EMBEDDINGS FOR INFINITE-DIMENSIONAL INTEGRATION AND $L_2$-APPROXIMATION WITH INCREASING SMOOTHNESS

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Abstract. We study integration and $L_2$-approximation on countable tensor products of function spaces of increasing smoothness. We obtain upper and lower bounds for the minimal errors, which are sharp in many cases including, e.g., Korobov, Walsh, Haar, and Sobolev spaces. For the proofs we derive embedding theorems between spaces of increasing smoothness and appropriate weighted function spaces of fixed smoothness.

1. Introduction

We study integration and $L_2$-approximation for functions of infinitely many variables. The complexity of computational problems of this kind has first been analyzed in [15, 19, 25]; for further contributions we refer to, e.g., [4, 8, 10, 11, 20, 24, 30, 46, 49]. First of all, this line of research may be viewed as the limit of tractability analysis of multivariate problems, where the number of variables tends to infinity. Furthermore, computational problems with infinitely many variables naturally arise in a number of different applications. One example are stochastic differential equations, since the driving processes, often a finite- or infinite-dimensional Brownian motion, is canonically represented in terms of a sequence of independent and identically distributed random variables. Another example are partial differential equations with random coefficients, where similar representations are employed for the underlying random fields.

Roughly speaking, problems with a large or infinite number of variables are computationally tractable if the variables may be arranged in such a way that their impact decays sufficiently fast.

The first, and still most popular approach to capture this phenomenon are weighted function spaces, where the weights directly moderate the influence of groups of variables. We refer to [39] as the pioneering paper and, e.g., to [4, 52, 32] for further results and references in the multivariate case. For problems with infinitely many variables weighted function spaces have first been studied in [19], and the structure of the corresponding spaces is analyzed in [13]. See, e.g., [11] for recent results and references on infinite-dimensional integration.

As an alternative concept, an increasing smoothness with respect to the properly ordered variables has first been studied in tractability analysis in [35], and further results in this setting have been derived in, e.g., [7, 14, 22, 26, 38]. We add that this kind of smoothness phenomenon is present for most of the partial differential equations with random coefficients that have been studied in the literature from...
a computational point of view, see [7,14] for further information. Moreover, increasing smoothness is a particular instance of anisotropic smoothness, as studied in approximation theory, see, e.g., [8, Sec. 10.1] for further information.

The function spaces under consideration in the present paper are tensor products

\[ H := \bigotimes_{j \in \mathbb{N}} H_j \]

for scales of Hilbert spaces \( H_j \) of functions of a single variable, defined on any domain \( D \). Accordingly, the elements of \( H \) are functions on the domain \( E := D^\mathbb{N} \).

For integration and \( L_2 \)-approximation the underlying probability measure \( \mu \) on \( E \) is the countable product of an arbitrary probability measure \( \mu_0 \) on \( D \).

Originally, we are interested in the case of spaces \( H_j \) of increasing smoothness in the sense that

\[ H_1 \supset H_2 \supset \ldots \]

with compact embeddings. The main aim of this paper is to show that this setting may be reduced to tensor products of suitable weighted function spaces \( H_j \) via embeddings. Reductions of this type lead to sharp upper and lower bounds for minimal errors for integration and \( L_2 \)-approximation, despite the fact that the weighted spaces \( H_j \) are isomorphic as Banach spaces, while we have compact embeddings in the case of increasing smoothness.

The embeddings between the two kinds of rather different tensor product spaces allow to derive new results for tensor products of spaces of increasing smoothness from known results for tensor products of weighted spaces that have a fixed smoothness. We carry out this program for Korobov spaces, Walsh spaces, Haar spaces, and Sobolev spaces of functions with derivatives in weighted \( L_2 \)-spaces.

The embedding approach, which has first been developed in [15], has meanwhile been applied to a number of different settings also beyond the Hilbert space and the tensor product case, see [11,12,16,20,21,24,25]. Embeddings between spaces of increasing smoothness and weighted function spaces have first been observed and exploited in [30].

For integration we wish to approximate \( \int_E f \, d\mu \) for \( f \in H \), and for \( L_2 \)-approximation we wish to recover \( f \in H \) with error measured in \( L_2(E,\mu) \). We are primarily interested in algorithms that use standard information, i.e., algorithms that may only use a finite number of function values of any \( f \), which requires \( H \) to be a reproducing kernel Hilbert space.

Since the functions \( f \in H \) depend on infinitely-many variables, it is unreasonable to assume that they may be evaluated at any point \( y \in E \) at unit cost. Instead we employ the so-called unrestricted subspace sampling model, which has been introduced in [28]. For a fixed nominal value \( a \in D \) function values are only available at points \( y = (y_j)_{j \in \mathbb{N}} \in E \) with

\[ \text{Act}(y) := \# \{ j \in \mathbb{N} : y_j \neq a \} < \infty, \]

and \( \text{Act}(y) \) (or a function thereof) is the cost of function evaluation at such an admissible point \( y \). Accordingly, the cost of a linear deterministic algorithm

\[ A(f) = \sum_{i=1}^m f(y_i) \cdot z_i \]

with admissible points \( y_i \in E \) and with scalars \( z_i \) for integration and \( z_i \in L_2(E,\mu) \) for \( L_2 \)-approximation is given by \( \text{cost}(A) := \sum_{i=1}^m \text{Act}(y_i) \).
The key quantities in the worst case analysis on the unit ball \( B(H) \subset H \) are the \( n \)-th minimal errors
\[
\text{err}_n(H, \text{Int}, A_{std}) := \inf_{\text{cost}(A) \leq n} \sup_{f \in B(H)} \left| \int_E f \, d\mu - A(f) \right|
\]
for integration and
\[
\text{err}_n(H, \text{App}, A_{std}) := \inf_{\text{cost}(A) \leq n} \sup_{f \in B(H)} \| f - A(f) \|_{L^2(E, \mu)}
\]
for \( L^2 \)-approximation.

Let us describe the function space setting in more detail. We focus on scales of function spaces \( H_j \) with the following structure, later on called the standard setting, which is based on an orthonormal basis \( (e_{\nu})_{\nu \in \mathbb{N}_0} \) of \( H_0 := L^2(D, \mu_0) \) with \( e_0 = 1 \) and on a family \( (\alpha_{\nu,j})_{\nu,j \in \mathbb{N}} \) of positive Fourier weights. With \( \langle \cdot, \cdot \rangle_0 \) denoting the scalar product on \( H_0 \), we define \( H_j \) to be the Hilbert space of all \( f \in H_0 \) such that
\[
\| f \|_j^2 := |\langle f, e_0 \rangle_0|^2 + \sum_{\nu \in \mathbb{N}} \alpha_{\nu,j} \cdot |\langle f, e_{\nu} \rangle_0|^2 < \infty.
\]

Typically, the asymptotic properties of the Fourier weights ensure that \( (H_j)_{j \in \mathbb{N}_0} \) is a scale of spaces of increasing smoothness. In any case, \( H \subset L^2(E, \mu) \) by assumption.

To give a flavor of our results, let us consider the uniform distribution \( \mu_0 \) on \( D := [0, 1] \) and the trigonometric basis given by \( e_{\nu}(x) := \exp(2\pi i(-1)^{\nu}[\nu/2]x) \), together with the Fourier weights
\[
a_{\nu,j} := (1 + |(\nu + 1)/2|)^{r_j},
\]
where
\[
0 < r_j < r_{j+1}
\]
for all \( j \in \mathbb{N} \). In this case the space \( H_j \) is the Korobov space with smoothness parameter \( r_j \). As a well-known fact, \( H_j \) is a reproducing kernel Hilbert space if and only if \( r_j > 1 \), and for an even integer \( r_j \geq 2 \) the elements of \( H_j \) have a weak derivative of order \( r_j/2 \) in \( L^2(D, \mu_0) \). Given \( r_1 > 1 \),
\[
\rho := \liminf_{j \to \infty} \frac{r_j}{\ln(j)} > \frac{1}{\ln(2)}
\]
is a sufficient condition for \( H \) to be a reproducing kernel Hilbert space of functions on the domain \( [0, 1]^N \). A necessary condition also permits \( \rho = 1/\ln(2) \). See Example 3.1 and 3.6. We determine the decay of the \( n \)-th minimal error for \( S = \text{Int} \) and \( S = \text{App} \) in Corollary 4.7. It turns out that this decay is equal to
\[
dec = \frac{1}{2} \cdot \min(r_1, \rho \cdot \ln(2) - 1)
\]
for both problems, i.e., for every \( \varepsilon > 0 \) there exists a constant \( c > 0 \) such that
\[
\text{err}_n(H, S, A_{std}) \leq c \cdot n^{-(\text{dec} - \varepsilon)}
\]
for all \( n \in \mathbb{N} \), and \( \text{dec} \) is minimal with this property. We observe, in particular, that the minimal smoothness \( r_1 \) with respect to a single variable and the increase of the smoothness along the variables, as quantified by \( \rho \), are the crucial parameters: together they determine whether \( H \) is a reproducing kernel Hilbert space as well as the asymptotic behavior of the \( n \)-th minimal errors.

Let us provide some details of our proof strategy, which applies to the standard setting in general, see Section 3.5 for the embeddings and Section 4.2 for the results.
on integration and approximation. The reproducing kernel \( K \) of the Hilbert space \( H = H(K) \) is the tensor product

\[
K := \bigotimes_{j \in \mathbb{N}} k_j
\]

of the reproducing kernels \( k_j \) of the spaces \( H_j = H(k_j) \), see Section 2.2. For the proof of the upper bound (1), we determine a sequence of weights \( \theta_j > 0 \), as small as possible, and show the existence of a reproducing kernel \( m \) for functions of a single variable with the following properties, see Theorem 3.19. The space \( H(K) \) is continuously embedded into the Hilbert space \( H(M) \) with reproducing kernel

\[
M := \bigotimes_{j \in \mathbb{N}} (1 + \theta_j \cdot m),
\]

and \( H(1 + m) = H(k_1) \) as vector spaces. Furthermore, \( m \) is anchored at a given point \( a \in D \), i.e., \( m(a, a) = 0 \). It follows that, \( \text{err}_n(H(K), S, A^{\text{std}}) \) is at most of the order of \( \text{err}_n(H(M), S, A^{\text{std}}) \). In this way we relate the tensor product space \( H(K) \) of spaces of increasing smoothness to the tensor product space \( H(M) \), which is based on weighted anchored kernels. A reverse embedding with a two-dimensional space \( H(1 + \ell) \subset H(k_1) \) is part of the proof that dec is maximal with the property (1).

Integration and \( L^2 \)-approximation is thoroughly studied in the literature for tensor products of weighted anchored spaces, where the multivariate decomposition method has been established as a powerful generic algorithm, see, e.g., [9]. In particular, it is known in this setting how the asymptotic behavior of the minimal errors depends on summability properties of the sequence \( (\theta_j)_{j \in \mathbb{N}} \) of weights and on the minimal errors for the univariate problem, see [10, 30, 47]. Interestingly, we obtain sharp results via embeddings in this way, although \( H(1 + \theta_j \cdot m) = H(k_1) \) as vector spaces for every \( j \in \mathbb{N} \), so that we embed \( H(K) \) into the much larger space \( H(M) \) as we trade increasing smoothness for decaying weights.

This paper is organized as follows. In Section 2 we determine when a Hilbert space may be canonically identified with a reproducing kernel Hilbert space; here subspaces of \( L^2 \)-spaces and countable tensor products are particularly relevant for the present paper. In Sections 3.1 and 3.2 we present the function space framework to introduce and study Hilbert spaces of increasing smoothness. Classical examples are given by Korobov spaces, Walsh spaces, Haar spaces, and Sobolev spaces with derivatives in weighted \( L^2 \)-spaces, see Section 3.3. In Sections 3.4 and 3.5 we construct the appropriate tensor products of weighted (anchored) spaces and provide embedding theorems between these spaces and tensor products of spaces of increasing smoothness. The embeddings are applied in Section 4 to determine the decay of the minimal errors for integration and \( L^2 \)-approximation. For the latter problem we actually compare two classes of algorithms that may either use standard information, as outlined above, or, potentially more powerful, use arbitrary bounded linear functionals at cost one. In Appendix A we recall basic properties of countable tensor products of Hilbert spaces, and Appendix B contains some facts on summability and decay of sequences of real numbers. In Appendix C we consider \( L^2 \)-approximation in Haar spaces of functions of a single variable.
2. Tensor Products and Reproducing Kernels

2.1. Reproducing Kernels. Consider a separable Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) over \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}\) with an orthonormal basis \((h_\nu)_{\nu \in N}\) for some countable set \(N\). Moreover, let \(E \neq \emptyset\) be any set. For any injective linear mapping \(\Phi : \mathcal{H} \to \mathbb{K}^E\) we define a scalar product \(\langle \cdot, \cdot \rangle_\Phi\) on \(\Phi(\mathcal{H})\) by

\[
\langle \Phi f, \Phi g \rangle_\Phi := \langle f, g \rangle
\]

for all \(f, g \in \mathcal{H}\). In this way we may identify the (abstract) Hilbert space \(\mathcal{H}\) with the Hilbert space \(\Phi(\mathcal{H})\) of real- or complex-valued functions on the domain \(E\).

The following two lemmata provide a necessary and a sufficient condition for the function space \(\Phi(\mathcal{H})\) to be a reproducing kernel Hilbert space.

Lemma 2.1. Suppose that \(\Phi : \mathcal{H} \to \mathbb{K}^E\) is linear and injective. Moreover, assume that \((\Phi(\mathcal{H}), \langle \cdot, \cdot \rangle_\Phi)\) is a reproducing kernel Hilbert space. Then we have

\[
\forall y \in E : \quad \sum_{\nu \in N} |\Phi h_\nu(y)|^2 < \infty
\]

and

\[
\forall y \in E \forall f \in \mathcal{H} : \quad \Phi f(y) = \sum_{\nu \in N} \langle f, h_\nu \rangle \cdot \Phi h_\nu(y)
\]

with absolute convergence. Furthermore, the reproducing kernel \(K\) of this space is given by

\[
K(x, y) = \sum_{\nu \in N} \Phi h_\nu(x) \cdot \overline{\Phi h_\nu(y)}
\]

with absolute convergence for all \(x, y \in E\).

