MANIFOLDS WITH $4\frac{1}{2}$-POSITIVE CURVATURE OPERATOR OF THE SECOND KIND

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Dedicated to Professor Peter Li on the occasion of his 70th birthday.

Abstract. We show that a closed four-manifold with $4\frac{1}{2}$-positive curvature operator of the second kind is diffeomorphic to a spherical space form. The curvature assumption is sharp as both $\mathbb{CP}^2$ and $S^3 \times S^1$ have $4\frac{1}{2}$-nonnegative curvature operator of the second kind. In higher dimensions $n \geq 5$, we show that closed manifolds with $4\frac{1}{2}$-positive curvature operator of the second kind are homeomorphic to spherical space forms. These results are proved by showing that $4\frac{1}{2}$-positive curvature operator of the second kind implies both positive isotropic curvature and positive Ricci curvature.

1. Introduction

In 1986, Nishikawa [Nis86] conjectured that a closed Riemannian manifold with positive (respectively, nonnegative) curvature operator of the second kind is diffeomorphic to a spherical space form (respectively, Riemannian locally symmetric space). Recall that the curvature tensor acts on the space of symmetric two-tensors $S^2(T_pM)$ via

$$\bar{R}(e_i \otimes e_j) = \sum_{k,l} R_{iklj} e_k \otimes e_l,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space $T_pM$ at $p$ and $\otimes$ denotes the symmetric product. In Nishikawa’s conjecture, curvature operator of the second kind, denoted by $\bar{R}$ in this paper, refers to the bilinear form

$$\bar{R} : S^2_0(T_pM) \times S^2_0(T_pM) \to \mathbb{R}$$

obtained by restricting the symmetric linear map $\bar{R} : S^2(T_pM) \to S^2(T_pM)$ defined in [L1] to the space of traceless symmetric two-tensors $S^2_0(T_pM)$. We refer the reader to Section 2 for a detailed discussion on curvature operator of the second kind.

Recently, the positive case of Nishikawa’s conjecture was proved by Cao, Gursky and Tran [CGT21] and the nonnegative case was settled by the author [Li21]. In fact, the assumption can be weakened.

Theorem 1.1. Let $(M^n, g)$ be a closed Riemannian manifold of dimension $n \geq 3$. 

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(1) If \( M \) has three-positive curvature operator of the second kind, then \( M \) is diffeomorphic to a spherical space form; 

(2) If \( M \) has three-nonnegative curvature operator of the second kind, then \( M \) is either flat, or diffeomorphic to a spherical space form, or isometric to a quotient of a compact irreducible symmetric space.

The key observation is that if the curvature operator of the second kind of \( M \) is three-positive (respectively, three-nonnegative), then \( M \times \mathbb{R} \) has positive (respectively, nonnegative) isotropic curvature. This was first proved in [C GT21] under the stronger assumption of two-positivity/two-nonnegativity of curvature operator of the second kind and was then weakened to three-positivity/three-nonnegativity in [Li21]. Part (1) of Theorem 1.1 then follows from a result of Brendle [Bre08], which asserts that the normalized Ricci flow evolves an initial metric satisfying the property that \( M \times \mathbb{R} \) has positive isotropic curvature into a limit metric with constant sectional curvature, thus implying that \( M \) must be diffeomorphic to a spherical space form. The key observation in proving part (2) of Theorem 1.1 is that an \( n \)-manifold with \( n \)-nonnegative curvature operator must be either flat or locally irreducible (see [Li21, Theorem 1.8]). This allows one to invoke the classification of simply-connected locally irreducible manifolds satisfying the property that \( M \times \mathbb{R} \) has nonnegative isotropic curvature (see for example [Bre10a] or [Bre10b]). The last step is to rule out Kähler manifolds by showing that a Kähler manifold with four-nonnegative curvature operator of the second kind must be flat (see [Li21, Theorem 1.9]).

Another important result obtained by Cao, Gursky and Tran in [CGT21] states that

**Theorem 1.2.** A closed simply-connected Riemannian manifold of dimension \( n \geq 4 \) with four-positive curvature operator of the second kind is homeomorphic to the \( n \)-sphere.

The proof of Theorem 1.2 relies on the observation that four-positive curvature operator of the second kind implies positive isotropic curvature. Theorem 1.2 then follows immediately from the work of Micallef and Moore [MM88]. The curvature assumption in Theorem 1.2 is sharp in the sense that it cannot weakened to five-positive curvature operator of the second kind in general. Indeed, both the complex projective space \( \mathbb{C}P^2 \) and the cylinder \( S^3 \times S^1 \) has five-positive curvature operator of the second kind.

In this paper, we prove a sharper result by weakening the assumption in Theorem 1.2 to \( 4 \frac{1}{2} \)-positive curvature operator of the second kind, whose meaning we explain below (see also Definitions 2.1 and 2.2).

