Abstract In this paper, we develop a Bernstein dual-Petrov-Galerkin method for the numerical simulation of two-dimensional subdiffusion equation. The equation is first discretized in time using the L1 approximation. Then, a spectral discretization is applied by introducing suitable combinations of dual Bernstein polynomials as the test functions and the Bernstein polynomials as the trial ones. We derive the exact sparse operational matrix of differentiation for the dual Bernstein basis which provides a matrix based approach for the spatial discretization. It is also shown that the proposed method leads to banded linear systems that can be solved efficiently. Finally, the stability and convergence of the proposed method is discussed theoretically. Some numerical examples are provided to support the theoretical claims and to show the accuracy and efficiency of the method.

Keywords Fractional PDEs · Bernstein polynomials · 2D Subdiffusion · dual-Petrov-Galerkin · Dual Bernstein basis · Operational matrix

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1 Introduction

Bernstein polynomial basis plays an important role in computer graphics for geometric modeling, curve and surface approximation. Some interesting features have been investigated for this basis in the last decades; for instance, it is proven to be an optimal stable basis among nonnegative bases in a sense described in [7]. Also, it provides some optimal shape preserving features [2]. We refer to [4] for detailed properties and applications in computer aided geometric design (CAGD).

Bernstein basis has also been used for the numerical solution of differential, integral, integro-differential and fractional differential equations, see e.g. [1, 12, 19, 24] and the references therein. However, it is not orthogonal and so leads to dense linear systems when using in numerical methods. Some numerical approaches implement the orthogonalized Bernstein basis. However, as we will see in the next section, it fails to keep some interesting properties of the Bernstein basis. Another approach uses the dual Bernstein polynomials (DBP) introduced by Juttler in 1998 [16]. To the best of our knowledge, the DBP basis has
been only discussed from the CAGD point of view (see the works of Lewanowicz and Wozny e.g. [17, 29]). So it is of interest to explore some new aspects of this basis in order to facilitate the numerical methods for differential equations that are based on Bernstein polynomials and to present a method for time fractional diffusion equation in two dimensions.

Fractional partial differential equations (FPDEs) have been widely used for the description of some important physical phenomena in many applied fields including viscoelastic materials, control systems, polymer, electrical circuits, continuum and statistical mechanics, etc. The subdiffusion equation is a FPDE describing the behavior of anomalous diffusive systems with the probability density of particles diffusing proportional to the mean square displacement \( \chi^2(t) \propto t^\alpha \) with \( 0 < \alpha < 1 \) [8]. Anomalous diffusion equations have been used for modeling transport dynamics, especially the continuous time random walk, the contamination in porous media, viscoelastic diffusion, etc [8, 9, 10, 20, 28]. For the numerical solution of the one-dimensional problem, we refer to [13, 23, 27, 36] and the references therein. Some classic numerical methods for PDEs have been developed for the simulation of two-dimensional subdiffusion equation, for example the finite difference schemes [9, 21, 23], meshless methods [26, 30], finite element method [35], alternating direction implicit methods [33, 34], etc.

In this paper, deriving some new aspects of DBPs, we present suitable combinations of these functions in order to develop a dual-Petrov-Galerkin method for solving the following 2D subdiffusion equation [28, 30, 31, 33, 34, 35]

\[
D_{t}^{\alpha} u (x, y, t) = \kappa \Delta u (x, y, t) + S(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \tag{1.1}
\]

with the following initial and boundary conditions

\[
\begin{align*}
  u(x, y, 0) &= g(x, y), \quad (x, y) \in \Omega, \tag{1.2} \\
  u(x, y, t) &= 0, \quad (x, y, t) \in \partial \Omega \times (0, T], \tag{1.3}
\end{align*}
\]

where \( \Omega = (0, 1)^2 \subset \mathbb{R}^2 \), \( \Delta \) is the Laplacian operator, \( T > 0 \), \( \kappa \) is the diffusion coefficient and \( S \) is the source function. Here, \( D_{t}^{\alpha} u \) denotes the Caputo fractional derivative of order \( \alpha \), \( 0 < \alpha < 1 \), with respect to \( t \) defined as

\[
D_{t}^{\alpha} u (x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \frac{\partial u (x, s)}{\partial s} ds, \quad 0 < \alpha < 1. \tag{1.4}
\]

The main contribution of our work is the development of an accurate Bernstein dual-Petrov-Galerkin method and the application for the numerical simulation of the 2D subdiffusion equation. It is shown the method leads to sparse linear systems. To give a matrix approach of the method, we present some results concerning the DBPs including a recurrence formula for the derivative, constructing a new basis using DBPs, deriving the operational matrix of differentiation and also providing the transformation matrices between the DBPs and the new basis.

The paper is organized as follows: Section 2 presents some new aspects of DBPs and provides modal basis functions and the associated transformation matrices between the bases. Section 3 is devoted to the Bernstein-spectral formulation of the subdiffusion problem (1.1)-(1.3) and the stability and convergence results are discussed in Section 4. Numerical examples are provided in Section 5. The paper ends with some concluding remarks in Section 6.

### 2 Bernstein polynomials and DBPs

The Bernstein polynomials with degree \( N \) on the unit interval are defined by

\[
\phi_i(x) = \binom{N}{i} x^i (1 - x)^{N-i}, \quad 0 \leq i \leq N.
\]
The set $\{\phi_i(x) : i = 0, \ldots, N\}$ forms a basis for $\mathbb{P}_N$, the space of polynomials of degree not exceeding $N$. These polynomials possess end-point interpolation property, i.e.,
\begin{equation}
\phi_i(0) = \delta_{i,0}, \quad \phi_i(1) = \delta_{i,N}, \quad 0 \leq i \leq N, \quad N > 0.
\end{equation}

Also, the summation is one and the integral over the unit interval is constant, namely
\begin{equation}
\sum_{i=0}^{N} \phi_i(x) \equiv 1, \quad \int_0^1 \phi_i(x) = \frac{1}{N+1}, \quad i = 0, 1, \ldots, N.
\end{equation}

The derivative enjoys the three-term recurrence relation [11]
\begin{equation}
\phi'_i(x) = (N - i + 1) \phi_{i-1}(x) - (N - 2i) \phi_i(x) - (i + 1) \phi_{i+1}(x), \quad 0 \leq i \leq N,
\end{equation}
where we adopt the convention that $\phi_i(x) \equiv 0$ for $i < 0$ and $i > N$.

