Exact closed-form analytic wave functions in two dimensions: Contact-interacting fermionic spinful ultracold atoms in a rapidly rotating trap

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Exact two-dimensional analytic wave functions for an arbitrary number \( N \) of contact-interacting lowest-Landau-level (LLL) spinful fermions are derived with the use of combined numerical and symbolic computational approaches via analysis of exact Hamiltonian numerical diagonalization data. Closed-form analytic expressions are presented for two families of zero-interaction-energy states at given total angular momentum and total spin \( 0 \leq S \leq N/2 \) in the neighborhood of the \( \nu = 1 \) filling, covering the range from the maximum density droplet to the first quasihole. Our theoretical predictions for higher-order spatial and momentum correlations reveal intrinsic polygonal, multi-ring crystalline-type structures, which can be tested with ultracold-atom experiments in rapidly rotating traps, simulating quantum Hall physics (including quantum LLL skyrmions).

I. INTRODUCTION

Exact analytic solutions for the quantum many-body problem, whether in a closed-form algebraic expression or in the form of the Bethe ansatz, are highly coveted and sought-after; however, they are available only for a few cases. Among this select group (for early pioneering studies see Refs. [1–7]), one-dimensional (1D) assemblies of strongly contact-interacting ultracold atoms have attracted much attention in the last few years [8–17], motivated by rapid experimental advances in the field of trapped ultracold atoms that allow direct \( \text{situ} \) and time-of-flight single-atom measurements of real-space and momentum-space higher-order correlations, respectively, hold a great promise [18–31].

Here we derive closed-form exact analytic wave functions (EAWFs) for two-dimensional (2D) systems of spinful contact-interacting lowest-Landau-level (LLL) fermions that simulate fractional quantum Hall (FQH) physics [32–39] with trapped ultracold atoms. We first introduce a novel approach for the extraction of EAWFs from the digital information provided via numerical exact-diagonalization (i.e., the configuration interaction, CI [29, 38, 58]) of the many-body LLL Hamiltonian. Subsequently, we present illustrative examples, showing that such EAWFs exhibit intrinsic geometric structures (ultracold Wigner molecules, UCWMs) in their higher-order correlations, in line with earlier findings using numerical CI solutions (see, e.g., Ref. [38]). The compact EAWFs enable consideration of larger assemblies compared to the CI-computed UCWMs [38].

Starting with the Laughlin trial wave function [41], compact algebraic forms have been extensively considered [33, 42–48] as approximations to the exact diagonalization solutions, both for electrons in semiconductors [42–46] and for ultracold bosons in rotating traps [33, 47, 48]. In several instances, like the Laughlin wave functions, it was shown that the variational trial functions [33, 44, 46] may be exact solutions, with zero-interaction energy (0IE states), of specific short-range pseudopotential-type parent Hamiltonians [33, 44].

Because of the fermionic statistics, this paper relates to electronic 2D quantum LLL skyrmions [45, 49–51, 53], multicomponent quantum Hall systems [53], 2D anyons [37], and rotating electronic [54–57] and ultracold-atom [29, 38, 58] Wigner molecules. Experimental realization of such 2D systems (including bosonic analogs [34–37]) with a few ultracold fermionic atoms (e.g., \( ^6\text{Li} \)) in rapidly rotating harmonic traps is currently pursued [39]. Importantly, unlike the skyrmion wave functions used in the literature [45, 49–52], which are not eigenstates of the total spin (see particularly Ref. [51], the Appendix), and the Supplemental Material (SM) [59]), the EAWFs introduced here provide total-spin preserving symmetric polynomials for the quantum LLL skyrmions; for other spin-preserving polynomials (restricted to the spin-singlet state), see Ref. [60].

II. METHODOLOGY

Extensions of Girardeau’s mapping between impenetrable bosons and non-interacting spinless fermions [1], and similar mappings [8, 10] applied to spinful and spin-parallel fermions, led to the formulation of a hard-core boundary condition for strongly-repelling 1D fermions [11, 12]. This entails vanishing of the many-body wave functions when two fermions with antiparallel spins are at the same position (in addition to the vanishing for parallel spins due to the Pauli exclusion principle). Concomitant of this condition is the appearance of 0IE states. In CI calculations and for a given number \( N \) of spinful LLL fermions, the 0IE states emerge in each spin sector \( (S, S_z) \); see Fig. 1 for the case of \( N = 9 \) LLL fermions interacting with a \( \sum_{i<j} \delta(z_i - z_j) \) two-body potential, where \( z_i = x_i + iy_i \) (with \( i = 1, 2, \ldots, N \)). Interest in such 0IE states arises from: (i) They can be prepared

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The many-body Hamiltonian describing ultracold neutral atoms in a rapidly rotating trap [29, 33–35, 39, 58] is given by

$$\frac{H_{\text{LLL}}}{\hbar \omega} = N + (1 - \frac{\Omega}{\omega})L + 2\pi R_0 \sum_{i<j}^N \delta(z_i - z_j), \quad (1)$$

where $\omega$ and $\Omega$ are, respectively, the parabolic trapping and rotational frequencies of the trap, and $L$ denotes the total angular momentum. $L = \sum_{i=1}^N l_i$, normal to the rotating-trap plane; the energies are in units of $\hbar \omega$ and the lengths in units of the oscillator length $\Lambda = \sqrt{\hbar/(M \omega)}$, with $M$ being the fermion mass. The first and second express the LLL kinetic energy, $H_K$, and the third term represents the contact interaction, $H_{\text{int}}$.

