Sample-based high-dimensional convexity testing

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Abstract

In the problem of high-dimensional convexity testing, there is an unknown set $S \subseteq \mathbb{R}^n$ which is promised to be either convex or $\varepsilon$-far from every convex body with respect to the standard multivariate normal distribution $\mathcal{N}(0, 1)^n$. The job of a testing algorithm is then to distinguish between these two cases while making as few inspections of the set $S$ as possible.

In this work we consider sample-based testing algorithms, in which the testing algorithm only has access to labeled samples $(x, S(x))$ where each $x$ is independently drawn from $\mathcal{N}(0, 1)^n$. We give nearly matching sample complexity upper and lower bounds for both one-sided and two-sided convexity testing algorithms in this framework. For constant $\varepsilon$, our results show that the sample complexity of one-sided convexity testing is $2^{\tilde{O}(n)}$ samples, while for two-sided convexity testing it is $2^{\tilde{O}((\sqrt{n})}$.

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1 Introduction

Over the past few decades the field of property testing has developed into a fertile area with many different branches of active research. Several distinct lines of work have studied the testability of various kinds of high-dimensional objects, including probability distributions (see e.g. [BKR04, RS05, AAK+07, RX10, ACS10, BFRV11, ADK15]), Boolean functions (see e.g. [BLR93, PRS02, Bla09, MORS10, KMS15] and many other works), and various types of codes and algebraic objects (see e.g. [AKK+05, GS06, KS08, BKS+10] and many other works). These efforts have collectively yielded significant insight into the abilities and limitations of efficient testing algorithms for such high-dimensional objects. A distinct line of work has focused on testing (mostly low-dimensional) geometric properties. Here too a considerable body of work has led to a good understanding of the testability of various low-dimensional geometric properties, see e.g. [CSZ00, CS01, Ras03, BMR16c, BMR16b, BMR16a].

This paper is about a topic which lies at the intersection of the two general strands (high-dimensional property testing and geometric property testing) mentioned above: we study the problem of high-dimensional convexity testing.

Convexity is a fundamental property which is intensively studied in high-dimensional geometry (see e.g. [GW93, Bal97, Sza06] and many other references) and has been studied in the property testing of images (the two-dimensional case) [Ras03, BMR16c, BMR16b, BMR16a], but as we discuss in Section 1.2 below, very little is known about high-dimensional convexity testing.

We consider $\mathbb{R}^n$ endowed with the standard normal distribution $N(0,1)^n$ as our underlying space, so the distance $\text{dist}(S,C)$ between two subsets $S,C \subseteq \mathbb{R}^n$ is $\Pr_{x \sim N(0,1)^n}[x \in S \triangle C]$, where $S \triangle C$ denotes their symmetric difference. The standard normal distribution is arguably one of the most natural, and certainly one of the most studied, distributions on $\mathbb{R}^n$. Several previous works have studied property testing over $\mathbb{R}^n$ with respect to the standard normal distribution, such as the work on testing halfspaces of [MORS10, BBBY12] and the work on testing surface area of [KNOW14, Nee14].

1.1 Our results

In this paper we focus on sample-based testing algorithms for convexity. Such an algorithm has access to independent draws $(x, S(x)) \in \mathbb{R}^n \times \{0, 1\}$, where $x$ is drawn from $N(0,1)^n$ and $S \subseteq \mathbb{R}^n$ is the unknown set being tested for convexity (so in particular the algorithm cannot select points to be queried) with $S(x) = 1$ if $x \in S$. We say such an algorithm is an $\varepsilon$-tester for convexity if it accepts $S$ with probability at least $2/3$ when $S$ is convex and rejects with probability at least $2/3$ when it is $\varepsilon$-far from convex, i.e., $\text{dist}(S,C) \geq \varepsilon$ for all convex sets $C \subseteq \mathbb{R}^n$. The model of sample-based testing was originally introduced by Goldreich, Goldwasser, and Ron almost two decades ago [GGR98], where it was referred to as “passive testing;” it has received significant attention over the years [KR00, GGL+00, BBBY12, GR16], with an uptick in research activity in this model over just the past year or so [AHW16, BY16, BMR16c, BMR16b, BMR16a].

We consider sample-based testers for convexity that are allowed both one-sided (i.e., the algorithm always accepts $S$ when it is convex) and two-sided error. In each case, for constant $\varepsilon > 0$ we give nearly matching upper and lower bounds on sample complexity. Our results are as follows:

**Theorem 1** (One-sided lower bound). Any one-sided sample-based algorithm that is an $\varepsilon$-tester for convexity over $N(0,1)^n$ for some $\varepsilon < 1/2$ must use $2^{\Omega(n)}$ samples.
Model | Sample complexity bound | Reference
--- | --- | ---
One-sided | $2^{\Omega(n)}$ samples (for $\varepsilon < 1/2$) | Theorem 1
 | $2^{O(n \log(n/\varepsilon))}$ samples | Theorem 2
Two-sided | $2^{\Omega(\sqrt{n})}$ samples (for $\varepsilon < \varepsilon_0$) | Theorem 3
 | $2^{O(\sqrt{n} \log(n)/\varepsilon^2)}$ samples | Theorem 4

Table 1: Sample complexity bounds for sample-based convexity testing. In line four, $\varepsilon_0 > 0$ is some absolute constant.

**Theorem 2** (One-sided upper bound). For any $\varepsilon > 0$, there is a one-sided sample-based $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^n$ which uses $(n/\varepsilon)^O(n)$ samples.

**Theorem 3** (Two-sided lower bound). There exists a positive constant $\varepsilon_0$ such that any two-sided sample-based algorithm that is an $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^n$ for some $\varepsilon \leq \varepsilon_0$ must use $2^{\Omega(\sqrt{n})}$ samples.

**Theorem 4** (Two-sided upper bound). For any $\varepsilon > 0$, there is a two-sided sample-based $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^n$ which uses $n^{O(\sqrt{n}/\varepsilon^2)}$ samples.

We will prove Theorems 1, 2, 3 and 4 in Sections 6, 4, 5 and 7 respectively. These results are summarized above in Table 1.

1.2 Related work

**Convexity testing.** As mentioned above, [Ras03, BMR16a, BMR16b, BMR16c] studied the testing of 2-dimensional convexity under the uniform distribution, either within a compact body such as $[0,1]^2$ [BMR16a, BMR16b] or over a discrete grid $[n]^2$ [Ras03, BMR16c]. The model of [BMR16a, BMR16b] is more closely related to ours: [BMR16b] showed that $\Theta(\varepsilon^{-4/3})$ samples are necessary and sufficient for one-sided sample-based testers, while [BMR16a] gave a one-sided general tester (which can make adaptive queries to the unknown set) for 2-dimensional convexity with only $O(1/\varepsilon)$ queries.

The only prior work that we are aware of that deals with testing high-dimensional convexity is that of [RV05]. However, the model considered in [RV05] is different from ours in the following important aspects. First, the goal of an algorithm in their model is to determine whether an unknown $S \subseteq \mathbb{R}^n$ is not convex or is $\varepsilon$-close to convex in the following sense: the (Euclidean) volume of $S \triangle C$, for some convex $C$, is at most an $\varepsilon$-fraction of the volume of $S$. Second, in their model an algorithm both can make membership queries (to determine whether a given point $x$ belongs to $S$), and can receive samples which are guaranteed to be drawn independently and uniformly at random from $S$. The main result of [RV05] is an algorithm which uses $(cn/\varepsilon)^n$ many random samples drawn from $S$, for some constant $c$, and poly$(n)/\varepsilon$ membership queries.

**Sample-based testing.** A wide range of papers have studied sample-based testing from several different perspectives, including the recent works [BMR16c, BMR16b, BMR16a] which study...
sample-based testing of convexity over two-dimensional domains. In earlier work on sample-based testing, [BBBY12] showed that the class of linear threshold functions can be tested to constant accuracy under $\mathcal{N}(0,1)^n$ with $\tilde{O}(n^{1/2})$ samples drawn from $\mathcal{N}(0,1)^n$. (Note that a linear threshold function is a convex set of a very simple sort, as every convex set can be expressed as an intersection of (potentially infinitely many) linear threshold functions.) The work [BBBY12] in fact gave a characterization of the sample complexity of (two-sided) sample-based testing, in terms of a combinatorial/probabilistic quantity called the “passive testing dimension.” This is a distribution-dependent quantity whose definition involves both the class being tested and the distribution from which samples are obtained; it is not a priori clear what the value of this quantity is for the class of convex subsets of $\mathbb{R}^n$ and the standard normal distribution $\mathcal{N}(0,1)^n$. Our upper and lower bounds (Theorems 4 and 3) may be interpreted as giving bounds on the passive testing dimension of the class of convex sets in $\mathbb{R}^n$ with respect to the $\mathcal{N}(0,1)^n$ distribution.

1.3 Our techniques

One-sided lower bound. Our one-sided lower bound has a simple proof using only elementary geometric and probabilistic arguments. It follows from the fact (see Lemma 29) that if $q = 2^{\Theta(n)}$ many points are drawn independently from $\mathcal{N}(0,1)^n$, then with probability $1 - o(1)$ no one of the points lies in the convex hull of the $q - 1$ others. This can easily be shown to imply that more than $q$ samples are required (since given only $q$ samples, with probability $1 - o(1)$ there is a convex set consistent with any labeling and thus a one-sided algorithm cannot reject).

Two-sided lower bound. At a high-level, the proof of our two-sided lower bound uses the following standard approach. We first define two distributions $D_{\text{yes}}$ and $D_{\text{no}}$ over sets in $\mathbb{R}^n$ such that (i) $D_{\text{yes}}$ is a distribution over convex sets only, and (ii) $D_{\text{no}}$ is a distribution such that $S \leftarrow D_{\text{no}}$ is $\varepsilon_0$-far from convex with probability at least $1 - o(1)$ for some positive constant $\varepsilon_0$. We then show that every sample-based, $q$-query algorithm $A$ with $q = 2^{0.01n}$ must have

$$\Pr_{S \leftarrow D_{\text{yes}}; x} [A \text{ accepts } (x, S(x))] - \Pr_{S \leftarrow D_{\text{no}}; x} [A \text{ accepts } (x, S(x))] \leq o(1),$$

where $x$ denotes a sequence of $q$ points drawn from $\mathcal{N}(0,1)^n$ independently and $(x, S(x))$ denotes the $q$ labeled samples from $S$. Theorem 3 follows directly from (1).

To draw a set $S \leftarrow D_{\text{yes}}$, we sample a sequence of $N = 2^{\sqrt{n}}$ points $y_1, \ldots, y_N$ from the sphere $S^{n-1}(r)$ of radius $r$ for some $r = \Theta(n^{1/4})$. Each $y_i$ defines a halfspace $h_i = \{ x : x \cdot y_i \leq r^2 \}$. $S$ is then the intersection of all $h_i$’s. (This is essentially a construction used by Nazarov [Naz03] to exhibit a convex set that has large Gaussian surface area, and used by [KOS07] to lower bound the sample complexity of learning convex sets under the Gaussian distribution.) The most challenging part of the two-sided lower bound proof is to show that, with $q$ points $x_1, \ldots, x_q \leftarrow \mathcal{N}(0,1)^n$, the $q$ bits $S(x_1), \ldots, S(x_q)$ with $S \leftarrow D_{\text{yes}}$ are “almost” independent. More formally, the $q$ bits $S(x_1), \ldots, S(x_q)$ with $S \leftarrow D_{\text{yes}}$ have $o(1)$-total variation distance from $q$ independent bits with the $i$th bit drawn from the marginal distribution of $S(x_i)$ as $S \leftarrow D_{\text{yes}}$. On the other hand, it is relatively easy to define a distribution $D_{\text{no}}$ that satisfies (ii) and at the same time, $S(x_1), \ldots, S(x_q)$ when $S \leftarrow D_{\text{no}}$ has $o(1)$-total variation distance from the same product distribution. (1) follows by combining the two parts.

Structural result. Our algorithms rely on a new structural result which we establish for convex sets in $\mathbb{R}^n$. Roughly speaking, this result gives an upper bound on the Gaussian volume of the
“thickened surface” of any bounded convex subset of $\mathbb{R}^n$; it is inspired by, and builds on, the classic result of Ball [Bal93] that upperbounds the Gaussian surface area of any convex subset of $\mathbb{R}^n$.

**One-sided upper bound.** Our one-sided testing algorithm employs a “gridding-based” approach to decompose the relevant portion of $\mathbb{R}^n$ (namely, those points which are not too far from the origin) into a collection of disjoint cubes. It draws samples and identifies a subset of these cubes as a proxy for the “thickened surface” of the target set; by the structural result sketched above, if the Gaussian volume of this thickened surface is too high, then the one-sided algorithm can safely reject (as the target set cannot be convex). Otherwise the algorithm does random sampling to probe for points which are inside the convex hull of positive examples it has received but are labeled negative (there should be no such points if the target set is indeed convex, so if such a point is identified, the one-sided algorithm can safely reject). If no such points are identified, then the algorithm accepts.

