On spectral decomposition of Smale–Vietoris axiom A diffeomorphisms

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ABSTRACT
We introduce Smale–Vietoris diffeomorphisms that include the classical DE mappings with Smale solenoids. We describe the correspondence between basic sets of axiom A Smale–Vietoris diffeomorphisms and basic sets of non-singular axiom A-endomorphisms. For Smale–Vietoris diffeomorphisms of 3-manifolds, we prove the uniqueness of non-trivial solenoidal basic set. We construct a bifurcation between different types of solenoidal basic sets which can be considered as a destruction (or birth) of Smale solenoid.

1. Introduction

Stephen Smale, in his celebrated paper [1], introduced the so-called DE maps which arise from expanding maps (the abbreviation DE is formed by first letters of Derived from Expanding map). Let $T$ be a closed manifold of dimension at least 1, and $N$ an $n$-disk of dimension $n \geq 2$. Omitting details, one can say that a DE map is the skew map:

$$f : T \times N \rightarrow T \times N, \quad (x, y) \mapsto (g_1(x), g_2(x, y)),$$

where $g_1 : T \rightarrow T$ is an expanding map of degree $d \geq 2$, and

$$g_2|_{\{x\} \times N} : \{x\} \times N \rightarrow \{g_1(x)\} \times N,$$

an uniformly attracting map of $n$-disk $\{x\} \times N$ into $n$-disk $\{g_1(x)\} \times N$ for every $x \in T$. In addition, $f$ must be a diffeomorphism onto its image $T \times N \rightarrow f(T \times N)$. In the particular case, when $T = S^1$ is a circle and $N = D^2$ is a 2-disk with the uniformly attracting $g_2$, one gets a classical Smale solenoid $\bigcap_{l \geq 0} f(T \times D^2) = \mathcal{S}$ (see Figure 1), which is a topological solenoid.

Recall that a topological solenoid was introduced by Vietoris [2] in 1927 (independently, a solenoid was introduced by van Danzig [3] in 1930, see review in [4]). Smale [1] proved that $\mathcal{S}(f)$ is a hyperbolic expanding attractor. This construction was generalized by Williams [5,6] who defined $g_1$ to be expansion mappings of branch manifolds (this allows to Williams to classify interior dynamics of expanding attractors) and by Block [7] who...
defined $g_1$ to be an Axiom A-endomorphism. The last paper concerns to the $\Omega$-stability and the proving of decomposition of non-wandering set into the so-called basic sets (Spectral Decomposition Theorem for A-endomorphisms). Ideologically, our paper is a continuation of Block, where it was proved the following result (Theorem A). Let $f : M^n \to M^n$ be a Smale–Vietoris diffeomorphism of closed $n$-manifold $M^n$ and $\mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f |_{\mathfrak{B}}$ (see the notations below). Then $f |_{\mathfrak{B}}$ satisfies axiom A on $\mathfrak{B}$ if and only if $g$ does on $T$.

Let us mention that in the frame of Smale–Williams construction, the interesting examples of expanding attractors was obtained in [8–12]. Bothe [13] classified the purely Smale solenoids on 3-manifolds. He was the first who also proved that a DE map $S^1 \times D^2 \to S^1 \times D^2$ can be extended to a diffeomorphism of some closed 3-manifold $M^3 \supset S^1 \times D^2$ (see also [14–16]). Ya. Zeldovich and others (see [17]) conjectured that Smale-type mappings could be responsible for the so-called fast dynamos. Therefore, it is natural to consider various generalizations of classical Smale mapping.

In a spirit of Smale construction of DE maps, we here introduce diffeomorphisms called Smale–Vietoris that are derived from non-singular endomorphisms. A non-wandering set of Smale–Vietoris diffeomorphisms belongs to an attractive invariant set of solenoidal type. In the classical case, the invariant set coincides with the non-wandering set consisting of a unique basic set. In general, the non-wandering set does not coincide with the invariant set, and divides into basic sets provided the non-singular endomorphism is an A-endomorphism.

Let $N$ be $(n - k)$-dimensional compact Riemannian manifold with a non-empty boundary where $n - k \geq 1$. For a subset $N_1 \subset N$, we define the diameter $\text{diam}
_1 = \max_{a, b \in N_1} \{\rho_N(a, b)\}$ of $N_1$ where $\rho_N$ is the metric on $N$. Denote by $\mathbb{T}^k = S^1 \times \cdots \times S^1$ the $k$-dimensional torus, $k \in \mathbb{N}$. A surjective mapping $g : \mathbb{T}^k \to \mathbb{T}^k$ is called a $d$-cover if $g$ is a preserving orientation local homeomorphism of degree $d$. A good example is the preserving orientation linear expanding mapping $E_d : \mathbb{T}^k \to \mathbb{T}^k$ defined by an integer $k \times k$ matrix with the determinant equals $d$. Certainly, $E_d$ is a $d$-cover.

A skew-mapping

$$ F : \mathbb{T}^k \times N \to \mathbb{T}^k \times N, \quad (t, z) \mapsto ((g(t), \omega(t, z)) \quad (2) $$

is called a Smale skew-mapping, if the following conditions hold:

- $F : \mathbb{T}^k \times N \to F(\mathbb{T}^k \times N)$ is a diffeomorphism on its image;
- $g : \mathbb{T}^k \to \mathbb{T}^k$ is a $d$-cover, $d \geq 2$;
- Given any $t \in \mathbb{T}^k$, the restriction $\omega|_{\{t\} \times N} : \{t\} \times N \to \mathbb{T}^k \times N$ is the uniformly attracting embedding,

$$ \{t\} \times N \to \text{int}\left(\{g(t)\} \times N\right), \quad (3) $$

Figure 1. DE map by S. Smale.
i.e. there are $0 < \lambda < 1$, $C > 0$, such that
\[ \text{diam} \left( F^n(\{t\} \times N) \right) \leq C\lambda^n \text{diam} \left( \{t\} \times N \right), \ \forall n \in \mathbb{N}. \]  

When $g = E_d$, Smale skew-mapping is a DE mapping (1) introduced by Smale [1].

A-diffeomorphism $f: M^n \to M^n$ is called a Smale–Vietoris diffeomorphism if there is the $n$-submanifold $\mathbb{T}^k \times N \subset M^n$ such that the restriction $f|_{\mathbb{T}^k \times N} \overset{\text{def}}{=} F$ is a Smale skew-mapping. The sub-manifold $\mathbb{T}^k \times N \subset M^n$ is called a support of Smale skew-mapping.

