) \circ (\Delta_B \otimes \Delta_B) \) the proof is given by the following graphical equalities.

\[
B_1 B_2 B_1 B_2 \quad = \quad \quad \quad \quad = \quad \quad \quad \quad f_2
\]

where the first identity requires the algebra-coalgebra compatibility, the second uses the module-algebra and the comodule-coalgebra compatibility, as well as the (co-)module properties of \( B_1 \) and \( B_2 \). In the third equation we applied associativity and again the (co-)module properties of \( B_1 \) and \( B_2 \).

Remark 3. Observe that we did not need the complete list of defining relations of a Hopf datum for the deduction of Proposition 2.9. We only needed that \( B_1 \) and \( B_2 \) are both algebras and coalgebras, \( B_1 \) is a left \( B_2 \)-module, \( B_2 \) is a right \( B_1 \)-module, the algebra-coalgebra compatibility, the module-coalgebra compatibility, the comodule-algebra compatibility, the module-algebra compatibility, and the comodule-coalgebra compatibility. In particular we did not use the module-comodule compatibility.

From Propositions 2.6 and 2.1 we conclude that Hopf data are more general objects than BATs and therefore do not correspond to them directly. In the following we will restrict our considerations to so-called recursive Hopf data. They do not necessarily imply universal
characterization. For that reason a more special kind of recursive Hopf data, so-called trivalent Hopf data, will be introduced below. We will verify that trivalent Hopf data and trivalent cross product bialgebras are indeed equivalent notions which provide a (co-)modular and universal description of cross product bialgebras. The resulting theory will turn out to be general enough to unify all the “classical” cross products of [34, 28, 22]. And it will even generate a new family of cross product bialgebras which can be described in this manner. The equivalent formulation of cross product bialgebras either by interrelated (co-)module structures or by certain universal projections and injections therefore will be provided by our theory.

**Definition 2.10** Let \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) be a Hopf datum in \(C\) and define the idempotent \(\pi := \eta_1 \circ \varepsilon_1 \otimes \text{id}_{B_2 \otimes B_1} \otimes \eta_2 \circ \varepsilon_2\). Suppose that for every endomorphism \(f\) of \(B_1 \otimes B_2 \otimes B_1 \otimes B_2\) there exists a non-negative integer \(n \in \mathbb{N}_0\) such that \(\Phi^n(f) = \Phi^n(\pi \circ f \circ \pi)\). Then the Hopf datum \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is called recursive. If \(\Lambda := \{n \in \mathbb{N}_0 | \Phi^n(f) = \Phi^n(\pi \circ f \circ \pi) \forall f \in \text{End}_C(B_1 \otimes B_2 \otimes B_1 \otimes B_2)\}\) is a non-empty set such that \(n_0 := \min \{n \in \Lambda\}\) exists, we call \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) a recursive Hopf datum of order \(n_0\).

A consequence of the structure of \(\Phi\) is the following lemma.

**Lemma 2.11** For every Hopf datum \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) it holds \(\pi \circ \Phi(f) \circ \pi = \pi \circ f \circ \pi\) for any \(f \in \text{End}_C(B_1 \otimes B_2 \otimes B_1 \otimes B_2)\). If \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is recursive of order \(n\) then \(\Phi^n(f) = \Phi^n(f) \quad \forall m \geq n\).

**Proof.** The first statement is verified straightforwardly using Hopf datum properties and the structure of \(\Phi\) according to Definition 2.8. Then it follows for a Hopf datum of order \(n\) that \(\Phi^n(f) = \Phi^n(\pi \circ f \circ \pi) = \Phi^n(\pi \circ \Phi(f) \circ \pi) = \Phi^n(\Phi(f)) = \Phi^{n+1}(f)\).

A Hopf datum \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is non-trivial if \(B_1\) or \(B_2\) is not isomorphic to \(\mathbb{I}_C\).

**Lemma 2.12** A recursive Hopf datum \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) of finite order is non-trivial if and only if its order is greater than 0.

**Proof.** If the order is 0 then \(f = \pi \circ f \circ \pi\) and in particular for \(f = \text{id}_{B_1 \otimes B_2 \otimes B_1 \otimes B_2}\) one shows that \(\text{id}_{A_i} = \eta_i \circ \varepsilon_i\). Therefore \(B_i \cong \mathbb{I}_C\) for \(i = \{1, 2\}\). Conversely if \(B_1\) and \(B_2\) are isomorphic to \(\mathbb{I}_C\) then \(\pi = \text{id}_{B_1 \otimes B_2 \otimes B_1 \otimes B_2}\) and one concludes that the order of the Hopf datum is 0.

**Remark 4** We will henceforth assume that Hopf data are non-trivial so that the order is greater than 0 if it exists.

For special categories recursivity of a Hopf datum implies its finite order.

**Proposition 2.13** Suppose that \(C\) is a category of modules over a commutative ring \(\sigma\), and \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is a Hopf datum in \(C\) with \(B_1\) and \(B_2\) free \(\sigma\)-modules of finite rank. Then \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is recursive if and only if it is recursive of finite order.
PROOF. Suppose that \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is recursive and let \(\{v_i\}_{i \in I}\) be a (finite) basis of \(B_1 \otimes B_2 \otimes B_1 \otimes B_2\). Then every endomorphism \(f \in \text{End}_C(B_1 \otimes B_2 \otimes B_1 \otimes B_2)\) can be written as \(f = \sum_{i,j,k} f_{i,j,k} \delta_{i,j,k}\) where \(f(v_i) = \sum_i f_{i,j} v_j\) and \(\delta_{i,j,k}(v_k) = \delta_{i,k} v_j\). Define \(n_0 = \min \{ n \in \mathbb{N} \mid \Phi^n(\delta_{i,j}) = \Phi^n(\pi \circ \delta_{i,j} \circ \pi) \forall i,j \in I \} \) which exists since \(|I|\) is finite and the Hopf datum is supposed to be recursive. Then one immediately concludes that the order of the Hopf datum is \(n_0\).

The most striking aspect of recursive Hopf data is the bialgebra structure of the corresponding tensor product (co-)algebra \(B_1 \otimes B_2\).

**Theorem 2.14** If \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is a recursive Hopf datum then \(B = B_1 \otimes B_2\) equipped with the structure morphisms \(m_B, \eta_B, \Delta_B, \varepsilon_B\) according to Proposition 2.6 is a bialgebra. It will be denoted by \(B = B_1^{\mu_1} \Join_{\mu_r, \nu_r} B_2\).

**Proof.** Because of Proposition 2.6, we only have to prove that \(\Delta_B\) is an algebra morphism. With the help of Proposition 2.3 and the recursivity of the Hopf datum we derive for some \(n \in \mathbb{N}\) the identities \(f_1 = \Phi^n(f_1) = \Phi^n(\pi \circ f_1 \circ \pi)\) and \(f_2 = \Phi^n(f_2) = \Phi^n(\pi \circ f_2 \circ \pi)\) where \(f_1 = \Delta_B \circ m_B\) and \(f_2 = (m_B \otimes m_B) \circ (\text{id}_B \otimes \Psi_B \otimes B_2 \otimes \text{id}_B) \circ (\Delta_B \otimes \Delta_B)\).

In the next proposition we will introduce trivalent Hopf data.

**Definition and Proposition 2.15** Suppose that \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is a Hopf datum in \(C\). Then one of the (co-)actions of \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) is trivial if and only if one of the morphisms \(\eta_1 \otimes \text{id}_{B_2}, \text{id}_{B_1} \otimes \eta_2, \varepsilon_1 \otimes \text{id}_{B_2}\) or \(\text{id}_{B_1} \otimes \varepsilon_2\) is both algebra and coalgebra morphism. If these equivalent conditions hold, we call \((B_1, B_2, \mu_1, \nu_1, \mu_r, \nu_r)\) a trivalent Hopf datum. Trivalent Hopf data are recursive of order \(\leq 2\). Therefore the corresponding bialgebra \(B = B_1^{\nu_1} \Join_{\mu_r, \nu_r} B_2\) exists and will be denoted by \(B = B_1^{\nu_1} \Join_{\mu_r, \nu_r} B_2\).

