NON-SPANNING LATTICE 3-POLYTOPES

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Abstract. We completely classify non-spanning 3-polytopes, by which we mean lattice 3-polytopes whose lattice points do not affinely span the lattice. We show that, except for six small polytopes (all having between five and eight lattice points), every non-spanning 3-polytope $P$ has the following simple description: $P \cap \mathbb{Z}^3$ consists of either (1) two lattice segments lying in parallel and consecutive lattice planes or (2) a lattice segment together with three or four extra lattice points placed in a very specific manner.

From this description we conclude that all the empty tetrahedra in a non-spanning 3-polytope $P$ have the same volume and they form a triangulation of $P$, and we compute the $h^*$-vectors of all non-spanning 3-polytopes.

We also show that all spanning 3-polytopes contain a unimodular tetrahedron, except for two particular 3-polytopes with five lattice points.

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1. Introduction and statement of results

A lattice $d$-polytope is a polytope $P \subseteq \mathbb{R}^d$ with vertices in $\mathbb{Z}^d$ and with $\text{aff}(P) = \mathbb{R}^d$. We call size of $P$ its number of lattice points and width the minimum length of the image $f(P)$ when $f$ ranges over all affine non-constant functionals $f : \mathbb{R}^d \to \mathbb{R}$ with $f(\mathbb{Z}^d) \subseteq \mathbb{Z}$. That is, the minimum lattice distance between parallel lattice hyperplanes that enclose $P$.

In our papers [2, 3, 4] we have enumerated all lattice 3-polytopes of size 11 or less and of width greater than one. This classification makes sense thanks to the

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following result [2, Theorem 3]: for each \( n \in \mathbb{N} \) there are only finitely many lattice 3-polytopes of width greater than one and with exactly \( n \) lattice points. Here and in the rest of the paper we consider lattice polytopes modulo unimodular equivalence or lattice isomorphism. That is, we consider \( P \) and \( Q \) isomorphic (and write \( P \cong Q \)) if there is an affine map \( f : \mathbb{R}^d \to \mathbb{R}^d \) with \( f(\mathbb{Z}^d) = \mathbb{Z}^d \) and \( f(P) = Q \).

As a by-product of the classification we noticed that most lattice 3-polytopes are “lattice-spanning”, according to the following definition:

**Definition.** Let \( P \subset \mathbb{R}^d \) be a lattice \( d \)-polytope. We call sublattice index of \( P \) the index, as a sublattice of \( \mathbb{Z}^d \), of the affine lattice generated by \( P \cap \mathbb{Z}^d \). \( P \) is called lattice-spanning if it has index 1. We abbreviate sublattice index and lattice-spanning as index and spanning.

In this paper we completely classify non-spanning lattice 3-polytopes. Part of our motivation comes from the recent results of Hofscheier et al. [6, 7] on \( h^* \)-vectors of spanning polytopes (see Theorem 7.1). In particular, in Section 7 we compute the \( h^* \)-vectors of all non-spanning 3-polytopes and show that they still satisfy the inequalities proved by Hofscheier et al. for spanning polytopes, with the exception of empty tetrahedra that satisfy them only partially.

In dimensions 1 and 2, every lattice polytope contains a unimodular simplex, i. e., a lattice basis, and is hence lattice-spanning. In dimension 3 it is easy to construct infinitely many lattice polytopes of width 1 and of any index \( q \in \mathbb{N} \), generalizing White’s empty tetrahedra ([9]). Indeed, for any positive integers \( p, q, a, b \) with \( \gcd(p, q) = 1 \) the lattice tetrahedron

\[ T_{p,q}(a, b) := \text{conv}\{(0,0,0), (a,0,0), (0,0,1), (bp,bq,1)\} \]

has index \( q \), width 1, size \( a+b+2 \) and volume \( abq \) (see a depiction of it in Figure 1). Here and in the rest of the paper we consider the volume of lattice polytopes normalized to the lattice, so that it is always an integer and the normalized volume of a simplex \( \text{conv}(v_0, \ldots, v_d) \) equals its determinant \( \det \left( \begin{array}{c} v_0 \cdots v_d \end{array} \right) \).

**Lemma 1.1** (Corollary 3.3). Every non-spanning 3-polytope of width one is isomorphic to some \( T_{p,q}(a, b) \).

For larger width, the complete enumeration of lattice 3-polytopes of width larger than one up to size 11 shows that their index is always in \{1, 2, 3, 5\}. The numbers of them for each index and size are as given in Table 1 (copied from Table 6 in [4]). This data seems to indicate that apart from a few small exceptions there

\[ \text{Figure 1. A polytope } T_{p,q}(a, b). \text{ An empty tetrahedron in it is highlighted.} \]
Table 1. Lattice 3-polytopes of width > 1 and size up to 11, classified according to sublattice index. Table taken from [4].

| size | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|------|----|----|----|----|----|----|----|
| index 1 | 7  | 71 | 486 | 2658 | 11680 | 45012 | 156436 |
| index 2 | 0  | 2  | 8  | 14 | 19 | 24 |
| index 3 | 1  | 3  | 2  | 3  | 4  | 4 |
| index 5 | 1  | 0  | 0  | 0  | 0  | 0 |

Table 2. The number of polytopes of size ≤ 11 in the families of Lemma 1.2

| size | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|------|----|----|----|----|----|----|----|
| index 2 | 0  | 2  | 7  | 11 | 15 | 19 | 24 |
| index 3 | 1  | 2  | 2  | 3  | 4  | 4 |

are about \( \Theta(n^3) \) polytopes of index three, about \( \Theta(n^2) \) of index two, and none of larger indices. Infinite families of lattice 3-polytopes of indices two and three that match these asymptotics are given in the following statement:

**Lemma 1.2** (See Section 3). Let \( S_{a,b} := \text{conv}\{(0,0,a),(0,0,b)\} \) be a lattice segment, with \( a,b \in \mathbb{Z}, a \leq 0 < b \). Then the following are non-spanning lattice 3-polytopes of width > 1 (see pictures of them in Figure 2):

- \( \tilde{F}_1(a,b) := \text{conv}\left( S_{a,b} \cup \{-1,-1,0\},\{2,-1,1\},\{-1,2,-1\}\right) \), of index 3.
- \( \tilde{F}_2(a,b) := \text{conv}\left( S_{a,b} \cup \{(-1,-1,1),(-1,1,0),(-1,1,0)\}\right) \), with \( (a,b) \neq (0,1) \), of index 2.
- \( \tilde{F}_3(a,b,k) := \text{conv}\left( S_{a,b} \cup \{(-1,-1,1),(1,-1,0),(-1,1,0),(1,1,2k-1)\}\right) \), for any \( k \in \{0,\ldots,b\} \) and \( (a,b,k) \neq (0,1,1) \), of index 2.
- \( \tilde{F}_4(a,b) := \text{conv}\left( S_{a,b} \cup \{-1,-1,1),(1,-1,0),(-1,1,0),(3,-1,-1)\}\right) \), of index 2.

Observe that in all cases of Lemma 1.2 the lattice points in the polytope are those in the segment \( S_{a,b} \) plus three or four additional points. We call \( S_{a,b} \) (or, sometimes, the line containing it) the spike of \( \tilde{F}_1(a,b,) \), because it makes these polytopes very closely related to (some of) the spiked polytopes introduced in [4] (see the proof of Theorem 5.2 for some properties of them).

