Thermodynamics of the bosonic randomized Riemann gas

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Abstract
The partition function of a bosonic Riemann gas is given by the Riemann zeta function. We assume that the Hamiltonian of this gas at a given temperature \( \beta^{-1} \) has a random variable \( \omega \) with a given probability distribution over an ensemble of Hamiltonians. We study the average free energy density and average mean energy density of this arithmetic gas in the complex \( \beta \)-plane. Assuming that the ensemble is made by an enumerable infinite set of copies, there is a critical temperature where the average free energy density diverges due to the pole of the Riemann zeta function. Considering an ensemble of non-enumerable set of copies, the average free energy density is non-singular for all temperatures, but acquires complex values in the critical region. Next, we study the mean energy density of the system which depends strongly on the distribution of the non-trivial zeros of the Riemann zeta function. Using a regularization procedure we prove that this quantity is continuous and bounded for finite temperatures.

Keywords: number theory, Riemann gas, random systems

1. Introduction
Arithmetic quantum theory has been developed to establish connections between number theory and quantum field theory [1–7]. A bosonic Riemann gas is a second quantized mechanical system at temperature \( \beta^{-1} \), with partition function given by the Riemann zeta function [8, 9]. As was discussed by Weiss and collaborators, the Hamiltonian for the Riemann gas can in principle be produced in a Bose–Einstein condensate [10]. The purpose of this article is to explore connections between the Riemann zeta function and thermodynamics using the arithmetic bosonic gas with quenched disorder. Introducing randomness in the bosonic Riemann gas and studying the thermodynamic variables of this arithmetic gas in the...
complex $\beta$-plane, the connection between the zeros of the Riemann zeta function and physics is established allowing new and interesting developments.

In equilibrium statistical mechanics, macroscopic systems can develop phase transitions when external parameters change. To give a mathematical description of these phenomena it is usual to define the free energy, $F_N(\beta)$, where $N$ is the number of particles of the system. For finite systems, the free energy and the partition function are analytic in the entire complex $\beta$-plane. In the thermodynamic limit, at points where analyticity is not preserved, one says that a phase transition occurs. The singularities in the free energy density corresponds to the zeros of the partition function in the complex $\beta$-plane.

In two papers, Lee and Yang [11, 12] studied the zeros of the partition function of the Ising model in the complex magnetic field. They obtained the following theorem: in the Ising model in a complex magnetic field $h$, the complex zeros of the partition function are located on the unit circle in the complex activity plane. Discussing the zeros in the complex temperature plane, Fisher studied a Ising model on a square two-dimensional lattice and showed that there is an accumulation of these complex zeros close to the critical point [13]. Systems with disorder and randomness also have been considered [14, 15]. A very simple example of a disordered system which exhibits a phase transition is the random energy model. The zeros of the partition function in the complex temperature plane in this model has been studied numerically and also analytically [16]. In disordered systems, Matsuda et al studied the distribution of zeros of the partition function in Ising spin glasses on the complex field plane [17]. More recently Takahashi and collaborators studied the zeros of the partition function in the mean-field spin-glass models [18–20].

For all these systems the disorder is a random variable in the Hamiltonian where the probability distribution is known. In the spin-glass the disorder appears in the coupling between neighbor spins, and in the random field Ising model the random variable is a quenched magnetic field. One way to investigate such models is to replace the original Hamiltonian of each model by an effective Hamiltonian of the Landau–Ginzburg model where the order parameter is a continuous $d$-dimensional field. In such case the disorder appears as a random temperature or a random external field. For the simplest case $d = 1$ we get the anharmonic oscillator with a random frequency $\omega$. The quenched free energy for this system without the anharmonic contribution can be calculated in terms of the derivative of a particular spectral zeta function $\zeta(s, \omega)$ at $s = 0$. This derivative has to be determined by the analytic continuation from the domain where the series actually converges. Due the particular spectrum of the arithmetic gas, the introduction of randomness in the Hamiltonian leads us to a quite interesting situation where the argument of the Riemann zeta function is a random variable with some probability distribution. Therefore, this perspective provides a connection between the theory of the Riemann zeta function and the physics of disorder systems. Although a probabilistic approach in number theory is not new in the literature [21, 22], as far as we know the approach presented here is discussed for the first time.

