I. INTRODUCTION

Open systems described by non-Hermitian Hamiltonians have drawn increasing attention in recent years [1-40] because of their academic interest and importance/relevance to reality. Unlike Hermitian Hamiltonians whose eigenenergies are real, eigenenergies of non-Hermitian Hamiltonians are, in general, complex numbers whose real parts are interpreted as quasiparticle energies and the imaginary parts are the inverse of quasiparticle lifetimes [28, 29, 32, 40]. It is known that the level spacing distribution of random Hermitian systems is replaced by the Poisson distribution for quasiparticle level spacing of non-Hermitian disordered metals in the thermodynamic limit of infinite system size. This is a very surprising result because Poisson statistics is universally true for the Anderson insulators where energy eigenstates do not overlap with each other so that energy levels are independent from each other. For disordered metals where different eigenstates overlap with each other, one should expect different levels trying to stay away from each other so that the Poisson distribution should not apply there. Our results show that the larger non-Hermitian energy (dissipation) can invalidate level repulsion principle that holds dearly in quantum mechanics. Thus, our theory provides a unified picture for recent discovery of so called “level attraction” in various systems. It provides also a theoretical basis for manipulating energy levels.

Level statistics of extended states in random non-Hermitian Hamiltonians

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Absence of level repulsion between extended states in random non-Hermitian systems is demonstrated. As a result, the general Wigner-Dyson distributions of level spacing of diffusive metals in the usual Hermitian systems is replaced by the Poisson distribution for quasiparticle level spacing of non-Hermitian disordered metals in the thermodynamic limit of infinite system size. This is a very surprising result because Poisson statistics is universally true for the Anderson insulators where energy eigenstates do not overlap with each other so that energy levels are independent from each other. For disordered metals where different eigenstates overlap with each other, one should expect different levels trying to stay away from each other so that the Poisson distribution should not apply there. Our results show that the larger non-Hermitian energy (dissipation) can invalidate level repulsion principle that holds dearly in quantum mechanics. Thus, our theory provides a unified picture for recent discovery of so called “level attraction” in various systems. It provides also a theoretical basis for manipulating energy levels.

In this work, we study a disordered two-dimensional electron gas (2DEG) subjected to a perpendicular imaginary magnetic field that models the finite lifetime of electronic levels due to the electron-electron, or electron-phonon or electron-impurity interactions [28, 29, 32, 40]. It is well known that disordered Hermitian 2DEG can support extended states in the absence of a magnetic field only when spin-orbit interaction is present [52]. In order to facilitate a metal-insulator transition, the model Hamiltonian contains also a Rashba/Dresselhaus or SU(2) spin-orbit coupling (SOC) that widely exists in 2DEGs, especially in semiconductor heterostructures. This non-Hermitian model supports the Anderson localization transitions (ALTs), similar to its Hermitian counterparts [53]. Surprisingly, spacings of quasiparticle energies Re[E] of extended states follow the Poisson distribution P(s) = exp[−s] in the thermodynamic limit of infinite system size no matter whether the system preserves time-reversal (TR) symmetry or not. For a finite system when the non-Hermiticity energy is smaller than mean level spacing, quasiparticle level spacings follow the Wigner-Dyson distribution P_D(s). On the other hand, in both limits, spacing distributions of the imaginary parts of the complex eigenenergies Im[E] of the extended states are also universal in the sense that they do not depend on the models and model parameters.

The paper is organized as follows. The model and numerical methods are described in Sec. II while the existence of ALTs is substantiated in Sec. III. Various results of level statistics are presented in Sec. IV. A discussion of the experimental relevance and a summary are given in Sec. V and VI respectively.
II. MODEL AND METHODS

Our model is non-interacting electrons on a square lattice subjected to an imaginary magnetic perpendicular field \[\mathbf{B}\] that generates a non-Hermitian term \(i\gamma\sigma_z\), without skin effect \[\mathbf{C}\].

\[
H = \sum_i c_i^\dagger (\epsilon_i \sigma_0 + \eta \sigma_2 + i\gamma \sigma_z) c_i + \left\{ \sum_{\langle ij \rangle} c_i^\dagger V_{ij} c_j + h.c. \right\},
\]

