Higher twist nucleon distribution amplitudes in Wandzura-Wilczek approximation

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(Dated: November 15, 2013)

We derive the higher twist (four and five) nucleon distribution amplitudes in the Wandzura-Wilczek approximation. Our method is based on an analysis of the conformal expansion of nonlocal operators in the spinor formalism.

PACS numbers: 12.38.Bx, 12.39.St
Keywords: higher twist, conformal symmetry, distribution amplitudes

I. INTRODUCTION

Hard exclusive processes provide an opportunity to access the internal structure of hadrons. As usual the processes with nucleons are of special importance due to their experimental availability. The theoretical description of exclusive processes is based on the QCD factorization approach [1–5]. It introduces a notion of the hadron distribution amplitudes (DAs) which can be thought of as momentum fraction distributions of partons in configurations with a fixed number of Fock constituents. Hadron DAs are customary defined as hadron matrix elements of the corresponding nonlocal operators which can be classified according to their twist. In the factorization approach the dominant contribution to an amplitude of exclusive processes with the conformal wave expansion for nonlocal operators it allows us to simplify substantially the derivation of the twist–4 WW terms given in [26].

In the present paper we develop a technique for calculation of the WW corrections to higher twist DAs. Our analysis relies on the spinor formalism which proves to be very effective for studies of the nucleon DAs, see Ref. [26]. In combination with the conformal wave expansion for nonlocal operators it allows us to simplify substantially the derivation of the twist–4 WW terms given in [26] and calculate the WW corrections to the twist–5 DAs. Making use of a new summation formula for conformal series we avoid calculation of certain normalization coefficients that requires knowledge of the evolution Hamiltonian for the corresponding operators.

The paper is organized as follows: Sect. II is introductory. We fix the notations and provide definitions for various nucleon DAs in the spinor formalism. In sect. III we explain our approach on the example of the WW contribution to the twist–4 DAs. Sect. IV contains details of the calculation of the twist–5 DAs. The results for all nucleon DAs up to twist five are collected in sect. V. In several Appendices we explain some technical issues.

II. NUCLEON DISTRIBUTION AMPLITUDES

The three-quark nucleon DAs of different twist are defined by matrix elements of the corresponding three quark operators, see Refs. [14, 26]. Below we present definitions for the relevant DAs in the spinor formalism. We will follow closely the notations of Ref. [27].

The leading twist DA is defined as

$$
\langle 0 | e^{ij} u_i^j(z_1n) u_i^j(z_2n) d_i^k(z_3n) | P \rangle = -\frac{1}{2} (pn) N^z \int \mathcal{D}x e^{-i(pn) \Sigma_{\alpha \beta} \Phi_3(x)}.
$$

(1)
The integration in (1) goes over the simplex, i.e.
\[ \mathcal{D}x = dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3). \] (2)

The up and down spinors, \( q^+(i) \), are the two component Weyl spinors,
\[ q = \left( q^+, q^- \right), \quad q^+(i) = \frac{1}{2}(1 \pm \gamma_5)q, \] (3)
and similarly for the nucleon spinor \( N \). The vector \( n \) is an auxiliary light-like vector \( (n^2 = 0) \). It can be parameterized by a Weyl spinor \( \lambda \) as follows
\[ n^\mu \sigma^\mu_{\alpha\alpha} = \lambda_\alpha \bar{\lambda}_\alpha, \] (4)
where \( \bar{\lambda} = \lambda \). For later convenience we introduce the second light-like vector \( \bar{n} \), such that \( (\bar{n}n) \neq 0 \) and denote the corresponding auxiliary spinor by \( \mu \), \( (\bar{n}\sigma^\mu_{\alpha\alpha} = \mu_\alpha \bar{\mu}_\alpha) \). We introduce shorthand notations for projections of the quark fields onto the auxiliary spinors \( \lambda \) and \( \mu \)
\[ q^+_\alpha = \lambda^\alpha q^\alpha, \quad q^-_\alpha = \mu^\alpha q^\alpha, \quad q^+_\nu = \bar{\lambda}^\nu q^\nu, \quad q^-_\nu = \bar{\mu}^\nu q^\nu. \]
We accept the conventions of Refs. [28, 29] for rising and lowering spinor indices.

The nucleon DAs of twist–4, \( \Phi_4, \Psi_4, \Xi_4 \), and of twist–5, \( \Phi_5, \Psi_5, \Xi_5 \) were defined in Refs. [14, 26]. Below we present definitions for these DAs in the spinor formalism that is more convenient for further analysis. We get
\[ \langle 0 | e^{ijk} u^{i(\xi_1)}(\xi_2) u^{j(\xi_2)}(\xi_3) d^{k(\xi_3)} | P \rangle = \frac{1}{4}(\mu \lambda) m_N N^+ \int dx e^{-i(p\xi)} \Phi_4(x), \]
\[ \langle 0 | e^{ijk} u^{i(\xi_1)}(\xi_2) u^{j(\xi_2)}(\xi_3) d^{k(\xi_3)} | P \rangle = \frac{1}{4}(\mu \lambda) m_N N^+ \int dx e^{-i(p\xi)} \Psi_4(x), \]
\[ \langle 0 | e^{ijk} u^{i(\xi_1)}(\xi_2) u^{j(\xi_2)}(\xi_3) d^{k(\xi_3)} | P \rangle = \frac{1}{4}(\mu \lambda) m_N N^+ \int dx e^{-i(p\xi)} \Xi_4(x) \] (5)
for the twist–4 DAs and
\[ \langle 0 | e^{ijk} u^{i(\xi_1)}(\xi_2) u^{j(\xi_2)}(\xi_3) d^{k(\xi_3)} | P \rangle = \frac{1}{4}(\mu \lambda) m_N N^+ \int dx e^{-i(p\xi)} \Phi_5(x), \]
\[ \langle 0 | e^{ijk} u^{i(\xi_1)}(\xi_2) u^{j(\xi_2)}(\xi_3) d^{k(\xi_3)} | P \rangle = \frac{1}{4}(\mu \lambda) m_N N^+ \int dx e^{-i(p\xi)} \Psi_5(x), \]
\[ \langle 0 | e^{ijk} u^{i(\xi_1)}(\xi_2) u^{j(\xi_2)}(\xi_3) d^{k(\xi_3)} | P \rangle = \frac{1}{4}(\mu \lambda) m_N N^+ \int dx e^{-i(p\xi)} \Xi_5(x). \]
for the twist–5 DAs. Here \( m_N \) stands for the nucleon mass and \( q_{\pm}(\xi) \equiv q_{\pm}(n\xi) \). Notice that these definitions differ by a sign from those given in Ref. [28] due to the nonstandard charge conjugation matrix \( C \) used there.