As we discussed above, we studied the thermodynamics of an arithmetic gas introducing randomness. We assume that the Hamiltonian of the bosonic gas contains a random variable with some probability distribution. The thermodynamic quantities must be calculated averaging over an ensemble of realizations of the random quantity. Since the random variable is a parameter in an extensive quantity, we have to perform ensemble average of some extensive quantity of interest, in our case the free energy [23, 24].

Assuming that the ensemble is made by a enumerable infinite set of copies, characterized by the parameter $\omega_k$ defined in the interval $\{\omega : 0 < \omega_k < 0}\) \in \mathbb{N}$, the first copy the ensemble with parameter $\omega_k$ provides a logarithmic singular contribution to the free energy density due to the pole of the Riemann zeta function. This temperature, where the average free
energy density diverges is the Hagedorn temperature \[25\] of the random system. On the other hand, considering an ensemble of non-enumerable set of copies, the singular behavior of the average free energy density disappears. Meanwhile, due to the behavior of the Riemann zeta function in the critical region, the average free energy density acquires complex values. Finally, we show that the non-trivial zeros of the Riemann zeta function, which are the Fisher zeros of the system, contribute to the average energy density of the system. For other approaches discussing the physics of Riemann zeros see \[26–29\]. Also, in the literature we can find other recent results connecting number theory and quantum field theory \[30–34\].

Note that it is possible to obtain the Riemann zeta function as the partition function of a prime membrane \[35\]. Let \( \mathcal{I} \) be the adelic infinite dimensional torus defined as the adelic product of circles of lengths \( \frac{1}{\ln 2}, \frac{1}{\ln 3}, \ldots, \frac{1}{\ln p}, \ldots \), where \( p \) ranges through the sequence of prime numbers. The normalized eigenfrequencies of \( \mathcal{I}_p \) are given by \( \nu_{m,p} = m \ln p \), with \( m = 0, 1, 2, \ldots \). The spectral partition function of the \( \mathcal{I}_p \) is \( Z_p(s) = (1 - p)^{-1} \), which is exactly the factor that appears in the Euler product for the Riemann zeta function.

The organization of the paper is as follows. In section 2 we discuss bosonic and fermionic arithmetic gases and discuss also the singularity structure for the average free energy density of the bosonic Riemann random gas on the complex \( \beta \)-plane. In section 3 the average energy density for the ensemble of Riemann random gas is presented. In section 4 we use superzeta functions in order to define the average energy density of the Riemann random gas. In section 5 the regularized average energy density is presented. Conclusions are given in section 6. In this paper we use \( k_B = c = \hbar = 1 \).

2. The partition function of the Riemann gas and the average free energy density of the system

Let us assume a non-interacting bosonic field theory defined in a volume \( V \) with Hamiltonian given by

\[
H_B = \alpha \sum_{k=1}^{\infty} \ln(p_k) b_k^\dagger b_k, \tag{1}
\]

where \( b_k^\dagger \) and \( b_k \) are respectively the creation and annihilation operators and \( \{p_k\}_{k \in \mathbb{N}} \) is the sequence of prime numbers. Since the energy of each mode is \( \nu_k = \alpha \ln p_k \) the partition function of this system is exactly the Riemann zeta function, i.e., \( Z_B = \zeta(\beta \omega) \).

Let us consider now the same situation for a system composed by fermions. The Hamiltonian of a free fermionic arithmetic gas is

\[
H_F = \alpha \sum_{k=1}^{\infty} \ln(p_k) c_k^\dagger c_k, \tag{2}
\]

where \( c_k^\dagger \) and \( c_k \) are respectively the creation and annihilation operators of quanta associated to the fermionic field and \( \{p_k\}_{k \in \mathbb{N}} \) is again the sequence of prime numbers.

To proceed, let us introduce the Möbius function \( \mu(n) \) defined by \[36\],

\[
\mu(n) = \begin{cases} 
1, & \text{if } n = 1, \\
(-1)^r, & \text{if } n \text{ is the product of } r (\geq 1) \text{ distinct primes,} \\
0 & \text{otherwise, i.e., if the square of at least one prime divides } n.
\end{cases}
\]

Using the Möbius function it is possible to show that the partition function of the fermionic system is given by \( \zeta(\beta \omega)\zeta(2\beta \omega) \) \[3, 6\]. As discussed in the literature, from the partition
function associated to the Hamiltonians \( H_B \) and \( H_F \) we get
\[
Z_F(\beta \omega)Z_B(2\beta \omega) = Z_B(\beta \omega).
\] (3)

The noninteracting mixture of a bosonic and a fermionic gas with respective temperatures \( \beta^{-1} \) and \( (2\beta)^{-1} \) is equivalent to another bosonic gas with temperature \( \beta^{-1} \).

