ON \textit{n}-PUNCTURED BALL TANGLES

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\textbf{Abstract.} We consider a class of topological objects in the 3-sphere \(S^3\) which will be called \textit{n}-punctured ball tangles. Using the Kauffman bracket at \(A = e^{i\pi/4}\), an invariant for a special type of \(n\)-punctured ball tangles is defined. The invariant \(F\) takes values in \(PM_{2 \times 2^n}(\mathbb{Z})\), that is the set of \(2 \times 2^n\) matrices over \(\mathbb{Z}\) modulo the scalar multiplication of \(\pm 1\). This invariant leads to a generalization of a theorem of D. Krebes which gives a necessary condition for a given collection of tangles to be embedded in a link in \(S^3\) disjointly. We also address the question of whether the invariant \(F\) is surjective onto \(PM_{2 \times 2^n}(\mathbb{Z})\). We will show that the invariant \(F\) is surjective when \(n = 0\). When \(n = 1\), \(n\)-punctured ball tangles will also be called \textit{spherical tangles}. We show that \(\det F(S) \equiv 0 \text{ or } 1 \mod 4\) for every spherical tangle \(S\). Thus, \(F\) is not surjective when \(n = 1\).

1. Introduction

In this paper, we will work in either the smooth or the piecewise linear category. For basic terminologies of knot theory, see [1, 2].

The notion of \textit{tangles} was introduced by J. Conway [3] as the basic building blocks of links in the 3-dimensional sphere \(S^3\). Slightly abusing the notation, a tangle \(T\) is a pair \((B^3, T)\), where \(B^3\) is a 3-dimensional ball and \(T\) is a proper 1-dimensional submanifold of \(B^3\) with 2 non-circular components. The points in \(\partial T \subset \partial B^3\) will be fixed once and for all. Recall that a link \(L\) is a submanifold of \(S^3\) homeomorphic to a disjoint union of several copies of the circle \(S^1\). A tangle \(T = (B^3, T)\) can be embedded in a link \(L\) in \(S^3\) if there is an embedding \(\phi : B^3 \rightarrow S^3\) such that \(\phi(B^3) \cap L = \phi(T)\). Using the Kauffman bracket at \(A = e^{i\pi/4}\), a necessary condition that one can embed a tangle \(T\) in a link \(L\) is given by D. Krebes in [4].

One of the purposes of this paper is to give a generalization to Krebes’ theorem. Suppose that \(k\) tangles \(T_i = (B^3, T_i)\), \(i = 1, 2, \ldots, k\), are given. They can be embedded disjointly in a link \(L\) if there are embeddings \(\phi_i : B^3 \rightarrow S^3\) such that \(\phi_i(B^3) \cap L = \phi_i(T_i)\) for all \(i\) and \(\phi_i(B^3) \cap \phi_j(B^3) = \emptyset\) for all \(i, j\) with \(i \neq j\). A necessary condition similar to that of Krebes’ will be given for the existence of such a disjoint embedding of tangles in a link (see Theorem 3.7).

In order to prove this generalization of Krebes’ theorem, we will study a class of topological objects in \(S^3\) called \textit{n}-punctured ball tangles. This class of topological

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objects has rich contents in the theory of operads \[6\], which is beyond the scope of this paper. Our main interest lies in a special type of \(n\)-punctured ball tangles, which in the case of \(n = 0\), corresponds exactly to Conway’s notion of tangles in the 3-ball \(B^3\). Using the Kauffman bracket at \(A = e^{i\pi/4}\), we will define an invariant for this special type of \(n\)-punctured ball tangles. For an \(n\)-punctured ball tangle \(T\), this invariant \(F(T)\) is an element in \(PM_{2\times 2^n}(\mathbb{Z})\), that is the set of \(2 \times 2^n\) matrices over \(\mathbb{Z}\) modulo the scalar multiplication of \(\pm 1\). When \(n = 0\), \(F(T)\) is Krebes’ invariant.

Suppose now that we have \(k\) tangles \(T_i\), \(i = 1, 2, \ldots, k\), embedded disjointly in a link \(L\). Let

\[
F(T_i) = \begin{bmatrix} p_i \\ q_i \end{bmatrix}, \quad i = 1, 2, \ldots, k,
\]

and let \(\langle L \rangle\) be the Kauffman bracket of \(L\) at \(A = e^{i\pi/4}\). Then Theorem 3.7 says that

\[
\prod_{i=1}^{k} \text{g.c.d.} (p_i, q_i)
\]

divides \(|\langle L \rangle|\). When \(k = 1\), this is exactly Krebes’ theorem.

The proof of Theorem 3.7 is based on the fact that the invariant \(F\) behaves well under the operadic composition of \(n\)-punctured ball tangles.

In the second part of this paper, we study the invariant \(F\) in some more details when \(n = 1\). In this case, \(n\)-punctured ball tangles are called spherical tangles. For a given spherical tangle \(S\), \(\det F(S)\) is a well-defined integer. Using a theorem of S. Matveev, H. Murakami and Y. Nakanishi in \[5, 7\], we will show that \(\det F(S)\) is either 0 or 1 modulo 4 (Theorem 4.29). Thus, not every element in \(PM_{2\times 2}(\mathbb{Z})\) can be realized as \(F(S)\) for some spherical tangle \(S\). This is in contrary with the case of \(n = 0\), where the invariant \(F\) is onto.

We organize the paper as follows: In Section 2, we formally define the notion of \(n\)-punctured ball tangles. We also recall the Kauffman bracket at \(A = e^{i\pi/4}\) and Krebes’ theorem in this section. In Section 3, we define our invariant \(F\) for a special class of \(n\)-punctured ball tangles. A key result is about the behave of the invariant \(F\) under operadic composition of \(n\)-punctured ball tangles (Theorems 3.6). Our generalization of Krebes’ theorem (Theorem 3.7) will follow easily from this result. Finally, in Section 4, we study the surjectivity of the invariant \(F\) in the case of \(n = 0, 1\). As mentioned before, we will show that \(F\) is surjective when \(n = 0\) but not surjective when \(n = 1\). In the final section, we pose some questions related with this work which we do not know how to answer at this moment.

Notice that D. Ruberman has given a topological interpretation of Krebes’ theorem \[8\]. We don’t know if our generalization of Krebes’ theorem could have a similar topological interpretation. In particular, it will be very nice if there is a topological interpretation of the restriction on \(\det F(S)\) for spherical tangles \(S\) (Theorem 4.29).
2. General definitions

2.1. $n$-punctured ball tangles. We define a topological object in the 3-dimensional sphere $S^3$ called an $n$-punctured ball tangle or, simply, an $n$-tangle. To study this object, we consider a model for a class of objects and an equivalence relation on it.

**Definition 2.1.** Let $n$ be a nonnegative integer, and let $H_0$ be a 3-dimensional closed ball, and let $H_1, \ldots, H_n$ be pairwise disjoint 3-dimensional closed balls contained in the interior $\operatorname{Int}(H_0)$ of $H_0$. For each $k \in \{0, 1, \ldots, n\}$, take $2m_k$ distinct points $a_{k1}, \ldots, a_{k2m_k}$ of $\partial H_k$ for some positive integer $m_k$. Then a 1-dimensional proper submanifold $T$ of $H_0 - \bigcup_{i=1}^{n} \operatorname{Int}(H_i)$ is called an $n$-punctured ball tangle with respect to $\{H_k\}_{0 \leq k \leq n}$ if $\partial T = \bigcup_{k=0}^{n} \{a_{k1}, \ldots, a_{k2m_k}\}$. Hence, $\partial T \cap \partial H_k = \{a_{k1}, \ldots, a_{k2m_k}\}$ for each $k \in \{0, 1, \ldots, n\}$. Note that an $n$-tangle $T$ with respect to $\{H_k\}_{0 \leq k \leq n}$, $\{m_k\}_{0 \leq k \leq n}$, and $\{(a_{k1}, \ldots, a_{k2m_k})\}_{0 \leq k \leq n}$ can be regarded as a 5-tuple $(n, \{H_k\}_{0 \leq k \leq n}, \{m_k\}_{0 \leq k \leq n}, \{(a_{k1}, \ldots, a_{k2m_k})\}_{0 \leq k \leq n}, T)$.

**Proposition 2.2.** Let $n \in \mathbb{N} \cup \{0\}$, and let $\mathbf{nPBT}$ be the class of all $n$-punctured ball tangles with respect to $\{H_k\}_{0 \leq k \leq n}$, $\{m_k\}_{0 \leq k \leq n}$, and $\{(a_{k1}, \ldots, a_{k2m_k})\}_{0 \leq k \leq n}$, and let $X = H_0 - \bigcup_{i=1}^{n} \operatorname{Int}(H_i)$. Define $\cong$ on $\mathbf{nPBT}$ by $T_1 \cong T_2$ if and only if there is a homeomorphism $h : X \to X$ such that $h|_{\partial X} = Id_X|_{\partial X}$, $h(T_1) = T_2$, and $h$ is isotopic to $Id_X$ relative to the boundary $\partial X$ for all $T_1, T_2 \in \mathbf{nPBT}$. Then $\cong$ is an equivalence relation on $\mathbf{nPBT}$, where $Id_X$ is the identity map from $X$ to $X$.

**Proof.** Note that $T_1 \cong T_2$ if and only if there are a homeomorphism $h : X \to X$ with $h|_{\partial X} = Id_X|_{\partial X}$ and $h(T_1) = T_2$ and a continuous function $H : X \times I \to X$ such that $H(\_ , t) : X \to X$ is a homeomorphism with $H(\_ , t)|_{\partial X} = Id_X|_{\partial X}$ for each $t \in I$ and $H(\_ , 0) = Id_X$ and $H(\_ , 1) = h$, where $I = [0, 1]$. Let us denote $H(x, t)$ by $H_t(x)$ for all $x \in X$ and $t \in I$, so $H(t, \_ ) = H_t$ for each $t \in I$.

For every $T \in \mathbf{nPBT}$, $T \cong T$ since $Id_X$ and the 1st projection $\pi_1 : X \times I \to X$ satisfy the condition. Suppose that $T_1 \cong T_2$ and $h : X \to X$ is a homeomorphism with $h|_{\partial X} = Id_X|_{\partial X}$ such that $h(T_1) = T_2$ and $H : X \times I \to X$ is a continuous function such that $H_t : X \to X$ is a homeomorphism with $H_t|_{\partial X} = Id_X|_{\partial X}$ for each $t \in I$ and $H_0 = Id_X$ and $H_1 = h$. Define $H' : X \times I \to X$ by $H'(x, t) = H_t^{-1}(x)$ for all $x \in X$ and $t \in I$. Then $h^{-1}$ and $H'$ make $T_2 \cong T_1$. To show the transitivity of $\cong$, suppose that $T_1 \cong T_2$ and $h : X \to X$ is a homeomorphism with $h|_{\partial X} = Id_X|_{\partial X}$ such that $h(T_1) = T_2$ and $H : X \times I \to X$ is a continuous function such that $H_t : X \to X$ is a homeomorphism with $H_t|_{\partial X} = Id_X|_{\partial X}$ for each $t \in I$ and $H_0 = Id_X$ and $H_1 = h$ and $T_2 \cong T_3$ and $h' : X \to X$ is a homeomorphism with $h'|_{\partial X} = Id_X|_{\partial X}$ such that $h'(T_2) = T_3$ and $H' : X \times I \to X$ is a continuous function such that $H'_t : X \to X$ is a homeomorphism with $H'_t|_{\partial X} = Id_X|_{\partial X}$ for each $t \in I$ and $H'_0 = Id_X$ and $H'_1 = h'$. Define $H'' : X \times I \to X$ by $H''(x, t) = (H_t \circ H_t)(x)$ for all $x \in X$ and $t \in I$. Then $h' \circ h$ and $H''$ make $T_1 \cong T_3$. Therefore, $\cong$ is an equivalence relation on $\mathbf{nPBT}$. $\square$
Definition 2.3. Let $T_1$ and $T_2$ be $n$-punctured ball tangles in $nPBT$. Then $T_1$ and $T_2$ are said to be equivalent or of the same isotopy type if $T_1 \cong T_2$. Also, for each $n$-punctured ball tangle $T$ in $nPBT$, the equivalence class of $T$ with respect to $\cong$ is denoted by $[T]$. By the context, without any confusion, we will also use $T$ for $[T]$.

There are many models for a class of $n$-punctured ball tangles. It is convenient to use normalized ones. One model for a class of $n$-punctured ball tangles is as follows:

1. $H_0 = B((\frac{n+1}{2},0,0), \frac{n+1}{2})$ and $H_i = B((i,0,0), \frac{1}{3})$ for each $i \in \{1, \ldots , n\}$. Here $B((x,y,z), r)$ is the 3-dimensional ball in $\mathbb{R}^3$ with center $(x,y,z)$ and radius $r$.