Proof. For every \(f \in \mathcal{H}\) we have

\[
\Phi f = \sum_{\nu \in N} \langle \Phi f, \Phi h_\nu \rangle_\Phi \cdot \Phi h_\nu = \sum_{\nu \in N} \langle f, h_\nu \rangle \cdot \Phi h_\nu
\]

with convergence in \(\Phi(\mathcal{H})\). By assumption, point evaluations are continuous on the latter space, which yields \((3)\). In particular, for \(\Phi f = K(\cdot, y)\) with \(y \in E\) we obtain

\[
K(\cdot, y) = \sum_{\nu \in N} \langle K(\cdot, y), \Phi h_\nu \rangle_\Phi \cdot \Phi h_\nu = \sum_{\nu \in N} \overline{\Phi h_\nu(y)} \cdot \Phi h_\nu,
\]

which yields \((4)\). Choose \(x := y\) to derive \((2)\) from \((4)\). The Cauchy-Schwarz inequality and \((2)\) guarantee the absolute convergence in \((3)\) and \((4)\). \(\square\)

Every mapping \(\Phi\) that leads to a reproducing kernel Hilbert space \(\Phi(\mathcal{H})\) is already determined by the values \(\Phi h_\nu\) for \(\nu \in N\), see \((3)\). In the construction of such a mapping we therefore start with an injective mapping \(\Phi : \{h_\nu : \nu \in N\} \to \mathbb{K}^E\), and we assume that \((2)\) is satisfied. The mapping \(\Phi\) is extended to a linear mapping \(\Phi : \mathcal{H} \to \mathbb{K}^E\) by

\[
\Phi f(y) := \sum_{\nu \in N} \langle f, h_\nu \rangle \cdot \Phi h_\nu(y).
\]
Assumption (2) yields the absolute convergence of the right-hand side in (5) for all \( f \in \mathcal{H} \) and \( y \in E \). Actually we have

\[
\sum_{\nu \in \mathbb{N}} |\langle f, h_\nu \rangle \cdot \Phi h_\nu(y)| \leq \left( \sum_{\nu \in \mathbb{N}} |\langle f, h_\nu \rangle|^2 \right)^{1/2} \cdot \left( \sum_{\nu \in \mathbb{N}} |\Phi h_\nu(y)|^2 \right)^{1/2}
\]

for \( f \in \mathcal{H} \) and \( y \in E \).

**Lemma 2.2.** Suppose that (2) is satisfied and that \( \Phi \) given by (5) is injective. Then \( (\Phi(\mathcal{H}), \langle \cdot, \cdot \rangle_{\Phi}) \) is a reproducing kernel Hilbert space.

**Proof.** Let \( \| \cdot \|_\Phi \) denote the norm that is induced by \( \langle \cdot, \cdot \rangle_\Phi \). Observe that

\[
\| \Phi f \|_\Phi = \left( \sum_{\nu \in \mathbb{N}} |\langle f, h_\nu \rangle|^2 \right)^{1/2}
\]

for \( f \in \mathcal{H} \). Use (2) and (6) to conclude that \( \Phi f \mapsto \Phi f(y) \) defines a bounded linear functional on \( \Phi(\mathcal{H}) \) for every \( y \in E \). \( \square \)

**Remark 2.3.** In general, (2) does not imply that \( \Phi \) defined according to (5) is injective. An obvious necessary assumption is that the set \( \{ \Phi h_\nu : \nu \in \mathbb{N} \} \) is linearly independent in \( \mathbb{K}^E \). The following example shows that even this is not sufficient.

Let \( N := \mathbb{N} \), and let \( \mathcal{H} := \ell_2 \) with the canonical unit vector basis \( \{ h_\nu : \nu \in \mathbb{N} \} \) and define \( \Phi : \{ h_\nu : \nu \in \mathbb{N} \} \rightarrow \mathbb{K}^N \) by \( \Phi h_\nu := \nu(h_\nu - h_{\nu-1}) \) for \( \nu \in \mathbb{N} \) with the convention \( h_0 := 0 \). For each \( y \in \mathbb{N} \) the sum in (2) is a finite sum, so (2) is satisfied. It is also easy to see that \( \{ \Phi h_\nu : \nu \in \mathbb{N} \} \) is linearly independent in \( \mathbb{K}^N \). For \( f \in \mathcal{H} \) with \( \langle f, h_\nu \rangle = \frac{1}{\nu} \) and \( y \in \mathbb{N} \) we obtain from (6) that

\[
\Phi f(y) = \sum_{\nu \in \mathbb{N}} \langle f, h_\nu \rangle \cdot \Phi h_\nu(y) = 0.
\]

Hence \( \Phi \) is not injective.

**Remark 2.4.** Assume that \( \Phi \) is linear and injective. Then, in general, condition (2) is not sufficient to guarantee that \( (\Phi(\mathcal{H}), \langle \cdot, \cdot \rangle_{\mathcal{H}}) \) is a reproducing kernel Hilbert space. We present a general counterexample.

We start with a reproducing kernel Hilbert space \( \mathcal{H} \subseteq \mathbb{K}^E \) with an orthonormal basis \( \{ h_\nu \}_{\nu \in \mathbb{N}} \) and consider \( \Phi : \mathcal{H} \rightarrow \mathbb{K}^E, f \mapsto f \). Due to Lemma 2.1 we have

\[
\forall y \in E : \sum_{\nu \in \mathbb{N}} |h_\nu(y)|^2 < \infty.
\]

Let \( E_* = E \cup \{ z \} \) with a point \( z \notin E \). Choose an arbitrary discontinuous linear functional \( \zeta \) on \( \mathcal{H} \) satisfying \( \zeta(h_\nu) = 0 \) for all \( \nu \in \mathbb{N} \). Let \( \Psi f \in \mathbb{K}^{E_*} \) be the extension of \( f \in \mathcal{H} \) to \( E_* \) with \( \Psi f(z) = \zeta(f) \). Obviously, \( \Psi \) is a linear and injective mapping from \( \mathcal{H} \) to \( \mathbb{K}^{E_*} \). It follows that \( \mathcal{H}_* := \Psi(\mathcal{H}) \), equipped with the scalar product \( \langle \cdot, \cdot \rangle_\Phi \) induced by \( \Psi \), is a Hilbert space, too, with orthonormal basis \( \{ \Psi h_\nu \}_{\nu \in \mathbb{N}} \). Since \( \Psi h_\nu(z) = 0 \) for all \( \nu \in \mathbb{N} \), we have

\[
\forall y \in E_* : \sum_{\nu \in \mathbb{N}} |\Psi h_\nu(y)|^2 < \infty.
\]

But \( \mathcal{H}_* \) is not a reproducing kernel Hilbert space since, by construction, the function evaluation \( \Psi f \mapsto \Psi f(z) = \zeta(f) \) is discontinuous on \( \mathcal{H}_* \).
Notice that $Ψ$ is not of the form (5). Indeed, $\sum_{\nu \in \mathbb{N}} \langle f, h_{\nu} \rangle \Psi h_{\nu}(z) = 0$ for all $f \in \mathcal{H}$, but since $\zeta$ is discontinuous, there has to exist at least one $g \in \mathcal{H}$ satisfying $Ψg(z) = \zeta(g) \neq 0$.

**Remark 2.5.** The particular case where $\mathcal{H}$ already consists of real- or complex-valued functions on $E$ with the natural choice of $Φ_f := f$ for every $f \in \mathcal{H}$ is also studied in [22, Rem. 1]. It is shown that $\mathcal{H}$ is a reproducing kernel Hilbert space if and only if (2) and (3) are satisfied.

Lemma 2.2 allows to go beyond the setting from Remark 2.5 in order to cover the most important case of $\mathcal{H}$ being a subspace of an $L^2$-space. Here it turns out that (2) already implies that the pointwise limits of the Fourier partial sums form a reproducing kernel Hilbert space.

**Remark 2.6.** Consider the space $L^2(E, \mu)$ with respect to any measure $\mu$ on any $\sigma$-algebra on $E$, and assume that $\mathcal{H}$ is a linear subspace of $L^2(E, \mu)$ with a continuous embedding. Consider a sequence of square-integrable functions $h_{\nu}$ on $E$ with the following properties: The corresponding equivalence classes $h_{\nu} \in L^2(E, \mu)$ form an orthonormal basis of $\mathcal{H}$, and

$$\forall y \in E : \sum_{\nu \in \mathbb{N}} |h_{\nu}(y)|^2 < \infty,$$

cf. (2). We claim that $Φ$ given by (5) with $Φ_{h_{\nu}} := h_{\nu}$, i.e.,

$$Φf(y) := \sum_{\nu \in \mathbb{N}} \langle f, h_{\nu} \rangle \cdot h_{\nu}(y),$$

is injective.

In fact, consider a square-integrable function $f$ on $E$, whose corresponding equivalence class $f \in L^2(E, \mu)$ satisfies $f \in \mathcal{H}$ and $Φf = 0$. The partial sums of the series $\sum_{\nu \in \mathbb{N}} \langle f, h_{\nu} \rangle \cdot h_{\nu}$ converge in mean-square to $f$. Due to the Fischer-Riesz Theorem there exists a subsequence of partial sums that converges almost everywhere to $f$. Since $Φf = 0$ means that the partial sums converge to zero at every point in $E$, we get $f = 0$ almost everywhere, i.e., $f = 0$.

Apply Lemma 2.2 to conclude that $(Φ(\mathcal{H}), \langle \cdot, \cdot \rangle_{Φ})$ is a reproducing kernel Hilbert space. We add that the inverse $Φ^{-1} : Φ(\mathcal{H}) \to L^2(E, \mu)$ of $Φ$ is continuous and maps $f \in Φ(\mathcal{H})$ to its equivalence class.

### 2.2. Countable Tensor Products.

Consider a sequence of separable Hilbert spaces $(H_j, \langle \cdot, \cdot \rangle_j)$ with $j \in \mathbb{N}$ together with orthonormal bases $(h_{\nu,j})_{\nu \in \mathbb{N}_j}$ with countable sets $N_j$. For notational convenience assume that $N_j \subseteq \mathbb{N}_0$ and $0 \in N_j$. Later on we will have $N_j = N_0$ for all $j \in \mathbb{N}$ most of the time. However, we also consider the case $N_j = \{0, 1\}$ for all $j \in \mathbb{N}$.

The countable tensor product

$$H := \bigotimes_{j \in \mathbb{N}} H_j$$

that is studied in this paper is the so-called incomplete tensor product introduced by von Neumann in [44] with the particular choice of the unit vector $h_{0,j}$ in the space $H_j$. The choice of $\nu = 0$ is without loss of generality at this point. The construction of this tensor product and the properties we use are summarized in Appendix A. Here we only mention two facts. First of all, $H$ is a complete space, i.e., a Hilbert space. Moreover, let $N$ denote the set of all sequences $\nu := (\nu_j)_{j \in \mathbb{N}}$.
in \( N_0 \) such that \( \nu_j \in N_j \) for every \( j \in \mathbb{N} \) and \( \sum_{j \in \mathbb{N}} \nu_j < \infty \). Then the elementary tensors
\[
h_\nu := \bigotimes_{j \in \mathbb{N}} h_{\nu_j, j}
\]
with \( \nu \in N \) form an orthonormal basis of the space \( H \).

In the sequel, we use \( \langle \cdot, \cdot \rangle \) to denote the scalar product on the tensor product space \( H \). Of course, the results from Section 2.1 are applicable with any set \( E \) and any injective linear mapping \( \Phi : H \to \mathbb{K}^E \). In the present setting it is reasonable, however, to require that \( \Phi \) respects the tensor product structure. Hence we assume in particular that
\[
E := D^N
\]
with a set \( D \neq \emptyset \).

If we have reproducing kernels \( k_j : D \times D \to \mathbb{K} \) for \( j \in \mathbb{N} \) such that
\[
K(x, y) := \prod_{j \in \mathbb{N}} k_j(x_j, y_j)
\]
converges for all \( x, y \in E \), we write
\[
K := \bigotimes_{j \in \mathbb{N}} k_j.
\]

We adapt Lemma 2.1 and Lemma 2.2 to the tensor product setting.

**Lemma 2.7.** Suppose that \( \Phi : H \to \mathbb{K}^E \) is linear and injective and that there exist mappings \( \Phi_j : \{ h_{\nu_j} : \nu \in N_j \} \to \mathbb{K}^D \) such that
\[
\forall j \in \mathbb{N} : \quad \Phi_j h_{0,j} = 1
\]
and
\[
\forall \nu \in N \forall y \in E : \quad \Phi h_\nu(y) = \prod_{j \in \mathbb{N}} \Phi_j h_{\nu_j, j}(y_j).
\]
Furthermore, assume that \( (\Phi(H), \langle \cdot, \cdot \rangle_{\Phi}) \) is a reproducing kernel Hilbert space. Then we have
\[
\forall y \in E : \quad \sum_{j \in \mathbb{N}} \sum_{\nu \in N_j \setminus \{0\}} |\Phi_j h_{\nu,j}(y_j)|^2 < \infty
\]
and
\[
\forall y \in E \forall f \in H : \quad \Phi f(y) = \sum_{\nu \in N} \langle f, h_\nu \rangle \cdot \prod_{j \in \mathbb{N}} \Phi_j h_{\nu_j, j}(y_j)
\]
with absolute convergence. Moreover, the reproducing kernel \( K \) of this space is given by
\[
K = \bigotimes_{j \in \mathbb{N}} k_j,
\]
where
\[
k_j(x_j, y_j) := 1 + \sum_{\nu \in N_j \setminus \{0\}} \Phi_j h_{\nu,j}(x_j) \cdot \overline{\Phi_j h_{\nu,j}(y_j)},
\]
with absolute convergence for all \( x_j, y_j \in D \).

**Proof.** Combine (2) and Lemma B.1 with \( \beta_{\nu,j} := |\Phi_j h_{\nu,j}(y_j)|^2 \) to obtain (9). In the same way we get (11) with absolute convergence from (4). Finally, (10) with absolute convergence follows immediately from (3). \( \square \)
Remark 2.8. Under the assumptions of Lemma 2.7 every mapping $\Phi_j$ can be extended to a linear injective mapping $\Phi_j : H_j \to \mathbb{K}^D$ analogously to (9), and $\Phi_j(H_j)$ is a reproducing kernel Hilbert space. Moreover, $k_j$ is the reproducing kernel of $\Phi_j(H_j)$, and $\Phi_j$ is an isometric isomorphism between $H_j$ and $H(k_j)$ mapping the unit vector $h_{0,j} \in H_j$ to the function $1 \in H(k_j)$. As noted in Appendix A, this implies that the tensor product of the mappings $\Phi_j$ is an isometric isomorphism between $H_j$ and $\bigotimes_{j \in \mathbb{N}} H(k_j)$ with unit vectors $\Phi_j h_{0,j} := 1$. In particular, $H(K)$ and $\bigotimes_{j \in \mathbb{N}} H(k_j)$ are canonically isometrically isomorphic.

In the construction of a mapping $\Phi : H \to \mathbb{K}^E$ we start with injective mappings $\Phi_j : \{h_{\nu,j} : \nu \in N_j\} \to \mathbb{K}^D$, and we assume that (7) and (9) are satisfied. Due to Lemma 2.1, the right-hand side in (10) may be used to define a linear mapping $\Phi : H \to \mathbb{K}^E$, satisfying (8), and Lemma 2.2 immediately carries over to the present setting.

Lemma 2.9. Suppose that (7) and (9) are satisfied and that $\Phi$, defined via (10), is injective. Then $(\Phi(H), \langle \cdot, \cdot \rangle_\Phi)$ is a reproducing kernel Hilbert space.

Next, we adapt Remark 2.6 which deals with $L_2$-spaces, to the tensor product setting.