**Proof.** Suppose that \(\mu_1\) is trivial. Then from (2.8) we conclude \(m_B = (m_1 \otimes m_2) \circ (\text{id}_{B_1} \otimes (\text{id}_{B_2} \otimes \mu_r) \circ (\Psi_{B_2, B_2} \otimes \text{id}_{B_2}) \circ (\text{id}_{B_2} \otimes \Delta_1) \otimes \text{id}_{B_2})\). Therefore \((\text{id}_{B_1} \otimes \varepsilon_2) \circ m_B = m_1 \circ (\text{id}_{B_1} \otimes \varepsilon_2 \otimes \text{id}_{B_1} \otimes \varepsilon_2)\) and \((\text{id}_{B_1} \otimes \varepsilon_2) \circ \eta_B = \eta_1\) which shows that \((\text{id}_{B_1} \otimes \varepsilon_2)\) is an algebra morphism. Because of Proposition 2.6, \((\text{id}_{B_1} \otimes \varepsilon_2)\) is also a coalgebra morphism. Conversely if \((\text{id}_{B_1} \otimes \varepsilon_2)\) is an algebra morphism, then in particular \((\text{id}_{B_1} \otimes \varepsilon_2) \circ m_B \circ (\eta_1 \otimes \text{id}_{B_2} \otimes \text{id}_{B_1} \otimes \eta_2) = m_1 \circ (\text{id}_{B_1} \otimes \varepsilon_2 \otimes \text{id}_{B_1} \otimes \varepsilon_2)\) from which the triviality \(\mu_1 = \varepsilon_2 \otimes \text{id}_{B_1}\) of \(\mu_1\) is derived. Since \(\mu_1\) is trivial the identity

\[
\Phi(f) = \Phi((\text{id}_{B_1} \otimes B_2 \otimes B_1 \otimes \eta_2 \otimes \varepsilon_2) \circ f \circ (\eta_1 \otimes \varepsilon_1 \otimes \text{id}_{B_2} \otimes B_1 \otimes B_2))
\] (2.15)

can be verified directly for any endomorphism \(f\). From the general structure of \(\Phi\) as given in Definition 2.8, we derive

\[
(\text{id}_{B_1} \otimes B_2 \otimes B_1 \otimes \eta_2 \otimes \varepsilon_2) \circ \Phi(f) \circ (\eta_1 \otimes \varepsilon_1 \otimes \text{id}_{B_2} \otimes B_1 \otimes B_2) = (\text{id}_{B_1} \otimes B_2 \otimes B_1 \otimes \eta_2 \otimes \varepsilon_2) \circ \Phi((\eta_1 \otimes \varepsilon_1 \otimes \text{id}_{B_2} \otimes B_1 \otimes B_2) \circ f \circ (\text{id}_{B_1} \otimes B_2 \otimes B_1 \otimes \eta_2 \otimes \varepsilon_2)) \circ (\eta_1 \otimes \varepsilon_1 \otimes \text{id}_{B_2} \otimes B_1 \otimes B_2).
\] (2.16)

Using (2.15) two times, (2.16) and then again (2.15) one obtains the result \(\Phi^2(f) = \Phi^2(\pi \circ f \circ \pi)\) for any \(f \in \text{End}_C(B_1 \otimes B_2 \otimes B_1 \otimes B_2)\). Hence the order of the Hopf datum is
≤ 2. Because of the dual and mirror symmetries of Hopf data the proof of the proposition follows analogously for any other (co-)action being trivial.

**Remark 5** One verifies easily that \((2.15)\) can be obtained under the following conditions which are weaker than the assumption of triviality of one of the (co-)actions.

\[
\begin{align*}
\left\{ \begin{array}{c}
B_2 B_1 \\
B_1 B_2
\end{array} \right\} = B_2 B_1 \\
\left\{ \begin{array}{c}
B_2 B_2 \\
B_1 B_1
\end{array} \right\} = B_2 B_2
\end{align*}
\]

or similar conditions involving \(\nu_l\) and \(\mu_r\). Then it follows quite analogously as in Proposition \(2.15\) that the Hopf datum is recursive of order \(≤ 2\). Using the notion of Proposition \(2.6\) these conditions can be reformulated as conditions of \(\phi_{1,2}\) and \(\phi_{2,1}\). These are conditions of a BAT since the Hopf datum is recursive. Therefore a generalization of Theorem \(2.16\) below can be obtained in this way (see also the notion “strong Hopf datum” in \(3\)).

The following theorem demonstrates that trivalent Hopf data and trivalent cross product bialgebras coincide. This shows that a unified theory of cross product bialgebras has been found which provides universal and (co-)modular characterization equivalently.

**Theorem 2.16** Suppose that \(A\) is a bialgebra in \(C\). Then the following statements are equivalent.

1. There is a trivalent Hopf datum \((B_1, B_2, \mu_1, \nu_l, \nu_r)\) such that the corresponding bialgebra \(B_1^\nu_l \otimes \nu_r B_2\) is bialgebra isomorphic to \(A\).

2. \(A\) is bialgebra isomorphic to a trivalent cross product bialgebra \(B_1 \varphi_{1,2} \otimes \varphi_{2,1} B_2\).

3. There are algebra morphisms \(i_j: B_j \to A\) and coalgebra morphisms \(p_j: A \to B_j\) such that \(p_j \circ i_j = \text{id}_{B_j}\) for \(j \in \{1, 2\}\), \(m_A \circ (i_1 \otimes i_2) = \left( (p_1 \otimes p_2) \circ \Delta_A \right)^{-1}\), and one of the morphisms \(i_1, i_2, p_1, p_2\) is both algebra and coalgebra morphism.

4. There are idempotents \(\Pi_1, \Pi_2 \in \text{End}(A)\) such that

\[
\begin{align*}
m_A \circ (\Pi_j \otimes \Pi_j) &= \Pi_j \circ m_A \circ (\Pi_j \otimes \Pi_j), \\
(\Pi_j \otimes \Pi_j) \circ \Delta_A &= (\Pi_j \otimes \Pi_j) \circ \Delta_A \circ \Pi_j, \\
\epsilon_A \circ \Pi_j &= \epsilon_A
\end{align*}
\]

for every \(j \in \{1, 2\}\), the sequence \(A \otimes A \xrightarrow{m_A \circ (\Pi_1 \otimes \Pi_2)} A \xrightarrow{(\Pi_1 \otimes \Pi_2) \circ \Delta_A} A \otimes A\) is a splitting of the idempotent \(\Pi_1 \otimes \Pi_2\) of \(A \otimes A\), and one of the idempotents \(\Pi_1, \Pi_2\) is either algebra or coalgebra morphism.

**Proof.** Essentially the proof of the theorem has been done in Propositions \(2.2, 2.6, 2.7, \) and \(2.15\). We only have to show the additional (co-)algebra properties of the corresponding morphisms or the triviality of the corresponding (co-)actions.

