COEXISTING HIDDEN ATTRACTORS IN A 5D SEGMENTED DISC DYNAMO WITH THREE TYPES OF EQUILIBRIA

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Abstract. Little seems to be known about coexisting hidden attractors in hyperchaotic systems with three types of equilibria. Based on the segmented disc dynamo, this paper proposes a new 5D hyperchaotic system which possesses the properties. This new system can generate hidden hyperchaos and chaos when initial conditions vary, as well as self-excited chaotic and hyperchaotic attractors when parameters vary. Furthermore, the paper proves that the Hopf bifurcation and pitchfork bifurcation occur in the system. Numerical simulations demonstrate the emergence of the two bifurcations. The MATLAB simulation results are further confirmed and validated by circuit implementation using NI Multisim.

1. Introduction. The problem of analyzing hidden attractors first arose in the second part of Hilbert’s 16th problem (1900), which considered the number and mutual disposition of limit cycles in two-dimensional polynomial systems [3, 5]. The surge of study in dynamic systems in recent years has opened avenues for the exploration of coexisting attractors and their applications. Many dynamical systems with coexisting attractors model financial markets [9, 18], ecosystems (e.g. the Amazon rainforest) [6, 25], arrays of coupled lasers [4, 19], the human brain [1, 17], etc..

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Multiple attractors indicate the existence of multistability in the dynamical systems [14, 15]. In multistable systems, we can observe the sudden switch to unexpected attractors, which may bring about such catastrophic events as sudden climate changes, serious diseases, financial crises and so on [24]. The spectacular example of the disaster caused by the sudden switch to the undesired attractor is the crash of aircraft YF-22 Boeing in April 1992 [3, 13].

Multistability, i.e. coexisting attractors, is often connected with the occurrence of unpredictable attractors which are called hidden attractors [3, 23]. The hidden attractors are mainly observed in the systems with a stable equilibrium [20, 28], a line equilibrium [7, 16] or no equilibria [8, 22], whereas a self-excited attractor has a basin of attraction that is associated with an unstable equilibrium.

Identifying and locating coexisting hidden attractors are vital to understand the dynamic behavior of systems, which becomes more difficult for hidden states. Here we propose a 5D segmented disc dynamo with three positive finite-time local Lyapunov exponents. There are coexisting hidden attractors with three types of equilibria in the system. It is remarkable that self-excited chaotic and hyperchaotic attractors are also generated when parameters vary. As far as we know, it is new. We hope that the work will shed light on more systematic studies of 5D systems and leading to final revealing the true geometrical structure of lower dimensional chaotic and hyperchaotic attractors.

According to Ref.[27], pitchfork bifurcation and Hopf bifurcation may be associated with the birth of a hidden attractor. Thus, we further study the Hopf bifurcation and pitchfork bifurcation of the new system by bifurcation theory [12, 29]. Numerical investigations are performed to verify the corresponding theoretical results for the two bifurcations. The circuit of system (2.2) is also designed and simulated by NI Multisim. The obtained results accord with the MATLAB simulation ones.

The paper is organized as follows. Sect. 2 introduces the new system and discusses coexisting attractors for different types of equilibria. Sect. 3 investigates the Hopf bifurcation and Sect. 4 analyzes the pitchfork bifurcation. The circuit design and simulation using NI Multisim are discussed in Sect. 5. Finally, Sect. 6 concludes the paper.

2. 5D segmented disc dynamo. Moffatt proposed the segmented disc dynamo in which the current associated with the radial diffusion of the magnetic field could be included in a simple way [21]:

\[
\begin{align*}
\dot{x}(t) &= r(y - x), \\
\dot{y}(t) &= mx - (1 + m)y + xz, \\
\dot{z}(t) &= g(mx^2 + 1 - (1 + m)xy).
\end{align*}
\] (2.1)

Based on (2.1), we introduce a 5D segmented disc dynamo

\[
\begin{align*}
\dot{x}(t) &= r(y - x) + k_1 u, \\
\dot{y}(t) &= mx - (1 + m)y + xz + v, \\
\dot{z}(t) &= g(mx^2 + 1 - (1 + m)xy) - k_2 - k_3 z, \\
\dot{u}(t) &= k_4 xy - k_5 u, \\
\dot{v}(t) &= k_6 x + xz - u,
\end{align*}
\] (2.2)

where \( r > 0, m > 0, k_3 \geq 0, k_5 \geq 0, g, k_1, k_2, k_4, k_6 \in \mathbb{R} \). The divergence of the system is \( \nabla \cdot V = -r - 1 - m - k_3 - k_5 \), and the system is dissipative.

