MULTIPLICITY RESULTS FOR FRACTIONAL SYSTEMS CROSSING HIGH EIGENVALUES

Fábio R. Pereira

Departamento de Matemática - Instituto de Ciências Exatas
Universidade Federal de Juiz de Fora
30161-970, Juiz de Fora - MG, Brazil

(Communicated by Wenxiong Chen)

Abstract. We investigate the existence of solutions for a system of nonlocal equations involving the fractional Laplacian operator and with nonlinearities reaching the subcritical growth and interacting, in some sense, with the spectrum of the operator.

1. Introduction. Let \( s \in (0,1) \), \( N > 2s \) and \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain. In this paper we study the possibility of existence of solutions for the following fractional system

\[
\begin{aligned}
(-\Delta)^s u &= au + bv + \frac{2\alpha}{\alpha + \beta} u^{\alpha-1} v^\beta + 2\xi_1 u^{\alpha+\beta-1} + f \quad \text{in } \Omega, \\
(-\Delta)^s v &= bu + cv + \frac{2\beta}{\alpha + \beta} u^\alpha v^{\beta-1} + 2\xi_2 v^{\alpha+\beta-1} + g \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

where \((-\Delta)^s\) is defined by

\[
(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,
\]

\(\xi_1, \xi_2 \geq 0\), \(f, g \in C_0^{0,\sigma}(\overline{\Omega})\) with \(\sigma \in (0,1)\), \(\alpha, \beta > 1\) are real constants such that the sum \(\alpha + \beta\) is compared with the fractional critical Sobolev exponent \(2^*_s := \frac{2N}{N-2s}\) and \(C(N,s)\) is a positive dimensional constant that depends on \(N\) and \(s\) given by

\[
C(N,s) = \left( \int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1 \zeta)}{|\zeta|^{N+2s}} d\zeta \right)^{-1}.
\]

Let us denote by \(0 < \lambda_1,s < \lambda_2,s \leq \lambda_3,s \leq \ldots\) the sequence of eigenvalues of the operator \((-\Delta)^s\) with homogeneous Dirichlet boundary condition and denote by \(\varphi_{1,s}\) the positive eigenfunction associated to \(\lambda_{1,s}\) normalized with respect to \(L^2(\Omega)\) norm. We may write the functions \(f\) and \(g\) as \(f = t \varphi_{1,s} + f_1\) and \(g = r \varphi_{1,s} + g_1\), in such a way that \(f_1, g_1 \in C_0^{0,\sigma}(\overline{\Omega})\), the pair \((t, r) \in \mathbb{R}^2\) and \(\int_{\Omega} f_1 \varphi_{1,s} dx = \int_{\Omega} g_1 \varphi_{1,s} dx = 0\).

2000 Mathematics Subject Classification. Primary: 35R11; Secondary: 35B33, 35J50, 35B34.

Key words and phrases. Fractional systems, Mountain Pass Theorem, Linking Theorem, resonance, variational method.

F. R. Pereira was supported by the Program CAPES/Brazil (Proc 99999.007090/2014-05) at the Granada University and partially by Fapemig/Brazil (CEX APQ 00972/13).

2069
With the above decomposition, in order to state and compare our results to the scalar case, it is convenient to rewrite system (1.1) as

\[
\begin{align*}
\left\{ \begin{array}{l}
(-\Delta)^s U &= AU + \nabla F(U) + T\varphi_{1,s} + F_1 \quad \text{in } \Omega, \\
U &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{array} \right.
\end{align*}
\]  

(1.2)

where \( U = \begin{pmatrix} u \\ v \end{pmatrix}, (-\Delta)^s U = \begin{pmatrix} (-\Delta)^s u \\ 0 \end{pmatrix}, (-\Delta)^s v \end{pmatrix}, A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_{2 \times 2}(\mathbb{R}), \nabla \) is the gradient operator, \( F(U) = \frac{2}{\alpha+\beta} \left( u_+^\alpha v_+^{\beta} + \xi_1 u_+^{\alpha+\beta} + \xi_2 v_+^{\alpha+\beta} \right) \), \( T = \begin{pmatrix} t \\ r \end{pmatrix} \) and \( F_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \).

Let \( \mu_1, \mu_2 \) be real eigenvalues of the symmetric matrix \( A \), which will assume \( \mu_1 \leq \mu_2 \). The interaction of these eigenvalues with the spectrum of the \((-\Delta)^s\) will play an important role in the study of the existence of solutions for the nonlocal gradient system (1.1).

The purpose of this work is to prove the existence of solutions of this class of nonlocal gradient systems of elliptic equations on the hypothesis of an interaction of the eigenvalues \( \mu_1, \mu_2 \) of the matrix \( A \) with eigenvalues of the fractional Laplacian operator \((-\Delta)^s\). When \( \mu_2 < \lambda_{1,s} \), this system belongs to the class of the so called Ambrosetti-Prodi type problems \([2]\) which have been studied by several authors in the last decades with different approaches. We also note that the fractional operators appears in phenomena in physics ([10, 28]), in stochastic processes ([22, 35]), in fluid dynamics, dynamical systems, elasticity, obstacle problems and others (see [8, 9, 12, 14, 32, 33, 34] and references therein).

Problem (1.1) is an extension to systems involving fractional Laplacian operator of the equation considered in \([27]\), in which (1.1) was studied in the local operators case \((s = 1)\) and with the particular matrix \( A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \) \( \in M_{2 \times 2}(\mathbb{R}) \). In \([27]\), the authors established a multiplicity result for the problem \(-\Delta u = \lambda u + u_+^p + f(x) \) in \( \Omega, u = 0 \) on \( \partial \Omega \) provided that the non-homogeneous term \( f \) has the form \( f(x) = h(x) + t\varphi_1(x) \) \((h \in L^r(\Omega) \) with \( r > N) \), \( \lambda \) is not an eigenvalue of \((-\Delta, H^1_0(\Omega)) \) and \( t > T \), for some sufficiently large number \( T = T(h) \). Still in the local scalar case, but with nonlinearity in the critical growth \((p = 2^* - 1)\), the problem above mentioned has been studied in \([16]\), the authors proved the existence of two solutions when \( N \geq 7 \). This result was extended in \([11]\) using the technique developed in \([20]\). Works related to this subject in the local scalar case, we recommend \([4]\) and the references therein. For the critical system in the local operators situation, problem (1.1) was studied, for instance, in \([17]\) and \([25]\) when \( \mu_2 < \lambda_1 \) and by \([23]\) in the uncoupled case. In this paper, we complement the results achieved in \([27]\), proving that the system (1.1) (or (1.2)) has at least two solutions for sufficiently large values of parameters \((t, r)\), the first solution is negative and obtained explicitly depending on the non-homogeneous terms \( f \) and \( g \). The second solution is obtained via the Mountain Pass Theorem when \( \mu_2 < \lambda_{1,s} \), or applying the Linking Theorem in the case \( \lambda_{k,s} < \mu_1 < \mu_2 < \lambda_{k+1,s} \) if \( k \geq 1 \). The resonant case \( \lambda_{k,s} = \mu_1 \) for \( k > 1 \) is also treated here.

To show the existence of solution difficulties arise when we consider fractional operators. Due to the lack of regularity, we impose that \( s \geq 1/2 \) and we develop similar techniques to these known for the Laplacian operator. In addition to these obstacles, further complications arise due to the presence of the mathematical term
\[ F(u, v) = \frac{2}{s + \beta} \left[ u_+^{\alpha} v_+^{\beta} + \xi_1 u_+^{\alpha + \beta} + \xi_2 v_+^{\alpha + \beta} \right] \] that includes either an uncoupled or a coupled nonlinearity. In the case \( \lambda_k, s \leq \mu_1 \leq \mu_2 < \lambda_{k+1, s} \), it is necessary to require the hypothesis that the constants \( \xi_j \) are strictly positive. It is important to point out that, with the aid of [18], our results are still valid for the general case \( \nabla F(u, v) \) when \( F \) is a \((\alpha + \beta)\)-homogeneous nonlinearity, which includes a larger class of functions.

Our main results are

**Theorem 1.1** (Existence of a negative solution). Let \( A \in M_{2 \times 2}(\mathbb{R}) \) be a symmetric matrix, \( s \geq 1/2 \) and \( F_1 = (f_1, g_1) \in C_0^0(\Omega) \times C_0^0(\Omega) \) with \( 0 < \sigma < 1 \). Consider

\[
\mathbf{R} = \left\{ (t, r) \in \mathbb{R}^2 : \begin{array}{l}
br + (\lambda_{1, s} - c)t < \eta \det(\lambda_{1, s}I - A) \\
(\lambda_{1, s} - a)r + bt < \vartheta \det(\lambda_{1, s}I - A) \end{array} \right\}
\]

and assume that

\[
\det(\lambda_{j, s}I - A) \neq 0, \forall j = 1, 2, \ldots
\]

Then there exist \( \eta, \vartheta \ll 0 \) such that system \((1.2)\) has a solution \((u_T, v_T)\) (with \( u_T < 0 \) and \( v_T < 0 \) in \( \Omega \)) for every \( T \in \mathbf{R} \).