For all polytopes in the same family \( \tilde{F}_1 \) the projection of the lattice points along the direction of the spike is the same lattice point configuration, the \( F_i \) depicted in the bottom of Figure 2. The spike projects to the origin, and the other three or four points in each \( \tilde{F}_1(a,b,) \) project bijectively to three or four points in \( F_i \).

The number of polytopes in the families of Lemma 1.2 taking redundancies into account, is computed in Corollary 4.2. Table 2 shows the result for sizes 5 to 11.

Comparing those numbers to Table 1 we see that the only discrepancies are six missing polytopes in sizes five to eight. Our main result in this paper is that indeed, Lemma 1.2 together with six exceptions gives the full list of non-spanning lattice 3-polytopes of width larger than one.
Theorem 1.3 (Classification of non-spanning 3-polytopes). Every non-spanning lattice 3-polytope either has width one (hence it is isomorphic to a $T_{p,q}(a,b)$ by Lemma 1.1) or is isomorphic to one in the infinite families of Lemma 1.2, or is isomorphic to one of the following six polytopes, depicted in Figure 3:

- Two tetrahedra of sizes 5 and 6, with indices 5 and 3 respectively:
  \[ E_{(5,5)} := \text{conv}\{(0, -2, 1), (1, 0, -1), (1, 1, 1), (-2, 1, -1)\}. \]
  \[ E_{(6,3)} := \text{conv}\{(1, 0, 0), (-1, -1, 0), (1, 2, 3), (-1, 1, -3)\}. \]
- A pyramid and two double pyramids over the following triangle of size 6:
  \[ B := \text{conv}\{(-1, -1, 0), (2, 0, 0), (1, 2, 0)\}. \]
  Namely, the following polytopes of sizes 7 and 8, all of index 2:
  \[ E_{(7,2)} := \text{conv}(B \cup \{(0, -1, 2)\}) \]
  \[ E_{(8,2)}^1 := \text{conv}(B \cup \{(0, -1, 2), (0, 1, -2)\}), \]
  \[ E_{(8,2)}^2 := \text{conv}(B \cup \{(0, -1, 2), (-2, -1, -2)\}). \]
- A tetrahedron of size 8 and index 2:
  \[ E_{(8,2)}^3 := \text{conv}\{(0, -1, -1), (2, 0, 2), (1, 2, -1), (-1, 1, 2)\}. \]
Corollary 1.4 (Corollary 4.2). For any \( n \geq 9 \) all non-spanning 3-polytopes of size \( n \) and width larger than one belong to the families of Lemma 1.2. There are \( \lfloor (n-3)(n+1)/4 \rfloor \) of index two, \( \lceil n-3 \rceil / 2 \) of index three, and none of larger index. □

As a consequence of our classification for non-spanning 3-polytopes we have the following statement. Recall that an empty tetrahedron is a lattice tetrahedron whose only lattice points are its vertices.

Corollary 1.5. Let \( P \) be a non-spanning lattice 3-polytope. Then the collection of all empty tetrahedra in \( P \) is a triangulation, and all such tetrahedra have volume equal to the index of \( P \).

Observe that the triangulation in the statement is necessarily the unique triangulation of \( P \) with vertex set equal to \( P \cap \mathbb{Z}^3 \).
Proof. For \( T_{p,q}(a, b) \), since all lattice points lie on two lines, every empty tetrahedron has as vertices two consecutive lattice points on each of the lines. These tetrahedra have volume equal to the index (in this case \( q \)) and, together, form a triangulation: the join of the two paths along the lines.

- Every polytope in the families \( \tilde{F}_i \) has the property that the triangle formed by any three lattice points outside the spike meets the spike at a lattice point. Thus, every empty tetrahedron has two vertices along the spike and two outside the spike. The collection of all of such tetrahedra forms a triangulation: the join of the path along the spike and a path (for the polytopes in the family \( \tilde{F}_2 \)) or a cycle (for the rest) outside the spike. Also, these tetrahedra have the volume of a triangle with vertices in the projection \( F_i \) that has the origin as a vertex. Any such triangle in \( F_i \) has the same volume as the index of the polytopes in the family \( \tilde{F}_i \).

- For the six exceptions of Theorem 1.3, the statement can be checked with geometric arguments or with computer help. Details are left to the reader. □

Corollary 1.5 is not true in dimension 4, as the following example shows:

Example 1.6. Let

\[
P = \text{conv}\left\{ \begin{array}{c}
(1, 0, 0, 0), \\
(0, 1, 0, 0), \\
(0, 0, 1, 0), \\
(-2, -1, -1, 0), \\
(1, 1, 1, 2)
\end{array} \right\}
\]

This is a lattice 4-simplex with six lattice points: its five vertices plus the origin, which lies in the relative interior of the facet given by the first four vertices. Thus, the volumes of the empty tetrahedra in \( P \) are the absolute values of the \( 4 \times 4 \) minors in the matrix (excluding the zero minor). These values are four, two, two and two, and \( P \) has index two. We thank Gabriele Balletti for providing (a variation of) this counterexample.

In the same vein, one can ask whether all spanning 3-polytopes have a unimodular tetrahedron. The answer is that only two do not.

**Theorem 1.7** (See Section 6). The only lattice-spanning 3-polytopes not containing a unimodular tetrahedron are the following two tetrahedra of size five:

- \( E_{(5, 1)}^1 := \text{conv}\{ (1, 0, 0), (0, 0, 1), (2, 7, 1), (-1, -2, -1) \} \) with four empty tetrahedra of volumes 2, 3, 5 and 7.
- \( E_{(5, 1)}^2 := \text{conv}\{ (1, 0, 0), (0, 0, 1), (3, 7, 1), (-2, -3, -1) \} \) with four empty tetrahedra of volumes 3, 4, 5 and 7.

The structure of the paper is as follows. After a brief introduction and remarks about the sublattice index in Section 2, Section 3 proves the (easy) classification of non-spanning 3-polytopes of width one (Lemma 1.1), and Section 4 is devoted to the study of the infinite families of non-spanning 3-polytopes of Lemma 1.2.

Section 5 proves our main result, Theorem 1.3. This is the most complicated part of the paper, relying substantially in our results from 4. A sketch of the proof is as follows: For small polytopes, Theorem 1.3 follows from comparing Tables 1 and 3 (together with the easy observation that the six polytopes described in Theorem 1.3...
are not isomorphic to one another or to the ones in Lemma 1.2). For polytopes of larger size we use induction on the size (taking the enumeration of size \( \leq 11 \) as the base case) and we prove the following:

- In Section 5.1 we look at polytopes that cannot be obtained \textit{merging} two smaller ones, in the sense of [4]. These polytopes admit quite explicit descriptions that allow us to prove that the only non-spanning ones (for sizes \( \geq 8 \)) are those of the form \( \tilde{F}_i(0,b) \) for \( i \in \{1,2\} \) (Theorem 5.2). Since merging can only decrease the index, this immediately implies that all non-spanning 3-polytopes of width \( >1 \) and size \( \geq 8 \) have index at most three (Corollary 5.3).

- In Sections 5.2 and 5.3 we look at \textit{merged} polytopes of indices three and two, respectively, and prove that, with the five exceptions mentioned in Theorem 1.3, they all belong to the families of Lemma 1.2.

Section 6 proves Theorem 1.7.

