ON THE HOMOTOPY OF CLOSED MANIFOLDS AND FINITE CW-COMPLEXES

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ABSTRACT. We study the finite generation of homotopy groups of closed manifolds and finite CW-complexes by relating it to the cohomology of their fundamental groups. Our main theorems are as follows: when $X$ is a finite CW-complex of dimension $n$ and $\pi_1(X)$ is virtually a Poincaré duality group of dimension $\geq n-1$, then $\pi_i(X)$ is not finitely generated for some $i$ unless $X$ is homotopy equivalent to the Eilenberg-Maclane space $K(\pi_1(X),1)$; when $M$ is an $n$-dimensional closed manifold and $\pi_1(M)$ is virtually a Poincaré duality group of dimension $\geq n-1$, then for some $i \leq \lfloor n/2 \rfloor$, $\pi_i(M)$ is not finitely generated, unless $M$ itself is an aspherical manifold. These generalize theorems of M. Damian from polycyclic groups to any virtually Poincaré duality groups. When $\pi_1(X)$ is not a virtually Poincaré duality group, we also obtained similar results. As a by-product of our results, we show that if a group $G$ is of type F and $H^i(G,\mathbb{Z})$ is finitely generated for any $i$, then $G$ is a Poincaré duality group. This recovers partially a theorem of Farrell.

INTRODUCTION

The homotopy groups $\pi_i(X)$ are important algebraic topological invariants associated to a space $X$. The study of their properties is a major topic in topology. The first concern is whether these abelian groups are finitely generated. Let $X$ be a finite connected CW-complex of dimension $n$. A celebrated theorem of Serre [23] says that when $X$ is simply connected, then all the homotopy groups of $X$ are finitely generated. So it is a natural question to consider the case when $X$ is not simply connected. When $\pi_1(X)$ is finite, one can pass to the universal cover of $X$ which is still a finite CW-complex, hence Serre’s theorem applies. When $\pi_1(X)$ is not finite, M. Damian did some interesting work on this problem [6, 7]. One of his main results [7, Theorem 1.2] says that when $X$ is a finite CW-complex of dimension $n$ and $\pi_1(X)$ is a polycyclic group of Hirsch length $\geq n-1$, then $\pi_i(X)$ is not finitely generated for some $i \geq 2$ unless $X$ is homotopy equivalent to the Eilenberg-Maclane space $K(\pi_1(X),1)$.

In this paper, we relate the (non)-finite generation of homotopy groups of closed manifolds and finite complexes to the cohomology of their fundamental groups. The main results extend Damian’s theorems [7, Theorem 1.2, 1.3, 1.4] to a much broader class of
groups. For example, our theorems apply to the case when the fundamental group is virtually a Poincaré duality group (see Definition 1.3), in particular it holds for the fundamental group of any closed aspherical manifold. Recall that a manifold (or more generally a topological space) is aspherical if its universal cover is contractible.

It is well-known that any finitely presented group can be realized as the fundamental group of a closed manifold of dimension ≥ 4. The following theorem considers the case when the fundamental group is of type F (see Definition 1.2).

**Theorem A.** Let \( M^n \) be a closed \( n \)-dimensional manifold with \( \pi_1(M) = G \), suppose \( G \) is of type F, then

(a) if \( G \) is not a Poincaré duality group, then \( \pi_i(M) \) is not finitely generated for some \( i \geq 2 \). Furthermore, if \( H^i(G, \mathbb{Z}G) \) is finitely generated for all \( i \leq \lfloor n/2 \rfloor \), then \( \pi_i(M) \) is not finitely generate for some \( 2 \leq i \leq \lfloor n/2 \rfloor \);

(b) if \( G \) is a duality group of dimension \( d \) such that \( d \geq n - 1 \), then either \( \pi_i(M) \) is not finitely generated for some \( 2 \leq i \leq \lfloor n/2 \rfloor \), or \( M \) itself is aspherical.

**Remark 0.1.** Note that the above theorem also holds when the assumption is virtually satisfied, i.e. if some finite index subgroup of \( \pi_1(M) \) satisfies the assumption. In particular, this applies to polycyclic groups since they are virtually Poincaré duality groups ([14, Theorem 2] and [1]).

**Remark 0.2.** If \( G \) is a Poincaré duality group of dimension \( d \) then there exists a closed manifold \( M \) with \( \pi_1(M) = G \) and \( \pi_i(M) \) finitely generated for all \( i \geq 2 \) (see Theorem E below). From Theorem A (b) we see that either \( M \) is aspherical with fundamental group \( G \), or \( \dim M \geq d + 2 \). In the first case the Borel conjecture predicts that \( M \) is topologically rigid. It would be interesting to have a structure theorem for the manifolds in the second case. There are several works in this direction, including the classical fibration theorem of Browder-Levine [3] saying that if \( G = \mathbb{Z} \) then \( M \) is always a fiber bundle over \( S^1 \) with simply-connected fiber, and the analysis of the topological rigidity of some classes of these manifolds by Kreck and Lück [19].

