On the Uniqueness of Kantorovich Potentials

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Abstract

Kantorovich potentials denote the dual solutions of the renowned optimal transportation problem. Uniqueness of these solutions is relevant from both a theoretical and an algorithmic point of view, and has recently emerged as a necessary condition for asymptotic results in the context of statistical and entropic optimal transport. In this work, we challenge the common perception that uniqueness in continuous settings is reliant on the connectedness of the support of at least one of the involved measures, and we provide mild sufficient conditions for uniqueness even when both measures have disconnected support. Since our main finding builds upon the uniqueness of Kantorovich potentials on connected components, we revisit the corresponding arguments and provide generalizations of well-known results. Several auxiliary findings regarding the continuity of Kantorovich potentials, for example in geodesic spaces, are established along the way.

Keywords: Kantorovich potentials, uniqueness, optimal transport, duality theory, regularity

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1 Introduction

Optimal transport theory addresses the question how mass can be moved from a source to a target distribution in the most cost-efficient way. While the history of this mathematical quest is long and rich, dating back to Monge[1781] and then Kantorovich[1942] who framed the theory in its modern formulation, it has drawn an enormous amount of attention in the past decades. In-depth monographs cover analytical, probabilistic, and geometric (Rachev and Rüschendorf
For given probability distributions $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ on measurable spaces $X$ and $Y$, the optimal transport problem is to find the minimal transportation cost

$$T_c(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int c \, d\pi,$$

(1a)

where $c : X \times Y \to \mathbb{R}_+$ denotes a non-negative measurable cost function that quantifies the effort of moving one unit of mass between elements in $X$ and $Y$, and where $\mathcal{C}(\mu, \nu) \subset \mathcal{P}(X \times Y)$ is the set of probability measures with marginal distributions $\mu$ and $\nu$. Any solution $\pi$ to (1a) is called an optimal transport plan. Conditions under which optimal transport plans exist are well-known, see for example Villani 2008 for the general framework of Polish spaces and lower-semicontinuous cost functions. Under these assumptions, the optimal transport problem (1a) also admits a dual formulation that reliably serves as a fertile ground for investigating structural properties of $T_c$. In fact, it holds that

$$T_c(\mu, \nu) = \sup_{f \in L^1(\mu)} \int f \, d\mu + \int f^c \, d\nu,$$

(1b)

where $f^c$, the $c$-transform of $f$, denotes the largest function satisfying $f(x) + f^c(y) \leq c(x, y)$ for all $x \in X$ and $y \in Y$. Specific optimizers $f$ for the dual problem – namely, those which can be written as the $c$-transform of a function $g$ on $Y$ – are called Kantorovich potentials, which exist under mild conditions (Villani 2008, Theorem 5.10).

While our efforts in this work focus on Kantorovich potentials, most of the foundational research on the optimal transport problem (1) has targeted properties of the primal solutions. Significant advances, which provided sufficient conditions for optimal plans to be concentrated on the graph of a uniquely determined function (the optimal transport map), have been achieved in Euclidean spaces (Smith and Knott 1987; Cuesta and Matrán 1989; Brenier 1991; Gangbo and McCann 1996), on manifolds (McCann 2001; Figalli 2007; Villani 2008; Figalli and Gigli 2011), and more recently also in more general metric spaces (Bertrand 2008; Gigli et al. 2012; Ambrosio and Rajala 2014). Strong regularity properties of optimal transport maps under squared Euclidean costs have first been established by the seminal work of Caffarelli (1990,1991,1992) for probability measures with bounded convex support (with recent extensions to unbounded settings by Cordero-Erausquin and Figalli 2019). Further insights were obtained by Ma et al. 2005 and Loeper 2009 for $C^1$-costs that satisfy a certain differential inequality (the Ma-Trudinger-Wang condition). Later, De Philippis and Figalli 2015 demonstrated regularity of optimal maps outside of “bad sets” of measure zero under more general conditions. A related line of research is devoted to the analysis of optimal transport plans that are not necessarily induced by a transport map. Uniqueness results for the primal solution in this context were, for example, obtained by Ahmad et al. 2011 or McCann and Rifford 2016 and more recently by Moameni and Rifford 2020.
Many of the techniques employed to characterize the primal solutions of \((1a)\) crucially depend on the duality theory \((1b)\). In fact, the gradients of dual solutions are intimately related to optimal transport maps (see Villani 2008, Chapter 10, for an in-depth treatment). Still, dual solutions are commonly not studied as objects of interest in their own right, and certain properties, such as the uniqueness of Kantorovich potentials, have received considerably less attention when compared to their primal counterparts. Recent developments, however, have emphasized the utility of dual uniqueness. For example, in the context of statistical optimal transport, uniqueness of Kantorovich potentials ensures a Gaussian limit distribution for the empirical optimal transport cost (Sommerfeld and Munk 2018; Del Barrio and Loubes 2019; Tameling et al. 2019; Del Barrio et al. 2021a; Del Barrio et al. 2021b). Furthermore, recent results on the convergence of entropically regularized optimal transport to its vanilla counterpart as the regularization tends to zero utilize dual uniqueness as a critical assumption as well (Altschuler et al. 2021; Bercu and Bigot 2021; Bernton et al. 2021; Nutz and Wiesel 2021). On a more general note, uniqueness is also required for the meaningful and efficient computation of optimal transport gradient flows in \(P¹(\mathcal{X})\), since Kantorovich potentials coincide with the subgradients of the functional \(\mu \mapsto T_c(\mu, \nu)\) (see Santambrogio 2015, Section 7.2).

For continuous measures in Euclidean spaces, several sufficient criteria for unique Kantorovich potentials are available. The involved arguments are well-known from the above mentioned literature on optimal transport maps, and depend on (i) sufficient local regularity of dual solutions \(f\) (e.g., local Lipschitz continuity) and (ii) exploiting that the gradient of \(f\) is (where it exists) determined by the cost function and an (arbitrary) optimal transport plan. Notable results that adopt this strategy include Proposition 7.18 in Santambrogio 2015, which is applicable in compact settings, or Appendix B of Bernton et al. 2021 and Corollary 2.7 of Del Barrio et al. 2021b, both relying on regularity properties of dual solutions derived by Gangbo and McCann 1996 for a certain family of strictly convex costs. Meanwhile, Remark 10.30 in Villani 2008 sketches a general argument for uniqueness of Kantorovich potentials on Riemannian manifolds. A commonality of these (and, to our knowledge, all other related) results in this vein is the requirement that the support of at least one probability measure \(\mu\) or \(\nu\) is connected. This requirement is usually taken as self-evident, since the standard proof technique – concluding uniqueness of a function (up to constants) from uniqueness of its gradients – naturally cannot bridge separated connected components. In fact, it is easy to construct trivial counter examples, like \(X = Y = [0, 1] \cup [2, 3]\) with uniformly distributed \(\mu = \nu\) on \(X\), where uniqueness of the Kantorovich potentials is bound to fail for a wide range of cost functions (see Lemma \[11\] in Appendix \[A\]).

At the same time, on finite spaces, dual uniqueness results for transportation problems have long been established via methods from finite linear programming (Klee and Witzgall 1968; Hung et al. 1986). Even though these settings naturally involve disconnected supports, they still feature unique Kantorovich potentials whenever the measures \(\mu\) and \(\nu\) are non-degenerate, meaning that all proper subsets \(X' \subset X\) and \(Y' \subset Y\) are assigned different masses \(\mu(X') \neq \nu(Y')\). The main conceptual contribution of our work is the formulation of an analogon of this observation accessible to the continuous world. This is realized by lifting the uniqueness of dual solutions on connected components to their uniqueness on the full space, a strategy that works in
The following statement is an informal version of our central result (Theorem 1).

**Theorem (Informal):** Assume that $\mu$ and $\nu$ are non-degenerate (in a suitable sense) and that each optimal potential $f^c$ is continuous. If the optimal transport problems restricted to the connected components of the support of $\mu$ have unique Kantorovich potentials, then the full optimal transport problem has unique Kantorovich potentials as well.

Clearly, this statement is mainly useful if combined with dual uniqueness results for measures with connected support. This motivates us to revisit the common proof strategy for uniqueness in the connected setting and present a formulation (Theorem 2) that is more general than the ones we have cited above. Theorem 2 covers settings where $X$ is a smooth manifold and $Y$ is allowed to be a generic Polish space, and we carefully discuss which properties of $\mu$, $\nu$, and $c$ are actually necessary for the argumentation. Particular scenarios where the assumptions of Theorem 2 can easily be checked include settings where the space $Y$ is compact (Corollary 2) or where the cost function satisfies certain growth and regularity conditions outside of compact sets (Corollary 3 and Section 4). In fact, we will learn that the requirements on $\mu$ are primarily of topological nature (i.e., concerning the shape of its support), while the actual distribution of mass can often be quite arbitrary. For example, in the setting $X = Y = [0, 1] \cup [2, 3]$ mentioned earlier, combining Theorem 1 and Corollary 2 establishes that Kantorovich potentials are unique if $\mu$ and $\nu$ are supported on all of $X$ and satisfy $\mu([0, 1]) \neq \nu([0, 1])$. This holds for general differentiable costs without further assumptions on $\mu$ or $\nu$ such as the existence of a Lebesgue density. In this sense, failure of uniqueness due to disconnected supports is typically an exception caused by a specific symmetry, and not the rule.

**Outline.** The notion of $c$-concavity is introduced and discussed in Section 2. Kantorovich potentials are then defined as $c$-concave solutions of the optimal transport problem in its dual formulation (1b). We proceed to discuss the regularity of such potentials, with a focus on the connection between continuity and transport towards infinity. Some technical results that cope with restrictions of the base spaces $X$ and $Y$ are emphasized as well. Section 3 opens with a clarification of equivalent ways to define almost surely unique Kantorovich potentials. Afterwards, our main results on uniqueness of Kantorovich potentials for probability measures with disconnected support (Theorem 1 and Corollary 1) are presented and discussed, including its consequences for the semi-discrete setting and countably discrete spaces. We then turn to uniqueness statements for measures with connected support on smooth manifolds (Theorem 2, Corollary 2, and Corollary 3). Section 4 contributes some findings on continuity properties of Kantorovich potentials for fast-growing cost functions (particularly for geodesic metric spaces, Theorem 3), revealing that discontinuities are often confined to the boundary of the support. Finally, Section 5 contains the proof of Theorem 1 and Appendix A documents auxiliary observations and arguments that have been omitted in the main text.

**Notation.** Throughout the manuscript, $X$ and $Y$ denote Polish spaces, i.e., completely metrizable and separable topological spaces. For $A \subset X$, we write $\text{int}(A)$ for its interior, $\text{cl}(A)$ for its closure, and $\partial A = \text{cl}(A) \setminus \text{int}(A)$ for its boundary. The Cartesian projection from a product of
spaces to a component \( X \) is denoted by \( p_X \). Real-valued functions \( f \) and \( g \) on spaces \( X \) and \( Y \) can be lifted to \( X \times Y \) via the operation \((x, y) \mapsto f(x) + g(y)\), which we denote by \( f \oplus g \).

The set of Borel probability measures, or distributions, on a Polish space \( X \) are called \( \mathcal{P}(X) \). The support of a probability distribution \( \mu \in \mathcal{P}(X) \), which is the smallest closed set \( A \subset X \) such that \( \mu(A) = 1 \), is denoted by \( \text{supp} \mu \). We write \( \mu \otimes v \in \mathcal{P}(X \times Y) \) to denote the product of probability measures on Polish spaces \( X \) and \( Y \). Integration \( \int f \, d\mu \) of a real-valued function \( f \) on \( X \) is abbreviated by juxtaposition \( \mu \).

If \( M \) is a smooth manifold (without boundary), we call \( f: M \to \mathbb{R} \) locally Lipschitz if \( f \circ \varphi^{-1} \) is locally Lipschitz for every chart \( \varphi \) of an atlas of \( M \). Similarly, we call \( f \) locally semiconcave if \( f \circ \varphi^{-1} \) is locally semiconcave for every chart \( \varphi \) of an atlas of \( M \). By this we mean that each point in range \( \varphi \) admits \( \lambda > 0 \) and a convex neighbourhood \( V \subset \text{range} \varphi \) such that

\[
\nu \mapsto f(\varphi^{-1}(\nu)) - \lambda \|\nu\|^2
\]

is concave on \( V \). A family of functions \( f_y: M \to \mathbb{R} \) for \( y \in Y \) is called \textit{locally Lipschitz (or locally semiconcave) uniformly in} \( y \) if \( f_y \) is locally Lipschitz (or locally semiconcave) with neighborhoods and constants that do not depend on \( y \). Similarly, the functions \( f_y \) are \textit{locally Lipschitz locally uniformly in} \( y \) if the functions \( f_y \) are locally Lipschitz uniformly in \( y \in K \) for each compact set \( K \subset Y \). We furthermore say that a Borel set \( A \) has \textit{full Lebesgue measure in charts of} \( M \) if range \( \varphi \setminus \varphi(A \cap \text{domain} \varphi) \) is a Lebesgue null set for each chart \( \varphi \) of an atlas of \( M \).