2. Sublattice index

By a \textit{lattice point configuration} we mean a finite subset \( A \subset \mathbb{Z}^d \) that affinely spans \( \mathbb{R}^d \). We denote by \( \langle A \rangle_{\mathbb{Z}} \) the affine lattice generated by \( A \) over the integers:

\[
\langle A \rangle_{\mathbb{Z}} := \left\{ \sum_{i} \lambda_i a_i \mid a_i \in A, \lambda_i \in \mathbb{Z}, \sum_{i} \lambda_i = 1 \right\}
\]

Since \( A \) is affinely spanning, \( \langle A \rangle_{\mathbb{Z}} \) has finite index as a sublattice of \( \mathbb{Z}^d \).

**Definition 2.1.** The \textit{sublattice index} of \( A \) is the index of \( \langle A \rangle_{\mathbb{Z}} \) in \( \mathbb{Z}^d \). We say that \( A \) is \textit{lattice-spanning} if its sublattice index equals 1. That is, \( \langle A \rangle_{\mathbb{Z}} = \mathbb{Z}^d \).

**Remark 2.2.** If \( A = P \cap \mathbb{Z}^d \) for \( P \) a lattice \( d \)-polytope, we call sublattice index of \( P \) the sublattice index of \( A \), and say that \( P \) is lattice-spanning if \( A \) is.

**Lemma 2.3.** Let \( A \subset \mathbb{Z}^d \) be a lattice point configuration.

1. The sublattice index of \( A \) divides the sublattice index of every subconfiguration \( B \) of \( A \).

2. Let \( \pi : \mathbb{Z}^d \to \mathbb{Z}^s \), for \( s < d \), be a lattice projection. Then the sublattice index of \( \pi(A) \) divides the sublattice index of \( A \).

**Proof.** In part (1), the injective homomorphism \( \langle B \rangle_{\mathbb{Z}} \to \langle A \rangle_{\mathbb{Z}} \) induces a surjective homomorphism \( \mathbb{Z}^d / \langle B \rangle_{\mathbb{Z}} \to \mathbb{Z}^d / \langle A \rangle_{\mathbb{Z}} \). In part (2), the lattice projection \( \pi \) induces a surjective homomorphism \( \mathbb{Z}^d / \langle A \rangle_{\mathbb{Z}} \to \mathbb{Z}^s / \langle \pi(A) \rangle_{\mathbb{Z}} \).

It is very easy to relate the index of a lattice polytope or point configuration to the volumes of (empty) simplices in it.

**Lemma 2.4.** Let \( A \) be a lattice point configuration of dimension \( d \). Then, the sublattice index of \( A \) equals the \( \gcd \) of the volumes of all the lattice \( d \)-simplices with vertices in \( A \).

**Remark 2.5.** Observe that the \( \gcd \) of volumes of all simplices in \( A \) equals the \( \gcd \) of volumes of simplices empty in \( A \), by which we mean simplices \( T \) such that \( T \cap A = \text{vert}(T) \). Indeed, if \( T \) is a non-empty simplex, then \( T \) can be triangulated into empty simplices \( T_1, \ldots, T_k \). Since \( \text{vol}(T) = \sum \text{vol}(T_i) \), we have that \( \gcd(\text{vol}(T_1), \ldots, \text{vol}(T_n)) = \gcd(\text{vol}(T_1), \ldots, \text{vol}(T_n), \text{vol}(T)) \).
Proof. Without loss of generality assume that the origin is in $A$. Then, the sublattice index of $A$ equals the gcd of all maximal minors of the $d \times |A|$ matrix having the points of $A$ as columns. These minors are the (normalized) volumes of lattice $d$-simplices with vertices in $A$. \hfill \Box

3. Non-spanning 3-polytopes of width one

Lemma 3.1. Let $P$ be a lattice 3-polytope of width one. Then $P$ either contains a unimodular tetrahedron or it equals the convex hull of two lattice segments lying in consecutive parallel lattice planes.

Proof. Since $P$ has width one, its lattice points are distributed in two consecutive parallel lattice planes. If $P$ has three non collinear lattice points in one of these planes we can assume without loss of generality that they form a unimodular triangle. Then, these three points together with any point in the other plane (there exists at least one) form a unimodular tetrahedron.

If $P$ does not have three non-collinear points in one of the two planes then all the points in each of the planes are contained in a lattice segment. \hfill \Box

Proposition 3.2. The convex hull of two lattice segments lying in consecutive parallel lattice planes is equivalent to

$$T_{p,q}(a,b) := \text{conv}\{(0,0,0), (a,0,0), (0,0,1), (bp,bq,1)\}$$

for some $0 \leq p < q$ with $\gcd(p,q) = 1$, and $a,b \geq 1$.

The sublattice index of $T_{p,q}(a,b)$ is $q$ and its size is $a + b + 2$.

See Figure 1 for an illustration of $T_{p,q}(a,b)$.

Proof. Let $P$ be the convex hull of two lattice segments lying in consecutive parallel lattice planes. Without loss of generality we assume that the segments are contained in the planes $\{z = 0\}$ and $\{z = 1\}$, respectively. Let $a$ and $b$ be the lattice length of the two segments, so that the size of $P$ is indeed $a + b + 2$.

By a unimodular transformation there is no loss of generality in assuming the first segment to be $\text{conv}\{(0,0,0), (a,0,0)\}$ and the second one to contain $(0,0,1)$ as one of its endpoints. Then the second endpoint is automatically of the form $(bp,bq,1)$ for some coprime $p,q \in \mathbb{Z}$ and with $q \neq 0$ in order for $P$ to be full-dimensional. We can assume $q > 0$ since the case $q < 0$ is symmetric. Also, since the unimodular transformation $(x,y,z) \mapsto (x \pm y,y,z)$ fixes $(0,0,0), (a,0,0)$ and $(0,0,1)$ and sends $(p,q,1) \mapsto (p \pm q,q,1)$, there is no loss of generality in assuming $0 \leq p < q$.

Since all lattice points of $P$ lie in the lattice $y \equiv 0 \mod q$ its index is a multiple of $q$. Since $P$ contains the tetrahedron $\text{conv}\{(0,0,0)(1,0,0), (0,0,1), (p,q,1)\}$, of determinant $q$, its index is exactly $q$. \hfill \Box

Corollary 3.3. Let $P$ be a lattice 3-polytope of width one. Then

- $P$ is lattice-spanning if, and only if, it contains a unimodular tetrahedron.
- $P$ has index $q > 1$ if, and only if, $P \cong T_{p,q}(a,b)$ for some $0 \leq p < q$ with $\gcd(p,q) = 1$ and $a, b \geq 1$. \hfill \Box
4. Four infinite non-spanning families

In this section we study the infinite families of non-spanning 3-polytopes introduced in Lemma 1.2 whose definition we now recall:

\[ \bar{F}_1(a, b) := \text{conv} (S_{a,b} \cup \{(-1, -1, 0), (2, -1, 1), (-1, 2, -1)\}), \]
\[ \bar{F}_2(a, b) := \text{conv} (S_{a,b} \cup \{(-1, -1, 1), (1, -1, 0), (-1, 1, 0)\}), \]
\[ \bar{F}_3(a, b, k) := \text{conv} (S_{a,b} \cup \{(-1, -1, 1), (1, -1, 0), (-1, 1, 0), (1, 1, 2k - 1)\}), \]
\[ \bar{F}_4(a, b) := \text{conv} (S_{a,b} \cup \{(-1, -1, 1), (1, -1, 0), (-1, 1, 0), (3, -1, -1)\}). \]

In all cases, \( S_{a,b} := \text{conv}(\{(0, 0, a), (0, 0, b)\}) \) with \( a, b \in \mathbb{Z} \) and \( a \leq 0 < b \), and in the third family, \( k \in \{0, \ldots, b\} \).

