Exclusion statistics and lattice random walks

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Abstract

We establish a connection between exclusion statistics with arbitrary integer exclusion parameter $g$ and a class of random walks on planar lattices. Square lattice random walks, described in terms of the Hofstadter Hamiltonian, correspond to $g = 2$. In the $g = 3$ case we construct a corresponding chiral random walk on a triangular lattice, and we point to potential random walk models for higher $g$. In this context, we also derive the form of the microscopic cluster coefficients for arbitrary exclusion statistics.

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1 Introduction

The enumeration of closed random walks of a given length with a given algebraic area on a planar lattice is a challenging subject. Algebraic area is defined as the oriented area spanned by the walk as it traces the lattice. A unit lattice cell enclosed in the counterclockwise (positive) way has an area +1, whereas when enclosed in the clockwise (negative) way it has an area −1. The total algebraic area is the area enclosed by the walk weighted by the winding number: if the walk winds around more than once, the area is counted with multiplicity.

It is well known that the algebraic area enumeration can be mapped to the quantum problem of a particle hopping on the lattice pierced by a perpendicular homogeneous magnetic field. Indeed, in quantum mechanics, the magnetic field is coupled to the area spanned by the particle. In [1] a closed formula for the enumeration on a square lattice...
was proposed. The relevant quantum model in that case is the celebrated Hofstadter model \([2]\) of a particle hopping on a square lattice in a perpendicular magnetic field.

The enumeration of closed walks of (necessarily even) length \(n\) for the square lattice was achieved by studying the secular determinant of the corresponding Hofstadter Hamiltonian. This determinant was calculated in \([3]\) for a rational flux per plaquette in units of the elementary flux quantum, in which case the determinant becomes finite dimensional. The coefficients of the expansion of this determinant in powers of the energy, called Kreft coefficients, are given by certain multiple nested trigonometric sums that are reminiscent of partition functions. The expression of the area enumeration generating function for the lattice walks of a given length was derived in \([1]\) in terms of a different set of coefficients, extracted from the Kreft coefficients, themselves reminiscent of cluster coefficients. These facts hinted to an interpretation in terms of statistical mechanics of many-body systems.

Motivated by these observations, we revisit the enumeration of closed lattice random walks enclosing a given algebraic area and demonstrate that it does admit a statistical mechanical interpretation, but in terms of particles obeying generalized exclusion statistics with exclusion parameter \(g\) (\(g = 0\) for bosons, \(g = 1\) for fermions and higher \(g\) means a stronger exclusion beyond Fermi).

Exclusion statistics was proposed by Haldane \([4]\) as a distillation of the statistical mechanics properties of Calogero-like spin systems. The relation of Calogero particles and fractional statistics was first pointed out in \([5]\). Exclusion statistics also emerges in the context of anyons projected on the lowest Landau level of a strong magnetic field \([6]\), and has further been extended to other situations \([7]\). It is a remarkable fact that algebraic area considerations in lattice walks also end up amounting to particular many body systems with exclusion statistics.

One payoff of the uncovered connection is that, once it is established from its defining properties that a random walk corresponds to specific exclusion statistics, the expression for the number of walks with given algebraic area can be straightforwardly extracted from statistical mechanical expressions. In this context, square lattice walks correspond to \(g = 2\), where the Kreft coefficients appear as many-body partition functions of exclusion-2 particles and the algebraic area generating functions as the corresponding cluster coefficients. We will also give an explicit construction of a triangular lattice walk realizing \(g = 3\) statistics and hint at other generalized walks corresponding to statistics with higher values of the exclusion parameter.

The algebraic area generating function of a lattice walk is determined by the exclusion statistics parameter \(g\) as well as a quantity (which we call the spectral function) derived from the properties of the walk. We will demonstrate that walks of different statistics and spectral functions can nevertheless be equivalent, a “fermionization” result mapping systems with different spectra and statistics.

Finally, an additional bonus of our analysis is an explicit expression for the cluster coefficients of particles with exclusion statistics \(g\) in a collection of single-particle quantum...
states with arbitrary energies. These coefficients were derived before in the thermodynamic limit, but their exact expressions for a set of microscopic states (i.e., not in the thermodynamic limit) were not known [8].

2 Lattice algebraic area enumeration and the Hofstadter model: a review

We will start by presenting a review of the Hofstadter model and the results in [1] in order to fix ideas and establish notations.

Consider closed random walks of length \( n \) on a square lattice (\( n = 2, 4, 6, \ldots \) is necessarily even) and denote by \( C_n(A) \) the number of such walks enclosing an algebraic area \( A \) (\( A \) is between \( -\lfloor (n/4)^2 \rfloor \) and \( \lfloor (n/4)^2 \rfloor \) where \( \lfloor \rfloor \) denotes the integer part). Obviously \( C_n(A) = C_n(-A) \).

The enumeration \( C_n(A) \) is achieved by establishing a relation between random lattice walks and the Hofstadter system of a charged particle hopping on a square lattice with a magnetic flux per unit cell \( \phi \). The Hofstadter Hamiltonian is

\[
H = u + u^{-1} + v + v^{-1}
\]

where \( u \) and \( v \) are respectively the hopping operators on the horizontal and vertical axis. Denoting \( Q = \exp(2i\pi\phi/\phi_0) \), with \( \phi_0 \) the unit of flux quantum, the hopping operators obey the “noncommutative torus” relation

\[
v u = Q u v
\]

due to the noncommutativity of magnetic translations when a flux is piercing the lattice. The connection to random walks and their algebraic area \( A \) is established through

\[v^{-1} u^{-1} v u = Q\]

The operators (in the sequence they act on the right) represent a walk enclosing one elementary square cell in the counterclockwise (positive) sense. Clearly the power of \( Q \) represents the enclosed area, and the above relation generalizes to products of \( u, v, u^{-1}, v^{-1} \) representing arbitrary (closed) walks on the lattice, their algebraic area appearing as \( Q^A \).

The generating function for closed walks of length \( n \) is then given by \( \text{Tr} \ H^n \) under an appropriate normalization of the trace such that \( \text{Tr} \ 1 = 1 \).

When the flux per square \( \phi \) is a rational multiple of the flux quantum \( \phi_0 \), that is, \( \phi/\phi_0 = p/q \) with \( p \) and \( q \) co-prime, the representation of \( u \) and \( v \) becomes finite dimensional

\[
u = e^{ik\phi}
\begin{pmatrix}
Q & 0 & 0 & \cdots & 0 & 0 \\
0 & Q^2 & 0 & \cdots & 0 & 0 \\
0 & 0 & Q^3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & Q^{q-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & Q^q
\end{pmatrix}
\]
where \( k_x \) and \( k_y \) are the quasimomenta in the \( x \) and \( y \) directions. \( u \) and \( v \) satisfy the commutation relation (2). The operators
\[
u^q = e^{i q k_y} , \quad v^q = e^{i q k_x}
\]
commute with \( u \) and \( v \) and constitute Casimirs of the algebra, their values in the above representation determined by the quasimomenta.

