TWISTED QUANTUM AFFINIZATIONS AND QUANTIZATION OF EXTENDED AFFINE LIE ALGEBRAS

FULIN CHEN\textsuperscript{1}, NAIHUAN JING\textsuperscript{2}, FEI KONG\textsuperscript{3}, AND SHAOBIN TAN\textsuperscript{4}

Abstract. In this paper, for an arbitrary Kac-Moody Lie algebra $\mathfrak{g}$ and a diagram automorphism $\mu$ of $\mathfrak{g}$ satisfying two linking conditions, we introduce and study a $\mu$-twisted quantum affinization algebra $U_\hbar(\hat{\mathfrak{g}}_\mu)$ of $\mathfrak{g}$. When $\mathfrak{g}$ is of finite type, $U_\hbar(\hat{\mathfrak{g}}_\mu)$ is Drinfeld’s current algebra realization of the twisted quantum affine algebra. And, when $\mu = \text{Id}$, $U_\hbar(\hat{\mathfrak{g}}_\mu)$ is the quantum affinization algebra introduced by Ginzburg-Kapranov-Vasserot. As the main results of this paper, we first prove a triangular decomposition of $U_\hbar(\hat{\mathfrak{g}}_\mu)$. Second, we give a simple characterization of the affine quantum Serre relations on restricted $U_\hbar(\hat{\mathfrak{g}}_\mu)$-modules in terms of “normal order products”. Third, we prove that the category of restricted $U_\hbar(\hat{\mathfrak{g}}_\mu)$-modules is a monoidal category and hence obtain a topological Hopf algebra structure on the “restricted completion” of $U_\hbar(\hat{\mathfrak{g}}_\mu)$. Fourth, we study the classical limit of $U_\hbar(\hat{\mathfrak{g}}_\mu)$ and abridge it to the quantization theory of extended affine Lie algebras. In particular, based on a classification result of Allison-Berman-Pianzola, we obtain the $\hbar$-deformation of nullity 2 extended affine Lie algebras.

1. Introduction

In this paper, we denote by $U_\hbar(\mathfrak{g})$ the quantum enveloping algebra associated with a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ over the ring $\mathbb{C}[[\hbar]]$ of complex formal series in one variable $\hbar$.

1.1. Twisted quantum affinizations. The quantum affine algebras $U_\hbar(\hat{\mathfrak{g}})$ ($\mathfrak{g}$ of finite type and $\hat{\mathfrak{g}}$ the affine Lie algebra of $\mathfrak{g}$) are one of the most important subclasses of quantum enveloping algebras introduced independently by Drinfeld \cite{Drinfeld} and Jimbo \cite{Jimbo}. Drinfeld’s current algebra realization for $U_\hbar(\hat{\mathfrak{g}})$ \cite{Drinfeld, Jimbo} is of fundamental importance in this area as it provides an explicit construction of $U_\hbar(\hat{\mathfrak{g}})$ as quantum affinization of $U_\hbar(\mathfrak{g})$. It is remarkable that the Drinfeld quantum affinization process can be extended to any symmetrizable quantum Kac-Moody algebras $U_\hbar(\mathfrak{g})$. One obtains in this way a new class of $\mathbb{C}[[\hbar]]$-algebras $U_\hbar(\hat{\mathfrak{g}})$, called the quantum affinization algebras. The algebras $U_\hbar(\hat{\mathfrak{g}})$ ($\mathfrak{g}$ not of finite type) were first introduced by Ginzburg-Kapranov-Vasserot \cite{Ginzburg-Kapranov-Vasserot} for the untwisted affine case, and then in \cite{Ginzburg-Kapranov-Vasserot, Vasserot} for the general case. (In the case that \(\mathfrak{g}\) is symmetric but not simply-laced, the quantum affinization algebra $U_\hbar(\hat{\mathfrak{g}})$ defined in \cite{Ginzburg-Kapranov-Vasserot, Vasserot} is slightly different from that in \cite{Vasserot}, and in this paper we shall exploit the latter definition). When $\mathfrak{g}$ is of (untwisted) affine type, $U_\hbar(\hat{\mathfrak{g}})$ is often referred as the quantum toroidal algebra. The

2010 Mathematics Subject Classification. 17B37, 17B67.

Key words and phrases. twisted quantum affinization, extended affine Lie algebra, quantum Kac-Moody algebra, triangular decomposition, Hopf algebra.

\textsuperscript{1}Partially supported by the Fundamental Research Funds for the Central Universities (No.20720190069) and NSF of China (No.11971397).

\textsuperscript{2}Partially supported by NSF of China (No.11531004, No.11726016) and Simons Foundation (No.529868).

\textsuperscript{3}Partially supported by NSF of China (No.11701183).

\textsuperscript{4}Partially supported by NSF of China (Nos.11531004, 11971397).
case $g = \hat{sl}_{\ell + 1}$ is particularly interesting: an additional parameter $p$ can be added in the quantum affinization process [34] so one gets a two parameter quantum toroidal algebra $U_{h,p}(\hat{g})$. One notices that such a two-parameter quantum toroidal algebra is yet to be defined for the other affine types.

The theory of general quantum affinizations has been intensively studied. In particular, the representation theory of quantum toroidal algebras is very rich: the Schur-Weyl duality with elliptic Cherednik algebras [69]; the action on $q$-Fock spaces [64, 65]; the vertex representation constructions [63, 63]; the connection with McKay correspondence [31]; the quantum fusion tensor category constructions [37, 38]; the geometric representation constructions [60] and some other applications [57]-[59], [26]-[30]. For a survey on the representation theory of quantum toroidal algebras, one may consult [40]. One of the main features of the quantum toroidal algebra is that it contains two remarkable subalgebras both isomorphic to a quotient of untwisted quantum affine algebras (one comes from the Drinfeld-Jimbo’s presentation and another from the Drinfeld’s new presentation).

When $g$ is of finite $ADE$ type, Drinfeld [21] also formulated a twisted quantum affinization of $U_{h}(g)$ associated with diagram automorphisms $\mu$ of $g$. It was announced in [21] and proved in [17, 18] (see also [45, 46]) that Drinfeld’s twisted quantum affinization algebras gave a new realization of twisted quantum affine algebras $U_{h}(\hat{g}_{\mu})$ ($\hat{g}_{\mu}$ the $\mu$-twisted affine Lie algebra of $g$). Similar to the untwisted case, Drinfeld’s new realization theorem is of fundamental importance in the theory of twisted quantum affine algebras [16, 13, 39]. It has been a question whether the twisted quantum affinization process introduced by Drinfeld can be generalized to diagram automorphisms of any symmetrizable Kac-Moody Lie algebra in a similar way to that of the untwisted case? In [9], by using vertex operator calculations, we generalized the twisted quantum affinization to a class of diagram automorphisms on simply-laced Kac-Moody Lie algebras. One notices that in general there exist nontrivial diagram automorphisms on non-simply-laced Kac-Moody Lie algebras that are not from finite types. For example, when $g$ is of affine type $B_{\ell}^{(1)}$ or $C_{\ell}^{(1)}$, there exists such an order 2 diagram automorphism on $g$. Currently, the question is still to be answered for such Kac-Moody Lie algebras and it except that this could give rise to new type of quantized algebras.

In this paper, we give a general affirmative answer to this question by defining a twisted quantum affinization algebra $U_{h}(\hat{g}_{\mu})$ for any symmetrizable Kac-Moody algebra $g$ and any diagram automorphism $\mu$ of $g$ with two natural linking conditions (LC1) and (LC2). The condition (LC1) was first recognized in [32] so that the $\mu$-folded Cartan matrix of $g$ is still a generalized Cartan matrix and the condition (LC2) appears naturally for establishing the Drinfeld-type affine quantum Serre relations. One notices that, except the transitive diagram automorphisms on $\hat{sl}_{\ell + 1}$, all the diagram automorphisms of finite and affine Kac-Moody Lie algebras satisfy these two linking conditions. When $g$ is of finite type, $U_{h}(\hat{g}_{\mu})$ is just the Drinfeld’s current algebra realization for twisted quantum affine algebras. When $\mu = Id$, it coincides with the quantum affinization algebra $U_{h}(\hat{g})$. When $g$ is of simply-laced type, $U_{h}(\hat{g}_{\mu})$ has been realized in [9] by vertex operators. Especially, in the case when $g$ is of untwisted affine type $X_{\ell}^{(1)}$ and $\mu$ fixes an additional node of $g$ [17], $U_{h}(\hat{g}_{\mu})$ contains two subalgebras which are both isomorphic to a quotient of the twisted quantum affine algebra of type $X_{\ell}^{(N)}$, where $N$ is the order of $\mu$.

The theory of quantum Kac-Moody algebras has been a tremendously successful story. Certainly, the most important aspect of the structures of $U_{h}(g)$ is that it is a Hopf algebra over $\mathbb{C}[[h]]$, which specializes to the universal enveloping algebra $\mathcal{U}(g)$ of $g$ and admits a
canonical triangular decomposition. A next natural question is whether these fundamental algebraic properties of quantum Kac-Moody algebras (triangular decomposition, Hopf algebra structure, the specialization and etc) can be extended to general twisted quantum affinizations?

In Section 4, we prove a triangular decomposition for $U_{\hbar}(\hat{g}_{\mu})$. When $\mu = \text{Id}$, this decomposition was obtained in [9] for the finite type and in [37] for the general case. When $g$ is of finite type, this decomposition was given in [18]. In analogy to the untwisted case [37, 38], the triangular decomposition of $U_{\hbar}(\hat{g}_{\mu})$ is essential in the studying of representation theory of $U_{\hbar}(\hat{g}_{\mu})$. For example, one can define a notion of $l$-highest weight module for $U_{\hbar}(\hat{g}_{\mu})$ and then study the properties of (simple) integrable $l$-highest weight modules.

Unlike quantum Kac-Moody algebras, no Hopf algebra structure is known for general quantum affinizations. According to an unpublished note of Drinfeld, certain completion of the quantum affine algebra has a Hopf algebra structure and this (infinite) coproduct is conjugate to the usual Drinfeld-Jimbo coproduct under a twist via the universal $R$-matrix. For the untwisted quantum affine algebras, a proof was given in [19] (the simply-laced case) and [23] (the other cases). One notices that Drinfeld’s new coproduct involves infinite sums and so cannot be defined directly on $U_{\hbar}(\hat{g}_{\mu})$. However, it makes sense on the so-called restricted (topologically free) $U_{\hbar}(\hat{g}_{\mu})$-modules. When $g$ is of finite type, the notion of restricted $U_{\hbar}(\hat{g}_{\mu})$-modules coincides with the usual one [35] and it is known that all irreducible highest weight $U_{\hbar}(\hat{g}_{\mu})$-modules are restricted [34, 23]. When $g$ is of simply-laced type, the vertex representations for $U_{\hbar}(\hat{g}_{\mu})$ constructed in [9] are also restricted. In Section 6, we prove that for any restricted $U_{\hbar}(\hat{g}_{\mu})$-modules $U$ and $V$, Drinfeld’s new coproduct affords a (restricted) $U_{\hbar}(\hat{g}_{\mu})$-module structure on the $b$-adically completed tensor product space $U \hat{\otimes} V$. In particular, this implies that the category $\mathcal{R}$ of restricted $U_{\hbar}(\hat{g}_{\mu})$-modules is a monoidal category. Let $F$ be the forgetful functor from $\mathcal{R}$ to the category of topologically free $\mathbb{C}[[h]]$-modules. Then the closure $\hat{U}_{\hbar}(\hat{g}_{\mu})$ of $U_{\hbar}(\hat{g}_{\mu})$ in the algebra of endomorphisms of $F$, called the restricted completion of $U_{\hbar}(\hat{g}_{\mu})$, is naturally a topological Hopf algebra. As usual, the key step is to check the compatibility between Drinfeld’s new coproduct and affine quantum Serre relations. For this purpose, we present a simple characterization of affine quantum Serre relations on restricted $U_{\hbar}(\hat{g}_{\mu})$-modules in terms of “normal order products”, which enables us to overcome this difficulty.

More precisely, let $A = (a_{ij})_{i,j \in I}$ be the generalized Cartan matrix associated to $g$ and let $x_{i}^{\pm}(z)$, $i \in I$ be the defining currents of $U_{\hbar}(\hat{g}_{\mu})$. Then the untwisted affine quantum Serre relations in $U_{\hbar}(\hat{g})$ take the form $(i, j \in I \text{ with } a_{ij} < 0)$:

$$\sum_{\sigma \in S_{m}} \sum_{r=0}^{m} (-1)^{r} \sum_{q_{i}} x_{i}^{\pm}(z_{\sigma(1)}) \cdots x_{i}^{\pm}(z_{\sigma(r)}) x_{j}^{\pm}(w) x_{i}^{\pm}(z_{\sigma(r+1)}) \cdots x_{i}^{\pm}(z_{\sigma(m)}) = 0,$$

where $m = 1 - a_{ij}$ and $q_{i}$ an invertible element in $\mathbb{C}[[h]]$. Comparing with the untwisted case, one of the main features in $U_{\hbar}(\hat{g}_{\mu})$ is that the affine quantum Serre relations contain certain (Drinfeld) polynomials (see Definition 3.2):

$$\sum_{\sigma \in S_{m}} \left\{ \prod_{1 \leq a < b \leq m} p_{ij}^{\pm}(z_{\sigma(a)}, z_{\sigma(b)}) \left( \sum_{r=0}^{m} (-1)^{r} \sum_{q_{i}} x_{j}^{\pm}(w) x_{i}^{\pm}(z_{\sigma(r+1)}) \cdots x_{i}^{\pm}(z_{\sigma(m)}) \right) \right\} = 0,$$

where $p_{ij}^{\pm}(z_{a}, z_{b})$ are some polynomials and $d_{ij}$ a positive integer. This distinguishes the theory of twisted quantum affinizations (in particular, the theory of twisted quantum
affine algebras) quite different from that of the untwisted one. In Section 5, we introduce a notion of normal order product $a_i^0 \circ a_j^0$ for the currents $x_i^\pm(z)$ on restricted $\mathcal{U}_\hbar(\hat{g}_\mu)$-modules $W$. When $W$ is the vertex representation constructed in [9] and so $x_i^\pm(z)$ are realized as certain vertex operators, $x_i^\pm(z) x_j^\pm(z) \circ \cdots \circ x_j^\pm(z) \circ x_i^\pm(z) \circ 0 = 0$.

In fact a more general and stronger version (see Theorem 5.13) will be given, and Theorem 5.14 also prove that for any finite or affine type Lie algebra $\mathfrak{g}$, the following relations hold on restricted $\mathcal{U}_\hbar(\hat{g}_\mu)$-modules:

$$\sum_{\sigma \in S_m} \sum_{r=0}^m (-1)^r \left( \sum_{i=1}^m \right) x_i^\pm(z_{\sigma(1)}) \cdots x_i^\pm(z_{\sigma(r)}) x_j^\pm(w) x_i^\pm(z_{\sigma(r+1)}) \cdots x_i^\pm(z_{\sigma(m)}) = 0,$$

where $m = 1 - \tilde{a}_{ij}$, $\tilde{q}_i \in \mathbb{C}[\hbar]$ and $(\tilde{a}_{ij})$ is the $\mu$-folded matrix of $A$. When $\mathfrak{g}$ is of finite type, the above relations are proved in [17], which plays a key role in understanding the isomorphism between Drinfeld-Jimbo and Drinfeld’s realizations for twisted quantum affine algebras.

When $\mathfrak{g}$ is of finite type, by taking classical limit, there is a natural vertex algebraic interpretation of the relations (1.1): let $W$ be a restricted module for the twisted affine Lie algebra $\tilde{\mathcal{g}}_\mu$ and let $V_\hbar$ be the universal affine vertex algebra associated to $\mathfrak{g}$. It is known [51] that the currents $x_i^\pm(z)$ on $W$ generate a vertex algebra, say $V_W$, in the space $\text{Hom}(W, W((z)))$ with $W$ as a quasi module. Furthermore, there is a surjective homomorphism from $V_\hbar$ to $V_W$ sending $x_i^\pm \mapsto x_i^\pm(z)$, where the Chevalley generators $x_i^\pm$ of $\mathfrak{g}$ are viewed as elements of $V_\hbar$ in the usual way. One can check directly that $(x_i^\pm(z))_0 \cdots (x_i^\pm(z))_0 x_j^\pm(z) = d_{ij}^{a_{ij} - 1} z^{(1 - a_{ij})(1 - d_{ij})} x_i^\pm(z) \cdots x_i^\pm(z) x_j^\pm(z) \circ 0$ in $V_W$ and hence

$$(\text{ad} x_i^\pm)^{1 - a_{ij}} x_j^\pm = ((x_i^\pm)_0)^{1 - a_{ij}} x_j^\pm \mapsto d_{ij}^{a_{ij} - 1} z^{(1 - a_{ij})(1 - d_{ij})} x_i^\pm(z) \cdots x_i^\pm(z) x_j^\pm(z) \circ 0.$$
and an invariant form \(|\cdot|\), that satisfies a list of conditions. The form \(|\cdot|\) induces a semipositive form on the \(\mathbb{R}\)-span of the root system \(\Phi\) of \(\mathcal{E}\) (relative to \(\mathcal{H}\)) and so \(\Phi\) divides into a disjoint union of the sets of isotropic and nonisotropic roots. Roughly speaking, the structure of \(\mathcal{E}\) is determined by its core \(\mathcal{E}_c\), the subalgebra generated by nonisotropic root vectors. Following [63], we say that an EALA \(\mathcal{E}\) is maximal if \(\mathcal{E}_c\) is centrally closed. Set \(\hat{\mathcal{E}} = \mathcal{E}_c + \mathcal{H}\), which we call the extended core of \(\mathcal{E}\). One of the conditions for \(\mathcal{E}\) requires that the group generated by isotropic roots is of finite rank and this rank is called its nullity. Nullity 0 EALAs are exactly finite dimensional simple Lie algebras, while nullity 1 EALAs are precisely affine Kac-Moody algebras [2]. Meanwhile, nullity 2 EALAs are of particular interest: they are closely related to the Lie algebras studied by Saito and Slowdoy in the work of elliptic singularities [63]. As an important achievement in the theory of EALAs, the centerless cores of nullity 2 EALAs have been completely classified by Allison-Berman-Pianzola in [3] (see also [35]).

We remark that quantum finite, affine and (one or two parameters) toroidal algebras can be related to EALAs in a uniform way. More precisely, let \(\mathcal{U}_h\) be an arbitrary quantum finite (resp. affine; resp. toroidal) algebra. Then there is an EALA \(\hat{\mathcal{E}}\) of nullity 0 (resp. 1; resp. 2) such that the classical limit of \(\mathcal{U}_h\) is isomorphic to the universal enveloping algebra of \(\hat{\mathcal{E}}\). Furthermore, except the quantum toroidal algebra of type \(A^{(1)}_1\), \(\mathcal{E}\) are always maximal. (Note that \(A^{(1)}_1\) is the unique symmetric but not simply-laced affine GCM and so in this case the quantum toroidal algebra defined in [41, 60] is slightly different from that in [37]. In the former case, the corresponding EALA is also maximal.) Therefore, an eminent problem in the theory of EALAs is to establish and explore natural connections of quantum algebras with all extended cores of (maximal) EALAs. The theory of quantum toroidal algebras suggests that the nullity 2 case is of particular importance.

As an application of our general twisted quantum affinization theory, we give an answer to this problem for the nullity 2 case. More precisely, denote by \(\hat{E}_2\) the class of Lie algebras which are isomorphic to the extended core of a maximal EALA with nullity 2. Let \(\mu\) be a diagram automorphism of an affine Kac-Moody algebra \(g\) and let \(\hat{t}(g, \mu)\) be the twisted toroidal Lie algebra associated to \((g, \mu)\), that is, the universal central extension of the \(\mu\)-twisted loop algebra of \([g, g]\). By adding two canonical derivations to \(\hat{t}(g, \mu)\), one obtains a semi-product Lie algebra \(\hat{t}(g, \mu)\). Then we have \(\hat{t}(g, \mu) \in \hat{E}_2\) if and only if \(\mu\) satisfies the linking conditions (LC1) and (LC2) [3]. On the other hand, let \(\mathbb{C}_p\) be the quantum 2-torus associated to a nonzero complex number \(p\) and let \(\hat{sl}_{\ell+1}(\mathbb{C}_p)\) be the universal central extension of the special linear Lie algebra over \(\mathbb{C}_p\) [8]. Again by adding two derivations to \(\hat{sl}_{\ell+1}(\mathbb{C}_p)\), we have a Lie algebra \(\hat{sl}_{\ell+1}(\mathbb{C}_p) \in \hat{E}_2\). According to a result of Allison-Berman-Pianzola [3], any algebra in \(\hat{E}_2\) is either isomorphic to \(\hat{t}(g, \mu)\) with \(\mu\) nontransitive, or isomorphic to \(\hat{sl}_{\ell+1}(\mathbb{C}_p)\) with \(p\) generic. It was known that the quantization of \(\hat{sl}_{\ell+1}(\mathbb{C}_p)\) is \(\mathcal{U}_{h,p}(\hat{sl}_{\ell+1})\) [67], while for the case \(g = \hat{sl}_2\), we define in [12] a new quantum toroidal algebra \(\mathcal{U}_{h}^{new}\) whose classical limit is \(\mathcal{U}(\hat{sl}_2, \text{Id})\). Just like the algebra \(\mathcal{U}_{h}(\hat{sl}_2)\), \(\mathcal{U}_{h}^{new}\) has a canonical triangular decomposition and a Hopf algebra structure [12]. Thus, it remains to treat the algebras \(\hat{t}(g, \mu)\) with \(g\) not of type \(A^{(1)}_1\) and \(\mu\) nontransitive. In this case, by using the Drinfeld type presentations of \(\hat{t}(g, \mu)\) established in [11], we obtain that the classical limit of the twisted quantum affinization algebra \(\mathcal{U}_{h}(\hat{g}_\mu)\) is isomorphic to \(\mathcal{U}(\hat{t}(g, \mu))\).

The root system of EALAs have been axiomatized under the name of extended affine root systems (EARSs for short) and characterized in [1]. As in the classical Lie theory, for any EARS \(\Phi\) of nullity 2, there exist algebras in \(\hat{E}_2\) with \(\Phi\) as its root system [3].
Such an algebra is unique (up to isomorphism) except \( \Phi \) is of type \( A_1^{(1,1)} \), in which case there are infinitely many nonisomorphic algebras in \( \hat{E}_2 \) that are parameterized by generic numbers (i.e., the algebras \( \mathfrak{sl}_{k+1}(\mathbb{C}_0) \)). By taking classical limit, we think this gives a natural explanation why two-parameter quantum toroidal algebras should only exist in type \( A \) case. On the other hand, this shows that it is reasonable to view quantum toroidal algebras as \( h \)-deformation of algebras in \( \hat{E}_2 \).

The layout of the paper is set as follows. In Section 2 we introduce a class of diagram automorphisms on any symmetrizable Kac-Moody algebra \( \mathfrak{g} \) which satisfies two linking conditions. Starting with such a diagram automorphism \( \mu \), in Section 3 we define a \( \mu \)-twisted quantum affinization algebra \( \mathcal{U}_h(\hat{\mathfrak{g}}_\mu) \) of \( \mathcal{U}_h(\mathfrak{g}) \), and in Section 4 we prove a triangular decomposition of \( \mathcal{U}_h(\hat{\mathfrak{g}}_\mu) \). In Section 5 we give a simple characterization of the affine quantum Serre relations on restricted \( \mathcal{U}_h(\hat{\mathfrak{g}}_\mu) \)-modules. As an application, in Section 6 we prove that there is a topological Hopf algebra structure on the restrict completion of \( \mathcal{U}_h(\hat{\mathfrak{g}}_\mu) \). In Section 7 we study the classical limit of \( \mathcal{U}_h(\hat{\mathfrak{g}}_\mu) \) and also link the algebra \( \mathcal{U}_h(\hat{\mathfrak{g}}_\mu) \) with the quantization theory of nullity 2 EALAs. Finally, Section 8 is devoted to a proof of Theorem 5.13 on general affine quantum Serre relations.

Throughout this paper, we denote the group of non-zero complex numbers, the set of non-zero integers, the set of positive integers and the set of non-negative integers by \( \mathbb{C}^\times \), \( \mathbb{Z}^\times \), \( \mathbb{Z}_+ \) and \( \mathbb{N} \), respectively. For any \( m \in \mathbb{Z}_+ \), we set \( \xi_m = e^{2\pi \sqrt{-1}/m} \) and \( \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} \).

2. Automorphisms of generalized Cartan matrices

In this section, we introduce a class of automorphisms on generalized Cartan matrices (GCMs) which satisfy two linking conditions.

2.1. Automorphisms of generalized Cartan matrices. Here we give some general backgrounds about automorphisms on GCMs. One may see [47, 50] for details.

Throughout this paper, let \( I \) be a finite subset of \( \mathbb{Z} \) and let \( A = (a_{ij})_{i,j \in I} \) be a symmetric GCM. Namely, \( A \) is a square matrix such that

\[
a_{ij} \in \mathbb{Z}, \quad a_{ii} = 2, \quad i \neq j \Rightarrow a_{ij} < 0 \quad \text{and} \quad a_{ij} = 0 \iff a_{ji} = 0
\]

for \( i, j \in I \), and that there exists an invertible diagonal matrix

\[
(2.1) \quad D = \text{diag}\{r_i\}_{i \in I} \quad (r_i \in \mathbb{Z}_+)
\]

such that \( DA \) is symmetric. Let \( (\mathfrak{h}, \Pi, \Pi^\vee) \) be a realization of the GCM \( A \) ([47]). Explicitly, \( \mathfrak{h} \) is a \((|I| - \ell)\)-dimensional \( \mathbb{C} \)-space, \( \Pi = \{\alpha_i\}_{i \in I} \) is a set of linearly independent elements in the dual space \( \mathfrak{h}^* \) of \( \mathfrak{h} \), \( \Pi^\vee = \{\alpha_i^\vee\}_{i \in I} \) is a set of linearly independent elements in \( \mathfrak{h} \) and \( \alpha_j(\alpha_i^\vee) = a_{ij} \) for \( i, j \in I \), where \( \ell \) is the rank of \( A \). Denote by \( \mathfrak{g} = \mathfrak{g}(A) \) the Kac-Moody Lie algebra associated with the quadruple \((A, \mathfrak{h}, \Pi, \Pi^\vee)\), and set \( \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \). By definition, \( \mathfrak{g} \) is the Lie algebra generated by the elements \( h \in \mathfrak{h}, e_i^\pm, i \in I \), and subject to the relations

\[
[h, h'] = 0, \quad [e_i^+, e_i^-] = \alpha_i^\vee, \quad [h, e_i^\pm] = \pm \alpha_i(h)e_i^\pm, \quad \text{ad}(e_i^\pm)^{1-a_{ij}}(e_j^\pm) = 0,
\]

where \( h, h' \in \mathfrak{h} \) and \( i, j \in I \) with \( i \neq j \).

Let \( \text{Aut}(A) \) be the group of automorphisms on the GCM \( A \). That is, it is the group of permutations \( \mu \) on the index set \( I \) such that \( a_{ij} = a_{\mu(i)\mu(j)} \) for \( i, j \in I \). Note that \( \text{Aut}(A) \) acts naturally on the root lattice \( Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \) of \( \mathfrak{g} \) so that \( \mu(\alpha_i) = \alpha_{\mu(i)} \) for \( \mu \in \text{Aut}(A) \) and \( i \in I \). Set

\[
\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}' = \bigoplus_{i \in I} \mathbb{C}\alpha_i^\vee \quad \text{and} \quad \mathfrak{c} = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0, i \in I\} \subset \mathfrak{h}'.
\]
Then we may view $\mathfrak{h}/\mathfrak{c}$ as the dual space of $\mathbb{C} \otimes \mathbb{Z} Q$. Hence, by duality, there is an $\text{Aut}(A)$-action on it. Moreover, as $\text{Aut}(A)$ is a finite group, there exists a subspace $\mathfrak{h}''$ of $\mathfrak{h}$ such that \((\mathfrak{h}'' + \mathfrak{c}) / \mathfrak{c} \) is $\text{Aut}(A)$-stable.

From now on let us fix such a choice of the complementary space $\mathfrak{h}''$ (which is not unique in general). Then there is a unique action of $\text{Aut}(A)$ on $\mathfrak{h}$ such that $\mu(\alpha_i^\vee) = \alpha_{\mu(i)}^\vee$, $\mu(\mathfrak{h}'') = \mathfrak{h}''$, and $\alpha_{\mu(i)}(\mu(\mathfrak{h}'')) = \alpha_i(h'')$ for $\mu \in \text{Aut}(A)$, $i \in I$ and $h'' \in \mathfrak{h}''$. Furthermore, the action of $\text{Aut}(A)$ on $\mathfrak{h}$ extends uniquely to an action of $\text{Aut}(A)$ on $\mathfrak{g}$ by automorphisms such that $\mu(e_i^\pm) = e_{\mu(i)}^\pm$ for $\mu \in \text{Aut}(A)$ and $i \in I$.

As in [50], we regard the elements of $\text{Aut}(A)$ as automorphisms on $\mathfrak{g}$ in this way, and call them diagram automorphisms. It is straightforward to see that the complementary space $\mathfrak{h}''$ for $\mathfrak{h}'$ in $\mathfrak{h}$ induces a non-degenerate symmetric bilinear form $\langle \cdot | \cdot \rangle$ on $\mathfrak{h}$ such that $r_i(\alpha_i^\vee | h) = \alpha_i(h)$ and $\langle h'' | h'' \rangle = 0$ for $i \in I$ and $h \in \mathfrak{h}$.

Moreover, the bilinear form $\langle \cdot | \cdot \rangle$ on $\mathfrak{h}$ induces a unique invariant non-degenerate symmetric bilinear form, still denoted as $\langle \cdot | \cdot \rangle$, on $\mathfrak{g}$. It is a simple fact [50] that the diagram automorphisms of $\mathfrak{g}$ preserve the bilinear form $\langle \cdot | \cdot \rangle$.

### 2.2. Linking conditions.

In this subsection, we introduce a class of automorphisms on the GCM $A$ which satisfy two linking conditions.

Given a $\mu \in \text{Aut}(A)$ with order $N$. For each $i \in I$, we write $O(i) = \{\mu^k(i) \mid k = 0, \ldots, N - 1\} \subset I$ for the orbit containing $i$, and $N_i$ the cardinality of $O(i)$. Following [32], we define the first linking condition on $\mu$:

\[ \sum_{p \in O(i)} a_{pi} > 0 \quad \text{for all } i \in I. \]

The condition (LC1) can be reformulated as follows:

**Lemma 2.1.** [32, Sect. 2.2] The automorphism $\mu$ on $A$ satisfies the condition (LC1) if and only if for every $i \in I$, the Dynkin subdiagram of $O(i)$ is either

(i) a direct sum of type $A_1$, or

(ii) a direct sum of type $A_2$ with $a_{\mu^{N_i/2} i, i} = -1$.

For $i, j \in I$, set

\[ \tilde{a}_{ij} = s_i \sum_{p \in O(i)} a_{pj} \in s_i \mathbb{Z}, \]

where

\[ s_i = \begin{cases} 1, & \text{if (i) holds in Lemma 2.1} \\ 2, & \text{if (ii) holds in Lemma 2.1} \end{cases} \]

For convenience, we fix a set of representatives for the orbits of $\mu$:

\[ \hat{I} = \{i \in I \mid \mu^k(i) \geq i \text{ for } k = 0, \ldots, N - 1\}. \]

It was known [32, Sect. 2.2] that the folded matrix

\[ \hat{A} = (\tilde{a}_{ij})_{i,j \in \hat{I}} \]
of $A$ associated with $\mu$ is a symmetrizable GCM. Moreover, $\tilde{A}$ is of finite (resp. affine; resp. indefinite) type if and only if $A$ is of finite (resp. affine; resp. indefinite) type.

Denote by $\mathfrak{h}$ the subspace of $\mathfrak{h}$ fixed by the isometry $\mu$. For $i \in I$, set

$$\tilde{\alpha}_i = \frac{1}{N_i} \sum_{p \in O(i)} \alpha_p \quad \text{and} \quad \tilde{\alpha}_i^\vee = s_i \sum_{p \in O(i)} \alpha_p^\vee.$$  

Then we have $\mathfrak{h} = \tilde{\mathfrak{h}} \oplus (1 - \mu)\mathfrak{h}$, $\tilde{\alpha}_i|_{\tilde{\mathfrak{h}}} = \alpha_i|_{\tilde{\mathfrak{h}}}$ and $\tilde{\alpha}_i|_{(1 - \mu)\mathfrak{h}} = 0$ for $i \in I$. This gives that the $\tilde{\alpha}_i$’s can be canonically identified with $\alpha_i|_{\tilde{\mathfrak{h}}}$. Thus, the triple

$$(\tilde{\mathfrak{h}}, \tilde{\Pi} = \{\tilde{\alpha}_i\}_{i \in I}, \tilde{\Pi}^\vee = \{\tilde{\alpha}_i^\vee\}_{i \in I})$$

is a realization of the folded GCM $\tilde{A}$. Moreover, set

$$\tilde{r}_i = \frac{N}{s_i N_i} r_i, \quad i \in I \quad \text{and} \quad \tilde{D} = \text{diag}\{\tilde{r}_i\}_{i \in I}.$$  

Then the matrix $\tilde{D} \tilde{A}$ is symmetric. Denote by $\mathfrak{g} = \mathfrak{g}(\tilde{A})$ the Kac-Moody Lie algebra associated with the quadruple $(A, \tilde{\mathfrak{h}}, \tilde{\Pi}, \tilde{\Pi}^\vee)$, which is called the orbit Lie algebra of $g$ associated with $\mu$ (2.2).

Note that the condition (LC1) on $\mu$ is used to control the edges joining the vertices in a same orbit so that the correspond folded matrix is again a GCM. In what follows, we introduce another linking condition on $\mu$ which controls the edges joining the vertices in different orbits. Set

$$\mathbb{I} = \{(i, j) \in I \times I \mid i \notin O(j) \text{ and } a_{ij} < 0\},$$

and for $i, j \in I$, set

$$\Gamma_{ij} = \{k \in \mathbb{Z}_N \mid a_{ijk(j)} \neq 0\} \quad \text{and} \quad \Gamma^*_{ij} = \{k \in \mathbb{Z}_N \mid a_{ijk(j)} = a_{ij}\}.$$  

The second linking condition on $\mu$ assumed in this paper is as follows:

(LC2) for any $(i, j) \in \mathbb{I}$, $\Gamma_{ij}$ is a subgroup of $\mathbb{Z}_N$ and coincides with $\Gamma^*_{ij}$.

We say that an automorphism of $A$ is transitive if it acts transitively on the set $I$. Note that when $A$ is of affine type, an automorphism of $A$ is transitive if and only if $A$ is of type $A_\ell^{(1)}$ and it has order $\ell + 1$ by rotating the Dynkin diagram. Here and henceforth, when $A$ is of finite type or affine type, we will label $A$ (or $\mathfrak{g}$) using the Tables Fin and Aff 1-3 of [47, Chap 4]. We have (cf. [32]):

**Lemma 2.2.** Assume that the GCM $A$ is of finite type or affine type. Then an automorphism $\mu$ of $A$ does not satisfy the condition (LC1) if and only if $A$ is of affine type $A_\ell^{(1)}$ and $\mu$ is transitive. Moreover, all the automorphisms of $A$ satisfy the condition (LC2).

**Proof.** The lemma is proved by checking the claim for each possible $A$ and each diagram automorphism $\mu$ on $A$. For a list of automorphisms on affine GCMs, see [3, Tables 2, 3] for example.  

