Defects in $G/H$ coset, $G/G$ topological field theory and discrete Fourier-Mukai transform.

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Abstract
In this paper we construct defects in coset $G/H$ theory. Canonical quantization of the gauged WZW model $G/H$ with $N$ defects on a cylinder and a strip is performed and the symplectomorphisms between the corresponding phase spaces and those of double Chern-Simons theory on an annulus and a disc with Wilson lines are established. Special attention to topological coset $G/G$ has been paid. We prove that a $G/G$ theory on a cylinder with $N$ defects coincides with Chern-Simons theory on a torus times the time-line $R$ with $2N$ Wilson lines. We have shown also that a $G/G$ theory on a strip with $N$ defects coincides with Chern-Simons theory on a sphere times the time-line $R$ with $2N + 4$ Wilson lines. This particular example of topological field theory enables us to penetrate into a general picture of defects in semisimple 2D topological field theory. We conjecture that defects in this case described by a 2-category of matrices of vector spaces and that the action of defects on boundary states is given by the discrete Fourier-Mukai transform.

Keywords: conformal field theory, gauged WZW models, defects, Fourier-Mukai transform, 2-categories, topological field theory, D-branes.

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1 Introduction

In this paper we study defects in the gauged WZW model. We construct the phase spaces of the gauged WZW model in the presence of defects and relate them to the moduli spaces of flat connections on punctured Riemann surfaces. We generalize here the results of the paper [37] on defects in the WZW model to the coset model, using canonical quantization technique of the gauged WZW model developed in [23]. Let us briefly remind the principal results of these papers.

In the paper [37], canonical quantization of the WZW model with defects has been performed. Using the Lagrangian formulation of the WZW model with defects and boundaries the following symplectomorphisms have been established:

1. The symplectic phase space of the WZW model with $N$ defects on a cylinder is isomorphic to that of a Chern-Simons theory on the annulus $\mathcal{A}$ times the time-line $\mathbb{R}$ with $N$ time-like Wilson lines.

2. The symplectic phase space of the WZW model with $N$ defects on a strip is isomorphic to that of Chern-Simons theory on a disc $D$ times the time-line $\mathbb{R}$ with $N + 2$ time-like Wilson lines.

In the paper [23], using the Lagrangian formulation of the gauged WZW $G/H$ model on a cylinder and on a strip the following symplectomorphisms have been established:

3. The phase space of the gauged WZW $G/H$ model on a cylinder is isomorphic to the phase space of the double Chern-Simons theory on the annulus $\mathcal{A}$ times the time-line $\mathbb{R}$.

4. The phase space of the gauged WZW $G/H$ model on a strip is isomorphic to the phase space of the double Chern-Simons theory on $D \times \mathbb{R}$ with $G$ and $H$ gauge fields both coupled to two time-like Wilson lines.

In the special case of topological coset $G/G$ these isomorphisms take the form:

5. The phase space of the gauged WZW $G/G$ model on a cylinder is isomorphic to the phase space of the Chern-Simons theory on the torus $T^2 = \mathcal{A} \cup (-\mathcal{A})$ times the time-line $\mathbb{R}$.
6. The phase space of the gauged WZW $G/G$ model on a strip is isomorphic to the phase space of the Chern-Simons theory on the sphere $S^2 = D \cup (-D)$ times $R$ with four time-like Wilson lines.

We show here that the combined versions of the symplectomorphisms 1-2 and 3-4 for the gauged WZW model in the presence of defects take the form:

7. The phase space of the gauged WZW $G/H$ model on a cylinder with $N$ defects is isomorphic to the phase space of the double Chern-Simons theory on $A \times R$ with $G$ and $H$ gauge fields both coupled to $N$ Wilson lines.

8. The phase space of the gauged WZW $G/H$ model on a strip with $N$ defects is isomorphic to the phase space of the double Chern-Simons theory on $D \times R$ with $G$ and $H$ gauge fields both coupled to $N + 2$ time-like Wilson lines.

In the special case of the topological coset $G/G$ these isomorphisms take the form:

9. The phase space of the gauged WZW $G/G$ model on a cylinder with $N$ defects is isomorphic to the phase space of the Chern-Simons theory on $T^2 \times R$ with $2N$ Wilson lines.

10. The phase space of the gauged WZW $G/G$ model on a strip with $N$ defects is isomorphic to the phase space of the Chern-Simons theory on $S^2 \times R$ with $2N + 4$ time-like Wilson lines.

The last two isomorphisms allow us to achieve a very detailed picture of defects in this particular example of topological field theory. This picture enables us to conjecture that in general defects in semisimple 2D TFT should be described by means of a 2-category of matrices of vector spaces. Previously the relation of defects and 2-categories in conformal field theory was discussed in [9,15,38].

The paper is organised in the following way.

In the second section we review the canonical quantization of the WZW model. In the third section we review the canonical quantization of the gauged WZW model. In section 4 we present defects in gauged WZW model, perform canonical quantization and establish the isomorphism 7 in the list above. In section 5 we consider defects in the gauged WZW model on a strip and establish the isomorphism number 8. In section 6 we consider defects in the topological coset $G/G$,
establish isomorphisms 9 and 10, and make conjectures on defects properties in
general 2D semisimple TFT.

2 Bulk WZW model

In this section we review the canonical quantization of the WZW model with
compact, simple, connected and simply connected group $G$ on the cylinder $\Sigma = R \times S^1 = (t, x \mod 2\pi)$ [6,13,20]. The world-sheet action of the bulk WZW model is [40]

\[
S^{WZW}(g) = \frac{k}{4\pi} \int_{\Sigma} \text{Tr}(g^{-1} \partial_+ g)(g^{-1} \partial_- g) dx^+ dx^- + \frac{k}{4\pi} \int_B \frac{1}{3} \text{tr}(g^{-1} dg)^3 \quad (1)
\]

\[
\equiv \frac{k}{4\pi} \left[ \int_{\Sigma} dx^+ dx^- L^{\text{kin}} + \int_B \omega^{WZ} \right],
\]

where $x^\pm = x \pm t$. The phase space of solutions $\mathcal{P}$ can be described by the Cauchy data at $t = 0$:

\[
g(x) = g(0, x) \quad \text{and} \quad \xi_0(x) = g^{-1} \partial_t g(0, x). \quad (2)
\]

The corresponding symplectic form is [20]:

\[
\Omega^{\text{bulk}} = \frac{k}{4\pi} \int_0^{2\pi} \Pi^G(g) dx, \quad (3)
\]

where

\[
\Pi^G(g) = \text{tr} \left( -\delta \xi_0 g^{-1} \delta g + (\xi_0 + g^{-1} \partial_x g)(g^{-1} \delta g)^2 \right). \quad (4)
\]

The $\delta$ denotes here the exterior derivative on the phase space $\mathcal{P}$. It is easy to check that the symplectic form density $\Pi(g)$ has the following exterior derivative

\[
\delta \Pi^G(g) = \partial_x \omega^{WZ}(g), \quad (5)
\]

what implies closedness of the $\Omega$:

\[
\delta \Omega^{\text{bulk}} = 0. \quad (6)
\]

The classical equations of motion are

\[
\partial_- J_L = 0 \quad \text{and} \quad \partial_+ J_R = 0, \quad (7)
\]