\("(3) =⇒ (1)\)". From Proposition \(2.2\) it follows especially that \(A\) is isomorphic to a cross product bialgebra \(B = B_1 \varphi_{1,2} \otimes \varphi_{2,1} B_2\) through the bialgebra isomorphism \(\phi : \)
$B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2 \to A$, $\phi = m_A \circ (i_1 \otimes i_2)$. Then there is a Hopf datum such that $B = B_1 \nu_1 \bowtie \mu_0 \nu_2 B_2 = B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2$ because of Proposition 2.7. Suppose that $p_1$ is algebra and coalgebra morphism. Then $p_1 \circ \phi \circ m_B = m_1 \circ (p_1 \otimes p_1) \circ (i_1 \otimes i_2) \circ m_B = p_1 \circ m_A \circ (\phi \otimes \phi)$. Therefore $m_1 \circ (p_1 \circ i_1 \otimes p_1 \circ i_2) \circ m_B = m_1 \circ (m_1 \otimes m_1) \circ (p_1 \circ i_1 \otimes p_1 \circ i_2 \circ p_1 \circ i_2)$. Since $p_1 \circ i_1 = \text{id}_{B_1}$ and $p_1 \circ i_2 = (\text{id}_{B_1} \otimes \varepsilon_2) \circ \phi^{-1} \circ \phi \circ (\eta_1 \otimes \text{id}_{B_1}) = \eta_1 \circ \varepsilon_2$ the identity $\mu_1 = (\text{id}_{B_1} \otimes \varepsilon_2) \circ \phi = \varepsilon_2 \circ \text{id}_{B_1}$ follows then from (2.9) and (2.11).

“(1) $\Rightarrow$ (2)”: If there is a trivalent Hopf datum with trivial left action $\mu_1$ then $\mu_1 = (\text{id}_{B_1} \otimes \varepsilon_2) \circ \phi = \varepsilon_2 \circ \text{id}_{B_1}$ and therefore $(\text{id}_{B_1} \otimes \varepsilon_2) \circ m_B = m_1 \circ (\text{id}_{B_1} \otimes \varepsilon_2 \otimes \text{id}_{B_1} \otimes \varepsilon_2)$ and $(\text{id}_{B_1} \otimes \varepsilon_2) \circ \eta_B = \eta_1$ which implies that $(\text{id}_{B_1} \otimes \varepsilon_2)$ is an algebra morphism. Because of Proposition 2.6 it is also a coalgebra morphism.

“(2) $\Rightarrow$ (4)”: Suppose that $(\text{id}_{B_1} \otimes \varepsilon_2)$ is an algebra morphism. Using Proposition 2.2 and the bialgebra isomorphism $\phi : B_1 \varphi_{1,2} \bowtie \varphi_{2,1} B_2 \to A$ we obtain $\Pi_1 = i_1 \circ p_1 = \phi \circ (\text{id}_{B_1} \otimes \eta_2) \circ (\text{id}_{B_1} \otimes \varepsilon_2) \circ \phi^{-1}$ and therefore $\Pi_1$ is an algebra morphism.

“(4) $\Rightarrow$ (3)”: If $\Pi_1$ is an algebra morphism then $\Pi_1 \circ m_1 = i_1 \circ p_1 \circ m_A = m_A \circ (i_1 \circ p_1 \otimes i_1 \circ p_1) = i_1 \circ m_1 \circ (p_1 \otimes p_1)$ and $\Pi_1 \circ \eta_A = i_1 \circ p_1 \otimes \eta_A = \eta_A = i_1 \circ \eta_1$ because $i_1$ is an algebra morphism. Since $i_1$ is monomorphic one concludes that $p_1$ is an algebra morphism and also a coalgebra morphism by Proposition 2.2. Thus the Theorem is proved for a particular case. Because of dual and mirror symmetry all other cases can be verified analogously.

Theorem 2.10 shows that there exists a one-to-one correspondence of trivalent cross product bialgebras and trivalent Hopf data. In addition both notations are equivalent to a description in terms of a certain projector decomposition. Since Definition 2.4 of a trivalent cross product bialgebra is a generalization of the cross products with bialgebra structure according to [34, 28, 22], we can express all of them in a unified manner through trivalent Hopf data. Moreover the most general trivalent Hopf data give rise to a new family of (trivalent) cross product bialgebras.

For a better understanding of the theory we list five special examples of trivalent Hopf data in the sequel which cover all the other cases because of the dual and mirror symmetries. The discussion of the different types of trivalent Hopf datum $(B_1, B_2, \mu_1, \nu_1, \nu_2)$ will be taken up with the help of the table [22, 28, 22] where the particular entries take values 0 or 1 dependent on the (co-)action is trivial or not. Thus by definition maximally three entries in the table take the value 1.

**Corollary 2.17**

: All the actions and coactions are trivial. Then the corresponding Hopf datum is equivalent given by the following data. $B_1$ and $B_2$ are bialgebras in $C$, $\Psi_{B_1, B_2} \circ \Psi_{B_1, B_2} = \text{id}_{B_1 \otimes B_2}$, and $B_1 \nu_1 \bowtie \mu_0 \nu_2 B_2$ is the canonical tensor product bialgebra $B_1 \otimes B_2$.

: The trivalent Hopf datum is given through the following data. $B_2$ is a bialgebra in $C$ and $B_1$ is a $B_2$-crossed comodule bialgebra in $B_2 \mathcal{D} Y (C)$ [3, 4]. Then $B_1 \nu_1 \bowtie \mu_0 \nu_2 B_2 = B_1 \nu_1 \bowtie \nu_2 B_2$ is the braided version of the crossed product or biproduct [23, 4].

: The trivalent Hopf datum $(B_1, B_2, \mu_1, \mu_2)$ is a braided version of the matched pair [22]. Explicitly, $B_1$ and $B_2$ are bialgebras in $C$, $B_1$ is a left $B_2$-module coalgebra,
$B_2$ is a right $B_1$-module coalgebra, and the following defining relations are fulfilled.

\[
\begin{align*}
\psi &= \begin{bmatrix} x \\ y \\ z \\ \nu \end{bmatrix}, & \mu_r \circ (\eta_2 \otimes \text{id}_{B_1}) &= \eta_2 \circ \varepsilon_1, \\
\chi &= \begin{bmatrix} x \\ y \\ z \\ \mu \end{bmatrix}, & \mu_l \circ (\text{id}_{B_2} \otimes \eta_1) &= \eta_1 \circ \varepsilon_2,
\end{align*}
\] (2.17)

The corresponding bialgebra $B_1 \mu_l \bowtie \mu_r B_2$ is a braided version of the double cross product [22]. The bialgebra structure reads as follows.

\[
\begin{align*}
m_{B_1 \mu_l \bowtie \mu_r B_2} &= \begin{bmatrix} x \\ y \\ z \\ \mu \end{bmatrix}, & \Delta_{B_1 \mu_l \bowtie \mu_r B_2} &= \begin{bmatrix} x \\ y \\ z \\ \nu \end{bmatrix}
\end{align*}
\]

This is the most general trivalent Hopf datum $(B_1, B_2, \nu_l, \mu_r, \nu_r)$. $B_1$ is a bialgebra in $C$ and a left $B_2$-comodule. $B_2$ is a counital right $B_1$-module-comodule algebra and
a unital coalgebra, and the following defining identities hold.

\[ \varepsilon_2 \circ \mu_r = \varepsilon_2 \otimes \varepsilon_1, \quad \nu_l \circ \eta_1 = \eta_2 \otimes \eta_1 \]

\[ (\varepsilon_2 \otimes \text{id}_{B_1}) \circ \nu_r = \eta_1 \circ \varepsilon_2, \quad (\text{id}_{B_2} \otimes \varepsilon_1) \circ \nu_l = \eta_2 \circ \varepsilon_1 \]

The structure of the resulting bialgebra \( B_1 \overset{\mu_r}{\rightarrow} \varepsilon_2 \otimes \varepsilon_1 \) is given by

\[ \Delta(B_1) = \Delta(B_2) = \varepsilon(B_1) = \varepsilon(B_2) = c \]

**Proof.** Straightforward evaluations of Definition 2.5 using the particular trivial (co-)actions.

In the following we will discuss two examples which are closely related to Hopf algebras constructed by Ore extensions [1]. The first example is an infinite Hopf algebra whereas the second example is Radford’s finite-dimensional 4-parameter Hopf algebra [29]. Both Hopf algebra types turn out to be trivalent cross product bialgebras of type \( 0 \) and therefore are biproduct Hopf algebras according to [28]. Closely related are the examples studied in the forthcoming paper [9] where the universal aspects have been considered.