Remark 2.10. Consider a probability measure $\mu_0$ on any $\sigma$-algebra on $D$ and the corresponding space $L_2(D, \mu_0)$. Then the tensor product space $\bigotimes_{j \in \mathbb{N}} L_2(D, \mu_0)$ is canonically isometrically isomorphic to the space $L_2(E, \mu)$ with respect to the probability measure $\mu = \mu_0 \times \mu_0 \times \ldots$ on the product $\sigma$-algebra on $E$. Assume that for every $j \in \mathbb{N}$ the space $H_j$ is a subspace of $L_2(D, \mu_0)$ with a continuous embedding of norm one and $h_{0,j} = 1$. Consequently, $H$ is a subspace of $\bigotimes_{j \in \mathbb{N}} L_2(D, \mu_0)$ with a continuous embedding of norm one.

Consider sequences of square-integrable functions $h_{\nu,j}$ on $D$ with the following properties: For every $j \in \mathbb{N}$ we have $h_{0,j} = 1$, the corresponding equivalence classes $h_{\nu,j} \in L_2(D, \mu_0)$ with $\nu \in N_j$ form an orthonormal basis of $H_j$, and

$$\forall y \in E : \sum_{j \in \mathbb{N}} \sum_{\nu \in N_j} |h_{\nu,j}(y_j)|^2 < \infty.$$  

According to Remark 2.6 and Lemma 2.1 the linear mapping $\Phi$ given by

$$\Phi f(y) := \sum_{\nu \in \mathbb{N}} \langle f, h_\nu \rangle \cdot \prod_{j \in \mathbb{N}} h_{\nu,j}(y_j)$$

for $f \in H$ and $y \in E$ is injective. We apply Lemma 2.9 to conclude that $(\Phi(H), \langle \cdot, \cdot \rangle_\Phi)$ is a reproducing kernel Hilbert space.

3. Increasing Smoothness and Weights

3.1. The Function Spaces: Abstract Setting. The abstract setting is given by a separable Hilbert space $(H_0, \langle \cdot, \cdot \rangle_0)$ with an orthonormal basis $(e_\nu)_{\nu \in \mathbb{N}_0}$ and a family $(\alpha_{\nu,j})_{\nu, j \in \mathbb{N}}$ of Fourier weights such that

(C1) \(\forall \nu, j \in \mathbb{N} : \alpha_{\nu,j} \geq \max(\alpha_{\nu,1}, \alpha_{1,j})\)

and

(C2) \(\alpha_{1,1} > 1\).
We define

$$H_j := \{ f \in H_0 : \sum_{\nu \in \mathbb{N}} \alpha_{\nu,j} \cdot |\langle f, e_\nu \rangle_0|^2 < \infty \}$$

and

$$\langle f, g \rangle_j := \langle f, e_0 \rangle_0 \cdot \langle e_0, g \rangle_0 + \sum_{\nu \in \mathbb{N}} \alpha_{\nu,j} \cdot \langle f, e_\nu \rangle_0 \cdot \langle e_\nu, g \rangle_0$$

for \( j \in \mathbb{N} \) and \( f, g \in H_j \) to obtain a sequence of Hilbert spaces \((H_j, \langle \cdot, \cdot \rangle_j)\). For notational convenience we put \( \alpha_{\nu,j} := 1 \) for \( j = 0 \) and \( \nu \in \mathbb{N}_0 \) as well as for \( j \in \mathbb{N} \) and \( \nu = 0 \). Clearly

$$\langle f, e_\nu \rangle_j = \alpha_{\nu,j} \cdot \langle f, e_\nu \rangle_0$$

for \( \nu, j \in \mathbb{N}_0 \) and \( f \in H_j \).

We state some basic properties of the spaces \( H_j \). Let \( i, j \in \mathbb{N}_0 \). We have a continuous embedding \( H_i \hookrightarrow H_j \) if and only if

$$\sup_{\nu \in \mathbb{N}} \alpha_{\nu,i} \alpha_{\nu,j} < \infty,$$

and in the case of a continuous embedding its norm is given by

$$\sup_{\nu \in \mathbb{N}_0} \sqrt{\frac{\alpha_{\nu,i}}{\alpha_{\nu,j}}} \geq 1.$$

In particular, (C1) and (C2) imply \( 1 \leq \alpha_{\nu,1} \leq \alpha_{\nu,j} \) for \( \nu, j \in \mathbb{N} \), and the latter is equivalent to \( H_0 \hookrightarrow H_1 \hookrightarrow H_j \) with continuous embeddings of norm one for every \( j \geq 1 \). Furthermore, we have a compact embedding \( H_i \hookrightarrow H_j \) if and only if

$$\lim_{\nu \to \infty} \frac{\alpha_{\nu,i}}{\alpha_{\nu,j}} = 0.$$

Throughout this paper, increasing smoothness is understood in this sense, i.e., \( H_i \supset H_j \) for \( i < j \) with a compact embedding.

Let \( j \in \mathbb{N} \) and \( f \in H_j \). The elements \( \alpha_{\nu,j}^{-1/2} e_\nu \) with \( \nu \in \mathbb{N}_0 \) form an orthonormal basis of the Hilbert space \( H_j \). Let \( S_j \) denote the embedding of \( H_j \) into \( H_0 \). Since \( \langle e_\nu, e_\mu \rangle_j = \alpha_{\nu,j} \cdot \langle S_j^* e_\nu, e_\mu \rangle_j \) for \( \nu, \mu \in \mathbb{N}_0 \), we obtain

$$\sum_{\nu \in \mathbb{N}_0} \alpha_{\nu,j}^{-1} \cdot \langle f, e_\nu \rangle_0 \cdot e_\nu.$$

Consequently, the singular values of \( S_j \) are given by \( \alpha_{\nu,j}^{-1/2} \) with \( \nu \in \mathbb{N}_0 \).

In the abstract setting we consider the tensor product space

$$H := \bigotimes_{j \in \mathbb{N}} H_j,$$

based on the choice of the unit vector \( e_0 \).

3.2. The Function Spaces: Standard Setting. Most often, we consider the following special case of the abstract setting. This standard setting is given by

$$H_0 := L_2(D, \mu_0)$$

for some probability measure \( \mu_0 \) on a \( \sigma \)-algebra on any set \( D \neq \emptyset \), by a linear and injective mapping

$$\Phi_1 : H_1 \to \mathbb{R}^D$$

that satisfies

$$\forall h \in H_1 : \Phi_1(h) \in h$$
and 

\[ \Phi_1(e_0) = 1, \]

and by 

\[ \Phi_j = \Phi_1|_{H_j} \]

for \( j \geq 2 \).

Consequently, the condition (2) reads

\[ \forall y \in D : \sum_{\nu \in \mathbb{N}} \alpha_{\nu,j}^{\nu-1} |\Phi_1 e_\nu(y)|^2 < \infty \]

for the space \( H_j \), and due to (C1) this condition is most restrictive in the case \( j = 1 \). Analogously, (9) reads

\[ \forall y \in D^\mathbb{N} : \sum_{\nu,j \in \mathbb{N}} \alpha_{\nu,j}^{\nu-1} |\Phi_1 e_\nu(y_j)|^2 < \infty \]

for the space \( H \). Here it is crucial that the tensor product is based on the choice of the unit vectors \( e_0 \). Henceforth we typically will not stress this point anymore.

In the standard setting the conditions (13) and (14) are necessary and sufficient for \( \Phi_1(H_j) \) and \( \Phi(H) \), respectively, to be reproducing kernel Hilbert spaces, see Remarks 2.6 and 2.10.

Subsequently we identify \( \Phi_1 f \) and \( f \) for \( f \in H_1 \), \( \Phi f \) and \( f \) for \( f \in H \), \( \Phi_1(H_j) \) and \( H_j \), and \( \Phi(H) \) and \( H \), if the respective spaces are reproducing kernel Hilbert spaces. Furthermore, we do no longer distinguish between square-integrable functions on \( D \) and elements of \( H_0 \). In this sense, we take \( \Phi_1 e_\nu := e_\nu \), so that, in particular, 

\[ e_0 := 1. \]

In the standard setting the space \( \bigotimes_{j \in \mathbb{N}} H_0 \) is canonically isometrically isomorphic to the space \( L_2(E, \mu) \), where \( \mu \) denotes the product of the probability measure \( \mu_0 \) on the product \( \sigma \)-algebra on \( E := D^\mathbb{N} \). Obviously, \( H \) is a subspace of \( L_2(E, \mu) \) with a continuous embedding of norm one.

3.3. Examples. In all the examples to be presented below, we consider the standard setting with a Borel probability measure \( \mu_0 \) on an interval \( D \subseteq \mathbb{R} \). We separate the choice of the Hilbert space \( H_0 \) and its orthonormal basis \( (e_\nu)_{\nu \in \mathbb{N}_0} \) from the selection of the Fourier weights \( (\alpha_{\nu,j})_{\nu,j \in \mathbb{N}} \).

See, e.g., [22] and the references therein, for the following example in the context of tractability analysis of high-dimensional problems. For further information about Korobov spaces see, e.g., [32, App. A.1], and about Walsh functions see, e.g., [5, App. A].

Example 3.1. Consider the uniform distribution \( \mu_0 \) on \( D := [0, 1] \) together with the trigonometric basis \( (e_\nu)_{\nu \in \mathbb{N}_0} \), given by \( e_\nu(x) := \exp(2\pi i (-1)^\nu [\nu/2] x) \), or with the Walsh basis \( (e_\nu)_{\nu \in \mathbb{N}_0} \), see [15]. Since \( |e_\nu(x)| = 1 \) for all \( \nu \in \mathbb{N}_0 \) and \( x \in D \), we conclude that

\[ \sum_{\nu \in \mathbb{N}} \alpha_{\nu,1}^{-1} < \infty \]

is equivalent to \( H_1, H_2, \ldots \) being reproducing kernel Hilbert spaces. Furthermore, \( H \) is a reproducing kernel Hilbert space if and only if

\[ \sum_{\nu,j \in \mathbb{N}} \alpha_{\nu,j}^{-1} < \infty. \]
If the spaces $H_j$ stem from the trigonometric basis, then they are known as Korobov spaces. If they stem from the Walsh basis, then they are often called Walsh spaces.

For the next example see, for instance, [17] and the references therein.

**Example 3.2.** Consider the uniform distribution $\mu_0$ on $D := [0,1]$ together with the $L_2$-normalized Haar basis $(e_\nu)_{\nu \in \mathbb{N}_0}$. Put $I_\ell := \{2^\ell, \ldots, 2^{\ell+1} - 1\}$ for $\ell \in \mathbb{N}_0$, and assume that

$$
\alpha_{2^\ell,j} = \cdots = \alpha_{2^{\ell+1}-1,j}
$$

for $\ell \in \mathbb{N}_0$ and $j \in \mathbb{N}$. Since

$$
\sum_{\nu \in I_\ell} |e_\nu(x)|^2 = 2^\ell
$$

for all $x \in D$ and $\ell \in \mathbb{N}_0$, the conclusions from Example 3.1 are also valid in the present case. Since the Haar functions $e_\nu$ as well as the Walsh functions $e_\nu$ from Example 3.1 with $\nu \in I_\ell$ are an orthonormal basis of the same finite-dimensional subspace of $L_2([0,1], \mu_0)$, condition (16) ensures that in both cases we obtain the same sequence of Hilbert spaces $H_j$.

**Example 3.3.** Consider the uniform distribution $\mu_0$ on $D := [-1,1]$ together with the $L_2$-normalized Legendre polynomials $e_\nu$. Here we have

$$
e_\nu(1) = \|e_\nu\|_{\infty} := \sup_{x \in D} |e_\nu(x)| = \sqrt{2\nu + 1} \propto \max(\nu^{1/2}, 1),
$$

see, e.g., [37, Ex. 2.20]. It follows that $H_1, H_2, \ldots$ are reproducing kernel Hilbert spaces if and only if

$$
\sum_{\nu \in \mathbb{N}} \alpha_\nu^{-1} \nu < \infty,
$$

while $H$ is a reproducing kernel Hilbert space if and only if

$$
\sum_{\nu,j \in \mathbb{N}} \alpha_{\nu,j}^{-1} \nu < \infty.
$$

**Example 3.4.** Now we consider a generalization of Example 3.3. Let $\mu_0$ be defined by the Lebesgue density $x \mapsto c(\alpha,\beta) \cdot (1-x)^\alpha (1+x)^\beta$ on $D := [-1,1]$ for some $\alpha, \beta > -1/2$, where

$$
e(\alpha,\beta) := \frac{\alpha + \beta + 1}{2^{\alpha+\beta+1}} \cdot \frac{(\alpha + \beta)}{\alpha}.
$$

The orthogonal polynomials associated to this weight function are the Jacobi polynomials $P_\nu^{(\alpha,\beta)}$, usually normalized such that $P_\nu^{(\alpha,\beta)}(1) = \binom{\nu+\alpha}{\nu}$, see, e.g., [40 Eqn. (4.1.1)]. The special case $\alpha = \beta = 0$ yields the Legendre polynomials. The $L_2$-normalized version is

$$
e_\nu := c(\alpha,\beta) \cdot P_\nu^{(\alpha,\beta)}
$$

with

$$
c(\alpha,\beta) := (c(\alpha,\beta))^{-1/2} \cdot \left(\frac{(2\nu + \alpha + \beta + 1) \cdot \Gamma(\nu + 1) \cdot \Gamma(\nu + \alpha + \beta + 1)}{2^{\alpha+\beta+1} \cdot \Gamma(\nu + \alpha + 1) \cdot \Gamma(\nu + \beta + 1)}\right)^{1/2}
$$

$$
\propto \max(\nu^{1/2}, 1),
$$
The Jacobi polynomials $P_{\nu}^{(\alpha, \beta)}$ attain their supremum norm in $-1$ or in $+1$ with
\[ \|P_{\nu}^{(\alpha, \beta)}\|_{\infty} = \left( \nu + \max(\alpha, \beta) \right)^{\frac{1}{\nu}} \max(\nu^{\max(\alpha, \beta)}, 1), \]
see, e.g., [40, Thm. 7.32.1]. Altogether we obtain
\[ \max(|e_{\nu}(1)|, |e_{\nu}(-1)|) = \sup_{x \in D} |e_{\nu}(x)| \approx \max(\nu^{\sigma}, 1) \]
with
\[ \sigma := \max(\alpha, \beta) + \frac{1}{2} > 0. \]

It follows that $H_1, H_2, \ldots$ are reproducing kernel Hilbert spaces if and only if
\[ \sum_{\nu \in \mathbb{N}} \alpha_{\nu,1}^{-1} \cdot \nu^{2\sigma} < \infty, \]
while $H$ is a reproducing kernel Hilbert space if and only if
\[ \sum_{\nu, j \in \mathbb{N}} \alpha_{\nu,j}^{-1} \cdot \nu^{2\sigma} < \infty. \]

**Remark 3.5.** In the Examples 3.1–3.4 the summability of $(\alpha_{\nu,j}^{-1} \cdot \nu^{\sigma})_{\nu,j \in \mathbb{N}}$ for some $\sigma \geq 0$ determines whether $H$ is a reproducing kernel Hilbert space. According to Lemma B.2 this summability already follows from the summability of $(\alpha_{\nu,1}^{-1} \cdot \nu^{\sigma})_{\nu \in \mathbb{N}}$ and $(\alpha_{1,j}^{-1})_{j \in \mathbb{N}}$, if
\[ (17) \quad \liminf_{\nu, j \to \infty} \frac{\ln(\alpha_{\nu,j})}{\ln(\nu) \cdot \ln(j)} > 0. \]

Next, we turn to two important classes of Fourier weights. At first we introduce some notation. The decay of any sequence $x = (x_i)_{i \in \mathbb{N}}$ of positive reals is defined by
\[ \text{decay}(x) := \sup \left\{ \tau > 0 : \sum_{i \in \mathbb{N}} x_i^{1/\tau} < \infty \right\} \]
with the convention that $\sup \emptyset := 0$, see [47, p. 311]. As a well-known fact
\[ \text{decay}(x) = \liminf_{i \to \infty} \frac{\ln(x_i^{-1})}{\ln(i)} \]
if the decay or the limes inferior is positive, which follows, e.g., from Lemma B.3 in Appendix B.