In general, when the fundamental group is not necessarily of type F, we have the following.

**Theorem B.** Let \( M^n \) be a closed \( n \)-dimensional manifold with \( \pi_1(M) = G \).

(a) Suppose that \( H^i(G, \mathbb{Z}G) = 0 \) for any \( i \leq d \). If \( n \leq d \), then for some \( i \), \( \pi_i(M) \) is not finitely generated. If \( n = d + 1 \) or \( d + 2 \), then either \( \pi_i(M) \) is not finitely generated for some \( i \), or \( M \) is aspherical and \( G \) is a Poincaré duality group of dimension \( n \).

(b) if \( H^i(G, \mathbb{Z}G) \) is not finitely generated for some \( i \), then \( \pi_i(M) \) is not finitely generated for some \( j \).
Example 0.3. The Thompson group $F$ is a finitely presented group of infinite cohomological dimension, but with the property that $H^i(F, \mathbb{Z}F) = 0$ for all $i$ [5, Theorem 7.2]. Now part (a) of Theorem B implies that any closed manifold with fundamental group $F$ can not have finitely generated homotopy group in each dimension.

Similar to Theorem A, in the finite CW-complex case, we have the following (compare [7, Theorem 1.2]).

**Theorem C.** Suppose that $X$ is a finite CW-complex of dimension $n$ and that $\pi_1(X)$ is a virtually Poincaré duality group of dimension $d$. Then

(a) If $d > n$, then $\pi_i(X)$ is not finitely generated for some $i \geq 2$.
(b) If $d = n$ or $n - 1$, then $\pi_i(X)$ is not finitely generated for some $i \geq 2$ unless $X$ is a $K(\pi_1(X), 1)$ space.

In particular, when $\pi_1(X)$ has torsion and $d \geq n - 1$, the conclusion of (a) holds.

In general, if $\pi_1(X)$ is not a virtually Poincaré duality group, we have the following theorem for finite CW-complexes.

**Theorem D.** Let $G$ be a group and $n$ be the smallest integer such that $H^{n+1}(G, \mathbb{Z}G)$ is not finitely generated and $X$ is a finite CW-complex of dimension $\leq n$ with fundamental group $G$. Then for some $i \geq 2$, $\pi_i(X)$ is not finitely generated.

Both Theorem C and Theorem D are proved by transforming the problem into the manifold case. In general, it is an interesting problem to realize a given group as the fundamental group of manifolds satisfying certain topological conditions, such as knot complements [15] or homology spheres [16]. From this point of view Theorem B leads to the following question

**Question I.** Given a finitely presented group $G$ of finite cohomological dimension, such that $H^i(G, \mathbb{Z}G)$ is finitely generated for all $i$. Then does there exist a closed manifold $M$ with $\pi_1(M) = G$ and $\pi_i(M)$ finitely generated for all $i \geq 2$?

**Remark 0.4.** Note that Example 0.3 of the Thompson group shows that the condition that $G$ has finite cohomological dimension is necessary. Also, it seems that the only groups we are aware of satisfying the conditions in Question I are the Poincaré duality groups (see part (b) of the Theorem E).

We have an affirmative answer to this question in the case when $G$ is of type F.

**Theorem E.** Let $G$ be a group of type F, with $H^i(G, \mathbb{Z}G)$ finitely generated for all $i$. Then

(a) there exists a closed manifold $M$ with $\pi_1(M) = G$ and $\pi_i(M)$ finitely generated for all $i \geq 2$;
(b) $G$ is a Poincaré duality group.
The second part of Theorem E recovers partially a theorem of Farrell [9, Theorem 3], see Remark 4.4.

Remark 0.5. It is an open question whether a finitely presented Poincaré duality group is always the fundamental group of a closed aspherical manifold, see [8] and [21] for more information.

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1. Basic definitions and results

In this section we collect some basic definitions and results that we may need later. For more details see [4] and [17].

Definition 1.1. A finite CW-complex $X$ is called a Poincaré duality space of dimension $n$ if there is a $\mathbb{Z}\pi_1(X)$-module structure on $\mathbb{Z}$ and $e \in H_n(X, \mathbb{Z})$ such that the cap-product $e \cap -$ induces isomorphisms for all $i$ and all $\mathbb{Z}\pi_1(X)$-modules $A$.

Definition 1.2. A group $G$ is called of type $F$ if it has a finite CW-complex model for the Eilenberg-Maclane space $K(G, 1)$. A group $G$ is called of type $F_\infty$ if it has a CW-complex model for the Eilenberg-Maclane space $K(G, 1)$ with finitely many cells in each dimension. A group $G$ is called of type $FP$ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution of finite type over $\mathbb{Z}G$.

