A FRAMORIZATION OF THE HECKE ALGEBRA OF TYPE B

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Abstract. In this article we introduce a framization of the Hecke algebra of type B. For this framization we construct a faithful tensorial representation and two linear bases. We finally construct a Markov trace on these algebras and from this trace we derive isotopy invariants for framed and classical knots and links in the solid torus.

Introduction

The idea of framization of a knot algebra (Hecke algebras and BMW–algebra among others) was introduced by the two last authors in [22] and consists in adding framing generators to the defining generators of the knot algebra with the aim of finding new invariants of classical links or, more generally, invariants of knot-like objects. The Yokonuma–Hecke algebra is the prototype of framization; indeed this algebra, introduced by T. Yokonuma [29] in the context of representations of Chevalley groups, can be thought of as a framization of the Hecke algebra of type A.

More precisely, the Yokonuma–Hecke algebra supports a Markov trace [17] and then it becomes a peculiar knot algebra considering that, by using the Jones’ recipe, one can construct invariants for: framed links [21], classical links [20] and singular links [19]. It is worth mentioning that recently it was proved that the invariants for classical links

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constructed in [20] are not topologically equivalent either to the Homflypt polynomial or to the Kauffman polynomial, see [4].

On the other hand, Jones raised that his recipe for the construction of the Homflypt polynomial might be used for Hecke algebra not only of type $\mathbb{A}$, cf. [14, p.336]. Then, the third author studied the Jones recipe by using the Hecke algebra of type $\mathbb{B}$; namely, in [25] she constructed the analogue of the Homflypt polynomial for oriented knots and links inside the solid torus, see also [9]. Further, in [24] Lambropoulou constructed all possible analogues of the Homflypt polynomial in the solid torus from the Ariki-Koike algebras and the affine Hecke algebras of type $\mathbb{A}$.

The purpose of this article is to introduce and to start a systematic study of a framization of the Hecke algebras of type $\mathbb{B}$, denoted by $Y_{d,n}^B(u,v)$, with the principal objective to explore their usefulness in knot theory. Thus, having in mind both the role of the Hecke algebra of type $\mathbb{B}$ [9] and the Yokonuma–Hecke algebra of type $\mathbb{A}$ [21, 20, 19] in knot theory, it is natural to define, by using the Jones’ recipe applied to $Y_{d,n}^B(u,v)$, invariants in the solid torus for: classical knots, framed knots and singular knots. For these purposes a first key point is to prove that the algebra $Y_{d,n}^B(u,v)$ supports a Markov trace. In fact, this is one of the main results proved in the present article.

In [2], M. Chlouveraki and L. Poulain D’Andecy have introduced the affine and cyclotomic Yokonuma–Hecke algebras. In the context of framization [22], the definition of the algebras introduced by Chlouveraki and Poulain D’Andecy can be understood by adding framing generators and making the framization according to the formula of framization of the generators of the Yokonuma–Hecke algebra of type $\mathbb{A}$, that is, only of the braiding generators. Now, the Hecke algebra of type $\mathbb{B}$ is a particular case of cyclotomic Hecke algebra. The framization of the Hecke algebra of type $\mathbb{B}$ proposed here makes the framization of all generators of the Hecke algebra of type $\mathbb{B}$; in particular, also of the special ‘loop’ generator of the algebra.

Before giving the organization of the article we note that, by taking into account the various articles generated recently from the algebra of Yokonuma–Hecke of type $\mathbb{A}$ (view for example [8, 5, 6, 3] among others), the framization proposed here indicates that the algebra $Y_{d,n}^B(u,v)$ should be interesting in itself.

The article is organized as follows. In Section 1 we introduce our notation and explain the background notions. In Section 2 we define our framizations for the Coxeter group of type $\mathbb{B}$, for the Artin braid group of type $\mathbb{B}$ and for the Hecke algebra of type $\mathbb{B}$, $Y_{d,n}^B$. In Section 3 we construct a tensorial representation for the algebra $Y_{d,n}^B := Y_{d,n}^B(u,v)$. In Section 4 we find linear bases for $Y_{d,n}^B$, one of which is used in Section 5 for constructing a Markov trace on the algebras $Y_{d,n}^B$. Finally, in Section 6 necessary and sufficient conditions are given for the trace parameters in order to proceed with the construction of topological invariants of framed and classical knots and links in the solid torus (Section 7).
1. Notation and background

In this section we review known results, necessary for this paper, and we also fix the following terminology and notations that will be used along the paper:

- The letters \( u \) and \( v \) denote two indeterminates. And we denote by \( \mathbb{K} \), the field of rational functions \( \mathbb{C}(u,v) \).
- The term algebra means unital associative algebra over \( \mathbb{K} \).
- For a finite group \( G \), \( \mathbb{K}[G] \) denotes the group algebra of \( G \).
- The letters \( n \) and \( d \) denote two fixed positive integers.
- We denote by \( \omega \) a fixed primitive \( d \)-th root of unity.
- We denote by \( \mathbb{Z}/d\mathbb{Z} \) the group of integers modulo \( d \), \( \{0, 1, \ldots, d-1\} \), and by \( C_d \) the cyclic group of order \( d \), \( \langle t; t^d = 1 \rangle \). Note that \( C_d \cong \mathbb{Z}/d\mathbb{Z} \).
- As usual, we denote by \( \ell \) the length function associated to the Coxeter groups.

1.1. Braid groups of type \( A \). The finite Coxeter group of type \( A_n \) (\( n \geq 2 \)) can be realized as the symmetric group on the set \( \{1, \ldots, n\} \). Set \( s_i \) the elementary transposition \((i, i+1)\), so the Coxeter presentation of \( S_n \) is encoded in the following Dynkin diagram:

\[
\begin{array}{cccccc}
& s_1 & s_2 & \cdots & s_{n-2} & s_{n-1} \\
\end{array}
\]

Then, the Artin braid group, \( B_n \), associated to \( S_n \) is generated by the elementary braidings \( \sigma_1, \ldots, \sigma_{n-1} \), which satisfy the following relations:

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| > 1 \\
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1.
\end{align*}
\]

The framed braid group, \( F_n \), is defined as the group presented by the braiding generators \( \sigma_1, \ldots, \sigma_{n-1} \) and the framing generators \( t_1, \ldots, t_n \) subject to the relations \( [1] \) together with the relations:

\[
\begin{align*}
t_i t_j &= t_j t_i & \text{for all } i, j \\
t_j \sigma_i &= \sigma_i t_{s(i)} & \text{for all } i, j.
\end{align*}
\]

Notice that \( F_n \) is isomorphic to the wreath product \( \mathbb{Z} \wr B_n \). The \( d \)-modular framed braid group \( F_{d,n} \) is defined by adding to the above defining presentation of \( F_n \) the relations \( t_i^d = 1 \). Hence, \( F_{d,n} \cong (\mathbb{Z}/d\mathbb{Z}) \wr B_n \).

It is convenient to write the elements of \( F_n \) in the split form \( \sigma t_1^{m_1} \ldots t_n^{m_n} \), where the \( m_i \)'s are integers (called the framings) and \( \sigma \in B_n \). A framed braid can be represented as a usual geometric braid on \( n \) strands by attaching the respective framing to each strand.

For details and the above geometric interpretation of the framed braid group see \[24\]. See also \[18, 21\].
1.2. Braid groups of type $B$. Set $n \geq 2$. Let us denote by $W_n$ the Coxeter group of type $B_n$. This is the finite Coxeter group associated to the following Dynkin diagram.

Define $r_k = s_{k-1} \ldots s_1 r_1 s_1 \ldots s_{k-1}$ for $2 \leq k \leq n$. It is known, see [9], that every element $w \in W_n$ can be written uniquely as $w = w_1 \ldots w_n$ with $w_k \in N_k$, $1 \leq k \leq n$, where

$$N_k := \{1, r_k, s_{k-1} \ldots s_i, s_{k-1} \ldots s_i r_i; 1 \leq i \leq k-1\}$$

Furthermore, this expression for $w$ is reduced. Hence, we have $\ell(w) = \ell(w_1) + \cdots + \ell(w_n)$.

The corresponding braid group of type $B_n$ associated to $W_n$, is defined as the group $\tilde{W}_n$ generated by $\rho_1, \sigma_1, \ldots, \sigma_{n-1}$ subject to the following relations

$$\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for} \quad |i-j| > 1 \\
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j \quad \text{for} \quad |i-j| = 1 \\
\rho_1 \sigma_i &= \sigma_i \rho_1 \quad \text{for} \quad i > 1 \\
\rho_1 \sigma_1 \rho_1 &= \sigma_1 \rho_1 \sigma_1 \rho_1.
\end{align*}$$

Geometrically, braids of type $B_n$ can be viewed as classical braids of type $A_{n+1}$ with $n+1$ strands, such that the first strand is identically fixed. This is called ‘the fixed strand’. The 2nd, ..., $(n+1)$st strands are renamed from 1 to $n$ and they are called ‘the moving strands’. The 'loop' generator $\rho_1$ stands for the looping of the first moving strand around the fixed strand in the right-handed sense. In Figure 1 we illustrate a braid of type $B_5$.

**Figure 1.** A braid of $B_5$-type.

1.3. The group $W_n$ can be realized as a subgroup of the permutation group of the set $X_n := \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$. More precisely, the elements of $W_n$ are the permutations $w$ such that $w(-m) = -w(m)$, for all $m \in X_n$. For example

$s_i$ is realized as $(-n \ldots -i-1 -i -i+1 \ldots i-1 i i+1 \ldots n)$

and

$r_1$ is realized as $(-n \ldots -2 -1 1 2 \ldots n)$.

Further, the elements of $W_n$ can be parameterized by the elements of $X_n^n$ (see [12, Lemma 1.2.1]). More precisely, the element $w \in W_n$ corresponds to the element $(m_1, \ldots, m_n) \in X_n^n$. 
such that $m_i = w(i)$. Then, we have that $s_i$ is parameterized by $(1, 2, \ldots, i+1, i, \ldots, n)$ and $r_1$ is parameterized by $(-1, 2, \ldots, n)$. More generally, if $w \in W_n$ is parameterized by $(m_1, \ldots, m_n) \in X^n_n$, then

$$
egin{align*}
wr_1 & \quad \text{is parameterized by} \quad (-m_1, m_2, \ldots, m_n) \\
ws_i & \quad \text{is parameterized by} \quad (m_1, \ldots, m_{i+1}, m_i, \ldots, m_n).
\end{align*}
$$

Lemma 1. [12, Lemma 1.2.2] Let $w \in W_n$ parameterized by $(m_1, \ldots, m_n) \in X^n_n$. Then $\ell(ws_i) = \ell(w) + 1$ if and only if $m_i < m_{i+1}$ and $\ell(wr_1) = \ell(w) + 1$ if and only if $m_1 > 0$.

Example 1. Set $n = 3$ and $w = s_1r_1$. Then we have that $\ell(wr_1) < \ell(w)$ and $\ell(ws_1) > \ell(w)$, and $w$ is represented by $(-2, 1, 3)$.

Remark 1 (Symmetric braids). The above realization of the $W_n$ can be lifted also to the braid level. Indeed in [27] tom Dieck defines the symmetric braids (denoted by $ZB_n$) and proves that $\widehat{W}_n$ is isomorphic to this group of braids. Specifically tom Dieck considers the braids in $\mathbb{R} \times [0,1]$ with strands between $X_n \times \{0\} \times \{0\}$ and $X_n \times \{0\} \times \{1\}$ which are symmetric about the axis $(0,0) \times [0,1]$. Moreover, the group of the symmetric braids is generated by the elements $z_0, z_1, \ldots, z_{n-1}$ represented graphically as follows.

![Figure 2: The generators of the symmetric braid group.](image)

An isomorphism between the groups $\widehat{W}_n$ and $ZB_n$ is induced by the mapping: $\rho_1 \mapsto z_0$, $\sigma_i \mapsto z_i$ for $1 \leq i \leq n - 1$.

1.4. The Hecke algebra of type $B_n$, denoted $H_n(u, v)$, is the algebra generated by $h_0, h_1, \ldots, h_{n-1}$ subject to the following relations:

$$
egin{align*}
h_i h_j & = h_j h_i \quad \text{for all} \quad |i - j| > 1 \\
h_i h_{i+1} h_i & = h_{i+1} h_i h_{i+1} \quad \text{for all} \quad i = 1, \ldots, n-2 \\
h_1 h_0 h_1 h_0 & = h_0 h_1 h_0 h_1 \\
h_i^2 & = 1 + (u - u^{-1}) h_i \quad \text{for all} \quad i \\
h_0^2 & = 1 + (v - v^{-1}) h_0.
\end{align*}
$$

It is well known that the dimension of $H_n(u, v)$ is $2^n n!$ and clearly for $u = v = 1$ it coincides with $\mathbb{K}[W_n]$. 
2. Framizations of type B

In this section we introduce the main object studied in the paper, that is, a framization of the Hecke algebra of type B. To do that, previously we introduce a framed version of both, the braid group and the Coxeter group of type B. At the end of the section we include some useful relations derived directly from the defining relations of our framization algebra.

2.1. We start with the definition of a $d$–framed version of $W_n$.

**Definition 1.** The $d$–modular framed Coxeter group of type $B_n$, $W_{d,n}$, is defined as the group generated by $r_1, s_1, \ldots, s_{n-1}$ and $t_1, \ldots, t_n$ satisfying the Coxeter relations of type $B_n$ among $r_1$ and the $s_i$'s, the relations $t_i t_j = t_j t_i$ for all $i, j$, the relations $t_j^d = 1$ for all $j$, together with the following relations:

$$
t_j r_1 = r_1 t_j \quad \text{for all } j
$$

The analogous group defined by the same presentation, where only relations $t_j^d = 1$ are omitted, shall be called framed Coxeter group of type $B_n$ and will be denoted as $W_{\infty,n}$.

**Definition 2.** The framed braid group of type $B_n$, denoted $F_{B_n}$, is the group presented by generators $\rho_1, \sigma_1, \ldots, \sigma_{n-1}, t_1, \ldots, t_n$ subject to the relations (2) and (4), together with the following relations:

$$
t_i \rho_1 = \rho_1 t_i \quad \text{for all } i.
$$

The $d$–modular framed braid group, denoted $F_{B_{d,n}}$, is defined as the group obtained by adding the relations $t_i^d = 1$, for all $i$, to the above defining presentation of $F_{B_n}$.

In Figure 3 the generators of the groups $F_{B_n}$ and $F_{B_{d,n}}$ are illustrated.

