Fuzzy Topology, Quantization and Gauge Invariance

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Abstract

Quantum space-time with Dodson-Zeeman topological structure is studied. In its framework the states of massive particle \(m\) correspond to elements of Poset called fuzzy points. Due to their weak (partial) ordering, \(m\) space coordinate \(x\) acquires principal uncertainty \(\sigma_x\). Quantization formalism is derived from consideration of \(m\) evolution in fuzzy phase space with minimal number of additional assumptions. It’s argued that particle’s interactions on fuzzy manifold should be gauge invariant.

1 Introduction

Importance of geometric methods in Quantum Physics is duly acknowledged now [1]. Currently the additional interest to them arises by the hypothesis that at very small (Plank) distances the standard Riemannian space-time geometry isn’t applicable and should be replaced by some principally new theory [2, 3]. In particular, it was proposed that such fundamental properties of space-time manifold as its set structure, topology and metrics can differ significantly from standard Riemannian formalism [3, 4, 5].

Nowadays, the new branch of mathematical theories, classified generally as quantum geometry, extensively developed for that purposes. The feature which is common for all of them is that the main impact is done on the use of various operator algebras which should induce the space-time geometric properties. The popular example is Connes noncommutative geometry which attempts to describe the fundamental interactions at Plank distances [2]. It’s worth to mention also the extensive studies of noncommutative fuzzy spaces with finite (sphere, tori) and infinite discrete structure [3, 4]. The general feature of such approach is that the space coordinates are principally fuzzy, the reason of that is noncommutativity of corresponding space coordinate observables \(x_{1,2,3}\).

Here we exploit the alternative approach to the microscopic space-time structure based on the modification of space topological properties. Some time ago it was shown that Posets and the fuzzy ordered sets (Fosets) can be used for the construction of fuzzy topology (FT), basing on which the consistent novel geometry with fuzzy features was formulated [6, 7]. In our previous papers it was shown that in its framework the quantization procedure by itself can be defined as the transition from the classical phase space to fuzzy one. Therefore, the quantum properties of particles and fields can be deduced directly from the geometry of
phase space induced by underlying FT and don’t need to be postulated separately of it [8, 9]. In particular, the space coordinate uncertainty is the consequence of fuzzy properties of such space geometry. As the example, the quantization of massive particles was considered; it was shown that the geometry equipped with FT induces the particle’s dynamics which is equivalent to the state evolution in quantum mechanics (QM) [8, 9]. Yet in these calculations some phenomenological assumptions were used, here the new and simple formalism which permits to drop them will be described. It will be argued also that the interactions on such fuzzy manifold should possess the local gauge invariance and under simple assumptions would correspond to Yang-Mills theory [9]. Note that the fuzzy structures were used earlier for the development of QM axiomatic in operator algebra setting, yet in such formalism the quantum dynamics is always postulated, no attempts to derive it were published [10].

2 Topological Fuzzy Structures

Here we consider only the most important steps in construction of mechanics on fuzzy manifold called fuzzy mechanics (FM), the details can be found in [8, 9]. For the start we consider the geometries for which its fundamental set is unambiguously defined, later this assumption, in fact, will be dropped.

To illustrate FT formalism let’s consider it first for 1-dimensional discrete structure. If its fundamental set of elements $D$ is the ordered discrete set, then for its elements $\{a_i\}$, the ordering relation between its elements $a_k \leq a_l$ (or vice versa) is fulfilled. But if $D$ is the partial-ordered set (Poset), then beside the relation $a_j \leq a_k$, some its elements can admit the incomparability (equivalence) relations (IR) between them: $a_j \sim a_k$. If this is the case, then both $a_j \leq a_k$ and $a_k \leq a_j$ propositions are false, and $a_j$ acquires some nontrivial properties.