Disorder can now be introduced simply assuming that the parameter \( \omega \) that appears in the Hamiltonian for the arithmetic gas given by equation (1) is a random variable. We take \( \{\omega_k\}_{k \in \mathbb{N}} \) a set of uncorrelated random variables with some probability distribution \( P(\omega_k) \) over an ensemble of Hamiltonians. To proceed we have to perform ensemble averages of some extensive quantity of interest. Let us define the average free energy of the Riemann gas
\[
\langle F(\beta) \rangle = -\frac{1}{\beta} \langle \ln \zeta(\beta \omega) \rangle,
\] (4)
where \( \langle (...) \rangle \) denotes the averaging over an ensemble of realizations of random variable with a given discrete probability distribution function. The average free energy density, for the case where the ensemble consists of an enumerable infinite set of copies of the system is
\[
\langle f(\beta) \rangle = -\frac{1}{\beta V} \sum_{k=1}^{\infty} P(\omega_k) \ln \zeta(\omega_k \beta),
\] (5)
where \( V \) is the volume of the system and \( P(\omega_k) \) is a given one-dimensional discrete distribution function defined in the interval \( \{\omega : \omega_1 \leq \omega_k < \infty \}_{k \in \mathbb{N}} \). Note that for low temperatures all of the copies of the ensemble contribute to the average free energy density of the system. Nevertheless, due to the pole of the Riemann zeta function there is a critical temperature where the first copy of the ensemble gives a singular contribution to the average free energy density of the system. This is the Hagedorn temperature of the random arithmetic gas.

In the next section, we first show that considering an ensemble made by a non-enumerable set of copies, the singular behavior of the average free energy density disappears. We are also interested in studying the average energy density of the system, which is related to the logarithmic derivative of the zeta function \( \zeta'(s) \). As we will see, the divergent contributions that appear in the average energy density of the system can be circumvented using an analytic regularization procedure.

3. The singular structure for the average free energy density and the average energy density for the system

In this section we will show that the thermodynamic quantities associated with the Riemann random gas are defined in terms of some number theoretical formulas. With this aim, our next task is to calculate relevant thermodynamic physical quantities as the average energy density \( \langle \varepsilon \rangle \) and the average entropy density \( \langle s \rangle \). These thermodynamic quantities are given respectively by
\[
\langle \varepsilon \rangle = -\frac{1}{V} \frac{d}{d\beta} \sum_{k=1}^{\infty} P(\omega_k) \ln \zeta(\omega_k \beta)
\] (6)
The properties of the model depend strongly on the analytic structure of the Riemann zeta function. In the following, instead of considering an ensemble made by an enumerable infinite set of copies, we extend these definitions for a non-enumerable set of copies. This approach is analogous to the one that makes the classical Gibbs ensemble in phase space a continuous fluid [37]. Therefore the average over an ensemble of realizations can be represented by an integral with \( \omega \) defined in the continuum i.e. \( \{ \omega : \omega \in \mathbb{R}^+ \} \). The average free energy density and average energy density can be written as

\[
\langle f(\beta, \lambda) \rangle = -\frac{1}{\beta V} \int_0^\infty d\omega \, P(\omega, \lambda) \ln \zeta(\omega \beta)
\]

and

\[
\langle e(\beta, \lambda) \rangle = -\frac{1}{V} \int_0^\infty d\omega \, P(\omega, \lambda) \frac{\partial}{\partial \beta} \ln \zeta(\omega \beta),
\]

where \( \lambda \) is a parameter with length dimension that we have to introduce to give the correct dimension in the expressions. For simplicity let us assume that the probability density distribution is given by \( P(\omega, \lambda) = \lambda e^{-\lambda \omega} \). At this point, a comment may be useful. It is important to stress that the choice of the \( P(\omega) \) does not affect the conclusions of the paper. By changing the variable \( s = \omega \beta \) we can write the average free energy density as

\[
\langle f(\beta, \lambda) \rangle = -\frac{\lambda}{\beta^2 V} \int_0^\infty ds \, e^{-s} \ln \zeta(s).
\]

Although there is a singularity in the integrand at \( s = 1 \), it is easy to show that the integral is bounded. On the neighborhood of \( s = 1 \) we can substitute the function \( \log \zeta(s) \approx \log \frac{1}{s-1} = -\log(s-1) \). This is an integrable singularity, therefore the average free energy density is non-singular and continuous for all temperatures. Note that overcomes the Hagedorn temperature implies to go into the critical region of the Riemann zeta function, i.e., the region \( 0 < s < 1 \), that may lead to complex values of the average free energy density.

