Symbolic Dynamics Generated by a Combination of Graphs

Vasileios Basios¹, Gian-Luigi Forti¹,² and Gregoire Nicolis¹

March 30, 2022

¹ Interdisciplinary Center for Nonlinear Phenomena and Complex Systems, C.P. 231, Université Libre de Bruxelles, B-1050 Brussels, Belgium. Contact: vbasios@ulb.ac.be, gnicolis@ulb.ac.be

² Dipartimento di Matematica, Università degli Studi di Milano, via C. Saldini 50, I-20133, Milano, Italy. Contact: forti@mat.unimi.it

Abstract

In this paper we investigate the growth rate of the number of all possible paths in graphs with respect to their length in an exact analytical way. Apart from the typical rates of growth, i.e. exponential or polynomial, we identify conditions for a stretched exponential type of growth. This is made possible by combining two or more graphs over the same alphabet, in order to obtain a discrete dynamical system generated by a triangular map, which can also be interpreted as a discrete non-autonomous system. Since the vertices and the edges of a graph usually are used to depict the states and transitions between states of a discrete dynamical system, the combination of two (or more) graphs can be interpreted as the driving, or perturbation, of one system by another.

PACS: 75.40Gb, 05.45.-a, 03.67.-a, 89.75.Da
MSC2000: 37B10
Keywords: Entropy, Graphs, Symbolic Dynamics, Non-autonomous systems

1 Introduction.

Strings consisting of sequences of symbols play a central role in many natural phenomena. For instance, the DNA and RNA are linear strings carrying
the genetic code written in an alphabet consisting of four letters A, C, G and T (or U) according to whether the nucleotide subunit contains the base adenine, cytosine, guanine and thymine (or uracil). Furthermore most of the communication processes generating and transporting information or having a cognitive dimension such as computer programs, the electrical activity of the brain, texts or music, are implemented in one way or the other in symbolic terms.

With the advent of nonlinear dynamics and in particular chaos theory, it has been realized that the evolution of large classes of dynamical systems can be also described, under certain conditions, by a sequence of symbols. The existence of such a symbolic dynamics related, in addition, to the principal indicators of the complexity of the underlying system, provides a unique opportunity to understand the mathematical and physical mechanisms at the basis of information generation and transduction. In this context, a central question is how to enumerate and to characterize the full set of possible sequences generated by a dynamical system or, in a more dynamic language, the paths leading from one state to another.

A variety of approaches to this major problem have been reported, but several aspects remain only partially understood. A measure of the dependence of the number of sequences of a given length (to which we will refer hereafter as words) with respect to the length is the block entropy, defined by

\[ H(n) = - \sum_{w(n)} P[w(n)] \ln (P[w(n)]) \] (1)

where the summation is over all allowed words \( w(n) \) of length \( n \) and \( P[w(n)] \) is the associated probability (a word is allowed if it represents a possible path leading from one state to another; in general not all conceivable paths are possible for the system under investigation). Note that for uniformly distributed probability, \( H(n) \) is exactly the logarithm of the number of words of length \( n \). For sequences generated by Bernoulli or Markov processes the Shannon-McMillan theorem \[\text{Applebaum, 1996}\] asserts that

\[ P[w(n)] \approx e^{-H(n)} = e^{-nh} \] (2)

where \( h \), the entropy of the source, is a discrete analog of the Kolmogorov-Sinai entropy. This entails that (a) \( H(n) \) scales linearly with the word length and (b) that long words are extremely improbable, as they are exponentially penalized. The penalization is maximal for a Bernoulli processor where \( h \) takes its largest value for a given alphabet \[\text{Applebaum, 1996}\].
Although still exponential, it is milder for a Markov processor since $h$ is in this case less than the maximum. As it turns out holds true for sequences generated by chaotic dynamical systems as well in the domain of fully developed chaos \cite{Gaspard_1998}. \cite{Nicolis_1995}, \cite{Hao_1990}.

In many real world systems we do not observe exponential rates of growth. This means that there exists a procedure of selection of a subset of sequences possessing some prescribed properties out of the total number of sequences. In this context Ebeling and Nicolis \cite{Ebeling_1992} proposed the following generic scaling law for the block entropy

\[ H(n) = nh + gn^\mu (\log n)^\nu + e, \quad h \geq 0, \quad 0 < \mu < 1, \quad \nu \leq 0 \]  

(3)

and showed that classical literature texts or music obey to Eq.\(\text{(3)}\) with $h = 0$, $\nu = 0$ and $\mu < 1$. Dynamical systems showing weak chaos in the form of intermittency, and sporadic systems have also been shown to give rise to a sublinear scaling of $H(n)$, \cite{Gaspard_1998}, \cite{Nicolis_1995}. The question has therefore been raised, under what conditions could the block entropy of a dynamical system scale sublinearly with $n$, thereby rendering possible the selection of long sequences with a non-negligible probability. Obviously one can think of a number of constraints external to the dynamics (for instance, the meaning of a string of letters in a natural language). A natural and interesting problem concerns the possibility of constructing a dynamical system including in itself a selection rule capable of producing a sublinear scale. Our principal goal in this paper is to propose the construction of higher dimensional dynamical systems obtained by suitable combinations of simple systems, each of them generated by a single graph.

The idea of our construction comes from the theory of two-variable triangular maps, i.e., maps where the first component depends only on the first variable. In our particular case, actually, the system generated by these triangular maps can also be interpreted as a non-autonomous discrete dynamical system. We are able to show that this mechanism can generate a subclass of scalings of the form of Eq.\(\text{(3)}\), leading at the level of Eq.\(\text{(2)}\) to a stretched exponential dependence of $P[w(n)]$ on $n$.