2 Kantorovich Potentials

Let \( c: X \times Y \to \mathbb{R}_+ \) be a non-negative cost function that compares elements of Polish spaces \( X \) and \( Y \). As laid out comprehensively in Villani [2008] or Santambrogio [2015], a central part of the duality theory of optimal transport is the notion of \( c \)-conjugacy. For any \( g: Y \to \mathbb{R} \cup \{ -\infty \} \), its associated \textit{\( c \)-transform} is defined via

\[
g^c: X \to \mathbb{R} \cup \{ -\infty \}, \quad g^c(x) = \inf_{y \in Y} c(x, y) - g(y). \tag{3a}
\]

Any function \( f: X \to \mathbb{R} \cup \{ -\infty \} \) that coincides with \( g^c \) for some \( g: Y \to \mathbb{R} \cup \{ -\infty \} \) and that is not equal \(-\infty\) everywhere is called \textit{\( c \)-concave on} \( X \). The set of all functions that are \( c \)-concave on \( X \) is denoted by \( S_c \). Since the roles of \( f \) and \( g \) can easily be exchanged in these definitions, we also write

\[
f^c: Y \to \mathbb{R} \cup \{ -\infty \}, \quad f^c(y) = \inf_{x \in X} c(x, y) - f(x) \tag{3b}
\]

for the \( c \)-transform of a function \( f: X \to \mathbb{R} \cup \{ -\infty \} \). Any \( g: Y \to \mathbb{R} \cup \{ -\infty \} \) that originates from a \( c \)-transform and that is not equal \(-\infty\) everywhere is called \textit{\( c \)-concave on} \( Y \). Since \( f = f^c \) and \( g = g^c \) for any \( c \)-concave \( f \) or \( g \) (see Santambrogio [2015] Proposition 1.34), the set \( S^c_c \) of pointwise \( c \)-transformed elements of \( S_c \) equals the set of functions that are \( c \)-concave on \( Y \). Under continuity of the cost function \( c \), all functions in \( S_c \) and \( S^c_c \) are upper-semicontinuous and thus Borel measurable. Note that our notation accentuates the asymmetry in the operations \( \oplus \) and \( \otimes \) less explicitly than Santambrogio [2015], who denotes \( (3a) \) as \( \hat{c} \)-transform, or Villani [2008] who picks a different sign convention and contrasts \( c \)-concavity to \( c \)-convexity.
For any given $f : X \to \mathbb{R} \cup \{-\infty\}$, the $c$-transform $g = f^c$ designates the largest function that satisfies $f \circ g \leq c$. The set of points in $X \times Y$ where equality holds is denoted as $c$-subdifferential of $f$, and we write

$$\partial_c f = \{(x, y) \in X \times Y \mid f(x) + f^c(y) = c(x, y)\}. \quad (4)$$

This set is closed when $c$ is continuous and $f$ is upper-semicontinuous (so in particular when $f$ is $c$-concave). If $f$ is a solution of the dual optimal transport problem (1b), it is clear that any optimal transport plan has to be concentrated on $\partial_c f$. The following statement, which is a special case of Theorem 5.10 (ii) in Villani [2008], establishes that (generalized) $c$-concave dual solutions exist under mild conditions.

**Theorem (Existence of optimal solutions):** Let $X$ and $Y$ be Polish and $c : X \times Y \to \mathbb{R}_+$ continuous. For any $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ with $T_c(\mu, \nu) < \infty$, there exists an optimal transport plan $\pi \in C(\mu, \nu)$ and a $c$-concave function $f \in S_c$ such that

$$T_c(\mu, \nu) = \pi c = \pi(f \circ f^c). \quad (5)$$

We emphasize that the function $f$ in this statement does not have to be $\mu$-integrable, nor does $f^c$ have to be $\nu$-integrable. Ensuring integrability requires further conditions (Villani [2008], Remark 5.14), for instance $(\mu \otimes \nu) c < \infty$. Only then can $f$ be viewed as a dual optimizer of (1b) in the strict sense

$$T_c(\mu, \nu) = \pi c = \mu f + \nu f^c.$$

For our ends, however, the more general solutions provided by (5) are sufficient. We call these solutions (generalized) Kantorovich potentials, and we write $f \in S_c(\mu, \nu) \subset S_c$ or $f^c \in S_c^c(\mu, \nu) \subset S_c^c$ to emphasize their dependence on $\mu$ and $\nu$. We stress that any $f \in S_c(\mu, \nu)$ satisfies (5) for all optimal transport plans $\pi$, so $f$ does not favor a particular primal solution (Beiglböck and Schachermayer [2011], Lemma 1.1).

Note that the existence of solutions as well as duality statements for optimal transportation problems have also been established for non-continuous cost functions (Villani [2008], Beiglböck and Schachermayer [2011]) or more general spaces (Rüschendorf [2007]). Two major advantages of working with continuous costs are the closedness of the $c$-subdifferential $\partial_c f$ for any $f \in S_c(\mu, \nu)$ and the (related) upper-semicontinuity of $c$-conjugate functions. The former implies

$$\text{supp} \pi \subset \partial_c f$$

for any optimal transport plan $\pi$, a property which we will often resort to, while the latter is needed for continuity results and permits sidestepping measurability issues.

**Regularity**

Due to their nature as $c$-concave functions, Kantorovich potentials inherit certain regularity properties from the cost function $c$. For example, if $c$ is concave in its first argument, then $f \in S_c$ is also concave as an infimum over concave functions. Similarly, if the family $\{c(\cdot, y) \mid y \in Y\}$ of partially evaluated costs is (locally) equicontinuous, then $f$ shares the respective (local) modulus
of continuity (Santambrogio 2015, Section 1.2). Imposing conditions of this form is hence a convenient way to guarantee continuity of \( c \)-concave functions, which are in general only upper-semicontinuous (for continuous costs). In the following, we introduce tools that give us a more fine-grained control over the continuity of Kantorovich potentials. We begin with continuity along sequences in \( \partial_c f \).

**Lemma 1:** Let \( X \) and \( Y \) be Polish, \( c : X \times Y \to \mathbb{R}_+ \) continuous, and \( f \in S_c \). If \((x_n, y_n)_{n \in \mathbb{N}}\) is a sequence in \( \partial_c f \) that converges to \((x, y) \in \partial_c f\), then \( f(x_n) \to f(x) \) and \( f^c(y_n) \to f^c(y) \) as \( n \to \infty \).

**Proof.** Both \( f \) and \( f^c \) are upper-semicontinuous. Since \((x_n, y_n)\) and \((x, y)\) are elements in \( \partial_c f \),

\[
f^c(y) \geq \limsup_{n \to \infty} f^c(y_n) \geq c(x, y) - \limsup_{n \to \infty} f(x_n) \geq c(x, y) - f(x) = f^c(y).
\]

Therefore, \( \limsup_{n \to \infty} f(x_n) = f(x) \) and \( \limsup_{n \to \infty} f^c(y_n) = f^c(y) \) has to hold. \( \square \)

We next show that Kantorovich potentials can only be discontinuous at points that are “sent to infinity” by all optimal transport plans. To put this more precisely, given a relatively open set \( U \subset \text{supp } \mu \), we say that a transport plan \( \pi \in C(\mu, \nu) \) induces regularity on \( U \) if there exists a compactum \( K \subset Y \) such that

\[
p_X(\text{supp } \pi) \cap U = p_X(\text{supp } \pi \cap (U \times K)),
\]

where \( p_X \) denotes the coordinate projection onto \( X \). We also say that \( \pi \) induces regularity at \( x \in \text{supp } \mu \), if there is a (relatively) open neighborhood \( U \subset \text{supp } \mu \) of \( x \) such that \( \pi \) induces regularity on \( U \) (see Figure 1). Similar definitions with reversed roles are deployed for subsets or points in \( \text{supp } \nu \), to which all of the following statements can be adjusted as well.
**Lemma 2**: Let $X$ and $Y$ be Polish, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and $c : X \times Y \to \mathbb{R}_+$ continuous with $T_c(\mu, \nu) < \infty$. Let $\pi \in C(\mu, \nu)$ be optimal and $f \in S_c(\mu, \nu)$. If $\pi$ induces regularity on $U \subset \text{supp} \, \mu$ with respect to the compact set $K \subset Y$, then $U \subset p_X(\text{supp} \, \pi)$ and

$$f(x) = \inf_{y \in K} c(x, y) - f^c(y)$$

for all $x \in U$. In particular, $f|_{\text{supp} \, \mu}$ is continuous on $U$.

**Proof.** Let $\pi$ induce regularity on the relatively open set $U \subset \text{supp} \, \mu$ with respect to the compact set $K \subset Y$ as defined in [6]. Restricted to the domain $X \times K$, the projection $p_X$ is a closed map. Condition [6] thus establishes that $A = p_X(\text{supp} \, \pi) \cap U = p_X(\text{supp} \, \pi \cap (X \times K) \cap U$ is relatively closed in $U$. Since $p_X(\text{supp} \, \pi)$ is dense in $\text{supp} \, \mu$, any point in $U$ is a limit point of $A$. Therefore, $U = A \subset p_X(\text{supp} \, \pi)$. Furthermore, each $x \in U$ admits a partner $y \in K$ such that $(x, y) \in \text{supp} \, \pi \subset A$. This establishes (7), since

$$f(x) = c(x, y) - f^c(y) = \inf_{y' \in K} c(x, y') - f^c(y').$$

To show continuity of $f|_{\text{supp} \, \mu}$ on $U$, it is enough to show that each sequence $(x_n)_{n \in \mathbb{N}} \subset \text{supp} \, \mu$ converging to $x \in U$ has a subsequence attaining the limit $f(x)$. Since $U$ is relatively open, we can assume $(x_n)_{n} \subset U$. Thus, each $x_n$ admits $y_n \in K$ such that $(x_n, y_n) \in \text{supp} \, \pi$. After taking a suitable subsequence, we may assume that $y_n \to y \in K$ due to the compactness of $K$. Thus, $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Since $\text{supp} \, \pi \subset \partial_c f$ is closed in $X \times Y$, it contains $(x, y)$, and so Lemma [1] can be applied to establish $\lim_{n \to \infty} f(x_n) = f(x)$. \hfill $\square$

**Remark 1** (Projections and measurability): We commonly formulate our results in terms of the projected sets $p_X(\text{supp} \, \pi) \subset \text{supp} \, \mu$ and $p_Y(\text{supp} \, \pi) \subset \text{supp} \, \nu$, where $\pi \in C(\mu, \nu)$ denotes an (arbitrary) optimal transport plan. Note that the inclusions can be strict, for which case Lemma [2] establishes the relation

$$\text{supp} \, \mu \setminus p_X(\text{supp} \, \pi) \subset \{ x \in \text{supp} \, \mu \mid \pi \text{ does not induce regularity at } x \}$$

and vice versa for $\nu$. Projected sets of the form $p_X(\text{supp} \, \pi)$ or $p_Y(\text{supp} \, \pi)$ are not Borel measurable in general, but they are analytic and thus contain Borel subsets of full $\mu$- or $\nu$-measure (see Lemma [9] in Appendix A for references). We usually prefer the explicit formulation via these sets (instead of writing “almost surely”) to emphasize that the domain of a property does not depend on the choice of a specific Kantorovich potential.

Some consequences of Lemma [2] deserve to be highlighted. First, the Kantorovich potentials $S_c(\mu, \nu)$ are always continuous if the space $Y$ is compact. If $Y$ is not compact, the intuition fostered by Lemma [2] is that discontinuities can only occur at points from whose immediate vicinity some mass is sent towards infinity, in the sense that this mass leaves any compactum in $Y$. Relation (7) is particularly useful for transferring properties of the cost function to Kantorovich.
potentials, like a modulus of continuity of \( c(\cdot, y) \) that holds only locally in \( y \). This observation becomes crucial for Theorem 2 in Section 3.

Examples where induced regularity fails at some points can easily be found, and include settings where \( \mu \) is compactly supported but \( \nu \) is not. At the offending points, \( f \) can turn out to be both continuous or discontinuous, depending on the regularity of the cost function as well as the specific behavior of \( \mu \) and \( \nu \) (this can already be observed for \( X = Y = \mathbb{R} \)). The following example anticipates that points of discontinuity are often restricted to the boundary of the support, a phenomenon that we more closely study in Section 4.

Example 1 (Continuity in geodesic spaces): Let \( X = Y \) for a locally compact complete geodesic space \((X, d)\) and consider a cost function of the form \( c(x, y) = h(d(x, y)) \) with convex and differentiable \( h: \mathbb{R}_+ \to \mathbb{R}_+ \). Then every Kantorovich potential \( f \in S_c(\mu, \nu) \) is continuous on the interior of the support of \( \mu \) and discontinuities at the boundary are only possible if \( h'(a) \to \infty \) as \( a \to \infty \). Proofs and more details are provided in Section 4.

