Languages ordered by the subword order

Dietrich Kuske
TU Ilmenau, Germany
dietrich.kuske@tu-ilmenau.de

Georg Zetzsche
MPI-SWS, Germany
georg@mpi-sws.org

Abstract
We consider a language together with the subword relation, the cover relation, and regular predicates. For such structures, we consider the extension of first-order logic by threshold- and modulo-counting quantifiers. Depending on the language, the used predicates, and the fragment of the logic, we determine four new combinations that yield decidable theories. These results extend earlier ones where only the language of all words without the cover relation and fragments of first-order logic were considered.

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1 Introduction

The subword relation (sometimes called scattered subword relation) is one of the simplest nontrivial examples of a well-quasi ordering [6]. This property allows its prominent use in the verification of infinite state systems [3]. The subword relation can be understood as embeddability of one word into another. This embeddability relation has been considered for other classes of structures like trees, posets, semilattices, lattices, graphs etc. [12, 14, 13, 7, 8, 9, 10, 19, 20].

In this paper, we study logical properties of a set of words ordered by the subword relation. We are mainly interested in general situations where we get a decidable logical theory. Regarding first-order logic, we already have a rather precise picture about the border between decidable and undecidable fragments: For the subword order alone, the $\exists^*$-theory is decidable [15] and the $\exists^*\forall^*$-theory is undecidable [11]. For the subword order together with regular predicates, the two-variable theory is decidable [11] and the three-variable theory [11] as well as the $\exists^*$-theory are undecidable [5] (these two undecidabilities already hold if we only consider singleton predicates, i.e., constants).

Thus, to get a decidable theory, one has to restrict the expressiveness of first-order logic considerably. For instance, neither in the $\exists^*$- nor in the two-variable fragment of first-order logic, one can express the cover relation $\sqsubseteq$ (i.e., “$u$ is a proper subword of $v$ and there is no word properly between these two”). As another example, one cannot express threshold properties like “there are at most $k$ subwords with a given property” in any of these two logics (for $k > 2$).

In this paper, we refine the analysis of logical properties of the subword order in three aspects:

- We restrict the universe from the set of all words to a given language $L$.
- Besides the subword order, we also consider the cover relation $\sqsubseteq$.
- We add threshold and modulo counting quantifiers to the logic.
Also as before, we may or may not add regular predicates or constants to the structure. In other words, we consider reducts of the structure

\[(L, \subseteq, \sqcap, (K \cap L)_{K \text{ regular}}, (w)_{w \in L})\]

with \(L\) some language and fragments of the logic C+MOD that extends first-order logic by threshold- and modulo-counting quantifiers.

In this spectrum, we identify four new cases of decidable theories:

1. The C+MOD-theory of the whole structure is decidable provided \(L\) is bounded and context-free (Theorem 4). A rather special case of this result follows from [5, Theorem 4.1]: If \(L = (a_1^* a_2^* \cdots a_m^*)^k\) and if only regular predicates of the form \((a_1^* a_2^* \cdots a_m)^k\) are used, then the FO-theory is decidable (with \(\Sigma = \{a_1, \ldots, a_m\}\)).

2. The C+MOD^2-theory (i.e., the 2-variable-fragment of the C+MOD-theory) of the whole structure is decidable whenever \(L\) is regular (Corollary 17). The decidability of the FO^2-theory without the cover relation is [11, Theorem 5.5].

3. The \(\Sigma_1\)-theory of the structure \((L, \subseteq)\) is decidable provided \(L\) is regular (Theorem 18). For \(L = \Sigma^*\), this is [15, Prop. 2.2].

4. The \(\Sigma_1\)-theory of the structure \((L, \subseteq, (w)_{w \in L})\) is decidable provided \(L\) is regular and almost every word from \(L\) contains a non-negligible number of occurrences of every letter (see below for a precise definition, Theorem 25). Note that, by [5, Theorem 3.3], this theory is undecidable if \(L = \Sigma^*\).

Our first result is shown by an interpretation of the structure in \((\mathbb{N}, +)\). Four ingredients are essential here: Parikh’s theorem [10], the rationality of the subword relation [11], Nivat’s theorem characterising rational relations [2], and the decidability of the C+MOD-theory of \((\mathbb{N}, +)\) [1, 18, 4] (note that this decidability does not follow directly from Presburger’s result since in his logic, one cannot make statements like “the number of witnesses \(x \in \mathbb{N}\) satisfying . . . is even”).

Our second result extends a result from [11] that shows decidability of the FO^2-theory of the structure \((\Sigma^*, \subseteq, (L)_{L \text{ regular}})\). They provide a quantifier elimination procedure which relies on two facts:

- The class of regular languages is closed under images of rational relations.
- The proper subword relation and the incomparability relation are rational.

Here, we follow a similar proof strategy. But while that proof had to handle the existential quantifier, only, we also have to deal with counting quantifiers. This requires us to develop a theory of counting images under rational relations, e.g., the set of words \(u\) such that there are at least two words \(v\) in the regular language \(K\) with \((u, v)\) in the rational relation \(R\). We show that the class of regular languages is closed under such counting images provided the rational relation \(R\) is unambiguous, a proof that makes heavy use of weighted automata [17].

To apply this to the subword and the cover relation, this also requires us to show that the proper subword, the cover, and the incomparability relations are unambiguous rational.

Our third result extends the decidability of the \(\Sigma_1\)-theory of \((\Sigma^*, \subseteq)\) from [15]. The main point there was to prove that every finite partial order can be embedded into \((\Sigma^*, \subseteq)\) if \(|\Sigma| \geq 2\). This is certainly false if we restrict the universe, e.g., to \(L = a^*\). However, such bounded regular languages are already covered by the first result, so we only have to handle unbounded regular languages \(L\). In that case, we prove nontrivial combinatorial results regarding primitive words in regular languages and prefix-maximal subwords. These considerations then allow us to prove that, indeed, every finite partial order embeds into \((L, \subseteq)\). Then, the decidability of the \(\Sigma_1\)-theory follows as in [15].
Regarding our fourth result, we know from [11] that decidability of the $\Sigma_1$-theory of $(L, \subseteq, (w)_{w \in L})$ does not hold for every regular $L$. Therefore, we require that a certain fraction of the positions in a word carries the letter $a$ (for almost all words from the language and for all letters). This allows us to conclude that every finite partial order embeds into $(L, \subseteq)$ above each word. The second ingredient is that, for such languages, any $\Sigma_1$-sentence is effectively equivalent to such a sentence where constants are only used to express that all variables take values above a certain word $v$. These two properties together with some combinatorial arguments from the theory of well-quasi orders then yield the decidability.

In summary, we identify four classes of decidable theories related to the subword order. In this paper, we concentrate on these positive results, i.e., we did not try to find new undecidable theories. It would, in particular, be nice to understand what properties of the regular language $L$ determine the decidability of the $\Sigma_1$-theory of the structure $(L, \subseteq, (w)_{w \in L})$ (it is undecidable for $L = \Sigma^* \ [12]$ and decidable for, e.g., $L = \{ab, baa\}^* \cup bb\{abb\}^*$ by our third result). Another open question concerns the complexity of our decidability results.

2 Preliminaries

Throughout this paper, let $\Sigma$ be some alphabet. A word $u = a_1 a_2 \ldots a_m$ with $a_1, a_2, \ldots, a_m \in \Sigma$ is a subword of a word $v \in \Sigma^*$ if there are words $v_0, v_1, \ldots, v_m \in \Sigma^*$ with $v = v_0 a_1 v_1 a_2 v_2 \ldots a_m v_m$.
In that case, we write $u \sqsubseteq v$; if, in addition, $u \neq v$, then we write $u \subset v$ and call $u$ a proper subword of $v$. If $u, w \in \Sigma^*$ such that $u \subseteq w$ and there is no word $v$ with $u \subset v \subset w$, then we say that $w$ is a cover of $u$ and write $u \sqsupseteq w$. This is equivalent to saying $u \subset w$ and $|u| + 1 = |w|$ where $|u|$ is the length of the word $u$. If, for two words $u$ and $v$, neither $u$ is a subword of $v$ nor vice versa, then the words $u$ and $v$ are incomparable and we write $u \parallel v$.

For instance, $aa \sqsubseteq babba$, $aa \sqsubseteq aba$, and $aba \parallel aab$.

Let $S = (L, (R_i)_{i \in I}, (w_j)_{j \in J})$ be a structure, i.e., $L$ is a set, $R_i \subseteq L^{n_i}$ is a relation of arity $n_i$ (for all $i \in I$), and $w_j \in L$ for all $j \in J$. Then, formulas of the logic $C+\text{MOD}$ are built from the atomic formulas

- $s = t$ for $s, t$ variables or constants $w_j$ and
- $R_i(s_1, s_2, \ldots, s_{n_i})$ for $i \in I$ and $s_1, s_2, \ldots, s_{n_i}$ variables or constants $w_j$

by the following formation rules:

1. If $\alpha$ and $\beta$ are formulas, then so are $\neg \alpha$ and $\alpha \land \beta$.
2. If $\alpha$ is a formula and $x$ a variable, then $\exists x \alpha$ is a formula.
3. If $\alpha$ is a formula, $x$ a variable, and $k \in \mathbb{N}$, then $\exists^{\geq k} x \alpha$ is a formula.
4. If $\alpha$ is a formula, $x$ a variable, and $p, q \in \mathbb{N}$ with $p < q$, then $\exists^{\text{mod } q} x \alpha$ is a formula.

We call $\exists^{\geq k}$ a threshold counting quantifier and $\exists^{\text{mod } q} x \alpha$ a modulo counting quantifier. The semantics of these quantifiers is defined as follows:

- $S \models \exists^{\geq k} x \alpha$ iff $|\{w \in L \mid S \models \alpha(w)\}| \geq k$
- $S \models \exists^{\text{mod } q} x \alpha$ iff $|\{w \in L \mid S \models \alpha(w)\}| \in p + q\mathbb{N}$

For instance, $\exists^{\text{mod } 2} x \alpha$ expresses that the number of elements of the structure satisfying $\alpha$ is even. Then $(\exists^{\text{mod } 2} x \alpha) \lor (\exists^{\text{mod } 2} x \alpha)$ holds iff only finitely many elements of the structure satisfy $\alpha$. The fragment $\text{FO}^{\text{mod}}\text{MOD}$ of $\text{C+MOD}$ comprises all formulas not containing any threshold counting quantifier $\exists^{\geq k}$. First-order logic FO is the set of formulas from C+MOD not mentioning any counting quantifier, i.e., neither $\exists^{\geq k}$ nor $\exists^{\text{mod } q} x \alpha$. Let $\Sigma_i$ denote the set of first-order formulas of the form $\exists x_1 \exists x_2 \ldots \exists x_n : \psi$ where $\psi$ is quantifier-free; these formulas are also called existential.
Note that the formulas
\[ \exists^2 x_1 x = x \] (1)
and
\[ \exists x_1 \exists x_2 \ldots \exists x_k : \bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \land \bigwedge_{1 \leq i \leq k} x_i = x_i \] (2)
are equivalent (they both express that the structure contains at least \( k \) elements). Generalising this, the threshold quantifier \( \exists^k \) can be expressed using the existential quantifier, only. Consequently, the logics \( \text{FO}+\text{MOD} \) and \( \text{C}+\text{MOD} \) are equally expressive. The situation changes when we restrict the number of variables that can be used in a formula: Let \( \text{FO}^2 \) and \( \text{C}+\text{MOD}^2 \) denote the set of formulas from \( \text{FO} \) and \( \text{C}+\text{MOD} \), respectively, that use the variables \( x \) and \( y \), only. Note that the formula from (1) belongs to \( \text{C}+\text{MOD}^2 \), but the equivalent formula from (2) does not belong to \( \text{FO}+\text{MOD}^2 \).

