The 1/e-strategy is the unique optimal strategy for the best-choice problem under no information.

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In Memory of
Professor Larry Shepp

Abstract This paper answers a long-standing open question (Bruss (1984)) concerning the optimality of the 1/e-strategy for the problem of best choice under no information on the number of options. We give the background of this open problem and explain why the no-information hypothesis has intriguing aspects, suggesting that the problem may be ill-posed. Moreover, we will explain why these aspects had distracted for a long time from the essence of the problem. Then comes the main contribution of this paper. We show that the open problem is well-posed and then prove that the 1/e-strategy is indeed optimal, and even the only optimal strategy. The main tools for the affirmative answer are generalized versions of the Odds-Theorem which we prove first, and, in particular, the ageless beautiful theorem of relative ranks of Rényi (1962).

In a short discussion we also argue that problems of optimal control and/or optimal stopping under the no-information hypothesis may lead to another interesting question: What would be a versatile and solid definition of optimality for more general stopping problems including those which, by definition, do not have a value?

Keywords Optimal stopping, Secretary problem, Stopping times, No-information, Well-posed problem, Proportional increments, Rényi’s theorem of relative ranks, Generalised Odds-theorem, Poisson process, k-record process, Pascal process.

MSC 2010 Subject Codes: Primary: 60G40

1 Dedication and background

At the evening of Professor Larry Shepp’s talk “Reflecting Brownian Motion” at Cornell University on July 11, 1983 (13th Conference on Stochastic Processes and
Applications), Larry and I ran into each other in front of the Ezra Cornell statue. I was honored to meet him in person, and Larry replied “What are you working on?” And so Professor Shepp was the very first person with whom I could discuss the $1/e$-law of best choice resulting from the Unified Approach (B. (1984)) which had been accepted for publication shortly before. I was glad to see the true interest Professor Shepp showed for the $1/e$-law. As many of us know, when Larry was interested in a problem, elementary or not, then he was deeply interested.

This article deals with an open question concerning the optimality of the so-called $1/e$-strategy for the problem of best choice under no information on the number $N$ of options. I drew again attention to this open question in my own talk “The $e^{-1}$-law in best choice problems” at Cornell on July 14, 1983, and re-discussed it with Larry at several later occasions. The first written record of this question appears on page 885 of B.(1984): “It is conjectured that the $e^{-1}$-strategy is even the best of all conceivable strategies under the given assumptions.” As far as I am aware, the last time the conjecture was addressed was in Bruss and Yor (2012).

The present article proves the conjecture, thus closing a 36-years gap, and even more: The $e^{-1}$-strategy is the unique optimal strategy under no information.

2 The Unified Approach

We begin with a review of the Unified Approach-model and previously known results.

**Unified Approach:** Suppose $N > 0$ points are i.i.d. with a continuous distribution function $F$ on some interval $[0, T]$. Points are marked with qualities which are supposed to be uniquely rankable from 1 (best) to $N$ (worst), and all rank arrival orders are supposed to be equally likely.

The goal is to maximize the probability of stopping online and without recall on rank 1. (B. (1984))

This model was suggested for the best choice problem (secretary problem) for an unknown number $N$ of candidates. Recall that, by Rényi’s theorem of relative ranks (Rényi (1962)), the $k$th candidate has relative rank $j$ with probability $1/k$ for all $1 \leq j \leq k$ whenever all rank arrival orders are equally likely.

Previous models for unknown $N$ had shown that the price for not knowing $N$ can be high. The influential paper by Presman and Sonin (1972) which modelled the unknown $N$ via the hypothesis of a known distribution $\{P(N = n)\}$, displayed the intricacies arising by the possible appearance of so-called stopping islands. Moreover, Abdel-Hamid et al. (1982) showed that the $N$-unknown problem may have several solutions, and, much worse, that for any $\epsilon > 0$ there exists a sufficiently unfavorable distribution $\{P(N = n)\}_{n=1,2,...}$ to reduce the optimal success probability.
to a value smaller than $\epsilon$. In other words, if $N$ is modelled via $\{P(N = n)\}$, optimality may mean in some cases almost nothing. This contrasts with the well-known lower bound $1/e$ which holds in the classical model for known $N = n \geq 1$. These discouraging facts for unknown $N$ instigated efforts to find more tractable models, as e.g. the model of Stewart (1981), or the one of Cowan and Zabczyk (1978) and its generalization studied in Bruss (1987), and also others.

The unified approach of B. (1984) was different. The idea behind it was that it is typically easier to estimate - and this is where the time distribution $F$ comes in - when options are more likely to arrive conditional on knowing that they do arrive than making hypotheses about the distribution of its number. No assumption at all was made about the distribution of $N$.

The continuous arrival time distribution $F$ is the crucial part with respect to applications. As one referee pointed out, the model itself was, in different terms, already in the abstract of Rubin (1966). For our open problem the form of $F$ is irrelevant, however. If we transform the unordered i.i.d. arrival times of the best, the second best ... , $T_1, T_2, \ldots$ say, by $T^*_k := F(T_k)$, then the $T^*_k$ are i.i.d. $U[0, 1]$ random variables and, since $F$ is continuous and increasing, the time transformation maintains the arrival order of the different relative ranks. Thus, if we know the optimal strategy for dealing with i.i.d. $U[0, 1]$ random arrivals on $[0, 1]$, then we know it as well for i.i.d. $F$-distributed arrival times on the original horizon $[0, T]$. In all what follows we therefore confine our interest to uniformly distributed arrival times in $[0, 1]$-time.

### 2.1 Related problems

A related problem, to which we will return in Subsection 2.6, is the so-called last-arrival-problem under no information (l.a.p.) studied by Bruss and Yor (2012).

In this model an unknown number $N$ of points are i.i.d. $U[0, 1]$ random variables, and an observer, inspecting the interval $[0, 1]$ sequentially from left to right, wants to maximise the probability of stopping on the very last point. No information about $N$ whatsoever is given. Only one stop is allowed, and this again without recall on preceding observations (online). Thus the only difference of the l.a.p. model of B. and Yor (2012) to the Unified Approach model of B. (1984) is that no ranks are attributed to the observations (points).

Other related problems, now again with the objective to get rank 1 of uniquely ranked candidates, arise by combining the Unified Approach model and the model of Presman and Sonin (1972) for different types of distributions of $N$. If $(P(N = n))_{n=1,2,\ldots}$ is known, then one is in the setting of a model with a prior. The i.i.d. $U[0, 1]$ arrival-times can then be used as an additional means of statistical inference.
to update the posterior distribution of $N$. Stopping islands, as observed in the paper of Presman and Sonin (1972), bear over to corresponding islands in continuous time. The optimal strategy may thus become very complicated, and we would typically not like to compute it, but, in principle, it can be computed.

For the latter class of problems, what would be a good alternative? Moreover, and in particular, what can one do if one has absolutely no information about $N$?

### 2.2 The 1/e-law

The answer given by the unified approach (B., 1984) was that, as far as applications are concerned, we need not care much. For ease of reference we recall these results summarised as the 1/e-law. Here we follow the meanwhile established tradition (see e.g. Goldie and Rogers, 1984) to call an observation of relative rank 1 a *record value*, or simply *record*, and the time when a record appears a *record time*. Rényi (1962) had called a record an *élément saillant*.

The 1/e-law says:

1. The strategy to wait (in $[0,1]$-time) up to time $1/e \approx 0.3678$, and then to select the first record (if any from time $1/e$ onward), called the 1/e-strategy, succeeds for all $N$ with probability at least $1/e$.

2. There exists no strategy which would be better for all $N$.

3. The 1/e-strategy selects no candidate with precise probability $1/e$.

Note also that 1. and 3. imply that a non-best option is selected with probability smaller than $1 - 2/e \approx 0.2642$. This multiple role of the number $1/e$ gave rise to the name 1/e-law, and Table 1 (B., 1984, p. 336) shows how good the lower bound $1/e$ for the success probability actually is. Taking also into account the minimax optimality stated in 2. we can conclude that the 1/e-strategy is a convenient and convincing alternative for all practical purposes. See e.g. the comments of Samuels (Math. Reviews: 1985).

But then, the following question arises:

*Is the 1/e-strategy optimal if we have no prior information at all on $N$?*

This is thus the open problem of B. (1984). As mentioned before, the last written attempt to draw attention to it seems to be the paper by B. and Yor (2012; see subsection 6.4). These authors studied the mentioned l.a.p., that is, the problem of stopping on the very last point. As a by-product of their method, they could give an independent proof of the 1/e-law. This proof did not provide insight into the open question, however.
We have to return to what is known.

What is known?