The statement of Lemma 1.2 is that all these polytopes are non-spanning and have width \( > 1 \), with the exceptions of \( \bar{F}_2(0, 1) \) and \( \bar{F}_3(0, 1, 1) \), which clearly have width one with respect to the third coordinate. For the proof, recall that we call the lattice segment \( S_{a,b} \) the spike, and that apart of the \( b - a + 1 \) lattice points in the spike these polytopes only have three (in the families \( \bar{F}_1 \) and \( \bar{F}_2 \)) or four (in the families \( \bar{F}_3 \) or \( \bar{F}_4 \)) other lattice points.

**Proof of Lemma 1.2.** The lattice points in \( \bar{F}_1(a, b) \) generate the sublattice \( \{(x, y, z) \in \mathbb{Z}^3 : x - y \equiv 0 \pmod{3}\} \), of index 3. Those in \( \bar{F}_2(a, b), \bar{F}_3(a, b, k) \) and \( \bar{F}_4(a, b) \) generate the sublattice \( \{(x, y, z) \in \mathbb{Z}^3 : x + y \equiv 0 \pmod{2}\} \), of index 2.

For the width we consider two cases: Functionals constant on \( S_{a,b} \) project to functionals in \( Z \) (that is, not depending on the third coordinate) project to functionals in \( Z^2 \) on one of the configurations \( F_i \) of Figure 2 of which \( F_1 \) has width 3 and the others width 2.

Functionals that are non-constant along \( S_{a,b} \) produce width at least \( b - a \), so the only possibility for width one would be \( a = 0 \) and \( b = 1 \). It is clear that \( \bar{F}_2(0, 1) \) and \( \bar{F}_3(0, 1, 1) \) have width one with respect to the functional \( z \) but it is easy to check (and left to the reader) that the rest have width at least two even when \( (a, b) = (0, 1) \).

The list in Lemma 1.2 contains some redundancy, but not much.

**Proposition 4.1.** The only isomorphic polytopes among the list of Lemma 1.2 are:

1. For \( i \in \{1, 2, 4\} \) we have that \( \bar{F}_i(a, b) \) is isomorphic to \( \bar{F}_i(-b, -a) \).
2. \( \bar{F}_3(a, b, k) \) is isomorphic to \( \bar{F}_3(k - b, k - a, k) \).
3. In size six:
   \[ \bar{F}_2(0, 2) \cong \bar{F}_4(0, 1) \quad \text{and} \quad \bar{F}_2(-1, 1) \cong \bar{F}_3(0, 1, 0). \]
4. In size seven:
   \[ \bar{F}_3(0, 2, 1) \cong \bar{F}_4(-1, 1). \]

**Proof.** The following maps show the isomorphisms in parts (1) and (2) among polytopes within each family:

\[
\begin{align*}
(x, y, z) \mapsto (y, x, -z) & \quad \Rightarrow \quad \bar{F}_1(a, b) \cong \bar{F}_1(-b, -a). \\
(x, y, z) \mapsto (x, y, -x - y - z) & \quad \Rightarrow \quad \bar{F}_2(a, b) \cong \bar{F}_2(-b, -a). \\
(x, y, z) \mapsto (y, -x, k + y(k - 1) - z) & \quad \Rightarrow \quad \bar{F}_3(a, b, k) \cong \bar{F}_3(k - b, k - a, k). \\
(x, y, z) \mapsto (x, y, -x - y - z) & \quad \Rightarrow \quad \bar{F}_4(a, b) \cong \bar{F}_4(-b, -a).
\end{align*}
\]
Whenever \( b - a \geq 3 \) these are the only isomorphisms, by the following arguments:

- The spike contains \( b - a + 1 \geq 4 \) collinear lattice points and is the only collinearity of at least four lattice points. Thus, every isomorphism sends the spike to the spike and, in particular, \( b - a \) is an invariant (modulo unimodular equivalence).

- Polytopes in different families \( \tilde{F}_i \) cannot be isomorphic: the polytopes in the family \( \tilde{F}_1 \) are the only ones of index three; those in \( \tilde{F}_2 \) are the only ones of index two with only three points outside the spike; and those in \( \tilde{F}_4 \) are the only ones with three collinear lattice points outside the spike.

- For \( i = 1, 2, 4 \) the pair \( \{ -a, b \} \) is an invariant since the plane spanned by lattice points not in the spike intersects the spike at the point \((0, 0, 0)\), which is at distances \( -a \) and \( b \) of the end-points of the spike.

- For \( i = 3 \) the pair \( \{ -a, b - k \} \) is an invariant since the midpoints of the pairs of opposite lattice points around the spike have as midpoints \((0, 0, 0)\) and \((0, 0, k)\), which are on the spike and at distances \( -a \) and \( b - k \) from the end-points.

It only remains to see that, when \( b - a \leq 2 \), the only isomorphisms that appear are those of parts (3) and (4) of the statement. By parts (1) and (2) we can assume that \(-a \leq b\) in all cases and that \(-a \leq b - k\) in the family \( \tilde{F}_3 \). That is, \((a, b) \in \{(0, 1), (0, 2), (-1, 1)\}\) in \( \tilde{F}_1 \), \( \tilde{F}_2 \) and \( \tilde{F}_4 \), and \((a, b, k) \in \{(0, 1, 0), (0, 2, 0), (0, 2, 1), (0, 2, 2), (-1, 1, 0)\}\) for \( \tilde{F}_3 \) (remember that \( \tilde{F}_2(0, 1) \) and \( \tilde{F}_3(0, 1, 1) \) are excluded). Let us separate the cases by index and size:

- In the family \( \tilde{F}_1 \), the only one of index 3, the three possibilities are distinguished by the fact that \( \tilde{F}_1(0, 1) \) is has size five, \( \tilde{F}_1(0, 2) \) is a tetrahedron of size six, and \( \tilde{F}_1(-1, 1) \) is a triangular bipyramid of size six.

- For index 2 and size six there are only \( \tilde{F}_2(-1, 1), \tilde{F}_3(0, 1, 0), \tilde{F}_2(0, 2) \) and \( \tilde{F}_4(0, 1) \). The first two are square pyramids, and isomorphic via \((x, y, z) \mapsto (x + z, y + z, -x - y - z)\). The last two are tetrahedra, isomorphic via \((x, y, z) \mapsto (-x + y - z + 1, z - 1, -y)\).

- For index 2 and size seven there are four possibilities in the family \( \tilde{F}_3 \) and two in the family \( \tilde{F}_4 \). All of them happen to have three collinear triples of lattice points. Looking at the distribution of the seven lattice points in collinear triples is enough to distinguish among five of the six possibilities, as the following diagram shows. (Each diagram shows what lattice points form collinear triples but also the relative order of points along each triple):

\[
\begin{align*}
\tilde{F}_3(0, 2, 0), & \quad \tilde{F}_3(0, 2, 1), & \quad \tilde{F}_3(0, 2, 2), & \quad \tilde{F}_3(-1, 1, 0), & \quad \tilde{F}_4(0, 2).
\end{align*}
\]

The unimodular transformation \((x, y, z) \mapsto (-x + y - z + 1, z - 1, x)\) maps \( \tilde{F}_3(0, 2, 1) \) to the remaining polytope of size seven, \( \tilde{F}_4(-1, 1) \).