Definition 1.3. A group $G$ is called a Poincaré duality group of dimension $n$ if $K(G, 1)$ is a Poincaré duality space of dimension $n$. A group $G$ of type $FP$ is called a duality group if there is an integer $n$ such that $H^i(G, \mathbb{Z}G) = 0$ for all $i \neq n$ and $H^n(G, \mathbb{Z}G)$ is a torsion-free abelian group.

Note that a Poincaré duality group is a duality group with $H^n(G, \mathbb{Z}G) \cong \mathbb{Z}$ [4, Section VIII.10].

Definition 1.4. A CW-complex is called finitely dominated if there is a finite CW-complex $K$ and two maps $i : X \rightarrow K$, $r : K \rightarrow X$ such that $r \circ i$ is homotopic to $id_X$.

Lemma 1.5. A simply connected finite dimensional CW-complex $X$ is finitely dominated if and only if $X$ is homotopy equivalent to a finite CW-complex, if and only if $\pi_i(X)$ is finitely generated for all $i$. 

Proof The first “if and only if” follows from Wall’s finiteness theorem as $X$ is simply connected. Now if $X$ is homotopy equivalent to a finite CW-complex, then by Serre’s mod $C$ theory ([23] or [24, Chapter 9 Section 6]), we have $\pi_i(X)$ is finitely generated for all $i$. For the other direction, if $\pi_i(X)$ is finitely generated for all $i$, again by Serre’s Theorem, we have $H_i(X)$ is finitely generated for all $i$. In this case $X$ has a minimal cell structure consisting of finitely many cells in each dimension [13, Proposition 4C.1]. Since $X$ is finite dimensional, it is homotopy equivalent to a finite CW-complex. □

2. A technical theorem

In this section, we prove a technical theorem, which provides the algebraic ingredients for the proof of our main results. We first need a sequence of lemmas for calculating the Ext functor.

Lemma 2.1. Let $G$ be a group such that $H^i(G, \mathbb{Z}G)$ is finitely generated for a given $i$, $A$ be a $\mathbb{Z}G$-module whose underlying abelian group is finitely generated free. Then $\text{Ext}^i_G(A, \mathbb{Z}G)$ is also finitely generated.

Proof Note first that when $A = \mathbb{Z}$ and $G$ acts trivially, $\text{Ext}^i_G(A, \mathbb{Z}G) \cong H^i(G, \mathbb{Z}G)$ which is finitely generated by assumption.

In general we have $\text{Ext}^i_G(A, \mathbb{Z}G) \cong H^i(G, \text{Hom}(A, \mathbb{Z}G))$ (c. f. [4, III.2.2, p.61]), where $G$ acts diagonally on $\text{Hom}(A, \mathbb{Z}G)$. There are the following $\mathbb{Z}G$-module isomorphisms

$$\text{Hom}(A, \mathbb{Z}G) \cong \text{Hom}(A, \mathbb{Z}) \otimes \mathbb{Z}G \cong \text{Hom}(A, \mathbb{Z})_0 \otimes \mathbb{Z}G \cong (\mathbb{Z}G)^r$$

where $\text{Hom}(A, \mathbb{Z})_0$ is the trivial $\mathbb{Z}G$-module and the second isomorphism is by [4, III.5.7, p.69], and $r$ is the rank of $A$ as a free abelian group. Therefore

$$\text{Ext}^i_G(A, \mathbb{Z}G) \cong H^i(G, (\mathbb{Z}G)^r) \cong H^i(G, \mathbb{Z}G)^r$$

and the lemma now follows. □

Lemma 2.2. Let $G$ be a group such that $H^i(G, \mathbb{Z}G)$ is finitely generated for $i$ and $i-1$, $A$ be a $\mathbb{Z}G$-module whose underlying abelian group is finite. Then $\text{Ext}^i_G(A, \mathbb{Z}G)$ is also finitely generated.

Proof We first show the lemma when $G$ acts on $A$ trivially. For that, we only need to prove the case when $A$ is a finite cyclic group. Suppose $A \cong \mathbb{Z}/k$, we have a short exact sequence (of trivial $G$-modules)

$$0 \rightarrow \mathbb{Z} \xrightarrow{k} \mathbb{Z} \rightarrow \mathbb{Z}/k \rightarrow 0$$

Apply the functor $\text{Hom}_G(-, \mathbb{Z}G)$, we get a long exact sequence

$$\cdots \rightarrow \text{Ext}^i_G(\mathbb{Z}/k, \mathbb{Z}G) \rightarrow \text{Ext}^i_G(\mathbb{Z}, \mathbb{Z}G) \rightarrow \text{Ext}^i_G(\mathbb{Z}, \mathbb{Z}G) \rightarrow \text{Ext}^{i+1}_G(\mathbb{Z}/k, \mathbb{Z}G) \cdots$$
By the assumption $\text{Ext}^i_G(\mathbb{Z}/k, \mathbb{Z}G)$ is finitely generated for $i$ and $i - 1$. Thus $\text{Ext}^i_G(\mathbb{Z}, \mathbb{Z}G)$ is also finitely generated.

