Phase transitions of bosonic fractional quantum Hall effect in topological flat bands

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We study the phase transitions of bosonic $\nu = 1/2$ fractional quantum Hall (FQH) effect in different topological lattice models under the interplay of onsite periodic potential and Hubbard repulsion. Through exact diagonalization and density matrix renormalization group methods, we demonstrate that the many-body ground state undergoes a continuous phase transition between bosonic FQH liquid and a trivial (Mott) insulator induced by the periodic potential, characterized by the smooth crossover of energy and entanglement entropy. When the Hubbard repulsion decreases, we claim that this bosonic FQH liquid would turn into a superfluid state with direct energy level crossing and a discontinuous leap of off-diagonal long-range order.

I. INTRODUCTION

In the past several decades the emergence of topological phases of matter beyond the Landau’s paradigm open up an innovatory chapter in modern condensed matter physics [1]. One of the paramount topics is related to the phase transitions among these different topological ordered phases such as fractional quantum Hall (FQH) effect [2] and spin liquid [3]. These topological ordered phases, which are characterized by long-range entanglement without Landau’s symmetry-breaking order parameters, usually host a well-defined topological invariant [4], e.g. fractional Hall conductance of the quantum Hall systems. Therefore it is a intricate problem about the phase transition between a topological ordered phase and a symmetry-broken phase, which inspires much interest in its transition nature.

Indeed, a lot of theoretical studies on quantum Hall transitions between quantum Hall phases and topologically trivial phases, which have enlarged the domain of phase transition physics, are in bloom [4–10]. Generally, there exist two possible scenarios for phase transitions separating a topological ordered phase with another trivial phase by (i) a first-order transition such as a transition between two distinct ground states of an Ising quantum Hall ferromagnet [11], and (ii) a second-order phase transition, for instance Landau-forbidden transitions between a bosonic integer quantum Hall liquid and trivial insulator in two dimensions [12, 13]. Of particular interest, under the periodic chemical potential, a continuous transition between a fractional quantum Hall liquid at weak potentials and a Mott insulator at strong potentials is claimed in Refs. [14, 15] on the basis of effective field theory. In contrast for disordered potentials, the localization/delocalization transition between plateaux in the fermionic integer quantum Hall system is shown to exhibit the universality class [16, 17]. In the presence of a spatial symmetry, it is argued that there may be a direct continuous transition between the bosonic $\nu = 1/2$ FQH state and the bosonic superfluid [18, 20].

However much less knowledge is gained about the quantum Hall physics in lattice. The topological flat bands without magnetic fields have become an exciting platform for studying the quantum Hall effect, with fractionalised topological phases predicted at partial fillings, dubbed “fractional Chern insulators” [21–25]. In cold atomic gases, topological Haldane-honeycomb lattice has been obtained in periodically driven optical lattice [26], and topological Harper-Hofstadter Hamiltonian is obtained using either laser-assisted tunneling in neutral $^{87}$Rb atoms [27, 28] or synthetic dimension in alkaline-earth-metal-like atoms with multiple internal degrees of freedom [29, 30]. Moreover in multi-layer systems, tunable Chern insulators under moiré superlattice potential have been achieved [31, 32]. In Ref. [33], the existence of fractionalised interacting phases is experimentally confirmed, and the phase transitions between these quantized phases are mapped out. Such continuous transitions between different fractional quantum Hall states induced by periodic potentials have been intensively discussed regarding different Chern insulators in Ref. [34] with fascinating QED properties at the critical points. These experimental progresses hold promise for exploring exotic phase transition physics in fractional quantum Hall systems, such as the phase transition between FQH and other trivial phases in interacting Harper-Hofstadter model [35].

In this work, we focus on the phase transition physics for softcore bosons in concrete topological lattice models at a filling factor $\nu = 1/2$ under the interplay of periodic potential and Hubbard repulsion. For strong Hubbard repulsion, Laughlin-like fractional quantum Hall effect is believed to emerge [23]. Upon the addition of periodic potential or the softening of Hubbard repulsion, we elucidate the phase transition nature between different competing phases through state-of-the-art density-matrix renormalization group (DMRG) and exact diagonalization (ED) simulations.