where \(c_i^\dagger = (c_{i1}^\dagger, c_{i2}^\dagger)\) and \(c_i\) are electron creation and annihilation operators at lattice site \(i = (x_i, y_i)\). \(\sigma_0\) and \(\sigma_{x,y,z}\) are respectively the two-by-two identity matrix and Pauli matrices acting on the spin space. \(\eta = 1\) is used as the energy unit. Randomness is introduced through \(\epsilon_i\) that randomly distributes in \([-W/2, W/2]\) with \(W\) measuring disorder strength. Rashba SOC \[\mathbf{D}\] of strength \(\alpha = 0.1\) encoded in two-by-two matrices of \(V_{ij} = V_x = \sigma_0 + i\alpha \sigma_y\) and \(V_{ij} = V_y = \sigma_0 - i\alpha \sigma_x\) for \(\langle ij \rangle\) along the \(x-\) and the \(y-\)directions, respectively, is used in this study. Note that Hamiltonian \(\mathbf{E}\) preserves the TR symmetry if \(\eta = 0\) while the TR symmetry is broken for \(\eta \neq 0\). This can easily be checked from the TR operator \(\mathcal{T} = -i\sigma_y K\) that commutes with the Hamiltonian \(\mathcal{T}\mathcal{H}^{-1} = H\) for \(\eta = 0\) and does not commutes with \(H\) for \(\eta \neq 0\), \(\mathcal{T}\mathcal{H}^{-1} \neq H\), where \(K\) is the complex conjugation \[\mathbf{F}\].

The eigenstates of Hamiltonian \(\mathbf{G}\) can be either localized or extended, and these two groups of states form separated bands. This can be seen from the inverse participation ratio (IPR) of a right eigenstate \(\psi_E\) defined as \(p_2(E, W) = \sum_i |\psi_E(i)|^4\)^{-1}, where \(\psi_E(i)\) is the wave function amplitude at site \(i\). \(\psi_E\) satisfies \(H|\psi_E\rangle = E|\psi_E\rangle\) and \(\langle\psi_E|\psi_E\rangle = 1\). \(p_2\) measures how many lattice sites are occupied by the wave function. If there exists an ALT from extended states to localized states when disorder strength \(W\) varies for a fixed \(E\), the correlation length \(\xi\) diverges at the critical value \(W_c\) as \(\xi(W) \propto |W - W_c|^{-\nu}\). \(p_2\) near \(W_c\) satisfies the following one-parameter scaling function \[\mathbf{H}\].

\[
p_2(W) = L^D [f(L/\xi) + C/L^\nu].
\]

Here \(f(x)\) is an unknown scaling function to be determined, \(C\) is a constant, and \(y > 0\) is the exponent for the irrelevant variable. \(D\) is the fractal dimension of critical wave functions which occupy a subspace of dimensionality smaller than the embedded space dimension \(d = 2\). The critical exponent \(\nu\), together with the fractal dimension \(D\), characterizes the universality class of ALTs according to the quantum phase transition ansatz \[\mathbf{I}\].

The following criteria are used to identify an ALT: (1) \(Y_L(W) = p_2 L^{-\nu} = CL^{-\nu}\) increases and decreased with \(L\) for an extended and a localized state, respectively. (2) Near \(W_c\), \(Y_L(W)\) of different system sizes \(L\) collapse into two branches of a smooth function (one for localized states and the other for extended states). The implementation of the finite-size scaling analysis is illustrated in detail in Appendix \[\mathbf{J}\].

To compute the level statistics of the real (quasiparticle energies) and imaginary parts of eigenenergy \(E\), we diagonalize exactly the Hamiltonian with periodic boundary conditions in both directions to obtained all \(E\)'s. \(\text{Re}[E]\) is sorted in the ascending order. The diagonalization is performed by using Scipy library \[\mathbf{K}\]. We consider the eigenenergies in a very narrow energy window for many realizations. The ensemble-averaged level spacing distribution for both \(\text{Re}[E]\) and \(\text{Im}[E]\), denoted as \(p_{Re}(s)\) and \(p_{Im}(s)\), respectively, can be described by the histogram plot, where the systematic error in the histogram plots is eliminate to increase the accuracy \[\mathbf{L}\]. We also exclude the Kramers double degeneracy when calculating \(p_{Re}(s)\) for systems with the TR symmetry.

III. EXISTENCE OF ALTS

We first identify the ALTs from the finite-size scaling of the IPR. Similar to its Hermitian counterparts \[\mathbf{M}\], an ALT of system \(\mathbf{N}\) occurs at a critical disorder strength \(W_c\) at which all curves of \(\ln Y_L(W)\) as a function \(W\) for a state with given energy \(E\) and for different system size \(L\) cross as shown in Fig. \[\mathbf{O}\] for \(E = 0\), \(\gamma = \eta = 0.1\) and \(L\) ranging from 140 to 420. Indeed, data in Fig. \[\mathbf{O}\]a) gives \(W_c = 1.90 \pm 0.02\), and \(\ln Y_L(W)/dL\) is positive for \(W < W_c\) and negative for \(W > W_c\). These features clearly support the occurrence of an ALT: The state of \(E = 0\) is extended for \(W < W_c\), and becomes localized for \(W > W_c\). We also plot the wave functions distribution \(\log_{10}(|\psi_i|^2)\) for three disorder strengths: \(W = 1 < W_c = W_c\) and \(W = 5 > W_c\), as shown in Fig. \[\mathbf{O}\]b) where the degree of red color encodes probability density. Apparently, the wave function spread uniformely over the whole lattice at a length scale larger than \(\xi\) for \(W < W_c\) while it is highly localized on the lattice for \(W > W_c\). At \(W = W_c\), the state is critical that occupies a much sparsr space than those of \(W < W_c\) and resemble a fractal object \[\mathbf{P}\].

