A MULTI-STAGE SIR MODEL FOR RUMOR SPREADING

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Abstract. We propose a multi-stage structured rumor spreading model that consists of ignorant, new spreader, old spreader, and stifler. We derive a mean field equation to obtain the multi-stage structured model on homogeneous networks. Since rumors spread from a few people, we consider a large population by setting the number of initial spread to one in total population $n$ and limiting $n$ to $\infty$. We investigate a threshold phenomenon of rumor outbreak in the sense of the large population limit by studying the driven multi-stage structured model. The main conclusion of this paper is that the proposed model has a threshold phenomenon in terms of a basic reproduction number which is similar to the SIR epidemic model. We present numerical simulations to show the developed theory numerically.

1. Introduction. The purpose of this paper is to study the large population limit, or macroscopic limit, on a multi-stage rumor spreading model,

$$
\begin{align*}
\dot{I} &= -k\lambda_1 S_1 I - k\lambda_2 S_2 I, \\
\dot{S}_1 &= k\lambda_1 S_1 I + k\lambda_2 S_2 I - \delta_1 S_1, \\
\dot{S}_2 &= \delta_1 S_1 - k\sigma S_2 (S_1 + S_2 + R) - \delta_2 S_2, \\
\dot{R} &= k\sigma S_2 (S_1 + S_2 + R) + \delta_2 S_2,
\end{align*}
$$

where $k$ is the average degree of the network. We assume that, at the beginning, there is one new spreader. Since each variable is the fraction of populations, for a large number $n$ ($n \to \infty$), initial data satisfy

$$
I(0) = I_0^n = \frac{n-1}{n}, \quad S_1(0) = (S_1^0)_n = \frac{1}{n}, \quad S_2(0) = (S_2^0)_n = 0, \quad R(0) = R_0^n = 0.
$$

In this model, the population group is divided into four groups: ignorants ($I$), new spreaders ($S_1$), old spreaders ($S_2$), and stiflers ($R$). What distinguishes old spreaders from new spreaders here is the relative time that has passed since the infection. In other words, our model is structured by age of infection. The system has been obtained from the following rules:

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(1) Ignorants contact both spreaders. Some of them accept the rumor and become new spreaders and then spread the rumor to other ignorants. We assume that the acceptance rates of ignorant from new spreader and old spreader are $\lambda_1$ and $\lambda_2$, respectively.

(2) A new spreader becomes an old spreader with transition rate $\delta_1$ and does not become a stifler directly.

(3) When old spreaders lose their interest in the rumor, they become stiflers. We assume that if an old spreader contacts other spreaders and stiflers, then he or she loses interest in spreading. For simplicity, we assume the contact rates are all the same at $\sigma$. Additionally, we assume that an old spreader becomes a stifler with transition rate $\delta_2$.

The entire population is divided into three groups: ignorants ($I$), spreaders ($S$), stiflers ($R$). Ignorant group is people who have never heard of rumors and spreader group is people who spread rumor. Stifler group represents people who are aware of the rumor but do not spread it. To derive a mean field equation, we assume that ignorants $I$ contact a spreader, and then one ignorant accepts the rumor with some probability. Since among the communicators, their respective influence and propagation power are different [6, 4, 3], we additionally assume that spreaders have a multi-stage structure, i.e., two types of spreaders. $S_1$ is the fraction of populations for relatively new spreaders who has recently heard rumors and $S_2$ is the fraction of populations for old spreaders who has been exposed to rumors relatively long ago. We assume that ignorants who meet new or old spreaders become spreaders. As time goes on, new spreaders become old spreaders with some probability, and old spreaders lose interest in the rumor and become stiflers. Moreover, if spreaders contact other spreaders or stiflers, then they become stiflers. See Figure 1.

Various theoretical and experimental aspects of rumor diffusion have been recently reported [4, 5, 20, 35]. For example, authors in [4] considered the case that an ignorant can spread the rumor. In the real world, it is hard to distinguish ignorant and spreader precisely, and spreaders have various kinds of spreading rate. Spreaders spread rumors aggressively when they first hear rumors, and spreaders will spread the rumors less aggressively if they get used to it over time. From this simple observation, we consider an SIR type rumor spreading model with multi-stage structure on a homogeneous network.

![Figure 1. The rumor spreading process of the multi-stage SIR model](image)

Rumor spreading is a dynamic process that occurs often due to the characteristics of human society and is closely related to the phenomenon of information homogenization [2] in the macroscopic viewpoint. Ranging from global rumor to
regional rumor in small village units, there are various forms of rumor spreading phenomena, and they have various effects on human society [22]. Rumors have had a great impact on human society for a long time; for instance, rumor has been used as a means of war, governance, and propaganda [17]. Rumors are now being transmitted more rapidly [11] and more extensively [24, 29] due to the development of media and social network service; therefore, they are expected to exert more influence, and their role and essence are getting more attention from researchers.

Based on Daley and Kendall [7, 8], a lot of researchers have tried to build mathematical models of rumor spreading [21, 28]. Researchers have been trying to understand rumor spreading qualitatively. In [33, 34], the author observed the existence of a critical threshold for a rumor spreading model numerically in small-world networks. In [23], the authors derived a mean-field equation for rumor spreading in complex heterogeneous networks. In [37], the authors considered an SIR type rumor spreading model with a forgetting mechanism. See also [26], for a spreading model in telecommunication networks where the structure of the network affects the model equations through the largest eigenvalue instead of the average degree.

As most scientific and theoretical studies, researchers have studied rumor spreading phenomena through simplified models. Research on rumor spreading has been also able to successfully derive mathematical and physical properties through such simplification, and it was possible to describe and study various derivative phenomena based on the modeling. As the research progresses, many authors provide rumor spreading models that are more realistic. See [30, 32, 36]. In this paper, we study a rumor model with multi-stage structure from this perspective. As mentioned earlier, the rumor spreading phenomenon in the real world has complex dynamics and various structures including multi-stage structure, which would have been omitted in the simplification process. Therefore, we try to understand in this paper how multi-stage structure affects rumor spreading models.

In general, an multi-stage structured model enables a more realistic representation for some dynamics phenomena [27]. In SIR models, many authors have considered a variety of equations to represent epidemic phenomena with the multi-stage structure. Their models are based on a system of partial differential equations with SIR structure; for example, [13, 15, 18]. For some threshold phenomena on an age structured epidemic model, see [16]. In this paper, we consider a simple SIR model for rumor spreading with two stages. See [19] for a similar discrimination approach on the epidemic model and [25] for an age structured model for rumor spreading.

The paper is organized as follows. In Section 2, we present our main result and provide a reduction of the multi-stage structured model to a system of two equations for rumor sizes $\phi_1$ and $\phi_2$. In Section 3, we provide a proof of the main theorem. In Section 4, we present numerical simulations to ensure our result. Finally, we summarize our results in Section 5.

2. Main theorem for rumor outbreak and equation reduction with auxiliary variables. In this section, we define rumor outbreak and present our main theorem. We consider new variables to prove our main theorem and derive reduced equations from (1). The equation reduction allows us to use steady state analysis. However, since there are two kinds of spreaders, we cannot deduce one equation that is equivalent with (1). Motivated by [12, 31], we define the rumor sizes $\phi_1$ and $\phi_2$ of $S_1$ and $S_2$, respectively:
\[ \phi_1(t) = \int_0^t S_1(\tau)d\tau, \quad \phi_2(t) = \int_0^t S_2(\tau)d\tau. \]

From this notion, we can define a rumor outbreak state in the sense of the large population limit.