Put by definition,
\[ \cap_{l \geq 0} F^l(\mathbb{T}^k \times N) \overset{\text{def}}{=} \mathcal{G}(f). \]

One can easy to see that the set $\mathcal{G}(f) = \mathcal{G}$ is attractive, invariant and closed, so that the restriction
\[ f|_{\mathcal{G}} : \mathcal{G} \to \mathcal{G} \]

is a homeomorphism.

The following theorem shows that there is an intimate correspondence between basic sets of $f|_{\mathcal{B}}$ and basic sets of the $A$-endomorphism $g$.

**Theorem 1.1:** Let $f: M^n \to M^n$ be a Smale–Vietoris $A$-diffeomorphism of closed $n$-manifold $M^n$ and $\mathbb{T}^k \times N \subset M^n$ be a support of the Smale skew-mapping $f|_{\mathcal{B}} = F$ (see (2)). Let $\Omega$ be a basic set of $g: \mathbb{T}^k \to \mathbb{T}^k$ and $\mathcal{G} = \cap_{l \geq 0} F^l(\mathbb{T}^k \times N)$. Then $\mathcal{G} \cap p_1^{-1}(\Omega)$ contains a unique basic set $\Omega_\mathcal{G}$ of $f$. Here, $p_1: \mathbb{T}^k \times N \to \mathbb{T}^k$ is the natural projection on the first factor. Moreover,

1. if $\Omega$ is a trivial basic set (isolated periodic orbit) of $g$, then $\Omega_\mathcal{G}$ also is a trivial basic set;
2. if $\Omega$ is a non-trivial basic set of $g$, then $\Omega_\mathcal{G}$ also is a non-trivial basic set;
3. if $\Omega$ is a backward $g$-invariant basic set of $g$, $\Omega = g^{-1}(\Omega)$, (hence, $\Omega$ is non-trivial), then $\Omega_\mathcal{G} = \mathcal{G} \cap p_1^{-1}(\Omega)$.

For $k = 1$, when $\mathbb{T}^1 = S^1$ is a circle, the following result says that $\text{NW}(F)$ contains a unique non-trivial basic set that is either Smale (one-dimensional) solenoid or a non-trivial zero-dimensional basic set.

**Theorem 1.2:** Let $f: M^n \to M^n$ be a Smale–Vietoris $A$-diffeomorphism of closed $n$-manifold $M^n$ and $\mathbb{T}^1 \times N \subset M^n$ the support of Smale skew-mapping $f|_{\mathcal{B}} = F$. Then, the non-wandering set $\text{NW}(F)$ of $F$ belongs to $\mathcal{G} = \cap_{l \geq 0} F^l(\mathbb{T}^1 \times N)$, and $\text{NW}(F)$ contains a unique non-trivial basic set $\Lambda(f)$ that is either

- a one-dimensional expanding attractor, and $\Lambda(f) = \mathcal{G}$, or
- a zero-dimensional basic set, and $\text{NW}(F)$ consists of $\Lambda(f)$ and finitely many (non-zero) isolated attracting periodic points plus finitely many (possibly, zero) saddle-type isolated periodic points of co-dimension one stable Morse index.

The both possibilities hold.

It is natural to consider bifurcations from one type of dynamics to another which can be thought of as a destruction (or, a birth) of Smale solenoid. For simplicity, we represent two such bifurcations for $n = 3$ and $M^3 = S^3$ a 3-sphere. Recall that a diffeomorphism $f$:}
\( M \to M \) is \( \Omega \)-stable if there is a neighbourhood \( U(f) \) of \( f \) in the space of \( C^1 \) diffeomorphisms \( \text{Diff}^1(M) \) such that \( f|_{NW(f)} \) conjugate to every \( g|_{NW(g)} \) provided \( g \in U(f) \).

**Theorem 1.3:** There is the family of \( \Omega \)-stable Smale–Vietoris diffeomorphisms \( f_\mu : S^3 \to S^3 \), \( 0 \leq \mu \leq 1 \), continuously depending on the parameter \( \mu \) such that the non-wandering set \( NW(f_\mu) \) of \( f_\mu \) is the following:
- \( NW(f_0) \) consists of a one-dimensional expanding attractor (Smale solenoid attractor) and one-dimensional contracting repeller (Smale solenoid repeller);
- For \( \mu > 0 \), \( NW(f_\mu) \) consists of two non-trivial zero-dimensional basic sets and finitely many isolated periodic orbits.

**Remark 1.1:** Theorem 1.3 is also true for every lens space (see the proof of Theorem 1.3).

### 2. Definitions

A mapping \( F : M \times N \to M \times N \) of the type \( F(x, y) = (g(x), h(x, y)) \) is called a skew-mapping. One says also a skew product transformation over \( g \) or simply, a skew product. Denote by \( \text{End}^1(M) \) the space of \( C^1 \) endomorphisms \( M \to M \), i.e. the \( C^1 \) maps of \( M \) onto itself. An endomorphism \( g \) is non-singular if the Jacobian \( |Dg| \neq 0 \). This means that \( g \) is a local diffeomorphism. In particular, \( g \) is a \( d \)-cover mapping. In this paper, we consider non-singular \( g \in \text{End}^1(M) \), so that \( Dg \neq 0 \) and \( g \) is not a diffeomorphism.

Fix \( g \in \text{End}^1(M) \). A point \( x \in M \) is said to be non-wandering if given any neighbourhood \( U(x) = U \) of \( x \), there is \( m \in \mathbb{N} \) such that \( g^m(U) \cap U \neq \emptyset \). Denote by \( NW(g) \) the set of non-wandering points. Clearly, \( NW(g) \) is a closed set and \( g(NW(g)) \subset NW(g) \), i.e. \( NW(g) \) is a forward \( g \)-invariant set. The set \( \{x_i\}_{i=-\infty}^{\infty} \) denoted by \( O(x_0) \) is called a \( g \)-orbit of \( x_0 \) if \( g(x_i) = x_{i+1} \) for every integer \( i \). A subset \( \{x_i, x_{i+1}, \ldots, x_{i+r}\} \subset O(x_0) \) consisting of a finitely many points of \( O(x_0) \) is called a compact part of \( O(x_0) \). A \( g \)-orbit \( \{x_i\}_{i=-\infty}^{\infty} \) is periodic if there is an integer \( p \geq 0 \) such that \( g^p(x_i) = x_{i+p} \) for each \( i \in \mathbb{Z} \). Certainly, \( NW(g) \) contains all periodic \( g \)-orbits.