In the quantum problem, \( k_x \) and \( k_y \) span the range \((-\pi/q, \pi/q)\) and produce the Hofstadter energy bands. In the mapping to lattice walks, the full trace involves also integrating over \( k_x, k_y \), and this results in the elimination of terms containing \( u^q \) and \( v^q \) which do not correspond to closed walks but contribute to \( \text{tr}H^n \). We will call such unwanted terms “umklapp” effects. A simpler approach is to simply set \( k_x = k_y = 0 \) and consider walks of length less than \( q \), or otherwise remove the above umklapp terms. In the following we will use (1) and \( u \) and \( v \) in (3,4) with \( k_x = k_y = 0 \) and still call this simplified Hamiltonian the “Hofstadter Hamiltonian”.

The Hofstadter spectrum stems from the secular determinant of the \( q \times q \) matrix
\[
\det(1-zH) = \begin{vmatrix}
1 - z(Q + \frac{1}{Q}) & -z & 0 & \cdots & 0 \\
-z & 1 - z(Q^2 + \frac{1}{Q^2}) & -z & \cdots & 0 \\
0 & -z & (\pmb{1}) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (\pmb{1}) \\
-z & 0 & 0 & \cdots & -z 1 - z(Q^q + \frac{1}{Q^q})
\end{vmatrix} = -\sum_{j=0}^{[q/2]} a_{p,q}(2j) z^{2j} - 4z^q
\]

The \( a_{p,q}(2j) \) are the so-called Kreft coefficients [3],
\[
a_{p,q}(2j) = (-1)^{j+1} \sum_{k_1=0}^{q-2j} \sum_{k_2=0}^{q-2j} \cdots \sum_{k_{j-1}=0}^{q-2j} 4 \sin^2 \left( \frac{\pi(k_1+2j-1)p}{q} \right) \cdots 4 \sin^2 \left( \frac{\pi(k_{j-1}+3)p}{q} \right) \cdots \\
4 \sin^2 \left( \frac{\pi(k_{j-1}+3)p}{q} \right) \cdots 4 \sin^2 \left( \frac{\pi(k_{j-1}+3)p}{q} \right)
\]

with \( a_{p,q}(0) = -1 \), while \(-4z^q\) is an umklapp term.

By scaling, the Kreft coefficient \( a_{p,q}(2j) = q[q]a_{p,q}(2j) + \ldots + q^j[q^j]a_{p,q}(2j) \) turns out to be a polynomial in \( q \) of order \( j \) with the \( k^{th} \) order coefficients \( [q^k]a_{p,q}(2j) \), \( 1 \leq k \leq j \) being linear combinations of the \( \cos(2A\pi p/q) \)'s.
The enumeration of closed lattice walks of length \( n \) with algebraic area \( A \) is possible \([1]\) since the generating function for the \( C_n(A) \)'s coincides with the \( 1^{\text{st}} \) order (i.e. proportional to \( q \)) term in the above polynomial, which represents \( \text{Tr} \ H^n \)

\[
\frac{1}{n} \sum_A C_n(A) Q^A = [q]a_{p,q}(n)
\]

The Kreft coefficients appear in the secular determinant of the Hofstadter Hamiltonian and therefore can be expressed in terms of traces of powers of this Hamiltonian. As a consequence, higher order terms in the \( q \) expansion of \( a_{p,q}(n) \) are also given in terms of the linear terms of lower-order coefficients \([q]a_{p,q}(2n), n \leq n/2\).

E.g.,

\[
a_{p,q}(2) = q[q]a_{p,q}(2)
\]

\[
a_{p,q}(4) = q[q]a_{p,q}(4) - \frac{q^2}{2!} ([q]a_{p,q}(2))^2
\]

\[
a_{p,q}(6) = q[q]a_{p,q}(6) - q^2[q]a_{p,q}(2)[q]a_{p,q}(4) + \frac{q^3}{3!} ([q]a_{p,q}(2))^3
\]

\[
a_{p,q}(8) = q[q]a_{p,q}(8) - q^2[q]a_{p,q}(2)[q]a_{p,q}(6) - \frac{q^2}{2!} ([q]a_{p,q}(4))^2
\]

\[
+ \frac{q^3}{2!} ([q]a_{p,q}(2))^2 [q]a_{p,q}(4) - \frac{q^4}{4!} ([q]a_{p,q}(2))^4
\]

etc.,

i.e.,

\[
a_{p,q}(n) = - \sum_{\sum_n n k_n = n/2} \frac{1}{\prod_n (-1)^n n!} (q[q]a_{p,q}(2n))^{k_n}
\]

Denoting

\[
4 \sin^2(\pi kp/q) = \tilde{b}_{p/q}(k)
\]

\([q]a_{p,q}(n)\) can be expressed \([1]\) as a linear combination of the building blocks

\[
\sum_{k=1}^q \tilde{b}_{p/q}^1(k)\tilde{b}_{p/q}^2(k-1)\ldots\tilde{b}_{p/q}^j(k-j+1)
\]

labeled by the composition\(^{1}\) of \( n/2 \), that is, the ordered set of any number of positive integers \( (l_1, \ldots, l_j) \) satisfying \( n/2 = l_1 + l_2 + \ldots + l_j, 1 \leq j \leq n/2 \), namely

\[
[q]a_{p,q}(n) = \sum_{l_1, l_2, \ldots, l_j, \text{composition of } n/2} c(l_1, l_2, \ldots, l_j) \frac{1}{q} \sum_{k=1}^q \tilde{b}_{p/q}^1(k)\tilde{b}_{p/q}^2(k-1)\ldots\tilde{b}_{p/q}^j(k-j+1)
\]

\(^{1}\)There are \( 2^{n/2-1} \) such compositions; for example, when \( n = 8 \), one has the 8 compositions \( 8/2 = 4 = 3 + 1 = 1 + 3 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 1 + 1 + 1 \).
where
\[ c(l_1, l_2, \ldots, l_j) = \frac{(l_1+l_2)}{l_1 + l_2} \frac{(l_2+l_3)}{l_2 + l_3} \ldots \frac{(l_{j-1}+l_j)}{l_{j-1} + l_j} \]
(13)
and
\[ \sum_{l_1, l_2, \ldots, l_j} c(l_1, l_2, \ldots, l_j) = \left(\frac{n}{n/2}\right) \cdot \left(\frac{n}{n/2}\right) \]
(14)
On the other hand \[ [1] \], \[ \tilde{b}_{p/q}(k) \] in (11) is such that the coefficients in the \( \cos(2\pi p/q) \) expansion of \( \sum_{k=1}^{q} \tilde{b}_{p/q}^{l_1}(k)\tilde{b}_{p/q}^{l_2}(k-1)\ldots\tilde{b}_{p/q}^{l_j}(k-j+1) \)'s also add up to \( \left(\frac{n}{n/2}\right) \) for any composition \( l_1, l_2, \ldots, l_j \) of \( n/2 \). These two factors concur to give the number of closed random walks of (even) length \( n \) to be
\[ \left(\frac{n}{n/2}\right)^2 \]
as it should. Note that this overall counting can directly be retrieved by taking the limit \( q \to \infty \) so that \( Q \to 1 \). In that limit all the \( \tilde{b}_{p/q}(k) \) factors in each building block become equal, so the sum in (12) factorizes into two sums
\[ [q]a_{p,q}(n) = \sum_{l_1, l_2, \ldots, l_j} c(l_1, l_2, \ldots, l_j) \cdot \frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^{n/2}(k) \]
and the second sum goes over to the integral
\[ \frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^{n/2}(k) \to \int_{0}^{1} \tilde{b}_{p/q}^{n/2}(qs) \, ds = \int_{0}^{1} (2\sin \pi ps)^n \, ds = \left(\frac{n}{n/2}\right) \]
reproducing the second factor \( \left(\frac{n}{n/2}\right) \).