To the degree that the states of a system can be encoded via a reasonably small finite alphabet and as long as the alphabet remains constant, it is natural to identify the states of the system as vertices of a graph and the edge between two vertices as a transition between these two states. The connectivity of a graph reflects, then, the possible transitions between states, and a path traversing through the vertices of the graph describes a possible trajectory of the system. In the present work we shall adopt this representation.
In this representation the upper bound of the block entropy (see Eq. (1)) computed over all possible Markov measures [Gaspard, 1998] gives the topological entropy of the system, which indicates what paths are possible to be realized 'in principle' independently of any probabilistic consideration.

More technically, [Douglas & Markus, 1995], [Kitchens, 1998], the topological (block) entropy of a sequence $X$ (or "the entropy of the shift $X$", if the sequence comes from a shift) is defined as

$$h_{\text{top}} = \lim_{n \to \infty} \left( \frac{1}{n} \log_2 |B_n(X)| \right)$$

where $|B_n(X)|$ is the number of n-blocks appearing in $X$. Obviously, since all the blocks counted cannot be more than all the combinations of the letters of the alphabet at hand, $|B_n(X)| = |A|^c n \leq |A|^n$, where $|A|$ is the cardinal number of the set of letters $A$ which constitute the alphabet in use. Usually the parameter $c$ is called entropy’s "growth rate". In this formulation the analogue of Eq. (3) would be the presence of a more intricate relationship, e.g. in the form $|B_n(X)| = |A|^{c(n)}$.

In Sect. 2 we review some results for the number of symbol sequences obtained by following all paths over a graph. Then in Sect. 3 we investigate the possibility of deviating from this standard symbol sequence generation by combination of graphs which correspond to a kind of perturbation of the original dynamical system. We report two examples which give a stretched exponential growth rate for the resulting symbol sequences. We conclude in Sect. 4 by an overview of the results and an outlook of further possible developments.

## 2 Symbolic Dynamics of a single graph

In all generality, the technique of associating a dynamical system to a symbol-sequence producing system consists of applying a suitable partition of the phase space of the system (coarse graining) to a finite number of cells [Nicolis, 1995], [Douglas & Markus, 1995] and then to associate the cells of the partition to the letters of an alphabet. A trajectory of the system produces then a symbol sequence as it visits the different cells of the partition. Relevant partitioning of the phase space reflects the ability of recording relevant features of the dynamics.

A key point in the foregoing is to associate a directed graph $G$ over the alphabet of the (coarse grained) states with the transitions from each state to another. The adjacency matrix, $M$, of the graph $G$ has elements...
$m_{i,j} = 1$ if the transition from state $i \to j$ is possible and $m_{i,j} = 0$ otherwise. Providing, in addition, a Markov measure over the states will give the transition probability matrix of the system.

Evidently all information about the possible transitions is encoded in the adjacency matrix of the graph and the growth of the block entropy of the resulting symbol sequences will be given by the first eigenvalue of $M$. Any path of transitions associated to a succession of states can be envisioned as a path on the graph $G$. We shall be interested here in the characterization of possible transitions, and therefore restrict this work on simple connected graphs associated with dynamical systems, leaving the graphs equipped with a Markov measure for later work. In order to proceed some definitions and a review of known results are necessary.

Let us consider an alphabet $A = \{X_1, X_2, \ldots, X_k\}$ and denote by $\mathcal{A}_k$ the set of all finite words generated by $A$. By $\mathcal{A}_n^k$ we denote the set of all words of length $n$, so we have

$$\mathcal{A}_k = \bigcup_{n=1}^{\infty} \mathcal{A}_n^k.$$

Let $G$ be a connected directed graph (from now on all graphs here considered are connected and directed) having $A$ as set of vertices and with at most two edges with different directions connecting two vertices; thus, there is at most one arrow from a vertex $X_i$ to another vertex $X_j$. Given a word $s \in \mathcal{A}_k$, a 1-letter extension of $s$ generated by $G$ is defined as follows: if $X_i$ is the last letter of $s$, we add a letter $X_j$ if and only if the graph $G$ has an arrow from $X_i$ to $X_j$. Depending on the number of arrows in $G$ starting from $X_i$ we have the same number of 1-letter extensions of $s$.

As a first step we formulate the generation of words by a graph in terms of a discrete dynamical system, i.e., the iterated application of a suitably defined function over a certain set. To do this, we define a map $G$ from the family $2^{\mathcal{A}_k}$ of all subset of $\mathcal{A}_k$ into itself in the following way:

For $S \subset \mathcal{A}_k$, $G(S)$ is the set of all 1-letter extensions of the words in $S$, generated by $G$.

The set $G^{n-1}(A)$ ($G^{n-1}$ is the $n - 1$-iterate of $G$) is the set of the words of length $n$ generated by $G$. In the following we will denote this set by $W_n^G$, and by $W_G$ the set of all finite words generated by $G$ or admissible words. Here, by an admissible word we mean a finite string constructed with the alphabet $\{X_1, X_2, \ldots, X_k\}$ such that if a pair $X_i, X_j$ appears in the string, then $G$ has an arrow from $X_i$ to $X_j$.

