Local SGD Converges Fast and Communicates Little

Sebastian U. Stich
EPF Lausanne (EPFL)

Abstract

Mini-batch stochastic gradient descent (SGD) is the state of the art in large scale parallel machine learning, but its scalability is limited by a communication bottleneck. Recent work proposed local SGD, i.e. running SGD independently in parallel on different workers and averaging only once in a while. This scheme shows promising results in practice, but eluded thorough theoretical analysis.

We prove concise convergence rates for local SGD on convex problems and show that it converges at the same rate as mini-batch SGD in terms of number of evaluated gradients, that is, the scheme achieves linear speed-up in the number of workers and mini-batch size. Moreover, the number of communication rounds can be reduced up to a factor of $T^{1/2}$—where $T$ denotes the number of total steps—compared to mini-batch SGD.

1 Introduction

Stochastic Gradient Descent (SGD) \[20\] consists of iterations of the form

$$x_{t+1} := x_t - \eta_t g_t,$$

for iterates $x_t, x_{t+1} \in \mathbb{R}^d$, stepsize (or learning rate) $\eta_t > 0$, and stochastic gradient $g_t$ with the property $\mathbb{E} g_t = \nabla f(x_t)$, for a loss function $f: \mathbb{R}^d \to \mathbb{R}$. This scheme can easily be parallelized by replacing $g_t$ in (1) by an average of stochastic gradients that are independently computed in parallel on separate workers (parallel SGD). This simple scheme has a major drawback: in each iteration the results of the local computations have to be shared with the other workers in order to compute the next iterate $x_{t+1}$. Communication has been reported to be a major bottleneck for many large scale deep learning applications, see e.g. [3, 11, 24, 31].

Mini-batch parallel SGD addresses this issue by increasing compute before communication. Each worker now computes a mini-batch of size $b$ before communication. This scheme is implemented in state-of-the-art distributed deep learning frameworks [1, 17, 23]. Recent work in [7, 30] explores the current limitations of this approach, as in general it is reported that performance degrades for too large mini-batch sizes [2].

As a remedy, [13, 35] propose to parallelize SGD in a different way: instead of keeping the sequences on different machines in sync, let them evolve locally on each machine, independent from each other, and only average the solutions at the end. Zhang and coauthors [33] show statistical convergence (see also [6]), but the analysis restricts the algorithm to at most one pass over the data, which is in general not enough for the training error to converge. Current work explores the benefits of more frequent averaging of the parallel sequences [4, 32], but thus far the question of how often communication rounds need to be initiated has eluded a concise theoretical answer. Indeed, the lack in the theoretical understanding of local SGD is astounding: the literature does not even resolve the question whether averaging helps, i.e. concretely, whether running local SGD on $K$ workers is $K$ times better than running just a single instance of SGD on one worker.

We fill this gap in the literature and provide a concise convergence analysis of local SGD. We show that averaging helps, i.e. by frequently synchronizing $K$ local sequences, the convergence rate increases by a factor of $K$, i.e. a linear speed-up can be attained. This shows that local SGD is as efficient as parallel mini-batch SGD, but the communication cost can be drastically reduced.

1On convex functions, the average of the $K$ local solutions can of course only decrease the objective value, but convexity does not imply that the averaged point is $K$ times better.
1.1 Contributions

We consider finite-sum convex optimization problems $f: \mathbb{R}^d \to \mathbb{R}$ of the form

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x), \quad x^* := \arg\min_{x \in \mathbb{R}^d} f(x), \quad f^* := f(x^*),$$

where $f$ is $L$-smooth\footnote{$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2, \forall x, y \in \mathbb{R}^d.$} and $\mu$-strongly convex\footnote{$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2, \forall x, y \in \mathbb{R}^d.$} We consider $K$ parallel mini-batch SGD sequences with mini-batch size $b$ that are synchronized after at most every $H$ iterations. For appropriate chosen stepsizes and an averaged iterate $\hat{x}_T$ after $T$ steps, we show convergence

$$\mathbb{E} f(\hat{x}_T) - f^* = O \left( \frac{1}{\mu KT} + \frac{\kappa + H}{\mu KT^2} \right) \sigma^2 + O \left( \frac{\kappa H^2}{\mu T^2} + \frac{\kappa^3 + H^3}{T^3} \right) G^2,$$

for $\kappa = \frac{L}{\mu}$, variance bound $\sigma^2 \geq \text{Var}(\nabla f_i(x))$ and second moment bound $G^2 \geq \mathbb{E} \|\nabla f_i(x)\|^2$. For $T$ large enough, the very first term is dominating and we see that both, increasing the number of parallel workers $K$ and increasing the mini-batch size $b$, yield a linear reduction in the number of iterations. This holds for communication rounds as few\footnote{Convergence can be shown even for larger $H$, but at a slower rate. See discussion in Section \ref{section:related_work}.} as every $H = O(T^{1/2} K^{-1/2} b^{-1/2})$ steps. Thus we see that compared to parallel mini-batch SGD the communication rounds can be reduced by a factor $O(T^{1/2} K^{-1/2} b^{-1/2})$ without hampering the asymptotic convergence.

Our proof is simple and straightforward, and we imagine that—with slight modifications of the proof—the technique can also be used to analyze other variants of SGD that evolve sequences on different worker that are not perfectly synchronized. For instance as it is the case in sparsified gradient methods \cite{2, 5, 24, 26} that aim to reduce the communication cost by sharing only a few coordinates of the stochastic gradients in each communication round.

Although we do not yet provide convergence guarantees for the non-convex setting we feel that the positive results presented here will spark further investigation of local SGD for this important application.

