Accelerated Search for Gaussian Generator Based on Triple Prime Integers

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Abstract: Problem statement: Modern cryptographic algorithms are based on complexity of two problems: Integer factorization of real integers and a Discrete Logarithm Problem (DLP). Approach: The latter problem is even more complicated in the domain of complex integers, where Public Key Cryptosystems (PKC) had an advantage over analogous encryption-decryption protocols in arithmetic of real integers modulo p: The former PKC have quadratic cycles of order O(p^2) while the latter PKC had linear cycles of order O(p). Results: An accelerated non-deterministic search algorithm for a primitive root (generator) in a domain of complex integers modulo triple prime p was provided in this study. It showed the properties of triple primes, the frequencies of their occurrence on a specified interval and analyzed the efficiency of the proposed algorithm. Conclusion: Numerous computer experiments and their analysis indicated that three trials were sufficient on average to find a Gaussian generator.

Key words: Communication network security, crypto-immunity, primitive root, public-key cryptography

INTRODUCTION

A Discrete Logarithm problem, {DLP, for short}, is defined as follows: For real integers g>1, p and h>0 to find an integer x such that satisfies the equation:

\[ g^x \mod p = h \] (1)

This is a computationally formidable problem especially if the integer g is a primitive root (generator). The complexity of the DLP is the basis for secret-key establishment in modern cryptography. An RSA cryptographic algorithm in the domain of complex integers is described in[9]. The DLP in the domain of complex integers (called Gaussian integers) is an extension of the problem (1): To find a real integer x such that holds

\[ G^x \mod p = h \] (2)

Where:
- G and H = Gaussian integers
- p = A prime

As in (1), a solution of Eq. 2 is computationally intense especially if the Gaussian integer G is a primitive root or generator as described in Definition 2 below.

Definition 1: If X is a Gaussian integer and m is the smallest positive integer, for which the following relation holds:

\[ X^m \mod p = (1,0) = 1 \] (3)

then m is defined as the order of X:

\[ \text{ord}(X) = m \] (4)

Definition 2: If the order of Gaussian integer G equals:

\[ m = p^2 - 1 \] (5)

then G is called a Gaussian primitive root or generator, (GG, for short).

There are currently no known deterministic algorithms that compute a GG. However, if p is appropriately selected, then the search for a GG can be substantially simplified.

Definition 3: A prime p is called a Triple Prime (TP) if both integers:

\[ q := (p+1)/4 \] and \[ r := (p-1)/6 \] (6)

are also primes.

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Table 1: TPs and corresponding primes q and r

| p   | 19 | 43 | 67 | 283 | 787 | 907 | 1997 | 9991 | 1998643 |
|-----|----|----|----|-----|-----|-----|------|------|---------|
| q   | 5  | 11 | 17 | 27  | 41  | 71  | 157  | 7323 | 499661  |
| r   | 3  | 11 | 17 | 41  | 113 | 131 | 173  | 197  | 61      |

Table 2: Frequencies of TPs on intervals [10^m, 10^{m+1}]

| m   | 8  | 6  | 8  | 3  | 5  | 7  | 4  | 6  | 7  | 3  | 7  | 5  | 3  | 1  | 3  | 3  | 5  | 2  |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| No. of TP | 8  | 6  | 8  | 3  | 5  | 7  | 4  | 6  | 7  | 3  | 7  | 5  | 3  | 1  | 3  | 3  | 5  | 2  |

Remark: If both p and q are primes, then neither (p-1)/2 nor (p-1)/3 are primes; [for details Lemma 2].

The Table 1 provides examples of several triple primes.

**MATERIALS AND METHODS**

**Triple primes and their properties:** The algorithm searching for Gaussian generator (9-12) is based on the following properties of triple primes.

**Lemma 1:** If a prime p ≥ 43, then for every TP the following condition holds:

\[ p \mod 60 = 7 \text{ or } 43 \]  

(7)

If (7) does not hold, then either q or r are not integers or not primes. For applications, it is necessary to know the occurrence of the TP (their density) for large p. The Table 2 provides such information.

**Remark:** CPU times T required to find a D digit-long TP are provided in milliseconds (ms).

**Lemma 2:** If n is an odd integer and:

- n ≥ 5 is not divisible by 3, then 24 divides n^2 - 1
- n ≥ 23 and (n+1)/4 is a prime, then (n-1)/6 is an integer

**Proof:** n-1 and n+1 are two consecutive even integers, hence one of them is divisible by 4 and their product is divisible by 8. In addition, either n-1 or n+1 is divisible by 3. Therefore, if n is a Blum prime, then:

\[ (n^2-1)/24 = [(n+1)/4][(n-1)/6] \]  

(8)

where both factors in (8) are integers.