\footnote{Surely we can choose any time slice, but for simplicity we always below take the slice $t = 0$.}
where
\[ J_L = -ik \partial_+ g g^{-1} \quad \text{and} \quad J_R = ik g^{-1} \partial_- g. \quad \tag{8} \]
The general solution of (7) satisfying the boundary conditions:
\[ g(t, x + 2\pi) = g(t, x) \quad \tag{9} \]
is
\[ g(t, x) = g_L(x^+) g_R^{-1}(x^-) \quad \tag{10} \]
with \( g_{L,R} \) satisfying the monodromy conditions:
\[ g_L(x^+ + 2\pi) = g_L(x^+) \gamma, \quad \tag{11} \]
\[ g_R(x^- + 2\pi) = g_R(x^-) \gamma \quad \tag{12} \]
with the same matrix \( \gamma \). Expressing the symplectic form density \( \Pi^G(g) \) in the terms of \( g_{L,R} \) we obtain:
\[ \Pi^G = \text{tr} \left[ g_L^{-1} \delta g_L \partial_x (g_L^{-1} \delta g_L) - g_R^{-1} \delta g_R \partial_x (g_R^{-1} \delta g_R) + \partial_x (g_L^{-1} \delta g_L g_R^{-1} \delta g_R) \right]. \quad \tag{13} \]
Using (13) and (11), (12) one derives for \( \Omega^{\text{bulk}} \):
\[ \Omega^{\text{bulk}} = \Omega^{\text{chiral}}(g_L, \gamma) - \Omega^{\text{chiral}}(g_R, \gamma), \quad \tag{14} \]
where
\[ \Omega^{\text{chiral}}(g_L, \gamma) = \frac{k}{4\pi} \int_0^{2\pi} \text{tr} \left( g_L^{-1} \delta g_L \partial_x (g_L^{-1} \delta g_L) \right) dx + \frac{k}{4\pi} \text{tr} (g_L^{-1} \delta g_L(0) \delta g_L). \quad \tag{15} \]
The chiral field \( g_L \) can be decomposed into the product of a closed loop in \( G \), a multivalued field in the Cartan subgroup and a constant element in \( G \):
\[ g_L(x^+) = h(x^+) e^{ix^+ / k g_0^{-1}}, \quad \tag{16} \]
where \( h \in LG, \tau \in t \) (the Cartan algebra) and \( g_0 \in G \). For the monodromy of \( g_L \) we find:
\[ \gamma = g_0 e^{2\pi \tau / k g_0^{-1}}. \quad \tag{17} \]
The parametrization (16) induces the following decomposition of \( \Omega^{\text{chiral}}(g_L, \gamma) \):
\[ \Omega^{\text{chiral}}(g_L, \gamma) = \Omega^{LG}(h, \tau) + \frac{k}{4\pi} \omega(\gamma) + \text{tr}[i \delta \tau g_0^{-1} \delta g_0], \quad \tag{18} \]
where $\Omega^{LG}(h, \tau)$ is:

$$\Omega^{LG}(h, \tau) = \frac{k}{4\pi} \int_0^{2\pi} \text{tr}[h^{-1}\delta h \partial_x (h^{-1}\delta h) + \frac{2i}{k}\tau(h^{-1}\delta h)^2 - \frac{2i}{k}\delta\tau h^{-1}\delta h] \, dx$$

(19)

and $\omega_{\tau}(\gamma)$ is:

$$\omega_{\tau}(\gamma) = \text{tr}[g_0^{-1}\delta g_0 e^{2i\pi\tau/k} g_0^{-1}\delta g_0 e^{-2i\pi\tau/k}]$$

(20)

Comparing (14) and (19) to the formulae in appendix C we see that the symplectic phase space of the WZW model on a cylinder coincides with that of Chern-Simons theory on the annulus times the time-line $A \times R$.

### 3 Gauged WZW model

Here we review quantization of the gauged WZW model on the cylinder $\Sigma = R \times S^1 = (t, x \mod 2\pi)$ as it is done in [23].

The action of the gauged WZW model is [3, 18, 19, 29]:

$$S^{G/H} = S^{WZW} + S^{gauge},$$

(21)

where

$$S^{gauge} = \frac{k}{2\pi} \int_\Sigma L^{gauge},$$

(22)

$$L^{gauge}(g, A) = -\text{tr}[-\partial_+ gg^{-1}A_- + g^{-1}\partial_- gA_+ + gA_+g^{-1}A_- - A_+A_-].$$

(23)

With the help of the Polyakov-Wiegmann identities:

$$L^{\text{kin}}(gh) = L^{\text{kin}}(g) + L^{\text{kin}}(h) + \text{Tr}(g^{-1}\partial_+ g\partial_+hh^{-1}) + \text{Tr}(g^{-1}\partial_- g\partial_-hh^{-1}),$$

(24)

$$\omega^{\text{WZ}}(gh) = \omega^{\text{WZ}}(g) + \omega^{\text{WZ}}(h) - d\left(\text{Tr}(g^{-1}dg dhh^{-1})\right),$$

(25)

it is easy to check that the action (21) is invariant under the gauge transformation:

$$g \rightarrow hgh^{-1}, \quad A \rightarrow hAh^{-1} - dhh^{-1}$$

(26)

for $h : \Sigma \rightarrow H$.

The equations of motions are:

$$D_+(g^{-1}D_-g) = 0, \quad \text{Tr}(g^{-1}D_-gT_H) = \text{Tr}(gD_+g^{-1}T_H) = 0, \quad F(A) = 0,$$

(27)

where $D_\pm g = \partial_\pm g + [A_\pm, g]$ and $T_H$ is any element in the $H$ Lie algebra.
The flat gauge field $A$ can be written as $h^{-1} dh$ for $h : R^2 \to H$ and satisfying:

$$h(t, x + 2\pi) = \rho^{-1} h(t, x)$$

for some $\rho \in H$.

Define $\tilde{g} = hgh^{-1}$. Note that $\tilde{g}$ satisfies

$$\tilde{g}(t, x + 2\pi) = \rho^{-1} \tilde{g}(t, x) \rho .$$

In the terms of $\tilde{g}$ equations (27) take the form:

$$\partial_+ (\tilde{g}^{-1} \partial_- \tilde{g}) = 0, \quad \text{Tr}(\tilde{g}^{-1} \partial_- \tilde{g} T_H) = \text{Tr}(\tilde{g} \partial_+ \tilde{g}^{-1} T_H) = 0 .$$

The canonical symplectic form density, obtained following the general prescription [7, 8, 20], is given by:

$$\Pi^{G/H}(g, h) = \Pi^G(\tilde{g}) + \partial_x \Psi(h, g) ,$$

where

$$\Psi(h, g) = \text{tr} h^{-1} dh (g^{-1}dg + dgg^{-1} + g^{-1}h^{-1}dhg) .$$

Equations (30) can be solved in the terms of the chiral fields:

$$\tilde{g} = g_L(x^+) g_L^{-1}(x^-) , \quad \text{Tr}(\partial_y g_L g_L^{-1} T_H) = \text{Tr}(\partial_y g_R g_R^{-1} T_H) = 0$$

with the monodromy properties:

$$g_L(y + 2\pi) = \rho^{-1} g_L(y) \gamma , \quad g_R(y + 2\pi) = \rho^{-1} g_R(y) \gamma .$$

The monodromy properties (35) imply that the chiral fields $g_{L,R}$ should be written as products of fields as well:

$$g_L = h_B^{-1} g_A , \quad g_R = h_D^{-1} g_C ,$$

7
where $h_B, h_D \in H$ and $g_A, g_C \in G$. The fields in \((36)\) should additionally satisfy: \[ \text{tr}[T_H(\partial_y h_B h_B^{-1} - \partial_y g_A g_A^{-1})] = 0, \quad \text{tr}[T_H(\partial_y h_D h_D^{-1} - \partial_y g_C g_C^{-1})] = 0 \] (37)

and
\[
\begin{align*}
    h_B(y + 2\pi) &= h_B(y)\rho, & g_A(y + 2\pi) &= g_A(y)\gamma, \\
    h_D(y + 2\pi) &= h_D(y)\rho, & g_C(y + 2\pi) &= g_C(y)\gamma.
\end{align*}
\] (38)

(39)

Using (37) one can show:
\[
\text{tr}[g_L^{-1} \delta g_L \partial_y (g_L^{-1} \delta g_L)] = \text{tr}[g_A^{-1} \delta g_A \partial_y (g_A^{-1} \delta g_A) - h_B^{-1} \delta h_B \partial_y (h_B^{-1} \delta h_B)]
\]
\[
+ \partial_y (\delta h_B h_B^{-1} \delta g_A g_A^{-1})]
\] (40)

and similarly for $g_R$ and $h_D, g_C$.