With the symbol $W_G^n(X_i, X_j)$ we denote the set of admissible words of
length \( n \) starting with \( X_i \) and ending with \( X_j \). Moreover, we set

\[
W^n_G(X_i, \cdot) = \bigcup_{j=1}^{k} W^n_G(X_i, X_j), \quad W^n_G(\cdot, X_j) = \bigcup_{i=1}^{k} W^n_G(X_i, X_j).
\]

The following theorem is well known [Douglas & Markus, 1995]:

**Theorem 1** Let \( M^{n-1} = [m_{ij}^{(n-1)}] \) be the \((n-1)\)-th power of the adjacency matrix \( M \).

The cardinality \( \omega^n_G(X_i, X_j) \), that is the number of words in \( W^n_G(X_i, X_j) \), is then

\[
\omega^n_G(X_i, X_j) = m_{ij}^{(n-1)}
\]

and

\[
\omega^n_G = \sum_{i,j=1}^{k} m_{ij}^{(n-1)}.
\]

Next we can state a theorem which provides a linear difference equation for \( \omega^n_G \).

**Theorem 2** Let \( P_M(\lambda) = \lambda^k + \sum_{r=0}^{k-1} a_r \lambda^r \) be the characteristic polynomial of \( M \) \( (P_M(\lambda) = \det(\lambda I - M)) \). Then, for every \( n > k \), we have

\[
\omega^n_G = - \sum_{r=0}^{k-1} a_r \omega^{n-k+r}_G.
\]

**Proof**- Since any matrix is a root of its characteristic polynomial, we have

\[
M^k = - \sum_{r=0}^{k-1} a_r M^r.
\]

Multiplying both sides by \( M^{n-k-1} \) we get

\[
M^{n-1} = - \sum_{r=0}^{k-1} a_r M^{n-k+r-1}.
\]

Thus, we immediately have

\[
\omega^n_G = - \sum_{r=0}^{k-1} a_r \omega^{n-k+r}_G \tag{4}
\]

QED.
As an obvious consequence of Theorem 2 we obtain the following corollary:

**Corollary 3** Assume that $\mu_1, \cdots, \mu_s$ are the non zero distinct roots of the characteristic polynomial of $M$, with respective multiplicities $r_1, \cdots, r_s$. Then

$$\omega^n_G = c_0 + \sum_{h=1}^{s} \left( \sum_{q=1}^{r_h} c_{hq} n^{q-1} \mu_h^n \right),$$

where the coefficients $c_0$ and $c_{hq}$ are determined by the values $\omega^n_p$ for

$p = 1, \cdots, k$.

We remark that not only $\omega^n_G$ satisfies the previous difference equation (4), but also $\omega^n_G(X_i, X_j)$, for every $i, j = 1, \cdots, k$.

We conclude this section by highlighting an important consequence stemming from Corollary 3. Let us define $\rho = \max \Re(\mu_i)$, then the rate of growth of $\omega^n_G$ with respect to the length $n$ of the words can only have one of the following three forms:

(i) exponential, $\omega^n_G \simeq \rho^n$, if $\rho > 1$,

(ii) polynomial, $\omega^n_G \simeq p(n)$, with $p(n)$ a polynomial with $\deg p(n) \geq 1$, if $\rho = 1$ or with $\deg p(n) = 0$ if $\rho = 0$

(iii) a combination of polynomials and exponentials, $\omega^n_G \simeq p(n) \rho^n$, if $\rho > 1$ and its corresponding root has multiplicity greater than one.

The graphs used in the following section are examples of case (i) and (ii). To get a polynomial of degree zero, it is enough to consider the graph on a two-letter alphabet with the following adjacency matrix $M$,

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To our knowledge, and according to some preliminary numerical investigation, it seems that case (iii) cannot arise.

The previous remark shows that there is simply no possibility for paths over any graph to generate symbol-sequences with stretched exponential growth rates, i.e., $\omega^n_G \simeq \rho^n \nu$ with $0 < \nu < 1$. We shall see, presently, that this kind of growth may result from a combination of graphs.
3 Combination of graphs.

Let us now turn to symbol sequences generated by a collection of graphs. Assume we are given \( \ell \) different graphs \( G_1, G_2, \ldots, G_\ell \) on the same alphabet \( A = \{X_1, X_2, \ldots, X_k\} \). We also equip this collection of graphs with a transition rule among them. This rule might represent, for instance, a 'superselection' rule [Nicolis, 2005] not encoded within each graph but governing the transition among different domains of distinct dynamics, as in the case of hybrid systems, see for example [Basios, 2001] and references therein. We shall assume that this rule is a deterministic one given by an increasing sequence of non-negative integers, \( \{g_i\}, i \in \mathbb{N}, g_0 = 0 \).

We construct on \( \mathbb{N} \times 2^A \) the following dynamical system:

\[
F(m, S) = (m + 1, G_t(S)) \quad \text{if} \quad g_{t+1} - 1 \leq m < g_{t+2},
\]

\( t = 0, 1, 2, \ldots, r = 1, 2, \ldots, \ell \).

(5)

This amounts to using repeatedly in sequence the graphs \( G_1, G_2, \ldots, G_\ell \), the duration of the use of a single graph being controlled by the given deterministic sequence \( \{g_i\} \).

The above construction (analogously to that presented in the previous section giving a dynamical description of the generation of words by a graph) gives a formulation in terms of a discrete dynamical system of the combination of the graphs \( G_1, G_2, \ldots, G_\ell \) driven by the sequence \( \{g_i\} \). This can be understood as a two-variable map with the first component depending only on the first variable known as a triangular map. The dynamical system is generated by iteration of this map. Due to the fact that the first variable is discrete, this construction has also a natural interpretation as a non-autonomous dynamical system. In any case, what is important and is in a certain sense the goal of the present investigation, is to embed the transition rule into the dynamics.

