DYNAMICAL PROPERTIES OF A LESLIE-GOWER PREY-PREDATOR MODEL WITH STRONG ALLEE EFFECT IN PREY

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ABSTRACT. This paper is devoted to study the dynamical properties of a Leslie-Gower prey-predator system with strong Allee effect in prey. We first give some estimates, and then study the dynamical properties of solutions. In particular, we mainly investigate the unstable and stable manifolds of the positive equilibrium when the system has only one positive equilibrium.

1. Introduction. In recent years, population models appearing in various fields of mathematical biology have been proposed and studied extensively due to their universal existence and importance [2]. A typical one is the prey-predator model, and such a type model has played the major role in the studies of biological invasion of foreign species, epidemics spreading, extinction/spread of flame balls in combustion or autocatalytic chemical reaction. Many researchers have been interested in the prey-predator models together with several functional responses.

Among the widely used mathematical models in theoretical ecological, the Leslie-Gower prey-predator model plays a special role in view of the interesting dynamics it possesses. The classical Leslie-Gower prey-predator model takes the form [7]

\[
\begin{align*}
    u' &= u(1-u) - \beta uv, \quad t > 0, \\
    v' &= \mu v(1-v/u), \quad t > 0.
\end{align*}
\]

where \(u\) and \(v\) represent the densities of prey and predator, respectively, parameters \(\mu\) and \(\beta\) are positive constants. Several ecologists regard (1) as a prototypical prey-predator system [8,9]. It is known that system (1) has a globally asymptotically stable equilibrium [5].

In most works for prey-predator models, the prey is assumed to grow at a logistic pattern. But in recent years it was recognized that the prey species may have a growth rate of Allee effect, as a result of mate limitation, cooperative defense, cooperative feeding, and environmental conditioning [6,10]. The Allee effect named after W.C. Allee [11], has significant contribution to population dynamics. Allee effect mainly classified into two ways: strong and weak Allee effect [4,12,13]. There is a threshold population level for the strong Allee effect such that the species will
become extinct below this threshold population density. However, when the growth rate decreases but remains positive at low population density, it is called the weak Allee effect.

In consideration of the strong Allee effect, in this paper we study the dynamical properties of the following Leslie-Gower prey-predator model with the strong Allee effect in the prey

\[
\begin{align*}
    u' &= u(1 - u)(u/b - 1) - \beta uv, \quad t > 0, \\
    v' &= \mu v(1 - v/u), \quad t > 0, \\
    u(0) &= u_0 > 0, v(0) = v_0 \geq 0,
\end{align*}
\]

(2)

The Allee threshold is \( b \in (0, 1) \): a strong Allee effect introduces a population threshold, and the population must surpass this threshold to grow ([3, 11, 13]).

Because of the reaction term \([u(1 - u)(u/b - 1) - \beta uv](t) < 0\) for \( 0 < u(t) < b \) and \( v(t) > 0 \), the component \( u(t) \) may tend to zero and thus the reaction term \( v(t)/u(t) \) may be unbounded. Such a bad structure will bring a lot of difficulties in the analysis.

The main purpose of this paper is to study the dynamical properties of the problem (2). Specifically, if (1.2) has no positive equilibrium, the positive solution converges to \((0, 0)\). That is, both the predator and prey will extinct under this situation. Then, if (1.2) has two positive equilibria, we obtain that the two species may persist when the intrinsic growth rate of predator is large. Later in this paper, we focus on the study of the unstable and stable manifolds of the positive equilibrium when the system has only one positive equilibrium. And in such case we find a triangular attraction basin of this equilibrium when the intrinsic growth rate of predator is large. The paper is organized as follows. In section 2, some estimates of global solutions are given. In section 3, we study the dynamical properties of solutions.

2. Existence and estimates of global solution to (2). In this section, we give some estimates for global solutions and verify the population threshold.

**Theorem 2.1.** (i) The problem (2) has a unique global solution \((u(t), v(t))\) satisfying \(u(t) > 0\) and \(v(t) \geq 0\) for \( t \geq 0 \), and

\[
\limsup_{t \to \infty} u(t) \leq 1, \quad \limsup_{t \to \infty} v(t) \leq 1,
\]

\[
\sup_{t > 0} \frac{v(t)}{u(t)} \geq 1;
\]

(3)

(ii) If \( u_0 \leq b \) and \((u_0, v_0) \neq (b, 0)\), then \( \lim_{t \to \infty} (u(t), v(t)) = (0, 0) \).

**Proof.** (i) Clearly, \( u \) satisfies

\[
\begin{align*}
    u' &= u(1 - u)(u/b - 1), \\
    u(0) &= u_0 > 0.
\end{align*}
\]

(4)

Hence, \( \limsup_{t \to \infty} u(t) \leq 1 \). And for any \( \varepsilon > 0 \), there is \( T > 0 \) such that \( u(t) < 1 + \varepsilon \) for all \( t \geq T \). From the second equation of (2), we have
\[ v' < \mu v \left( 1 - \frac{v}{1 + \varepsilon} \right), \quad t > T. \]  
Obviously, \( \limsup_{t \to \infty} v(t) \leq 1 + \varepsilon. \) The arbitrariness of \( \varepsilon \) leads to \( \limsup_{t \to \infty} v(t) \leq 1. \)

If (3) does not hold, then there exists a constant \( 0 < \alpha < 1 \) such that \( \frac{v(t)}{u(t)} < \alpha \) for all \( t > 0. \) Therefore, by the second equation of (2),
\[ v' = \mu v(1 - v/u) \geq \mu v(1 - \alpha), \quad \forall \ t > 0, \]
which implies \( \lim_{t \to \infty} v(t) = \infty. \) This is a contradiction to \( \limsup_{t \to \infty} v(t) \leq 1. \) Hence, (3) holds.