**Example 1** Suppose \( C \) is an abelian group, \( k \) is an algebraically closed field with \( \text{char}(k) = 0 \). Let \( C^* \) be the characteter group of \( C \) and let \( t \in \mathbb{N} \). Assume further that \( g = (g_1, \ldots, g_t) \in C^t \) and \( g^* = (g_1^*, \ldots, g_t^*) \in (C^*)^t \) where at least one of the \( g_i \) and one of the \( g_i^* \) is non-trivial, and \( g_{lr} := g^*_l(g_r) \) with \( g_{lr} \cdot g_{rl} = 1 \) for any \( l, r \in \{1, \ldots, t\} \). The algebra \( H(C, t, g, g^*) \) be generated by the group \( C \) and the generators \( \{x_i\}_{i=1}^t \) subject to the additional relations

\[ x_j \cdot c = g_j^*(c) \cdot x_j \quad \text{and} \quad x_j \cdot x_k = g_{jk} x_k \cdot x_j \quad (2.18) \]

for any \( j, k \in \{1, \ldots, t\} \) and \( c \in C \). Then the following relations define a Hopf algebra structure on \( H(C, t, g, g^*) \).

\[ \Delta(c) = c \otimes c, \quad \varepsilon(c) = 1, \quad S(c) = c^{-1}, \quad (2.19) \]

\[ \Delta(x_j) = x_j \otimes g_j + \mathbb{I} \otimes x_j, \quad \varepsilon(x_j) = 0, \quad S(x_j) = -x_j \cdot g_j^{-1} \]
for all \( j, k \in \{1, \ldots, t\} \) and \( c \in C \). Every element of \( H(C, t, g, g^*) \) is a finite sum of the form \( h = \sum_{c \in C} c \cdot f(x_j)_c \) where \( f(x_j)_c \) is a (non-commutative) polynomial in \( \{x_j\} \). The Hopf algebra is finite-dimensional if \( C \) is finite and \( g_{jj} = -1 \) for all \( j \in \{1, \ldots, t\} \).

Now we define \( B_1 := kC \) the group Hopf algebra of \( C \) with comultiplication \( \Delta_1 \) and counit \( \varepsilon_1 \) on \( C \). Let \( B_2 \) be the algebra \( B_2 := k\langle \{x_j\}\rangle/(x_j x_k = g_{jk} x_k x_j \forall j, k \in \{1, \ldots, t\} \). Then the following definitions yield \( k \)-linear morphisms.

\[
\begin{align*}
i_1 : & \quad \begin{cases} B_1 \to H(C, t, g, g^*) \\ c \mapsto c \end{cases} & \quad i_2 : & \quad \begin{cases} B_2 \to H(C, t, g, g^*) \\ x \mapsto x \end{cases} \\
p_1 : & \quad \begin{cases} H(C, t, g, g^*) \to B_1 \\ c \cdot f(x_j) \mapsto \varepsilon(f(x_j))c \end{cases} & \quad p_2 : & \quad \begin{cases} H(C, t, g, g^*) \to B_2 \\ c \cdot f(x_j) \mapsto \varepsilon(c) f(x_j) \end{cases}
\end{align*}
\]

(2.20)

Straightforward calculations show that \( i_1 \) is a Hopf algebra morphism, \( i_2 \) is an algebra morphism, and \( p_1 \) is an algebra morphism. Then one concludes easily that \( p_1 \) is coalgebra morphism since \( (\Delta_1 \otimes \Delta_1) \circ p_1 \) and \( p_1 \circ \Delta \) are algebra morphism, and the equality of both morphisms has to be proven on the generators only. According to [9] we define the comultiplication \( \Delta_2 \) and counit \( \varepsilon_2 \) of \( B_2 \) as \( \Delta_2 := (p_2 \otimes p_2) \circ \Delta \circ i_2 \) and \( \varepsilon_2 := \varepsilon \circ i_2 \). Using again the fact that every element of \( H(C, t, g, g^*) \) is a finite sum of the form \( h = \sum_{c \in C} c \cdot f(x_j)_c \), one finds

\[
\Delta_2 \circ p_2(h) = \sum_{c \in C} \varepsilon(c) \Delta_2(f(x_j)_c)
\]

\[
= \sum_{c \in C} \Delta(f(x_j)_c)
\]

\[
= \sum_{c \in C} (p_2 \otimes p_2) \circ \Delta(f(x_j)_c)
\]

and on the other hand

\[
(p_2 \otimes p_2) \circ \Delta(h) = \sum_{c \in C} (p_2 \otimes p_2)((c \otimes c) \cdot \Delta(f(x_j)_c))
\]

\[
= \sum_{c \in C} (p_2 \otimes p_2)(\Delta(f(x_j)_c)).
\]

Hence \( \Delta_2 \circ p_2 = (p_2 \otimes p_2) \circ \Delta \). Then it follows that \( \Delta_2 \) is coassociative and \( p_2 \) is coalgebra morphism. Furthermore \( p_2 \) is not an algebra morphism. Suppose the converse, then \( p_2(c \cdot x_j) = p_2(c) \cdot p_2(x_j) = \varepsilon(c) x_j \). On the other hand \( p_2(c \cdot x_j) = p_2(g_j^*(c) x_j \cdot c) = g_j^*(c) \varepsilon(c) x_j \). Therefore \( \varepsilon(c) x_j = g_j^*(c) \varepsilon(c) x_j \). But by assumption there exists a non-trivial \( g_j^* \) which then leads to a contradiction. Similarly, by the existence of a non-trivial \( g_j^* \) it can be shown that \( i_2 \) is not a coalgebra morphism.

Now we prove that \( (p_1 \otimes p_2) \circ \Delta = (\text{id} \otimes (i_1 \otimes i_2))^{-1} \). Observe that \( (p_1 \otimes \text{id}) \circ \Delta \circ i_2 = (\text{id} \otimes i_2) \circ i_2 \) because the corresponding identity holds on the generators and then the statement follows since \( (p_1 \otimes \text{id}) \circ \Delta \circ i_2 \) and \( (\text{id} \otimes i_2) \circ i_2 \) are algebra morphisms. Then for
any finite sum $\sum_{c \in C} c \otimes f(x)_c \in B_1 \otimes B_2$ we have

$$(p_1 \otimes p_2) \circ \Delta \circ m \circ (i_1 \otimes i_2) \sum_{c \in C} c \otimes f(x)_c = \sum_{c \in C} (p_1 \otimes p_2) \circ \Delta(c \cdot f(x)_c)$$

$$= \sum_{c \in C} (c \otimes \mathbb{1}) \cdot (p_1 \otimes p_2) \circ \Delta(f(x)_c)$$

$$= \sum_{c \in C} (c \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes p_2(f(x)_c))$$

$$= \sum_{c \in C} c \otimes f(x)_c.$$  

Thus $(p_1 \otimes p_2) \circ \Delta$ is a left inverse of $m \circ (i_1 \otimes i_2)$. Similarly the right invertibility can be proven. Therefore all conditions of Theorem 2.11 hold, and the morphisms $\varphi_{1,2}$ and $\varphi_{2,1}$ of the corresponding cross product bialgebra are given by $\varphi_{1,2} = (\varepsilon_1 \otimes \text{id}_{B_2} \otimes \text{id}_{B_1} \otimes \varepsilon_2) \circ \Delta_B$ and $\varphi_{2,1} = m_B \circ (\eta_1 \otimes \text{id}_{B_2} \otimes \text{id}_{B_1} \otimes \eta_2)$ (see the proof of Proposition 2.11). Using (2.11) yields the (non-trivial) (co-)actions

$$\mu_r(x \otimes c) = p_2(x \cdot c) \quad \text{and} \quad \nu_r(x) = (p_2 \otimes p_1) \circ \Delta(x).$$

From Corollary 2.17 it follows eventually

**Proposition 2.18** The Hopf algebra $H(C, t, g, g^*)$ is isomorphic to the biproduct bialgebra $B_1 \bowtie_{\mu, \nu} B_2$. In particular $B_2$ is a right $B_1$-crossed module bialgebra.  