(I) **Optimal** \(x\)-strategies given \(N = n\). First, suppose that \(N\) were known, say \(N = n\), and that we want to determine the optimal strategy in the class of so-called \(x\)-strategies, that is to wait until time \(x \in [0, 1]\) and then to select, if any, the first record from time \(x\) onward. It is not difficult to find, conditioned on \(\{N = n\}\), the optimal waiting time \(x_n\) and its performance in this class of \(x\)-strategies, namely (see B.(1984), p. 884, (2)-(7)),

\[
x_1 = 0; \quad x_n = \arg \left\{ 0 \leq x \leq 1 : \sum_{k=1}^{n-1} \frac{(1-x)^k}{k} = 1 \right\}, \quad n = 2, 3, \ldots. \tag{1}
\]

Note that the \(x_n\)-strategy is suboptimal since it does not fully use the knowledge \(N = n\), as it is the case for the optimal strategy for the classical secretary problem for \(n\) candidates.

(II) **Monotonicity results.** We can derive from (1) that

\[
p_n(x) := P(\text{x-strategy succeeds}|N = n) = \frac{(1-x)^n}{n} + x \sum_{k=1}^{n-1} \frac{(1-x)^k}{k}, \tag{2}
\]

and also that \(p_n(x) \geq p_{n+1}(x)\) for all \(x \in [0, 1]\). This implies

\[
\forall x \in [0, 1] : p_n(x) \downarrow \lim_{n \to \infty} p_n(x) = -x \log(x). \tag{3}
\]

Moreover, it follows from (2) and (3) that the optimal waiting time \(x_n\) and the corresponding optimal win probability \(p_n(x_n)\) satisfy, respectively,

\[
x_n \uparrow \frac{1}{e} \quad \text{and} \quad p(x_n) \downarrow \frac{1}{e}, \quad \text{as } n \to \infty. \tag{4}
\]

(III) **Asymptotic optimality.** The \(1/e\)-strategy is, as \(n \to \infty\), asymptotically optimal with win probability \(1/e\). This follows from (3) and (4), showing that the limiting performance of the \(1/e\)-strategy is the same as that of the well-known optimal strategy for the classical secretary problem for known \(n\) as \(n \to \infty\), namely \(1/e\). Clearly one cannot do better than in the case that one knows \(N\).

2.3 Interest versus challenge

Apart from the challenge of getting a complete answer, the question of optimality of the \(1/e\)-strategy in the case of no information is, in the light of the \(1/e\)-law, of
little interest in practice. The reason is that, in real world problems, we typically have an idea about bounds for $N$, and then we can look up the mentioned Table 1 of B. (1984) to assess the likely range of the win probability.

Moreover, if we have an idea about an upper bound of $N$, then this opens already the way to more standard arguments. For instance, if we see no point yet up to time $t = 0.33$, say, we expect $N$ around 3, say, that is to be rather small. Although we cannot use Bayes’ formula to obtain without a prior a corresponding posterior distribution of $N$, we would find it improbable that $N$ would be large. But since, given $N = n$, the optimal waiting time $x_n$ satisfies $x_n < 1/e$ for all $n$ (see (4)), the $1/e$-strategy cannot be optimal if we see any upper bound $n_u$ for $N$. Even the simple $x_{n_u}$-strategy to wait until $x_{n_u}$ and then to accept the first record (if any) is, according to (3), already better.

**Challenge and Intuition**

The mathematical challenge to have a complete answer for the case of no-information remains however, in particular because the unified approach model was created in order to deal with any $N$. But then, what attempts were made before, and why?

Looking in (II) of Subsection 2.2 closely at (2), (3) and (4), the open problem comes up quite naturally. Things become intriguing. For any $N = n$ there is a better strategy since the optimal $x_n$-waiting time strategy turns out strictly better than the $1/e$-strategy. Thus one gets the feeling that if there were a way of collecting information about $N$ sufficiently quickly, whatever this may mean, then this may be sufficient to prove that the $1/e$-strategy cannot be optimal. Viewing to disprove optimality, it seems promising to assume certain types and amounts of weak information about $N$, still strong enough to imply that the $1/e$-strategy is not optimal, and then to weaken the information.

Interestingly, as soon as one lets information about $N$ become weaker and weaker, and finally fade away towards no-information, the $1/e$-waiting time seems to become a miraculous ”fixpoint” of optimal thresholds. According to III, this would surprise us much less if no-information on $N$ implied in any way that $N$ is likely to be large, but of course it does not! What remained was the tantalizing question whether the attempts to disprove optimality in a constructive way by collecting information sufficiently quickly was possibly not clever enough.

What about trying to find other types of counterexamples?

The challenge remains. It is not easy to do this without leaving the framework of no-information. Arguing for example “If we have no information on $\{P(N = n)\}_{n=1,2,...}$, then let us for instance suppose that this distribution turns out such and such, and that we have seen a history of points such and such, ...” and then imply
that the 1/e-strategy is not optimal, would not be correct. Proofs by contradiction are only valid within the same logical framework, i.e. no-information. Arguments implying initial information whatsoever on $N$ would not be rigorous. For the same reason, simulations are meaningless as they require parameters to randomize $N$, and thus information on $N$ must be inputted. Looking for counterexamples cannot be expected to help. Knowing this increases the challenge.

2.4 Ill-posed or well-posed problem?

Is the question possibly ill-posed?

This question was asked repeatedly by several peers, and, during certain periods, the author also shared these doubts. Indeed, the notion of “no-information” requires clarification.

Can one interpret no-information in the sense that all possible values of $N$ are in an unknown interval $\{1, 2, \cdots, n\}$ with no value of $N$ being more likely than others, and then let $n$ tend to infinity?

No. This is equivalent to the improper Laplace prior for $N$. It is true that this prior is the prime candidate for no-information, and very often used to express the lack of knowledge about a parameter. However, this prior implies that $N$ is likely to be very large, and this is information. Therefore, when Bruss (1988) pp. 312-314, used this prior to prove that the 1/e-strategy is optimal “in a Bayesian sense,” he sees his statement today as not being sufficiently precise. His answer was only a partial answer. After all, “no information” on $N$ should mean that at time 0 we know really nothing at all about $N$.

Now, more importantly, can we assure that the no-information hypothesis is a honest hypothesis in the sense that it is contradiction-free? If it is not contradiction-free, then of course we must declare the open problem ill-posed.

2.5 Formalising no-information

When B. and Yor (2012) studied the last-arrival problem (l.a.p.) under the no-information hypothesis, they faced a similar difficulty of knowing whether their problem is well-posed. These authors found a simple argument to prove that it is impossible to prove that the no-information hypothesis may imply contradictions! Their argument was that there is only one contradiction-free way to define no-information, namely indirectly by saying where no-information must “live” and how it must compare to its surrounding in the space where it lives. B. and Yor (2012) concluded that, whatever a hypothetical information space $\mathcal{H}$ may look like for the unknown parameter or random variable $N$, no-information is bound to be a singleton in that space $\mathcal{H}$.
This definition may first sound like a formalism to prevent saying something wrong, but there is more to it. It implies that, as a singleton, the no-information hypothesis cannot lead to contradictory implications. A singleton in \( \mathcal{H} \) has, by definition of a singleton, nothing in common with other points in \( \mathcal{H} \), whereas contradicting implications cannot come out of nothing. They would need different sources of information giving rise to (at least two) different implications.

B. and Yor (2012) therefore concluded that they should, a priori, take a constructive attitude and try to find a solution. And so they did. But then the question was to know whether their solution is the solution of a well-posed problem. Hadamard’s criteria (Hadamard (1902)) were the only criteria B. and Yor could find for the solution of a well-posed problem, and they agreed with these criteria. This is why they were glad to see that their solution fully satisfied Hadamard’s criteria. According to these criteria, the solution given in B. and Yor (2012, subsection 5.3) is the solution of a well-posed problem.

Despite similarities between the l.a.p. and our open problem, the situation in the latter will turn out quite differently, however. We will see that the open problem turns out a well-posed self-contained problem, and that this fact is mainly a consequence of Rényi’s Theorem of relative ranks. We will solve the open problem, and, as we shall see, no external criteria will be needed.

One part of the approach of B. and Y. (2012), following next, remains however very helpful for our problem.