**Example 3.6.** We consider
\[ \alpha_{\nu,j} := \alpha_{\nu}^{r_j}, \]
where
\[ (18) \quad \forall j \in \mathbb{N} : \quad 0 < r_1 \leq r_j \]
as well as
\[ \forall \nu \in \mathbb{N} : \quad 1 < a_1 \leq a_\nu \]
and
\[ (19) \quad a_\nu \asymp \nu. \]
Put \( r_0 := 0 \). For \( j \in \mathbb{N}_0 \) the space \( H_{j+1} \) is continuously embedded into \( H_j \) (with norm one) if and only if \( r_j \leq r_{j+1} \), and in this case \( r_j < r_{j+1} \) is equivalent to the compactness of this embedding.

Obviously \( (C1) \) and \( (C2) \) hold true, and \( (17) \) is equivalent to
\[
\rho > 0
\]
for
\[
\rho := \liminf_{j \to \infty} \frac{r_j}{\ln(j)}.
\]

Note that
\[
\text{decay}((\alpha^{-1}_\nu,j)_{\nu \in \mathbb{N}}) = r_1
\]
and
\[
\text{decay}((\alpha^{-1}_j)_{j \in \mathbb{N}}) = \rho \cdot \ln(a_1).
\]

Let \( \sigma \geq 0 \). Observe that
\[
(20) \sum_{\nu \in \mathbb{N}} \alpha^{-1}_{\nu,j} \cdot \nu^\sigma < \infty
\]
is actually equivalent to
\[
r_1 > \sigma + 1,
\]
while
\[
r_1 > \sigma + 1 \wedge \rho > \frac{1}{\ln(a_1)}
\]
is a sufficient condition for
\[
(21) \sum_{\nu,j \in \mathbb{N}} \alpha^{-1}_{\nu,j} \cdot \nu^\sigma < \infty
\]
to hold. A necessary condition also permits \( \rho = 1 / \ln(a_1) \).

**Remark 3.7.** The exponents \( r_j \) in Example 3.6 may be regarded as smoothness parameters. To illustrate this point, we first consider the complex \( L_2 \)-space and the complex exponentials according to Example 3.1. Up to equivalence of norms, the Korobov spaces \( H_j \) with parameters \( r_j \) may be defined by any choice of \( a_\nu > 0 \) such that \( (19) \) is satisfied. Specifically
\[
(22) a_\nu := 2\pi \lceil \nu+1/2 \rceil
\]
is considered in, e.g., \([35, 38]\) and
\[
(23) a_\nu := 1 + \lceil \nu+1/2 \rceil
\]
is considered in, e.g., \([7]\). Observe that the index set \( Z \) instead of \( \mathbb{N}_0 \) is considered in \([7, 35, 38]\). Furthermore, the parameters \( 2r_j \) instead of \( r_j \) are used in \([35, 38]\). See \([27]\) for a generalization of this type of Fourier weights, which involves an additional fine parameter.

Secondly, we consider the smoothness spaces based on Legendre polynomials and, more general, on Jacobi polynomials in Examples 3.3 and 3.4 which are related to weighted Sobolev spaces. Such spaces were considered in, e.g., \([31]\). We discuss one special case where the relation can be directly explained. Corollary 2.6 and Theorem 2.7 from \([31]\) show that, if \( r_j \) is an even integer and \( \alpha = \beta > -1/2 \), then the space \( H_j \) with respect to the Jacobi polynomials \( P_{\nu}^{(\alpha,\beta)} \) can be identified (with equivalent norms) with the Sobolev space of all functions on \((-1,1)\) with weak
derivatives up to order \( r_j/2 \) in the weighted \( L_2 \)-space of functions on \((-1, 1)\) with respect to the weight function \( \varrho_{\alpha,r_j}(x) = (1 - x^2)^{\alpha+r_j/2} \).

More formally, let \( L_2(\varrho_{\alpha,r_j}) \) be the Hilbert space of all functions \( f : (-1, 1) \to \mathbb{R} \) with
\[
\|f\|_{L_2(\varrho_{\alpha,r_j})}^2 = \int_{-1}^{1} |f(x)|^2 \varrho_{\alpha,r_j}(x) \, dx < \infty.
\]

Let \( W_{r_j/2}(\varrho_{\alpha,r_j}) \) be the Hilbert space of all functions \( f \) on \((-1, 1)\) with weak derivatives up to order \( r_j/2 \) in \( L_2(\varrho_{\alpha,r_j}) \) with norm given by
\[
\left( \sum_{k=0}^{r_j/2} \|f^{(k)}\|_{L_2(\varrho_{\alpha,r_j})}^2 \right)^{1/2}.
\]
Then
\[
H_j = W_{r_{j+1}/2}(\varrho_{\alpha,r_j})
\]
with equivalent norms.

**Example 3.8.** Choose \( a > 1 \) and consider
\[
\alpha_{\nu,j} := a^{r_j \cdot \nu^b_j}
\]
with (18) being satisfied and with
\[
\forall j \in \mathbb{N} : \quad 0 < b_1 \leq b_j.
\]
See, e.g., [22] and the references therein for this type of Fourier weights.

Put \( r_0 := 0 \) as previously. For \( j \in \mathbb{N}_0 \) we have a compact embedding of \( H_{j+1} \) into \( H_j \) if and only if \( b_j < b_{j+1} \) or \( b_j = b_{j+1} \) and \( r_j < r_{j+1} \). Furthermore, we have a continuous, non-compact embedding only in the trivial case \( b_j = b_{j+1} \) and \( r_j = r_{j+1} \).

Obviously (C1) and (C2) hold true, and (17) follows from
\[
\rho > 0,
\]
where \( \rho \) is defined as in Example 3.6. In contrast to Example 3.6, we now have (sub-)exponentially growing Fourier weights for every space \( H_j \) with \( j \in \mathbb{N} \). In particular,
\[
\text{decay}((\alpha_{\nu,1})_{\nu \in \mathbb{N}}) = \infty,
\]
while
\[
\text{decay}((\alpha_{1,j})_{j \in \mathbb{N}}) = \rho \cdot \ln(a).
\]
Hence (20) is satisfied for every \( \sigma \geq 0 \). A sufficient condition for (21) to hold is
\[
\rho > \frac{1}{\ln(a)}.
\]
Again a necessary condition also permits equality.
3.4. The Embeddings: Abstract Setting. Consider the abstract setting. Let
\[ \gamma_j := \sup_{\nu \in \mathbb{N}} \alpha_{\nu,j} \]
for \( j \in \mathbb{N} \), and observe that \( 0 < \gamma_j \leq 1 \) due to (C1) and (C2).

For the first kind of embedding we use the sequence \((\alpha_{\nu,1})_{\nu \in \mathbb{N}}\) of Fourier weights of the space \((H_1, \langle \cdot, \cdot \rangle_1)\) and the sequence \((\gamma_j)_{j \in \mathbb{N}}\) of positive weights to construct a new sequence of Hilbert spaces \((G_j, \langle \cdot, \cdot \rangle_{G_j})\) in the following way. We take
\[ G_j := H_1 \]
and
\[ \langle f, g \rangle_{G_j} := \langle f, e_0 \rangle_0 \cdot \langle e_0, g \rangle_0 + \frac{1}{\gamma_j} \cdot \sum_{\nu \in \mathbb{N}} \alpha_{\nu,1} \cdot \langle f, e_\nu \rangle_0 \cdot \langle e_\nu, g \rangle_0 \]
for \( j \in \mathbb{N} \) and \( f, g \in H_1 \). Of course, this is a particular case of the construction of the spaces \((H_j, \langle \cdot, \cdot \rangle_j)\), where the Fourier weights are now of the form \( \alpha_{\nu,j} := \frac{\alpha_{\nu,1}}{\gamma_j} \).

In addition to \( H \) we consider the tensor product space
\[ G := \bigotimes_{j \in \mathbb{N}} G_j \]
with the corresponding scalar product.

**Remark 3.9.** The embeddings \( G_j \hookrightarrow G_{j+1} \) and \( G_j \hookrightarrow G_{j+1} \) are continuous with norms \( \max(1, \sqrt{\gamma_{j+1}/\gamma_j}) \) and \( \max(1, \sqrt{\gamma_j/\gamma_{j+1}}) \), respectively. In particular, we have equivalence of the norms on all spaces \((G_j, \langle \cdot, \cdot \rangle_{G_j})\), which is in sharp contrast to spaces of increasing smoothness, where we have compact embeddings \( H_j \hookrightarrow H_{j+1} \).

For the second kind of embedding we take
\[ F_j := \text{span}\{e_0, e_1\} \]
as well as
\[ \langle f, g \rangle_{F_j} := \langle f, e_0 \rangle_0 \cdot \langle e_0, g \rangle_0 + \alpha_{1,j} \cdot \langle f, e_1 \rangle_0 \cdot \langle e_1, g \rangle_0 \]
for \( j \in \mathbb{N} \) and \( f, g \in F_1 \), and we consider the tensor product space
\[ F := \bigotimes_{j \in \mathbb{N}} F_j \]
with the corresponding scalar product.

Our analysis is based on the following simple observation.

**Theorem 3.10.** In the abstract setting we have
\[ F \hookrightarrow H \hookrightarrow G \]
with embeddings of norm one.

**Proof.** The norm of the embeddings \( H_j \hookrightarrow G_j \) and \( F_j \hookrightarrow H_j \) is one. \qed
of recent results on infinite-dimensional integration on such spaces can be found in [11].

In the present setting the weighted tensor products are based on the spaces $(H^1, \langle \cdot, \cdot \rangle_1)$ and $(\text{span} \{e_0, e_1\}, \langle \cdot, \cdot \rangle_1)$ and on the weights $\gamma_j$ and $\alpha_{1,1}/\alpha_{1,j}$, respectively. Theorem 3.10 allows to transfer results from weighted tensor product spaces to tensor products of spaces of increasing smoothness and vice versa.

The results that will be derived in the subsequent sections depend on the family $(\alpha_{\nu,j})_{\nu,j \in \mathbb{N}}$ of Fourier weights via the decays of the sequences $(\alpha_{\nu,1})_{\nu \in \mathbb{N}}, (\alpha_{1,j})_{j \in \mathbb{N}},$ and $(\gamma_j)_{j \in \mathbb{N}}$.

**Example 3.11.** In the situation of Example 3.6 we have $\gamma_j = \sup_{\nu \in \mathbb{N}} a^{r_1 - r_j}_\nu = a^{r_1 - r_j}_1$, and therefore

$$\text{decay} \left( (\gamma_j)_{j \in \mathbb{N}} \right) = \rho \cdot \ln(a_1) = \text{decay} \left( (\alpha_{1,j})_{j \in \mathbb{N}} \right).$$

Analogously, in the situation of Example 3.8 we have $\gamma_j = \sup_{\nu \in \mathbb{N}} a^{r_{1,\nu_1} - r_{1,j}}_{\nu_1} = a^{r_{1,\nu_1} - r_{1,j}}_{1}$, and therefore

$$\text{decay} \left( (\gamma_j)_{j \in \mathbb{N}} \right) = \rho \cdot \ln(a) = \text{decay} \left( (\alpha_{1,j})_{j \in \mathbb{N}} \right).$$

**Remark 3.12.** The family $(\alpha_{\nu,1}/\gamma_j)_{\nu,j \in \mathbb{N}}$ of Fourier weights satisfies [(C1) and (C2)] as well. However, if also [(A)] holds for $(\alpha_{\nu,j})_{\nu,j \in \mathbb{N}}$, we do not necessarily have this property for $(\alpha_{\nu,1}/\gamma_j)_{\nu,j \in \mathbb{N}}$. Nevertheless, it is easy to see that the conclusion of Lemma 3.7 still holds true for the latter family of Fourier weights.

### 3.5 The Embeddings: Standard Setting

Now we turn to the standard setting, and we assume that $G$ is a reproducing kernel Hilbert space (in the sense of the study from Section 2.2). It follows that each of the spaces $H_J, G_J, \text{ or } F_j$ is a Hilbert space with a reproducing kernel of the form $1 + m$, where $m$ is a reproducing kernel as well and $H(1) \cap H(m) = \{0\}$.

Consider any reproducing kernel $m$ on $D \times D$. If there exists a point $a \in D$ such that $m(a, a) = 0$, then $m$ is called an anchored kernel with anchor $a$. The latter is equivalent to $f(a) = 0$ for every $f \in H(m)$. Next, consider the reproducing kernel $1 + m$, and suppose that $H(1) \cap H(m) = \{0\}$. Then $m$ is an anchored kernel with anchor $a$ if and only if the orthogonal projection onto the subspace $H(1)$ of constant functions in $H(1 + m)$ is given by $f \mapsto f(a)$. An anchored kernel induces an anchored function space decomposition on $\otimes_{j=1}^d H(1 + \gamma_j m)$ with $d \in \mathbb{N}$, see [29], and on $\otimes_{j \in \mathbb{N}} H(1 + \gamma_j m)$, see [13]. Individual components of this decomposition can be evaluated efficiently using function values only; see again [29].

We stress that for each of the spaces $H_J, G_J, \text{ or } F_j$ the respective kernel $m$ is not necessarily anchored. Actually, all the spaces $H_j$ and $G_j$ that we obtain in the Examples 3.1 to 3.4 do not have a reproducing kernel $1 + m$ with an anchored kernel $m$. This is easily verified: Since $H(m)$ is the orthogonal complement of $H(1)$ in $H(1 + m)$, we have that $e_1, e_2 \in H(m)$. If $m(a, a) = 0$ for some $a \in D$, then necessarily $e_1(a) = 0 = e_2(a)$. But in the Examples 3.1 and 3.2 we have $|e_1(x)| = 1 = |e_2(x)|$ for all $x \in D$. In Example 3.3 (and thus also in Example 3.3, which is a special case of the former example) the only zero of $e_1$ is $a := (\beta - \alpha)/(\alpha + \beta + 2), \ldots}$
and it is easily checked that \( c_2(a) \neq 0 \). Furthermore, we have that the kernel
\[
m(x, y) = \alpha_{x, y}^1 e_1(x) e_1(y)
\]
corresponding to \( F_j \) is not anchored in the Examples 3.1 and 3.2 and anchored in \( a := (\beta - \alpha)/(\alpha + \beta + 2) \) in Example 3.4 and, consequently, in \( a := 0 \) in Example 3.3.