1.2 Related Work

A parallel line of work reduces the communication cost by compressing the stochastic gradients before communication. For instance by limiting the number of bits in the floating point representation \cite{8, 15, 22}, or random quantization \cite{3, 29}. The ZipML framework applies this technique also to the data \cite{31}. Sparsification methods reduce the number of non-zero entries in the stochastic gradient \cite{3, 28}.

A very aggressive—and promising—sparsification method is to keep only very few coordinates of the stochastic gradient by considering only the coordinates with the largest magnitudes \cite{2, 3, 11, 24, 26, 27}.

Asynchronous updates provide an alternative solution to disguise the communication overhead to a certain amount \cite{16, 22}.

Convergence proofs for SGD \cite{20} typically rely on averaging of the iterates \cite{14, 18, 21}, tough also convergence of the last iterate can be proven \cite{25}. For our convergence proof we rely on averaging techniques that give more weight to more recent iterates \cite{10, 13, 25}. The first steps in the proof were inspired by the the perturbed iterate framework from Mania et al. \cite{12}.

1.3 Outline

We formally introduce local SGD in Section \ref{section:local_sgd} and sketch the convergence proof in Section \ref{section:convergence}. The proof of the technical results are deferred to Appendix \ref{section:technical_results}.\footnote{Convergence can be shown even for larger $H$, but at a slower rate. See discussion in Section \ref{section:related_work}.}
2 Local SGD

The algorithm local SGD (depicted in Algorithm 1) generates in parallel $K$ sequences $\{x^k_t\}_{t=0}^T$ of iterates, $k \in [K]$. Here $K$ denotes the level of parallelization, i.e. the number of distinct parallel sequences and $T$ the number of steps (i.e. the total number of stochastic gradient evaluations is $TK$). Let $\mathcal{I}_T \subseteq [T]$ with $T \in \mathcal{I}_T$ denote a set of synchronization indices. Then local SGD evolves the sequences $\{x^k_t\}_{t=0}^T$ in the following way:

$$x^k_{t+1} := \begin{cases} x^k_t - \eta_t \nabla_i^k f(x^k_t), & \text{if } t+1 \notin \mathcal{I}_T \\ \frac{1}{K} \sum_{k=1}^K (x^k_t - \eta_t \nabla_i^k f(x^k_t)) & \text{if } t+1 \in \mathcal{I}_T \end{cases}$$

(4)

where indices $i^k_t \sim \text{a.a.r.} \ [n]$ and $\{\eta_t\}_{t \geq 0}$ denotes a set of stepsizes. If $\mathcal{I}_T = [T]$ then the synchronization of the sequences is performed every iteration. In this case, (4) amounts to parallel or mini-batch SGD with mini-batch size $K$. On the other extreme, if $\mathcal{I}_T = \{T\}$, the synchronization only happens at the end, which is known as one-shot averaging.

**Algorithm 1 Local SGD**

1: Initialize variables $x^k_0 = x_0$ for $k \in [K]$
2: for $t$ in $0, \ldots, T-1$ do
3: \hspace{1em} parallel for $k \in [K]$ do
4: \hspace{2em} Sample $i^k_t$ uniformly in $[n]$
5: \hspace{2em} $x^k_{t+1} \leftarrow x^k_t - \eta_t \nabla f(x^k_t)$ \quad \triangleright \text{local update}
6: \hspace{1em} end parallel for
7: \hspace{1em} if $t+1 \in \mathcal{I}_T$ then
8: \hspace{2em} $x^k_{t+1} \leftarrow \frac{1}{K} \sum_{k=1}^K x^k_t$ \quad \triangleright \text{global synchronization}
9: \hspace{1em} end if
10: end for

In order to measure the longest interval between subsequent synchronization steps, we introduce the gap of a set of integers.

**Definition 2.1 (gap).** The gap of a set $\mathcal{P} := \{p_0, \ldots, p_t\}$ of $t+1$ integers, $p_i \leq p_{i+1}$ for $i = 0, \ldots, t-1$, is defined as

$$\text{gap}(\mathcal{P}) := \max_{i=1, \ldots, t} p_i - p_{i-1}.$$  

(5)

2.1 Variance reduction in local SGD

Before jumping to the convergence result, we first discuss a guiding example.

**Parallel SGD.** For carefully chosen stepsizes $\eta_t$, SGD converges at rate $\mathcal{O}(\sigma^2/T)$ on strongly convex and smooth functions $f$, where $\sigma^2 \geq \mathbb{E}||\nabla f(x^k_t) - \nabla f(x^k_t)||^2$ for $t > 0, k \in [K]$ is an upper bound on the variance, see for instance [34]. By averaging $K$ stochastic gradients—such as in parallel SGD—the variance decreases by a factor of $K$, and we conclude that parallel SGD converges at a rate $\mathcal{O}(\sigma^2/K)$, i.e. achieves a linear speed-up.

**Towards local SGD.** For local SGD such a simple argument is elusive. For instance, just capitalizing the convexity of the objective function $f$ is not enough: this will show that the averaged iterate of $K$ independent SGD sequences converges at rate $\mathcal{O}(\sigma^2/K)$, i.e. no speed-up can be shown in this way. Thus, it seems local SGD cannot profit from parallelization.

---

5For the ease of presentation, we assume here that each worker in local SGD only processes a mini-batch of size $b = 1$. This can be done without loss of generality, as we discuss later in Remark 2.3.
The variance by a factor of \(b\) gradient. In mini-batch local SGD, each worker computes a mini-batch of size \(\hat{\sigma}\) where \(W \in \mathbb{R}\). We now discuss the savings in communication compared to parallel SGD and mini-batch SGD. We omit the dependency on \(\sqrt{G}\) and \(\sigma\) for stepsizes \(\eta\) and \(\mu\). Indeed, we will make this statement precise in the proof below. By synchronizing the sequences sufficiently often, their diversity can be controlled.