(ii) The proof is divided into two cases.

\textbf{Case 1.} \( u_0 < b \) and \( v_0 \geq 0. \) By (4), we obtain that \( u(t) \) converges to 0 as \( t \to \infty. \)
For any \( \varepsilon > 0, \) there is \( T > 0, \) such that \( u(t) < \varepsilon \) for all \( t \geq T. \) Replacing \( 1 + \varepsilon \) with \( \varepsilon \) in (5), similarly to the above,
\[ \limsup_{t \to \infty} v(t) \leq \varepsilon. \]
The arbitrariness of \( \varepsilon \) implies \( \lim_{t \to \infty} v(t) = 0. \)

\textbf{Case 2.} \( u_0 = b \) and \( v_0 > 0. \) From the first equation of (2), we get \( u'(0) = -\beta u_0 v_0 < 0. \)
Hence \( u(t_0) < b \) and \( v(t_0) \geq 0 \) for some small \( t_0 > 0. \) Using the result of Case 1, we conclude that
\[ \lim_{t \to \infty} u(t) = \lim_{t \to \infty} v(t) = 0. \]
The proof is completed.

\hfill \Box

3. **Dynamical properties of (2).** Obviously, \( (b, 0) \) and \( (1, 0) \) are nonnegative equilibria of (2). On the other hand, the positive equilibria of (2) has the form \( (\bar{u}, \bar{v}), \) where \( u \) satisfies
\[ \bar{u}^2 + (\beta b - 1 - b)\bar{u} + b = 0. \]

The following results concerning with positive equilibria are obvious:

(i) If \( \beta b - 1 - b > -2\sqrt{b}, \) then (2) has no positive equilibrium;

(ii) If \( \beta b - 1 - b < -2\sqrt{b}, \) then (2) has two positive equilibria: \( \hat{u}_1 = (\tilde{u}_1, \tilde{u}_1), \)
\[ \hat{u}_2 = (\tilde{u}_2, \tilde{u}_2) \]
with
\[ \tilde{u}_1 = \frac{1}{2} \left( 1 + b - \beta b - \sqrt{(1 + b - \beta b)^2 - 4b} \right), \]
\[ \tilde{u}_2 = \frac{1}{2} \left( 1 + b - \beta b + \sqrt{(1 + b - \beta b)^2 - 4b} \right). \]

(iii) If \( \beta b - 1 - b = -2\sqrt{b}, \) then (2) has a unique positive equilibrium \( \bar{u}_3 = (\tilde{u}_3, \tilde{u}_3) \)
with \( \tilde{u}_3 = \sqrt{b}. \)

For the simplicity of the notations, we denote \( u = (u, v) \) and
\[ G(u) = \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix} = \begin{pmatrix} u(1 - u)(u/b - 1) - \beta uv \\ \mu v(1 - v/u) \end{pmatrix}. \]

By direct computation, the linearization of \( G(u) \) at \( \bar{u} = (\tilde{u}, \tilde{v}) \) is
\[ G_u(\bar{u}) = \begin{pmatrix} -3\tilde{u}^2/b + 2(1 + 1/b)\tilde{u} - 1 - \beta \tilde{v} & -\beta \tilde{u} \\ \mu \tilde{v}^2/\tilde{u}^2 & \mu - 2\mu \tilde{v}/\tilde{u} \end{pmatrix}. \]
Denote
\[ A_i = (1 + 1/b)u_i - 2u_i^2/b, \tag{6} \]
than we get
\[ G_u(\bar{u}) = \begin{pmatrix} A_i & -\beta u_i \\ \mu & -\mu \end{pmatrix}, \tag{7} \]
and
\[ A_1 > \beta \bar{u}_1, \quad A_2 < \beta \bar{u}_2, \quad A_3 = \beta \bar{u}_3. \]

Let \((u(t), v(t))\) be the unique solution of (2) and define
\[ s(t) = \frac{v(t)}{u(t)}, \quad t \geq 0. \]

For the positive constant \(\lambda\), we denote
\[ \mathcal{R}_\lambda := \{ (u, v) : v \geq \lambda u > 0 \}. \]

We call that \(\mathcal{R}_\lambda\) is an invariant region of (2) if \((u_0, v_0) \in \mathcal{R}_\lambda\) implies \((u(t), v(t)) \in \mathcal{R}_\lambda\)
for all \(t \geq 0\).