**Definition 2.1.** Let \( \{I_n, (S_1)_n, (S_2)_n, R_n\} \) be a sequence of solutions to system (1) subject to initial data \( I_0^n, (S_0^1)_n \) and \( (S_0^2)_n = (R_0^n)_n = 0 \), respectively. We define the final size of the rumor:

\[ \phi^\infty(n) := \lim_{t \to \infty} \phi^n_1(t), \]

where

\[ \phi^n_1(t) = \int_0^t (S_1)_n(\tau)d\tau. \]

**Definition 2.2.** Let \( I_0^n > 0, (S_0^1)_n > 0, (S_0^2)_n \) and \( R_0^n \) be sequences satisfying

\[ I_0^n \to 1, \quad (S_1)_n \to 0^+ \quad \text{as} \quad n \to \infty \]

and

\[ (S_2)_n = R_0^n = 0 \quad \text{for} \quad n \in \mathbb{N}, \quad 1 = I_0^n + (S_1)_n. \]

Then we say that a rumor outbreak occurs if \( \phi_e > 0 \), where

\[ \phi_e = \lim \inf_{n \to \infty} \phi^\infty(n). \]

For the multi-stage model (1), we define the corresponding basic reproduction number such as

\[ \mathcal{R}_0 = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2}. \]

Next, we present threshold phenomena with respect to \( \mathcal{R}_0 \) for the asymptotic behavior of the multi-stage model, the so called rumor outbreak state. For the computation of \( \mathcal{R}_0 \) in general structured models, we refer to \([1, 10, 9]\).

**Remark 1.** In the case without multi-stage structure, there is a threshold phenomenon of rumor spreading dynamics for the basic reproduction number \( k\lambda/\delta \), where \( \delta \) is the forgetting rate and \( \lambda \) is the acceptance rate. If \( k\lambda/\delta > 1 \), then rumor outbreak occurs; if \( k\lambda/\delta \leq 1 \), then rumor outbreak does not occur. For details, see \([12, 31]\). Similar to a mono-stage case, we can observe a threshold phenomenon in the multi-stage structured model.

**Remark 2.** For the next generation matrix, we get

\[ \mathcal{F} = \begin{bmatrix} 0 \\
 k\lambda_1 S_1 I + k\lambda_2 S_2 I \\
 0 \end{bmatrix} \]

and

\[ \mathcal{V} = \begin{bmatrix} k\lambda_1 S_1 I + k\lambda_2 S_2 I \\
 \delta_1 S_1 \\
 k\sigma S_2 (S_1 + S_2 + R) + \delta_2 S_2 \end{bmatrix} - \begin{bmatrix} 0 \\
 0 \\
 \delta_1 S_1 \end{bmatrix}. \]
For the infected compartments $S_1$ and $S_2$, the Jacobian matrices for $F$ and $V$ at the disease free equilibrium are given by

$$F = \begin{bmatrix} k\lambda_1 & k\lambda_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \delta_1 & 0 \\ -\delta_1 & \delta_2 \end{bmatrix}. $$

Therefore,

$$\rho(FV^{-1}) = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2}. $$

This basic reproduction number coincides with the number which we proposed. However, this basic reproduction number obtaining from the next generation matrix explains the linear stability of the disease free equilibrium. As we can see in [10], there are five assumptions for the global stability. However, the model (1) does not satisfy the assumptions in [10]. In particular, the fifth assumption is that ‘if $F$ is set to zero, then all eigenvalues of $Df(x_0)$ have negative real parts’, where $f = F - V$ and $x_0$ is the disease free equilibrium. In our model, if we set $F = 0$, then the eigenvalues of $Df(x_0)$ are $-\delta_1$, $-\delta_2$ and 0 with multiplicity 2. Therefore, we investigate the asymptotic stability of the system (1) by employing the following method.

The following is the main theorem of this paper.

**Theorem 2.3.** Let $\{I_n(t), (S_1)_n(t), (S_2)_n(t), R_n(t)\}$ be a sequence of solutions to system (1) subject to initial data $I_0 > 0$, $(S_1)_0 > 0$, $(S_2)_0 = 0$ and $R_0 = 0$, respectively. Assume that

$$I_n^0 \to 1^-, \quad (S_1^0)_n \to 0^+ \quad \text{as} \quad n \to \infty, \quad 1 = I_n^0 + (S_1^0)_n. $$

Then,

rumor outbreak occurs if and only if $R_0 > 1$.

**Remark 3.** Similar two-stage rumor spreading model was considered in [14]. Authors in [14] studied the locally asymptotic stability of equilibrium by using Routh-Hurwitz criteria and the global stability of internal equilibrium of their model by Lasalle’s invariance principle. In this paper, they found five equilibrium depending on inflow and outflow rates. However, we assume that there is no inflow and outflow in our model. Thus, for any $0 \leq I_e \leq 0$, $(I, S_1, S_2, R) = (I_e, 0, 0, 1 - I_e)$ is equilibrium point of our model. Moreover, the stability of equilibrium points is not our scope. Here, we consider the thermodynamic limit or large population limit of rumor spreading model.

Let $(I, S_1, S_2, R)$ be the solution to (1) subject to initial data $(I^0, S_1^0, S_2^0, R^0)$.

Since $\dot{I}(t) + \dot{S}_1(t) + \dot{S}_2(t) + \dot{R}(t) = 0$ and $I(0) + S_1(0) + S_2(0) + R(0) = 1$, total population is conserved:

$$1 = I(t) + S_1(t) + S_2(t) + R(t). \quad (3)$$

Next, we derive a system of differential equations for $\phi_1$ and $\phi_2$. We assume that $S_2^0 = R^0 = 0$, i.e.,

$$I^0 + S_1^0 = 1, \quad 0 < I^0 < 1, \quad S_1^0 > 0. $$

From (1), we can get

$$\frac{\dot{I}(t)}{I(t)} = -k\lambda_1 S_1(t) - k\lambda_2 S_2(t). $$
From integrating the above, it follows that
\[
\log I(t) - \log I^0 = \int_0^t \frac{\dot{I}(\tau)}{I(\tau)} d\tau \\
= -k\lambda_1 \int_0^t S_1(\tau) d\tau - k\lambda_2 \int_0^t S_2(\tau) d\tau \\
= -k\lambda_1 \phi_1(t) - k\lambda_2 \phi_2(t),
\]
this implies
\[
I(t) = I^0 e^{-k\lambda_1 \phi_1(t) - k\lambda_2 \phi_2(t)}.
\] (4)

