We investigate equations of motion and future singularities of $f(R, T)$ gravity where $R$ is the Ricci scalar and $T$ is the trace of stress-energy tensor. Future singularities for two kinds of equation of state (barotropic perfect fluid and generalized form of equation of state) are studied. While no future singularity is found for the first case, some kind of singularity is found to be possible for the second. Using the method of fixed points and certain assumptions, a large number of singularities can be removed. Finally, the effect of the Noether symmetry on $f(R, T)$ is studied and the consistent form of $f(R, T)$ function is found using the symmetry and the conserved charge.

I. INTRODUCTION

Recent cosmological observations show that our universe has an accelerating expansion [1]. Two groups of solutions are available that can be invoked for explaining the phenomenon. The first one is based on the belief that some exotic matter exist within the framework of General Relativity (GR) known as dark energy that has the parameter $\omega < 0$ in its equation of motion. Such a matter raises some fundamental questions such as the existence of negative entropy, future singularities, and the violation of some energy conditions.

On the other hand, some authors have generalized GR to some new theories of gravity in which the standard Einstein-Hilbert action is replaced with an arbitrary function of Richi scalar $R$ known as $f(R)$ gravity [2]. In addition to its capability to describe the expansion of the universe without introducing any dark energy [3], this generalized theory of gravity has other advantages. For example, it can explain the dynamics of galaxies without recourse to the concept of dark matter [4] and unifies inflation with dark energy [5, 6].

A further generalization of $f(R)$ is $f(R, T)$ gravity where $T$ is the trace of stress-energy tensor [7]. As a consequence of using stress-energy tensor as a source, the motion of the particles does not take place along a geodesic path because there is an extra force perpendicular to the four-velocity unless we add the constraint of conservation of stress-energy tensor (unlike GR and $f(R)$ theories, the continuity equation is independent of equations of motion in this case). It is shown in [8] that due to the conservation of stress-energy tensor, $T$ sector of $f(R, T)$ cannot be chosen arbitrarily but it has a special form. The thermodynamics of this model is studied in [9], and the possibility of wormhole geometry is examined in [10]. In this article, we keep the conservation of energy and assume that the motion of the particles still takes place along a geodesic path.

There are lots of theories that use one single fluid with different equations of state to explain dark energy. So, we also assumed there are some fixed points in our theory. We will use this assumption and some other conditions in section 4 to show that it is possible to eliminate a large number of singularities.

The concept of symmetry has always been an attractive subject in physics. Noether symmetry attracts more attention because it helps find constants of motion (like energy and momentum) from continuous symmetries of the system. Some efforts have been done to look for such conserved quantities in cosmological models [11]. Some authors have considered the effect of Noether symmetry in extended theories of gravity such as $f(R)$ [12] and $f(T)$ theories [13]. We investigate the effect of Noether symmetry on $f(R, T)$ theory to see if it is possible to make a consistent form of $f(R, T)$ by Noether symmetry.

This article is organized as follows. In Sec. 2, we briefly explain the action of the $f(R, T)$ model and obtain the gravitational field equations. In Section 3, we consider singularities for dark energy. In Sec. 4, we consider the method of fixed points and elimination of singularities. The aim of Sec. 5 is to consider Noether symmetry in $f(R, T)$ theory and Sec. 6 concludes with a summary and discussion.

II. EQUATIONS OF MOTION

The general form of the action for the $f(R, T)$ model is as follows:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2k} f(R, T) + \mathcal{L}_m \right\}$$  \hspace{1cm} (1)

where $k = 8\pi G$, $G$ is the Newtonian constant, $R$ is the Richi scalar, $T$ is the trace of the stress-energy tensor, and $\mathcal{L}_m$ is matter lagrangian. By varying the action with respect to the metric, we obtain the equations of motion:

$$f_R R_{\mu\nu} - \frac{1}{2} f(R, T) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R = 8\pi T_{\mu\nu} - f_T (T_{\mu\nu} + \Theta_{\mu\nu})$$  \hspace{1cm} (2)