Our methodology integrating both numerical (e.g., fortran) and symbolic (algebraic, e.g., MATHEMATICA [64]) languages consists of two steps: (1) numerical diagonalization of the Hamiltonian matrix problem employing the ARPACK solver [65, 66] of large-scale sparse eigenvalue problems, followed by step (2) where the numerically exact CI wave functions

$$\Phi_{\text{CI}}(z_1\sigma_1, \ldots, z_N\sigma_N) = \sum_I c_{\text{CI}}(I) \Psi_I(z_1\sigma_1, \ldots, z_N\sigma_N), \quad (2)$$

are analyzed and processed using symbolic scripts targeting extraction of the corresponding exact analytical wave functions.

The basis Slater determinants that span the Hilbert space are

$$\Psi_I = \text{Det}[\varphi_{j_1}(z_s)\sigma_{j_1}(s)]/\sqrt{N!}, \quad (3)$$

where $r, s = 1, \ldots, N$, the LLL single-particle orbitals are

$$\varphi_j(z) = z^j e^{-zz'^2}/\sqrt{\pi l_j!}, \quad (4)$$

and $\sigma$ signifies an up ($\alpha$) or a down ($\beta$) spin. The master index $I$ counts the number of ordered arrangements (lists) $\{j_1, j_2, \ldots, j_N\}$ under the restriction that $1 \leq j_1 < j_2 < \ldots < j_N \leq K$; $K \in \mathbb{N}$ is chosen large enough to provide numerical convergence. Below, explicit mention of the Gaussian factor is omitted.

Step (2) starts with the rewriting of the CI wave function $\Phi_{\text{CI}}$ in Eq. (2) as

$$\Phi_{\text{alg}}(z_1\sigma_1, \ldots, z_N\sigma_N) = \sum_I c_{\text{alg}}(I) \Psi_I(z_1\sigma_1, \ldots, z_N\sigma_N), \quad (5)$$

where the replacement of the subscript “CI” by “alg” corresponds to the fact that, using the symbolic language code, one obtains an equivalent multivariate homogeneous polynomial $\Phi_{\text{alg}}$ with algebraic coefficients $c_{\text{alg}}$; see the transcription of coefficients for $N = 4$ and $N = 9$ in Tables STI and STII in the SM [59].

Validation of our closed-form analytic wave functions (see below) is achieved via direct comparison of the numerical CI coefficients, $c_{\text{CI}}$, with those in $\Phi_{\text{alg}}$ [Eq. (5)], thus circumventing uncertainties, associated with the common use of wave function overlap [39, 41, 43, 44, 46], due to the van Vleck-Anderson orthogonality catastrophe [67–73].

Invariably, the symbolic code is able to simplify the derived multivariate polynomial in Eq. (5) to the compact form of a product of a Vandermonde determinant (VDet), $\prod_{i<j}^N (z_i - z_j)$, involving the space coordinates only, with a symmetric polynomial (under two-particle exchange) with mixed space and spin coordinates [see Eq. (6) below]. The factoring out of the VDet reflects the fact that $\Phi_{\text{alg}}$ represents a 0IE LLL state.

Using symbolic scripts, we verify further that the fully-algebraic $\Phi_{\text{alg}}$ [Eq. (6)] is indeed an eigenstate of the total spin, obeying the Fock condition [74]. The final closed form expressions [see Eq. (8) below] are derived for $N \leq 9$, but they are valid for any $N$, thus circumventing the CI numerical diagonalization of large matrices, which is not feasible for $N \geq 10$. 

![FIG. 1. CI-calculated, relative-ground-state LLL energy spectra for $N = 9$ fermions associated with the contact-interaction term only; see third term of $H_{\text{LLL}}$ in Eq. (1). Spectra in a given spin sector $S = 1/2, 3/2, 5/2, 7/2$ and $S = 9/2$ are explicitly denoted. The spectra were calculated for $S_z = 1/2$; however, they are independent of the precise value of $S_z$. The 0IE states of family A are colored in red; those of family B are colored in white. The first quasi-hole state is also explicitly denoted. $L$ is the total angular momentum.](image-url)
For the CI diagonalization, a small perturbing term \( V_p \)
(e.g., a small trap deformation [38], or a small hard-wall boundary [75])
needs to be added to the LLL Hamiltonian in Eq. (1). This has a
negligible influence on the numerical eigenvalues, but it is instrumental in
lifting the degeneracies among the 0IE states, and thus produce
CI states whose total spin \( S \) is a good quantum number.

III. TARGETED TOTAL SPINS AND
ANGULAR MOMENTA

For each size \( N \), we provide analytic expressions for the
maximum-spin \((S = S_z)\) 0IE ground states with angular
momenta \( L = L_0 + \Delta L \) [with \( L_0 = N(N - 1)/2 \)] from
\( \Delta L = 0 \) (maximum density droplet) to \( \Delta L = N \) (first
quasihole, 1QH); they form two families \( A \) and \( B \) (see
Fig. 1 for an illustration).

Using \( k \) to denote the number of spin-up fermions and
\( p \) that of spin-down fermions, and focusing on the case
with \( k \geq p \) (or equivalently \( p \leq N/2 \)), the states in both
families are associated with the same set of total spins
specified as \( S = S_z = (k - p)/2 = N/2 - p \). Furthermore,
given a pair \((k, p)\):

(A) The states in family \( A \) have \( \Delta L = p \), with \( \Delta L \)
varying from 0 to \( N/2 \) for even \( N \), and from 0 to \((N - 1)/2\)
for odd \( N \).

