DEFORMATIONS OF REDUCIBLE SL\((n, \mathbb{C})\) REPRESENTATIONS OF FIBERED 3-MANIFOLD GROUPS

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Abstract. Let \(M_\phi\) be a surface bundle over a circle with monodromy \(\phi : S \to S\). We study deformations of certain reducible representations of \(\pi_1(M_\phi)\) into \(\text{SL}(n, \mathbb{C})\), obtained by composing a reducible representation into \(\text{SL}(2, \mathbb{C})\) with the irreducible representation \(\text{SL}(2, \mathbb{C}) \to \text{SL}(n, \mathbb{C})\). In particular, we show that under certain conditions on the eigenvalues of \(\phi^*\), the reducible representation is contained in a \((n + 1 + k)(n - 1)\) dimensional component of the representation variety, where \(k\) is the number of components of \(\partial M_\phi\). This result applies to mapping tori of pseudo-Anosov maps with orientable invariant foliations whenever \(1\) is not an eigenvalue of the induced map on homology, where the reducible representation is also a limit of irreducible representations.

1. Introduction

Suppose that \(S = S_{g,p}\) is a surface of genus \(g\) with \(p \geq 1\) punctures, where \(2g + p > 2\). Then \(S\) admits a hyperbolic structure. If \(\phi : S \to S\) is a homeomorphism, we can form the mapping torus \(M_\phi = S \times [0, 1]/(x, 1) \sim (\phi(x), 0)\). Whenever \(\lambda^2\) is an eigenvalue of \(\phi^* : H^1(S) \to H^1(S)\) with eigenvector \((a_1, \ldots, a_{2g+p-1})^T\) with respect to a generating set \(\{[\gamma_1], \ldots, [\gamma_{2g+p-1}]\}\) of \(H^1(S)\), we obtain a reducible representation \(\rho_\lambda : \pi_1(M_\phi) \to \text{SL}(2, \mathbb{C})\) by defining,

\[
\rho_\lambda(\gamma_i) = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix},
\rho_\lambda(\tau) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},
\]

where \(\tau\) is the generator of the fundamental group of the \(S^1\) base of the fiber bundle \(S \to M_\phi \to S^1\). (Recall that a representation \(\rho : G \to \text{GL}(n, \mathbb{C})\) is reducible if the image \(\rho(G)\) preserves a proper subspace of \(\mathbb{C}^n\), and otherwise is called irreducible.)

When \(M_\phi\) is the complement of a knot \(K\) in \(S^3\), this observation was originally made by Burde [1] and de Rham [3]. Furthermore, the Alexander polynomial is the characteristic polynomial of \(\phi^*\), so the condition on \(\lambda\) is equivalent to the condition that \(\lambda^2\) is a root of the Alexander polynomial \(\Delta_K(t)\). It was shown in [6] that the non-abelian, metabelian, reducible representation \(\rho_\lambda\) is the limit of irreducible representations if \(\lambda^2\) is a simple root.
of $\Delta_K(t)$. Heusener and Medjerab [5] have also shown using an inductive argument that the conclusion still holds in $\text{SL}(n, \mathbb{C})$, $n \geq 3$, if $\rho_{\lambda}$ is composed with the irreducible representation $r_n : \text{SL}(2, \mathbb{C}) \to \text{SL}(n, \mathbb{C})$. These results apply even if the knot complement is not fibered, as long as $\lambda^2$ is a simple root of $\Delta_K(t)$.

In this paper, we show that reducible $\text{SL}(n, \mathbb{C})$ representations of fibered 3-manifolds groups obtained as the composition $\rho_{\lambda,n} = r_n \circ \rho_{\lambda}$ can be deformed to irreducible representations using a more direct calculation of the deformation space using coordinates for $\mathfrak{sl}(n, \mathbb{C})$. If the punctures form a single orbit under $\phi$ and the mapping torus $M_\phi$ is the complement of a fibered knot, then the results of [6] and [5] apply. The main result in Theorem 1.1 also covers the cases where $M_\phi$ is the complement of a fibered link $L$ with $k \geq 2$ components $L_1, \ldots, L_k$, or a $k$-cusped fibered manifold which is not a link complement. In the statement of Theorem 1.1, $\bar{\phi}$ is the homeomorphism on $\bar{S} = S_{g,0}$ obtained from $\phi$ by filling in the punctures of $S_{g,p}$. This defines a homeomorphism $\bar{\phi} : \bar{S} \to \bar{S}$.

**Theorem 1.1.** Suppose that $\lambda^2$ is a simple eigenvalue of $\bar{\phi}^*$. If $|\lambda| \neq 1$, $\bar{\phi}^* : H^1(S) \to H^1(\bar{S})$ does not have 1 as an eigenvalue, and if for each $2 \leq j \leq n - 1$, we have that $\lambda^{2j}$ is not an eigenvalue of $\bar{\phi}^*$, then $\rho_{\lambda,n}$ is a smooth point of the representation variety $R(\pi_1(M_\phi), \text{SL}(n, \mathbb{C}))$, contained in a unique component of dimension $(n + 1 + k)(n - 1)$.

Note that for a knot complement, the Alexander polynomial satisfies $\Delta_K(1) = \pm 1$. Hence for a knot complement, the condition that $\bar{\phi}^* : H^1(S) \to H^1(\bar{S})$ does not have 1 as an eigenvalue (in the fibered case) or the corresponding condition that 1 is not a root of $\Delta_K(t)$ (in the non-fibered case) is automatically satisfied. For a generic mapping torus, a fixed point of $\bar{\phi}^*$ implies that the closed manifold obtained as the mapping torus of $\bar{\phi}$ has second Betti number at least 2, in which case the manifold fibers over a circle in infinitely many ways [17]. Heuristically, this leads to more infinitesimal deformations. When the local dimension of infinitesimal dimensions is higher than half the dimension of $H^1(\partial M_\phi)$, the standard techniques using Poincaré duality to show smoothness of the space of representations cannot be used. Whether the reducible representation can be obtained as a limit of irreducible representations in this case is unknown.