Since the computational time is finite in numerical experiments, Leonov, Kuznetsov, et al. developed the concept of finite-time Lyapunov dimension and an approach
to its reliable numerical computation \[10, 11, 15\]. Along with widely used numerical methods for estimating and computing the Lyapunov dimension, the advantage of the approach is that it allows one to estimate the Lyapunov dimension of invariant sets without localization of the set in the phase space and, in many cases, to get effectively an exact Lyapunov dimension formula \[10, 11\]. In the paper, we compute the finite-time local Lyapunov exponents (FTLEs) and finite-time local Lyapunov dimension (FTLD) via the corresponding MATLAB realization from Ref.\[15\] with \(t_{\text{Step}} = 0.1, n_{\text{Factors}} = 10000, n_{\text{SvdIterations}} = 3\). For \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.05, 12.0, 11.9, 0.01, 0, -0.01, -99)\) and initial condition \((0, 0, 0, 0, 0)\), system (2.2) has three positive FTLEs \((0.5631, 0.5601, 0.0100, -0.0100, -2.1933)\), and the FTLD is 4.5121. Fig. 1 shows the spectrum of finite-time local Lyapunov exponents versus \(r \in (0, 3)\).

**Figure 1.** For parameters \((m, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 12, 0, 11.9, 0.01, 0, -0.01, -99)\) and initial condition \((0, 0, 0, 0, 0)\), the finite-time local Lyapunov exponents spectrum in system (2.2) versus \(r \in (0, 3)\)

### 2.1. Equilibria and stability

If \(k_1 = k_3 = 0\) and \(g(g - k_2) < 0\), system (2.2) has no equilibria. If \(k_2 = g, k_3 = 0\) and \(k_1 k_4 = 0\), system (2.2) only has a line equilibrium at \((0, 0, z, 0, 0)\), where \(z \in \mathbb{R}\). If \(k_3 = 0, k_3 \neq 0\) and \(4gk_5^2(g - k_2 + k_3k_6) + k_3^2k_4^2 < 0\), system (2.2) only has an equilibrium \(P(0, 0, \frac{g-k_2}{k_3}, 0, 0)\). The corresponding characteristic equation at \(P\) is

\[
(\lambda + k_3)(\lambda + k_5) \left( \lambda^3 + (m + r + 1) \lambda^2 + \left( 1 + \frac{k_2 - g}{k_3} \right) r \lambda + \left( \frac{k_2 - g}{k_3} - k_6 \right) r \right) = 0.
\]

(2.3)

According to the Routh-Hurwitz criterion, the real parts of all the roots \(\lambda\) are negative if and only if

\[
\Delta_1 = m + r + 1 > 0,
\]

\[
\Delta_2 = \left| \begin{array}{c} m + r + 1 \\ 1 \\ \frac{k_2 - g}{k_3} \\ \left( 1 + \frac{k_2 - g}{k_3} \right) r \end{array} \right| = \left( m + r + k_6 + 1 + \frac{(m + r)(k_2 - g)}{k_3} \right) > 0,
\]

(2.4)
\[
\Delta_3 = \begin{vmatrix}
m + r + 1 & \frac{(k_2 - g - k_6)r}{k_3} & 0 \\
1 & \frac{(1 + \frac{k_2}{k_3})r}{k_3} & 0 \\
0 & m + r + 1 & \frac{(k_2 - g - k_6)r}{k_3} & 0 \\
\end{vmatrix} = \left(\frac{k_2 - g}{k_3} - k_6\right) r \Delta_2 > 0.
\]

From these inequalities, there are
\( (m + r) (k_3 + k_2 - g) + k_3 (k_6 + 1) > 0, \quad k_2 - g - k_3 k_6 > 0. \)

When \( k_2 = g \), the above inequalities are simplified as follows:
\[-(m + r + 1) < k_6 < 0. \quad (2.4)\]

The equilibrium \( P \) is asymptotically stable with (2.4).

2.2. Coexistence of hidden hyperchaotic and chaotic attractors with only one stable equilibrium. System (2.2) only has one stable equilibrium \( P = (0, 0, -3000, 0, 0) \) for \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12.3, 0.0001, 0.01, 0.01, -100)\). The eigenvalues at \( P \) are \(-0.0001, -0.01, -1.0331, -0.0034 \pm 7.7468i\).

System (2.2) is sensitive to initial conditions.

(a) For initial condition \((0.8147, 0.9058, 0.1270, 0.9134, 0.6324)\), the FTLEs of system (2.2) are \((0.0534, -0.0017, -0.0098, -0.0352, -1.0568)\), and the FTLD is 4.0063. A chaotic attractor with a stable equilibrium \( P \) can be obtained (see Fig. 2(a)). Fig. 2(b) shows the Poincaré map. The time series of \( x \) is shown in Fig. 2(c), and Fig. 2(d) shows the finite-time local Lyapunov dimension.

