Lepton Flavour Violating Decays $\tau \to \bar{\ell}\ell\ell$ and $\mu \to e\gamma$

in the Higgs Triplet Model

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Abstract

Singly and doubly charged Higgs bosons in the Higgs Triplet Model mediate the lepton flavour violating (LFV) decays $\tau \to \bar{\ell}\ell\ell$ and $\mu \to e\gamma$. The LFV decay rates are proportional to products of two triplet Yukawa couplings ($h_{ij}$) which can be expressed in terms of the parameters of the neutrino mass matrix and an unknown triplet vacuum expectation value. We determine the parameter space of the neutrino mass matrix in which a signal for $\tau \to \bar{\ell}\ell\ell$ and/or $\mu \to e\gamma$ is possible at ongoing and planned experiments. The conditions for respecting the stringent upper limit for $\mu \to eee$ are studied in detail, with emphasis given to the possibility of $|h_{ee}| \simeq 0$ which can only be realized if Majorana phases are present.

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I. INTRODUCTION

The now-established evidence that neutrinos oscillate and possess a small mass below the eV scale \([1]\) necessitates physics beyond the Standard Model (SM), which could manifest itself at the CERN Large Hadron Collider (LHC) and/or in low energy experiments which search for lepton flavour violation (LFV) \([2]\). Consequently, models of neutrino mass generation which can be probed at present and forthcoming experiments are of great phenomenological interest.

Neutrinos may obtain mass via the vacuum expectation value (vev) of a neutral Higgs boson in an isospin triplet representation \([3, 4, 5, 6, 7]\). A particularly simple implementation of this mechanism of neutrino mass generation is the “Higgs Triple Model” (HTM) in which the SM Lagrangian is augmented solely by an \(SU(2)\) triplet of scalar particles with hypercharge \(Y = 2\) \([3, 6, 7]\). In the HTM neutrinos acquire a Majorana mass given by the product of a triplet Yukawa coupling \((h_{ij})\) and a triplet vev \((\nu_\Delta)\). Consequently, there is a direct connection between \(h_{ij}\) and the neutrino mass matrix which gives rise to phenomenological predictions for processes which depend on \(h_{ij}\). A distinctive signal of the HTM would be the observation of doubly charged Higgs bosons \((H^{\pm\pm})\) whose mass \((m_{H^{\pm\pm}})\) may be of the order of the electroweak scale. Such particles can be produced with sizeable rates at hadron colliders in the processes \(q\bar{q} \rightarrow H^{++}H^{--}\) \([8]\) and \(qq' \rightarrow H^{++}H^{--}\) \([9, 10]\). Direct searches have been carried out at the Fermilab Tevatron in the production channel \(q\bar{q} \rightarrow H^{++}H^{--}\) and decay \(H^{\pm\pm} \rightarrow l_i^\pm l_j^\pm\), with mass limits of the order \(m_{H^{\pm\pm}} > 110 \rightarrow 150\) GeV \([11]\). The branching ratios (BRs) for \(H^{\pm\pm} \rightarrow l_i^\pm l_j^\pm\) depend on \(h_{ij}\) and are predicted in the HTM in terms of the parameters of the neutrino mass matrix \([10, 12, 13]\). Detailed quantitative studies of BR\((H^{\pm\pm} \rightarrow l_i^\pm l_j^\pm)\) in the HTM have been performed in \([14, 15, 16, 17]\) with particular emphasis given to their sensitivity to the Majorana phases and the absolute neutrino mass, i.e., parameters which cannot be probed in neutrino oscillation experiments. Recent simulations \([17, 18]\) of the discovery prospects of \(H^{\pm\pm}\) at the LHC now include the mechanism \(qq' \rightarrow H^{\pm\pm}H^\mp\), which plays a crucial role in extracting the parameters of the neutrino mass matrix.

The Yukawa couplings \(h_{ij}\) also mediate low energy lepton flavour violating (LFV) processes. In this paper we study the BRs of the LFV decays \(\tau \rightarrow l_i l_j l_k\) and \(\mu \rightarrow e\bar{e}e\) (which are mediated by \(H^{\pm\pm}\)) and \(\mu \rightarrow e\gamma\) (which is mediated by \(H^{\pm\pm}\) and \(H^\pm\)). Previous studies of such decays in the HTM were performed in \([13, 19]\), and it was shown that specific patterns of LFV are predicted in analogy with the prediction for BR\((H^{\pm\pm} \rightarrow l_i^\pm l_j^\pm)\). Experimental prospects for \(\mu \rightarrow e\gamma\) are bright with the recent commencement of the MEG experiment which will probe BR\(\sim 10^{-13}\), two orders of magnitude beyond the current upper limit \([20]\). The decay \(\tau \rightarrow l_i l_j l_k\) is currently being searched for at the \(e^+e^-\) B factories with upper limits in the range BR\((\tau \rightarrow l_i l_j l_k) < 2 \rightarrow 8 \times 10^{-8}\) \([21, 22]\). Simulations of the detection prospects at a proposed high luminosity \(e^+e^-\) B factory with \(\mathcal{L} = 5 \rightarrow 75\) ab\(^{-1}\) anticipate sensitivity to BR\(\sim 10^{-9} - 10^{-10}\) \([23, 24, 25, 26]\). Searches for \(\tau \rightarrow \mu\bar{\mu}\mu\) can be performed at the LHC where \(\tau\) leptons are copiously produced from the decays of \(W, Z, B, D\, D\), with anticipated sensitivities to BR\(\sim 10^{-8}\) \([27, 28]\).

The decay \(\mu \rightarrow e\bar{e}e\), for which there is a strict bound BR\(< 10^{-12}\) \([29]\), is a strong constraint on the parameter space of \(h_{ij}\) in the HTM. Obtaining BR\((\tau \rightarrow l_i l_j l_k) > 10^{-9}\) together with compliance of the above bound on BR\((\mu \rightarrow e\bar{e}e)\) is possible but can only be realized in specific regions of the parameter space for \(h_{ij}\) (e.g., \(|h_{e\mu}| \simeq 0\) \([13]\)). In this paper we perform a detailed quantitative study in order to find the parameter space of the neutrino mass
matrix where a signal for $\tau \rightarrow \bar{l}_i l_j l_k$ and/or $\mu \rightarrow e\gamma$ is possible at ongoing and planned experiments. We study in detail a novel way to satisfy the constraint from $\mu \rightarrow e\gamma$, namely $|h_{ee}| \simeq 0$, which can only be realized if Majorana phases are present \cite{30}. The pattern of LFV violation for $|h_{ee}| \simeq 0$ is studied and shown to differ from that for the case of $|h_{e\mu}| \simeq 0$. We also discuss the prediction for $\text{BR}(H^{\pm \pm} \rightarrow l_i^+ l_j^-)$ in the HTM if a signal for $\tau \rightarrow \bar{l}_i l_j l_k$ and/or $\mu \rightarrow e\gamma$ is observed.

Our work is organized as follows. In section II the HTM is briefly reviewed. In section III the theoretical basis for the decays $\text{BR}(\tau \rightarrow \bar{l}_i l_j l_k)$ and $\text{BR}(\mu \rightarrow e\gamma)$ is presented. The numerical analysis is contained in section IV with conclusions given in section V.

II. THE HIGGS TRIPLET MODEL

In the Higgs Triplet Model (HTM) \cite{3, 6, 7} a $I = 1, Y = 2$ complex $SU(2)_L$ isospin triplet of scalar fields is added to the SM Lagrangian. Such a model can provide a Majorana mass for the observed neutrinos without the introduction of a right-handed neutrino via the gauge invariant Yukawa interaction:

$$L = h_{ij} \psi^T_i C \tau_2 \Delta \psi_j + \text{h.c}$$

Here $h_{ij}(i, j = e, \mu, \tau)$ is a complex and symmetric coupling, $C$ is the Dirac charge conjugation operator, $\tau_2$ is a Pauli matrix, $\psi_{iL} = (\nu_i, l_i^T)$ is a left-handed lepton doublet, and $\Delta$ is a $2 \times 2$ representation of the $Y = 2$ complex triplet fields:

$$\Delta = \begin{pmatrix} \Delta^+ / \sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+ / \sqrt{2} \end{pmatrix}$$

A non-zero triplet vacuum expectation value $\langle \Delta^0 \rangle$ gives rise to the following mass matrix for neutrinos:

$$m_{ij} = 2h_{ij} \langle \Delta^0 \rangle = \sqrt{2}h_{ij} v_\Delta$$

The necessary non-zero $v_\Delta$ arises from the minimization of the most general $SU(2) \otimes U(1)_Y$ invariant Higgs potential, which is written as follows \cite{12, 13} (with $\Phi = (\phi^+, \phi^0)^T$):

$$V = m^2 (\Phi^\dagger \Phi) + \lambda_1 (\Phi^\dagger \Phi)^2 + M^2 \text{Tr}(\Delta^\dagger \Delta) + \lambda_2 [\text{Tr}(\Delta^\dagger \Delta)]^2 + \lambda_3 \text{Det}(\Delta^\dagger \Delta) + \lambda_4 (\Phi^\dagger \Phi) \text{Tr}(\Delta^\dagger \Delta) + \lambda_5 (\Phi^\dagger \tau_1 \Phi) \text{Tr}(\Delta^\dagger \tau_1 \Delta) + \left( \frac{1}{\sqrt{2}} \mu (\Phi^T i \tau_2 \Delta^\dagger \Phi) + \text{h.c} \right)$$

Here $m^2 < 0$ in order to ensure $\langle \phi^0 \rangle = v / \sqrt{2}$, which spontaneously breaks $SU(2) \otimes U(1)_Y$ to $U(1)_Q$, and $M^2 > 0$ is the mass term for the triplet scalars. In the model of Gelmini-Roncadelli \cite{31} the term $\mu (\Phi^T i \tau_2 \Delta^\dagger \Phi)$ is absent, which leads to spontaneous violation of lepton number for $M^2 < 0$. The resulting Higgs spectrum contains a massless triplet scalar (majoron, $J$) and another light scalar ($H^0$). Pair production via $e^+ e^- \rightarrow H^0 J$ would give a large contribution to the invisible width of the $Z$ and this model was excluded at the CERN Large Electron Positron Collider (LEP). The inclusion of the term $\mu (\Phi^T i \tau_2 \Delta^\dagger \Phi)$ explicitly breaks lepton number when $\Delta$ is assigned $L = 2$, and eliminates the Majoron \cite{3, 6, 7}. Thus the scalar potential in eq. (4) together with the triplet Yukawa interaction of eq. (1) lead
to a phenomenologically viable model of neutrino mass generation. For small $v_\Delta/v$, the expression for $v_\Delta$ resulting from the minimization of $V$ is:

$$v_\Delta \sim \frac{\mu v^2}{2M^2 + (\lambda_4 + \lambda_5)v^2}.$$  \hspace{1cm} (5)

For large $M$ compared to $v$, one has $v_\Delta \simeq \mu v^2/2M^2$ which is sometimes referred to as the “Type II seesaw mechanism” and would naturally lead to a small $v_\Delta$. Recently there has been much interest in the scenario of light triplet scalars ($M \approx v$) within the discovery reach of the LHC, for which eq. (5) leads to $v_\Delta \approx \mu$. In extensions of the HTM the term $\mu(\Phi^T i \tau_2 \Delta^\dagger \Phi)$ may arise in various ways: i) it could be from the vev of a Higgs singlet field [32]; ii) it could be generated at higher orders in perturbation theory [13]; iii) it could originate in the context of extra dimensions [12].

