Target Space Duality as a Symmetry of String Field Theory

Taichiro Kugo

Department of Physics
Kyoto University, Kyoto 606, Japan

and

Barton Zwiebach *

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606, Japan

ABSTRACT

Toroidal backgrounds for bosonic strings are used to understand target space duality as a symmetry of string field theory and to study explicitly issues in background independence. Our starting point is the notion that the string field coordinates $X(\sigma)$ and the momenta $P(\sigma)$ are background independent objects whose field algebra is always the same; backgrounds correspond to inequivalent representations of this algebra. We propose classical string field solutions relating any two toroidal backgrounds and discuss the space where these solutions are defined. String field theories formulated around dual backgrounds are shown to be related by a homogeneous field redefinition, and are therefore equivalent, if and only if their string field coupling constants are identical. Using this discrete equivalence of backgrounds and the classical solutions we find discrete symmetry transformations of the string field leaving the string action invariant. These symmetries, which are spontaneously broken for generic backgrounds, are shown to generate the full group of duality symmetries, and in general are seen to arise from the string field gauge group.

* Present Address: Institute for Advanced Study, Princeton, NJ 08540. Permanent Address: Center for Theoretical Physics, MIT, Cambridge, Mass. 02139.
1. Introduction and Summary

One of the most interesting properties of string theory is target space duality invariance [1–5]. It means that strings cannot tell the difference between backgrounds that appear to be quite different as far as particle field theory is concerned. The simplest example is that of compactification of a single space-like dimension into a circle of radius $R$. The physics of strings remains unchanged if the circle becomes of radius $1/R$. Given such a striking behavior it is only natural that there has been much discussion about the way in which duality would manifest itself in the context of a field theory of strings [6,7]. There have been also proposals for low energy effective actions for string theory having duality invariance [8, 9].

We have found that our current understanding of string field theory (SFT) is sufficient to discuss quite effectively the issue of target space duality. Since the first priority in the present paper has been the clarification of the physics of duality, we have used the form of string field theory that, at the present time, appears to be easiest to use for our purposes. This is a variant of the original closed string field theory formulated by the Kyoto group [10,11], in which the string length parameter, which was an unphysical parameter, is now taken to be equal to the $+$ component of the momentum. The string length thus becomes physical, and gives no problems at the loop level. The price one pays is that the theory, while gauge covariant, is not fully Lorentz covariant. This, however, is no serious problem for us since we will always consider the case where at least two coordinates $X^0, X^1$ (the first of them time) are not curled up. The $+$ component of momentum refers to $X^+ = (X^0 + X^1)/\sqrt{2}$, and all the curled up coordinates can be treated in the same footing. The simplicity of using this theory, refered to as the $\alpha = p^+$ HIKKO theory, is that one can find explicitly exact classical solutions of string field theory. A large part of the technical tools necessary for our analysis have been developed in studies by Yoneya [12], Itoh [13], Maeno and Takano [14] and Hata and Nagoshi [15]. The light-cone string field theory [16] could have been used for everything except discussing gauge invariance. Much of our discussion applies directly to the nonpolynomial closed string field theory [17–19]. Possibly the hardest point there is to find exact classical solutions. The methods developed by A. Sen [20–22] in his study of background independence of closed string field theory may help give a self-contained explicit discussion of duality in the context of the nonpolynomial string field theory.

The present work explains in very general grounds how duality transformations turn out to be discrete symmetries of string field theory. In making this clear we have had to address some of the issues of background independence of string field theory. Toroidal backgrounds are an ideal setting since the analysis can be done very explicitly. The basic points we have understood will be summarized now.

Aspects of Background Independence. For a space of backgrounds that corresponds to sigma models with a fixed number of two dimensional fields we suggest the idea of
universal coordinates and background dependent representations. For our case, namely toroidal backgrounds, we think of the coordinate \(X^i(\sigma)\) as a universal object, in fact, for all compactifications we take it to be periodic with period \(2\pi\). The momentum operator \(P_i(\sigma)\) is also universal, since it just represents functional differentiation \(-i\delta/\delta X^i(\sigma)\). Oscillator expansions simply furnish convenient background dependent representations of these objects. Thus we think of the oscillators \([\alpha, \bar{\alpha}]\) as background dependent objects and we write this explicitly as \([\alpha(E), \bar{\alpha}(E)]\). The vacuum state is also background dependent. Oscillators for different backgrounds can therefore be related to each other, and these relations are Bogoliubov type transformations. This viewpoint gives us a way to relate operators in different conformal field theories (cft’s). We believe this is a consistent viewpoint since in this context one can verify explicitly the background independence of string field vertices defined by overlap conditions. This is something we would expect to be true on intuitive grounds. Vertices look background dependent in oscillator form, but this is only an appearance. Two vertices written for two different backgrounds are verified to be identical when the oscillators and vacuum of one background are expressed in terms of those of the second background.

Classical Solutions of String Field Theory. Using the above ideas operators defined in string field theories at nearby backgrounds can be related and one can find the infinitesimal string field classical solution that shifts the background. While such shift was found earlier by Sen [20] in the context of the nonpolynomial closed string field theory, we are able to show explicitly that the new string field theory after shifting is of the same form as the original one. This is due to our use of the light-cone type vertex. We then give an expression for a classical solution corresponding to a finite shift of background. While this solution has some shortcomings arising from the singular nature of the light cone vertex (which we expect would disappear in the nonpolynomial SFT), it suggests quite strongly that the classical solution does not live in the Hilbert space of the original string field theory, and thus the classical string field cannot be thought in terms of the component fields of the string field theory. The classical solution is a superposition (integral) of Fock space states around different vacua (cft’s) and due to the infinite number of oscillators these vacua are orthogonal making it apparently impossible to describe the solution in terms of a single Fock space. This point requires far more investigation, and as a first step we give an alternative form for the string field classical solution corresponding to an exactly marginal perturbation. This solution is found solving the string field theory perturbatively and applies to any covariant string field theory, in particular to the nonpolynomial closed string field theory. The solution is given as an infinite series, each term defined by an off-shell string amplitude. The convergence of the series is controlled by the off-shell behavior of the theory, and it may be a tractable problem.

Duality Implies Discrete String Field Symmetries. The main question we discuss is why and how target space duality turns out to be a string field symmetry. A priori duality relates two different looking theories that turn out to be physically equivalent. This
relation is clearly not in the form of an invariance. The strategy we follow is quite general and might have further applications. If two different background field configurations correspond to the same conformal field theory, it is clear that the corresponding string field theories must describe the same physics. We show that this implies the existence of a homogeneous string field redefinition relating these two string field actions. Moreover, the two string field theories can also be related via condensation, or a classical solution that takes us from one background to the other. If we start from one of the string field theories the composition of a string field shift plus the field redefinition brings us back to the original theory and gives rise to a field transformation corresponding to an invariance. Thus the string action for any fixed background has a discrete symmetry corresponding to each possible duality transformation. The symmetries are exact, but are spontaneously broken unless we are at a background invariant under duality. This analysis is carried out explicitly for the complete discrete group of dualities $O(d,d;\mathbb{Z})$. The discrete duality symmetries turn out to correspond in general to finite global gauge transformations, as was predicted earlier on the basis of conformal field theory arguments [6, 7]. We show how to identify them in the context of the string field theory. We find it quite interesting both that string field theory is essentially manifestly dual, and that the string field gauge transformations contain already target space duality.

**The String Field Coupling Constant Does not Change Under Duality.** A curious fact about the discussion of duality from the viewpoint of first quantization has been the understanding that duality invariance requires a shift in the dilaton field [23]. We emphasize here that this shift does not involve a change of the physical string coupling constant. In string field theory it is manifest that two string theories formulated around dual backgrounds could not possibly be equivalent unless the string field coupling constants are identical. String field theory necessarily uses two zero modes $(x,q)$ for the compactified space coordinates, conjugate to the momentum and winding numbers $(p,w)$. The volumes in which the zero modes $(x,q)$ live are inversely related, and their product is a constant. This makes it unnecessary to rescale the coupling constant as we exchange winding and momentum modes. We define the string field dilaton $\Phi_s$ to be the field whose condensation changes the string field coupling constant. We can then show that the string field condensation that takes us from one background to the physically equivalent dual background does not involve the string field dilaton. There is no actual disagreement with the results of first quantization, and we explain how this happens by using path integral methods. The extra factors necessary for duality, which are introduced in first quantization by giving the sigma model dilaton $\Phi_\sigma$ a background dependent expectation value, are seen to arise automatically when the string field theory amplitudes are rewritten as first quantized path integrals. The string field theory is essentially manifestly dual, and one can choose whether to give a path integral expression by Fourier transforming either the momentum modes, to obtain the usual sigma model action for the coordinates $X$, or the winding modes, to obtain a sigma model action for the dual coordinate $Q$. If we Fourier transform both the momentum and winding modes we obtain an interesting
dual-symmetric first quantized action involving $X$ and $Q$, closely related, though not identical to the one proposed by Tseytlin [24]. All this implies that the sigma model dilaton $\Phi_\sigma$, defined as the field coupling to the Euler number of the surface, and the string field dilaton $\Phi_s$ are not the same. One must have $\Phi_\sigma = \Phi_s + \ln \det G$, where $G$ is the background metric for the compactified dimensions. The necessary shift of the sigma model dilaton simply reflects the change of the background metric. Relations between alternative definitions of the dilaton field in somewhat more general backgrounds have been discussed recently [25].

Let us now give a brief description of the contents of the various sections of this paper. In Sect. 2 we set up our conventions for toroidal compactification, discuss the universal objects and their oscillator expansions, and give the expressions for the BRST operator and its variation due to a change in background. In Sect. 3 we give the $\alpha = p^+$ HIKKO string field theory for toroidal backgrounds. We show explicitly the background independence of string field overlap vertices. We observe the interesting fact that the string vertex fails to be dual only due to a phase factor involving a product of momenta and winding. In Sect. 4 we discuss the relations between string field theories formulated around backgrounds related by a discrete duality transformation $g$ in the complete group of dualities $O(d,d;\mathbb{Z})$. We give the explicit form of the operator $U_g$ that defines the homogeneous redefinition relating dual string field theories. These operators are seen to give a projective representation of the group of dualities. The operators satisfy the group multiplication rules up to extra parity like operators that turn out to be symmetries of the string field theory. In Sect. 5 we discuss the condensation of the dilaton extending the light cone results of Yoneya [12] to the $\alpha = p^+$ HIKKO string field theory. We then give the classical solutions for infinitesimal backgrounds shifts, and their form for finite shifts. Finally we derive the series form of the classical solution, applicable to the nonpolynomial closed string field theory. In Sect. 6 we show explicitly how the asymmetric looking first quantized path integrals (satisfying duality) arise from manifestly dual symmetric string field theory in the passage to path integrals. Sect. 7 deals with the string field symmetry of duality, its algebra, and its relation with gauge transformations. In Sect. 8 we offer some comments related to background independence of string field theory and summarize the main open questions.

We include three Appendices. In Appendix A we give some notations and definitions for the ingredients entering the three string vertex of the theory. In Appendix B we show that the $\alpha = p^+$ HIKKO theory reproduces the light-cone string field theory amplitudes for processes involving physical states, thus establishing the correctness of the theory at the loop level. Finally in Appendix C we show how to calculate the overlap of exactly marginal states with the three string vertex, thus deriving the shift of the BRST operator under an infinitesimal string condensation. We also include in this Appendix C a derivation of the SU(2) current algebra at the selfdual radius. The charges are found by contracting a BRST cohomology class of ghost number one, representing a suitable
global gauge parameter, against the three string vertex. This shows how global unbroken symmetries arise in string field theory.

Closed string field theory has been used recently [26] as a tool to understand cosmological solutions to string theory, scale factor duality [27], and to generate new classical solutions.

2. String Theory in Toroidal Backgrounds and Universal Objects

Our objective in this section is to set up the formalism that will enable us to discuss the string field interpretation of duality. We begin by giving the first quantized action describing bosonic string propagation in a general toroidal background (we follow the conventions of Ref. 4.):

$$S = -\frac{1}{4\pi} \int_0^{2\pi} d\sigma \int d\tau \left( \sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j B_{ij} \right)$$  \hspace{1cm} (2.1)

where we take the world sheet metric to be of signature (−, +), $\epsilon^{01} = -1$, and we have given just the part of the action corresponding to the compactified dimensions. There are $d$ such dimensions, and thus the indices $i, j$ run from 1 to $d$. The $d \times d$ constant matrices $G_{ij}$ and $B_{ij}$ represent the background metric and antisymmetric tensor respectively. Note that the strings are parameterized by $\sigma \in [0, 2\pi]$. Our compactification hypothesis reads

$$X^i \equiv X^i + 2\pi,$$  \hspace{1cm} (2.2)

and will not be background dependent. All of the background dependence (such as the radii of the tori, etc.) is encoded in $G$ and $B$; so, in particular, the physical length of the period in the $i$-th direction is $2\pi R_i$ with $R_i = \sqrt{G_{ii}}$. It will be convenient to define the matrices $E_\pm \equiv G \pm B$. The matrix $E_+$ alone (or, $E_-$ alone) contains the full information about the background fields, $G$ is the symmetric part of $E_+$ and $B$ the antisymmetric part. Whenever we write $E$ without a subscript, we mean $E_+$: namely,

$$E \equiv G + B, \hspace{1cm} E^t = G - B,$$  \hspace{1cm} (2.3)

with superscript $t$ denoting transposed matrix. From the action one finds that the momentum conjugate to $X$ is given by

$$2\pi P = G_{ij} \dot{X}^i \dot{X}^j + B_{ij} X^j.$$  \hspace{1cm} (2.4)

The Hamiltonian density $H(\sigma, \tau)$ is given by

$$4\pi H = (\dot{X}^i \dot{X}^j + X^i X^j)G_{ij},$$  \hspace{1cm} (2.5)
which, written in terms of proper canonical variables, takes the form

$$4\pi H = (2\pi)^2 P_i G^{ij} P_j + X^{ij}(G - BG^{-1}B)_{ij} X^{j'} + 4\pi X^{ij} B_{ik} G^{kj} P_j.$$  \tag{2.6}

It is convenient to use matrix notation for this equation, one writes

$$4\pi H = (X', 2\pi P) \mathcal{R}(E) \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}$$  \tag{2.7}

where the matrix $\mathcal{R}(E)$ is defined by

$$\mathcal{R}(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}.$$  \tag{2.8}

Another convenient rewriting of the Hamiltonian is obtained by defining left and right components of the momentum $((-, +) \text{ respectively})$:

$$P_{\mp i} = \frac{1}{2} \left( P_i \pm \frac{1}{4\pi} E_{\mp ij} X^{j'} \right),$$  \tag{2.9}

one then finds

$$H = 2\pi (P_{-i} G^{ij} P_{-j} + P_{+i} G^{ij} P_{+j}),$$  \tag{2.10}

The Hamiltonian equations of motion can be solved as usual to give oscillator expansions:

$$X^i(\sigma, \tau) = x^i + w^i \sigma + \tau G^{ij} (p_j - B_{jk} w^k)$$
$$+ \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^i e^{-in(\tau-\sigma)} + \bar{\alpha}_n^i e^{-in(\tau+\sigma)}]$$  \tag{2.11}

$$2\pi P_i(\sigma, \tau) = p_i + \frac{1}{\sqrt{2}} \sum_{n \neq 0} [E_{ij} \alpha_n^j e^{-in(\tau-\sigma)} + E_{ij} \bar{\alpha}_n^j e^{-in(\tau+\sigma)}],$$  \tag{2.12}

It should be noted that from the periodicity of $x^i$, $x^i = x^i + 2\pi$, both the momentum $p_i$ and the winding number $w^i$ take integer eigenvalues.
2.1. Universal Objects and Oscillator Expansions

As is necessary for field theory of strings we look at functionals of the string coordinates at \( \tau = 0 \). The coordinates for the string field are just \( X^i(\sigma) \equiv X^i(\sigma, \tau = 0) \). Corresponding to these coordinates, we have the operation of functional differentiation, which is realized by \( P_i(\sigma) \equiv P_i(\sigma, \tau = 0) = -i\frac{\delta}{\delta X^i(\sigma)} \). We must think of \( X^i(\sigma) \) and \( P_i(\sigma) \) as background independent notions. In string field theory the background dependence comes in when in constructing a kinetic operator and the vertices, one uses the background field \( E = G + B \). Schematically, a string field theory looks like

\[
S = \int dX_1 dX_2 \Psi(X_1) \mathcal{V}_2(E, X_r, P_r) \Psi(X_2) \\
+ \int (\prod_{r=1}^3 dX_r \Psi(X_r)) \mathcal{V}_3(E, X_r, P_r) + \ldots
\]

and all the background dependence is concentrated on the vertices \( \mathcal{V} \). In a string field theory the background field \( E \) is fixed. Oscillator expansions are a convenience to study such actions. As it turns out the expansions of equations (2.11) and (2.12), restricted to \( \tau = 0 \) are extremely convenient. These read:

\[
X^i(\sigma) = x^i + w^i \sigma + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} [\alpha^i_n(E)e^{in\sigma} + \bar{\alpha}^j_n(E)e^{-in\sigma}],
\]

\[
2\pi P_i(\sigma) = p_i + \frac{1}{\sqrt{2}} \sum_{n \neq 0} [E_{ij}^l \alpha^j_n(E)e^{in\sigma} + E_{ij} \bar{\alpha}^j_n(E)e^{-in\sigma}].
\]

Here we have introduced the \( E \) label to the oscillators to emphasize that they depend on background field \( E = G + B \), since they appear in an explicitly background dependent expansion of the background independent objects \( X(\sigma), P(\sigma) \). The commutation relations arise from \([X^i(\sigma), P_j(\sigma')] = i\delta_j^i \delta(\sigma - \sigma')\)

\[
[x^i, p_j] = i\delta_j^i, \\
[\alpha^i_m(E), \alpha^j_n(E)] = [\bar{\alpha}^i_m(E), \bar{\alpha}^j_n(E)] = mG^{ij}\delta_{m+n,0}.
\]

As expected, the commutation relations for the oscillators are background dependent. We have not introduced a background label for the zero modes since they will actually turn out to be background independent, as their commutation relation suggests.

For discussions of target space duality, it is convenient to introduce another coordi-
nate \( Q_i(\sigma) \), dual to \( X^i(\sigma) \), by the relation \( Q'_i(\sigma) = 2\pi P_i(\sigma) \): that is,

\[
Q_i(\sigma) \equiv \text{constant} + \int_0^\sigma d\sigma' 2\pi P_i(\sigma')
\]

\[
= q_i + p_i \sigma + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} [-E_{ij} \alpha_n(E) e^{i n \sigma} + E_{ij} \bar{\alpha}_n(E) e^{-i n \sigma}].
\]

(2.16)

The new zero-mode variable \( q_i \) is introduced as a CM coordinate conjugate to \( w^i \):

\[
[q_i, w^j] = i \delta^i_j, \quad [q_i, x^j] = [q_i, p_j] = 0.
\]

(2.17)

Since \( w^i \) takes integer values, the \( q_i \), and hence \( Q_i(\sigma) \) also, must be periodic variables with period \( 2\pi \) just like \( x^i \). The coordinate \( Q_i(\sigma) \) is also background independent.

If we have a string field theory written for a fixed background \( E_0 \), it is most convenient to expand the background independent objects \( X(\sigma) \) and \( P(\sigma) \), and the string field, using oscillators \( \alpha(E_0), \bar{\alpha}(E_0) \) corresponding to that particular background. This leads to a kinetic operator with a diagonal mass operator, enabling one to read the spectrum easily. It is important to realize, however, that this is not required, one may expand a string field theory written around some background using oscillators that correspond to a different background. This possibility is essential to understand explicitly the meaning of a shift of background in string field theory.

The universality of \( X(\sigma) \) and \( P(\sigma) \) allows us to define a relation between oscillators that correspond to different backgrounds. We simply equate the two different expansions of the universal coordinates. It should be noted that we have precisely the right number of conditions to determine the relations between the oscillators uniquely. We have two sets of Fourier coefficients \( (\alpha, \bar{\alpha}) \) and precisely two functions of \( \sigma \), namely, \( X \) and \( P \). One easily finds that the zero modes must be identical; this is the reason we did not introduce a background label for them. For the oscillators we get the following relations

\[
\alpha_n(E) - \bar{\alpha}_{-n}(E) = \alpha_n(E') - \bar{\alpha}_{-n}(E'),
\]

\[
E^t \alpha_n(E) + E \bar{\alpha}_{-n}(E) = E^{t'} \alpha_n(E') + E' \bar{\alpha}_{-n}(E'),
\]

(2.18)

where we have ommitted, for brevity, the indices \( i, j \) both in the oscillators and in the backgrounds. In fact, the above relations hold for \( n = 0 \) too. Indeed, with our normalization convention for the oscillators one must have

\[
\alpha_0(E) \equiv \frac{1}{\sqrt{2}} G^{-1}(p - E w)
\]

\[
\bar{\alpha}_0(E) \equiv \frac{1}{\sqrt{2}} G^{-1}(p + E^t w).
\]

(2.19)
Inverting the above relations one has

\begin{align}
  w &= \frac{1}{\sqrt{2}} (\bar{\alpha}_0 - \alpha_0), \\
  p &= \frac{1}{\sqrt{2}} (E^t \alpha_0 + E\bar{\alpha}_0),
\end{align}

(2.20)

and one now sees that (2.18), for \( n = 0 \) simply says that \( p \) and \( w \) are background independent.

Solving equations (2.18) for the \( \alpha(E') \), \( \bar{\alpha}(E') \) in terms of the \( \alpha(E) \), \( \bar{\alpha}(E) \) oscillators one finds

\begin{align}
  2G' \alpha_n(E') &= (E^t + E') \alpha_n(E) + (E - E') \bar{\alpha}_{-n}(E), \\
  2G' \bar{\alpha}_n(E') &= (E^t - E'^t) \alpha_{-n}(E) + (E + E'^t) \bar{\alpha}_n(E).
\end{align}

(2.21)

As a first application of the above results, let us find the change in the oscillators under a small fluctuation of the background fields from \( E \) to \( E' = E + \delta E \). Defining \( \delta \alpha_n \equiv \alpha_n(E + \delta E) - \alpha_n(E) \) and similarly for \( \delta \bar{\alpha}_n \), we find

\begin{align}
  2G \delta \alpha_n &= - \left( \delta E^t \alpha_n(E) + \delta E \bar{\alpha}_{-n}(E) \right), \\
  2G \delta \bar{\alpha}_n &= - \left( \delta E^t \alpha_{-n}(E) + \delta E \bar{\alpha}_n(E) \right).
\end{align}

(2.22)

These equations will allow us to relate operators at nearby values for the background fields.