## 3. Twisted quantum affinizations

In the rest of the paper, we will always assume that $\mu$ is an automorphism of the symmetrizable GCM $A$, which has order $N$ and satisfies the linking conditions (LC1) and (LC2). In this section, we introduce a notion of $\mu$-twisted quantum affinization algebras.
3.1. Twisted quantum affinizations. We start with some conventions. In this paper, by a $\mathbb{C}[[\hbar]]$-algebra, we mean a topological algebra over $\mathbb{C}[[\hbar]]$, equipped with its canonical $\hbar$-adic topology. For two $\mathbb{C}[[\hbar]]$-modules $V$ and $W$, we denote by $V \hat{\otimes} W$ the $\hbar$-adically completed tensor product of $V$ and $W$. For any invertible element $v$ in $\mathbb{C}[[\hbar]]$ and $n, k, s \in \mathbb{Z}$ with $0 \leq k \leq s$, we define the usual quantum numbers as follows

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [s]_v! = [s]_v[s-1]_v \cdots [1]_v, \quad \text{and} \quad \left(\begin{array}{c}s \\ k\end{array}\right)_v = \frac{[s]_v!}{[s-k]_v!(k)_v!}.$$ 

Throughout this paper, we set (see (2.1) and (2.7))

$$(3.1) \quad q = e^\hbar, \quad q_i = q^{r_i} = e^{r_i \hbar}, \quad \text{and} \quad \check{q}_i = q^{\check{r}_i} = q_i^{-1} \quad \text{for} \quad i \in I.$$

The following notion was introduced independently by Drinfeld and Jimbo (cf. [48]).

**Definition 3.1.** The quantum Kac-Moody algebra $U_q(\mathfrak{g})$ is the $\mathbb{C}[[\hbar]]$-algebra topologically generated by the elements $\hbar \in \mathfrak{h}, e_i^\pm, i \in I$, and subject to the relations $(h, h' \in \mathfrak{h}, i, j \in I)$

$$(3.2) \quad [h, h'] = 0, \quad [h, e_i^+] = \pm \alpha_i(h) e_i^+, \quad [e_i^+, e_j^-] = \delta_{ij} \left( q_i^{\alpha_j} - q_i^{-\alpha_j} \right),$$

$$(3.3) \quad \sum_{r=0}^{1-\alpha_{ij}} (-1)^r \left( \frac{1 - \alpha_{ij}}{r} \right) q_i^{(e_i^+) r} (e_i^+) \left( e_i^- \right)^{-1-\alpha_{ij}-r} = 0, \quad \text{if} \quad i \neq j.$$

For $i, j \in I$, set

$$(3.4) \quad d_{ij} = \text{Card } \Gamma_{ij} \quad \text{and} \quad d_i = N/N_i.$$

One can check that both $d_i$ and $d_{ij}$ divide $d_{ij}$ for $i, j \in I$, and that

$$(3.5) \quad \tilde{a}_{ij} = \frac{s_{ij} d_{ij}}{d_i} a_{ij} \quad \text{for} \quad (i, j) \in I.$$

We introduce the $(\mathbb{C}[[\hbar]]$-valued) polynomials:

$$(3.6) \quad F_{ij}^\pm(z, w) = \prod_{k \in \Gamma_{ij}} \left( z - \xi^{k} q_i^{\pm a_{w(i)} z} w \right),$$

$$(3.7) \quad G_{ij}^\pm(z, w) = \prod_{k \in \Gamma_{ij}} \left( q_i^{a_{w(i)} z} - \xi^{k} w \right),$$

$$(3.8) \quad p_{ij}^\pm(z_1, z_2, z_3) = q_i^{\pm a_{w(i)} z_1} - (q_i^{a_{w(i)} z_2} + q_i^{-a_{w(i)} z_2}) z_2 + q_i^{a_{w(i)} z_3} z_3, \quad \text{if} \quad s_i = 2,$$

$$(3.9) \quad p_{ij}^\pm(z, w) = \left( z^{d_i} + q_i^{-a_{w(i)} z} w^{d_i} \right) s_i - 1 \frac{z^{d_i + 2a_{w(i)} z} w^{d_i} - z^d_i w^{d_i}}{q_i^{a_{w(i)} z} w^{d_i} - w^{d_i}}, \quad \text{if} \quad \tilde{a}_{ij} < 0,$$

and the formal series

$$(3.10) \quad g_{ij}(z) = \prod_{k \in \mathbb{Z}_N} \frac{q_i^{a_{w(i)} z} - \xi^{k} z}{1 - \xi^{k} q_i^{a_{w(i)} z}},$$

which is expanded for $|z| < 1$, where

$$(3.11) \quad \xi = \xi_N = e^{2\pi \sqrt{-1}/N}.$$

Now we introduce the quantum algebras concerned about in this paper.
where $h_{i,0} = r_i \sum_{k \in \mathbb{Z}_N} \alpha_{\mu(i)} \in \mathfrak{h}$. The relations are $(i, j, h, h' \in \mathfrak{h})$:

\begin{align}
(0) & \quad \phi_{\mu(i)}^\pm(z) = \phi_i^\pm(\xi^{-1}z), \quad x_{\mu(i)}^\pm(z) = x_i^\pm(\xi^{-1}z), \\
(1) & \quad [d, h] = 0 = [d, c], \quad q^d \phi_i^\pm(z)q^{-d} = \phi_i^\pm(q^{-1}z), \\
(2) & \quad [h, h'] = 0 = [c, h] = [c, \phi_i^\pm(z)] = [\phi_i^\pm(z), \phi_i^\pm(w)] = [h, \phi_i^\pm(z)], \\
(3) & \quad \phi_i^\pm(z)\phi_j^-(w) = \phi_j^-(w)\phi_i^+(z)g_{ij}(q^w/z)^{-1}g_{ij}(q^w/z), \\
(4) & \quad [h, x_i^\pm(z)] = \pm \alpha_i(h)x_i^\pm(z), \quad q^d x_i^\pm(z)q^{-d} = x_i^\pm(q^{-1}z), [c, x_i^\pm(z)] = 0, \\
(5) & \quad \phi_i^+(z)x_j^\pm(w) = x_j^\pm(w)\phi_i^+(z)g_{ij}(q^z/w)^\pm, \\
(6) & \quad \phi_i^-(z)x_j^\pm(w) = x_j^\pm(w)\phi_i^-(z)g_{ij}(q^z/w)^\pm, \\
(7) & \quad [x_i^\pm(z), x_j^-(w)] = \frac{1}{q_i - q_j} \sum_{k \in \mathbb{Z}_N} \delta_{i, \mu(k)} \\
& \quad \times \left( \phi_i^+(zq^{-c})\delta\left(\frac{\xi^cwq^c}{z}\right) - \phi_i^-(zq^{-c})\delta\left(\frac{\xi^cwq^{-c}}{z}\right) \right), \\
(8) & \quad F_{ij}^\pm(z, w)x_i^\pm(z)x_j^\pm(w) = G_{ij}^\pm(z, w)x_j^\pm(w)x_i^\pm(z), \\
(9) & \quad \sum_{\sigma \in S_3} p_i^\pm(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})x_i^\pm(z_{\sigma(1)})x_i^\pm(z_{\sigma(2)})x_i^\pm(z_{\sigma(3)}) = 0, \quad \text{if } s_i = 2, \\
(10) & \quad \sum_{\sigma \in S_m_{ij}} \left\{ \prod_{1 \leq a < b \leq m_{ij}} p_i^\pm(z_{\sigma(a)}, z_{\sigma(b)}) \left( \sum_{r=0}^{m_{ij}} (-1)^r \binom{m_{ij}}{r} q_i^{-r} x_i^\pm(z_{\sigma(1)}) \cdots x_i^\pm(z_{\sigma(r)}) \cdot x_j^\pm(w)x_i^\pm(z_{\sigma(m_{ij})}) \right) \right\} = 0, \quad \text{if } a_{ij} < 0,
\end{align}

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ is the usual delta function and for $i, j \in I$ with $a_{ij} < 0$,

\begin{equation}
(3) m_{ij} = \max\{1 - a_{ij}, k \in \mathbb{Z}_N\}.
\end{equation}

It is straightforward to see that the relations (Q1)-(Q10) are compatible with (Q0) and so the $\mathbb{C}[[h]]$-algebra $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ is well-defined. When $\mathfrak{g}$ is of finite type, the algebra $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ was first introduced by Drinfeld [21] for the purpose of giving a current presentation of quantum affine algebras:

**Theorem 3.3.** [21] Assume that $\mathfrak{g}$ is of finite type $X_\ell$. Then $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ is isomorphic to the quantum affine algebra of type $X_\ell^{(N)}$.

When $\mu = \text{Id}$, $\mathcal{U}_h(\hat{\mathfrak{g}}) = \mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ was called the quantum affinization of $\mathcal{U}_h(\mathfrak{g})$ [34, 60, 10]. And when $\mathfrak{g}$ is of simply-laced type, $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ was realized in [15] in terms of vertex operators.
Remark 3.4. When \( g \) is of non-simply-laced type and \( \mu \neq \text{Id} \), the affine quantum Serre relations (Q10) are new. According to the structure theory of \( \mathcal{U}_h(\tilde{g}_\mu) \) developed later, it seems that our generalization is natural.

Alternatively we can write the defining relations of \( \mathcal{U}_h(\tilde{g}_\mu) \) in terms of the generators given in (3.12). For the defining relations (Q0)-(Q7), we have (i, j) \( \in I, h, h' \in \mathfrak{h}, m, n \in \mathbb{Z} \):

\[
\begin{align*}
(\text{Q}0') & \quad h_{\mu(i),m} = \xi^{-m}h_{i,m}, \quad x_{\mu(i),n}^+ = \xi^{-n}x_{i,n}^+, \\
(\text{Q}1') & \quad [d, h] = [d, c], \quad [d, h_{i,m}] = mh_{i,m}, \\
(\text{Q}2') & \quad [h, h'] = 0 = [c, h] = [c, h_{i,\pm m}], [h, h_{i,\pm m}] = [h_{i,\pm m}, h_{i,\pm n}], m, n \geq 0, \\
(\text{Q}3') & \quad [h, x_{i,n}^+] = \pm \alpha_i(h)x_{i,n}^+, \quad [d, x_{i,n}^+] = nx_{i,n}^+, \quad [c, x_{i,n}^+] = 0, \\
(\text{Q}4') & \quad [h_{i,m}, h_{j,-n}] = \delta_{m,n} \frac{1}{m} \sum_{k \in \mathbb{Z}_N} \xi^{mk}[mr_i\alpha_{\mu(k)}(j)]q^{\frac{mc - q^{-mc}}{q - q^{-1}}}, m, n > 0, \\
(\text{Q}5') & \quad [h_{i,m}, x_{j,n}^-] = \pm \frac{1}{m} \sum_{k \in \mathbb{Z}_N} \xi^{mk}[mr_i\alpha_{\mu(k)}(j)]q^{\frac{mc - q^{-mc}}{q - q^{-1}}}x_{j,m+n}^-, m > 0, \\
(\text{Q}6') & \quad [h_{i,m}, x_{j,n}^+] = \pm \frac{1}{m} \sum_{k \in \mathbb{Z}_N} \xi^{mk}[mr_i\alpha_{\mu(k)}(j)]q^{\frac{mc - q^{-mc}}{q - q^{-1}}}x_{j,m+n}^+, m < 0, \\
(\text{Q}7') & \quad [x_{i,m}^+, x_{j,-n}^-] = \sum_{k \in \mathbb{Z}_N} \delta_{i,m} \xi^{mk}[\phi_{j,m+n}^+q^{\frac{mc - q^{-mc}}{q - q^{-1}}}] \phi_{j,m+n}^-, m \geq 0,
\end{align*}
\]

where the elements \( \phi_j^+ (m) \) and \( \phi_j^- (-m) \) (\( m \geq 0 \)) are defined respectively by

\[
\phi_j^+(z) = \sum_{m \geq 0} \phi_j^+(m)z^{-m} \quad \text{and} \quad \phi_j^-(z) = \sum_{m \geq 0} \phi_j^-(m)z^m.
\]

For the relations (Q8), note that for \( i, j \in I \) we have

\[
(3.14) \quad F_{ij}^\pm(z, w) = \begin{cases} 
(z_{i}^+ + q_i^{ \pm d_i}w_i^{d_i})s_i - 1(z_{i}^+ - q_i^{ \pm 2d_i}w_i^{d_i}), & \text{if } i = j; \\
1, & \text{if } \delta_{ij} = 0; \\
z_{i}^{d_i} - q_i^{ \pm d_i}a_{ij}w_i^{d_i}, & \text{if } (i, j) \in \mathbb{I}.
\end{cases}
\]

\[
(3.15) \quad G_{ij}^\pm(z, w) = \begin{cases} 
(q_i^{ \pm d_i}z_i^{d_i} + w_i^{d_i})s_i - 1(q_i^{ \pm 2d_i}z_i^{d_i} - w_i^{d_i}), & \text{if } i = j; \\
1, & \text{if } \delta_{ij} = 0; \\
q_i^{ \pm d_i}a_{ij}z_i^{d_i} - w_i^{d_i}, & \text{if } (i, j) \in \mathbb{I}.
\end{cases}
\]

Then the relations (Q8) are equivalent to the following relations:

\[
(3.16) \quad [x_{i,m}^+, x_{j,n}^+] = 0, \text{ if } \delta_{ij} = 0,
\]

\[
(3.17) \quad [x_{i,m+d_i}, x_{j,n}^+] q_i^{\pm \delta_{ij}a_{ij}} + [x_{i,n+d_i}, x_{j,m}^+] q_i^{\pm \delta_{ij}a_{ij}} = 0, \text{ if } (i, j) \in \mathbb{I},
\]

\[
(3.18) \quad \sum_{\sigma \in S_2} [x_{i,m(1)+d_1}, x_{i,m(2)}^+] q_i^{\pm s_1} = 0, \text{ if } s_1 = 1,
\]

\[
(3.19) \quad \sum_{\sigma \in S_2} \{ [x_{i,m(1)+2d_1}, x_{i,m(2)}^+] q_i^{\pm s_1} - p_i^{\pm 2d_1} [x_{i,m(1)}, x_{i,m(2)}^+] q_i^{\pm s_1} \} = 0, \text{ if } s_1 = 2,
\]

11
where \([a, b]_v = ab - vba\) for \(0 \neq v \in \mathbb{C}[[\hbar]]\) and \(a, b \in \mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\). Similarly, the relations \((Q9)\) are equivalent to (cf. [17])

\[(Q9') \quad \sum_{\sigma \in S_3} \left[ [x_{i,m(1)}^\pm \xi, x_{i,m(2)}^\pm \eta]_{q_i}, x_{i,m(3)}^\pm \eta \right]_{q_i} = 0.\]

Finally, as the relations \((Q10)\) depend on the polynomials \(p_{ij}^\pm(z, w)\), one needs a case by case argument. For some special cases, one may see [17].

**Remark 3.5.** One of the main features in the definition of \(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\) is the existence of the Drinfeld polynomials \(p_{ij}^\pm(z, w)\) in the affine quantum Serre relations \((Q10)\). When \(\mathfrak{g}\) is of finite type, it was proved in [17] that the relations \((Q10)\) in \(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\) are equivalent to the following affine quantum Serre relations (without polynomials):

\[(Q11) \quad \sum_{\sigma \in S_3} \left( \sum_{r=0}^{\hat{m}_{ij}} (-1)^r \left( \hat{m}_{ij} \right)_r \right) x_i^{\pm} (z_{\sigma(1)}) \cdots x_i^{\pm} (z_{\sigma(r)}) \cdots x_j^{\pm} (z_{\sigma(r+1)}) \cdots x_i^{\pm} (z_{\sigma(\hat{m}_{ij})}) = 0, \quad \text{if} \quad \hat{a}_{ij} < 0,
\]

where \(\hat{m}_{ij} = 1 - \hat{a}_{ij}\). Thus, for any \(\mu\)-invariant subset \(J\) of \(I\) such that the GCM \((a_{ij})_{i,j \in J}\) is a direct sum of GCMs of finite type, the relations \((Q11)\) hold in \(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\) when \(i, j \in J\). We conjecture that the relation \((Q11)\) hold in \(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\) for the general \(\mathfrak{g}\) and \(\mu\).

### 3.2. Twisted quantum toroidal algebras.

As in the untwisted case, we define the horizontal subalgebra \(\mathcal{U}_h^h\) of \(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\) to be the closed subalgebra generated by \(h, x_{i,0}^\pm\) for \(h \in \mathfrak{h}\) and \(i \in I\). When \(\mathfrak{g}\) is of affine type and \(\mu\) fixes the additional node of \(\mathfrak{g}\) ([17]), we further define the vertical subalgebra \(\mathcal{U}_h^v\) of \(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\) to be the closed subalgebra generated by \(h_{i,n}, x_{i,n}^\pm, c, d\) for \(n \in \mathbb{Z}\) and \(i \in I\) not equal to the additional node of \(\mathfrak{g}\).

It was known that ([32]) the orbit Lie algebra \(\hat{\mathfrak{g}}\) can be realized as the subalgebra of \(\mathfrak{g}\) generated by the elements \(h, \hat{e}^\pm = \sum_{p \in \mathbb{Z}} e_{\mu(p)}^\pm\), for \(h \in \mathfrak{h}\) and \(i \in I\). However, in the quantum case, \(\mathcal{U}_h(\hat{\mathfrak{g}})\) is not simply a subalgebra of \(\mathcal{U}_h(\mathfrak{g})\). One may see [39] Section 2.6] for details. We expect that there is an algebra morphism from \(\mathcal{U}_h(\hat{\mathfrak{g}})\) to \(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\). Explicitly, noting that \(\hat{\phi}_{i,0}^\pm = q^{\pm h_{i,0}}\) and so from \((Q7')\) we have

\[
(3.20) \quad [x_{i,0}^+, x_{j,0}^-] = \sum_{k \in \mathbb{Z}} \delta_{i,\mu^0(j)} \frac{q^{h_{i,0}} - q^{-h_{i,0}}}{q_i - q_i^{-1}} = \sum_{k \in \mathbb{Z}} \delta_{i,\mu^0(j)} \frac{q_i^{\hat{a}^\vee} - q_i^{-\hat{a}^\vee}}{q_i - q_i^{-1}}.
\]

Assume now that the relations \((Q11)\) hold on in \(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)\). Then we have

\[
\sum_{r=0}^{1-\hat{a}_{ij}} (-1)^r \left( \hat{m}_{ij} \right)_r \left( x_{i,0}^\pm, x_{j,0}^\pm, x_{i,0}^\pm, x_{j,0}^\pm, x_{i,0}^\pm, x_{j,0}^\pm, \cdots \right) = 0, \quad \text{for} \quad i \neq j \in \hat{I}.
\]

This together with \((Q3')\) and \((3.20)\) gives that there is a surjective algebra morphism \(\mathcal{U}_h(\hat{\mathfrak{g}}) \rightarrow \mathcal{U}_h^h\) defined by

\[
(3.21) \quad h \mapsto h, \quad e_i^\pm \mapsto x_{i,0}^\pm / \sqrt{d_i} \left[ \frac{d_i}{s_i} \right]_{q_i}, \quad \text{for} \quad h \in \hat{\mathfrak{g}}, \; i \in \hat{I}.
\]

In particular, we have (see Remark [26] and Theorem [33]):
Proposition 3.6. Assume that \( g \) is of untwisted affine type \( X^{(1)}_\ell \) and \( \mu \) fixes the additional node of \( g \). Then \( \hat{A} \) is of affine type \( X^{(N)}_\ell \) and both \( U^h \) and \( U^r \) are isomorphic to a quotient of the twisted quantum affine algebra \( U_h(\hat{g}) \).

4. Triangular decomposition

In this section, we prove a triangular decomposition of \( U_h(\hat{g}_\mu) \).

4.1. Triangular decomposition of \( U_h(\hat{g}_\mu) \).

Definition 4.1. Let \( A \) be a completed and separated \( \mathbb{C}[[h]] \)-algebra. By a triangular decomposition of \( A \), we mean a data of three closed \( \mathbb{C}[[h]] \)-subalgebras \( (A^-, H, A^+) \) of \( A \) such that the multiplication \( x^- \otimes h \otimes x^+ \mapsto x^- h x^+ \) induces an isomorphism from the \( h \)-adically completed tensor product \( \mathbb{C}[[h]] \)-module \( A^- \otimes H \otimes A^+ \) to \( A \).

Let \( U_h(\hat{g}_\mu)^+ \) (resp. \( U_h(\hat{g}_\mu)^- \); resp. \( U_h(\hat{h}_\mu) \)) be the closed subalgebra of \( U_h(\hat{g}_\mu) \) generated by \( x^+_{i,n} \) (resp. \( x^-_{i,n} \); resp. \( h, h_{i,m}, c, d \)). The following is the main result of this section.

Theorem 4.2. \( (U_h(\hat{g}_\mu)^-, U_h(\hat{h}_\mu), U_h(\hat{g}_\mu)^+) \) is a triangular decomposition of \( U_h(\hat{g}_\mu) \). Moreover, \( U_h(\hat{g}_\mu)^+ \) (resp. \( U_h(\hat{g}_\mu)^- \); resp. \( U_h(\hat{h}_\mu) \)) is isomorphic to the \( \mathbb{C}[[h]] \)-algebra topologically generated by \( x^+_{i,n} \) (resp. \( x^-_{i,n} \), \( h, h_{i,m}, c, d \)), and subject to the relations \((Q0), (Q8)-(Q10)\) with \( + \), \( - \) (resp. \( (Q6), (Q8)-(Q10) \) with \( + \)), \( - \) (resp. \( (Q0)-(Q2) \)).

The rest of this section is devoted to proving Theorem 4.2. When \( \mu = Id \), Theorem 4.2 was proved in \( [5] \) for the finite type and in \( [37] \) for the general case. In addition, when \( g \) is of finite type, Theorem 4.2 was proved in \( [18] \). We adopt a similar method of \( [37] \) to show the theorem. To prove Theorem 4.2 and for later use, in the following definition we collect some other \( \mathbb{C}[[h]] \)-algebras related to \( U_h(\hat{g}_\mu) \).

Definition 4.3. We denote by \( U_{i,h}^{(\mu)} \) the \( \mathbb{C}[[h]] \)-algebra topologically generated by the elements in \( U_{i,h}^{(\mu)} \) with relations \((Q0)-(Q7)\), denote by \( U_{i,h}^{(\mu)} \) the quotient algebra of \( U_{i,h}^{(\mu)} \) modulo its closed ideal generated by relations \((Q8)\), and denote by \( U_{i,h}^{(\mu)} \) the quotient algebra of \( U_{i,h}^{(\mu)} \) modulo its closed ideal generated by relations \((Q9)\). Let \( U_{i,h}^{(\mu)} \) (resp. \( U_{i,h}^{(\mu)} \); resp. \( U_{i,h}^{(\mu)} \)) be the closed subalgebra of \( U_{i,h}^{(\mu)} \) generated by \( x^+_{i,n} \) (resp. \( x^-_{i,n} \), \( h, h_{i,m}, c, d \)). Similarly, we have the closed subalgebras \( U_{i,h}^{(\mu)} \), \( U_{i,h}^{(\mu)} \), and \( U_{i,h}^{(\mu)} \), \( U_{i,h}^{(\mu)} \), \( U_{i,h}^{(\mu)} \), \( U_{i,h}^{(\mu)} \), respectively.

Before proving Theorem 4.2 we mention one of its consequences. For \( i \in I \), define \( U_i \) to be the closed subalgebra of \( U_h(\hat{g}_\mu) \) generated by \( h_{i,n}, x^+_{i,n}, c, d \) for \( n \in \mathbb{Z} \). Denote by \( t \) the simple finite dimensional Lie algebra of type \( A_1 \) and denote by \( \theta \) the diagram automorphism of \( t \) with order \( s_i \). Then there is a \( \mathbb{C} \)-algebra morphism from \( U_h(\hat{g}_\mu) \) to \( U_i \) given by

\[
(4.1) \quad h_{1,0} \mapsto h_{r_i d_i}, h_{1,m} \mapsto h_{i,d,m} \frac{m}{r_i d_i}, x^+_{1,n} \mapsto x^+_{i,d,n} \sqrt{d_i |d|}, c \mapsto c \frac{d}{r_i}, d \mapsto \frac{d}{d_i}, h \mapsto d r_i h
\]

where \( 0 \neq m \in \mathbb{Z}, n \in \mathbb{Z} \). Just as untwisted quantum affinization algebras are glued by copies of quantum affine algebras of type \( A_1^{(1)} \) (see \( [37] \) Corollary 3.3)), from Theorem 4.2 we have

Corollary 4.4. For \( i \in I \), \( U_i \) is isomorphic to the quantum affine algebra of type \( A_1^{(s_i)} \) as \( \mathbb{C} \)-algebras with the isomorphism given by \( (4.1) \).
4.2. Technical lemmas. In this subsection we prove two lemmas.

Lemma 4.5. Let \( i \in I \) with \( s_i = 2 \), \( l = 1, 2 \) or 3 and \( \eta = + \) or \(-\). Then the following hold in \( U_1^{\pm}(\tilde{\mu}) \):

\[
\sum_{\sigma \in S_3} p_i^{\pm}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) \xi_{i,l}^{\pm,\eta}(z_{\sigma(1)}) \xi_{i,l}^{\pm,\eta}(z_{\sigma(2)}) \xi_{i,l}^{\pm,\eta}(z_{\sigma(3)}) = 0,
\]

where for \( k = 1, 2, 3 \),

\[
\xi_{i,l}^{\pm,\eta}(z_k) = \begin{cases} x_i^{\pm}(z_k) & \text{if } k \neq l; \\ \phi_i^{\eta}(q^{-\eta}z^{\frac{1}{2}} - z_k) & \text{if } k = l. \end{cases}
\]

Proof. We denote by \( T_{i,l}^{\pm,\eta}(z_1, z_2, z_3) \) the LHS of (4.2). It is clear that

\[
T_{i,l}^{\pm,\eta}(z_1, z_2, z_3) = T_{i,3}^{\pm,\eta}(z_1, z_3, z_2) = T_{i,1}^{\pm,\eta}(z_3, z_2, z_1).
\]

Thus it suffices to check the case \( l = 3 \). Assume first that \( \eta = + \). By using the relations (Q5), we have

\[
T_{i,3}^{\pm,\eta}(z_1, z_2, z_3) = \sum_{\sigma \in S_2} \frac{A_i^{\pm}(z_{\sigma(1)}, z_{\sigma(2)}, z_3)}{F_{i}^{\pm}(z_3, z_1)F_{ii}^{\pm}(z_3, z_2)} x_i^{\pm}(z_{\sigma(1)}) x_i^{\pm}(z_{\sigma(2)}) \phi_i^{\eta}(q^{-\frac{1}{2}}z^{\frac{1}{2}} z_3),
\]

where

\[
A_i^{\pm}(z_1, z_2, z_3) = F_{ii}^{\pm}(z_3, z_1) F_{i}^{\pm}(z_3, z_2) p_i^{\pm}(z_1, z_2, z_3)
\]

\[
+ F_{ii}^{\pm}(z_3, z_1) G_i^{\pm}(z_3, z_2) p_i^{\pm}(z_1, z_3, z_2) + G_{ii}^{\pm}(z_3, z_1) G_{i}^{\pm}(z_3, z_2) p_i^{\pm}(z_3, z_1, z_2).
\]

Using (2.11), (3.14) and (5.15), one gets that

\[
A_i^{\pm}(z_1, z_2, z_3) = (z_3^{d_1} + q_i^{\pm d_i} z_1^{d_i})(z_3^{d_1} - q_i^{\mp d_i} z_1^{d_i})(z_3^{d_1} + q_i^{\pm d_i} z_2^{d_i})(z_3^{d_1} - q_i^{\mp d_i} z_2^{d_i})
\]

\[
\times (q_i^{\pm d_i} z_1^{d_1} - (q_i^{d_1} + q_i^{\mp d_i} z_1^{d_1} - q_i^{\pm d_i} z_1^{d_1}))
\]

\[
+ (z_3^{d_1} + q_i^{\pm d_i} z_1^{d_i})(z_3^{d_1} - q_i^{\mp d_i} z_1^{d_i})(z_3^{d_1} + z_2^{d_i} z_3^{d_1} + q_i^{\pm d_i} z_3^{d_1} - z_2^{d_i})
\]

\[
\times (q_i^{\pm d_i} z_1^{d_1} - (q_i^{d_1} + q_i^{\mp d_i} z_1^{d_1} + q_i^{\pm d_i} z_1^{d_1}))
\]

\[
+ (q_i^{\pm d_i} z_3^{d_1} + q_i^{d_1})(q_i^{\pm d_i} z_3^{d_1} - z_1^{d_1})(q_i^{\pm d_i} z_3^{d_1} + z_2^{d_i}) (q_i^{\pm d_i} z_3^{d_1} - z_2^{d_i})
\]

\[
\times (q_i^{\pm d_i} z_3^{d_1} - (q_i^{d_1} + q_i^{\mp d_i} z_1^{d_1} + q_i^{\pm d_i} z_2^{d_i})).
\]

We can verify that (by using Maple for example)

\[
A_i^{\pm}(z_1, z_2, z_3) = F_{ii}^{\pm}(z_1, z_2) B_i^{\pm}(z_1, z_2, z_3),
\]

where

\[
B_i^{\pm}(z_1, z_2, z_3) = q_i^{d_i} (q_i^{d_i} - 1) z_3^{d_1} ((q_i^{d_1} + q_i^{d_1} + q_i^{d_1}) z_1^{d_1} z_2^{d_1}
\]

\[
- (q_i^{4d_i} + q_i^{3d_i} + q_i^{2d_i} + q_i^{d_i} + 1) z_3^{d_1} + q_i^{d_1} z_3^{d_1} z_2^{d_1}).
\]

Note that \( B_i^{\pm}(z_1, z_2, z_3) = B_i^{\pm}(z_2, z_1, z_3) \) and \( F_{ii}^{\pm}(z_2, z_1) = -G_{ii}^{\pm}(z_1, z_2) \) (see (3.14) and (5.15)). Then one can conclude from (4.3), (4.4) and the relations (Q8) that

\[
T_{i,3}^{\pm,\eta}(z_1, z_2, z_3)
\]
\[
\frac{B^\pm(z_1, z_2, z_3)}{F^\pm_{ii}(z_3, z_1)F^\pm_{ii}(z_3, z_1)} \left( \sum_{\sigma \in S_2} F^\pm_{ii}(z_{\sigma(1)}, z_{\sigma(2)}; x^\pm_i(z_{\sigma(1)})x^\pm_i(z_{\sigma(2)})) \phi^+_i(q^{-\frac{1}{2}}z_3) \right)
= \frac{B^\pm(z_1, z_2, z_3)}{F^\pm_{ii}(z_3, z_1)F^\pm_{ii}(z_3, z_1)} \left( F^\pm_{ii}(z_1, z_2)x^\pm_i(z_1)x^\pm_i(z_2) - G^\pm_{ii}(z_1, z_2)x^\pm_i(z_1)x^\pm_i(z_2) \right) \phi^+_i(q^{-\frac{1}{2}}z_3)
= 0,
\]
as desired. The proof of the case \(\eta = -\) is similar and omitted. \(\square\)

The following lemma can be viewed as a twisted analogue of [37, Lemma 9].

**Lemma 4.6.** Let \((i, j) \in \mathbb{I}\). Then in \(\mathcal{U}_i^j(\mathfrak{g}_\mu)\) we have:

\[
\sum_{\sigma \in S_m} \sum_{r=0}^m \prod_{1 \leq a < b \leq m} P_{ij}^\sigma(z_{\sigma(a)}, z_{\sigma(b)}) (-1)^r \left( \sum_{a_i} \phi^+_i(q^{-\frac{1}{2}}w)x^\pm_i(z_{\sigma(1)}) \cdots x^\pm_i(z_{\sigma(r)}) \right)(x_{\sigma(a)}) \cdot x^\pm_i(z_{\sigma(m)}) = 0,
\]

\[
\sum_{\sigma \in S_m} \sum_{r=0}^m \prod_{1 \leq a < b \leq m} P_{ij}^\sigma(z_{\sigma(a)}, z_{\sigma(b)}) (-1)^r \left( \sum_{a_i} \phi^+_i(q^{-\frac{1}{2}}w)x^\pm_i(z_{\sigma(1)}) \cdots x^\pm_i(z_{\sigma(r)}) \right)(x_{\sigma(a)}) \cdot x^\pm_i(z_{\sigma(m)}) = 0,
\]

where \(\eta = \pm\), \(m = m_{ij}\), \(\xi_i(z_p) = x^\pm_i(z_p)\) if \(p \neq 1\) and \(\xi_i(z_1) = \phi^+_i(q^{-\frac{1}{2}}z_1)\).

**Proof.** We prove (4.6) and (4.7) for the case \(\eta = +\), as the proof of \(-\) case is similar. Denote by \(T^\pm\) the LHS of (4.6) (with \(\eta = +\)). Recall that for \((i, j) \in \mathbb{I}\), we have (see (3.14) and (3.15))

\[
g_{ij}(w/z)^{\pm} = \frac{G_{ij}^\pm(z, w)}{F_{ij}^\pm(z, w)} = \frac{z^{\pm d_{ij}a_{ij}} - z^{d_{ij}}}{z^d_{ij} - q_{ij} z^{d_{ij}}},
\]

Using this, it follows from the relations (Q5) that

\[
T^\pm = \sum_{\sigma \in S_m} B^\pm(z_{\sigma(1)}, \ldots, z_{\sigma(m)}, w) x^\pm_i(z_{\sigma(1)}) \cdots x^\pm_i(z_{\sigma(m)}) \phi^+_i(q^{-\frac{1}{2}}w),
\]

where

\[
B^\pm(z_1, \ldots, z_m, w) = \prod_{1 \leq a < b \leq m} P_{ij}^\sigma(z_a, z_b) \prod_{s=1}^m \frac{1}{w^{d_{ij}} - q_{ij} z^{\pm d_{ij}a_{ij}} z_s^{d_{ij}}}
\]

\[
\times \frac{1}{\sum_{r=0}^m \left( \sum_{a_i} \phi^+_i(q^{-\frac{1}{2}}w)x^\pm_i(z_{\sigma(1)}) \cdots x^\pm_i(z_{\sigma(r)}) \right)(x_{\sigma(a)})} \frac{1}{\prod_{r=0}^m \left( \sum_{a_i} \phi^+_i(q^{-\frac{1}{2}}w)x^\pm_i(z_{\sigma(1)}) \cdots x^\pm_i(z_{\sigma(r)}) \right)(x_{\sigma(a)})}.
\]

It was proved in [37, Lemma 5] that there exist polynomials \(f_{\pm, 1}, f_{\pm, 2}, \ldots, f_{\pm, m-1}\) in \(m - 1\) variables such that

\[
B^\pm(z_1, \ldots, z_m, w) = \prod_{1 \leq a < b \leq m} P_{ij}^\sigma(z_a, z_b) \prod_{s=1}^m \frac{1}{z^{\pm d_{ij}a_{ij}} z_s^{d_{ij}} - w^{d_{ij}}}
\]

\[
\sum_{1 \leq r \leq m-1} (z_r^{d_{ij}} - q_{ij} z^{\pm 2d_{ij}a_{ij}} z_r^{d_{ij}}) f_{\pm, r}(z_1, \ldots, z_{r-1}, z_{r+2}, \ldots, z_m, w).
\]
For \( r = 1, \ldots, m - 1 \), define an equivalent relation in \( S_m \) by \( \sigma \sim_r \sigma' \) if and only if 
\( \sigma(r) = \sigma'(r+1), \sigma(r+1) = \sigma'(r), \) and \( \sigma(r') = \sigma'(r') \) for all \( r' \neq r, r + 1 \). Let \( S_m^{(r)} \) be a complete set of equivalence class representatives. Then from (4.9) and (4.10) we have

\[
T^\pm = \sum_{r=1}^{m-1} \sum_{\sigma \in S_m^{(r)}} \prod_{1 \leq a < b \leq m, (a,b) \neq (r,r+1)} \frac{P_{ij}^+(z_\sigma(a), z_\sigma(b))}{q_i z_i \sum_{d=1}^a \alpha_i z_i - u_i z_i}
\]

\[
\times \int_{\pm} \left( z_{\sigma(1)}, \ldots, z_{\sigma(r-1)}, z_{\sigma(r+1)}, \ldots, z_{\sigma(m)}, w \right) x_i^\pm \left( z_{\sigma(1)} \right) \cdots x_i^\pm \left( z_{\sigma(r-1)} \right) \phi_j^+(q^{-\frac{1}{2}} c, w),
\]

where

\[
A^\pm \left( z_{\sigma(r)}, z_{\sigma(r+1)} \right) = p_{ij}^+ \left( z_{\sigma(r)}, z_{\sigma(r+1)} \right) \left( z_{\sigma(r)}^\pm - q_i z_i \sum_{d=1}^a \alpha_i z_i - u_i z_i \right) x_i^\pm \left( z_{\sigma(r)} \right) x_i^\pm \left( z_{\sigma(r+1)} \right)
\]

\[
+ p_{ij}^+ \left( z_{\sigma(r+1)}, z_{\sigma(r)} \right) \left( z_{\sigma(r+1)}^\pm - q_i z_i \sum_{d=1}^a \alpha_i z_i - u_i z_i \right) x_i^\pm \left( z_{\sigma(r+1)} \right) x_i^\pm \left( z_{\sigma(r)} \right).
\]

On the other hand, by using (Q8) we have

\[
p_{ij}^+(z, w) \left( z_{\sigma(i)}, z_{\sigma(j)} \right) x_i^\pm \left( z \right) x_j^\pm \left( w \right)
\]

\[
= \frac{q_i^{\pm 2d_{ij}} z_i^{d_{ij}} - w_{d_{ij}} z_i^{d_{ij}} - q_i^{\pm 2d_{ij}} w_{d_{ij}}}{z_i^{d_{ij}} - q_i^{\pm 2d_{ij}} w_{d_{ij}}} \frac{G_{ij}^+ \left( z, w \right) x_i^\pm \left( z \right) x_j^\pm \left( w \right)}{G_{ij}^+ \left( z, w \right) x_i^\pm \left( z \right) x_j^\pm \left( w \right)}
\]

\[
= - \left( w_{d_{ij}} - q_i^{\pm 2d_{ij}} z_i^{d_{ij}} \right) p_{ij}^+ \left( w, z \right) x_i^\pm \left( w \right) x_j^\pm \left( z \right).
\]

This gives that \( A^\pm \left( z_{\sigma(r)}, z_{\sigma(r+1)} \right) = 0 \) for all \( r \) and hence \( T^\pm = 0 \), as required.

