A MODEL OF THE CUBIC CONNECTEDNESS LOCUS

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ABSTRACT. We construct a locally connected model of the boundary of the cubic connectedness locus. The model is obtained by constructing a decomposition of the space of critical portraits and collapsing elements of the decomposition into points. This model is similar to a quotient of the quadratic combinatorial locus where all baby Mandelbrot sets are collapsed to points. All fibers of the model, possibly except one, are connected.

1. Introduction

The quadratic connectedness locus $C_2$ is defined as the family of all quadratic polynomials $Q_c(z) = z^2 + c$ with connected Julia set. The set $C_2$ identifies with the Mandelbrot set $M = \{ c \in \mathbb{C} \mid Q_c(0) \not\to \infty \}$, studying $M$ is one of the major tasks of the complex dynamics. Among the tools used for that the concepts of renormalization and tuning play a central role as they allow one to uncover the structural properties of $M$; more precisely, they serve to explain the self-similar structure of $M$. Self-similarity of $M$ means, in particular, that there are infinitely many homeomorphic copies of $M$ in $M$, the so-called baby Mandelbrot sets. Baby Mandelbrot sets accumulate to any boundary point of $M$. If $c$ is in a baby Mandelbrot set, then, basically, $Q_c$ is obtained from another quadratic polynomial $P$ by “tuning”, i.e. consistently modifying (“pinching”) closures of Fatou domains of $P$.

By definition, the principal hyperbolic domain $\text{PHD}_2$ of $M$ consists of all $c \in M$ such that $Q_c$ has an attracting fixed point. The attracting basin of this point is an invariant Fatou domain homeomorphic to the disk; its closure is a closed Jordan disk coinciding with the filled Julia set of $Q_c$.

The boundary of $\text{PHD}_2$ is called the Main Cardioid.

A baby Mandelbrot set consists of all tunings of some polynomial $Q_c$, where $c \notin \text{PHD}_2$, and is contained in a unique maximal baby Mandelbrot
set. If we collapse the closure of \( \text{PHD}_2 \) and all maximal baby Mandelbrot sets, we obtain a dendrite \( D(M) \) that reveals the macro-structure of \( M \). A self-similar description of \( M \) may involve knowing \( D(M) \) together with a subset of marked points in \( D(M) \), where each marked point is a collapsed maximal baby Mandelbrot set.

The combinatorics of \( M \) can be understood through the “pinched disk” model \([13, 14, 27]\). The latter is more detailed than \( D(M) \) but also is more complicated. Ideas from \([13, 14, 27]\) can be used to construct a precise combinatorial model for \( D(M) \). However the structure of \( D(M) \) and its combinatorial model can be uncovered independently, is simpler than that of \( M \), and, therefore, should be easier to extend onto higher degrees.

We set out to do exactly that and to model the cubic connectedness locus \( C_3 \) by identifying subsets of \( C_3 \) similar to maximal baby Mandelbrot sets in the quadratic case, and collapsing such sets to points. This gives a macro-model of \( C_3 \), similar to \( D(M) \).

Thurston \([27]\) gave a detailed, conjecturally homeomorphic, model of the quadratic connected locus \( C_2 = M \). The situation with the cubic connectedness locus \( C_3 \) is different. Indeed, \( C_3 \) is complex 2-dimensional. Cubic polynomials are richer dynamically than quadratic ones (critical points are essential for the dynamics of polynomials, and cubic polynomials generically have two critical points) which makes the cubic case highly intricate combinatorially \([7, 10]\) and results into a breakdown of crucial steps of \([27]\) (e.g., cubic invariant laminations admit wandering triangles \([4, 5]\)). Also, \( C_3 \) is not locally connected \([23]\) and contains copies of various non-locally connected quadratic Julia sets \([12]\). All this makes the cubic case harder and complicates a complete description of \( C_3 \).

Our model adds new information about the self-similar structure of \( C_3 \) and can be viewed as a step in the right direction. We are not aware of any other models of \( C_3 \).

1.1. **Statement of the main results.** We assume familiarity with complex polynomial dynamics (Julia sets, external rays, etc). Unless specified otherwise, we will call external rays simply rays, sometimes specifying the polynomial. Let \( \text{Poly}_d \) be the space of all monic centered polynomials of degree \( d > 1 \); these are maps \( f : \mathbb{C} \to \mathbb{C} \) of the form

\[
f(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_1z + a_0, \quad a_0, a_1, \ldots, a_{d-2} \in \mathbb{C}.
\]

For \( f \in \text{Poly}_d \), write \( K_f \) for the filled Julia set of \( f \), and \( J_f = \text{Bd}(K_f) \) for the Julia set of \( f \) (here \( \text{Bd}(X) \) is the topological boundary of a set \( X \subset \mathbb{C} \)). By definition, the connectedness locus \( C_d \) consists of all \( f \in \text{Poly}_d \) with connected \( K_f \). Note that \( C_2 = M \). In \([17]\), a combinatorial setup
is developed to study renormalization of higher degree polynomials. This approach is based on rational laminational equivalences [19].

Given \( f \in \text{Poly}_d \), the rational laminational equivalence \( \lambda(f) \) is an equivalence relation on \( \mathbb{Q}/\mathbb{Z} \) such that two arguments \( \alpha, \beta \in \mathbb{Q}/\mathbb{Z} \) are equivalent if and only if the rays \( R_f(\alpha), R_f(\beta) \) land at the same point (these two rays are said to form a (rational) cut). Observe that \( \lambda(f) \) is closed in \( \mathbb{Q} \times \mathbb{Q} \): if the arguments of rational cuts converge to two rational angles then the corresponding rays form a rational cut, too.

For \( f_0 \in \text{Poly}_d \), the combinatorial renormalization domain \( C(f_0) \) is defined [17] as \( \{ f \in \text{Poly}_d \mid \lambda(f) \supset \lambda(f_0) \} \). Thus, \( C(f_0) \) is the family of all polynomials \( f \) such that if the rays \( R_{f_0}(\alpha) \) and \( R_{f_0}(\beta) \) land at a point \( z_{f_0} \) for some \( \alpha, \beta \in \mathbb{Q} \), then the rays \( R_f(\alpha) \) and \( R_f(\beta) \) land at a point \( z_f \). If \( f_0 \) is hyperbolic, then \( C(f_0) \) is an analog of a baby Mandelbrot set, except that there are several types of domains \( C(f_0) \), and domains of different types have different models (see Section 1.2). The internal structure of \( C(f_0) \) can be studied through a renormalization operator defined in [17]. In this paper, however, we are concerned with the way different combinatorial renormalization domains fit together. A central objective is a model, in which every \( C(f_0) \) is collapsed to a point provided that \( \lambda(f_0) \) is nonempty. Such a model is similar to \( D(M) \).

As a topological space that eventually provides the desired model we choose the space \( \mathcal{C}_P \) of all cubic critical portraits, i.e., basically, pairs of critical chords disjoint in the open unit disk \( \mathbb{D} \) (for definitions see Section 3). This is a natural choice as behavior of critical points essentially defines the dynamics of the entire polynomial. On the other hand, once the basin of attraction of infinity is foliated by the circle of rays, critical portraits become a combinatorial counterpart of critical points of polynomials. We want to define an upper semi-continuous partition of \( \mathcal{C}_P \) whose elements are called alliances so that for every \( P \in \mathcal{C}_3 \), there is the corresponding alliance \( \mathcal{A}_P \), which is a closed subset of \( \mathcal{C}_P \).

There are several considerations to take into account here. First, we adopt the point of view that rational laminations are “tags” of polynomials. This is reasonable since pairs of rational rays mostly land at repelling periodic points and are, therefore, stable so that one can expect continuity of rational laminations as “tags”, and, hence, continuity of the desired model. The choice of critical portraits is then limited to the ones compatible with rational laminations (two collections of chords are compatible if no representatives of the collections intersect inside \( \mathbb{D} \) while not being equal).

In this paper we generalize \( D(M) \) to the cubic case. Hence if for polynomials \( P_1, P_2 \) we have \( \lambda(P_1) \supset \lambda(P_2) \} \) then we should allow for critical portraits compatible with \( P_2 \) to be contained in the alliance of \( P_1 \). In other words, similar to \( D(M) \), we want alliances to be maximal sets of critical
portraits that can possibly be associated with a given polynomial $P$. This ensures that distinct alliances are disjoint.

In the end the model of $C_3$ is the space of all alliances, i.e. the quotient space $\mathbb{C}P^r/\{A_P\}$ with the quotient topology; it is the quotient space of $\mathbb{C}P^r$ with respect to the equivalence relation whose classes are exactly alliances. The modeling map is the quotient map associating $P \in C_3$ to $A_P$ as an element of $\mathbb{C}P^r/\{A_P\}$. Our main result is stated below.

**Main Theorem.** The map $\psi : P \mapsto A_P$ is continuous and takes $C_3$ onto the quotient space $\mathbb{C}P^r/\{A_P\}$. There is one special alliance, called the central alliance, so that all other alliances (called regular alliances) are of the form $C(P)$ for certain polynomials $P$. All fibers of $\psi$ except possibly for the preimage of the central alliance (called the central fiber), are connected.

### 1.2. Models for $C(P)$

Following [17], we now describe models for $C(P)$; we understand “models” in the sense of Milnor [25], who calls them reduced mapping schemata. In certain cases, the spaces $C(P)$ are naturally bijective to the models (by results of Inou, Kiwi, Shen, Wang cited below), and, in some other cases, this is an open question and an active area of research. The results of this paper are complementary to those cited below.