\begin{itemize}
\item \textbf{Remark.} Let \( L_1 \) be a set with two elements and let \( L_2 \) be a set with \( k \cdot q + 2 \) elements (where \( k > 2 \)). Furthermore, let \( \varphi \in \text{C}+\text{MOD}^2 \) be a formula such that all moduli appearing in \( \varphi \) are divisors of \( q \). By induction on the construction of the formula \( \varphi \), one can show the following for any \( x, y \in L_1 \) and \( x', y' \in L_2 \):
\begin{itemize}
\item If \( x \neq y \) and \( x' \neq y' \), then \( L_1 \models \varphi(x, y) \iff L_2 \models \varphi(x', y') \).
\item \( L_1 \models \varphi(x, x) \iff L_2 \models \varphi(x', x') \).
\end{itemize}
Consequently, there is no \( \text{C}+\text{MOD}^2 \)-formula expressing that the number of elements of a structure is \( \geq k \).
\end{itemize}

In this paper, we will consider the following structures:
\begin{itemize}
\item The largest one is \( (L, \subseteq, (K \cap L)_L \text{regular}, (w)_{w \in L}) \) for some \( L \subseteq \Sigma^* \). The universe of this structure is the language \( L \), we have two binary predicates \( (\subseteq \text{ and } \sqsubseteq) \), a unary predicate \( K \cap L \) for every regular language \( L \), and we can use every word from \( L \) as a constant.
\item The other extreme is the structure \( (L, \subseteq) \) for some \( L \subseteq \Sigma^* \) where we consider only the binary predicate \( \subseteq \).
\item Finally, we will also prove results on the intermediate structure \( (L, \subseteq, (w)_{w \in L}) \) that has a binary relation and any word from the language as a constant.
\end{itemize}
For any structure \( S \) and any of the above logics \( \mathcal{L} \), we call
\[ \{ \varphi \mid \varphi \in \mathcal{L} \text{ sentence and } S \models \varphi \} \]
the \( \mathcal{L} \)-theory of \( S \).

A language \( L \subseteq \Sigma^* \) is \textit{bounded} if there are a number \( n \in \mathbb{N} \) and words \( w_1, w_2, \ldots, w_n \in \Sigma^* \) such that \( L \subseteq w_1^* w_2^* \cdots w_n^* \). Otherwise, it is \textit{unbounded}.

For an alphabet \( \Gamma \), a word \( w \in \Gamma^* \), and a letter \( a \in \Gamma \), let \( |w|_a \) denote the number of occurrences of the letter \( a \) in the word \( w \). The \textit{Parikh vector} of \( w \) is the tuple \( \Psi_\Gamma(w) = ([|w|_a])_{a \in \Gamma} \in \mathbb{N}_\Gamma^\ast \). Note that \( \Psi_\Gamma \) is a homomorphism from the free monoid \( \Gamma^\ast \) onto the additive monoid \( (\mathbb{N}_\Gamma^\ast, +) \).

## 3 The FO+MOD-theory with regular predicates

The aim of this section is to prove that the full FO+MOD-theory of the structure
\[ (L, \subseteq, (K \cap L)_K \text{regular}) \]
Lemma 1. Let $K \subseteq \Sigma^*$ be context-free, $w_1, \ldots, w_n \in \Sigma^*$, and $g : \mathbb{N}^n \to \Sigma^*$ be defined by $g(\overline{m}) = w_1^{m_1} w_2^{m_2} \cdots w_n^{m_n}$ for all $\overline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$. The set $g^{-1}(K) = \{ \overline{m} \in \mathbb{N}^n \mid g(\overline{m}) \in K \}$ is effectively semilinear.

Proof. Let $\Gamma = \{a_1, a_2, \ldots, a_n\}$ be an alphabet and define the monoid homomorphism $f : \Gamma^* \to \Sigma^*$ by $f(a_i) = w_i$ for all $i \in [1, n]$.

Since the class of context-free languages is effectively closed under inverse homomorphisms, the language $K_1 = f^{-1}(K) = \{ u \in \Gamma^* \mid f(u) \in K \}$ is effectively context-free. Since $a_1^* a_2^* \cdots a_n^*$ is regular, also the language $K_2 = K_1 \cap a_1^* a_2^* \cdots a_n^*$ is effectively context-free. By Parikh’s theorem [14], the Parikh-image $\Psi_f(K_2) \subseteq \mathbb{N}^n$ of this intersection is effectively semilinear.

Now let $\overline{m} \in \mathbb{N}^n$. Then $\overline{m} \in \Psi_f(K_2)$ iff there exists a word $u \in K_1 \cap a_1^* a_2^* \cdots a_n^*$ with Parikh image $\overline{m}$. But the only word from $a_1^* a_2^* \cdots a_n^*$ with this Parikh image is $a_1^m a_2^{m_2} \cdots a_n^{m_n}$, i.e., $\overline{m} \in \Psi_f(K_2)$ iff $a_1^m a_2^{m_2} \cdots a_n^{m_n} \in K_1$. Since $f(a_1^m a_2^{m_2} \cdots a_n^{m_n}) = g(\overline{m})$, this is equivalent to $g(\overline{m}) \in K$. Thus, the semilinear set $\Psi_f(K_2)$ equals the set $g^{-1}(K)$ from the lemma.

Lemma 2. Let $w_1, \ldots, w_n \in \Sigma^*$ and $g : \mathbb{N}^n \to \Sigma^*$ be defined by $g(\overline{m}) = w_1^{m_1} w_2^{m_2} \cdots w_n^{m_n}$ for all $\overline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$. The set $\{ (\overline{m}, \overline{n}) \in \mathbb{N}^n \times \mathbb{N}^n \mid g(\overline{m}) \subsetneq g(\overline{n}) \}$ is semilinear.

Proof. Let $\Gamma = \{a_1, a_2, \ldots, a_n\}$ be an alphabet and define the monoid homomorphism $f : \Gamma^* \to \Sigma^*$ by $f(a_i) = w_i$ for all $i \in [1, n]$.

In this proof, we construct an alphabet $\Delta$ and homomorphisms $g$, $h_1$, $h_2$, $p_1$, and $p_2$, such that the diagrams (for $i \in \{1, 2\}$) from Fig. 1 commute. In addition, we will construct a regular language $R \subseteq \Delta^*$ with $U \subseteq V \iff \exists w \in R : U = f \circ h_1(w)$ and $V = f \circ h_2(w)$ for all $U, V \in \Sigma^*$.

The subword relation $S = \{(U, V) \in \Sigma^* \times \Sigma^* \mid U \subseteq V \}$ on $\Sigma^*$ is rational [11]. Since the class of rational relations is closed under inverse homomorphisms [2], also the relation $S_1 = \{(u, v) \in \Gamma^* \times \Gamma^* \mid f(u) \subsetneq f(v) \}$ is rational. While the class of rational relations is not closed under intersections, it is at least closed under intersections with direct products of regular languages. Hence also $S_2 = \{(u, v) \mid u, v \in a_1^* a_2^* \cdots a_n^*, f(v) \subseteq f(v) \}$
is rational. By Nivat’s theorem [2], there are a regular language $R$ over some alphabet $\Delta$ and two homomorphisms $h_1, h_2: \Delta^* \to \Gamma^*$ with

$S_2 = \{(h_1(w), h_2(w)) \mid w \in R\}$.

Since $R$ is regular, its Parikh-image

$\Psi_{\Delta}(R) = \{\Psi_{\Delta}(w) \mid w \in R\}$

is semilinear [16].

Note that the additive monoid $(\mathbb{N}_{\Delta}, +)$ is the commutative monoid freely generated by the vectors $\Psi_{\Delta}(b) = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $b \in \Delta$. Since also $(\mathbb{N}^n, +)$ is commutative, we can define monoid homomorphisms $p_1, p_2: \mathbb{N}_\Delta \to \mathbb{N}^n$ by $p_1(\Psi_{\Delta}(b)) = \Psi_{\Gamma}(h_1(b))$ and $p_2(\Psi_{\Delta}(b)) = \Psi_{\Gamma}(h_2(b))$ for $b \in \Delta$. Then also $(p_1, p_2): \mathbb{N}_\Delta \to \mathbb{N}^n \times \mathbb{N}^n: \mathcal{X} \mapsto (p_1(\mathcal{X}), p_2(\mathcal{X}))$ is a monoid homomorphism.

Let $w = b_1 b_2 \cdots b_m$ with $b_i \in \Delta$ for all $1 \leq i \leq m$. Then we have

$\Psi_{\Gamma}(h_1(w)) = \Psi_{\Gamma}(h_1(b_1 b_2 \cdots b_m))
= \Psi_{\Gamma}(h_1(b_1)) h_1(b_2) \cdots h_1(b_m))
= \sum_{1 \leq j \leq m} \Psi_{\Gamma}(h_1(b_j))
= \sum_{1 \leq j \leq m} p_1(\Psi_{\Delta}(b_j))
= p_1(\sum_{1 \leq j \leq m} \Psi_{\Delta}(b_j))
= p_1(\Psi_{\Delta}(w))$

and similarly $\Psi_{\Gamma}(h_2(w)) = p_2(\Psi_{\Delta}(w))$.

Since the class of semilinear sets is closed under monoid homomorphisms, the image of the semilinear set $\Psi_{\Delta}(R)$ under $(p_1, p_2)$, i.e.,

$H = \{(p_1(\Psi_{\Delta}(w)), p_2(\Psi_{\Delta}(w))) \mid w \in R\}$

is semilinear.
Lemma 3. Let \( m, n \in \mathbb{N}^n \). Then \( (\overline{m}, \overline{n}) \in H \) iff there exists \( w \in R \) with \( \overline{m} = p_1(\Psi_H(w)) = \Psi_T(h_1(w)) \) and \( \overline{n} = p_2(\Psi_H(w)) = \Psi_T(h_2(w)) \). The existence of such a word \( w \in R \) is equivalent to the existence of a pair \((u, v) \in S_2\) with \( \overline{m} = \Psi_T(u) \) and \( \overline{n} = \Psi_T(v) \).

Since all words appearing in \( S_2 \) belong to \( a_1^n a_2^n \cdots a_m^n \), this last statement is equivalent to saying
\[
(a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}, a_1^{n_1} a_2^{n_2} \cdots a_n^{n_n}) \in S_2.
\]

But this is equivalent to saying
\[
f(a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}) \subseteq f(a_1^{n_1} a_2^{n_2} \cdots a_n^{n_n}).
\]

Note that \( g(\overline{m}) = f(a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}) \) and similarly \( g(\overline{n}) = f(a_1^{n_1} a_2^{n_2} \cdots a_n^{n_n}) \). Thus, the last claim is equivalent to \( g(\overline{m}) \subseteq g(\overline{n}) \).

In summary, we showed that the semilinear set \( H \) is the set from the lemma.◼

Lemma 3. Let \( w_1, w_2, \ldots, w_n \in \Sigma^* \), \( L \subseteq w_1^* w_2^* \cdots w_n^* \) be context-free, and \( g: \mathbb{N}^n \rightarrow \Sigma^* \) be defined by \( g(\overline{m}) = w_1^{m_1} w_2^{m_2} \cdots w_n^{m_n} \) for every tuple \( \overline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n \). Then there exists a semilinear set \( U \subseteq \mathbb{N}^n \) such that \( g \) maps \( U \) bijectively onto \( L \).