2. $a_{k1}, \ldots , a_{k2m_k}$ are $2m_k$ distinct points of $\partial H_k$ in the $xy$-plane for each $k \in \{0,1, \ldots , n\}$.

3. $T$ is a 1-dimensional proper submanifold of $H_0 - \bigcup_{i=1}^{n} \text{Int}(H_i)$ such that $\partial T = \bigcup_{k=0}^{n} \{a_{k1}, \ldots , a_{k2m_k}\}$.

In order to study an $n$-punctured ball tangle $T$ through its diagram $D$, we consider the $xy$-projection $P_{xy} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $P_{xy}(x,y,z) = (x,y,0)$ for all $x,y,z \in \mathbb{R}$. A point $p$ of the image $P_{xy}(T)$ is called a multiple point of $T$ if the cardinality of $P_{xy}^{-1}(p) \cap T$ is greater than 1. In particular, $p$ is called a double point of $T$ if the cardinality of $P_{xy}^{-1}(p) \cap T$ is 2. If $p$ is a double point of $T$, then $P_{xy}^{-1}(p) \cap T$ is called the crossing of $T$ corresponding to $p$ and the point in the crossing whose $z$-coordinate is greater is called the overcrossing of $T$ corresponding to $p$ and the other is called the undercrossing.

An $n$-punctured ball tangle $T$ is said to be in regular position if the only multiple points of $T$ are double points and each double point of $T$ is a transversal intersection of the images of two arcs of $T$ and $P_{xy}(T - \partial T) \cap (\partial P_{xy}(H_0)) \cup \bigcup_{i=1}^{n} P_{xy}(H_i)) = \emptyset$.

Note that for each $T \in nPBT$, there is $T' \in nPBT$ such that $T'$ is in regular position and $T' \cong T$. Furthermore, $T'$ has a finite number of crossings.

Consider the image $P_{xy}(T)$ of an $n$-punctured ball tangle $T$ in regular position. For each double point of $T$, take a sufficiently small closed ball centered at the double point such that the intersection of $P_{xy}(T)$ and the closed ball is an X-shape on the $xy$-plane. We may assume that the closed balls are pairwise disjoint. Now, modify the interiors of the closed balls keeping the image $P_{xy}(T)$ to assign crossings corresponding to the crossings of $T$. As a result, we have a representative $D$ of $T$ which is ‘almost planar’ and $P_{xy}(D) = P_{xy}(T)$. $D$ is called a diagram of $T$ and we usually use this representative.

To deal with diagrams of $n$-punctured ball tangles in the same isotopy type, we need Reidemeister moves among them. For link diagrams or ball tangle diagrams, we have 3 kinds of Reidemeister moves. However, we need one and only one more kind of moves which are called the Reidemeister moves of type IV.
The Reidemeister moves for diagrams of $n$-punctured ball tangles are illustrated in the following figure.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{reidemeister_moves.png}
\caption{Tangle Reidemeister moves.}
\end{figure}

Like link diagrams, tangle diagrams also have Reidemeister Theorem involving the Reidemeister moves of type IV. Let us call Reidemeister moves including type IV Tangle Reidemeister moves.

**Theorem 2.4.** Let $n$ be a nonnegative integer, and let $D_1$ and $D_2$ be diagrams of $n$-punctured ball tangles. Then $D_1 \cong D_2$ if and only if $D_2$ can be obtained from $D_1$ by a finite sequence of Tangle Reidemeister moves.
Remark that, even though we may have different models for $n$-punctured ball tangles, we may regard them as the same $n$-punctured ball tangle if there are suitable model equivalences among them.

2.2. Kauffman bracket at $A = e^{i\pi/4}$ and monocyclic state of link diagram.

Our invariant is based on the Kauffman bracket at $A = e^{i\pi/4}$. In this section, we recall the Kauffman bracket which is a regular isotopy invariant of link diagrams. That is, it will not be changed under Reidemeister moves of type II and III.

Assume that $L$ is a link diagram with $n$ crossings and $c$ is a crossing of $L$. Take a sufficiently small disk at the projection of $c$ to get an X-shape on the projection plane of $L$. Now, we have 4 regions in the disk. Rotate counterclockwise the projection of the over-strand in the disk which is an arc of $L$ containing the overcrossing for $c$ to pass over 2 regions. These 2 regions and the other 2 regions are called the $A$-regions and the $B$-regions of $c$, respectively. We consider 2 ways of splitting the double point in the disk. $A$-type splitting is to open a channel between the $A$-regions so that we have 1 $A$-region and 2 $B$-regions in the disk and $B$-type splitting is to open a channel between the $B$-regions so that we have 2 $A$-regions and 1 $B$-region in the disk. A choice of how to destroy all of $n$ double points in the projection of $L$ by $A$-type or $B$-type splitting is called a state of $L$.

Notice that we regard a state $\sigma$ of the link diagram $L$ with $n$ crossings as a function $\sigma : \{c_1, \ldots, c_n\} \to \{A, B\}$, where $\{c_1, \ldots, c_n\}$ is the set of all crossings of $L$ and $\{A, B\}$ is the set of $A$-type and $B$-type splitting functions, respectively. Therefore, a link diagram $L$ with $n$ crossings has exactly $2^n$ states of it. Apply a state $\sigma$ to $L$ in order to change $L$ to a diagram $L_\sigma$ without any crossing.

**Definition 2.5.** Let $L$ be a link diagram. Then the Kauffman bracket $\langle L \rangle_A$, or simply, $\langle L \rangle$, is defined by

$$\langle L \rangle_A = \sum_{\sigma \in S} A^{\alpha(\sigma)} (A^{-1})^{\beta(\sigma)} (-A^2 - A^{-2})^{d(\sigma)-1},$$

where $S$ is the set of all states of $L$, $\alpha(\sigma) = |\sigma^{-1}(A)|$, $\beta(\sigma) = |\sigma^{-1}(B)|$, and $d(\sigma)$ is the number of circles in $L_\sigma$.

We have the following *skein relation of the Kauffman bracket*.

**Proposition 2.6.** Let $L$ be a link diagram, and let $c$ be a crossing of $L$. Then if $L_A$ and $L_B$ are link diagrams obtained from $L$ by $A$-type splitting and $B$-type splitting only at $c$, respectively, then $\langle L \rangle = A \langle L_A \rangle + A^{-1} \langle L_B \rangle$.

The following lemma is useful in our discussion of the Kauffman bracket.

**Lemma 2.7.** Let $L$ be a link diagram. Then states $\sigma$ and $\sigma'$ of $L$ are of the same parity, i.e., $d(\sigma) \equiv d(\sigma') \mod 2$, if and only if $\sigma$ and $\sigma'$ differ at an even number of crossings, where $d(\sigma)$ and $d(\sigma')$ are the numbers of circles in $L_\sigma$ and $L_{\sigma'}$, respectively.
Proof. Let $\sigma$ be a state of a link diagram $L$ with $n$ crossings $c_1, \ldots, c_n$. Change the value of $\sigma$ at only one crossing $c_i$ to get another state $\sigma_i$ and observe what happens to $d(\sigma_i)$, where $1 \leq i \leq n$. We claim that $\sigma$ and $\sigma_i$ have different parities, more precisely, $d(\sigma) = d(\sigma_i) \equiv 1 \mod 2$. Hence, we will have $d(\sigma) \equiv d(\sigma_i) + 1 \mod 2$. Now, to consider $\sigma(c_i)$ and $\sigma_i(c_i)$, take a sufficiently small neighborhood $B_i$ at the projection of $c_i$ so that the intersection of $\text{Int}(B_i)$ and the set of all double points of $L$ is the projection of $c_i$ and the intersection of $\partial B_i$ and the projection of $L$ has exactly 4 points on the projection plane of $L$ which are not double points of $L$.

Case 1. If these 4 points are on a circle in $L_\sigma$, then

$$d(\sigma_i) = d(\sigma) + 1.$$ 

Case 2. If two of 4 points are on a circle and the other points are on another circle in $L_\sigma$, then

$$d(\sigma_i) = d(\sigma) - 1.$$ 

Now, it is easy to show the lemma. Suppose that $\sigma$ and $\sigma'$ are states of $L$ which differ at $k$ crossings of $L$ for some $k \in \{0, 1, \ldots, n\}$. Then $d(\sigma') \equiv d(\sigma) + k \mod 2$. If $d(\sigma) \equiv d(\sigma') \mod 2$, then $k$ is even. Conversely, if $d(\sigma) \equiv d(\sigma') + 1 \mod 2$, then $k + 1$ is even, that is, $k$ is odd. This proves the lemma. \hfill $\Box$

Following [4], a state $\sigma$ of a link diagram $L$ is called a monocyclic state of $L$ if $d(\sigma) = 1$. That is, we have only one circle when we remove all crossings of $L$ by $\sigma$.

From now on, we consider only the Kauffman brackets at $A = e^{i\pi/4}$. Since $|A| = 1$, the determinant $|\langle L \rangle|$ of $L$ is an isotopy invariant. It is easy to show that Reidemeister move of type I dose not change $|\langle L \rangle|$ by the skein relation of Kauffman bracket and $|A| = 1$.

Notice that $-A^2 - A^{-2} = 0$ if $A = e^{i\pi/4}$. Therefore,

$$\langle L \rangle = \sum_{\sigma \in M} A^{\alpha(\sigma) - \beta(\sigma)},$$

where $M$ is the set of all monocyclic states of $L$.

As a corollary of Lemma 2.7, monocyclic states $\sigma$ and $\sigma'$ of $L$ differ at an even number of crossings.

Lemma 2.8. If $L$ is a link diagram, then there are $p \in \mathbb{Z}$ and $u \in \mathbb{C}$ such that $u^8 = 1$ and $\langle L \rangle = pu$.

Proof. Suppose that $\sigma$ and $\sigma'$ are states of a link diagram $L$ such that $\sigma$ and $\sigma'$ differ at only one crossing. Then either $\alpha(\sigma') - \beta(\sigma') = (\alpha(\sigma) + 1) - (\beta(\sigma) - 1) = (\alpha(\sigma) - \beta(\sigma)) + 2$ or $\alpha(\sigma') - \beta(\sigma') = (\alpha(\sigma) - 1) - (\beta(\sigma) + 1) = (\alpha(\sigma) - \beta(\sigma)) - 2$. Hence, either $A^{\alpha(\sigma') - \beta(\sigma')} = +iA^{\alpha(\sigma) - \beta(\sigma)}$ or $A^{\alpha(\sigma') - \beta(\sigma')} = -iA^{\alpha(\sigma) - \beta(\sigma)}$. That is, $A^{\alpha(\sigma') - \beta(\sigma')} = \pm iA^{\alpha(\sigma) - \beta(\sigma)}$. 

ON n-PUNCTURED BALL TANGLES
If $0 \leq k \leq c(L)$ and $\sigma''$ is a state of $L$ such that $\sigma$ and $\sigma''$ differ at $k$ crossings, where $c(L)$ is the number of crossings of $L$, then $A^\alpha(\sigma'') - \beta(\sigma'') = \pm i^k A^\alpha(\sigma) - \beta(\sigma)$ because there are exactly $2^k$ sequences with $k$ terms consisting of $+i$ and $-i$ and the product of all terms of each of the sequences is either $+i^k$ or $-i^k$. Hence, $A^\alpha(\sigma'') - \beta(\sigma'') = \pm iA^\alpha(\sigma) - \beta(\sigma)$ if $k$ is odd and $A^\alpha(\sigma'') - \beta(\sigma'') = \pm A^\alpha(\sigma) - \beta(\sigma)$ if $k$ is even.

Now, let us take a monocyclic state $\sigma_0$ of $L$, and let $u = A^\alpha(\sigma_0) - \beta(\sigma_0)$. Then $u^8 = 1$ and $\langle L \rangle = pu$ for some $p \in \mathbb{Z}$ by the corollary above. This proves the lemma. \hfill \Box

2.3. Krebes’ Theorem. In this subsection, we introduce some notations and Krebes’ Theorem [4].

Definition 2.9. A ball tangle $B$, which is a 0-punctured ball tangle with $m_0 = 2$, is said to be embedded in a link $L$ if there are a diagram $D_B$ of $B$, a diagram $D_L$ of $L$, and a 3-dimensional closed ball $H$ such that $H \cap D_L$ and $D_B$ are of the same isotopy type.

Given a ball tangle diagram $B$, we consider 3 kinds of closures as in Figure 2 a).