### 2.2. The BRST Operator

We will use indices \( \mu, \nu, \cdots \) to label the noncompact \((D - d)\) dimensions and indices \( i, j, \cdots \) to label the \(d\) dimensions that have been curled up into tori (of course, \( D = 26 \)). The BRST operator on the constant background is given by \((\eta_{\mu\nu}, G_{ij}, B_{ij})\):

\begin{equation}
  Q_B = - \sum_n : c_{-n} \left( L_n^X(E) + \frac{1}{2} L_n^{FP} - \alpha(0) \delta_{n,0} \right) : + a.h., \tag{2.23}
\end{equation}

where the Virasoro operators are:

\begin{align}
  L_n^X(E) &= \sum_m \frac{1}{2} : (\alpha_{n-m}^\mu \eta_{\mu\nu} \alpha_m^\nu + \alpha_{n-m}^i G_{ij} \alpha_m^j(E)) : \\
  L_n^{FP} &= \sum_m (n + m) : b_{n-m} c_m :
\end{align}

(2.24)

The zero-modes \( p, w \) appear in the BRST operator in the form

\begin{equation}
  Q_B = - c_0^+ \left[ \frac{1}{2} (w, p) \mathcal{R}(E) \begin{pmatrix} w \\ p \end{pmatrix} + \cdots \right] - \frac{1}{2} c_0^- \left[ - pw + \cdots \right] + \cdots, \tag{2.25}
\end{equation}
where and henceforth we use the following notation for ghost zero modes:

\[ c_0^+ \equiv \frac{1}{2}(c_0 + \bar{c}_0), \quad c_0^- \equiv c_0 - \bar{c}_0, \]

\[ b_0^+ \equiv b_0 + \bar{b}_0, \quad b_0^- \equiv \frac{1}{2}(b_0 - \bar{b}_0). \tag{2.26} \]

Let us understand how the BRST operator changes under a shift of background. It follows from (2.23) that the only change comes from the contribution of the compact dimensions to the matter Virasoro generators. Consider the Virasoro operator

\[ \delta L_0^X \equiv L_0^X(E + \delta E) - L_0^X(E) \]

\[ = \frac{1}{2} \alpha_{-n}(E + \delta E)(G + \delta G)\alpha_n(E + \delta E) - \frac{1}{2} \alpha_{-n}(E)G\alpha_n(E) \]

\[ = -\frac{1}{2} \alpha_n(E)\delta E \bar{\alpha}_n(E). \tag{2.27} \]

A small identical calculation shows that \( \delta L_0^X = \delta L_0^X \), and this is quite essential since it implies that the operator \( \Delta N \equiv L_0 - \bar{L}_0 \) is unchanged under a shift of background, and therefore it is background independent. Note that the hamiltonian \( L_0 + \bar{L}_0 \) is not background independent. Since we have that the BRST operator commutes with \( \Delta N \), namely \([Q_B(E), \Delta N] = 0\), under variation we must find that \([\delta Q_B, \Delta N] = 0\). This is indeed what one finds after a small calculation.

\[ \delta Q_B = \frac{1}{2} \sum_{\ell + n + m = 0} (c_\ell + \bar{c}_{-\ell}) \left( \alpha_n^i(E)\delta E_{ij}\bar{\alpha}_{-m}^j(E) \right). \tag{2.28} \]

This will be of utility later.

3. String Field Theory in Toroidal Backgrounds

In this section we set up completely the \( \alpha = p^+ \) HIKKO string field theory in toroidal backgrounds. We then turn to the explicit analysis of the background independence of the three string vertex. Much of our discussion below applies to the nonpolynomial closed string field theory.
3.1. The $\alpha = p^+$ HIKKO String Field Theory

The string field is denoted in our notation by

$$|\Psi\rangle = c_0^- (|\phi\rangle + c_0^+ |\psi\rangle) + (i|\chi\rangle + c_0^+ |\eta\rangle),$$

and it is a Grassmann odd object with ghost number $+3$ (with respect to the $SL(2;C)$-invariant vacuum). Actually, in writing an action, only the field $b_0^-|\Psi\rangle$, which is even and of ghost number $+2$ appears. The above component string fields $\phi, \psi, \chi, \eta$ are constructed on the “down-down” ghost zero-mode vacuum $|--,--\rangle$ defined by

$$b_0^+|--,--\rangle = 0, \quad b_0^-|--,--\rangle = 0, \quad \langle--|--,--|c_0^-c_0^+|--,--\rangle = 1,$$

Let us henceforth denote $|--,--\rangle$ simply $|0\rangle$, we then have

$$\langle0|c_0^-c_0^+|0\rangle = 1. \quad (3.3)$$

The $|0\rangle$ vacuum is related with the $SL(2;C)$-invariant vacuum $|1\rangle$ via $|1\rangle = -b_{-1}\bar{b}_{-1}|0\rangle$, and $|0\rangle = c_1\bar{c}_1|1\rangle$. *

The action of $\alpha = p^+$ HIKKO SFT is given by

$$S = \frac{1}{12}\langle R||\Psi\rangle_1 Q_B^{(2)}b_0^-(2)|\Psi\rangle_2 + \frac{g}{3} \langle V||\Psi\rangle_1|\Psi\rangle_2|\Lambda\rangle_3,$$

which is invariant under the following gauge transformation:

$$\delta(b_0^-|\Psi\rangle) = Q_Bb_0^-|\Lambda\rangle + g|\Psi\star\Lambda\rangle$$

with $|\Psi\star\Lambda\rangle_1 \equiv 1^{123}\langle V||R\rangle_1|\Psi\rangle_2|\Lambda\rangle_3. \quad (3.5)$

The inner-product of states implied by the repeated string labels 1, 2, $\cdots$ also implies integration over the noncompact zero-modes and summation over the compact ones, as

* There is an apparent inconsistency concerning this notation. Take the hermitian conjugate of (3.3), to get $\langle0|c_0^+c_0^-|0\rangle = 1$, in contradiction with (3.3). An easy way out of this difficulty is to adopt a coordinate representation for $c_0^+$ and $c_0^-$. Then we regard $\langle0|c_0^-c_0^+|0\rangle$ as an abbreviation standing for $\int dc_0^+dc_0^-c_0^-c_0^+$. The above difficulty is avoided by the presence of the integration measure $\int dc_0^+dc_0^-$ which changes sign under hermitian conjugation since the order of $dc_0^+$ and $dc_0^-$ is interchanged. The operators $b_0^+$ and $b_0^-$ are then the differential operators $\partial/\partial c_0^+$ and $\partial/\partial c_0^-$, respectively. This interpretation is only necessary when dealing with hermitian conjugation.
\[ \int \frac{d^{D-d}p}{(2\pi)^{D-d}} \sum_{p_i} \sum_{w_i}. \]

i) **2-point vertex.** This state, in the direct product of two Hilbert spaces, and denoted as the reflector \( _{12} \langle R \mid \rangle \), is given by

\[ _{12} \langle R \mid \rangle = \delta(1, 2) _{12} \langle 0 \mid \exp \left( E_{12} \right) \left( c_0^{(1)} + c_0^{(2)} \right) \left( c_0^{(1)} + c_0^{(2)} \right) e^{-i\pi p_2 w_2} \mathcal{P}_{12}. \] (3.6)

where the exponent \( E_{12} \) is defined by

\[ E_{12} = (-)^{n+1} \sum_{n \geq 1} \left( \frac{1}{n} \alpha_n^{(1)} \cdot \alpha_n^{(2)} + \alpha_n^{(1)} b_n^{(2)} - b_n^{(1)} c_n^{(2)} \right) + \text{a.h.} \] (3.7)

with

\[ \alpha_n^{(1)} \cdot \alpha_n^{(2)} \equiv \alpha_n^{(1)} \eta_{\mu\nu} \alpha_n^{(2)} + \alpha_n^{(1)} G_{ij} \alpha_n^{(2)}(E), \] (3.8)

and the delta functions and projectors defined by

\[ \delta(1, 2, \cdots, n) \equiv (2\pi)^{D-d} \delta^D \left( \sum_{r=1}^{n} p_r^\mu \right) \cdot \delta^d \left( \sum_{r=1}^{n} p_{ri} \right) \cdot \delta^d \left( \sum_{r=1}^{n} w_r^i \right), \] (3.9)

\[ \mathcal{P}_{12 \cdots n} \equiv \prod_{r=1}^{n} \mathcal{P}^{(r)}, \quad \mathcal{P} \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \exp i\theta (L - \bar{L}) \]

where the two \( \delta^d \)'s are Kronecker deltas, \( L = L_0^X(E) + L_0^{FP} - 1 \) and \( \bar{L} \) is its antiholomorphic counterpart. The hermitian conjugate of \( \langle R \rangle \) coincides with the minus of the ket reflector \( | \tilde{R} \rangle \):

\[ (\langle R \rangle)^\dagger \equiv | \tilde{R} \rangle = -| \tilde{R} \rangle, \] (3.10)

where the ket reflector \( | \tilde{R} \rangle \) is defined by the property

\[ _{12} \langle R \mid \tilde{R} \rangle_{23} | \Phi \rangle_1 = | \Phi \rangle_3 \] (3.11)

for arbitrary \( | \Phi \rangle \). The string field \( | \Psi \rangle \) satisfies the following reality condition:

\[ _{12} \langle R \mid \Psi \rangle_1 = \langle \Psi | \text{ or } \langle \Psi | \tilde{R} \rangle_{12} = | \Psi \rangle_2 \] (3.12)

The reflector has been written in momentum representation, as we can see from the fact that \( p \) and \( w \) appear, instead of \( x \) and \( q \). The \( p \)'s and \( w \)'s are \( c \)-numbers and they
are taken to have the value of the momenta of the states that eventually appear to the right. The vacuum, appearing in the reflector has nothing to do with momenta. An alternative notation, preferred by some physicists would be to let the \( p \)'s and \( w \)'s that appear on the reflector to be operators, and to replace

\[
\delta(1, 2)_{12}|0| \rightarrow \sum_{p_1; p_2} \delta(1, 2)\langle p_1, p_2|,
\]

(3.13)

where the sum extends over all possible values of the momenta \( p = (p_\mu, p_i, w^i) \) for each of the strings, the delta function constraining the sum to the momentum conserving combinations, and the vacua representing the corresponding momentum eigenstates.

Note the presence in the reflector of the phase factor \( e^{-i\pi p_2 w_2} \). It is the unique possible sign factor of the form \( pw \), as can be checked using momentum conservation. It is important to note that the phase factor is invariant under the exchange of \( p \) and \( w \). This implies that the reflector treats in the same way the coordinates \( X(\sigma) \) and \( Q(\sigma) \). Let us understand why this phase factor is essential in getting the expected type of connection conditions from the reflector. A straightforward calculation gives the following continuity conditions on the reflector

\[
\begin{align*}
12\langle R| \left( \alpha_n^{(1)} + (-)^n \alpha_{-n}^{(2)}, c_n^{(1)} + (-)^n c_{-n}^{(2)}, b_n^{(1)} - (-)^n b_{-n}^{(2)} \right) &= 0,
\end{align*}
\]

(3.14)

and the same ones for the anti-holomorphic oscillators. The above hold for all \( n \) different from zero. Consider now the expansion for \( X(\sigma) \) in (2.14) written as

\[
X(\sigma) = x + w\sigma + \tilde{X}(\sigma),
\]

(3.15)

where we explicitly separate out the oscillators. It follows from (3.14) that

\[
12\langle R| \left( \tilde{X}_{(1)}(\sigma) - \tilde{X}_{(2)}(\pi - \sigma) \right) = 0.
\]

(3.16)

It is clear that the full coordinate must be connected in a similar fashion. Let us therefore consider the zero modes \( x \) and \( w \) for the compactified coordinates. We must be careful since the zero mode operator \( \hat{x} \) is not a well defined operator, due to the periodicity condition on the torus. Due to the periodicity of \( x \), the momentum \( p \) takes the integer eigenvalues \( \hat{p}|n\rangle = n|n\rangle \) (here \( n \) is a vector of integers). Rather than trying to define a coordinate operator, we define a coordinate eigenstate via

\[
|x\rangle \equiv \sum_n \frac{e^{-inx}}{\sqrt{(2\pi)^d}} |n\rangle.
\]

(3.17)

Then, as desired, the state label \( x \) becomes the label of the point on the torus since \( |x\rangle = |x + 2\pi e^{(i)}\rangle \) where \( e^{(i)} \) is a unit vector in the \( i \) direction. The inner product of two
coordinate eigenstates is given by
\[ \langle x | y \rangle = \sum_n \frac{1}{(2\pi)^d} e^{in(x-y)} = \sum_m \delta(x - y + 2\pi m) \equiv \delta(x - y), \] (3.18)

where \( \delta \) is a periodic delta function. In order to understand what type connection the reflector gives we must evaluate the overlap
\[ _{12} \langle R || x_1, w_1 \rangle_1 | x_2, w_2 \rangle_2, \]
where the second label on the kets is the winding eigenvalue. As far as the zero modes are concerned the above is equals
\[ \sum_{n_1, n_2} \delta^d(n_1 + n_2) \delta^d(w_1 + w_2) e^{-in_1 x_1} e^{-in_2 x_2} \]
where the delta functions and the phase factor \( e^{-in_2 w_2} \) came from the reflector. It then follows that
\[ _{12} \langle R || x_1, w_1 \rangle_1 | x_2, w_2 \rangle_2 = \delta(x_1 - [x_2 + \pi w_2]) \delta(w_1 + w_2), \] (3.19)
which means that the vertex “connects” zero modes as
\[ x_1 \approx x_2 + \pi w_2, \quad w_1 \approx -w_2, \] (3.20)
where the first one is modulo \( 2\pi \). Actually the connection of the windings is a true operator relation when acting on the reflector, since the winding operator is well defined. The above implies that
\[ x_1 + w_1 \sigma \approx x_2 + \pi w_2 - w_2 \sigma \approx x_2 + w_2(\pi - \sigma). \] (3.21)
This fits nicely with equations (3.15) and (3.16) to give the connection condition
\[ X_1(\sigma) \approx X_2(\pi - \sigma), \] (3.22)
for the full coordinate. This is the expected result, it shows the relevance of the phase factor. For the dual coordinate \( Q(\sigma) \) one finds a similar result, namely
\[ Q_1(\sigma) \approx Q_2(\pi - \sigma), \] (3.23)
due to the symmetry of the reflector under the exchange of \( p \) and \( w \). We also use the \( \approx \) symbol for this coordinate because the zero mode \( q \) is not a well defined operator.
ii) 3-point vertex. The three string vertex can be given in two useful forms:

\[ 123(V) = \mu_{123}^2 \delta(1, 2, 3) \prod_{r=1}^{3} c_0^{(r)}(E_{123}) \]

\[ \times \left( \sum_{r=1}^{3} \frac{b_0^{(r)}}{p_0^r} \right) G(\sigma_f) e^{-i\pi(p_{3w_2} - p_{1w_1})} \mathcal{P}_{123}, \]

(3.24)

or, equivalently, by

\[ 123(V) = \mu_{123}^2 \delta(1, 2, 3) \prod_{r=1}^{3} c_0^{(r)} \exp(F_{123}) \]
It is therefore not dual symmetric. In fact, this phase factor, which is either one or minus one, is the only factor that prevents the string field theory from being completely dual symmetric. It is not hard to check that the factor cannot be made dual symmetric by a redefinition of the string field. We therefore expect the connection conditions on the vertex not to be dual symmetric. This expression for the sign factor was first given by Maeno and Takano [14]. We refer to this sign factor as a vertex cocycle factor henceforth.

The following symmetry and Grassmann even-odd properties are worth remembering:

\begin{align}
1_{2}\langle R \rangle & : \text{Grassmann even, symmetric under } 1 \leftrightarrow 2 \\
1_{23}\langle V \rangle & : \text{Grassmann odd, anti-symmetric under interchange of } 1,2,3 \\
|\Psi \rangle & : \text{Grassmann odd} \\
|\Lambda \rangle & : \text{Grassmann even}
\end{align}

(3.29)

3.2. Universality of the Three String Vertex

The above expression for the 3-string vertex was obtained by Maeno and Takano [14] starting from the following naive delta functional expression for the vertex:

\begin{align}
V[X^{(1)}, X^{(2)}, X^{(3)}] & \sim \prod_{-\pi |p_i^+| \leq \sigma \leq \pi |p_i^+|} \delta \left( \Theta_1 X^{(1)}(\sigma_1) + \Theta_2 X^{(2)}(\sigma_2) - X^{(3)}(\sigma_3) \right), \\
\Theta_1(\sigma) & \equiv \theta(\pi p_1^+ - |\sigma|), \quad \Theta_2(\sigma) \equiv 1 - \Theta_1(\sigma), \\
\sigma_1(\sigma) & \equiv \sigma, \quad \sigma_2(\sigma) \equiv \frac{\sigma - \pi p_1^+ \text{sgn}(\sigma)}{p_2^+}, \quad \sigma_3(\sigma) \equiv \frac{\pi |p_3^+| \text{sgn}(\sigma) - \sigma}{p_3^+}.
\end{align}

(3.30)

[This overlapping pattern is for the case $p_1^+, p_2^+ > 0, p_3^+ < 0$ ($p_1^+ + p_2^+ = |p_3^+|$); other cases are similar.] The delta functions for compact coordinates are the periodic ones defined in (3.18). As expected from this derivation, they proved that the following Goto-Naka type connection conditions are satisfied by the above vertex:

\begin{align}
1_{23}\langle V \rangle \left( \Theta_1 X^{(1)}(\sigma_1) + \Theta_2 X^{(2)}(\sigma_2) - X^{(3)}(\sigma_3) \right) & \approx 0 \quad (\text{mod } 2\pi), \\
1_{23}\langle V \rangle \left( \Theta_1 P^{(1)}(\sigma_1) + \Theta_2 P^{(2)}(\sigma_2) + P^{(3)}(\sigma_3) \right) & = 0.
\end{align}

(3.31)

The dual coordinate $Q$ (in (2.16)), however, does not connect the way the $X$ coordinate does, one finds

\begin{align}
1_{23}\langle V \rangle \left( \Theta_1 Q^{(1)}(\sigma_1) + \Theta_2 Q^{(2)}(\sigma_2) - Q^{(3)}(\sigma_3) \right) & \approx 1_{23}\langle V \rangle \pi(\Theta_1 p_2 + \Theta_2 p_1) \quad (\text{mod } 2\pi).
\end{align}

(3.32)

We should note that the expressions (3.24) or (3.25) for the 3-string vertex $1_{23}\langle V \rangle$ apparently depend on the background fields $E$, but always satisfy the Goto-Naka equa-
tions (3.31) and (3.32) irrespectively of \( E \). Note also that the coordinates \( X^{(r)}(\sigma) \) and \( Q^{(r)}(\sigma) \) give a complete set of operators in the three string Hilbert space. Namely there is no operator which commutes with all the \( X^{(r)}(\sigma) \) and \( Q^{(r)}(\sigma) \), and hence the Goto-Naka connection equations (3.31) and (3.32) uniquely specify the 3-string vertex up to an overall normalization. Therefore, despite its appearance, the 3-string vertex (3.24) or (3.25) gives in fact a unique object that does not depend on the background at all. [The coincidence of the normalization will be checked explicitly.]

This argument proves the universality of the 3-string vertex. But it is also very illuminating to confirm it directly for the explicit expression given in (3.24) or (3.25). The apparently background dependent part of the vertex is given by

\[
E^{(0)} \exp(E_{123}) ,
\]

where \( E_{123} \) here is the \( E_{123}^{\text{compact}}(E) \) given above in (3.28), which may be rewritten more concisely as

\[
E_{123} = \frac{1}{2} (\alpha^t, \tilde{\alpha}_0^t) \begin{pmatrix} N & N_0 \\ N_0^t & N_{00} \end{pmatrix} G \begin{pmatrix} \tilde{\alpha} \\ \tilde{\alpha}_0 \end{pmatrix} + \text{a.h.} ,
\]

using the following condensed vector- and matrix-notations:

\[
\begin{pmatrix} \tilde{\alpha} \\ \tilde{\alpha}_0 \end{pmatrix} \equiv \begin{pmatrix} \alpha^{i(r)}(E) (n \geq 1) \\ \alpha_0^{i(r)}(E) \end{pmatrix} , \quad \begin{pmatrix} \tilde{\alpha} \\ \tilde{\alpha}_0 \end{pmatrix} \equiv \begin{pmatrix} \tilde{\alpha}_n^{i(r)}(E) (n \geq 1) \\ \tilde{\alpha}_0^{i(r)}(E) \end{pmatrix} ,
\]

\[
N \equiv \begin{bmatrix} N_{rs}^{rs} \\ N_{nm}^{rs} \end{bmatrix} , \quad N_0 \equiv \begin{bmatrix} N_{n0}^{rs} \\ N_{00}^{rs} \end{bmatrix} , \quad N_{00} \equiv \begin{bmatrix} N_{00}^{rs} \end{bmatrix} .
\]

Note the suffix \( E \) on the vacuum \( \langle 0 | \) in (3.33) to emphasize that it is the vacuum of the background dependent oscillators \( \alpha_n(E) \), \( \tilde{\alpha}_n(E) \).

Now let us show that (3.33) is indeed independent of the background \( E \). Under an arbitrary infinitesimal change of \( E \) to \( E + \delta E \), the oscillators \( \alpha_n(E) \), \( \tilde{\alpha}_n(E) \) change, from (2.22), by

\[
\delta \alpha_n = -\frac{1}{2} G^{-1} (\delta E^t \alpha_n + \delta E \tilde{\alpha}_n) ,
\]

\[
\delta \tilde{\alpha}_n = -\frac{1}{2} G^{-1} (\delta E \tilde{\alpha}_n + \delta E^t \alpha_n) .
\]

Here we have omitted the background label \( E \) from \( \alpha_n(E) \) for brevity. The vacuum corresponding to the changed oscillators \( \alpha'_n \equiv \alpha_n(E+\delta E) = \alpha_n + \delta \alpha_n \) and \( \tilde{\alpha}'_n = \tilde{\alpha}_n + \delta \tilde{\alpha}_n \)
is also infinitesimally shifted from the original one $|0\rangle_E$:

$$|0\rangle_{(E+\delta E)} = |0\rangle_E - \mathcal{B}|0\rangle_E. \quad (3.37)$$

$\mathcal{B}$ is easily found to be given by

$$\mathcal{B} = \frac{1}{2} \left( \bar{\alpha}^T \frac{\delta E}{n} \alpha - \bar{\alpha}^T \frac{\delta E}{n} \alpha^\dagger \right), \quad (3.38)$$

with condensed notation again:

$$\frac{\delta E}{n} = \left[ \delta E_{ij} \overline{\delta_{nm}} \delta_{rs} \right]. \quad (3.39)$$

Indeed $\mathcal{B}$ is an anti-hermitian generator of the Bogoliubov transformation

$$[\mathcal{B}, \alpha_n] = \frac{1}{2} G^{-1} \delta E \bar{\alpha}_{-n}, \quad [\mathcal{B}, \bar{\alpha}_n] = \frac{1}{2} G^{-1} \delta E^t \alpha_{-n}, \quad (3.40)$$

for all $n \neq 0$, from which, together with (3.36), one can see that the vacuum (3.37) is really annihilated by the changed oscillators $\alpha'_n = \alpha_n + \delta \alpha_n$ and $\bar{\alpha}'_n = \bar{\alpha}_n + \delta \bar{\alpha}_n$ with $n \geq 1$.