The orbit \( O(x_0) \) is said to be hyperbolic if there is a continuous splitting of the tangent bundle

\[
\mathbb{T}^1_{O(x_0)}M = \bigcup_{i=-\infty}^{\infty} \mathbb{T}_{x_i}M = \mathbb{E}^s \oplus \mathbb{E}^u = \bigcup_{i=-\infty}^{\infty} \mathbb{E}^s_{x_i} \oplus \mathbb{E}^u_{x_i}
\]

which is preserved by the derivative \( Dg \) such that

\[
||Dg^m(v)|| \leq c\mu^m||v||, \quad ||Dg^m(w)|| \geq c^{-1}\mu^{-m}||w|| \quad \text{for } v \in \mathbb{E}^s, \ w \in \mathbb{E}^u, \ \forall m \in \mathbb{N}
\]

for some constants \( c > 0, \ 0 < \mu < 1 \) and a Riemannian metric on \( \mathbb{T}M \). Note that \( \mathbb{E}^s(x_0) \) depends on the negative semi-orbit \( \{x_i\}_{i=-\infty}^{0} \). It may happen that \( \mathbb{E}^u(x_0) \neq \mathbb{E}^u(y_0) \) though \( x_0 = y_0 \) but \( O(x_0) \neq O(y_0) \). Such a phenomenon is impossible for \( \mathbb{E}^s(x_0) \), it depends only on \( x_0 \) [18].

We say that a non-singular \( g \in \text{End}^1(M) \) satisfies axiom A, in short, \( f \) is an A-endomorphism if
- the periodic \( g \)-orbits are dense in \( NW(g) \) (it follows that \( g(NW(g)) = NW(g) \));
all g-orbits of $NW(g)$ are hyperbolic, and the corresponding splitting of the tangent bundle $\mathbb{T}^{NW(g)}$ depends continuously on the compact parts of the g-orbits.

Recall that Smale’s Spectral Decomposition Theorem says that for every Axiom A-diffeomorphisms the non-wandering set partitions into finitely many non-empty-closed invariant sets each of which is transitive. Similar theorem for A-endomorphisms was proved in [7, Theorem C] and [18, Theorem 3.11 and Proposition 3.13]). Thus, if g is a non-singular A-endomorphism, then the non-wandering set $NW(g)$ is the disjoint union $\Omega_1 \cup \cdots \cup \Omega_k$ such that each $\Omega_i$ is closed and invariant, $g(\Omega_i) = \Omega_i$, and $\Omega_i$ contains a point whose g-orbit is dense in $\Omega_i$. Each $\Omega_i$ is called a basic sets.

Following Williams [5,6], we introduce an inverse limit for $g: T \to T$ as follows. Put by definition, $\prod g = \{ (t_0, t_1, \ldots, t_i, \ldots) \in T^\mathbb{N} : g(t_{i+1}) = t_i, \ i \geq 0 \}$. This set is endowed by the product topology of countable factors. This topology has a basis generating by $(\varepsilon, r)$-neighbourhoods

$$U = \left\{ \{x_i\}_0^\infty \in \prod g : x_i \in U_i(t_i), \ 0 \leq i \leq r \text{ for some } \varepsilon > 0, \ r \in \mathbb{N} \right\},$$

where $\{t_0, t_1, \ldots, t_i, \ldots\} \in \prod g$. Define the shift map

$$\hat{g} : \prod g \to \prod g, \quad \hat{g}(t_0, t_1, \ldots, t_i, \ldots) = (g(t_0), t_0, t_1, \ldots, t_i, \ldots), \quad (t_0, t_1, \ldots, t_i, \ldots) \in \prod g.$$

This map $\hat{g} : \prod g \to \prod g$ is called the inverse limit of g. Indeed, g is a homeomorphism.[6,19]

### 3. Proofs of main results

We denote by $p_1 : \mathbb{T}^k \times N \to \mathbb{T}^k$, $p_2 : \mathbb{T}^k \times N \to N$ the natural projections $p_1(t, z) = t$ and $p_2(t, z) = z$. A fibre $\{t\} \times N$ def $N_t$ of the trivial fibre bundle $p_1$ is called a t-leaf. It follows from (2) that $F = f|_\mathbb{B}$ takes a t-leaf into $g(t)$-leaf.

Let $t \in \mathbb{T}^k$ and $\varepsilon > 0$. We denote by $U_\varepsilon(t)$ the $\varepsilon$-neighbourhood of the point t, i.e. $U_\varepsilon(t) = \{x \in \mathbb{T}^k : g(x, t) < \varepsilon \}$ where $g$ is a metric on $\mathbb{T}^k$.

The following technical lemma describes the symbolic model of the restriction $f|_\mathbb{S}$. This lemma is a generalization of the similar classical result by Williams [5,6].

**Lemma 3.1:** Let $f: M^n \to M^n$ be a Smale–Vietoris diffeomorphism of closed n-manifold $M^n$ and $\mathbb{T}^k \times N = \mathbb{B} \subset M^n$ the support of Smale skew-mapping $f|_\mathbb{B} = F$. Then the restriction $f|_\mathbb{S}$ is conjugate to the inverse limit of the mapping $g : \mathbb{T}^k \to \mathbb{T}^k$, where $\mathbb{S} = \cap_{i \geq 0} F^i(\mathbb{T}^k \times N)$.