### 2.2 Convergence Result and Discussion

**Theorem 2.2.** Let \(f\) be \(L\)-smooth and \(\mu\)-strongly convex, \(\mathbb{E}_i \|\nabla f_i(x^k) - \nabla f(x^k)\|^2 \leq \sigma^2\), \(\mathbb{E}_i \|\nabla f_i(x^k)\|^2 \leq G^2\), for \(t = 0, \ldots, T - 1\), where \(\{x^k\}_{i=0}^T\) for \(k \in [K]\) are generated according to (4) with gap \(\mathcal{I}_T \leq H\) and for stepsizes \(\eta = \frac{1}{\mu(a + t)}\) with shift parameter \(a > \max\{16\kappa, H\}\), for \(\kappa = \frac{2}{\mu}\). Then

\[
\mathbb{E} f(x_T) - f^* \leq \frac{\mu^3}{25T} \|x_0 - x^*\|^2 + \frac{4T(T + 2a)}{\mu K S_T} \sigma^2 + \frac{256T}{\mu^2 S_T} G^2 H^2 L,
\]

where \(x_T = \frac{1}{K S_T} \sum_{k=1}^K \sum_{t=0}^{T-1} w_t x^k\), for \(w_t = (a + t)^2\), and \(S_T = \sum_{t=0}^{T-1} w_t \geq \frac{1}{3} T^3\).

We were not especially careful to optimize the constants (and the lower order terms) in (6), so we now state the asymptotic result.

**Corollary 2.3.** Let \(x_T\) be as defined in Theorem 2.2 for parameter \(a = \max\{16\kappa, H\}\). Then

\[
\mathbb{E} f(x_T) - f^* = O \left( \frac{1}{\mu KT} + \frac{\kappa + H}{\mu KT^2} \right) \sigma^2 + O \left( \frac{\kappa H^2}{\mu T^2} \right) G^2 + O \left( \frac{\mu(\kappa + H)^3}{T^3} \right) \|x_0 - x^*\|^2
\]

\[
= O \left( \frac{1}{\mu KT} + \frac{\kappa + H}{\mu KT^2} \right) \sigma^2 + O \left( \frac{\kappa H^2}{\mu T^2} + \frac{\kappa^3 + H^3}{\mu T^3} \right) G^2.
\]

For the last estimate we used \(\mathbb{E} \mu \|x_0 - x^*\| \leq 2G\) for \(\mu\)-strongly convex \(f\), as derived in [19, Lemma 2].

**Remark 2.4** (Mini-batch local SGD). So far we assumed that each worker only computes a single stochastic gradient. In mini-batch local SGD, each worker computes a mini-batch of size \(b\) in each iteration. This reduces the variance by a factor of \(b\), and thus Theorem 2.2 gives the convergence rate of mini-batch local SGD when \(\sigma^2\) is replaced by \(\frac{\sigma^2}{b}\).

We now state some consequences of equation (7). For the ease of the exposition we omit sometimes the dependency on \(L\) and \(H\) below.

**Convergence rate.** For large \(T\), i.e. \(T = \Omega \left( \kappa + H + \frac{\mu KT^2}{\sigma^2} + \frac{(\kappa^3 + H^3) K^{1/2} G}{\sigma} \right)\) the very first term is dominating in (7). For constant \(K, H\), local SGD converges at the rate \(O\left(\frac{\sigma^2}{\mu KT}\right)\). That is, local SGD achieves a linear speed-up in the number of workers.

**Global synchronization steps.** It needs to hold \(H = O(T^{1/2} G)\) to attain the \(O\left(\frac{\sigma^2}{\mu KT}\right)\) convergence rate.

Yet, local SGD converges for any \(H = o(T)\), though at a lower rate.

We now discuss the savings in communication compared to parallel SGD and mini-batch SGD. We omit the dependency on \(\sigma^2\) and \(G^2\), but depict the dependency on the local mini-batch size \(b\). That is, we assume both, mini-batch SGD and local SGD use a batch size of \(b\) per worker (see Remark 2.3).

**Saving in communication.** It suffices to set \(H = O\left(\frac{T^{1/2} G}{\sqrt{K} b} \right)\) to attain the same convergence rate as parallel mini-batch SGD. The number of global synchronization rounds, \(\frac{T}{\sqrt{b}}\), can therefore be as small as \(o\left((TKb)^{1/2}\right)\). This yields a reduction of the number of communication rounds by a factor \(O\left(T^{1/2} K^{-1/2} b^{-1/2}\right)\) compared to parallel mini-batch SGD without hurting the convergence rate.
Extreme Cases. We have not optimized the result for extreme settings of $H$, $K$, $L$ or $\sigma$. For instance, we do not recover convergence for the one shot averaging, i.e. the setting $H = T$ (though convergence for $H = o(T)$). For large $K$ or small $\sigma^2$ we get the rate $O\left(\frac{K^2}{T^2}\right)$. However, this is not optimal, as for $\mu > 0$ gradient descent (with constant stepsize) converges linearly.

3 Proof Outline

We now give the outline of the proof. The proofs of the lemmas are given in Appendix A.

