MAXIMAL SUBGROUPS OF $^2E_6(2)$ AND ITS AUTOMORPHISM GROUPS

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Abstract. We give a new computer-assisted proof of the classification of maximal subgroups of the simple group $^2E_6(2)$ and its extensions by any subgroup of the outer automorphism group $S_3$. This is not a new result, but no earlier proof exists in the literature. A large part of the proof consists of a computational analysis of subgroups generated by an element of order 2 and an element of order 3. This method can be effectively automated, and via statistical analysis also provides a sanity check on results that may have been obtained by delicate theoretical arguments.

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1. Introduction

The maximal subgroups of $2^{E_6}(2)$ and its automorphism groups are part of the folklore, but, as far as I am aware, no proof has ever been published. I believe the original result was obtained by some subset of Peter Kleidman, Simon Norton and myself, some time around 1989, but I cannot be entirely certain of that. In any case, it seems to be worthwhile to provide a new proof, in the interests of increasing confidence in the result.

Throughout, let $G$ be the simple group $2^{E_6}(2)$ of order $2^{36}3^95^27^211.13.17.19$. Its automorphism group has shape $G.S_3$, in which the diagonal automorphism group has order 3, and the field automorphism has order 2. We aim to prove that the maximal subgroups of $G$, $G.2$, $G.3$ and $G.S_3$ are as listed in the Atlas [2], apart from one or two minor errors in the structure of certain subgroups.

In Section 2 we prove the existence of the subgroups listed in the Atlas, with the required corrections, and deduce the existence of 39 isomorphism types of proper non-abelian simple subgroups of $G$. In Sections 3, 4 and 5 we classify the $p$-local subgroups in $G.S_3$ for all relevant primes $p$. In Section 6 we list as many conjugacy classes of simple subgroups as we can, with justification given in Sections 7, 8, 9, 10 and 11. This includes a complete classification of simple subgroups centralized by elements of order 5 or 7, and hence yields complete classification of non-abelian characteristically simple subgroups that are not simple.

We begin the non-local analysis in Section 12 by determining the proper non-abelian simple subgroups up to isomorphism. We then embark on the main part of the classification, first using structure constant analysis. Subgroups generated by $(2, 3, n)$ triples for $n = 5, 7, 11, 13, 17, 19$ are classified in Sections 13, 14, 15, 16, 17, 18 respectively. These results, summarized in Section 19, give sufficient information in 23 of the 39 cases. We then move on in Section 20 to methods using the embedding of $2^{E_6}(2):S_3$ in the Monster. Essentially we use Norton's extensive work on subgroups of the Monster to restrict the possibilities for maximal subgroups of $G$. This deals with a further 12 cases. The final four cases are dealt with in Section 21 and use detailed knowledge of subgroups of the Baby Monster, much of it obtained by computational means. The final Section 22 includes alternative, computer-free, proofs for some of the results.

Our notation throughout follows the Atlas [2]. In particular, $O$ is used for the generically simple groups of orthogonal type, for example $O^-_{16}(2)$ denotes the simple group of order 25920.

2. Existence of the Known Maximal Subgroups

There are four conjugacy classes of maximal parabolic subgroups of $G$, all of which extend to $G.S_3$, as follows:

- the centralizer $2^{1+20}:U_6(2)$ of a 2A-involution (a $\{3, 4\}$-transposition), extending to $2^{1+20}:U_6(2):S_3$ in $G.S_3$;
- a four-group normalizer $2^{2+9+18}:(L_3(4) \times S_3)$, extending to a group of shape $2^{2+9+18}:(L_3(4) \times S_3 \times S_3)$ in $G.S_3$;
- a 23-normalizer $2^{3+4+12+12}:(A_5 \times L_3(2))$: note that the structure of this group is given incorrectly in the Atlas; it extends to a group of shape $2^{3+4+12+12}:(A_5 \times L_3(2) \times S_3)$ in $G.S_3$;
- a group of shape $2^{8+16}:O^-_8(2)$, extending to $2^{8+16}:(O^-_8(2) \times 3):2$ in $G.S_3$. 

The following are maximal rank subgroups of $G$ which can be read off from the Dynkin diagram, and also extend to $G.S_3$.

- $S_3 \times U_6(2)$, of type $A_1 + 2A_5$, extending to $S_3 \times U_6(2):S_3$ in $G.S_3$;
- $O_{10}(2)$, of type $2D_5$, extending to $(O_{10}(2) \times 3):2$ in $G.S_3$;
- $L_3(2) \times L_3(4)$, of type $A_1 + A_1(q^2)$: this acquires an extra automorphism, giving $(L_3(2) \times L_3(4)):2_1$ in $G$, and extending to $(L_3(2) \times L_3(4):S_3):2$ in $G.S_3$; by looking at centralizers of outer element of order 14 in $G.2$, we can see that the $L_3(4):S_3$ that centralizes $L_3(2)$ contains automorphisms of type 22, in Atlas notation.

There are also the following subgroups of $G$ with their Lie type names:

- $F_4(2)$, in three conjugacy classes in $G$, extending to $F_4(2) \times 2$ in $G.S_3$;
- $(3 \times O^+_8(2)):3$:2, of type $T_1 + D_4$, extending to $(3^2:2 \times O^+_8(2)):S_3$ in $G.S_3$;
- $3D_4(2):3$, extending to $3D_4(2):3 \times S_3$ in $G.S_3$;
- $U_3(8):3$, of type $2A_1(q^3)$, extending to $(3 \times U_3(8)):3:2$ in $G.S_3$;
- $2^3:Q_8 \times U_3(3):2$, of type $2A_1 + G_2$, extending to $2^3:2S_4 \times U_3(3):2$ in $G.S_3$;
- $3^{1+6}:2^3+6:3^{1+2}:2^2$, of type $3^2:A_1$, extending to $3^{1+6}:2^3+6:3^{1+2}:2^2$ in $G.S_3$;
- $3^5:O_5(3):2$, of the normalizer of a maximal torus, extending to $3^6:(2 \times O_5(3):2)$ in $G.S_3$;

The normalizers of the groups $3D_4(2):3$ and $3^5:O_5(3):2$ are not maximal in $G$ or $G.2$, as they are contained in $F_4(2)$ and $O_7(3)$ respectively. As we shall see later on, their normalizers are however maximal when the diagonal automorphism of order 3 is adjoined.

Finally, as shown by Fischer, $G$ contains

- $Fi_{22}$, in three conjugacy classes in $G$, extending to $Fi_{22}:2$ in $G.S_3$;
- $O_7(3)$, in three conjugacy classes in $G$, extending to $O_7(3):2$ in $G.S_3$.

It turns out that the normalizer of $O_7(3)$ is maximal only in $G.2$.

In particular, $G$ contains subgroups isomorphic to $U_6(2)$, $O_{10}(2)$, $F_4(2)$, $U_3(8)$, and $Fi_{22}$. Using knowledge of the maximal subgroups of these subgroups [11 7 5], we obtain the following list of 39 isomorphism types of known simple subgroups of $G$:

- $A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}$,
- $L_2(7), L_2(8), L_2(11), L_2(13), L_2(16), L_2(17), L_2(25), L_3(3), L_3(4), L_4(3), L_5(3)$,
- $U_3(3), U_3(8), U_4(2), U_4(3), U_5(2), U_6(2)$,
- $O_7(3), O^+_8(2), O^-_8(2), O^-_{10}(2), S_4(4), S_6(2), S_4(2)$,
- $2F_4(2), 3D_4(2), G_2(3), F_4(2), M_{11}, M_{12}, M_{22}, Fi_{22}$.

We shall show in Section 3 below that every nonabelian simple proper subgroup of $G$ is isomorphic to one of these 39 groups.

### 3. Centralizers of outer automorphisms

The outer automorphism group of $G$ is $S_3$, and a number of maximal subgroups may be obtained as centralizers of outer automorphisms of $G$, of order 2 or 3. The outer automorphisms of order 2 are given in the Atlas [2], namely the elements in classes $2D$ and $2E$. Those of order 3 are not listed there, but are available in the GAP [3] character table of $G:3$. In Atlas notation these are elements in classes $3D_1$, $3E$, $3F$, $3G$ and their inverses. Piecing together information from these various sources, we find the structures of the centralizers in $G$ as follows:

- $C_G(2D) \cong F_4(2)$:
• \(C_G(2E) \cong [2^{15}]:S_6(2) < F_4(2)\);
• \(C_G(3D) \cong O'_{10}(2)\);
• \(C_G(3E) \cong 3D_4(2):3\);
• \(C_G(3F) \cong U_5(2) \times S_3 < S_3 \times U_6(2)\);
• \(C_G(3G) \cong U_3(8):3\).

4. \(p\)-LOCAL ANALYSIS FOR \(p \neq 3\)

In the simple group \(G\), and in the group \(G:3\) containing the diagonal automorphisms, the maximal 2-local subgroups are, by the Borel–Tits theorem, just the maximal parabolic subgroups, which are well-known. Since all outer automorphisms of \(E_6(2)\) are of diagonal or field type, all the parabolic subgroups are normalized by the full outer automorphism group, and no more maximal subgroups arise in any extension of \(G\) as normalizers of 2-subgroups of \(G\). In \(G:2\) the 2-local subgroups include the centralizers of outer automorphisms of \(G\) of order 2, which were considered in Section 3 above.

For the cyclic Sylow \(p\)-subgroups, that is, for \(p = 19, 17, 13, \) or 11, we have the following normalizers in \(G\):

• \(N(19) \cong 19:9 < U_3(8):3\);
• \(N(17) \cong 17:8 < O'_{10}(2)\);
• \(N(13) \cong 13:12 < 3D_4(2):3 < F_4(2)\);
• \(N(11) \cong S_3 \times 11:5 < S_3 \times U_6(2)\).

Extending to \(G.S_3\) we obtain the following.

• \(N(19) \cong (19:9 \times 3):2 < (3 \times U_3(8):3):2\);
• \(N(17) \cong 17:8 \times S_3 < (3 \times O'_{10}(2)):2\);
• \(N(13) \cong 13:12 \times S_3 < S_3 \times 3D_4(2):3\);
• \(N(11) \cong S_3 \times (3 \times 11:5):2 < S_3 \times U_6(2):S_3\).

We turn next to the Sylow subgroups of order \(p^2\), that is, \(p = 7\) or 5. The relevant normalizers in \(G\) as follows.

• \(N(7A) \cong (7:3 \times L_3(4)):2 < (L_3(2) \times L_3(4)):2\);
• \(N(7B) \cong (7:3 \times L_3(2)):2 < (L_3(2) \times L_3(4)):2\);
• the subgroup \(3D_4(2):3\) contains \(7^2:(3 \times 2A_4)\), which is the full \(7^2\)-normalizer, since the \(7^2\) is self-centralizing, and the stabilizer of any one of the cyclic 7-subgroups is \((7:3 \times 7:3):2\), and there are two classes, each of four such cyclic subgroups.
• \(N(5) \cong (D_{10} \times A_8):2 < O^{-}_{10}(2)\).
• the centralizer of the Sylow 5-subgroup is \(5^2 \times 3\), so the normalizer lies in \((3 \times O_4^+(2):3):2\), and therefore has shape \((3 \times 5^2:4A_4):2\).

These normalizers extend to \(G.S_3\) as follows.

• \(N(7A) \cong (7:3 \times L_3(4):S_3):2\);
• \(N(7B) \cong (7:3 \times S_3 \times L_3(2)):2\);
• \(N(7^2) \cong S_3 \times 7^2:(3 \times 2A_4)\);
• \(N(5) \cong (D_{10} \times (3 \times A_8)):2\);
• \(N(5^2) \cong (3^2:2 \times 5^2:4A_4):2\).

In particular, in this section we have proved the following.

**Theorem 1.** If \(p \geq 5\), then no \(p\)-local subgroup is maximal in \(G\) or any extension of \(G\) by outer automorphisms.
This leaves just the 3-local subgroups to determine.

5. The 3-local subgroups

Since we have already dealt with the centralizers of outer automorphisms of $G$ of order 3, we may restrict attention to the normalizers of elementary abelian 3-groups inside $G$. The three classes of subgroups of order 3 in $G$ have the following normalizers:

- $N(3A) \cong S_3 \times U_6(2)$;
- $N(3B) \cong (3 \times O_8^+(2)):3; 2$;
- $N(3C) \cong 3^{1+6} 2^{3+6} 3^2 \cdot 2 = 3 \cdot (3^2:Q_8 \times 3^2:Q_8 \times 3^2:Q_8) : 3^2 : 2$.

Note that in this last case, the Atlas \cite[p. 191]{Atlas} claims the quotient $3^2 : 2$ is isomorphic to $3 \times S_3$. That this is not the case is easily seen by comparing with the subgroup $3^{1+6} (2A_4 \times A_4) : 2$ of $O_7(3)$.

Next we classify the elementary abelian subgroups of order 9 in $G$. Note first that there is a maximal torus with normalizer $3^5 : O_5(3) : 2$, in which the isotropic points are of type $3C$ and the $+$ points of type $3B$, and the $-$ points of type $3A$. Hence we get the fusion of classes of elements of order 3 in $3 \times O_8^+(2)$ and $3 \times U_6(2)$. Alternatively, observe that the restriction of the 2-modular Brauer character of degree 78 to the former has constituents of degrees $2 + 28 + 48$, from which the fusion of outer elements of order 3 can also be read off. The full fusion of elements of order 3 is given in the following tables.

| $O_8^+(2)$: | 3ABC | 3D | 3E | 3F | 3G |
|------------|------|----|----|----|----|
| diagonal:  | 3AA  | 3BB | 3CC| 3BB| 3CC|
| $U_6(2)$:  | 3A   | 3B | 3C |    |    |
| diagonal:  | 3BB  | 3AA| 3CC|    |    |

Thus there are 6 classes of $3^2$ that contain $3A$ or $3B$ elements, and they have one of the types $3AABB$, $3AAAC$, $3ABCC$, $3BBBC$, $3BBBB$, or $3CCCC$. Note also that the centralizer in $O_8^+(2)$ of an element of class $3F$ is $U_5(3):2$, while the centralizer of an element of class $3G$ is $3^2 : 2A_4$, in which the normal $3^2$ consists of $3D$-elements, and all other 3-elements are in $3E$.

In order to determine the class fusion from $N(3C)$ we need to describe the structure of this group in some detail. There is a normal subgroup of index 18 formed from the central product of three copies of $3^{1+2} : Q_8$. Acting on this is (a) an element of order 3 extending each copy to $3^{1+2} : 2A_4$, (b) and element of order 3 permuting the three copies, and (c) an involution which swaps two copies and extends the third to $3^{1+2} : 2A_4$. Now the outer automorphism of order 3 of $G$ conjugates the elements of type (b) to the products of (b) with (a). Hence we do not need to consider these other two cosets separately.

As we know the $3^2$ groups of type $3AAAC$ and $3BBBC$, we can identify these with elements in one copy of $3^{1+2}$, and elements diagonal between two copies, respectively. Hence the elements diagonal between all three copies are in $3C$. Now in $3^{1+2} : 2A_4$, there are two types of outer elements: one is of order 3, and centralizes $3 \times 3^2 : 2$, while the other is of order 9, and is self-centralizing. It follows that in the coset of type (a) we see four types of groups of order $3^2$, according to how many of the factors $3^{1+2}$ contribute something of order 9. If none, we obtain a
$3^2$ with centralizer $3^2 \times 3^3:2^4:3$. If one, we obtain an element of order 9 with centralizer of order $3^4:2^2$, and if two, an element of order 9 and centralizer of order $3^4:2$. If all three, then we again obtain an elementary abelian $3^2$, whose centralizer is elementary abelian of order $3^4$. In the coset of type (b) we see two types of elementary abelian groups of order $3^2$, one with centralizer $3^2 \times 3^2:2A_4$, the other with centralizer elementary abelian of order $3^4$.

The first $3^2$ of type (a) must be of type $3ABCC$, and the $3^5$ normal in its centralizer is exactly the $3^5$ with normalizer $3^5:O_5(3):2$. The other $3^2$ of type (a) is necessarily pure $3C$. The first $3^2$ of type (b) is then forced to be of type $3BCCC$, and the second is again pure $3C$. In particular, there are just two classes of $3^2$ of pure $3C$ type in $G.S_3$. However, the second class splits into three classes in $G$.

We are now ready to classify the maximal 3-local subgroups. The strategy is to deal first with the elementary abelian groups of pure $3C$ type, then those that contain $3A$ elements, and finally those that contain $3B$ elements but not $3A$ elements.

**Lemma 1.** Every $3C$ pure elementary abelian group has normalizer in $G$ contained in one of the following groups:

- $3^{1+6}:2^{1+6}:3^2:2$;
- $3^5:O_5(3):2$;
- $3^{3+3}:L_3(3)$ (three classes).

**Proof.** Suppose we have an elementary abelian 3-group of pure $3C$ type containing the normal 3 in $N(3C)$. If it does not lie in $3^{1+6}$, then its centralizer is elementary abelian of order $3^4$, and the centralizer contains $3B$ elements. Moreover, the subgroup generated by these $3B$-elements has order $3^3$, and contains a unique $3C$-pure $3^2$, which lies inside $3^{1+6}$. Now some straightforward calculations show that the latter group contains three classes of $3C$-pure $3^3$ under the action of the group $(Q_8 \times Q_8 \times Q_8):3^2:2$, each with centralizer of order $3^6$. Such a centralizer consists of $3^2 \times 3^1:3$ inside $3^{1+6}$, together with an outer element which is of type (b) in one case, and a product of type (a) and type (b) in the other two cases. Thus these three classes are fused in $G.3$. Hence each such group has normalizer $3^{1+3}:L_3(3)$, which is already visible inside $O_7(3)$. Moreover, any $3^2$ subgroup of this $3^3$ has centralizer of shape $(3 \times 3^{1+4}).3^2$ which is the centralizer of an isotropic 2-space in $3^5:O_5(3):2$. Hence the normalizer of this $3^2$ is contained in the latter group, and has shape $3^5:3^{1+2}:2S_4$. $\square$

We turn next to the $3A$ elements.

**Lemma 2.** The normalizer of every elementary abelian subgroup of $G$ generated by $3A$ elements lies in one of the following:

- the normalizer of a pure $3C$ type elementary abelian group;
- $(S_3 \times S_3 \times U_4(2)):2$, contained in $O_{10}(2)$;
- the group $3^5:O_5(3):2$.

**Proof.** From the above class fusion, we see that any elementary abelian 3-group generated by $3A$ elements either contains $3B$ elements, or has a unique subgroup of index 3 that is pure $3C$. The latter case corresponds to subgroups of $U_6(2)$ of pure $3B$ type, and is covered by Lemma[II]. In $U_6(2)$ the centralizer of a $3A$ element is $3 \times U_4(2)$, and the centralizer of a $3C$ element is a soluble group of order $2^4.3^5$, and shape $3^4:(2 \times A_4)$. 


There is a unique $3^2$ of type $3AABB$, and it has normalizer $(S_3 \times S_3 \times U_4(2)):2$, contained in $O_{10}^-(2)$. Any $3^3$ generated by $3A$ elements lies inside $3^2 \times U_4(2)$, corresponding to one of the three classes of subgroups of order 3 in $U_4(2)$. Hence there are three types of $3^3$ generated by $3A$ elements and containing $3B$ elements. One has a unique subgroup of order 3 containing $3C$ elements, so we can ignore this case. The other two have either 3 cyclic subgroups of type $3A$, and 6 of type $3B$, or vice versa, and in both cases the centralizer contains a unique Sylow 3 subgroup, which is the torus of order $3^2$ described above.

Hence we reduce to considering elementary abelian 3-groups which contain $3B$ elements but no $3A$ elements.