The following example is Radford’s 4-parameter Hopf algebra [23]. Its structure resembles the one of the Hopf algebra $H(C, t, g, g^*)$ discussed in Example 1.

**Example 2 (Radford’s 4-Parameter Hopf Algebra)** Suppose again that $k$ is an algebraically closed field with $\text{char}(k) = 0$. Let $n, N, \nu$ be positive integers such that $n | N$, and $\nu < n$. Let $q$ be a primitive $n$th root of unity and $q^\nu$ be an $r$th root of unity, where $r = n/(n, \nu)$. Set $C_N$ to be the cyclic group of order $N$ and $g$ a generating element of $C_N$. Then the 4-parameter Hopf algebra $H_{n,q,N,\nu}$ is generated by the group algebra $kC_N$ and the generator $x$ subject to the additional relations

$$x^r = 0 \quad \text{and} \quad x \cdot g = g \cdot x.$$  

On the generators the comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$ are given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1},$$

$$\Delta(x) = x \otimes \mathbb{1} + g^{-\nu} \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -g^{-\nu} \cdot x.$$  

Similarly as in the previous example one shows that every element of $H_{n,q,N,\nu}$ can be represented in the form $b = \sum_{m=0}^{r-1} \sum_{l=0}^{N-1} \lambda_{m,l} x^m \cdot g^l$. In particular $H_{n,q,N,\nu}$ is finite-dimensional. Let $B_1$ be the algebra $B_1 := k(x)/(x^r)$ and $B_2$ be the group Hopf algebra $B_2 := kC_N$. Then $B_1$ becomes a coalgebra through

$$\Delta_1(x^m) = \sum_{l=0}^{m} \binom{m}{l} q^\nu x^l \otimes x^{m-l} \quad \text{and} \quad \varepsilon_1(x^m) = \delta_{m,0}$$  

(2.22)
where the $q$-binomial is \[ \binom{m}{j}_q := \frac{(m)_q!}{(j)_q!(m-j)_q!}, \quad (s)_p! := (1)_p \cdot (2)_p \cdot \cdots \cdot (s)_p, \quad (0)_p := 1, \]
and \[ (s)_p := \frac{1-q^s}{1-q}. \] Using the properties of the $q$-binomials it is evident that (2.22) renders $B_1$ a coalgebra. Then it can be proven that the following definitions yield $k$-linear homomorphisms $i_1, i_2, p_1$ and $p_2$.

\[
i_1 : \begin{cases} B_1 \hookrightarrow H_{n,q,N,\nu} \\ x^m \mapsto x^m \end{cases}, \quad i_2 : \begin{cases} B_2 \hookrightarrow H_{n,q,N,\nu} \\ g^j \mapsto g^j \end{cases}
\]

\[
p_1 : \begin{cases} H_{n,q,N,\nu} \rightarrow B_1 \\ x^m \cdot g^j \mapsto x^m \end{cases}, \quad p_2 : \begin{cases} \{H(C,t,g,g^*) \rightarrow B_2 \\ x^m \cdot g^j \mapsto \delta_{m0} g^j \end{cases}
\]

\[ (2.23) \]

Obviously $i_1, i_2$ and $p_2$ are algebra morphisms since they preserve the relations of the algebras $H_{n,q,N,\nu}, B_1$ and $B_2$. Furthermore $i_2$ and $p_2$ are coalgebra homomorphisms since the corresponding identities hold for the generators $g$ and $x$. Finally, $p_1$ is algebra morphism because $\Delta_1 \circ p_1(x^m \cdot g^j) = \Delta_1(x^m)$ and

\[ (p_1 \otimes p_1) \circ \Delta(x^m \cdot g^j) = (p_1 \otimes p_1)((x \otimes m \cdot (g^j \otimes g^j)) = (p_1 \otimes p_1)((x \otimes m + g^{-\nu} \otimes x)^m \cdot (g^j \otimes g^j)) = \sum_{j=0}^{m} \binom{m}{j} q^\nu (p_1 \otimes p_1)(x^j \cdot g^{j-\nu (m-j)} \otimes x^{m-j} \cdot g^j) = \sum_{j=0}^{m} \binom{m}{j} q^\nu x^j \otimes x^{m-j} = \Delta_1(x^m) \]

where we used the $q$-binomial identity $(a + b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j \cdot b^{m-j}$ if $a \cdot b = \lambda^{-1} b \cdot a$. This proves that $p_1$ is a coalgebra morphism. Since by assumption $q \neq 1$ and $g^* \neq 1$ one concludes similarly as in Example [1] that $i_1$ in no coalgebra morphism and $p_1$ is no algebra morphism. Because $H_{n,q,N,\nu}$ is finite-dimensional the subsequent relations prove that $(p_1 \otimes p_2) \circ \Delta = (m \circ (i_1 \otimes i_2))^{-1}$.

\[ (p_1 \otimes p_2) \circ \Delta \circ m \circ (i_1 \otimes i_2)(x^m \otimes g^j) = (p_1 \otimes p_2) \circ \Delta(x^m \cdot g^j) = (p_1 \otimes p_2)((x \otimes m \cdot (g^j \otimes g^j)) = \sum_{j=0}^{m} \binom{m}{j} q^\nu (p_1 \otimes p_2)(x^j \cdot g^{j-\nu (m-j)} \otimes x^{m-j} \cdot g^j) = \sum_{j=0}^{m} \binom{m}{j} q^\nu x^j \otimes \varepsilon(x^{m-j} \cdot g^j) = x^m \otimes g^j. \]

Hence again all conditions of Theorem [2.16] hold, and the morphisms $\varphi_{1,2}$ and $\varphi_{2,1}$ of the corresponding cross product bialgebra are given by $\varphi_{1,2} = (\varepsilon_1 \otimes \text{id}_{B_2} \otimes \text{id}_{B_1} \otimes \varepsilon_2) \circ \Delta_B$ and $\varphi_{2,1} = m_B \circ (\eta_1 \otimes \text{id}_{B_2} \otimes \text{id}_{B_1} \otimes \eta_2)$, or explicitly $\varphi_{1,2}(x^m \otimes g^j) = g^{-\nu m} \otimes x^m$ and $\varphi_{2,1}(g^j \otimes x^m) = q^{-ml} x^m \otimes g^j$. Using (2.11) yields the (non-trivial) (co-)actions

\[ \mu_l(g^j \otimes x^m) = q^{-ml} x^m \quad \text{and} \quad \nu_l(x^m) = g^{-\nu m} \otimes x^m. \]
We collect the previous results in the next proposition.

**Proposition 2.19** The 4-parameter Hopf algebra $H_{n,q,N,\nu}$ is isomorphic to the biproduct bialgebra $B_1 \mu, \nu \bowtie B_2$. In particular $B_1$ is a left $B_2$-crossed module bialgebra.