The chi-square fit of \(p_2(W)\) with a satisfactory goodness-of-fit of \(Q = 0.2\) yields the critical exponent \(\nu = 0.83 \pm 0.06\), the fractal dimension of \(D = 1.60 \pm 0.05\), the irrelevant exponent of \(\gamma = 0.10 \pm 0.03\), and \(C = 0.5 \pm 0.1\). Fig. \[\mathbf{O}\]c) shows the scaling functions of \(f(x)\) obtained by collapsing all curves in Fig. \[\mathbf{O}\]a) into a single one. We also plot \(\ln p_{Re}(W = W_c) vs \ln L\) in Fig. \[\mathbf{O}\]d), and the curve is a straight line of a slope \(\nu\) of \(D = 1.60 \pm 0.05\) \[\mathbf{Q}\], the same value as that from the scaling function analysis. Interestingly, it agrees with an analytical result obtained from the non-Hermitian XY model \[\mathbf{R}\].

The important feature or the fingerprint of a quantum phase transition is the universality concept. It says that critical exponents such as correlation length exponent \(\nu\) and fractal dimension \(D\) do not depend on model parameters. We carried out more calculations of IPR to show that \(\nu\) and \(D\) for the case without TR symmetry (\(\eta = 0.1\)) do not depend (within numerical errors) on the strength of Rashba SOC \(\alpha\), the complex eigenenergy \(E\), and the form of disorders for \(\gamma = 0.1\). The results are summarized in Table \[\mathbf{S}\].

Figures \[\mathbf{O}\]c) and \(i)\) show how the critical disorder \(W_c\) changes with the complex energy \(E\): \(W_c\) varies with \(\text{Re}[E]\) for
FIG. 1. (a) $\ln Y_L$ vs $W$ for state of $E = 0$. (b) Spatial distributions $\log_{10}|\Psi(x,y)|^2$ of wave function of state of $E = 0$ in a typical realization for $W = 1$ (extended), 1.9 (critical), and 5 (localized). The degree of red encodes the probability density as indicated by the color bar. (c) Scaling function $\ln f(L/\xi)$ vs $L/\xi$. (d) $p_L(W = W_c)$ as a function of $\ln L$. The solid line is a linear fit with slope $D = 1.60 \pm 0.05$. (e) $W_c$ vs $\Re[E]$ for $\Im[E] = 0$. (f) $W_c$ vs $\Im[E]$ for $\Re[E] = 0$. (g) Phase diagram in the complex eigenenergy plane $E$ at a fixed disorder strength $W$. Colour encodes the fractal dimension $D$. The red line is the mobility boundary with $D = 1.6 \pm 0.1$. Each point is averaged over 200 samples.

TABLE I. Critical exponent $\nu$, fractal dimension $D$ of wave functions at the ALT, and the goodness-of-fit $Q$ for different model parameters (Rashba SOC strength $\alpha$ and eigenenergy $E$) at a fixed non-Hermicity energy $\gamma = 0.1$. We consider two different types of disorders: (i) Independent uniform distribution (as those in the main text) of $\epsilon_i$ in the window of $[-W/2, W/2]$; (ii) Independent Gaussian distribution (used in Ref. [62]) of $\epsilon_i$ with zero mean and the variance of $W^2$.

| $\nu$  | $D$     | $Q$ |
|-------|---------|-----|
| $E = 0.0, \alpha = 0.2$ | $0.80 \pm 0.05$ | $1.65 \pm 0.03$ | $0.1$ |
| $E = 0.0, \alpha = 0.3$ | $0.7 \pm 0.1$ | $1.7 \pm 0.1$ | $0.05$ |
| $E = 0.01i, \alpha = 0.1$ | $0.6 \pm 0.2$ | $1.6 \pm 0.1$ | $0.08$ |
| $E = 0.1 + 0.01i, \alpha = 0.1$ | $0.85 \pm 0.09$ | $1.63 \pm 0.8$ | $0.1$ |
| Gaussian distribution | $E = 0.0, \alpha = 0.1$ | $0.8 \pm 0.1$ | $1.6 \pm 0.2$ | $0.04$ |

$\Im[E] = 0$ (e) and with $\Im[E]$ for $\Re[E] = 0$ (f). All states are localized for $|\Re[E]| < 4$, and one needs the largest disorder strength (maximal $W_c$) to localize states around $\Re[E] = \pm 1.6$. Different from its $\Re[E]$-dependence, $W_c$ is monotonic in $|\Im[E]|$.