**Remark 1.** Suppose that \( \det(\lambda_{1, s}I - A) > 0 \) and

\[
\lambda_{1, s} > \max\{a, c\},
\]

then the set \( \mathbf{R} \) is a region between lines satisfying:

(i) If \( b = 0 \),

\[
\mathbf{R} = (-\infty, \eta \frac{\det(\lambda_{1, s}I - A)}{\lambda_{1, s} - c}) \times (-\infty, \vartheta \frac{\det(\lambda_{1, s}I - A)}{\lambda_{1, s} - a}) \subset \mathbb{R}^2.
\]

(ii) If \( b > 0 \),

\[
\mathbf{R} = \left\{ (t, r) \in \mathbb{R}^2 : \begin{array}{l}
\frac{\det(\lambda_{1, s}I - A)}{b} \frac{\lambda_{1, s} - a}{t} - (\lambda_{1, s} - c) t \text{ and } \\
\frac{\det(\lambda_{1, s}I - A)}{b} \frac{\lambda_{1, s} - a}{t} - \frac{\lambda_{1, s} - c}{t} \end{array} \right\}.
\]

(iii) If \( b < 0 \),

\[
\mathbf{R} = \left\{ (t, r) \in \mathbb{R}^2 : \begin{array}{l}
\frac{\det(\lambda_{1, s}I - A)}{b} \frac{\lambda_{1, s} - a}{t} - (\lambda_{1, s} - c) t \text{ and } \\
\frac{\det(\lambda_{1, s}I - A)}{b} \frac{\lambda_{1, s} - a}{t} - \frac{\lambda_{1, s} - c}{t} \end{array} \right\}.
\]

On the other hand, if \( \det(\lambda_{1, s}I - A) > 0 \) and

\[
\lambda_{1, s} < \min\{a, c\},
\]

then the set \( \mathbf{R} \) satisfies:

(i) If \( b = 0 \),

\[
\mathbf{R} = (\eta \frac{\det(\lambda_{1, s}I - A)}{\lambda_{1, s} - c}, +\infty) \times (\vartheta \frac{\det(\lambda_{1, s}I - A)}{\lambda_{1, s} - a}, +\infty) \subset \mathbb{R}^2.
\]

(ii) If \( b > 0 \),

\[
\mathbf{R} = \left\{ (t, r) \in \mathbb{R}^2 : \begin{array}{l}
\frac{\det(\lambda_{1, s}I - A)}{b} \frac{\lambda_{1, s} - a}{t} - (\lambda_{1, s} - c) t \text{ and } \\
\frac{\det(\lambda_{1, s}I - A)}{b} \frac{\lambda_{1, s} - a}{t} - \frac{\lambda_{1, s} - c}{t} \end{array} \right\}.
\]
(iii) If $b < 0$,

$$
R = \left\{ (t, r) \in \mathbb{R}^2 : \begin{array}{l}
  r > \eta \frac{\det(\lambda_1, I - A)}{b} - \frac{(\lambda_1 - c)}{b} t \\
  r > \vartheta \frac{\det(\lambda_1, I - A)}{(\lambda_1 - a)} - \frac{(\lambda_1 - a)}{b} t
\end{array} \right\}.
$$

Note that, since $\det(\lambda_1, I - A) \neq 0$, the lines that define the region $R$ are not parallel. Moreover, if $\det(\lambda_1, I - A) < 0$ a similar result can be obtained as in the Remark 1.

**Theorem 1.2.** Let $s \geq 1/2$ and assume that $N > 2s$, $\alpha + \beta < 2s$ and that one of the following conditions hold,

$$
\xi_1, \xi_2 \geq 0 \text{ and } \mu_2 < \lambda_{1,s},
$$

(1.6)

$$
\xi_1, \xi_2 \geq 0 \text{ and } \lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s}, \text{ for some } k \geq 1.
$$

(1.7)

Then, system (1.2) has a second solution.

**Remark 2.** It is important to note that the hypothesis (1.6) implies that the conditions (1.3) and (1.4) are verified and the hypothesis (1.7) implies in (1.3) and (1.5). In both cases, $\det(\lambda_1, I - A) > 0$.

**Theorem 1.3.** Suppose

$$
\xi_1, \xi_2 > 0 \text{ and } \lambda_{k,s} = \mu_1 \leq \mu_2, \text{ for some } k > 1.
$$

In addition assume that

$$
F_1 = (f_1, g_1) \in (Ker((-\Delta)^s - \lambda_{k,s} I))^+.
$$

(1.8)

Then system (1.2) has a second solution.

2. **Notations and preliminary stuff.** For any measurable function $u : \mathbb{R}^N \to \mathbb{R}$ the Gagliardo seminorm is defined by

$$
[u]_s := \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2} = \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx \right)^{1/2}.
$$

The second equality follows by [19, Proposition 3.6] when the above integrals are finite. Then, we consider the fractional Sobolev space

$$
H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : [u]_s < \infty \}, \quad \| u \|_{H^s} = (\| u \|_{L^2}^2 + [u]_s^2)^{1/2},
$$

which is a Hilbert space. We use the closed subspace

$$
X(\Omega) := \{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.
$$

By Theorems 6.5 and 7.1 in [19], the imbedding $X(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, 2^+_s]$ and compact for $r \in [1, 2^+_s]$. Due to the fractional Sobolev inequality, $X(\Omega)$ is a Hilbert space with inner product

$$
\langle u, v \rangle_X := C(N, s) \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy,
$$

which induces the norm $\| \cdot \|_X = [\cdot]_s$. Observe that by Proposition 3.6 in [19], we have the following identity

$$
\| u \|^2_X = \frac{2}{C(N, s)} \| (-\Delta)^{s/2} u \|^2_{L^2} \quad u \in X(\Omega).
$$
Then it is proved that for \( u, v \in X(\Omega), \)
\[
\frac{2}{C(N, s)} \int_{\mathbb{R}^N} u(x)(-\Delta)^s v(x)dx = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy,
\]
(2.1)
in particular, \((-\Delta)^s\) is self-adjoint in \( X(\Omega). \)

Now, we consider the Hilbert space given by the product space
\[
Y(\Omega) := X(\Omega) \times X(\Omega),
\]
equipped with the inner product
\[
\langle (u, v), (\varphi, \psi) \rangle_Y := \langle u, \varphi \rangle_X + \langle v, \psi \rangle_X
\]
and the norm
\[
\|(u, v)\|_Y := (\|u\|_X^2 + \|v\|_X^2)^{1/2}.
\]
The space \( L^r(\Omega) \times L^r(\Omega) \) \((r > 1)\) is considered with the standard product norm
\[
\|(u, v)\|_{L^r \times L^r} := (\|u\|_{L^r}^2 + \|v\|_{L^r}^2)^{1/2}.
\]

Besides, we recall that
\[
\mu_1|U|^2 \leq (AU, U)_{x^2} \leq \mu_2|U|^2, \quad \text{for all } U := (u, v) \in \mathbb{R}^2,
\]
(2.2)
where \( \mu_1, \mu_2 \) are the eigenvalues of the symmetric matrix \( A. \) In this paper, we consider the following notation for product space \( S \times S := S^2 \) and
\[
w^+(x) := \max\{w(x), 0\}, \quad w^-(x) := \max\{-w(x), 0\}
\]
for positive and negative part of a function \( w. \) Consequently we get \( w = w^+ - w^-\).

**Definition 2.1.** By a solution of (1.1) we mean a weak solution, that is, a pair of functions \((u, v) \in Y(\Omega)\) such that
\[
\langle (u, v), (\varphi, \psi) \rangle_Y - \int_{\Omega} A(u, v), (\varphi, \psi) \rangle_{x^2} dx = \int_{\Omega} \frac{\partial H}{\partial u} \varphi dx - \int_{\Omega} \frac{\partial H}{\partial v} \psi dx = 0,
\]
for all \((\varphi, \psi) \in Y(\Omega),\) where
\[
H(u, v) = \frac{2}{\alpha + \beta} \left[ u^+^{\alpha} v^+^{\beta} + \xi_1 u^+^{\alpha+\beta} + \xi_2 v^+^{\alpha+\beta} \right] + fu + gv.
\]

2.1. A regularity result. Consider the following system
\[
\begin{cases}
(-\Delta)^s u = G_u(u, v) & \text{in } \Omega, \\
(-\Delta)^s v = G_v(u, v) & \text{in } \Omega, \\
u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
(2.3)
where \( G_u, G_v : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) are Carathéodory functions which satisfy respectively, the following growth conditions
\[
|G_u(z, w)| \leq C(1 + |z|^\alpha + |w|^\beta), \quad \text{for all } (z, w) \in \mathbb{R}^2,
\]
(2.4)
\[
|G_v(z, w)| \leq C(1 + |z|^\alpha + |w|^\beta), \quad \text{for all } (z, w) \in \mathbb{R}^2.
\]
(2.5)
The next result can be found in [15, Lemma 3.1].

**Lemma 2.2.** Let \( \Omega \) be a bounded \( C^{1,1} \) domain and let \( G \in C^1(\mathbb{R} \times \mathbb{R}) \) be such that its partial derivatives \( G_u \) and \( G_v \) satisfy the growth conditions (2.4) and (2.5). Let \((u, v) \in Y(\Omega)\) be a solution to system (2.3), then \( u, v \in C^s(\mathbb{R}^N) \). Furthermore, if \( s \in (1/2, 1) \), then \( u, v \in C^0_{\text{loc}}(\Omega) \) and
\[
\frac{u}{\delta^s} |_{\partial \Omega}, \frac{v}{\delta^s} |_{\partial \Omega} \in C^{0, \nu}(\overline{\Omega}) \quad \text{for some } \nu \in (0, 1), \quad \text{where } \delta(x) := \text{dist}(x, \partial \Omega).
Remark 3. If \( \alpha + \beta < 2^*_s \) and \( U = (u, v) \) is a solution to the system (1.2), then, by Lemma 2.2, \( u, v \in C^{2,s}_{\text{loc}}(\Omega) \) for \( s \in (1/2, 1) \). In particular \( U \) solves (1.2) in the classical sense.

2.2. An eigenvalue problem. For \( \lambda \in \mathbb{R} \), we consider the problem with homogeneous Dirichlet boundary condition

\[
\begin{align*}
(-\Delta)^s u &= \lambda u \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{align*}
\]

(2.6)

If (2.6) admits a weak solution \( u \in X(\Omega) \setminus \{0\} \), then \( \lambda \) is called an eigenvalue and \( u \) a \( \lambda \)-eigenfunction. The set of all eigenvalues is referred as the spectrum of \((-\Delta)^s \) in \( X(\Omega) \) and denoted by \( \sigma((-\Delta)^s) \). Since \( K = [(-\Delta)^s]^{-1} \) is a compact operator, the problem (2.6) can be written as \( u = \lambda Ku \) with \( u \in L^2(\Omega) \), hence the following results are true (see [30], [31]).