We stress that regularity properties of Kantorovich potentials that go beyond mere continuity, like their degree of differentiability, are extensively studied in the literature on the optimal transport map (see Villani 2008, Chapter 10, for a detailed exposition). To mention a common argument in this context, one possible way to enforce twofold differentiability of each \( f \in S_c \) is to work with semiconcave cost functions.

Example 2 (Continuity under semiconcavity): Let \( X = \mathbb{R}^d \) for some \( d \in \mathbb{N} \) with the Euclidean norm \( \| \cdot \| \) and assume that the function class \( \{ c(\cdot, y) \mid y \in Y \} \) has uniformly bounded second derivatives. Then there exists \( \lambda > 0 \) such that \( x \mapsto c(x, y) - \lambda \| x \|^2 \) is concave for each \( y \in Y \), which implies concavity of \( x \mapsto f(x) - \lambda \| x \|^2 \) for \( f \in S_c(\mu, \nu) \) as well. In particular, \( f \) is continuous and Lebesgue-almost everywhere twice differentiable in the domain \( \Omega = \text{int}(\{ x \mid f(x) > -\infty \}) \subset \mathbb{R}^d \), which is convex and contains the interior of \( \text{supp} \mu \). On the boundary of \( \Omega \), the potential may assume finite values or \( -\infty \).

Restrictions

A valuable trait of optimal transport theory is that both primal and dual solutions behave consistently when the base spaces \( X \) and \( Y \) are restricted to subspaces. General results in this direction can be found in Villani 2008, Theorem 4.6 and Theorem 5.19. In the following, we stress some selected statements that complement our assertions on continuity and uniqueness of Kantorovich potentials. We begin with a technical observation that restrictions of Kantorovich potentials of the form \( f|_{\text{supp} \mu} \), as they appear in Lemma 2, can (almost) be understood as Kantorovich potentials of a suitably restricted problem. The proof is delegated to Appendix A.
We next address the behavior of Kantorovich potentials when transportation is restricted to disconnected spaces. As a consequence of Lemma 4, it is always possible to decompose Kantorovich potentials defined on disconnected spaces in a natural way. Indeed, if \( \mu = \bigcup_{i \in I} X_i \) is a countable partitioning into connected components (which are always closed) with \( \mu(X_i) > 0 \) for each \( i \in I \), then any \( f \in S_c(\mu, \nu) \) admits restricted potentials \( f_i \in S_{c_{X_i}}(\mu_{X_i}, \nu_{X_i}) \) such that

\[
 f = \sum_{i \in I} 1_{X_i} \cdot f_i \quad \text{on } \rho_X(\text{supp } \pi) \tag{8}
\]

for any optimal \( \pi \in C(\mu, \nu) \). This simple but crucial observation lies at the heart of the uniqueness result for probability measures with disconnected support discussed in the next section.

Lemma 3 (Restriction to sets of full mass): Let \( X \) and \( Y \) be Polish and \( c: X \times Y \to \mathbb{R}_+ \) continuous. Suppose \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \) such that \( T_c(\mu, \nu) < \infty \). Let \( \pi \in C(\mu, \nu) \) be an optimal plan and let \( \tilde{c} \) denote the restriction of \( c \) to the set \( \tilde{X} \times \tilde{Y} \), where \( \tilde{X} \subset X \) and \( \tilde{Y} \subset Y \) are Borel and Polish subspaces with \( \mu(\tilde{X}) = \nu(\tilde{Y}) = 1 \). Let \( \tilde{\Gamma} = \text{supp } \pi \cap (\tilde{X} \times \tilde{Y}) \).

(Restrict) Every \( f \in S_c(\mu, \nu) \) admits \( \tilde{f} \in S_{c_{\tilde{X}}}(\mu_{\tilde{X}}, \nu_{\tilde{Y}}) \) that agrees with \( f \) on \( \rho_X(\tilde{\Gamma}) \).

(Extend) Every \( f \in S_{c}(\mu, \nu) \) admits \( \tilde{f} \in S_{c_{\tilde{X}}}(\mu_{\tilde{X}}, \nu_{\tilde{Y}}) \) that agrees with \( f \) on \( \rho_X(\tilde{\Gamma}) \).

In both cases, the conjugates \( f^c \) and \( \tilde{f}^c \) agree on \( \rho_Y(\tilde{\Gamma}) \).

Remark 2 (Ambiguity of extensions): Depending on the setting, there can be distinct ways of extending Kantorovich potentials from the support to the whole space. For example, if \( X = Y \) are equal and \( c \) is a metric, then the \( c \)-concave functions are exactly the 1-Lipschitz functions with respect to \( c \), and it is easy to see that \( f^c = -f \) holds for any \( f \in S_c \). In this situation, ambiguous extensions are common if \( \text{supp } \mu \cup \text{supp } \nu \) does not cover the whole space \( X \).

We next address the behavior of Kantorovich potentials when transportation is restricted to a part of \( X \) that does not necessarily occupy full \( \mu \)-mass. Let \( \pi \in C(\mu, \nu) \) be an optimal plan and suppose that \( \mu(\tilde{X}) > 0 \) for some closed subset \( \tilde{X} \subset X \). We denote the optimal transport problem between the probability measures \( \mu_{\tilde{X}} = \mu|_{\tilde{X}}/\mu(\tilde{X}) \) and \( \nu_{\tilde{X}} = \pi(\tilde{X} \times \cdot)/\mu(\tilde{X}) \) under the cost function \( c_{\tilde{X}} = c|_{X \times Y} \) as the \( \tilde{X} \)-restricted problem (with respect to \( \pi \)). As an application of Villani [2008] Theorem 5.19, we note that the restriction of Kantorovich potentials in the original problem yields Kantorovich potentials in the restricted problem.

Lemma 4 (Restriction to sets of partial mass): Let \( X \) and \( Y \) be Polish and \( c: X \times Y \to \mathbb{R}_+ \) continuous. Suppose \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \) such that \( T_c(\mu, \nu) < \infty \) and let \( \tilde{X} \subset X \) be closed with \( \mu(\tilde{X}) > 0 \). Then any \( f \in S_c(\mu, \nu) \) admits \( \tilde{f} \in S_{c_{\tilde{X}}}(\mu_{\tilde{X}}, \nu_{\tilde{X}}) \) that agrees with \( f \) on \( \rho_X(\text{supp } \pi) \cap \tilde{X} \).

As a consequence of Lemma 4, it is always possible to decompose Kantorovich potentials defined on disconnected spaces in a natural way. Indeed, if \( \mu = \bigcup_{i \in I} X_i \) is a countable partitioning into connected components (which are always closed) with \( \mu(X_i) > 0 \) for each \( i \in I \), then any \( f \in S_c(\mu, \nu) \) admits restricted potentials \( f_i \in S_{c_{X_i}}(\mu_{X_i}, \nu_{X_i}) \) such that

\[
 f = \sum_{i \in I} 1_{X_i} \cdot f_i \quad \text{on } \rho_X(\text{supp } \pi) \tag{8}
\]
3 Uniqueness

In a strict sense, Kantorovich potentials are never unique. Indeed, it is easy to see that \( f \in S_c(\mu, \nu) \) implies \( f + a \in S_c(\mu, \nu) \) for any \( a \in \mathbb{R} \). Therefore, statements about uniqueness are generally only reasonable up to constant shifts. Besides this ambiguity, it is often too restrictive to require uniqueness to hold outside of the supports of the involved measures (see Remark 2 on ambiguous extensions). We will therefore focus on the notion of almost surely unique Kantorovich potentials (up to constant shifts), by which we mean that \( f_1 - f_2 \) is \( \mu \)-almost surely constant for all \( f_1, f_2 \in S_c(\mu, \nu) \). Due to the regularizing nature of the \( c \)-transform, almost sure uniqueness of \( S_c(\mu, \nu) \) is actually equivalent to almost sure uniqueness of \( S(\mu, \nu) \).

Lemma 5: Let \( X \) and \( Y \) be Polish, \( \mu \in \mathcal{P}(X) \), \( \nu \in \mathcal{P}(Y) \), and \( c : X \times Y \rightarrow \mathbb{R} \), continuous such that \( T_c(\mu, \nu) < \infty \). For any optimal transport plan \( \pi \) and any \( f_1, f_2 \in S_c(\mu, \nu) \)

1) \( f_1 = f_2 \) \( \mu \)-almost surely iff \( f_1 = f_2 \) on \( p_X(\text{supp} \ \pi) \),
2) \( f_1^c = f_2^c \) \( \nu \)-almost surely iff \( f_1^c = f_2^c \) on \( p_Y(\text{supp} \ \pi) \),
3) \( f_1 = f_2 \) on \( p_X(\text{supp} \ \pi) \) iff \( f_1^c = f_2^c \) on \( p_Y(\text{supp} \ \pi) \).

Proof. We begin with the first assertion and assume that \( f_1 = f_2 \) holds on a Borel set \( A \subset X \) with \( \mu(A) = 1 \). The set \( B = \text{supp} \ \pi \cap (A \times Y) \) is dense in \( \text{supp} \ \pi \), so that there is a convergent sequence in \( B \) to any \( (x, y) \in \text{supp} \ \pi \). Lemma 1 then asserts \( f_1(x) = f_2(x) \) for all \( x \in p_X(\text{supp} \ \pi) \). Conversely, Lemma 9 in Appendix A shows that \( p_X(\text{supp} \ \pi) \) contains a Borel set of full \( \mu \)-measure. Assertion 2 follows similarly. To show assertion 3, it is sufficient to observe \( c(x, y) = f_1(x) + f_1^c(y) = f_2(x) + f_2^c(y) \) and thus \( f_1(x) - f_2(x) = f_2^c(y) - f_1^c(y) \) for any \( (x, y) \in \text{supp} \ \pi \). 

Disconnected support

Our contributions regarding the uniqueness of Kantorovich potentials for measures with disconnected support are inspired by well-known results from the theory of finite linear programming. In a nutshell, we will show that unique Kantorovich potentials on the connected components of the support are sufficient to imply the uniqueness on the whole support, as long as we have continuous \( c \)-transformed potentials and so-called non-degenerate optimal plans. If

\[
\text{supp} \ \mu = \bigcup_{i \in I} X_i \quad \text{and} \quad \text{supp} \ \nu = \bigcup_{j \in J} Y_j
\]

are (at most countable) decompositions of the supports of \( \mu \) and \( \nu \) into connected components, then \( \pi \in C(\mu, \nu) \) is called degenerate if there exist subsets \( I' \subset I \) and \( J' \subset J \) such that

\[
0 < \sum_{i \in I'} \mu(X_i) = \sum_{i \in I'} \sum_{j \in J'} \pi(X_i \times Y_j) = \sum_{j \in J'} \nu(Y_j) < 1.
\]

This definition allows for the following sufficient criterion for non-degeneracy, which has the advantage that it can easily be checked on the basis of \( \mu \) and \( \nu \) alone.
Lemma 6: If all nonempty proper \( I' \subset I \) and \( J' \subset J \) satisfy \( \sum_{i \in I'} \mu(X_i) \neq \sum_{j \in J'} \nu(Y_j) \), then no transport plan \( \pi \in C(\mu, \nu) \) is degenerate.

Under suitable conditions, non-degenerate optimal transport plans make it possible to uniquely link together Kantorovich potentials of the \( X_i \)-restricted transport problems (recall this notion from Section 2) to assert uniqueness of Kantorovich potentials on the full support. For the following result, note that \( \mu(X_i) > 0 \) for all \( i \in I \) if \( I \) is finite, since each \( X_i \subset \text{supp} \mu \) is open in this case.

Theorem 1 (Uniqueness under disconnected support): Let \( X \) and \( Y \) be Polish, \( \mu \in \mathcal{P}(X) \), \( \nu \in \mathcal{P}(Y) \), and \( c: X \times Y \to \mathbb{R}_+ \) continuous with \( T_c(\mu, \nu) < \infty \). Assume decomposition \( \mu = \sum_i \mu_i(X_i) \) for \( I \) finite and \( J \) (at most) countable and assume that for all \( f^c \in S_c^*(\mu, \nu) \) either

1) \( f^c|\text{supp} \nu \) is continuous, or
2) \( f^c|Y_j \) is continuous and \( \text{supp} \nu|Y_j \) is connected for all \( j \in J \) with \( \nu(Y_j) > 0 \).

If there exists a non-degenerate optimal transport plan \( \pi \in C(\mu, \nu) \) with respect to which the \( X_i \)-restricted Kantorovich potentials \( S_{c\mid X_i}^{\mu(X_i), \nu(X_i)} \) are almost surely unique for all \( i \in I \), the Kantorovich potentials \( S_c(\mu, \nu) \) are also almost surely unique.