where

$$\Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}} = -2T_{\mu\nu} + g_{\mu\nu} \mathcal{L}_m - 2g^{\alpha\beta} \frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}$$ \hspace{1cm} (3)

$$f_R = \frac{\partial f(R, T)}{\partial R}, \quad f_T = \frac{\partial f(R, T)}{\partial T}$$ \hspace{1cm} (4)
From a purely geometrical ground and the conservation of energy-momentum tensor, we know:
\[ \nabla^\mu G_{\mu\nu} = 0 \]  
\[ \nabla^\mu(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box)f(R, T) = R_{\mu\nu} \nabla^\mu f(R, T) \]  
\[ \nabla^\mu T_{\mu\nu} = 0 \]
where, \( G_{\mu\nu} \) is the Einstein tensor and \( T_{\mu\nu} \) is the energy-momentum tensor. By contracting (2) into \( \nabla^\mu \) and using \([5],[6], \) and \([7]\), we obtain a constraint equation on \( f(R, T) \):
\[ (T_{\mu\nu} + \Theta_{\mu\nu}) \nabla^\mu f_T + f_T \nabla^\mu \Theta_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(\nabla^\mu T) f_T = 0 \]  
We see in equation (8) that by changing \( f(R, T) \) to \( f(R) \) the left hand side of the equation vanishes, leaving no constraint on \( f(R) \). So, the conservation of energy does not impose any new relation in the \( f(R) \) theory but it does so in the \( f(R, T) \) theory. Now, we concentrate on the case \( f(R, T) = R + g(T) \). Assuming the matter content of the universe as a perfect fluid \( (T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}) \) and having a flat FRW universe, i.e.:
\[ ds^2 = dt^2 - a^2(t) dx^2. \]
the Friedman equations will take the following form
\[ 3H^2 = [k + g'(T)]\rho + g'(T)p + \frac{1}{2}g(T) \]  
\[ -2\dot{H} - 3H^2 = kp - \frac{1}{2}g(T) \]
In General Relativity and \( f(R) \) theory, the continuity equation is not independent of Friedman equation. This is not the case in \( f(R, T) \) theory in which the equations of motion and continuity equation for matter make three independent equations. One of three options might be entertained at this point. Assuming that there is no continuity equation for matter, particle motion is not along the geodesic path because of an extra force perpendicular to the four velocity \([10]\). Another option is to include the conservation of energy as a constraint in the lagrangian whose Lagrange multiplier should be determined in the theory \([17]\). In this paper, we choose a third way and preserve the continuity equation by assuming the particle motion to be along a geodesic path. We may now find the consistent form of \( f(R, T) \). The continuity equation in question is independent from the Friedman equations:
\[ \dot{\rho} + 3H(\rho + p) = 0. \]
Adding the equation of state \( p = \rho(\rho) \) to the three equations \([10],[11]\) and \([12]\), we have four independent equations which can be used to obtain the time dependence of four parameters of \( \rho, p, a, \) and \( g(T) \). To do this, we add the Friedman equations \([10]\) and \([11]\):
\[ -2\dot{H} = [k + g'(T)](\rho + p) \]  
Eliminating the term \( \rho + p \) from \([12]\) and \([13]\), we have:
\[ 6\dot{H} - 3H^2 = |k + g'(T)| \dot{\rho} \]
By differentiating \([10]\) and substituting \([11]\), we have:
\[ \dot{g}(T)(\rho + p) + \frac{1}{2}g'(T) = 0 \]
By solving this equation for a perfect barotropic fluid \( (p = \omega \rho) \), we have for \( \omega \neq \pm \frac{1}{3}, -1 \):
\[ g(T) = g_0 T^\alpha, \quad \alpha = \frac{1 + 3\omega}{2(1 + \omega)} \]
Thus, the Friedman equations become:
\[ 3H^2 = kp + g_0(1 - 3\omega)^{\alpha - 1}\rho^\alpha = k(\rho + p_{DE}) \]  
\[ 2\dot{H} + 3H^2 = -kp + \frac{1}{2}g_0(1 - 3\omega)^{\alpha} \rho^\alpha = -(k(p + p_{DE})) \]
Both fluids are, therefore, perfect with the equation of state parameter \( \omega \) and \( \omega_{DE} = -\frac{1}{2}(1 - 3\omega) \) \([11]\). To investigate the future singularities of this model, we first solve the continuity equation \([12]\) as follows:
\[ \rho = \rho_0 a^{3(1 + \omega)} \]
By substituting this equation in Friedman equation \([10]\), we have:
\[ \pm(t - t_0) = \int \frac{a^{\frac{1 + 3\omega}{3}} da}{d_1 + d_2 a^{3(1 + \omega) - 2}} \]
where, \( d_1 = \frac{8\pi G \rho_0}{3} \) and \( d_2 = \frac{2g_0 c_0 (1 - 3\omega)^{\alpha - 1}}{3} \). By substituting different allowed values of \( \omega \) (\( \omega \neq \pm \frac{1}{3}, -1 \)) in the above equation, we find no future singularities in this model. Below, we will consider the Friedman equations and future singularities for a more general equation of state.
We can interpret the Friedman equations \([10]\) and \([11]\) in a different way. If we define \( \rho_{de} \) and \( p_{de} \) and \( k \) as:
\[ \rho_{de} = -p_{de} \leq \frac{pg'(T) + \frac{1}{2}g(T)}{k} \]
\[ \dot{k} \equiv k + g'(T) \equiv 8\pi \dot{G} \]
then, the Friedman equations become:
\[ 3H^2 = \dot{k}(\rho + \rho_{de}) \]  
\[ -3H^2 - 2\dot{H} = \dot{k}(p + p_{de}) \]
Adding \( g(T) \) to the action will have two effects: the coupling constant is not constant anymore and it has running with energy like the field theories of QED and QCD. It should be noted that \( \rho_{de} \) and \( p_{de} \) behave in the same way as the dark energy so that we can explain expansion of the universe without having to introduce any exotic matter like dark energy.
III. GENERALIZED EQUATION OF STATE

