Confined-exotic-matter wormholes with no gluing effects – Imaging supermassive type (1) wormholes and black holes

Mustapha Azreg-Aïnou

1Baskent University, Faculty of Engineering, Bağlıkça Campus, 06810 Ankara, Turkey

We classify wormholes endowed with redshift effects and finite mass into three types. Type (1) wormholes have their radial pressure dying out faster, as one moves away from the throat, than any other component of the stress-energy and thus violate the least the local energy conditions. We introduce a novel and generalizable method for deriving, with no cutoff in the stress-energy or gluing, the three types of wormholes. We focus on type (1) wormholes and construct different asymptotically flat solutions with finite, upper- and lower-bounded, mass \( M \). It is observed that the radial pressure is negative, and the null energy condition is violated, only inside a narrow shell, adjacent to the throat, of relative spatial extent \( \epsilon \). Reducing the relative size of the shell, without harming the condition of traversability, yields an inverse square law of \( \epsilon \) versus \( M \) for supermassive wormholes. We show that the diameter of the shadow of this type (1) supermassive wormhole overlaps with that of the black hole candidate at the center of the Milky Way and that the recent derivation, using the up-to-date millimeter-wavelength very long baseline interferometry made in Astrophys. J. 795 (2014) [ArXiv:1409.4690], remains inconclusive.

We show that redshift-free wormholes, with positive energy density, have one of their barotropic equations of state in the phantom regime, have their stress energy tensor traceless, and are anisotropic. They are all type (3) wormholes having their variable equations of state approaching 1 and \(-1\) at spatial infinity. We also introduce a new approach for deriving new redshift-free wormholes.

I. INTRODUCTION

How exotic is exotic matter? Do young and old galaxies harbor exotic matter? So far there has been no simple or advanced theory about exotic matter nor a prediction and all we know about it is its mathematical definition: it violates our perception of energy. That is, if an observer measures some negative local amount of energy density, we say that that corresponds to exotic matter.

The other thing we know about exotic matter is its possible wormhole sustainability [1, 2]. While they are of exotic nature, wormholes may interact with ordinary matter and may be indirectly observed through the effects they have on light and particle paths as well as on fields [3–5], on falling hot objects and spots [8] and so on.

The field equations of classical general relativity do not fix the topology of its solutions nor do they fix the amount of exotic matter need to sustain the throat of a wormhole. Quantum effects allow for violations of the local and averaged null energy condition (NEC) and might be used to support and stabilize wormholes.

Since exotic matter remains still a mystery, workers, using different techniques, have ever strived hard to derive wormhole solutions that minimize its use [1, 2] and [10, 11]. To the best of our knowledge, no classification of wormholes has been performed so far. Observers of events are usually located far away from the sources, say, at spatial infinity, where observable entities may behave differently. The only distinctions among wormholes, which are widely used by workers, are finiteness of the mass, traversability, and stability. Other observable entities that may distinguish between wormholes are the components of the stress energy tensor (SET). It is the duty of this paper to perform this classification based on the relative behavior of the components of the SET at spatial infinity.

Another, but implicit, classification of wormholes concerns redshift-free wormholes and wormholes endowed with it. The radial and transverse pressures of redshift-free wormholes, with positive energy density and finite mass, behave the same way at spatial infinity, so there is no classification added for these solutions. This fact could be announced as a uniqueness theorem. This is no longer the case for wormholes endowed with redshift effects where three types of solutions, having positive energy density and finite mass, emerge.

The classification of wormholes motivates a new mathematical quest for theoretical wormholes fueled by the recent activities [4, 8] to whether the observations of the shadow or hot spots are able to distinguish between a supermassive black hole (SMBH), located at Sagittarius A* (Sgr A*), and a supermassive wormhole (SMWH). Questioning if the SMBH candidate at the center of the Milky Way is a SMWH is right but trying to answer it is hard. In fact, we have noticed that the wormhole solutions used in these investigations are the types that demand the most exotic matter. We will show that it is possible, without using the cut and paste technique, to construct their counterparts which violate the least the NEC and yield a value 46.4024 \( \mu \)as — 48.3931 \( \mu \)as for the diameter of the shadow.

This is the same value derived very recently [13] using the millimeter-wavelength very long baseline interferometry (VLBI). The image of the emission surrounding the SMBH candidate in the center of the Milky Way reveals, at 1.3 mm VLBI, the same features of general relativity including that of a shadow of diameter \( \sim 50 \mu \)as. Knowing that Sgr A* along with M87 are on the list of the main
targets of the Event Horizon Telescope \[20\], the sensitivity of these measurements will increase with the inclusion of Atacama Large Millimeter/Submillimeter Array VLBI-station \[21\].

In Sec. \[11\] we review the field equations and the local energy condition's (LEC's). In Sec. \[11\] we consider redshift-free static wormholes and construct by a new procedure some new exact solutions. Those redshift-free wormholes, with positive energy density, have one of their barotropic equations of state in the phantom regime, have their stress energy tensor traceless, and are anisotropic. In Sec. \[11\] we focus on solutions with variable redshift function and finite mass. We classify them into three types (1), (2), and (3). Type (1) [respectively type (3)] wormholes violate the least [respectively the most] the LEC's. The importance of type (1) and type (3) solutions is that they can be used by distant observers for testing hypotheses and in computer simulations. We introduce a 3-parameter approach to derive, without gluing, all types of wormholes. The approach splits into two directions, in the one of which only one parameter remains free, and in the other one two parameters remain free to confine the exotic matter. We discuss the violations of the LEC's and traversability.

Sec. \[1\] is devoted to an application. First, we show that the wormhole solution that has been used \[3\] for evaluating the shadow of the SMBH candidate is type (3). We use, instead, a type (1) solution and show that the evaluation of the shadow is inconclusive. Said otherwise, the outcome of the observation is such that (a) the candidate might either be a (Schwarzschild or Kerr) SMBH or a type (1) SMWH, (b) the candidate is a type (3) SMWH with relatively large amounts of exotic matter in the center of the galaxy. Based on the recent results of Ref \[19\], this last possibility is ruled out.