Let \( R \) be an algebraic curvature operator on a Euclidean vector space of dimension \( n \). Denote by \( S^2_0(V) \) the space of traceless symmetric two-tensors on \( V \) and \( N = \text{dim}(S^2_0(V)) = \frac{(n-1)(n+2)}{2} \). Let \( \lambda_1 \leq \cdots \leq \lambda_N \) be the eigenvalues of the curvature operator of the second kind \( R \). We say \( R \) has \( \frac{1}{2} \)-positive curvature operator of the second kind if

\[
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \frac{1}{2} \lambda_5 > 0.
\]
Clearly, $\frac{4}{12}$-positive curvature operator of the second kind is weaker than four-positive curvature operator of the second kind but stronger than five-positive curvature operator of the second kind. More generally, for $1 \leq k \leq N$ and $0 \leq \alpha \leq 1$ satisfying $k + \alpha \leq N$, we say $R$ has $(k + \alpha)$-positive (respectively, $(k + \alpha)$-nonnegative) curvature operator of the second kind if

$$
\lambda_1 + \cdots + \lambda_k + \alpha \lambda_{k+1} > (\text{respectively,} \geq) 0.
$$

The main result of this paper states the following.

**Theorem 1.3.** A closed Riemannian manifold of dimension $n \geq 4$ with $\frac{4}{12}$-positive curvature operator of the second kind is homeomorphic to spherical space form. Moreover, if either $n = 4$ or $n \geq 12$, then the “homeomorphism” can be upgraded to “diffeomorphism”.

We would like to point out that the curvature assumption in Theorem 1.3 in general cannot be weakened to $(4 + \alpha)$-positive curvature operator of the second kind for any $\alpha > \frac{1}{2}$. For the complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric, the eigenvalues of $\tilde{R}$ are given by $\{-2, -2, 4, 4, 4, 4, 4, 4\}$, up to scaling (see [BK78]). For $S^3 \times S^1$ with the product metric, the eigenvalues of $R$ are given by $\{-\frac{1}{2}, 0, 0, 1, 1, 1, 1, 1\}$, up to scaling (see [Li21]). Therefore, both $\mathbb{CP}^2$ and $S^3 \times S^1$ have $(4 + \alpha)$-positive curvature operator of the second kind for any $\alpha \in (\frac{1}{2}, 1]$ but not for any $\alpha \in [0, \frac{1}{2}]$.

We also prove a rigidity result.

**Theorem 1.4.** Let $(M^n, g)$ be a closed non-flat Riemannian manifold of dimension $n \geq 4$. Suppose that $M$ has $\frac{4}{12}$-nonnegative curvature operator of the second kind. Then one of the following statements holds:

1. $M$ is homeomorphic (diffeomorphic if $n = 4$ or $n \geq 12$) to a spherical space form;
2. $n = 2m$ and the universal cover of $M$ is a Kähler manifold biholomorphic to $\mathbb{CP}^m$;
3. $n = 4$ and the universal cover of $M$ is diffeomorphic to $S^3 \times \mathbb{R}$;
4. $n \geq 5$ and $M$ is isometric to a quotient of a compact irreducible symmetric space.

The key ingredients in proving Theorem 1.3 and Theorem 1.4 are the following two results.

**Theorem 1.5.** Let $R$ be an algebraic curvature operator on a Euclidean vector space $V$ of dimension $n \geq 4$. If $R$ has $\frac{4}{12}$-positive (respectively, $\frac{4}{12}$-nonnegative) curvature operator of the second kind, then $R$ has positive (respectively, nonnegative) isotropic curvature.

**Theorem 1.6.** Let $R$ be an algebraic curvature operator on a Euclidean vector space $V$ of dimension $n \geq 3$. If $R$ has $(n + \frac{n-2}{n})$-positive (respectively, $(n + \frac{n-2}{n})$-nonnegative) curvature operator of the second kind, then $R$ has positive (respectively, nonnegative) Ricci curvature.

In dimension four, the assumption in Theorem 1.5 cannot be weakened to $(4 + \alpha)$-positive (respectively, $(4 + \alpha)$-nonnegative) curvature operator of the second kind.
for any $\alpha > \frac{1}{2}$ in view of the fact that $\mathbb{CP}^2$ has $4\frac{1}{2}$-nonnegative (but not $4\frac{1}{2}$-positive) curvature operator of the second kind and nonnegative (but not positive) isotropic curvature. However, it is unclear in higher dimensions.

The assumption in Theorem 1.6 is sharp in all dimensions, as $\mathbb{S}^{n-1} \times S^1$ has $(n + \frac{n-2}{n})$-nonnegative (but not $(n + \frac{n-2}{n})$-positive) curvature operator of the second kind (see [Li21, Example 2.5]).