As we mentioned in the preceding section, the Bernstein basis is not orthogonal. The corresponding orthogonalized basis, obtained e.g., by the Gram-Schmidt process fails to keep some interesting aspects of the original basis. We will not consider this basis in the present work. Instead we turn to the dual basis.

The DBPs are defined as
\begin{equation}
\tilde{\psi}_i(x) = \sum_{j=0}^{N} c_{i,j} \phi_j(x),
\end{equation}
with the coefficients given by
\begin{equation}
c_{i,j} = \frac{(-1)^{i+j}}{N(r)!} \sum_{r=0}^{\min(i,j)} (2r+1) \left( \frac{N+r+1}{N} \right)^{N-i} \left( \frac{N-r}{N-j} \right)^{N-j}.
\end{equation}

It is verified that they satisfying the biorthogonal system [16, Theorem 3]
\begin{equation}
\int_0^1 \phi_i(x) \tilde{\psi}_j(x) dx = \delta_{ij}, \quad 0 \leq i, j \leq N.
\end{equation}

It is worth noting that less than a quarter of the entries of transformation matrix between the Bernstein and dual Bernstein basis $C = [c_{i,j}]$, are to be computed directly by (2.5); for it is bisymmetric, i.e., symmetric about both of the main diagonal and antidiagonal.

Another property which is used later is that the sum of the entries for each row (column) is equal to the order of the matrix, i.e.,
\begin{equation}
\sum_{i=0}^{N} c_{i,j} = \sum_{j=0}^{N} c_{i,j} = N + 1.
\end{equation}

In the next lemma, we present some properties of the DBPs.

**Lemma 1** Let $N$ be a fixed nonnegative integer. The following statements hold.
\begin{enumerate}
\item[(i)] For all $x \in [0, 1]$, $\tilde{\psi}_{N-i}(x) = \tilde{\psi}_i(1-x)$.
\item[(ii)] For all $x \in [0, 1]$, $\sum_{i=0}^{N} \tilde{\psi}_i(x) = N + 1$.
\item[(iii)] The basis functions have the same definite integral, i.e., $\int_0^1 \tilde{\psi}_i(x) dx = 1$, $0 \leq i \leq N$.
\end{enumerate}
Proof The first statement is an immediate consequence of the similar formula for Bernstein polynomials, i.e., \( \phi_{N-i}(x) = \phi_i(1-x) \). From (2.4), (2.7) and (2.2), we have

\[
\sum_{i=0}^{N} \tilde{\psi}_i (x) = \sum_{i=0}^{N} \sum_{j=0}^{N} c_{i,j} \phi_j (x) = \sum_{j=0}^{N} \phi_j (x) \sum_{i=0}^{N} c_{i,j} = N + 1.
\]

statement (iii) is also verified similarly. \( \square \)

The property (i) implies that \( \tilde{\psi}_i \), for \( \left[ \frac{N}{2} \right] + 1 \leq i \leq N \), need not to be computed by (2.4)-(2.5). It especially gives \( \tilde{\psi}_i (0) = \tilde{\psi}_{N-i} (1) \).

2.1 Modal basis functions

One may use compact combinations of orthogonal polynomials as a basis in the Galerkin methods for boundary value problems which leads to sparse linear systems in some problems (see e.g., [15, 32]). Here, we use this idea for the non-orthogonal Bernstein polynomials to present a simple and accurate dual-Petrov-Galerkin spectral method for two-dimensional subdiffusion equation. Compact combinations of the basis functions are referred to as the modal basis functions (see [25, Section 1.3]).

**Proposition 1** Let \( N \geq 2 \) be an integer, \( \{ \tilde{\psi}_i : 0 \leq i \leq N \} \) be the dual Bernstein basis and \( \mathbb{P}_N^0 = \{ u \in \mathbb{P}_N : u(0) = 0, u(1) = 0 \} \). Set

\[
\psi_i (x) = \tilde{\psi}_i (x) + a_i \tilde{\psi}_{i+1} (x) + b_i \tilde{\psi}_{i+2} (x),
\]

for \( 0 \leq i \leq N - 2 \), where

\[
a_i = \frac{2i + 4}{N - i + 1}, \quad b_i = \frac{(i + 2)(i + 3)}{(N - i)(N - i + 1)}.
\]

Then, the polynomials \( \tilde{\psi}_i(x) \) vanish at 0 and 1, so the set \( \{ \psi_i (x) \}_{i=0}^{N-2} \) forms a basis for \( \mathbb{P}_N^0 \).

**Proof** By (2.4) and (2.1), we have

\[
\tilde{\psi}_i (0) = \sum_{j=0}^{N} c_{i,j} B_{j,N} (0) = c_{i,0} = (-1)^i (N + 1) \binom{N + 1}{i + 1}.
\]

From Lemma 1, we infer

\[
\tilde{\psi}_i (1) = \tilde{\psi}_{N-i} (0) = (-1)^{N-i} (N + 1) \binom{N + 1}{i}.
\]
By (2.10) and (2.11), we obtain

\[
\psi_i(0) = (-1)^i(N+1) \left( \binom{N+1}{i+1} - a_i \binom{N+1}{i+2} + b_i \binom{N+1}{i+3} \right) = 0, \\
\psi_i(1) = (-1)^{N-i}(N+1) \left( \binom{N+1}{N-i+1} - a_i \binom{N+1}{N-i} + b_i \binom{N+1}{N-i-1} \right) = 0,
\]

for \(0 \leq i \leq N-2\). It is easy to see \(\{\psi_i(x)\}_{i=0}^{N-2}\) is linearly independent. Since \(\dim P_0^N = N-1\), this set is a basis for \(P_0^N\). This completes the proof. \(\square\)

Figure 2.1 illustrates the DBPs and the modal basis functions for \(6 \leq N \leq 8\). It is seen that the modal basis functions have less values than the corresponding dual functions on the unit interval, expecting less round-off errors.
2.2 Transformation matrices and the operational matrix for derivatives

For \( N \geq 2 \), consider the \((N + 1)\)-vector \( \tilde{\Psi} \) and the \((N - 1)\)-vector \( \Psi \) consisting of dual functions given by (2.4) and the modal basis functions given by (2.8), respectively:

\[
\tilde{\Psi}(\cdot) = [\tilde{\psi}_i(\cdot) : 0 \leq i \leq N]^T, \tag{2.12}
\]
\[
\Psi(\cdot) = [\psi_i(\cdot) : 0 \leq i \leq N - 2]^T. \tag{2.13}
\]

For simplicity, we ignore the dependence of the vectors on variable. First, note that

\[
\Psi = G \tilde{\Psi}, \tag{2.14}
\]

where \( G = [g_{i,j}] \) is an \( (N - 1) \times (N + 1) \) matrix with three diagonals as

\[
g_{i,j} = \begin{cases} 1, & i - j = 0, \\
    a_i, & j = i + 1, \quad 0 \leq i \leq N - 1, 0 \leq j \leq N. \\
    b_i, & j = i + 2, \end{cases}
\]

To derive a formula for the derivative of the modal basis functions, we first prove the following result.