In Sec. \[1\] we show how the approach introduced in Sec. \[1\] can be generalized and provide two more wormhole solutions. We conclude in Sec. \[1\].

II. FIELD EQUATIONS AND LEC'S

The metric of a static, spherically symmetric, wormhole is better brought to the form \[1\]

\[
ds^2 = A(r)dt^2 - \frac{dr^2}{1 - b(r)/r} - r^2d\Omega^2,
\]

in Schwarzschild coordinates. The throat is located at \( r = r_0 \). We assume symmetry of the two asymptotically flat regions, which particularly implies that if the mass of the wormhole is finite then it is the same as seen from both spatial infinities. The functions \( A \) and \( b \) are constrained by \[1,2\]

\[
\lim_{r \to \infty} A = \text{finite} = 1,
\]

\[
0 < b < r \text{ if } r > r_0 \text{ and } b(r_0) = r_0,
\]

\[
\lim_{r \to \infty} (b/r) = 0,
\]

\[
r b' < b \text{ (at least near, but not on, the throat)},
\]

\[
b'(r_0) \leq 1.
\]

Notice that \( r b' = b \) may hold on the throat. The value of the limit in the first line \[2\] is set to 1 by rescaling \( A \) and redefining \( t \). These constraints hold even if the mass of the wormhole is not finite. If the latter is finite, we have the further constraint

\[
\lim_{r \to \infty} b \equiv b_\infty = 2GM = 2M.
\]

The SET is usually taken anisotropic of the form \[1,2\],

\[
T^\mu_{\nu} = \text{diag}(\rho(r), -p_r(r), -p_t(r), -p_t(r)), \quad \rho, \quad p_r, \quad p_t
\]

\( \rho \) being the energy density and \( p_r \) and \( p_t \) are the radial and transverse pressures. The filed equations \( G^\mu_\nu = 8\pi T^\mu_\nu \) and the identity \( T^\mu_{\nu,\mu} \equiv 0 \) yield, respectively

\[
4p_t = 4p_r + 2rp'_t + r(p_r + \rho)(\ln A)',
\]

where a prime denotes derivation with respect to \( r \).

The SET is subject to the requirements of the LEC's, known as null, weak (WEC), strong (SEC), and dominant (DEC) conditions. These requirements read, respectively \[2\]

\[
\text{NEC: } \quad \rho + p_r \geq 0, \quad \rho + p_t \geq 0,
\]

\[
\text{WEC: } \quad \rho \geq 0, \quad \rho + p_r \geq 0, \quad \rho + p_t \geq 0,
\]

\[
\text{SEC: } \quad \rho + p_r \geq 0, \quad \rho + p_t \geq 0, \quad \rho + p_r + 2p_t \geq 0,
\]

\[
\text{DEC: } \quad \rho \geq 0, \quad p_r \in [-\rho, \rho], \quad p_t \in [-\rho, \rho].
\]

III. REDSHIFT-FREE STATIC WORMHOLES

If no redshift effects occur (\( A = 1 \)) Eqs. \[1\] take the forms

\[
4p_t = 4p_r + 2rp'_t + r(p_r + \rho)(\ln A)',
\]

Since the the energy density and the pressures depend only on \( r \), we can always assume two barotropic equations of state of the form

\[
p_r(r) = \alpha(r)\rho(r), \quad p_t(r) = \beta(r)\rho(r).
\]

These barotropic assumptions are valid for any static, spherically symmetric, solution be it redshift-free or
other. With a positive energy density \( \rho \), the second line (6) shows that \( p_r \) and \( \alpha \) are negative. Combining the last two lines (8) yields

\[
p_t = \frac{b - rb'}{16\pi r^4},
\]

which is positive by the fourth line (2). Thus, \( \beta \) and the dimensionless anisotropy parameter \( \Delta \equiv (p_t - p_r)/\rho = \beta - \alpha \) are both positive.

Using (7), the first two lines (6) and (8) yield

\[
\begin{align*}
    b(r) &= r_0 \exp \left( -\int_{r_0}^{r} \frac{d\tilde{r}}{\tilde{r}^{\alpha}} \right), \\
    \rho &= \frac{b'}{(8\pi r^2)}, \\
    2\beta &= - (\alpha + 1),
\end{align*}
\]

where we have used the second line (2). We see that in the absence of redshift effects the knowledge of \( \alpha(r) \) suffices to determine all the necessary functions \((b, \rho, p_r, p_t)\). This consists the method of resolution we introduce to construct wormhole solutions with no redshift effects.

Equivalently, one may use \( \beta \), instead of \( \alpha \), to determine all the other functions.

The third line (9) implies that the SET is traceless \( \rho + p_r + 2p_t = 0 \).

The coefficient \( \beta(r) \) being positive, the last line (9) results in

\[
\alpha(r) < -1,
\]

which lies in the phantom regime. Notice that in this regime it is not possible to have \( \beta = -\alpha \), that is, an isotropic solution, since in this case the third line (9) would imply \( \alpha = 1 \), which is not allowed by (10). The conclusion is even stronger than that: For a given solution, there is no sphere of radius \( r \geq r_0 \) where all the components of the pressure, in absolute value, are equal to each other. A similar conclusion concerning gravastars was drawn in \[22\] where it was shown that such objects cannot be perfect fluids.

Using (6) and (10) it is straightforward to show that all the local (null, weak, strong, and dominant) energy conditions are violated (2), for \( \rho > 0 \) implies \( \rho + p_r = (1 + \alpha)\rho < 0 \) and \( p_r \notin [-\rho, +\rho] \); the condition \( p_t \in [-\rho, +\rho] \) is satisfied only if \( \beta \leq 1 \) (\( -3 \leq \alpha < -1 \)). All the other constraints of the energy conditions are satisfied: \( \rho + p_t = (1 + \beta)\rho > 0 \) and \( \rho + p_r + 2p_t = 0 \).

