EMBEDDABILITY OF MULTIPLE CONES

D. REPOVŠ, W. ROSICKI, A. ZASTROW, AND M. ŽELJKO

ABSTRACT. The main result of this paper is that if $X$ is a Peano continuum such that its $n$-th cone $C^n(X)$ embeds into $\mathbb{R}^{n+2}$ then $X$ embeds into $S^2$. This solves a problem proposed by W. Rosicki.

1. INTRODUCTION

The classical Lefschetz-Nöbeling-Pontryagin Embedding Theorem [10] asserts that every compact metric space $X$ of dimension $n$ embeds into $\mathbb{R}^{2n+1}$. We are interested in the relationship between the embeddability of $X$ and embeddability of its Cartesian product $X \times I^n$ with a cube $I^n$ (resp. its cone $C(X)$, iterated cone $C^n(X) = C(\ldots (C(X))\ldots)$, suspension $\Sigma(X)$). Clearly, if $X$ embeds in $\mathbb{R}^m$, then $X \times I^n$ and $C^n(X)$ embed into $\mathbb{R}^{n+m}$. However, sometimes they embed into lower-dimensional Euclidean space. Such is the case for the spheres $S^n$, where $S^n$, $C(S^n) \cong B^{n+1}$ and $S^n \times I$ all embed into $\mathbb{R}^{n+1}$.

Let $X$ be a Peano continuum. It was proved in [14] that if the cone $C(X)$ of $X$ embeds into $\mathbb{R}^3$, then $X$ embeds into $S^2$. As a consequence, if the suspension $\Sigma(X)$ of $X$ embeds into $\mathbb{R}^3$, then $X$ is planar. Note that for each $n \geq 3$, there exists a Peano continuum $X_n$ such that $X_n$ is not embeddable in $S^n$, whereas the cone $C(X_n)$ of $X_n$ is embeddable in $\mathbb{R}^{n+1}$ (see [14]).

The main result of this paper is Theorem 1.1 which solves a problem from [14]. Our proof is based on the methods of [4] and [14].

Theorem 1.1. Let $X$ be a Peano continuum. Suppose that for some $n \in \mathbb{N}$, $C^n(X)$ is embeddable in $\mathbb{R}^{n+2}$. Then $X$ is embeddable in $S^2$.

Let $X$ be a Peano continuum. Claytor [7] proved that $X$ is embeddable in $S^2$ if and only if $X$ does not contain any of the Kuratowski curves $K_1$, $K_2$, $K_3$, $K_4$ (see Figure 1).

2. PRELIMINARIES

A space $X$ is said to be planar if $X$ is embeddable in $\mathbb{R}^2$. We say that $X$ is locally planar if for every point $x \in X$ there exists a neighbourhood $U_x$ of $x$ in $X$ such that $U_x$ is embeddable in $\mathbb{R}^2$. Rosicki [13, Theorem 1.1] proved that if a Peano continuum $X$ is embeddable in $\mathbb{R}^3$ and $X$ is a nontrivial Cartesian product $X = Y \times Z$ then one of the factors is either an arc or a simple closed curve.

Rosicki [13] also proved that if a Peano continuum $X$ is embeddable in $\mathbb{R}^3$ and is homeomorphic to the product $Y \times S^1$ then the factor $Y$ must be planar. Alternatively, if $X = Y \times [0, 1]$ is embeddable in $\mathbb{R}^3$ and $\hat{H}^1(X) = \hat{H}^2(X) = 0$ then $Y$ must be planar. Cauty [4], generalizing Rosicki [13], proved that for every $n > 3$ and every Peano continuum $X$ such that $X \times I^{n-2}$ is embeddable into an $n$-manifold, it follows that $X$ must be locally planar. This theorem was stated earlier by Stubblefield [15]. However, Burgess [2] found a mistake in his proof.

Borsuk [1] constructed an example of a locally connected, locally planar continuum $X$ which is not embeddable into any surface. This continuum contains a sequence $(X_n)$ of subsets

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homeomorphic to Kuratowski curve $K_1$ which converge to an arc. Cauty \cite{4} proved that $X \times I^{n-2}$ is not embeddable into any $n$-manifold so the converse to his theorem does not hold.