In this section, we consider a more general equation of state:

\[ p = -\rho - f(\rho) \]  

(27)

This kind of equation of state leads to five types of singularities [18, 19]:

- Type I ("Big Rip"): \( t \to t_s, a \to \infty, \rho \to \infty, \) and \( |P| \to \infty \)
- Type II ("Sudden"): \( t \to t_s, a \to a_s, \rho \to \rho_s, \) and \( |P| \to \infty \)
- Type III: \( t \to t_s, a \to a_s, \rho \to \infty, \) and \( |P| \to \infty \)
- Type IV: \( t \to t_s, a \to \infty, \rho \to 0, \) and \( |P| \to 0, \) but higher derivatives of \( H \) diverges.
- Type V: In this type of singularity, \( \omega = \frac{P}{\rho} \) diverges and it is possible that none of the other parameters has a singularity.

In the following, we will try to see if similar types of singularity exist in the \( f(R, T) \) model. Assuming (27) and from the continuity equation (12), we have:

\[ \dot{\rho} = 3Hf(\rho) \]  

(28)

and from (15), we have:

\[ g'(T) = g'_0 \sqrt{f(\rho)}a^3 \]  

(29)

Eliminating \( g'(T) \) between (13) and (29) we have:

\[ 2H = \left( k + g'_0 \sqrt{f(\rho)}a^3 \right)f(\rho) \]  

(30)

Having \( a(t) \), we can now get the behavior of \( \rho(t), p(t), g(t) \) and \( f(\rho(t)) \) from (27), (28), (29) and (30). If we assume \( H(t) \) to have a singular form as:

\[ H(t) = h(t_s - t)^{-m} \]  

(31)

where \( t_s \) is the time of future singularity then for \( m = 1, \) \( a(t) \) becomes:

\[ a(t) = a_0(t_s - t)^{-h}. \]  

(32)

The behavior of other functions near \( t_s \) will be as follows:

\[ \rho, p \propto (t_s - t)^{\alpha} \]  