Proof. By Lemma 1 the set \( g^{-1}(L) \) is semilinear and, by its definition, \( g(g^{-1}(L)) = L \cap w_1^* w_2^* \cdots w_n^* = L \).

Let \( T \) denote the semilinear set from Lemma 2.

Then let \( U \) denote the set of \( n \)-tuples \( \overline{m} \in g^{-1}(L) \) such that the following holds for all \( \overline{m} \in \mathbb{N}^n \):

If \( (\overline{m}, \overline{n}) \in T \) and \( (\overline{m}, \overline{m}) \in T \), then \( \overline{m} \) is lexicographically smaller than or equal to \( \overline{n} \).

This set \( U \) is semilinear since the class of semilinear relations is closed under first-order definitions. Now let \( u \in L \). Since \( g \) maps \( g^{-1}(L) \) onto \( L \), there is \( \overline{m} \in g^{-1}(L) \) with \( g(\overline{m}) = u \).

Since, on \( g^{-1}(L) \subseteq \mathbb{N}^n \), the lexicographic order is a well-order, there is a lexicographically minimal such tuple \( \overline{m} \). This tuple belongs to \( U \) and it is the only tuple from \( U \) mapped to \( u \).◼

Now we can prove the main result of this section.

Theorem 4. Let \( L \subseteq \Sigma^* \) be context-free and bounded. Then the FO+MOD-theory of \( S = (L, \subseteq, (K \cap L)_K \text{ regular}) \) is decidable.

Proof. Since \( L \) is bounded, there are words \( w_1, w_2, \ldots, w_n \in \Sigma^* \) such that \( L \subseteq w_1^* w_2^* \cdots w_n^* \).

For an \( n \)-tuple \( \overline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n \) we define \( g(\overline{m}) = w_1^{m_1} w_2^{m_2} \cdots w_n^{m_n} \in \Sigma^* \).

1. By Lemma 3, there is a semilinear set \( U \subseteq \mathbb{N}^n \) that is mapped by \( g \) bijectively onto \( L \).

   From this semilinear set, we obtain a first-order formula \( \lambda(\overline{m}) \) in the language of \( (\mathbb{N}, +) \) such that, for any \( \overline{m} \in \mathbb{N}^n \), we have \( (\mathbb{N}, +) \models \lambda(\overline{m}) \iff \overline{m} \in U \).

2. The set \( \{ (\overline{m}, \overline{m}) \mid g(\overline{m}) \subseteq g(\overline{m}) \} \) is semilinear by Lemma 2.

   From this semilinear set, we obtain a first-order formula \( \sigma(\overline{m}, g) \) in the language of \( (\mathbb{N}, +) \) such that \( (\mathbb{N}, +) \models \sigma(\overline{m}, g) \iff g(\overline{m}) \subseteq g(\overline{m}) \).

3. For any regular language \( K \subseteq \Sigma^* \) the set \( \{ \overline{m} \in \mathbb{N}^n \mid g(\overline{m}) \in K \} \subseteq \mathbb{N}^n \) is effectively semilinear by Lemma 1.

   From this semilinear set, we can compute a first-order formula \( \kappa_K(\overline{m}) \) in the language of \( (\mathbb{N}, +) \) such that \( (\mathbb{N}, +) \models \kappa_K(\overline{m}) \iff g(\overline{m}) \in K \).
Languages ordered by the subword order – long version

We now define, from an FO+MOD-formula \( \varphi(x_1, \ldots, x_k) \) in the language of \( S \), an FO+MOD-formula \( \varphi'(\overline{x_1}, \ldots, \overline{x_k}) \) in the language of \((\mathbb{N}, +)\) such that

\[
(\mathbb{N}, +) \models \varphi'(\overline{m_1}, \ldots, \overline{m_k}) \iff S \models \varphi(g(\overline{m_1}), \ldots, g(\overline{m_k})).
\]

If \( \varphi = (x \subseteq y) \), then \( \varphi' = \sigma(\overline{x, y}) \). If \( \varphi = (x \in K) \), then \( \varphi' = \kappa_K(\overline{x}) \). Furthermore, \((\alpha \wedge \beta)' = \alpha' \wedge \beta' \) and \((\neg \varphi)' = \neg \varphi' \).

For \( \varphi = \exists x : \psi \), we set \( \varphi' = \exists \overline{x_1} \exists x_2 \ldots \exists x_n : \lambda(x_1, x_2, \ldots, x_n) \wedge \psi' \).

Finally, let \( \varphi = \exists p \mod q : \psi \). Intuitively, one is tempted to set \( \varphi' = \exists p \mod q : \lambda(\overline{x}) \wedge \psi' \), but this is not a valid formula since \( \overline{x} \) is not a single variable, but a tuple of variables. To rectify this, we define FO+MOD-formulas \( \alpha^k_p \) for \( p \in [0, q - 1] \) and \( k \in [0, n - 1] \) as follows:

\[
\alpha^k_p(x_1, \ldots, x_k) = \begin{cases} 
\exists p \mod q x_{k+1} : \lambda(x_1, \ldots, x_k, x_{k+1}) \wedge \psi' & \text{if } k = n - 1 \\
\bigvee_{i=0}^{q-1} \exists f(i) \mod q x_{k+1} : \alpha^{k+1}_i(x_1, \ldots, x_k, x_{k+1}) & \text{otherwise}
\end{cases}
\]

where the disjunction \((*)\) extends over all functions \( f : \{0, 1, \ldots, q - 1\} \to \{0, 1, \ldots, q - 1\}\) with \( \sum_{0 \leq i < q} i \cdot f(i) \equiv p \mod q \).

By induction, one obtains

\[
(\mathbb{N}, +) \models \alpha^k_p(m_1, \ldots, m_k)
\]

iff

\[
\left\{(m_{k+1}, m_{k+2}, \ldots, m_n) \mid (\mathbb{N}, +) \models \lambda(\overline{m}) \wedge \psi'(\overline{m})\right\} \in p + q\mathbb{N}.
\]

Recall that \( g \) maps the tuples satisfying \( \lambda \) bijectively onto \( L \). Hence, the above is equivalent to

\[
\left\{w \in L \mid \exists m_{k+1}, m_{k+2}, \ldots, m_n \in \mathbb{N} : w = w_1^{m_1} \cdots w_n^{m_n} \in L \text{ and } S \models \psi(w)\right\} \in p + q\mathbb{N}.
\]

Setting \( \varphi' = \alpha^0_0 \) therefore solves the problem.

Consequently, any sentence \( \varphi \) from FO+MOD in the language of \( S \) is translated into an equivalent sentence \( \varphi' \) in the language of \((\mathbb{N}, +)\). By [11][13][4], validity of the sentence \( \varphi' \) in \((\mathbb{N}, +)\) is decidable.

4 The C+MOD\(^2\)-theory with regular predicates

By [11], the FO\(^2\)-theory of \((\Sigma^\ast, \subseteq, (L)_L\text{ regular})\) is decidable. This two-variable fragment of first-order logic has a restricted expressive power since, e.g., the following two properties cannot be expressed:

1. \( x \not\subseteq y = (x \not\subseteq y \land x \neq y \land \forall z : (x \subseteq z \subseteq y \rightarrow (x = z \lor z = y)) \).
2. \( \exists x_1, x_2, x_3 : \land_{1 \leq i \leq 3} x_i \neq x_j \land \land_{1 \leq i \leq 3} x_i \in L \).

To make the first property accessible, we add the cover relation to the structure. The logic C+MOD\(^2\) allows to express the second property with only two variables by \( \exists \exists^3 x : x \in L \) (in addition, it can express that the regular language \( L \) contains an even number of elements which is not expressible in first-order logic at all).

It is the aim of this section to show that the C+MOD\(^2\)-theory of the structure

\[
S = (\Sigma^\ast, \subseteq, (L)_L\text{ regular})
\]
is decidable. This decidability proof extends the proof from [11] for the decidability of the FO^{2}_{\Sigma_{L}}-theory of \((\Sigma_{L}^{*}, \subseteq, L)\). That proof provides a quantifier-elimination procedure that relies on two facts, namely

1. that the class of regular languages is closed under images under rational relations and
2. that the proper subword relation and the incomparability relation are rational.

Similarly, our more general result also provides a quantifier-elimination procedure that relies on the following extensions of these two properties:

1. The class of regular languages is closed under counting images under unambiguous rational relations (Section 4.2) and
2. the proper subword, the cover, and the incomparability relation are unambiguous rational (Section 4.1).

The actual quantifier-elimination is then presented in Section 4.3.

4.1 Unambiguous rational relations

Recall that, by Nivat’s theorem [2], a relation \(R \subseteq \Sigma^{*} \times \Sigma^{*}\) is rational if there exist an alphabet \(\Gamma\), a homomorphism \(h: \Gamma^{*} \rightarrow \Sigma^{*} \times \Sigma^{*}\), and a regular language \(S \subseteq \Gamma^{*}\) such that \(h\) maps \(S\) surjectively onto \(R\). We call \(R\) unambiguous rational relation if, in addition, \(h\) maps \(S\) injectively (and therefore bijectively) onto \(R\).

Example 5. The relations \(R_{1} = \{(a^{m}ba^{n}, a^{m}) \mid m, n \in \mathbb{N}\}\) and \(R_{2} = \{(a^{m}ba^{n}, a^{n}) \mid m, n \in \mathbb{N}\}\) are unambiguous rational: take \(\Gamma = \{x, y, z\}\), \(S = x^{*}yz^{*}\) and the homomorphisms

\[
x \mapsto (a, a) \quad y \mapsto (b, \varepsilon) \quad z \mapsto (a, \varepsilon)
\]

(for the first relation) and

\[
x \mapsto (a, \varepsilon) \quad y \mapsto (b, \varepsilon) \quad z \mapsto (a, a)
\]

Note that the intersection \(R_{1} \cap R_{2}\) is not even rational, while the union \(R = R_{1} \cup R_{2}\) is rational (since the union of rational language is always rational) [2]. But this union is not unambiguous rational: If it were unambiguous rational, then the set

\[
\{u \in a^{*}ba^{*} \mid \exists^{\geq 2}v: (u, v) \in R\} = \{a^{m}ba^{n} \mid m \neq n\}
\]

would be regular by Prop. 14 below.

Lemma 6. Let \(R_{1}, R_{2} \subseteq \Sigma^{*} \times \Sigma^{*}\) be unambiguous rational and disjoint. Then \(R_{1} \cup R_{2}\) is unambiguous rational.

Proof. There are disjoint alphabets \(\Gamma_{1} \text{ and } \Gamma_{2}\), regular languages \(S_{i} \subseteq \Gamma_{i}\) and homomorphisms \(h_{i}: \Gamma_{i}^{*} \rightarrow \Sigma^{*} \times \Sigma^{*}\) such that \(S_{i}\) is mapped bijectively onto \(R_{i}\). Let \(\Gamma = \Gamma_{1} \cup \Gamma_{2}\) and \(S = S_{1} \cup S_{2}\). Let the homomorphism \(h: \Gamma^{*} \rightarrow \Sigma^{*} \times \Sigma^{*}\) be given by

\[
h(a) = \begin{cases} h_{1}(a) & \text{if } a \in \Gamma_{1} \\ h_{2}(a) & \text{if } a \in \Gamma_{2} \end{cases}
\]

for \(a \in \Gamma\). Then \(h\) maps the regular language \(S\) bijectively onto \(R_{1} \cup R_{2}\).