Now we deal with the general case. Since $A$ is a finite group, its automorphism group is also finite. Thus, we can choose a finite index subgroup $H$ of $G$ such that $H$ acts trivially on $A$. Note that since $H$ is a finite index subgroup of $G$, $\text{Hom}_H(\mathbb{Z}G, \mathbb{Z}H) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}H \cong \mathbb{Z}G$ (c. f. [4, III.5.9]). By Eckmann-Shapiro Lemma (c. f. [2, Corollary 2.8.4]), we have

$$\text{Ext}^i_H(A, \mathbb{Z}H) \cong \text{Ext}^i_G(A, \text{Hom}_H(\mathbb{Z}G, \mathbb{Z}H)) \cong \text{Ext}^i_G(A, \mathbb{Z}G)$$

therefore $\text{Ext}^i_G(A, \mathbb{Z}G)$ is finitely generated. □

Note that the arguments in the proof also show the following lemma, since by definition $H^i_G(\mathbb{Z}G, \mathbb{Z}G) = \text{Ext}^i_G(\mathbb{Z}, \mathbb{Z}G)$ where $\mathbb{Z}$ is a constant $\mathbb{Z}G$-module.

**Lemma 2.3.** Let $H$ be a finite index subgroup in $G$, then for any given $i$, $H^i_G(\mathbb{Z}G, \mathbb{Z}G)$ is finitely generated if and only if $H^i_H(\mathbb{Z}H, \mathbb{Z}H)$ is finitely generated.

**Proposition 2.4.** Let $G$ be a group such that $H^i_H(\mathbb{Z}G, \mathbb{Z}G)$ is finitely generated for $i$ and $i - 1$, $A$ be a $\mathbb{Z}G$-module whose underlying abelian group is finitely generated. Then $\text{Ext}^i_G(A, \mathbb{Z}G)$ is also a finitely generated abelian group.

**Proof.** Since the torsion subgroup $T$ of $A$ is a $\mathbb{Z}G$ submodule of $A$, we have the follow short exact sequence

$$0 \rightarrow T \rightarrow A \rightarrow A/T \rightarrow 0$$

Now as abelian groups, $T$ is finite, $A/T$ is finitely generated free. Apply the functor $\text{Hom}_G(-, \mathbb{Z}G)$, we get a long exact sequence, and the proposition follows now from Lemma 2.1 and Lemma 2.2. □

Note that the proof of Proposition 2.4 also shows the following which will be useful later to prove part (a) of Theorem B.

**Proposition 2.5.** Let $G$ be a group such that $H^i_H(\mathbb{Z}G, \mathbb{Z}G) = 0$ for any $i \leq n$ and $A$ is a finitely generated abelian group with a $\mathbb{Z}G$-module structure. Then $\text{Ext}^i_G(A, \mathbb{Z}G) = 0$ for any $i \leq n$.

We are now ready to show the following.

**Theorem 2.6.** Let $M$ be a closed manifold of dimension $n$ with fundamental group $G$, and $H^i(G, \mathbb{Z}G)$ is finitely generated for all $i \leq [n/2]$. If for $2 \leq i \leq [n/2]$, $\pi_i(M)$ is finitely generated, then $\pi_i(M)$ is finitely generated for all $i$.

**Proof.** We may assume $M$ is orientable by passing to an index two cover. In doing so the finite generation of $H^i(G, \mathbb{Z}G)$ is not affected by Lemma 2.3. By Serre’s theorem the universal cover $\widetilde{M}$ has finitely generated $\pi_i(M) \cong \pi_i(\widetilde{M})$ for all $i \geq 2$ if and only if $H_i(\widetilde{M}, \mathbb{Z})$ is finitely generated for all $i \geq 2$. Let $\Lambda = \mathbb{Z}\pi_1(M)$ be the group ring, fix a
CW-structure on $M$ (see for example [13, Corollary A.12]), then $H_i(\tilde{M}, \mathbb{Z})$ is isomorphic to the cellular homology $H_i(M, \Lambda)$. Since $M$ is a closed manifold, by Poincaré duality, we have $H_i(M, \Lambda) \cong H^{n-i}(M, \Lambda)$. So we only need to show for $i \leq \lfloor n/2 \rfloor$, $H_i(M, \Lambda)$ is finitely generated.