### 2.6 Proportional increments

For \( N \) i.i.d. \( U([0,1]) \) arrival points, let

\[
N_t = \# \text{ arrivals up to time } t, \quad t \in [0,1].
\]

B. and Yor (2012, subsection 1.1 and pp. 3242-46) showed that the counting process \( (N_t)_{0 \leq t \leq 1} \) of incoming points on \([0,1]\) with \( N := N_1 \) can be seen as a history-driven process with, what they called, *proportional increments*. This means that the process \( (N_t) \) must satisfy

\[
\forall 0 < t < 1 \text{ with } N_t > 0 : \quad E(N_{t+\Delta t} - N_t | \mathcal{F}_t) = \frac{\Delta t}{t} N_t \text{ a.s., } 0 < \Delta t \leq 1 - t,
\]

where the condition \( N_t > 0 \) is crucial. Such a process \( (N_t) \) will be said to have the property of proportional increments, in short *p.i.-property*.

Conditioned on \( N = N_1 > 0 \), let \( T_1 \) be the first arrival time. The definition of the p.i.-property implies then that, given \( N > 0 \), the process \( (N_t/t) \) is a martingale
on $[T_1, 1]$, as shown in B. and Yor (2012, p. 3245). This clearly holds also under the stronger assumption that $(N_t)$ is a Poisson process on $[0, 1]$. However, B. and Yor (2012, see p. 3255) saw a true benefit in not imposing that $(N_u)$ be Poisson.

To be complete on this, we should mention that in our open problem we could, from a purely decision-theoretic point of view, suppose right away that the process $(N_t)_{0 \leq t \leq 1}$ is a Poisson point process with unknown rate. Indeed, this cannot make a difference for decisions because we cannot tell a counting process which leaves a pattern of arrival times of a homogeneous Poisson process from another counting process leaving, in distribution, the same pattern. Doing so would have the advantage to be able to use the same compensator on the whole interval $[0, 1]$. However, we will not need the Poisson process assumption.

### 2.7 Towards suitable odds-theorems

Recall that our open problem is different from the l.a.p. of B. and Yor (2012) since in the Unified Approach model we would like to stop on the very last record, not on the very last point, and thus our approach must also be different.

The very first arrival time $T_1$ in the counting process $(N_t)$, is the time when $(N_t)$ makes its first jump, and the processes $(N_t)$ and $(N_t/t)$ have exactly the same jump times. $T_1$ is also the birth time of the record process $(R_t)$, say, defined by

$$R_t = \# \text{ records on } [0, t], \ 0 \leq t \leq 1.$$ 

Since in our open problem any strategy is equivalent for $N = 0$ we may and do suppose that $N > 0$ almost surely, and thus $T_1 < 1$ almost surely. $N$ is unknown at time 0, but at time 1 we know that, by definition, $N = N_1 \geq 1$ almost surely.

Since the first arrival is also the first record, we have $N_{T_1^-} = R_{T_1^-} = 0$ and $N_{T_1} = R_{T_1} = 1$. Thus the two processes $(N_t)$ and $(R_t)$ have the same random birth time $T_1$. On the interval $[T_1, 1]$, the process $(N_t)$ has proportional increments, i.e. dependent increments, whereas $(R_t)$ has, as we shall see later, independent non-homogeneous increments.

To prepare for these properties of $(R_t)$, the idea is to first concentrate on its increments (after time $T_1$). For this purpose we prove two suitably extended versions the Odds-Theorem of optimal stopping. We should also mention here that Ferguson (2016) gave several interesting extensions of the Odds-theorem in other directions. Moreover, Matsui and Ano (2016) studied in another extension lower bounds of the optimal success probability for the case of multiple stops. However, here we will here new extensions which are specifically tailored for our open problem.

We begin with an extension in discrete time.
2.8 Odds-Theorems for delayed stopping

Let $n$ be a positive integer, and let $X_1, X_2, \cdots, X_n$ be independent Bernoulli random variables with success parameters $p_k = P(X_k = 1) = 1 - P(X_k = 0), k = 1, 2, \cdots, n$. Suppose our goal is to maximize the probability of stopping online on the last success, i.e. on the last $X_k = 1$. The optimal strategy to achieve this goal is immediate from the Odds-Theorem (Bruss (2000)) which we recall for convenience of reference. Let

$$q_k = 1 - p_k; \ r_k = \frac{p_k}{q_k}; \ R(k, n) = \sum_{j=k}^{n} r_j, \ k = 1, 2, \cdots, n$$

and let the integer $s \geq 1$ (called threshold index) be defined by

$$s = \begin{cases} 1, \text{if } R(1, n) < 1 \\ \sup \{1 \leq k \leq n : R(k, n) \geq 1\}, \text{otherwise.} \end{cases}$$

The strategy to stop on the first index $k$ with $k \geq s$ and $X_k = 1$ (if such a $k$ exists) maximises the probability of stopping on the very last success (B.2000). If no such $k$ exists, then it is understood that we have to stop at time $n$ and lose by definition.

**Delayed stopping in discrete time**

Let us now consider the new case that there is a deterministic or a random delay imposed by a random variable $W$ with values in $\{1, 2, \cdots, n\}$ in the sense that stopping is not allowed before time $W$. Our objective, as before, is to maximize the probability of stopping on the last success. Does it suffice to replace simply the threshold $s$ defined in (6) by $\tilde{s} := \max\{W, s\}$ to obtain an optimal strategy? This seems trivial (and is true) if $W$ is deterministic.

In general this is not true, of course, even not true if $W$ is a stopping time on $X_1, X_2, \cdots, X_n$, unless we can guarantee that the knowledge of $W$ has no effect on the laws of $X_{W+1}, X_{W+2}, \ldots$ and their independence. The following is a more tractable formulation.

**Theorem 2.1.** Let $X_1, X_2, \cdots, X_n$ be Bernoulli random variables defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_k), P)$ where $\mathcal{A}_k = \sigma(\{X_j : 1 \leq j \leq k\})$. Suppose there exists a random time $W$ for $X_1, X_2, \cdots, X_n$ on the same probability space such that the $X_j$ with $j \geq W$ are independent random variables satisfying

$$p_j(w) := P(X_j = 1 | W \leq w), \ 1 \leq w \leq j \leq n.$$

Then, putting $r_j(w) = p_j(w)/(1 - p_j(w))$, it is optimal to stop at the random time

$$\tau := \inf \left\{ k \in \{W, W + 1, \cdots, n\} : \{X_k = 1\} \& \sum_{j=k+1}^{n} r_j(W) \leq 1 \right\},$$

with the understanding that we stop at time $n$ and lose by definition, if $\{\cdots\} = \emptyset$.
**Remark 2.2.** We note that no (initial) independence hypothesis is assumed for the $X_1, X_2, \cdots$ but only for those $X_j$’s with $j \geq W$.

**Proof of Theorem 2.1**

Our proof will profit from the proof of the Odds-Theorem (B. (2000)) if we rewrite the threshold index (6) in an equivalent for $m$.

Recall the definition of $R(k, n)$ in (5). If we define, as usual, an empty sum as zero, then $s$ defined in (6) can be written as

$$s' = \inf \left\{ 1 \leq k \leq n : R(k+1, n) := \sum_{j=k+1}^{n} r_j \leq 1 \right\}. \quad (8)$$

This is straightforward: If $R(1, n) \leq 1$ then $R(2, n) \leq 1$ so that from (8) $s' = 1$, and $s = 1$, as stated in (6). Otherwise, if $R(1, n) > 1$, then there exists a unique $k$ where $R(k+1, n)$ drops for the first time below the value 1 since $R(k, n)$ decreases in $k$, and $R(n+1, n) = 0$. The first such $k$ is the $s'$ defined in (8). The definitions (6) and (8) are thus equivalent. (See also Stirzaker (2015, p. 50))

Let now $p_j(w)$ as defined in Theorem 2.1, and let for $j = w, w+1, \cdots, n$

$$q_j(w) = 1 - p_j(w) = P(X_j = 0 | W \leq w).$$

It follows from the assumptions concerning $W$ that $X_w, X_{w+1}, \cdots, X_n$ are independent random variables with laws only dependent on the event $\{W \leq w\}$. If we think of $w$ as being fixed, then we can and do define $p_j := p_j(w)$ for all $w \leq j \leq n$ and use the same notation as before defined in (5) with the corresponding odds $r_j(w) = p_j(w)/q_j(w) =: r_j$. Accordingly, we have for $k \geq w$ the same simple monotonicity property $R(k, n) \geq R(k + 1, n)$.

It is easy to check that this monotonicity property is equivalent to the unimodality property proved in B. (2000, p.1386, lines 3-12). The latter implies that the optimal rule is a monotone rule in the sense that, once it is optimal to stop on a success at index $k$, then it is also optimal to stop on a success after index $k$. (See e.g. section 5 of Chow et al. (1971). For a convenient criterion for a stopping rule in the discrete setting being monotone, see Ferguson (2016, p. 49)).