In dimension three, Theorem 1.6 says that $3\frac{1}{3}$-positive (respectively, $3\frac{1}{3}$-nonnegative) curvature operator of the second kind implies positive (respectively, nonnegative) Ricci curvature. Using Hamilton’s classification of three-manifolds with positive or nonnegative Ricci curvature in [Ham82, Ham86], we have the following theorem for three-manifolds, which improves Theorem 1.1.

**Theorem 1.7.** Let $(M^3, g)$ be a three-dimensional closed Riemannian manifold. If $M$ has $3\frac{1}{3}$-positive curvature operator of the second kind, then $M$ is diffeomorphic to $S^3$ or its quotient. If $M$ has $3\frac{1}{3}$-nonnegative curvature operator of the second kind, then $M$ is diffeomorphic to a quotient of one of the spaces $S^3$ or $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by a group of fixed point free isometries in the standard metrics.

The referee kindly pointed out that, by appealing to the result of Liu [Liu13] on the classification of complete noncompact three-manifolds with nonnegative Ricci curvature, we have the following theorem.

**Theorem 1.8.** Let $(M^3, g)$ be a complete noncompact three-manifold with $3\frac{1}{3}$-nonnegative curvature operator of the second kind. Then either $M$ is diffeomorphic to $\mathbb{R}^3$ or the universal cover of $M$ is isometric to a Riemann product $N^2 \times \mathbb{R}$ where $N^2$ is a complete 2-manifold with nonnegative sectional curvature.

It was shown by Micallef and Wang [MW93] for $n = 4$ and Brendle [Bre10a] for $n \geq 5$ that Einstein manifolds with positive isotropic curvature have constant sectional curvature and closed Einstein manifolds with nonnegative isotropic curvature are locally symmetric. Thus Theorem 1.5 implies the following theorem for Einstein manifolds.

**Theorem 1.9.** Let $(M^n, g)$ be an Einstein manifold of dimension $n \geq 4$. If $M$ has $4\frac{1}{2}$-positive curvature operator of the second kind, then $M$ has constant sectional curvature. If $M$ is closed and has $4\frac{1}{2}$-nonnegative curvature operator of the second kind, then $M$ is locally symmetric.

Finally, we propose the following conjecture.

**Conjecture 1.10.** A closed $n$-dimensional Riemannian manifold with $(n + \frac{n-2}{n})$-positive curvature operator of the second kind is diffeomorphic to a spherical space form.

When $n = 2$, it is easy to see that two-positive curvature operator of the second kind is equivalent to positive scalar curvature. So Conjecture 1.10 holds in dimension two by the uniformization theorem. What we prove in this paper is that Conjecture 1.10 holds in three and four dimensions in Theorem 1.7 and Theorem 1.3 respectively. Note that the assumption cannot be weakened to $(n+\alpha)$-positive curvature operator of the second kind for any $\alpha > \frac{n-2}{n}$ as the product manifold $\mathbb{S}^{n-1} \times S^1$ has $(n + \alpha)$-positive curvature operator of the second kind for any $\alpha > \frac{n-2}{n}$ (see [Li21, Example 2.5]).
Finally, we would like to point out that Nienhaus, Petersen and Wink have shown that compact, $n$-dimensional Riemannian manifolds with $\left\lfloor \frac{n+2}{2} \right\rfloor$-nonnegative curvature operators of the second kind are either rational homology spheres or flat.

The paper is organized as follows. In Section 2, we give an introduction to curvature operator of the second kind. In Section 3, we prove Theorem 1.5. In Section 4, we prove Theorem 1.6. The proofs of Theorem 1.3 and Theorem 1.4 are presented in Section 5.

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## 2. Curvature operator of the second kind

The purpose of this section is to give an introduction to curvature operator of the second kind, as it is significantly less investigated than curvature operator (of the first kind) in the literature. Along the way, we also fix our notations, state our conventions and make some definitions. Most of this section is extracted from [Li21]. The reader is also referred to [OT79, BK78, Nis86, Kas93, CGT21] for more information as well as previous results concerning curvature operator of the second kind.

Let $(V, g)$ be a Euclidean vector space of dimension $n \geq 2$. We always identify $V$ with its dual space $V^*$ via the metric $g$. The space of bilinear forms on $V$ is denoted by $T^2(V)$, and it splits as

$$T^2(V) = S^2(V) \oplus \Lambda^2(V),$$

where $S^2(V)$ is the space of symmetric two-tensors on $V$ and $\Lambda^2(V)$ is the space of two-forms on $V$. Our conventions on symmetric products and wedge products are that, for $u$ and $v$ in $V$,

- $\odot$ denotes the symmetric product defined by $u \odot v = u \otimes v + v \otimes u$, and
- $\wedge$ denotes the wedge product defined by $u \wedge v = u \otimes v - v \otimes u$.