Now we turn to prove (4.7). Denote by \( R^\pm \) the LHS of (4.7). Recall that

\[
g_{ij} \left( w/z \right)^\pm = \frac{G_{ij}^+ \left( z, w \right)}{F_{ij}^+ \left( z, w \right)} = \frac{q_i^{\pm 2d_{ij}} z_i^{d_{ij}} + w_{d_{ij}} \left( q_i^{\pm 2d_{ij}} z_i^{d_{ij}} - u_i^{d_{ij}} \right)}{z_i^{d_{ij}} + q_i^{\pm 2d_{ij}} w_{d_{ij}} \left( z_i^{d_{ij}} - q_i^{\pm 2d_{ij}} w_{d_{ij}} \right)}
\]

Then it follows from (4.8), (14.11) and the relation (Q5) that

\[
R^\pm = \sum_{r=1}^{m-1} \sum_{\sigma \in S_m^{(r)}} C_{ij(r)}^+ \left( z_1, z_2, \ldots, z_{\sigma(m-1)}^r, z_{\sigma(m)}^r, w \right) x_i^\pm \left( z_{\sigma(1)}^r \right) \cdots x_i^\pm \left( z_{\sigma(r)}^r \right) \phi_j^+ \left( q^{-\frac{1}{2}} c, z_1^r \right),
\]

where \( S_{m-1} \) acts on the set \( \{2, \ldots, m \} \) and for \( 1 \leq r \leq m \),

\[
C_{ij(r)}^+ \left( z_1, z_2, \ldots, z_m, w \right) = D_{ij}^+ \left( z_1, \ldots, z_m, w \right) P_{ij}^\left( \right) \left( z_{d_{ij}}^1, z_{d_{ij}}^2, \ldots, z_{d_{ij}}^m, w_{d_{ij}} \right),
\]

\[
D_{ij}^+ \left( z_1, z_2, \ldots, z_m, w \right) = \prod_{2 \leq a < b \leq m} \frac{z_a^{d_{ij}} + q_i^{\pm 2d_{ij}} z_b^{d_{ij}}} {q_i^{\pm 2d_{ij}} z_a^{d_{ij}} - z_b^{d_{ij}}},
\]

16
and
\[ P^{(r)}_{ij}(z_1, z_2, \ldots, z_m, w, q) = \binom{m}{r} (1 - 1)^r \sum_{p=1}^{r} \prod_{1 \leq a \leq p} (z_1 - q^2 a) \prod_{p < a \leq m} (q^2 z_1 - a)(q^{a+j} z_1 - w) \]
\[ + \binom{m}{r - 1} (1 - 1)^{r-1} \sum_{p=r}^{m} \prod_{1 \leq a \leq p} (z_1 - q^2 a) \prod_{p < a \leq m} (q^2 z_1 - a)(z_1 - q^{a+j} w). \]

It was proved in [37] Lemma 6] that
\[ P^{(1)}_{ij}(z_1, z_2, \ldots, z_m, w, q) = (z_2 - q^{-ai} w) f_{i}^{(1)}(z_1, z_2, \ldots, z_m, w, q), \]
\[ + \sum_{2 \leq a \leq m-1} (z_{a+1} - q^{-2} z_a) f_{a}^{(1)}(z_1, \ldots, z_{a-1}, z_{a+2}, \ldots, z_m, w, q), \]
\[ P^{(r)}_{ij}(z_1, z_2, \ldots, z_m, w, q) = (w - q^{-ai} z_r) f_{r}^{(r)}(z_1, \ldots, z_{r-1}, z_{r+1}, \ldots, z_m, w, q), \]
\[ + (z_{r+1} - q^{-a+1} w) f_{m}^{(r)}(z_1, \ldots, z_r, z_{r+2}, \ldots, z_m, w, q), \]
\[ + \sum_{2 \leq a \leq m-1} (z_{a+1} - q^{-2} z_a) f_{a}^{(r)}(z_1, \ldots, z_{a-1}, z_{a+2}, \ldots, z_m, w, q), \]
\[ P^{(m)}_{ij}(z_1, z_2, \ldots, z_m, w, q) = (w - q^{-ai} z_m) f_{m}^{(m)}(z_1, \ldots, z_{m-1}, z_m, w, q), \]
\[ + \sum_{2 \leq a \leq m-1} (z_{a+1} - q^{-2} z_a) f_{a}^{(m)}(z_1, \ldots, z_{a-1}, z_{a+2}, \ldots, z_m, w, q), \]

where \( f_{a}^{(r)}, a = 2, \ldots, m \) are some polynomials of \( m - 1 \) variables, of degree at most 1 in each variable.

For \( a = 2, \ldots, m - 1 \) and \( \pi, \pi' \in S_{m-1}, \) we define the equivalent relation \( \pi \sim_a \pi' \) and the equivalence class \( S^{(a)}_{m-1} \) as above. Then for any \( \pi, \pi' \in S_{m-1} \) with \( \pi \sim_a \pi', \) we have from (Q8) that
\[ (z_{\pi(a)}^{d_i} + q_i z_{\pi'(a+1)}^{d_i}) \prod_{1 \leq j \leq m} \left( \frac{1}{z_{\pi(a+1)}^{d_j}} - \frac{z_{\pi(a)}^{d_j}}{z_{\pi'(a+1)}^{d_j}} \right) \]
\[ = (z_{\pi(a)}^{d_i} + q_i z_{\pi'(a+1)}^{d_i}) \prod_{1 \leq j \leq m} \left( \frac{1}{z_{\pi(a+1)}^{d_j}} - \frac{z_{\pi(a)}^{d_j}}{z_{\pi'(a+1)}^{d_j}} \right). \]

This gives
\[ D_{ij}^{+}(z_1, z_2, \ldots, z_m, w) (z_{\pi(a+1)}^{d_i} - q_i z_{\pi(a)}^{d_i} x_i^{+}(z_{\pi(a)}^{d_i}) x_i^{+}(z_{\pi(a+1)}^{d_i})) \]
\[ = D_{ij}^{+}(z_1, z_2, \ldots, z_m, w) (z_{\pi'(a+1)}^{d_i} - q_i z_{\pi'(a)}^{d_i} x_i^{+}(z_{\pi'(a)}^{d_i}) x_i^{+}(z_{\pi'(a+1)}^{d_i})), \]

In view of (4.12), (4.13), (4.14) and (4.15), for \( r = 1, \ldots, m \) and \( a = 2, \ldots, m - 1 \) with \( a \neq r, \) we have
\[ \sum_{\pi \in S_{m-1}} D_{ij}^{+}(z_1, z_2, \ldots, z_m, w) P^{(r)}_{ij}(z_1, z_2, \ldots, z_m, w) \]
\[ x_i^{+}(z_{\pi(2)}) x_j^{+}(w) x_i^{+}(z_{\pi(r+1)}) \cdots x_i^{+}(z_{\pi(m)}) = 0. \]
This implies that all the terms in \( R^\pm \) which contain the polynomials \( f_a^{(r)} \) with \( a \neq r, m \) can be erased. Thus, we obtain

\[
R^\pm = \sum_{\tau \in S_{m-1}} D_{ij}^\pm (z_1, z_{\tau(2)}, \ldots, z_{\tau(m)}, w) \sum_{r=2}^m (q_i^{d_{ij}a_{ij}(0)} w_{d_{ij}} - z_{d_{ij}(r)}) \times (f_r^{(r)} - f_m^{(r-1)}) (z_1^{d_{ij}}, z_{\tau(2)}^{d_{ij}}, \ldots, z_{\tau(r-1)}^{d_{ij}}, z_{\tau(r+1)}^{d_{ij}}, \ldots, z_{\tau(m)}^{d_{ij}}, w_{d_{ij}}, q_i^{\pm d_{ij}}) \times x_i^+(z_{\tau(2)}) \cdots x_i^+(z_{\tau(r-1)}) x_i^+(w) x_i^+(z_{\tau(r)}) \cdots x_i^+(z_{\tau(m)}) \phi_i^+(q_i^{-\sum} z_1).
\]

Recall from [37] Lemma 7 that

\[
f_r^{(r)} - f_m^{(r-1)} = \sum_{a=1}^{m-2} (z_r - q^2 z_{r+1}) g_a^{(r)} (z_1, \ldots, z_{r-1}, z_{r+2}, \ldots, z_m, w, q),
\]

where \( g_a^{(r)} \) are some polynomials. This together with (4.15) gives \( R^\pm = 0. \]

4.3. Proof of Theorem 4.2. We start with two propositions which are about the compatibility with affine quantum Serre relations (Q8), (Q9), and (Q10).

**Proposition 4.7.** For \( i, j \in I \), the following hold in \( U_\hbar^I (\mathfrak{g}_\mu) \):

\[
[F_{ij}^\pm (z, w) x_i^+(z) x_j^+(w) - G_{ij}^\pm (z, w) x_j^+(z) x_i^+(w), x_i^+(w_0)] = 0.
\]

**Proof.** From the relations (Q5)-(Q7), it follows that

\[
\pm (q_k - q_k^{-1}) [F_{ij}^\pm (z, w) x_i^+(z) x_j^+(w), x_k^+(w_0)]
\]

\[
= F_{ij}^\pm (z, w) \sum_{p \in \mathbb{Z}} \delta_{i, \mu^p(k)} \left( \phi_i^+(q_i^{\pm \frac{1}{2}} z) \delta \left( \frac{q_i^{(\pm c) k w_0}}{z} \right) - \phi_i^-(q_i^{\pm \frac{1}{2}} z) \delta \left( \frac{q_i^{(\pm c) k w_0}}{z} \right) \right) x_j^+(w)
\]

\[
+ F_{ij}^\pm (z, w) x_i^+(z) \sum_{p \in \mathbb{Z}} \delta_{i, \mu^p(k)} \left( \phi_j^+(q_i^{\pm \frac{1}{2}} w) \delta \left( \frac{q_i^{(\pm c) k w_0}}{w} \right) - \phi_j^-(q_i^{\pm \frac{1}{2}} w) \delta \left( \frac{q_i^{(\pm c) k w_0}}{w} \right) \right) x_j^+(z)
\]

\[
= G_{ij}^\pm (z, w) \sum_{p \in \mathbb{Z}} \delta_{i, \mu^p(k)} \left( \phi_i^+(q_i^{\pm \frac{1}{2}} z) \delta \left( \frac{q_i^{(\pm c) k w_0}}{z} \right) - \phi_i^-(q_i^{\pm \frac{1}{2}} z) \delta \left( \frac{q_i^{(\pm c) k w_0}}{z} \right) \right) x_i^+(z)
\]

\[
+ G_{ij}^\pm (z, w) \sum_{p \in \mathbb{Z}} \delta_{j, \mu^p(k)} \left( \phi_j^+(q_i^{\pm \frac{1}{2}} w) \delta \left( \frac{q_i^{(\pm c) k w_0}}{w} \right) - \phi_j^-(q_i^{\pm \frac{1}{2}} w) \delta \left( \frac{q_i^{(\pm c) k w_0}}{w} \right) \right) x_i^+(z)
\]

where we have used the facts:

\[
G_{ij}^\pm (z, w) = F_{ij}^\pm (z, w) g_{ij}(z/w) \pm^1 \quad \text{and} \quad F_{ij}^\pm (z, w) = G_{ij}^\pm (z, w) g_{ji}(z/w) \pm^1.
\]

**Proposition 4.8.** Let \( i, j, k \in I \). Then in \( U_\hbar^I (\mathfrak{g}_\mu) \) we have:

\[
\left[ \sum_{\sigma \in S_3} p_i^\pm (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) x_i^+(z_{\sigma(1)}) x_i^+(z_{\sigma(2)}) x_i^+(z_{\sigma(3)}), x_j^+(w) \right] = 0, \quad \text{if} \ s_i = 2,
\]

18
\[
\sum_{\sigma \in S_{m_j}} \left\{ \prod_{1 \leq a < b \leq m_j} p_{ij}^{\pm}(z_{\sigma(a)}, \hat{z}_{\sigma(b)}) \left( \sum_{r=0}^{m_j} (-1)^r \binom{m_j}{r} q_{ij}^{\pm} x_{ij}^{\pm}(z_{\sigma(1)}) \right) \right. \\
\left. \cdots x_{i}^{\pm}(z_{\sigma(r)}) x_{i}^{\pm}(\hat{z}_{\sigma(r+1)}) \cdots x_{i}^{\pm}(z_{\sigma(m_j)}) \right\} x_{k}^{\pm}(w_{0}) = 0, \text{ if } \hat{a}_{ij} < 0.
\]

(4.18)

Proof. We first prove (4.17). Note that it suffices to treat the case \(i = j\). In this case, it follows from (Q7) that the LHS of (4.17) equals to

\[
\pm \frac{1}{q_{i}^{\pm}} \sum_{k \in \mathbb{Z}_{N}} \delta_{i,j}^{\pm}(i) \sum_{\sigma \in S_{3}} p_{i}^{\pm}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) C_{i,k}^{\pm}(z_{1}, z_{2}, z_{3}),
\]

where \(C_{i,k}^{\pm}(z_{1}, z_{2}, z_{3})\) stands for the following formal series:

\[
x_{i}^{\pm}(z_{\sigma(1)}) x_{i}^{\pm}(z_{\sigma(2)}) \left( \hat{z}_{\sigma(3)} \right) \delta \left( \frac{\xi^{k} w_{q}^{\pm c}}{z_{\sigma(3)}} \right) - \hat{z}_{\sigma(1)} \left( \hat{z}_{\sigma(2)} \right) \delta \left( \frac{\xi^{k} w_{q}^{\pm c}}{z_{\sigma(2)}} \right) x_{i}^{\pm}(z_{\sigma(3)}) + \hat{z}_{\sigma(1)} \left( \hat{z}_{\sigma(2)} \right) \delta \left( \frac{\xi^{k} w_{q}^{\pm c}}{z_{\sigma(1)}} \right) x_{i}^{\pm}(z_{\sigma(3)}).
\]

It is straightforward to see that

\[
\sum_{\sigma \in S_{3}} p_{i}^{\pm}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) C_{i,k}^{\pm}(z_{1}, z_{2}, z_{3})
\]

\[
= \sum_{l=1}^{3} T_{i,l}^{\pm}(z_{1}, z_{2}, z_{3}) \delta \left( \frac{\xi^{k} w_{q}^{\pm c}}{z_{l}} \right) - T_{i,l}^{-}(z_{1}, z_{2}, z_{3}) \delta \left( \frac{\xi^{k} w_{q}^{\pm c}}{z_{l}} \right),
\]

recalling that \(T_{i,l}^{\pm}(z_{1}, z_{2}, z_{3})\) stands for the LHS of (4.12). Thus the relation (4.17) follows from Lemma 4.5. Next, by a same argument as that in the proof of [37, Lemma 10], (4.18) follows from Lemma 4.6. □

Proof of Theorem 4.2: We would use a general proof of triangular decompositions (cf. [37, Lemma 3.5]). Let \(A\) be a completed and separated \(\mathbb{C}[[\hbar]]\)-algebra and \((A^{-}, H, A^{+})\) a triangular decomposition of \(A\). Let \(B^{+}\) and \(B^{-}\) be respectively a closed two-sided ideal of \(A^{-}\) and \(A^{+}\), and let \(B\) be the closed ideal of \(A\) generated by \(B^{+} + B^{-}\). Set \(C = A/B\) and denote by \(C^{\pm}\) the image of \(B^{\pm}\) in \(C\). Assume that \(AB^{+} \subset B^{+} A\) and \(B^{-} A \subset AB^{-}\). Then \((C^{+}, H, C^{-})\) is a triangular decomposition of \(C\) and \(C^{\pm}\) are isomorphic to \(A^{\pm}/B^{\pm}\).

Note that \((U_{h}^{1}(\hat{g}_{\mu})^{-}, U_{h}^{1}(\hat{g}_{\mu})^{+})\) is a triangular decomposition of \(U_{h}^{1}(\hat{g}_{\mu})\). Moreover, \(U_{h}^{1}(\hat{g}_{\mu})^{+}\) (resp. \(U_{h}^{1}(\hat{g}_{\mu})^{-}\)) is isomorphic to the \(\mathbb{C}[[\hbar]]\)-algebras topologically freely generated by \(x_{i,n}^{1}(\hat{g}_{\mu})\) (resp. \(x_{i,n}^{-}(\hat{g}_{\mu})\)) and \(U_{h}^{1}(\hat{g}_{\mu})^{+}\) (resp. \(U_{h}^{1}(\hat{g}_{\mu})^{-}\)) is isomorphic to the \(\mathbb{C}[[\hbar]]\)-algebra topologically generated by \(h, h_{i,m}, c, d\) with relations (Q0)-(Q2). In view of the above criteria, it follows from Proposition 4.7 that \(U_{h}^{1}(\hat{g}_{\mu})\) admits a triangular decomposition \((U_{h}^{1}(\hat{g}_{\mu})^{-}, U_{h}^{1}(\hat{g}_{\mu}), U_{h}^{1}(\hat{g}_{\mu})^{+})\) induced from the triangular decomposition \((U_{h}^{1}(\hat{g}_{\mu})^{-}, U_{h}^{1}(\hat{g}_{\mu}), U_{h}^{1}(\hat{g}_{\mu})^{+})\) of \(U_{h}^{1}(\hat{g}_{\mu})\). Furthermore, due to Proposition 4.8, the triangular decomposition \((U_{h}^{1}(\hat{g}_{\mu})^{-}, U_{h}^{1}(\hat{g}_{\mu}), U_{h}^{1}(\hat{g}_{\mu})^{+})\) of \(U_{h}^{1}(\hat{g}_{\mu})\) induces a triangular decomposition of \(U_{h}^{1}(\hat{g}_{\mu})\) as stated in Theorem 4.2. This completes the proof of Theorem 4.2.
5. Affine quantum Serre relations

In this section, we introduce a notion of “normal ordered products” of currents on restricted modules for $\mathcal{U}_h^\dagger(\hat{g}_\mu)^\pm$ and then reformulate the affine quantum Serre relations (Q9) and (Q10) in terms of normal ordered products.

5.1. Normal ordered products. We start with some notations. Recall that a $\mathbb{C}[[\hbar]]$-module $W$ is called topologically free if $W = W^0[[\hbar]]$ for some vector space $W^0$ over $\mathbb{C}$. We denote by $\mathcal{M}_f$ the category of topologically free $\mathbb{C}[[\hbar]]$-modules. For $W \in \mathcal{M}_f$ and $m, n \in \mathbb{Z}_+$, there is a $\mathbb{C}[[\hbar]]$-module map

$$\tilde{\pi}_n^{(m)} : \text{End}_{\mathbb{C}[[\hbar]]}(W)[[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]] \to \text{End}(W/\hbar^nW)[[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]]$$

induced by the canonical $\mathbb{C}[[\hbar]]$-map $\text{End}_{\mathbb{C}[[\hbar]]}(W) \to \text{End}(W/\hbar^nW)$. As usual, for a $\mathbb{C}$-vector space $W_0$, we denote by $W_0((z_1, \ldots, z_m))$ the space of lower truncated (infinite) integral power series in the variables $z_1, \ldots, z_m$ with coefficients in $W_0$. Set

$$\mathcal{E}^{(m)}(W_0) = \text{Hom}(W_0, W_0((z_1, \ldots, z_m))).$$

The following notion is an $h$-analogue of $\mathcal{E}^{(m)}(W_0)$ introduced in [52].

Definition 5.1. Let $W$ be a topologically free $\mathbb{C}[[\hbar]]$-module and $m$ a positive integer. Define $\mathcal{E}^{(m)}_h(W)$ to be the $\mathbb{C}[[\hbar]]$-submodule of $\text{End}(W)[[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]]$, consisting of each formal series $\psi(z_1, \ldots, z_m)$ such that for every $n \in \mathbb{Z}_+$,

$$\tilde{\pi}_n^{(m)}(\psi(z_1, \ldots, z_m)) \in \mathcal{E}^{(m)}(W/\hbar^nW),$$

or equivalently, for every $v \in W$ and $n \in \mathbb{Z}_+$,

$$\psi(z_1, \ldots, z_m)v \in \hbar^nW((z_1, \ldots, z_m)).$$

Let $W = W_0[[\hbar]] \in \mathcal{M}_f$. For convenience, we will also write $\mathcal{E}_h(W) = \mathcal{E}^{(1)}_h(W)$. One notices that for any $m \in \mathbb{Z}_+$, $\mathcal{E}^{(m)}_h(W) = \mathcal{E}^{(m)}(W_0[[\hbar]])$ is also a topologically free $\mathbb{C}[[\hbar]]$-module. Recall from [52, Remark 4.7] that for every $n \in \mathbb{Z}_+$, there is a surjective $\mathbb{C}[[\hbar]]$-map

$$\tilde{\pi}_n^{(m)} : \mathcal{E}^{(m)}_h(W) \to \mathcal{E}^{(m)}(W/\hbar^nW)$$

induced by $\tilde{\pi}_n^{(m)}$ with kernel $\hbar^n\mathcal{E}^{(m)}_h(W)$. Furthermore, there is an inverse system

$$0 \leftarrow \mathcal{E}^{(m)}(W/\hbar^nW) \leftarrow \mathcal{E}^{(m)}(W/\hbar^{n+1}W) \leftarrow \mathcal{E}^{(m)}(W/\hbar^{n+2}W) \leftarrow \cdots$$

with $\mathcal{E}^{(m)}_h(W)$ equipped with $\mathbb{C}[[\hbar]]$-maps $\tilde{\pi}_n^{(m)}$ as an inverse limit.

Definition 5.2. Let $\mathcal{U}$ be one of the algebras $\mathcal{U}_h^\dagger(\hat{g}_\mu)^\pm, \mathcal{U}_h^\dagger(\hat{g}_\nu)^\pm, \mathcal{U}_h(\hat{g}_\mu)^\pm, \mathcal{U}_h(\hat{g}_\nu)^\pm$, or $\mathcal{U}_h(\hat{g}_\mu)$. We say that a (left) $\mathcal{U}$-module $W$ is restricted if $W$ is topologically free as a $\mathbb{C}[[\hbar]]$-module, and for each $i \in I, w \in W$, $x^+_\i n w \to 0$ as $n \to +\infty$, that is, for every $m \in \mathbb{Z}_+$, there exists $m' \in \mathbb{Z}$, such that

$$x^+_\i n w \in \hbar^mW \quad \text{for } n \geq m'.$$

We denote by $\mathcal{R}_{\pm, i}^1, \mathcal{R}_{\pm, i}^2, \mathcal{R}_{\pm}$, and $\mathcal{R}$ the categories of restricted modules for $\mathcal{U}_h^\dagger(\hat{g}_\mu)^\pm$, $\mathcal{U}_h(\hat{g}_\mu)^\pm$, $\mathcal{U}_h(\hat{g}_\nu)^\pm$, and $\mathcal{U}_h(\hat{g}_\mu)$, respectively.

Note that a $\mathcal{U}_h^\dagger(\hat{g}_\mu)^\pm$-module $W$ is restricted if and only if $W \in \mathcal{M}_f$ and $x^\pm_i(z) \in \mathcal{E}_h(W)$ for $i \in I$. The following argument is standard in the vertex algebra theory.

Lemma 5.3. For $W \in \mathcal{R}_{\pm, i}^1$ and $i, j \in I$, we have

$$F_{ij}^\pm(z_1, z_2)x^\pm_i(z_1)x^\pm_j(z_2) \in \mathcal{E}^{(2)}_h(W).$$
Proof. Let \( v \in W \) and \( n \in \mathbb{Z}_+ \). Then by definition we have \( F^0_{ij}(z_1, z_2)x^+_i(z_1)x^+_j(z_2)v \in W((z_1))((z_2)) + h^0W[[z_1^{\pm 1}, z_2^{\pm 1}]] \). On the other hand, in view of (Q8), we have
\[
F^+_{ij}(z_1, z_2)x^+_i(z_1)x^+_j(z_2)v = G^+_{ij}(z_1, z_2)x^+_i(z_1)x^+_j(z_2)v \in W((z_1))((z_2)) + h^0W[[z_1^{\pm 1}, z_2^{\pm 1}]].
\]
This forces that \( F^+_{ij}(z_1, z_2)x^+_i(z_1)x^+_j(z_2)v \in \mathcal{E}^{(2)}_h(W) \), as required.

Let \( \mathbb{C}_s[[z_1, \ldots, z_m, h]] \) denote the algebra extension of \( \mathbb{C}[[z_1, \ldots, z_m, h]] \) by inverting \( z_a, z_a - cz_b \) with \( 1 \leq a \neq b \leq m \) and \( c \) invertible in \( \mathbb{C}[h] \). Denote by
\[
\iota_{z_1, \ldots, z_m} : \mathbb{C}_s[[z_1, \ldots, z_m, h]] \to \mathbb{C}[h][((z_1)) \cdots ((z_m))],
\]
the canonical algebra embedding that preserves each elements of \( \mathbb{C}[[z_1, \ldots, z_m, h]] \).

**Lemma 5.4.** For \( W \in \mathcal{R}_\pm^l \) and \( i \in I \), we have
\[
\iota_{z_1, z_2}(z_i^{\pm d_i} - z_2^{\pm d_i})^{-1}F^+_{ii}(z_1, z_2)x^+_i(z_1)x^+_i(z_2) = \iota_{z_2, z_1}(z_i^{\pm d_i} - z_2^{\pm d_i})^{-1}F^+_{ii}(z_1, z_2)x^+_i(z_1)x^+_i(z_2).
\]
In particular, we have \( \iota_{z_1, z_2}(z_i^{\pm d_i} - z_2^{\pm d_i})^{-1}F^+_{ii}(z_1, z_2)x^+_i(z_1)x^+_i(z_2) \in \mathcal{E}^{(2)}_h(W) \).

**Proof.** Set \( x^+_i(z_1, z_2) = F^+_{ii}(z_1, z_2)x^+_i(z_1)x^+_i(z_2) \in \mathcal{E}^{(2)}_h(W) \) (see Lemma 8.3). Recall from (8.14) and (8.15) that
\[
F^+_{ii}(z_1, z_2) = \left( z_i^{d_i} - q_i^{2d_i} z_2^{d_i} \right) \left( z_1^{d_i} + q_i^{-d_i} z_2^{d_i} \right)^{s_i^{-1}} = -G^+_{ii}(z_2, z_1).
\]
This together with (Q8) gives
\[
x^+_i(z_1, z_2) = G^+_{ii}(z_1, z_2)x^+_i(z_1) = -F^+_{ii}(z_2, z_1)x^+_i(z_2) = -x^+_i(z_2, z_1).
\]
So we have \( x^+_i(z_1, z_1) = 0 \) on \( W \). Note that for \( k \in \mathbb{Z}_{d_i} \), we have \( F^+_{ii}(z_1, z_2) = F^+_{ii}(z_1, z_2) \) and \( x^+_i(\xi_{d_i} z_1) = x^+_i(z_1) \) (see (Q0)). This implies that for every \( k \in \mathbb{Z}_{d_i} \),
\[
(\iota_{z_1, z_2}(z_i^{d_i} - z_2^{d_i})^{-1} - \iota_{z_2, z_1}(z_1^{d_i} - z_2^{d_i})^{-1})x^+_i(z_1, z_2)v = z_1^{-d_i}\delta(z_1^{d_i} / z_2^{d_i})x^+_i(z_1, z_2)v = 0. \tag{5.2}
\]
In view of this, for any \( v \in W \), we have
\[
(\iota_{z_1, z_2}(z_i^{d_i} - z_2^{d_i})^{-1} - \iota_{z_2, z_1}(z_1^{d_i} - z_2^{d_i})^{-1})x^+_i(z_1, z_2)v = z_1^{-d_i}\delta(z_1^{d_i} / z_2^{d_i})x^+_i(z_1, z_2)v = 0.
\]
This proves the lemma. \( \square \)

For \( i, j \in I \), set
\[
\iota^{(\pm)}_{i, j}(z) = \prod_{k \in \mathbb{Z}_{N^0_{i, j}}} \left( z - \xi^{k}w \right)^{-1} \cdot F^\pm_{i, j}(z, w) \in \mathbb{C}_s[[z, w, h]]. \tag{5.3}
\]

Now we introduce a notion of “normal ordered products” of currents on \( \mathcal{U}^l_h(\mathfrak{g}_m)^\pm \).

**Definition 5.5.** For \( W \in \mathcal{R}_\pm^l \) and \( i_1, \ldots, i_m \in I \), we define a normal ordered product
\[
\hat{x}^\pm_{i_1}(z_1)x^\pm_{i_2}(z_2) \cdots x^\pm_{i_m}(z_m) \in \mathrm{End}(W)((z_1)) \cdots ((z_m))
\]
of the currents \( x^\pm_{i_1}(z_1), \ldots, x^\pm_{i_m}(z_m) \) to be
\[
\prod_{1 \leq r < s \leq m} \iota_{z_r, z_s}(\hat{f}^\pm_{i_r i_s}(z_r, z_s)) x^\pm_{i_1}(z_1)x^\pm_{i_2}(z_2) \cdots x^\pm_{i_m}(z_m).
\]
For $i, j \in I$, set
\begin{equation}
C_{ij} = \prod_{k \in \mathbb{Z}_N, a_{i,m(k)} = 0} (-\xi^k).
\end{equation}

We have the following properties of the normal order products.

**Proposition 5.6.** Let $W \in \mathcal{R}_\pm^l$ and $i_1, i_2, \ldots, i_m \in I$. Then we have
\begin{equation}
\circ x_{i_1}^\pm(z_1)x_{i_2}^\pm(z_2) \cdots x_{i_m}^\pm(z_m)^0 \in \mathcal{E}_h^{(m)}(W),
\end{equation}
and for $k_1, \ldots, k_m \in \mathbb{Z}_N$,
\begin{equation}
\circ x_{\mu^k(i_1)}^\pm(z_1) \cdots x_{\mu^k(i_m)}^\pm(z_m)^0 = \circ x_{z_i}^\pm(\xi^{-k_1}z_1) \cdots x_{z_m}^\pm(\xi^{-k_m}z_m)^0.
\end{equation}
Furthermore, for $\sigma \in S_m$, we have
\begin{equation}
\circ x_{i_1}^\pm(z_1) \cdots x_{i_m}^\pm(z_m)^0 = \left( \prod_{1 \leq s < t \leq m, \sigma(s) > \sigma(t)} C_{\sigma(st)} \right) \circ x_{\sigma(1)}^\pm(z_{\sigma(1)}) \cdots x_{\sigma(m)}^\pm(z_{\sigma(m)})^0.
\end{equation}

**Proof.** We first treat the case that $m = 2$. From Lemma 5.3 and Lemma 5.4 it follows that $\circ x_{i_1}^\pm(z_1)x_{i_2}^\pm(z_2)^0 \in \mathcal{E}_h^{(2)}(W)$. Note that the identity (Q8) follows directly from definition and the relations (Q0). For the third assertion, for $i, j \in I$ we have
\begin{equation}
G_{ij}^+(z_1, z_2) = \prod_{k \in \Gamma_{ij}} (-\xi^k) \cdot F_{ji}^+(z_2, z_1) = \prod_{k \in \mathbb{Z}_N, a_{i,k} = 0} (-\xi^k) \cdot C_{ij} \cdot F_{ji}^+(z_2, z_1).
\end{equation}
This together with Lemma 5.4 and the relations (Q8) gives that
\begin{align*}
t_{z_1, z_2} \left( f_{i_1}^+(z_1, z_2) \right) & x_{i_2}^+(z_1)x_{i_2}^+(z_2) \\
&= t_{z_1, z_2} \prod_{k \in \mathbb{Z}_N, a_{i, k} = 0} \left( z_1 - \xi^k z_2 \right)^{-1} F_{i_1}^+(z_1, z_2)x_{i_2}^+(z_1)x_{i_2}^+(z_2) \\
&= t_{z_2, z_1} \prod_{k \in \mathbb{Z}_N, a_{i, k} = 0} \left( z_1 - \xi^k z_2 \right)^{-1} F_{i_2}^+(z_1, z_2)x_{i_2}^+(z_1)x_{i_2}^+(z_2) \\
&= t_{z_2, z_1} \prod_{k \in \mathbb{Z}_N, a_{i, k} = 0} \left( z_1 - \xi^k z_2 \right)^{-1} G_{i_2}^+(z_1, z_2)x_{i_2}^+(z_2)x_{i_2}^+(z_1) \\
&= C_{i_2} t_{z_2, z_1} \prod_{k \in \mathbb{Z}_N, a_{i, k} = 0} \left( z_2 - \xi^k z_1 \right)^{-1} F_{i_2}^+(z_2, z_1)x_{i_2}^+(z_2)x_{i_2}^+(z_1) \\
&= C_{i_2} t_{z_2, z_1} \left( f_{i_2}^+(z_2, z_1) \right) x_{i_2}^+(z_2)x_{i_2}^+(z_1).
\end{align*}
Thus we have $\circ x_{i_1}^\pm(z_1)x_{i_2}^\pm(z_2)^0 = C_{ij} \circ x_{i_2}^\pm(z_2)x_{i_2}^\pm(z_1)^0$, as required.

