ON A DEGREE ASSOCIATED WITH THE GROSS-PITAEVSKII SYSTEM WITH A LARGE PARAMETER

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Abstract. In a number of cases we calculate the sum of the degrees of the small positive solutions of the Gross-Pitaevskii system when the interaction is strong.

1. Introduction. We want to study the following system for large positive $k$

$$
-\Delta u = f(u) - kv^2u \quad \text{in } \Omega,
$$

$$
-\Delta v = g(v) - ku^2v \quad \text{in } \Omega,
$$

where $u = v = 0$ on $\partial \Omega$ and $u, v \geq 0$ in $\Omega$. Here $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$. We assume $f(0) = g(0) = 0$, $f$ and $g$ are $C^1$, and $f(y)$ and $g(y)$ are negative for large positive $y$. This trivially ensures that we have bounds in $L^\infty(\Omega) \times L^\infty(\Omega)$ for non-negative solutions.

We further assume that the problem

$$
-\Delta w = aw^+ + dw^- \quad \text{in } \Omega
$$

$$
w = 0 \quad \text{on } \partial \Omega
$$

has only the trivial solution where $a = f'(0)$ and $d = g'(0)$. (This means $(a,d)$ is not in the Fučík spectrum.)

Under these assumptions, it was proved in [2] and [6] that positive solutions of (1) for large $k$ are of two types: $(u,v)$ is close to $(z^+, z^-)$ where $z$ is a non-trivial sign changing solution of

$$
-\Delta y = f(y^+) - g(y^-) \quad \text{in } \Omega
$$

$$
y = 0 \quad \text{on } \partial \Omega,
$$

or $u$ and $v$ are both of order $k^{-1/2}$ and $U = k^{1/2}u$ and $V = k^{1/2}v$ are close in $L^\infty(\Omega)$ to a non-trivial solution of

$$
-\Delta U = aU - V^2U \quad \text{in } \Omega
$$

$$
-\Delta V = dV - U^2V \quad \text{in } \Omega
$$

$$
U = V = 0 \quad \text{on } \partial \Omega.
$$

Conversely, under a natural degree condition, isolated solutions of (4) lead to a small positive solution to (1) for each large positive $k$. (Here we need to work with the degree on the cone $\{ (u,v) \in C_0(\Omega) \times C_0(\Omega) : u, v \geq 0 \text{ in } \Omega \}$.)

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Thus it is natural to try to evaluate the degree of (4) on large balls centred on zero in $C_0(\Omega) \times C_0(\Omega)$ intersected with the cone. In particular, we conjecture that the degree of (4) is equal to the degree of (2).

Here by “the degree of (2)”, we mean the degree of the map
$$u \mapsto (-\Delta)^{-1}(au^+ + du^-)$$
on the unit ball in $C_0(\Omega)$. By “the degree of (4)”, we mean the degree of the map
$$(u, v) \mapsto (-\Delta + \alpha I)^{-1}((a + \alpha)u - v^2 u, (d + \alpha)v - u^2 v),$$
on a large ball $B$, with centre 0, in $C_0(\Omega) \oplus C_0(\Omega)$ intersected with the natural cone, where $\alpha$ is chosen suitably large so that the above map sends nonnegative $(u, v) \in B$ to the positive cone in $C_0(\Omega) \oplus C_0(\Omega)$.

In fact, it is more convenient to evaluate the degree of the map
$$-\Delta u = au - kv^2 u$$
$$-\Delta v = dv - ku^2 v$$
for large $k$ on the ball with centre 0 and radius 1 in $K \oplus K$, where $K$ is the cone of non-negative functions in $C_0(\Omega)$. It is shown in [2] that is suffices to evaluate the degree for one $(a, d)$ in each component of the complement of the Fučík spectrum. In particular, as in [2], this is true for a component of the complement of the Fučík spectrum which intersects the diagonal $a = d$.

Moreover, we suspect (and this seem rather harder to prove) that, if $Y$ is an isolated changing sign solution of (3) of non-zero degree (for the natural map), then for large $k$ there is a positive solution of (1) with $u$ near $Y^+$ and $v$ near $-Y^-$. We will not discuss this last problem further. We briefly discuss later why this problem is more difficult.

There is a known related theory for the system where both $kv^2 u$ and $ku^2 v$ are replaced by $kuv$. In this case, all our results and conjectures above can be proved. See [3]. This problem is much easier because if we use $u$ and $u - v$ as variables, only one of the equations is singularly perturbed. (There is difficulty in keeping $v$ positive but, as in [3], this problem can be overcome).

Our main result here is the following.

**Theorem 1.** The conjecture above is true if $\Omega$ is one dimensional, or if $\Omega$ is a two dimensional ball or a true annulus (i.e. both balls have the same centre), or where $\Omega$ is a three dimensional ball or true annulus (with some extra conditions to be specified later).

2. **Proofs.** We prove these results by deforming the problem (1) to the population model (where $kv^2 u$ and $ku^2 v$ are both replaced by $kuv$) and using degree results for mappings with a rotational symmetry. We first consider the case of one space dimension.