### 3.1. The case: \(\beta b - 1 - b > -2\sqrt{b}\).

**Theorem 3.1.** Assume \(\beta b - 1 - b > -2\sqrt{b}\), and denote \(\lambda_0 = (1 - \sqrt{b})^2/(\beta b) < 1\).

(i) If \(\lambda_0 \leq \lambda \leq 1\), then the set \(\mathcal{R}_\lambda\) is an invariant region of (2), and \(s(t)\) is strictly increasing provided \(\lambda_0 \leq s(t) < 1\);

(ii) \(\lim_{t \to \infty} (u(t), v(t)) = (0, 0)\).

**Proof.** (i) Let \((u(t), v(t))\) be the unique solution of (2) with \((u_0, v_0) \in \mathcal{R}_\lambda\). Set
\[ w(t) = v(t) - \lambda u(t) \quad \text{and} \quad h(w, t) = g(u(t), w + \lambda u(t)) - \lambda f(u(t), w + \lambda u(t)). \]

Then \(w\) satisfies
\[ w' = h, \quad t > 0; \quad w(0) = v_0 - \lambda u_0, \]
and
\[ h(0, t) = g(u, \lambda u) - \lambda f(u, \lambda u) \]
\[ = \mu \lambda u(1 - \lambda) - \lambda u(1 - u)(u/b - 1) + \beta \lambda^2 u^2 \]
\[ = \lambda u[\mu(1 - \lambda) + \beta u - (1 - u)(u/b - 1)] \]
\[ = \lambda u[(\beta u - \mu)\lambda + \mu - (1 - u)(u/b - 1)]. \]

Set
\[ k(\lambda, u) = (\beta u - \mu)\lambda + \mu - (1 - u)(u/b - 1). \]

Notice \(\lambda_0 = (1 - \sqrt{b})^2/(\beta b), \beta b - 1 - b > -2\sqrt{b}\) and \(u > 0\), we have
\[ k(\lambda_0, u) = (1 - \lambda_0)\mu + \lambda_0 \beta u - (1 - u)(u/b - 1) > 0, \]
\[ k(1, u) = \beta u - (1 - u)(u/b - 1) > 0. \]

It follows that \(k(\lambda, u) > 0\) for all \(\lambda_0 \leq \lambda \leq 1\) and \(u > 0\), which leads to
\[ h(0, t) > 0, \quad \forall \ t \geq 0. \]

This means that \(w(t) = v(t) - \lambda u(t)\) won’t reach 0 for \(t \geq 0\) except \(w(0) = v_0 - \lambda u_0 = 0\). Hence \(\mathcal{R}_\lambda\) is an invariant region for each \(\lambda_0 \leq \lambda \leq 1\) and
\[ w(t) > 0, \quad \text{i.e.,} \quad v(t) > \lambda u(t), \quad \forall \ t > 0. \]
This formula imply that if \( \lambda_0 < s(0) \leq 1 \), then
\[
v(t) > s(0)u(t), \quad \forall \ t > 0,
\]
i.e., \( s(t) > s(0) \) for all \( t > 0 \). Similarly, we can show that if \( \lambda_0 < s(t_1) \leq 1 \) for some \( t_1 > 0 \), then \( s(t) > s(t_1) \) for all \( t > t_1 \). Therefore, \( s(t) \) is strictly increasing provided \( \lambda_0 < s(t) < 1 \).

(ii) In view of \( \lambda_0 < 1 \) and \([3]\), there exist \( T > 0 \) and \( \lambda_0 < \lambda < 1 \) such that
\[
\frac{v(T)}{u(T)} > \lambda.
\]
Applying the conclusion (i) we can show that \( v(t) \geq \lambda u(t) \) for all \( t > T \). Consequently, \( u \) satisfies
\[
u' = u(1-u)(u/b-1) - \beta uv
\leq u[(1-u)(u/b-1) - \lambda \beta u] \leq ku, \quad \forall \ t > T,
\]
where \( k = \max_{y\geq 0}[(1-y)(y/b-1) - \lambda \beta y] < 0 \). Therefore,
\[
\lim_{t\to\infty} u(t) = 0.
\]
Similarly to the proof of Theorem 2.1(ii), we have \( \lim_{t\to\infty} v(t) = 0 \). The proof is finished.

3.2. The case: \( \beta b - 1 - b < -2\sqrt{b} \).

**Theorem 3.2.** Assume \( \beta b - 1 - b < -2\sqrt{b} \).

(i) If \( u_0 \leq \min\{v_0, \tilde{u}_1\} \) and \( (u_0, v_0) \neq (\tilde{u}_1, \tilde{u}_1) \), then \( \lim_{t\to\infty} (u(t), v(t)) = (0,0) \).

(ii) \( (\tilde{u}_2, \tilde{u}_2) \) is asymptotically locally stable for \( \mu > A_2 \), while it is unstable for \( \mu < A_2 \), where \( A_2 \) is given by \([6]\).