We integrate the second equation in (1) to obtain
\[
S_1(t) - S_1^0 = \int_0^t \dot{S}_1(\tau) d\tau \\
= \int_0^t \left[ k\lambda_1 S_1(\tau) I(\tau) + k\lambda_2 S_2(\tau) I(\tau) - \delta_1 S_1(\tau) \right] d\tau \\
= \int_0^t \left( k\lambda_1 S_1(\tau) + k\lambda_2 S_2(\tau) \right) I(\tau) d\tau - \delta_1 \phi_1.
\]
By (4), the above result yields
\[
S_1(t) - S_1^0 = \int_0^t \left( k\lambda_1 S_1(\tau) + k\lambda_2 S_2(\tau) \right) I^0 e^{-k\lambda_1 \phi_1(\tau) - k\lambda_2 \phi_2(\tau)} d\tau - \delta_1 \phi_1.
\]
Note that, by the definitions of \( \phi_1 \) and \( \phi_2 \),
\[
\frac{d}{dt} \left( k\lambda_1 \phi_1(t) + k\lambda_2 \phi_2(t) \right) = k\lambda_1 S_1(t) + k\lambda_2 S_2(t).
\]
The change of variables leads us to
\[
S_1(t) - S_1^0 = \int_0^t \left( k\lambda_1 S_1(\tau) + k\lambda_2 S_2(\tau) \right) I^0 e^{-k\lambda_1 \phi_1(\tau) - k\lambda_2 \phi_2(\tau)} d\tau - \delta_1 \phi_1 \\
= \int_0^{k\lambda_1 \phi_1(t) + k\lambda_2 \phi_2(t)} I^0 e^{-\eta} d\eta - \delta_1 \phi_1 \\
= I^0 - I^0 e^{-k\lambda_1 \phi_1(t) - k\lambda_2 \phi_2(t)} - \delta_1 \phi_1.
\]
By the initial data condition in (2) and \( \dot{\phi}_1(t) = S_1(t) \), we can get
\[
\dot{\phi}_1 = 1 - I^0 e^{-k\lambda_1 \phi_1(t) - k\lambda_2 \phi_2(t)} - \delta_1 \phi_1.
\] (5)
Next we derive a equation for \( \dot{\phi}_2 \). We integrate the third equation (1) to obtain
\[
S_2(t) - S_2^0 = \int_0^t \dot{S}_2(\tau) d\tau \\
= \int_0^t \left[ \delta_1 S_1(\tau) - k\sigma S_2(\tau)(S_1(\tau) + S_2(\tau) + R(\tau)) - \delta_2 S_2(\tau) \right] d\tau.
\]
By (3), \( S_1(\tau) + S_2(\tau) + R(\tau) = 1 - I(\tau) \), and this implies
\[
S_2(t) - S_2^0 = \int_0^t \left[ \delta_1 S_1(\tau) - k\sigma S_2(\tau)(1 - I(\tau)) - \delta_2 S_2(\tau) \right] d\tau.
\]
We use (4) to yield
\[ S_2(t) - S_2^0 = \int_0^t \left( \delta_1 S_1(\tau) - k\sigma S_2(\tau) \left[ 1 - \int_0^\tau e^{-k\lambda_1 \phi_2(\tau)} - k\lambda_2 \phi_2(\tau) \right] - \delta_2 S_2(\tau) \right) d\tau \]
\[ = \delta_1 \phi_1(t) - \int_0^t k\sigma S_2(\tau) \left[ 1 - \int_0^\tau e^{-k\lambda_1 \phi_2(\tau)} - k\lambda_2 \phi_2(\tau) \right] d\tau - \delta_2 \phi_2(t) \]
\[ = \delta_1 \phi_1(t) - \delta_2 \phi_2(t) - k\sigma \phi_2(t) + k\sigma I_0 \int_0^t S_2(\tau)e^{-k\lambda_1 \phi_2(\tau)} - k\lambda_2 \phi_2(\tau) d\tau. \]

From \( S_2^0 = 0 \) and \( \dot{\phi}_2(t) = S_2(t) \), it follows that
\[ \dot{\phi}_2(t) = \delta_1 \phi_1(t) - \delta_2 \phi_2(t) - k\sigma \phi_2(t) + k\sigma I_0 \int_0^t S_2(\tau)e^{-k\lambda_1 \phi_2(\tau)} - k\lambda_2 \phi_2(\tau) d\tau. \]

We note that \( \dot{\phi}_1(t) = S_1(t) \) and \( \dot{\phi}_2(t) = S_2(t) \). Therefore, both are increasing functions, and for a fixed \( I_0 > 0 \), there is a function \( \phi_2^{-1}(\cdot) = \phi_2^{-1}(I^0(\cdot)) \) such that
\[ t = \phi_2^{-1}(\phi_2(t)). \]
Here, \( \phi_2^{-1} \) depends on initial data \( I^0 > 0 \). We define a function \( \Phi_1 = \Phi_1^0(\cdot) \) such that
\[ \Phi_1^0(\cdot) = \phi_1(\phi_2^{-1}(\cdot)). \]
Similarly, there exists \( \phi_1^{-1}(\cdot) = \phi_1^{-1}(I^0(\cdot)) \) such that
\[ t = \phi_1^{-1}(\phi_1(t)), \]
and we define \( \Phi_2 = \Phi_2^0(\cdot) \) such that
\[ \Phi_2^0(\cdot) = \phi_2(\phi_1^{-1}(\cdot)). \]

Therefore,
\[ \dot{\phi}_2(t) = \delta_1 \phi_1(t) - \delta_2 \phi_2(t) - k\sigma \phi_2(t) + k\sigma I_0 \int_0^t S_2(\tau)e^{-k\lambda_1 \Phi_1(\phi_2(t)) - k\lambda_2 \phi_2(t)} d\tau, \]
and
\[ \dot{\phi}_2(t) = \delta_1 \phi_1(t) - \delta_2 \phi_2(t) - k\sigma \phi_2(t) \]
\[ + k\sigma I_0 \int_0^t \left( \frac{\lambda_1}{\lambda_2} S_1(\tau) + S_2(\tau) \right)e^{-k\lambda_1 \phi_2(\tau) - k\lambda_2 \phi_2(t)} d\tau \]
\[ - k\sigma I_0 \int_0^t \frac{\lambda_1}{\lambda_2} S_1(\tau)e^{-k\lambda_1 \phi_2(\tau) - k\lambda_2 \phi_2(\phi_1(t))} d\tau. \]

By the change of variables, we have
\[ \dot{\phi}_2 = \delta_1 \phi_1 - \delta_2 \phi_2 - k\sigma \phi_2 + k\sigma I_0 \int_0^{\phi_2} e^{-k\lambda_1 \Phi_1(\eta) - k\lambda_2 \eta} d\eta, \quad (6) \]
and
\[ \dot{\phi}_2 = \delta_1 \phi_1 - \delta_2 \phi_2 - k\sigma \phi_2 + \frac{\sigma I_0}{\lambda_2} \int_0^{\phi_1 + k\lambda_2 \phi_2} e^{-\eta} d\eta \]
\[ - \frac{k\sigma I_0}{\lambda_2} \int_0^{\phi_1} e^{-k\lambda_1 \eta - k\lambda_2 \phi_2(\eta)} d\eta. \quad (7) \]

In summary, by (5) and (6), \( \phi_1 \) and \( \phi_2 \) satisfy the following system of differential equations:
\[ \dot{\phi}_1 = F_1^0(\phi_1, \phi_2), \quad \dot{\phi}_2 = F_2^0(\phi_1, \phi_2), \quad (8) \]
where

\[ F_1^0(x, y) = 1 - I^0 e^{-k \lambda_1 x - k \lambda_2 y} - \delta_1 x, \]

\[ F_2^0(x, y) = \delta_1 x - \delta_2 y - k \sigma y + k \sigma I^0 \int_0^y e^{-k \lambda_1 (\eta - k \lambda_2 \eta) d\eta}. \]

Simultaneously, by (5) and (7), \( \phi_1 \) and \( \phi_2 \) also satisfy

\[ \dot{\phi}_1 = \xi_1^0 (\phi_1, \phi_2), \quad \dot{\phi}_2 = \xi_2^0 (\phi_1, \phi_2), \]

where

\[ \xi_1^0 (x, y) = 1 - I^0 e^{-k \lambda_1 x - k \lambda_2 y} - \delta_1 x, \]

\[ \xi_2^0 (x, y) = \delta_1 x - \delta_2 y - k \sigma y + \frac{k \sigma I^0}{\lambda_2} \int_0^x e^{-k \lambda_1 \eta - k \lambda_2 \eta \Phi_2(\eta) d\eta}. \]

Figure 2. Zero sets of \( F_1(x, y) \) and \( F_2(x, y) \) on the \( xy \) plane with \( \sigma = 1 \)

3. Steady state analysis and threshold phenomena. In this section, we prove the main theorem by using the reduced equations. We proceed to the steady state analysis based on (8) and (9). Note that on an infinite time scale (\( t \to \infty \)), formally the following holds:

\[ \lim_{t \to \infty} \dot{\phi}_1 = \lim_{t \to \infty} \dot{\phi}_2 = 0, \]

since \( \dot{\phi}_1 = S_1, \dot{\phi}_2 = S_2 \), and \((S_1, S_2) = (0, 0)\) is the unique steady state solution to the original system (1). From this simple observation, we can expect that \((\phi_1(t), \phi_2(t))\) converges to a solution to (8) and (9).