We are mainly concerned with the trajectory starting with \( (0, A) \) and, by denoting with \( pr_2 \) the projection on the second coordinate, the set

\[
W_F^n := pr_2(F^{n-1}(0, A))
\]

will be called the set of words of length \( n \) generated by the dynamical system (5). In the two following subsections we shall demonstrate how non-standard, e.g. stretched exponentials growth rates can result by such a combination of graphs. First, we will give an example where both graphs have an absorbing state (for our purposes, an absorbing state is a vertex with no outcoming edges other than its own self). Secondly, we give an
example of a combination of a fully connected graph and a graph with an absorbing state.

3.1 Combination of two graphs: An example with absorbing states

The two graphs to be combined are defined over the three-letter alphabet \{X,Y,Z\} as follows. The first graph \(G_1\) has the form:

\[
\begin{array}{c}
\uparrow \\
X \\
\downarrow \\
\downarrow \\
\uparrow \\
Z \\
\rightarrow \\
\uparrow \\
Y \\
\leftarrow \\
\end{array}
\]

Its adjacency matrix is

\[
M_1 = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

and the corresponding characteristic polynomial is

\[P_{M_1} = \lambda^3 - 2\lambda^2 + 1,\]

which has 1, \((1 + \sqrt{5})/2\) =: \(\mu\) and \((1 - \sqrt{5})/2\) =: \(1 - \mu\) as roots.

By applying the results presented in Section 2, we obtain the following formulas (the subscript 1 to all \(\omega\)'s refers to the graph \(G_1\)):

\[
\begin{align*}
\omega_1^n(X,X) &= \frac{1}{\sqrt{5}}[\mu^n - (1 - \mu)^n] \\
\omega_1^n(X,Y) &= -2 + \frac{5 + 3\sqrt{5}}{10}\mu^n + \frac{5 - 3\sqrt{5}}{10}(1 - \mu)^n \\
\omega_1^n(X,Z) &= \frac{5 - \sqrt{5}}{10}\mu^n + \frac{5 + \sqrt{5}}{10}(1 - \mu)^n \\
\omega_1^n(X,\cdot) &= -2 + \frac{5 + 2\sqrt{5}}{5}\mu^n + \frac{5 - 2\sqrt{5}}{5}(1 - \mu)^n \\
\omega_1^n(Y,X) &= \omega_1^n(Y,Z) = 0, \quad \omega_1^n(Y,Y) = 1 \\
\omega_1^n(Y,\cdot) &= 1
\end{align*}
\]
\[
\omega^1_n(Z, X) = \frac{5 - \sqrt{5}}{10} \mu^n + \frac{5 + \sqrt{5}}{10} (1 - \mu)^n \\
\omega^1_n(Z, Y) = -1 + \frac{5 + \sqrt{5}}{10} \mu^n + \frac{5 - \sqrt{5}}{10} (1 - \mu)^n \\
\omega^1_n(Z, Z) = \frac{3\sqrt{5} - 5}{10} \mu^n - \frac{3\sqrt{5} + 5}{10} (1 - \mu)^n \\
\omega^1_n(Z, \cdot) = -1 + \frac{5 + 3\sqrt{5}}{10} \mu^n + \frac{5 - 3\sqrt{5}}{10} (1 - \mu)^n \\
\omega^1_n(\cdot, X) = \frac{5 + \sqrt{5}}{10} \mu^n + \frac{5 - \sqrt{5}}{10} (1 - \mu)^n \\
\omega^1_n(\cdot, Y) = -2 + \frac{10 + 4\sqrt{5}}{10} \mu^n + \frac{10 - 4\sqrt{5}}{10} (1 - \mu)^n \\
\omega^1_n(\cdot, Z) = \frac{1}{\sqrt{5}} [\mu^n - (1 - \mu)^n] \\
\omega^1_n = -2 + \frac{15 + 7\sqrt{5}}{10} \mu^n + \frac{15 - 7\sqrt{5}}{10} (1 - \mu)^n
\]

The following two inequalities will be useful for further developments:
\[
\omega^1_n(X, X) > \omega^1_n(X, Z), \quad \omega^1_n(Z, X) > \omega^1_n(Z, Z).
\]  

(6)

The second graph \(G_2\) is the following:

![Graph](image)

Its adjacency matrix is
\[
M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\]

and the corresponding characteristic polynomial is
\[
P_{M_2} = \lambda^3 - 2\lambda^2 + \lambda,
\]
which has 0 and 1 (with multiplicity 2) as roots. We obtain the following formulas:

\[
\begin{align*}
\omega_2^n(X, X) &= \omega_2^n(X, Z) = 0, \quad \omega_2^n(X, Y) = 1 \\
\omega_2^n(X, \cdot) &= 1 \\
\omega_2^n(Y, X) &= \omega_2^n(Y, Z) = 0, \quad \omega_2^n(Y, Y) = 1 \\
\omega_2^n(Y, \cdot) &= 1 \\
\omega_2^n(Z, X) &= \omega_2^n(Z, Z) = 1, \quad \omega_2^n(Z, Y) = 2n - 3 \\
\omega_2^n(Z, \cdot) &= 2n + 1 \\
\omega_2^n(\cdot, X) &= \omega_2^n(\cdot, Z) = 1, \quad \omega_2^n(\cdot, Y) = 2n - 1 \\
\omega_2^n &= 2n + 1
\end{align*}
\]