**Lemma 3.** The normalizer in $G$ of every elementary abelian group that contains $3B$ elements but no $3A$ elements lies in one of the following:

- $N(3B)$;
- $3^2:Q_8 \times U_3(3):2$;
- $N(3A)$;
- $N(3C)$.

**Proof.** If the elementary abelian group lies in $3 \times O^+_8(2)$, then essentially the same argument as in Lemma 2 with $3A$ and $3B$ interchanged, proves that we are in one of the cases already considered. This is because in $O^+_8(2)$ we have $C(3A/B/C) \cong 3 \times U_4(2)$, while $C(3E)$ contains a unique elementary abelian $3^4$. If it contains an outer element of $O^+_8(2):3$, then either this is in class $3F$ or class $3G$ of $O^+_8(2):3$, in Atlas notation. In the $3F$ case, we obtain a pure $3B$ type $3^2$, with normalizer $3^2:Q_8 \times U_3(3):2$. Now the class fusion from $U_3(3)$ to $O^+_8(2)$ goes via $S_6(2)$ classes $3B$ and $3C$ respectively, so $O^+_8(3)$ classes $3D$ and $3E$. Thus any larger elementary abelian 3-group containing the $3^2$ of type $3BBBB$ either contains a unique cyclic subgroup containing $3A$ elements, or contains a unique cyclic subgroup containing $3C$ elements.

In the $3G$ case, the $3^2$ has type $3BCCC$, so its normalizer lies in $N(3B)$. Its centralizer in $O^+_8(2)$ is a group of shape $3^2:2A_4$, in which the normal $3^2$ consists of $3D$ elements, and all other 3-elements are in class $3E$. Again, $3E$ elements fuse to $3A$ in $G$, so can be excluded. We can also assume that our elementary abelian 3-group contains no elements of class $3F$ in $O^+_8(2):3$. But this implies that any remaining elementary abelian 3-group contains $3B$ elements but is not generated by them. Hence its normalizer is contained in a case already considered.

This concludes the proof of the following theorem.

**Theorem 2.** Every 3-local subgroup of $G$ is contained in one of the following subgroups:

- $N(3A) = S_3 \times U_6(2)$;
- $N(3B) = (3 \times O^+_8(2):3):2$;
- $N(3C) = 3^{1+6}:Q_8^3;3^2:2$;
- $N(3AABB) = (S_3 \times S_3 \times U_4(2)):2$, contained in $O^-_{10}(2)$;
- $N(3B^2) = 3^2:Q_8 \times U_3(3):2$;
- $3^5:O_5(3):2$, contained in $O_7(3)$;
- $3^3:3A3$, contained in $O_7(3)$ (three conjugacy classes).
6. List of Known Conjugacy Classes of Simple Subgroups

The main part of the proof below is a classification of simple subgroups up to conjugacy. In order to facilitate this proof, we first list as many conjugacy classes of simple subgroups as we can. Tables 1, 2, and 3 contain one row for each conjugacy class in $G.S_3$, of known simple subgroups $S$. We give the normalizer $N$ in $G.S_3$, and the number $n$ of conjugacy classes in $G$ (which may be 1, 2, 3 or 6), as well as a maximal overgroup $M$ of $N$ in many cases. The next several sections are devoted to proving the results contained in these tables.

In fact, most of the known simple subgroups lie in the centralizer of some (inner or outer) automorphism, so we deal with these first. We start by centralizing elements of order 7, followed by 5, 3 and 2. We conclude with some subgroups of $Fi_{22}$. Structures of normalizers are given in $G.S_3$ unless otherwise stated.

7. Simple Subgroups Centralizing an Element of Order 7

The centralizers of elements of order 7 in $G$ are $C(7A) \cong 7 \times L_3(4)$ and $C(7B) \cong 7 \times L_3(2)$. Power maps show that a 7A element commutes with elements in classes $2B, 3B, 4D, 4E, 4F, 5A$ and that a 7B element commutes with elements of classes $2A$ and $3A$, and $4A$. It is clear from the 7-local analysis that in $L_3(2) \times L_3(4)$ the $L_3(2)$ factor contains 7A elements and the $L_3(4)$ contains 7B elements. It follows immediately that every $L_3(2) \times L_3(2)$ contains one factor with 7A-elements and the other with 7B-elements, so there is no automorphism swapping the two factors.

We have $N(7B) = (7.3 \times S_3 \times L_3(2)):2$ in $G.S_3$, so the only simple group centralizing a 7B element is an $L_3(2)$ with normalizer $(L_3(2) \times L_3(4):2):S_3$. Also $N(7A) = (7.3 \times L_3(4):3:2):2$, and the normalizers of the simple subgroups of $L_3(4):D_{12}$ are

- One class of $A_6.2^2$.
- One class of $L_3(2):2 \times 2$.
- One class of $S_3 \times S_5$.
- One class of $S_5$.

All these simple groups centralize $L_3(2)$ in $G$, but may centralize more. First, the $L_3(2)$ contains a 7B element, so the centralizer does not grow, and the normalizer in $G.S_3$ is $L_3(2):2 \times L_3(2):2$. The other groups all contain elements of order 5, so the centralizer lies in $(3 \times A_5):2$. The $A_5$ in $S_3 \times S_5$ must be the one with normalizer $(A_5 \times 3 \times A_5):2^2$. The other two normalizers do not contain the full $A_5$, but in the case of $A_6$ the normalizer contains $L_3(2)$ and an involution that normalizes the $A_6$ and centralizes the element of order 5, and hence the normalizer is $(L_3(2) \times A_6.2):2$.

In the case of the last $S_5$, the normalizer is in fact $2^4.L_3(2) \times S_5$.

Remark 1. From the class fusion to the factors of $L_3(2) \times L_3(4)$ we can compute the restriction of the character of degree 1938 to these factors. There is then only one way to fit them together into characters of the direct product, thus:

$$1 \otimes 1 + 3a \otimes 45a + 3b \otimes 45b + 6 \otimes (1^2 + 20^2) + 7 \otimes 35abc + 8 \otimes (1 + 20 + 64).$$

Hence we can compute the character value on the diagonal involutions to be 18, and on the diagonal elements of order 3 to be −6. It follows that these diagonal elements are in classes $2C$ and $3C$. In particular, the diagonal copies of $L_3(2)$ in $L_3(2) \times L_3(4)$ are of type $(2C, 3C, 7A)$ and $(2C, 3C, 7B)$. It is not immediately
MAXIMAL SUBGROUPS OF \( ^2E_6(2) \) AND ITS AUTOMORPHISM GROUPS

Table 1. Some simple subgroups, I

(a) with elements of order 19

| S     | N               | M                | n | Notes |
|-------|-----------------|------------------|---|-------|
| \( U_3(8) \) | \((3 \times U_3(8):3):2\) | 1 | 7A |

(b) with elements of order 17 but not 19

| S     | N               | M                | n | Notes |
|-------|-----------------|------------------|---|-------|
| \( O_{10}(2) \) | \((3 \times O_{10}(2)):2\) | 1 | 7A |
| \( F_4(2) \) | \((3 \times F_4(2)):2\) | 3 | 7B |
| \( S_8(2) \) | \((3 \times S_8(2)):2\) | 3 | 7B |
| \( O_8^-(2) \) | \((3 \times O_8^-(2)):2\) | 1 | 7A |
| \( O_8(2) \) | \((3 \times O_8(2)):2\) | 3 | 7B |
| \( S_4(4) \) | \((3 \times S_4(4)):2\) | 1 | |
| \( L_2(17) \) | \((3 \times L_2(17)):2\) | 1 | |
| \( L_2(17) \) | \((3 \times L_2(17)):2\) | 3 | |
| \( L_2(16) \) | \((3 \times L_2(16)):2\) | 1 | |
| \( L_2(16) \) | \((3 \times L_2(16)):2\) | 1 | |

(c) with elements of order 13 but not 17 or 19

| S     | N               | M                | n | Notes |
|-------|-----------------|------------------|---|-------|
| \( 3D_4(2) \) | \((3 \times 3D_4(2)):2\) | 1 | 7A |
| \( 3D_4(2) \) | \((3 \times 3D_4(2)):2\) | 3 | 7B |
| \( 2F_4(2)' \) | \((2 \times 2F_4(2)):2\) | 3 | 7B |
| \( L_2(25) \) | \((2 \times L_2(25)):2\) | 3 | 7B |
| \( L_4(3) \) | \((2 \times L_4(3)):2\) | 3 | 7B |
| \( L_3(3) \) | \((2 \times L_3(3)):2\) | 3 | 7B |
| \( Fi_{22} \) | \((2 \times Fi_{22}):2\) | 3 | 7B |
| \( O_7(3) \) | \((2 \times O_7(3)):2\) | 3 | 7B |
| \( G_2(3) \) | \((2 \times G_2(3)):2\) | 3 | 7B |
| \( L_2(13) \) | \((2 \times L_2(13)):2\) | 3 | 7B |

(d) with elements of order 11 but not 13, 17 or 19

| S     | N               | M                | n | Notes |
|-------|-----------------|------------------|---|-------|
| \( A_{12} \) | \((3 \times A_{12}):2\) | 1 | 7A |
| \( A_{11} \) | \((3 \times A_{11}):2\) | 1 | 7A |
| \( M_{12} \) | \((3 \times M_{12}):2\) | 2 | |
| \( M_{11} \) | \((3 \times M_{11}):2\) | 2 | |
| \( M_{12} \) | \((3 \times M_{12}):2\) | 2 | |
| \( L_2(11) \) | \((3 \times L_2(11)):2\) | 1 | |
| \( L_2(11) \) | \((3 \times L_2(11)):2\) | 1 | |
| \( M_{22} \) | \((3 \times M_{22}):2\) | 3 | |
| \( U_6(2) \) | \((3 \times U_6(2)):2\) | 1 | |
| \( U_5(2) \) | \((3 \times U_5(2)):2\) | 1 | |

Note: in fact this is a complete list of conjugacy classes of the given simple groups. This fact is proved in this paper.
Table 2. Some simple subgroups, II

(e) with elements of order 7 but not 11, 13, 17, 19

| S   | N               | M                   | n | Notes |
|-----|-----------------|----------------------|---|-------|
| $O_8^+(2)$ | $O_8^+(2):3^{\frac{1+2}{2}}2^2$ | $1$ | 7A    |
| $O_8^+(2)$ | $2 \times O_8^+(2):S_3$ | $2 \times F_4(2)$ | $3$ | 7B    |
| $A_{10}$ | $S_3 \times S_{10}$ | $(3 \times O_{10}(2)):2$ | $1$ | 7A    |
| $A_{10}$ | $2 \times S_{10}$ | $2 \times F_4(2)$ | $3$ | 7B    |
| $A_9$ | $(A_9 \times 3^2):2$ | $O_8^+(2):3^{\frac{1+2}{2}}2^2$ | $1$ | 7A    |
| $A_9$ | $S_9 \times 2$ | $2 \times F_4(2)$ | $3$ | 7B    |
| $A_8$ | $(A_8 \times A_5 \times 3):2^2$ | $(3 \times O_{10}(2)):2$ | $1$ | 7A    |
| $A_8$ | $(A_8 \times A_4 \times 3):2^2$ | $(3 \times O_{10}(2)):2$ | $1$ | 7A    |
| $A_7$ | $(A_7 \times A_5 \times 3):2^2$ | $(3 \times O_{10}(2)):2$ | $1$ | 7A    |
| $A_7$ | $S_7 \times S_3$ | $N(3A)$ | $3$ | 3A, 3C, 7B |
| $A_7$ | $S_7 \times S_3$ | $N(3A)$ | $3$ | 3C, 3C, 7B |
| $S_6(2)$ | $S_3 \times S_3 \times S_6(2)$ | $(3 \times O_{10}(2)):2$ | $1$ | 7A    |
| $S_6(2)$ | $2 \times S_3 \times S_6(2)$ | $2 \times F_4(2)$ | $3$ | 7B    |
| $S_6(2)$ | $2 \times S_6(2)$ | $2^{1+20}.U_6(2):S_3$ | $6$ | 7B    |
| $L_2(8)$ | $S_3 \times S_3 \times L_2(8):3$ | $S_3 \times 3D_4(2):3$ | $1$ | 7A    |
| $L_2(8)$ | $2 \times L_2(8):3$ | $2 \times F_4(2)$ | $3$ | 7A    |
| $L_2(8)$ | $S_3 \times S_3 \times L_2(8):3$ | $S_3 \times U_6(2):S_3$ | $1$ | 7B    |
| $L_2(8)$ | $2 \times L_2(8):3$ | $N(2A)$ | $3$ | 7B    |
| $L_2(8)$ | $2 \times L_2(8):3$ | $N(2A)$ | $2$ | 7B    |
| $L_2(8)$ | $2^2 \times L_2(8)$ | $N(2A)$ | $3$ | 7B    |
| $L_2(8)$ | $2^2 \times L_2(8)$ | $N(2A)$ | $3$ | 7B    |
| $L_2(8)$ | $2 \times L_3(8)$ | $N(2A)$ | $6$ | 7B    |
| $L_2(8)$ | $2 \times L_3(8)$ | $N(2A)$ | $6$ | 7B    |
| $U_4(3)$ | $S_3 \times U_4(3).2^2$ | $S_3 \times U_6(2):S_3$ | $3$ | 7B    |
| $L_3(4)$ | $(L_3(2) \times L_3(4)):S_3$ | $1$ | 2B, 3B, 7B |
| $L_3(2)$ | $L_3(2):2 \times L_3(2):2$ | $(L_3(2) \times L_3(4)):S_3$ | $2$ | 2B, 3B, 7B |
| $L_3(2)$ | $(L_3(2) \times L_3(4)):S_3$ | $1$ | 2B, 3A, 7A |
| $L_3(2)$ | $L_3(2) \times 2^{1+3}(3 \times A_3):2$ | $N(2A^3)$ | $1$ | 2B, 3A, 7A |
| $L_3(2)$ | $L_3(2) \times 2^1 \times S_3$ | $(3 \times O_{10}(2)):2$ | $2$ | 2C, 3C, 7B |
| $L_3(2)$ | $L_3(2):2$ | $(L_3(2) \times L_3(4)):S_3$ | $2$ | 2C, 3C, 7B |
| $U_3(3)$ | $U_3(3):2 \times 3^2.2S_4$ | $1$ | 7A    |
| $U_3(3)$ | $U_3(3):2 \times 2 \times S_3$ | $2 \times F_4(2)$ | $3$ | 7B    |

Note: This list is claimed to be complete in the cases $O_8^+(2)$, $A_{10}$, $A_9$, $A_8$, $A_7$, $S_6(2)$, $L_2(8)$, but not necessarily in the cases $L_2(7)$, $L_3(4)$, $U_3(3)$, and $U_4(3)$.

obvious what the centralizers of these copies of $L_3(2)$ are, and we shall come back to this problem later.

We conclude this section by summarizing the consequences for the classification of maximal subgroups with non-simple minimal normal subgroups.

**Theorem 3.** There is exactly one class of characteristically simple subgroup of $G, S_3$ that has order divisible by 7 and is not simple. Such groups are isomorphic to $L_3(2) \times L_3(2)$ and have normalizers $L_3(2):2 \times L_3(2):2$, all of which are contained in $(L_3(2) \times L_3(4)):S_3$. The class splits into three classes in $G$. 
Table 3. Some simple subgroups, III

\[
\begin{array}{|ccc|c|}
\hline
S & N & M & n & Notes \\
\hline
U_4(2) & 3^2 : D_8 \times (3 \times U_4(2)):2 & (3 \times O_{10}(2)):2 & 1 & U_4(2) < U_6(2) \\
U_4(2) & 2 \times S_3 \times U_4(2):2 & S_3 \times U_6(2); S_3 & 3 & O_6^-(2) < U_6(2) \\
A_6 & S_6 \times S_6 \times S_3 & (3 \times O_{10}(2)):2 & 1 & \\
A_6 & (A_6 \times L_3(2)):2 & (L_3(4); S_3 \times L_3(2)):2 & 3 & 2B, 3B \\
A_5 & S_5 \times S_6 \times S_3 & (3 \times O_{10}(2)):2 & 1 & 2B, 3A \\
A_5 & (A_5 \times 3 \times A_8):2^2 & (3 \times O_{10}(2)):2 & 1 & 2B, 3B \\
A_5 & S_5 \times 2^3; L_3(2) & N(2A^3) & 3 & 2B, 3B \\
A_5 & S_5 \times S_3 & (3 \times O_{10}(2)):2 & 1 & 2C, 3C \\
A_5 & (A_5 \times 2^3; S_3):2 & 3 & 2C, 3B \\
A_5 & (A_5 \times 2^4; 3^2; 2):2 & 1 & 2C, 3A \\
A_5 & (A_5 \times [2; 3]):2 & 3 & 2B, 3B \\
A_5 & (A_5 \times [2^5; 3^2]):2 & 1 & 2B, 3B \\
A_5 & A_5 \times [2^9; 3] & 3 & 2C, 3B \\
A_5 & (A_5 \times [2^4]):2 & 3 & 2C, 3B \\
\hline
\end{array}
\]

Note: This list is claimed to be complete in the cases \(A_5\) and \(U_4(2)\), but not necessarily in the case \(A_6\).

8. Simple subgroups centralizing an element of order 5

In \(G.S_3\) we have \(N(5A) = (D_{10} \times 3 \times A_8):2^2\), and the normalizers of the simple subgroups of \(S_8\) are

- One class of \(S_7\).
- One class of \(S_6 \times 2\).
- One class of \(S_5 \times 3\).
- One class of \(S_5 \times 2\).
- One class of \(L_3(2):2\).
- One class of \(L_3(2)\).

All these simple groups centralize \(3 \times A_5\), but may centralize more. The ones with order divisible by 7 have centralizers inside \(L_3(4); S_3\). Clearly the \(L_3(2)\) in \(L_3(2):2\) is the one which centralizes \(L_3(4)\). The other \(L_3(2)\) centralizes \(2^4; A_5\), and has normalizer \(L_3(2) \times 2^4; (3 \times A_5):2\), lying inside the maximal parabolic subgroup of shape \(2^4; 2^4; 2^12; 2^{12}.(S_5 \times L_3(2) \times S_3)\) in \(G.S_3\), and remaining a single class in \(G\).

Since there is no parabolic subgroup containing \(A_7 \times A_5\), the \(A_7\) has normalizer just \((3 \times A_5 \times A_7):2^2\). The \(A_5\) in \(S_5 \times S_3\) centralizes a \(3B\) element and is therefore conjugate in \(O_5^+(2):S_3\) to the one that centralizes \(A_5\). The \(S_6\) lies in \(O_{10}(2)\) acting as \(S_4(2)\), so centralizes \(S_6\). Hence the normalizer of the \(A_6\) is \(S_6 \times S_6 \times S_3\) in \(G.S_3\). The remaining \(A_5\) also centralizes the same \(S_6\), so has normalizer \(S_5 \times S_6 \times S_3\).

Power maps give the class fusion from \(A_8\) to \(G\). The elements of \(A_8\)-classes \(2A, 2B, 3A, 3B\) fuse to \(G\)-classes \(2A, 2B, 3B, 3A\) respectively. In particular, the 5-point \(A_5\) in \(A_8\) is of type \((2B, 3B)\), while the 6-point \(A_5\) is of type \((2B, 3A)\).

Since \(S_6\) is maximal in \(A_8\), it follows that the centralizer in both \(O_{10}(2)\) and \(G\) of this \(A_5\) of type \((2B, 3A)\) is exactly \(S_6\). It follows that there are just two classes of \(A_5 \times A_5\) in which the two factors are conjugate in \(G\). One of these has factors of type \((2B, 3B)\) and centralizer of order 3, so its normalizer in \(G\) is contained in
The other has factors of type $(2B, 3A)$, and has trivial centralizer, and its normalizer in $G$ is contained in $(S_6 \times S_6):2$ in $O^-_{10}(2)$. Similarly, there is a unique conjugacy class of $A_6 \times A_6$, with normalizer $(S_6 \times S_6):2$ in $G$, contained in $O^-_{10}(2)$.