There is a variety of factors \( \alpha(r) \) leading to closed-form expressions for all the functions \((b, \rho, p_r, p_t)\). These can be easily seen from the first line (4) and are investigated in the following two subsections. So, in the remaining part of this section, we fix the expression of \( \alpha(r) \) and use (4) to determine the functions \((b, \rho, p_r, p_t)\) and the metric.

### A. \( \alpha \) is constant

Let \( \nu \equiv -1/\alpha \) yielding \( 0 < \nu < 1 \). The metric and the necessary functions read

\[
\begin{align*}
    ds^2 &= dt^2 - \frac{dr^2}{1 - (r_0/r)^{1-\nu}} - r^2 d\Omega^2, \\
    b &= r_0(r/r_0)\nu, \\
    \rho &= \frac{\nu}{8\pi r_0^2(r/r_0)^{3-\nu}}, \\
    p_r &= -\rho/\nu, \\
    p_t &= \frac{1 - \nu}{2\nu} \rho \quad (0 < \nu < 1).
\end{align*}
\]

It is easy to check that the limits of \((\rho, p_r, p_t)\), as \( r \to \infty \), vanish but that of \( b \) diverges, so the mass is infinite. It is also easy to check that all the constraints (2) are satisfied.

The special case \( \nu = 1/2 \) was discussed in Ref. [1]. This solution has been rederived in [23, 24].

### B. \( \alpha + 1 \propto -(r/r_0)^\mu, \mu > 0 \)

Another closed-form solution is derived taking \( \alpha + 1 \propto -(r/r_0) \). Since \( \alpha < -1 \), we write \( \alpha \) as

\[
\alpha = -1 - k^2 \frac{T}{r_0} \quad \text{and} \quad k^2 > 0.
\]

Direct integration yields

\[
b = \frac{(1 + k^2)r_0r}{k^2 r + r_0}.
\]

Since \( b_v = (1 + k^2)r_0/k^2 \) is finite we introduce the mass parameter defined in (4):

\[
2M = (1 + k^2)r_0/k^2.
\]

This implies \( M > r_0/2 \).

In terms of \((M, r_0)\) we obtain the following solution

\[
\begin{align*}
    ds^2 &= dt^2 - \left( 1 - \frac{2M}{r + 2M - r_0} \right)^{-1} dr^2 - r^2 d\Omega^2, \\
    b &= \frac{2Mr}{r + 2M - r_0}, \\
    \rho &= \frac{M(2M - r_0)}{4\pi r^2(r + 2M - r_0)^2} > 0, \\
    p_r &= -\frac{M}{4\pi r^2(r + 2M - r_0)}, \\
    p_t &= \frac{M}{8\pi r(r + 2M - r_0)^2}.
\end{align*}
\]

The components of the SET vanish at spatial infinity.

The constraints (2) are all satisfied. For instance, the last two lines (2) read, respectively

\[
-\frac{2Mr^2}{(r + 2M - r_0)^2} < 0, \quad -\frac{r_0}{2M} < 0.
\]

The above solution generalizes easily to the case\(^1 \) \( \alpha = -1 - k^2(r_0/r)^\mu \) with \( \mu > 0 \) do not satisfy the third line (2).

\(^1\) Solutions of the form \( \alpha = -1 - k^2(r_0/r)^\mu \) with \( \mu > 0 \) do not satisfy the third line (2).
$1 \propto -(r/r_0)^\mu$ and $\mu > 0$

\[ ds^2 = dt^2 - \left(1 - \frac{2M}{R^{1/\mu}}\right)^{-1} dr^2 - r^2d\Omega^2, \]
\[ b = \frac{2Mr}{R^{1/\mu}}, \quad \rho = \frac{M(2^\mu M^\mu - r_0^\mu)}{4\pi r^2 R^{(\mu+1)/\mu}} > 0, \]
\[ p_r = -\frac{M}{4\pi r^2 R^{1/\mu}}, \quad p_t = \frac{M}{8\pi r^2 - \sigma R^{1/\mu}}, \quad R \equiv r^\mu + 2^\mu M^\mu - r_0^\mu, \quad M > r_0/2, \quad \mu > 0. \tag{15} \]

According to the analysis made in [1], these wormholes are traversable. At spatial infinity the energy density dies out as fast as $r^{-3-\mu}$ and the pressures as $r^{-3}$ where $\mu$ is an arbitrary positive constant. This behavior is general and applies to all redshift-free static wormholes with finite mass parameter. In fact, if $\rho \sim \rho_\infty r^{-3-\mu} (\mu > 0)$ as $r \to \infty$, then the first, second, and third lines in [6] yield, respectively, $b \sim b_\infty - 8\pi \rho_\infty r^{-\mu}/\mu$, $p_r \sim -b_\infty r^{-3}/(8\pi)$, and $p_t \sim -b_\infty r^{-3}/(16\pi)$. The solutions (15), as well as the special case (16), are the simplest ones with these properties. Since the violations of the LEC’s are attributable to $p_r$, which is negative, this dashes any hope for obtaining redshift-free solutions with $p_r$ dying out faster than $r^{-3}$.

Now, we consider the limiting case $M = r_0/2$. We obtain the wormhole solution [1]

\[ ds^2 = dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2d\Omega^2, \]
\[ b = 2M, \quad M = r_0/2, \quad \rho \equiv 0, \]
\[ p_r = -\frac{M}{4\pi r^3}, \quad p_t = \frac{M}{8\pi r^3}, \tag{16} \]

which can also be derived from [6] and [8] taking $b = constant = 2M$ [It is also derived from (15) taking the limit $\mu \to \infty$]. With $\rho \equiv 0$ and $p_r < 0$, this represents the most exotic matter distribution. This is not a Schwarzschild wormhole since the latter has $\rho = p_r = p_t \equiv 0$.

Had we assumed the fluid isotropic, such a solution would not exist even in the full regime where $A$ is not constant. In fact, a solution which behaves at spatial infinity as $\rho \sim r^{-3-\sigma}$ and $p_r \sim p_t \sim r^{-3}$ yields a nonasymptotically flat solution.