The twist–3 DA \( \Phi_3 \) can be represented as a series
\[ \Phi_3(x) = x_1 x_2 x_3 \sum_{N,q} C_{Nq} \phi_{Nq}(\mu_R) P_{Nq}(x_1, x_2, x_3), \] (7)
where \( \mu_R \) is the renormalization scale. The common prefactor is dictated by the conformal symmetry. \( P_{Nq} \) are the homogeneous polynomials of degree \( N \), \( P_{Nq}(xx) = x^N P_{Nq}(x) \), that form an orthogonal system
\[ \int \mathcal{D}x_1 x_2 x_3 P_{Nq}(x) P_{Nq}'(x) = \delta_{qq} c_{Nq}^{-1}. \] (8)
The index \( q \) enumerates different polynomials of the same degree. The polynomials \( P_{Nq} \) can be obtained as solutions of one-loop RG equation for twist–3 three quark operators. The expansion coefficients \( \phi_{Nq}(\mu_R) \) are related to the nucleon matrix element of local three quark operators
\[ \langle 0 | \bar{Q}^{(3)}_{Nq}(\mu_R) | \bar{O}_{Nq}(\mu_R) \rangle = \delta_{qq} c_{Nq}^{-1}. \]
Here, \( \bar{O}_{Nq}(\mu_R) \) is the (renormalized) light-ray operator
\[ \bar{O}_{Nq}(z) = e^{ijk} u^{i(\xi_1)}(\xi_2) u^{j(\xi_2)}(\xi_3) d^{k(\xi_3)}. \] (10)
Throughout this paper \( z = \{ z_1, z_2, z_3 \} \) and from now on we do not show the scale dependence.

The matrix element of the operator (9) can be parameterized as follows
\[ \langle 0 | \bar{Q}^{(3)}_{Nq} | \bar{N}_q \rangle = -\frac{i}{2} N^+_q (\bar{n}d)^{N+1} \phi_{Nq}. \] (11)
The reduced matrix element \( \phi_{Nq} \) can be expressed as a convolution integral
\[ \phi_{Nq} = \int \mathcal{D}x P_{Nq}(x) \Phi_3(x). \] (12)
To the one-loop accuracy the reduced matrix elements \( \phi_{Nq}(\mu_R) \) have an autonomous scale dependence. More details can be found in Refs. [28, 30].

The expansion of the higher twist DAs has a similar form
\[ \mathcal{F}(x) = \omega(x) x_1 x_2 x_3 + \ldots, \] (13)
where dots stand for the contribution of quark-gluon operators which are irrelevant for our purposes. For the twist–4 DAs, \( \mathcal{F} = \{ \Phi_4, \Psi_4, \Xi_4 \} \), the weight function takes the form \( \omega(x) = \{ x_1 x_2, x_1 x_3, x_2 x_3 \} \), respectively, and for the twist–5 DAs, \( \mathcal{F} = \{ \Phi_5, \Psi_5, \Xi_5 \} \) one finds \( \omega(x) = \{ x_3, x_1, x_2 \} \). The expansion coefficients \( \eta_{Nq}(\mu) \) are related to the nucleon matrix elements of the (multiplicatively renormalized) local operators of a collinear twist four and five, respectively. The corresponding polynomials \( \mathcal{F}_{Nq} \) can be obtained by solving RG equations. (For the twist four functions, this expansion was worked out in detail in Ref. [28].)
Since DAs $\Phi_4$ and $\Psi_4$ have the *collinear* twist four they receive contributions from the operators of both the *geometrical* twist four and three. The part due to twist−3 operators is nothing else than the Wandzura-Wilczek contribution

$$\Phi_4 = \Phi_4^{eff} + \Phi_4^{WW}, \quad \Psi_4 = \Psi_4^{eff} + \Psi_4^{WW}. \quad (14)$$

The chiral DA

$$\Phi_5 = \Phi_5^{eff} + \Phi_5^{WW}, \quad \Psi_5 = \Psi_5^{eff} + \Psi_5^{WW}, \quad (15)$$

$$_3\Sigma$$ does not receive contributions from twist−3 operators, i.e. $X_3 = X_3^{eff}$. Conformal expansion for the functions $\Phi_4^{WW}, \Psi_4^{WW}$ was obtained in Ref. [28].

Quite similarly, the twist−5 DAs can be represented as

$$\Phi_5 = \Phi_5^{eff} + \Phi_5^{WW}, \quad \Psi_5 = \Psi_5^{eff} + \Psi_5^{WW}, \quad \Xi_5 = \Xi_5^{eff} + \Xi_5^{WW}.$$ (15)

The functions $\Phi_5^{eff}, \Psi_5^{eff}$ and $\Xi_5^{eff}$ contain the contributions from the local operators of geometrical twist−5, while the WW parts entail the contributions from the operators of geometrical twist three and four.

III. TWIST−3 CONTRIBUTION TO $\Psi_4$.

The WW contribution to the DAs $\Phi_4, \Psi_4$ was originally derived in [28]. Here we streamline the derivation and obtain some results which will be used in the next section.

The nonlocal three-quark operator of twist−3 was defined in Eq. (19). Here we introduce two operators of collinear twist−4

$$\Omega_4(z, \mu) = e^{ijk}u^{ij}_3(z_1)u^{ij}_3(z_2)d^{ij}_3(z_3), \quad (16a)$$

$$\bar{\Omega}_4(z, \bar{\mu}) = e^{ijk}u^{ij}_3(z_1)u^{ij}_3(z_2)d^{ij}_3(z_3). \quad (16b)$$

The dependence of light-ray operators on the auxiliary spinors $\lambda, \bar{\lambda}$ is always implied and we will not display it explicitly.

The matrix element of the operator $\Omega_4(z, \mu)$ defines the DA $\Psi_4$, while the nucleon matrix element of the operator $\bar{\Omega}_4(z, \bar{\mu})$ vanishes because its helicity is equal to $3/2$. Nevertheless, we consider this operator since it will be necessary for our analysis of twist−5 functions.

The nonlocal operators $\Omega_3(z), \Omega_4(z, \mu), \bar{\Omega}_4(z, \bar{\mu})$ transform in a proper way under the transformations of the collinear $SL(2, R)$ subgroup of the conformal group. The representation $T^j$ of the $SL(2, R)$ group ($j$ is called a conformal spin) is defined by a transformation law

$$[T^j(g^{-1})f(z)] = \frac{1}{(cz+d)^{j+1}} f\left(\frac{ax+b}{cz+d}\right), \quad (17)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a unimodular real matrix. The generators of infinitesimal transformations $S_\pm, S_0$ have the form

$$S_+ = c^2 \partial_z + 2jz, \quad S_0 = z \partial_z + j, \quad S_- = - \partial_z \quad (18)$$

and obey the standard $sl(2)$ commutation relations

$$[S_+, S_-] = 2S_0, \quad [S_0, S_\pm] = \pm S_\pm. \quad (19)$$

The ‘$+$’ projections of the quark field, $q_+ (z)$ and $q_- (z)$, transform according to the representations $T^{j=1}$ and $T^{j=1/2}$, respectively. Thus, the operator $\bar{\Omega}_3(z)$ transforms according to the tensor product of representations, $T^1 \otimes T^1 \otimes T^1$, while the operators $\bar{\Omega}_4(z, \bar{\mu})$ transform according to the tensor products $T^{1/2} \otimes T^1 \otimes T^1$ and $T^1 \otimes T^{1/2} \otimes T^1$, respectively.