An upper limit on $v_\Delta$ can be obtained from considering its effect on the parameter $\rho(= M_W^2/M_Z^2 \cos^2 \theta_W)$. In the SM $\rho = 1$ at tree-level, while in the HTM one has (where $x = v_\Delta/v$):

$$\rho \equiv 1 + \delta \rho = \frac{1 + 2x^2}{1 + 4x^2}.$$  \hspace{1cm} (6)

The measurement $\rho \approx 1$ leads to the bound $v_\Delta/v \lesssim 0.03$, or $v_\Delta < 8$ GeV. At the 1-loop level $v_\Delta$ must be renormalized, and explicit analyses lead to bounds on its magnitude similar to those derived from the tree-level analysis, e.g. see [33]. The HTM has seven Higgs bosons ($H^{++,H^{--},H^+,H^-,H^0,A^0,h^0}$). The doubly charged $H^{\pm\pm}$ is entirely composed of the triplet scalar field $\Delta^{\pm\pm}$, while the remaining eigenstates are, in general, mixtures of the doublet and triplet fields. Such mixing is proportional to the triplet vev, and hence small even if $v_\Delta$ assumes its largest value of a few GeV. Therefore $H^\pm,H^0,A^0$ are predominantly composed of the triplet fields, while $h^0$ is predominantly composed of the doublet field and plays the role of the SM Higgs boson. The squared masses of $H^{\pm\pm},H^\pm,H^0,A^0$ are of order $M^2$ with splittings of order $\lambda_5 v^2$. The mass hierarchy $m_{H^{\pm\pm}} < m_{H^\pm} < m_{H^0,A^0}$ is obtained for $\lambda_5 > 0$.

The phenomenologically attractive feature of the HTM is the direct connection between the triplet Yukawa coupling $h_{ij}$ and the neutrino mass matrix ($m_{ij}$) shown in eq. (3). The mass matrix for three Dirac neutrinos is diagonalized by the MNS (Maki-Nakagawa-Sakata) matrix $V_{\text{MNS}}$ [34] for which the standard parametrization is:

$$V_{\text{MNS}} = \begin{pmatrix} 
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}c_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},$$  \hspace{1cm} (7)

where $s_{ij} \equiv \sin \theta_{ij}$ and $c_{ij} \equiv \cos \theta_{ij}$, and $\delta$ is the Dirac phase. The ranges are chosen as $0 \leq \theta_{ij} \leq \pi/2$ and $0 \leq \delta < 2\pi$. For Majorana neutrinos, two additional phases appear, and then the mixing matrix $V$ becomes

$$V = V_{\text{MNS}} \times \text{diag}(1,e^{i\varphi_1/2},e^{i\varphi_2/2}),$$  \hspace{1cm} (8)

where $\varphi_1$ and $\varphi_2$ are referred to as the Majorana phases [6,35] and $0 \leq \varphi_1, \varphi_2 < 2\pi$. One has the freedom to work in the basis in which the charged lepton mass matrix is diagonal,
and then the neutrino mass matrix is diagonalized by $V$. Using eq. (3) one can write the couplings $h_{ij}$ as follows [12, 13]:

$$h_{ij} = \frac{m_{ij}}{\sqrt{2}v_\Delta} \equiv \frac{1}{\sqrt{2}v_\Delta} [V_{\text{MNS}} \text{diag}(m_1, m_2 e^{i\varphi_1}, m_3 e^{i\varphi_2}) V_{\text{MNS}}^T]_{ij}$$  

(9)

Neutrino oscillation experiments involving solar [36], atmospheric [37], accelerator [38], and reactor neutrinos [39, 40] are sensitive to the mass-squared differences and the mixing angles, and give the following preferred values and ranges:

$$\Delta m^2_{21} \equiv m_2^2 - m_1^2 \simeq 7.6 \times 10^{-5} \text{eV}^2, \quad |\Delta m^2_{31}| \equiv |m_3^2 - m_1^2| \simeq 2.4 \times 10^{-3} \text{eV}^2,$$

$$\sin^2 2\theta_{12} \simeq 0.87, \quad \sin^2 2\theta_{23} \simeq 1, \quad \sin^2 2\theta_{13} \lesssim 0.14.$$  

(10)  

(11)

We use these values in our numerical analysis unless otherwise mentioned. The small mixing angle $\theta_{13}$ has not been measured yet and hence the value of $\delta$ in completely unknown. Sensitivity to $\sin^2 2\theta_{13} \sim 0.01$ is expected from various forthcoming experiments [41, 42]. Probing $\sin^2 2\theta_{13} \ll 0.01$ would require construction of a neutrino factory or beta beam experiment [43]. Since the sign of $\Delta m^2_{31}$ is also undetermined at present, distinct neutrino mass hierarchy patterns are possible. The case with $\Delta m^2_{31} > 0$ is referred to as Normal hierarchy (NH) where $m_1 < m_2 < m_3$ and the case with $\Delta m^2_{31} < 0$ is known as Inverted hierarchy (IH) where $m_3 < m_1 < m_2$. Information on the mass $m_0$ of the lightest neutrino and the Majorana phases cannot be obtained from neutrino oscillation experiments. This is because the oscillation probabilities are independent of these parameters, not only in vacuum but also in matter. If $m_0 \gtrsim 0.2\text{eV}$, a future $^3\text{H}$ beta decay experiment [44] can measure $m_0$. Experiments which seek neutrinoless double beta decay [45] are only sensitive to a combination of neutrino masses and phases. Certainly, extracting information on Majorana phases alone from these experiments seems extremely difficult, if not impossible [46]. Therefore it is worthwhile to consider other possibilities in the context of the HTM. One method is the branching ratio of the doubly charged Higgs boson to two leptons, $\text{BR}(H^{\pm\pm} \to l_i^\pm l_j^\pm)$, which is determined by $|h_{ij}|^2$ and has been studied in detail in [14, 15, 16, 17]. An alternative method is $\text{BR}(\tau \to l_i l_j l_k)$ which depends on $|h_{ik}^* h_{jk}|^2$ and will be studied in detail in this work.

### III. LFV DECAYS IN THE HTM

In this section we introduce the theoretical framework for the LFV decays $\tau \to l_i l_j l_k$, $\mu \to eee$ and $\mu \to e\gamma$ in the HTM. Early studies of the effect of $H^{\pm\pm}$ on these decays were performed in [47]. The works [13, 19] are the only ones which address the prediction for such LFV decays in terms of the parameters of the neutrino mass matrix, which is a unique feature of the HTM. We note here that analogous studies of these LFV decays have been performed for the supersymmetric type I seesaw in which the Higgs triplet has a mass of the grand unification (GUT) scale [48, 49]. In such models the LFV decays are mediated via TeV scale SUSY particles but the dependence of $\mu \to e\gamma$ on the neutrino mass matrix parameters is identical to that for the TeV scale triplet model (HTM) in [13, 19]. However, for the LFV $\tau$ decays there is a significant difference between models with a GUT scale triplet and a TeV scale triplet. In the former the tree-level contribution to $\tau \to l_i l_j l_k$ from $H^{\pm\pm}$ is negligible and the dominant contribution to these decays is from higher order diagrams. Consequently, one expects $\text{BR}(\tau \to l\gamma) > \text{BR}(\tau \to l_i l_j l_k)$, and the predictions for
$\tau \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$ constitute a probe of such models. The distinctive feature of the HTM is the prediction for the decays $\tau \rightarrow \bar{l}_i l_j l_k$ which are mediated by a TeV scale $H^{\pm\pm}$, and the opposite hierarchy $\text{BR}(\tau \rightarrow \bar{l}_i l_j l_k) > \text{BR}(\tau \rightarrow l\gamma)$. The LFV processes we study are not mediated by the neutral scalar triplet fields because $\Delta^0$ does not couple to charged leptons. In other models (e.g. the Two Higgs Doublet Model) neutral Higgs bosons can contribute significantly to LFV decays \cite{50}.

In our numerical analysis the stringent constraint from $\mu \rightarrow e\bar{e}ee$ is imposed, while $\tau \rightarrow \bar{l}_i l_j l_k$ and $\mu \rightarrow e\gamma$ offer the possibility to observe a LFV signal in the HTM. Other constraints on $h_{ij}$ (e.g. the anomalous magnetic moment $(g-2)$ of $\mu$, Bhabha scattering and other LFV processes - reviewed in \cite{51}) are considerably weaker and are neglected. We note that $\mu \rightarrow e$ conversion in the HTM was also studied in \cite{12,19} and can give a constraint similar to that of $\mu \rightarrow e\gamma$. However, we choose not to impose the constraint from $\mu \rightarrow e$ conversion, which will be considerably weaker than that from $\mu \rightarrow e\bar{e}ee$. In addition, an improvement of the current bounds \cite{52} on $\mu \rightarrow e$ conversion is not likely in the near future (in contrast to the situation for $\mu \rightarrow e\gamma$ and $\tau \rightarrow \bar{l}_i l_j l_k$).