3.1. Case I \( (R_0 \leq 1) \): In this subsection, we will present an estimate for \( \phi_1(t) \) for the case \( (R_0 \leq 1) \) via auxiliary equation based on (8). The asymptotic state \( (\phi_1^\infty, \phi_2^\infty) := \lim_{t \to \infty} (\phi_1(t), \phi_2(t)) \) will be the solution to the following system of equations(See Subsection 3.3):

\[ F_1^0(x, y) = 0, \]

\[ F_2^0(x, y) = 0. \]
The curve \((x, f(x))\) satisfies (10), where
\[
f(x) = \frac{k\lambda_1}{k\lambda_2}x - \frac{1}{k\lambda_2} \log \frac{1 - \delta_1 x}{I^0}, \quad \text{on} \ x \in (0, 1/\delta_1).
\]

Similar to (10), the graph of the following function \(g(y)\) is the curve satisfying (11).
\[
g(y) = \frac{1}{\delta_1} \left( \delta_2 y + k\sigma y - k\sigma I^0 \int_y^0 e^{-k\lambda_1 \Phi_1(\eta) - k\lambda_2 \eta} d\eta \right), \quad y \geq 0.
\]

See Figure 2.

In summary,
\[
F_1(x, f(x)) = 0 \quad \text{and} \quad F_2(g(y), y) = 0.
\]

Note that, if \(y \geq 0\),
\[
\frac{dg}{dy} = \frac{\delta_2 + k\sigma - k\sigma I^0 e^{-k\lambda_1 \Phi_1(y) - k\lambda_2 y}}{\delta_1}.
\]

This yields
\[
\frac{\delta_2}{\delta_1} \leq \frac{dg}{dy} \leq \frac{\delta_2 + k\sigma}{\delta_1}.
\]

Therefore, the inverse \(g^{-1}(x)\) exists on \(x \geq 0\). Consider
\[
h(x) = f(x) - g^{-1}(x).
\]

We notice that \(h(x)\) is defined on \(x \in [0, 1/\delta_1]\).

**Proposition 1.** Let \(h(x)\) be the function defined in (12). We assume that
\[
R_0 = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2} \leq 1.
\]

Then we have
\[
0 < x_0 \leq \sqrt{-\frac{2 \log I^0}{\delta_1}},
\]
where \(x_0 > 0\) is the unique zero satisfying
\[
h(x_0) = 0.
\]

Proof. Note that \(h(0) = f(0) - g^{-1}(0) = \frac{1}{k\lambda_2} \log I^0 < 0\).

From simple computations, it follows that
\[
\frac{dg^{-1}}{dx} = \frac{1}{dx/dy} = \frac{\delta_1}{\delta_2 + k\sigma - k\sigma I^0 e^{-k\lambda_1 \Phi_1(y) - k\lambda_2 y}}
\]
and
\[
\frac{df}{dx} = -\frac{k\lambda_1}{k\lambda_2} + \frac{1}{k\lambda_2} \frac{\delta_1}{1 - \delta_1 x},
\]
where \(y = g^{-1}(x)\).

By the definition of \(h\),
\[
h'(x) = \frac{d}{dx} (f(x) - g^{-1}(x))
\]
\[
= -\frac{k\lambda_1}{k\lambda_2} + \frac{1}{k\lambda_2} \frac{\delta_1}{1 - \delta_1 x} - \frac{\delta_1}{\delta_2 + k\sigma - k\sigma I^0 e^{-k\lambda_1 \Phi_1(y) - k\lambda_2 y}}.
\]
For \(0 \leq \delta_1 x < 1\),
\[
\frac{1}{1 - \delta_1 x} \geq 1 + \delta_1 x.
\]
Since \(y \geq 0\), \(\Phi_1(y) \geq 0\) and \(0 < I^0 < 1\),
\[
0 \leq k\sigma - k\lambda_1 I^0 e^{-k\lambda_1 \Phi_1(y) - k\lambda_2 y}.
\]
Therefore,
\[
h'(x) \geq -\frac{k\lambda_1}{k\lambda_2} \frac{\delta_1}{\delta_2} (1 + \delta_1 x) - \frac{\delta_1}{\delta_2}
\]
\[
= -\frac{k\lambda_1}{k\lambda_2} + \frac{\delta_1}{\delta_2} - \frac{\delta_1}{\delta_2} + \frac{\delta_1^2 x}{k\lambda_2}
\]
\[
= \frac{\delta_1}{k\lambda_2} \left(1 - \frac{k\lambda_1}{\delta_1} - \frac{k\lambda_2}{\delta_2}\right) + \frac{\delta_1^2 x}{k\lambda_2}.
\]
By (13), we have
\[
h'(x) \geq \frac{\delta_1^2 x}{k\lambda_2}.
\]
Thus,
\[
h(x) \geq h(0) + \int_0^x \frac{\delta_1^2 s}{k\lambda_2} ds
\]
\[
\geq \frac{1}{k\lambda_2} \log I^0 + \frac{\delta_1^2 x^2}{2k\lambda_2}.
\]
(14)

Since \(h(0)\) is a negative real number, \(\lim_{x \to \frac{1}{\delta_1^{-1}}} h(x) = \infty\) and \(h'(0) > 0\), there exists the unique zero \(x_0 > 0\) satisfying
\[
h(x_0) = 0.
\]
Moreover, by (14),
\[
h\left(\sqrt{-\frac{2 \log I^0}{\delta_1^2}}\right) \geq 0.
\]
This implies that
\[
0 < x_0 \leq \sqrt{-\frac{2 \log I^0}{\delta_1^2}}.
\]

Remark 4. We note that it is difficult to obtain an upper bound of \(\Phi_1(y)\) in \(h'(x)\). Therefore, we cannot use this framework to obtain an lower bound of solution to \(h(x) = 0\) for other case \(R_0 > 1\).

3.2. Case II \((R_0 > 1)\): Since we cannot apply the framework which used in the previous subsection to this case, we need another auxiliary equation based on (9) to derive an estimate for \(\phi_1(t)\) for the case \((R_0 > 1)\). Although this framework seems more complicated to obtain the desired estimate for \(\phi_1(t)\), we can calculate the fixed lower bound of \(\lim_{t \to \infty} \phi_1(t)\) which is independent of initial data \(I^0\).