We have a Universal Coefficient Spectral Sequence (see [11, Chapter I, Theorem 5.5.1] or [20, Theorem (2.3)]), which converges to $H_i(M, \Lambda)$, with $E_2$-terms

$$E_2^{p,q} = \text{Ext}_\Lambda^q(H_p(\tilde{M}), \Lambda)$$

Since we know already that $\pi_p(M)$ is finitely generated for $2 \leq p \leq \lfloor n/2 \rfloor$, by Serre’s theorem $H_p(\tilde{M})$ as an abelian group is finitely generated for $p \leq \lfloor n/2 \rfloor$. Now since $H^i(G, \mathbb{Z}G)$ is finitely generated for any $i \leq \lfloor n/2 \rfloor$, by Proposition 2.4, each term in the $E_2$-page of the spectral sequence is also finitely generated as long as $p \leq \lfloor n/2 \rfloor$ and $q \leq \lfloor n/2 \rfloor$. Thus for $i \leq \lfloor n/2 \rfloor$, the limit group $H^i(M, \Lambda)$ is finitely generated. □

Remark 2.7. Note that Poincaré duality groups satisfy the condition that $H^i(G, \mathbb{Z}G)$ is finitely generated for any $i$; duality groups also satisfy the condition in our theorem if the dimension of the group is $> \lfloor n/2 \rfloor$ (see Definition 1.3).

Remark 2.8. When $G$ does not satisfy the condition that $H^i(G, \mathbb{Z}G)$ is finitely generated for any $i \leq \lfloor n/2 \rfloor$, then Theorem 2.6 is not true in general. For example, take $M$ to be the connected sum of two copies of $S^1 \times S^d$ for some $d \geq 3$. Then $\pi_i(M) = 0$ for $2 \leq i \leq d - 1$ but $\pi_d(M)$ is a free abelian group of infinite rank.

3. The manifold case

In this section we prove our main theorems in the manifolds case. There are two main ingredients in the proof. The algebraic one we have already presented in Section 2. The geometric one is the following theorem on fibration of Poincaré duality spaces, first announced by Quinn [22, Remark 1.6], for a proof see Gottlieb [12] or Klein [18, Corollary F].

Theorem 3.1. Let $F \to E \to B$ be a fibration such that $F, E, B$ are homotopy equivalent to finite CW-complexes. Then $E$ is a Poincaré duality space if and only if $F$ and $B$ are Poincaré duality spaces. When $E$ is a Poincaré duality space of dimension $n$, the sum of the duality dimension of $F$ and $B$ is also $n$.

Corollary 3.2. Let $M^n$ be a closed manifold such that $\pi_1(M)$ is the fundamental group of a finite aspherical CW-complex $B$.

(a) If $B$ is not a Poincaré duality space, then $\pi_i(M)$ is not finitely generated for some $i \geq 2$;

(b) If $B$ is a Poincaré duality space of dimension $d$ with $d \geq n - 1$, then either $\pi_i(M)$ is not finitely generated for some $i \geq 2$, or $M$ is homotopy equivalent to $B$. 
Proof. Since $\pi_1(M) \cong \pi_1(B)$, we have a map $f : M \to B$ which induces isomorphism on $\pi_1$. Let $F_f$ be its homotopy fiber, then $F_f$ is homotopy equivalent to the universal cover $\tilde{M}$ of $M$. Assume now $\pi_i(M)$ is finitely generated for all $i \geq 2$, by Lemma 1.5, $\tilde{M}$ is homotopy equivalent to a finite CW-complex. Now by Theorem 3.1, we have $F_f$ and $B$ are Poincaré duality spaces and the duality dimension of $F_f$ is $n - d$. Note that by assumption $n - d \leq 1$. If $n - d = 0$, then $F_f$ is homotopy equivalent to a point since it is simply connected. Therefore, $M$ and $B$ are homotopy equivalent. When $n - d = 1$, then $H_1(F_f, \mathbb{Z}) \cong \mathbb{Z}$ since it is a Poincaré duality space, which is contradicting to the fact that $F_f$ is simply connected. □

Note that now Theorem A follows from Corollary 3.2 and Theorem 2.6, as duality groups of dimension $d \geq n - 1$ satisfy the condition that $H^i(G, \mathbb{Z}G)$ is finitely generated for $i \leq \lfloor n/2 \rfloor$. The only exceptional case is when $n = 2$. But in this case the theorem follows directly from the classification of surfaces.

We now proceed to prove Theorem B.