**Proof.** (i) The proof is divided into two cases.

**Case 1.** \( u_0 \leq v_0 \) and \( u_0 < \tilde{u}_1 \). Set
\[
\mathcal{A} := \{(u, v) : 0 < u \leq v, \ u < \tilde{u}_1 - \varepsilon\},
\]
where \( \varepsilon > 0 \) is a small constant. First, we will prove that \( \mathcal{A} \) is an invariant region. Let
\[
w(t) = v(t) - u(t), \quad h(w, t) = g(u(t), w + u(t)) - f(u(t), w + u(t)).
\]
Then \( w \) satisfies
\[
w'(t) = h(w, t), \quad t > 0; \quad w(0) = v_0 - u_0.
\]
By directly computation, we get
\[
\begin{aligned}
h(0, t) &= g(u, u) - \lambda f(u, u) = -f(u, u)
\leq u[\beta u - (1-u)(u/b-1)] > 0, & \text{ when } u(t) \leq \tilde{u}_1 - \varepsilon,
\end{aligned}
\]
and
\[
u' \leq u[(1-u)(u/b-1) - \beta u] < 0, \quad \text{ when } v(t) - u(t) \geq 0, \ u(t) \leq \tilde{u}_1 - \varepsilon.
\]
This two inequality show that \( (u(t), v(t)) \) won’t arrive at the boundary of \( \mathcal{A} \) except \( (u_0, v_0) \in \partial \mathcal{A} \). Therefore \( \mathcal{A} \) is an invariant region. Notice \( (u_0, v_0) \in \mathcal{A} \), we have
\[
u' \leq u[(1-u)(u/b-1) - \beta u] < Cu, \quad \forall \ t \geq 0,
\]
where \( C = (1-u_0)(u_0/b - 1) - \beta u_0 < 0 \). This leads to \( \lim_{t \to \infty} u(t) = 0 \). Similarly, we can conclude that \( \lim_{t \to \infty} v(t) = 0 \).

**Case 2.** \( \tilde{u}_1 = u_0 < v_0 \). Because of

\[
\begin{align*}
u'(0) &= u_0[(1-u_0)(u_0/b - 1) - \beta u_0] + \beta v_0(u_0 - v_0) \\
&= \beta u_0(u_0 - v_0) < 0,
\end{align*}
\]

there is a \( t_0 > 0 \) such that

\[
u(t_0) \leq v(t_0) \text{ and } u(t_0) < \tilde{u}_1.
\]

By the conclusion of Case 1, we have \( \lim_{t \to \infty} (u(t), v(t)) = (0, 0) \).

(ii) Let \( G_u(\tilde{u}_2) \) be defined by (7). We have

\[
|\lambda - G_u(\tilde{u}_2)| = \lambda^2 + (\mu - A_2)\lambda - \mu A_2 + \mu \beta \tilde{u}_2.
\]

Since \( A_2 < \beta \tilde{u}_2 \), the real part of eigenvalues of \( G_u(\tilde{u}_2) \) are negative if \( \mu > A_2 \), which means that \( (\tilde{u}_2, \tilde{u}_2) \) is locally stable. The real part of eigenvalues of \( G_u(\tilde{u}_2) \) are positive if \( \mu < A_2 \), which implies that \( (\tilde{u}_2, \tilde{u}_2) \) is unstable.

**3.3. The case:** \( \beta b - 1 - b = -2\sqrt{b} \).

**Theorem 3.3.** Assume \( \beta b - 1 - b = -2\sqrt{b} \).

(i) The set \( R_1 \) is an invariant region of (2);

(ii) \( (u(t), v(t)) \) converges to \((0, 0)\) or \((\tilde{u}_3, \tilde{u}_3)\) as \( t \to \infty \);

(iii) If \( u_0 \leq \min\{v_0, \tilde{u}_3\} \) and \( u_0, v_0 \neq (\tilde{u}_3, \tilde{u}_3) \), then \( \lim_{t \to \infty} (u(t), v(t)) = (0, 0) \).

**Proof.** (i) We also use the notations in the proof of Theorem 3.1. Let \( (u, v) \) be the unique solution of (2). According to \( \beta b - 1 - b = -2\sqrt{b} \), we have \( k(1, u) \geq 0 \). Similar to the proof of Theorem 3.1, we can show that \( R_1 \) is an invariant region of (2).

(ii) In view of (9), we obtain that either

\[
v(T_0) \geq u(T_0) \text{ for some } T_0 \geq 0,
\]

or

\[
\lim_{t \to \infty} \frac{v(t)}{u(t)} = 1, \quad \frac{v(t)}{u(t)} < 1, \quad \forall \ t > 0.
\]

**Step 1.** We will prove that \((u(t), v(t))\) converges to \((0, 0)\) or \((\tilde{u}_3, \tilde{u}_3)\) if (8) holds. In fact, by use of the conclusion (i) we have that

\[
v(t) \geq u(t), \quad \forall \ t > T_0.
\]

It follows from (2) that, for all \( t > T_0 \),

\[
u' = u(1-u)(u/b - 1) - \beta uv \leq u[(1-u)(u/b - 1) - \beta u] \leq 0,
\]

and

\[
v' = \mu v(1 - v/u) \leq 0.
\]