We would like to point out that there is an alternative formulation of quantum mechanics and quantum field theory with non-Hermitian Hamiltonians [38]. The main problem to go into the critical region is the existence of a branch point of \( \zeta(s) \). This fact generates an ambiguity in the free energy density. It is clear that this problem disappear if we deal with the logarithmic derivative of the zeta function \( \zeta'(s) \). The picture that emerges from this discussion is that the mean energy is a well behaved function of the temperature.

As we discussed above, the logarithmic derivative of the zeta function \( \zeta'(s) \), which is fundamental in the study of the density of non-trivial zeros of the zeta function, must be used in the definition of the average energy density. Let us briefly discuss the symmetries of this set of numbers. Using the function \( \xi(s) \), the functional equation for \( \zeta(s) \) given by equation (11) takes the form \( \xi(s) = \xi(1-s) \). Therefore, if \( \rho \) is a zero of \( \xi(s) \), then so is \( 1 - \rho \). Since \( \xi(\rho) = \xi(\bar{\rho}) \) we have that \( \rho \) and \( 1 - \bar{\rho} \) are also zeros. The zeros are symmetrically arranged about the real axis and also about the critical line. Let us write the complex zeros of the zeta function, which are the Fisher zeros of the system, as \( \frac{i}{2} \gamma_k, \gamma_k \in \mathbb{C} \). The Riemann hypothesis is the statement that all \( \gamma \) are real. We assume the Riemann hypothesis. If the zeros \( \rho = \frac{i}{2} + i\gamma, \gamma > 0 \) are arranged in a sequence \( \rho_k = \frac{i}{2} + i\gamma_k \) so that \( \gamma_{k+1} > \gamma_k \).
The Riemann zeta function $\zeta(s)$ satisfies the functional equation
\[
\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s),
\]
for $s \in \mathbb{C} \setminus \{0, 1\}$. Let us define the entire function $\xi(s)$ as
\[
\xi(s) = \frac{1}{2} s(s - 1) \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s).
\]
This function $\xi(s)$ has an infinitely many zeros. If they are denoted by $\rho$, Hadamard product for $\xi(s)$ is of the form
\[
\xi(s) = e^{b_0 + b_1 s} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{\frac{s}{\rho}},
\]
where $b_0$ and $b_1$ are given constants. Combining both equations we get
\[
\frac{\zeta'(s)}{\zeta(s)} = C_1 - \frac{1}{s - 1} + \sum_{\rho} \frac{1}{s - \rho} + \sum_{\rho} \frac{1}{s + \rho} + \sum_{n=1}^{\infty} \frac{1}{s + 2n} - \sum_{n=1}^{\infty} \frac{1}{2n},
\]
where $C_1 = -1 - \frac{\zeta(0)}{\zeta'}$ is an absolute constant, and $\rho$ is the set of the nontrivial zeros of the Riemann zeta function [39]. Substituting the equation (15) in equation (9) we can write the average energy density as
\[
\langle \varepsilon(\beta, \lambda) \rangle = \varepsilon_1(\lambda) + \varepsilon_2(\beta, \lambda) + \varepsilon_3(\beta, \lambda) + \varepsilon_4(\lambda) + \varepsilon_5(\beta, \lambda) + \varepsilon_6(\lambda),
\]
where each of these terms are given by
\[
\varepsilon_1(\lambda) = -\frac{C_1}{V} \int_0^\infty d\omega \omega P(\omega, \lambda),
\]
\[
\varepsilon_2(\beta, \lambda) = \frac{1}{V} \int_0^\infty d\omega \omega P(\omega, \lambda) \frac{1}{\beta \omega - 1},
\]
\[
\varepsilon_3(\beta, \lambda) = -\frac{1}{V} \int_0^\infty d\omega \omega P(\omega, \lambda) \sum_{\rho} \frac{1}{\beta \omega - \rho},
\]
\[
\varepsilon_4(\lambda) = -\frac{1}{V} \int_0^\infty d\omega \omega P(\omega, \lambda) \sum_{\rho} \frac{1}{\rho},
\]
\[
\varepsilon_5(\beta, \lambda) = -\frac{1}{V} \int_0^\infty d\omega \omega P(\omega, \lambda) \sum_{n=1}^{\infty} \frac{1}{(\beta \omega + 2n)}
\]
and finally
\[
\varepsilon_6(\lambda) = \frac{1}{2V} \int_0^\infty d\omega \omega P(\omega, \lambda) \sum_{n=1}^{\infty} \frac{1}{n}.
\]
Usually, the information of the thermodynamics of the system is contained in derivatives of the mean free energy density. With the exception of the contribution coming from the pole of
the zeta function, the above expressions for the mean free energy density are divergent series. We can use a standard regularization procedure to give meaning to these divergent terms [40]. Here, we choose to use an analytic regularization procedure introduced in quantum field theory in [41] and used extensively since then.