The rest of this paper is organized as follows. In Sec. 1 we give a description of the Bose-Hubbard model Hamiltonian with periodic potential in different topological lattice models, such as π-flux checkerboard and Haldane-honeycomb lattices. In Sec. II we present the numerical results for the FQH-insulator transition induced by periodic potential. Further in Sec. III we present the numerical results for the FQH-superfluid transition when
Hubbard repulsion is decreased. Finally, in Sec. VI we summarize our results and discuss the prospect of investigating topological phase transitions in flat band systems.

II. MODELS AND METHODS

We begin with the following noninteracting hopping Hamiltonian in two typical topological lattice models, as shown in Fig. 1(a) and 1(b),

\[
H_{CB}^0 = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \left[ b^\dagger_{\mathbf{r}} b_{\mathbf{r}'} \exp(i \phi_{\mathbf{r}, \mathbf{r}'} + H.c.] 
- \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} t'_{\mathbf{r}, \mathbf{r}'} b^\dagger_{\mathbf{r}} b_{\mathbf{r}'} - t'' \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} b^\dagger_{\mathbf{r}} b_{\mathbf{r}'} + H.c.,
\]

\[
H_{HC}^0 = -t' \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \left[ b^\dagger_{\mathbf{r}} b_{\mathbf{r}'} \exp(i \phi_{\mathbf{r}, \mathbf{r}'} + H.c.] 
- t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} b^\dagger_{\mathbf{r}} b_{\mathbf{r}'} - t'' \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} b^\dagger_{\mathbf{r}} b_{\mathbf{r}'} + H.c.,
\]

where \( H_{CB}^0 \) denotes the \( \pi \)-flux checkerboard (CB) lattice and \( H_{HC}^0 \) the Haldane-honeycomb (HC) lattice. Here \( b^\dagger_{\mathbf{r}}(b_{\mathbf{r}}) \) is the particle creation (annihilation) operator at site \( \mathbf{r} \), \( \langle \ldots \rangle \) and \( \langle\langle \ldots \rangle\rangle \) denote the nearest-neighbor, the next-nearest-neighbor, and the next-next-nearest-neighbor pairs of sites, respectively. In the flat band parameters, we choose \( t' = 0.3t, t'' = -0.2t, \phi = \pi/4 \) for checkerboard lattice, while \( t' = 0.6t, t'' = -0.58t, \phi = 2\pi/5 \) for honeycomb lattice, as in Refs. 25, 39.

We take the Bose-Hubbard repulsion as \( V_{int} = U/2 \sum_{\mathbf{r}} n_\mathbf{r} (n_\mathbf{r} - 1) \), where \( U \) is the onsite interaction strength and \( n_\mathbf{r} = b^\dagger_{\mathbf{r}} b_{\mathbf{r}} \) is the particle number operator.

![FIG. 1. (Color online) The schematic plot of (a) The \( \pi \)-flux checkerboard lattice model in Eq. \ref{eq:cb} and (b) the Haldane-honeycomb lattice model in Eq. \ref{eq:hc}. The arrow directions present the signs of the phases \( \phi \) in the hopping terms. Sublattice A (B) is labeled by blue (red) filled circles. The arrow link shows the hopping direction carrying chiral flux \( \phi_{\mathbf{r}, \mathbf{r}'} \). For the checkerboard lattice, the next-nearest-neighbor hopping amplitudes are \( t'_{\mathbf{r}, \mathbf{r}'} = \pm t' \) along the solid (dotted) lines. \( e_{x,y} \) indicate the real-space lattice translational vectors. The yellow dotted box depicts the periodic potential \( \mu_j \) with \( \mu_A = 0, \mu_A' = \mu_B = \mu_B' = \mu \).](image)

In what follows, we would numerically address the many-body ground states of interacting bosons in the presence of periodic potentials, and the full Hamiltonian is written as

\[
H_{CB} = H_{CB}^0 + U \sum_{\mathbf{r}} n_\mathbf{r} (n_\mathbf{r} - 1) + \sum_{\mathbf{r}} \mu_\mathbf{r} n_\mathbf{r}, \quad (1)
\]

\[
H_{HC} = H_{HC}^0 + U \sum_{\mathbf{r}} n_\mathbf{r} (n_\mathbf{r} - 1) + \sum_{\mathbf{r}} \mu_\mathbf{r} n_\mathbf{r}, \quad (2)
\]

where periodic potential \( \mu_\mathbf{r} \) is chosen with commensurate period two: \( \mu_\mathbf{r} = 0 \) for A sites while \( \mu_\mathbf{r} = \mu \) for B, A', B' sites within each unit cell, as shown in Fig. 1(a) and 1(b).