Hence, \( u(t) \) and \( v(t) \) are decreasing for \( t > T_0 \). Since, in the case \( \beta b - 1 - b = -2\sqrt{b} \), the problem (2) only has three nonnegative equilibriums \((b, 0)\), \((1, 0)\) and \((\tilde{u}_3, \tilde{u}_3)\), applying (10) and Theorem (2.1) (ii) we have that

\[
\lim_{t \to \infty} (u(t), v(t)) = (0, 0), \quad \text{or} \quad \lim_{t \to \infty} (u(t), v(t)) = (\tilde{u}_3, \tilde{u}_3).
\]
Step 2. We shall show that \((u, v)\) converges to \((\tilde{u}_3, \tilde{u}_3)\) if (9) holds. By the second equation of (2), \(v(t)\) satisfies
\[
v' = \mu v(1 - v/u) > 0, \quad \forall \; t > 0.
\]
This together with \(\lim_{t \to \infty} v(t) \leq 1\) yields that \(\lim_{t \to \infty} v(t) = \overline{v}\) exists and \(0 < \overline{v} \leq 1\).

Using the first conclusion of (9) we deduce that \(\lim_{t \to \infty} u(t) = \overline{v}\). Because \((\tilde{u}_3, \tilde{u}_3)\) is the unique positive equilibrium of (2), it is derived that \(\overline{v} = \tilde{u}_3\), and so \((u(t), v(t))\) converges to \((\tilde{u}_3, \tilde{u}_3)\).

(iii) As \(u_0 \leq v_0\), the conclusion (i) asserts \(v(t) \geq u(t)\) for \(t \geq 0\). This implies
\[
u' \leq u[(1 - u)(u/b - 1) - \beta u] \leq 0, \quad \forall \; t \geq 0.
\]

The following proof will be divided into two cases.

Case 1. \(u_0 \leq v_0\) and \(u_0 < \tilde{u}_3\). In this situation, it is easy to see that
\[
C := (1 - u_0)(u_0/b - 1) - \beta u_0 < 0.
\]

According to (11), we have that \(u \leq u_0\) and \(u' \leq Cu\) for all \(t \geq 0\). This leads to \(\lim_{t \to \infty} u(t) = 0\). Similarly to Theorem 2.1 we can get \(\lim_{t \to \infty} v(t) = 0\).

Case 2. \(\tilde{u}_3 = u_0 < v_0\). In such case we have
\[
u'(0) = \tilde{u}_3[(1 - \tilde{u}_3)(\tilde{u}_3/b - 1) - \beta \tilde{u}_3] + \beta \tilde{u}_3(\tilde{u}_3 - v_0) = \beta \tilde{u}_3(\tilde{u}_3 - v_0) < 0.
\]

Therefore, there exists \(0 < t_0 \ll 1\) such that \(u(t_0) < v(t_0)\) and \(u(t_0) < \tilde{u}_3\). The same as Case 1, \(\lim_{t \to \infty} u(t) = \lim_{t \to \infty} v(t) = 0\). The proof is complete. 

Theorem 3.3 ii) asserts that \((u(t), v(t))\) converges to \((0,0)\) or \((\tilde{u}_3, \tilde{u}_3)\) as \(t \to \infty\), and Theorem 3.3 iii) gives a sufficient condition to guarantee \(\lim_{t \to \infty} (u(t), v(t)) = (0,0)\). In the following we will show that if \(\mu > A_3\), then there exists a triangular attraction basin of \((\tilde{u}_3, \tilde{u}_3)\).

In the proof of Theorem 3.3 ii) we see that \((u(t), v(t))\) converges to \((\tilde{u}_3, \tilde{u}_3)\) if (9) holds. In the following we just need to consider the situation that (8) holds.

Set
\[
\Gamma_{\lambda} = \{(u, v) : v - \lambda u \geq (1 - \lambda)\sqrt{b}, \; v \geq u \geq 0\}.
\]

Theorem 3.4. Assume \(\beta b - 1 - b = -2\sqrt{b}\).

(i) If \(\mu \leq \beta \sqrt{b}\), then \(\Gamma_{\lambda}\) is an invariant region when \(\lambda > 1\);

(ii) If \(\mu \leq \beta \sqrt{b}/2\), then \(\lim_{t \to \infty} (u(t), v(t)) = (0,0)\) when \(v_0 \geq u_0\) and \((u_0, v_0) \neq (\tilde{u}_3, \tilde{u}_3)\);

(iii) If \(\mu > \beta \sqrt{b}\), then there exists a triangular region which is a attraction basin of \((\tilde{u}_3, \tilde{u}_3)\).

Proof. We define
\[
w(t) = v(t) - \lambda u(t), \quad h(w, t; \lambda) = g(u(t), w + \lambda u(t)) - \lambda f(u(t), w + \lambda u(t)).
\]
Then \(w\) satisfies
\[
w' = h(w, t; \lambda), \quad \forall \; t > 0; \quad w(0) = v(0) - \lambda u(0).
\]

(12)
By directly computation, we get
\[ h(w, t; \lambda) = \mu(w + \lambda u)(1 - \frac{w + \lambda u}{u}) - \lambda u(1 - u)(\frac{u}{b} - 1) + \lambda \beta u(w + \lambda u) \]
\[ = \lambda u((1 - \lambda)\mu + \lambda \beta u - (1 - u)(\frac{u}{b} - 1) + \beta w) + \mu w(1 - \frac{w}{u} - 2\lambda). \]

