Counting and Enumerating Independent Sets with Applications to Knapsack Problems*

Frank Gurski¹ and Carolin Rehs¹

¹University of Düsseldorf, Institute of Computer Science, Algorithmics for Hard Problems Group, 40225 Düsseldorf, Germany. {frank.gurski, carolin.rehs}@hhu.de

Abstract

We introduce methods to count and enumerate all maximal independent, all maximum independent sets, and all independent sets in threshold graphs and k-threshold graphs. Within threshold graphs and k-threshold graphs independent sets correspond to feasible solutions in related knapsack instances. We give several characterizations for knapsack instances and multidimensional knapsack instances which allow an equivalent graph. This allows us to solve special knapsack instances as well as special multidimensional knapsack instances for fixed number of dimensions in polynomial time. We also conclude lower bounds on the number of necessary bins within several bin packing problems.

Keywords: knapsack problem; multidimensional knapsack problem; threshold graphs

1 Introduction

The knapsack problem is one of the most famous NP-hard tasks in combinatorial optimization. Within the knapsack problem (KP) there is given a set $A$ of $n$ items. Every item $j$ has a profit $p_j$ and a size $s_j$. Further there is a capacity $c$ of the knapsack. The task is to choose a subset $A'$ of $A$ such that the total profit of $A'$ is maximized and the total size of $A'$ is at most $c$. The d-dimensional knapsack problem (d-KP) generalizes the knapsack problem by using items of sizes within a number $d$ of dimensions.

Different techniques for solving hard problems were successfully applied on the knapsack problem. Among these are pseudo-polynomial algorithms, approximation algorithms, and integer programming, see [Fre04] for a survey. These results lead to several parameterized algorithms [AGRY16]. In this paper we show how to solve special knapsack instances as well as special multidimensional knapsack instances for fixed number of dimensions in polynomial time. A related approach to solve special instances for d-KP was suggested by Chvátal and Hammer in [CH77]. They consider d-KP instances allowing only zero-one sizes given by some matrix $A$ such that some graph model $G(A)$ is a threshold graph. Every such instance was solved in time $O(d \cdot n^2)$ by using a split partition of graph $G(A)$.

This paper is organized as follows. In Section 2 we give preliminaries on graphs and special vertex sets in graphs. In Section 3 we give a graph theoretic approach to solve special knapsack instances. For this purpose we apply threshold graphs, which have the useful property, that they are equivalent to some knapsack instance, i.e. their independent sets correspond to feasible solutions in some knapsack instance. We give characterizations for knapsack instances which allow an equivalent threshold graph. Further we give methods in order to count and enumerate all maximal independent sets in threshold graphs. These approaches improve the result of [ZVI04]. This allows us to solve every KP instance on $n$ items which has an equivalent graph in time $O(n^2)$. Furthermore we show how to count and enumerate all maximum independent sets and independent sets in threshold graphs. We also give lower bounds on the solutions for the bin packing problem using equivalent graphs. The idea comes from [CLR04] but the authors did not characterize which instances of the bin packing problem can be treated in this way. In Section 4 we consider k-threshold graphs to generalize our results for d-KP. We characterize d-KP instances which allow

* A short version of this paper has been presented at the International Conference on Operations Research (OR 2017) [GR17].
an equivalent $k$-threshold graph. By combining the maximal independent sets for $k$ covering threshold graphs we give a method to count and enumerate all maximal independent sets in a $k$-threshold graph in time $O(n^{2k+1})$. Our method implies new bounds on the number of maximal independent sets in $k$-threshold graphs using the clique number of $k$ covering threshold graphs. For $k = 2$ we can improve our results by considering split partitions for the two covering threshold graphs and obtain all maximal independent sets in a 2-threshold graph in time $O(n^3)$. This allows us to solve every d-KP instance on $n$ items in which every dimension has an equivalent graph in time $O(n^{2d+1})$. This result generalizes the above mentioned method by Chvátal and Hammer in [CH77] for solving d-KP instances since we consider instances using non-negative real valued sizes and capacities as well as $k$-threshold graphs as graph models. Since every maximum independent set is also a maximal independent set our results improve existing solutions for the maximum independent set problem on $k$-threshold graphs from [CLR04] if we can bound the vertex degree of the threshold graphs. In the final Section 5 we survey our results on counting and enumerating special independent sets within threshold graphs and $k$-threshold graphs (cf. Table I). Further we give some conclusions on counting and enumerating special cliques in graphs of bounded threshold intersection dimension.

2 Preliminaries

We work with finite undirected graphs $G = (V, E)$, where $V$ is a finite set of vertices and $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ is a finite set of edges. For $v \in V$ and $S \subseteq V$ we define the neighbourhood of $v$ in $S$ by $N(v, S) = \{u \in S \mid \{u, v\} \in E\}$ and the non-neighbourhood of $v$ in $S$ by $N(v, S) = S - (N(v, S) \cup \{v\})$. The (edge) complement graph $\overline{G}$ of graph $G$ has the same vertex set as $G$ and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in $G$, i.e. $\overline{G} = (V, \{\{u, v\} \mid u, v \in V, u \neq v, \{u, v\} \not\in E\})$.

An independent set of $G$ is a subset $V'$ of $V$ such that there is no edge in $G$ between two vertices from $V'$. A maximum independent set is an independent set of largest size. A maximal independent set is an independent set that is not a proper subset of any other independent set. Note that a maximum independent set is maximal but in general the converse is not true. While computing a maximum independent set is a hard problem, a maximal independent set can always be found in polynomial time. The family of all independent sets (maximum independent sets and maximal independent sets, respectively) of some graph $G$ is denoted by $I(G)$ ($\text{IM}(G)$ and $\text{MIS}(G)$, respectively). The cardinalities of these families are denoted by $i(G)$ ($\text{im}(G)$ and $\text{mis}(G)$, respectively). These notations imply the following relations for every graph $G$ it holds

$$\text{IM}(G) \subseteq \text{MIS}(G) \subseteq I(G) \quad \text{and} \quad \text{im}(G) \leq \text{mis}(G) \leq i(G) \quad (1)$$

Enumerating and counting maximal independent sets in graphs are often studied problems in the field of special graph classes. In general every graph on $n$ vertices has at most $3^{n/3}$ maximal independent sets [MM65] and each of these sets can be generated in time $O(n^2)$ [LU50]. Thus all maximal independent sets can be found in time $O^*(3^{n/3}) = O^*(1.4423^n)$. This number can be achieved as the example of $\frac{3^n}{2}$ triangles shows. For $k$-threshold graphs we will show how to compute efficiently better bounds on the number of maximal independent sets.

For several natural counting problems it is known that they are #$P$-complete [Val79]. Even for special graph classes counting the number of maximal independent sets is #$P$-complete, such as for regular graphs and planar bipartite graphs of maximum degree 5 [Vad01], and for chordal graphs [OUU08]. For subclasses of chordal graphs, namely threshold graphs and split graphs we will show linear time solutions for counting the number of maximal independent sets.

Enumeration problems are often easier than counting problems when considering the running time w.r.t. the size of the output. There are special graphs for which enumerating all maximal independent sets can be done in polynomial time or even constant time w.r.t. the size and the number of independent sets of the input graph such as interval, circular arc, and chordal graphs [Leu84].

The family of all cliques (maximum cliques and maximal cliques, respectively) of some graph $G$ is denoted by $C(G)$ ($\text{CM}(G)$ and $\text{MC}(G)$, respectively). The cardinality of these families are

---

1The class of #$P$-complete problems is a class of computationally equivalent counting problems which are at least as difficult as NP-complete problems.
denoted by \( c(G) \) (\( \text{cm}(G) \) and \( \text{mc}(G) \), respectively). For every graph \( G \) it holds

\[
\text{CM}(G) \subseteq \text{MC}(G) \subseteq C(G) \quad \text{and} \quad \text{cm}(G) \leq \text{mc}(G) \leq c(G)
\]

(2)

As usual let \( \alpha(G) \) be the size of a largest independent set and \( \omega(G) \) be the size of a largest clique.

### 3 Knapsack Problem and Threshold Graphs

#### 3.1 Knapsack Problem

We recall the definition of Max Knapsack, see [KPP10] for a survey.

**Name:** Max Knapsack (Max KP)

**Instance:** A set \( A = \{a_1, \ldots, a_n\} \) of \( n \) items, for every item \( a_j \), there is a size of \( s_j \) and a profit of \( p_j \). Further there is a capacity \( c \) for the knapsack.

**Task:** Find a subset \( A' \subseteq A \) such that the total profit of \( A' \) is maximized and the total size of \( A' \) is at most \( c \).

In this paper, the parameter \( n \) is assumed to be a positive integer and parameters \( p_j, s_j, \) and \( c \) are assumed to be non-negative reals. Let \( I \) be an instance for Max KP. Every subset \( A' \) of \( A \) such that \( \sum_{a_j \in A'} s_j \leq c \) is a feasible solution of \( I \). A feasible solution which is not the subset of another feasible solution is called maximal.

**Definition 1.** An instance \( I \) for Max KP and a graph \( G = (V, E) \) are equivalent, if there is a bijection \( f : A \to V \) such that \( A' \subseteq A \) is a feasible solution of \( I \) if and only if \( f(A') := \{f(a_j) \mid a_j \in A'\} \) is an independent set of \( G \).

Next we give some examples of graphs which possess an equivalent instance for Max KP. As usual we denote by \( P_n \) the path on \( n \) vertices, by \( C_n \) the cycle on \( n \) vertices, by \( K_n \) the complete graph on \( n \) vertices, and by \( K_{n,m} \) the complete bipartite graph on \( n + m \) vertices.

**Example 2.**

1. Every clique \( K_n \) on \( n \) vertices is equivalent to every\(^2\) instance for Max KP with sizes \( s_j = 1 \) for \( 1 \leq j \leq n \) and capacity \( c = 1 \).

2. Every edgeless graph \( I_n \) on \( n \) vertices is equivalent to every instance for Max KP with sizes \( s_j = 1 \) for \( 1 \leq j \leq n \) and capacity \( c = n \).

3. Every star \( K_{1,p} \) is equivalent to every instance for Max KP with sizes \( s_j = 1 \) for the \( p \) items corresponding to the \( p \) vertices of degree one, \( s_j = p \) for the item corresponding to the vertex of degree \( p \), and capacity \( c = p \).

Further examples of graphs which possess an equivalent instance for Max KP are shown in Table 1.

![Special Threshold Graphs](image)

Table 1: Special threshold graphs

Next we want to characterize graphs for which there is an equivalent instance for Max KP.

---

\(^2\)Since the notation of equivalence between graphs and instances of (cf. Definition 1) does not consider the profits, there are several instances with a given list of sizes and a capacity.
3.2 Threshold Graphs

We will use the following characterizations for threshold graphs, see [MP95] for a survey.

**Theorem 3** (Theorem 1.2.4 of [MP95]). For every graph $G = (V, E)$ the following statements are equivalent.

1. There exist non-negative reals $w_v, v \in V$, and $T$ such that for every $U \subseteq V$ it holds $\sum_{v \in U} w_v \leq T$ if and only if $U$ is an independent set of $G$.

2. Graph $G$ contains no $C_4$, no $P_4$, and no $2K_2$ as induced subgraph.

3. There exist non-negative reals $w_v, v \in V$, and $T$ such that for every two vertices $u, v$ it holds $w_u + w_v > T$ if and only if $\{u, v\} \in E$.

4. Graph $G$ can be constructed from the one-vertex graph $K_1$ by repeatedly adding an isolated vertex or a dominating vertex.

5. Graph $G$ is a split graph with vertex partition $V = S \cup K$ and the neighbourhoods of the vertices of the independent set $S$ are nested.

Condition (1) of Theorem 3 was used by Chvátal and Hammer in the 1970s to define threshold graphs [CH73, CH77]. By [Gol80, page 221] one can assume that all $w_v$ and $T$ are non-negative integers. On the other hand, the possibility of choosing arbitrary (also negative) reals leads a larger graph class, namely generalized threshold graphs, see [RRu89]. Furthermore condition (1) of Theorem 3 implies a characterization for graphs which are equivalent to instances for $\text{Max } KP$.

**Observation 4.** A graph is a threshold graph if and only if it has an equivalent instance for $\text{Max } KP$.

Threshold graphs can be recognized in linear time [CH73, HIS78]. Recently a linear time recognition algorithm was found which also gives a forbidden induced subgraph from $\{2K_2, P_4, C_4\}$ if the input is not a threshold graph [HK07]. For a set of graphs $F$ we denote by $F$-free graphs the set of all graphs that do not contain a graph of $F$ as an induced subgraph.

**Proposition 5** ([BLS99]). We have the following properties for threshold graphs.

1. threshold $\subset$ trivially perfect $\subset$ co-graphs
2. threshold $\subset$ split $\subset$ chordal
3. threshold $\subset$ split $\subset \{2K_2, C_4\}$-free graphs

By condition (1) of Theorem 3 every threshold graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ can be constructed from the one-vertex graph by repeatedly adding an isolated vertex or a dominating vertex. A creation sequence for $G$ (cf. [HIS90]) is a binary string $t = t_1, \ldots, t_n$ of length $n$ such that there is a bijection $v: \{1, \ldots, n\} \rightarrow V$ with

- $t_i = 1$ if $v(i)$ is a dominating vertex for the graph induced by $\{v(1), \ldots, v(i)\}$ and
- $t_i = 0$ if $v(i)$ is an isolated vertex for the graph induced by $\{v(1), \ldots, v(i)\}$.