Example 3 (Semi-discrete optimal transport): Let \( X \) be a finite set with \( n \in \mathbb{N} \) elements and \( Y \) be a general Polish space. Questions concerning dual uniqueness in this setting, which is referred to as semi-discrete optimal transport, have recently been raised by Altschuler et al. [2021] and Bercu and Bigot [2021]. Theorem 1 provides simple and general answers in this context, since the uniqueness of the \( X_i \)-restricted Kantorovich potentials and the continuity of all \( f^c \in S_c^* \) turn out to be trivial (for continuous \( c \)). For example, if \( \nu \in \mathcal{P}(Y) \) has connected support, Kantorovich potentials are always unique in the above sense for any \( \mu \in \mathcal{P}(X) \). If the support of \( \nu \) is disconnected with (at most) countably many components \( Y_j \), uniqueness holds if the measures \( \mu \) and \( \nu \) are non-degenerate in the sense of Lemma 6. In particular, when \( \mu \) is the uniform distribution on \( X \), Kantorovich potentials are guaranteed to be unique unless \( \nu \) assigns a multiple of mass \( 1/n \) to individual connected components of \( \text{supp} \nu \).

A sketch that assists in the interpretation of Theorem 1 is provided by Figure 2. At the heart of the proof lies observation (8), which ensures that each Kantorovich potential in \( S_c(\mu, \nu) \) assumes the form

\[
f_a = \sum_{i \in I} 1_{X_i} \cdot (f_i + a_i) \quad \text{on } p_X(\text{supp } \pi)
\]

for some \( a \in \mathbb{R}^{|I|} \), where representatives \( f_i \in S_{c\mid X_i}^{\mu(X_i), \nu(X_i)} \) of the \( X_i \)-restricted problems have been fixed. Not each choice of \( a \) leads to a viable optimal solution \( f_a \in S_c(\mu, \nu) \), however. Due to the continuity of \( f^c_a \), one can show that a value \( a_{i_1} \) uniquely determines \( a_{i_2} \) for \( i_1, i_2 \in I \) if the masses transported from \( X_{i_1} \) and \( X_{i_2} \) to \( Y \) touch one another, meaning that their topological closures have a common contact point (see Section 5 for formal definitions). The non-degeneracy
Figure 2: Uniqueness of Kantorovich potentials in disconnected spaces. In sketch (a), transport between $X_1 \cup X_2$ and $Y_1$ is decoupled from transport between $X_3$ and $Y_2$. As the proof of Theorem 1 shows, the existence of a contact point $y_1$ links the (restricted) Kantorovich potentials $f_1 : X_1 \to \mathbb{R}$ and $f_2 : X_2 \to \mathbb{R}$. However, since $f_3 : X_3 \to \mathbb{R}$ is linked to neither $f_1$ nor $f_2$, uniqueness of the full Kantorovich potential $f : X \to \mathbb{R}$ is not guaranteed. In sketch (b), $f_1$ is linked to $f_2$ via $y_1$ and $f_2$ is linked to $f_3$ via $y_2$. Therefore, uniqueness of $f$ follows from uniqueness of the restricted Kantorovich potentials if $f^\pi$ is continuous at $y_1$ and $y_2$.

of $\pi$ then ensures the existence of suitable contact points such that fixing $a_{i_0}$ for an arbitrary $i_0 \in I$ actually determines the whole vector $a$, which in consequence implies dual uniqueness. The full proof is documented in Section 5 while the following paragraphs discuss the assumptions in Theorem 1.

Degeneracy. The uniqueness of Kantorovich potentials can break down if the condition of non-degeneracy of $\pi$ is not satisfied. For a simple family of examples, consider non-negative continuous and symmetric costs with $c(x,x) = 0$ for $x \in X = Y$ in a setting where $\mu = \nu$ with $\text{supp}(\mu) = X_1 \cup X_2$. If the components $X_1$ and $X_2$ are strictly cost separated, $\Delta = \inf_{x_1 \in X_1, x_2 \in X_2} c(x_1, x_2) > 0$,

each optimal plan $\pi$ satisfies $\pi(X_1 \times X_2) = \pi(X_2 \times X_1) = 0$. In particular, no mass is transported between $X_1$ and $X_2$. In this situation, any pair $a, b \in \mathbb{R}$ with $|a - b| \leq \Delta$ defines a Kantorovich potential $f_{a,b}$ via $f_{a,b} = a$ on $X_1$ and $f_{a,b} = b$ on $X_2$. A proof of this observation is provided in Appendix A, Lemma 11.

Continuity. Even in the presence of non-degenerate optimal transport plans, the uniqueness of Kantorovich potentials can in principle still break down due to discontinuities of the cost function or the potentials. For instance, the construction of non-unique potentials $f_{a,b}$ in Lemma 11 under a $\Delta$-separation between components of a disconnected support can easily be carried over to cost functions that exhibit a jump by $\Delta$ between two disjoint subsets of $\text{supp} \mu$, even if the support is connected.
For continuous costs, we discussed in Section 2 that the c-transformed Kantorovich potentials \( f^c \in S^c(\mu, \nu) \) are always continuous when the family \( \{c(x, \cdot) \mid x \in X\} \) of partially evaluated costs is (locally) equicontinuous, or when the space \( X \) is compact (Lemma 2). We also note that conditions 1 and 2 in Theorem 1 can actually be relaxed, and it would in both cases suffice to require continuity only at the finite number of contact points (implicitly) constructed in the proof. According to Lemma 2, this is for example guaranteed if \( \pi \) induces regularity at each contact point \( y \in Y \). In settings with \( \nu(\partial \text{supp} \nu) = 0 \), we can sometimes even drop the additional continuity assumption altogether: if the functions \( f^c \) are known to be continuous in the interior of \( \text{supp} \nu \), which is true for a wide range of superlinear costs (see Section 4, which in particular covers Example 1) or costs with uniformly bounded second derivatives (see Example 2), then we can apply Lemma 3 to transition to the restricted problem with \( \tilde{X} = X \) and \( \tilde{Y} = \text{int}(\text{supp} \nu) \subset Y \). This reformulation, where one only has to heed possible changes in decomposition (9) when replacing \( Y \) by \( \tilde{Y} \) (which may affect the degeneracy of optimal transport plans), makes sure that suitable contact points \( y \) can always be found in the interior of \( \text{supp} \nu \).

**Corollary 1:** Under either of the following additional assumptions, Theorem 1 remains valid for countable index sets \( I \), where uniqueness of the \( X_i \)-restricted Kantorovich potentials is only required for \( i \in I \) with \( \mu(X_i) > 0 \).

1) Condition 2 in Theorem 1 holds and \(|\{i \in I \mid \pi(X_i \times Y_j) > 0\}| < \infty \) for all \( j \in J \).

2) \( Y_j \) consists of a single point for each \( j \in J \) with \( \nu(Y_j) > 0 \) (then condition 2 in Theorem 1 is always satisfied).

In the special case of statement 2 of Corollary 1, where all components \( X_i \) are also single points, we conclude that non-degeneracy of the vectors \( (\mu(X_i))_{i \in I} \) and \( (\nu(Y_j))_{j \in J} \) as in Lemma 6 is already sufficient to imply uniqueness of Kantorovich potentials without further assumptions. This criterion has long been established for finite transportation problems via the theory of finite linear programming (Klee and Witzgall 1968; Hung et al. 1986), but we are not aware of a comparable result that covers probability measures with countable support. In particular, we note that Corollary 1 yields sufficient conditions for Gaussian distributional limits in the countable discrete settings studied by Tameling et al. 2019.

**Connected support**

One piece that is still missing to fully utilize Theorem 1 is a set of criteria for the uniqueness of Kantorovich potentials on the individual connected components of the support. In Euclidean settings, results in this regard are readily available, see for example Proposition 7.18 in Santam-
brogio 2015 (compactly supported measures) or more recently Appendix B of Bernton et al. 2021 and Corollary 2.7 of Del Barrio et al. 2021b (possibly non-compactly supported measures). The necessary techniques for these statements have long been established (Brenier 1991; Gangbo and McCann 1996) and have been extended to the more general setting of manifolds (McCann 2001; Villani 2008; Fathi and Figalli 2010; Figalli and Gigli 2011). In the following, we briefly revisit the underlying arguments and spell out a general uniqueness result together with some of its consequences for probability measures with connected support. To our knowledge, no formal statement with comparable scope has yet been assembled in this form, even though the involved arguments are well known (see for example Villani 2008, Remark 10.30).

Let \( \pi \) denote an optimal transport plan under a continuous cost function \( c \). Recalling the properties of the \( c \)-transform, any Kantorovich potential \( f \) satisfies \( f(x) + f^*(y) \leq c(x, y) \) for all \( (x, y) \in X \times Y \) with equality if \( (x, y) \in \partial_c f \). Fixing \( (x, y) \in \text{supp} \pi \subset \partial_c f \), this implies

\[
x' \mapsto f(x') - c(x', y) \quad \text{is minimal at } x' = x.
\]

Therefore, if \( X \) is a smooth manifold (without boundary) and the functions \( f \) as well as \( x' \mapsto c(x', y) \) are both differentiable at \( x \), it has to hold that

\[
\nabla f(x) = \nabla_x c(x, y), \tag{12}
\]

by which we mean equality of the respective gradients in charts of \( X \). This relation determines the derivatives of Kantorovich potentials in the set \( p_X(\Gamma) \subset X \), where we define

\[
\Gamma = \{ (x, y) \in \text{supp} \pi \mid \nabla_x c(x, y) \text{ exists} \}. \tag{13}
\]

In order to conclude uniqueness of \( f \) up to constants from characterization (12), we will make use of the following auxiliary result. A proof is provided in Appendix A.

**Lemma 7**: Let \( M \subset X \) be an open and connected subset of a smooth manifold \( X \) and let \( f_1, f_2 : M \to \mathbb{R} \) be locally Lipschitz. If \( \nabla f_1 = \nabla f_2 \) on a set that has full Lebesgue measure in charts of \( M \), then \( f_1 - f_2 \) is constant on \( M \).

At this point, several considerations have to be taken into account.

1. The region \( M \subset X \) chosen for Lemma 7 should have full \( \mu \)-measure, or it should at least be possible to uniquely recover the function \( f \) from \( f \mid_M \) on a set with full \( \mu \)-measure.

2. The Kantorovich potential \( f \) must be locally Lipschitz on \( M \) in order for Lemma 7 to be applicable. This also implies the existence of \( \nabla f \) in a set of full Lebesgue measure in each chart (via Rademacher’s theorem).

3. The cost function has to be sufficiently smooth in its first argument along the transport. More precisely, \( p_X(\Gamma) \) must have full Lebesgue measure in charts of \( M \). Otherwise, relation (12) would not determine the gradients of \( f \) suitably for application of Lemma 7.
One immediate conclusion of the first point is the necessity of the condition cl(M) = supp µ, without which some mass would be out of reach from the region M controlled by the gradients. The second and third points stress that the cost function should be locally Lipschitz in its first argument, in a way that is inherited to S.c. In order to choose M properly for a general uniqueness statement, we recall the notion of induced regularity defined in Section 2 and set

\[\Sigma = \{ x \in \text{supp } \mu \mid \pi \text{ does not induce regularity at } x \}. \quad (14)\]

We know that this set is closed (see Lemma 10 in Appendix A), that supp µ \(\Sigma \subset p_X(\text{supp } \pi)\), and that \(\Sigma\) contains all points of discontinuity of \(f|_{\text{supp } \mu}\) for any \(f \in S_c(\mu, \nu)\) (see Lemma 2).

**Theorem 2 (Uniqueness under connected support):** Let \(X\) be a smooth manifold, \(Y\) be Polish, \(\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)\), and \(c : X \times Y \to \mathbb{R}_+\) continuous such that \(c(\cdot, y)\) is locally Lipschitz locally uniformly in \(y \in Y\) with \(T_c(\mu, \nu) < \infty\). Let \(\Gamma\) and \(\Sigma\) be as in (13) and (14) for an optimal \(\pi \in C(\mu, \nu)\). If

1. \(\mu(\text{supp } \mu \setminus \Sigma) = 1,\)
2. \(M = \text{int}(\text{supp } \mu \setminus \Sigma)\) is connected with \(\text{cl}(M) = \text{supp } \mu,\)
3. \(p_X(\Gamma)\) has full Lebesgue measure in charts of \(M,\)

then the Kantorovich potentials \(S_c(\mu, \nu)\) are almost surely unique.