Suppose there is n1 ≥ 23, for which q1 = (n1+1)/4 is a prime, but r1 = (n1-1)/6 is not an integer. Since n1-1 is even, hence 3 does not divide n1-1. Therefore three divides the prime q1. This contradiction proves Lemma 2. More details are provided in Fig. 1.

**RESULTS AND DISCUSSION**

**Algorithm searching for generator:**

**Step 1:** Select a triple prime p ≥ 19:

compute q := (p+1)/4

and

r := (p-1)/6  

(9)

**Step 2:** Select integers such that hold:

a ≠ b; 1 ≤ a, b ≤ p-1; a+b ≠ p

and

\[(a^2 + b^2) \mod p ≠ 1\]

**Step 3:** For k = {2, 3, q} compute:

\[(c,d) := (a,b)^k \mod p\]  

(10)

**Step 4:** If k ≠ q and \{c = 0 or d = 0 or |c| = |d|\}, then goto Step 2; \{(a, b) is not a generator\}.

**Step 5:** If k = q and \{c = 0 or d = 0\}, then goto Step 2; \{(a, b) is not a generator\}.
Step 6: Compute:

\[ e := -4c^4 \mod p \] (11)

Step 7: If \( e \mod p = \pm 1 \) (12)

then go to Step 2; \{ (a, b) is not a generator \};
else output \( G = (a, b); \{ (a, b) is a generator \} \).

Analysis of basic results: Numerous computer experiments demonstrated that the algorithm (9-12) finds a Gaussian generator after three trials of (a, b) on average.

Algorithm validation:

Lemma 3: Suppose (a, b) is a Gaussian integer (Gaussian, for short), p is a TP and:

\[(c,d):= (a,b)^q \mod p \] (13)

If a component in (c, d) equals zero, then (a, b) is not a GG.

Proof: If \((a,b)^q \mod p = (c,0)\), then:

\[(a,b)^{p^2-1/q^2} \mod p = e^{p-1} \mod p = 1 \] (14)

Hence, the order of (a, b) is smaller than \( p^2 - 1 \):

therefore, \( \text{ord}(a,b) \leq (p^2 - 1)/4 \) (15)

If \((a,b)^q \mod p = (0,d)\), then:

\[(a,b)^{p^2-1/q^2} \mod p = (-d^2)^{p-1} \mod p = 1 \] (16)

Therefore:

\[(a,b)^{(p^2-2)/q^2} \mod p = (-d^2)^{p-1} \mod p = 1 \] (17)

Thus, the order of (a, b) is smaller than \( p^2 - 1 \); as a result:

\[ \text{ord}(a,b) \leq (p^2 - 1)/2 \] (18)

Hence in both cases, (14) and (16), (a, b) is not a GG.

Theorem 4: Suppose (a, b) is a Gaussian integer, p is a TP:

\[(a,b)^{q^4} = (c,p\pm c) \mod p \] (19)

Let:

\[ e := -4c^4 \mod p = -(\pm 2c^2)^2 \mod p \] (20)

If:

\[ e^{(p-1)/6} \mod p = \pm 1 \] (21)

then \( (a, b) \) is not a GG.

Proof: First of all:

\[(a,b)^q = (c\pm c)^4 = - (\pm 2c^2)^2 (\mod p) = e \] (22)

Let us define:

\[ v_k = e^{(p-1)/k} \mod p \] (23)

then \( v_1 = 1 \) and \( v_1 = \pm 1 \). Therefore (21) implies that:

\[ (-4c^4)^q = (c,\pm c)^4 = (a,b)^{q^4} \] (24)

Hence:

\[ \text{ord}(a,b) \leq (p^2 - 1)/3 \] (25)

i.e., \( (a, b) \) is not a generator.

Suppose that condition (21) does not hold, i.e., let:

\[ v_6 \neq \pm 1 \] (26)

Let us analyze two sub-cases:

Case A: \( v_6^2 \mod p = v_3 = -1 \) is impossible, otherwise \( v_1 = -1 \) and that contradicts Fermat’s theorem. On the other hand \( v_6^2 \mod p = v_3 = 1 \) implies that \( v_6 = \pm 1 \), which contradicts the assumption (26).

Case B: Let us demonstrate that:

\[ v_2 = 1 \] (28)

is also impossible. Indeed, consider:

\[ v_2 = e^{(p-1)/2} = (-1)^{(p-1)/2} \left[(2c^2)^2 \right]^{(p-1)/2} \] (29)

Therefore, this implies that (a, b) is a GG. Q.E.D.