The inner product $g$ on $V$ naturally induces inner products on $S^2(V)$ and $\Lambda^2(V)$. To be consistent with [CGT21], the inner product on $S^2(V)$ is defined as

$$\langle u, v \rangle = \text{tr}(u^T v),$$

and the inner product on $\Lambda^2(V)$ is defined as

$$\langle u, v \rangle = \frac{1}{2} \text{tr}(u^T v).$$

In particular, if $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $V$, then $\{e_i \wedge e_j\}_{1 \leq i \leq j \leq n}$ is an orthonormal basis for $\Lambda^2(V)$ and $\left\{ \frac{1}{\sqrt{2}} e_i \odot e_j \right\}_{1 \leq i \leq j \leq n} \cup \left\{ \frac{1}{2} e_i \odot e_i \right\}_{1 \leq i \leq n}$ is an orthonormal basis for $S^2(V)$.

The space of symmetric two-tensors on $\Lambda^2(V)$ has the orthogonal decomposition

$$S^2(\Lambda^2(V)) = S^2_B(\Lambda^2(V)) \oplus \Lambda^4(V),$$
where $S_B^2(\Lambda^2(V))$ consists of all tensors $R \in S^2(\Lambda^2(V))$ that also satisfy the first Bianchi identity. Any $R \in S_B^2(\Lambda^2(V))$ is called an algebraic curvature operator.

By the symmetries of $R \in S_B^2(\Lambda^2(V))$ (not including the first Bianchi identity), there are (up to sign) two ways that $R$ can induce a symmetric linear map $\hat{R} : T^2(V) \to T^2(V)$. The first one, denoted by $\hat{R} : \Lambda^2(V) \to \Lambda^2(V)$ in this paper, is the so-called curvature operator defined by

$$\hat{R}(e_i \wedge e_j) = \frac{1}{2} \sum_{k,l} R_{ijkl} e_k \wedge e_l$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $V$. Note that if the eigenvalues of $\hat{R}$ are all greater than or equal to $\kappa \in \mathbb{R}$, then all the sectional curvatures of $R$ are bounded from below by $\kappa$.

The second one, denoted by $\check{R} : S^2(V) \to S^2(V)$, is defined by

$$\check{R}(e_i \odot e_j) = \sum_{k,l} R_{ijkl} e_k \odot e_l.$$ 

However, on contrary to the case of $\hat{R}$, all eigenvalues of $\check{R}$ being nonnegative implies all the sectional curvatures of $R$ are zero, that is, $R \equiv 0$. The new feature here is that $S^2(V)$ is not irreducible under the action of the orthogonal group $O(V)$ of $V$. The space $S^2(V)$ splits into $O(V)$-irreducible subspaces as

$$S^2(V) = S_0^2(V) \oplus \mathbb{R} g,$$

where $S_0^2(V)$ denotes the space of traceless symmetric two-tensors on $V$. The map $\check{R}$ defined in (2.2) then induces a bilinear form

$$\check{R} : S_0^2(V) \times S_0^2(V) \to \mathbb{R}$$

by restriction to $S_0^2(V)$. Note that if all the eigenvalues of $\check{R}$ restricted to $S_0^2(V)$ are bounded from by $\kappa \in \mathbb{R}$, then the sectional curvatures of $R$ are bounded from below by $\kappa$. It should be noted that $\check{R}$ does not preserve the subspace $S_0^2(V)$ in general, but it does, for instance, when $R$ has constant Ricci curvature.

Following [Nis80], we call $\check{R}$ in (2.2) (restricted to $S_0^2(V)$) the curvature operator of the second kind, to distinguish it from the map $\hat{R}$ defined in (2.1), which he called the curvature operator of the first kind.

We make the following definitions.

**Definition 2.1.** Let $R \in S_B^2(\Lambda^2(V))$ be an algebraic curvature operator. Denote by $\lambda_1 \leq \cdots \leq \lambda_N$ the eigenvalues of $\hat{R}$ (restricted to $S_0^2(V)$), where $N = \dim(S_0^2(V)) = \frac{(n-1)(n+2)}{2}$. For $1 \leq k \leq N$ and $0 \leq \alpha \leq 1$ satisfying $k + \alpha \leq N$, we say $R$ has $(k + \alpha)$-nonnegative curvature operator of the second kind if

$$\lambda_1 + \cdots + \lambda_k + \alpha \lambda_{k+1} \geq 0.$$ 

If the above inequality is strict, $R$ is said to have $(k + \alpha)$-positive curvature operator of the second kind.

**Definition 2.2.** Let $(M^n, g)$ be a Riemannian manifold of dimension $n$. For $1 \leq k \leq N$ and $0 \leq \alpha \leq 1$ satisfying $k + \alpha \leq N$, we say $M$ has $(k + \alpha)$-positive
(respectively, \((k + \alpha)\)-nonnegative) curvature operator of the second kind if the Riemannian curvature tensor at each point \(p \in M\) has \((k + \alpha)\)-positive (respectively, \((k + \alpha)\)-nonnegative) curvature of the second kind.