Suppose first that $P \in C_3$ is hyperbolic. According to the classification of Milnor [24], it has type A, B, C, or D. If $P$ has type D (Disjoint), then it has two disjoint cycles of attracting bounded Fatou domains. The model for $C(P)$ is the space $M \times M$. If $P$ has type C (Capture), then $P$ has one cycle of attracting domains, and a critical strictly preperiodic Fatou domain that does not belong to the cycle but eventually maps to it. In this case, the model space for $C(P)$ is

$$MK = \{(c,z) \in \mathbb{C}^2 \mid c \in M, z \in K_{Q_c}\}.$$ 

If $P$ has type B (Bi-transitive), then $P$ has one cycle of bounded Fatou domains, and critical points of $P$ are in distinct domains of this cycle. Then the model space is the connectedness locus in the parameter space of polynomials of the form $Q_{c_1} \circ Q_{c_0}$ for $(c_0, c_1) \in \mathbb{C}^2$ (bi-quadratic polynomials). Finally, if $P$ has type A (Adjacent), then both critical points of $P$ are in the same bounded Fatou domain. The model space for $C(P)$ is $C_3$ in this case.

If $P$ is primitive (the closures of all bounded Fatou domains of $P$ are disjoint), then $C(P)$ is in a natural one-to-one correspondence with the model (the latter, unfortunately, fails to be continuous, see [16]). This follows from [17, Theorems E and F] for types D and C, respectively, and from [26] for all types.

Suppose now that $P$ is not hyperbolic but does not have neutral cycles. In this case, at least one critical point of $P$ belongs to $J_P$. Suppose additionally that the critical points of $P$ in $J_P$ are not renormalizable (this is equivalent
We parameterize the rays of a polynomial \( f \in \text{Poly}_d \) by angles, i.e., elements of \( \mathbb{R}/\mathbb{Z} \). The ray of argument \( \theta \in \mathbb{R}/\mathbb{Z} \) is denoted by \( R_f(\theta) \).

A chord \( \overline{ab} \) is a closed segment connecting points \( a, b \) of the unit circle \( S = \{ z \in \mathbb{C} \mid |z| = 1 \} \). If \( a = b \), then \( \overline{ab} \) is degenerate. Distinct chords cross if they intersect in \( \mathbb{D} \) (alternatively, they are called linked). Chords that do not cross are said to be unlinked. Sets of chords are compatible if chords from distinct sets do not cross. Write \( \sigma_d \) for the self-map of \( S \) that takes \( z \) to \( z^d \). A chord \( \overline{ab} \) is \((\sigma_d)\) critical if \( \sigma_d(a) = \sigma_d(b) \).

2.1. Laminational equivalence relations. Given \( f \in \text{Poly}_d \), write \( K_f \) for the filled Julia set of \( f \), and \( J_f = \text{Bd}(K_f) \) for the Julia set of \( f \). For \( f \in \mathcal{C}_d \) with locally connected \( J_f \), let \( \psi(e^{2\pi i \theta}) \) be the landing point of \( R_f(\theta) \); then \( \psi : S \to J_f \) is a semi-conjugacy between \( \sigma_d : S \to S \) and \( f : J_f \to J_f \) called the Caratheodory loop. Define an equivalence relation \( \sim_f \) on \( S \) by \( x \sim_f y \) iff \( \psi(x) = \psi(y) \) and call \( \sim_f \) the laminational equivalence relation (generated by \( f \)). The relation \( \sim_f \) is \( \sigma_d \)-invariant; \( \sim_f \)-classes have pairwise disjoint convex hulls. The quotient space \( S/\sim_f = J_{\sim_f} \) is called a topological Julia set. Clearly, \( J_{\sim_f} \) is homeomorphic to \( J_f \). The map \( f_{\sim_f} : J_{\sim_f} \to J_{\sim_f} \), induced by \( \sigma_d \) and called a topological polynomial, is topologically conjugate to \( f|_{J_f} \).

As a visual counterpart of \( \sim_f \), consider the family of all edges of convex hulls of \( \sim_f \)-classes. This set of chords, together with \( S \), is called the (invariant) lamination generated by \( f \) and is denoted \( \mathcal{L}_f \); these chords then are said to be the leaves of \( \mathcal{L}_f \).

2.2. General properties of laminations. Thurston [27] defined invariant laminations as families of chords with certain dynamical properties resembling properties of \( \mathcal{L}_f \). He did it abstractly (without invoking polynomials). We use a slightly different approach (see [3]).

**Definition 2.1** (Laminations). A prelamination is a family \( \mathcal{L} \) of chords called leaves such that distinct leaves are unlinked and all points of \( S \) are leaves. If, moreover, the set \( \mathcal{L}^+ = \bigcup_{\ell \in \mathcal{L}} \ell \) is compact, then \( \mathcal{L} \) is called a lamination.
Lemma 2.3. \( q_{ab} q_{bc} q_{ab} q_{bc} \)

Definition 2.2. \( l: \text{clos-sibl} \)

From now on \( \mathcal{L} \) denotes a lamination (unless we specify that it is a pre-lamination). \( \text{Gaps of } \mathcal{L} \) are the closures of components of \( \mathbb{D} \setminus \mathcal{L}^+ \). A gap \( G \) is countable (finite, uncountable) if \( G \cap \mathbb{S} \) is countable infinite (finite, uncountable). Uncountable gaps are called Fatou gaps. For a closed convex set \( H \subset \mathbb{C} \), edges of \( H \) are maximal straight segments in \( \text{Bd}(H) \).

Convergence of pre laminations \( \mathcal{L}_i \) to a set of chords \( \mathcal{E} \) is understood as Hausdorff convergence of leaves of \( \mathcal{L}_i \) to chords from \( \mathcal{E} \). Evidently, then \( \mathcal{E} \) is a lamination. A lamination \( \mathcal{L} \) is nonempty if it has nondegenerate leaves and empty otherwise (the empty lamination is denoted by \( \mathcal{L}_\emptyset \)). Say that \( \mathcal{L} \) is countable if it has countably many nondegenerate leaves and uncountable otherwise; \( \mathcal{L} \) is perfect if it has no isolated leaves (thus, Fatou gaps of perfect laminations have no critical edges).

If \( G \subset \text{conv}(G \cap \mathbb{S}) \), define \( \sigma_d(G) \) as the convex hull of \( \sigma_d(G \cap \mathbb{S}) \). A sibling of a leaf \( \ell \in \mathcal{L} \) is a leaf \( \ell' \in \mathcal{L} \) different from \( \ell \) with \( \sigma_d(\ell') = \sigma_d(\ell) \). Call a leaf \( \ell^* \) such that \( \sigma_d(\ell^*) = \ell \) a pullback of \( \ell \). The map \( \sigma_d \) can be extended continuously over \( \mathcal{L}^+ \) by extending linearly over all leaves of \( \mathcal{L} \). We also denote this extended map by \( \sigma_d \).

**Definition 2.2**. \( \text{d:sibl} \) A (pre) lamination \( \mathcal{L} \) is sibling (\( \sigma_d \))-invariant if

1. for each \( \ell \in \mathcal{L} \), we have \( \sigma_d(\ell) \in \mathcal{L} \),
2. for each \( \ell \in \mathcal{L} \) there exists \( \ell^* \in \mathcal{L} \) with \( \sigma_d(\ell^*) = \ell \),
3. for each non-critical \( \ell \in \mathcal{L} \) there exist pairwise disjoint leaves \( \ell_1, \ldots, \ell_d \) in \( \mathcal{L} \) such that \( \ell_1 = \ell \) and \( \sigma_d(\ell_1) = \cdots = \sigma_d(\ell_d) \).

Leaves from (3) above form full sibling collections. Their leaves cannot intersect even on \( \mathbb{S} \). Here is a useful property of such collections.

**Lemma 2.3.** The following properties hold.

1. Let \( \ell_1, \ldots, \ell_d \) be the limit of a sequence of full sibling collections and \( \ell_1 \) is not critical. Then \( \ell_1, \ldots, \ell_d \) is a full sibling collection.
2. The family of all non-isolated leaves of a sibling invariant lamination is a sibling invariant lamination.

Proof. (1) We claim that \( \ell_1 \) and \( \ell_2 \) are disjoint. If \( \ell_1 = \overline{ab} \) and \( \ell_2 = \overline{bc} \), then \( \sigma_d(a) = \sigma_d(c) \neq \sigma_d(b) \). A full sibling collection approximating the given one has a pair of leaves \( ab'c' \) and \( b''c'' \) with \( b', b'' \) close to \( b \) and \( \sigma_d(ab'c') = \sigma_d(b''c'') \), a contradiction.

(2) All non-isolated leaves in \( \mathcal{L} \) form a forward invariant closed family of leaves. If \( \ell \) is non-isolated, choose a sequence of leaves \( \overline{q_i} \rightarrow \ell \) with \( \sigma_d(\overline{q_i}) \rightarrow \ell \) so that \( \sigma(\overline{q}) = \ell \). Now, let \( \ell \) be non-isolated and non-critical. Choose \( \ell_i \rightarrow \ell \) so that \( \ell_i \)'s belong to their full sibling collections. We may assume that these collections of leaves converge; by (1) they converge to a full sibling collection that includes \( \ell \). This completes the proof. \( \square \)
These are properties of sibling invariant laminations [3]:

**gap invariance:** if \( G \) is a gap of \( L \), then \( H = \sigma(G) \) is a leaf of \( L \) (possibly degenerate), or a gap of \( L \), and in the latter case, the map \( \sigma|_{Bd(G)} : Bd(G) \to Bd(H) \) is an orientation preserving composition of a monotone map and a covering map (gap invariance is a part of Thurston’s original definition, and sibling invariant laminations are invariant in the sense of Thurston [27]);

**compactness:** if a sequence of sibling invariant prelaminations converges to a set of chords \( A \), then \( A \) is a sibling invariant lamination.

A motivation behind introducing sibling invariant laminations was that it is easier to deal with leaves and their sibling collections than with gaps. As a consequence, studying families of laminations became more transparent. In particular, the following theorem holds (recall that we always consider spaces of compact sets in Hausdorff topology).

