Abstract—Big data consists of large multidimensional datasets that would often be difficult to analyze if working with the original tensor. There is a rising interest in the use of tensor decompositions to approximate large tensors in order to reduce their dimensions by selecting important features for classification. Of particular interest is the Tucker decomposition (TD), which has already been applied in neuroscience, geoscience, signal processing, pattern and image recognition. However, the decomposition itself leads to exponential computational time for high-order tensors. To circumvent this obstacle we propose an alternative known as the matrix product state (MPS) decomposition for the data representation of big data tensors. This decomposition has been used extensively in quantum physics within the last decade and its benefit has surprisingly not been seen in other areas of research. We prove that the MPS decomposition for feature extraction and classification in supervised learning can be implemented efficiently with high classification rates in pattern and image recognition.

Keywords—Matrix product state, Tucker decomposition, high-order tensor, feature extraction, feature classification, image recognition, pattern classification, tensor-train

I. INTRODUCTION

There is an increasing need to handle large multidimensional datasets that cannot easily be analyzed or processed using modern day computers. Due to the curse of dimensionality researchers need to investigate mathematical tools which can evaluate information beyond the properties of large matrices. The essential goal is to reduce the dimensionality of big data with minimal information loss. One such method is to approximate multidimensional datasets in terms of tensor decompositions [1]. This method has been of great interest within the last decade and successfully applied in a diverse range of research areas such as data classification [2], [3], computer vision [4], quantum many-body physics [5]–[10] and signal processing [11], [12].

A popular method for tensor decomposition is the Tucker decomposition, which is an important tool for problems considering feature extraction, feature selection and classification of multilinear structures in large-scale multidimensional datasets. It has present applications in various fields such as neuroscience, pattern analysis, image classification and signal processing [2], [12], [13]. The central concept is to decompose a large multidimensional tensor into a set of common factors and a single core tensor of reduced dimension which approximately describes the features of the original tensor. The main disadvantages of using this decomposition is the computational exponential growth with the order of the tensor and the core tensor itself is still a multidimensional dataset. Hence, TD might be hampered when applied to study feature extraction and classification of big datasets represented by high-order tensors.

In this paper we solve the feature extraction and classification problem utilizing the matrix product state decomposition [8], [10] (also known as the tensor-train (TT) decomposition [14]). Note that previous literature has categorized that the MPS and TT decompositions [1] are equivalent, however the concept of MPS has already been used for decades in the field of quantum physics [15]–[17] prior to its introduction in the mathematics community. Since the concept of MPS was introduced from the point of view of quantum information theory [18], more specifically the quantum entanglement theory [3]–[8], it has been broadly applied to study problems in one-dimensional quantum many-body physics with great success.

When comparing the MPS decomposition with the TD, the MPS decomposition has some significant advantages over the TD that can be employed to study the feature extraction and classification problem of big datasets. For instance, an MPS can be employed to represent a tensor which computational complexity increases polynomially with the increasing of the number of tensor orders as compared to the exponential growth in the case of using TD. In addition, due to its natural structure, an MPS is commonly represented by a set of local tensors of low orders, i.e. the maximum order of a local tensor is usually three, it allows us to update the local tensors modified by some local transformation without affecting the others. This is again a huge advantage in terms of saving computational complexity.

To benchmark our state-of-the-art method we have studied the pattern and image classification for a few well-known big datasets, e.g. the Columbia University Image Libraries COIL-20 [19] and COIL-100 [20], [21], and the Extended Yale B dataset [22] from the Computer Vision Laboratory located at the University of California San Diego. The results can be comparable with the ones obtained from other well-known methods. To our knowledge this is the
first paper applying the MPS decomposition in computer vision, pattern recognition and data classification which rely on feature extraction, feature selection and classification. However, without any restriction, the method can be potentially applied to study other data mining problems such as compressive sensing or unsupervised learning.