Figure 2. a) Closures, b) Diagrams by a numerator state and a denominator state.
The link diagrams $B_1$ and $B_2$ are called the numerator closure and the denominator closure of $B$, respectively. A monocyclic state of $B_1$ is called a numerator state of $B$ and that of $B_2$ is a denominator state of $B$.

Notice that a numerator state $\sigma$ and a denominator state $\sigma'$ of a ball tangle diagram $B$ differ at an odd number of crossings. To see this, we think of a diagram of another closure $L$ of $B$ which has only one more crossing $c$ at the outside of ball containing $B$ (See $L$ in Figure 2). We have two monocyclic states of $L$ from the numerator state $\sigma$ and the denominator state $\sigma'$, respectively, which differ at $c$. Hence, $\sigma$ and $\sigma'$ differ at an odd number of crossings.

The following notations throughout the rest of the paper:

- $\Phi = \{ z \in \mathbb{C} \mid z^8 = 1 \} = \{ A^k \mid k \in \mathbb{Z} \}$, i.e. $\Phi$ is the set of 8-th roots of unity; and $\mathbb{Z}\Phi = \{ k\mathbb{z} \mid k \in \mathbb{Z}, z \in \Phi \}$.
- $M_{n \times m}(\mathbb{Z})$ is the set of all $n \times m$ matrices over $\mathbb{Z}$, and $PM_{n \times m}(\mathbb{Z})$ is the quotient of $M_{n \times m}(\mathbb{Z})$ under the scalar multiplication by $\pm 1$.
- $BT$ is the class of diagrams of 0-punctured ball tangles with $m_0 = 2$ (i.e. ball tangles).
- $ST$ is the class of diagrams of 1-punctured ball tangles with $m_0 = m_1 = 2$ (they will be called spherical tangles).

**Proposition 2.10.** If $a, b, k, l \in \mathbb{Z}$, then $aA^k + bA^l \in \mathbb{Z}\Phi$ if and only if $ab = 0$ or $k \equiv l \mod 4$.

**Proof.** Suppose that $ab \neq 0$ and $k - l \equiv t \mod 4$ for some $t \in \{1, 2, 3\}$. then $k - l = 4m + t$ for some $m \in \mathbb{Z}$, hence, $A^k = (-1)^m A^t A^l$. Therefore, $aA^k + bA^l$ is not in $\mathbb{Z}\Phi$. Conversely, if $ab = 0$ or $k \equiv l \mod 4$, then $aA^k + bA^l \in \mathbb{Z}\Phi$. \hfill \qed

We have

$$\langle L \rangle = A\langle B_1 \rangle + A^{-1}\langle B_2 \rangle \in \mathbb{Z}\Phi$$

for the links $L$, $B_1$, and $B_2$ in Figure 2. If $\langle B_1 \rangle = pA^k$ and $\langle B_2 \rangle = qA^l$, by Proposition 2.10, we have $l \equiv k + 2 \mod 4$. So there is a unique $(\alpha, \beta) \in \mathbb{Z}^2$ such that

$$\left\{ \left( \frac{z\langle B_1 \rangle}{i\bar{z}\langle B_2 \rangle} \right) \mid z \in \Phi \right\} \cap M_{2 \times 1}(\mathbb{Z}) = \left\{ \left( \begin{array}{c} \alpha \\ \beta \end{array} \right), \left( \begin{array}{c} -\alpha \\ -\beta \end{array} \right) \right\} := \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \in PM_{2 \times 1}(\mathbb{Z}).$$

**Definition 2.11.** Define $f : BT \to PM_{2 \times 1}(\mathbb{Z})$ by

$$f(B) = \left\{ \left( \frac{z\langle B_1 \rangle}{i\bar{z}\langle B_2 \rangle} \right) \mid z \in \Phi \right\} \cap M_{2 \times 1}(\mathbb{Z}) \in PM_{2 \times 1}(\mathbb{Z})$$

for each $B \in BT$. This is Krebes’ tangle invariant.

Notice that Reidemeister move of type I does not change $f(B)$. So $f(B)$ is a ball tangle invariant. The following lemmas about the ball tangle invariant $f$ are proved in [3].
Lemma 2.12. If $B^{(1)}$ and $B^{(2)}$ are diagrams of ball tangles with $f(B^{(1)}) = \begin{bmatrix} p \\ q \end{bmatrix}$ and $f(B^{(2)}) = \begin{bmatrix} r \\ s \end{bmatrix}$, then $f(B^{(1)} +_h B^{(2)}) = \begin{bmatrix} ps + qr \\ qs \end{bmatrix}$, where $B^{(1)} +_h B^{(2)}$ stands for the horizontal addition of ball tangles (see Figure 8 a)).

Lemma 2.13. If $B$ is a diagram of ball tangle with $f(B) = \begin{bmatrix} p \\ q \end{bmatrix}$, then we have $f(B^*) = \begin{bmatrix} p \\ -q \end{bmatrix}$ and $f(B^R) = \begin{bmatrix} -q \\ -p \end{bmatrix}$, where $B^*$ is the mirror image of $B$ and $B^R$ is the $90^\circ$ counterclockwise rotation of $B$ on the projection plane.

Theorem 2.14. (Krebes [4]) If $L$ is a link and $B$ is a ball tangle embedded in $L$ with $f(B) = \begin{bmatrix} p \\ q \end{bmatrix}$, then $\gcd((p, q))$ divides $|\langle L \rangle|$.

3. The special case of $n$-punctured ball tangles

Let $n$ be a positive integer. Then an $n$-punctured ball tangle $T^n$ with $\{H_k\}_{0 \leq k \leq n}$ and $\{m_k\}_{0 \leq k \leq n}$ can be regarded as an $n$-variable function $T^n : A_1 \times \cdots \times A_n \to T$ defined as $T^n(X_1, \ldots, X_n)$ is a tangle filled up in the $i$-th hole $H_i$ of $T^n$ by $X_i \in A_i$ for each $i \in \{1, \ldots, n\}$, where $A_i$ is a class of $t_i$-punctured ball tangles with $\{m^i_k\}_{0 \leq k \leq t_i}$ such that $m_i = m^i_0$ for each $i \in \{1, \ldots, n\}$ and $T$ is a class of tangles. However, this representation of $n$-punctured ball tangles as $n$-variable functions is not perfect in the sense that $n$-punctured ball tangles are equivalent only if they induce the same function. $n$-punctured ball tangles which induce the same function need not be equivalent. That is, we can say that tangles are stronger than functions.

Roughly speaking, the class of $n$-punctured ball tangles as only $n$-variable functions gives us an operad, a mathematical device which describes algebraic structure of many varieties and in various categories. See [3].

3.1. $n$-punctured ball tangles with $m_k = 2$ for each $k \in \{0, 1, \ldots, n\}$ and their invariants $F^n$. From now on, we consider only $n$-punctured ball tangles with $m_0 = m_1 = \cdots = m_n = 2$.

To construct the invariant $F^n$ of $n$-punctured ball tangle $T^n$, let us regard $T^n$ as a ‘hole-filling function’, in sense described as above $T^n : BT_1 \times \cdots \times BT_n \to BT$, where $BT_1 = \cdots = BT_n = BT$.

To construct our invariant of $n$-punctured ball tangles with $m_0 = m_1 = \cdots = m_n = 2$, we need to use some quite complicated notations. Let us start with a gentle introduction to our notations:
(1) For a diagram of 0-punctured ball tangle $T^0$ (a ball tangle), we can produce 2 links $T_{1}^0$ and $T_{2}^0$, which are the numerator closure and the denominator closure of $T^0$, respectively.

(2) For a diagram of 1-punctured ball tangle $T^1$ (a spherical tangle), we can produce $2^{1+1}$ links $T_{1(1)}^1$, $T_{1(2)}^1$, $T_{2(1)}^1$, $T_{2(2)}^1$, where the subscript 1(1) means to take the numerator closure of $T$ with its hole filled by the fundamental tangle 1.

(3) For a diagram of 2-punctured ball tangle $T^2$, we can produce $2^{2+1}$ links $T_{1(11)}^2$, $T_{1(12)}^2$, $T_{1(21)}^2$, $T_{1(22)}^2$, $T_{2(11)}^2$, $T_{2(12)}^2$, $T_{2(21)}^2$, $T_{2(22)}^2$.

Figure 3. An $n$-punctured ball tangle with $m_0 = m_1 = \cdots = m_n = 2$.

Figure 4. A hole-filling function $T^n$, a) $T_{1\alpha_1}^n$, b) $T_{2\alpha_2}^n$, c) $T^n(B^{(1)}, \ldots, B^{(n)})$. 
If \( n \) is a positive integer, \( J_1 = \cdots = J_n = \{1,2\} \), and \( J(n) = \prod_{k=1}^n J_k \), then \( J(n) \) is linearly ordered by a dictionary order, or lexicographic order, consisting of \( 2^n \) ordered \( n \)-tuples each of whose components is either 1 or 2. That is, if \( x, y \in J(n) \) and \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \), then \( x < y \) if and only if \( x_1 < y_1 \) or there is \( k \in \{1, \ldots, n-1\} \) such that \( x_1 = y_1, \ldots, x_k = y_k, x_{k+1} < y_{k+1} \).

(4) \( J(n) = \{\alpha_i^n | 1 \leq i \leq 2^n\} \) and \( \alpha_1^n < \alpha_2^n < \cdots < \alpha_{2^n}^n \), where < is the dictionary order on \( J(n) \). Hence, \( \alpha_1^n \) is the least element \( (1,1,\ldots,1) \) and \( \alpha_{2^n}^n \) is the greatest element \( (2,2,\ldots,2) \) of \( J(n) \). Let us denote \( \alpha_i^n = (\alpha_{i1}^n, \ldots, \alpha_{in}^n) \) for each \( i \in \{1, \ldots, 2^n\} \).

(5) For a diagram of \( n \)-punctured ball tangle \( T^n \), we can produce \( 2^{n+1} \) links \( T_{1\alpha_1^n}; \ldots, T_{1\alpha_{2^n}^n}; T_{2\alpha_1^n}; \ldots, T_{2\alpha_{2^n}^n} \).

(6) The sequence \((a_n)_{n \geq 0} = ((t_k)_{1 \leq k \leq 2^n})_{n \geq 0}\) is defined recursively as follows:

1) \( a_0 = (0) \);

2) If \( a_{k-1} = (t_1, \ldots, t_{2k-1}) \), then \( a_k = (t_1, \ldots, t_{2k-1}, t_1 + 1, \ldots, t_{2k-1} + 1) \) for each \( k \in N \). Note that \( t_{2^n} = n \) for each \( n \in N \cup \{0\} \).

Now, we define our invariant of \( n \)-punctured ball tangles with \( m_0 = m_1 = \cdots = m_n = 2 \) inductively.

**Theorem 3.1.** For each \( n \in N \), define \( F^n: nPBT \to PM_{2 \times 2^n}(\mathbb{Z}) \) by

\[
F^n(T^n) = \left\{ \left. \left( -i \right)^{t_1} z \langle T_{1\alpha_1^n}^n \rangle \cdots \left( -i \right)^{t_{2^n}} z \langle T_{1\alpha_{2^n}^n}^n \rangle \right| z \in \Phi \right\} \cap M_{2 \times 2^n}(\mathbb{Z})
\]

for each \( T^n \in nPBT \). Then \( F^n \) is an isotopy invariant of \( n \)-punctured ball tangle diagrams. In particular, \( F^n \) is Krebes’ ball tangle invariant \( f \).

**Proof.** Let \( X(T^n) = \left( -i \right)^{t_1} z \langle T_{1\alpha_1^n}^n \rangle \cdots \left( -i \right)^{t_{2^n}} z \langle T_{1\alpha_{2^n}^n}^n \rangle \right\} \cap M_{2 \times 2^n}(\mathbb{Z}) \) consists of two elements differ by a scalar multiplication of \(-1\). By Lemma 2.8, for each \( k \in \{1, \ldots, 2^n\} \), there are \( p_k, q_k \in \mathbb{Z} \) and \( u_k, v_k \in \Phi \) such that \( \langle T_{1\alpha_k^n}^n \rangle = p_k u_k \) and \( \langle T_{2\alpha_k^n}^n \rangle = q_k v_k \). Notice that, for each \( k \in \{1, \ldots, 2^n\} \), \( \alpha_1^n \) and \( \alpha_2^n \) differ at only \( t_k \) coordinates. Hence, \( T_{1\alpha_1^n}^n \) and \( T_{2\alpha_1^n}^n \) differ at only \( t_k \) holes. If \( l, m \in \{1, \ldots, 2^n\} \) and \( T_{1\alpha_l^n}^n \) and \( T_{1\alpha_m^n}^n \) differ at only 1 hole, then \( u_m = \pm tu_l \) by Proposition 2.6, Lemma 2.8, and Proposition 2.10. Thus, \( u_k = \pm t^k u_1 \). Since \( v_k = \pm u_k \) for each \( k \in \{1, \ldots, 2^n\} \), \( X(T^n) \in u_1 M_{2 \times 2^n}(\mathbb{Z}) \). This shows that \( F^n(T^n) := \{ z X(T^n) | z \in \Phi \} \cap M_{2 \times 2^n}(\mathbb{Z}) = \{ u_1^{-1} X(T^n), -u_1^{-1} X(T^n) \} \).