We summarize the consequences for maximal subgroups with non-simple minimal normal subgroups.

**Theorem 4.** There are four classes of characteristically simple subgroups of $G.S_3$ that have order divisible by 5 and are not simple, as follows.

- One class of $A_6 \times A_6$, with normalizer $S_6!2 \times S_3$, contained in $(3 \times O^-_{10}(2)):2$.
- One class of $A_5 \times A_5$ with normalizer contained in $N(3B)$.
- Two classes of $A_5 \times A_5$ with normalizer contained in $S_6!2 \times S_3$.

**9. Simple subgroups centralizing an element of order 3**

**9.1. Groups centralizing a 3G element.** We have $N(3G) = (3 \times U_3(8)):3):2$, and the only proper non-abelian simple subgroup of $U_3(8)$ is a single class of $L_2(8)$.

Now a 7A element commutes with $L_3(4):S_3$, which has two classes outer 3-elements, fusing to 3D and 3E respectively. Similarly, a 7B element commutes with $(L_2(7) \times 3):2$ which also has two classes of outer 3-elements, fusing to 3E and 3G respectively. In particular, the 7-elements in $U_3(8)$ lie in 7B, so the $L_2(8)$ is of type $(2C, 3C, 7B)$. Hence the $L_2(8)$ centralizes in $G.S_3$ a proper subgroup of $(L_2(7) \times 3):2$, containing $3^2:2$. The normalizer therefore contains $L_2(8):3 \times 3^2:2$, in which the centralizing $3^2$ contains one cyclic subgroup of type $3A$, one of type $3E$, and two of type $3G$.

Hence there is also an element of the normalizer interchanging the two subgroups of type $3G$, so the normalizer in $G.S_3$ contains $S_3 \times S_3 \times L_2(8):3$. But $S_3 \times S_3$ is maximal in $(L_2(7) \times 3):2$, so this is the full normalizer.

**9.2. Groups centralizing a 3E element.** Next consider the case 3E, where we have $N(3E) = 3^2 D_4(2):3 \times S_3$. Now $3^2 D_4(2)$ contains just three classes of proper non-abelian simple subgroups, with normalizers $S_3 \times L_2(8)$, $U_3(3):2$, and $(7 \times L_2(7)):2$.

The 7-elements in $L_2(7)$ and $U_3(3)$ are in $3 D_4(2)$ class 7D, so $G$-class 7A, while those in $L_2(8)$ are in 7B. (Warning: there are two classes of $3D_4(2)$, with opposite fusion of 7-elements. The one under consideration here contains 3A-elements, so 7ABC fuses to 7B while 7D fuses to 7A. The other one contains 3B-elements. These facts are verified below using structure constants.) The above group $L_2(8)$ therefore has type $(2C, 3C, 7B)$ and its normalizer in $G.S_3$ is $S_3 \times S_3 \times L_2(8):3$.

This is the same as the one in $U_3(8)$. The above group $L_2(7)$ has type $(2A, 3A, 7A)$ and centralizes $L_4(4)$. Its full normalizer in $G.S_3$ is $(L_2(2) \times L_2(4):S_3):2$. The centralizer of the $U_3(3)$ is a proper subgroup of $L_3(4):S_3$, and is therefore $3^2:2 S_4$, since we know that there is a subgroup $3^2:2 S_4 \times U_4(3):2$ in $G.S_3$.

**9.3. Groups centralizing a 3F element.** Next consider the case 3F, where we have $N(3F) = S_3 \times (3 \times U_5(2)):2$. Now $U_5(2)$ contains one class each of $U_4(2)$, $L_2(11)$ and $A_6$, and three classes of $A_5$. This $L_2(11)$ has normalizer $S_3 \times (3 \times L_2(11)):2$ in $G.S_3$. The other groups could have larger normalizers in $G.S_3$ than they do in $N(3F)$. In any case, the centralizer in $G$ of any of these groups is a subgroup of $A_8$, containing the given $S_3$. Note that the normal $3^2$ in $N(3F)$ contains one cyclic subgroup of type $3A$, one of type $3F$, and two of type $3D$.

In the case of $U_4(2)$, we already see inside the group $S_3 \times U_4(2):S_3$ a subgroup $S_3 \times S_3 \times (3 \times U_4(2)):2$, which contains a subgroup $S_3 \times S_3$ of $A_8$. This $S_3 \times S_3$ contains...
two cyclic subgroups of type 3A and two of type 3B, and an extra automorphism of order 2 is realised inside \( N(3D) \). The only proper subgroup of \( A_8 \) containing \( 3^2: D_8 \) is \( S_6 \), but \( U_4(2) \) does not centralize an element of order 5. Hence the full \( U_4(2) \)

normalizer in \( G.S_3 \) is \( 3^2: D_8 \times (3 \times U_4(2)):2 \). The normal \( 3^2 \) is of type 3AABB, and normal 3 is of type 3D. The diagonal elements are of type 3D and 3F, corresponding to the 3B and 3A elements respectively. The 13 subgroups of order \( 3^2 \) are one of type 3AABB, two each of types 3ADFF and 3BDFF, and four each of types 3ADFF and 3BDFF.

The same argument applied to \( A_6 \) allows the possibility that the centralizer of the \( A_6 \) grows to \( S_6 \). Indeed, we can see this centralizer inside \( O_{16}^-(2) \), so we have \( S_6 \times S_6 \times S_3 \) as the normalizer of the \( A_6 \) in \( G.S_3 \). Moreover, there is also an automorphism swapping the two factors of \( S_6 \), also realised inside \( (3 \times O_{10}^-):2 \).

Of the two types of \( A_5 \) in \( A_6 \), one centralizes the full \( A_8 \), while the other centralizes only \( S_6 \). Hence we obtain normalizers \((3 \times A_5 \times A_8):2^2 \) and \( S_6 \times S_6 \times S_3 \) respectively.

9.4. Groups centralizing a 3D or 3B element. The 3D-normalizer \((3 \times O_{16}^-(2)):2 \) contains a subgroup of index 3 of the 3B-normalizer, so in particular contains all non-abelian simple subgroups of \( N(3B) \).

The group \( O_{16}^-(2) \) contains a large number of classes of simple subgroups. Consider first those with order divisible by 17, that is \( S_6(2), O_6^-(2), S_4(4), L_2(16) \) and \( L_2(17) \). There are two classes of \( L_2(16) \) in \( O_{16}^-(2) \), and one class of each of the other groups. Since the Sylow 17-normalizer lies inside \( N(3D) \), the normalizers are all easy to write down, and they are \( S_6(2) \times S_3, O_6^-(2):2 \times S_3, S_4(4):2 \times S_3, L_2(17) \times S_3, \) and two classes of \( L_2(16):2 \times S_3 \).

Next consider the simple subgroups with order divisible by 11, that is \( A_{12}, A_{11}, M_{12}, M_{11}, U_5(2) \) and \( L_2(11) \). There is a single class each of \( A_{12}, A_{11} \) and \( U_5(2) \), with normalizers in \( N(3D) \) of shape \((3 \times A_{12}):2, (3 \times A_{11}):2 \) and \((3^2 \times U_5(2)):2 \). The last of these has a larger normalizer in \( G.S_3 \), as already discussed in Subsection [9.3]. Since \( A_{11} \) does not lie in \( U_6(2) \), it does not centralize an involution, so the above groups are the full normalizers of \( A_{11} \) and \( A_{12} \) in \( G.S_3 \).

Now \( O_{16}^-(2) \) contains two classes of \( M_{12} \), fused by the outer automorphism. Hence there is one such class of \( M_{12} \) in \( G.S_3 \), each with normalizer \( 3 \times M_{12} \), and splitting into two classes in \( G \). In the case of \( M_{11} \), there are two classes fixing one of the 12 points on which \( A_{12} \) acts, and two acting transitively, each pair fused by the outer automorphism. Hence there are two classes of \( M_{11} \) in \( G.S_3 \), each with normalizer \( 3 \times M_{11} \), and splitting into four classes in \( G \). The subgroups \( L_2(11) \) are again of two types: transitive and intransitive on 12 points. The former also lies in \( U_5(2) \), so has normalizer \( S_3 \times (3 \times L_2(11)):2 \) as discussed in Subsection [9.3] above. The latter has normalizer \((3 \times L_2(11)):2 \) contained in \( N(3D) \). (We need to prove that it does not also centralize a 2A-element: a somewhat subtle question. In fact, there is a unique class of \( L_2(11) \) in \( U_6(2) \), and such an \( L_2(11) \) acts on the \( 2^{20} \)

factor of \( 2^{20} + 20 \) as two copies of an absolutely irreducible 10-dimensional module. The 1-cohomology of this module is trivial, so there is only one conjugacy class of \( L_2(11) \) in \( 2^{20}, L_2(11) \).

Next consider subgroups with order divisible by 7 but not by 11 or 17. This includes \( A_{10}, A_9, A_8, A_7, L_2(7), L_2(8), O_8^+(2), S_6(2), U_3(3) \). The 7-elements here are in class 7A, since they centralize elements of order 5. There is a unique class of \( O_8^+(2) \) in \( O_{10}^-(2) \), and such a group centralizes a unique 3B element in \( G \). The normalizer is therefore equal to \( N(3B) \). Since this normalizer includes a triality
automorphism of $O^+_8(2)$, we can use this automorphism to prove conjugacy in $G.S_3$ of various groups. In particular there is a unique class of $S_6(2)$ that centralizes a 3D element.

Since the 7A element centralizes $L_3(4):S_3$, in which the 3D element we are centralizing lies in class 3B, it makes sense to classify the subgroups of $L_3(4):S_3$ that contain a 3B-element. Now in $L_3(4):3$, the maximal subgroups containing an outer element of order 3, are $2^{4}: (3 \times A_5)$ (two classes), $7.3 \times 3$ and $3^2:2A_4$. There are two classes of outer 3-elements in $3^2:2A_4$ (up to inversion), of which those that centralize involutions are in $L_3(4)$-class 3B. All outer elements of order 3 in 7:3 \times 3 lie in $L_3(4)$ class 3C. Notice also that the 7-centralizer in $(3 \times O_{10}^+(2))$:2 is $7 \times (3 \times A_5)$:2.

There is a unique class of $A_{10}$ in $N(3D)$, with normalizer $S_3 \times A_{10}$. Since the centralizer of a 5-cycle therein is $5^2 \times S_3$, this centralizer cannot grow in $G.S_3$. There are two classes of $A_5$ in $N(3D)$, but these are conjugate by a triality automorphism of $O^-_8(2)$. The normalizer is $(3^2:2 \times A_5)$:2 contained in $(3^2:2 \times O^+_8(2)):2$. The precise structure of this normalizer can be seen from the subgroup $(A_5 \times A_{12})$:2 of the Monster, and is the unique subgroup of index 2 in $S_3 \times S_3 \times S_9$ that is not a direct product.

Similarly for the case of $S_6(2)$, there are two classes in $N(3D)$, fused by triality. Since $S_6(2)$ contains elements of order 15, the centralizer in $G$ lies somewhere between $S_3$ and $A_5$. But it is not $A_5$, and $S_3$ is maximal in $A_5$, so it is $S_3$. Hence the normalizer in $G.S_3$ is $S_3 \times S_3 \times S_6(2)$.

This leaves $A_7$, $A_8$, $L_2(7)$, $L_2(8)$, $U_3(3)$, which are significantly harder. We shall not attempt a complete classification at this stage. Note however that there is a maximal subgroup $L_2(7)$:2 of $O^-_8(2)$:2. This $L_2(7)$ is of type $(2C, 3C, 7A)$ in $G$ and does not centralize any involution in $G$. Hence its normalizer in $G.S_3$ is exactly $L_2(7):2 \times S_3$.

Finally, we consider the simple groups divisible by no prime greater than 5, that is $A_5$, $A_6$, and $U_4(2)$. There are two classes of $U_4(2)$: those that act on the 10-space as $O^-_6(2)$, and those that act as $U_4(2)$. But by triality of $O^+_8(2)$, these are conjugate in $G.S_3$. Moreover, they are conjugate to the subgroup $U_4(2)$ of $U_5(2)$ already considered above.

We leave $A_5$ and $A_6$ for the time being.

9.5. **Groups centralizing a 3A element.** The maximal subgroup $N(3A) = S_3 \times U_6(2)$:3 contains many interesting subgroups. As it contains 7B-elements, many of these are not visible elsewhere. In particular $U_6(2)$ contains $U_4(3)$. The normalizer in $G.S_3$ is $S_3 \times U_4(3).2^2$, and the class splits into three classes in $G$. It also contains $M_{22}$, with normalizer $S_3 \times M_{22}:2$, and splitting into three classes in $G$. It also contains $S_6(2)$, with normalizer $S_3 \times 2 \times S_6(2)$, splitting into three classes in $G$, and similarly $U_3(3)$, with normalizer $S_3 \times 2 \times U_3(3):2$, also splitting into three classes in $G$.

In addition to these four groups, there are unique classes in $U_6(2)$:3 of each of $A_8$, $L_2(11)$, $U_5(2)$ and $L_2(8)$, all of which we have already seen. There are two classes of $U_4(2)$, one acting as $U_4(2)$, the other as $O^-_6(2)$. The former we have already seen; the latter group centralizes only $S_3 \times 2$ in $G.S_3$, and the class splits into three classes in $G$. There are two classes of $A_7$ in $U_6(2)$:3, one inside $S_6(2)$, the other inside $U_4(3)$. In both cases the normalizer is $S_3 \times S_7$. The first contains
both $3A$ and $3C$ elements, the other only $3C$ elements. There are also subgroups $A_5$, $A_6$, $L_3(2)$, $L_3(4)$, which we shall not completely classify here.

10. Simple subgroups centralizing an involution

10.1. Groups centralizing a 2D element. The centralizer of a 2D-element in $G.S_3$ is $2 \times F_4(2)$. Now $F_4(2)$ has an outer automorphism of order 2 that is not realised in $G.S_3$. Hence there are pairs of automorphic subgroups of $F_4(2)$ that behave completely differently in $G$. This applies in particular to the subgroups $S_8(2)$, $O^+_8(2)$, and $^3D_4(2)$, but also to their subgroups $O^-_8(2)$, $A_{10}$, $A_9$, $L_2(17)$, and perhaps others. In each case, one of the subgroups is centralised by a 3D or 3E element and the other is not. Hence we obtain another class of each of these six groups in $G.S_3$, and in each case the class splits into three classes in $G$. Note however that this does not apply to the subgroups $S_4(4)$ and $L_2(16)$, which are normalized by the outer automorphism of $F_4(2)$.

There are four further isomorphism types of simple subgroups of $F_4(2)$ with order divisible by 13, namely $^2F_4(2)'$, $L_2(25)$, $L_4(3)$ and $L_3(3)$. It is easy to see that there is a unique class in each case, except possibly $L_3(3)$, which is a subgroup of both $^2F_4(2)$ and $L_4(3)$. But both these copies of $L_3(3)$ lie in the centralizer of an outer automorphism of $F_4(2)$, so they are indeed conjugate. Since the 13-element is self-centralizing in $G$, it is easy to write down the normalizers in each case. They are $2 \times ^2F_4(2)$, $2 \times L_2(25):2$, $2 \times L_4(3):2$, and $2 \times L_3(3):2$.

The remaining simple subgroups with order divisible by 7 are $A_8$, $A_7$, $L_2(8)$, $L_2(7)$, $S_4(2)$, $U_3(3)$. Finally there are the simple groups with order divisible by no prime bigger than 5, namely $A_5$, $A_6$, $U_4(2)$. We make no attempt at a complete classification in these cases.

10.2. Groups centralizing an inner involution. The centralizer in $G.S_3$ of a 2C involution is soluble. The centralizer of a 2B involution in $G.S_3$ has shape $2^8.2^{16}:(S_6(2) \times S_3)$, and contains 7A elements. As subgroups containing 7A elements turn out to be relatively easy to classify, we shall not need to consider this case much further here.

The centralizer of a 2A involution is $2^{1+20}.U_6(2):S_3$, and contains 7B elements. Now in $2^{1+20}.U_6(2)$ we see a subgroup $2^{1+20}.L_2(8).3$, and the representation of $L_2(8)$ on $2^{20}$ is the direct sum of the Steinberg module and two copies of the natural module. Since the latter has 1-dimensional 1-cohomology over the field of order 8, there are in total 64 conjugacy classes of $L_2(8)$ in $2^{1+20}.L_2(8)$. One of these has normalizer $S_3 \times L_2(8):3$, and three more have normalizer $2 \times L_2(8):3$. The remainder are fused into 20 classes of groups all with normalizer $2 \times L_2(8)$. Together these 24 classes of $L_2(8)$ account for the full structure constant of $1/6 + 3(1/2) + 20(3/2) = 95/3$. Under the action of the outer automorphism group $S_3$, the three with normalizer $2 \times L_2(8):3$ are fused into one class, while the last 20 appear to be fused into orbits of sizes $3 + 3 + 2 + 6 + 6$, with normalizers either $2^2 \times L_2(8)$ or $2 \times L_2(8):3$ or $2 \times L_2(8)$.

11. Simple subgroups of $Fi_{22}$

In $G.S_3$ there is a class of subgroups $Fi_{22}$, each with normalizer $Fi_{22}:2$, and splitting into three classes in $G$. In $Fi_{22}:2$ there is a unique class of subgroups $O_{7}(3)$. Since $O_{7}(3)$ is not a subgroup of $F_4(2)$ or of any of the 3-element centralizers,
it follows that the normalizer in $G.S_3$ is $O_7(3):2$, and that there are three classes of $O_7(3)$ in $G$.

In $O_7(3):2$ there is a unique class of $G_2(3)$, and the normalizer in $G.S_3$ of any such $G_2(3)$ is therefore $G_2(3):2$, contained in $Fi_{22}:2$. In $G_2(3)$ there is a unique class of $L_2(13)$, and again the normalizer in $G.S_3$ of such an $L_2(13)$ is $L_2(13):2$, contained in $Fi_{22}:2$.

12. Eliminating other isomorphism types of simple subgroups

First note that the largest order of an element of $G$ is 35. Since $L_2(q)$ has an element of order $(q+1)/2$ we can disregard $L_2(q)$ whenever $q > 70$. Similarly, $L_3(q)$ has an element of $(q^2 + q + 1)/3$ and $U_3(q)$ has an element of order $(q^2 − q + 1)/3$ so we can disregard $L_3(q)$ and $U_3(q)$, and also $G_2(q)$, when $q > 10$. Also, $L_4(q)$, $U_4(q)$ and $S_4(q)$ contains elements of order $(q^2 + 1)/2$, so can be disregarded when $q > 8$. All remaining simple groups of small enough order are explicitly given in the Atlas list [2, pp. 239–242], and, using Lagrange’s Theorem and CFSG, we obtain in addition to the known subgroups above, just 16 more possibilities, as follows:

- $A_{13}$, $A_{14}$, $L_2(19)$, $L_2(27)$, $L_2(49)$, $L_2(64)$, $L_3(9)$, $L_4(4)$,
- $S_4(8)$, $S_6(3)$, $G_2(4)$, $Sz(8)$, $J_1$, $J_2$, $J_3$, $Suz$.