W.l.o.g. we define a single vertex to be a dominating vertex, i.e. $t_1 = 1$.

**Example 6.** In Table 2 we list all creation sequences for threshold graphs on four vertices and the defined graphs.

Using the method in Figure 1.4 of [MP95] a creation sequence can be found in linear time.

---

3Robinson [Rob97] has introduced knapsack graphs as graphs whose independent sets correspond to feasible solutions of a knapsack instance. In [Rob97] knapsack graphs are characterized as graphs that do not contain a cycle of four vertices $C_4$, a path on four vertices $P_4$, or two disjoint edges $2K_2$ as an induced subgraph. Further it is shown that every knapsack graph can be constructed from the one-vertex graph $K_1$ by repeatedly adding an isolated or a dominating vertex. Further it was shown that the edge complement of a knapsack graphs is also a knapsack graph. These results have already been shown in the 1970s by Chvátal and Hammer in [CH73, CH77] which implies that the class of knapsack graphs equals the well known class of threshold graphs.
**Lemma 7** ([MP95]). Given some threshold graph $G$ on $n$ vertices and $m$ edges, a creation sequence for $G$ can be found in time $O(n + m)$.

The following properties of creation sequences will be useful in Sections 3.5, 3.6, and 3.7.

**Observation 8.** Let $G$ be a threshold graph and $t = t_1, \ldots, t_n$ be a creation sequence for $G$.

1. Every vertex corresponding to a 1 in the sequence is adjacent to all vertices corresponding to a 1 or 0 on the left and to all vertices corresponding to a 1 on the right.
2. Every vertex corresponding to a 0 in the sequence is adjacent to all vertices corresponding to a 1 on the right.
3. For every $j$ such that $t_j = 1$ the vertex set $\{v(j)\} \cup \{v(i) \mid j < i, t_i = 0\}$ leads a maximal independent set.
4. For every $j$ such that $t_j = 0$ the vertex set $\{v(j)\} \cup \{v(i) \mid j < i, t_i = 1\}$ and set $\{v(1)\} \cup \{v(i) \mid j < i, t_i = 1\}$ lead a maximal clique.
5. The vertex set $\{v(i) \mid i = 1 \vee t_i = 0\}$ leads a maximum independent set.
6. The vertex set $\{v(i) \mid t_i = 1\}$ leads a maximum clique.
7. A creation sequence $t'$ for the complement graph $\overline{G}$ can be obtained by $t'_1 = 1$ and $t'_i = 1 - t_i$ for $1 < i \leq n$.

**Corollary 9.** Let $G$ be a threshold graph on $n$ vertices and $m$ edges. The size of a maximum clique $\omega(G)$ and the size of a maximum independent set $\alpha(G)$ can be found in time $O(n + m)$.

By their characterization as $\{2K_2, P_4, C_4\}$-free graphs and also by Observation 8 we conclude that threshold graphs are closed under taking edge complements.

**Lemma 10.** The complement of a threshold graph is a threshold graph.

The next result immediately follows by the characterization using creation sequences.

**Observation 11.** There are $2^{n-1}$ many threshold graphs on $n$ vertices.

### 3.3 Creating knapsack problems from threshold graphs

Given some threshold graph $G = (V, E)$ we know by definition, that there is an equivalent MAX KP instance for $G$, which can be identified in several ways. Using a creation sequence $t_1, \ldots, t_n$, which can be found in linear time by Lemma 7 we can define an equivalent instance for MAX KP as shown in Figure 1.

**Lemma 12.** For some given threshold graph on $n$ vertices an equivalent instance for MAX KP can be constructed in time $O(n^2)$.

**Proof.** Let $G$ some threshold graph on $n$ vertices and $m$ edges. By Lemma 7 we can find a creation sequence for $G$ in time $O(n + m)$ and by the algorithm given in Figure 1 we obtain an instance $I$ for MAX KP in time $O(n^2)$. 

Obviously we can start with an arbitrary value for $c$ and $s_1$ in Figure 1 such that there are several equivalent instances for MAX KP for some fixed threshold graph. In general the sizes found by the algorithm given in Figure 1 are not minimal, see Examples 2.2 and 2.3. By combining consecutive isolated and dominating vertices one can produce smaller sizes in many cases. In [Orl77] (see also Section 1.3 of [MP95]) it is shown how to find minimal sizes.
Lemma 14. If some instance $I$ of Max KP has an equivalent graph, then it satisfies the property $P(I)$.\

Proof. Let $I$ be some instance $I$ of Max KP. In order to give a proof by contradiction we assume that $I$ does not satisfy property $P(I)$. That is, there is some $A' \subseteq A$ such that $\forall a_j, a_j' \in A': s_j + s_j' \leq c$ and $\sum_{a_j \in A'} s_j > c$. Then there is some set $A'' \subseteq A'$ and $a_\ell \in A''$ such that $\sum_{a_j \in A''} s_j > c$ and $\sum_{a_j \in A'' \setminus \{a_\ell\}} s_j \leq c$. That means that for every $a_j \in A''$ set $\{a_j, a_\ell\}$ is a feasible solution for $I$ and $A'' \setminus \{a_\ell\}$ is also a feasible solution for $I$, but $A''$ is not a feasible solution for $I$.

Every bijection $f : A \rightarrow V$ certifying an equivalent graph $G = (V, E)$ for $I$ has to map for every $a_j \in A''$ item set $\{a_j, a_\ell\}$ onto an independent set $\{f(a_j), f(a_\ell)\}$ and item set $A'' \setminus \{a_\ell\}$ onto an independent set $f(A'' \setminus \{a_\ell\}) = f(A'') - f(\{a_\ell\})$. But then also $f(A'')$ is an independent set in $G$ while $A''$ is not a feasible solution for $I$.

Next we want to show the reverse direction of Lemma 14. For some instance $I$ of Max KP we define the graph $G(I) = (V(I), E(I))$ by

$$V(I) = \{v_j \mid a_j \in A\} \quad \text{and} \quad E(I) = \{\{v_j, v_j'\} \mid s_j + s_j' > c\}.$$

For $A = A' = \{a_1, a_2\}$, $s_1 = 2$, $s_2 = 7$, $c = 10$, the right condition is true, but for $j \neq j'$ the left condition is not true.
In general $G(I)$ is not equivalent to $I$, see Example 13. But by Condition 8 of Theorem 8 and by our assumptions for instances of Max KP graph $G(I)$ is even a threshold graph.

**Observation 15.** For every instance $I$ for Max KP graph $G(I)$ is a threshold graph.

Next we give a tight connection between the property $P(I)$ and graph $G(I)$. Since both are defined by the sizes of $I$, the result follows straightforward.

**Lemma 16.** Let $I$ be some instance $I$ for Max KP. Then $I$ satisfies property $P(I)$, if and only if $I$ is equivalent to graph $G(I)$.

**Proof.** Let $I$ be an instance for Max KP.

First assume that $I$ satisfies property $P(I)$. Let $A' \subseteq A$ be a feasible solution for $I$, i.e. $\sum_{a_j \in A'} s_j \leq c$. Then obviously also $\forall a_j, a_j' \in A' \, s_j + s_j' \leq c$ holds and by the definition of $G(I)$ the vertex set $V = \{v_j \mid a_j \in A'\}$ is an independent set in graph $G(I)$. For the reverse direction let $V' \subseteq V$ be an independent set in $G(I)$. Then for $A' = \{a_j \mid v_j \in V'\}$ it holds $\sum_{a_j, a_j' \in A'} s_j + s_j' \leq c$ and since property $P(I)$ is satisfied this implies that $\sum_{a_j \in A'} s_j \leq c$, i.e. $A'$ is a feasible solution for $I$. That is, the bijection defined by the definition of $V(I) = \{v_j \mid a_j \in A\}$ verifies that $I$ is equivalent to $G(I)$.

Next assume that $I$ is equivalent to graph $G(I)$. Let $f$ be the bijection between the item set $A$ and the vertex set $V(I)$ which exists by Definition 1. Let $A' \subseteq A$. If $\forall a_j, a_j' \in A' \, s_j + s_j' \leq c$, then by the definition of $E(I)$ for all $a_j, a_j' \in A'$ it holds $\{f(a_j), f(a_j')\} \notin E$ and thus set $V'^\prime = \{f(a_j) \mid a_j \in A'\}$ is an independent set of $G(I)$, which corresponds to a feasible solution $A'$ by the equivalence of $I$ and $G(I)$ via bijection $f$. Thus $\sum_{a_j \in A'} s_j \leq c$, which implies that $P(I)$ holds true.

Now we can state a result corresponding to the reverse direction of Lemma 12.

**Lemma 17.** Let $I$ be some instance for Max KP on $n$ items which has an equivalent graph. Then $I$ is equivalent to graph $G(I)$, which can be constructed from $I$ in time $O(n^2)$.

**Proof.** Let $I$ be some instance for Max KP on $n$ items which has an equivalent graph. By Lemma 11 we know that $I$ satisfies property $P(I)$ and by Lemma 10 we know that $I$ is equivalent to graph $G(I)$. Furthermore we can construct $G(I)$ in time $O(n^2)$ by its definition.

We obtain the following characterizations for instances $I$ of Max KP allowing an equivalent graph.

**Theorem 18.** For every instance $I$ of Max KP the following conditions are equivalent.

1. Instance $I$ has an equivalent graph.
2. Instance $I$ satisfies property $P(I)$.
3. Instance $I$ is equivalent to graph $G(I)$.
4. Instance $I$ has an equivalent threshold graph.

**Proof.** (1)⇒(2) by Lemma 13 (2)⇔(3) by Lemma 15 (3)⇒(4) by Observation 15 (4)⇒(1) obvious.

**Corollary 19.** Let $I$ be some instance for Max KP which has two equivalent graphs $G_1$ and $G_2$. Then $G_1$ is isomorphic to $G_2$.

**Proof.** Let $G_i = (V_i, E_i)$ for $i = 1, 2$ be an equivalent graph for $I$ and $f_i : A \to V_i$ the thereby existing bijections. Then by $\{v_j, v_j'\} \in E_1 \iff \{f_1^{-1}(v_j), f_1^{-1}(v_j')\}$ is no feasible solution for $I \iff \{f_2(f_1^{-1}(v_j)), f_2(f_1^{-1}(v_j'))\} \in E_2$ the isomorphy of $G_1$ and $G_2$ follows.

Whenever some instance $I$ for Max KP has an equivalent graph, then by Theorem 18 graph $G(I)$ is also an equivalent graph for $I$ thus we obtain the next result.
Corollary 20. Let \( I \) be some instance for Max KP which has an equivalent graph \( G \). Then \( G \) is isomorphic to \( G(I) \).

Although not every instance for Max KP allows an equivalent graph there are several such instances. Since there are \( 2^n-1 \) threshold graphs on \( n \) vertices (cf. Observation \ref{obs:threshold_graphs}) the given algorithm given in Figure \ref{fig:algorithm} leads \( 2^n-1 \) equivalent instances for Max KP on \( n \) items. Further every of these instances leads further instances allowing an equivalent graph by multiplying every size and the capacity by some common real number. Furthermore the profits can be chosen arbitrary in such instances.

### 3.5 Counting and enumerating maximal independent sets

The maximal independent sets in threshold graphs represent maximal feasible solutions, i.e. non-extensible subsets \( A' \subseteq A \) such that \( \sum_{x \in A'} s_j \leq c \), of corresponding instances for Max KP. Every optimal solution is among one of these sets. Therefore we want to count and enumerate these sets.

#### 3.5.1 Split Graphs

In order to list all maximal independent sets of a threshold graph, in \cite{Zylicz1998} a method using their relation to split graphs (cf. Proposition \ref{prop:split_graphs}) and the hereby existing split partition was mentioned without any bound on the running time. Since the structure of split graphs will also be useful for our results in Section \ref{sec:split_graphs} we want to show how to list all maximal independent sets of a split graph.

A **split graph** is a graph \( G \) whose vertex set \( V \) can be partitioned into a clique \( K \) and an independent set \( S \) (either of which may be empty). For the edges between vertices of \( K \) and vertices of \( S \) there is no restriction.

For some split graph with a partition of its vertex set in a clique \( K \) and an independent set \( S \) we denote the pair \((K, S)\) as a **split partition**. In general a split partition \((K, S)\) is not unique and \( S \) is not a maximum independent set and \( K \) is not a maximum clique in \( G \). But there is at most one vertex which missing to obtain a maximum independent set or clique, which is even maximal, respectively.

**Theorem 21** (\cite{HSS}). Let \( G \) be a split graph with split partition \((K, S)\). Then exactly one of the three conditions hold true.