Therefore, \( F^n(T_1) = F^n(T_2) \) if \( T_1 \) and \( T_2 \) are isotopic \( n \)-punctured ball tangle diagrams. \( \square \)
Definition 3.2. For each nonnegative integer \( n \), \( F^n \) is called the \( n \)-punctured ball tangle invariant, simply, the \( n \)-tangle invariant.

Now, in order to think of an \( n \)-punctured ball tangle \( T^n \) as a ‘hole-filling function’, we define a function which makes a dictionary order on complex numbers.

Let \( n \) be a positive integer, and let \((k_1, \ldots, k_n)\) be an \( n \)-tuple of positive integers, and let \( J(n, k_1, \ldots, k_n) = \prod_{i=1}^{n} I_{k_i} \). Then \( J(n, k_1, \ldots, k_n) \) is linearly ordered by a dictionary order, where \( I_k = \{1, \ldots, k\} \) for each \( k \in \mathbb{N} \).

\( J \) is ordered by \( \alpha_{i}^{n,k_1,\ldots,k_n} \) for each \( i \in \{1, \ldots, k_1 \cdot \ldots \cdot k_n\} \). Hence, \( J \) is well-defined and called the dictionary order function on \( \mathbb{C}^{k_1} \times \ldots \times \mathbb{C}^{k_n} \rightarrow \mathbb{C}^{k_1 \cdot \ldots \cdot k_n} \).

Definition 3.3. For each \( n \in \mathbb{N} \) and \( n \)-tuple \((k_1, \ldots, k_n)\) of positive integers, define \( \xi_{n,k_1,\ldots,k_n} : \mathbb{C}^{k_1} \times \ldots \times \mathbb{C}^{k_n} \rightarrow \mathbb{C}^{k_1 \cdot \ldots \cdot k_n} \) by

\[
\xi_{n,k_1,\ldots,k_n}((v_1^1, \ldots, v_1^{k_1}), \ldots, (v_n^1, \ldots, v_n^{k_n})) = \left( \prod_{j=1}^{n} v_{i,j}^{1, k_1, \ldots, k_n}, \ldots, \prod_{j=1}^{n} v_{i,j}^{n, k_1, \ldots, k_n} \right)
\]

for all \((v_1^1, \ldots, v_1^{k_1}) \in \mathbb{C}^{k_1}, \ldots, (v_n^1, \ldots, v_n^{k_n}) \in \mathbb{C}^{k_n} \). Then \( \xi_{n,k_1,\ldots,k_n} \) is well-defined and called the dictionary order function on \( \mathbb{C} \) with respect to \( k_1, \ldots, k_n \). Also, the \( i \)-th projection of \( \xi_{n,k_1,\ldots,k_n} \) is denoted by \( \xi_{i,n,k_1,\ldots,k_n} \) for each \( i \in \{1, \ldots, k_1 \cdot \ldots \cdot k_n\} \). In particular, we simply denote \( \xi_{n,k_1,\ldots,k_n} \) by \( \xi_n \) when \( k_1 = \ldots = k_n = 2 \).
Denote by $\mathbb{C}^{k\dagger}$ the $k$-dimensional column vector space over $\mathbb{C}$, so the map
\[
(v_1, \ldots, v_k) \mapsto (v_1, \ldots, v_k)^\dagger : \mathbb{C}^k \to \mathbb{C}^{k\dagger}
\]
is to transpose row vectors to column vectors. Let $P\mathbb{C}^{k\dagger} = \mathbb{C}^{k\dagger} / \pm 1$. If \((v_1, \ldots, v_k)^\dagger \in \mathbb{C}^{k\dagger}\), then we denote by
\[
[v_1, \ldots, v_k]^\dagger = \{(v_1, \ldots, v_k)^\dagger, (-v_1, \ldots, -v_k)^\dagger\}
\]
the corresponding element in $P\mathbb{C}^{k\dagger}$.

Remark that, we may extend the above notation to matrices modulo $\pm 1$. Under this extension, matrix multiplication is well-defined. That is, if $A$ and $B$ are matrices and $AB$ is defined, then $[A][B] = [A][-B] = [-A][B] = [-A][-B] = [AB]$.

**Lemma 3.4.** For each $n \in \mathbb{N}$ and $n$-tuple $(k_1, \ldots, k_n)$ of positive integers, define
\[
[\xi^{n,k_1,\ldots,k_n}] : P\mathbb{C}^{k_1\dagger} \times \cdots \times P\mathbb{C}^{k_n\dagger} \to P\mathbb{C}^{k_1\cdots k_n\dagger}
\]
by
\[
[\xi^{n,k_1,\ldots,k_n}](v_1^1, \ldots, v_1^n) = \left[ \prod_{j=1}^n v_j^{\alpha_{n,k_1,\ldots,k_n}} \right]
\]
for all $(v_1^1, \ldots, v_1^n) \in \mathbb{C}^{k_1}, \ldots, (v_1^n, \ldots, v_n^n) \in \mathbb{C}^{k_n}$. Then $[\xi^{n,k_1,\ldots,k_n}]$ is well-defined and called the dictionary order function induced by $\xi^{n,k_1,\ldots,k_n}$.

**Proof.** Suppose that $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ are in $\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n}$ such that\(([X_1]^\dagger, \ldots, [X_n]^\dagger) = ([Y_1]^\dagger, \ldots, [Y_n]^\dagger) \in P\mathbb{C}^{k_1\dagger} \times \cdots \times P\mathbb{C}^{k_n\dagger}\). Then $X_1 = \pm Y_1, \ldots, X_n = \pm Y_n$ and $\xi^{n,k_1,\ldots,k_n}(X_1, \ldots, X_n) = \pm \xi^{n,k_1,\ldots,k_n}(Y_1, \ldots, Y_n)$. Hence,
\[
[\xi^{n,k_1,\ldots,k_n}(X_1, \ldots, X_n)^\dagger] = [\xi^{n,k_1,\ldots,k_n}(Y_1, \ldots, Y_n)^\dagger] .
\]

Therefore, $[\xi^{n,k_1,\ldots,k_n}([X_1]^\dagger, \ldots, [X_n]^\dagger)] = [\xi^{n,k_1,\ldots,k_n}([Y_1]^\dagger, \ldots, [Y_n]^\dagger)]$. This shows that $[\xi^{n,k_1,\ldots,k_n}]$ is well-defined. \hfill $\Box$

As another notation, if $L$ is a link diagram and $T^n$ is a diagram of $n$-punctured ball tangle for some $n \in \mathbb{N} \cup \{0\}$, then the sets of all crossings of $L$ and $T^n$ are denoted by $c(L)$ and $c(T^n)$, respectively.

**Lemma 3.5.** If $n \in \mathbb{N}$ and $T^n$ is an $n$-punctured ball tangle diagram and $B^{(1)}, \ldots, B^{(n)}$ are ball tangle diagrams, then
\[
\begin{pmatrix}
\langle T^n(B^{(1)}, \ldots, B^{(n)}) \rangle_{11} \\
\langle T^n(B^{(1)}, \ldots, B^{(n)}) \rangle_{21} \\
\end{pmatrix}
= \left( \sum_{i=1}^{2^n} \langle T^n_{1\alpha_i^n} \rangle \langle B^{(1)}_{\alpha_i^n} \rangle \cdots \langle B^{(n)}_{\alpha_i^n} \rangle \right),
\]
\[
\begin{pmatrix}
\langle T^n(B^{(1)}, \ldots, B^{(n)}) \rangle_{12} \\
\langle T^n(B^{(1)}, \ldots, B^{(n)}) \rangle_{22} \\
\end{pmatrix}
= \left( \sum_{i=1}^{2^n} \langle T^n_{2\alpha_i^n} \rangle \langle B^{(1)}_{\alpha_i^n} \rangle \cdots \langle B^{(n)}_{\alpha_i^n} \rangle \right).
\]
Theorem 3.6. For each $F_0$.

Proof. We denote the set of all monocyclic states of a link diagram $L$ by $M(L)$. Let $B = T^n(B^{(1)}, \ldots, B^{(n)})$. Then $\sigma$ is a monocyclic state of $B_1$ if and only if there is a unique $i \in \{1, \ldots, 2^n\}$ such that $\sigma|_{c(T^n)} \in M(T^n_{\alpha_1^{(i)}}, \sigma|_{c(B^{(1)})} \in M(B^{(1)}_{\alpha_1^{(i)}}, \ldots, \sigma|_{c(B^{(n)})} \in M(B^{(n)}_{\alpha_1^{(i)}})$. Note that $c(B_1) = c(T^n) \prod c(B^{(1)}) \prod \cdots \prod c(B^{(n)})$. Let us denote a state $\sigma$ of $B_1$ by $\sigma_0 \sigma_1 \cdots \sigma_n$, where $\sigma_0 = \sigma|_{c(T^n)}$, $\sigma_1 = \sigma|_{c(B^{(1)})}, \ldots, \sigma_n = \sigma|_{c(B^{(n)})}$. Then

$$M(B_1) = \prod_{i=1}^{2^n} M(T^n_{\alpha_1^{(i)}} M(B^{(1)}_{\alpha_1^{(i)}} \cdots M(B^{(n)}_{\alpha_1^{(i)}})$$

and

$$\sum_{\sigma \in M(B_1)} A^{\alpha(\sigma) - \beta(\sigma)}$$

$$= \sum_{i=1}^{2^n} \sum_{\sigma_0 \sigma_1 \cdots \sigma_n \in M(T^n_{\alpha_1^{(i)}} M(B^{(1)}_{\alpha_1^{(i)}} \cdots M(B^{(n)}_{\alpha_1^{(i)}}) A^{\alpha(\sigma_0) + \alpha(\sigma_1) + \cdots + \alpha(\sigma_n) - \beta(\sigma_0) - \beta(\sigma_1) - \cdots - \beta(\sigma_n)}$$

$$= \sum_{i=1}^{2^n} \sum_{\sigma_0 \in M(T^n_{\alpha_1^{(i)}} A^{\alpha(\sigma_0) - \beta(\sigma_0)} \sum_{\sigma_1 \in M(B^{(1)}_{\alpha_1^{(i)}} A^{\alpha(\sigma_1) - \beta(\sigma_1)} \cdots \sum_{\sigma_n \in M(B^{(n)}_{\alpha_1^{(i)}} A^{\alpha(\sigma_n) - \beta(\sigma_n)}$$

$$= \sum_{i=1}^{2^n} \langle T^n_{\alpha_1^{(i)}} \rangle \langle B^{(1)}_{\alpha_1^{(i)}} \rangle \cdots \langle B^{(n)}_{\alpha_1^{(i)}} \rangle$$

Similarly, $\langle B_2 \rangle = \sum_{i=1}^{2^n} \langle T_{2\alpha_1^{(i)}} \rangle \langle B^{(1)}_{\alpha_1^{(i)}} \rangle \cdots \langle B^{(n)}_{\alpha_1^{(i)}} \rangle$. This proves the lemma. \qed

Theorem 3.6. For each $n \in \mathbb{N}$, $F_n$ is an $n$-punctured ball tangle invariant such that $F^0(T^n(B^{(1)}, \ldots, B^{(n)})) = F^n(T^n)[\xi^n](F^0(B^{(1)}), \ldots, F^0(B^{(n)}))$ for all $B^{(1)}, \ldots, B^{(n)} \in BT$.