Lemma 7. For any alphabet \(\Sigma\), the cover relation \(\subseteq\) and the relation \(\subseteq \setminus \subseteq\) are unambiguous rational.
Note that the set $R$ is unambiguous rational.

Proof. For $i \in \{1, 2\}$, let $\Sigma_i = \Sigma \times \{i\}$ and $\Gamma = \Sigma_1 \cup \Sigma_2$. Furthermore, let the homomorphism $\proj_i : \Gamma^* \rightarrow \Sigma^*$ be defined by $\proj_i(a, i) = a$ and $\proj_i(a, 3 - i) = \varepsilon$ for all $a \in \Sigma$. Finally, let the homomorphism $\proj : \Gamma^* \rightarrow \Sigma^* \times \Sigma^*$ be defined by $\proj(w) = (\proj_1(w), \proj_2(w))$.

Now consider the regular language
\begin{equation*}
\Sub = \left( \bigcup_{a \in \Sigma} \left( \{a, 2\} \right)^* \right)^* \Sigma_2^*.
\end{equation*}

Let $w \in \Sub$. Since any occurrence of a letter $(a, 1)$ in $w$ is immediately preceded by an occurrence of $(a, 2)$, we get $\proj_1(w) \subseteq \proj_2(w)$.

Conversely, if $u = a_1 a_2 \cdots a_n \subseteq v$, then $v$ can be written (uniquely) as $v = v_1 \cdot v_2 a_2 \cdots v_n a_n v_{n+1}$ such that $a_i$ does not occur in $v_i$ (for all $i \in \{1, 2, \ldots, n\}$). For $i \in [1, n+1]$ let $w_i \in \Gamma^*$ be the unique word from $\Sigma_2^*$ with $\proj_2(w_i) = v_i$. Then
\begin{equation*}
w = w_1(a_1, 2)(a_1, 1) w_2(a_2, 2)(a_2, 1) w_3(a_3, 2)(a_3, 1) \cdots w_n(a_n, 2)(a_n, 1) w_{n+1}
\end{equation*}

belongs to $\Sub$ and satisfies $\proj(w) = (u, v)$.

Since the factorization of $v$ is unique, we have that $\proj$ maps $\Sub$ bijectively onto the subword relation $\sqsubseteq$.

Let $S$ denote the intersection of $\Sub$ with $\{\Sigma_2 \Sigma_1\}^* \Sigma_2 \{\Sigma_2 \Sigma_1\}^*$, i.e., the regular language of words from $\Sub$ with precisely one more occurrence of letters from $\Sigma_2$ than from $\Sigma_1$. Then $S$ is mapped bijectively onto the relation $\sqsubseteq$, hence this relation is unambiguous rational.

Similarly, let $S'$ denote the regular language of all words from $\Sub$ with at least two more occurrences of letters from $\Sigma_2$ than from $\Sigma_1$. It is mapped bijectively onto the relation $\sqsubseteq \setminus \sqsubseteq$. Hence this relation is unambiguous rational.

Lemma 8. For any alphabet $\Sigma$, the incomparability relation
\begin{equation*}
\parallel = \{(u, v) \in \Sigma^* \times \Sigma^* \mid \text{neither } u \sqsubseteq v \text{ nor } v \sqsubseteq u\}
\end{equation*}
is unambiguous rational.

Proof. Note that the set $\parallel$ is the disjoint union of the following three relations:
1. $R_1 = \{(u, v) \mid |u| < |v| \text{ and not } u \sqsubseteq v\}$,
2. $R_2 = \{(u, v) \mid |u| = |v| \text{ and } u \neq v\}$, and
3. $R_3 = \{(u, v) \mid |u| > |v| \text{ and not } v \sqsubseteq u\}$.

As in the previous proof, let $\Sigma_i = \Sigma \times \{i\}$ and $\Gamma = \Sigma_1 \cup \Sigma_2$. Furthermore, let the homomorphism $\proj_i : \Gamma^* \rightarrow \Sigma^*$ be defined by $\proj_i(a, i) = a$ and $\proj_i(a, 3 - i) = \varepsilon$ for all $a \in \Sigma$.

We prove that the relations $R_1, R_2,$ and $R_3$ are all unambiguous rational. From Lemma 6, we then get that $R$ is unambiguous rational since it is the disjoint union of these three relations.

We start with the simple case: $R_2$. Consider the regular language
\begin{equation*}
\Inc_2 = (\Sigma_1 \Sigma_2)^* \cdot \{a, 1\}(b, 2) \cdot \{a, b \in \Sigma, a \neq b\} \cdot (\Sigma_1 \Sigma_2)^*.
\end{equation*}
This is the set of sequences of words of the form $(a, 1)(b, 2)$ such that, at least once, $a \neq b$. Hence, $\proj$ maps the regular language $\Inc_2$ bijectively onto $R_2$.

Next, we handle the relation $R_1 \cup R_2$. Correcting Lemma 5.2 slightly, we learn that $(u, v) \in R_1 \cup R_2$ if, and only if,
In the literature, a realizable function is often called recognizable formal power series. Since, in this paper, we will not encounter any operations on formal power series (like addition, Cauchy product etc), we use the (in this context) more intuitive notion of a “realizable function”.  

Example 9. Consider the rational relation $R = \{(a^kb^{\ell}, a^\ell) : k = m \text{ or } \ell = m\}$ and the regular language $L = \Sigma^*$. With $k = 2$, the language $\{a^kb^{\ell} : k \neq \ell\}$ equals the non-regular set $\{a^kb^{\ell} : k \neq \ell\}$. Thus, to prove effective regularity, we need to restrict the rational relation $R$.  

For these proofs, we need the following classical concepts. Let $S$ be a semiring. A function $r : \Sigma^* \to S$ is \textit{realizable over} $S$, if there are $n \in \mathbb{N}$, $\lambda \in S^{1 \times n}$, a homomorphism $\mu : \Sigma^* \to S^{n \times 1}$, and $\nu \in S^{n \times 1}$ with $r(w) = \lambda \cdot \mu(w) \cdot \nu$ for all $w \in \Sigma^*$ \footnote{In the literature, a realizable function is often called recognizable formal power series. Since, in this paper, we will not encounter any operations on formal power series (like addition, Cauchy product etc), we use the (in this context) more intuitive notion of a “realizable function.”}. The triple $(\lambda, \mu, \nu)$ is a \textit{presentation} or a \textit{weighted automaton} for $r$.  

In the following, we consider the semiring $\mathbb{N}^\infty$, i.e., the set $\mathbb{N} \cup \{\infty\}$ together with the commutative operations $+$ and $\cdot$ (with $x + \infty = \infty$ for all $x \in \mathbb{N} \cup \{\infty\}$, $x \cdot \infty = \infty$ for all $x \in (\mathbb{N} \cup \{\infty\}) \setminus \{0\}$, and $0 \cdot \infty = 0$). On this set, we define (in a natural way) an infinite sum setting 

$$
\sum_{i \in I} x_i = \begin{cases} 
\infty & \text{if there are infinitely many } i \in I \text{ with } x_i > 0 \\
\sum_{i \in I} x_i & \text{otherwise.}
\end{cases}
$$

for any family $(x_i)_{i \in I}$ with entries in $\mathbb{N}^\infty$.  

Our first aim in this section is to prove the following
Proposition 10. Let $\Gamma$ and $\Sigma$ be alphabets, $f: \Gamma^* \to \Sigma^*$ a homomorphism, and $\chi: \Gamma^* \to \mathbb{N}^\infty$ a realizable function over $\mathbb{N}^\infty$. Then the function
\[
r = \chi \circ f^{-1} : \Sigma^* \to \mathbb{N}^\infty : u \mapsto \sum_{w \in \Gamma^* : f(w) = u} \chi(w)
\]
is effectively realizable over $\mathbb{N}^\infty$.

Before we can do this in all generality, we first consider two special cases: A monoid homomorphism $f: \Gamma^* \to \Sigma^*$ between free monoids is non-expanding if $|f(w)| \leq |w|$ for all $w \in \Gamma^*$, i.e. $f(a) \in \Sigma \cup \{\varepsilon\}$ for all $a \in \Gamma$. It is non-erasing if, dually, $|f(w)| \geq |w|$ for all $w \in \Gamma^*$, i.e., $f(a) \in \Sigma^+$ for all $a \in \Gamma$.

Lemma 11. Let $\Gamma$ and $\Sigma$ be alphabets, $f: \Gamma^* \to \Sigma^*$ a non-expanding homomorphism, and $\chi: \Gamma^* \to \mathbb{N}^\infty$ a realizable function over $\mathbb{N}^\infty$. Then the function
\[
r = \chi \circ f^{-1} : \Sigma^* \to \mathbb{N}^\infty : u \mapsto \sum_{w \in \Gamma^* : f(w) = u} \chi(w)
\]
is effectively realizable over $\mathbb{N}^\infty$.

Proof. Let $(\lambda, \mu, \nu)$ be a presentation of dimension $n$ for $\chi$.

For $\sigma \in \Sigma \cup \{\varepsilon\}$, let
\[
\Gamma_\sigma = \{b \in \Gamma \mid f(b) = \sigma\}.
\]
Since $f$ is non-expanding, $\Gamma$ is the disjoint union of these subalphabets. Furthermore, let $M \in (\mathbb{N}^\infty)^{n \times n}$ be the matrix defined by
\[
M_{ij} = \sum_{w \in \Gamma_j^*} \mu(w)_{ij}
\]
for all $i, j \in [1, n]$.

To define a presentation for the function $r$, we first define a homomorphism $\mu': \Sigma^* \to (\mathbb{N}^\infty)^{n \times n}$ by
\[
\mu'(a) = \sum_{b \in \Gamma_a} (\mu(b) \cdot M)
\]
for all $a \in \Sigma$. Setting
\[
\lambda' = \lambda \cdot M \quad \text{and} \quad \nu' = \nu
\]
defines the presentation $(\lambda', \mu', \nu')$ of dimension $n$. Now let $u = a_1 a_2 \ldots a_m \in \Sigma^*$ with $a_i \in \Sigma$ for all $1 \leq i \leq m$. Then we get
\[
\begin{align*}
\lambda' \cdot \mu'(u) \cdot \nu' &= \lambda \cdot M \cdot \left( \prod_{1 \leq i \leq m} \sum_{b_i \in \Gamma_{a_i}} (\mu(b_i) \cdot M) \right) \cdot \nu \\
&= \lambda \cdot \sum_{w_1 \in \Gamma_{a_1}^*} (\mu(w_1) \cdot \left( \prod_{1 \leq i \leq m} \sum_{w_i \in \Gamma_{a_i} \cap \Gamma^{\infty}_{a_i}} (\mu(w_i) \cdot \nu) \right) \cdot \nu \\
&= \lambda \cdot \sum_{w \in \Gamma^* : f(w) = u} (\mu(w) \cdot \nu) \\
&= r(u).
\end{align*}
\]
Hence, \((λ', μ', ν')\) is a presentation for the function \(r\), i.e., \(r\) is realizable.

It remains to be shown that the presentation \((λ', μ', ν')\) is computable from the presentation \((λ, μ, ν)\) and the homomorphism \(f\). For this, it suffices to construct the matrix \(M\) effectively, i.e., to compute the infinite sum in Eq. (5). Using a pumping argument, one first shows the equivalence of the following two statements for all \(i, j \in \{1, 2, \ldots, n\}\):

(a) There are infinitely many words \(w ∈ Γ^*\) with \(μ(w)_{ij} > 0\).
(b) There is a word \(w ∈ Γ^*\) with \(n < |w| ≤ 2n\) and \(μ(w)_{ij} > 0\).