1. \(|S| = \alpha(G)\) and \(|K| = \omega(G)|. In this case the partition is unique.
2. \(|S| = \alpha(G)\) and \(|K| = \omega(G) - 1\|. In this case there is one vertex \( x \in S \), such that \( K \cup \{x\} \) is a clique.
3. \(|S| = \alpha(G) - 1\) and \(|K| = \omega(G)|. In this case there is one vertex \( x \in K \), such that \( S \cup \{x\} \) is an independent set.

The special structure of split graphs allows us to count and enumerate all maximal independent sets as follows. The result can also be applied to threshold graphs and even gives ideas for the problem on \( k \)-threshold graphs (cf. Theorem \ref{thm:k_threshold_graphs}).

**Theorem 22.** For every split graph \( G \) the number of maximal independent sets is \( \omega(G) \) or \( \omega(G) + 1 \). For every split graph \( G \) on \( n \) vertices which is given by a split partition all maximal independent sets can be counted and enumerated in time \( O(\omega(G) \cdot n) \).

**Proof.** We determine the family of all maximal independent sets of of a split graph \( G \) with split partition \((K, S)\). In order to count the maximal independent sets we assume that \( K \) is a maximum clique, i.e. \(|K| = \omega(G)\), which easily can be achieved by Theorem \ref{thm:split_partition}. We define \( K' \) as the subset of the vertices in \( K \) which are not adjacent to any vertex in \( S \). Then the maximal independent sets of \( G \) can be obtained as follows.

- If \( K' = \emptyset \), set \( S \) is a maximal independent set.
- If \( K' \neq \emptyset \), every vertex \( v \in K' \) leads a maximal independent set \( S \cup \{v\} \).


Every proper subset $S'$ of $S$ is an independent set but not a maximal independent set. To go through all these proper subsets is very inefficient. But we know that these sets can be extended by exactly one vertex from $K$. Thus every vertex $v \in K - K'$ together with its non-neighbours in $S$ leads a maximal independent set, i.e. every vertex $v \in K - K'$ leads the maximal independent set $N(v, S) \cup \{v\}$.

The union of all sets obtained in both steps leads the family of all maximal independent sets of $G$. Since we assume that we have given a split partition $(K, S)$ for $G$ the running time for every of the three steps is $O(n)$.

In order to justify the number of maximal independent sets we consider the three steps given above. The first step leads at most one set and the second and third step lead $|K| = \omega(G)$ sets together.

Please note that both numbers of maximal independent sets given in Theorem 22 are possible. The $P_3$ has $3 = \omega(P_3) + 1$ maximal independent sets and the $K_4$ has $4 = \omega(K_4)$ maximal independent sets.

By [HS81] split graphs can be recognized and a split partition can be found in linear time. In [HK07] a $O(n + m)$ time recognition algorithm was found which also gives a forbidden induced subgraph from $\{2K_2, C_5, C_4\}$ if the input is not a split graph.

**Corollary 23.** For every split graph $G$ on $n$ vertices and $m$ edges all maximal independent sets can be counted and enumerated in time $O(\omega(G) \cdot n + m)$. After $O(n + m)$ time precomputation all independent sets in $G$ can be enumerated in constant time per output.

### 3.5.2 Threshold Graphs

Since threshold graphs are split graphs (cf. Proposition 4), we can use the method of Theorem 22 to enumerate all maximal independent sets in a threshold graph. From a given creation sequence a split partition can found as follows.

**Remark 24.** For every threshold graph $G$ on $n$ vertices a split partition $(K, S)$ can be found in time $O(n)$ from a creation sequence $t = t_1 \ldots t_n$ for $G$. Vertex $v(1)$ can be chosen into $K$ or $S$. For $i > 1$ if $t_i = 0$ vertex $v(i)$ will be chosen into $S$ and if $t_i = 1$ vertex $v(i)$ will be chosen into $K$. Furthermore, if we choose $v(1)$ into $K$ this set leads a maximum clique, i.e. $|K| = \omega(G)$ and $|S| = \alpha(G) - 1$. If we choose $v(1)$ into $S$ this set leads a maximum independent set, i.e. $|S| = \alpha(G)$ and $|K| = \omega(G) - 1$.

Next we give a more simple method to enumerate all maximal independent sets in a threshold graph.

```plaintext
A = \emptyset;
MIS(G) = \emptyset;
for (i = n; i ≥ 1; i - -)
if (t_i = 1)
    MIS(G) = MIS(G) \cup \{A \cup \{v(i)\}\};
else
    A = A \cup \{v(i)\};
```

Figure 2: Enumerating all maximal independent sets in a threshold graph.

**Theorem 25.** The number of all maximal independent sets in a threshold $G$ equals $\omega(G)$. Let $G$ be a threshold graph $G$ on $n$ vertices which is given by a creation sequence. Then all maximal independent sets in $G$ can be counted in time $O(n)$ and enumerated in time $O(\omega(G) \cdot n)$.

**Proof.** Let $G$ be a threshold graph and $t = t_1 \ldots t_n$ be a creating sequence for $G$, i.e. $t_1 = 1$ and $t_i \in \{0, 1\}$. In Observation 34 we mentioned how we can obtain maximal independent sets using a creating sequence, which is realized in the method given in Figure 2 in order to generate all maximal independent sets in $G$. Since the vertices corresponding to a 1 in the creation sequence lead a maximum clique (Observation 34) our method implies every threshold graph $G$ has exactly $\omega(G)$ maximal independent sets.
Example 26. We apply the method given in Figure 3 we generate all maximal independent sets in the paw graph, which can be defined by the creation sequence $t = 1101$.

$$
\begin{array}{c|c|c|c|c}
\text{i} & v(i) & t_i & A & \text{MIS(paw)} \\
\hline
4 & v_1 & 1 & \emptyset & \{\{v_1\}\} \\
3 & v_4 & 0 & \{v_4\} & \{\{v_1\}\} \\
2 & v_2 & 1 & \{v_4\} & \{\{v_1\}, \{v_2, v_4\}\} \\
1 & v_3 & 1 & \{v_4\} & \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_4\}\} \\
\end{array}
$$

So we obtain three maximal independent sets which equals the clique size of the paw graph.

For of graphs $G$ of special graph classes the value of $\omega(G)$ can be bounded, which implies that we only have a small number of maximal independent sets. Therefore we recall that a graph $G$ on $n$ vertices and $m$ edges is $\ell$-sparse if $m \leq \ell \cdot n$. It is uniformly $\ell$-sparse if every subgraph of $G$ is $\ell$-sparse.

Corollary 27. Let $G$ be a threshold graph.

1. If $G$ is planar, then $\text{mis}(G) \leq 4$.
2. If $G$ is uniformly $\ell$-sparse, then $\text{mis}(G) \leq 2\ell + 1$.
3. If $G$ has maximum degree at most $d$, then $\text{mis}(G) \leq d + 1$.

Proof. (1.) Planar graphs do not contain the $K_5$ as a subgraph, thus $\omega(G) \leq 4$. (2.) If graph $G$ is uniformly $\ell$-sparse then the complete graph $K_{2\ell+2}$ is not a subgraph of $G$, thus $\omega(G) \leq 2\ell + 1$. (3.) If graph $G$ has maximum degree at most $d$ then the complete graph $K_{d+2}$ is not a subgraph of $G$, thus $\omega(G) \leq d + 1$.

Corollary 28. Let $G$ be a threshold graph on $n$ vertices and $m$ edges. All maximal independent sets in $G$ can be counted in time $O(n + m)$ and enumerated in time $O(n + m)$ time precomputation all independent sets in $G$ can be enumerated in constant time per output.

Proof. Since a creation sequence can be found in time $O(n + m)$ by Lemma 7 we obtain the result by Theorem 24. Since the time of the computation is at most to the size of the output, all independent sets in $G$ can be enumerated in constant time per output.

In [OUU08] results on counting and enumeration problems for chordal graphs, especially concerning independent sets, are given. Since threshold graphs are chordal, the results concerning upper bounds also hold for threshold graphs, see Table 4. While counting all maximal independent set in chordal graphs is \#P-complete [OUU08], for threshold graphs the problem is easy by Corollary 28.

3.5.3 Solutions for the Knapsack Problem

Theorem 29. Let $I$ be an instance for MAX KP on $n$ items which has an equivalent graph. Then $I$ can be solved in time $O(n^2)$.

Proof. Let $I$ be some instance for MAX KP on $n$ items which has an equivalent graph. By Lemma 17 instance $I$ is equivalent to graph $G(I)$, which can be constructed from $I$ in time $O(n^2)$. Graph $G(I)$ is a threshold graph by Observation 15.

Thus the $\omega(G(I)) \leq n$ maximal independent sets in $G(I)$ can be found in $O(n^2)$ by Corollary 28 and correspond to the maximal feasible solutions of $I$. For every of these solutions we can compute its profit in time $O(n)$.
Since there are several instances for Max KP which do not allow an equivalent graph (cf. Example 13) this does not imply that we can solve all instances for Max KP in polynomial time.

Knapsack problems have also been studied in connection with conflict graphs [PS09]. The solution of knapsack problems with conflict graphs (KPC) are independent sets in the conflict graph. For example for chordal conflict graphs a pseudo-polynomial solution is known from [PS09]. Since maximal independent sets in the conflict graph do not necessarily correspond to feasible solutions in the knapsack instance, our methods do not imply solutions for knapsack problems with conflict graphs.

3.6 Counting and enumerating maximum independent sets

Since every maximum independent set is a maximal independent set, our results given in Section 3.5 also can be applied to list all maximum independent sets in threshold graphs. By the method given in Figure 2 and removing non-maximum sets we obtain a method for listing all maximum independent sets in a threshold graph.

Corollary 30. All maximum independent sets in a threshold graph on \(n\) vertices and \(m\) edges can be counted and enumerated in time \(O(\omega(G) \cdot n + m)\).

Next we give a method to enumerate all maximum independent sets in a threshold graph.

\[
\begin{align*}
IM(G) &= \emptyset; \\
&j = 1; \text{ while } (t_j = 1) j + +; \quad \triangleright \text{find first 0 in } t \text{ if exists} \\
&\text{let } A = \{v(i) \mid t_i = 0\} \\
&\text{for } (i = 1; i < j; i + +) \\
&\quad IM(G) = IM(G) \cup \{\{v(i)\} \cup A\}; \\
\end{align*}
\]

Figure 4: Enumerating all maximum independent sets in a threshold graph.

Theorem 31. Let \(G\) be a threshold graph \(G\) on \(n\) vertices which is given by a creation sequence. Then all maximum independent sets in \(G\) can be counted in time \(O(n)\) and enumerated in time \(O(\omega(G) \cdot \alpha(G))\).

Proof. The number of maximum independent sets is equal to the value of \(j - 1 \leq \omega(G)\) (cf. Figure 4) and each set has \(\alpha(G)\) elements. \(\square\)

Since a creation sequence can be found in time \(O(n + m)\) by Lemma 7 we obtain the following result.

Corollary 32. Let \(G\) be a threshold graph on \(n\) vertices and \(m\) edges. The number of all maximum independent sets in \(G\) equals \(\omega(G)\). All maximum independent sets in \(G\) can be counted in time \(O(n + m)\). All maximum independent sets in \(G\) can be enumerated in time \(O(\omega(G) \cdot n + m)\). After \(O(n + m)\) time precomputation all independent sets in \(G\) can be enumerated in constant time per output.

The latter result can also be obtained from the fact that threshold graphs are chordal graphs and for these graphs in [OUU08] it has been shown that all maximum independent sets in a chordal graph on \(n\) vertices and \(m\) edges can be counted in time \(O(n + m)\) and enumerated in time \(O(1)\) per output. The related problem of finding one maximum independent set in a threshold graph was solved in [CLR04] in time \(O(n \log n)\). This problem can also be solved by Corollary 9 or by Remark 24. A remarkable difference between our solutions and these of [CLR04] is that we work with graph representations and the authors of [CLR04] use the coefficients occurring in the knapsack instance. Each of these versions can transformed into the other in quadratic time by Lemma 12 and 17.
Theorem 33. Let \( G \) be a threshold graph on \( n \) vertices which is given by a creation sequence. All independent sets in \( G \) can be counted in time \( \mathcal{O}(n) \). All independent sets in \( G \) can be enumerated in time \( \mathcal{O}(n \cdot 2^{n-1}) \).

Proof. Let \( G \) be given by creation sequence \( t = t_1 \ldots t_n \) and \( G_i \) be the subgraph of \( G \) induced by \( \{v(1), \ldots, v(i)\} \). Then \( i(G_i) = 1 \). For \( i > 1 \) and for \( t_i = 1 \) we have \( i(G_i) = i(G_{i-1}) + 1 \) and for \( t_i = 0 \) we have \( i(G_i) = 2 \cdot i(G_{i-1}) + 1 \). Thus we can count all independent sets in \( G \) in time \( \mathcal{O}(n) \). The time for listing all independent sets in \( G \) depends on the size and this is at most the size \( \sum_{i=1}^{n} i \cdot \binom{n}{i} = n \cdot 2^{n-1} \) of the independent sets in an edgeless graph.

Since a creation sequence can be found in time \( \mathcal{O}(n + m) \) by Lemma 7 we obtain the following result.