As the illustrative example, let’s consider Poset $D^F = A^p \cup B$, which includes the subset of ‘incomparable’ elements $A^p = \{a_j\}$, and ordered subset $B = \{b_i\}$. For the simplicity suppose that in $B$ the element’s indexes grow correspondingly to their ordering, so that $\forall i, b_i \leq b_{i+1}$. Let’s consider some $B$ interval $\{b_i, b_n\}$ and suppose that some $A^p$ element $a_j$ is confined in it: $a_j \in \{b_i, b_n\}$, i.e. $\forall k \geq 0 b_{i-k} \leq a_j; a_j \leq b_{n+k}$, and that $a_j$ is incomparable with all $\{b_i, b_n\}$ elements: $b_j \sim a_j; l \leq i \leq n$. In this case $a_j$ is’smeared’ over $\{b_i, b_n\}$ interval, which is rough analogue of $a_j$ coordinate uncertainty relative to $B$ ‘coordinate axe’.

It’s possible to detalize such smearing introducing the fuzzy relations, for that purpose one can put in correspondence to each $a_j, b_i$ pair the weight $w_i^j \geq 0$ with the norm $\sum_j w_i^j = 1$. In this case $D^F$ is fuzzy ordered set (Foset), $A^p$ element $a_j$ called the fuzzy point (FP) [6, 7]. In the simplest case the continuous 1-dimensional Foset $C^F$ is defined analogously to discrete one: $C^F = A^p \cup X$ where $A^p$ is the same discrete subset, $X$ is the continuous ordered subset, which is equivalent to $R^1$ axe of real numbers. Correspondingly, fuzzy relations between elements $a_j, x$ are described by real function $w_j^i(x) \geq 0$ with the norm $\int w_j^i dx = 1$.

Remind that in 1-dimensional Euclidian geometry, the elements of its manifold $X$ are the points $x_n$ which constitute the ordered continuum set. $x_n$ position on $X$ in 1-dimensional Euclidian geometry is characterized by the real number $x_n$ in some coordinate system. Yet in 1-dimensional geometry equipped with FT the position of fuzzy point $a_j$ becomes the positive normalized function $w_j^i(x)$ on $X$; $w_j^i$ dispersion $\sigma_x$ characterizes $a_j$ coordinate uncertainty on $X$. Note that in such geometry $w_j^i(x)$ doesn’t have any probabilistic meaning but only the algebraic one [6]. To characterize the distinction between the fuzzy structure and probabilistic one, the correlation $K_f(x, x')$ defined over $w_j$ support can be introduced; thus if
w(x_1, x_2) \neq 0, then \forall x_1, x_2; K_f(x_1, x_2) = 1 for FP a_j and K_f(x_1, x_2) = 0 for probabilistic a_j distribution. Thus a_j 'state' G on X is described by two functions \( G = \{w(x), K_f(x, x')\} \).

3 Linear Model of Fuzzy Dynamics

In the described terms the massive particle of 1-dimensional classical mechanics corresponds to the ordered point \( x(t) \) on X. By the analogy, we suppose that in 1-dimensional fuzzy mechanics (FM) the particle \( m \) corresponds to fuzzy point \( a(t) \) in \( C^F \) characterized by normalized positive density \( w(x, t) \). However, \( m \) physical fuzzy state \( |g\} \) can also depend on other \( m \) degrees of freedom (DFs), which are \( g \) free parameters characterizing \( w(x, t) \) evolution. For \( m \) free evolution the obvious candidate for that is \( m \) average velocity \( \bar{v} \):

\[
\bar{v} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} xw(x)dx = \int_{-\infty}^{\infty} x\frac{\partial w}{\partial t}(x, t)dx
\]

We shall look for such parameters in form of real functions on X: \( q_1(x, t), ..., q_n(x, t) \). In this vein consider \( m \) free evolution and suppose that in FM it's local, i.e.:

\[
\frac{\partial w}{\partial t}(x, t) = -f(x, t)
\]

where \( f \) is an arbitrary function which can depend on \( x, w(x, t), q_1(x, t), ..., q_n(x, t) \) only. Such locality is essential assumption of our theory, however, the alternative nonlocal variants are quite complicated and demand to use the additional constants. Then from \( w \) norm conservation:

\[
\int_{-\infty}^{\infty} f(x, t)dx = -\int_{-\infty}^{\infty} \frac{\partial w}{\partial t}(x, t)dx = -\frac{\partial}{\partial t} \int_{-\infty}^{\infty} w(x, t)dx = 0
\]