**Lemma 2** The operational matrix for derivative of the DBPs, \( P \) satisfies

\[
\tilde{\Psi}' = P \tilde{\Psi}, \tag{2.15}
\]

where the matrix \( P = [p_{i,j} : 0 \leq i, j \leq N] \) is given by

\[
p_{i,j} = \begin{cases} (-1)^j(N + 1)(N + 1)(i+1) + N\delta_{i,0} + \delta_{i,1}, & j = 0, \\
    -p_{N-i,0}, & j = N, \\
    i, & j = i - 1, j \neq 0, \\
    N - 2i, & j = i, j \neq 0, N \\
    -N + i, & j = i + 1, j \neq N. \end{cases}
\]

**Proof** The DBPs \( \tilde{\Psi} \) is a basis for \( P_N \), so we expand \( \tilde{\psi}'_i(x) \) for \( 0 \leq i \leq N \), as

\[
\tilde{\psi}'_i(x) = \sum_{j=0}^{N} p_{i,j} \tilde{\psi}_j(x).
\]

Integration by parts and (2.3) imply that

\[
p_{i,j} = \int_{0}^{1} \tilde{\psi}'_i(x) B_{j,N}(x) \, dx \\
= \tilde{\psi}_i(1) \delta_{j,N} - \tilde{\psi}_i(0) \delta_{j,0} - \int_{0}^{1} \tilde{\psi}_i(x) \left((N - j + 1)B_{j-1,N}(x) - (N - 2j)B_{j,N}(x) - (j + 1)B_{j+1,N}(x)\right) \, dx.
\]

The biorthogonality (2.6) gives

\[
p_{i,j} = \tilde{\psi}_i(1) \delta_{j,N} - \tilde{\psi}_i(0) \delta_{j,0} - ((N - j + 1)\delta_{i,j-1} - (N - 2j)\delta_{i,j} - (j + 1)\delta_{i,j+1}).
\]

Now, the result is proved by considering (2.10) and (2.11). \( \square \)
Remark 1 The matrix $P$ is a sparse matrix of order $N+1$ with $p_{i,j} = 0$ for $|i-j| > 1$, $j \neq 0, N$; for instance, see the matrix given below.

Corollary 1 Set $\alpha_{i,0} = -(1)^i (N+1)_{i+1} + N \delta_{i,0} + \delta_{i,1}$ for $0 \leq i \leq N$. Then, from (2.15), we infer the following five-term recurrence relation is deduced

$$
\tilde{\psi}_i(x) = \alpha_{i,0} \tilde{\psi}_0(x) + (1 - \delta_{i,1}) \tilde{\psi}_{i-1}(x) + (1 - \delta_{i,0}) (1 - \delta_{i,N}) (N-2i) \tilde{\psi}_i(x)
$$

$$(1 - \delta_{i,N-1}) (N-i) \tilde{\psi}_{i+1}(x) = \alpha_{N-i,0} \tilde{\psi}_{N}(x),$$

where we set $\tilde{\psi}_i \equiv 0$ for $i < 0$ and $i > N$.

We derive the transformation matrices that map the Bernstein and Chebyshev coefficients

Now we derive the transformation matrix that maps the derivative of modal basis functions to DBPs. This facilitates the use of Galerkin method in the next section. In the following, $(p,q) - band$ matrix stands for a matrix with $p$ and $q$ nonzero diagonals below and above the main diagonal, respectively.

Lemma 3 Let the vectors $\Psi$ and $\tilde{\Psi}$ be defined as in (2.12) and (2.13), respectively. Then,

$$
\Psi' = Q \tilde{\Psi},
$$

where $Q$ is an $(N-1) \times (N+1)$, $(1,3) - band$ matrix given by $Q = GP$.

Proof Combining (2.14) with (2.15), implies $Q = GP$. To prove that $Q$ is a $(1,3) - band$ matrix, it is sufficient to show that $q_{i,0} = 0$ for $i > 1$ and $q_{i,N} = 0$ for $i < N-2$,

$$
q_{i,0} = (GP)_{i,0} = p_{i,0} + a_i p_{i+1,0} + b_i p_{i+2,0}
$$

$$= - (\tilde{\psi}_i(0) + a_i \tilde{\psi}_{i+1}(0) + b_i \tilde{\psi}_{i+2}(0)) = - \psi_i(0) = 0,
$$

and for $i < N-2$, by (1)

$$
q_{i,N} = p_{i,N} + a_i p_{i+1,N} + b_i p_{i+2,N} = \tilde{\psi}_{N-i}(0) + a_i \tilde{\psi}_{N-i+1}(0) + b_i \tilde{\psi}_{N-i+2}(0)
$$

$$= \tilde{\psi}_i(1) + a_i \tilde{\psi}_{i+1}(1) + b_i \tilde{\psi}_{i+2}(1) = \psi_i(1) = 0.
$$

Note that $\psi_i$’s vanish at the boundary values according to Proposition 1. The proof is complete. □

To see the sparsity of the transformation matrices, $P$, $G$ and $Q$ for $N = 6$ are shown in the following.