As the previous subsection, we consider
\[
\mathcal{F}_1^0(x, y) = 0, \quad \mathcal{F}_2^0(x, y) = 0.
\]
(15)
Note that the asymptotic state \((\phi_1^\infty, \phi_2^\infty) := \lim_{t \to \infty} (\phi_1(t), \phi_2(t))\) is the solution to (15). Let \(\bar{f}\) and \(\bar{g}\) be functions satisfying
\[
F_I^0 \left( x, \bar{f}(x) \right) = 0 \quad \text{and} \quad F_I^0 \left( \bar{g}(y), y \right) = 0.
\]
Thus,
\[
\bar{f}(x) = \frac{k \lambda_1}{k \lambda_2} x - \frac{1}{k \lambda_2} \log \frac{1 - \delta_1 x}{I^0}, \quad \text{on} \quad x \in (0, 1/\delta_1),
\]
and
\[
0 = \delta_1 \bar{g}(y) - \delta_2 y - k \sigma y + \frac{\sigma I^0}{\lambda_2} \int_0^{k \lambda_1 \bar{g}(y) + k \lambda_2 y} e^{-\eta} d\eta
\]
\[
- \frac{k \sigma I^0 \lambda_1}{\lambda_2} \int_0^{\bar{g}(y)} e^{-k \lambda_1 \eta - k \lambda_2 \Phi_2(\eta)} d\eta, \quad y \geq 0.
\]
By implicit differentiation, we have
\[
0 = \delta_1 \frac{d \bar{g}}{dy} - \delta_2 - k \sigma + \frac{\sigma I^0}{\lambda_2} \left( k \lambda_1 \frac{d \bar{g}}{dy} + k \lambda_2 \right) e^{-k \lambda_1 \bar{g}(y) - k \lambda_2 y}
\]
\[
- \frac{k \sigma I^0 \lambda_1}{\lambda_2} \frac{d \bar{g}}{dy} e^{-k \lambda_1 \bar{g}(y) - k \lambda_2 \Phi_2(\bar{g}(y))}, \quad y \geq 0.
\]
Equivalently,
\[
\frac{d \bar{g}}{dy} = \frac{\delta_2 + k \sigma - k \sigma I^0 e^{-k \lambda_1 \bar{g}(y) - k \lambda_2 y}}{\delta_1 + \frac{k \sigma I^0 \lambda_1}{\lambda_2} e^{-k \lambda_1 \bar{g}(y) - k \lambda_2 y} - \frac{k \sigma I^0 \lambda_1}{\lambda_2} e^{-k \lambda_1 \bar{g}(y) - k \lambda_2 \Phi_2(\bar{g}(y))}}, \quad y \geq 0. \quad (16)
\]
This yields
\[
\frac{\delta_2}{\delta_1 + \frac{k \sigma I^0 \lambda_1}{\lambda_2}} \leq \frac{d \bar{g}}{dy}. \quad (17)
\]
Therefore, the inverse \(\bar{g}^{-1}(x)\) exists on \(x \geq 0\). As Case I, we consider
\[
\bar{h}(x) = \bar{f}(x) - \bar{g}^{-1}(x), \quad x \in [0, 1/\delta_1). \quad (18)
\]
**Proposition 2.** Let \(\bar{h}(x)\) be the function defined in (18) and we assume that
\[
R_0 = \frac{k \lambda_1}{\delta_1} + \frac{k \lambda_2}{\delta_2} > 1.
\]
Then there is a positive constant \(x_1 > 0\) which is independent of initial data \(I^0\) satisfying
\[
0 < x_1 < x_0,
\]
for \(\frac{k \sigma \delta_1 S^0}{\delta_2} \leq \frac{\epsilon_1}{4}\). Here, \(x_0 > 0\) is any positive solution to
\[
\bar{h}(x_0) = 0,
\]
and
\[
\epsilon_1 = \frac{\delta_1}{k \lambda_2} \left( \frac{k \lambda_1}{\delta_1} + \frac{k \lambda_2}{\delta_2} - 1 \right).
\]
**Proof.** Note that
\[
\frac{k \lambda_1}{\delta_1} + \frac{k \lambda_2}{\delta_2} - 1 = R_0 - 1 > 0.
\]
Therefore, $\epsilon_1$ is a positive real number. We define a positive constant $C_1 > 0$ depending on $k, \sigma, \lambda_1, \lambda_2, \delta_1$ and $\delta_2$:

$$C_1 = \max\left\{ \frac{k^2\sigma\lambda_1\delta_1}{\delta_2} + \frac{k^3\sigma^2\lambda_1^2}{\lambda_2\delta_2}, \frac{k^2\sigma\lambda_2\delta_1}{\delta_2} + \frac{k^3\sigma^2\lambda_2^2}{\delta_2\lambda_2} \right\}.$$ 

Clearly, $\bar{h}(0) = \bar{f}(0) - \bar{g}^{-1}(0) = \frac{1}{k\lambda_2}$ log $I^0 < 0$.

By the definition of $\bar{f}$ and (16), we have

$$\frac{d\bar{f}}{dx} = \frac{k\lambda_1}{k\lambda_2} + \frac{1}{k\lambda_2} \frac{\delta_1}{1 - \delta_1 x}.$$

and

$$\frac{d\bar{g}^{-1}}{dx} = \frac{\delta_1 + \frac{k\sigma I^0 \lambda_1}{\lambda_2} e^{-k\lambda_1 x - k\lambda_2 \bar{g}^{-1}(x)} - \frac{k\sigma I^0 \lambda_1}{\lambda_2} e^{-k\lambda_1 x - k\lambda_2 \Phi_2(x)}}{\delta_2 + k\sigma - k\sigma I^0 e^{-k\lambda_1 x - k\lambda_2 \bar{g}^{-1}(x)}}.$$

Therefore,

$$\bar{h}'(x) = \frac{d}{dx}(\bar{f}(x) - \bar{g}^{-1}(x)) = \frac{-k\lambda_1 + \frac{1}{k\lambda_2} \frac{\delta_1}{1 - \delta_1 x} - \delta_1 + \frac{k\sigma I^0 \lambda_1}{\lambda_2} e^{-k\lambda_1 x - k\lambda_2 \bar{g}^{-1}(x)} - \frac{k\sigma I^0 \lambda_1}{\lambda_2} e^{-k\lambda_1 x - k\lambda_2 \Phi_2(x)}}{\delta_2 + k\sigma - k\sigma I^0 e^{-k\lambda_1 x - k\lambda_2 \bar{g}^{-1}(x)}}.$$ 

Since $\Phi_2(x) \geq 0,$

$$\bar{h}'(x) = \frac{-k\lambda_1 + \frac{1}{k\lambda_2} \frac{\delta_1}{1 - \delta_1 x} - \delta_1 + \frac{k\sigma I^0 \lambda_1}{\lambda_2} e^{-k\lambda_1 x - k\lambda_2 \Phi_2(x)}}{\delta_2 + k\sigma - k\sigma I^0 e^{-k\lambda_1 x - k\lambda_2 \Phi_2(x)}} .$$ 

Let

$$N(x) = \frac{k\sigma I^0 \lambda_1}{\lambda_2} e^{-k\lambda_1 x}(1 - e^{-k\lambda_2 \bar{g}^{-1}(x)}), \quad D(x) = k\sigma(1 - I^0 e^{-k\lambda_1 x - k\lambda_2 \bar{g}^{-1}(x)}).$$

Then, we have $N(x) > 0$, $D(x) > 0$ and

$$\bar{h}'(x) \leq \frac{-k\lambda_1 + \frac{1}{k\lambda_2} \frac{\delta_1}{1 - \delta_1 x} - \delta_1 - N(x)}{\delta_2 + D(x)} .$$

In order to obtain an upper bound of $\bar{h}'(x)$, we need upper bounds of $N(x)$ and $D(x)$.

We now assume that $0 \leq x \leq x_1$, where

$$x_1 = \min\left\{ \frac{1}{2\delta_1}, \frac{k\lambda_2 \epsilon_1}{8\delta_1^2}, \frac{\delta_2^2 \epsilon_1}{4C_1(\delta_1 + \delta_2)} \right\} .$$

By (17),

$$0 \leq \bar{g}^{-1}(x) \leq \frac{\delta_1 + \frac{k\sigma I^0 \lambda_1}{\lambda_2}}{\delta_2} x_1 = \left( \frac{\delta_1}{\delta_2} + \frac{k\sigma I^0 \lambda_1}{\delta_2 \lambda_2} \right) x_1 .$$
Therefore,

\[ 0 \leq N(x) \leq \frac{k \sigma I^0 \lambda_1}{\lambda_2} e^{-k \lambda_1 x_1} \left( 1 - e^{-\left( \frac{k \lambda_2 \delta_1}{\lambda_2} + \frac{k \sigma I^0 \lambda_1}{\delta_2} \right) x_1} \right) \]

\[ \leq \frac{k \sigma I^0 \lambda_1}{\lambda_2} \left( 1 - e^{-\left( \frac{k \lambda_2 \delta_1}{\lambda_2} + \frac{k \sigma I^0 \lambda_1}{\delta_2} \right) x_1} \right) \]

\[ \leq \frac{k \sigma I^0 \lambda_1}{\lambda_2} \left( k \lambda_2 \delta_1 + \frac{k \sigma I^0 \lambda_1}{\delta_2} \right) x_1 \]

\[ = \frac{k \sigma I^0 \lambda_1 \delta_1}{\delta_2} + \frac{k \sigma I^0 \lambda_1}{\delta_2} \lambda_2 \delta_2 \]

\[ \leq \left( \frac{k \sigma \lambda_1 \delta_1}{\delta_2} + \frac{k \sigma \lambda_1 \lambda_2^2}{\delta_2} \right) x_1 \]

\[ \leq C_1 x_1, \]

