Nonassociativity as gravity

V. Yu. Dorofeev

Abstract
Gravitational interactions are treated as a nonassociative part of the field interactions. The vector-potential of Dirac equation is extended to octonionic algebra. The solution is considered as an element of the matrix set $BS \times O_4$.

Introduction
Generalization of physical theories to octonions has its own history. Since octonions form an algebra and have $U(1)$, $SU(2)$, $SU(3)$ automorphism groups, there were attempts to use this algebra as a basis for the theory of electroweak and strong interactions.

In this article we propose to consider nonassociativity as gravity. On this way the all-in-one picture of all interactions arises.

1 Dirac equation in GRG and on the octonions
The interaction of $A_a(x)$ with Dirac particle is introduced as minimal:

$$ (i\gamma^a(\partial_a - iA_a) - m)\Psi(x) = 0 $$

(1)

where $\gamma^a$ – Dirac matrices:

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}
$$

(2)

and $\sigma^a$, $a = 1, 2, 3$ – Pauli hermitian matrices.

If there is $SU(2)$ symmetry then:

$$ (i\gamma^a(\partial_a - iA_a^{(b)}\sigma^b) - m)\Psi(x) = 0. $$

(3)

According to Utiyama, Dirac equation with minimal coupling in curved coordinates leads to the appearance of compensating field $\Gamma_\mu(x)$ in the following form

$$ p_\mu \rightarrow (\partial_\mu - \Gamma_\mu)\Psi(x) = (\partial_\mu - \omega_\mu^{ab}(x)\sigma^{ab})\Psi(x), $$

(4)
where $\omega_{\mu}^{ab}(x)$ – real functions determined by vierbeins $e_{\mu}^{a} = \partial x^{a}/\partial u_{\mu}$ ($x^{a}$ – coordinates in flat space and $u_{\mu}$ – in curved coordinate system), and $\sigma^{ab}$ – matrix as $\sigma^{ab} = \frac{1}{2}[\gamma^{a}, \gamma^{b}]$.

The important difference between (3) and (4) is the reality of the coefficient $\omega_{\mu}^{ab}(x)$ and formally we ”unify” gravity and electroweak interactions by real $\omega_{\mu}^{ab}(x)$ and imaginary $iA_{\mu}^{a}(x)$. But in fact there is another way to unify all interactions, i.e., to go beyond the associative algebra.

Really, let the field $A_{\mu}^{a}(x) = \tilde{A}_{\mu}^{a}(x)\Sigma^{\tilde{a}}$ is defined on the nonassociative algebra with generating $\Sigma^{\tilde{a}}$. We select octonionic algebra as nonassociative algebra:

\[
\Sigma^{0} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \Sigma^{\tilde{a}} = \begin{pmatrix} 0 & -\sigma^{\tilde{a}} \\ \sigma^{\tilde{a}} & 0 \end{pmatrix}, \quad \tilde{a} = 1, 2, 3,
\]

\[
\Sigma^{4} = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}, \quad \Sigma^{4+\tilde{a}} = \begin{pmatrix} 0 & i\sigma^{\tilde{a}} \\ i\sigma^{\tilde{a}} & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

and introduce special multiplication ”*” [4]:

\[
\left( \begin{array}{cc} \lambda & A \\ B & \xi \end{array} \right) \ast \left( \begin{array}{cc} A' & \lambda' \\ B' & \xi' \end{array} \right) = \left( \begin{array}{cc} \lambda\lambda' + \frac{1}{2}tr(AB') & \lambda\xi' + \xi\lambda' + \frac{1}{2}[B, A'] \\ \lambda'B + \xi'B' - \frac{1}{2}[A, A'] & \xi\xi' + \frac{1}{4}tr(BA') \end{array} \right)
\]

(6)

where $A, A', B, B'$ – matrices $(2 \times 2)$ and $\lambda, \lambda', \xi, \xi'$ – scalar matrices $(2 \times 2)$.

Let the vector-potential $A_{\mu}^{a}(x)$ belongs to nonassociative algebra, i.e. to octonionic algebra. So we come to the matrix representation:

\[
(i\gamma^{a}(\partial_{a} - iA_{a}^{b}\Sigma^{b}) - m)\Psi(x) = 0.
\]

Then in case $A_{a}^{4}(x) = (0, A(r)\vec{r}/r)$ and $A_{a}^{i}(x) = 0, i \neq 4$ in spherical coordinates we have:

\[
(\gamma^{0}\partial_{t} + \gamma\partial_{r} - \gamma\vec{\Sigma} \cdot \vec{L} + im)\Psi(x) = -\gamma A(r)\Psi(x),
\]

(8)

where $\vec{L} = \vec{r} \times \vec{p}$ is the angular momentum operator,

\[
\vec{\gamma} = \gamma^{1}\sin\theta\cos\varphi + \gamma^{2}\sin\theta\sin\varphi + \gamma^{3}\cos\theta, \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.
\]

Let’s remember this equation and come to Dirac equation on the other hand.