Collecting (34), (35), (36), (38), (39), (40) and (13) one can show that
\[
\Omega^{G/H} = \Omega^\text{chiral}(g_A, \gamma) - \Omega^\text{chiral}(g_C, \gamma) - \Omega^\text{chiral}(h_B, \rho) + \Omega^\text{chiral}(h_D, \rho)
\] (41)

Comparing (41) with (14) and remembering that the latter is the symplectic form of the Chern-Simons theory on $A \times R$, we arrive at the conclusion that the phase space of the gauged WZW model on a cylinder coincides with that of double Chern-Simons theory [23, 32] on $A \times R$.

### 4 Defects in the gauged WZW model G/H

Let us assume that one has a defect line $S$ separating the world-sheet into two regions $\Sigma_1$ and $\Sigma_2$. In such a situation the WZW model is defined by pairs of maps $g_1$ and $g_2$. On the defect line itself one has to impose conditions that relate the two maps. The necessary data are captured by the geometrical structure of a bibrane: a bibrane is in particular a submanifold of the Cartesian product of the group $G$ with itself: $Q \subset G \times G$. The pair of maps $(g_1, g_2)$ are restricted by the requirement that the combined map
\[
S \to (G \times G) : s \to (g_1(s), g_2(s)) \in Q
\] (42)
takes its value in the submanifold $Q$. Additionally one should require, that on the submanifold $Q$ a two-form $\varpi(g_1, g_2)$ exists satisfying the relation
\[
d\varpi(g_1, g_2) = \omega^{WZ}(g_1)|_Q - \omega^{WZ}(g_2)|_Q.
\] (43)

---

One can arrive at the decomposition (36) with the properties (38) and (39) in the following way: taking, say, a field $h_B$ satisfying the first part of (38), one can then define $g_A$ as $g_A \equiv h_B g_L$, satisfying the second part of (38).
To write the action of the WZW model with defect one should introduce an auxiliary disc $D$ satisfying the conditions:

$$\partial B_1 = \Sigma_1 + D \quad \text{and} \quad \partial B_2 = \Sigma_2 + \bar{D}, \quad (44)$$

and continue the fields $g_1$ and $g_2$ on this disc always holding the condition $(42)$. After this preparations the topological part of the action takes the form $[16]$:

$$S_{\text{top-def}} = \frac{k}{4\pi} \int_{B_1} \omega_W(g_1) + \frac{k}{4\pi} \int_{B_2} \omega_W(g_2) - \frac{k}{4\pi} \int_D \varpi(g_1, g_2). \quad (45)$$

Equation $(43)$ guarantees that $(45)$ is well defined.

Denote by $C_G^\mu$ a conjugacy class in group $G$:

$$C_G^\mu = \{ \beta f_\mu \beta^{-1} = \beta e^{2i\pi \mu / k} \beta^{-1}, \ \beta \in G \}, \quad (46)$$

where $\mu \equiv \mu \cdot H$ is a highest weight representation integrable at level $k$, taking value in the Cartan subalgebra of the $G$ Lie algebra.

To construct the gauged WZW model with defects we take for bibrane the following ansatz:

$$(g_1, g_2) = (C_2 C_1 p, L p L^{-1}). \quad (47)$$

Here $C_1 \in C_G^{\mu_1}, C_2 \in C_H^{\mu_2}, p \in G$ and $L \in H$.

The Polyakov-Wiegmann identity $[25]$ implies that the bibrane $(47)$ satisfies the condition $(43)$ with the following $\varpi$:

$$\varpi(L, p, C_2, C_1) = \Omega^{(2)}(C_2, C_1) - \text{tr}((C_2 C_1)^{-1} d(C_2 C_1) d p p^{-1}) + \Psi(L, p), \quad (48)$$

where

$$\Omega^{(2)}(C_2, C_1) = \omega_{\mu_2}(C_2) - \text{tr}(C_2^{-1} dC_2 dC_1 C_1^{-1}) + \omega_{\mu_1}(C_1) \quad (49)$$

and $\omega_{\mu}(C)$ is defined in $[20]$ and $\Psi(L, p)$ is defined in $[32]$.

In the following we show that this bibrane provides a geometric realization of the Cardy defect $[33]$ corresponding to the primary $(\mu_1, \mu_2)$.

Some comments are in order at this point:

1. In fact this is a folded version of the permutation brane for the gauged WZW model suggested in $[35]$, and can be derived following the same lines as for the permutation branes.
2. Recall that primaries of the coset $G/H$ in the presence of the common center $C$ of the $G$ and $H$ are given by the pairs of the $G$ and $H$ primaries up to the field identification and selection rules [14, 24, 25]. It was shown in [12, 23] that the product of the conjugacy classes $C_2 C_1$ provides geometric realization of the field identification and selection rules. Remind briefly the arguments. Field identification follows immediately from the fact that given an element of the common center $z$ one has the same object for $(f_{\mu_1}, f_{\mu_2})$ and $(zf_{\mu_1}, z^{-1}f_{\mu_2})$. Selection rule is a consequence of the global issues [2, 16, 21, 30]. Recall that when the second homotopy group of the bibrane worldvolume is not trivial the action is defined up to the multiples of $2\pi$ only for the special values of $(\mu_1, \mu_2)$. In the absence of the common center it is enough to require that $(\mu_1, \mu_2)$ is a pair of $G$ and $H$ primaries. The common center $C$ makes the gauge group $H/C$, and the above mentioned process of the continuation of the gauged transformed values of the $g_1$ and $g_2$ into the disc $D$ becomes non-trivial. Resolution of this problem leads to the selection rule, requiring $\mu_2$ to lie somewhere in the integrable representation $\mu_1$, after some projection.\footnote{See [14] for details.}

Now we can define the action of the gauged WZW model with defect as

$$S^{G/H-def}(g_1, g_2, A_1, A_2) = S^{kin-def}(g_1, g_2) + S^{gauge-def}(g_1, g_2, A_1, A_2) + S^{top-def},$$

(50)

where

$$S^{kin-def}(g_1, g_2) = \frac{k}{4\pi} \int_{\Sigma_1} L^{kin}(g_1) dx^+ dx^- + \frac{k}{4\pi} \int_{\Sigma_2} L^{kin}(g_2) dx^+ dx^-$$

(51)

and

$$S^{gauge}(g_1, g_2, A_1, A_2) = \frac{k}{2\pi} \int_{\Sigma_1} L^{gauge}(g_1, A_1) + \frac{k}{2\pi} \int_{\Sigma_2} L^{gauge}(g_2, A_2).$$

(52)

The gauge fields $A_1$ and $A_2$ are not restricted to the defect line.

One can check that the action (50) is invariant under the gauge transformations:

$$g_1 \rightarrow h_1 g_1 h_1^{-1}, \quad A_1 \rightarrow h_1 A_1 h_1^{-1} - dh_1 h_1^{-1},$$

$$g_2 \rightarrow h_2 g_2 h_2^{-1}, \quad A_2 \rightarrow h_2 A_2 h_2^{-1} - dh_2 h_2^{-1},$$

(53)
where \( h_1 : \Sigma \to H, \ h_2 : \Sigma \to H \). For this purpose note that under (53) the boundary parameters transform in the following way:

\[
p \to h_1 p h_1^{-1}, \ C_1 \to h_1 C_1 h_1^{-1}, \ C_2 \to h_1 C_2 h_1^{-1}, \ L \to h_2 L h_2^{-1}.
\]  

(54)

The gauge invariance follows from the Polyakov-Wiegmann identities and the transformation properties of \( \tilde{\varpi} \) (58) which can be obtained using formulae of appendix A.

Now we consider the gauged WZW model on the cylinder \( \Sigma = R \times S^1 = (t, x \mod 2\pi) \) and put defect line at \( x = a \) in parallel to the time line.