Perturbed iterate analysis. Inspired by the perturbed iterate framework in [12] we first define a virtual sequence $\{\bar{x}_t\}_{t \geq 0}$ in the following way:

$$\bar{x}_0 = x_0, \quad \bar{x}_t = \frac{1}{K} \sum_{k=1}^{K} x_t^k, \tag{8}$$

where the sequences $\{x_t^k\}_{t \geq 0}$ for $k \in [K]$ are the same as in (4). Notice that this sequence never has to be computed, it is just a tool that we use in the analysis. Further notice that $\bar{x}_t = x_t^k$ for $k \in [K]$ whenever $t \in \mathcal{I}_T$. Especially, when $\mathcal{I}_T = [T]$, then $\bar{x}_t \equiv x_t^k$ for every $k \in [K], t \in [T]$.

It will be useful to define

$$g_t := \frac{1}{K} \sum_{k=1}^{K} \nabla f_t(x_t^k), \quad \bar{g}_t := \frac{1}{K} \sum_{k=1}^{K} \nabla f(x_t^k). \tag{9}$$

Observe $\bar{x}_{t+1} = \bar{x}_t - \eta_t g_t$ and $E g_t = \bar{g}_t$.

Lemma 3.1. Let $\{x_t\}_{t \geq 0}$ and $\{\bar{x}_t\}_{t \geq 0}$ for $k \in [K]$ be defined as in (4) and (8) and let $f$ be $L$-smooth and $\mu$-strongly convex and $\eta_t \leq \frac{4}{\mu L T}$. Then

$$E \|\bar{x}_{t+1} - \bar{x}^*\|^2 \leq (1 - \mu \eta_t) E \|\bar{x}_t - \bar{x}^*\|^2 + \eta_t^2 E \|g_t - \bar{g}_t\|^2 - \frac{1}{2} \eta_t E(f(\bar{x}_t) - f^*) + 2\eta_t L \frac{1}{K} \sum_{k=1}^{K} E \|x_t - x_t^k\|^2. \tag{10}$$

Bounding the variance. From equation (10) it becomes clear that we should derive an upper bound on $E \|g_t - \bar{g}_t\|^2$. We will relate this to the variance $\sigma^2$.

Lemma 3.2. Let $\sigma^2 \geq E \|\nabla f_t(x_t^k) - \nabla f(x_t^k)\|^2$ for $k \in [K], t \in [T]$. Then $E \|g_t - \bar{g}_t\|^2 \leq \frac{\sigma^2}{T^2}$.

Bounding the deviation. Further, we need to bound $\frac{1}{K} \sum_{k=1}^{K} E \|x_t - x_t^k\|^2$. For this we impose a condition on $\mathcal{I}_T$ and an additional condition on the stepsize $\eta_t$.

Lemma 3.3. If $\text{gap}(\mathcal{I}_T) \leq H$ and $\eta_t := \frac{c}{a+t}$ for $a \geq H$, $c > 0$, (i.e. $H \leq \frac{c}{\eta_t}, \forall t \geq 0$), then

$$\frac{1}{K} \sum_{k=1}^{K} E \|x_t - x_t^k\|^2 \leq 4\eta_t^2 G^2 H^2, \tag{11}$$

where $G^2$ is a constant such that $E \|\nabla f_t(x_t^k)\|^2 \leq G^2$ for $k \in [K], t \in [T]$. 

5
We prove convergence of local SGD and are the first to show that local SGD attains linear as speed-up when parallelized among K workers. We show that local SGD saves up to a factor of $O(T^{1/2})$ in global communication rounds compared to mini-batch SGD, while still converging at the same rate in terms of total stochastic gradient computations. This result shows that averaging helps in the convex setting, as observed in [32]. However, convergence results for the non-convex setting are still absent. We feel that the positive results shown here motivate to intensify the research in this direction—that could in turn have major impacts on the distributed training of neural networks—and suppose that the analysis presented here can serve as a guide for the first steps. Moreover, the presented proof techniques seem to be applicable to analyze other variants of SGD where the iterates are not entirely synchronized, for instance as it happens to be the case in sparsified gradient methods [2, 5, 24, 26].

**Optimal Averaging.** Similar as discussed in [10, 19, 25] we have to define a suitable averaging scheme for the iterates $\{\bar{x}_t\}_{t \geq 0}$ to get the optimal convergence rate. In contrast to [10] that use linearly increasing weights, we use quadratically increasing weights, as for instance [25].

**Lemma 3.4.** Let $\{a_t\}_{t \geq 0}$, $a_t \geq 0$, $\{e_t\}_{t \geq 0}$, $e_t \geq 0$ be sequences satisfying

$$a_{t+1} \leq (1 - \mu \eta_t) a_t - \eta_t e_t A + \eta_t^2 B + \eta_t^3 C,$$

for $\eta_t = \frac{4}{\mu (a+t)}$ and constants $A > 0$, $B, C \geq 0$, $\mu > 0$, $a > 1$. Then

$$\frac{A}{S_T} \sum_{t=0}^{T-1} w_t e_t \leq \frac{\mu a^3}{4S_T} a_0 + \frac{2T(T+2a)}{\mu S_T} B + \frac{16T}{\mu^2 S_T} C,$$

for $w_t = (a + t)^2$ and $S_T := \sum_{t=0}^{T-1} w_t = \frac{T}{6} (2T^2 + 6aT - 3T + 6a^2 - 6a + 1) \geq \frac{1}{3} T^3$.

**Proof of Theorem 2.2.** The proof of the theorem immediately follows from the four lemmas that we have presented and convexity of $f$, i.e. we have $\mathbb{E} f(\bar{x}_T) - f^* \leq \frac{1}{S_T} \sum_{t=0}^{T-1} w_t e_t$ in [13], for constants $A = \frac{2}{3}$, (Lemma 3.1), $B = \frac{a^2}{2}$, (Lemma 3.2) and $C = 8G^2 H^2 L$, (Lemma 3.3).