When $\phi$ is a pseudo-Anosov element of the mapping class group, $\lambda^2$ is the dilatation factor of $\phi$, and the $p$ punctures are exactly the singular points of the invariant foliations of $\phi$, $\rho_\lambda = \rho_{\lambda,2}$ is shown to have deformations to irreducible representations under some additional conditions on the eigenvalues of $\bar{\phi}^*$, the map on the closed surface $S_{g,1}$, in [8]. We show that under the same hypotheses, the same holds for $\rho_{\lambda,n}$ when $n > 2$.

**Theorem 1.2.** Suppose that $\lambda^2$ is the dilatation of a pseudo-Anosov map $\phi$ such that the stable and unstable foliations are orientable, and the singular points coincide with the punctures of $S$. Suppose also that 1 is not an eigenvalue of $\bar{\phi}^*$. Then $\rho_{\lambda,n}$ is a limit of irreducible $\text{SL}(n, \mathbb{C})$ representations and
is a smooth point of $R(\pi_1(M_\phi), \text{SL}(n, \mathbb{C}))$, contained in a unique component of dimension $(n + 1 + k)(n - 1)$.

In Section 2 we give the basic definitions and background about representations of $\text{SL}(2, \mathbb{C})$ into $\text{SL}(n, \mathbb{C})$. Section 3 discusses the general theory of deformations, and Section 4 contains the main results, including relevant cohomological calculations.

2. Representations into $\text{SL}(n, \mathbb{C})$

For notational convenience, we denote $\text{SL}(n) = \text{SL}(n, \mathbb{C})$, $\mathfrak{sl}(n) = \mathfrak{sl}(n, \mathbb{C})$, $\text{GL}(n) = \text{GL}(n, \mathbb{C})$, and $\Gamma_\phi = \pi_1(M_\phi)$. Note that we have the following identities in $\text{SL}(2)$:

\[
\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a + b \\ 0 & 1 \end{pmatrix},
\]

\[
\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \lambda^2 a \\ 0 & 1 \end{pmatrix}.
\]

Thus, if $\lambda^2$ is an eigenvalue of $\phi^*: H^1(S) \to H^1(S)$, \{[\gamma_1], \ldots, [\gamma_{2g\gamma + p - 1}]\} generate $H^1(S)$, and $(a_1, \ldots, a_{2g\gamma + p - 1})^T$ is an eigenvector for $\lambda^2$, we can define

\[
\rho_\lambda(\gamma_i) = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}
\]

\[
\rho_\lambda(\tau) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.
\]

Since $\pi_1(\Gamma_\phi)$ is a semi-direct product of the free group $\pi_1(S) = \langle \gamma_1, \ldots, \gamma_{2g\gamma + p - 1} \rangle$ with $\pi_1(S^1) = \langle \tau \rangle$ satisfying the relations $\tau\gamma_i\tau^{-1} = \phi(\gamma_i)$ and $\phi^*$ maps the vector $(a_1, \ldots, a_{2g\gamma + p - 1})^T$ to $\lambda^2(a_1, \ldots, a_{2g\gamma + p - 1})^T$, the identities (2.1) imply that this defines a representation $\rho_\lambda : \Gamma_\phi \to \text{SL}(2)$.

We now describe representations of $\text{SL}(2)$ into $\text{SL}(n)$, which we will compose with $\rho_\gamma$ to obtain representations $\Gamma_\phi \to \text{SL}(n)$. A more general version of the discussion in this section can be found in [5, Section 4].

Let $R = \mathbb{C}[X,Y]$ be the polynomial algebra on two variables. We have an action of $\text{SL}(2)$ on $R$ by,

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = dX - bY,
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = -cX + aY,
\]

for $(a, b, c, d) \in \text{SL}(2)$. Let $R_{n-1} \subset R$ denote the $n$-dimensional subspace of homogenous polynomials of degree $n - 1$, generated by $X^\ell Y^{n-\ell}, 1 \leq \ell \leq n$. The action of $\text{SL}(2)$ leaves $R_{n-1}$ invariant, turning $R_{n-1}$ into a $\text{SL}(2)$ module, and we obtain a representation $r_n : \text{SL}(2) \to \text{GL}(R_{n-1})$. We can identify $R_{n-1}$ with $\mathbb{C}^n$ by identifying the basis elements \{X^\ell Y^{n-\ell}\} with
the standard basis elements \( \{e_\ell\} \) of \( \mathbb{C}^n \). The induced isomorphism turns \( r_n \) into a representation \( \text{SL}(2) \to \text{GL}(n) \cong \text{GL}(R_{n-1}) \), which we will also call \( r_n \). The representation \( r_n \) is \textit{rational}, that is the coefficients of the matrix coordinates of \( r_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) are polynomials in \( a, b, c, d \).

We have the following two well-known results about \( r_n \).

**Lemma 2.1.** \cite{16} Lemma 3.1.3(ii) The representation \( r_n \) is irreducible.

**Lemma 2.2.** \cite{16} Lemma 3.2.1] Any irreducible rational representation of \( \text{SL}(2, \mathbb{C}) \) is conjugate to some \( r_n \).

It is easy to check that \( r_n \) maps the unipotent matrices \( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \) to unipotent elements of \( \text{SL}(R_{n-1}) \), and the diagonal element \( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \) is mapped to the diagonal element \( \text{diag}(a^{n-1}, a^{n-3}, \ldots, a^{-n+1}) \). Since these elements generate \( \text{SL}(2) \), the image of \( r_n \) lies in \( \text{SL}(R_{n-1}) \cong \text{SL}(n) \).

We now define \( \rho_{\lambda, n} = r_n \circ \rho_\lambda \). As we will only be considering the case when \( \lambda^2 \) is a simple eigenvalue of \( \phi^* \) and the above lemmas imply the uniqueness of \( r_n \), this gives a well-defined and unique (up to conjugation) representation \( \rho_{\lambda, n} : \Gamma_\phi \to \text{SL}(n) \).