The variational equation \( \delta S^{G/H-def}(g_1, g_2, A_1, A_2) = 0 \) implies the bulk equations (56), (57), (58) for \( g_1, A_1 \) and \( g_2, A_2 \) separately supplemented by the defect equations at \( x = a \):

\[
\begin{align*}
g_1^{-1} D_- g_1 - L^{-1} g_2^{-1} D_- g_2 L &= 0, \quad (56) \\
C_2^{-1} g_1 D_+ g_1^{-1} C_2 - L^{-1} g_2 D_+ g_2^{-1} L &= 0, \quad (57) \\
L^{-1} D_t L &= 0, \quad C_2^{-1} D_t C_2 = 0, \quad (58)
\end{align*}
\]

where \( D_t = D_+ - D_- \), \( D_{\pm} L = \partial_{\pm} L + A_{2\pm} L - L A_{1\pm}, \ D_{\pm} g_1 = \partial_{\pm} g_1 + [A_{1\pm}, g_1], \ D_{\pm} g_2 = \partial_{\pm} g_2 + [A_{2\pm}, g_2], \ D_{\pm} C_2 = \partial_{\pm} C_2 + [A_{1\pm}, C_2] \).

Derivation of the equations (56), (57), (58) is outlined in the appendix B.

Flat gauge fields can be parameterised as before:

\[
A_1 = h_1^{-1} dh_1, \quad A_2 = h_2^{-1} dh_2.
\]  

(59)

Defining as before:

\[
\begin{align*}
\tilde{g}_1 &= h_1 g_1 h_1^{-1}, \quad \tilde{g}_2 = h_2 g_2 h_2^{-1}, \\
\tilde{C}_1 &= h_1 C_1 h_1^{-1}, \quad \tilde{C}_2 = h_1 C_2 h_1^{-1}, \\
\tilde{p} &= h_1 p h_1^{-1}, \quad \tilde{L} = h_2 L h_2^{-1},
\end{align*}
\]

we have the bulk equations (30) for \( \tilde{g}_1 \) and \( \tilde{g}_2 \) and the defect equations (56), (57), (58) take the form:

\[
\begin{align*}
\tilde{g}_1^{-1} \partial_- \tilde{g}_1 - \tilde{L}^{-1} \tilde{g}_2^{-1} \partial_- \tilde{g}_2 \tilde{L} &= 0,  \\
\tilde{C}_2^{-1} \tilde{g}_1 \partial_+ \tilde{g}_1^{-1} \tilde{C}_2 - \tilde{L}^{-1} \tilde{g}_2 \partial_+ \tilde{g}_2^{-1} \tilde{L} &= 0,
\end{align*}
\]

(61)

(62)
\[ \tilde{L}^{-1} \partial_t \tilde{L} = 0, \quad \tilde{C}_2^{-1} \partial_t \tilde{C}_2 = 0. \]  

Equation (63) implies that \( \tilde{L} \) and \( \tilde{C}_2 \) are constant along the defect line.

Using that, the bulk-defect equations can be solved in the terms of the chiral fields:

\[
\begin{align*}
\tilde{g}_1 &= g_{1L}g_{1R}^{-1}, & \text{Tr}(\partial_y g_{1L}g_{1L}^{-1}T_H) &= \text{Tr}(\partial_y g_{1R}g_{1R}^{-1}T_H) = 0, \\
\tilde{g}_2 &= g_{2L}g_{2R}^{-1}, & \text{Tr}(\partial_y g_{2L}g_{2L}^{-1}T_H) &= \text{Tr}(\partial_y g_{2R}g_{2R}^{-1}T_H) = 0,
\end{align*}
\]

and

\[ g_{2L} = \tilde{L}\tilde{C}_2^{-1}g_{1L}n^{-1}, \quad g_{2R} = \tilde{L}g_{1R}m^{-1}, \]

with \( m \) and \( n \in G \). Equations (66) imply

\[ (\tilde{g}_1(t,a),\tilde{g}_2(t,a)) = (\tilde{C}_2\tilde{C}_1\tilde{p},\tilde{L}\tilde{p}\tilde{L}^{-1}), \]

where

\[
\begin{align*}
\tilde{p} &= \tilde{C}_2^{-1}g_{1L}(a+t)n^{-1}mg_{1R}^{-1}(a-t), \\
\tilde{C}_1 &= \tilde{C}_2^{-1}g_{1L}(a+t)m^{-1}ng_{1L}^{-1}(a+t)\tilde{C}_2.
\end{align*}
\]

To have that \( \tilde{C}_1 \in C_G^{\mu_1} \) we should require that \( d \equiv m^{-1}n \in C_G^{\mu_1} \).

Given that we consider GWZW model on a cylinder we should additionally require:

\[ g_1(t,0) = g_2(t,2\pi), \quad h_1(t,0) = \rho h_2(t,2\pi). \]

From (70) and (71) one obtains:

\[ \tilde{g}_1(t,0) = \rho\tilde{g}_2(t,2\pi)\rho^{-1}, \]

and

\[ g_{1L}(y+2\pi) = \tilde{C}_2\tilde{L}^{-1}\rho^{-1}g_{1L}(y)\gamma_L, \quad g_{1R}(y+2\pi) = \tilde{L}^{-1}\rho^{-1}g_{1R}(y)\gamma_R, \]

with \( \gamma_L \) and \( \gamma_R \) satisfying the relation:

\[ \gamma_R^{-1}\gamma_L = d. \]

Comparing (73) with (35) we see that the presence of the defect leads to the relative shifts between the left and right monodromies, equal to the defect conjugacy classes.
The monodromies (73) as before can be realized in the terms of the decomposition of the fields $g_{1L}$ and $g_{1R}$ as products:

$$\begin{align*}
g_{1L} &= h_B^{-1}g_A, \\
g_{1R} &= h_D^{-1}g_C
\end{align*}$$  \hspace{1cm} (75)

of the new fields $h_B$, $g_A$, $h_D$, $g_C$ possessing the monodromy properties:

$$\begin{align*}
h_B(2\pi) &= h_B(0)\rho\tilde{L}\tilde{C}_2^{-1}, \\
g_A(2\pi) &= g_A(0)\gamma_L, \\
h_D(2\pi) &= h_D(0)\rho\tilde{L}, \\
g_C(2\pi) &= g_C(0)\gamma_R
\end{align*}$$  \hspace{1cm} (76)

and satisfying (37).

The symplectic form of the gauged WZW model with a defect can be written using the symplectic form density (31) and the form $\varpi$:

$$\Omega^{G/H-def} = \frac{k}{4\pi} \left[ \int_0^a \Pi^{G/H}(g_1, h_1)dx + \int_a^{2\pi} \Pi^{G/H}(g_2, h_2)dx - \varpi(g_1(a), g_2(a)) \right].$$  \hspace{1cm} (78)

Substituting in (78) the symplectic form density (31) and using the transformation property (55) we obtain:

$$\Omega^{G/H-def} = \frac{k}{4\pi} \left[ \int_0^a \Pi(\tilde{g}_1)dx + \int_a^{2\pi} \Pi(\tilde{g}_2)dx - \varpi(\tilde{L}, \tilde{p}, \tilde{C}_2, \tilde{C}_1) - \Psi(\rho, \tilde{g}_2(2\pi)) \right],$$  \hspace{1cm} (79)

where $\tilde{p}$ and $\tilde{C}_1$ defined in (68) and (69).