3. Local separation

We say that a subset $D \subset \mathbb{R}^n$ locally separates $\mathbb{R}^n$ at the point $x_0 \in D$ into $k \in \mathbb{N}$ components if there exists $\varepsilon > 0$ such that for all $0 < \delta < \varepsilon$, the set $B(x_0, \delta) \setminus D$ has exactly $k$ components $A_1, \ldots, A_k$ for which $x_0 \in \overline{A_i}$, for all $i \in \{1, \ldots, k\}$.

It is easy to prove the following lemma using similar methods as in the proof of Lemma 1 in \cite{14}.

Lemma 3.1. A homeomorphic image of any $n$-disk locally separates $\mathbb{R}^{n+1}$ at any point of its interior into two components.

Note that $C^n(X) = \sigma^{n-1} * X = \{xt + y(1-t); x \in \sigma^{n-1}, y \in X, t \in [0,1]\}$, where $\sigma^{n-1}$ is an $(n-1)$-simplex. Then $\sigma^{n-1} * \{x\}$ is an $n$-ball and $\sigma^{n-1} * I$ is an $(n+1)$-ball. We consider $\sigma^{n-1}$ as a subset of $\sigma^{n-1} * X$.

Lemma 3.2. Let $I_i, i \in \{1, \ldots, k\}$, $k > 1$ be arcs with common endpoints and pairwise disjoint interiors and $C_k = C^n(\bigcup_{i=1}^{k} I_i) = \sigma^{n-1} * (\bigcup_{i=1}^{k} I_i)$. Let $h: C_k \to \mathbb{R}^{n+2}$ be an embedding. Then $h(C_k)$ locally separates $\mathbb{R}^{n+2}$ at any point $h(x_0)$, where $x_0$ is an interior point of $\sigma^{n-1}$, into $k$ components (where $\sigma^{n-1}$ is considered as a subset of $C_k$).

Proof. The proof is by induction on $k$. If $k = 2$, then $C_2 = \sigma^{n-1} * S^0 * S^0$ hence $h(C_2)$ locally separates $\mathbb{R}^{n+2}$ at $h(x_0)$ into two components, by Lemma 3.1.

Assume that Lemma 3.2 holds for $k - 1$. Choose $\varepsilon > 0$ smaller than the distance between $h(x_0)$ and the image of $\partial \sigma^{n-1} * (\bigcup_{i=1}^{k} I_i)$. Let $\delta > 0$ be so small that

$$D_k = h(C_k \cap B(x_0, \delta)) \subset B(h(x_0), \varepsilon).$$

There exists an open connected set $U_k \subset \mathbb{R}^{n+2}$ such that $D_k = U_k \cap h(C_k)$. Consider the exact sequence of the pair $(U_k, U_k \setminus D_k)$:

$$\to H_1(U_k) \to H_1(U_k, U_k \setminus D_k) \to H_0(U_k \setminus D_k) \to H_0(U_k) \to H_0(U_k, U_k \setminus D_k) \to 0.$$  