We eliminate these as follows:

- $L_2(49)$ has elements of order 25.
- $L_2(64)$ has elements of order 65, and is contained in $S_4(8)$.
- $L_3(9)$ has elements of order 91.
- $L_4(4)$ has elements of order 85.
- $S_6(3)$ has elements of order 36.
- $J_1$ contains 19:6, whereas the Sylow 19-normalizer in $G$ is 19:9.
- $J_2$ contains a triple cover 3-$A_6$, but the centralizers of elements of order 3 in $G$ are either soluble, in the case 3-$C$, or direct products $3 × U_6(2)$ or $3 × O_8^+(2):3$ in the cases 3-$A$ and 3-$B$. This argument also eliminates $G_2(4)$ and $Suz$, which contain $J_2$.
- $A_{13}$ can be generated by taking a subgroup $A_5 × A_8$, restricting to $A_5 × A_5 × 3$, and extending to another $A_5 × A_8$ normalizing the other $A_5$ factor. But the 5-centralizer is $5 × A_8$, and is contained in $O_{10}^-(2)$, so the entire construction takes place inside $O_{10}^+(2)$, and therefore cannot generate $A_{13}$. This argument eliminates also $A_{14}$.
- The elements of order 7 and 13 in $G$ are rational. The elements of order 5 have fixed space of dimension 7 in the action of the triple cover 3-$G$ on the 27-dimensional module over $F_4$. Using the Brauer character table for $Sz(8)$ in characteristic 2, in [4], it is easy to check that these properties cannot be achieved in any restriction of the Brauer character to $Sz(8)$.
- $L_2(19)$ contains elements of order 10, so the involutions are in class 2-$A$ or 2-$B$; and elements of order 9, so the elements of order 3 are in 3-$C$. But the structure constants of type (2-$A$, 3-$C$, 5-$A$) and (2-$B$, 3-$C$, 5-$A$) are zero, so there is no $A_5$ of this type in $G$. Since $L_2(19)$ is a subgroup of $J_3$, this argument eliminates also $J_3$.
- $L_2(27)$ contains elements of order 14, so either 2-$A$- and 7-$B$-elements, or 2-$B$- and 7-$A$-elements. All such (2, 3, 7) structure constants are zero except for (2-$B$, 3-$A$, 7-$A$), where the value is 1/480. This implies every $L_2(27)$ has non-trivial centralizer, which is impossible.
13. Subgroups isomorphic to $A_5$

There is a well-known character formula for the number of ways a given element in a specified conjugacy class $C_3$ is the product of an element $x$ from class $C_1$ and an element $y$ from class $C_2$. Dividing this by the order of the centralizer of an element of class $C_3$ gives the symmetrized structure constant

$$\xi(C_1, C_2, C_3) = \frac{|C_1| |C_2| |C_3|}{|G|^2} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(x)\chi(y)\chi(y^{-1}x^{-1})}{\chi(1)},$$

which can also be interpreted as the sum over conjugacy classes of such triples $(x, y, z)$ of the inverse of the order of the centralizer of $(x, y)$. Putting this together with the fact that

$$\langle x, y, z \mid x^2 = y^3 = z^5 = 1, xy = z \rangle$$

is a presentation for $A_5$, leads to the well-known method of determining $A_5$ subgroups using structure constants for triples of type $(2, 3, 5)$.

We calculate the structure constants using GAP [3]. It turns out that just 5 of the 9 structure constants of type $(2, 3, 5)$ in $G$ are non-zero, as follows:

- $\xi(2B, 3A, 5A) = 1/720.$
- $\xi(2B, 3B, 5A) = 59/1440 = 1/20160 + 1/224 + 1/64 + 1/48.$
- $\xi(2C, 3A, 5A) = 1/48.$
- $\xi(2C, 3B, 5A) = 9/32 = 1/32 + 1/16 + 3/16.$
- $\xi(2C, 3C, 5A) = 1.$

In particular, every $A_5$ centralizes an inner or outer automorphism, and hence its normalizer lies in one of the known maximal subgroups.

A classification of the subgroups isomorphic to $A_5$ up to conjugacy is more difficult. We use knowledge of the subgroups isomorphic to $A_5$ in the Monster [6].

Now the elements of classes $3A, 3B, 3C, 5A$ lift to Monster classes $3A, 3A, 3B, 5A$ respectively, so the Monster $A_5$s that live in $2^6.G.S_3$ are as follows:

- Monster type $2A, 3A, 5A$, normalizer $(A_5 \times A_{12}):2$.
- Monster type $2B, 3A, 5A$, normalizer $(A_5 \times 2M_{22}):2$.
- Monster type $2B, 3B, 5A$, normalizer $S_6.2 \times M_{11}$.

Since there is a unique class of $2^2$ in $M_{11}$, and the normalizer is $S_4$, we obtain a single class of $A_5$ in $G$ of the third type, and the normalizer in $G.S_3$ is $S_3 \times S_5$. The outer automorphism of order 3 centralizing this $A_5$ cannot be in $3E, 3F, 3G$, so must lie in $3D$.

Turning now to $A_{12}$, the involutions of Monster class $2A$ are the ones of cycle types $2^6$ and $2^2.1^8$. Hence the $2A^2$ subgroups are of the following types:

- Moving 4 points: $(12)(34), (13)(24)$.
- Moving 6 points: $(12)(34), (12)(56)$.
- $(12)(34)(56)(78)(9X)(ET), (13)(24)$.
- $(12)(34)(56)(78)(9X)(ET), (13)(24)(57)(68)(9E)(XT)$.

In the third case, there is only an $S_2$ of automorphisms, not $S_3$. The $2^2$ centralizer in $S_{12}$ is $2^2 \times 2^3:S_4$, so there is a unique type of such $A_5$ in $G.S_3$, with normalizer $(A_5 \times 2^1:S_4):2$, and splitting into three classes in $G$. The attributable structure constant is $6/2^7.3 = 1/64$. In the first case, the $A_5$ centralizes $A_8$, and this is a case we know about, contributing $1/20160$ to the structure constant, and having type $(2B, 3B)$. In the second case, the $A_5$ centralizes $S_6$, and this is another case.
we know about, contributing 1/720 to the structure constant, and having type 
\((2B, 3A)\). In the last case, the normalizer of the 2\(^2\) in \(S_{12}\) is 2\(^6\cdot(S_\delta \times S_\delta)\), so giving an \(A_5\) in \(G.S_3\) with normalizer \((A_5 \times 2^4:3^2:2):2\), contributing 6/25.3\(^2\) = 1/48 to the structure constant.

Finally we need to consider the case 2.\(M_{22}\).2, which is rather more intricate. This group has 6 classes of involutions, which in Atlas notation are \(-1A, +2A, -2A, +2B, -2B\) and 2\(C\). These fuse into 2.\(HS.2\) classes \(-1A, -2A, +2A, 2C, 2C,\) and 2\(D\) respectively. Hence they fuse in \(HN\), and also in the Monster, to 2\(A, 2B, 2A, 2A, 2B\) respectively. We therefore need to classify 2\(^2\) subgroups consisting of elements of classes \(-1A, -2A,\) and \(\pm 2B\) only. We obtain the following cases:

- \(-1A, +2B, -2B\), with centralizer 2\(^2\times 2^3\cdot L_3(2)\), and normalized only by an involution that also normalizes the \(A_5\).
- two of type \(-2A, -2A, -2A\) with normalizers in \(M_{22}\) of shape 2\(^4\cdot 3^2\).2 and 2\(^4\).S\(_3\) respectively.
- one each of types \(-2A, \pm 2B, \pm 2B\).

The first case gives an \(A_5\) with normalizer \((2^3\cdot L_3(2) \times A_5)\cdot 2\), contributing 6/1344 = 1/224 to the structure constant, and being of type 2\(B, 3B\).

The remaining contributions to the structure constants are as follows. The case given above as 2\(^4\cdot 3^2\).2 in \(M_{22}\) becomes 2\(^2\).S\(_3\).S\(_3\) in 2.\(M_{22}\).2, and factoring out the normal 2\(^2\) as well as taking the S\(_3\) off the top appears to leave us with a structure constant of 1/48. The other case of this type differs by a factor of 3, so gives 1/16.

In the remaining cases we have types \((-2A, +2B, +2B)\) and \((-2A, -2B, -2B)\) which are swapped by an element normalizing the \(A_5\): hence these cases do not extend to \(S_5\) in \(G.S_3\). There is also the type \((-2A, +2B, -2B)\) which does extend to \(S_5\) in \(G.S_3\). The centralizer of the 2\(^2\) in the first two cases would seem to be 2\(^2\times 2^3\).S\(_4\) in 2.\(M_{22}\).2, but there is also an element swapping the two elements of type 2\(B\) in the 4-group, hence giving a total contribution to the structure constant of 1/32. In the other case the centralizer in 2.\(M_{22}\).2 is smaller by a factor of 6, leaving us with 3/16.

This accounts exactly for the structure constants, and also gives all the class fusions, except that there is an ambiguity as to which of the two contributions of 1/48 to attribute to \(\xi(2B, 3B, 5A)\) and which to \(\xi(2C, 3A, 5A)\). To summarize:

**Theorem 5.** There are ten classes of \(A_5\) in \(G.S_3\), becoming 18 classes in \(G\). The normalizers in \(G.S_3\) are as follows.

- \(S_3 \times S_3\), type (2\(C, 3C\)), contributing 1 to the structure constant;
- \((A_5 \times (A_5 \times 3):2):2\), type (2\(B, 3B\)), contributing 1/20160;
- \(S_5 \times S_6 \times S_3\), type (2\(B, 3A\)), contributing 1/720;
- \((A_5 \times 2^4:3^2:2):2\), type (2\(C, 3A\)), contributing 1/48;
- \((A_5 \times [2^7:3]):2\), type (2\(B, 3B\)), splitting into 3 classes in \(G\), contributing 1/64;
- \(S_5 \times 2^3\cdot L_3(2)\), type (2\(B, 3B\)), splitting into 3 classes in \(G\), contributing 1/224;
- \((A_5 \times [2^5:3^2]):2\), type (2\(B, 3B\)), contributing 1/48;
- \((A_5 \times [2^4:2^3]):2\), type (2\(C, 3B\)), contributing 1/32;
- \(A_5 \times [2^6:3]\), type (2\(C, 3B\)), splitting into 3 classes in \(G\), contributing 1/16;
- \((A_5 \times [2^4]):2\), type (2\(C, 3B\)), splitting into 3 classes in \(G\), contributing 3/16.
14. Hurwitz groups

A Hurwitz group is a group generated by an element \( x \) of order 2 and an element \( y \) of order 3 such that \( xy \) has order 7. Among the simple groups we are interested in here, \( L_2(7), L_2(8), L_2(13), 3D_4(2) \) and \( Fi_{22} \) are Hurwitz groups. Another group which turns up is the non-split extension \( 2^3L_3(2) \). To aid in identification of the group generated by \( x \) and \( y \), we compute a fingerprint, consisting of the orders of the elements

\[ s, st, sst, ssts, ssstst, ssststt \]

where \( s = xy \) and \( t = xyy \). The last of these words is the only one that does not necessarily have the same order as its reverse, so we shall often augment the fingerprint with the order of \( sssttst \) in order to distinguish a pair \((x, y)\) from its reciprocal \((x^{-1}, y^{-1}) = (x, yy)\). Note that this fingerprint is significantly redundant when \( s \) has order 7, but is more discriminating for larger orders.

We pre-compute the fingerprints for the \((2, 3, 7)\) generators of all the above groups. In the list below we adopt an obvious shorthand for the four reciprocal pairs of Hurwitz generators for \( Fi_{22} \). All generating pairs not so marked are either self-reciprocal, or automorphic to their reciprocals. The generating pairs in \( Fi_{22} \) are of type \((2C, 3D)\) while those in \( 3D_4(2) \) are of type \((2B, 3B, 7D)\).

\[
\begin{align*}
L_2(7) & : 7 \ 4 \ 4 \ 7 \ 4 \ 3 \ 3 \\
L_2(8) & : 7 \ 9 \ 9 \ 7 \ 7 \ 9 \ 7 \\
3D_4(2) & : 7 \ 14 \ 14 \ 7 \ 21 \ 13 \ 18 \\
L_2(13) & : 7 \ 6 \ 6 \ 7 \ 7 \ 7 \ 13 \\
& \quad \quad \quad 7 \ 7 \ 7 \ 6 \ 6 \ 13 \\
& \quad \quad \quad 7 \ 13 \ 13 \ 7 \ 3 \ 7 \ 7 \\
Fi_{22} & : 7 \ 13 \ 13 \ 7 \ 21 \ 30 \ 8/12 \\
& \quad \quad \quad 7 \ 18 \ 18 \ 7 \ 12 \ 24 \ 12/12 \\
& \quad \quad \quad 7 \ 20 \ 20 \ 7 \ 15 \ 8 \ 9/13 \\
& \quad \quad \quad 7 \ 21 \ 21 \ 7 \ 12 \ 21 \ 30 \\
& \quad \quad \quad 7 \ 24 \ 24 \ 7 \ 30 \ 15 \ 13/20
\end{align*}
\]

In order to classify the Hurwitz subgroups of \( G \), we relate them to Hurwitz subgroups of the Monster, as follows. First, note that all involutions in \( G \) lift to involutions in \( 2^2G \), which is a subgroup of the Monster, and all elements of order 7 in \( G \) fuse to class \( 7A \) in the Monster. Moreover, the classes \( 3A, 3B, 3C \) in \( G \) fuse to \( 3A, 3A, 3B \) respectively in the Monster. It is therefore possible to use Norton’s analysis \([6]\) of the corresponding structure constants in the Monster, to write down a complete list of Hurwitz subgroups of \( G \). Of help also is the corresponding analysis in the Baby Monster \([14]\).

First we compute the structure constants of type \((2, 3, 7)\) in \( G \). Exactly 7 of the 18 structure constants are non-zero, as follows:

\[
\begin{align*}
\xi(2A, 3A, 7A) &= 1/20160 \\
\xi(2B, 3A, 7A) &= 1/480 \\
\xi(2C, 3A, 7A) &= 3/64 \\
\xi(2C, 3C, 7A) &= 11/3 \\
\xi(2B, 3B, 7B) &= 15/56 \\
\xi(2C, 3B, 7B) &= 43/8 \\
\xi(2C, 3C, 7B) &= 329/3
\end{align*}
\]
Theorem 6. \textbf{The 2A, 3A, 7A case.} We have already discussed the $L_3(2)$ of type $(2A, 3A, 7A)$, which has centralizer $L_3(4)$, and embeds in $A_5 \times A_8$ permuting the 8 points transitively. In particular, since the normalizer is $(L_3(2) \times L_3(4)) : 2$, this accounts for the full structure constant.

14.2. The 2B, 3A, 7A case. The group $A_8$ also contains two classes of $L_3(2)$ acting on 7 points, and therefore of type $(2B, 3A, 7A)$. The visible normalizer is just $A_5 \times L_3(2)$, but the structure constant is $\xi(2B, 3A, 7A) = 1/480$, so this cannot be the whole normalizer. Since the centralizer lies inside $L_3(4)$, the only possibility is $2^4 A_5 \times L_3(2)$, which exactly accounts for the structure constant. In particular, this $L_3(2)$ does not extend to $L_3(2):2$ inside $G$.

14.3. The 2C, 3C, 7A case. To account for $\xi(2C, 3C, 7A) = 11/3$, note that 1 is attributable to $3^3 D_4(2)$, and 1 to $L_3(2)$, and $3/2$ to an $L_2(8)$ with normalizer $2 \times L_2(2)$, and 1/6 to an $L_2(8)$ with normalizer $S_3 \times L_2(8):2$.

14.4. The 2C, 3C, 7B case. To account for $\xi(2C, 3C, 7B) = 329/3$, note that 54 is attributable to $F_{i22}$, since there are three classes of $F_{i22}$ in $G$, each with two (automorphic) classes of generating triples of each of 9 types. A further 3 is attributable to three more classes of $3^3 D_4(2)$, and 18 to three classes of $L_2(13)$, and 3 to three classes of $L_3(2)$; the remainder $95/3 = 1/6 + 213/2$ is attributable to the seven classes of $L_2(8)$ in $G.S_3$ identified in Sections 9.5 and 10.2.

14.5. The 2B, 3B, 7B case. There are three classes of $L_3(2)$ with normalizer $(L_3(2) \times L_3(2)) : 2$, accounting for an amount $3/168 = 1/56$ of the structure constant, leaving an amount $14/56 = 1/4$.

14.6. The 2C, 3A, 7A case. Since the structure constant is $3/64$, every such group is centralized by an involution, necessarily of class 2B.

14.7. The 2C, 3B, 7B case. In this case an explicit computer search found 36 such triples of elements in $G$, of which 6 generate $L_3(2)$ and 30 generate $2^3 L_3(2)$. Now the latter group has four classes of $(2, 3, 7)$ generating triples.

It would seem likely therefore that the structure constant of $53 \over 8$ should be attributed as 4 for $2^3 L_3(2)$ and $14 \over 8$ for at least two classes of $L_3(2)$ in $G.S_3$. In any case, since the total structure constant is less than 6, it follows that any $L_3(2)$ of this type has non-trivial centralizer in $G.S_3$, and hence its normalizer cannot be maximal.

14.8. Conclusion. In this section we have proved the following theorem. Apart from the calculation of structure constants, the proof is mostly computer-free. However, we used computational methods to check the results, by collecting large numbers of $(2, 3, 7)$ triples and using statistical analysis of frequencies to show that the above allocation of structure constants to isomorphism types of groups is the only plausible one.

Theorem 6. \textbf{In $G.S_3$ there is a unique class of each of $L_2(13)$ and $F_{i22}$, two classes of $3^3 D_4(2)$, and nine classes of $L_2(8)$, as listed in Tables 1 and 2. The ones whose normalizers are maximal are just $F_{i22}$ (in $G$ and $G.2$) and one class of $3^3 D_4(2)$ (in $G.3$ and $G.S_3$).}

\textbf{Every $L_3(2)$ in $G$ has non-trivial centralizer in $G.S_3$. Only one class of $L_3(2)$ has maximal normalizer (in all of $G$, $G.2$, $G.3$ and $G.S_3$).}
MAXIMAL SUBGROUPS OF $^2E_6(2)$ AND ITS AUTOMORPHISM GROUPS

Table 4. Some fingerprints of type $(2, 3, 11)$

| Group | Fingerprints |
|-------|--------------|
| $L_2(11)$ | $2A, 3A$  | 11 5 5 6 3 11 11 |
| $A_{11}$ | $2B, 3C$ | 11 8 11 8 12 11 11/21 |
|        |            | 11 12 9 14 9 9 8  |
|        |            | 11 14 10 11 11 14 8  |
|        |            | 11 20 12 11 5 8 12/12  |
| $O_{10}(2)$ | $2D, 3F$ | 11 9 15 8 12 35 30  |
|        |            | 11 12 30 18 17 12 12/17  |
|        |            | 11 17 12 6 18 18 17  |
|        |            | 11 20 17 24 21 9 11/18  |
|        |            | 11 21 21 24 5 18 30  |
|        |            | 11 24 12 30 15 21 12/17  |
| $M_{12}$ | $2A, 3A$ | 11 6 10 6 10 6 8  |
|         | $2B, 3B$ | 11 6 6 11 6 11 8  |
| $A_{12}$ | $2C, 3C$ | 11 14 9 20 11 35 4/9  |
|         |            | 11 21 35 35 10 8 11  |
| $U_6(2)$ | $2C, 3C$ | 11 9 18 12 7 12 18  |
|         |            | 11 12 11 8 12 8 18  |

15. Classifying $(2, 3, 11)$ triples

We next analyse the structure constants of type $(2, 3, 11)$ in a similar manner. First we pre-compute the fingerprints for simple groups we know to be generated by such a triple, including $L_2(11)$, $M_{12}$, $A_{11}$, $A_{12}$, $U_6(2)$, $O_{10}(2)$, $Fi_{22}$. Note that the groups $2^{10}.L_2(11)$ and $2^2.U_6(2)$ are also generated by $(2, 3, 11)$ triples. The fingerprints for the smaller groups are listed in Table 4. The fingerprints of the 89 triples which generate $Fi_{22}$ are listed in Table 5. (For practical purposes, especially for distinguishing between a triple and its reciprocal, these fingerprints were then extended by the orders of $ssssstst$ and $sssssttt$ to give extra discriminating power.)