Since statement (b) is decidable, we can evaluate Eq. (5) calculating

\[
M_{ij} = \begin{cases} \infty & \text{if (b) holds} \\ \sum \{μ(w)_{ij} \mid w ∈ Γ^*, |w| ≤ n \} & \text{otherwise.} \end{cases}
\]

Lemma 12. Let \(Γ\) and \(Σ\) be alphabets, \(f : Γ^* → Σ^*\) a non-erasing homomorphism, and \(χ : Γ^* → \mathbb{N}^∞\) a realizable function over \(\mathbb{N}^∞\). Then the function

\[
r = χ \circ f^{-1} : Σ^* → \mathbb{N}^∞ : u ↦ \sum_{f(w) = u} χ(w)
\]

is effectively realizable over \(\mathbb{N}^∞\).

Proof. Since \(χ\) is realizable, it can be constructed from functions \(s : Γ^* → \mathbb{N}^∞\) with \(s(w) \neq 0\) for at most one \(w ∈ Γ^*\) using addition, Cauchy-product, and iteration applied to functions \(t\) with \(t(ε) = 0\) [17, Theorem 3.11]. Replacing, in this construction, the basic function \(s\) by \(s' : Σ^* → \mathbb{N}^∞\) with

\[
f'(x) = \sum_{w ∈ Γ^* \mid f(w) = x} s(w)
\]

yields a construction of \(r\). Since \(f\) is non-erasing, also in this construction, iteration is only applied to functions \(t\) with \(t(ε) = 0\). Hence, by [17, Theorem 3.1] again, \(r\) is realizable. Analysing the proof of that theorem, one even obtains that a presentation for \(r\) can be computed from \(f\) and a presentation of \(χ\).

Proof of Prop. 10 Let \(σ ∈ Σ\) be arbitrary. Then define homomorphisms \(f_1 : Γ^* → Γ^*\) and \(f_2 : Γ^* → Σ^*\) by

\[
f_1(a) = \begin{cases} ε & \text{if } f(a) = ε \\ a & \text{otherwise} \end{cases}\quad \text{and}\quad f_2(a) = \begin{cases} f(a) & \text{if } f(a) \neq ε \\ σ & \text{otherwise} \end{cases}
\]

for all letters \(a ∈ Γ\). Then \(f_1\) is non-expanding, \(f_2\) is non-erasing, and \(f = f_2 \circ f_1\).

By Lemma 11 the function \(χ \circ f_1^{-1}\) is effectively realizable. By Lemma 12 also \(χ \circ f_1^{-1} \circ f_2^{-1} = χ \circ f^{-1}\) is effectively realizable.

Lemma 13. Let \(R ⊆ Σ^* × Σ^*\) be an unambiguous rational relation and \(L ⊆ Σ^*\) be regular. Then the function

\[
r : Σ^* → \mathbb{N}^∞ : u ↦ \# \{v ∈ L \mid (u, v) ∈ R\}
\]

is effectively realizable.
Proof. Since $R$ is unambiguous rational, there are an alphabet $\Gamma$, homomorphisms $f, g : \Gamma^* \to \Sigma^*$, and a regular language $S \subseteq \Gamma^*$ such that

$$(f, g) : \Gamma^* \to \Sigma^* \times \Sigma^* : w \mapsto (f(w), g(w))$$

maps $S$ bijectively onto the relation $R$. Let $S_L = S \cap g^{-1}(L)$. Since $L$ is regular, the language $S_L$ is effectively regular. Furthermore, $(f, g)$ maps $S_L$ bijectively onto $R \cap (\Sigma^* \times L)$.

Since $S_L$ is regular, the characteristic function

$$\chi : \Gamma^* \to \mathbb{N}^\infty : w \mapsto \begin{cases} 1 & \text{if } w \in S_L \\ 0 & \text{otherwise} \end{cases}$$

is effectively realizable over $\mathbb{N}^\infty$. By Proposition 13, also the function

$$r' : \Sigma^* \to \mathbb{N}^\infty : u \mapsto \sum_{w \in \Gamma^*} \chi(w)$$

is effectively realizable over $\mathbb{N}^\infty$. Note that, for $u \in \Sigma^*$, we get

$$r'(u) = \sum_{w \in \Gamma^*} \chi(w)$$

$$= |\{ w \in S_L \mid f(w) = u \}|$$

since $\chi$ is the characteristic function of $S_L$

$$= |\{ g(w) \mid w \in S_L, f(w) = u \}|$$

since $(f, g)$ is injective on $S_L \subseteq S$

$$= |\{ v \mid (u, v) \in R \cap (\Sigma^* \times L) \}|$$

since $(f, g)$ maps $S_L$ onto $R \cap (\Sigma^* \times L)$

$$= |\{ v \in L \mid (u, v) \in R \}|.$$  

In other words, the effectively realizable function $r'$ is the function $r$ from the statement of the lemma. \hfill \Box

\textbf{Proposition 14.} Let $R \subseteq \Sigma^* \times \Sigma^*$ be an unambiguous rational relation and $L \subseteq \Sigma^*$ be regular.

1. For $k \in \mathbb{N}$, the set of words $u \in \Sigma^*$ with

$$|\{ v \in L \mid (u, v) \in R \}| \geq k$$

is effectively regular.

2. For $p, q \in \mathbb{N}$ with $p < q$, the set $H$ of words $u \in \Sigma^*$ with

$$|\{ v \in L \mid (u, v) \in R \}| \in p + q\mathbb{N}$$

is effectively regular.

Proof. By Lemma 13, the function $r : \Sigma^* \to \mathbb{N}^\infty : u \mapsto |\{ v \in L \mid (u, v) \in R \}|$ is effectively realizable over $\mathbb{N}^\infty$.

We construct a semiring $S_k^\infty = (\{0, 1, \ldots, k, \infty\}, \oplus, \odot, 0, 1)$ setting

$$x \oplus y = \begin{cases} x + y & \text{if } x + y \in S_k^\infty \\ k & \text{otherwise} \end{cases} \quad \text{and} \quad x \odot y = \begin{cases} x \cdot y & \text{if } x \cdot y \in S_k^\infty \\ k & \text{otherwise} \end{cases}$$

for $x, y \in S_k^\infty$ (note that $x + y \not\in S_k^\infty$ is equivalent to $k < x + y < \infty$).
Then the mapping

\[ h : \mathbb{N}^\infty \rightarrow S_k^\infty : n \mapsto \begin{cases} \min \{k, n\} & \text{if } n \in \mathbb{N} \\ \infty & \text{if } n = \infty \end{cases} \]

is a semiring homomorphism. It follows that the function

\[ h \circ r : \Sigma^* \rightarrow S_k^\infty : u \mapsto \min \{k, |\{v \in L | (u, v) \in R\}|\} \]

is effectively realizable over \( S_k^\infty \) [17, Prop. 4.5]. Since the semiring \( S_k^\infty \) is finite, the language \( (h \circ r)^{-1}(k) = \{u \in \Sigma^* | r(u) \geq k\} \)

is effectively regular [17, Prop. 6.3]. Since \( r(u) \geq k \iff |\{v \in L | (u, v) \in R\}| \geq k \), this proves the first statement.

The second statement is shown similarly. We construct the semiring \( \mathbb{Z}_q = (\{0, 1, \ldots, q, \infty\}, \oplus, \odot, 0, 1) \) with

\[
x \oplus y = \begin{cases} x + y & \text{if } x + y \in \mathbb{Z}_q^\infty \\ x + y \mod q & \text{otherwise} \end{cases}
\]

and

\[
x \odot y = \begin{cases} x \cdot y & \text{if } x \cdot y \in \mathbb{Z}_q^\infty \\ x \cdot y \mod q & \text{otherwise} \end{cases}
\]

for \( x, y \in S_k^\infty \). Note that, with \( q = 6 \), we get \( 4 \oplus 2 = 6 \neq 0 = (1 \oplus (q - 1)) \odot \infty \) and similarly

\[
3 \odot 2 = 6 \neq 0 = (3 \cdot 2) \mod q^2
\]

Let \( \eta \) denote the semiring homomorphism from \( \mathbb{N}^\infty \) to \( \mathbb{Z}_q^\infty \) with

\[
\eta(n) = \begin{cases} \infty & \text{if } n = \infty \\ q & \text{if } n \in q\mathbb{N} \setminus \{0\} \\ n \mod q & \text{otherwise.} \end{cases}
\]

It follows that the function

\[ \eta \circ r : \Sigma^* \rightarrow \mathbb{Z}_q^\infty : u \mapsto \eta(|\{v \in L | (u, v) \in R\}|) \]

is effectively realizable over \( \mathbb{Z}_q^\infty \) [17, Prop. 4.5]. Since the semiring \( \mathbb{Z}_q^\infty \) is finite, the language

\[ (\eta \circ r)^{-1}(x) \]

is effectively regular for all \( x \in \mathbb{Z}_q^\infty \) [17, Prop. 6.3]. Then the claim follows since

\[
H = \begin{cases} (\eta \circ r)^{-1}(p) & \text{if } 1 \leq p < q \\ (\eta \circ r)^{-1}(0) \cup (\eta \circ r)^{-1}(q) & \text{if } p = 0. \end{cases}
\]

---

2 The reader might wonder why both, 0 and \( q \), belong to \( \mathbb{Z}_q^\infty \). Suppose we identified them, i.e., considered \( S = \{0, 1, \ldots, q - 1, \infty\} \). Then we would get \( 0 = 0 \odot \infty = (1 \oplus (q - 1)) \odot \infty = 1 \odot \infty \oplus (q - 1) \odot \infty = \infty \oplus \infty = \infty \).
Our decision procedure employs a quantifier alternation procedure, i.e., we will transform an arbitrary formula into an equivalent one that is quantifier-free. As usual, the heart of this procedure handles formulas $\psi = Q \varepsilon \phi$ where $Q$ is a quantifier and $\phi$ is quantifier-free. Since the logic $C+MOD^2$ has only two variables, any such formula $\psi$ has at most one free variable. In other words, it defines a language $K$. The following lemma shows that this language is effectively regular, such that $\psi$ is equivalent to the quantifier-free formula $x \in K$.

**Lemma 15.** Let $\phi(x, y)$ be a quantifier-free formula from $C+MOD^2$. Then the sets

$$\{x \in \Sigma^* | S \models \exists^k y \phi\} \text{ and } \{x \in \Sigma^* | S \models \exists^p \text{mod} q y \phi\}$$

are effectively regular for all $k \in \mathbb{N}$ and all $p, q \in \mathbb{N}$ with $p < q$.

**Proof.** Without changing the meaning of the formula $\phi$, we can do the following replacements of atomic formulas:

- $x = y$ can be replaced by $x \subseteq y \land y \subseteq x$,
- $x \subseteq x$ and $y \subseteq y$ by $x \in \Sigma^*$, and
- $x \sqsubseteq x$ and $y \sqsubseteq y$ by $x \in \emptyset$.

Since $\phi$ is quantifier-free, we can therefore assume that it is a Boolean combination of formulas of the form

- $x \in K$ for some regular language $K$,
- $y \in L$ for some regular language $L$,
- $x \subseteq y$,
- $y \subseteq x$,
- $x \sqsubseteq y$, and
- $y \sqsubseteq x$.