**Proof:** Recall that given any point $t_0 \in \mathbb{T}^k$, $g^{-1}(t_0)$ consists of d points, one says $t_0^1, t_0^2, \ldots, t_0^d \in \mathbb{T}^k$. Since F is a diffeomorphism on its image, the sets $F(N_{t_0^i}), \ldots, F(N_{t_0^j})$ are pairwise disjoint,

$$F(N_{t_0^i}) \cap F(N_{t_0^j}) = \emptyset, \quad i \neq j, \quad 1 \leq i, \ j \leq d.$$  \hspace{1cm} (6)

Now, for the sake of simplicity, we divide the proof into several steps. The end of each step of the proof will be denoted by $\square$.
Step 1: Given any point \( p \in \mathcal{G} \), there is a unique sequence of points \( \{t_i\}_{i=0}^{\infty}, t_i \in \mathbb{T}^k \), and the corresponding sequence of the leaves \( \{N_{t_i}\}_{i=0}^{\infty} \), such that

- \( p \in \cdots \subset F^i(N_{t_i}) \subset F^{i-1}(N_{t_{i-1}}) \cdots \subset F(N_{t_0}) \subset N_0, p = \cap_{i \geq 0} F^i(N_{t_i}) \);

- \( t_i = g(t_{i+1}), i \geq 0 \).

**Proof of Step 1:** Put \( t_0 = p_1(p) \in \mathbb{T}^k \). Let \( g^{-1}(t_0) = \{t_0, t_1, \ldots, t_d\} \). By (6), there is a unique \( t_i^j \) such that \( p \in F(N_{t_i}) \). Put by definition \( t_0^j = t_1 \). Note that \( F(N_{t_1}) \subset N_0 \). Similarly, \( g^{-1}(t_1) \) consists of \( d \) points \( t_1^1, t_1^2, \ldots, t_1^d \). By (6), the sets \( F(N_{t_1}), \ldots, F(N_{t_d}) \) are pairwise disjoint. Since \( p \in F^2(\mathbb{T}^k \times N) \), there is a unique \( t_i^j \) such that \( p \in F^2(N_{t_i}) \). Put by definition \( t_1^j = t_2 \). Note that \( p \in F^2(N_{t_2}) \subset F(N_{t_1}) \subset N_0 \). Continuing by this way, one gets the sequences \( \{t_i^j\}_{i=0}^{\infty}, \{N_{t_i}\}_{i=0}^{\infty} \) desired. It follows from (4) that \( \text{diam} F^i(N_{t_i}) = \text{diam} (F^i(t_i) \times N) \to 0 \) as \( i \to \infty \). Hence, \( p = \cap_{i \geq 0} F^i(N_{t_i}) \).

Let \( \hat{g} : \prod_g \to \prod_g \) be the inverse limit of \( g : \mathbb{T}^k \to \mathbb{T}^k \) where \( \prod_g = \{ (t_0, t_1, \ldots, t_d) \} \subset T^N \). For a point \( p \in \mathcal{G} \), denote by \( P(t_0, t_1, \ldots, t_d), t_i \in \mathbb{T}^k \), the sequence due to Step 1. Define the mapping

\[
\theta : \mathcal{G} \to \prod_g, \quad p \mapsto P(t_0, t_1, \ldots, t_d), \quad p \in \mathcal{G}.
\]

Step 2: The mapping \( \theta \) is a homeomorphism.

**Proof of Step 2:** It follows from (4) that \( \theta \) is injective. Since the intersection of nested sequence of closed subsets is non-empty, \( \theta \) is surjective. One remains to prove that \( \theta \) and \( \theta^{-1} \) are continuous. Take a neighbourhood \( U \) of \( \theta(p), p \in \mathcal{G} \). We can assume that \( U \) is an \((\varepsilon, r)\)-neighbourhood \( (5) \), where \( \theta(p) = \{t_0, t_1, \ldots, t_d, \ldots\} \subset \prod_g \). Moreover, one can assume that \( g^{-1}(U_{t_i}(t_j)) \) consists of \( d \) pairwise disjoint domains for every \( 0 \leq i \leq r \). Recall that \( t_i = g(t_{i+1}), i \geq 0 \). Therefore, \( t_{r-j} = g(t_r) \) for all \( 1 \leq j \leq r \). Similarly, \( x_{r-j} = g(t_r) \) for all \( 1 \leq j \leq r \). Since \( g \) is continuous, there exists \( 0 < \delta \leq \varepsilon \) such that the inclusion \( x_i \in U_{t_i}(t_j) \) for all \( i = 0, \ldots, r \). The restriction \( F|_\mathcal{G} : \mathcal{G} \to \mathcal{G} \) is a homeomorphism. To prove Theorem 1.1, we need some previous results.
Lemma 3.2: Let \( T = \{t_0, t_1, \ldots, t_i, \ldots \} \in \prod g, g(t_{i+1}) = t_i, i \geq 0. \) Suppose that \( t_i \in NW(g) \) for all \( i \geq 0. \) Then, \( T \in NW(\hat{g}) \) and \( \theta^{-1}(T) \in NW(F). \)

Proof: Since \( T = \{t_0, t_1, \ldots, t_i, \ldots \} = \{g^r(t_r), g^{-1}(t_r), \ldots, t_r, \ldots \}, \) we can take the \((\varepsilon, r)-\)neighbourhood \( V_T \) as follows:

\[
V = \{ g^r(x_r), g^{-1}(x_r), \ldots, x_r, \ldots \} : g^r(x_r) \in U_\varepsilon(g(t_r)), 0 \leq i \leq r \}.
\]

Since \( g, g^2, \ldots, g^r \) are uniformly continuous, there is \( 0 < \delta \leq \varepsilon \) such that \( x \in U_\delta(y) \) implies that \( g^0(x) \in U_\varepsilon(g^0(y)) \) for all \( 0 \leq i \leq r. \) By condition, \( t_r \in NW(g). \) Hence, there exists \( n_0 \in \mathbb{N} \) such that \( g^{n_0}(V_\delta(t_r)) \cap V_\delta(t_r) \neq \emptyset. \) It follows that there is a point \( x_0 \in V_\delta(t_r) \) such that \( g^{n_0}(x_0) \in V_\delta(t_r). \)

Take \( \bar{x}_0 = \{g^r(x_0), g^{-1}(x_0), \ldots, x_0, \ldots \} \in \prod g. \) Since \( x_0 \in V_\delta(t_r), g^i(x_0) \in U_\varepsilon(g^i(t_r)) \) for all \( 0 \leq i \leq r. \) Therefore, \( \bar{x}_0 \in V. \) Since \( g^{n_0}(x_0) \in V_\delta(t_r), g^{n_0+i}(x_0) \in U_\varepsilon(g^i(t_r)) \) for all \( 0 \leq i \leq r. \) Therefore,

\[
\hat{g}^{n_0}(\bar{x}_0) = \{g^{n_0+r}(x_0), g^{n_0+r-1}(x_0), \ldots, g^{n_0}(x_0), \ldots \} \in V.
\]