3. Positive Isotropic Curvature

The notion of positive isotropic curvature was introduced by Micallef and Moore [MM88] in their study of minimal two-spheres in Riemannian manifolds. They proved that a simply-connected closed Riemannian manifold with positive isotropic curvature is homeomorphic to the sphere. We recall the following definition for the reader’s convenience.

**Definition 3.1.** Let \(R \in S^2_B(\Lambda^2 V)\) be an algebraic curvature operator on a Euclidean vector space \(V\) of dimension \(n \geq 4\). We say \(R\) has nonnegative isotropic curvature if for all orthonormal four-frames \(\{e_1, e_2, e_3, e_4\} \subset V\), it holds that

\[
R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0.
\]

If the inequality is strict, \(R\) is said to have positive isotropic curvature.

In this section, we prove Theorem 1.5, which is restated below.

**Theorem 3.2.** Suppose \(n = \dim(V) \geq 4\) and \(R \in S^2_B(\Lambda^2(V))\) has \(4 \frac{1}{2}\)-positive (respectively, \(4 \frac{1}{2}\)-nonnegative) curvature operator of the second kind. Then for any orthonormal four-frame \(\{e_1, \cdots, e_4\} \subset V\), we have

\[
(3.1) \quad R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > (\text{respectively,} \geq) 0.
\]

**Proof.** Let \(\{e_1, e_2, e_3, e_4\}\) be an orthonormal four-frame in \(V\). We define the following symmetric two-tensors on \(V\):

\[
\begin{align*}
\varphi_1 &= \frac{1}{2}(e_1 \circ e_1 + e_2 \circ e_2 - e_3 \circ e_3 - e_4 \circ e_4), \\
\varphi_2 &= \frac{1}{2}(e_1 \circ e_1 - e_2 \circ e_2 + e_3 \circ e_3 - e_4 \circ e_4), \\
\varphi_3 &= \frac{1}{2}(e_1 \circ e_1 - e_2 \circ e_2 - e_3 \circ e_3 + e_4 \circ e_4), \\
\varphi_4 &= e_1 \circ e_4 + e_2 \circ e_3, \\
\varphi_5 &= e_1 \circ e_4 - e_2 \circ e_3, \\
\varphi_6 &= e_1 \circ e_3 + e_2 \circ e_4, \\
\varphi_7 &= e_1 \circ e_3 - e_2 \circ e_4, \\
\varphi_8 &= e_1 \circ e_2 + e_3 \circ e_4, \\
\varphi_9 &= e_1 \circ e_2 - e_3 \circ e_4.
\end{align*}
\]

It is easy to verify that these tensors are traceless, mutually orthogonal in \(S^2_0(V)\) and of the same magnitude 2.
If $R$ has $4\frac{1}{2}$-nonnegative curvature operator of the second kind, we get
\[
\hat{R}(\varphi_1, \varphi_1) + \hat{R}(\varphi_5, \varphi_5) + \hat{R}(\varphi_6, \varphi_6) + \hat{R}(\varphi_2, \varphi_2) + \frac{1}{2}\hat{R}(\varphi_3, \varphi_3) \geq 0,
\]
\[
\hat{R}(\varphi_1, \varphi_1) + \hat{R}(\varphi_5, \varphi_5) + \hat{R}(\varphi_6, \varphi_6) + \hat{R}(\varphi_3, \varphi_3) + \frac{1}{2}\hat{R}(\varphi_2, \varphi_2) \geq 0,
\]
\[
\hat{R}(\varphi_1, \varphi_1) + \hat{R}(\varphi_5, \varphi_5) + \hat{R}(\varphi_6, \varphi_6) + \hat{R}(\varphi_4, \varphi_4) + \frac{1}{2}\hat{R}(\varphi_7, \varphi_7) \geq 0,
\]
\[
\hat{R}(\varphi_1, \varphi_1) + \hat{R}(\varphi_5, \varphi_5) + \hat{R}(\varphi_6, \varphi_6) + \hat{R}(\varphi_4, \varphi_4) + \frac{1}{2}\hat{R}(\varphi_7, \varphi_7) \geq 0,
\]
\[
\hat{R}(\varphi_1, \varphi_1) + \hat{R}(\varphi_5, \varphi_5) + \hat{R}(\varphi_6, \varphi_6) + \hat{R}(\varphi_8, \varphi_8) + \frac{1}{2}\hat{R}(\varphi_9, \varphi_9) \geq 0,
\]
\[
\hat{R}(\varphi_1, \varphi_1) + \hat{R}(\varphi_5, \varphi_5) + \hat{R}(\varphi_6, \varphi_6) + \hat{R}(\varphi_9, \varphi_9) + \frac{1}{2}\hat{R}(\varphi_8, \varphi_8) \geq 0.
\]
Adding the above six inequalities together yields
\[
6 \left( \hat{R}(\varphi_1, \varphi_1) + \hat{R}(\varphi_5, \varphi_5) + \hat{R}(\varphi_6, \varphi_6) \right) + \frac{3}{2} \left( \hat{R}(\varphi_2, \varphi_2) + \hat{R}(\varphi_3, \varphi_3) \right) + \frac{3}{2} \left( \hat{R}(\varphi_4, \varphi_4) + \hat{R}(\varphi_7, \varphi_7) \right) + \frac{3}{2} \left( \hat{R}(\varphi_8, \varphi_8) + \hat{R}(\varphi_9, \varphi_9) \right) \geq 0.
\]
(3.2)