Proof of part (a) of Theorem B. Suppose $\pi_i(M)$ is finitely generated for any $i$, by Serre’s theorem, $H_i(\tilde{M})$ is finitely generated for any $i$. We have a Universal Coefficient Spectral Sequence, which converges to $H^i(M, \mathbb{Z}G)$, with $E_2$-terms

$$E_2^{p,q} = \text{Ext}^q_{\mathbb{Z}G}(H_p(\tilde{M}), \mathbb{Z}G)$$

Since $H^i(G, \mathbb{Z}G) = 0$ for any $i \leq d$, we have $\text{Ext}^q_{\mathbb{Z}G}(H_p(\tilde{M}), \mathbb{Z}G) = 0$ for any $q \leq d$ by Proposition 2.5. Hence $H^i(M, \mathbb{Z}G) = 0$ for any $i \leq d$.

If $n = d$, then $H_i(\tilde{M}, \mathbb{Z}) \cong H_i(M, \mathbb{Z}G) \cong H^{d-i}(G, \mathbb{Z}G) = 0$ for any $i$. But this is a contradiction since $H_0(\tilde{M}, \mathbb{Z}) \cong \mathbb{Z}$.

Now we assume $n = d + 1$. Then $H_i(\tilde{M}, \mathbb{Z}) \cong H_i(M, \mathbb{Z}G) \cong H^{d-i}(M, \mathbb{Z}G) = 0$ for any $i \geq 1$. This implies $M$ is aspherical, in particular, $G$ is a Poincaré duality group. The same argument works for $n = d + 2$ as we know already $\tilde{M}$ is simply connected. □

Remark 3.3. Note that for a finite group $G$, we have $H^p(G, \mathbb{Z}G) \cong \mathbb{Z}$ [10, Proposition 13.2.11] and $H^i(G, \mathbb{Z}G) = 0$ for any $i > 0$ [10, Proposition 13.3.1]. On the other hand, when $G$ is not finite, $H^p(G, \mathbb{Z}G) = 0$ [10, Proposition 13.2.11].

Remark 3.4. Part (a) of Theorem A now also follows from part (a) of Theorem B and Theorem 2.6 which are independent of Theorem 3.1.

Part (b) of Theorem B is a special case of the following theorem using Lemma 1.5.

Theorem 3.5. Let $X$ be an $n$-dimensional finite Poincaré complex. Let $N$ be a normal subgroup of $\pi_1(X)$ with quotient $G$, and $X_N$ be the corresponding cover. If $X_N$ is finitely dominated, then $H^i(G, \mathbb{Z}G)$ is a finitely generated abelian group for all $i$. 
Proof. If \( G \) is a finite group, the theorem automatically holds. So we assume from now on \( G \) is an infinite group, in particular \( H_n(X, \mathbb{Z}) = 0 \).

Apply the Leray–Serre spectral sequence to the fibration \( X_N \to X \to BG \), we have \( E_2^{p,q} = H^p(G, H^q(X_N, \mathbb{Z})G)) \) and the spectral sequence converges to the graded groups of a filtration of \( H^i(X; \mathbb{Z})G \). By Poincaré duality and [10, Corollary 13.2.3] \( H^i(X; \mathbb{Z})G) \cong H_{n-i}(X, \mathbb{Z}G) \cong H_{n-i}(X_N) \) which is a finitely generated abelian group since \( X_N \) is finitely dominated. Note also that \( H^p(G, H^0(X_N; \mathbb{Z})G)) \cong H^p(G, \mathbb{Z}G). \)

For \( i = 0 \), we have \( 0 = H_0(X, \mathbb{Z}) = E_\infty^{0,0} = E_2^{0,0} = H^0(G, H^0(X_N; \mathbb{Z})G)) \cong H^0(G, \mathbb{Z}G). \)

For \( i = 1 \), we have \( E_\infty^{1,0} = E_2^{1,0} = H^1(G, H^0(X_N; \mathbb{Z})G)) \cong H^1(G, \mathbb{Z}G) ). \) \( E_\infty^{1,0} \) is a subgroup of \( H_{n-1}(X_N) \) hence finitely generated.

We proceed by induction. Assume that \( H^i(G, \mathbb{Z}G) \) is finitely generated for \( i < k \). Now \( E_\infty^{k,0} \) is a subgroup of \( H_{n-k}(X_N) \), hence finitely generated. Note that \( E_2^{k,0} \) is a quotient of \( E_\infty^{k,0} = H^k(G, H^0(X_N; \mathbb{Z})G)) \cong H^k(G, \mathbb{Z}G). \) The differentials ending at the position \((k,0)\) come from the line \( p + q = k - 1 \). Hence we only need to show that all the \( E_2 \)-terms \( E_2^{p,q} = H^p(G, H^q(X_N; \mathbb{Z}G)) \) are finitely generated for \( p < k - 1 \).