As a consequence, \( \hat{g}^{n_0}(V) \cap V \neq \emptyset \) and \( T \in NW(\hat{g}). \) A conjugacy map takes a non-wandering set onto a non-wandering set. By Lemma 3.1, \( \theta^{-1}(T) \in NW(F). \)

Corollary 3.1: The following qualities hold \( p_1[NW(f_{2\Omega})] = p_1[NW(F)] = NW(g). \)

Proof: Since the projection \( p_1 \) is continuous, \( p_1[NW(F)] \subset NW(g). \) Take a point \( t_0 \in NW(g). \) Since \( g \) is an \( A \)-endomorphism, \( g[NW(g)] = NW(g). \) Therefore, there is a sequence \( \{t_i\}_{i = 0, 1, \ldots} \subset NW(g) \) such that \( g(t_{i+1}) = t_i \) for every \( i \geq 0. \) It follows from Lemma 3.2 that \( \bar{T} = \{t_0, t_1, \ldots, t_i, \ldots \} \in NW(\hat{g}) \) and \( \theta^{-1}(T) \in NW(F). \) By definition of the mapping \( \theta, \) \( \theta^{-1}(T) \in p_1^{-1}(t_0). \) Hence, \( NW(g) \subset p_1[NW(F)]. \)

Lemma 3.3: Let \( (t_0, z_0) \in \mathcal{S} \) be a non-wandering point of \( f, \) and \( \theta(t_0, z_0) = \{t_i\}_{i \geq 0}. \) Then, \( t_i \in NW(g) \) for all \( i \geq 0. \)

Proof: According to Corollary 3.1, \( p_1[NW(f_{2\Omega})] = p_1[NW(F)] = NW(g). \) Therefore, \( t_0 \in NW(g). \) Since \( F_\mathcal{S} : \mathcal{S} \rightarrow \mathcal{S} \) is a diffeomorphism, \( F^{-1}(NW(F)) = NW(F) \) and \( F^{-1}(t_0, z_0) = (t_1, z_1) \in NW(F) = NW(f_{2\Omega}). \) Hence, \( t_1 \in NW(g) \) by Step 1. Continuing this way, one gets that \( t_i \in NW(g) \) for all \( i \geq 0. \)

Corollary 3.2: Let \( (t_0, z_0) \in \mathcal{S} \) be a non-wandering point of \( f, \) and \( \theta(t_0, z_0) = \{t_i\}_{i \geq 0}. \) Suppose that \( t_0 \) belongs to a basic set \( \Omega \) of \( g. \) Then, \( t_i \in \Omega \) for all \( i \geq 0. \)

Proof: By Lemma 3.3, \( t_i \in NW(g) \) for all \( i \geq 0. \) Since \( \Omega \) is forward \( g \)-invariant, \( t_i \in \Omega \) for all \( i \geq 0. \)

Lemma 3.4: Let \( \Omega \) be a non-trivial basic set of \( g, \) and \( t_0 \in \Omega. \) Suppose that two points \( (t_0, z_1), (t_0, z_2) \in \mathcal{S} \) are non-wandering under \( f. \) Then, both \( (t_0, z_1) \) and \( (t_0, z_2) \) belong to the same basic set of \( f. \)

Proof: Denote by \( \Omega_j \) the basic set of \( F \) containing the point \( (t_0, z_j), j = 1, 2. \) Clearly, \( \Omega_j \subset \mathcal{S}. \) We have to prove that \( \Omega_1 = \Omega_2. \) It is sufficient to show that there is a non-wandering point \( q \in NW(F) \) such that each point \( (t_0, z_1) \) and \( (t_0, z_2) \) belongs to the \( \omega \)-limit set of \( q. \)
Let $\bar{t}_j = \theta(t_0, z_j) = \{t_0, t_1^{(j)}, \ldots, t_i^{(j)}, \ldots\}, j = 1, 2$. By Corollary 3.2, $t_i^{(j)} \in \Omega$ for all $i \geq 0, j = 1, 2$. Since the basic set $\Omega$ is transitive, there is a point $x_0 \in \Omega$ such that its positive semi-orbit $O^+_x(x_0)$ is dense in $\Omega$, $\text{cl}(O^+_x(x_0)) = \Omega$.

It follows from Corollary 3.1 that there is a point $\bar{x}_0 = \{x_0, x_1, \ldots, x_i, \ldots\} \in \prod_{\delta}$ such that $x_i \in \Omega$ for all $i \geq 0$. Take arbitrary $(e, r)$-neighbourhood $U(\bar{t}_j)$ of $\bar{t}_1$. Since $g, g^2, \ldots, g^r$ are uniformly continuous, there exists $\delta > 0$ such that the inequality $x \in U_\delta(y)$ implies $g^r(x) \in U_\delta(y)$ for all $0 \leq i \leq r$. Because of the semi-orbit $O^+_x(x_0)$ is dense in $\Omega$, there is $n_0 \in \mathbb{N}$ such that $g^{n_0}(x_0) \in U_\delta(t^{(i)})$. Hence, $g^{n_0}(\bar{x}_0) \in U(\bar{t}_1)$. Therefore, $\bar{t}_1 = \theta(t_0, z_1)$ belongs to the $\omega$-limit set of $\bar{x}_0$. Similarly, one can prove that $\bar{t}_2 = \theta(t_0, z_2)$ belongs to the $\omega$-limit set of $\bar{x}_0$ as well. Since $\theta$ is a conjugacy mapping, the points $(t_0, z_1) = \theta^{-1}(\bar{t}_1)$ and $(t_0, z_2) = \theta^{-1}(\bar{t}_2)$ belongs to the $\omega$-limit set of the point $q = \theta^{-1}(\bar{x}_0) \in NW(F)$.

**Proof of Theorem 1.1:** We know that $p_1[NW(F)] = NW(g)$. Hence, $\mathcal{G} \cap p_1^{-1}(\Omega)$ contains basic sets of $f$. Suppose that $\Omega$ is trivial, i.e. $\Omega$ is an isolated periodic orbit:

$$\Omega = Orb_{g^0}(q) = \{q, g(q), \ldots, g^{p-1}(q), g^p(q) = q\}, \text{ where } q \in \mathbb{T}^k \text{ and } p \in \mathbb{N} \text{ are a period of } q.$$

By definition of Smale skew-mapping, the restriction of $F = f|_{\Omega}$ on the second factor $N$ is the uniformly attracting embedding. Therefore,

$$N_q \supset f^p(N_q) \supset \cdots \supset f^{mp}(N_q) \supset \cdots \text{ and the intersection } \bigcap_{m \geq 0} f^{mp}(N_q)$$

is a unique point, say $Q$.