On the other hand, direct calculation shows
\[
\hat{R}(\varphi_1, \varphi_1) = 2(-R_{1212} - R_{3434} + R_{1313} + R_{2424} + R_{1414} + R_{2323}),
\]
\[
\hat{R}(\varphi_2, \varphi_2) = 2(-R_{1313} - R_{2424} + R_{1212} + R_{3434} + R_{1414} + R_{2323}),
\]
\[
\hat{R}(\varphi_3, \varphi_3) = 2(-R_{1414} - R_{2323} + R_{1212} + R_{3434} + R_{1313} + R_{2424}),
\]
and
\[
\hat{R}(\varphi_4, \varphi_4) = 2(R_{1414} + R_{2323} + 2R_{1234} - 2R_{1342}),
\]
\[
\hat{R}(\varphi_5, \varphi_5) = 2(R_{1414} + R_{2323} - 2R_{1234} + 2R_{1342}),
\]
\[
\hat{R}(\varphi_6, \varphi_6) = 2(R_{1313} + R_{2424} - 2R_{1234} + 2R_{1423}),
\]
\[
\hat{R}(\varphi_7, \varphi_7) = 2(R_{1313} + R_{2424} + 2R_{1234} - 2R_{1423}),
\]
\[
\hat{R}(\varphi_8, \varphi_8) = 2(R_{1212} + R_{3434} + 2R_{1423} - 2R_{1342}),
\]
\[
\hat{R}(\varphi_9, \varphi_9) = 2(R_{1212} + R_{3434} - 2R_{1423} + 2R_{1342}).
\]

Therefore, we obtain
\[
\hat{R}(\varphi_1, \varphi_1) + \hat{R}(\varphi_5, \varphi_5) + \hat{R}(\varphi_6, \varphi_6) = 4(R_{1313} + R_{1414} + R_{2323} + R_{2424}) - 2(R_{1212} + R_{3434}) - 12R_{1234},
\]
and
\[
\hat{R}(\varphi_2, \varphi_2) + \hat{R}(\varphi_3, \varphi_3) = 4(R_{1212} + R_{3434}),
\]
\[
\hat{R}(\varphi_4, \varphi_4) + \hat{R}(\varphi_7, \varphi_7) = 2(R_{1313} + R_{1414} + R_{2323} + R_{2424}) + 12R_{1234}.
\]
Substituting the above four identities into (3.2) produces
\[
27(R_{1313} + R_{1414} + R_{2323} + R_{2424}) - 54R_{1234} \geq 0.
\]
Hence $R$ has nonnegative isotropic curvature. Similarly, if $R$ has $4\frac{1}{2}$-positive curvature operator of the second kind, then all the above inequalities become strict and we conclude that $R$ has positive isotropic curvature.
Remark 3.3. On the complex projective space $\mathbb{CP}^2$ with its Fubini-Study metric satisfying $\text{Ric} = 3g$, all the inequalities in the above proof become identities if the orthonormal frame is suitably chosen. More precisely, if we choose the orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that the two-plane spanned by $e_1$ and $e_2$ maximizes the sectional curvature among all two-planes, then all the nonzero components of curvature tensors are given by

$$
R_{1212} = R_{3434} = 2, \\
R_{1313} = R_{2424} = 1/2, \\
R_{1414} = R_{2323} = 1/2, \\
R_{1234} = R_{1423} = 1/2, \\
R_{1342} = -1,
$$

and their variants by symmetry. Therefore, we have that $\check{R}(\varphi_i, \varphi_i) = -4$ for $i = 1, 5, 6$ and $\check{R}(\varphi_i, \varphi_i) = 8$ for $i = 2, 3, 4, 7, 8, 9$.

We are ready to prove Theorem 1.9.

Proof of Theorem 1.9. As explained in the Introduction, it follows from Theorem 1.5 and the result of Micallef and Wang [MW93] when $n = 4$ and that of Brendle [Bre10a] when $n \geq 5$. □

4. Positive Ricci Curvature

In this section, we prove Theorem 1.6.