**Remark 3 (Uniqueness of optimal transport maps):** Solving equation (12) for \(y \in Y\) is a standard method to construct and study optimal transport maps \(t : X \to Y\), see Villani 2008, Chapter 10. In this context, a natural requirement is the injectivity of \(\nabla_x c(x, \cdot)\), denoted as the *twist condition*, which also implies that an optimal map is uniquely defined wherever (12) holds. To make sure that \(t\) is determined by (12) \(\mu\)-almost surely, one usually imposes some form of regularity on \(c\) (such as local semiconcavity) and requires the probability measure \(\mu\) to assign no mass to sets on which Kantorovich potentials may be non-differentiable (e.g., by assuming a Lebesgue density). Theorem 2 helps clarify to which extent similar assumptions on \(c\) and \(\mu\) are necessary if we are only interested in uniqueness of the Kantorovich potentials, and not in the uniqueness of optimal maps.

**Proof.** According to Lemma 2, each point of \(M\) admits an open neighborhood \(U\) and a compactum \(K \subset Y\) such that \(f(x) = \inf_{y \in K} c(x, y) - f^*(y)\) for all \(x \in U\) and \(f \in S_c(\mu, \nu)\). Since \(c(\cdot, y)\) is locally Lipschitz uniformly in \(y \in K\) by assumption, we conclude that \(f\) is locally Lipschitz on \(M\). Now, let \(f_1, f_2 \in S_c(\mu, \nu)\) and let \(A\) be the subset of \(M\) where both functions are differentiable. Set \(B = A \cap p_X(\Gamma)\), which is the set where the gradients of \(f_1\) and \(f_2\) have to coincide via (12). Due to Rademacher’s Theorem (see, e.g., Federer 2014, Theorem 3.1.6) and assumption 3, the set \(B\) has full Lebesgue measure in each chart of \(M\). We can thus apply Lemma 2 under assumption 2 and conclude that \(f_1 = f_2\) (up to a constant) on \(M\). Since any \(f \in S_c(\mu, \nu)\) is continuous at each point in \(\text{supp } \mu \setminus \Sigma \subset \text{cl}(M)\) via Lemma 2, the function \(f\) is uniquely determined on \(\text{supp } \mu \setminus \Sigma\) by its values on \(M\). This shows that \(f_1 = f_2\) (up to a constant) on \(\text{supp } \mu \setminus \Sigma\), which is a set of full \(\mu\)-measure by condition 1. \(\square\)
Without further context, the assumptions in this theorem may seem to be fairly opaque, as they rely on specific details of an optimal transport plan \( \pi \), like the \( \mu \)-measure of \( \Sigma \) and topological properties of \( \text{supp} \mu \setminus \Sigma \). In more specialized settings, however, the requirements of Theorem 2 can often be checked easily. As a first example, we assume \( Y \) to be compact. If \( c \) is differentiable in its first component, then all assumptions of Theorem 2 that depend on \( \pi \) are automatically satisfied and only conditions on the topology of \( \text{supp} \mu \) remain.

**Corollary 2:** Let \( X \) be a smooth manifold, \( Y \) be compact Polish, \( \mu \in \mathcal{P}(X) \), \( \nu \in \mathcal{P}(Y) \), and \( c : X \times Y \to \mathbb{R}_+ \) continuous such that \( c(\cdot, y) \) is differentiable and locally Lipschitz uniformly in \( y \in Y \) with \( T_c(\mu, \nu) < \infty \). If \( M = \text{int}(\text{supp} \mu) \) is connected and \( \text{cl}(M) = \text{supp} \mu \), then the Kantorovich potentials \( S_c(\mu, \nu) \) are almost surely unique.

**Proof.** For compact \( Y \), it holds by definition that \( \Sigma = \emptyset \). Thus, conditions 1 and 2 of Theorem 2 are satisfied. Condition 3 is ensured since \( c \) is differentiable in the first component and we thus find \( \Gamma = p_X(\text{supp} \pi) = \text{supp} \mu \) (since the projection \( p_X \) is a closed map for compact \( Y \)).

If \( Y \) is not compact, uniqueness statements based on Theorem 2 hinge on the behavior of the cost function outside of \( Y \)-compacta. The simplest setting of this kind is the one where \( c(\cdot, y) \) is (locally) Lipschitz uniformly in \( y \in Y \). Then, a statement analogous to Corollary 2 is possible, where the only obstacle is to assert condition 3 of Theorem 2. To provide a convenient sufficient criterion to this end, we work with the assumption that

\[
\lambda \text{ is absolutely continuous w.r.t. } \varphi_*\mu \text{ on range } \varphi \tag{15}
\]

for any chart \( \varphi \) of \( M = \text{int}(\text{supp} \mu) \), where \( \varphi_*\mu := \mu \circ \varphi^{-1} \) corresponds to the push-forward measure of \( \mu \) under \( \varphi \) and \( \lambda \) denotes the Lebesgue measure. Loosely speaking, this property states that the mass of \( \mu \) can be placed quite arbitrarily on \( \text{supp} \mu \), as long as it contains a continuous component everywhere on its support. As an example where condition (15) fails for \( X = \mathbb{R} \), consider a measure \( \mu \) that is concentrated on the rational numbers \( \mathbb{Q} \) but where \( \text{supp} \mu \) contains an open set.

**Corollary 3:** Let \( X \) be a smooth manifold, \( Y \) Polish, \( \mu \in \mathcal{P}(X) \), \( \nu \in \mathcal{P}(Y) \), and \( c : X \times Y \to \mathbb{R}_+ \) continuous such that \( c(\cdot, y) \) is differentiable and locally Lipschitz uniformly in \( y \in Y \) with \( T_c(\mu, \nu) < \infty \). If \( M = \text{int}(\text{supp} \mu) \) is connected such that \( \text{cl}(M) = \text{supp} \mu \) and condition (15) holds, then the Kantorovich potentials \( S_c(\mu, \nu) \) are almost surely unique.

**Proof.** Since \( c(\cdot, y) \) is assumed to be locally Lipschitz uniformly in \( y \in Y \), every Kantorovich potential \( f \in S_c(\mu, \nu) \) is locally Lipschitz on \( \text{supp} \mu \). Looking at the proof of Theorem 2 we furthermore note that the set \( \Sigma \) can actually be replaced by any other subset of \( X \) that contains all points at which some Kantorovich potential fails to be locally Lipschitz. Therefore, we may functionally assume \( \Sigma = \emptyset \) in conditions 1 and 2 of Theorem 2 and only have to show that condition 3 holds with \( M = \text{int}(\text{supp} \mu) \). Since \( \Gamma = \text{supp} \pi \) due to differentiability of \( c \), we find a
Remark 4 (local semiconcavity): In Example 2 we pointed out that Kantorovich potentials can inherit semiconcavity from the cost function. In fact, it is possible to formulate Corollary 3 for costs where \( c(\cdot, y) \) is locally semiconcave (instead of locally Lipschitz) uniformly in \( y \in Y \), in the sense of equation (2). This alternative formulation, whose proof is documented in Appendix A, can be put to use in settings where Corollary 3 might not apply directly. For example, let \( X \) and \( Y \) be two (possibly distinct) affine subspaces of \( \mathbb{R}^d \) equipped with an atlas of linear charts. Then the squared Euclidean cost function \( c(x, y) = \|x - y\|^2 \) is differentiable and locally semiconcave (but not necessarily locally Lipschitz) in \( x \) uniformly in \( y \). Consequently, the Kantorovich potentials in this setting are unique under the mild conditions on \( \mu \) imposed by Corollary 3. If \( \text{supp} \mu \) and \( \text{supp} \nu \) are separated sets, the same holds for Euclidean cost functions \( c(x, y) = \|x - y\|^p \) with \( 0 < p \leq 2 \).

One question largely unaddressed by the previous results is how Theorem 2 fares in the general setting that \( c(\cdot, y) \) is actually no more than locally Lipschitz locally uniformly in \( y \in Y \), which is typically the case for rapidly growing cost functions. In the next section, we show that such cost functions often confine the set \( \Sigma \) to the boundary of \( \text{supp} \mu \), which makes the application of Theorem 2 particularly simple: condition 1 collapses into \( \mu(\partial \text{supp} \mu) = 0 \) and condition 2 only relies on the topology of \( \text{supp} \mu \). Furthermore, condition 3 is always satisfied when \( c \) is differentiable in the first component, since then \( \Gamma = \text{supp} \pi \) and \( M \subset \text{supp} \mu \setminus \Sigma \subset p_X(\Gamma) \) via Lemma 2.

4 Interior regularity

We now investigate conditions on the cost function under which each optimal transport plan \( \pi \) induces regularity in the interior of the support of \( \mu \). In other words, we search for criteria that ensure

\[
\Sigma \subset \partial \text{supp} \mu,
\]

where \( \Sigma \subset \text{supp} \mu \) was defined in (14) and denotes the set of points where induced regularity fails. This property is of interest for both Theorem 1 (applied to \( \nu \) in place of \( \mu \)) and Theorem 2, as it guarantees continuity of Kantorovich potentials in the interior of the support. Instead of working with definition (6) of induced regularity directly, we will show the slightly stronger result that only boundary points can be “sent towards infinity”, which implies (16). To achieve this, the cost function has to behave in a certain way if \( y \) leaves all compacta in \( Y \). For convenience, we write \( y_n \to \infty \) to denote that each compact set in \( Y \) contains only a finite number of elements of the sequence \( (y_n)_{n \in \mathbb{N}} \subset Y \). Despite the suggestive notation, we want to point out that \( y_n \to \infty \) does not necessarily imply that \( y_n \) leaves all bounded sets if \( Y \) is not a proper metric space. We
also define the region of dominated cost

\[ C(x, y) = \{ x' \in X \mid c(x', y) \leq c(x, y) \} \tag{17} \]

for any \((x, y) \in X \times Y\). In Euclidean settings, for example, under costs \(c(x, y) = \|x - y\|^p\) for \(x, y \in \mathbb{R}^d\) and \(p \geq 1\), the set \(C(x, y)\) is the closed ball centered at \(y\) with radius \(\|x - y\|\). As we will see next, the geometry of \(C(x, y)\) as \(y \to \infty\) shapes the region where Kantorovich potentials do not attain finite values.

**Lemma 8** (Interior regularity): Let \(X\) and \(Y\) be Polish, \(c : X \times Y \to \mathbb{R}_+\) be continuous, and \(\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)\) with \(T_c(\mu, \nu) < \infty\). Let \(\pi \in C(\mu, \nu)\) be optimal and \(x \in \text{supp} \mu\) such that there exists \((x_n, y_n)_{n \in \mathbb{N}} \subset \text{supp} \pi\) with \(x_n \to x\) and \(y_n \to \infty\) as \(n \to \infty\). If

1) there is \((\tilde{x}_n)_{n \in \mathbb{N}} \subset X\) converging to \(x\) with \(\lim_{n \to \infty} c(\tilde{x}_n, y_n) - c(x_n, y_n) = -\infty\), then all \(f \in S_c(\mu, \nu)\) assume the value \(-\infty\) on \(C_\infty = \limsup_{n \to \infty} C(\tilde{x}_n, y_n)\). If additionally

2) \(C_\infty\) contains an open subset \(U\) that touches \(x\), meaning \(x \in \text{cl}(U)\), then \(x \in \partial \text{supp} \mu\).

**Proof.** Let \(x' \in C_\infty\). After a suitable subsequence has been taken, we may assume that \(x' \in C(\tilde{x}_n, y_n)\) for all \(n \in \mathbb{N}\). For \(f \in S_c(\mu, \nu)\), note \(f(x_n) = c(x_n, y_n) - f^c(y_n)\) and observe

\[
\begin{align*}
  f(x') &\leq c(x', y_n) - f^c(y_n) \\
          &\leq c(\tilde{x}_n, y_n) - f^c(y_n) \\
          &= c(\tilde{x}_n, y_n) - c(x_n, y_n) + f(x_n) \to -\infty
\end{align*}
\]

due to condition 1 and the upper-semicontinuity of \(f\), which implies \(\limsup_{n \in \mathbb{N}} f(x_n) < \infty\). This shows the first claim. To show the second claim, we lead \(x \in \text{int}(\text{supp} \mu) \cap \text{cl}(U)\) to a contradiction: by density of \(p_X(\text{supp} \pi)\) in \(\text{supp} \mu\), such an \(x\) would imply \(p_X(\text{supp} \pi) \cap U \neq \emptyset\). Thus, there would exist \((x', y') \in \text{supp} \pi \subset \partial f\) with \(-\infty = f(x') = c(x', y') - f^c(y') > -\infty\). \(\square\)

The assumptions in Lemma 8 deserve some more context and discussion. First, note that any \(x \in \Sigma\) admits a suitable sequence \((x_n, y_n)_{n \in \mathbb{N}} \subset \text{supp} \pi\) with \(x_n \to x\) and \(y_n \to \infty\), meaning that Lemma 8 can be applied to all points at which \(\pi\) fails to induce regularity. This is a direct implication of definition 6. Furthermore, we observe that conditions 1 and 2 in the presented formulation of Lemma 8 rely not only on the cost function \(c\), but also on the support of \(\pi\) (via the points \(x_n\) and \(y_n\)). Typically, however, these two conditions can be shown to hold for any sequences \(x_n \to x \in X\) and \(y_n \to \infty\), which makes them an assumption on the cost function only. In fact, condition 1 is implied by property (H\(_{1\infty}\)) in Villani [2008] Chapter 10, which can be viewed as a condition on the growth behavior of the cost function as \(y_n \to \infty\).