A. The decays $\tau \rightarrow \bar{l}_i l_j l_k$ and $\mu \rightarrow e\bar{e}e$

Mere observation of the LFV decays $\tau \rightarrow \bar{l}_i l_j l_k$ and $\mu \rightarrow e\bar{e}e$ would constitute a spectacular signal of physics beyond the SM. In the HTM these decays are mediated at tree level by $H^{\pm\pm}$ and provide sensitive probes of the $h_{ij}$ couplings if $m_{H^{\pm\pm}}$ is of the order of the electroweak scale. There are six distinct decays for $\tau^- \rightarrow \bar{l}_i l_j l_k$ (likewise for $\tau^+$): $\tau^- \rightarrow \mu^+\mu^-\mu^-$, $\tau^- \rightarrow e^+e^-\mu^-$, $\tau^- \rightarrow \mu^+\mu^-\mu^-$, $\tau^- \rightarrow e^+e^-\mu^-$, $\tau^- \rightarrow \mu^-\mu^-\mu^-$, $\tau^- \rightarrow e^+e^-\mu^-$. Searches for all six decays have been performed by BELLE \cite{21} and BABAR \cite{22}. Upper limits of the order $\text{BR}(\tau \rightarrow \bar{l}_i l_j l_k) < 2 \times 10^{-8}$ were obtained. For the decay $\mu \rightarrow e\bar{e}e$ there exists the stringent bound $\text{BR}(\mu \rightarrow e\bar{e}e) < 10^{-12}$ \cite{29}. For a given $m_{H^{\pm\pm}}$ these LFV $\tau$ and $\mu$ decays constrain many combinations of the $h_{ij}$ couplings in the HTM. Regarding the experimental prospects, the sensitivity to $\text{BR}(\mu \rightarrow e\bar{e}e)$ will not improve in the foreseeable future and presumably can only be improved at a $\mu$ storage ring at a future neutrino factory \cite{43}. In contrast, greater sensitivity to $\tau \rightarrow \bar{l}_i l_j l_k$ is expected at the ongoing $B$ factories, and a proposed Super $B$ factory could probe $\text{BR}(\tau \rightarrow \bar{l}_i l_j l_k) < 10^{-9}$ for luminosity $> 10^8$ ab$^{-1}$ \cite{23,21,25,26}. At the LHC, $\tau$ can be copiously produced from several sources i.e., from $B/D$ decay and direct production via $pp \rightarrow W \rightarrow \tau\nu$, $pp \rightarrow Z \rightarrow \tau^+\tau^-$. Sensitivity to $\text{BR}(\tau \rightarrow \bar{\mu}\mu\mu) > 10^{-8}$ is claimed \cite{27,28}, which is similar to the sensitivity already reached at the $B$ factories. It is important to note that all the above experiments can probe smaller BRs for $\tau \rightarrow \bar{l}_i l_j l_k$ than for $\tau \rightarrow l\gamma$ and this is due to the smaller SM background for the former. Hence the hierarchy $\text{BR}(\tau \rightarrow \bar{l}_i l_j l_k) > \text{BR}(\tau \rightarrow l\gamma)$ in the HTM is favourable from the standpoint of experimental sensitivity.

The virtual exchange of $H^{\pm\pm}$ induces an effective interaction of four charged leptons for $l_m \rightarrow \bar{l}_i l_j l_k$ decay ($j \neq k$) as follows:

$$
\mathcal{L} = \frac{(h^*)_{mi}(h)_{jk}}{4\sqrt{2} G_F m_{H^{\pm\pm}}^2} \left\{ 2\sqrt{2} G_F \left( l_m^\gamma \gamma^\mu P_L l_k \right) \left( \bar{l}_i^\gamma \gamma^\mu P_L l_j \right) \right\} + \text{(three terms with } m \leftrightarrow i \text{ and } j \leftrightarrow k) + h.c.
$$

(12)

Here $m, i, j, k$ are fixed and we used the Fierz transformation. $G_F = 1.17 \times 10^{-5}$ GeV$^{-2}$ is the Fermi constant. Neglecting the masses of the final-state particles, the branching ratio
for $\tau \to \bar{l}_l l_j l_k$ is given by:

$$\text{BR}(\tau \to \bar{l}_l l_j l_k) = \frac{S |h_{\tau i}|^2 |h_{jk}|^2}{4G_F^2 m_H^{\pm \pm}} \text{BR}(\tau \to \mu \bar{\nu}_\mu)$$  \hspace{1cm} (13)

$$\approx 0.19 S |h_{\tau i}|^2 |h_{jk}|^2 \left(\frac{200 \text{GeV}}{m_H^{\pm \pm}}\right)^4,$$  \hspace{1cm} (14)

where $\text{BR}(\tau \to \mu \bar{\nu}_\mu) \approx 17\%$. Here $S=1$ (2) for $j = k$ ($j \neq k$). The branching ratio for $\mu \to e e e$ is given by:

$$\text{BR}(\mu \to e e e) = \frac{|h_{\mu e}|^2 |h_{e e}|^2}{4G_F^2 m_H^{\pm \pm}} \text{BR}(\mu \to e \bar{\nu}_e)$$  \hspace{1cm} (15)

$$\approx 1.1 |h_{\mu e}|^2 |h_{e e}|^2 \left(\frac{200 \text{GeV}}{m_H^{\pm \pm}}\right)^4,$$  \hspace{1cm} (16)

where $\text{BR}(\mu \to e \bar{\nu}_e) \approx 100\%$.

Previous studies of the above decays in the context of the HTM were performed in \cite{13, 19}. In \cite{13} only the specific case of $|h_{e\mu}| = 0$ was studied, which automatically suppresses $\text{BR}(\mu \to e e e)$, and predictions for the ratios of $\text{BR}(\tau \to \bar{l}_l l_j l_k)$ were given. In \cite{19} it was mentioned that an observable $\text{BR}(\tau \to \bar{l}_l l_j l_k)$ would require accidental cancellations to suppress $\text{BR}(\mu \to e e e)$, although there was no quantitative analysis.

We note that studies of $\tau \to \bar{l}_l l_j l_k$ (and $\mu \to e e e$) in other models which contain $H^{\pm \pm}$ have been performed (hereafter the subscripts $L$ and $R$ indicate the chirality of the leptons in the interaction of eq. (1), and $H^{\pm \pm}$ in the HTM corresponds to $H_L^{\pm \pm}$), i) Zee-Babu model ($H_R^{\pm \pm}$ only) \cite{53}; ii) Left-Right symmetric model ($H_L^{\pm \pm}$ and $H_R^{\pm \pm}$) \cite{30, 54}; iii) other models which contain a $H^{\pm \pm}$ \cite{55}. Analyses using effective Lagrangians have been performed \cite{56} and can be applied to models with $H^{\pm \pm}$. Angular asymmetries can also be defined which can probe the relative strength of the contributions from $H_L^{\pm \pm}$ and $H_R^{\pm \pm}$ \cite{28, 30, 57}. In the HTM such an asymmetry would be maximal since $H^{\pm \pm}$ only interacts with left-handed leptons. The distinctive phenomenological feature of $H^{\pm \pm}$ in the HTM is the specific relationship between $h_{ij}$ and the neutrino mass matrix given by eq. (3), which is not realized in the above models.

\subsection{B. $\mu \to e \gamma$}

Searches for $\mu \to e \gamma$ have a long history (e.g., see \cite{2}), and the most stringent upper bound is from the MEGA Collaboration which obtained $\text{BR}(\mu \to e \gamma) < 1.2 \times 10^{-11}$ \cite{58}. The ongoing MEG experiment anticipates sensitivity to $\text{BR}(\mu \to e \gamma) \sim 10^{-13}$ \cite{20}. The effective Lagrangian for $\mu \to e \gamma$ is as follows:

$$\mathcal{L} = -2\sqrt{2} G_F \left\{ m_\mu A_R \bar{\psi}_\mu \sigma^{\mu\nu} P_L F_{\mu \nu} + m_\mu A_L \bar{\psi}_\mu \sigma^{\mu\nu} P_R F_{\mu \nu} + h.c. \right\},$$  \hspace{1cm} (17)

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. In the HTM $A_L = 0$ but $A_R$ receives contributions from $H^{\pm \pm}$ and $H^\pm$. Explicit expressions for $A_R$ have been obtained, e.g., following \cite{59}:

$$A_R \simeq \frac{-q_e (h^\dagger h)_{e\mu}}{48\sqrt{2} \pi^2 G_F} \left( \frac{1}{m_{H^{\pm \pm}}} + \frac{1}{8m_{H^\pm}^2} \right),$$  \hspace{1cm} (18)
where $q_e$ is the positron charge and we have neglected the electron mass in the final state and all lepton masses in the loop. If $m_{H^±} \simeq m_{H^±}$ the dominant contribution is from the loops involving virtual $H^{±±}$. The decay rate is determined by the combination $(h^†h)_{eμ}$, and the branching ratio for $μ \rightarrow eγ$ with $m_{H^±} \simeq m_{H^{±±}}$ is given by:

$$BR(μ \rightarrow eγ) \simeq 384π^2|A_R|^2 \simeq \frac{27α|(h^†h)_{eμ}|^2}{64πG_F^2m_{H^{±±}}^4} \simeq 4.5 \times 10^{-3} |(h^†h)_{eμ}|^2 \left(\frac{200\text{GeV}}{m_{H^{±±}}}\right)^4.$$  \hspace{1cm} (19)

Here $α \equiv q_e^2/4π = 1/137$ is the fine structure constant. It was noted in [13, 19] that the decay rate for $μ \rightarrow eγ$ is very sensitive to $s_{13}$, but it is not so sensitive to the neutrino mass spectrum because it is independent of the absolute neutrino mass. Importantly (and stressed in [49]), the coupling $(h^†h)_{eμ}$ has no dependence on the Majorana phases either. Hence the prediction for $BR(μ \rightarrow eγ)$ in the HTM is much sharper than that for $τ \rightarrow l_1l_2l_3$, but the former is unable to probe those neutrino parameters which cannot be probed in neutrino oscillation experiments (i.e., absolute neutrino mass and Majorana phases). If polarized muons are available, then an angular asymmetry can be defined for $μ \rightarrow eγ$ [61]. In the HTM such an asymmetry would be maximal [30] because $H^{±±}$ and $H^±$ only interact with left-handed leptons.