(b) For initial condition \((0, 0, 0, 0, 0)\), the FTLEs are \((0.2052, 0.2006, -0.0001, -0.0100, -1.4458)\), and the FTLD is 4.2737. The trajectories of system (2.2) converge on a hyperchaotic attractor.

\[ \begin{array}{ll}
\text{(a)} & \text{(b)} \\
\text{(c)} & \text{(d)} \\
\end{array} \]

**Figure 2.** Parameters \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12.3, 0.0001, 0.01, 0.01, -100)\) and initial condition \((0.8147, 0.9058, 0.1270, 0.9134, 0.6324)\); (a) chaotic attractor of system (2.2); (b) Poincaré map on the \(x-z\) plane; (c) time series of \(x\); (d) finite-time local Lyapunov dimension.
2.3. Coexistence of hidden hyperchaotic and chaotic attractors with only a line equilibrium. For \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (1.1, 6.1, 12, 0, 12, 0, 0, 0, -100)\), system (2.2) only has a line equilibrium at \((0, 0, z, 0, 0)\), where \(z \in \mathbb{R}\), with no other equilibria (in other words the z-axis is the line equilibrium of this system).

(a) For initial condition \((21, 0.1, 1, 0, 0)\), the FTLEs of system (2.2) are \((0.0019, -0.0006, -2.5915, -2.7987, -2.8111)\), and the FTLD is 2.0005. A chaotic attractor appears (see Fig. 3).

(b) For initial condition \((0, 0, 0, 0, 0)\), the FTLEs of system (2.2) are \((1.8804, 1.8779, 0.0000, 0.0000, -11.9583)\), and the FTLD is 4.3143. A hyperchaotic attractor can be obtained.

![Figure 3. Chaotic attractor of system (2.2) for initial condition (21, 0.1, 1, 0, 0) and parameters (m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (1.1, 6.1, 12, 0, 12, 0, 0, 0, -100)](image)

2.4. Coexistence of hidden hyperchaotic and chaotic attractors with no equilibria. When \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.01, 0.02, 12, 0, 13, 0, 0, 0.1, -100)\), (2.2) has no equilibria.

(a) For initial condition \((0, 0, 0, 0, 0)\), the FTLEs of system (2.2) are \((0.1089, 0.1022, 0.0000, -0.1000, -1.2410)\), and the FTLD is 4.0895. A hyperchaotic attractor is observed.

(b) For initial condition \((0, -10, 1, -100, -10)\), the FTLEs of system (2.2) are \((0.0433, -0.0016, -0.0276, -0.1000, -1.0441)\), and the FTLD is 3.1408. A chaotic attractor can be obtained (see Fig. 4(a)). Fig. 4(b) shows the Poincaré map.

2.5. Coexistence of self-excited hyperchaotic and chaotic attractors associated with only one unstable equilibrium. System (2.2) only has an unstable equilibrium \(P [0, 0, 0, 0, 0, 0]\) for \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12, 0.001, 0.01, 0.01, -100)\). The eigenvalues at \(P\) are \(-0.001, -0.01, 0.3356 \pm 1.0277i, -1.7113\).

(a) For initial condition \((0.5268, 7.3786, 2.6912, 4.2284, 0.8147)\), the FTLEs of system (2.2) are \((0.0517, 0.0027, -0.0092, -0.0315, -1.0647)\), and the FTLD is 4.0129. A hyperchaotic attractor can be obtained (see Fig. 5).

(b) For initial condition \((0, -1000, 100, -10, 0)\), the trajectories of (2.2) converge on a chaotic attractor (see Fig. 6(a)), and Fig. 6(b) shows the Poincaré map. The FTLEs are \((0.0466, -0.0015, -0.0098, -0.0240, -1.0623)\), and the FTLD is 4.0106.

(c) In order to show that the obtained set is a transient chaotic attractor, we have to show that all trajectories starting from the vicinities of the unstable equilibrium \(P\)
Figure 4. Initial condition \((0, -10, 1, -100, -10)\) and parameters \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.01, 0.02, 12, 0, 13, 0, 0, 0.1, -100)\); (a) chaotic attractor of system (2.2); (b) Poincaré map on the \(x-y\) plane

Figure 5. Hyperchaotic attractor of system (2.2) for parameters \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12, 0.001, 0.01, 0.01, -100)\) and initial condition \((0.5268, 7.3786, 2.6912, 4.2284, 0.8147)\)

Figure 6. Parameters \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12, 0.001, 0.01, 0.01, -100)\) and initial condition \((0, -1000, 100, -10, 0)\); (a) chaotic attractor of system (2.2); (b) Poincaré map on the \(y-z\) plane

tend to infinity \([2, 11]\), because now system (2.2) only has one unstable equilibrium \(P\).