To the one loop accuracy (which we restrict ourselves to), the expansion of nonlocal operators in terms of local multiplicatively renormalized operators reads as follows

$$\Omega_3(z) = \sum_{N, k, q} a_{Nk} \Phi_{Nq}(z) \partial^k \Omega^{3-3}_{Nq}, \quad (20)$$

Here $\partial_n = (n \partial)$ is the derivative along the $n$ direction. The operator $S_+ = S_{1+} + S_{2+} + S_{3+}$ is the sum of one-particle generators. (In what follows we will assume that the conformal spins of the generators are always determined by the transformation properties of the objects they act on.) The coefficient $a_{Nk}$ in the expansion (20) has the form

$$a_{Nk} = \frac{\Gamma(2N + 6)}{k! \Gamma(2N + 6 + k)}, \quad (21)$$

which follows immediately from the consistency equation

$$\left( \partial_+ + S_- \right) \Omega_3(z) = 0. \quad (22)$$

To the one loop accuracy multiplicatively renormalized operators $\Omega^{3-3}_{Nq}$ are known to be the conformal operators that is they obey the following equation

$$[K, \Omega^{3-3}_{Nq}] = 0, \quad (23)$$

where $K = \bar{n}^p K_p$ and $K_p$ is the generator of special conformal transformations. We note also that the functions $\Phi_{Nq}(z)$ are shift invariant polynomials while the polynomials $P_{Nq}(x)$ form a biorthogonal basis $[30, 32]$

$$P_{Nq}(\partial_z \phi_n(z \mid z = 0) = \delta_{NN'} \delta_{qq'}. \quad (24)$$

The conformal expansion for the twist−4 operators $\Omega_4(z, \mu)$ and $\bar{\Omega}_4(z, \bar{\mu})$ reads

$$\Omega_4(z, \mu) = \sum_{N, k, q} b_{Nk} \Phi_{Nq}(z) \partial^k \Omega^{4-4}_{Nq}(\mu), \quad (25a)$$

$$\bar{\Omega}_4(z, \bar{\mu}) = \sum_{N, k, q} b_{Nk} \Phi_{Nq}(z) \partial^k \bar{\Omega}^{4-4}_{Nq}(\bar{\mu}), \quad (25b)$$

where

$$b_{Nk} = \frac{\Gamma(2N + 5)}{k! \Gamma(2N + 5 + k)} \quad (26)$$
and we tacitly assumed that the operator $S_+$ involves proper conformal spins that correspond to the transformation properties of these operators. The operators $O_{Nq}^{\mu}(\mu)$ are the conformal (lowest weight) operators, i.e.

$$[K_{-}, O_{Nq}^{\mu}(\mu)] = [\bar{K}_{-}, \bar{O}_{Nq}^{\mu}(\bar{\mu})] = 0.$$  \hspace{1cm} (27)

Among such lowest-weight operators there are descendants of the twist–3 operators, i.e. the operators of the collinear twist four which can be expressed of terms of the operators $O_{Nq}^{\mu}$. We refer to Ref. [31] where it was shown that this effect is related to the spin rotation of hadron in the rest frame.

The operators of geometric twist–4 contain the factor $(\mu \lambda)$ or $(\bar{\lambda} \bar{\mu})$. Indeed, to construct the operator of geometric twist–4 one has to antisymmetrize a pair of indices and symmetrize all others. After contraction with auxiliary spinors $\mu (\bar{\mu})$ and $\lambda , \bar{\lambda}$ the antisymmetrized pair produces the factor $(\mu \lambda)$ $(\bar{\lambda} \bar{\mu})$.

Our next task is to determine the functions, $\Psi^{(1)}_{Nq}, \bar{\Psi}^{(1)}_{Nq}$ and $\Psi^{(2)}_{Nq}, \bar{\Psi}^{(2)}_{Nq}$ which accompany the operators (28) in the expansion (25). To do this we note that replacing $\mu \rightarrow \mu$ and $\lambda \rightarrow \lambda$ in the definition (16a) for the nonlocal operator $O_{k}(z, \mu)$ one gets the twist–3 operator $O_{3}(z)$. Formally, it can be written as

$$\lambda^\alpha \frac{\partial}{\partial \mu^\alpha} O_{3}(z, \mu) = (\lambda \partial_\mu) O_{3}(z, \mu) = O_{3}(z),$$

and

$$\bar{\lambda}^\alpha \frac{\partial}{\partial \bar{\mu}^\alpha} \bar{O}_{3}(z, \bar{\mu}) = (\bar{\lambda} \partial_{\bar{\mu}}) \bar{O}_{3}(z, \bar{\mu}) = \bar{O}_{3}(z).$$  \hspace{1cm} (30)

First of all, we stress that the operation $(\lambda \partial_\mu)$, $(\bar{\lambda} \partial_{\bar{\mu}})$ kills all operators of the geometric twist–4 in the expansion (25). Indeed, as was discussed above, they contain the factor $(\mu \lambda)$ and $(\lambda \bar{\lambda}) (\mu \lambda) = 0$. Then, taking into account

$$(\lambda \partial_\mu) O_{Nq}^{\mu(\lambda)}(\mu) = (\bar{\lambda} \partial_{\bar{\mu}}) \bar{O}_{Nq}^{\lambda(\bar{\mu})}(\bar{\mu}) = \bar{O}_{Nq}^{\lambda(\bar{\mu})}$$

and

$$\bar{\lambda}^\alpha \frac{\partial}{\partial \bar{\mu}^\alpha} \bar{O}_{Nq}^{\lambda(\bar{\mu})}(\bar{\mu}) = \frac{1}{2(N+3)(N+5)} \partial_{+} \bar{O}_{Nq}^{\lambda(\bar{\mu})},$$  \hspace{1cm} (32)

we can bring Eqs. (30) into the following form

$$\sum_{N, k, q} \left\{ a_{nk} O_{Nq}^{\lambda(\bar{\mu})} - b_{nk} \bar{O}_{Nq}^{\lambda(\bar{\mu})} \right\} \frac{\partial_{+} \bar{O}_{Nq}^{\lambda(\bar{\mu})}}{2(N+3)(N+5)} = 0,$$

and similar for the second equation. The generators acting on the polynomials $\Phi_{Nq}, \Psi_{Nq}^{(a)}$ correspond to different conformal spins, $S_{+} = S_{+}^{(1)}$ and $\bar{S}_{+} = \bar{S}_{+}^{(1)}$ since the operators $O_{3}$ and $O_{4}$ transform according to different representations of the $SL(2, \mathbb{R})$ group. The coefficients at $\partial_{+} \bar{O}_{Nq}^{\lambda(\bar{\mu})}$ have to vanish identically for arbitrary $k$. To fix the functions $\Psi_{Nq}^{(1)}$ and $\Psi_{Nq}^{(2)}$ it is sufficient to consider the equations for $k = 0, 1$. Indeed, the equation for $k = 0$ results in

$$\Psi_{Nq}^{(1)}(z) = \Phi_{Nq}(z),$$  \hspace{1cm} (34)

while the equation for $k = 1$ gives

$$\Psi_{Nq}^{(2)}(z) = \left[ (2N+5)S_{+} - (2N+6)\bar{S}_{+} \right] \Phi_{Nq}(z).$$  \hspace{1cm} (35)
Using the representation \((\text{B.7})\) for the function \(S^{k}_{\pm} \Psi^{(i)}_{Nq}\), one can check that all equations for \(k > 1\) are satisfied provided \(\Psi^{(i)}_{Nq}\) are given by Eqs. \((\text{34})\), \((\text{35})\).