(i) problem (2.6) admits an eigenvalue \( \lambda_{1,s} = \min \sigma((-\Delta)^s) > 0 \) that can be characterized as follows

\[
\lambda_{1,s} = \min_{u \in X \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 \, dx}{\int_{\mathbb{R}^N} |u(x)|^2 \, dx}; 
\]

(ii) there exists a non-negative function \( \varphi_{1,s} \in X(\Omega) \), which is an eigenfunction corresponding to \( \lambda_{1,s} \), attaining the minimum in (2.7);

(iii) all \( \lambda_{1,s} \)-eigenfunctions are proportional, and if \( u \) is a \( \lambda_{1,s} \)-eigenfunction, then either \( u(x) > 0 \) a.e. in \( \Omega \) or \( u(x) < 0 \) a.e. in \( \Omega \);

(iv) the set of the eigenvalues of problem (2.6) consists of a sequence \( \{\lambda_{k,s}\} \) satisfying

\[
0 < \lambda_{1,s} < \lambda_{2,s} \leq \lambda_{3,s} \leq \ldots \leq \lambda_{j,s} \leq \lambda_{j+1,s} \leq \ldots, \lambda_{k,s} \to \infty, \text{ as } k \to \infty,
\]

which is characterized by

\[
\lambda_{k+1,s} = \min_{u \in P_{k+1} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |u(x)|^2 \, dx}{\int_{\mathbb{R}^N} |u(x)|^2 \, dx};
\]

where

\[
P_{k+1} = \{ u \in X(\Omega) : \langle u, \varphi_{j,s} \rangle_\chi = 0, \ j = 1, 2, \ldots, k \};
\]

(v) if \( \lambda \in \sigma((-\Delta)^s) \setminus \{\lambda_{1,s}\} \) and \( u \) is a \( \lambda \)-eigenfunction, then \( u \) changes sign in \( \Omega \).

(vi) Denote by \( \varphi_{k,s} \) the eigenfunction associated to the eigenvalue \( \lambda_{k,s} \), for each \( k \in \mathbb{N} \). The sequence \( \{\varphi_{k,s}\} \) is an orthonormal basis either of \( L^2(\Omega) \) or of \( X(\Omega) \).

Remark 4. Every eigenfunction of \((-\Delta)^s \) is in \( C^{0,\sigma}(\overline{\Omega}) \) for some \( \sigma \in (0,1) \) (see Theorem 1 of [30] or Proposition 2.4 of [29]).

3. Proof of Theorem 1.1. The following result was proved in [13, Lemma 1.1] in the local operator case. The proof follows arguing as was made there.

Lemma 3.1. The problem

\[
\begin{align*}
(-\Delta)^s U &= AU \quad \text{in } \Omega, \\
U &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{align*}
\]

(3.1)

has a solution \( U \neq 0 \) in \( Y(\Omega) \) if, and only if, the matrix \( \lambda_{k,s} I - A \) is singular for some \( \lambda_{k,s} \).
Proof. Let $U = (u, v) \in Y(\Omega)$. Using that the eigenfunctions $\varphi_{k,s}$ constitute an orthonormal basis of $X$, we can write the Fourier developments

$$u = \sum_k u_k \varphi_{k,s} \quad \text{and} \quad v = \sum_k v_k \varphi_{k,s}.$$ 

If $z_k := (u_k, v_k) \in \mathbb{R}^2$, we obtain $U = \sum_k z_k \varphi_{k,s}$. With this notation, consider $\Psi = \sum_k w_k \varphi_{k,s}$ with $w_k = (\omega_k, \psi_k) \in \mathbb{R}^2$. Hence, if $U$ is a solution of the problem (3.1), we get

$$\langle U, \Psi \rangle_Y = \int_{\Omega} (AU, \Psi)_{\mathbb{R}^2} dx.$$ 

From (2.1),

$$\langle U, \Psi \rangle_Y = \int_{\mathbb{R}^N} (U, (-\Delta)^s \Psi)_{\mathbb{R}^2} dx = \int_{\Omega} \left( \sum_k z_k \varphi_{k,s}, \sum_k w_k \lambda_{k,s} \varphi_{k,s} \right)_{\mathbb{R}^2} dx \quad \text{(3.2)}$$

and

$$\int_{\Omega} (AU, \Psi)_{\mathbb{R}^2} dx = \int_{\Omega} \left( A \left( \sum_k z_k \varphi_{k,s}, \sum_k w_k \varphi_{k,s} \right) \right)_{\mathbb{R}^2} dx = \sum_k (\lambda_{k,s}, w_k)_{\mathbb{R}^2}, \quad \text{(3.3)}$$

we have $\lambda_{k,s} z_k = A(z_k)$, $\forall k = 1, 2, \ldots$, i.e. $(\lambda_{k,s} I - A) z_k = 0$, for every $k$. Hence, $U = \sum_k z_k \varphi_{k,s} \neq 0$ if, and only if, $\det(\lambda_{k,s} I - A) = 0$ for some $k$. \ \Box

To prove Theorem 1.1, we need the following lemma.

Lemma 3.2. If (1.3) hold and $F_1 \in L^2(\Omega) \times L^2(\Omega)$, then the system

$$\begin{align*}
(-\Delta)^s U &= AU + F_1 \quad \text{in } \Omega, \\
U &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*} \quad \text{(3.4)}$$

has a unique solution $U_0 = (u_0, v_0) \in Y(\Omega)$.

Proof. Consider the modified problem

$$\begin{align*}
(-\Delta)^s U - AU + \gamma U &= F_1 \quad \text{in } \Omega, \\
U &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*} \quad \text{(3.5)}$$

where $\gamma > \mu_2$. Define the bilinear form $\mathcal{B} : Y(\Omega) \times Y(\Omega) \to \mathbb{R}$ by:

$$\mathcal{B}(U, \Phi) = \langle U, \Phi \rangle_Y - \int_{\Omega} (AU, \Phi)_{\mathbb{R}^2} dx + \gamma \int_{\Omega} (U, \Phi)_{\mathbb{R}^2} dx,$$

for every $U = (u, v)$, $\Phi = (\varphi, \psi) \in Y(\Omega)$. So,

$$\mathcal{B}(U, U) \geq \|U\|_Y^2 + (\gamma - \mu_2) \int_{\Omega} |U|^2 dx \geq \|U\|_Y^2,$$

which shows that $\mathcal{B}$ is coercive, and that

$$|\mathcal{B}(U, \Phi)| \leq \|U\|_Y \|\Phi\|_Y + C \int_{\Omega} |U|^2 dx \leq \left(1 + \frac{C}{\lambda_{1,s}}\right) \|U\|_Y \|\Phi\|_Y,$$

where $C = \max\{|a|, |b|, |c|, |\gamma|\} > 0$, proving that $\mathcal{B}$ is a continuous bilinear form on $Y(\Omega)$. Therefore, by the Lax-Milgram Theorem, for every $F_1 \in L^2(\Omega) \times L^2(\Omega)$, there is a unique $U$ in $Y(\Omega)$, such that $\mathcal{B}(U, \Phi) = \int_{\Omega} (F_1, \Phi)_{\mathbb{R}^2} dx$, for all $\Phi \in Y(\Omega)$,
i.e., $U$ is the unique weak solution for the problem (3.5). Moreover, taking $\Phi = U$ in the above equation,

$$\|U\|_Y^2 \leq (\mu_2 - \gamma)\|U\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{\lambda_{1,s}}} \|F_1\|_{L^2(\Omega)}^2 \|U\|_Y$$

and consequently, the solution $U$ satisfies $\|U\|_Y \leq \frac{1}{\sqrt{\lambda_{1,s}}} \|F_1\|_{L^2(\Omega)}^2$. This estimate shows that the operator $S : L^2(\Omega) \times L^2(\Omega) \to Y(\Omega)$, which maps the inhomogeneity $F_1$ to the solution $U$ of (3.5), is a linear continuous operator. Since we have the compactness of the embedding $Y(\Omega) \hookrightarrow L^2(\Omega) \times L^2(\Omega)$, the operator $S : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ is compact.

On the other hand, $U$ is a weak solution, the problem (3.4), if and only if, $U$ satisfies $U - K(U) = S(F_1)$, where $K = \gamma S$ is a compact operator. Therefore, by the Fredholm alternative, one of the following conditions hold, either $U - K(U) = S(F_1)$ has an unique solution or $U - K(U) = 0$ has a solution $U \neq 0$. If occurs the second alternative, there exists $U \neq 0$ such that $S(U) = \frac{1}{r}U$, i.e., the system (3.1) has a solution $U \neq 0$ in $Y(\Omega)$, which is a contradiction with the Lemma 3.1.

**Remark 5.** If (1.8) holds, using the Fredholm alternative, we have that (3.4) has a unique solution.

**Remark 6.** If $s = 1/2$ and $F_1 \in C^0(\Omega) \times C^0(\Omega)$, with $0 < \sigma < 1$ and $N > 2s$, then $U_0 \in C^1(\Omega) \times C^1(\Omega)$ and $\|U_0\|_{C^1(\Omega)} \leq c\|F_1\|_{C^0(\Omega)}$, see [7] Proposition 3.1 and if $s > 1/2$, arguing as in [5], we have that $U_0 \in C^1,2s-1(\Omega) \times C^{1,2s-1}(\Omega)$. Moreover, a bootstrap argument ensures that if the function $F_1 \in C^{0}(\Omega) \times C^{0}(\Omega)$ and $N > 2s$, then the solution $U_0$ given by Lemma 3.2 satisfies $\|U_0\|_{C^0(\Omega)} \leq c\|F_1\|_{L^2(\Omega)}$, where $\sigma = \text{min}\{s,2s - \frac{N}{q}\}$, for some constant depending only on $N,s,q$ and $\Omega$ (see [26] Proposition 1.4).

We are ready to prove the existence of a negative solution for the system (1.2).

**Proof of Theorem 1.1.** We will prove the theorem when the conditions (1.3) and (1.5) hold (other cases (1.3) and (1.4) or (1.8)) are analogous to this and left to the reader.

By Lemma 3.2 and Remark 6, the system

$$\begin{cases}
(-\bar{\Delta})^s U = AU + F_1 & \text{in } \Omega, \\
U = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}$$

has a unique solution $U_0 = (u_0,v_0) \in C^1(\Omega) \times C^1(\Omega)$.