In the ED study, we study a finite system of \( N_x \times N_y \) unit cells (the total number of sites is \( N_s = 2 \times N_x \times N_y \)). The total filling of the lowest Chern band is \( \nu = N/(N_x N_y) \) with global \( U(1) \)-symmetry. With the translational symmetry, the energy states are labeled by the total momentum \( K = (K_x, K_y) \) in units of \( (2\pi/N_x, 2\pi/N_y) \) in the Brillouin zone. While the ED calculations on the periodic lattice are limited to a small system, we exploit infinite DMRG for larger systems on cylinder geometry with open boundary conditions in the \( x \)-direction and periodic boundary conditions in the \( y \)-direction. We keep the bond dimension up to 3600 to obtain accurate results for different system sizes.

III. NUMERICAL RESULTS FOR FQH-INSULATOR TRANSITION

In this section, we present our numerical results for the transition between a FQH liquid and a Mott insulator induced by periodic potential \( \mu \) at a given filling \( \nu = 1/2 \) for bosons. For strong Hubbard repulsion the system is known to fall into the \( \nu = 1/2 \) FQH state at \( \mu = 0 \), as demonstrated in Ref. 25. When \( \mu \) increases, the \( \nu = 1/2 \) FQH would be overwhelmed by a trivial insulator where the particles are localized at strong \( \mu/t \gg 1 \). In the following parts, we will discuss topological phase transition from several aspects including ground state degenerate manifold and entanglement entropy.

A. ED results

We first present an ED study of the ground state properties in Eqs. \ref{eq:cb} and \ref{eq:hc} with hardcore limit \( U/t = \infty \). In Fig. 2(a) and 2(b), we plot the low energy evolution as a function of on-site periodic potential \( \mu \) for various system sizes in different topological lattices. For weak potentials, there always exists two nearly-degenerate ground states with a large gap separated from higher excited levels, which is the hallmark characteristic of a FQH liquid. By tuning \( \mu \) from weak to strong, the ground states do not undergo the level crossing with excited levels for different system sizes, signaling a continuous phase transition nature. Across a threshold value \( \mu = \mu_c \), these
two-fold ground states split and a unique ground state is left in the Mott insulator for \( \mu > \mu_c \).

Next, we examine the change of single-particle entanglement entropy for interacting \( N \)-particle systems as a function of on-site periodic potential \( \mu \). Here we diagonalize the reduced single-particle density matrix \( \hat{\rho}_{r,r'} = \langle \phi_r | \hat{h}^\dagger \hat{h} | \phi_{r'} \rangle \) with \( N_s \times N_s \) elements, and obtain single-particle eigenstates \( |\phi_{j}\rangle = n_j |\phi_j\rangle \) where \( |\phi_j\rangle (j = 1, \ldots, N_s) \) are the natural orbitals and \( n_j \) (\( n_1 \geq \ldots \geq n_{N_s} \)) are interpreted as occupations. And the single-particle entanglement entropy is defined as

\[
S(N) = - \sum_{j=1}^{N_s} n_j \ln n_j. \tag{3}
\]

For \( \nu = 1/2 \) FQH liquid, the particles uniformly occupy the lowest Chern band with the occupations \( n_j \approx 1/2 \) for \( j \leq N_s/2 \) and \( n_j \ll 1 \) for \( j > N_s/2 \). Thus \( S(N) \) becomes a universal constant \( N \times \ln 2 \). However, increasing periodic potential \( \mu \) leads to the splitting of the lowest band into two subbands, and the particles tend to occupy the lower subband with \( n_j \approx 1 \) for \( j \leq N = N_s/4 \) and \( n_j \ll 1 \) for \( j > N \), in order to minimize the total energy. For the Mott insulator, the \( N \) particles are localized at certain \( N \) sites, and \( S(N) \) tends to zero for strong \( \mu/|t| \geq 1 \). As shown in Fig. 3(a), the single-particle entanglement entropy \( S(N) \) evolves smoothly from weak \( \mu \) to strong \( \mu \), which serves as another signature of continuous FQH-insulator transition.