(B) The states in family \( B \) have \( \Delta L = k \), with \( \Delta L \)
varying from \( N/2 \) to \( N \) for even \( N \), and from \((N + 1)/2\)
to \( N \) for odd \( N \).

The states in family \( A \) are unique ground states,
whereas those in family \( B \) are part of degenerate manifolds. (This degeneracy is lifted as described above.)

IV. THE EXACT 0IE LLL WAVE FUNCTIONS

1. Mathematical preliminary

The quantity \( k \)-subset (list) is a subset containing exactly \( k \) elements out of the set of \( n \) elements (named list).
The number of \( k \)-subsets on \( n \) elements is given by
\( \binom{n}{k} = \frac{n!}{k!(n-k)!} \). The set represented by list is taken to
be a list of cardinally ordered positive integers. For example, there are 6 2-subsets when list = \{1,2,3,4\}, namely
\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, and \{3,4\}.

2. General form of the 0IE LLL wave functions

The compact algebraic expression has the general form
\[
\Phi_{\text{alg}}(z_1 \chi(1), \ldots, z_N \chi(N)) \propto \Phi_V(l_1, \ldots, l_N; z_1, \ldots, z_N) \\
\times \Phi_{\text{sym}}(z_1 \chi(1), \ldots, z_N \chi(N)),
\]
where \( \chi(i) \) denotes an up spin, \( \alpha \), or down spin, \( \beta \), and
\( i = 1, \ldots, N \).

\( \Phi_V \) is a Vandermonde determinant,
\[
\Phi_V([l]; [z]) = \det[z_i^j] = \prod_{i<j} (z_i - z_j),
\]
where \( l_j = (j - 1) \) and \( i, j = 1, 2, \ldots, N \). The product of Jastrow factors above reflects the fact that the wave
function in Eq. (6) is a 0IE eigenstate of the contact-
interaction term, \( H_{\text{int}} \), in Eq. (1).

Due to the fermionic symmetry of the \( \Phi_{\text{alg}}, \Phi_{\text{sym}} \) has
to be symmetric under the exchange of any pair of indices
\( i \) and \( j \). Furthermore, \( \Phi_{\text{sym}} \) can be written as
\[
\Phi_{\text{sym}}(z_1 \chi(1), \ldots, z_N \chi(N)) = \sum_{m=1}^{K} \mathcal{P}_m^n[z] Z_m,
\]
where \( \mathcal{P}_m^n \) (defined below) are homogeneous multivariate polynomials of order \( \alpha = p \) (family \( A \)) or \( \alpha = k \) (family \( B \)), and
\[
Z_m = \alpha(i_1)\alpha(i_2)\ldots\alpha(i_K)\beta(j_{k+1})\ldots\beta(j_N),
\]
is one of the \( K = N!/(k!p!) \) distinct spin primitives having
\( k \leq N \) up and \( p = N - k \) down spins. The set of indices \( \{i_1, \ldots, i_k\} \) is the \( m \)-th element \((m = 1, 2, \ldots, K)\)
of the \( k \)-subsets of the cardinal list (top-level list, see
below) specified as list = \{1,2,...,N\}. The set of indices
\( \{j_{k+1}, \ldots, N\} \) is complementary to the \( \{i_1, \ldots, i_k\} \) set.

The \( \Phi_V \) [Eq. (7)] corresponds to a filling factor \( \nu = 1 \),
whereas the filling fraction corresponding to Eq. (6) with \( \Phi_{\text{sym}} \) given in Eq. (8) through polynomials of order \( \alpha \) is
near \( \nu = 1 \). These fractions are indeed the ones most
likely to be accessed first in upcoming experiments [39].

3. Algebraic expressions for the polynomials \( \mathcal{P}_m^n([z]) \)

For each \( S = S_z = (k - p)/2 \), except when \( k = p \) which
has a single state, there exists a pair of targeted LLL
states, with one state of the pair belonging to family \( A \)
and the other to family \( B \) (see Fig. 1 for an example).

Family \( A \): First, the following square matrices of rank
\( p \) (the number of spin-down fermions) need to be considered:
\[
M_{q,m} = \begin{bmatrix}
z_{i_1} - z_{j_{k+1}} & \cdots & z_{i_1} - z_{j_N} \\
\vdots & \ddots & \vdots \\
z_{i_p} - z_{j_{k+1}} & \cdots & z_{i_p} - z_{j_N}
\end{bmatrix},
\]
where the dummy indices \( i_1, \ldots, i_p \) here are associated with
spin-up fermions, and the set \( \{i_1, \ldots, i_p\} \) denotes the \( q \)-th subset among the \( p \)-subsets on a second-level list-2,
with list-2 being the \( m \)-th element among the \( k \)-subsets
on the \{1,2,...,N\} top-level list. The number of \( p \)-subsets of any second-level list-2 is \( K_2 = k!/((p!(k-p)!)) \),
and thus the \( q \)-subscript runs from 1 to \( K_2 \). The set of indices
\( \{j_{k+1}, \ldots, j_N\} \) is complementary to the \( \{i_1, \ldots, i_k\} \) set,
and thus it remains constant for a given \( m \) index in
the matrices defined in Eq. (10). (Recall that $k$ is the total number of spin-up fermions, and that $\{i_1, \ldots, i_k\}$ is also referred to as a second-level list.)