The boundary that separates the extended states from the localized states is a closed curve in the complex energy plane as shown in Fig. 1(g) obtained from extensive numerical calculations of the IPR for different $E$ and system sizes $L$ (ranging from $L = 160$ to $L = 320$) at $W = 2$. The wave functions at the mobility boundary (the red line in Fig. 1(g)) are fractals with the same fractal dimension $D = 1.6$.

**IV. LEVEL STATISTICS**

After establishing the ALTs for Hamiltonian (1), we are now in the position to discuss the level statistics of the extended states. Figures 2(a) and 2(d) are $P_{\beta=2}(s)$ (the cyan squares) and $P_{\beta=3}(s)$ (the purple cross) for systems without TR symmetry for $\eta = 0.1$ (a) and with TR symmetry for $\eta = 0$ (b) within $|E| < 0.01$ for $L = 160$, $W = 1$, and $\gamma = 0.1$, where all states are extended (see Fig. 1). Surprisingly, the level-spacing distribution of $\Re[E]$ is well described by the Poisson function $P_P(s)$ no matter with or without the TR symmetry, instead of the Wigner-Dyson distributions of $P_{\beta=2}(s)$ or $P_{\beta=4}(s)$ that would be the case for an Hermitian Hamiltonian when $\gamma = 0$. This is surprising because the Poisson distribution is not normally for extended states, but for the localized states whose eigenenergies distribute independently and randomly in certain energy ranges. Similarly, $P_{\beta}(s)$ is universally described by an unknown function in the sense that it does not depend on models with different forms of SOCs, disorders, and dimensionality, see Appendix B). This unknown function shows a “level repulsion”, i.e., $P_{\beta}(s = 0) = 0$.

However, for very small non-Hermicity of $\gamma = 10^{-7}$ and the same $W = 1$ and $L = 160$, $P_P(s)$, obtained from those extended states within the window of $|E| < 0.01$, follows perfectly with the Wigner-Dyson distributions of $P_{\beta=2}(s)$ and $P_{\beta=4}(s)$ as shown in Figs. 2(b) and 2(e), respectively for the
cases without and with TR symmetry. At the same time, $P_\gamma(s)$ is universally described by the Gaussian function for $\eta \neq 0$ or by an unknown function with a universal non-zero constant $P_\gamma(s = 0)$, or non-level-repulsion, in the sense that the distribution are model-independent, see Appendix B. For the intermediate non-Hermicity energy of $2\gamma = 10^{-4}$, some parameter-dependent distributions of $P_\gamma(s)$ and $P_\nu(s)$ are seen, as shown in Fig. 2 (c) for $\eta = 0.1$ and (f) for $\eta = 0$.

To obtain the insight of the dramatical change in level statistics from the Wigner-Dyson distribution of $\gamma = 0$ to the Poisson distribution of non-zero $\gamma$, we follow the wisdom of Wigner by considering the two-by-two non-Hermitean random matrix [41]

$$\mathcal{H} = \begin{pmatrix} \epsilon_1 + \epsilon_2 & h_{12} \\ h_{21} & \epsilon_1 - \epsilon_2 \end{pmatrix} + i\gamma \sigma_\xi. \quad (3)$$

$\epsilon_{1,2}$ and $h_{12}$ are independent random variants of Gaussian distribution of zero mean and variance $\sigma^2$, i.e., $f(x, \sigma) \sim \exp[-x^2/\sigma^2]$. $\gamma$ is of the non-Hermicity energy. Hamiltonian (3) breaks both spin-rotation symmetry and TR symmetry. The difference of the two eigenenergies (level spacing) is