For the functions \(\bar{\Psi}^{(a)}_{Nq}\), the equations take the form

\[
\bar{\Psi}^{(1)}_{Nq}(z) = \Phi_{Nq}(z), \tag{36}
\]

\[
\bar{\Psi}^{(2)}_{N+1,q}(z) = \left[ (2N+5)S_{+}^{(11)} - (2N+6)S_{+}^{(1)} \right] \Phi_{Nq}(z).
\]

Note that the coefficient functions \(\Phi_{Nq}(\Psi_{Nq})\) which accompany the conformal (lowest weight) operators in the expansion of nonlocal operators are the shift invariant polynomials, \(S_{-}\Phi_{Nq}(z) = 0 (S_{-}\Psi_{Nq}(z) = 0)\) (see Refs. \([30,32]\) for details). One can easily check that the polynomials \((35), (36)\) indeed satisfy this condition.

Thus one gets the following expression for the contributions of the descendants of the twist–3 operators to the light-ray operators \(\bar{O}_{d}(z)\) (and similarly for \(\bar{O}_{d}(z)\))

\[
\bar{O}^{WW}_{d}(z) = \sum_{N,k,q} b_{Nk} \left\{ \Psi^{(1)}_{Nq}(z) \partial_{x}^{k} \bar{O}^{l=4,(1)}_{Nq}(\mu) + \Psi^{(2)}_{Nq}(z) \partial_{x}^{k} \bar{O}^{l=4,(2)}_{Nq}(\mu) \right\}.
\]

In order to find \(\Psi^{WW}_{d}(x)\) we take the nucleon matrix elements of both sides of Eq. \((37)\). By definition

\[
\langle 0 | \bar{O}^{WW}_{d}(z) | P \rangle = \frac{1}{4} (\mu \lambda) m_{N} N_{d}^{\uparrow} \int dx e^{-i(mx)} \bar{\Psi}^{WW}_{d}(x_{2},x_{1},x_{3}).
\]

In its turn, for the matrix elements of the operators \(\bar{O}^{l=4,(1)}_{Nq}(\mu), \bar{O}^{l=4,(2)}_{Nq}(\mu)\) one derives

\[
\langle 0 | \bar{O}^{l=4,(1)}_{Nq}(\mu) | P \rangle = \frac{1}{4} (\mu \lambda) m_{N} N_{d}^{\uparrow} \frac{-i \phi_{Nq}}{N+2},
\]

\[
\langle 0 | \bar{O}^{l=4,(2)}_{N+1,q} | P \rangle = -\frac{1}{8} (\mu \lambda) m_{N} N_{d}^{\uparrow} \frac{-i \phi_{Nq}}{N+3} \frac{1}{(2N+5)}.
\]

Here we take into account that nucleon has zero transverse momentum, \(p_{\mu \lambda} = p_{\lambda \mu} = 0\), and use the equation of motion relation \(2(p_{N} N_{d}^{\uparrow} = - (\mu \lambda) m_{N} N_{d}^{\uparrow}\).

Using Eqs. \((36)\) and the summation formula Eq. \((\text{B.10})\), one can bring the matrix element of the rhs of Eq. \((37)\) to the form

\[
\frac{1}{4} (\mu \lambda) m_{N} N_{d}^{\uparrow} \sum_{Nq} \Gamma(2N+5) \phi_{Nq} \int dx_{2} x_{3} e^{-i(px)} \bar{\Psi}^{WW}_{d}\left( x_{2},x_{1},x_{3} \right) \times \left\{ \frac{1}{N+2} P^{(1)}_{N,q}(x) - \frac{1}{N+3} P^{(2)}_{N+1,q}(x) \right\},
\]

where the polynomials \(P^{(1)}_{N,q}(x), P^{(2)}_{N+1,q}(x)\) are given by the \(sl(2)\) Fourier transform

\[
P^{(k)}_{N}(x) = \left\langle e^{\Sigma_{j=1}^{k} \xi_{j}} | \Phi^{(k)}_{Nq} \right\rangle \frac{1}{\xi_{1}}.
\]

Here \(\langle *, * \rangle\) stands for the \(sl(2)\) invariant scalar product that is defined in Appendix \(\text{B}\). We have to express these polynomials in terms of \(P_{Nq}\) which enter the expansion for twist–3 nucleon DA, \(\Phi_{3}(x)\). It follows from Eqs. \((1), \langle \text{7} \rangle, \langle \text{20} \rangle\)

\[
c_{Nq} P_{Nq} = \frac{\Gamma(2N+6)}{k! \Gamma(2N+4+k)} \left( e^{\Sigma_{j=1}^{k} \xi_{j}} | \Phi_{Nq} \right\rangle \langle 111 \rangle.
\]

Taking into account Eq. \((\text{B.11})\), one gets for \(P^{(1)}_{Nq}\)

\[
P^{(1)}_{Nq}(x) = r_{Nq} \partial_{x_{1}} P_{Nq}(x),
\]

where \(r_{Nq} = c_{Nq}/\Gamma(2N+6)\). Since the generators \(S_{+}^{(j)}\) and \(S_{+}^{(j)}\) (here \(j\) is multindex \(j = (j_{1}, j_{2}, j_{3})\)) are conjugated with respect to the corresponding scalar product, \(\langle \Psi | S_{+}^{(j)} \Phi \rangle = -\langle \Phi | S_{-}^{(j)} \Psi \rangle\), we obtain

\[
P^{(2)}_{N+1,q}(x) = r_{Nq} \left( (2N+5) - x_{123} \partial_{x_{1}} \right) \bar{x}_{123} P_{Nq}(x_{2},x_{1},x_{3}),
\]

where \(x_{123} = x_{1} + x_{2} + x_{3}\). Inserting \((43), (44)\) into \((40)\) and comparing the result with \((38)\), we derive the following expression

\[
\bar{\Psi}^{WW}_{d}(x) = -\sum_{Nq} c_{Nq} \frac{n_{Nq}}{(N+2)(N+3)} \times [N+2 - \partial_{x_{2}}] x_{123} x_{3} P_{Nq}(x_{2},x_{1},x_{3}).
\]

For \(\Phi^{WW}_{3}\) does not require a new calculation. It can be obtained from \((45)\) by a permutation of arguments, see sect. \(\text{V}\).

IV. TWIST–3 CONTRIBUTION TO \(\Phi^{WW}_{3}\).

The function \(\Phi_{3}\) receives the WW contributions from the operators of the geometric twist three and four. We discuss only the twist–3 contribution because it is a bit involved. The calculation of twist–4 part is straightforward and follows the lines of the previous section. The function \(\Psi^{WW}_{3}\) can be obtained from \(\Phi^{WW}_{3}\) by a simple permutation of arguments.