We integrate numerically the trajectory with initial condition \((0.5, 0.5, 0.5, 0.5, 0.5)\) in the vicinity of \(P\). The trajectory computed on the time interval \([0, 10000]\)
traces a chaotic set in the phase space, which looks like an “attractor” (see Fig. 7(a)). Further integration to $t = 20000$, the chaotic set is shown in Fig. 7(b). The time series of $x$ is shown in Fig. 7(c), and Fig. 7(d) shows the finite-time local Lyapunov dimension. From Fig. 7 the chaotic set is not a transient chaotic set.

3. Hopf bifurcation in system (2.2). We utilize the projection method [12] to calculate the first Lyapunov coefficient associated with Hopf bifurcation.

Let

$$f(\lambda) = \lambda^3 + (m + r + 1)\lambda^2 + \left(1 + \frac{k_2 - g}{k_3}\right) r\lambda + \left(\frac{k_2 - g}{k_3} - k_6\right)r,$$

$$\omega = \sqrt{-r(m + r)(k_6 + 1)}\frac{m + r}{m + r},$$

$$k_2^0 = g - k_3 - \frac{k_3(k_6 + 1)}{m + r},$$

$$S = \left\{(m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) \mid m > 0, r > 0, g > 0, k_1 = k_4 = 0, k_2 = k_2^0, k_3 > 0, k_5 > 0, k_6 < -1 \right\}.$$

For $(m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) \in S$, system (2.2) has only one equilibrium $P \left(0, 0, \frac{2 - k_2}{k_3}, 0, 0\right)$, and the corresponding eigenvalues are $-k_3, -k_5, -m - r - 1$ and conjugate purely imaginary roots $\pm \omega i$. 

Figure 7. Parameters $(m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12, 0.001, 0.01, 0.01, -100)$ and initial condition $(0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)$; (a) trajectory for $t \in [0, 10000]$; (b) trajectory for $t \in [0, 20000]$; (c) time series of $x$; (d) finite-time local Lyapunov dimension.
Let \( \lambda = \alpha (k_2) + i \omega (k_2) \). We substitute \( \lambda = \alpha (k_2) + i \omega (k_2) \) into \( f(\lambda) = 0 \) and differentiate two sides of \( f(\lambda) = 0 \) with respect to \( k_2 \), and it follows that

\[
\frac{d\lambda(k_2)}{dk_2} = -\frac{r(\lambda + 1)}{3k_3\lambda^2 + 2k_3(m + r + 1)\lambda - \lambda (g - k_2 - k_3)}.
\]

Considering that \( \pm \omega i \) are the roots of \( f(\lambda) = 0 \), we have

\[
\frac{d\lambda(k_2)}{dk_2} |_{k_2 = k_2^0} = \frac{r(\omega i + 1)}{2k_3(\omega^2 - (m + r + 1)\omega)}.
\]

Hence the transversality condition

\[
\text{Re}(\lambda'(k)) |_{k = k_2^0} = -\frac{r(m + r)}{2k_3(m + r + 1)^2 + \omega^2} < 0 \quad (3.1)
\]

is also satisfied, and Hopf bifurcation at \( P \) occurs. We have the following theorem.

**Theorem 3.1.** Consider system (2.2), the first Lyapunov coefficient at \( P(0, 0, \frac{g - k_2}{k_3}, 0, 0) \) for parameter values \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) \in S \) is given by

\[
l_1 = -\frac{gr^2 (b + 2k_3 (m \omega^2 - r) (m + r + 1) + 8\omega^2 r (m + r))}{2k_3 \omega (k_3^2 + 4\omega^2) ((m + r + 1)^2 + \omega^2)},
\]

where

\[
b = k_3^3 (1 + m) (\omega^2 + m + 1) + r (4m + 3r + 1).
\]

1. If \( l_1 > 0 \), system (2.2) has a transversal Hopf point at \( P \) and the Hopf point is unstable (weak repelling focus). Moreover, for each \( k_2 > k_2^0 \), but close to \( k_2^0 \), there exists an unstable limit cycle near the asymptotically stable equilibrium \( P \).

2. If \( l_1 < 0 \), system (2.2) has a transversal Hopf point at \( P \) and the Hopf point is stable (weak attractor focus). Moreover, for each \( k_2 < k_2^0 \), but close to \( k_2^0 \), there exists a stable limit cycle near the unstable equilibrium \( P \).