If \( w \) free evolution possesses \( x, t \)-shift invariance, then \( f \) can’t depend on \( x, t \) directly, but only on \( w(x, t) \) and \( q_i(x, t) \). If to substitute \( f = \frac{\partial J}{\partial x} \), then eq. (2) demands:

\[
J(\infty, t) - J(-\infty, t) = 0
\]

If this condition is fulfilled, \( J \) can be regarded as \( w \) flow (current), so eq. (1) can be transformed to 1-dimensional flow continuity equation [11]:

\[
\frac{\partial w}{\partial t} = -\frac{\partial J}{\partial x}
\]

Below it will be shown that in our theory \( J(\pm\infty, t) = 0 \). \( J(x) \) can be decomposed formally as: \( J = w(x)v(x) \) where \( v(x) \) corresponds to 1-dimensional \( w \) flow velocity. In these terms \( w \) evolution equation transforms to:

\[
\frac{\partial w}{\partial t} = -v \frac{\partial w}{\partial x} - \frac{\partial v}{\partial x}w
\]

Thus in 1-dimensional FM \( w \) evolution obeys to 1-dimensional flow continuity equation, which derived here just from the locality axiom. Note that such equation in classical kinetics or hydrodynamics obtained from additional assumptions about properties of evolving media
It’s sensible to suppose that \( v(x, t) \) can be considered as \( |g\) free parameter \( q_1 \), yet we substitute it by the related parameter \( \gamma(x) \) defined as:

\[
\gamma(x, t) = \mu \int_{-\infty}^{x} v(\xi, t) d\xi + c_{\gamma}
\]  

(6)

where \( \mu \) is theory constant, \( c_{\gamma} \) is an arbitrary real value.

If we assume that \( m \) state \( |g\) doesn’t depend on any other DFs, i.e. \( |g\} = \{w(x), \gamma(x)\} \), then obviously it can be expressed also as some complex function \( g(x) \), the simplest example is: \( g(x) = w(x) + i\gamma(x) \). However, we are interested to find such dynamical \( g \) representation for which \( |g\) evolution equation would acquire most simple form, such condition imposes some constraints on \( g(x) \) ansatz. First of all, we admit that \( g(x) \) is normalized by the relation \( g^* g = w(x) \), in that case \( |g\) can be expressed as:

\[
g(x) = w(x) e^{i\eta w(x, \gamma(x))}
\]  

(7)

where \( \eta \) is arbitrary real function. We postulate also that \( m \) observables are the linear, self-adjoint operators on the space of \( g \) states; plainly, for such \( g(x) \) ansatz it coincides with QM Hilbert space \( \mathcal{H} \). In addition, the expectation values of all \( m \) observables shouldn’t depend on \( c_{\gamma} \) value. This condition is fulfilled only if \( \eta(w, \gamma) = \gamma(x) \), the proof of that will be given below, for the moment we shall admit it ad hoc. In this case the pure \( m \) states described by \( g(x) \) are equivalent to dirac vectors (rays) \( |\psi\rangle \) of QM [12].

Evolution equation for \( g \) is supposed to be of the first order in time, i.e.:

\[
i \frac{\partial g}{\partial t} = \hat{H} g.
\]  

(8)

In general \( \hat{H} \) is nonlinear operator, for the simplicity we shall consider first the linear case and turn to nonlinear one afterwards. The free \( m \) evolution is invariant relative to \( x \) space shifts performed by the operator \( \hat{W}(a) = \exp(a \frac{\partial}{\partial x}) \). Because of it, \( \hat{H} \) should commute with \( \hat{W}(a) \) for the arbitrary \( a \), i.e. \([\hat{H}, \frac{\partial}{\partial x}] = 0\). It holds only if \( \hat{H} \) is differential polinonm, which can be written as:

\[
\hat{H} = -\sum_{l=0}^{n} b_l \frac{\partial^l}{\partial x^l}
\]  