Since $U_k$ is an open $(n+2)$-manifold, $H_1(U_k) \cong \check{H}_c^{n+1}(U_k)$ by the Poincaré duality, where $\check{H}_c$ denotes the Čech cohomology with compact supports. Also $H_1(U_k, U_k \setminus D_k) \cong \check{H}_c^{n+1}(D_k)$ (see \cite{9} VIII, 7.14], where $L = \emptyset$, $K = D_k$ and $X = U_k$.
We can consider the exact sequence
\[ \to \hat{H}_{c}^{n+1}(U_k) \to \hat{H}_{c}^{n+1}(D_k) \to H_{0}(U_k \setminus D_k) \to H_{0}(U_k) \to 0. \]
Next we show by induction that the map \( \hat{H}_{c}^{n+1}(U_k) \to \hat{H}_{c}^{n+1}(D_k) \) is trivial. If \( k = 2 \) then \( D_k \) is an open \((n + 1)\)-ball. Then \( H_{0}(U_k \setminus D_k) \cong \mathbb{Z}^2 \), by Lemma 3.1. Since \( \hat{H}_{c}^{n+1}(D_k) \cong \mathbb{Z} \) and \( H_{0}(U_k) \cong \mathbb{Z} \), we obtain the exact sequence
\[ \hat{H}_{c}^{n+1}(U_k) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0. \]
Hence the map \( \hat{H}_{c}^{n+1}(U_k) \to \hat{H}_{c}^{n+1}(D_k) \) is indeed trivial, as asserted.
Since \( \hat{H}_{c}^{n+1}(D_2) \cong \mathbb{Z} \), we obtain by induction that \( \hat{H}_{c}^{n+1}(D_k) \cong \hat{H}_{c}^{n+1}(D_{k-1}) \oplus \hat{H}_{c}^{n+1}(D'_2) \cong \mathbb{Z}^{k-2} \oplus \mathbb{Z} \), where \( D'_2 = h(C^n(I_1 \cup I_2) \cap B(x_0, \delta)) \).
The map \( \hat{H}_{c}^{n+1}(U_k) \to \hat{H}_{c}^{n+1}(D_k) \cong \hat{H}_{c}^{n+1}(h(D_{k-1})) \oplus \hat{H}_{c}^{n+1}(D'_2) \) is trivial because both of its coordinates are trivial, by inductive hypothesis.
Therefore the sequence
\[ 0 \to \hat{H}_{c}^{n+1}(D_k) \to H_{0}(U_k \setminus D_k) \to H_{0}(U_k) \to 0 \]
is exact. So the sequence
\[ 0 \to \mathbb{Z}^{k-1} \to H_{0}(U_k \setminus D_k) \to \mathbb{Z} \to 0 \]
is also exact. Hence \( H_{0}(U_k \setminus D_k) \cong \mathbb{Z}^k \) and \( U_k \setminus D_k \) has \( k \) components.
The point \( h(x_0) \) belongs to the closure of each of them. Indeed, if \( X_k \) is \( D_k \) with a small open neighbourhood of \( h(x_0) \) removed then \( \hat{H}_{c}^{n+1}(X_k) \cong 0 \) and the sequence
\[ 0 \to H_{0}(U_k \setminus X_k) \to H_{0}(U_k) \to 0 \]
is exact, therefore \( H_{0}(U_k \setminus X_k) \cong \mathbb{Z} \).
\[ \square \]

4. Proof of Theorem 1.1

We shall need two more lemmata:

**Lemma 4.1.** Consider the Kuratowski curve \( K_1 \) and let \( n \in \mathbb{N} \). Then \( C^n(K_1) \) is not embeddable in \( \mathbb{R}^{n+2} \).

**Proof.** Suppose to the contrary, that there exists an embedding \( h : C^n(K_1) \to \mathbb{R}^{n+2} \). Consider \( K_1 \subset \mathbb{R}^3 \) and denote (see Figure 2)
\[ I_1 = [c, a] \cup [a, b], \quad I_2 = [c, p] \cup [p, b], \quad \text{and} \quad I_3 = [c, d] \cup [d, b]. \]

![Figure 2. Kuratowski curve K1](image)