**Two-sided upper bound.** Finally, the main tool we use to obtain our two-sided testing algorithm is a learning algorithm for convex sets with respect to the normal distribution over $\mathbb{R}^n$. The main result of [KOS07] is an (improper) algorithm which learns the class of all convex subsets of $\mathbb{R}^n$ to accuracy $\varepsilon$ using $n^{O(\sqrt{n}/\varepsilon^2)}$ independent samples from $\mathcal{N}(0,1)^n$. Using the structural result mentioned above, we show that this can be converted into a proper algorithm for learning convex sets under $\mathcal{N}(0,1)^n$, with essentially no increase in the sample complexity. Given this proper learning algorithm, a two-sided algorithm for testing convexity follows from the well-known result of [GGR98] which shows that proper learning for a class of functions implies (two-sided) testability.

## 2 Preliminaries and Notation

**Notation.** We use boldfaced letters such as $\mathbf{x}$, $\mathbf{f}$, $\mathbf{A}$, etc. to denote random variables (which may be real-valued, vector-valued, function-valued, set-valued, etc; the intended type will be clear from the context). We write “$\mathbf{x} \leftarrow \mathcal{D}$” to indicate that the random variable $\mathbf{x}$ is distributed according to probability distribution $\mathcal{D}$. Given $a,b,c \in \mathbb{R}$ we use $a = b \pm c$ to indicate that $b - c \leq a \leq b + c$.

**Geometry.** For $r > 0$, we write $S^{n-1}(r)$ to denote the origin-centered sphere of radius $r$ in $\mathbb{R}^n$ and $\text{Ball}(r)$ to denote the origin-centered ball of radius $r$ in $\mathbb{R}^n$, i.e.,

$$S^{n-1}(r) = \{ x \in \mathbb{R}^n : \|x\| = r \} \quad \text{and} \quad \text{Ball}(r) = \{ x \in \mathbb{R}^n : \|x\| \leq r \},$$

where $\|x\|$ denotes the $\ell_2$-norm $\| \cdot \|_2$ of $x \in \mathbb{R}^n$. We also write $S^{n-1}$ for the unit sphere $S^{n-1}(1)$.

Recall that a set $C \subseteq \mathbb{R}^n$ is convex if $x,y \in C$ implies $\alpha x + (1 - \alpha)y \in C$ for all $\alpha \in [0,1]$. We write $C_{\text{convex}}$ to denote the class of all convex sets in $\mathbb{R}^n$. Recall that convex sets are Lebesgue measurable. Given a set $C \subseteq \mathbb{R}^n$ we write $\text{Conv}(C)$ to denote the convex hull of $C$.

For sets $A, B \subseteq \mathbb{R}^n$, we write $A + B$ to denote the Minkowski sum $\{a + b : a \in A \text{ and } b \in B\}$. For a set $A \subseteq \mathbb{R}^n$ and $r > 0$ we write $rA$ to denote the set $\{ra : a \in A\}$. Given a point $a$ and a set $B \subseteq \mathbb{R}^n$, we use $a + B$ and $B - a$ to denote $\{a\} + B$ and $B + \{-a\}$ for convenience. For a convex set $C$, we write $\partial C$ to denote its boundary, i.e. the set of points $x \in \mathbb{R}^n$ such that for all $\delta > 0$, the set $x + \text{Ball}(\delta)$ contains at least one point in $C$ and at least one point outside $C$.

**Probability.** We use $\mathcal{N}(0,1)^n$ to denote the standard $n$-dimensional Gaussian distribution with zero mean and identity covariance matrix. We also recall that the probability density function for
the one-dimensional Gaussian distribution is
\[ \varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right). \]

Sometimes we denote \(\mathcal{N}(0,1)^n\) by \(\mathcal{N}^n\) for convenience. The squared norm \(\|x\|^2\) of \(x \sim \mathcal{N}(0,1)^n\) is distributed according to the chi-squared distribution \(\chi_n^2\) with \(n\) degrees of freedom. The following tail bound for \(\chi_n^2\) (see [Joh01]) will be useful:

**Lemma 5** (Tail bound for the chi-squared distribution). Let \(X \sim \chi_n^2\). Then we have
\[ \Pr\left[|X - n| \geq tn\right] \leq e^{-(3/16)n^2}, \quad \text{for all } t \in [0,1/2). \]

All target sets \(S \subseteq \mathbb{R}^n\) to be tested for convexity are assumed to be Lebesgue measurable and we write \(\text{Vol}(S)\) to denote \(\text{Pr}_{x \sim \mathcal{N}^n}[x \in S]\), the Gaussian volume of \(S \subseteq \mathbb{R}^n\). Given two Lebesgue measurable subsets \(S, C \subseteq \mathbb{R}^n\), we view \(\text{Vol}(S \triangle C)\) as the distance between \(S\) and \(C\), where \(S \triangle C\) is the symmetric difference of \(S\) and \(C\). Given \(S \subseteq \mathbb{R}^n\), we abuse the notation and use \(S\) to denote the indicator function of the set, so we may write “\(S(x) = 1\)” or “\(x \in S\)” to mean the same thing.

We say that a subset \(C \subseteq \mathcal{C}_{\text{convex}}\) is a \(\tau\)-cover of \(\mathcal{C}_{\text{convex}}\) if for every \(C \in \mathcal{C}_{\text{convex}}\), there exists a set \(C' \in \mathcal{C}\) such that \(\text{Vol}(C \triangle C') \leq \tau\).

Given a convex set \(C\) and a real number \(h > 0\), we let \(C_h\) denote the set of points in \(\mathbb{R}^n\) whose distance from \(C\) do not exceed \(h\). We recall the following theorem of Ball [Bal93] (also see [Naz03]).

**Theorem 6** ([Bal93]). For any convex set \(C \subseteq \mathbb{R}^n\) and \(h > 0\), we have
\[ \frac{\text{Vol}(C_h \setminus C)}{h} \leq 4n^{1/4}. \]

**Sample-based property testing.** Given a point \(x \in \mathbb{R}^n\), we refer to \((x, S(x)) \in \mathbb{R}^n \times \{0,1\}\) as a labeled sample from a set \(S \subseteq \mathbb{R}^n\). A sample-based testing algorithm for convexity is a randomized algorithm which is given as input an accuracy parameter \(\varepsilon > 0\) and access to an oracle that, each time it is invoked, generates a labeled sample \((x, S(x))\) from the unknown (Lebesgue measurable) target set \(S \subseteq \mathbb{R}^n\) with \(x\) drawn independently each time from \(\mathcal{N}(0,1)^n\). When run with any Lebesgue measurable \(S \subseteq \mathbb{R}^n\), such an algorithm must output “accept” with probability at least \(2/3\) (over the draws it gets from the oracle and its own internal randomness) if \(S \in \mathcal{C}_{\text{convex}}\) and must output “reject” with probability at least \(2/3\) if \(S\) is \(\varepsilon\)-far from being convex, meaning that for every \(C \in \mathcal{C}_{\text{convex}}\) it is the case that \(\text{Vol}(S \triangle C) \geq \varepsilon\). (We also refer to an algorithm as an \(\varepsilon\)-tester for convexity if it works for a specific accuracy parameter \(\varepsilon\).) Such a testing algorithm is said to be one-sided if whenever it is run on a convex set \(S\) it always outputs “accept,” equivalently, such an algorithm can only output “reject” if the labeled samples it receives are not consistent with any convex set. A testing algorithm which is not one-sided is said to be two-sided.

Throughout the rest of the paper we reserve the symbol \(S\) to denote the unknown target set (a measurable subset of \(\mathbb{R}^n\)) that is being tested for convexity. If \(S(x) = 1\) then we say that \(x\) is a positive point, and if \(S(x) = 0\) we say \(x\) is a negative point.

Given a finite set \(T\) of labeled samples \((x, b)\) with \(x \in \mathbb{R}^n\) and \(b \in \{0,1\}\), we say \(x\) is a positive point in \(T\) if \((x, 1) \in T\) and is a negative point in \(T\) if \((x, 0) \in T\). We use \(T^+\) to denote the set of positive points \(\{x : (x, 1) \in T\}\), and \(T^-\) to denote the set of negative points \(\{x : (x, 0) \in T\}\).
3 A useful structural result: Bounding the volume of the thickened boundary of bounded convex bodies

For a bounded convex set $C$ in $\mathbb{R}^n$ (i.e., $\sup_{c \in C} ||c|| \leq K$ for some real $K$) we may view $\partial C + \text{Ball}(\alpha)$ as the “$\alpha$-thickened boundary” of $C$. In this section, we use Theorem 6 of [Bal93] to give an upper bound on the volume of the $\alpha$-thickened boundary of such a set:

**Theorem 7.** If $C \subset \mathbb{R}^n$ is convex and $\sup_{c \in C} ||c|| \leq K$ for some $K > 1$, then we have

$$\text{Vol}(\partial C + \text{Ball}(\alpha)) \leq 20n^{5/8}K\sqrt{\alpha}, \text{ for any } 0 < \alpha < n^{-3/4}.$$

Having such a bound will be useful to us in two different contexts. First, it plays an important role in the proof of correctness of our one-sided algorithm for testing convexity (see Section 4). Second, as an easy consequence of the theorem, we get an algorithm which, for any $\tau > 0$, constructs a $\tau$-cover of $C_{\text{convex}}$ (this is Corollary 33, which we defer to later as its proof employs a “gridding” argument which we introduce in Section 4). This cover construction algorithm plays an important role in our two-sided algorithm for testing convexity (see Section 7).

### 3.1 Proof of Theorem 7

Let $C \subset \mathbb{R}^n$ be a bounded convex set that satisfies $\sup_{c \in C} ||c|| \leq K$ for some $K > 1$.

The proof has two cases and uses Lemmas 34, 35, and 36 to be proved later.

**Case I:** $C$ contains no ball of radius $\rho := \sqrt{\alpha}/n^{3/8}$. In this case we have

$$\text{Vol}(\partial C + \text{Ball}(\alpha)) \leq \text{Vol}(C + \text{Ball}(\alpha)) \leq 2(n\rho + \alpha)$$

(Lemma 34)

$$\leq 3n^{5/8}\sqrt{\alpha}$$

(using $\alpha < n^{-3/4}$)

$$< 20n^{5/8}K\sqrt{\alpha}$$

(using $K > 1$)

**Case II:** $C$ contains some ball of radius $\rho$. We let $z^*$ be the center of such a ball and let $D = \partial C + \text{Ball}(\alpha)$. To upperbound $\text{Vol}(D)$, we define a set that contains $D$ and then upperbound its volume.

To this end, we first shift $C$ to get $C' = C - z^*$ (so that the ball of radius $\rho$ is now centered at the origin). By triangle inequality we have $\sup_{c \in C'} ||c|| \leq 2K$. Let $\beta = n^{3/8}\sqrt{\alpha} = \alpha/\rho$, and observe that since $\alpha < n^{-3/4}$ we have $\beta < 1$. Let $D' = D - z^* = \partial C' + \text{Ball}(\alpha)$. By Lemma 35, we have

$$C'_0 := (1 - \beta)C' = (1 - \beta)(C - z^*)$$

contains no point of $D'$, and then by Lemma 36 the set $C'_1 := (C'_0)_h$ with $h = 4\beta K + \alpha$ contains all of $D'$. As a result, $D' \subseteq C'_1 \setminus C'_0$ and it suffices to upperbound $\text{Vol}(z^* + C'_1 \setminus C'_0)$, which is at most $4hn^{1/4}$ by Theorem 6 (since $C'_0$ is convex). Combining everything together, we have

$$\text{Vol}(D) \leq \text{Vol}(z^* + C'_1 \setminus C'_0) \leq (4\beta K + \alpha)(4n^{1/4}) \leq 20n^{5/8}K\sqrt{\alpha}.$$

(again using $K > 1$ and $\alpha < n^{-3/4}$ for the last inequality).

It remains to prove Lemmas 34, 35, and 36. We prove these lemmas in Appendix A.

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1 Recall that $(C'_0)_h$ is the set of all points that have distance at most $h$ to $C'_0$. Also note that the coefficient of $\beta K$ in our choice of $h$ is 4 instead of 2 since we have $\sup_{c \in C'} ||c|| \leq 2K$ instead of $K$. 

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6
4 One-sided upper bound: Proof of Theorem 2

Recall Theorem 2:

**Theorem 2.** For any $\varepsilon > 0$, there is a one-sided sample-based $\varepsilon$-tester for convexity over $\mathcal{N}(0, 1)^n$ which uses $(n/\varepsilon)^{O(n)}$ samples.

In Section 4.1 we show that it suffices to test convex bodies contained in a large ball $B$ centered at the origin (rather than all of $\mathbb{R}^n$) and give some useful preliminaries. Section 4.2 then builds on Theorem 7 (the upper bound on the volume of the “thickened boundary” of any bounded convex body) to give an upper bound, in the case that $S$ is convex and contained in $B$, on the total volume of certain “boundary cubes” (defined in Section 4.1). In Section 4.3 we present the one-sided testing algorithm and establish its correctness, thus proving Theorem 2.