We first use the natural homotopy
$$-u'' = u - tkv^2 u - (1 - t)ku - v'' = \alpha v - tku^2 v - (1 - t)ku,$$
on $[0,\pi]$ for $0 \leq t \leq 1$ and for $k$ large positive (plus the boundary conditions). Here $u, v$ non-negative, $\|u\|_\infty \leq m$ and $\|v\|_\infty \leq m$ where $m$ is fixed and reasonably small. Note that $t$ is the homotopy parameter and $x$ is the space variable and we work on the unit ball, centred at 0, of the cone of non-negative functions in $C[0,\pi] \oplus C[0,\pi]$. \[5\]
We aim to prove that if $0 \leq t \leq 1$ and $k$ is large then (5) has no solution $u, v$ with $\|u\|_\infty + \|v\|_\infty = 1$ and $u, v \geq 0$. This and homotopy invariance gives the result (by using [3]). From the first equation $u$ is the sum of a $C^2$-bounded concave function $u_1$ and a negative convex function $u_2$ (which is bounded since $u$ and $u_1$ are bounded). Here both components vanish at the ends. Hence $u'(0) = u_1'(0) + u_2'(0)$ is bounded above and similarly $u_1'(\pi)$ is bounded below. Thus by integrating the first equation on $[0, \pi]$ we easily see that

$$tkv^2u + (1 - t)kvu$$

is bounded in $L^1[0, \pi]$ uniformly in $k$ and $t$. This implies a Lipschitz bound for $u$ and $v$ (uniform in $k$). It follows easily that if $k$ is large, $uv$ is uniformly small and if $\alpha > 0$ and $u(x) \geq \alpha$, then $v(x)$ is small (and vice versa). Moreover if $u(x_0) \geq \alpha$ and $k$ is large, then $u(x) \geq \frac{1}{2}\alpha$ on a neighbourhood $N$ of $x_0$ independent of $k$ (by the Lipschitz bounds) and hence $kv^2u, kvu$ and $kuv$ are uniformly small (and even exponentially small in $k$) on $N$. This follows because $v$ is exponentially small in $k$, since $v'' \geq \left(\frac{1}{4}kx^2 + O(1)\right)v$ on $N$ and $v$ is bounded. (We construct supersolutions.) Now, after taking subsequences $u \to \bar{u}$ and $v \to \bar{v}$ uniformly as $k \to \infty$ and $\|\bar{u}\|_\infty + \|\bar{v}\|_\infty = 1$.

Hence we see that $tkv^2u$ converges weakly (in $W^{-1,2}[0, 1]$) to a non-negative measure $m_1(t)$ supported on $\bar{u} = \bar{v} = 0$. Similarly $(1 - t)kvu$ converges to a non-negative measure $m_3(t)$ and $tkv^2v$ converges to a non-negative measure $m_2(t)$ supported on $\bar{u} = \bar{v} = 0$. Thus we easily see that in the limit we have $\bar{w} = 0, \bar{u} \geq 0, \bar{v} \geq 0$ and if $\bar{w} = \bar{u} - \bar{v}$

$$-\bar{u}'' = a\bar{u} - m_1(t) - m_3(t)$$

$$-\bar{v}'' = d\bar{v} - m_2(t) - m_3(t)$$

$$-(\bar{u} - \bar{v})'' = a\bar{u} - d\bar{v} + m_2(t) - m_1(t) = a\chi_{\bar{w} > 0}\bar{w} + d\chi_{\bar{w} < 0}\bar{w} + m_2(t) - m_1(t),$$

and so

$$-\bar{w}'' = a\chi_{\bar{w} > 0}\bar{w} + d\chi_{\bar{w} < 0}\bar{w} - \bar{m}(t)$$

$$\bar{w}(0) = 0, \quad \bar{w}(\pi) = 0,$$

where $\bar{m}(t)$ is a measure supported on $\bar{w} = 0$. ($\bar{m}(t) = m_2(t) - m_1(t)$).

We need to check that $\bar{w}$ cannot vanish on any non-trivial interval and $\bar{w}$ changes sign at any interior zero. Here we use the first integral (or Pokojaev identities). If $\bar{u}$ is zero on both sides of $\alpha$

$$\left(\frac{1}{2}(u'^2 + \frac{1}{2}u^2 + \frac{1}{2}v'^2 + \frac{1}{2}v^2 + tkv^2u^2)\right)' = (1 - t)kvu(u' + v')$$

(6)

We need to check that $\bar{u}$ can not vanish on both sides of a positive zero $\tau$ of $\bar{w}$, where $\tau < \pi$. Suppose $\bar{u}$ has a zero on both sides of $\tau$. Now

$$-\bar{u}'' = a\bar{u} - m_1(t) - m_3(t)$$

and thus if $\bar{u}$ vanishes near $\tau$, $m_1(t) = m_3(t) = 0$. Hence $(1 - t)kvu \to 0$ as a measure and since $v'$ and $u'$ are bounded, $(1 - t)kvu(u' + v') \to 0$ as $k \to \infty$.