If \( X = \bigcup I_i \), then \( \sigma^{n-1} \ast X = \bigcup (\sigma^{n-1} \ast I_i) \) is a union of \((n + 1)\)-disks. Let \( x_0 \in \text{Int} \sigma^{n-1} \) and choose \( \varepsilon > 0 \) so that (see Figure 3)
\[ C_1 = h(\sigma^{n-1} \ast (I_1 \cup I_3)) \text{ locally separates } B(h(x_0), \varepsilon) \text{ into } B_1, A_1 \text{ at } h(x_0), \]
\[ C_2 = h(\sigma^{n-1} \ast (I_1 \cup I_2)) \text{ locally separates } B(h(x_0), \varepsilon) \text{ into } B_2, A_2 \text{ at } h(x_0), \]
\[ C_3 = h(\sigma^{n-1} \ast (I_2 \cup I_3)) \text{ locally separates } B(h(x_0), \varepsilon) \text{ into } B_3, A_3 \text{ at } h(x_0). \]
By Lemma \[3.2\] we have that \( C = h(C^n(I_1 \cup I_2 \cup I_3)) = h(\sigma^{n-1} \ast \bigcup_{i=1}^{3} I_i) = h(\bigcup_{i=1}^{3} \sigma^{n-1} \ast I_i) \) locally separates \( B(h(x_0), \varepsilon) \) into three components. We will show that we can adopt the notation for these three components to be \( B_1, A_2 \) and \( A_3 \).

We use abstract linear combinations for describing our joins, e.g.,

\[
\sigma^{n-1} \ast K_1 = \{xt + y(1-t); \ x \in \sigma^{n-1}, y \in K_1, t \in [0,1]\}.
\]

For \( \sigma^{n-1} \subset \sigma^{n-1} \ast K_1 \), we have that \( h(\sigma^{n-1}) \) is a subset of \( C_1 \), but that \( h|_{\sigma^{n-1} \ast I_2} \) maps all linear combinations with \( t \neq 1 \), but sufficiently close to 1, to a subset that is connected but disjoint from \( C_1 \). Hence this subset can only be contained either in \( A_1 \) or in \( B_1 \). We may assume that it is in \( A_1 \). Since the entire neighbourhood of \( \sigma^{n-1} \ast I_2 \) is mapped by \( h \) into \( A_1 \), we have \( h(\sigma^{n-1} \ast I_2) \cap B_1 = \emptyset \), provided \( \varepsilon > 0 \) is small enough. Then \( B_1 \) is not divided by \( C \), so it is one of the three components.

Analogously, by considering \( C_2 \) (resp. \( C_3 \)) we can make sure that \( A_2 \) and \( A_3 \) are the other two components and that \( h(\sigma^{n-1} \ast I_3) \cap A_2 = \emptyset \) and \( h(\sigma^{n-1} \ast I_3) \cap A_3 = \emptyset \). Since \( C \cup B_1 \cup A_2 \cup A_3 \) and \( C \cup A_1 \cup B_1 \) are both disjoint decompositions of a neighbourhood of \( h(x_0) \), the set \( h(\sigma^{n-1} \ast I_2) \cup C_1 \) separates the component \( A_1 \) into components \( A_2 \) and \( A_3 \).

Note that

\[
x_0 \ast K_1 = \{x_0t + x(1-t); \ x \in K_1, t \in [0,1]\} \subset C^n(K_1).
\]

Choose \( t_0 \) near 1 so that

\[
h(\{x_0t + x(1-t); \ x \in K_1, t \geq t_0\}) \subset B(h(x_0), \varepsilon).
\]

Let \( p' = h(x_0t_0 + p(1-t_0)) \in A_1 \). The arc \( H = h(\{x_0t_0 + x(1-t_0); \ x \in (p,q)\}) \) is contained in \( B(h(x_0), \varepsilon) \setminus h(C) \). Therefore points \( p' \) and \( q' = h(x_0t_0 + q(1-t_0)) \) are in the same component. Hence \( q' \in A_2 \) or \( q' \in A_3 \). So the arc \( I = h(\{x_0t_0 + x(1-t_0); \ x \in (a, q]\cup[a, d)\}) \) is contained either in \( A_2 \) or in \( A_3 \). But this yields a contradiction since \( a' = h(x_0t_0 + a(1-t_0)) \notin A_3 \) (so \( I \notin A_3 \)) and \( d' = h(x_0t_0 + a(1-t_0)) \notin A_2 \) (so \( I \notin A_2 \)).

The proof of the next lemma can be obtained by changing the proof of [14, Lemma 4] in the same way as we did it for the proof of Lemma 2.3 using the proof of [14, Lemma 3].