**IV. STATIC WORMHOLES WITH REDSHIFT EFFECTS**

If the redshift effects are present ($A' \neq 0$), wormholes with finite positive mass and $p_t$, dying out faster than any other component of the SET may exist. In this case, however, the radial gravitational tidal forces, which vanish if $A' = 0$ [1], constrain, and may prevent, traversibility of the wormhole.

Asymptotic treatment of (1) reveals the following results. Wormhole solutions, with finite positive mass, that are of the form

$\rho \sim \rho_\infty r^{-3-\sigma}$ and $p_r \sim p_r \sim r^{-3-\eta}$ (as $r \to \infty$) \tag{17}

Notice that, in the solutions of type (1), $p_r$ vanishes asymptotically faster than the other components of the SET; these are the best solutions minimizing the use of exotic matter. In the solutions of type (2), the pressures have the same asymptotic behavior and vanish faster than the energy density. To the best of our knowledge, no solutions of type (1) and (2) are available in the literature. A solution of type (3), with $\eta = \sigma = 1$ and thus $p_{t,\infty} = -p_{r,\infty}$, was derived in Eqs. (35) to (40) of Ref. [24].

The aim of this section is to derive closed-form wormhole solutions of type (1) and (2). The barotropic equations (17) no longer are suitable as ansatzes, so we will introduce a new systematic approach.

FIG. 1: A surface plot of $r_0^2p_t$ against $(x/r)/r_0$ for the case $n = 6$ [Eq. (19)]. Here $x \equiv 8\pi r_0^2p_0 = (2M - r_0)/r_0$ and $y \equiv r/r_0$. where $(\sigma, \eta)$ are assumed to be positive numbers, may exist if

**type (1):** $\eta \sim \sigma > 1$ ($\Rightarrow r^{-4-\sigma} > r^{-3-\eta}$) yielding

\[ 4p_t \sim b_\infty \rho_\infty r^{-4-\sigma} \Rightarrow 4p_{t,\infty} = b_\infty \rho_\infty; \tag{18} \]

**type (2):** $\eta \sim \sigma = 1$ yielding

\[ 4p_t \sim b_\infty \rho_\infty - 2(1 + \eta) p_{r,\infty} r^{-3-\eta} \Rightarrow 4p_{t,\infty} = b_\infty \rho_\infty - 2(1 + \eta) p_{r,\infty}; \tag{19} \]

**type (3):** $\eta \leq \sigma$ yielding

\[ 2p_t \sim -(1 + \eta)p_{r,\infty} r^{-3-\eta} \Rightarrow 2p_{t,\infty} = -(1 + \eta)p_{r,\infty}. \tag{20} \]

In all three cases, $A$ and $b$ behave asymptotically as

$$ A \sim 1 - \frac{b_\infty}{r}, \quad b \sim b_\infty - \frac{8\pi \rho_\infty}{\sigma r^\sigma}. \tag{21} $$

Notice that, in the solutions of type (1), $p_r$ vanishes asymptotically faster than the other components of the SET; these are the best solutions minimizing the use of exotic matter. In the solutions of type (2), the pressures have the same asymptotic behavior and vanish faster than the energy density. To the best of our knowledge, no solutions of type (1) and (2) are available in the literature. A solution of type (3), with $\eta = \sigma = 1$ and thus $p_{r,\infty} = -p_{t,\infty}$, was derived in Eqs. (35) to (40) of Ref. [24].

The aim of this section is to derive closed-form wormhole solutions of type (1) and (2). The barotropic equations (17) no longer are suitable as ansatzes, so we will introduce a new systematic approach.
Assume we look for a wormhole solution of type (1) of the form \( \rho = \rho_0 r^m / r^m \) \((m = 3 + \sigma)\) and \( \rho_\infty = \rho_0 r^m \).

We start with the case

\[ \rho = \frac{\rho_0 r^4}{r^4} = \frac{\rho_\infty}{r^4} \quad (\sigma = 1). \tag{22} \]

The first line yields

\[ b = (1 + x)r_0 - \frac{x r_0^2}{r} \quad \text{with} \quad x \equiv 8 \pi r_0^2 \rho_0 > 0, \tag{23} \]

from which we obtain

\[ b_\infty = (1 + x)r_0 = 2M, \tag{24} \]

and then

\[ b = 2M - \frac{(2M - r_0)r_0}{r} \quad \text{with} \quad x = \frac{2M - r_0}{r_0}. \tag{25} \]

In order to satisfy all the constraints on \( b \) we must have

\[ x \leq 1, \tag{26} \]

or equivalently,

\[ \frac{r_0}{2} < M \leq r_0. \tag{27} \]

The next step is to choose a form for \( p_r \) yielding a type (1) solution and determine \( A \). Our approach consists in taking \( p_r \) as a two-term polynomial in \( 1/r \) of the form

\[ p_r = \frac{c_n}{r^n} + \frac{c_{n+1}}{r^{n+1}} \quad (n = 3 + \eta > 3), \tag{28} \]

yielding

\[ \frac{8 \pi r^3 p_r + b}{r(r - b)} = \frac{8 \pi (c_{n+1} + c_n r + r_0[(1 + x)r - xr_0]r^{n-3}}{(r - r_0)(r - xr_0)r^{n-2}} \]

\[ = \frac{N(r)}{(r - r_0)(r - xr_0)r^{n-2}}. \tag{29} \]

In order to not have a horizon at \( r_0 \) we set \( N(r_0) = 0 \), where \( N(r) \) is the numerator of the r.h.s. of (29).

\[ 8 \pi (c_{n+1} + c_n r) + r_0^n - 1 = 0. \tag{30} \]

The remaining equation could be integrated and leads to no horizon at \( r_0 \) if \((0 < x < 1)\). In this case, one of the constants \((c_n, c_{n+1})\) remains undetermined and the case \( x = 1 \) would not yield a wormhole solution.