Corollary 34. Let \( G \) be a threshold graph on \( n \) vertices and \( m \) edges. All independent sets in \( G \) can be counted in time \( \mathcal{O}(n + m) \). All independent sets in \( G \) can be enumerated in time \( \mathcal{O}(n \cdot 2^{n-1}) \).

After \( \mathcal{O}(n + m) \) time precomputation all independent sets in \( G \) can be enumerated in constant time per output.

3.8 Bin Packing

In [CLR04] threshold graphs were briefly considered in order to give lower bounds on solutions for bin packing problems.\(^5\) But in [CLR04] it is not discussed whether the instances of bin packing problems posses equivalent graphs.

Name: MIN BIN PACKING (MIN BP)

Instance: A set \( A = \{a_1, \ldots, a_n\} \) of \( n \) items and a number \( n \) of bins. For every item \( a_j \), there is a positive rational size of \( s_j \).

Task: Find \( n \) disjoint (possibly empty) subsets \( A_1, \ldots, A_n \) of \( A \) such that the number of non-empty subsets is minimized and the sum of the sizes of the items in each subset is at most 1.

For some instance \( I \) for MIN BP two items \( a_j \) and \( a_{j'} \) can be chosen into the same subset (bin) if and only if

\[
s_j + s_{j'} \leq 1.
\]

(5)

This motivates to model the compatibleness of the items in \( I \) by threshold graphs. Therefore we consider the inequality

\[
s_1x_1 + s_2x_2 + \ldots + s_nx_n \leq 1.
\]

(6)

An instance \( I \) for MIN BP and a graph \( G = (V, E) \) are equivalent, if there is a bijection \( f : A \to V \) such that for every \( A' \subseteq A \) the characteristic vector of \( A' \) satisfies inequality (6) if and only if \( f(A') := \{f(a_i) \mid a_i \in A'\} \) is an independent set of \( G \).

\(^5\)Please note the different but equivalent definition of edges within threshold graphs in [CLR04]. By using values \( w_i \) for the vertices \( v_i \) such that \( 0 < w_i \leq 1 \) they define edges \( \{v_i, v_j\} \) whenever \( w_i + w_j \leq 1 \). For our notations see Definition 8 and 9.
If we assume that instance $I$ has an equivalent graph $G = (V, E)$, then we know by condition (i) of Theorem 3 that $G$ is a threshold graph. Further by (3) and (4) two items $a_i$ and $a_j$ can be chosen into the same bin if and only if $\{v_i, v_j\} \notin E$. That is any two items corresponding to the vertices of a clique in $G$ can not be chosen into the same bin. Thus $\omega(G)$ leads a lower bound on the number of bins needed for the considered instance $I$ for Min BP. The value $\omega(G)$ can be computed in linear time from threshold graph $G$, see Corollary 9.

Let $\Pi$ be some optimization problem and $I$ be some instance of $\Pi$. By $OPT(I)$ we denote the value of an optimal solution for $\Pi$ on input $I$.

**Theorem 35.** Let $I$ be an instance for Min BP which has an equivalent graph $G$. Then $OPT(I) \geq \omega(G)$.

In order to know an equivalent graph we can proceed as in Section 3.4 and use graph $G(I)$ defined by the values of inequality (9).

**Theorem 36.** Let $I$ be an instance for Min BP which has an equivalent graph. Then $OPT(I) \geq \omega(G(I))$.

### 4 Multidimensional Knapsack Problem and $k$-Threshold Graphs

#### 4.1 Multidimensional Knapsack Problem

Next we consider the knapsack problem for multiple dimensions, see [KPP10] for a survey.

**Name:** MAX d-DIMENSIONAL KNAPSACK (MAX d-KP)

**Instance:** A set $A = \{a_1, \ldots, a_n\}$ of $n$ items and a number $d$ of dimensions. Every item $a_j$ has a profit $p_j$ and for dimension $i$ the size $s_{i,j}$. Further for every dimension $i$ there is a capacity $c_i$.

**Task:** Find a subset $A' \subseteq A$ such that the total profit of $A'$ is maximized and for every dimension $i$ the total size of $A'$ is at most the capacity $c_i$.

In the case of $d = 1$ the MAX d-KP problem corresponds to the MAX KP problem considered in Section 3. The parameters $n$ and $d$ are assumed to be positive integers and $p_j, s_{i,j}$, and $c_i$ are assumed to be non-negative reals. Let $I$ be an instance for MAX d-KP. Every subset $A'$ of $A$ such that $\sum_{a_j \in A'} s_{i,j} \leq c_i$ for every $i \in [d]$ is a feasible solution of $I$. A feasible solution which is not the subset of another feasible solution is called maximal.

**Definition 37.** An instance $I$ for MAX d-KP and a graph $G = (V, E)$ are equivalent, if there is a bijection $f : A \to V$ such that $A' \subseteq A$ is a feasible solution of $I$ if and only if $f(A') := \{f(a_j) | a_j \in A'\}$ is an independent set of $G$.

#### 4.2 $k$-Threshold Graphs

In order to characterize graphs which are equivalent to instances for MAX d-KP, we recall the notation of the threshold dimension from [Go80]. The **threshold dimension** of a graph $G = (V, E)$ on $n$ vertices, $t(G)$ for short, is the minimum number $k$ of linear inequalities

$$a_{i,1}x_1 + \ldots + a_{i,n}x_n \leq T_i$$

such that $X \subseteq V$ is an independent set in $G$ if and only if the characteristic vector $(x_1, \ldots, x_n)$ of $X$ satisfies for $i = 1, \ldots, k$ the inequalities of type (7). By [Go80, page 221] we can assume that all $a_{i,j}$ and $T_i$ are non-negative integers.

**Theorem 38 (CH77).** For every graph $G = (V, E)$ the following statements are equivalent.

1. $G$ has threshold dimension at most $k$.
2. There are at most $k$ of threshold graphs $G_i = (V, E_i)$ such that $E = E_1 \cup \ldots \cup E_k$. 

13
Graphs $G$ such that $t(G) = 0$ are exactly the edgeless graphs and graphs $G$ such that $t(G) \leq 1$ are exactly threshold graphs. A complete characterization for graphs $G$ such that $t(G) \leq 2$ is unknown.

A graph is denoted as $k$-threshold if and only if its threshold dimension is at most $k$.

For $k = 1$ the set of $k$-threshold graphs is closed under edge complements (Lemma 10). This does not hold for $k \geq 2$ since $t(K_{n,n}) = n \neq 2 = t(K_{n,n})$ for $n \geq 3$. That is, in general the threshold dimension of a graph $G$ is not equal to the threshold dimension of its edge complement $\overline{G}$. But there is an interesting relation between the threshold dimension and the threshold intersection dimension of the edge complement graph. For some graph $G$ its threshold intersection dimension, $ti(G)$ for short, is the minimum number of threshold graphs whose intersection is $G$ [Riv85]. We (did not find some assigned notation and) call a graph $k$-threshold intersection if and only if its threshold intersection dimension is at most $k$.

**Lemma 39.** For every graph $G$ it holds $t(G) = ti(\overline{G})$.

*Proof.* The complement of a graph covered by $k$ graphs $G_1, \ldots, G_k$ is the intersection of the complements. If the graphs $G_1, \ldots, G_k$ are threshold graphs then their complements are also threshold graphs.

**Lemma 40.** For every graph $G$ the threshold dimension $t(G)$ is the minimum number $k$ such that $\overline{G}$ is the intersection of $k$ threshold graphs.

Threshold dimension 1 can be decided in linear time and threshold dimension 2 can be decided in polynomial time [SR98].

**Theorem 41 (Yan82).** For every graph $G$ it is NP-complete to determine whether $t(G) \leq k$ where $k \geq 3$.

The number of $k$-threshold graphs can be bounded by the union of one of $2^{n-1}$ (cf. Observation 14) threshold graphs for every of $k$ dimensions.

**Observation 42.** There are at most $2^{k(n-1)}$ many $k$-threshold graphs on $n$ vertices.

### 4.3 Creating multidimensional knapsack problems from $k$-threshold graphs

Let $G = (V, E)$ be a $k$-threshold graph. Then we can define an equivalent instance for MAX d-KP as follows. Because of the hardness mentioned in Theorem 41 we assume that we have given $k$ threshold graphs which cover the edge set of $G$.

**Lemma 43.** Given is some $k$-threshold graph $G = (V, E)$ on $n$ vertices which is given by $k$ threshold graphs $G_j = (V, E_j)$, $1 \leq j \leq k$, which cover the edge set of $G$. An equivalent instance $I$ for MAX d-KP can be constructed in time $O(d \cdot n^2)$.

*Proof.* We assume that we have given $k$ threshold graphs $G_j = (V, E_j)$, $1 \leq j \leq k$, which cover the edge set of $G$. For each of these graphs $G_j$ we can find a creation sequence in $O(n^2)$. Then we define by the method given in Figure 11 sizes $s_{j,i}$ and a capacity $c_j$ to obtain an instance for MAX KP. The union of all these $k$ instances (inequalities) is an equivalent instance for MAX d-KP. □

### 4.4 Creating $k$-threshold graphs from multidimensional knapsack problems

Next we consider the problem of defining a graph from an instance for MAX d-KP. First we look at the relation between the existence of an equivalent graph for some MAX d-KP instance $I$ and the existence of an equivalent graph for the $d$ corresponding MAX KP instances, which are defined as follows. Let $I$ be some MAX d-KP instance on item set $A = \{a_1, \ldots, a_n\}$ with sizes $s_{1,i}, \ldots, s_{d,n}$, capacities $c_1, \ldots, c_d$ and $i \in [d]$. Then for $i \in [d]$ we define by $I_i$ the MAX KP instance on item set $A = \{a_1, \ldots, a_n\}$ with profits $p_1, \ldots, p_n$, sizes $s_{1,i}, \ldots, s_{i,n}$, and capacity $c_i$.

**Lemma 44.** Let $I$ be some instance of MAX d-KP. If for every dimension $i \in [d]$ the corresponding MAX KP instance $I_i$ possesses an equivalent graph $G_i = (V, E_i)$, then $G = (V, E)$ where $E = E_1 \cup \ldots \cup E_d$ leads an equivalent graph for instance $I$. □
Proof. Let \( I \) be some instance of \( \text{Max } d\text{-KP} \) on item set \( A = \{a_1, \ldots, a_n\} \) with sizes \( s_{1,1}, \ldots, s_{d,n} \), capacities \( c_1, \ldots, c_d \), and \( i \in [d] \), and \( I_1, \ldots, I_d \) the corresponding \( \text{Max } KP \) instances. Assume that for every dimension \( i \in [d] \) \( \text{Max } KP \) instance \( I_i \) possesses an equivalent graph \( G_i \). Every \( G_i \) is a threshold graph by definition. Since all graphs are defined on the item set of \( I \) they can be defined on the same vertex set, i.e., \( G_i = (V, E_i) \). Then we obtain a \( d \)-threshold graph \( G = (V, E) \) by \( E = E_1 \cup \ldots \cup E_d \). Then \( A' \subseteq A \) is a feasible solution for \( I \) if and only if \( A' \) is a feasible solution for every \( I_i \), if and only if \( f(A') \) is an independent set of of every \( G_i \) if and only if \( f(A') \) is an independent set of \( G \). Thus \( G \) is equivalent to \( I \). \( \square \)

On the other hand, it is not possible to carry over the existence of an equivalent graph from \( I \) to the instances \( I_1, \ldots, I_d \).

Example 45. We consider \( d = 2 \) dimensions. First let \( s_{1,1} = 3, s_{1,2} = 1, s_{1,3} = 2, s_{1,4} = 4, s_{1,5} = 5 \) and \( c_1 = 5 \). Further let \( s_{2,1} = 5, s_{2,2} = 5, s_{2,3} = 5, s_{2,4} = 1, s_{2,5} = 1 \) and \( c_2 = 5 \). Then \( I_1 \) does not allow an equivalent graph and \( I_2 \) allows an equivalent graph \((K_5 - e)\). Instance \( I \) allows an equivalent graph \((K_5)\).

That is, if for some dimension \( i \) the instance \( I_i \) does not allow an equivalent graph, this not necessary implies that \( I \) does not have an equivalent graph. But this assumption also can imply that \( I \) does not have an equivalent graph by the following example.

Example 46. We consider \( d = 2 \) dimensions. First let \( s_{1,1} = 12, s_{1,2} = 10, s_{1,3} = 11, s_{1,4} = 8, s_{1,5} = 9 \) and \( c_1 = 26 \). Further let \( s_{2,1} = 2, s_{2,2} = 1, s_{2,3} = 2, s_{2,4} = 4, s_{2,5} = 5 \) and \( c_2 = 5 \). Then \( I_1 \) does not allow an equivalent graph. and \( I_2 \) allows an equivalent graph \( (\text{"dart" graph}) \). Instance \( I \) does not allow an equivalent graph, since for \( A' = \{a_1, a_2, a_3\} \) we every two different items of \( A' \) form a feasible solution of \( I \) which would imply an independent set of size three in a corresponding graph, but \( A' \) itself is not a feasible solution.

If none of the instances \( I_1, \ldots, I_d \) has an equivalent graph, for instance \( I \) both situations are possible. By the following two examples.