**Proof.** By the changes

\[
\begin{align*}
x &= x, \\
y &= y, \\
z_1 &= z - \frac{g - k_2}{k_3}, \\
u &= u, \\
v &= v,
\end{align*}
\]

system (2.2) becomes the following system (still denoted by \( x, y, z, u, v \) )

\[
\begin{align*}
\dot{x}(t) &= r(y - x) + k_1 u, \\
\dot{y}(t) &= m x - (1 + m) y + x \left( z + \frac{g - k_2}{k_3} \right) + v, \\
\dot{z}(t) &= g(m x^2 + 1 - (1 + m) x y) - k_2 - k_3 \left( z + \frac{g - k_2}{k_3} \right), \\
\dot{u}(t) &= k_4 x y - k_5 u, \\
\dot{v}(t) &= k_6 x + x \left( z + \frac{g - k_2}{k_3} \right) - u,
\end{align*}
\]

and the equilibrium \( P \left(0, 0, \frac{g - k_2}{k_3}, 0, 0\right) \) is moved to \( O(0, 0, 0, 0, 0) \).

From (3.1), the transversality condition holds. Now we calculate the Lyapunov coefficient, which shows the stability of the equilibrium and the periodic orbit which appears.
According to Ref. [12], for the parameters \((m,r,g,k_1, k_2, k_3, k_4, k_5, k_6) \in S\), we have \(\lambda_1 = -m - r - 1, \lambda_2 = -k_3, \lambda_3 = -k_4, \lambda_{4,5} = \pm \omega i\), and

\[
A = \begin{pmatrix}
-\frac{m + 1 - \omega^2}{r} & -1 - m & 0 & 0 & 0 \\
0 & 0 & -k_3 & 0 & 0 \\
0 & 0 & 0 & -k_5 & 0 \\
-\frac{(m + r + 1)\omega^2}{r} & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad q = \begin{pmatrix}
r \\
r + \omega i \\
0 \\
0 \\
(m + r + 1)\omega i \\
\end{pmatrix},
\]

\[
p = \frac{i}{2\omega(m + r + 1 - \omega i)} \begin{pmatrix}
0 \\
\frac{\omega i}{\omega^2 + k^2} \\
0 \\
\frac{1}{\omega^2 + k^2} \\
\end{pmatrix}, \quad C(x,y,z) = \begin{pmatrix} 0 \\
0 \\
0 \\
0 \\
\end{pmatrix},
\]

\[
B(x,y) = \begin{pmatrix}
x_1 y_1 + x_3 y_1 \\
g(2m x_1 y_1 - (1 + m)(x_1 y_2 + x_2 y_1)) \\
0 \\
x_1 y_3 + x_3 y_1 \\
\end{pmatrix}.
\]

Then we compute the following values:

\[
A^{-1} B (q, \bar{q}) = \begin{pmatrix}
0 \\
0 \\
\frac{2g r^2}{k_3 r} \\
0 \\
0 \\
\end{pmatrix}, \quad (2\omega i E - A)^{-1} B (q, q) = \begin{pmatrix}
0 \\
0 \\
\frac{2g r (m + 1) \omega^2 - r}{k_3 r^2} \\
0 \\
0 \\
\end{pmatrix}.
\]

Therefore

\[
l_1 = \frac{1}{2\omega} \text{Re} \left[ \left( p, C(q, q, \bar{q}) \right) - 2 \left( p, B(q, A^{-1} B(q, \bar{q})) \right) + \left( p, B\left( \bar{q}, (2\omega i E - A)^{-1} B(q, q) \right) \right) \right]
= \frac{-g r^2 (b + 2k_3 (m \omega^2 - r) (m + r + 1) + 8 \omega^2 r (m + r))}{2k_3 \omega (k_3^2 + 4\omega^2) (m + r + 1)^2 + \omega^2},
\]

where

\[
b = k_3^2 \left( (1 + m) \omega^2 + m + 1 \right) + r (4m + 3r + 1).
\]

**Numerical simulation** For \(m = 1.3, r = g = k_3 = k_5 = 0.1, k_1 = k_4 = 0\) and \(k_6 = -2\), we have \(k_2 = 0.0614\) and \(l_1 = -0.0015\). A stable limit cycle is obtained with initial condition \((0.0026, -0.3011, 0.2967, 0, -0.7291)\) (See Fig. 8).

4. **Pitchfork bifurcation in system (2.2).** We utilize the center manifold theorem and the bifurcation theory [12, 29] to study pitchfork bifurcation of system (2.2).