Suppose that \( N(p) \) is the number of Gaussian generators.
Table 4: Required number of Real Integer Multiplications (RIM)

| Operations          | (c,d):= (a,b) \mod p | c' \mod p |
|---------------------|----------------------|-----------|
| Squarings           | log q complex squarings | Requires logq RIM |
| require 2logq RIM   |                       |           |
| Multiplications     | On average \((\log q)/2\) complex | Requires on average \((\log r)/2\) RIM |
| Overall             | 3.5 log q RIM         | 1.5 log r RIM |

Theorem 5: For large p:

\[ N(p) \rightarrow (p^2 - 1)/3 \] (30)

Proof: If G is a GG, then \(G^z \mod p\) is also a GG if \(z\) is co-prime with \(p^2-1\).

Euler's totient function shows how many integers \(z\) satisfy this requirement:

\[ \varphi(p^2-1) = \varphi(24qr) = \varphi(24)\varphi(q-1)\varphi(r-1) = (p-3)(p-7)/3 = \left(\frac{p^2-1}{3}\right)-o(p). \] (31)

Therefore, \(N(p) = \left(\frac{p^2-1}{3}\right) = \left(\frac{p^2-1}{3}\right)-o(p). \)

Corollary: For a large p, if a Gaussian (a, b) is selected randomly, then the probability of it being a primitive root (generator) is close to \(1/3\). After t trials, (a, b) is likely to be a GG with probability \(1-(2/3)^t\).

If \(t = 3\), then the probability is \(1-(2/3)^3 = 0.703703\ldots\) that it is a GG.

Numerous computer experiments demonstrate that, on average, the algorithm (9-12) finds a GG after a mere three trials, with a standard deviation of 2.44.

Complexity analysis of algorithm: The following Table 4 facilitates the analysis.

Indeed, from (32)

\[ (x,y)^2 = ((x+y)(x-y),2xy)(\mod p) \]

and \((u,w)(x,y)=(ux-wy,(u+w)(x+y)-ux-wy)\mod p\) (33)

therefore, the squaring requires two RIM and the product of two complex integers requires three RIM (32). Thus, the total number of required RIM is equal to \(3.5\log(p+1)/4+1.5\log(p-1)/6<5\log(p+1)/4 = \Theta(\log p). \)

Further reduction of complexity can be achieved via application of Toom’s algorithm for computation of multi-digit long integers[9].

Illustrative example: The following numeric example demonstrates the most important features of the search algorithm and quadratic order of the GG.

Suppose a triple prime \(p = 11443\).

Step 1: \(p = 11443, q = 2861\) and \(r = 1907\); \{all three integers are primes\}.

Step 2: {First trial}:

Select randomly (a, b) = (2446, 1893);

Step 3: Repeat for \(k = \{2, 3, q\}\)

Step 4 {iteration for \(k = 2\)}:

Compute (c, d) = (7880, 3169);

if \(c = 0\) or \(d = 0\) or \(|c| = |d|\),

then goto the next trial;

Step 4 {iteration for \(k = 3\)}:

Compute (c, d) = (10173, 12371);

Step 4 {iteration for \(k = q\)}:

Compute (c, d) = (9571, 9571);

if \(c = 0\) or \(d = 0\), then goto the next trial;

Step 5: Compute \(e = 938\);

Step 6: \(f = 2932\); {see (11) and (12)};

Step 7: If \(f \neq \pm 1\), then output (a, b) = (3801, 7240) is a Gaussian generator.

It is easy to verify that the order of the GG equals:

\[ \text{ord } (3801, 7240) = p^2-1 = 11443^2 - 1 = 130,942,248; \]

(Table 5).

Remark: CPU times \(T\) required to compute a Gaussian generator are provided in micro-seconds (\(\mu s\)).
All computations were performed on a PC with the following specifications: Intel Pentium dual-core processor; 2.16 GHz, 1 MB L2 cache and 2GB DDR2 Main Memory.

Table 5 shows that if a GG=(574467, 342161) modulo triple prime p=2,004,763, then ord(GG) = 4,019,074,686,168.

**CONCLUSION**

In various public-key cryptographic protocols users select a large prime and a corresponding generator g that are computationally-intense problems. Selection of a triple prime p is a computationally challenging task. Fortunately, the triple prime p must be selected only on a system-design level. After the triple prime p of a specified size is computed, the system designer can periodically change Gaussian generators. This policy provides additional cyber-immunity to cryptographic protocols.

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