Note that, whatever $W = w \in \{1, 2, \cdots, n\}$, the odds $r_j := r_j(w)$ are deterministic functions of the $p_j := p_j(w)$, and so the future odds $(r_j)_{j \geq W + 1}$ are also known and will not change. The only restriction we have to keep in mind for the simplified notation is that $k \geq w$ on the set $\{W \leq w\}$. But then the monotonicity property of $R(\cdot, \cdot)$ is also not affected, that is

$$\forall \ell \geq j : R(W + j, n) \leq 1 \implies R(W + \ell, n) \leq 1.$$
Since the latter implies the uni-modality property of the resulting win probability on $W \leq j \leq n$, the monotone rule property is again maintained for the optimal rule after the random time $W$, exactly as in B. (2000). Therefore the optimal strategy is to stop on the first success (if it exists) from time $\tau$ onwards where $\tau$ satisfies

$$\tau \geq W \text{ and } \sum_{j=\tau+1}^{n} r_j(W) \leq 1. \quad (9)$$

This is the threshold index $\tau$ of Theorem 2.1, and hence the proof. \hfill \Box

**Remark 2.3.** Note that Theorem 2.1 is intuitive. Its applicability, nevertheless, can be delicate. It depends on the $p_j$'s being predictable for all $j \geq W$. Often this is not the case. For instance, we may have (conditionally) independent random variables, but, if we collect information about the $p_j$ from observations then the distributions of the future values of $X_{j+1}, X_{j+2}, \cdots$ typically depend on $X_k$, $1 \leq k \leq j$, on which, for a stopping time $W$, the event \{W = j\} is allowed to depend! (For our purpose of settling the open question the Theorem will turn out to be the perfect tool, however.)

**Remark 2.4.** (Side-remark). Given that (8) is a one-line definition whereas (6) needs two lines, some readers ask why B. (2000) used definition (6). The answer is that it is (6) which points to the odds-algorithm (subsection 2.1, p.1386) which works backwards until the stopping time $s$ with $r_n, r_{n-1}, \cdots$ to give optimal strategy and value at the same time. No other algorithm can be quicker since it computes exactly those $r_j$ which produce both answers. If we used instead the odds beginning with $r_1, r_2, \cdots$ and (8) we would first need $R(1,n)$, implying in general redundant calculations. For the preceding theorem, however, we clearly needed (8).

**Delayed stopping in continuous time**

We now state and prove a continuous-time analogue of the Theorem 2.1 which plays an important role in the proof of the open conjecture. We state and prove it in a slightly more general form than what we need for the conjecture, because it may be also of interest for other problems of optimal stopping.

**Theorem 2.5.** Suppose $(C_t)$ is a counting process on $[0, 1]$ for which there exists a random time $T$ such that the confined process $(C_t)_{\tau<t\leq1}$ has independent increments according to a predictable (non-random) intensity measure $\eta(t)_{\tau<t\leq1}$. We suppose that $\eta(t)$ is Riemann integrable on $[0, 1]$ with $E(C_1) < \infty$. Then the optimal strategy to stop on the last jump-time of $(C_t)$ is to select, if it exists, the first arrival time $\tau \geq T$ with $\tau$ satisfying

$$E(C_1 - C_\tau) = \int_\tau^1 \eta(u) \leq 1. \quad (10)$$

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Remark 2.6. In the special case when the process \((C_t)\) is a Poisson process on \([0, 1]\) the conditions of Theorem 2.6 are clearly satisfied everywhere on \([0, 1]\). See also subsection 4.1 of B. (2000).

**Proof of Theorem 2.6**

Consider a partition \(\{u_0 < u_1 < \cdots < u_m\}, \ m \in \{1, 2, \cdots\}\), of the random sub-interval \([T, 1] \subseteq [0, 1]\) with \(u_0 = T\) and \(u_m = 1\). Let the index \(j\) be thought of as depending on \(m\), thus \(j := j(m)\) and \(u_j := u_j(m)\). Put

\[ p_j := p_j(m) = \int_{u_{j-1}}^{u_j} \eta(u) du, \ j = 1, 2, \cdots, m, \]  

(11)

where \([u_{j-1}, u_j]\) is by definition the \(j\)th sub-interval of the partition, \(j = 1, 2, \cdots, m\). It follows that \(p_j\) is the expected number of points of the process \((C_u)\) in the \(j\)th sub-interval, and thus by additivity from (11)

\[ \sum_{j=1}^{m} p_j = \int_{T}^{1} \eta(u) du = E(C_1 - C_T) \leq E(C_1) < \infty. \]  

(12)

Since all \(p_j\) in (11) are non-negative, and \(E(C_1)\) is finite, we can interpret them all as probabilities of certain events as soon as we choose sufficiently fine partitions to have the the \(p_j\) less or equal to 1. This is always possible since, as we see in (11), \(p_j \to 0\) as \(\Delta_j = u_j - u_{j-1} \to 0\). For the following it is understood that we only speak of such sufficiently fine partitions. Since the counting process \((C_u)\) has independent increments, this allows us at the same time to see the \(p_j\) as the success probabilities of independent Bernoulli random variables, namely as the indicators

\[ I_j := I_{j(m)} = 1{\{[u_{j-1}, u_j[ \text{ contains jump times of } (C_u)_{T \leq u \leq 1}\}} \]

for \(j = 1, 2, \cdots, m\). The success probability of the \(j\)th Bernoulli experiment is then given by \(p_j = E(I_j)\). Let us call this interpretation the "Bernoulli model" for increments of the process \((C_u)\) for the chosen partition of \([T, 1]\).

To be definite we now confine our interest to equidistant partitions, and in this class to those such that all \(p_j < 1\). Let

\[ s(m) = \sup_{j \in \{1, 2, \cdots, m\}} \{p_j(m)\}. \]

From (11) we obtain \(p_j \sim \Delta_j \eta(u_j)\) and thus, as \(\Delta_j \to 0\), we have \(E(I_j) \to 0\) and also

\[ E(I_{j(m)}) / P([u_j, u_{j+1[ \text{ contains exactly one jump time}) \to 1. \]  

(13)
The idea is now the following: First, if we can interpret any increment $C_{uk} - C_{uj}, j \leq k \leq m$ as a sum of odds in our Bernoulli models, then the optimal odds-rule for stopping on the last success identifies the optimal rule for stopping on the last sub-interval of the partition containing jump-times. Note that for any fixed $m$, the last Bernoulli success may correspond to more than one point in the last sub-interval containing points (i.e. jump-times of $(C_u)$). Second, in a limiting Bernoulli model defined by letting $m \to \infty$, the last success corresponds, according to (13), with probability 1 to the very last jump in $(C_u)$. Hence, provided that the notion of limiting odds is meaningful for the limiting Bernoulli model, the optimal rule in the latter identifies the optimal rule for stopping on the last jump of $(C_u)$.

We will combine both parts by showing that the continuous-time analogue of odds in the limiting Bernoulli model is an intensity measure of a counting process, and we will adapt it to become the process $(C_u)$.

Let $\rho$ be a real-valued non-negative Riemann integrable function $\rho : [0, 1] \to \mathbb{R}^+$, and let

$$\Psi(x, \Delta x) := \int_x^{x+\Delta x} \rho(u) du.$$ 

We now chose a function $\rho$ in such a way that all $\Psi(u_j, \Delta_j)$ satisfy the equation

$$\Psi_j := \Psi(u_j, \Delta_j) = \frac{p_j}{1-p_j} = r_j, \ j = 1, 2, \ldots, m.$$  

(14)

Note that the existence of such a function $\rho$ is evident for any finite partition since the class of Riemann integrable functions contains already infinitely many. If we choose $\rho$ in this class we have $\lim_{\Delta u \to 0} \Psi(u, \Delta u) / \Delta u$ exists almost everywhere on $[\mathcal{T}, 1]$, and this derivative coincides with $\rho(u)$ on $[\mathcal{T}, 1]$.

Now we must check whether such a function $\rho$ exists if we let the caliber of the partition tend to 0. We shall now prove that the $\rho$ is unique in the limiting Bernoulli model, and that $\eta$ and $\rho$ coincide almost everywhere on $[\mathcal{T}, 1]$. It will thus be justified to call the function $\rho$ the odds-intensity associated with the (identical) intensity $\eta$ of the process $(C_u)$ on $[\mathcal{T}, 1]$.