We now state a few observations that become clear after inspecting the proof.

**Weighted Averaging.** Uniform averaging of the local sequences in [4] is not in particular important. A weighted average would yield the same result.

**Conditions on $a$.** The dependence of $a_t$ and thus the initial stepsize, on $L$ is due to a technical condition in Lemma 3.1 that requires $\eta_t \leq \frac{1}{4T}$, $\forall t > 0$. As the sequence $\eta_t$ is decreasing the condition $\eta_t \leq \frac{1}{4T}$ will eventually hold for $t$ large enough, without imposing a condition on $\eta_0$. Thus, it is possible to revoke this condition by treating the first iterates separately.

**Adaptive communication frequency.** The literature discusses schemes with adaptive communication frequency, i.e. it is suggested to communicate more frequently at the beginning of the optimization [32]. We did not consider such schemes here, but we observe that for small $T$, the estimate in (7) suffers from choosing $H$ too large, as the third term depends on $\frac{H^2}{T^2} G^2$. Thus, adaptively increasing $H$ seems to be a good strategy for the first few epochs.

**4 Conclusion**

We prove convergence of local SGD and are the first to show that local SGD attains linear as speed-up when parallelized among $K$ workers. We show that local SGD saves up to a factor of $O(T^{1/2})$ in global communication rounds compared to mini-batch SGD, while still converging at the same rate in terms of total stochastic gradient computations. This result shows that averaging helps in the convex setting, as observed in [32]. However, convergence results for the non-convex setting are still absent. We feel that the positive results shown here motivate to intensify the research in this direction—that could in turn have major impacts on the distributed training of neural networks—and suppose that the analysis presented here can serve as a guide for the first steps. Moreover, the presented proof techniques seem to be applicable to analyze other variants of SGD where the iterates are not entirely synchronized, for instance as it happens to be the case in sparsified gradient methods [2, 5, 24, 26].

**Acknowledgements**

The author thanks Tao Lin and Jean-Baptiste Cordonnier for spotting various typos in the first version of this manuscript.
References

[1] Martín Abadi, Ashish Agarwal, Paul Barham, Eugene Brevdo, Zhifeng Chen, Craig Citro, Greg S Corrado, Andy Davis, Jeffrey Dean, Matthieu Devin, et al. Tensorflow: Large-scale machine learning on heterogeneous distributed systems. arXiv preprint arXiv:1603.04467, 2016.

[2] Alham Fikri Aji and Kenneth Heafield. Sparse communication for distributed gradient descent. In Proceedings of the 2017 Conference on Empirical Methods in Natural Language Processing, pages 440–445. Association for Computational Linguistics, 2017.

[3] Dan Alistarh, Demjan Grubic, Jerry Li, Ryota Tomioka, and Milan Vojnovic. QSGD: Communication-efficient SGD via gradient quantization and encoding. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information Processing Systems 30, pages 1709–1720. Curran Associates, Inc., 2017.

[4] Avleen S Bijral, Anand D Sarwate, and Nathan Srebro. On Data Dependence in Distributed Stochastic Optimization. arXiv.org, 2016.

[5] N. Dryden, T. Moon, S. A. Jacobs, and B. V. Essen. Communication quantization for data-parallel training of deep neural networks. In 2016 2nd Workshop on Machine Learning in HPC Environments (MLHPC), pages 1–8, Nov 2016.

[6] Antoine Godichon-Baggioni and Sofiane Saadane. On the rates of convergence of parallelized averaged stochastic gradient algorithms. arXiv preprint arXiv:1710.07926, 2017.

[7] Priya Goyal, Piotr Dollár, Ross B. Girshick, Pieter Noordhuis, Lukasz Wesolowski, Aapo Kyrola, Andrew Tulloch, Yangqing Jia, and Kaiming He. Accurate, large minibatch SGD: training ImageNet in 1 hour. CoRR, abs/1706.02677, 2017.

[8] Suyog Gupta, Ankur Agrawal, Kailash Gopalakrishnan, and Pritish Narayanan. Deep learning with limited numerical precision. In Proceedings of the 32Nd International Conference on International Conference on Machine Learning - Volume 37, ICML’15, pages 1737–1746. JMLR.org, 2015.

[9] Nitish Shirish Keskar, Dheevatsa Mudigere, Jorge Nocedal, Mikhail Smelyanskiy, and Ping Tak Peter Tang. On large-batch training for deep learning: Generalization gap and sharp minima. arXiv preprint arXiv:1609.04836, 2016.

[10] Simon Lacoste-Julien, Mark W. Schmidt, and Francis R. Bach. A simpler approach to obtaining an $O(1/t)$ convergence rate for the projected stochastic subgradient method. CoRR, abs/1212.2002, 2012.

[11] Yujun Lin, Song Han, Huizi Mao, Yu Wang, and Bill Dally. Deep gradient compression: Reducing the communication bandwidth for distributed training. In ICLR 2018 - International Conference on Learning Representations, 2018.

[12] Horia Mania, Xinghao Pan, Dimitris Papailiopoulos, Benjamin Recht, Kannan Ramchandran, and Michael I. Jordan. Perturbed iterate analysis for asynchronous stochastic optimization. SIAM Journal on Optimization, 27(4):2202–2229, 2017.