Now we can evaluate the change of the vertex (3.33) under the change of $E$. Working with ket state representation for convenience of writing, and noting that $\delta(|0\rangle_E) = -\mathcal{B}|0\rangle_E$, we have

$$\delta \left( e^{E^t_{123}|0\rangle_E} \right) = \delta \left( e^{E^t_{123}} |0\rangle_E - e^{E^t_{123}} \mathcal{B}|0\rangle_E \right).$$

(3.41)

To evaluate the first term we need know the change of $E^t_{123}$, which is calculated by using (3.36), (3.34) and the property $\bar{\alpha}^T \delta B \gamma \bar{\alpha} = \bar{\alpha}^T \delta B N \bar{\alpha} = 0$ owing to the antisymmetry of the $\delta B$ matrix. One finds

$$\delta E^t_{123} = -\frac{1}{2} \left( \bar{\alpha}^T \alpha^0, \alpha^0 \right) \left( \begin{array}{cc} N & N^0 \\ N^0 & N \end{array} \right) \left( \begin{array}{c} \delta E \bar{\alpha} \\ \delta E \alpha^0 \end{array} \right) + \text{a.h.}(E \rightarrow E^t). \quad (3.42)$$

Here a.h.$(E \rightarrow E^t)$ denotes the anti-holomorphic term which is obtained by making substitutions $\bar{\alpha} \rightarrow \alpha$ and $E \rightarrow E^t$ in the first term. Since the variation $\delta E \bar{\alpha}$ consists of annihilation operators, it does not commute with $E^t_{123}$ and could make the calculation of $\delta \left( \exp(E^t_{123}) \right)$ complicated. Fortunately that part of the change is just identical with
the one given by the Bogoliubov transformation (3.40), so that (3.42) can be written in the form

$$\delta E_{123} = -[\mathcal{B}, E_{123}^\dagger] + \delta_0 E_{123}^\dagger,$$

(3.43)

with $\delta_0 E_{123}^\dagger$ denoting the change in the zero-mode part:

$$\delta_0 E_{123}^\dagger = -\frac{1}{2} \left( (\vec{\alpha}^T N_0 + \bar{\alpha}_0^T N_{00}) \delta E \bar{\alpha}_0 + a.h. (E \to E^t) \right).$$

(3.44)

Since $\delta_0 E_{123}^\dagger$ commutes with $E_{123}^\dagger$, we have

$$\delta \left( e^{E_{123}^\dagger} \right) = -[\mathcal{B}, e^{E_{123}^\dagger}] + \delta_0 E_{123}^\dagger e^{E_{123}^\dagger},$$

(3.45)

and hence the vertex change (3.41) becomes

$$\delta \left( e^{E_{123}^\dagger} |0\rangle_E \right) = -\mathcal{B} e^{E_{123}^\dagger} |0\rangle_E + \delta_0 E_{123}^\dagger e^{E_{123}^\dagger} |0\rangle_E.$$

(3.46)

We can now evaluate the first term and show that it cancels exactly the second term. Using the expression (3.38) of $\mathcal{B}$ and making the annihilation operators in $\mathcal{B}$ act on $e^{E_{123}^\dagger} |0\rangle_E$, we evaluate the first term and find

$$-\mathcal{B} e^{E_{123}^\dagger} |0\rangle_E$$

$$= -\frac{1}{2} \left\{ \bar{\alpha}^T \left( -\frac{1}{n} + N^t n N \right) \delta E \bar{\alpha} + \bar{\alpha}^T (N^t n N_0) \delta E \bar{\alpha}_0 
+ \bar{\alpha}_0^T (N^t n N_0) \delta E \bar{\alpha}_0 + 2 \bar{\alpha}_0 (N^t n N_0) \delta E \bar{\alpha}_0 \right\} e^{E_{123}^\dagger} |0\rangle_E.$$

(3.47)

Here we can use the following identities [12] for the Neumann coefficients of the light-cone type three-string vertex

$$N^t n N = \frac{1}{n},$$

$$N^t n N_0 = -N_0,$$

$$N_0^t n N_0 = -2N_{00},$$

(3.48)

where the last two equalities hold in the presence of conservation delta-functions (or Kronecker deltas) for the zero-modes. Using these identities, we find that (3.47) becomes

$$-\mathcal{B} e^{E_{123}^\dagger} |0\rangle_E$$

$$= \frac{1}{2} \left( \bar{\alpha}^T N_0 \delta E \bar{\alpha}_0 + \bar{\alpha}_0^T N_0 \delta E \bar{\alpha}_0 + 2 \bar{\alpha}_0 (N^t n N_0) \delta E \bar{\alpha}_0 \right) e^{E_{123}^\dagger} |0\rangle_E.$$

(3.49)

But we immediately see that the quantity in parenthesis equals $-\delta_0 E_{123}^\dagger$ (see (3.44)), and therefore the first term in (3.46) cancels the second term as desired. We thus have shown
directly that the 3-string vertex is actually independent of the background $E$ despite its apparent dependence.

4. Equivalence of String Field Theories around Dual Backgrounds

In order to begin our study of duality in string field theory we need to understand why string field theory formulated around backgrounds related by duality transformations describe physically equivalent theories. Duality transformations are discrete transformations, and for the case of toroidal compactification of $d$ space dimensions, they form the group $O(d,d;\mathbb{Z})$. Consider two backgrounds $E$ and $E'$ related by a duality transformation. As we have discussed in a previous section we can write a string field theory $S_{E}(\Psi)$ around the background $E$, and a string field theory $S_{E'}(\Psi)$ around the background $E'$. We can also choose arbitrarily the string field coupling constant. Let $g_0$ denote the coupling constant for the $E$-theory and $g'_0$ denote the coupling constant for the $E'$-theory. These string field actions are manifestly different, in particular, the kinetic terms are defined by $Q_B(E)$ and $Q_B(E')$ respectively, and these two BRST operators are different. The purpose of the present section is to show that these two actions describe the same physics if and only if $g_0 = g'_0$.

As we will see, in string field theory it is manifest that two theories written around dual backgrounds could only be equivalent if their string field coupling constants are identical. The string field coupling constant is defined from the three-point couplings of states of the theory. If the spectra of two theories are identical, and the perturbative $S$-matrix is identical, the three point couplings ought to be the same, thus the coupling constants must be identical. Duality therefore does not involve a shift in the string field dilaton. This result is in agreement with first quantization analysis, if this analysis is properly interpreted. That will be the subject of section 6.

The concrete way of proving the physical equivalence of the two theories will pave the way for our writing of the symmetry transformations that leave the string action invariant (Sect. 7). In the present section we will find, for each discrete symmetry transformation $g \in O(d,d;\mathbb{Z})$ a unitary operator $U_g$ that will have the following fundamental property

$$S_{E}(U_{g}|\Psi\rangle) = S_{g(E)}(|\Psi\rangle),$$

for any background $E$, where $g(E)$ denotes the background obtained by acting with the transformation $g$ on the background $E$. Equation (4.1) shows that these two string field theories are related by the homogeneous invertible field redefinition $|\Psi\rangle \rightarrow U_{g}|\Psi\rangle$, and therefore the field theories are physically equivalent.

It should be emphasized that finding this operator $U_g$ relating two apparently different theories does not yet give us a symmetry transformation. A symmetry transformation corresponds to an invariance of an action, and the above operator does not yet give us
any such invariance. The operator $U_g$, however, will be a key element in the symmetry transformation to be constructed in Sect. 7.

This section is divided into three parts. In the first one we review the necessary properties of the group $O(d, d; \mathbb{Z})$ and the definition of its action on the backgrounds. In the second part we construct the operator $U_g$, show it is universal, prove (4.1) and show that the operators $U_g$ form a representation of the discrete group $O(d, d; \mathbb{Z})$.

4.1. $O(d, d; R)$ and Background Fields

The group $O(d, d; R)$ is defined by its elements, real matrices $g$ of size $2d \times 2d$, such that $g^t J g = J$, where $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. We can spell out explicitly the conditions for a matrix to belong to $O(d, d; R)$. Denote $g$ by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow g^t = \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix}. \quad (4.2)$$

where $a, b, c, d$ are $d \times d$ matrices. The conditions for $g \in O(d, d; R)$ are

$$a^t c + c^t a = b^t d + d^t b = 0, \quad \text{and} \quad a^t d + c^t b = I. \quad (4.3)$$

These relations tell, in particular, that $(a^t c)$ and $(b^t d)$ are antisymmetric matrices. Useful consequences of the above conditions are derived next. If $g \in O(d, d; R)$, then $g^t \in O(d, d; R)$. This is proven as follows: begin with $g^t J g = J$, then take the inverse in both sides to find $g^{-1} J g^{t-1} = J$ (since $J^{-1} = J$). Now multiply from the left by $g$, and from the right by $g^t$ to find $J = g J g^t$, which shows that $g^t \in O(d, d; R)$. If we now apply the conditions in (4.3) to $g^t$ we find

$$a b^t + b a^t = c d^t + d c^t = 0, \quad \text{and} \quad a d^t + b c^t = I, \quad (4.4)$$

which means that $(a b^t)$ and $(c d^t)$ are also antisymmetric matrices. With all this information, it is possible now to check that

$$g^{-1} = \begin{pmatrix} d^t & b^t \\ c^t & a^t \end{pmatrix}. \quad (4.5)$$

In fact, this follows directly from $g^t J g = J \rightarrow (J g^t J) g = J^2 = I \rightarrow g^{-1} = J g^t J$, which is the result quoted above.

Let us now review the action of $O(d, d; R)$ on the background field $E = G + B$. In order to have $2d \times 2d$ matrices in $O(d, d; R)$ act on the $d \times d$ matrix $E$ one uses
linear fractional transformations. Let $g \in O(d,d;R)$ be given by (4.2). We denote by $E' = g(E)$ the new background obtained by acting with $g$ on the background $E$. The background $E'$ is given by

$$E' = g(E) \equiv (aE + b)(cE + d)^{-1}. \tag{4.6}$$

This definition can be checked to be consistent with the group property: $g(g'(E)) = gg'(E)$. Let us now derive a few useful relations that arise from (4.6), which we now denote as

$$E' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} E. \tag{4.7}$$

Solving for $E$ from (4.6) and transposing, one finds

$$E^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} E'^t, \tag{4.8}$$

where the matrix above is readily verified to belong to $O(d,d;R)$. Taking inverses to the previous two equations one also finds the useful relations (using (4.5))

$$E = \begin{pmatrix} d^t & b^t \\ c^t & a^t \end{pmatrix} E', \quad E'^t = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} E^t. \tag{4.9}$$

An extra pair of relations will be useful:

$$(d + cE)^t G' (d + cE) = G, \quad (d - cE^t)^t G' (d - cE^t) = G. \tag{4.10}$$

The first relation is derived by writing $G' = (E' + E'^t)/2$, using the expression for $E'$ from (4.6) and evaluating the left hand side. The second equation is derived similarly beginning with the expression for $E'^t$ given in (4.9).

Backgrounds related by generic $O(d,d;R)$ do not give equivalent physics. We need to restrict ourselves to $O(d,d;Z)$. At the level of the spectrum this follows from the form of the first quantized hamiltonian $H(E)$

$$H(E) = \frac{1}{2} \bar{Z}^t R(E) \bar{Z} + N + \bar{N} + \cdots \tag{4.11}$$

where $N$ and $\bar{N}$ denote the number operators, the dots represent terms irrelevant for our discussion, and $\bar{Z}$ denotes a $2d$-column vector with integer entries $\bar{Z} = \begin{pmatrix} \bar{m} \\ \bar{n} \end{pmatrix}$, where
the integers \( n_i \) and \( m^j \) (with \( i, j = 1, \cdots, d \)) represent momentum and winding quantum numbers respectively. The matrix \( R \) was defined in (2.8) and has the property that \( R(E') = g R(E) g^t \) if \( E' = g(E) \). There is a further condition on the spectrum, one must have
\[
\frac{1}{2} \tilde{Z}^t J \tilde{Z} = N - \bar{N}. \tag{4.12}
\]
Consider now a background \( E' = g(E) \), with \( g \in O(d, d; R) \). One then has
\[
H(E') = \frac{1}{2} \tilde{Z}'^t R(E') \tilde{Z}' + N' + \bar{N}' + \cdots = \frac{1}{2} \tilde{Z}^t g R(E) g^t \tilde{Z}' + N' + \bar{N}' + \cdots. \tag{4.13}
\]
Equations (4.11) and (4.13) can define the same spectrum if we can consistently set \( \tilde{Z} = g^t \tilde{Z}' \), and thus think of the two spectra as identical, although labeled by different momentum and winding quantum numbers. Two requirements are enough, \( g^t \) must be invertible (it is so), and all its entries must be integer (otherwise there would exist some integer vectors \( \tilde{Z}' \) that would be mapped into non integer vectors). There is one extra condition coming from eqn. (4.12), we need \( \tilde{Z}^t J \tilde{Z} = \tilde{Z}'^t J \tilde{Z}' \) which guarantees that an allowed state remains allowed after the relabeling of the quantum numbers (and without change of the oscillator excitations). This requires \( gJg^t = J \), which is satisfied since \( g \in O(d, d; R) \). Thus, all our discussion just shows that backgrounds related by \( O(d, d; R) \) transformations with integer entries, give an identical spectrum.

There is one extra discrete symmetry beyond \( O(d, d; Z) \). It corresponds to taking \( B \to -B \). From the form of the hamiltonian this is seen to be a symmetry of the spectrum which is taken care by letting \( m^i \to -m^i \) (or \( n_i \to -n_i \)). Since this change alters the sign in the constraint (4.12), one must also have \( N \leftrightarrow \bar{N} \), by exchanging the right moving and left moving oscillators. This clearly does not change the contribution of \( N + \bar{N} \) to the hamiltonian.

### 4.2. The Unitary Operator \( U_g \)

We have seen that the definition of the universal objects \( X, P \) in term of background dependent oscillators led to definite relations between any two sets of oscillators corresponding to two different backgrounds. Those relations were given in equation (2.18). Such relations, of course, are consistent with the background dependent commutation relations of the oscillators. They always mix mode numbers, in particular, oscillators of mode number \( +n \) are related to oscillators of mode numbers \( +n \) and \( -n \). The only way to avoid mode number mixing is to have identical backgrounds. This is sensible because different backgrounds correspond to different string field theory vacua.

The above arguments do not rule out the possibility that the physics of different vacua is the same. This is actually a well known fact in first quantization analysis of toroidal compactification. In our language the key idea is that we can define maps (not equalities)
between sets of oscillators, and these maps will respect the commutation relations. These maps will be realized by the operators we are after. Let us begin by recalling that the commutation relations of the oscillators \( \{\alpha, \bar{\alpha}\} \) are conveniently summarized by

\[
[X'_{ij}(\sigma), P_{j'}(\sigma')] = i\delta_{ij} \frac{d}{d\sigma} \delta(\sigma - \sigma'),
\]

(4.14)

These commutation relations, however, are left unchanged under the following replacement

\[
\begin{pmatrix} X' \\ 2\pi P \end{pmatrix} \to \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} X' \\ 2\pi P \end{pmatrix},
\]

(4.15)

if the above matrix performing the map belongs to \( O(d,d; R) \). Since this map implies that the zero modes \( (w, p) \) are transformed as

\[
\begin{pmatrix} w \\ p \end{pmatrix} \to \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} w \\ p \end{pmatrix},
\]

(4.16)

and the eigenvalues of \( (w, p) \) are integers, the matrix performing the map must actually belong to \( O(d,d; Z) \).

Note that the map, which may be labeled by the group element, is by definition background independent (since \( X' \) and \( P \) are). Since the map preserves commutation relations it must be possible to obtain via a unitary operator. In order to find the implications of the map for oscillators we have a choice of backgrounds to make on the left and on the right of the arrow in (4.15). Let us choose the background \( E \) for the left hand side and the background \( E' \) (which may or may not be different for the right hand side), the map (4.15) then implies

\[
[\bar{\alpha}_{-n} - \alpha_n](E) \to a^t[\bar{\alpha}_{-n} - \alpha_n](E') + c^t[E'^t \alpha_n + E'\bar{\alpha}_{-n}](E'),
\]

\[
[E'^t \alpha_n + E\bar{\alpha}_{-n}](E) \to b^t[\bar{\alpha}_{-n} - \alpha_n](E') + d^t[E'^t \alpha_n + E'\bar{\alpha}_{-n}](E').
\]

(4.17)

From the above maps, a small calculation gives

\[
2G\alpha_n(E) \to \left[ E(-c^t E'^t + a^t) + (d^t E'^t - b^t) \right] \alpha_n(E'),
\]

\[
+ \left[ -E(c^t E' + a^t) + (d^t E' + b^t) \right] \bar{\alpha}_{-n}(E'),
\]

\[
2G\bar{\alpha}_{-n}(E) \to \left[ E'(c^t E'^t - a^t) + (d^t E'^t - b^t) \right] \alpha_n(E'),
\]

\[
+ \left[ E'(c^t E' + a^t) + (d^t E' + b^t) \right] \bar{\alpha}_{-n}(E').
\]

(4.18)

We now want to think of the \( O(d,d; Z) \) matrix as fixed, and find if the above maps become diagonal in mode number for a particular choice of \( E' \). This requires the following
conditions
\[-E(c^t E' + a^t) + (d^t E' + b^t) = 0,\]
\[E^t(c^t E'^t - a^t) + (d^t E'^t - b^t) = 0.\]  (4.19)

It follows from (4.8) and (4.9), that the above conditions are simultaneously satisfied if

\[E' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} E = g(E),\]  (4.20)

and therefore for this choice of \(E'\) the maps do not change mode number. A little calculation gives

\[G\alpha_n(E) \rightarrow (d^t - E c^t)G'\alpha_n(E'),\]
\[G\bar{\alpha}_n(E) \rightarrow (d + cE)^tG'\bar{\alpha}_n(E'),\]  (4.21)

and using (4.10) we obtain the simplest form of the map

\[\alpha_n(E) \rightarrow (d - cE^t)^{-1}\alpha_n(E'),\]
\[\bar{\alpha}_n(E) \rightarrow (d + cE)^{-1}\bar{\alpha}_n(E').\]  (4.22)

One can verify explicitly that the above maps hold also for \(n = 0\), namely, their action on \(\alpha_0, \bar{\alpha}_0\), is consistent with (4.16) via (2.20). We want to define now the unitary operator that performs the above map. We will denote this operator by \(U_g\). Rather than trying to construct the operator explicitly in terms of the operators \(X, P\), we will define the operator by describing how it acts on states (this defines completely the operator). We therefore write

\[U_g^\dagger \alpha_n(E) U_g = (d - cE^t)^{-1}\alpha_n(E'),\]
\[U_g^\dagger \bar{\alpha}_n(E) U_g = (d + cE)^{-1}\bar{\alpha}_n(E').\]  (4.23)

Since we require that the operator be unitary, the above relations determine \(U_g\) up to phases. We fix those phases now:

\[U_g |w, p\rangle_E = |a^t w + c^t p, b^t w + d^t p\rangle_{E'}.\]  (4.24)

One can verify that the state on the right hand side must be the one shown (up to a phase) by acting on the left hand side with various operators, for example

\[\hat{p} U_g |w, p\rangle_{E'} = U_g U_g^\dagger \hat{p} U_g |w, p\rangle_{E'} = U_g (b^t \hat{w} + d^t \hat{p}) |w, p\rangle_{E'} = (b^t w + d^t p) U_g |w, p\rangle_{E'},\]  (4.25)

where from the quoted result follows. The fact that the operator \(U_g\) turns the \(E'\) vacuum
into the $E$ vacuum follows from
\[ \alpha_n(E) U_g |0\rangle_{E'} = U_g U_g^\dagger \alpha_n(E) U_g |0\rangle_{E'} \propto U_g \alpha_n(E') |0\rangle_{E'} = 0, \]
which holds for all positive $n$. We therefore have
\[ U_g |0\rangle_{E'} = |0\rangle_E \iff U_g^\dagger |0\rangle_{E'} = |0\rangle_{E'}. \] (4.27)

Note that the action of $U_g$ on operators was defined in a background independent way via (4.15). It should be emphasized that $U_g$ is an operator relating states in different Hilbert spaces, unless the original state is in a Hilbert space corresponding to a background that is invariant under the group element $g$.

Let us now find the action of $U_g$ on the BRST operator and on the vertex. Most results will follow from the action of $U_g$ on oscillator bilinears
\[ U_g^\dagger \alpha_n(E) G \alpha_m(E) U_g = \alpha_n(E') (d - c E')^{-1t} G (d - c E')^{-1} \alpha_m(E') \]
\[ = \alpha_n(E') G' \alpha_m(E'), \] (4.28)
where use was made of (4.10). The same equation holds for the antiholomorphic oscillators. It therefore follows that we have a very simple action on the Virasoro generators:
\[ U_g^\dagger \left( \begin{array}{c} L^X(E) \\ \bar{L}^X(E) \end{array} \right) U_g = \left( \begin{array}{c} L^X(E') \\ \bar{L}^X(E') \end{array} \right), \] (4.29)
and this result implies that
\[ U_g^\dagger Q_B(E) U_g = Q_B(E'), \] (4.30)
namely, that the operator $U_g$ changes the BRST operator from that corresponding to the original background $E$ into that corresponding to the background $g(E)$.