(33)

\[ g \propto (t_s - t)^{\beta} \]  

(34)

\[ g' \propto (t_s - t)^{\gamma} \]  

(35)

The values of \( \alpha, \beta \) and \( \gamma \) for different values of \( h \) are shown in Table 1.

| \( h \) | \( \alpha \) | \( \beta \) | \( \gamma \) |
|---|---|---|---|
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| -1 | -6 | 0 | 8 |
| -2 | -12 | 0 | 17 |
| -3 | -18 | 0 | 26 |

with the exponent \( \alpha \) while \( g \) and \( g' \) are finite at \( t_s \). The following relations holds between the exponents:

\[ h > 0 : \beta_h = \frac{\alpha_h}{2} - 3h, \quad \alpha_h = -\beta_{h-1}, \quad \gamma_h = \beta_h - 1 \]  

(36)

\[ h < 0 : \beta_h = 0, \quad \alpha_h = 6h, \quad \gamma_h = 3h - 2 \]  

(37)

For \( m \neq 1, \) \( a(t) \) has the following different form:

\[ a(t) = a_0\exp \left[ \frac{h(t_s - t)^{-m}}{m - 1} \right] \]  

(38)

Depending on \( m \), we have different results near \( t_s \):

1. \( m < -1 \): It is obvious from the form of \( a, H, \dot{H} \) that none of them are singular near \( t_s \) and it is also clear from (28), (29) and (30) that \( f(\rho), g'(T), \dot{\rho} \) are finite, too. In this case, we have no singularity in \( t_s \) except for the higher derivatives of \( H \).

2. \( -1 < m < 0 \): In this case, at \( t_s, a \) and \( H \) are finite and just \( \dot{H} \) is singular, it is obvious from (28), (29) and (30) that \( f(\rho), g'(T), \dot{\rho} \) are singular. We can determine the behavior of \( \rho, p \) and \( g \) by numerical solution of equations (27), (28), (29) and (30). The behavior of the parameters for \( m = -\frac{1}{2} \) are plotted in figure 1. All the parameters go to infinity near \( t_s \).

3. \( 0 < m < 1 \): In this case, \( H \) and \( \dot{H} \) are singular but \( a \) is finite. From (29) and (30) \( f \) and \( g' \) must be singular. The behavior of the parameters are plotted in figure 2 for \( m = \frac{1}{2} \) by numerical solution. Again, we see that all the parameters have singular behavior near \( t_s \) but \( p \) is negative in this case.

4. \( m > 1 \): In this case, all the three \( a, H, \dot{H} \) are singular. The behavior of the parameters are plotted in figure 3 for \( m = 2 \). In this case, just \( g \) and \( g' \) are singular near \( t_s \), \( \rho \) and \( p \) tend to zero from below.

We have summarized the above discussion about singularity in Table I. So we have following types of singularity in \( f(R, T) \) model:

- Type \( \Gamma \): \( t \to t_s; H \to \infty, g \to \infty, \rho \to \infty, p \to \infty, g' \to \infty; \) (for \( 0 < m < 1 \)).
- Type \( \Gamma \Gamma \): \( t \to t_s; g \to \infty, \rho \to \infty, p \to \infty, g' \to \infty; \) (for \( -1 < m < 0 \)).
TABLE II. Singularity of cosmological parameters for the Huble parameter $H(t) = h(t_s - t)^{-m}$

| values of $m$ | values of scale factor | values of other parameters |
|---------------|------------------------|---------------------------|
| $m < -1$      | $a$, $H$ and $\dot{H}$ are finite. | $\rho$, $p$, $g$ and $g'$ are finite. |
| $-1 < m < 0$  | $\dot{H}$ is singular. | $\rho$, $p$, $g$ and $g'$ are singular. |
| $0 < m < 1$   | $H$ and $\dot{H}$ are singular. | $\rho$, $p$, $g$ and $g'$ are singular. |
| $m = 1$       | $h > 0$ | $a$, $H$ and $\dot{H}$ are singular. | $g$ and $g'$ are singular, $\rho$ and $p$ are finite. |
| $m = 1$       | $h < 0$ | $H$ and $\dot{H}$ are singular. | $g$ and $g'$ are finite, $\rho$ and $p$ are finite. |
| $m > 1$       | $a$, $H$ and $\dot{H}$ are singular. | $g$ and $g'$ are singular, $\rho$ and $p$ are finite. |