Let

\[
S = \left\{ (m,r,g,k_1,k_2,k_3,k_4,k_5,k_6) \mid k_1 = k_4 = k_6 = 0, \quad k_2 = g, \quad m > 0, \quad r > 0, \quad g > 0, \quad k_3 > 0, \quad k_5 > 0 \right\}.
\]
When \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) \in S\), system (2.2) only has one equilibrium \(O(0,0,0,0,0)\).

The Jacobian matrix at \(O\) is

\[
J = \begin{pmatrix}
-r & r & 0 & 0 & 0 \\
m & -1 - m & 0 & 0 & 1 \\
0 & 0 & -k_3 & 0 & 0 \\
0 & 0 & 0 & -k_5 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix},
\]

and the corresponding characteristic equation is

\[
\lambda(\lambda + k_3)(\lambda + k_5)(\lambda^2 + (m + r + 1)\lambda + r) = 0.
\]

System (2.2) has a zero eigenvalue \(\lambda_1=0\) and the other four eigenvalues

\[
\lambda_2 = -k_3, \lambda_3 = -k_5, \lambda_{4,5} = -\frac{1}{2}(m + r + 1) \pm \frac{1}{2} \sqrt{(m + r)^2 + 2(m - r) + 1}.
\]

\(O(0,0,0,0,0)\) is nonhyperbolic, and then we can get the following theorem.

**Theorem 4.1.** For \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) \in S\), system (2.2) undergoes a pitchfork bifurcation at \(O(0,0,0,0,0)\). Furthermore, when \(k_6 < 0\), there is only one equilibrium \(O(0,0,0,0,0)\) which is stable near the left-hand side of \(k_6 = 0\); when \(k_6 > 0\), \(O(0,0,0,0,0)\) becomes unstable and the other two equilibria are stable near the right-hand side of \(k_6 = 0\).

**Proof.** The corresponding eigenvectors are

\[
\eta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} \frac{r}{k_5(k_6-m-r-1)+r} \\ \frac{r}{k_5-k_6} \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

\[
\eta_4 = \begin{pmatrix} \frac{r}{\lambda_4+r} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \eta_5 = \begin{pmatrix} \frac{r}{\lambda_5+r} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]
Let

\[ k_6 = \varepsilon, \quad T = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5), \quad (x, y, z, u, v)^T = T(x_1, y_1, z_1, u_1, v_1)^T. \]  

By (4.1), system (2.2) becomes

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
\dot{z}_1 \\
\dot{u}_1 \\
\dot{v}_1
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -k_3 & 0 & 0 & 0 \\
0 & 0 & -k_5 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
y_1 \\
z_1 \\
u_1 \\
v_1
\end{pmatrix} +
\begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4 \\
g_5
\end{pmatrix},
\]

where

\[
a = x_1 + \frac{r}{k_5 (k_5 - m - r - 1)} + r z_1 + \frac{r}{\lambda_4 + r} u_1 + \frac{r}{\lambda_5 + r} v_1, \\
g_1 = a (y_1 + \varepsilon), \\
g_2 = a g \left( \frac{k_5 (m + 1) - r}{k_5 (k_5 - m - r - 1)} + r z_1 - x_1 - \left( 1 + \frac{m \lambda_4}{\lambda_4 + r} \right) u_1 - \left( 1 + \frac{m \lambda_5}{\lambda_5 + r} \right) v_1 \right), \\
g_3 = 0, \\
g_4 = \frac{a (\lambda_4 + r) (\lambda_5 \varepsilon + (\lambda_5 + r) y_1)}{r (\lambda_4 - \lambda_5)}, \\
g_5 = \frac{a (\lambda_5 + r) (\lambda_4 \varepsilon + (\lambda_4 + r) y_1)}{r (\lambda_5 - \lambda_4)}.
\]

From the center manifold theorem, there exists a center manifold for Eqs. (4.2), which can be expressed locally as the following set through the variable \( x_1 \) and \( \varepsilon \):

\[ W_\varepsilon (0) = \{ (x_1, y_1, z_1, u_1, v_1, \varepsilon) \in \mathbb{R}^5 : y_1 = h_1 (x_1, \varepsilon), z_1 = h_2 (x_1, \varepsilon), u_1 = h_3 (x_1, \varepsilon), v_1 = h_4 (x_1, \varepsilon), |x_1| < \delta, |\varepsilon| < \delta, h_i (0, 0) = 0, Dh_i (0, 0) = 0, i = 1, 2, 3, 4 \}, \]

where \( \delta \) and \( \delta \) are sufficiently small.