Let us now consider the three string vertex. Recall it is built of a vacuum, oscillator bilinears and a cocycle factor. Up to the cocycle factor, equations (4.28) and (4.27) imply that
\[ 123 \langle V(E) | U_g^{(1)} U_g^{(2)} U_g^{(3)} = 123 \langle V(E'). \] (4.31)

The cocycle factor is conveniently written as follows
\[ \exp(i\pi [p_3^t w_2 - p_1^t w_1]) = \exp(i\pi [p_3^t P p_2 - p_1^t J p_1]) \] (4.32)
where the matrix $P$ and the vector $p$ are defined by
\[ P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} w \\ p \end{pmatrix}, \] (4.33)
and $J$ is the $O(d,d;R)$ metric matrix. It follows that

$$
\exp(i\pi[p_3^t P p_2 - p_1^t J p_1])U^{(1)}_g U^{(2)}_g U^{(3)}_g
= U^{(1)}_g U^{(2)}_g U^{(3)}_g \exp(i\pi[p_3^t g P g^t p_2 - p_1^t J p_1]),
$$

which shows that the form of the cocycle factor is not preserved. The solution to this is to modify the operator $U_g$ by including an extra phase factor

$$
U_g = U_g \Upsilon(g, p)
$$

In order to give a simple description of $\Upsilon$ let us introduce some notation. For any matrix $A$, $A_u$ and $A_l$ are defined to be the upper and lower triangular part matrices of $A$, respectively; namely, for $A = (a_{ij})$

\[
(A_u)_{ij} = \begin{cases} a_{ij} & \text{for } i < j \\ 0 & \text{for } i \geq j \end{cases}, \quad (A_l)_{ij} = \begin{cases} 0 & \text{for } i \leq j \\ a_{ij} & \text{for } i > j \end{cases}
\]

Then, clearly, $A_l = ((A^t)_{ul})^t$, and, if $A$ is an antisymmetric matrix, $A = A_u + A_l$ and $A_l = -A_u^t$. Let us now give the form for the $\Upsilon$ factor:

$$
\Upsilon(g, p) = \exp(i\pi p^t A_u(g)p),
$$

where $A$ is an antisymmetric matrix given by

$$
A(g) = gPg^t - P = \begin{pmatrix} bat & bc^t \\ -cb^t & dc^t \end{pmatrix}.
$$

We now verify that the product $\Upsilon^{(1)} \Upsilon^{(2)} \Upsilon^{(3)}$, in the presence of the momentum conservation Kronecker deltas (for $p$) of the vertex, restores the cocycle to its original form. Begin with

$$
\Upsilon^{(1)} \Upsilon^{(2)} \Upsilon^{(3)} = \exp(i\pi p_3^t A_u p_2 + i\pi p_2^t A_u p_3)
= \exp(i\pi p_3^t A_u p_2 - i\pi p_3^t A_l p_2)
= \exp(i\pi p_3^t[A_u + A_l]p_2)
= \exp(i\pi p_3^t A p_2),
$$

where use was made of momentum conservation to eliminate $p_1$, and of the identities given above (4.36). It therefore follows that indeed the vertex cocycle is restored to its
Thus, we have finally obtained the proper unitary operator $U_g$ that leaves invariant the three string vertex

$$123\langle V(E)|U_g^{(1)}U_g^{(2)}U_g^{(3)} = 123\langle V(E')|.$$ (4.40)

[We say “invariant” because the vertex $\langle V(E)|$ is actually independent of $E$ as shown in Sect. 3.2.] This proves that the interaction term of the action does not change under the homogeneous redefinition of the string field induced by $U$. We must now check that the kinetic term is changed from that corresponding to the background $E$ to that corresponding to the background $E' = g(E)$. For this purpose one first verifies that the operator $U$ acts on the reflector state $12\langle R(E)|$ as expected

$$12\langle R(E)|U_g^{(1)}U_g^{(2)} = 12\langle R(E')|,$$ (4.41)

since the vertex cocycle of $\langle R|$ is not changed by the $U$ operators, and the $\Upsilon$ factors vanish by momentum conservation. Equation (4.41) implies that

$$12\langle R(E)|U_g^{(1)} = 12\langle R(E')|U_g^{(2)\dagger}, \quad U_g^{(1)}|R(E)\rangle_{12} = U_g^{(2)\dagger}|R(E')\rangle_{12}. \quad (4.42)$$

With this information, we immediately find that the string kinetic term becomes

$$12\langle R(E)||\Psi\rangle_1 Q_{B}^{(2)}(E)b_0^{-(2)}|\Psi\rangle_2 \rightarrow 12\langle R(E)|U_g^{(1)}|\Psi\rangle_1 Q_{B}^{(2)}(E)b_0^{-(2)}U_g^{(2)}|\Psi\rangle_2$$

$$= 12\langle R(E')||\Psi\rangle_1 U_g^{(2)\dagger} Q_{B}^{(2)}(E)\Upsilon U_g^{(2)}b_0^{-(2)}|\Psi\rangle_2$$

$$= 12\langle R(E')||\Psi\rangle_1 Q_{B}^{(2)}(g(E))b_0^{-(2)}|\Psi\rangle_2$$ (4.43)

as desired. This completes our proof of $S_{E}(U|\Psi\rangle) = S_{g(E)}(|\Psi\rangle)$.

The action of $U$ on the star product will be of utility later; using (4.40) and (4.42) one finds

$$U|A \star B\rangle = |UA \star UB\rangle,$$ (4.44)

where $A$ and $B$ are arbitrary string fields.
4.3. **Group Properties of** $U$

Having constructed the operator $U$ and established both its background independence, and how it relates string field theories on dual backgrounds, we now establish the group properties of the operators $U$. We will show that

$$U_g U_g' = U_{gg'} \exp[i \pi C(p, g', g)],$$  \hspace{1cm} (4.45)

namely, that the operators $U$ form a projective representation of the discrete group of dualities, with $\exp(i \pi C)$ a nontrivial cocycle factor. It follows directly from our expression for $U$ that

$$\exp[i \pi C(p, g', g)] = \exp (i \pi p^t [A_u(g) + gA_u(g')g^t - A_u(gg')]p).$$  \hspace{1cm} (4.46)

We will concentrate on the cocycle factor only since it is clear from sect. 4.2 that the operators $U$ satisfy $U_g U_g = U_{gg'}$. It is also clear that the above cocycle factor satisfies the conditions that arise from the associativity of the $U$ operators, since this cocycle was derived from operators that associate.

Let us show that only the diagonal piece of $C$ is relevant. It follows from (4.37) that

$$A(gg') = gg' P(gg')^t - P = g[g' P g'^t] g^t - P = g[A(g') + P] g^t - P = g A(g') g^t + A(g),$$

and taking the upper triangular part of this matrix equation we have

$$A_u(gg') = (g A(g') g^t)_u + A_u(g).$$

Using this in (4.46), we have now

$$\exp[i \pi C(p, g', g)] = \exp (i \pi p^t (g A_u(g') g^t - (g A(g') g^t)_u) p).$$  \hspace{1cm} (4.47)

Note that inside the square bracket $[\cdots]$ above, any matrix can be transposed freely and its sign can be changed at will (the matrices are integer-valued). We denote equality in this sense by $\sim$. Then, for the first term, we have

$$g A_u g^t = (g A_u g^t)_u + (g A_u g^t)_l + (g A_u g^t)_d$$

$$= (g A_u g^t)_u + ((g A_u g^t)_u)^{\dagger} + (g A_u g^t)_d$$

$$\sim (g A_u g^t)_u - [(g A_u g^t)_u] u + (g A_u g^t)_d$$

where $A_u$ here means $A_u(g')$ and $A_d$ denotes a diagonal matrix obtained by setting all the entries of $A$ other than the diagonal elements equal to zero. For the second term in
Since $A$ is an antisymmetric matrix, we have
\[(gAg^t)_u = [g(Au - (Au)^t)g^t]_u = (gAu^t)_u - [(gAu^t)^t]_u.\]

This shows that only the diagonal matrix part $(gAu(g')g^t)_d$ survives, and this gives us the simplest form for the cocycle factor:
\[\exp[i\pi C(p, g', g)] = \exp\left(i\pi \sum_{i=1}^{d} (a_iw_i^2 + b_ip_i^2)\right).\] 

If we write $(gAu(g')g^t)_d \equiv \text{diag}(a_i, b_i)$ the above expression becomes
\[\exp[i\pi C(p, g', g)] = \exp\left(i\pi \sum_{i=1}^{d} (a_iw_i^2 + b_ip_i^2)\right).\] 

We can see now why the cocycle appearing in the composition of two $U$ operators is irrelevant when acting on the three string vertex: momentum conservation shows that $\sum_{r=1}^{3} w_r^2$ and $\sum_{r=1}^{3} p_r^2$ are necessarily even integers and hence $\prod_{r=1}^{3} \exp(i\pi C^{(r)}) = 1$. Incidentally, this also shows that the string field theory is invariant under the following field transformations:
\[\Psi \to \Psi' = \exp\left(i\pi \sum_{i=1}^{d} (a_iw_i^2 + b_ip_i^2)\right)\Psi,\]

with $a_i$ and $b_i$ taking the values of 0 or 1. These are parity-like transformations on the component fields; for instance, in the case of $a_i = \delta_{11}$ and $b_i = 0$, the component fields corresponding to odd $w_1$ eigenvalues change their signs and those corresponding to even eigenvalues remain unchanged. The complete set of such parity transformations form a discrete closed group with $2^d$ elements in all.

Before closing this section let address the issue of the nontriviality of the cocycle in (4.45). If the cocycle were trivial, a suitable redefinition of the $\Upsilon$ factor would eliminate it. So consider redefining our $\Upsilon$ factor (4.36) into
\[\Upsilon(g, p) = \exp [i\pi p^t (Au(g) + M(g))p]\] 

by adding an integer-valued matrix function $M(g)$ of $g$. This new factor $\Upsilon$ must still preserve the form of the three string vertex (otherwise we could just take $M = -Au$). Following our previous analysis we conclude that $M$ must be diagonal. If this were not the case, then in addition to the diagonal piece, which preserves the vertex cocycle, $M$ would have an extra piece $M = Au$ for some antisymmetric integer matrix $A$. Then
the vertex cocycle would acquire the extra factor \( \exp(i\pi p_3^t A p_2) \). Since \( p_3 \) and \( p_2 \) are independent, and this factor must be unity always, \( A \) must be zero. This shows \( M \) is diagonal. Then, in view of Eq. (4.48), the condition for the cocycle factor to vanish for this \( \Upsilon \) is given by the equation

\[
M(gg') \sim M(g) + gM(g')g^t - (gA_u(g')g^t)_d
\]

(4.52)

where \( \sim \) means equality when used inside the brackets in \( \exp(i\pi p^t[\cdots]p) \). In fact, inside the brackets any symmetric integer matrix can be replaced by its diagonal part, so the condition for the triviality of the cocycle reduces to

\[
M(gg') \sim M(g) + (gM(g')g^t)_d - (gA_u(g')g^t)_d
\]

(4.53)

We do not know if a diagonal matrix \( M(g) \) satisfying this equation exists.

5. Condensation of States and Classical Solutions

In this section we will discuss classical solutions of string field theory, in particular, classical solutions that correspond to changes of backgrounds. It is important to emphasize that the classical solutions that change backgrounds need not involve change of coupling constant of the theory. To make that clear we will first consider the condensation of the dilaton, and explain how it induces a change of coupling constant. In this respect, our work is an extension of that of Refs. [12,15].

Consider a state \( |S\rangle \) which condensates or acquires a vacuum expectation value. Assume that when contracted with the three-string vertex it gives the following result

\[
_{123}\langle V||S\rangle_3 = \ _{12}\langle R|[O_S^{(2)}, Q^{(2)}_B]b_{0}^{-}\rangle_2,
\]

(5.1)

namely, it can be written as a commutator of the BRST operator with a (Grassman even) differential \( O_S \). Such property implies that the condensation of \( |S\rangle \) converts the three-string interaction term into kinetic-like term:

\[
_{123}\langle V||\Psi_1|\Psi_2|S\rangle_3 = \ _{12}\langle R|[\Psi_1, Q^{(2)}_S, Q^{(2)}_B]b_{0}^{-}\rangle_2|\Psi_2\rangle
\]

(5.2)

\[
= \langle \Psi|[O_S, Q_B]b_{0}^{-}\Psi\rangle.
\]

If the operator \( O_S \) is anti-hermitian, this change of kinetic term can be cancelled by making a homogeneous field redefinition by \( O_S \). Indeed, the following (infinitesimal)
inhomogeneous field transformation
\[ \delta |\Psi\rangle = \frac{1}{g} |S\rangle + \mathcal{O}_S |\Psi\rangle \] (5.3)
gives the following change of the action (3.4):
\[ \delta S = \frac{2}{g} \langle \Psi | Q_B b_0^- | S \rangle 
+ g \cdot \frac{1}{g} 1_{23} \langle V | |\Psi\rangle_1 |\Psi\rangle_2 |S\rangle_3 
+ \langle \Psi | [Q_B, \mathcal{O}_S] b_0^- |\Psi\rangle 
+ g \cdot \frac{1}{g} 1_{23} \langle V | (\sum_{r=1}^3 \mathcal{O}_S^{(r)}) |\Psi\rangle_1 |\Psi\rangle_2 |\Psi\rangle_3. \] (5.4)

The first term in the right hand side vanishes if the equation of motion
\[ Q_B b_0^- |S\rangle = 0 \] (5.5)
is satisfied, and the two terms in the second line cancel each other out because of eq.(5.2).
Therefore, if the vertex \( \langle V \rangle \) is an eigenstate of the operator \( (\sum_{r=1}^3 \mathcal{O}_S^{(r)}) \),
\[ \langle V | (\sum_{r=1}^3 \mathcal{O}_S^{(r)}) = \lambda_S \langle V \rangle, \] (5.6)
with eigenvalue \( \lambda_S \), then the total change of the action is simply given by
\[ \delta S = \frac{\lambda_S g}{3} 1_{23} \langle V | |\Psi\rangle_1 |\Psi\rangle_2 |\Psi\rangle_3, \] (5.7)
implying that the string field coupling constant \( g \) is changed by the field condensation \( g^{-1} |S\rangle \) by the amount
\[ \delta g = \lambda_S g. \] (5.8)

It should be noted that the role of the homogeneous part of the field transformation was that of a field redefinition useful to bring the action to a form where one could read directly the fact that the string coupling constant had changed.
5.1. Dilaton condensation

We now apply the above discussion to the condensation of the dilaton state

\(|D\rangle = c_0^{-} \left[ \alpha_{-1}^{-} \eta_{\mu,\nu} \alpha_{-1}^{-} + c_{-1}^{-} \delta_{-1}^{-} - b_{-1}^{-} \bar{\delta}_{-1}^{-} \right] |0\rangle \delta_\epsilon(p, w),

\(\delta_\epsilon(p, w) = \lim_{\epsilon \to 0} \frac{1}{2}[\delta(p^+ - \epsilon) + \delta(p^+ + \epsilon)](2\pi)^{-d} \left( \prod_{\mu \neq +} \delta(p^\mu) \right) \cdot \delta^d(p) \delta^d(w), \quad (5.9)\)

in the \(\alpha = p^+\) HIKKO theory. A calculation similar to that of Hata and Nagoshi [15] shows that condensation of this dilaton yields the following differential operator:

\(\mathcal{O}_D = \frac{1}{2} \left[ \mathcal{D}_X + \mathcal{D}_{FP} + N_{FP} - \left\{ p^+, \frac{\partial}{\partial p^+} \right\} \right]. \quad (5.10)\)

where the dilation operators \(\mathcal{D}_X, \mathcal{D}_{FP}\) and ghost number operator are defined as:

\(\mathcal{D}_X = \frac{1}{2} \left\{ p_\mu, \frac{\partial}{\partial p_\mu} \right\} + \sum_{n \neq 0} \frac{1}{n} \alpha_n \eta_{\mu,\nu} \alpha_n^\prime\)

\(\mathcal{D}_{FP} = \frac{1}{2} \left[ b_0^+, c_0^+ \right] + \sum_{n \neq 0} b_n \bar{c}_n + \bar{b}_n c_n \quad (5.11)\)

\(N_{FP} = \frac{1}{2} \left[ c_0^+, b_0^+ \right] + \sum_{n \neq 0} : c_n b_n + \bar{c}_n \bar{b}_n :\)

Since the operator \(\mathcal{O}_D\) is anti-hermitian and the dilaton state \(|D\rangle\) satisfies the on-shell equation (5.5), the dilaton condensation

\(\delta |\Psi\rangle = \frac{1}{g} |D\rangle + \mathcal{O}_D |\Psi\rangle, \quad (5.12)\)

yields a change of the coupling constant ((5.8)):

\(\delta g = \lambda_D g, \quad \text{where} \quad \langle V | \mathcal{O}_D = \lambda_D \langle V |, \quad (5.13)\)

where \(\mathcal{O}_D\) stands for the sum \(\sum_{r=1}^{3} \mathcal{O}_D^{(r)}\). The eigenvalue \(\lambda_D\) of the operator \(\mathcal{O}_D\) defined in (5.10) is obtained as follows. Noting that \(\mathcal{D}_X\) and \(\mathcal{D}_{FP}\) act on oscillators as

\(\mathcal{D}_X : \quad \alpha_n \rightarrow \bar{\alpha}_{-n}, \quad \bar{\alpha}_n \rightarrow \alpha_{-n},\)

\(\mathcal{D}_{FP} : \quad c_n \leftrightarrow -\bar{c}_{-n}, \quad b_n \leftrightarrow \bar{b}_{-n}, \quad (5.14)\)

we find the following eigenvalues for each factor appearing in the vertex (3.25) under the action of the operators \(\mathcal{D}_X, \mathcal{D}_{FP}, N_{FP}\) and \(- \left\{ p^+, \frac{\partial}{\partial p^+} \right\}\):

\* Dilaton condensation is difficult to analyze in the original HIKKO string field theory due to the fact that changes in the unphysical string length parameter change the effective coupling constant of the theory.
\[ D_X = \frac{d_0}{2} + p^\mu \frac{\partial}{\partial p^\mu} + (\text{oscillators}) \]

\[ D_{FP} = \frac{1}{2} - c_0^+ b_0^+ + (\text{oscillators}) \]

\[ N_{FP} = -\frac{1}{2} + c_0^+ b_0^+ + (\text{oscillators}) \]

\[ - \left\{ p^+, \frac{\partial}{\partial p^\tau} \right\} = -1 - 2p^+ \frac{\partial}{\partial p^\tau} \]

where \( d_0 \equiv D - d \) is the number of uncompactified coordinates. Note that these are quantum numbers for the \textit{ket-state} vertex \( |V\rangle \). For the desired bra \( \langle V| \), they change signs because of anti-hermiticity of those operators. The factor of three in the last column comes from the fact that these operators actually stand for sums over the three strings. So from eq.(5.10) for \( O_D \), we have

\[ \langle V|O_D = \lambda_D \langle V|, \quad \lambda_D = -\frac{1}{4}(d_0 - 2). \] (5.15)

Note that the coupling constant change we have obtained is proportional to the transversal dimensions \( (d_0 - 2) \) and the \(-2\) contribution came from \(- \left\{ p^+, \frac{\partial}{\partial p^\tau} \right\} \).

It may be interesting to note that the condensation of the following “transversal dilaton” \( |trD\rangle \) also gives the same change of coupling constant:

\[ |trD\rangle = c_0^- \left[ \sum_{\mu,\nu \neq \pm} \alpha_{-1}^{\mu} \eta_{\mu\nu} \bar{\alpha}_{-1}^\nu \right] |0\rangle \delta_\epsilon(p, w), \] (5.16)

\[ \longrightarrow O_{trD} = \frac{1}{2} D_X^{\text{transverse}} \longrightarrow \delta g = -\frac{1}{4}(d_0 - 2)g, \]

where

\[ D_X^{\text{transverse}} = \frac{1}{2} \sum_{\mu \neq \pm} \left\{ p_\mu, \frac{\partial}{\partial p_\mu} \right\} + \sum_{n \neq 0} \frac{1}{n} \alpha_{n}^{\mu} \eta_{\mu\nu} \bar{\alpha}_{n}^\nu. \] (5.17)

Having understood how dilaton condensation changes the coupling constant we now turn to condensation of states that do not change the coupling constant but rather the string background.
5.2. Condensation of Exactly Marginal States

Our discussion of target space duality as a string field symmetry relies on the existence of suitable classical solutions that shift the backgrounds. Even more, the statement of background independence in the space of toroidal compactifications is essentially the statement that there are classical solutions that move us in this space.

Our discussion of condensation of states in this section begins at the infinitesimal level. Here the convenience of using the light cone style vertex will be manifest. In contrast with the less direct calculation which is possible in the covariant closed string field theory [20], we will be completely explicit. We will show how the change of the BRST operator comes about directly by the condensation. Moreover, after the infinitesimal condensation the action has the correct form at the new background. Our analysis benefitted from the earlier discussion of Itoh [13]. We then turn to finite condensation. We give an expression for the classical solution that corresponds to a finite change of background. Here several issues arise, having to do with the space where the solution lives, and with the singular nature of the light cone vertex. These will be discussed explicitly.

Let us begin by considering the condensation of the following exactly marginal state:

\[
|EM\rangle = c_{-1} \left[ \alpha_{-1} (E) a_{ij} \bar{\alpha}_{-1} (E) \right] |0\rangle \delta_{\epsilon} (p, w),
\]  

(5.18)

which is a state with a tensor coefficient \( a_{ij} \) in the curled up dimensions. Let us denote \( \alpha_{n} (E) \) simply as \( \alpha_{n} \) by omitting the background label \( (E) \), whenever confusion would not occur. The condensation of this state yields a contribution to the kinetic term of the theory coming from the contraction of \( |EM\rangle \) against the three string vertex. This is actually an interesting calculation, and therefore we give its complete details in Appendix C. The result of the condensation is

\[
123 \langle V || EM \rangle_3 = - \frac{1}{2} \sum_{\ell+n+m=0} (c_{\ell} + \bar{c}_{\ell}) (\alpha_{n} a_{ij} \bar{\alpha}_{-m}) b_{0}^{-2}. 
\]

(5.19)

Therefore, under the following infinitesimal transformation \((a_{ij} \ll 1)\):

\[
|\Psi\rangle \longrightarrow |\Psi\rangle' = \frac{1}{g'} |EM\rangle + |\Psi\rangle.
\]

(5.20)

it follows from eq.(5.19) that the kinetic operator is now changed into

\[
Q_{B}' = Q_{B} (E) - \frac{1}{2} \sum_{\ell+n+m=0} (c_{\ell} + \bar{c}_{\ell}) (\alpha_{n} a_{ij} \bar{\alpha}_{-m}).
\]

(5.21)

Comparing this with (2.28), we see that this transformed BRST operator \( Q_{B}' \) is just the
BRST operator on the new background \( E' = E - a \); i.e., \( \delta E = -a \).

\[
Q'_B = Q_B(E' = E - a). \tag{5.22}
\]

Conversely, the change of the background metric by an amount \( \delta E \) is realized by the following string field condensation

\[
\delta |\Psi_0\rangle = -\frac{1}{g}c_0^{-1} \left[ \alpha^i_{-1}(E) (\delta E)_{ij} \bar{\alpha}^j_{-1}(E) \right] |0\rangle_{\alpha(E)} \delta \epsilon(p, w). \tag{5.23}
\]

This is the form of the infinitesimal string field condensation we were looking for. Let us see what has happened with the complete action after the transformation indicated in (5.20). The kinetic term has become that of the new background and the three-string vertex now couples three \(|\Psi\rangle\)’s. Since the vertex is background independent the whole result is simply the string field theory around the new background, namely

\[
S_E(\delta |\Psi_0\rangle + |\Psi\rangle) = S_{E + \delta E}(|\Psi\rangle) + \mathcal{O}( (\delta |\Psi_0\rangle)^2 ). \tag{5.24}
\]

Let us now consider a finite change of the background fields from \( E_0 = G_0 + B_0 \) to \( E_1 = G_1 + B_1 \). We take an arbitrary interpolating path \( E(t) = G(t) + B(t) \) satisfying \( E(0) = E_0 \) and \( E(1) = E_1 \). Then we simply integrate the infinitesimal string field condensation along the path to obtain the following state:

\[
|\Psi_0\rangle \equiv |E_0; E_1\rangle = -\frac{1}{g}c_0^{-1} \int_0^1 dt \alpha^i_{-1}(t) \left( dE_{ij} \right)_t(t) \bar{\alpha}^j_{-1}(t) |0\rangle_{\alpha(t)} \delta \epsilon(p, w), \tag{5.25}
\]

where \( \alpha^i_{n}(t) \equiv \alpha^i_{n}(E(t)) \). Formally, this state is expected, as a consequence of (5.24) to shift the string action from the background \( E_0 \) to the background \( E_1 \), namely

\[
S_{E_0}(|\Psi_0\rangle + |\Psi\rangle) = S_{E_1}(|\Psi\rangle). \tag{5.26}
\]

Note that this equation does not involve the value of the original action \( S_{E_0}(|\Psi_0\rangle) \) at the classical solution because this constant is zero. Indeed, let \( S(t) \equiv S(|\Psi\rangle = |\Psi_0(t)\rangle) \), where \( |\Psi_0(t)\rangle \) denotes the state indicated in equation (5.25) integrated only up to \( t \). The constant in question is \( S(t = 1) \). It follows that

\[
\frac{dS}{dt}(t) = \left( \frac{\delta S}{\delta \Psi} \right)_{\Psi=\Psi_0(t)} \cdot \frac{d\Psi_0}{dt} = 0, \tag{5.27}
\]

since \( \Psi_0(t) \) is a classical solution for all \( t \). Since \( S(t = 0) = 0 \), then \( S(t) \equiv 0 \), and we verify that there is no constant term in (5.26).
Why didn’t we consider the differential operator \( \mathcal{O}_{\text{EM}} \) in analogy to the dilaton case? There is an “operator” \( \mathcal{O}_{\text{EM}} \) for which eq.(5.1) holds. It is given by

\[
\mathcal{O}_{\text{EM}} = \frac{1}{2} \left( \frac{\partial}{\partial \alpha^0_i} (G^{-1} a^i_j \bar{\alpha}^0_j + \alpha^0_i (aG^{-1})^j_i \frac{\partial}{\partial \bar{\alpha}^0_j} + \sum_{n \neq 0} \frac{1}{n} \alpha^i_n a^i_j \bar{\alpha}^n_j) \right),
\]

(5.28)

but this operator is completely ill-defined because it contains the differential operators \( \frac{\partial}{\partial \alpha^0_i} \) and \( \frac{\partial}{\partial \bar{\alpha}^0_j} \) that correspond to the zero-mode coordinate operators \( x^i \) and \( q_i \) in the compactified directions. (It is, however, interesting to note the similarity of the non-zero mode part of \( \mathcal{O}_{\text{EM}} \) and the generator \( \mathcal{B} \) of the Bogoliubov transformation in (3.40). Furthermore, when the zero modes take continuous values, \( \mathcal{O}_{\text{EM}} \) becomes a generator of \( O(d, d; R) \).)