**IV. NUMERICAL RESULTS**

In this section we present our numerical results for the BRs of the decays $τ \rightarrow l_il_jl_k$ and $μ \rightarrow eγ$ as a function of the parameters of the neutrino mass matrix. In eq. (9) one can express $m_1, m_2, m_3$ in terms of two neutrino mass-squared differences ($Δm^2_{21}, Δm^2_{31}$) and the mass of the lightest neutrino $m_0$. The couplings $h_{ij}$ are functions of nine parameters: $Δm^2_{21}, Δm^2_{31}, m_0, θ_{12}, θ_{13}, θ_{23}$, and three complex phases ($δ, ϕ_1, ϕ_2$).

The work of [13] focused on the case of suppressing $BR(μ \rightarrow eee)$ by small $|h_{eμ}|$. For the special case of $|h_{eμ}| = 0$ the ratios of $τ \rightarrow l_il_jl_k$ and $μ \rightarrow eγ$ were studied, normalizing to a specific decay $τ \rightarrow l_il_jl_k$. In analogy with the analysis of [13] our numerical analysis will study the values of ratios of LFV decays in which the arbitrary triplet vacuum expectation value cancels out. However, in difference to [13], some of our numerical analysis will normalize to $BR(μ \rightarrow eee)$ (which is not set to zero) and will quantify the parameter space of the neutrino mass matrix where a signal for $τ \rightarrow l_il_jl_k$ and/or $μ \rightarrow eγ$ could be seen at ongoing and planned experiments. The condition

$$\frac{BR(τ \rightarrow l_il_jl_k)}{BR(μ \rightarrow eee)} > 10^3$$  \hspace{1cm} (21)

signifies that $BR(τ \rightarrow l_il_jl_k) > 10^{-9}$ is possible while satisfying the bound $BR(μ \rightarrow eee) < 10^{-12}$. $BR(τ \rightarrow l_il_jl_k) > 10^{-9}$ is within the sensitivity of proposed high luminosity $B$ factories. Furthermore, $BR(τ \rightarrow l_il_jl_k)/BR(μ \rightarrow eee) > 10^4$ signifies that $τ \rightarrow l_il_jl_k$ can be

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1 In our numerical analysis we do not include a suppression factor of $\sim 15\%$ arising from electromagnetic corrections [60].
observed even in the current runs of the $B$ factories with integrated luminosities of the order of $1 \text{ab}^{-1}$. The condition
\[ \frac{\text{BR}(\mu \to e\gamma)}{\text{BR}(\mu \to eee)} > 10^{-1} \] (22)
corresponds to $\text{BR}(\mu \to e\gamma) > 10^{-13}$, which is within the sensitivity of the MEG experiment. These are necessary but not sufficient conditions for observation of $\tau \to \bar{l}_i l_j l_k$ and/or $\mu \to e\gamma$ in the HTM.

Once $\text{BR}(\mu \to eee)$ is sufficiently suppressed to satisfy both eq. (21) and (22), there is the possibility of observing multiple LFV signals while respecting current bounds on $\tau \to \bar{l}_i l_j l_k$ and $\mu \to e\gamma$ if the following conditions are satisfied:
\[ 10^2 < \frac{\text{BR}(\tau \to \bar{l}_i l_j l_k)}{\text{BR}(\mu \to e\gamma)}, \] (23)
\[ \frac{\text{BR}(\tau \to \bar{l}_a b l_c)}{\text{BR}(\mu \to e\gamma)} < 10^5, \] (24)
where $\text{BR}(\tau \to \bar{l}_a b l_c)$ is the largest among $\text{BR}(\tau \to \bar{l}_i l_j l_k)$. For the ratio $< 10^2$ in eq. (23) only a signal of $\mu \to e\gamma$ is possible, corresponding to $\text{BR}(\tau \to \bar{l}_i l_j l_k) \lesssim 10^{-9}$ with the current bound $\text{BR}(\mu \to e\gamma) \lesssim 10^{-11}$ even if eq. (21) is satisfied. Only $\tau \to \bar{l}_i l_j l_k$ can be observed for the ratio $> 10^5$ in eq. (24), for which the current bound $\text{BR}(\tau \to \bar{l}_a b l_c) \lesssim 10^{-8}$ gives $\text{BR}(\mu \to e\gamma) \lesssim 10^{-13}$ even if eq. (22) is satisfied. If more than two $\text{BR}(\tau \to \bar{l}_i l_j l_k)$ satisfy eq. (23), a condition for the ratio of $\tau$ LFV decays
\[ \frac{\text{BR}(\tau \to \bar{l}_i l_j l_k)}{\text{BR}(\tau \to \bar{l}_a b l_c)} > 10^{-1} \] (25)
makes it possible to observe $\tau \to \bar{l}_i l_j l_k$ in addition to $\tau \to \bar{l}_a b l_c$. The above ratios will be used to interpret our numerical results and find the regions of parameter space for possible LFV signals. Note again that conditions (23)–(24) are necessary but not sufficient. Once these conditions are satisfied, the arbitrary $v_\Delta$ and $m_{H^{\pm \pm}}$ can be freely chosen to provide an observable BR because such conditions do not depend on these parameters.

We present below explicit expressions for $h_{ee}$ and $h_{em}$ (which determine the decay rate for $\mu \to eee$), and the remaining $h_{ij}$ can be found in [14, 15, 16, 17].
\[ h_{ee} = \frac{1}{\sqrt{2}v_\Delta} \left( m_1 c_{12}^2 c_{13}^2 + m_2 s_{12}^2 c_{13}^2 e^{i\phi_1} + m_3 s_{13}^2 e^{i(\varphi_2 - 2\delta)} \right), \] (26)
\[ h_{em} = \frac{1}{\sqrt{2}v_\Delta} \left\{ m_1 (-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta}) c_{12} c_{13} + m_2 (c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta}) s_{12} c_{13} e^{i\varphi_1} + m_3 s_{23} c_{13} s_{13} e^{-i\delta} e^{i\varphi_2} \right\}, \] (27)

Four cases corresponding to no CP violation from Majorana phases can be defined as follows: Case I ($\varphi_1 = 0, \varphi_2 = 0$); Case II ($\varphi_1 = 0, \varphi_2 = \pi$); Case III ($\varphi_1 = \pi, \varphi_2 = 0$); Case IV ($\varphi_1 = \pi, \varphi_2 = \pi$). These four cases have been studied in [13] for values of $m_0 = 0$ or $\mathcal{O}(1)$ eV. In this work we will study in detail the dependence of $\tau \to \bar{l}_i l_j l_k$ and $\mu \to e\gamma$ on the neutrino mass matrix parameters (in particular $s_{13}, \delta, m_0, \varphi_1$ and $\varphi_2$) in order to find the regions which can provide an observable signal at the $B$ factories (for $\tau \to \bar{l}_i l_j l_k$) and the MEG experiment (for $\mu \to e\gamma$). We consider two distinct cases for the necessary suppression of $\mu \to eee$, corresponding to $|h_{em}| \simeq 0$ and $|h_{ee}| \simeq 0$. 


A. Case of $|h_{e\mu}| \simeq 0$

1. Normal Hierarchy

In the scenarios with no CP violation from Majorana phases, the special case of $|h_{e\mu}| = 0$ is achieved at a specific (“magic”) value of $\theta_{13}$ for all values of $m_1 (= m_0)$. In the normal hierarchy, the magic value is given by

$$s^\text{mge}_{13} \equiv \sin \theta^\text{mge}_{13}$$

The value of $\delta$ is chosen to give $|h_{e\mu}| = 0$. For all four cases, $\sin^2 2\theta^\text{mge}_{13}$ is about 0.02 for $m_1 = 0$; for Case I and II $\sin^2 2\theta^\text{mge}_{13}$ is a decreasing function of $m_1$ and converges to $9 \times 10^{-4}$ and 0, respectively. On the other hand, $\sin^2 2\theta^\text{mge}_{13}$ for Case I and II is an increasing function of $m_1$, and then $\sin^2 2\theta^\text{mge}_{13} < 0.14$ can be satisfied only for $m_1 \lesssim 0.01eV$. If $\sin^2 2\theta^\text{mge}_{13}$ is tuned completely to the magic value, the constraint from $\mu \rightarrow \bar{e}ee$ is automatically satisfied and one cannot normalize BRs to $\text{BR}(\mu \rightarrow \bar{e}ee)$.

In the following we discuss in detail the values of ratios of LFV decays in the case of $|h_{e\mu}| = 0$. In this case $\text{BR}(\tau \rightarrow \bar{\mu}\mu\mu)$ is the largest one among $\text{BR}(\tau \rightarrow l_i l_j l_k)$. Fig. 1 shows contours of the ratio $\text{BR}(\tau \rightarrow \bar{\mu}\mu\mu)/\text{BR}(\mu \rightarrow \bar{e}ee)$. This ratio is displayed in the plane $[m_1, \sin^2 2\theta^\text{mge}_{13}]$ in Fig. 1(a). It is evident that a wide range of $\sin^2 2\theta^\text{mge}_{13}$ around its magic value for small $m_1$ gives a ratio $> 10^3$, which is the necessary condition to provide a signal of $\tau \rightarrow \bar{\mu}\mu\mu$ high luminosity B factories. This can be traced to the fact that $\text{BR}(\mu \rightarrow \bar{e}ee)$ depends on the second power of the coupling $|h_{e\mu}|$, and so obtaining a ratio $> 10^3$ does not constitute a fine-tuning when expressed in terms of the neutrino parameters which determine $h_{e\mu}$. In Fig. 1(b) the ratio is displayed in the plane $[\delta, \sin^2 2\theta^\text{mge}_{13}]$ with $m_1 = 0$. One sees that achieving a ratio $> 10^3$ can be obtained in the interval $0.5 < \delta/\pi < 1.5$ for a wide range of $\sin^2 2\theta^\text{mge}_{13}$ around its magic value. Since only the relative Majorana phase $\varphi_2 - \varphi_1$ is physical for $m_1 = 0$, Fig. 1(b) is also the result for Case IV. Roughly speaking, the region of large ratio in Fig. 1(b) shifts horizontally for different values of the Majorana phases. Consequently, the $\delta$-dependence for Case II and III is similar to the result in Fig. 1(b) with a shift of $\delta$ by $\pi$. If $m_1$ is not very small, $|h_{e\mu}| \simeq 0$ with $s^\text{mge}_{13}$ of eq. (28) requires considerable fine-tuning of three phases. Even for large $m_1$, $s^\text{mge}_{13}$ of eq. (28) gives $|h_{e\mu}| \simeq 0$ in a sizeable region of $\delta$ and $\varphi_2$.