**Remark 5:** Lemma 8 shows that optimal transport towards infinity places conditions on the set \(\{f = -\infty\} \subset X\) for \(f \in S_c(\mu, \nu)\), at least for rapidly growing cost functions.
In a certain sense, this relation can be reversed. If \( f(x) = \inf_{y \in Y} c(x, y) - f^e(y) = -\infty \) for any \( x \in X \) (not necessarily in \( \text{supp} \mu \)), it follows that there exists \( (y_n)_{n \in \mathbb{N}} \) with \( c(x, y_n) - f^e(y_n) \to -\infty \) as \( n \to \infty \). Due to the upper-semicontinuity of \( f^e \), this implies \( y_n \to \infty \). Moreover, it is straightforward to see that \( f \) has to assume the value \(-\infty\) on the set \( C_\infty = \limsup_{n \to \infty} C(x, y_n) \). Therefore, if condition 2 of Lemma 8 holds, this set has to be disjoint from the interior of \( \text{supp} \mu \).

Example 4: Let \( X = Y = \mathbb{R}^d \) with squared Euclidean costs \( c(x, y) = \|x - y\|^2 \). Given any sequence \( x_n \to x \in \mathbb{R}^d \) and \( y_n \to \infty \), the choice \( \tilde{x}_n = x_n + (y_n - x_n)/\|y_n - x_n\|^{3/2} \) provides a perturbation of \( x_n \) that satisfies condition 1 in Lemma 8. To see that condition 2 is also satisfied, note that the asymptotic set \( C_\infty \) will contain an open half-space anchored at the point \( x \) (see Figure 3a for an illustration). The (inwards pointing) normal direction is given by an (arbitrary) limit point \( u \) of the directions \( u_n = (y_n - \tilde{x}_n)/\|y_n - \tilde{x}_n\| \). Such a limit exists, since the unit sphere in \( \mathbb{R}^d \) is compact. Therefore, Lemma 8 lets us conclude \( \|u\| = 1 \) and provides additional insights about the set \( \{f = -\infty\} \) and its relation to the direction \( u \) of transport towards infinity. Indeed, the fact that \( C_\infty \) always contains half-spaces (which is also true for all other \( l_p \) costs for \( p > 1 \), but not necessarily for \( p = 1 \), see Figure 3b) implies that the interior of the convex hull of \( \text{supp} \mu \) is contained in the set \( \{f < \infty\} \) (see Remark 5).

---

**Figure 3:** Asymptotic regions of dominated cost. Both figures depict the sets \( C(\tilde{x}_n, y_n) \) as defined in (17) for a sequence \( (\tilde{x}_n, y_n)_{n \in \mathbb{N}} \) with \( \tilde{x}_n \to x \) and \( y_n \to \infty \) as \( n \to \infty \), where \( u = \lim_{n \to \infty} (y_n - \tilde{x}_n)/\|y_n - \tilde{x}_n\| \). In sketch (a), the cost function \( c(x, y) = h(\|x - y\|) \) for the Euclidean norm and some strictly increasing \( h \) is chosen, while sketch (b) shows a similar setting for costs based on the \( l_1 \) norm. In both examples, condition 1 of Lemma 8 is always satisfied if \( h \) is differentiable and \( h'(a) \to \infty \) as \( a \to \infty \) (see Theorem 3). Therefore, all Kantorovich potentials assume the value \(-\infty\) on the region \( C_\infty \) if mass is transported from \( x \) towards infinity in direction \( u \). For illustration, (b) shows the rather exceptional case where \( C_\infty \) is not a half space, which (in this example) can only happen if \( u \) is aligned with one of the coordinate axes. More precisely, the depicted shape is only possible if the vertical coordinates of \( y_n \) converge to the vertical coordinate of the apex of \( C_\infty \).
We stress that the reasoning in this example can easily be extended to other costs on \( \mathbb{R}^d \) and even non-Euclidean spaces. For instance, Gangbo and McCann\[1996\] work with cost functions of the form \( c(x, y) = g(x - y) \) on \( \mathbb{R}^d \) for a convex and superlinear \( g \), which automatically implies condition 1 of Lemma\[8\] They also consider a geometric monotonicity condition on \( g \), which ensures that, for \( y \) large, the sets \( C(x, y) \) contain broad cones with apex \( x \) (implying condition 2). Similarly, the validity of \( (16) \) can be exposed in geodesic spaces as well.

**Theorem 3 (Interior regularity in geodesic spaces):** Let \( X = Y \) be a locally compact complete geodesic space with metric \( d \) and let \( c: X^2 \to \mathbb{R}_+ \) be of the form \( c(x, y) = h(d(x, y)) \), where \( h: \mathbb{R}_+ \to \mathbb{R}_+ \) is differentiable with \( \lim_{a \to \infty} h'(a) = \infty \). Then

\[
\Sigma \subset \partial \text{supp } \mu
\]

for any \( \mu, v \in \mathcal{P}(X) \) with \( T_c(\mu, v) < \infty \), where \( \Sigma \) is defined in \( (14) \) for \( \pi \in C(\mu, v) \) optimal.

**Proof.** Let \( x_n \to x \in X \) and \( y_n \to \infty \). Since \( X \) is a proper metric space (e.g., by the Hopf-Rinow theorem as stated in Bridson and Haefliger 2013, Proposition 3.7), each closed ball in \( X \) is compact. This implies \( r_n = d(x_n, y_n) \to \infty \). Next, let \( g: \mathbb{R}_+ \to \mathbb{R} \) be a function that satisfies \( g \leq h' \) and \( g(a) \to \infty \) as \( a \to \infty \). For example, one can pick \( g(a) = \inf_{b \geq a} h'(b) \). We also let \( g_{x',x''} : [0, d(x', x'')] \to X \) denote a geodesic connecting \( x' \) to \( x'' \) in \( X \).

To show condition 1 of Lemma\[8\] let \( \tilde{x}_n = y_{x_n y_{n}} (t_n) \) for \( 0 < t_n < 1 \), meaning that \( x_n \) is pushed towards \( y_n \) along a geodesic to generate perturbations \( \tilde{x}_n \). The amount \( t_n \) by which \( x_n \) is pushed is chosen to satisfy \( t_n \to 0 \) and \( t_n g(r_n - 1) \to \infty \) as \( n \to \infty \). Since \( d(x, \tilde{x}_n) \leq d(x, x_n) + t_n \), the points \( \tilde{x}_n \) indeed converge to \( x \). We also note that \( d(\tilde{x}_n, y_n) = r_n - t_n \) and observe, as \( n \to \infty \),

\[
c(\tilde{x}_n, y_n) - c(x_n, y_n) = h(r_n - t_n) - h(r_n) \leq -t_n g(r_n - 1) \to -\infty.
\]

To verify condition 2, let \( u_n = y_{\tilde{x}_n y_{n}} (1) \) and pick a limit point \( u \in X \) of this sequence. Such a point exists, since the points \( u_n \) are bounded and \( X \) is proper. By selecting a suitable subsequence, we may assume \( u_n \to u \). Let \( U \) be the open unit ball at \( u \). We will show \( U \subset \lim \sup \{ \tilde{x}_n, y_n \} \) and \( x \in \text{cl}(U) \). The latter is evident, since \( d(x, u) \leq d(x, \tilde{x}_n) + d(\tilde{x}_n, u_n) + d(u_n, u) \to 1 \) as \( n \to \infty \). To see the former, let \( x' \in U \) and note that \( d(x', u) < 1 = d(\tilde{x}_n, u_n) \) for large \( n \). Then

\[
d(x', y_n) \leq d(x', u_n) + d(u_n, y_n) < d(\tilde{x}_n, u_n) + d(u_n, y_n) = d(\tilde{x}_n, y_n).
\]

For \( n \) large enough such that \( h \) can be assumed to be increasing, this implies \( h(d(x', y_n)) \leq h(d(\tilde{x}_n, y_n)) \) and thus \( x' \in C(\tilde{x}_n, y_n) \). \( \square \)

The proof above shows that the asymptotic region \( C_\infty \) in Lemma\[8\] at the very least contains the unit ball touching \( x \) centered at a suitable \( u \in X \). Of course, this approach can also be extended to balls of arbitrary radius \( r > 1 \), which provides additional insight about the geometry of \( C_\infty \). Like in Example\[4\] a central argument is the compactness of closed balls, which guarantees the existence of asymptotic directions \( u \) of transport towards infinity.
5 Proofs of the main results

In the following, we formulate the proofs of the uniqueness statements for Kantorovich potentials under disconnected support, Theorem 1 and Corollary 1. For reasons of exposition, we start with Theorem 1 under continuity assumption 2, before we document the adjustments necessary to prove the theorem under the alternative assumption 1, which requires a slightly different strategy. Afterwards, we provide the arguments to extend Theorem 1 to countable I as claimed in Corollary 1. For notational convenience, we denote the topological closure of a set A by A instead of cl(A) in this section.

Proof of Theorem 1 under assumption 2. Recall decomposition (9) of the support of μ and ν into connected components (X_i)_{i \in I} and (Y_j)_{j \in J} for finite I and countable J. Since we consider the second condition of Theorem 1 first, we can assume for each f^c \in S^c(\mu, \nu) and j \in J with ν(Y_j) > 0 that f^c|_{Y_j} is continuous and that the set \tilde{Y}_j = supp ν|_{Y_j} is connected. As I is finite, each X_i is open in supp μ and consequently satisfies μ(X_i) > 0. Therefore, the X_i-restricted optimal transport problem is well defined and we can fix a representative f_i \in S_{cx_i}(μ_{X_i}, ν_{X_i}) for each i \in I. The uniqueness assumption in Theorem 1 together with Lemma 5 implies that f_i is uniquely determined on p_X(supp π) ∩ X_i (up to an additive offset), so its actual choice does not matter to us. Applying Lemma 4, we conclude that each f \in S_c(\mu, \nu) can be assigned a unique offset vector a = (a_i)_{i \in I} \in \mathbb{R}^{|I|} with components a_i = f(x_i) - f_i(x_i), where the point x_i \in p_X(supp π) ∩ X_i can be chosen arbitrarily. We suggestively write f = f_a if f \in S_c(\mu, \nu) has offset vector a and emphasize that the equality

\[ f_a = \sum_{i \in I} 1_{X_i} \cdot (f_i + a_i) \] (18)

holds on p_X(supp π). Clearly, two Kantorovich potentials in S_c(\mu, \nu) have identical offset vectors if and only if they coincide on p_X(supp π). Therefore, almost sure uniqueness of S_c(\mu, \nu) follows if we can show that there is only a single feasible offset vector a (up to an additive constant that is the same in each component). To formalize this idea, we divide the support of π into closed disjoint pieces Γ_{ij} = supp π ∩ (X_i \times Y_j), and we say that two indices i_1 and i_2 in I are linked if there exists a contact index j \in J with ν(Y_j) > 0 and a contact point y \in Y_j such that

\[ y \in p_Y(Γ_{i_1,j}) \cap p_Y(Γ_{i_2,j}). \] (19)

Intuitively, two indices in I are linked if the masses transported from X_{i_1} and X_{i_2} to Y_j touch one another at a common point y \in Y_j. In a first step, we establish that

\[ i_1 \text{ and } i_2 \text{ are linked } \implies a_{i_1} - a_{i_2} \text{ is fixed} \]

under the adopted continuity assumptions on S^c_c(\mu, \nu), where the right hand side indicates that the difference a_{i_1} - a_{i_2} has to be the same for all feasible offset vectors. In a second step, we then show that non-degeneracy of the optimal plan π guarantees that there are enough contact points to connect all indices in I, which will conclude the proof.

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Step 1. Let $i_1$ and $i_2$ be indices in $I$ that are linked through a contact point $y \in Y_j$ for $j \in J$. According to (19), there are sequences $(x_n, y_n)_n \subset \Gamma_{i,j}$ and $(x'_n, y'_n)_n \subset \Gamma_{i,j}$ such that $y_n \to y$ and $y'_n \to y$ in $Y_j$ as $n \to \infty$. Since $f_a \oplus f_a^\circ = c$ on $\text{supp} \pi$ as well as $f_a = f_{i_1} + a_i$ and $f_a = f_{i_2} + a_i$ on $p_X(\Gamma_{i,j})$ respectively due to relation (18), we find

$$a_{i_1} - a_{i_2} = (c(x_n, y_n) - f_{i_1}(x_n) - f_{i_2}^\circ(y_n)) - (c(x'_n, y'_n) - f_{i_1}(x'_n) - f_{i_2}^\circ(y'_n))$$

for all $n \in \mathbb{N}$. Exploiting the continuity of $f_a^\circ | y_j$ at the contact point $y \in Y_j$, we thus obtain

$$a_{i_1} - a_{i_2} = \lim_{n \to \infty} (c(x_n, y_n) - f_{i_1}(x_n) - f_{i_2}^\circ(y_n)) - (c(x'_n, y'_n) - f_{i_1}(x'_n) - f_{i_2}^\circ(y'_n)) = \lim_{n \to \infty} c(x_n, y_n) - c(x'_n, y'_n) - f_{i_1}(x_n) + f_{i_2}(x_n).$$

Crucially, the limit in the second line exists and does not depend on $a$ anymore. It only depends on the cost function $c$, the restricted potentials $f_{i_k}$, and the sets $\Gamma_{i,j}$ (determined by $\pi$), whose topologies decide the contact point $y$ and the involved sequences. Hence, knowing the value of $a_{i_k}$ determines the one of $a_{i_{2-k}}$ and vice versa.