There is actually one difficulty with the above classical solution arising from our use of the light-cone vertex. It can be verified that this solution is not path-independent. The simplest way to check this is to perform two infinitesimal string field condensations \( \delta E_1 \) and \( \delta E_2 \) successively, but in two different orders. These would read

\[
|\Psi\rangle = \alpha_{-1}^\star (E) \delta E_1 \bar{\alpha}_{-1}^\star (E) |0\rangle_E + \alpha_{-1}^\star (E + \delta E_1) \delta E_2 \bar{\alpha}_{-1}^\star (E + \delta E_1) |0\rangle_{E + \delta E_1},
\]

(5.29)

and a similar one with the labels 1 and 2 exchanged. A simple calculation, using the Bogoliubov transformed vacua to first order, shows they are not equal. By performing the shifts in the order \( \delta E_1, \delta E_2, -\delta E_1, -\delta E_2 \), this gives us a nonvanishing string field corresponding to a condensation that should not change the background. This would seem impossible on account that the BRST operator should be shifted as \( Q \rightarrow Q + \Psi_0^\star \), where \( \Psi_0 \) is the classical solution. But in the light cone field theory there exist nonvanishing string fields whose product with any ordinary string field (of nonzero \( p_+ \)) is zero (see Ref. [28]). Such pathology is not expected to occur for a string field theory whose star product does not admit singular configurations, as is the case of the nonpolynomial closed string field theory.

Let us now discuss the most important issue, that of the space where the classical solutions are expected to live. At face value one may think that our finite classical solution should live in the Hilbert space of the original string theory. We could then speak of the string classical solution as a collection of classical solutions for the component fields of the theory. This does not seem to be possible, according to (5.25). The classical solution is a sum (actually integral) of very simple Fock space vectors, but each on a different vacuum. Since we are dealing with a system with infinite number of degrees of freedom (oscillators) it turns out that the different vacua, as related formally by Bogoliubov transformations are actually orthogonal. Their inner product is always zero! There is no way we can perform the sum in a single Hilbert space, unless we cut-off the number of oscillators. As the cutoff is removed the difficulties reappear. We may be
forced to admit that nontrivial classical solutions must live outside the Hilbert space of
the original background, but this will demand that we learn how to define string field
theory beyond the usual methods based on oscillator expansions. The natural language
for string field theory, at any rate, is likely to be that of functionals, and we may be
learning that restricting ourselves to functionals corresponding to a single vacuum is a
very unnatural thing to do. Understanding the implications of this fact is possibly the
most important issue that we face in string field theory.

5.3. Solving the Classical Equations Recursively

In this section we find an explicit expression for the classical string field solution
according to an exactly marginal operator. The solution will be expressed as a
series, and if the series converges, it will define a classical solution corresponding to a
finite shift of background. This solution applies to any form of closed string field theory
using symmetric vertices. We have in mind, of course, the nonpolynomial closed string
field theory. The convergence of the series depends on the off-shell behavior of the theory.

String field condensation that changes the toroidal backgrounds corresponds to ex-
actly marginal operators of the conformal field theory. Such operators are dimension
(1,1) primary fields of the form current-current $J \bar{J}$. The corresponding BRST invariant
states, which are used in the string field theory, are the states created by the dimension
(0,0) operator $cJ\bar{c}\bar{J}$. Such states, for our case can be chosen to have zero momentum
and zero winding.

Consider a conformal field theory, and denote by $\phi_i, i = 1, 2, \cdots$ the dimension (1,1)
primary fields of the theory, and by $\lambda_i$ their corresponding couplings. We thus consider
deformation of the conformal field theory via the perturbations

$$S_{cft}(\lambda) = S_{cft} + \sum_i \lambda_i \int d^2z \phi_i(z, \bar{z})$$

If one of the dimension (1,1) operators above, say $\phi_A$, is exactly marginal then it must
happen that the operator product coefficient $c_{iAA}$ must vanish for all $i$ (including $A$)
[29–32]. As we will see, the recursive solution of the string field equations without
obstructions will demand the above condition and, in addition, higher order requirements.
These state that the string scattering amplitude, in genus zero, of any dimension (1,1)
operator $\phi_i$ with $n \geq 2$ copies of the exactly marginal operator $\phi_A$ vanishes:

$$\int_{M_{0,n+1}} <\phi_i \phi_A \phi_A \cdots \phi_A > = 0$$

We will not attempt to give a conformal field theory derivation of this statement. We
will simply assume that the obstructions vanish and find the string field solution. The
work of Mukherji and Sen [21] provides evidence that string field theory obstructions correspond to conformal field theory nonzero beta functions. Our analysis in this section parallels that of [21] in general strategy.

Consider now the equations of motion of the nonpolynomial closed string field theory:

\[
Qb_0^- |\Psi\rangle + \frac{1}{2!}|\Psi^2\rangle + \frac{1}{3!}|\Psi^3\rangle + \cdots = 0.
\] (5.32)

We now attempt a perturbative solution of this field equation via the expansion

\[
|\Psi\rangle = \sum_{n=1}^{\infty} \epsilon^n |\Psi_n\rangle = \epsilon |\Psi_1\rangle + \epsilon^2 |\Psi_2\rangle + \cdots.
\] (5.33)

The equations that we must solve recursively read:

\[
Qb_0^- |\Psi_1\rangle = 0
\]
\[
Qb_0^- |\Psi_2\rangle = -\frac{1}{2} |\Psi_1^2\rangle,
\] (5.34)

\[
Qb_0^- |\Psi_3\rangle = -|\Psi_1 \Psi_2\rangle - \frac{1}{3!} |\Psi_1^3\rangle,
\]

and so on. Note that the structure of the equations is such that the right hand sides must correspond to BRST trivial states. Therefore the BRST operator must annihilate every right hand side. The identities relating the BRST operator and the string products guarantee that this condition is satisfied automatically to every order if the lower order equations are satisfied. The only obstruction to solving these equations is that a state corresponding to a BRST cohomology class may appear in the right hand side (a state annihilated by Q which is not of the form Q|\alpha\rangle for any state |\alpha\rangle). Since the string field has ghost number +3 and Qb_0^- has ghost number zero, the terms in the right hand sides must have ghost number +3. Thus the obstructions are the BRST cohomology classes at ghost number +3. In critical string theory, a full copy of the physical cohomology appears at ghost number +3 [33], so there exist potential obstructions.

Our ansatz is that to leading order the string field is the BRST invariant version of the marginal operator \(\phi_A\), namely

\[
|\Psi_1\rangle = |\phi_A\rangle \rightarrow Qb_0^- |\phi_A\rangle = 0
\] (5.35)

We will now try to solve all the higher order equations. Note that all the higher order corrections |\Psi_n\rangle (n \geq 2) to the string field correspond to unphysical states. This is so because they must not be annihilated by Q, as is seen in (5.34). The effect of this is that string field condensation of the massless fields is not sufficient to change the background, we must also give expectation values to unphysical fields, namely, to the
zero momentum components of massive fields in the string field theory. While at each stage we could add some BRST invariant physical field to $|\Psi_n\rangle$ we will not do so. The states in the right hand side must be annihilated by $b^-_0$ and by $L^+_0$, and these conditions are guaranteed by the structure of the string field theory. In order to solve the equations we use the following Lemma.

**Lemma.** Consider a state $|A\rangle$ such that

$$Q|A\rangle = 0, \quad L^-_0|A\rangle = 0, \quad L^+_0|A\rangle \neq 0.$$  \hspace{1cm} (5.36)

If the state is a linear superposition of Fock space states, we require that all those states have nonzero $L^+_0$ eigenvalue. Then one can solve

$$Qb^-_0|\psi\rangle = b^-_0|A\rangle$$  \hspace{1cm} (5.37)

with

$$b^-_0|\psi\rangle = -\frac{b^+_0}{L^+_0}b^-_0|A\rangle.$$  \hspace{1cm} (5.38)

**Proof.** This is just proven by calculation. We use

$$Q = -c^+_0L^+_0 + c^-_0L^-_0 + b^+_0M^+ + b^-_0M^- + \hat{Q}$$

and the generic expression for the field $b^-_0|A\rangle$

$$b^-_0|A\rangle = A_0|+ -\rangle + A_1| - -\rangle$$

which given the odd statistics of the string field $|\Psi\rangle$, and the even statistics of the SL(2,C) vacuum (convention), we have that $A_0$ is even and $A_1$ odd. One then finds

$$0 = Qb^-_0|A\rangle = (L^+_0A_1 + \hat{Q}A_0)|+ -\rangle + (M^+A_0 + \hat{Q}A_1)| - -\rangle$$  \hspace{1cm} (5.39)

Now verify that the solution given in (5.38) is correct

$$b^-_0|\psi\rangle = -\frac{b^+_0}{L^+_0}b^-_0|A\rangle = -\frac{b^+_0}{L^+_0}A_0|+ -\rangle = -\frac{1}{L^+_0}A_0| - -\rangle,$$

and upon acting with the BRST operator one gets

$$Qb^-_0|\Psi\rangle = -Q\frac{1}{L^+_0}A_0| - -\rangle$$

$$= (c^+_0L^+_0 - \hat{Q})\frac{1}{L^+_0}A_0| - -\rangle$$

$$= A_0|+ -\rangle - \frac{1}{L^+_0}\hat{Q}A_0| - -\rangle$$

$$= A_0|+ -\rangle + A_1| - -\rangle = b^-_0|A\rangle,$$

where use was made of (5.39). This proves the lemma. Note that the reason we had to do an explicit check was that $b^+_0$ annihilates part of $|A\rangle$. 

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Before beginning to consider the recursive solution, let us establish one more useful result. We are going to solve equations of the type

\[ Qb_0 |\Psi\rangle = A_0 |+\rangle + A_1 |-\rangle \]  

(5.41)
as indicated before. The right hand side is annihilated by \( Q \) and conformal field theory should imply that the right hand side must not contain a nontrivial BRST class, since otherwise the recursive procedure falls flat on its face. Thus it must only contain BRST trivial states. Let us show that it \textit{cannot} contain BRST trivial states of dimension (0,0). While such states would present no obstruction to the recursive procedure, it is useful to show they are not present since this will simplify considerably our results and enable us to use the lemma. Consider all possible type of states that can appear for the right hand side of (5.41). Since they must have dimension (0,0) and the momentum must be zero, they must be made by one holomorphic and one antiholomorphic oscillator. Taking into account ghost number the only possible states are

\[ c_{-1} \bar{\alpha}^\mu_{-1} |+\rangle - \rangle, \quad \bar{c}_{-1} \alpha^\mu_{-1} |-\rangle - \rangle, \]  

(5.42)
and

\[ c_{-1} \bar{b}_{-1} |+\rangle - \rangle, \quad \bar{c}_{-1} \bar{b}_{-1} |+\rangle - \rangle, \quad \alpha^\mu_{-1} a_{\mu\nu} \bar{\alpha}^\nu_{-1} |+\rangle - \rangle. \]  

(5.43)
From the latter group the combination \((\bar{c}_{-1} \bar{b}_{-1} + c_{-1} \bar{b}_{-1}) |+\rangle - \rangle\) is BRST invariant, but also nontrivial, as is checked in a straightforward way by writing states of suitable ghost number. The last state in (5.43) is also BRST invariant and nontrivial. The states in (5.42) are more delicate. They are BRST invariant and actually BRST trivial (the first one is \(-Qb_{-1} \bar{\alpha}^\mu_{-1} |+\rangle - \rangle\), for example). They cannot arise in the right hand side of (5.41) because of Lorentz invariance of the string field theory. Since the right hand side is built of string products, which are manifestly Lorentz invariant, an \( \alpha^\mu \) oscillator can only appear contracted with a momenta. Since all momenta are zero, it cannot appear at all. This concludes our proof that all states appearing in the right hand side of (5.41) will satisfy the conditions of the lemma.

The first nontrivial equation to solve is

\[ Qb_0^- |\Psi_2\rangle = -\frac{1}{2} |\phi_A^2\rangle \]  

(5.44)
where \( |\phi_A^2\rangle = |\phi_A \star \phi_A\rangle \). Indeed, \( Q \) acting on the right hand side vanishes on the account that \( Q \) acts as a derivation on the star product, and the fact that \( |\phi_A\rangle \) is BRST invariant.

\* The “graviton trace state” which is unphysical for nonzero momentum, becomes physical at zero momentum. Even though it can then be written as \( Q|\alpha\rangle \) [34], it should not be considered a trivial state because the ket \( |\alpha\rangle \) involves the \( X \) field, which is not a conformal field.
Now this is solved by
\[ b_0^- |\Psi_2\rangle = \frac{b_0^+}{2L_0^+} |\phi_A \star \phi_A\rangle, \]
\[ = \frac{b_0^+}{2L_0^+} \sum_r <f_1 \circ \Phi_c^r(0) f_2 \circ (b_0^- \phi_A(0)) f_3 \circ (b_0^- \phi_A(0)) > b_0^- |\Phi_r\rangle, \quad (5.45) \]
\[ = b_0^+ \sum_r \frac{1}{2L_{0r}^+} \mathcal{A}(\Phi_c^r, \phi_A, \phi_A) b_0^- |\Phi_r\rangle. \]

Here we have used the conformal field theory definition of the string product (see [20]). The bra \( \langle \Phi_c^r | \) denotes the state conjugate to \( |\Phi_r\rangle \). This means that \( \langle \Phi_c^r | \Phi_s \rangle = \delta_{rs} \). Note how ghost number works. The star product of two fields of ghost number three must give a field of ghost number three; indeed, it follows from the expression that \( \Phi_c^r \) must have ghost number two (to add up to six in the correlator), and thus \( \Phi \) has ghost number four, and finally \( b_0^- \Phi \) has ghost number three. Note that while our earlier arguments show that \( b_0^- |\Phi_r\rangle \) must be BRST trivial, that does not imply that \( \Phi_c^r \) is a BRST trivial operator, thus the correlator above does not vanish. The correlator, with the string field theoretic functions \( f_i \) telling us how to insert the states, is giving us the off-shell amplitude for scattering of the marginal operators into the \( \Phi_c^r \) operator, and we have denoted, for simplicity, the correlator, by the letter \( \mathcal{A} \). The sum over \( r \) runs over all states of the Hilbert space, satisfying \( L_0^- = 0 \), but only states \( |\Phi_r\rangle \) of ghost number four, that are not annihilated by \( b_0^\pm \) will contribute. If \( |\Phi_r\rangle = \alpha_{-1}^\mu a_{\mu\nu} \bar{\alpha}_{-1}^\nu |+\rangle \), which is the BRST nontrivial marginal perturbation, it better be that the correlator with \( \Phi_c^r \) vanish. Here \( |\Phi_c^r\rangle = \alpha_{-1}^\mu a_{\mu\nu} \bar{\alpha}_{-1}^\nu |-\rangle \) and the correlator is simply (since all fields are on shell) \( C_{rAA}, \) which is required to vanish in the conformal field theory.

We now generalize. Consider the equation for \( |\Psi_3\rangle \). We solve it as
\[ b_0^- |\Psi_3\rangle = \frac{b_0^+}{L_0^+} |\Psi_1 \star \Psi_2\rangle + \frac{b_0^+}{3!L_0^+} |\Psi_1^3\rangle, \]
\[ = \frac{b_0^+}{L_0^+ 3!} \left( 3|\phi_A \star \alpha_{c_0} b_0^+ \frac{1}{L_0^+} |\phi_A \star \phi_A\rangle \right). \quad (5.46) \]

But it is clear now that the quantity inside parenthesis is building a four point amplitude, the first term corresponding to the three Feynman diagrams with an intermediate propagator, as the presence of \( (b_0^+/L_0^+) \) indicates, and the last term being the product that defines the four point function. Thus the above result can be written as
\[ b_0^- |\Psi_3\rangle = b_0^+ \sum_r \frac{1}{3!L_{0r}^+} \mathcal{A}(\Phi_c^r, \phi_A, \phi_A) b_0^- |\Phi_r\rangle, \quad (5.47) \]
where \( \mathcal{A} \) here denotes the off-shell four external state amplitude (with the integral over moduli space understood) calculated using the string diagrams of the corresponding
string field theory. It is clear what is now the complete generalization. The final solution
is therefore
\[ b_0^-|\Psi\rangle = b_0^-|\phi_A\rangle + b_0^+ \sum_{n \geq 2, r} \frac{\epsilon^n}{n!} \frac{1}{L_0^+} A(\Phi^r_c, \phi_A, \cdot \cdot \cdot \phi_A) b_0^-|\Phi_r\rangle. \] (5.48)

where \( A \) denotes the off-shell scattering amplitude (summed over moduli space) for the
field \( \Phi^c_r \) with \( n \) marginal fields. This formula suggests that to every order in \( \epsilon \) the string
field components of the solution are finite. It is nice that classical solutions and off-shell
amplitudes are related like this, it indeed indicates that a good string field theory must
have nice off-shell structure. The question of whether or not the finite classical solution
is in the Hilbert space of the theory becomes just the issue of convergence of the whole
series.

6. Relation with First Quantization

It is well-known that the usual path-integral expression for the partition function of
a free scalar living on a circle of radius \( R \) (\( X \equiv X + 2\pi R \)) must be multiplied by a radius
dependent factor, if one wishes to have a duality invariant expression [35]. Namely, the
amplitude at the \( L \)-loop level is given by
\[ Z_L(R) = g^{-2(g^2R)^L} \int_{x: \text{ fixed}} DX \exp(-S[X]), \] (6.1)

where \( S[X] \) is the usual sigma model action
\[ S[X] = \frac{1}{\pi} \int d^2z \partial X(z, \bar{z}) \bar{\partial} X(z, \bar{z}) \] (6.2)

with \( d^2z \equiv dzd\bar{z}/(-2i) = dt\sigma \). The path-integral contains one trivial integration
\( \int_{0}^{2\pi R} dx \) over the zero mode \( x \) of the position \( X \) at a time. Due to translational invariance
the integrand does not depend on \( x \) and one gets a factor of \( R \) that has been extracted
explicitly in (6.1)(thus the label “\( x: \text{ fixed} \)”).

For \( d \)-dimensional compactification, the formulas (6.1) and (6.2) are replaced by
\[ Z_L(E) = g^{-2(g^2\sqrt{G})^L} \int_{x: \text{ fixed}} (\sqrt{G}DX) \exp(-S[X; E]) \] (6.3)

\[ S[X; E] = \frac{1}{\pi} \int d^2z \partial X(z, \bar{z}) E^t \bar{\partial} X(z, \bar{z}). \]

Here \( X \equiv X + 2\pi \), and the path-integration measure \( (\sqrt{G}DX) \) means that each inte-
gration has a volume factor \( \sqrt{G} \). (This is due to the different periodicity conventions in
(6.1) and (6.3).)
The purpose of this section is to show that the prefactor $R^L$ or $\sqrt{G}^L$ in these path-integral expressions automatically appears in the string field theory and does imply neither coupling constant change nor dilaton condensation. To show this, we first discuss the path-integral for a point particle moving on a torus, which contains some of the features of the string case. We then turn to strings moving on a torus, and to higher loop amplitudes.

6.1. Particle Moving on a Target Space Torus

We consider a particle on a $d$ dimensional target space torus, namely, the position $x^i$ ($i = 1, 2, \cdots, d$) of the particle obeys the identifications $x^i \equiv x^i + 2\pi$. The Hamiltonian will be given by.

$$\hat{H} = \frac{1}{2} \hat{p}_i G^{-1} \hat{p}_j.$$  \hspace{1cm} (6.4)

Henceforth we omit vector indices: e.g., $x^i G x^i = g_{ij} x^j$. From the periodicity of $x$, the momentum $p$ takes the integer eigenvalues:

$$\hat{p} |n\rangle = n |n\rangle, \quad n = (n_i) \in \mathbb{Z}$$  \hspace{1cm} (6.5)

with normalization and completeness relations

$$\langle n| m\rangle = \delta_{n,m}, \quad \sum_n |n\rangle \langle n| = 1.$$  \hspace{1cm} (6.6)

The coordinate eigenstate, however, cannot be defined by $\hat{x} |x\rangle = x |x\rangle$, since the eigenvalue $x$ is defined only modulo the periodicity and hence the operator $\hat{x}$ is not a well-defined operator. As we explained in (3.17), we define the coordinate eigenstate via momentum eigenstate as

$$|x\rangle \equiv \sum_n e^{-inx} \frac{1}{\sqrt{(2\pi)^d}} |n\rangle.$$  \hspace{1cm} (6.7)

Then the state label $x$ actually becomes the label of the point on the torus, satisfying $|x\rangle = |x + 2\pi e^{(i)}\rangle$ ($e^{(i)}$ : a unit vector in the $i$ direction). The inner product of the

* The authors learned the derivation of the path-integral formula presented in this subsection from T. Kashiwa, whom they would like to thank. See also Ref. [36].