The distribution amplitude \(\Phi_{3}\) is determined by the nucleon matrix element of the light-ray operator

\[
\bar{O}_{S}(z) = e^{\Sigma_{j} u_{j}^{+} z_{j}} \bar{u}_{j}^{+} | (z_{2}) d^{+}_{j} | (z_{3})\tag{46}\]

This operator transforms according to the tensor product of \(SL(2,R)\) representations, \(T^{1/2} \otimes T^{1/2} \otimes T^{1}\). The expansion of the light-ray operator \((46)\) over local operators reads

\[
\bar{O}_{S}(z) = \sum_{Nq,k} d_{Nk} S^{k}_{+} N_{q}(z) \partial_{x}^{k} \bar{O}^{sl=5}_{Nq}.
\]

We emphasize that the three particle generator \(S_{+}\) in this expression carries the conformal spins corresponding to the transformation properties of the operator \(\bar{O}_{S}(z)\). \(j_{1} = j_{2} = 1/2, j_{3} = 1\). The coefficient \(d_{Nk}\) have the following form

\[
d_{Nk} = \frac{\Gamma(2N+4)}{k! \Gamma(2N+4+k)}\tag{48}.
\]
Again, among the lowest weight operators \( \mathcal{O}_{Nq}^{t=5} \) contributing to (47) there are descendants of the twist–3 and twist–4 operators \( \mathcal{O}_{Nq}^{t=3}, \mathcal{O}_{Nq}^{t=4} \). The descendants of twist–3 operators can be chosen as follows:

(i) the first descendant of \( \mathcal{O}_{Nq}^{t=3} \) takes the form

\[
\mathcal{O}_{Nq}^{t=5,(1)}(\mu, \bar{\mu}) = \frac{1}{(N+1)(N+2)}(\mu \partial_\lambda)(\bar{\mu} \partial_{\bar{\lambda}})\mathcal{O}_{Nq}^{t=3}, \tag{49}
\]

(ii) the two more are related to the operators defined in (29)

\[
\begin{align*}
\mathcal{O}_{N+1,q}^{t=5,(2)}(\mu, \bar{\mu}) &= \frac{1}{N+2}(\mu \partial_\lambda)(\mathcal{O}_{N+1,q}^{t=4,(2)}(\lambda) \mid \mu), \\
\mathcal{O}_{N+1,q}^{t=5,(3)}(\mu, \bar{\mu}) &= \frac{1}{N+3}(\mu \partial_\lambda)(\mathcal{O}_{N+1,q}^{t=4,(2)}(\lambda) \mid \bar{\mu})), \tag{50}
\end{align*}
\]

(iii) the last two operators are

\[
\begin{align*}
\mathcal{O}_{N+1,q}^{t=5,(4)}(\mu, \bar{\mu}) &= i[\mathcal{P}\bar{\mu}, \mathcal{O}_{Nq}^{t=3}], \\
\mathcal{O}_{N+2,q}^{t=5,(5)}(\mu, \bar{\mu}) &= \frac{1}{2N+3}
\begin{cases}
(\mu \partial_{\lambda})[\mathcal{P}\bar{\mu}, \mathcal{O}_{Nq}^{t=3}] \\
- (\mu \partial_{\lambda})[\bar{\mu}\partial_{\bar{\lambda}}(\mathcal{O}_{Nq}^{t=3})] \\
+ (\mu \partial_{\lambda})(\bar{\mu}\partial_{\bar{\lambda}}[\mathcal{O}_{Nq}^{t=3}])
\end{cases} \\
&\hspace{1cm} + \frac{1}{2N+7}[\mathcal{P}\bar{\mu}, \mathcal{O}_{Nq}^{t=3}]. \tag{51}
\end{align*}
\]

All these operators are the lowest weight (conformal) operators, \( \mathcal{K}, \mathcal{O}_{Nq}^{t=5,(a)} \mid a = 1, \ldots, 5 \). This fact is obvious for the first four operators while for the last one it can be checked with a help of the commutation relations given in Appendix 3.

The next step is to determine the coefficient functions \( \gamma_{Nq}^{a} \) in the expansion (47). To this end we consider the following two equations

\[
\begin{align*}
(\bar{\lambda} \partial_\beta)\mathcal{O}_{5}(z) &= \mathcal{O}_{4}(z), \tag{52a} \\
(\lambda \partial_{\mu})\mathcal{O}_{5}(z) &= \mathcal{O}_{4}(z). \tag{52b}
\end{align*}
\]

Let us substitute the expansions (47) and (28) into Eqs. (52). The derivatives \( (\bar{\lambda} \partial_\beta)\mathcal{O}_{Nq}^{t=5,(a)}, (\lambda \partial_{\mu})\mathcal{O}_{Nq}^{t=5,(a)} \) of the local operators can be expressed in terms of the operators (28), (29) and their \( \partial_\beta \), derivatives. For the sake of completeness, we collect the corresponding expressions in Appendix 3. Note that the derivatives \( (\bar{\lambda} \partial_\beta), (\lambda \partial_{\mu}) \) annihilate all operators of geometric twist–5 by the same reason as discussed in sect. III.

As a result, Eqs. (52) take the form:

\[
\begin{align*}
\sum_{Nqk} \left( C_{Nqk}^{(1)} \mathcal{O}_{N+1,q}^{t=4,(4)} + C_{Nqk}^{(2)} \mathcal{O}_{N+1,q}^{t=4,(2)} \right) &= 0, \\
\sum_{Nqk} \left( \bar{C}_{Nqk}^{(1)} \mathcal{O}_{N+1,q}^{t=4,(4)} + \bar{C}_{Nqk}^{(2)} \mathcal{O}_{N+1,q}^{t=4,(2)} \right) &= 0. \tag{53}
\end{align*}
\]

The coefficients \( C_{Nqk}^{(1)}, \bar{C}_{Nqk}^{(1)} \) are given by some linear combinations of the polynomials \( \gamma_{Nq}^{a} \). Since the operators \( \partial_\beta \mathcal{O}_{Nq}^{t=4,(a)}, \partial_{\bar{\beta}} \mathcal{O}_{N+1,q}^{t=4,(a)} \) are independent all coefficients \( C_{Nqk}^{(1)}, \bar{C}_{Nqk}^{(1)} \) have to vanish. This requirement results in an infinite number of equations on the \( \gamma_{Nq}^{a} \)–functions. Only a few of them, however, are independent. To fix the functions \( \gamma_{Nq}^{a} \), it is sufficient to consider the equations \( \gamma_{Nq}^{(i)} = 0, \gamma_{Nq}^{(i)} = 0 \) for \( k = 0, 1 \). All other equations will be satisfied automatically that can be checked with the help of Eq. (B.7).

The equations \( \gamma_{Nq}^{(1)} = 0, k = 0, 1 \) and \( \gamma_{Nq}^{(2)} = 0, k = 0 \) take the form

\[
\begin{align*}
\sum_{a=2}^{4} B_{Nq}^{a} \gamma_{N+1,q}^{(a)}(z) &= \frac{S_{a}^{(4,1)} S_{a}^{(4,1)}}{2N+5} \Phi_{Nq}(z), \\
\sum_{a=2}^{4} \bar{B}_{Nq}^{a} \gamma_{N+1,q}^{(a)}(z) &= \frac{S_{a}^{(4,1)} S_{a}^{(4,1)}}{2N+5} \Phi_{Nq}(z), \\
\sum_{a=2}^{4} A_{Nq}^{a} \gamma_{N+1,q}^{(a)}(z) &= \Psi_{N+1,q}^{(2)}(z), \\
\sum_{a=2}^{4} \bar{A}_{Nq}^{a} \gamma_{N+1,q}^{(a)}(z) &= \Psi_{N+1,q}^{(2)}(z). \tag{54}
\end{align*}
\]

The coefficients \( A_{Nq}^{a}, \bar{A}_{Nq}^{a}, B_{Nq}^{a}, \bar{B}_{Nq}^{a} \) can be found in Appendix 3. These equations are sufficient to fix the functions \( \gamma_{N+1,q}^{(a)} \mid a = 2, 3, 4 \). In fact, it is sufficient to consider any three of them, the last equation provides a consistency check.