Performing similar steps as before we arrive at the following expression for the symplectic form of the gauged WZW model with defects:

$$\Omega^{G/H-def} = \Omega^{\text{chiral}}(g_A, \gamma_L) - \Omega^{\text{chiral}}(g_C, \gamma_R) - \Omega^{\text{chiral}}(h_B, \rho\tilde{L}\tilde{C}_2^{-1}) + \Omega^{\text{chiral}}(h_D, \rho\tilde{L}) + \left[ -\omega_{\mu_2}(\tilde{C}_2) - \omega_{\mu_1}(d) - \text{tr}(d^{-1}\delta d^{-1}\delta\gamma_L) - \text{tr}(\tilde{C}_2^{-1}\delta\tilde{C}_2(\rho\tilde{L})^{-1}\delta(\rho\tilde{L})) \right].$$  \hspace{1cm} (80)

Recalling the decomposition (18) of $\Omega^{\text{chiral}}$ and comparing with the corresponding formulae in appendix C we arrive at the conclusion that the phase space of the gauged WZW model on a cylinder with a defect line coincides with that of double Chern-Simons theory on $A \times R$ with gauge fields of groups $G$ and $H$ coupled to a Wilson line. This result can be straightforwardly generalized to the presence of the $N$ defect lines.
5 Defects in open coset model $G/H$

Let us at the beginning remind some facts on boundary coset model $G/H$ [12][23].

Boundary condition corresponding to a Cardy state $(\mu, \nu)$ is given by the product of the conjugacy classes

$$g|_{\text{boundary}} = bc,$$  \hfill (81)

where $b \in C^\mu_G$ and $c \in C^\nu_H$. As explained in the previous section in the presence of the common center $C$, $\mu$ and $\nu$ should satisfy the selection rule.

To write the action one should introduce an auxiliary disc $D$ satisfying the condition $\partial B = \Sigma + D$, and continue the field $g$ on this disc, always taking value in product of conjugacy classes.

The action with the boundary conditions (81) has the form:

$$S^{G/H-\text{bndry}} = S^{G/H} - \frac{k}{4\pi} \int_D \Omega^{(2)}(b, c),$$  \hfill (82)

where $\Omega^{(2)}(b, c)$ is defined in [49].

Consider a WZW model with a defect on the strip $R \times [0, \pi]$. Assume again that we have a defect at the point $x = a$ in parallel to the time line. The strip is divided into two parts with the fields $g_1, A_1$ and $g_2, A_2$. We impose a Cardy boundary condition (81) at $x = 0$ on $g_1$ requiring:

$$g_1(t, 0) = C_3 C_4, \quad C_3 \in C^\mu_3, \quad C_4 \in C^\mu_4,$$  \hfill (83)

a defect condition (47) at $x = a$:

$$(g_1, g_2) = (C_2 C_1 p, L p L^{-1}),$$  \hfill (84)

and again a Cardy boundary condition (81) at $x = \pi$:

$$g_2(t, \pi) = C_5 C_6, \quad C_5 \in C^\mu_5, \quad C_6 \in C^\mu_6.$$  \hfill (85)

Let us analyze first the consequences of the boundary condition (83) at the point $x = 0$.

The boundary equations of motion resulting from the action (82) at $x = 0$ are derived in [23]:

$$g_1^{-1} D_- g_1 + C_4^{-1} g_1 D_+ g_1^{-1} C_4 = 0, \quad C_4^{-1} D_4 C_4 = 0.$$  \hfill (86)
Representing again the flat gauge field $A_1 = h_1^{-1} dh_1$, and again defining $\tilde{g}_1 = h_1 g_1 h_1^{-1}$, $\tilde{C}_3 = h_1 C_3 h_1^{-1}$, $\tilde{C}_4 = h_1 C_4 h_1^{-1}$ one can write (86) as:

$$\tilde{g}_1^{-1} \partial_- \tilde{g}_1 + \tilde{C}_4^{-1} \tilde{g}_1 \partial_+ \tilde{g}_1^{-1} \tilde{C}_4 = 0,$$

(87)

$$\tilde{C}_4^{-1} \partial_- \tilde{C}_4 = 0.$$

(88)

The last equation implies that $\tilde{C}_4$ is constant on the boundary. Therefore using the chiral decomposition (64) $\tilde{g}_1 = g_1 L g_1^{-1} R$ one can solve (87):

$$g_{1R}(y) = \tilde{C}_4^{-1} g_{1L}(-y) R_0^{-1}$$

(89)

with $R_0 \in G$. Now we get that :

$$\tilde{g}_1(t, 0) = g_{1L}(t) R_0 g_{1L}^{-1}(t) \tilde{C}_4.$$

(90)

The boundary condition (83) implies:

$$\tilde{g}_1(0, t) = \tilde{C}_3 \tilde{C}_4,$$

(91)

$$\tilde{C}_3 \in C_G^{\mu_3}, \quad \tilde{C}_4 \in C_H^{\mu_4}.$$

We find that

$$\tilde{C}_3 = g_{1L}(t) R_0 g_{1L}^{-1}(t).$$

(92)

To be in agreement with the requirement that $\tilde{C}_3 \in C_G^{\mu_3}$ one should demand:

$$R_0 \in C_G^{\mu_3}.$$

(93)

The defect condition as before implies:

$$g_{1L} = \tilde{C}_2 \tilde{L}^{-1} g_{2L} n, \quad g_{1R} = \tilde{L}^{-1} g_{2R} m,$$

(94)

where $g_{2L}$, $g_{2R}$ are fields of the chiral decomposition (65): $\tilde{g}_2 = g_{2L} g_{2R}^{-1}$.

From the boundary condition (83) we conclude:

$$\tilde{g}_2(t, \pi) = \tilde{C}_5 \tilde{C}_6,$$

(95)

where $\tilde{C}_5 = h_2 C_5 h_2^{-1}$, $\tilde{C}_6 = h_2 C_6 h_2^{-1}$.

To satisfy (95) we assume the following monodromy behaviour of $g_{1L}$:

$$g_{1L}(y + 2\pi) = \rho^{-1} g_{1L}(y) \gamma.$$

(96)

From relations (89), (94) and (93) we derive:

$$\tilde{g}_2(t, \pi) = \tilde{L} \tilde{C}_2^{-1} g_{1L}(\pi + t)n^{-1}R_0 \gamma (\tilde{L} \tilde{C}_2^{-1} g_{1L}(\pi + t))^{-1} \tilde{L} \tilde{C}_2^{-1} \rho^{-1} \tilde{C}_4 \tilde{L}^{-1}.$$

(97)
We see that
\[
\tilde{C}_5 = \tilde{L}\tilde{C}_2^{-1}g_{1\ell}(\pi + t)n^{-1}mR_0\gamma(\tilde{L}\tilde{C}_2^{-1}g_{1\ell}(\pi + t))^{-1},
\]
(98)
and
\[
\tilde{C}_6 = \tilde{L}\tilde{C}_2^{-1}\rho^{-1}\tilde{C}_4\tilde{L}^{-1}.
\]
(99)
To satisfy (95) we should demand:
\[
d^{-1}R_0\gamma = R_\pi \in C^\mu_G,
\]
(100)
\[
\tilde{C}_2^{-1}\rho^{-1}\tilde{C}_4 = S_\pi \in C^\mu_H.
\]
(101)
The symplectic form is
\[
\Omega^{G/H-def-bndry} = \frac{k}{4\pi} \left[ \int_0^a \Pi(\tilde{g}_1) + \int_0^\pi \Pi(\tilde{g}_2) - \varpi(\tilde{L}, \tilde{p}, \tilde{C}_2, \tilde{C}_3) + \Omega(\tilde{C}_3, \tilde{C}_4) - \Omega(\tilde{C}_5, \tilde{C}_6) \right].
\]
(102)
In formula (102) \(\tilde{p}, \tilde{C}_1, \tilde{C}_3, \tilde{C}_5, \tilde{C}_6\) are given by the equations (68), (69), (92), (98), (99) correspondingly. Representing again
\[
g_{1\ell} = h_B^{-1}g_A,
\]
(103)
with \(h_B\) and \(g_A\) possessing the monodromy properties:
\[
h_B(y + 2\pi) = h_B(y)\rho,
\]
(104)
\[
g_A(y + 2\pi) = g_A(y)\gamma,
\]
(105)
and repeating the same steps as before we obtain:
\[
\frac{4\pi}{k} \Omega^{G/H-def-bndry} = \frac{4\pi}{k} \Omega^{chiral}(g_A, \gamma) - \frac{4\pi}{k} \Omega^{chiral}(h_B, \rho) + \omega_{\mu_3}(R_0) + \omega_{\mu_4}(\tilde{C}_4) - \omega_{\mu_5}(R_\pi) - \omega_{\mu_6}(S_\pi) - \omega_{\mu_2}(\tilde{C}_2) - \omega_{\mu_1}(d)
\]
\[
-\text{tr}(R_0^{-1}\delta R_0 \delta \gamma \gamma^{-1}) + \text{tr}(\delta d^{-1} \delta R_0 R_0^{-1}) + \text{tr}(\delta d^{-1} R_0 \delta \gamma \gamma^{-1} R_0^{-1})
\]
\[
-\text{tr}(\delta \tilde{C}_4 \tilde{C}_4^{-1} \delta \rho \rho^{-1}) - \text{tr}(\delta \tilde{C}_2 \tilde{C}_2^{-1} \rho^{-1} \delta \rho) + \text{tr}(\delta \tilde{C}_2 \tilde{C}_2^{-1} \rho^{-1} \delta \tilde{C}_4 \tilde{C}_4^{-1} \rho).
\]
(106)
Recalling again the decomposition (18) of \(\Omega^{chiral}\) and comparing with the corresponding formulae in appendix C we arrive at the conclusion that the phase space of the gauged WZW model on a strip with a defect line coincides with that of the double Chern-Simons theory on \(D \times R\) with gauge fields of groups \(G\) and \(H\) coupled to three Wilson lines. This result can be straightforwardly generalized to the presence of the \(N\) defect lines.
6 Topological $G/G$ coset