Besides, $(w,z) = \left(\frac{(\lambda_1,s - c)t + br}{\det(\lambda_1,s I - A)} \varphi_{1,s}, \frac{bt + (\lambda_1,s - a)r}{\det(\lambda_1,s I - A)} \varphi_{1,s}\right)$ is the unique solution of the system

$$\begin{cases}
(-\bar{\Delta})^s U = AU + T\varphi_{1,s} & \text{in } \Omega, \\
U = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}$$

Consequently, if

$$w_T = \frac{(\lambda_1,s - c)t + br}{\det(\lambda_1,s I - A)} \varphi_{1,s} + u_0, \quad v_T = \frac{bt + (\lambda_1,s - a)r}{\det(\lambda_1,s I - A)} \varphi_{1,s} + v_0,$$
then \( U_T = (u_T, v_T) \) is a solution of the system

\[
\begin{cases}
(-\Delta)^s U = AU + T \varphi_{1,s} + F_1 & \text{in } \Omega, \\
U = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Clearly if \( u_T \) and \( v_T \) are negative in \( \Omega \), we deduce also that \( U_T \) is a solution of (1.2). Therefore, to conclude the proof under the conditions (1.3) and (1.5) (see Remark 1), it suffices to show the existence of an unbounded region \( R \in \mathbb{R}^2 \) where \( u_T \) and \( v_T \) are negative in \( \Omega \) for every \( T = (t, r) \in R \).

Indeed, since \( \varphi_{1,s} \in C^1(\overline{\Omega}) \) is strictly positive in \( \Omega \), \( \frac{\partial \varphi_{1,s}}{\partial \nu} > 0 \) on \( \partial \Omega \) where \( \frac{\partial g}{\partial \nu} \) denote the inner normal derivative of \( g \) on \( \partial \Omega \) (see Lemma 1.2 and Appendix A in [21], Lemma 4.3 in [7]) and \( u_0, v_0 \in C^1(\overline{\Omega}) \), there exists \( \eta, \vartheta \ll 0 \) such that

- \( \eta \varphi_{1,s} + u_0 < 0 \) in \( \Omega \),
- \( \vartheta \varphi_{1,s} + v_0 < 0 \) in \( \Omega \).

Then \( u_T \) and \( v_T \) are negative in \( \Omega \) for every \( T = (t, r) \in R \) and the proof of theorem is concluded. \( \square \)

4. Proof of Theorems 1.2 and 1.3. Let \( U_T := (u_T, v_T) \) be the solution with \( u_T, v_T < 0 \) in \( \Omega \) given by Theorem 1.1 for \( T \in R \). Notice that if \( \overline{U} \neq (0,0) \) is a solution of

\[
\begin{cases}
(-\Delta)^s U = AU + \nabla F(U + U_T) & \text{in } \Omega, \\
U = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

then \( U = \overline{U} + U_T \) is a (second) solution of the system (1.2) with \( \overline{U} + U_T \neq U_T \).

Therefore, to prove Theorems 1.2 and 1.3 we only have to show that under the hypotheses of these theorems, the system (4.1) has a nonzero solution for every \( T \in R \).

Observe that the weak solutions of (4.1) are the critical points of the functional \( I_s : Y(\Omega) \to \mathbb{R} \) given by

\[
I_s(U) = \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x - y|^{N + 2s}} dxdy
- \frac{1}{2} \int_{\Omega} (AU, U)_{\mathbb{R}^2} dx - \int_{\Omega} F(U + U_T) dx,
\]

where

\[
F(U) := \frac{2}{\alpha + \beta} \left[ u_+^{\alpha} v_+^{\beta} + \xi_1 u_+^{\alpha+\beta} + \xi_2 v_+^{\alpha+\beta} \right], \quad \text{for every } U = (u, v) \in \mathbb{R}^2
\]

and that \( U = 0 \) is a critical point for \( I_s \) with \( I_s(0) = 0 \).

Remark 7. Notice that every nonzero critical point \( U \) of \( I_s \) has positive part \( U_+ \neq 0 \). Indeed, if \( U = (u, v) \) is critical point with \( u_+ = v_+ = 0 \), then \( u, v \leq 0 \) and consequently \( U \) satisfies the system (3.1). Since \( \det(\lambda_j I - A) \neq 0 \) for all \( j \), by Lemma 3.1, we have that \( U \equiv 0 \).

Remark 8. The nonlinearity \( F \) is \((\alpha + \beta)\)-homogeneous, i.e.

\[
F(\lambda U) = \lambda^{\alpha+\beta} F(U), \quad \forall U \in \mathbb{R}^2, \quad \forall \lambda \geq 0.
\]

In particular:

(i) \( (\nabla F(U), U)_{\mathbb{R}^2} = uF_u(U) + vF_v(U) = (\alpha + \beta) F(U), \quad \forall U = (u, v) \in \mathbb{R}^2 \).

(ii) \( F_u \) and \( F_v \) are \((\alpha + \beta - 1)\)-homogeneous.
There exists $K > 0$ such that
\[ F_u(U) \leq K(|u|^{\alpha + \beta} + |v|^{\alpha + \beta}) \]
and
\[ F_v(U) \leq K(|u|^{\alpha + \beta} + |v|^{\alpha + \beta}), \]
for all $U = (u, v) \in \mathbb{R}^2$.

Since $F(U) = F(u_+, v_+)$, $\forall U = (u, v) \in \mathbb{R}^2$, we deduce that
\[ |\nabla F(U)| \leq K(u_+^{\alpha + \beta} + v_+^{\alpha + \beta}) \]
for some constant $K > 0$.

4.1. The Palais-Smale condition for the functional $\mathcal{I}_s$. In this subsection we discuss a compactness property for the functional $\mathcal{I}_s$, given by the Palais-Smale condition.

**Lemma 4.1.** If $\mu_2 < \lambda_1$, then the functional $\mathcal{I}_s$ satisfies the (PS) condition.

**Proof.** The Fréchet derivative of the functional $\mathcal{I}_s$ is given by
\[
\mathcal{I}_s'(u, v)(\varphi, \psi) = C(N, s) \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y)) + (v(x) - v(y))(\psi(x) - \psi(y))}{|x - y|^{N + 2\alpha}} dx dy
\]
\[ - \int_{\Omega} (A(u, v), (\varphi, \psi))_{\mathbb{R}^2} dx - \int_{\Omega} \langle \nabla F(u + u_T, v + v_T), (\varphi, \psi) \rangle_{\mathbb{R}^2} dx, \]
for every $(u, v), (\varphi, \psi) \in \mathcal{Y}(\Omega)$.

Let $(U_n) \subset \mathcal{Y}(\Omega)$ be a (PS)-sequence, i.e. satisfying $|\mathcal{I}_s(U_n)| \leq M$ for some positive real constant $M$ and $\mathcal{I}_s'(U_n) \rightarrow 0$ in the dual space $\mathcal{Y}(\Omega)'$. We are going to prove that there is a convergent subsequence of $U_n$ in two steps:

**Step 1:** $U_n$ is bounded. Indeed, observe that:
\[
\mathcal{I}_s(U_n) - \frac{1}{2} \mathcal{I}'_s(U_n)U_n = \frac{1}{2} \int_{\Omega} (\nabla F(U_n + U_T), U_n)_{\mathbb{R}^2} dx - \int_{\Omega} F(U_n + U_T) dx
\]
\[ = \frac{1}{2} \int_{\Omega} (\nabla F(U_n + U_T), U_n + U_T)_{\mathbb{R}^2} dx + \frac{1}{2} \int_{\Omega} (\nabla F(U_n + U_T), -U_T)_{\mathbb{R}^2} dx
\]
\[ - \int_{\Omega} F(U_n + U_T) dx. \]

Now, by Remark 8 (i),
\[
\left(\frac{\alpha + \beta}{2} - 1\right) \int_{\Omega} F(U_n + U_T) dx + \frac{1}{2} \int_{\Omega} (\nabla F(U_n + U_T), -U_T)_{\mathbb{R}^2} dx
\]
\[ \leq |\mathcal{I}_s(U_n)| + \frac{1}{2} \|\mathcal{I}'_s(U_n)\|_{\mathcal{Y}'} \|U_n\|_{\mathcal{Y}} \leq M + \varepsilon_n \|U_n\|_{\mathcal{Y}}, \]
with $\varepsilon_n = \|\mathcal{I}'_s(U_n)\|_{\mathcal{Y}'} \rightarrow 0$. Therefore, since $u_T$ and $v_T$ are negative, $\xi_1, \xi_2 > 0$ and $\alpha, \beta > 1$, we have that the left hand side of the above inequality is positive, and consequently
\[
\int_{\Omega} F(U_n + U_T) dx \leq C + C\|U_n\|_{\mathcal{Y}} \quad (4.2)
\]
and
\[
\int_{\Omega} (\nabla F(U_n + U_T), -U_T)_{\mathbb{R}^2} dx \leq C + C\|U_n\|_{\mathcal{Y}}. \quad (4.3)
\]
On the other hand, by Remark 8 (i) again, \[
    \mathcal{I}_s'(U_n)U_n = \|U_n\|^2_Y - \int_\Omega \langle AU_n, U_n \rangle dx - \int_\Omega (\nabla F(U_n + U_T), U_n + U_T) dx - \int_\Omega (\nabla F(U_n + U_T), -U_T) dx \\
    \geq \left(1 - \frac{\mu_2}{\lambda_{1,s}}\right)\|U_n\|^2_Y - (\alpha + 3) \int_\Omega F(U_n + U_T) dx \\
    \geq \left(1 - \frac{\mu_2}{\lambda_{1,s}}\right)\|U_n\|^2_Y - (\alpha + 3) \int_\Omega F(U_n + U_T) dx.
\]

Since \((U_n)\) is a \((PS)\)-sequence, by (4.2) and (4.3), we get
\[
    \varepsilon_n \|U_n\|_Y \geq \mathcal{I}_s'(U_n)U_n \geq \left(1 - \frac{\mu_2}{\lambda_{1,s}}\right)\|U_n\|^2_Y - C\|U_n\|_Y - C
\]
and consequently, using that \(\mu_2 < \lambda_{1,s}\), we conclude that \((U_n) = (u_n, v_n)\) is bounded in \(Y(\Omega)\).