We conclude by noting that the expressions in (9) and (10) are essentially cluster expansions of \( a_{p,q}(n) \) viewed as a partition function in terms of the \( [q]a_{p,q}(2n) \)'s, \( n \leq n/2 \), which then play the role of cluster coefficients. This is the first hint that there is a statistical mechanical interpretation at hand. In the next section we will make this correspondence explicit.

3 Lattice random walks and \( g = 2 \) exclusion statistics

Our key observation is that, looking at (9) and (10), the area generating function for random walks \( [q]a_{p,q}(n) \) in (8) can be viewed as the cluster coefficient for the partition function \( a_{p,q}(n) \) of a system of particles with exclusion statistics of order \( g = 2 \). To this end, let us interpret \( \tilde{b}_{p/q}(k) \) as a spectral function
\[ \tilde{b}_{p/q}(k) := \exp(-\beta \epsilon_k) \]
then \( a_{p/q}(2j) \) in (7) becomes (up to an alternating sign) the \( j \)-body partition function for quantum particles with spectrum \( \epsilon_k \) and exclusion statistics parameter \( g = 2 \), because of the +2 shifts in the nested multiple sum (7).

Indeed, consider now in place of \( \hat{b}_{p/q}(k) \) in (11) a general spectral function \( s(k) \) and in place of \( a_{p,q}(2j) \) in (7) a general \( n \)-body partition function \( Z(n) \) that rewrites as

\[
Z(n) = \sum_{k_1=1}^{q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s(k_1 + 2n - 2)s(k_2 + 2n - 4) \cdots s(k_{n-1} + 2)s(k_n) \quad (15)
\]

Note that its relation to the Kreft coefficient in (7) is \( Z(j) = (-1)^{j-1}a_{p,q}(2j) \).

Again, due to the +2 shifts in the nested multiple sum (15), the arguments of the Boltzman factors \( s(k) \) in the above expression differ by at least 2; that is, no terms with particles in adjacent energy levels \( \epsilon_k \) and \( \epsilon_{k+1} \) are admitted. This is precisely the definition of the partition function for \( n \) particles on the line with energies \( \epsilon_k \) obeying exclusion statistics of order \( g = 2 \). Explicitly, for the first few \( Z(n) \) we have

\[
Z(1) = \sum_{k_1=1}^{q} s(k_1) ,
\]
\[
Z(2) = \sum_{k_1=1}^{q-2} \sum_{k_2=1}^{k_1} s(k_1 + 2)s(k_2) ,
\]
\[
Z(3) = \sum_{k_1=1}^{q-4} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} s(k_1 + 4)s(k_2 + 2)s(k_3) ,
\]
\[
Z(4) = \sum_{k_1=1}^{q-6} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \sum_{k_4=1}^{k_3} s(k_1 + 6)s(k_2 + 4)s(k_3 + 2)s(k_4) ,
\]

etc.

If we set \( s(k) = 1 \) – i.e., we consider a degenerate quantum gas at energy 0 with \( q \) single-particle quantum states – we get

\[
Z(n) = \frac{(q - n + 1)!}{n!(q - 2n + 1)!} = \left( \frac{q - (g - 1)(n - 1)}{n} \right)_{g=2}
\]

which is the number of ways to arrange \( n \) particles in \( q \) single-particle states on a line with exclusion statistics \( g = 2 \). This is the “linear counting” as originally proposed by Haldane [4] where single-particle states are arranged on an open line segment.

\[\text{2The notations \( n \) for the length –the number of steps– of closed lattice walks on the one hand, and \( n \) for the number of particles in statistical mechanics on the other hand, should not be confused.}\]
In an alternative “periodic counting” one starts instead with single-particle states on a circle. \( g = 2 \) exclusion statistics in this case implies

\[
Z(n) = \sum_{k_1=1}^{q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s(k_1 + 2n - 2)s(k_2 + 2n - 4) \cdots s(k_n) \cdot 
\left( 1 - \delta(k_1 - (q - 2n + 2)) \delta(k_n - 1) \right) \tag{16}
\]

Again if we set \( s(k) = 1 \) we obtain

\[
Z(n) = \frac{q(q - n - 1)!}{n!(q - 2n)!} = \frac{q}{n} \binom{q - (g - 1)n - 1}{n - 1} \bigg|_{g=2}
\]

which is the number of ways to arrange \( n \) particles in \( q \) quantum states on a circle with exclusion statistics \( g = 2 \). This is the “cyclic counting” that appeared in [7] and is at work in the thermodynamic limit of the LLL-anyon model [6]. Note that for the specific spectral function (11) of the square lattice walk problem the two partition functions (open and periodic) are identical, since \( s(q) = b_{p/q}(q) = 0 \) and the extra terms in (15) relative to (16) vanish.

As is standard in many-body statistical mechanics, we introduce the cluster coefficients \( b(n) \) through the grand partition function

\[
\log \left( \sum_{n=0}^{\infty} Z(n)z^n \right) = \sum_{n=1}^{\infty} b(n)z^n \tag{17}
\]

with \( z \) playing the role of fugacity; i.e.,

\[
Z(n) = \sum_{k_j \geq 0, \sum_j jk_j = n} \prod_{j=1}^{n} \frac{1}{k_j!} b^{k_j}(j) \tag{18}
\]

Explicitly, for the first few \( n \),

\[
\begin{align*}
Z(1) &= b(1) \\
Z(2) &= b(2) + \frac{1}{2!} b^2(1) \\
Z(3) &= b(3) + b(1)b(2) + \frac{1}{3!} b^3(1) \\
Z(4) &= b(4) + b(1)b(3) + \frac{1}{2!} b^2(2) + \frac{1}{2!} b^2(1)b(2) + \frac{1}{4!} b^4(1)
\end{align*}
\]

The similarity to (9) and (10) is obvious – as already noticed above the cluster coefficients in the Hofstadter case are (up to alternating signs) the \( q[q]a_{p,q}(2n) \)'s, \( n \leq n/2 \).

In the fermionic \( (g = 1) \) and bosonic \( (g = 0) \) cases the cluster coefficients \( b(n) \) are (up to sign) single-particle partition functions with temperature parameter \( n\beta \), that is,
\[ \sum_{k=1}^{q} s^a(k). \] In the \( g = 2 \) exclusion case the corresponding expressions for the cluster coefficients become

\[ b(1) = \sum_{k=1}^{q} s(k), \]
\[ -b(2) = \frac{1}{2} \sum_{k=1}^{q} s^2(k) + \sum_{k=1}^{q} s(k + 1)s(k), \]
\[ b(3) = \frac{1}{3} \sum_{k=1}^{q} s^3(k) + \sum_{k=1}^{q} s^2(k + 1)s(k) + \sum_{k=1}^{q} s(k + 1)s^2(k) + \sum_{k=1}^{q} s(k + 2)s(k + 1)s(k), \]
\[ -b(4) = \frac{1}{4} \sum_{k=1}^{q} s^4(k) + \sum_{k=1}^{q} s^3(k + 1)s(k) + \sum_{k=1}^{q} s(k + 1)s^3(k) + \frac{3}{2} \sum_{k=1}^{q} s^2(k + 1)s^2(k) + 2 \sum_{k=1}^{q} b(k + 2)s^2(k + 1)s(k) + \sum_{k=1}^{q} s^2(k + 2)s(k + 1)b(k_1) + \sum_{k=1}^{q} s(k + 2)s(k + 1)s^2(k) + \sum_{k=1}^{q} s(k + 3)s(k + 2)s(k + 1)s(k) \]