By composing \( \rho_{\lambda, n} \) with the adjoint representation, we also obtain an action of \( \Gamma_\phi \) on \( \mathfrak{sl}(n) \), turning it into a \( \Gamma_\phi \) module. The following decomposition is a consequence of the Clebsch-Gordan formula (see, for example, \cite{11} Lemma 1.4).

**Lemma 2.3.** With the \( \Gamma_\phi \) module structure, \( \mathfrak{sl}(n) \cong \bigoplus_{j=1}^{n-1} R_{2j} \).

This decomposition will be used to calculate the infinitesimal deformations of \( \rho_{\lambda, n} \).

### 3. Infinitesimal Deformations

In this section, let \( M \) be a 3-manifold with finitely many torus boundary components \( \partial M = \sqcup_{i=1}^k T_i \) and \( \Gamma = \pi_1(M) \). For each boundary torus \( T_i \), the inclusion map \( \iota : T_i \to M \) induces a map from \( \pi_1(T_i) \) to a conjugacy class of subgroups isomorphic to \( \pi_1(T_i) \cong \mathbb{Z} \times \mathbb{Z} \) in \( \pi_1(M) \). To each boundary component \( T_i \), we associate \( \pi_1(T_i) \) with a representative subgroup \( \Delta_i \) in \( \Gamma \). Let \( R(\Gamma, \text{SL}(n)) = \text{Hom}(\Gamma, \text{SL}(n)) \) be the variety of representations of \( \Gamma \) into \( \text{SL}(n) \) and \( X(\Gamma, \text{SL}(n)) = R(\Gamma, \text{SL}(n)) / \text{SL}(n) \) be the \( \text{SL}(n) \) character variety, where the quotient is the GIT quotient as \( \text{SL}(n) \) acts by conjugation.

Suppose \( \rho : \Gamma \to \text{SL}(n) \) is a representation. The group of twisted cocycles \( Z^1(\Gamma; \mathfrak{sl}(n)_\rho) \) is defined as the set of maps \( z : \Gamma \to \mathfrak{sl}(n) \) that satisfy the twisted cocycle condition

\[
(3.1) \quad z(ab) = z(a) + \text{Ad}_{\rho(a)} z(b),
\]

which can be interpreted as the derivative of the homomorphism condition for a smooth family of representation \( \rho_t \) at \( \rho \). The derivative of the triviality condition that \( \rho_t \) is a smooth family of representations obtained by
conjugating \( \rho \) gives the coboundary condition,
\[
(3.2) \quad z(\gamma) = u - \text{Ad}_{\rho(\gamma)}u,
\]
and \( B^1(\Gamma; \mathfrak{sl}(n)) \) is defined as the set of coboundaries, or the cocycles satisfying Equation (3.2). The quotient is defined to be
\[
H^1(\Gamma; \mathfrak{sl}(n)) = Z^1(\Gamma; \mathfrak{sl}(n))/B^1(\Gamma; \mathfrak{sl}(n)).
\]
Weil [9,18] has noted that \( Z^1(\Gamma; \mathfrak{sl}(n)) \) contains the tangent space to \( R(\Gamma, \text{SL}(n)) \) at \( \rho \) as a subspace. The following tools can be used to determine if the representation variety is smooth at \( \rho \) so that we can study the space of cocycles to determine the first order behavior of deformations of a representation \( \rho \). In the following proposition, \( C^1(\Gamma; \mathfrak{sl}(n)) \) denotes the set of cochains \( \{c: \Gamma \to \mathfrak{sl}(n)\} \).

**Proposition 3.1** ([5], Lemma 3.2; [6], Proposition 3.1). Let \( \rho \in R(\Gamma, \text{SL}(n)) \), \( u_i \in C^1(\Gamma; \mathfrak{sl}(n)), 1 \leq i \leq j \) be given, and \( \mathbb{C}[[t]] \) denote the set of formal power series in \( t \) with coefficients in \( \mathbb{C} \). If
\[
\rho_j(\gamma) = \exp(\sum_{i=1}^{j} t^i u_i(\gamma)) \rho(\gamma)
\]
is a homomorphism into \( \text{SL}(n, \mathbb{C}[[t]]) \) modulo \( t^{j+1} \), then there exists an obstruction class \( \zeta_{j+1}^{(u_1, \ldots, u_j)} \in H^2(\Gamma; \mathfrak{sl}(n)) \) such that:

1. There is a cochain \( u_{j+1}: \Gamma \to \mathfrak{sl}(n) \) such that
\[
\rho_{j+1}(\gamma) = \exp(\sum_{i=1}^{j+1} t^i u_i(\gamma)) \rho(\gamma)
\]
is a homomorphism modulo \( t^{j+2} \) if and only if \( \zeta_{j+1} = 0 \).

2. The obstruction \( \zeta_{j+1} \) is natural, i.e. if \( f \) is a homomorphism then \( f^* \rho_j =: \rho^{j_1}_j \circ f \) is also a homomorphism modulo \( t^{j+1} \) and \( f^* \zeta_{j+1}^{(u_1, \ldots, u_j)} = \zeta_{j+1}^{(f^*u_1, \ldots, f^*u_j)} \).

We will apply the previous proposition to the restriction map \( \iota^* \) on cohomology, which is induced by the inclusion map \( \iota: \partial M \to M \). As \( \partial M \) consists of a disjoint union of tori, we will need to understand \( H^1(\Delta_i; \mathfrak{sl}(n)) \). Recall that a hyperbolic element of \( \text{SL}(2) \) is an element that acts on \( \mathbb{H}^3 \) with no fixed points in \( \mathbb{H}^3 \) and two fixed points on \( \partial \mathbb{H}^3 \). Such elements are characterized by being conjugate in \( \text{SL}(2) \) to a diagonal matrix with distinct eigenvalues that are not on the unit circle.