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coordinate eigenstates is given by the periodic delta function,
\[
\langle x | y \rangle = \sum_n \frac{1}{(2\pi)^d} e^{in(x-y)} = \sum_m \delta(x - y + 2\pi m) \equiv \delta(x - y),
\] (6.8)
and the completeness relation reads
\[
\int_C dx \langle x | x \rangle = 1, \quad \text{ (torus } C : 0 \leq x^i < 2\pi \text{).}
\] (6.9)

Let us now derive the path-integral formula for the transition amplitude
\[
T = \langle x_F | e^{-\hat{H}T} | x_I \rangle, \quad (0 \leq x_F, x_I < 2\pi).
\] (6.10)
For an infinitesimal time interval \(\Delta t \equiv T/(N+1)\) \((N \gg 1)\), we evaluate the transition amplitude as follows:
\[
\langle x_{j+1} | e^{-\hat{H}\Delta t} | x_j \rangle = \sum_{n_j} \langle x_{j+1} | n_j \rangle \langle n_j | e^{-\hat{H}\Delta t} | x_j \rangle
= \sum_{n_j} \frac{1}{(2\pi)^d} \exp\left(-\frac{1}{2} n_j G^{-1} n_j \Delta t\right) e^{in_j(x_{j+1} - x_j)}
= \int \frac{dp_j}{(2\pi)^d} \exp \left(-\frac{1}{2} p_j G^{-1} p_j \Delta t + ip_j(x_{j+1} - x_j)\right) \sum_{n_j} \delta(p_j - n_j)
\] (6.11)
where in the last step we used Poisson's formula to replace a sum of delta functions with a sum of exponentials. We thus get for the transition amplitude \(T\),
\[
T = \lim_{N \to \infty} \left( \prod_{j=1}^N \int_C dx_j dp_j \frac{1}{(2\pi)^d} \sum_{m_j} \right) \left( \int \frac{dp_0}{(2\pi)^d} \sum_{m_0} \right) \times \exp \left\{ -\frac{1}{2} p_j G^{-1} p_j \Delta t + ip_j(x_{j+1} - x_j + 2\pi m_j) \right\}
\] (6.12)
where \(x_0 = x_I\) and \(x_{N+1} = x_F\). Now we define the following new coordinates:
\[
\tilde{x}_0 = x_0; \quad \tilde{x}_j \equiv x_j + 2\pi \ell_j, \quad \ell_j \equiv \sum_{k=0}^{j-1} m_k, \quad (j = 1, \cdots, N + 1)
\] (6.13)
The sums over \(m_j\) \((j = 0, \cdots N)\) can now be traded for sums over the \(\ell_j\) \((j = 1, \cdots, N+1)\), and the \(x\)-integration region, restricted to the torus, is combined with the \(\ell\) sums so that
the restriction to the torus disappears for $\tilde{x}$:

$$\int_C dx_j \sum_{\ell_j} = \int_{-\infty}^{\infty} d\tilde{x}_j \quad \text{for} \quad j = 1, 2, \cdots, N. \quad (6.14)$$

Denoting $\ell_{N+1} \equiv n$, ($\ell_{N+1}$ was not used in (6.14) ) the transition amplitude becomes

$$T = \lim_{N \to \infty} \sum_{n=-\infty}^{\infty} \left( \prod_{j=1}^{N} \int_{-\infty}^{\infty} \frac{d\tilde{x}_j dp_j}{(2\pi)^d} \right) \int \frac{dp_0}{(2\pi)^d} \times \exp \left\{ \Delta t \left[ \sum_{j=0}^{N} \frac{1}{2} \left( \tilde{x}_{j+1} - \tilde{x}_j \right) \Delta t - \frac{1}{2} p_j G^{-1} p_j \right] \right\},$$

with boundary conditions $\tilde{x}_0 = x_I$ and $\tilde{x}_{N+1} = x_F + 2\pi n$. If we path-integrate out the momentum variables, we obtain

$$T = \lim_{N \to \infty} \sum_{n=-\infty}^{\infty} \left( \prod_{j=1}^{N} \int_{-\infty}^{\infty} \frac{\sqrt{G} d\tilde{x}_j}{\sqrt{(2\pi \Delta t)^d}} \right) \int \frac{\sqrt{G}}{\sqrt{(2\pi \Delta t)^d}} \times \exp \left\{ -\Delta t \left[ \sum_{j=0}^{N} \frac{1}{2} \left( \tilde{x}_{j+1} - \tilde{x}_j \right) \Delta t \right] \right\}. \quad (6.16)$$

In the limit $N \to \infty$, omitting the tilde of $x$, one writes

$$T = \sqrt{G} \sum_{n=-\infty}^{\infty} \int \left( \sqrt{G} D x \right) \exp \left\{ -\int_0^T dt \frac{1}{2} \dot{x}(t) G \dot{x}(t) \right\}, \quad (6.17)$$

where the prefactor $\sqrt{G}$ came from the factor $\sqrt{G}/\sqrt{(2\pi \Delta t)^d}$ and the singular factor $1/\sqrt{(2\pi \Delta t)^d}$ was omitted, as usual in the path-integral formulas. This is the desired formula for the particle moving on a torus. The prefactor $\sqrt{G}$ in this formula will play a key role below. It is useful to understand in simple terms how the main features of (6.17) arise. For every interval $\Delta t$, we introduced an integral over $p$ and a sum over $m$. The number of intermediate $x$ integrations, however, is one less than the number of intervals, thus, when reassembling the result one is left with an extra sum over $m$, giving rise to the winding of the particle as it moves in time, and an extra $p$ integral that gives rise to the $\sqrt{G}$ prefactor.
6.2. String Moving on a Target Space Torus

Now we consider a string moving on a target space $d$ dimensional torus. The dynamical variable is $X(\tilde{\sigma})$ with identifications $X \equiv X + 2\pi$ (the target space index will be omitted). This string coordinate is expanded as (see (2.11))

$$X(\tilde{\sigma}) = x + w\tilde{\sigma} + \text{(oscillators)}.$$  \hspace{1cm} (6.18)

Corresponding to the dynamical variable $x$ we have the momentum operator $\hat{p}$, and corresponding to the operator $\hat{w}$, whose eigenvalues $w$ appear above, we have the dynamical variable $q$. In operator language, the dynamics of the string will be determined by

$$L + \bar{L} = \frac{1}{2}(\hat{p} - B\hat{w})G^{-1}(\hat{p} - B\hat{w}) + \frac{1}{2}\hat{w}G\hat{w} + \text{(oscillators)},$$

$$L - \bar{L} = -\hat{p}\hat{w} + \text{(oscillators)},$$  \hspace{1cm} (6.19)

where the first term corresponds to the generator of time translations, and the second is the generator of rotations of the string. Our expressions will concentrate on the zero-mode pieces, the full expressions will be written when necessary. Using the $(p,w)$ eigenstate, given by

$$\hat{p}|n,m\rangle = n|n,m\rangle, \quad \hat{w}|n,m\rangle = m|n,m\rangle,$$

$$\langle n,m|k,\ell\rangle = \delta_{n,k}\delta_{m,\ell},$$  \hspace{1cm} (6.20)

the $x$- and $q$-eigenstates are defined by

$$|x,m\rangle \equiv \sum_n \frac{e^{-inx}}{\sqrt{(2\pi)^d}} |n,m\rangle,$$

$$|n,q\rangle \equiv \sum_m \frac{e^{-imq + i\pi nm}}{\sqrt{(2\pi)^d}} |n,m\rangle,$$

$$|x,q\rangle \equiv \sum_{n,m} \frac{e^{-inx-imq+i\pi nm}}{(2\pi)^d} |n,m\rangle,$$  \hspace{1cm} (6.21)

and again, $x$ and $q$ become labels on the points on tori of unit radii: e.g., $|x,q\rangle = |x + 2\pi e^{(i)}, q + 2\pi e^{(j)}\rangle$. Note that the $q$-eigenstate is defined with an additional sign factor $\exp(i\pi nm)$, in order to compensate for the asymmetry of the vertex due to the cocycle factor. [See the end of this section.] Omission of this sign factor would not affect the results of the present subsection.
Let us evaluate the partition function
\[ Z_{T, \theta} \equiv \text{tr} \left( e^{-(T-i\theta)L-(T+i\theta)\bar{L}} \right) = \text{tr} \ e^{-\hat{\mathcal{H}} T}, \] (6.22)
where \( \hat{\mathcal{H}} \) takes the form, omitting the oscillator parts,
\[ \hat{\mathcal{H}} = \frac{1}{2} (\hat{p} - B\hat{w})G^{-1}(\hat{p} - B\hat{w}) + \frac{1}{2} \hat{w}G\hat{w} + i\varphi\hat{p}\hat{w}, \] (6.23)
with \( \varphi \equiv \theta/T \). Evaluation of \( Z_{T, \theta} \) can be done in four ways by using either momentum- or coordinate-representations:
\[ Z_{T, \theta} = \sum_{n,m} \langle n, m | e^{-\hat{\mathcal{H}} T} | n, m \rangle \]
\[ = \int_C dx \sum_m \langle x, m | e^{-\hat{\mathcal{H}} T} | x, m \rangle \]
\[ = \int_C dq \sum_n \langle n, q | e^{-\hat{\mathcal{H}} T} | n, q \rangle \]
\[ = \int_C dx \int_C dq \langle x, q | e^{-\hat{\mathcal{H}} T} | x, q \rangle. \] (6.24)

Duality is manifest in the first expression. It is also manifest in the last expression, which also leads to an interesting path-integral expression as we shall see at the end of this section. To reach the \( x \)-space sigma model path-integral expression, it is quickest to start with the second representation. Following the same procedure as in particle case, we find
\[ Z_{T, \theta} = \left( \int_C \sqrt{G} dx \right) \sum_{n=-\infty}^{\infty} \int_{x(0): \text{fixed}} x(T) = x(0) + 2\pi n \] \( \infty \)
\[ \times \exp \left\{ - \int_0^T dt \left[ \frac{1}{2}(\dot{x}(t) - \varphi m) G (\dot{x}(t) - \varphi m) + \frac{1}{2} m \dot{G} m - i\dot{x} B m \right] \right\} \] (6.25)

A few comments are in order. The sum over \( m \), corresponding to the winding of the string at any time, is the same sum we began with, since the hamiltonian is diagonal in winding eigenstates (as well as momentum eigenstates). The integral over \( x \) is also the same one we started with; the extra factor of \( \sqrt{G} \) arises because of an unmatched \( p \) integration, as in the particle case. The winding in time, described by the integer \( n \), arises from an unmatched \( m_j \) sum, as in the particle case.
Let us now translate this result into a string path integral with a sigma model lagrangian. Note that the coordinate $\tilde{\sigma}$ in $X(\tilde{\sigma}, \tilde{t})$ is a co-moving coordinate fixed to the string, which is different from the coordinate $\sigma \equiv \text{Re } z$ on the Riemann surface with metric $ds^2 = |dz|^2$. The relation is

$$\sigma = \tilde{\sigma} + \varphi t \quad (\varphi = \frac{\theta}{T}). \quad (6.26)$$

(Recall the usual description of a torus with moduli $2\pi \tau = \theta + iT$ on the complex $z$-plane as the parallelogram with corners $(0, 2\pi, \theta + iT, 2\pi + \theta + iT)$.) Therefore, the above together with (6.18) gives us

$$X(t, \sigma) = x(t) + w(\sigma - \varphi t) + \text{(oscillator modes)}, \quad (6.27)$$

so that $\dot{X}(t, \sigma) = \dot{x}(t) - \varphi w + \text{oscillator modes}$; and $X'(t, \sigma) = w$. Then we see that, when the oscillator modes are taken into account, the action functional in the exponent in eq.(6.25) takes the form:

$$\int dt \int d\sigma \left[ \frac{1}{2} \dot{X} G \dot{X} + \frac{1}{2} X' G X' - i \dot{X} B X' \right]. \quad (6.28)$$

This is just identical with $i$ times the sigma model action $S$ given in (2.1) with $\gamma^{\alpha\beta} = \eta^{\alpha\beta}$, if we go back to the original 2D Minkowski world sheet with identification $t = i\tau$ ($\tau$: Minkowski time). If we use the complex coordinate (in Euclidean space), $z = t + i\sigma$, $\bar{z} = t - i\sigma$, then the above (6.28) is also seen to agree with the action given in (6.3):

$$-S[X; E] = -\frac{1}{\pi} \int_{0 \leq t \leq T} d^2 z \partial X(z, \bar{z}) E^t \partial X(z, \bar{z}). \quad (6.29)$$

Noting that the boundary condition $x(T) = x(0) + 2\pi n$ implies, via (6.27), the condition $X(T, \sigma + \theta) = X(0, \sigma) + 2\pi n$ for the string coordinate, we finally find that eq.(6.25) gives the following path-integral expression for the string partition function:

$$Z_{T, \theta} = \left( \int \sqrt{G} dx \right) \sum_{n, m} \int_{X(T, \sigma + \theta) = X(0, \sigma) + 2\pi n \atop X(t, 2\pi) = X(t, 0) + 2\pi m} (\sqrt{G} DX)' \exp(-S[X; E]) \right), \quad (6.30)$$

where the prime in $(\sqrt{G} DX)'$ means that the integration over the CM coordinate $x(0) = X(0, 0)$ is omitted. Note that the factor $\sqrt{G}$ appeared as promised. This proves the formula (6.3) for the $L = 1$ case.
6.3. Higher Loop Amplitudes

The amplitudes at any loop level are constructed in SFT by the products of vertices connected by the string propagators. Each propagator is written in the form:

\[
\frac{\mathcal{P}}{L + \bar{L}} = \int_0^\infty dT e^{-T(L + \bar{L})} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(L - \bar{L})} = \int_0^\infty dT \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-\hat{H}T},
\]

(6.31)

where \(\mathcal{P}\) is the projector to rotational invariant states, and \(\hat{H}\) was defined before. The amplitude corresponding to a Riemann surface with definite moduli is given by the product of “finite moduli propagators” \(e^{-\hat{H}T}\) and vertices \(\langle V \rangle\).

We now consider a generic loop diagram with no external legs. The expression for this amplitude given by SFT will be converted into an \(X\)-space sigma model path integral.

From the previous two results, (6.17) for the point particle transition amplitude and (6.30) for the string partition function, we see that the “finite moduli propagator” \(e^{-\hat{H}T}\) is given by

\[
\langle X_F | e^{-\hat{H}T} | X_I \rangle = \sqrt{G} \sum_n \int \langle X_I | e^{-i\hat{H}x} \rangle \exp(-S[X; E]) \).
\]

(6.32)

There is also the boundary condition \(X(t, 2\pi) = X(t, 0) + 2\pi m\), where \(m\) is the winding number of \(X_I(\sigma)\) (or \(X_F(\sigma)\)). There is no sum \(\sum_m\) because the winding must remain fixed. One should note that this propagator is associated with the factor \(\sqrt{G}\).

Now we come to the vertex. Again the essential part is the zero-modes \(p\) and \(w\), so let us concentrate on those modes alone. Generally any \(N\)-string vertex is of the form

\[
\langle V_N \rangle \sim \sum_{\mathbf{n}} \sum_{\mathbf{m}} \delta(\sum_{r=1}^N n_r) \delta(\sum_{r=1}^N m_r) \prod_{r=1}^N \langle n_r, m_r \rangle
\]

(6.33)

for the relevant zero-mode parts, where \(n_r\) and \(m_r\) denote eigenvalues of \(p\) and \(w\) as before and \(\mathbf{n} \equiv (n_r), \mathbf{m} \equiv (m_r)\). In the \(x\)-representation the basis \(\langle n | x \rangle\) are Fourier transformed into the \(x\) basis using \(\langle n | x \rangle = e^{-inx}/\sqrt{(2\pi)^d}\) and the vertex becomes a vertex function coupling several wavefunctions:

\[
\int C \prod_{r=1}^N dx_r \langle V_N | x, m \rangle \sim \int C \prod_{r=1}^N dx_r \sum_{\mathbf{n}} \sum_{\mathbf{m}} \delta(\sum_{r=1}^N n_r) \delta(\sum_{r=1}^N m_r) \exp(-i \sum_{r} n_r x_r).
\]

(6.34)

The integrations \(\int C dx_r\) came from the insertion of completeness relation \(1 = \int C dx | x \rangle \langle x |\). The summation over \(\mathbf{n}\) with conservation factor \(\delta(\sum_{r=1}^N n_r)\) gives
\[ \prod_{r=1}^{N} \delta(x_r - x_N) \text{ up to irrelevant factors of } \sqrt{(2\pi)^d}. \]

Multiplying \( 1 = \int_C dx \delta(x - x_N) \),
the vertex function becomes
\[
\int \prod_{r=1}^{N} dx_r \langle V_N | x, m \rangle \sim \int dx \cdot \left[ \prod_{r=1}^{N} \int dx_r \delta(x_r - x) \right] \cdot \sum_m \delta(\sum_{r=1}^{N} m_r) \quad (6.35)
\]

Note that essentially a single integral \( \int_C dx \) exists at each vertex, since all the other integrals over \( x_r \) are trivial; they simply set \( x_r = x \). This \( x \) is the position of the vertex.

We are now almost finished. As we saw in (6.32), each propagator has a prefactor \( \sqrt{G} \), so a factor \( \sqrt{G}^P \) appears for a Feynman diagram with \( P \) propagators. At each vertex, however, there is an integration \( \int_C dx \), which is to be included as a part of the path-integral over the 2D world sheet spanned by the diagram. But the integration measure in the path-integral is \( (\sqrt{G} D x) \) and accordingly the integral \( \int_C dx \) at each vertex should be multiplied by \( \sqrt{G} \) so as to construct the path-integral correctly. Since for each vertex we need a factor \( \sqrt{G} \), the overall left over factor relating the \( L \)-loop diagram with \( P \) propagators and \( V \) vertices to a sigma model \( x \)-path integral is
\[
(\sqrt{G})^{P-V} = (\sqrt{G})^{L-1}, \quad (6.36)
\]

and thus we end up with the
\[
(g^2 \sqrt{G})^{(L-1)} \int (\sqrt{G} D X) \exp(-S[X; E]). \quad (6.37)
\]

Note, however, that this path-integral still contains an integration \( \int_C \sqrt{G} dx \) over the zero mode \( x \) of the \( X \) coordinate (at a time) on which the action does not depend. So extracting that factor we finally obtain the following expression for the general \( L \)-loop amplitude
\[
Z_L(E) = g^{-2} (g^2 \sqrt{G})^L \int \limits_{x: \text{fixed}} (\sqrt{G} D X) \exp(-S[X; E]), \quad (6.38)
\]

and finish the proof of (6.3).

### 6.4. Dual Sigma Models

In the above we derived the sigma model path-integral expression (6.3) from string field theory. The final expression is very asymmetric from the viewpoint of duality. But note that the starting set up of string field theory in the \( p \)-\( w \) momentum representation

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is manifestly dual-symmetric (aside from the cocycle factor in the vertex). In particular, the operator $\hat{\mathcal{H}}$ in the propagator satisfies the duality relation

$$\hat{\mathcal{H}}(P, X'; E) = \hat{\mathcal{H}}(P_Q, Q'; \tilde{E}) ,$$

where $\tilde{E}$ is the dual background $\tilde{E} = E^{-1}$ and

$$2\pi P_Q(\sigma) \equiv X'(\sigma), \quad Q'(\sigma) \equiv 2\pi P(\sigma) .$$

Therefore the apparent asymmetry in the above path-integral formula resulted simply because we chose the $x$-coordinate representation. In fact, we could have chosen the $q$-representation by Fourier-transforming the $w$-eigenstates $| m \rangle$ but keeping the momentum representation for $p$-freedom. Then, as is clear from the duality relation (6.39), we would have obtained the following $q$-space sigma model path-integral formula for the same $L$-loop amplitude:

$$Z_L(E) = g^{-2}(g^2\sqrt{\tilde{G}})^L \int_{\text{fixed} q} (\sqrt{\tilde{G}} DQ) \exp(-S[Q; \tilde{E}]).$$

Note also that $\sqrt{\tilde{G}} = 1/\sqrt{G - BG^{-1}B}$.

One might notice here that the dual coordinate $Q(\sigma)$ does not connect smoothly on our vertex, as the Goto-Naka conditions (3.32) shows, and wonder what happened in obtaining the $q$-sigma model path-integral formula (6.41). The asymmetry in the $X(\sigma)$ and $Q(\sigma)$ connection conditions is a reflection of the asymmetry in the vertex cocycle factor under the exchange $p \leftrightarrow w$. But this asymmetry is compensated by the additional sign factor $\exp(i\pi nm)$ put in the definition of $| n, q \rangle$ eigenstate (6.21), and we can get the same vertex factors for this $q$ case as for the $x$ case and obtain (6.41). The reason why this happens is easy to understand: putting the sign factor $\exp(i\pi nm)$ in (6.21) is equivalent to giving the coordinate $q$ the meaning that it stands for the eigenvalue of the operator $Q(\sigma) + \pi p$ instead of $Q(\sigma)$. But the operator $Q(\sigma) + \pi p$ is just the coordinate which is smoothly connected (mod 2$\pi$) on our vertex as is seen in the Goto-Naka conditions (3.32).