The remaining function \( \gamma_{N+2,q}^{(5)} \) can be determined from the equation \( \gamma_{N+2,q}^{(2)} = 0 \) which implies

\[
\begin{align*}
C_{N} \gamma_{N+2,q}^{(5)}(z) &= \frac{S_{a}^{(4,1)} S_{a}^{(4,1)}}{2N+7} \Psi_{N+1,q}^{(2)}, \tag{55}
\end{align*}
\]

where the coefficient \( C_{N} \) is given in Eq. (C3). The functions \( \gamma_{Nq}^{a} \) are shift invariant, \( S_{a} \gamma_{Nq}^{a} = 0, \) as they should be.

To proceed further, we need to calculate the \(SL(2)\) Fourier transform of the functions \( \gamma_{Nq}^{a} \)

\[
\mathcal{F}_{Nq}^{(a)}(x) = \left\{ \Sigma_{Nq}^{a} \gamma_{Nq}^{a}(z) \right\}_{x+1}. \tag{57}
\]

Using the method described in the previous section we obtain

\[
\begin{align*}
\mathcal{F}_{Nq}^{(1)}(x) &= r_{Nq} \partial_{x_1} \partial_{x_2} x_{1} x_{2} P_{Nq}(x), \\
\mathcal{F}_{N+2,q}^{(5)}(x) &= r_{Nq} \partial_{x_1} \partial_{x_2} \left( 2N+6 - x_{123} \partial_{x_2} \right) \\
&\times (2N+5 - x_{123} \partial_{x_1}) x_{1} x_{2} P_{Nq}(x), \tag{58}
\end{align*}
\]
where, again, $x_{123} = x_1 + x_2 + x_3$ and
\[ x_N = (4(N + 3)(N + 4)(2N + 5)(2N + 6))^{-1}. \]  
(59)

The expressions for the polynomials $\mathcal{B}_{N+1,q}^{(2,3,4)}$ are a bit more complicated
\[
\mathcal{B}_{N+1,q}^{(2)}(x) = r_{Nq} \left\{ \frac{2N + 5}{(N + 1)} \left[ \frac{3N + 5}{2(N + 2)} x_{123} \partial x_1 \partial x_2 ight. \right.
\left. + (N + 1) \partial x_1 + 2(N + 2) \partial x_2 \right\} x_1 x_2 P_{Nq}(x),
\]
\[
\mathcal{B}_{N+1,q}^{(3)}(x) = r_{Nq} \left\{ \frac{(2N + 5)(N + 3)}{(N + 1)(N + 2)} \left[ - \frac{3N + 4}{2(N + 2)} x_{123} \partial x_1 \partial x_2 ight. \right.
\left. + 2(N + 1) \partial x_1 + (N + 2) \partial x_2 \right\} x_1 x_2 P_{Nq}(x),
\]
\[
\mathcal{B}_{N+1,q}^{(4)}(x) = \frac{r_{Nq}}{2(N + 1)(N + 2)} \left\{ \frac{1}{2} x_{123} \partial x_1 \partial x_2 ight.
\left. - (N + 1) \partial x_1 - (N + 2) \partial x_2 \right\} x_1 x_2 P_{Nq}(x). \]  
(60)

Next, we represent the nucleon matrix element of the operators $\mathcal{O}_{Nq}^{(5,6)}$ in the form
\[ \langle 0 | \mathcal{O}_{Nq}^{(5,6)}(a) | P \rangle = - \frac{1}{4} m_N (\mu \lambda) N^\dagger (-ip)^N \Theta_{Nq}^{(a)}, \]  
(61)

Taking into account Eqs. (49), (50), (51) and Eq. (11), one obtains for the reduced matrix elements $\Theta_{Nq}^{(a)}$
\[
\Theta_{Nq}^{(1)} = \frac{\phi_{Nq}}{(N + 2)},
\]
\[
\Theta_{N+1,q}^{(2)} = \frac{(N + 4)\phi_{Nq}}{2(N + 2)(N + 3)^2(2N + 5)},
\]
\[
\Theta_{N+1,q}^{(3)} = \frac{\phi_{Nq}}{(N + 2)(N + 3)^2(2N + 5)},
\]
\[
\Theta_{N+1,q}^{(4)} = \frac{2\phi_{Nq}}{(N + 2)(N + 3)^2(2N + 5)},
\]
\[
\Theta_{N+2,q}^{(5)} = - \frac{2(N + 4)\phi_{Nq}}{(N + 3)(2N + 7)}. \]  
(62)

Using the summation formula (15,19), one obtains for the Wandzura-Wilczek contribution to the matrix element of the operator $\mathcal{O}_5(z)$
\[ \langle 0 | \mathcal{O}_5(z) | P \rangle = - \frac{1}{4} m_N (\mu \lambda) N^\dagger \sum_{Nq} \int \mathcal{D} x_3 e^{-ip_3 \cdot x_3} \times \]
\[ \left\{ \zeta_N \Theta_{Nq}^{(1)} \Theta_{Nq}^{(1)}(x) + \zeta_{N+2} \Theta_{N+2,q}^{(5)}(x) \right\}, \]  
(63)

where $\zeta_N = \Gamma(2N + 4)$. Substituting the explicit expressions for the polynomials $\mathcal{B}_{Nq}$ and the reduced matrix elements $\Theta_{Nq}$, one finds after some algebra
\[ \langle 0 | \mathcal{O}_5^{(5)}(z) | P \rangle = - \frac{1}{4} m_N (\mu \lambda) N^\dagger \int \mathcal{D} x e^{-i(p \cdot x)} \sum_{Nq} \times \sum_{Nq} \mathcal{B}_{Nq}(x), \]  
(64)

where
\[ \mathcal{B}_{Nq}(x) = \left\{ (N + 2 - \partial x_1)(N + 1 - \partial x_2) \right\} \]  
\[ \times - (N + 2)^2 x_1 x_2 x_3 P_{Nq}(x). \]  
(65)

Comparing (62) with the definition (6), one gets
\[ \Phi_3^{WW}(x) = \sum_{Nq} \frac{c_{Nq} \phi_{Nq}(\mu)}{(N + 2)(N + 3)} \mathcal{B}_{Nq}(x). \]  
(66)

The expression for $\Psi_3^{WW}(x)$ is given in sect. IV and can be restored from (66) by a simple permutation of arguments. The calculation of the WW contribution from twist–4 operators to the DAs $\Phi_3$, $\Psi_3$ and $\Xi_3$ is straightforward and we only present the final result in sect. IV.