There are two possible directions which we shall follow: Case (1), treated in Sec. IV A one may add another constraint to fix both constants \((c_n, c_{n+1})\). To ease the calculations and obtain a simple closed-form metric and SET, it would be better to set \( N(xr_0) = 0 \), which would allow for an equal treatment of the cases \( x < 1 \) and \( x = 1 \) and yields a polynomial in \( 1/r \) on the r.h.s. of (29) if \( n \) is an integer. For \( x = 1 \), the constraint \( N(xr_0) = 0 \) is the same as \( N_0(r_0) = 0 \) allowing \( N(r) \) to have a double root at \( r = r_0 \), as is the denominator of the r.h.s. of (29). Case (2), treated in Sec. IV B one adds no further constraint. We will use \( c_n \) as a free parameter and the wormhole solution will be valid only for \((0 < x < 1)\).

### A. Case (1): A further constraint \( 0 < x \leq 1 \)

The constraint \( N(xr_0) = 0 \) reads

\[ 8 \pi (c_{n+1} + c_n x r_0) + x^n - 1 r_0^{n-1} = 0. \tag{31} \]

Equations (29) and (31) are linear in \((c_n, c_{n+1})\), so one can always solve them analytically. They are identical for \( x = 1 \). We solve them for \( x < 1 \)

\[ c_n = -\frac{1 - x^{n-1}}{8 \pi (1 - x)} r_0^{n-2}, \quad c_{n+1} = \frac{x(1 - x^{n-2})}{8 \pi (1 - x)} r_0^{n-1}, \tag{32} \]

and we analytically extend them to \( x = 1 \) since the limit, as \( x \to 1 \), in each r.h.s. of (32) exists. If \( n \) is an integer\(^2\), the extension is done on introducing the partial sums

\[ S_k(x) \equiv \sum_{i=0}^{k} x^i = \begin{cases} 1 - x^{k+1}/k+1, & \text{if } x < 1, \\ 1, & \text{if } x = 1, \end{cases} \tag{33} \]

\((S_0(x) \equiv 1)\) and the extended expressions of \((c_n, c_{n+1})\) read

\[ c_n = \frac{S_n(x) r_0^{n-2}}{8 \pi}, \quad c_{n+1} = \frac{xS_{n-1}(x) r_0^{n-1}}{8 \pi} \tag{34} \]

For \( x = 1 \), these expressions coincide with those we would obtain on solving (30), \( N(r_0) = 0 \), along with \( N'(r_0) = 0 \).

\(^2\) If \( n \) is not integer, the extension is still possible on introducing the two variable function \( S(x, \ell) = (1 - x^{\ell+1})/(1 - x) \) if \( x < 1 \) and \( S(1, \ell) = \ell + 1 \) where \( \ell \) is some positive real number. This also leads to integrable expressions, but sometimes sizeable, for \( A \) and the components of the SET. So, we will not consider this extension here.
From now on we will omit to write the argument of $S_t$ unless there is a confusion. We can now write explicitly the expression of $p_r$

$$p_r = -\frac{S_{n-2}r_0^{n-2}}{8\pi r^n} + \frac{xS_{n-3}r_0^{n-1}}{8\pi r^{n+1}}$$

$$= \frac{[r(r_0) - 1]S_{n-2} + 1}{8\pi r_0^2[r(r_0)]^{n+1}} < 0, \quad (35)$$

where we have used $xS_{n-3} = S_{n-2} - 1$. This is manifestly negative for all $r \geq r_0$. With this expression of $p_r$ the factors $r - r_0$ and $r - xr_0$ in the r.h.s. of (34) cancel out and the remaining expression, which is equal to $A'/A$ by the second line (4), reduces to a polynomial in $1/r$ given by

$$A' = \sum_{i=1}^{n-3} \frac{S_i r_0^i}{r^{i+1}} + 1 = 2M \quad (36)$$

Using the first constraint in (2), we are led to

$$A = \exp( -\sum_{i=1}^{n-3} \frac{S_i r_0^i}{r^{i+1}} ) \quad (37)$$

In the limit $r \to \infty$, we obtain

$$A \sim 1 - \frac{S_1 r_0}{r} = 1 - \frac{2M}{r} \quad (38)$$

where we have used (22).

Introducing the dimensionless variable $y = r/r_0$ [already used in (33)] and re-expressing (22) as $\rho = x/(8\pi r_0^2 y^4)$, Eq. 4 yields

$$p_t = \frac{2[(n-2) y + 1 - n]S_{n-2} + 2(n-1)}{32\pi r_0^2 y^{n+1}}$$

$$+ \frac{[xy^{n-3} - (y - 1)S_{n-2} - 1(\sum_{i=1}^{n-3} \frac{S_i}{y^i})]}{32\pi r_0^2 y^{n+1}} \quad (39)$$

This is a polynomial in $1/r$ the highest power of which is $xS_{n-2}^2/(32\pi r_0^2 y^{2(n-1)})$ and its lowest power depends on $n$. The solution is of type (3) if $n = 4 (\eta = 1)$, of type (2) if $n = 5 (\eta = 2)$, and of type (1) if $n \geq 6 (\eta \geq 3)$. Given $b_\infty = S_1 r_0$, $\rho_\infty = x r_0^2/(8\pi)$, and $p_\infty = -S_{n-2} r_0^{-n-2} / (8\pi)$, it is straightforward to check Eqs. [15], [16], and [20]. For instance, for $n = 5$ we find

$$p_{\infty} = (x S_1 + 6 S_4) r_0^3 / (32\pi),$$

which is the coefficient of $1/\pi^3 = 1/\pi^{4+n}$ [the lowest power in (35)], this is conform with (19). For $n \geq 6$ we find

$$p_{\infty} = x S_1 r_0^3 / (32\pi),$$

which is the coefficient of $1/\pi^3 = 1/\pi^{4+n}$ [the lowest power in (35)], this is conform with (18).