Assume that

\[
y_1 = h_1 (x_1, \varepsilon) = a_1 x_1^2 + a_2 x_1 \varepsilon + a_3 \varepsilon^2 + o(3), \\
z_1 = h_2 (x_1, \varepsilon) = b_1 x_1^2 + b_2 x_1 \varepsilon + b_3 \varepsilon^2 + o(3), \\
u_1 = h_3 (x_1, \varepsilon) = c_1 x_1^2 + c_2 x_1 \varepsilon + c_3 \varepsilon^2 + o(3), \\
v_1 = h_4 (x_1, \varepsilon) = d_1 x_1^2 + d_2 x_1 \varepsilon + d_3 \varepsilon^2 + o(3).
\]

Considering \( \varepsilon \equiv 0 \), the center manifold should satisfy

\[
N (h (x_1, \varepsilon)) \Delta = D_x h \cdot g_1 - Bh - g_1 \equiv 0, \quad \text{(4.4)}
\]

where

\[
h (x_1, \varepsilon) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \quad D_x h = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} \\ \frac{\partial h_2}{\partial x_1} \\ \frac{\partial h_3}{\partial x_1} \\ \frac{\partial h_4}{\partial x_1} \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}, \quad B = \begin{pmatrix} -k_3 & 0 & 0 & 0 \\ 0 & -k_5 & 0 & 0 \\ 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & \lambda_5 \end{pmatrix}.
\]
Substituting Eqs. (4.3) to (4.4) gives

\[
\begin{align*}
a_1 &= -\frac{g}{k_3}, a_2 = 0, a_3 = 0, \\
b_1 &= 0, b_2 = 0, b_3 = 0, \\
c_1 &= 0, c_2 = \frac{\lambda_5(\lambda_4+r)}{\lambda_4(\lambda_5-r)}, c_3 = 0, \\
d_1 &= 0, d_2 = \frac{\lambda_4(\lambda_5+r)}{\lambda_4(\lambda_5-r)}, d_3 = 0,
\end{align*}
\]

and we obtain

\[
\begin{align*}
y_1 &= h_1(x_1, \varepsilon) = -\frac{g}{k_3} x_1^2 + o(3), \\
z_1 &= h_2(x_1, \varepsilon) = o(3), \\
u_1 &= h_3(x_1, \varepsilon) = \frac{\lambda_5(\lambda_4+r)}{\lambda_4(\lambda_5-r)} x_1 \varepsilon + o(3), \\
v_1 &= h_4(x_1, \varepsilon) = -\frac{\lambda_4(\lambda_5+r)}{\lambda_4(\lambda_5-r)} x_1 \varepsilon + o(3).
\end{align*}
\]

Applying Eqs. (4.5) into \( \dot{x}_1 = g_1 \) of (4.2) and reducing the vector field to the center manifold, we can get

\[
\begin{align*}
\dot{x}_1 &= F(x_1, \varepsilon) + o(4), \\
\dot{\varepsilon} &= 0,
\end{align*}
\]

where

\[
F(x_1, \varepsilon) = \left(1 - \frac{r + m + 1}{r} \varepsilon \right) \left( \varepsilon - \frac{g}{k_3} x_1^2 \right) x_1.
\]

\( F(x_1, \varepsilon) \) satisfies

\[
\begin{align*}
F(0, 0) &= 0, \quad \frac{\partial F}{\partial x_1}(0, 0) = 0, \quad \frac{\partial F}{\partial \varepsilon}(0, 0) = 0, \\
\frac{\partial^2 F}{\partial x_1 \partial r}(0, 0) &= 0, \quad \frac{\partial^2 F}{\partial x_1 \partial \varepsilon}(0, 0) = 1 \neq 0, \\
\frac{\partial^2 F}{\partial x_1^2}(0, 0) &= -\frac{6g}{k_3} \neq 0,
\end{align*}
\]

which indicates that the equilibrium \((x_1, \varepsilon) = (0, 0)\) of Eqs. (4.6) undergoes a pitchfork bifurcation at \( \varepsilon = 0 \) \((k_6 = 0)\). Since \( -\frac{\partial^2 F}{\partial x_1 \partial \varepsilon}(0, 0) > 0 \), the bifurcation direction is near the right-hand side \( \varepsilon = 0 \) \((k_6 = 0)\). So Theorem 4.1 is proved.

**Numerical simulation** For \( r = m = g = k_2 = k_3 = k_5 = 1, \ k_1 = k_4 = 0 \), (4.7) becomes

\[
F(x_1, \varepsilon) = x_1 (3\varepsilon - 1) (x_1^2 - \varepsilon).
\]

As shown in Fig. 9, system (2.2) undergoes a pitchfork bifurcation, which accords with Theorem 4.1.