\[ (19) \]

etc. The above expressions generalize the formula \[ (12) \]

\[ b(n) = (-1)^{n-1} \sum_{l_1, l_2, \ldots, l_j} c(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q} s^{l_1}(k + j - 1) \ldots s^{l_j}(k + 1)s^{l_1}(k) \]

\[ (20) \]

where the coefficients \( c(l_1, l_2, \ldots, l_j) = c(l_j, l_{j-1}, \ldots, l_1) \) are given in \[ (13) \]. All the above formulae hold both in the non-periodic (linear) counting and the periodic counting (cyclic), provided that we put \( s(k) = 0 \) for \( k > q \) in the non-periodic case and \( s(k + q) = s(k) \) in the periodic case.

Formulae \[ (13) \] and \[ (20) \] were derived in [1] for the "building blocks" of the algebraic area generating function \( q|a_{p,q}(n) \) with the spectral function \( s(k) \) being the Hofstadter \( b_{p/q}(k) \) in \[ (11) \]. Here we demonstrated that they admit an interpretation in terms of \( g = 2 \) exclusion statistics. In fact, \[ (13) \] and \[ (20) \] provide the microscopic (that is, not the thermodynamic limit) \( g = 2 \) cluster coefficients for an arbitrary number \( q \) of energy levels \( \epsilon_k \). Their form is intuitively clear: they "correct" the \( n \)-body Boltzmann partition function \( (\sum_{k=1}^{q} s(k))^{n}/n! \) by subtracting terms forbidden by statistics. In the fermionic case these are terms where particles occupy the same level \( k \), and they come with coefficient \( \pm 1/n \). In the \( g = 2 \) case these are terms where the particles occupy the same or neighboring levels, as is clear by the terms in \( b(n) \) which are all in clusters of neighboring levels, and they come with nontrivial combinatorial coefficients \( \pm c(l_1, \ldots, l_j) \).

Setting as above \( s(k) = 1 \) we obtain in the non-periodic case

\[ b(n) = (-1)^{n-1} \frac{1}{2n} \left[ (q + 2) \binom{2n}{n} - 2^{2n} \right] \]
and in the periodic case
\[ b(n) = q(-1)^{n-1} \frac{1}{2n} \binom{2n}{n} \]

The large-\(q\) limit of \(b(n)\) gives the corresponding cluster coefficients in the thermodynamic limit, \(q\) playing the role of volume. (Note that for the periodic case no limit is necessary.) The result agrees with the known expressions obtained in [6, 7]

\[ b(n)_{\text{therm}} = q \frac{1}{n} \prod_{k=1}^{n-1} (1 - g n/k) \] (21)

which, for \(g = 2\), reduces to \(q(-1)^{n-1} \binom{2n}{n} / 2n\).

4 General \(g\) microscopic cluster coefficients

We aim to generalize the statistics-lattice walks connection to a general integer \(g\). This goal involves two aspects: deriving the expression for the microscopic cluster coefficients for exclusion \(g\) statistics, which were crucial in the statistics–Hofstadter model connection, and then devising related random walk models on general lattices (not necessarily square) that realize these statistics in terms of algebraic area enumerations. In this section we address the first component.

4.1 \(g = 3\)

As a first step let us consider the case \(g = 3\) and its corresponding cluster coefficients. Focusing on the periodic case (the non periodic case can be treated along the same lines) the \(n\)-body partition function becomes

\[ Z(n) = \sum_{k_1=1}^{q-3n+3} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} s(k_1 + 3n - 3)s(k_2 + 3n - 6) \cdots s(k_{n-1} + 3)s(k_n) \]
\[ \left( 1 - \delta(k_1 - (q - 3n + 3)) \delta(k_{n-1}) \right) \left( 1 - \delta(k_1 - (q - 3n + 2)) \delta(k_{n-1}) \right) \]
\[ \left( 1 - \delta(k_1 - (q - 3n + 3)) \delta(k_{n-2}) \right) \] (22)

(the awkward \(\delta\)-factors are there to eliminate terms that by "umklapp" would become nearest or next-to-nearest neighbors). The cluster coefficients are defined again through
the grand partition function \([17]\), the first few being

\[
b(1) = \sum_{k=1}^{q} s(k) ,
\]

\[-b(2) = \frac{1}{2} \sum_{k=1}^{q} s^2(k) + \sum_{k=1}^{q} s(k + 1) s(k) + \sum_{k=1}^{q} s(k + 2) s(k) ,
\]

\[b(3) = \frac{1}{3} \sum_{k=1}^{q} s^3(k) + \sum_{k=1}^{q} s^2(k + 1) s(k) + \sum_{k=1}^{q} s^2(k + 2) s(k) 
+ \sum_{k=1}^{q} s(k + 1) s^2(k) + \sum_{k=1}^{q} s(k + 2) s^2(k) 
+ 2 \sum_{k=1}^{q} s(k + 2) s(k + 1) s(k) + \sum_{k=1}^{q} s(k + 3) s(k + 1) s(k) 
+ \sum_{k=1}^{q} s(k + 3) s(k + 2) s(k) + \sum_{k=1}^{q} s(k + 4) s(k + 2) s(k)
\]

(23)

e etc., where now jumps up to 2 appear in the \(k\)-summation. The terms in the cluster coefficients correct for terms in the Boltzmann partition function where particles are in the same, nearest neighbor and next-to-nearest neighbor states that are excluded by the \(g = 3\) statistics rule. Their general expression is still given by \([20]\), but now in the compositions of \(n\) the occurrence of isolated 0’s (i.e., no two or more successive 0’s and no 0 in the first and last entry) among the \(l_i\)’s is accepted

In the \(g = 2\) case the coefficients \(c(l_1, l_2, \ldots, l_j)\) given in \([13]\) can be rewritten as

\[
c(l_1, l_2, \ldots, l_j) = \frac{1}{l_1} \prod_{i=1}^{j-1} l_i \frac{(l_i + l_{i+1})}{l_i + l_{i+1}}.
\]

In the \(g = 3\) case one finds the new set of coefficients \(c(l_1, \ldots, l_j)\)’s to be

\[
c(l_1, l_2, \ldots, l_j) = \frac{(l_1 + l_2 - 1)!}{l_1! l_2!} \prod_{i=1}^{j-2} l_i! l_{i+1}! \frac{(l_i + l_{i+1} + l_{i+2})}{l_i + l_{i+1} + l_{i+2}}.
\]

(24)

Again, the \(c(l_1, l_2, \ldots, l_j)\)’s sum up to the \(n\)-th thermodynamic cluster coefficient for particles with exclusion statistics \(g = 3\) (up to an alternating sign)

\[
b(n) = q(-1)^{n-1} \sum_{l_1, l_2, \ldots, l_j} c(l_1, l_2, \ldots, l_j) = q \frac{1}{n} \prod_{k=1}^{n-1} (1 - gn/k)
\]