(9)

where \( b_l \) are an arbitrary constants, \( n \geq 1 \). From \( X- \)reflection invariance \( b_l = 0 \) for noneven \( l \), and \( b_0 = 0 \) can also be settled. If to substitute \( v(x) \) by \( \gamma(x) \) in eq. (5) and transform it to \( \sqrt{w} \) time derivative, then left part of (8) is equal to:

\[
i \frac{\partial g}{\partial t} = -(\frac{i}{\mu} \frac{\partial w^{1/2}}{\partial x} \frac{\partial \gamma}{\partial x} + \frac{i}{2\mu} w^{1/2} \frac{\partial^2 \gamma}{\partial x^2} + w^{1/2} \frac{\partial \gamma}{\partial t}) e^{i\gamma} = e^{i\gamma} \hat{F}(g)
\]  

(10)

As follows from eq. (8), the operator \( \hat{F}(g) \) is also equal to :

\[
\hat{F} = e^{-i\gamma} \hat{H} g
\]

Then, \( \hat{H} \) can be calculated from the comparison of the imaginary terms in brackets of (10) and the same terms of \( \hat{F}(g) \). Really, the imaginary part of \( e^{-i\gamma} \hat{H} g \) includes the highest \( \gamma \) derivative as the term \( ib_n w^{1/2} \frac{\partial^{n+1}}{\partial x^n} \), yet for eq. (10) and its imaginary term in brackets the
higher $\gamma$ derivative is proportional to $\frac{\partial^2 \gamma}{\partial x^2}$. Hence for $l > 2$ all $b_l = 0$, and thus $g$ evolution is described by the only $\hat{H}$ term with $b_2 = \frac{1}{2\mu}$. It means that $m$ free evolution is described by free Schrödinger equation for particle with mass $\mu$. The possible interpretation of this result is that the flow continuity equation of (5) is incompatible with $g(x)$ dynamics which depends on high $x$-derivatives of $g$. The obtained ansatz gives also $J(\pm \infty, t) = 0$ for $w$ flow of eq. (2), in accordance with our expectations. Note also that the flow velocity $v(x, t)$ can be formally defined as the ratio of $J(x)$, $w(x)$ observable expectation values, where $w$ observable is described by the projection operator $\hat{P}(x)$ [12]. In this approach the evolution of extended object $m$ in some aspects is similar to the potential motion of continuous media [11]. In particular, $\gamma(x)$ is analogue of hydrodynamical velocity potential.

In this framework the evolution equation for $g$ is equivalent to the system of two equations, one for $\frac{\partial w}{\partial t}$ and other for $\frac{\partial \gamma}{\partial t}$, the important features of such representation will be discussed below. Plainly, $\gamma(x)$ corresponds to quantum phase, so that

$$\Delta(x, x') = \gamma(x) - \gamma(x')$$

describes the dynamical or phase correlation between the state components in $x, x'$. The mixed states in FM are defined exactly like in QM formalism, i.e. are positive, trace one operators $\rho$ on $\mathcal{H}$. The purity rate of mixed states in $x$-representation $\rho(x, x')$ corresponds to geometric correlation $K_f(x, x')$ defined in sect. 2. For pure $m$ states:

$$\rho(x, x') = g(x)g^*(x') = [w(x)w(x')]\frac{1}{2}e^{\Delta(x, x')}$$

is equivalent to $g(x)$, yet this $g$ representation demonstrates in the open the correlation structure of $m$ pure states which induces, in fact, the interference effects between $|g\rangle$ components in $x, x'$.

### 4 General Fuzzy Dynamics

In the previous section FM dynamics was obtained assuming that $g$ evolution is linear, and $g$ is defined by standard QM ansatz for QM wave function $\psi(x)$. Here it will be shown that both these assumptions can be dropped one by one. Concerning with nonlinear case, the conditions of QM dynamics linearity were investigated by Jordan, and turn out to be essentially weaker than Wigner theorem asserts [13]. In particular, it was proved that if the evolution maps the set of all pure states one to one onto itself, and for arbitrary mixture of orthogonal states $\rho(t) = \sum P_i(t) \rho_i(t)$ all $P_i$ are independent of time, then such evolution is linear. Yet for considered FM formalism first condition is, in fact, generic: no mixed state can appear in the free evolution of pure fuzzy state. The second condition involves the probabilistic mixture of such orthogonal states and seems to be rather weak assumption also.