4.1 Setup

Let $n'$ be the following parameter (that depends on both $n$ and $\varepsilon$):

$$n' := \left( n + 4\sqrt{n \ln(4/\varepsilon)} \right)^{1/2}.$$

Let $C_{\text{convex}}'$ denote the set of convex bodies in $\mathbb{R}^n$ that are contained in $\text{Ball}(n')$, equivalently,

$$C_{\text{convex}}' = \{ C \cap \text{Ball}(n') : C \in C_{\text{convex}} \}.$$

We prove the following claim that helps us focus on testing of $C_{\text{convex}}'$ instead $C_{\text{convex}}$.

**Claim 8.** Suppose that there is a one-sided sample-based $\varepsilon$-testing algorithm $A'$ which, given any Lebesgue measurable target set $S$ contained in $\text{Ball}(n')$, uses $(n/\varepsilon)^{O(n)}$ samples drawn from $\mathcal{N}(0, 1)^n$ to test whether $S \in C_{\text{convex}}'$ versus $S$ is $\varepsilon$-far from $C_{\text{convex}}'$. Then this implies Theorem 2.

**Proof.** Given $A'$ for $C_{\text{convex}}'$, we consider an algorithm $A$ which works as follows to test whether an arbitrary Lebesgue measurable subset $S$ of $\mathbb{R}^n$ is convex or $\varepsilon$-far from $C_{\text{convex}}$: algorithm $A$ runs $A'$ with parameter $\varepsilon/2$, but with the following modification: each time $A'$ receives from the oracle a labeled sample $(x, b)$ with $x \notin \text{Ball}(n')$, it replaces the label $b$ with 0 and gives the modified labeled sample to $A'$. When the run of $A'$ is complete $A$ returns the output of $A'$.

If $S \subseteq \mathbb{R}^n$ is the target set, then it is clear that the above modification results in running $A'$ on $S \cap \text{Ball}(n')$. If $S$ is convex, then $S \cap \text{Ball}(n')$ is also convex. As $A'$ commits only one-sided error, it will always output “accept,” and hence so will $A$. On the other hand, suppose that $S$ is $\varepsilon$-far from $C_{\text{convex}}$. We claim that $\text{Vol}(\text{Ball}(n')) \geq 1 - \varepsilon/4$ (this will be shown below); given this claim, it must be the case that $S \cap \text{Ball}(n')$ is at least $(3\varepsilon/4)$-far from $C_{\text{convex}}$ and at least $(3\varepsilon/4)$-far from $C_{\text{convex}}'$ as well. Consequently $A'$ will output “reject” with probability at least $2/3$, and hence so will $A$.

To bound $\text{Vol}(\text{Ball}(n'))$, observe that it is the probability that an $x \sim \mathcal{N}(0, 1)^n$ has

$$\|x\|^2 \leq n + 4\sqrt{n \ln(4/\varepsilon)}.$$

It follows from Lemma 5 that the probability is at least $1 - \varepsilon/4$ as claimed. \qed

Given Claim 8, it suffices to prove the following slight variant of Theorem 2:
Theorem 9. There is a one-sided sample-based $\varepsilon$-testing algorithm $A'$ which, given any Lebesgue measurable target set $S$ contained in $\text{Ball}(n')$, uses $(n/\varepsilon)^{O(n)}$ samples from $\mathcal{N}(0, 1)^n$ to test whether $S \in C'_{\text{convex}}$ versus $S$ is $\varepsilon$-far from $C'_{\text{convex}}$.

In the rest of this section we prove Theorem 9. We start with some terminology and concepts that we use in the description and analysis of our algorithm. Some of the notions that we introduce below, such as the notions of “boundary” cubes and “internal” cubes, are inspired by related notions that arise in earlier works such as [Ker92, Ras03].

Fix $\ell := \varepsilon^3/n^4$ in the rest of the section, and let $\text{Cube}_0$ denote the following set

$$\text{Cube}_0 := [-\ell/2, \ell/2)^n \subset \mathbb{R}^n$$

of side length $\ell$ that is centered at the origin. We say that a cube is a subset of $\mathbb{R}^n$ of the form $\text{Cube}_0 + \ell \cdot (i_1, \ldots, i_n)$, where each $i_j \in \mathbb{Z}$, which contains at least one point of $\text{Ball}(2n')$. We use $\text{CubeSet}$ to denote the set of all such cubes.

It is easy to see that

$$\text{Ball}(n') \subset \text{union of all cubes in } \text{CubeSet} \subset \text{Ball}(2n' + \ell\sqrt{n}) \subset \text{Ball}(3n').$$

Fix an $S \subseteq \text{Ball}(n')$ as the target set being tested for membership in $C'_{\text{convex}}$. Additionally fix a finite set $T = \{(x^1, S(x^1)), \ldots, (x^M, S(x^M))\}$ of labeled samples according to $S$, for some positive integer $M$. (The set $T$ will correspond to the set of labeled samples that the testing algorithm receives.) We classify cubes in the $\text{CubeSet}$ based on $T$ in the following way:

- A cube $\text{Cube}$ is said to be an external cube if $\text{Cube} \cap T^+ = \emptyset$ (i.e., no positive point of $T$ lies in $\text{Cube}$). We let $\text{EC}$ denote the union of all the external cubes.
• Any cube which is not an external cube (equivalently, any cube that contains at least one positive point of $T$) is said to be a positive cube.

• We say that two cubes $\text{Cube}$, $\text{Cube}'$ are adjacent if for any $\kappa > 0$ there exist $x \in \text{Cube}$ and $y \in \text{Cube}'$ that have Euclidean distance at most $\kappa$ (in other words, two cubes are adjacent if their closure “touch anywhere, even only at a vertex;” note that each cube is adjacent to itself). If a cube is both (i) a positive cube and (ii) is adjacent to a cube (including itself) that contains at least one negative point of $T$, then we call it a boundary cube. We use $BC$ to denote the union of all boundary cubes.

• We say that a positive cube which is not a boundary cube is an internal cube. (Equivalently, a cube is internal if and only if it contains at least one positive point and all the points in $T$ that are contained in any of its adjacent cubes, including itself, are positive.) We use $IC$ to denote the union of all internal cubes.

We note that since each cube is either external, internal, or boundary, the set $\text{Ball}(n')$ is contained in the (disjoint) union of $EC, BC$ and $IC$. Figure 1 illustrates the different types of cubes.

We will use the following useful property of internal cubes:

**Lemma 10.** Suppose a finite set of labeled samples $T$ is such that every cube in CubeSet contains at least one point of $T$. Then every internal cube is contained in $\text{Conv}(T^+)$.

The lemma is a direct consequence of the following claim by setting $H = T^+$:

**Claim 11.** Let $H \subseteq \mathbb{R}^n$ be any set that contains at least one point in each cube that is adjacent to $\text{Cube}_0$. Then $\text{Cube}_0$ is contained in $\text{Conv}(H)$.

**Proof.** We prove the claim by induction on the dimension $n$. When $n = 1$ the claim is trivial since $\text{Cube}_0$ is simply the interval $[-\ell/2, \ell/2]$ and by assumption, there is at least one point of $H$ in $[-3\ell/2, -\ell/2]$ and at least one point of $H$ in $[\ell/2, 3\ell/2]$.

For $n > 1$, let $P = \{p \in H \mid p_n \geq \ell/2\}$ and $P' = \{p' \in H \mid p'_n \leq -\ell/2\}$ be two subsets of $H$. Intuitively, the convex hulls of $P$ and $P'$ “cover” $\text{Cube}_0$ on both sides (by induction), so the convex hull of their union will contain the whole $\text{Cube}_0$. More formally, let $x$ be any point in $\text{Cube}_0$. By projecting $P, P'$ and $x$ onto the first $n-1$ dimensions and using the inductive hypothesis\(^2\), we can find points $y \in \text{Conv}(P)$ and $y' \in \text{Conv}(P')$ such that $y_i = y'_i = x_i$ for all $i \in [n-1]$. Since we have $p_n \geq 1/2$ and $p'_n \leq -1/2$ for all $p \in P$ and $p' \in P'$, respectively, it follows directly that $y_n \geq 1/2$ and $y'_n \leq -1/2$. As $x \in \text{Cube}_0$, $x$ is on the line segment between $y$ and $y'$ and thus is in the convex hull of $H$. Hence all of $\text{Cube}_0$ is contained in $\text{Conv}(H)$. \qed

4.2 Bounding the total volume of boundary cubes

Before presenting our algorithm we record the following useful corollary of Theorem 7, which allows the one-sided tester to reject bodies as non-convex if it detects too much volume in boundary cubes. (Note that we do not assume below that $T$ satisfies the condition of Lemma 10, i.e., that $T$ has at least one point in each cube in CubeSet, though this will be the case when we use it later.)

\(^2\)Observe that after projecting out the last coordinate, the assumed property of $H$ (that it has at least one sample point in each adjacent cube) will still hold in $n-1$ dimensions.
Algorithm $A^*$: Given access to independent draws $(x, S(x))$ where $x \sim \mathcal{N}(0,1)^n$ and the target set $S$ is a Lebesgue measurable set that is contained in $\text{Ball}(n')$.

1. Draw a set $T$ of $s := (n/\varepsilon)^{O(n)}$ labeled samples $(x, S(x))$, where each $x \sim \mathcal{N}(0,1)^n$.
2. If any cube does not contain a point of $T$, then halt and output “accept.”
3. If $\text{Vol}(BC) \geq \varepsilon/4$ (the volume of the union of boundary cubes), halt and output “reject.”
4. Define $I \subseteq \mathbb{R}^n$ to be $\text{Conv}(T^+)$, the convex hull of all positive points in $T$.
5. Draw a single fresh labeled sample $(y, S(y))$, where $y \sim \mathcal{N}(0,1)^n$. If $y \in I$ but $S(y) = 0$ then halt and output “reject.” Otherwise, halt and output “accept.”

Figure 2: Description of the algorithm $A^*$

Corollary 12. Let $S$ be a convex set in $\mathcal{C}_{\text{convex}}'$ and $T$ be any finite set of labeled samples according to $S$, which defines sets $EC, IC$ and $BC$ as discussed earlier. Then we have

$$\text{Vol}(BC) \leq 20n^{5/8}n' \sqrt{2\ell \sqrt{n}} = o(\varepsilon).$$

Proof. Let Cube be a boundary cube. Then by definition, there is a positive point of $T$ (call it $t$) in Cube, and there is a Cube' adjacent to Cube that contains a negative point of $T$ (call it $t'$). It follows that there must be a boundary point of $\partial S$ (call it $t^*$) in the segment between $t$ and $t'$, and we have $\text{Cube} \in t^* + \text{Ball}(2\ell \sqrt{n})$. It follows that $BC \subseteq \partial S + \text{Ball}(2\ell \sqrt{n})$, and hence

$$\text{Vol}(BC) \leq \text{Vol}(\partial S + \text{Ball}(2\ell \sqrt{n})) \leq 20n^{5/8}n' \sqrt{2\ell \sqrt{n}} = o(\varepsilon)$$

by Theorem 7 (and using $\ell \sqrt{n} \ll n^{-3/4}$ by our choice of $\ell = \varepsilon^3/n^4$).

4.3 The one-sided testing algorithm

Now we describe and analyze the one-sided testing algorithm $A'$ mentioned in Theorem 9. Algorithm $A'$ works by performing $O(1/\varepsilon)$ independent runs of the algorithm $A^*$, which we describe in Figure 2. If any of the runs of $A^*$ output “reject” then algorithm $A'$ outputs “reject,” and otherwise it outputs “accept.”

In words, Algorithm $A^*$ works as follows: first, in Step 1 it draws enough samples so that (with very high probability) it will receive at least one sample in each cube (if the low-probability event that this does not occur takes place, then the algorithm outputs “accept” since it can only reject if it is impossible for $S$ to be convex). If the region “close to the boundary” of $S$ (as measured by $\text{Vol}(BC)$ in Step 3) is too large, then the set cannot be convex (by Corollary 12) and the algorithm rejects. Finally, the algorithm checks a freshly drawn point; if this point is in the convex hull of the positive samples but is labeled negative, then the set cannot be convex and the algorithm rejects. Otherwise, the algorithm accepts.
To establish correctness and prove Theorem 9 we must show that (i) algorithm $A^*$ never rejects if the target set $S$ is a Lebesgue measurable set that belongs to $C_{\text{convex}}'$ and (ii) if $S$ is $\varepsilon$-far from $C_{\text{convex}}'$ then algorithm $A^*$ rejects with probability at least $\Omega(\varepsilon)$. Part (i) is trivial as $A^*$ only rejects if either (a) $\text{Vol}(BC) \geq \varepsilon/4$ or (b) step 5 identifies a negative point in the convex hull of the positive points in $T$. For both cases we conclude (using Corollary 12 for (a)) that $S \notin C_{\text{convex}}'$.

For (ii) suppose that $S$ is $\varepsilon$-far from $C_{\text{convex}}'$. Let $E$ be the following event (over the draw of $T$):

Event $E$: Every cube in CubeSet contains at least one point of $T$ (so the algorithm does not accept in Step 2) and moreover, every Cube with

$$\frac{\text{Vol}(\text{Cube} \cap S)}{\text{Vol}(\text{Cube})} \geq \varepsilon/4$$

contains at least one positive point in $T$ and thus, is not external.