In Fig. 2 the ratio $\text{BR}(\mu \rightarrow e\gamma)/\text{BR}(\mu \rightarrow \bar{e}ee)$ is plotted for Case I. Fig. 2(a) shows that a ratio $> 10^{-1}$, which is necessary for discovery of $\mu \rightarrow e\gamma$, is obtained without fine-tuning of $\sin^2 2\theta^\text{mge}_{13}$ to the magic value for small $m_1$. In contrast, for large $m_1$ fine-tuning seems to be necessary for a ratio $> 10^{-1}$. Since $\text{BR}(\mu \rightarrow e\gamma)$ does not depend on the absolute neutrino mass, the behaviour in Fig. 2(a) is a result of the $m_1$ dependence of $\text{BR}(\mu \rightarrow \bar{e}ee)$. We will see below (Fig. 3) that $\mu \rightarrow e\gamma$ cannot be observed in the region of large $m_1$ even if $\theta_{13}$ is fine-tuned to $\theta^\text{mge}_{13}$ since current limits on LFV $\tau$ decays would be violated. The ratio becomes
very small at \( \sin^2 2\theta_{13} \simeq 0.001 \) because \(|(hh^1)_{e\mu}| \simeq 0\). Fig. 2(b) shows the \( \delta \)-dependence of the ratio for \( m_1 = 0 \). Most of the plane above \( \sin^2 2\theta_{13} \simeq 10^{-3} \) allows the ratio > \( 10^{-1} \), and the large ratio is achieved at any value of \( \delta \). The circle of ratio < \( 10^{-2} \) in Fig. 2(b) exists around the point of \(|(hh^1)_{e\mu}| = 0\), which does not depend on Majorana phases. Therefore the ratio is small for \( \sin^2 2\theta_{13} \lesssim 10^{-3} \) even with different values of Majorana phases. Comparing Fig. 1 with Fig. 2 one can see the parameter regions which allow a signal for one or both of \( \tau \rightarrow \bar{\mu}\mu\mu \) and \( \mu \rightarrow e\gamma \). Most notably, a signal for at least one LFV decay is possible in a large part of the parameter space for small \( m_1 \). In the region where signals for both \( \tau \rightarrow \bar{\mu}\mu\mu \) and \( \mu \rightarrow e\gamma \) are possible, the current bounds on these decays must also be respected.

Once \( s_{13}^{\text{mge}} \) is taken, \( \text{BR}(\tau \rightarrow \bar{l}_i l_j l_k)/\text{BR}(\mu \rightarrow e\gamma) > 10^2 \text{ and BR}(\mu \rightarrow e\gamma)/\text{BR}(\mu \rightarrow e\gamma) > 10^{-1} \) are satisfied automatically. Then current bounds on \( \text{BR}(\tau \rightarrow \bar{l}_i l_j l_k) \) and \( \text{BR}(\mu \rightarrow e\gamma) \) should be considered. Fig. 3 shows the \( m_1 \)-dependence of several ratios of LFV BRs normalized to \( \text{BR}(\mu \rightarrow e\gamma) \) with \( \delta = \pi \) and \( s_{13} = s_{13}^{\text{mge}} \) of eq. (2g) for Case I. Both \( \tau \rightarrow \bar{\mu}\mu\mu \) and \( \mu \rightarrow e\gamma \) vanish for \( s_{13} = s_{13}^{\text{mge}} \). For \( m_1 \gtrsim 0.03 \text{ eV} \), the ratio for \( \tau \rightarrow \bar{\mu}\mu\mu \) exceeds \( 10^5 \) and so \( \mu \rightarrow e\gamma \) is difficult to observe. Although Fig. 3(a) shows that \( \text{BR}(\tau \rightarrow \bar{l}_i l_j l_k)/\text{BR}(\mu \rightarrow e\gamma) > 10^2 \) is satisfied for several \( \tau \rightarrow \bar{l}_i l_j l_k \) (e.g. all four \( \tau \rightarrow \bar{l}_i l_j l_k \) decays satisfy this condition for \( m_1 \sim 0.03 \text{ eV} \)) one can see in Fig. 3(b) that only the decay \( \tau \rightarrow \bar{\mu}ee \) can satisfy \( \text{BR}(\tau \rightarrow \bar{l}_i l_j l_k)/\text{BR}(\tau \rightarrow \bar{\mu}\mu\mu) > 10^{-1} \) and be observed in addition to \( \tau \rightarrow \bar{\mu}\mu\mu \) for \( m_1 \gtrsim 0.01 \text{ eV} \). For \( m_1 \lesssim 0.01 \text{ eV} \), \( \tau \rightarrow \bar{\mu}\mu\mu \) is the only LFV \( \tau \) decay which can be observed. Table IV summarizes the LFV decays which can be measured for the case with eq. (2g).

2. Inverted Hierarchy

In the inverted hierarchy scenario the condition for \( |h_{e\mu}| = 0 \) differs from that in the hierarchical scenario, as was noted in [13]. In Case I and II, \( |h_{e\mu}| = 0 \) is achieved when \( \delta = 0 \).
with a magic value of $s_{13}$,

$$s_{13}^{mgc} = \frac{(m_2 - m_1)c_{23} \sin 2\theta_{12}}{2(m_1c_{12}^2 + m_2s_{12}^2 \mp m_3)s_{23}},$$

(32)

where the upper and lower signs of the $m_3(= m_0)$ term are for Case I and II respectively. At $m_3 = 0$ the value of $\sin^2 2\theta_{13}^{mgc}$ is $2 \times 10^{-4}$ and converges to $9 \times 10^{-4}$ (0) for large $m_3$ in Case I (II). No acceptable $s_{13}^{mgc}$ exists for Case III and IV. In the inverted hierarchy, $\text{BR}(\tau \to \bar{\mu} \bar{e}e)$ is the largest one among $\text{BR}(\tau \to \bar{l}_i l_j l_k)$. Only a small region of $s_{13}$ can give $\text{BR}(\tau \to \bar{\mu} \bar{e}e)/\text{BR}(\mu \to \bar{e}ee) > 10^3$ for Case I in IH. Fine tunings of three phases are
\[ (\delta, \varphi_1, \varphi_2) = (\pi, 0, 0) \]

\[
\begin{array}{ccc}
  & m_1 \lesssim 0.01 \text{ eV} & 0.01 \text{ eV} \lesssim m_1 \lesssim 0.03 \text{ eV} & 0.03 \text{ eV} \lesssim m_1 \\
\hline
\text{NH} & \tau \to \bar{\mu}\mu & \tau \to \bar{\mu}\mu & \tau \to \bar{\mu}\mu \\
 & \mu \to e\gamma & \tau \to \bar{\mu}ee & \tau \to \bar{\mu}ee \\
\hline
\text{IH} & \tau \to \bar{\mu}\mu & \tau \to \bar{\mu}ee & \\
\end{array}
\]

TABLE I: Table of LFV decays which can be within the future experimental sensitivity with \(|h_{e\mu}| = 0\) for \(\delta = \pi\) in Case I. Other decay modes are rather unlikely to be observed.

FIG. 4: Contours of \(\text{BR}(\tau \to \bar{\mu}\mu)/\text{BR}(\mu \to \bar{e}ee)\) in the normal hierarchy. a) \(m_1\)-dependence with \(\varphi_1 = \varphi_1^{\text{mge}} = \pi\) for \((\delta, \varphi_2) = (\pi, 0)\), which corresponds to Case III. b) \(\varphi_1\)-dependence with \(m_1 = m_1^{\text{mge}}\) for \((\delta, \varphi_2) = (\pi, 0)\), where \(m_1^{\text{mge}}\) is a function of \(\sin^2 2\theta_{13}\). A signal for \(\tau \to \bar{\mu}\mu\) is possible if the ratio \(> 10^3\) (depicted by the shaded region).

required except for very small \(m_3\). The parameter space in the plane \([m_1, \sin^2 2\theta_{13}]\) for \(\text{BR}(\mu \to e\gamma)/\text{BR}(\mu \to \bar{e}ee) > 0.1\) is similar to that for \(\text{BR}(\tau \to \bar{\mu}ee)/\text{BR}(\mu \to \bar{e}ee) > 10^3\). Taking \(s_{13} = s_{13}^{\text{mge}}\) of eq. (32) for Case I results in an unobservable \(\mu \to e\gamma\) and only \(\tau \to \bar{\mu}\mu\mu\) can be observed in addition to \(\tau \to \bar{\mu}ee\) for all values of \(m_3\).

B. Case of \(|h_{ee}| = 0\)

An alternative way of suppressing \(\text{BR}(\mu \to \bar{e}ee)\) is via small \(|h_{ee}|\). This scenario has been discussed [30] in the context of the Left-Right symmetric model in which the light neutrino mass matrix \(m_{ij}\) in the expression for \(h_{ij}\) in eq. (9) is replaced by an arbitrary mass matrix for heavy Majorana neutrinos. We point out that \(|h_{ee}| \simeq 0\) is also a viable option in the more restricted framework of the HTM. The fact that \(|m_{ee}| = 0\) is possible for normal hierarchy is well known in studies of neutrinoless double beta decay [62]. However, this possibility seems to have been overlooked in the context of the HTM and its predictions for LFV decays. Neutrinoless double beta decay in the HTM was studied in [62, 64, 65].
FIG. 5: Same as Fig. 4 but for $\mu \rightarrow e\gamma$. A signal for $\mu \rightarrow e\gamma$ is possible if the ratio $> 0.1$ (depicted by the shaded region).