\(\square\)
Corollary 4.2. The number of isomorphism classes of polytopes in the families $\tilde{F}_i$ are as given in Table 3.

| Size | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|------|----|----|----|----|----|----|----|
| Family $\tilde{F}_1$ | 1  | 2  | 2  | 3  | 3  | 4  | 4  |
| Total Index 3 | 1  | 2  | 2  | 3  | 3  | 4  | 4  |
| Family $\tilde{F}_2$ | 0* | 2  | 2  | 3  | 3  | 4  | 4  |
| Family $\tilde{F}_3$ | 1* | 4  | 6  | 9  | 12 | 16 |  |
| Family $\tilde{F}_4$ | 1  | 2  | 2  | 3  | 3  | 4  |  |
| Total Index 2 | 0  | 2* | 7* | 11 | 15 | 19 | 24 |

Table 3. The number of non-spanning 3-polytopes in the infinite families of Lemma 1.2. The two entries marked with * do not coincide with the formula for general $n$ because in each of them one of the configurations counted by the formula has width one. The entries marked ** are less than the sum of the three above them (and, in particular, do not coincide with the general formula) because of the isomorphisms in parts (3) and (4) of Proposition 4.1.

Proof. For size up to seven, the counting is implicit in the proof of parts (3) and (4) of Proposition 1.1. Thus, in the rest of the proof we assume size $n \geq 8$ and the only isomorphisms we need to take into account are those in parts (1) and (2) of Proposition 4.1.

In the families $\tilde{F}_1$ and $\tilde{F}_2$ we have $b - a + 1 = n - 3$ lattice points along the spike and apart of size the only invariant is on which of them does the plane of the other lattice points intersect. This gives $n - 3$ possibilities, but opposite ones are isomorphic so the count is $\lceil (n - 3)/2 \rceil$. For $\tilde{F}_4$ we have the same count except that now $b - a + 1 = n - 4$, so we get $\lceil (n - 4)/2 \rceil = \lceil (n - 3)/2 \rceil$.

For $\tilde{F}_3$ we have $b - a + 1 = n - 4$ lattice points along the spike, and we have to choose (perhaps with repetition) two of them to be mid-points of non-spike pairs of lattice points. This gives $(n - 3)$ possibilities. To mod out symmetric choices we divide that number by two but then have to add one half of the self-symmetric choices, of which there are $\lceil (n - 4)/2 \rceil = \lceil (n - 3)/2 \rceil$. The count, thus, is

$$\frac{1}{2} \left( \binom{n - 3}{2} + \frac{n - 3}{2} \right) = \frac{1}{2} \left( \frac{n^2 - 6n - 9}{2} \right) = \frac{(n - 3)^2}{4}.$$

□

5. Proof of Theorem 1.3

5.1. Merged and non-merged 3-polytopes. For a lattice $d$-polytope $P \subset \mathbb{R}^d$ of size $n$ and a vertex $v$ of $P$ we denote

$$P^v := \text{conv}(P \cap \mathbb{Z}^d \setminus \{v\}),$$
Theorem 5.2. Let $P$ be a lattice 3-polytope of width $> 1$ and size $n$. We say that $P$ is merged if there exist at least two vertices $u, v \in \text{vert}(P)$ such that $P^u$ and $P^v$ have width larger than one and such that $P^{uv}$ is 3-dimensional.

Loosely speaking, we call a polytope of size $n$ merged if it can be obtained merging two subpolytopes $Q_1 \cong P^u$ and $Q_2 \cong P^v$ of size $n - 1$ and width $> 1$ along their common (full-dimensional) intersection $P^{uv}$. This merging operation is the basis of the enumeration algorithm in [4] and to make it work, a complete characterization of the polytopes that are not merged was undertaken. Combining several results from [4] we can prove that:

**Theorem 5.2.** Let $P$ be a lattice 3-polytope of size $n \geq 8$ and suppose that $P$ is not merged. Then, one of the following happens:

1. $P$ contains a unimodular tetrahedron, and in particular it is spanning.
2. $P$ has sublattice index 2 and is isomorphic to

\[ \tilde{F}_2(0, n - 4) = \text{conv}\{(1, -1, 0), (-1, 1, 0), (-1, -1, 1), (0, 0, n - 4)\} . \]

3. $P$ has sublattice index 3 and is isomorphic to

\[ \tilde{F}_1(0, n - 4) = \text{conv}\{(2, -1, 1), (-1, 2, -1), (-1, -1, 0), (0, 0, n - 4)\} . \]

**Proof.** This follows from Theorems 2.9, 2.12, 3.4 and 3.5 in [4], but let us briefly explain how.

- [4] Theorem 2.12] shows that every non-merged 3-polytope of size $\geq 7$ is quasi-minimal, according to the following definition: it has at most one vertex $v$ such that $P^v$ has width larger than one.
- [4] Theorem 2.9] says that every quasi-minimal 3-polytope is either spiked or boxed.
- By [4] Proposition 4.6], boxed polytopes of size $\geq 8$ contain $n - 3 \geq 5$ vertices of the unit cube, so in particular they contain a unimodular tetrahedron.

Thus, for the rest of the proof we can assume that $P$ is spiked. Theorems 3.4 and 3.5 in [4] contain a very explicit description of all spiked 3-polytopes, which in particular implies that every spiked 3-polytope $P$ of size $n$ projects to one of the ten 2-dimensional configurations $A'_1, \ldots, A'_{10}$ of Figure 1 (notation taken from [4]), where the number next to each point in $A'_i$ indicates the number of lattice points in $P$ projecting to it.

An easy inspection shows that all $A'_i$ except $A'_1$ and $A'_4$ have a unimodular triangle $T$ with the property that (at least) one vertex of $T$ has at least two lattice points of $P$ in its fiber. Then $T$ is the projection of a unimodular tetrahedron in $P$. It thus remains only to show that projections to $A'_1$ and $A'_4$ correspond exactly to cases (2) and (3) in the statement. For this we use the coordinates shown in Figure 5. Observe that $A'_1$ and $A'_4$ are exactly the $F_2$ and $F_1$ of Lemma 1.2 respectively.