**Step 2:** Conclusion: Passing to a subsequence, we can assume that \(U_n \rightharpoonup U\) in \(Y(\Omega)\), \(U_n \rightarrow U\) a.e. in \(\Omega\), \(U_n \rightarrow U\) in \(L^p(\Omega) \times L^p(\Omega)\), for all \(p \in [1, 2^*_s)\) and \(|u_n| + |v_n| \leq \psi \in L^{\alpha + \beta}(\Omega)\).

On the other hand, we have the convergence to zero of \(\mathcal{I}_s'(U_n)(U_n - U)\); i.e.
\[
    \langle U_n, U_n - U \rangle_Y - \int_\Omega \langle AU_n, U_n - U \rangle dx - \int_\Omega (\nabla F(U_n + U_T), U_n - U) dx \rightarrow 0, \hspace{1cm} (4.4)
\]
as \(n \rightarrow \infty\).

**Claim:** \(\int_\Omega (\nabla F(U_n + U_T), U_n - U) dx \rightarrow 0\), as \(n \rightarrow \infty\).

Indeed, by Remark 8 (iii), there exists a constant \(K > 0\) such that
\[
    |\nabla F(U_n + U_T)| \leq K[(u_n + u_T)^{\alpha + \beta - 1} + (v_n + v_T)^{\alpha + \beta - 1}]
\]
and consequently
\[
    (\nabla F(U_n + U_T), U_n - U) \leq C[(u_n + u_T)^{\alpha + \beta - 1} + (v_n + v_T)^{\alpha + \beta - 1}] |u_n - u| + |v_n - v|). \hspace{1cm} (4.5)
\]

Since \(u_T < 0\) and \(v_T < 0\) are negative, we have that \((u_n + u_T)^{\alpha + \beta - 1} \leq |u_n|^\alpha + \beta - 1\) and \((v_n + v_T)^{\alpha + \beta - 1} \leq |v_n|^\alpha + \beta - 1\). So
\[
    (u_n + u_T)^{\alpha + \beta - 1}(|u_n - u| + |v_n - v|) \leq \psi^{\alpha + \beta - 1}(|u_n - u| + |v_n - v|) \text{ a.e. in } \Omega
\]
and
\[
    (v_n + v_T)^{\alpha + \beta - 1}(|u_n - u| + |v_n - v|) \leq \psi^{\alpha + \beta - 1}(|u_n - u| + |v_n - v|) \text{ a.e. in } \Omega.
\]

Therefore, by (4.5),
\[
    (\nabla F(U_n + U_T), U_n - U) \leq \Phi \in L^1(\Omega).
\]

Besides, \((\nabla F(U_n + U_T), U_n - U) \rightarrow 0\) a.e. in \(\Omega\), so the use of the Dominated Convergence Theorem concludes the claim.

Now, by (4.4) and the claim, it follows that \(\|U_n\|^2_Y \rightarrow \|U\|^2_Y\) and consequently \(\|U_n - U\|_Y \rightarrow 0\).

**Lemma 4.2.** If \(k \geq 1\) and \(\lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s}\), the functional \(\mathcal{I}_s\) satisfies the \((PS)\) condition.
Proof. Define the following subspaces
\[ E_k^- = \text{span}(0, \varphi_1, \ldots, \varphi_k, 0) \]
and \[ E_k^+ = (E_k^-)^\perp, \] for \( 1 \leq k \in \mathbb{N} \). Note that \( E_k^+ = (\mathbb{R}^{k+1})^2 \) and \( Y(\Omega) = E_k^- \oplus E_k^+ \).

Let \( U_n \in Y(\Omega) \) be a (PS)-sequence. We split
\[ U_n = U_n^- + U_n^+, \]
where \( U_n^- = (u_n^-, v_n^-) \in E_k^- \) and \( U_n^+ = (u_n^+, v_n^+) \in E_k^+ \).

Taking \( (\varphi, \psi) = U_n^- \) as test function, since \( T'_0(U_n) \to 0 \) as \( n \to \infty \), it follows that
\[ \|U_n\|^2_Y - \int_{\Omega} (AU_n^-, U_n^-)_{{\mathbb{R}^2}} dx - \int_{\Omega} (\nabla F(U_n + U_T), U_n^-)_{{\mathbb{R}^2}} dx \leq C\|U_n^-\|_Y. \] 

Now, using (2.2) and that \( U_n^- = (u_n^-, v_n^-) \in E_k^- \), we obtain
\[ \left( \frac{\mu_1}{\lambda_{k,s}} - 1 \right) \|U_n^-\|^2_Y \leq \int_{\Omega} (AU_n^-, U_n^-)_{{\mathbb{R}^2}} dx - \|U_n^-\|^2_Y \]
\[ = - \int_{\Omega} (\nabla F(U_n + U_T), U_n^-)_{{\mathbb{R}^2}} dx - T'_0(U_n)(U_n^-) \quad (4.6) \]
\[ \leq \int_{\Omega} F_u(U_n + U_T)|u_n^-| dx + \int_{\Omega} F_v(U_n + U_T)|v_n^-| dx + C\|U_n^-\|_Y. \]

Hence, by Remark 8 (iii), there exists a constant \( K > 0 \) such that
\[ |\nabla F(U_n + U_T)| \leq K[(u_n + u_T)^{\alpha + \beta - 1} + (v_n + v_T)^{\alpha + \beta - 1}] \]
and using Hölder’s inequality with \( p = \frac{\alpha + \beta}{\alpha + \beta - 1} \) and \( q = \alpha + \beta \), Young’s inequality, we deduce that
\[ \int_{\Omega} F_u(U_n + U_T)|u_n^-| dx + \int_{\Omega} F_v(U_n + U_T)|v_n^-| dx \]
\[ \leq K \left\{ \varepsilon \|u_n^-\|_{{L^{\alpha + \beta}}}^2 + C_\varepsilon \left[ \|(u_n + u_T)^{\frac{2(\alpha + \beta - 1)}{\alpha + \beta}} + \|(v_n + v_T)^{\frac{2(\alpha + \beta - 1)}{\alpha + \beta}} \right] \right\} \]
\[ + K \left\{ \varepsilon \|v_n^-\|^2_{{L^{\alpha + \beta}}} + C_\varepsilon \left[ \|(u_n + u_T)^{\frac{2(\alpha + \beta - 1)}{\alpha + \beta}} + \|(v_n + v_T)^{\frac{2(\alpha + \beta - 1)}{\alpha + \beta}} \right] \right\}. \]

Using (4.2), in view of the embedding \( X(\Omega) \hookrightarrow L^r(\Omega), \forall r \leq 2^\alpha \), we get
\[ \int_{\Omega} F_u(U_n + U_T)|u_n^-| dx + \int_{\Omega} F_v(U_n + U_T)|v_n^-| dx \]
\[ \leq \varepsilon C_1\|u_n^-\|^2_Y + C_2(1 + \|u_n\|_Y) \frac{2^2(\alpha + \beta - 1)}{\alpha + \beta}. \]

By (4.6), taking \( \varepsilon > 0 \) small enough, we conclude that
\[ \|U_n^-\|^2_Y \leq C_3(1 + \|u_n\|_Y) \frac{2^2(\alpha + \beta - 1)}{\alpha + \beta} + C_4\|U_n^-\|_Y. \] 

(4.7)

Analogously, the following estimate is valid
\[ \|U_n^+\|^2_Y \leq C_5(1 + \|u_n\|_Y) \frac{2^2(\alpha + \beta - 1)}{\alpha + \beta} + C_6\|U_n^+\|_Y. \] 

(4.8)

Using the estimates (4.7) and (4.8), we get
\[ \|U_n\|^2_Y \leq C(1 + \|u_n\|_Y) \frac{2^2(\alpha + \beta - 1)}{\alpha + \beta} + C\|U_n\|_Y. \]

Since \( \frac{2^2(\alpha + \beta - 1)}{\alpha + \beta} < 2 \), we conclude that \( (U_n) \) is bounded in \( Y(\Omega) \) and reasoning as in Step 2 of the proof of Lemma 4.1, it follows that \( (U_n) \) admits a convergent subsequence. \( \square \)
Lemma 4.3. If $k > 1$ and $\lambda_{k,s} = \mu_1 \leq \mu_2 < \lambda_{k+1,s}$, the functional $\mathcal{I}_s$ satisfies the $(PS)$ condition.

Proof. We follow the notations of the previous proof.

Let $U_n \in Y(\Omega)$ such that $[\mathcal{I}_s(U_n)] \leq C$ and $\mathcal{I}_s'(U_n) \to 0$ in the dual space $Y(\Omega)'$.

Writing $Y(\Omega) = E_{k-1}^- + E_k^+ \oplus Z_k$, consequently we have

$$U_n = U_n^- + U_n^+ + \beta_n Y_n := W_n + \beta_n Y_n,$$

where $U_n^- \in E_{k-1}^-$, $U_n^+ \in E_k^+$ and $Y_n \in Z_k = \text{span}\{ (\varphi_{k,s}, 0), (0, \varphi_{k,s}) \}$ with $\|Y_n\|_Y = 1$. Using similar arguments as in (4.7) and (4.8), we obtain

$$\|W_n\|^2_\beta \leq C(1 + \|U_n\|_Y) + C\|W_n\|_Y, \quad (4.9)$$

where $\tau = \frac{\alpha + \beta - 1}{\alpha + \beta}$. We can assume $\|U_n\|_Y \geq 1$ (if $\|U_n\|_Y \leq 1$, the sequence $(U_n)$ is bounded in $Y(\Omega)$). Then, since $\|U_n\|_Y \leq \|W_n\|_Y + \|\beta_n\|$, from (4.9), we have

$$\|W_n\|^2_\beta \leq C_1(\|W_n\|_Y + \|\beta_n\|) + C\|W_n\|_Y. \quad (4.10)$$

If $\beta_n$ is bounded, since $\tau < 1$, by (4.10) we conclude that $(U_n)$ is bounded. Otherwise, we may assume $\beta_n \to +\infty$, therefore, from (4.10), it follows that

$$\|W_n\|^2_\beta \leq C_1 \left( \frac{\|W_n\|_Y + \|\beta_n\|}{\|\beta_n\|} \right)^\tau + C \frac{1}{\beta_n} \|W_n\|_Y$$

$$\leq C_1 \left( \frac{1}{\|\beta_n\|^{\tau - 1}} \|W_n\|_Y + \frac{1}{\|\beta_n\|^{\tau - 1}} \right)^\tau + C \frac{1}{\beta_n} \|W_n\|_Y.$$

Using again the fact that $\tau < 1$, the above estimate yields that

$$\|W_n\|^2_\beta \leq C_2 \|\beta_n\|^{2\tau} + C_3 \frac{\|W_n\|_Y}{\|\beta_n\|} + C_4$$

and consequently the sequence $\left\{ \frac{W_n}{\beta_n} \right\}$ is bounded in $Y(\Omega)$ and $\frac{W_n}{\beta_n} \to 0$.