Proof. Suppose that $T^n$ is an $n$-punctured ball tangle such that $F^n(T^n) = [zX(T^n)]$ for some $z \in \Phi$ and $B^{(1)}, \ldots, B^{(n)}$ are ball tangles such that

$$F^0(B^{(1)}) = \begin{bmatrix} z_1 \langle B^{(1)}_1 \rangle \\ i z_1 \langle B^{(1)}_2 \rangle \end{bmatrix}, \ldots, F^0(B^{(n)}) = \begin{bmatrix} z_n \langle B^{(1)}_1 \rangle \\ i z_n \langle B^{(1)}_2 \rangle \end{bmatrix}$$
for some $z_1, \ldots, z_n \in \Phi$, where $\langle B^{(i)} \rangle$ and $\langle B^{(i)}_2 \rangle$ are the numerator closure and the denominator closure of $B^{(i)}$, respectively, for each $i \in \{1, \ldots, n\}$. Then

$$F^n(T^n)[z^n](F^0(B^{(1)}), \ldots, F^0(B^{(n)}))$$

\[
= \begin{bmatrix}
(-i)^{t_1} z \langle T_{10}^n \rangle & \cdots & (-i)^{t_m} z \langle T_{10}^n \rangle \\
(-i)^{t_1} i z \langle T_{20}^n \rangle & \cdots & (-i)^{t_m} i z \langle T_{20}^n \rangle \\
\vdots & & \vdots \\
(-i)^{t_1} z_1 \cdots z_n \langle B^{(1)}_{\alpha_1} \rangle \cdots \langle B^{(n)}_{\alpha_n} \rangle & \cdots & (-i)^{t_1} z_1 \cdots z_n \langle B^{(1)}_{\alpha_1} \rangle \cdots \langle B^{(n)}_{\alpha_n} \rangle \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
zzz1 \cdots z_n (\langle T_{10}^n \rangle \langle B^{(1)}_{\alpha_1} \rangle \cdots \langle B^{(n)}_{\alpha_n} \rangle) + \cdots + \langle T_{10}^n \rangle \langle B^{(1)}_{\alpha_2} \rangle \cdots \langle B^{(n)}_{\alpha_n} \rangle \\
izzz1 \cdots z_n (\langle T_{20}^n \rangle \langle B^{(1)}_{\alpha_1} \rangle \cdots \langle B^{(n)}_{\alpha_n} \rangle) + \cdots + \langle T_{20}^n \rangle \langle B^{(1)}_{\alpha_2} \rangle \cdots \langle B^{(n)}_{\alpha_2} \rangle \\
zzz1 \cdots z_n (\sum_{i=1}^{2^n} T_{10}^n \langle B^{(1)}_{\alpha_1} \rangle \cdots \langle B^{(n)}_{\alpha_n} \rangle) \\
izzz1 \cdots z_n (\sum_{i=1}^{2^n} T_{20}^n \langle B^{(1)}_{\alpha_1} \rangle \cdots \langle B^{(n)}_{\alpha_n} \rangle) \\
\end{bmatrix}
\]

\[
= F^0(T^n(B^{(1)}, \ldots, B^{(n)}))
\]

by Lemma 3.5.

\[\square\]

3.2. **A Generalization of Krebes’ Theorem.** Suppose that a ball tangle diagram $B$ is embedded in a link diagram $L$. Since the complement of a ball in $S^3$ is still a ball, we may think of $B' = \overline{L - B}$ as another ball tangle diagram embedded in $L$ and $L = (B + h B')_1$, that is, $L$ is the numerator closure of horizontal addition of $B$ and $B'$. If $F^0(B) = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix}$ and $F^0(B') = \begin{bmatrix} p \\ q \end{bmatrix}$, then $F^0(B + h B') = \begin{bmatrix} p q + q p \\ q q \end{bmatrix}$ and $|\langle L \rangle| = |\langle (B + h B')_1 \rangle| = |p q + q p|$. Hence, g.c.d. $(p_i, q_i)$ divides $|\langle L \rangle|$. This is Krebes’ Theorem (see Theorem 2.14). We have the following generalization of Krebes’ theorem.

**Theorem 3.7.** Let $L$ be a link, and let $B^{(1)}, \ldots, B^{(n)}$ be ball tangles with the invariants $\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}, \ldots, \begin{bmatrix} p_n \\ q_n \end{bmatrix}$, respectively. If $B^{(1)}, \ldots, B^{(n)}$ are embedded in $L$ disjointly, then $\prod_{i=1}^{k} \text{g.c.d.} (p_i, q_i)$ divides $|\langle L \rangle|$.

**Proof.** Denote by $d_i = \text{g.c.d.} (p_i, q_i)$ for each $i \in \{1, \ldots, n\}$. Let $B^{(n)}' = \overline{L - B^{(n)}}$, and let $F^0(B^{(n)}') = \begin{bmatrix} p \\ q \end{bmatrix}$. Then $L = (B^{(n)} + h B^{(n)}')_1$ and $F^0(B^{(n)} + h B^{(n)}') = \begin{bmatrix} p q + q p \\ q q \end{bmatrix}$, hence, $|\langle L \rangle| = |\langle (B^{(n)} + h B^{(n)}')_1 \rangle| = |p q + q p|$. Notice that we can regard $B^{(n)}'$ as an $(n - 1)$-punctured ball tangle with its holes filled up by $B^{(1)}, \ldots, B^{(n-1)}$. Hence, $B^{(n)}' = T^{n-1}(B^{(1)}, \ldots, B^{(n-1)})$ for some $(n - 1)$-punctured ball tangle $T^{n-1}$. 
Let $F^{n-1}(T^{n-1}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & a_{22} & \cdots & a_{2n-1} \end{bmatrix}$. Then, by Lemma 3.4 and Theorem 3.6, we have

$$F^0(B^{(n)}) = F^0(T^{n-1}(B^{(1)}, \ldots, B^{(n-1)})) = \left( 1 \sim a \right) \left( \delta \right)$$

Let $d'$ be g.c.d. $(p, q)$. Then

$$d' = \text{g.c.d.} \left( a_{11}p_1 + \cdots + a_{1n-1}q_1, a_{21}p_2 + \cdots + a_{2n-1}q_2 \right)$$

and there are $k, l \in \mathbb{Z}$ such that $d' = k(a_{11}p_1 + \cdots + a_{1n-1}q_1) + l(a_{21}p_2 + \cdots + a_{2n-1}q_2)$. Since $d_1 \cdots d_{n-1}$ divides each term of the above linear combination, $d_1 \cdots d_{n-1}$ divides $d'$. Hence, $d_1 \cdots d_{n-1}d_n$ divides $d'd_n$ and $d'd_n$ divides $\langle L \rangle$. Therefore, $\prod_{i=1}^k \text{g.c.d.} (p_i, q_i)$ divides $\langle L \rangle$.

4. Surjectivity of invariants

We use the following notation throughout this section:

(1) The subscripts 1,2 of ball tangles will no longer used to denote different kinds of closures. They will be used simply to distinguish different ball tangles.

(2) The ball tangle invariant $F^0$ will be denoted by $f$ with values in $PM_2 = PM_{2 \times 1}(\mathbb{Z})$ and the spherical tangle invariant $F^1$ will be denoted by $F$ with values in $PM_{2 \times 2} = PM_{2 \times 2}(\mathbb{Z})$.

**Definition 4.1.** Let $a_1, a_2, a_3, a_4$ be 4 points in $S^2$, and let $x_i = \{a_i\} \times \{0\}$ and $y_i = \{a_i\} \times \{1\}$ for each $i \in \{1, 2, 3, 4\}$. Then a 1-dimensional proper submanifold $S$ of $S^2 \times I$, $I = [0, 1]$, is called a spherical tangle about $a_1, a_2, a_3, a_4$ (or simply, spherical tangle) if $\partial S \cap (S^2 \times \{0\}) = \{x_1, x_2, x_3, x_4\}$ and $\partial S \cap (S^2 \times \{1\}) = \{y_1, y_2, y_3, y_4\}$.

Note that $B(0, 0) - \text{Int}(B((0, 0), 1))$ is homeomorphic to $S^2 \times I$.

**Definition 4.2.** Define $\cong$ on the class of all spherical tangles about $a_1, a_2, a_3, a_4$ by $S_1 \cong S_2$ if and only if there is a homeomorphism $h : S^2 \times I \to S^2 \times I$ such that $h(S_1) = S_2$ and $h \simeq Id_{S^2 \times I}$ rel $\partial$ for spherical tangles $S_1$ and $S_2$. Then $\cong$ is an
equiv-alence relation on it. \( S_1 \) and \( S_2 \) are said to be isotopic, or of the same isotopy type, if \( S_1 \cong S_2 \) and, for each spherical tangle \( S \), the equivalence class \([S]\) is called the isotopy type of \( S \).

Remark that we usually use \( S \) for \([S]\) and consider only diagrams for spherical tangles and ball tangles. Now, let us define the product of spherical tangle diagrams as follows:

\[ [S_2] \circ [S_1] = [S_2(S_1)], \]

or simply, \( S_2 \circ S_1 = S_2(S_1) \) for all spherical tangle diagrams \( S_1 \) and \( S_2 \), where, roughly speaking, \( S_2(S_1) \) means to put \( S_1 \) inside of \( S_2 \), using the identification

\[
\frac{(S^2 \times [0, 1])_1 \coprod (S^2 \times [0, 1])_2}{(S^2 \times 1)_1 \coprod (S^2 \times [0, 1])_2} = S^2 \times [0, 1].
\]

It is clear that \( \circ \) is associative and \( I = \bigoplus_{i=1}^{1} a_i \times I \) is the identity spherical tangle for \( \circ \). Thus, the class \( ST \) of spherical tangle diagrams with \( \circ \) forms a monoid.

4.1. **Surjectivity of the ball tangle invariant** \( f \). Recall Lemma 2.12, and Lemma 2.13:

1. If \( B_1, B_2 \in BT \) and \( f(B_1) = \begin{bmatrix} p \\ q \end{bmatrix}, f(B_2) = \begin{bmatrix} r \\ s \end{bmatrix} \), then \( f(B_1 +_h B_2) = \begin{bmatrix} ps + qr \\ qs \end{bmatrix} \).

So if we denote \( \begin{bmatrix} ps + qr \\ qs \end{bmatrix} \) by \( \begin{bmatrix} p \\ q \end{bmatrix} +_h \begin{bmatrix} r \\ s \end{bmatrix} \), then we have \( f(B_1 +_h B_2) = f(B_1) +_h f(B_2) \).

2. If \( B \in BT \) and \( f(B) = \begin{bmatrix} p \\ q \end{bmatrix} \), then \( f(B^*) = \begin{bmatrix} p \\ -q \end{bmatrix} \) and \( f(B^R) = \begin{bmatrix} q \\ -p \end{bmatrix} \), where \( B^* \) is the mirror image of \( B \) and \( B^R \) is the 90\(^\circ\) rotation of \( B \) counterclockwise on the projection plane. So if we denote \( \begin{bmatrix} p \\ -q \end{bmatrix} \) by \( \begin{bmatrix} p \\ q \end{bmatrix}^* \) and \( \begin{bmatrix} q \\ -p \end{bmatrix} \) by \( \begin{bmatrix} p \\ q \end{bmatrix}^R \), then we have \( f(B^*) = f(B)^* \) and \( f(B^R) = f(B)^R \).

To avoid complication, we use the same notations for \( +_h, ^* \), and \( ^R \) applied to \( BT \) and \( PM_2 \). We shall be able to understand the meaning of different operations by their contexts.

3. If \( B \in BT \), then \( B^{**} = B \) but \( B^{RR} \) need not be the same as \( B \).

4. If \( A \in PM_2 \), then \( A^{**} = A \) and \( A^{RR} = A \).

Notice that, if an element \( A \) in \( PM_2 \) can be obtained by applying \( +_h, ^* \), and \( ^R \) to finitely many invariants of ball tangles, then \( A \) itself is the invariant of a ball tangle.

Let us calculate \( f \) for ball tangles in Figure 6.

1. The ball tangles \( b \) and \( c \) have invariants \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), respectively.
2. The ball tangle $a$ has invariant \[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \] because $a = b + h b$ and \[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + h \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \]

3. The ball tangles $d$ and $e$ have invariants \[ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \] and \[ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \], respectively. They are the mirror images each other.

4. The ball tangle $f$ has invariant \[ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + h \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \], and the ball tangle $g$ has invariant \[ \begin{bmatrix} p \\ 1 \end{bmatrix} \], where $p$ is the number of horizontal twists in $g$.

5. Ball tangles $j$ and $k$ have invariants \[ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \] and \[ \begin{bmatrix} 1 \\ q \end{bmatrix} \], respectively, where $q$ is the number of vertical twists in $k$.

6. The ball tangle $h$ has invariant \[ \begin{bmatrix} 3 \\ 0 \end{bmatrix} \] because \[ \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + h \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \] The ball tangle $l$ is $h^R$ and has invariant \[ \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} . \]

7. The ball tangle $m$ has invariant \[ \begin{bmatrix} 3 \\ 3 \end{bmatrix} \] because $m = d + h l$ and \[ \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + h \begin{bmatrix} 0 \\ 3 \end{bmatrix} . \] Hence, the ball tangles $i$ and $m$ have the same invariant but they are apparently not isotopic.