In the limit $n \to \infty$, the graph of $p_r$ approaches that of the semi-step function $\Theta(r)$ defined by

$$\Theta(r) = \begin{cases} -1/(8\pi r_0^2), & \text{if } r = r_0, \\ 0, & \text{if } r > r_0. \end{cases} \quad (40)$$

The solution derived in this section has the property that the scaled functions $\{b/r_0, r_0^{-2}p, r_0^{-2}p_t, r_0^{-2}p_t \}$ and $A$ do depend only on $(x, y)$. Using this property, it is possible to show that $r_0^2 p_t$ may undulate for fixed $0 < x \leq 1$ and $y \geq 1$, as depicted in Fig. 1 and Fig. 2. So, it is not possible to prove analytically the positiveness of $p_t$ because of the existence of local extreme values the critical points of which depend on $n$. On the throat, $p_t$ is positive and vanishes only in the special case $x = 1$. This is obvious from its value at the point $(x, y) = (1, 1)$

$$p_t(x, y) = \frac{n - x - S_{n-2}(x)}{32\pi r_0^2} \quad (41)$$

which is 0 if $x = 1$ knowing that $S_{n-2}(1) = n - 1$. For $x < 1$, $p_t(x, 1) > 0$ since $S_{n-2}(x) < n - 1$. This is confirmed graphically for the cases $n = 6$ and $n = 10$, as depicted in Fig. 1 and Fig. 2 where $r_0^2 p_t$ read, respectively

$$r_0^2 p_t = x S_1 + \frac{S_2 + 8 S_4}{32\pi y^2} - \frac{9 x S_3 + S_1 S_4}{32\pi y^2} - \frac{S_6}{32\pi y^4} + \frac{16 S_8 + x S_6}{32\pi y^10} \quad (42)$$

$$r_0^2 p_t = \frac{x S_1 + \sum_{i=1}^{5} S_i y^{i-1}}{32\pi y^{11}} - \frac{1}{32\pi y^{12}} \sum_{i=1}^{6} S_i y^{i-1} + \frac{x S_2 y^2}{32\pi y^{18}} \quad (43)$$

where we have used $x S_1 S_3 - S_2 S_4 = -S_6$ and $x S_2 - S_4 = -(1 + x^2)$ in (42) and $x S_1 S_3 - S_2 S_4 = -S_6$ in (43).

Fig. 1 and Fig. 2 show only a portion of the $y$ axis where $p_t \geq 0$, however, we have numerically checked that the equation $p_t(\bar{r}) = 0$, where $p_t$ is given by (42) ($n = 6$) or by (43) ($n = 10$), has no root $\bar{r} > r_0$ for $0 < x \leq 1$.

Fig. 3 shows clearly how the violation of the NEC is narrowed as $n$ increases. The solution we provide in the Case (2) will do better, in that, for the same value of $n$ the region of violation of the NEC gets narrower.

B. Case (2): No further constraints ($0 < x < 1$)

The only constraint one needs to solve is (40) yielding

$$c_{n+1} = -r_0 c_n - \frac{r_0^{n-1}}{8\pi}. \quad (44)$$

Introducing the dimensionless constant $c$ defined by

$$c_n = r_0^{-2+n} c, \quad (45)$$

we obtain

$$p_r = \frac{8\pi r_0^{-n-2}(r - r_0) - r_0^{-n-1}}{8\pi r_0^{n+1}} = \frac{8\pi c(y - 1) - 1}{8\pi r_0^{n+1}} \quad (46)$$

With this expression of $p_r$, $A$ has no horizon at $r = r_0$ for all $c_n$ at the expense of having $p_t$ negative in the vicinity of the throat: $p_t(r_0) = -1/(8\pi r_0^2)$. The only
constraints we may impose on \( c_n \) is to ensure positivity of \( p_t \). It may seem possible to generate wormhole solutions where \( p_t \) is negative only in the vicinity of the throat. In fact, from the asymptotic behavior \([18]\) of type (1) wormholes, we see that \( p_{t\infty} \) is positive without constraining \( c_n = p_{t\infty} \). Thus, for type (1) wormholes, \( c_n \) may a priori assume positive values, depending on \( x \), provided \( p_t \) remains positive everywhere. When this is the case—it is the case indeed as we shall see below—\( p_t \) vanishes at some point \( r_1 \) then becomes positive. At \( r_1 \), \( p_t \) is certainly positive by \([4]\). From the end-behavior \([19]\) of type (2) wormholes, we see that \( c_n \) may also assume positive values constrained by \( c_n < x(1 + x) r_0^3/[16\pi(n - 2)] \), where we have used \( b_{\infty} = S_1 r_0 \), \( \rho_{\infty} = x r_0^2/(8\pi) \), and \( n = n_3 + 3 \).

In this Case (2), however, it is not possible to derive general solutions valid for all \( n > 3 \) \([23]\), so we provide an example of resolution for \( n = 6 \) which will yield a type (1) solution:

\[
p_t = \frac{8\pi r_0^4 c (r - r_0) - r_0^5}{8\pi r_0^5} = \frac{8\pi c (y - 1) - 1}{8\pi r_0^2 r_0^2 y^2}. \tag{47}
\]

In this case also the graph of \( p_t \) approaches that of the semi-step function \([10]\) in the limit \( n \to \infty \). With this expression of \( p_t \), \([17]\) the factor \( r - r_0 \) in the r.h.s. of \([29]\) cancels out and the remaining expression, which is equal to \( A'/A \) by the second line \([4]\), reduces to

\[
\frac{A'}{A} = \frac{8\pi r_0^4 c + r_0[(1 + x) r_0^3 + r_0^2 r^2 + r_0^2 r + r_0^3]}{r^4(r - x r_0)}. \tag{48}
\]

Performing the elementary integrals, we arrive at

\[
A = \left(1 - \frac{x r_0}{r}\right)^a \exp \left(\sum_{i=1}^{3} \frac{(8\pi c + S_{3-i}) r_0^i}{r^{4-i} r^i}\right),
\]

\[
a(x, r_0, c) = \frac{8\pi c + S_1}{x^2} \quad (0 < x < 1), \tag{49}
\]

\[
r_0^3 p_t = \frac{1}{y - x} \left[ \frac{x(1 + x)}{32\pi y^4} + \frac{x - 64\pi c}{32\pi y^3} + \frac{10 + x + 8\pi c(11 + 9x)}{32\pi y^2} - \frac{1}{32\pi y^2} - \frac{16\pi c^2}{32\pi y^3} - \frac{(1 + 8\pi c)^2}{32\pi y^4} \right]. \tag{50}
\]

The other functions \( (\rho, b) \) keep their expressions as given in \([22\text{ and }29]\). In the limit \( r \to \infty \), \((A, p_t)\) behave as in \([88\text{ and }18]\), respectively.