We establish, however, relations between the spaces \( H_j \), \( G_j \) and \( F_j \) and spaces with anchored kernels via suitable embeddings.

To this end, we fix a point \( a \in D \), and for \( j \in \mathbb{N} \) and \( c > 0 \) we define
\[
G_j^c := G_j = H_1
\]
and
\[
(f, g)_{G_j^c} := f(a) \cdot g(a) + \frac{1}{c \gamma_j} \cdot \sum_{\nu \in \mathbb{N}} \alpha_{\nu, 1} \cdot |\langle f, e_\nu \rangle_0| \cdot |\langle e_\nu, g \rangle_0|
\]
where \( f, g \in H_1 \).

In the sequel we employ results from [11], which have been formulated for reproducing kernel Hilbert spaces of real-valued functions. These results may be extended to complex-valued functions in a canonical way and are thus applicable in the present setting.

**Lemma 3.13.** For all \( j \in \mathbb{N} \) and \( c > 0 \) the space \((G_j^c, \langle \cdot, \cdot \rangle_{G_j^c})\) is a reproducing kernel Hilbert space of functions with domain \( D \), and its norm is equivalent to \( \| \cdot \|_{G_1} \). Moreover, there exists a (uniquely defined) reproducing kernel \( m \) on \( D \times D \) such that \( 1 + c \gamma_j \cdot m \) is the reproducing kernel of \((G_j^c, \langle \cdot, \cdot \rangle_{G_j^c})\) for all \( j \in \mathbb{N} \) and \( c > 0 \), and
\[
m(a, a) = 0.
\]

**Proof.** Put
\[
\|f\|_{1,1} := |\langle f, e_0 \rangle_0|
\]
and
\[
\|f\|_{1,\|} := |f(a)|
\]
as well as
\[
\|f\|_{2,1} := \|f\|_{2,\|} := \sum_{\nu \in \mathbb{N}} \alpha_{\nu, 1} \cdot |\langle f, e_\nu \rangle_0|^2
\]
for \( f \in H_1 \). According to [11] Rem. 2.1, the vector space \( H_1 \) together with the seminorms \( \| \cdot \|_{1,1} \) and \( \| \cdot \|_{2,1} \) satisfies the conditions [11] (A1)–(A3). The same holds true for the seminorms \( \| \cdot \|_{1,\|} \) and \( \| \cdot \|_{2,\|} \), see [11] Rem. 2.5.

By definition of [11] (A3) this ensures, in particular, that \( \langle \cdot, \cdot \rangle_{G_j^c} \) is a scalar product on \( G_j^c \) that turns the latter space into a reproducing kernel Hilbert space. The closed graph theorem yields the equivalence of norms as claimed.

Let \( m \) denote the reproducing kernel of \( \{f \in H_1 : f(a) = 0\} \) in \((G_j^c, \langle \cdot, \cdot \rangle_{G_j^c})\) in the particular case \( c \gamma_j = 1 \). By definition, we have [24], and [11] Lem. 2.1, Rem. 2.2] imply that the reproducing kernel of \((G_j^c, \langle \cdot, \cdot \rangle_{G_j^c})\) is given by \( 1 + c \gamma_j \cdot m \) for all \( j \in \mathbb{N} \) and \( c > 0 \).

We stress the following important differences between the spaces \((G_j, \langle \cdot, \cdot \rangle_{G_j})\) and \((G_j^c, \langle \cdot, \cdot \rangle_{G_j^c})\). In the latter case the orthogonal projection onto the space of constant functions is easy to compute, but \( \langle e_\nu \rangle_{\nu \in \mathbb{N}_0} \) is an orthogonal system only in the trivial case that the two scalar products of \( G_j \) and \( G_j^1 \) coincide.
Lemma 3.14. There exists a constant $0 < c_0 < 1$ such that
\begin{equation}
(1 + c_0^{-1} \gamma_j)^{-1/2} \cdot \| f \|_{G_j^c}^{-1} \leq \| f \| \leq (1 + \gamma_j)^{1/2} \cdot \| f \|_{G_j^c_0}
\end{equation}
and
\[ \| f \|_0 \leq (1 + c_0^{-2} \gamma_j) \cdot \| f \|_{G_j^c_0}^{-1} \]
for all $j \in \mathbb{N}$ and $f \in H_1$.

Proof. According to the first paragraph of the proof of Lemma 3.13 we are in the situation from [11]. The inequality (25) follows directly from [11, Thm. 2.1] and Lemma 3.13.

Analogously, the norm of the embedding $G_j^c_0 \hookrightarrow \tilde{G}_j$ is bounded from above by $(1 + c_0^{-2} \gamma_j)^{1/2}$, where $\tilde{G}_j$ is defined as $G_j$, however with new weights $c_0^{-2} \cdot \gamma_j$ instead of $\gamma_j$. Furthermore, the norm of the embedding $\tilde{G}_j \hookrightarrow H_0$ is given by
\[ \max \left( \sqrt{c_0^{-2} \gamma_j / \alpha_{1,1,1}}, 1 \right) \cdot (1 + c_0^{-2} \gamma_j)^{1/2} \leq 1 + c_0^{-2} \gamma_j. \]
\[ \square \]

Condition (14) for the space $G$ reads
\[ \forall y \in D^N : \sum_{\nu,j \in \mathbb{N}} (\alpha_{\nu,1}/\gamma_j)^{-1} \cdot |e_\nu(y_j)|^2 < \infty. \]
Considering $\nu = 1$ and some $y \in D$ such that $e_1(y) \neq 0$ yields
\begin{equation}
\sum_{j \in \mathbb{N}} \gamma_j < \infty.
\end{equation}

For $c > 0$ we define
\[ G^c := \bigotimes_{j \in \mathbb{N}} G_j^c. \]

Note that different values of $c$ may lead to different spaces and not just to different norms, see [15], and the spaces do not necessarily fit into the setting of Section 3.1.

Lemma 3.15. For every $c > 0$ the space $G^c$ is a reproducing kernel Hilbert space of functions with domain $D^N$ and reproducing kernel given by $\bigotimes_{j \in \mathbb{N}} (1 + c \gamma_j \cdot m)$ with $m$ according to Lemma 3.13. Furthermore, there exists a constant $0 < c_0 < 1$ such that we have continuous embeddings
\[ G_j^c_0 \hookrightarrow G \hookrightarrow G_j^c \hookrightarrow \bigotimes_{j \in \mathbb{N}} H_0. \]

Proof. According to the first paragraph of the proof of Lemma 3.13 we are in the situation from [11].

Combining (26) with [11, Thm. 2.3] yields the first claim. The second claim follows directly from Lemma 3.14 and (26).

We proceed in the same way for the space $F$. For $j \in \mathbb{N}$ and $c > 0$ we define
\[ F_j^c := F_j = F_1 \]
and
\[ \langle f, g \rangle_{F_j^c} := f(a) \cdot g(a) + \frac{\alpha_{1,j}}{c} \cdot \langle f, e_1 \rangle_0 \cdot \langle e_1, g \rangle_0. \]
where $f, g \in F_1$.

**Lemma 3.16.** For all $j \in \mathbb{N}$ and $c > 0$ the space $(F^c_j, \langle \cdot, \cdot \rangle_{F^c_j})$ is a reproducing kernel Hilbert space of functions with domain $D$, and its norm is equivalent to $\| \cdot \|_{F^1}$. Moreover, there exists a (uniquely defined) reproducing kernel $\ell$ on $D \times D$ such that $1 + c\alpha^{-1}_{1,j} \cdot \ell$ is the reproducing kernel of $(F^c_j, \langle \cdot, \cdot \rangle_{F^c_j})$ for all $j \in \mathbb{N}$ and $c > 0$, and

$$
\ell(a, a) = 0.
$$

**Lemma 3.17.** There exists a constant $0 < c_0 < 1$ such that

$$(1 + c_0^{-1}\alpha^{-1}_{1,j})^{-1/2} \cdot \|f\|_{F^c_j} \leq \|f\|_{F^c_j} \leq (1 + \alpha^{-1}_{1,j})^{1/2} \cdot \|f\|_{F^c_j}$$

and

$$
\|f\|_0 \leq (1 + c_0^{-2}\alpha^{-1}_{1,j}) \cdot \|f\|_{F^c_j}^{-1},
$$

for all $j \in \mathbb{N}$ and $f \in F_1$.

Observe that $F$ is a reproducing kernel Hilbert space. Furthermore, from (26) we get

$$
\sum_{j \in \mathbb{N}} \alpha^{-1}_{1,j} < \infty.
$$

For $c > 0$ we define

$$
F^c := \bigotimes_{j \in \mathbb{N}} F^c_j.
$$

**Lemma 3.18.** For every $c > 0$ the space $F^c$ is a reproducing kernel Hilbert space of functions with domain $D^{\mathbb{N}}$ and reproducing kernel given by $\bigotimes_{j \in \mathbb{N}} (1 + c\alpha^{-1}_{1,j} \cdot \ell)$ with $\ell$ according to Lemma 3.16. Furthermore, there exists a constant $0 < c_0 < 1$ such that we have continuous embeddings

$$
F^{c_0} \hookrightarrow F \hookrightarrow F^{c_0^{-1}} \hookrightarrow \bigotimes_{j \in \mathbb{N}} H_0.
$$

Combining Theorem 3.10, Lemma 3.15, and Lemma 3.18 yields the following result.

**Theorem 3.19.** Consider the standard setting, and assume that $G$ is a reproducing kernel Hilbert space. Then there exists a constant $0 < c_0 < 1$ with the following properties. We have continuous embeddings

$$
\begin{align*}
F^{c_0} & \hookrightarrow F \hookrightarrow F^{c_0^{-1}} \hookrightarrow \bigotimes_{j \in \mathbb{N}} H_0, \\
G^{c_0} & \hookrightarrow G \hookrightarrow G^{c_0^{-1}} \hookrightarrow \bigotimes_{j \in \mathbb{N}} H_0.
\end{align*}
$$

and $F^{c_0^{-1}}$ as well as $G^{c_0^{-1}}$ are reproducing kernel Hilbert spaces.
4. Infinite-Dimensional Approximation and Integration

Consider a bounded linear operator $S : \mathcal{X} \to \mathcal{Z}$ between two $\mathbb{K}$-Hilbert spaces as well as a non-decreasing sequence $\mathcal{A} = (\mathcal{A}_n)_{n \in \mathbb{N}}$ of sets $\mathcal{A}_n$ of bounded linear operators between $\mathcal{X}$ and $\mathcal{Z}$. We study the corresponding $n$-th minimal worst case error

$$\text{err}_n(S, \mathcal{A}) := \inf_{A \in \mathcal{A}_n} \sup \{ \| S(f) - A(f) \|_Z : f \in \mathcal{X}, \| f \|_X \leq 1 \};$$

more precisely, we determine

$$\text{dec}(S, \mathcal{A}) := \text{decay}((\text{err}_n(S, \mathcal{A}))_{n \in \mathbb{N}}).$$

To stress the dependence on $\mathcal{X}$ we often write $\text{err}_n(\mathcal{X}, S, \mathcal{A})$ and $\text{dec}(\mathcal{X}, S, \mathcal{A})$. Observe that

$$\text{decay}(x) = \sup \{ \tau > 0 : \sup_{i \in \mathbb{N}} x_i \cdot i^\tau < \infty \}$$

for any non-increasing sequence $x = (x_i)_{i \in \mathbb{N}}$ of positive reals, cf. [47, p. 311], and note that lower bounds for $\text{dec}(S, \mathcal{A})$ correspond to upper bounds for the $n$-th minimal errors $\text{err}_n(S, \mathcal{A})$ and vice versa.

For the approximation problem we have $\mathcal{X} \subseteq \mathcal{Z}$ with a continuous embedding, and $S = \text{App}$ is the corresponding embedding operator

$$\text{App} : \mathcal{X} \hookrightarrow \mathcal{Z}.$$  

For the integration problem we have $\mathcal{Z} = \mathbb{K}$, and $S = \text{Int}$ is a bounded linear functional

$$\text{Int} : \mathcal{X} \to \mathbb{K},$$

which is defined by means of integration with respect to a probability measure.

Theorems 3.10 and 3.19 allow us to derive results for linear problems, like approximation and integration, on the tensor product $H$ of spaces of increasing smoothness from known results for the weighted tensor product spaces $F$ and $G$ or $F^{\circ 0}$ and $G^{\circ -1}$, where the latter pair of spaces is, additionally, based on anchored kernels. Under the corresponding assumptions we have

(27) \[ \text{dec}(G, S, \mathcal{A}) \leq \text{dec}(H, S, \mathcal{A}) \leq \text{dec}(F, S, \mathcal{A}) \]

and

(28) \[ \text{dec}(G^{\circ -1}, S, \mathcal{A}) \leq \text{dec}(H, S, \mathcal{A}) \leq \text{dec}(F^{\circ 0}, S, \mathcal{A}). \]

Furthermore, $H_1$ can be isometrically embedded into $H$ via $f \mapsto f \otimes (\otimes_{n \geq 2} e_0)$. If we identify $H_1$ with its image under this embedding, we may consider a non-decreasing sequence $\mathcal{B} = (\mathcal{B}_n)_{n \in \mathbb{N}}$ of sets $\mathcal{B}_n$ of bounded linear operators between $H_1$ and $\mathcal{Z}$ that satisfies $A|_{H_1} \in \mathcal{B}_n$ for every $A \in \mathcal{A}_n$ and every $n \in \mathbb{N}$. Then we have

(29) \[ \text{dec}(H, S, \mathcal{A}) \leq \text{dec}(H_1, S, \mathcal{B}), \]

where, by definition, $\text{dec}(H_1, S, \mathcal{B})$ is the decay of the $n$-th minimal errors

$$\text{err}_n(H_1, S, \mathcal{B}) := \inf_{A \in \mathcal{B}_n} \{ \| S(f) - A(f) \|_Z : f \in H_1, \| f \|_{H_1} \leq 1 \}$$

for the corresponding univariate problem.
4.1. Approximation with Unrestricted Linear Information. We consider the abstract setting from Section 3.1. We are primarily interested in the case $X = H$ and

$$Z := \bigotimes_{j \in \mathbb{N}} H_0,$$

but for comparison we also consider $Z$ together with $X = F$ or $X = G$. In all these cases the embedding operator $\text{App}$, which is well defined since $\alpha_{\nu,j} \geq 1 \geq \gamma_j$ for $\nu, j \in \mathbb{N}$ yields embeddings $G_j \hookrightarrow H_0$ of norm one for every $j \in \mathbb{N}$, defines an infinite-dimensional approximation problem.