We define the following formulas $\theta_i(x, y)$ for $1 \leq i \leq 6$:

$$\theta_i(x, y) = \begin{cases} 
  x = y & \text{if } i = 1 \\
  x \subseteq y & \text{if } i = 2 \\
  x \subseteq y \land \neg(x = y \lor x \sqsubseteq y) & \text{if } i = 3 \\
  y \sqsubseteq x & \text{if } i = 4 \\
  y \sqsubseteq x \land \neg(y = x \lor y \sqsubseteq x) & \text{if } i = 5 \\
  \neg(x \sqsubseteq y \lor y \sqsubseteq x) & \text{if } i = 6 
\end{cases}$$

Note that any pair of words $x$ and $y$ satisfies precisely one of these six formulas. Hence $\phi$ is equivalent to

$$\bigvee_{1 \leq i \leq 6} (\theta_i \land \phi).$$

In this formula, any occurrence of $\phi$ appears in conjunction with precisely one of the formulas $\theta_i$. Depending on this formula $\theta_i$, we can simplify $\phi$ to $\phi_i$ by replacing the atomic subformulas that compare $x$ and $y$ as follows:

- If $i \in \{1, 2, 3\}$, we replace $x \sqsubseteq y$ by the valid formula $\top = (x \in \Sigma^*)$.
- If $i \in \{1, 4, 5\}$, we replace $y \sqsubseteq x$ by $\top$.
- If $i = 2$, we replace $x \sqsubseteq y$ by $\top$.
- If $i = 4$, we replace $y \sqsubseteq x$ by $\top$. 


All remaining comparisions are replaced by \( \bot = (x \in \emptyset) \).

As a result, the formula \( \varphi \) is equivalent to

\[
\bigvee_{1 \leq i \leq 6} (\theta_i \wedge \varphi_i)
\]

where the formulas \( \varphi_i \) are Boolean combinations of formulas of the form \( x \in K \) and \( y \in L \) for some regular languages \( K \) and \( L \).

Now let \( k \in \mathbb{N} \). Since the formulas \( \theta_i \) are mutually exclusive (i.e., \( \theta_i(x,y) \land \theta_j(x,y) \) is satisfiable iff \( i = j \)), we get

\[
\exists^k y \varphi \equiv \exists^k y \bigvee_{1 \leq i \leq 6} (\theta_i \wedge \varphi_i) \equiv \bigvee_{\ast} \bigwedge_{1 \leq i \leq 6} \exists^k y (\theta_i \wedge \varphi_i)
\]

where the disjunction (\( \ast \)) extends over all tuples \( (k_1, \ldots, k_6) \) of natural numbers with \( \sum_{1 \leq i \leq 6} k_i = k \).

Hence it suffices to show that

\[
\{ x \in \Sigma^* \mid \exists^k y (\theta_i \wedge \varphi) \}
\]

is effectively regular for all \( 1 \leq i \leq 6 \), all \( k \in \mathbb{N} \), and all Boolean combinations \( \varphi \) of formulas of the form \( x \in K \) and \( y \in L \) where \( K \) and \( L \) are regular languages. Since the class of regular languages is closed under Boolean operations, we can find regular languages \( K_i \) and \( L_i \) such that \( \varphi \) is equivalent to

\[
\bigvee_{1 \leq i \leq n} (x \in K_i \wedge y \in L_i).
\]

Note that this formula is equivalent to

\[
\bigvee_{M \subseteq \{1, \ldots, n\}} \left( x \in \bigcap_{i \in M} K_i \setminus \bigcup_{i \notin M} K_i \wedge y \in \bigcup_{i \in M \setminus \{K_M\}} L_i \right).
\]

Since this disjunction is exclusive (i.e. any pair of words \( (x,y) \) satisfies at most one of the cases), the set from (6) equals the union of the sets

\[
\{ x \in \Sigma^* \mid \exists^k y (\theta_i \wedge x \in K_M \wedge y \in L_M) \}
\]

for \( M \subseteq \{1, 2, \ldots, n\} \). Observe that for \( k = 0 \), this set equals \( \Sigma^* \) and we are done. So let us assume \( k \geq 1 \) from now on. Note that in that case, the set from (7) equals

\[
K_M \cap \{ x \in \Sigma^* \mid \exists^k y \in L_M : \theta_i \}
\]

This set, in turn, equals

- \( K_M \cap L_M \) if \( i = 1 \) and \( k = 1 \),
- \( \emptyset \) if \( i = 1 \) and \( k > 1 \),
- \( K_M \cap \{ x \in \Sigma^* \mid \exists^k y \in L_M : x \sqsubseteq y \} \) if \( i = 2 \),
- \( K_M \cap \{ x \in \Sigma^* \mid \exists^k y \in L_M : (x,y) \in \sqsubseteq \} \) if \( i = 3 \),
- \( K_M \cap \{ x \in \Sigma^* \mid \exists^k y \in L_M : y \sqsubseteq x \} \) if \( i = 4 \),
- \( K_M \cap \{ x \in \Sigma^* \mid \exists^k y \in L_M : (y,x) \in \sqsubseteq \} \) if \( i = 5 \), and
- \( K_M \cap \{ x \in \Sigma^* \mid \exists^k y \in L_M : x \sqsupseteq y \} \) if \( i = 6 \).
In any case, it is effectively regular by Prop. 14, Lemma 7, and Lemma 8. Since the language from the claim of the lemma is a Boolean combination of such languages, the first claim is demonstrated.

To also demonstrate the regularity of the second language, let \( p,q \in \mathbb{N} \) with \( p < q \). Then \( \exists p \mod q \varphi \) is equivalent to the disjunction of all formulas of the form

\[
\bigwedge_{1 \leq i \leq 6} \exists p_{i} \mod q (\theta_{i} \land \varphi_{i})
\]

where \((p_{1},...,p_{6})\) is a tuple of natural numbers from \{0,1,...,q - 1\} with \( \sum_{1 \leq i \leq 6} p_{i} \equiv p \mod q \). The rest of the proof proceeds \textit{mutatis mutandis}.

\[\blacksquare\]

\[\blacksquare\]

\[\square\]

Theorem 16. Let \( S = (\Sigma^{*}, \sqsubseteq, (L)_{L \text{ regular}}) \). Let \( \varphi(x) \) be a formula from C+MOD\(^{2}\). Then the set

\[
\{ x \in \Sigma^{*} \mid S \models \varphi(x) \}
\]

is effectively regular.

\textbf{Proof.} The claim is trivial if \( \varphi \) is atomic. For more complicated formulas, the proof proceeds by induction using Lemma 15 and the effective closure of the class of regular languages under Boolean operations.

\[\blacksquare\]

\[\blacksquare\]

\[\square\]

Corollary 17. Let \( L \subseteq \Sigma^{*} \) be a regular language. Then the C+MOD\(^{2}\)-theory of the structure \( \mathcal{S} = (L, \sqsubseteq, (K \cap L)_{K \text{ regular}}, (w)_{w \in L}) \) is decidable.

\textbf{Proof.} Let \( \varphi \in \text{C+MOD}^{2} \) be a sentence. By the previous theorem, the set

\[
\{ x \in L \mid S \models \varphi \}
\]

is regular. Hence \( \varphi \) holds iff this set is nonempty, which is decidable.

\[\blacksquare\]

\[\blacksquare\]

\[\square\]

5 The \( \Sigma_{1} \)-theory

Let \( L \) be regular and bounded. Then, by Theorem 4, we obtain in particular that the \( \Sigma_{2} \)-theory of \((L, \sqsubseteq)\) is decidable. Note that the regular language \( L = \{a,b\}^{*} \) is not covered by this result since it is unbounded. And, indeed, the \( \Sigma_{2} \)-theory of \((\{a,b\}^{*}, \sqsubseteq)\) is undecidable [5]. On the positive side, we know that the \( \Sigma_{1} \)-theory of \((\{a,b\}^{*}, \sqsubseteq)\) is decidable [15].

In this section, we generalize this positive result to arbitrary regular languages, i.e., we prove the following result:

\[\blacksquare\]

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\[\square\]

Theorem 18. Let \( L \subseteq \Sigma^{*} \) be regular. Then the \( \Sigma_{1} \)-theory of \( \mathcal{S} = (L, \sqsubseteq) \) is decidable.

\textbf{Theorem 18.} Let \( L \subseteq \Sigma^{*} \) be regular. Then the \( \Sigma_{1} \)-theory of \( \mathcal{S} = (L, \sqsubseteq) \) is decidable.

The proof for the case \( L = \{a,b\}^{*} \) in [15] essentially relies on the fact that each order \((\mathbb{N}^{k}, \leq)\), and thus every finite partial order, embeds into \((\{a,b\}^{*}, \sqsubseteq)\).

In the general case here, the situation is more involved. Take, for example, \( L = \{ab, ba\}^{*} \). Then, orders as simple as \((\mathbb{N}^{2}, \leq)\) do not embed into \((L, \sqsubseteq)\): This is because the downward closure of any infinite subset of \( L \) contains all of \( L \), but \( \mathbb{N}^{2} \) contains a downwards closed infinite chain. Nevertheless, we will show, perhaps surprisingly, that every finite partial order embeds into \((L, \sqsubseteq)\). In fact, this holds whenever \( L \) is an unbounded regular language. The latter requires two propositions that we shall prove only later. Recall that a word \( w \in \Sigma^{+} \) is called \textit{primitive} if there is no \( r \in \Sigma^{+} \) with \( w = rr^{+} \).
Proof of Theorem 18. By Theorem 4, we may assume that \( L \) is unbounded.

By Proposition 19 below, there are words \( x, y, u, v \in \Sigma^* \) with \( |u| = |v| \), \( uv \) primitive, and \( x \{ u, v \}^* y \subseteq L \). By Proposition 23 below, any finite partial order embeds into \((\{u, v\}^*, \sqsubseteq)\) and therefore into \((x \{ u, v \}^* y, \sqsubseteq)\) which is a substructure of \((L, \sqsubseteq)\), i.e., every finite partial order embeds into \((L, \sqsubseteq)\).

Hence \( \varphi = \exists x_1, x_2, \ldots, x_n : \psi \) with \( \psi \) quantifier-free holds in \((L, \sqsubseteq)\) if and only if it holds in some finite partial order whose size can be bounded by \( n \). Since there are only finitely many such partial orders, the result follows. \( \square \)

The first proposition used in the above proof deals with the existence of certain primitive words for every unbounded regular language.

Proposition 19. For every unbounded regular language \( L \subseteq \Sigma^* \), there are words \( x, u, v, y \in \Sigma^* \) so that

1. \( |u| = |v| \),
2. the word \( uv \) is primitive, and
3. \( x \{ u, v \}^* y \subseteq L \).

Proof. Since \( L \) is unbounded and regular, there are words \( x, y, p, q \in \Sigma^* \) with \( |p| = |q| \), \( p \neq q \), and \( x \{ p, q \}^* y \subseteq L \). Set \( r = pq \) and \( s = pp \).

Then \( |r| = |s| \) and \( x \{ r, s \}^* y \subseteq x \{ p, q \}^* y \subseteq L \). Suppose \( r \) and \( s \) are conjugate. Since \( s = p^2 \), this implies \( r = yxyx \) with \( p = yr \), i.e., \( r \) is the square of some word \( yr \) of length \( |p| = |q| \). But this contradicts \( r = pq \) and \( p \neq q \). Hence \( r \) and \( s \) are not conjugate.

Next let \( n = |r|, u = rsn^{-1} \) and \( v = sn \).

By contradiction, we show that \( uv \) is primitive.