Table \( I \) provides, in terms of \( x \), the limiting values \( c_{\text{lim}}(x) \) of \( c_t \) at, or below, which \( p_t \) is positive for all \( r \geq r_0 \).

Now, for \( c \leq c_{\text{lim}}(x) \), the transverse pressure being positive, the radial one \( p_t \) is negative only near the throat, vanishes at

\[
r_1 = r_0 + \frac{r_0}{8\pi c}. \tag{51}
\]

then remains positive for \( r > r_1 \). \( r_1 = 1.4r_0 \) for the largest value of \( c_{\text{lim}} = 0.098 \) given in Table \( I \) corresponding to \( x = 0.9 \), and \( r_1 = 4.3r_0 \) for the smallest one. This shows that the field equations \([1]\) admit a simple, with no gluing process, wormhole solution satisfying all requirements \([2]\) where the exotic matter can be made confined in a region around the throat not exceeding 1.4 times the radius of the latter.

Moreover, as Fig. \( 4 \) depicts, the requirement \( \rho + p_r \geq 0 \) imposed by the NEC, WEC, and SEC and the requirement \( p_r \in [-\rho, 0] \) imposed by the DEC \([9]\), are violated only within a shell of outer and inner radii \( 1.019r_0 \) and \( r_0 \), respectively, and all the other requirements imposed by the LEC’s are satisfied by the wormhole solution corresponding to \( x = 0.9 \) and \( c = 0.098 \). We have thus reached the conclusion that violations of the LEC’s occur partly in a narrow spherical shell (of relative extent \( \epsilon = 0.019 \)) around the throat that might not be cumbersome for an extended object crossing the throat.

This relative extent of 0.019 can be improved, in that, reduced to much lower values on increasing \( n \) and, most likely, \( x \). In fact, for a generic value of \( n \), we obtain
FIG. 4: Plots, for \( x = 0.9 \) and \( c = 0.098, \) (a) \( r_0^2 p_r, \) (b) \( r_0^2(\rho + p_r), \) and (c) \( r_0^2(\rho + p_r + p_t) \) versus \( y = r/r_0 \) where \( \rho, p_r, \) and \( p_t \) are given by (22), (47), and (50), respectively. We have numerically checked that the equations \( p_t = 0 \) and \( \rho + p_r + p_t = 0 \) have no roots larger than, or equal to, \( r_0 \). The equation \( \rho + p_r = 0 \) has a single root larger than \( r_0 \) given by \( r = 1.019r_0. \)

The saturation in (50) results in \( r_0 \approx 5.7 \times 10^5 \) km, which is a bit smaller than the radius of the Sun \( R_\odot = 695,800 \) km. Roughly speaking, if all other parameters are held constant, \( n \) increases linearly with \( r_0^2 \) without modifying the value of the l.h.s. of (59). To design a wormhole, say of two times the Sun’s radius \( r_0 = 2R_\odot, \) without violating the traversability condition (59) and with maximum confinement of the negative radial pressure around the throat we need to take \( n \approx 24. \) The condition (59) remains, however, satisfied for \( n \lesssim 24. \)

Here we have reached the same conclusion drawn in [23], in that, the geometry of the wormhole has very different length scales if the exotic matter is to be confined within a narrow shell adjacent to the throat. For the sake of example, compare the inverses of the relative rates of \( b \) (23) and \( A \) (54) on the throat to find

\[
\frac{b}{A} \sim r_0, \quad \frac{A}{A'} \sim \frac{r_0}{n-2}
\]

This results in a discrepancy in the two scales if \( n \) is large, which is the value ensuring maximum confinement. This discrepancy is obvious from Fig. (b), and Fig. 3 where the graph intersects the vertical axis at the same point \(-(1-x)/(8\pi)\) independently of \( n. \) Hence, increasing \( n \) will shift to the left the point of intersection with the \( r/r_0 \) axis and thus reduces the scale of variation of \( \rho + p_r. \)

V. TYPE (1) WORMHOLES FOR TESTING THE NATURE OF THE SMBH CANDIDATES

We have seen that for large \( n, \) \( \epsilon \) varies as \( 1/n \) to confine the violation of the NEC, and \( r_0^2 \sim M^2 \) (27) vary as \( n \) to not violate the traversability condition. For SMWH this yields an inverse square law of \( \epsilon \) versus \( M. \)

SMWH: \( \epsilon \propto 1/M^2 \)

For such large values of \( r_0 \) and \( M \) the geometry of the SMWH, where \( \rho \to \text{const.}, \) \( p_r \to 0, \) and \( p_t \to 0, \) approaches that of a SMBH, but the topology remains different. This has raised the question whether such two suppermassive objects (SMWH and SMBH) can be distinguished through astrophysical observations [4, 5].
An instance of such a SMBH is the one located at Sgr $A^*$. The calculation of the photon trajectories yields the determination of the shape of the shadow of the emitting central object. For a static solution, this amounts to find the photon spheres which are unstable circular paths separating the absorbed paths (captured photons) and scattering ones. The apparent dividing line between black hole and sky is the apparent position of the photon sphere, which is the limiting value of the impact parameter $b_{\text{lim}}$ of the absorbed paths. It can be shown that $b_{\text{lim}}$ is related to the radius of the photon sphere $r_{\text{ps}}$ by (see, for instance, [6])

$$b_{\text{lim}} = r_{\text{ps}}/\sqrt{A(r_{\text{ps}})}, \quad (\ln A)' = 2/r_{\text{ps}}. \quad (59)$$