- Type $\overline{III}$: $t \to t_s; H \to \infty, g \to \infty, g' \to \infty; \rho, p$ finite (for $m = 1, h > 0$ and $m > 1$).
- Type $\overline{IV}$: $t \to t_s; H \to \infty, \rho \to \infty, p \to \infty; g, g'$ finite (for $m = 1, h < 0$).

It should be noted that the behaviors of $g$ and $g'$ exhibit the new characteristic of these types of singularities.

**IV. $f(R,T)$ AND FIXED POINTS**

To explain the meaning of a fixed point, we start with a simple model in the framework of GR in a flat FRW universe. The related Friedman equations are:

$$H^2 = \frac{\rho}{3}$$  \hspace{1cm} (39)

$$\dot{H} = -\frac{1}{2}(\rho + p)$$  \hspace{1cm} (40)

We may infer from (39) that $\rho = \rho(H)$. If we assume to have an equation of state of the form $p = p(H)$, we have:

$$\dot{H} = -\frac{1}{2}(\rho(H) + p(H)) \equiv f(H)$$  \hspace{1cm} (41)

As we see from Eq. (41), the sign of $f(H)$ determines whether $H$ increases or decreases with time. We define fixed points as zeros of $f(H)$, which is equivalent to a constant $H$. The neighboring behavior of $H$ in a fixed point is determined by the next nonzero term in the expansion of $f(H)$, i.e. $f'(H)$. We can write Eq. (41) as:

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1} = f'(H_1)(H_2 - H_1) + \ldots$$  \hspace{1cm} (42)

As we see from Fig. 4 and Eq. (42), when $f'(H_1)$ is negative, the system approaches $H_1$ from both sides so we have a stable fixed point. When $f'(H_1)$ is zero, the sign of $f'(H)$ is negative on one side and positive on the other. So, $H(t)$ approaches $H_1$ from one side and distance away from it from the other so that we have a halfstable fixed point. Similarly, we have unstable fixed points for positive $f'(H)$. It is known that fixed points and the behavior of $f(H)$ help us to predict the behavior of the system during the time (for a relevant study, see Ref. [20]).

For the $f(R,T)$ model, we start with a FRW flat universe with the following equations of motion:

$$3H^2 = [k + g'(T)]\rho + g'(T)p + \frac{1}{2}g(T) = \tilde{k}p_{eff}$$  \hspace{1cm} (43)

$$-2\dot{H} - 3H^2 = kp - \frac{1}{2}g(T) = \tilde{k}p_{eff}$$  \hspace{1cm} (44)

where,

$$\rho_{eff} \equiv \rho + \frac{g'(T)p + \frac{1}{2}g(T)}{k}$$  \hspace{1cm} (45)

$$p_{eff} \equiv p - \frac{g'(T)p + \frac{1}{2}g(T)}{k}$$  \hspace{1cm} (46)

We assume to have an equation of state of the form:

$$\tilde{k}p_{eff} = u(H)$$  \hspace{1cm} (47)

where, $u(H)$ is a continuous function of $H$. Then, equation (44) becomes:

$$\dot{H} = -\frac{1}{2}\tilde{k}(\rho_{eff} + p_{eff}) = -\frac{1}{2}\tilde{k}(\rho + p) = f(H)$$  \hspace{1cm} (48)

Now let us make the following two assumptions:

1. $f(H)$ is continuous and differentiable, and

2. $f(H)$ has a future fixed point $H_1$; this means if $f(H_0) > 0$, then $H_1 > H_0$ and if $f(H_0) < 0$, then $H_0 > H_1$, where $H_0$ is the present value of $H$.  
FIG. 1. The behavior of $g$, $\frac{dg}{dt'}$, $\rho$ and $p$ versus $t'$ ($t' = t_s - t$) for $m = -\frac{1}{2}$ in figures (a), (b), (c) and (d), respectively. All the parameters have singular behaviors at $t_s$.