$$\Delta = \sqrt{\Delta_0^2 - 4\gamma^2 + i8\gamma\epsilon_2}, \quad (4)$$

with $\Delta_0 = 2{\sqrt{\epsilon_1^2 + |h_{12}|^2}}$ being the mean level spacing of the Hermitian part of Hamiltonian [3]. If $\gamma = 0$, the eigenenergies are real, and its level spacing distribution is $P(s) = \int \delta(s - \Delta_0) \exp[-(\epsilon_1^2 + |h_{12}|^2)/\sigma^2] d\epsilon_1 d\epsilon_2$, where $\Delta_0 = \sqrt{\epsilon_1^2 + |h_{12}|^2}$, $|h_{12}|^2 = \epsilon_1^2$; $\epsilon_1^2 + \epsilon_2^2$; $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2$ for the Gaussian orthogonal ensemble ($\beta = 1$, real matrix elements), the Gaussian unitary ensemble ($\beta = 2$, complex matrix elements), and the Gaussian symplectic ensemble ($\beta = 4$, quaternion matrix elements) respectively. Here $\xi_i (i = 1, 2, 3, 4)$ are real. Thus, $P(s) = C_{1,\beta} \exp[-C_2 s^2]$ is exactly the well-known Wigner-Dyson distribution. The prefactor is proportional to the area of equal-$\Delta_0$ hyper-surface in the $\epsilon_2 - \epsilon_3$ space. If $\gamma = 0$ in the current case, level spacing $\Delta = \Delta_0$ is non-negative. Any coupling (non-zero $\xi_1$ and $\xi_2$) tends to push two levels apart. The probability of having zero level spacing is the probability to have $\epsilon_2 = \epsilon_1 = \epsilon_3 = 0$, which is vanishingly small and gives rise to the Wigner-Dyson distributions. However, if $|\gamma|$ is of the order of $\Delta_0$, the real part of $\Delta$ is possible to be negative, zero, and positive. In this case, two levels can freely cross each other, and are, in principle, independent from each other. This is our understanding of why $P_\gamma(s)$ follows the Poisson function (see derivation later).

Above poor-man’s analysis reveals two relevant energy scales for the level statistics: The mean level spacing $\langle \Delta_0 \rangle$ of the Hermitian part of the model and the non-Hermicity energy $2\gamma$. We expect three different regimes. (i) Strong non-Hermicity limit $2\gamma \gg \langle \Delta_0 \rangle$: Level repulsion is invalid, and two quasiparticle levels can freely cross each other such that the quasiparticle level spacing distribution follows the universal Poisson function that is for independent random level distribution. The spacings of the imaginary part of the complex eigenenergies follow an unknown universal distribution function. (ii) Weak non-Hermicity limit $2\gamma \ll \langle \Delta_0 \rangle$: The non-Hermicity energy is much smaller than the average level spacings between two Hermitian modes. Therefore, the non-Hermicity is not enough to induce level crossing so that quasiparticle level spacing of extended states follows still the Wigner-Dyson statistics. (iii) Intermediate non-Hermicity: The level spacings follow some non-universal distributions that are sensitive to the details of a model. This explains well the changes of level statistics when the ratio of non-Hermicity energy to $\langle \Delta_0 \rangle$ is tuned by fixing lattice size $L$ and varying $\gamma$.

We further verify above picture by noticing that the ratio of non-Hermicity energy to $\langle \Delta_0 \rangle$ can also be tuned by fixing $\gamma$ and varying lattice size $L$ because the mean level spacing is inversely proportional to the number of lattice sites as $\langle \Delta_0 \rangle \approx 0.22(W + 8)/L^2$, see Appendix C for the clarification. We compute $P_\nu(s)$ and $P_\gamma(s)$ in the energy range of $|E| < 0.01$ for the cases with and without TR symmetry and for $W = 1$, $\gamma = 10^{-2}$ and three different system sizes: $L = 200$, $L = 20$ ($\langle \Delta_0 \rangle = 0.5\gamma$), and $L = 10$ ($\langle \Delta_0 \rangle = 2\gamma$). The results are plotted in Fig. 3 for the cases with (a,b,c) and without (d,e,f) TR symmetry. Similar to the results for the cases of fixing $L$ and varying $\gamma$ above, $P_\gamma(s)$ follows either the Poisson or Wigner-Dyson distribution while $P_\nu(s)$ follows either an unknown universal or the Gaussian distribution when lattice size are respectively of $L = 200$ and $L = 10$. It should be noted that the system is always in the strong non-Hermicity limit at fixed $\gamma \neq 0$ and in the thermodynamic limit of $L \to \infty$ so that the quasiparticle energy level spacing distribution is Poissonian. All our results show that analysis based on the
random matrix $\mathcal{P}$ can explain the results shown in Figs. 2 and 3 for strong and weak non-Hermicity limits.

Before ending this section, we would like to point out that the Poisson level statistics is universal for all systems without level repulsion, i.e. level cross each other independently. Obviously, our results of non-Hermicity Hamiltonians due to the inevitable gain/loss in open systems.