It is easy to show that the probability mass of each cube in CubeSet is at least $(\varepsilon/n)^{O(n)}$ (since its volume is $(\varepsilon/n)^{O(n)}$ and the density function of the Gaussian is at least $1/\varepsilon^{O(n)}$ using our choice of $n$), it follows from a union bound over CubeSet that, for a suitable choice of $s = (n/\varepsilon)^{O(n)}$ (with a large enough coefficient in the exponent), $E$ occurs with probability $1 - o(1)$. Assuming that $E$ occurs, we show below that either $\text{Vol}(BC) \geq \varepsilon/4$ or $A^*$ rejects in Step 5 with probability $\Omega(\varepsilon)$.

For this purpose, we assume below that both $E$ occurs and $\text{Vol}(BC) < \varepsilon/4$. Note that the set $I$ is convex and is contained in $\text{Ball}(n')$. Thus it belongs to $C_{\text{convex}}'$ and consequently $\text{Vol}(I \triangle S) \geq \varepsilon$ (since $S$ is assumed to be $\varepsilon$-far from $C_{\text{convex}}'$), which implies that

$$\text{Vol}(S \setminus I) + \text{Vol}(I \setminus S) \geq \varepsilon.$$

It suffices to show that $\text{Vol}(S \setminus I) \leq \varepsilon/2$, since $\text{Vol}(I \setminus S)$ is exactly the probability that algorithm $A^*$ rejects in Step 5. To see that $\text{Vol}(S \setminus I) \leq \varepsilon/2$, observe that by Lemma 10, $\text{Vol}(S \setminus I)$ is at most $\text{Vol}(S \cap BC) + \text{Vol}(S \cap EC)$. On the one hand, $\text{Vol}(S \cap BC) \leq \text{Vol}(BC) < \varepsilon/4$ by assumption. On the other hand, given the event $E$, every external cube has at most $(\varepsilon/4)$-fraction of its volume in $S$ and thus, $\text{Vol}(S \cap EC) \leq \varepsilon/4$ (as the total volume of $EC$ is at most 1). Hence $\text{Vol}(S \setminus I) \leq \varepsilon/2$.

This concludes the proof of Theorem 9.

5 Two-sided lower bound

We recall Theorem 3:

**Theorem 3.** There exists a positive constant $\varepsilon_0$ such that any two-sided sample-based algorithm that is an $\varepsilon$-tester for convexity over $\mathcal{N}(0, 1)^n$ for some $\varepsilon \leq \varepsilon_0$ must use $2^{\Omega(\sqrt{n})}$ samples.

Let $q = 2^{0.01\sqrt{n}}$ and let $\varepsilon_0$ be a positive constant to be specified later. To prove Theorem 3, we show that no sample-based, $q$-query (randomized) algorithm $A$ can achieve the following goal:

Let $S \subset \mathbb{R}^n$ be a target set that is Lebesgue measurable. Let $x_1, \ldots, x_q$ be a sequence of $q$ samples drawn from $\mathcal{N}(0, 1)^n$. Upon receiving $((x_i, S(x_i)) : i \in [q])$, $A$ accepts with probability at least $2/3$ when $S$ is convex and rejects with probability at least $2/3$ when $S$ is $\varepsilon_0$-far from convex.

Recall that a pair $(x, b)$ with $x \in \mathbb{R}^n$ and $b \in \{0, 1\}$ is a labeled sample. Thus, a sample-based algorithm $A$ is simply a randomized map from a sequence of $q$ labeled samples to $\{\text{"accept"}, \text{"reject"}\}$. 
5.1 Proof Plan

Assume for contradiction that there is a $q$-query (randomized) algorithm $A$ that accomplishes the task above. In Section 5.2 we define two probability distributions $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ such that (1) $\mathcal{D}_{\text{yes}}$ is a distribution over convex sets in $\mathbb{R}^n$ ($\mathcal{D}_{\text{yes}}$ is a distribution over certain convex polytopes that are the intersection of many randomly drawn halfspaces), and (2) $\mathcal{D}_{\text{no}}$ is a probability distribution over sets in $\mathbb{R}^n$ that are Lebesgue measurable ($\mathcal{D}_{\text{no}}$ is actually supported over a finite number of measurable sets in $\mathbb{R}^n$) such that $S \leftarrow \mathcal{D}_{\text{no}}$ is $\varepsilon_0$-far from convex with probability at least $1 - o(1)$.

Given a sequence $x = (x_1, \ldots, x_q)$ of points, we abuse the notation and write

$$S(x) = (S(x_1), \ldots, S(x_q))$$

and use $(x, S(x))$ to denote the sequence of $q$ labeled samples $(x_1, S(x_1)), \ldots, (x_q, S(x_q))$. It then follows from our assumption on $A$ that

$$\Pr_{S \leftarrow \mathcal{D}_{\text{yes}}; x \leftarrow (\mathcal{N}^n)^q} [A \text{ accepts } (x, S(x))] \geq 2/3 \quad \text{and}$$

$$\Pr_{S \leftarrow \mathcal{D}_{\text{no}}; x \leftarrow (\mathcal{N}^n)^q} [A \text{ accepts } (x, S(x))] \leq 1/3 + o(1).$$

where we use $x \leftarrow (\mathcal{N}^n)^q$ to denote a sequence of $q$ points sampled independently from $\mathcal{N}^n$ and we usually skip the $\leftarrow (\mathcal{N}^n)^q$ part in the subscript when it is clear from the context. Since $A$ is a mixture of deterministic algorithms, there exists a deterministic sample-based, $q$-query algorithm $A'$ (equivalently, a deterministic map from sequences of $q$ labeled samples to \{“Yes”, “No”\}) with

$$\Pr_{x \leftarrow (\mathcal{N}^n)^q} [A' \text{ accepts } (x, S(x))] - \Pr_{S \leftarrow \mathcal{D}_{\text{no}}; x \leftarrow (\mathcal{N}^n)^q} [A' \text{ accepts } (x, S(x))] \geq 1/3 - o(1). \tag{2}$$

Let $\mathcal{E}_{\text{yes}}$ (or $\mathcal{E}_{\text{no}}$) be the distribution of $(x, S(x))$, where $x \leftarrow (\mathcal{N}^n)^q$ and $S \leftarrow \mathcal{D}_{\text{yes}}$ (or $S \leftarrow \mathcal{D}_{\text{no}}$, respectively). Both of them are distributions over sequences of $q$ labeled samples. Then the LHS of (2), for any deterministic sample-based, $q$-query algorithm $A'$, is at most the total variation distance between $\mathcal{E}_{\text{yes}}$ and $\mathcal{E}_{\text{no}}$. We prove the following key lemma, which leads to a contradiction.

**Lemma 13.** The total variation distance between $\mathcal{E}_{\text{yes}}$ and $\mathcal{E}_{\text{no}}$ is $o(1)$.

To prove Lemma 13, it is convenient for us to introduce a third distribution $\mathcal{E}_{\text{no}}^*$ over sequences of $q$ labeled samples, where $(x, b) \leftarrow \mathcal{E}_{\text{no}}^*$ is drawn by first sampling a sequence of $q$ points $x = (x_1, \ldots, x_q)$ from $\mathcal{N}^n$ independently and then for each $x_i$, its label $b_i$ is set to be 1 independently with a probability that depends only on $\|x_i\|$ (see Section 5.2). Lemma 13 follows from the following two lemmas by the triangle inequality.

**Lemma 14.** The total variation distance between $\mathcal{E}_{\text{no}}$ and $\mathcal{E}_{\text{no}}^*$ is $o(1)$.

**Lemma 15.** The total variation distance between $\mathcal{E}_{\text{yes}}$ and $\mathcal{E}_{\text{no}}^*$ is $o(1)$.

The rest of the section is organized as follows. We define the distributions $\mathcal{D}_{\text{yes}}, \mathcal{D}_{\text{no}}$ (which are used to define $\mathcal{E}_{\text{yes}}$ and $\mathcal{E}_{\text{no}}$) as well as $\mathcal{E}_{\text{no}}^*$ in Section 5.2 and prove the necessary properties about $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ as well as Lemma 14. We prove Lemma 15 in Sections 5.3 and 5.4.
5.2 The Distributions

Let \( r = \Theta(n^{1/4}) \) be a parameter to be fixed later, and let \( N = 2\sqrt{n} \). We start with the definition of \( \mathcal{D}_{\text{yes}} \). A random set \( S \subset \mathbb{R}^n \) is drawn from \( \mathcal{D}_{\text{yes}} \) using the following procedure:

1. We sample a sequence of \( N \) points \( y_1, \ldots, y_N \) from \( S^{n-1}(r) \) independently and uniformly at random. Each point \( y_i \) defines a halfspace
   \[
   h_i = \{ x \in \mathbb{R}^n : x \cdot y_i \leq r^2 \}.
   \]

2. The set \( S \) is then the intersection of \( h_i, i \in [N] \) (this is always nonempty as indeed \( \text{Ball}(r) \) is contained in \( S \)).

It is clear from the definition that \( S \leftarrow \mathcal{D}_{\text{yes}} \) is always a convex set.

Next we define \( \mathcal{E}_{\text{no}}^* \) (instead of \( \mathcal{D}_{\text{no}} \)), a distribution over sequences of \( q \) labeled samples \( (x, b) \). To this end, we use \( \mathcal{D}_{\text{yes}} \) to define a function \( \rho : \mathbb{R}_0^+ \rightarrow [0, 1] \) as follows:

\[
\rho(t) = \mathbf{Pr}_{S \leftarrow \mathcal{D}_{\text{yes}}} \{ (t, 0, \ldots, 0) \in S \}.
\]

Due to the symmetry of \( \mathcal{D}_{\text{yes}} \) and \( \mathcal{N}^n \), the value \( \rho(t) \) is indeed the probability that a point \( x \in \mathbb{R}^n \) at distance \( t \) from the origin lies in \( S \leftarrow \mathcal{D}_{\text{yes}} \). To draw a sequence of \( q \) labeled samples \( (x, b) \leftarrow \mathcal{E}_{\text{no}}^* \), we first independently draw \( q \) random points \( x_1, \ldots, x_q \leftarrow \mathcal{N}^n \) and then independently set each \( b_i = 1 \) with probability \( \rho(\|x_i\|) \) and \( b_i = 0 \) with probability \( 1 - \rho(\|x_i\|) \).

Given \( \mathcal{D}_{\text{yes}} \) and \( \mathcal{E}_{\text{no}}^* \), Lemma 15 shows that information-theoretically no sample-based algorithm can distinguish a sequence of \( q \) labeled samples \( (x, b) \leftarrow \mathcal{E}_{\text{no}}^* \) from a sequence of \( q \) labeled samples drawn from \( \mathcal{E}_{\text{no}}^* \). While the marginal distribution of each labeled sample is the same for the two cases, the former is generated in a correlated fashion using the underlying random convex \( S \leftarrow \mathcal{D}_{\text{yes}} \) while the latter is generated independently.

Finally we define the distribution \( \mathcal{D}_{\text{no}} \), prove Lemma 14, and show that a set drawn from \( \mathcal{D}_{\text{no}} \) is far from convex with high probability. To define \( \mathcal{D}_{\text{no}} \), we let \( M \geq 2\sqrt{n} \) be a large enough integer to be specified later. With \( M \) fixed, we use

\[
0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 2\sqrt{n}
\]

to denote a sequence of numbers such that the origin-centered ball \( \text{Ball}(2\sqrt{n}) \) is partitioned into \( M \) shells \( \text{Ball}(t_i) \setminus \text{Ball}(t_{i-1}), i \in [M] \), and all the \( M \) shells have the same probability mass under \( \mathcal{N}^n \). By spherical coordinates, it means that the following integral takes the same value for all \( i \):

\[
\int_{t_{i-1}}^{t_i} \phi(x, 0, \ldots, 0)x^{n-1}dx,
\]

where \( \phi \) denotes the density function of \( \mathcal{N}^n \). We show below that when \( M \) is large enough, we have

\[
|\rho(x) - \rho(t_i)| \leq 2^{-\sqrt{n}},
\]

for any \( i \in [M] \) and any \( x \in [t_{i-1}, t_i] \). We will fix such an \( M \) and use it to define \( \mathcal{D}_{\text{no}} \). (Our results are not affected by the size of \( M \) as a function of \( n \); we only need it to be finite, given \( n \).)

To show that (4) holds when \( M \) is large enough, we need the continuity of the function \( \rho \), which follows directly from the explicit expression for \( \rho \) given later in (6).
Lemma 16. The function $\rho : R_{\geq 0} \to [0,1]$ is continuous.

Since $\rho$ is continuous, it is continuous over $[0,2\sqrt{n}]$. Since $[0,2\sqrt{n}]$ is compact, $\rho$ is also uniformly continuous over $[0,2\sqrt{n}]$. Also note that max$_{i \in [M]}(t_i - t_{i-1})$ goes to 0 as $M$ goes to $+\infty$. It follows that (4) holds when $M$ is large enough.

With $M \geq 2\sqrt{n}$ fixed, a random set $S \leftarrow D_{no}$ is drawn as follows. We start with $S = \emptyset$ and for each $i \in [M]$, we add the $i$th shell $\text{Ball}(t_i) \setminus \text{Ball}(t_{i-1})$ to $S$ independently with probability $\rho(t_i)$. Thus an outcome of $S$ is a union of some of the shells and $D_{no}$ is supported over $2^M$ different sets.