Now we can return to initial $g(x)$ ansatz of (7) and demonstrate that $\eta = \gamma(x)$. Really, $\gamma(x)$ is defined up to real constant $c_\gamma$, yet for any $m$ observable $\hat{Q}$ its expectation value $\bar{Q}$ shouldn’t depend on $c_\gamma$. For the proof, it’s enough to consider the flow (current) operator $\hat{J}(x)$ where $x$ is operator parameter. Its expectation value is equal to [12]:

$$\bar{J}(x) = \frac{i}{2\mu}(\frac{\partial g^*}{\partial x} g - \frac{\partial g}{\partial x} g^*) = \frac{w^2}{2\mu} \left( \frac{\partial \eta}{\partial w} \frac{\partial w}{\partial x} + \frac{\partial \eta}{\partial \gamma} \frac{\partial \gamma}{\partial x} \right)$$ (11)
As follows from the simple calculations omitted here, \( \mathcal{J}(x) \) is independent of \( \zeta, \gamma \), if \( \frac{\partial n}{\partial \chi} = \zeta \) where \( \zeta \) is arbitrary real constant. Hence \( \eta = \chi(w) + \zeta(\gamma + c_{\gamma}) \), here \( \chi \) is some real function, since \( \gamma(x) \) scale is undefined, one can put \( \zeta = 1 \). Then for \( \eta \) of (7) \( w \) and \( \gamma \) are factorized, so \( g = e^{i\chi}g' \), where \( g'(x) \) is standard QM wave function. Note that if \( \langle g_i'|g_j' \rangle = \delta_{ij} \), then \( \langle g_i|g_j \rangle = \delta_{ij} \) and vice versa, and if \( g' \) is the pure state, then \( g \) is also pure and vice versa. As was argued above, in FM any pure state \( g'(t_0) \) should evolve to pure state \( g'(t) \) for arbitrary \( t \), so the same should be true for any \( g(t_0) \). Hence \( g \) evolution equation can be also expressed as: \( i\frac{\partial g}{\partial t} = \hat{H}g \), yet here \( \hat{H} \) isn’t necessarily linear. Now Jordan theorem can be applied to \( g \) evolution. Let’s decompose \( i\frac{\partial g}{\partial t} \) analogously to eq. (10):

\[
i\frac{\partial g}{\partial t} = i\frac{\partial}{\partial t}[w^{\frac{1}{2}}e^{i\chi(x)+\gamma}] = (\frac{i}{\mu} \frac{\partial w^{\frac{1}{2}}}{\partial t} + \frac{iw^{\frac{1}{2}}}{2\mu} \frac{\partial \chi}{\partial w} \frac{\partial w}{\partial t} - w^{\frac{1}{2}} \frac{\partial \gamma}{\partial t})e^{i\chi(x)+\gamma}
\]

(12)

From it one can come to the evolution equation for \( g' \), for that the term containing \( \frac{\partial w^{\frac{1}{2}}}{\partial t} \) can be rewritten according to (5) and added to \( \hat{H}g \). As the result, it gives:

\[
i\frac{\partial g'}{\partial t} = e^{-i\chi} \hat{H}(g' e^{i\chi}) + \frac{\partial \chi}{\partial w} \frac{\partial}{\partial x}(w \frac{\partial \gamma}{\partial x})g' = \hat{H}'g'
\]

(13)

But for \( g' \) the conditions of Jordan theorem are fulfilled, hence \( \hat{H}' \) should be the linear operator. Therefore, for arbitrary \( \chi(w) \) any \( g(x) \) is equivalent to corresponding \( g'(x) \) which evolves linearly.

Beside the locality, space and time shift invariance the proposed FM formalism exploits only three axioms. First one defines the object states as the evolving points on fuzzy manifold, which characterized by two DFs only: the spatial density \( w(x,t) \) and \( \gamma(x,t) \) which is the integral function of \( w \) flow velocity \( v(x,t) \). It follows that for massive particle states the resulting state space coincides with QM Hilbert space \( \mathcal{H} \). Second axiom defines the physical observables as the linear self-adjoint operators on \( \mathcal{H} \). Third one is Reduction (Projection) postulate of observable measurement taken copiously from standard QM. For the comparison remind that the standard (Schroedinger) QM formalism is based on seven axioms [14].