**Theorem 2.4.** [3, Corollary 3.20, Theorem 3.21] The closure of a sibling invariant prelamination is a sibling invariant lamination. The space \( \text{Lam}_d \) of all sibling invariant laminations of degree \( d \) is compact.

Since (pre)laminations are collections of chords, the concept of compatibility applies to them, and we can talk about compatible laminations. Then a useful fact that follows from Definition 2.2 is the following lemma.

**Lemma 2.5.** [3] If invariant laminations \( L, L' \) are compatible, then \( L \cup L' \) is an invariant lamination. If \( L, L' \) are in addition perfect, then so is \( L \cup L' \).

From now on, unless stated otherwise, all laminations are \( \sigma_d \)-invariant for some \( d \geq 2 \).

### 2.3. Gaps and laps of arbitrary laminations.

A chord \( \ell \) is **inside** a gap \( G \) if, except for the endpoints, \( \ell \) is in the interior of \( G \); if \( \ell \subset G \), then we say that \( \ell \) is **contained in** \( G \). A gap \( G \) of \( L \) is **critical** if either all edges of \( G \) are critical, or there is a critical chord **inside** \( G \). A **critical set** of \( L \) is a critical leaf or a critical gap. A **lap** of \( L \) is either a finite gap of \( L \) or a nondegenerate leaf of \( L \) not on the boundary of a finite gap. By the period we mean the **minimal** period.

The following is known (see, e.g., [2] or [11]). Fatou gaps of \( \sigma_d \)-invariant laminations are (pre)periodic. If \( U \) is a \( \sigma_d \)-periodic Fatou gap of period \( n \) and the map \( \sigma^n_d : Bd(U) \to Bd(U) \) has topological degree \( k \geq 1 \), then \( U \) is called a **periodic gap of degree** \( k \). If \( k > 1 \), then the monotone map from \( Bd(U) \) to \( \mathbb{S} \) collapsing all edges of \( U \) also semi-conjugates \( \sigma^n_d|_{Bd(U)} \) to \( \sigma_k \).

The case of infinite gaps of degree one is more delicate (see Lemma 2.7).

**Lemma 2.6.** Suppose that \( \ell = \overline{xy} \) is a critical leaf of a \( \sigma_d \)-invariant lamination \( L \) with a periodic endpoint \( x \). Then \( \ell \) is isolated in \( L \).
Proof. We may assume that \( x \) is fixed. If a leaf \( \ell' \) is close to \( \ell \) and \( x \notin \ell' \), then \( \sigma_d(\ell') \) crosses \( \ell' \), a contradiction. Now, let a leaf \( \overline{xy} \) be very close to \( \overline{xy} \), the points \( \sigma_d(z) \) and \( z \) are separated in \( \mathbb{D} \) by \( \overline{xy} \). By [3, Lemma 3.8] the leaf \( \overline{xt} \) has a sibling leaf \( \overline{yt} \) where \( t \) and \( \sigma_d(z) \) are separated in \( \mathbb{D} \) by \( \overline{xy} \), that implies that \( \overline{xt} \) crosses \( \overline{xz} \), a contradiction. Thus, \( \ell \) is isolated in \( \mathcal{L} \) as desired. \( \square \)

In Lemma 2.7 and in what follows, we often consider the linear extension of \( \sigma_d \) over leaves of laminations described right before Definition 2.2.

**Lemma 2.7.** Suppose that \( G \) is a degree one \( k \)-periodic infinite gap of a \( \sigma_d \)-invariant lamination \( \mathcal{L} \) for some \( d \geq 2 \). Then there are two possibilities.

1. There is a monotone semi-conjugacy between \( \sigma_d^k|_{\text{Bd}(G)} \) and an irrational rotation of \( \mathbb{S} \) that collapses all edges of \( G \) to points; moreover, if there are concatenations of edges of \( G \), each concatenation consists of at most \( d \) leaves.
2. A critical edge of \( \sigma_d^m(G) \), for some \( m \), has a periodic endpoint.

Thus, in either case some gaps from the orbit of \( G \) have critical edges. Also, \( \mathcal{L} \) has isolated leaves with both endpoints non-preperiodic, or with one periodic and one non-periodic endpoints. In particular, (a) the lamination \( \mathcal{L} \) is not perfect, and (b) it cannot have a dense subset of (pre)periodic leaves whose endpoints have equal preperiods.

**Proof.** Recall that every edge of \( G \) eventually maps to a critical or a periodic leaf. Also, since \( \sigma_d^k|_{\text{Bd}(G)} \) has degree one, it is associated with a rotation number \( \rho \in [0, 1) \).

(1) If \( \rho \) is irrational, then \( \sigma_d^k|_{\text{Bd}(G)} \) is monotonically semiconjugate to an irrational rotation. Consider maximal by inclusion concatenations of edges of \( G \). Note that the image of a concatenation is a (possibly degenerate) concatenation as the only thing that may happen is that some critical leaves collapse. Because the rotation number is irrational, it follows that any maximal concatenation is wandering in the strong sense: all images of a maximal concatenation are pairwise disjoint. Now, let \( A \) be a concatenation like that. Then \( A \) has well-defined endpoints, say, \( a \) and \( b \). Connect them with a chord; then the resulting gap is such that all its images have pairwise disjoint interiors which implies by [20, Theorem 1.1] that the concatenation \( A \) can consist of at most \( d \) edges of \( G \) as claimed.

Since there are no periodic edges of \( G \), by [11, Lemma 2.28], then all its edges are eventually mapped to one of finitely many \( \sigma_d^k \)-critical edges of \( G \). Since concatenations of edges of \( G \) are wandering, then after finitely many iterations a concatenation becomes a point. Hence by collapsing all concatenations to points we monotonically semiconjugate \( \sigma_d^k|_{\text{Bd}(G)} \) to an
irrational rotation of the circle as otherwise there will exist wandering and never collapsing concatenations which is impossible as we have just seen.

If now $\rho$ is rational, then $\text{Bd}(G)$ contains periodic points. Considering a suitable iterate of $\sigma_d$, we may assume that they all are $\sigma_d^k$-fixed. However then it follows that some of them must be attracting for $\sigma_d^k|_{\text{Bd}(G)}$ from at least one side, and this is only possible if there is a $\sigma_d^k$-critical edge of $G$ with a $\sigma_d^k$-fixed endpoint. Evidently, this proves 2).

Now, let $\ell$ be a critical edge of an infinite gap $U$ of $\mathcal{L}$. We need to show that $\mathcal{L}$ has isolated leaves with both endpoints non-preperiodic, or with one periodic and one non-periodic endpoints. Clearly, $\ell$ is not the limit of leaves approaching $\ell$ from the outside of $U$ and disjoint from $\ell$ as otherwise $x = \sigma_d(\ell) \in \mathbb{S}$ is the limit of their image leaves, each separating a small neighborhood of $x$ in $\mathbb{S}$ from the rest of the circle and hence cutting through $\sigma_d(U)$, a contradiction. Suppose that $\ell = \overline{xy}$ is the limit of a sequence of leaves $\overline{xy}_i$ approaching $\ell$ from the outside of $U$. The (infinite) collection of leaves coming out of $x$ is called an infinite cone; infinite cones were defined in [3] and studied in detail in [9]. By [3, Lemma 4.7] it follows that $x$ is (pre)periodic (and, hence, $U$ is not Siegel). Mapping forward and passing to an iterate, assume that $U$ and $x$ are $\sigma_d$-fixed. If there are critical leaves coming out of $x$, then they are isolated by Lemma 2.6 and we are done. Suppose that there are no critical leaves with endpoint $x$ and consider this situation closely.

Thus, $\overline{xy}$ is invariant. We may assume that there are invariant leaves $\overline{uv}$ and $\overline{uv}$ (one of which can be degenerate) so that there are no invariant leaves with endpoint $x$ between them, but there are infinitely (countably) many non-invariant leaves between them. Denote this family of leaves by $Z$ and include $\overline{uv}$ and $\overline{uv}$ in it; then $\sigma_d(Z) = Z$ by [9, Lemma 2.14]. Since by the assumption there are no critical leaves with endpoint $x$, [9, Lemma 2.15] implies that all leaves of $Z$ but $\overline{uv}$ and $\overline{uv}$ map under $\sigma_d$ in one direction, say, from $\overline{uv}$ to $\overline{uv}$, it follows that there are isolated leaves among leaves of $Z$, and these isolated leaves are not $\overline{uv}$ or $\overline{uv}$ as desired. Claims (a) and (b) of the lemma easily follow.

Gaps in case (1) are called Siegel gaps. In case (2) gaps are said to be of caterpillar type. Indeed, if a critical edge $\ell$ of $G$ from above has a $\sigma_d^k$-fixed point, then it follows that there exists a countable concatenation of edges of $G$ consisting of $\ell$ and its consecutive $\sigma_d^k$-pullbacks. Observe that this phenomenon (having a countable concatenation of leaves that begins with a critical leaf with a periodic endpoint) is not confined to gaps of degree one. For example, a gap of degree greater than one may have, say, a periodic concatenation like that mapping on top of itself under an appropriate
iterate of the map. In these cases we will still refer to such gaps as gaps of caterpillar type or just caterpillar gaps.

In general, if $\ell$ is a critical edge of a Fatou gap $U$, then $\ell$ is isolated, and another gap $U'$ on the other side of $\ell$ has the same image as $U$. Since cycles of Siegel (caterpillar) gaps include gaps with critical edges, it follows that in a lamination such gaps have edges at which other infinite gaps are attached on the opposite side.

**Lemma 2.8.** Suppose that $L_i \rightarrow L$ are $\sigma_d$-invariant laminations, and let $G$ be a periodic lap of $L$. Then $G$ is also a lap of $L_i$ for all sufficiently large $i$.