By (19) and (20),

Here we used the assumption \( 0 < I^0 < 1 \) and the following simple inequality:

\[ 1 - e^{-x} \leq x. \]

Similarly,

\[ 0 \leq D(x) = k \sigma S^0 + k \sigma I^0 \left( 1 - e^{-k \lambda_1 x - k \lambda_2 \bar{y}^{-1}(x)} \right) \]

\[ \leq k \sigma S^0 + k \sigma I^0 \left( 1 - e^{-\left( k \lambda_1 + \frac{k \lambda_2 \delta_1}{\delta_2} + \frac{k \sigma I^0 \lambda_1 \lambda_2}{\delta_2} \right) x_1} \right) \]

\[ \leq k \sigma S^0 + k \sigma I^0 \left( 1 - e^{-\left( k \lambda_1 + \frac{k \lambda_2 \delta_1}{\delta_2} + \frac{k \sigma I^0 \lambda_1 \lambda_2}{\delta_2} \right) x_1} \right) \]

\[ \leq k \sigma S^0 + \left( k^2 \sigma \lambda_1 + \frac{k^2 \sigma \lambda_2 \delta_1}{\delta_2} + \frac{k \sigma I^0 \lambda_1 \lambda_2}{\delta_2} \right) x_1 \]

\[ \leq k \sigma S^0 + C_1 x_1. \]

By (19) and (20),

\[ \bar{h}'(x) \leq -\frac{k \lambda_1}{k \lambda_2} + \frac{1}{k \lambda_2 \left( 1 - \delta_1 x_1 \right)} - \frac{\delta_1}{\delta_2 + k \sigma S^0 + C_1 x_1} \]

\[ = -\frac{k \lambda_1}{k \lambda_2} + \frac{1}{k \lambda_2 \left( 1 - \delta_1 x_1 \right)} - \frac{\delta_1}{\delta_2 + k \sigma S^0 + C_1 x_1} + \frac{\delta_1}{\delta_2} - \frac{\delta_1}{\delta_2} \]

\[ = \left( -\frac{k \lambda_1}{k \lambda_2} + \frac{1}{k \lambda_2 \left( 1 - \delta_1 x_1 \right)} - \frac{\delta_1}{\delta_2 + k \sigma S^0 + C_1 x_1} \right) \]

\[ + \left( -\frac{\delta_1}{\delta_2} + \frac{\delta_1}{\delta_2 + k \sigma S^0 + C_1 x_1} \right) \]

\[ = \frac{\delta_1}{k \lambda_2} \left( 1 - \frac{k \lambda_1}{\delta_1} - \frac{k \lambda_2}{\delta_2} \right) + \frac{\delta_1}{k \lambda_2} \frac{\delta_1}{\delta_2} + \frac{\delta_1}{\delta_2 + k \sigma S^0 + C_1 x_1} \]

\[ \leq \frac{\delta_1}{k \lambda_2} \left( 1 - \frac{k \lambda_1}{\delta_1} - \frac{k \lambda_2}{\delta_2} \right) \]

\[ \leq \frac{k \sigma \delta_1 S^0}{\delta_2} \leq \frac{c_1}{4}. \]
then, by the definition of $x_1$,
\[ \hat{h}'(x) \leq -\epsilon_1 + \frac{\epsilon_1}{4} + \frac{\epsilon_1}{4} = -\frac{\epsilon_1}{4}, \quad \text{on } x \in [0, x_1]. \]

Since $\hat{h}(0) < 0$, we can conclude that
\[ \hat{h}(x) \neq 0, \quad \text{on } x \in [0, x_1]. \]

Therefore, the positive constant $x_1 > 0$ is independent of initial data $I^0, S^0$ and the following holds:
\[
0 < x_1 < x_0, \quad \text{if } \frac{k\sigma_1S^0}{\delta_2} \leq \frac{\epsilon_1}{4},
\]
where $x_0$ is any zero satisfying
\[ \hat{h}(x_0) = 0. \]

\[ \square \]

**3.3. The proof of the main theorem:** In this subsection, we provide the proof of the main theorem.

Assume that
\[ \mathcal{R}_0 = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2} \leq 1. \]

For fixed $n \in \mathbb{N}$, let $I^n_0$ be initial data of ignorant $I_n(t)$ that is the solution to (1). Then $\{\phi^n_1(t), \phi^n_2(t)\}$ is the corresponding solution to the following equation:
\[
\phi^n_1(t) = F^n_1(\phi^n_1(t), \phi^n_2(t)), \quad \phi^n_2(t) = F^n_2(\phi^n_1(t), \phi^n_2(t)).
\]

By the definition of $\phi^n_1$ and $\phi^n_2$:
\[
\phi^n_1(t) = \int_0^t (S_1)_n(\tau)d\tau, \quad \phi^n_2(t) = \int_0^t (S_2)_n(\tau)d\tau,
\]
are increasing functions with respect to $t > 0$ and
\[
\phi^n_1(t) \geq 0, \quad \phi^n_2(t) \geq 0.
\]

From an elementary result of ordinary differential equations, it follows that $\phi^n_1(t)$ and $\phi^n_2(t)$ converge to $x^n_0 > 0$ and $y^n_0 > 0$, where $\{x^n_0, y^n_0\}$ is a solution to
\[
0 = F^n_1(x^n_0, y^n_0), \quad 0 = F^n_2(x^n_0, y^n_0). \tag{21}
\]

Let $\{x^n_0, y^n_0\}$ be any solution to (21). Then
\[
0 = F^n_1(x^n_0, y^n_0) = F^n_1(x^n_0, f(x^n_0))
\]
and
\[
0 = F^n_2(x^n_0, y^n_0) = F^n_2(g(y^n_0), y^n_0).
\]

By the definition of $f$ and $g$ in Subsection 3.1,
\[
y^n_0 = f(x^n_0), \quad x^n_0 = g(f(x^n_0)),
\]
and
\[
g^{-1}(x^n_0) = f(x^n_0).
\]

This yields that $h(x^n_0) = 0$. By Proposition 1,
\[
0 < x^n_0 \leq \sqrt{-\frac{2 \log I^n_0}{\delta_1^2}}.
\]
Thus,

$$0 \leq \phi_e = \lim_{n \to \infty} \phi^\infty(n) = \lim_{n \to \infty} \lim_{t \to \infty} \phi^n_1(t) \leq \lim_{t_n \to 1^-} \sqrt{-\frac{2 \log I_0^n}{\delta_1^n}} = 0.$$ 

For the remaining case, we assume that

$$R_0 = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2} > 1.$$ 

Similar to the previous case, \{\phi^n_1(t), \phi^n_2(t)\} is the solution to the following equation:

$$\dot{\phi}^n_1(t) = \Phi^n_1(\phi^n_1(t), \phi^n_2(t)), \quad \dot{\phi}^n_2(t) = \Phi^n_2(\phi^n_1(t), \phi^n_2(t)),$$

and \phi^n_1(t) and \phi^n_2(t) converge to \(x^n_0 > 0\) and \(y^n_0 > 0\), where \{\(x^n_0, y^n_0\)\} is a solution to

$$0 = \Phi^n_1(x^n_0, y^n_0), \quad 0 = \Phi^n_2(x^n_0, y^n_0). \tag{22}$$

For any solution \{\(x^n_0, y^n_0\)\} to (22),

$$0 = \tilde{\Phi}^{n}_1(x^n_0, y^n_0) = \tilde{\Phi}^{n}_1(x^n_0, \bar{f}(x^n_0))$$

and

$$0 = \tilde{\Phi}^{n}_2(x^n_0, y^n_0) = \tilde{\Phi}^{n}_2(\bar{g}(y^n_0), y^n_0).$$