There are \(3^{n-1}\) such compositions \(l_1 + l_2 + \ldots + l_j = n\), \(1 \leq j \leq 2n - 1\) — for example, for \(n = 3\) one has the \(3^2 = 9\) compositions \(3 = 2 + 1 = 2 + 0 + 1 = 1 + 2 = 1 + 0 + 2 = 1 + 1 + 1 = 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1\). with \(2 + 0 + 1\) leading to the term \(\sum_{k=1}^{q} s^2(k + 2) s(k), 1 + 1 + 0 + 1\) to \(\sum_{k=1}^{q} s(k + 3) s(k + 2) s(k), 1 + 0 + 1 + 0 + 1\) to \(\sum_{k=1}^{q} s(k + 4) s(k + 2) s(k), \) etc.
which for \( g = 3 \) reduces to
\[
b(n)_{\text{therm}} = q(-1)^{n-1} \frac{1}{3n} \binom{3n}{n}.
\] (25)

4.2 Generalization to higher \( g \)

The above results can be generalized to higher integer exclusion statistics \( g \). The interesting and novel quantities are the microscopic cluster coefficients \( b(n) \). These are given by (20), as for \( g = 2 \) and \( g = 3 \), but now with a generalized definition of compositions and new expressions for the \( c(l_1, l_2, \ldots, l_j) \)'s.

Let us define a \( g \)-composition of a positive integer \( n \) as the ordered set of any number of positive or zero integers \( l_1, l_2, \ldots, l_j \) such that \( l_1 + \cdots + l_j = n \) and where at most \( g - 2 \) consecutive numbers can be zero, therefore extending the compositions admitted in the \( g = 2 \) and \( g = 3 \) cases (2-compositions are the standard compositions with nonzero entries, while 1-compositions would be defined as the unique integer \( n \)). There are \( g^{n-1} \) such compositions in total.

The \( g \)-cluster coefficients \( b_g(n) \), then, are given by the general formula
\[
b_g(n) = (-1)^{n-1} \sum_{l_1, l_2, \ldots, l_j} c_g(l_1, l_2, \ldots, l_j) \sum_{k=1}^{g} s^j(k+j-1) \ldots s^2(k+1) s^1(k) \] (26)

with the coefficients \( c_g(l_1, l_2, \ldots, l_j) \), \( 1 \leq j \leq (g - 1)(n - 1) + 1 \), given by
\[
c_g(l_1, l_2, \ldots, l_j) = \frac{(l_1 + \cdots + l_{g-1} - 1)!}{l_1! \cdots l_{g-1}!} \prod_{i=1}^{j-g+1} \frac{(l_i + \cdots + l_{i+g-1} - 1)!}{l_{i+g-1}!} \]
\[
= \frac{\prod_{i=1}^{j-g+1}(l_i + \cdots + l_{i+g-1} - 1)!}{\prod_{i=1}^{j-g}(l_{i+1} + \cdots + l_{i+g-1} - 1)!} \prod_{i=1}^{j} \frac{1}{l_i!} \] (27)

The second expression makes explicit the fact that the coefficients are symmetric under reversal of order \( c_g(l_1, l_2, \ldots, l_j) = c_g(l_j, \ldots, l_2, l_1) \) since each product in it is manifestly invariant under this reversal. It is the generalization to any higher \( g \) of (13) for \( g = 2 \) and (24) for \( g = 3 \). Again, as for \( g = 2, 3 \) in (21) and (25), the sum of these coefficients gives (up to an alternating sign) the thermodynamic cluster coefficient [6,7] of \( n \) particles with exclusion statistics \( g \), that is,
\[
b_g(n)_{\text{therm}} = q(-1)^{n-1} \frac{1}{gn} \binom{gn}{n} \] (28)
5 Matrix formulation of exclusion statistics

We now turn to the connection between exclusion statistics and various lattice models with appropriate random walk algebraic area enumeration. Since the connection between the Hofstadter model and random walks on the square lattice for $g = 2$ exclusion statistics was established from the Hofstadter matrix and its secular determinant, we focus on the structure of this and related matrices.

5.1 $g = 2$: Hofstadter-like matrix

The Hofstadter matrix has a form not very convenient for computing its determinant. In particular, the spectral function $\tilde{b}_{pq}(k)$ does not show up explicitly in the elements of the Hamiltonian $(k_x = k_y = 0$ understood).

In an alternative form of the Hamiltonian was used, which consisted of a transformation of the matrices $u$ and $v$ that preserves their algebra. It corresponds to the mapping

$$u \rightarrow -uv, \quad v \rightarrow v$$

The above is essentially a unitary transformation. This is guaranteed by the facts that it preserves the commutation relation $vu = Quv$ and that the given matrix realization of $u$ and $v$ is an irreducible representation of this relation. The Casimir $u^q$, however, is mapped to

$$u^q \rightarrow (-uv)^q = (-1)^q Q^{(q-1)/2} = -(1)^{(p+1)(q-1)} = -1$$

(the last equality follows from the fact that at least one of $p, q$ must be odd, since they are co-prime, and therefore $(p+1)(q-1)$ is always even). The transformation $u \rightarrow -uv$ is thus unitary up to a phase. This, and any similar change of the Casimirs, has no effect on the counting of closed walks and only affects walks with at least $q$ steps in the horizontal direction that are not closed but are still counted by the Hofstadter Hamiltonian, that is, spurious “umklapp” terms.

Transformations preserving the commutation relation are, in general, area-preserving lattice automorphisms that deform individual random walks but leave their algebraic area, and the number of walks corresponding to a given area, invariant.

Under the above mapping the Hofstadter Hamiltonian rewrites as

$$H' = -u v - v^{-1} u^{-1} + v + v^{-1}$$

In this new representation the matrix

$$1 - zH' = \begin{pmatrix}
1 & -(1 - Q)z & 0 & \cdots & 0 & -(1 - Q^{-q})z \\
-(1 - Q^{-1})z & 1 & -(1 - Q^2)z & \cdots & 0 & 0 \\
0 & -(1 - Q^{-2})z & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -(1 - Q^{q-1})z \\
-(1 - Q^q)z & 0 & 0 & \cdots & -(1 - Q^{1-q})z & 1
\end{pmatrix}$$
possesses two important properties:
1. It has unit diagonal elements
2. It is tridiagonal, since $Q^q = 1$ and thus the corner elements $1 - Q^{-q} = 1 - Q^q = 0$
The latter property facilitates the calculation of the determinant, yielding
\[
\det(1 - z H') = -\sum_{j=0}^{\lfloor q/2 \rfloor} a_{p,q}(2j) z^{2j}
\]
where the umklapp term $-4z^q$ present in (6) is now absent. Also, the Hofstadter spectral function $\tilde{b}_{p/q}(k)$ is actually recovered as the product of the off-diagonal elements
\[
\tilde{b}_{p/q}(k) = (1 - Q^k)(1 - Q^{-k}) = 4 \sin^2(\pi kp/q)
\]
Motivated by these observations, let us define in the $g = 2$ case a general Hamiltonian $H_2$ of the above cyclic type but with general off-diagonal elements
\[
(H_2)_{jk} = f(j) \delta_{j+1,k} + g(k) \delta_{k+1,j}, \quad \text{with} \quad \delta_{j,j+q} = 1
\]
or, explicitly,
\[
H_2 = \begin{pmatrix}
0 & f(1) & 0 & \cdots & 0 & g(q) \\
g(1) & 0 & f(2) & \cdots & 0 & 0 \\
0 & g(2) & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & f(q-1) \\
f(q) & 0 & 0 & \cdots & g(q-1) & 0
\end{pmatrix}
\]
(30)
Hermiticity of $H_2$ would require $f(k) = g(k)^*$ but we allow in general for non-Hermitian Hamiltonians, so $f(k)$ and $g(k)$ are arbitrary complex numbers. Note, further, that $f(q) = 0$ or $g(q) = 0$ is not required.