For the general case, we prove it by induction on $m$. Indeed, by induction assumption we have $\circ x_{i_1}^\pm(z_1)x_{i_2}^\pm(z_2) \cdots x_{i_m}^\pm(z_m)^0 \in \text{Hom}(W, W((z_1, \ldots, z_{m-1}))(z_m)))$. 

\(22\)
On the other hand, by using the fact \((k = 1, \ldots, m - 1)\)
\[
\iota_{z_k,z_m} \left( f_{i_k,i_m}^\pm (z_k, z_m) \right) x_{i_k}^\pm (z_k) x_{i_m}^\pm (z_m) = C_{i_k,i_m} \iota_{z_m,z_k} \left( f_{i_m,i_k}^\pm (z_m, z_k) \right) x_{i_m}^\pm (z_m) x_{i_k}^\pm (z_k),
\]
we can move the term \(x_{i_m}^\pm (z_m)\) in \(\circ x_{i_1}^\pm (z_1) \cdots x_{i_m}^\pm (z_m)^0\) to the left so that
\[
\circ x_{i_1}^\pm (z_1) \cdots x_{i_m}^\pm (z_m)^0 = \prod_{1 \leq r < s \leq m} f_{i_r,i_s}^\pm (z_r, z_s) x_{i_r}^\pm (z_1) \cdots x_{i_m}^\pm (z_m),
\]
which is independent with the expansions of \(f_{i_r,i_s}^\pm (z_r, z_s)\).

This gives that \(\circ x_{i_1}^\pm (z_1) \cdots x_{i_m}^\pm (z_m)^0 \in \mathcal{E}_h^m(W)\). Similarly, the remaining two assertions in proposition follow by an induction argument.

\begin{remark}
From the proof of Proposition \ref{remark5.6} we see that in the definition of the normal order product \(\circ x_{i_1}^\pm (z_1) \cdots x_{i_m}^\pm (z_m)^0\), the iota-maps \(\iota_{i_s,i_r}\) can be replaced with \(\iota_{i_r,i_s}\) for \(1 \leq r < s \leq m\). That is,
\[
\circ x_{i_1}^\pm (z_1) \cdots x_{i_m}^\pm (z_m)^0 = \prod_{1 \leq r < s \leq m} f_{i_r,i_s}^\pm (z_r, z_s) x_{i_r}^\pm (z_1) \cdots x_{i_m}^\pm (z_m),
\]
which is independent with the expansions of \(f_{i_r,i_s}^\pm (z_r, z_s)\).
\end{remark}

5.2. **On the relations (Q9)**. This subsection is devoted to prove the following result:

**Proposition 5.8.** Let \(W \in \mathcal{R}_\pm^l\) and \(i \in I\) with \(s_i = 2\). Then as operators on \(W\),
\[
\sum_{\sigma \in S_3} p_i^\pm (\sigma_1, \sigma_2, \sigma_3) x_i^\pm (\sigma_1) x_i^\pm (\sigma_2) x_i^\pm (\sigma_3) = 0
\]
in \(W\) if and only if
\[
\circ x_i^\pm (z) x_i^\pm (q_i z) x_i^\pm (\xi_{2d}, q_i z)^0 = 0.
\]

To prove Proposition 5.8 and for later use, we need the following notion.

**Definition 5.9.** For \(W \in \mathcal{R}_\pm^l\) and \(i_1, \ldots, i_m \in I\), we define the \(g\)-commutator
\[
[x_{i_1}^\pm (z_1), \ldots, x_{i_m}^\pm (z_m)]_g \in \text{End}(W)[[z_1, \ldots, z_m]]
\]
inductively such that \([x_{i_1}^\pm (z_1)]_g = x_{i_1}^\pm (z_1)\) and for \(r = m - 1, \ldots, 1\),
\[
[x_{i_r}^\pm (z_r), \ldots, x_{i_m}^\pm (z_m)]_g = x_{i_r}^\pm (z_r) [x_{i_{r+1}}^\pm (z_{r+1}), \ldots, x_{i_m}^\pm (z_m)]_g
\]
\begin{align*}
&- \left( \prod_{r+1 \leq a \leq m} g_{a,i_r} (z_r/z_a)^{\mp 1} \right) [x_{i_{r+1}}^\pm (z_{r+1}), \ldots, x_{i_m}^\pm (z_m)]_g x_{i_r}^\pm (z_r).
\end{align*}

Recall from (3.10) that for \(i, j \in I\),
\[
g_{ji}(z/w)^{\mp 1} = \iota_{w,z} \left( C_{ij}^\pm \left( z, \omega \right) \right) = C_{ij} \iota_{w,z} \left( f_{ij}^\pm (w, z) \right).
\]
This implies that
\[
[x_i^\pm (z), x_j^\pm (w)]_g = \circ x_i^\pm (z) x_j^\pm (w)^0 \left( \iota_{z,w} (f_{ij}^\pm (z, w)^{-1}) - \iota_{w,z} (f_{ij}^\pm (z, w)^{-1}) \right),
\]
which is an \(\mathcal{E}_h(W)\)-linear combination of \(\delta\)-functions. In general, the \(g\)-commutator \([x_{i_1}^\pm (z_1), \ldots, x_{i_m}^\pm (z_m)]_g\) is also an \(\mathcal{E}_h(W)\)-linear combination of products of \(\delta\)-functions (along with their partial differentials).

The following two elementary lemmas will be used later on.
Lemma 5.10. Let \( c_1, \ldots, c_n, d_1, \ldots, d_m \) be distinct invertible elements in \( \mathbb{C}[[h]] \). Then

\[
(t_{z,w} - t_{w,z}) \left( \prod_{i=1}^{n} (z - c_iw)^{-1} \prod_{j=1}^{m} (z - d_jw)^{-2} \right)
\]

(5.14)

\[
= \sum_{i=1}^{n} \lim_{z \to c_i w} \left( \prod_{a \neq i}^{n} (z - c_a w)^{-1} \prod_{j=1}^{m} (z - d_jw)^{-2} \right) z^{-1} \delta \left( \frac{c_i w}{z} \right)
\]

+ \sum_{j=1}^{m} \lim_{z \to d_j w} \left( \prod_{i=1}^{n} (z - c_i w)^{-1} \prod_{b \neq j}^{m} (z - d_b w)^{-2} \right) \frac{1}{d_j} \partial_{dw} z^{-1} \delta \left( \frac{d_j w}{z} \right)

- \sum_{j=1}^{m} \lim_{z \to d_j w} \frac{\partial}{\partial w} \left( \prod_{i=1}^{n} (z - c_i w)^{-1} \prod_{b \neq j}^{m} (z - d_b w)^{-2} \right) \frac{1}{d_j} \partial_{dw} z^{-1} \delta \left( \frac{d_j w}{z} \right).
\]

Lemma 5.11. Let \( m, s \in \mathbb{Z}_+ \) and let \( (c_1, \ldots, c_m, (n_1), \ldots, (c_m, \ldots, (n_{m-1}), \ldots, (n_{m-1})^s) \) be pairwise distinct elements in \( (\mathbb{C}[[h]] \setminus \{0\})^s \times \mathbb{N}^s \). Then for any \( f_1(z), f_2(z), \ldots, f_m(z) \in \mathcal{E}_h(W) \), we have

\[
\sum_{i=1}^{m} f_i(z) \prod_{j=1}^{s} \left( \frac{\partial}{\partial z} \right)_{n_{ij}} z_j^{-1} \delta \left( \frac{c_{ij} z_j}{z_j} \right) = 0
\]

if and only if \( f_i(z) = 0 \) for all \( i = 1, 2, \ldots, m \).

Now we calculate the \( g \)-commutators that are related to (Q9).

Lemma 5.12. For \( W \in \mathcal{R}_l^d \) and \( i \in I \) with \( s_i = 2 \), we have

\[
[x_i^+(z_1), x_i^+(z_2), x_i^+(z_3)]_g = 0\]  

\[
\begin{align*}
\left(1 + q_i^{-d_i} \right) (1 + q_i^{+3d_i}) (1 + q_i^{+5d_i}) z_1^{-2d_i} \delta \left( \frac{-d_i}{z_3} \right) & \left( \frac{q_i^{+2d_i} z_3}{z_3^{d_i}} \right) \\
\frac{1}{\left(1 + q_i^{+3d_i}\right) (1 + q_i^{+5d_i})} z_1^{-2d_i} \delta \left( \frac{-d_i}{z_3} \right) & \left( \frac{q_i^{+2d_i} z_3}{z_3^{d_i}} \right) \\
\left(1 + q_i^{+2d_i} \right) (1 + q_i^{+4d_i}) z_1^{-2d_i} \delta \left( \frac{-d_i}{z_3} \right) & \left( \frac{q_i^{+2d_i} z_3}{z_3^{d_i}} \right) \\
\left(1 + q_i^{+4d_i} \right) (1 + q_i^{+5d_i}) z_1^{-2d_i} \delta \left( \frac{-d_i}{z_3} \right) & \left( \frac{q_i^{+2d_i} z_3}{z_3^{d_i}} \right) \\
\left(1 + q_i^{+2d_i} \right) (1 + q_i^{+4d_i}) z_1^{-2d_i} \delta \left( \frac{-d_i}{z_3} \right) & \left( \frac{q_i^{+2d_i} z_3}{z_3^{d_i}} \right) \\
\left(1 + q_i^{+4d_i} \right) (1 + q_i^{+5d_i}) z_1^{-2d_i} \delta \left( \frac{-d_i}{z_3} \right) & \left( \frac{q_i^{+2d_i} z_3}{z_3^{d_i}} \right)
\end{align*}
\]

Proof. Firstly, it follows from (5.13) and (5.14) that

\[
[x_i^+(z_2), x_i^+(z_3)]_g
\]
Then by definition we have

\[
[x^+_i(z_1), x^+_i(z_2), x^+_i(z_3)]_g = (x^+_i(z_2, z_3) - x^+_i(z_1, z_2) + f^+_i(z_1, z_2) - f^+_i(z_2, z_3) - f^+_i(z_3, z_1) - f^+_i(z_1, z_3)).
\]

Now the assertion is implied by the following two facts, which can be proved directly by using (5.14):

\[
\lim_{\substack{z_i \to q_i^\pm d_i, z_2 \to z_2 \to \infty}} \frac{1}{z_i^2} = \frac{1}{z_2} \left( \frac{z_i^2 - q_i^\pm d_i z_2}{z_2} \right) \left( \frac{z_i^2 + q_i^\pm d_i z_2}{z_2} \right) = \frac{1}{z_i^2} \left( \frac{z_i^2 - q_i^\pm d_i z_2}{z_2} \right) \left( \frac{z_i^2 + q_i^\pm d_i z_2}{z_2} \right)
\]

and

\[
\lim_{\substack{z_i \to -q_i^\pm d_i, z_2 \to z_2 \to \infty}} \frac{1}{z_i^2} = \frac{1}{z_2} \left( \frac{z_i^2 - q_i^\pm d_i z_2}{z_2} \right) \left( \frac{z_i^2 + q_i^\pm d_i z_2}{z_2} \right) = \frac{1}{z_i^2} \left( \frac{z_i^2 - q_i^\pm d_i z_2}{z_2} \right) \left( \frac{z_i^2 + q_i^\pm d_i z_2}{z_2} \right).
\]

**Proof of Proposition 5.8** Denote by \( R^\pm_{ii} \) the LHS of (5.14). Note that for \( \sigma \in S_3 \), we can write \( x^\pm_i(z^\sigma(1))x^\pm_i(z^\sigma(2))x^\pm_i(z^\sigma(3)) \in \text{End}(W)[z_1, z_2, z_3] \) as a sum of the \( g \)-commutators and the currents \( x^\pm_i(z_3)x^\pm_i(z_2)x^\pm_i(z_1) \). For example,

\[
x^\pm_i(z_1)x^\pm_i(z_2)x^\pm_i(z_3) = [x^\pm_i(z_1), x^\pm_i(z_2), x^\pm_i(z_3)]_g + \text{gi}(z_1/z_3)^{\pm1} \text{gi}(z_1/z_2)^{\pm1} [x^\pm_i(z_2), x^\pm_i(z_3)]_g x^\pm_i(z_1)
\]

\[
+ \text{gi}(z_2/z_3)^{\pm1} [x^\pm_i(z_1), x^\pm_i(z_3)]_g x^\pm_i(z_2) + \text{gi}(z_2/z_3)^{\pm1} [x^\pm_i(z_1, z_2)]_g x^\pm_i(z_3) + \text{gi}(z_1/z_3)^{\pm1} x^\pm_i(z_1) x^\pm_i(z_2) x^\pm_i(z_3).
\]

\[
[x^\pm_i(z_1), x^\pm_i(z_2)]_g + \text{gi}(z_2/z_3)^{\pm1} \text{gi}(z_1/z_2)^{\pm1} [x^\pm_i(z_2), x^\pm_i(z_3)]_g x^\pm_i(z_1)
\]

\[
+ \text{gi}(z_2/z_3)^{\pm1} [x^\pm_i(z_1), x^\pm_i(z_3)]_g x^\pm_i(z_2) + \text{gi}(z_2/z_3)^{\pm1} [x^\pm_i(z_1, z_2)]_g x^\pm_i(z_3) + \text{gi}(z_1/z_3)^{\pm1} x^\pm_i(z_1) x^\pm_i(z_2) x^\pm_i(z_3).
\]
Then it is straightforward to see that $R_{\text{iii}}^\pm$ can be rewritten as (recalling (5.12))

$$p_i^\pm(z_1, z_2, z_3)[x_{i_1}^\pm(z_1), x_{i_2}^\pm(z_2), x_{i_3}^\pm(z_3)]g + p_i^\pm(z_2, z_1, z_3)[x_{i_1}^\pm(z_2), x_{i_2}^\pm(z_1), x_{i_3}^\pm(z_3)]g$$

$$+ t_{z_3, z_2, z_1}^\pm A_{\text{iii}}^\pm(z_1, z_2, z_3) \frac{F_{\text{ii}}^\pm(z_1, z_3)}{F_{\text{ii}}^\pm(z_2, z_3)} x_{i_1}^\pm(z_3)[x_{i_2}^\pm(z_1), x_{i_3}^\pm(z_2)]g$$

$$+ t_{z_3, z_1, z_2}^\pm A_{\text{iii}}^\pm(z_1, z_3, z_2) \frac{F_{\text{ii}}^\pm(z_2, z_3)}{F_{\text{ii}}^\pm(z_1, z_3)} [x_{i_1}^\pm(z_1), x_{i_3}^\pm(z_3)]g x_{i_2}^\pm(z_2)$$

$$+ t_{z_3, z_2, z_1}^\pm A_{\text{iii}}^\pm(z_1, z_2, z_3) \frac{F_{\text{ii}}^\pm(z_1, z_3)}{F_{\text{ii}}^\pm(z_2, z_3)} [x_{i_1}^\pm(z_2), x_{i_2}^\pm(z_3)]g x_{i_3}^\pm(z_1)$$

$$+ t_{z_3, z_2, z_1}^\pm G_{\text{iii}}^\pm(z_1, z_2, z_3) A_{\text{iii}}^\pm(z_1, z_2, z_3) + F_{\text{ii}}^\pm(z_1, z_2) A_{\text{iii}}^\pm(z_2, z_1, z_3) \frac{F_{\text{ii}}^\pm(z_1, z_2)}{F_{\text{ii}}^\pm(z_2, z_3)} x_{i_1}^\pm(z_3) x_{i_2}^\pm(z_2) x_{i_3}^\pm(z_1),$$

where

$$A_{\text{iii}}^\pm(z_1, z_2, z_3) = G_{\text{iii}}^\pm(z_1, z_2, z_3) G_{\text{iii}}^\pm(z_2, z_3) p_i^\pm(z_1, z_2, z_3)$$

$$+ G_{\text{iii}}^\pm(z_1, z_3) F_{\text{ii}}^\pm(z_2, z_3) p_i^\pm(z_1, z_3, z_2) + F_{\text{ii}}^\pm(z_1, z_3) F_{\text{ii}}^\pm(z_2, z_3) p_i^\pm(z_3, z_1, z_2).$$

are as in (5.10) (noting that $F_{\text{ii}}^\pm(z, w) = -G_{\text{ii}}^\pm(w, z)$).

Recall from (4.4) that $A_{\text{iii}}^\pm(z_1, z_2, z_3) = F_{\text{ii}}^\pm(z_1, z_2) B_{\text{ii}}^\pm(z_1, z_2, z_3)$, where $B_{\text{ii}}^\pm(z_1, z_2, z_3)$ is a polynomial satisfying the symmetry: $B_{\text{ii}}^\pm(z_1, z_2, z_3) = B_{\text{ii}}^\pm(z_2, z_1, z_3)$. This implies that $G_{\text{iii}}^\pm(z_1, z_2) A_{\text{iii}}^\pm(z_1, z_2, z_3) + F_{\text{ii}}^\pm(z_1, z_2) A_{\text{iii}}^\pm(z_2, z_1, z_3) = 0$. Furthermore, since $F_{\text{ii}}^\pm(z_1, z_2)$ $[x_{i_1}^\pm(z_1), x_{i_2}^\pm(z_2)]g = 0$, we obtain

$$P_{\text{iii}}^\pm = \sum_{\sigma \in S_2} p_i^\pm(z_{\sigma(1)}, z_{\sigma(2)}, z_3)[x_{i_1}^\pm(z_{\sigma(1)}), x_{i_2}^\pm(z_{\sigma(2)}), x_{i_3}^\pm(z_3)]g.$$  

It is straightforward to check that

$$\lim_{z_1 \to q_i^{\pm d_i}} \lim_{z_2 \to q_i^{\pm d_i}} p_i^\pm(z_1, z_2, z_3) = 0,$$

$$= q_i^{\pm d_i} \frac{1 + q_i^{\pm d_i}}{1 + q_i^{\pm 3d_i}}$$

$$= \lim_{z_1 \to q_i^{\pm 2d_i}} \lim_{z_2 \to q_i^{\pm d_i}} p_i^\pm(z_1, z_2, z_3) \frac{(1 + q_i^{\pm d_i})^{\frac{q_i^{\pm 2d_i}}{q_i^{\pm d_i}}}}{(1 + q_i^{\pm 3d_i})(1 + q_i^{\pm 4d_i})} z_1^{-d_i}.$$ 

Recalling Proposition 5.4, we have

$$p_i^\pm(z_1, z_2, z_3) = p_i^\pm(z_2, z_1, z_3) = 0.$$ 

$$\partial x_{i_1}^\pm(z_1) x_{i_2}^\pm(z_2) x_{i_3}^\pm(z_3) = \partial x_{i_1}^\pm(z_2) x_{i_2}^\pm(z_1) x_{i_3}^\pm(z_3).$$
Note that by specializing and the polynomials conjecture given in Remark 3.5.

Finally, the proposition follows from the following simple fact:

\[ R_{i;i} = \sum_{\sigma \in S_{\ell}} p_{\ell}^{+}(z(\sigma(1)), z(\sigma(2)), z_{3}) \circ x_{i}^{+}(z(\sigma(1))) x_{i}^{+}(z(\sigma(2))) x_{i}^{+}(z_{3}) \circ \]

\[
(1 + q_{i}^{-d_{i}})(1 + q_{i}^{z_{3}d_{i}})(1 + q_{i}^{z_{3}d_{i}}) z^{2d_{i}} = \delta(z_{\sigma(1)}) \delta(z_{\sigma(2)}) \delta(z_{3})
\]

\[
(5.19)
\]

Thus, by using Lemma 5.11 and (5.19), we find that \( R_{i;i} \) is 0 on \( W \) if and only if

\[
\delta x_{i}^{+}(\xi_{2d_{i}q_{i}^{z+1}}z_{3}) x_{i}^{+}(q_{i}^{z+2}z_{3}) x_{i}^{+}(z_{3}) \circ = \delta x_{i}^{+}(\xi_{2d_{i}q_{i}^{z+1}}z_{3}) x_{i}^{+}(q_{i}^{z+2}z_{3}) x_{i}^{+}(z_{3}) \circ = 0.
\]

Finally, the proposition follows from the following simple fact:

\[
\delta x_{i}^{+}(\xi_{2d_{i}q_{i}^{z+1}}z_{3}) x_{i}^{+}(q_{i}^{z+2}z_{3}) x_{i}^{+}(z_{3}) \circ = \lim_{z_{3} \to q_{i}^{z+1}z} \delta x_{i}^{+}(\xi_{2d_{i}q_{i}^{z+1}}z_{3}) x_{i}^{+}(q_{i}^{z+2}z_{3}) x_{i}^{+}(z_{3}) \circ
\]

\[
= \lim_{z_{3} \to q_{i}^{z+1}z} \delta x_{i}^{+}(\xi_{2d_{i}q_{i}^{z+1}}z_{3}) x_{i}^{+}(q_{i}^{z+2}z_{3}) x_{i}^{+}(z_{3}) \circ
\]

5.3. **On affine quantum Serre relations.** In this subsection we study the affine quantum Serre relations on restricted \( U_{h}^{+}(\mathfrak{g}_{\mu})^{+} \)-modules in a general setting. As applications, we give a simple characterization of the relations (Q10) and also a partial answer of the conjecture given in Remark 3.5.

Let \((i, j) \in \Pi, m \in \mathbb{Z}_{+}, f^{\pm} = f^{\pm}(z, w) \in \mathbb{C}[\mathbb{H}][z^{d_{i}}, w^{d_{j}}] \) be homogenous and \( B = (B_{0, B_{1}, \ldots, B_{m}}) \in (\mathbb{C}[\mathbb{H}])^{m+1} \). Associated to these data, we introduce the currents

\[
D_{ij}^{\pm}(m, f^{\pm}, B) = \sum_{\sigma \in S_{m}} \left\{ \prod_{1 \leq a < b \leq m} f^{\pm}(z(\sigma(a)), z(\sigma(b))) \left( \sum_{r=0}^{m} (-1)^{r} B_{r} x_{i}^{+}(z(\sigma(1))) \cdots x_{i}^{+}(z(\sigma(r+1))) x_{j}^{+}(w) x_{i}^{+}(z(\sigma(m))) \right) \right\}
\]

and the polynomials

\[
P_{ij}^{\pm}(m, f^{\pm}, B) = \sum_{\sigma \in S_{m}} (-1)^{|\sigma|} \left( \sum_{r=0}^{m} (-1)^{r} B_{r} \prod_{1 \leq a < b \leq m} f^{\pm}(z(\sigma(a)), z(\sigma(b))) G_{i}^{\pm}(z(\sigma(a)), z(\sigma(b))) \right)
\]

\[
\times \prod_{a=0}^{r} G_{ij}^{\pm}(z(\sigma(a)), w) \prod_{b=r+1}^{m} F_{ij}^{\pm}(z(\sigma(b)), w).
\]

Note that by specializing

\[
m = m_{ij}, f^{\pm}(z, w) = p_{ij}^{\pm}(z, w) \text{ and } B_{r} = \left( \frac{m_{ij}}{q_{i}^{r}}, r = 0, \ldots, m_{ij}, \right)
\]

(5.20)
the relations $D_{ij}^+(m, f^\pm, B) = 0$ are nothing but the relations (Q10). On the other hand, by specializing
\begin{equation}
(5.21) \quad m = \hat{m}_{ij}, \ f^\pm(z, w) = 1 \text{ and } B_r = \left( \begin{array}{c}
\hat{m}_{ij} \\
q_i
\end{array} \right), \ r = 0, \ldots, \hat{m}_{ij},
\end{equation}
the relations $D_{ij}^+(m, f^\pm, B) = 0$ are precisely the relations (Q11).

The following result will be proved in Section 8.

**Theorem 5.13.** Let $W$ be a restricted $U_\hbar^\pm(\hat{g}_\mu)$-module and let $(i, j) \in I$, $m \in \mathbb{Z}_+$, $f^\pm \in \mathbb{C}[[h]][z^{d_i}, w^{d_j}]$ be homogenous, $B \in \mathbb{C}[[h]]^{m+1}$ such that the polynomial $P_{ij}^+(m, f^\pm, B) = 0$. If
\begin{equation}
(5.22) \quad \circ\, x_i^{\pm}(q_i^{a_{ij}} \xi_{d_{ij}} w)x_i^{\pm}(q_i^{a_{ij}+2} \xi_{d_{ij}} w) \cdots x_i^{\pm}(q_i^{-a_{ij}} \xi_{d_{ij}} w)x_i^{\pm}(w)_o = 0 \text{ on } W
\end{equation}
for all $0 \leq l < d_{ij}/d_i$, then $D_{ij}^+(m, f^\pm, B) = 0$ on $W$.

Conversely, if $m = m_{ij}$, $f^\pm(w, q_i^{\pm 2n} w) \neq 0$ for $n = 1, \ldots, -a_{ij}$, $n_1 \neq n_2$ and $D_{ij}^+(m, f^\pm, B) = 0$ on $W$, then (5.22) holds for all $0 \leq l < d_{ij}/d_i$.

In view of Theorem 5.13 we can prove the following main result of this section.

**Theorem 5.14.** Let $W$ be a restricted $U_\hbar^\pm(\hat{g}_\mu)$-module. Then the relations (Q9) and (Q10) hold on $W$ if and only if for all $i, j \in I$ with $a_{ij} < 0$,
\begin{equation}
(5.23) \quad \circ\, x_i^{\pm}(q_i^{a_{ij}} w)x_i^{\pm}(q_i^{a_{ij}+2} w) \cdots x_i^{\pm}(q_i^{-a_{ij}+2} w)x_i^{\pm}(w)_o = 0 \text{ on } W.
\end{equation}

**Proof.** Note that for $i, j \in I$ with $a_{ij} < 0$ and $i \in \mathcal{O}(j)$, it follows from (5.6) that (5.10) is equivalent to the relations $\circ\, x_i^{\pm}(q_i^{-2} w)x_i^{\pm}(q_i w)x_i^{\pm}(w)_o = 0$. In view of Proposition 5.8, it remains to prove that for $(i, j) \in I$, the relations (Q10) are equivalent to (5.22) on $W$. Notice that now we may view $W$ as a restricted $U_\hbar^\pm(\hat{g}_\mu)$-module. Take the triple $(m, f^\pm, B)$ as in (5.20) so that the relations $D_{ij}^+(m, f^\pm, B) = 0$ are exactly the relations (Q10). In this case, it was proved in [44] (see also [37]) that $P_{ij}^+(m, f^\pm, B) = 0$. Furthermore, one can easily check that $p_{ij}^+(w, q_i^{\pm 2n} w) \neq 0$ for $n = 1, \ldots, -a_{ij}$. Then the assertion follows from Theorem 5.14.

Combining Theorem 5.13 with Theorem 5.14 one immediately gets

**Corollary 5.15.** Let $W \in \mathcal{R}_+$, $(i, j) \in I$, $m \in \mathbb{Z}_+$, $f^\pm \in \mathbb{C}[[h]][z^{d_i}, w^{d_j}]$ be homogenous and $B \in \mathbb{C}[[h]]^{m+1}$ if $P_{ij}^+(m, f^\pm, B) = 0$, then $D_{ij}^+(m, f^\pm, B) = 0$ on $W$.

In particular, we have

**Corollary 5.16.** Assume that $\mathfrak{g}$ is of finite or affine type. Then the relations (Q11) hold on any restricted $U_\hbar(\hat{g}_\mu)$-module.

**Proof.** Take $(m, f^\pm, B)$ as in (5.21) so that the relations (Q11) are just the relations $D_{ij}^+(m, f^\pm, B) = 0$. In view of Corollary 5.15 it suffices to check that $p_{ij}^+(\hat{m}_{ij}, 1, B) = 0$, which can be checked directly case by case (by using Maple for example). In fact, the polynomial $P_{ij}^+(\hat{m}_{ij}, 1, B)$ depends only on the positive integers $\hat{m}_{ij}, d_{ij}/d_i$ and $s_i$. Furthermore, when $\mathfrak{g}$ is of finite type or affine type, one has $\hat{m}_{ij} \leq 5$, $d_{ij}/d_i \leq 4$ and $s_i \leq 2$.

**Remark 5.17.** We conjecture that $P_{ij}^+(m, f^\pm, B) = 0$ for any triple $(m, f^\pm, B)$ as in (5.21), which depends on the positive integers $\hat{m}_{ij}, d_{ij}/d_i$ and $s_i$. When $\mu = \text{Id}$ (and so $d_{ij}/d_i = 1 = s_i$), this combinatorial identity was discovered in [44].
6. Hopf algebra structure

In this section, we define a Hopf algebra structure on the restricted completion \( \tilde{U}_h (\hat{g}_\mu) \) of \( \hat{U}_h (\hat{g}_\mu) \).

6.1. Restricted \( \mathcal{U}_h (\hat{g}_\mu) \)-modules. In this subsection, we show that the category \( \mathcal{R} \) of restricted \( \mathcal{U}_h (\hat{g}_\mu) \)-modules is a monoidal category.

Let \( U \) and \( V \) be two topologically free \( \mathbb{C}[h] \)-modules. Recall from [6.1] that \( \mathcal{E}_h (U) \otimes \mathcal{E}_h (V) \) is the limit of the inverse system

\[
0 \leftarrow \mathcal{E}(U/hU) \otimes \mathcal{E}(V/hV) \leftarrow \mathcal{E}(U/h^2U) \otimes \mathcal{E}(V/h^2V) \leftarrow \cdots,
\]

noting that \( \mathcal{E}(U/h^nU) \otimes \mathcal{E}(V/h^nV) \cong \mathcal{E}(V/h^n) \otimes \mathcal{E}(U/h^n) \). On the other hand, \( \mathcal{E}_h (U \hat{\otimes} V) \) is the limit of the inverse system

\[
0 \leftarrow \mathcal{E}(U/hU) \otimes \mathcal{E}(V/hV) \leftarrow \mathcal{E}(U/h^2U) \otimes \mathcal{E}(V/h^2V) \leftarrow \cdots,
\]

noting that \( \mathcal{E}(U/h^nU) \otimes \mathcal{E}(V/h^nV) \cong \mathcal{E}(U \otimes V)/h^n(U \otimes V) \cong \mathcal{E}((U \hat{\otimes} V)/h^n(U \hat{\otimes} V)) \). In view of these two inverse limits, we have a \( \mathbb{C}[h] \)-map

\[
(6.1) \quad \theta_{U,V} = \lim_{n \in \mathbb{Z}_+} \theta_{U/h^nU,V/h^nV} : \mathcal{E}_h (U) \hat{\otimes} \mathcal{E}_h (V) \rightarrow \mathcal{E}_h (U \hat{\otimes} V),
\]

where for two vector \( \mathbb{C} \)-spaces \( U^0 \) and \( V^0 \), \( \theta_{U^0,V^0} \) denotes the \( \mathbb{C} \)-map from \( \mathcal{E}(U^0) \otimes \mathcal{E}(V^0) \) to \( \mathcal{E}(U^0 \otimes V^0) \) defined by

\[
\sum_{m,n \in \mathbb{Z}} a_m z^{-m} \otimes b_n z^{-n} \mapsto ((u \otimes v) \mapsto \sum_{m,n \in \mathbb{Z}} a_m (u) \otimes b_{m-n} (v)) z^{-m}.
\]

Without confusion, for \( a(z) \in \mathcal{E}_h (U) \) and \( b(z) \in \mathcal{E}_h (V) \), we shall denote \( \theta_{U,V}(a(z) \hat{\otimes} b(z)) \) by \( a(z) \otimes b(z) \) for simplicity in this section.

Note that for \( W \in \mathcal{R} \) and \( i \in I \), one has \( x_i^+(z), \phi_i^+(z) \in \mathcal{E}_h (W) \). As an application of Theorem [6.14] we have

**Proposition 6.1.** Let \( U \) and \( V \) be two restricted \( \mathcal{U}_h (\hat{g}_\mu) \)-modules. Then there is a restricted \( \mathcal{U}_h (\hat{g}_\mu) \)-module structure on the \( h \)-adically completed tensor product space \( U \hat{\otimes} V \) with the action \( \Delta \) defined by (i.e., \( h \in \mathcal{R} \)): \n
\[
\begin{align*}
(\text{Co1}) \quad & \Delta(x_i^+(z)) = x_i^+(z) \otimes 1 + \phi_i^-(z q^2) \otimes x_i^+(z q^2), \\
(\text{Co2}) \quad & \Delta(x_i^-(z)) = 1 \otimes x_i^-(z) + x_i^-(z q^2) \otimes \phi_i^+(z q^2), \\
(\text{Co3}) \quad & \Delta(\phi_i^+(z)) = \phi_i^+(z q^2) \otimes \phi_i^+(z q^2), \\
(\text{Co4}) \quad & \Delta(h) = h \otimes 1 + 1 \otimes h, \quad \Delta(c) = c_1 + c_2, \quad \Delta(d) = d \otimes 1 + 1 \otimes d,
\end{align*}
\]

where \( c_1 = c \otimes 1 \) and \( c_2 = 1 \otimes c \).

**Proof.** We need to show that the action \( \Delta \) is compatible with the relations (Q0-Q10). The compatibility with the relations (Q0-Q8) can be proved in a similar way to that in [14]. Then \( U \hat{\otimes} V \) is naturally a restricted \( \mathcal{U}_h (\hat{g}_\mu) \)-module. Due to Theorem [5.14] it suffices to check the compatibility between \( \Delta \) and the relations (5.23).