VI. CONCLUSION

In conclusion, 2DEGs subjected to an imaginary magnetic field, random on-site energies, and SOCs undergo an ALT at a finite disorder $W_c$. Near $W_c$, correlation lengths diverge as $\xi(W) \propto |W - W_c|^{1/\nu}$ with $\nu = 0.83 \pm 0.05$. A mobility boundary separating the extended from the localized states exists in the complex energy plane. In the thermodynamic limit of infinity system size, the quasiparticle level spacing $P_R(s)$ in the metallic phase is universally described by the Poisson distribution no matter whether the system has the time-reversal symmetry or not, while the spacing of the imaginary part of the complex eigenenergies $P_I(s)$ is also universal, exhibits “level repulsion”, and is sensitive to the TR symmetry. For a finite system when the non-Hermicity energy $\gamma$ is smaller than the mean level spacing, $P_R(s)$ can be described by the Wigner-Dyson distribution $P_R(s)$ and $P_I(s)$ is universal with a universal non-zero constant.

VII. ACKNOWLEDGMENTS

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Appendix A: Finite-size scaling analysis

To extract the fractal dimension, the critical disorder, and the critical exponent $\nu$ at the quantum phase transitions defined in the scaling function Eq. (2) with $\xi = \xi_0 |W - W_c|^{1/\nu}$.
i.e.,
\[ p_{2}(L, W) = L^{D}[f(L|W - W_{c}^\prime/\xi_{0}) + CL^{-\gamma}], \quad (A1) \]
\[
\begin{align*}
 f(L|W - W_{c}^\prime/\xi_{0}) &= F_{0} + F_{1}L|W - W_{c}^\prime/\xi_{0} + F_{2}(L|W - W_{c}^\prime/\xi_{0})^{2} + F_{3}(L|W - W_{c}^\prime/\xi_{0})^{3} \\
 &= F_{0} + \tilde{F}_{1}L|W - W_{c}^\prime + \tilde{F}_{2}(L|W - W_{c}^\prime)^{2} + \tilde{F}_{3}(L|W - W_{c}^\prime)^{3},
\end{align*}
\]
(A2)

with \( D, C, \gamma, \nu, W_c, F_0, F_1, F_2, F_3 \) being fitting parameters. Then we adjust those parameters to minimize the chi square
\[
\chi^{2} = \sum_{i}^{N_{w}} \sum_{j}^{N_{c}} \left( \frac{p_{2}(W_{i}, L_{j}) - L^{D}[f(L_{j}|W_{i} - W_{c}^\prime/\xi_{0}) + CL^{-\gamma}]}{\sigma_{ij}} \right)^{2},
\]
(A3)

following the approach illustrated in the appendix of Ref. \([69]\), where \( N_{w} \) and \( N_{c} \) are the number of \( W \) and \( L \), respectively. The fitting process yields the critical disorder \( W_c \), the fractal dimension \( D \), and the critical exponent \( \nu \). After determining the minimal chi square, we calculate the goodness-of-fit \( Q \) by the standard algorithm suggested in Ref. \([70]\), which measures how well our numerical data of \( p_{2} \) fit to the model of Eq. (A1). Take data in Fig. \( 1a \) as examples: Following the above process, we obtain \( Q = 0.2 > 10^{-3} \), a satisfactory number that says the fit acceptable.

**Appendix B: Model-independence of level statistics**

To demonstrate that \( P_{R}(s) \) and \( P_{I}(s) \) are universal in the strong and weak non-Hermicity limits, we study level statistics of extended states for other random non-Hermitian models with different forms of SOCs, disorders, and dimensionalities.

Firstly, we study Hamiltonian \([1\text{I}]\) with different forms of SOCs. The first one is the random SU(2) model subjected to an imaginary perpendicular magnetic field \((0,0,i\gamma)\) \([52]\).

\[
H = \sum_{i} c_{i}^{\dagger}(\epsilon_{i}\sigma_{0} + \eta\sigma_{z} + i\gamma\sigma_{x})c_{i} + \left( t \sum_{(ij)} c_{i}^{\dagger}V_{ij}c_{j} + h.c. \right),
\]
(B1)

with

\[
V_{ij} = \begin{bmatrix}
    e^{-i\alpha_{ij}} \cos(\beta_{ij}/2) & e^{-i\gamma_{ij}} \sin(\beta_{ij}/2) \\
    e^{i\gamma_{ij}} \sin(\beta_{ij}/2) & e^{i\alpha_{ij}} \cos(\beta_{ij}/2)
\end{bmatrix}.
\]
(B2)

Here \( \alpha_{ij} \) and \( \gamma_{ij} \) distribute randomly and uniformly in the range of \([0,2\pi)\), \( \sin(\beta_{ij}/2) \) distributes uniformly in \([0,1)\). The second model is to replace the Rashba SOC in model \([1\text{I}]\) by the Dresselhaus SOC \([71]\), where the matrices \( V_{ij} \) are parametrized as \( V_{x} \) and \( V_{y} \) for the \( x \)- and \( y \)-direction hopping, respectively,

\[
V_{x} = \sigma_{0} + i\xi\sigma_{z}, \quad V_{y} = \sigma_{0} - i\xi\sigma_{y}.
\]
(B3)

we perform a Taylor expansion of the scaling function \( f(x) \) up to the third order in \( |W - W_{c}^\prime| \) near \( W = W_{c} \).