**Proof.** Let $\ell$ be an edge of $G$; write $k$ for the period of $\ell$. Then $L_i$, for large $i$, must have a lap $G_i$ with $G_i \rightarrow G$. Choose an edge $\ell_i$ of $G_i$ so that $\ell_i \rightarrow \ell$. Then $\ell_i$ does not cross $\ell$ for large $i$ as otherwise the leaves $\sigma^k_d(\ell_i)$ and $\ell_i$ cross. Moreover, $\ell_i$ is disjoint from the interior of $G$ for large $i$ as otherwise $\sigma^k_d(\ell_i)$ intersect the interior of $G_i$ (note that $\ell_i$ is repelled away from $\ell$ by $\sigma^k_d$). By way of contradiction assume that $L_i$ do not contain $G_i$. Then $G_i \not\supset G$ and $\ell_i \neq \ell$ for at least one edge $\ell$ of $G$. It follows that $\sigma^k_d(G_i) \not\supset G_i$, a contradiction. \qed

2.4. **Laminations and their chiefs.** A lamination $L$ is clean if any pair of distinct non-disjoint leaves of $L$ is on the boundary of a finite gap. Clean laminations give rise to equivalence relations: $a \sim L b$ if either $a = b$ or $a, b$ are in the same lap of $L$. In that case, if $L$ is $\sigma_d$-invariant, the quotient $\mathbb{S}/ \sim_L = J_L$ is called a topological Julia set and the map $f_L : J_L \rightarrow J_L$, induced by $\sigma_d$, is called a topological polynomial. By [8, Lemma 3.16], any clean lamination has the following property: if one endpoint of a leaf is periodic, then the other endpoint is also periodic with the same period.

A maximal by inclusion perfect sublamination $L^p$ of $L$ is called the perfect part of $L$. Equivalently, one can define $L^p$ as the set of all leaves $\ell \in L$ such that, arbitrarily close to $\ell$, there are uncountably many leaves of $L$. Evidently, perfect laminations are clean.

**Definition 2.9 (Chiefs).** A chief is a minimal, by inclusion, nonempty lamination. A chief of $L \neq L\emptyset$ is a sublamination of $L$ that is a chief.

Consider two examples of quadratic chiefs. First, take the lamination $L_P$ associated with a polynomial $P(z) = e^{2\pi i \alpha}z + z^2$ with $\alpha \in \mathbb{Q}$ (then $J_P$ is locally connected, and $L_P$ is constructed in Subsection 2.1). Then $L_P$ is a chief as it consists of the grand orbit of one leaf (e.g., of the minor of $L_P$).

Another quadratic example is as follows. Take a $\sigma_2$-critical leaf $\overline{xy}$ not equal to the horizontal diameter such that $\sigma_2^N(x) = \sigma_2^N(y) = a_0$ is the unique $\sigma_2$-fixed point of $\mathbb{S}$ (i.e., the point with argument 0), consider iterated pullbacks of $\overline{xy}$ compatible with $\overline{xy}$, and close the resulting chords to obtain
a lamination $\mathcal{L}$ (the construction of a pullback lamination is due to Thurston [27]). To show that $\mathcal{L}$ is a chief, assume that there exists $\mathcal{L}' \subsetneq \mathcal{L}$, where $\mathcal{L}'$ has non-degenerate leaves. Then $\overline{xy}$ is contained in a gap $G$ of $\mathcal{L}'$ as otherwise $\mathcal{L}'$ contains all pullbacks of $\overline{xy}$ and their closure, i.e. $\mathcal{L}' \supset \mathcal{L}$, a contradiction. The gap $G$ cannot contain $a_0$ as then it must be an invariant gap of $\mathcal{L}'$ that maps onto itself two-to-one which implies that $G = \overline{xy}$ and that $\mathcal{L}'$ is degenerate, a contradiction.

Consider $\sigma^N(G)$. Observe that $\sigma^N_2(x) = a_0 \in \sigma^N(G) \cap \mathcal{S}$. Also, $\sigma^N_2(G)$ is not degenerate as this is only possible if an iterated image of $G$ equals $\overline{xy}$, a contradiction. Thus, $\mathcal{L}'$ has a non-degenerate leaf or gap $\sigma^N_2(G)$ containing $a_0$. Since $\overline{xy}$ is a leaf of $\mathcal{L}$, all points of $G \cap \mathcal{S}$ have orbits contained in the half-circle $S_0$ with endpoints $x, y$ containing $a_0$, and it is easy to see that this is impossible. We claim that $\mathcal{L}$ is perfect. Indeed, repeatedly pulling $\overline{xy}$ back towards $a_0$ one can find a leaf $\ell$ of $\mathcal{L}$ that is arbitrarily close to $a_0$ and separates $a_0$ from $\overline{xy}$. Since $\sigma^N_2(\overline{xy}) = a_0$, we can pull $\ell$ back $N$ steps along the backward orbit of $a_0$ that leads to $\overline{xy}$. In this way, one obtains two leaves with the same images enclosing $\overline{xy}$ in a narrow strip. Thus, $\overline{xy}$ is not isolated in $\mathcal{L}$. Hence pullbacks of $\overline{xy}$ are not isolated in $\mathcal{L}$ either. Since by definition any leaf of $\mathcal{L}$ is either a pullback of $\overline{xy}$ or a limit of such pullbacks, $\mathcal{L}$ is perfect as claimed.

More generally, the concept of a chief is related to that of renormalization. Here is a heuristic explanation of this fact. Consider a quadratic non-renormalizable polynomial $P$ without neutral periodic points. By a theorem of Yoccoz [18], such a polynomial has a locally connected Julia set. Moreover, it is easy to see that the associated lamination $\mathcal{L}$ is perfect. Suppose that $\mathcal{L}$ is not a chief. Then there exists a lamination $\mathcal{L}' \subsetneq \mathcal{L}$. The only way it can happen is when $\mathcal{L}'$ has an $n$-periodic Fatou gap $G$ with the first return map of degree greater than 1. Hence $P$ is renormalizable; the corresponding renormalization is generated by the restriction of $P^n$ onto a domain naturally associated with $G$. This contradiction implies that $\mathcal{L}$ is a chief. However, there are renormalizable quadratic polynomials whose associated laminations are chiefs, e.g. hyperbolic polynomials $z^2 + c$ such that $c$ comes from within copies of the filled Main Cardioid attached to the Main Cardioid itself.

**Lemma 2.10.** The set $\mathcal{L}^p$ is an invariant lamination. If $\mathcal{L}$ is uncountable, then $\mathcal{L}^p \subset \mathcal{L}$ is nonempty. A chief is either perfect or countable. In the latter case all nondegenerate leaves are isolated.

**Proof.** By [11, Lemma 3.12], the set $\mathcal{L}^p$ is an invariant lamination. If $\mathcal{L}$ is uncountable, then it is easy to see that $\mathcal{L}^p \subset \mathcal{L}$ is nonempty. The last claim holds by Lemma 2.3(2).
Given a chord \( \ell = \overline{ab} \), let \( |\ell| \) denote the length of the smaller circle arc with endpoints \( a \) and \( b \) (computed with respect to the Lebesgue measure on \( S \) such that the total length of \( S \) is 1); call \( |\ell| \) the length of \( \ell \).

**Lemma 2.11.** If \( 0 < |\ell| < \frac{1}{d+1} \), then \( |\sigma_d(\ell)| = d|\ell| \) or \( |\sigma_d(\ell)| \geq \frac{1}{d+1} \). The \( \sigma_d \)-orbit of any chord has a chord of length \( \geq \frac{1}{d+1} \). In any nonempty \( \sigma_d \)-invariant lamination, there are leaves of length \( \geq \frac{1}{d+1} \).

The proof of Lemma 2.11 is straightforward; it is left to the reader.

**Lemma 2.12.** If \( \mathcal{L} \) is nonempty, then \( \mathcal{L} \) contains a chief.

**Proof.** Let \( L_\alpha \) be a nested family of laminations. Definition 2.2 implies that then \( \bigcap L_\alpha \) is a sibling invariant lamination too. If all \( L_\alpha \) are nonempty, then by Lemma 2.11 each of them has a leaf of length at least \( \frac{1}{d+1} \) and so \( \bigcap L_\alpha \) is nonempty. Now the desired statement follows from Zorn’s lemma.

Given \( \mathcal{L} \) and a nondegenerate leaf \( \ell \in \mathcal{L} \), let \( G(\ell) \subset \mathcal{L} \) be the set of all iterated pullbacks of \( \ell \) and all its nondegenerate iterated images. Lemma 2.13 now follows from the compactness property of invariant laminations.

**Lemma 2.13.** Let \( \mathcal{L} \) be a chief. If \( \ell \in \mathcal{L} \) is a nondegenerate leaf, then \( G(\ell) \) is dense in \( \mathcal{L} \).

2.5. **Invariant objects.** Let \( \Delta \subset \mathbb{C} \) be an open Jordan disk. Recall [15, Definition 3.6] that a continuous map \( f : \Delta \rightarrow \mathbb{C} \) is weakly polynomial-like (weakly PL for short) of degree \( d \) if \( f(\text{Bd}(\Delta)) \cap \Delta = \emptyset \), and the induced map on integer homology

\[
f_* : H_2(\Delta, \text{Bd}(\Delta)) \cong \mathbb{Z} \rightarrow H_2(\mathbb{C}, \mathbb{C} \setminus \{z_0\}) \cong \mathbb{Z}
\]

is the multiplication by \( d \), where \( z_0 \) is any base point in \( \Delta \).

**Lemma 2.14.** [15, Lemma 3.7] If \( f : \overline{D} \rightarrow \mathbb{C} \) is weakly PL with isolated fixed points, then the degree of \( f \) equals the sum of the Lefschetz indices over all fixed points of \( f \) in \( \overline{D} \).