Let \( i, j \in I \) with \( a_{ij} < 0 \). Set \( m = 1 - a_{ij} \) and \( \overline{m} = \{ 1, 2, \ldots, m \} \). Then we have

\[
\Delta\left( \cdot \otimes x_i^+(z_1) \cdots x_i^+(z_m) x_j^+(w) \right) = \Delta \left( \prod_{1 \leq r < s \leq m} f_{ii}(z_s, z_r) \prod_{1 \leq r \leq m} f_{ij}^+(z_r, w) x_i^+(z_1) \cdots x_i^+(z_m) x_j^+(w) \right).
\]
\[
\begin{align*}
&= \sum_{J \subset \mathcal{m}} \ell_{z_1, \ldots, z_m, w} \prod_{1 \leq r < s \leq m} f_{i_{ii}}^+(z_s, z_r) \prod_{1 \leq r \leq m} f_{i_{ij}}^+(z_r, w) \xi(z_1) \cdots \xi(z_m) x_j^+(w) \\
&\quad \otimes x_i^+(z_j, q^{i^1}) \cdots x_i^+(z_{j-1}, q^{i^{j-2}}) \\
&+ \sum_{J \subset \mathcal{m}} \ell_{z_1, \ldots, z_m, w} \prod_{1 \leq r < s \leq m} f_{i_{ii}}^+(z_s, z_r) \prod_{1 \leq r \leq m} f_{i_{ij}}^+(z_r, w) \xi(z_1) \cdots \xi(z_m) \phi_j^-(w q^{\frac{c}{2}}) \\
&\quad \otimes x_i^+(z_j, q^{c^1}) \cdots x_i^+(z_{j-1}, q^{c^{j-2}}) x_j^+(w q^{c_j}),
\end{align*}
\]

where \(\xi(z_a) = \phi_i^+(z_a q^{\frac{c}{2}})\) if \(a \in J\), \(\xi(z_a) = x_i^+(z_a)\) if \(a \notin J\) and \(J = \{j_1, \ldots, j_r\}\) with \(j_1 < \cdots < j_r\). Note that for \(k, l \in I\), we have \(\phi_k^-(z q^{\frac{c}{2}}) x_j^+(w) \otimes 1 = \mathcal{C}_h^{(2)}(U \otimes V)\). This together with the relations (Q6) gives

\[
\ell_{w, z} f_{ik}^-(z, w) \phi_k^-(z q^{\frac{c}{2}}) x_i^+(w) \otimes 1 = C_{ki} \ell_{w, z} f_{ik}^+(w, z) x_i^+(w) \phi_k^-(z q^{\frac{c}{2}}) \otimes 1.
\]

In view of this and Proposition 5.6, we can move those \(\phi_j^-(z_j q^{\frac{c}{2}})\) to the left so that

\[
\Delta \left( \otimes x_i^+(z_1) x_i^+(z_2) \cdots x_i^+(z_m) x_j^+(w) \otimes 1 \right)
\]

\[
= \sum_{J \subset \mathcal{m}} \ell_{z_1, \ldots, z_m, w} \left( \prod_{a \leq r < b} f_{i_{ii}}^+(z_{r+1}, z_{r+2}) \prod_{r \leq a \leq r} f_{i_{ij}}^+(z_a, w) \right) q^{r(1-r)d_i c_i / 2} \times \phi_i^-(z_1, q^{c_1}) \cdots \phi_i^-(z_{j-1}, q^{c_{j-2}}) \phi_j^-(z_j, q^{c_{j-1}}) x_i^+(z_{j+1}) \cdots x_i^+(z_m) x_j^+(w) \otimes 1 \\
+ \sum_{J \subset \mathcal{m}} (-1)^{m-r} \ell_{z_1, \ldots, z_m, w} \left( \prod_{a \leq r < b} f_{i_{ii}}^+(z_{r+1}, z_{r+2}) \prod_{b=r+1}^m f_{i_{ij}}^+(w, z_{j+1}) \right) q^{r(1-r)d_i c_i / 2} \times \phi_i^-(z_1, q^{c_1}) \cdots \phi_i^-(z_{j-1}, q^{c_{j-2}}) \phi_j^-(z_j, q^{c_{j-1}}) x_i^+(z_{j+1}) \cdots x_i^+(z_m) \otimes 1,
\]

noting that \(C_{ii} = 1\) and \(C_{ij} = C_{ji} = -1\), where \(r = |J|\) and \(j_1, \ldots, j_m\) are distinct numbers in \(\mathcal{m}\) such that

\[J = \{j_1, \ldots, j_r\}, \quad j_1 < \cdots < j_r \quad \text{and} \quad j_{r+1} < \cdots < j_m.\]

Now by taking \(z_a = q_i^{a_i^1 + (a-1)^2} w\) for \(a = 1, \ldots, m\), we obtain

\[
\Delta \left( \otimes x_i^+(q_i^{a_i^1} w) x_i^+(q_i^{a_i^2 + 2} w) \cdots x_i^+(q_i^{a_i^m} w) x_j^+(w) \otimes 1 \right)
\]

\[
= \otimes x_i^+(q_i^{a_i^1} w) x_i^+(q_i^{a_i^2 + 2} w) \cdots x_i^+(q_i^{a_i^m} w) x_j^+(w) \otimes 1 \\
+ \phi_i^-(q_i^{a_i^1} w q^{\frac{c_i}{2}}) \phi_i^-(q_i^{a_i^2 + 2} w q^{c_2}) \cdots \phi_i^-(q_i^{a_i^m} w q^{c_m}) \phi_j^-(w q^{\frac{c}{2}}) q^m \prod_{i \neq j} (q_i^{a_i^1} w q^{c_1}) \cdots x_i^+(q_i^{a_i^m} w q^{c_m}) x_j^+(w q^{c_j}) \otimes 1.
\]

This implies that \(\Delta\) is compatible with the relation \(5.23\), as desired. \(\square\)

It is obvious that the canonical \(\mathbb{C}[[h]]\)-isomorphism

\[
a : (U \otimes V) \otimes W \cong U \otimes (V \otimes W)
\]

30
is a $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$-module isomorphism. Furthermore, we endow $\mathbb{C}[[\hbar]]$ with a $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$-module structure given by $(i \in I, h \in \hat{\mathfrak{h}})\nabla^I \epsilon(x_i^H(z)) = 0 = \epsilon(h) = \epsilon(c) = \epsilon(d), \quad \epsilon(\hat{\phi}_i^\pm(z)) = 1.$

Then $\mathbb{C}[[\hbar]] \in \mathcal{R}$ and for every $U \in \mathcal{R}$, the natural $\mathbb{C}[[\hbar]]$-isomorphisms

$l : \mathbb{C}[[\hbar]] \otimes U \cong U \quad \text{and} \quad r : U \otimes \mathbb{C}[[\hbar]] \cong U$

are $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$-module isomorphisms. In view of Proposition 6.1 we have the following straightforward result.

**Theorem 6.2.** The category $\mathcal{R}$ of restricted $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$-modules, together with the completed tensor product $\otimes$, the trivial module $\mathbb{C}[[\hbar]]$, the associativity constraint $a$, the left constraint $l$ and the right constraint $r$ form a monoidal category.

### 6.2. Hopf structure on $\tilde{\mathcal{U}}_h(\hat{\mathfrak{g}}_\mu)$

In this subsection we define a topological Hopf algebra structure on a completion of $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$.

Let $\mathcal{F} : \mathcal{R} \to \mathcal{M}_f$ be the forgetful functor, and let $\mathcal{A} = \text{End}_{\mathbb{C}[[\hbar]]}(\mathcal{F})$ be the algebra of endomorphisms of the functor $\mathcal{F}$. For each $W \in \mathcal{R}$, $\text{End}_{\mathbb{C}[[\hbar]]}(W)$ is a topological algebra over $\mathbb{C}[[\hbar]]$ such that

$$\{(K : h^nW) \mid K \subset W, |K| < \infty, n \in \mathbb{Z}_+\}$$

form a local basis at 0, where

$$(K : h^nW) = \{\varphi \in \text{End}_{\mathbb{C}[[\hbar]]}(W) \mid \varphi(K) \subset h^nW\}.$$  

Equip $\mathcal{A}$ with the weakest topology such that for any $W \in \mathcal{R}$, the canonical $\mathbb{C}[[\hbar]]$-algebra epimorphism $\pi_W$ from $\mathcal{A}$ to $\text{End}_{\mathbb{C}[[\hbar]]}(W)$ is continuous. Note that as a topological algebra over $\mathbb{C}[[\hbar]]$, $\text{End}_{\mathbb{C}[[\hbar]]}(W)$ (and so is $\mathcal{A}$) is complete and separated. This particular shows that $\text{End}_{\mathbb{C}[[\hbar]]}(W)$ and $\mathcal{A}$ are topologically free.

Recall from Theorem 6.2 that $\mathcal{R}$ is a monoidal category. For each $U, V \in \mathcal{R}$, let $J_{U,V} : \mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \otimes V)$ be the canonical $\mathbb{C}[[\hbar]]$-isomorphism. Then $\{J_{U,V} \mid U, V \in \mathcal{R}\}$ defines a tensor structure on the functor $\mathcal{F}$ [22, Section 2.4]. Let $\mathcal{F}^2 : \mathcal{R} \times \mathcal{R} \to \mathcal{M}_f$ be the bifunctor defined by $\mathcal{F}^2(U, V) = \mathcal{F}(U) \otimes \mathcal{F}(V)$, and set $\mathcal{A}^2 = \text{End}_{\mathbb{C}[[\hbar]]}(\mathcal{F}^2)$. Similar to $\mathcal{A}$, we endow $\mathcal{A}^2$ a weakest topological structure so that it is a complete separated topological $\mathbb{C}[[\hbar]]$-algebra. Following [22, Section 9.1], $\mathcal{A}$ admits a natural “coproduct” $\Delta : \mathcal{A} \to \mathcal{A}^2$ defined by $\Delta(a)(U,V)(u \otimes v) = J_{U,V}^{1} a_{U \otimes V} J_{U,V}(u \otimes v), a \in \mathcal{A}, u, v \in \mathcal{F}(V)$, where $a_{U \otimes V}$ denotes the action of $a$ on $\mathcal{F}(U)$. We also define the counit on $\mathcal{A}$ by $\epsilon(a) = a_{\mathbb{C}[[\hbar]]}(1) \in \mathbb{C}[[\hbar]]$, where $\mathbb{C}[[\hbar]]$ is the trivial $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$-module in $\mathcal{R}$.

Notice that each $a \in \mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ naturally defines an endomorphism of $\mathcal{F}$, so we have a canonical $\mathbb{C}[[\hbar]]$-algebra homomorphism $\psi$ from $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ to $\mathcal{A}$.

**Definition 6.3.** We define the restricted completion $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ of $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ to be the closure of $\psi(\mathcal{U}_h(\hat{\mathfrak{g}}_\mu))$ in $\mathcal{A}$.

It is straightforward to see that $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ is complete, separated and topologically free. Furthermore, the following standard result is clear.

**Proposition 6.4.** The continuous $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$-modules in $\mathcal{M}_f$ are exactly restricted $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$-modules.
Denote by $\tilde{O}$ the set of open left ideals of $\tilde{U}_h(\hat{g}_\mu)$. Set $\tilde{O}_1 = \tilde{O}$ and 
$$\tilde{O}_{m+1} = \left\{ L_1 \otimes \tilde{U}_h(\hat{g}_\mu) \otimes^m + \tilde{U}_h(\hat{g}_\mu) \otimes L_m \mid L_1 \in \tilde{O}, L_m \in \tilde{O}_m \right\}, \quad m \in \mathbb{Z}_+.$$ 

It is easy to verify that $\tilde{U}_h(\hat{g}_\mu) \otimes^m$ is a topological $\mathbb{C}[[\hbar]]$-algebra with $\tilde{O}_m$ a local basis at 0. For $m \in \mathbb{Z}_+$, set

$$\lim_{L \in \tilde{O}_m} \tilde{U}_h(\hat{g}_\mu) \otimes^m / L.$$ 

One notices that $\tilde{U}_h(\hat{g}_\mu) \otimes^2$ is a closed subalgebra of $\mathcal{A}^2$.

For each element $a$ in $[\mathbb{R}]$, we still denote its image in $\tilde{U}_h(\hat{g}_\mu)$ by $a$. From Theorem 6.5, we have that $\Delta$ satisfies the conditions (Co1-Co4) and $\epsilon$ satisfies the condition (CoU). Then we have that $\Delta(\tilde{U}_h(\hat{g}_\mu)) \subset \tilde{U}_h(\hat{g}_\mu) \otimes^2$. Following [22], Proposition 9.1, one immediately gets the following result.

**Proposition 6.5.** $(\tilde{U}_h(\hat{g}_\mu), \Delta, \epsilon)$ is a topological bialgebra over $\mathbb{C}[[\hbar]]$.

By a similar argument of [19], Theorem 2.1, one can verify that there is a continuous anti-homomorphism $S : \tilde{U}_h(\hat{g}_\mu) \rightarrow \tilde{U}_h(\hat{g}_\mu)$ determined by $(i \in I, h \in \mathbb{R})$

$$S(c) = -c, \quad S(d) = -d, \quad S(h) = -h, \quad S(x_i^+(z)) = -\phi_i^-(zq^{-2})^{-1}x_i^+(zq^{-c}),$$

$$S(x_i^-(z)) = -x_i^-(zq^{-c})\phi_i^+(zq^{-2})^{-1}, \quad S(\phi_i^+(z)) = \phi_i^+(z)^{-1}.$$

In summary, we have obtained the following main result of this section:

**Theorem 6.6.** $(\tilde{U}_h(\hat{g}_\mu), \Delta, \epsilon, S)$ is a topological Hopf algebra over $\mathbb{C}[[\hbar]]$.

7. **Specialization and Quantization of EALAs**

In this section we study the classical limit of $U_h(\hat{g}_\mu)$ and establish its connection with the quantization theory of extended affine Lie algebras.

7.1. **Specialization.** By specializing $\hbar = 0$ in the definition of $U_h(\hat{g}_\mu)$, we have:

**Definition 7.1.** Define $\hat{g}_\mu$ to be the Lie algebra over $\mathbb{C}$ generated by the set

$$\left\{ L_h, H_{i,m}, X_{i,m}^\pm, C, D \mid h \in \mathbb{R}, i \in I, m \in \mathbb{Z} \right\}$$

and subject to the following relations $(h, h' \in \mathbb{R}, i, j \in I, m, n \in \mathbb{Z})$

(L0) $H_{\mu(i),n} = \xi^n H_{i,n}, \quad L_{ri} \sum_{k \in \mathbb{Z}} a_{\mu(i)}^k = H_{i,0},$

(L1) $[D, L_h] = 0, \quad [D, H_{i,m}] = m H_{i,m}^\pm, \quad [L_h, H_{i,m}] = 0 = [L_h, L_{h'}],$

(L2) $H_{i,m}, H_{j,n} = \sum_{k \in \mathbb{Z}} r_i a_{\mu(j)}(j) \delta_{m+n,0} m \xi^C, \quad C$ is central,

(L3) $[D, X_{i,m}^\pm] = m X_{i,m}^\pm, \quad [L_h, X_{j,n}^\pm] = \pm \alpha_j(h) X_{j,n}^\pm,$

(L4) $H_{i,m}, X_{j,n}^\pm = \sum_{k \in \mathbb{Z}} r_j a_{\mu(k)}(j) X_{j+m+n}^\pm \xi^m,$

(L5) $X_{i,m}^+, X_{j,n}^- = \sum_{k \in \mathbb{Z}} \delta_{i,j} \left( \frac{H_{j,m+n}}{r_j} + \frac{m}{r_j} \delta_{m+n,0} C \right) \xi^m.$
(L6) $X_{\mu,n}^\pm = \xi^n X_{\mu,n}^\pm; \quad F_{ij}(z,w)[X_i^\pm(z),X_j^\pm(w)] = 0,$

(L7) $\sum_{\sigma \in S_2} p_i(z_\sigma(1),z_\sigma(2),z_3) [X_i^\pm(z_\sigma(1)),[X_i^\pm(z_\sigma(2)),X_i^\pm(z_3)]] = 0,$ if $s_i = 2,$

(L8) $\prod_{1 \leq s < t \leq m_{ij}} p_{ij}(z_s,z_t) [X_i^\pm(z_1), \ldots, [X_i^\pm(z_{m_{ij}}),X_j^\pm(w)]] = 0,$ if $a_{ij} < 0,$

where $X_i^\pm(z) = \sum_{m \in Z} X_{i,m}^\pm z^{-m}$ and

$$F_{ij}(z,w) = F_{ij}(z,w)|_{h=0} = \prod_{k \in \Gamma_{ij}} (z - \xi^k w),$$

$$p_{ij}(z,w) = p_{ij}(z,w)|_{h=0} = (z^{d_i} + w^{d_j}) s_i - 12 z^{d_i}w^{d_j} - z^{d_i} w^{d_j},$$

$$p_i(z_1, z_2, z_3) = p_i(z_1, z_2, z_3)|_{h=0} = z_i^{d_i} - 2z_2^{d_i} + 2z_3^{d_i}.$$  

By definition, one immediately gets the following result.

**Proposition 7.2.** The classical limit $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)/h\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ of $\mathcal{U}_h(\hat{\mathfrak{g}}_\mu)$ is isomorphic to the universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}}_\mu)$ of $\hat{\mathfrak{g}}_\mu.$

Let $\hat{\mathfrak{g}}_\mu^+$ (resp. $\hat{\mathfrak{g}}_\mu^-$; resp. $\hat{\mathfrak{h}}_\mu$) be the subalgebra of $\hat{\mathfrak{g}}_\mu$ generated by the elements $X_{i,m}^+$ (resp. $X_{i,m}^-$; resp. $H_{i,m}, L_h, C, D$). By specializing $h = 0$ in the proof of Theorem 1.2 one also obtains the following result.

**Proposition 7.3.** $\hat{\mathfrak{g}}_\mu = \hat{\mathfrak{g}}_\mu^+ \oplus \hat{\mathfrak{h}}_\mu \oplus \hat{\mathfrak{g}}_\mu^-$ (resp. $\hat{\mathfrak{g}}_\mu^-; \hat{\mathfrak{h}}_\mu$ is the subalgebra of $\hat{\mathfrak{g}}_\mu$ abstractly generated by $X_{i,m}^+$ (resp. $X_{i,m}^-$; resp. $H_{i,m}, L_h, C, D$) subject to the relations (L5)-(L7) with “+” (resp. (L5)-(L7) with “−”; resp. (L0)-(L2)).

Let $\hat{\mathfrak{g}}'_\mu$ be the subalgebra of $\hat{\mathfrak{g}}_\mu$ generated by the elements $X_{i,m}^\pm, H_{i,m}, C.$ Then it follows from Proposition 7.3 that $\hat{\mathfrak{g}}'_\mu$ is abstractly generated by these elements with relations (L0),(L2) and (L4)-(L8). Furthermore, we have

$$\hat{\mathfrak{g}}_\mu = \hat{\mathfrak{g}}'_\mu \oplus \sum_{h \in h'/c} \mathbb{C} L_h \oplus \mathbb{C} D.$$  

Let Aff($\mathfrak{g}$) = $(\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C} c \oplus \mathbb{C} d$ be the affinization of $\mathfrak{g}$ [17], where

$$[t^m \otimes x + a_1 c + b_1 d, t^n \otimes y + a_2 c + b_2 d] = t^{m+n} \otimes [x,y] + (t^{n-1} | y) \delta_{m+n,0} c + b_1 n(t^n \otimes y) - b_2 m(t^m \otimes x),$$

for $x \in \mathfrak{g}, m \in \mathbb{Z},$ and $a_1, b_1, a_2, b_2 \in \mathbb{C}.$ We extend the bilinear form on $\mathfrak{g}$ to Aff($\mathfrak{g}$) by decreeing that

$$(t^m \otimes x + a_1 c + b_1 d | t^n \otimes y + a_2 c + b_2 d) = \delta_{m+n,0} (x | y) + a_1 b_2 + b_1 a_2.$$  

Recall from [17] that any finite order automorphism $\sigma$ of $\mathfrak{g}$ can be extended to an automorphism, say $\hat{\sigma},$ of Aff($\mathfrak{g}$) with

$$\hat{\sigma}(t^m \otimes x) = \xi_m^\sigma (t^m \otimes \mu(x)); \quad \hat{\sigma}(c) = c \quad \text{and} \quad \hat{\sigma}(d) = d,$$

for $x \in \mathfrak{g}$ and $m \in \mathbb{Z},$ where $M$ is the order of $\sigma.$ Denote by Aff($\mathfrak{g}, \sigma$) the subalgebra of Aff($\mathfrak{g}$) fixed by $\hat{\sigma}.$ Let $\mathfrak{b}$ be a $\sigma$-invariant subspace of $\mathfrak{g}.$ For $x \in \mathfrak{b}$ and $m \in \mathbb{Z},$ set

$$x_{(m)} = \sum_{\mathfrak{p} \in \mathbb{Z}_M} \xi_{\mathfrak{p}M}^\sigma \mu^\mathfrak{p}(x) \quad \text{and} \quad b_{(m)} = \text{Span}_\mathbb{C} \{x_{(m)} | x \in \mathfrak{b}\}.$$  

33
Proposition 7.4. The assignment \((\text{subalgebras by})\)

Note that both \(\hat{k}\) the relations (L6) hold for \(\{1\}\) if both

This claim together with (7.4) gives that \([\sum\limits_{m \in \mathbb{Z}} t^m \otimes g_{(m)}] \oplus \mathbb{C}c \oplus \mathbb{C}d\).

We have:

\textbf{Proposition 7.4.} The assignment \((h \in \mathfrak{h}, i \in I \text{ and } m \in \mathbb{Z})\)

\begin{equation}
L_h \mapsto h_{(0)}, \quad H_{i,m} \mapsto t^m \otimes r_i \alpha^\vee_{i(m)}, \quad X_{i,m}^\pm \mapsto t^m \otimes e^\pm_{i(m)}, \quad C \mapsto \frac{c}{N}, \quad D \mapsto d,
\end{equation}

defines a surjective Lie homomorphism from \(\hat{\mathfrak{g}}_\mu\) to \(\hat{\mathfrak{L}}(\mathfrak{g}, \mu)\).

\textbf{Proof.} Note that \(\hat{\mathfrak{L}}(\mathfrak{g}, \mu)\) is generated by the elements \(h_{(0)}, t^m \otimes r_i \alpha^\vee_{i(m)}, t^m \otimes e^\pm_{i(m)}, c, d,\)

where \(h \in \mathfrak{h}, i \in I, m \in \mathbb{Z} \). Thus it suffices to verify that, under the correspondence \((7.3)\)

defines the generating functions \(x(z) = \sum_{n \in \mathbb{Z}} (t^m \otimes x(n))z^{-n}\) for \(x \in \mathfrak{g}\). And, for \(i, j \in I, m \in \mathbb{Z}_+\) and \(k = (k_1, \ldots, k_m) \in (\mathbb{Z}_N)^m\),

\begin{align*}
\epsilon_{ij}^+(k) &= [\epsilon_{\mu^k_1(i)}^+, \epsilon_{\mu^k_2(i)}^+, \cdots, \epsilon_{\mu^k_m(i)}^+], \\
\alpha_{ij}(k) &= \alpha_{\mu^k_1(i)} + \alpha_{\mu^k_2(i)} + \cdots + \alpha_{\mu^k_m(i)} + \alpha_j.
\end{align*}

Note that \(\epsilon_{ij}^+(k) \neq 0\) only if \(\alpha_{ij}(k) \in \Delta\), the root system of \(\mathfrak{g}\). Furthermore, it is straightforward to see that

\begin{equation}
\sum_{k=(k_1, \ldots, k_m) \in (\mathbb{Z}_N)^m} e_{ij}^+(k)(w) \delta \left(\frac{\xi^{-k_1 w}}{z_1}\right) \delta \left(\frac{\xi^{-k_2 w}}{z_2}\right) \cdots \delta \left(\frac{\xi^{-k_m w}}{z_m}\right).
\end{equation}

Note that the relations (L6) follow from \((7.3)\). We now prove the relations (L7). So, assume that \(i = j \in I, s_i = 2\) and \(m = 2\). In this case we claim that

\begin{equation}
\alpha_{ii}(k) \notin \Delta, \quad \forall \ k = (k_1, k_2) \in (\mathbb{Z}_N)^2.
\end{equation}

This claim together with \((7.4)\) gives that \([\epsilon_i^+(z_1), [\epsilon_i^+(z_2), \epsilon_i^+(w)]] = 0\) and in particular the relations (L6) hold for \(\epsilon_i^+(z)\). We divide the proof of the claim \((7.5)\) into three cases: (1) if both \(k_1\) and \(k_2\) can be divided by \(N_i/2\), then it follows from Lemma \((2.1)\) (ii) that \(\{\alpha_{\mu^k_1(i)}, \alpha_{\mu^k_2(i)}, \alpha_i\}\) form a base for the root system of type \(A_1\) or \(A_2\). This implies that \(\alpha_{ii}(k) \notin \Delta\) as any root in such a root system has height \(< 3\); (2) if exactly one of \(k_1, k_2\), say \(k_1\), can be divided by \(N_i/2\), then \(\alpha_{\mu^k_2(i)}\) is orthogonal to \(\alpha_{\mu^k_1(i)}, \alpha_i\) and so \(\alpha_{ii}(k) \notin \Delta\);
(3) if neither \( k_1 \) nor \( k_2 \) can be divided by \( N_i/2 \), then \( \alpha_i \) is orthogonal to \( \alpha_{\mu k_1(i)}, \alpha_{\mu k_2(i)} \) and so \( \alpha_{ij}(k) \notin \Delta \), as desired.

Next we prove the relations (L8). Let \( i, j \in I \) with \( \bar{a}_{ij} < 0 \), \( m = 1 - m_{ij} \) and \( k = (k_1, \ldots, k_m) \in (\mathbb{Z}_N)^m \). In view of (7.3), it suffices to prove the following assertion:

(a) if \( \alpha_{ij}(k) \in \Delta \), then \( \xi^{k_s z_s - \xi^{k_t z_t}} \) divides \( p_{ij}(z_s, z_t) \) for some \( 1 \leq s < t \leq m \).

So let us assume that \( \alpha_{ij}(k) \notin \Delta \). We first prove the following assertion:

(b) \( \xi^{k_s z_s - \xi^{k_t z_t}} \) does not divide \( z_s^{d_{ij}} - z_t^{d_{ij}} \) for some \( 1 \leq s < t \leq m \).

Otherwise, it follows that \( N_i \) divides \( k_s - k_t \) for all \( s, t \). Then \( \mu k_1(i) = \cdots = \mu k_m(s) = i' \) for some \( i' \in O(i) \). In view of (LC2), we see either \( a_{i'j} = 0 \) or \( a_{i'j} = m_{ij} \) and in each case \( \alpha_{ij}(k) = (1 - m_{ij})\alpha_i + \alpha_j \notin \Delta \), as required.

Suppose now that \( k_r \in \Gamma_{ij} \) for all \( r = 1, \ldots, m \). In view of the linking condition (LC2), this implies that \( N/d_{ij} \) divides \( k_s - k_t \) for all \( s, t = 1, \ldots, m \). So we have \( \xi^{k_s z_s - \xi^{k_t z_t}} \) divide \( z_s^{d_{ij}} - z_t^{d_{ij}} \) for all \( s, t = 1, \ldots, m \). This together with the assertion (b) gives the following assertion:

(c) if \( k_r \in \Gamma_{ij} \) for all \( r = 1, \ldots, m \), then \( \xi^{k_s z_s - \xi^{k_t z_t}} \) divides \( z_s^{d_{ij}} - z_t^{d_{ij}} \) for some \( 1 \leq s < t \leq m \).

On the other hand, assume that \( k_{s_0} \notin \Gamma_{ij} \) for some \( s_0 = 1, \ldots, m \). Recall that \( k_{s_0} \notin \Gamma_{ij} \) implies that \( a_{\mu k_{s_0}(i)j} = 0 \). For the case \( s_i = 1 \), it follows from Lemma 2.1 (i) that \( a_{\mu k_{s_0}(i)j} = 0 \) for any \( i' \neq \mu k_{s_0}(i) \in O(i) \). Thus, \( a_{\mu k_{s_0}(i)} \) is orthogonal to all the elements in the set

\[
\{ \alpha_{\mu k_{s_0}(i)}(k) : 1 \leq s \leq m \} \setminus \{ \alpha_{\mu k_{s_0}(i)} \},
\]

which gives \( \alpha_{ij}(k) \notin \Delta \), a contradiction. And for the case \( s_i = 2 \), if \( k_{s_0} - k_t \neq \frac{N_i}{2} \) (mod \( N_i \)) for all \( t = 1, \ldots, m \), then by Lemma 2.1 (ii), \( a_{\mu k_{s_0}(i)} \) is orthogonal to all the elements in (7.6) and hence \( \alpha_{ij}(k) \notin \Delta \), a contradiction too. In summary, we obtain that

(d) if \( k_{s_0} \notin \Gamma_{ij} \) for some \( s_0 = 1, \ldots, m \), then \( s_i = 2 \) and there exists a \( t_0 = 1, \ldots, m \) such that \( k_{s_0} - k_{t_0} \equiv \frac{N_i}{2} \) (mod \( N_i \)).

Recall that \( z_s^{d_{ij}} - z_t^{d_{ij}} \) divides \( p_{ij}(z_s, z_t) \). Thus, from the assertions (c) and (d), it implies that the assertion (a) holds except the case that \( s_i = 2 \) and there exist \( s_0, t_0 \) such that \( k_{s_0} \notin \Gamma_{ij} \) and \( k_{s_0} - k_{t_0} \equiv \frac{N_i}{2} \) (mod \( N_i \)). But in this case, we have that \( \xi^{k_{s_0} z_{s_0} - \xi^{k_{t_0} z_{t_0}}} \) divides \( z_s^{d_{ij}} + z_t^{d_{ij}} \) and hence divides \( p_{ij}(z_s, z_t) \). This completes the proof of the proposition.

We denote the resulting surjective homomorphism given in Proposition 7.3 by \( \psi_{\hat{g}, \mu} \). The following result was proved in [18] (see also [11]).

**Theorem 7.5.** Assume that the GCM A is of finite type. Then the homomorphism \( \psi_{\hat{g}, \mu} : \hat{g} \rightarrow \hat{L}(\hat{g}, \mu) \) is an isomorphism.

In general, \( \psi_{\hat{g}, \mu} \) is not injective and when \( g \) is of affine type, we will determine the kernel of \( \psi_{\hat{g}, \mu} \) in §7.3.

**7.2. Basics on EALAs.** We start with the definition of an extended affine Lie algebra (EALA for short). Let \( \mathcal{E} \) be a Lie algebra equipped with a nontrivial finite-dimensional self-centralizing ad-diagonalizable subalgebra \( \mathcal{H} \) and a nondegenerate invariant symmetric bilinear form \( (\cdot, \cdot) \). Let \( \mathcal{E} = \oplus_{\alpha \in \mathcal{H} \cdot \mathcal{E}_\alpha} \) be the root space decomposition of \( \mathcal{E} \) with respect
to $\mathcal{H}$, and let $\Phi = \{ \alpha \in \mathcal{H}^* \mid \mathcal{E}_\alpha \neq 0 \}$ be the corresponding root system. The form $( \mid \ )$ restricted to $\mathcal{H} = \mathcal{E}_0$ is nondegenerate, and hence induces a nondegenerate symmetric bilinear form on $\mathcal{H}^*$. Set

$$\Phi^\times = \{ \alpha \in \Phi \mid (\alpha \mid \alpha) \neq 0 \} \quad \text{and} \quad \Phi^0 = \{ \alpha \in \Phi \mid (\alpha \mid \alpha) = 0 \}. $$

Let $\mathcal{E}_c$ be the subalgebra of $\mathcal{E}$ generated by the root spaces $\mathcal{E}_\alpha$, $\alpha \in \Phi^\times$, called the core of $\mathcal{E}$. Following [1], we have:

**Definition 7.6.** The triple $(\mathcal{E}, \mathcal{H}, (\mid \ )$) is called an extended affine Lie algebra if

1. $\text{ad}(x)$ is locally nilpotent for $x \in \mathcal{E}_\alpha$, $\alpha \in \Phi^\times$.
2. $\Phi^\times$ cannot be decomposed as a union of two orthogonal nonempty subsets.
3. The centralizer of $\mathcal{E}_c$ in $\mathcal{E}$ is contained in $\mathcal{E}_c$.
4. $\Phi$ is a discrete subset of $\mathcal{H}^*$.

The condition (4) in Definition 7.6 implies that the subgroup $(\Phi^0)$ of $\mathcal{H}^*$ generated by $\Phi^0$ is a free abelian group of finite rank. This rank is called the nullity of $\mathcal{E}$. Nullity 0 EALAs are exactly finite dimensional simple Lie algebras, while nullity 1 EALAs are exactly affine Kac-Moody algebras [2]. Set $\mathcal{H}_\sigma = \mathfrak{h}_{(0)} \oplus \mathbb{C}c \oplus \mathbb{C}d$, an abelian subalgebra of $\text{Aff}(\mathfrak{g}, \sigma)$. Then we have

**Lemma 7.7.** Assume that $\mathfrak{g}$ is of finite type or affine type. Then for any diagram automorphism $\sigma$ of $\mathfrak{g}$, the triple $(\text{Aff}(\mathfrak{g}, \sigma), \mathcal{H}_\sigma, (\mid \ )$) is an EALA if and only if $\sigma$ satisfies the linking conditions (LC1) and (LC2).

**Proof.** It was proved in [3] that $(\text{Aff}(\mathfrak{g}, \sigma), \mathcal{H}_\sigma, (\mid \ )$) is an EALA if and only if either $\mathfrak{g}$ is of finite type, or $\mathfrak{g}$ is of affine type and $\sigma$ is nontransitive. Then the assertion is implied by Lemma 2.2. \qed

**Remark 7.8.** In the Definition 7.6, the choice of the invariant bilinear form is not important: let $(\mathcal{E}, \mathcal{H}, (\mid \ )$) and $(\mathcal{E}, \mathcal{H}, (\mid \ )'$ be two EALA structures on $\mathcal{E}$. Then $\Phi = \Phi'$, $\Phi^\times = \Phi'^\times$ and $\Phi^0 = \Phi'^0$, where we distinguish the notations for $(\mathcal{E}, \mathcal{H}, (\mid \ )$) by $. One may see [15 Corollary 3.3] for details. Thus, as in [62, 15], from now on we will denote EALAs as couples $(\mathcal{E}, \mathcal{H})$.

**Remark 7.9.** In a series of papers [8, 13, 15, 14], the conjugacy theorem of Cartan subalgebras for EALAs was proved. Explicitly, let $(\mathcal{E}, \mathcal{H})$ and $(\mathcal{E}, \mathcal{H}')$ be two EALA structures on a Lie algebra $\mathcal{E}$. Then there is an automorphism of $\mathcal{E}$ that maps $\mathcal{H}$ onto $\mathcal{H}'$. In particular, root system is an invariant of $\mathcal{E}$.

The structure of EALAs is intimately connected to Lie torus introduced by Yoshii [68]. Let $S$ be a finite irreducible root system, which is not necessarily reduced and contains 0, and let $Q(S)$ be the corresponding root lattice. Let $\Lambda$ be a free abelian group of finite type, and let $L$ be a Lie algebra graded by $(Q(S), \Lambda)$, that is, $L = \oplus_{\lambda \in Q(S)} L^\lambda_{\alpha \in \Lambda}$ such that $[L^\lambda_{\alpha}, L^\gamma_{\beta}] \subset L^\lambda_{\alpha+\beta}$. For convenience, we set

$$L^\lambda_{\alpha} = \oplus_{\lambda \in \Lambda} L^\lambda_{\alpha \in \Lambda}, \alpha \in Q(S) \quad \text{and} \quad L^\lambda = \oplus_{\alpha \in Q(S)} L^\lambda_{\alpha \in \Lambda}, \lambda \in \Lambda.$$ 

A Lie torus of type $(S, \Lambda)$ is by definition a $(Q(S), \Lambda)$-graded Lie algebra satisfying certain conditions [68], and its nullity is defined as the rank of $\Lambda$.

**Remark 7.10.** Let $L$ be a Lie torus. Then $L$ is perfect and so it has a universal central extension $f : u(L) \rightarrow L$. It is known that $u(L)$ is a Lie torus of the same type with $f(u(L)^{\lambda}_\alpha) = L^\lambda_{\alpha}$ for $\alpha \in Q(S)$, $\lambda \in \Lambda$ (cf. [62]).
Lemma 7.13. A Lie torus $L$ is called centerless if its center $Z(L) = 0$. For example, the centerless core $E_{cc} = E_c/Z(E_c)$ of an EALA $E$ is a centerless Lie torus. Let $L$ be a centerless Lie torus of type $(S, \Lambda)$. Then $u(L)$ is a central closed Lie torus. Conversely, let $L$ be a Lie torus of type $(S, \Lambda)$, which is central closed. Then $L/Z(L)$ a centerless Lie torus of type $(S, \Lambda)$ such that $L \cong u(L/Z(L))$.

It is known that the core of an EALA is naturally a Lie torus \[4 \ 62\]. More precisely, let $(E, \mathcal{H})$ be an EALA of nullity $n$ with the root system $\Phi$. Set $X = \text{span}_R(\Phi)$ and $X^0 = \text{span}_R(\Phi^0)$. Then the image $S$ of $\Phi$ in the quotient map $\pi: X \to Y = X/X^0$ is a finite irreducible root system, and there is a linear map $f: Y \to X$ such that $f \circ \pi = \text{Id}_Y$ and $f(S_{\text{ind}}) \subset \Phi$, where $S_{\text{ind}} = \{0\} \cup \{ \beta \in S \mid \frac{1}{2} \alpha \notin S \}$. Furthermore, the core $E_c$ of $E$ is a nullity $n$ Lie torus of type $(S, \Lambda)$ with $(E_c)^\Lambda = E_c \cap \mathcal{E}_{f(\alpha) + \lambda}$, where $\Lambda = (\Phi^0)$. In particular, the $\Lambda$-grading on $E_c$ is determined by

$$(E_c)^\lambda = \bigoplus_{\beta \in \Phi, p(\beta) = \lambda} E_c \cap E_c \beta,$$

where $p$ is the projection map from $X = f(Y) \oplus \Lambda$ to $\Lambda$.