Similarly, $\cap_{m \geq 0} f^{mp}(N_{g^0(q)})$ is a unique point $f^i(Q)$ for every $0 \leq i \leq p - 1$. It follows from (2) that $\{Q, f(Q), \ldots, f^{p-1}(Q), f^p(Q) = Q\}$ is an isolated periodic orbit $\text{Orb}_{g^0}(Q)$ such that $NW(F) \cap p_1^{-1}(\Omega) = \text{Orb}_{g^0}(Q)$. Therefore, $\text{Orb}_f(Q) = \Omega_{\mathcal{G}}$ is a unique basic set of $F$ that belongs to $\mathcal{G} \cap p_1^{-1}(\Omega)$.

Let $\Omega$ be a non-trivial basic set. It follows from Lemma 3.4 that all basic sets of $F$ that is contained in $\mathcal{G} \cap p_1^{-1}(\Omega)$ are coincide. Hence, $\Omega_{\mathcal{G}}$ is a unique non-trivial basic set of $f$ contained in $\mathcal{G} \cap p_1^{-1}(\Omega)$.

Now let $\Omega$ be a backward $g$-invariant basic set of $g$. Note that the equality $\Omega = g^{-1}(\Omega)$ implies that $\Omega$ cannot be a trivial basic set, since $g$ is a $d$-cover, $d \geq 2$. It follows from Lemma 3.2 that every point of $\mathcal{G} \cap p_1^{-1}(\Omega)$ is a non-wandering point of $f$. By Lemma 3.4, $\mathcal{G} \cap p_1^{-1}(\Omega)$ is a unique basic set. Theorem 1.1 is proved.

**Example:** Let us consider three endomorphisms $g_i : \mathbb{T}^2 \to \mathbb{T}^2, i = 1, 2, 3$, that are $2$-covers. $g_1$ is defined by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Clearly, $g_1$ is an expanding A-endomorphism, and $\mathbb{T}^2$ is a unique basic set of $g_1$. The corresponding diffeomorphism $f$ has a unique basic set, say $\Omega_1$, that is locally homeomorphic to the product of $\mathbb{R}$ and Cantor set. Thus, $\Omega_1$ is two-dimensional.

Now, let us consider the case when $\mathbb{T}^1 = S^1$ is a circle, and $d$-cover $g : \mathbb{T}^1 \to \mathbb{T}^1$ is a non-singular endomorphism of $S^1$. The crucial step of the proof of Theorem 1.2 is the following result.
Lemma 3.5: Let \( g : T^1 \to T^1 \) be a non-singular \( A \)-endomorphism, and \( NW(g) \) a non-wandering set of \( g \). Then \( NW(g) \) is either \( T^1 \) or \( NW(g) \) is the union of the Cantor type set \( \Sigma \) and finitely many (non-zero) isolated attracting periodic orbits plus finitely many (possibly, zero) repelling isolated periodic orbits. Moreover, in the last case, \( \Sigma \) is backward \( g \)-invariant.

Proof: Suppose that \( NW(g) \neq T^1 \). By [20], \( g \) is semi-conjugate to the expanding linear mapping \( E_d, E_d(t) = dt \mod 1 \), i.e. there is a continuous map \( h : T^1 \to T^1 \) such that \( g \circ h = h \circ E_d \). Moreover, \( h \) is monotone.[21] As a consequence, given any point \( t \in T^1 \), \( h^{-1}(t) \) is either a point or a closed segment. Since \( NW(g) \neq T^1 \), \( h \) is not a homeomorphism. Hence, there are points \( t \in T^1 \) for which \( h^{-1}(t) \) is a (non-trivial) closed segment. Denote the set of such points by \( \chi \). The set \( \chi \) is countable and invariant under \( E_d, E_d(\chi) = E_d^{-1}(\chi) = \chi \).[21,22] Therefore, \( h^{-1}(\chi) \) is also invariant under \( g \).

Let as prove that \( \Sigma \) is totally discontinuous. Since \( h \) is a semi-conjugacy and \( \chi \) is invariant, \( \Sigma \) is an invariant set under \( g \). Moreover, \( h \) is monotone and \( \chi \) contains every dense orbits. It follows that \( \Sigma \) is totally discontinuous. As a consequence, \( \Sigma = T^1 \setminus \text{clos}(h^{-1}(\chi)) \) is the Cantor set consisting on non-wandering points of \( g \). Moreover, \( \Sigma \) is invariant under \( g \) (in particular, backward \( g \)-invariant). It follows from [23] that the part of \( NW(g) \) that different from \( \Sigma \) consists of finitely many (non-zero) isolated attracting periodic orbits and finitely many (possibly, zero) repelling isolated periodic orbits. \( \square \)

Now, Theorem 1.2 except the realisation part immediately follows from Theorem 1.1 and Lemma 3.5. It remains to construct a Smale–Vietoris \( A \)-diffeomorphism whose non-wandering set consists of a non-trivial zero-dimensional basic set and a finitely many (non-zero) isolated periodic orbits. It follows from [13,14] for \( n = 3 \) and [5,7] for \( n \geq 4 \) that it is sufficient to construct Smale skew-mapping \( F : S^1 \times D^n - 1 \to S^1 \times D^n - 1 \) with the non-wandering set desired because of Smale skew-mapping can be extended to a diffeomorphism of some closed \( n \)-manifolds. Moreover, according to Robinson–Williams[12] construction of classical Smale solenoid, we can suppose \( n = 3 \).