Let \(\bar{f}\) and \(\bar{g}\) be functions defined in Subsection 3.2. Then

$$y^n_0 = \bar{f}(x^n_0), \quad x^n_0 = \bar{g}(\bar{f}(x^n_0)),$$

and

$$\bar{g}^{-1}(x^n_0) = \bar{f}(x^n_0).$$

This also yields that \(\bar{h}(x^n_0) = 0\). From Proposition 2, it follows that

$$x^n_0 > x_1, \quad \text{for} \quad t^n > 1 - \frac{\delta_2^n \epsilon_1}{4k\sigma \delta_1},$$

where \(x_1 > 0\) is a positive constant which is independent of initial data \(t^n_0\). Thus,

$$\phi_e = \lim_{n \to \infty} \phi^\infty(n) \geq \lim_{t^n_0 \to 1^-} x_1 = x_1 > 0.$$

![Figure 3. Numerical solutions of (1) when \(k = 0.5, \sigma = 1\), and \(n = 10^6\)](image)
4. Numerical simulations. In this section, we provide numerical simulations to guarantee our main result. To carry out numerical simulations for several cases, we use the fourth order Runge-Kutta method and Matlab with time step size $\Delta t = 0.01$. As we mentioned before, for a sufficiently large number $n \gg 1$, there is a threshold phenomenon for long time asymptotics of the solution to (1). The equivalent condition for rumor outbreak is

$$R_0 = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2} > 1.$$  

We assume that the average degree of the network $k$ is 0.5, and the contact rate $\sigma$ is 1. Additionally, we assume that $n = 10^6$. In Figure 3 (A), we take $\delta_1 = 0.1$, $\delta_2 = 0.1$, $\lambda_1 = 3$, and $\lambda_2 = 2$. Then,

$$R_0 = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2} = \frac{1.5}{0.1} + \frac{1}{0.1} = 25 > 1.$$  

Figure 3 (A) shows that, in this case, rumor outbreak occurs, and the results of the simulation and the theory are consistent.

For the opposite case, we take $\delta_1 = 0.3$, $\delta_2 = 0.3$, $\lambda_1 = 0.2$, and $\lambda_2 = 0.2$. Then,

$$R_0 = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2} = \frac{0.1}{0.3} + \frac{0.1}{0.3} = \frac{2}{3} < 1.$$  

In this case, rumor outbreak does not occur, and the results of simulation and theory are also consistent. Please see Figure 3 (B).

In the rest of this section, we will consider two phase diagrams to investigate the threshold phenomena on the asymptotic behavior of the solution ($I, S_1, S_2, R$).

- Case 1 (phase diagram $I$ on $(x, y) = (\lambda_1, \lambda_2)$ with fixed $\delta_1$ and $\delta_2$): To clarify the threshold phenomena, we simulate several numerical experiments. We fix the values of $\delta_1$ and $\delta_2$ and change the values of $\lambda_1$ and $\lambda_2$. We choose $\delta_1 = 0.2$ and $\delta_2 = 0.2$. For $(\lambda_1, \lambda_2) \in [0, 0.3] \times [0, 0.3]$ with fixed $\delta_1$ and $\delta_2$, we numerically compute the solutions $I(t)$ to (1). We note that the mesh size on $(\lambda_1, \lambda_2) \in [0, 0.3] \times [0, 0.3]$ is $(0.001, 0.001)$. Figure 4 is a contour plot of limit values $\lim_{t \to \infty} I(t) \approx I(10000)$ on

![Figure 4. Phase diagram $I(\infty)$ on $(x, y) = (\lambda_1, \lambda_2)$ with $\delta_1 = \delta_2 = 0.2$](image)
Figure 5. Phase diagram $R(\infty)$ on $(x, y) = (\lambda_1, \lambda_2)$ with $\delta_1 = \delta_2 = 0.2$

Figure 6. Time evolution of the solutions $(I, S_1, S_2, R)$ with $k = 0.5$, $\sigma = 1$, $n = 10^6$, $\delta_1 = \delta_2 = 0.2$ and $\lambda_1 = 0.1$
\((\lambda_1, \lambda_2) \in [0, 0.3] \times [0, 0.3]\). In Figure 4, we can check that the yellow area expresses the rumor non-outbreak territory that overlaps the region \(R_0 \leq 1\) exactly. The rest of the graph expresses the rumor outbreak territory and is the region \(R_0 > 1\). Moreover, we can see that, in the upper region of the straight line

\[
\frac{x}{\delta_1} + \frac{y}{\delta_2} = \frac{x}{0.2} + \frac{y}{0.2} = 1,
\]

the limit values \(\lim_{t \to \infty} I(t)\) are less than one, and as the distance from the line to the point \((\lambda_1, \lambda_2)\) increases, the values of \(\lim_{t \to \infty} I(t)\) gradually decrease. Figure 5 is a graph of \(R(10000)\) with the same parameters \(\delta_1, \delta_2, \lambda_1\) and \(\lambda_2\), as in Figure 4.

We easily verify that the outbreak interface takes place through the line \(R_0 = 1\).

Figure 6 shows solutions \((I, S_1, S_2, R)\) to (1) for several cases by choosing

\[
\lambda_1 = 0.1 \quad \text{and} \quad \lambda_2 = 0.10, 0.15, 0.2, 0.25, 0.3.
\]

We can check that the solutions are getting closer to the trivial solution as the value \(R_0\) approaches 1.

- Case 2 (phase diagram \(I\) on \((x, y) = (\delta_1, \delta_2)\) with fixed \(\lambda_1\) and \(\lambda_2\)): In Figure 7, we can verify the threshold phenomenon of rumor outbreak.

If we choose \(\lambda_1 = 0.2\) and \(\lambda_2 = 0.2\), then the interface between the region of outbreak and non-outbreak is located in

\[
\frac{\lambda_1}{x} + \frac{\lambda_2}{y} = \frac{0.2}{x} + \frac{0.2}{y} = 1.
\]

Thus, equivalently, we have

\[
y = \frac{1}{5(5x - 1)} + \frac{1}{5}.
\]

For \((\delta_1, \delta_2) \in [0.2, 1] \times [0.2, 1]\) with fixed \(\lambda_1\) and \(\lambda_2\), we numerically compute the solutions \(I(t)\) to (1). The mesh size on \((\delta_1, \delta_2) \in [0.2, 1] \times [0.2, 1]\) is \((0.002, 0.002)\). We draw a contour plot of limit values \(\lim_{t \to \infty} I(t) \approx I(10000)\) in Figure 7.
Figure 7 shows that in the lower region of a part of the hyperbola, the limit values of \( \lim_{t \to \infty} I(t) \) are less than 1. Similar to Case 1, the values \( \lim_{t \to \infty} I(t) \) gradually decrease as the distance from the interface line to the point \((\delta_1, \delta_2)\) increases.

![Graphs of population densities](image)

**Figure 8.** Time evolution of the solutions \((I, S_1, S_2, R)\) with \(k = 0.5, \sigma = 1, n = 10^6, \lambda_1 = \lambda_2 = 0.2, \) and \(\delta_1 = 0.4\)

Figure 8 displays several solutions \((I, S_1, S_2, R)\) for \(\delta_1 = 0.4, \delta_2 = 0.40, 0.35, 0.3, 0.25, 0.2\). We also show that the solutions \((I, S_1, S_2, R)\) to (1) are getting close to the trivial solution as \(R_0 > 1\) goes to 1 in Figure 8.