Example 47. We consider \( d = 2 \) dimensions. First let \( s_{1,1} = 3, s_{1,2} = 1, s_{1,3} = 2, s_{1,4} = 5, s_{1,5} = 5, s_{1,6} = 5 \), and \( c_1 = 5 \). Further let \( s_{2,1} = 5, s_{2,2} = 5, s_{2,3} = 5, s_{2,4} = 3, s_{2,5} = 1, s_{2,6} = 2 \), and \( c_2 = 5 \). Then \( I_1 \) and \( I_2 \) do not allow an equivalent graph. Instance \( I \) allows an equivalent graph \((K_5)\).

Example 48. We consider \( d = 2 \) dimensions. First let \( s_{1,1} = 3, s_{1,2} = 1, s_{1,3} = 2, s_{1,4} = 5, s_{1,5} = 5, s_{1,6} = 5 \), and \( c_1 = 5 \). Further let \( s_{2,1} = 3, s_{2,2} = 1, s_{2,3} = 2, s_{2,4} = 3, s_{2,5} = 1, s_{2,6} = 2 \), and \( c_2 = 5 \). Then \( I_1 \) and \( I_2 \) do not allow an equivalent graph. But now also instance \( I \) also does not allow an equivalent graph.

Next we want to characterize \( \text{Max } d\text{-KP} \) instances allowing an equivalent graph. Therefor let \( \mathcal{I}_d \) be the set of all instances of \( \text{Max } d\text{-KP} \). We define a Boolean property \( P_d : \mathcal{I}_d \to \{\text{true}, \text{false}\} \) for some instance \( I \in \mathcal{I}_d \) by

\[
P_d(I) = \bigvee_{A \subseteq A'} \left( \left( \bigwedge_{i \in [d]} \bigwedge_{a_i,j \in A} s_{i,j} + s_{i,j'} \leq c_i \right) \Rightarrow \left( \bigwedge_{i \in [d]} \sum_{a_j \in A'} s_{i,j} \leq c_i \right) \right).
\] (8)

The idea of this property is to ensure feasibility in the case of independence within \( A \). Although independence has to be valid within a graph, this will be useful. For subsets \( A' \) with \( |A'| \leq 2 \) the property is always true. Since we assume that \( j \neq j' \) the implication from the right to the left is always true. Comparing property \( P_d \) with property \( P \) defined in Section 3.4 we conclude that if property \( P(I) \) holds for every \( i \in [d] \) then also property \( P_d(I) \) holds, but the reverse direction does not hold in general (see Examples 45 and 47).

Lemma 49. If some instance \( I \) for \( \text{Max } d\text{-KP} \) has an equivalent graph, then \( I \) satisfies property \( P_d(I) \).

Proof. Let \( I \) be some instance for \( \text{Max } d\text{-KP} \). In order to give a proof by contradiction we assume that \( I \) does not satisfy property \( P_d(I) \). That is there is some \( A' \subseteq A \) such that for every \( i \in [d] \) and \( \forall a_j, a_{j'} \in A' : s_{i,j} + s_{i,j'} \leq c_i \) and there is some dimension \( i' \in [d] \) such that \( \sum_{a_j \in A'} s'_{i',j} > c_{i'} \). Then
there is some set \( A'' \subseteq A' \), \( a_\ell \in A'' \) and \( i'' \in [d] \) such that \( \sum_{a_j \in A''} s_{i''j} > c_{i''} \) and for all \( i \in [d] \) it holds \( \sum_{a_j \in A'' - \{a_\ell\}} s_{ij} \leq c_i \). That means that for every \( a_j \in A'' \) set \( \{a_j, a_\ell\} \) is a feasible solution for \( I \) and \( A'' - \{a_\ell\} \) is also a feasible solution for \( I \), but \( A'' \) is not a feasible solution for \( I \).

Every bijection \( f : A \rightarrow V \) certifying an equivalent graph \( G = (V,E) \) for \( I \) has to map for every \( a_j \in A'' \) item set \( \{a_j, a_\ell\} \) onto an independent set \( \{f(a_j), f(a_\ell)\} \) and item set \( A'' - \{a_\ell\} \) onto an independent set \( f(A'' - \{a_\ell\}) = f(A'') - f(\{a_\ell\}) \). But then also \( f(A'') \) is an independent set in \( G \) while \( A'' \) is not a feasible solution for \( I \).

Next we want to show the reverse direction of Lemma 49. For some instance \( I \) of MAX d-KP we define the graph \( G_d(I) = (V(I), E(I)) \) by \n
\[ V(I) = \{v_j \mid a_j \in A\} \quad \text{and} \quad E(I) = \{\{v_j, v_{j'}\} \mid s_{1,j'} + s_{1,j'} > c_1 \lor \ldots \lor s_{d,j} + s_{d,j'} > c_d\}. \quad (9) \]

In general \( G(I) \) is not equivalent to \( I \), see Example 16. Graph \( G_d(I) \) can also be obtained from the \( d \) threshold graphs \( G(I_i) = (V(I_i), E(I_i)) \), \( 1 \leq i \leq d \), by \( E(I) = E(I_1) \cup \ldots \cup E(I_d) \), which implies that \( G_d(I) \) is a \( d \)-threshold graph by Theorem 38.

**Observation 50.** For every instance \( I \) for MAX d-KP graph \( G_d(I) \) is a \( d \)-threshold graph.

Next we give a tight connection between the property \( P_d(I) \) and graph \( G_d(I) \). Since both are defined by the sizes of \( I \), the result follows straightforward.

**Lemma 51.** Let \( I \) be some instance \( I \) for MAX d-KP. Then \( I \) satisfies property \( P_d(I) \), if and only if \( I \) is equivalent to graph \( G_d(I) \).

**Proof.** Let \( I \) be an instance for MAX d-KP.

First assume that \( I \) satisfies property \( P_d(I) \). Let \( A' \subseteq A \) be a feasible solution for \( I \), i.e. for every \( i \in [d] \) it holds \( \sum_{a_j \in A'} s_{ij} \leq c_i \). Then obviously also for every \( i \in [d] \) it holds \( \forall_{a_j, a_j' \in A'} s_{ij} + s_{ij'} \leq c_i \) holds and by the definition of edge set \( E(I) \) the vertex set \( V' = \{v_j \mid a_j \in A'\} \) is an independent set in graph \( G_d(I) \). For the reverse direction let \( V' \subseteq V \) be an independent set in \( G_d(I) \). Then for \( A' = \{a_j \mid v_j \in V'\} \) for every \( i \in [d] \) it holds \( \forall_{a_j, a_j' \in A'} s_{ij} + s_{ij'} \leq c_i \) and since property \( P_d(I) \) is satisfied this implies that for every \( i \in [d] \) it holds \( \sum_{a_j \in A'} s_{ij} \leq c_i \), i.e. \( A' \) is a feasible solution for \( I \). That is, the bijection defined by the definition of \( V(I) = \{v_j \mid a_j \in A\} \) verifies that \( I \) is equivalent to \( G(I) \).

Next assume that \( I \) is equivalent to graph \( G_d(I) \). Let \( f \) be the bijection between the item set \( A \) and the vertex set \( V(I) \), which exists by Definition 57. Let \( A' \subseteq A \). If for every \( i \in [d] \) it holds \( \forall_{a_j, a_j' \in A'} s_{ij} + s_{ij'} \leq c_i \) then by the definition of \( E(I) \) for all \( a_j, a_j' \in A' \) it holds \( \{f(a_j), f(a_j')\} \notin E \) and thus set \( V' = \{f(a_j) \mid a_j \in A'\} \) is an independent set of \( G_d(I) \), which corresponds to a feasible solution \( A' \) by the equivalence of \( I \) and \( G_d(I) \) via bijection \( f \). Thus for every \( i \in [d] \) it holds \( \sum_{a_j \in A'} s_{ij} \leq c_i \), which implies that \( P_d(I) \) holds true.

Now we can state a result corresponding to the reverse direction of Lemma 49.

**Lemma 52.** Let \( I \) be some instance for MAX d-KP on \( n \) items which has an equivalent graph. Then \( I \) is equivalent to graph \( G_d(I) \), which can be constructed from \( I \) in time \( O(d \cdot n^3) \).

**Proof.** Let \( I \) be some instance for MAX d-KP on \( n \) items which has an equivalent graph. By Lemma 49 we know that \( I \) satisfies property \( P_d(I) \) and by Lemma 51 we know that \( I \) is equivalent to graph \( G_d(I) \). Furthermore we can construct \( G_d(I) \) in time \( O(d \cdot n^2) \) by its definition.

We obtain the following characterizations for instances \( I \) of MAX d-KP allowing an equivalent graph.

**Theorem 53.** For every instance \( I \) of MAX d-KP the following conditions are equivalent.

1. Instance \( I \) has an equivalent graph.
2. Instance \( I \) satisfies property \( P_d(I) \).
3. Instance \( I \) is equivalent to graph \( G_d(I) \).
4. Instance \( I \) has an equivalent \( d \)-threshold graph.
Proof. (1) \Rightarrow (2) by Lemma \ref{lem:19} (2) \Leftrightarrow (3) by Lemma \ref{lem:21} (3) \Rightarrow (4) by Observation \ref{obs:40} (4) \Rightarrow (1) obvious \hfill \Box

The proof of the following result runs similar to that of Corollary \ref{cor:49}.

**Corollary 54.** Let $I$ be some instance for MAX d-KP which has two equivalent graphs $G_1$ and $G_2$. Then $G_1$ is isomorphic to $G_2$.

Whenever some instance $I$ for MAX d-KP has an equivalent graph, then by Theorem \ref{thm:53} graph $G_d(I)$ is also an equivalent graph for $I$ thus we obtain the next result.

**Corollary 55.** Let $I$ be some instance for MAX d-KP which has an equivalent graph $G$. Then $G$ is isomorphic to $G_d(I)$.

### 4.5 Counting and enumerating maximal independent sets

#### 4.5.1 k-threshold graphs

Next we show how to enumerate and count the maximal independent sets in a $k$-threshold graph $G$ using these sets for $k$ covering threshold graphs of $G$.

\
\begin{align*}
\text{MIS}(G) &= \emptyset; \mathcal{I} = \emptyset \\
\text{for } (i = 1; i \le k; i++) & \quad \triangleright \text{see Figure } 3 \\
& \quad \text{compute MIS}(G_i) \\
& \quad \mathcal{I} = \mathcal{I} \cup \{M_1 \cap \ldots \cap M_k\} \\
& \quad \triangleright \text{compute } \mathcal{I} = \{M_1 \cap \ldots \cap M_k \mid M_i \in \text{MIS}(G_i)\} \\
& \quad \text{for each } (I, J) \in \mathcal{I} \times \mathcal{I} \\
& \quad \text{if } (I, J) \text{ is a subset of } (I - I) \\
& \quad \mathcal{I} = \mathcal{I} - I \\
\text{MIS}(G) &= \mathcal{I};
\end{align*}

Figure 6: Enumerating all maximal independent sets in a $k$-threshold graph.

**Theorem 56.** All maximal independent sets in a $k$-threshold graph $G$ on $n$ vertices given by an edge set covering of $k$ threshold graphs $G_i = (V, E_i)$ on $m_i$ edges, $1 \le i \le k$, can be enumerated and counted in time $\mathcal{O}(\sum_{i=1}^{k} m_i \cdot (\prod_{i=1}^{k} \omega(G_i))^2) \subseteq \mathcal{O}(n^{2k+1})$.

**Proof.** Let $G$ be a $k$-threshold graph on $n$ vertices and $m$ edges. Further let $G_i = (V, E_i)$, $1 \le i \le k$, be a covering by $k$ threshold graphs for $G$ and $m_i$ denote the number of edges in $G_i$. By the method given in Figure 6 we generate all maximal independent sets in $G$.

The running time for computing $\text{MIS}(G_i)$ for $1 \le i \le k$ can be bounded using Corollary \ref{cor:28} by $\mathcal{O}(\sum_{i=1}^{k} \omega(G_i) \cdot n + m_i) = \mathcal{O}(n \cdot \sum_{i=1}^{k} \omega(G_i) + \sum_{i=1}^{k} m_i)$.

Since every family $\text{MIS}(G_i)$ consists of $\omega(G_i) \le n$ sets, there are at most $\prod_{i=1}^{k} \omega(G_i) \le n^k$ tuples $(M_1, \ldots, M_k)$ in $\text{MIS}(G_1) \times \ldots \times \text{MIS}(G_k)$. For every tuple $(M_1, \ldots, M_k)$ the intersection $M_1 \cap \ldots \cap M_k$ can be computed as follows. We have given $k$ subsets of a set (the vertex set of $G$) of $n$ elements. These $k$ subsets are merged into one set $M$ of at most $k \cdot n$ elements. We can sort $M$ using counting sort in time $\mathcal{O}(k \cdot n + n) = \mathcal{O}(k \cdot n)$. Then an element of $M$ belongs to the intersection $I$ if and only if it occurs $k$ times consecutively in the sorted list $M$. This can be checked by comparing every first element on position $i$ with the element on position $i + k - 1$. Thus one intersection $M_1 \cap \ldots \cap M_k$ can be computed in time $\mathcal{O}(k \cdot n)$ and the set of all intersections $\mathcal{I}$ can be computed in time $\mathcal{O}(k \cdot n \cdot (\prod_{i=1}^{k} \omega(G_i))$.