3 Applications

In Section 3 we discuss further applications of the results of Section 2. The first example shows that Majid’s double biproduct [26] essentially is a cross product bialgebra construction in the braided category $H_C^H$ of Hopf bimodules over a Hopf algebra $H$. In the second subsection we show that the Drinfel’d double in a braided category can be reconstructed as a matched pair if and only if the braiding of the two tensor factors is involutive. This confirms in a certain sense the statements of [26].

Double Biproduct Bialgebras

We consider a Hopf algebra $H$. From Theorem 1.2 and its mirror symmetric version we know that every right $H$-crossed module $B \in \text{Obj}(DY(C^H))$ and every left $H$-crossed module $C \in \text{Obj}(DY(C))$ yield Hopf bimodules $X = H \ltimes B$ and $Y = C \rtimes H$ in $H_C^H$ respectively. These special types of Hopf bimodules will be used later for the construction of the double biproduct as a twist of a tensor product bialgebra in $H_C^H$ considered as bialgebra in the category $C$ according to Proposition 1.4.

**Lemma 3.1** Let $B \in \text{Obj}(DY(C^H))$ and $C \in \text{Obj}(DY(C))$ be $H$-crossed modules. Then for the objects $X = H \ltimes B$ and $Y = C \rtimes H$ in the category $H_C^H$ the identity $H_C^H \Psi_{X,Y} \circ H_C^H \Psi_{Y,X} = \text{id}_{X \otimes_H Y}$ holds if and only if

$$\psi_{C,B} \circ \psi_{B,C} = (\mu_B \otimes \mu_C^l) \circ (\text{id}_B \otimes \psi_{H,H} \otimes \text{id}_C) \circ (\nu^B \otimes \nu^C).$$

**Proof.** From (1.3) and its dual version one deduces

$$(\chi \Pi \otimes \Pi_Y) \circ \rho_{X,Y} \circ H_C^H \Psi_{Y,X} \circ H_C^H \Psi_{X,Y} \circ \lambda_{X,Y} \circ (\chi \Pi \otimes \Pi_Y)$$

$$= (\chi \Pi \otimes \Pi_Y) \circ \psi_{Y,X} \circ \rho^H_{X,Y} \circ \lambda^H_{X,Y} \circ \psi_{X,Y} \circ (\chi \Pi \otimes \Pi_Y).$$

Condition (3.1) means that the right hand side of (3.2) equals to $(\chi \Pi \otimes \Pi_Y) \circ \rho_{X,Y} \circ \lambda_{X,Y} \circ (\chi \Pi \otimes \Pi_Y)$. Then we use Lemma 1.3 to derive $\rho_{X,Y} \circ H_C^H \Psi_{X,Y} \circ \lambda_{X,Y} = \rho_{X,Y} \circ \lambda_{X,Y}$. Since $\lambda^H_{X,Y}$ is an epimorphism and $\rho^H_{X,Y}$ is a monomorphism the sufficient part of the lemma is proved. Conversely if $H_C^H \Psi_{X,Y} \circ H_C^H \Psi_{Y,X} = \text{id}_{X \otimes_H Y}$ then equation (1.1) proves (3.1).

Lemma 3.1 is a braided version of the identity (58) in [26] which was one of the compatibility conditions for the construction of the double biproduct bialgebra. We suppose henceforth that $B$ and $C$ are bialgebras in $DY(C^H)$ and $DY(C)$ respectively. Then according to Theorem 1.2 the objects $X = H \ltimes B$ and $Y = C \rtimes H$ are bialgebras in $H_C^H$. From (1) (and Proposition 1.4) we know that the multiplications $m_X$ and $m_Y$ are uniquely determined. In particular $m_{H \ltimes B} = m_X \circ \lambda^H_{X,X}$ and $m_X = m_{H \rtimes B} \circ (\text{id}_H \otimes X)$ where $m_{H \ltimes B}$ is the multiplication of the crossed product algebra in $C$ (see Corollary
Proof. We denote by $Z$ the object in $\mathcal{C}$ given by Proposition 1.4 the multiplication of $\mathcal{H}^cH$ according to Corollary 2.17. Explicitly

$$m_Y \otimes_{H^X} = (m_Y \otimes_{H^X}) \circ (id_Y \otimes_m \Psi_{Y,X} \otimes_{H} id_X),$$

$$\Delta_Y \otimes_{H^X} = (id_Y \otimes \Psi_{Y,X} \otimes_{H} id_X) \circ (\Delta_Y \otimes_{H^X} \Delta_X).$$

Lemma 3.2 Suppose that the conditions of Lemma 3.1 are fulfilled for the bialgebras $X$ and $Y$ in $\mathcal{H}^cH$. Then the Hopf bimodule tensor product bialgebra $\mathcal{H}^cH$ is identified with the object $\mathcal{H}^cH$ through the canonical morphisms given by

$$\lambda_{Y,X} = id_C \otimes m_B \otimes id_B, \quad \rho_{Y,X} = id_C \otimes \Delta_B \otimes id_B.$$  \hfill (3.4)

We denote by $Z := Y \otimes_{H} X = C \otimes H \otimes B$ the bialgebra in $\mathcal{H}^cH$. Then

$$\lambda_{Z} = id_C \otimes \lambda_{X,Y} \otimes id_B, \quad \rho_{Z} = id_C \otimes \rho_{X,Y} \otimes id_B.$$  \hfill (3.5)

Proof. Without problems one verifies that $\lambda_{Y,X}$ and $\rho_{Y,X}$ according to (3.4) fulfill the required properties of Theorem 1.1. Using the identification $\mathcal{H}^cH$ induced from (3.4) the proof of (3.5) follows.

Lemma 3.2 one obtains a bialgebra $Z = C \otimes H \otimes B$ in $\mathcal{C}$ from the $H$-Hopf bimodule bialgebra $Z = Y \otimes_{H} X$. The explicit structure of $Z$ is presented in the subsequent proposition.

Proposition 3.3 Multiplication and comultiplication of the bialgebra $Z = C \otimes H \otimes B$ are given by

$$m_Z = \begin{array}{ccc}
\begin{array}{ccc}
C & H & B \\
C & H & B \\
C & H & B \\
\end{array}
\end{array} \quad \text{and} \quad \Delta_Z = \begin{array}{ccc}
\begin{array}{ccc}
C & H & B \\
\end{array}
\end{array}$$  \hfill (3.6)

The unit and counit are $\eta_Z = \eta_{C} \otimes \eta_{H} \otimes \eta_{B}$ and $\epsilon_Z = \epsilon_{C} \otimes \epsilon_{H} \otimes \epsilon_{B}$ respectively. There are canonical bialgebra monomorphisms $id_{C}\otimes H \otimes \eta_{B}: C \times B \to Z$, $\eta_{C} \otimes id_{B}: H \otimes B \to Z$, and bialgebra epimorphisms $id_{C} \otimes \epsilon_{B}: Z \to C \times B$, $\epsilon_{C} \otimes id_{H}: Z \to H \times B$. Additionally $id_{C} \otimes \eta_{H} \otimes id_{B}: C \otimes B \to Z$ is an algebra monomorphism, and $id_{C} \otimes \epsilon_{H} \otimes id_{B}$ is a coalgebra epimorphism.