Therefore, possibly up to a subsequence, $W_n/\beta_n \to 0$ a.e. in $\Omega$ and strongly in $L^q(\Omega) \times L^q(\Omega)$, $1 \leq q < 2^*_s$; $Y_n \to Y_0 \in Z_k$ a.e. in $\Omega$ and strongly in $Y(\Omega)$ and $L^q(\Omega) \times L^q(\Omega)$, $1 \leq q < 2^*_s$.

Taking $\beta_n Y_n \in Z_k$ as test function, we get

$$\mathcal{I}_s'(U_n) Y_n = \beta_n \left( \|Y_n\|^2_Y - \int_\Omega (AY_n, Y_n)_{\mathbb{R}^2} dx \right) - \int_\Omega (\nabla F(U_n + U_T), Y_n)_{\mathbb{R}^2} dx.$$

Since $(U_n)$ is a $(PS)$-sequence and $\frac{1}{(\sigma_n)^{\alpha + \beta - 1}} (\|Y_n\|^2_Y - \int_\Omega (AY_n, Y_n)_{\mathbb{R}^2} dx) \to 0$, as $n \to \infty$, we obtain that

$$o(1) = \frac{1}{(\beta_n)^{\alpha + \beta - 1}} \mathcal{I}_s'(U_n) Y_n \to -\frac{1}{(\beta_n)^{\alpha + \beta - 1}} \int_\Omega (\nabla F(U_n + U_T), Y_n)_{\mathbb{R}^2} dx.$$

Now, from remark 8 (ii),

$$\int_\Omega (\nabla F(U_n + U_T), Y_n)_{\mathbb{R}^2} dx = \frac{1}{(\beta_n)^{\alpha + \beta - 1}} \int_\Omega \nabla F(U_n + U_T) Y_n_{\mathbb{R}^2} dx \to 0. \quad (4.11)$$

On the other hand, since $U_n = W_n + \beta_n Y_n$, we have that $\frac{U_n}{\beta_n} \to Y_0$ in $L^q(\Omega) \times L^q(\Omega)$ for all $1 \leq q < 2^*_s$ a.e. in $\Omega$. So, by the Dominated Convergence Theorem and by (4.11), it follows that

$$\int_\Omega (\nabla F(U_n + U_T), Y_n)_{\mathbb{R}^2} dx \to \int_\Omega (\nabla F(Y_0), Y_0)_{\mathbb{R}^2} dx = 0.$$
Now, from remark 8 (i), we concluded that $\int_\Omega F(Y_0)dx = 0$.

Finally, using the notation $Y_0 = (y_1^0, y_2^0)$, it follows that $(y_1^0)_+ = 0 = (y_2^0)_+$, contradicting $\|Y_0\|_Y = 1$ and $Y_0 \in Z_k$ with $k > 1$, which ensures that at least one of the functions is not null and changes sign. Thus $(U_n)$ is bounded and as in the proof of Lemma 4.1, we have that $(U_n)$ admits a convergent subsequence.

5. Geometry of the functional $I_s$. In this section, we show that the hypothesis $\mu_2 < \lambda_{1,s}$ ensures that the functional $I_s$ satisfies the geometrical conditions of the Mountain Pass Theorem, and has the geometric structure required by the Linking Theorem when $\lambda_{k,s} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,s}$, for some $k \geq 1$.

**Proposition 5.1.** Suppose $\mu_2 < \lambda_{1,s}$. The functional $I_s$ satisfies the following:

a) there exist $\sigma, \rho > 0$ such that $I_s(u, v) \geq \sigma$ if $\|(u, v)\|_Y = \rho$;

b) there exists $(e_1, e_2) \in Y(\Omega)$ with $\|(e_1, e_2)\|_Y > \rho$ such that $I_s(e_1, e_2) \leq 0$.

**Proof.**
a) Since $u_T, v_T < 0$, using (2.2), the estimate $|u|^\alpha|v|^\beta \leq |u|^{\alpha+\beta} + |v|^{\alpha+\beta}$ and the Poincaré inequality, we have

$$I_s(u, v) \geq \frac{1}{2} \left(1 - \frac{\mu_2}{\lambda_{1,s}}\right)\|\|(u, v)\|_Y^2 - C\|\|(u, v)\|_Y^{\alpha+\beta},$$

where $C > 0$ is a constant. Taking $\rho = \left(\frac{4C}{1-\frac{\mu_2}{\lambda_{1,s}}}\right)^{\frac{1}{2-2(\alpha+\beta)}}$, we have that

$$I_s(u, v) \geq \left[\frac{1}{4} (1 - \frac{\mu_2}{\lambda_{1,s}})\right]^{\frac{\alpha+\beta}{2-2(\alpha+\beta)}} \left(\frac{1}{C}\right)^{\frac{2}{2-2(\alpha+\beta)}}, \quad \forall\|\|(u, v)\|_Y = \rho.$$

b) Fix $t_0 > \rho$ and choose a function $(u_0, v_0) \in Y(\Omega) \setminus \{(0, 0)\}$ satisfying $u_0 \geq \frac{2\|u_T\|_{L^2}}{t_0}$ and $v_0 \geq \frac{2\|v_T\|_{L^2}}{t_0}$ a.e. in some subset $C \subset \Omega$ such that $|C| > 0$. Then, for all $t > t_0$,

$$I_s(t(u_0, v_0)) \leq \frac{t^2}{2} \|\|(u_0, v_0)\|_Y^2 - \frac{t^2}{2} \mu_1 \|\|(u_0, v_0)\|_{(L^2)^2}^2$$

$$- \frac{2t^{\alpha+\beta}}{\alpha+\beta} \left[\int_\Omega \left(u_0 + \frac{u_T}{t}\right)^\alpha + \left(v_0 + \frac{v_T}{t}\right)^\beta dx\right].$$

Since

$$\int_\Omega \left(u_0 + \frac{u_T}{t}\right)^\alpha + \left(v_0 + \frac{v_T}{t}\right)^\beta dx \geq \int_\Omega \left(u_0 - \frac{\|u_T\|_{C^0}}{t_0}\right)^\alpha + \left(v_0 - \frac{\|v_T\|_{C^0}}{t_0}\right)^\beta dx$$

$$\geq \int_C \left(\frac{\|u_T\|_{C^0}}{t_0}\right)^\alpha + \left(\frac{\|v_T\|_{C^0}}{t_0}\right)^\beta dx$$

$$= \left(\frac{\|u_T\|_{C^0}}{t_0}\right)^\alpha + \left(\frac{\|v_T\|_{C^0}}{t_0}\right)^\beta |C| := C > 0,$$

we conclude that

$$I_s(t(u_0, v_0)) \leq \frac{t^2}{2} \|\|(u_0, v_0)\|_Y^2 - \frac{t^2}{2} \mu_1 \|\|(u_0, v_0)\|_{(L^2)^2}^2 - \frac{2t^{\alpha+\beta}}{\alpha+\beta} C < 0,$$

if $t > t_0$ is chosen sufficiently large. Therefore we conclude b).
Consider the following orthogonal decomposition of \( Y(\Omega) \):
\[
Y(\Omega) = E^-_k \oplus E^+_k.
\]

**Proposition 5.2.** Suppose \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^N \), \( \alpha + \beta < 2^*_s \) and \( \lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s} \), for some \( k \in \mathbb{N} \). Then
a) there exist \( \sigma, \rho > 0 \) such that \( I_s(U) \geq \sigma \) for all \( U \in E^+_k \) with \( \|U\|_Y = \rho \),
b) there exists \( E \in E^+_k \) and \( R > 0 \) such that \( R\|E\|_Y > \rho \) and if
\[
Q = (\overline{B}_R \cap E^-_k) \oplus [0, R],
\]
then \( I_s(U) \leq 0 \), for all \( U \in \partial Q \).

**Proof.**

a) Let \( U = (u, v) \in E^+_k \), using the fact that \( u_T, v_T < 0 \), estimate \( |u|^\alpha |v|^{\beta} \leq |u|^{\alpha+\beta} + |v|^{\alpha+\beta} \), and the fractional imbedding \( X \hookrightarrow L^{\alpha+\beta} \), by (2.2), we have
\[
I_s(U) \geq \frac{1}{2} \|U\|^2_Y - \frac{\mu_2}{2} \|U\|^2_{L^2} - C \int_\Omega (|u|^{\alpha+\beta} + |v|^{\alpha+\beta}) \, dx
\geq \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k+1,s}} \right) \|U\|^2_Y - C \|U\|^{\alpha+\beta}_Y,
\]
where \( C > 0 \) is a constant. Since \( \mu_2 < \lambda_{k+1,s} \) and \( \alpha + \beta > 2 \), for \( \|U\|_Y = \rho \) small enough, we get \( I_s(U) \geq \sigma \).

b) Now consider the following decomposition \( \partial Q = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), where
\[
\Gamma_1 = \{ U \in Y(\Omega); U = U_1 + rE, \text{ with } U_1 \in E^-_k, \|U_1\|_Y = R, 0 \leq r \leq R \},
\Gamma_2 = \{ U \in Y(\Omega); U = U_1 + RE, \text{ with } U_1 \in E^-_k, \|U_1\|_Y \leq R \},
\Gamma_3 = \overline{B}_R(0) \cap E^-_k.
\]

Let us show that on each set \( \Gamma_i \) we have \( I_s|_{\Gamma_i} \leq 0 \), \( i = 1, 2, 3 \).