![Figure 3. The generators of the groups $F_{B_n}$ and $F_{B_{d,n}}$.](image)

The mapping that acts as the identity on the generators $r_1$ and the $s_i$'s and maps the $t_j$'s to 1 defines a morphism from $W_{d,n}$ onto $W_n$. Also, we have the natural epimorphism from $F_{B_{d,n}}$ onto $W_{d,n}$ defined as the identity on the $t_j$'s and mapping $\rho_1$ to $r_1$ and $\sigma_i$ to $s_i$, for all $i$. Thus, we have the following sequence of epimorphisms.

$$
F_{B_n} \longrightarrow F_{B_{d,n}} \longrightarrow W_{d,n} \longrightarrow W_n
$$

where the first arrow is the natural projection of $F_{B_n}$ to $F_{B_{d,n}}$. 
Finally, we can lift trivially the length function \( \ell \) on \( W_n \) to \( W_{d,n} \). Indeed, we deduce from relations \( (6) \), that every \( x \in W_{d,n} \) can be written in the form \( x = wt_1^{a_1} \cdots t_n^{a_n} \), with \( w \in W_n \). Thus, we define the length of \( x \) by \( \ell(w) \). We denote again by \( \ell \) the length function on \( W_{d,n} \).

Geometrically, elements in \( \mathcal{F}_n \) (resp. \( \mathcal{F}_{d,n} \)) are braids of type \( B \) (recall Figure 1) such that each one of the \( n \) moving strands has a framing in \( \mathbb{Z} \) (resp. \( \mathbb{Z}/d\mathbb{Z} \)) attached.

**Remark 2** (Symmetric framed braids). Following Remark 1, one can define analogously the symmetric framed braids resp. \( d \)-modular symmetric framed braids (denoted by \( Z\mathcal{F}_n \) resp. \( Z\mathcal{F}_{d,n} \)) and prove that the liftings \( \tilde{W}_{\infty,n} \) resp. \( \tilde{W}_{d,n} \) are isomorphic to these groups of braids. In terms of geometric realizations, a symmetric framed braid is a braid in \( ZB_n \) with an integer resp. \( d \)-modular framing attached on each strand, such that the strands \( i \) and \( -i \) have the same framing, for all \( i \).

Note that \( W_n \) can be regarded naturally as a subgroup of \( W_{d,n} \); indeed, the elements of \( W_n \) correspond to the elements of \( W_{d,n} \) having all framings equal to 0. We will proceed now to lift the sets \( N_k \) of \( W_n \), introduced in Subsection 1.2, to subsets \( N_{d,k} \) of the group \( W_{d,n} \), with the aim to give a standard writing for the elements of \( W_{d,n} \) which will be useful to parameterize later a basis of the framization of the Hecke algebra of type \( B_n \) defined here.

We define inductively the subsets \( N_{d,k} \) of \( W_{d,n} \) as follows:

\[
N_{d,1} := \{t_1^m, r_1t_1^m, 0 \leq m \leq d - 1\}
\]

and

\[
N_{d,k} = \{t_k^m, r_k t_k^m, s_{k-1}x ; x \in N_{d,k-1}, 0 \leq m \leq d - 1\} \quad \text{for all } 2 \leq k \leq n.
\]

Note that, for all \( k \) and \( d \), we have \( N_k \subseteq N_{d,k} \) and for \( d = 1 \) the sets \( N_k \) and \( N_{1,k} \) coincide. Also, every element \( x \in N_{d,k} \) can be written as \( x = yt_k^m \), with \( y \in N_k \). Further, we have the following proposition.

**Proposition 1.** Every element of \( W_{d,n} \) can be expressed in the standard form, that is, as a product \( m_1 \cdots m_n \), where \( m_i \in N_{d,i} \).

**Proof.** Set \( x = w_1 \cdots w_n t_1^{a_1} \cdots t_n^{a_n} \in W_{d,n} \). We will prove the claim by induction on \( n \). For \( n = 2 \), it is straightforward to check that \( x \) can be written in standard form. E.g. if \( w_1 = r_1 \) and \( w_2 = s_1r_1 \), we have

\[
x = w_1w_2t_1^{a_1}t_2^{b_2} = (r_1)(s_1r_1)t_1^{a_1}t_2^{b_2} = (r_1t_1^{a_1})(s_1r_1t_1^{b_2}) = m_1m_2
\]

where \( m_1 = r_1t_1^{a_1} \) and \( m_2 = s_1r_1t_1^{b_2} \). Suppose now that the proposition is true for all positive integers less than \( n \). Then, if \( w_n \) in \( x \) is equal to 1 or \( r_n \), we have:

\[
x = (w_1 \cdots w_{n-1}t_1^{a_1} \cdots t_{n-1}^{a_{n-1}})(w_n t_n^{a_n}).
\]

Now, \( w_n t_n^{a_n} \in N_{d,n} \) and applying the induction hypothesis on the word inside the first parenthesis above, we deduce that \( x \) can be written in the standard form.
If \( w_n \) is equal to \( s_{n-1} \ldots s_i \) or \( s_{n-1} \ldots s_i x_i \), we have:
\[
\begin{align*}
    x &= w_1 \ldots w_{n-1} (w_n t_i^{a_i}) t_1^{a_1} \ldots t_{i-1}^{a_{i-1}} t_{i+1}^{a_{i+1}} \ldots t_n^{a_n} \\
    &= (w_1 \ldots w_{n-1} t_1^{a_1} \ldots t_{i-1}^{a_{i-1}} t_{i+1}^{a_{i+1}} \ldots t_{n-1}^{a_{n-1}})(w_n t_i^{a_i}).
\end{align*}
\]
Again, noting that \( w_n t_i^{a_i} \in \mathbb{N}_{d,n} \) and applying the induction hypothesis on the word inside the first parenthesis above, it follows that \( x \) can be written in the standard form.

2.2. In order to define a framization of the Hecke algebra of type \( B \), we need to introduce the following elements \( f_1 \) and \( e_i \), for \( i = 1, \ldots, n-1 \), in \( \mathbb{K}[\mathcal{F}_d] \),
\[
    f_1 := \frac{1}{d} \sum_{m=0}^{d-1} t_i^{m} \quad \text{and} \quad e_i := \frac{1}{d} \sum_{m=0}^{d-1} t_i^{m} t_i^{d-m} \quad \text{for all} \quad 1 \leq i \leq n-1.
\]
Notice that the \( f_1 \) and the \( e_i \)'s are idempotents, cf [21] for the \( e_i \)'s.

**Definition 3.** Let \( n \geq 2 \). The algebra \( Y_{d,n}^B = Y_{d,n}^B(u, v) \) is defined as the quotient of \( \mathbb{K}[\mathcal{F}_d] \) over the two–sided ideal generated by the following elements:
\[
    \rho_1^2 - 1 - (v - v^{-1}) f_1 \rho_1 \quad \text{and} \quad \sigma_i^2 - 1 - (u - u^{-1}) e_i \sigma_i \quad \text{for all} \quad 1 \leq i \leq n-1.
\]
We shall denote the corresponding to \( \sigma_i \) (respectively, to \( \rho_1 \)) in \( Y_{d,n}^B \) by \( g_i \) (respectively, by \( b_1 \)) and we shall keep the same notation for the \( t_j \)'s (respectively, the \( e_i \)'s and \( f_1 \)) in \( Y_{d,n}^B \). Hence, equivalently, the algebra \( Y_{d,n}^B \) can be defined by generators \( 1, b_1, g_1, \ldots, g_{n-1}, t_1, \ldots, t_n \) and relations as follows.

\[
\begin{align*}
    g_i g_j &= g_j g_i \quad \text{for} \quad |i - j| > 1 \\
    g_i g_j &= g_i g_j \quad \text{for} \quad |i - j| = 1 \\
    b_1 g_i &= g_i b_1 \quad \text{for all} \quad i \neq 1 \\
    b_1 g_i b_1 g_i &= g_i b_1 g_i b_1 \\
    t_i t_j &= t_j t_i \quad \text{for all} \quad i, j \\
    t_j g_i &= g_i t_{s(i, j)} \quad \text{for all} \quad i, j \\
    t_i b_1 &= b_1 t_i \quad \text{for all} \quad i \\
    t_i^{d_i} &= 1 \quad \text{for all} \quad i \\
    g_i^2 &= 1 + (u - u^{-1}) e_i g_i \quad \text{for all} \quad 1 \leq i \leq n-1 \\
    b_1^2 &= 1 + (v - v^{-1}) f_1 b_1. 
\end{align*}
\]

**Note.** We extend the above definition for \( n = 1 \) by defining \( Y_{d,1}^B \) as the algebra generated by \( t_1 \) and \( b_1 \) subject to the relations (14), (15) and (17).

**Remark 3.** Note that \( Y_{d,n}^B \) is different from the algebra \( Y(d, m, n) \), for \( m = 2 \) and suitable parameters \( \lambda_1 \) and \( \lambda_2 \), defined by M. Chlouveraki and L. Poulain d’Andecy in [2]. Indeed, they differ in the quadratic relation of the generator \( b_1 \), since in \( Y_{d,n}^B \) the relation...
(17) involves framing generators, meanwhile the quadratic relation defined in $Y(d, 2, n)$ doesn’t.

From the above description by generators and relations of the algebras $Y^B_{d,n}$ we have $Y^B_{d,n} \subseteq Y^B_{d,n+1}$, for all $n \geq 1$. Thus, by taking $Y^B_{d,0} := \mathbb{K}$, we have that following tower of algebras.

$$Y^B_{d,0} \subseteq Y^B_{d,1} \subseteq \cdots \subseteq Y^B_{d,n} \subseteq Y^B_{d,n+1} \subseteq \cdots$$  (18)

It is clear that $f_1$ commutes with $b_1$ and $e_i$ commutes with $g_i$. These facts implies that the generators $b_1$ and $g_i$’s are invertible. Moreover, we have:

$$b_1^{-1} = b_1 - (v - v^{-1})f_1 \quad \text{and} \quad g_i^{-1} = g_i - (u - u^{-1})e_i.$$  (19)

**Remark 4.** Notice that, by taking $d = 1$, the algebra $Y^B_{1,n}$ becomes $H_n(u, v)$. Further, by mapping $g_i \mapsto h_i$ and $t_i \mapsto 1$, we obtain an epimorphism from $Y^B_{d,n}$ to $H_n(u, v)$. Moreover, if we map the $t_i$’s to a fixed non–trivial $d$–th root of the unity, we have an epimorphism from $Y^B_{d,n}$ to $H_n(u, 1)$.

**Remark 5.** Notice that the relations (8)–(14) are the defining relations of $F^B_n$ and the relations (8)–(15) are the defining relation of $F^B_{d,n}$. Then, $Y^B_{d,n}$ can be obviously defined as a quotient of the group algebra $\mathbb{K}[F^B_n]$ or $\mathbb{K}[F^B_{d,n}]$. Alternatively, $Y^B_{d,n}$ can be regarded as a $(u, v)$–deformation of the group algebra $\mathbb{K}[W_{d,n}]$ of the $d$–modular framed braid group of type B.

2.3. We also have the following relations which are deduced easily and will be used frequently in the sequel.

$$e_i b_1 = b_1 e_i \quad \text{for all } i, \quad \text{and} \quad f_j g_i = g_i f_j \quad \text{for } |i - j| > 1$$

where $f_j$ is the natural generalization of $f_1$,

$$f_j := \frac{1}{d} \sum_{m=0}^{d-1} t_j^m.$$  

Notice that the $f_j$’s are idempotents.

Finally, we finish the section by introducing certain elements $b_i \in Y^B_{d,n}$ and proving some algebraic identities which will be used along the paper. Set

$$b_i := g_{i-1} \cdots g_1 b_1 g_1^{-1} \cdots g_{i-1}^{-1} \quad \text{for all} \quad 2 \leq i \leq n.$$
Further, for all \(i\) we have \(b_i f_i = f_i b_i\) and a direct computation shows that,

\[
 b_i^2 = 1 + (v - v^{-1}) f_i b_i \quad \text{and} \quad b_i^{-1} = b_i - (v - v^{-1}) f_i.
\]

**Proposition 2.** For \(n \geq 2\) and \(1 \leq i, k \leq n - 1\), the following relations hold in \(Y_{d,n}^\mathbb{B}\):

(i) \(b_k t_j = t_j b_k\), for all \(j\)

(ii) \(b_k g_i = g_i b_k\), for \(i \leq k - 2\) or \(i \geq k + 1\)

(iii) \(g_k b_k g_i b_k = b_k g_k b_k g_k\)

(iv) \(g_k b_k b_{k+1} = b_k g_k b_k\).

**Proof.** The proof of relations (i) and (ii) follows directly by using the defining relations of \(Y_{d,n}^\mathbb{B}\). The proof of relations (iii) is by induction on \(k\). For \(k = 1\), the relation in question is the defining relation \(1\). Let us suppose now that relation (iii) holds for all positive integers less than \(k + 1\). Then for \(k + 1\) we have:

\[
 g_{k+1} b_{k+1} g_{k+1} b_{k+1} = g_{k+1}(g_k b_k g_k^{-1}) g_{k+1}(g_k b_k g_k^{-1})
\]

\[
 = g_{k+1} g_k b_k g_{k+1} g_k g_{k+1} b_k g_k^{-1}
\]

\[
 = g_{k+1} g_k g_{k+1} b_k g_k b_k g_k^{-1} g_k
\]

\[
 = g_k g_{k+1} b_k g_k b_k g_k g_{k+1} g_k^{-1}
\]

\[
 = g_k g_{k+1} b_k g_k b_k g_k g_{k+1} g_k^{-1}
\]

\[
 = g_k b_k g_{k+1} g_k b_k g_k g_{k+1} g_k^{-1}
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]

\[
 = b_{k+1} g_k g_{k+1} b_k g_k g_{k+1} g_k^{-1} g_k
\]
We prove now relation (iv). We have $g_kb_kb_{k+1} = g_k(b_kb_kg_k^{-1}) = (g_kb_kb_kg_k^{-1})g_k^{-1}$. Then by using relation (iii) we obtain $g_kb_kb_{k+1} = (b_kb_kb_kg_k^{-1}) = b_kb_kb_k$, so relation (iv) is true.

3. A TENSORIAL REPRESENTATION FOR $Y^B_{d,n}$

We will define now a tensorial representation for the algebra $Y^B_{d,n}$. The definition of this representation is based on the tensorial representation constructed by Green in [12] for the Hecke algebra of type $A$ and following the idea of an extension of the Jimbo representation of the Hecke algebra of type $B$ to the Yokonuma-Hecke algebra proposed by Espinoza and Ryom-Hansen in [5]. The tensorial representation constructed here will be used to prove that a set of linear generators, denoted $D_n$, is a basis for $Y^B_{d,n}$ (Theorem [2]). Further, as a corollary, we obtain that this tensorial representation is faithful (Corollary [1]).