Step 2. It is left to show that all indices are linked, at least indirectly, such that the vector $a$ is in fact determined by fixing a single component. To do so, we consider an arbitrary decomposition $I = I_1 \cup I_2$ of the index set $I$ into a disjoint union of non-empty subsets, and show that there always exist $i_1 \in I_1$ and $i_2 \in I_2$ that are linked. First, define

$$J_1 = \{ j \in J \mid \pi(\Gamma_{i,j}) > 0 \text{ for some } i \in I_1 \}$$

and analogously $J_2$. Intuitively, an index $j$ is in $J_1$ (or $J_2$) if $\pi$ transports mass between $Y_j$ and some component $X_i$ with $i \in I_1$ (or $i \in I_2$). We note that the sets $J_1$ and $J_2$ cannot be disjoint: if they were, then $\pi$ would transport all mass in $\bigcup_{i \in I_1} X_i$ to $\bigcup_{j \in J_1} Y_j$ and vice versa, contradicting the condition of non-degeneracy. Formally, this follows from $0 < \sum_{i \in I_1} \mu(X_i) < 1$ and

$$\sum_{i \in I_1} \mu(X_i) = \sum_{i \in I_1} \pi(\text{supp} \pi \cap (X_i \times Y)) = \sum_{i \in I_1} \sum_{j \in J_1} \pi(\Gamma_{i,j}) \quad \text{(definition of } J_1)$$

$$= \sum_{j \in J_1} \sum_{i \in I_1} \pi(\Gamma_{i,j}) = \sum_{j \in J_1} \pi(\text{supp} \pi \cap (X \cap Y_j)) = \sum_{j \in J_1} \nu(Y_j),$$

the former of which holds since $I_1$ is nonempty and a proper subset of $I$. Therefore, $J_1$ and $J_2$ are not disjoint and we find some $j \in J_1 \cap J_2$. By definition of $J_1$ and $J_2$, this implies that the two sets

$$B_1 = \bigcup_{i \in I_1} \overline{p_Y(\Gamma_{i,j})} \subset Y_j \quad \text{and} \quad B_2 = \bigcup_{i \in I_2} \overline{p_Y(\Gamma_{i,j})} \subset Y_j$$

have positive $\nu$-mass and are thus non-empty. Since $\pi\left( \bigcup_{i \in I_1} \Gamma_{i,j} \right) = \pi(X \times Y_j) = \nu(Y_j) > 0$, we can apply (a suitably restricted version of) Lemma 9 to conclude that $p_Y\left( \bigcup_{i \in I_1} \Gamma_{i,j} \right) = \bigcup_{i \in I_1} p_Y(\Gamma_{i,j})$ contains a subset that is dense in $\tilde{Y}_j = \text{supp } \nu | y_j$. Thus, we observe

$$\tilde{Y}_j \subset \bigcup_{i \in I_1} p_Y(\Gamma_{i,j}) \cup \bigcup_{i \in I_2} p_Y(\Gamma_{i,j}) = B_1 \cup B_2,$$
where the last equality hinges on the fact that $B_1$ and $B_2$ are closed (this is where we need the assumption that $I$ is finite). Since $v(B_k) = v|_{Y_j}(B_k) > 0$, we find that $\bar{B}_k = B_k \cap \bar{Y}_j$ is non-empty for $k \in \{1, 2\}$. Together with the connectedness of $\bar{Y}_j = \bar{B}_1 \cup \bar{B}_2$, this implies that $\bar{B}_1$ and $\bar{B}_2$ are not disjoint (since closed disjoint sets can be separated by open neighborhoods in metric spaces) and the intersection $\bar{B}_1 \cap \bar{B}_2$ hence contains at least one element $y \in Y_j$. In particular, there also exist $i_1 \in I_1$ and $i_2 \in I_2$ such that

$$y \in p_Y(\Gamma_{i_1,j}) \cap p_Y(\Gamma_{i_2,j}),$$

which means that $i_1$ and $i_2$ are linked with contact index $j$ and contact point $y$. We have thus shown that any proper decomposition $I = I_1 \cup I_2$ admits links between the components $I_1$ and $I_2$, implying that all indices in $I$ can be connected by a chain of links. As discussed above, this makes the Kantorovich potentials $f^\pi_x \in S_x(\mu, \nu)$ almost surely unique and finishes the proof of Theorem 1 under assumption 2.

\[\square\]

Proof of Theorem 1 under assumption 1. The preceding proof has to be adapted to some degree if we work with the slightly stronger continuity requirement that $f^\pi|_{\text{supp} \nu}$ is continuous for each $f^\pi \in S^2_x(\mu, \nu)$, but in turn do not require any topological features of $\text{supp} v|_{Y_j}$. The main difference is that we now allow a contact point $y \in \text{supp} \nu$ to be reached along sequences that hop through different components $Y_j$ (while $j$ was considered fixed for such sequences before). Thus, we let $\Gamma_j = \text{supp} \pi \cap (X_i \times Y) = \bigcup_{j \in g} \Gamma_{i,j}$ and this time define $i_1, i_2 \in I$ to be linked if there exists a contact point $y \in \text{supp} \nu$ such that

$$y \in \overline{p_Y(\Gamma_{i_1,j})} \cap \overline{p_Y(\Gamma_{i_2,j})},$$

which replaces definition (19). In particular, we do not care about the contact index anymore. It is now easy to check that continuity of $f^\pi_x|_{\text{supp} \nu}$ is sufficient for step 1 of the proof above to work as before, and we find that $a_{i_1} - a_{i_2}$ is fixed if $i_1$ and $i_2$ are linked in the sense of (23).

For step 2, we choose the same approach as above and again exploit the non-degeneracy of $\pi$ to find a suitable index $j \in J_1 \cap J_2$ with $J_1$ and $J_2$ defined as in (20). Then, however, we define the sets

$$B_1 = Y_j \cap \bigcup_{i \in I_1} \overline{p_Y(\Gamma_i)} \quad \text{and} \quad B_2 = Y_j \cap \bigcup_{i \in I_2} \overline{p_Y(\Gamma_i)},$$

somewhat differently, which is better aligned with (23). These sets are again closed (use that $I$ is finite) and have positive $\nu$-mass (follows from the definition of $J_1$ and $J_2$). Furthermore, $p_Y(\text{supp} \pi) = p_Y(\bigcup_{i \in I} \Gamma_i)$ is dense in $Y_j$, which leads to $Y_j = B_1 \cup B_2$ along similar lines as in equation (22). Connectedness of $Y_j$ thus shows that $B_1 \cap B_2$ cannot be empty, from which the existence of $y \in Y_j$ as well as $i_1 \in I_1$ and $i_2 \in I_2$ that satisfy (23) follows. By the same argument as before, the claim of the theorem is established.

\[\square\]

Proof of Corollary 1. There are two issues that arise in the proof of Theorem 1 when $I$ is allowed to be countable. The first is that some components $X_i$ might now have a $\mu$-measure of zero, for which the notion of the $X_i$-restricted transport problem ceases to make sense. This can be reconciled by replacing the index set $I$ by $I_* = \{i \in I \mid \mu(X_i) > 0\}$ throughout the proof.
Then, representation (18) of $f_\alpha$ only works on the set $p_X(\text{supp} \pi) \cap \bigcup_{i \in I} X_i$, which is, however, sufficient for almost sure uniqueness.

The second issue concerns the sets $B_1$ and $B_2$ constructed in equation (21). For countable $I$, these sets do in general not have to be closed, which would invalidate the ensuing argumentation. For the two settings described in Corollary 1 however, this can easily be fixed. First, if the index set $I' = \{i \in I \mid \pi(X_i \times Y_j) > 0\}$ has finite cardinality, then we may as well work with the alternative sets

$$B_1 = \bigcup_{i \in I' \cap I} p_Y(\Gamma_{ij}) \subset Y_j \quad \text{and} \quad B_2 = \bigcup_{i \in I' \cap I} p_Y(\Gamma_{ij}) \subset Y_j,$$

for which the remainder of the proof works just as before. Since the unions are finite, these sets are closed. Secondly, if $Y_j = \{y_j\}$ for $y_j \in Y$ consists of a single point only, then noting that $B_1$ and $B_2$ in (21) are both non-empty already establishes $B_1 = B_2 = Y_j$, directly yielding the desired contact point $y = y_j$. In this case, the continuity of $f^\cdot|_{Y_j}$ and the connectedness of $\text{supp} \nu|_{Y_j}$ are trivially true.

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A Auxiliary results and omitted proofs

In a brief assertion of his 1905 memoir, Henry Lebesgue famously sketched an erroneous proof claiming that the projection of Borel subsets of the plane $\mathbb{R}^2$ onto one of the coordinate axes is again Borel. The invalidity of this claim was uncovered in 1916 by Mikhail Suslin, which inspired the study of what is now called analytic sets or Suslin sets (Kanamori 1995). Placed into the context of our work, we learn that it can happen that projected sets of the form $p_X(A)$ and $p_Y(A)$ for Borel sets $A \subset X \times Y$ are not Borel again. However, these sets are analytic and thus universally measurable. In particular, they can be approximated from within and without by Borel sets whose difference is a null set. This settles potential measurability issues in a satisfactory manner.

**Lemma 9:** Let $X$ and $Y$ be Polish and $\pi \in C(\mu, \nu)$ be a transport plan between $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$. Suppose $A \subset X \times Y$ is Borel with $\pi(A) = 1$. Then there exist Borel sets

$$A_\mu \subset p_X(A) \quad \text{and} \quad A_\nu \subset p_Y(A)$$

such that $\mu(A_\mu) = \nu(A_\nu) = 1$. In particular, $A_\mu$ and $A_\nu$ are dense in $\text{supp} \mu$ and $\text{supp} \nu$.

**Proof of Lemma 9:** We only show the statement for $\mu$ since the one for $\nu$ follows equivalently. According to Kechris (2012) Exercise 14.3, the set $p_X(A) \subset X$ is analytic, i.e., the continuous image of a Polish space. By Kechris (2012) Theorem 21.10, every analytic set is universally measurable (see Definition 12.5 for the term standard Borel space), which implies that $p_X(A) = A_\mu \cup N$ where $A_\mu \subset X$ is Borel and $N$ is a subset of a Borel $\mu$-null set $M \subset X$ (see Kechris (2012) Section 17.A, for respective definitions). It is left to show that $\mu(A_\mu) = \mu(A_\mu \cup M) = 1$, which follows from observing that $A \subset p_X(A) \times Y \subset (A_\mu \cup M) \times Y$ and thus

$$\mu(A_\mu \cup M) = \pi((A_\mu \cup M) \times Y) \geq \pi(A) = 1.$$  

Most of the time, we employ Lemma 9 with the choice $A = \text{supp} \pi$ for an optimal transport plan $\pi$, making sure that properties on the sets $p_X(\text{supp} \pi)$ and $p_Y(\text{supp} \pi)$ are valid $\mu$- and $\nu$-almost surely. We next highlight a simple consequence of Lemma 9 relating the points $x \in \text{supp} \mu$ with $x \notin p_X(\text{supp} \pi)$ to the ones where $\pi$ does not induce regularity.

**Lemma 10:** Let $X$ and $Y$ be Polish, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and $c : X \times Y \to \mathbb{R}_+$ continuous such that $T_c(\mu, \nu) < \infty$. Let $\pi \in C(\mu, \nu)$ be an optimal transport plan. Then

$$\text{cl}(\text{supp} \mu \setminus p_X(\text{supp} \pi)) \subset \{ x \in \text{supp} \mu \mid \pi \text{ does not induce regularity at } x \} =: \Sigma$$

and the set $\Sigma$ is closed in $\text{supp} \mu$.

**Proof:** To see that $\Sigma$ is closed, assume $\pi$ to induce regularity at $x \in \text{supp} \mu$ with relatively open $U \subset \text{supp}(\mu)$ and compact $K \subset Y$. Then $\pi$ is also inducing regularity at any other
We extend ŵ with the same U and K, showing that supp μ \ \ Σ is open. It now suffices to note that supp μ \ \ Σ ⊂ pX(supp π) according to Lemma 2 which, together with closedness of Σ, implies the inclusion.