Let $X^*$ denote the dual space of $X$. We are interested in approximating $\text{App}$ using $n$ bounded linear functionals on $X$, i.e., we consider

$$A_n^\text{all} := \left\{ \sum_{i=1}^n \lambda_i \cdot z_i : \lambda_i \in X^*, \ z_i \in Z \right\}.$$  

Observe that in the standard setting we actually deal with $L_2$-approximation.

Theorem 4.1. Consider the abstract setting, and assume that (17) is satisfied. We have

$$\text{dec}(H, \text{App}, A_n^\text{all}) = \frac{1}{2} \cdot \min \left\{ \text{decay} \left( (\alpha_{\nu,1})_{\nu \in \mathbb{N}} \right), \text{decay} \left( (\alpha_{1,j}^{-1})_{j \in \mathbb{N}} \right) \right\},$$

$$\text{dec}(F, \text{App}, A_n^\text{all}) = \frac{1}{2} \cdot \text{decay} \left( (\alpha_{1,j}^{-1})_{j \in \mathbb{N}} \right),$$

and

$$\text{dec}(G, \text{App}, A_n^\text{all}) = \frac{1}{2} \cdot \min \left\{ \text{decay} \left( (\alpha_{\nu,1})_{\nu \in \mathbb{N}} \right), \text{decay} \left( (\gamma_j)_{j \in \mathbb{N}} \right) \right\}.$$  

Proof. First we consider $\text{dec}(H, \text{App}, A_n^\text{all})$. The singular values of $\text{App}$ on $H$ are given by $\alpha_{\nu}^{-1/2}$ with $\nu \in \mathbb{N}$, where

$$\alpha_{\nu} := \prod_{j \in \mathbb{N}} \alpha_{\nu,j},$$

see (12). Let $\xi := (\xi_i)_{i \in \mathbb{N}}$ denote the sequence of these singular values, arranged in non-increasing order. Due to a general result for linear problems on Hilbert spaces

$$\text{err}_n(H, \text{App}, A_n^\text{all}) = \xi_{n+1},$$

see [43, Thm. 5.3.2]. Hence Lemma B.1 and Lemma B.2 yield

$$\text{dec}(H, \text{App}, A_n^\text{all}) = \text{decay}(\xi) = \sup \left\{ \tau > 0 : \sum_{\nu \in \mathbb{N}} \alpha_{\nu}^{-1/(2\tau)} < \infty \right\} = \frac{1}{2} \cdot \min \left\{ \text{decay} \left( (\alpha_{\nu,1})_{\nu \in \mathbb{N}} \right), \text{decay} \left( (\alpha_{1,j})_{j \in \mathbb{N}} \right) \right\}.$$  

The results for $\text{dec}(G, \text{App}, A_n^\text{all})$ and $\text{dec}(F, \text{App}, A_n^\text{all})$ are established in the same way. For the spaces $G$ we only have to observe that the singular values of $\text{App}$ are given by $\prod_{j \in \mathbb{N}} (\alpha_{\nu,j}/\gamma_j)^{-1/2}$ for $\nu \in \mathbb{N}$ and to use Remark 3.12 instead of Lemma B.2. The singular values of $\text{App}$ on $F$ are given by $\prod_{j \in \mathbb{N}} (\alpha_{\nu,j})^{-1/2}$ for $\nu \in \mathbb{N} \cap \{0, 1\}^\mathbb{N}$, which immediately yields the claim.  

$\square$
Corollary 4.2. Consider the abstract setting. For the Fourier weights according to Example 3.6 we have
\[ \text{dec}(H, \text{App}, \mathcal{A}^{\text{all}}) = \frac{1}{2} \cdot \min(r_1, \rho \cdot \ln(a_1)), \]
and for the Fourier weights according to Example 3.8 we have
\[ \text{dec}(H, \text{App}, \mathcal{A}^{\text{all}}) = \frac{1}{2} \cdot \rho \cdot \ln(a). \]

Proof. Recall that
\[ \text{decay}((\alpha_1^{-1})_{\nu \in \mathbb{N}}) = r_1 \]
and
\[ \text{decay}((\alpha_{1,j}^{-1})_{j \in \mathbb{N}}) = \text{decay}((\gamma_j)_{j \in \mathbb{N}}) = \rho \cdot \ln(a_1) \]
for the first kind of Fourier weights, and with \( r_1 = \infty \) and \( a_1 = a \) we have the same decays for the second kind of Fourier weights, see Examples 3.6, 3.8, and 3.11. If \( \rho > 0 \), then (17) is satisfied for both types of Fourier weights, and both of the claims follow from Theorem 4.1. If \( \rho = 0 \) we get both of the claims from the fact that \( \text{decay}((\alpha_{1,j}^{-1})_{j \in \mathbb{N}}) = 0 \). \( \square \)

Remark 4.3. Let us also consider the finite-dimensional approximation problem that is given by the embedding operator
\[ \text{App}: H^{(d)} := \bigotimes_{j=1}^{d} H_j \hookrightarrow \bigotimes_{j=1}^{d} H_0. \]

Determining \( \text{dec}(H, \text{App}, \mathcal{A}^{\text{all}}) \) or determining the rate of strong polynomial tractability of \( \text{err}_n(H^{(d)}, \text{App}, \mathcal{A}^{\text{all}}) \) are equivalent problems. More precisely,
\[ \text{err}_n(H, \text{App}, \mathcal{A}^{\text{all}}) = \sup_{d \in \mathbb{N}} \text{err}_n(H^{(d)}, \text{App}, \mathcal{A}^{\text{all}}), \]
and hence
\[ \text{dec}(H, \text{App}, \mathcal{A}^{\text{all}}) = \text{decay} \left( \left( \sup_{d \in \mathbb{N}} \text{err}_n(H^{(d)}, \text{App}, \mathcal{A}^{\text{all}}) \right)_{n \in \mathbb{N}} \right). \]

The quantity on the right hand side of (34) is called the rate of strong polynomial tractability, and its reciprocal is called the exponent of strong polynomial tractability. In this sense (30) is due to [35, Thm. 1], who study Korobov spaces and the case (22), where \( a_1 = 2\pi \), and (31) is due to [26, Thm. 5.2], who derive this result for Korobov spaces under the additional assumption of convergence of \( (r_j / \ln(j))_{j \in \mathbb{N}} \); furthermore \( \rho > 0 \) is established as a sufficient condition for strong polynomial tractability.

Our version of this result shows that \( \rho > 0 \) is also a necessary condition and thus settles an open problem from [26].

Remark 4.4. We consider a particular case of the setting from [7], namely \( m = 1 \) and \( \beta = 0 \) in their notation. This means that the domain is of the form \( E := D_1 \times D \times D \times \ldots \) with closed intervals \( D_1, D \subseteq \mathbb{R} \) and that the space \( H_1 \) may be defined in terms of an orthonormal basis that is different from the basis used to defined the other spaces \( H_j \) with \( j \geq 2 \). Furthermore, \( \text{App} \) maps \( H \) into the \( L^2 \)-space with respect to a product probability measure of the form \( \mu_1 \times \mu_0 \times \mu_0 \times \ldots \). Our results extend to this setting in a straight-forward way.
In [7] the Fourier weights from Example 3.6 with (23) are considered. It is shown that

\[(35) \quad \text{dec}(H, \text{App}, A^{\text{all}}) = \frac{1}{2} \cdot r_1 \]

holds, if the requirement

\[(36) \quad \sum_{j \in \mathbb{N}} \frac{1}{r_j} \cdot \eta^{r_j} < \infty \]

for \( \eta = (2/3)^{1/r_1} \) is satisfied, see [7, Thm. 4.1]. Observe that \( \rho \cdot \ln(3/2) > r_1 \) is a sufficient condition for (36) to hold, and a necessary condition for (36) also permits equality, cf. Lemma B.3 of Appendix B.

Notice that (23) implies \( a_1 = 2 \). Thus our result (30) improves on the findings in [7] as it shows that the weaker condition

\[ \rho \cdot \ln(2) \geq r_1 \]

is necessary and sufficient for (35) to hold. Nevertheless, we stress that in [7] explicit error bounds are derived, while our Theorem 4.1 only determines the decay of the minimal errors.

4.2. Approximation and Integration with Standard Information. Now, we investigate the approximation and the integration problem in the standard setting.

In the sequel \( \mathcal{X} \) typically will be one of the tensor product spaces \( F, H, G, F^c, \) or \( G^c \). For approximation we have

\[ Z := \bigotimes_{j \in \mathbb{N}} L_2(D, \mu_0) = L_2(D^\mathbb{N}, \mu) \]

with the respective embedding \( S = \text{App} \). For integration we have \( Z = \mathbb{K} \), and \( S = \text{Int} \) is given by

\[ \text{Int}(f) = \int_{D^\mathbb{N}} f \, d\mu \]

for \( f \in \mathcal{X} \).

Again, we are primarily interested in the case \( \mathcal{X} = H \), but in our analysis it is crucial to consider \( \mathcal{X} = F^c \) and \( \mathcal{X} = G^c \) with suitably chosen \( c > 0 \) as well. The latter enables us to apply general results from [36, 47] for weighted tensor product spaces based on anchored kernels. For comparison we also consider \( \mathcal{X} = F \) and \( \mathcal{X} = G \) as before.

Assume that \( \mathcal{X} \) is a reproducing kernel Hilbert space of functions on the domain \( D^\mathbb{N} \), so that \( \delta_y(f) := f(y) \) defines a bounded linear functional on \( \mathcal{X} \) for every \( y \in D^\mathbb{N} \). We consider a class \( A^{\text{std}}_{n} \) of bounded linear operators that is much smaller than \( A^{\text{all}}_{n} \). First of all, \( A \in A^{\text{std}}_{n} \) is only based on function evaluations \( \delta_y \) instead of arbitrary bounded linear functionals \( \lambda \in \mathcal{X}^* \). Furthermore, we do not permit evaluations at any point \( y \in D^\mathbb{N} \) and also do not just take into account the total number of function evaluations that is used by \( A \). Instead, we employ the unrestricted subspace sampling model, which has been introduced in [28]. This model is based on a non-decreasing cost function \( $: \mathbb{N}_0 \cup \{\infty\} \to [1, \infty] \) and some nominal
value $a \in D$ in the following way. For $y = (y_j)_{j \in \mathbb{N}} \in D^\mathbb{N}$ the number of active variables is given by
\[(37) \quad \text{Act}(y) := \{ j \in \mathbb{N} : y_j \neq a \},\]
and
\[(38) \quad A_n^{\text{std}} := \left\{ \sum_{i=1}^m \delta_{y_i} \cdot z_i : m \in \mathbb{N}_0, y_i \in D^\mathbb{N}, \sum_{i=1}^m \delta(\text{Act}(y_i)) \leq n, z_i \in Z \right\} \]
is the class of algorithms with the cost bounded by $n$. For the univariate approximation and integration problems, where $Z := L_2(D, \mu_0)$ or $Z := K$ and $\text{Int}(f) := \int_D f \, d\mu_0$, respectively, we simply take as sets $E_n^{\text{std}}$ of bounded linear operators between $H_1$ and $Z$
\[(39) \quad E_n^{\text{std}} := \left\{ \sum_{i=1}^n \delta_{y_i} \cdot z_i : y_i \in D, z_i \in Z \right\} \].

In the sequel, we assume that
\[(40) \quad \delta(n) = \Omega(n) \text{ and } \delta(n) = O(e^{\zeta n}) \text{ for some } \zeta \in (0, \infty).\]

**Theorem 4.5.** Consider the standard setting, and assume that $G$ is a reproducing kernel Hilbert space. For $S = \text{App}$ and $S = \text{Int}$ we have
\[(41) \quad \text{dec}(F, S, A_n^{\text{std}}) = \frac{1}{2} \cdot (\text{decay}((\alpha_{1,j})_{j \in \mathbb{N}}) - 1)
\]
and
\[(42) \quad \text{dec}(G, S, A_n^{\text{std}}) = \min \left( \text{dec}(H_1, S, B_n^{\text{std}}), \frac{1}{2} \cdot (\text{decay}((\gamma_j)_{j \in \mathbb{N}}) - 1) \right),\]

implying
\[(43) \quad \text{min} \left( \text{dec}(H_1, S, B_n^{\text{std}}), \frac{1}{2} \cdot (\text{decay}((\gamma_j)_{j \in \mathbb{N}}) - 1) \right) \leq \text{dec}(H, S, A_n^{\text{std}}) \leq \text{dec}(H_1, S, B_n^{\text{std}}), \frac{1}{2} \cdot (\text{decay}((\alpha_{1,j})_{j \in \mathbb{N}}) - 1) \right).\]

**Proof.** At first, we derive (42). For every $c > 0$ the reproducing kernel of $G^c$ is a weighted tensor product that is based on an anchored kernel, see Lemma 3.15. We claim that
\[(44) \quad \text{dec}(G^c, S, A_n^{\text{std}}) = \min \left( \text{dec}(H_1, S, B_n^{\text{std}}), \frac{1}{2} \cdot (\text{decay}((\gamma_j)_{j \in \mathbb{N}}) - 1) \right).\]

In fact, if $\text{decay}((\gamma_j)_{j \in \mathbb{N}}) > 1$, then [47] Cor. 9 yields this claim for $S = \text{App}$, while we employ [36] Thm. 2 and Sec. 3.3 for $S = \text{Int}$. Otherwise we have $\text{decay}((\gamma_j)_{j \in \mathbb{N}}) = 1$, see (26), and this case may be easily reduced to the previous one. Indeed, this can be done by making the weights smaller such that their decay $\delta$ is strictly larger than one. Making the weights smaller leads to smaller $n$-th minimal errors and thus to a larger decay of the minimal errors. Using the claim for the case $\text{decay}((\gamma_j)_{j \in \mathbb{N}}) > 1$ and letting $\delta$ tend to one establishes our claim $\text{dec}(G^c, S, A_n^{\text{std}}) = 0$ in the case $\text{decay}((\gamma_j)_{j \in \mathbb{N}}) = 1$.

With the help of our claim and the embedding result from Lemma 3.15 we obtain (40).

The proof of (39) is similar. Here we only have to observe that $e_0 = 1$ and $e_1(x) \neq e_1(y)$ for some $x, y \in D$, which yields $\text{err}_2(F_1, S, B_n^{\text{std}}) = 0$, and to apply Lemma 3.18 instead of Lemma 3.15.

Finally, (41) follows from (39) and (40) together with (27).
Remark 4.6. In the proof of Theorem 4.5 we rely on results from [36,47] that were actually proved under slightly stronger assumptions than the ones we make in the theorem. It is assumed in [36,47] that $D$ is a Borel measurable subset of $\mathbb{R}$ and that the probability measure $\mu_0$ has a Lebesgue density. The proofs are applicable, however, in the setting of the present paper, cf. [2,13].