Proof. From the considerations before Lemma 3.2 we know that the multiplication of $Z = Y \otimes_{H} X$ is given by $m_Z = (m_Y \otimes_{H} m_X) \circ (id_Y \otimes_{H} \Psi_{Y,X} \otimes_{H} id_X)$. and according to Proposition 1.4 the multiplication of $Z$ reads as

$$m_Z = m_Z \circ \lambda_{Z}$$

$$= (m_Y \otimes_{H} m_X) \circ (id_Y \otimes_{H} \Psi_{Y,X} \otimes_{H} id_X) \circ \lambda_{Z}$$

$$= (m_C \otimes H \otimes m_B) \circ (id_C \otimes \Psi_{X,Y} \otimes_{H} id_B)$$  \hfill (3.7)
where we used (3.5) in the third equation. From (1.3) and from the Hopf bimodule property of $\rho^H$ and $\lambda^H$ (see Theorem 1.1) it follows

\begin{equation}
\begin{aligned}
\rho^H \Psi_{X,Y} \circ \lambda^H_{X,Y} & = (\mu^Y \circ (\mu \otimes X) \otimes \mu^X) \otimes \left( (\mu^Y \circ (\Pi_Y \otimes \mu_X) \otimes \mu_Y) \circ \nu_r^Y \right) \\
\rho^H \Psi_{X,Y} \circ \lambda^H_{B,C} & = (\mu^Y \circ (\mu \otimes X) \otimes \mu^X) \bigg( (\mu^Y \circ (\Pi_Y \otimes \mu_X) \otimes \mu_Y) \circ \nu_r^Y \bigg) \\
& = M \circ (\mu^Y \circ (\mu \otimes X) \otimes \mu^X) \bigg( (\mu^Y \circ (\Pi_Y \otimes \mu_X) \otimes \mu_Y) \circ \nu_r^Y \bigg)
\end{aligned}
\end{equation}

where $M = \mu^Y \circ (\mu^Y \circ (\Pi_Y \otimes \mu_X) \otimes \mu_Y) \circ \nu_r^Y$. Inserting (3.8) into (3.7) yields the final result for $m_Z$. Dually $\Delta_Z$ can be derived. 

If it is clear from the context we will henceforth denote the bialgebra $Z$ in $C$ by $C \otimes H \otimes B$.

**Proposition 3.4** Let $B$ and $C$ be as in Lemma 3.2. Suppose that $\rho : B \otimes C \to \mathbb{I}_C$ is a morphism in $C$ satisfying the identities

\begin{equation}
\begin{aligned}
\rho \circ (\mu^B \otimes \text{id}_B) & = \rho \circ (\text{id}_C \otimes \mu^C), \\
\rho \circ (\text{id}_B \otimes m_C) & = \rho^{(2)} \circ (\Psi^C_{B,B} \otimes \Delta_B \otimes \text{id}_C), \\
\rho \circ (m_B \otimes \text{id}_C) & = \rho^{(2)} \circ (\Delta_C \otimes \Psi^C_{B,B} \otimes \text{id}_C)
\end{aligned}
\end{equation}

where $\rho^{(2)} = \rho \circ (\text{id}_B \otimes \rho \otimes \text{id}_C)$ and $\Delta^M \Psi_{B,B} = (\text{id}_B \otimes \mu_B^B) \circ (\Psi^C_{B,B} \otimes \text{id}_B) \circ (\text{id}_B \otimes \nu^B)$. Then $\hat{\rho} := \epsilon_C \otimes H \otimes \rho \otimes \epsilon_H \otimes B$ is a 2-cocycle of the bialgebra $C \otimes H \otimes B$ from Proposition 2.3. If $\rho$ is convolution invertible then $\hat{\rho}$ is convolution invertible with inverse $\hat{\rho}^{-1} = \epsilon_C \otimes H \otimes \rho^{-1} \otimes \epsilon_H \otimes B$.

**Proof.** The convolution invertibility and the cocycle property (1.3) for $\hat{\rho}$, $m_Z$ and $\Delta_Z$ can be proven easily. The calculation of the left and the right hand side of (1.8) respectively yields

\begin{equation}
\begin{aligned}
\rho \circ (\mu^B \otimes \text{id}_C) & \circ (\text{id}_B \otimes \rho) \circ (\Psi^C_{B,B} \otimes \text{id}_C) \circ (\Delta_B \otimes \text{id}_C) \\
& \otimes (\rho \otimes \text{id}_H) \circ (\text{id}_B \otimes \Psi^C_{H,C} \otimes \text{id}_C) \\
& \circ (\text{id}_B \otimes m_H) \circ (\Psi^C_{H,B} \otimes \text{id}_H) \circ (\text{id}_B \otimes \nu^B) \otimes \Delta_C
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\rho \circ (m_B \otimes \text{id}_C) & \circ (\mu^B \otimes \text{id}_B \otimes \text{id}_C) \\
& \circ ((\text{id}_B \otimes \rho) \circ (\text{id}_B \otimes \Delta_B \otimes \text{id}_C) \otimes \text{id}_H \otimes B \otimes C)
\end{aligned}
\end{equation}

where the first and the second defining property of $\rho$ in (3.3) have been used. With the help of the third equation of (3.9) one can show that (3.10) equals (3.11). 

**Remark 6** The identities (3.9) are braided versions of [26 eqs. (56)].
The following corollary is a straightforward consequence of Propositions 3.4.4 and 3.4.

**Corollary 3.5** Suppose that the pairing $\rho$ in Proposition 3.4 is invertible. Then according to Proposition 1.5, the multiplication $m_{C \otimes H \otimes B}^\hat{\rho}$ of the twisted bialgebra $(C \otimes H \otimes B)_\hat{\rho}$ is given by

$$m_{C \otimes H \otimes B}^\hat{\rho} = \begin{array}{cccc}
  C & H & B & C \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  C & H & B & C \\
\end{array}$$

(3.12)

where the pairing $\rho$ is presented by $\sum_i^+$ and its convolution inverse $\rho^-$ by $\sum_i^-$.

**Proof.** Because of Proposition 3.4 we can apply Definition 1.5 and Proposition 1.6 to $\rho$ where the pairing $\rho := \lambda \triangleright y$ pairing bidualgebra according to Corollary 3.5 with $H$. The following corollary is a straightforward consequence of Propositions 1.6, 3.3 and 3.4. A straightforward calculation then yields the result.

We will discuss in the following an example in which unavoidably cross product bialgebras in certain braided categories emerge.

**Example 3** Corollary 3.5 is a (braided) generalization of Majid’s double biproduct construction [28]. It has been shown in [26, 33] that the quantum enveloping algebra $U$ in terms of Lusztig’s construction [17] is a double biproduct bialgebra. Explicitly, let $(I, \cdot)$ be a Cartan datum, and $(X, Y, \langle \cdot, \cdot \rangle)$ be a root datum of type $(I, \cdot)$. Given the commutative ring $k = \mathbb{Q}(q)$, let $f$ be the $k$-algebra generated by $I$, factorized by the annihilator radical of the unique pairing $\langle \cdot, \cdot \rangle : k(I) \times k(I) \to k$ given in Proposition 1.2.3 in [17]. Furthermore let $U_0$ be the group algebra of $Y$ over $k$. Then [17], $U \cong f \otimes U_0 \otimes f$ are isomorphic Hopf algebras. The algebra $f$ is both left and right $U_0$-crossed module bialgebra. The bialgebra structure of $f$ is induced by the algebra structure of $f$ and by primitivity of all elements of $k(I)$. From [17] we know that $f = \bigoplus_{\nu \in \mathbb{N}[I]} f_\nu$ is an $\mathbb{N}[I]$-graded algebra. The root datum provides embeddings $x : I \hookrightarrow X$, $i \mapsto x_i$ and $y : I \hookrightarrow Y$, $i \mapsto y_i$ which in turn canonically induce homomorphisms of abelian groups $x : \mathbb{N}[I] \to X$, $\nu \mapsto x_\nu$ and $y : \mathbb{N}[I] \to Y$, $\nu \mapsto y_\nu$. Then the left $U_0$-crossed module structure of $f$ is given by $\lambda \triangleright y := q^{-\langle y, x|\lambda \rangle} \lambda$ and $\nu_\lambda(\nu) := -\sum_{i \in I}(\frac{\lambda}{i}) \cdot |\lambda||\lambda| \cdot y_i \otimes \lambda$ where $y \in Y$ and $\lambda \in f$ is homogeneous of degree $\nu = |\lambda| = \sum_{i \in I} |\lambda||i \in \mathbb{N}[I]$, i.e. $\lambda \in f_{\langle \lambda \rangle}$. The right $U_0$-crossed module structure of $f$ is $\lambda \triangleright y := q^{-\langle y, x|\lambda \rangle} \lambda$ and $\nu_\lambda(\nu) := -\sum_{i \in I}(\frac{\lambda}{i}) \cdot |\lambda||\lambda| \cdot y_i$. Then $U \cong f \otimes U_0 \otimes f$ is a double biproduct bialgebra according to Corollary 3.5 with $H = U_0$, $C = f$ and $B = f$, and the pairing $\rho = (\langle \cdot, \cdot \rangle)$.