[13] Ryan McDonald, Mehryar Mohri, Nathan Silberman, Dan Walker, and Gideon S. Mann. Efficient large-scale distributed training of conditional maximum entropy models. In Y. Bengio, D. Schuurmans, J. D. Lafferty, C. K. I. Williams, and A. Culotta, editors, Advances in Neural Information Processing Systems 22, pages 1231–1239. Curran Associates, Inc., 2009.

[14] Eric Moulines and Francis R. Bach. Non-asymptotic analysis of stochastic approximation algorithms for machine learning. In J. Shawe-Taylor, R. S. Zemel, P. L. Bartlett, F. Pereira, and K. Q. Weinberger, editors, Advances in Neural Information Processing Systems 24, pages 451–459. Curran Associates, Inc., 2011.

[15] T. Na, J. H. Ko, J. Kung, and S. Mukhopadhyay. On-chip training of recurrent neural networks with limited numerical precision. In 2017 International Joint Conference on Neural Networks (IJCNN), pages 3716–3723, May 2017.

[16] Feng Niu, Benjamin Recht, Christopher Re, and Stephen J. Wright. Hogwild!: A lock-free approach to parallelizing stochastic gradient descent. In Proceedings of the 24th International Conference on Neural Information Processing Systems, NIPS’11, pages 693–701, USA, 2011. Curran Associates Inc.
[17] Adam Paszke, Sam Gross, Soumith Chintala, Gregory Chanan, Edward Yang, Zachary DeVito, Zeming Lin, Alban Desmaison, Luca Antiga, and Adam Lerer. Automatic differentiation in pytorch. 2017.

[18] B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM Journal on Control and Optimization*, 30(4):838–855, 1992.

[19] Alexander Rakhlin, Ohad Shamir, and Karthik Sridharan. Making gradient descent optimal for strongly convex stochastic optimization. In *Proceedings of the 29th International Conference on International Conference on Machine Learning*, ICML’12, pages 1571–1578, USA, 2012. Omnipress.

[20] Herbert Robbins and Sutton Monro. A Stochastic Approximation Method. *The Annals of Mathematical Statistics*, 22(3):400–407, September 1951.

[21] David Ruppert. Efficient estimations from a slowly convergent robbins-monro process. Technical report, Cornell University Operations Research and Industrial Engineering, 1988.

[22] Christopher De Sa, Ce Zhang, Kunle Olukotun, and Christopher Ré. Taming the wild: A unified analysis of HOG WILD!-style algorithms. In *Proceedings of the 28th International Conference on Neural Information Processing Systems - Volume 2*, NIPS’15, pages 2674–2682, Cambridge, MA, USA, 2015. MIT Press.

[23] Frank Seide and Amit Agarwal. CNTK: Microsoft’s open-source deep-learning toolkit. In *Proceedings of the 22nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 2135–2135. ACM, 2016.

[24] Frank Seide, Hao Fu, Jasha Droppo, Gang Li, and Dong Yu. 1-bit stochastic gradient descent and its application to data-parallel distributed training of speech DNNs. In Haizhou Li, Helen M. Meng, Bin Ma, Eungsiong Chung, and Lei Xie, editors, *INTERSPEECH*, pages 1058–1062. ISCA, 2014.

[25] Ohad Shamir and Tong Zhang. Stochastic gradient descent for non-smooth optimization: Convergence results and optimal averaging schemes. In Sanjoy Dasgupta and David McAllester, editors, *Proceedings of the 30th International Conference on Machine Learning*, volume 28 of *Proceedings of Machine Learning Research*, pages 71–79, Atlanta, Georgia, USA, 17–19 Jun 2013. PMLR.

[26] Nikko Strom. Scalable distributed DNN training using commodity GPU cloud computing. In *INTERSPEECH*, pages 1488–1492. ISCA, 2015.

[27] Xu Sun, Xuancheng Ren, Shuming Ma, and Houfeng Wang. meProp: Sparsified back propagation for accelerated deep learning with reduced overfitting. In Haizhou Li, Helen M. Meng, Bin Ma, Eungsiong Chung, and Lei Xie, editors, *INTERSPEECH*, pages 1058–1062. ISCA, 2014.

[28] Jianqiao Wang, Ji Liu, and Tong Zhang. Gradient sparsification for communication-efficient distributed optimization. *CoRR*, abs/1710.09584, 2017.

[29] Wei Wen, Cong Xu, Feng Yan, Chunpeng Wu, Yandan Wang, Yiran Chen, and Hai Li. Terngrad: Ternary gradients to reduce communication in distributed deep learning. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, *Advances in Neural Information Processing Systems 30*, pages 1509–1519. Curran Associates, Inc., 2017.

[30] Yang You, Igor Gitman, and Boris Ginsburg. Scaling SGD batch size to 32k for ImageNet training. *CoRR*, abs/1708.03888, 2017.

[31] Hantian Zhang, Jerry Li, Kaan Kara, Dan Alistarh, Ji Liu, and Ce Zhang. ZipML: Training linear models with end-to-end low precision, and a little bit of deep learning. In Doina Precup and Yee Whye Teh, editors, *Proceedings of the 34th International Conference on Machine Learning*, volume 70 of *Proceedings of Machine Learning Research*, pages 4035–4043, International Convention Centre, Sydney, Australia, 06–11 Aug 2017. PMLR.

[32] Jian Zhang, Christopher De Sa, Ioannis Mitliagkas, and Christopher Ré. Parallel SGD: When does averaging help? *arXiv*, 2016.

[33] Yuchen Zhang, John C. Duchi, and Martin J. Wainwright. Communication-efficient algorithms for statistical optimization. *Journal of Machine Learning Research*, 14:3321–3363, 2013.