Let \( P_\lambda(u) = h(w_0, t; \lambda), \) where \( w_0 = (1 - \lambda)\sqrt{b}, \) then we have
\[ P_\lambda(u) = \lambda u \left[ (1 - \lambda)(\mu + \beta\sqrt{b}) + \lambda \beta u - (1 - u)(\frac{u}{b} - 1) \right] \]
\[ + \mu(1 - \lambda)\sqrt{b} \left( 1 - \frac{(1 - \lambda)\sqrt{b}}{u} - 2\lambda \right). \]
\[ P''_\lambda(u) = \lambda(1 - \lambda)(\mu + \beta\sqrt{b}) + 2\lambda^2 \beta u + \lambda \left( \frac{3u^2}{b} - 2(1 + \frac{1}{b})u + 1 \right) + \frac{\mu(1 - \lambda)\sqrt{b}}{u^2}, \]
and
\[ P''_\lambda(u) = 2\lambda^2 \beta + \lambda \left( \frac{6}{b} - 2(1 + \frac{1}{b}) \right) - \frac{2\mu(1 - \lambda)^2 b}{u^2}. \] (13)

It is easy to see that \( P_\lambda(\sqrt{b}) = 0, \) and
\[ P''_\lambda(\sqrt{b}) = (\lambda - 1)(\lambda \beta \sqrt{b} - \mu). \] (14)

(i) Let \( (u(0), v(0)) \in \Gamma_\lambda, \) we will prove that \( (u(t), v(t)) \in \Gamma_\lambda \) for all \( t \geq 0. \) Thanks to Theorem [3.3(i)], it follows that
\[ v(t) \geq u(t), \quad \forall t \geq 0. \] (15)

Clearly, when \( (u(0), v(0)) = (\tilde{u}_3, \tilde{v}_3), \) we have \( (u(t), v(t)) = (\tilde{u}_3, \tilde{v}_3), \) and so \( (u(t), v(t)) \in \Gamma_\lambda. \)

For the case \( (u(0), v(0)) \neq (\tilde{u}_3, \tilde{v}_3), \) the uniqueness yields
\[ (u(t), v(t)) \neq (\tilde{u}_3, \tilde{v}_3), \quad \forall t \geq 0. \] (16)

Moreover, making use of (15) and (16) we get that, for any \( t \geq 0, \)
\[ w(t) = v(t) - \lambda u(t) = w_0, \quad \implies \quad u(t) > \sqrt{b}. \] (17)

As \( \mu \leq \beta \sqrt{b}, \) it then follows, upon using (13) and (14), that
\[ P''_\lambda(\sqrt{b}) > 0, \] (18)
\[ P''_\lambda(u) = 2\lambda^2 \beta + \lambda \left[ \frac{6}{b} \sqrt{b} - 2(1 + \frac{1}{b}) \right] - \frac{2\mu(1 - \lambda)^2 \sqrt{b}}{b} \]
\[ = \frac{2\lambda^2 (1 - \sqrt{b})^2 + 6\lambda \sqrt{b} - 2\lambda(1 + b) - 2\mu(1 - \lambda)^2 \sqrt{b}}{b} \]
\[ \geq \frac{2\lambda^2 (1 - \sqrt{b})^2 + 6\lambda \sqrt{b} - 2\lambda(1 + b) - 2(1 - \lambda)^2 (1 - \sqrt{b})^2}{b} \]
\[ = \frac{2\lambda \sqrt{b} + 2(\lambda - 1)(1 - \sqrt{b})^2}{b} > 0. \] (19)

From (13), we know that \( P''_\lambda(u) \) is increasing when \( u \geq \sqrt{b}. \) This together with (18) and (19) imply that
\[ P_\lambda(u) > 0, \quad \forall u > \sqrt{b}. \]
Applying this inequality, \(P_\lambda(u) = h(w_0, t; \lambda)\) and (17), we arrive at, for any \(t \geq 0\),
\[ w(t) = w_0, \quad \Longrightarrow \quad h(w_0, t; \lambda) > 0. \] (20)
Hence \(w(t)\) won’t arrive at \(w_0\) for all \(t \geq 0\) except \(w(0) = w_0\). Therefore \(\Gamma_\lambda\) is an invariant region.

(ii) From Theorem [3.3][iii], we know that \((u(t), v(t))\) converges to \((0,0)\) as \(t \to \infty\) if \(u_0 \leq \min\{v_0, \tilde{u}_3\}\) and \((u_0, v_0) \neq (\sqrt{b}, \sqrt{b})\). Consequently, in order to prove the desired conclusion we just need to show that \(\lim_{t \to \infty} (u(t), v(t)) = (0,0)\) when \(v_0 \geq u_0 > \sqrt{b} = \tilde{u}_3\). In the following, the proof is divided into two cases.