Figure 6. Ball tangle diagrams.
To prove the surjectivity of \( f \), we use Euclidean Algorithm.

**Proposition 4.3.** (Euclidean Algorithm) If \( a, b \in \mathbb{N} \) and \( a < b \), then there are uniquely \( k \in \mathbb{N} \) and \( r_0, r_1, \ldots, r_k, r_{k+1} \in \mathbb{N} \cup \{0\} \) and \( q_1, \ldots, q_{k+1} \in \mathbb{N} \) such that \( r_0 = a \) and \( r_{k+1} = 0 \) and \( r_0 > r_1 > \cdots > r_k > r_{k+1} \) and \( b = q_1 a + r_1 \) and \( r_{i-2} = q_i r_{i-1} + r_i \) for each \( i \in \{2, \ldots, k+1\} \).

**Theorem 4.4.** The ball tangle invariant \( f : BT \rightarrow PM_2 \) is onto.

*Proof.* It is enough to show that there is \( B \in BT \) such that \( f(B) = \begin{bmatrix} b \\ a \end{bmatrix} \) if \( a, b \in \mathbb{N} \) and \( a < b \).

Suppose that \( a, b \in \mathbb{N} \) and \( a < b \). Then, by Euclidean Algorithm, there are uniquely \( k \in \mathbb{N} \) and \( r_0, r_1, \ldots, r_k, r_{k+1} \in \mathbb{N} \cup \{0\} \) and \( q_1, \ldots, q_{k+1} \in \mathbb{N} \) such that \( r_0 = a \) and

\[
\begin{align*}
b & = q_1 a + r_1, \quad \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} q_1 \\ 1 \end{bmatrix} + h \begin{bmatrix} r_1 \\ a \end{bmatrix}, \quad 0 < r_1 < a, \\
a & = q_2 r_1 + r_2, \quad \begin{bmatrix} a \\ r_1 \end{bmatrix} = \begin{bmatrix} q_2 \\ 1 \end{bmatrix} + h \begin{bmatrix} r_2 \\ r_1 \end{bmatrix}, \quad 0 < r_2 < r_1, \\
r_1 & = q_3 r_2 + r_3, \quad \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} q_3 \\ 1 \end{bmatrix} + h \begin{bmatrix} r_3 \\ r_2 \end{bmatrix}, \quad 0 < r_3 < r_2, \\
\vdots \quad \vdots \\
r_{k-2} & = q_k r_{k-1} + r_k, \quad \begin{bmatrix} r_{k-2} \\ r_{k-1} \end{bmatrix} = \begin{bmatrix} q_k \\ 1 \end{bmatrix} + h \begin{bmatrix} r_k \\ r_{k-1} \end{bmatrix}, \quad 0 < r_{k-1} < r_{k-2}, \\
r_{k-1} & = q_{k+1} r_k + r_{k+1}, \quad \begin{bmatrix} r_{k-1} \\ r_k \end{bmatrix} = \begin{bmatrix} q_{k+1} \\ 1 \end{bmatrix} + h \begin{bmatrix} 0 \\ r_k \end{bmatrix}, \quad r_{k+1} = 0.
\end{align*}
\]

Since \( \begin{bmatrix} q_1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} q_{k+1} \\ 1 \end{bmatrix} \), and \( \begin{bmatrix} 0 \\ r_k \end{bmatrix} \) are realizable by ball tangles and \( \begin{bmatrix} r_i \\ r_{i-1} \end{bmatrix} = \begin{bmatrix} r_{i-1} \\ r_i \end{bmatrix}^{R^*} \) for each \( i \in \{1, \ldots, k\} \), \( \begin{bmatrix} b \\ a \end{bmatrix} \) corresponds a ball tangle.

Therefore, there is \( B \in BT \) such that \( \begin{bmatrix} b \\ a \end{bmatrix} = f(B) \). This proves the theorem. \( \square \)

We can define vertical connect sum of ball tangle diagrams by horizonal connect sum and rotations.

**Definition 4.5.** Define \(+_v\) on \( BT \) by \( B_1 +_v B_2 = (B_1^R + h B_2^R)^{R^R} \) for all \( B_1, B_2 \in BT \). Then \(+_v\) is called the vertical connect sum on \( BT \).
Lemma 4.8. If $B_1, B_2 \in \mathbf{BT}$ and $f(B_1) = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$, then $f(B_1 + v B_2) = \begin{bmatrix} p r & r s \\ q r + p s & s \end{bmatrix}$.

So if we denote $\begin{bmatrix} pr \\ q r + ps \end{bmatrix} \text{ by } \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} r \\ s \end{bmatrix}$, we have $f(B_1 + v B_2) = f(B_1) + v f(B_2)$.

Note that $(\mathbf{BT}, +_h)$ and $(\mathbf{BT}, +_v)$ are noncommutative monoids with identities $c$ and $b$ in Figure 6, respectively. On the other hand, $(\mathbf{PM}_2, +_h)$ and $(\mathbf{PM}_2, +_v)$ are commutative monoids with identities $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, respectively. The ball tangle invariant $f$ is a monoid epimorphism from $(\mathbf{BT}, +_h)$ and $(\mathbf{BT}, +_v)$ to $(\mathbf{PM}_2, +_h)$ and $(\mathbf{PM}_2, +_v)$, respectively.

4.2. The invariant $F$ of spherical tangles, connect sums, and determinants.

The following lemma tells us a unique commutative square.

Lemma 4.6. For each $S \in \mathbf{ST}$, there is a unique function $S_* : \mathbf{PM}_2 \to \mathbf{PM}_2$ such that $f \circ S = S_* \circ f$. Furthermore, $S_*$ is the function from $\mathbf{PM}_2$ to $\mathbf{PM}_2$ defined by $S_*(A) = F(S) A$ for each $A \in \mathbf{PM}_2$.

Proof. Let $S \in \mathbf{ST}$. Then $f(S(B)) = F(S) f(B)$ for each $B \in \mathbf{BT}$ (Theorem 3.6). Hence, $f \circ S = S_* \circ f$. The uniqueness of $S_*$ follows from the surjectivity of $f$. To show the uniqueness of $S_*$, suppose that $F_1$ and $F_2$ are functions from $\mathbf{PM}_2$ to $\mathbf{PM}_2$ such that $f \circ S = F_1 \circ f$ and $f \circ S = F_2 \circ f$, respectively, and $A \in \mathbf{PM}_2$. Then there is $B \in \mathbf{BT}$ such that $A = f(B)$ by the surjectivity of $f$ (Theorem 4.4) and $F_1(A) = F_1(f(B)) = (F_1 \circ f)(B) = (F_2 \circ f)(B) = F_2(f(B)) = F_2(A)$. Hence, $F_1 = F_2$, that is, $S_*$ is unique. This proves the lemma. \qed

Lemma 4.7. If $S_1, S_2 \in \mathbf{ST}$, then $(S_2 \circ S_1)_* = S_2_* \circ S_1*$.

Proof. Suppose that $S_1, S_2 \in \mathbf{ST}$. Then $f \circ S_1 = S_1_* \circ f$ and $f \circ S_2 = S_2_* \circ f$. Hence, $f \circ (S_2 \circ S_1) = (f \circ S_2) \circ S_1 = (S_2_* \circ f) \circ S_1 = S_2_* \circ (f \circ S_1) = S_2_* \circ (S_1_* \circ f) = (S_2_* \circ S_1)_* \circ f$. Therefore, by the uniqueness of $(S_2 \circ S_1)_*$, $(S_2 \circ S_1)_* = S_2_* \circ S_1_*$. \qed

Let us identify $S_*$ with $F(S)$ for each $S \in \mathbf{ST}$. Since $S_2 \circ S_1$ is the composition of $S_1$ and $S_2$, we have the following lemma immediately.

Lemma 4.8. If $S_1, S_2 \in \mathbf{ST}$, then $F(S_2 \circ S_1) = F(S_2) F(S_1)$.

Notice that Lemma 4.8 does not depend on the surjectivity of $f$. We can prove Lemma 4.8 by Theorem 3.6 and the following lemma without using the surjectivity of $f$.

Lemma 4.9. If $A, B \in \mathbf{PM}_{2 \times 2}$ and $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = B \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = B \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $A = B$.\[\]
By Theorem 3.6, we have that $F(S_2 \circ S_1)$ and $F(S_2)F(S_1)$ are matrices in $PM_{2 \times 2}$ such that $F(S_2 \circ S_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = F(S_2)F(S_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $F(S_2 \circ S_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = F(S_2)F(S_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $F(S_2 \circ S_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = F(S_2)F(S_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence, $F(S_2 \circ S_1) = F(S_2)F(S_1)$ by Lemma 4.9.

Let us introduce the elementary operations on $ST$.

**Definition 4.10.** Let $S$ be a spherical tangle diagram. Then

1. $S^*$ is the mirror image of $S$,
2. $S^-$ is the spherical tangle diagram obtained by interchanging the inside hole with the outside hole of $S$,
3. $S^{r_1}$ is the spherical tangle diagram obtained by only rotating inside hole of $S$ $90^\circ$ counterclockwise on the projection plane,
4. $S^{r_2}$ is the spherical tangle diagram obtained by only rotating outside hole of $S$ $90^\circ$ counterclockwise on the projection plane,
5. $S^R$ is the spherical tangle diagram obtained by the $90^\circ$ rotation of $S$ itself counterclockwise on the projection plane.

![Figure 7. Elementary operations on $ST$.](image)

Note that $S^{r_2} = S^{-r_1}$, $S^{r_1} = S^{-r_2}$, and $S^R = S^{r_1}r_2 = S^{r_2}r_1$ for each $S \in ST$.

**Lemma 4.11.** If $S \in ST$ with the invariant $F(S) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, then

1. $F(S^*) = \begin{bmatrix} \alpha & -\gamma \\ -\beta & \delta \end{bmatrix}$,  
2. $F(S^-) = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix}$,  
3. $F(S^{r_1}) = \begin{bmatrix} -\gamma & \alpha \\ -\delta & \beta \end{bmatrix}$,
(4) \( F(S^{r_2}) = \begin{bmatrix} -\beta & -\delta \\ \alpha & \gamma \end{bmatrix} \), (5) \( F(S^R) = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \).

**Proof.** Let \( S \in ST \) with \( F(S) = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \). Then there is \( u \in \Phi \) such that \( \langle S_{11} \rangle = \alpha u, \langle S_{12} \rangle = \gamma iu, \langle S_{21} \rangle = \beta (-i) u, \langle S_{22} \rangle = \delta u \). Here the link \( S_{ij}, i, j \in \{1, 2\} \), is obtained by taking the numerator closure \((i = 1)\) or the denominator closure \((i = 2)\) of \( S \) with its hole filled by the fundamental tangle \( j \). Therefore,

\[
\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} u^{-1} \alpha u & u^{-1} (-i) \gamma i u \\ u^{-1} i \beta (-i) u & u^{-1} \delta u \end{bmatrix}.
\]

Now we have

1. \( \langle S_{11}^* \rangle = \alpha u^{-1}, \langle S_{12}^* \rangle = \gamma (iu)^{-1}, \langle S_{21}^* \rangle = \beta (-iu)^{-1}, \langle S_{22}^* \rangle = \delta u^{-1} \),
2. \( \langle S_{11}^- \rangle = \langle S_{22} \rangle, \langle S_{12}^- \rangle = \langle S_{12} \rangle, \langle S_{21}^- \rangle = \langle S_{21} \rangle, \langle S_{22}^- \rangle = \langle S_{21} \rangle \),
3. \( \langle S_{11}^r \rangle = \langle S_{12} \rangle, \langle S_{12}^r \rangle = \langle S_{12} \rangle, \langle S_{21}^r \rangle = \langle S_{22} \rangle, \langle S_{22}^r \rangle = \langle S_{22} \rangle \).

Hence, \( F(S^*) = \begin{bmatrix} \alpha & -\gamma \\ -\beta & \delta \end{bmatrix} \), \( F(S^-) = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} \), \( F(S^{r_1}) = \begin{bmatrix} \gamma & -\alpha \\ -\delta & \beta \end{bmatrix} \).

Since \( S^{r_2} = S^{-r_1} \) and \( S^R = S^{r_1 r_2} \), (4) and (5) are easily proved by (2) and (3). \( \square \)

Like the case of ball tangle operations and invariants, it is convenient to use the following notations.