5. **Discussion and conclusion.** In this paper, we present a mathematically rigorous proof for threshold phenomena on a simple multi-stage structured rumor spreading model. On the derivation of the model, we divided the spreaders into two groups: new \((S_1)\) and old \((S_2)\) spreaders. Motivated by the steady state analysis with multiple variables, we rigorously prove that threshold phenomena of the rumor outbreak exist in the large population limit sense (see the phase diagram for \(R\) in Figure 9). The basic reproduction number

\[
R_0 = \frac{k\lambda_1}{\delta_1} + \frac{k\lambda_2}{\delta_2}
\]

of our model is an extension of the classical basic reproduction number for the SIR rumor spreading model without multi-stage structure [31]. If the basic reproduction number is greater than one, i.e., \(R_0 > 1\), then rumor outbreak occurs. On the other
hand, if $R_0 \leq 1$, then rumor outbreak does not occur. In fact, our model can be derived from a system of stage-structured partial differential equations; similarly we can derive a multi-stage structured model for rumor spreading. In the future, we will extend the result of this paper to multiple stages and the original system of partial differential equations.

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REFERENCES

[1] C. Barril, Calsina and J. Ripoll, A practical approach to $R_0$ in continuous time ecological models, *Mathematical Methods in the Applied Sciences*, 41 (2018), 8432–8445.
[2] P. Bordia and N. DiFonzo, Problem solving in social interactions on the Internet: Rumor as social cognition, *Social Psychology Quarterly*, 67 (2004), 33–49.
[3] P. Bordia and R. L. Rosnow, Rumor rest stops on the information highway transmission patterns in a computer mediated rumor chain, *Human Communication Research*, 25 (1998), 163–179.
[4] J. Borge-Holthoefer, S. Meloni, B. Goncalves and Y. Moreno, Emergence of influential spreaders in modified rumor models, *Journal of Statistical Physics*, 151 (2013), 383–393.
[5] J. Borge-Holthoefer and Y. Moreno, Absence of influential spreaders in rumor dynamics, *Physical Review E*, 85 (2012), 026116.
[6] J. Borge-Holthoefer, A. Rivero and Y. Moreno, Locating privileged spreaders on an online social network, *Physical Review E*, 85 (2012), 066123.
[7] D. J. Daley and D. G. Kendall, Epidemics and rumours, *Nature*, 204 (1964), 1118.
[8] D. J. Daley and D. G. Kendall, Stochastic rumours, *IMA Journal of Applied Mathematics*, 1 (1965), 42–55.
[9] O. Diekmann, J. A. P. Heesterbeek and J. A. Metz, On the definition and the computation of the basic reproduction ratio $R_0$ in models for infectious diseases in heterogeneous populations, *Journal of Mathematical Biology*, 28 (1990), 365–382.
[10] V. Driessche, Pauline and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Mathematical Biosciences, 180 (2002), 29–48.

[11] N. Fountoulakis, K. Panagiotou and T. Sauerwald, Ultra-fast rumor spreading in social networks, in Proceedings of the Twenty-third Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, 2012, 1642–1660.

[12] B. I. Hong, N. Hahm and S.-H. Choi, SIR rumor spreading model with trust rate distribution, Networks and Heterogeneous Media, 13 (2018), 515–530.

[13] G. Huang, X. Liu and Y. Takeuchi, Lyapunov functions and global stability for age-structured HIV infection model, SIAM Journal on Applied Mathematics, 72 (2012), 25–38.

[14] L. A. Huo, L. Wang, N. Song, C. Ma and B. He, Rumor spreading model considering the activity of spreaders in the homogeneous network, Physica A: Statistical Mechanics and its Applications, 468 (2017), 855–865.

[15] H. Inaba, Mathematical analysis of an age-structured SIR epidemic model with vertical transmission, Discrete and Continuous Dynamical Systems Series B, 6 (2006), 69–96.

[16] H. Inaba, Threshold and stability results for an age-structured epidemic model, Journal of Mathematical Biology, 28 (1990), 411–434.

[17] R. H. Knapp, A psychology of rumor, Public Opinion Quarterly, 8 (1944), 22–37.

[18] T. Kuniya, Existence of a nontrivial periodic solution in an age-structured SIR epidemic model with time periodic coefficients, Applied Mathematics Letters, 27 (2014), 15–20.

[19] T. Kuniya, Global stability analysis with a discretization approach for an age-structured multigroup SIR epidemic model, Nonlinear Analysis: Real World Applications, 12 (2011), 2640–2655.

[20] S. Kwon, M. Cha, K. Jung, W. Chen and Y. Wang, Prominent features of rumor propagation in online social media, in 2013 IEEE 13th International Conference on Data Mining, 2013, 1103–1108.

[21] D. Maki and M. Thomson, Mathematical Models and Applications, Prentice-Hall, Englewood Cliffs, 1973.

[22] M. McDonald, O. Suleman, S. Williams, S. Howison and N. F. Johnson, Impact of unexpected events, shocking news, and rumors on foreign exchange market dynamics, Physical Review E, 77 (2008), 046110.

[23] Y. Moreno, M. Nekovee and A. Pacheco, Dynamics of rumor spreading in complex networks, Physical Review E, 69 (2004), 066130.

[24] M. Nagao, K. Suto and A. Ohuchi, A media information analysis for implementing effective countermeasure against harmful rumor, Journal of Physics, Conference Series, 221 (2010), 012004.

[25] A. Noymer, The transmission and persistence of urban legends Sociological application of agestructured epidemic models, Journal of Mathematical Sociology, 25 (2001), 299–323.

[26] J. Ripoll, M. Manzano and E. Calle, Spread of epidemic-like failures in telecommunication networks, Physica A: Statistical Mechanics and its Applications, 410 (2014), 457–469.

[27] N. Sherborne, K. B. Blyuss, I. Z. Kiss, Dynamics of Multi-stage Infections on Networks, Bulletin of Mathematical Biology, 77 (2015), 1909–1933.

[28] A. Sudbury, The proportion of population never hearing a rumour, Journal of Applied Probability, 22 (1985), 443–446.

[29] S. A. Thomas, Lies, damn lies, and rumors: An analysis of collective efficacy, rumors, and fear in the wake of Katrina, Sociological Spectrum, 27 (2007), 679–703.

[30] J. Wang, L. Zhao and R. Huang, 2SIR rumor spreading model in homogeneous networks, Physica A: Statistical Mechanics and its Applications, 413 (2014), 153–161.

[31] Y.-Q. Wang, X.-Y. Yang, Y.-L. Han and X.-A. Wang, Rumor spreading model with trust mechanism in complex social networks, Communications in Theoretical Physics, 59 (2013), 510–516.

[32] Y. Zan, J. Wu, P. Li and Q. Yua, SICR rumor spreading model in complex networks: Counter-attack and self-resistance, Physica A: Statistical Mechanics and its Applications, 405 (2014), 159–170.

[33] D. H. Zanette, Critical behavior of propagation on small-world networks, Physical Review E, 64 (2001), 050901.

[34] D. H. Zanette, Dynamics of rumor propagation on small-world networks, Physical Review E, 65 (2002), 041908.
[35] L. Zhang, Q. Zhong and W. Qi, Two-stage dynamics of crisis information dissemination on social network, in 2009 International Conference on Management Science and Engineering, IEEE, 2009, 2117–2122.

[36] L. Zhao, J. Wang, Y. Chen, Q. Wang, J. Cheng and H. Cui, SIHR rumor spreading model in social networks, Physica A: Statistical Mechanics and its Applications, 391 (2012), 2444–2453.

[37] L. Zhao, Q. Wang, J. Cheng, Y. Chen, J. Wang and W. Huang, Rumor spreading model with consideration of forgetting mechanism: A case of online blogging LiveJournal, Physica A: Statistical Mechanics and its Applications, 390 (2011), 2619–2625.

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