$$
G = \begin{bmatrix}
1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{2}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{3}{4} & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{5}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{6}{7} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad P = \begin{bmatrix}
-43 & -6 & 0 & 0 & 0 & 0 & 7 \\
148 & 4 & -5 & 0 & 0 & 0 & -49 \\
-245 & 2 & 2 & -4 & 0 & 0 & 147 \\
-147 & 0 & 0 & 4 & -2 & -2 & 245 \\
49 & 0 & 0 & 0 & 5 & -4 & -148 \\
-7 & 0 & 0 & 0 & 0 & 6 & 43
\end{bmatrix}, \quad Q = \begin{bmatrix}
46 & -24 & -18 & -\frac{4}{3} & 0 & 0 & 0 \\
-2 & -\frac{2}{3} & 0 & -\frac{4}{3} & -\frac{7}{3} & 0 & 0 \\
0 & 2 & -\frac{2}{3} & 0 & -\frac{4}{3} & -2 & 0 \\
0 & 0 & 3 & 10 & -\frac{9}{2} & -15 & -\frac{5}{2} \\
0 & 0 & 0 & 4 & 18 & 24 & -46
\end{bmatrix}.
$$

3 Variational formulation

In this section, at first the problem (1.1)-(1.3) is discretized in time. Then we develop a Bernstein dual-Petrov-Galerkin method using ?. The matrix formulation and the error estimations are also provided in this section.
3.1 Time discretization

Consider the subdiffusion equation (1.1) at \( t = t_{k+1} \), \( k \geq 0 \) as

\[
D^\alpha_t u(x, y, t_{k+1}) = \kappa \Delta u(x, y, t_{k+1}) + S(x, y, t_{k+1}).
\]

(3.1)

Let \( u^k \) be an approximation of \( u \) at \( t = t_k = k\tau \) for \( k = 0, 1, \ldots, M \), where \( \tau = \frac{T}{M} \) is the time step length. The time fractional derivative can be approximated by definition (1.4) and using forward difference for the derivative inside as

\[
D^\alpha_t u(x, y, t_{k+1}) \approx \mu(u^{k+1} - (1 - b_1)u^k - \sum_{j=1}^{k-1}(b_j - b_{j+1})u^{k-j} - b_ku^0) + r^{k+1}_\tau, \quad k \geq 1,
\]

(3.2)

where \( \mu = \frac{1}{\tau^{\alpha(2-\alpha)}} \) and \( b_j = (j+1)^{1-\alpha} - j^{1-\alpha} \) for \( k \geq 0 \) and \( 0 \leq j \leq k \). The error is bounded by

\[
|r^{k+1}_\tau| \leq \tilde{c}_u \tau^{2-\alpha},
\]

(3.3)

where the coefficient \( \tilde{c}_u \) depends only on \( u \) [3]. The time discretization (3.2) is referred to as L1 approximation (see e.g. [3, 22]). Substituting from (3.2) into (3.1) and multiplying both sides by \( \tau^\alpha \Gamma(2 - \alpha) \) and dropping \((x, y)\), the following time-discrete scheme is obtained

\[
u^{k+1} - \alpha_0 \Delta u^{k+1} = f^{k+1}, \quad k \geq 0,
\]

\[
f^{k+1} := (1 - b_1)u^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})u^{k-j} + b_ku^0 + \frac{1}{\mu}S^{k+1},
\]

(3.4)

with \( \alpha_0 = \frac{k}{\mu} \) and \( u^0 = g \) is given by the initial condition (1.2) with the error

\[
r^{k+1} \leq \tau^\alpha \Gamma(2 - \alpha) |r^{k+1}_\tau| \leq \tilde{c}_u \tau^2.
\]

(3.5)

For \( k = 0 \), it reads as

\[
u^1 - \alpha_0 \kappa \Delta u^1 = (1 - b_1)u^1 + b_1u^0 + \frac{1}{\mu}S^1.
\]

(3.6)

The boundary conditions for the semidiscrete problem is \( u^{k+1} = 0 \) on \( \partial \Omega \).

3.2 Weak and spectral formulation

Consider the problem (3.4) with \( \Omega = I^2 \), \( I = (0, 1) \) and the homogeneous Dirichlet boundary conditions \( u^{k+1}|_{\partial \Omega} = 0 \). We seek an approximate solution in the Sobolev space \( H_0^1(\Omega) = \{ u \in H^1(\Omega); u = 0 \text{ on } \partial \Omega \} \). A weak formulation of the problem (3.4) is to find \( u^{k+1} \in H_0^1(\Omega) \) such that \( \forall v \in H_0^1(\Omega) \):

\[
(u^{k+1}, v) + \alpha_0(\nabla u^{k+1}, \nabla v) = ((1 - b_1)u^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})u^{k-j} + b_ku^0 + \frac{1}{\mu}S^{k+1}, v).
\]

(3.7)

Let \( P_N \) be the space of polynomials over \( I \) with degree not exceeding \( N \) and \( (P_N^0)^2 = \{ v \in (P_N)^2 : v = 0 \text{ on } \partial \Omega \} \). The Galerkin formulation of the (3.7) is to find \( u^{k+1}_N \in (P_N^0)^2 \) such that \( \forall v_N \in (P_N^0)^2 \):

\[
(u^{k+1}_N, v_N) + \alpha_0(\nabla u^{k+1}_N, \nabla v_N) = ((1 - b_1)u^k_N + \sum_{j=1}^{k-1}(b_j - b_{j+1})u^{k-j}_N + b_ku^0_N + \frac{1}{\mu}I_NS^{k+1}, v_N),
\]

(3.8)

with \((f, g)\) being the standard \( L^2 \)-norm and \( I_N \) an interpolation operator.
3.2.1 Bernstein dual-Petrov-Galerkin method

Since \( \dim \mathbb{P}_N^0 = N - 1 \), and due to (2.1), we choose a basis for it by removing the first and last Bernstein polynomials of degree \( N \), i.e.,

\[
\Phi = [\phi_i(x) : 1 \leq i \leq N - 1]^T.
\] (3.9)

Using (2.3), it is easy to verify

\[
\Phi' = D\Phi + d,
\] (3.10)

where \( D = \text{tridiag}(N - i + 1, 2i - N, -(i + 1)) \) is a tridiagonal matrix of order \( N - 1 \) and \( d = N [\phi_0, 0, \ldots, 0, -\phi_N]^T \) is an \( (N - 1) \)-vector.