Proof of Theorem 1.6. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $V$. We define the following symmetric two-tensors on $V$:

$$
\varphi_1 = \frac{1}{2\sqrt{n(n-1)}} \left( (n-1)e_1 \otimes e_1 - \sum_{p=2}^{n} e_p \otimes e_p \right), \\
\varphi_i = \frac{1}{\sqrt{2}} e_1 \otimes e_i \text{ for } 2 \leq i \leq n, \\
\psi_{kl} = \frac{1}{\sqrt{2}} e_k \otimes e_l \text{ for } 2 \leq k < l \leq n, \\
\xi_j = \frac{1}{2\sqrt{j(j-1)}} \left( \sum_{p=2}^{j} e_p \otimes e_p - (j-1)e_{j+1} \otimes e_{j+1} \right), \text{ for } 2 \leq j \leq n-1.
$$

One easily verifies that these tensors form an orthonormal basis of $S^2_0(V)$.

If $\check{R}$ has $(n + \frac{n-2}{2})$-nonnegative curvature operator of the second kind, then we get that for $2 \leq k < l \leq n$,

$$
\sum_{i=1}^{n} \check{R}(\varphi_i, \varphi_i) + \frac{n-2}{n} \check{R}(\psi_{kl}, \psi_{kl}) \geq 0,
$$
and for $2 \leq j \leq n$, 
\[
\sum_{i=1}^{n} \hat{R}(\varphi_i, \varphi_i) + \frac{n-2}{n} \hat{R}(\xi_j, \xi_j) \geq 0.
\]
Adding the above $\frac{(n-2)(n+1)}{2}$-many inequalities together yields 
\[
\frac{(n-2)(n+1)}{2} \sum_{i=1}^{n} \hat{R}(\varphi_i, \varphi_i) + \frac{n-2}{n} \sum_{2 \leq k < l \leq n} \hat{R}(\psi_{kl}, \psi_{kl}) 
\]
\[(4.1)\]
\[+ \frac{n-2}{n} \sum_{j=2}^{n-1} \hat{R}(\xi_j, \xi_j) \geq 0.
\]

On the other hand, direct calculation using $(\varphi_1)_{11} = \sqrt{\frac{n-1}{n}}$ and $(\varphi_1)_{jj} = -\frac{1}{\sqrt{n(n-1)}}$ for $2 \leq j \leq n$ shows that 
\[
\hat{R}(\varphi_1, \varphi_1) = \sum_{i,j=1}^{n} R_{ijjj}(\varphi_1)_{ii}(\varphi_1)_{jj} 
\]
\[= \frac{2}{n} \sum_{j=2}^{n} R_{1j1j} - \frac{1}{n(n-1)} \sum_{i,j=2}^{n} R_{ijjj} 
\]
\[= \frac{2}{n} R_{11} - \frac{1}{n(n-1)} (S - 2R_{11}) 
\]
\[(4.2)\]
where $R_{11} = \text{Ric}(e_1, e_1)$ and $S$ denotes the scalar curvature. We also calculate that 
\[
\sum_{i=2}^{n} \hat{R}(\varphi_i, \varphi_i) = \sum_{i=2}^{n} R_{ii1} = R_{11} 
\]
\[(4.3)\]
and 
\[
\sum_{2 \leq k < l \leq n} \hat{R}(\psi_{kl}, \psi_{kl}) = \sum_{2 \leq k < l \leq n} R_{klkl} = \frac{S}{2} - R_{11}. 
\]
\[(4.4)\]
Finally, we compute that 
\[
\sum_{j=2}^{n-1} \hat{R}(\xi_j, \xi_j) = \sum_{j=2}^{n-1} \sum_{p,q=1}^{n} R_{pqqp}(\xi_j)_{pp}(\xi_j)_{qq} 
\]
\[= \sum_{j=2}^{n-1} \left( \sum_{j=2}^{j} \frac{j}{j(j-1)} \sum_{p,q=2}^{j} R_{ppqq} \right) 
\]
\[= \frac{2}{n-1} \sum_{2 \leq p < q \leq n} R_{pqpq} 
\]
\[(4.5)\]
\[= \frac{1}{n-1} (S - 2R_{11}).
\]
Plugging the above four identities (4.2), (4.3), (4.4) and (4.5) into (4.1), we obtain that

\[
0 \leq \frac{(n-2)(n+1)}{2} \left( \frac{2}{n-1} R_{11} - \frac{1}{n(n-1)} S + R_{11} \right) + \frac{n-2}{n} \left( \frac{S}{2} - R_{11} \right) + \frac{n-2}{n(n-1)} (S - 2R_{11})
\]

\[
= \frac{(n-2)(n+1)(n+2)}{2n} R_{11}.
\]

Hence \( R \) has nonnegative Ricci curvature. Similarly, if \( R \) has \( (n+\frac{n-2}{n}) \)-positive curvature operator of the second kind, then all the above inequalities become strict and we get that \( R \) has positive Ricci curvature. \( \square \)

**Remark 4.1.** On the product manifold \( S^{n-1} \times S^1 \), all the inequalities in the above proof become identities if the orthonormal frame is chosen such that \( e_1 \in T_p S^1 \) and \( e_2, \ldots, e_n \in T_p S^{n-1} \).