Here the constant \( \zeta \) measures the strength of the Dresselhaus SOC.

The case without TR symmetry \( (\eta = 0.1) \) and the case with TR symmetry \( (\eta = 0) \) are investigated. \( P_{R}(s) \) and \( P_{I}(s) \) within the energy window of \(|E| < 0.01 \) for \( W = 1, L = 160, \) and \( \gamma = 0.1 \) (strong non-Hermicity limit) or \( \gamma = 10^{-7} \) (weak non-Hermicity limit) for all three models are plotted in Fig. \( 4 \). It is clear that all three models (Rashba, Dresselhaus and random SU(2) SOCs) give identical \( P_{R}(s) \) and \( P_{I}(s) \). Within the symbol size, we cannot see any difference in both \( P_{R}(s) \) and \( P_{I}(s) \) for all three models. Thus, these results provide strong evidence that the new distributions are independent of the forms of SOCs.

**FIG. 4.** \( P_{R}(s) \) (the filled symbols) and \( P_{I}(s) \) (the open symbols) within \(|E| < 0.01 \) in the cases without TR symmetry ((a,b) for \( \eta = 0.1 \)) and with TR symmetry ((c,d) for \( \eta = 0 \)) for \( W = 1, L = 160, \) and \( \gamma = 0.1 \) (a,c), \( \gamma = 1 \times 10^{-7} \) (b,d). The black solid lines in (a) and (c) are \( P_{P}(s) \). The red and the orange solid lines in (b) and (d) are \( P_{R}=2(s) \) and \( P_{R}=4(s) \), respectively. The green solid line in (b) is the Gaussian function. The squares, triangles and circles are respectively for the Rashba SOC, the Dresselhaus SOC, and the SU(2) SOC.

Secondly, we show that the level statistics do not depend on the forms of disorders by considering the following model,

\[
H = \sum_{i} c_{i}^{\dagger}(\epsilon_{i}\sigma_{0} + \eta\sigma_{z} + i\gamma_{i}\sigma_{x})c_{i} + \left( \sum_{(ij)} c_{i}^{\dagger}V_{ij}c_{j} + h.c. \right),
\]
(B4)
where $\epsilon_i$ and $\gamma_i$ are independent random numbers that distribute in the range of $[-W/2, W/2]$ and $[-\Theta/2, \Theta/2]$, respectively. $V_{ij} = V_{ji} = \sigma_0 + ia_\sigma \epsilon_i$ and $V_{r} = \sigma_0 - ia_\sigma \epsilon_i$ for $(ij)$ along the $x$- and the $y$-directions. $\alpha$ and $\eta$ are two constants measuring SOC strength and the degree of TR symmetry violation. Different from model (1), with the constant non-Hermicity, both the Hermitian and the non-Hermitic parts are random here. All states of this model within the energy window of $|E| < 0.01$ for $W = 1, \alpha = \eta = 0.1, L = 160$ (system sizes), and $\Gamma = 0.1$ (strong non-Hermicity) and $10^{-7}$ (weak non-Hermicity) are extended. The corresponding $P_R(s)$ and $P_I(s)$ of those states are plotted in Figs. (a-c) (consist of three figures) with TR symmetry and $\eta = 0$. They are the same as those of Model (B1).

Thirdly, we investigate the level statistics of a three-dimensional non-Hermitian Anderson model

$$H = \sum_i \epsilon_i c_i^\dagger c_i + t \sum_{ij} c_i^\dagger c_j + h.c., \quad (B5)$$

where $c_i^\dagger$ and $c_i$ are the creation and annihilation operator of a single electron at site $i = (l, m, n)$ with $l, m, n$ being integers and $1 \leq l, m, n \leq L$. The hopping energy $t$ is chosen as the energy unit, i.e., $t = 1$. Randomness is introduced through random real numbers $\epsilon_i$ and $\theta_l$ uniformly and independently distributed in $[-W/2, W/2]$ and $[-\Theta/2, \Theta/2]$, respectively. Periodic boundary conditions are applied in all directions to avoid the non-Hermitian skin effect. The obtained $P_R(s)$ and $P_I(s)$ in the energy interval of $|E| \in [-0.01, 0.01]$ and $W = 1$ are shown in Figs. (a-c) (three figures). Clearly, they also follow the same level statistics as those of states of Hamiltonian (1).