Indeed, recalling $\Delta_j = 1/m$, we will first show that

(i) $\rho(u_j) = \lim_{\Delta_j \to 0} \frac{1}{\Delta_j} r_j = \eta(u_j), \ j = 1, 2, \ldots$

(ii) $\lim_{m \to \infty} \sum_{j=1}^{m} \Psi(u_j, \Delta_j) = \sum_{j=1}^{\infty} \lim_{m \to \infty} \Psi(u_j, \Delta_j).$

The limiting equation (i) follows from the definition of odds in the Bernoulli models,
and from (11), since
\[ \frac{p_j}{1 - p_j} \sim \frac{1}{\Delta_j (1 - \Delta_j \eta(u))} = \frac{\eta(u)}{1 - \Delta_j \eta(u)} \to \eta(u) \text{ as } \Delta_j \to 0. \]

To see (ii), we first recall that for all \( j = 1, 2, \ldots, m \) we have \( p_j < 1 \) and thus from (14)
\[ p_j \leq \Psi_j = \frac{p_j}{(1 - p_j)}. \]

For fixed \( \epsilon \) with \( 0 < \epsilon < 1 \) we now choose an integer \( m := m(\epsilon) \) large enough so that \( s(m) := \sup\{p_k : 1 \leq k \leq m\} < \epsilon \). This is trivially always possible for a finite number \( m \) of \( p_k \), since, again seen as a function of \( \Delta_k \), in (11) each \( p_k \to 0 \) as \( \Delta_k \to 0^+ \), that is as \( m \to \infty \). Then we obtain
\[ p_j \leq \Psi_j \leq \frac{p_j}{(1 - s(m))} \leq \frac{p_j}{(1 - \epsilon)}, \]
or with (11) more explicitly,
\[ \int_{u_{j-1}}^{u_j} \eta(u) \, du \leq \Psi_j \leq \frac{1}{1 - \epsilon} \int_{u_{j-1}}^{u_j} \eta(u) \, du. \] (15)

Since this inequality holds for all \( j = 1, 2, \ldots, m(\epsilon) \), it must hold also for any sum of these terms (column-wise) taken over the same set of indices. In particular this includes tail sums beginning at an arbitrary time \( x \geq \mathcal{T} \). Hence, by bounded convergence, (ii) is true.

But then the latter also holds for any random time \( x := \tau \geq \mathcal{T} \), since, by the hypothesis stated in Theorem 2.5, the intensity measure \( \eta \) is supposed to be non-random from time \( \mathcal{T} \) onwards. Thus for any set of sub-intervals of \( [\mathcal{T}, 1] \), the limiting odds sum for the limiting Bernoulli model, corresponds to the integral of \( \rho \) over the same set of intervals. Therefore, in particular, \( \rho \) satisfying (14) must satisfy for any \( \epsilon \) and equidistant partition with caliber \( \Delta_j = 1/m(\epsilon) \)
\[ \mathbb{E}(C_1 - C_\tau) = \int_{\tau}^{1} \eta(u) \, du \leq \int_{\tau}^{1} \rho(u) \, du \leq \frac{1}{1 - \epsilon} \mathbb{E}(C_1 - C_\tau). \] (16)

Since \( \epsilon \) can be chosen arbitrarily close to 0 in the inequality (16) it follows from the squeezing theorem that the inner integral is bound to coincide with \( \mathbb{E}(C_1 - C_\tau) \). According to (ii), this inner integral is however the limiting tail sum of odds for the limiting Bernoulli model, and (i) implies thus \( \rho(u) = \eta(u) \).

Finally, letting \( \epsilon \to 0^+ \) in (16) that the inner integral, that is, the limiting tail sum of odds in the limiting Bernoulli model, drops below 1 if and only if \( \mathbb{E}(C_1 - C_\tau) \) does so. Hence the proof.
Remark 2.7. The preceding criterion is valid independently of whether \( \tau \) is a jump-time of \((C_u)\) or not. Indeed, if \( \eta(u) > 0 \) on \([\tau, 1]\) then for all \( 0 < \epsilon < 1 - \tau \) we have \( \mathbb{E}(C_1 - C_{\tau + \epsilon}) < 1 \). Therefore, if \( \tau \) happens to be a jump-time of \((C_u)_{t \leq u \leq 1}\) we must also stop on \( \tau \).

We are now ready to tackle our main problem.

3 Optimality of the \( 1/e \)-strategy

3.1 Preview and visualisation of our approach

If the optimal strategy exists, then it must solely be based on all the sequential information we can have, that is, on the information stemming from the history of arrivals (points) and their relative ranks.

Clearly, any strategy is trivially optimal if there are no points so that we can confine our interest to the case \( N > 0 \). Denote by \( N_u \) the number of arrivals up to time \( u \). If \( N > 0 \), there is at least one arrival on \([0, 1]\), and the first one is a record by definition.

Due to the i.i.d. structure of points on \([0, 1]\), if the decision maker looks back at time \( t \in [0, 1] \), and if there are preceding arrivals, then he or she knows that their pattern is the outcome of i.i.d. uniformly distributed points on \([0, t]\). The same will hold by looking forward, that is, if there are arrivals then their unordered arrival times are i.i.d. on \([t, 1]\). This is true since i.i.d. uniform random variables on a given interval \( I \), say, stay i.i.d. conditioned on their location in sub-intervals of \( I \). This is illustrated in the figure below (Fig.1) where arrivals are denoted by * , and where the first * is meant to indicate the arrival time \( T_1 \).

![Fig. 1](image)

Decision-maker’s perception

From the first arrival time \( T_1 \) onwards (\( 0 < T_1 < 1 \) a.s.) the decision maker has the information that the counting process \((N_u)_{u \geq T_1}\) is a process with proportional increments as defined in Subsection 2.6. See Fig. 2. Accordingly, given \( N_u \), the expected value of the number of points in \([u, u + \Delta u]\) equals \( (N_u \Delta u)/u \) almost surely, and it is important to note that no \( o(\Delta u) \) is added here.
The relevant stochastic process for stopping on rank 1 is then the record process
\((R_u)_{u \geq T_1}\) which is a sub-process of \((N_u)_{u \geq T_1}\) (see Fig. 3).

This thinning is by Rényi’s Theorem such that if \(J \geq T_1\) is a jump-time of the
process \((N_u)\) then it is retained for the record process \((R_u)_{u \geq T_1}\) with probability \(1/N_J\)
independently of retained preceding points. By this inverse-proportional thinning of
\((N_u)_{u \geq T_1}\), the increments of the resulting sub-process \((R_u)_{u \geq T_1}\) become independent
with a known distribution depending only on time.

With independent increments having a predictable non-random intensity mea-
sure, the process \((R_u)\) can now play the role of \((C_u)\) in Theorem 2.5. Stopping online
on the desired rank 1 means stopping online on the very last record, i.e. on the last
jump of \((R_u)_{u \geq T_1}\), and this will lead to the desired result.

We are now ready for the main result and its proof. We prove the conjecture in
an even stronger form, namely the \(1/e\)-strategy is the unique optimal strategy for
the best-choice problem under no information.

**Theorem 3.1.** The \(1/e\)-strategy is the unique optimal strategy for the best-choice
problem under no information on the number \(N\) of options.

**Proof** Recall the unified approach model in Section 2.1. After time transformation
via the continuous distribution function \(F\), time runs from 0 to 1. As before we
denote the process counting the arrivals on \([0,1]\) by \((N_u)_{0 \leq u \leq 1}\), and its sequential
arrival times by \(T_1, T_2, \ldots\). The sub-process counting the corresponding records is
denoted by \((R_u)_{0 \leq u \leq 1}\). The last jump time of \((R_u)\) will thus be the location of the
last record, i.e., of absolute rank 1 (the best).

Therefore our Theorem requires to prove that, under the no-information hypo-
thesis on \(N\), it is optimal to stop on the very first record time greater or equal time
\(1/e\) provided that it exists (otherwise we have to stop at time 1 and loose), and that
this is the unique optimal stopping time. Since we know already from the 1/e-law that the 1/e-strategy is uniquely optimal for \( N = N_1 = \infty \) (see (III) in Subsection 2.2) we can confine our interest to \( 1 \leq N_1 < \infty \) without, however, knowing an upper bound for \( N_1 \).

Our proof will use Theorem 2.5 by showing that the record process \((R_u)\) satisfies the hypotheses of the process \((C_u)\) with \( T := T_1 \), which is the first arrival time in both processes \((N_u)_{0 \leq u \leq 1}\) and \((R_u)_{0 \leq u \leq 1}\). Since \( R_1 \leq N_1 \) we have \( \mathbb{E}(R_1) < \infty \).

Let \( \mathcal{F}_t \) denote the filtration generated by \( \{N_s : 0 \leq s \leq t\} \), and denote by \( \mathcal{G}_t \) the one generated by both \( \{N_s : 0 \leq s \leq t\} \) and \( \{R_s : 0 \leq s \leq t\} \) together. Since both fields are clearly increasing we have \( \mathcal{G}_t \subseteq \mathcal{G}_u \) for \( t \leq u \leq 1 \).