By the universal coefficient theorem there is a short exact sequence

\[
0 \to \text{Ext}^1(H_{q-1}(X_N), \mathbb{Z})G)) \to H^p(X, \mathbb{Z}G) \to \text{Hom}(H_q(X, \mathbb{Z})) \to 0
\]

which is an exact sequence of \( G \)-modules by naturality. This induces a long exact sequence

\[
\cdots \to H^p(G, \text{Ext}^1(H_{q-1}(X_N), \mathbb{Z})) \to H^p(G, H^0(X_N, \mathbb{Z}G)) \to H^p(G, \text{Hom}(H_q(X, \mathbb{Z}G)) \cdots
\]

We have \( \text{Hom}(H_q(X, \mathbb{Z}G)) \cong \text{Hom}(H_q(X_N), \mathbb{Z}G) \cong (\mathbb{Z}G)^r \) for some integer \( r \). Therefore by the inductive assumption \( H^p(G, \text{Hom}(H_q(X, \mathbb{Z}G)) \) is finitely generated for \( p < k \) and any \( q \). We only need to show \( H^p(G, \text{Ext}^1(H_{q-1}(X_N), \mathbb{Z}G)) \) is finitely generated for \( p < k - 1 \).

Notice that \( H_{q-1}(X_N) \) is a finitely generated abelian group, we have \( \text{Ext}^1(H_{q-1}(X_N), \mathbb{Z}G) = \text{Ext}^1(A, \mathbb{Z}G) \), where \( A \) is the torsion part of \( H_{q-1}(X_N) \). Here \( A \) as an abelian group is finite. Let \( A_0 \) be the underlying abelian group of \( A \). Then as \( G \)-modules \( \text{Ext}^1(A, \mathbb{Z}G) \cong \text{Ext}^1(A, \mathbb{Z}G) \cong A \cong \mathbb{Z}G \cong A_0 \otimes \mathbb{Z}G \). Let \( 0 \to \mathbb{Z}^r \to \mathbb{Z}^r \to A_0 \to 0 \) be a free resolution of \( A_0 \) over \( \mathbb{Z} \), then from the short exact sequence of \( G \)-modules

\[
0 \to (\mathbb{Z}G)^r \to (\mathbb{Z}G)^r \to A_0 \otimes \mathbb{Z}G \to 0
\]

and the assumption that \( H^p(G, \mathbb{Z}G) \) is finitely generated for \( p < k \), it’s easy to see that \( H^p(G, A_0 \otimes \mathbb{Z}G) \) is finitely generated for \( p < k - 1 \). Therefore \( H^p(G, \text{Ext}^1(H_{q-1}(X_N), \mathbb{Z}G)) \) is finitely generated for \( p < k - 1 \) by Proposition 2.4. This finishes the proof.

4. The finite CW-complex case

In this section, we prove our main results for finite CW-complexes. We also discuss the problem of constructing manifolds with finitely dominated universal cover.
We first generalize [7, Theorem 1.2] from polycyclic groups to any virtually Poincaré duality groups.

**Proof of Theorem C.** With the help of Theorem 3.1 and 2.6, the proof now follows similarly to the arguments in [7, p.1797-1798]. We will assume \(\pi_1(M)\) is a Poincaré duality group. The general case follows easily from this. In fact, we will assume that \(\pi_1(M)\) is an orientable Poincaré duality group (i.e. the \(ZG\)-module structure on \(Z\) is trivial in Definition 1.3) as we can always pass to an orientable index two subgroup.

Suppose \(\pi_i(X)\) is finitely generated for all \(i \geq 2\). By simplicial approximation we may assume that \(X\) is a finite simplicial complex. Embed \(X\) in an Euclidean space \(\mathbb{R}^{2n+r+1}\) for some \(r \geq 0\) which will be fixed later in the proof. Let \(W\) be a regular neighbourhood of \(X\) and denote by \(M^{2n+r}\) the boundary of \(W\). Then it is a standard fact that \(X\) is a deformation retract of \(W\), and the inclusion map \(M \to W\) is \((n + r)\)-connected. Therefore \(\pi_i(M)\) is finitely generated for \(i \leq n + r - 1\). If \(r \geq 2\), then \(n + r - 1 \geq \lfloor (2n + r)/2 \rfloor\), by Theorem 2.6 \(\pi_i(M)\) is finitely generated for all \(i\). Therefore \(\tilde{M}\) is homotopy equivalent to a finite CW-complex. Apply Theorem 3.1 to the fibration sequence \(\tilde{M} \to M \to B\pi_1(X)\), where \(B\pi_1(X)\) is a model for \(K(\pi_1(X), 1)\), we see that \(\tilde{M}\) is homotopy equivalent to a Poincaré duality complex of dimension \(2n + r - d\), in particular \(H_{2n+r-d}(\tilde{M}) \cong \mathbb{Z}\).