We now give the proofs of Theorem 1.7 and Theorem 1.8.

**Proof of Theorem 1.7.** As explained in the Introduction, it follows from Theorem 1.6 and Hamilton’s classification of closed three-manifolds with positive or nonnegative Ricci curvature in [Ham82, Ham86]. \( \square \)

**Proof of Theorem 1.8.** It follows immediately by combining Theorem 1.6 and Liu’s classification [Liu13] of complete noncompact three-manifolds with nonnegative Ricci curvature. \( \square \)

5. Proofs

We present the proofs of Theorem 1.3 and Theorem 1.4 in this section.

**Proof of Theorem 1.3.** By Theorem 1.6, \( M \) has positive Ricci curvature. Thus the fundamental group of \( M \) is finite and \( \tilde{M} \), the universal cover of \( M \), is also a closed manifold. Since \( M \) also has \( 4\frac{1}{2} \)-positive curvature operator of the second kind, it has positive isotropic curvature by Theorem 1.5. By the work of Micallef and Moore [MM88], \( \tilde{M} \) is homeomorphic to \( S^n \).

On the other hand, since \( \tilde{M} \) has trivial fundamental group, it does not contain nontrivial incompressible \( (n-1) \)-dimensional space forms. Therefore, we can use the work of Hamilton [Ham97] if \( n = 4 \) and that of Brendle [Bre19] if \( n \geq 12 \) to conclude that \( \tilde{M} \) must be diffeomorphic to \( S^n \).

**Proof of Theorem 1.4.** By Theorem 1.6, \( M \) has nonnegative Ricci curvature. Let \( \tilde{M} \) be the universal cover of \( M \). By the Cheeger-Gromoll splitting theorem, \( \tilde{M} \) is isometric to a product of the form \( N^{n-k} \times \mathbb{R}^k \), where \( N \) has positive Ricci curvature and is compact.
If \( n \geq 5 \), we can use Theorem 1.8 in [Li21] to conclude that \( M \) is locally irreducible, which implies that \( k = 0 \). Hence \( \tilde{M} \) is compact. Since \( \tilde{M} \) has \( 4\frac{1}{2} \)-nonnegative curvature operator of the second kind, it also has nonnegative isotropic curvature by Theorem 1.5. One can then invoke the classification of simply-connected locally irreducible closed Riemannian manifolds with nonnegative isotropic curvature (see [Bre10b, Theorem 9.30] and [Ses09]) to conclude that one of the following statement holds:

1. \( \tilde{M} \) is homeomorphic to \( S^n \);
2. \( n = 2m \) and \( \tilde{M} \) is a Kähler manifold biholomorphic to \( \mathbb{CP}^m \);
3. \( n \geq 5 \) and \( \tilde{M} \) is isometric to a compact irreducible symmetric space.

Similar as in the proof of Theorem 1.3, the “homeomorphism” case (1) can be upgraded to a “diffeomorphism” if \( n \geq 12 \) in view of Brendle’s work [Bre19].

Finally, we treat the \( n = 4 \) case by dividing it into two subcases.

**Case A:** \( M \) is locally irreducible. In this case, the above argument remains valid and we conclude that \( \tilde{M} \) is either diffeomorphic to \( S^4 \) using Hamilton’s work [Ham97] or \( M \) is a Kähler manifold biholomorphic to \( \mathbb{CP}^2 \) using the classification of closed four-manifolds with nonnegative isotropic curvature (see [MW93, Theorem 4.10]).

**Case B:** \( M \) is locally reducible. We observe that in this case \( \tilde{M} \) must be isometric to \( N^3 \times \mathbb{R} \), as it cannot split as the product of two manifolds of dimension two by Theorem 5.1 in [Li21]. Now we have that \( N \) is a closed and simply-connected three-manifold which is locally irreducible. Since \( N \times \mathbb{R} \) has nonnegative isotropic curvature, we conclude that \( N \) must have nonnegative Ricci curvature. Thus \( N \) is diffeomorphic to \( S^3 \) by Hamilton’s classification of closed three-manifolds with nonnegative Ricci curvature [Ham80]. Hence \( \tilde{M} \) is diffeomorphic to \( S^3 \times \mathbb{R} \). This finishes the proof.

\[\square\]

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