We apply Theorem 4.5 to the trigonometric basis and to the Haar basis. For the univariate problem on the corresponding space $H_1$ the asymptotic behavior of the $n$-th minimal errors $err_n(H_1,S,B^{std})$ is known for $S = \text{App}$ and $S = \text{Int}$ in the case of the trigonometric basis. In the case of the Haar basis we are only aware of a lower bound for $S = \text{Int}$. A matching upper bound for $S = \text{App}$ is established in Appendix C.

Corollary 4.7. Assume that $H_0 = L_2([0,1],\mu_0)$ for the uniform distribution $\mu_0$. Consider the trigonometric or the Haar basis $(e_\nu)_{\nu \in \mathbb{N}_0}$ according to Example 3.1 or Example 3.2 respectively. For the Fourier weights according to Example 3.6 we have

$$r_1 > 1 \land \rho \cdot \ln(a_1) > 1 \Rightarrow H \text{ is a RKHS} \Rightarrow (r_1 > 1 \land \rho \cdot \ln(a_1) \geq 1),$$

as well as

$$\text{dec}(H,S,A^{\text{std}}) = \frac{1}{2} \cdot \min(r_1, \rho \cdot \ln(a_1) - 1)$$

for $S = \text{App}$ and $S = \text{Int}$ if $H$ is a reproducing kernel Hilbert space.

Proof. Examples 3.1, 3.2, and 3.6 with $\sigma = 0$ yield the necessary and the sufficient condition for $H$ to be a reproducing kernel Hilbert space. Recall that $G$ is based on the Fourier weights $(\alpha_\nu,1/\gamma_j)_{\nu,j \in \mathbb{N}}$. We proceed as for the space $H$ to establish the same pair of conditions for $G$ to be a reproducing kernel Hilbert space.

In the sequel we therefore assume that $r_1 > 1$. Then we have

$$\text{(42)}$$

The lower bound for $S = \text{App}$ in the case of the trigonometric basis follows from the well-known approximation error estimates of Dirichlet or de la Vallée-Poussin means, see, e.g., [23,42]. The case of the Haar basis is studied in Theorem C.1. The upper bound for $S = \text{Int}$ in the case of the trigonometric basis follows from equally well-known constructions of fooling functions that are trigonometric polynomials, see, e.g., [41]. The case of the Haar basis follows from [6, Thm. 41].

If $\rho \cdot \ln(a_1) > 1$ is valid, then we apply (33) and (41) to determine $\text{dec}(H,S,A^{\text{std}})$ as claimed, and the remaining case $\rho \cdot \ln(a_1) = 1$ is easily reduced to the previous one.

In a similar way we may handle the Fourier weights according to Example 3.8 instead of Example 3.6. Since this type of Fourier weights does never satisfy (16), we only consider the trigonometric basis.

Corollary 4.8. Assume that $H_0 = L_2([0,1],\mu_0)$ for the uniform distribution $\mu_0$. Consider the trigonometric basis $(e_\nu)_{\nu \in \mathbb{N}_0}$ according to Example 3.1. For the Fourier weights according to Example 3.8 we have

$$\rho \cdot \ln(a) > 1 \Rightarrow H \text{ is a RKHS} \Rightarrow \rho \cdot \ln(a) \geq 1,$$

as well as

$$\text{dec}(H,S,A^{\text{std}}) = \frac{1}{2} \cdot (\rho \cdot \ln(a) - 1)$$

for $S = \text{App}$ and $S = \text{Int}$ if $H$ is a reproducing kernel Hilbert space.
Proof. Proceed as in the proof of Corollary 4.7. □

4.3. Concluding Remarks.

Remark 4.9. Consider the approximation or the integration problem in the standard setting, and assume that $G$ is a reproducing kernel Hilbert space. For the trigonometric basis and the polynomial or (sub-)exponential Fourier weights and for the Haar basis and the polynomial Fourier weights we obtain sharp results via embeddings: It turns out that

$$\text{dec}(G^{-1}, S, A_{\text{std}}) = \text{dec}(H, S, A_{\text{std}}) = \min\left(\text{dec}(H_1, S, A_{\text{std}}), \text{dec}(F, S, A_{\text{std}})\right)$$

for $S = \text{App}$ and $S = \text{Int}$, see Corollaries 4.7 and 4.8 and their proofs.

The analysis on the tensor product $H$ of spaces of increasing smoothness is therefore reduced to the analysis on its first factor $H_1$ and on the weighted tensor product spaces $G^{-1}$ and $F$, which are based on anchored kernels. We stress that already the space $G$ is typically much larger than $H$, while already the space $F$ is always much smaller than $H$.

A similar conclusion holds true for the approximation problem with unrestricted linear information in the abstract setting. For the polynomial and the (sub-)exponential Fourier weights

$$\text{dec}(G, \text{App}, A_{\text{all}}) = \text{dec}(H, \text{App}, A_{\text{all}}) = \min\left(\text{dec}(H_1, \text{App}, A_{\text{all}}), \text{dec}(F, \text{App}, A_{\text{all}})\right),$$

see Theorem 4.1, Corollary 4.2 as well as (32) and (33).

Remark 4.10. Consider the approximation problem in the setting from Corollary 4.7. Combining the latter with Corollary 4.2 reveals that, at least with respect to the decay of the $n$-th minimal errors, the class $A_{\text{std}}$ is as powerful as the class $A_{\text{all}}$ if and only if $\rho \cdot \ln(a_1) \geq r_1 + 1$. Furthermore, $\text{dec}(H, \text{App}, A_{\text{std}})$ and $\text{dec}(H, \text{App}, A_{\text{all}})$ differ at most by $1/2$.

In the setting from Corollary 4.8 we always have

$$\text{dec}(H, \text{App}, A_{\text{std}}) = \text{dec}(H, \text{App}, A_{\text{all}}) - \frac{1}{2}.$$

Appendix A. Countable Tensor Products

Let $(H_j, \langle \cdot, \cdot \rangle_j)_{j \in \mathbb{N}}$ be a sequence of Hilbert spaces and fix, for each $j \in \mathbb{N}$, a unit vector $u_j \in H_j$. If it is clear from the context we sometimes omit to name the unit vectors $u_j \in H_j$ explicitly. In the setting of Section 3 it is natural to choose $u_j = e_0$ for all $j \in \mathbb{N}$. Then the incomplete tensor product

$$H := \bigotimes_{j \in \mathbb{N}} H_j$$

is the completion of the linear span of elementary infinite tensors $\otimes_{j \in \mathbb{N}} f_j$, for which only finitely many $f_j$ are different from $u_j$. Here the completion is taken with respect to the inner product given by

$$\langle \otimes_{j \in \mathbb{N}} f_j, \otimes_{j \in \mathbb{N}} g_j \rangle := \prod_{j \in \mathbb{N}} \langle f_j, g_j \rangle_j$$
for elementary infinite tensors and extended linearly to finite sums of elementary infinite tensors. The abstract completion can be replaced by a concrete description of elements in the incomplete tensor product via linear functionals, see [44].

This notion of an infinite tensor product is the natural one for our purpose since the incomplete tensor product of spaces $L_2(D_j, \mu_j)$ is in a canonical way isometrically isomorphic to $L_2(D, \mu)$, where $\mu := \times_{j \in \mathbb{N}} \mu_j$ is the product measure of the probability measures $\mu_j$ on $D := \times_{j \in \mathbb{N}} D_j$.

We freely used the following facts, which can be found in [44]. Each $H_j$ is isometrically embedded in $H$ by identifying $h_j \in H_j$ with the tensor $\otimes_{j \in \mathbb{N}} f_j$ with $f_j = h_j$ and $f_j = u_j$ for $j \neq j_0$. Similarly, the finite Hilbert space tensor products $\otimes_{j=1}^d H_j$ are isometrically embedded in $H$. If we have another incomplete tensor product $G := \otimes_{j \in \mathbb{N}} G_j$ with unit vectors $v_j \in G_j$ and a sequence of bounded linear operators $T_j : H_j \to G_j$ with $T_j u_j = v_j$ such that $C := \prod_{j \in \mathbb{N}} \| T_j \| < \infty$, then there exists a unique linear bounded operator $T : H \to G$ acting on elementary tensors as

$$T(\otimes_{j \in \mathbb{N}} f_j) := \otimes_{j \in \mathbb{N}} T_j f_j.$$ 

Moreover, $\| T \| = C$.

APPENDIX B. SUMMABILITY AND DECAY OF SEQUENCES

As in Section 2.2 we consider sets $N_j \subseteq \mathbb{N}_0$ such that $0 \in N_j$ for $j \in \mathbb{N}$, and we let $N = \{ \nu : (\nu_j)_{j \in \mathbb{N}} \in \mathbb{N}_0 \text{ such that } \nu_j \in N_j \text{ for every } j \in \mathbb{N} \}$ and $\sum_{j \in \mathbb{N}} \nu_j < \infty$.

**Lemma B.1.** Let $\beta_{\nu} := \prod_{j \in \mathbb{N}} \beta_{\nu,j}$ for $\nu \in N$ and $\beta_{\nu,j} \in \mathbb{R}$ for $j \in \mathbb{N}$ and $\nu \in N_j$ with $\beta_{0,j} = 1$ for every $j \in \mathbb{N}$. Then $\sum_{\nu \in N} \beta_{\nu}$ is absolutely convergent if and only if

$$\sum_{j \in \mathbb{N}} \sum_{\nu \in N \setminus \{0\}} |\beta_{\nu,j}| < \infty,$$

in which case

$$\sum_{\nu \in N} \beta_{\nu} = \prod_{j \in \mathbb{N}} \left(1 + \sum_{\nu \in N_j \setminus \{0\}} \beta_{\nu,j} \right).$$

**Proof.** Without loss of generality we may assume that $N_j = \mathbb{N}_0$ for every $j \in \mathbb{N}$. Let

$$N_k := \{ \nu \in N : \nu_j = 0 \text{ for } j > k \}$$

for $k \in \mathbb{N}$. It is easy to prove by induction that

$$\sum_{\nu \in N_k} |\beta_{\nu}| = \prod_{j=1}^k \left(1 + \sum_{\nu \in N \setminus \{0\}} |\beta_{\nu,j}| \right).$$
for every \( k \in \mathbb{N} \). Therefore

\[
\sum_{\nu \in \mathbb{N}} |\beta_\nu| = \prod_{j \in \mathbb{N}} \left( 1 + \sum_{\nu \in \mathbb{N}} |\beta_{\nu,j}| \right),
\]

so that \( \sum_{\nu \in \mathbb{N}} |\beta_\nu| < \infty \) and \( \sum_{\nu,j \in \mathbb{N}} |\beta_{\nu,j}| < \infty \) are equivalent. Similarly, we get

\[
\sum_{\nu \in \mathbb{N}} \beta_\nu = \prod_{j \in \mathbb{N}} \left( 1 + \sum_{\nu \in \mathbb{N}} \beta_{\nu,j} \right),
\]

if \( \sum_{\nu \in \mathbb{N}} |\beta_\nu| < \infty \).

\[ \square \]

Lemma B.2. Assume that (17) is satisfied, in addition to (C1) and (C2). For every \( \tau > 0 \) and every \( \sigma \geq 0 \) we have

\[
\sum_{\nu,j \in \mathbb{N}} \alpha_{\nu,j}^{-\tau} \cdot \nu^\sigma < \infty \quad \iff \quad \left( \sum_{\nu \in \mathbb{N}} \alpha_{\nu,1}^{-\tau} \cdot \nu^\sigma < \infty \wedge \sum_{j \in \mathbb{N}} \alpha_{1,j}^{-\tau} < \infty \right).
\]

Proof. It suffices to verify the implication ‘\( \iff \)’ for \( \tau = 1 \). Accordingly, we assume that \( \sum_{\nu \in \mathbb{N}} \alpha_{\nu,1}^{-1} \cdot \nu^\sigma < \infty \) and \( \sum_{j \in \mathbb{N}} \alpha_{1,j}^{-1} < \infty \), and we show that

\[
\sum_{\nu,j \in \mathbb{N}} \alpha_{\nu,j}^{-1} \cdot \nu^\sigma < \infty.
\]

Put \( \beta_{\nu,j} := \alpha_{\nu,j} \cdot \nu^{-\sigma} \). For any choice of \( n \in \mathbb{N} \) we have

\[
\sum_{\nu,j \in \mathbb{N}} \beta_{\nu,j}^{-1} \leq \sum_{j=1}^{n} \sum_{\nu \in \mathbb{N}} \beta_{\nu,j}^{-1} + \sum_{\nu=1}^{n} \sum_{j=n+1}^{\infty} \beta_{\nu,j}^{-1} \leq n \cdot \sum_{\nu \in \mathbb{N}} \alpha_{\nu,1}^{-1} \cdot \nu^\sigma + n^\alpha \sum_{j \in \mathbb{N}} \alpha_{1,j}^{-1} \sum_{\nu=1}^{\infty} \sum_{j=n+1}^{\infty} \beta_{\nu,j}^{-1},
\]

see (C1), where the two single sums are finite by assumption. Choose \( \varepsilon > 0 \) and \( n \geq \exp(2/\varepsilon) \) such that

\[
\ln(\beta_{\nu,j}) \geq \varepsilon \cdot \ln(\nu) \cdot \ln(j)
\]

for all \( \nu,j \geq n \), see (17). For \( \nu,j \) as before we obtain

\[
\beta_{\nu,j}^{-1} = \exp(-\ln(\beta_{\nu,j})) \leq \exp(-\varepsilon \cdot \ln(\nu) \cdot \ln(j)).
\]

Hence

\[
\sum_{j=1}^{n} \beta_{i,j}^{-1} \leq \sum_{j=n+1}^{\infty} j^{-\varepsilon \cdot \ln(\nu)} \leq n^{-\varepsilon \cdot \ln(n) + 1}
\]

for every \( i \geq n \), and therefore

\[
\sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} \beta_{i,j}^{-1} \leq n \cdot \sum_{i=n+1}^{\infty} i^{-\varepsilon \cdot \ln(n)} < \infty.
\]

\[ \square \]

Lemma B.3. Let \( q_j > 0 \) for \( j \in \mathbb{N} \) and

\[
q := \liminf_{j \to \infty} \frac{q_j}{\ln(j)}.
\]

Then we have

\[
q > 1 \quad \Rightarrow \quad \sum_{j \in \mathbb{N}} \exp(-q_j) < \infty \quad \Rightarrow \quad q \geq 1.
\]
Proof. Assume that $0 \leq q < 1$. Then there exists a sequence of integers $j_m$ such that $q_{j_m} \leq \ln(j_m)$ and $j_{m+1} \geq 2j_m$. Consequently,

$$\sum_{j \in \mathbb{N}} \exp(-q_j) \geq \sum_{m=2}^{\infty} \exp(-q_{j_m}) \cdot (j_m - j_{m-1}) \geq \sum_{m=2}^{\infty} j_{m-1} \cdot (j_m - j_{m-1}) = \infty,$$

where in the last step we used that $1 - j_m/j_{m-1} > 1/2$ for all $m \geq 2$.