In this approach the state space is defined by geometry and corresponding dynamics i.e. is derivable concept. For pure states of free nonrelativistic particle \( m \) it corresponds to \( \mathcal{H} \), but, in principle, it can be different for other systems. The similar features possess the formalism of algebraic quantum mechanics where the state space is defined by the observable algebra and system dynamics [14]. It important to notice also that in such formalism the commutation relation \( [\hat{x},\hat{p}_x] = i \) results, in fact, from the geometry and topology of fuzzy manifold.

## 5 3-dimensional FM Dynamics

Here we shall consider FM in 3 dimensions, in fact, it doesn’t demand any serious modification of described formalism. In 3 dimensions our fundamental set \( C^F = \mathcal{A}^0 \cup R^3 \), hence for any fuzzy point \( a_j \) its ordering properties should be defined relative to \( X,Y,Z \) coordinate axes separately. Assuming FM rotational invariance, \( a_j \) fuzzy weight can be described by the positive function \( w^j(\vec{r}) \) with norm \( \int w^j d^3r = 1 \). If the particle \( m \) is the fuzzy point \( a(t) \) characterized by \( w(\vec{r},t) \), then analogously to sect. 2 \( w \) evolution in \( R^3 \) can be expressed as:

\[
\frac{\partial w}{\partial t}(\vec{r},t) = -f(\vec{r},t)
\]

(14)
where $f$ is an arbitrary function. Then from $w$ norm conservation:

$$
\int_V f(\vec{r}, t) d^3r = \int_V \frac{\partial w}{\partial t}(\vec{r}, t) d^3r = \frac{\partial}{\partial t} \int_V w(\vec{r}, t) d^3r = 0
$$

(15)

where $V$ denotes the infinite volume with $|\vec{r}| \to \infty$ in all directions. Analogously to sect. 2, we shall use $f$ counterpart defined via the relation: $f = \text{div} \vec{J}$ where $\vec{J}$ is some vector function. Naturally, such equality has many solutions, of which we choose the symmetric one:

$$
J_x(\vec{r}, t) = \frac{1}{3} \int_{-\infty}^{\infty} f(\xi, y, z, t) d\xi
$$

(16)

whereas $J_y, z$ are defined by the obvious replacement of $x, y, z$ in this formulae. From this equality one can obtain 3-dimensional flow continuity equation:

$$
\frac{\partial w}{\partial t} = -\text{div} \vec{J}
$$

(17)

with additional condition:

$$
\int_S \vec{J} \vec{n} ds = 0
$$

(18)

where $S$ is the surface surrounding the infinite volume $V$, $\vec{n}$ is the vector normal to the given surface element. After that one can decompose formally $\vec{J} = w \vec{v}$ and regard $w$ flow velocity $\vec{v}(\vec{r})$ as independent $|g\rangle$ parameter. $|g\rangle$ phase $\gamma(\vec{r})$ is related to it via the equality $\mu \vec{v} = \text{grad} \gamma$. To guarantee the formalism consistency, we assume that the phase correlation $\Delta_\gamma(\vec{r}, \vec{r}')$ is independent of the path $l$ which connects $\vec{r}, \vec{r}'$, i.e.:

$$
\Delta_\gamma(\vec{r}, \vec{r}') = \int_{\vec{r}}^{\vec{r}'} \text{grad} \gamma dl
$$

has the same value for arbitrary path $l$. Note that in hydrodynamics this condition corresponds to the fluid potential motion [11]. Then, from the similar sequence of calculations, as for 1-dimensional case, free Schroedinger equation can be derived in 3 dimensions. Note that for all normalized states the condition (18) is fulfilled, because for them $\vec{J}(\vec{r}) \to 0$ at $\vec{r} \to \infty$.

Planck constant $\hbar = 1$ in our FM ansatz, but the same value ascribed to it in relativistic unit system together with velocity of light $c = 1$; in FM framework $\hbar$ only connects $x, p$ geometric scales and doesn't have any other meaning.