The expression for the polynomial is given by

$$ P^p_m([z]) = \sum_{q=1}^{K_2} \text{Perm}[M_{q,m}], \tag{11} $$

where the symbol "Perm" denotes a Permanent.

The analytic expressions of the states with $S_z < S$, in a given spin multiplicity $2S+1$, are obtained by repeated application of the spin lowering operator.

**Example.** We consider the state associated with $N = 5$, $S = S_z = 1/2$, and $L = 12$. Note that $L_0 = N(N + 1)/2 = 10$ in the corresponding fully polarized case. There are $K = 5!/(3!2!) = 10$ spin primitives $Z_m$, with $m = 1, 2, \ldots, 10$; they correspond to the ten 3-subsets on the top-level list $\{1, 2, 3, 4, 5\}$, i.e., $\{1, 2, 3\}$ ($m = 1$), $\{1, 2, 4\}$ ($m = 2$), $\{1, 2, 5\}$ ($m = 3$), $\{1, 3, 4\}$ ($m = 4$), $\{1, 3, 5\}$ ($m = 5$), $\{1, 4, 5\}$ ($m = 6$), $\{2, 3, 4\}$ ($m = 7$), $\{2, 3, 5\}$ ($m = 8$), $\{2, 4, 5\}$ ($m = 9$), $\{3, 4, 5\}$ ($m = 10$).

Here $k = 3$, $p = 2$, and there are $K_2 = 3$ 2-subsets for each $(m)$-th 3-subset listed above. $K_2 = 3$ is also the number of permanents entering in expression (11), i.e., $q = 1, \ldots, 3$. Choosing $m = 10$ as an example, the three 2-subsets are $\{3, 4\}$, $\{3, 5\}$, and $\{4, 5\}$, and the three associated matrices $M_{q,10}$ are given by:

$$ M_{q,10} = \begin{bmatrix} \eta(q, 1) - z_1 & \eta(q, 1) - z_2 \\ \eta(q, 2) - z_1 & \eta(q, 2) - z_2 \end{bmatrix}, \tag{12} $$

with $q = 1, 2, 3$; $\eta(1, 1) = z_2$, $\eta(1, 2) = z_4$, $\eta(2, 1) = z_3$, $\eta(2, 2) = z_5$, and $\eta(3, 1) = z_4$, $\eta(3, 2) = z_5$.

An additional example is presented in the SM [59].

**Family B:** Similarly, we found that the symmetric polynomials $P^p_m([z])$ related to the ground states of family $B$ consist always [for any $m$ in the summation of Eq. (8)] of a single permanent associated with a matrix of rank $k$ (the number of spin-up fermions). Namely

$$ P^p_m([z]) = \text{Perm}[M^B_m], \tag{13} $$

with

$$ M^B_m = \begin{bmatrix} z_{i_1} - z_{j_{k+1}} & \cdots & z_{i_1} - z_{j_N} & z_{i_1} - z_{j_{N+1}} & \cdots & z_{i_1} - z_{j_{2k}} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ z_{i_k} - z_{j_{k+1}} & \cdots & z_{i_k} - z_{j_N} & z_{i_k} - z_{j_{N+1}} & \cdots & z_{i_k} - z_{j_{2k}} \end{bmatrix}. \tag{14} $$

Above, the set of indices $\{i_1, \ldots, i_k\}$ is the $m$th element of the $k$-subsets associated with the spin-up fermions [see Eq. (9)]. Because $k > p$, the complimentary set of the $p$ spin-down indices $\{j_{k+1}, \ldots, j_N\}$ has been expanded to contain exactly $k$ elements, through the introduction of virtual fermion coordinates such that $z_{j_s} = 0$ for all $s > N$; see specific matrices $M^B_m$, as well as a comparison with the wave functions in Ref. [45], in the Appendix and the SM [59].

Note that the first quasi-hole state (1QH) [41, 75] coincides with the analytic expression associated with family $B$ above for $L = L_0 + N$.

**V. HIGHER-ORDER CORRELATIONS**

We used the analytic wave functions above to calculate spin-unresolved higher-order correlations for $N = 19$, 25, and 27 fermions; see Fig. 2 (for completeness, see Fig. SF3 for $N = 9$ in the SM [59]). The $n$-body correlations for spinful fermions were defined in detail in Sec. II C of Ref. [38]. For the $N$-body $\Phi_{\text{alg}}$ [Eq. (6)], the spatial $n$-body correlation is given in a compact form by

$$ G^n(N) = (1 - \delta_{n,N}) \int |\Phi_{\text{alg}}(z_1\chi(1), \ldots, z_N\chi(N))|^2 \times dz_{n+1}d\chi(n+1) \ldots dz_Nd\chi(N) + \delta_{n,N}|\Phi_{\text{alg}}|^2, \tag{15} $$
with $n = 2, \ldots, N$. $G^n(N)$ gives the conditional probability to find particles $n, \ldots, N$ anywhere, for prespecified (fixed) locations of particles $1, \ldots, n - 1$ with pre-determined (resolved) or unspecified (unresolved) spins.