Let now \( \{g_t\}, i \in \mathbb{N}, g_0 = 0 \), be a given increasing sequence of non-negative integers. We construct on \( \mathbb{N} \times 2^{A_3} \) the dynamical system:

\[
\mathcal{F}(m, S) = \begin{cases} 
(m + 1, G_1(S)) & \text{if } g_{2t} \leq m < g_{2t+1}, \ t = 0, 1, 2, \cdots \\
(m + 1, G_2(S)) & \text{if } g_{2t+1} \leq m < g_{2t+2}, \ t = 0, 1, 2, \cdots 
\end{cases}
\]

and we denote by \( \omega^n_\mathcal{F} \) the cardinality of the set \( \mathcal{W}^n_\mathcal{F} \). Our aim is to give a lower and an upper bound for \( \omega^n_\mathcal{F} \) and then to prove that, for suitable choices of the sequence \( \{g_t\} \), we may obtain a growth given by stretched exponentials.

Given two square matrices \( M \) and \( N \) of the same dimension, we say that \( M \leq N \) if this order relation is valid entry by entry. We write \( M < N \) if \( M \leq N \) and \( M \neq N \). We denote by \( |M| \) the sum of all entries of the matrix \( M \). Clearly, if \( M < N \), then \( |M| < |N| \). Moreover, if \( M < N \), \( L < K \) and all entries are non-negative, then \( ML \leq NK \) and \( |ML| \leq |NK| \).

If we define \( s_t = g_t - g_{t-1} \), we have the following theorem.

**Theorem 3** For every \( t \geq 1 \) we have

\[
5^{-1/2} \prod_{i=1}^{t} \left[ \mu^{s_{2i-1}} - (1 - \mu)^{s_{2i-1}} \right] < \omega_\mathcal{F}^{2t} < \omega_1^{s_1 + s_3 + \cdots + s_{2t-1}} [1 + 2 \sum_{i=1}^{t} s_{2i}] \quad (7)
\]
Proof – We begin with the second inequality. As a first step we prove that

\[
M_1^{s_1-1}M_2^{s_2}M_1^{s_3} \cdots M_2^{s_{2t}} < M_1^{s_1+s_3+\cdots+s_{2t-1}+1} + C(s_1, s_2) + C(s_1 + s_3, s_4) + \cdots + C(s_1 + s_3 + \cdots + s_{2t-1}, s_{2t}),
\]

where

\[
C(a, b) = \begin{bmatrix}
0 & 2b\omega_1^a (X, Z) & 0 \\
0 & 0 & 0 \\
0 & 2b\omega_1^a (Z, Z) & 0
\end{bmatrix}.
\]

The matrices \(M_1^r\) and \(M_2^r\) have the following forms:

\[
M_1^r = \begin{bmatrix}
\omega_1^{r+1}(X, X) & \omega_1^{r+1}(X, Y) & \omega_1^{r+1}(X, Z) \\
0 & 1 & 0 \\
\omega_1^{r+1}(Z, X) & \omega_1^{r+1}(Z, Y) & \omega_1^{r+1}(Z, Z)
\end{bmatrix}
\]

\[
M_2^r = \begin{bmatrix}
0 & 1 & 0 \\
1 & 2r - 1 & 1
\end{bmatrix}.
\]

The proof is by induction on \(t\). For \(t = 1\) we have:

\[
M_1^{s_1-1}M_2^{s_2} = \begin{bmatrix}
\omega_1^{s_1}(X, X) & \omega_1^{s_1}(X, Y) & \omega_1^{s_1}(X, Z) \\
0 & 1 & 0 \\
\omega_1^{s_1}(Z, X) & \omega_1^{s_1}(Z, Y) & \omega_1^{s_1}(Z, Z)
\end{bmatrix} +
\begin{bmatrix}
0 & \omega_1^{s_1}(X, X) + (2s_2 - 1)\omega_1^{s_1}(X, Z) & 0 \\
0 & 0 & 0 \\
0 & \omega_1^{s_1}(Z, X) + (2s_2 - 1)\omega_1^{s_1}(Z, Z) & 0
\end{bmatrix} <
\begin{bmatrix}
\omega_1^{s_1}(X, X) & \omega_1^{s_1}(X, Y) & \omega_1^{s_1}(X, Z) \\
0 & 1 & 0 \\
\omega_1^{s_1}(Z, X) & \omega_1^{s_1}(Z, Y) & \omega_1^{s_1}(Z, Z)
\end{bmatrix} +
\begin{bmatrix}
0 & 2s_2\omega_1^{s_1}(X, Z) & 0 \\
0 & 0 & 0 \\
0 & 2s_2\omega_1^{s_1}(Z, Z) & 0
\end{bmatrix} = M_1^{s_1-1} + C(s_1, s_2).
\]

In order to obtain the previous inequality we used inequalities (6).
Now, assume that the relation is true for \( t \) and consider \( t + 1 \):

\[
M_1^{s_1-1} M_2^{s_2} \cdots M_t^{s_{2t-1}} M_2^{s_{2t}} M_1^{s_{2t+1}} M_2^{s_{2t+2}} < \\
\left[ M_2^{s_1+s_3+\cdots+s_{2t}-1} + C(s_1, s_2) + \cdots + C(s_1 + s_3 + \cdots + s_{2t-1}, s_{2t}) \right] \times \\
\left[ M_t^{s_{2t+1}} + C(s_{2t+1} + 1, s_{2t+2}) \right].
\]