Recall the definition of $E_{yes}$ and $E_{no}$ using $D_{yes}$ and $D_{no}$. We now prove Lemma 14.

Proof of Lemma 14. Let $x = (x_1, \ldots, x_q)$ be a sequence of $q$ points in $R^n$. We say $x$ is bad if either (1) at least one point lies outside of Ball$(2\sqrt{n})$ or (2) there are two points that lie in the same shell of $D_{no}$; we say $x$ is good otherwise. We first claim that $x \leftarrow (N^n)^q$ is good with probability $o(1)$. To see this, we have from Lemma 5 that event (1) occurs with probability $o(1)$, and from $M \geq 2\sqrt{n}$ and $q = 2^{0.01\sqrt{n}}$ that event (2) occurs with probability $o(1)$. The claim follows from a union bound.

Given that $x \leftarrow (N^n)^q$ is good with probability $1 - o(1)$, it suffices to show that for any good $q$-tuple $x$, the total variation distance between (1) $S(x)$ with $S \leftarrow D_{no}$ and (2) $b = (b_1, \ldots, b_q)$ with each bit $b_i$ being 1 with probability $\rho(||x_i||)$ independently, is $o(1)$. Let $\ell_i \in [M]$ be the index of the shell that $x_i$ lies in. Since $x$ is good (and thus, all points lie in different shells), $S(x)$ has the $i$th bit being 1 independently with probability $\rho(t_{\ell_i})$; for the other distribution, the probability is $\rho(||x_i||)$. Using the subadditivity of total variation distance (i.e., the fact that the $d_{TV}$ between two sequences of independent random variables is upper bounded by the sum of the $d_{TV}$ between each pair) as well as (4), we have $d_{TV}(S(x), b) \leq q \cdot 2^{-\sqrt{n}} = o(1)$. This finishes the proof. \hfill \Box

The next lemma shows that $S \leftarrow D_{no}$ is $\varepsilon_0$-far from convex with probability $1 - o(1)$, for some positive constant $\varepsilon_0$. In the proof of the lemma we fix both the constant $\varepsilon_0$ and our choice of $r = \Theta(n^{1/4})$. (We remind the reader that $\rho$ and $D_{no}$ both depend on the value of $r$.)

Lemma 17. There exist a real value $r = \Theta(n^{1/4})$ with $e^{r^2/2} \geq N/n$ and a positive constant $\varepsilon_0$ such that a set $S \leftarrow D_{no}$ is $\varepsilon_0$-far from convex with probability at least $1 - o(1)$.

Proof. We need the following claim but delay its proof to the end of the subsection:

Claim 18. There exist an $r = \Theta(n^{1/4})$ with $e^{r^2/2} \geq N/n$ and a constant $c \in (0,1/2)$ such that $c < \rho(x) < 1 - c$, for all $x \in [\sqrt{n} - 10, \sqrt{n} + 10]$.

Let $K \subset [M]$ denote the set of all integers $k$ such that $[t_{k-1}, t_k] \subseteq [\sqrt{n} - 10, \sqrt{n} + 10]$ (note that $K$ is a set of consecutive integers). Observe that (1) the total probability mass of all shells $k \in K$ is at least $\Omega(1)$ (by Lemma 5), and (2) the size $|K|$ is at least $\Omega(M)$ (which follows from (1) and the fact that all shells have the same probability mass).

Consider the following 1-dimensional scenario. We have $|K|$ intervals $[t_{k-1}, t_k]$ and draw a set $T$ by including each interval independently with probability $\rho(t_k)$. We prove the following claim:

Claim 19. The random set $T$ satisfies the following property with probability at least $1 - o(1)$: For any interval $I \subseteq R_{\geq 0}$, either $I$ contains $\Omega(M)$ intervals $[t_{k-1}, t_k]$ that are not included in $T$, or $\bar{I}$ contains $\Omega(M)$ intervals $[t_{k-1}, t_k]$ included in $T$. 

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Proof. First note that it suffices to consider intervals \( I \subseteq \cup_{k \in K} [t_{k-1}, t_k] \) and moreover, we may further assume that both endpoints of \( I \) come from endpoints of \([t_{k-1}, t_k], k \in K\). (In other words, for a given outcome \( T \) of \( T \), if there exists an interval \( I \) that violates the condition, i.e., both \( I \) and \( T \) contain fewer than \( \Omega(M) \) intervals, then there is such an interval \( I \) with both ends from end points of \([t_{k-1}, t_k]\). This assumption allows us to focus on \( |K|^2 \leq M^2 \) many possibilities for \( I \) (as we will see below, our argument applies a union bound over these \( K^2 \) possibilities).

Given a candidate such interval \( I \), we consider two cases. If \( I \) contains \( \Omega(M) \) intervals \([t_{k-1}, t_k], k \in K\), then it follows from Claim 18 and a Chernoff bound that \( I \) contains at least \( \Omega(M) \) intervals not included in \( T \) with probability \( 1 - 2^{-\Omega(M)} \). On the other hand, if \( T \) contains \( \Omega(M) \) intervals, then the same argument shows that \( T \) contains \( \Omega(M) \) intervals included in \( T \) with probability \( 1 - 2^{-\Omega(M)} \). The claim follows from a union bound over all the \( |K|^2 \) possibilities for \( I \).

We return to the \( n \)-dimensional setting and consider the intersection of \( S \leftarrow D_{no} \) with a ray starting from the origin. Note that the intersection of the ray and any convex set is an interval on the ray. As a result, Claim 19 shows that with probability \( 1 - o(1) \) (over the draw of \( S \leftarrow D_{no} \)), the intersection of any convex set with any ray either contains \( \Omega(M) \) intervals \([t_{k-1}, t_k]\) such that shell \( k \in K \) is not included in \( S \), or misses \( \Omega(M) \) intervals \([t_{k-1}, t_k]\) such that shell \( k \in K \) is included in \( S \). Since by (1) above shells \( k \in K \) together have \( \Omega(1) \) probability mass under \( N^n \) and each shell contains the same probability mass, we have that with probability \( 1 - o(1) \), \( S \) is \( \varepsilon_0 \)-far from any convex set for some constant \( \varepsilon_0 > 0 \). (A more formal argument can be given by performing integration using spherical coordinates and applying (3).)

Proof of Claim 18. We start with the choice of \( r \). Let

\[
\alpha = \sqrt{n} - 10 \quad \text{and} \quad \beta = \sqrt{n} + 10.
\]

Let \( \text{cap}(t) \) denote the fractional surface area of the spherical cap \( S^{n-1} \cap \{x : x_1 \geq t\} \), i.e.,

\[
\text{cap}(t) = \Pr_{x \leftarrow S^{n-1}} \left[ x_1 \geq t \right].
\]

So \( \text{cap} \) is a continuous, strictly decreasing function over \([0, 1]\). Since \( \text{cap}(0) = 1/2 \) and \( \text{cap}(1) = 0 \), there is a unique \( r \in (0, \alpha) \) such that \( \text{cap}(r/\alpha) = 1/N = 2^{-\sqrt{n}} \). Below we show that \( r = \Theta(n^{1/4}) \) and fix it in the rest of the proof. First recall the following explicit expression (see e.g. [KOS07]):

\[
\text{cap}(t) = a_n \int_t^1 \left( \sqrt{1 - z^2} \right)^{n-3} \, dz,
\]

where \( a_n = \Theta(n^{1/2}) \) is a parameter that only depends on \( n \). Also recall the following inequalities from [KOS07] about \( \text{cap}(t) \):

\[
\text{cap}(t) \leq e^{-nt^2/2}, \quad \text{for all } t \in [0, 1]; \quad \text{cap}(t) \geq \Omega \left( t \cdot e^{-mt^2/2} \right), \quad \text{for } t = O(1/n^{1/4}). \tag{5}
\]

By our choice of \( \alpha \) and the monotonicity of the cap function, this implies that \( r = \Theta(n^{1/4}) \) and

\[
1/N = \text{cap}(r/\alpha) \geq \Omega(1/n^{1/4}) \cdot e^{-n(r/\alpha)^2/2} \geq \Omega(1/n^{1/4}) \cdot e^{-(r^2/2)(1+O(1/\sqrt{n}))} = \Omega(1/n^{1/4}) \cdot e^{-r^2/2}
\]

(using \( r = \Theta(n^{1/4}) \) for the last inequality), and thus, we have \( e^{r^2/2} \geq N/n. \)
Next, using the function \( \text{cap} \) we have the following expression for \( \rho \):

\[
\rho(x) = \left( 1 - \text{cap} \left( \frac{r}{x} \right) \right)^N. \tag{6}
\]

As a side note, \( \rho \) is continuous and thus, Lemma 16 follows. Since \( \text{cap} \) is strictly decreasing, we have that \( \rho \) is strictly decreasing as well. To finish the proof it suffices to show that there is a constant \( c \in (0, 1/2) \) such that \( \rho(\alpha) < 1 - c \) and \( \rho(\beta) \geq c \). The first part is easy since

\[
\rho(\alpha) = (1 - 1/N)^N \approx e^{-1}
\]

by our choice of \( r \). In the rest of the proof we show that

\[
\text{cap} \left( \frac{r}{\beta} \right) \leq a \cdot \text{cap} \left( \frac{r}{\alpha} \right) = \frac{a}{N}, \tag{7}
\]

for some positive constant \( a \). It follows immediately that

\[
\rho(\beta) = \left( 1 - \text{cap} \left( \frac{r}{\beta} \right) \right)^N \geq \left( 1 - \frac{a}{N} \right)^N \geq \left( e^{-2a/N} \right)^N = e^{-2a},
\]

using \( 1 - x \geq e^{-2x} \) for \( 0 \leq x \ll 1 \), and this finishes the proof of the claim.

Finally we prove (7). Let

\[
w = \frac{r}{\alpha} - \frac{r}{\beta} = \Theta \left( \frac{1}{n^{3/4}} \right)
\]

since \( r = \Theta(n^{1/4}) \). Below we show that

\[
\int_{r/\beta}^{r/\alpha} (\sqrt{1 - z^2})^{n-3} dz \leq a' \cdot \int_{r/\alpha}^{r/\alpha+w} (\sqrt{1 - z^2})^{n-3} dz, \tag{8}
\]

for some positive constant \( a' \). It follows that

\[
\text{cap} \left( \frac{r}{\beta} \right) - \text{cap} \left( \frac{r}{\alpha} \right) \leq a' \cdot \text{cap} \left( \frac{r}{\alpha} \right)
\]
and implies (7) by setting \( a = a' + 1 \). For (8), note that the ratio of the \([r/\beta, r/\alpha]\)-integration over the \([r/\alpha, r/\alpha + w]\)-integration is at most

\[
\left( \frac{\sqrt{1 - (r/\beta)^2}}{\sqrt{1 - (r/\beta + 2w)^2}} \right)^{n-3}
\]

as the length of the two intervals are the same and the function \((\sqrt{1 - z^2})^{n-3}\) is strictly decreasing. Figure 3 illustrates this calculation. Let \( \tau = r/\beta = \Theta(1/n^{1/4}) \). We can rewrite the above as

\[
\left( \frac{1 - \tau^2}{1 - (\tau + 2w)^2} \right)^{(n-3)/2} = \left( 1 + \frac{4\tau w + 4w^2}{1 - (\tau + 2w)^2} \right)^{(n-3)/2} = \left( 1 + O\left( \frac{1}{n} \right) \right)^{(n-3)/2} = O(1).
\]

This finishes the proof of the claim. \( \square \)

### 5.3 Distributions \( \mathcal{E}_{\text{yes}} \) and \( \mathcal{E}_{\text{no}}^* \) are close

In the rest of the section we show that the total variation distance between \( \mathcal{E}_{\text{yes}} \) and \( \mathcal{E}_{\text{no}}^* \) is \( o(1) \) and thus prove Lemma 15. Let \( z = (z_1, \ldots, z_q) \) be a sequence of \( q \) points in \( \mathbb{R}^n \). We use \( \mathcal{E}_{\text{yes}}(z) \) to denote the distribution of labeled samples from \( \mathcal{E}_{\text{yes}} \), conditioning on the samples being \( z \), i.e., \( (z, S(z)) \) with \( S \leftarrow D_{\text{yes}} \). We let \( \mathcal{E}_{\text{no}}^*(z) \) denote the distribution of labeled samples from \( \mathcal{E}_{\text{no}}^* \), conditioning on the samples being \( z \), i.e., \( (z, b) \) where each \( b_i \) is 1 independently with probability \( \rho(\|z_i\|) \). Then

\[
d_{\text{TV}}(\mathcal{E}_{\text{yes}}, \mathcal{E}_{\text{no}}^*) = \mathbb{E}_{z \leftarrow (\mathbb{N}^n)^q} \left[ d_{\text{TV}}(\mathcal{E}_{\text{yes}}(z), \mathcal{E}_{\text{no}}^*(z)) \right]. \tag{9}
\]

We split the proof of Lemma 15 into two steps. We first introduce the notion of typical sequences \( z \) of \( q \) points and show in this subsection that with probability \( 1 - o(1) \), \( z \leftarrow (\mathbb{N}^n)^q \) is typical. In the next subsection we show that \( d_{\text{TV}}(\mathcal{E}_{\text{yes}}(z), \mathcal{E}_{\text{no}}^*(z)) \) is \( o(1) \) when \( z \) is typical. It follows from (9) that \( d_{\text{TV}}(\mathcal{E}_{\text{yes}}, \mathcal{E}_{\text{no}}^*) \) is \( o(1) \). We start with the definition of typical sequences.

