Chromatic number of Euclidean plane

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Abstract: If the chromatic number of Euclidean plane is larger than four, but it is known that the chromatic number of planar graphs is equal to four, then how does one explain it? In my opinion, they are contradictory to each other. This idea leads to confirm the chromatic number of the plane about its exact value.

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1 Introduction

One of the most interesting unsolved problems in geometric graph theory or Euclidean Ramsey theory is as below:

How many colors are needed to color the plane so that no two points at unit distance are the same color?

which is also called the Hadwiger-Nelson problem, named after Hugo Hadwiger and Edward Nelson. [1,2,3]

As for the age of the emergence of it, according to Jensen and Toft’s investigation (1995), the problem was first formulated by E. Nelson in 1950, and first published by Gardner in 1960. Hadwiger in 1945 published a related result, showing that any cover of the plane by five congruent closed sets contains a unit distance in one of the sets, and he also mentioned the problem in a later paper (Hadwiger 1961). A. Soifer (2003,2008,2011) discusses the problem and its origin extensively.

Although the answer of this problem is unknown, it has been narrowed down to one of the numbers 4, 5, 6 or 7. The lower bound of four follows from a seven-vertex unit distance graph with chromatic number four discovered by brothers William and Leo Moser in 1961, named the Moser spindle. An alternative lower bound in the form of a ten-vertex four-chromatic unit distance graph was discovered at around the same time by Solomon W. Golomb. The upper bound of seven follows from the existence of a tessellation of the plane by regular hexagons, with diameter slightly less than one, that can be assigned seven colors in a repeating pattern to form a 7-coloring of the plane; according to A. Soifer (2003,2008,2011), this upper bound was first observed by John R. Isbell.

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As variants of the problem, one of extensions is that one can consider coloring the plane in which the sets of points of each color are restricted to sets of some particular type.\cite{10-20} Such restrictions may cause the required number of colors to change, either increase or decrease. For instance, if all color classes are required to be the rational points in the plane, then it is known that 2 colors are only required, rather than 3, 4 and so on;\cite{15,16} if all color classes are required to be Lebesgue measurable, it is known that at least five colors are required. In the Solovay model of set theory,\cite{11} all point sets are measurable, so this result implies that in this model the chromatic number of the plane is at least five.\cite{8,10} If a coloring of plane consists of regions bounded by Jordan curves, then at least six colors are required.\cite{14,18,19}

Another extension of the problem is to higher dimensions. In particular, finding the chromatic number of space usually refers to the 3-dimensional version. As with the version on the plane, the answer is not known, but has been shown to be at least 6 and at most 15.\cite{19,20}

The most significant extension leads to found Euclidean Ramsey theory from the beginning of a series of papers published jointly by six mathematicians: P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, from 1973 to 1975.\cite{21}

The question can be phrased in the term of graph theory. Let $E^2$ be the unit distance graph of the plane: an infinite graph with all points of the plane as vertices and with an edge between two vertices if and only if there is unit distance between the two points. Then the Hadwiger-Nelson problem is to find the chromatic number of $E^2$. As a consequence, the problem is usually called ”finding the chromatic number of the plane”, i.e.

$$\chi(E^2) = ?$$

By the de Bruijn-Erdős theorem (1951),\cite{12,22} the problem is equivalent (under the assumption of the axiom of choice) to that of finding the largest possible chromatic number of a finite unit distance graph.

In the inspiration of the above de Bruijn-Erdős compactness principle, according to A. Soifer, et al’s researches,\cite{5-8,10,12,23} the determination of the chromatic number of the plane about its correct value is viewed as to depend on the choice of axioms for set theory.

Based on the axiom of choice here, we will prove that the chromatic number of the plane is four, just as E. Nelson predicted initially.\cite{6,7}

## 2 Main result and proof of it

Now we determine which of 4, 5, 6 and 7 is the value of the chromatic number of the plane $\chi(E^2)$.

**THEOREM.** $\chi(E^2) = 4$.

**Proof.** If $\chi(E^2) = k > 4$, $k = 5, 6$, or 7 possibly, by the definition of chromatic number of the plane, that is,

$$E^2 \not\rightarrow (S_2)_k,$$
and
\[ E^2 \to (S_2)_{k-1}, \]
where \( S_2 \) is a set of unit-distance two points in \( E^2 \), then by the de Bruijn-Erdős theorem, there is some finite unit-distance plane subgraph \( G \) in Euclidean plane \( E^2 \) such that
\[ G \not\to (S_2)_k, \]
but
\[ G \to (S_2)_{k-1}. \]

By the hypothesis of the chromatic number \( k > 4 \), it is known no matter how the \( G \) is 4-colored, it will contain same color two points with distance 1, that is that
\[ G \to (S_2)_4. \]

But on the other hand, we may prove the following result via mathematical induction on the order of the graph \( G \), i.e. the number of \( G \)'s vertices.

It is easily verified that every unit distance plane graph \( G \) with the vertices not exceeding 10, such as Moser spindle or Golomb graph, has the chromatic number at most 4. Suppose that for some integer \( n \) not smaller than 10, as the vertex number of such a graph \( G \), denoted by \( G_n \), \( \chi(G_n) \leq 4 \). Since every such finite graph contains at least one point whose degree is not exceeding 3, from this and by the inductive hypothesis it follows that \( \chi(G_{n+1}) \leq 4 \) for every unit distance plane graph of order \( n + 1 \). The induction is finished.

This implies that each finite unit distance plane graph \( G \) can be normally colored with four sorts of colors, contradictory to \( G \to (S_2)_4 \). In other words, our initial hypothesis is incorrect. Hence the desired result holds. \( \square \)

In Euclidean Ramsey theoretic terms, \([12,21]\) it is stated as a Euclidean-Ramsey number:

Corollary 1. \( R(S_2, 4) = 3 \).

A related Erdős’ open problem (in 1958) \([6,7]\) is that

**What is the smallest number of colors needed for coloring the plane in such a way that no color realizes all distances?**

This invariant was named by A. Soifer in 1992 \([23]\) as the polychromatic number of the plane, and denoted by \( \chi_p(E^2) \). It is clear that \( \chi_p(E^2) \leq \chi(E^2) \).

From the above result, and by Raiskii’s lower bound, \([24]\) it follows that

Corollary 2. \( \chi_p(E^2) = 4 \).
3 Conclusion

The above assertion, i.e. each unit distance finite plane graph contains at least one point whose degree is not exceeding three, is based on such a basic fact that for a given point on a circle of radius 1, there are just two distinct points on it such that distance respectively unit from the point. Otherwise it leads to create an infinite unit distance plane graph.

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