FIG. 2. The behavior of $g$, $\frac{dg}{dt'}$, $\rho$ and $p$ versus $t'$ ($t' = t_s - t$) for $m = \frac{1}{2}$ in figures (a), (b), (c) and (d) respectively. All the parameters have singular behaviors at $t_s$. 
FIG. 3. The behavior of $g$, $g'$, $\rho$ and $p$ versus $t'$ ($t' = t_s - t$) for $m = 2$ in figures (a), (b), (c) and (d), respectively. $g$ and $g'$ are singular but $\rho$ and $p$ are zero at $t_s$.

FIG. 4. Behavior of $f(H)$ in a stable fixed point.

It is simple to demonstrate that by these two assumptions and the equation of state (47) in a flat FRW universe, there exists a unique solution $H(t)$ for $t > 0$, and that it takes an infinite time to reach the future fixed point $H_1$. As a result, for $t > 0$, $H$ remains bounded and finite between $H_0$ and $H_1$. Also, $f(H)$ is finite since every continuous and differentiable function is finite in a bounded regime. From (48), $H$ becomes finite, too. There are six parameters in equations (43) and (44) that can go to infinity; $H, \dot{H}, \rho, p, g, g'$. It seems at first sight that we can have $2^6 = 64$ types of singularity. Since $H$ and $\dot{H}$ are finite, just 16 cases remain. However, ten cases are forbidden because just one side of the equality in (43) and (44) diverge. Finally, only just 6 types of singularity remain out of the original 64:

1. $\rho \to \infty, p \to \infty, g \to \infty, g' \to \infty, H$ finite
2. $\rho \to \infty, p \to \infty, g \to \infty, H, g'$ finite
3. $\rho \to \infty, g' \to \infty, H, p, g$ finite
4. $p \to \infty, g \to \infty, g' \to \infty, H, \rho$ finite
5. $p \to \infty, g \to \infty, H, p, g'$ finite
6. just higher derivatives of $H$ (except $\dot{H}$) diverge.

It is interesting that some mild assumptions lead to just a few kinds of singularity.

V. $f(R, T)$ AND NOETHER SYMMETRY

One tool that can be used to find the symmetries of a theory is the Noether symmetry approach. From the Noether theorem, we know that there is a conserved charge related to every continuous symmetry of the theory that could be used to find the cyclic variables and reduce the dynamics of the system. Let $L$ be a lagrangian defined on tangent space $TQ = \{q_i, \dot{q}_i\}$. A vector field on the tangent space can be represented by:

$$X = \alpha^i(q) \frac{\partial}{\partial q^i} + \dot{\alpha}_i(q) \frac{\partial}{\partial \dot{q}^i}$$ (49)
where, dot means the derivative with respect to time. Lie derivative of Lagrangian $\mathcal{L}$ in the direction of $X$ is defined as:

$$L_X \mathcal{L} = X \mathcal{L} = \alpha^i(q) \frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}_i(q) \frac{\partial \mathcal{L}}{\partial q^i}$$  \hspace{1cm} (50)$$

The condition:

$$L_X \mathcal{L} = 0$$  \hspace{1cm} (51)$$

implies that $\mathcal{L}$ is conserved along the direction of $X$; in other words, $X$ is a symmetry of $\mathcal{L}$. We have Euler-Lagrange equations too:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0$$  \hspace{1cm} (52)$$

Contracting the above equation with $\alpha^i$, we have:

$$\alpha^i \left( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} \right) = 0 \Rightarrow \frac{d}{dt} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L}$$  \hspace{1cm} (53)$$

We see from the above equation that if $X$ is a symmetry of $L$ (i.e. $L_X \mathcal{L} = 0$), the function:

$$A = \alpha^i \frac{\partial \mathcal{L}}{\partial q^i}$$  \hspace{1cm} (54)$$

is the constant of the motion or the conserved charge. This is the Noether theorem.