Given a point \( z \in \mathbb{R}^n \), we are interested in the fraction of points \( y \) (in terms of the area) in \( S^{n-1}(r) \) such that \( z \cdot y > r^2 \). This is because if any such point \( y \) is sampled in the construction of \( S \leftarrow D_{\text{yes}} \), then \( z \notin S \). This is illustrated in Figure 4. We refer to the set of such points \( y \) as the (spherical) cap covered by \( z \) and we write \( \text{cap}(z) \) to denote it. (Note that \( \text{cap}(z) = \emptyset \) if \( \|z\| \leq r \).)

Given a subset \( H \) of \( S^{n-1}(r) \) (such as \( \text{cap}(z) \)), we use \( \text{fsa}(H) \) to denote the fractional surface area of \( H \) with respect to \( S^{n-1}(r) \). Using Figure 4 and elementary geometry, we have the following connection between the fractional surface area of \( \text{cap}(z) \) and the cap function (for \( S^{n-1} \)):

\[
\text{fsa}(\text{cap}(z)) = \text{cap}(r/\|z\|). \tag{10}
\]

We are now ready to define typical sequences.

**Definition 20.** We say a sequence \( z = (z_1, \ldots, z_q) \) of \( q \) points in \( \mathbb{R}^n \) is typical if

1. For every point \( z_i \), we have

\[
\text{fsa}(\text{cover}(z_i)) \in \left[ e^{-0.51r^2}, e^{-0.49r^2} \right]. \tag{11}
\]

2. For every \( i \neq j \), we have

\[
\text{fsa}(\text{cover}(z_i) \cap \text{cover}(z_j)) \leq e^{-0.96r^2}.
\]


The fractional surface area of \( \text{cover}(z) \), \( \text{fsa}(\text{cover}(z)) \), is the fraction of \( S^{n-1}(r) \) to the right of the dashed line. By similarity of triangles \( 0az \) and \( 0ba \), scaling down to the unit sphere, we get (10).

The first condition of typicality essentially says that every \( z_i \) is not too close to and not too far away from the origin (so that we have a relatively tight bound on the fractional surface area of the cap covered by \( z_i \)). The second condition says that the caps covered by two points \( z_i \) and \( z_j \) have very little intersection. We prove the following lemma:

**Lemma 21.** \( z \leftarrow (N^n)^q \) is typical with probability at least \( 1 - o(1) \).

**Proof.** We show that \( z \) satisfies each of the two conditions with probability \( 1 - o(1) \). The lemma then follows from a union bound.

For the first condition, we let \( c^* = 0.001 \) be a sufficiently small constant. We have from Lemma 5 and a union bound that every \( z_i \) satisfies \( (1 - c^*)\sqrt{n} \leq \|z_i\| \leq (1 + c^*)\sqrt{n} \) with probability \( 1 - o(1) \). When this happens, we have (11) for every \( z_i \) using (5) and the upper bound of \( \text{cap}(t) \leq e^{-nt^2/2} \).

For the second condition, we first note that the argument used in the first part implies that

\[
E_{z_i \leftarrow N^n} \left[ \text{fsa}(\text{cover}(z_i)) \right] \leq e^{-0.49r^2}.
\]

Let \( x_0 \) be a fixed point in \( S^{n-1}(r) \). Viewing the fractional surface area as the following probability

\[
\text{fsa}(\text{cover}(z_i)) = \Pr_{x \leftarrow S^{n-1}(r)} [x \in \text{cover}(z_i)],
\]

we have

\[
e^{-0.49r^2} \geq E_{z_i \leftarrow N^n} \left[ \text{fsa}(\text{cover}(z_i)) \right] \\
= E_{z_i} \left[ \Pr_{x \leftarrow S^{n-1}(r)} [x \in \text{cover}(z_i)] \right] \\
= \Pr_{x,z_i} [x \in \text{cover}(z_i)] = \Pr_{z_i} [x_0 \in \text{cover}(z_i)],
\]

where the last equation follows by sampling \( x \) first and spherical and Gaussian symmetry.

Similarly we can express the fractional surface area of \( \text{cover}(z_i) \cap \text{cover}(z_j) \) as

\[
\text{fsa}(\text{cover}(z_i) \cap \text{cover}(z_j)) = \Pr_{x \leftarrow S^{n-1}(r)} [x \in \text{cover}(z_i) \text{ and } x \in \text{cover}(z_j)].
\]
We consider the expectation over $z_i$ and $z_j$ drawn independently from $\mathcal{N}^n$:

$$
\mathbb{E}_{z_i, z_j} \left[ \text{fsa}(\text{cover}(z_i) \cap \text{cover}(z_j)) \right]
= \mathbb{E}_{z_i, z_j} \left[ \frac{1}{2} \mathbb{P}_{x \leftarrow S^{0-1}(r)} [x \in \text{cover}(z_i) \text{ and } x \in \text{cover}(z_j)] \right]
= \mathbb{P}_{x \leftarrow S^{0-1}(r)} [x \in \text{cover}(z_i) \text{ and } x \in \text{cover}(z_j)] = \mathbb{P}_{z_i} [x_0 \in \text{cover}(z_i)] \cdot \mathbb{P}_{z_j} [x_0 \in \text{cover}(z_j)],
$$

where the last equation follows by sampling $x$ first, independence of $z_i$ and $z_j$, and symmetry.

By (12), the expectation of $\text{fsa}(\text{cover}(z_i) \cap \text{cover}(z_j))$ is at most $e^{-0.98r^2}$, and hence by Markov’s inequality, the probability of it being at least $e^{-0.96r^2}$ is at most $e^{-0.92r^2}$. Using $e^{2r} \geq (N/n)^2$ and a union bound, the probability of one of the pairs having the fsa at least $e^{-0.96r^2}$ is at most

$$
q^2 \cdot e^{-0.02r^2} \leq 2^{0.02\sqrt{n}} \cdot (n/N)^{0.04} = o(1),
$$

since $q = 2^{0.01\sqrt{n}}$ and $N = 2\sqrt{n}$. This finishes the proof of the lemma.

We prove the following lemma in Section 5.4 to finish the proof of Lemma 15.

**Lemma 22.** For every typical sequence $z$ of $q$ points, we have $d_{TV}(\mathcal{E}_{\text{yes}}(z), \mathcal{E}_{\text{no}}^*(z)) = o(1)$.

**5.4 Proof of Lemma 22**

Fix a typical $z = (z_1, \ldots, z_q)$. Our goal is to show that the total variation distance of $\mathcal{E}_{\text{yes}}(z)$ and $\mathcal{E}_{\text{no}}^*(z)$ is $o(1)$. To this end, we define a distribution $\mathcal{F}$ over pairs $(b, d)$ of strings in $\{0,1\}^q$ (as a coupling of $\mathcal{E}_{\text{yes}}(z)$ and $\mathcal{E}_{\text{no}}^*(z)$), where the marginal distribution of $b$ as $(b, d) \leftarrow \mathcal{F}$ is the same as $\mathcal{E}_{\text{yes}}(z)$ and the marginal distribution of $d$ is the same as $\mathcal{E}_{\text{no}}^*(z)$. Our goal follows by establishing

$$
\mathbb{P}_{(b, d) \leftarrow \mathcal{F}} [b \neq d] = o(1).
\tag{13}
$$

To define $\mathcal{F}$, we use $M$ to denote the $q \times N$ $\{0,1\}$-valued random matrix derived from $z$ and $S \leftarrow \mathcal{D}_{\text{yes}}$ (recall that $S$ is the intersection of $N$ random halfspaces $h_j, j \in [N]$): the $(i, j)$th entry $M_{i,j}$ of $M$ is 1 if $h_j(z_i) = 1$ (i.e., $z_i \in h_j$) and is 0 otherwise. We use $M_{i,*}$ to denote the $i$th row of $M$, $M_{*,j}$ to denote the $j$th column of $M$, and $M^{(i)}$ to denote the $i \times N$ sub-matrix of $M$ that consists of the first $i$ rows of $M$. (We note that $M$ is derived from $S$ and they are defined over the same probability space. So we may consider the (conditional) distribution of $S \leftarrow \mathcal{D}_{\text{yes}}$ conditioning on an event involving $M$, and we may consider the conditional distribution of $M$ conditioning on an event involving $S$.)

We now define the distribution $\mathcal{F}$. A pair $(b, d) \leftarrow \mathcal{F}$ is drawn using the following randomized procedure. The procedure has $q$ rounds and generates the $i$th bits $b_i$ and $d_i$ in the $i$th round:

1. In the first round, we draw a random real number $r_1$ from $[0,1]$ uniformly at random. We set $b_1 = 1$ if $r_1 \leq \mathbb{P}_{S \leftarrow \mathcal{D}_{\text{yes}}} [S(z_1) = 1]$ and set $b_1 = 0$ otherwise. We then set $d_1 = 1$ if $r_1 \leq \rho(\|z_1\|)$ and set $d_1 = 0$ otherwise. (Note that for the first round, the two thresholds are indeed the same so we always have $b_1 = d_1$.) At the end of the first round, we also draw a row vector $N_{1,*}$ according to the distribution of $M_{1,*}$ conditioning on $S(z_1) = b_1$. 


2. In the $i$th round, for $i$ from 2 to $q$, we draw a random real number $r_i$ from $[0, 1]$ uniformly at random. We set $b_i = 1$ if we have

$$r_i \leq \Pr_{S \sim \mathcal{D}_{yes}} \left[ S(z_i) = 1 \mid M^{(i-1)} = N^{(i-1)} \right]$$

and set $b_i = 0$ otherwise. We then set $d_i = 1$ if $r_i \leq \rho(\|z_i\|)$ and set $d_i = 0$ otherwise. At the end of the $i$th round, we also draw a row vector $N_i,*$ according to the distribution of $M_{i,*}$ conditioning on $M^{(i-1)} = N^{(i-1)}$ and $S(z_i) = b_i$.

It is clear that the marginal distributions of $b$ and $d$, as $(b, d) \leftarrow \mathcal{F}$, are $\mathcal{E}_{yes}$ and $\mathcal{E}_{no}^*$ respectively.

To prove (13), we introduce the following notion of nice and bad matrices.

**Definition 23.** Let $M$ be an $i \times N \{0, 1\}$-valued matrix for some $i \in [q]$. We say $M$ is nice if

1. $M$ has at most $\sqrt{N}$ many 0-entries; and
2. Each column of $M$ has at most one 0-entry.

We say $M$ is bad otherwise.

We prove the following two lemmas and use them to prove (13).

**Lemma 24.** $\Pr_{S \sim \mathcal{D}_{yes}} \left[ M \text{ is bad} \right] = o(1/q)$.

Note that when $M$ is nice, we have by definition that $M^{(i)}$ is also nice for every $i \in [q]$.

**Lemma 25.** For any nice $(i-1) \times N \{0, 1\}$-valued matrix $M^{(i-1)}$, we have

$$\Pr_{S \sim \mathcal{D}_{yes}} \left[ S(z_i) = 1 \mid M^{(i-1)} = M^{(i-1)} \right] = \rho(\|z_i\|) \pm o(1/q). \tag{14}$$

Before proving Lemma 24 and 25, we first use them to prove (13). Let $I_i$ denote the indicator random variable that is 1 if $(b, d) \leftarrow \mathcal{E}$ has $b_i \neq d_i$ and is 0 otherwise, for each $i \in [q]$. Then (13) can be bounded from above by $\sum_{i \in [q]} \Pr[I_i = 1]$. To bound each $\Pr[I_i = 1]$ we split the event into

$$\sum_{M^{(i-1)}} \Pr \left[ N^{(i-1)} = M^{(i-1)} \right] \cdot \Pr \left[ I_i = 1 \mid N^{(i-1)} = M^{(i-1)} \right],$$

where the sum is over all $(i-1) \times N \{0, 1\}$-valued matrices $M^{(i-1)}$, and further split the sum into two sums over nice and bad matrices $M^{(i-1)}$. As $N^{(i-1)}$ has the same distribution as $M^{(i-1)}$, it follows from Lemma 24 (and the fact that $M$ is bad when $M^{(i-1)}$ is bad) that the sum over bad $M^{(i-1)}$ is at most $o(1/q)$. On the other hand, it follows from Lemma 25 that the sum over nice $M^{(i-1)}$ is $o(1/q)$. As a result, we have $\Pr[I_i = 1] = o(1/q)$ and thus, $\sum_{i \in [q]} \Pr[I_i = 1] = o(1)$.

We prove Lemmas 24 and 25 in the rest of the section.

**Proof of Lemma 24.** We show that the probability of $M$ violating each of the two conditions in the definition of nice matrices is $o(1/q)$. The lemma then follows by a union bound.