We now turn to Lemma 3, which states that Kantorovich potentials behave as expected when restricted to subsets X ⊂ X and Y ⊂ Y with full μ- and ν-mass. The proof of this statement follows the reasoning behind Villani [2008] Lemma 5.18 and Theorem 5.19. Cases of particular interest to us are restrictions to the (interior of the) support of μ or ν (if the boundary carries no mass). Then the subsets X and Y are either closed or open, and so they are always Borel and Polish.

**Proof of Lemma 3.** By assumption, X ⊂ X, Y ⊂ Y, and X × Y ⊂ X × Y are Borel and Polish subsets with μ(X) = ν(Y) = 1. Since the Borel σ-algebra of a restricted spaces coincides with the respective subspace σ-algebra (see, e.g., Kallenberg [1997] Lemma 1.6), it is easy to recognize that Tε(μ, ν) = Tε(μ, ν) with equal optimal plans π ∈ S(μ, ν), where we permissively identify the measures μ, ν, and π with their restrictions to the Borel sets X, Y, and X × Y of full mass.

Recall that Π = supp π ∩ (X × Y) and observe π(Π) = 1, which implies that Π is dense in the support of π. We begin with the claim on restrictions. Let f ∈ Sε(μ, ν) and define the c-concave function ̃f : X → R ∪ {-∞} via

\[ ̃f(x) = (f^c)^\pi (x) = \inf_{y ∈ Y} c(x, y) - f^c(y). \]

For any (x₀, y₀) ∈ Π, we calculate

\[ ̃f(x₀) = \inf_{y ∈ Y} c(x₀, y) - f^c(y) ≤ c(x₀, y₀) - f^c(y₀) = f(x₀), \]

\[ ̃f(x₀) = \inf_{y ∈ Y} c(x₀, y) - \inf_{x ∈ X} c(x, y) + f(x) ≥ f(x₀), \]

which shows that ̃f = f on pX(Π). Similarly,

\[ ̃f^c(x₀) = \inf_{x ∈ X} c(x, y₀) - ̃f(x) ≤ c(x₀, y₀) - f(x₀) = f^c(y₀), \]

\[ ̃f^c(y₀) = \inf_{x ∈ X} c(x, y₀) - \inf_{y ∈ Y} c(x, y) + f^c(y) ≥ f^c(y₀), \]

which asserts ̃f^c = f^c on pY(Π). Since the set Π has full π-measure, it follows that Tε(μ, ν) = Tε(μ, ν) = π(f ⊕ f^c) = π(f ⊕ ̃f^c). This shows ̃f ∈ Sε(μ, ν) and thus proves the first statement.

We next turn towards the claim on extending potentials, where we begin with ̃f ∈ Sε(μ, ν) and observe Π ⊂ ∂(̃f) (which is true since Π equals the support of the measure π restricted to X × Y).

We extend ̃f to a function f : X → R on all of X via defining

\[ f(x) = \inf_{y ∈ Y} c(x, y) - ̃f^c(y). \]
This function is $c$-concave, since it is the $c$-transform of a function that equals $\tilde{f}^{c}$ on $\tilde{Y}$ and $-\infty$ on $Y \setminus \tilde{Y}$. Furthermore, since $\tilde{f}$ is $\hat{c}$-concave, the definition of $f$ directly shows that it coincides with $\tilde{f}$ ($= \tilde{f}^{c}$) on $\tilde{X}$. For $(x_0, y_0) \in \tilde{\Gamma}$, we furthermore note that

$$f^{c}(y_0) = \inf_{x \in \tilde{X}} c(x, y_0) - f(x) \leq c(x_0, y_0) - f(x_0) = \tilde{f}^{c}(y_0),$$

$$f^{c}(y_0) = \inf_{x \in \tilde{X}} c(x, y_0) - \inf_{y \in \tilde{Y}} c(x, y) + \tilde{f}^{c}(y) \geq \tilde{f}^{c}(y_0),$$

and thus $f^{c} = \tilde{f}^{c}$ on $\rho_{Y}(\tilde{\Gamma})$. The optimality of $f$ is checked like above, yielding $f \in S_{c}(\mu, v)$. □

Note that the sets of consensus $\rho_{X}(\tilde{\Gamma})$ and $\rho_{Y}(\tilde{\Gamma})$ in Lemma 3 can be enlarged to the (potentially slightly bigger) sets $\rho_{X}(\text{supp } \pi) \cap \tilde{X}$ and $\rho_{Y}(\text{supp } \pi) \cap \tilde{Y}$ when restricting Kantorovich potentials. To prove this, the density of $\tilde{\Gamma}$ in $\text{supp } \pi$ can be exploited in combination with Lemma 1. For extending a potential $\tilde{f}$, the proof above shows that it is always possible to pick an extension that agrees with $\tilde{f}$ on all of $\tilde{X}$, or alternatively to pick an extension whose $c$-transform agrees with $\tilde{f}^{c}$ on all of $\tilde{Y}$.

We next prove the claim stated in the context of equation (11), which provides an example where degeneracy of the optimal transport plan leads to non-unique Kantorovich potentials.

**Lemma 11:** Let $X = Y$ be Polish, $\mu = v \in \mathcal{P}(X)$ with $\text{supp}(\mu) = X_1 \cup X_2$, and $c : X^2 \to \mathbb{R}_+$ be continuous and symmetric with $c(x, x) = 0$ for all $x \in X$. If

$$\Lambda = \inf_{x_1 \in X_1, x_2 \in X_2} c(x_1, x_2) > 0,$$

then for all $a, b \in \mathbb{R}$ with $|a - b| \leq \Delta$, there exists $f_{a,b} \in S_{c}(\mu, \mu)$ such that $f_{a,b} = a$ on $X_1$ and $f_{a,b} = b$ on $X_2$.

**Proof.** It is apparent that $T_{c}(\mu, \mu) = 0$. For real numbers $a, b \in \mathbb{R}$ with $|a - b| \leq \Delta$, we define the map $g : X \to \mathbb{R} \cup \{-\infty\}$ via

$$g(x) = \begin{cases} -a & \text{if } x \in X_1, \\ -b & \text{if } x \in X_2, \\ -\infty & \text{if } x \notin X_1 \cup X_2. \end{cases}$$

For $x \in X_1$, the $c$-concave function $f_{a,b} := g^{c}$ fulfills

$$f_{a,b}(x) = \inf_{y \in X} c(x, y) - g(y) = \begin{cases} a & \text{for } y = x \in X_1, \\ c(x, y) + a & \text{for } y \in X_1, \\ c(x, y) + b & \text{for } y \in X_2, \\ \infty & \text{for } y \notin X_1 \cup X_2. \end{cases}$$

Since $c \geq 0$ and $a \leq \Delta + b \leq c(x, y) + b$ for $x \in X_1, y \in X_2$, we find $f_{a,b} = a$ on $X_1$. Likewise, we also find $f_{a,b} = b$ on $X_2$. By a similar argument, it follows that $f_{a,b}^{c} \leq -a$ on $X_1$ and $f_{a,b}^{c} \leq -b$ on $X_2$. 

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We first show that $f^c = -f$ on $X_1 \cup X_2$. This implies $T_\gamma(\mu, \mu) = 0 = \pi(f_{\mu, \mu} \oplus f^c_{\mu, \mu})$ for any optimal $\pi$, asserting that $f_{\mu, \mu}$ is indeed a Kantorovich potential.

Next, we address the claim raised in Lemma 7 which states that (locally) Lipschitz continuous functions on connected manifolds coincide (up to constants) if their gradients coincide almost surely (in charts). This is a simple generalization of corresponding results on $\mathbb{R}^d$, which can, for example, be gathered from considerations in Qi 1989.

**Proof of Lemma 7** Let $f_1, f_2 : M \to \mathbb{R}$ be locally Lipschitz on a $d$-dimensional smooth manifold $M$, and let $(\varphi_x)_{x \in M}$ be a family of charts with $x \in U_x = \text{domain} \varphi_x$ for all $x \in M$. By translating, restricting, and rescaling the charts, we can assume that range $\varphi_x = B_1 \subset \mathbb{R}^d$ is the unit ball and that $\varphi_x^{-1}(B_1) \subset \mathbb{R}^d$ is Lipschitz for each $x \in M$ and $i \in \{1, 2\}$. Since $\nabla f_{i,x} = \nabla f_{2,x}$ holds by assumption on a set with full Lebesgue measure, we can conclude (e.g., by formula (2) of Qi 1989) that $f_{i,x} - f_{2,x} = c_x$ on all of $B_1$, where $c_x \in \mathbb{R}$ is a constant. This implies $f_{i}|_{U_x} - f_{2}|_{U_x} = c_x$ as well. To see that $c_x$ is actually independent of $x$, note that $c_{x'} = c_x$ holds for any $x' \in U_x$, since then $U_x \cap U_{x'} \neq \emptyset$. Thus, $x \mapsto c_x$ is a locally constant function. The connectedness of $M$ implies that it is also constant on the whole space. To see this, just note that the set $V = \{x \in X \mid c_x = c_{x_0} \} \neq \emptyset$ and its complement are both open (for some fixed $x_0 \in X$), which makes $V$ open and closed. Hence, $V = M$ and the claim $f_1 - f_2 = c_{x_0}$ is established.

To conclude this appendix, we show that Corollary 3 is also true if $c(\cdot, \cdot)$ is assumed to be locally semiconcave uniformly in $y \in Y$ (instead of locally Lipschitz uniformly in $y$), which is claimed in Remark 4.

**Proof of Remark 4** We adhere to the general proof strategy of Theorem 2 but will provide additional details for some of the arguments. For every $x \in X$, let $\varphi_x : U_x \to V_x \subset \mathbb{R}^d$ denote a chart around $x$. By restricting and translating, we can assume that $V_x$ is convex with $\varphi(x) = 0 \in V_x$. Recalling equation (3), we may also assume that $v \mapsto c(\varphi_x^{-1}(v), y) - \lambda_x \|v\|^2$ is concave on $V_x$ for some $\lambda_x > 0$ and all $y \in Y$. Due to the nature of $c$-conjugates as infima, we find that the function $v \mapsto h_x(v) = f(\varphi_x^{-1}(v)) - \lambda_x \|v\|^2$ is concave as well for any $f \in S_\gamma(\mu, \nu)$ (Rockafellar 2015 Theorem 5.5).

We first show that $f > -\infty$ on $M = \text{int}(\text{supp} \mu)$. Indeed, assume that there existed a point $x \in M$ with $f(x) = -\infty$. Then $h_x(0) = -\infty$, and, since the effective domain $h_x^{-1}(\mathbb{R}) \subset V_x$ is convex, it followed that $h_x^{-1}\{\{\infty\}\}$ contained at least an open half-space (intersected with $V_x$) touching the origin. Hence, $f = -\infty$ would have to hold on an open subset of $M \subset \text{supp} \mu$, which cannot be true, since $p_x(\text{supp} \pi)$ is dense in $\text{supp} \mu$ and $f > -\infty$ on $p_x(\text{supp} \pi)$ due to supp $\pi \subset \partial f$.

Since (locally semi-)concave functions are locally Lipschitz in the interior of their effective domain (Rockafellar 2015 Theorem 10.4), we can conclude that each Kantorovich potential is locally Lipschitz on all of $M$.

Next, since $\Gamma = \text{supp} \pi$, we find a Borel set $A \subset p_X(\Gamma) \subset X$ that satisfies $\mu(A) = 1$ (Lemma 9). Hence, $B = \text{range} \varphi \setminus \varphi(A \cap \text{domain} \varphi)$ is a $\varphi_y \mu$-null set for any chart $\varphi$ of $M$. Due to condition 15,
it is also a Lebesgue-null set and we conclude that $p_X(\Gamma)$ has full Lebesgue measure in charts of $M$. We can now argue as in Theorem 2 to find that any two Kantorovich potentials $f_1, f_2 \in S_c(\mu, \nu)$ coincide on $M$. It remains to show that $f_1 = f_2$ holds on the boundary of $\text{supp } \mu$ as well. Let $x \in \partial \text{supp } \mu$ and let $(x_n)_{n \in \mathbb{N}} \subset M$ be a sequence converging to $x$. Then it holds that $\lim_{n \to \infty} f_i(x_n) = f_i(x)$ for $i \in \{1, 2\}$. This can be seen as follows: if $f_i(x) > -\infty$, then the limit holds since (locally semi-)concave functions are continuous on their effective domain (Rockafellar 2015, Theorem 10.1), and if $f_i(x) = -\infty$, then the limit holds due to upper-semicontinuity of $f_i$. Since $f_1(x_n) = f_2(x_n)$ for all $n$, this establishes equality of $f_1$ and $f_2$ on the full support. □