We calculate the following structure constants of type $(2, 3, 11)$ in $G$ (all others are zero):

- $\xi(2B, 3B, 11A/B) = 1/6$ each. This is accounted for by $L_2(11)$.
- $\xi(2B, 3C, 11A/B) = 1$ each. Such a subgroup must be centralized by a non-trivial automorphism, and one of the two types of $(2, 3, 11)$ generating triples for the subgroup $M_{12}$ of $O_{10}(2)$ accounts for the full structure constant.
- $\xi(2C, 3B, 11A/B) = 37/2$ each.
- $\xi(2C, 3C, 11A/B) = 1650$ each.

15.1. The $(2C, 3B, 11)$ case. The known simple subgroups make the following contributions to each structure constant of $18^{12}$:

- 1 from the other type of $(2, 3, 11)$ generators for the subgroup $M_{12}$, with normalizer $M_{12} \times 3$ in $G.S_3$;
- 3 from $A_{12}$, with normalizer $(A_{12} \times 3):2$ in $G.S_3$;
- 1 from $U_6(2)$, with normalizer $S_3 \times U_6(2):S_3$ in $G.S_3$. 
Now the 1-cohomology of $U_6(2)$ on the 20-dimensional module is 2-dimensional, so there are four classes of $U_6(2)$ in $2^{20}:U_6(2)$, which fuse in $2^{20}:U_6(2)$.S_3$ into one class of $U_6(2)$ with normalizer $U_6(2).S_3$, and one class with normalizer $U_6(2):2$. The former lifts to $2 \times U_6(2).S_3$, and the corresponding normalizer in $G.S_3$ is $S_3 \times U_6(2).S_3$. On the other hand, there is a subgroup $2-U_6(2)$ inside $F_{22}$, and therefore the other case must lift to $2-U_6(2).2$. This group does not centralize any element of order 3 in $G.S_3$, so this is the full normalizer. Hence the three classes of $2-U_6(2)$ in $G$ account for a contribution of 9 to each of the two $(2,3,11)$ structure constants.

Also within $2^{1+20}:U_6(2)$ there is a subgroup $2^{1+20}.L_2(11)$, which contains three conjugacy classes of $2 \times 2^{10}.L_2(11)$, permuted by the $S_3$ of outer automorphisms. Each $(2,3,11)$ generating triple for $L_2(11)$ lifts to 4 triples of type $(2,3,11)$ in $2^{10}.L_2(11)$, since the elements of orders 2 and 3 each centralize a 6-space in the $2^{10}$. One of these triples generates the complementary $L_2(11)$, which is of type $(2B,3B)$ in $G$, while the other three must generate the whole of $2^{10}.L_2(11)$. Extending to $2^{1+20}:U_6(2).S_3$ we have a subgroup $2^{1+20}.(U_5(2) \times 3):2$ and hence a subgroup $2^{1+20}.(L_2(11) \times 3):2$. Altogether, therefore, the contribution to each $(2,3,11)$ structure constant is $9/2$.

Altogether we have accounted for an amount $1 + 3 + 9 + 4\frac{1}{2} = 18\frac{1}{2}$, that is the whole structure constant.

### Table 5. The 89 fingerprints of type $(2,3,11)$ for $F_{22}$

| 11 6 11 18 22 18 13/21 | 11 13 13 13 13 24 12 12/15 |
|-------------------------|-----------------------------|
| 11 6 14 8 13 8 9/15 | 11 13 13 13 18 10 12 9/24 |
| 11 7 15 12 13 12 11/16 | 11 13 13 22 13 20 12/18 |
| 11 8 12 10 22 20 13/21 | 11 13 21 20 16 18 18/21 |
| 11 9 10 11 12 13 15/24 | 11 13 24 15 8 13 12/13 |
| 11 9 11 13 24 8 11/12 | 11 13 24 30 12 21 11/12 |
| 11 9 11 13 24 11 12/16 | 11 14 8 11 9 12 13/13 |
| 11 9 21 11 10 8 22 | 11 18 9 13 15 14 12/16 |
| 11 9 21 30 12 22 12/14 | 11 18 12 9 13 13 14/22 |
| 11 9 24 22 16 22 16/21 | 11 18 12 22 16 9 14/15 |
| 11 10 12 22 13 13 18/18 | 11 18 14 18 9 8 13/15 |
| 11 10 13 11 22 11 20/24 | 11 18 15 15 14 10 10/13 |
| 11 10 14 18 20 22 13/15 | 11 18 22 20 11 22 18/24 |
| 11 10 20 16 13 22 12/18 | 11 20 12 18 9 16 16/22 |
| 11 12 20 14 6 20 12/22 | 11 20 22 8 13 14 13/30 |
| 11 12 21 11 21 12 20 | 11 21 7 24 18 21 12 |
| 11 12 21 30 11 13 9/9 | 11 24 8 16 15 30 14/22 |
| 11 12 22 13 30 13 14/24 | 11 24 9 22 13 22 10/10 |
| 11 12 22 14 12 11 13/21 | 11 24 11 12 16 11 8/9 |
| 11 13 9 20 8 20 8/22 | 11 24 13 11 14 22 5/7 |
| 11 13 11 20 22 9 13/14 | 11 30 16 18 13 20 11/24 |
| 11 13 13 11 12 14 8/21 | 11 30 21 8 14 30 10/18 |
| 11 13 13 11 18 8 13/21 | 11 30 24 11 11 30 13/14 |
15.2. The \((2C, 3C, 11)\) case. In this case, it turns out that most of the triples generate the whole of \(G\), and a computer search to collect and identify these. The total number of \((2C, 3C, 11A/B)\) fingerprints collected was 562. Of these, 1 generates \(L_2(11)\), and 8 generate \(A_{11}\), while 9 generate \(O_{10}^-\). These subgroups all centralize an outer automorphism of order 3, so each fingerprint contributes 1 to each structure constant. The remaining 544 fingerprints correspond to generators of either \(F_{i22}\) or \(G\), and in either case contribute 3 to each structure constant. Thus we have accounted for the full structure constant of \(18 + 3 \times 544 = 1650\).

As noted above, there are 89 fingerprints for generators of \(F_{i22}\), listed in Table 1. The 455 fingerprints for \((2, 3, 11)\) triples that generate \(G\) are listed in the Appendix.

15.3. Conclusion. In this section we have proved the following result. The proof depends crucially on the computational analysis of the \((2, 3, 11)\) triples of type \((2C, 3C)\).

Theorem 7. There is a unique class of each of the groups \(O_{10}^-\), \(A_{12}\), \(A_{11}\), \(M_{12}\) and \(U_6\), and two classes of \(L_2(11)\) in \(G.S_3\). In every case the centralizer in \(G.S_3\) is non-trivial, and the only cases in which the normalizer is maximal are \(O_{10}^-\) and \(U_6\) (in all of \(G, G.2, G.3\) and \(G.S_3\)).

16. Classifying \((2, 3, 13)\) Triples

Groups generated by \((2, 3, 13)\) triples include \(L_2(13)\), \(L_2(25)\), \(L_3(3)\), \(L_4(3)\), \(G(2, 3)\), \(2F_4(2)'\), \(3D_4(2)\), \(F_4(2)\) and \(F_{i22}\). Of these, we have already classified \(L_2(13)\), \(3D_4(2)\) and \(F_{i22}\) using Hurwitz generators. The group \(3^3;L_3(3)\) is also generated by \((2, 3, 13)\) triples, and so is the maximal subgroup \(3^3+3;L_3(3)\) of \(O_7(3)\), a fact which initially caused me a significant amount of difficulty in accounting for the structure constants of type \((2C, 3C, 13)\).

We pre-compute the fingerprints of the smaller simple groups above, and list them in Table 1. There are also 109 fingerprints for generators of type \((2, 3, 13)\) for \(F_{i22}\), which are of \(F_{i22}\)-type \((2C, 3D, 13)\) and are listed in Table 7 as well as 261 fingerprints for \((2, 3, 13)\) generators of \(F_4(2)\) that are of \(F_4(2)\)-type \((2D, 3C, 13)\) and are listed in Table 8.

In fact the non-zero \((2, 3, 13)\) structure constants in \(F_4(2)\) are

- \(\xi(2C, 3C, 13A) = 5\), of which 1 is from \(2F_4(2)'\) and 4 is from \(L_4(3)\);
- \(\xi(2D, 3A, 13A) = \xi(2D, 3B, 13A) = 3\), from generators of \(3D_4(2)\);
- \(\xi(2D, 3C, 13A) = 552\), of which 14 is from \(3D_4(2)\), and 2 is from \(L_3(3)\), and 8 is from \(3^3;L_3(3)\), and 6 is from \(L_2(25)\), leaving 522 for 261 automorphic pairs of generators for \(F_4(2)\).

There are only four non-zero structure constants of type \((2, 3, 13)\) in \(G\), that is

- \(\xi(2B, 3C, 13A) = 15\);
- \(\xi(2C, 3A, 13A) = 3\);
- \(\xi(2C, 3B, 13A) = 63\);
- \(\xi(2C, 3C, 13A) = 9658\).

The first three of these can be easily accounted for by known subgroups. To increase the reliability of the results, however, we also computed explicitly triples of group elements of these types.
Table 6. Some fingerprints of type (2, 3, 13)

|          | fingerprints |
|----------|--------------|
| $L_2(13)$ | 13 7 6 7 7 2 7 |
| $L_2(25)$ | 13 4 13 13 3 6 |
|          | 13 12 6 13 3 13 |
|          | 13 13 12 5 13 13 |
| $L_3(3)$ | 13 4 8 13 8 3 13 |
|          | 13 6 13 13 8 13 |
| $2F_4(2)'$ | 2A,3A 13 5 12 8 8 13 |
| $L_4(3)$ | 2A,3D 13 5 12 20 13 13 |
|          | 13 5 20 6 6 13 |
| $G_2(3)$ | 2A,3C 13 13 8 7 6 12 8 |
|          | 2A,3E 13 9 13 12 7 9 8 |
|          | 13 12 12 7 8 8 7/8 |
| $3D_4(2)$ | 2B,3A 13 21 12 13 6 21 13 |
|          | 13 28 18 21 6 13 12/18 |
|          | 2B,3B 13 7 18 18 21 8 12/14 |
|          | 13 9 12 13 21 13 12/14 |
|          | 13 18 28 13 14 18 12/21 |
|          | 13 21 28 18 28 21 12 |
| $Fi_{22}$ | 2C,3C 13 12 12 22 16 13 11/30 |
|          | 13 18 14 22 9 20 13/20 |
|          | 13 21 18 11 9 15 16/16 |

16.1. **The (2B, 3C, 13) case.** An explicit search of the 2B,3C case yields three fingerprints with $xy$ of order 13, namely those for $2F_4(2)'$ and $L_4(3)$. There are three known conjugacy classes of $2F_4(2)'$ in $G$, one in each of the three copies of $F_4(2)$, and each with normalizer $2F_4(2)$. Each is centralised by an outer automorphism of order 2, so in total this accounts for an amount 3 of the structure constant. There are also three known conjugacy classes of subgroups $L_4(3)$ in $G$, again one in each of the three copies of $F_4(2)$. In this case, the outer automorphism group of $L_4(3)$ is $2^2$, of which only 2 is realised in $G$. It follows that the amount of structure constant accounted for here is $2 \times 2 \times 3 = 12$. Hence we have accounted for the full structure constant of $3 + 12 = 15$.

16.2. **The (2C, 3A, 13) case.** An explicit search of the (2C, 3A) case yields three fingerprints with $xy$ of order 13, equal to the three fingerprints for $3D_4(2)$ generators of type (2B, 3A, 13). Since there is a subgroup $3D_4(2)$ of this type, it accounts for the full structure constant.

16.3. **The (2C, 3B, 13) case.** A explicit search of the (2C, 3B) case yields 12 fingerprints where $xy$ has order 13, equal to the pre-computed fingerprints for $Fi_{22}$ (six fingerprints of type (2C, 3C, 13)), $G_2(3)$ (three of type (2A,3E,13)), and $3D_4(2)$ (three of type (2B,3A,13)). There are three conjugacy classes in $G$ of self-normalizing subgroups $Fi_{22}$, which account for an amount $6 \times 6 = 36$ of the structure constant. Also there are three classes of self-normalizing subgroups $G_2(3)$, accounting for an amount $3 \times 6 = 18$.
of the structure constant. There is also a generating triple of $G_2(3)$-type $(2A,3C)$, which turns out to be of type $(2C,3C)$ in $G$ (see below).

Recall that there are two classes of $^3D_4(2)$ inside $F_4(2)$, differing in the fusion of $7$-elements to $F_4(2)$, and hence to $G$. One class has been already found above, in Subsection 16.2, generated by $(2C,3A)$-triples. These groups lie in centralizers of outer automorphisms of order 3. The three triples in the present case therefore generate a $^3D_4(2)$ with normalizer $^3D_4(2):3 \times 2$ in $G.S_3$, so account for an amount $3 \times 3 = 9$ of the structure constant.

This accounts for the full amount $36 + 18 + 9 = 63$ of structure constant.

16.4. **The $(2C,3C,13)$ case.** The case $(2C,3C)$ is analyzed as follows. We collected 1599 distinct fingerprints, of which 1213 generated $G$, hence accounting for an amount 7278 of the structure constant. There were 261 generating $F_4(2)$, and 109 generating $F_{i22}$, accounting for a further 2220, making a running total of 9498. Two copies of $^3D_4(2)$ account for $7 + 21$ of the structure constant, while single
Table 8. \((2D, 3C, 13)\) generators for \(F_4(2)\)

| 13  | 6  | 10 | 13 | 16 | 13 | 28 |
| 13  | 6  | 13 | 12 | 21 | 12 | 17/18 |
| 13  | 6  | 16 | 28 | 12 | 28 | 17 |
| 13  | 6  | 16 | 30 | 30 | 30 | 16/17 |
| 13  | 6  | 17 | 18 | 28 | 18 | 24 |
| 13  | 6  | 24 | 18 | 24 | 18 | 28 |
| 13  | 8  | 12 | 10 | 13 | 20 | 6 |
| 13  | 8  | 13 | 9  | 28 | 30 | 17/24 |
| 13  | 8  | 13 | 30 | 18 | 21 | 12/24 |
| 13  | 8  | 17 | 17 | 28 | 28 | 17/21 |
| 13  | 8  | 17 | 21 | 9  | 17 | 21/30 |
| 13  | 8  | 20 | 17 | 13 | 8  | 6/18 |
| 13  | 8  | 21 | 28 | 24 | 24 | 28/30 |
| 13  | 8  | 24 | 28 | 30 | 28 | 13/21 |
| 13  | 8  | 30 | 28 | 21 | 24 | 18/21 |
| 13  | 10 | 10 | 12 | 12 | 15 | 16 |
| 13  | 10 | 12 | 8  | 8  | 15 | 12 |
| 13  | 10 | 13 | 10 | 13 | 13 | 21/30 |
| 13  | 10 | 14 | 16 | 12 | 8  | 16 |
| 13  | 10 | 24 | 24 | 16 | 30 | 14 |
| 13  | 12 | 9  | 18 | 17 | 24 | 21/30 |
| 13  | 12 | 12 | 17 | 21 | 13 | 20/28 |
| 13  | 12 | 12 | 24 | 17 | 21 | 17/20 |
| 13  | 12 | 12 | 13 | 12 | 28 | 12 |
| 13  | 12 | 14 | 16 | 21 | 17 | 24/24 |
| 13  | 12 | 16 | 13 | 30 | 9  | 12/28 |
| 13  | 12 | 16 | 16 | 12 | 21 | 12/21 |
| 13  | 12 | 16 | 18 | 13 | 17 | 21/21 |
| 13  | 12 | 17 | 12 | 12 | 12 | 12/20 |
| 13  | 12 | 17 | 16 | 16 | 13 | 9/18 |
| 13  | 12 | 17 | 20 | 20 | 18 | 9/24 |
| 13  | 12 | 17 | 21 | 30 | 18 | 16/17 |
| 13  | 12 | 18 | 12 | 21 | 17 | 12/16 |
| 13  | 12 | 18 | 13 | 21 | 12 | 21/28 |
| 13  | 12 | 18 | 17 | 30 | 20 | 13/18 |
| 13  | 12 | 20 | 28 | 24 | 13 | 10/12 |
| 13  | 12 | 20 | 18 | 12 | 30 | 17/24 |
| 13  | 12 | 21 | 9  | 20 | 30 | 12/18 |
| 13  | 12 | 21 | 17 | 12 | 13 | 21/30 |
| 13  | 12 | 21 | 18 | 20 | 21 | 16/21 |
| 13  | 12 | 24 | 28 | 12 | 30 | 18/28 |
| 13  | 12 | 28 | 17 | 18 | 15 | 13/18 |
| 13  | 12 | 30 | 17 | 30 | 18 | 12/24 |
| 13  | 12 | 30 | 21 | 28 | 24 | 12/16 |
copies of $G_2(3)$, $L_2(13)$ and $L_3(3)$ account for a further 18, for a running total of 9544. The subgroup $L_2(25)$ also accounts for 18, making a total of 9562. Out of the total of 9658, therefore, there is still 96 to account for.

Now there are three classes of $3^3:3^3.L_3(3)$ in $G$. The total (2, 3, 13) structure constant in these groups is $3.4.3.3 = 108$. Of this, 12 comes from triples generating (three classes of) $L_3(3)$, while 24 is attributable to (three classes of) $3^3:3^3.L_3(3)$ and the remaining 72 comes from generators for the whole group. Hence we exactly account for the remaining structure constant of $24 + 72 = 96$.

16.5. Conclusion. In this section we have completely classified subgroups isomorphic to $L_3(3)$, $L_4(3)$, $G_2(3)$, $L_2(25)$, $F_4(2)$ and $2F_4(2)'$, as well as verifying the earlier results for $L_2(13), 3D_4(2)$ and $F_{22}$.

Theorem 8. In $G.S_3$ there is a unique class of each of $G_2(3)$, $L_4(3)$, $F_4(2)$, $2F_4(2)'$, $L_2(25)$ and $L_3(3)$. In the case of $F_4(2)$, the normalizer in $G$ and $G.2$ is maximal, and there are three conjugacy classes. In all other cases the normalizer is not maximal in any of $G$, $G.2$, $G.3$ or $G.S_3$.