Assuming \( N_x = N_y = N \), we use the tensor product of the basis functions of \( \mathbb{P}_N^0 \) as a basis for two dimensional case, \( \{ \phi_i(x)\phi_j(y) : 1 \leq i, j \leq N - 1 \} \) and consider an approximate solution of (3.4) as

\[
u^{k+1}_N = \sum_{i,j=1}^{N-1} u_{i,j}^{k+1} \phi_i(x)\phi_j(y) = \Phi^T(x)U^{k+1}\Phi(y), \quad (x, y) \in \Omega,
\] (3.11)

where \( \Phi(\cdot) = [\phi_i(\cdot) : 1 \leq i \leq N - 1]^T \) and \( U^{k+1} = [u_{i,j}^{k+1}] \). Let us use the following notations.

\[
a_{i,j} = \int \int \phi_j'(x)\psi_i(x)dx, \quad A = [a_{i,j}],
\]

\[
b_{i,j} = \int \int \phi_j(x)\psi_i(x)dx, \quad B = [b_{i,j}],
\]

\[
f_{i,j}^{k+1} = \int \int I_N f^{k+1}(x,y)\psi_j(x)\psi_i(y)d\Omega,
\] (3.12)

for \( 1 \leq j \leq N - 1, 0 \leq i \leq N - 2 \).

Taking the test functions of (3.8) as \( v = \psi_l(x)\psi_m(y) \) for \( l, m = 0, 1, \ldots, N - 2 \), it is seen that the spectral form (3.8) is equivalent to the following linear system:

\[
\mu^2 Bu^{k+1} + \kappa \left( AU^{k+1}B^T + BU^{k+1}A^T \right) = F^{k+1}, \quad k \geq 0,
\]

that can be equivalently written as a Sylvester equation but it requires computing the inverse of \( B \). Although \( B \) has only three nonzero diagonals, it can be shown that its inverse is a dense matrix and so we avoid transforming to Sylvester equation. Instead we use the equivalent tensor product form

\[
(\mu^2 B \otimes B + \kappa (B \otimes A + A \otimes B))u^{k+1} = f^{k+1},
\] (3.13)

with \( f = [f_{1,0}, \ldots, f_{q,0}; f_{1,1}, \ldots, f_{q,1}; \ldots; f_{1,q-1}, \ldots, f_{q,q-1}]^T \), \( q = N - 1 \). It is worth to note that the coefficient matrix of the above system is the same for all time steps and to be evaluated just once for all \( k \geq 0 \).

In terms of the trial vector (3.9), and test vector (2.13), we may write

\[
A = \int \int \Psi'\Phi'^T dx, \quad B = \int \int \Psi\Phi^T.
\]
To facilitate the computations, in what follows, these matrices are related to the transformation matrices introduced in Section 2.2. First, note that by the biorthogonality (2.6), we have

\[
\int I \hat{\Psi} \Phi^T dx = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 
\end{bmatrix} =: \tilde{I}.
\] (3.14)

Now from (2.14), and writing \( G = [g_0, g_1, \ldots, g_N] \), we get

\[
B = \int G \hat{\Psi} \Phi^T dx = G \tilde{I} = [g_1, g_2, \ldots, g_{N-1}].
\]

So \( B \) is a tridiagonal matrix whose entries are given by

\[
b_{i,j} = \begin{cases} 
1, & j = i + 1, \\
a_i, & j = i, \\
b_i, & j = i - 1, \\
0, & \text{otherwise},
\end{cases}
\] (3.15)

where \( a_i \)'s and \( b_i \)'s are easily computed by (2.9). On the other hand, from Lemma 3, the Bernstein operational matrix of differentiation (3.10) and (3.14), we obtain

\[
A = \int Q \hat{\Psi} (\Phi^T D^T + d^T) dx \\
= Q \tilde{I} D^T + N[q_0, 0, \ldots, 0, q_N] \\
= [q_1, \ldots, q_{N-1}] D^T + N[q_0, 0, \ldots, 0, q_N],
\] (3.16)

where \( Q = [q_0, q_1, \ldots, q_N] \) is a \((1, 3)\) – band matrix introduced Lemma 3. Hence, \( Q \tilde{I} \) is a pentadiagonal matrix and \( A \) is the product of a pentadiagonal and a tridiagonal matrix plus a sparse matrix. From Lemma 3 and (3.16), it is seen that \( A \) is a seven-diagonal matrix.

Notice that the solution of linear system (3.13) requires the matrices \( A \) and \( B \). \( A \) is obtained by a sparse matrix-matrix multiplication (3.16) and entries of \( B \) are given by (3.15).

**Remark 2** Since the coefficient matrix of the linear system (3.13) remains intact for a fixed \( \tau \), only the RHS vector to be computed for different time steps, \( k = 0, 1, \ldots \) up to a desired time. So it is efficient to use a band-LU factorization for solving the system. It is remarkable that for a \((2p+1)\) – band matrix, the LU-factorization can be done with \( O(Np^2) \) flops and backward substitutions require \( O(Np) \) flops [14, Section 4.3].

## 4 Error estimation

For the error analysis, we assume the problem (1.1) to be homogeneous, \( S = 0 \).

For \( \alpha \geq 0 \), the bilinear form \( a(u, v) = (\nabla u, \nabla v) + \alpha(u, v) \) in (3.8) is continuous and coercive in \( H^1_0(\Omega) \times H^1_0(\Omega) \). The existence and uniqueness of the solution for both the weak form (3.8) and the Galerkin form (3.8) is guarantied by the well-known Lax-Milgram lemma.
We define the following inner product and the associated energy norm on $H^1_0(\Omega)$:

$$(u, v) = \int_{\Omega} uv \, d\Omega, \quad (u, v)_1 = (u, v) + \alpha_0 (\nabla u, \nabla v), \quad \|u\|_1 = (u, u)^{\frac{1}{2}}.$$ \hspace{1cm} (4.1)

**Theorem 1** The weak form (3.7) is unconditionally stable:

$$\|u_k\|_1 \leq \|u_0\|, \quad k = 1, \ldots, M.$$ \hspace{1cm} (4.2)

**Proof** Let $v = u^1$ in (3.7). Then,

$$(u^1, u^1) + \alpha_0 (\nabla u^1, \nabla u^1) = (u^0, u^1),$$

giving (4.2) for $k = 1$, by the definition (4.1), the Schwarz inequality and the inequality $\|v\| \leq \|v\|_1$. By mathematical induction, assume (4.2) holds for $k = 0, \ldots, n$. Let $v = u^{n+1}$ in (3.7), i.e.,

$$(u^{n+1}, u^{n+1}) + \alpha_0 (\nabla u^{n+1}, \nabla u^{n+1}) = (u^n, u^{n+1})
+ \sum_{j=1}^{n-1} (b_j - b_{j+1})(u^{n-j}, u^{n+1}) + b_n (u^0, u^{n+1}).$$