3.1. Let $V$ be a $\mathbb{K}$-vector space with basis $B = \{v_i^r ; i \in X_n, 0 \leq r \leq d - 1\}$. As usual we denote by $B^\otimes k$ the natural basis of $V^\otimes k$ associated to $B$. That is, the elements of $B^\otimes k$ are of the form:

$$v_{i_1}^{m_1} \otimes \cdots \otimes v_{i_k}^{m_k}$$

where $(i_1, \ldots, i_k) \in X_n^k$ and $(m_1, \ldots, m_k) \in (\mathbb{Z}/d\mathbb{Z})^k$.

We define the endomorphism $T$ of $V$ by

$$(v_i^r)T = \omega^r v_i^r$$

and the endomorphism $G$ of $V \otimes V$ by

$$(v_i^r \otimes v_j^s)G = \begin{cases} uv_j^s \otimes v_i^r & \text{for } i = j \text{ and } r = s \\ v_j^s \otimes v_i^r & \text{for } i < j \text{ and } r = s \\ v_j^s \otimes v_i^r + (u - u^{-1})v_i^r \otimes v_j^s & \text{for } i > j \text{ and } r = s \\ v_j^s \otimes v_i^r & \text{for } r \neq s. \end{cases}$$

For all $1 \leq i \leq n - 1$, we extend these endomorphisms to the endomorphisms $T_i$ and $G_i$ of the $n$th tensor power $V^\otimes n$ of $V$, as follows:

$$T_i := 1_V^\otimes(i-1) \otimes T \otimes 1_V^\otimes(n-i) \quad \text{and} \quad G_i := 1_V^\otimes(i-1) \otimes G \otimes 1_V^\otimes(n-i-1)$$

where $1_V^\otimes k$ denotes the endomorphism identity of $V^\otimes k$. Further we define the endomorphism $B_1$ of $V^\otimes n$ by:

$$(v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n})B_1 = \begin{cases} v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n} & \text{for } i_1 > 0 \text{ and } r_1 = 0 \\ v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n} + (v - v^{-1})v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n} & \text{for } i_1 < 0 \text{ and } r_1 = 0 \\ v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n} & \text{for } r_1 \neq 0. \end{cases}$$

The main goal of this section is to prove that these endomorphisms define a representation of the $Y^B_{d,n}$ in the algebra of endomorphisms $\text{End}(V^\otimes n)$ of $V^\otimes n$. To do that, we will
need certain endomorphisms $E_i$ of $V^\otimes n$, introduced in [8], which are defined by,

$$E_i = \frac{1}{d} \sum_{m=0}^{d-1} T_i^m T_{i+1}^m \quad (1 \leq i \leq n - 1).$$

Also, we will need the following element $F \in \text{End}(V^\otimes n)$,

$$F := \frac{1}{d} \sum_{m=0}^{d-1} T_i^m.$$

**Lemma 2.** We have:

1. $$(v_i^r \otimes v_j^s)E_i = \begin{cases} 0 & r \neq s \\ v_i^r \otimes v_j^s & r = s \end{cases}$$

2. $$(v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n})F = \begin{cases} 0 & r_1 > 0 \\ v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n} & r_1 = 0. \end{cases}$$

**Proof.** Claim (1) is [8, Lemma 3]. To prove (2), we note that $(v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n})T_i^m = \omega^{mr_1} v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n}$. Hence,

$$(v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n})F = (\frac{1}{d} \sum_{m=0}^{d-1} \omega^{mr_1}) v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n}.$$
On the other hand, if $r \neq 0$ it is clear that $B_1^2 = 1^2$. Then by Lemma 2 relation (14) holds in both cases.

Finally, we will prove that the identity (11) holds if we replace $b_1$ by $B_1$ and $g_1$ by $G_1$. To do that, we will distinguish first the cases according to the following exhaustive values of $r$ and $s$:

(a) Case $r = s = 0$
(b) Case $r \neq 0$ and $r \neq s$
(c) Case $r = 0$ and $s \neq 0$
(d) Case $r \neq 0$ and $r = s$.

Case (a) holds by (12), and case (b) is easy to check. We distinguish now according the values of $i$ and $j$ in the remaining cases (c) and (d). For item (c) we have four cases and eight cases for item (d). We will check only the most representative cases by evaluating in $x = v_i^s \otimes v_j^s$.

Case: $r = 0$, $s \neq 0$, $i > 0$ and $j < 0$. We have:

$$(x)G_1B_1G_1B_1 = (v_i^s \otimes v_i^r)B_1G_1B_1$$
$$= (v_{-j}^s \otimes v_i^r + (v - v^{-1})v_j^s \otimes v_i^r)G_1B_1$$
$$= (v_i^r \otimes v_{-j}^s + (v - v^{-1})v_i^r \otimes v_j^s)B_1$$
$$= v_{-i}^r \otimes v_{-j}^s + (v - v^{-1})v_{-i}^r \otimes v_j^s$$
$$= (v_i^s \otimes v_{-i}^r)G_1B_1 = (v_{-i}^r \otimes v_j^s)G_1B_1 = (x)B_1G_1B_1G_1.$$

Case: $r = 0$, $s \neq 0$, $i < 0$ and $j < 0$. We have:

$$(x)G_1B_1G_1B_1 = (v_i^s \otimes v_i^r)B_1G_1B_1$$
$$= (v_{-j}^s \otimes v_i^r + (v - v^{-1})v_j^s \otimes v_i^r)G_1B_1$$
$$= (v_i^r \otimes v_{-j}^s + (v - v^{-1})v_i^r \otimes v_j^s)B_1$$
$$= v_{-i}^r \otimes v_{-j}^s + (v - v^{-1})v_{-i}^r \otimes v_j^s + (v - v^{-1})v_{-i}^r \otimes v_j^s + (v - v^{-1})^2 v_i^r \otimes v_j^s$$
$$= (v_{-j}^s \otimes v_{-i}^r)G_1B_1$$
$$= (v_j^s \otimes v_{-i}^r + (v - v^{-1})v_j^s \otimes v_i^r)B_1G_1$$
$$= (v_{-i}^s \otimes v_j^r + (v - v^{-1})v_{-i}^s \otimes v_j^r)G_1B_1G_1 = (x)B_1G_1B_1G_1.$$
Case: \( r \neq 0, s = r, i > j \) and \(-j < i\). We have:

\[
(x)G_1B_1G_1B_1 = (v^s_j \otimes v^r_i + (u - u^{-1})v^r_i \otimes v^s_j)B_1G_1B_1
\]
\[
= (v^s_j \otimes v^r_i + (u - u^{-1})v^r_i \otimes v^s_j)G_1B_1
\]
\[
= (v^r_i \otimes v^{s_j} + (u - u^{-1})v^s_j \otimes v^r_i)B_1
\]
\[
= v^r_i \otimes v^{s_j} + (u - u^{-1})v^s_j \otimes v^r_i
\]
\[
= (v^s_j \otimes v^r_i)G_1 = (v^s_j \otimes v^r_i)B_1G_1 = (v^s_j \otimes v^r_i)G_1B_1G_1
\]
\[
= (x)B_1G_1B_1G_1.
\]

Case: \( r \neq 0, s = r, i > j \) and \(-j > i\). We have:

\[
(x)G_1B_1G_1B_1 = (v^s_j \otimes v^r_i + (u - 1)v^r_i \otimes v^s_j)B_1G_1B_1
\]
\[
= (v^s_j \otimes v^r_i + (u - u^{-1})v^r_i \otimes v^s_j)G_1B_1
\]
\[
= (v^r_i \otimes v^{s_j} + (u - u^{-1})v^s_j \otimes v^r_i + (u - u^{-1})v^s_j \otimes v^r_i)
\]
\[
+ (u - u^{-1})^2v^r_i \otimes v^s_j)B_1
\]
\[
= v^r_i \otimes v^{s_j} + (u - u^{-1})v^s_j \otimes v^r_i + (u - u^{-1})v^s_j \otimes v^r_i
\]
\[
= (v^s_j \otimes v^r_i)G_1 = (v^s_j \otimes v^r_i)B_1G_1
\]
\[
= (v^s_j \otimes v^r_i)G_1B_1G_1 = (x)B_1G_1B_1G_1.
\]

Case: \( r \neq 0, s = r, i > j \) and \(-j = i\). We have:

\[
(x)G_1B_1G_1B_1 = (v^s_j \otimes v^r_i + (u - u^{-1})v^r_i \otimes v^s_j)B_1G_1B_1
\]
\[
= (v^s_j \otimes v^r_i + (u - u^{-1})v^r_i \otimes v^s_j)G_1B_1
\]
\[
= (uv^r_i \otimes v^{s_j} + u(u - u^{-1})v^s_j \otimes v^r_i)B_1
\]
\[
= uv^r_i \otimes v^{s_j} + u(u - u^{-1})v^s_j \otimes v^r_i
\]
\[
= (uv^s_j \otimes v^r_i)G_1 = (uv^s_j \otimes v^r_i)B_1G_1 = (v^s_j \otimes v^r_i)G_1B_1G_1
\]
\[
= (x)B_1G_1B_1G_1.
\]

\[\square\]

3.2. We shall finish the section by proving Proposition 3 which is an analogue of [12, Lemma 3.1.4]. This proposition will be used in the proof of Theorem 2 and describes, through \( \Phi \), the action of \( W_n \) on the basis \( S^\otimes n \).

The defining generators \( b_i \) and \( g_i \) of the algebra \( Y^B_{d,n} \) satisfy the same braid relations as the Coxeter generators \( r \) and \( s_i \) of the group \( W_n \). Thus, the well–known Matsumoto Lemma implies that if \( w_1 \ldots w_m \) is a reduced expression of \( w \in W_n \), with \( w_i \in \{r, s_1, \ldots, s_{n-1}\} \), then the following element \( g_w \) is well–defined:

\[
g_w := g_{w_1} \cdots g_{w_m}
\]
where $g_{w_i} = b_i$, if $w_i = r$ and $g_{w_i} = g_j$, if $w_i = s_j$.

The notation $\Phi_w$ stands for the image by $\Phi$ of $g_w \in Y^B_{d,n}$. Note that, for $w, w' \in W_n$ such that $\ell(ww') = \ell(w) + \ell(w')$, we have $\Phi_{ww'} = \Phi_w \Phi_{w'}$.

**Proposition 3.** Let $w \in W_n$ parameterized by $(m_1, \ldots, m_n) \in X_n^n$. Then
\[
(v_1^{r_1} \otimes \cdots \otimes v_n^{r_n})\Phi_w = v_{m_1}^{r_{m_1}} \otimes \cdots \otimes v_{m_n}^{r_{m_n}}.
\]

**Proof.** The proof follows by induction on the length of $w$. For $l(w) = 1$ we have that $w \in \{r_1, s_1, \ldots, s_{n-1}\}$, then the result is direct from the definition of $\Phi$. Now suppose that the induction hypothesis holds for any $w' \in W_n$ with $l(w') = n - 1$ and let $w$ be an element with length $n$. Then we have two cases: $w = w' r$ or $w = w's_i$ for some $w' \in W_n$ with $l(w') = n - 1$. We only present the proof of the case $w = w's_i$, as the proof of the other case is analogous. Suppose $w'$ is parameterized by $(m_1, \ldots, m_n) \in X_n^n$. Then by the induction hypothesis we obtain:
\[
(v_1^{r_1} \otimes \cdots \otimes v_n^{r_n})\Phi_w = (v_1^{r_1} \otimes \cdots \otimes v_n^{r_n})\Phi_{w'}G_i = (v_{m_1}^{r_{m_1}} \otimes \cdots \otimes v_{m_n}^{r_{m_n}})G_i.
\]

Now, in Lemma 1 we have $m_i < m_{i+1}$, therefore from the definition of $G_i$’s we obtain
\[
(v_1^{r_1} \otimes \cdots \otimes v_n^{r_n})\Phi_w = v_{m_1}^{r_{m_1}} \otimes \cdots \otimes v_{m_{i+1}}^{r_{m_{i+1}}} \otimes v_{m_i}^{r_{m_i}} \otimes \cdots \otimes v_{m_n}^{r_{m_n}}.
\]

Finally, from (5) we have that $w$ is parameterized by $(m_1, \ldots, m_{i+1}, m_i, \ldots, m_n)$. Hence the claim follows. \qed

### 4. Linear bases for $Y^B_{d,n}$

We introduce here two linear bases $C_n$ and $D_n$ for $Y^B_{d,n}$. The first one is used for defining in the next section a Markov trace on $Y^B_{d,n}$, the second one plays a technical role for proving that $C_n$ is a linearly independent set.