**Lemma 3.2.** Suppose \( \rho: \mathbb{Z} \times \mathbb{Z} \to \text{SL}(2) \) contains a hyperbolic element in its image. Then \( \dim H^1(\mathbb{Z} \times \mathbb{Z}; \mathfrak{sl}(n)) = 2(n-1) \).

**Proof.** Suppose \( \gamma \in \mathbb{Z} \times \mathbb{Z} \) such that \( \rho(\gamma) \) is a hyperbolic element in \( \text{SL}(2) \). Then, up to conjugation,
\[
\rho(\gamma) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},
\]
for some \(|a| > 1\). The image of such an element under the irreducible representation \(r_n : \text{SL}(2) \to \text{SL}(n)\) is conjugate to a diagonal matrix with \(n\) distinct eigenvalues. Hence any nearby representation \(\rho' : \mathbb{Z} \times \mathbb{Z} \to \text{SL}(n)\) is conjugate to a diagonal matrix with distinct entries. In other words, up to coboundary, we can assume that any class \([z] \in H^1(\mathbb{Z} \times \mathbb{Z}; \mathfrak{sl}(n)_{r_n,\rho})\) has the form of a diagonal matrix \(z(\gamma) = \text{diag}(y_1, y_2, \ldots, y_n)\) where \(\text{tr} z(\gamma) = 0\). Since for any other \(\gamma' \in \mathbb{Z} \times \mathbb{Z}\), we have that \(\gamma'\) commutes with \(\gamma\), \(z(\gamma')\) must also be diagonal, so the dimension of \(H^1(\mathbb{Z} \times \mathbb{Z}; \mathfrak{sl}(n)_{r_n,\rho})\) is \(2(n - 1)\). \(\square\)

**Lemma 3.3.** Let \(\rho : \pi_1(M) \to \text{SL}(2)\) be a non-abelian representation such that \(\rho(\Delta_i)\) contains a hyperbolic element for each subgroup \(\Delta_i\) of \(\pi_1(M)\) associated to a boundary component \(T_i\) of \(\partial M\). If \(\dim H^1(T; \mathfrak{sl}(n)_{r_n,\rho}) = k(n - 1)\) where \(k\) is the number of components of \(\partial M\), then \(\iota^* : H^2(M; \mathfrak{sl}(n)_{r_n,\rho}) \to H^2(\partial M; \mathfrak{sl}(n)_{r_n,\rho})\) is injective.

**Proof.** We have the cohomology exact sequence for the pair \((M, \partial M)\)

\[
\begin{align*}
H^1(M, \partial M) & \to H^1(M) \xrightarrow{\alpha} H^1(\partial M) \\
& \xrightarrow{\beta} H^2(M, \partial M) \to H^2(M) \\
& \xrightarrow{\iota^*} H^2(\partial M) \to H^3(M, \partial M) \to \cdots
\end{align*}
\]

where all cohomology groups are taken to be with the twisted coefficients \(\mathfrak{sl}(n)_{r_n,\rho}\). A standard Poincaré duality argument \([6][7][13]\) implies that \(\alpha\) has half-dimensional image in \(H^1(\partial M)\). By Lemma 3.2,

\[
\dim H^1(\Delta_i) = 2(n - 1),
\]

as long as \(\rho(\Delta_i)\) contains a hyperbolic element for each \(i\). We can identify \(H^1(\partial M) \cong \bigoplus_{i=1}^k H^1(\Delta_i)\), which has dimension \(2k(n - 1)\). Since \(H^1(M) \cong H^1(\Gamma)\) has dimension \(k(n - 1)\), then \(\alpha\) is injective. Since \(\beta\) is dual to \(\alpha\) under Poincaré duality, then \(\beta\) is surjective. This implies that \(\iota^*\) is injective. \(\square\)

We now utilize these facts to determine sufficient conditions for deforming representations.

**Proposition 3.4.** Let \(\rho : \Gamma \to \text{SL}(2)\) be a non-abelian representation such that \(\rho(\Delta_i)\) contains a hyperbolic element for each subgroup \(\Delta_i\). If \(H^1(\Gamma; \mathfrak{sl}(2)_{r_n,\rho}) = k(n - 1)\) where \(k\) is the number of components of \(\partial M\), then \(r_n \circ \rho\) is a smooth point of the representation variety \(R(\Gamma, \text{SL}(n))\), and it is contained in a unique component of dimension \((n + 1 + k)(n - 1) - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n,\rho})\).

**Proof.** We begin by showing that every cocyle in \(Z^1(\Gamma; \mathfrak{sl}(n)_{r_n,\rho})\) is integrable.

Suppose we have \(u_1, \ldots, u_j : \Gamma \to \mathfrak{sl}(n)\) such that

\[
\rho^j_n(\gamma) = \exp \left( \sum_{i=1}^j t^iu_i(\gamma) \right) \rho(\gamma)
\]
is a homomorphism modulo $\rho^{j+1}$. By Lemma 3.2 and 14, the restriction of $\rho_n$ to $\Delta_i$ is a smooth point of the representation variety $R(\Delta_i, \text{SL}(n))$. Hence $\rho_n|_{\pi_1(T_i)}$ extends to a formal deformation of order $j+1$ by the formal implicit function theorem (see [6], Lemma 3.7). This implies that the restriction of $\zeta_{j+1}^{(u_1,\ldots,u_j)}$ to each component $H^2(T_i) < H^2(\partial M)$ vanishes.

As $H^2(\partial M) = \bigoplus_{i=1}^k H^2(T_i)$, hence, $\iota^*\zeta_{j+1}^{(u_1,\ldots,u_j)} = \zeta_{j+1}^{(u_1,\ldots,u_j)} = 0$. The injectivity of $\iota^*$ follows from Lemma 3.3 and implies that $\iota^*(\zeta_{j+1}^{(u_1,\ldots,u_j)}) = 0$. Hence, the homomorphism can be extended to a deformation $(r_n \circ \rho)^{j+1}$ of order $j + 1$, and inductively to a formal deformation $(r_n \circ \rho)^\infty$.