Then we have to eliminate non-maximal subsets in $\mathcal{I}$, where $|\mathcal{I}| \le \prod_{i=1}^{k} \omega(G_i) \le n^k$. For every pair $(I, J) \in \mathcal{I} \times \mathcal{I}$ we can sort $I$ and $J$ and then verify whether $I$ is a subset of $J$ in time $\mathcal{O}(n)$. Thus we can obtain $\text{MIS}(G)$ from $\mathcal{I}$ in time $\mathcal{O}(n \cdot (\prod_{i=1}^{k} \omega(G_i))^2)$. 

17
By assuming \( m_i \geq 1 \) and thus \( \omega(G_i) \geq 2 \) for \( 1 \leq i \leq k \) the overall running time is in

\[
O(n \cdot \sum_{i=1}^{k} \omega(G_i) + \sum_{i=1}^{k} m_i + k \cdot n \cdot (\prod_{i=1}^{k} \omega(G_i)) + n \cdot (\prod_{i=1}^{k} \omega(G_i))^2) \\
\leq O(n \cdot \prod_{i=1}^{k} \omega(G_i) + \sum_{i=1}^{k} m_i + (\prod_{i=1}^{k} \omega(G_i)) \cdot n \cdot (\prod_{i=1}^{k} \omega(G_i)) + n \cdot (\prod_{i=1}^{k} \omega(G_i))^2) \\
\leq O\left(\sum_{i=1}^{k} n \cdot m_i + n \cdot (\prod_{i=1}^{k} \omega(G_i))^2\right) \\
\subseteq O(n^k+1).
\]

The correctness holds as follows. Every independent set \( S \) in \( G \) is also an independent set in graph \( G_i \) for every \( 1 \leq i \leq k \) by the definition of \( G \). Thus every independent set \( S \) in \( G \) is a subset of some maximal independent set \( M_i \) in graph \( G_i \) for every \( 1 \leq i \leq k \). Thus every independent set \( S \) in \( G \) is a subset of the intersection \( M_1 \cap \ldots \cap M_k \) for some maximal independent sets \( M_i \) in graph \( G_i \) for every \( 1 \leq i \leq k \). Further since every such intersection \( M_1 \cap \ldots \cap M_k \) is an independent set in \( G \) and we remove the non-maximal independent sets from the set of all these intersections in the last step of our method, we create exactly the set of all maximal independent sets of \( G \). \( \square \)

![Figure 7: Covering the edge set of the house graph by two threshold graphs paw \( \cup \) \( K_1 \) and \( P_5 \cup 2K_1 \) which is used in Example 57](image)

**Example 57.** We apply the algorithm given in Figure 6 to enumerate maximal independent sets in the 2-threshold graph “house”. By Figure 7 the edge set of the house can be covered by two threshold graphs \( G_1 \) and \( G_2 \).

1. The algorithm shown in Figure 6 leads \( \text{MIS}(G_1) = \{\{2, 5\}, \{1, 4, 5\}, \{1, 3, 5\}\} \) and \( \text{MIS}(G_2) = \{\{2, 3, 5\}, \{1, 2, 3, 4\}\} \).
2. \( \mathcal{I} = \{\{2, 5\}, \{2\}, \{5\}, \{1, 4\}, \{3, 5\}, \{1, 3\}\} \)
3. \( \text{MIS}(\text{house}) = \{\{2, 5\}, \{1, 4\}, \{3, 5\}, \{1, 3\}\} \)

Thus it holds \( \text{IM}(\text{house}) = \text{MIS}(\text{house}) \). In order to give an example for a 2-threshold graph where this equality does not hold we consider the gem graph which is a split graph as well as a 2-threshold graph.

![Figure 8: Covering the edge set of the gem graph by two threshold graphs paw \( \cup \) \( K_1 \) and \( K_3 \cup 2K_1 \)](image)

**Example 58.** We apply the algorithm given in Figure 6 to enumerate maximal independent sets in the 2-threshold graph “gem”. By Figure 8 the edge set of the house can be covered by two threshold graphs \( G_1 \) and \( G_2 \).

1. The algorithm shown in Figure 6 leads \( \text{MIS}(G_1) = \{\{3, 4, 5\}, \{1, 3, 4\}, \{2, 4\}\} \) and \( \text{MIS}(G_2) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\} \).
2. \( \mathcal{I} = \{\{3\}, \{4\}, \{5\}, \{1, 3\}, \{1, 4\}, \{1\}, \{2\}, \{2, 4\}\} \)
3. \( \text{MIS}(\text{gem}) = \{\{5\}, \{1, 3\}, \{1, 4\}, \{2, 4\}\} \)

The method given in Figure 6 implies the following bound.

**Corollary 59.** Let \( G \) be a \( k \)-threshold graph and \( G_i = (V, E_i), 1 \leq i \leq k \), a covering by \( k \) threshold graphs for \( G \). Then \( G \) has at most \( \prod_{i=1}^{k} \omega(G_i) \) maximal independent sets.
In order to show that the bound \( \text{mis}(G) \leq \prod_{i=1}^{k} \omega(G_i) \) can be achieved, we consider \( G \) as the disjoint union of \( k \) complete graphs \( K_i \). Then for \( \ell \geq 2 \) the graph \( G \) has has threshold dimension \( k \). Further \( G \) leads \( t^k = \prod_{i=1}^{k} \omega(K_i) \) maximal independent sets.

For several graphs we can bound the size of the largest complete subgraph.

**Corollary 60.** Let \( G \) be a \( k \)-threshold graph and \( G_i = (V, E_i) \), \( 1 \leq i \leq k \), some \( k \) covering threshold graphs for \( G \).

1. If every \( G_i \), \( 1 \leq i \leq k \), is planar, then \( \text{mis}(G) \leq 4^k \).
2. If every \( G_i \), \( 1 \leq i \leq k \), is uniformly \( \ell \)-sparse, then \( \text{mis}(G) \leq (2\ell + 1)^k \).
3. If every \( G_i \), \( 1 \leq i \leq k \), has maximum degree at most \( d \), then \( \text{mis}(G) \leq (d+1)^k \).

**Proof.** (1.) Planar graphs do not contain the \( K_3 \) as a subgraph, thus \( \omega(G_i) \leq 4 \) for \( 1 \leq i \leq k \).
(2.) If a graph \( G_i \) is uniformly \( \ell \)-sparse then the complete graph \( K_{2\ell+2} \) is not a subgraph of \( G_i \), thus \( \omega(G_i) \leq 2\ell + 1 \) for \( 1 \leq i \leq k \).
(3.) If a graph \( G_i \) has maximum degree at most \( d \) then the complete graph \( K_{d+2} \) is not a subgraph of \( G_i \), thus \( \omega(G_i) \leq d + 1 \) for \( 1 \leq i \leq k \).

If some \( k \)-threshold graph has one of the discussed properties, then this also holds for all of the \( k \) subgraphs \( G_i \).

**Corollary 61.** Let \( G \) be a \( k \)-threshold graph.

1. If \( G \) is planar, then \( \text{mis}(G) \leq 4^k \).
2. If \( G \) is uniformly \( \ell \)-sparse, then \( \text{mis}(G) \leq (2\ell + 1)^k \).
3. If \( G \) has maximum degree at most \( d \), then \( \text{mis}(G) \leq (d+1)^k \).

The method given in Figure 6 leads all maximal independent sets for \( k \)-threshold graphs and an interesting bound on the number of maximal independent sets using the clique number of \( k \) suitable covering threshold graphs (Corollary 59). The main drawback of the method given in Figure 6 is the last step in which we have to remove non-maximal subsets. For better solutions on enumerating covering threshold graphs (Corollary 59). The main idea is to partition the vertex set of a \( 2 \)-threshold graph into three cliques and one independent set. This allows us to generalize the idea for split graphs given in the proof of Theorem 22.

**Theorem 62.** All maximal independent sets in a \( 2 \)-threshold graph on \( n \) vertices can be enumerated in time \( \mathcal{O}(n^3) \) and counted in time \( \mathcal{O}(n^2) \).

**Proof.** Let \( G = (V, E) \) be a \( 2 \)-threshold graph. Then there are two threshold graphs \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) that cover the edge set of \( G \), i.e. \( E = E_1 \cup E_2 \). Let \( (K_1, S_1) \) be a split partition of \( G_1 \) and \( (K_2, S_2) \) be a split partition of \( G_2 \). Such partitions are easy to find by Remark 24. We consider the intersections \( K = K_1 \cap K_2 \), \( S = S_1 \cap S_2 \), \( A = K_1 \cap S_2 \), and \( B = S_1 \cap K_2 \). Then \( K \), \( A \), and \( B \) are cliques in \( G \) and \( S \) is an independent set in \( G \). Further it holds \( V = K \cup S \cup A \cup B \), see Table 3. We define \( K' \) (\( A' \), \( B' \), respectively) as the subset of the vertices in \( K \) (\( A \), \( B \), respectively) which are not adjacent to any vertex in \( S \). The maximal independent sets of \( G \) can be found as follows.

1. Set \( S \) is an independent set. But \( S \) has not be maximal (even not if \( S_1 \) and \( S_2 \) are maximal).
   That is, we have to verify whether we can add vertices from \( A' \), \( B' \), or \( K' \) to \( S \). Since \( K_1 \) and \( K_2 \) are cliques in \( G_1 \) and \( G_2 \) they are also cliques in \( G \) and thus \( A \cup K \) is a clique in \( G \) as
well $B \cup K$. Thus we cannot add a vertex from $A'$ and a vertex from $K'$ to an independent set and the same holds for $B'$ and $K'$ and for all three sets $A'$, $B'$, and $K'$. We can either add a vertex from $K'$ or at most two nonadjacent vertices from $A' \cup B'$.

(a) If $K' = A' = B' = \emptyset$, then $S$ is a maximal independent set.

(b) If $K'$ is non-empty, for every $v \in K'$ we obtain a maximal independent set $S \cup \{v\}$.

(c) If either $A'$ or $B'$ is empty, the other one will be treated as $K'$.

(d) If $A'$ and $B'$ are non-empty, for every $(v_1, v_2) \in A' \times B'$ such that $\{v_1, v_2\} \not\in E$ we obtain a maximal independent set $S \cup \{v_1, v_2\}$.

(e) If $A'$ and $B'$ are non-empty, for every $v_1 \in A'$ such that $v_1$ is adjacent to every vertex in $B'$, we obtain a maximal independent set $S \cup \{v_1\}$.

(f) If $A'$ and $B'$ are non-empty, for every $v_2 \in B'$ such that $v_2$ is adjacent to every vertex in $A'$, we obtain a maximal independent set $S \cup \{v_2\}$.

In this cases we have to add up to two vertices into $S$ to obtain a maximal independent set.

2. Every proper subset of $S$ is an independent set but not a maximal independent set. But the possibilities how we can add vertices from $V - S$ to subsets of $S$ are restricted as mentioned in case (1.). Obviously a vertex $v \in V - S$ can only be added to the (possibly empty) set of its non-neighours in $S$, which we denote by $N(v, S)$.

(a) If $K - K'$ is non-empty, for every $v \in K - K'$ we obtain a maximal independent set $N(S, v) \cup \{v\}$.

(b) If either $A - A'$ or $B - B'$ is empty, the other one will be treated as $K - K'$.

(c) If $A - A'$ and $B - B'$ are non-empty, for every $(v_1, v_2) \in A - A' \times B - B'$ such that $\{v_1, v_2\} \not\in E$ we obtain a maximal independent set by $(N(v_1, S) \cap N(v_2, S)) \cup \{v_1, v_2\}$.

(d) For every $v_1 \in A - A'$ such that there is no $v_2 \in B - B'$ such that $\{v_1, v_2\} \not\in E$ and $N(v_2, S) \subseteq N(v_1, S)$ (in order to ensure maximality of the sets) we obtain a maximal independent set $N(v_1, S) \cup \{v_1\}$.

(e) For every $v_2 \in B - B'$ such that there is no $v_1 \in A - A'$ such that $\{v_1, v_2\} \not\in E$ and $N(v_1, S) \subseteq N(v_2, S)$ (in order to ensure maximality of the sets) we obtain a maximal independent set $N(v_2, S) \cup \{v_2\}$.

In this cases we have to add up to two vertices into a subset of $S$ to obtain a maximal independent set.

By our construction all independent sets obtained in both steps are maximal. The union of all these leads the family of all maximal independent sets of $G$. Every of the eleven conditions can be verified in time $O(n^2)$ and leads $O(n^2)$ maximal independent sets of size $O(n)$.

The partition of the vertex set of a 2-threshold graph in the proof of Theorem 62 motivates to look at the following generalization of split graphs which was introduced by Brandstädt in [Bra96]. A graph is a $(k, \ell)$-graph if its vertex set can be partitioned into $k$ independent sets and $\ell$ cliques. For the edges between vertices of these sets there is no restriction. Thus $(2, 0)$-graphs are exactly bipartite graphs, $(k, 0)$ are exactly $k$-colorable graphs, and $(1, 1)$-graphs are exactly split graphs. The given partition the proof of Theorem 62 shows that 2-threshold graphs are special $(1, 3)$-graphs and more generally $k$-threshold graphs are special $(1, 2^{k-1})$-graphs.