#### 4.1. The basis $D_n$. (Cf. [2, Sec. 4.1].)

Set $\bar{b}_1 := b_1$, and
\[
\bar{b}_k := g_{k-1} \cdots g_1 b_1 g_1 \cdots g_{k-1} \quad \text{for all } 2 \leq k \leq n.
\]

For all $1 \leq k \leq n$, let us define inductively the set $N_{d,k}$ by
\[
N_{d,1} := \{t_1^m, \bar{b}_1 t_1^m ; 0 \leq m \leq d - 1\}
\]
and
\[
N_{d,k} := \{t_k^m, \bar{b}_k t_k^m, g_{k-1} x ; x \in N_{d,k-1}, 0 \leq m \leq d - 1\} \quad \text{for all } 2 \leq k \leq n.
\]

**Definition 4.** We define $D_n$ as the subset of $Y^B_{d,n}$ formed by the following elements
\[
n_1n_2 \cdots n_n
\]
where $n_i \in N_{d,i}$. (21)
We will prove first that $D_n$ is a linearly spanning set for $Y^B_{d,n}$. To do that we need some formulas of multiplication among the defining generators of $Y^B_{d,n}$ and the elements $N_{d,k}$. These are given in Lemmas 3 and 4 below. Notice that every element of $N_{d,k}$ has the form $n_{k,j,m}^+$ or $n_{k,j,m}^-$, with $j \leq k$ and $0 \leq m \leq d - 1$, where

$$n_{k,k,m}^+ := t^m_k, \quad n_{k,j,m}^+ := g_{k-1} \cdots g_j t_j^m \quad \text{for} \quad j < k$$

and

$$n_{k,k,m}^- := b_k t^m_k, \quad n_{k,j,m}^- := g_{k-1} \cdots g_j b_j t_j^m \quad \text{for} \quad j < k.$$

**Lemma 3.** In $Y^B_{d,n}$ the following relations holds:

(i) $(\overline{t}_n^m t_n^a) b_j = b_1 (\overline{t}_n^m t_n^a)$, for all $j < n - 1$

(ii) $(\overline{t}_n^m t_n^a) g_j = g_j (\overline{t}_n^m t_n^a)$, for all $j < n - 1$

(iii) $(\overline{t}_n^m t_n^a) g_{n-1} = n_{n,n-1,a} + d^{-1}(u - u^{-1}) \sum_s \overline{t}^{a-s-n}_{n-1} t^n_s$

(iv) $n_{n,k,a}^{+} b_1 = b_1 n_{n,k,a}$, if $k \neq 1$

(v) $n_{n,k,a}^{+} b_1 = n_{n,k,a}$, if $k = 1$

(vi) $n_{n,k,a}^{-} b_1 = n_{n,k,a}^{+} + d^{-1}(v - v^{-1}) \sum_s n_{n,k,s}^{-}$, for $k = 1$

(vii) $n_{n,k,a}^{+} t_j = \begin{cases} n_{n,k,a}^{+} & \text{for } j > k \\ n_{n,k,a}^{+} + d^{-1}(u - u^{-1}) \sum_s n_{n,k,s}^{-} & \text{for } j = k \\ t_j n_{n,k,a}^{+} & \text{for } j < k. \end{cases}$

**Proof.** All relations follow from direct computations. 

**Lemma 4.** In $Y^B_{d,n}$ we have:

(i) $n_{n,k,a}^{-} g_j = \begin{cases} g_{j-1} n_{n,k,a}^{-} & \text{for } j > k \\ n_{n,k+1,a}^{-} & \text{for } j = k \\ n_{n,k-1,a}^{-} + d^{-1}(u - u^{-1}) \sum_s t^{a-s}_{j} n_{n,k,s}^{-} & \text{for } j = k - 1 \\ g_{j} n_{n,k,a}^{-} & \text{for } j < k - 1. \end{cases}$

(ii) $n_{n,k,a}^{+} g_j = \begin{cases} g_{j-1} n_{n,k,a}^{+} & \text{for } j > k \\ n_{n,k+1,a}^{+} + d^{-1}(u - u^{-1}) \sum_s t^{a-s}_{j} n_{n,k,s}^{+} & \text{for } j = k \\ n_{n,k-1,a}^{+} & \text{for } j = k - 1 \\ g_{j} n_{n,k,a}^{+} & \text{for } j < k - 1. \end{cases}$
Proof. In claim (i) we will check only the case \( j = k - 1 \), since the other cases are clear. We have

\[
\begin{align*}
n_{n,k,\alpha}g_{k-1} &= g_{n-1} \cdots g_1 b_1 g_1 \cdots g_{k-1} t_{k}^{\alpha} g_{k-1} \\
&= g_{n-1} \cdots g_1 b_1 g_1 \cdots g_{k-1} t_{k-1}^{\alpha} \\
&= g_{n-1} \cdots g_1 b_1 g_1 \cdots g_{k-2} t_{k-1}^{\alpha} + (u - u^{-1}) g_{n-1} \cdots g_1 b_1 g_1 \cdots g_{k-1} c_{k-1} t_{k-1}^{\alpha} \\
&= g_{n-1} \cdots g_1 b_1 g_1 \cdots g_{k-2} t_{k-1}^{\alpha} + d^{-1}(u - u^{-1}) \sum_{s} g_{n-1} \cdots g_1 b_1 g_1 \cdots g_{k-1} t_{k}^{s} t_{k-1}^{\alpha-s} \\
&= n_{n,k-1,\alpha} - d^{-1}(u - u^{-1}) \sum_{s} t_{k-1}^{\alpha-s} n_{n,k,s}.
\end{align*}
\]

The only non-trivial case in claim (ii) is whenever \( j = k \). We have

\[
\begin{align*}
n_{n,k,\alpha}g_{k} &= (g_{n-1} \cdots g_{k} t_{k}^{\alpha}) g_{k} \\
&= g_{n-1} \cdots g_{k} t_{k+1}^{\alpha} \\
&= g_{n-1} \cdots g_{k+1} t_{k+1}^{\alpha} + (u - u^{-1}) g_{n-1} \cdots g_{k+1} c_{k} t_{k+1}^{\alpha} \\
&= g_{n-1} \cdots g_{k+1} t_{k+1}^{\alpha} + d^{-1}(u - u^{-1}) \sum_{s} g_{n-1} \cdots g_{k+1} t_{k+1}^{s} t_{k+1}^{\alpha-s} \\
&= n_{n,k+1,\alpha} + d^{-1}(u - u^{-1}) \sum_{s} t_{k+1}^{\alpha-s} n_{n,k,s}.
\end{align*}
\]

\[\Box\]

Proposition 4. The set \( D_n \) is a spanning set for \( Y^B_{d,n} \).

Proof. The proof is by induction on \( n \). Let \( D_n \) be the linear subspace of \( Y^B_{d,n} \) spanned by \( D_n \). The assertion is true for \( n = 1 \), since \( D_1 = N_1 \) and obviously \( Y^B_{d,1} \) is equal to the space spanned by \( N_{d,1} \). Assume now that \( Y^B_{d,n-1} \) is spanned by \( D_{n-1} \). Notice that \( 1 \in D_n \). This fact and proving that \( D_n \) is a right ideal, implies the proposition. Now, we deduce that \( D_n \) is a right ideal from the hypothesis induction and Lemmas 3 and 4. Indeed, the multiplication of \( n \in N_{d,n} \) from the right by all defining generators of \( Y^B_{d,n} \) results in a linear combination of elements of the form \( w n' \), with \( n' \in N_{d,n} \) and \( w \in Y^B_{d,n-1} \). \( \Box \)

In order to prove now that \( D_n \) is a linearly independent set, we will firstly rewrite its elements in split form, that is, as the product between the braiding part and the framing part. More precisely, given an element in \( D_n \), then by using the relations (13)–(14), the framing elements (every power of the \( t_j \)'s) that appears in this given element, can be moved to the right. Thus, we deduce that the elements in \( D_n \) can be written in the following form:

\[
\mathbf{r}_1 \cdots \mathbf{r}_n t_1^{m_1} \cdots t_n^{m_n}
\]

with \( m_k \in \mathbb{Z}/d\mathbb{Z} \) and \( \mathbf{r}_k \in N_k \), where the sets \( N_k \) are defined inductively as follows: \( N_1 := \{1, b_1\} \) and

\[
N_k := \{1, b_k, g_{k-1} x ; x \in N_{k-1}\} \quad \text{for all } 2 \leq k \leq n.
\]
Recall now that \( g_k = g_i \) and notice that \( r_k \) is reduced, so \( g_{r_k} = t_k \) (see (20)). These facts and noting that the elements of the sets \( N_k \) (see (3)) are reduced, imply that \( N_k = \{ g_w ; w \in N_k \} \).

Then, the set \( D_n \) can be described by:

\[
D_n = \{ g_w t_1^{m_1} \cdots t_n^{m_n} ; w \in W_n, (m_1, \ldots, m_n) \in (\mathbb{Z}/d\mathbb{Z})^n \}.
\]

Secondly, we shall use a certain basis \( D \) of \( V \) introduced by Espinoza and Ryom–Hansen in [8]. More precisely, \( D \) consist of the following elements:

\[
u^r_k = \sum_{i=0}^{d-1} \omega^{ir} v^i_k \]

where \( k \) is running \( X_n \) and \( 0 \leq r \leq d - 1 \).

Notice that \( D \) is a basis for \( V \), since for any fixed \( k \) the base change matrix between \( \{ u^r_k ; 0 \leq r \leq d - 1 \} \) and \( \{ v^s_k ; 0 \leq s \leq d - 1 \} \) is non–singular, see [8]. Further, it is easy to see that

\[(u^r_k)^T = u^{r+1}_k. \quad (23)\]

We are now in the position to prove that \( D_n \) is a basis for \( Y^B_{d,n} \).

**Theorem 2.** \( D_n \) is a linear basis for \( Y^B_{d,n} \). Hence the dimension of \( Y^B_{d,n} \) is \( 2^n d^m n! \).

**Proof.** According to Proposition [4] we only need to prove that \( D_n \) is a linearly independent set. Indeed, suppose that we have a linear combination in the form:

\[\sum_{c \in D_n} \lambda_c c = 0.\]

The proof follows by proving that \( \lambda_c = 0 \) for all \( c \in D_n \). Now, using the expression (22) for the elements of \( D_n \) and applying \( \Phi \) to the above equation, we obtain the following equation:

\[
\sum_{m,w} \lambda_{m,w} \Phi_w T_1^{m_1} \cdots T_n^{m_n} = 0 \quad (24)
\]

where \( w \) runs in \( W_n \) and \( m = (m_1, \ldots, m_n) \) runs in \((\mathbb{Z}/d\mathbb{Z})^n\).

Now, set \( w \in W_n \) parameterized by \((i_1, \ldots, i_n) \in X_n^n \). Then from Lemma [3] and the definition of the elements \( u^r_k \)'s, we get

\[(u^0_{i_1} \otimes \cdots \otimes u^0_{i_n}) \Phi_w = u^0_{i_1} \otimes \cdots \otimes u^0_{i_n}.\]

On the other hand, by using (23) we have:

\[(u^0_{i_1} \otimes \cdots \otimes u^0_{i_n}) T_1^{m_1} \cdots T_n^{m_n} = u^m_{i_1} \otimes \cdots \otimes u^m_{i_n} \]

where \( m := (m_1, \ldots, m_n) \) runs in \((\mathbb{Z}/d\mathbb{Z})^n\). Thus, evaluating equation (24) in \( u^0_{i_1} \otimes \cdots \otimes u^0_{i_n} \) we obtain

\[\sum_{m,i} \lambda_{m,i} u^m_{i_1} \otimes \cdots \otimes u^m_{i_n} = 0\]
where $i := (i_1, \ldots, i_n)$ runs in $X_0^n$ and $m := (m_1, \ldots, m_n)$ runs in $(\mathbb{Z}/d\mathbb{Z})^n$. Therefore, $\lambda_{m,i} = 0$ for all $i$ and $m$ since the left side of the last equation is a linear combination of elements of the basis $D^\otimes n$.

In particular, the above theorem implies the following corollary.

**Corollary 1.** The representation $\Phi$ is faithful.

4.2. **The basis $C_n$.** For all $1 \leq k \leq n$, let us define inductively the sets $M_{d,k}$ by

$$M_{d,1} = \{ t_1^m, t_1^m b_1 ; 0 \leq m \leq d - 1 \}$$

and

$$M_{d,k} = \{ t_k^m, t_k^m b_k, g_k x ; x \in M_{d,k-1}, 0 \leq m \leq d - 1 \} \quad \text{for all } 2 \leq k \leq n.$$ 

**Definition 5.** We define $C_n$ as the subset of $Y_{d,n}$ formed by the following elements:

$$m_1 m_2 \cdots m_n$$

where $m_i \in M_{d,i}$.

To prove that $C_n$ is a linearly spanning set we will need some formulas of multiplication among the defining generators of $Y_{d,n}$ and the elements $M_{d,k}$. These are given in Lemmas 5–7 below. Now notice that every element of $M_{d,k}$ has the form $m_{k,j,m}$ or $m_{k,j,m}$ with $j \leq k$ and $0 \leq m \leq d - 1$, where

$$m_{k,k,m}^+ := t_k^m, \quad m_{k,j,m}^+ := g_k \cdots g_j t_j^m \quad \text{for } j < k$$

and

$$m_{k,k,m}^- := t_k^m b_k, \quad m_{k,j,m}^- := g_k \cdots g_j b_j t_j^m \quad \text{for } j < k.$$ 

**Lemma 5.** The following hold:

(i) $m_{n,k,m}^\pm t_j = \begin{cases} t_{j-1} m_{n,k,m}^\pm & \text{for } j > k \\ m_{n,k,m+1}^\pm & \text{for } j = k \\ t_j m_{n,k,m}^\pm & \text{for } j < k \end{cases}$

(ii) $t_n m_{n,k,m}^\pm = m_{n,k,m+1}^\pm$.

**Proof.** The proof is straightforward.

**Lemma 6.** The following hold:

$$m_{n,k,m}^\pm g_j = \begin{cases} g_{j-1} m_{n,k,m}^\pm & \text{for } j > k \\ m_{n,k+1,m}^\pm + d^{-1} (u - u^{-1}) \sum_s t_j^{m-s} m_{n,k,s}^\pm & \text{for } j = k \\ m_{n,k-1,m}^\pm & \text{for } j = k - 1 \\ g_j m_{n,k,m}^\pm & \text{for } j < k - 1. \end{cases}$$
Proof. The positive case follows directly from Lemma 4 (ii), since $m^+_{n,k,m} = n^+_{n,k,m}$. For the negative case we have:

\[
m^-_{n,k,m}g_k &= g_{n-1} \cdots g_k b_i t_k^m g_k \\
&= g_{n-1} \cdots g_k b_i g_i^{-1} \cdots g_{k-1} t_k^m g_k \\
&= g_{n-1} \cdots g_k b_i g_i^{-1} \cdots g_{k-1} g_k t_{k+1}^m \\
&= g_{n-1} \cdots g_k b_i g_i^{-1} \cdots g_{k-1} g_k t_{k+1}^m + (u - u^{-1}) g_{n-1} \cdots g_k b_i g_i^{-1} \cdots g_{k-1} e_k t_{k+1}^m \\
&= g_{n-1} \cdots g_k b_i g_i^{-1} \cdots g_{k-1} g_k t_{k+1}^m + \frac{1}{d} (u - u^{-1}) \sum_s g_{n-1} \cdots g_k b_i g_i^{-1} \cdots g_{k-1} e_k t_{k+1}^m \\
&= m^-_{n,k+1,m} + \frac{1}{d} (u - u^{-1}) \sum_s t_k^{m-s} m_{n,k,s}.
\]

Lemma 7. The following hold:

(i) $m^+_{n,k,m} b_1 = \begin{cases} m^-_{n,k,m} & \text{for } k = 1 \\ b_1 m^+_{n,k,m} & \text{for } k > 1 \end{cases}$

(ii) $m^-_{n,k,m} b_1 = \begin{cases} m^+_{n,k,m} + d^{-1}(v - v^{-1}) \sum_s m^+_{n,k,s} & \text{for } k = 1 \\ b_1 m^-_{n,k,m} + d^{-1}(u - u^{-1}) \sum_s p_{k,s} & \text{for } k > 1 \end{cases}$

where $p_{k,s} = b_1 t_1^{m-s} g_1^{-1} \cdots g_{k-2}^{-1} m_{n,1,s} - t_1^{m-s} g_1^{-1} \cdots g_{k-2}^{-1} (m_{n,1,s} b_1)$. In particular we have:

\[(b_n t_n^m) b_1 = b_1 (b_n t_n^m) + \frac{1}{d} (u - u^{-1}) \sum_s p_{n,s}.\]