The rest of the paper is structured as follows. Section III reviews mathematical foundation that will be used throughout the paper. In Section III we state the problem of pattern recognition and data classification of tensors and provide a mathematical analysis comparing both TD and MPS decomposition. In Section IV we describe the MPS algorithm in detail. In Section V we showcase the algorithm with several experiments in pattern and image classification. Lastly, Section VI concludes the paper.

II. MATHEMATICAL FOUNDATION

A tensor is a multidimensional array and its order (also known as ways or modes) is the number of dimensions it contains. Zero-order tensors are scalars and denoted by lowercase letters, e.g., a. A first-order tensor is a vector, which we denote by boldface lowercase letters, e.g., a. A matrix is a tensor of order two and is defined by boldface capital letters, e.g., A. A higher-order tensor (tensors of order three and above) are denoted by boldface calligraphic letters, e.g., \( \mathcal{X} \). Therefore a general Nth-order tensor can be defined as \( \mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \), where each \( I_i \) is the dimension of the local subspace \( i \).

We denote \( a_i \) as the \( i \)th entry of a vector \( a \) and \( a_{ij} \) as an element of a matrix \( A \). The element of a third-order tensor \( \mathcal{X} \) is denoted as \( x_{ijk} \) and thus defined similarly for a general Nth-order tensor. Indices will range from 1 to their capital version, e.g., \( i = 1, \ldots, I \) or \( \delta = 1, \ldots, \Delta \) for Greek letters. The \( n \)th element in a sequence of tensors is denoted with a superscript in parentheses, e.g., \( A^{(n)} \) is the \( n \)th matrix. The \( n \)-mode matrix product of a tensor \( \mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \) and a matrix \( A \in \mathbb{R}^{J \times I_n} \) is denoted by \( \mathcal{X} \times_n A \), and results in an \( N \)-th-order tensor of size \( I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N \). The index notation is given by

\[
(\mathcal{X} \times_n A)_{i_1 \ldots i_{n-1}j_in_{n+1} \ldots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 \ldots i_{n-1}j_in_{n+1} \ldots i_N} a_{ji_n},
\]  

The multiplication in all possible modes \( (n = 1, \ldots, N) \) of the tensor \( \mathcal{X} \) with a set of matrices \( A^{(n)} \) is denoted as

\[
\mathcal{X} \times \{A\} = \mathcal{X} \times_1 A^{(1)} \times_2 A^{(2)} \cdots \times_N A^{(N)}.
\]

The matricization, also known as unfolding or flattening, is the procedure to transform a tensor into a matrix. The mode-\( n \) matricization of a tensor \( \mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \) is denoted by \( X^{(n)} \in \mathbb{R}^{I_n \times (I_1 \cdots I_{n-1} I_{n+1} \cdots I_N)} \), which arranges the mode-\( n \) fibers to be the columns of the resulting matrix. The definition of reshaping a tensor \( \mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \) is changing from the \( N \)-th-order to another order as long as the exact number of parameters is kept, e.g. the matricization of \( \mathcal{X} \) to \( X_{i_1 \ldots i_n} \) is said to be reshaping \( \mathcal{X} \) to a \( (i_1 \cdots i_m) \times (i_{m+1} \cdots i_N) \) matrix.

Graphical notation has been used extensively [10] in order to communicate more efficiently the ideas of tensor decompositions. In a tensor network diagram (TND), tensors are represented as shapes and their corresponding indices are lines protruding from these shapes as shown in Fig 1. A tensor contraction or Einstein summation can be performed by connecting common indices in the diagram. For example the \( n \)-mode matrix product defined in Eq. (1) can be easily converted to a TND shown in Fig 2. For the remainder of this paper we will use TND’s to support explanations.

III. TENSOR DECOMPOSITIONS FOR MULTILINEAR CLASSIFICATION

A. Problem formulation

In this section we present the general problem of feature extraction and classification [2] for a set of \( K \) tensors and derive a new concept based on MPS to address the problem.