**Proposition 63.** We have the following properties for $k$-threshold graphs and for $k$-threshold intersection graphs.

1. $k$-threshold $\subset (1, 2^{k-1})$-graphs

2. $k$-threshold intersection $\subset (2^{k-1}, 1)$-graphs
4.5.2 Solutions for the Multidimensional Knapsack Problem

We have shown how to solve every instance for MAX KP on \(n\) items which has an equivalent graph in time \(O(n^2)\) (cf. Theorem 49) using the method given in Figure 2. Next we want to give a similar result for instances for MAX d-KP using our method for enumerating maximal independents in a \(d\)-threshold graph shown in Figure 6.

**Theorem 64.** Let \(I\) be an instance for MAX d-KP on \(n\) items which has an equivalent graph. Then \(I\) can be solved in time \(O(n^{2d+1})\).

**Proof.** Let \(I\) be some instance for MAX d-KP on \(n\) items such that \(I\) has an equivalent graph. By Theorem 58 instance \(I\) is equivalent to graph \(G_d(I)\). Graph \(G_d(I)\) is a \(d\)-threshold graph by Observation 50. In order to apply Theorem 56 on \(G_d(I)\) we need \(d\) covering threshold graphs \(G_i, 1 \leq i \leq d\), for \(G_d(I)\). Such graphs can be obtained by \(G_i = G(I_i)\), i.e. the graphs defined in Section 3.4 for every of the \(d\) instances \(I_i\) for MAX KP defined in Section 4.4. Every graph \(G_i\) is a threshold graph by Observation 15 and can be constructed in time \(O(n^2)\).

Thus the \(\prod_{i=1}^d \omega(G(I_i)) \leq n^d\) maximal independent sets in \(G\) can be found in time \(O(n^{2d+1})\) by Theorem 56 and correspond to the maximal feasible solutions of \(I\). For every of these solutions we can compute its profit in time \(O(n)\).

A related approach to solve special for MAX d-KP instances was suggested by Chvátal and Hammer in [CH77]. They consider \(m \times n\) zero-one matrices \(A\) for which there is a single inequality

\[
\sum_{j=1}^n a_j x_j \leq b
\]

whose zero-one solutions are exactly the zero-one solutions of the \(m\) inequalities \(i = 1, \ldots, m\)

\[
\sum_{j=1}^n a_{i,j} x_j \leq 1.
\]

For such a matrix \(A\) they define a graph \(G(A)\) by representing the columns of \(A\) as vertices and two vertices are adjacent if and only if the dot product of the corresponding vectors is positive. If \(G(A)\) is a threshold graph the MAX d-KP instance \(I_A\) using zero-one sizes with respect to \(A\), capacities \(c_I = 1\) and \(d = m\) dimensions was solved in [CH77] within time \(O(m \cdot n^2)\) by using the split partition of graph \(G(A)\).

The sketched results of [CH77] motivate us to consider MAX d-KP instances \(I\) which have an equivalent threshold graph \(G\). By Corollary 55 we know that \(G\) is isomorphic to \(G_d(I)\). Since \(G\) is a threshold graph in this case \(G_d(I)\) is also a threshold graph. Thus instead of the method shown in Figure 6 we now can apply the method given in Figure 2 on graph \(G_d(I)\) in order to list all maximal feasible solutions for \(I\). Since \(G_d(I)\) can be defined from \(I\) in time \(O(d \cdot n^2)\) we have shown the following result.

**Corollary 65.** Let \(I\) be an instance for MAX d-KP on \(n\) items such that \(I\) has an equivalent threshold graph. Then \(I\) can be solved in time \(O(d \cdot n^2)\).

Comparing the running \(O(d \cdot n^2)\) time with the result in [CH77] we obtain the same running time but we even can handle instances using positive integer valued sizes and capacities. Further \(O(d \cdot n^2)\) is no longer exponential within \(d\) as in Theorem 64 but only solves much more restricted MAX d-KP instances.

Theorem 62 allows to improve Theorem 64 for \(d = 2\) dimensions:

**Theorem 66.** Let \(I\) be an instance for MAX 2-KP on \(n\) items which has an equivalent graph. Then \(I\) can be solved in time \(O(n^3)\).

4.6 Counting and enumerating maximum independent sets

Since every maximum independent set is a maximal independent set, our results given in Section 4.5 also can be applied to list all maximum independent sets in \(k\)-threshold graphs. By omitting the last step of the method given in Figure 6 and removing non-maximum sets we obtain a method of running time \(O(\sum_{i=1}^k m_i + k \cdot n \cdot (\prod_{i=1}^k \omega(G_i))) \leq O(k \cdot n^{k+1})\) for listing all maximum independent sets in a \(k\)-threshold graph.

21
Corollary 67. All maximum independent sets in a \(k\)-threshold graph \(G\) on \(n\) vertices given by an edge set covering of \(k\) threshold graphs \(G_i = (V, E_i)\) on \(m_i\) edges, \(1 \leq i \leq k\), can be enumerated and counted in time \(O(\sum_{i=1}^{k} m_i + k \cdot n \cdot \left(\prod_{i=1}^{k} \omega(G_i)\right)) \subseteq O(k \cdot n^{k+1})\). The size of a maximum independent set, i.e. \(\alpha(G)\), in a \(k\)-threshold graph \(G\) can be computed in the same time.

The related problem of finding one independent set of maximum size in a \(k\)-threshold graph was solved in [CLR04] in time \(O(n \log n + n^{k-1})\). Comparing our solutions and those of [CLR04] we observe that we require graph representations and the authors of [CLR04] use the coefficients occurring in the multiple knapsack instance. Each of these versions can be transformed into the other by Lemma 43 and 52. Especially when we can bound the vertex degree of the threshold graphs (cf. Corollary 60) our results are much better.

4.7 Counting and enumerating independent sets

Next we show how to enumerate and count the maximal independent sets in a \(k\)-threshold graph \(G\) using these sets for \(k\) covering threshold graphs of \(G\).

\[
\begin{align*}
I(G) & = \emptyset; \\
\text{for } (i = 1; i \leq k; i + +) & \quad \text{compute } I(G_i) \quad \triangleright \text{see Figure}\ [5] \\
\text{for each } (M_1, \ldots, M_k) & \in I(G_1) \times \cdots \times I(G_k) \\
I(G) & = I(G) \cup \{M_1 \cap \cdots \cap M_k\} \quad \text{compute } I(G) = \{M_1 \cap \cdots \cap M_k \mid M_i \in I(G_i)\}
\end{align*}
\]

Figure 9: Enumerating all independent sets in a \(k\)-threshold graph.

Corollary 68. All independent sets in a \(k\)-threshold graph \(G\) on \(n\) vertices given by an edge set covering of \(k\) threshold graphs \(G_i = (V, E_i)\) on \(m_i\) edges, \(1 \leq i \leq k\), can be enumerated and counted in time \(O(k \cdot n \cdot 2^{n-k})\).

Proof. Let \(G\) be a \(k\)-threshold graph on \(n\) vertices. Further let \(G_i = (V, E_i)\), \(1 \leq i \leq k\), be a covering by \(k\) threshold graphs for \(G\). By the method given in Figure 9 we generate all independent sets in \(G\). The running time for computing \(I(G_i)\) for \(1 \leq i \leq k\) can be bounded using Corollary 60 by \(O(k \cdot n \cdot 2^{n-1})\). Since every family \(I(G_i)\) consists of at most \(2^n\) sets, there are at most \(2^{n-k}\) tuples \((M_1, \ldots, M_k)\) in \(I(G_1) \times \cdots \times I(G_k)\). For every tuple \((M_1, \ldots, M_k)\) the intersection \(M_1 \cap \cdots \cap M_k\) can be computed in time \(O(k \cdot n)\) (cf. proof of Theorem 50) and the set of all intersections \(\mathcal{I}\) can be computed in time \(O(k \cdot n \cdot 2^{n-k})\). Identical sets can be observed by a suitable data structure. The overall running time can be bounded by \(O(k \cdot n \cdot 2^{n-k})\).

4.8 \(d\)-dimensional Vector Packing

Next we consider the generalization of Min BP where every item \(a_j\) corresponds to a \(d\)-dimensional vector \(s_j := (s_{1,j}, \ldots, s_{d,j})\) and every bin \(i\) to the unit \(d\)-dimensional vector \((1, \ldots, 1)\). A set of items \(A'\) fits into a bin if each component of the vector \(\sum_{a_j \in A'} s_{j}\) does not exceed 1.

Name: \textsc{Min }\(d\)-dimensional Vector Packing (\textsc{Min d-VP})

Instance: A set \(A = \{a_1, \ldots, a_n\}\) of \(n\) items, a number \(d\) of dimensions, and a number \(n\) of \(d\)-dimensional bins. Every item \(a_j\) has \(d\)-dimensional positive rational size vector \(s_j := (s_{1,j}, \ldots, s_{d,j})\).

Task: Find \(n\) disjoint (possibly empty) subsets \(A_1, \ldots, A_n\) of \(A\) such that the number of non-empty subsets is minimized and the items of each subset fit into a bin.

For some instance \(I\) for \textsc{Min d-VP} two items \(a_j\) and \(a_{j'}\) can be chosen into the same subset (bin) if and only if

\[
s_{1,j} + s_{1,j'} \leq 1 \land s_{2,j} + s_{2,j'} \leq 1 \land \ldots \land s_{d,j} + s_{d,j'} \leq 1.
\]
This also motivates to model the compatibleness of the items in \( I \) by \( d \)-threshold graphs. Therefor we consider for every dimension \( i, 1 \leq i \leq d \), the inequality
\[
s_{i,1} x_1 + s_{i,2} x_2 + \ldots + s_{i,n} x_n \leq 1. \tag{13}
\]

An instance \( I \) for Min D-VP and a graph \( G = (V, E) \) are equivalent, if there is a bijection \( f : A \to V \) such that for every \( A' \subseteq A \) the characteristic vector of \( A' \) satisfies for every dimension \( i, 1 \leq i \leq d \), inequality \( \textbf{[13]} \) if and only if \( f(A') := \{ f(a_j) \mid a_j \in A' \} \) is an independent set of \( G \).

If for every dimension \( i, 1 \leq i \leq d \), the Min BP instance defined by inequality \( \textbf{[13]} \) has an equivalent graph \( G_i = (V_i, E_i) \), see Section \( \textbf{[33]} \) we know that \( G_i \) is a threshold graph. Further two items \( a_j \) and \( a_{j'} \) can be chosen into the same bin if and only if
\[
\{v_j, v_{j'}\} \not\in E_1 \land \{v_j, v_{j'}\} \not\in E_2 \land \ldots \land \{v_j, v_{j'}\} \not\in E_d. \tag{14}
\]
Then \( G = (V, E) \) where \( E = E_1 \cup \ldots \cup E_d \) leads a \( d \)-threshold graph. By \( \textbf{[14]} \) two items \( a_j \) and \( a_{j'} \) can be chosen into the same bin if and only if there is no edge \( \{v_j, v_{j'}\} \) in \( G \). That is any two items corresponding to the vertices of a clique in \( G \) can not be chosen into the same bin. Thus \( \omega(G) \) leads a lower bound on the number of bins needed for the considered instance \( I \) for Min D-VP.

**Theorem 69.** Let \( I \) be an instance for Min D-VP such that for every dimension \( i \) instance \( I_i \) has an equivalent graph \( G_i = (V_i, E_i) \). Then graph \( G = (V, E_1 \cup \ldots \cup E_d) \) leads the bound \( \text{OPT}(I) \geq \omega(G) \).

In order to know an equivalent graph for \( I \) we can proceed as in Section \( \textbf{[22]} \) and use graph \( G_d(I) \) defined by the values of the \( d \) inequalities \( \textbf{[13]} \).

**Theorem 70.** Let \( I \) be an instance for Min D-VP such that for every dimension \( i \) instance \( I_i \) has an equivalent graph. Then \( \text{OPT}(I) \geq \omega(G_d(I)) \).

### 4.9 \( d \)-dimensional Bin Packing

Next we consider the generalization of Min BP where we have to pack items corresponding to \( d \)-dimensional parallelepipeds into bins corresponding to \( d \)-dimensional cubes of size 1 on every dimension. Each of the items \( a_j \) has for dimension \( i \) the size \( s_{i,j} \). The items have to be packed without overlapping and without rotations into the cubes, such that the faces of the items are parallel to those of the cubes.

**Name:** Min D-dimensional Bin Packing (Min D-BP)

**Instance:** A set \( A = \{ a_1, \ldots, a_n \} \) of \( n \) items, a number \( d \) of dimensions, and a number \( n \) of \( d \)-dimensional bins. Every item \( a_j \) has for dimension \( i \) the positive rational size \( s_{i,j} \).

**Task:** Find \( n \) disjoint (possibly empty) subsets \( A_1, \ldots, A_n \) of \( A \) such that the number of non-empty subsets is minimized and the items of each subset can be packed into a different \( d \)-dimensional cube.