Finally in this subsection, let us comment on a manifestly dual-symmetric sigma model which automatically results if we use coordinate representations both for the $p$ and $w$ degrees of freedom. Consider the following (Minkowskian) transition amplitude in the $x, q$-coordinate representation:

$$T = \langle x_F, q_F, X_F | e^{-i\hat{T}} | x_I, q_I, X_I \rangle, \quad (0 \leq x_F, x_I, q_F, q_I < 2\pi).$$

The $X$ denotes $X(\sigma)$ with the zero-mode parts omitted. We are considering the $\theta = 0$ case, for simplicity, and then $\hat{\mathcal{H}}$ reduces to $\hat{H}$ given in (2.7). Performing the same
procedure as in the particle case to reach (6.15) for the $x, q$, and $X$ degrees of freedom, and using the expression of $\hat{H}$ in (2.7), we clearly obtain

$$
T = \sum_{n,m=-\infty}^{\infty} \int_{x(T)=x(0)+2\pi n}^{\infty} D\nu Dq Dw D\nu Dq \exp(iS),
$$

(6.43)

with an action functional $S$ given by

$$
S = \int_{0}^{T} dt \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \left[ 2\pi P\dot{X} + p\dot{x} + w\dot{q} - \frac{1}{2}(X', 2\pi P) R(E) \left( \begin{array}{c} X' \\ 2\pi P \end{array} \right) \right].
$$

(6.44)

It is amusing to note that this action takes a manifestly dual-symmetric form if we use the $Q(\sigma)$ coordinate defined in (2.16) instead of $P(\sigma)$ and perform a suitable partial integration:

$$
S = \int_{0}^{T} dt \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \left[ (w, p) J \left( \begin{array}{c} \dot{x} \\ \dot{q} \end{array} \right) + \frac{1}{2}(X', Q') J \left( \begin{array}{c} \dot{X} \\ \dot{Q} \end{array} \right) \\
- \frac{1}{2}(X', Q') R(E) \left( \begin{array}{c} X' \\ Q' \end{array} \right) \right]
$$

(6.45)

$$
+ \text{surface term},
$$

where $J$ is the $O(d,d; R)$ metric matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). As for the non-zero mode parts, this action happens to coincide exactly with the dual-symmetric action which was proposed by Tseytlin [24] some time ago. But there are some differences for the zero-mode parts; for instance, the $\sigma$-linear terms $p\dot{\sigma}$ in $\dot{X}$ and $w\dot{\sigma}$ in $\dot{Q}$ do not appear here while they did in Ref. [24]. The surface term in (6.45), which appeared as a result of partial integration, is given by

$$
\int_{0}^{2\pi} \frac{d\sigma}{2\pi} \left[ XQ' \right]_{t=T} = \int_{0}^{2\pi} \frac{1}{2} \left[ P(\sigma, T)X(\sigma, T) - P(\sigma, 0)X(\sigma, 0) \right].
$$

(6.46)

This is not dual-symmetric but it simply reflects the asymmetry of the initial and final states, specified by the $X$ eigenvalues. It should be noted that the path-integral measure also takes the dual-symmetric form $Dx Dq Dw Dp D\nu DQ$.  

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7. String Field Duality Transformations

In this section we begin by deriving discrete symmetry transformations of the string field that are invariances of the string action. They arise due to the physical equivalence of string field theories written around dual backgrounds, plus the existence of classical solutions that connect those dual backgrounds. We verify that they generate the discrete group of dualities $O(d,d;Z)$. This full group of symmetry transformations exists for any possible background $E$, and it leaves the action invariant. All symmetries, except those corresponding to group elements $g$ that leave the background invariant ($g(E) = E$), are spontaneously broken. Dine et. al. [6] anticipated from conformal field theory arguments that duality must correspond to global gauge transformations in a field theory description. This result was generalized by Giveon et. al. [7] for the case of generalized discrete dualities. We will indeed show that the discrete symmetries we have obtained, arise mostly from the string field gauge group.

While in string field theory duality turns naturally into a symmetry transformation of the string field leaving the action invariant and existing for all backgrounds, in conformal field theory duality is generically thought as a relation between two apparently different conformal field theories that are actually identical. A general way to obtain dual sigma models corresponding to the same conformal field theory by starting with a self-dual sigma model and gauging different combinations of chiral currents has been given recently [37].

Let us now begin our derivation by finding the discrete global symmetry corresponding to a generic $O(d,d;Z)$ transformation $g$. We have shown that corresponding to any such group element there is a unitary operator $U_g$ such that for any background $E$ one has

$$S_E(\Psi) = S_{g(E)}(\Psi) = S_{g(E)}(U_g^\dagger \Psi),$$

(7.1)

(While we will write, for brevity, the string field as a functional, it is convenient to think of it as a ket, in order to use the equations derived earlier.) Consider now the classical solution $\Psi(E;g(E))$. We have established in section 5, equation (5.26) that

$$S_E(\Psi(E;g(E)) + \Psi) = S_{g(E)}(\Psi),$$

(7.2)

namely, that the classical solution shifts the theory precisely to the final background. It follows from the above two equations that

$$S_E(\Psi(E;g(E)) + U_g^\dagger \Psi) = S_{g(E)}(U_g^\dagger \Psi) = S_E(\Psi),$$

(7.3)

which shows that $S_E$ is invariant under the following string field discrete transformation

$$D_g : \Psi \rightarrow D_g \Psi \equiv \Psi(E;g(E)) + U_g^\dagger \Psi.$$

(7.4)

The discrete symmetry transformation $D_g$ is the symmetry we were after. It consists
of an inhomogeneous term, given by the classical solution, plus a homogeneous term in which the operator $U$ acts on the field. The symmetry is spontaneously broken unless the first term vanishes, and this only happens if the background $E$ is invariant under $g$. Let us derive now the group properties of the discrete transformations, consider a further discrete transformation

$$D_g' : \Psi \rightarrow D_g' \Psi = \Psi(E; g'(E)) + U_{g'}^\dagger \Psi. \quad (7.5)$$

and now consider

$$D_g D_g \Psi = D_g[\Psi(E; g(E)) + U_g^\dagger \Psi],$$

$$= \Psi(E; g(E)) + U_g^\dagger [\Psi(E; g'(E)) + U_g^\dagger \Psi], \quad (7.6)$$

$$= \Psi(E; g(E)) + U_g^\dagger \Psi(E; g'(E)) + U_g^\dagger U_g^\dagger \Psi.$$

In order to simplify further we note that classical solutions have a simple behaviour under the action of $U$:

$$U_g^\dagger \Psi(E_0; E_1) = \Psi(g(E_0); g(E_1)), \quad (7.7)$$

as one easily verifies using equation (5.25) (note that the phase factor in $U^\dagger$ is irrelevant because the classical solution ket has zero momentum and zero winding). It thus follows that (7.6) simplifies to

$$= \Psi(E; g(E)) + \Psi(g(E); gg'(E)) + \exp(-i\pi C(p, g', g)) U_{gg'}^\dagger \Psi$$

$$= \Psi(E; gg'(E)) + \exp(-i\pi C(p, g', g)) U_{gg'}^\dagger \Psi, \quad (7.8)$$

$$= \exp(-i\pi C(p, g', g)) D_{gg'} \Psi,$$

which shows that the second quantized operators $D$ satisfy the algebra

$$D_{g'} D_g = \exp(-i\pi C(p, g', g)) D_{gg'} \quad (7.9)$$

Note that the action of the operators $D$ on the string field is background dependent, it depends on $E$ via the classical solution. Operators $D_g$ referring to different backgrounds are simply related by a shift in the string field. The algebra of the operators is clearly background independent.

A natural question that comes to mind is whether these operators commute with gauge transformations of the string field theory. We represent the gauge transformations
as
\[ G(\Lambda) : \Psi \to G(\Lambda)\Psi \equiv \Psi + Q(E)\Lambda + g_0 \Psi \star \Lambda, \quad (7.10) \]
(note that \( g_0 \) is the coupling constant) one can show that
\[ D_{g^{-1}} G(\Lambda) D_g = G(U_g^\dagger \Lambda), \quad (7.11) \]
where use was made of (4.44), and of the equation
\[ Q(g(E)) + g_0 \Psi(g(E); E) \star = Q(E). \quad (7.12) \]

Equation (7.11) shows that the discrete symmetries generate automorphisms of the gauge group. This suggests strongly that the discrete symmetries correspond to large gauge transformations. In the remaining of this section we will show explicitly how this is obtained in the string field theory for the case of the standard \( R \to 1/R \) duality. This will illustrate how the conformal field theory arguments of Ref. [6] apply. For the case of the more general symmetry transformations one may not have a background that they leave invariant, and the arguments of [6] do not tell us what is the connection with gauge transformations. For example, the composition of two discrete transformations, each having a fixed point background, may not have a fixed background (in the space of backgrounds we are considering). In this case, however, it is clear that the resulting transformation is a gauge transformation, which is never unbroken, but can be identified at any background. Reference [7] shows that this is essentially the generic case, and that all discrete symmetries can be written as products of symmetry transformations at special backgrounds with extended symmetry, plus permutations of spacetime coordinates. These permutations are clearly symmetries of string field theory, but it is not clear to us if they belong to the string field gauge group. The complication arises because we only know the infinitesimal string field gauge transformations, and permutations cannot be built from infinitesimal rotations, due to the compactification of the extra coordinates.

The standard duality inversion is defined by the \( O(d, d; Z) \) matrix \( g_D \) given by
\[ g_D = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (7.13) \]
It follows that \( \det g_D = (-1)^d \). Acting on backgrounds the transformation \( g_D \) is recognized to give the well-known action on backgrounds (Ref. [4]), indeed
\[ E' = G' + B' = g_D(E) = [0E + I][IE + 0]^{-1} = E^{-1} = (G + B)^{-1}. \quad (7.14) \]
The background invariant under the duality trasformation is \( E = I \), and we will therefore discuss string field theory around it. The oscillators corresponding to this
background will be simply denoted as $\alpha, \bar{\alpha}$ and the operator $U_{gD}$ will just be denoted as $U$. It follows from equations (4.36) and (4.37) that

$$U = U \exp(i\pi p \cdot w) \quad (7.15)$$

and the action of $U$ on the oscillators and zero modes is given by (see (4.23))

$$U^\dagger \begin{pmatrix} \alpha_n \\ \bar{\alpha}_n \end{pmatrix} U = \begin{pmatrix} -\alpha_n \\ -\bar{\alpha}_n \end{pmatrix}, \quad U^\dagger \begin{pmatrix} w \\ p \end{pmatrix} U = \begin{pmatrix} p \\ w \end{pmatrix} \quad (7.16)$$

which says that all the $\alpha_n$ oscillators, including $n = 0$, are changed sign, and the bar oscillators are left unchanged. In more geometrical terms

$$U^\dagger \begin{pmatrix} X(\sigma) \\ Q(\sigma) \end{pmatrix} U = \begin{pmatrix} Q(\sigma) \\ X(\sigma) \end{pmatrix} \quad (7.17)$$

It is convenient to introduce the general decomposition $X^i(\sigma) = X^i_+ (\sigma) + X^i_- (\sigma)$ with

$$X^i_+ (\sigma) = x^i_+ - G^{ij} p^j_+ \sigma + \frac{i}{\sqrt{2}} \sum_n \frac{1}{n} \alpha_n^i e^{in\sigma}$$

$$X^i_- (\sigma) = x^i_- + G^{ij} p^j_- \sigma + \frac{i}{\sqrt{2}} \sum_n \frac{1}{n} \bar{\alpha}_n^i e^{-in\sigma} \quad (7.18)$$

where the momentum zero modes $p_+, p_-$ are given by

$$p_+ = \frac{1}{2} (p_i - E_{ij} w^j)$$

$$p_- = \frac{1}{2} (p_i + E_{ij}^t w^j) \quad (7.19)$$

which for the case at hand ($E = I$) reduce to

$$p_{\pm} = \frac{1}{2} (p_i \mp w^i) \quad (7.20)$$

The mass-shell conditions read

$$\frac{1}{2} M^2 = N + \bar{N} + p^2_+ + p^2_- - 2, \quad N - \bar{N} = p^2_- - p^2_+ \quad (7.21)$$

and we will denote the momentum eigenstates by $|p_+, p_-\rangle$. It is well known that at this background one has an $SU(2)^d \otimes SU(2)^d$ symmetry. The gauge bosons for the
parameters are given by states are BRST invariant. The gauge transformations associated with these gauge bosons cannot give a massless state). It follows now from the mass-shell conditions that the desired states are given by

\[ (\alpha^\mu_1, 0, \pm k^i), \quad \bar{\alpha}_i^\mu_1, \pm k^i, 0) \quad (7.22) \]

where the \( k^i \) is a d-component vector whose \( i \)-th entry is +1 and all others are zero. Now we want to find the global transformations associated with such gauge bosons. From the standard string field gauge transformations

\[ \delta(b_0^- | \Psi) = Q_B b_0^- | \Lambda) + g_0 | \Psi \star \Lambda) \]

we must require, in order to have an unbroken symmetry, that \( Q_B b_0^- | \Lambda) = 0 \), and for the symmetry to be global the momentum for the open coordinates \( p_\mu = 0 \), which implies \( M^2 = 0 \). Moreover, the ghost number of \( b_0^- | \Lambda) \) must be \(-1\) (with respect to the vacuum state \( |0\)\)). In order to get this ghost number we need an antighost oscillator, and the only two possibilities are \( b_{-1} \) and \( \bar{b}_{-1} (b_0^\pm \) annihilates \( |0\), and \( b_{-n} \) is ruled out since it cannot give a massless state). It follows now from the mass-shell conditions that the desired states are given by

\[ b_0^- | \Lambda_{\pm} \rangle = \bar{b}_{-1} | \pm k^i, 0) \quad b_0^- | \Lambda^i_\pm \rangle = \bar{b}_{-1} \alpha_i^- | 0, 0) \]

\[ b_0^- | \Lambda^- \rangle = b_{-1} | 0, \pm k^i \rangle, \quad b_0^- | \Lambda^i \rangle = b_{-1} \bar{\alpha}^i_\pm | 0, 0) \quad (7.23) \]

Here we must take the string length \( \alpha \) equal to zero. One easily verifies that the above states are BRST invariant. The gauge transformations associated with these gauge parameters are given by

\[ \delta(b_0^- | \Psi) = g_0 | \Psi \star \Lambda) = -\frac{g_0}{\sqrt{2}} E b_0^- | \Psi), \quad (7.24) \]

where the operator \( E \) arises from the contraction of \( \Lambda \) against the vertex. The calculation of the operator \( E \) is familiar from Hata et. al. [38] and is explained in Appendix C. One obtains

\[ E_{\pm} = \frac{e^{i\pi p k^i}}{\sqrt{2}} \int \frac{d\sigma}{2\pi} \exp(\pm 2i k^i \cdot X_+ (\sigma)) \cdot E_{\pm} = p_+ = \alpha_0^i / \sqrt{2}, \]

\[ \bar{E}_{\pm} = \frac{e^{i\pi p k^i}}{\sqrt{2}} \int \frac{d\sigma}{2\pi} \exp(\pm 2i k^i \cdot X_- (\sigma)) \cdot \bar{E}_{\pm} = p_- = \bar{\alpha}_0^i / \sqrt{2}, \quad (7.25) \]

As is easily confirmed, these operators give generators of the gauge group \( SU(2)^d \otimes SU(2)^d \), e.g., \([E_{+}^i, -E_{-}^j] = E_{3}^i \delta_{ij}, [E_{3}^i, E_{\pm}^j] = \pm E_{\pm}^i \delta_{ij}\). If we define

\[ E_{\pm} = \frac{e^{i\pi p k^i}}{\sqrt{2}} \int \frac{d\sigma}{2\pi} e^{-i\sigma} \exp(\pm 2i k^i \cdot X_+ (\sigma)) \cdot, \quad (7.26) \]

we then have \([\alpha_0^i / \sqrt{2}, E_{\pm}^j] = \pm E_{\pm}^i \delta_{ij}\), and this implies that the operators \((\alpha_0^i / \sqrt{2})\),
$E^\perp_{\pm,n}$ form a spin one representation of the $SU(2)$ we are considering. Thus via a global rotation we can indeed make $\alpha_n \rightarrow -\alpha_n$. This shows our $U$ operator performing the duality rotation is just a global $SU(2)$ gauge transformation.

8. Conclusions and Open Questions

We believe that string field theory, as presently formulated, is powerful enough to give useful insights into the basic issues of target space duality. As we have seen it affords a manifestly dual formulation of the theory, where basic physical facts, such as the invariance of the string coupling constant are completely clear. The string field picture explains the origin of the discrete symmetries as a simple consequence of the facts that two different backgrounds lead to the same physics, and that there are classical solutions shifting us from one background to the others.

The most important questions left open by our work have to do with background independence of string field theory and classical solutions. Our notion of universal coordinates $X(\sigma)$ and $P(\sigma)$ is basically the idea that these are field operators whose existence is independent of the background and whose (field) algebra is always the same. The various backgrounds correspond to inequivalent representations of this unique algebra. In this way we learned how to relate different theories corresponding to different backgrounds, and how to write operators in one background in terms of operators in another background. One feels that there should be more understanding of how this fits together with studies of deformations of conformal field theories, and possibly with geometrical approaches to the study of the space (or subspaces) of conformal field theories. Our notion of universal coordinates applies only to conformal field theories with two-dimensional field theory Lagrangians. It is not clear to us how to extend these ideas to conformal field theories described more abstractly in terms of their operator content.

One of the most puzzling aspects of our results is the indication that classical solutions corresponding to finite changes in the background may not live in the conventional Hilbert space of the theory. If this is really the case, the idea of component fields loses meaning beyond perturbation theory, and a classical string field solution will not correspond to a classical solution for the component fields. It would also mean that we need to learn how to define string field theory for a class of functional fields larger than the conventional one, which corresponds to Fock space states. As a way to test these ideas we explored a recursive solution of the string field equations, in the spirit of Ref. [21]. The solution is written as an infinite series of vectors in the Hilbert space of the original theory. For this finite solution to make sense the series must converge. Each term of the series corresponds to an off-shell amplitude of the string field theory, and we hope it will be possible to reach a conclusion on the issue of convergence in the near future.
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APPENDIX A. Quantities appearing in the vertex of the $\alpha = p^+$ HIKKO theory

Here we give some explicit expressions for the quantities which appear in the three-string vertex (3.24) or (3.25) of the $\alpha = p^+$ HIKKO theory:

$$E_{123}^{\text{ordinary}} = \sum_{r,s} \sum_{n,m \geq 0} \bar{N}_{nm}^{rs} \left( \frac{1}{2} \alpha_n^{\mu(r)} \eta_{\mu \nu} \alpha_m^{\nu(s)} + i \gamma_n^{(r)} \beta_m^{(s)} + \text{a.h.} \right)$$

$$F_{123}^{\text{ordinary}} = \frac{1}{2} \sum_{r,s} \sum_{n,m \geq 0} \bar{N}_{nm}^{rs} \alpha_n^{\mu(r)} \eta_{\mu \nu} \alpha_m^{\nu(s)}$$

$$\gamma_n^{(r)} = i p_r^+ c_n^{(r)}, \quad \beta_n^{(r)} = b_n^{(r)}/p_r^+$$

$$P^\mu = p_r^+ P_r^\mu - p_r^{+\mu} P_r$$

$$\mu_{123} = \exp \left( -\tau_0 \sum_{r=1}^3 (1/p_r^+) \right), \quad \tau_0 = \sum_{r=1}^3 p_r^+ \ln |p_r^+|,$$

$$G(\sigma_I) = \frac{p_r^+}{2} \sum_{n=-\infty}^\infty \left( c_n^{(r)} + c_n^{(r)} \right) \cos n\sigma_I^{(r)}, \quad r = 1 \text{ or } 2 \text{ or } 3$$

$$W_I^{(r)} = -\frac{i}{\sqrt{2}} \sum_{s \geq 1} \sum_{n \geq 1} \left( \chi^{rs} \bar{N}_n^{s} + \sum_{m=1}^{n-1} \bar{N}_{n-m,m}^{rs} / p_r^+ \right) (\gamma_n^{(s)} + \bar{\gamma}_n^{(s)}),$$

The Neumann coefficients $\bar{N}_{nm}^{rs}$ and $\bar{N}_n^s$ as well as coefficients $\chi^{rs}$ are the same as defined by [11] with the understanding that $\alpha = p^+$.

When $\alpha_3 \equiv \epsilon$ becomes very small compared with $\alpha_2$ and $\alpha_1 = -(\alpha_2 + \epsilon)$, the measure factor $\mu_{123}^2$ has a singularity

$$\mu_{123}^2 = \left( \frac{e \alpha_2}{\epsilon} \right)^2 \left( 1 + O(\epsilon) \right).$$

In the calculations of string field condensation in Sect. 5, we need several formulas
showing how the various quantities in the vertex behave in this limit. Such detailed formulas can be found in Hata and Nagoshi [15]. Here we only cite

\[
\bar{N}_{1n}^3 = \left( \frac{\epsilon \text{sgn}(\epsilon \alpha_2)}{\epsilon \alpha_2} \right) \times \begin{cases}
1 & \text{for } r = 1 \\
(-1)^{n+1} & \text{for } r = 2
\end{cases} \quad (n \geq 1), \\
\bar{N}_{10}^3 = \left( \frac{\epsilon \text{sgn}(\epsilon \alpha_2)}{\epsilon \alpha_2} \right) \times \begin{cases}
1 & \text{for } r = 1 \\
0 & \text{for } r = 2
\end{cases}
\]

which will be used in deriving Eq. (5.19). The latter formula for the \( n = 0 \) case is valid only in the presence of zero-mode conservation factor.

APPENDIX B. Physical equivalence of \( \alpha = p^+ \) HIKKO and light-cone SFTs

In this appendix we explain why the \( \alpha = p^+ \) HIKKO theory correctly reproduces the light-cone string field theory amplitudes at any loop order. Of course, this is the case only for processes with external states of physical polarizations.

The vertex in the gauge-fixed \( \alpha = p^+ \) HIKKO theory takes the form

\[
1_{123}(v) = \mu^{123} \delta(1, 2, 3) 1_{123}(0) \exp(F_{123}) P_{123}
\]

For notational simplicity we consider the case in which all the coordinates are uncompactified. Then the exponent \( F_{123} \) in the vertex is the same as \( F^{\text{ordinary}}_{123} \) given in (A.1). An important fact is that, when \( \alpha = p^+ \), the momentum appearing in \( F_{123} \) does not contain the + component: \( P^+ = p_r^+ p_{r+1}^- - p_{r+1}^+ p_r^- = 0 \).

Moreover, when \( \alpha = p^+ \), the momentum-square term in (3.1) becomes purely transversal:

\[
\tau_0 \sum_r \frac{1}{p_r^+} p_r^2 = \tau_0 \sum_r \frac{1}{p_r^+} p_r^2 + \tau_0 \sum_r p_r^- = \tau_0 \sum_r \frac{1}{p_r^+} p_r^2 \quad (B.3)
\]

due to the conservation of \( p^- \). We will use boldface letters to denote transverse vectors. Now the exponent of the vertex takes the form

\[
F_{123} = F^{\text{LC}}_{123} + F^{\text{extra}}_{123}
\]

where the first part \( F^{\text{LC}}_{123} \) is exactly the same one as in the light-cone SFT,

\[
F^{\text{LC}}_{123} = \sum_{r,s} \sum_{n,m \geq 1} \bar{N}_{nm}^{rs} \left( \frac{1}{2} \alpha_n^r \cdot \bar{\alpha}_m^s + \text{a.h.} \right)
\]

\[
+ \frac{1}{\sqrt{2}} \sum_r \sum_{n \geq 1} \bar{N}_n^r (\bar{\alpha}_n^r + \bar{\alpha}_n^r) \cdot P + \tau_0 \sum_r \frac{1}{\alpha_r} p_r^2
\]

and the second part \( F^{\text{extra}}_{123} \) contains the extra modes \( \alpha_n^+, \alpha_n^-, \gamma_n, \beta_n \) of the covariant
F^{\text{extra}}_{123} = \sum_{r,s} \sum_{n,m \geq 1} \bar{N}_{nm} \left( \alpha_n^{(r)} - \alpha_m^{(s)} + i \gamma_n^{(r)} \beta_m^{(s)} + \text{a.h.} \right) \\
+ \frac{1}{\sqrt{2}} \sum_r \sum_{n \geq 1} \bar{N}_n^r (\alpha_n^{(r)} + \bar{\alpha}_n^{(r)}) P^-.