Problem statement: Given a set of \( K \) training samples represented by \( K \)-th order tensors \( \mathcal{X}^{(k)} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \) \( (k = 1, \ldots, K) \) corresponding to \( Q \) categories, and a set of \( T \) test data \( \mathcal{X}^{(t)} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \) \( (t = 1, \ldots, T) \), classify the test data into the categories \( Q \) with high accuracy.

This can be achieved by the following steps:

- **Step 1**: Find the set of common factors and corresponding core feature tensor from the training data \( \mathcal{X}^{(k)} \).
- **Step 2**: Perform feature extraction on the test data \( \mathcal{X}^{(t)} \) using the basis factors found from the training data.
- **Step 3**: Classify the test data.

The classification problem is a supervised problem where the categories \( Q \) are defined according to the problem
and the data provided. Our main result utilizes the MPS decomposition for the training data in Step 1 to obtain the common factors and core tensor. The features from the training and test data can be used with any classification method such as Support Vector Machine (SVM) or K-Nearest Neighbours (KNN). The following subsections describe how to mathematically model the problem in TD and MPS decomposition.

B. Tucker decomposition for feature extraction and classification

Given an $N$th-order tensor $X \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, the Tucker decomposition is defined as

$$X = \mathcal{G} \times_1 \mathbf{A}^{(1)} \times_2 \mathbf{A}^{(2)} \cdots \times_N \mathbf{A}^{(N)} + \mathcal{E}$$

(3)

where $\mathcal{G} \in \mathbb{R}^{\Delta_1 \times \Delta_2 \times \cdots \times \Delta_N}$ is known as the core tensor and each $I_n \times \Delta_n$ matrix $\mathbf{A}^{(n)}$ is known as a factor matrix, $\mathcal{E}$ denotes the approximation error. For simplicity if we assume $I_1 = I_2 = \cdots = I_N = I$ and $\Delta_1 = \Delta_2 = \cdots = \Delta_N = \Delta$, then the TD representation of $X$ consists of $O(NI\Delta^2)$ parameters, which is exponential in $N$ and thus is only suitable for small $N$. In feature extraction, for sets of $K$ $N$th-order tensors $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, the TD can be considered as

$$\mathbf{X}^{(k)} = \mathcal{G}^{(k)} \times_1 \mathbf{A}^{(1)} \times_2 \mathbf{A}^{(2)} \cdots \times_N \mathbf{A}^{(N)} + \mathcal{E}^{(k)}$$

(4)

where $\{\mathbf{A}^{(i)}\}_{i=1}^N$ is the common set of factor matrices obtained for each core tensor $\mathbf{X}^{(k)}$, for $k = 1, \ldots, K$. However, one can choose a more effective way to represent the TD for the set of all $K$ $N$th-order tensors in a single equation by concatenating all tensors $\mathbf{X}^{(k)}$ such that

$$\mathbf{Y} = [\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(K)}] \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N \times K}$$

(5)

then its TD is

$$\mathbf{Y} = \mathcal{G} \times_1 \mathbf{A}^{(1)} \times_2 \mathbf{A}^{(2)} \cdots \times_N \mathbf{A}^{(N)} + \mathcal{E},$$

(6)

where $\mathbf{Y}, \mathcal{E} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N \times K}$ and $\mathcal{G} \in \mathbb{R}^{\Delta_1 \times \Delta_2 \times \cdots \times \Delta_N \times K}$ is an $(N+1)$th-order tensor, see Fig. 3 for a TND of the core tensor and factor matrices. As we can see, the number of parameters in the core tensor increases exponentially with the number of orders $N$ in the tensor $\mathbf{Y}$. Accordingly, this representation can make simulation intractable by increasing $N$. Alternatively, in what follows we apply a different decomposition to represent the tensor $\mathbf{Y}$ in terms of a MPS decomposition such that the number of parameters only depends polynomially in $N$, and more importantly the number of parameters in the core tensor does not directly depend on $N$.