For the first condition, since $z$ is typical the probability of $M_{i,j} = 0$ is

$$\text{fsa}(\text{cover}(z_i)) \leq e^{-0.49 r^2}.$$
By linearity of expectation, the expected number of 0-entries in $M$ is at most 

$$qN \cdot e^{-0.49r^2} = o(\sqrt{N}/q),$$

using $e^{r^2/2} \geq N/n$, $N = 2\sqrt{n}$ and $q = 2^{0.01}\sqrt{n}$. It follows directly from Markov’s inequality that the probability of $M$ having more than $\sqrt{N}$ many 0-entries is $o(1/q)$.

For the second condition, again since $z$ is typical, the probability of $M_{i,j} = M_{i',j} = 1$ is

$$\text{fsa}(\text{cover}(z_i) \cap \text{cover}(z_i')) \leq e^{-0.96r^2}.$$ 
By a union bound, the probability of $M_{i,j} = M_{i',j} = 1$ for some $i, i', j$ is at most 

$$q^2N \cdot e^{-0.96r^2} = o(1/q).$$

This finishes the proof of the lemma.

Finally we prove Lemma 25. Fix a nice $(i-1) \times N$ matrix $M$ (we henceforth omit the superscript $(i-1)$ since the number of rows of $M$ is fixed to be $i-1$). Recall that $S(z_i) = 1$ if and only if $h_j(z_i) = 1$ for all $j \in [N]$. As a result, we have

$$\Pr_{S \sim D_{\text{yes}}} \left[ S(z_i) = 1 \mid M^{(i-1)} = M \right] = \prod_{j \in [N]} \Pr_{h_j} \left[ h_j(z_i) = 1 \mid M^{(i-1)}_{\ast,j} = M_{\ast,j} \right].$$

On the other hand, letting $\tau = \text{fsa}(\text{cover}(z_i)) = \text{cap}(r/\|z_i\|)$, we have $\rho(\|z_i\|) = (1 - \tau)^N$.

In the next two claims we compare

$$\Pr_{h_j} \left[ h_j(z_i) = 1 \mid M^{(i-1)}_{\ast,j} = M_{\ast,j} \right]$$

with $1 - \tau$ for each $j \in [N]$ and show that they are very close. The first claim works on $j \in [N]$ with no 0-entry in $M_{\ast,j}$ and the second claim works on $j \in [N]$ with one 0-entry in $M_{\ast,j}$. (These two possibilities cover all $j \in [N]$ since the matrix $M$ is nice.) Below we omit $M^{(i-1)}_{\ast,j}$ in writing the conditional probabilities.

Claim 26. For each $j \in [N]$ with no 0-entry in the $j$th column $M_{\ast,j}$, we have

$$\Pr_{h_j} \left[ h_j(z_i) = 1 \mid M_{\ast,j} \right] = (1 - \tau) \left( 1 \pm \frac{o(1)}{qN} \right).$$

Proof. Let $\delta$ be the probability of $h_j(z_i) = 0$ conditioning on $M_{\ast,j}$ (which is all-1). Then

$$\delta = \frac{\text{fsa}(\text{cover}(z_i) - \bigcup_{j < i} \text{cover}(z_j))}{1 - \text{fsa}(\bigcup_{j < i} \text{cover}(z_j))}.$$ 

Using $e^{-0.51r^2} \leq \text{fsa}(\text{cover}(z_j)) \leq e^{-0.49r^2}$ and $\text{fsa}(\text{cover}(z_i) \cap \text{cover}(z_j)) \leq e^{-0.96r^2}$, we have

$$\delta \leq \frac{\tau}{1 - q \cdot e^{-0.49r^2}} < \tau(1 + 2q \cdot e^{-0.49r^2}) = \tau + 2\tau q \cdot e^{-0.49r^2}. $$

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Using $\tau \leq e^{-0.49r^2}$ and $e^{r^2/2} \geq N/n$, we have
\[
1 - \delta \geq 1 - \tau - 2\tau q \cdot e^{-0.49r^2} \geq 1 - \tau - o(1/(qN)) \geq (1 - \tau)(1 - o(1/(qN))).
\]

On the other hand, we have
\[
\delta \geq \tau - q \cdot e^{-0.96r^2}
\]
and thus,
\[
1 - \delta \leq 1 - \tau + q \cdot e^{-0.96r^2} \leq 1 - \tau + o(1/(qN)) = (1 - \tau)(1 + o(1/(qN))).
\]
This finishes the proof of the claim.

\begin{proof}

\begin{claim}
For each $j \in [N]$ with one 0-entry in the $j$th column $M_{s,j}$, we have
\[
\Pr_{h_j} \left[ h_j(z_i) = 1 \mid M_{s,j} \right] \geq 1 - O\left(e^{-0.45r^2}\right).
\]
\end{claim}

\begin{proof}
Let $i'$ be the point with $M_{i',j} = 1$ and $\delta$ be the conditional probability of $h_j(z_i) = 0$. Then
\[
\delta \leq \frac{\text{fsa}(\text{cover}(z_i) \cap \text{cover}(z_{i'}))}{\text{fsa}(\text{cover}(z_i) - \bigcup_{j < i, j \neq i'} \text{cover}(z_j))} \leq \frac{e^{-0.96r^2}}{e^{-0.51r^2} - q \cdot e^{-0.96r^2}} = O\left(e^{-0.45r^2}\right),
\]
by our choice of $q$. This finishes the proof of the claim.
\end{proof}

We combine the two claims to prove Lemma 25.

\begin{proof}[Proof of Lemma 25]
Let $h$ be the number of 0-entries in $M$. We have $h \leq \sqrt{N}$ since $M$ is nice. By Claims 26, the conditional probability of $S(z_i) = 1$ is at most
\[
\left( (1 - \tau) \left( 1 + o\left(\frac{1}{qN}\right) \right) \right)^{N-h} = \rho(\|z_i\|) \cdot \frac{1}{(1 - \tau)^h} \cdot \left( 1 + o\left(\frac{1}{qN}\right) \right)^{N-h} \leq \rho(\|z_i\|) \cdot (1 + 2\tau)^h \cdot \left( 1 + o\left(\frac{1}{qN}\right) \right)^N \leq \rho(\|z_i\|) \cdot \exp\left(2\tau h + o(1/q)\right) = \rho(\|z_i\|) \cdot \exp\left(o(1/q)\right) = \rho(\|z_i\|) + o(1/q).
\]
Similarly, the conditional probability of $S(z_i) = 1$ is at least
\[
\left( (1 - \tau) \left( 1 - o\left(\frac{1}{qN}\right) \right) \right)^{N-h} \left( 1 - O\left(e^{-0.45r^2}\right) \right)^h \geq \rho(\|z_i\|) \cdot \left( 1 - o\left(\frac{1}{qN}\right) \right)^{N-h} \left( 1 - O\left(e^{-0.45r^2}\right) \right)^h \geq \rho(\|z_i\|) \cdot (1 - o(1/q)) \geq \rho(\|z_i\|) - o(1/q).
\]
This finishes the proof of the lemma.
\end{proof}
6 One-sided lower bound

We recall Theorem 1:

**Theorem 1.** Any one-sided sample-based algorithm that is an \( \varepsilon \)-tester for convexity over \( \mathcal{N}(0,1)^n \) for some \( \varepsilon < 1/2 \) must use \( 2^{\Omega(n)} \) samples.

We say a finite set \( \{x^1, \ldots, x^M\} \subset \mathbb{R}^n \) is shattered by \( C_{\text{convex}} \) if for every \( (b_1, \ldots, b_M) \in \{0,1\}^M \) there is a convex set \( C \in C_{\text{convex}} \) such that \( C(x^i) = b_i \) for all \( i \in [M] \). Theorem 1 follows from the following lemma:

**Lemma 28.** There is an absolute constant \( c > 0 \) such that for \( M = 2^c n \), it holds that

\[
\Pr_{x^1 \sim \mathcal{N}(0,1)^n} \left[ \{x^1, \ldots, x^M\} \text{ is shattered by } C_{\text{convex}} \right] \geq 1 - o(1).
\]

**Proof of Theorem 1 using Lemma 28.** Suppose that \( A \) were a one-sided sample-based algorithm for \( \varepsilon \)-testing \( C_{\text{convex}} \) using at most \( M \) samples. Fix a set \( S \) that is \( \varepsilon \)-far from \( C_{\text{convex}} \) to be the unknown target subset of \( \mathbb{R}^n \) that is being tested.\(^3\) Since \( S \) is \( \varepsilon \)-far from convex, it must be the case that

\[
\Pr_{x^i \sim \mathcal{N}(0,1)^n} \left[ A \text{ rejects when run on } (x^1, S(x^1)), \ldots, (x^M, S(x^M)) \right] \geq 2/3. \quad (15)
\]

But Lemma 28 together with the one-sidedness of \( A \) imply that

\[
\Pr_{x^i \sim \mathcal{N}(0,1)^n} \left[ \text{for any } (b^1, \ldots, b^M) \in \{0,1\}^M, \ A \text{ rejects when run on } (x^1, b^1), \ldots, (x^M, b^M) \right] \leq o(1),
\]

as \( A \) can only reject if the labeled samples are not consistent with any convex set, which implies that \( A \) cannot reject when \( \{x^1, \ldots, x^M\} \) is shattered by \( C_{\text{convex}} \). This contradicts with (15). \( \square \)

In the next subsection we prove Lemma 28 for \( c = 1/500 \).

6.1 Proof of Lemma 28

Let \( M = 2^c n \) with \( c = 1/500 \). We prove the following lemma:

**Lemma 29.** For \( x^1, \ldots, x^M \) drawn independently from \( \mathcal{N}(0,1)^n \), with probability 1 \(- o(1) \) it is the case that for all \( i \in [M] \), no \( x^i \) lies in \( \text{Conv}(\{x^j : j \in [M] \setminus i\}) \).

If \( x^1, \ldots, x^M \) are such that no \( x^i \) lies in \( \text{Conv}(\{x^j : j \in [M] \setminus i\}) \), then given any \( (b^1, \ldots, b^M) \), by taking \( C = \text{Conv}(\{x^j : b^j = 1\}) \) we see that there is a convex set \( C \) such that \( C(x^i) = b^i \) for all \( i \in [M] \). Thus to establish Lemma 28 it suffices to prove Lemma 29.

To prove Lemma 29, it suffices to show that for each fixed \( j \in [M] \) we have

\[
\Pr_{x^i \sim \mathcal{N}(0,1)^n} \left[ x^j \in \text{Conv}(\{x^k : k \in [M] \setminus \{j\}\}) \right] \leq M^{-2} \quad (16)
\]

\(^3\)An example of such a subset \( S \) is as follows (we define it as a function \( S : \mathbb{R}^n \to \{0,1\} \)): Given an odd integer \( N > (1/2-\varepsilon)^{-1} - 1 \), let \( -\infty = \tau_0 < \tau_1 < \cdots < \tau_N < \tau_{N+1} = +\infty \) be values such that \( \Pr_{z \sim \mathcal{N}(0,1)}[z \leq \tau_i] = i/(N+1) \), and let \( S : \mathbb{R}^n \to \{0,1\} \) be the function defined by \( S(x_1, \ldots, x_n) = 1[i \text{ is even}] \), where \( i \in \{0, \ldots, N\} \) is the unique value such that \( \tau_i \leq x_i < \tau_{i+1} \). Fix any \( z = (z_2, \ldots, z_n) \in \mathbb{R}^{N-1} \) and we let \( S_z : \mathbb{R} \to \{0,1\} \) be the function defined as \( S_z(x_1) = S(x_1, z_2, \ldots, z_n) \). An easy argument gives that \( S_z \) is \((1/2-1/(N+1))-\text{far} \) (and hence \( \varepsilon \)-far) from every convex subset of \( \mathbb{R} \), and it follows by averaging (using the fact that the restriction of any convex subset of \( \mathbb{R} \) to a line is a convex subset of \( \mathbb{R} \)) that \( S \) is \( \varepsilon \)-far from \( C_{\text{convex}} \).
since given this a union bound implies that
\[
\Pr_{x^j \leftarrow N(0, 1)^n} \left[ \text{for some } j \in [M], \, x^j \text{ lies in } \text{Conv}(\{x^k : k \in [M] \setminus \{j\}\}) \right] \leq M^{-1} = o(1).
\]

By symmetry, to establish (16) it suffices to show that
\[
\Pr_{x^i \leftarrow N(0, 1)^n} \left[ x^M \in \text{Conv}(\{x^1, \ldots, x^{M-1}\}) \right] \leq M^{-2}.
\]

In turn (17) follows from the following inequalities (\(v \in \mathbb{R}^n\) is a fixed unit vector in the second)
\[
\Pr_{x \leftarrow N(0, 1)^n} \left[ \|x\| \leq \sqrt{n}/10 \right] < \frac{1}{2} M^{-2} \quad \text{and} \quad \Pr_{x \leftarrow N(0, 1)^n} \left[ x \cdot v \geq \sqrt{n}/10 \right] < \frac{1}{2} M^{-3}.
\]

The first inequality follows directly from Lemma 5 using \(c = 1/500\). For the second, by the spherical symmetry of \(N(0, 1)^n\) we may take \(v = (1, 0, \ldots, 0)\).