We introduce curved homogeneous isotropic space. Then the general form of static metric looks like:

\[
ds^{2} = K^{2}(r)dt^{2} - B^{2}(r)dr^{2} - r^{2}d\Omega^{2}.
\]

(9)

Particularly, if $K^{2} = B^{2} = f^{2} = 1 - r_{g}/r, r_{g}$ – gravitational radius – this is the Schwarzschild metric:

\[
ds^{2} = (1 - \frac{r_{g}}{r})dt^{2} - \frac{dr^{2}}{1 - \frac{r_{g}}{r}} - r^{2}(\sin^{2}\theta d\varphi^{2} + d\theta^{2}).
\]

(10)
In the Dirac equation (4) we put Ricci rotation coefficients of curved space (10) [5]:

\[
(\gamma^0 \frac{1}{f} \partial_t + \tilde{\gamma} f \partial_r - \tilde{\gamma} \Sigma \cdot \tilde{L} + im)\Psi = -\tilde{\gamma}(\sqrt{f}(\sqrt{f})_r + \frac{1}{r}(\sqrt{f} - 1))\Psi.
\] (11)

Thus we get likeness of Dirac equation in the Schwarzschild metric and Dirac equation on the nonassociative algebra which in this case is octonionic algebra. Formally nonassociativity is necessary to relieve imaginarity in the long derivative. Indicated likeness allows us to make supposition: the nonassociativity should be considered as a local demonstration of the curvature of space.

As we go to the curved space, we have to introduce vierbein variables:

\[
\Delta t \rightarrow \Delta \tau = K(r)\Delta t, \quad \Delta r \rightarrow \Delta R = B(r)\Delta r
\]

and to give physical meaning to the new variables.

Remind that the time dilation and the length contraction in the special theory of relativity (STR) have the form

\[
dt^2 = (1 - v^2)dt_0^2, \quad dl^2 = (1 - v^2)dl_0^2.
\]

It is natural to suppose: if \( r \rightarrow \infty \) then functions \( K(r) \) and \( B(r) \rightarrow 1 \). On the other hand, the transfer of the body moving on a circle in central symmetric homogeneous static gravitational field from large distance to short distance according to Kepler’s law means transformation from zero velocity to nonzero velocity because of

\[
\Delta t \rightarrow \Delta \tau = f(r)\Delta t, \quad \Delta r \rightarrow \Delta R = \Delta r/f(r).
\]

At last according to geodesic equation

\[
\frac{du^i}{ds} + \Gamma^i_{kl}u^ku^l = 0,
\]

where \( \Gamma^i_{kl} \) – Christoffel symbol in the metric (10), from Newton equation we get

\[
f^2(r) = 1 - \frac{r_g}{r}
\]

Thus the extention of field to the octonionic algebra and the principles of STR give all the effects of GRG.

2 Octonionic states

The solution of the Dirac equation requires extension of bispinor to the new algebra. For the sake of justice it should be noted that this is a very big problem. The author reported the new solution of this problem on the Conf. "RUSGRAV-13" [6].
According to the principle of that extension the state is determinated as a bispinor $\Psi(x)$ from the set $BS \times O_4$:

$$
\Psi(x) = \begin{pmatrix}
\Psi(x)_{i_n}^1 \\
\Psi(x)_{i_n}^2 \\
\Psi(x)_{i_n}^3 \\
\Psi(x)_{i_n}^4
\end{pmatrix}, \quad \Psi(x)_{i_n}^{a'} = \begin{pmatrix}
u(x)^{a'}_{i_n} \\
A(x)^{a'}_{i_n}
\end{pmatrix}, a' = 1, 2, 3, 4 \quad (12)
$$

or

$$
\Psi(x) = \begin{pmatrix}
u(x)^1_{i_n} \\
A(x)^1_{i_n} \\
u(x)^2_{i_n} \\
A(x)^2_{i_n} \\
u(x)^3_{i_n} \\
A(x)^3_{i_n} \\
u(x)^4_{i_n} \\
A(x)^4_{i_n}
\end{pmatrix} = \begin{pmatrix}
u(x)^{1,2}_{i_n} \\
A(x)^{1,2}_{i_n} \\
u(x)^{1,3}_{i_n} \\
A(x)^{1,3}_{i_n} \\
u(x)^{1,4}_{i_n} \\
A(x)^{1,4}_{i_n} \\
u(x)^{2,3}_{i_n} \\
A(x)^{2,3}_{i_n} \\
u(x)^{2,4}_{i_n} \\
A(x)^{2,4}_{i_n} \\
u(x)^{3,4}_{i_n} \\
A(x)^{3,4}_{i_n}
\end{pmatrix} = \begin{pmatrix} \varphi \\
\psi
\end{pmatrix}. \quad (13)
$$

$v(x)^{a'}$, $u(x)^{a'}$—scalar complex matrices $(2 \times 2)$ and $A(x)^{a'}$, $B(x)^{a'}$—matrices: $B(x)^{a'} = \vec{b}^{a'}(x) \cdot \vec{\sigma}$, $\vec{b}^{a'}(x) \in C$ (sign "." is simple multiplication).