C. MPS decomposition for feature extraction and classification

The MPS decomposition of the tensor $\mathbf{Y}$ in Eq. (5) is defined as

$$y_{i_1 \cdots k \cdots i_N} = a_{i_1}^{(1)} \cdots a_{i_{n-1}}^{(n-1)} a_k^{(n)} a_{i_{n+1}}^{(n+1)} \cdots a_{i_M}^{(M)}$$

(7)

where the $k$th index is positioned at $n$ and $M = N + 1$. For each $k$ and $i_j$ ($j \neq 1, N$), the corresponding matrix $\mathbf{a}^{(k)}$ has the size $\Delta_{j-1} \times \Delta_j$. For each $i_1, i_M$, $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(M)}$ correspond to row and column vectors of size $\Delta_1$ and $\Delta_{M-1}$, respectively. See Fig. 4 for a TND of Eq. (7).

Given the assumptions of index sizes stated previously in Section III-B, the MPS consists of $O((M-2)I\Delta^2 + 2I\Delta)$ parameters, which is polynomial in $M$. Also, if we choose $\Delta = I^{M/2}$, the MPS can exactly represent the tensor $\mathbf{Y}$ without any approximation. However, this will result into a large computational complexity. In practice, when dealing with high dimensional tensors, the MPS decomposition is employed as an approximation representation of the tensor such that $\Delta$ can be chosen to be much smaller than $I^{M/2}$.

In the above MPS representation one can easily represent it in such a way that each link between the tensors can be described by an orthonormal base, this is known as a canonical MPS decomposition. For instance, an MPS is in a left-canonical form if it satisfies the following conditions

$$\sum_{i_n} (\mathbf{a}^{(i_n)})^\dagger \mathbf{a}^{(i_n)} = \mathbf{I}, \quad (n = 2, \ldots, M - 1)$$

(8)

$$\sum_{i_n} (\mathbf{a}^{(i_n)})^\dagger \mathbf{a}^{(i_n)} = \mathbf{I}, \quad (n = 1)$$

(9)

and right-canonical form if

$$\sum_{i_n} \mathbf{C}^{(i_n)} (\mathbf{C}^{(i_n)})^\dagger = \mathbf{I}, \quad (n = 2, \ldots, M - 1)$$

(10)

$$\sum_{i_n} \mathbf{C}^{(i_n)} (\mathbf{C}^{(i_n)})^\dagger = \mathbf{I}, \quad (n = M)$$

(11)
core tensor and then proceed to classify the test data based on the test core tensor.

**A. Mixed-canonical form of the MPS**

Suppose that the training data is represented by an \((N+1)\)th-order tensor \(\mathbf{Y}\) defined in Eq. (5). The algorithm to obtain the mixed-canonical MPS decomposition of \(\mathbf{Y}\) is divided into two sweeps (left-to-right and right-to-left) of an iterative SVD algorithm. These sweeps are needed to find each of the left-canonical and right-canonical common factors of the MPS decomposition. Each sweep is completed when the next index to be evaluated is \(k\), which will be the core tensor.

Assume that the \(k\) index is positioned at \(n\). The following two steps are performed consecutively to obtain the mixed-canonical form:

1) **Left-to-right sweep:** The left-to-right sweep involves acquiring common factors \(1, \ldots, n - 1\) in left-canonical form. First perform a mode-\(i_1\) matricization on \(\mathbf{Y}\) to obtain the matrix \(\mathbf{Z} = \mathbf{Y}_{(1)}\) of size \(I_1 \times (I_2 \times \cdots \times I_M)\). In terms of elements, \(y_{i_1 \cdots k_1M} = z_{i_1 \cdots i_2 \cdots k_1 \cdots k_2 \cdots k_M}\), then the SVD of \(\mathbf{Z}\) gives

\[
y_{i_1 \cdots k_1M} = z_{i_1 \cdots i_2 \cdots k_1 \cdots k_2 \cdots k_M} = \sum_{\delta_1=1}^{\Delta_1} a_{i_1 \delta_1}^1 \delta_1 \delta_1^* (i_2 \cdots k_1 \cdots k_2 \cdots k_M) = \sum_{\delta_1=1}^{\Delta_1} a_{i_1 \delta_1}^1 \delta_1 \delta_1^* (i_2 \cdots k_1 \cdots k_2 \cdots k_M).
\]