Proof. The claim of (i) is straightforward. To prove claim (ii), we note first that $m^-_{n,k,m} b_1 = g_{n-1} \cdots (g_1 b_i g_i^{-1} b_1) g_2^{-1} \cdots g_{k-1} t_k^m$. Then, splitting $g_1^{-1}$ according to (19) and invoking (11) we deduce:

\[
m^-_{n,k,m} b_1 &= g_{n-1} \cdots g_2 (b_1 g_i b_1 g_i) g_2^{-1} \cdots g_{k-1} t_k^m - (u - u^{-1}) g_{n-1} \cdots g_i b_i e_1 g_2^{-1} \cdots g_{k-1} t_k^m \\
&= b_1 g_{n-1} \cdots g_2 (g_1 b_i g_1) g_2^{-1} \cdots g_{k-1} t_k^m - (u - u^{-1}) g_{n-1} \cdots g_i b_i e_1 g_2^{-1} \cdots g_{k-1} t_k^m b_1.
\]

By using again (19), we write the second $g_1$ that appears inside the parenthesis above in terms of $g_1^{-1}$. So, we obtain:

\[
m^-_{n,k,m} b_1 = b_1 g_{n-1} \cdots g_1 b_i g_i^{-1} g_2^{-1} \cdots g_{k-1} t_k^m + (u - u^{-1}) b_1 g_{n-1} \cdots g_i b_i e_1 g_2^{-1} \cdots g_{k-1} t_k^m - (u - u^{-1}) g_{n-1} \cdots g_i b_i e_1 g_2^{-1} \cdots g_{k-1} t_k^m b_1.
\]

Hence

\[
m^-_{n,k,m} b_1 = (u - u^{-1}) [b_1 g_{n-1} \cdots g_i b_i e_1 t_2^m g_2^{-1} \cdots g_{k-1} - g_{n-1} \cdots g_i b_i e_1 t_2^m g_2^{-1} \cdots g_{k-1} b_1] + b_1 m_-^{n,k,m}.
\]
Now, by using the definition of \( e_i \), we obtain:
\[
b_1g_{n-1} \ldots g_1b_1e_1t_{12}^m g_2 \ldots g_{k-1} = \sum_s b_1g_{n-1} \ldots g_1b_1s_{I_1}^m g_2 \ldots g_{k-1} = \sum_s b_1t_{12}^m g_1 \ldots g_{k-2}m_{n,1,s}.
\]
In the same way we obtain:
\[
g_{n-1} \ldots g_1b_1s_{I_1}^m g_2 \ldots g_{k-1}b_1 = \sum_s g_{n-1} \ldots g_1b_1s_{I_1}^m g_2 \ldots g_{k-1}b_1 = t_{12}^m g_1 \ldots g_{k-2}m_{n,1,s}b_1.
\]
Therefore, the proof follows.

**Proposition 5.** The set \( C_n \) is a basis for \( Y_{d,n}^B \).

**Proof.** We can prove that \( C_n \) is a spanning set for \( Y_{d,n}^B \) analogously to the proof of Proposition [4] but by using now Lemmas [5] [7] instead of Lemmas [3] and [4]. Now, the cardinal of \( C_n \) is \( 2^n d^n n! \), hence the proof follows.

We shall close the subsection with a lemma, which will be used in Section 6.

**Lemma 8.** For \( k \geq 2 \) and \( X \in Y_{d,n}^B \) the following identities hold:

(i) \( m_{k,j,m}^k b_k = b_{k-1}m_{k,j,m}^k \) for \( j \leq k - 1 \)

(ii) \( g_{n-1}Xg_{n-1} = g_{n-1}Xg_{n-1} + (u - u^{-1})(e_{n-1}Xg_{n-1} - g_{n-1}Xe_{n-1}) \)

(iii) \( g_{k+1}^2b_{k-1}g_{k-1}^1 = b_{k-1}g_{k-1} - (u - u^{-1})b_{k-1}e_{k-1} + (u - u^{-1})e_{k-1}b_k \)

**Proof.** To prove claim (i) we use Lemmas [5] [7]. More precisely, we have:
\[
m_{k,j,m}^k b_k = g_{k-1}g_{k-2} \ldots g_1b_1g_1^{-1} \ldots g_{j-1}^{-1}t_j(g_{k-1} \ldots g_1b_1g_1^{-1} \ldots g_{k-1}^{-1})
= g_{k-1} \ldots g_1b_1g_1^{-1} \ldots g_{j-1}^{-1}(g_{k-1} \ldots g_1b_1g_1^{-1} \ldots g_{k-1}^{-1})t_j
= g_{k-1} \ldots g_1b_1(g_{k-1} \ldots g_1b_1g_1^{-1} \ldots g_{k-1}^{-1})g_1^{-1} \ldots g_{j-1}^{-1}t_j
= g_{k-1} \ldots g_1b_1g_1^{-1} \ldots g_2g_1b_1g_1^{-1} \ldots g_{k-1}^{-1}g_1^{-1} \ldots g_{j-1}^{-1}t_j
= g_{k-1} \ldots g_1b_1g_1^{-1} \ldots g_2g_1b_1g_1^{-1} \ldots g_{k-1}^{-1}g_1^{-1} \ldots g_{j-1}^{-1}t_j
= g_{k-1} \ldots g_1b_1g_1^{-1} \ldots g_2g_1b_1g_1^{-1} \ldots g_{k-1}^{-1}g_1^{-1} \ldots g_{j-1}^{-1}t_j
= (g_{k-1} \ldots g_1b_1g_1^{-1} \ldots g_{j-1}^{-1})(g_{k-1}g_{k-2} \ldots g_1b_1g_1^{-1} \ldots g_{j-1}^{-1})t_j
= b_{k-1}m_{k,j,m}^k.
\]
For claim (ii) we have by [19]:
\[
g_{n-1}Xg_{n-1} = (g_{n-1}^{-1} + (u - u^{-1})e_{n-1})X(g_{n-1} - (u - u^{-1})e_{n-1})
= g_{n-1}Xg_{n-1} + (u - u^{-1})e_{n-1}Xg_{n-1} - (u - u^{-1})g_{n-1}Xe_{n-1}
- (u - u^{-1})^2 e_{n-1}Xe_{n-1}.
\]
Then, by expanding the $g_{n-1}$ above, the result follows. Finally, we can prove claim (iii) similarly, but by using now [16] and [19].

\[ \Box \]

5. A Markov trace on $Y_{d,n}^B$

The section is devoted to proving that the tower of algebras [18] associated to the algebras $Y_{d,n}^B$ supports a Markov trace (Theorem 3). This fact is proved by using the method of relative traces, cf. [11,2]. Probably this method is due to A. P. Isaev and O. V. Ogievetsky, see for example [13]. In few words, the method consists in constructing a certain family of linear maps $\text{tr}_n : Y_{d,n}^B \to Y_{d,n-1}^B$, called relative traces, which builds step by step the desired Markov properties. The Markov trace on $Y_{d,n}^B$ is defined by $\text{Tr}_n := \text{tr}_1 \circ \cdots \circ \text{tr}_n$.

5.1. Let $z$ be an indeterminate and denote by $\mathbb{L}$ the field of rational functions $\mathbb{K}(z) = \mathbb{C}(u,v,z)$. We work now on the algebra $\mathbb{L} \otimes \mathbb{K} Y_{d,n}^B$, which, for simplicity, we denote again by $Y_{d,n}^B$. Notice that $\mathbb{L} \otimes \mathbb{K} \mathbb{K} = \mathbb{L}$. Consequently, $Y_{d,0}^B$ is taken as $\mathbb{L}$.

We set $x_0 := 1$ and from now on we fix non–zero parameters $x_1, \ldots, x_{d-1}, y_0, \ldots, y_{d-1}$ in $\mathbb{L}$.

**Definition 6.** For $n \geq 1$, we define the linear functions $\text{tr}_n : Y_{d,n}^B \to Y_{d,n-1}^B$ as follows. For $n = 1$, $\text{tr}_1(t_1^{a_1}) = x_{a_1}$ and $\text{tr}_1(b_1t_1^{a_1}) = y_{a_1}$. For $n \geq 2$, we define $\text{tr}_n$ on the basis $C_n$ of $Y_{d,n}^B$ by:

\[
\text{tr}_n(wm_n) = \begin{cases} 
  x_mw & \text{for } m_n = t_n^m \\
  y_mw & \text{for } m_n = b_n^m \\
  zwm_{n-1,k,m} & \text{for } m_n = m_{n,k,m}
\end{cases} \quad (26)
\]

where $w := m_1 \cdots m_{n-1} \in C_{n-1}$. Note that (26) also holds for $w \in Y_{d,n-1}^B$, since $C_{n-1}$ is a basis for $Y_{d,n-1}^B$.

**Lemma 9.** For all $X, Z \in Y_{d,n-1}^B$ and $Y \in Y_{d,n}^B$, we have:

(i) $\text{tr}_n(YZ) = \text{tr}_n(Y)Z$

(ii) $\text{tr}_n(XY) = X\text{tr}_n(Y)$

(iii) $\text{tr}_n(XYZ) = X\text{tr}_n(Y)Z$.

**Proof.** For proving claim (i) notice that, due to the linearity of $\text{tr}_n$, we can suppose that $Z$ is a defining generator of $Y_{d,n-1}^B$ and $Y = wm_n \in C_n$, with $w \in C_{n-1}$. Further, to prove the claim we shall distinguish the $Y$’s according to the possibilities of $m_n$.

- For $m_n = t_n^m$, we have $YZ = wt_n^mZ = wZt_n^m$, then $\text{tr}_n(wZt_n^m) = x_mwZ$ since $wZ \in Y_{d,n-1}^B$. Hence, $\text{tr}_n(YZ) = \text{tr}_n(Y)Z$.

- For $m_n = b_n^m$, we consider first $Z \in \{t_1, \ldots, t_{n-1}, g_1, \ldots, g_{n-2}\}$. Then using Lemma 5 and if we have: $YZ = wbt_n^mZ = wZb_n^m$. Hence,

\[
\text{tr}_n(YZ) = \text{tr}_n(wZb_n^m) = y_mwZ = \text{tr}_n(Y)Z.
\]
Suppose now $Z = b_1$. By definition, \( \text{tr}_n(wb_n t^m_n b_1) = w \text{tr}_n(b_n t^m_n b_1) \). Then by Lemma 7,

\[
\text{tr}_n(wb_n t^m_n b_1) = w \text{tr}_n \left( b_n t^m_n + \frac{1}{d} (u - u^{-1}) A \right)
\]

where

\[
A := \sum_s \left( b_1 t_1^{m-s} g_1^{-1} \ldots g_{n-2}^{-1} m_{n,1,s}^{-} - t_1^{m-s} g_1^{-1} \ldots g_{n-2}^{-1} (m_{n,1,s}^{-} b_1) \right)
\]

But, we have

\[
\text{tr}_n(A) = z \sum_s \left( b_1 t_1^{m-s} g_1^{-1} \ldots g_{n-2}^{-1} m_{n-1,1,s}^{-} - t_1^{m-s} g_1^{-1} \ldots g_{n-2}^{-1} (m_{n-1,1,s}^{-} b_1) \right)
\]

\[
= z \sum_s \left( b_1 t_1^{m-s} b_1 t_1^s - t_1^{m-s} b_1 t_1^s b_1 \right) = 0.
\]

Therefore,

\[
\text{tr}_n(Y Z) = \text{tr}_n(wb_n t^m_n b_1) = \text{tr}_n(wb_1 b_n t^m_n) = y_m w b_1 = \text{tr}_n(Y) Z.
\]

- For \( m_n = m_{n,k,m}^\pm \), with \( k < n \), we have:
  - If \( Z = t_j \) with \( j \in \{1, \ldots, n - 1\} \), the claim follows directly from (i) Lemma 5. For example, for \( j > k \), we have

\[
\text{tr}_n(wm_{n,k,m}^\pm t_j) = \text{tr}_n(w t_{j-1} m_{n,k,m}^\pm) = zw t_{j-1} m_{n-1,k,m}^\pm = zw m_{n-1,k,m}^\pm t_j = \text{tr}_n(Y) Z.
\]

We can proceed in similar way for the other cases for \( j \).

- If \( Z = g_j \) with \( j \in \{1, \ldots, n - 2\} \). The claim follows by using the formulas of Lemma 6.

Below, we show only the prove of the case \( j = k \), since that the other cases for \( j \) follows easily.

We have:

\[
\text{tr}_n(wm_{n,k,m}^\pm g_j) = \text{tr}_n \left( w \left[ m_{n,k+1,m}^\pm + \frac{1}{d} (u - u^{-1}) \sum_s t_j^{m-s} m_{n,k,s}^\pm \right] \right)
\]

\[
= \left( zw \left[ m_{n-1,k+1,m}^\pm + \frac{1}{d} (u - u^{-1}) \sum_s t_j^{m-s} m_{n-1,k,s}^\pm \right] \right)
\]

\[
= zw m_{n-1,k,m}^\pm g_j = \text{tr}_n(Y) Z.
\]

- If \( Z = b_1 \), we deduce the claim directly from (ii) Lemma 5.

To prove (ii), by using the linearity of \( \text{tr}_n \), we can suppose again that \( X \) stands for the defining generators of \( Y^B_{d,n-1} \) and \( Y = w m_n \in C_n \), with \( w \in C_{n-1} \). Note now that \( X w \in Y^B_{d,n-1} \). Hence claim (ii) follows directly from the definition of \( \text{tr}_n \).

Finally, claim (iii) is a combination of claims (i) and (ii). □
Lemma 10. For every \( n \geq 1 \) and \( X \in Y^B_{d,n} \), we have that

\[
\text{tr}_n(Xt_n) = \text{tr}_n(t_nX)
\]

Proof. As we know, from linearity of the trace is enough consider \( X \) in \( C_n \), then we have \( X = wm_n \), with \( w \in C_{n-1} \). Whenever \( m_n = b_n t_n^m \) or \( t_n^m \) the result is clear, since \( t_n \) commute with \( X \). So, suppose \( m_n = m_{n,k,m} \). Then, from Lemma 5, we obtain

\[
\text{tr}_n(Xt_n) = \text{tr}_n(wm_{n,k,m}^\pm t_n) = zwt_{n-1}m_{n-1,k,m}^\pm = zm_{n-1,k,m+1}^\pm.
\]

On the other hand, we have

\[
\text{tr}_n(t_nX) = \text{tr}_n(wm_{n,k,m}^\pm n) = \text{tr}_n(wm_{n,k,m+1}^\pm) = zm_{n-1,k,m+1}^\pm.
\]

Thus, the proof of the lemma follows.