Applying [6, Proposition 3.6] to the formal deformation $(r_n \circ \rho)^\infty$ results in a convergent deformation. Hence, $r_n \circ \rho$ is a smooth point of the representation variety.

As in [6], we note that the exactness of

$$1 \to H^0(\Gamma; \mathfrak{sl}(n)_{r_n,\rho}) \to \mathfrak{sl}(n)_{r_n,\rho} \to B^1(\Gamma; \mathfrak{sl}(n)_{r_n,\rho})$$

implies that

$$\dim B^1(\Gamma; \mathfrak{sl}(n)_{r_n,\rho}) = n^2 - 1 - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n,\rho}).$$

Thus, we conclude that the local dimension of $R(\Gamma, \text{SL}(n))$ is

$$\dim Z^1(\Gamma; \mathfrak{sl}(n)_{r_n,\rho}) = (n + 1 + k)(n - 1) - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n,\rho}).$$

That it is in a unique component follows from [6, Lemma 2.6].

4. Deforming $\rho_{\lambda,n}$

We will now show that $\rho_{\lambda,n}$ satisfies the conditions in Proposition 3.4 so that $\rho_{\lambda,n}$ can be deformed within a neighborhood of representations. This will entail a computation of the dimension of the cohomology group $H^1(\Gamma; \mathfrak{sl}(n)_{\rho_{\lambda,n}})$. By the decomposition in Lemma 2.3 the cohomology group $H^1(\Gamma; \mathfrak{sl}(n)_{\rho_{\lambda,n}})$ is a direct sum

$$H^1(\Gamma; \mathfrak{sl}(n)_{\rho_{\lambda,n}}) \cong \bigoplus_{j=1}^{n-1} H^1(\Gamma; R_{2j}),$$

so it suffices to compute the dimensions of $H^1(\Gamma; R_{2j})$, for $1 \leq j \leq n - 1$.

To simplify the computations which follow, we give a presentation of $\Gamma$ with an additional generator $\gamma_{2g+p}$. We will choose $\gamma_1, \ldots, \gamma_{2g}$ to be standard generators of the fundamental group for the closed surface $S_g$, and $\gamma_{2g+1}, \ldots, \gamma_{2g+p}$ to be curves around the $p$ punctures of $S$. Then $\pi_1(\Gamma)$ has a presentation of the form:

$$\langle \gamma_1, \ldots, \gamma_{2g+p}, \tau \gamma_i \tau^{-1} = \phi(\gamma_i), \Pi_{t=1}^p [\gamma_{2t-1}, \gamma_{2t}] = \Pi_{s=1}^p [\gamma_{2g+s}] \rangle.$$

With these generators for $\pi_1(S)$, $\phi^*: H^1(S) \to H^1(S)$ can be written as a block matrix

$$\begin{pmatrix} [\phi^*] & [\ast] \\ 0 & [P] \end{pmatrix}$$
where $\tilde{\phi}^* : H^1(\tilde{S}) \to H^1(\tilde{S})$ is the induced map on the first cohomology of the closed surface $\tilde{S}$ obtained by filling in the $p$ punctures of $S$, and $P = (p_{i,j})$ is a permutation matrix denoting the permutation of the punctures of $S$ under $\phi$. In particular, $p_{jk} = 1$ if and only if $\tau \delta_j \tau^{-1}$ is conjugate to $\delta_k$, and $p_{jk} = 0$ otherwise. The matrix $\tilde{\phi}^*$ is a symplectic matrix preserving the intersection form $\omega$ on $\tilde{S}$. The eigenvalues of $P$ are roots of unity, with 1 occurring as an eigenvalue once for each cycle in the permutation.

We now compute the cohomological dimension of $H^1(\Gamma_\phi; R_{2j})$. The argument uses similar ideas to [8, Theorem 4.1] using the generators $X^\ell \cdot Y^{2j-\ell}$, $\ell = 0, \ldots, 2j$, of $R_{2j}$ and is equivalent up to a coordinate change when $j = 1$.

**Proposition 4.1.** Let $\phi : S \to S$ be a homeomorphism, with $\lambda^2$ a simple eigenvalue of $\phi^*$. Suppose also that $|\lambda| \neq 1$, $\tilde{\phi}^* : H^1(\tilde{S}) \to H^1(\tilde{S})$ does not have 1 as an eigenvalue, and for each $2 \leq j \leq n-1$, we have that $\lambda^{2j}$ is not an eigenvalue of $\phi^*$. Then for each $j$, $1 \leq j \leq n-1$, $\dim H^1(\Gamma_\phi; R_{2j}) = k$ where $k$ is the number of components of $\partial M_\phi$.

**Proof.** Let $z \in Z^1(\Gamma_\phi, R_{2j})$. Then $z$ is determined by its values on $\gamma_1$, $\ldots$, $\gamma_{2g+p}$, and $\tau$, subject to the cocycle condition [8.1] imposed by the relations in $\Gamma_\phi$. These can be computed via the Fox calculus [9, Chapter 3]. Differentiating the relations

$$\tau \gamma_i \tau^{-1} = \phi(\gamma_i),$$

yields

$$\begin{align*}
\frac{\partial[\phi(\gamma_i)\tau \gamma_i^{-1} \tau^{-1}]}{\partial \gamma_i} &= \frac{\partial \phi(\gamma_i)}{\partial \gamma_i} - \phi(\gamma_i) \tau \gamma_i^{-1} = \frac{\partial \phi(\gamma_i)}{\partial \gamma_i} - \tau \\
\frac{\partial[\phi(\gamma_i)\tau \gamma_i^{-1} \tau^{-1}]}{\partial \gamma_h} &= \frac{\partial \phi(\gamma_i)}{\partial \gamma_h}, i \neq h \\
\frac{\partial[\phi(\gamma_i)\tau \gamma_i^{-1} \tau^{-1}]}{\partial \tau} &= \phi(\gamma_i) - \phi(\gamma_i) \tau \gamma_i^{-1} \tau^{-1} = \phi(\gamma_i) - 1.
\end{align*}$$