Case (a): when \(d > n\), we have \(2n + r - d \leq n + r - 1\), hence \(H_{2n+r-d}(\tilde{M}) \cong H_{2n+r-d}(\tilde{W}) \cong H_{2n+r-d}(\tilde{X})\) by the Whitehead theorem. On the other hand, if we choose \(r\) to be bigger than \(d - n\), we have \(2n + r - d > n = \dim X\), so \(H_{2n+r-d}(\tilde{X})\) vanishes. This leads to a contradiction.

Case (b): when \(d = n\) or \(n - 1\), the Poincaré duality dimension of \(\tilde{M}\) is \(n + r + 1\) or \(n + r\).

Since the map \(M \to W\) is \((n + r)\)-connected and \(X\) is an \(n\)-dimensional complex, \(H_i(\tilde{M}, \mathbb{Z})\) vanishes for \(i = n + 1, n + 2, \ldots, n + r - 1\). By the universal coefficient theorem and Poincaré duality it is easy to see that \(H_i(\tilde{M}, \mathbb{Z})\) also vanishes for \(i = n - d + 1, \ldots, n - d + r - 2\). Now \(n - d + 1 = 1\) or 2, and \(\tilde{M}\) is simply connected, hence for \(r\) sufficiently large \(H_i(\tilde{M}) = 0\) for all \(1 \leq i \leq n\). But \(H_i(\tilde{X}) \cong H_i(\tilde{M})\) for \(i \leq n\), this implies that \(\tilde{X}\) is contractible. Therefore \(X\) is a \(K(\pi_1(X), 1)\) space. \(\square\)

The first part of Theorem E is a special case of the following theorem.

**Theorem 4.1.** Let \(X\) be a \(n\)-dimensional finite CW-complex such that \(\pi_1(X) = G\) and \(\pi_i(X)\) is finitely generated for any \(i\). If \(H^i(G, \mathbb{Z})\) is finitely generated for all \(i \leq n\), then we can find a closed manifold \(M\) of dimension \(2n + 1\) such that \(\pi_1(M) = G\) and \(\pi_i(M)\) is finitely generated for any \(i\).

**Proof** We can assume \(n \geq 2\). Embed \(X\) into an Euclidean space \(\mathbb{R}^{2n+2}\). Let \(W\) be a regular neighbourhood of \(X\) and denote by \(M^{2n+1}\) the boundary of \(W\). Similar to the proof of Theorem C, \(W\) is a deformation retract of \(X\) and the inclusion map \(M \to W\) is \((n + 1)\)-connected, therefore \(\pi_i(M)\) is finitely generated for any \(2 \leq i \leq n\). Note that \(n \geq \lfloor \frac{2n+1}{2} \rfloor\), by Theorem 2.6, we have \(\pi_i(M)\) is finitely generated for all \(i \geq 2\). \(\square\)
Remark 4.2. Note that Theorem 4.1 implies that, to answer Question I, it suffices to find a finite CW-complex $X$ with fundamental group $G$ such that $\pi_i(X)$ finitely generated for all $i$.

Corollary 4.3. Let $X$ be a finite CW-complex with fundamental group the Thompson group $F$, then for some $i \geq 2$, $\pi_i(X)$ is not finitely generated.

Proof Suppose $\pi_i(X)$ is finitely generated for any $i$. Since $H^i(F, \mathbb{Z}) = 0$ for any $i$ [5, Theorem 7.2], by Theorem 4.1, we have a manifold $M$ with fundamental group $F$ such that $\pi_i(M)$ is finitely generated for all $i$. Now by Theorem B (a), we have $M$ is aspherical and $F$ is a Poincaré duality group which is a contradiction. □

Proof of Theorem D. Suppose $\pi_i(X)$ is finitely generated for any $i \geq 2$, Theorem 4.1 implies there is a closed manifold $M^{2n+1}$ with fundamental group $G$ such that $\pi_i(M)$ is finitely generated for all $i \geq 2$. But then part (b) of Theorem B says $H^i(G, \mathbb{Z})$ must be finitely generated for all $i$. □

Proof of Theorem E (b). By part (a), we can find a closed manifold $M$ such that $\pi_i(M)$ is finitely generated for all $i$. Hence the universal cover $\widetilde{M}$ of $M$ is homotopy equivalent to a finite CW-complex. Now consider the following fibration sequence

$$\widetilde{M} \longrightarrow M \longrightarrow B\pi_1(M)$$

Where $B\pi_1(M)$ is a model for $K(\pi_1(M), 1)$. By Theorem 3.1, we have $B\pi_1(M)$ is a Poincaré duality space. Hence $G$ is a Poincaré duality group. □

Remark 4.4. This recovers partially a result of Farrell [9, Theorem 3]. Recall Farrell’s theorem says the following: let $G$ be a group of type F and $n$ be the smallest integer such that $H^n(G, \mathbb{Z}) \neq 0$, if $H^n(G, \mathbb{Z})$ is a finitely generated abelian group, then $G$ is Poincaré duality group.

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