[34] Peilin Zhao and Tong Zhang. Stochastic optimization with importance sampling for regularized loss minimization. In Francis Bach and David Blei, editors, *Proceedings of the 32nd International Conference on Machine Learning*, 8
[35] Martin Zinkevich, Markus Weimer, Lihong Li, and Alex J. Smola. Parallelized stochastic gradient descent. In J. D. Lafferty, C. K. I. Williams, J. Shawe-Taylor, R. S. Zemel, and A. Culotta, editors, Advances in Neural Information Processing Systems 23, pages 2595–2603. Curran Associates, Inc., 2010.
A Proofs

Proof of Lemma 3.1. Using the update equation (8) we have
\[ \|\bar{x}_{t+1} - x^*\|^2 = \|\bar{x}_t - x^* - \eta_t g_t\|^2 = \|\bar{x}_t - x^* - \eta_t g_t + \eta_t g_t\|^2 \]
\[ = \|\bar{x}_t - x^*\|^2 + \eta_t^2 \|g_t\|^2 + 2\eta_t \langle \bar{x}_t - x^*, g_t \rangle. \tag{14} \]

Observe that
\[ \|\bar{x}_t - x^*\|^2 = \|\bar{x}_t - x^*\|^2 + \eta_t^2 \|g_t\|^2 - 2\eta_t \langle \bar{x}_t - x^*, \nabla f(x_t^\gamma) \rangle \]
\[ = \|\bar{x}_t - x^*\|^2 + \eta_t^2 \frac{1}{K} \sum_{k=1}^{K} \|\nabla f(x_t^\gamma)\|^2 \]
\[ - 2\eta_t \frac{1}{K} \sum_{k=1}^{K} \langle \bar{x}_t - x_t^\gamma + x_k^\gamma - x^*, \nabla f(x_t^\gamma) \rangle \]
\[ = \|\bar{x}_t - x^*\|^2 + \eta_t^2 \frac{1}{K} \sum_{k=1}^{K} \|\nabla f(x_t^\gamma) - \nabla f(x^*)\|^2 \]
\[ - 2\eta_t \frac{1}{K} \sum_{k=1}^{K} \langle x_k^\gamma - x^*, \nabla f(x_t^\gamma) \rangle - 2\eta_t \frac{1}{K} \sum_{k=1}^{K} \langle \bar{x}_t - x_k^\gamma, \nabla f(x_t^\gamma) \rangle, \tag{15} \]

where we used the inequality \( \sum_{i=1}^{K} a_i \|^2 \leq K \sum_{i=1}^{K} \|a_i\|^2 \).

By L-smoothness,
\[ \|\nabla f(x_t^\gamma) - \nabla f(x^*)\|^2 \leq 2L \langle f(x_t^\gamma) - f^* \rangle, \tag{20} \]
and by \( \mu \)-strong convexity
\[ - \langle x_t^k - x^*, \nabla f(x_t^\gamma) \rangle \leq - (f(x_t^\gamma) - f^*) - \frac{\mu}{2} \|x_t^k - x^*\|^2. \tag{21} \]

To estimate the last term in (19) we use 2 \( \langle a, b \rangle \leq \gamma \|a\|^2 + \gamma^{-1} \|b\|^2 \), for \( \gamma > 0 \).
\[ -2 \langle \bar{x}_t - x_k^\gamma, \nabla f(x_t^\gamma) \rangle \leq 2L \|\bar{x}_t - x_k^\gamma\|^2 + \frac{1}{2L} \|\nabla f(x_t^\gamma)\|^2 \]
\[ = 2L \|\bar{x}_t - x_k^\gamma\|^2 + \frac{1}{2L} \|\nabla f(x_t^\gamma) - \nabla f(x^*)\|^2 \]
\[ \leq 2L \|\bar{x}_t - x_k^\gamma\|^2 + (f(x_t^\gamma) - f^*), \tag{22} \]
where we have again used (20) in the last inequality.

By applying these three estimates to (19) we get
\[ \|\bar{x}_t - x^* - \eta_t g_t\|^2 \leq \|\bar{x}_t - x^*\|^2 + 2\eta_t \frac{1}{K} \sum_{k=1}^{K} \|\bar{x}_t - x_k^\gamma\|^2 \]
\[ + 2\eta_t \frac{1}{K} \sum_{k=1}^{K} \left( \left( \eta_t L - \frac{1}{2} \right) (f(x_t^\gamma) - f^*) - \frac{\mu}{2} \|x_k^\gamma - x^*\|^2 \right). \tag{25} \]
For $\eta_t \leq \frac{1}{\mu}$ it holds $(\eta_t L - \frac{1}{4}) \leq -\frac{1}{4}$. By convexity of $a(f(x) - f^*) + b\|x - x^*\|^2$ for $a, b \geq 0$:

$$-\frac{1}{K} \sum_{k=1}^{K} \left( a(f(x^k) - f^*) + b\|x^k - x^*\|^2 \right) \leq - \left( a(f(\bar{x}_t) - f^*) + b\|\bar{x}_t - x^*\|^2 \right),$$

hence we can continue in (25) and obtain

$$\|\bar{x}_t - x^* - \eta_t \bar{g}_t\|^2 \leq (1 - \mu \eta_t) \|\bar{x}_t - x^*\|^2 - \frac{1}{2} \eta_t (f(\bar{x}_t) - f^*)$$

$$+ 2\eta_t L \frac{1}{K} \sum_{k=1}^{K} \|\bar{x}_t - x^k\|^2 .$$

(27)