1. Normal Hierarchy

The conditions for $|h_{ee}| \simeq 0$ are different to those for $|h_{e\mu}| \simeq 0$, and hence this possibility enlarges the parameter space for an observable signal for $\tau \rightarrow \bar{l}_j l_k$ and $\mu \rightarrow e\gamma$. Let us start with $m_1 = 0$ for simplicity. The complete elimination of $|h_{ee}|$ is possible, in principle, with $\varphi_2 - 2\delta - \varphi_1 = \pi$ at $s_{13}$ given by

$$s^2_{13} = \frac{s^2_{12} \sqrt{\Delta m^2_{21}} + \Delta m^2_{31}}{s^2_{12} \sqrt{\Delta m^2_{21}} + \sqrt{\Delta m^2_{31}}} \simeq 0.05. \quad (33)$$

However, this value of $s^2_{13}$ violates the bound $\sin^2 2\theta_{13} < 0.14$ ($s^2_{13} \lesssim 0.04$). On the other hand, $|h_{ee}| = 0$ is achieved for given values of $s_{13}$ and $\varphi_2 - 2\delta$ by the magic values of $\varphi_1$ and $m_1$:

$$\sin \varphi^\text{mge}_1 \equiv -\sqrt{(m^\text{mge}_1)^2 + \Delta m^2_{31}} \frac{t^2_{13} \sin(\varphi_2 - 2\delta), \cos \varphi^\text{mge}_1 \leq 0,} {s^2_{12} \sqrt{(m^\text{mge}_1)^2 + \Delta m^2_{21}}} \quad (34)$$

$$(m^\text{mge}_1)^2 \equiv \frac{1}{\cos^2 2\theta_{12} - 2(s^4_{12} + c^4_{12} \cos 2(\varphi_2 - 2\delta)) t^4_{13} + t^6_{13}} \times \left[ s^4_{12} \cos 2\theta_{12} \Delta m^2_{21} + \left( s^4_{12} \Delta m^2_{21} + (s^4_{12} + c^4_{12} \cos 2(\varphi_2 - 2\delta)) \Delta m^2_{31} \right) t^4_{13} \right. \left. - \Delta m^2_{31} t^8_{13} - 2c^2_{12} t^2_{13} \cos(\varphi_2 - 2\delta) \sqrt{A + B t^4_{13}} \right], \quad (35)$$

$$A \equiv (s^4_{12} \Delta m^2_{21} + \cos 2\theta_{12} \Delta m^2_{31}) s^4_{12} \Delta m^2_{21},$$

$$B \equiv \left\{ (s^4_{12} - c^4_{12} \sin^2(\varphi_2 - 2\delta)) \Delta m^2_{31} - s^4_{12} \Delta m^2_{21} \right\} \Delta m^2_{31}, \quad (36)$$

where we define $t_{13} \equiv s_{13}/c_{13}$. We have $|h_{ee}| = 0$ at $s_{13} = 0$ with $m^\text{mge}_1 \simeq 4.6 \times 10^{-3}\text{eV}$. In contrast, $|h_{e\mu}| = 0$ can only be obtained at $s_{13} = 0$ in the unphysical limit of infinite $m_1$. Elimination of $h_{ee}$ is possible in Case III and IV of $\varphi_1 = \pi$ but Case I and II cannot realize $|h_{ee}| = 0$ within the bound $\sin^2 2\theta_{13} < 0.14$. 

\[14\]
BR(\tau \rightarrow l_i l_j l_k)/BR(\tau \rightarrow e\gamma)$; b) $BR(\tau \rightarrow l_i l_j l_k)/BR(\tau \rightarrow \mu\mu\mu)$.

| $\sin^2 2\theta_{13}$ | \(\delta, \varphi_1, \varphi_2 = (\pi, \pi, 0)\) |
|------------------------|-----------------------------------------------|
| \(2 \times 10^{-4}\)  | $\tau \rightarrow \mu\mu\mu$ |
| \(2 \times 10^{-3}\)  | $\tau \rightarrow \mu\mu\mu$ |
| \(2 \times 10^{-3}\)  | $\tau \rightarrow \mu\mu\mu$ |

TABLE II: LFV decays which can be measured for $|h_{ee}| = 0$ in Case III. $\delta = \pi$ is used. $\tau \rightarrow \mu\mu\mu$ with parentheses exists just below the condition (25).

Fig. 4(a) shows the ratio $BR(\tau \rightarrow \mu\mu\mu)/BR(\mu \rightarrow \bar{e}ee)$ in the plane $[m_1, \sin^2 2\theta_{13}]$ with $\varphi_1 = \varphi_1^{mgc}$ for $\delta = \pi$ and $\varphi_2 = 0$. This case corresponds to Case III because these values of $\delta$ and $\varphi_2$ give $\varphi_1^{mgc} = \pi$. The vertical funnel in the figure is caused by $|h_{ee}| \simeq 0$ around $m_1^{mgc}$ of eq. (35). One sees that the ratio $> 10^3$ can be obtained without fine-tuning of $m_1$ to $m_1^{mgc}$, especially for very small $\sin^2 2\theta_{13}$. Since $|h_{ee}| = 0$ can be achieved for $s_{13} = 0$ also, there is no tuning of $\delta$ and $\varphi_2$, which appear only with $s_{13}$ in $h_{ee}$. Fig. 4(b) shows the $\varphi_1$-dependence of the ratio with $m_1 = m_1^{mgc}$. Two isolated regions which give the ratio $> 10^3$ exist around $\varphi_1 = \pi$ and $0$. The large ratio around $\varphi_1 = \pi$ is caused by $|h_{ee}| \simeq 0$. The other large ratio around $\varphi_1 = 0$ corresponds to the suppression of $|h_{e\mu}|$ in Case I (see eq. (28)) and can also be seen in Fig. 4(a) at $m_1 = m_1^{mgc}$. It is evident that sufficient suppression of $|h_{ee}|$ (and $|h_{e\mu}|$ also) is achieved in a sizable range of $\varphi_1$.

The ratio $BR(\mu \rightarrow e\gamma)/BR(\mu \rightarrow \bar{e}ee)$ is presented in Fig. 5(a) in the plane $[m_1, \sin^2 2\theta_{13}]$ with $\varphi_1 = \varphi_1^{mgc}$ for $\delta = \pi$ and $\varphi_2 = 0$. It can be seen that the regions of large ratio roughly correspond to the regions of large ratio in Fig. 4(a). In Fig. 5(b) the ratio becomes large for $\varphi_1 \simeq 0$ and $\pi$ because of $|h_{ee}| \simeq 0$ and $|h_{e\mu}| \simeq 0$ respectively (as in Fig. 4(b)). In both figures the ratio becomes very small for $\sin^2 2\theta_{13} \simeq 10^{-3}$ because $|(h_{hh}^{ee})_{e\mu}| = 0$.

Fig. 6 shows the $\theta_{13}$ dependence of ratios of BRs for several LFV decays. We take the magic value for $m_1$ of eq. (35) in Case III and the Dirac phase is taken as $\delta = \pi$. Fig. 6(a) shows $BR(\tau \rightarrow l_i l_j l_k)/BR(\mu \rightarrow e\gamma)$ and (b) presents $BR(\tau \rightarrow l_i l_j l_k)/BR(\tau \rightarrow \mu\mu\mu)$. The
solid, dashed, dash-dotted, and dotted lines show results for $\tau \to \mu\mu\mu$, $\tau \to \mu\mu\mu$, $\tau \to \tau\mu\mu$, and $\tau \to \tau\mu\mu$, respectively. The decays $\tau \to \tau\ee$ and $\tau\ee$ are forbidden via $|h_{ee}| = 0$. In Fig. 6(a), the ratio for $\tau \to \mu\mu\mu$ (the largest among $\tau \to \bar{l}_i l_j l_k$) becomes larger than $10^5$ for $\sin^2 2\theta_{13} \simeq 10^{-3}$ because of $|(h h^\dagger)_{e\mu}| \simeq 0$ and $\mu \to e\gamma$ cannot be measured in this region. For $\sin^2 2\theta_{13}$ near its maximum value, only the ratio for $\tau \to \mu\mu\mu$ stays $> 10^2$. Ratios for several $\tau \to \bar{l}_i l_j l_k$ satisfy $> 10^2$ for most values of $\sin^2 2\theta_{13}$. However, Fig. 5(b) shows that only $\tau \to \mu\mu\mu$ can be measured in addition to $\tau \to \mu\mu\mu$ because the other decays do not satisfy $\BR(\tau \to \bar{l}_i l_j l_k)/\BR(\tau \to \mu\mu\mu) > 0.1$. The decays $\tau \to \bar{e}\mu\mu$ and $\tau \to \bar{e}\mu\mu$ are vanishing around $\sin^2 2\theta_{13} = 7 \times 10^{-2}$ because of $|h_{\tau e}| \simeq 0$. Table III summarizes the LFV decays which can be measured in this case. We emphasize again that a signal of $\tau \to \mu\mu\mu$ is impossible for the case where $\mu \to e\ee$ is suppressed by $|h_{\mu\mu}| \simeq 0$.

Fig. 7(a) shows the ratio $\BR(\tau \to \mu\mu\mu)/\BR(\mu \to e\gamma)$ in the plane $[\delta, \sin^2 2\theta_{13}]$ for $\varphi_2 = 0$ with $|h_{ee}| = 0$, which is achieved by $\phi_1^{mgc}$ and $m_1^{mgc}$. The ratio becomes very large around $\delta = \pi$ and $\sin^2 2\theta_{13} \simeq 1 \times 10^{-3}$ because $|(h h^\dagger)_{e\mu}| \simeq 0$. Except for this tiny region where the ratio is very large, the ratio satisfies $\lesssim 10^5$ and $\gtrsim 10^2$ and therefore it is possible to observe a signal for both $\mu \to e\gamma$ and $\tau \to \mu\mu\mu$ in a very large part of the parameter space. As shown in Table III, $\tau \to \mu\mu\mu$ can be measured in addition to $\tau \to \mu\mu\mu$. The ratios for $\BR(\tau \to \mu\mu\mu)$ normalized by $\BR(\mu \to e\gamma)$ and $\BR(\tau \to \mu\mu\mu)$ are presented in Figs. 7(b) and (c), respectively. Note that a non-negligible rate for $\BR(\tau \to \mu\mu\mu)$ is possible only if $\mu \to e\ee$ is suppressed by $|h_{ee}| \simeq 0$. In Fig. 7(c) the ratio becomes very tiny around $\delta = 0$ and $\sin^2 2\theta_{13} \simeq 7 \times 10^{-2}$ because $|h_{\mu\mu}| \simeq 0$. The ratio in Fig. 7(b) exceeds $10^2$ in almost all of the plane $[\delta, \sin^2 2\theta_{13}]$, and the ratio in Fig. 7(c) is larger than 0.1 for $\sin^2 2\theta_{13} \gtrsim 10^{-4}$ and for a wide range of values of $\delta$ around $\delta = \pi$. Even below the line for ratio = 0.1 in Fig. 7(c), the ratio is very close to 0.1 in most of the parameter space as we have seen in Fig. 6(b) for $\delta = \pi$. Therefore there are good prospects for observing $\tau \to \mu\mu\mu$ in addition to $\tau \to \mu\mu\mu$ in this scenario.