(9)

Since we have the following equations:

\[
C(a, b)C(c, d) = 0, \quad C(a, b)M_1^r = C(a, b), \\
M_1^{s_1+s_3+\cdots+s_{2t-1}+1} C(s_{2t+1} + 1, s_{2t+2}) = C(s_1 + s_3 + \cdots + s_{2t+1}, s_{2t+2}),
\]

the last term in Eq. (9) becomes

\[
M_1^{s_1+s_3+\cdots+s_{2t+1}-1} + C(s_1, s_2) + \cdots + C(s_1 + s_3 + \cdots + s_{2t+1}, s_{2t+2}).
\]

Thus the inequality (8) is proved.

Since \(|C(a, b)| < 2k\omega_1^s\), we obtain

\[
\omega_{2t}^g < \omega_1^{s_1+s_3+\cdots+s_{2t-1}} + 2s_1\omega_1^{s_1} + \cdots + 2s_{2t}\omega_1^{s_1+s_3+\cdots+s_{2t-1}}.
\]

Since, obviously,

\[
\omega_1^{s_1+s_3+\cdots+s_{2j-1}} \leq \omega_1^{s_1+s_3+\cdots+s_{2t-1}},
\]

for any \( j \leq t \), we finally get

\[
\omega_{2t}^g \omega_1^{s_1+s_3+\cdots+s_{2t-1}} \left[ 1 + 2 \sum_{i=1}^{t} s_{2i} \right],
\]

i.e., the right-hand side of the above is nothing more than the right-hand side of Eq. (7). The other inequality follows easily, since we have

\[
M_1^{s_1-1} M_2^{s_2} \geq \\
\begin{bmatrix}
\omega_1^{s_1}(X, X) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and so

\[
|M_1^{s_1-1} M_2^{s_2} M_1^{s_3} \cdots M_2^{s_{2t}}| > \omega_1^{s_1}(X, X)\omega_1^{s_3+1}(X, X) \cdots \omega_1^{s_{2t-1}+1}(X, X).
\]
Since $\omega_{1}^{r+1}(X, X) \geq \omega_{1}^{r}(X, X)$, we have

$$\omega_{1}^{s_{1}}(X, X) \omega_{1}^{s_{3}+1}(X, X) \cdots \omega_{1}^{s_{2t-1}+1}(X, X) \geq \omega_{1}^{s_{1}}(X, X) \omega_{1}^{s_{3}}(X, X) \cdots \omega_{1}^{s_{2t-1}}(X, X)$$

and

$$\omega_{1}^{s_{1}}(X, X) \omega_{1}^{s_{3}}(X, X) \cdots \omega_{1}^{s_{2t-1}}(X, X) = 5^{-t/2} \prod_{i=1}^{t} [\mu^{s_{2i-1}} - (1 - \mu)^{s_{2i-1}}].$$

QED

Now we specify the sequence $\{s_{t}\}$ (or $\{g_{t}\}$) in order to obtain the desired result. Choosing

$$s_{1} = 4, \quad s_{2t-1} = 2t + 1, \quad s_{2t} = (t + 1)^{4} - t^{4} + t^{2} - (t + 1)^{2}$$

we have:

$$s_{1} + s_{3} + \cdots + s_{2t-1} = (t + 1)^{2}$$
$$s_{2} + s_{4} + \cdots + s_{2t} = (t + 1)^{4} - (t + 1)^{2}$$
$$n := g_{2t} = s_{1} + s_{2} + \cdots + s_{2t} = (t + 1)^{4}.$$

Substituting in Eq. (7) we obtain the inequalities:

$$5^{-(\frac{1}{\sqrt{n}} - 1/2)} \prod_{i=1}^{\sqrt{n}} [\mu^{s_{2i-1}} - (1 - \mu)^{s_{2i-1}}] < \omega_{1}^{n} < \omega_{1}^{\sqrt{n}} [1 + 2n - 2\sqrt{n}].$$

Denoting by $f_{1}(n)$ and $f_{2}(n)$ the left- and right-hand sides of (7) respectively, we obtain the following asymptotic evaluations:

$$f_{1}(n) \asymp 5^{-(\frac{1}{\sqrt{n}} - 1/2)} \mu^{\sqrt{n}}, \quad f_{2}(n) \asymp n \mu^{\sqrt{n}}.$$

At this point it is worthwhile to specify the role of the absorbing states in the graphs $G_{1}$ and $G_{2}$. In both graphs the absorbing state if given by $Y$ and the consequence of this is that $\omega_{1}^{n}(Y, \cdot) = \omega_{2}^{n}(Y, \cdot) = 1$, while $\omega_{1}^{(\cdot, Y)} = -2 + \frac{5 + 4\sqrt{5}}{10} \mu^{n} + \frac{5 - 4\sqrt{5}}{10} (1 - \mu)^{n}$ and $\omega_{2}^{(\cdot, Y)} = 2n - 1$. This implies that when passing from one graph to the other the quite large number of words ending by $Y$ in not changed along the application of the new graph. The times of application of each graph are given by $\{s_{t}\}$ and in our case this sequence increases sufficiently fast to permit a strong reduction of the number of generated words.
3.2 Combination of two graphs: A second example with a fully connected graph

For this second example we take as graph $G_1$ the complete graph on three letters i.e., the graph:

![Graph](image)

Clearly this graph has no absorbing states. Its adjacency matrix is

$$M_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and we immediately have

$$\omega^n_1(X, X) = \omega^n_1(X, Y) = \omega^n_1(X, Z) = \omega^n_1(Y, X) = \omega^n_1(Y, Z) = \omega^n_1(Z, X) = \omega^n_1(Z, Y) = \omega^n_1(Z, Z) = 3^n$$

We remark that $M_1^r = 3^{r-1}M_1$.