The notion of the Lefschetz index adapted to the present context is discussed in [15]. A gap \( G \) of a lamination is invariant if \( \sigma_d(G) = G \) (with “=” rather than “\( \subset \)”).

**Lemma 2.15.** A \( \sigma_d \)-invariant lamination has an invariant lap or an invariant infinite gap.

**Proof.** Consider a \( \sigma_d \)-invariant lamination \( \mathcal{L} \). Extend \( \sigma_d \) linearly over each simplex in the barycentric subdivision of \( \mathcal{L} \) (cf. [27]), and denote the extended map by \( f : \overline{D} \rightarrow \mathbb{C} \); this map is weakly PL. If there are invariant non-degenerate leaves of \( \mathcal{L} \) or invariant gaps with fixed points on the boundary, then we are done. Otherwise, \( f \) has isolated fixed points. There are \( d-1 \)
fixed points on the unit circle, all with Lefschetz index one. Thus, there is a fixed point \( a \) inside \( \mathbb{D} \). If \( G \) is a gap of \( \mathcal{L} \) containing \( a \), then \( G \) is clearly invariant. □

3. INVARIANT GAPS AND THEIR CANONICAL LAMINATIONS, CRITICAL PORTRAITS, AND FLOWER-LIKE SETS

By cubic (resp., quadratic) laminations, we always mean sibling \( \sigma_3 \)- (resp., \( \sigma_2 \)) invariant laminations. When dealing with cubic laminations, we write \( \sigma \) instead of \( \sigma_3 \). For brevity we denote the horizontal diameter of the unit disk by \( \overline{H \mathbb{D}} \). From now on \( \mathcal{L} \) (possibly with sub- and superscripts) denotes a cubic sibling invariant (pre)lamination.

This section is based upon [8]. An invariant gap is an invariant gap of a cubic lamination (the latter may be unspecified). An infinite invariant gap is quadratic if it has degree 2. By [8, Section 3], a quadratic invariant gap is as follows. A critical chord \( \tau \) gives rise to an open circle arc \( L(\tau) \) of length \( 2/3 \) with the same endpoints as \( \tau \); let \( I(\tau) \) be the complement of \( L(\tau) \). The set \( \Pi(\tau) \) of all points with orbits in \( L(\tau) \) is nonempty, closed and forward invariant. Let \( \Pi'(\tau) \) be the maximal perfect subset of \( \Pi(\tau) \). The convex hulls \( G(\tau) \) of \( \Pi(\tau) \) and \( G'(\tau) \) of \( \Pi'(\tau) \) are invariant quadratic gaps, and any invariant quadratic gap is like that.

If \( \tau \) has non-periodic endpoints, or both eventually map to \( I(\tau) \), then \( \Pi(\tau) = \Pi'(\tau) \) and \( G(\tau) = G'(\tau) \). Moreover, if the entire orbit of \( \sigma(\tau) \) is contained in \( L(\tau) \), then \( \tau \) is an edge of \( G(\tau) \), and \( G(\tau) \) is said to be of regular critical type. If an endpoint of \( \tau \) is periodic, and the entire orbit of \( \tau \) is contained in \( L(\tau) \), then \( \Pi'(\tau) \subseteq \Pi(\tau) \), and \( G(\tau) \) is said to be of caterpillar type. In this case, \( G'(\tau) \) is said to be of periodic type.

For any invariant gap \( G \), finite or infinite, a major of \( G \) is an edge \( M = \overline{ab} \) of \( G \), for which there is a critical chord \( \overline{ax} \) or \( \overline{by} \) disjoint from the interior of \( G \). By Section 4.3 of [8], a degree 1 invariant gap \( G \) has one or two majors; every edge of \( G \) eventually maps to a major and, if \( G \) is infinite, at least one of its majors is critical. An invariant gap \( G \) is rotational if \( \sigma \) acts on its vertices (i.e., on \( G \cap S \)) as a combinatorial rotation. A chord is compatible with a finite collection of gaps if it does not cross edges of these gaps.

Quadratic invariant gaps give rise to canonical laminations. Defining those is easy for quadratic invariant gaps of regular critical and caterpillar type as in those cases (i.e., if the major of the quadratic invariant gap \( U \) is critical) there is a unique lamination \( \mathcal{L}_U \) that has the gap \( U \). Indeed, under the assumptions we already have two critical sets of \( \mathcal{L}_U \), namely \( U \) and its (critical) major, therefore, all iterated pullbacks of \( U \) that are gaps of \( \mathcal{L}_U \) are well-defined; these pullbacks of \( U \) “tile” the entire \( \overline{\mathbb{D}} \) and give rise to \( \mathcal{L}_U \).
For a gap $U$ of periodic type, $\mathcal{L}_U$ is as follows. Add to $U$ a critical quadrilateral $Q_U = Q$ which is the convex hull of the major $M_U = M$ of $U$ and its sibling $M'_U = M'$ located outside of $U$. Form the pullback lamination with critical sets $U$ and $Q$; this lamination is now well defined. After removing the edges $\ell$ and $\ell'$ of $Q$ distinct from $M$ and $M'$, and all iterated pullbacks of $\ell$ and $\ell'$, we obtain the canonical lamination $L_U$ which, unlike before, has two cycles of Fatou gaps. Indeed, $L_U$ has a gap $V \supset Q$ with $M$ and $M'$ as its edges, and $\sigma|_{\partial(V)}$ is two-to-one. The entire $\mathbb{D}$ is “tiled” by concatenated pullbacks of $U$ and $V$. In this context $U$ is said to be the senior (gap), and $V$ is called the vassal (gap).

For sets $A$, $B$, let $A \lor B$ be the set of all unordered pairs $\{a, b\}$ with $a \in A, b \in B$. Let $CCh$ be the set of all individual $\sigma_3$-critical chords with the natural topology; $CCh$ is homeomorphic to $\mathbb{S}$. A critical portrait is a pair $\{\overline{c}, \overline{y}\} \in CCh \lor CCh$ such that $\overline{c}$ and $\overline{y}$ do not cross. Let $CrP$ be the space of all critical portraits. It is homeomorphic to the Möbius band $[28]$. For an invariant $\mathcal{L}$, let $CrP(\mathcal{L})$ be the family of all critical portraits compatible with $\mathcal{L}$; if $K \in CrP(\mathcal{L})$, call $K$ a critical portrait of $\mathcal{L}$.

**Definition 3.1** (Flower-like sets). Suppose that $\mathcal{L}$ has an infinite invariant gap $U$ or an invariant lap $G$ and an infinite gap $U$ that shares an edge with $G$ (in the latter case it follows that $U$ is periodic). Then $\{U\}$ (in the former case) or the set consisting of $G$ and all periodic Fatou gaps attached to it (in the latter case) is said to be a flower-like set. Thus, a flower-like set is a certain set of gaps or laps. Flower-like sets can be viewed as standing alone (i.e., without specifying a lamination but with the understanding that such a lamination exists).

By saying that $\mathcal{L}$ has a flower-like set $F$, we mean that all gaps/laps from $F$ are gaps/laps of $\mathcal{L}$. Say that $\mathcal{L}$ is compatible with a flower-like set $F$ if no leaf of $\mathcal{L}$ crosses an edge of a gap from $F$. Heuristically, flower-like sets represent dynamics of polynomials with a non-repelling fixed point. They are a convenient object of study as shown by Lemma 3.2.

**Lemma 3.2.** If $F_i$ are flower-like sets, and $F_i \to F$ then $F$ contains a flower-like set.

By $F_i \to F$, we mean that $F_i = \{F_i^1, \ldots, F_i^k\}$ and $F = \{F^1, \ldots, F^k\}$ so that $F^j_i \to F^j$ as $i \to \infty$ for each $j = 1, \ldots, k$.

**Proof.** If periodic gaps/laps $H_i$ of period $m$ converge to a gap/lap $H$, then $H$ is periodic of period $m$. If all $H_i$’s are infinite, then, by Lemma 2.11 for any integer $N > 0$, there is $\varepsilon > 0$ such that $N$ edges of $H_i$ are longer than $\varepsilon$ for any $i$. Choosing a subsequence, we see that $H$ has at least $N$ non-degenerate edges, and $H$ is an infinite periodic gap. Thus, we may assume
that each $F_i$ consists of a finite invariant lap $G_i$ and the cycle of an infinite gap $U_i$ sharing an edge $\ell_i$ with $G_i$, and that $G_i$'s converge to an invariant gap $G$. If $G$ is infinite, we are done. Assume that $G$ is finite. Then $G_i = G$ for large $i$ (by Lemma 2.8), and by the above $U_i \to U$ where $U$ is infinite and periodic, which completes the proof. □

4. TWO TYPES OF CUBIC CHIEFS

**Definition 4.1.** A chief is **central** if it is compatible with a flower-like set; a chief is **regular** otherwise.

### 4.1. Central chiefs.

Countable chiefs are rather specific.

**Lemma 4.2.** Let $\mathcal{L}$ be a cubic countable chief. Then:

1. all nondegenerate leaves of $\mathcal{L}$ are isolated;
2. for any nondegenerate leaf $\ell \in \mathcal{L}$, the set of all nondegenerate leaves in $\mathcal{L}$ coincides with $G(\ell)$;
3. the lamination $\mathcal{L}$ has a flower-like set.

**Proof.** (1) This claim follows from Lemma 2.10.

(2) By Lemma 2.13, the set $G(\ell)$ is dense in $\mathcal{L}$. Since each leaf is of $\mathcal{L}$ is isolated, then $G(\ell)$ equals the set of all nondegenerate leaves in $\mathcal{L}$.