![Figure 9. Pitchfork bifurcation diagram in system (2.2) near \( k_6 = 0 \)](image-url)
5. Circuit design and implementation of system (2.2). Electronic circuit provides an alternative approach to explore system (2.2). A possible electronic circuit to realize the system is proposed in Fig. 10. The circuit equations have the following form

\[
\begin{align*}
\frac{dV_{c1}}{dt} &= \frac{R_4}{R_3 R_6 C_1} (V_{c2} - V_{c1}), \\
\frac{dV_{c2}}{dt} &= \frac{1}{R_1^2 C_2} \left( R_{34} V_{c1} - \frac{R_4}{R_3} V_{c2} + \frac{0.1 R_{18}}{R_{21}} V_{c1} V_{c3} + \frac{R_6}{R_7} V_{c5} \right), \\
\frac{dV_{c3}}{dt} &= \frac{1}{R_1 C_3} \left( \frac{0.1 R_{18}}{R_2} V_{c1}^2 - \frac{0.1 R_{18}}{R_{21}} V_{c1} V_{c2} - \frac{R_{18}}{R_{19}} V_{c3} \right), \\
\frac{dV_{c4}}{dt} &= \frac{1}{R_2 C_4} \left( \frac{0.1 R_{18}}{R_{25}} V_{c1} V_{c2} - \frac{R_{25}}{R_{26}} V_{c4} \right), \\
\frac{dV_{c5}}{dt} &= \frac{1}{R_3 C_5} \left( \frac{0.1 R_{18}}{R_{25}} V_{c1} V_{c3} - \frac{R_{25}}{R_{31}} V_{c1} - \frac{R_{25}}{R_{32}} V_{c4} \right),
\end{align*}
\]

where \( V_{ci} \) are voltages of capacitors \( C_i \) \( (i = 1, 2, 3, 4, 5) \) respectively.

For \((m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12, 0.001, 0.01, 0.01, -100)\), the circuit components in Fig. 10 are as follows: \( R_4 = 100k\Omega \) \( (i=1, 2, 7, 8, 15, 16, 22, 23, 26, 28, 29) \), \( R_p = 1k\Omega \) \( (p=4, 10, 12, 18, 25, 31, 33) \), \( R_q = 1\Omega \) \( (q=3, 9, 17, 24, 30) \), \( R_5 = R_6 = R_{11} = 50k\Omega \), \( R_{27} = 10k\Omega \), \( R_{13} = R_{34} = 0.1k\Omega \), \( R_{14} = 0.98k\Omega \), \( R_{19} = 1.01\Omega \), \( R_{20} = 0.417k\Omega \), \( R_{21} = 0.098k\Omega \), \( R_{32} = 0.01k\Omega \) and \( C_q = 1mF \) \( (q = 1, 2, 3, 4, 5) \), while the power supplies of all active devices are 15V\(_{DC}\).

It is observed from Fig. 11 that the results obtained by circuit implementation of systems (2.2) confirm the MATLAB simulation results in Fig. 12. NI Multisim components are based on actual circuit components. Simulation results obtained using NI Multisim are in consistence with the actual circuit results [26, 30].

6. Conclusions. The paper presents a new 5D segmented disc dynamo. Of particular interest is the observation that there are coexisting hidden attractors with three types of equilibria in the system, which is reported rarely. Hidden hyperchaos and chaos have been displayed. Self-excited hyperchaotic and chaotic attractors are also observed when parameters vary. By choosing an appropriate bifurcation parameter, the paper proves that the Hopf bifurcation and pitchfork bifurcation occur in the system. The simulation results demonstrate the correctness of the Hopf and pitchfork bifurcations analysis. Finally, the circuit is designed and simulated by NI Multisim. The obtained results have good agreement with the MATLAB simulation results.

This work on coexisting hidden attractors may make the chaotic and hyperchaotic theories more rich. The study on such models is useful to further analyze and suggest practical applications of some phenomena. A future work will be explored on more aspects of hidden hyperchaos.

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Figure 11. Hyperchaotic attractor of system (2.2) obtained using NI Multisim circuit implementation for $(m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12, 0.001, 0.01, 0.01, -100)$

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Figure 12. 2D projections of hyperchaotic attractor of system (2.2) with parameters $(m, r, g, k_1, k_2, k_3, k_4, k_5, k_6) = (0.02, 0.02, 12, 0, 12, 0.001, 0.01, 0.01, -100)$ and initial condition $(0.5268, 7.3786, 2.6912, 4.2284, 0.8147)$.

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