Finally, we can plug (27) back into (15). By taking expectation we get

$$E \|\bar{x}_{t+1} - x^*\|^2 \leq (1 - \mu \eta_t) E \|\bar{x}_t - x^*\|^2 + \eta_t^2 E \|\bar{g}_t - \bar{g}_t\|^2$$

$$- \frac{1}{2} \eta_t E(f(\bar{x}_t) - f^*) + 2\eta_t L \frac{1}{K} \sum_{k=1}^{K} E \|\bar{x}_t - x^k\|^2 .$$

□

**Proof of Lemma 3.2.** By definition of $\bar{g}_t$ and $\bar{g}_t$ we have

$$E \|\bar{g}_t - \bar{g}_t\|^2 = E \left[ \left( \frac{1}{K} \sum_{k=1}^{K} (\nabla f_{i(t)}(x_k^t) - \nabla f(x_k^t)) \right)^2 \right] = \frac{1}{K^2} \sum_{k=1}^{K} E \left( \nabla f_{i(t)}(x_k^t) - \nabla f(x_k^t) \right)^2 \leq \frac{g^2}{K},$$

(28)

where we used $\text{Var}(\sum_{k=1}^{K} X_k) = \sum_{k=1}^{K} \text{Var}(X_k)$ for independent random variables. □

**Proof of Lemma 3.3.** As the gap $(\mathcal{I}_T) \leq H$, there is an index $t_0$, $t - t_0 \leq H$ such that $\bar{x}_{t_0} = x_k^k$ for $k \in [K]$. Observe, using $E\|X - E X\|^2 = E\|X\|^2 - \|E X\|^2$ and $\|\sum_{i=1}^{H} a_i\|^2 \leq H \sum_{i=1}^{H} \|a_i\|^2$,

$$\frac{1}{K} \sum_{k=1}^{K} E \|\bar{x}_t - x^k\|^2 = \frac{1}{K} \sum_{k=1}^{K} E \|x^k_t - \bar{x}_t - (\bar{x}_t - x^k_{t_0})\|^2$$

$$= \frac{1}{K} \sum_{k=1}^{K} E \|x^k_t - x^k_{t_0}\|^2$$

$$\leq \frac{1}{K} \sum_{k=1}^{K} H \eta_{t_0}^2 \sum_{h=t_{k_0}}^{t-1} E \|\nabla f_{i(h)}(x_k^h)\|^2$$

$$\leq \frac{1}{K} \sum_{k=1}^{K} H^2 \eta_{t_0}^2 G^2 ,$$

(29)

(30)

(31)

(32)

where we used $\eta_t \leq \eta_{t_0}$ for $t \geq t_0$ and the assumption $E\|\nabla f_{i(h)}(x_k^h)\|^2 \leq G^2$. Finally, the claim follows by $\frac{\eta_{t_0}}{\eta_t} \leq \frac{a + H}{a} \leq 2$. □

**Proof of Lemma 3.4.** Observe

$$(1 - \mu \eta_t) \frac{w_t}{\eta_t} = \left( \frac{a + t - 4}{a + t} \right) \frac{\mu(a + t)^3}{4} = \frac{\mu(a + t - 4)(a + t)^2}{4} \leq \frac{\mu(a + t - 1)^3}{4} = \frac{w_{t-1}}{\eta_{t-1}},$$

(33)

where the inequality is due to

$$(a + t - 4)(a + t)^2 = (a + t - 1)^3 + 1 - 3a - a^2 - 3t - 2at - t^2 \leq (a + t - 1)^3,$$

(34)

$\leq 0$.
for $a \geq 1, t \geq 0$.

We now multiply equation (12) with $\frac{w_t}{\eta_t}$, which yields

$$a_{t+1} \frac{w_t}{\eta_t} \leq a_t \left(1 - \mu \eta_t\right) \frac{w_t}{\eta_t} - w_t \eta_t A + w_t \eta_t B + w_t \eta_t^2 C.$$ (35)

and by recursively substituting $a_t \frac{w_t}{\eta_{t-1}}$ we get

$$a_T \frac{w_{T-1}}{\eta_{T-1}} \leq \left(1 - \mu \eta_0\right) \frac{w_0}{\eta_0} a_0 - \sum_{t=0}^{T-1} w_t \eta_t A + \sum_{t=0}^{T-1} w_t \eta_t B + \sum_{t=0}^{T-1} w_t \eta_t^2 C,$$ (36)

i.e.

$$A \sum_{t=0}^{T-1} w_t \eta_t e_t \leq \frac{w_0}{\eta_0} a_0 + B \sum_{t=0}^{T-1} w_t \eta_t + C \sum_{t=0}^{T-1} w_t \eta_t^2 .$$ (37)

We will now derive upper bounds for the terms on the right hand side. We have $\frac{w_0}{\eta_0} = \frac{\mu a^3}{4}$,

$$\sum_{t=0}^{T-1} w_t \eta_t = \sum_{t=0}^{T-1} \frac{4(a + t)}{\mu} = \frac{2T^2 + 4aT - 2T}{\mu} \leq \frac{2(T + 2a)}{\mu},$$ (38)

and

$$\sum_{t=0}^{T-1} w_t \eta_t^2 = \sum_{t=0}^{T-1} \frac{16}{\mu^2} = \frac{16T}{\mu^2}.$$ (39)

Let $S_T := \sum_{t=0}^{T-1} \frac{w_t}{\nu} \left(2T^2 + 6aT - 3T + 6a^2 - 6a + 1\right)$. Observe

$$S_T \geq \frac{1}{3} \left[ T^3 + aT^2 - \frac{1}{2} T^2 + a^2 T - aT \right] = \frac{1}{3} T^3 .$$ (40)

for $a \geq 1, T \geq 0$. \qed