In our derivation of evolution equation we don't assume Galilean invariance of FM, rather in our approach it follows itself from the obtained evolution equation, if the observer reference frame (RF) is regarded as the physical object with mass $\mu \to \infty$ [8]. For the transition to the relativistic covariant description the linearity of state evolution becomes the important criteria for the choice of consistent ansatz. If to demand also that $m$ density $w(\vec{r}, t)$ is normalized and nonnegative in any RF, then for massive particle $m$ the simplest extension of FM state $|g\rangle$ is 4-spinor $g_i(\vec{r}, t); i = 1, 4$; its evolution is described by Dirac equation for spin-$\frac{1}{2}$, i. e. such particle is fermion.

Now we shall consider the interaction between fuzzy states in nonrelativistic FM and discuss their possible generalization for the relativistic case. Note first that by derivation in FM the free Hamiltonian $H$ induces, in fact, $\mathcal{H}$ dynamical asymmetry between $|\vec{r}\rangle$ and $|\vec{p}\rangle$.
'axes' which *apriori* is absent in standard QM formalism. As was shown above, in FM, as well in QM, $m$ free dynamics is described by the system of two equations which define $\frac{\partial w}{\partial t}$ and $\frac{\partial \gamma}{\partial t}$:

$$\frac{\partial w}{\partial t}(\vec{r}) = -\frac{1}{\mu} \frac{\partial w}{\partial \vec{r}} \frac{\partial \gamma}{\partial \vec{r}} - \frac{1}{2\mu} w^{\frac{1}{2}} \frac{\partial^{2} \gamma}{\partial x^{2}}$$

$$\frac{\partial \gamma}{\partial t} = -\frac{1}{2\mu} [(\frac{\partial \gamma}{\partial \vec{r}})^{2} - \frac{1}{w^{\frac{1}{2}} \partial \vec{r}^{2}}]$$

(19)

Yet the first of them is equivalent to eq. (17) which describes just $w(\vec{r})$ balance and so is, in fact, kinematical one and can’t depend on any interactions directly. Namely, under some external influence the values of $w, \gamma$ variables can change, but no new terms can appear in the equation. Note that in QED $e\vec{A}$ term formally appears in it, but it’s just the part of the expression for kinematic momentum [12]. Hence $m$ interactions can be accounted only via the modification of second equation of this system:

$$\frac{\partial \gamma}{\partial t} = -\frac{1}{2\mu} [(\frac{\partial \gamma}{\partial \vec{r}})^{2} - \frac{1}{w^{\frac{1}{2}} \partial \vec{r}^{2}}] + H_{int}$$

(20)

where $H_{int}$ is possible interaction term. Since $\gamma$ corresponds to the quantum phase, it supposes that in FM all $m$ interactions should possess some form of local gauge invariance [16]. Despite that the fermion state is described by several quantum phases, the same invariance fulfilled for it and can be extended also on relativistic case. Of course, one can just postulate the gauge interactions of certain kind, yet it seems worth to explore whether such dynamics can be obtained from some considerations related to FM or some other fundamental principles. In our previous paper the abelian toy-model of gauge interactions on fuzzy manifold was formulated which in the main aspects is similar to QED [9]. Preliminary results for interactions of fermion multiplets show that their interactions can also possess local $SU(n)$ gauge invariance and to be transferred by corresponding Yang-Mills fields.

In conclusion, we have shown that the quantization of elementary systems can be derived directly from axiomatic of Set theory and topology together with the natural assumptions about system evolution. It allows to suppose that the quantization phenomenon has its roots in foundations of mathematics and logics [14]. In the same time the considered fuzzy manifold describes the possible variant of fundamental pregeometry which is basic component of some quantum gravity theories [3, 5]. Note also that in such geometry the fundamental set of elements is, in fact, absent, it replaced by the set of positive functions, which makes it similar to the basic structure of noncommutative geometry [2, 3]. The main aim of our theory, as well as other studies of fuzzy spaces, is the construction of nonlocal QFT (or other more general theory) [15]. In this vein, FM provides the interesting opportunities, being generically nonlocal theory which, in the same time, can possess Lorentz covariance and the local gauge invariance.

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