Writing schematically $F^{\text{extra}}_{123} = \alpha^{+} N \alpha^{-} + i \gamma N \beta + \alpha^{+} P^{-} + \text{a.h.}$, the vertex takes the form

$$\langle v | = \langle v_{\text{LC}} | \otimes_{\text{EX}} \langle 0 | e^{\alpha^{+} N \alpha^{-} + i \gamma N \beta + \alpha^{+} P^{-} + \text{a.h.}}.$$ (B.7)

Here $|0\rangle_{\text{EX}}$ denotes the vacuum for the modes $\alpha_n^{+}, \alpha_n^{-}, \gamma_n, \beta_n$ and $\langle v_{\text{LC}} |$ is just the vertex appearing in the light-cone SFT. It is also important to remember that the $\alpha^{+} \alpha^{-} + i \gamma \beta$ term has $OSp(1,1|2)$ symmetry.

The general (tree or loop) amplitude in this theory is calculated by evaluating an expression of the form

$$\mathcal{M} = (\prod \int d\ell) (\prod \langle v |) (\prod e^{-L_{\tau} - \bar{L}_{\bar{\tau}}}) (\prod |R \rangle) |\text{external}\rangle \quad \text{(B.8)}$$

where $\langle v |, |R \rangle, |\text{external}\rangle$ are vertices, reflectors and external states, respectively, $e^{-L_{\tau} - \bar{L}_{\bar{\tau}}}$ are propagators with definite moduli and $\prod \int d\ell$ stands for the integration over the loop momenta $\ell$. The physical external states are constructed by using the DDF modes $A_n^i$ alone which are given by

$$A_n^i = \oint \frac{dz}{2\pi i} z^{n-1} (\sum_m \alpha_m^i z^{-m}) \exp \left( -\frac{n}{p^+} \sum_{\ell \neq 0} \frac{1}{\ell} \alpha_\ell^+ z^{-\ell} \right). \quad \text{(B.9)}$$

So we write the physical external state schematically as

$$|\text{external}\rangle = |\varphi_{\text{LC}}\rangle \otimes e^{-\alpha^{+} z} |0\rangle_{\text{EX}}, \quad \text{(B.10)}$$

where the state $|\varphi_{\text{LC}}\rangle$ is a state written in terms of transverse modes alone which reduces to the same state as in the light-cone SFT after $z$-integration if the factor $e^{-\alpha^{+} z}$ can be replaced by 1. The Klein-Gordon-Virasoro operator $L$ is written as a sum of that of the light-cone SFT and an extra piece:

$$L = L_{\text{LC}} + L_{\text{extra}},$$

$$L_{\text{LC}} = \frac{1}{2} p^2 + p^+ p^- + \sum_{n \geq 1} \vec{\alpha}_{-n} \cdot \vec{\alpha}_n,$$ (B.11)

$$L_{\text{extra}} = \sum_{n \geq 1} (\alpha_n^+ \alpha_n^- + \alpha_{-n}^+ \alpha_{-n}^- + i \gamma_{-n} \beta_n - i \beta_{-n} \gamma_n).$$
We write again schematically
\[ L_{\text{extra}} = \alpha^+ \alpha^- + \alpha^{-\dagger} \alpha^+ + i \gamma^\dagger \beta + i \gamma \beta^\dagger. \] (B.12)

The reflector in the gauge-fixed theory is given by
\[
\langle R | = \delta(1,2) \langle 0 \vert \exp (E_{12}) P_{12},
\]
\[
E_{12} = (-)^{n+1} \sum_{n \geq 1} \left( -\frac{1}{n} \alpha_n^{(1)} \cdot \alpha_n^{(2)} + i \gamma_n^{(1)} \beta_n^{(2)} + i \gamma_n^{(2)} \beta_n^{(1)} \right) + \text{a.h.} \quad (B.13)
\]

Again we write the ket reflector schematically
\[
|R\rangle = |R_{LC}\rangle \otimes e^{(\alpha^+ \alpha^- + i \gamma \beta)} |0\rangle_{\text{EX}} \quad (B.14)
\]

where \(|R_{LC}\rangle\) is the reflector in the light-cone SFT. Note again that the extra mode parts of \(L\) and the reflector are \(OSp(1,1|2)\) invariant.

Now we can evaluate the amplitude (B.8): substituting the above schematic expressions for the external states, reflectors, vertices and \(L\), we find
\[
\mathcal{M} = \left( \prod \int d\ell \right) \mathcal{M}_{\text{LC}} \cdot \mathcal{M}_{\text{extra}} \quad (B.15)
\]

where
\[
\mathcal{M}_{\text{LC}} = \left( \prod \langle v_{\text{LC}} \vert \right) \left( \prod e^{-L_{\text{LC}} \tau - \text{a.h.}} \right) \left( \prod |R_{\text{LC}}\rangle \right) |\varphi_{\text{LC}}\rangle \quad (B.16)
\]
is the amplitude in the light-cone SFT before the loop-integration, and \(\mathcal{M}_{\text{extra}}\) is the similar one for the extra modes which can be schematically written in the following form (omitting the anti-holomorphic parts):
\[
\mathcal{M}_{\text{extra}} = \left( \prod_{\text{EX}} e^{(\alpha^+ N \alpha^- + i \gamma N \beta + \alpha^+ P^-)} \right) \left( \prod e^{-(\alpha^+ \alpha^- + \alpha^{-\dagger} \alpha^+ + i \gamma^\dagger \beta + i \gamma \beta^\dagger) \tau} \right) \times \left( \prod e^{(\alpha^+ \alpha^- + i \gamma \beta)} |0\rangle_{\text{EX}} \right) e^{-\alpha^{++} z} |0\rangle_{\text{EX}}. \quad (B.17)
\]

Let us evaluate this amplitude \(\mathcal{M}_{\text{extra}}\) for extra mode part. We claim that the momentum dependent factor \(\exp(\alpha^+ P^-)\) in the vertex can be set equal to one. This is seen as follows: since the \(\alpha^+\) oscillators are contracted with \(P^-\) and have non-zero commutator only with \(\alpha^{-\dagger}\), which in turn appears in (B.17) contracted only with \(\alpha^+\), or \(\alpha^{+\dagger}\), the momenta \(P^-\) must appear always in the form \(\alpha^+ P^-\) or \(\alpha^{+\dagger} P^-\) at any stage of the calculation of (B.17). But those oscillators are eventually eliminated on the bra or ket vacuum. Thus the terms containing a \(P^-\) factor can give no contribution to the amplitude \(\mathcal{M}_{\text{extra}}\), and we can set \(P^-\) equal to zero in (B.17). (Note that, if there were a term of the form \(\alpha^{-\dagger} K^+\) or \(\alpha^- K^+\) with some momentum \(K^+\), then the term \(\alpha^+ P^-\)
could have given a finite contribution proportional to $P^{-K^+}$.) For the same reason we can set the factor $\exp(-\alpha^+ z)$ in the external state equal to one. Thus the amplitude $M_{\text{extra}}$ becomes

\[ M_{\text{extra}} = \text{EX} \langle 0 | (\prod e^{\alpha^+ N_{\alpha^+}} + i \gamma N_{\beta}) \times (\prod e^{-(\alpha^+ \alpha^- + \alpha^- \alpha^+ + i \gamma^+ \beta + i \gamma \beta^+)} r_{\text{LC}} | 0 \rangle_{\text{EX}}. \]  

(B.18)

Note that this is completely $OSp(1, 1|2)$ symmetric. Therefore it has to be one, since whatever factor is given by the $\alpha^+, \alpha^-$ oscillators, it is cancelled by the contribution of the $\gamma, \beta$ oscillators. We thus find

\[ M = (\prod \int d\ell) M_{\text{LC}}. \]

This coincides with the amplitude in the light-cone SFT. (Recall that the external states also reduced to the light-cone ones since the factor $\exp(-\alpha^+ z)$ was replaced by one.) Namely we have proven that the physical amplitudes in the $\alpha = p^+$ HIKKO theory indeed agree with those in the light-cone SFT.

A comment may be in order. In the light-cone SFT there is only propagation forward in time due to the structure of the kinetic term plus the independence of the vertices on $p^-$, which implies locality in the light cone time. The kinetic term in the $\alpha = p^+$ HIKKO theory has the same structure as the light cone theory, and we have shown that the $P^-$ dependence of the vertices dissapeared for physical amplitudes. Therefore the string diagrams for physical amplitudes agree.

APPENDIX C. Derivation of Eqn. (5.19) and Eqn. (7.25).

First we briefly explain how Eq. (5.19) is derived. We have to evaluate

\[ 123\langle V || EM \rangle_3 = \lim_{\epsilon \to 0} 123\langle V | c_0^{-(3)} \left[ \alpha^{-1}_i(E) a_{ij} \bar{\alpha}^j_{-1}(E) \right]^{(3)} | 0 \rangle_3 \delta_\epsilon(p_3, w_3), \]

where integrations (or summations) over $p_3$ and $w_3$ are implied. We omit the background label $(E)$ from $\alpha^+_i(E)$ henceforth. Since the ghost oscillator dependence is trivial here, we first calculate the ghost zero-mode part substituting the vertex expression (3.24) and find

\[ 123\langle V || EM \rangle_3 \sim \mu^2_{123} \delta(1, 2, 3) 123\langle 0 | \frac{1}{\alpha_1 \alpha_2} (\alpha_1 c_0^{+(1)} - \alpha_2 c_0^{+(2)}) \exp(E_{123}) \times G(\sigma_f) e^{-i \pi(p_3 w_2 - p_1 w_1)} \mathcal{P}_{123}(\alpha^{-1}_i a_{ij} \bar{\alpha}^j_{-1})^{(3)} | 0 \rangle_3 \delta_\epsilon(p_3, w_3). \]  

(C.1)

where use was made of Eqn. (3.3) and the $b$ zero modes in the vertex were moved towards the vacuum on the left. Since $\mu^2_{123} \sim (e\alpha_2/\epsilon)^2$, we have to evaluate the rest of
the expression to $O(\epsilon^2)$. But we have

$$123|0\rangle \exp(123)(\alpha_{i}^{i}a_{ij}\bar{\alpha}_{-1}^{j}(3)|0\rangle = 12(0)|0\rangle \exp(12') \sum_{n,m \geq 0} \bar{N}_{1n}^{3n} \bar{N}_{1m}^{3m} a_{ij}\bar{\alpha}_{m}^{j}^{(n)}$$

(C.2)

where use was made of Eqn. (3.27), and with $E_{12}'$ denoting $E_{123}$ with string-three oscillators eliminated. Since the Neumann coefficient factor $\bar{N}_{1n}^{3n} \bar{N}_{1m}^{3m}$ is already of $O(\epsilon^2)$ as is seen in (A.5), we have only to calculate the $O(1)$ part for all the other quantities in Eq.(C.1). Then $\exp(E_{12}')$ becomes the exponent $\exp(E_{12})$ of the 2-point vertex $12|R'|$ in (3.6) and so we have

$$\lim_{\epsilon \to 0} \delta(1, 2)12(0)|0\rangle \frac{1}{\alpha_{1}\alpha_{2}}(\alpha_{1}c_{0}^{(1)} - \alpha_{2}c_{0}^{(2)}) \exp(E_{12}')e^{i\pi p_{i}w_{1}} = \frac{1}{\alpha_{1}}12'|R'|b_{0}^{\pm (1)},$$

(C.3)

where $\langle R' \rangle$ denotes the reflector, but without the rotational projector $\mathcal{P}_{12} = \mathcal{P}(1)\mathcal{P}(2)$. The ghost prefactor $G(\sigma_{1})$ in (C.1) yields in this limit

$$G(\sigma_{1}) = \frac{\alpha_{1}}{2} \sum_{\ell = -\infty}^{\infty} (c_{\ell}(1) + c_{\ell}^{(1)}),$$

(C.4)

using Eq.(A.3) and the fact that the interaction point $\sigma_{1}^{1}$ for string one becomes zero for $\alpha_{3} \to 0$. Now using Eqs. (C.2), (C.4), (A.4) and (A.5), we find that Eq.(C.1) becomes

$$123\langle V|\mathcal{E}M\rangle_{3} = 12\langle R'|\frac{1}{2} \sum_{\ell = -\infty}^{\infty} (c_{\ell}(1) + c_{\ell}^{(1)})
\times \left[ \sum_{n,m \geq 0} \alpha_{n}^{(1)} \alpha_{m}^{(1)} + \sum_{n,m \geq 1} (-)^{n+m} \alpha_{n}^{(2)} \alpha_{m}^{(2)} \right]
+ \sum_{n \geq 0} (-)^{n+1} \alpha_{n}^{(1)} \alpha_{m}^{(1)} + \sum_{n \geq 1} (-)^{n+1} \bar{\alpha}_{n}^{(1)} \bar{\alpha}_{m}^{(1)} \right]b_{0}^{\pm (1)}\mathcal{P}_{12},$$

(C.5)

with the abbreviation $\alpha_{n} * \bar{\alpha}_{m} \equiv \alpha_{n}^{i}a_{ij}\bar{\alpha}_{m}^{j}$. We can now use the following continuity conditions on $12|R'|$,

$$12\langle R'| \left( \alpha_{n}^{(1)} + (-)^{n} \alpha_{-n}^{(2)}, c_{n}^{(1)} + (-)^{n}c_{-n}^{(2)}, b_{n}^{(1)} - (-)^{n}b_{-n}^{(2)} \right) = 0,$$

(C.6)

and the analogous ones for the anti-holomorphic oscillators to find

$$123\langle V|\mathcal{E}M\rangle_{3} = 12\langle R'|\frac{1}{2} \sum_{\ell \in \mathbb{Z}} (c_{\ell}(1) + c_{\ell}^{(1)}) \sum_{n,m} \alpha_{n}^{(1)} \bar{\alpha}_{m}^{(1)} b_{0}^{(1)} \mathcal{P}_{12}
= 12\langle R'|\frac{1}{2} \sum_{\ell \in \mathbb{Z}} (c_{\ell}(1) + c_{\ell}^{(1)}) (\alpha_{n}^{(1)} \bar{\alpha}_{m}^{(1)} b_{0}^{(1)}).$$

(C.7)
In going to the second expression we have first moved the operator $P^{(2)}$ all the way up to $\langle R' \rangle$ and used $\langle R' | P^{(2)} = \langle R' | P_{12} = \langle R \rangle$ as follows from \((C.6)\). Then the projector $P^{(1)}$, that appears actually both to the left and to the right of the prefactor picks up only the terms in which the separate mode number sums of the holomorphic and anti-holomorphic oscillators are equal. The second expression is seen to imply Eq. \((5.19)\) after use of \((C.6)\).

The calculation of dilaton condensation \((5.9)\) is somewhat more complicated because of the presence of ghost oscillators in the dilaton state. This time it is easier to use the vertex expression \((3.25)\) rather than \((3.24)\). We then use the various $\epsilon$-expansion formulas for the Neumann coefficients $N_{nm}^{rs}$ and the coefficients in $W_I^{(r)}$ which are given in Ref. \([15]\). We here only cite a particularly useful formula which we learned from Hata [39]:

$$
123(0|\exp(F_{123})|0)_{3}^{p_3=w_3=0} = 12(0)|\exp(H_{12})\left[1 - \frac{\epsilon}{2\alpha_1} \sum'_{n,m} \frac{1}{n + m} \left(\alpha_n^{(1)} \cdot \alpha_m^{(1)} + 2i\gamma_n^{(1)} \beta_m^{(1)} + \text{a.h.}\right) - \frac{\epsilon}{\alpha_1} \sum_{n \neq 0} \left( : c_n^{(1)} b_{-n}^{(1)} : + \text{a.h.} \right) + O(\epsilon^2)\right],
$$

(C.8)

with

$$
\alpha_n \cdot \alpha_n \equiv \alpha_n^\mu \eta_{\mu\nu} \alpha_n^\nu + \alpha_n^i (E) G_{ij} \alpha_n^j (E).
$$

Here $p_3$ is set equal to zero except for the $p_3^+$ component, of course, and $E_{12}$ is the exponent of the 2-point vertex $12(R)$. The primed summation $\sum'_{n,m}$ means the summation excluding the $n = m = 0$ or $n + m = 0$ terms for $\alpha_n^{(1)} \cdot \alpha_m^{(1)}$ part and the $nm = 0$ or $n + m = 0$ terms for the $2i\gamma_n^{(1)} \beta_m^{(1)}$ part. For our case ($\alpha = p^+$ HIKKO theory) the terms containing $\alpha_n^+ = \alpha_n^- = p^+ / \sqrt{2}$ should also be excluded from the summation.

The second $OSp(1,1|2)$ asymmetric term in \((C.8)\) comes from the $\epsilon$ difference between $\alpha_1$ and $-\alpha_2$ contained in $\gamma_1^{(1)} \beta_2^{(2)} + \gamma_2^{(2)} \beta_1^{(1)}$.

Finally we explain how \((7.25)\) is derived. For the case of the $E_{\pm}^i$ generators we have to evaluate

$$
123(V|\Lambda^\pm_{1})_{3} = \lim_{\epsilon = \alpha_3 \to 0} 123(V|c_0^{(1)} \delta_{-1}^{(3)} | \pm k^i, 0)_{3} = \mu_{123}^2 \delta(1, 2, 3) 123(0)|\exp(F_{123})(c_0^{(1)} + \frac{1}{\sqrt{2}} W_{I}^{(1)}) \times (c_0^{(2)} + \frac{1}{\sqrt{2}} W_{I}^{(2)}) e^{-i\pi (p_1 w_3 - p_2 w_2)} P_{123} b_{-1}^{(3)} \pm k^i, 0)_{3},
$$

(C.9)

where we have used the vertex expression \((3.25)\) and the cyclic symmetry of the vertex cocycle factor $\exp(-i\pi (p_3 w_2 - p_1 w_1)) = \exp(-i\pi (p_1 w_3 - p_2 w_2))$ for later convenience.
Since the momentum \( p_{+3} = \pm k^i \) is non-zero and the exponent \( F_{123} \) contains a singular (zero-mode)\(^2 \) term \( \tau_0 \sum_{r=1}^3 (p_{+r}^2 + p_{-r}^2)/\alpha_r \), we have the factor

\[
\mu_{123}^2 \exp(\tau_0 \sum_{r=1}^3 \frac{p_{+r}^2 + p_{-r}^2}{\alpha_r}) \sim \mu_{123} \sim \frac{e\alpha_2}{\epsilon} \text{sgn}(\epsilon\alpha_2). \tag{C.10}
\]

Since this is \( O(1/\epsilon) \), we have to evaluate the other terms up to \( O(\epsilon) \). The oscillator \( \beta_{-1}^{(3)} \) can be contracted with \( F_{123} \) or \( W_I^{(r)} \). But, since \( \beta_{-1}^{(3)} = \epsilon \beta_{-1}^{(3)} \) is already of \( O(\epsilon) \), the contraction with \( F_{123} \) does not contribute. The contraction with \( W_I^{(r)} \) gives

\[
3(0|c_0^{+(1)} + \frac{1}{\sqrt{2}} W_I^{(1)}(c_0^{+(2)} + \frac{1}{\sqrt{2}} W_I^{(2)})\beta_{-1}^{(3)}|0)_3 = \left( c_0^{+(1)} + c_0^{+(2)} + \frac{1}{\sqrt{2}} (W_I^{(1)} + W_I^{(2)}) \right) \frac{1}{2} \left( \frac{\epsilon}{e\alpha_2} \text{sgn}(e\alpha_2) \right) \tag{C.11}
\]

by the help of expression (A.3) for \( W_I^{(r)} \) and the related limiting formulas of Ref. [15], where \( W_I^{(r)} \) denotes \( W_I^{(r)} \) with string-three oscillators eliminated. The second term \( (W_I^{(1)} + W_I^{(2)}) \) vanishes on the reflector \( 12|\langle R \rangle \) or on \( 12|\langle 0 \rangle e_E \). Noting the presence of the term linear in \( p_{+3} = \pm k^i \) in \( F_{123} \), we see that the exponent \( F_{123}' \) \( (F_{123} \) with the (zero-mode)\(^2 \) term omitted) approaches

\[
\lim_{\epsilon \to 0} 1_{123} (0|\exp(F_{123})|0)_3 = 1_{12} (0|\exp \left( E_{12} \pm \sqrt{2} \alpha_2 \sum_{r=2,3} \sum_{n \geq 1} \tilde{N}_{n}^r k^i \cdot \alpha_n^{(r)} \right) \tag{C.12}
\]

Since \( \alpha_2 \tilde{N}_{n}^2 = (-)^n \alpha_2 \tilde{N}_{n}^3 = -1/n \) in this limit, we find

\[
1_{123}(V||\Lambda^i_{\pm})_3 = \frac{1}{2} \delta(1,2,3)|_{p_{+3} = \pm k^i} 1_{12} (0|e^{E_{12} e^{i\pi p_{2} w_2}} c_0^{+(1)} + c_0^{+(2)}) \times \exp \left( \mp \sqrt{2} \sum_{n \geq 1} \frac{1}{n} (\alpha_n^{(1)} + (-)^n \alpha_n^{(2)}) \cdot k^i \right) e^{-i\pi p_{1} w_3} \mathcal{P}_{12}. \tag{C.13}
\]

We note that the equality

\[
\delta(p_{+1} + p_{+2} \pm k^i) = \delta(p_{+1} + p_{+2}) \exp(\pm 2ik^i \cdot x_{+1})
\]

holds since \( [x^i_{+}, p_{+j}] = (i/2) \delta^i_j \) (although the \( \delta \) here is a Kronecker’s delta). Using this
and \( w_3 = \mp k^i \), we find

\[
123\langle V||\Lambda_+^i\rangle_3 = \frac{1}{2} 12\langle R' b_0^{-1} \rangle e^{\pm 2ik^i x_{+1}}
\times \exp \left( \mp \sqrt{2} \sum_{n \geq 1} \frac{1}{n} (\alpha_n^{(1)} + (-)^n \alpha_n^{(2)}) \cdot k^i \right) e^{\pm i\pi p_1 k^i} \mathcal{P}_{12}. \tag{C.14}
\]

Owing to the connection condition (C.6), this equals

\[
e^{\pm i\pi p_1 k^i} \int \frac{d\sigma}{2\pi} 12\langle R|b_0^{-1} \rangle \exp \left( \pm 2ik^i \cdot X^{(1)}_+ (\sigma = 0) \right) : \mathcal{P}^{(1)} e^{\pm i\pi p_1 k^i} : \tag{C.15}
\]

This gives rise to the desired result for \( E^i_\pm \) in (7.25). The calculation for the case of \( E^i_3 \) is much simpler and can be carried out similarly.

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