(12)

In Eq. (12), \(\mathbf{T} = \mathbf{DV}^\dagger\) and since \(\mathbf{U}\) is left-orthogonal and contains the index \(i_1\), we let \(a^{(1)} = \mathbf{U}\) be the first common factor. Next reshape \(\mathbf{T}\) to a \((\Delta_1 I_2) \times (I_3 \times \cdots \times I_M)\) matrix, then the SVD of \(\mathbf{T}\) gives,

\[
y_{i_1 \cdots k_1M} = \delta_1 \delta_1^* (i_2 \cdots k_1 \cdots k_2 \cdots k_M) = \sum_{\delta_1=1}^{\Delta_1,\Delta_2} a_{i_1 \delta_1}^1 \delta_1 \delta_1^* (i_2 \cdots k_1 \cdots k_2 \cdots k_M) = \sum_{\delta_1=1}^{\Delta_1,\Delta_2} a_{i_1 \delta_1}^1 \delta_1 \delta_1^* (i_2 \cdots k_1 \cdots k_2 \cdots k_M).
\]

(13)

Similar to Eq. (12), in Eq. (13) we had \(\mathbf{T} = \mathbf{DV}^\dagger\) and since \(\mathbf{U}\) is left-orthogonal, we obtain common factor \(a^{(2)}\) of dimension \(\Delta_1 \times I_2 \times \Delta_2\) from reshaping \(\mathbf{U}\) to a third-order tensor. The process above is repeated until we obtain all common factors up to \(A^{(n-1)}\) and the last \(\mathbf{T}\) matrix has dimension \(\Delta_{n-1} \times (K \times I_M)\).

2) **Right-to-left sweep:** The right-to-left sweep involves acquiring common factors \(n+1, \ldots, M\) in right-canonical form. The last \(\mathbf{T}\) matrix from the left-to-right sweep is reshaped to \((\Delta_{n-1} K \times I_M) \times I_M\) and its SVD leads to a right-orthogonal common factor \(c^{(M)} = \mathbf{V}^\dagger\). Consequently we acquire common factors \(n+1, \ldots, M-1\) by performing successive SVD’s and at each instance reshaping the right
orthogonal matrix $\mathbf{V}^\dagger$ to a third-order tensor. Finally, reshape the last $\mathbf{T}$ matrix of dimension $(\Delta_{n-1}K) \times \Delta_{n}$ to a third order tensor to obtain the core tensor $\mathcal{G}$ as shown in the following equations,

$$y_{l_1,\ldots,l_M} = \sum_{\delta_{n-1},\delta_{n}=1}^{\Delta_{n-1},\Delta_{n}} \mathcal{L}(\delta_{n-1}k)\delta_{n} \mathcal{R}$$

with

$$\mathcal{L} = \sum_{\delta_{1},\ldots,\delta_{n-2}=1}^{\Delta_{1},\ldots,\Delta_{n-2}} a_{(1)}^{(1)} \cdots a_{(n-2,i_{n-2},\ldots,\delta_{n-1})}^{(n-1)}$$

and

$$\mathcal{R} = \sum_{\delta_{n+1},\ldots,\delta_{M-1}=1}^{\Delta_{n+1},\ldots,\Delta_{M-1}} a_{(n+1)}^{(n+1)} \cdots a_{(M-1,i_{n+1},\ldots,\delta_{M-1})}^{(M)}$$

Eq. (15) is the mixed-canonical form of $\mathbf{Y}$ and an example of the two-step process for a $(6 + 1)$th-order tensor can be seen in Fig. 7. Hence we have completed Step 1 of the feature extraction and classification problem.