Let \( g : S^1 \to S^1 \) be a \( C^\infty \) non-singular \( A \)-endomorphism that is a \( d \)-cover \( (d \geq 2) \) with the non-wandering set \( NW(g) \) consisting of a unique attracting fixed point \( x_0 \) and a Cantor set \( \Omega \). Moreover, one can assume that \( Dg|_\Omega = 2d - 1, Dg(x_0) = \lambda < 1 \) where \( \lambda \) will be specified below. Such endomorphism was constructed by Shub [20]. Hirsch [24] has noticed that such endomorphism can be smoothed to be analytical. Now, the circle \( S^1 \) is endowed with the parameter inducing by the natural projection \( [0, 1] \to [0, 1]/(0 \sim 1) = S^1 \). We can assume that the restriction \( g|_{[0, \frac{1}{2}]} \) is a diffeomorphism \( [0, \frac{1}{2}] \to [0, \frac{1}{2}] \) with the attracting fixed point \( x_0 = \frac{1}{2} \) and two repelling fixed points \( 0, \frac{1}{2} \). Without loss of generality, one can also assume that \( g|_{[\frac{1}{2}, 1]}(x) = (2d - 1)x \mod 1 \). By construction, \( \cup_{n \geq 0} g_d^{\frac{n}{2}}(0, \frac{1}{2}) \) is the stable manifold \( W^s(x_0) \) of \( x_0 \), and \( \Omega = S^1 \setminus W^s(x_0) \) is Cantor set belonging to \( NW(g) \). Clearly, given any \( y \in S^1 \), \( \min_{k \neq j} |t_k - t_j| = \frac{1}{2d - 1} \) where \( t_k \neq t_j \) and \( g(t_k) = g(t_j) = y \). We take \( 0 < \lambda < \frac{1}{4} \sin \frac{\pi}{2d - 1} \). After this specification, we denote \( g \) by \( g_d \). Put by definition,

\[
F(t, z) = \left( g_d(t), \lambda z + \frac{1}{2} \exp 2\pi it \right), \quad F : B = S^1 \times D^2 \to B, \tag{7}
\]
where $D^2 \subset \mathbb{R}^2$ is the unit disk, and $z = x + iy$, and $B$ is a support of Smale skew-mapping. Since $\lambda < \frac{1}{4}$, $F(B) \subset \text{int} B$. The Jacobian of $F$ equals

$$DF(t, z) = \begin{pmatrix} Dg_d(t) & 0 \\ \pi i \exp 2\pi it \lambda Id_2 \end{pmatrix},$$

(8)

where $Id_2$ is the identity matrix on $\mathbb{C}$ or $\mathbb{R}^2$. Since $Dg_d > 0$ and $\lambda > 0$, $F$ is a local diffeomorphism. It follows from $\lambda < \frac{1}{4} \sin \frac{\pi}{2d-1}$ that $F$ is a (global) diffeomorphism on its image.

Since $g_d$ is an $A$-endomorphism, the periodic points of $g_d$ are dense in $NW(g_d)$. By Lemma 3.2, the periodic points of $F$ are dense in $NW(F)$. Thus, it remains to prove the $NW(F)$ has a hyperbolic structure. We follow [19, Proposition 8.7.5]. Clearly, the tangent bundle $T(B) = T(S^1 \times D^2)$ is the sum $T(B) = T(S^1) \oplus T(D^2)$, and the fibre $T_{(t, z)}(B)$ at each point $(t, z) \in B$ is the sum of one-dimensional and two-dimensional tangent spaces $T_t(S^1) = \mathbb{E}^1 \cong \mathbb{R}$, $T_z(T^2) = \mathbb{E}^2 \cong \mathbb{R}^2$, respectively. It follows from (8) that $\mathbb{E}^2$ is invariant under $DF$:

$$DF\left(\begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix}\right) = \begin{pmatrix} \bar{v}_1 \\ \lambda \bar{v}_{23} \end{pmatrix}, \quad \bar{v}_{23} \in \mathbb{E}^2.$$

Moreover, since $|\lambda| < 1$, $\mathbb{E}^2$ is the stable bundle, $E' = \mathbb{E}^2$.

Take $q = (t, z) \in NW(F)$. Then $p_1(q) = t \in NW(g_d)$. If $t = x_0$, then $q$ is a hyperbolic (attractive) fixed point of $F$. For $t \in \Omega$, we consider the cones

$$C^u_q = \left\{ \begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix} : \bar{v}_1 \in T_t(S^1), \bar{v}_{23} \in \mathbb{E}^2, \quad |\bar{v}_1| \geq \frac{2d-1}{4} |\bar{v}_{23}| \right\} \subset T(B) = \mathbb{E}^1 \oplus \mathbb{E}^2.$$

For $\begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix} \in C^u$, it follows from (8) that

$$\begin{align*}
DF\left(\begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix}\right) &= \begin{pmatrix} \bar{v}_1' \\ \bar{v}_{23}' \end{pmatrix} = \begin{pmatrix} (2d-1)\bar{v}_1 \\ \pi i \bar{v}_1 \exp 2\pi it + \lambda \bar{v}_{23} \end{pmatrix}. \\
\end{align*}$$

Hence, $|\bar{v}_{23}'| \leq |\pi i \exp 2\pi it \bar{v}_1| + \lambda |\bar{v}_{23}| = \pi |\bar{v}_1| + \lambda |\bar{v}_{23}|$. Taking in mind $\lambda \leq \frac{1}{4}$, one gets

$$|\bar{v}_1'| = (2d-1)|\bar{v}_1| = \frac{2d-1}{4} |\pi |\bar{v}_1|| + \frac{2d-1}{4} \left(\pi |\bar{v}_1| + \frac{1}{2} |\bar{v}_1|\right) \geq \frac{2d-1}{4} \left(\pi |\bar{v}_1| + \frac{1}{2} |\bar{v}_1|\right) \geq \frac{2d}{4} |\bar{v}_1|^2,$$

since $\frac{2d-1}{8} \geq \frac{1}{4}$. Therefore, $\begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix} \in C^u_{F(q)}$ and $DF(C^u_q) \subset C^u_{F(q)}$. As a consequence,

$$DF^k(C^u_{F-k(q)}) \subset DF^{k-1}(C^u_{F-k+1(q)}) \subset \cdots \subset DF(C^u_{F-1(q)}) \subset C^u_q \quad \text{for any} \quad k \in \mathbb{N}.$$

To prove that the intersection of this nested cones is a line, take

$$\begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix}, \begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix} \in C^u_{F-k(q)}, \begin{pmatrix} \bar{v}_1^k \\ \bar{v}_{23}^k \end{pmatrix} = DF^k\left(\begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix}\right), \begin{pmatrix} \bar{v}_1^k \\ \bar{v}_{23}^k \end{pmatrix} = DF^k\left(\begin{pmatrix} \bar{v}_1 \\ \bar{v}_{23} \end{pmatrix}\right).$$
Lemma 3.6: to a line, say which goes to 0 as $k$.