6.1 Bulk $G/G$ coset

In this section we consider the bulk $G/H$ model studied in section 2 for the special case $G = H$. It was shown in section 2 that the phase space of the bulk $G/H$ model is symplectomorphic to that of the double Chern-Simons theory on $R \times A$. In the special case, when $G = H$ it becomes a Chern-Simons theory on the torus times $R : R \times (A \cup (-A)) = R \times T^2$. This result can be obtained also by a direct calculation.

In the case when $G = H$ the equations of motion (30) imply that $\tilde{g}$ is $(t, x)$ independent and therefore the symplectic form $\Omega^{G/H}$ (33) reduces to

$$\Omega^{G/G} = \frac{k}{4\pi} \Psi(\rho, \tilde{g}^{-1}).$$

(107)

The fact that $\tilde{g}$ is constant on a cylinder and the relation (29) also imply

$$\rho\tilde{g}\rho^{-1}\tilde{g}^{-1} = I.$$

(108)

Comparing (107) and (108) with the corresponding formulae reviewed in appendix C we arrive at the conclusion that the phase space of a bulk $G/G$ theory on a cylinder is symplectomorphic to that of a Chern-Simons theory on $T^2 \times R$. The quantization of the latter gives rise to the space of the 0-point conformal blocks of the WZW theory on the torus. The dimension of the space of conformal blocks on a Riemann surface of genus $g$ with insertion of the primary fields with labels $\mu_n$ is:

$$N_{\mu_n}(g) = \sum_{\alpha} (S_0^\alpha)^{2g - 2g} \prod_n (S_{\mu_n}^\alpha / S_0^\alpha).$$

(109)

This implies that the Hilbert space of the quantized $G/G$ theory on a cylinder has dimension equals to the number of the integrable primaries. The equivalence of the topological $G/G$ coset on a cylinder $R \times S^1$ with a Chern-Simons on $R \times T^2$ demonstrated here is actually a particular case of the more general equivalence of the topological $G/G$ coset on a Riemann surface $\Sigma$ and the Chern-Simons theory on $\Sigma \times S^1$ established in [4, 39, 42].

6.2 A defect in a closed topological model $G/G$

We have established in section 3 that the phase space of the coset $G/H$ on a cylinder with a defect is symplectomorphic to that of a double Chern-Simons
theory on $R \times \mathcal{A}$ with $G$ and $H$ gauge fields both coupled to a time like Wilson line. In the case when $G = H$ we again arrive at the conclusion that the topological coset $G/G$ on a cylinder with a defect line is equivalent to the Chern-Simons theory on $R \times T^2$ with two time like Wilson lines. This again can be verified by a direct calculation. For the case $G = H$ the bulk equations of motion imply that $\tilde{g}_1$ and $\tilde{g}_2$ are $(t, x)$ independent.

Therefore one has:

$$\tilde{g}_1(0) = \tilde{g}_1(a) = \tilde{C}_2 \tilde{C}_1 \tilde{p}, \quad \tilde{L} \tilde{p} \tilde{L}^{-1} = \tilde{g}_2(a) = \tilde{g}_2(2\pi). \quad (110)$$

From (72) we also obtain:

$$\tilde{g}_1(0) = \rho \tilde{g}_2(2\pi) \rho^{-1}. \quad (111)$$

Inserting (111) in (110) we get:

$$\tilde{C}_2 \tilde{C}_1 \tilde{p} = \rho \tilde{L} \tilde{p} \tilde{L}^{-1} \rho^{-1}. \quad (112)$$

The symplectic form (79) now takes the form:

$$\Omega^{G/G-def} = -\frac{k}{4\pi} \omega(\rho \tilde{L}, \tilde{p}, \tilde{C}_2, \tilde{C}_1). \quad (113)$$

Comparing (112) and (113) with the corresponding formulae in appendix C we arrive at the conclusion that the topological coset $G/G$ on a cylinder with a defect is symplectomorphic with that of a Chern-Simons theory on $T^2 \times R$ with two Wilson lines. The quantization of the latter gives rise to the space of the 2-point conformal blocks of the WZW theory on a torus. Using equation (109) we can compute the dimension of the Hilbert space of the quantized topological coset $G/G$ on cylinder with a defect line $(\mu_1, \mu_2)$:

$$\dim H_{d\mu_1, \mu_2} = \sum_{\alpha \beta} N_{\alpha \mu_1}^\alpha N_{\beta \mu_2}^\beta. \quad (114)$$

### 6.3 Defects in the open topological model $G/G$

Previously we have seen that the phase space of $G/H$ coset on a strip with a defect is symplectomorphic to that of the double Chern-Simons theory on $D \times R$ with gauge fields $G$ and $H$ both coupled to three Wilson lines. In the case when $G = H$ we arrive at the conclusion that the $G/G$ topological coset on a strip with a defect line is equivalent to the Chern-Simons theory on sphere times $R$:
\((D \cup (-D)) \times R = S^2 \times R\) with six time-like Wilson lines. This can be verified also directly. In this case \(\tilde{g}_1\) and \(\tilde{g}_2\) are \((t, x)\) independent and therefore one has:

\[
\tilde{g}_1(0) = \tilde{C}_3 \tilde{C}_4 = \tilde{C}_2 \tilde{C}_1 \tilde{p} = \tilde{g}_1(a),
\]

\[
\tilde{g}_2(a) = \tilde{L} \tilde{p} \tilde{L}^{-1} = \tilde{C}_5 \tilde{C}_6 = \tilde{g}_2(2\pi).
\]

From equations (115) and (116) one obtains:

\[
(\tilde{L} \tilde{C}_1^{-1} \tilde{L}^{-1})(\tilde{L} \tilde{C}_2^{-1} \tilde{L}^{-1})(\tilde{L} \tilde{C}_3 \tilde{L}^{-1})(\tilde{L} \tilde{C}_4 \tilde{L}^{-1}) \tilde{C}_6^{-1} \tilde{C}_5^{-1} = I,
\]

and from (102) one derives:

\[
\Omega^{G/G-\text{def-bdry}} = -\frac{k}{4\pi} \Omega(\tilde{L}, \tilde{p}, \tilde{C}_1) + \frac{k}{4\pi} \Omega(\tilde{C}_3, \tilde{C}_4) - \frac{k}{4\pi} \Omega(\tilde{C}_5, \tilde{C}_6).
\]

Comparing (117) and (118) with the corresponding equations in appendix C we arrive at the mentioned symplectomorphism of the phase space of \(G/G\) topological coset on a strip with a defect and a Chern-Simons theory on \(S^2 \times R\) with six Wilson lines. The quantization of the latter gives rise to the space of the 6-point conformal blocks of the WZW theory on a sphere. Using equation (109) we can compute the dimension of the Hilbert space of the quantized topological coset \(G/G\) on a strip with a defect line:

\[
N^{\lambda_1}_{\mu_3 \mu_4} N^{\lambda_2}_{\lambda_1 \mu_1} N^{\lambda_3}_{\lambda_2 \mu_2} N^{\mu_6}_{\lambda_3 \mu_5}.
\]

Recall that here \((\mu_3, \mu_4)\) are labels of the Cardy state on the first end of the strip, \((\mu_5, \mu_6)\) are labels of the Cardy state on the second end of the strip, and \((\mu_1, \mu_2)\) are the labels of the defect.