Since \( \lambda_{k,s} < \mu_1 \leq \mu_2 < \lambda_{k+1,s} \) with \( k \geq 1 \), choose \( E \) as follows:
fixed \( R_0 > \rho \), take \( E = (e_1, e_2) \in E^+_k = (E^-_k)^\perp \) satisfying
\[
(1) \ \|E\|^2_Y < \frac{\mu_1}{\lambda_{k,s}} - 1.
\]
\[
(2) \ e_1 \geq \frac{2}{K + \|v_r\|_{C^0}} \text{ and } e_2 \geq \frac{2}{K + \|v_r\|_{C^0}} \text{ a.e. in some } C \subset \Omega \text{ with } |C| > 0, \text{ where } K > 0 \text{ satisfies } \|V\|_{(C^0)^2} \leq K \|V\|_Y, \text{ for all } V \in E^-_k.
\]

Note that this choice is possible because \( (E^-_k)^\perp \) has unbounded functions; \( E^-_k \) has finite dimension and \( K = \sup_{V \in E^-_k} \|V\|_Y = 1 \|V\|_{(C^0)^2} \).

i) **Estimates on \( \Gamma_1 \):**

If \( U \in \Gamma_1 \), by (1)
\[
I_s(U) \leq \frac{1}{2} \|U_1\|^2_Y + \frac{r^2}{2} \|E\|^2_Y - \frac{\mu_1 r^2}{2} \|U_1\|^2_{L^2} - \frac{\mu_1 r^2}{2} \|E\|^2_{L^2}
\leq \frac{1}{2} R^2 + \frac{R}{2} \|E\|^2_Y - \frac{\mu_1}{2 \lambda_{k,s}} R^2
\leq \frac{1}{2} R^2 \left( 1 - \frac{\mu_1}{\lambda_{k,s}} + \|E\|^2_Y \right) < 0. \quad (5.1)
\]
For $U = U_1 + RE \in \Gamma_2$, we have
\begin{equation}
\mathcal{I}_s(U_1 + RE) \leq \frac{1}{2} \|U_1\|_Y^2 + \frac{R^2}{2} \|E\|_Y^2 - \frac{\mu_1 R^2}{2} \|U_1\|_{(L^2)^2}^2 - \frac{\mu_1 R^2}{2} \|E\|_{(L^2)^2}^2 \tag{5.2}
\end{equation}
\[ - \int_\Omega F(U_1 + RE + U_T)dx \]
\[ \leq \frac{1}{2} \|U_1\|_Y^2 (1 - \frac{\mu_1}{\lambda_{k,s}}) + \frac{R^2}{2} \|E\|_Y^2 - \int_\Omega F(U_1 + RE + U_T)dx. \]

Since $\lambda_{k,s} < \mu_1$,
\begin{equation}
\mathcal{I}_s(U_1 + RE) \leq \frac{R^2}{2} \|E\|_Y^2 - \int_\Omega F(U_1 + RE + U_T)dx. \tag{5.3}
\end{equation}

Now, to estimate the last integral, note that, if $U_1 = (u_1, u_2)$,
\[ \int_\Omega F(U_1 + RE + U_T)dx \]
\[ \geq \frac{2}{\alpha + \beta} \left[ \xi_1 \int_\Omega (u_1 + R e_1 + u_T)^{\alpha+\beta} dx + \xi_2 \int_\Omega (u_2 + R e_2 + v_T)^{\alpha+\beta} dx \right] \]
\[ = \frac{2}{\alpha + \beta} \left[ \xi_1 R^{\alpha+\beta} \int_\Omega \left( e_1 + \frac{u_1 + u_T}{R} \right)^{\alpha+\beta} dx + \xi_2 R^{\alpha+\beta} \int_\Omega \left( e_2 + \frac{u_2 + v_T}{R} \right)^{\alpha+\beta} dx \right] \]
for $R \geq R_0$, and by (II) each integral on the right can be estimated as follows
\[ \int_\Omega \left( e_i + \frac{u_i + w_T}{R} \right)^{\alpha+\beta} dx \geq \int_\Omega \left( e_i - \frac{\|u_i\|_{C^0} + \|w_T\|_{C^0}}{R} \right)^{\alpha+\beta} dx \]
\[ \geq \int_\Omega \left( e_i - \left( K + \frac{\|w_T\|_{C^0}}{R_0} \right) \right)^{\alpha+\beta} dx \]
\[ \geq \int_\Omega \left( K + \frac{\|w_T\|_{C^0}}{R_0} \right)^{\alpha+\beta} dx \]
\[ = \left( K + \frac{\|w_T\|_{C^0}}{R_0} \right)^{\alpha+\beta} |\mathcal{Y}|. \tag{5.4} \]
for $i = 1, 2$ and $w = u, v$.

Therefore, by (5.3) and by above estimates,
\begin{equation}
\mathcal{I}_s(U_1 + RE) \leq \frac{R^2}{2} \|E\|_Y^2 - \xi_1 R^{\alpha+\beta} \int_\Omega \left( e_1 + \frac{u_1 + u_T}{R} \right)^{\alpha+\beta} dx
\end{equation}
\[ - \xi_2 R^{\alpha+\beta} \int_\Omega \left( e_2 + \frac{u_2 + v_T}{R} \right)^{\alpha+\beta} dx \]
\[ \leq \frac{R^2}{2} \|E\|_Y^2 - C R^{\alpha+\beta}. \]

Since $\alpha + \beta > 2$, for $R \geq R_0$, we have $\mathcal{I}_s(U) < 0$ for all $U \in \Gamma_2$.

iii) Estimates on $\Gamma_3$:
For $U \in \Gamma_3$, it follows the estimate
\[ \mathcal{I}_s(U) \leq \frac{1}{2} \|U\|_Y^2 - \frac{1}{2} \int_\Omega (AU, U)_{(H_0^1)} dx \leq \frac{1}{2} \|U\|_Y^2 - \frac{\mu_1}{2} \|U\|_{(L^2)^2}^2 \]
\[ \leq \frac{1}{2} \left( 1 - \frac{\mu_1}{\lambda_{k,s}} \right) \|U\|_Y \leq 0. \]

Therefore, for all $R \geq R_0 > 0$, follows that $\mathcal{I}_s(U) \leq 0$ for all $U \in \partial Q$, concluding the desired result. \hfill \Box
Proposition 5.3. Suppose \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^N \), \( \alpha + \beta < 2 \) and \( \lambda_{k,s} = \mu_1 \leq \mu_2 < \lambda_{k+1,s} \) for some \( k > 1 \). Then
a) there exist \( \sigma, \rho > 0 \) such that \( \mathcal{I}_s(U) \geq \sigma \) for all \( U \in E_k^\pm \) with \( \| U \|_Y = \rho \),
b) there exists \( E \in E_k^\pm \) and \( R > 0 \) such that \( R \| E \|_Y > \rho \) and \( \mathcal{I}_s(U) \leq 0 \), for all \( U \in \partial Q \), where \( Q = (B_R \cap E_k^-) \oplus [0, R]E \).

Proof. Choose \( E = (e_1, e_2) \in E_k^\pm \) (with \( e_i \geq 0 \), \( i = 1, 2 \)) satisfying the condition (II) as in the Proposition 5.2 and \( \| E \|_Y^2 < \left( \frac{\mu_1}{\lambda_{k-1,s}} - 1 \right) \delta^2 \), where \( \delta > 0 \) is a constant to be obtained forward.

In this case, it is sufficient to estimate the functional on \( \Gamma_1 \), since the estimates on \( \Gamma_2 \) and \( \Gamma_3 \), and the proof of a) are made as in the previous proposition.

Therefore, for \( U = U_1 + rE \in \Gamma_1 \), we consider \( U_1 = R\hat{U}_1 \in E_k^- \) with \( \| \hat{U}_1 \|_E = 1 \) and we set \( \hat{U}_1 = c_1 Y + c_2 E_k \), where \( E_k \in Z_k = \text{span} \{ (\varphi_{k,s}, 0), (0, \varphi_{k,s}) \} \) and \( Y \in E_{k-1}^- \) with \( \| Y \|_Y = 1 \). Then,

\[
\mathcal{I}_s(U) \leq \frac{1}{2} \| U_1 \|^2_Y + \frac{r^2}{2} \| E \|^2_Y - \frac{\mu_1}{2} \| U_1 \|^2_{(L^2)^2} - \frac{\mu_1}{2} r^2 \| E \|^2_{(L^2)^2} - \int_{\Omega} F(U + UT) dx
\]

\[
\leq \frac{R^2}{2} \| \hat{U}_1 \|^2_Y + \frac{r^2}{2} \| E \|^2_Y - \frac{\mu_1 R^2}{2} \| \hat{U}_1 \|^2_{(L^2)^2} - \int_{\Omega} F(U + UT) dx
\]

\[
= \frac{R^2}{2} \| c_1 Y + c_2 E_k \|^2_Y + \frac{r^2}{2} \| E \|^2_Y - \frac{\mu_1 R^2}{2} \| c_1 Y + c_2 E_k \|^2_{(L^2)^2} - \int_{\Omega} F(U + UT) dx
\]

\[
= \frac{R^2}{2} c_1^2 (\| Y \|^2_Y - \mu_1 \| Y \|^2_{(L^2)^2}) + \frac{r^2}{2} \| E_k \|^2_Y - \mu_1 \| E_k \|^2_{(L^2)^2})
\]

\[
+ \frac{R^2}{2} \| E \|^2_Y - \int_{\Omega} F(U + UT) dx.
\]

Consequently

\[
\mathcal{I}_s(U) \leq \frac{R^2}{2} c_1^2 \left( 1 - \frac{\mu_1}{\lambda_{k-1,s}} \right) \| Y \|^2_Y + \frac{r^2}{2} \| E \|^2_Y - \int_{\Omega} F(U + UT) dx.
\]