For $N = 19$. Figs. 2(a,b,c) display structured $N$th-order correlations for the spin state with $S = S_z = 1/2$ and total angular momentum $L = 190$. Extending Ref. [38], we found similar crystalline structures also in the $N$th-order correlations of the associated fully polarized, single VDdet state with $S = 19/2$, $S_z = 19/2$, and $L_0 = 171$ (Pauli-exclusion-only case, experimentally investigated [28]). Fig. 2(d) displays the structured $N$th-order correlations for $N = 25$ with $L = 301$ and $S = S_z = 23/2$, whereas Fig. 2(e) presents the structured $N$th-order correlations for the 1QH for $N = 27$ (with $L = 378$ and $S = S_z = 27/2$). The implied intrinsic geometric structure (UCWM) in Fig. 2 is a polygonal triple ring $(n_1, n_2, n_3)$ of localized fermions (with $n_1 + n_2 + n_3 = N$); specifically (1,6,12), (3,9,13), and (4,9,14) for Figs. 2(a,b,c), Fig. 2(d), and Fig. 2(e), respectively. We note that in the LLL neighborhood of $\nu = 1$ (expected in experiments with trapped ultracold fermions [39]), the intrinsic ring geometry can be probed only with higher-order correlations. Indeed in this case, the second-order correlations are structureless; see the findings for $N = 4$ [[0,4) single ring] and $N = 6$ [[1,5) double ring] in Ref. [38].

VI. CONCLUSION

A novel approach for deriving exact closed-form analytic expressions for the wave functions (beyond the Jastrow-factors paradigm) of an assembly of 2D contact-alytic expressions for the wave functions (beyond the double ring) in Ref. [38]. The polynomial in Eq. (A.3) is clearly different from the MFB one [Eq. (A.2)]. We verified that the wave functions derived here are eigenfunctions of the square, $S^2$, of the total-spin operator [with eigenvalue 15/4 and $S = 3/2$ for the case in this Appendix], whereas the MFB ones are not (see also Ref. [51]); for details see Ref. [59].
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SUPPLEMENTAL MATERIAL

Exact closed-form analytic wave functions in two dimensions: Contact-interacting fermionic spinful ultracold atoms in a rapidly rotating trap

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Comparison with symmetric polynomials for skyrmions in previous literature

We compare here with the symmetric polynomials for the seed skyrmions specified in Eq. (6) of (R1) A. H. MacDonald, H. A. Fertig, and Luis Brey, Skyrmions without Sigma Models in Quantum Hall Ferromagnets, Phys. Rev. Lett. 76, 2153 (1996) (https://doi.org/10.1103/PhysRevLett.76.2153); see also Eq. (2) in (R2) R.K. Kamilla, X.G. Wu, and J.K. Jain, Skyrmions in the fractional quantum Hall effect, Solid State Commun. 99, 289 (1996) (https://doi.org/10.1016/0038-1098(96)00126-3).

Omitting the trivial Gaussian functions, these polynomials are given by the single formula

$$\Phi_p^{sk,MFB} = \sum_{m=1}^{K} z_{i_1} z_{i_2} \cdots z_{i_k} Z_m, \quad (S5)$$

where $Z_m$ are the spin primitives defined in Eq. (9) of the main text, i.e.,

$$Z_m = \alpha(i_1)\alpha(i_2) \cdots \alpha(i_k)\beta(j_{k+1}) \cdots \beta(j_N). \quad (S6)$$

The index $m$ runs also over the $k$-subsets $\{i_1, \ldots, i_k\}$ of the list $\{1, 2, 3, \ldots, N\}$, $k = N \uparrow$ being the number of spin-up fermions, with $p = N - k = N \downarrow$ being that of the spin-down fermions. The number of $k$-subsets is $K = N!/(k!p!(N-k)!)$.

As is the case in Ref. R1, one can take the index $m$ as running over the $p$-subsets associated with the spin-down fermions, because there is a one-to-one correspondence to the $k$-subsets of the spin-up fermions. Note that Ref. R1 (Ref. R2) uses the capital letter $K$ ($R$) in place of our $p$.

We present here a comparison for the case $N = 5$, $k = 4$, $p = 1$, $S = S_z = 3/2$, and $\Delta L = 4$, a state belonging to family B in our exposition.

According to Eq. (S5), the corresponding MacDonald-Fertig-Brey (MFB) symmetric polynomial is

$$\Phi_p^{sk,MFB} = z_{1z} z_{2z} z_{3z} Z_1 + z_{1z} z_{2z} z_{5z} Z_2 + z_{1z} z_{4z} z_{5z} Z_3 + z_{1z} z_{3z} z_{5z} Z_4 + z_{2z} z_{3z} z_{5z} Z_5. \quad (S7)$$

The corresponding polynomial derived in this paper is given by Eqs. (8) and (13) in the main text. Expanding the permanent, one obtains for the polynomial with $k = 4$ and $m = 1$ (in front of the $Z_1$ spin primitive):

$$P_1^4[z] = 4z_{1z} z_{2z} z_{3z} z_{4z} - z_{1z} z_{2z} z_{4z} z_{5z} - z_{1z} z_{3z} z_{4z} z_{5z} - z_{1z} z_{3z} z_{3z} z_{5z}, \quad (S8)$$

For the $P_2^4$ polynomial in front of $Z_2$, one obtains similarly:

$$P_2^4[z] = -z_{1z} z_{2z} z_{3z} z_{4z} + 4z_{1z} z_{2z} z_{3z} z_{5z} - z_{1z} z_{3z} z_{4z} z_{5z} - z_{2z} z_{3z} z_{4z} z_{5z}. \quad (S9)$$

In general, one has

$$\Phi_{sym}^{exact}(N = 5, N \uparrow = 4, \Delta L = 4) = \sum_{m=1}^{5} P_m^4[z] Z_m, \quad (S10)$$

with

$$P_m^4[z] = c_1 z_{1z} z_{2z} z_{3z} z_{4z} + c_2 z_{1z} z_{2z} z_{3z} z_{5z} + c_3 z_{1z} z_{2z} z_{4z} z_{5z} + c_4 z_{1z} z_{3z} z_{4z} z_{5z} + c_5 z_{2z} z_{3z} z_{4z} z_{5z}, \quad (S11)$$

and $c_i = 4$ when $i = m$ and $c_i = -1$ otherwise.