Note that the position of the core tensor $n$ can be varied. Intuitively it is better to position the core tensor between two indices where it can achieve the maximum bond dimension so as to increase the number of features available to the core tensor.

An ALS method can be used to variationally optimize each of the common factors. After the ALS procedure is completed, to ensure we can extract the training core tensor with minimal error it is important to be certain that the MPS decomposition is in mixed-canonical form. This can be achieved by using an iterative sweep on the MPS as explained in detail by Schollwöck [9].

B. Feature extraction of test data

To solve Step 2 we can directly apply the common factors obtained from the mixed-canonical MPS decomposition to extract the core tensor for the test data tensor. However, for more optimized results you can use the common factors obtained after the ALS method.

Consider the concatenation of test data $\mathbf{X}^{(t)}$, then we can also represent this as a tensor

$$\mathcal{W} = [\mathbf{X}^{(1)} \mathbf{X}^{(2)} \cdots \mathbf{X}^{(T)}] \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N \times T}.$$ (18)

Subsequently the test core tensor can be extracted from this concatenated tensor by projecting common factors. The resultant test core tensor will also be third-order and consist of the same dimension of features as the training core tensor. An example for the extraction of the test core tensor from a $(6 + 1)$th-order test data tensor can be seen in the TND in Fig. 8.

V. EXPERIMENTS

The algorithm was tested with three datasets: The Columbia University Image Libraries COIL-20 [19] and COIL-100 [20], [21], and the Extended Yale B dataset [22]. Fig 9 displays the images of all classes in each dataset.

A. Experimental conditions

1) COIL-20: The database consists of 1440 grayscale images of 20 objects (72 images per object) with different reflectance and complex geometric characteristics. Each object was rotated 360 degrees as 72 images were taken per object; one every 5 degrees of rotation. All images used were initially $128 \times 128$ pixels and then downsampled to $32 \times 32$ grayscale (0-255).

Images were randomly divided into two partitions according to a hold/out ratio. For our the experiment we tested with hold/out ratios of 10%, 30%, 50%, 70%, 90% and 95%, e.g. a 10% hold/out ratio accounts to 10% test data and 90% training data. The training and test data were structured as third-order tensors of dimensions $K \times 32 \times 32$ and $T \times 32 \times 32$. 
The test core tensor extracted was of size $32 \times T \times 32$. This experiment was conducted to initially test the performance of the MPS algorithm for training data with trivial matrix form. The results were averaged over 10 trials.

2) **COIL-100**: This database has 7200 color images of 100 objects (72 images per object) with different reflectance and complex geometric characteristics. Similar to COIL-20 each object was rotated 360 degrees as 72 images were taken per object; one every 5 degrees of rotation. The hold/out ratios for training and test data were also tested with 10%, 30%, 50%, 75%, 90% and 95%. Images were originally $128 \times 128 \times 3$ pixels and downsampled to $32 \times 32 \times 3$ pixels. The training and test data were constructed as fourth-order tensors of dimensions $K \times 32 \times 32 \times 3$ and $T \times 32 \times 32 \times 3$, respectively. The test core tensor extracted was of size $32 \times T \times 32$. The results were averaged over 10 trials.

3) **Extended Yale Face database B**: The database contains 16128 grayscale images with 28 human subjects under 9 poses, where for each pose there is 64 illumination conditions. Similar to [5], to improve computational time each image was cropped to keep only the center area containing the face, then resized to $73 \times 55$. Training data and test data was not selected randomly but partitioned according to poses. For training and test data we selected poses 0, 2, 4, 6 and 8 and 1, 3, 5, and 7, respectively. For a single subject the training tensor was size $5 \times 73 \times 55 \times 64$ and the test tensor was size $4 \times 73 \times 55 \times 64$. Hence for all 28 subjects we had a training fourth-order tensors $140 \times 73 \times 55 \times 64$ and $112 \times 73 \times 55 \times 64$ for training and test data, respectively. The test core tensor extracted was of size $55 \times T \times 55$.