It is easy to see that the RHS coefficients in (3.4) are positive. So we obtain

$$\|u^{n+1}\|_1 \leq (1 - b_1)\|u^n\| + \sum_{j=1}^{n-1} (b_j - b_{j+1})\|u^{n-j}\| + b_n \|u^0\|
\leq \left( (1 - b_1) + \sum_{j=1}^{n-1} (b_j - b_{j+1}) + b_n \right) \|u^0\| = \|u^0\|.$$ 

So the proof is done. \qed

**Theorem 2** Let $u$ be the solution of the equation (1.1) with conditions (1.2)-(1.3) and $u^k$ be the solution of the semidiscrete problem (3.4). Then,

$$\|u(t_k) - u^k\|_1 \leq \frac{c_1 \tau^2}{\tau^2 - \alpha}, \quad 0 < \alpha < 1, \hspace{1cm} (4.3)$$

$$\|u(t_k) - u^k\|_1 \leq c_n \tau, \quad \text{as } \tau \to 1. \hspace{1cm} (4.4)$$

**Proof** The idea of the proof comes from [18]. We first prove

$$\|u(t_k) - u^k\|_1 \leq \frac{c_n \tau^2}{b_{k-1}}, \quad k = 1, \ldots, M.$$ \hspace{1cm} (4.5)

By (1.1) and (3.6), we have

$$(e^1, v) + \alpha_0 (\nabla e^1, \nabla v) = (e^0, v) + (r^1, v), \quad \forall v \in H^1_0(\Omega),$$

in which $e^k := u(t_k) - u^k$. For $v = e^1$ and by using $e^0 = 0$, $\|v\| \leq \|v\|_1$ and (3.5), we get

$$\|e^1\|_1 \leq c_n \tau^2.$$ \hspace{1cm} (4.6)
i.e., (4.5) holds for \( k = 1 \). By induction, assume (4.5) holds for \( k \leq n \). Using (1.1) and (3.4), we get

\[
\begin{align*}
(e^{n+1}, v) + \alpha_1(\nabla e^{n+1}, \nabla v) &= (1 - b_1)(e^n, v) \\
+ \sum_{j=1}^{n-1} (b_j - b_{j+1})(e^{n-j}, v) + b_n(e^0, v) + (r^{n+1}, v), & \forall v \in H^1_0(\Omega).
\end{align*}
\]

For \( v = e^{n+1} \), it reads as

\[
\|e^{n+1}\|_1^2 \leq (1 - b_1)\|e^n\|_1 \|e^{n+1}\|_1 + \sum_{j=1}^{n-1} (b_j - b_{j+1})\|e^{n-j}\|_1 \|e^{n+1}\|_1 + \|r^{n+1}\|_1 \|e^{n+1}\|_1,
\]

\[
\Rightarrow \|e^{n+1}\|_1 \leq (1 - b_1) \frac{c_u}{b_{n-1}} \tau^2 + \sum_{j=1}^{n-1} (b_j - b_{j+1}) \frac{c_u}{b_{n-j-1}} \tau^2 + c_u \tau^2 \\
\leq \left( (1 - b_1) + \sum_{j=1}^{n-1} (b_j - b_{j+1}) + b_n \right) \frac{c_u}{b_n} \tau^2 = \frac{c_u}{b_n} \tau^2,
\]

proving (4.5) for \( k = n + 1 \) that completes the proof of (4.5).

Consider \( f(t) = t^{1-\alpha} \), then there exists \( a, k, k - 1 < \xi < k \leq M \) such that

\[
b_{k-1} \tau^{-\alpha} = \frac{(k\tau)^{1-\alpha} - (\tau(k - 1))^{1-\alpha}}{\tau} = (1 - \alpha)(\xi)^{-\alpha} \geq (1 - \alpha)(\tau)^{-\alpha} \geq (1 - \alpha)(T)^{-\alpha},
\]

which gives

\[
\frac{c_u}{b_{k-1}} \tau^2 \leq \frac{c_u}{1 - \alpha} \tau^{\alpha} \tau^{2-\alpha}.
\]

Now using this along with (4.5) proves (4.3).

In order to derive (4.4), we first prove

\[
\|u(t_k) - u^*_k\|_1 \leq c_u k \tau^2, \quad k = 1, \ldots, M.
\]

By (4.6), the inequality (4.8) holds for \( k = 1 \). Assume (4.8) holds for \( k = 1, \ldots, n, n \leq M - 1 \). Then, from (1.1), (3.4) and (3.5), we obtain

\[
\|e^{n+1}\|_1 \leq (1 - b_1)\|e^n\| + \sum_{j=1}^{n-1} (b_j - b_{j+1})\|e^{n-j}\| + \|r^{n+1}\| \\
\leq \left( (1 - b_1) \frac{n}{n+1} + \sum_{j=1}^{n-1} (b_j - b_{j+1}) \frac{n-j}{n+1} + \frac{1}{n+1} \right) c_u(n+1) \tau^2 \\
\leq \left( (1 - b_1) \frac{n}{n+1} + (b_1 - b_n) \frac{1}{n+1} - (b_1 - b_n) \frac{1}{n+1} + \frac{1}{n+1} \right) c_u(n+1) \tau^2 \\
= \left( 1 - b_n \frac{n}{n+1} - (b_1 - b_n) \frac{1}{n+1} \right) c_u(n+1) \tau^2 \leq c_u(n+1) \tau^2.
\]

So (4.8) holds for \( k = n + 1 \). From \( k \tau \leq T \) and (4.8), we get (4.4). □
4.1 Convergence of the full discretization scheme

Let \( \pi_N^{1,0} \) be the \( H^1 \)-orthogonal projection operator from \( H^1_0(\Omega) \) into \( (\mathbb{P}_N^0)^2 \) associated with the energy norm \( \| \cdot \|_1 \) defined in (4.1). Due to the equivalence of this norm with the standard \( H^1 \) norm, we have the following error estimation [18, Relation (4.3)]

\[
\| u - \pi_N^{1,0} u \|_1 \leq c N^{1-m} \| u \|_m, \quad u \in H^m_0(\Omega) \cap H^1_0(\Omega), \quad m \geq 1. \tag{4.9}
\]

The idea of the proof for the following result comes from the paper [18].