(3) By Lemma 2.15 find an invariant lap or infinite gap $G$ of $\mathcal{L}$. If $G$ is infinite, then by definition $\mathcal{L}$ has a flower-like set. Assume that $G$ is finite. Let $\ell$ be an edge of $G$; it is isolated by (1). Let $H$ be a gap of $\mathcal{L}$ attached to $G$ along $\ell$. If $H$ is infinite, then, again by definition, $\mathcal{L}$ has a flower-like set. Assume that $H$ is finite. If $n$ is the period of $\ell$, then there are two cases: $\sigma^n(H) = H$ and $\sigma^n(H) = \ell$. The former case contradicts (2), hence we may assume that $H \cap G = \ell$ is a leaf and $\sigma(H) = \sigma(\ell)$, i.e. that $H$ is a collapsing gap that shares an edge with $G$. Then there are several cases: 1) the gap $H$ is a collapsing quadrilateral with a sibling gap of $G$ attached to $H$ at the edge of $H$ that is a sibling of $\ell$; 2) the gap $H$ is a collapsing hexagon with two more edges $\ell_1$ and $\ell_2$ so that $\ell$, $\ell_1$ and $\ell_2$ are pairwise disjoint and at $\ell_1$ and $\ell_2$ sibling gaps of $G$ are attached to $H$ (the fact that $\ell$, $\ell_1$ and $\ell_2$ are pairwise disjoint uniquely defines $\ell_1$ and $\ell_2$ since $\ell$ is given); 3) a modified version of the second case is possible when $H$ is a hexagon from the second case but there is also a diagonal of this $H$ that creates two collapsing quadrilaterals inside $H$ and is the only common edge of these two quadrilaterals. Consider case 1); then there are edges $\overline{q}_1$ and $\overline{q}_2$ of $H$ that share endpoints with $\ell$. They are isolated in $\mathcal{L}$. It follows by Definition 2.2 of a sibling invariant lamination that we can remove these leaves and their pullbacks and still have a smaller sibling invariant lamination than $\mathcal{L}$, a contradiction with the fact that $\mathcal{L}$ is a chief. Similarly we show that in cases 2) and 3) the lamination $\mathcal{L}$ is not a chief. □
There are also uncountable (hence, by Lemma 2.10 perfect) chiefs compatible with flower-like sets, but not having them. Here is heuristic example. Take a perfect non-renormalizable quadratic lamination $\mathcal{L}_2$ with critical diameter $\ell$. Choose a non-periodic non-precritical leaf $\overline{xy} \in \mathcal{L}_2$ and blow up $y$ to create a regular critical major $\overline{y}$ which results into a triangle contained in a quadratic invariant gap $U$, then reflect this triangle with respect of $y$ and erase $\overline{y}$ to create a critical quadrilateral $Q$. Together with the leaf $\ell$ that used to be $\ell$ before the transformations, we have two critical sets $Q$ and $\ell$ that define a cubic lamination $\mathcal{L}_3$. We claim that this is a perfect chief. Indeed, suppose that there is a non-empty lamination $\hat{\mathcal{L}} \subset \mathcal{L}_3$. If $\hat{\ell} \notin \hat{\mathcal{L}}$, then $\hat{\ell}$ is inside a gap $G$ of $\hat{\mathcal{L}}$. By the assumptions pullbacks of $\hat{\ell}$ in $\mathcal{L}_3$ approach $\hat{\ell}$ from both sides; hence $G$ cannot be finite and so $G$ must be infinite, a contradiction with the fact that $\mathcal{L}$ is non-renormalizable. Since pullbacks of $\hat{\ell}$ approximate $\overline{xy}$, the set $Q$ survives, too, and in the end $\mathcal{L}_3 = \hat{\mathcal{L}}$, a contradiction.

**4.2. Regular chiefs.** Let us prove a useful claim concerning regular chiefs.

**Lemma 4.3.** A regular chief $\mathcal{L}$ is perfect, has infinitely many periodic laps, and the family of all iterated pullbacks of a periodic leaf of $\mathcal{L}$ is dense in $\mathcal{L}$.

**Proof.** By Lemma 2.10 a chief is countable or perfect. By Lemma 4.2 and since $\mathcal{L}$ is regular, $\mathcal{L}$ is perfect, and, hence, clean. Consider the associated topological polynomial $f_\mathcal{L} : J_\mathcal{L} \to J_\mathcal{L}$ to which $\sigma_3|_S$ is semiconjugate by a map $\varphi$. Since $\mathcal{L}$ is perfect, there are uncountably many grand orbits of non-degenerate leaves of $\mathcal{L}$ containing no leaves of critical sets of $\mathcal{L}$. Choose a leaf $\ell$ from one of these grand orbits; then $\varphi(\ell) = x$ is a cutpoint of $J_\mathcal{L}$, and all points of the $f_\mathcal{L}$-orbit of $x$ are cutpoints of $J_\mathcal{L}$. Such dynamics was studied in [6] where, in Theorem 3.8, it was proven that then $f_\mathcal{L}$ has infinitely many periodic cutpoints. Taking their $\varphi$-preimages, we see that $\mathcal{L}$ has infinitely many periodic laps. The last claim holds by Lemma 2.13.

**5. The space of alliances**

We will need the next lemma (here we state a weaker version of Lemma 3.53 of [11]).

**Lemma 5.1.** [11] Lemma 3.53] If there is a critical portrait compatible with cubic laminations $\mathcal{L}$ and $\mathcal{L}'$, then any leaf of $\mathcal{L}$ crosses at most countably many leaves of $\mathcal{L}'$, and vice versa.

Recall that two distinct chords cross if they have common points in $\mathcal{D}$.

**Definition 5.2.** The set $\text{CrP}(\mathcal{L})$ of all critical portraits compatible with a regular chief $\mathcal{L}$ is called a regular alliance of critical portraits (generated
Critical portraits from regular alliances of critical portraits are said to be \textit{regular}. All laminations whose chiefs are regular are said to be \textit{regular}. Regular laminations with the same chief form a \textit{regular alliance of laminations}. The set \( \mathcal{A}_0 \) of all critical portraits compatible with central chiefs is called the \textit{central alliance of critical portraits}. Critical portraits from \( \mathcal{A}_0 \) are called \textit{central}. All laminations with central chiefs form the \textit{central alliance of laminations}.

Theorem \( 5.3 \) shows that alliances are pairwise disjoint.

**Theorem 5.3.** Let \( \mathcal{L} \neq \mathcal{L}' \) be chiefs. Then \( \text{CrP}(\mathcal{L}) \cap \text{CrP}(\mathcal{L}') = \emptyset \) unless \( \mathcal{L} \) and \( \mathcal{L}' \) are central. Thus, two distinct alliances of critical portraits are disjoint, and two distinct alliances of laminations are disjoint.

**Proof.** If \( \mathcal{L} \) is regular, then \( \mathcal{L} \) is perfect by Lemma \( 4.3 \). Let \( \mathcal{K} = \{\tau, \gamma\} \) be a critical portrait compatible with \( \mathcal{L} \) and \( \mathcal{L}' \). Since \( \mathcal{L} \) is perfect, \( \mathcal{L} \) and \( \mathcal{L}' \) are compatible by Lemma \( 5.1 \). If \( \mathcal{L}' \) has a flower-like set, then \( \mathcal{L} \) is compatible with it, a contradiction with \( \mathcal{L} \) being regular. Hence \( \mathcal{L}' \) has no flower-like sets. Hence \( \mathcal{L}' \) is perfect by Lemma \( 4.2 \). Also, by Lemma \( 2.15 \) there is an invariant lap or infinite gap \( G' \) of \( \mathcal{L}' \), and \( G' \) is a lap, again because \( \mathcal{L} \) is regular and compatible with \( G' \). If a leaf \( \ell \in \mathcal{L} \) is inside \( G' \), then, since \( \mathcal{L} \) is perfect, other leaves of \( \mathcal{L} \) approximate \( \ell \) and, hence, cross leaves of \( \mathcal{L}' \), a contradiction. Hence \( G' \subset G \), where \( G \) is an invariant lap of \( \mathcal{L} \) (since \( \mathcal{L} \) is regular, \( G \) is finite). Since no leaf of \( \mathcal{L}' \) can be inside \( G \), then \( G = G' \).

We claim that each edge of \( G \) is a limit of leaves of \( \mathcal{L} \). By way of contradiction, this claim holds if \( G \) is a gap or a “flipping” leaf (whose endpoints map to each other under \( \sigma_3 \)), since \( \mathcal{L} \) is perfect. Suppose that \( G = \text{Hdi} \) is isolated in \( \mathcal{L} \) from below. Notice that perfect laminations cannot have collapsing quadrilaterals with a periodic edge; hence the only gap of \( \mathcal{L} \) that can be attached to \( G \) from below is the (infinite) gap \( FG_{b} \), a contradiction with \( \mathcal{L} \) being regular. Thus, each edge of \( G \) is a limit of leaves of \( \mathcal{L} \). Similarly, each edge of \( G' = G \) is a limit of leaves of \( \mathcal{L}' \).

If iterated images of \( \tau \) and \( \gamma \) avoid \( G \), then iterated \( \mathcal{L} \)-pullbacks of \( G \) and iterated \( \mathcal{L}' \)-pullbacks of \( G' \) are the same. Hence \( \mathcal{L} = \mathcal{L}' \) since the iterated pullbacks of \( G \) are dense in both \( \mathcal{L} \) and \( \mathcal{L}' \) by Lemma \( 2.13 \). Let for some minimal \( n \geq 0 \) the point \( \sigma^n(\tau) \) be a vertex of \( G \). Let \( C, C' \) be the critical sets of \( \mathcal{L}, \mathcal{L}' \), resp., containing \( \tau \). By the previous paragraph, the only way this can happen is when \( C = C' \) is the appropriate \( \sigma^n \)-pullback of \( G = G' \) (in particular, neither \( C \) nor \( C' \) can be a collapsing quadrilateral because both laminations are perfect). Similarly, we see that either \( \gamma \) never maps to \( G \) or the critical sets of \( \mathcal{L}, \mathcal{L}' \) containing \( \gamma \) coincide. Thus, iterated pullbacks of \( G \) in \( \mathcal{L} \) are the same as iterated pullbacks of \( G \) in \( \mathcal{L}' \), and \( \mathcal{L} = \mathcal{L}' \). The remaining two claims of the theorem immediately follow. \( \square \)
Let us prove a combinatorial analog of the Main Theorem. Denote the space of all (sibling) invariant cubic laminations by $\mathcal{Lam} = \mathcal{Lam}_3$. For $L \in \mathcal{Lam}$, let $\mathcal{A}(L)$ be the alliance of laminations to which $L$ belongs; for a critical portrait $K$, let $\mathcal{A}(K)$ be the alliance of critical portraits to which $K$ belongs. By Theorem 5.3, these sets are well-defined.