A cocycle $z$ then must satisfy the set of equations for $1 \leq i \leq 2g + p$ of the form

$$\sum_{h=1}^{2g+p} \frac{\partial[\phi(\gamma_i)\tau \gamma_i^{-1} \tau^{-1}]}{\partial \gamma_h} \cdot z(\gamma_h) + \frac{\partial[\phi(\gamma_i)\tau \gamma_i^{-1} \tau^{-1}]}{\partial \tau} \cdot z(\tau) = 0. \tag{4.2}$$

With respect to the basis $X^0 Y^{2j}$, $X^1 Y^{2j-1}$, $\ldots$, $X^{2j} Y^0$ for $R_{2j}$, the values $z(\gamma_i)$ can be expressed in coordinates $(x_{i,\ell})$, where $x_{i,\ell}$ is the coefficient of $X^\ell Y^{2j-\ell}$ for $z(\gamma_i)$. We similarly express $z(\tau)$ in the coordinates $x_{0,\ell}$, $0 \leq \ell \leq 2j$ with $x_{0,\ell}$ being the $X^\ell Y^{2j-\ell}$ coefficient of $z(\tau)$. Direct calculation
shows that
\[(4.3) \quad \rho(\gamma_i) \cdot X^\ell Y^{2j-\ell} = (X - a_i Y)^\ell Y^{2j-\ell} = \sum_{m=0}^\ell (-a_i)^m \binom{\ell}{m} X^{\ell-m} Y^{2j-\ell+m}, \]
\[\rho(\tau) \cdot X^\ell Y^{2j-\ell} = (\lambda^{-1} X)^\ell (\lambda Y)^{2j-\ell} = \lambda^{2j-2\ell} X^\ell Y^{2j-\ell}. \]

The set of coboundaries can be computed from Equation (3.2) as the set of cocycles \(z'\) satisfying,
\[
z'(\gamma_i) = \sum_{\ell=0}^{2j} b_\ell X^\ell Y^{2j-\ell} - b_\ell (X - a_i Y)^\ell Y^{2j-\ell} = \sum_{\ell=0}^{2j} \sum_{m=1}^\ell -b_\ell (-a_i)^m \binom{\ell}{m} X^{\ell-m} Y^{2j-\ell+m},
\]
\[z'(\tau) = \sum_{\ell=0}^{2j} (b_\ell - \lambda^{2j-2\ell} b_\ell) X^\ell Y^{2j-\ell}, \]

where \(b_0, \ldots, b_{2j} \in \mathbb{C}\) parametrize the set \(B^1(\Gamma_\phi, R_{2j})\) of coboundaries. In particular, adding the appropriate coboundary \(z'\) to \(z\), we can assume \(x_{0,\ell} = 0\) for \(\ell \neq j\), so that \(z(\tau)\) has the form
\[
z(\tau) = x_{0,j} X^j Y^j. \]

Then \(z\) is determined by a vector
\[
\vec{v} = (x_{1,0}, \ldots, x_{2g+p,0}, \ldots, x_{0,j}, x_{1,j}, \ldots, x_{2g+p,j}, \ldots, x_{1,2j}, \ldots, x_{2g+p,2j})^T
\]
in the kernel of a block matrix \(A = (A_{\alpha,\beta})\) where the entries in the \(i\)-th row of \(A_{\alpha,\beta}\) are the coefficients of the terms \(x_{s,\beta} X^\alpha Y^{2j-\alpha}\) in Equation (4.2).

Since the image under \(\rho\) of any word \(w\) in \(\{\gamma_i, \gamma_i^{-1}\}^{2g+1}\) has the form
\[
\rho(w) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}
\]
for some \(a \in \mathbb{C}\), then the previous calculations in Equations (4.3) imply that \(A_{\alpha,\beta} = 0\) for \(\beta < \alpha\). Moreover, when \(\alpha \neq j\), \(A_{a,\alpha}\) is a square matrix, and we note that the coefficient of \(X^\alpha Y^{2j-\alpha}\) in \(\rho(\gamma_i) \cdot X^\alpha Y^{2j-\alpha}\) is 1, so that in Equation (4.2), the coefficient of \(x_{b,a}\) in the \(X^\alpha Y^{2j-\alpha}\) term is the signed number of times that \(\gamma_i\) appears in the word \(\phi(\gamma_i)\). In addition, Equation (4.2) will contain a single \(-\tau \cdot z(\gamma_i)\) term, so that \(A_{a,a} = \phi^s - \lambda^{2j-2\alpha} I\) when \(\alpha \neq j\). We also see that
\[
A_{j,j} = \begin{pmatrix} 0 & \phi^s - I \\ \vdots & \phi^s - I \\ 0 & \phi^s - I \end{pmatrix}.
\]
Since \( z(\tau) = x_{0,j}X^jY^j \), direct calculation also shows that for some matrix \( K \),
\[
A_{j-1,j} = \begin{pmatrix}
-j\lambda^2a_1 & 0 \\
\vdots & K \\
-j\lambda^2a_{2g+p} & 0
\end{pmatrix}.
\]

As \( \lambda^2 \) is a simple eigenvalue, \( \phi^* \) is symplectic, and the eigenvalues of \( P \) are roots of unity, \( \phi^* - \lambda^2I \) and \( \phi^* - \lambda^{-2}I \) have 1 dimensional kernel. Furthermore, since 1 is not an eigenvalue of \( \bar{\phi}^* \), \( \phi^* - I \) has kernel whose dimension is equal to the number of disjoint cycles of the permutation of the punctures. This is equal to the number of components of \( \partial M_\phi \). In addition, since \( \lambda^{2j-2\alpha} \) is not an eigenvalue of \( \phi^* \) for \( \alpha \neq j-1, j, 1 \), the kernel of \( A_{\alpha,\alpha} \) is trivial in these cases. Hence, the kernel of \( A \) has dimension at most \( 2 + k + 1 \), where

\[
k = \# \text{ of components of } \Sigma = \# \text{ of components of } \partial M_\phi.
\]

The additional dimension comes from the possible contribution to the kernel from the first column of \( A_{(j-1),j} \). Consider the submatrix
\[
U = \begin{pmatrix}
A_{j-1,j-1} & A_{j-1,j} \\
0 & A_{j,j}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\phi^* - \lambda^2I & -j\lambda^2a_1 & K \\
\vdots & \vdots & \vdots \\
0 & -j\lambda^2a_{2g+n} & 0 \\
0 & \vdots & \phi^* - I
\end{pmatrix}.
\]

If \( \text{null}(A) > 2 + k \), then we must have that \( \text{null}(U) > k + 1 \).