Moreover we continue to discuss the insulating behavior against particle excitations across the FQH-insulator transition. As is well-known, both \( \nu = 1/2 \) FQH liquid and Mott insulator are incompressible phases characterized by the presence of the charge-hole gap. Here we calculate the charge-hole gap \( \Delta_c(\mu) = \langle E_{N+1}(\mu) + E_{N-1}(\mu) - 2E_N(\mu) \rangle/2 \) where \( E_N(\mu) \) is the ground energy for interacting \( N \)-particle systems. As shown in Fig. 3(b), \( \Delta_c \) shrinks as \( \mu \) increases from zero to \( \mu_c \), implying the softening of \( \nu = 1/2 \) FQH liquid, while it tends to dilate for strong \( \mu > \mu_c \), which scales as \( \Delta_c(\mu) \propto \mu \) when the particle excitation is controlled by the periodic potential. Nevertheless, \( \Delta_c(\mu) \) hosts a nonzero minimum cusp at the threshold point \( \mu = \mu_c \) for different system sizes, indicating the continuous transition. However, it should be careful to extract the excitation information in the thermodynamic limit. According to the construction of effective QED\(_3\)-Chern-Simons theory \(^{37}\), this critical point should be described by one Dirac fermion coupled to a gauge field and the gap closing should happen at certain high symmetry momentum point like the \( \Gamma \) point \( K = (0,0) \) in the Brillouin zone as indicated in Fig. 2 for all system sizes. Thus we expect the exciton gap in Figs. 2 and 3(b), due to finite size effect, would collapse at the critical point \( \mu = \mu_c \) in the two-dimensional limit, which may be explored in future.

**B. DMRG results**

Following the last section, we move on to discuss the continuous transition between \( \nu = 1/2 \) FQH liquid and Mott insulator from the perspective of DMRG simulation. Here we exploit an unbiased DMRG approach for larger system sizes, using a cylindrical geometry up to a maximum width \( N_y = 8 \). For topological ordered phases like \( \nu = 1/2 \) FQH liquid, they host nonlocal anyons with long-range entanglement in the ground state, reflected in the topological entanglement entropy. As pointed out in Refs. \(^{40, 41}\), the entanglement entropy \( S_L \) of a partitioned subsystem of a gapped two-dimensional system.
satisfies the volume area
\[ S_L(\ell) = \alpha \ell - \gamma + \cdots , \tag{4} \]
where \( \ell \) is the boundary length of the subsystem. The topological entanglement entropy \( \gamma = \ln(D) \) is a universal constant with \( D \) the total quantum dimension, i.e. \( D = \sqrt{2} \) for Laughlin \( \nu = 1/2 \) FQH liquid. In our infinite DMRG, we divide the cylinder into left and right parts along the \( x \) direction \( 42 \), and calculate the entanglement entropy of the left part as \( S_L(\ell) \) with the boundary length \( \ell = 2N_y \).

We calculate the entanglement entropy \( S_L \) for three different widths \( N_y = 4, 6, 8 \), which varies smoothly as \( \mu \) increases. In the Mott insulator, there is no fractionalization of anyons, and the topological entanglement entropy \( \gamma = 0 \). As shown in Fig. 4(a), we plot the first-order derivative of \( S_L \) as function of \( \mu \), which exhibits a local minimum at \( \mu = \mu_c \), indicating a continuous phase transition \( 43, 44 \).

Meanwhile, we scale \( S_L \) as a function of \( N_y \), and obtain the topological entanglement entropy \( \gamma \) for a given \( \mu \). As shown in Fig. 4(b), for \( \mu < \mu_c \), \( \gamma \) remains close to the theoretical value \( \sqrt{2} \), consistent with the prediction of \( \nu = 1/2 \) FQH liquid. However for \( \mu > \mu_c \), \( \gamma \) decreases monotonically down to zero as \( \mu \) increases. Together with \( S_L \), the analytic behavior of \( \gamma \) demonstrates the continuous quantum phase transition from \( \nu = 1/2 \) FQH liquid to a Mott insulator.