4. The superzeta function and the average energy density for the arithmetic random gas

In this section we will discuss each of the terms that contributes to the average energy density given by equation (16). The contribution of the first term given by equation (17) to the average energy density can be written as

$$\epsilon_1(\lambda) = -\frac{C_1}{\lambda V}.$$  \hspace{1cm} (23)

The second term that contributes to the average energy density \(\epsilon_2(\beta, \lambda)\) can be written as

$$\epsilon_2(\beta, \lambda) = \frac{1}{\beta V} - \frac{\lambda}{\beta^2 V} e^{-\frac{1}{\beta}} Ei(\lambda/\beta),$$  \hspace{1cm} (24)

where \(Ei(x)\) is the exponential-integral function [42, 43] defined by

$$Ei(x) = -\int_{-\infty}^{\infty} dt \frac{e^{-t}}{t} x < 0,$$  \hspace{1cm} (25)

and

$$Ei(x) = -\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} dt \frac{e^{-t}}{t} + \int_{-\infty}^{\infty} dt \frac{e^{-t}}{t} x > 0.$$  \hspace{1cm} (26)

For the third term that contributes to the average energy density \(\epsilon_3(\beta, \lambda)\) we have

$$\epsilon_3(\beta, \lambda) = -\frac{\lambda}{V} \int_0^{\infty} d\omega \omega e^{-\lambda \omega} \sum_{\rho} \frac{1}{\beta \omega - \rho}. \hspace{1cm} (27)$$

As we will see, the contributions given by \(\epsilon_3(\beta, \lambda)\) and \(\epsilon_4(\lambda)\) can be written in terms of superzeta functions. A straightforward calculation give us for \(\epsilon_4(\lambda)\)

$$\epsilon_4(\lambda) = -\frac{1}{\lambda V} \sum_{\rho} \frac{1}{\rho}. \hspace{1cm} (28)$$

Let us discuss the construction of the so-called superzeta or secondary zeta function built over the Riemann zeros, i.e., the nontrivial zeros of the Riemann zeta function [44]. As was discussed by Voros, in view of the central symmetry of the Riemann zeros \(\rho \leftrightarrow 1 - \rho\), leads us to generalized zeta functions of several kinds. Each one of these superzeta functions reflects our choice of set of numbers to built zeta functions over the Riemann zeros. The first family that we call \(G_1(s, t)\) is defined by

$$G_1(s, t) = \sum_{\rho} \frac{1}{\left(\frac{1}{2} + t - \rho\right)^s}, \quad \Re(s) > 1, \hspace{1cm} (29)$$

valid for \(t \in \Omega_t = \{ t \in \mathbb{C} \mid \frac{1}{2} + t - \rho \not\in \mathbb{R}_{-}(\forall \rho) \}. \) This is the simplest generalized zeta function over the Riemann zeros. The sum runs over all zeros symmetrically and \(t\) is just a shift parameter. In view of the fact that
the second generalized superzeta function is defined as

$$G_2(\sigma, t) = \sum_{k=1}^{\infty} \left( \tau_k^2 + t^2 \right)^{-\sigma} \quad \Re(\sigma) > \frac{1}{2},$$

valid for \( t \in \Omega_2 = \{ t \in \mathbb{C} \mid t \pm i r_k \not\in \mathbb{R} \quad (\forall k) \} \). The central symmetry \( \tau_k \leftrightarrow -\tau_k \), is preserved in the family of superzeta functions \( G_2(\sigma, t) \). Using the above definitions, the contributions to the average energy density given by \( \epsilon_3(\beta, \lambda) \) and \( \epsilon_4(\lambda) \) can be written respectively as