Now, we want to find the consistent form of $f(R, T)$ using the Noether symmetry and its conserved charge. The general action of the $f(R, T)$ theory is:

$$S = \int d^4x \sqrt{-g} f(R, T) + S_m$$  \hspace{1cm} (55)$$

where, $S_m$ is the pressureless action of the matter. We use Lagrange multipliers to set $R$ and $T$ as constraints of the motion. Integrating by parts, we have:

$$S = 2\pi^2 \int dt a^3 \left\{ f(R) - \lambda \left[ \dot{R}^2 + \frac{\dot{a}^2}{a^2} + k \frac{a^2}{a^2} \right] \right. \right.$$

$$\left. - \lambda' \left[ T - \frac{\rho_m a}{a^3} \right] - \frac{\rho_m a}{a^3} \right\}$$  \hspace{1cm} (56)$$

With the continuity equation (12) in the pressureless universe, the density scales is defined as $\rho = \frac{\rho_m a}{a^3}$, where $\rho_m$ is the present value of density and we have set $a_0 = 1$. By varying the action with respect to $R$ and $T$, we have $\lambda = f_R$ and $\lambda' = f_T$. Therefore,

$$S = 2\pi^2 \int dt a^3 \left\{ f(R) - f_R \left[ \dot{R}^2 + \frac{\dot{a}^2}{a^2} + k \frac{a^2}{a^2} \right] \right. \right.$$

$$\left. - f_T \left[ T - \frac{\rho_m a}{a^3} \right] - \frac{\rho_m a}{a^3} \right\}$$  \hspace{1cm} (57)$$

Integrating by parts, for point like lagrangian of $f(R, T)$ model we have:

$$\mathcal{L} = a^3 \left( f - f_R R - f_T T \right) + 6a^2 f_R \dot{R} \dot{a} + 6a^2 f_R a \dot{\dot{a}} - 6k f_R a + f_T \rho_m a - \rho_m$$  \hspace{1cm} (58)$$

The tangent space for the above lagrangian is $\mathcal{T}_q = \{ a, \dot{a}, \dot{R}, \dot{T}, T \}$. As discussed above, the generator of the Noether symmetry is:

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \gamma \frac{\partial}{\partial T} + \delta \frac{\partial}{\partial \dot{a}} + \dot{\delta} \frac{\partial}{\partial \dot{R}} + \dot{\gamma} \frac{\partial}{\partial \dot{T}}$$  \hspace{1cm} (59)$$

The Noether symmetry exists if at least one of the functions $\alpha$, $\beta$ and $\gamma$ is non-zero. So, we should solve the equation:

$$L_X \mathcal{L} = X \mathcal{L} = 0$$  \hspace{1cm} (60)$$

By replacing $\frac{d}{dt}$ in $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ as follows:

$$\frac{d}{dt} = \frac{d}{da} \dot{a} + \frac{d}{dR} \dot{R} + \frac{d}{dT} \dot{T}$$  \hspace{1cm} (61)$$

and setting the coefficients $\dot{a}^2$, $\dot{R}^2$, $\dot{T}$, etc. equal to zero, the following equations obtain:

$$f_R \dot{R} \dot{\alpha} = 0$$  \hspace{1cm} (62)$$

$$2a f_R \dot{R} + a^2 f_R \dot{R} \dot{\beta} + \gamma a^2 \dot{R} T + \delta a^2 f_R = 0$$  \hspace{1cm} (63)$$

$$\alpha f_R + \beta f_R \dot{R} + \gamma f_R T + 2 f_R \dot{a} \dot{\alpha} = 0$$  \hspace{1cm} (64)$$

$$2 \dot{\beta} \dot{R} + a^2 \dot{R} \dot{\beta} \dot{f}_R = 0$$  \hspace{1cm} (65)$$

$$\dot{\gamma} f_R = 0$$  \hspace{1cm} (66)$$

and the Noether constraint:

$$3a^2 (f - f_R R - f_T T) - 6k f_R$$

$$- \beta a^3 (f_R + f_R \dot{R} + f_T (T - \frac{a m}{a}))-6 \beta k f_R a + \beta a^3 f_R + \gamma \beta (a^3 - f_T R) - f_T T (T - \frac{a m}{a}) - 6k f_R a = 0$$  \hspace{1cm} (67)$$