We will now show is that \( T_1 \) is a \( \mathcal{G}_t \)-measurable stopping time \( T \) from which onward \((R_t)\) has independent increments. Clearly \( T_1 \) is a \( \mathcal{G}_t \)-measurable stopping time since \( \mathcal{F}_t \subseteq \mathcal{G}_t \).

Given \( T_1 \), choose \( t \in [T_1, 1] \) and define for fixed \( m \in \{2, 3, \cdots\} \) and \( k = 0, 1, 2, \cdots m - 1 \),

\[
     u_k := u_k(t) = t + \frac{k(1 - t)}{m},
\]

\[
     \Delta_k := \Delta_k(t) = u_{k+1} - u_k = \frac{1 - t}{m}.
\]

It follows that for any \( \mathcal{G}_u \)-measurable random variable \( X \) and \( 0 \leq t \leq u \leq 1 \),

\[
     \mathbb{E}(\mathbb{E}(X \mid \mathcal{G}_u) \mid \mathcal{G}_t) = \mathbb{E}(X \mid \mathcal{G}_t). \tag{17}
\]

Let now \( X \) denote the number of records in \([t, 1]\), that is \( X = R_1 - R_t \). From the linearity of the expectation operator we obtain

\[
     \mathbb{E}(R_1 - R_t \mid \mathcal{G}_t) = \mathbb{E} \left( \sum_{k=0}^{m-1} (R_{uk+1} - R_{uk}) \mid \mathcal{G}_t \right) = \sum_{k=0}^{m-1} \mathbb{E} \left( R_{uk+1} - R_{uk} \mid \mathcal{G}_t \right), \tag{18}
\]

and then from (18) used in (17)

\[
     \mathbb{E}(R_1 - R_t \mid \mathcal{G}_t) = \sum_{k=0}^{m-1} \mathbb{E} \left( R_{uk+1} - R_{uk} \mid \mathcal{G}_u \right) \mid \mathcal{G}_t. \tag{19}
\]

Let \( \lambda(u) \) denote the rate of the point process \((N_t)\) at time \( u \), and \( h(u) \) be the conditional probability of a point appearing at time \( u \) being a record. The process \((N_u)\) inherits history-dependence from the p.i.-property so that \( \lambda(u) \) is also history-dependent, namely a \( \mathcal{F} \)-predictable intensity process for \((N_t)\) relative to the filtration \((\mathcal{F}_t)\). The function \( h(u) \) acts like a thinning on the counting process \((N_t)\), retaining
only its record-times as events. The resulting record process has an intensity, \( \eta \) say, which may depend on both \( \lambda \) and \( h \), and which we write formally as

\[
\eta(u) := g_{\lambda,h}(u), T_1 \leq u \leq 1.
\]  

Note that this formal definition is a step of caution because \( h(u) \) and \( \lambda(u) \) are history-dependent random variables, and dependent on each other. Thus we do not assume so far that \( \eta(u) = g_{\lambda,h}(u) \) factorises into \( \lambda(u)h(u) \) over sub-intervals we will consider. Of course we know it does so point-wise because \( h \) is defined as the conditional probability of a point being retained as a record.

Now consider the inner conditional expectation on the r.h.s. of (19). Since \( (N_u)_{T_1 \leq u \leq 1} \) is a p.i.-process, and \( N_u \) is \( \mathcal{G}_u \)-measurable, we have correspondingly

\[
\mathbb{E}(N_{u_{k+1}} - N_{u_k} \mid \mathcal{G}_{u_k}) = \Delta_k N_{u_k} / u_k \text{ a.s.,}
\]

and thus \( \lambda(u_k) = N_{u_k} / u_k \text{ a.s.} \). Moreover, if \( u_k \) were a jump-time for \( (N_u) \) it would be according to Rényi’s Theorem a record time with probabilty \( 1 / N_{u_k} \) which shows that \( h \) in (20) is also history-dependent.

We now show the central fact that the increments of the record process \( (R_u) \) on \([u, u + \Delta u]\) given \( \mathcal{G}_u \) will never depend on the locations of jump-times in \([u, u + \Delta u]\), but only on the number of jumps in there. Indeed, if we denote the \( j \)th jump-time in \([u_k, u_{k+1}]\) by \( A_j := T_{j+N_{u_k}} \), then

\[
\mathbb{E}
\left(
R_{u_{k+1}} - R_{u_k} \mid N_{u_{k+1}} - N_{u_k} = J; A_1, A_2, \ldots, A_J \right)
= \mathbb{E}
\left(
\sum_{j=1}^J 1\{A_j \text{ is a record time}\}
\right).
\]

Since \( J \leq N_1 < \infty \) we can exchange the operators expectation and summation, and then use Rényi’s Theorem. Therefore, by the definition of the \( A_j \), the latter equals

\[
\sum_{j=1}^J P(A_j \text{ is a record time} \mid \mathcal{G}_{u_k}) = \sum_{j=1}^J \frac{1}{N_{u_k} + j} \text{ a.s.,}
\]

which is understood as being zero if \( J = 0 \). Given \( \mathcal{G}_{u_k} \), the value \( N_{u_k} \) is a constant, and \( J \) is \( \mathcal{F}_{u_k} \)-predictable. Hence the r.h.s. of (20) is \( \mathcal{F}_{u_k} \)-predictable and does not depend on the location of jumps.

But then, given any interval \([u, u + du]\), we can imagine these jump times (if any) to be located where we want them to be within this interval, and we are entitled to think of the first one (if any) as being in \( u \). This implies from (21) that \( g_{\lambda,h} \) in (20) must factorize on the sub-interval \([u, u + du]\) into the intensity of \((N_u)\), namely \( \lambda(u) \),
and the inverse proportional thinning \( h(u) = 1/N_u \). Now recall that the p.i.-property of \((N_u)\) for \( u \geq T_1 \) implies

\[
\lambda(u) du := E(dN_u | \mathcal{F}_u) = E(N_{u+du} - N_u | \mathcal{F}_u) = \frac{N_u}{u} du \text{ a.s.} \tag{22}
\]

Since the inverse-proportional thinning on \((N_u)\) is \((\mathcal{F}_u)\)-predictable and \( \mathcal{F}_u \subseteq \mathcal{G}_u \) we have correspondingly

\[
E\left( dR_u \bigg| \mathcal{G}_u \right) := E\left( R_{u+du} - R_u \bigg| \mathcal{G}_u \right) = \frac{N_u}{u} \frac{1}{N_u} du = \frac{du}{u} \text{ a.s.,} \tag{23}
\]

which depends on \( \mathcal{G}_u \) only through the time \( u \). This means that the stopping time \( T_1 \) fulfills in Theorem 2.5 the conditions that \((C_u) := (R_u)\) has, for \( u \geq T_1 \), independent increments with the predictable non-random intensity measure \( \eta(u) := g_{\lambda,h}(u) = 1/u \).

Since Theorem 2.5 is the continuous-time version of Theorem 2.1 which implies the monotone-rule property of the optimal rule, we know that the optimal rule must be a monotone rule. This means we are ready to accept any record after some fixed, or random, threshold time. The expected increment (23) tells us now that this optimal threshold time \( t \) cannot be random if \( T_1 \leq t \) because the relevant decision function \( E(R_1 - R_u | \mathcal{G}_u) \) at time \( u \geq T_1 \) does not depend on the history \( \mathcal{G}_u \). This is all what counts because, before time \( T_1 \), no decision must be taken. Therefore the optimal threshold time must be either \( T_1 \) itself, or else a fixed threshold time \( t \geq T_1 \).

Thus, from Theorem 2.5, if

\[
E(R_1 - R_{T_1}) = \int_{T_1}^{1} \frac{1}{u} du = -\log (T_1) \leq 1, \tag{24}
\]

then optimal behaviour forces us to stop on the very first arrival, i.e. at time \( T_1 \) itself. If not, then, again from Theorem 2.5, optimal behaviour imposes to accept the first available record from that time \( t \) onwards which satisfies \( E(R_1 - R_t) = -\log t \leq 1 \).

Since \( \log(t) = 1 \) if and only if \( t = 1/e \) and \( -\log t \) decreases strictly from 1 down to 0 on \([1/e, 1]\) it is thus optimal to stop on the first record (if any) from time \( \tau = \max\{T_1, 1/e\} \) onwards.