Since we assume \( uv = rsn^{-1} \) not to be primitive, there is a word \( w \in \Sigma^* \) with \( rs^{2n-1} \in uw^+ \). Observe that there is a \( t \in \mathbb{N} \) such that \( n \leq |w^t| \leq n^2 \): If \( |w| \geq n \), we can choose \( t = 1 \) since \( |w| \leq \frac{1}{2} |rs^{2n-1}| = n^2 \) and if \( |w| < n \), we can take \( t = n \).

Observe that \( r \) and \( w^t \) are prefixes of \( uv = rsn^{-1} \) of length \( n \) and \( \geq n \), respectively. Hence \( r \) is a prefix of \( w^t \).

On the other hand, \( v = sn \) and \( w^t \) are suffixes of \( uv \) of length \( n^2 \) and \( \leq n^2 \), respectively. Hence \( w^t \) is a suffix of \( v = sn \).

Taking these two facts together, we obtain that \( r \) is a factor of \( sn \). Since \( r \) and \( s \) are not conjugate, this implies \( pq = r = s = pp \) which contradicts \( p \neq q \). \( \square \)

The second proposition used above talks about the embeddability of every finite partial order into certain regular languages of the form \( \{ u, v \}^* \) where the words \( u \) and \( v \) originate from the previous proposition. The proof of this embeddability requires a good deal of preparation that deals with the combinatorics of subwords, more precisely with the properties of “prefix-maximal subwords”.

Let \( x = a_1a_2\ldots a_m \) and \( y = b_1b_2\ldots b_n \) with \( a_i, b_j \in \Sigma \). An embedding of \( x \) into \( y \) is a mapping \( \alpha : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\} \) with \( a_i = b_{\alpha(i)} \) and \( i < j \iff \alpha(i) < \alpha(j) \) for all \( i, j \in \{1, 2, \ldots, m\} \). Note that \( x \sqsubseteq y \) iff there exists an embedding of \( x \) into \( y \). This embedding is called initial if \( \alpha(1) = 1 \), i.e., if the left-most position in \( x \) hits the left-most position in \( y \). Symmetrically, the embedding \( \alpha \) is terminal if \( \alpha(m) = n \), i.e., if the right-most position in \( x \) hits the right-most position in \( y \).

We write \( x \leftrightarrow y \) if \( x \sqsubseteq y \) and every embedding of \( x \) into \( y \) is terminal. This is equivalent to saying that \( x \), but no word \( xa \) with \( a \in \Sigma \) is a subword of \( y \). In other words, \( x \leftrightarrow y \) if \( x \) is a prefix-maximal subword of \( y \).
Lemma 20. Let \( w \) be primitive and \( n > |w| \). Then, every embedding of \( w^n \) into \( w^{n+1} \) is either initial or terminal.

Proof. Let \( \alpha \) be an embedding of \( w^n \) into \( w^{n+1} \) that is neither initial nor terminal. Consider the \( n \) copies of \( w \) in the word \( w^n \). We call such a copy gapless if its image in \( w^{n+1} \) under \( \alpha \) is contiguous. Since the length difference between \( w^n \) and \( w^{n+1} \) is only \( |w| < n \), there has to be at least one gapless copy of \( w \), say the \( i \)th copy. The image of this copy is a contiguous subword of \( w^{n+1} \) that spells \( w \) and occurs at some position \( i \cdot |w| + j \) with \( j \in \{0, \ldots, |w|\} \).

If \( j = 0 \), then \( \alpha \) is initial and if \( j = |w| \), then \( \alpha \) is terminal. This means \( j \in \{1, \ldots, |w| - 1\} \). However, since \( w \) is primitive, it can occur as a contiguous subword of \( w^{n+1} \) only at positions that are divisible by \( |w| \), which is a contradiction.

Lemma 21. The ordering \( \leftrightarrow \) is multiplicative: If \( x, x', y, y' \in \Sigma^* \) with \( x \leftrightarrow y \) and \( x' \leftrightarrow y' \), then \( xy \leftrightarrow x'y' \).

Proof. Suppose \( xy \not\leftrightarrow x'y' \). Since \( xy \subseteq x'y' \), there is \( a \in \Sigma \) such that \( xya \subseteq x'y' \). Then, either \( zb \subseteq x' \) where \( b \) is the first letter of \( ya \) (contradicting \( x \leftrightarrow y' \)), or \( ya \subseteq y' \) (contradicting \( y \leftrightarrow y' \)).

Lemma 22. Let \( u, v \in \Sigma^* \) be words such that \( |u| = |v| \) and \( uv \) is primitive. Then, for all \( \ell, n \in \mathbb{N} \) with \( n > |uv| + \ell + 2 \), we have

(i) \( (uv)^n \leftrightarrow v(uv)^n \),
(ii) \( (uv)^1v(uv)^{n-\ell-1} \leftrightarrow (uv)^n \), and
(iii) \( (uv)^{1+\ell}v(uv)^{n-\ell-2} \leftrightarrow v(uv)^n \).

Proof. For claim (i), suppose \( \alpha \) is an embedding of \( (uv)^n \) into \( v(uv)^n \). Then \( \alpha \) induces an embedding \( \beta \) of \( (uv)^{n-1} \) into \( (uv)^n \). Note that \( \beta \) cannot be initial because otherwise \( \alpha \) would embed \( uv \) into \( v \). Thus, \( \beta \) is terminal by Lemma 20. Hence, \( \alpha \) is terminal.

For claim (ii), suppose \( \alpha \) is an embedding of \( (uv)^1v(uv)^{n-\ell-1} \) into \( (uv)^n \). Since \( (uv)^n = (uv)^1v(uv)^{n-\ell-1} \), \( \alpha \) induces an embedding \( \beta \) of \( (uv)^{n-\ell-1} \) into \( (uv)^n \). Again, \( \beta \) cannot be initial because otherwise \( \alpha \) would embed \( (uv)^1v \) into \( (uv)^\ell \). Therefore, \( \beta \) is terminal according to Lemma 20 meaning that \( \alpha \) is terminal as well.

Finally, for claim (iii), suppose \( \alpha \) is an embedding of \( (uv)\gamma(v(uv)^{n-\ell-2} \leftrightarrow (uv)^n \). Since \( v(uv)^\gamma = v(uv)^{1+\ell}v(uv)^{n-\ell-1} \), \( \alpha \) induces an embedding \( \beta \) of \( (uv)^{n-\ell-2} \) into \( (uv)^n \). Again, \( \beta \) cannot be initial because otherwise \( \alpha \) would embed \( (uv)^{1+\ell}v \) into \( (uv)^{1+\ell} \), but these are distinct words of equal length. Thus, Lemma 20 tells us that \( \beta \) must be terminal and hence also \( \alpha \).

Proposition 23. Let \( u, v \in \Sigma^* \) be distinct such that \( uv \) is primitive and \( |u| = |v| \). Then every finite partial order embeds into \( (\{u, v\}^* \cup \subseteq) \).

Proof. For \( m \in \mathbb{N} \), let \( \leq \) denote the componentwise order on the set \( \{0, 1\}^m \) of \( m \)-dimensional vectors over \( \{0, 1\} \). Note that every finite partial order with \( m \) elements embeds into \( (\{0, 1\}^m \cup \subseteq) \). Hence, it suffices to embed this partial order into \( (\{u, v\}^*, \subseteq) \).

We define the map \( \varphi_m : \{0, 1\}^m \to \{u, v\}^* \) as follows. Set \( n = |uv| + m + 3 \). Then, for a tuple \( t = (t_1, \ldots, t_m) \in \{0, 1\}^m \), let

\[ \varphi_m(t_1, \ldots, t_m) = v^{t_1}(uv)^n \ldots v^{t_m}(uv)^n. \]

It is clear that for \( s, t \in \{0, 1\}^m \), \( s \leq t \) implies \( \varphi_m(s) \subseteq \varphi_m(t) \).

Now let \( s = (s_1, \ldots, s_m) \) and \( t = (t_1, \ldots, t_m) \) be two vectors from \( \{0, 1\}^m \) with \( \varphi_m(s) \subseteq \varphi_m(t) \). Towards a contradiction, suppose \( s \not\subseteq t \). Then there is an \( i \in [1, m] \) with \( s_i = 1, \ldots, s_m = 0 \), and \( t_i = 0, \ldots, t_m = 1 \). The case \( s_i = 0, \ldots, s_m = 1 \), and \( t_i = 1, \ldots, t_m = 0 \) is symmetric. Then, we have

\[ \varphi_m(s) \subseteq \varphi_m(t) \]

which is a contradiction.
$t_i = 0$ and $s_j \leq t_j$ for all $j \in [1, i - 1]$. Since $(uv)^n \Rightarrow v(uv)^n$ by Lemma 22(i) and clearly also $(uv)^n \Rightarrow (uv)^n$, we have $v^s_i(uv)^n \Rightarrow v_i(uv)^n$ for every $j \in [1, i - 1]$. Furthermore, since $s_i = 1$ and $t_i = 0$, we have $v^s_i(uv)^{n-1} \Rightarrow v_i(uv)^{n-1}$ according to Lemma 22(ii) with $\ell = 0$. Therefore, Lemma 21(i) yields
\[ v^{s_1}(uv)^n \ldots v^{s_i}(uv)^{n-1} \Rightarrow v^i_1(uv)^n \ldots v^i_n(uv)^n. \] (8)
We show by induction on $k$ that for every $k \in [i, n]$, there is an $\ell \in [1, k]$ with
\[ v^{s_1}(uv)^n \ldots v^{s_k}(uv)^{n-\ell} \Rightarrow v^{i_1}(uv)^n \ldots v^{i_{\ell+k}}(uv)^n. \] (9)
Of course, (8) is the base case. So let $k \in [i, n - 1]$ and $\ell \in [1, k]$ such that (9) holds. We distinguish three cases.

1. Suppose $s_{k+1} = 0$. Then
\[ (uv)^n \Rightarrow v^{s_{k+1}}(uv)^n \]

since either $t_{k+1} = 0$ and the two words are the same, or $t_{k+1} = 1$ and $(uv)^n \Rightarrow v(uv)^n$ by Lemma 22(i). So together with the induction hypothesis (9), Lemma 21(i) yields
\[ v^{s_1}(uv)^n \ldots v^{s_k}(uv)^{n-\ell} = v^{s_1}(uv)^n \ldots v^{s_k}(uv)^{n-\ell} 
\Rightarrow v^{i_1}(uv)^n \ldots v^{i_{\ell+k}}(uv)^n, \]

where the equality is due to $s_{k+1} = 0$. Since $\ell \in [1, k] \subseteq [1, k+1]$, this proves (9) for $k+1$.

2. Suppose $s_{k+1} = 1$ and $t_{k+1} = 0$. By Lemma 22(ii), we have
\[ (uv)^\ell v(uv)^{n-(\ell+1)} \Rightarrow (uv)^n. \]
So together with the induction hypothesis (9), Lemma 21(i) implies
\[ v^{s_1}(uv)^n \ldots v^{s_k}(uv)^{n-(\ell+1)} = v^{s_1}(uv)^n \ldots v^{s_k}(uv)^{n-\ell} 
\Rightarrow v^{i_1}(uv)^n \ldots v^{i_{\ell+k}}(uv)^n \]
\[ = v^{i_1}(uv)^n \ldots v^{i_{\ell+k}}(uv)^n, \]

where the second equality is due to $t_{k+1} = 0$. Since $\ell + 1 \in [1, k+1]$, this proves (9) for $k+1$.