As second graph we use the graph $G_2$ of the previous example. We have the following:

**Theorem 4** For every $t \geq 1$ we have

$$3^{s_1+s_3+ \cdots +s_{2t-1}} < \omega_{F}^{2t} < 3^{s_1+s_3+ \cdots +s_{2t-1}} \prod_{i=1}^{t}(2s_{2i} + 1).$$

**Proof** We begin by computing $M_1^{s_1-1}M_2^{s_2}$. First we have the following:

$$M_1M_2^{s_2} = \begin{bmatrix} 1 & 2s_2 + 1 & 1 \\ 1 & 2s_2 + 1 & 1 \\ 1 & 2s_2 + 1 & 1 \end{bmatrix}.$$ 

Thus, we obtain the inequalities

$$M_1 < M_1M_2^{s_2} < (2s_2 + 1)M_1.$$
Hence, we easily get
\[ M_1^{s_1-1}M_2^{s_2} > 3^{s_1-2}M_1 \]
and, by induction,
\[ M_1^{s_1-1}M_2^{s_2}M_1^{s_3} \cdots M_2^{s_{2t}} > 3^{s_1+s_3+\cdots+s_{2t-1}-2}M_1. \]
So we have
\[ |M_1^{s_1-1}M_2^{s_2}M_1^{s_3} \cdots M_2^{s_{2t}}| = \omega^{2t} \cdot F > 3^{s_1+s_3+\cdots+s_{2t-1}}. \]

To prove the other inequality, we see that
\[ M_1^{s_1-1}M_2^{s_2} < 3^{s_1-2}(2s_2+1)M_1 \]
and, again by induction,
\[ M_1^{s_1-1}M_2^{s_2}M_1^{s_3} \cdots M_2^{s_{2t}} < 3^{s_1+s_3+\cdots+s_{2t-1}-2}\left[ \prod_{i=1}^{t}(2s_{2i}+1) \right]M_1, \]
thus
\[ M_1^{s_1-1}M_2^{s_2}M_1^{s_3} \cdots M_2^{s_{2t}} = \omega^{2t} \cdot F < 3^{s_1+s_3+\cdots+s_{2t-1}}\left[ \prod_{i=1}^{t}(2s_{2i}+1) \right]. \]

QED

Choosing now the sequence \( \{s_t\} \) as in the previous example, we remark that
\[ s_{2t} = (t+1)^4 - t^4 + t^2 - (t+1)^2 \leq 4(t+1)^3. \]
Thus we have:
\[ \prod_{i=1}^{t}(2s_{2i}+1) < \prod_{i=1}^{t}(3s_{2i}) < 12^t \prod_{i=1}^{t}(i+1)^3 =
\]
\[ 12^t \left( \frac{(t+1)!}{2} \right)^3 = \frac{12 \sqrt{\sqrt{n}}}{96} [((\sqrt{n})!)^3]. \]
By using Stirling’s formula we obtain for the upper bound \( f_2(n) \) of \( \omega^n \) the relation
\[ f_2(n) \asymp \sqrt{n}^{3\sqrt{n}+3/4} \sqrt{n} \log n \log 3. \]
For the lower bound \( f_1(n) \) of \( \omega_n \), we have
\[
f_1(n) = 3\sqrt{n}.
\]

The choice of the graphs is obviously crucial for getting this result. Clearly, by combining graphs each of which produces an exponential growth rate, we cannot obtain a growth rate given by a stretched exponential. The two examples presented herein permit to compute analytically the bounds for the number of words generated by their combination. In general one can realistically expect only an estimation obtained by numerical simulation. Furthermore, the choice of the sequence \( \{s_t\} \) has been carefully made, in order to have the graph with the polynomial growth rate acting for a longer time. By combining more than two graphs one has much more freedom in the choice of the controlling sequence.

4 Conclusions and Outlook

We have investigated the growth rate of the number of all possible paths in a graph with respect to their length in an exact analytical way. By combining two or more graphs over the same alphabet, in order to obtain a discrete dynamical system generated by a triangular map, we were able to demonstrate stretched exponential growth of the resulting symbol-sequences with respect to their length. The combination of two graphs may be interpreted as the driving of one system by another through parametric perturbations. In this view a sub-exponential growth rate indicates 'persistent memory effects'.

In most cases persistent memory makes any finite graph description only approximate as in the case of zero Kolmogorov-Sinai entropy systems. The best studied example is that of the Fibonacci system, and the binary (coarse grained) representation of the circle-map [Berthe, 1994], where the 'memory effect' in the language of dynamical systems excludes any simple graph generating mechanism. Indeed, symbol sequence of the Fibonacci system can only afford a description by an \textit{infinite}, but countable, family (the leaves of a rooted tree) of Rauzy graphs, see for example [Casaigne, 1997], [Dekking, 1992], [Dekking, 1982]. Scaling behavior of entropy estimates and compressibility issues of such systems with their well known \( \propto \log(n) \) behavior, have been extensively studied [Queffelec, 1987], [Schurman, 2002], in a non-probabilistic framework. A combination of graphs might also be useful.
in this context. It might account for another description of such systems with 'long memory'.