**Lemma 4.2.** Consider the Kuratowski curve \( K_2 \) and let \( n \in \mathbb{N} \). Then \( C^n(K_2) \) is not embeddable in \( \mathbb{R}^{n+2} \).

**Proof of Theorem 1.1** By Claytor’s theorem (see [6], [7]), it suffices to show that \( C^n(K_i) \) is not embeddable into \( \mathbb{R}^{n+2} \) for any \( i \in \{1,2,3,4\} \). Now, Cauty [4] proved that \( K_i \times I^n \) is not embeddable into \( \mathbb{R}^{n+2} \) for any \( i \in \{3,4\} \). Therefore also \( C^n(K_i) \) is not embeddable into \( \mathbb{R}^{n+2} \) for any \( i \in \{3,4\} \). Hence we only have to consider the cases \( i = 1 \) and \( i = 2 \). The proof is now completed by application of Lemmata 4.1 and 4.2. \( \square \)
5. Epilogue

Repovš, Skopenkov and Ščepin [12] proved that if \( X \times I \) PL embeds into \( \mathbb{R}^{n+1} \), where \( X \) is either an acyclic polyhedron and \( \dim X \leq \frac{2n}{3} - 1 \) or a homologically \((2 \dim X - n - 1)\)-connected manifold and \( \dim X \leq \frac{2n}{3} - 1 \) or a collapsible polyhedron, then \( X \) PL embeds into \( \mathbb{R}^n \).

**Question 5.1.** What can one say about embeddability of \( X \) into Euclidean spaces if one considers \( C(X) \) or \( C^n(X) \) or \( \Sigma(X) \) or \( \Sigma^n(X) \) instead of \( X \times I \) for \( X \) in [12]?

It follows by [12] that if \( X \) is a contractible polyhedron such that \( X \times I \) embeds into \( \mathbb{R}^{n+1} \) then \( X \) embeds into \( \mathbb{R}^n \). So if \( X \) is contractible and \( C(X) \subset \mathbb{R}^{n+1} \) then \( X \) embeds into \( \mathbb{R}^n \).

Note that there exists a polyhedron \( P_n \) such that \( P_n \) is not embeddable into \( \mathbb{R}^n \) but \( C^2(P_n) \) is embeddable in \( \mathbb{R}^{n+2} \). Namely, Cannon [3] proved that if \( H^n \) is a homology \( n \)-sphere then its double suspension \( \Sigma^2(H^n) \) is the \((n + 2)\)-sphere (see [8] [11] for a far reaching generalization of this result). So if \( P_n = H^n \setminus B^n \) where \( B^n \) is an \( n \)-ball then the double cone \( C^2(P_n) \) embeds in \( \mathbb{R}^{n+2} \). The polyhedron \( P_n \) is acyclic but not contractible.

**Question 5.2.** Does there exist a contractible \( n \)-dimensional polyhedron \( X^n \) such that \( C^k(X^n) \) embeds into \( \mathbb{R}^{n+k} \), but \( X^n \) does not embed into \( \mathbb{R}^n \)?

In [14] Theorem 2] contractible continua \( X_n \) were constructed, such that \( X_n \) is not embeddable in \( \mathbb{R}^n \), \( C(X_n) \) is embeddable in \( \mathbb{R}^{n+1} \), and \( X_n \) is not a polyhedron. By [12], if \( X \) is an \( n \)-polyhedron then \( X \times I \) embeds in \( \mathbb{R}^{2n+1} \). If \( X \) is an \( n \)-polyhedron then \( C(X) \) need not embed into \( \mathbb{R}^{2n+1} \). For example, the Kuratowski curves \( K_1 \) and \( K_2 \) are 1-polyhedra but the cones \( C(K_1) \) and \( C(K_2) \) do not embed into \( \mathbb{R}^3 \).

**Question 5.3.** Suppose that \( X \) is a compact contractible \( n \)-dimensional polyhedron. Does the cone \( C(X) \) embed into \( \mathbb{R}^{2n+1} \)? Does the same hold if \( X \) is only acyclic?

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