We note that the expressions associated with the $Z_i$, $i = 1, \ldots, 5$ in the MFB polynomial consist only of a single term with a numerical factor $+1$ in front. This contrasts with our expressions in Eqs. (S8), (S9), and (S11) which have five terms each with factors of $+4$ and $-1$ in front of them.

For $p$ (spin-down fermions) $> k$ (spin-up fermions), the MFB expression in Eq. (S5) is associated with a negative total-spin projection $S_z = (k-p)/2 < 0$. In this case, the indices for the corresponding wave function in this paper are found by reversing all $N$ spins, i.e., by considering the case with $p \rightarrow k$, $k \rightarrow p$, and $S_z = \lceil(k-p)/2\rceil$.

Using our algebraic scripts, we readily verified that the wave functions derived in this work are indeed eigenfunctions of the total-spin square operator [with eigenvalue $15/4$ and $S = 3/2$ for the case in this section], whereas the MFB ones are not (as indeed has been discussed by M. Abolfath et al., Phys. Rev. B 56, 6795 (1997) (https://doi.org/10.1103/PhysRevB.56.6795).

In particular, applying the spin-square, $\hat{S}^2$, operator, one gets

$$\left(\hat{S}^2 - \frac{15}{4}\right)\Phi_{sym}^{exact}(N = 5, N \uparrow = 4, \Delta L = 4) = 0. \quad (S12)$$
On the contrary, for the MFB wave function, one gets
\[
\left( S^2 - \frac{15}{4} \right) \Phi^{sk, MFB}_{p=1} = (z_1z_2z_3z_4 + z_1z_2z_3z_5 + z_1z_2z_4z_5 + z_1z_4z_4z_5 + z_2z_3z_4z_5) \sum_{m=1}^5 Z_m.
\] (S13)

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**Additional examples for the wave functions in families A and B**

**Family A.** As another example from family A, we consider the spin singlet state associated with \( N = 4, S = 0, S_z = 0 \), and \( L = 8 \). Note that \( L_0 = N(N - 1)/2 = 6 \) in the corresponding fully spin-polarized case. There are \( K = 4!(2!2!) = 6 \) spin primitives \( Z_m \), with \( m = 1, 2, \ldots, 6 \); they correspond to the six 2-subsets on the top-level list \( \{1, 2, 3, 4\} \), i.e., \( \{1, 2\} \) (\( m = 1 \), \( \{1, 3\} \) (\( m = 2 \), \( \{1, 4\} \) (\( m = 3 \), \( \{2, 3\} \) (\( m = 4 \), \( \{2, 4\} \) (\( m = 5 \), \( \{3, 4\} \) (\( m = 6 \).

Because \( k = 2 \) and \( p = 2 \), there is only one (\( K_2 = 1 \)) 2-subset for each (\( m \)th) 2-subset listed above. \( K_2 = 1 \) is also the number of permanents entering in expression (11) of the main text, i.e., the index \( q \) takes only the value of one. Choosing \( m = 2 \) as an example, the single matrix \( M_{1,2} \) is:

\[
M_{1,2} = \begin{bmatrix}
z_1 - z_2 & z_1 - z_4 \\
z_3 - z_2 & z_3 - z_4
\end{bmatrix}.
\] (S14)

**Family B. First example.** As a first example from family B, we consider the spin \( S = S_z = 3/2 \) state associated with \( N = 9 \) and \( L = 42 \). Note that \( L_0 = N(N - 1)/2 = 36 \). There are \( k = 6 \) up spins, \( p = 3 \) down spins, and \( K = 9!/6(3!) = 84 \) spin primitives \( Z_m \), with \( m = 1, 2, \ldots, 84 \).

Focusing on the \( m = 1 \) spin primitive, we have a subset of indices \( \{i_1, i_2, \ldots, i_6\} \) for the spin up fermions and a subset of indices \( \{j_7, j_8, j_9\} \) for the spin down fermions. The subset of the spin down indices needs to be augmented by introducing three additional indices \( \{j_{10}, j_{11}, j_{12}\} \), which specify virtual fermions with \( z_{j_{10}} = z_{j_{11}} = z_{j_{12}} = 0 \). Then the corresponding matrix \( M_1^B \) [see Eq. (14) in the main text] is given by

\[
M_1^B = \begin{bmatrix}
z_{i_1} - z_{j_7} & z_{i_1} - z_{j_8} & z_{i_1} - z_{j_9} & z_{i_1} & z_{i_1} & z_{i_1} \\
z_{i_2} - z_{j_7} & z_{i_2} - z_{j_8} & z_{i_2} - z_{j_9} & z_{i_2} & z_{i_2} & z_{i_2} \\
z_{i_3} - z_{j_7} & z_{i_3} - z_{j_8} & z_{i_3} - z_{j_9} & z_{i_3} & z_{i_3} & z_{i_3} \\
z_{i_4} - z_{j_7} & z_{i_4} - z_{j_8} & z_{i_4} - z_{j_9} & z_{i_4} & z_{i_4} & z_{i_4} \\
z_{i_5} - z_{j_7} & z_{i_5} - z_{j_8} & z_{i_5} - z_{j_9} & z_{i_5} & z_{i_5} & z_{i_5} \\
z_{i_6} - z_{j_7} & z_{i_6} - z_{j_8} & z_{i_6} - z_{j_9} & z_{i_6} & z_{i_6} & z_{i_6}
\end{bmatrix}.
\] (S15)