**Notation:** Let \( \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in PM_{2 \times 2} \). Then

1. \( \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^* = \begin{bmatrix} \alpha & -\gamma \\ -\beta & \delta \end{bmatrix} \),
2. \( \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^- = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} \),
3. \( \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^r_1 = \begin{bmatrix} -\gamma & \alpha \\ -\delta & \beta \end{bmatrix} \),
4. \( \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^r_2 = \begin{bmatrix} -\beta & -\delta \\ \alpha & \gamma \end{bmatrix} \),
5. \( \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^R = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \).

With these notations, we can write: \( F(S^*) = F(S^*), F(S^-) = F(S^-), F(S^{r_1}) = F(S)^{r_1}, F(S^{r_2}) = F(S)^{r_2}, F(S^R) = F(S)^R \) if \( S \in ST \).

The determinant function \( \det \) is well-defined on \( PM_{2 \times 2} \) since \( \det (-A) = (-1)^2 \det A \) for each \( A \in PM_{2 \times 2} \). Notice that the 5 elementary operations on \( ST \) do not change the determinant of invariants of spherical tangles.

**Lemma 4.12.** If \( S_1, S_2 \in ST \), then

1. \( (S_1 \circ S_2)^* = S_1^* \circ S_2^* \),
2. \( (S_1 \circ S_2)^- = S_2^- \circ S_1^- \),
3. \( (S_1 \circ S_2)^{r_1} = S_1 \circ S_2^{r_1} \),
4. \( (S_1 \circ S_2)^{r_2} = S_1^{r_2} \circ S_2 \),
5. \( (S_1 \circ S_2)^R = S_1^R \circ S_2^R \).

Notice that a spherical tangle has exactly 2 holes which are inside and outside.
Definition 4.13. Let $B \in BT$, and let $S \in ST$. Then

(1) the 1st and the 2nd outer horizontal connect sums of $B$ and $S$ are the spherical tangle diagrams denoted by $B +_h S$ and $S +_h B$, respectively,

(2) the 1st and the 2nd outer vertical connect sums of $B$ and $S$ are the spherical tangle diagrams denoted by $B +_v S$ and $S +_v B$, respectively (See Figure 8).

We also define the connect sums at the inside hole by $+_h, +_v, -$ as follows.

Definition 4.14. Let $B \in BT$, and let $S \in ST$. Then

(1) the 1st and the 2nd inner horizontal connect sums of $B$ and $S$ are the spherical tangle diagrams $B +^h S$ and $S +^h B$ defined by $B +^h S = (S - +^h B^h)^-$ and $S +^h B = (B^h - +^h S^-)^-$, respectively,

(2) the 1st and the 2nd inner vertical connect sums of $B$ and $S$ are the spherical tangle diagrams $B +^v S$ and $S +^v B$ defined by $B +^v S = (S - +^v B^v)^-$ and $S +^v B = (B^v - +^v S^-)^-$, respectively,

where $B^h$ and $B^v$ are the 180° rotation of $B$ with respect to the vertical axis and the horizontal axis of the projection plane, respectively (See Figure 9).

Figure 8. Connect sums of ball tangles and outer connect sums.
Figure 9. Inner connect sums and rotations of ball tangles about axes.

Let us give definitions of monoid actions. This is just a generalization of group actions.

**Definition 4.15.** Let \( M \) be a monoid with the identity \( e \), and let \( X \) be a nonempty set. Then

1. a function \( \ast_l : M \times X \to X \) is called a left monoid action of \( M \) on \( X \) if \( \ast_l(m_1 m_2, x) = \ast_l(m_1, \ast_l(m_2, x)) \) and \( \ast_l(e, x) = x \) for all \( m_1, m_2 \in M \) and \( x \in X \),

2. a function \( \ast_r : X \times M \to X \) is called a right monoid action of \( M \) on \( X \) if \( \ast_r(x, m_1 m_2) = \ast_r(\ast_r(x, m_1), m_2) \) and \( \ast_r(x, e) = x \) for all \( m_1, m_2 \in M \) and \( x \in X \).

In this sense, the connect sums of diagrams of ball tangles and spherical tangles are monoid actions on \( ST \). Hence, we have 8 monoid actions on \( ST \) by \( BT \) which are similar. Also, the composition on \( ST \) induces a left monoid action and a right monoid action on \( BT \). In particular, a monoid action is onto like a group action because of identity.

**Lemma 4.16.** Let \( B \in BT \), and let \( S \in ST \). Then

1. \((B +_h S)^R = S^R +_v B^R \),
2. \((S +_h B)^R = B^R +_v S^R \),
3. \((B +_v S)^R = B^R +_h S^R \),
4. \((S +_v B)^R = S^R +_h B^R \).

Note that \( B^{RRRR} = B \) for each \( B \in BT \) and \( S^{RRRR} = S \) for each \( S \in ST \).
The following lemma tells us that the other outer horizontal sum and two outer vertical sums can be expressed in terms of the 1st horizontal sum and $^R$ which is the rotation for ball tangle diagrams or spherical tangle diagrams.

**Lemma 4.17.** Let $B \in BT$, and let $S \in ST$. Then

1. $S +_h B = (B^{RR} +_h S^{RR})^{RR}$,
2. $B +_v S = (B^R +_h S^R)^{RRR}$,
3. $S +_v B = (B^{RRR} +_h S^{RRR})^R$.

**Proof.** Let $B \in BT$, and let $S \in ST$. Then

1. $S +_h B = (B' +_h S')^{RR}$ for some $B' \in BT$, $S' \in ST$. Hence, $(B' +_h S')^{RR} = (S^{R} +_v B'^{R})^{RR} = S'^{RR} +_h B'^{RR}$. Take $B' = B^{RR}$ and $S' = S^{RR}$. Then $S +_h B = (B^{RR} +_h S^{RR})^{RR}$,
2. $B +_v S = (B' +_h S')^{RRR}$ for some $B' \in BT$, $S' \in ST$. Hence, $(B' +_h S')^{RRR} = (S'^{R} +_v B'^{R})^{RRR} = (S'^{RR} +_h B'^{RR})^{RR} = B'^{RRR} +_v S'^{RRR}$. Take $B' = B^R$ and $S' = S^R$. Then $B +_v S = (B^R +_h S^R)^{RRR}$,
3. $S +_v B = (B' +_h S')^R$ for some $B' \in BT$, $S' \in ST$. Hence, $(B' +_h S')^R = S'^{R} +_v B'^{R}$. Take $B' = B^{RRR}$ and $S' = S^{RRR}$. Then $S +_v B = (B^{RRR} +_h S^{RRR})^R$. □

Now, we consider the invariants of spherical tangles obtained from the various connect sums with ball tangles and their determinants.

Note that $f(B) = f(B^{h-}) = f(B^{v-})$ for each $B \in BT$ and $S^{--} = S$ for each $S \in ST$. Also, $S^{**} = S$ for each $S \in ST$.

**Lemma 4.18.** If $B \in BT$ with $f(B) = \left[ \begin{array}{c} p \\ q \end{array} \right]$ and $S \in ST$ with $F(S) = \left[ \begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array} \right]$, then

1. $F(B +_h S) = F(S +_h B) = \left[ \begin{array}{cc} p\beta + q\alpha & p\delta + q\gamma \\ q\beta & q\delta \end{array} \right]$, det $F(B +_h S) = q^2$ det $F(S)$,
2. $F(B +_v S) = F(S +_v B) = \left[ \begin{array}{cc} p\alpha & p\gamma \\ q\alpha + p\beta & q\gamma + p\delta \end{array} \right]$, det $F(B +_v S) = p^2$ det $F(S)$,
3. $F(B^{h-}hS) = F(S^{h-}hB) = \left[ \begin{array}{cc} q\alpha & p\alpha + q\gamma \\ q\beta & p\beta + q\delta \end{array} \right]$, det $F(B^{h-}hS) = q^2$ det $F(S)$,
4. $F(B^{v-}hS) = F(S^{v-}hB) = \left[ \begin{array}{cc} q\gamma + p\alpha & p\gamma \\ q\delta + p\beta & p\delta \end{array} \right]$, det $F(B^{v-}hS) = p^2$ det $F(S)$.
Proof. (1) Let \( \begin{bmatrix} x \\ y \end{bmatrix} \in PM_2 \). Then there is \( X \in BT \) such that \( f(X) = \begin{bmatrix} x \\ y \end{bmatrix} \in PM_2 \).

We have
\[
F(B + h S)f(X) = f(B + h S(X)) = \begin{bmatrix} p \\ q \end{bmatrix} + h \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p\beta x + p\delta y + q\alpha x + q\gamma y \\ q\beta x + q\delta y \end{bmatrix} = \begin{bmatrix} p\beta + q\alpha \\ q\beta \end{bmatrix} + \begin{bmatrix} p\delta + q\gamma \\ q\delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = f(X).
\]

By Lemma 4.9, \( F(B + h S) = \begin{bmatrix} p\beta + q\alpha & p\delta + q\gamma \\ q\beta & q\delta \end{bmatrix} \). Hence, \( \det F(B + h S) = q^2 \det F(S) \).

Also, \( F(S + h B)f(X) = f(S(X) + h B) = f(B + h S(X)) = F(B + h S)f(X) \).

Therefore, \( F(S + h B) = F(B + h S) \). This proves (1).

(3) Since \( B + h S = (S^- + h B^-)^- \), \( F(S^-) = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} \), and \( f(B^-) = f(B) \), we have
\[
F(B + h S) = F((S^- + h B^-)^-) = F(S^- + h B^-)^- = F(B^- + h S^-)^- = F(S + h B^- + h S^-) = F(S + h B^-)B.
\]

Since \( F(S^- + h B^-) = \begin{bmatrix} p\beta + q\delta & p\alpha + q\gamma \\ q\beta & q\alpha \end{bmatrix} \), we have
\[
F(B + h S) = \begin{bmatrix} q\alpha & p\alpha + q\gamma \\ q\beta & p\beta + q\delta \end{bmatrix}
\]
and \( \det F(B + h S) = q^2 \det F(S) \). This proves (3).

Similarly, (2) and (4) can be proved. \( \square \)

Definition 4.19. A spherical tangle diagram \( S \) is said to be \( I \)-reducible if there are \( n \in \mathbb{N}, A_1, \ldots, A_{n+1} \in BT \cup \{ I \}, *_1, \ldots, *_n \in \{ +h, +_h^o, +v, +_v^o, +_h^o, +_v^o, +v^o, +_h^o \} \) such that \( S = (\cdots (A_1 *_1 A_2) *_2 \cdots A_n *_n A_{n+1}) \) and there is only one of \( A_1, \ldots, A_{n+1} \) equal to \( I \), where \( A +_h^o B = B +h A, A +_v^o B = B +v A, A +h^o B = B +h A, \) and \( A +v^o B = B +v A \). In general, a spherical tangle \( S \) is \( J \)-reducible if, in above definition, we replace \( I \) by another spherical tangle \( J \).

In other words, a spherical tangle diagram \( S \) is \( I \)-reducible if \( S \) can be decomposed by finitely many ball tangle diagrams and only one identity spherical tangle diagram with respect to the inner and the outer connect sums and their opposite operations. Note that, in Definition 4.19, the expression of \( S \) can be written as \( S = A_1 *_1 \cdots *_n A_{n+1} \). In this case, the order of operations in the expression is important.

Theorem 4.20. If a spherical tangle \( S \) is \( I \)-reducible, then \( \det F(S) = n^2 \) for some \( n \in \mathbb{Z} \). Furthermore, if \( S \) is \( J \)-reducible, then \( \det F(S) = n^2 \det F(J) \) for some \( n \in \mathbb{Z} \).
Proof. It follows from Lemma 4.18 immediately. By Lemma 4.18, we know that a ball tangle connected to a spherical tangle in the sense of Definition 4.13 and Definition 4.14 contributes a square of integer to the determinant of the invariant of connect sum.

Some examples of spherical tangles are given in Figure 10.

1. The spherical tangle $a$ is $I$ and has invariant $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
2. The spherical tangle $b$ has invariant $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.
3. The spherical tangle $c$ is $b \circ b$ and has invariant $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.
4. The spherical tangle $d$ has invariant $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
5. The spherical tangle $e$ has invariant $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + h_{1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. (See lemma 4.18)
6. The spherical tangle $f$ has invariant $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
7. The spherical tangle $b \circ e$ has invariant $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $(b \circ e)^{r_{1}}$ has invariant $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Figure 10. Spherical tangle diagrams.
4.3. **Nonsurjectivity of the spherical tangle invariant** $F$. Recall the $\Delta$-move on knot diagrams introduced in [7]. It is illustrated in Figure 11 (a). If we apply the Kauffman states to the diagrams involved in the $\Delta$-move, we get the 5 basis diagrams without closed components as shown in Figure 11 (b).

![Diagram](image)

Figure 11. (a) A $\Delta$-move on a diagram; (b) The 5 basis diagrams.