Lemma 11. For \( n \geq 2 \), \( X \in Y^B_{d,n-1} \) and \( Y \in Y^B_{d,n} \), we have:

(i) \( \text{tr}_n(e_{n-1}Xg_{n-1}) = \text{tr}_n(g_{n-1}Xe_{n-1}) \)

(ii) \( \text{tr}_{n-1}\text{tr}_n(e_{n-1}Y) = \text{tr}_{n-1}\text{tr}_n(Ye_{n-1}) \).

Proof. We prove (i). Expanding the left side and using Lemma 9 we have:

\[
\text{tr}_n(e_{n-1}Xg_{n-1}) = \frac{1}{d} \sum_s \text{tr}_n(t_{s-1}^n Yt_n^s Xg_{n-1}) = \frac{1}{d} \sum_s \text{tr}_n(t_{s-1}^n Xg_{n-1}t_n^s) = \frac{1}{d} \sum_s zt_n^{s-1}Xt_n^s.
\]

Similarly, we expand the right side obtaining:

\[
\text{tr}_n(g_{n-1}Xe_{n-1}) = \frac{1}{d} \sum_s zt_n^{s-1}Xt_n^s.
\]

Hence, claim (i) is true.

To prove claim (ii) we use Lemmas 9 and 10. Indeed, we have:

\[
\text{tr}_{n-1}(\text{tr}_n(e_{n-1}Y)) = \frac{1}{d} \sum_s \text{tr}_{n-1}(\text{tr}_n(t_{n-1}^s t_n^{-s} Y)) = \frac{1}{d} \sum_s \text{tr}_{n-1}(t_{n-1}^s \text{tr}_n(t_n^{-s} Y)) = \frac{1}{d} \sum_s \text{tr}_{n-1}(t_{n-1}^s \text{tr}_n(Y t_n^{-s})) = \frac{1}{d} \sum_s \text{tr}_{n-1}(\text{tr}_n(Y t_n^{-s}) t_{n-1}^s) = \frac{1}{d} \sum_s \text{tr}_{n-1}(\text{tr}_n(Y t_n^{-s} t_{n-1}^s)) = \text{tr}_{n-1}(\text{tr}_n(Y e_{n-1})).
\]
Lemma 12. For \( n \geq 2 \) and \( X \in Y_{d,n-1}^B \). We have
\[
\text{tr}_n(g_{n-1}Xg_{n-1}^{-1}) = \text{tr}_{n-1}(X) = \text{tr}_n(g_{n-1}^{-1}Xg_{n-1}).
\]

Proof. As before, we can suppose \( X = w\mathfrak{m}_{n-1} \) with \( w \in \mathfrak{C}_{n-2} \). We will check the first equality by distinguishing the possibilities for \( \mathfrak{m}_{n-1} \). For \( \mathfrak{m}_{n-1} = t_{n-1}^m \) the claim is only a direct computation. For \( \mathfrak{m}_{n-1} = b_{n-1}t_{n-1}^m \), we have
\[
\text{tr}_n(g_{n-1}wb_{n-1}t_{n-1}^m g_{n-1}^{-1}) = \text{tr}_n(wb_{n}t_{n-1}^m) = y_m w = \text{tr}_{n-1}(X).
\]

Finally, for \( \mathfrak{m}_{n-1} = \mathfrak{m}_{n-1,k,m}^\pm \), we have:
\[
\text{tr}_n(g_{n-1}w\mathfrak{m}_{n-1,k,m}^\pm g_{n-1}^{-1}) = \text{tr}_n(w\mathfrak{m}_{n,k,m}^\pm g_{n-1}^{-1}) = \text{tr}_n(wg_{n-2}^{-1}\mathfrak{m}_{n,k,m}^\pm) = zwg_{n-2}^{-1}\mathfrak{m}_{n-2,k,m}^\pm = zw\mathfrak{m}_{n-2,k,m}^\pm = \text{tr}_{n-1}(X).
\]

Thus, the proof of the first equality is done.

From a combination of Lemma 8 (ii) and Lemma 11 (i), we deduce
\[
\text{tr}_n(g_{n-1}Xg_{n-1}^{-1}) = \text{tr}_n(g_{n-1}^{-1}Xg_{n-1}).
\]

Hence, from (i) the second equality follows. \( \square \)

Lemma 13. For all \( X \in Y_{d,n}^B \), we have
\[
\text{tr}_{n-1}(\text{tr}_n(Xg_{n-1})) = \text{tr}_{n-1}(\text{tr}_n(g_{n-1}X)).
\]

Proof. Again, from the linearity of \( \text{tr}_n \) it is enough to consider \( X \) in the basis \( \mathfrak{C}_n \). Set \( X = w\mathfrak{m}_n \), with \( w \in \mathfrak{C}_{n-1} \). We are going to prove the statement by distinguishing according to the possibilities for \( \mathfrak{m}_n \).

- For \( \mathfrak{m}_n = t_n^m \), the claim follows from Lemma 10
- For \( \mathfrak{m}_n = \mathfrak{m}_{n,k,m}^\pm \), we note that, using the formula 19 for the inverse of \( g_{n-1} \) and Lemma 11 (ii), we obtain the following:
\[
\text{tr}_{n-1}(\text{tr}_n(Xg_{n-1}^{-1})) = \text{tr}_{n-1}(\text{tr}_n(Xg_{n-1})).
\]

Now, for the left side of this equality, we have:
\[
\text{tr}_{n-1}(\text{tr}_n(Xg_{n-1}^{-1})) = \text{tr}_{n-1}(\text{tr}_n(w\mathfrak{m}_{n,k,m}^\pm g_{n-1}^{-1})) = \text{tr}_{n-1}(w\text{tr}_n(g_{n-1}^{-1}\mathfrak{m}_{n-1,k,m}^\pm g_{n-1})) = \text{tr}_{n-1}(w\text{tr}_n(\mathfrak{m}_{n-1,k,m}^\pm)) = \text{tr}_{n-1}(w)\text{tr}_{n-1}(\mathfrak{m}_{n-1,k,m}^\pm).
\]
In the same manner one obtains this last expression for \( \text{tr}_{n-1}(\text{tr}_n(X g_{n-1}^{-1})) \). In consequence the claim holds.

- Finally, for \( m_n = b_n t_n^m \) we separate the proof depending on the form of \( w \) in \( X = w m_n \).

* Suppose that \( w = w' t_{n-1}^m \), with \( w' \in C_{n-2} \). Then, we have:

\[
\text{tr}_{n-1}(\text{tr}_n(X g_{n-1}^{-1})) = \text{tr}_{n-1}(\text{tr}_n(w' t_{n-1}^m b_n t_n^m g_{n-1}^{-1})) = \text{tr}_{n-1}(\text{tr}_n(w' t_{n-1}^m b_n g_{n-1}^{-1} t_n^m)) = \text{tr}_{n-1}(\text{tr}_n(w' t_{n-1}^m b_n^{-1} t_{n-1}^m)) = zw_{n+\beta} w'.
\]

On other hand:

\[
\text{tr}_{n-1}(\text{tr}_n(g_{n-1} X)) = \text{tr}_{n-1}(\text{tr}_n (g_{n-1} w' t_{n-1}^m b_n t_n^m g_{n-1}^{-1})) = \text{tr}_{n-1}(\text{tr}_n(w' g_{n-1} t_{n-1}^m)) = \text{tr}_{n-1}(\text{tr}_n(w' g_{n-1}^{-1} t_{n-1}^m))
\]

Then using Lemma 8 (iii), we obtain

\[
\text{tr}_{n-1}(\text{tr}_n(g_{n-1} X)) = A - (u - u^{-1}) B + (u - u^{-1}) C
\]

where

\[
A := \text{tr}_{n-1}(\text{tr}_n(w' b_{n-1} g_{n-1} t_{n-1}^m)) \quad \quad \quad B := \text{tr}_{n-1}(\text{tr}_n(w' b_{n-1}^{-1} t_{n-1}^m)) \quad \quad \quad C := \text{tr}_{n-1}(\text{tr}_n(w' e_{n-1} t_{n-1}^m)).
\]

We will compute the values of \( A, B \) and \( C \). A direct computation shows that:

\[
A = \text{tr}_{n-1}(\text{tr}_n(w' b_{n-1} t_{n-1}^m g_{n-1}^{-1})) = zw_{n+\beta} (tr_{n-1}(w' b_{n-1} t_{n-1}^m)) = zw_{n+\beta} w'.
\]

Expanding \( e_{n-1} \) in \( B \), we get:

\[
B = \frac{1}{d} (u - u^{-1}) \sum_s \text{tr}_{n-1}(\text{tr}_n(w' b_{n-1} t_{n-1}^{-s} t_n^m t_{n-1}^m)) = \frac{1}{d} \sum_s \text{r}_{m-s} w_{n+\beta} w'.
\]

Then

\[
B = \frac{1}{d} \sum_s \text{r}_{m-s} w_{n+\beta} w'.
\]

By expanding also \( e_{n-1} \) in \( C \), we have:

\[
C = \frac{1}{d} \sum_s \text{tr}_{n-1}(\text{tr}_n(w' t_n^s b_n t_{n-1}^m)) = \frac{1}{d} \sum_s y_{m-s} w_{n+\beta} w' = \frac{1}{d} \sum_r y_{n-r} w_{n-r} w'.
\]

\[\text{where} \quad m_n = b_n t_n^m \quad \quad \text{and} \quad w = w' t_{n-1}^m \quad \quad \text{with} \quad w' \in C_{n-2} \text{.}
\]
(notice that the last equality is obtain by making \( s = -r + m - \beta \)). Thus \( B = C \), this imply 
\[
\text{tr}_{n-1}(\text{tr}_n(Xg_{n-1})) = A = \text{tr}_{n-1}(\text{tr}_n(g_{n-1}X))
\]

* Suppose \( m_{n-1} = w'b_{n-1}t^{\beta}_{n-1} \), with \( w' \in C_{n-2} \). We have:

\[
Xg_{n-1} = w'b_{n-1}t^\beta_{n-1}b_{n}t^n g_{n-1} = w't^\beta_{n-1}b_{n-1}b_{n}g_{n-1}t^{m}_{n-1} = w't^\beta_{n-1}b_{n-1}g_{n-1}b_{n-1}t^{m}_{n-1}
\]

Then

\[ \text{tr}_{n-1}(\text{tr}_n(Xg_{n-1})) = z\text{tr}_{n-1}(w'b_{n-1}t^{m+\beta}_{n-1}) \]

On the other hand:

\[
g_{n-1}X = g_{n-1}w'b_{n-1}t^\beta_{n-1}b_{n}t^n = w'g_{n-1}b_{n-1}b_{n}t^\beta_{n-1}t^n
\]

\[ = w't^{m}_{n-1}g_{n-1}b_{n-1}b_{n}t^\beta_{n-1} = w't^{m}_{n-1}b_{n-1}g_{n-1}b_{n-1}t^\beta_{n-1} \]

(in the last equality we have used (iv) Proposition[2]. Hence

\[ \text{tr}_{n-1}(\text{tr}_n(g_{n-1}X)) = z\text{tr}_{n-1}(w'b_{n-1}t^{m+\beta}_{n-1}) = \text{tr}_{n-1}(\text{tr}_n(Xg_{n-1})). \]

* Suppose \( m_{n-1} = w'm^\pm_{n-1,j,\beta} \). We have:

\[
Xg_{n-1} = w'm^{-}_{n-1,j,\beta}b_{n}t^n g_{n-1} = w'm^{-}_{n-1,j,\beta}g_{n-1}b_{n-1}t^{m}_{n-1} = w'm^{-}_{n-1,j,\beta}g_{n-1}b_{n-1}t^{m}_{n-1}
\]

Then

\[ \text{tr}_{n-1}(\text{tr}_n(Xg_{n-1})) = z\text{tr}_{n-1}(w'm^{-}_{n-1,j,\beta}b_{n-1}t^{m}_{n-1}) \]

\[ = z\text{tr}_{n-1}(w'b_{n-2}t^{m}_{n-2}m^{-}_{n-1,j,\beta}) \]

\[ = z^2 w'b_{n-2}t^{m}_{n-2}m^{-}_{n-2,j,\beta} \]

On the other hand, we note that

\[
g_{n-1}X = g_{n-1}w'm^{-}_{n-1,j,\beta}b_{n}t^n = w'g_{n-1}m^{-}_{n-1,j,\beta}b_{n}t^n = w'm^{-}_{n-1,j,\beta}b_{n}t^n = w'b_{n-1}t^{m}_{n-1}m^{-}_{n-1,j,\beta}.
\]

Then

\[ \text{tr}_{n-1}(\text{tr}_n(g_{n-1}X)) = z\text{tr}_{n-1}(w'b_{n-1}t^{m}_{n-1}m^{-}_{n-1,j,\beta}) \]

\[ = z\text{tr}_{n-1}(w'b_{n-1}g_{n-2}t^{m}_{n-2}m^{-}_{n-2,j,\beta}) \]

\[ = z\text{tr}_{n-1}(w'g_{n-2}b_{n-2}t^{m}_{n-2}m^{-}_{n-2,j,\beta}) \]

\[ = z^2 w'b_{n-2}t^{m}_{n-2}m^{-}_{n-2,j,\beta}. \]

Hence, \( \text{tr}_{n-1}(\text{tr}_n(Xg_{n-1})) = \text{tr}_{n-1}(\text{tr}_n(g_{n-1}X)). \)

* Finally, let us suppose that \( m_{n-1} = m^+_{n-1,j,\beta} \). We have

\[
Xg_{n-1} = w'm^+_{n-1,j,\beta}b_{n}t^n g_{n-1} = w'm^+_{n-1,j,\beta}g_{n-1}b_{n-1}t^{m}_{n-1}
\]
Then
\[
\text{tr}_{n-1}(\text{tr}_n(Xg_{n-1})) = z\text{tr}_{n-1}(w'm^+_{n-1,j,\beta}b_{n-1}t^m_{n-1})
\]
\[
= z\text{tr}_{n-1}(w'g_{n-2}m^+_{n-2,j,\beta}b_{n-1}t^m_{n-1})
\]
\[
= zw\text{tr}_{n-1}(g_{n-2}b_{n-1}t^m_{n-1})m^+_{n-2,j,\beta}
\]
\[
= zw\text{tr}_{n-1}(g_{n-2}^2b_{n-2}g_{n-2}^{-1}t^m_{n-1})m^+_{n-2,j,\beta}
\]