For Schwarzschild black hole, $r_{\text{ps}} = 3M$ yielding

$$b_{\text{lim}}/M = 3\sqrt{3} \simeq 5.196. \quad (60)$$

For wormholes one usually takes $A = \exp(-2r_0/r)$ yielding

$$b_{\text{lim}}/r_0 = e \simeq 2.718. \quad (61)$$

For massive wormholes, with finite mass parameter $M$, $b \sim 2M - k_1 r^{-\sigma}$ yielding, using the first line [4], $\rho \sim k_2 r^{-3-\sigma}$ [17], where $(\sigma, k_1, k_2)$ are positive constants. Hence, if (1) $r_0 \neq M$, $|p_r| \propto r^{-3} > \rho$ as $r \to \infty$ [type (3)], if (2) $r_0 = M$ and $0 < \sigma \leq 1$, $|p_r| \propto r^{-3-\sigma} \sim \rho$ as $r \to \infty$ [type (3)], and if (3) $r_0 = M$ and $\sigma > 1$, $|p_r| \propto r^{-4} > \rho$ as $r \to \infty$ [type (3)].

At spatial infinity (here the Earth’s surface), where observations are performed in the absence of exotic matter, the wormhole solution selected to represent the SMWH, thought to inhabit the center of the Milky Way near Sgr $A^*$, should be type (1), which minimizes the use of exotic matter. In the previous section, we have developed enough tools to generate this class of massive solutions. We set $x = 1 \Rightarrow S_i = i + 1$ and select the solution given by [36] and [37], which we rewrite as

$$A' = \sum_{i=1}^{n-3} \frac{i+1}{ry^i} \Rightarrow A = \exp \left( -\sum_{i=1}^{n-3} \frac{i+1}{iy^i} \right), \quad (62)$$

so that the second Eq. [60] reads

$$\sum_{i=1}^{n-3} \frac{i+1}{y^i} = 2. \quad (63)$$

The sum in (63) evaluates to $\partial_z [(z^2 - z^{n-1})/(1 - z)]$, with $z = 1/y$, finally reducing to

$$y^{n-3}(2y^2 - 6y + 3) + (n - 1)y + 2 - n = 0. \quad (64)$$

Solving numerically either (63) or (64) for different values of $n$, we find

$$n = 6: \quad r_{\text{ps}} = 2.13865 r_0, \quad b_{\text{lim}}/r_0 = 4.30548,$$
$$n = 10: \quad r_{\text{ps}} = 2.35652 r_0, \quad b_{\text{lim}}/r_0 = 4.48567,$$
$$n = 14: \quad r_{\text{ps}} = 2.36561 r_0, \quad b_{\text{lim}}/r_0 = 4.49003, \quad (65)$$
$$n \to \infty: \quad r_{\text{ps}} = 2.36603 r_0, \quad b_{\text{lim}}/r_0 = 4.49017.$$

In the limit $n \to \infty$, the graph of $p_r$ approaches that of the semi-step function [40].

The values of $b_{\text{lim}}/r_0$ given in (65), which have been derived using type (1) wormholes ($n \geq 6$) are much closer to the black hole value [60] than the value of $b_{\text{lim}}/r_0 = e \simeq 2.718$ derived with a type (3) wormhole. Since the ratio of the apparent diameters of the shadows is equal to the ratio of the $b_{\text{lim}}$’s, with $M = r_0$ ($x = 1$), we have $\theta_S/\theta_W = 5.196/4.49017 = 1.15719$ for the lowest ratio and $5.196/4.30548 = 1.20683$ for the highest one, where $\theta_S$ and $\theta_W$ are the diameters of the Schwarzschild black hole and the wormhole, respectively. Now, $\theta_S = 56\mu\text{s}$, we obtain

$$\theta_W = 46.4024\mu\text{s} - 48.3931\mu\text{s}. \quad (66)$$

Including the 14% absolute uncertainty on $\theta_S$ [4], which is $8\mu\text{s}$, we see that $\theta_S$ and $\theta_W$ overlap. The value $\theta_W$ also overlaps with the corresponding values of the Kerr solution as derived in [28]. We have thus reached the conclusion that the observation of the shadow is inconclusive, in that, the distinction between a (Schwarzschild or Kerr) black hole and a wormhole, as harbored candidates at Sgr $A^*$, is not possible within today’s limits of the VLBI facilities, very recently the director team of which has reported a value of the diameter $\sim 50\mu\text{s}$ [19].

However, if the observed value of the diameter were much lower than 46 $\mu\text{s}$, say 30 $\mu\text{s}$, this would be an indication that the Sgr $A^*$ might harbor a type (3) SMWH as well as large amounts of exotic matter. If that were the case, the difference in the diameters could be used as a measure of the amount of exotic matter harbored at Sgr $A^*$.

Since the external geometric properties of SMWH and SMBH are similar, this leaves open the question whether a SMWH may evolve to a SMBH.

VI. GENERALIZATION

There are two possible directions to generalize the method introduced in [4]. One consists in generalizing

---

3 For a discussion using the Weierstrass elliptic functions see [34].

4 For $n = 5$ or type (2) wormholes we obtain $r_{\text{ps}} = 1.82288 r_0$, $b_{\text{lim}}/r_0 = 3.95392$. For $n = 4$ or type (3) wormholes we obtain $r_{\text{ps}} = r_0$, $b_{\text{lim}}/r_0 = e \simeq 2.718 (n = 4)$, which is already known.
the expression (28) of \( p_r \) to

\[
p_r = \frac{c_n}{r^n} + \frac{c_{n+1}}{r^{n+1}} + \frac{c_{n+2}}{r^{n+2}} \quad (n = \eta + 3 > 3),
\]

(67)

which after imposing the constraint \( N(r_0) \equiv 0 \), generalizing (31), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.

The second possibility amounts to consider higher values of \( m \) (or \( \sigma \)) (30), yields a solution with two free parameters \( c_n, c_{n+1} \) to confine the exotic matter. We will not pursue this program here.