**Definition 4.21.** Link diagrams $D_1$ and $D_2$ are said to be $\Delta$-equivalent if $D_2$ can be obtained from $D_1$ by a finite sequence of $\Delta$-moves and Reidemeister moves.

For a spherical tangle $S$, we can get four links $S_{11}, S_{12}, S_{21}, S_{22}$ by taking closures of $S$ with its hole filled by fundamental tangles. We say that two spherical tangles $S$ and $S'$ are $\Delta$-equivalent if each $S'_{ij}$ can be obtained from $S_{ij}$ by a finite sequence of $\Delta$-moves and Reidemeister moves.

**Lemma 4.22.** Let $S$ and $S'$ be spherical tangles such that $S$ and $S'$ are $\Delta$-equivalent. If $F(S) = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ and $F(S') = \begin{bmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{bmatrix}$, then $\alpha \equiv \epsilon \alpha' \mod 4$, $\beta \equiv \epsilon \beta' \mod 4$, $\gamma \equiv \epsilon \gamma' \mod 4$, $\delta \equiv \epsilon \delta' \mod 4$, for some $\epsilon = \pm 1$. 
Proof. Suppose that \( L \) is a link diagram and \( L_\Delta \) is a link diagram obtained from \( L \) by a single \( \Delta \)-move. Then
\[
\langle L \rangle = A^3 \langle C_2 \rangle + A \langle C_4 \rangle + A \langle C_3 \rangle + A^{-1} \langle C_1 \rangle + A \langle C_5 \rangle + A^{-1} \langle C_1 \rangle
\]
\[
= 3A^{-1} \langle C_1 \rangle + A^3 \langle C_2 \rangle + A \langle C_3 \rangle + A \langle C_4 \rangle + A \langle C_5 \rangle
\]
and
\[
\langle L_\Delta \rangle = A^3 \langle C_1 \rangle + A \langle C_3 \rangle + A \langle C_5 \rangle + A^{-1} \langle C_2 \rangle + A \langle C_4 \rangle + A^{-1} \langle C_2 \rangle + A^{-1} \langle L_2 \rangle
\]
\[
= A^3 \langle C_1 \rangle + 3A^{-1} \langle C_2 \rangle + A \langle C_3 \rangle + A \langle C_4 \rangle + A \langle C_5 \rangle.
\]
Hence,
\[
\langle L \rangle - \langle L_\Delta \rangle = 4A^{-1} \langle C_1 \rangle - 4A^{-1} \langle C_2 \rangle.
\]

Suppose that \( \langle L \rangle = aA^k, \langle L_\Delta \rangle = a' A'^k, \langle C_1 \rangle = bA', \langle C_2 \rangle = b'A'' \) for some \( a, a', b, b', k, k', l, l' \in \mathbb{Z} \). Then \( aA^k - a' A'^k = 4bA'^{l-1} - 4b'A''^{l-1} \).

Case 1. If \( A^k = A'^k \), then \( (a-a')A^k = 4bA'^{l-1} - 4b'A''^{l-1} \). So we must have \( a - a' \equiv 0 \mod 4 \).

Case 2. If \( A^k = -A'^k \), then \( (a + a')A^k = 4bA'^{l-1} - 4b'A''^{l-1} \). So we must have \( a + a' \equiv 0 \mod 4 \).

Case 3. If \( A^k \neq \pm A'^k \), then \( aA^k - a' A'^k = 4bA'^{l-1} - 4b'A''^{l-1} \) implies \( a \equiv 0 \mod 4 \) and \( a' \equiv 0 \mod 4 \). Therefore, we always have \( a \equiv ea' \mod 4 \) with \( e = \pm 1 \).

In general, suppose that a link \( L' \) can be obtained from \( L \) by a finite sequence of \( \Delta \)-moves and Reidemeister moves of type II and III. If \( \langle L \rangle = aA^k \) and \( \langle L' \rangle = a' A'^k \), then \( a \equiv ea' \mod 4 \) and \( e = \pm 1 \) depends only on the powers \( k \) and \( k' \).

For the spherical tangle \( S \), we need to consider 4 links \( S_{11}, S_{12}, S_{21}, S_{22} \). If \( F(S) = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \), then \( \langle S_{11} \rangle = \alpha A^k, \langle S_{12} \rangle = \gamma A^{k+2}, \langle S_{21} \rangle = \beta A^{-k-2}, \langle S_{22} \rangle = \delta A^k \). Also, if \( F(S') = \begin{bmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{bmatrix} \), then \( \langle S'_{11} \rangle = \alpha' A^{k'}, \langle S'_{12} \rangle = \gamma' A^{k'+2}, \langle S'_{21} \rangle = \beta' A^{k'-2}, \langle S'_{22} \rangle = \delta' A^{k'} \). Notice that since a \( \Delta \)-move will not change the writhe and we can postpone all Reidemeister moves of type I in any finite sequence of diagram moves to the end of that sequence, we do not need to worry that the Kauffman bracket is only a regular isotopy invariant.

Thus, for the four corresponding links we obtained from \( S' \), the sign \( e \) is a constant. Thus, we have \( \alpha \equiv ee' \mod 4, \beta \equiv e\beta' \mod 4, \gamma \equiv e\gamma' \mod 4, \delta \equiv e\delta' \mod 4 \). \( \square \)

We will make use of the following theorem.

Theorem 4.23. (Matveev \[5\] and Murakami-Nakanishi \[7\]): Oriented links \( L = L_1 \sqcup \cdots \sqcup L_n \) and \( L' = L'_1 \sqcup \cdots \sqcup L'_n \) are \( \Delta \)-equivalent if and only if \( \text{lk}(L_i, L_j) = \text{lk}(L'_i, L'_j) \) for all \( i, j \) with \( 1 \leq i < j \leq n \).
Suppose that a spherical tangle diagram $S$ has no circle components. It has 4 components $K_1, K_2, K_3, K_4$. We will look at a diagram of $S$ and orient each $K_i$ arbitrarily. We use $-K_i$ to mean to reverse the orientation of $K_i$. We define $\text{lk}(K_i, K_j)$ to be a half of the sum of the signs of crossings between $K_i$ and $K_j$. We have $\text{lk}(-K_i, K_j) = -\text{lk}(K_i, K_j)$.

Note: For each $S_{11}, S_{12}, S_{21}, S_{22}$, we get a link whose components are unions of some of $K_1, K_2, K_3, K_4$. If components of $S_{11}$ etc. are oriented, they are unions of some of $\varepsilon K_1, \varepsilon K_2, \varepsilon K_3, \varepsilon K_4$, where $\varepsilon_i = \pm 1$.

Let $S'$ be another spherical tangle with components $K'_1, K'_2, K'_3, K'_4$. Suppose the end points of $K'_i$ are the same as the end points of $K_i$, i.e., $\partial K'_1 = \partial K_1, \partial K'_2 = \partial K_2, \partial K'_3 = \partial K_3, \partial K'_4 = \partial K_4$. So we can orient each $K_i$ and $K'_i$ consistently. In this case, we can orient the links $S_{ij}$ and $S'_{ij}$ consistently in the sense that the corresponding components of $S_{ij}$ and $S'_{ij}$ are the same union of $\varepsilon_i K_i$ and $\varepsilon_i K'_i$, respectively.

**Lemma 4.24.** The linking numbers of $S_{ij}$ and $S'_{ij}$ are equal for all $i, j \in \{1, 2\}$ if $\text{lk}(K_i, K_j) = \text{lk}(K'_i, K'_j)$ for all $i, j \in \{1, 2, 3, 4\}$.

**Proof.** This is obvious from the definition of consistent orientations of $S_{ij}$ and $S'_{ij}$ and 

$$\text{lk}(\varepsilon_i K_i, \varepsilon_j K_j) = \varepsilon_i \varepsilon_j \text{lk}(K_i, K_j).$$

\[\square\]
Lemma 4.25. Let $J$ be the spherical tangle shown in Figure 12. Let $p_1, p_2, p_3, p_4$ be the number of half twists inside of the balls marked by 1, 2, 3, 4, respectively. Then

$$\det F(J) = (p_1 p_4 - p_2 p_3)^2.$$  

Proof. This is by a direct calculation. We have

$$F(J) = \begin{bmatrix} p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4 & -p_1 p_3 - p_1 p_4 - p_2 p_4 - p_2 p_3 \\ p_1 p_2 + p_1 p_4 + p_2 p_3 + p_3 p_4 & -p_1 - p_2 - p_3 - p_4 \end{bmatrix}.$$

We calculate the determinant of $F(J)$ and get $(p_1 p_4 - p_2 p_3)^2$. □

Lemma 4.26. Let $S$ be a spherical tangle without closed components. Then either there is a spherical tangle $S'$ which is either $I$-reducible or $J$-reducible, with $J$ as shown in Figure 12 or that $J$ after some possible operations of $(\cdot)^r_1$ and/or $(\cdot)^r_2$, such that $\partial K'_i = \partial K_i$ and $\text{lk}(K_i, K_j) = \text{lk}(K'_i, K'_j)$ for all $i, j \in \{1, 2, 3, 4\}$.

Proof. If $S$ has one component whose end points lie on different boundary components of $S^2 \times I$, then we can find such a spherical tangle $S'$ that is $I$-reducible. Suppose now $S$ has no such components. The linking number of two components whose end points lie on the same boundary component of $S^2 \times I$ can be realized by adding ball tangles. So we may assume that there is no linking between such components. Then, after some possible operations of $(\cdot)^r_1$ and/or $(\cdot)^r_2$, we can take $S'$ as $J$ with the number of full twistings equal to the linking numbers of $S$. □

Theorem 4.27. If $S$ is a spherical tangle diagram without closed components, then

$$\det F(S) \equiv n^2 \mod 4$$

for some integer $n$.

Proof. Let $S'$ be the spherical tangle in Lemma 4.26. Then $S_{ij}$ and $S'_{ij}$ have the same linking numbers, for each $ij = 11, 12, 21, 22$. By Theorem 4.23, $S$ and $S'$ are $\Delta$-equivalent. The theorem then follows from Lemma 4.22, Theorem 4.20, and Lemma 4.25. □

Suppose now that a spherical tangle $S$ has closed components, we have the following theorem.

Theorem 4.28. If $S \in ST$ with $F(S) = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ and $S$ has closed components, then

$$\alpha \equiv 0 \mod 2, \beta \equiv 0 \mod 2, \gamma \equiv 0 \mod 2, \delta \equiv 0 \mod 2.$$

Proof. See Figure 13 (top left), where a closed component of $S$ is hooked with another component of $S$ as shown. Applying the Kauffman skein relation to the local picture there, we get two diagrams with coefficients $A^2$ and 2, respectively. The diagram with coefficient 2 has one less closed component than $S$ and the diagram with coefficient $A^2$ is obtained from $S$ by unhook the closed component at that place. We keep performing this unhooking process until the closed component is hooked with another component only once, as illustrated in Figure 13 (bottom left). Applying the Kauffman skein
relation again, we can unhook this closed component entirely and we end up with a diagram having one less closed component and a coefficient 2. Note that this closed component may itself being knotted. But this is not important since a knotted closed component separating from other components will make no contribution to the Kauffman bracket when $A = e^{i\pi/4}$.

What we have shown is the fact that when $S$ has a closed component, then 2 divides $\langle S_{ij} \rangle$ for all $ij = 11, 12, 21, 22$. This proves the theorem.

Figure 13. The case that $S$ has closed components.

Note that $n^2 \equiv 0$ or 1 mod 4. So combine Theorem 4.27 and Theorem 4.28, we get the following theorem.

**Theorem 4.29.** For every $S \in ST$, either $\det F(S) \equiv 0 \mod 4$ or $\det F(S) \equiv 1 \mod 4$.

From this theorem, we can conclude that there is no spherical tangle $S$ such that

$$F(S) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

since the determinant of the matrix above is not equal to 0 or 1 mod 4.
5. Open questions

Here are some open questions that we are unable to answer at this moment.

(1) Can one describe exactly the image of the invariant $F$ in $PM_{2\times 2}$?

The following two questions make the above question more specific

(1a) Is it true that $\det F(S)$ is the square of an integer for any spherical tangle $S$?

(1b) Is it true that if a matrix $[A]$ in $PM_{2\times 2}$ has its determinant equal to the square of an integer, then there is a spherical tangle $S$ such that $F(S) = [A]$?

The spherical tangle $J$ does not look like $I$-reducible. But we do not know how to verify this observation.

(2) How to show that $J$ is not $I$-reducible? In general, given spherical tangles $S$ and $S'$, how to show that $S$ is not $S'$-reducible?

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