The two examples presented here are part of a wide class which can produce stretched exponential growth. Indeed, it is clear that by combining more graphs, some of them with polynomial rate of growth of the number of words, we gain in degrees of freedom in the choice of the controlling sequence \( \{g_t\} \) and thus in lowering the number of admissible words. It should be noted that in general a subword of an admissible word generated by a combination of graphs is not an admissible word. This contrasts with the case of a single graph and may be thought as a "mark" of this type of construction. It is only a matter of direct observation to realize that in natural languages, too, a subword is not, in general, a word of the language.

Although a great deal of attention has been devoted to the scaling laws associated with the growth of networks, attention on the dynamical basis of the generator of networks from a dynamical systems’ point of view has been reported only quite recently [Nicolis et al., 2005]. Now, since the edges of \( G \), connecting two states/letters of the alphabet can be labeled by these states: i.e. the edge from \( i \) to \( j \) has the label "\( ij \)”, then from the list of all edges of \( G \) we can construct another graph \( G^{(2)} \) having the edges of \( G \) as its vertices. This way we can designate the 3-path "\( ijk \)" describing the transition \( i \rightarrow j \rightarrow k \), as an edge from \( ij \) \( jk \). We label this edge from vertex \( ij \) \( G^{(2)} \) to the vertex \( jk \) \( G^{(2)} \) as the edge \( ij(j)k \equiv "ijk" \) of \( G^{(2)} \). One can continue in this way produce higher order graphs from the original graph. The study of the growth of block entropy of the original sequence, i.e. the proliferation law of words of length \( n \) can then be performed by studying the growth of the associated higher order graphs of \( G \). The same argument goes for the Markov chains associated with \( G \) and its associated measures, if we are interested not in the possible transitions but rather in the probable ones. Our work opens therefore the perspective that a link between the concepts of block entropy and network growth. The key observation here is that by a combination of graphs, the necessary increase of the number of non admissible words, for example via suitable "super-selection rules" as in [Nicolis, 2005], is achieved to such a degree that leads to the establishment of a stretched exponential growth rates. This sub-exponential growth is not due to a limited sample or system size but can be thought of as integral part of the topological description of the system. The presence of specific constraints, super-selection rules or parameter forcing might play exactly this role in a more realistic modeling of chaotic or stochastic systems.
Acknowledgements

We thank J.S. Nicolis, J.P. Boon and A. Garcia-Cantù for fruitful discussions. G.L.F. is supported by COFIN–MIUR, V.B. thanks ESF–EPEAEK II and particularly the Program PYTHAGORAS II, for funding the above work.

References

[Applebaum, 1996] Applebaum, D. [1996] "Information and Probability: an Integrated Approach", Cambridge University Press, Cambridge, UK.

[Basios, 2001] Basios, V. [2001] "Symbolic Dynamics and Control of Complex, Chaotic systems: a Probabilistic Approach", PhD Thesis, Centre for nonlinear phenomena and complex systems, Université Libre de Bruxelles, 2001.

[Berthe, 1994] Berthé, V. [1994] "Conditional entropy of some automatic sequences", J. Phys. A: Math; Gen., 27, 7993 – 8006.

[Casaigne, 1997] Casaigne, J. [1997] Complexité et facteurs spéciaux. Bulletin of the Belgian Mathematical Society, 4, 67–68.

[Dekking, 1982] Dekking, F. M. [1982] "Recurrent sets" Advances in Mathematics, 44, 78–104.

[Dekking, 1992] Dekking, F. M. [1992] "On the Thue–Morse measure", Acta Universitatis Carolinae - Mathematica et Physica, 33, 35–40.

[Douglas & Markus, 1995] Douglas, L. and Marcus, B. [1995] "An Introduction to Symbolic Dynamics and Coding", Cambridge University Press, Cambridge, UK.

[Ebeling & Nicolic, 1992] Ebeling, W. and Nicolis, G. [1992] "World frequency and entropy of symbolic sequences : a dynamical perspective", Chaos, Solitons and fractals, 2, 635–650.

[Gaspard, 1998] Gaspard, P. [1998] "Chaos, Scattering and Statistical Mechanics", Cambridge University Press, Cambridge, UK.

[Hao, 1990] Hao, B.-L. (editor), [1990] "Chaos II: a reprint selection", Singapore, World Scientific.
[Kitchens, 1998] Kitchens, B.P. [1998] "Symbolic Dynamics: One-sided, Two-sided and Countable State Markov Shifts", Springer Verlag, Berlin.

[Nicolis, 1995] Nicolis, G. [1995] "Introduction to Nonlinear Science", Cambridge University Press, Cambridge, UK.

[Nicolis et al, 2005] Nicolis, G. Garcia, A. and Nicolis, C. [2005] "Dynamical aspects of interaction networks", International Journal of Bifurcation and Chaos, 15, 3467–3480.

[Nicolis, 2005] Nicolis, J.S. [2005] "Super-selection rules modulating complexity: an overview", Chaos, Solitons and Fractals, 24, 1159–1163.

[Queffelec, 1987] Queffélec, M. [1987] "Substitution Dynamical Systems - Spectral Analysis", Springer Verlag Lecture Notes in Mathematics No 1294, NY.

[Schurman, 2002] Schurmann, T. [2002] "Scaling behaviour of entropy estimates", Journal of Physics A: Math. Gen., 35, 1589–1596.