Now using the notation \( \hat{U}_1 = (\hat{u}_1, \hat{v}_1) = (c_1 y_1 + c_2 e_1^k, c_1 y_2 + c_2 e_2^k) \), where \( Y = (y_1, y_2) \in E_{k-1} \cap B_1 \) and \( E_k = (e_1, e_2) \in Z_k \cap B_1 \), we will prove that there exist \( \delta > 0 \) and \( \eta > 0 \) such that

\[
\max_{i=1,2} \left\{ \max_{\Omega} \{ c_1 y_i + c_2 e_i^k ; \ | c_1 | \leq \delta \} \right\} \geq \eta > 0.
\]

Indeed, by contradiction, assume that there exist sequences \( (c_1^n, c_2^n) \subset \mathbb{R} \) and \( Y_n = (y_1^n, y_2^n) \subset Y(\Omega) \) with \( \| Y_n \|_Y = 1 \) such that \( c_1^n \to 0 \), \( | c_2^n | = \sqrt{1 - (c_1^n)^2} \to 1 \) and

\[
\max_{i=1,2} \left\{ \max_{\Omega} \{ c_1^n y_i^n + c_2^n e_i^k \} \right\} \to 0, \text{ as } n \to \infty.
\]

Therefore, \( c_1^n y_i^n \to 0 \) and \( c_2^n e_i^k \to e_i^k \) and consequently

\[
\max_{i=1,2} \left\{ \max_{\Omega} e_i^k(x) \right\} = 0.
\]

Hence, we conclude that \( e_1^k \leq 0 \) and \( e_2^k \leq 0 \) in \( \Omega \), which is a contradiction, because \( E_k = (e_1, e_2^k) \in Z_k \) and \( \| E_k \|_Y = 1 \) imply that at least one of the coordinate
functions must change sign.
So, we conclude that there exist $\delta > 0$, $\eta > 0$ such that
\[
\max \left\{ \max_{\Omega} \tilde{u}_1; \max_{\Omega} \hat{v}_1 : |c_1| \leq \delta \right\} \geq \eta > 0, \ \forall \ \tilde{U}_1 = c_1 Y + c_2 E_k \in E_k^-
\]
with $\|\tilde{U}_1\|_Y = 1$.
Denoting $\Omega_+ = \left\{ x \in \Omega : (\tilde{u}_1)(x) \geq \eta/2 \text{ or } (\hat{v}_1)(x) \geq \eta/2 \right\}$, by equicontinuity of the functions $\tilde{U}_1$, we have that $|\Omega_+| \geq \nu > 0$, $\forall \tilde{U}_1 \in E_k^- \cap B_1$ and $|c_1| \leq \delta$.
Moreover
\[
u_T \geq - \frac{\|u_T\|_{C^0}}{R} > - \frac{\eta}{4} \text{ and } \frac{v_T}{R} \geq - \frac{\|v_T\|_{C^0}}{R} > - \frac{\eta}{4} \text{ for } R \text{ sufficiently large.}
\]
Then
\[
\begin{align*}
\int_{\Omega} F(U + U_T) dx &\geq \frac{2}{\alpha + \beta} \xi_1 R^{\alpha + \beta} \int_{\Omega} \left( \tilde{u}_1 + \frac{r}{R} \varepsilon_1 + u_T \right)^{\alpha + \beta} dx \\
&\quad + \frac{2}{\alpha + \beta} \xi_2 R^{\alpha + \beta} \int_{\Omega} \left( \hat{v}_1 + \frac{r}{R} \varepsilon_2 + v_T \right)^{\alpha + \beta} dx \\
&\geq CR^{\alpha + \beta} \left[ \int_{\Omega} \left( \tilde{u}_1 + \frac{u_T}{R} \right)^{\alpha + \beta} dx + \int_{\Omega} \left( \hat{v}_1 + \frac{v_T}{R} \right)^{\alpha + \beta} dx \right] \\
&\geq CR^{\alpha + \beta} \left[ \int_{\Omega_+} \left( \tilde{u}_1 - \frac{\eta}{4} \right)^{\alpha + \beta} dx + \int_{\Omega_+} \left( \hat{v}_1 - \frac{\eta}{4} \right)^{\alpha + \beta} dx \right] \\
&\geq CR^{\alpha + \beta} \left( \frac{\eta}{4} \right)^{\alpha + \beta} |\Omega_+| \\
&= \tilde{C} R^{\alpha + \beta},
\end{align*}
\]
for all $R$ sufficiently large. Thus, from (5.6) we can conclude that there exists $R_1 > 0$ such that
\[
\mathcal{I}_s(U) \leq \frac{R^2}{2} \delta^2 \left( 1 - \frac{\mu_1}{\lambda_{k-1,s}} \right) + \frac{R^2}{2} \|E\|_Y^2 - \tilde{C} R^{\alpha + \beta} < 0,
\]
for all $R \geq R_1$.

On the other hand, if $|c_1| \geq \delta > 0$, by the choose of $E$, we get
\[
\mathcal{I}_s(U) \leq - \frac{R^2}{2} \xi_1 \left( \frac{\mu_1}{\lambda_{k-1,s}} - 1 \right) + \frac{R^2}{2} \|E\|_Y^2 \\
\leq - \frac{R^2}{2} \delta^2 \left( \frac{\mu_1}{\lambda_{k-1,s}} - 1 \right) - \|E\|_Y^2 < 0.
\]

\[\square\]

**Proof of Theorems 1.2 and 1.3.** If $\mu_2 < \lambda_{1,s}$, by Lemma 4.1 and Proposition 5.1, the functional $\mathcal{I}_s$ satisfies the hypothesis of Mountain Pass Theorem [3], and when $\lambda_{k,s} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,s}$, Lemma 4.2 and Proposition 5.2 (or Lemma 4.3 and Proposition 5.3) ensure the application of Linking Theorem [24]. Thus, in both cases, there exists a non-null solution $\overline{U}$ for the problem (4.1). By Remark 7, it follows that $\overline{U}_+ \neq 0$ and therefore, $U_T$ and $\overline{U} + T$ are distinct solutions for the problem (1.2). \[\square\]
Acknowledgments. The author would like to thank David Arcoya for helpful discussions and suggestions.

REFERENCES

[1] C. O. Alves, D. C. de Morais Filho and O. H. Miyagaki, Multiple solutions for an elliptic system on bounded and unbounded domains, Nonlinear Anal., 56 (2004), 555–568.

[2] A. Ambrosetti and G. Prodi, On the inversion of some differential mappings with singularities between Banach spaces, Ann. Mat. Pura. Appl., 93 (1972), 231–246.

[3] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal., 14 (1973), 349–381.

[4] D. Arcoya and S. Villegas, Nontrivial solutions for a Neumann problem with a nonlinear term asymptotic linear at $-\infty$ and superlinear at $+\infty$, Math. Z., 219 (1995), 499–513.

[5] B. Barrios, E. Colorado, A. De Pablo and U. Sanchez, On Some Critical Problems for the fractional Laplacian operator, arXiv:1106.6081.

[6] M. Bouchekif and Y. Nasri, On a singular elliptic system at resonance, Annali Mat., 189 (2010), 227–240.

[7] X. Cabrè and J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, Adv. Math., 224 (2010), 2052–2093.

[8] L. Caffarelli and P. H. Rabinowitz, Critical superlinear Ambrosetti-Prodi problems, Top. Methods Nonlinear Anal., 14 (1999), 59–80.

[9] D. C. de Morais Filho and M. A. S. Souto, Systems of p-Laplacean equations involving homogeneous nonlinearities with critical Sobolev exponent degrees, Commun. in Partial Diff. Equations, 24 (1999), 1537–1553.

[10] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math., 136 (2012), 521–573.

[11] F. Gazzola and B. Ruf, Lower order perturbations of critical growth nonlinearities in semilinear elliptic equations, Adv. Diff. Equations, 4 (1999), 555–572.

[12] A. Greco and R. Servadei, Hopf’s lemma and constrained radial symmetry for the fractional Laplacian, to appear in Math. Res. Lett.

[13] K. Ito, Lectures on stochastic processes, Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Notes by K. Muralidhar Rao. 2nd edition, Bombay, 24 1984.

[14] F. R. Pereira, Multiple solutions for a class of Ambrosetti-Prodi type problems for systems involving critical Sobolev exponents, Comm. Pure Appl. Analysis., 7 (2008), 355–372.

[15] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS, American Mathematical Society, 65, 1986.

[16] B. Ribeiro, The Ambrosetti-Prodi problem for gradient elliptic systems with critical homogeneous nonlinearity, J. Math. Anal. Appl., 363 (2010), 606–617.

[17] X. Ros-Oton and J. Serra, The extremal solution for the fractional Laplacian, Calc. Var., 50 (2014), 723–750.
[27] B. Ruf and P. N. Srikanth, Multiplicity results for superlinear elliptic problems with partial interference with the spectrum, *J. Math. Anal. App.*, **118** (1986), 15–23.

[28] O. Savin and E. Valdinoci, Γ-convergence for nonlocal phase transitions, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **29** (2012), 479–500.

[29] R. Servadei, The Yamabe equation in a non-local setting, *Adv. Nonlinear Anal.*, **2** (2013), 235–270.

[30] R. Servadei and E. Valdinoci, On the spectrum of two different fractional operators, *Proc. Roy. Soc. Edinburgh Sect. A.*, **144** (2014), 831–855.

[31] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Cont. Dyn. Syst.*, **33** (2013), 2105–2137.

[32] M. F. Shlesinger, G. M. Zaslavsky and J. Klafter, Strange kinetics, *Nature*, **363** (1993), 31–37.

[33] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.*, **60** (2006), 67–112.

[34] T. H. Solomon, E. R. Weeks and H. L. Swinney, Observation of anomalous diffusion and Lévy flights in a two-dimensional rotating flow, *Physic. Rev. Lett.*, **71** (1993), 3975–3978.

[35] G. M. Viswanathan, E. P. Raposo and M. G. E. Da Luz, Lévy flights and superdiffusion in the context of biological encounters and random searches, *Phys. Life Rev.*, **5** (2008), 133–150.

Received December 2016; revised May 2017.

*E-mail address: fabio.pereira@ufjf.edu.br*