17. Classifying (2, 3, 17) Triples

Known subgroups with (2, 3, 17) generators include $L_2(16), O_8^-(2), O_{10}^-(2), S_8(2), L_2(17), F_4(2)$, and $2^6:O_8^-(2)$ and $2^{8+16}:O_8^-(2)$. We precompute the fingerprints in Table 6.
Table 9. Some fingerprints of type (2, 3, 17)

| Table 9 | Some fingerprints of type (2, 3, 17) |
|---------|------------------------------------|
| $O_{10}(2)$ | $2D, 3E$ |
| 17, 30, 33, 15, 35, 33, 18 |
| 17, 12, 33, 11, 18, 30, 11/21 |
| 17, 17, 20, 30, 6, 8, 9 |

$O_{8}(2)$

| 17, 21, 9, 9, 7, 6, 8/21 |
| 17, 15, 17, 12, 5, 9, 17 |
| 17, 21, 7, 17, 10, 10, 30 |

$2C, 3C$

| 17, 8, 30, 12, 8, 21, 10/17 |
| 17, 10, 21, 17, 21, 12, 10/17 |
| 17, 15, 10, 17, 21, 17, 6/10 |
| 17, 17, 17, 8, 15, 30, 7 |
| 17, 17, 30, 8, 7, 6, 9 |
| 17, 21, 12, 17, 10, 17/30 |

$L_2(16)$

| 17, 17, 17, 5, 17, 15, 17 |
| 17, 15, 17, 17, 17, 17, 5 |

$L_2(17)$

| 17, 9, 9, 8, 9, 9, 9 |

$S_8(2)$

| 2D, 3D |
| 17, 6, 12, 12, 21, 21, 21 |
| 17, 6, 30, 24, 21, 21, 30 |

$2E, 3D$

| 17, 7, 12, 24, 15, 4, 17 |
| 17, 8, 17, 15, 18, 12, 8/21 |
| 17, 9, 20, 15, 6, 12, 12 |
| 17, 12, 12, 18, 20, 30, 10/17 |
| 17, 12, 12, 24, 20, 10, 17/20 |
| 17, 15, 20, 30, 15, 15, 10/12 |

$2F, 3D$

| 17, 6, 21, 24, 12, 24, 20 |
| 17, 7, 30, 14, 21, 12, 12/17 |
| 17, 9, 14, 17, 17, 15, 5/17 |
| 17, 12, 12, 9, 12, 30, 12 |
| 17, 12, 18, 14, 30, 14, 10/12 |
| 17, 17, 15, 18, 21, 30, 12/17 |
| 17, 20, 30, 17, 14, 17, 17/17 |
| 17, 30, 24, 12, 12, 15, 4/21 |

$F_4(2)$

| 2C, 3C |
| 17, 6, 30, 12, 24, 12, 13/16 |
| 17, 6, 30, 12, 28, 12, 12 |

The structure constants we need to account for are the following:

- $\xi(2B, 3C, 17A + B) = 13 + 13 = 26$;
- $\xi(2C, 3A, 17A + B) = 3 + 3 = 6$;
- $\xi(2C, 3B, 17A + B) = 35 + 35 = 70$;
- $\xi(2C, 3C, 17A + B) = 7614 + 7614 = 15228$.

17.1. **Triples inside** $F_4(2)$. We shall also need information about (2, 3, 17) generating triples for $F_4(2)$. To determine these, we compute the structure constants in $F_4(2)$ as follows.
• \(\xi(2C, 3C, 17A + B) = 5 + 5\), of which 2 + 2 are \((2D, 3D)\) generators for two classes of \(S_8(2)\) and 3 + 3 are generators for \(F_4(2)\) whose fingerprints are given above;

• \(\xi(2D, 3A, 17A + B) = 3 + 3\), of which 2 + 2 are \((2C, 3B)\) generators for one class of \(O_8^-(2)\) and 1 + 1 are generators for one class of \(L_2(16)\);

• \(\xi(2D, 3B, 17A + B) = 3 + 3\), similarly accounted for by the other class of \(O_8^-(2)\) and of \(L_2(16)\);

• \(\xi(2D, 3C, 17A + B) = 420 + 420\), of which 10 + 10 are the other generators for both classes of \(O_8^-(2)\), and 24 + 24 are the other generators for both classes of \(S_8(2)\), and 2 + 2 are generators for \(L_2(16)\).

This leaves 3 + 3 of type \((2C, 3C)\) and 384 + 384 of type \((2D, 3C)\) all of which must generate the whole of \(F_4(2)\). The fingerprints for the former are given in Table 9 and for the latter in Table 10.

17.2. The \((2B, 3C, 17)\) case. In the case \((2B, 3C)\) we found five fingerprints with \(xy\) of order 17. The first three are for generators of \(F_4(2)\). The three classes of (self-normalizing) \(F_4(2)\) therefore account for an amount 9 + 9 of the structure constant.

The other two cases are generators of type \((2D, 3D, 17)\) for \(S_8(2)\). Now \(F_4(2)\) contains two conjugacy classes of \(S_8(2)\), which can be distinguished by the class of 7-elements they contain. In one case, the \(S_8(2)\) lies in \(O_{10}^-\), so is centralized by a non-inner automorphism. In the other case there are three conjugacy classes in \(G\), fused in \(G.3\). Hence these four classes of \(S_8(2)\) account for an amount 2 + 6 = 8 of the structure constant. Together these subgroups account for the full structure constant 26.

17.3. The \((2C, 3A, 17)\) case. In the case \((2C, 3A)\), there are four fingerprints with \(xy\) of order 17 that generate \(O_8^-\). One of the known classes of \(O_8^-\) is centralized by an \(S_3\) of outer automorphisms, so each fingerprint for this group accounts for 1 of the structure constant, making 4 altogether. There are two fingerprints that generate \(L_2(16)\). One of the known classes of \(L_2(16):4\) therefore accounts for the remaining 2 of the structure constant.

17.4. The \((2C, 3B, 17)\) case. In the case \((2C, 3B)\) we found 18 fingerprints when the order of \(xy\) is 17. Four are generators of type \((2D, 3E)\) for \(O_{10}^-\), four are generators of type \((2C, 3B)\) for \(O_8^-\), and two are generators for \(L_2(16)\), while the remaining 8 generate \(G\). The fingerprints for the generators for \(G\) are as follows:

\[
\begin{array}{cccccccc}
17 & 12 & 33 & 22 & 33 & 22 & 17/18 \\
17 & 18 & 22 & 12 & 13 & 19 & 13/18 \\
17 & 28 & 16 & 17 & 16 & 12 & 14/24 \\
17 & 28 & 20 & 20 & 8 & 18 & 15/30 \\
\end{array}
\]

The generators for \(G\) account for an amount 48 of the structure constant. The generators for \(O_{10}^-\) account for an amount 8. Now there are two classes of \(O_8^-\) in \(F_4(2)\), one of which has already been counted in the enumeration of the \((2C, 3A)\) case. The other gives rise to three classes of \(O_8^-\) in \(G\), each extending to \(O_8^-(2):2\), and each centralized by an outer automorphism of order 2. Together these account for an amount 12 of the structure constant.
Table 10. \((2D, 3C, 17)\) generators for \(F_4(2)\)

| 17 | 4 | 28 | 17 | 28 | 3 | 10/21 | 17 | 12 | 12 | 30 | 21 | 13 | 9/20 |
|----|---|----|----|----|---|-------|----|----|----|----|----|----|-----|
| 17 | 6 | 9 | 21 | 12 | 21 | 16/21 | 17 | 12 | 13 | 9 | 8 | 20 | 12/24 |
| 17 | 6 | 17 | 20 | 21 | 20 | 9/20 | 17 | 12 | 13 | 13 | 21 | 16 | 16/18 |
| 17 | 6 | 24 | 21 | 20 | 21 | 14/17 | 17 | 12 | 13 | 28 | 12 | 18 | 21/24 |
| 17 | 7 | 18 | 21 | 20 | 8 | 17/28 | 17 | 12 | 16 | 12 | 28 | 12 | 13 |
| 17 | 7 | 28 | 12 | 15 | 4 | 12 | 17 | 12 | 16 | 16 | 9 | 9 | 21/24 |
| 17 | 8 | 9 | 21 | 30 | 12 | 8/18 | 17 | 12 | 16 | 20 | 18 | 18 | 13/16 |
| 17 | 8 | 10 | 16 | 28 | 13 | 12/17 | 17 | 12 | 17 | 24 | 28 | 21 | 12/21 |
| 17 | 8 | 13 | 9 | 13 | 17 | 16/21 | 17 | 12 | 18 | 13 | 20 | 30 | 12 |
| 17 | 8 | 13 | 13 | 16 | 14 | 13/16 | 17 | 12 | 18 | 13 | 30 | 17 | 13/14 |
| 17 | 8 | 13 | 18 | 13 | 28 | 8/17 | 17 | 12 | 18 | 16 | 16 | 17 | 12/15 |
| 17 | 8 | 13 | 21 | 13 | 21 | 10/30 | 17 | 12 | 18 | 18 | 9 | 17 | 16/17 |
| 17 | 8 | 14 | 12 | 17 | 18 | 8/18 | 17 | 12 | 18 | 21 | 12 | 17 | 16 |
| 17 | 8 | 16 | 17 | 17 | 20 | 13/21 | 17 | 12 | 18 | 28 | 21 | 28 | 18/18 |
| 17 | 8 | 16 | 18 | 28 | 20 | 17/24 | 17 | 12 | 21 | 13 | 13 | 30 | 15/16 |
| 17 | 8 | 17 | 18 | 21 | 28 | 17/24 | 17 | 12 | 21 | 17 | 13 | 17 | 12/18 |
| 17 | 8 | 17 | 24 | 30 | 12 | 12/16 | 17 | 12 | 21 | 17 | 24 | 16 | 28/30 |
| 17 | 8 | 21 | 14 | 13 | 21 | 18/24 | 17 | 12 | 21 | 18 | 9 | 24 | 18/18 |
| 17 | 8 | 21 | 16 | 30 | 12 | 9/12 | 17 | 12 | 24 | 30 | 17 | 12 | 13/30 |
| 17 | 8 | 21 | 21 | 13 | 24 | 12/18 | 17 | 12 | 28 | 17 | 17 | 13 | 24/24 |
| 17 | 8 | 21 | 21 | 14 | 21 | 13/13 | 17 | 12 | 28 | 18 | 12 | 17 | 20/24 |
| 17 | 8 | 24 | 18 | 13 | 21 | 13/16 | 17 | 12 | 28 | 21 | 12 | 24 | 12/24 |
| 17 | 8 | 28 | 17 | 12 | 21 | 21/30 | 17 | 12 | 28 | 24 | 28 | 17 | 13 |
| 17 | 8 | 30 | 9 | 6 | 24 | 17 | 17 | 12 | 28 | 29 | 9 | 21 | 9/13 |
| 17 | 8 | 30 | 18 | 12 | 21 | 28/30 | 17 | 12 | 30 | 17 | 21 | 12 | 12/18 |
| 17 | 9 | 12 | 9 | 20 | 24 | 16 | 17 | 13 | 18 | 9 | 14 | 21 | 8/13 |
| 17 | 9 | 14 | 20 | 12 | 15 | 18 | 17 | 13 | 20 | 24 | 18 | 8 | 17/21 |
| 17 | 9 | 17 | 12 | 24 | 17 | 12/24 | 17 | 13 | 21 | 12 | 14 | 28 | 12/15 |
| 17 | 9 | 17 | 28 | 20 | 18 | 17 | 17 | 13 | 21 | 13 | 30 | 10 | 18/30 |
| 17 | 9 | 24 | 28 | 20 | 9 | 28 | 17 | 14 | 12 | 13 | 16 | 12 | 30 |
| 17 | 9 | 28 | 16 | 9 | 18 | 30 | 17 | 14 | 12 | 24 | 12 | 12 | 13 |
| 17 | 10 | 10 | 13 | 24 | 30 | 21/24 | 17 | 14 | 13 | 9 | 12 | 14 | 21/28 |
| 17 | 10 | 12 | 21 | 17 | 8 | 9/18 | 17 | 14 | 20 | 24 | 14 | 18 | 12 |
| 17 | 10 | 17 | 24 | 16 | 18 | 17/28 | 17 | 14 | 28 | 24 | 28 | 10 | 18/24 |
| 17 | 10 | 20 | 28 | 14 | 30 | 18/21 | 17 | 14 | 30 | 16 | 12 | 14 | 24 |
| 17 | 10 | 21 | 14 | 28 | 16 | 21/30 | 17 | 14 | 30 | 28 | 21 | 24 | 9/30 |
| 17 | 10 | 24 | 21 | 18 | 18 | 12/21 | 17 | 15 | 18 | 17 | 9 | 28 | 17/24 |
| 17 | 10 | 30 | 12 | 24 | 18 | 14/24 | 17 | 15 | 18 | 17 | 18 | 9/15 |
| 17 | 12 | 8 | 24 | 21 | 12 | 10 | 17 | 15 | 18 | 28 | 12 | 21 | 12/30 |
| 17 | 12 | 9 | 17 | 12 | 12 | 13/17 | 17 | 15 | 21 | 20 | 18 | 24 | 16/24 |
| 17 | 12 | 9 | 18 | 17 | 17 | 12/13 | 17 | 15 | 21 | 20 | 20 | 24 | 21/28 |
| 17 | 12 | 12 | 10 | 20 | 13 | 12/16 | 17 | 15 | 30 | 13 | 18 | 13 | 21/30 |
| 17 | 12 | 12 | 13 | 15 | 17 | 16/18 | 17 | 16 | 8 | 14 | 21 | 12 | 30/30 |
| 17 | 12 | 12 | 21 | 21 | 17/24 | 17 | 16 | 12 | 9 | 12 | 18 | 9/13 |
| 17 | 12 | 12 | 17 | 16 | 21 | 8/28 | 17 | 16 | 14 | 16 | 24 | 21 | 18/24 |
| 17 | 16 | 14 | 21 | 21 | 24 | 12/24 |
| 17 | 16 | 17 | 14 | 14 | 24 | 9/10  |
| 17 | 16 | 17 | 17 | 21 | 21 | 15/18 |
| 17 | 16 | 20 | 9  | 24 | 9  | 9/12  |
| 17 | 16 | 20 | 30 | 21 | 14 | 12/28 |
| 17 | 16 | 21 | 12 | 17 | 18 | 18/21 |
| 17 | 16 | 21 | 13 | 21 | 16 | 30/30 |
| 17 | 16 | 21 | 16 | 28 | 24 | 9/17  |
| 17 | 16 | 28 | 20 | 17 | 13 | 6/21  |
| 17 | 17 | 6  | 14 | 24 | 24 | 8     |
| 17 | 17 | 8  | 16 | 17 | 20 | 14    |
| 17 | 17 | 12 | 14 | 13 | 12 | 12    |
| 17 | 17 | 21 | 16 | 28 | 14 | 14    |
| 17 | 17 | 21 | 28 | 18 | 18 | 10/28 |
| 17 | 17 | 21 | 28 | 24 | 21 | 17/18 |
| 17 | 17 | 24 | 16 | 16 | 18 | 24    |
| 17 | 17 | 28 | 12 | 9  | 18 | 24/24 |
| 17 | 17 | 28 | 12 | 18 | 16 | 20/3  |
| 17 | 18 | 9  | 12 | 13 | 14 | 17/21 |
| 17 | 18 | 9  | 20 | 20 | 21 | 12/15 |
| 17 | 18 | 12 | 18 | 12 | 17 | 10/28 |
| 17 | 18 | 13 | 17 | 18 | 28 | 13/17 |
| 17 | 18 | 15 | 17 | 12 | 12 | 14/20 |
| 17 | 18 | 17 | 9  | 13 | 14 | 13/14 |
| 17 | 18 | 18 | 20 | 12 | 16 | 20/24 |
| 17 | 18 | 21 | 12 | 18 | 13 | 12/24 |
| 17 | 18 | 24 | 16 | 8  | 8  | 18/18 |
| 17 | 20 | 13 | 24 | 18 | 18 | 24/30 |
| 17 | 20 | 14 | 18 | 24 | 21 | 17/30 |
| 17 | 20 | 15 | 30 | 21 | 9  | 16/24 |
| 17 | 20 | 16 | 15 | 17 | 30 | 21/24 |
| 17 | 20 | 17 | 16 | 18 | 12 | 30/30 |
| 17 | 20 | 18 | 18 | 12 | 13 | 15/21 |
| 17 | 20 | 24 | 12 | 28 | 12 | 17/18 |
| 17 | 20 | 24 | 17 | 13 | 12 | 13/20 |
| 17 | 20 | 28 | 30 | 16 | 28 | 13/24 |
| 17 | 20 | 30 | 12 | 18 | 24 | 16/20 |
| 17 | 21 | 8  | 30 | 12 | 14 | 17/21 |
| 17 | 21 | 9  | 12 | 20 | 17 | 16/17 |
| 17 | 21 | 13 | 12 | 12 | 16 | 18/24 |
| 17 | 21 | 16 | 24 | 28 | 13 | 17/21 |
| 17 | 21 | 17 | 13 | 30 | 28 | 18/30 |
| 17 | 21 | 18 | 18 | 16 | 28 | 16/28 |
| 17 | 21 | 20 | 17 | 20 | 13 | 10/13 |
| 17 | 21 | 20 | 18 | 24 | 24 | 15/28 |
| 17 | 21 | 20 | 28 | 21 | 13 | 17/30 |
| 17 | 21 | 21 | 8  | 14 | 16 | 12/30 |
Finally, the second known class of $L_2(16):4$ centralizes an outer $S_2$ (but not $S_3$) and accounts for an amount 2 of the structure constant. This accounts for the full amount $48 + 8 + 12 + 2 = 70$ of the structure constant.

17.5. The $(2C, 3C, 17)$ case. The $(2C, 3C)$ case is considerably harder to analyse. The subgroups which turn out to be generated in this way are $L_2(17), O^-_8(2), S_8(2), F_4(2)$, as well as $2^8:O^-_8(2)$ and $2^{8+16}:O^-_8(2)$. The latter two 2-local subgroups have many classes of $(2, 3, 17)$ generating triples, and it is hard to distinguish many of them using fingerprints of the type we have been using. We found only 86 distinct fingerprints, but it would appear that the number of distinct (i.e. non-automorphic) such triples (using an element of $O^-_8(2)$ class $3C$) is 200 in the case of $2^{8+16}:O^-_8(2)$, and 30 in the case of $2^8:O^-_8(2)$.

To prove this, observe that the $2C$ elements in $O^-_8(2)$ centralize $2^4$ of the $2^8$, while the $3C$ elements centralize just $2^2$. Hence each $(2C, 3C, 17)$ triple in $O^-_8(2)$ lifts to four $(2, 3, 17)$ triples in $2^8:O^-_8(2)$, of which three generate the whole group. Similarly, lifting to $2^{8+16}:O^-_8(2)$ we acquire a factor of $2^6$, so each triple for $O^-_8(2)$ lifts to 64 triples, of which 60 generate the whole group. However, there is an outer automorphism of order 3 which effectively reduces this number to 20. We have therefore 1200 generating triples, falling into 200 types under the action of the outer automorphism group $S_3$. Therefore this contributes 600 to each of the two structure constants. Similarly for $2^8:O^-_8(2)$ we have 60 generating triples, each centralized by an outer automorphism of order 3, and swapped in pairs by the outer automorphism of order 2, so this group contributes 30 to each structure constant.

Now the fingerprint collection found in total 1919 distinct fingerprints of type $(2C, 2C, 17A/B)$ for generators of $^2E_6(2)$. However, it is not possible to have an odd number of such fingerprints, as there are equal numbers with $xy$ in $17A$ and in $17B$. (This argument depends crucially on the fact that $17A$ and $17B$ are not fused by the outer automorphism group.) Therefore there must be at least 1920 such generating triples, up to automorphisms. These account for an amount $3 \times 1920 = 5760$ of each structure constant. (Running total 6390.)

Next, we have shown that $F_4(2)$ has 384 generating triples of this type, for each of the two classes of 17-elements. Hence the three classes of $F_4(2)$ together contribute $3 \times 384 = 1152$ to each structure constant. (Running total 7542.)

For each of the groups $L_2(17), O^-_8(2)$ and $S_8(2)$, there are two known classes in $^2E_6(2):S_3$, one centralized by an outer $S_3$, the other centralized by only an outer 2. The counting is slightly different in each case. In the case of $L_2(17)$, the outer automorphism of $L_2(17)$ is not realised, and therefore the unique fingerprint
contributes a total of $1 + 3 = 4$ to each structure constant. (Running total 7546.)