Thus, by (6), in the limit of the first integral

$$\frac{1}{2}u'^2 + \frac{1}{2}v'^2 + \frac{1}{2}u^2 + \frac{1}{2}v^2 + tkv^2u^2$$
is continuous across $\tau$ and hence $|\bar{v}'|$ is continuous at $\tau$ and is non-zero at $\tau$. (Since otherwise $\bar{v}$ would vanish identically.) Note that $tk\tilde{v}^2\tilde{u} - tk\tilde{v}^2\bar{u}$ must be uniformly small since $tk\tilde{v}^2\bar{u}$ is a measure and $\bar{u}$ is small nearby.

Hence $|u|$ is close to $\bar{v} = \alpha|x - x_0|$ for $x$ near $x_0$ where $\alpha > 0$. We now look at the $u$ equation

$$-u'' = u \left(\tilde{f}(u) - kv^2\right)$$

and $\tilde{f}(u) - kv^2$ is large negative except for $x$ close to $x_0$ and we easily see that

$$\tilde{f}(u) - kv^2 \leq -k^{\frac{3}{2}}$$

for $x$ close to $x_0$ but not too close and is bounded above. Since $u$ is bounded, it follows that $u$ is exponentially small in $k$ close to $x_0$. Hence $kv^2u$ is exponentially small in $k$ close to $x_0$ and hence it is impossible for $\bar{v}'$ to have a jump at $x_0$. Thus this case does not occur.

Since we can interchange the role of $u$ and $v$, the only possibility is that $\bar{u}$ and $\bar{v}$ are non-zero on alternate intervals, and to have positive right derivative at the right hand end of a zero interval. Now (and thus is where we essentially use that we are in one space dimension and the linearity) the distance to the next zero is independent of the size of the jump in the derivative. Thus the measures have no effect on the zeros and in particular the zeros of $\bar{w}$ are exactly the same as the zeros of a non-trivial solution

$$-z'' = ax^+ + dz^-, \quad z(0) = 0, \quad z(\pi) = 0. \quad (7)$$

Thus we have a contradiction and our result is proved. (Note that $\bar{w}$ is non-trivial since $sup\{||\bar{u}||_{\infty}, ||\bar{v}||_{\infty}\} = 1$.)

**Remark.** Note that the argument still applies if the coefficients have some space dependence as it only affects higher order terms when we cross a thin layer. Secondly if our equation is singular at $r = 0$, we can easily prove that one component of the solution is bounded away from zero near $r = 0$ and the argument only needs minor modifications. This proves our main result in the one dimensional case.

The result in the two dimensional case where $\Omega$ is a ball or an annulus in $\mathbb{R}^2$ follows because by a theorem of Nussbaum[7], in the ball or annulus the degree of (1) (or (4)) on the unit ball in $(C_0(\Omega) \oplus C_0(\Omega))^+$ (i.e. the non-negative functions on $C_0(\Omega) \oplus C_0(\Omega)$) is equal to the degree restricted to the space of radial functions. This uses the $S^1$ symmetry (from rotations in $\theta$). Thus our argument above proves the theorem in this case. Note that we apply Nussbaum’s theorem to both the system and the limiting scalar equation.

Lastly, we consider a three dimensional ball or a true annulus $\tilde{\Omega}$ in $\mathbb{R}^3$. Once again, we use the $S^1$ group of rotations in the first two coordinates. It is a good group. As before, by Nussbaum’s theorem, we see that to prove the degrees of (2) and (4) are equal reduces to proving the result in the space $\{u \in W^{1,2}(\Omega) : u(x_1, x_2, x_3) = u(r, x_3)\}$ where $r = \sqrt{x_1^2 + x_2^2}$. Note that both equations respect the symmetries in this space. Note also that the Fučík spectrum in this space tends to be much sparser. In particular, the condition that there is at most 1 eigenvalue of $-\Delta$ on $\Omega$ between $a$ and $d$ with an eigenfunction in this space is a sufficient condition for the two degrees on this space to be equal (and hence for the degree result in three dimensions to be true for these domains and for these $a$ and $d$).

I want finally to make some comments on our homotopy in the proof of our main theorem. If $t$ is bounded away from 1, then since $u^2v$ and $\bar{v}^2u$ are both much smaller
than $uv$ when $u$ and $v$ are both small, we deduce that $m_1 = m_3 = 0$ if $t < 1$ and hence $-\Delta (\bar{u} - \bar{v}) = a\bar{u} - d\bar{v}$ and $\bar{u}\bar{v} = 0$. Thus $-\Delta \bar{w} = a\bar{w}^+ + d\bar{w}^-$. Hence we only have difficulties for $t$ close to 1. In fact, we conjecture that this argument can be refined to show that it is a true homotopy in all dimensions. This is non-trivial to prove because in higher dimensions there are rather more possibilities for the measures (and the measures can possibly affect the zero set more drastically in higher dimensions). Moreover, in more than one dimension, the shape of $\Omega$ affects the eigenvalues which seems to make proofs much more difficult.

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