Suppose that $P$ projects to $F_2$. By the assumption on the number of lattice points of $P$ projecting to each point in $F_2$, $P$ has exactly three lattice points outside the spike $\{(0, 0, t) : t \in \mathbb{R}\}$, and they have coordinates $(1, -1, h_1), (-1, 1, h_2)$ and $(-1, -1, h_3)$. In order not to have lattice points
Figure 4. Possible projections of a non-merged 3-polytope $P$ of size $n \geq 8$ along its spike. A black dot with a label $j$ is the projection of exactly $j$ lattice points in $P$. White dots are lattice points in the projection of $P$ which are not the projection of any lattice point in $P$.

\begin{align*}
A_1' &= \begin{pmatrix} 1 \\ n-3 \\ 1 \end{pmatrix} \\
A_2' &= \begin{pmatrix} 1 \\ n-4 \\ 1 \end{pmatrix} \\
A_3' &= \begin{pmatrix} 1 \\ n-5 \\ 1 \end{pmatrix} \\
A_4' &= \begin{pmatrix} 1 \\ n-3 \\ 1 \end{pmatrix} \\
A_5' &= \begin{pmatrix} 1 \\ n-3 \\ 1 \end{pmatrix} \\
A_6' &= \begin{pmatrix} 1 \\ n-3 \\ 1 \end{pmatrix} \\
A_7' &= \begin{pmatrix} 1 \\ n-4 \\ 1 \end{pmatrix} \\
A_8' &= \begin{pmatrix} 1 \\ n-4 \\ 1 \end{pmatrix} \\
A_9' &= \begin{pmatrix} 1 \\ n-4 \\ 1 \end{pmatrix} \\
A_{10}' &= \begin{pmatrix} 1 \\ n-k-3 \\ 1 \end{pmatrix}
\end{align*}

Figure 5. Possible projections of non-merged non-spanning 3-polytopes of size $\geq 8$.

\begin{align*}
A_1' &= F_2 \\
&= \begin{pmatrix} -1,1 \\ 0,0 \end{pmatrix} \\
&\quad (0,-1,0) \\
A_4' &= F_1 \\
&= \begin{pmatrix} -1,2 \\ 0,0 \end{pmatrix} \\
&\quad (2,-1,0)
\end{align*}

in $P$ projecting to the white dots in the figure for $F_2$, $h_1$ and $h_2$ must be of the same parity and $h_3$ of the opposite parity. There is no loss of generality in assuming $h_1$ and $h_2$ even, in which case the unimodular transformation

$$(x,y,z) \rightarrow \left( x, y, \frac{h_3 - h_1 - 1}{2} x + \frac{h_3 - h_2 - 1}{2} y + \frac{h_1 + h_2}{2} \right)$$

sends $P$ to $\widetilde{F}_2(a,b)$ for some $a \leq 0 < b$ and $b - a = n - 4$.

If both $a$ and $b$ are non-zero, then $(0,0,a)$ and $(0,0,b)$ are vertices of $P$ and $P$ is merged from $P^{(0,0,a)}$ and $P^{(0,0,b)}$ (of width > 1 since $n \geq 8$), which is a contradiction. Thus, one of $a$ or $b$ is zero, and by the isomorphism in part (1) in Proposition 4.1 there is no loss of generality in assuming it to be $a$.

- For $F_1$ the arguments are essentially the same, and left to the reader.

\begin{flushright}
\square
\end{flushright}

Corollary 5.3. With the only exception of the tetrahedron $E_{(5,5)}$ of Theorem 1.3 (of size 5 and index 5) every lattice 3-polytope of width $> 1$ has index at most 3.

Proof. Let $P$ be a lattice 3-polytope of width $> 1$. If $P$ has size at most 7 (or at most 11, for that matter), the statement follows from the enumerations in [4].
as seen in Table 1. For size $n \geq 8$ we use induction, taking $n = 7$ as the base case. Either $P$ is non-merged, in which case Theorem 5.2 gives the statement, or $P$ is merged, in which case it has a vertex $u$ such that $P_u$ still has width $> 1$. By inductive hypothesis $P_u$ has index at most three, and by Lemma 2.3(1), the index of $P$ divides that of $P_u$. □

5.2. Lattice 3-polytopes of index 3.

Theorem 5.4. Let $P$ be a lattice 3-polytope of width $> 1$ and of index three. Then $P$ is equivalent to either the tetrahedron $E_{(6,3)}$ of Theorem 1.3 (of size six) or to a polytope in the family $\tilde{F}_1$ of Lemma 1.2.

Proof. Let $n$ be the size of $P$. The statement is true for $n \leq 11$ by the enumerations in [4], as seen comparing Tables 1 and 3. For size $n > 11$ we use induction, taking $n = 11$ as the base case.

Let $n > 11$. If $P$ is not merged, then the result holds by Theorem 5.2. So for the rest of the proof we assume that $P$ is merged. Then there exist vertices $u, v \in \text{vert}(P)$ such that $P_u$ and $P_v$ (of size $n - 1$) have width $> 1$, and such that $P_{uv}$ (of size $n - 2$) is full-dimensional. By Lemma 2.3(1), the sublattice indices of $P_u$ and $P_v$ are multiples of 3 and by Corollary 5.3 they equal 3. Thus, by induction hypothesis, both $P_u$ and $P_v$ are in the family $\tilde{F}_1$.

Since both $P_u$ and $P_v$ have a spike with $n - 4 > 7$ lattice points and the spike is the only collinearity of more than three lattice points, $P_u$ and $P_v$ have their spikes along the same line. If $u \in P_v$ lies along the spike of $P_v$ (resp. if $v$ lies along the spike of $P_u$), then $P$ is obtained from $P_u$ (resp. $P_v$) by extending its spike by one point, which implies $P$ is also in the family $\tilde{F}_1$.

Hence, we only need to study the case where both $u$ and $v$ are outside the spike of $P_v$ and $P_u$, respectively.

In this case, $P_{uv}$ consists of a spike of length $n - 5$ plus two lattice points outside of it. Let $w_1$ and $w_2$ be these two lattice points. We use the following observation about the polytopes in the family $\tilde{F}_1$: the barycenter of the three lattice points outside the spike is a lattice point in the spike. That is, we have that both $\frac{1}{3}(u + w_1 + w_2)$ and $\frac{1}{3}(v + w_1 + w_2)$ are lattice points in the spike. In particular, $u - v$ is parallel to the spike and of length a multiple of three, which is a contradiction to the fact that $u$ and $v$ are the only lattice points of $P$ not in $P_{uv}$. See Figure 6. □

![Figure 6](image-url)
5.3. Lattice 3-polytopes of index 2. The case of index 2 uses the same ideas as in index 3 but is a bit more complicated, since there are more cases to study. As a preparation for the proof, the following two lemmas collect properties of the polytopes $\tilde{F}_i(a, b, k)$ of index two.

**Lemma 5.5.**

1. Let $Q = \tilde{F}_3(a, b, k)$ and $w$ be any vertex of $Q$ not in the spike. Then $Q^w$ is isomorphic to either $\tilde{F}_2(a, b)$ or $\tilde{F}_2(k - b, k - a)$.

2. Let $Q = \tilde{F}_4(a, b)$ and $w \in \{(-1, -1, 1), (3, -1, -1)\}$. Then $Q^w \cong \tilde{F}_2(a, b)$.

**Proof.** For part (2) simply observe that $(x, y, z) \mapsto (-x - 2y, y, x + y + z)$ is an automorphism of $\tilde{F}_4(a, b)$ that swaps $(-1, -1, 1)$ and $(3, -1, -1)$. Thus $Q^{(-1, -1, 1)} \cong Q^{(3, -1, -1)} = \tilde{F}_2(a, b)$.