Now we assume that $q > 1$. Choose $\varepsilon > 0$ such that $1 + \varepsilon < q$ and $j_0 \in \mathbb{N}$ such that $q_j \geq (1 + \varepsilon) \cdot \ln(j)$ for every $j \geq j_0$. Then we get

$$\sum_{j \in \mathbb{N}} \exp(-q_j) \leq j_0 - 1 + \sum_{j=j_0}^{\infty} j^{-(1+\varepsilon)} < \infty. \quad \square$$

Example B.4. Consider the limiting case $q = 1$ in Lemma [B.3]. For $q_j = \ln(j)$ we have $q = 1$ and

$$\sum_{j \in \mathbb{N}} \exp(-q_j) = \sum_{j \in \mathbb{N}} j^{-1} = \infty.$$

For $q_j = \ln(j) + 2 \ln \ln(j)$ we have $q = 1$ as well, but

$$\sum_{j \in \mathbb{N}} \exp(-q_j) = \sum_{j \in \mathbb{N}} j^{-1} \cdot (\ln(j))^{-2} < \infty.$$

Appendix C. $L_2$-Approximation in Haar Spaces

For $n \in \mathbb{N}_0$ we put $M := \{0, \ldots, 2^n - 1\}$, and for $m \in M$ we consider the intervals

$$I_m := \left[m/2^n, (m + 1)/2^n\right], \quad m < 2^n - 1,$$

as well as

$$I_{2^n-1} := [(2^n - 1)/2^n, 1].$$

Moreover, let $T_n(f)$ be the piecewise constant interpolation of $f : [0, 1] \to \mathbb{C}$ on these intervals, based on the values of $f$ at the respective midpoints.

Theorem C.1. Assume that $H_0 = L_2([0, 1], \mu_0)$ for the uniform distribution $\mu_0$. Consider the Haar basis $(e_\nu)_{\nu \in \mathbb{N}_0}$ according to Example [B.2] and the Fourier weights according to Example [B.7]. Furthermore, assume that $H_1$ is a reproducing kernel Hilbert space, i.e., $r_1 > 1$. Then there exists a constant $C > 0$ such that

$$\sup \{\|f - T_n(f)\|_0 : f \in H_1, \|f\|_1 \leq 1\} \leq C \cdot 2^{-n \cdot r_1/2}$$

for all $n \in \mathbb{N}_0$. Furthermore,

$$\text{err}_n(H_1, S, B^{\text{std}}) \asymp \text{err}_n(H_1, S, A^{\text{all}}) \asymp n^{-r_1/2}$$

for the embedding $S : H_1 \hookrightarrow H_0$.

Proof. Fix $n \in \mathbb{N}_0$, and let $f := \sum_{\nu \in \mathbb{N}_0} a_\nu e_\nu$ with $a_\nu \in \mathbb{C}$ such that $a_\nu \neq 0$ for only finitely many $\nu \in \mathbb{N}_0$. Put

$$k(\ell, m) := 2^\ell + m2^{\ell-n} + \begin{cases} 0, & \text{if } \ell = n, \\ 2^\ell - 1, & \text{if } \ell > n, \end{cases}$$

as well as

$$c(\ell) := 2^{\ell/2} \cdot \begin{cases} -1, & \text{if } \ell = n, \\ +1, & \text{if } \ell > n, \end{cases}$$
for \( \ell \geq n \) and \( m \in M \). Observe that

\[
T_n(e_\nu) = \begin{cases} 
\ell, & \text{if } \nu \leq 2^n - 1, \\
(\ell) \cdot 1_{I_m}, & \text{if } \nu = k(\ell, m) \text{ for } \ell \geq n \text{ and } m \in M, \\
0, & \text{otherwise.}
\end{cases}
\]

From (45) we get

\[
\|f - T_n(f)\|_0 \leq \|\sum_{\nu \geq 2^n} a_\nu e_\nu\|_0 + \|\sum_{\nu \geq 2^n} a_\nu T_n(e_\nu)\|_0,
\]

and obviously

\[
\|\sum_{\nu \geq 2^n} a_\nu e_\nu\|_0^2 = \sum_{\nu \geq 2^n} |a_\nu|^2 \leq 2^{-n} \sum_{\nu \in N_0} |a_\nu|^2 \nu^{r_1}.
\]

Furthermore, (46) yields

\[
\sum_{\nu \geq 2^n} a_\nu T_n(e_\nu) = \sum_{m \in M} \sum_{\ell \geq n} a_{k(\ell, m)} T_n(e_{k(\ell, m)}) = \sum_{m \in M} \left( \sum_{\ell \geq n} a_{k(\ell, m)} (\ell) \right) \cdot 1_{I_m}.
\]

Note that \( (1_{I_m}, 1_{I_{m'}}) = 0 \) for \( m, m' \in M \) with \( m \neq m' \). Therefore

\[
\|\sum_{\nu \geq 2^n} a_\nu T_n(e_\nu)\|_0^2 = \sum_{m \in M} \left( \sum_{\ell \geq n} a_{k(\ell, m)} (\ell) \right)^2 2^{-n}.
\]

Hence the Cauchy-Schwarz inequality and the fact \( k(\ell, m) \geq 2^\ell \) yield

\[
\begin{align*}
\|\sum_{\nu \geq 2^n} a_\nu T_n(e_\nu)\|_0^2 &\leq 2^{-n} \sum_{m \in M} \left( \sum_{\ell \geq n} |a_{k(\ell, m)}|^2 |k(\ell, m)|^{r_1} \cdot \sum_{\ell \geq n} |k(\ell, m)|^{-r_1} 2^{\ell} \right) \\
&\leq 2^{-n} \sum_{\ell \geq n} 2^{(1 - r_1)} \cdot \sum_{\nu \in N_0} |a_\nu|^2 \nu^{r_1} \\
&= c^2 \cdot 2^{-n} \sum_{\nu \in N_0} |a_\nu|^2 \nu^{r_1}
\end{align*}
\]

with \( c = (1 - 2^{1 - r_1})^{-1/2} \). Combing this estimate with (46) and (47) yields

\[
\|f - T_n(f)\|_0 \leq (1 + c) \cdot 2^{-n} \left( \sum_{\nu \in N_0} |a_\nu|^2 \nu^{r_1} \right)^{1/2},
\]

which shows (44). It remains to observe that

\[
\text{err}_n(H_1, S, B^{\text{std}}) \geq \text{err}_n(H_1, S, A^{\text{all}}) \geq n^{-r_1/2}
\]

to complete the proof of (44).

\[\square\]

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REFERENCES

[1] J. Baldeaux and M. Gnewuch, Optimal randomized multilevel algorithms for infinite-dimensional integration on function spaces with ANOVA-type decomposition, SIAM J. Numer. Anal., 52 (2014), pp. 1128–1155.

[2] J. Dick and M. Gnewuch, Infinite-dimensional integration in weighted Hilbert spaces: anchored decompositions, optimal deterministic algorithms, and higher order convergence, Found. Comput. Math., 14 (2014), pp. 1027–1077.

[3] ———, Optimal randomized changing dimension algorithms for infinite-dimensional integration on function spaces with ANOVA-type decomposition, J. Approx. Theory, 184 (2014), pp. 111–145.

[4] J. Dick, F. Y. Kuo, and I. H. Sloan, High dimensional integration – the quasi-Monte Carlo way, Acta Numer., 22 (2013), pp. 133–288.

[5] J. Dick and F. Pillichshammer, Digital Nets and Sequences, Cambridge University Press, New York, 2010.

[6] ———, Discrepancy theory and quasi-Monte Carlo integration, in A panorama of discrepancy theory, W. Chen, A. Srivastav, and G. Travaglini, eds., vol. 2107 of Lecture Notes in Math., Springer, Cham, 2014, pp. 539–619.

[7] D. Düng and M. Griebel, Hyperbolic cross approximation in infinite dimensions, J. Complexity, 33 (2016), pp. 55–88.

[8] D. Düng, V. Temlyakov, and T. Ullrich, Hyperbolic Cross Approximation, Advanced Courses in Mathematics – CRM Barcelona, Birkhäuser, Basel, 2018.

[9] A. D. Gilbert, F. Y. Kuo, D. Nuyens, and G. W. Wasilkowski, Efficient implementations of the multivariate decomposition method for approximating infinite-variate integrals, SIAM J. Scient. Comput., (2018). To appear.

[10] M. Gnewuch, Lower error bounds for randomized multilevel and changing dimension algorithms, in Monte Carlo and Quasi-Monte Carlo Methods 2012, J. Dick, F. Y. Kuo, G. W. Peters, and I. H. Sloan, eds., Springer, Heidelberg, 2013, pp. 399–415.

[11] M. Gnewuch, M. Hefter, A. Hinrichs, and K. Ritter, Embeddings of weighted Hilbert spaces and applications to multivariate and infinite-dimensional integration, J. Approx. Theory, 222 (2017), pp. 8–39.

[12] M. Gnewuch, M. Hefter, A. Hinrichs, K. Ritter, and G. W. Wasilkowski, Equivalence of weighted anchored and ANOVA spaces of functions with mixed smoothness of order one in $L_p$, J. Complexity, 40 (2017), pp. 78–99.

[13] M. Gnewuch, S. Mayer, and K. Ritter, On weighted Hilbert spaces and integration of functions of infinitely many variables, J. Complexity, 30 (2014), pp. 29–47.

[14] A.-L. Haji-Ali, H. Harbrecht, M. D. Peters, and M. Siebenmorgen, Novel results for the anisotropic sparse grid quadrature, J. Complexity, 47 (2018), pp. 62–85.

[15] M. Hefter and K. Ritter, On embeddings of weighted tensor product Hilbert spaces, J. Complexity, 31 (2015), pp. 405–423.

[16] M. Hefter, K. Ritter, and G. W. Wasilkowski, On equivalence of weighted anchored and ANOVA spaces of functions with mixed smoothness of order one in $L_1$ or $L_\infty$, J. Complexity, 32 (2016), pp. 1–19.

[17] S. Heinrich, F. J. Hickernell, and R. X. Yue, Optimal quadrature for Haar wavelet spaces, Math. Comp., 73 (2004), pp. 259–277.

[18] F. J. Hickernell, T. Müller-Gronbach, B. Niu, and K. Ritter, Multi-level Monte Carlo algorithms for infinite-dimensional integration on $R^H$, J. Complexity, 26 (2010), pp. 229–254.

[19] F. J. Hickernell and X. Wang, The error bounds and tractability of quasi-Monte Carlo algorithms in infinite dimension, Math. Comp., 71 (2001), pp. 1641–1661.

[20] A. Hinrichs, P. Kritzer, F. Pillichshammer, and G. W. Wasilkowski, Truncation dimension for linear problems on multivariate function spaces, Numer. Algorithms, (2018). To appear.

[21] A. Hinrichs and J. Schneider, Equivalence of anchored and ANOVA spaces via interpolation, J. Complexity, 33 (2016), pp. 190–198.
[22] C. Irrgeher, P. Kritzer, F. Pillichshammer, and H. Woźniakowski, Tractability of multivariate approximation defined over Hilbert spaces with exponential weights, J. Approx. Theory, 207 (2016), pp. 301–338.

[23] V. K. Khristov, Convergence of certain interpolation processes in integral and discrete norms, in Constructive function theory ‘81 (Varna, 1981), Publ. House Bulgar. Acad. Sci., Sofia, 1983, pp. 185–188.

[24] P. Kritzer, F. Pillichshammer, and G. W. Wasilkowski, Very low truncation dimension for high dimensional integration under modest error demand, J. Complexity, 35 (2016), pp. 63–85.

[25] A note on equivalence of anchored and ANOVA spaces; lower bounds, J. Complexity, 38 (2017), pp. 31–38.

[26] P. Kritzer, F. Pillichshammer, and H. Woźniakowski, Tractability of multivariate analytic problems, Radon Ser. Comput. Appl. Math., 15 (2014), pp. 147–170.

[27] T. Kühn, W. Sickel, and T. Ullrich, Approximation numbers of embeddings of anisotropic Sobolev spaces of dominating mixed smoothness - preasymptotics and asymptotics, 2018. In progress.

[28] F. Y. Kuo, I. H. Sloan, G. W. Wasilkowski, and H. Woźniakowski, Liberating the dimension, J. Complexity, 26 (2010), pp. 422–454.

[29] On decompositions of multivariate functions, Math. Comp., 79 (2010), pp. 953–966.

[30] H. Lê-Oüey, Derivative Based Quasi-Monte Carlo Constructions and Sensitivity Estimations, PhD thesis, Humboldt Universität Berlin, 2015.

[31] S. Nicaise, Jacobi polynomials, weighted Sobolev spaces and approximation results of some singularities, Math. Nachr., 213 (2000), pp. 117–140.

[32] E. Novak and H. Woźniakowski, Tractability of Multivariate Problems. Vol. 1: Linear Information, vol. 6 of EMS Tracts in Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[33] Tractability of Multivariate Problems. Vol. 2: Standard Information for Functionals, vol. 12 of EMS Tracts in Mathematics, European Mathematical Society (EMS), Zürich, 2010.

[34] Tractability of Multivariate Problems. Vol. 3: Standard Information for Operators, vol. 18 of EMS Tracts in Mathematics, European Mathematical Society (EMS), Zürich, 2012.

[35] A. Papageorgiou and H. Woźniakowski, Tractability through increasing smoothness, J. Complexity, 26 (2010), pp. 409–421.

[36] L. Plaskota and G. W. Wasilkowski, Tractability of infinite-dimensional integration in the worst case and randomized settings, J. Complexity, 27 (2011), pp. 505–518.

[37] T. J. Rivlin, An Introduction to the Approximation of Functions, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1969.

[38] P. Siedlecki, Uniform weak tractability of multivariate problems with increasing smoothness, J. Complexity, 30 (2014), pp. 716–734.

[39] I. H. Sloan and H. Woźniakowski, When are quasi-Monte Carlo algorithms efficient for high dimensional integrals?, J. Complexity, 14 (1998), pp. 1–33.

[40] G. Szegő, Orthogonal Polynomials, American Mathematical Society, Providence, R.I., fourth ed., 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[41] J. von Neumann, On infinite direct products, Compositio Math., 6 (1939), pp. 1–77.

[42] J. L. Walsh, A closed set of normal orthogonal functions, Amer. J. Math., 45 (1923), pp. 5–24.

[43] G. W. Wasilkowski, Liberating the dimension for function approximation and integration, in Monte Carlo and quasi-Monte Carlo Methods 2010, L. Plaskota and H. Woźniakowski, eds., Springer-Verlag, 2012, pp. 211–231.

[44] Liberating the dimension for $L_2$-approximation, J. Complexity, 28 (2012), pp. 304–319.

[45] On tractability of linear tensor product problems for infinite-variate classes of functions, J. Complexity, 29 (2013), pp. 351–369.
[49] G. W. Wasilkowski and H. Woźniakowski, Liberating the dimension for function approximation: standard information, J. Complexity, 27 (2011), pp. 417–440.

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