2. Inverted Hierarchy

The case of $|h_{ee}| \simeq 0$ cannot be realized for the inverted hierarchy scenario under the constraints on the neutrino oscillation parameters. In the expression for $h_{ee}$ in eq. (27) one sees that an exact cancellation between the $m_1$ term and the $m_2$ term cannot be achieved because $s_{12}^4 / \cos 2\theta_{12} \simeq 0.28$ cannot be equal to $m_1^2 / \Delta m_{21}^2$, which is greater than $|\Delta m_{31}^2| / \Delta m_{21}^2$ in the inverted hierarchy scenario. Furthermore, $\theta_{13}$ is too small to cancel the remaining difference between the $m_1$ term and the $m_2$ term. In Fig. 8(below), we will numerically reconfirm that $|h_{ee}| \simeq 0$ is not possible in the inverted hierarchy scenario.

C. Branching ratios of $H^{\pm\pm} \to l^\pm l^\pm$ with observable LFV $\mu/\tau$ decay

Finally, we discuss the impact of the observation of a LFV lepton decay on the leptonic branching ratios of $H^{\pm\pm}$ in the HTM. The LHC has sensitivity up to $m_{H^{\pm\pm}} \sim 1$TeV if $H^{\pm\pm}$ decays leptonically to $e^\pm e^\pm$, $e^\pm \mu^\pm$ or $\mu^\pm \mu^\pm$ with sizeable BRs $\frac{8}{3}$ $\frac{10}{10}$ $\frac{17}{17}$ $\frac{18}{18}$. As can be seen from the Appendix, observation of a LFV decay with $100 \text{ GeV} < m_{H^{\pm\pm}} < 1000 \text{ GeV}$ requires $h_{ij}$ of the order $10^{-2} - 10^{-3}$, which corresponds to $1 \text{ eV} \lesssim v_\Delta \lesssim 1000 \text{ eV}$ (see eq. (3)) with a naive bound $\sqrt{|\Delta m_{31}^2|} \lesssim m_{ij} \lesssim 1 \text{ eV}$. For these values of $h_{ij}$ the decay width
for $H^{\pm\pm} \to l_i^{\pm} l_j^{\pm}$ (which is proportional to $v_\Delta^{-2}$) is much larger than the decay width for the competing decay $H^{\pm\pm} \to W^+ W^-$ (which is proportional to $v_\Delta^2$) and $H^{\pm\pm} \to H^\pm W^\pm$ (which is independent of $v_\Delta$ and potentially sizeable, but is very suppressed for $m_{H^{\pm\pm}} > m_{H^\pm}$ if the mass splitting is small, and absent if $m_{H^{\pm\pm}} < m_{H^\pm}$). The decay branching ratios for $H^{\pm\pm} \to l_i^{\pm} l_j^{\pm}$ depend on just one $h_{ij}$ coupling, while the LFV decays depend on the product of two $h_{ij}$ couplings. Detailed studies of $BR(H^{\pm\pm} \to l_i^{\pm} l_j^{\pm})$ have been performed in [14, 15, 16, 17].

We now impose the condition for an observable LFV decay on the allowed regions of $BR(H^{\pm\pm} \to l_i^{\pm} l_j^{\pm})$ in the HTM. If $H^{\pm\pm}$ is observed at the LHC before a $\tau$ or $\mu$ LFV signal, and if $BR(H^{\pm\pm} \to l^\pm l^\pm)$ is consistent with the HTM prediction, then measurements of $BR(H^{\pm\pm} \to l^\pm l^\pm)$ will determine whether a signal for LFV violation is possible in the framework of the HTM. Conversely, if a $\tau$ or $\mu$ LFV signal is observed first (which would also have an interpretation in many models without $H^{\pm\pm}$), then the possible regions for $BR(H^{\pm\pm} \to l^\pm l^\pm)$ in the HTM would be constrained.

In Fig. 8(a) the conditions $BR(\tau \to \mu \mu)/BR(\mu \to e e e) > 10^3$ and $BR(\tau \to \mu \mu)/BR(\mu \to e \gamma) > 10^2$ are imposed on the plane of $BR_{\epsilon \mu}/BR_{\mu \mu}$, where

$$BR_{ij} \equiv BR(H^{\pm\pm} \to l_i^{\pm} l_j^{\pm}) = \frac{S|m_{ij}|^2}{\sum_k m_k^2}.$$  (38)

In the figure $\sin^2 2\theta_{23} > 0.94$ [37] was used. The thin solid and thin dashed lines show the possible regions in the HTM for the normal and inverted hierarchies, respectively, and correspond to our previous result in [13]. The region above the dotted line is unphysical because the sum of BRs exceeds unity. The bold solid and bold dashed lines correspond to the boundaries of regions in which an observable signal for $\tau \to \mu \mu \mu$ is possible for the normal and inverted hierarchies respectively. The HTM predicts $BR_{\epsilon \mu} \approx 0$ if $\mu \to e e e$ is suppressed by $h_{\epsilon \mu} \approx 0$. In the inverted hierarchy it is clear that $BR_{\epsilon \mu} \approx 0$ is required for an observable signal for $\tau \to \mu \mu \mu$, i.e. $|h_{ee}| / |h_{\epsilon \mu}| \approx 0$ cannot be realized. In contrast, Fig. 8 shows that in the normal hierarchy scenario $BR_{\epsilon \mu}$ can reach $\approx 10%$ because $\mu \to e e e$ can be suppressed by $|h_{ee}| \approx 0$. However, the maximum value for $BR_{\epsilon \mu}$ (for a given $BR_{\mu \mu}$) is still considerably smaller than the maximum value for $BR_{\epsilon \mu}$ without imposing $BR(\tau \to \mu \mu \mu)/BR(\mu \to e e e) = 10^3$. The reason is because the condition $|h_{ee}| \approx 0$ constrains $m_0$ which has a large effect on $BR_{ij}$. In contrast, $BR_{ee}$ is not constrained so much (beyond the allowed region in HTM) because any value of $m_0$ is allowed if $\mu \to e e e$ is suppressed by $|h_{ee}| \approx 0$. Small $BR_{\mu \mu}$ is not preferred for $\tau \to \mu \mu \mu$ signal with $|h_{ee}| = 0$. The result for $\mu \to e \gamma$ is shown in Fig. 8(b). Inside the bold solid and bold dashed lines one can have $BR(\mu \to e \gamma)/BR(\mu \to e e e) > 0.1$ (and $BR(\mu \to e \gamma)/BR(\tau \to \mu \mu \mu) < 10^5$ in order to satisfy the bound on $\tau \to \mu \mu \mu$) for normal and inverted hierarchy respectively. After imposing the condition for an observable $BR(\mu \to e \gamma)$ one can see from the figure that the maximum value of $BR_{\epsilon \mu}$ is about 20%. Observation of $\mu \to e \gamma$ does not prefer small $BR_{\mu \mu}$ with $h_{ee} = 0$ either.

V. CONCLUSIONS

The lepton flavour violating (LFV) decays $\tau \to l_i l_j l_k$ and $\mu \to e \gamma$ were studied in the Higgs Triplet Model (HTM). The stringent constraint from non-observation of $\mu \to e e e$ was discussed in detail, and corresponds to distinct scenarios of $|h_{ee}| \approx 0$ and $|h_{\epsilon \mu}| \approx 0$. The case of $|h_{\epsilon \mu}| = 0$ can be realized at a specific ("magic") value of $\theta_{13}$ [13, 19] which is obtained as a function of the lightest neutrino mass ($m_0$) for both normal and inverted neutrino mass.
hierarchies, and for each of the four distinct cases for no CP violation from Majorana phases. The scenario of $|h_{ee}| = 0$ is only possible for normal neutrino mass hierarchy and requires magic values of $\varphi_1$ and $m_1$, which are functions of $\theta_{13}$ and $\varphi_2 - 2\delta$. Observation of $\tau \rightarrow l_i l_j l_k$ at a proposed high luminosity $B$ factory requires its branching ratio to be $10^3$ larger than that of $\mu \rightarrow \bar{e} e e$. It was shown that this can be realized for both scenarios $|h_{e\mu}| \simeq 0$ and $|h_{ee}| \simeq 0$ for a sizeable range of $m_0$, $\theta_{13}$, and phases, and thus a signal for $\tau \rightarrow l_i l_j l_k$ does not require fine-tuning to the magic values of these parameters. The pattern of branching ratios of $\tau \rightarrow l_i l_j l_k$ decays is different for these two cases. A distinctive signal of the scenario $|h_{ee}| \simeq 0$ would be the observation of $\tau \rightarrow \bar{\mu} e e$, while the observation of $\tau \rightarrow \bar{\mu} e e$ indicates the scenario $|h_{e\mu}| \simeq 0$. An observable signal for $\mu \rightarrow e \gamma$ at the ongoing MEG experiment requires a branching ratio $> 10^{-1}$ times that of $\mu \rightarrow \bar{e} e e$, a condition which can be achieved in a wide region of the parameter space of the neutrino mass matrix. Finally, it was shown that detection of any of the above LFV decays in the HTM would significantly constrain the possible branching ratios for the doubly charged Higgs boson to two leptons ($H^{\pm \pm} \rightarrow l_i^\pm l_j^\pm$). If $H^{\pm \pm}$ is observed at the Large Hadron Collider then measurements of the branching ratios of $H^{\pm \pm} \rightarrow l_i^\pm l_j^\pm$ determine whether a signal of LFV in $\tau$ and/or $\mu$ decay is possible or not.