Our basic observation is that the secular determinant of the matrix $1 - z H_2$ gives the grand partition function for particles of exclusion statistics $g = 2$ with spectral function $s_2(k)$ and fugacity $z_2$ given by
\[
s_2(k) = f(k)g(k), \quad z_2 = -z^2
\]
up to a spurious umklapp term at order $z^q$. Specifically, we get
\[
\det(1 - z H_2) = 1 - z^2 \sum_{k_1=1}^{q} s_2(k_1) + z^4 \sum_{k_1=1}^{q-2} \sum_{k_2=1}^{k_1} s_2(k_1 + 2) s_2(k_2) \left(1 - \delta(k_1 - (q - 2)) \delta(k_2 - 1)\right)
\]
\[-z^6 \sum_{k_1=1}^{q-4} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} s_2(k_1 + 4) s_2(k_2 + 2) s_2(k_3) \left(1 - \delta(k_1 - (q - 4)) \delta(k_3 - 1)\right)
\]
\[+ \ldots
\]
\[-(-z)^q \left(\prod_{k=1}^{q} f(k) + \prod_{k=1}^{q} g(k)\right)
\]
(31)
as in (16), or, in another writing,

\[
\det(1 - zH_2) = 1 - z^2 \sum_{k_1=1}^{q} s_2(k_1) + z^4 \sum_{k_1=1}^{q-2} \sum_{k_2=\max(1,k_1-4)}^{k_1} s_2(k_1 + 2) s_2(k_2) \\
- z^6 \sum_{k_1=1}^{q-4} \sum_{k_2=1}^{k_1} \sum_{k_3=\max(1,k_1-6)}^{k_2} s_2(k_1 + 4) s_2(k_2 + 2) s_2(k_3) \\
+ \ldots \\
- (-z)^q \left( \prod_{k=1}^{q} f(k) + \prod_{k=1}^{q} g(k) \right) 
\] (32)

The term of order \((-z^2)^n\) is the partition function of \(n\) particles on \(q\) energy levels \(\epsilon_k\) determined by \(e^{-\beta \epsilon_k} = s_2(k)\) with \(g = 2\) exclusion on the circle. The last term is the spurious umklapp one, the only term in which \(f(k)\) and \(g(k)\) do not appear in the combination \(f(k)g(k) = s_2(k)\).

Notice that if at least one \(f(k)\) and one \(g(k)\) vanish the umklapp term disappears. Further, if at least one \(s_2(k)\) vanishes, which can be chosen to be \(s_2(q)\) by a cyclic renaming of states, then the cyclic constraint in the summation becomes irrelevant and the result coincides with the linear exclusion counting. Both the above conditions hold for the Hofstadter spectral function \(s_2(k) = \tilde{b}_{p/q}(k)\) given in (11).

5.2 Generalization to higher \(g\)

The construction of the previous section can be generalized to matrices that reproduce higher \(g\) exclusion statistics: consider the cyclic \(H_g\) matrix

\[
(H_g)_{jk} = f(j) \delta_{j+1,k} + g(k) \delta_{k+g-1,j}, \quad \text{with} \quad \delta_{j,j+q} = 1 
\] (33)

with two nonzero off-diagonals (plus their ‘tails’ as they wrap around the length of the matrix), one immediately above the main diagonal and the other \(g\) positions below it. The matrix (33) obviously cannot be Hermitian, but it is acceptable for generating lattice random walks, as we shall see.

The basic fact is that the secular determinant of the matrix \(1 - zH_g\) reproduces, up to umklapp terms, the grand partition function of exclusion-\(g\) particles with spectral function \(s_g(k)\) and fugacity \(z_g\) given by

\[
s_g(k) = g(k)f(k)f(k+1)\cdots f(k+g-2), \quad z_g = -z^g
\]
that is,

$$\det(1 - z H_g) = 1 - z^g \sum_{k_1=1}^{q} s_g(k_1) + z^{2g} \sum_{k_1=1}^{q-g} \sum_{k_2=\max(1,k_1-q+2g)}^{k_1} s_g(k_1 + g)s_g(k_2)$$

$$- z^{3g} \sum_{k_1=1}^{q-2g} \sum_{k_2=1}^{k_1} \sum_{k_3=\max(1,k_1-q+3g)}^{k_2} s_2(k_1 + 4)s_2(k_2 + 2)s_2(k_3)$$

$$+ \ldots$$

$$+ z^{n_c} \text{ (umklapp terms)}$$

It is of no interest to give an explicit expression for the umklapp terms. Such terms appear at power $z_c^n$ with

$$n_c = r + \epsilon , \quad \text{where} \quad q = (g - 1)r - \epsilon , \quad 0 \leq \epsilon \leq g - 2$$

so that the largest number of particles in the grand partition function reproduced faithfully by the determinant is

$$n_{\text{max}} = \left\lfloor \frac{q}{g(g-1)} \right\rfloor .$$

Note that for the $g = 2$ Hofstadter-like case, $n_c = q$, as explicitly displayed in (31), while the total number of particles $n_{\text{max}}$ is the same as the maximum allowed by exclusion statistics, namely $\lfloor q/2 \rfloor$, while for $g \geq 3$ umklapp terms appear before the maximum number of particles, $\lfloor q/g \rfloor$, is reached.

We also point out that if $g - 1$ successive values of $s_g(k)$ vanish, say $s_g(1) = \cdots = s_g(g - 1) = 0$, the cyclic and linear exclusion (i.e., periodic versus non periodic) statistics countings coincide. If, moreover, the stronger condition

$$g(1) = \cdots = g(g-1) = 0 , \quad f(g-1) = 0$$

holds, the umklapp terms actually vanish. (Note that $f(g-1) = 0$ is the only $f(k)$ appearing in the definition of $s_g(1), \ldots, s_g(g - 1)$ and no other $s_g(k)$.) In that case the secular determinant has only two nonzero off-diagonals (the terms in the wraparound ‘tails’ vanish) and gives the full exact exclusion-$g$ grand partition function.

Finally, we point out that the secular determinant $\det(1 - z H_g)$ admits two distinct statistical mechanical interpretations:

1. As a fermionic grand partition function with fugacity parameter $z_1 = -z$ and particles occupying energy levels $\epsilon_k$ given by $e^{-\beta \epsilon_k} = E_k$, where $E_k$ are the eigenvalues of $H_g$.

2. As an exclusion statistics $g$ grand partition function with fugacity parameter $z_g = -z^g$ and levels $e^{-\beta \epsilon_k} = s_g(k)$, as above.