### IV. NUMERICAL RESULTS FOR FQH-SUPERFLUID TRANSITION

In this section, we move on to discuss the numerical results for the transition between a FQH liquid and bosonic superfluid induced by the softening of Hubbard repulsion \( U \) at a given filling \( \nu = 1/2 \) for bosons. For weak Hubbard repulsion \( U/t \ll 1 \), the \( \nu = 1/2 \) FQH should be destroyed and the weakly interacting bosons would condense into the lowest kinetic single-particle orbital, where the system becomes a gapless superfluid. In the following parts, we will discuss topological phase transition from several aspects including ground state degenerate manifold and the off-diagonal long-range order \( \langle b^*_r b_s \rangle \). Here to bypass the numerical difficulty, we take the maximum particle occupation per site \( N_{\text{max}} = 2 \) for Bose-Hubbard model in Eqs. 1 and 2, which is a very good approximation for the low lattice filling \( N/N_s = 1/4 \ll 1 \).

As indicated in Fig. 5(a) and 5(b) for different system sizes, we plot the low energy evolution at \( \nu = 1/2 \) on topological \( \pi \)-flux checkerboard lattice as a function of Hubbard repulsion \( U \). When \( U \) decreases, the energy gap protecting the two-fold ground state degeneracy diminishes, and finally near the transition point \( U \simeq U_c \).
FIG. 6. (Color online) Numerical ED results for the FQH-superfluid transition at $\nu = 1/2, \mu = 0$ on topological $\pi$-flux checkerboard lattice as Hubbard repulsion $U$ is decreased from strong to weak. The system size $N = 2 \times 4 \times 4$. The black dashed line labels the discontinuous transition point $U_c$ of off-diagonal long-range order (ODLRO).

the ground state undergoes the direct level crossing with another excited level, signaling a first-order phase transition nature. Across the transition point $U < U_c$, there are many low-lying excited energy levels and the system enters into a bosonic superfluid. Similarly, we also confirm that the level crossing is observed on topological Haldane-honeycomb lattice with a smaller $U_c$, regardless of detailed lattice geometry.

In order to demonstrate the first-order transition, we investigate the change behavior of the off-diagonal long-range order (ODLRO) related to bosonic superfluid. For finite system sizes, we define ODLRO as $\langle b_s^\dagger b_s \rangle$ where $s = |r|_{\text{max}}$ is the most remote distance relative to the original point $r = 0$ in periodic lattice. As shown in Fig. 6, when $U$ weakens, bosonic phase coherence is enhanced and a discontinuous jump of ODLRO appears at the transition point $U = U_c$, consistent with the level crossing in Fig. 6(b). Further, we calculate the charge-hole gap $\Delta_c(U) = |E_{N+1}(U) + E_{N-1}(U) - 2E_N(U)|/2$ which serves as a characterization tool between incompressible and compressible liquids. $\Delta_c$ decreases monotonously in the FQH region as $U$ is tuned down to $U_c$, and then for $U < U_c$, $\Delta_c$ hosts a very small value, i.e. of the order 0.01t limited by our finite system size. The nonanalytic behavior of $\Delta_c$ at $U = U_c$ is also consistent with the first-order transition.

Finally we remark that our detailed lattice models give a contrary deduction to the conclusion drawn by Ref. [19] where instead an effective continuous transition is derived. In the thermodynamic limit, it may be likely that the weakly first-order transition is smeared by disorder in realistic experimental environments, and a continuous transition intervenes.

V. SUMMARY AND DISCUSSIONS

In summary, we have studied the different competing phases under the interplay of interaction and periodic potential. We show that the continuous phase transition between bosonic $\nu = 1/2$ FQH liquid and a featureless (Mott) insulator is induced by the periodic potential, characterized by the smooth crossover of ground state energy and entanglement entropy. In contrast as the Hubbard repulsion decreases, we find that this bosonic FQH liquid would undergo a weakly first-order transition into a gapless superfluid with a sudden change of off-diagonal long-range order. Our softcore bosonic flat band models would be generalized to a larger class of interacting Hamiltonian featuring $\nu = 1/2$ FQH effect, such as interacting Harper-Hofstadter models, which is of sufficient feasibility to be realized for current experiments in cold atoms. [19] We emphasize that, actually the phase transitions between $\nu = 1/2$ FQH liquid and other competing phases tuned by periodic potential in interacting Harper-Hofstadter models are of the same nature as those in our flat band models, regardless of any details in lattice structure, demonstrating the universal physical picture in these phase transitions.

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