In the case of $S_8(2)$, there is no outer automorphism, and therefore each fingerprint (of which there are 24) contributes just 2 to each structure constant. (Running total 7594.) Finally, in the case of $O^-_8(2)$ the outer automorphism is realised inside $^2E_6(2)$, so each of the 10 fingerprints again contributes 2. (Total so far 7614, of an expected 7614.)

The analysis of $(2,3,17)$ triples in this section proves that there is no subgroup isomorphic to one of $L_2(16)$, $L_2(17)$, $O^-_8(2)$, $O^-_{10}(2)$ or $S_8(2)$ other than those contained in known maximal subgroups.

18. Classifying $(2,3,19)$ triples

Analysis of these triples proves only one thing: that there is a unique conjugacy class of $U_3(8)$. However, there does not seem to be any easier way to prove this result. The structure constants that we need to account for are as follows:

- $\xi(2B,3C,19A+B) = 9 + 9 = 18$;
- $\xi(2C,3B,19A+B) = 15 + 15 = 30$;
- $\xi(2C,3C,19A+B) = 6126 + 6126 = 12252$.

The expected numbers of fingerprints therefore are respectively 3, 5 and 2042.

In the case $(2B,3C)$ we find three fingerprints, all for triples that generate $G$, as follows:

19 5 16 28 28 19 24
19 5 30 12 12 19 19
19 5 33 30 15 19 10

In the case $(2C,3B)$, similarly, we find five fingerprints, all for triples that generate $G$, as follows:

19 9 30 13 13 19 8/16
19 18 12 13 10 30 24/30
19 21 24 21 16 19 30

In the $(2C,3C)$ case when $xy$ has order 19, we found 2041 distinct fingerprints. Since we expecting 2042, we looked more closely, and found that one fingerprint which appeared to be from a self-reciprocal generating set was in fact from a pair of mutually reciprocal generating sets, but the fingerprint was not sufficiently discriminating to pick this up. Of these 2042 distinct types of triples, 2041 generate $G$, so contribute 3 to each structure constant, making a total contribution of 6123 to the structure constant of 6126. The remaining fingerprint corresponds to subgroups $U_3(8)$. Now the known subgroup $U_3(8)$ has normalizer $(U_3(8):3 \times 3):2$ in $G.S_3$, so contributes 3 to each structure constant. This fully accounts for the structure constant, and proves that there is a unique class of $U_3(8)$ in $^2E_6(2)$.

19. Status report

Using computational analysis of $(2,3,n)$ triples for $n = 7,11,13,17,19$ we have robust classifications for simple subgroups of the following isomorphism types

- $L_2(8)$, $L_2(13)$, $^3D_4(2)$, $F_{i22}$;
- $L_2(11)$, $M_{12}$, $A_{11}$, $A_{12}$, $O^-_{10}(2)$, $U_6(2)$;
- $L_2(25)$, $L_3(3)$, $L_4(3)$, $G_2(3)$, $^2F_4(2)'$;
- $L_2(16)$, $L_2(17)$, $O^-_8(2)$, $S_8(2)$, $F_4(2)$;
- $U_3(8)$;
as well as incomplete classifications of $A_5$ and $L_3(2)$ which are nevertheless sufficient for the purposes of determining maximal subgroups. We can therefore say that we have dealt with 23 of the 39 cases. The remaining 16 cases require other methods, more group theoretic than character theoretic.

The groups we need to deal with are
- $A_6, A_7, A_8, A_9, A_{10}$;
- $L_3(4), U_3(3), U_4(2), U_4(3), U_5(2)$;
- $O_7(3), O_8^+(2), S_4(4), S_6(2)$;
- $M_{11}, M_{22}$.

Most of these contain $A_5$, and therefore can be analyzed using Norton’s classification [6] of subgroups of the Monster containing 5A-type $A_5$. To ensure that Norton’s methods carry through, we need to be sure that in every case where the simple group $H$ he is interested in has a double cover $2^2 \cdot H$, he is using a copy of $A_5$ that is doubly covered in $2^2 \cdot H$. This is the case for all the alternating groups, and $U_4(2)$, and therefore $S_6(2), O_8^+(2)$ and $O_7(3)$; but not for $L_3(4), U_4(3)$ or $M_{22}$.

Hence the cases that require other methods are $U_3(3)$, which does not contain $A_5$, and $L_3(4), M_{22}$ and $U_4(3)$, all of which have proper double covers in which all involutions lift to involutions. For all except $L_3(4)$, it is sufficient to use known results on subgroups of the Baby Monster. The case $L_3(4)$ is problematic, since there is a subgroup $2^2 \cdot L_3(4)$ in $2^2 \cdot E_6(2)$, and there may potentially be more than one conjugacy class, and such groups will not be detected by existing work on the Monster and Baby Monster. In this last case we need to extend Norton’s methods to include classifications of $2^2 \cdot L_3(4)$ and $2 \cdot 2^2 \cdot L_3(4)$ in the Monster, as well as the simple group $L_3(4)$.

20. Using the Monster

The fact that the Monster contains a subgroup $2^2 \cdot G:S_3$ means that what is known about subgroups of the Monster implies facts about subgroups of $G$. Of particular interest is Norton’s list [6] of simple subgroups of the Monster containing 5A-elements. Now all elements of order 5 in $G$ fuse to 5A in the Monster, and all involutions in $G$ lift to involutions in $2^2 \cdot G$, so this effectively deals with almost all simple subgroups of $G$ which contain $A_5$. The ones which require extra care are those which have a double cover in which all involutions lift to involutions. There is no such group on Norton’s list, but in principle we need to consider the following possibilities:
- $L_3(4), M_{22}, U_4(3), U_6(2), Fi_{22}, F_4(2)$.

We shall not consider the cases $U_6(2), Fi_{22}$ and $F_4(2)$, however, as these have been adequately dealt with by other methods.

In this section we go through the whole of Norton’s list, including the groups we have already dealt with. This is partly in order to provide alternative, computer-free, proofs in many cases, but also to demonstrate the reliability of Norton’s work, which is published essentially without proof.

20.1. Six very easy cases. From Norton’s list we pick out first the following, which are the simple groups that centralize a unique class of $2^2$ in the Monster. In each case we write down the subgroup of the Monster that is the direct product of the simple group under consideration with its centralizer (which Norton calls its Monstralizer).
• $O_{10}^-(2) \times A_4$
• $O_7(3) \times S_3 \times S_3$
• $A_{12} \times A_5$
• $A_{11} \times A_5$
• $2F_4(2)' \times 2 \cdot S_4$
• $L_2(25) \times 2 \cdot S_4$

It follows that there is a unique class of the corresponding simple group in $G.S_3$. In the cases $H \cong O_{10}^-(2)$, $A_{12}$ and $A_{11}$ the centralizer contains $A_4$, and the normalizers in $G.S_3$ are of the shape $(H \times 3):2$. In particular, there is a single conjugacy class of $H$ in $G$.

In the case $O_7(3)$, the normalizer in $G.S_3$ is $O_7^-(3):2$, and there are three conjugacy classes of $O_7(3)$ in $G$. These are not maximal in $G$, since they are contained in $F_{22}$. However, in $G.2$ the normalizer is $O_7(3):2$, which is not contained in $F_{22}:2$, so both groups are maximal.

In the cases $H \cong L_2(25)$ and $2F_4(2)'$, the centralizer is $2 \cdot S_4$, containing a unique class of $2^2$. On factoring out by this $2^2$, an extra centralizing involution appears, and in each case we have normalizer in $G.S_3$ of shape $2 \times H.2$. Again we obtain three conjugacy classes of $H$ in $G$.

For our proof we only require the case $O_7(3)$ and summarize as follows.

**Theorem 9.** There is a unique class of $O_7(3)$ in $G.S_3$. The normalizer is $O_7(3):2$, and the class splits into three classes in $G.S_3$. In particular, there is a single conjugacy class of $H$ in $G$.

20.2. **Six more easy cases.** Next we pick from Norton’s list the following five groups, whose centralizers in the Monster contain two distinct classes of $2^2$:

• $S_6(2) \times S_4$
• $O_8^-(2) \times S_4$
• $O_8^+(2) \times (3 \times A_4).2$
• $A_{10} \times S_5$
• $A_9 \times (3 \times A_5).2$

In all these cases, one of the two classes of $2^2$ in the centralizer is normalized to $S_4$, the other only to $D_8$. Thus we obtain two classes of $H$ in $G.S_3$, one of which splits into three classes in $G$. It is easy to see that the normalizers are as given in Table 1 or 3. The case $S_6(2)$, with centralizer $S_4 \times S_3$, is similar. Note that all involutions in $S_4 \times S_3$ are in the Monster class $2A$, except for those of shape $(12)(34)(ab)$, which are in $2B$. There are therefore three types of $2A$-pure $2^2$, and hence three conjugacy classes of $S_6(2)$ in $G.S_3$. In $G$, the normalizers are as follows:

• one class of $S_3 \times S_6(2)$, of type $7A$;
• three classes of $S_3 \times S_6(2)$, of type $7B$;
• six classes of $2 \times S_6(2)$, of type $7B$.

The normalizers in $G.S_3$ are therefore

• $S_3 \times S_3 \times S_6(2)$, contained in $(3 \times O_{10}^-(2)):2$;
• $2 \times S_3 \times S_6(2)$, contained in $2 \times F_4(2)$;
• $2 \times S_6(2)$, contained in $N(2A)$.

We remark that the remaining lifts of $S_6(2)$ in $2^{1+10}:U_6(2)$ must therefore be $2 \cdot S_6(2)$, in three conjugacy classes.

For our proof we need the cases $O_8^+(2)$, $S_6(2)$, $A_{10}$ and $A_9$. 

20.3. **Three quite easy cases.** Now consider the following cases, where again there are two types of $2^2$ in the centralizer in the Monster.

- $M_{12} \times L_2(11)$
- $S_4(4) \times L_3(2)$
- $L_2(16) \times L_3(2)$

Here both classes of $2^2$ in the centralizer extend to $A_4$. In the case of $M_{12}$, there is an element of the Monster extending the group to $(M_{12} \times L_2(11)):2$. This swaps the two classes of $2^2$, while also effecting the outer automorphism of $M_{12}$. It follows that there is a single class of $M_{12}$ in $G.S_3$, with normalizer $3 \times M_{12}$, and this class splits into two classes in $G$.

Similarly in the second case we have $(S_4(4) \times L_3(2)):2$. Hence there is a unique class of $S_4(4)$ in $G$, with normalizer $S_4(4) \times S_3$ in $G.S_3$, contained in $(3 \times O_{10}^-(2)):2$.

In the third case we have $L_2(16):4 \times L_3(2)$ instead, so there are two conjugacy classes of $L_2(16)$ in $G.S_3$, with normalizer $L_2(16):4 \times S_3$ in each case.

For our proof, we need only the case $S_4(4)$, and summarize as follows.

**Theorem 10.** There is a unique class of $S_4(4)$ in $G.S_3$, and the normalizer is $S_3 \times S_4(4):2$, contained in $(3 \times O_{10}^-(2)):2$. The class remains a single class in $G$.

20.4. **Five tougher groups.**

- $A_7 \times (A_5 \times A_5):2.2$
- $A_8 \times (A_5 \times A_4):2$
- $U_5(2) \times S_3 \times A_4$
- $L_4(3) \times 3^2.D_8$
- $M_{11} \times S_6:2$
- $M_{11} \times L_2(11)$

We do not need the case $L_4(3)$, where we see two classes of $2^2$ in the centralizer. These are swapped by an outer automorphism of $L_4(3)$ realised in the Monster. Hence there is a unique class of $L_4(3)$ in $G.S_3$.

The $U_5(2)$ centralizer is $S_3 \times A_4$. Embedding the latter in $11 \times M_{12}$, we see that the only Monster $2A$-elements are in the $A_4$. Hence there is a unique class of $U_5(2)$ in $G.S_3$, and any $U_5(2)$ centralizes an element of order 3 in $G$. It follows that it lies in $S_3 \times U_5(2)$, as does its normalizer in $G$.

In the second $M_{11}$ case, the centralizer in the Monster is $L_2(11)$, whose centralizer is $M_{12}$. The same argument as for $M_{12}$, therefore, shows that there are just two classes of $M_{11}$ of this type in $G.S_3$, each with normalizer $3 \times M_{11}$, contained in $3 \times M_{12}$.

In the other $M_{11}$ case, the Monstralizer is $S_6:2$, contained in $11 \times M_{12}$. But the involutions in the $A_6$ lie in $M_{12}$-class $2B$, and therefore class $2B$ in the Monster. Hence there is no pure $2A^2$ subgroup in $S_6:2$, so this case does not arise in $G$.

We next take the group $A_8$ with centralizer $(A_5 \times A_5):2$ in the Monster. From the embedding of the latter in $A_{12}$ we see that the only $2A$-elements are in one of the factors $A_4$ or $A_5$. Hence there are exactly two classes of $A_8$ in $G$, and the normalizers in $G$ are respectively $(A_8 \times A_5):2$ and $(A_8 \times A_4):2$. Both groups lie inside $2^8.O_8^-(2)$ in $O_{10}^-(2)$.

Finally we take the $A_7$ with centralizer $(A_5 \times A_5):2.2$. In the latter group, the elements of Monster class $2A$ either lie in one of the two $A_5$ factors, or swap the two factors. Hence there is a unique class of pure $2A$-type $2^2$. Therefore there is a unique class of $A_7$ in $G$, and the normalizer is $(A_7 \times A_5):2$. 


From this list we need the cases $A_7$, $A_8$, $M_{11}$, and $U_5(2)$, and summarize as follows:

**Theorem 11.**

- In $G.S_3$ there is a unique class of $A_7$; each $A_7$ has normalizer $(A_7 \times (A_5 \times 3):2):2$;
- in $G.S_3$ there is a unique class of $U_5(2)$, with normalizer $S_3 \times (3 \times U_5(2)):2$;
- in $G.S_3$ there are two classes of $M_{11}$, each with normalizer $3 \times M_{11}$;
- in $G.S_3$ there are two classes of $A_8$; in one case each $A_8$ has normalizer $(A_8 \times (A_5 \times 3):2):2$, and in the other $(A_8 \times (A_4 \times 3):2):2$.

20.5. The last two. The remaining items on Norton’s list are:

- $A_6 \times (A_6 \times A_6).2.2$
- $A_6 \times 2.L_3(4).2$
- $A_6 \times M_{11}$
- $U_4(2) \times (A_4 \times S_3 \times S_3).2$

In the cases $A_6$ and $U_4(2)$, we have some difficulty in getting the complete list of conjugacy classes. However, it is quite straightforward to show that there is no maximal subgroup which is the normalizer of an $A_6$ or $U_4(2)$.

For example, every $2^2$ in $(A_4 \times S_3 \times S_3).2$ centralizes a further involution, and therefore every $U_4(2)$ in $G$ centralizes an involution. Similarly, it is easy to see that in the first two $A_6$ cases in the list, the centralizer of any $2^2$ in the $A_6$-centralizer is larger than the $2^2$ itself.

In the third $A_6$ case, the centralizer is $M_{11}$, which contains a unique conjugacy class of $2^2$, whose normalizer is $S_4$. It follows that there is a unique class of such $A_6$ in $G.S_3$, centralizing an $S_3$ of outer automorphisms. Its normalizer therefore lies in the normalizer of $O_{10}(2)$.

20.6. Conclusion. In this section we have dealt with the twelve cases $O_T(3)$, $O_{10}(2)$, $A_{10}$, $A_9$, $S_4(4)$, $S_6(2)$, $U_5(2)$, $U_4(2)$, $M_{11}$, $A_6$, $A_7$, $A_8$. This leaves just the four cases $U_3(3)$, $M_{22}$, $U_4(3)$ and $L_3(4)$, where we use properties of the Baby Monster as well.

21. Using the Baby Monster

21.1. The case $M_{22}$. It is shown in [13] that there is a unique class of $M_{22}$ containing $5A$-elements in the Baby Monster. The normalizer is $S_5 \times M_{22}:2$. Only the transpositions in $S_6$ fuse to class $2A$ in the Monster. Hence every $M_{22}$ in $G$ centralizes $S_3$ in $G$, so centralizes a $3A$ element. It follows that there are three conjugacy classes of $S_3 \times M_{22}$ in $G$, lying inside $S_3 \times U_6(2)$, and extending to a single class of $S_3 \times M_{22}:2$ in $G.S_3$.

21.2. The case $U_3(3)$. It is shown in [15] that every $U_3(3)$ in the Baby Monster is conjugate in the Monster to the one with centralizer $(2^2 \times 3^2.Q_8):S_3$ in the Monster. This centralizer contains four classes of involutions, with centralizers $(2^2 \times 3^2.Q_8):2$, $(2^2 \times Q_8):S_3$, $2 \times SD_{16}$, and $2^2 \times S_3$ respectively. The first must be of Monster type $2A$, since it gives rise to the subgroup $3^2.Q_8 \times U_3(3):2$ of $G$.

In any case, no $2^2$ is self-centralizing in $(2^2 \times 3^2.Q_8):S_3$, and indeed every such $2^2$ centralizes at least a group $2^3$, so every $U_3(3)$ in $G$ centralizes an involution.
21.3. The case $U_4(3)$. Every $U_4(3)$ in $G$ must lift to $2 \times 2 \cdot U_4(3)$ in $2^2 \cdot G$, and therefore lifts to $2 \times U_4(3)$ in one of the three copies of the Baby Monster, containing one of the three copies of $2 \cdot G$.

Now it is shown in [13, Theorem 11.3] that any $U_4(3)$ in the Baby Monster has non-trivial centralizer. Looking at the proof in more detail, we see that any $2 \cdot U_4(3)$ in the Monster has centralizer which is the intersection of two copies of $2 \times S_6$ in $2 \cdot L_3(4):2_2$. But the action has rank 3 and it is easy to see that the intersections are $2 \times 3^2 \cdot D_8$ and $2^2 \times D_8$. In particular, every $2^2$ in either of these possibilities centralizes a further involution. Hence every $U_4(3)$ in $G$ centralizes an involution.

21.4. The case $L_3(4)$. This last case is problematical because there is an embedding of $2^2 \cdot L_3(4)$ in $2^2 \cdot G$, and hence the enumeration of subgroups $L_3(4)$ in the Baby Monster and the Monster is not in itself sufficient to deal with the problem. However, we can modify the argument used in [13, Theorem 11.2]. Note however that although each of the groups $L_3(4)$ and $2 \cdot L_3(4)$ can be generated by two copies of $A_6$ intersecting in $3^2 \cdot 4$, such that $3^2 \cdot Q_8$ interchanges these two copies of $A_6$, this is no longer true in $2^2 \cdot L_3(4)$, where the intersection is only $3^2 \cdot 2$.

Now there are three different types of $A_6$ that need to be considered. The first has centralizer $(A_6 \times A_6) \cdot 2.2$ in the Monster, and contains elements of Monster class $2A$. These necessarily map to $2A$ elements in $G$. Now the only non-zero structure constants of type $(2A, 4, 7)$ in $G$ are

- $\xi(2A, 4A, 7A) = 1/20160$, fully accounted for by the $L_3(2)$ with normalizer $(L_3(2) \times L_3(4)) : 2$.
- $\xi(2A, 4H, 7A) = 1/480$. Consider the subgroup $2^3 \cdot L_3(2)$ of $A_8$ in $A_5 \times A_5$. The centralizer lies between $A_5$ and $L_3(4)$, and has order at least 480, so is $2^4 \cdot A_5$. Hence the normalizer is $2^3 \cdot L_3(2) \times 2^4 \cdot A_5$, fully accounting for the structure constant.
- $\xi(2A, 4L, 7A) = 1/192$. There must be such a group inside the $3B$-centralizer, hence in $O_8^+(2)$. Moreover, its centralizer is in $L_3(4)$ but contains no elements of order 5, so has order at most 192, and therefore exactly 192. But it cannot be $L_3(4)$, since $L_3(4)$ is not a subgroup of $O_8^+(2)$.