From (62) and (63), we have:

$$\dot{\alpha} = \dot{\beta} = 0$$  \hspace{1cm} (68)$$

and from (65) and the above equation:

$$\dot{\gamma} = 0$$  \hspace{1cm} (69)$$

We have two equations for the three functions $\alpha$, $\beta$ and $\gamma$. So, one is free for which we can choose any value. The simplest case is $\gamma = 0$. So, equations (63) and (64) convert to the following form:

$$2a f_R \dot{R} + a^2 f_R \dot{R} \dot{\beta} + \frac{da}{dt} a^2 f_R + \partial_R a^2 f_R + 6k f_R a + f_T \partial_m a = 0$$  \hspace{1cm} (70)$$

The solutions are as follows:

$$\alpha = c_1 a + \frac{c_2}{a}, \quad \beta = - \left( 3c_1 + \frac{c_2}{a^2} \right) \frac{f_R f_R}{f_R} + \frac{\dot{e}}{a f_R}$$  \hspace{1cm} (71)$$

To solve the Noether constraint (67) for $f(R, T)$, we assume a special case:

$$f(R, T) = f_1(R) + g(T)$$  \hspace{1cm} (72)$$
So, equation (67) becomes:

\[
\frac{1}{3\alpha^2} \left[ 3\alpha^2 (f_1 - f_{1R} R) - 6k f_{1R} \alpha - 6\beta k f_{1R} a \right] = g(T) = \mathcal{A}
\]

where, \(\mathcal{A}\) is a constant. The above differential equation is separable and we can find equations for \(f_1(R)\) and \(g(T)\) by substituting \(\alpha\) and \(\beta\) in the above equation:

\[
\frac{df_1}{dR} = \frac{3(c_1 a + \frac{\alpha}{\alpha - 1}) (f_1 - c(R + \frac{\alpha}{\alpha - 1}) - 3A(c_1 a + \frac{\alpha}{\alpha - 1}))}{2(6k c_1 - c_2 R)}
\]

\[
g(T) = A \ln \frac{T}{T_0}
\]

We see that Eq. (76) is different from (16) for \(g(T)\) calculated for barotropic perfect fluid because our matter lagrangian is different from the perfect fluid lagrangian \((\mathcal{L}_m = -p)\). The conserved charge for the Noether symmetry can be obtained from Eq. (54):

\[
\mathcal{A} = \alpha \left( 6 f_{RR} a^2 \dot{R} + 12 f_R a \dot{a} \right) + \beta \left( 6 f_{RR} a^2 \dot{a} \right)
\]

Using this constant of motion, we can solve Eq. (75) for different special cases.

**VI. CONCLUDING REMARKS**

This paper dealt with the \(f(R, T)\) theory of gravity. We have studied equations of motion and future singularities for a barotropic perfect fluid and a dark energy like fluid. To keep the conservation of stress-energy tensor, the choice of \(f(R, T)\) is not completely arbitrary. It was found that there is no future singularity for the barotropic fluid while some kinds of singularity possibly exist for the dark energy like fluid due to the new degrees of freedom in choosing the equation of state. We found relationships between the exponents of \(\rho, p, g\) and \(g'\). The method of fixed points was also investigated and some interesting results were obtained for the types of singularities. We showed that it is possible to explain the expansion of the universe by an effective running coupling constant where the pressure and density produced in the equations have the same behavior as the dark energy. Finally, the effect of the Noether symmetry on \(f(R, T)\) was studied and the consistent form of this function was determined using the Noether symmetry and the conserved charge. For future research, it will be interesting to generalize this study to other types of gravitational theories.

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