**Theorem 5.4.** Alliances of laminations form a USC-partition of the space $\mathcal{Lam}$. Alliances of critical portraits form a USC-partition $\{A\}$ of the set $\mathcal{CrP}$. The map $\Psi : \mathcal{Lam} \to \mathcal{CrP}/\{A\}$ that associates to each lamination $L$ the alliance of critical portraits compatible with chief(s) from $\mathcal{A}(L)$ is well-defined and continuous. The union of regular alliances of critical portraits is open and dense in $\mathcal{CrP}$.

**Proof.** By Theorem 5.3, alliances of laminations partition $\mathcal{Lam}$ and alliances of critical portraits partition $\mathcal{CrP}$. Clearly, a regular alliance of laminations is closed in $\mathcal{Lam}$, and a regular alliance of critical portraits $\mathcal{CrP}(L)$, with $L$ a regular chief, is closed in $\mathcal{CrP}$.

We claim that the central alliance of laminations is closed. Let $L_i \to L$ where $L_i$ are central laminations with central chiefs $L_i''$; by Theorem 2.4 we may assume that $L_i''$ converge to an invariant lamination $L'' \subset L$. Choose flower-like sets $F_i'$ compatible with $L_i''$ for every $i$. By Lemma 3.2 we may assume that $F_i' \to F'$ with $F'$ containing a flower-like set $F'$ compatible with $L''$. Hence chiefs of $L''$ are central as desired. This implies that the central alliance of critical portraits is closed.

To show that alliances of laminations form a USC-partition of $\mathcal{Lam}$, let $L_i \to L$ and $L_i' \to L'$ be two sequences of laminations where $L_i$ and $L_i'$ belong to the same regular alliance of laminations with a chief $L_i''$ for every $i$. Assume that $L_i'' \to L''$; then $L'' \subset L \cap L'$ and, hence, that $L$ and $L'$ have a common chief, and belong to the same alliance (regular or central).

To show that alliances of critical portraits form a USC-partition of $\mathcal{CrP}$, let $K_i \to K$ and $K_i' \to K'$ be two sequences of critical portraits, where $K_i$ and $K_i'$ belong to the same regular alliance of critical portraits with chiefs $L_i''$ compatible with $K_i$ and $K_i'$ for every $i$. Assume that $L_i'' \to L''$. It follows that $L''$ is compatible with both $K$ and $K'$. Hence $K$ and $K'$ belong to the same alliance of critical portraits (regular or central). The same arguments show that the map $\Psi : \mathcal{Lam} \to \mathcal{CrP}/\{A\}$ is well defined and continuous.

We claim that the union $\mathcal{U}$ of regular alliances of critical portraits is open and dense in $\mathcal{CrP}$. The set $\mathcal{U}$ is open since its complement is the central alliance of critical portraits which is closed. Let $K = \{\overline{\tau}, \overline{\eta}\}$ be a critical portrait such that the orbits of $\sigma(\overline{\tau})$ and $\sigma(\overline{\eta})$ are dense in $\mathbb{S}$. We claim that $K$ is regular. Indeed, if it is central, then there is a central chief $\mathcal{L}$ compatible with $K$. Let $C \supset \overline{\tau}$ and $Y \supset \overline{\eta}$ be the critical sets of $\mathcal{L}$. If $C$ is infinite, then $C$ is a (pre)periodic gap by [20], which contradicts the density of $\sigma(\overline{\tau})$. 

Thus, $C$ (and $Y$) are finite. On the other hand, $L$ is compatible with a flower-like set $F$. In particular, there is a cycle of infinite gaps compatible with $L$. Let $G$ be a gap from this cycle that contains a critical chord (as an edge or a diagonal). Since endpoints of $\tau$ and $\eta$ cannot be vertices of $G$, we may assume that $\tau = \overline{xy}$ where $x, y$ belong to distinct components $I, J$ of $\mathbb{S} \setminus G$. Since $L$ and $F$ are compatible, the finite concatenation of edges of $C$ that connects the endpoints of $c$ must pass through an endpoint of, say, $I$. As $\sigma(\tau)$ visits $I$ infinitely often, each time the corresponding image of a vertex of $C$ coincides with endpoint of $I$. The fact that $C$ has finitely many vertices implies now that a vertex of $C$ is preperiodic. To together with the density of the orbit of $\sigma(\tau)$ this yields a contradiction. □

6. The Model

Consider a polynomial $P \in \mathbb{C}^3$. Recall that $\lambda(P)$ is an equivalence relation on $\mathbb{Q}/\mathbb{Z}$ such that two arguments $\alpha, \beta \in \mathbb{Q}/\mathbb{Z}$ are equivalent if and only if the rays $R_P(\alpha), R_P(\beta)$ land at the same point.

Lemma 6.1. [19, Lemma 3.9] For any polynomial $P$, $\lambda(P)$ is closed in $\mathbb{Q} \times \mathbb{Q}$. If $xy \in L_P$ is (pre)periodic then $\{x, y\} \in \lambda(P)$ and the rays $R_P(x)$ and $R_P(y)$ form a cut.

We now define a stable version of $\lambda(P)$. Namely, a point $x$ is $(P\text{-})legal$ if it eventually maps to a repelling periodic point. An unordered pair of rational angles $\{\alpha, \beta\}$ is $(P\text{-})legal$ if the rays with arguments $\alpha$ and $\beta$ land at the same legal point of $P$. Let $\mathcal{Z}_P$ be the set of all $P$-legal pairs of angles.

Let $\sim_P$ be the equivalence relation on $\mathbb{S}$ defined by $e^{2\pi i \alpha} \sim_P e^{2\pi i \beta}$ if $\{\alpha, \beta\} \in \mathcal{Z}_P$ or $\alpha = \beta$. Unlike $\lambda(P)$, the equivalence relation $\sim_P$ deals with repelling cycles but not with parabolic cycles. Thus, $\sim_P$-classes are $\lambda(P)$-classes, yet some $\lambda(P)$-classes (namely, the ones related to parabolic points) are not $\sim_P$-classes. It follows that $\sim_P \subset \lambda(P)$. Let $\mathcal{Z}^{\text{lam}}_P$ be the set of all edges of the convex hulls in $D$ of all $\sim_P$-classes. Define $L'_P$ as this set of edges $Z^{\text{lam}}_P$ and the limits of these edges. Adding the limits may make the distinction between $\sim_P$ and $\lambda(P)$ less significant, as the closures of the family of convex hulls of $\sim_P$-classes and the family of convex hulls of $\lambda(P)$-classes may coincide. By Theorem [2.4], the set $L'_P$ is a cubic lamination (cf. [3]), and it is easy to see that it is clean.

The lamination $L'_P$ is associated with an equivalence relation $\sim_{L'_P}$ so that all laps of $L'_P$ are convex hulls of $\sim_{L'_P}$-classes. Since $\sim_{L'_P}$ is the closure of $\sim_P$, in what follows we simply denote it by $\sim_P$. If $P$ has no neutral periodic points, then $\sim_P$ coincides with the rational lamination [19] of $P$ while $\sim_{L'_P}$ coincides with the real lamination [21] of $P$. The next lemma summarizes some results of [19, 21] (see also [1]).
Lemma 6.2. Suppose that $P \in \mathcal{C}_3$ has no neutral cycles. Then the restriction $P|_{J_P}$ is monotonically semiconjugate to the topological polynomial $f_{\sim P} : J_{\sim P} \to J_{\sim P}$ on its topological Julia set so that fibers of this semiconjugacy are trivial at all (pre)periodic points of $J_{\sim P}$ (thus, for a periodic lap $G$ of $\mathcal{L}_P^r$, rays corresponding to vertices of $G$ land at the same legal point). Also, any clean lamination whose isolated critical leaves are strictly preperiodic has the form $\mathcal{L}_P^r$ for some $P$ without neutral cycles.

In this section by “alliance” we mean “alliance of critical portraits”. If $\mathcal{L}_P^r$ is regular, then by Theorem 5.3 it has a unique regular chief denoted by $\mathcal{L}_P^c$. Denote by $A_P$ the alliance $\text{CrP}(\mathcal{L}_P^r)$ generated by $\mathcal{L}_P^c$. However, if $\mathcal{L}_P^r$ is central (e.g., if $\mathcal{L}_P^r$ is empty), we let $A_P = A_0$ be the central alliance. Thus, $A_0$ serves all polynomials $P$ with empty $\mathcal{L}_P^r$, all polynomials with non-repelling fixed points, and some other polynomials. By Theorem 5.4, the alliance $A_0$ is a closed subset of $\text{CrP}$.

Lemma 6.3. A regular alliance $A_P$ has the form $\text{CrP}(\mathcal{L}_Q^r)$ for some polynomial $Q$, possibly different from $P$; here $\mathcal{L}_Q^r = \mathcal{L}_P^r$.