Case 1. \(v_0 > u_0 > \sqrt{b}\). Set
\[ \lambda = \frac{v_0 - \sqrt{b}}{u_0 - \sqrt{b}} > 1, \quad w(t) = v(t) - \lambda u(t). \]
Then \((u_0, v_0) \in \Gamma_\lambda\) and \((u_0, v_0) \neq (\tilde{u}_3, \tilde{u}_3)\). That is,
\[ v(t) - \lambda u(t) \geq (1 - \lambda)\sqrt{b} \quad \text{and} \quad v(t) \geq u(t), \quad \forall \ t \geq 0. \] (21)
Suppose on the contrary that
\[ \lim_{t \to \infty} (u(t), v(t)) = (\tilde{u}_3, \tilde{u}_3). \] (22)
We claim that
\[ u(t) > \sqrt{b}, \quad \forall \ t \geq 0. \] (23)
If this is not true, then \(u(t_0) \leq \sqrt{b} = \tilde{u}_3\) and \((u(t_0), v(t_0)) \neq (\tilde{u}_3, \tilde{u}_3)\) for some \(t_0 \geq 0\).
Note that \(v(t_0) \geq u(t_0)\). Using Theorem [3.3][iii] we see that \(\lim_{t \to \infty} (u(t), v(t)) = (0,0)\), which is a contradiction to (22). Thus, (23) holds.

Combining (21) and (23) we derive
\[ v(t) > u(t) > \sqrt{b}, \quad \forall \ t \geq 0. \] (24)
Make use of the following Lemma [3.5] we have that there exists a \(T \geq 0\) such that
\[ v(t) - \frac{v(T) - \sqrt{b}}{u(T) - \sqrt{b}} u(t) < 0, \quad \forall \ t \geq T. \] (25)
Define \(\tilde{\lambda} = \frac{v(T) - \sqrt{b}}{u(T) - \sqrt{b}}\) and
\[ \tilde{w}(t) = v(t) - \tilde{\lambda} u(t), \quad \tilde{w}_0 = (1 - \tilde{\lambda})\sqrt{b}. \]
It follows from (24) and (25) that
\[ \tilde{\lambda} > 1, \quad \tilde{w}(T) = \tilde{w}_0, \] (26)
\[ \tilde{w}(t) < 0, \quad \forall \ t \geq T. \] (27)
Making use of the conclusion (i), one can conclude that \(\Gamma(\tilde{\lambda})\) is an invariant region of problem (2), and
\[ \tilde{w}(t) = v(t) - \tilde{\lambda} u(t) \geq (1 - \tilde{\lambda})\sqrt{b} = \tilde{w}_0, \quad \forall \ t \geq T. \] (28)
Notice (20) and (23), we get that, for any \(t \geq T\),
\[ h(\tilde{w}_0, t; \tilde{\lambda}) > 0 \quad \text{if} \quad \tilde{w}(t) = \tilde{w}_0. \] (29)
Thanks to $\mu \leq \frac{1}{2} \beta \sqrt{b}$ and (27), it follows from the definition of $h(\tilde{w}, t; \tilde{\lambda})$ that

$$\frac{\partial h(\tilde{w}, t; \tilde{\lambda})}{\partial \tilde{w}} = \tilde{\lambda} \beta u + \mu (1 - 2 \tilde{\lambda}) - \frac{2 \mu \tilde{w}}{u} \geq \tilde{\lambda} (\beta u - 2 \mu) + \mu > 0, \quad \forall t \geq T. \quad (30)$$

Applying (28), (29) and (30), we have

$$h(\tilde{w}, t; \tilde{\lambda}) > 0, \quad \forall t \geq T. \quad (31)$$

Take advantage of (22) and (26), it yields

$$\lim_{t \to \infty} \tilde{w}(t) = (1 - \tilde{\lambda}) \sqrt{b} = \tilde{w}_0 = \tilde{w}(T).$$

Integrating the first equation of (12) and using (31) we derive

$$0 = (1 - \tilde{\lambda}) \sqrt{b} - \tilde{w}(T) = \lim_{t \to \infty} \tilde{w}(t) - \tilde{w}(T) = \lim_{t \to \infty} \int_T^t \tilde{w}'(\tau) d\tau = \lim_{t \to \infty} \int_T^t h(\tilde{w}, \tau; \tilde{\lambda}) d\tau > 0.$$

This contradiction shows that our assumption (22) does not hold. Thus, $\lim_{t \to \infty} (u(t), v(t)) = (0, 0)$ by Theorem 3.3(ii).

**Case 2.** $v_0 = u_0 > \sqrt{b}$. Then

$$v'(0) - u'(0) = 0 - u_0(1 - u_0)(\frac{u_0}{b} - 1) + \beta u_0 v_0 = \frac{u_0}{b} [u_0^2 + (\beta b - 1 - b)u_0 + b] > 0.$$

This together with $v(0) - u(0) = v_0 - u_0 = 0$ and $v_0 = u_0 > \sqrt{b}$ imply that there exists $0 < t_0 < 1$ such that

$$v(t_0) > u(t_0) > \sqrt{b}.$$ 

Similarly to Case 1, we obtain $\lim_{t \to \infty} (u(t), v(t)) = (0, 0)$.