**Family B. Second example.** As a second example from family B, we consider the 1QH state for \( N = 9 \). In this case, \( S = S_z = 9/2 \) and \( L = 45 \). Note that \( L_0 = N(N - 1)/2 = 36 \). There are \( k = 9 \) up spins, \( p = 0 \) down spins, and \( K = 9!/9(0!) = 1 \) spin primitive \( Z_1 \).

For this single spin primitive, we have a subset of indices \( \{i_1, i_2, \ldots, i_9\} \) for the spin up fermions and an empty subset of indices for the spin down fermions. The subset of the spin down indices needs to be augmented by introducing nine additional indices \( \{j_1, j_2, \ldots, j_9\} \), which specify virtual fermions with \( z_{j_1} = z_{j_2} = \ldots = z_{j_9} = 0 \). Then the corresponding matrix \( M_1^B \) [see Eq. (14) in the main text] is given by

\[
M_1^B = \begin{bmatrix}
z_{i_1} & z_{i_1} & z_{i_1} & z_{i_1} & z_{i_1} & z_{i_1} & z_{i_1} & z_{i_1} & z_{i_1} \\
z_{i_2} & z_{i_2} & z_{i_2} & z_{i_2} & z_{i_2} & z_{i_2} & z_{i_2} & z_{i_2} & z_{i_2} \\
z_{i_3} & z_{i_3} & z_{i_3} & z_{i_3} & z_{i_3} & z_{i_3} & z_{i_3} & z_{i_3} & z_{i_3} \\
z_{i_4} & z_{i_4} & z_{i_4} & z_{i_4} & z_{i_4} & z_{i_4} & z_{i_4} & z_{i_4} & z_{i_4} \\
z_{i_5} & z_{i_5} & z_{i_5} & z_{i_5} & z_{i_5} & z_{i_5} & z_{i_5} & z_{i_5} & z_{i_5} \\
z_{i_6} & z_{i_6} & z_{i_6} & z_{i_6} & z_{i_6} & z_{i_6} & z_{i_6} & z_{i_6} & z_{i_6} \\
z_{i_7} & z_{i_7} & z_{i_7} & z_{i_7} & z_{i_7} & z_{i_7} & z_{i_7} & z_{i_7} & z_{i_7} \\
z_{i_8} & z_{i_8} & z_{i_8} & z_{i_8} & z_{i_8} & z_{i_8} & z_{i_8} & z_{i_8} & z_{i_8} \\
z_{i_9} & z_{i_9} & z_{i_9} & z_{i_9} & z_{i_9} & z_{i_9} & z_{i_9} & z_{i_9} & z_{i_9}
\end{bmatrix}.
\] (S16)

When expanded, the associated permanent yields one term only, namely, \( 9!(z_{i_1}z_{i_2}z_{i_3}z_{i_4}z_{i_5}z_{i_6}z_{i_7}z_{i_8}z_{i_9}) \), and thus the 1QH state here agrees with the analytic form introduced by Laughlin; see https://doi.org/10.1103/PhysRevLett.50.1395.
TABLE STI. The 15 dominant numerical CI coefficients, $c_{CI}(J)$, in the CI expansion of the relative LLL ground state, and the corresponding extracted algebraic ones, $c_{alg}(J)$, for $N = 4$ fermions with total angular momentum $L = 8$ (first 0IE state) in the $(S = S_z = 0)$ spin sector of singlet states. The spinful-fermions Slater determinants are specified through the set of single-particle angular momenta and spins, $(l_1 \uparrow, l_2 \uparrow, l_3 \downarrow, l_4 \downarrow)$. The converged CI expansion included a much larger set of basis Slater determinants, but naturally the dominant ones are only relevant, namely those with coefficients $|c_{CI}(I)| > 0.001$.