Proof. By Lemma 4.3 $\mathcal{L}^r = \mathcal{L}_P^r$ is perfect. Then $\mathcal{L}$ has no infinite periodic gaps of degree 1 by Lemma 2.7. By Lemma 6.2, there is a polynomial $Q$ without neutral periodic points such that $\mathcal{L}_Q^r = \mathcal{L}_P^r$. Then by definition $A_P = \text{CrP}(\mathcal{L}) = \text{CrP}(\mathcal{L}_Q^r)$ as desired. □

The next lemma immediately follows and is stated without a proof.

Lemma 6.4. For any $P$ with nonempty $\mathcal{L}_P^r$, we have $\text{CrP}(\mathcal{L}_P^r) \subset A_P$.

A point $x$ is $(P)$-stable if its forward orbit is finite and contains no critical or non-repelling periodic points. The next lemma is proven in [14] (cf [15, Lemma B.1]).

Lemma 6.5. [14] Let $g$ be a polynomial, $g \in \mathcal{C}_3$, and $z$ be a stable point of $g$. If an ray $R_g(\theta)$ with rational argument $\theta$ lands at $z$, then, for every polynomial $\hat{g}$ sufficiently close to $g$, the ray $R_{\hat{g}}(\theta)$ lands at a stable point $\hat{z}$ close to $z$. Moreover, $\hat{z}$ depends holomorphically on $\hat{g}$.

The next lemma deals with the continuity of alliances in the regular case. Given families of sets $A_i$ and $A$ in the plane, we will use the notation $A_i \rightarrow A$ for lim sup $A_i \subset A$.

Lemma 6.6. Consider a sequence of polynomials $P_i \in \mathcal{C}_3$ converging to a polynomial $P \in \mathcal{C}_3$. If $\mathcal{L}_P^r$ is regular and $\mathcal{L}_{P_i}^r$ converge to some lamination $\mathcal{L}^r$, then $\mathcal{L}_P^r$ and $\mathcal{L}^r$ belong to the same regular alliance of laminations, $\mathcal{L}_{P_i}^r$ are regular for all sufficiently large $i$, and $A_{P_i} \rightarrow A_P$. 
Proof. Set $\mathcal{L} = \mathcal{L}_P$. By Lemma 4.3 it is perfect, has infinitely many periodic laps, and all iterated pullbacks of a periodic leaf of $\mathcal{L}$ are dense in $\mathcal{L}$. Choose a repelling $P$-periodic point $x$ that is not an eventual image of a critical point of $P$. Take any iterated $P$-preimage $y$ of $x$ and write $A_y(P)$ for the collection of arguments of all $P$-external rays landing at $y$. Then, by Lemma 6.5 there exists $N_y$ such that for every $i > N_y$, there is a point $y_i$ with $A_{y_i}(P_i) = A_y(P)$. It follows that $\mathcal{L} \subset \mathcal{L}'$. This implies the first claim of the lemma which, by Theorem 5.4, implies the other claims.

Definition 6.7. Define the map $\pi : C_3 \to CrP/\{A_P\}$ by the formula $\pi(P) = A_P$ (here $C_3$ is the cubic connectedness locus and $CrP/\{A_P\}$ is the quotient space of $CrP$ formed by alliances).

Theorem 6.8 is a part of the Main Theorem.

Theorem 6.8. The map $\pi$ is continuous.

Proof. Consider a sequence $P_i \to P$ of polynomials, and set $A_i = A_{P_i}$. If $A_i \not\to A_P$, then by Theorem 5.4 we may assume that $A_i \to A' \neq A_P$ where (1) the limit $A'$ is an alliance, and (2) all alliances $A_i$ are regular, or $A_i = A_0$ for every $i$. By Lemma 6.6 this is impossible if $A_P$ is regular, or if $A_i = A_P = A_0$. It remains to assume that all alliances $A_i$ are regular, $A'$ is regular, $A_P = A_0$ is central, and bring this to a contradiction. Denote by $\mathcal{L}'$ the regular chief corresponding to $A'$ so that $A' = CrP(\mathcal{L}')$.

Set $\mathcal{L}^r_{P_i} = \mathcal{L}_i$, and assume that $\mathcal{L}_i \to \mathcal{L}$ for a regular lamination $\mathcal{L}$. Then $\mathcal{L}' \subset \mathcal{L}$ by Theorem 5.4. By Lemma 4.3 there are infinitely many periodic laps of $\mathcal{L}'$; since $\mathcal{L}' \subset \mathcal{L}$, it is easy to see (using Lemma 2.8) that any such lap is a lap of $\mathcal{L}$. Denote by $B_P$ the family of vertices of laps of $\mathcal{L}_P$, associated with parabolic points. Choose a periodic lap $G$ of $\mathcal{L}'$ so that no vertex of $G$ belongs to $B_P$ (this is always possible as $B_P$ is finite). By Lemma 2.8 the set $G$ is a lap of $\mathcal{L}_i$ for sufficiently large $i$. By Lemma 6.2 by our assumptions, $G$ is associated with a repelling periodic point, say, $y_i$ of $P_i$.

It is easy to see that then $G$ is a lap of $\mathcal{L}'_{P_i}$, too. Indeed, $P$-rays corresponding to the vertices of $G$ land at repelling points by the choice of $G$. By the above, $P_i$-rays with the same arguments all land at $y_i$. If the $P_i$-rays mentioned above do not land at the same point, then by continuity (Lemma 6.5) neither do the corresponding $P_i$-rays, a contradiction. So, $G$ is a lap of $\mathcal{L}'_{P_i}$, too. Denote by $y$ the repelling periodic point of $P$ associated with $G$. By the choice of $G$, no critical point of $P$ ever maps to $y$, hence, by Lemma 6.5 all pullbacks of $G$ in $\mathcal{L}'_{P_i}$ eventually become laps of $\mathcal{L}_i$, and, therefore, of $\mathcal{L}'$. By the properties of regular chiefs listed in Lemma 4.3 it follows that $\mathcal{L}' \subset \mathcal{L}'_{P_i}$. This contradicts the assumption that $A_P = A_0$ and completes the proof.
We now complete the proof of the Main Theorem. For convenience let us recall some important definitions. Flower-like sets are defined in Definition 3.1. A chief is central if it is compatible with a flower-like set. A critical portrait is central if it is compatible with a central chief. The family of all central critical portraits is the central alliance (of critical portraits) $A_0$. The central fiber $\pi^{-1}(A_0)$ is the set of all cubic polynomials $P \in C_3$ that belongs to the central alliance of laminations. All other fibers of $\pi$ are called regular fibers. For $f_0 \in \text{Poly}_d$, the combinatorial renormalization domain $C(f_0)$ is defined \[17\] as \{ $f \in \text{Poly}_d \mid \lambda(f) \supset \lambda(f_0)$ \} (i.e., if two angles $\alpha$ and $\beta$ are $\lambda(f_0)$-equivalent, then they are $\lambda(f)$-equivalent).

**Lemma 7.1.** If $P, P_0 \in C_3$, every non-degenerate leaf of $L_{P_0}$ is in a lap of $L_P$, and $P_0$ has no neutral cycles, then $P \in C(P_0)$.

Note that Lemma 7.1 is applicable in the case $L_P \supset L_{P_0}$.

**Proof.** Since $P_0$ has no neutral cycles, then $\lambda(P_0) \sim_{P_0}$ on rational angles. If now $\alpha$ is $\lambda(P_0)$-equivalent to $\beta$, then $\alpha \sim_{P_0} \beta$. By the assumptions, if $\alpha \sim_{P_0} \beta$ then $\alpha \sim_P \beta$; by Lemma 6.1 then $\alpha$ is $\lambda(P)$-equivalent to $\beta$. Thus, $\lambda(P_0) \subset \lambda(P)$ as desired. \qed

The following theorem easily follows from much stronger results of Shen–Wang \[26\], Wang \[29\], and Kozlovsky–van Strien \[22\].

**Theorem 7.2.** Suppose that $L_P$ is perfect for some $P \in C_3$. If there exists no infinite sequence $P_n \in C_3$ such that $C(P_1) \supset C(P_2) \supset \ldots$ and $\bigcap_n C(P_n) \supset C(P)$ then the set $C(P)$ is connected. In particular, $C(P)$ is connected provided that $L_P$ is a perfect chief.

**Proof.** By Lemma 6.2 there is $P_0 \in C_3$ without neutral cycles such that $L_P = L_{P_0}$. It follows from Lemma 7.1 that $C(P) = C(P_0)$. Thus, we may assume that $P$ has no neutral cycles. If all periodic points of $P$ are repelling, then $C(P)$ is a singleton by \[22\] and the assumptions. Assume now that $P$ has (super)attracting domains. If $P$ is hyperbolic, the result follows from the main result of Shen and Wang \[26\]. Finally, if one critical point $c_1$ of $P$ lies in a (super)attracting periodic basin, the other critical point $c_2$ is in the Julia set, and $P$ has the properties from the statement of the theorem, then $C(P)$ is also connected by \[29\] (only the nonrenormalizable case is considered in \[29\] but the extension to the finitely renormalizable case is straightforward). \qed

Let $A$ be a regular alliance of critical portraits, and consider the corresponding regular fiber $\pi^{-1}(A)$. Evidently, Lemma 7.3 completes the proof of the Main Theorem.
Lemma 7.3. If $A$ is a regular alliance then the regular fiber $\pi^{-1}(A)$ equals $C(P)$ for some $P$. Any regular fiber is connected.

Proof. By Lemma 6.3 $A = \text{CrP}(L_P)$ for $P \in C$ without neutral cycles such that $L_P$ is a perfect chief. Then $\pi^{-1}(A) = C(P)$ by definition. Thus, by Theorem 7.2 a regular fiber is connected. □

Observe that for a regular fiber $\pi^{-1}(A) = C(P)$ the periodic Fatou domains of $P$ have pairwise disjoint closures as otherwise this would not be a regular fiber.

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