$$\epsilon_3(\beta, \lambda) = -\frac{\lambda}{V} \int_0^{\infty} d\omega \, e^{-i\omega \beta} G_1(1, \beta\omega - 1/2)$$

and

$$\epsilon_4(\lambda) = -\frac{1}{\lambda V} G_2(1, 1/2).$$

Since we are interested in the region \( \Re(\sigma) > \frac{1}{2} \) we are in the region of convergence of the series in equation (31). Therefore, assuming the Riemann hypothesis we can write \( \epsilon_4(\lambda) \) as

$$\epsilon_4(\lambda) = -\frac{1}{\lambda V} \sum_{n=1}^{\infty} \frac{1}{\gamma_n + \frac{1}{2}}.$$

After this discussion of the terms that involve superzeta functions, let us proceed with the contribution to the average energy density given by terms that involve the Hurwitz zeta function and the Riemann zeta function. The Hurwitz zeta function \( \zeta(z, q) \) is the analytic extension of the series

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(q + n)^z} \quad q \neq 0, -1, -2, \ldots, \Re(z) > 1,$$

which is a meromorphic function in the whole complex plane with a single pole at \( z = 1 \).

We can write \( \epsilon_5(\beta, \lambda) \) in terms of the Hurwitz zeta function as

$$\epsilon_5(\beta, \lambda) = \lim_{z \to 1} \left( \frac{1}{\beta V} - \frac{\lambda}{2V} \int_0^{\infty} d\omega \, e^{-i\omega \beta} \zeta(z, \beta\omega/2) \right).$$

Finally, let us discuss the last term given by \( \epsilon_6(\lambda) \). Using the zeta function \( \zeta(s) \) defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \Re(s) > 1,$$

and otherwise by its analytic extension, we can write \( \epsilon_6(\beta, \lambda) \) as

$$\epsilon_6(\lambda) = \lim_{s \to 1} \frac{1}{2\lambda V} \zeta(s).$$

In the next section we will discuss the structure of the singularities for \( \epsilon_3(\beta, \lambda), \epsilon_5(\beta, \lambda) \) and \( \epsilon_6(\lambda) \) to the average energy density.
5. The regularized average energy density

The aim of this section is to find the contributions given by \(\epsilon_3(\beta, \lambda), \epsilon_5(\beta, \lambda)\) and \(\epsilon_6(\lambda)\) to the average energy density. From the previous section we have that the contribution to the average energy density given by \(\epsilon_3(\beta, \lambda)\) can be written as

\[
\epsilon_3(\beta, \lambda) = -\frac{\lambda}{V} \int_0^\infty d\omega \omega e^{-i\omega\beta} G_1(1, \beta \omega - 1/2).
\]

(39)

The superzeta function \(G_1(s, t)\) admits an analytic extension to the half-complex plane given by

\[
G_1(s, t) = -Z(s, t) + \frac{1}{(t - 1/2)^s} + \frac{\sin \pi s}{\pi} J(s, t),
\]

(40)

where \(Z(s, t)\) is the superzeta function \(G_1(s, t)\) evaluated in the trivial zeroes of the Riemann zeta function [44]. It is possible to write \(Z(s, t)\) in terms of the Hurwitz zeta function, defined in equation (35), as

\[
Z(s, t) = \sum_{k=1}^\infty \left( \frac{1}{2} + t + 2k \right)^{-s} = 2^{-s} \gamma \left( s, \frac{1}{4} + \frac{1}{2} \right).
\]

(41)

The function \(J(s, t)\), a Mellin transform of the logarithmic derivative of the Riemann zeta function, is defined as

\[
J(s, t) = \int_0^\infty \frac{1}{\zeta} \left( \frac{1}{2} + t + y \right)^{-s} dy \quad \text{Re}(s) < 1.
\]

(42)

As discussed by Voros, all the discontinuities of \((t - 1/2)^{-s}\) and \(-Z(s, t)\) in the interval \((-\infty, 1/2]\) cancel against jumps of \(J(s, t)\). Therefore, the analytic continuation of \(G_1(s, t)\) is a regular function of \(t \in \Omega_t\). Using the analytical properties of the Mellin transformation of \((\zeta'/\zeta)(s)\) the function \(J(s, t)\) has a known global meromorphic structure which implies the meromorphic continuation to the whole plane of the superzeta function \(G_1(s, t)\). The function \(J(s, t)\) has simple poles at \(s = +1, +2, \ldots\) with residues given by \(\text{Res}(J(s, t))_{s=n} = -\frac{1}{(n-1)!} \frac{d^n}{dx^n} \log \left( \frac{1}{2} + t \right)\); and this structure makes the product \(\sin(\pi x) J(s, t)\) free of singularities. Hence, the singularity structure of the function \(G_1(s, t)\) is the same as the one of \(Z(s, t)\), i.e., of the Hurwitz zeta function.