For some instance \( I \) for Min D-BP two items \( a_j \) and \( a_{j'} \) can be chosen into the same subset (bin) if and only if
\[
s_{i_j,1} + s_{i_j,2} x_2 + \ldots + s_{i_j,n} x_n \leq 1. \tag{15}
\]

This motivates to model the compatibleness of the items in instance \( I \) by \( d \)-threshold graphs. Therefor we consider for every dimension \( i, 1 \leq i \leq d \), the inequality
\[
s_{i,1} x_1 + s_{i,2} x_2 + \ldots + s_{i,n} x_n \leq 1. \tag{16}
\]

An instance \( I \) for Min D-BP and a graph \( G = (V, E) \) are equivalent, if there is a bijection \( f : A \to V \) such that for every \( A' \subseteq A \) the characteristic vector of \( A' \) satisfies for every dimension \( i, 1 \leq i \leq d \), inequality \( \textbf{[16]} \) if and only if \( f(A') := \{ f(a_j) \mid a_j \in A' \} \) is an independent set of \( G \).

If for every dimension \( i, 1 \leq i \leq d \), the Min BP instance defined by inequality \( \textbf{[16]} \) has an equivalent graph \( G_i = (V_i, E_i) \), see Section \( \textbf{[33]} \) then we know that \( G_i \) is a threshold graph. Further two items \( a_j \) and \( a_{j'} \) can be chosen into the same bin if and only if
\[
\{v_j, v_{j'}\} \not\in E_1 \lor \{v_j, v_{j'}\} \not\in E_2 \lor \ldots \lor \{v_j, v_{j'}\} \not\in E_d. \tag{17}
\]
Then \( G = (V, E) \) where \( E = E_1 \cap \ldots \cap E_d \) leads a \( d \)-threshold intersection graph. By (17) two items \( a_j \) and \( a_{j'} \) can be chosen into the same bin if and only if there is no edge \( \{v_j, v_{j'}\} \) in \( G \). That is any two items corresponding to the vertices of a clique in \( G \) can not be chosen into the same bin. Thus \( \omega(G) \) leads a lower bound on the number of bins needed for the considered instance \( I \) for Min d-BP. The value \( \omega(G) \) can be computed in polynomial time from \( G \), see Section 5.

**Theorem 71.** Let \( I \) be an instance for Min d-BP such that for every dimension \( i \) instance \( I_i \) has an equivalent graph \( G_i = (V, E_i) \). Then graph \( G = (V, E_1 \cap \ldots \cap E_d) \) leads the bound OPT(\( I \)) \( \geq \omega(G) \).

In order to know a graph we can proceed similar to Section 4.4 and use graph \( G'_d(I) = (V(I), E(I)) \) defined by

\[
V(I) = \{v_j \mid a_j \in A\} \quad \text{and} \quad E(I) = \{\{v_j, v_{j'}\} \mid s_{1,j} + s_{1,j'} > c_1 \land \ldots \land s_{d,j} + s_{d,j'} > c_d\}
\]

(18) using the values of the \( d \) inequalities (16).

**Theorem 72.** Let \( I \) be an instance for Min d-BP such that for every dimension \( i \) instance \( I_i \) has an equivalent graph. Then OPT(\( I \)) \( \geq \omega(G'_d(I)) \).

## 5 Conclusions

We introduced methods to count and enumerate all maximal independent, all maximum independent sets, and all independent sets in threshold graphs and \( k \)-threshold graphs. These results allowed us to solve a large number of knapsack instances in polynomial time. Since we generate all maximal independent sets for our solutions we even can extend our results of Theorems 9 and 14 to generate all optimal solutions of the respective knapsack instances.

The related problem of counting and listing maximal cliques in a threshold graph \( G \) can be treated by considering maximal independent sets of the complement graph \( \overline{G} \) by Lemma 10 and \( \text{MC}(G) = \text{MIS}(\overline{G}) \). Using Observation 5 (2) the method shown in Figure 2 can be modified to enumerate all maximal independent sets in a threshold graph. We have to exchange the commands within the cases \( t_i = 1 \) and \( t_i = 0 \) and further we have to store a maximum clique for \( i = 1 \), since \( t_1 = 1 \) by our definition. This implies that the number of all maximal cliques in a threshold graph \( G \) equals \( \alpha(G) \). If \( G \) is a threshold graph \( G \) on \( n \) vertices which is given by a creation sequence, then all maximal cliques can be counted in time \( \mathcal{O}(n) \) and enumerated in time \( \mathcal{O}(\alpha(G) \cdot n) \) and in time \( \mathcal{O}(1) \) per output.

In Table 4 we survey our results on counting and enumerating special independent sets and cliques within threshold graphs and \( k \)-threshold graphs. Since threshold graphs are chordal (cf. Proposition 5) some results are known from [OUU08].

| \( \alpha(G) \) | \( \omega(G) \) | \( n \) \( m \) | Corollary 9 | \( \sum_{i=1}^{n} m_i + k \cdot n \cdot (\prod_{i=1}^{n} \omega(G_i)) \) | Corollary 52 | \( \sum_{i=1}^{n} m_i + n \cdot (\prod_{i=1}^{n} \omega(G_i)) \) \( \sum_{i=1}^{n} m_i + k \cdot n \cdot (\prod_{i=1}^{n} \omega(G_i)) \) | Corollary 52 |
|---|---|---|---|---|---|---|---|
| \( \omega(G) \) | \( n \) \( m \) | Corollary 68 | Corollary 67 | \( \omega(G) \) by \( \alpha(G) \) \( \omega(G) \) by \( \alpha(G) \) | Corollary 68 |
| \( \omega(G) \) | \( n \) \( m \) | Corollary 68 | Corollary 67 | \( \omega(G) \) by \( \alpha(G) \) \( \omega(G) \) by \( \alpha(G) \) | Corollary 68 |
| \( \omega(G) \) | \( n \) \( m \) | Corollary 68 | Corollary 67 | \( \omega(G) \) by \( \alpha(G) \) \( \omega(G) \) by \( \alpha(G) \) | Corollary 68 |
| \( \omega(G) \) | \( n \) \( m \) | Corollary 68 | Corollary 67 | \( \omega(G) \) by \( \alpha(G) \) \( \omega(G) \) by \( \alpha(G) \) | Corollary 68 |
| \( \omega(G) \) | \( n \) \( m \) | Corollary 68 | Corollary 67 | \( \omega(G) \) by \( \alpha(G) \) \( \omega(G) \) by \( \alpha(G) \) | Corollary 68 |

Table 4: Summary of results for counting and enumerating problems related to \( k \)-threshold graphs on \( n \) vertices and \( m \) edges. Running times for enumeration algorithms for threshold graphs are given in time per output.

The structure of \( k \)-threshold graphs has shown to be very useful for listing maximal independent sets. In a very similar way the structure of \( k \)-threshold intersection graphs, i.e. graphs whose
threshold intersection dimension is at most $k$, can be used to list maximal cliques. Therefore we just have to compute $\text{MC}(G_i)$ instead of $\text{MIS}(G_i)$ of the involved threshold graphs $G_i$ and combine them using the method shown in Figure 6. Thus all maximal cliques in a $k$-threshold intersection graph $G$ on $n$ vertices whose edge set is the intersection of those of $k$ threshold graphs $G_i = (V, E_i)$ on $m_i$ edges, $1 \leq i \leq k$, can be enumerated and counted in time $O(\sum_{i=1}^{k} m_i + n \cdot (\prod_{i=1}^{k} \alpha(G_i))^2) \subseteq O(n^{2k+1})$. Furthermore we know that $G$ has at most $\prod_{i=1}^{k} \alpha(G_i)$ maximal cliques.

The arguments given in Section 4.6 lead a method of running time $O(\sum_{i=1}^{k} m_i + k \cdot n \cdot (\prod_{i=1}^{k} \alpha(G_i))) \subseteq O(k \cdot n^{k+1})$ for listing all maximum independent sets in a $k$-threshold intersection graph. The size of a maximum clique, i.e. $\omega(G)$, in a $k$-threshold intersection graph $G$ can be computed in the same time.

6 Acknowledgements

The work of the second author was supported by the German Research Association (DFG) grant GU 970/7-1.

References

[AGRY16] C. Albrecht, F. Gurski, J. Rethmann, and E. Yilmaz. Knapsack Problems: A Parameterized Point of View. ACM Computing Research Repository (CoRR), abs/1611.07724:27 pages, 2016.

[BLS99] A. Brandstädt, V.B. Le, and J.P. Spinrad. Graph Classes: A Survey. SIAM Monographs on Discrete Mathematics and Applications. SIAM, Philadelphia, 1999.

[Bra96] A. Brandstädt. Partitions of graphs into one or two independent sets and cliques. Discrete Mathematics, 152:47–54, 1996.

[CH73] V. Chvátal and P.L. Hammer. Set-packing and threshold graphs. Technical Report CORR 73-21, Comp. Sci. Dept., Univ. of Waterloo, 1973.

[CH77] V. Chvátal and P.L. Hammer. Aggregation of inequalities in integer programming. Annals of Discrete Math., 1:145–162, 1977.

[CLR04] A. Caprara, A. Lodi, and R. Rizzi. On $d$-threshold graphs and $d$-dimensional bin packing. Networks, 44(4):266–280, 2004.

[Fré04] A. Fréville. The multidimensional 0-1 knapsack problem: An overview. European Journal of Operational Research, 155:1–21, 2004.

[Gol80] M.C. Golumbic. Algorithmic Graph Theory and Perfect Graphs. Academic Press, 1980.

[GR17] F. Gurski and C. Rehs. A graph theoretic approach to solve special knapsack problems in polynomial time (Abstract). International Conference on Operations Research (OR 2017), 2017.

[HIS78] P.L. Hammer, T. Ibaraki, and B. Simeone. Degree sequences of threshold graphs. Congressus Numerantium, 21:329–355, 1978.

[HK07] P. Hegavernes and D. Kratsch. Linear-time certifying recognition algorithms and forbidden induced subgraphs. Nord. J. Comput., 14(1-2):87–108, 2007.

[HS81] P.L. Hammer and B. Simeone. The splittance of a graph. Combinatorica, 1(3):275–284, 1981.

[HSS06] A. Hagberg, P.J. Swart, and D.A. Schult. Designing threshold networks with given structural and dynamical properties. Phys. Rev. E, 056116, 2006.

[KPP10] H. Kellerer, U. Pferschy, and D. Pisinger. Knapsack Problems. Springer-Verlag, Berlin, 2010.
[Leu84] J. Y.-T. Leung. Fast algorithms for generating all maximal independent sets of interval, circular-arc and chordal graphs. *Journal of Algorithms*, 5(1):22–35, 1984.

[MM65] J. Moon and L. Moser. On cliques in graphs. *Israel Journal of Mathematics*, 3:23–28, 1965.

[MP95] N.V.R. Mahadev and U.N. Peled. *Threshold Graphs and Related Topics*. Annals of Discrete Math. 56. Elsevier, North-Holland, 1995.

[Orl77] J. Orlin. The minimal integral separator of a threshold graph. In B.H. Korte P.L. Hammer, E.L. Johnson and G.L. Nemhauser, editors, *Studies in Integer Programming*, volume 1 of *Annals of Discrete Mathematics*, pages 415 – 419. Elsevier, 1977.

[OUU08] Y. Okamoto, T. Uno, and R. Uehara. Counting the number of independent sets in chordal graphs. *Journal of Discrete Algorithms*, 6(2):229–242, 2008.

[PS09] U. Pferschy and J. Schauer. The knapsack problem with conflict graphs. *Journal of Graph Algorithms and Applications*, 13(2):233–249, 2009.

[PU59] M. Paull and S. Unger. Minimizing the number of states in incompletely specified state machines. *IRE Transactions on Electronic Computers*, EC-8:356–367, 1959.

[Riv85] I. Rival, editor. *Graphs and Order: The Role of Graphs in the Theory of Ordered Sets and Its Applications*. Nato Science Series C: (Book 147). Springer, 1985.

[Rob97] T. Robinson. Knapsack graphs. *New Zealand Journal of Mathematics*, 26:107–123, 1997.

[RRa89] J. Reitermann, V. Rödl, and E. Šiňajová. Geometrical embeddings of graphs. *Discrete Mathematics*, 74:291–319, 1989.

[SR98] A. Sterbini and T. Raschle. An $O(n^3)$ time algorithm for recognizing threshold dimension 2 graphs. *Information Processing Letters*, 67(5):255–259, 1998.

[Vad01] S.P. Vadhan. The complexity of counting in sparse, regular, and planar graphs. *SIAM Journal on Computing*, 31(2):398–427, 2001.

[Val79] L.G. Valiant. The complexity of enumeration and reliability problems. *SIAM Journal on Computing*, 8(3):410–421, 1979.

[Yan82] M. Yannakakis. The complexity of the partial order dimension problem. *SIAM J. Algebraic Methods*, 3(3):351–358, 1982.

[ZVI04] C. Ortiz Z. and M. Villameca-Ilhu. Difficult problems in threshold graphs. *Electronic Notes in Discrete Mathematics*, 18:187 – 192, 2004.