The fact that the same system admits these two distinct interpretations is a kind of generalized “bosonization” (really a “superfermionization”) that maps an exclusion-1 (fermion) system to an exclusion-$g$ system. In that trade-off the spectrum of the exclusion $g$ system is simple and known while the spectrum of the fermionic system is nontrivial (e.g., the Hofstadter Hamiltonian spectrum in the $g = 2$ case).
6 Random walks corresponding to higher exclusion statistics

We proceed, now, to identify random walks that realize exclusion statistics $g$ through their algebraic area enumeration generating function, as was the case for square lattice walks with $g = 2$ statistics.

6.1 General construction

Let us rewrite the Hamiltonian (33) corresponding to exclusion statistics $g$ in terms of $u$ and $v$ in (3) and (4) ($k_x = k_y = 0$ understood)

$$H_g = F(u) \, v + v^{1-g} \, G(u) \quad (35)$$

where $F(u)$ and $G(u)$ are scalar functions of $u$ such that

$$F(Q^k) = f(k), \quad G(Q^k) = g(k)$$

This puts us in the context of random walks on the square lattice, since $u$ and $v$ represent unit hops in the horizontal and vertical directions, but with allowed steps as defined in the above Hamiltonian upon expanding $F(u)$ and $G(u)$ as a power series in $u$. E.g., choosing $F(u) = 1 - u$, $G(u) = 1 - u^{-1}$ and $g = 2$ we recover the Hofstadter Hamiltonian (29).

We also note that the results stated in section (5.2) can be systematically obtained in the above representation by writing

$$\det(1 - zH_g) = e^{\text{Tr} \ln(1 - zH_g)} = \exp \left( - \sum_{k=1}^{\infty} \frac{z^k}{k} \text{Tr} \, H_g^k \right) \quad (36)$$

and taking into account that $v^q = 1$ and that, for any scalar function $X(u)$,

$$\text{Tr} \, (X(u)v^n) = \sum_{k=-\infty}^{\infty} \delta_{n-kq} \, \text{Tr} \, X(u)$$

Terms with $n = 0$ reproduce the exclusion-$g$ grand partition function and start at power $z^g$, while terms with $n = \pm q, \ldots$ produce the umklapp effects and start at $z^{n\epsilon}$ as in (34).

The identification of lattice random walks that realize fractional statistics $g$ relies not only upon choosing specific spectral functions $F(u)$ and $G(u)$ in the above Hamiltonian, but also upon choosing specific realizations of the matrices $u$ and $v$ in terms of lattice hopping operators. The obvious choice is to interpret $u$ and $v$ as the vertical and horizontal hopping operators. However, other choices may yield a different lattice and a different set of walks. Although all these choices are related by area-preserving lattice mappings, and thus give the same area counting, some interpretations may be more symmetric, more physically motivated and more useful. We see this pattern in the Hofstadter case where (29) corresponds to vertical hops and hops in the $45^\circ$ diagonal, while the original realization (1) gives the standard random walks on the square lattice.
6.2 $g = 3$ and chiral walks on the triangular lattice

Let us illustrate the above considerations in the case $g = 3$ with a choice of spectral functions and realizations that will map to chiral walks on a triangular lattice.

The starting Hamiltonian is

$$H_t = U + V + Q^{1+a} U^{-1} V^{-1} , \quad (37)$$

with $a$ an arbitrary real number. The unitary matrices $U$, $V$ satisfy

$$V U = Q^2 U V \quad (38)$$

where the change from $Q$ to $Q^2$ is for later convenience. Note that any other phase factors introduced in the above Hamiltonian could be absorbed in appropriate redefinitions of the phases of $U$ and $V$, so $a$ is, along with $Q$, the only relevant parameter in the problem.

This Hamiltonian is clearly not (yet) of the general form (35). Before bringing it to that form let us give it a lattice interpretation as acting on a particle hopping on the vertices of a triangular lattice. $U$, $V$ and $W = Q^{1+a} U^{-1} V^{-1}$ correspond to motion in directions with angles $0$, $2\pi/3$ and $4\pi/3$, respectively, with respect to the horizontal axis. Therefore, $W V U = V U W$ corresponds to a closed counterclockwise walk along the three sides of an elementary up-vertex triangular cell, and it assigns to this cell an area $1 + a$. Conversely, $V W U = W V U = U V W = Q^{1-a}$ represents a closed clockwise walk around an elementary down-vertex triangular cell, and it assigns to this cell an area $1 - a$ (see Fig.1). So, in general, up-vertex and down-vertex triangular cells can have different areas, and $a$ is the half-difference of the area of the up-vertex and down-vertex triangular cells. Alternately, we can consider $Q_u = Q^{1+a}$ and $Q_d = Q^{1-a}$ as counting parameters for the algebraic number of up-vertex and down-vertex triangles enclosed by the paths. If we wish to assign up-vertex and down-vertex triangular cells the same area, so that they be counted on an equal footing, we must take $a = 0$.

Powers of the Hamiltonian (37) represent random chiral walks on the triangular lattice: from each lattice vertex the walk can proceed in only three of the possible six directions (see Fig.2). E.g., $U W U W V^2 = Q^{2(1+a)}$ is a closed walk enclosing two up-vertex triangular cells, each in the counterclockwise sense, while $U^2 W^2 V^2 = Q^{4+2a} = Q^{3(1+a)} Q^{1-a}$ is a closed walk enclosing 3 up-vertex and one down-vertex triangular cells, all in the counterclockwise sense (see Fig.3). Clearly only walks with a total number of steps $n$ a multiple of 3 can be closed. As in square lattice walks, $(1/q) \text{Tr} H_n^p$ is the generating function of closed walks of length $n = 3n$ weighted by the exponential of their algebraic area, up to umklapp effects.

To bring the triangular lattice Hamiltonian (37) to the standard exclusion form (35) we choose the representation

$$U = -i u v , \quad V = i u^{-1} v$$
Figure 1: walks going around *up-vertex* and *down-vertex* triangular cells starting from the black bullet lattice site.
Figure 2: Three of the 6 possible chiral walks starting from the same black bullet lattice site. Only the 3 outgoing arrows represent possible motions from the original site.
Figure 3: $UWUWV^2$ and $U^2W^2V^2$ walks.
with \( u, v \) the standard matrices (3) and (4) \((k_x = k_y = 0\) understood), which reproduces
the commutation relation (38). (It also makes the Casimirs \( U^q = i^q \) and \( V^q = (-i)^q \) but,
as stated before, this only affects umklapp terms.) \( H_t \) in (37) then becomes
\[
H_3 = i(-u + u^{-1}) \ v + Q^a \ v^{-2}
\]
which is of the exclusion form (35) with
\[
F = i(-u + u^{-1}) , \quad G = Q^a , \quad g = 3
\]
We obtain the spectral parameters
\[
f(k) = 2 \sin \frac{2\pi p k}{q} , \quad g(k) = e^{i2\pi ap/q}
\]
and the spectral function
\[
s_3(k) = g(k) f(k) f(k + 1) = 4 e^{i2\pi ap/q} \sin \frac{2\pi p k}{q} \sin \frac{2\pi p (k + 1)}{q}
\] (39)
while the corresponding secular matrix assumes the form
\[
1 - z H_3 = \begin{pmatrix}
1 & i(Q - \frac{1}{Q})z & 0 & \ldots & 0 & -Q^a z & 0 \\
0 & 1 & i(Q^2 - \frac{1}{Q^2})z & \ldots & 0 & 0 & -Q^a z \\
-Q^a z & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & i(Q^{q-2} - \frac{1}{Q^{q-2}})z & 0 \\
i(Q^q - \frac{1}{Q^q})z & 0 & 0 & \ldots & -Q^a z & 0 & 1
\end{pmatrix}
\]
Note that, since \( s_3(q-1) = s_3(q) = 0 \), the periodic and linear exclusion statistics countings coincide, as in the square lattice case.