We shall compute now \(\text{tr}_{n-1}(g_{n-2}^2b_{n-2}g_{n-2}^{-1}t^m_{n-1})\). To do that, we note that splitting the square and recalling the definition of \(b_{n-1}\), we can write
\[
\text{tr}_{n-1}(g_{n-2}^2b_{n-2}g_{n-2}^{-1}t^m_{n-1}) = A - (u - u^{-1})B + (u - u^{-1})C
\]
where
\[
A := \text{tr}_{n-1}(b_{n-2}g_{n-2}t^m_{n-1}) = zb_{n-2}t^m_{n-2}
\]
\[
B := \text{tr}_{n-1}(b_{n-2}e_{n-2}t^m_{n-1}) = \sum sx_{m-s}b_{n-2}t^s_{n-2}
\]
\[
C := \text{tr}_{n-1}(e_{n-2}b_{n-1}t^m_{n-1}) = \sum sy_{m-s}t^s_{n-2}.
\]

Hence
\[
\text{tr}_{n-1}(\text{tr}_n(g_{n-1}X)) = z^2w'g_{n-2}m^+_{n-2,j,\beta}t^m_{n} + z(u - u^{-1})w' (C - B) m^+_{n-2,j,\beta}
\] (27)

On the other side, we have
\[
g_{n-1}X = g_{n-1}w'm^+_{n-1,j,\beta}b_{n-1}t^m_{n} = w'g_{n-1}m^+_{n-1,j,\beta}b_{n-1}t^m_{n} = w'g_{n-1}b_{n}m^+_{n-1,j,\beta}t^m_{n}
\]
Then
\[
\text{tr}_{n-1}(\text{tr}_n(g_{n-1}X)) = w\text{tr}_{n-1}(\text{tr}_n(g_{n-1}b_{n}t^m_{n})m^+_{n-1,j,\beta})
\]
\[
= w\text{tr}_{n-1}(\text{tr}_n(g_{n-1}^2b_{n-1}g_{n-1}^{-1}t^m_{n})m^+_{n-1,j,\beta})
\]
As before, we split the square and then we deduce the following
\[
\text{tr}_{n-1}(\text{tr}_n(g_{n-1}X)) = A_1 + (u - u^{-1})(C_1 - B_1)
\] (28)
where
\[
A_1 := \text{tr}_{n-1}(\text{tr}_n(b_{n-1}g_{n-1}t^m_{n})m^+_{n-1,j,\beta}) = z^2w'b_{n-2}t^m_{n-2}m^+_{n-2,j,\beta}
\]
\[
B_1 := \text{tr}_{n-1}(\text{tr}_n(b_{n-1}e_{n-1}t^m_{n})m^+_{n-1,j,\beta}) = z[\sum sx_{m-s}b_{n-2}t^s_{n-2}] m^+_{n-2,j,\beta}
\]
\[
C_1 := \text{tr}_{n-1}(\text{tr}_n(e_{n-1}b_{n}t^m_{n})m^+_{n-1,j,\beta}) = z[\sum sy_{m-s}t^s_{n-2}] m^+_{n-2,j,\beta}.
\]

Hence, comparing (27) and (28), the claim follows. Therefore the lemma is proved. \(\square\)

5.2. In this subsection we prove that the family \(\{Y^B_{d,n}\}_{n \geq 1}\) supports a Markov trace. Let \(\text{Tr}_n\) be the linear map, from \(Y^B_{d,n}\) to \(L\), defined inductively by setting: \(\text{Tr}_1 = \text{tr}_1\) and
\[
\text{Tr}_n = \text{Tr}_{n-1} \circ \text{tr}_n \quad \text{for} \quad n \geq 2.
\]
The definition of \(\text{Tr}_n\) says that \(\text{Tr}_n(1) = 1\) and that
\[
\text{Tr}_n(x) = \text{Tr}_k(x) \quad \text{for} \quad x \in Y^B_{d,k} \quad \text{and} \quad n \geq k.
\] (29)

Let us denote \(\text{Tr}\) the family \(\{\text{Tr}_n\}_{n \geq 1}\). The following theorem is one of our main results.
Theorem 3. \( \text{Tr} \) is a Markov trace on \( \{Y^B_{d,n}\}_{n \geq 1} \). That is, for every \( n \geq 1 \) the linear map \( \text{Tr}_n : Y^B_{d,n} \to \mathbb{L} \) satisfies the following rules:

(i) \( \text{Tr}_n(1) = 1 \)

(ii) \( \text{Tr}_{n+1}(Xg_n) = z \text{Tr}_n(X) \)

(iii) \( \text{Tr}_{n+1}(Xb_{n+1}t_m) = y_m \text{Tr}_n(X) \)

(iv) \( \text{Tr}_{n+1}(Xt_m) = x_m \text{Tr}_n(X) \)

(v) \( \text{Tr}_n(XY) = \text{Tr}_n(YX) \)

where \( X, Y \in Y^B_{d,n} \).

Proof. Rules (ii)–(iv) are direct consequences of Lemma 9 (ii). Indeed, for example for (ii), we have:

\[
\text{Tr}_{n+1}(Xg_n) = \text{Tr}_n(\text{tr}_{n+1}(Xg_n)) = \text{Tr}_n(X \text{tr}_{n+1}(g_n)) = \text{Tr}_n(Xz) = z \text{Tr}_n(X).
\]

We prove rule (v) by induction on \( n \). For \( n = 1 \), the rule holds since \( Y^B_{d,1} \) is commutative. Suppose now that (v) is true for all \( k \) less than \( n \). We prove it first for \( Y \in Y^B_{d,n} \) and \( X \in Y^B_{d,n-1} \). We have

\[
\text{Tr}_n(XY) = \text{Tr}_{n-1}(\text{tr}_n(XY)) = \text{Tr}_{n-1}(\text{tr}_{n-1}(XY)) = \text{Tr}_{n-1}(\text{tr}_{n-1}(YX)) = \text{Tr}_{n-1}(\text{tr}_{n-1}(YX))
\]

Hence, \( \text{Tr}_n(XY) = \text{Tr}_n(YX) \) for all \( X \in Y^B_{d,n} \) and \( Y \in Y^B_{d,n-1} \). Now, we prove the rule for \( Y \in \{g_{n-1}, t_n\} \). By using Lemma 13, we get

\[
\text{Tr}_n(XY) = \text{Tr}_{n-2}(\text{tr}_{n-1}(\text{tr}_n(XY))) = \text{Tr}_{n-2}(\text{tr}_{n-1}(\text{tr}_n(YX)))
\]

Summarizing, we have

\[
\text{Tr}_n(XY) = \text{Tr}_n(YX)
\]

for all \( X \in Y^B_{d,n} \) and \( Y \in Y^B_{d,n-1} \cup \{g_{n-1}, t_n\} \). Clearly, having in mind the linearity of \( \text{Tr}_n \), this last equality implies that rule (v) holds.

The rules of the trace on the topological level are illustrated in the next figure.

\[
\text{Tr}_{n+1} \left( \begin{array}{ccc} a_1 & a_n & 0 \\ \vdots & & \vdots \\ \end{array} \right) = z \text{ Tr}_n \left( \begin{array}{cc} a_1 & a_n \\ \vdots & \vdots \\ \end{array} \right)
\]
6. THE E–CONDITION AND THE F–CONDITION

In this section we establish the necessary and sufficient conditions by which the parameters trace $x_1, \ldots, x_{d-1}, y_0, \ldots, y_{d-1} \in \mathbb{L}$ satisfies the following equation.

$$\text{Tr}_{n+1}(we_n) = \text{Tr}_n(w)\text{Tr}_{n+1}(e_n)$$

for all $w \in Y_{d,n}^B$.

This equation plays a key role for defining knot and link invariants in the next section. In this section we will prove that if the parameters satisfy the so–called E–conditon, and a set of new conditions, called F–condition, then the above equation holds; see Theorem 4.

Finally, we will compute such trace parameters, by using the method due to P. Gérardin to solve the so–called E–system, see [18, Appendix].

6.1. In [18] certain elements $E^{(k)}$ were introduced, associated to the trace parameters of the trace on the Yokonuma–Hecke algebra. With these $E^{(k)}$ the authors defined a non-linear system of equations called the E–system. We say that the solutions of this E–system satisfy the E–condition. Notably, whenever the trace parameters of the Markov trace on the Yokonuma–Hecke algebra satisfy the E–condition we have an invariant for framed and classical knots and links.
We consider here the same formal expressions of elements $E^{(k)}$ associated now to the trace parameters $x_1, \ldots, x_{d-1}$ of $\text{Tr}_n$. More precisely, we define

$$E^{(k)} := \frac{1}{d} \sum_m x_{k+m} x_{d-m} \quad \text{for} \quad 0 \leq k \leq d-1. \quad (30)$$

Note that $E^{(0)} = \text{Tr}_n(e_n)$. Also we need to introduce the following elements

$$F^{(k)} := \frac{1}{d} \sum_m x_{d-m} y_{k+m} \quad \text{for} \quad 0 \leq k \leq d-1. \quad (31)$$

In the summations above the m’s are regarded modulo $d$.

The $E$–system is defined as the non–linear system of equations in $x_1, \ldots, x_{d-1}$ formed by the following $d-1$ equations:

$$E^{(m)} = x_mE^{(0)}$$

where $1 \leq m \leq d-1$. Any solution $(x_1, \ldots, x_n)$ of the $E$–system is referred to by saying that it satisfies the $E$–condition.

Assume that $(x_1, \ldots, x_n)$ satisfies the $E$–condition. The $F$–system is the homogeneous linear system of equations in $y_0, \ldots, y_{d-1}$, formed by the following $d$ equations:

$$F^{(m)} = y_mE^{(0)}$$

where $0 \leq m \leq d-1$, and $E^{(0)}$ and $F^{(m)}$ are the elements that result from replacing $x_i$ by $x_i$ in (30) and (31) respectively, that is:

$$F^{(m)} := \frac{1}{d} \sum_m x_{d-m} y_{k+m} \quad \text{and} \quad E^{(0)} := \frac{1}{d} \sum_m x_m x_{d-m}.$$  

Also we have that $E^{(0)} = \frac{1}{|S|}$, see [21] Section 4.3]. Thus the $F$–system is formed by the following equations

$$\sum_m x_{d-m} y_{k+m} - \frac{d}{|S|} y_m = 0 \quad 0 \leq m \leq d-1. \quad (32)$$

Notice that the matrix associated to this linear system is given by:

$$\begin{pmatrix}
    x_0 - \frac{d}{|S|} & x_{d-1} & \ldots & x_1 \\
    x_1 & x_0 - \frac{d}{|S|} & \ddots & x_2 \\
    \vdots & \ddots & \ddots & x_{d-1} \\
    x_{d-1} & \ldots & x_1 & x_0 - \frac{d}{|S|}
\end{pmatrix}.$$  

Any solution $(y_0, \ldots, y_n)$ of the $F$–system is referred to saying that it satisfies the $F$–condition.

We have the following theorem.
Theorem 4. We assume that the trace parameters are specialize to complex numbers \((x_1, \ldots, x_n)\) and \((y_0, \ldots, y_n)\) that satisfy the \(E\)-condition and the \(F\)-condition respectively. Then

\[
    \text{Tr}_{n+1}(we_n) = \text{Tr}_n(w)\text{Tr}_{n+1}(e_n) \quad \text{for all } w \in \mathcal{Y}_{d,n}^B.
\]

We shall prove this theorem at the end of the subsection and using the Lemmas 14–16 below. We will introduce first the elements

\[
e_{(m)}_n := \frac{1}{d} \sum_{s=0}^{d-1} t_n^{m+s} t_{n+1}^{d-s}.
\]

Lemma 14. Let \(w = w't_n^k\), where \(w' \in \mathcal{Y}_{d,n-1}^B\). Then

\[
    \text{Tr}_{n+1}(we_{(m)}_n) = \frac{E^{(k+m)}}{x_k} \text{Tr}_n(w).
\]

Hence, \(\text{Tr}_{n+1}(we_{(m)}_n) = \frac{E^{(k)}}{x_k} \text{Tr}_n(w)\).

Proof. Splitting \(e_{(m)}_n\), we have:

\[
    \text{Tr}_{n+1}(we_{(m)}_n) = \frac{1}{d} \sum_s \text{Tr}_{n+1}(w' t_n^{m+s} t_{n+1}^{d-s}).
\]

Now, \(\text{Tr}_{n+1}(w' t_n^{m+s} t_{n+1}^{d-s}) = x_{d-s} \text{Tr}_n(w' t_n^{k+m+s}) = x_{d-s} x_{k+m+s} \text{Tr}_n(w')\). Then

\[
    \text{Tr}_{n+1}(we_{(m)}_n) = \frac{1}{d} \sum_s x_{d-s} x_{k+m+s} \text{Tr}_{n-1}(w') = E^{(k+m)} \text{Tr}_{n-1}(w') = \frac{E^{(k+m)}}{x_k} \text{Tr}_n(w).
\]

\[\square\]

Lemma 15. Let \(w = w'b_n t_n^k\), where \(w' \in \mathcal{Y}_{d,n-1}^B\). Then

\[
    \text{Tr}_{n+1}(we_{(m)}_n) = \frac{F^{(k+m)}}{y_k} \text{Tr}_n(w).
\]

In particular, we have \(\text{Tr}_{n+1}(we_{(m)}_n) = \frac{F^{(k)}}{y_k} \text{Tr}_n(w)\).
Proof. We have:

\[
\text{Tr}_{n+1}(w e_n^{(m)}) = \frac{1}{d} \sum_s \text{Tr}_{n+1}(w^' b_n t^{k+m+s} t_{n+1}^{d-s})
\]

\[
= \frac{1}{d} \sum_s x_{d-s} \text{Tr}_n(w^' b_n t^{k+m+s})
\]

\[
= \frac{1}{d} \sum_s x_{d-s} y_{k+m+s} \text{Tr}_{n-1}(w^')
\]

\[
= E^{(k+m)} \text{Tr}_{n-1}(w^')
\]

\[
= \frac{E^{(k+m)}}{y_k} \text{Tr}_n(w).
\]

Lemma 16. Let \( w = w^m_{n,k,\alpha} \), with \( w^' \in Y_{d,n-1}^B \). Then \( \text{Tr}_{n+1}(w e_n) = z \text{Tr}_n(x e_{n-1}) \), where \( x = m^\pm_{n-1,k,\alpha} w^' \).