In part (1), the automorphism of $\tilde{F}_3(a, b, k)$

$$(x, y, z) \mapsto (-y, -x, -(k - 1)x - (k - 1)y + z)$$

swaps $(-1, -1, 1)$ and $(1, 1, 2k - 1)$, and gives

$$Q^{(-1, -1, 1)} \cong Q^{(1, 1, 2k - 1)} = \tilde{F}_2(a, b).$$

But we have also the isomorphism in part (2) of Proposition 4.1 which sends $Q$ to $\tilde{F}_3(k - b, k - a, k)$ and maps $\{(-1, -1, 1), (1, 1, 2k - 1)\}$ to $\{(1, -1, 0), (-1, 1, 0)\}$. Thus

$$Q^{(1, -1, 0)} \cong Q^{(-1, 1, 0)} \cong \tilde{F}_2(k - b, k - a).$$

□

**Lemma 5.6.** Let $Q$ be of size $\geq 8$ and isomorphic to a polytope in one of the families $\tilde{F}_3$ or $\tilde{F}_4$ of Lemma 1.2. Let $w$ be a vertex of $Q$ such that $Q^w = \tilde{F}_3(a, b)$. Then, $w$ is one of $(3, -1, -1)$, $(-1, 3, -1)$, or $(1, 1, 2j - 1)$ for some $j \in \{a, \ldots, b\}$.

![Figure 7. Illustration of Lemma 5.6](image)

**Proof.** Since $Q$ has four lattice points outside the spike and $Q^w$ has three, their spikes have the same size. Also, since the size of $Q$ is $\geq 8$, the spike has at least four lattice points. In particular $Q$ and $Q^w$ have the same spike. The four points of $Q$ outside the spike are $(1, -1, 0)$, $(-1, 1, 0)$, $(-1, -1, 1)$ and $w$. The fact that $(1, -1, 0), (-1, 1, 0)$ have their midpoint on the spike implies that:

- If $Q$ is in the family $\tilde{F}_3$ then the midpoint of $w$ and $(-1, -1, 1)$ must also be a lattice point in the spike. That is, $w = (1, 1, 2j - 1)$ with $j \in \{a, \ldots, b\}$.
- If $Q$ is in the family $\tilde{F}_4$ then $w$ and $(-1, -1, 1)$ have as midpoint one of $(1, -1, 0)$ and $(-1, 1, 0)$, so that $w \in \{(3, -1, -1), (-1, 3, -1)\}$.

□
**Theorem 5.7.** Let $P$ be a lattice 3-polytope of width $> 1$ and of index two. Then $P$ is equivalent to either one of the four exceptions of index two in Theorem 1.3 or to one of the polytopes in the families $\tilde{F}_2$, $\tilde{F}_3$ or $\tilde{F}_4$ of Lemma 1.2.

**Proof.** With the same arguments as in the proof of Theorem 5.4 we can assume that $P$ has size $> 11$, is merged from $P^u$ and $P^v$, that $P^u$ and $P^v$ are in the families $\tilde{F}_2$, $\tilde{F}_3$ or $\tilde{F}_4$ and have the same spike, and that $u$ and $v$ lie outside the spike. In particular, $P^u$ and $P^v$ have the same number of lattice points outside the spike.

This number can be three or four, and we look at the two possibilities separately:

- **If** $P^u$ and $P^v$ **have three lattice points not on the spike then they are both in the family $\tilde{F}_2$.** The lattice points in $P^u$ are those in the spike (which is common to $P^u$ and $P^v$) together with two extra lattice points $w_1$ and $w_2$.

  Let $\phi_u : \tilde{F}_2(a, b) \to P^u$ and $\phi_v : \tilde{F}_2(a', b') \to P^v$ be isomorphisms. Then $\phi_u(-1, -1, 1) \neq v$ and $\phi_v(-1, -1, 1) \neq u$, since this would imply $P^uv$ to be 2-dimensional. Thus, $\phi_u(-1, -1, 1), \phi_v(-1, -1, 1) \in \{w_1, w_2\}$. This implies

  \[
  \{\phi_u(-1, 1, 0), \phi_u(1, -1, 0)\} = \{v, w_1\}, \quad \text{and} \quad \{\phi_v(-1, 1, 0), \phi_v(1, -1, 0)\} = \{u, w_2\}
  \]

  for some $i, j \in \{1, 2\}$. In particular the segments $vw_i$ and $uw_j$ have as midpoints the points $\phi_u(0, 0, 0)$ and $\phi_v(0, 0, 0)$, respectively, which are lattice points along the spike. This gives two possibilities for the merging, illustrated in Figure 8: if $i = j$ (top part of the figure) then $uv$ is parallel to the spike and of even length, which gives a contradiction: the midpoint of $u$ and $v$ would be a third lattice point of $P$ outside $P^uv$. If $i \neq j$ then $P^u \cup P^v$ are the lattice points in a polytope isomorphic to $\tilde{F}_3(a, b, k)$ (bottom row in the figure), so the statement holds.

- **If** $P^u$ and $P^v$ **have four lattice points not on the spike then they belong to the families $\tilde{F}_3$ or $\tilde{F}_4$.**

  Our first claim is that there is no loss of generality in assuming that $P^uv$ is in the family $\tilde{F}_2$. Indeed, if $P^u$ is in the family $\tilde{F}_3$ then this is automatically true by Lemma 5.5(1). If $P^u = \tilde{F}_4(a, b)$ then at least one of the vertices $(-1, -1, 1)$ and $(3, -1, -1)$ of $P^u$ is also a vertex of $P$ (e.g., because these two vertices are the unique minimum and maximum on $P^u$.
of the linear functional $f(x, y, z) = x + y$. Call that vertex $v'$, and consider $P$ as merged from $P^u$ and $P^v$ instead of the original $P^u$ and $P^v$. Then $P^{uv'}$ is in the family $F_2$ by Lemma 5.5\(^2\).

Thus, we can assume for the rest of the proof that $P^{uv} = \overline{F}_2(a, b)$. Lemma 5.6 (applied first with $Q = P^u$ and $w = v$ and then with $Q = P^v$ and $w = u$) implies that both $u$ and $v$ belong to

$$\{(3, -1, -1), (-1, 3, -1)\} \cup \{(1, 1, 2j - 1) : j = a, \ldots, b\}$$

which is contained in $\{(x, y, z) : x + y = 2, x \equiv y \equiv z \equiv 1 \pmod{2}\}$. But then the segment $uv$, lying in the hyperplane $\{x + y = 2\}$, contains at least an extra lattice point, namely $\frac{1}{2}(u + v)$. See Figure 7. This implies $P$ has at least three more lattice points than $P^{uv}$, a contradiction.

\[\square\]

6. (Almost All) Spanning 3-polytopes Have a Unimodular Tetrahedron

The following statement follows from Corollary 1.5.

**Lemma 6.1.** Let $P$ be a non-spanning lattice 3-polytope and let $v$ be a vertex of it such that $P^v$ is still 3-dimensional. Then $P$ and $P^v$ have the same index. \[\square\]

With this we can now prove Theorem 1.7. The only spanning 3-polytopes that do not contain a unimodular tetrahedron are the $E^1_{(5, 1)}$ and $E^2_{(5, 1)}$ from the statement of Theorem 1.7.

**Proof of Theorem 1.7.** Let $P$ be a spanning 3-polytope. If $P$ has width one, the statement is true by Corollary 3.3. So assume $P$ to be of width $> 1$ and let $n$ be its size. If $n \leq 7$ then the statement is true by the enumerations in [4], and if $P$ is not merged and of size $n \geq 8$ by Theorem 5.2. So, we suppose that $n \geq 8$ and $P$ is merged. That is, there exist $u, v \in \text{vert}(P)$ such that $P^u$ and $P^v$ have width $> 1$, and such that $P^{uv}$ is 3-dimensional.