Proof: For the parameters $\varepsilon, \delta \in \mathbb{R}$, represent the two-parameter family of circle endomorphisms $S^1 \to S^1$, $d \geq 2$. First, we represent the two-parameter family of circle endomorphisms $f_{\varepsilon, \delta}$ continuously depending on the parameters $\varepsilon \in (0, 1)$ and $\delta \in [0, \frac{1}{d}]$.

Let $U_\delta(x)$ be the bump function such that
- $U_\delta(x) = 1$ for $x \in \left[ -\frac{\delta}{2}, \frac{\delta}{2} \right]$, $0 < \delta \leq \frac{1}{4}$;
- $U_\delta(x) = 0$ for $|x| \geq \delta$;
- $U'_\delta(x) \geq 0$ for $x \in \left[ -\delta, -\frac{\delta}{2} \right]$, and $U'_\delta(x) \leq 0$ for $x \in \left[ \frac{\delta}{2}, \delta \right]$.

Lemma 3.6: Let

$$f_{\varepsilon, \delta}(x) = \begin{cases} 
  dx + (-d + \varepsilon)xU_\delta(x) \mod 1 & \text{for } \varepsilon \in (0, 1), \delta \in (0, \frac{1}{d}) \\
  dx \mod 1 & \text{for } \varepsilon = 0, \delta = 0 
\end{cases}$$

Then $f_{\varepsilon, \delta}$ is a structurally stable non-singular circle $d$-endomorphism such that the non-wandering set $NW(f_{\varepsilon, \delta})$ is the union of a unique hyperbolic attracting point $x = 0$ and a Cantor set provided $\varepsilon \neq 0$ and $\delta \neq 0$. Moreover, $NW(f_{0, 0}) = S^1$. In addition, $f_{\varepsilon, \delta} \to E_d$ as $\varepsilon \neq 0$ is fixed and $\delta \to 0$ in the $C^0$ topology.

Proof: For $\varepsilon \neq 0$ and $\delta \neq 0$, we see

$$f'_{\varepsilon, \delta}(x) = d + (-d + \varepsilon) \left[xU_\delta(x)' + U_\delta(x)\right] = d + (-d + \varepsilon)xU_\delta(x)' + (-d + \varepsilon)U_\delta(x).$$

Clearly, $d + (-d + \varepsilon)U_\delta(x) \geq \varepsilon$. Since $xU_\delta(x)' \leq 0$, $f'_{\varepsilon, \delta}(x) \geq \varepsilon$. Because of outside of the $\delta$-neighbourhood $V_\delta(0)$ of $x_0 = 0$, the mapping $f_{\varepsilon, \delta}$ coincides with the linear $d$-endomorphism $E_d(x) = dx \mod 1$, $f_{\varepsilon, \delta}$ is a non-singular $d$-endomorphism. Since $f'_{\varepsilon, \delta}(0) = \varepsilon \in (0, 1)$, $x = 0$ is a hyperbolic attracting point. Solving the equation $dx + (-d + \varepsilon)xU_\delta(x) = x$, one gets two fixed points $\pm x_\varepsilon \in V_\delta(0)$ such that $U_\delta(\pm x_\varepsilon) = \frac{d-1}{d-\varepsilon}$, where $\frac{\delta}{2} < x_\varepsilon < \delta$. Moreover, the $\omega$-limit set of any point from $(-x_\varepsilon, x_\varepsilon)$ is $x_0 = 0$. Hence, $NW(f_{\varepsilon, \delta})$ equals

$$NW(f_{\varepsilon, \delta}) = \{x_0\} \bigcup (S^1 \setminus \bigcup_{k \geq 0} f_{\varepsilon, \delta}^{-k}(-x_\varepsilon, x_\varepsilon)), \quad \omega(-x_\varepsilon) = \omega(x_\varepsilon) = \{x_0\} \bigcup (S^1 \setminus \bigcup_{k \geq 0} f_{\varepsilon, \delta}^{-k}(-x_\varepsilon, x_\varepsilon)).$$
where \( C = S^1 \setminus \bigcup_{k \geq 0} f_{\epsilon, \delta}^{-k} (-x_*, x_*) \) is Cantor set. For any \( x \in C \), one can prove that
\[
f_{\epsilon, \delta}'(x) = d + (-d + \epsilon) x U_{\delta}'(x) + (-d + \epsilon) U_{\delta}(x) \geq d + (-d + \epsilon) U_{\delta}(x_*) \\
+ (-d + \epsilon)x U_{\delta}'(x) \\
= 1 + (-d + \epsilon) x U_{\delta}'(x) > 1.
\]

It follows from [23] that \( f_{\epsilon, \delta} \) is structurally stable. At last, for \( x \in V_{\delta}(0) \), one gets
\[
\left| f_{\epsilon, \delta}(x) - E_d(x) \right| = |(-d + \epsilon)x U_{\delta}(x)| \leq \delta d \to 0 \text{ as } \delta \to 0.
\]

As a consequence, \( f_{\epsilon, \delta} \to E_d \) as \( \delta \to 0 \) in the \( C^0 \) topology. \( \Box \)

Recall that original Smale solenoid map is built by a skew map \( f: S^1 \times D^2 \mapsto S^1 \times D^2 \) by \( f(x, y) = (g_1(x), g_2(y)) \) so that \( g_1 \) is an expanding map on \( S^1 \) and \( g_2 \) is uniformly attracting. In [13], it was proved that \( f: S^1 \times D^2 \mapsto S^1 \times D^2 \) can be extended to a diffeomorphism of some lens space \( L_{p, q} \) (including \( S^3 \)). In [14], it was proved that for any given \( L_{p, q} \) there is a diffeomorphism of \( L_{p, q} \) with one Smale solenoid attractor and one Smale solenoid repeller. The analysis of [13,14] shows that the constructions above can be applied to non-singular endomorphisms \( g_1 \) as well. Thus, taking in mind Lemma 3.6 and the technics developed in [13,14] (see also [5,7,15]), one can prove Theorem 1.3.

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