It follows that on a triangular lattice the algebraic area enumeration generating function for closed chiral random walks is obtained as the grand partition function of exclusion-
\( g=3 \) particles with the spectral function \( s_3(k) \) given in (39). (The secular determinant
will also produce umklapp terms, but these correspond to walks that are not closed but
are counted as such and can simply be ignored.)

For closed walks of length \( n = 3n \), \( s_3(k) \) is such that the sum of the coefficients in
the \( Q^A \) expansion of each of the building blocks of the cluster coefficients in (23) gives a
counting identical to the Hofstadter case, that is \((2n)\). Together with the counting \((3n)\)
from the \( c(l_1, l_2, \ldots, l_j) \)'s in (25) this gives the total number of closed walks of length \( 3n \)
\[
\binom{3n}{n} \binom{2n}{n} = \binom{3n}{n, n, n} = \binom{n}{n/3, n/3, n/3}
\]
as can also be derived from combinatorial considerations.
The algebraic area enumeration follows from rewriting the cluster coefficients \( b(n) \) as linear combinations of \( Q^{1+a} A_u, Q^{1-a} A_d = Q^{A_u+A_d+a(A_u-A_d)} \), where \( A_u \) and \( A_d \) are the algebraic area of the enclosed up-vertex and down-vertex triangular cells. In the symmetric case \( a = 0 \), where only the total area \( A_u + A_d \) is counted, we get

\[
 b(1) = 2q \cos \left( \frac{2\pi p}{q} \right), \quad b(2) = -q(6 + 7 \cos(4\pi p/q) + 2 \cos(8\pi p/q)), \quad \text{etc} \tag{40}
\]

Since the number of walks of length \( n \) enclosing an algebraic area \( A \) is, up to an alternating sign, \( C_n(A) = n b(n/3)/q \) it follows that

- \( C_3(1) = C_3(-1) = 3 \); that is, among the 6 closed walks of length 3, 3 enclose an area \(+1\) (1 up-vertex triangular cell) and 3 enclose an area \(-1\) (1 down-vertex triangular cell) (see Fig.2)

- \( C_6(0) = 36, C_6(2) = C_6(-2) = 21, C_6(4) = C_6(-4) = 6 \); that is, among the 90 closed walks of length 6, 36 enclose an area 0, 21 an area +2 and 21 an area \(-2\), and 6 an area +4 and 6 an area \(-4\) (see Fig.3)

- etc.

We postpone the full analysis of the properties, statistics and complete algebraic enumeration \( C_n(A_u, A_d) \) of the chiral triangular walks for a future publication.

### 6.3 Examples of lattices for higher \( g \)

We conclude by briefly discussing an example of a \( g = 4 \) lattice model. The starting Hamiltonian is

\[
 H_4 = i(-u^2 + u^{-1}) v - iQ^{2a} v^{-3}
\]

with \( a \) a real number. Again, \( a \) is the only relevant parameter in the Hamiltonian.

The above is clearly a \( g = 4 \) model with spectral parameters

\[
 f_4(k) = -iQ^{2k} + iQ^{-k} = 2 e^{i\pi pk/q} \sin \frac{3\pi pk}{q}, \quad g_4(k) = -i e^{4i\pi ap/q}
\]

and spectral function

\[
 s_4(k) = g_4(k) f_4(k) f_4(k + 1) f_4(k + 2) \tag{41}
\]

The lattice walk extracted from \( H_4 \) by interpreting \( u \) and \( v \) as hops on the square lattice, however, is rather unnatural. Defining, now, the operators

\[
 U = i Q^{1+a} v^{-2} u^{-1}, \quad V = -i Q^a v^{-1} u, \quad VU = Q^a UV
\]

the Hamiltonian takes the form

\[
 H_4 = i(-U + U^{-1})V + Q^a V^{-1} \tag{42}
\]
which has a more natural interpretation on the square lattice as hops down, up-right and up-left by single steps. For generic \( a \) it is also chiral, as it assigns different areas to different triangular half-cells, namely, \( \frac{3}{2} + a \) for triangles with right angle in the first and fourth quadrants and \( \frac{3}{2} - a \) for mirror-image triangles with right angle in the second and third quadrants.

Interestingly, the form (42) of the Hamiltonian makes it now a \( g = 2 \) Hamiltonian with spectral parameters

\[
f_2(k) = -iQ^{3k} + iQ^{-3k}, \quad g_2(k) = Q^a\]

and spectral function

\[
s_2(k) = 2e^{2\pi op/q} \sin \frac{6\pi pk}{q}\]

We therefore obtain another example of “superfermionization,” specifically, a mapping of a \( g = 2 \) system with spectrum implied by (44) to a \( g = 4 \) system with spectrum implied by (41). An indication that the \( g = 2 \) system is secretly a \( g = 4 \) one is the fact that all odd-particle number partition functions of the \( g = 2 \) system vanish, as can be deduced from the form of \( s_2(k) \), or from the fact that the corresponding closed lattice walks must have \( 2n \) steps down, \( n \) steps up-right and \( n \) steps up-left, for a total of \( n = 4n \) steps. So the number of \( g = 2 \) particles is \( n/g = 2n \), which is always even.

The above gives a flavor of the possibilities for higher \( g \). The identification of interesting random walks corresponding to other values of \( g \) and the pattern of equivalences between models with different \( g \) is left for future work.

## 7 Conclusions and outlook

In conclusion, we demonstrated the equivalence between specific lattice walk models and particles with (integer) exclusion statistics and presented some examples of random walks, beyond the square lattice model, where this equivalence holds. We also obtained the exact microscopic cluster coefficients for exclusion \( g \) statistics.

Clearly there are several open questions and directions for further investigations. The full treatment of the \( g = 3 \) chiral triangular lattice model statistics and its algebraic enumeration \( C_n(A_u, A_d) \) is an immediate problem that will be addressed in an upcoming publication. Other useful models of physical relevance realizing higher-\( g \) statistics should also be sought and analyzed.

The lattice walk–exclusion statistics connection is somewhat mysterious. It would be useful to have a more intuitive understanding of this correspondence (if one can be had). Further, a general classification of the type of lattice walk models that admit an exclusion statistics treatment would be highly desirable, especially in the context of elucidating any geometric or topological properties of such walks. Such a classification would also clarify and organize the uncovered superfermionization phenomena, presumably revealing them
as a set of dualities between lattice models. This would also demonstrate whether physically relevant walks, such as random walks on the hexagonal lattice, admit a formulation that connects them to exclusion statistics.

It is also tempting to speculate about possible lattice walk realizations of statistics with fractional exclusion parameter $g$. A formulation departing from the matrix model presented here would be appropriate for such generalizations. The question of whether random walks corresponding to bosonic ($g = 0$) statistics exist is also an interesting one.

Finally, the “holy grail” of this quest would be the uncovering of random walks in higher dimensional lattices that admit a statistical interpretation. Generalizations of exclusion statistics to other types of counting, or to higher dimensional objects (strings etc.), might become relevant and necessary.

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