## B. Results

1) **COIL-20**: The classification algorithm used was KNN with correlational distance. The classification accuracy was plotted versus the bond dimension and the number of features used is the square of the bond dimension. Six plots are compared for hold/out ratios in Fig. 11 and the highest classification accuracy came from a 10% hold/out ratio as expected, with a 100% classification accuracy in several bond dimensions. With 30% hold/out ratio the highest accuracy was 99.79%. At 50% hold/out ratio the highest accuracy is 99.35%. The remaining hold/out ratios 70%, 90% and 95% had maximum classification accuracies of 97.64%, 89.95% and 81.18%, respectively.

2) **COIL-100**: Similar to COIL-20 the classification algorithm used was KNN with correlational distance. Six plots are compared for hold/out ratios in Fig. 10 and the highest classification accuracy came from a 10% hold/out ratio with a 99.85% classification accuracy. With 30% hold/out ratio the highest accuracy was 99.74%. At 50% hold/out ratio the highest accuracy is 99.22%. The remaining hold/out ratios 75%, 90% and 95% had maximum classification accuracies of 96.65%, 89.14% and 80.59%, respectively.

3) **Extended Yale Face database B**: Four classification methods were used to compare performance: SVM one-against-one (1v1), SVM one-against-all (1vall), KNN 1 and KNN 2 algorithms. Classification accuracy was plotted versus bond dimension as shown in Fig. [12] Specifically, the 1v1 SVM algorithm obtains the highest accuracy with 93.75% compared to the 1vall SVM with a maximum accuracy of 88.39%. Using KNN, we see that KNN 1 and KNN 2 have classification accuracy rates of 91.07% and 89.29%, respectively.
C. Analysis

The bond dimension has a direct affect on the classification of the test data. In all experiments a small bond dimension was only needed to achieve high classification accuracies, however the accuracy does not necessarily increase at higher bond dimensions. This is because a larger number of features does not necessarily correlate to increased classification performance.

In the COIL-20 and COIL-100 results it was expected that larger hold/out ratios decreases the classification performance and this can be seen by the decreased accuracies at a hold/out ratio of 95%. For COIL-100 our algorithm outperformed several methods [23–25] at the 75% hold/out ratio. This is also using the full color image as opposed to the converted grayscale image for these experiments.

The classification performance for the Extended Yale Face database B (EYFB) is much more stable, has lower computational complexity, and has higher classification accuracies from low to high bond dimensions compared to Direct General Tensor Discriminant Analysis (DGTDA) and Constrained Multilinear Discriminant Analysis (CMDA) proposed recently by Li & Schonfeld [3] as well as other methods mentioned in their paper.

It is imperative to highlight that the experiments conducted are not meant to outperform all current techniques in image and pattern analysis but to show that the MPS decomposition can be used to represent a generic multidimensional dataset using low-order tensors with polynomial computational complexity.

VI. Conclusion

The classification performance has shown that the MPS decomposition can be used as an efficient and simple mechanism in representing multidimensional datasets for feature extraction and classification with supervised learning. Furthermore the core tensor required to sufficiently classify data was of a greatly reduced order and dimension, showing that only a small number of features is necessary to classify multidimensional datasets in pattern and image analysis. From these promising results we hope to pursue multiple directions that can have a major impact in many fields of research. One direction is the investigation of state-of-the-art techniques to increase classification accuracies. Other directions can be the reformulation of research problems in signal processing, compressive sensing or remote sensing in terms of the MPS decomposition.

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