**Theorem 3** Let \( u^k, k = 0, \ldots, M \) be the solution of the variational formulation (3.7) and \( u_N^k \) be the solution of the scheme (3.8), assuming \( u^0 = \pi_N^{1,0} u^0 \) and \( u^k \in H^m(\Omega) \cap H^1_0(\Omega) \) for some \( m > 1 \). Then,

\[
\| u^k - u_N^k \|_1 \leq \frac{c}{1 - \alpha} \tau^{-\alpha} N^{1-m} \max_{0 \leq j \leq k} \| u^j \|_m, \quad 0 < \alpha < 1,
\]

\[
\| u^k - u_N^k \|_1 \leq c N^{1-m} \sum_{j=0}^k \| u^j \|_m, \quad \alpha \to 1,
\]

(4.10)

for \( k = 1, \ldots, M \), where \( c \) depends only on \( T^m \).

**Proof** We have \( (u^{k+1} - \pi_N^{1,0} u^{k+1}, v_N) = 0, \forall v_N \in (\mathbb{P}_N^0)^2 \) by the projection operator. By definition of the norm (4.1), we get

\[
(\pi_N^{1,0} u^{k+1}, v_N) + \alpha_1(\nabla \pi_N^{1,0} u^{k+1}, \nabla v_N) = (u^{k+1}, v_N) + \alpha_1(\nabla u^{k+1}, \nabla v_N), \quad \forall v_N \in (\mathbb{P}_N^0)^2.
\]

By the weak form (3.7), the RHS of the above equation is replaced as

\[
(\pi_N^{1,0} u^{k+1}, v_N) + \alpha_1(\nabla \pi_N^{1,0} u^{k+1}, \nabla v_N) = (1 - b_1)(u^k, v_N)
\]

\[
+ \sum_{j=1}^{k-1} (b_j - b_{j+1})(u^{k-j}, v_N) + b_k(u^0, v_N), \quad \forall v_N \in (\mathbb{P}_N^0)^2.
\]

(4.11)

Subtracting (4.11) from (3.8), we have

\[
(e_N^{k+1}, v_N) + \alpha_1(\frac{\partial e_N^{k+1}}{\partial x}, \frac{\partial v_N}{\partial x}) = (1 - b_1)(e_N^k, v_N)
\]

\[
+ \sum_{j=1}^{k-1} (b_j - b_{j+1})(e_N^{k-j}, v_N) + b_k(e_N^0, v_N), \quad \forall v_N \in (\mathbb{P}_N^0)^2,
\]

where \( e_N^{k+1} = u^{k+1} - u_N^{k+1} \) and \( e_N^{k+1} = \pi_N^{1,0} u^{k+1} - u_N^{k+1} \). Let \( v_N = e_N^{k+1} \), then

\[
\| e_N^{k+1} \|_1 \leq (1 - b_1)\| e_N^k \|_1 + \sum_{j=1}^{k-1} (b_j - b_{j+1})\| e_N^{k-j} \|_1 + b_k\| e_N^0 \|_1.
\]

With \( \| e_N^{k+1} \|_1 \leq \| e_N^{k+1} \|_1 + \| u^{k+1} - \pi_N^{1,0} u^{k+1} \|_1 \), we obtain

\[
\| e_N^{k+1} \|_1 \leq (1 - b_1)\| e_N^k \|_1 + \sum_{j=1}^{k-1} (b_j - b_{j+1})\| e_N^{k-j} \|_1 + b_k\| e_N^0 \|_1 + c N^{1-m} \| u^{k+1} \|_1.
\]
As in the proof of Theorem 2, it is first proved by induction that:

\[ \|e_{N}^{k+1}\|_{1} \leq \frac{1}{b_{k-1}} \max_{0 \leq j \leq k} \|u^{j} - \pi_{N}^{1,0}u^{j}\|_{1}, \quad 0 < \alpha < 1, \]

\[ \|e_{N}^{k+1}\|_{1} \leq \sum_{j=0}^{k} \|u^{j} - \pi_{N}^{1,0}u^{j}\|_{1}, \quad \alpha \to 1, \]

for \(0 \leq k \leq M\). Then, by using (4.7) and the projection error (4.9) the desired result is derived. \(\square\)

The following theorem is obtained by the triangle inequality \(\|u(\cdot, t_{k}) - u_{N_{k}}^{k}\|_{1} \leq \|u(\cdot, t_{k}) - u^{k}\|_{1} + \|u^{k} - u_{N}^{k}\|_{1}\) along with the inequalities (4.3) and (4.10).

**Theorem 4** Let \(u\) be the solution of the problem (1.1) with the initial and boundary conditions given by (1.2)-(1.3) and \(u_{N}^{k}\) be the solution of the scheme (3.8). Then, assuming \(u_{N}^{0} = \pi_{1,0}^{0}u^{0}\) and \(u \in H^{m}(\Omega) \cap H_{0}^{1}(\Omega)\), we have

\[
\|u(t_{k}) - u_{N}^{k}\|_{1} \leq \frac{CT^{\alpha}}{1 - \alpha} (c_{u}T^{2 - \alpha} + c_{T}^{-\alpha}N^{1-m}) \sup_{0 < t < T} \|u(x, t)\|_{m}, \quad k \leq M, \quad 0 < \alpha < 1, \quad (4.12)
\]

\[
\|u(t_{k}) - u_{N}^{k}\|_{1} \leq T^{\alpha} (c_{u}T^{1-m} \sup_{0 < t < T} \|u(x, t)\|_{m}), \quad k \leq M, \quad \alpha \to 1.
\]

The constants \(C\) and \(c\) are independent of \(\tau\), \(T\), \(N\).

It is seen that the method has the so-called spectral convergence in space and the order of convergence \(O(\tau^{2-\alpha})\) in time.