Since \( \lambda^2 \) is a simple eigenvalue of \( \phi^* \) and \( (a_1, \ldots, a_{2g+p})^T \) is an eigenvector of the \( \lambda^2 \) eigenspace, \( (a_1, \ldots, a_{2g+p})^T \) is not in the image of \( \phi^* - \lambda^2I \). Hence, for any \( x = (x_{1,j}, \ldots, x_{2g+p,j})^T \) in the kernel of \( \phi^* - I \), there is a unique \( x_{0,j} \) such that \( Kx - x_{0,j}(a_1, \ldots, a_{2g+p})^T \) is in the image of \( \phi^* - \lambda^2I \). Therefore, \( \text{null}(U) = k + 1 \).

Hence \( \text{null}(A) = 2 + k \). However, the solution arising from the kernel of \( \phi^* - \lambda^2I \) is the eigenvector
\[
(0, \ldots, 0, x_{1,j}, \ldots, x_{2g+p,j}, 0, \ldots, 0)^T = (0, \ldots, 0, a_1, \ldots, a_{2g+p}, 0, \ldots, 0)^T
\]
which is a coboundary. So we have that \( \dim H^1(\Gamma_\phi; R_{2j}) \leq k + 1 \). Finally, there is one further redundancy since
\[
\Pi^p_{i=1}[\gamma_{2i-1}, \gamma_{2i}] = \Pi^p_{s=1} \gamma_{2g+s}.
\]

From the \( \phi^* - I \) in \( A_{j,j} \), we can see that \( x_{j,2g+1}, \ldots, x_{j,2g+p} \) can be freely chosen as long as \( x_{j,2g+s} = x_{j,2g+t} \) whenever \( \gamma_{2g+s} \) and \( \gamma_{2g+t} \) are in the same
cycle of $P$. Since $|\lambda| \neq 1$, for any eigenvector of $\phi^*$, $a_{2g+1} = \cdots = a_{2g+p} = 0$, so the $X^jY^j$ coefficient of $z(\Pi_{s=1}^g \gamma_{2g+s})$ can be chosen to be any quantity

$$x_{j,2g+1} + \cdots + x_{j,2g+p}.$$ (4.4)

The relation $\Pi_{i=1}^g [\gamma_{2i}, \gamma_{2i+1}] = \Pi_{i=1}^g [\gamma_{2i}, \gamma_{2i+1}]$ relates the sum in Equation (4.4) to the $X^jY^j$ coefficient of $\Pi_{i=1}^g [\gamma_{2i}, \gamma_{2i+1}]$, which has no dependence on $x_{j,2g+s}$, for $1 \leq s \leq p$. This imposes a 1-dimensional relation on the space of cocycles, and we conclude that

$$\dim H^1(\Gamma_\phi, R_{2j}) = k.$$ □

We now prove Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** By Lemma 2.3, $sl(n)$ is the direct sum of $R_{2j}$, $j = 1, \ldots, n-1$. The conditions on the eigenvalues of $\phi^*$ and Proposition 4.1 imply that for each $j$, $\dim H^1(\Gamma_\phi; R_{2j}) = k$. Hence $\dim H^1(\Gamma_\phi, sl(n)_{\rho_{\lambda,n}}) = k(n-1)$. By Proposition 3.4, this implies smoothness of $R(\Gamma_\phi, SL(n))$ at $\rho_{\lambda,n}$. Since $\rho_{\lambda,n}$ is non-abelian, it has trivial infinitesimal centralizer, so $H^0(\Gamma_\phi; R_2) = 0$, so that the local dimension is $(n + 1 + k)(n - 1)$. □

We obtain the special case in Theorem 1.2 when $\lambda^2$ is the dilatation of a pseudo-Anosov map $\phi$.

**Proof of Theorem 1.2.** When the stable and unstable foliations of $\phi$ are orientable, it is a well-known fact that the dilatation is a simple eigenvalue and the largest eigenvalue of $\phi^*$ (see [4], [10], [12]). Hence, $\phi$ satisfies the conditions of Theorem 1.1.

From [8], we know that there are hyperbolic deformations of $\rho_\lambda = \rho_{\lambda,2}$, which are irreducible representations since they correspond to hyperbolic structures. The composition of these deformations with the irreducible representation $r_n$ then provides nearby deformations of $\rho_{\lambda,n}$ which are also irreducible. □

### 5. Description of Deformations

Recall that the action of $\Gamma_\phi$ on $sl(n)$ is given by composing $\rho_{\lambda,n}$ with the adjoint representation. That is, for $\gamma \in \Gamma_\phi$ and $c \in sl(n)$,

$$\gamma \cdot c = Ad_{\rho_{\lambda,n}(\gamma)}(c) = \rho_{\lambda,n}(\gamma)c\rho_{\lambda,n}(\gamma)^{-1}.$$ 