FIG. 7: Contours of ratios of BRs for $\varphi_2 = 0$ with $|h_{ee}| = 0$. (a) $\text{BR}(\tau \rightarrow \bar{\mu} \mu \mu)/\text{BR}(\mu \rightarrow e \gamma)$; the dark (light) shaded region signifies ratio $< 10^3$ ($> 10^5$). (b) $\text{BR}(\tau \rightarrow \bar{\mu} e e)/\text{BR}(\mu \rightarrow e \gamma)$; the light shaded region signifies ratio $> 10^2$. (c) $\text{BR}(\tau \rightarrow \bar{\mu} e e)/\text{BR}(\tau \rightarrow \bar{\mu} \mu \mu)$; the light shaded region signifies ratio $> 10^{-1}$. 
FIG. 8: Region of allowed $\text{BR}(H^{\pm \pm} \rightarrow e^\pm \mu^\pm)$ and $\text{BR}(H^{\pm \pm} \rightarrow \mu^\pm \mu^\pm)$ for normal and inverted hierarchy. The range $0.94 \leq \sin^2 2\theta_{23} \leq 1$ is used: a) Bold solid and bold dashed lines show the allowed BR regions after imposing the conditions for an observable signal for $\tau \rightarrow \bar{\mu}\mu\mu$ (also depicted by the shaded region); thin solid and thin dashed lines show the possible BR regions without imposing the conditions for an observable $\tau \rightarrow \bar{\mu}\mu\mu$. b) Same as a) but for $\mu \rightarrow e\gamma$.

in the HTM.

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APPENDIX

For comparison with the results of [13] we show explicit expressions for $h_{ij}$ and $(hh^\dagger)_{ij}$ using their approximations. The following seven distinct scenarios were considered in [13]:
HI : $m_1 \ll m_2 < m_3$, $\varphi_1 = \varphi_2 = 0$
IN1 : $m_3 \ll m_1 < m_2$, $\varphi_1 = \varphi_2 = 0$
IN2 : $m_3 \ll m_1 < m_2$, $\varphi_1 = \pi$, $\varphi_2 = 0$
DG1 : $\sqrt{|\Delta m^2_{21}|} \ll m_1$, $\varphi_1 = \varphi_2 = 0$
DG2 : $\sqrt{|\Delta m^2_{21}|} \ll m_1$, $\varphi_1 = 0$, $\varphi_2 = \pi$
DG3 : $\sqrt{|\Delta m^2_{21}|} \ll m_1$, $\varphi_1 = \pi$, $\varphi_2 = 0$
DG4 : $\sqrt{|\Delta m^2_{21}|} \ll m_1$, $\varphi_1 = \varphi_2 = \pi$

Note that

\[
\sqrt{2}v_\Delta h_{ij} = [V_{MNS} \text{diag}(m_1, m_2 e^{i\varphi_1}, m_3 e^{i\varphi_2})V_{MNS}^T]_{ij},
\]

\[
2v_\Delta^2 [h h^\dagger]_{ij} = m_i^2 \delta_{ij} + [V_{MNS} \text{diag}(0, \Delta m^2_{21}, \Delta m^2_{31})V_{MNS}^\dagger]_{ij}.
\]

We find good agreement with the analytical results of [13] except for a few cases which are highlighted below. We note that $f_{ij}$ in [13] (Yukawa coupling for the Higgs triplet) is related to $h_{ij}$ by $h_{ij} = \sqrt{2} f_{ij}$. For the bound on $|(h h^\dagger)_{\mu\mu}|$ from BR($\mu \rightarrow e\gamma$) we find:

\[\text{BR}(\mu \rightarrow e\gamma) < 1.2 \times 10^{-11} \Rightarrow |(h h^\dagger)_{\mu\mu}| < 5.2 \times 10^{-5} \left(\frac{m_{\mu\pm}}{200 \text{GeV}}\right)^2\]

This result disagrees with the bound for $(f f^\dagger)_{\mu\mu}$ given in Table 1 in [13]. If we neglect the $H^\pm$ contribution to $\mu \rightarrow e\gamma$ in eq. (18) we find better agreement with the bound on $(f f^\dagger)_{\mu\mu}$ in [13]. For specific $\tau \rightarrow \ell_l \ell_j \ell_k$ decays we find:

\[\text{BR}(\tau \rightarrow \ell_l \ell_j \ell_k) < 3.1 \times 10^{-7} \Rightarrow |h_{\ell_l \ell_j \ell_k}| < 8.9 \times 10^{-4} \left(\frac{m_{\mu\pm}}{200 \text{GeV}}\right)^2\]

\[\text{BR}(\tau \rightarrow \mu\mu\mu) < 3.8 \times 10^{-7} \Rightarrow |h_{\mu\mu\mu}| < 1.4 \times 10^{-3} \left(\frac{m_{\mu\pm}}{200 \text{GeV}}\right)^2\]

The bound on $|h_{\ell_l \ell_j \ell_k}|$ agrees with the result of [13] but the bound on $|h_{\mu\mu\mu}|$ disagrees. It seems that a factor of 2 has been used for BR($\tau \rightarrow \mu\mu\mu$) in [13] instead of 1/2 due to identical particles (two $\mu$‘s) in the final state.

For the case of $h_{\ell_l \ell_j \ell_k} = 0$ we find the following ratios of BRs:

(HI) $\text{BR}(\tau \rightarrow \mu\mu\mu) : \text{BR}(\mu \rightarrow e\gamma) = 1 : 4.6 \times 10^{-2} \frac{|\Delta m^2_{21}|}{|\Delta m^2_{31}|} \sin^2 2\theta_{12}$

(IN1) $\text{BR}(\tau \rightarrow \mu\mu\mu) : \text{BR}(\mu \rightarrow e\gamma) = 1 : 0.25 : 2.9 \times 10^{-3} \left(\frac{|\Delta m^2_{21}|}{|\Delta m^2_{31}|}\right)^2 \sin^2 2\theta_{12}$

(DG1) $\text{BR}(\tau \rightarrow \mu\mu\mu) : \text{BR}(\mu \rightarrow e\gamma) = 1 : 1 : 2.9 \times 10^{-3} \left(\frac{|\Delta m^2_{21}|}{|\Delta m^2_{31}|}\right)^2 \sin^2 2\theta_{12}$

(DG2) $\text{BR}(\tau \rightarrow \mu\mu\mu) : \text{BR}(\mu \rightarrow e\gamma) = 1 : 2.9 \times 10^{-3} \left(\frac{|\Delta m^2_{21}|}{|\Delta m^2_{31}|}\right)^2 \sin^2 2\theta_{12}$

A factor $8.6 \times 10^{-3}$ for HI appears in [13] instead of our $4.6 \times 10^{-2}$. This disagreement seems to be caused by the aforementioned differences in BR($\tau \rightarrow \mu\mu\mu$) (an extra factor of 4 compared to ours) and in BR($\mu \rightarrow e\gamma$) (an extra factor of $(8/9)^2$ compared to ours). The ratio $\text{BR}(\mu \rightarrow e\gamma) / \text{BR}(\tau \rightarrow \mu\mu\mu)$ is the same in DG1 and DG2, although there is no $\mu \rightarrow e\gamma$ in DG1 in [13].

In Tables [III] and [IV] we present expressions for $h_{ij}$ and $[h h^\dagger]_{ij}$. The Dirac phase is taken to be zero ($\delta = 0$). Explicit $\delta$ dependence is not shown but enters through $s_{13}$. We find
agreement with the results of [13] except for the following cases: $h_{\mu\mu}$ and $h_{\tau\tau}$ in IN2; $h_{e\mu}$ and $h_{e\tau}$ in DG2; $[hh^\dagger]_{e\mu}$ and $[hh^\dagger]_{e\tau}$ in IN.

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TABLE III: Approximate forms of $\sqrt{2}v_{\Delta h_{ij}}$.
| | $2\nu_\Delta^2 [hh^{\dagger}]_{ee}$ | $2\nu_\Delta^2 [hh^{\dagger}]_{\mu\mu}$ | $2\nu_\Delta^2 [hh^{\dagger}]_{\tau\tau}$ | $2\nu_\Delta^2 [hh^{\dagger}]_{e\mu}$ | $2\nu_\Delta^2 [hh^{\dagger}]_{e\tau}$ | $2\nu_\Delta^2 [hh^{\dagger}]_{\mu\tau}$ |
|---|---|---|---|---|---|---|
| HI | $|\Delta m_{31}^2| \left( s_{13}^2 + \frac{\Delta m_{21}^2}{2|\Delta m_{31}^2|} s_{12}^2 \right)$ | $\frac{\Delta m_{31}^2}{2}$ | $\frac{\Delta m_{31}^2}{\sqrt{2}} \left( s_{13} + \frac{1}{2} \frac{\Delta m_{21}^2}{|\Delta m_{31}^2|} \sin 2\theta_{12} \right)$ | $\frac{\Delta m_{31}^2}{\sqrt{2}} \left( s_{13} - \frac{1}{2} \frac{\Delta m_{21}^2}{|\Delta m_{31}^2|} \sin 2\theta_{12} \right)$ | $|\Delta m_{31}^2|$ |
| IN | $|\Delta m_{31}^2|$ | $\frac{\Delta m_{31}^2}{2}$ | $-\frac{\Delta m_{31}^2}{\sqrt{2}} \left( s_{13} - \frac{1}{2} \frac{\Delta m_{21}^2}{|\Delta m_{31}^2|} \sin 2\theta_{12} \right)$ | $-\frac{\Delta m_{31}^2}{\sqrt{2}} \left( s_{13} + \frac{1}{2} \frac{\Delta m_{21}^2}{|\Delta m_{31}^2|} \sin 2\theta_{12} \right)$ | $-|\Delta m_{31}^2|$ |
| DG | $m_1^2$ | $m_1^2$ | $m_1^2$ | $\frac{\Delta m_{31}^2}{\sqrt{2}} \left( s_{13} + \frac{1}{2} \frac{\Delta m_{21}^2}{|\Delta m_{31}^2|} \sin 2\theta_{12} \right)$ | $\frac{\Delta m_{31}^2}{\sqrt{2}} \left( s_{13} - \frac{1}{2} \frac{\Delta m_{21}^2}{|\Delta m_{31}^2|} \sin 2\theta_{12} \right)$ | $\frac{\Delta m_{31}^2}{2}$ |

TABLE IV: Approximate forms of $2\nu_\Delta (hh^{\dagger})_{ij}$.
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