Next consider the second type of $A_6$. In this case, the centralizer of the $A_6$ in the Monster is $2 \cdot L_3(4) : 2$, and the $A_6$ contains elements of Monster class $2B$. Hence the centralizer of $3^2 \cdot 2$ is $2 \cdot U_4(3) : 2^2$, that is an involution centralizer in $O_8^+(3)$, and the argument of [13, Theorem 11.2] then shows that any group we obtain in this way lies in $S_6 \times 2 \cdot F_{12}^2$. Hence, if it has shape $2^2 \cdot L_3(4)$ then it centralizes an element of order 3. In other words, any $L_3(4)$ of this type in $G$ lies in $S_3 \times U_6(2)$.

To put more detail into the argument, the intersection of two copies of $L_3(4)$ in $U_4(3)$ is either $S_6$ or $2^4 \cdot S_4$ (in the case when the two copies are conjugate), or $L_2(7) : 2$ or $2^4 \cdot A_5$ (when they are not). Lifting to the double covers we may lose a 2 from the top of the group. Now the centralizer of $2^2 \cdot L_3(4)$ in the Monster cannot contain elements of order 5, so this eliminates two of the cases. The $L_2(7) : 2$ case gives the well-known group $L_3(2) \times 2^2 \cdot L_3(4)$ which we have already seen. The final case may or may not be $L_3(4)$, but whatever it is has normalizer contained in a $2$-local subgroup of $G$.

Finally we consider the case of the third type of $A_6$. This case was omitted in the proof of Theorem 11.2 in [13], perhaps because it was considered obvious, but more likely due to oversight. The centralizer of this $A_6$ in the Monster is $M_{11}$. Then
from Norton’s Monstralizer list \[6\] we read off that the centralizer of the relevant $3^2:2$ is $3^5:M_{11}$. Now the intersection of two copies of $M_{11}$ in this $3^5:M_{11}$ is either $3^2:Q_8$ or $A_5$. In the former case, the Monstralizer of $3^2:Q_8$ is again $3^5:M_{11}$, which does not involve $L_3(4)$. In the latter case, the Monstralizer of $A_5$ is either $A_{12}$ or $M_{11}$ (neither of which involves $L_3(4)$), or $2.M_{22}.2$, in which the subgroup $2.L_3(4)$ does not centralize a $2A$-pure $2^2$, so does not lie in $2^2.2E_6(2)$.

21.5. Conclusion. In this section we have shown that there is a unique class of $M_{22}$ in $G.S_3$, and that the only case in which the normalizer of a group $L_3(4)$, $U_3(3)$ or $U_3(3)$ is maximal is the case of the $L_3(4)$ with normalizer $(L_3(2) \times L_3(4))::2$ in $G$. This concludes the proof of our main results.

Of the 39 simple groups we had to classify, in the following 33 cases a complete list up to conjugacy has been obtained:

- $A_{12}$, $A_{11}$, $A_{10}$, $A_9$, $A_8$, $A_7$, $A_5$,
- $L_2(8)$, $L_2(11)$, $L_2(16)$, $L_2(17)$, $L_2(25)$, $L_2(13)$,
- $L_3(3)$, $L_4(3)$, $U_3(8)$, $U_5(2)$, $U_6(2)$,
- $O_7(3)$, $O_8^+(2)$, $O_8^-(2)$, $S_4(4)$, $O_{10}^-(2)$, $S_6(2)$, $S_6(2)$,
- $G_2(3)$, $F_4(2)$, $3D_4(2)$, $2F_4(2)'$,
- $M_{11}$, $M_{12}$, $M_{22}$, $F_{122}$.

The remaining 6 have been dealt with to the extent that their normalizers are shown to be non-maximal, although a complete list of conjugacy classes and normalizers has not (yet) been obtained:

- $A_6$, $L_2(7)$, $U_3(3)$, $U_4(2)$, $U_4(3)$, $L_3(4)$.

22. Further remarks

In this section we provide an alternative proof of the following theorem, using neither computation nor Norton’s results:

**Theorem 12.**

1. There is a single conjugacy class of each of the groups $O_{10}^-(2)$, $A_{12}$ and $A_{11}$ in $G$, and the normalizers in $G.S_3$ are

   $$(3 \times A_{11})::2 < (3 \times A_{12})::2 < (3 \times O_{10}^-(2))::2.$$  

2. There is a single class of $F_4(2)$ in $G.S_3$, splitting into three classes in $G$. The normalizer in $G.S_3$ is $2 \times F_4(2)$.

3. There are two classes of each of the groups $S_6(2)$ and $3D_4(2)$ in $G.S_3$, splitting into four each classes in $G$. The normalizers in $G.S_3$ are

   - $S_8(2) \times S_3 < (O_{10}^-(2) \times 3) :: 2$;
   - $S_8(2) \times 2 < F_4(2) \times 2$;
   - $3D_4(2) : 3 \times S_3$;
   - $3D_4(2) : 3 \times 2 < F_4(2) \times 2$.

22.1. $F_4(2)$. Any subgroup isomorphic to $F_4(2)$ may be constructed by taking a group $L_3(2) \times L_3(2)$ and extending the Sylow 7-normalizer to $7^2:(3 \times 2A_4)$. Now $L_3(4)$ contains exactly three conjugacy classes of $L_3(2)$, so there are just three possibilities for the $L_3(2) \times L_3(2)$. The extension is to the full Sylow 7-normalizer in $G$, so is unique. Hence there are exactly 3 conjugacy classes of $F_4(2)$ in $G$, fused in $G.3$. The full normalizer in $G.S_3$ is $F_4(2) \times 2$. 

22.2. $O^-_{10}(2)$. Any subgroup $O^-_{10}(2)$ can be constructed from two copies of $A_5 \times A_8$ intersecting in $A_5 \times A_5 \times 3$. Since the 5-centralizer is $5 \times A_8$, it is obvious that the choices, of the first subgroup $A_5 \times A_8$, and the subgroup $A_5 \times A_5 \times 3$, and finally the second $A_8 \times A_5$ are unique up to relevant conjugacy at each stage. Hence there is exactly one conjugacy class of $O^-_{10}(2)$ in $G$, whose full normalizer in $G.S_3$ is $(O^-_{10}(2) \times 3):2$.

22.3. $A_{12}$. Any subgroup isomorphic to $A_{12}$ may be constructed from two groups $(A_5 \times A_7):2$ intersecting in $S_5 \times S_5$. Since the 5-centralizer $5 \times A_8$ lies in the group $(A_5 \times A_8):2$, there is a unique class of $(A_5 \times A_7):2$, and a unique class of $S_5 \times S_5$ in it. Now in the normalizer of the second $A_5$, we need to extend $S_5 \times 2$ to $S_7$ inside $S_8$, and there is obviously a unique way to do this. Hence there is exactly one conjugacy class of $A_{12}$ in $G$, whose full normalizer in $G.S_3$ is $(A_{12} \times 3):2$. This group is never maximal in any extension of $G$ by outer automorphisms.

22.4. $A_{11}$. A similar argument applies with $(A_5 \times A_8):2$. We restrict to a subgroup $(A_5 \times A_5):2$, in which the two factors are conjugate in $G$. At the final stage, we have to extend $S_5$ to $S_6$ in $A_8$, and again it is clear that there is only one way to do this. Hence there is exactly one conjugacy class of $A_{11}$ in $G$, whose full normalizer in $G.S_3$ is $(3 \times A_{11}):2$. This group is never maximal in any extension of $G$.

22.5. $3D_4(2)$. Every such group can be made from $7 \times L_3(2)$ by extending the Sylow 7-normalizer to $7^2:2A_4$. As shown above, there are exactly four classes of $7 \times L_3(2)$ in $G$, one containing a central 7B and three containing a central 7A. In each case the extension from $7^2:3$ to $7^2:2A_4$ is unique within the full Sylow 7-normalizer $7^2.(3 \times 2A_4)$. Hence there are exactly four conjugacy classes of $3D_4(2)$ in $G$, each with normalizer $3D_4(2):3$ in $G$. In $G.S_3$, three classes are fused and the other is centralized. Thus the normalizers in $G.S_3$ are respectively $3D_4(2):3 \times 2$ and $3D_4(2) \times S_3$. The latter is maximal in $G.S_3$ (and its intersection with $G.S_3$ is maximal therein).

22.6. $S_8(2)$. Any group $S_8(2)$ can be constructed by taking a group $S_3 \times S_6(2)$, restricting to $S_3 \times S_3 \times S_6$, and then extending to $S_6 \times S_6$. Now there are exactly four classes of $S_3 \times S_6(2)$ in $G$, in one of which the $S_3$ contains 3B-elements, while in the other three the $S_3$ contains 3A-elements. Then the restriction from $S_6(2)$ to $S_3 \times S_6$ is unique up to conjugacy, and the centralizer in $G$ of the $S_3$ is exactly $S_6$. Hence there is at most one copy of $S_8(2)$ containing any given $S_3 \times S_6(2)$. But we already know there are at least four conjugacy classes of $S_8(2)$ in $G$, one centralized by an outer $S_3$, and three more in $F_4(2)$ centralized by an outer involution, so there are exactly four. Three are fused in $G.S_3$, while the other is centralized by an element of class 3D. The normalizers in $G.S_3$ are respectively $S_8(2) \times 2$, contained in $F_4(2) \times 2$, and $S_8(2) \times S_3$, contained in $(O^-_{10}(2) \times 3):2$.

References

[1] J. N. Bray, D. F. Holt and C. Roney-Dougal, The maximal subgroups of the low-dimensional finite classical groups, LMS Lecture Notes Ser. 407, Cambridge UP, 2013.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, An Atlas of Finite Groups, Oxford University Press, 1985.

[3] The GAP group, GAP — Groups, Algorithms, and Programming, Version 4.8.10; 2018. (https://www.gap-system.org)
MAXIMAL SUBGROUPS OF $^2E_6(2)$ AND ITS AUTOMORPHISM GROUPS

[4] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An Atlas of Brauer Characters*, Oxford UP, 1995.

[5] P. B. Kleidman and R. A. Wilson, The maximal subgroups of $Fi_{22}$, *Math. Proc. Cambridge Philos. Soc.* 102 (1987), 17–23. Corrigendum, *ibid.* 103 (1988), 383.

[6] S. Norton, Anatomy of the Monster: I, in *Proceedings of the Atlas Ten Years On conference (Birmingham 1995)*, pp. 198–214, Cambridge Univ. Press, 1998.

[7] S. P. Norton and R. A. Wilson, The maximal subgroups of $F_4(2)$ and its automorphism group, *Comm. Algebra* 17 (1989), 2809–2824.

[8] S. P. Norton and R. A. Wilson, Anatomy of the Monster: II, *Proc. London Math. Soc.* 84 (2002), 581–598.

[9] R. A. Wilson, On maximal subgroups of the Fischer groups $Fi_{22}$, *Math. Proc. Cambridge Philos. Soc.* 95 (1984), 197–222.

[10] R. A. Wilson, Maximal subgroups of automorphism groups of simple groups, *J. London Math. Soc.* 32 (1985), 460–466.

[11] R. A. Wilson, The local subgroups of the Fischer groups, *J. London Math. Soc.* 36 (1987), 77–94.

[12] R. A. Wilson, Some subgroups of the Baby Monster, *Invent. Math.* 89 (1987), 197–218.

[13] R. A. Wilson, More on maximal subgroups of the Baby Monster, *Arch. Math. (Basel)* 61 (1993), 497–507.

[14] R. A. Wilson, The symmetric genus of the Baby Monster, *Quart. J. Math. (Oxford)* 44 (1993), 513–516.

[15] R. A. Wilson, The maximal subgroups of the Baby Monster, I, *J. Algebra* 211 (1999), 1–14.

[16] R. A. Wilson, *The finite simple groups*, Springer GTM 251, 2009.

[17] R. A. Wilson et al., *An Atlas of Group Representations*, [http://brauer.maths.qmul.ac.uk/Atlas/](http://brauer.maths.qmul.ac.uk/Atlas/)
### Appendix: \((2C, 3C, 11)\) generators for \(^2E_6(2)\)

| 11 | 6 | 24 | 20 | 19 | 20 | 22 |
|----|---|----|----|----|----|----|
| 11 | 8 | 11 | 17 | 20 | 12 | 12/19 |
| 11 | 8 | 11 | 22 | 35 | 12 | 15 |
| 11 | 8 | 18 | 28 | 17 | 17 | 11/30 |
| 11 | 8 | 22 | 16 | 13 | 22 | 19/28 |
| 11 | 8 | 24 | 20 | 19 | 12 | 18/35 |
| 11 | 8 | 28 | 17 | 14 | 17 | 19/24 |
| 11 | 8 | 35 | 8 | 17 | 14 | 19/35 |
| 11 | 9 | 13 | 17 | 17 | 20 | 19/28 |
| 11 | 9 | 19 | 11 | 33 | 17 | 9/22 |
| 11 | 9 | 19 | 35 | 13 | 19 | 20/22 |
| 11 | 9 | 21 | 17 | 17 | 19 | 20/20 |
| 11 | 9 | 22 | 35 | 33 | 21 | 12/33 |
| 11 | 9 | 24 | 12 | 13 | 16 | 35 |
| 11 | 9 | 24 | 35 | 12 | 17 | 11 |
| 11 | 9 | 33 | 13 | 19 | 30 | 17/28 |
| 11 | 10 | 19 | 19 | 19 | 33 | 11/19 |
| 11 | 10 | 22 | 18 | 10 | 10 | 24 |
| 11 | 10 | 22 | 20 | 28 | 16 | 13/35 |
| 11 | 10 | 28 | 17 | 13 | 19 | 13/16 |
| 11 | 10 | 35 | 11 | 28 | 17 | 13/20 |
| 11 | 12 | 11 | 24 | 22 | 24 | 17/24 |
| 11 | 12 | 13 | 24 | 18 | 21 | 22/33 |
| 11 | 12 | 17 | 17 | 14 | 19 | 15/16 |
| 11 | 12 | 18 | 10 | 30 | 28 | 17/24 |
| 11 | 12 | 18 | 19 | 13 | 19 | 22/30 |
| 11 | 12 | 21 | 12 | 10 | 16 | 19 |
| 11 | 12 | 22 | 21 | 22 | 35 | 13/19 |
| 11 | 12 | 30 | 20 | 11 | 9 | 33 |
| 11 | 12 | 33 | 13 | 19 | 11 | 16/35 |
| 11 | 12 | 33 | 17 | 22 | 17 | 22/35 |
| 11 | 12 | 35 | 24 | 11 | 17 | 17 |
| 11 | 13 | 12 | 21 | 33 | 22 | 19/35 |
| 11 | 13 | 16 | 28 | 28 | 17 | 16/19 |
| 11 | 13 | 16 | 30 | 35 | 16 | 21/35 |
| 11 | 13 | 17 | 30 | 17 | 35 | 13/20 |
| 11 | 13 | 17 | 30 | 33 | 18 | 16/19 |
| 11 | 13 | 19 | 17 | 19 | 19/28 |
| 11 | 13 | 20 | 21 | 20 | 17 | 17/24 |
| 11 | 13 | 21 | 21 | 30 | 18 | 13/17 |
| 11 | 13 | 21 | 24 | 22 | 12 | 19/22 |
| 11 | 13 | 22 | 22 | 22 | 22 | 20/22 |
| 11 | 13 | 24 | 12 | 17 | 13 | 18/22 |
| 11 | 13 | 28 | 21 | 22 | 19 | 17/21 |
| 11 | 13 | 30 | 17 | 22 | 16 | 18/33 |
| 11 | 13 | 30 | 22 | 19 | 24 | 17/35 |

The table above lists the generators for \(^2E_6(2)\). Each row represents a set of two generators, with the first being the \((2C, 3C, 11)\) generator and the second being the \(^2E_6(2)\) generator. The numbers correspond to specific entries in the \(E_6\) root system.
| 11 | 18 | 19 | 15 | 11 | 19 | 13/30 |
|----|----|----|----|----|----|--------|
| 11 | 18 | 19 | 18 | 17 | 19/30 |
| 11 | 18 | 20 | 19 | 17 | 22 | 20/22 |
| 11 | 18 | 20 | 28 | 15 | 17 | 17/18 |
| 11 | 18 | 21 | 16 | 33 | 12 | 13/24 |
| 11 | 18 | 21 | 28 | 16 | 14 | 21/22 |
| 11 | 18 | 22 | 19 | 19 | 20 | 17/33 |
| 11 | 18 | 24 | 22 | 21 | 13 | 17/30 |
| 11 | 18 | 24 | 30 | 19 | 28 | 18 |
| 11 | 18 | 28 | 18 | 12 | 24 | 33/35 |
| 11 | 18 | 28 | 28 | 33 | 22 | 16/18 |
| 11 | 18 | 30 | 12 | 17 | 17 | 12/17 |
| 11 | 18 | 30 | 19 | 17 | 19 | 19/28 |
| 11 | 18 | 30 | 35 | 17 | 19 | 12/33 |
| 11 | 18 | 33 | 18 | 20 | 24 | 17/18 |
| 11 | 18 | 35 | 11 | 19 | 18 | 14/17 |
| 11 | 18 | 35 | 22 | 17 | 22 | 30/33 |
| 11 | 20 | 10 | 11 | 21 | 19 | 16/19 |
| 11 | 20 | 11 | 4 | 14 | 33 | 12/17 |
| 11 | 20 | 11 | 16 | 17 | 13 | 13/13 |
| 11 | 20 | 12 | 13 | 33 | 16 | 9/19 |
| 11 | 20 | 13 | 17 | 18 | 20 | 10/19 |
| 11 | 20 | 13 | 35 | 33 | 21 | 17/18 |
| 11 | 20 | 16 | 13 | 17 | 13 | 18/24 |
| 11 | 20 | 16 | 17 | 30 | 18 | 17/22 |
| 11 | 20 | 17 | 19 | 13 | 22 | 17/21 |
| 11 | 20 | 17 | 19 | 17 | 22 | 8/17 |
| 11 | 20 | 17 | 22 | 15 | 18 | 13/19 |
| 11 | 20 | 17 | 24 | 28 | 33 | 22/24 |
| 11 | 20 | 17 | 30 | 24 | 28 | 8/24 |
| 11 | 20 | 17 | 35 | 19 | 22 | 35/35 |
| 11 | 20 | 19 | 11 | 16 | 9 | 12/33 |
| 11 | 20 | 19 | 33 | 30 | 24 | 19/24 |
| 11 | 20 | 22 | 17 | 13 | 22 | 22/33 |
| 11 | 20 | 22 | 19 | 15 | 17 | 21 |
| 11 | 20 | 22 | 19 | 17 | 22 |
| 11 | 20 | 22 | 33 | 15 | 11/18 |
| 11 | 20 | 22 | 30 | 10 | 30 | 20/33 |
| 11 | 20 | 28 | 19 | 14 | 20 | 13/17 |
| 11 | 20 | 30 | 24 | 13 | 19 | 9/18 |
| 11 | 20 | 33 | 16 | 30 | 22 | 22/28 |
| 11 | 21 | 9 | 16 | 16 | 17 | 35 |
| 11 | 21 | 12 | 17 | 12 | 13 | 18/28 |
| 11 | 21 | 13 | 22 | 19 | 30 | 11/17 |
| 11 | 21 | 16 | 16 | 10 | 33 | 22/24 |
| 11 | 21 | 16 | 20 | 24 | 17 | 20/30 |
| 11 | 21 | 18 | 17 | 30 | 20 | 14/35 |
| 11 | 21 | 18 | 33 | 17 | 28 | 20/30 |
| 11 | 21 | 18 | 35 | 18 | 13 | 18/19 |
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