We summarize: Under the no-information hypothesis we have by definition no knowledge about \( N \) and thus none about the distribution of \( T_1 \) either. However, on the one hand, we need not care about \( T_1 \) if \( T_1 < 1/e \) because we would not accept, knowing from Theorem 2.5 and (24) that it is then optimal to wait for the first record (if any) from time \( 1/e \) onwards. On the other hand we know that it is optimal to stop at time \( T_1 \) and accept, if \( T_1 \geq 1/e \). In simpler words this means that it is always optimal to accept the first record (if any) from the fixed threshold time \( 1/e \) onwards. Moreover, we see from (24) that this optimal threshold is unique.
In conclusion, the optimal strategy is unique and is, what we called in B. (2000), and recalled in Subsection 2.2, the \( x \)-strategy for \( x = 1/e \). This completes the proof.

3.1.1 Why now, and why not earlier?

I think it was immature intuition which had led me, again and again, into wrong directions. If one can solve an optimal stopping problem if a certain parameter is known in distribution, then, having no information on this parameter, it seems so persuading that one must learn as quickly as possible about this parameter before it is "too late" for stopping. The results (1), (2), (3) and (4) look strong and made me believe them strong enough to conclude, collecting information on \( N \), that the conjecture must be false or true. When I did not succeed, I believed that my "constructions" were not sufficiently sophisticated but that improvements should yield what I wanted. However, they did not, and with the omnipresent problem that counterexamples based on any type of prior information on \( N \) was inadmissible, ideas to try out became rare. Not knowing for sure whether the problem is well-posed made it worse.

It took time, and relevant questions from good referees, to ask oneself: Would it not be better to forget learning and concentrate on optimality, i.e. to find an optimality criterion for the \( 1/e \)-strategy which can be shown to be either satisfied or non-satisfiable? And then, would it not be better to look at what comes out of the criterion in spe and worry later about the problem being well-posed?

The last set of questions was the right one. Two modifications of the Odds-theorem (of which, ironically, I did not think before) showed what was needed. The independent increments of the record process generated by the decision process made then their application relatively easy. Rényi’s Theorem gave these from the stopping time \( T_1 \) onward, and the intensity measure of the record process at time \( u \geq T_1 \) turned out \( \eta(u) = 1/u \). Hence the question 'well-posed or not' did not even come up. No external confirmation such as Hadamard’s criteria was needed. The new Odds-Theorems showed that the \( 1/e \)-strategy is the unique optimal strategy under no information.

3.2 \( k \)-record processes

Rényi’s Theorem of relative ranks (1962), which is easy to prove, was central in our paper. For our purpose viewing the open question of optimality it was sufficient to look at the sub-process of records which we denoted (in continuous time) by \( (R_u) \).

If instead we consider the sub-process of those points which are the \( k \)th best so far, called \( k \)-records, then we obtain a corresponding \( k \)-record process \( (R_u^{(k)}) \).
Clearly, at the time of its appearance, a point cannot have different relative ranks at the same time. Hence, for fixed \( k \geq 1 \), the process \((R_u^{(k)})\) is well-defined and unique. We should mention here a result of Goldie and Rogers (1984) which may be seen as the strongest in the domain of record processes. Goldie and Rogers have shown that \( k \)-record processes are i.i.d. These authors speak here of the processes as such including the variables (marks) attached to the points which give rise to the relative ranks. These processes are i.i.d., and this result is stronger, less intuitive, and harder to prove.

Knowing this and returning to the \( 1/e \)-law and our result, does there exist for a fixed \( k > 1 \) a "\( 1/e \)-law" for getting (under no information in the Unified Approach model) the \( k \)th best observation? Our proof of the optimality of the \( 1/e \)-strategy, and the results above mentioned for \( k \)-record processes may suggest this, but we see easily that this is not true. For \( k = 1 \) the last 1-record (= record) is always the best, and vice versa, but the \( k \)th best for \( k > 1 \) can show up as a record, or 2-record, ..., or \( k \)-record. Only trying to get the last \( k \)-record should be a strictly sub-optimal strategy; and it is.

The optimal strategy to obtain the last \( k \)-record under no information is the same, however, if we suppose \( N \geq k \) (the problem is not defined otherwise). For the proof we simply use the \( k \)th arrival time \( T_k \) of the process \((N_t)\) as the right stopping time to activate \((R_u^{(k)})_{u \geq T_k}\) and put \( C_u := R_u^{(k)} \) in Theorem 2.5.

### 3.3 Return to the last-arrival problem

The solution of the last arrival problem (see Theorem 5.1 of Bruss and Yor (2012)) becomes more elegant in view of Theorem 2.5.

**Proof** Since \((N_t/t)\) is a \( \mathcal{F}_t \)-martingale from the arrival time \( T_1 \) onward, we have

\[
E\left( \frac{N_u}{u} \mid \mathcal{F}_t \right) = \frac{N_t}{t} \text{ a.s., } T_1 \leq t \leq u \leq 1.
\]

By definition of the l.a.p., any arrival point is of equal interest, so that the relevant decision process is \((N_t)_{t \leq 1}\) itself. Using the tower property of conditional expectation and \( \mathcal{F}_t \subseteq \mathcal{F}_u \) we see that, from time \( t \geq T_1 \) onwards, the expected increments \( E(dN_u \mid \mathcal{F}_t) \) equal \((N_t/t)du\) and depend on \( \mathcal{F}_u \) for \( u \geq t \) only through \( \mathcal{F}_t \). By the martingale property of \((N_u/u)\) on \([T_1, 1]\) this remains true if we replace \( t \) by any stopping time \( \tau \geq T_1 \), i.e. \( E(N_u u^{-1} \mid \mathcal{F}_\tau) = N_\tau \tau^{-1} \text{ a.s.} \). If an optimal stopping time \( \tau^* \) exists, then clearly \( \tau^* \geq T_1 \). Hence from Theorem 2.5, \( \tau^* \) must satisfy

\[
\int_{\tau^*}^1 E\left( \frac{N_u}{u} \mid \mathcal{F}_{\tau^*} \right) du = \int_{\tau^*}^1 \frac{N_\tau}{\tau^*} du \leq 1 \tag{25}
\]
which is equivalent to $N_{\tau^*} \leq \tau^*/(1 - \tau^*)$. Here it is understood that we lose by
definition, if no such jump time $\tau^*$ exists for the process $(N_u)_{T_1 \leq u \leq 1}$, but if it exists
then we must stop according to Theorem 2.5. This agrees with equation (5.11) of
B. and Yor (2012), and proves their Theorem 5.1. □

There is a major difference between the solution of this problem (l.a.p.), in which
each arrival time is a decision time, and the problem which we solved where only
record times are times where we have to take decisions. It is not hard to show that
the thinning in the latter implies nice continuity properties of our solution. If we
replace in $([0, 1]$-time) the fixed threshold time $1/e$ by a slightly wrong threshold
time $1/e + \epsilon$, then the loss is always small, namely $p_N(1/e) - p_N(1/e + \epsilon)$ (see (2)),
and we know this already at time 0. The random win probability in the l.a.p. with
the history-driven optimal threshold determined by $N_t \leq t(1 - t)^{-1}$ is more sensitive
than the random optimal win probability in the $1/e$-law, and lies, in contrast, always
below $1/e$ (for more details, see B. and Yor (2012)).

3.4 Implications

1. Applying the $1/e$-strategy gives a random value, namely $p_N(1/e)$, where $p_n(x)$ is
given in (2). If we consider the $N$ as a random variable in a well-defined (unknown)
probability space (and we are certainly free to do so), then the value $p_N(1/e)$ is for
us a random variable. Otherwise it is just an unknown value. We know from (3)
$p_N(1/e) > 1/e$ if $N < \infty$, and that $p_N(1/e) \geq 1/e$ always holds. It makes no sense
to try to replace the random optimal payoff $p_N(1/e)$ by $E(p_N(1/e))$. By definition
we cannot compute the latter.

One aspect of our result, and of our approach to obtain it, may attract our
attention. We are, for evident reasons, not used to speak of the maximization of a
random value or random variable, but here we are, at least to some extent, actually
forced to make sense out of it. The random variable or random value $p_N(1/e)$ is
the optimal payoff because we obtain it from the optimal strategy, which is, as we
proved, the $1/e$-strategy. An optimal strategy should give (by definition) at any
instance of time optimal decision instructions since otherwise this would lead to
contradictions to the optimality principle. These optimal instructions were implied
by extensions in discrete time, and then continuous time, of the Odds-Theorem
for which there are no logical problems to understand optimality. Thus, viewing
consistency, elementary logic forces us to accept, at least in this case, the notion of
a random value in an optimal stopping problem.