Recall the standard Gaussian tail bound
\[
\Pr_{z \leftarrow N(0, 1)} \left[ z \geq t \right] \leq e^{-t^2/2}
\]
for \(t \geq 0\). This gives us that
\[
\Pr_{x \leftarrow N(0, 1)^n} \left[ x \cdot v \geq \sqrt{n}/10 \right] \leq e^{-n/200} < \frac{1}{2} M^{-3},
\]
again using that \(M = 2^m\) and \(c = 1/500\).

Finally, to see that (17) follows from (18), we observe first that by the first inequality we may assume that \(\|x^M\| > \sqrt{n}/10\) (at the cost of failure probability at most \(M^{-2}/2\) towards (17)); fix any such outcome \(x^M\) of \(x^M\). By a union bound over \(x^1, \ldots, x^{M-1}\) and the second inequality, we have
\[
\Pr_{x^i \leftarrow N(0, 1)^n} \left[ \text{any } i \in [M-1] \text{ has } x^i \cdot \frac{x^M}{\|x^M\|} \geq \sqrt{n}/10 \right] < \frac{1}{2} M^{-2}.
\]

But if every \(x^i\) has \(x^i \cdot (x^M/\|x^M\|) < \sqrt{n}/10 < \|x^M\|\), then \(x^M \notin \text{Conv}(\{x^1, \ldots, x^{M-1}\})\).

### 7 Two-sided upper bound

Recall Theorem 4:

**Theorem 4.** For any \(\varepsilon > 0\), there is a two-sided sample-based \(\varepsilon\)-tester for convexity over \(N(0, 1)^n\) using \(n^{O(\sqrt{n}/\varepsilon^2)}\) samples.

We begin by recalling some definitions from learning theory. Let \(\mathcal{C}\) be a class of subsets of \(\mathbb{R}^n\) (such as \(\mathcal{C}_{\text{convex}}\)). We say an algorithm learns \(\mathcal{C}\) to error \(\varepsilon\) with confidence \(1 - \delta\) under \(N(0, 1)^n\) if, given a set of labeled samples \((x, S(x))\) from an unknown set \(S \in \mathcal{C}\) with \(x\)’s drawn independently from \(N(0, 1)^n\), the algorithm outputs with probability at least \(1 - \delta\) a hypothesis set \(H \subseteq \mathbb{R}^n\) with \(\text{Vol}(S \triangle H) \leq \varepsilon\). We say it is a *proper* learning algorithm if it always outputs a hypothesis \(H\) that belongs to \(\mathcal{C}\). Next we recall the main algorithmic result of [KOS07]:
Theorem 30 (Theorem 5 of [KOS07]). There is an algorithm $A$ that learns the class $C_{\text{convex}}$ of all convex subsets of $\mathbb{R}^n$ to error $\varepsilon$ with confidence $1 - \delta$ under $\mathcal{N}(0, 1)^n$ using

$$n^{O(\sqrt{n}/\varepsilon^2)} \cdot \log(1/\delta)$$

samples\(^4\) drawn from $\mathcal{N}(0, 1)^n$.

Next we recall the result of Goldreich, Goldwasser and Ron which relates proper learnability of a class $C$ to the testability of $C$.

Theorem 31 (Proposition 3.1.1 of [GGR98], adapted to our context). Let $C$ be a class of subsets of $\mathbb{R}^n$ that has a proper learning algorithm $A$ which uses $m_A(n, \varepsilon, \delta)$ samples from $\mathcal{N}(0, 1)^n$ to learn $C$ to error $\varepsilon$ with confidence $1 - \delta$. Then there is a property testing algorithm $A_{\text{test}}$ for $C$ under the distribution $\mathcal{N}(0, 1)^n$ that uses

$$m_A(n, \varepsilon/2, \delta/2) + O\left(\log(1/\delta)/\varepsilon\right)$$

samples drawn from $\mathcal{N}(0, 1)^n$.

By Theorem 31, to obtain Theorem 4 it suffices to have a proper learning analogue of Theorem 30. We establish the required result, as a corollary of Theorem 30, in the next subsection:

Corollary 32. There is a proper learning algorithm $A'$ for the class $C_{\text{convex}}$ of all convex subsets of $\mathbb{R}^n$ that uses $n^{O(\sqrt{n}/\varepsilon^2)} \cdot \log(1/\delta)$ samples from $\mathcal{N}(0, 1)^n$ to learn to error $\varepsilon$ with confidence $1 - \delta$.

We remark that while algorithm $A$ from Theorem 30 runs in time $n^{O(\sqrt{n}/\varepsilon^2)}$ and uses $n^{O(\sqrt{n}/\varepsilon^2)}$ samples, the algorithm $A'$ of Corollary 32 presented below has a much larger running time (at least $(n/\varepsilon)^{O(n)}$); however, its sample complexity is essentially no larger than that of algorithm $A$.

7.1 Proof of Corollary 32

The idea behind the proof of Corollary 32 is simple. Let $S \subseteq \mathbb{R}^n$ be the unknown target convex set that is to be learned. Algorithm $A'$ first runs algorithm $A$ with error parameter $\varepsilon/5$ and confidence parameter $\delta/2$ to obtain, with high probability a convex set $C'$ with $\text{Vol}(H \triangle C') \leq 4\varepsilon/5$ and thus, we have $\text{Vol}(S \triangle C') \leq \varepsilon/5 + 4\varepsilon/5 = \varepsilon$. (Note that this part of the algorithm does not require any labeled samples $(x, S(x))$ from the oracle for $S$.)

For this purpose let $C_{\text{cover}} \subseteq C_{\text{convex}}$ be a finite $(\varepsilon/5)$-cover of $C_{\text{convex}}$. (We show in Corollary 33 below that there is an algorithm the finds a finite $(\varepsilon/5)$-cover of $C_{\text{convex}}$.) Next, the algorithm $A'$ enumerates over all elements $C \in C_{\text{cover}}$ and for each such $C$ uses random sampling from $\mathcal{N}(0, 1)^n$ to estimate $\text{Vol}(H \triangle C)$ to within an additive error of $\varepsilon/5$, with success probability $1 - \delta/(2|C_{\text{cover}}|)$ for each $C$. (Note that this does not require any labeled samples $(x, S(x))$ from the oracle for $S$, since $A'$ can generate its own draws from $\mathcal{N}(0, 1)^n$ and for each such $x$ it can compute $H(x)$ and $C(x)$ on its own.) $A'$ outputs the $C^* \in C_{\text{cover}}$ for which the estimate of $\text{Vol}(H \triangle C^*)$ is smallest.

The fact that this works follows a standard argument. Since

$$\text{Vol}(H \triangle S) \leq \varepsilon/5 \quad \text{and} \quad \text{Vol}(S \triangle C') \leq \varepsilon/5$$

\(^4\)Theorem 5 as stated in [KOS07] gives a sample complexity upper bound of $n^{O(\sqrt{n}/\varepsilon^4)}$ for agnostic learning, but inspection of the proof gives the theorem as stated here, with an upper bound of $n^{O(\sqrt{n}/\varepsilon^2)}$ for non-agnostic learning.
for some set $C' \in C_{\text{cover}}$, it holds that $\text{Vol}(H \triangle C') \leq 2\varepsilon/5$ and hence the estimate of $\text{Vol}(H \triangle C')$ will be at most $3\varepsilon/5$. Thus the element $C^*$ of $C_{\text{cover}}$ that is selected will have its estimated value of $\text{Vol}(H \triangle C^*)$ being at most $4\varepsilon/5$ (since each estimate is within $\pm \varepsilon/5$ of the true value).

Given the above analysis, to finish the proof of Corollary 32 it suffices to establish the following corollary of structural results proved in Sections 3 and 4.1, which shows that indeed it is possible for $A'$ to enumerate over the elements of $C_{\text{cover}}$ as described above:

**Corollary 33.** There is an algorithm that, on inputs $\varepsilon$ and $n$, outputs a finite $\varepsilon$-cover of $C_{\text{convex}}$.

**Proof.** We recall the material and parameter settings from Section 4.1. Since every convex set in $\mathbb{R}^n$ is $(\varepsilon/4)$-close to a set in $C_{\text{convex}}'$, it suffices to describe a finite family $C$ of convex sets $C_1, C_2, \ldots$ such that every $C \in C_{\text{convex}}'$ is $(3\varepsilon/4)$-close to some $C_i$ in $C$. We claim that

$$C = \{ \text{Conv}(\cup_{Q \subseteq Q} \text{Cube}) \mid Q \subseteq \text{CubeSet} \}$$

is such a family. To see this, fix any convex body $C \in C_{\text{convex}}$. Let

$$Q_C = \{ \text{Cube} \in \text{CubeSet} \mid \text{Cube} \subseteq C \},$$

the set of cubes that are entirely contained in $C$. Note that $\text{Conv}(Q_C)$ is a subset of $C$. If a Cube contains at least one point in $C$ and at least one point outside $C$, then every point in Cube has distance at most $\ell \sqrt{n}$ from the boundary of $C$ (since any two points in a given Cube have distance at most $\ell \sqrt{n}$). Thus, the missing volume $C \setminus \text{Conv}(Q_C)$ is completely contained in $\partial C + \text{Ball}(\ell \sqrt{n})$, whose Gaussian volume, by Theorem 7, is at most $20n^{5/8} n' \ell \sqrt{n} \approx 3\varepsilon/4$. \qed

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A Proof of Lemmas 34, 35, and 36

Lemma 34. If $C \subset \mathbb{R}^n$ is convex and contains no ball of radius $\rho$, then we have
\[ \text{Vol}(C + \text{Ball}(\alpha)) \leq 2(n\rho + \alpha). \]

Proof. By the theorem of John [Joh48] (see also Theorem 3.1 of [Bal97]), there is a unique ellipsoid contained in $C$ that has maximal Euclidean volume; let us denote this by $E(C)$. Since $C$ does not contain a ball of radius $\rho$, $E(C)$ must have some axis $u$ which has length less than $\rho$. Let us translate $C$ so that the center of $E(C)$ lies at the origin. Again by the theorem from John (see the discussion in [Bal97] on pages 13 and 16), we have that $C \subseteq nE(C)$. Now consider the set $H$ of all points $v \in \mathbb{R}^n$ whose projection onto the $u$ direction has magnitude at most $n\rho + \alpha$. This is a “thickened hyperplane” which contains $C + \text{Ball}(\alpha)$, and its Gaussian volume is given by
\[ \text{Vol}(H) = \int_{-(n\rho + \alpha)}^{(n\rho + \alpha)} \varphi(x) \, dx, \]
where $\varphi(x)$ is the density function of a univariate normal distribution as defined in Section 2. We know that $\varphi$ is bounded from above by 1 so this integral is at most $2(n\rho + \alpha)$. It is also easy to see that the same volume upper bound must hold upon undoing the translation of $C$ back to its original position, and the lemma is proved. \hfill \Box

Lemma 35. Let $C$ be a bounded convex subset of $\mathbb{R}^n$ that contains $\text{Ball}(\rho)$, the origin-centered ball of radius $\rho$, for some $\rho > \alpha$. Then the distance between $(1 - (\alpha/\rho))C$ and $\partial C$ is at least $\alpha$. 

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Proof. This is essentially Lemma 2.2 of [Ker92]; for completeness we give the simple proof here.

Let \( \beta = \alpha/\rho \). Let \( z \in \partial C \) be a point on the boundary of \( C \). Since \( C \) is convex and contains the origin, there exists a vector \( v \) for which \( v \cdot z = 1 \) but for all \( x \in C \) we have \( v \cdot x \leq 1 \) (intuitively, one can think of \( v \) as defining the tangent hyperplane at \( z \)). Then for any \( y \in (1 - \beta)C \) we have \( v \cdot y \leq 1 - \beta \), which implies that \( v(z - y) \geq \beta \). Since \( \rho v/\|v\| \in \text{Ball}(\rho) \subseteq C \), it must be the case that \( v \cdot (\rho v/\|v\|) = \rho\|v\| \leq 1 \), which means that \( \|v\| \leq 1/\rho \) and thus (as \( v(z - y) \geq \beta \)) \( \|z - y\| \geq \alpha \).

\[\Box\]

**Lemma 36.** Let \( C \subseteq \mathbb{R}^n \) be a convex set that satisfies \( \sup_{c \in C}\|c\| \leq K \) for some \( K > 1 \). Then for any \( 0 < \beta < 1 \), every point \( v \in \partial C + \text{Ball}(\alpha) \) is within distance \( 2K\beta + \alpha \) of a point in \( (1 - \beta)C \).

Proof. We have that \( v = c + y \) for some \( c \in \partial C \) and \( y \) with \( \|y\| \leq \alpha \). While \( v \) may not lie in \( C \) (as \( C \) might be an open set), we know for any \( \varepsilon > 0 \) there is a point \( c' \in C \) and \( \|c' - c\| \leq \varepsilon \). Take such a point \( c' \) with \( \varepsilon = \beta K \). Then \( (1 - \beta)c' \in (1 - \beta)C \) and

\[
\|(1 - \beta)c' - v\| = \|(1 - \beta)c' - c - y\| \leq \|c' - c\| + \beta\|c'\| + \|y\| \leq \beta K + \beta K + \alpha = 2\beta K + \alpha.
\]

This finishes the proof of the lemma. \[\Box\]