V. RESULTS

In this section we collect all results for the nucleon DAs. The conformal expansion for twist–3 and genuine (geometric) twist–4 DAs reads
\[ \Phi_3(x) = x_1 x_2 x_3 \sum_{Nq} c_{Nq} \phi_{Nq}(\mu R) P_{Nq}(x), \]
\[ \Phi_4^{d=4}(x) = x_1 x_2 \sum_{Nq} \Lambda_{Nq} \eta_{Nq}(\mu R) R_{Nq}(x), \]
\[ \Psi_4^{d=4}(x) = x_1 x_2 \sum_{Nq} \Lambda_{Nq} \eta_{Nq}(\mu R) \bar{R}_{Nq}(x), \]
\[ \Xi_4(x) = x_2 x_3 \sum_{Nq} B_{Nq} \bar{\xi}_{Nq}(\mu R) \Pi_{Nq}(x). \]  
(67)

Explicit expressions for the first few functions $P_{Nq}, R_{Nq}, \bar{R}_{Nq}, \Pi_{Nq}$ can be found in Ref. [29]. The Wandzura-Wilczek contributions to the twist four DAs $\Phi_4$ and $\Psi_4$ take the form [29]
\[ \Phi_4^{WW}(x) = - \sum_{Nq} \frac{c_{Nq} \phi_{Nq}(\mu R)}{(N + 2)(N + 3)} [N + 2 - \partial x_1] \]  
\[ \times x_1 x_2 x_3 P_{Nq}(x_1, x_2, x_3), \]
\[ \Psi_4^{WW}(x) = - \sum_{Nq} \frac{c_{Nq} \phi_{Nq}(\mu R)}{(N + 2)(N + 3)} [N + 2 - \partial x_2] \]  
\[ \times x_1 x_2 x_3 P_{Nq}(x_2, x_1, x_3). \]  
(68)
The WW contributions to the twist five DAs $\Phi_5$ and $\Psi_5$ from the twist–3 operators read
\[
\Phi_5^{WW}(x) = \sum_{Nq} \frac{c_{Nq} \phi_{Nq}(\mu)}{(N+2)(N+3)} \left( (N + 2 - \partial_{x_3})(N + 1 - \partial_{x_2}) - (N + 2)^2 \right) x_1 x_2 x_3 P_{Nq}(x_1, x_2, x_3),
\]
\[
\Psi_5^{WW}(x) = \sum_{Nq} \frac{c_{Nq} \phi_{Nq}(\mu)}{(N+2)(N+3)} \left( (N + 2 - \partial_{x_3})(N + 1 - \partial_{x_2}) - (N + 2)^2 \right) x_1 x_2 x_3 P_{Nq}(x_1, x_2, x_3),
\]
\[
\Phi_5^{WW}(x) = \sum_{Nq} \frac{\bar{A}_{Nq} \eta_{Nq}(\mu)}{(N+1)(N+3)} \left[ N + 1 - \partial_{x_3} \right] x_1 x_2 x_3 \bar{R}_{Nq}(x_2, x_1, x_3),
\]
\[
\Psi_5^{WW}(x) = \sum_{Nq} \frac{\bar{A}_{Nq} \eta_{Nq}(\mu)}{(N+1)(N+3)} \left[ N + 1 - \partial_{x_2} \right] x_1 x_2 x_3 \bar{R}_{Nq}(x_2, x_1, x_3).
\]

Finally, for the chiral DA $\Xi_5$, we obtain
\[
\Xi_5^{WW}(x) = - \sum_{Nq} \frac{B_{Nq} \tilde{\xi}_{Nq}(\mu)}{(N+1)(N+3)} \left\{ \left[ N + 1 - \partial_{x_3} \right] x_1 x_3 \Pi_{Nq}(x_2, x_1, x_3) + \Pi_{Nq}(x_3, x_1, x_2) \right\}. 
\]

One can check that the first moments of the WW contribution to the twist–5 DAs agrees with the results of Refs. [14,15].

Acknowledgements

The authors are grateful to V. M. Braun for useful discussions. This work was supported by the German Research Foundation (DFG), grant BR2021/5-2, grant BR 2021/6-1 and in part by the RFBR (grant 12-02-00613) and the Heisenberg-Landau Program.

Appendices

Appendix A: Conformal group

In this Appendix we collect the commutators of the conformal group generators that are necessary to construct the lowest weight (conformal) operators. The generators in the spinor representations and their commutators can be found in Ref. [29]. We list below the relevant ones. Starting from
\[
[i K_{\alpha \alpha}, i P^{\beta \beta}] = 4 \left( \frac{\delta_{\alpha}^{\beta}}{\delta_{\alpha}^{\beta}} D + \delta^{\beta}_{\alpha} i M_{\alpha}^{\beta} + \delta_{\alpha}^{\beta} i M_{\alpha}^{\beta} \right),
\]
where $D$, $M_{\alpha \beta}$ and $\bar{M}_{\alpha \beta}$ are the generators of dilatations and Lorentz rotations, respectively, one gets
\[
\begin{align*}
[i K_{\mu \beta}, i P_{\lambda \lambda}] &= 4 i \left( (\mu \lambda)(\bar{\lambda} \bar{\mu}) D - (\bar{\lambda} \bar{\mu}) M_{\mu \lambda} - (\mu \lambda) \bar{M}_{\mu \lambda} \right), \\
[i K_{\mu \beta}, i P_{\lambda \lambda}] &= -4 (\bar{\lambda} \bar{\mu}) M_{\mu \lambda}, \\
[i K_{\mu \beta}, i P_{\lambda \lambda}] &= -4 (\mu \lambda) \bar{M}_{\mu \lambda},
\end{align*}
\]
where $P_{\mu \lambda} = \mu^{\alpha} P_{\alpha \alpha \lambda} \lambda$, etc. One also finds
\[
\begin{align*}
[i M_{\mu \lambda}, i P_{\mu \lambda}] &= \frac{(\mu \lambda)}{2} i P_{\mu \lambda}, \\
[i \bar{M}_{\mu \lambda}, i P_{\mu \lambda}] &= \frac{(\bar{\mu} \bar{\lambda})}{2} i P_{\mu \lambda}, \\
[i \bar{M}_{\mu \lambda}, i P_{\mu \lambda}] &= \frac{(\bar{\mu} \bar{\lambda})}{2} i P_{\mu \lambda}.
\end{align*}
\]
Let $\mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda})$ be an operator of Lorentz spin $j$, $\bar{j}$ at point $x = 0$ with all indices contracted with spinors $\lambda, \bar{\lambda}$. Such an operator transforms under Lorentz rotations as follows
\[
\begin{align*}
i [M_{\mu \lambda}, \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda})] &= - \frac{1}{2} \left( \lambda_{\mu} \frac{\partial}{\partial \lambda^\rho} + \lambda_{\rho} \frac{\partial}{\partial \lambda^\mu} \right) \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda}), \\
i [\bar{M}_{\mu \lambda}, \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda})] &= - \frac{1}{2} \left( \bar{\lambda}_{\rho} \frac{\partial}{\partial \lambda^\mu} + \bar{\lambda}_{\mu} \frac{\partial}{\partial \lambda^\rho} \right) \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda}).
\end{align*}
\]
One easily finds
\[
\begin{align*}
i [M_{\mu \lambda}, \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda})] &= - 2 j (\mu \lambda) \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda}), \\
i [\bar{M}_{\mu \lambda}, \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda})] &= - 2 \bar{j} (\bar{\lambda} \bar{\mu}) \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda}), \\
i [M_{\mu \lambda}, \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda})] &= -(\mu \lambda)(\mu \bar{\lambda}) \mathcal{O}^{ij}(\bar{\lambda}, \bar{\lambda}).
\end{align*}
\]