(iii) Since $\mu > \beta \sqrt{b}$, it follows from (14) that there exists $\tilde{\lambda} > 1$ such that

$$P'_\lambda(\sqrt{b}) < 0, \quad \forall 1 < \lambda < \tilde{\lambda}.$$ 

There exists $\epsilon_\lambda > 0$, such that

$$P'_\lambda(u) \leq 0, \quad \forall \sqrt{b} \leq u \leq \sqrt{b} + \epsilon_\lambda,$$ 

i.e., $h(w_0, t; \lambda) \leq 0$ when $w(t) = w_0$ and $\sqrt{b} \leq u(t) \leq \sqrt{b} + \epsilon_\lambda$. Define

$$\Lambda(\lambda) = \{(u, v) : v - \lambda u \leq (1 - \lambda) \sqrt{b}, \quad 0 < u \leq \min\{v, \sqrt{b} + \epsilon_\lambda\}\}.$$ 

Similarly to the above, we can show that $\Lambda(\lambda)$ is an invariant region of (2) for $1 < \lambda < \tilde{\lambda}$. According to Theorem 3.3(ii), $\Lambda(\lambda)$ is an attraction basin of $\tilde{u}_3, \tilde{u}_8$ for $1 < \lambda < \tilde{\lambda}$ since this point is the only equilibria of (2) in $\Lambda(\lambda)$.

**Lemma 3.5.** Assume that $v(t) > u(t) > \sqrt{b}$ for all $t \geq 0$. Then there exists a constant $T \geq 0$ such that

$$v(t) - \frac{v(T) - \sqrt{b}}{u(T) - \sqrt{b}} u(t) < 0, \quad \forall t \geq T.$$
Proof. Since \(v(t) > u(t) > \sqrt{b}\) for all \(t \geq 0\), it yields that

\[
u' \leq u[(1 - u)(u/b - 1) - \beta u] < 0, \quad v' = \mu v(1 - v/u) < 0, \quad \forall \ t \geq 0.
\]

Similarly to Step 2 in the proof of Theorem 3.3(ii), we have \(\lim_{t \to \infty} (u(t), v(t)) = (\hat{u}_3, \hat{u}_3) = (\sqrt{b}, \sqrt{b})\).

Without loss of generality we may think that \(u(t) \leq \sqrt{b} + \sqrt{b}/n_0\) for all \(t \geq 0\) and some \(n_0 > 1\). Assume on the contrary that the conclusions of this lemma do not hold. Set

\[
\lambda_0^* = \frac{\sqrt{b} - \sqrt{b}}{u(0) - \sqrt{b}} > 1, \quad w_0(t) = v(t) - \lambda_0^* u(t).
\]

Obviously, \(w_0(0) = (1 - \lambda_0^*) \sqrt{b} < 0\). Define

\[
T_0 = \sup \{\tau : w_0(t) < 0, \ 0 \leq t \leq \tau \}.
\]

Then \(T_0 < \infty\) and

\[
w_0(T_0) = 0, \quad \text{i.e., } v(T_0) = \lambda_0^* u(T_0) > \lambda_0^* \sqrt{b}.
\]

Denote

\[
\lambda_i^* = \frac{v(T_i) - \sqrt{b}}{u(T_i) - \sqrt{b}}, \quad w_1(t) = v(t) - \lambda_i^* u(t),
\]

\[
T_1 = \sup \{\tau : w_1(t) < 0, \ 0 \leq t \leq \tau \}.
\]

Similarly, we obtain \(\lambda_1^* > \lambda_0^*, \ T_1 < \infty\) and

\[
v(T_1) = \lambda_1^* u(T_1) > \lambda_1^* \sqrt{b}.
\]

Repeating this procedure we can get three sequences \(\{\lambda_i^*\}, \{u(T_i)\}\) and \(\{v(T_i)\}\) such that

\[
\lambda_{i+1}^* = \frac{v(T_i) - \sqrt{b}}{u(T_i) - \sqrt{b}}, \quad v(T_i) = \lambda_i^* u(T_i) > \lambda_i^* \sqrt{b}, \quad i = 0, 1, 2, \ldots,
\]

\[
1 < \lambda_0^* < \lambda_1^* < \lambda_2^* \ldots < \lambda_i^* < \ldots
\]

Set \(\varepsilon := \lambda_0^* - 1 > 0\), then we have

\[
\lambda_i^* = \frac{v(T_{i-1}) - \sqrt{b}}{u(T_{i-1}) - \sqrt{b}} = \frac{\lambda_{i-1}^* u(T_{i-1}) - \sqrt{b}}{u(T_{i-1}) - \sqrt{b}}
\]

\[
= \lambda_{i-1}^* + \frac{(\lambda_{i-1}^* - 1) \sqrt{b}}{u(T_{i-1}) - \sqrt{b}} \geq \lambda_{i-1}^* + n_0 (\lambda_{i-1}^* - 1)
\]

\[
\geq \lambda_{i-1}^* + n_0 \varepsilon \geq \lambda_{i-2}^* + 2n_0 \varepsilon \geq \ldots
\]

\[
\geq \lambda_0^* + in_0 \varepsilon \to \infty, \quad \text{as } i \to \infty,
\]

and so \(v(T_i) \geq \lambda_i^* \sqrt{b} \to \infty\) as \(i \to \infty\). This contradicts to \(\lim_{t \to \infty} v(t) = \sqrt{b}\).

\[\square\]

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