3. If $s_{k+1} = 1$ and $t_{k+1} = 1$, then Lemma 22(iii) tells us that
\[ (uv)^\ell v(uv)^{n-(\ell+1)} \Rightarrow v(uv)^n. \]
So together with the induction hypothesis (9), Lemma 21(i) implies
\[ v^{s_1}(uv)^n \ldots v^{s_k}(uv)^{n-(\ell+1)} = v^{s_1}(uv)^n \ldots v^{s_k}(uv)^{n-\ell} 
\Rightarrow v^{i_1}(uv)^n \ldots v^{i_{\ell+k}}(uv)^n \]
\[ = v^{i_1}(uv)^n \ldots v^{i_{\ell+k}}(uv)^n. \]

Since $\ell + 1 \in [1, k+1]$, this proves (9) for $k+1$. This completes the induction. Therefore, we have in particular
\[ v^{s_1}(uv)^n \ldots v^{s_m}(uv)^{n-\ell} \Rightarrow v^{i_1}(uv)^n \ldots v^{i_m}(uv)^n = \varphi_m(t) \]
for some $\ell > 0$. Since the left-hand side is a proper prefix of $\varphi_m(s)$, this contradicts $\varphi_m(s) \subseteq \varphi_m(t)$.

This completes the proof of the main result of this section, i.e., of Theorem 18.
6 The $\Sigma_1$-theory with constants

By Theorem [18], the $\Sigma_1$-theory of $(L, \sqsubseteq)$ is decidable for all regular languages $L$. If $L$ is bounded, then even the $\Sigma_1$-theory of $(L, \sqsubseteq, (w)_{w \in L})$ is decidable (Theorem [1]). This result does not extend to all regular languages since, e.g., the $\Sigma_1$-theory of $(\Sigma^*, \sqsubseteq, (w)_{w \in \Sigma^*})$ is undecidable [5]. In this section, we present another class of regular languages $L$ (besides the bounded ones) such that

$$S = (L, \sqsubseteq, (w)_{w \in L})$$

has a decidable $\Sigma_1$-theory.

Let $L \subseteq \Sigma^*$ be some language. Then almost all words from $L$ have a non-negligible number of occurrences of every letter if there exists a positive real number $\varepsilon$ such that for all $a \in \Sigma$ and all but finitely many words $w \in L$, we have

$$\frac{|w|_a}{|w|} > \varepsilon.$$ 

An example of such a regular language is $\{ab, ba\}^*$ (this class contains all finite languages, is closed under union and concatenation and under iteration, provided every word of the iterated language contains every letter).

For $w \in \Sigma^*$, let $w^\uparrow$ denote the set of superwords of $w$, i.e., the upward closure of $\{w\}$ in $(\Sigma^*, \sqsubseteq)$. The basic idea is, as in the proof of Theorem [18], to embed every finite partial order into $(L, \sqsubseteq)$. The following lemma refines this embedability. Furthermore, it shows that $L \setminus w^\uparrow$ is finite in this case.

Lemma 24. Let $L \subseteq \Sigma^*$ be an unbounded regular language such that almost all words from $L$ have a non-negligible number of occurrences of every letter. Let $w \in \Sigma^*$. Then every finite partial order $(P, \leq)$ can be embedded into $(L \cap w^\uparrow, \sqsubseteq)$. Furthermore, the set $L \setminus w^\uparrow$ is finite.

Note that $L \cap w^\uparrow$ is regular, but not necessarily unbounded (it could even be finite). Hence the first claim is not an obvious consequence of Propositions [23] and [18].

Proof. Since $L$ is regular and unbounded, there are words $x, u, v, y \in \Sigma^*$ with $|u| = |v| > 0$, $uv$ primitive, and $x\{u, v\}^* y \subseteq L$ (by Proposition [19]). In particular, $xu^* y \subseteq L$. Let $a \in \Sigma$ and suppose $|u|_a = 0$. Then

$$\lim_{n \to \infty} \frac{|xu^ny|_a}{|xu^ny|} = 0$$

contradicting that almost all words from $L$ have a non-negligible number of occurrences of every letter. Hence, $u$ contains every letter from $\Sigma$ implying $w \subseteq u|w|$. Set $x' = xu|w|$. Then we have $x'\{u, v\}^* y \subseteq L \cap w^\uparrow$. From Proposition [23] we learn that $(P, \leq)$ can be embedded into $(\{u, v\}^* y, \sqsubseteq)$. Hence, it can be embedded into $(x'\{u, v\}^* y, \sqsubseteq)$ and therefore into $(L \cap w^\uparrow, \sqsubseteq)$.

Next, we show that $L \setminus w^\uparrow$ is finite. Let $M = (Q, \Sigma, \iota, \delta, F)$ be the minimal deterministic finite automaton accepting $L$. Let $v \in L$ with $|v| \geq |Q| \cdot |w|$. Then we can factorize the word $v$ into $v = v_0 v_1 \cdots v_{|w|+1}$ such that $\delta(\iota, v_0) = \delta(\iota, v_0 v_1 \cdots v_i)$ for all $1 \leq i \leq |w|$. With $q = \delta(\iota, v_0)$, we obtain $\delta(q, v_i) = q_i$ for all $1 \leq i \leq |w|$ and therefore $vv_1 v_2 \cdots v_{|w|-1} v_{|w|+1} \subseteq L$. Since almost all words from $L$ have a non-negligible number of occurrences of every letter, this implies (as above) that $v_i$ contains all letters from $\Sigma$. Since this holds for all $1 \leq i \leq |w|$, we obtain $w \subseteq v_1 v_2 \cdots v_{|w|} \subseteq v$ and therefore $v \notin L \setminus w^\uparrow$. Hence, indeed, $L \setminus w^\uparrow$ is finite. ◀
Theorem 25. Let \( L \subseteq \Sigma^* \) be an unbounded regular language such that almost all words from \( L \) have a non-negligible number of occurrences of every letter. Then the \( \Sigma_1 \)-theory of \((L, \subseteq, (w)_{w \in L})\) is decidable.

Proof. We want to show that satisfiability in \((L, \subseteq, (w)_{w \in L})\) is decidable for quantifier-free formulas, i.e., for positive Boolean combinations \( \varphi \) of literals of the following forms (where \( x \) and \( y \) are arbitrary variables and \( w \) an arbitrary word from \( L \)):

- \( (i) \ x \subseteq w \)
- \( (ii) \ x \not\subseteq w \)
- \( (iii) \ w \subseteq x \)
- \( (iv) \ w \not\subseteq x \)
- \( (v) \ x \subseteq y \)
- \( (vi) \ x \not\subseteq y \)

Note that literals of the form \( x = y \) can be written as \( x \subseteq y \land y \subseteq x \), \( x \neq y \) as \( x \not\subseteq y \lor y \not\subseteq x \), and similarly \( x \neq w \) as \( x \not\subseteq w \lor w \not\subseteq x \). Furthermore, literals mentioning two words like \( u \subseteq v \) can be replaced by \( \top = (x \subseteq y \lor x \not\subseteq y) \) or \( \bot = (x \subseteq y \land x \not\subseteq y) \). By bringing the formula in disjunctive normal form, we may assume that we are given a disjunction of conjunctions of such literals.

Step 1. We first show that literals of types \( (i) \) and \( (iv) \) can be eliminated. To this end, observe that for each \( w \in L \), both of the sets

\[ \{ u \in L \mid u \subseteq w \} \quad \text{and} \quad \{ u \in L \mid w \not\subseteq u \} \]

are finite. In the case \( \{ u \in L \mid u \subseteq w \} \), this is trivial. In the case of \( \{ u \in L \mid w \not\subseteq u \} \), this is the second claim in Lemma 24. Thus, every conjunction that contains a literal \( x \subseteq w \) or \( w \not\subseteq x \), constrains \( x \) to finitely many values. Therefore, we can replace this conjunction with a disjunction of conjunctions that result from replacing \( x \) by one of these values. (Here, we might obtain literals \( u \subseteq v \) or \( u \not\subseteq v \), but those can be replaced by \( \bot \) and \( \top \) as above).

Note that such a replacement reduces the number of variables by one. We repeat this replacement until there are no more literals of the form \( (i) \) and \( (iv) \). Since we replace each conjunction with (a disjunction of) conjunctions that have fewer variable, this has to terminate. Thus, we arrive at a disjunction of conjunctions of literals of the forms \( (ii), (iii), (v), \) and \( (vi) \).

Step 2. In the second step, we will eliminate literals of the form \( (iii) \). Note that the language \( \{ u \in L \mid u \not\subseteq w \} \) is upward closed in \((L, \subseteq)\). Since \( L \) is regular, we can compute the finite set of minimal elements of this set. Thus, \( x \not\subseteq w \) is equivalent to a finite disjunction of literals of the form \( u' \subseteq x \). As a result, we get a positive Boolean combination \( \psi \) of literals of the form \( (iii), (v), (vi) \) that is equivalent to \( \varphi \).

Step 3. In the third step, we check whether our formula is satisfiable. We may assume that \( \psi \) is in disjunctive normal form. To verify whether \( \psi \) is satisfiable in \((L, \subseteq)\), it therefore suffices to verify satisfiability of conjunctions of literals of the form \( (iii), (v), (vi) \). So let \( \gamma \) be such a conjunction. It can be written as \( \gamma_1 \land \gamma_2 \) where \( \gamma_1 \) is a conjunction of literals of the form \( (iii) \) and \( \gamma_2 \) is a conjunction of literals of the form \( (v) \) and \( (vi) \).

Let \( n \) denote the number of variables appearing in \( \gamma_2 \). If \( \gamma \) is satisfiable in \((L, \subseteq)\), then \( \gamma_2 \) is satisfied by some partial order with at most \( n \) elements. Conversely, let \( \gamma_2 \) be satisfied by \((P, \leq)\) where \( P \) has at most \( n \) elements. Let, furthermore, \( w \) denote some concatenation of all words \( w \) appearing in the formula \( \gamma_1 \). By the first claim of Lemma 24, the finite partial order \((P, \leq)\) can be embedded into \((L \cap w^+, \subseteq)\). Consequently, \( \gamma_1 \land \gamma_2 \) is satisfiable in \((L \cap w^+, \subseteq)\) and therefore in \((L, \subseteq)\). In summary, \( \gamma \) is satisfiable in \((L, \subseteq)\) iff \( \gamma_2 \) holds in some finite partial order of size at most \( n \). Since there are only finitely many such finite partial orders, we get that satisfiability of \( \gamma \) in \((L, \subseteq)\) is decidable.

\( \blacksquare \)
Open questions

We did not consider complexity issues. In particular, from [11], we know that the FO²-theory of the structure \((Σ^*, \sqsubseteq, (w)_{w \in Σ^*})\) can be decided in elementary time. We currently work out the details for the extension of this result to the C+MOD²-theory of the structure \((L, \sqsubseteq, (w)_{w \in L})\) for \(L\) regular. We reduced the FO+MOD-theory of the full structure (for \(L\) context-free and bounded) to the FO+MOD-theory of \((\mathbb{N}, +)\) which is known to be decidable in elementary time [4]. Unfortunately, our reduction increases the formula exponentially due to the need of handling statements of the form “there is an even number of pairs \((x, y) \in \mathbb{N}^2\) such that ...” It should be checked whether the proof from [4] can be extended to handle such statements in FO+MOD for \((\mathbb{N}, +)\) directly.

Finally, we did not give any new undecidability results. For example, we know that the Σ₁-theory of \((L, \sqsubseteq, (w)_{w \in L})\) is undecidable for \(L = Σ^*\) [5] and decidable for \(L = \{ab, ba\}^*\) (Theorem 25). To narrow the gap between decidable and undecidable cases, one should find more undecidable cases.

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