Considering $A = \vec{a} \vec{\sigma}, B = \vec{b} \vec{\sigma}$, we find the probability density $\rho = \Psi^+ \Psi$ in the representation

$$
\frac{1}{4} tr \Psi^+ \Psi = \frac{1}{4} tr \left( \begin{pmatrix} v^* & -A \\
A^+ & u \end{pmatrix} \begin{pmatrix} v^* & -B \\
B^+ & v \end{pmatrix} \right) \ast \begin{pmatrix} u \\
A^+ \end{pmatrix} = |u|^2 + |v|^2 + |\vec{a}|^2 + |\vec{b}|^2.
$$

For example, we can consider the bispinor of electron in the form

$$
\Psi(x) = \begin{pmatrix}
u(x)^1 \\
0 \\
u(x)^2 \\
0 \\
u(x)^3 \\
0 \\
u(x)^4 \\
0
\end{pmatrix} = \begin{pmatrix} u(x)^1 \\
u(x)^2 \\
u(x)^3 \\
u(x)^4
\end{pmatrix} \cdot \begin{pmatrix} u(x)^1 \\
u(x)^2 \\
u(x)^3 \\
u(x)^4
\end{pmatrix}. \quad (14)
$$

and the field $A^{(0)}_{\mu}(x) = A_{\mu}(x), A^{(a)}_{\mu}(x) = 0, a = 1, 2, 3, \ldots, 7$.

The author believes that the weak and strong interactions are resulted as an extension of symmetry for $SU(2)$ and $SU(3)$ groups [1] where we can get $V - A$ interaction [6] but the author wants to consider this problem in the next article.

4
3 Free Lagrangian of this theory

We define the free Lagrangian of field as

$$L = \frac{1}{4} \text{tr} F_{ab} F^{ab}$$

$$F_{ab} = A_{b,a} - A_{a,b} - [A_a A_b - A_b A_a].$$

(15)

Notice that the Lagrangian doesn’t equal to zero when its associative part equals to zero because the nonassociative part exists:

$$L_{\text{gr.}} = -\frac{1}{2} \eta^{bd} \eta^{ac} \tilde{e}_{\tilde{a} \tilde{b} \tilde{c} \tilde{d}} \left( A^\tilde{a} \tilde{b} A^\tilde{c} A^\tilde{d} - A^\tilde{a} A^\tilde{b} \tilde{c} \tilde{d} \right).$$

(16)

The last three multipliers in the formula (16) are similar to Riemannian tensor in the vierbein representation. Particularly, the scalar curvature $R$ in the vierbein representation is determined as

$$R = \eta^{bd} \eta^{ac} R_{abcd}.$$  

(17)

The Ricci tensor $R_{abcd}$ has symmetry and antisymmetry properties for indices as in (16).

Thereby the analogy between nonassociative part of (15) and gravitation field Lagrangian is suggested. But it is necessary to outline another property: Jacoby identity and cyclic rearrangement by three indices don’t take place. Therefore we don’t come to Einstein theory, but we have a complete coincidence between the basic data experiments and our theory results.

In fact the equation on the octonionic algebra is equation for differentials on a map. In this regard GRG offers to result general equation in the whole atlas by the instrumentality of equation in the Riemannian space.

4 Acknowledgements

The author expresses his gratitude to the participants of the Friedmann Seminar for Theoretical Physics (St. Petersburg) for profound discussions. The work was carried with the support of the Russian Ministry of Education (grant RNP No 2.1.1.68.26).

References

[1] O. K. Kalashnikov, S. E. Bronstein, E. S. Fradkin. v. 29, 6, 1979.
[2] V. Dorofeev, gr-qc/0604024.
[3] R. Utiyama. Phys. Rev., 1597, 1956.
[4] J. Daboul, R. Delbourgo, hep-th/9906065.
[5] D. R. Brill, J. A. Wheeler. Rev. Mod. Phys., 29, 465, (1957).
[6] V. Yu. Dorofeev, Classical gravity on the octonionic algebra. Theses of XIII International Conf. RUSGRAV-13, Moscow, 23-28 June, 2008.