Although in the present example the base category $C = \text{k}-\text{mod}$ is symmetric, the categories of $H$-crossed modules and $H$-Hopf bimodules are braided and the cross product bialgebra construction is within these categories. This emphasizes once again that the double biproduct (even in ordinary symmetric categories) is a cross product bialgebra construction in a braided category.
Quantum Double Construction

In Corollary 2.17 we discussed braided versions of matched pairs leading to braided double cross products. From [25] we know that two dually paired bialgebras in a symmetric category yield a matched pair from which a generalization of the Drinfel’d double can be reconstructed. Such a procedure had been discussed for braided categories in [26]. It was announced that a similar construction fails there since the braiding twists up and can not be disentangled. The subsequent proposition confirms this observation in a certain sense.

**Proposition 3.6** Suppose that $A$ and $H$ are bialgebras in $C$ which are paired by the pairing $\langle.,.\rangle : H \otimes A \rightarrow \mathbb{I}_C$ subject to the defining identities

\[
\langle.,.\rangle \circ (m_H \otimes \text{id}) = \langle.,.\rangle \circ (\text{id} \otimes \langle.,.\rangle \otimes \text{id}) \circ (\Delta_H \otimes \text{id} \otimes \text{id}) \\
\langle.,.\rangle \circ (\text{id} \otimes m_A) = \langle.,.\rangle \circ (\text{id} \otimes \langle.,.\rangle \otimes \text{id}) \circ (\Delta_H \otimes \text{id} \otimes \text{id}) \\
\langle.,.\rangle \circ (\eta_H \otimes \text{id}) = \varepsilon_A \\
\langle.,.\rangle \circ (\text{id} \otimes \eta_A) = \varepsilon_H.
\]  

(3.13)

Then the following statements hold.

1. If $A$ and $H$ are Hopf algebras and $S_A$ is an isomorphism in $C$ then $\langle.,.\rangle$ is convolution invertible. Explicitely $\langle.,.\rangle^{-1} = \langle.,.\rangle \circ (S_H \otimes \text{id}) = \langle.,.\rangle \circ (\text{id} \otimes S_A^{-1})$.

2. If $\langle.,.\rangle$ is convolution invertible we define

\[
\langle.\rangle = (\langle.,.\rangle^{-1} \otimes \text{id} \otimes \langle.,.\rangle) \circ (\text{id} \otimes \Psi_{H \otimes H,A} \otimes \text{id}) \circ (\Delta_H^{(2)} \otimes \Delta_A) \\
\langle.\rangle = (\langle.,.\rangle^{-1} \otimes \text{id} \otimes \langle.,.\rangle) \circ (\text{id} \otimes \Psi_{H,A \otimes A} \otimes \text{id}) \circ (\Delta_H \otimes \Delta_A^{(2)})
\]  

(3.14)

where $\Delta^{(2)} = (\Delta \otimes \text{id}) \circ \Delta$. Then $(A,\langle.\rangle)$ is a left $H$-module and $(H,\langle.\rangle)$ is a right $A$-module. The tuple $(A,H,\langle.\rangle,\langle.\rangle)$ is a matched pair as in Corollary 2.17 if and only if $\Psi_{H,A} \circ \Psi_{A,H} = \text{id}$.

**Proof.** Statement 1 is proved analogously as in the standard symmetric case. Without problems one verifies that $\langle.\rangle$ and $\langle.\rangle$ in statement 2 define algebra actions. Now suppose that $\Psi_{H,A} \circ \Psi_{A,H} = \text{id}_{A \otimes H}$. It is not difficult to show that $(A,H,\langle.\rangle,\langle.\rangle)$ is a matched pair because nearly everything works like in the classical symmetric case [23, 25]. Conversely if $(A,H,\langle.\rangle,\langle.\rangle)$ defined by (3.14) is a matched pair then the following identity has to be fulfilled because of the last equation in (2.17).

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
H \rightarrow A \rightarrow H \leftarrow A \leftarrow H
\end{array}
\end{array}
\end{array}
\end{align*}
\]  

(3.15)

From (3.15) we obtain $\Psi_{A,H} \circ \Psi_{H,A} = \text{id}_{H \otimes A}$ by multiplying $\langle.,.\rangle$ to the left and $\langle.,.\rangle^{-1}$ to the right of (3.15) using the product given by (1.7).
4 Conclusions and Outlook

We defined Hopf data and cross product bialgebras very generally. Cross product bialgebras are Hopf data. A special class of Hopf data are recursive Hopf data probably with finite order. Recursive Hopf data are cross product bialgebras. We further restricted to trivalent Hopf data and trivalent cross product bialgebra. We showed the equivalence of both notions and provided a description of trivalent cross product bialgebras either through (co-)modular properties or by universal systems of certain projections and injections respectively. Therefore the classification of trivalent cross product bialgebras in terms of trivalent Hopf data has been achieved. The known cross products with bialgebra structure fit into this new classification scheme. In addition new types of trivalent cross product bialgebras have been found which generalize all other types. However explicit examples have not been found yet for these general types of trivalent cross product bialgebras.

We have been working throughout in a braided monoidal setting which allowed us to apply the machinery of Hopf data and cross product bialgebras to braided categories. In particular we showed that the double biproduct bialgebras come from a certain tensor product bialgebra in the braided category of Hopf bimodules over a given Hopf algebra. A more general study of recursive Hopf data in Hopf bimodule categories will be published elsewhere.

The structure of (recursive) Hopf data shows to be symmetric under duality and reflection at a vertical axis – if one considers the defining identities as graphics in three dimensional space. These symmetries will be somehow destroyed when considering trivalent Hopf data and trivalent cross product bialgebras and one might ask if such a breaking of symmetry is a generic feature of the theory of cross product bialgebras. Therefore it remains an open problem if the present setting is the most general one to describe cross product bialgebras by (co-)modular properties or by universal systems of projection and injection morphisms equivalently. One could think of certain types of recursive Hopf data (with finite order) or some other specializations of Hopf data which preserve the above mentioned symmetries, to be good candidates for a more general framework. A possible generalization has been presented in Remark 5 although the symmetries will be destroyed in this case, too.

In the present article we studied cross product bialgebras without cocycles and cycles (or dual cocycles). In a forthcoming paper we will apply similar techniques, and results from [1] [2] [3] [4] to describe certain types of cross product bialgebras with co-cycles in a co-cyclic (co-)modular way. Analogous statements as in Proposition 2.2 and Theorem 2.10 will be derived for rather general cross product bialgebras with co-cycles. But the co-cyclic (co-)modular scheme of classification turns out to be much more subtle than in the present case. There might be different ways of restricting the general set-up of co-cycle cross product bialgebra to achieve different sorts of classification schemes.

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