As far as the author is aware, classical definitions of optimal stopping (see e.g.
Chow et al. (1971), Ferguson (2008, 2016), Rüschendorf (2016), Stirzaker (2015))
do not include this notion. More breadth in the definition, or at least some more
flexibility, could possibly be obtained by embedding, or at least linking, optimal stopping problems, in, or with, other domains. Free boundary problems for instance (see the well-chosen examples in the introduction section of the book by Peskir and Shiryaev (2006)), do offer, as the author understands at least some flexibility towards such a goal.

Somewhat simplified the author would say that it would be nice to have a definition of optimal stopping problems which includes the notion of value-free ”optimal actions”. We pick up this discussion more generally in the forthcoming Section 4, where we give examples for (undeniably) optimal actions leading to (unknown) optimal payoffs.

**Remark 3.2.** We recall what we see in the proof of our main Theorem. The dependence of future records on $G_t$ reduces to dependence on $t$ only. Records behave in distribution as if we had started by assuming an improper uniform Laplace prior for $N$ leading to a Pascal process as shown in Bruss (1988). (For the characterization of these interesting Pascal processes see Bruss and Rogers (1991).)

Now, the no-information hypothesis insisted on the complete lack of information, whereas the improper Laplace prior for $N$ implies that $N$ is likely to be large. This is far off the non-information hypothesis. How come?

The answer is that the non-informative uniform prior for $N$ does not affect the proportional increments property of $\left( N_t \right)_{t \geq T_1}$ as long as we do not assume to know an upper bound $n_u$ for $N$, because then Rényi’s Theorem (1962) renders with $N \geq 1$ further information on $N$ irrelevant. Thus, more generally, no model of ”learning” about $N$ could give us for our problem a decisive conclusion, and now we understand why all previous attempts to solve the open question were bound to fail.

## 4 Optimal strategies without value

Having proven optimality of the $1/e$ strategy we explain why one must be careful in dealing with problems under the hypothesis of no information. Indeed, it may, as we have seen in the preceding Section, occur that a problem of optimal stopping and/or optimal control has no value. Moreover, as we will show below, it may resist any comparison of performance versus non-optimal strategies.

The following Lemma illustrates this in a simple form.

**Lemma 4.1.** In a model for problems of optimal stopping and/or optimal control in a no-information setting, the following features are possible:

(i) An optimal strategy $S$ solving the defined problem may exist independently of whether one can attribute a value to $S$. 

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(ii) If an optimal strategy $S$ exists, it need not be the limit of $\epsilon$-optimal strategies as $\epsilon \to 0+$.

**Remark 4.2.** In the way Lemma 4.1 is formulated, the statements (i) and (ii) can be proven by examples having properties (i) and (ii). As said before, the no-information last-arrival problem is such an example. However, the following simple example suffices to make the point. We keep it in form of a an optimal control problem in order to concentrate on the essence, but by adding costs for observations we can change the example easily into a stopping problem.

**Proof**

(i) Let $(I^{(1)}_j)_{j=1,2,...}$ and $(I^{(2)}_j)_{j=1,2,...}$ be two sequences of Bernoulli random variables, not necessarily independent of each other, and let $p^{(1)}_j = P(I^{(1)}_j = 1)$ and $p^{(2)}_j = P(I^{(2)}_j = 1)$. At each time $j$ the decision-maker (he, say) sees both $p^{(1)}_j$ and $p^{(2)}_j$ and decides on which Bernoulli experiment he wants to bet (see Fig. 4). If he bets on Line(1) he receives the random reward $I^{(1)}_j$, and, alternatively, if he bets on Line(2), he receives the random reward $I^{(2)}_j$. At time $j$ he sees only the two entries $p^{(1)}_j$ and $p^{(2)}_j$ but none of the future values for $j'>j$.

\[ \begin{align*}
\text{Line (1)} & : \quad p^{(1)}_1 \quad p^{(1)}_2 \quad p^{(1)}_3 \quad \ldots \quad p^{(1)}_j \quad \ldots \quad p^{(1)}_n \\
\text{Line (2)} & : \quad p^{(2)}_1 \quad p^{(2)}_2 \quad p^{(2)}_3 \quad \ldots \quad p^{(2)}_j \quad \ldots \quad p^{(2)}_n
\end{align*} \]

Fig. 4

Denoting by $\pi : \mathbb{N} \to \{\text{Line (1)}, \text{Line (2)}\}$ the decision policy at each step, his objective is to maximize for each $n$ the expected accumulated reward. The optimal strategy, if it exists, is defined by

$$
\pi^* = \arg \max_\pi \left\{ \mathbb{E} \left( \sum_{k=1}^n \pi(k) \right) \right\}.
$$

But this implies that it does exists: in order to play optimally, it suffices to bet at each step $j$ on $\max \{p^{(1)}_j, p^{(2)}_j\}$. Indeed, this strategy yields at each time $n$ the expected accumulated reward

$$
M(n) = \max \{p^{(1)}_1, p^{(2)}_1\} + \max \{p^{(1)}_1, p^{(2)}_2\} + \cdots + \max \{p^{(1)}_n, p^{(2)}_n\}
$$

upon which one cannot possibly improve because the maximum of a sum never exceeds the sum of the maxima. And thus we have

$$
M(n) = \sum_{k=1}^n \max \{p^{(1)}_k, p^{(2)}_k\} \geq \max_\pi \left( \sum_{k=1}^n \pi(k) \right).
$$

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(26) and (27) imply that the optimal strategy $S_n$ maximizing the accumulated reward until time $n$ exists, but nevertheless, before time $n$, no value can be attributed to the optimal $S_n$ because $M(n)$ is still unknown. This proves (i). (We note that if the corresponding values in Line (1) and Line (2) never coincide, all $S_n$ are moreover unique.)

(ii) To prove (ii), look at the following modification. Suppose that at some time $t \in \mathbb{N}$ a red light is switched on for Line (2), say, with probability $\delta$. If the light is switched on, the decision maker is supposed to be no longer entitled to bet on Line (2). No information is given how often, or how long, the red light may be switched on, given it is switched on at least once.

It is straightforward to check, similarly as above, that now the unique optimal strategy is to bet, whenever possible, on the Line with the entry $\max\{p_j^{(1)}, p_j^{(2)}\}$. If $\delta = 0$ then we are in the case (i). Further we see easily that, if

$$\ell = \lim_{n \to \infty} \sum_{j=1}^{n} |p_j^{(1)} - p_j^{(2)}| < \infty,$$

then, for any given $\epsilon > 0$, we can always choose $\delta$ sufficiently small so that the optimal strategy in this setting is $\epsilon$-optimal with respect to $S$. Indeed, for all $n$ the difference in the accumulated rewards is bounded above by $\delta \ell$. In this case, the optimal strategy can be seen as the limit of $\epsilon$-optimal strategies.

If the limit $\ell$ in (28) satisfies $\ell = \infty$, however, then this is not possible.

In conclusion, we simply do not know whether the existing optimal strategy can be seen as a limit of $\epsilon$-optimal strategies, at least not in this class of $\epsilon$-optimal strategies. This does of course not exclude that one may still be able to find other $\epsilon$-optimal strategies.

However, the point we want to make is that special circumstances in a given problem may naturally lead us to a certain class of $\epsilon$-optimal strategies with which we would like to study the problem. Then we should be able to count on some form of "closedness" as we know it from other domains of Mathematics. In Analysis for instance, we require for good reasons that a function $f : \mathbb{R}^n \to \mathbb{R}$ allows a limit in $x \in \mathbb{R}^n$ if and only if for all sequences $(x_m) \to x$ we have $f(x_m) \to f(x)$. As we have just seen, without knowing that $\ell$ defined in (24) satisfies $\ell < \infty$, we would not know whether all $\epsilon$-optimal strategies would do as $\epsilon \to 0+$.

4.1 Particularities of the no-information hypothesis

Lemma 4.1 tells us that we must (in contrast to the situation we found in the Unified Approach setting for the problem we solved) keep, in more general cases, something important in mind:
In a setting of an optimal stopping problem under no-information an optimal strategy need not have a "neighborhood" in the set of possible strategies in a more classical sense. An optimal expected payoff need not be a limit in an analytic sense of the corresponding expected payoffs.

But then, any argument based on a continuity assumption, or on the existence of a point of indifference for the optimal decision, etc., may become questionable. This implies that we may, in certain cases, be able to show the optimality of a certain strategy without being able to assess at the same time how a (slightly) suboptimal strategy, or in fact any other strategy, would compare to the optimal strategy with respect to performance. The modest content of what we say here is that we have to be careful when speaking about indifference values, limiting performances or any limit argument in the context of no-information.

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