For example, let $\mathcal{E} = \text{Aff}(g, \mu)$ with $\tilde{g}$ is of affine type (see Lemma 7.17). We view $(\tilde{g})^*$ as a subspace of $(\mathcal{H}_\mu)^*$ in a natural way, and define $\delta \in (\mathcal{H}_\mu)^*$ such that $\delta(\tilde{h}) = 0 = \delta(c)$ and $\delta(\tilde{d}) = 1$. Recall that $\tilde{g}$ is the Kac-Moody algebra associated with $\tilde{A}$ and $\tilde{a}_i, i \in \tilde{I}$ are simple roots of it (see §2.2). Then we have $(i, m \in \mathbb{Z})$:

$$t^m \otimes \epsilon^\pm_{i(m)} \in \text{Aff}(g, \mu)_{\tilde{A}_{i,m}}^\pm, \quad t^m \otimes \epsilon^\gamma_{i(m)} \in \text{Aff}(g, \mu)_{\tilde{A}_{i,m}}, \quad \tilde{D}^m \in \text{Aff}(g, \mu).$$

Since these elements generate the algebra $\text{Aff}(g, \mu)$, it follows that the root system $\Phi \subset \tilde{Q} \oplus \mathbb{Z} \tilde{d}$, where $\tilde{Q} = \oplus_{i \in I} \mathbb{C} \tilde{a}_i$ is the root lattice of $\tilde{g}$. On the other hand, we have $\Phi^0 \subset \mathbb{Z} \tilde{d}_2 \oplus \mathbb{Z} \tilde{n}$, where $\tilde{d}_2$ denotes the null root in $\tilde{g}$. In fact, it follows from \[1\] Corollary 2.31 that $\Phi^0 = \mathbb{Z} \tilde{d}_2 \oplus \mathbb{Z} \tilde{n}$. Thus, the core $\mathcal{L}(g, \mu)$ of $\text{Aff}(g, \mu)$ is a Lie torus of type $(S, \Lambda)$, where $\Lambda = \Phi^0$ and the root system $S$ is determined in \[3\] Theorem 12.2.1. Note that the elements $\pi(\tilde{a}_i), i \in \tilde{I} \setminus \{\omega\}$ form a basis of $Y$ and so we can define the section $f$ by

$$f(\pi(\tilde{a}_i)) = \tilde{a}_i$$

for $i \in \tilde{I} \setminus \{\omega\}$, where $\omega \in \tilde{I}$ is the additional node of $\tilde{g}$. Then we have

$$p(\tilde{a}_i) = \delta_{i,\omega} \tilde{d}_2$$

for $i \in \tilde{I}$, which implies $p(\alpha) = \alpha(\tilde{d}_2) \tilde{d}_2 + \alpha(\tilde{d}) \delta$ for $\alpha \in \Phi$, where $\tilde{d}_2$ is a scaling element in $\tilde{h}$ such that $\tilde{a}_i(\tilde{d}_2) = \delta_{i,\omega}$ for $i \in \tilde{I}$. In particular, the $\Lambda$-grading on $\mathcal{L}(g, \mu)$ is given by

$$\mathcal{L}(g, \mu)^{m \tilde{d}_2 + n \tilde{n}} = \{ x \in \mathcal{L}(g, \mu) \mid [\tilde{d}_2, x] = mx, \ |	ilde{d}, x\} = nx \}.$$

We define the extended core $\tilde{E}$ of an EALA $(E, \mathcal{H})$ to be the subalgebra $E_{cc} + \mathcal{H}$ of $E$. It follows from Remark 7.9 that, up to the isomorphism, the extended core of an EALA is independent of the choice of its Cartan subalgebra. Following \[6\], we say that an EALA is maximal if its core is central closed.

Definition 7.12. For a nonnegative integer $n$, denote by $\tilde{E}_n$ the class of Lie algebras which are isomorphic to the extended core of an maximal EALAs with nullity $n$.

Note that $\tilde{E}_0$ is the class of finite dimensional simple Lie algebras and $\tilde{E}_1$ is the class of affine Kac-Moody algebras. Let $L$ be a Lie torus of type $(S, \Lambda)$. Then any $\theta \in \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{C})$ induces a so-called degree derivation $\partial_\theta$ of $L$: $\partial_\theta(x) = \theta(\lambda)x$ for all $x \in L^\Lambda$. Form the semi-product Lie algebra $L = L \oplus D_L$, where $D_L = \{ \partial_\theta \mid \theta \in \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{C}) \}$. Note that for any $\lambda \in \Lambda, L^\lambda = \{ x \in L \mid [\partial_\theta, x] = \theta(\lambda)x \text{ for all } \theta \in \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{C}) \}$ \[61\]. We have:

Lemma 7.13. Let $L$ be a nullity $n$ Lie torus, which is central closed. Then $E_c \in \tilde{E}_n$. Conversely, any algebra in $\tilde{E}_n$ has this form.
Recall from [47, Exercise 2.5] that any two such forms on \( g \) with the Lie bracket given by (7.8) may (and do) fix the choice of \( (\cdot | \cdot) \). Note that (7.8) induces an invariant bilinear form on \( g' \) such that
\[(\cdot | \cdot) \}_{g'} \approx (\cdot | \cdot) \}_{g' \times g'}.

Proof. The proof follows Neher’s construction of maximal EALAs, and a sketch is given here; see \([61\) or \([62\) for details. Let \( L \) be a centerless Lie torus of type \((S, \Lambda)\) and \( \Gamma \) the central grading group of \( L \), which is a free subgroup of \( \Gamma \). Let \( D = SCDer(L) \) be the space of skew centroidal derivations on \( L \), which is \( \Gamma \)-graded with the degree zero space \( D_0 = D_L \). Let \( D_{gr} \) the \( \Gamma \)-graded dual of \( D \) and let \( \tau : D \times D \to D_{gr} \) be an affine two-cocycle. Then there is a maximal EALA structure on \( E(L, \tau) = L \oplus D^r_{gr} \oplus D \) with the Cartan subalgebra \( H(L) = L_0^0 \oplus D_{gr}^0 \oplus D_0^0 \) and the core \( E(L, \tau)_c = L \oplus D^r_{gr} \). Conversely, any maximal EALA arises in this way.

Note that one of the conditions for \( \tau \) requires that \( \tau(D_L, D) = 0 \). This gives that the Lie bracket on \( E(L, \tau) = L \oplus D^r_{gr} \oplus D_L \) is independent from the choice of \( \tau \). In particular, we have \( E(L, \tau) = (L \oplus D^r_{gr})_c \) as a Lie algebra. Now, let \( L \) be a central closed Lie torus of nullity \( n \). Then by Remark 7.11 there is a centerless Lie torus \( L \) such that \( L \cong u(L) \cong L \oplus D^r_{gr} \) and hence
\[ L_c \cong (L \oplus D^r_{gr})_c = \bar{E}(L, \tau_0) \in \bar{E}_n, \]
where \( \tau_0 \) denotes the trivial two-cocycle on \( D \). On the other hand, let \( \bar{E} \in \bar{E}_n \) and by Neher’s construction we may assume \( \bar{E} \) has the form \( E(L, \tau) \). Then we have \( \bar{E} = \bar{E}(L, \tau) = (L \oplus D^r_{gr})_c \), as required. \( \square \)

7.3. Quantization of nullity 2 EALAs. In this subsection, we first classify the algebras in \( \bar{E}_2 \) and then consider their quantization. Throughout this subsection we always assume that \( g \) is of affine type \( X_\varphi^{(r)} \), where \( \varphi \in \mathbb{Z}_+ \) and \( r = 1, 2, 3 \). For convenience, we set \( I = \{0, 1, \ldots, \ell \} \).

Let \( \hat{g} \) be a finite dimensional simple Lie algebra of type \( X_\varphi \), and \( \hat{\nu} \) a diagram automorphism of \( \hat{g} \) with order \( r \). For \( \hat{x} \in \hat{g} \) and \( m \in \mathbb{Z} \), we set
\[ \hat{x}[m] = \sum_{p \in \mathbb{Z}_r} \xi^{p-1} \hat{\nu}^{p}(\hat{x}) \quad \text{and} \quad \hat{g}[m] = \Span_{\mathbb{C}} \{ \hat{x}[m] \mid \hat{x} \in \hat{g} \}. \]
Then it was shown in [47, Chap. 8] that the affine Kac-Moody algebra \( g \) can be realized as the Lie algebra
\[ g = \left( \sum_{m \in \mathbb{Z}} \ell^m_2 \otimes \hat{g}[m] \right) \oplus \mathbb{C}k_2 \oplus \mathbb{C}d_2 \]
with the Lie bracket given by
\[ \ell^m_2 \otimes \hat{x} + a_1 k_2 + b_1 d_2, \ell^m_2 \otimes \hat{y} + a_2 k_2 + b_2 d_2 \]
and
\[ \ell^{m_1+m_2}_2 \otimes [\hat{x}, \hat{y}] + (\hat{x}|\hat{y})\delta_{m_1+m_2,0}m_1k_2 + b_1 m_2 \ell^m_2 \otimes \hat{y} - b_2 m_1 \ell^m_2 \otimes \hat{x}, \]
where \( m_1, m_2 \in \mathbb{Z} \), \( \hat{x} \in \hat{g}[m_1] \), \( \hat{y} \in \hat{g}[m_2] \), \( a_1, a_2, b_1, b_2 \in \mathbb{C} \) and \((\cdot | \cdot)\) is a nondegenerate invariant symmetric bilinear form on \( \hat{g} \). Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \hat{g} \) and take
\[ \mathfrak{h} = (\mathfrak{h} \cap \hat{g}[0]) \oplus \mathbb{C}k_2 \oplus \mathbb{C}d_2 \]
as the Cartan subalgebra of \( g \). We also realize the simple roots \( \alpha_i \)'s and simple coroots \( \alpha_i^\vee \)'s as in [47]. Note that \( d_2 \) in \( \mathfrak{h} \) is a scaling element such that \( \alpha_i(d_2) = \delta_{i,0} \) for \( i \in I \).

Remark 7.14. Note that \((\cdot | \cdot)\) induces an invariant bilinear form on \( g' \) such that
\[ (\ell^{m_1}_2 \otimes \hat{x} + a_1 k_2 | \ell^{m_2}_2 \otimes \hat{y} + a_2 k_2) = \delta_{m_1+m_2,0}(\hat{x}|\hat{y}). \]
Recall from [47, Exercise 2.5] that any two such forms on \( g' \) are proportional. Thus, we may (and do) fix the choice of \((\cdot | \cdot)\) on \( \hat{g} \) such that \((\cdot | \cdot)\) on \( g' \times g' \).
Let $\mathcal{K}$ be a vector space over $\mathbb{C}$ spanned by the symbols

$$t_i^{m_1}t_2^{m_2}k_i, \quad i = 1, 2, \ m_1 \in \mathbb{Z}, \ m_2 \in r\mathbb{Z}$$

and subject to the relations $m_1t_i^{m_1}t_2^{m_2}k_1 + m_2t_1^{m_1}t_2^{m_2}k_2 = 0$. Let

$$t(g) = \sum_{m_1,m_2 \in \mathbb{Z}} t_1^{m_1}t_2^{m_2} \otimes \hat{g}(m_2) \oplus \mathcal{K}$$

be the toroidal Lie algebra associated with $g$, where $\mathcal{K}$ is the center space and

$$(7.9) \quad [t_1^{m_1}t_2^{m_2} \otimes \dot{x}, t_1^{n_1}t_2^{n_2} \otimes \dot{y}] = t_1^{m_1+n_1}t_2^{m_2+n_2} \otimes [\dot{x}, \dot{y}] + (\dot{x}|\dot{y}) \sum_{i=1}^{2} m_i t_1^{m_1+n_1}t_2^{m_2+n_2}k_i$$

for $\dot{x} \in \hat{g}(m_2)$, $\dot{y} \in \hat{g}(n_2)$ and $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. With the identification $\{\xi\}$, for $x = t_1^{m_2} \otimes \dot{x} + ak_2 \in \hat{g}$ ($m_2 \in \mathbb{Z}, \dot{x} \in \hat{g}, a \in \mathbb{C}$) and $m_1 \in \mathbb{Z}$, let us set

$$(7.10) \quad t_1^{m_1} \otimes x := t_1^{m_1}t_2^{m_2} \otimes \dot{x} + at_1^{m_1}k_2 \in t(g).$$

Note that these elements together with the central elements $t_1^{m_1}t_2^{m_2}k_1$ ($m_1 \in \mathbb{Z}, m_2 \in r\mathbb{Z} \setminus \{0\}$), $k_1$ span the algebra $t(g)$. Denote by $\Delta, \Delta^{re}$ and $\Delta^{im}$ the sets of roots, real roots and imaginary roots in $g$, respectively. In view of Remark 7.13, the commutator (7.9) can be rewritten as follows:

**Lemma 7.15.** Let $\alpha, \beta \in \Delta, \ x \in g_\alpha, y \in g_\beta$ and $m_1, n_1 \in \mathbb{Z}$. If $\alpha + \beta \in \Delta^{re} \cup \{0\}$, then we have

$$(7.11) \quad [t_1^{m_1} \otimes x, t_1^{n_1} \otimes y] = t_1^{m_1+n_1} \otimes [x, y] + m_1\delta_{m_1,n_1}(x \mid y)k_1.$$  

If $x = t_1^{m_2} \otimes \dot{x}, y = t_2^{n_2} \otimes \dot{y}$ and $\alpha + \beta \in \Delta^{im} \setminus \{0\}$, then

$$(7.12) \quad [t_1^{m_1} \otimes x, t_1^{n_1} \otimes y] = t_1^{m_1+n_1} \otimes [x, y] + (\dot{x}|\dot{y}) \frac{m_1n_2 - m_2n_1}{m_2 + n_2} t_1^{m_1+n_1}t_2^{m_2+n_2}k_1.$$  

Note that the elements

$$(7.13) \quad t_1^{m} \otimes e_i^{\pm}, \ t_1^{m} \otimes \alpha_i^{\pm}, \ k_1, \ m \in \mathbb{Z}, \ i \in I$$

generate the algebra $t(g)$. From [10], there is an automorphism $\hat{\mu}$ of $t(g)$ with

$$(7.14) \quad t_1^{m} \otimes e_i^{\pm} \mapsto \xi^{-m}t_1^{m} \otimes e_{\mu(i)}^{\pm}, \ t_1^{m} \otimes \alpha_i^{\pm} \mapsto \xi^{-m}t_1^{m} \otimes \alpha_{\mu(i)}^{\pm}, \ k_1 \mapsto k_1$$

for $i \in I, \ m \in \mathbb{Z}$. Denote by $t(g, \mu)$ the subalgebra of $t(g)$ fixed by $\hat{\mu}$. Using (7.13) and (7.14), we see that $t(g, \mu)$ is generated by the elements $t_1^{m} \otimes e_{i(m)}^{\pm}, t_1^{m} \otimes \alpha_{i(m)}^{\pm}, k_1$ for $i \in I, m \in \mathbb{Z}$. We have

**Lemma 7.16.** The assignment $(i \in I, m \in \mathbb{Z})$

$$(7.15) \quad t_1^{m} \otimes e_{i(m)}^{\pm} \mapsto t_1^{m} \otimes e_{i(m)}^{\pm}, \ t_1^{m} \otimes \alpha_{i(m)}^{\pm} \mapsto t_1^{m} \otimes \alpha_{i(m)}^{\pm}, \ k_1 \mapsto c.$$  

determines a surjective homomorphism $\phi_{\theta, \mu} : t(g, \mu) \rightarrow L(g, \mu)$, which is a universal central extension.

**Proof.** From (7.2) and Lemma 7.15 it follows that the map

$$\phi : t(g) \rightarrow L(g), \quad t_1^{m} \otimes x \rightarrow t_1^{m} \otimes x, \ k_1 \rightarrow c, \ t_1^{m_1}t_2^{m_2}k_1 \rightarrow 0$$

is a central extension, where $m \in \mathbb{Z}, x \in g, m_1 \in \mathbb{Z}$ and $m_2 \in r\mathbb{Z} \setminus \{0\}$. Recall that $L(g, \mu)$ is the $\mu$-fix point subalgebra of $L(g)$. As $\phi \circ \hat{\mu} = \hat{\mu} \circ \phi$ on the generators in (7.13), $\phi$ induces a central extension $\phi_{\theta, \mu}$ from $t(g, \mu)$ to $L(g, \mu)$, which is determined by (7.15). The fact that $\phi_{\theta, \mu}$ is universal follows from [10, Theorem 3.3].
By adding two canonical derivations $d_1, d_2$ to $t(g)$, we obtain the Lie algebra 

$$
\hat{t}(g) = t(g) \oplus C d_1 \oplus C d_2,
$$

where $(i, j = 1, 2, \dot{x} \in \hat{g}[m_2]$ and $m_1, m_2 \in \mathbb{Z})$

\begin{equation}
(7.16) \quad [d_i, t_1^{m_1} t_2^{m_2} \otimes \dot{x}] = m_i t_1^{m_1} t_2^{m_2} \otimes \dot{x} \quad \text{and} \quad [d_i, t_1^{m_1} t_2^{m_2} k_j] = m_i t_1^{m_1} t_2^{m_2} k_j.
\end{equation}

With the notation given in (7.10), we in particular have

\begin{equation}
(7.17) \quad [d_2, t_1^{m_1} \otimes x] = t_1^{m_1} \otimes [d_2, x(m)], \quad [d_1, t_1^{m_1} \otimes x(m)] = m_1 t_1^{m_1} \otimes x.
\end{equation}

One notices that $\mu(d_2) \in d_2 + h'(\text{see the proof of Proposition 7.2.1})$, $h = h' \oplus C d_2 \oplus h''$ is $\mu$-invariant. Set

\begin{equation}
(7.18) \quad \hat{d}_2 = 1/N \sum_{p \in \mathbb{Z}_N} \mu^p(d_2) \in \hat{h}.
\end{equation}

Then we have $\hat{h} = h(0) \oplus C \hat{d}_2$ and $\hat{d}_2$ is a scaling element in $\hat{h}$ such that $\hat{d}_2 = 1/\mu$ for $i \in I$ (see §2.2). By (7.11) and (7.17), one immediately has

\begin{lemma}
For $i \in I$ and $m \in \mathbb{Z}$, one has

\begin{align*}
[d_2, t_1^m \otimes \alpha_{i(m)}] &= 0, \quad [d_2, t_1^m \otimes e_{i(m)}^\pm] = \pm \alpha_i(d_2)t_1^m \otimes e_{i(m)}^\pm, \quad [d_2, k_1] = 0, \\
[d_1, t_1^m \otimes \alpha_{i(m)}] &= m_1 t_1^m \otimes \alpha_{i(m)}, \quad [d_1, t_1^m \otimes e_{i(m)}^\pm] = m_1 t_1^m \otimes e_{i(m)}^\pm, \quad [d_1, k_1] = 0.
\end{align*}

In view of Lemma 7.17, we have the following subalgebra of $\hat{t}(g)$:

\begin{equation}
(7.19) \quad \hat{t}(g, \mu) = t(g, \mu) \oplus C d_1 \oplus C d_2.
\end{equation}

\begin{remark}
Recall that $\hat{L}(g, \mu) = L(g, \mu) \oplus C \hat{d}_2 \oplus C d_1$ is a Lie torus with the $\Lambda$-grading determined by $\phi_{g, \mu}(t(g, \mu)^\lambda) = L(g, \mu)^\lambda$ for $\lambda \in \Lambda$. Using Remark 7.17 and (7.7), we have

\begin{equation}
(7.20) \quad t(g, \mu)^m d_2 + n \delta = \{ x \in t(g, \mu) \mid [d_2, x] = m x, \quad [d_1, x] = n x \}.
\end{equation}

Namely, via the adjoint action, $C \hat{d}_2 \oplus C d_1$ is the space of degree derivations on $t(g, \mu)$. From Lemma 7.17, it follows that $\hat{t}(g, \mu) = t(g, \mu)_e \in \hat{E}_2$.

On the other hand, let $p$ be a generic complex number. Set $C_p$ to be the quantum 2-torus $C_p$ associated to $p$, which is the associative algebra with underlying space $\mathbb{C} t_1 t_2^{1/2}$ subject to the basic commutation relation $t_2 t_1 = p t_1 t_2$. \( \ell \) be a positive integer as before. Denote by $g_{\ell+1}(C_p)$ the matrix Lie algebra over $C_p$ and set $s_\ell t_1^{(C_p)} = [g_{\ell+1}(C_p), g_{\ell+1}(C_p)]$, the derived subalgebra. Furthermore, let $s_{\ell+1}(C_p)$ denote the universal central extension of $s_{\ell+1}(C_p)$. By adding two derivation $d_1, d_2$ to $s_{\ell+1}(C_p)$, one obtains a nullity 2 EALA $s_{\ell+1}(C_p)$ with Lie brackets as in (7.10) [3]. Note that the extended core of $s_{\ell+1}(C_p)$ is itself and hence $s_{\ell+1}(C_p) \in \hat{E}_2$. Following [3], we have the following classification result.

\begin{proposition}
Every algebra in $\hat{E}_2$ is either isomorphic to $\hat{t}(g, \mu)$ with $g$ affine, or isomorphic to $s_{\ell+1}(C_p)$ with $p$ generic.
\end{proposition}
Proof. From Remark 7.11 and Lemma 7.13 it follows that any algebra in \( \hat{E}_2 \) has the form \( u(L) \) for some centerless Lie torus \( L \) of nullity 2. It was proved in [3] that every centerless Lie torus of nullity 2 is either isomorphic to \( \text{Aff}(g, \mu) = \mathcal{L}(g, \mu)/\mathcal{Z}(\mathcal{L}(g, \mu)) \) or isomorphic to \( \mathfrak{sl}_{\ell+1}(\mathbb{C}_p) \). Note that \( \mathfrak{sl}_{\ell+1}(\mathbb{C}_p) = \mathfrak{sl}_{\ell+1}(\mathbb{C}_p)_e \cong u(\mathfrak{sl}_{\ell+1}(\mathbb{C}_p))_e \). Furthermore, we have \( u(\text{Aff}(g, \mu))_e \cong u(\mathcal{L}(g, \mu))_e \cong \hat{t}(g, \mu) \) and hence \( u(\text{Aff}(g, \mu))_e \cong \hat{t}(g, \mu) \), as desired. □

Now we are ready to give the quantization of the algebras classified in Proposition 7.19. Note that when \( g \) is of type \( A_1^{(1)} \), it follows from Lemma 2.22 that \( \mu = \text{id} \). In this case, the quantum affinization algebra \( U_h^{(1)} \) defined in [14] is slightly different from that of \( U_h^{(g, \mu)} \): the relations (Q8) with \( i \neq j \) is now replaced by the relations

\[
 F^\pm_{ij}(z, w)(z - w)x^\pm_i(z)x^\pm_j(w) = G^\pm_{ij}(z, w)(z - w)x^\pm_j(w)x^\pm_i(z).
\]

Define \( U_h^{\text{new}} \) to be the quotient \( \mathbb{C}[[\hbar]] \)-algebra of \( U_h^{(1)} \) by modulo the relations

\[
 [x^\pm_i(z_1), (F^\pm_{ij}(z_2, w)x^\pm_i(z_2)x^\pm_j(w) - G^\pm_{ij}(z_2, w)x^\pm_j(w)x^\pm_i(z_2))] = 0, \quad \text{for } i \neq j.
\]

Just as \( U_h^{(g, \mu)} \), it was proved in [12] that \( U_h^{\text{new}} \) has a triangular decomposition and a topological Hopf algebra structure. We also denote by \( U_{h, p}(\mathfrak{sl}_{\ell+1}) \) the two parameters quantum toroidal algebra defined in [33] Definition 2.2.1. As the main result of this section, we have:

**Theorem 7.20.** Let \( g \) be of affine type and \( p \) a generic number. Then we have

1. \( U_{h, p}(\mathfrak{sl}_{\ell+1})|_{h \to 0} \cong U(\mathfrak{sl}_{\ell+1}(\mathbb{C}_p)) \);
2. \( U_{h}^{\text{new}}|_{h \to 0} \cong U(\hat{t}(g)) \) if \( g \) is of type \( A_1^{(1)} \);
3. \( U_{h}^{(g, \mu)}|_{h \to 0} \cong U(\hat{t}(g, \mu)) \) if \( g \) is of non-\( A_1^{(1)} \) type.

**Proof.** The first assertion was proved in [67], the second assertion was proved in [12] and the third assertion was proved in [11]. In the last case, the isomorphism from \( \hat{g}_\mu \) to \( \hat{t}(g, \mu) \) is given by

\[
 H_{i, m} \mapsto t_{i, m}^m \otimes r_i a_{i(m)}^\vee, \quad X_{i, m}^\pm \mapsto t_{i, m}^m \otimes e_{i(m)}^\pm, \quad C \mapsto \frac{k_1}{N}, \quad L_d \mapsto \hat{d}_2, \quad D \mapsto \hat{d}_1.
\]

□

**Remark 7.21.** When \( g \) is of type \( A_1^{(1)} \) (and so \( \mu = \text{id} \)), \( \hat{g}_\mu \) is isomorphic to the quotient algebra \( \hat{t}(g)/\hat{K} \) (cf. [25]), where \( \hat{K} = \sum_{m \in \mathbb{Z}} (\mathbb{C}t_1^{m_1}t_2k_1 + \mathbb{C}t_1^{m_1}t_2^{-1}k_1) \).

8. **Proof of Theorem 5.13**

This section is devoted to the proof of Theorem 5.13. Throughout this section, let \((i, j) \in \bar{I} \) and \( W \in \mathcal{R}_1^2 \) be fixed. For a finite set \( J \) and a formal variable \( z \), we denote by \( \underline{z} \) the map from \( \hat{J} \) to the set \( \{z_a | a \in J\} \) of formal variables defined by \( a \mapsto z_a \). If there is no ambiguity, we simply denote \( \underline{z} \) by \( \underline{z} \). For simplify the notation, we also write \( \frac{1}{z - w} = t_{z, w}^{-1} \) as usual.
8.1. On $g$-commutators. In this subsection we calculate the $g$-commutators $[x^+_i(z_1), x^+_i(z_2), \ldots, x^+_i(z_r), x^+_j(w)]_g$ with $r \in \mathbb{N}$.

We start with some notations. Denote by $K = (-1, \xi_{d_i/d_j})$ the subgroup of $\mathbb{C}^\times$ generated by $-1$ and $\xi_{d_i/d_j}$, and denote by $K^\nu = (-1)^\nu (\xi_{d_i/d_j})$, where $\nu \in \mathbb{Z}_2 = \{0, 1\}$. For $c \in K^0$ and $r \in \mathbb{N}$, set $A^c_{c,r} = \emptyset$ if $r = 0$ and set

$$A^c_{c,r} = \{((-1)^\nu c, a_{ij} - \nu + 2p) \in K \times \mathbb{Z} \mid 0 \leq p \leq r - 1\} \quad \text{if } r > 0.$$

Let $\mathcal{M}'$ be the set consisting of subsets of $K \times \mathbb{Z}$ which are a finite union of the $A^c_{c,r}$'s with $c \in K^0$, $r \in \mathbb{N}$ and $\nu \in \mathbb{Z}_2$. For $M \in \mathcal{M}'$, $c \in K^0$ and $\nu \in \mathbb{Z}_2$, define

$$p^c = \max\{r - 1 \mid r \in \mathbb{N} \text{ with } A^c_{c,r} \subset M\}.$$

Form the sets

$$\mathcal{M} = \{M \in \mathcal{M}' \mid p^1 = -1 \text{ if } s_i = 1 \text{ and } p^1 \leq p^0 \text{ if } s_i = 2 \text{ for all } c \in K^0\},$$

$$\mathcal{M}_r = \{M \in \mathcal{M} \mid |M| = r\} \quad \text{for } r \in \mathbb{N}.$$  

For $M \in \mathcal{M}$ and $\nu \in \mathbb{Z}_2$, we further introduce the following sets:

$$M^\nu = \{((-1)^\nu c, a_{ij} - \nu + 2p) \in M \mid c \in K^0, \ 0 \leq p \leq p^c\},$$

$$\partial M = \{a \in (K \times \mathbb{Z}) \setminus M \mid M \cup \{a\} \in \mathcal{M}\},$$

$$\partial^* M = \{(-c, a_{ij} + 2p_c + 1) \in \partial M \mid c \in K^0, 0 \leq p_c \leq p^c\}.$$

We also define a map as follows:

$$(8.1) \quad \tau^\pm : K \times \mathbb{Z} \rightarrow \mathbb{C}[\hbar], \quad (c, n) \mapsto c^{1/d_i} q_i^{\pm n},$$

where $c^{1/d_i} = \xi_{d_i/K}$ if $c = \xi_{K} \in K$. For the case $s_i = 2$, we say that $(a_1, a_2, a_3) \in (K \times \mathbb{Z})^3$ is an $A^3_2$-triple if there is a $\sigma \in S_3$ such that

$$\tau^\pm(a_{\sigma(1)}) = \xi_{24}(a_{\sigma(1)}) = \xi_{24}(q_i^{\pm 1}(a_{\sigma(1)})).$$

In this case it follows from Proposition 5.3, 5.6 and 5.7 that for any $p \in Z_{d_i}$, we have

$$(8.2) \quad \tilde{\psi}(\xi_{d_i}^p \tau^+(a_1)w) x^+_{i} \tau^+(a_2)w x^+_{i} \tau^+(a_3)\tilde{\psi} = 0.$$

Remark 8.1. Let $J$ be a finite set with $|J| = m$ and let $\psi(w, J, w) \in \mathcal{E}^{(m+1)}_h(W)$. Assume that there is an $a \in J$ such that $\lim_{w \rightarrow \xi_{d_i}^p w} \psi(w, J, w) = 0$ for all $p \in Z_{d_i}$. Then for any $v \in W$, we have

$$(w_i^d - w_i^d)^{-1} \psi(w_i, J, w)v = (w_i^d - w_i^d)^{-1} \psi(w_i, J, w)v = \psi(w_i, J, w)v = 0.$$

This gives $(w_i^d - w_i^d)^{-1} \psi(w_i, J, w)v \in W((w_i, J, w)) + \hbar^n W[[w_{\pm 1}, w_{\mp 1} \mid b \in J]]$ for all $n \in \mathbb{Z}_+$. Thus one obtains

$$(8.3) \quad (w_i^d - w_i^d)^{-1} \psi(w, J, w) \in \mathcal{E}^{(m+1)}_h(W).$$

Assume further that there is another $b \in J$ such that $\lim_{w \rightarrow \xi_{d_i}^p w} \psi(w, J, w) = 0$ for all $p \in Z_{d_i}$. It follows from 5.3 that $\lim_{w \rightarrow \xi_{d_i}^p w} (w_i^d - w_i^d)^{-1} \psi(w_i, J, w)$ exists and is zero. Then we have

$$(w_i^d - w_i^d)^{-1} \psi(w_i, J, w) \in \mathcal{E}^{(m+1)}_h(W).$$

42
In general, if there is a subset $J'$ of $J$ such that $\lim_{w_n \to e^p_{d_i}} \psi(w_j, w) = 0$ for all $p \in \mathbb{Z}_{d_i}$ and $a \in J'$, then we have
\[
\prod_{a \in J'} (w_a^d - w_{d_i})^{-1} \psi(w_j, w) \in \mathcal{E}_h^{(m+1)}(W).
\]

Let $M \in \mathcal{M}_r$ with $r \in \mathbb{Z}_+$ and $M' \subset M$. In view of Proposition 5.4 and the fact that $C_{ii} = 1$, we can define the following currents on $W$:
\[
\tilde{X}_{ij,r}(M, M', \omega, w) = \lim_{\omega \to \omega, \omega' \in M'} \sum_{a \in M} x^+_{ij}(\tau^+(a) w_a) x^+_j(w) \circ \sum_{b \in M'} x^+_{ij}(\tau^+(b) w) x^+_j(w),
\]
\[
\tilde{x}_{ij,r}^+(z) \tilde{X}_{ij,r}(M, M', \omega, w) = \lim_{\omega \to \omega, \omega' \in M'} \sum_{a \in M} x^+_{ij}(\tau^+(a) w_a) x^+_j(w) \circ \sum_{b \in M'} x^+_{ij}(\tau^+(b) w) x^+_j(w).
\]
As a convention, when $r = 0$, we set $\tilde{X}_{ij,r}(M, M', \omega, w) = x^+_j(w)$.