### 5 Numerical examples

Here, some numerical experiments are provided to show the accuracy and efficiency of the proposed method. For the computations, we use Maple 18, on a laptop with CPU core i3 1.9 GHz and RAM 4 running Windows 8.1 platform. To compute the errors, we use the discrete \(L^{2}\) and \(L^{\infty}\) errors defined as

\[
L^{2} \approx \left( \frac{1}{N^{2}} \sum_{i,j=0}^{N-1} |u(x_{i}, y_{j}, T) - u_{N}^{M}(x_{i}, y_{j})|^{2} \right)^{1/2},
\]

\[
L^{\infty} \approx \max_{0 \leq i,j \leq N} |u(x_{i}, y_{j}, T) - u_{N}^{M}(x_{i}, y_{j})|,
\]

respectively, where \(u\) is the exact solution of the problem (1.1)-(1.3), \(u_{N}^{M}\) is the approximation solution (3.11) at \(T = t_{M} = 1\), \(x_{i} = y_{i} = \frac{i}{N}\) and \(N = 100\). Also, the convergence rates in space and time are respectively computed by

\[
\text{rate}_{N_{i}} = \frac{\log E(N, \tau_{i})}{\log \frac{N}{N_{i}}}, \quad \text{rate}_{\tau_{i}} = \frac{\log E(N, \tau_{i})}{\log \frac{\tau}{\tau_{i}}},
\]

where \(E(N, \tau)\) is the error with \(N\) stands for the dimension of basis and \(\tau\) is the time-step size. However, as it is common in the literature, we will show the spectral convergence of the proposed method by logarithmic scaled error plots.
Fig. 5.1: Convergence of the spectral method in terms of some norms with $\tau = 1/100$ and $\alpha = 0.25$

Table 1: Error and spatial convergence rate at $t = 1$ for Example 1.

| $N$ | $L^\infty$ | $L^2$ | $H^1$ | $L^\infty$ | $L^2$ | $H^1$ | $L^\infty$ | $L^2$ | $H^1$ |
|-----|------------|-------|-------|------------|-------|-------|------------|-------|-------|
| 2   | 7.53E-02  | 2.76E-02 | 2.81E-01 | 7.52E-02 | 2.75E-02 | 2.81E-01 | 7.49E-02 | 2.75E-02 | 2.81E-01 |
| 4   | 1.74E-03  | 5.60E-04 | 8.91E-03 | 1.72E-03 | 5.60E-04 | 8.91E-03 | 1.63E-03 | 5.62E-04 | 8.91E-03 |
| 6   | 1.78E-05  | 6.50E-06 | 1.34E-04 | 3.01E-05 | 1.29E-05 | 1.43E-04 | 9.98E-05 | 5.60E-05 | 2.86E-04 |

*Example 1* Consider the problem (1.1) with $\kappa = 1$ and the exact solution $u(x, y, t) = \sin(\pi x)\sin(\pi y)t^2$. Table 1 shows the convergence of the method for $\tau = \Delta t = 1/100$ for some fractional orders. Also Figure 5.1 demonstrate the logarithmic error plot for $\alpha = 0.25$. Table 2 illustrates the temporal rate of convergence with $N = 8$.

*Example 2* To see the method works for the case in which there is no source term, consider the problem (1.1)-(1.3) with the initial condition $u(x, y, 0) = x(x - 1)\sin(2\pi xy)$, $\kappa = 1$ and no source term [31]. The errors and also the rate of convergence are provided in Table 3 where the solution with $N = 8$ is treated as the exact solution.

The errors are reported at $t = T = 1$. We have used the eight point Gauss-Legendre quadrature rule to perform the integrals (3.12) in the right hand side of the linear system (3.13).

Numerical results confirm the convergence and the accuracy of the method.

6 Conclusion

In this paper, some new aspects of dual Bernstein polynomials have been discussed. A suitable compact combinations of these polynomials has been derived for developing a dual-Petrov-Galerkin variational
Table 2: Error and temporal rate of convergence for the time at $t = 1$ for Example 1.

| $N$ | $L^\infty$ | $L^2$ | $L^\infty$ | $L^2$ | $L^\infty$ | $L^2$ | $L^\infty$ | $L^2$ | $L^\infty$ | $L^2$ |
|-----|-------------|-------|-------------|-------|-------------|-------|-------------|-------|-------------|-------|
| 8   | 0.1000      | 9.25E-05 | 1.85E-04 | 16.922       | 3.41E-04 | 6.82E-04 | 16.671       | 9.80E-04 | 1.96E-04 | 16.906       |
| 0.0500 | 2.91E-05 | 1.67 | 5.80E-05 | 1.67 | 34.657 | 1.23E-04 | 1.47 | 2.46E-04 | 1.47 | 33.859 | 4.14E-04 | 1.24 | 8.28E-04 | 1.24 | 34.30       |
| 0.0250 | 9.04E-06 | 1.69 | 1.78E-05 | 1.69 | 68.422 | 4.41E-05 | 1.48 | 8.92E-05 | 1.48 | 69.141 | 1.75E-04 | 1.25 | 3.49E-04 | 1.25 | 70.156       |

Table 3: Error and spatial rate of convergence at $t = 1$ for Example 2.

| $N$ | $L^\infty$ | $L^2$ | $L^\infty$ | $L^2$ | $L^\infty$ | $L^2$ | $L^\infty$ | $L^2$ |
|-----|-------------|-------|-------------|-------|-------------|-------|-------------|-------|
| 2   | 3.95E-03 | 2.01E-03 | 2.54E-03 | 1.27E-03 | 1.17E-03 | 5.63E-04 |
| 4   | 5.87E-04 | 2.75 | 2.77E-04 | 2.85 | 3.97E-04 | 2.68 | 1.75E-04 | 2.86 | 1.97E-04 | 2.58 | 7.80E-05 | 2.85 |
| 6   | 3.79E-05 | 6.76 | 1.77E-05 | 6.79 | 2.62E-05 | 6.70 | 1.11E-05 | 6.79 | 1.31E-05 | 6.68 | 4.96E-06 | 6.80 |

The formulation for the numerical simulation of two-dimensional subdiffusion equation. It was shown that the method leads to sparse linear systems. The illustrated numerical examples have been provided to show the accuracy of the method. It is important to note that the transformation matrices and the operational matrix for differentiation of dual Bernstein polynomials that have been obtained in this work can be used similarly for developing Bernstein-based dual-Petrov-Galerkin Galerkin methods for other fractional partial differential equations on bounded domains.

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