To interpret this result let us remind some general facts on a semisimple 2D topological theory on a world-sheet with boundary [31]. First of all let us recall that the whole content of the 2D topological field theory is encoded in a finite-dimensional commutative Frobenius algebra \(\mathcal{C}\). In the case when \(\mathcal{C}\) is semisimple it can be realized as the algebra of complex-valued functions on a finite set \(X = \text{Spec} \mathcal{C}\), which can be considered as a toy "space-time". Using sewing constraints of open topological theory it was proved in [31] that every boundary condition \(a\) is realized by a collections of vector spaces corresponding to each point of \(X\): \(x \mapsto V_{x,a}\). This can be considered as a vector bundle over finite space-time, in agreement with the K-theory interpretation of boundary
conditions. The Hilbert space of open string with boundary conditions specified by $a$ and $b$ is given by the bundle morphism:

$$H_{ab} = \bigoplus_x \text{Hom}(V_{x,a}; V_{x,b}). \quad (120)$$

Consider now an open topological $G/G$ coset. Note that in this case the points of $X$ are labelled by integrable primaries. Let us remind first the situation without defect considered in [23]. The dimension of the Hilbert space for this case can be derived from (119) putting there $\mu_1$ and $\mu_2$ equal to vacuum state:

$$N^{\lambda}_{\mu_3 \mu_4} N^{\mu_6}_{\lambda \mu_5}. \quad (121)$$

This can be interpreted saying that the Hilbert space of the open string with the Cardy boundary conditions $(\mu_3, \mu_4)$ and $(\mu_5, \mu_6)$ at the ends is

$$H_{\mu_3,\mu_4; \mu_5,\mu_6} = \bigoplus_\lambda \text{Hom}(W_{\mu_3 \mu_4 \lambda}; W_{\mu_5 \mu_6 \lambda}), \quad (122)$$

where $W_{\mu_\nu \lambda}$ are spaces of three points conformal blocks. This implies that the Cardy state $(\mu, \nu)$ is given by the vector bundle

$$\lambda \rightarrow W_{\mu \nu \lambda}. \quad (123)$$

Now consider the case with a defect $(\mu_1, \mu_2)$.

It is well known (see e.g. [15][26][27][31][38]), that open string propagating with boundary conditions $a$ and $b$ with inserted defect $d$ can be considered, as propagating between one of the original boundary conditions, say $a$, and the second transformed by defect: $d \ast b$. According to formula (119) the transformed state corresponds to the spaces $V_{\lambda,\mu_1;\mu_2,\mu_5,\mu_6}$ with the dimensions

$$N^{\lambda_2}_{\lambda_1 \mu_1} N^{\lambda_3}_{\lambda_2 \mu_2} N^{\mu_6}_{\lambda_3 \mu_5}. \quad (124)$$

and therefore can be considered as transformed by tensoring and summing with the space of 4-point conformal blocks $W_{\mu_1 \mu_2 \lambda_1 \lambda_3}$:

$$V_{\lambda_1,\mu_1;\mu_2,\mu_5,\mu_6} = \bigoplus_{\lambda_3} W_{\mu_1 \mu_2 \lambda_1 \lambda_3} \otimes W_{\mu_5 \mu_6 \lambda_3}. \quad (125)$$

This suggests the following general description of defects in semisimple 2D TFT’s. It seems that to every defect separating 2D TFT’s with ”space-time”s $X$ and $Y$ corresponds a collection of spaces $V_{x,y}^D$ where $x \in X$ and $y \in Y$. This can be considered as a fibre bundle over $X \times Y$. Then the boundary condition
given by the fibre bundle \( V_y \) over \( Y \) is transformed to the boundary condition corresponding to the following bundle over \( X \):

\[
x \rightarrow \oplus_y V_{x,y}^D \otimes V_y.
\]

(126)

It is interesting to note that the transformation (126) can be viewed as a discrete Fourier-Mukai transform in agreement with the general interpretation of the defect worldvolume or bi-brane as kernel of the Fourier-Mukai transform suggested in [5, 28, 36].

Let us elaborate now on fusion of defects. For this purpose consider an open string with insertion of two defects. The Hilbert space in this case is given by the space of 8-point conformal blocks. Along the same lines we conclude that the fusion of two defects \( (\mu_1, \mu_2) \) and \( (\nu_1, \nu_2) \) is given by the space of 6-point conformal blocks: \( W_{\mu_1,\mu_2,\nu_1,\nu_2,\lambda_1,\lambda_2} \). According to the factorization properties of the space of conformal blocks this space can be expressed through the space of 4-point conformal blocks:

\[
W_{\mu_1,\mu_2,\nu_1,\nu_2,\lambda_1,\lambda_2} = \oplus_{\gamma} W_{\mu_1,\mu_2,\lambda_1,\gamma} \otimes W_{\nu_1,\nu_2,\lambda_2,\gamma}.
\]

(127)

This suggests that in general the fusion of two defects given by the bundles \( V_{x,y}^D \) and \( V_{y,z}^D \) over the spaces \( X \times Y \) and \( Y \times Z \) is given by the equation:

\[
V_{x,z}^{D_1 \star D_2} = \oplus_y V_{x,y}^{D_1} \otimes V_{y,z}^{D_2}.
\]

(128)

It is interesting to note that equation (128) appeared as a composition rule in the 2-category of matrices of vector spaces (see for example [17]). The relation with 2-categories actually can be traced further.

Note that equation (114) for the dimension of the \( G/G \) theory on a cylinder with a defect can be written as the dimension of the space \( \sum_{\lambda} W_{\mu,\nu,\lambda,\lambda} \):

\[
\dim H_{d_{\mu,\nu}} = \dim \sum_{\lambda} W_{\mu,\nu,\lambda,\lambda}.
\]

(129)

We can conclude that probably in the general case the dimension of the bulk theory with defect given by the collection of the spaces \( \{ V_{x_1,x_2}, \quad x_1, x_2 \in X \} \), is given by the dimension of the space \( \oplus_x V_{x,x}^D \):

\[
\dim H_d = \dim \oplus_x V_{x,x}^D.
\]

(130)

The space \( \oplus_x V_{x,x}^D \) appears in [17] as categorical trace.
The study of the conjectures (126), (128) and (130) for general 2D semisimple TFT is left for future work.