Let $\tau = (t_c | c \in K^0) \in \mathbb{N}^{d_i}$ be a tuple which satisfies the condition
\[
(8.4) \quad 0 \leq t_c \leq p^0_c \quad \text{if} \quad p^0_c > 0 \quad \text{and} \quad t_c = 0 \quad \text{if} \quad p^0_c = -1.
\]

We define a subset of $M$ associated to $\tau$ as follows:
\[
M_\tau = \{(c, a_{ij} + 2n) | c \in K^0 \text{ with } p^0_c > 0, t_c \leq n \leq p^0_c\}
\]
\[
\cup \{(c, a_{ij} - 1 + 2m) | c \in K^0 \text{ with } p^1_c \geq 0, 0 \leq m \leq t_c\}.
\]

Note that $M = M_\tau$ if $s_i = 1$. Furthermore, if $M \cap M_\tau \neq \emptyset$ (and so $s_i = 2$), then for any $a \in M \setminus M_\tau$ there exist $b, b' \in M_\tau$ such that $(a, b, b')$ is an $A^2_2$-triple. Indeed, if $a = (c, a_{ij} + 2n)$ with $c \in K^0$ and $0 \leq n \leq t_c - 1$, then we may take $b = (c, a_{ij} - 1 + 2n)$ and $b' = (c, a_{ij} + 1 + 2n)$. And, if $a = (c, a_{ij} - 1 + 2m)$ with $c \in K^0$ and $t_c < m \leq p^1_c$, then we may take $b = (c, a_{ij} + 2m - 2)$ and $b' = (c, a_{ij} + 2m)$. In view of [8.2] and Remark 8.3, we can continue to define the following currents on $W$:
\[
X_{ij,r}(M, M_\tau, \omega) = \lim_{\omega \to \omega, \omega' \in M_\tau} \prod_{a \in M \setminus M_\tau} x^+_{ij}(\tau^+(a) w_a) \circ \sum_{b \in M'} x^+_{ij}(\tau^+(b) w) x^+_j(w),
\]
\[
\tilde{x}_{ij,r}^+(z) X_{ij,r}(M, M_\tau, \omega) = \lim_{\omega \to \omega, \omega' \in M_\tau} \prod_{a \in M \setminus M_\tau} x^+_{ij}(\tau^+(a) w_a) \circ \sum_{b \in M'} x^+_{ij}(\tau^+(b) w) x^+_j(w) \circ \tilde{x}_{ij,r}^+(z).
\]

**Lemma 8.2.** Let $\tau = (t_c | c \in K^0)$ and $\tau' = (t'_c | c \in K^0)$ be two tuples which satisfy the condition (8.4). Then
\[
X_{ij,r}(M, M_\tau, \omega) = \left( \prod_{c \in K^0} (-1)^{|t_c - t'_c|} \right) \left( \prod_{a \in M_\tau} x^+_{ij}(\tau^+(a) w_a) \circ \sum_{b \in M'} x^+_{ij}(\tau^+(b) w) x^+_j(w) \right).
\]

**Proof.** We first consider the case that there is a $c \in K^0$ such that $t_c = t'_c + 1$ and $t_d = t'_d$ for any $d \neq c \in K^0$. Set $M_1 = M_\tau \cap M_{\tau'}$ and $M_2 = M_\tau \cup M_{\tau'}$. One can verify that $M_\tau = M_1 \cup \{b_1\}$ with $b_1 = (c, a_{ij} - 1 + 2t_c)$ and $M_{\tau'} = M_1 \cup \{b_2\}$ with $b_2 = (c, a_{ij} - 2 + 2t_c)$. As indicated before, for any $a \in M \setminus M_2$, there exist $a_1, a_2 \in M_1$ such that $(a, a_1, a_2)$ is an...
\(A^{(2)}\)-triple. From (8.2) and Remark 8.1 it follows that
\[
\psi(w, w) := \prod_{a \in M \setminus M_2} (w_a^{d_i} - w_a^{d_i})^{-1} \hat{X}_{ij}^{\pm} (M, M_1, w, w) \in \mathcal{E}_h^{(m+1)}(W).
\]
where \(m = |M \setminus M_1|\). One notices from (8.6) that
\[
\lim_{w \to \mathcal{E}_h^{(m)}(W)} \psi(w, w) = \psi(w, w) = \lim_{w \to \mathcal{E}_h^{(m)}(W)} \psi(w, w) \text{ for } p \in Z_{d_i}, \ a \in M \setminus M_2.
\]
Set \(b_3 = (c, a_{ij} + 2t_e)\) and \(b_4 = (-c, a_{ij} - 3 + 2t_e)\). Then \(b_3, b_4 \in M_1\) and \((b_1, b_2, b_3), (b_1, b_2, b_4)\) are both \(A^{(2)}\)-triples. Again by (8.2) and Remark 8.1 we see that the current
\[
(w_{b_3}^{d_i} - w_{b_4}^{d_i})^{-1}(w_{b_1}^{d_i} - w_{b_2}^{d_i})^{-1} \lim_{w_{b_3} \to w_{b_4}} \circ x_i^\pm (\tau^\pm(b_1)w_{b_1})x_i^\pm (\tau^\pm(b_2)w_{b_2})x_i^\pm (\tau^\pm(b_3)w_{b_3})x_i^\pm (\tau^\pm(b_4)w_{b_4})^o
\]
lies in \(\mathcal{E}_h^{(3)}(W)\). By taking limits \(w_{b_3} \to w\) and \(w_{b_4} \to w\), we have
\[
(w_{b_3}^{d_i} - w_{b_4}^{d_i})^{-2} \lim_{w_{b_3} \to w_{b_4}} \circ x_i^\pm (\tau^\pm(b_1)w_{b_1})x_i^\pm (\tau^\pm(b_2)w_{b_2})x_i^\pm (\tau^\pm(b_3)w_{b_3})x_i^\pm (\tau^\pm(b_4)w_{b_4})^o \in \mathcal{E}_h^{(2)}(W).
\]
Thus, one gets
\[
H_3(w, w) := (w_{b_3}^{d_i} - w_{b_4}^{d_i})^{-2} \lim_{w_{b_3} \to w_{b_4}} \psi(w, w) \in \mathcal{E}_h^{(m)}(W).
\]
By definition we have
\[
\lim_{w_{b_3} \to w_{b_4}} (\psi(w, w) - (w_{b_3}^{d_i} - w_{b_4}^{d_i})(w_{b_3}^{d_i} - w_{b_4}^{d_i}))H_3(w, w) = 0.
\]
Then we can continue to define the current
\[
H_4(w, w) = (w_{b_3}^{d_i} - w_{b_4}^{d_i})^{-1}(\psi(w, w) - (w_{b_3}^{d_i} - w_{b_4}^{d_i})(w_{b_3}^{d_i} - w_{b_4}^{d_i}))H_3(w, w) \in \mathcal{E}_h^{(m+1)}(W).
\]
Again by the fact that \((b_1, b_2, b_3)\) is an \(A^{(2)}\)-triple one obtains
\[
H_1(w, w) = (w_{b_2}^{d_i} - w_{b_3}^{d_i})^{-1} \lim_{w_{b_1} \to w} \psi(w, w) \in \mathcal{E}_h^{(m)}(W),
\]
\[
H_2(w, w) = (w_{b_1}^{d_i} - w_{b_2}^{d_i})^{-1} \lim_{w_{b_3} \to w} \psi(w, w) \in \mathcal{E}_h^{(m)}(W).
\]
Then we have \(\lim_{w_{b_1} \to w} \psi(w, w) = (w_{b_1}^{d_i} - w_{b_1}^{d_i})H_1(w, w)\), while we also have
\[
\lim_{w_{b_1} \to w} (w_{b_2}^{d_i} - w_{b_3}^{d_i})H_4(w, w) = (w_{b_2}^{d_i} - w_{b_3}^{d_i}) \lim_{w_{b_1} \to w} H_4(w, w).
\]
Thus we get \(H_1(w, w) = \lim_{w_{b_1} \to w} H_4(w, w)\). Similarly, one has
\[
(w_{b_3}^{d_i} - w_{b_4}^{d_i})H_2(w, w) = \lim_{w_{b_2} \to w} \psi(w, w) = (w_{b_2}^{d_i} - w_{b_1}^{d_i}) \lim_{w_{b_2} \to w} H_4(w, w),
\]
which implies \(H_2(w, w) = - \lim_{w_{b_2} \to w} H_4(w, w)\). In summary, we have obtained
\[
\lim_{w_{b_1} \to w} H_1(w, w) = \lim_{w_{b_2} \to w} H_4(w, w) = \lim_{w_{b_2} \to w} H_4(w, w),
\]
Note that \(\hat{X}_{ij}^{\pm}(M, M_1, w, w) = \lim_{w_{b_1} \to w} \hat{X}_{ij}^{\pm}(M, M_1, w, w)\) and so we have
\[
\prod_{a \in M \setminus M_2} \frac{1}{w_a^{d_i} - w_a^{d_i}} \hat{X}_{ij}^{\pm}(M, M_1, w, w) = (w_{b_2}^{d_i} - w_{b_1}^{d_i})^{-1} \lim_{w_{b_1} \to w} \psi(w, w) = H_1(w, w).
\]
This gives
\[ X^\pm_{ij,r}(M, M_t, w) = \lim_{w_2 \to w} \prod_{a \in M \setminus M_t} \frac{1}{\tau^\pm(a)^{d_i}} \cdot H_1(w, w) = \lim_{w_2 \to w} \prod_{a \in M \setminus M_t} \frac{1}{\tau^\pm(a)^{d_i}} \cdot H_1(w, w). \]

Similarly, one can get that
\[ X^\pm_{ij,r}(M, M_t, w) = \lim_{w_2 \to w} \prod_{a \in M \setminus M_t} \frac{1}{\tau^\pm(a)^{d_i}} \cdot H_2(w, w) = \lim_{w_2 \to w} \prod_{a \in M \setminus M_t} \frac{1}{\tau^\pm(a)^{d_i}} \cdot H_2(w, w). \]

In view of \([8.5]\), we obtain
\[ X^\pm_{ij,r}(M, M_t, w) = -\tau^\pm(b_1)^{d_i} X^\pm_{ij,r}(M, M_t, w), \]

as required. Finally, the assertion follows from an induction argument on the nonnegative integer \(\sum_{c \in K^0} |t_c - t_c'|\).

Let \(\hat{t} = (t_c | c \in K_0)\) be the tuple such that \(t_c = 0\) for all \(c \in K^0\). Set \(\hat{M} = M_\hat{t} = M^0 \cup \{(-c, a_{ij} - 1) | c \in K^0 \text{ with } p_c \geq 0\}\).

For convenience, we also write
\[ X^\pm_{ij,r}(M, w) = X^\pm_{ij,r}(M, \hat{M}, w) \quad \text{and} \quad \circ x^\pm_i(z) X^\pm_{ij,r}(M, w)^0 = \circ x^\pm_i(z) X^\pm_{ij,r}(M, \hat{M}, w)^0. \]

**Lemma 8.3.** If \(s_t = 2\), then \(\circ x^\pm_i(\tau^\pm(a)w) X^\pm_{ij,r}(M, w)^0 = 0\) for \(a \in \hat{M}_1 \cup \hat{M}_2\), where
\[
\hat{M}_1 = \{(-c, a_{ij} - 1 + 2n) | c \in K^0, 1 \leq n \leq p_c^0\} \cup \{(c, a_{ij} - 2) | c \in K^0 \text{ with } p_c^0 \geq 0\},
\]
\[
\hat{M}_2 = \{(c, a_{ij} + 2n) | c \in K^0, 1 \leq n \leq p_c^0 - 1\} \cup \{(-c, a_{ij} + 2p_c^0 + 1) | c \in K^0 \text{ with } p_c^0 = p_c^1\}.
\]

**Proof.** Assume first that \(a \in \hat{M}_1\). Then it is easy to see that there exist \(b_1, b_2 \in \hat{M}\) such that \((a, b_1, b_2)\) is an \(A_2^{(2)}\)-triple. This together with \([8.2]\) gives \(\circ x^\pm_i(\tau^\pm(a)w) \tilde{X}^\pm_{ij,r}(M, \hat{M}, w, w)^0 = 0\) and hence \(\circ x^\pm_i(\tau^\pm(a)w) X^\pm_{ij,r}(M, w)^0 = 0\). For the case that \(a \in \hat{M}_2\), we need to introduce another tuple \(\hat{t} = (t_c | c \in K_0)\), which is defined by \(t_c = p_c^1\) if \(p_c^1 \geq 0\) and \(t_c = 0\) if \(p_c^1 = -1\). Then it is obvious that there exist \(b_1, b_2 \in M_\hat{t}\) such that \((a, b_1, b_2)\) is an \(A_2^{(2)}\)-triple. This implies that \(\circ x^\pm_i(\tau^\pm(a)w) \tilde{X}^\pm_{ij,r}(M, M_\hat{t}, w, w)^0 = 0\). In view of Lemma \([8.2]\) we also have \(\circ x^\pm_i(\tau^\pm(a)w) X^\pm_{ij,r}(M, \hat{M}, w, w)^0 = 0\) and hence \(\circ x^\pm_i(\tau^\pm(a)w) X^\pm_{ij,r}(M, w)^0 = 0\), as required. \(\square\)

As in Definition \([8.9]\), we introduce the following \(g\)-commutator:
\[
[x^\pm_i(z), X^\pm_{ij,r}(M, w)]_g = x^\pm_i(z) X^\pm_{ij,r}(M, w) - g_{ji}(z/w)^{\mp 1} \prod_{a \in M} g_{ii}(z/\tau^\pm(a)w)^{\mp 1} X^\pm_{ij,r}(M, w)x^\pm_i(z),
\]
and for \( a \in \partial M \), we define
\[
\alpha^\pm(a, M, w) = \begin{cases} 
\lim_{z \to \tau^\pm(a)w} f_{ij}^\pm(z, w) \prod_{a \in M} f_{ii}^\pm(z, \tau^\pm(a)w) & \text{if } a \in \partial M \\setminus \partial^* M \\
\lim_{z \to \tau^\pm(a)w} f_{ij}^\pm(z, w) \prod_{a \in M} f_{ii}^\pm(z, \tau^\pm(a)w)^2 & \text{if } a \in \partial^* M.
\end{cases}
\]

**Proposition 8.4.** Let \( M \in \mathcal{M}_r \) with \( r \in \mathbb{N} \). Then for any \( a \in \partial M \), \( \alpha^\pm(a, M, w) \) lies in \( \mathbb{C}((w))[[[h]]] \) and is nonzero. Furthermore, we have
\[
[x_i^\pm(z), X_{ij,r}^\pm(M, w)]_g = \sum_{a \in \partial M} \alpha^\pm(a, M, w) X_{ij,r+1}^\pm(M \cup \{ a \}, w)z^{-d_i}\delta \left( \frac{\tau^\pm(a)d_iw^d_i}{z^d_d} \right).
\]

**Proof.** Set \( D(z, w) = f_{ij}^\pm(z, w) \prod_{a \in M} f_{ii}^\pm(z, \tau^\pm(a)w) \). Then it follows from (5.12) that
\[
[x_i^\pm(z), X_{ij,r}^\pm(M, w)]_g = \delta x_i^\pm(z) X_{ij,r}^\pm(M, w) \circ (t_{z, w} - t_{w, z}) (D(z, w)^{-1}).
\]

Assume first that \( s_i = 1 \). Note that in this case
\[
D(z, w) = \prod_{c \in K^0} (z_d - c q_i^{\pm a_{ij}d_i \pm 2 p_0^d_i \pm 2 d_i} w^{d_i}),
\]
and so for \( a = (c, a_{ij} + 2 p_0^d_i + 2) \in \partial M \) we have
\[
\alpha^\pm(a, M, w) = \prod_{c \notin c' \in K^0} \left( c q_i^{\pm a_{ij}d_i \pm 2 p_0^d_i \pm 2 d_i} w^{d_i} - c' q_i^{\pm a_{ij}d_i \pm 2 p_0^d_i \pm 2 d_i} w^{d_i} \right)^{-1},
\]
which is nonzero. Furthermore, by (5.14), (8.6), (8.7) and (8.8) we have
\[
[x_i^\pm(z), X_{ij,r}^\pm(M, w)]_g = \sum_{a \in \partial M} \alpha^\pm(a, M, w) X_{ij,r+1}^\pm(M \cup \{ a \}, w)z^{-d_i}\delta \left( \frac{\tau^\pm(a)d_iw^d_i}{z^d_d} \right).
\]

Now we turn to consider the case that \( s_i = 2 \). In this case, we have
\[
D(z, w) = \prod_{c \in K^0} (z_d - c q_i^{\pm a_{ij}d_i \pm 2 p_0^d_i \pm 2 d_i} w^{d_i}) \prod_{c \notin c' \in K^0} (z_d - c q_i^{\pm a_{ij}d_i \pm 2 p_0^d_i \pm 2 d_i} w^{d_i})
\]
\[
\times \prod_{c \in K^0 \setminus \{0\}} \left( z_d - c q_i^{\pm a_{ij}d_i \pm 2 p_0^d_i \pm 2 d_i} w^{d_i} \right) \prod_{c \notin c' \in K^0 \setminus \{0\}} \left( z_d - c q_i^{\pm a_{ij}d_i \pm 2 p_0^d_i \pm 2 d_i} w^{d_i} \right)
\]
\[
= \prod_{a \in \partial M \setminus \partial^* M} (z_d - \tau^\pm(a) w^d_a) \prod_{a \in \partial^* M} \left( z_d - \tau^\pm(a) w^d_a \right)^2.
\]

This shows that \( \alpha^\pm(a, M, w) \) is nonzero. Furthermore, one can conclude from (5.14), (8.6) and Lemma \( \text{**Lemma**} \)
\[ [x_i^{\pm}(z), X_{ij,r}^{\pm}(M, w)]_g = \sum_{a \in \partial M \setminus \partial^* M} z^{d_i - \tau^\pm(a) d_i w_i d_i} \circ x_i^{\pm}(z) X_{ij,r}^{\pm}(M, w) \circ \delta \left( \frac{\tau^\pm(a) d_i w_i}{z^{d_i}} \right) + \sum_{a \in \partial^* M} \frac{z^{d_i - \tau^\pm(a) d_i w_i d_i}}{D(z, w)} \circ x_i^{\pm}(z) X_{ij,r}^{\pm}(M, w) \circ \frac{\partial}{\partial \tau^\pm(a) d_i w_i d_i} z^{-d_i} \delta \left( \frac{\tau^\pm(a) d_i w_i}{z^{d_i}} \right). \]

Note that for \( a \in \partial M \setminus \partial^* M \), we have
\[
\lim_{z \to \tau^\pm(a) w} z^{d_i - \tau^\pm(a) d_i w_i d_i} \circ x_i^{\pm}(z) X_{ij,r}^{\pm}(M, w) \circ = \alpha^\pm(a, M, w) X_{ij,r+1}^{\pm}(M \cup \{a\}, w). \]

On the other hand, for \( a \in \partial^* M \), set \( E_a(z^{d_i}, w^{d_i}) = (z^{d_i} - \tau^\pm(a) d_i w_i d_i)^2 \cdot D(z, w)^{-1} \). Recall from Lemma 8.3 that \( \circ x_i^{\pm}(\tau^\pm(a)w) X_{ij,r}^{\pm}(M, w) \circ = 0 \) and so \( z^{d_i} - \tau^\pm(a) d_i w_i d_i \) divides \( \circ x_i^{\pm}(z) X_{ij,r}^{\pm}(M, w) \circ = 0 \). Furthermore, by definition we have
\[
X_{ij,r+1}^{\pm}(M \cup \{a\}, w) = \lim_{z \to \tau^\pm(a) w} \frac{1}{z^{d_i} - \tau^\pm(a) d_i w_i d_i} \circ x_i^{\pm}(z) X_{ij,r}^{\pm}(M, w) \circ.
\]

Now we have
\[
E_a(z^{d_i}, w^{d_i}) \circ x_i^{\pm}(z) X_{ij,r}^{\pm}(M, w) \circ \frac{\partial}{\partial \tau^\pm(a) d_i w_i d_i} z^{-d_i} \delta \left( \frac{\tau^\pm(a) d_i w_i}{z^{d_i}} \right) = - \frac{\partial}{\partial z^{d_i}} \left( E_a(z^{d_i}, w^{d_i}) \circ x_i^{\pm}(z) X_{ij,r}^{\pm}(M, w) \circ z^{-d_i} \delta \left( \frac{\tau^\pm(a) d_i w_i}{z^{d_i}} \right) \right) + \lim_{z \to \tau^\pm(a) w} \left( \frac{\partial}{\partial z^{d_i}} \left( E_a(z^{d_i}, w^{d_i}) \circ x_i^{\pm}(z) X_{ij,r}^{\pm}(M, w) \circ \right) \right) z^{-d_i} \delta \left( \frac{\tau^\pm(a) d_i w_i}{z^{d_i}} \right) = \alpha^\pm(a, M, w) X_{ij,r+1}^{\pm}(M \cup \{a\}, w) z^{-d_i} \delta \left( \frac{\tau^\pm(a) d_i w_i}{z^{d_i}} \right).
\]

This completes the proof of proposition. \( \square \)

Let \( r \) be a nonnegative integer. Set
\[
\bar{r} = 0 \text{ if } r = 0 \quad \text{and} \quad \bar{r} = \{1, 2, \ldots, r\} \text{ if } r > 0.
\]

For \( M \in \mathcal{M}_r \), a bijective map \( \theta: \bar{r} \to M \) is called a flag of \( M \) if \( r = 0 \) or \( r > 0 \) and for each \( 1 \leq k \leq r, \theta(\bar{r} \setminus \bar{k}) \in \mathcal{M}_{r-k} \) and \( \theta(k) \in \partial \theta(\bar{r} \setminus \bar{k}) \). Denote by \( \mathcal{F}_M \) the set of all flags of \( M \). For \( \theta \in \mathcal{F}_M \), set \( \alpha^\pm_{ij,r}(M, \theta, w) = 1 \) if \( r = 0 \) and
\[
\alpha^\pm_{ij,r}(M, \theta, w) = \prod_{k=1}^{r} \alpha^\pm(\theta(k), \theta(\bar{r} \setminus \bar{k}), w) \quad \text{if } r > 0.
\]
For \( r \in \mathbb{N}, M \in \mathcal{M}, \) and \( \theta \in \mathcal{F}_M, \) set \( \delta^\pm(M, \theta, \underline{z}, w) = 1 \) if \( r = 0 \) and set
\[
\delta^\pm(M, \theta, \underline{z}, w) = \prod_{a=1}^{r} z_a \cdot \delta^{\frac{(\tilde{\theta}^\pm(a) d_i + d_i)}{\varepsilon d_i}} \quad \text{if} \quad r > 0,
\]
where \( \tilde{\theta}^\pm \) denotes the composition map \( \tau^\pm \circ \theta. \) Note that for \( \sigma \in S_r, \) we have
\[
(8.9) \quad \delta^\pm(M, \theta, \sigma \underline{z}, w) = \delta^\pm(M, \theta \circ \sigma^{-1}, \underline{z}, w),
\]
where \( (\sigma z)(a) = z_{\sigma(a)} \) for \( a = 1, \ldots, r. \)

By using Proposition 8.4 and an induction argument, one can obtain the following main result of this subsection.

**Proposition 8.5.** For a nonnegative integer \( r, \) one has
\[
[x_i^+(z_1), x_i^+(z_2), \ldots, x_i^+(z_r), x_i^+(w)]_g = \sum_{M \in \mathcal{M}} \sum_{\theta \in \mathcal{F}_M} \alpha_{ij,r}^\pm(M, \theta, w) X_{ij,r}^\pm(M, \theta, w) \delta^\pm(M, \theta, \underline{z}, w).
\]

### 8.2. Relations among \( \alpha_{ij,r}^\pm(M, \theta, w). \) Throughout this subsection, we fix an \( M \in \mathcal{M} \) with \( r \in \mathbb{Z}_+. \) The main goal of this subsection is to establish some relations among \( \alpha_{ij,r}^\pm(M, \theta, w) \) with \( \theta \in \mathcal{F}_M. \)

We first define a partial order \( \prec \) on \( M \) as follows: \( a \prec b \) if and only if there exist a sequence \( a = a_0, a_1, \ldots, a_k = b \) in \( M \) such that \( a_p \prec a_{p+1} \) for \( p = 0, \ldots, k-1, \) where for \( (c_1, n_1), (c_2, n_2) \in M, (c_1, n_1) \prec (c_2, n_2) \) means that either \( c_1 = c_2 \) and \( n_1 = n_2 + 2 \) or \( c_2 \in K^0, c_1 = -c_2 \) and \( n_1 = -n_2 - 1. \) It is straightforward to see that a bijective map \( \theta : r \to M \) is a flag if and only if
\[
(8.10) \quad \theta(a) \prec \theta(b) \quad \text{implies} \quad a < b \quad \text{for} \ a, b \in r.
\]

Furthermore, let \( (c_1, n_2), (c_2, n_2) \in M. \) Then \( F_{ij}^\pm(\tau^\pm((c_1, n_1)) w, \tau^\pm((c_2, n_2)) w) = 0 \) if and only if
\[
(8.11) \quad \text{either} \quad (c_1, n_1) \prec (c_2, n_2) \quad \text{or} \quad c_2 \in K^1, \ c_1 = -c_2 \text{ and } n_1 = n_2 - 1.
\]

Note that in the latter case one has
\[
(8.12) \quad (c_2, n_2) \prec (c_2, n_2 + 1), \ (c_2, n_2 - 2) \prec (c_2, n_2 - 1) = (c_1, n_1).
\]

For \( \theta \in \mathcal{F}_M, \sigma \in S_r, k = 1, \ldots, r, \) set
\[
F_{\theta, \sigma, k}^\pm(w) = \prod_{k=1}^{a} F_{ii}^\pm(\tilde{\theta}^\pm(a)w, \tilde{\theta}^\pm(b)w),
\]
\[
G_{\theta, \sigma, k}^\pm(w) = \prod_{k=1}^{a} G_{ii}^\pm(\tilde{\theta}^\pm(a)w, \tilde{\theta}^\pm(b)w).
\]

**Lemma 8.6.** For \( \theta \in \mathcal{F}_M \) and \( \sigma \in S_r, \theta \circ \sigma \in \mathcal{F}_M \) if and only if \( F_{\theta, \sigma, 1}^\pm(w) \neq 0. \)

**Proof.** In view of (8.10), it suffices to prove that \( F_{\theta, \sigma, 1}^\pm(w) = 0 \) if and only if there exist \( 1 \leq a < b \leq \sigma^{-1}(a) \) such that \( \sigma^{-1}(a) > \sigma^{-1}(b) \) and \( \theta(a) \prec \theta(b). \) For \( 1 \leq a < b \leq \sigma^{-1}(a) \) and \( (c_1, n_1), (c_2, n_2) \) that \( F_{ij}^\pm(\tilde{\theta}^\pm(a)w, \tilde{\theta}^\pm(b)w) = 0 \) if and only if \( \theta(a) \prec \theta(b). \) This in particular implies that, if \( F_{\theta, \sigma, 1}^\pm(w) = 0, \) then there exist \( a < b \) such that \( \sigma^{-1}(a) > \sigma^{-1}(b) \) and \( \theta(a) \prec \theta(b). \) Conversely, let \( a < b \) such that \( \sigma^{-1}(a) > \sigma^{-1}(b) \) and \( \theta(a) \prec \theta(b). \) Then there exists a sequence \( a = a_1 < a_2 < \cdots < a_k = b \) such that
Thus the limit $(k < b)$

\[ L_k^\pm \left(z, w \right) = \lim_{\substack{t \to \theta(z) \left( l \right) \w \ t \to \sigma^{-1}(a) > \sigma^{-1}(b) \ 
\text{for all } a \leq b \leq r \}} \prod_{i=1}^{r} \lim_{\substack{t \to \theta(z) \left( l \right) \w \ t \to \sigma^{-1}(a) > \sigma^{-1}(b) \ 
\text{for all } a \leq b \leq r \}} \frac{F_i^\pm (z_a, z_b)}{G_i^\pm (z_a, z_b)} \]

exists in $\mathbb{C}((z_1))((z_2)) \cdots ((z_k-1))((w))[[h]]$. Furthermore, the limit $L_k^\pm \left(z, w \right) = 0$ if and only if $F_{\theta, \sigma, k}(w) = 0$.

**Proof.** We will prove the assertion by induction on $k$. The case $k = r$ is trivial and we assume that the assertion hold for all $s > k$. Set

\[ F_k(w) = \prod_{k < a \leq r, \sigma^{-1}(k) > \sigma^{-1}(a)} F_i^\pm (\theta^\pm(k)w, \bar{\theta}^\pm(a)w), \]

\[ G_k(w) = \prod_{k < a \leq r, \sigma^{-1}(k) > \sigma^{-1}(a)} G_i^\pm (\theta^\pm(k)w, \bar{\theta}^\pm(a)w). \]

Note that if $L_{k+1}^\pm \left(z, w \right) = 0$, then $L_k^\pm \left(z, w \right) = 0$ and $F_{\sigma, \theta, k}(w) = F_k^\pm (w)F_{\sigma, \theta, k+1}(w) = 0$. Thus it suffices to consider the case that $L_{k+1}^\pm \left(z, w \right) \neq 0$ (and so $F_{\sigma, \theta, k+1}(w) \neq 0$). Assume first that $G_k(w) \neq 0$. Then we have

\[ L_k^\pm \left(z, w \right) = 0 \iff \frac{F_k^\pm (w)}{G_k^\pm (w)} = 0 \iff \bar{F}_k^\pm (w) = 0 \iff F_{\sigma, \theta, k}(w) = 0. \]

Assume next that $G_k(w) = 0$. Then there exists $b \in \bar{r}$ such that $k < b$, $\sigma^{-1}(k) > \sigma^{-1}(b)$ and $G_i^\pm (\theta^\pm(k)w, \bar{\theta}^\pm(b)w) = 0$. Recall that $G_i^\pm (z, w) = -F_i^\pm (w, z)$. Then it follows from (8.11) that either $\theta(b) > \theta(k)$ or $\theta(k) = (c, n), \theta(b) = (-c, n + 1)$ with $c \in K^1$. Since $k < b$, it follows from (8.10) that the former case is impossible. Recall from (8.12) that $(c, n - 2), (-c, n + 1) \in M$ with $\theta(k) \neq (c, n + 2), (-c, n + 1) \neq \theta(b)$. Take $b_1, b_2 \in \bar{r}$ such that $\theta(b_1) = (c, n - 2)$ and $\theta(b_2) = (-c, n + 1)$. In view of (8.10) and (8.11), we obtain $k < b_l < b$ and $F_i^\pm (\bar{\theta}(k)w, \theta(b_l)w) = 0 = F_i^\pm (\bar{\theta}(b_l)w, \theta(b)w)$ for $l = 1, 2$. From the induction assumption $F_{\theta, \sigma, k+1}(w) \neq 0$, it follows that $\sigma^{-1}(b_l) < \sigma^{-1}(b)$ and hence $\sigma^{-1}(k) > \sigma^{-1}(b_l)$. Thus we find that $\bar{F}_k^\pm (w) = 0$ and hence $F_{\sigma, \theta, k}(w) = 0$. Finally, notice that

\[ F_i^\pm (z_k, \theta^\pm(b_1)w) = (z_k^{d_1} + q^{\pm d_1} \theta^\pm(b_1)w d_1)(z_k^{d_1} - q^{\pm 2d_1} \theta^\pm(b_1)w d_1), \]

\[ F_i^\pm (z_k, \theta^\pm(b_2)w) = (z_k^{d_1} - q^{\pm 4d_1} \theta^\pm(b_2)w d_1)(z_k^{d_1} + q^{\pm 4d_1} \theta^\pm(b_2)w d_1), \]

\[ G_i^\pm (z_k, \bar{\theta}(b)w) = (q^{\pm d_1} z_k^{d_1} + \bar{\theta}(b)w d_1)(q^{\pm 2d_1} z_k^{d_1} - \bar{\theta}(b)w d_1). \]

Thus the limit $L_k^\pm \left(z, w \right)$ exists and is zero, as required. \qed

For convenience, we set $\alpha_{ij, \theta}^\pm (M, \theta, w) = 0$ when $\theta : \bar{r} \to M$ is a bijective map such that $\theta \notin F_M$. The following result is the main result of this subsection.
Proposition 8.8. Let \( r \in \mathbb{Z}_+, \sigma \in S_r, M \in \mathcal{M}_r \) and \( \theta \in \mathcal{F}_M \). Then we have

\[
\alpha^\pm_{ij,r}(M, \theta \circ \sigma, w) = \alpha^\pm_{ij,r}(M, \theta, w) \lim_{l \to 1} \prod_{l=1}^{r} \lim_{z_i \to \theta z_i(l)w} \prod_{1 \leq a < b \leq r} \frac{F^\pm_{ii}(z_a, z_b)}{G^\pm_{ii}(z_a, z_b)}.
\]

Proof. In view of Lemmas 8.6 and 8.7, it suffices to prove the assertion for the case that \( \theta \circ \sigma \in \mathcal{F}_M \). For each \( 1 \leq a < r \), write \( \sigma_a = (a, a+1) \in S_r \). From the definition of \( g \)-commutators (see (5.11)) and the relations (Q8), it follows that

\[
F^\pm_{ii}(z_a, z_{a+1})[x_i^\pm(z_1), x_i^\pm(z_2), \ldots, x_i^\pm(z_r), x_j^\pm(w)]_g
=G^\pm_{ii}(z_a, z_{a+1})[x_i^\pm(z_{\sigma_a(1)}), x_i^\pm(z_{\sigma_a(2)}), \ldots, x_i^\pm(z_{\sigma_a(r)}), x_j^\pm(w)]_g.
\]

By an induction argument, we also have

\[
\prod_{1 \leq a < b \leq r} \prod_{\sigma^{-1}(a) > \sigma^{-1}(b)} F^\pm_{ii}(z_a, z_b)[x_i^\pm(z_1), x_i^\pm(z_2), \ldots, x_i^\pm(z_r), x_j^\pm(w)]_g
= \prod_{1 \leq a < b \leq r} \prod_{\sigma^{-1}(a) > \sigma^{-1}(b)} G^\pm_{ii}(z_a, z_b)[x_i^\pm(z_1), x_i^\pm(z_2), \ldots, x_i^\pm(z_r), x_j^\pm(w)]_g.
\]

Combining this with Proposition 8.5 and (8.9), we obtain:

\[
\sum_{M \in \mathcal{M}} \sum_{\theta \in \mathcal{F}_M} F^\pm_{\theta, \sigma, 1}(w) \alpha^\pm_{ij,r}(M, \theta, w) X^\pm_{ij,r}(M, w) \delta^\pm(M, \theta, z, w)
= \sum_{M \in \mathcal{M}} \sum_{\theta \in \mathcal{F}_M} G^\pm_{\theta, \sigma, 1}(w) \alpha^\pm_{ij,r}(M, \theta, w) X^\pm_{ij,r}(M, w) \delta^\pm(M, \theta, z, w)
= \sum_{M \in \mathcal{M}} \sum_{\theta \in \mathcal{F}_M} G^\pm_{\theta, \sigma, 1}(w) \alpha^\pm_{ij,r}(M, \theta, w) X^\pm_{ij,r}(M, w) \delta^\pm(M, \theta \circ \sigma^{-1}, z, w)
= \sum_{M \in \mathcal{M}} \sum_{\theta \in \mathcal{F}_M} G^\pm_{\theta, \sigma, 1}(w) \alpha^\pm_{ij,r}(M, \theta \circ \sigma, w) X^\pm_{ij,r}(M, w) \delta^\pm(M, \theta, z, w).
\]

Now the proposition follows from Lemma 5.11 and Lemma 8.7. \( \square \)

8.3. Proof of Theorem 5.13. In this subsection, we complete the proof of Theorem 5.13. Throughout this subsection, let \( W \in \mathcal{R}_\mathbb{Z}_+^\tau, (i, j) \in I, m \in \mathbb{Z}_+, f^\pm(z, w) \in \mathbb{C}[[\hbar]][z^{\pm}, w^\pm] \) and \( B = (B_0, \ldots, B_m) \in (\mathbb{C}[[\hbar]])^{m+1} \) be as in Theorem 5.13.