Appendix B

In this Appendix, we collect some useful representations for the coefficient functions $\Psi_{Nq}$. Let $\langle *, * \rangle_j$ be a $n$-particle $sl(2)$ invariant scalar product defined as
\[
\langle \Psi, \Phi \rangle_j = \prod_{k=1}^{n} \int_{|z_k| < 1} d^2 z_k \mu_k(z_k) \bar{\Omega}(z) \Phi(z),
\]
where $z_k$ are complex variables, $j = \{ j_1, \ldots, j_n \}$ is a multi-index and the weight function $\mu_k(z_k)$ is given by the following expression
\[
\mu_k(z_k) = \frac{2(2j_k - 1)}{\pi} (1 - |z_k|^2)^2 j_k - 2. 
\]
The generator $S_0 = \sum_i S_0^{(i)}$ is self-adjoint with respect to this scalar product, $S_0^{\dagger} = S_0$ and $S_0^2 = -S_0$. The unit operator (or the reproducing kernel) on this space takes the form

$$K_j(z, w) = \sum_{k=1}^n (1 - z_k w_k)^{-2j}. \quad (B.3)$$

One can easily check that

$$\Phi(z) = \langle K_j(z, w) \Phi(w) \rangle_j. \quad (B.4)$$

We define the $sl(2)$ Fourier transform of the function $\Psi_N(z_1, \ldots, z_n)$ as follows

$$P_N(u_1, \ldots, u_n) = \left< \sum_{n=1}^N \zeta_n u_n \right> \Psi_N(z), \quad (B.5)$$

Notice that

$$\left< \Psi_N(z) \right> = \left< \sum_{n=1}^N \zeta_n u_n \right> \Psi_N(z). \quad (B.6)$$

For an arbitrary polynomial $\Psi_N$, one gets the following relation between the polynomials $P_N$ and $\Psi_N$ \[ \frac{\langle \zeta_n \rangle}{\langle \zeta_n \rangle^2} \Psi_N(z) = \left< \sum_{n=1}^N \zeta_n u_n \right> \Psi_N(z). \quad (B.7) \]

Another useful relation holds for the shift invariant polynomials $\Psi_N$, $S_\Psi \Psi_N = 0$. (Note that the coefficient functions $\Psi_{Nq}$ of conformal operators entering the expansion of nonlocal operators are always shift invariant.) This new relation reads

$$S_N^{(k)} \Psi_N(z) = \frac{k!}{(N+k)!} \prod_{m=1}^n \Gamma(2j_m) \int_0^\infty du_m u^{2j_m-1} e^{-u_m} \sum_{n=1}^N \chi_n u_n \Psi_N(u_1, \ldots, u_n). \quad (B.8)$$

Taking into account that $P_N(u)$ is a polynomial of the degree $N$ and separating the so-called infinite volume integration, $\Lambda = \sum u_i$, we arrive at

$$a_{Nk}(j) S_N^{(k)} \Psi_N(z) = \frac{\Lambda_{N}(j)}{(N+k)!} \prod_{m=1}^n \Gamma(2j_m) \int_0^\infty du_m u^{2j_m-1} e^{-u_m} \sum_{n=1}^N \chi_n u_n \Psi_N(u_1, \ldots, u_n). \quad (B.8)$$

The integral goes on $N$–dimensional simplex, i.e.

$$\mathcal{D}_n \chi = \prod_{m=1}^n d\chi_m \delta \left( 1 - \sum_{i=1}^N \chi_i \right), \quad (B.9)$$

and the coefficients $a_{Nk}(j)$ and $\Lambda_{N}(j)$ read

$$a_{Nk}(j) = \frac{\Gamma(2N+2j)}{k! \Gamma(2N+2j+k)}, \quad \Lambda_{N}(j) = \frac{\Gamma(2N+2j)}{\Gamma(2j_1) \ldots \Gamma(2j_n)},$$

where $J = \sum_{i=1}^n j_i$ is the total conformal spin. The representation $\left< B.8 \right>$ allows us to resum a series over $k$ which appears in the expansion of nonlocal operators. Namely, we have

$$\sum_{k=0}^\infty (-i)^{\gamma_{n+k}} a_{Nk}(j) S_N^{(k)} \Psi_N(z) =$$

$$= \chi_{N}(j) \int \mathcal{D}x \prod_{m=1}^n \frac{2^{j_m-1} e^{-i(pm)\Sigma \chi_n p_n}}{\Gamma(n+2j)}, \quad (B.10)$$

where we take into account that an integral of the polynomial $P_N$ with any polynomial of less degree vanishes.

Finally, let us note that there exists the following relation between the $sl(2)$ Fourier transforms

$$\langle e^{i\zeta_2} \Psi \rangle_{j_2,j_3,...} = \partial_{\zeta_1} u_1 \langle e^{i\zeta_2} \Psi \rangle_{j_1,j_2,j_3,...} \quad (B.11)$$

which can easily be checked by expanding the exponent in a power series and taking into account that

$$|e^{i\beta}|^2 = \frac{\Gamma(2j+1)}{\Gamma(n+2j)}. \quad (B.12)$$

Eq. (B.11) allows one to express polynomials $\mathcal{D}_{Nq}$ entering WW part of higher twist DAs in terms of the twist–3 nucleon functions $P_{Nq}$.

**Appendix C**

In this Appendix we collect expressions for the derivatives of the twist–5 operators $\mathcal{O}_{Nq}^{t=5,(a)}$ that were used in sect. I[V] in order to derive the equations on the functions $Y_{Nq}$. One easily finds

$$\tilde{\lambda} \partial_\mu \mathcal{O}_{Nq}^{t=5,(1)} = \mathcal{O}_{Nq}^{t=4,(1)}, \quad (\tilde{\lambda} \partial_\mu \mathcal{O}_{Nq}^{t=5,(1)} = \mathcal{O}_{Nq}^{t=4,(1)}). \quad (C.1)$$

For $a = 2, \ldots, 4$, we obtain

$$\tilde{\lambda} \partial_\mu \mathcal{O}_{Nq}^{t=5,(a)} = A_{Nq}^{t=5,(a)} + B_{Nq}^{t=5,(a)} \mathcal{O}_{Nq}^{t=4,(1)}, \quad \tilde{\lambda} \partial_\mu \mathcal{O}_{Nq}^{t=5,(a)} = A_{Nq}^{t=5,(a)} + B_{Nq}^{t=5,(a)} \mathcal{O}_{Nq}^{t=4,(1)}, \quad (C.2)$$

where

$$A_{Nq} = \left\{ 1, (2N+5)^{-1}, 4(N+3)^2 \right\}, \quad B_{Nq} = \frac{N+2}{2N+5} \left\{ 0, ((2N+5)(N+3))^{-1}, 2 \right\}, \quad \tilde{A}_{Nq} = \left\{ \frac{N+4}{(N+2)(2N+5)}, 1, 4(N+3)(N+4) \right\}, \quad \tilde{B}_{Nq} = \frac{N+1}{2N+5} \left\{ ((2N+5)(N+2))^{-1}, 0, 2 \right\}. \quad (C.3)$$

Finally for the last operator we derive

$$\langle \tilde{\lambda} \partial_\mu \mathcal{O}_{Nq}^{t=5,(5)} = C_{Nq}^{t=5,(2)} \mathcal{O}_{Nq}^{t=4,(1)} \rangle, \quad \langle \tilde{\lambda} \partial_\mu \mathcal{O}_{Nq}^{t=5,(5)} = C_{Nq}^{t=5,(2)} \mathcal{O}_{Nq}^{t=4,(1)} \rangle. \quad (C.4)$$
where

\[ C_N = \frac{4(N + 3)(N + 4)(2N + 5)}{2N + 7} \quad \text{(C.5)} \]

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