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A Useful formulae

In this appendix we collect some useful properties of the two-form \( \Psi(h, g) \) defined by formula (32).

\[
\Psi(hL, p) = \Psi(L, p) + \Psi(h, LpL^{-1}).
\]

(131)

\[
\Psi(Lh^{-1}, hph^{-1}) = \Psi(L, p) - \Psi(h, p).
\]

(132)

\[
\omega_\mu(hCh^{-1}) - \omega_\mu(C) = -\Psi(h, C).
\]

(133)

\[
\Omega^{(2)}(hC_1h^{-1}, hC_2h^{-1}) - \Omega^{(2)}(C_1, C_2) = -\Psi(h, C_1C_2),
\]

(134)

\[
\Psi(h, C_1C_2) = \Psi(h, C_1) + \Psi(h, C_2) + (\tilde{C}_1^{-1}d\tilde{C}_1d\tilde{C}_2\tilde{C}_2^{-1} - C_1^{-1}dC_1dC_2C_2^{-1}),
\]

(135)

where \( \tilde{C}_1 = hC_1h^{-1} \) and \( \tilde{C}_2 = hC_2h^{-1} \).

\[
\omega^{WZW}(ghh^{-1}) - \omega^{WZW}(g) = -d\Psi(h, g).
\]

(136)

B Defect Equations of motion

Computing variation of the action (50) one obtains for the defect part:

\[
\begin{align*}
tr[g_1^{-1}\delta g_1(g_1^{-1}\partial_+g_1 + g_1^{-1}\partial_-g_1)]dt & - tr[g_2^{-1}\delta g_2(g_1^{-1}\partial_+g_2 + g_2^{-1}\partial_-g_2)]dt \\
+2tr[\delta g_1^{-1}A_1- - A_1+g_2^{-1}\delta g_1 - (\delta g_2g_2^{-1}A_2- - A_2+g_2^{-1}\delta g_2)]dt & + B = 0.
\end{align*}
\]

(137)

The last term \( B \) is a one-form satisfying the relation:

\[
tr(g_1^{-1}\delta g_1(g_1^{-1}dg_1)^2) - tr(g_2^{-1}\delta g_2(g_2^{-1}dg_2)^2) - \delta \omega = dB.
\]

(138)

Recalling that the first two terms come from the equation

\[
\delta \omega^{WZ} = d[tr(g^{-1}\delta g(g^{-1}dg)^2)],
\]

(139)

we see that the existence of the one-form \( B \) satisfying (138) is a consequence of the equation (143). Using (147) one can compute \( B \) explicitly:

\[
B = A_{\mu_1}(C_1) + A_{\mu_2}(C_2) + tr[C_2^{-1}\delta C_2dC_1C_1^{-1} - \delta C_1C_1^{-1}C_1^{-1}dC_2 - (140)
\delta pp^{-1}(C_2C_1)^{-1}d(C_2C_1) + (C_2C_1)^{-1}\delta(C_2C_1)dpp^{-1} - L^{-1}\delta Lp^{-1}dp + \]

\[
p^{-1}\delta pL^{-1}dL - L^{-1}\delta Ldpp^{-1} + \delta pp^{-1}L^{-1}dL - L^{-1}\delta Lp^{-1}L^{-1}dLp + \]

\[
L^{-1}\delta LpL^{-1}dLp^{-1}]
\]

(140)
The one-form $A_\mu(C)$ was defined in [11]:

$$A_\mu(C) = \text{tr}[h^{-1}\delta h(f_\mu^{-1}dhf_\mu - f_\mu h^{-1}dhf_\mu^{-1})], \quad (141)$$

where $C = hf_\mu h^{-1}$, $f_\mu = e^{2\pi i \mu/k}$, and satisfies:

$$\text{tr}(g^{-1}\delta g(g^{-1}dg)^2)|_{g=C} - \delta \omega_\mu(C) = dA_\mu(C). \quad (142)$$

$A_\mu(C)$ satisfies also another important relation on the time-line:

$$\text{tr}[g^{-1}\delta g(g^{-1}d)dt + A_\mu(C)] = \text{tr}[2\delta hh^{-1}(\partial_+gg^{-1} - g^{-1}\partial_-g)]dt, \quad (143)$$

where $g = C$. Let us explain the meaning of this equation.

The left hand side of the (143) is a particular case of (137) and describes boundary equation of motion of the WZW model with the boundary condition specified by the conjugacy class $C$, while the right hand side proportional to $J_L + J_R$, what is the condition for the diagonal chiral algebra preservation.

Now, using (47) and (140), one can show by a straightforward calculation, that (137) implies the equations (56), (57), (58) in section 4.

C  Symplectic forms of the moduli space of flat connections on a Riemann surface

In this appendix we briefly review the symplectic phase space of the Chern-Simons theory on the three-dimensional manifold of the form $M \times R$, where $M$ is two-dimensional Riemann surface, $R$ is time direction, with $n$ time-like Wilson lines assigned with representations $\lambda_i$. It was shown in [10,41] that the phase space of the Chern-Simons theory in such a situation is given by the moduli space of flat connections on the Riemann surface $M$ punctured at the points $z_i$ where Wilson lines hit $M$, with the holonomies around punctures belonging to the conjugacy classes $C^\lambda_i = \eta e^{2\pi i \lambda_i/k} \eta^{-1}$. Therefore denoting holonomies around handles $a_j$ and $b_j$ by $A_j$ and $B_j$, and around punctures by $M_i \in C^\lambda_i$ we arrive at the conclusion that the phase space of the Chern-Simons theory is

$$\mathcal{F}_{g,n} = G^{2g} \times \prod_{i=1}^n C^\lambda_i \quad (144)$$
subject to the relation
\[ [B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] M_n \cdots M_1 = I , \tag{145} \]
where
\[ [B_j, A_j] = B_j A_j B_j^{-1} A_j^{-1} , \tag{146} \]
and to the adjoint group action.

The symplectic form on \( \mathcal{F}_{g,n} \) was derived in [1] and has the form:
\[ \Omega_{\mathcal{M}_{g,n}} = \sum_{i=1}^{n} \Omega_{M_i} + \sum_{j=1}^{g} \Omega_{H_j} , \tag{147} \]
where
\[ \Omega_{M_i} = \frac{k}{4\pi} \omega_\lambda(M_i) + \frac{k}{4\pi} \text{tr}(K_{i-1}^{-1} \delta K_{i-1} K_i^{-1} \delta K_i) , \tag{148} \]
\[ \Omega_{H_j} = \frac{k}{4\pi} \Psi(B_j, A_j) + \frac{k}{4\pi} \left( \text{tr}(K_{n+2j-2}^{-1} \delta K_{n+2j-2} K_{n+2j-1}^{-1} \delta K_{n+2j-1}) + \text{tr}(K_{n+2j-1}^{-1} \delta K_{n+2j-1} K_{n+2j}^{-1} \delta K_{n+2j})) \right) , \tag{149} \]
and where
\[ K_i = M_i \cdots M_1 \quad i \leq n , \tag{150} \]
\[ K_{n+2j-1} = A_j [B_{j-1}, A_{j-1}^{-1}] \cdots [B_1, A_1^{-1}] K_n , \tag{151} \]
\[ K_{n+2j} = [B_j, A_j^{-1}] \cdots [B_1, A_1^{-1}] K_n \quad 1 \leq j \leq g . \]

\( K_0 \) can be chosen to be equal to the unity element. According to (145) also
\[ K_{n+2g} = I . \tag{152} \]
\( \omega_\lambda(M) \) and \( \Psi(B, A) \) are defined in equations (20) and (32) correspondingly.

It was also proved in [1] that quantization of the moduli space \( \mathcal{F}_{g,n} \) with
the symplectic form (147) leads to the space of \( n \)-point conformal blocks on a
Riemann surface of the genus \( g \).

The last piece of information which we need is the symplectic form on the
moduli space of flat connections on the punctured sphere with holes \( S^2_{n,m} \), where
\( n \) as before is number of punctures and \( m \) is number of holes. It was argued
in [10][22] that the corresponding symplectic form \( \Omega_{S^2_{n,m}} \) is given by:
\[ \Omega_{S^2_{n,m}} = \Omega_{S^2_{n+m,0}} + \sum_{i=1}^{m} \Omega_{I}^{LG} , \tag{153} \]
where \( \Omega_{I}^{LG} \) is defined in (19) and its geometrical quantization leads to the inte-
grable representation of the affine algebra \( \hat{\mathfrak{g}} \) at level \( k \).
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