For \( 0 \leq r \leq m \) and \( M \in \mathcal{M}_r \), we set

\[
D^\pm_{r,M} = \sum_{\sigma \in S_{m}} \sum_{s=r}^{m} \sum_{\{j_1 < \cdots < j_r\} \in \delta} \sum_{\{j_{r+1} < \cdots < j_s\} \in \delta \setminus J} (-1)^s B_s \prod_{1 \leq a < b \leq m} f^\pm(z_{\sigma(a)}, z_{\sigma(b)})
\times \prod_{1 \leq a < r \leq s} g_{ii}(z_{\sigma(j_a)}, z_{\sigma(j_b)}) \prod_{r < b \leq s} g_{ii}(z_{\sigma(j_b)/w}) \alpha^\pm_{ij,r}(M, \theta, w) X^\pm_{ij,r}(M, w)
\times \delta^\pm(M, \theta, z_{J_f}, w) x_i^\pm(z_{\sigma(j_{r+1})}) \cdots x_i^\pm(z_{\sigma(j_s)}) x_i^\pm(z_{\sigma(s+1)}) \cdots x_i^\pm(z_{\sigma(m+1)}).
\]

Recall the current \( D^\pm_{ij}(m, f^\pm, B) \) defined in Section 5.3 We have:
Lemma 8.9. As operators on $W$, one has

$$D_{ij}^+(m, f^\pm, B) = \sum_{r=0}^m \sum_{M \in \mathcal{M}_r} D_{r,M}^+.$$  

Proof. Take a subset $J = \{j_1 \prec \cdots \prec j_s\}$ of $\bar{r}$ and set $\bar{r} \setminus J = \{j'_1 \prec \cdots \prec j'_t\}$. By definition, it is straightforward to check that

$$x_i^+$$(z_0)\,[x_i^+(z_{j_1}), x_i^+(z_{j_2}), \cdots, x_i^+(z_{j_s})] \times \prod_{a \in J} g_{ij}(z_0/z_a)^{1/2} g_{ij}(z_0/w)^{1/2} \times x_i^+(z_0) x_i^+(z_{j'_1}) \cdots x_i^+(z_{j'_t}).$$

Using this and an induction argument on $r$, we have

$$x_i^+(z_1) x_i^+(z_2) \cdots x_i^+(z_r) x_i^+(w) = \sum_{J \subset \bar{r}} \left( \prod_{a \in J} g_{ij}(z_0/z_a)^{1/2} \prod_{b \notin J} g_{ij}(z_0/w)^{1/2} \times x_i^+(z_{j_1}) \times x_i^+(z_{j_2}) \cdots x_i^+(z_{j_s}) \right).$$

Then the assertion follows from the above formula and Proposition 8.5. □

For $r \in \mathbb{N}$ with $r \leq m$, let us set

$$(8.14) \quad \mathcal{N}_r = \{ M \in \mathcal{M}_r \mid p_c^0 < -a_{ij} \text{ for all } c \in K^0 \} \quad \text{and} \quad \mathcal{N}_r^c = \mathcal{M}_r \setminus \mathcal{N}_r.$$

Lemma 8.10. For each $r \in \mathbb{N}$, $M \in \mathcal{N}_r$ and $\theta \in \mathcal{F}_M$, the following holds

$$\lim_{z_s \rightarrow \theta^\pm(s)w} \prod_{1 \leq s \leq r} G_{ij}^+(z_a, w) \neq 0.$$

Proof. Notice that

$$\lim_{z_s \rightarrow \theta^\pm(s)w} \prod_{1 \leq s \leq r} G_{ij}^+(z_a, w) = \prod_{1 \leq a \leq r} \left( \theta^\pm(a)^{d_{ij}} q_i^{d_{ij}a_{ij}} w^{d_{ij}} - w^{d_{ij}} \right).$$

Assume that there exist some $a$ such that $\theta^\pm(a)^{d_{ij}} q_i^{d_{ij}a_{ij}} = 1$. Then $\theta(a) = (c, -a_{ij})$ for some $c \in K^0$. From the definition of $\mathcal{M}_r$, one has

$$(c, a_{ij} + 2p) \in M \quad \text{for } 0 \leq p \leq -a_{ij}.$$  

This implies $p_c^0 \geq -a_{ij}$, a contradiction with $M \in \mathcal{N}_r$. □

Let $r \in \mathbb{N}$ with $r \leq m$. As usual, we define an $(m, r)$-shuffle to be an element $\sigma \in S_m$ such that

$$(8.15) \quad \sigma(a) < \sigma(b), \quad \sigma(a') < \sigma(b') \quad \text{for } 1 \leq a < b \leq r < a' < b' \leq m.$$  

Denote by $S_{m,r}$ the set of all $(m, r)$-shuffles. And, for a subset $J$ of $\bar{m} = \{1, \ldots, m\}$, denote by

$$S_J = \{ \sigma \in S_m \mid \sigma(j) \in J \text{ and } \sigma(j') = j' \text{ for } j \in J, j' \notin J \}.$$  

Note that for $\sigma \in S_{m,r}$, we have

$$(8.16) \quad \sigma(a) \geq a \text{ if } a \leq r, \quad \sigma(b) \leq b \text{ if } b > r \quad \text{and} \quad \sigma(c) = c \text{ if } c > \sigma(r).$$
In particular, we have
\[(8.17) \quad \sigma(\tilde{s}) = \tilde{s} \quad \text{and} \quad \sigma(m \setminus \tilde{s}) = m \setminus \tilde{s} \quad \text{for} \quad \sigma \in S_{m,r} \quad \text{and} \quad s \geq \sigma(r).\]

It is straightforward to see that for \( \sigma \in S_m, \)
\[(8.18) \quad \prod_{1 \leq a < b \leq m} \frac{G_{ii}^+(z(\sigma(a), z(\sigma(b))))}{G_{ii}^-(z(a), z(b))} = (-1)^{|\sigma|} \prod_{1 \leq a < b \leq m, \sigma^{-1}(a) > \sigma^{-1}(b)} \frac{F_{ii}^+(z_a, z_b)}{F_{ii}^-(z_a, z_b)}.\]

Using (8.16) and (8.18), one can easily check that for \( \sigma \in S_{m,r} \) and \( s \geq \sigma(r), \)
\[(8.19) \quad \prod_{r < a \leq s} \frac{G_{ij}^+(z_a, w)}{F_{ij}^+(z_a, w)} = \prod_{1 \leq a < r} \frac{G_{ij}^+(z_a, w)}{F_{ij}^+(z_a, w)} \prod_{r < b \leq m} \frac{1}{F_{ij}^+(z_b, w)} \prod_{1 \leq a < s} \frac{G_{ij}^+(z_a, w)}{G_{ij}^+(z_a, w)} \prod_{s < b \leq m} \frac{F_{ij}^+(z_{\sigma^{-1}(a)}, w)}{F_{ij}^+(z_{\sigma^{-1}(a)}, w)}.
\]

For a subset \( H \) of \( S_m, \) we define the polynomial
\[P^\pm(H, \sigma, w) = \sum_{\sigma \in H} (-1)^{|\sigma|} Q^\pm(z_{\sigma^{-1}}, w),\]
where
\[Q^\pm(z, w) = \sum_{r=0}^{m} (-1)^r B_r \prod_{1 \leq a < b \leq m} f^+(z_a, z_b) G_{ij}^+(z_a, z_b) \prod_{a=0}^{r} G_{ij}^+(z_a, w) \prod_{b=r+1}^{m} F_{ij}^+(z_b, w).\]

Note that when \( H = S_m, \) the polynomials \( P^\pm(S_m, \sigma, w) \) coincide with \( P_{ij}^\pm(m, f^\pm, B) \) (see Section 5.3). It is straightforward to see that
\[(8.21) \quad P^\pm(H, \sigma, w) = (-1)^{|\sigma|} P^\pm(H \sigma^{-1}, \sigma, w) \quad \text{for} \quad \sigma \in S_m.\]

We have:

**Lemma 8.11.** Let \( r \in \mathbb{N} \) with \( r \leq m \) and \( M \in \mathcal{M}. \) Then
\[D_{r,M}^\pm = \sum_{\sigma \in S_m} \sum_{\sigma \in S_m} C_{r}^\pm(\sigma, w) P^\pm(S_{m,r}, \sigma, w) a_{ij,r}^\pm(M, \theta, w) X_{ij,r}^\pm(w)
\times \delta^\pm(M, \theta, \sigma, w) x_{i}^\pm(z_{\sigma(r+1)}) x_{i}^\pm(z_{\sigma(r+2)}) \ldots x_{i}^\pm(z_{\sigma(m)}),\]
where
\[C_{r}^\pm(\sigma, w) = \prod_{b=r+1}^{m} F_{ij}^+(z_b, w) \prod_{1 \leq b \leq r} G_{ij}^+(z_b, w) \prod_{1 \leq a < b \leq m} \frac{1}{G_{ij}^+(z_a, z_b)}.\]

**Proof.** We denote by \( T_{r,M}^\pm(\sigma, w) \) the currents:
\[\sum_{s=r}^{m} \sum_{J\cup\{j_1, \ldots, j_{s-1}\} \subseteq \mathcal{J}} (-1)^s B_s \prod_{1 \leq a < b \leq m} f^+(z_a, z_b)
\times \prod_{1 \leq a < b < s} g_{ij}(z_b / z_a)^{+1} \prod_{r < b \leq s} g_{ij}(z_b / z_a)^{\mp1} \alpha_{ij,r}^\pm(M, \theta, w) X_{ij,r}^\pm(M, w),\]

52
By using (8.17), $T_{r,M}^{±}(z, w)$ can be rewritten as:

$$
T_{r,M}^{±}(z, w) = \sum_{\sigma \in S_{m}, r} \sum_{s = \sigma(r)}^{m} \sum_{\theta \in \mathcal{F}_{M}} (-1)^{s} B_{s} \prod_{1 \leq a < b \leq m} f^{±}(z_{a}, z_{b})
\times \prod_{1 \leq a \leq r < b \leq s \leq \sigma(a) > \sigma(b)} g_{ij}(z_{\sigma(b)}/z_{\sigma(a)}) \prod_{r < b \leq s} g_{ji}(z_{\sigma(b)}/w)\alpha^{±}_{ij,r}(M, \theta, w)X_{ij,r}^{±}(M, w)
\times \delta^{±}(M, \theta, \bar{z}_{\sigma(r)}, w)x_{i}^{±}(z_{\sigma(r+1)}) \cdots x_{i}^{±}(z_{\sigma(m)}).
$$

This together with the fact $g_{ji}(z/w)\prod_{1 \leq a < b \leq m} f^{±}(z_{a}, z_{b})$ gives

$$
D_{r,M}^{±} = \sum_{\sigma \in S_{m}, r} \sum_{s = \sigma(r)}^{m} \sum_{\theta \in \mathcal{F}_{M}} (-1)^{s} B_{s} \prod_{1 \leq a < b \leq m} f^{±}(z_{a}, z_{b})
\times \prod_{1 \leq a \leq r < b \leq s \leq \sigma(a) > \sigma(b)} G_{si}^{±}(z_{\sigma(b)}/z_{\sigma(a)}) \prod_{r < b \leq s} G_{ji}^{±}(z_{\sigma(b)}/w)\alpha^{±}_{ij,r}(M, \theta, w)X_{ij,r}^{±}(M, w)
\times \delta^{±}(M, \theta, \bar{z}_{\sigma(r)}, w)x_{i}^{±}(z_{\sigma(r+1)}) \cdots x_{i}^{±}(z_{\sigma(m)}).
$$

Therefore, it follows from Lemma 8.10, 8.15, and 8.20 that
From Lemma 8.11, it follows that

\[ (8.22) \]

this with Lemma 8.9, one gets the well-defined current as desired. \( \square \)

**Lemma 8.12.** Let

\[ D_{r,M}^{\pm} = \sum_{s \in S_M} \sum_{\sigma \in S_m, r s = 0} \sum_{\theta \in F_M} (-1)^s B_s (-1)^{|\sigma|} \]

\[ \times \prod_{1 \leq a < b \leq m} f^{\pm} (\sigma_{a-1}(\sigma), \sigma_{b-1}(\sigma)) \]

\[ \times \prod_{1 \leq a < b \leq m} G_{ij}^{\pm} (\sigma_{a}(\sigma), w) \]

\[ \times \prod_{s < b \leq m} F_{ij}^{\pm} (\sigma_{a-1}(\sigma), w) \alpha_{ij,r}^{\pm} (M, \theta, w) X_{ij,r}^{\pm} (M, w) \delta^{\pm} (M, \theta, \sigma_{a-1}(\sigma), w) \]

\[ \times x_{i}^{\pm} (\sigma_{a+1}(\sigma)) \cdots x_{m}^{\pm} (\sigma_{m}(\sigma)) \]

\[ = \sum_{\theta \in F_M} \sum_{\sigma \in S_m} C_{r}^{\pm} (\sigma_{a}(\sigma), w) P^{\pm} (S_{m,r}, \sigma_{a}(\sigma), w) \alpha_{ij,r}^{\pm} (M, \theta, w) X_{ij,r}^{\pm} (w) \]

\[ \times \delta^{\pm} (M, \theta, \sigma_{a}(\sigma), w) x_{i}^{\pm} (\sigma_{a+1}(\sigma)) x_{m}^{\pm} (\sigma_{m}(\sigma)) \delta^{\pm} (M, \theta, \sigma_{a-1}(\sigma), w) \]

as desired.

It is straightforward to see that \( \prod_{1 \leq a < b \leq r} G_{ij}^{\pm} (\sigma_{a}(\sigma), \sigma_{b}(\sigma)) \) divides \( P^{\pm} (S_{m,r}, \sigma_{a}(\sigma), w) \). Combining this with Lemma 8.3, one gets the well-defined current

\[ (8.22) \]

\[ C_{r}^{\pm} (\sigma_{a}(\sigma), w) P^{\pm} (S_{m,r}, \sigma_{a}(\sigma), w) \delta^{\pm} (M, \theta, \sigma_{a}(\sigma), w) \]

where \( 0 \leq r \leq m, M \in M_r, \theta \in F_M \) and \( \sigma \in S_{r} \). Then we have

**Lemma 8.12.** Let \( 0 \leq r \leq m, M \in M_r \) and \( \theta \in M \). Then

\[ D_{r,M}^{\pm} = \sum_{\sigma \in S_m} C_{r}^{\pm} (\sigma_{a}(\sigma), w) P^{\pm} (S_{m,r}, \sigma_{a}(\sigma), w) \alpha_{ij,r}^{\pm} (M, \theta, \sigma, w) X_{ij,r}^{\pm} (M, w) \]

\[ \times x_{i}^{\pm} (\sigma_{a+1}(\sigma)) \cdots x_{m}^{\pm} (\sigma_{m}(\sigma)) \delta^{\pm} (M, \theta, \sigma_{a-1}(\sigma), w) \]

**Proof.** From Lemma 8.11 it follows that

\[ D_{r,M}^{\pm} = \sum_{\sigma \in S_m} \sum_{\theta \in F_M} C_{r}^{\pm} (\sigma_{a}(\sigma), w) P^{\pm} (S_{m,r}, \sigma_{a}(\sigma), w) \alpha_{ij,r}^{\pm} (M, \theta, \sigma, w) X_{ij,r}^{\pm} (M, w) \]

\[ \times x_{i}^{\pm} (\sigma_{a+1}(\sigma)) \cdots x_{m}^{\pm} (\sigma_{m}(\sigma)) \delta^{\pm} (M, \theta, \sigma_{a-1}(\sigma), w) \]
\[
\sum_{\sigma \in S_m} \sum_{\theta \in \mathcal{F}_M} C_r^\pm(\sigma z, w) P^\pm(S_{m, r}, \sigma z, w) \alpha_{ij,r}^\pm(M, \theta, w) X_{ij,r}^\pm(M, w)
\]
\[
\times x_i^\pm(\sigma(z_{r+1})) \cdots x_i^\pm(\sigma(z_m)) \delta^\pm(M, \theta M, \sigma^{-1} \theta M z, w)
\]
\[
= \sum_{\sigma \in S_m} \sum_{\theta \in \mathcal{F}_M} C_r^\pm(\sigma \theta^{-1} \theta z, w) P^\pm(S_{m, r}, \sigma \theta^{-1} \theta z, w) \alpha_{ij,r}^\pm(M, \theta M^{-1} \theta, w)
\]
\[
\times X_{ij,r}^\pm(M, w) x_i^\pm(\sigma(z_{r+1})) \cdots x_i^\pm(\sigma(z_m)) \delta^\pm(M, \theta M, \sigma z, w)
\]
\[
= \sum_{\sigma \in S_m} \sum_{\alpha_1 \in S_r} C_r^\pm(\sigma_1 z, w) P^\pm(S_{m, r}, \sigma_1 z, w) \alpha_{ij,r}^\pm(M, \theta M \circ \sigma_1, w)
\]
\[
\times X_{ij,r}^\pm(M, w) x_i^\pm(\sigma(z_{r+1})) \cdots x_i^\pm(\sigma(z_m)) \delta^\pm(M, \theta M, \sigma z, w)
\]
\]
where the second equation follows from Lemma 8.13 and (8.22), and the last one follows from (8.21) and the convention that
\[
\alpha_{ij,r}^\pm(M, \theta M \circ \sigma_1, w) = 0 \quad \text{if} \quad \theta M \circ \sigma_1 \notin \mathcal{F}_M.
\]

From Lemma 8.18 and Proposition 8.8 we have
\[
\alpha_{ij,r}^\pm(M, \theta M \circ \sigma_1, w) \delta^\pm(M, \theta M, \bar{z}, w) = (-1)^{\vert \sigma_1 \vert} \prod_{1 \leq a < b \leq m} \frac{G_{ij}^\pm(z_{\sigma_1(a)}, z_{\sigma_1(b)})}{G_{ij}^\pm(z_a, z_b)}
\]
\[
\times \alpha_{ij,r}^\pm(M, \theta M, w) \delta^\pm(M, \theta M, \bar{z}, w).
\]

(8.23)

Notice that
\[
C_r^\pm(\sigma_1 z, w) \prod_{1 \leq a < b \leq m} \frac{G_{ij}^\pm(z_{\sigma_1(a)}, z_{\sigma_1(b)})}{G_{ij}^\pm(z_a, z_b)} = C_r^\pm(z, w).
\]

Then one can conclude from (8.23) that
\[
C_r^\pm(\sigma_1 z, w) \alpha_{ij,r}^\pm(M, \theta M \circ \sigma_1, w) P^\pm(S_{m, r}, \sigma_1 z, w) \delta^\pm(M, \theta M, \bar{z}, w)
\]
\[
= (-1)^{\vert \sigma_1 \vert} C_r^\pm(z, w) \alpha_{ij,r}^\pm(M, \theta M, w) P^\pm(S_{m, r}, \sigma_1 z, w) \delta^\pm(M, \theta M, \bar{z}, w).
\]

Combining these with the equation (8.21), we have that
\[
D_{r, M}^\pm = \sum_{\sigma \in S_m} \sum_{\alpha \in S_r} C_r^\pm(\sigma z, w) (-1)^{\vert \sigma_1 \vert} P^\pm(S_{m, r}, \sigma z, w) \alpha_{ij,r}^\pm(M, \theta M, w)
\]
\[
\times X_{ij,r}^\pm(M, w) x_i^\pm(\sigma(z_{r+1})) \cdots x_i^\pm(\sigma(z_m)) \delta^\pm(M, \theta M, \sigma z, w)
\]
\[
= \sum_{\sigma \in S_m} C_r^\pm(\sigma z, w) P^\pm(S_{m, r}, \sigma z, w) \alpha_{ij,r}^\pm(M, \theta M, w)
\]
\[
\times X_{ij,r}^\pm(M, w) x_i^\pm(\sigma(z_{r+1})) \cdots x_i^\pm(\sigma(z_m)) \delta^\pm(M, \theta M, \sigma z, w).
\]

Therefore, we complete the proof of the lemma. \[\square\]

**Proposition 8.13.** If \(P^\pm(S_{m, z}, w) = 0\), then \(D_{r, M}^\pm = 0\) on \(W\) for all \(0 \leq r \leq m\) and \(M \in \mathcal{N}_r\).

**Proof.** From the assumption \(P^\pm(S_{m, z}, w) = 0\), we get
\[
P^\pm(S_{m, r}, \sigma_1 z, w) = \sum_{1 \neq \sigma_1 \in S_{m, \sigma} \setminus \sigma} (-1)^{\vert \sigma_1 \vert + 1} P^\pm(S_{m, r}, \sigma_1 z, w).
\]

55
Then it follows from Lemma 8.12 that

\[
D_{r,M}^\pm = \sum_{\sigma \in S_m} C_r^\pm(\sigma z, w)P_r^\pm(S_{m,r}S_r, \sigma z, w)\alpha_{ij,r}^\pm(M, \theta_M, w)X_{ij,r}^\pm(M, w)
\times x_i^\pm(z_{(r+1)}) \cdots x_i^\pm(z_{(m)})
\]

\[
= \sum_{\sigma \in S_m} \sum_{1 \neq \sigma_1 \in S_{m \setminus r}} (-1)^{\sigma_1 + 1} C_r^\pm(\sigma z, w)P_r^\pm(S_{m,r}S_r, \sigma z, w)\alpha_{ij,r}^\pm(M, \theta_M, w)X_{ij,r}^\pm(M, w)
\times x_i^\pm(z_{(r+1)}) \cdots x_i^\pm(z_{(m)})\delta^\pm(M, \theta_M, \sigma z, w).
\]

Notice that \(\prod_{r < a < b \leq m} F_{it}^\pm(z_a, z_b)\) divides \(P_{m}^\pm(S_{m,r}S_r, \sigma z, w)\). We deduce from (Q8) that

\[
C_r^\pm(z, w)P_r^\pm(S_{m,r}S_r, \sigma z, w)X_{ij,r}^\pm(M, w)\delta^\pm(M, \theta_M, \sigma z, w)
\times x_i^\pm(z_{(r+1)}) \cdots x_i^\pm(z_{(m)})
\]

\[
= (-1)^{\sigma_1} C_r^\pm(\sigma_1, w)P_r^\pm(S_{m,r}S_r, \sigma z, w)\alpha_{ij,r}^\pm(M, \theta_M, w)X_{ij,r}^\pm(M, w)
\times x_i^\pm(\sigma_1(z_{r+1})) \cdots x_i^\pm(\sigma_1(z_m)).
\]

Combining these with the fact \(\sigma z_1 = \sigma_1 z\), we get that

\[
D_{r,M}^\pm = - \sum_{\sigma \in S_m 1 \neq \sigma_1 \in S_{m \setminus r}} C_r^\pm(\sigma z, w)P_r^\pm(S_{m,r}S_r, \sigma z, w)
\times x_i^\pm(\sigma_1(z_{r+1})) \cdots x_i^\pm(\sigma_1(z_m))
\times x_i^\pm(z_{(r+1)}) \cdots x_i^\pm(z_{(m)})
\]

\[
= (1 - (m - r)!) \sum_{\sigma \in S_m} C_r^\pm(\sigma z, w)P_r^\pm(S_{m,r}S_r, \sigma z, w)
\times x_i^\pm(\sigma(z_{r+1})) \cdots x_i^\pm(\sigma(z_m))
\]

\[
= (1 - (m - r)!D_{r,M}^\pm.
\]

Therefore, we have that

\[
D_{r,M}^\pm = 0 \quad \text{for } 0 \leq r < m.
\]

For the case \(r = m\), it follows from the assumption of the proposition that \(D_{m,M}^\pm = 0\). This finishes the proof of proposition.

**Proof of Theorem 5.13.** First, in the setting of Theorem 5.13, one can conclude from Lemma 8.9 and Proposition 8.13 that

\[
D_{ij}^\pm(m, f^\pm, B) = \sum_{r = m_i}^m \sum_{M \in N_{rc}} D_{r,M}^\pm.
\]

Assume now that the relations (5.22) hold. Let \(M \in N_{rc}\) with \(r = m_i, \ldots, m\). Then there exists \(l = 0, \ldots, d_{ij}/d_i - 1\) such that \([l]\) is a subset of \(M\), where

\[
[l] = A^0_{d_{ij}/d_i - a_{ij}} = \left\{ (\xi_{d_{ij}/d_i}^l, a_{ij} + 2p) \mid 0 \leq p \leq -a_{ij} \right\}.
\]
One notices that
\[
\sum_{b \in [\ell]} x_i^+(\tau^+(b)w)x_j^-(w) = \circ x_i^+(q_i^{a_{ij}} \xi_i^{d_{ij}} w)x_i^+(q_i^{-a_{ij}}^{d_{ij}} \xi_i^{d_{ij}} w) \cdots x_i^+(q_i^{-a_{ij}}^{d_{ij}} \xi_i^{d_{ij}} w) = 0.
\]

This implies that \(\tilde{X}_{ij,M}(M, w) = 0\) and hence \(X_{ij,M}^\pm (M, w) = 0\). Then it follows from \([8, 24]\) that \(D_{ij}^\pm (M, f \pm, B) = 0\), which proves the first assertion in Theorem 5.13.

Assume next that \(m = m_{ij}, f^\pm (w, q_i^{\pm 2n} w) \neq 0\) for \(n = 1, \ldots, -a_{ij}\) with \(n_1 \neq n_2\) and \(D_{ij}^\pm (m, f \pm, B) = 0\). Note that with \(m = m_{ij}\) we have
\[
\mathcal{N}_m^\pm = \{[\ell] \mid 0 \leq \ell < d_{ij}/d_i - 1\},
\]
\[
\mathcal{F}_\ell = \{\theta_\ell : p \mapsto (\zeta_{d_{ij}/d_i}^{1/2} a_{ij} 2 - 2p)\}. 
\]

Then by applying \([8, 24]\) we have
\[
D_{ij}^\pm (m, f \pm, B) = (-1)^{m_{ij}} \sum_{\sigma \in S_{m_{ij}}} \sum_{\ell=0}^{d_{ij}/d_i-1} \prod_{1 \leq a < b \leq m_{ij}} f_{ij}^\pm (\zeta_{\sigma(a)}, \zeta_{\sigma(b)}) \alpha_{ij, m_{ij}}^\pm ([\ell], \theta_\ell, w) X_{ij, m_{ij}}^\pm ([\ell], w) \delta^\pm ([\ell], \theta_\ell, \sigma, w).
\]

From Lemma \([5, 13]\) we get that
\[
\prod_{1 \leq a < b \leq m_{ij}} f_{ij}^\pm (\zeta_{\sigma(a)}, \zeta_{\sigma(b)}) \alpha_{ij, m_{ij}}^\pm ([\ell], \theta_\ell, w) X_{ij, m_{ij}}^\pm ([\ell], w) \delta^\pm ([\ell], \theta_\ell, \sigma, w) \neq 0.
\]

Thus from the condition \(f^\pm (q_i^{\pm 2n_1} w, q_i^{\pm 2n_2} w) \neq 0\) for \(n_1, n_2 = 1, \ldots, -a_{ij}\) with \(n_1 \neq n_2\), it follows that
\[
X_{ij, m_{ij}}^\pm ([\ell], w) = 0 \quad \text{for } 0 \leq \ell < d_{ij}/d_i.
\]

Since
\[
X_{ij, m_{ij}}^\pm ([\ell], w) = \circ x_i^+(q_i^{a_{ij}} \xi_i^{d_{ij}} w)x_i^+(q_i^{a_{ij}} \xi_i^{d_{ij}} w) \cdots x_i^+(q_i^{a_{ij}} \xi_i^{d_{ij}} w) x_i^+(w) = 0,
\]
we complete the proof of Theorem 5.13.

\begin{thebibliography}{1}
\bibitem[1]{1} B. Allison, S. Azam, S. Berman, Y. Gao, and A. Pianzola. Extended affine Lie algebras and their root systems, \textbf{126}. Mem. Amer. Math. Soc., 1997.
\bibitem[2]{2} B. Allison, S. Berman, Y. Gao, and A. Pianzola. A characterization of affine Kac-Moody Lie algebras. \textit{Comm. Math. Phys.}, \textbf{185}, 671–688, 1997.
\bibitem[3]{3} B. Allison, S. Berman, and A. Pianzola. Multiloop algebras, iterated loop algebras and extended affine Lie algebras of nullity 2. \textit{J. Eur. Math. Soc.}, \textbf{16}, 327–385, 2014.
\bibitem[4]{4} B. Allison and Y. Gao. The root system and the core of an extended affine Lie algebra. \textit{Selecta Math.}, \textbf{7}, 149, 2001.
\bibitem[5]{5} J. Beck. Braid group action and quantum affine algebras. \textit{Comm. Math. Phys.}, \textbf{165}, 555–568, 1994.
\bibitem[6]{6} S. Berman, Y. Gao, and Y. Krylyuk. Quantum tori and the structure of elliptic quasi-simple Lie algebras. \textit{J. Funct. Anal.}, \textbf{135}, 339–389, 1996.
\bibitem[7]{7} F. Chen, S. Tan, and Q. Wang. Twisted \(\Gamma\)-algebras and their vertex operator representations. \textit{J. Alg.}, \textbf{442}, 202–232, 2015.
\bibitem[8]{8} V. Chernousov, P. Gille, and A. Pianzola. Conjugacy theorems for loop reductive group schemes and Lie algebras. \textit{Bull. Math. Sci.}, \textbf{4}, 281–324, 2014.
\end{thebibliography}
[9] F. Chen, N. Jing, F. Kong, and S. Tan. Twisted quantum affinizations and their vertex representations. *J. Math. Phys.*, **59**, 081701, 2018.

[10] F. Chen, N. Jing, F. Kong, and S. Tan. Twisted toroidal Lie algebras and Moody-Rao-Yokonuma presentation. to appear in *Sci. China Math.*. (arXiv:1901.00627).

[11] F. Chen, N. Jing, F. Kong, and S. Tan. Drinfeld type presentations of loop algebras. to appear in *Trans. Amer. Math. Soc.*. (arXiv:1902.00207).

[12] F. Chen, N. Jing, F. Kong, and S. Tan. On quantum toroidal algebra of type $A_1$. preprint, (arXiv:2006.14558).

[13] V. Chernousov, E. Neher, and A. Pianzola. On conjugacy of Cartan subalgebras in non-FGC Lie tori. *Transform. Groups*, **21**, 1003–1037, 2016.

[14] V. Chernousov, E. Neher, and A. Pianzola. Conjugacy of cartan subalgebras in ealas with a non-fgc centreless core. *Trans. Moscow Math. Soc.*, **78**, 235–256, 2017.

[15] V. Chernousov, E. Neher, A. Pianzola, and U. Yahorau. On conjugacy of cartan subalgebras in extended affine lie algebras. *Adv. Math.*, **290**, 260–292, 2016.

[16] V. Chari and A. Pressley. Twisted quantum affine algebras. *Comm. Math. Phys.*, **196**, 461–476, 1998.

[17] I. Damiani. Drinfeld realization of affine quantum algebras: the relations. *Publ. Res. Inst. Math. Sci.*, **48**, 661–733, 2012.

[18] I. Damiani. From the Drinfeld realization to the Drinfeld-Jimbo presentation of affine quantum algebras: injectivity. *Publ. Res. Inst. Math. Sci.*, **51**, 131–171, 2015.

[19] J. Ding and K. Iohara. Generalization of Drinfeld quantum affine algebras. *Lett. Math. Phys.*, **41**, 181–193, 1997.

[20] V. Drinfeld. Hopf algebras and quantum Yang-Baxter equation. *Soviet Math. Dokl.*, **283**, 1060–1064, 1985.

[21] V. Drinfeld. A new realization of Yangians and quantized affine algebras. *Soviet Math. Dokl.*, **36**, 212–216, 1988.

[22] P. Etingof and D. Kazhdan. Quantization of Lie bialgebras, I. *Selecta Math. (N.S.)*, **2**, 1–41, 1996.

[23] P. Etingof and D. Kazhdan. Quantization of Lie bialgebras, Part VI: quantization of generalized Kac–Moody algebras. *Groups, Groups*, **13**, 527–539, 2008.

[24] B Enriquez. On correlation functions of Drinfeld currents and shuffle algebras. *Transform. Groups*, **5**, 111–120, 2000.

[25] B. Enriquez. PBW and duality theorems for quantum groups and quantum current algebras. *J. Lie Theory*, **13**, 21–64, 2003.

[26] B. Feigin, M. Jimbo, T. Miwa and E. Mukhin, Representations of quantum toroidal $gl_n$. *J. Alg.*, **380**, 78–108, 2013.

[27] B. Feigin, M. Jimbo, T. Miwa and E. Mukhin, Branching rules for quantum toroidal $gl(n)$. *Adv. Math.*, **300**, 229–274, 2016.

[28] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, E. Mukhin. Quantum continuous $gl_{\infty}$: Tensor products of Fock modules and $V_q$-characters. *Kyoto J. Math.*, **51**, 365–392, 2011.

[29] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin. Quantum toroidal $gl_1$-algebra: Plane partitions. *Kyoto J. Math.*, **52**, 621–659, 2012.

[30] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. Representations of quantum toroidal $gl_n$. *J. Alg.*, **380**, 78–108, 2013.

[31] I. Frenkel, N. Jing, and W. Wang. Quantum vertex representations via finite groups and the McKay correspondence. *Comm. Math. Phys.*, **211**, 365–393, 2000.

[32] J. Fuchs, B. Schellekens, and C. Schweigert. From Dynkin diagram symmetries to fixed point structures. *Comm. Math. Phys.*, **180**, 39–97, 1996.

[33] Y. Gao and N. Jing. $U_q(gl_N)$ action on $gl_N$-modules and quantum toroidal algebras. *J. Alg.*, **273**, 320–343, 2004.

[34] V. Ginzburg, M. Kapranov, and E. Vasserot. Langlands reciprocity for algebraic surfaces. *Math. Res. Lett.*, **2**, 147–160, 1995.

[35] P. Gille and A. Pianzola. Torsors, reductive group schemes and extended affine Lie algebras. *Mem. Amer. Math. Soc.*, **226**, vi+112, 2013.

[36] P. Grossé. On quantum shuffle and quantum affine algebras. *J. Alg.*, **318**, 495–519, 2007.
[37] D. Hernandez. Representations of quantum affinizations and fusion product. *Transform. Groups*, **10**, 163–200, 2005.

[38] D. Hernandez. Drinfeld coproduct, quantum fusion tensor category and applications. *Proc. Lond. Math. Soc.*, **95**, 567–608, 2007.

[39] D. Hernandez. Kirillov–Reshetikhin conjecture: the general case. *Int. Mat. Res. Not.*, **2010**, 149–193, 2009.

[40] D. Hernandez. Quantum toroidal algebras and their representations. *Selecta Math.*, **14**, 701–725, 2009.

[41] R. Høegh-Krohn and B. Torresani. Classification and construction of quasisimple Lie algebras. *J. Funct. Anal.*, **89**, 106–136, 1990.

[42] M. Jimbo. A q-difference analogue of $U_q(g)$ and the Yang–Baxter equation. In *Yang–Baxter Equation In Integrable Systems*, 292–298. World Scientific, 1990.

[43] N. Jing. Twisted vertex representations of quantum affine algebras. *Invent. Math.*, **102**, 663–690, 1990.

[44] N. Jing. Quantum Kac-Moody algebras and vertex representations. *Lett. Math. Phys.*, **44**, 261–271, 1998.

[45] N. Jing and H. Zhang. Addendum to “Drinfeld realization of twisted quantum affine algebras”. *Commun. Algebra*, **38**, 3484–3488, 2010.

[46] N. Jing and H. Zhang. Drinfeld realization of quantum twisted affine algebras via braid group. *Adv. Math. Phys.*, 1–15, 2016.

[47] V. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, 1994.

[48] C. Kassel. *Quantum groups*. Graduate Texts in Mathematics, Springer-Verlag, New York, 1995.

[49] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. IV. *J. Amer. Math. Soc.*, **7**, 383–453, 1994.

[50] V. Kac and S. Wang. On automorphisms of Kac-Moody algebras and groups. *Adv. Math.*, **92**, 129–195, 1992.

[51] H. Li. A new construction of vertex algebras and quasi-modules for vertex algebras. *Adv. Math.*, **202**, 232–286, 2006.

[52] H. Li. ℏ-adic quantum vertex algebras and their modules. *Comm. Math. Phys.*, **296**, 475–523, 2010.

[53] J. Lepowsky and H. Li. *Introduction to vertex operator algebras and their representations*, **227**, Birkhäuser Boston Incorporation, 2004.

[54] G. Lusztig. Quantum deformations of certain simple modules over enveloping algebras. *Adv. Math.*, **70**, 237–249, 1988.

[55] G. Lusztig. *Introduction to quantum groups*. Springer Science & Business Media, 2010.

[56] E. Neher. Extended affine Lie algebras. *C.R. Math. Acad. Sci. R. Can.*, **26**, 90–96, 2004.

[57] E. Neher. Extended affine Lie algebras and other generalizations of affine Lie algebras—a survey. In *Developments and trends in infinite-dimensional Lie theory*, 53–126. Springer, 2011.

[58] Y. Saito. Quantum toroidal algebras and their vertex representations. *Publ. Res. Inst. Math. Sci.*, **34**, 155–177, 1998.

[59] Y. Saito, K. Takemura, and D. Uglov. Toroidal actions on level 1 modules of $U_q\left(\hat{sl}_n\right)$. *Transform. Groups*, **3**, 75–102, 1998.

[60] K. Takemura and D. Uglov. Level-0 action of $U_q\left(\hat{sl}_n\right)$ on the q-deformed Fock spaces. *Comm. Math. Phys.*, **190**, 549–583, 1998.
[66] M. Varagnolo and E. Vasserot. Schur duality in the toroidal setting. Comm. Math. Phys., 182, 469–483, 1996.

[67] M. Varagnolo and E. Vasserot. Double-loop algebras and the Fock space. Invent. Math., 133, 133–159, 1998.

[68] Y. Yoshii. Lie tori – a simple characterization of extended affine Lie algebras. Publ. Res. Inst. Math. Sci., 42, 739–762, 2006.

School of Mathematical Sciences, Xiamen University, Xiamen, China 361005
E-mail address: chenf@xmu.edu.cn

Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA
E-mail address: jing@math.ncsu.edu

Key Laboratory of Computing and Stochastic Mathematics (Ministry of Education), School of Mathematics and Statistics, Hunan Normal University, Changsha, China 410081
E-mail address: kongmath@hunnu.edu.cn

School of Mathematical Sciences, Xiamen University, Xiamen, China 361005
E-mail address: tans@xmu.edu.cn