Let $E_j$ denote the $j$-th standard basis vector for $\mathbb{C}^n$. Then every element of $sl(n)$ is a linear combination of the matrices $E_jE_\ell^T$. In order to obtain a useful description of the action of $\Gamma_\phi$ on $sl(n)$, it suffices to compute
the action of \( \gamma \) on \( E_{j,\ell} \) for a set of generators of \( \Gamma_\phi \). By direct calculation,

\[
\gamma_i \cdot E_{j,\ell} = r_n \left( \begin{array}{cc}
1 & a_i \\
0 & 1
\end{array} \right) E_j \cdot E^T \gamma \left( \begin{array}{cc}
1 & a_i \\
0 & 1
\end{array} \right)^{-1} = \lambda^{2(n-j+1)} E_j \cdot E^T \gamma \lambda^{-2(n-\ell+1)} = \lambda^{2(\ell-j)} E_{j,\ell}.
\]

Notably, the actions of \( \Gamma_\phi \) on the \((j, \ell)\)-coordinates of \( \mathfrak{sl}(n) \) have no contributions to all rows \( r \geq j \) and all columns \( c \leq \ell \). Applying analogous calculations as in the proof of Proposition \ref{proposition}, to the relations in \( \Gamma_\phi \), we find that if \( z : \Gamma_\phi \rightarrow \mathfrak{sl}(n)_{\rho_\lambda,n} \) is a cocycle and \( z_{j,\ell}(\gamma_i) \) is the \((j, \ell)\)-coordinate of \( z(\gamma_i) \), then the vector

\[
\vec{v}_{n,1} = \left( \begin{array}{c}
z_{n,1}(\gamma_1) \\
\vdots \\
z_{n,1}(\gamma_{2g+p})
\end{array} \right) = (z_{n,1}(\gamma_i))
\]

is a solution to \(( \phi^* - \lambda^{-2(n-1)} I ) \vec{v}_{n,1} = 0 \). Since \( \lambda^{-2(n-1)} \) is not an eigenvalue of \( \phi^* \), it follows that \( \vec{v}_{n,1} = 0 \).

Since \( \vec{v}_{n,1} = 0 \), when the relations in \( \Gamma_\phi \) applied to \( z \) are restricted to the \((n-1,1)\)-coordinate and the \((n,2)\)-coordinate, we obtain that \( \vec{v}_{n-1,1} = (z_{n-1,1}(\gamma_i)) \) and \( \vec{v}_{n,2} = (z_{n,2}(\gamma_i)) \) are solutions to \(( \phi^* - \lambda^{-2(n-2)} I ) \vec{v} = 0 \). A straightforward induction combined with Equations \ref{equation} then shows that \( z_{j,\ell}(\gamma_i) = 0 \) for all \( j \geq \ell + 1 \) while \( \vec{v}_{j,\ell} = (z_{j,\ell}(\gamma_i)) \) is a \( \lambda^{-2}\)-eigenvector of \( \phi^* \) when \( j = \ell + 1 \), i.e. the subdiagonal entries of \( z(\gamma_i) \in \mathfrak{sl}(n) \) are coordinates from eigenvectors of \( \phi^* \), and all other entries below the diagonal are 0. This provides \( n-1 \) generators of cocycles. The others come from the 1-eigenspaces of \( \phi^* \) when applying the cocycle conditions to the diagonal entries of \( z(\gamma_i) \).

We have that \( \mathfrak{sl}(n) \) can be associated with the tangent space to \( \text{SL}(n) \) at the identity, and multiplying \( z(\gamma_i) \) by \( \rho_{\lambda,n}(\gamma_i) \) gives the derivative at \( \rho_{\lambda,n}(\gamma_i) \). The previous calculations then imply that if \( \rho_t : \Gamma_\phi \rightarrow \text{SL}(n) \) is a path of representations such that \( \rho_0 = \rho_{\lambda,n} \), then the subdiagonal entries of \( \rho'_t(\gamma_i) \)
Figure 1. The curves $\alpha_1, \alpha_2, \beta_1, \beta_2$ which form the basis for $H_1(S)$, and $\gamma$.

at $t = 0$ are equal to the subdiagonal entries of $z(\gamma_i)$. Hence, for each $j, \ell$, there exists at least one $i$ for which $z_{j,\ell}(\gamma_i) \neq 0$.

Note that in the case that $\lambda^2$ is the dilatation of a pseudo-Anosov map $\phi$ as in Theorem 1.2, the subdiagonal entries of the irreducible representations obtained by deforming $\rho_\lambda$ in $\text{SL}(2)$ and composing with $r_n$ to obtain a deformation of $\rho_{\lambda,n}$ necessarily satisfy certain relations. In particular, the first derivatives of the subdiagonal entries would have to be fixed multiples of entries of the $\lambda^{-2}$-eigenvector determined by the irreducible representation $r_n$. As described above, the deformations in $\text{SL}(n)$ allow the derivatives to be freely chosen multiples of the $n - 1$ generators, so there are deformations which are not from deformations of $\rho_\lambda$ that are composed with $r_n$.

6. Example

The genus 2 example $\phi : S_{2,2} \to S_{2,2}$ from [8], obtained from taking the left Dehn twists $T_{\beta_1}, T_{\beta_2}, T_\gamma$, followed by the right Dehn twists $T_{\alpha_1}^{-1}, T_{\alpha_2}^{-1}$, satisfies the hypotheses of Theorem 1.2. Each component of $S_2 \setminus \{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma\}$ contains one of the two punctures. The map on cohomology $\phi^*$ has two simple eigenvalues $\lambda_1^2 = \frac{3 + \sqrt{5}}{2}$ and $\lambda_2^2 = \frac{3 - \sqrt{5}}{2}$, along with their reciprocals $\lambda_1^{-2}$ and $\lambda_2^{-2}$. The reducible representations $\rho_{\lambda_1,n}$ are smooth points of $R(\Gamma_\phi, \text{SL}(n))$, each on a component of dimension $(n + 3)(n - 1)$. There is a two-dimensional family of irreducible representations in $X(\Gamma_\phi, \text{SL}(n))$, which is the image of a two-dimensional family of irreducible representations in $X(\Gamma_\phi, \text{SL}(2))$ under $r_n$, limiting to $\rho_{\lambda_1,n}$.

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