**Appendix C: Mean level spacing of Hermitian part**

The mean level spacing $\langle \Delta_0 \rangle$ of the Hermitian part of Hamiltonian (1) is an important energy scale related different level statistics. In this section, we want to find an accurate estimate of $\langle \Delta_0 \rangle$ for a given system size $L$ and disorder strength $W$. For small disorders $W$, all eigenenergies should lie in the energy range of $[-(4 + W/2), (4 + W/2)]$ such that the energy bandwidth is about $8 + W$. Since the number of eigenstates is proportional to $L^2$, the mean level spacing should then satisfy

$$\langle \Delta_0 \rangle = \beta \frac{(W + 8)}{L^2}, \quad (C1)$$

with $\beta$ being a coefficient that is obtained below.

To numerically determine the coefficient $\beta$, we calculate $\langle \Delta_0 \rangle$ and plot them (symbols) against $L$ in Fig. 8. Here $\Delta_0$ is obtained from a small energy window $[-0.01, 0.01]$ around $E = 0$, and $(\cdots)$ is averaged over more than 200 ensembles. A fit of $\langle \Delta_0 \rangle$ to Eq. (C1) yields $\beta = 0.22$, which accords well with numerical data (up to $L = 200$), see the black line in Figs. Thus, the mean level spacing of the Hermitian part of Hamiltonian (1) can be obtained by formula $\langle \Delta_0 \rangle = 0.22(W + 8)/L^2$. 

![Figure 5](image1.png)

**Fig. 5.** Results of Hamiltonian (B4) without TR symmetry ($\eta = 0$). (a-c) Strong Non-Hermicity $\Gamma = 0.1$: (a) $\ln p_R(E = 0)$ as a function of $\ln L$. The red dash line is a linear fit with slope $D = 1.99 \pm 0.01$. (b) $P_R(s)$ (the black squares) within $|E| < 0.01$. The orange solid line is $P_I(s)$ (the filled squares) and for Hamiltonian (B4) (the empty squares). (d-f) Weak Non-Hermicity $\Gamma = 10^{-7}$: (d) $\ln p_R(E = 0)$ as a function of $\ln L$. The red dash line is a linear fit with slope $D = 1.99 \pm 0.01$. (e) $P_R(s)$ (the black squares) within $|E| < 0.01$. The magenta solid line is $P_I(s)$ (the filled squares) and for Hamiltonian (B4) (the empty squares). (f) $P_I(s)$ within $|E| < 0.01$ for Hamiltonian (B4) (the filled squares) and for Hamiltonian (B1) (the empty squares).

![Figure 6](image2.png)

**Fig. 6.** Results of Hamiltonian (B4) with TR symmetry ($\eta = 0$). (a-c) Strong Non-Hermicity $\Gamma = 0.1$: (a) $\ln p_R(E = 0)$ as a function of $\ln L$. The red dash line is a linear fit with slope $D = 1.99 \pm 0.01$. (b) $P_R(s)$ (the black squares) within $|E| < 0.01$. The orange solid line is $P_I(s)$ (the empty squares). (d-f) Weak Non-Hermicity $\Gamma = 10^{-7}$: (d) $\ln p_R(E = 0)$ as a function of $\ln L$. The red dash line is a linear fit with slope $D = 1.99 \pm 0.01$. (e) $P_R(s)$ (the black squares) within $|E| < 0.01$. The magenta solid line is $P_I(s)$ (the filled squares) and for Hamiltonian (B4) (the empty squares).
The results of Hamiltonian $H$ with eigenenergy $E$, then
\[
\hat{H}|\psi_E\rangle = \left( H - \sum_i e_i^2 \gamma_0 \sigma_0 c_i \right)|\psi_E\rangle \\
= \left( H - i\gamma_0 I \right)|\psi_E\rangle = (E - i\gamma_0)|\psi_E\rangle
\]
with $I$ being the identity matrix. Thus, $|\psi_E\rangle$ is also a right eigenstate of $\hat{H}$ with eigenenergy $E - i\gamma_0$. Since $|\psi_E\rangle$ is arbitrary, all the levels of $\hat{H}$ are the same as those of $H$ but shift by a constant imaginary value of $-i\gamma_0$. Obviously, the constant shift of all levels of complex energies does not change the distributions of level spacings, $P_R(s)$ and $P_I(s)$.

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