The analytic extension of the Hurwitz zeta function \(\zeta(s, q)\) can be performed as given in [39]. The result is a meromorphic function for \(\text{Re}(s) > 0\), with a simple pole at \(s = 1\) and of residue 1. It is possible to show that

\[
\lim_{s \to 1} \left( \zeta(s, q) - \frac{1}{s-1} \right) = -\psi(q),
\]

(43)

where the \(\psi(q)\) is the Euler’s Psi-function defined by

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x).
\]

(44)

Since \(\sin(\pi x)\) vanishes at integers, the contribution to the average energy density given by \(\epsilon_3(\beta, \lambda)\) can be written as
\[ \varepsilon_3(\beta, \lambda) = -\frac{\lambda}{V} \int_0^\infty \omega \omega e^{-i\omega \lambda} \left( \frac{1}{2} \psi \left( \frac{\beta \omega}{2} \right) + \frac{1}{\beta \omega - 1/4} \right) \]  

(45)

To perform the integral of the first term in the parenthesis, we can use the functional relation for the psi function \( \psi(x + 1) = \psi(x) + 1/x \) and the following series representation for \( \psi(x + 1) \):

\[ \psi(x + 1) = -C + \sum_{k=2}^{\infty} (-1)^{k-1}(k) x^{k-1}. \]  

(46)

The integral of the second term is similar to the one calculated for \( \varepsilon_2(\beta, \lambda) \) which can be expressed in terms of the exponential-integral function \( Ei(x) \). Accordingly, the contribution to the average energy density given by \( \varepsilon_3(\beta, \lambda) \) can be written as

\[ \varepsilon_3(\beta, \lambda) = \frac{C}{2V} - \frac{1}{\beta V} \sum_{k=2}^{\infty} g(k) \left( \frac{\beta}{\lambda} \right)^k - \frac{\lambda}{V} Ei \left( \frac{\lambda}{4\beta} \right). \]  

(47)

where \( g(k) = \frac{(-1)^k}{k!} \Gamma(k + 1) \zeta(k) \).

From the last section, the contribution to the average energy density given by \( \varepsilon_5(\beta, \lambda) \) can be written as

\[ \varepsilon_5(\beta, \lambda) = \lim_{s \to 1} \left( \frac{1}{\beta V} \sum_{k=2}^{\infty} g(k) \left( \frac{\beta}{\lambda} \right)^k - \frac{\lambda}{V} \int_0^\infty \omega \omega e^{-\omega \lambda} \zeta(s, \beta \omega/2) \right). \]

A similar procedure as the mentioned above in the analysis of the Hurwitz zeta function can be performed with the aid of equations (43), (44) and (46). In this case we can see that \( \varepsilon_5(\beta, \lambda) \) has three contributions, the first two are finite and the last one is singular. We have

\[ \varepsilon_5(\beta, \lambda) = \frac{1}{\beta V} + \frac{\lambda}{2V} \int_0^\infty \omega \omega e^{-\omega \lambda} \psi \left( \frac{a \beta}{2} \right) - h(\lambda, V). \]  

(48)

We can write \( \varepsilon_5(\beta, \lambda) \) as

\[ \varepsilon_5(\beta, \lambda) = -\frac{C}{2\lambda V} + \frac{1}{\beta V} \sum_{k=2}^{\infty} g(k) \left( \frac{\beta}{\lambda} \right)^k - h(\lambda, V), \]  

(49)

where \( h(\lambda, V) = \lim_{s \to 1} \left( \frac{1}{2V} \frac{1}{s-1} \right) \) and \( g(k) \) being the same coefficient as before. From the previous section we have that the contributions to the average energy density given by \( \varepsilon_6(\lambda) \) is

\[ \varepsilon_6(\lambda) = \lim_{s \to 1} \frac{1}{2\lambda V} \zeta(s). \]

The \( \varepsilon_6(\lambda) \) contribution is proportional to the Riemann zeta function. Using the analytic extension of the Riemann zeta function and the fact that

\[ \lim_{s \to 1} \left( \zeta(s) - \frac{1}{s-1} \right) = -\psi(1), \]  

(50)

where \( \psi(1) = -0.577215 \) is the Euler’s constant, we can write \( \varepsilon_6(\beta) \) as a finite contribution and again a singular part. We have

\[ \varepsilon_6(\lambda) = -\frac{1}{2\lambda V} \psi(1) + h(\lambda, V). \]  

(51)

Note that the singular contribution in \( \varepsilon_5(\beta, \lambda) \) and \( \varepsilon_6(\beta, \lambda) \) cancel each other.