If $P^u$ or $P^v$ are spanning, then by inductive hypothesis they contain a unimodular tetrahedron, and so does $P$. To finish the proof we show that it is impossible for $P^u$ and $P^v$ to both have index greater than one. If this happened, then Lemma 6.1 tells us that both $P^u$ and $P^v$ have the same index as $P^{uv}$. That is, $u$ and $v$ lie in the affine lattice spanned by $P^{uv} \cap \mathbb{Z}^3$, which implies the index of $P$ being the same (and bigger than one), a contradiction. \[\square\]

7. The $h^*$-Vectors of Non-spanning 3-polytopes

Part of our motivation for studying non-spanning 3-polytopes comes from results on $h^*$-vectors of spanning lattice polytopes recently obtained by Hofscheier et al. in [6, 7]. Recall that the $h^*$-polynomial of a lattice $d$-polytope $P$, first introduced by Stanley [8], is the numerator of the generating function of the sequence $(\text{size}(tP))_{t \in \mathbb{N}}$. That is, it is the polynomial

$$h^*_0 + h^*_1 z + \cdots + h^*_s z^s := (1 - z)^{d+1} \sum_{t=0}^{\infty} \text{size}(tP) z^t.$$ 

Its coefficient vector $h^*(P) := (h^*_0, \ldots, h^*_s)$ is the $h^*$-vector of $P$. All entries of $h^*(P)$ are known to be nonnegative integers, and the degree $s \in \{0, \ldots, d\}$ of the $h^*$-polynomial is called the degree of $P$. See [1] for more details.
For a lattice $d$-polytope with $n$ lattice points in total, $n_0$ of them in the interior, and with normalized volume $V$, one has $h_0^* = 1$, $h_1^* = n - d - 1$, $h_s^* = n_0$ and $\sum_i h_i^* = V$. In particular, in dimension three the $h^*$-vector can be fully recovered from the three parameters $(n, n_0, V)$ as follows:

1. $h_0^* = 1$, $h_1^* = n - 4$, $h_2^* = V + 3 - n_0 - n$, $h_3^* = n_0$.

From this we can easily compute the $h^*$-vectors of all non-spanning 3-polytopes. They are given in Table 4. For their computation, in the infinite families $\tilde{F}_i$ we use that:

- The total number of lattice points equals $b - a + 4$ in $\tilde{F}_1(a, b)$ and $\tilde{F}_2(a, b)$ and it equals $b - a + 5$ in $\tilde{F}_3(a, b, k)$ and $\tilde{F}_4(a, b)$.
- The number of interior lattice points is zero in $\tilde{F}_2(a, b)$ and $b - a - 1$ in the other three.
- The volume always equals the volume of the projection $F_i$ (see Figure 2) times the length $b - a$ of the spike. This follows from Corollary 1.6.

| $P$          | $(h_0^*, h_1^*, h_2^*, h_3^*)$ |
|--------------|----------------------------------|
| $T_{p,q}(a, b)$ | $(1, a + b - 2, abq - a - b + 1, 0)$ |
| $\tilde{F}_1(a, b)$ | $(1, n - 4, 7n - 28, n - 5)$ |
| $\tilde{F}_2(a, b)$ | $(1, n - 4, 3n - 13, 0)$ |
| $\tilde{F}_3(a, b, k)$ | $(1, n - 4, 6n - 31, n - 6)$ |
| $\tilde{F}_4(a, b)$ | $(1, n - 4, 6n - 31, n - 6)$ |
| $E^{(5,5)}_5$ | $(1, 1, 17, 1)$ |
| $E^{(6,3)}_3$ | $(1, 2, 19, 2)$ |
| $E^{(7,2)}_2$ | $(1, 3, 10, 0)$ |
| $E^{(8,2)}_2$ | $(1, 4, 20, 3)$ |
| $E^{(8,2)}_6$ | $(1, 4, 20, 3)$ |
| $E^{(8,2)}_8$ | $(1, 4, 21, 4)$ |

Table 4. The $h^*$-vectors of non-spanning 3-polytopes. In the infinite families (top table) $n$ is the size of the polytope. Remember that: in $T_{p,q}(a, b)$ we have $a,b \geq 1$, $n = a + b + 2$ and $q > 1$; in $\tilde{F}_1(a, b)$ we have $n \geq 5$ and in the rest $n \geq 6$.

The main results in [6, 7] are:

**Theorem 7.1** ([6] Theorem 1.3, [7] Theorem 1.2). Let $h^* = (h_0^*, \ldots, h_s^*)$ be the $h^*$-vector of a spanning lattice $d$-polytope of degree $s$. Then:

1. $h^*$ has no gaps, that is, $h_i^* > 0$ for all $i \in \{0, \ldots, s\}$.
2. For every $i, j \geq 1$ with $i + j < s$ one has

$$h_i^* + \cdots + h_s^* \leq h_{j+1}^* + \cdots + h_{j+i}^*.$$
In dimension three the only nontrivial inequality in part (2) is \( h_1^* \leq h_2^* \) for lattice polytopes of degree \( s = 3 \) (that is, for lattice polytopes with interior lattice points), which is true by Hibi’s Lower Bound Theorem \([5\), Theorem 1.3\]. This inequality fails for spanning polytopes without interior lattice points, as the prism \([0,1] \times [0,k] \) shows (its \( h^* \)-vector equals \((1, 4k, 2k−1)\)). The following easy argument implies that non-spanning 3-polytopes satisfy a stronger inequality, even if some of them do not have interior lattice points:

**Proposition 7.2.** Let \( P \) be a lattice 3-polytope of index \( q > 1 \). Then, \( h_2^*(P) \geq (q−1)(1 + h_1^*(P)) \).

**Proof.** Let \( \Lambda' \subset \mathbb{Z}^3 \) be the lattice spanned by \( P \cap \mathbb{Z}^3 \) and let \( P' \) be the polytope \( P \) considered with respect to \( \Lambda' \) (equivalently, let \( P' = \phi(P) \) where \( \phi : \mathbb{R}^3 \to \mathbb{R}^3 \) is an affine map extending a lattice isomorphism \( \Lambda' \cong \mathbb{Z}^3 \)).

Let \( V' \) and \( V = qV' \) be the volumes of \( P' \) and \( P \). Observe that the set of lattice points (in particular, the parameters \( n \) and \( n_0 \) in Equation (1)) does not depend on whether we look at one or the other lattice. Then, using the expression for \( h_2^* \) in Equation (1) we get:

\[
\begin{align*}
    h_2^*(P) &= h_2^*(P') - V' + V \\
    &= h_2^*(P') + (q−1)V' \\
    &= h_2^*(P') + (q−1)\sum_i h_i^*(P') \\
    &\geq (q−1)(1 + h_1^*(P')) = (q−1)(1 + h_1^*(P)).
\end{align*}
\]

Concerning gaps (part (1) of Theorem 7.1), empty tetrahedra of volume \( q > 1 \) do have gaps (their \( h^* \)-vectors are \((1, 0, q−1)\)). Clearly, every other non-spanning polytope has \( h_1^* = n − 4 > 0 \) and, by Proposition 7.2, it has \( h_2^* > 0 \) too. Thus, empty tetrahedra are the only lattice 3-polytopes with gaps.

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