| $J$ | $c_{CI}(J)$ | $c_{alg}(J)$ | $(l_1 \uparrow, l_2 \uparrow, l_3 \downarrow, l_4 \downarrow)$ | $\sum_{i=1}^{4} l_i$ |
|-----|-------------|--------------|---------------------------------|----------------|
| 1   | -0.28005598 | $\frac{7}{\sqrt{61}}$ | (0,1,3,4) | 8 |
| 2   | 0.34299719  | $-\sqrt{\frac{2}{17}}$ | (0,2,2,4) | 8 |
| 3   | -0.14002799 | $\frac{1}{\sqrt{21}}$ | (0,3,1,4) | 8 |
| 4   | -0.2970427  | $\frac{3}{\sqrt{34}}$ | (0,2,2,4) | 8 |
| 5   | 0.14002799  | $-\sqrt{\frac{1}{21}}$ | (1,2,1,4) | 8 |
| 6   | -0.24253563 | $\frac{1}{\sqrt{34}}$ | (1,2,2,3) | 8 |
| 7   | -0.17149861 | $-\sqrt{\frac{1}{7}}$ | (1,3,0,4) | 8 |
| 8   | 0.14002799  | $\frac{1}{\sqrt{34}}$ | (1,3,1,3) | 8 |
| 9   | 0.42068405  | $-\sqrt{\frac{3}{17}}$ | (1,4,0,3) | 8 |
| 10  | -0.14002799 | $\frac{1}{\sqrt{34}}$ | (1,4,1,2) | 8 |
| 11  | -0.24253563 | $\frac{1}{\sqrt{34}}$ | (1,4,1,2) | 8 |
| 12  | -0.2970427  | $\frac{3}{\sqrt{34}}$ | (2,3,0,3) | 8 |
| 13  | -0.17149861 | $-\sqrt{\frac{1}{7}}$ | (2,3,1,2) | 8 |
| 14  | 0.34299719  | $-\sqrt{\frac{1}{7}}$ | (2,4,0,2) | 8 |
| 15  | -0.28005598 | $\frac{7}{\sqrt{61}}$ | (3,4,0,1) | 8 |

TABLE STII. A sample of the 1551 dominant numerical CI coefficients, $c_{CI}(J)$, in the CI expansion of the relative LLL ground state, and the corresponding extracted algebraic ones, $c_{alg}(J)$, for $N = 9$ fermions with total angular momentum $L = 40$ (first 0IE state) in the $(S = S_z = 1/2)$ spin sector. The spinful-fermions Slater determinants are specified through the set of single-particle angular momenta and spins, $(l_1 \uparrow, l_2 \uparrow, l_3 \uparrow, l_4 \uparrow, l_5 \uparrow, l_6 \downarrow, l_7 \downarrow, l_8 \downarrow, l_9 \downarrow)$. The converged CI expansion included a much larger set of basis Slater determinants, but naturally the dominant ones are only relevant, namely those with coefficients $|c_{CI}(J)| > 0.001$.

| $J$ | $c_{CI}(J)$ | $c_{alg}(J)$ | $(l_1 \uparrow, l_2 \uparrow, l_3 \uparrow, l_4 \uparrow, l_5 \uparrow, l_6 \downarrow, l_7 \downarrow, l_8 \downarrow, l_9 \downarrow)$ | $\sum_{i=1}^{9} l_i$ |
|-----|-------------|--------------|---------------------------------|----------------|
| 1   | -0.10319005 | $4\sqrt{\frac{5}{6717}}$ | (0,1,2,3,4,6,7,8,9) | 40 |
| 2   | 0.11303906  | $-4\sqrt{\frac{6}{5617}}$ | (0,1,2,3,5,7,8,9) | 40 |
| 3   | -0.10063800 | $\frac{4}{\sqrt{37565}}$ | (0,1,2,3,6,4,7,8,9) | 40 |
| 4   | -0.10465386 | $\frac{6}{\sqrt{37565}}$ | (0,1,2,3,6,5,6,8,9) | 40 |
| 5   | 0.02063800  | $\frac{1}{\sqrt{37565}}$ | (0,1,2,3,7,4,6,8,9) | 40 |
| 6   | 0.09789477  | $\frac{1}{\sqrt{26295}}$ | (0,1,2,3,7,4,5,6,7,9) | 40 |
| ... | ...         | ...         | ...                            | ... |
| 1546 | 0.00872117  | $-\frac{2}{\sqrt{26295}}$ | (4,5,6,7,9,0,1,3,5) | 40 |
| 1547 | 0.00827362  | $-3\sqrt{\frac{6}{202895}}$ | (4,5,6,7,9,0,2,3,4) | 40 |
| 1548 | -0.02136239 | $2\sqrt{\frac{6}{55635}}$ | (4,5,6,8,9,0,1,2,5) | 40 |
| 1549 | -0.01103148 | $\frac{4}{\sqrt{202895}}$ | (4,5,6,8,9,0,1,3,4) | 40 |
| 1550 | 0.02527631  | $-2\sqrt{\frac{6}{55635}}$ | (4,5,7,8,9,0,1,2,4) | 40 |
| 1551 | -0.02063801 | $\frac{4}{\sqrt{37565}}$ | (4,6,7,8,9,0,1,2,3) | 40 |
FIG. SF3. Higher-order correlations for $N = 9$ ultracold LLL fermions. (a-d) 9th-order correlations for the 0IE lowest-spin state ($S = 1/2, S_z = 1/2$) with total angular momentum $L = 40$. Similar 9th-order correlations are found for the fully polarized state ($S = S_z = 9/2$) with $L_0 = 36$. (e,f) 8th-order correlations for the fully polarized state ($S = 9/2, S_z = 9/2$) with $L_0 = 36$, whose space part is a pure Vandermonde determinant. (g,h) 7th-order correlations for the fully polarized state ($S = 9/2, S_z = 9/2$) with $L_0 = 36$. The implied intrinsic geometric structure (ultracold Wigner molecule) is a (2,7) double ring, with 2 fermions in the inner ring and 7 fermions in the outer ring. The fixed fermions are marked by solid dots (black for those in the outer ring and red for those in the inner ring). The white numbers denote the remaining, beyond the fixed ones, fermions. Vertical axes: arbitrary units.