We can split the contribution of each term to the average energy density into two categories. The first one coming from \( \varepsilon_A(\lambda) = \varepsilon_1(\lambda) + \varepsilon_4(\lambda) + \varepsilon_6(\lambda) \). These terms are
temperature independent and, therefore, we can interpret them as coming from the vacuum modes associated to the arithmetic gas. The other terms $\varepsilon_1(\beta, \lambda), \varepsilon_2(\beta, \lambda)$ and $\varepsilon_3(\beta, \lambda)$ are temperature dependent contributions. We can write this thermal contribution $\varepsilon_4(\beta, \lambda)$ as

$$
\varepsilon_4(\beta, \lambda) = \frac{1}{\beta V} - \frac{\lambda}{\beta^2 V} e^{-\frac{\lambda}{\beta}} + \frac{\lambda}{4\beta^2 V} e^{-\frac{\lambda}{4\beta}}.
$$

We have shown that the thermodynamic quantities associated to the arithmetic gas can be calculated. A similar calculation can be performed to find the average entropy density. Since we have an ambiguity in the mean free energy density the mean entropy also keeps such ambiguity.

6. Conclusions

‘Do you know a physical reason that the Riemann hypothesis should be true?’ (Landau). Hilbert and Pólya suggested that there might be a spectral interpretation of the the non-trivial zeros of the Riemann zeta function. The corresponding operator must be self-adjoint. The existence of such operator may led to the proof of the Riemann hypothesis. In this paper we present a different scenario where it is possible to present some links between the Riemann zeta function theory and physics. We show that using an arithmetic quantum field theory with randomness it is possible to connect strongly the non-trivial zeros of the Riemann zeta function with a measurable physical quantity defined in the system.

Arithmetic quantum field theories establish a bridge between statistical mechanic systems with an exponential density of states and multiplicative number theory. In these theories, the partition function of hypothetical systems are related to the Riemann zeta function or other Dirichlet series. The partition function of a bosonic Riemann gas is given by $Z_{\beta} = \zeta(\beta \omega)$, where $\beta$ is the inverse of the temperature of the gas. Since the Riemann zeta function has a simple pole in $s = 1$, there is a Hagedorn temperature above which the system can not be heated up. In a bosonic Riemann gas with randomness, we show that the mean energy density depends strongly on the distribution of the Riemann zeros. Being more precise, the mean energy density of the system is defined in terms of the quantity $\frac{\zeta'}{\zeta}(s)$, which is related to the non-trivial zeros of the Riemann zeta function.

First, assuming that the ensemble is made by a enumerable infinite set of copies, there is a critical temperature where the average free energy density diverges due to the pole of the Riemann zeta function. This is the Hagedorn phenomenon in this random system. On the other hand, considering an ensemble of made by a non-enumerable set of copies, the singular behavior of the average free energy density disappears, i.e., it is non-singular for all temperatures. Due to the behavior of the Riemann zeta function in the critical region, the average free energy density acquires complex values and appears an ambiguity in the free energy density. Second, we study the average energy density of the system, where the above mentioned problem disappears, since is related to the logarithmic derivative of the zeta function $\frac{\zeta'}{\zeta}(s)$. We showed that the divergent contributions that appear in the average energy density of the system can be circumvented using an analytic regularization procedure.

The parafermion arithmetic gas of order $r$ is a quantum gas where the exclusion principle states that no more than $r - 1$ parafermions can have the same quantum numbers [45]. The partition function for an arithmetic parafermion gas or order $r$ is given by $Z_r(\beta, \omega) = \zeta(\beta \omega)\zeta(\beta^r \omega)$. A natural extension of this paper is to assume that, as the previous case, $\omega$ is a random variable with a given probability distribution over an ensemble of
Hamiltonians. The average energy density and average entropy density of this parafermion arithmetic gas must be investigated in a future work.

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