Proof. We have:

\[
\text{Tr}_{n+1}(w e_n) = \frac{1}{d} \sum_s \text{Tr}_{n+1}(w^m_{n,k,\alpha} t^s t_{n+1}^{d-s})
\]

\[
= \frac{1}{d} \sum_s x_{d-s} \text{Tr}_n(w^' g_{n-1} m^\pm_{n-1,k,\alpha} t^s)
\]

\[
= \frac{1}{d} \sum_s x_{d-s} \text{Tr}_n(w^' t_{n-1} g_{n-1} m^\pm_{n-1,k,\alpha})
\]

\[
= \frac{z}{d} \sum_s x_{d-s} \text{Tr}_{n-1}(w^' t_{n-1} m^\pm_{n-1,k,\alpha})
\]

\[
= \frac{z}{d} \sum_s x_{d-s} \text{Tr}_{n-1}(m^\pm_{n-1,k,\alpha} w^t_{n-1})
\]

\[
= \frac{z}{d} \sum_s \text{Tr}_n(m^\pm_{n-1,k,\alpha} w^t_{n-1} t_{n}^{d-s}) = z \text{Tr}_n(x e_{n-1}).
\]

Proof of Theorem 4. By the linearity of \( \text{Tr}_{n+1} \) we can assume that \( w \) is an element in the inductive basis \( C_n \). We proceed by induction on \( n \). For \( n = 1 \) we have two possibilities: \( w = t_1^k \) or \( w = b_1 t_1^k \). For \( w = t_1^k \), we have:

\[
\text{Tr}_{n+1}(w e_1) = \frac{1}{d} \sum_s x_{d-s} x_{k+s} = \frac{E^{(k)}}{x_k} \text{Tr}_n(w) = E^{(0)} \text{Tr}_n(w) = \text{Tr}_{n+1}(e_1) \text{Tr}_n(w).
\]
For \( w = b_1 t_1^k \), we have:
\[
Tr_{n+1}(we_1) = \frac{1}{d} \sum_s x_{d-s} y_{k+s} = \frac{F^{(k)}}{y_k} Tr_n(w) = E^{(0)} Tr_n(w) = Tr_{n+1}(e_1) Tr_n(w).
\]

Thus, for \( n = 1 \) the theorem is proved. Suppose now that the theorem is true for every positive integer less than \( n + 1 \). Set \( w \) be an element in \( C_n \). We shall prove the theorem by distinguishing the three types of form for \( w \).

- Suppose \( w = w't_n^k \), where \( w' \in Y_{d,n-1}^B \). By using the Lemma 14 and the fact that \( x_k \)'s satisfies the \( E \)-condition, we have:
\[
Tr_{n+1}(we_n) = \frac{E^{(k)}}{x_k} Tr_n(w) = E^{(0)} Tr_n(w) = Tr_n(w) Tr_{n+1}(e_n).
\]

- Suppose \( w = w'b_n t_n^k \), where \( w' \in Y_{d,n-1}^B \). Then, by using now Lemma 15 and the fact that the \( y_k \)'s satisfied the \( F \)-condition, we have:
\[
Tr_{n+1}(we_n) = \frac{F^{(k)}}{y_k} Tr_n(w) = E^{(0)} Tr_n(w) = Tr_n(w) Tr_{n+1}(e_n).
\]

- Finally, suppose \( w = w'm_{n,k,\alpha}^\pm \), where \( w' \in Y_{d,n-1}^B \). From Lemma 16 we have
\[
Tr_{n+1}(we_n) = z Tr_n(x e_{n-1}) \quad \text{where } x = m_{n-1,k,\alpha}^\pm w' \in Y_{d,n-1}^B.
\]

Now, by using the induction hypothesis, we get \( Tr_n(x e_{n-1}) = Tr_{n-1}(x) Tr_n(e_{n-1}) \). But, now \( Tr_{n+1}(e_n) = Tr_n(e_{n-1}) \) and
\[
z Tr_{n-1}(x) = Tr_n(g_{n-1} m_{n-1,k,\alpha}^\pm w') = Tr_n(m_{n,k,\alpha}^\pm w').
\]

Therefore
\[
Tr_{n+1}(we_n) = Tr_n(w) Tr_{n+1}(e_n).
\]

\[\square\]

6.2. Solving the \( F \)-system. The \( E \)-system was solved by P. Gérardin, by using some tools from the complex harmonic analysis on finite groups, see [21, Appendix]. However, his method works on any field having characteristic 0. We shall introduce now some notations and definitions, necessary to explain the method used by Gérardin, which will be used to solve the \( F \)-system as well. For more details on the tools of harmonic analysis used here, see [18, 11].

We shall regard the group algebra \( \Lambda := L[\mathbb{Z}/d\mathbb{Z}] \), as the algebra formed by all complex functions on \( \mathbb{Z}/d\mathbb{Z} \), where the product is the convolution product, that is:
\[
(f * g)(x) = \sum_{y \in \mathbb{Z}/d\mathbb{Z}} f(y) g(x - y) \quad \text{where } f, g \in \Lambda.
\]

As usual, we denote by \( \delta_a \in \Lambda \) the function with support \( \{a\} \). Recall that \( \delta_0 \) is the unity with respect to the convolution product and that \( \{ \delta_a ; a \in \mathbb{Z}/d\mathbb{Z} \} \) is a linear basis for
Λ. The algebra Λ is commutative and is the direct sum of the simple ideals \( \mathbb{K} e_a \), where \( a \in \mathbb{Z}/d\mathbb{Z} \) and the \( e_a \)'s are the characters of \( \mathbb{Z}/d\mathbb{Z} \), that is:

\[
e_a : b \mapsto \cos \left( \frac{2\pi ab}{d} \right) + i \sin \left( \frac{2\pi ab}{d} \right).
\]

In Λ we have another product, the punctual product, that is:

\[
fg : x \mapsto f(x)g(x) \quad \text{where} \quad f, g \in \Lambda.
\]

The algebra Λ with the punctual product has unity \( e_0 \) and is the direct sum of its simple ideals \( \mathbb{K} \delta_a \), where \( a \in \mathbb{Z}/d\mathbb{Z} \).

The Fourier transform \( \mathcal{F} \) on Λ is the automorphism defined by \( f \mapsto \hat{f} \), where

\[
\hat{f}(x) := (f * e_x)(0) = \sum_{y \in \mathbb{Z}/d\mathbb{Z}} f(y)e_x(-y).
\]

Recall that \( (\mathcal{F}^{-1}f)(x) = d^{-1}f(-u) \), where \( \hat{f}(u) = \sum_{u \in G} f(u)e_u(-u) \).

The following proposition collects the properties of the Fourier transform used here. These properties are well-known and can be found, for example, in [28].

**Proposition 6.** For every \( a \in \mathbb{Z}/d\mathbb{Z} \) and \( f, g \in \Lambda \). We have:

(i) \( \hat{\delta}_a = e_{-a} \)

(ii) \( \hat{e}_a = d\delta_a \)

(iii) \( \hat{f}(u) = df(-u) \)

(iv) \( \hat{f} * g = \hat{f} \hat{g} \)

(v) \( \hat{fg} = d^{-1}\hat{f} * \hat{g} \).

To solve the E–system, Gérardin considered the elements \( x \in \Lambda \), defined by \( x(k) = x_k \). Then, he interpreted the E–system as the functional equation \( x * x = (x * x)(0)x \) with the initial condition \( x(0) = 1 \). Now, by applying the Fourier transform on this functional equation we obtain \( \hat{x}^2 = (x * x)(0)\hat{x} \). This last equation implies that \( \hat{x} \) is constant on its support \( S \), where it takes the values \( (x * x)(0) \). Thus, we have

\[
\hat{x} = (x * x)(0) \sum_{s \in S} \delta_s.
\]

By applying \( \mathcal{F}^{-1} \) and the properties listed in the proposition above, Gérardin showed that the solutions of the E–system are parameterized by the non–empty subsets of \( \mathbb{Z}/d\mathbb{Z} \). More precisely, for such a subset \( S \), the solution \( x_S \) is given as follows.

\[
x_S = \frac{1}{|S|} \sum_{s \in S} e_s.
\]

Now, in order to solve the F–system with respect to \( x_S \), we define \( y \in \Lambda \) by \( y(k) = y_k \). Then we have \( F^{(k)} = d^{-1}(x * y)(k) \). So, to solve the F–system is equivalent to solving the
following functional equation:
\[ x \ast y = (x \ast x)(0)y. \]
which, applying the Fourier transform and Proposition 6 (iv), is equivalent to:
\[ \hat{x}\hat{y} = (x \ast x)(0)\hat{y}. \]
This equation implies that the support of \( \hat{y} \) is contained in the support of \( \hat{x} \). Now, set \( S \) the support of \( \hat{x} \). Then we can write \( \hat{y} = \sum_{s \in S} \lambda_s \delta_s \). In this last equation, by applying \( \mathcal{F}^{-1} \) and Proposition 6 (i) and (iv), we get:
\[ y = \frac{1}{d} \sum_{s \in S} \lambda_s e_s. \]
Thus, we have proved the following proposition.

**Proposition 7.** The solution of the F-system with respect to the solution \( x_S \) of the E-system is in the form:
\[ y_S = \sum_{s \in S} \alpha_s e_s \]
where the \( \alpha_s \)'s are complex numbers.

## 7. Knot and link invariants from \( Y_{d,n}^B \)

In this section we define invariants for knots and links in the solid torus, by using the Jones recipe applied to the pairs \( (Y_{d,n}^B, \text{Tr}_n) \) where \( n \geq 1 \). To do that, we fix from now on that the trace parameters \( x_k \) satisfy the E-condition and the trace parameters \( y_k \) satisfy the F-system, with respect to the \( x_k \)'s. The invariants constructed here will take values in \( \mathbb{L} \).

More precisely, the closure of a framed braid \( \alpha \) of type \( B \) (recall Section 2) is defined by joining with simple (unknotted and unlinked) arcs its corresponding endpoints and is denoted by \( \hat{\alpha} \). The result of closure, \( \hat{\alpha} \), is a framed link in the solid torus, denoted \( ST \). This can be understood by viewing the closure of the fixed strand as the complementary solid torus. For an example of a framed link in the solid torus see Figure 6. By the analogue of the Markov theorem for \( ST \) (cf. for example [25, 26]), isotopy classes of oriented links in \( ST \) are in bijection with equivalence classes of braids of type \( B \) and this bijection carries through to the class of framed links of type \( B \).

![Figure 6](image-url). A framed link in the solid torus.
We set
\[ \lambda_S := \frac{z - (u - u^{-1}) E_S}{z} \quad \text{and} \quad \Lambda_S := \frac{1}{z \sqrt{\lambda_S}}, \]
where \( E_S = 1/|S| \). We are now in the position to define link invariants in the solid torus.

**Definition 7.** For \( \alpha \) in \( \mathcal{F}^B_n \), the Markov trace \( \text{Tr} \) with the trace parameters specialized to solutions of the E–system and the F–system, and \( \pi \) the natural epimorphism of \( \mathcal{F}^B_n \) onto \( Y^B_{d,n} \) we define
\[ \mathcal{X}^B_S(\tilde{\alpha}) := \Lambda_S^{n-1} \left( \sqrt{\lambda_S} \right) e \text{Tr}(\pi(\alpha)), \]
where \( e \) is the exponent sum of the \( \sigma_i \)'s that appear in \( \alpha \). Then \( \mathcal{X}^B_S \) is a Laurent polynomial in \( u, v \) and \( z \) and it depends only on the isotopy class of the framed link \( \tilde{\alpha} \), which represents an oriented framed link in \( ST \).

**Remark 6.** The invariants \( \mathcal{X}^B_S \), when restricted to framed links with all framings equal to 0, give rise to invariants of oriented classical links in \( ST \). By the results in [4] and since classical knot theory embeds in the knot theory of the solid torus, these invariants are distinguished from the Lambropoulou invariants [9, 25]. More precisely, they are not topologically equivalent to these invariants on links.

**Remark 7.** As we have said previously the cyclotomic Yokonuma–Hecke algebra \( Y(d, m, n) \) provides a framization of the Hecke algebra of type \( B \) when \( m = 2 \). In [2] where this algebra was introduced, it was also proved that \( Y(d, m, n) \) supports a Markov trace, which will be denoted here by \( \text{Tr} \), for details see [2, Section 5]. Then using Jones’s recipe a new invariant for framed links in the solid torus is constructed, which is given by
\[ \Gamma_m(\tilde{\alpha}) := \Lambda_S^{n-1} \left( \sqrt{\lambda_S} \right) e \text{Tr}(\pi(\alpha)), \]
where \( \pi : \mathcal{F}^B_n \rightarrow Y(d, 2, n) \) is the natural algebra epimorphism given by
\[ \rho_1 \mapsto b_1, \quad \sigma_i \mapsto g_i, \quad i = 1, \ldots, n-1, \quad \text{and} \quad t_j \mapsto t_j, \]
see [2, Section 6.3]. As we see in the previous section, in order that this polynomial becomes an invariant, the trace parameters (of \( \text{Tr} \)) have to satisfy a non-linear system of equations, which for \( m = 2 \) is equivalent to the systems given here (E– and F–system).

Now, we would like to make some comparison between \( \Gamma_2 \) and \( \mathcal{X}_S \). At first sight the invariants look similar, but the structural differences between the \( Y^B_{d,n} \) and \( Y(d, 2, n) \) make them differ (see Remark 3). For example, for the loop generator twice, we have the following
\[
\begin{align*}
\text{In } Y^B_{d,n} & \quad \text{Tr}(\pi(b_1^2)) = \text{Tr}(1 + (v-v^{-1}) b_1 f_1) = 1 + \frac{(v-v^{-1})}{d} \sum_s \text{Tr}(b_1 t_1^s) = 1 + \frac{(v-v^{-1})}{d} \sum_s y_s \\
\text{In } Y(d, 2, n) & \quad \text{Tr}(\pi(b_1^2)) = \text{Tr}(1 + (v-v^{-1}) b_1) = 1 + (v-v^{-1}) y_0
\end{align*}
\]
Therefore
\[ \chi_S^g(\hat{b}_1^2) = 1 + \frac{(v - v^{-1})}{d} \sum_s y_s \quad \text{and} \quad \Gamma_2(\hat{b}_1^2) = 1 + (v - v^{-1})y_0 \]

Then clearly for the framed link \( \hat{b}_1^2 \), the two invariants have different values, nevertheless to do a proper comparison of these invariants it is necessary a deeper study.

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