Generation of multipole moments by external field in Born-Infeld non-linear electrodynamics

Dariusz Chruściński*
Fakultät für Physik, Universität Freiburg
Hermann-Herder-Str. 3, D-79104 Freiburg, Germany

Jerzy Kijowski
Centrum Fizyki Teoretycznej PAN
Aleja Lotników 32/46, 02-668 Warsaw, Poland

Abstract

The mechanism for generation of multipole moments due to an external field is presented for the Born-Infeld charged particle. The “polarizability coefficient” $\kappa_l$ for arbitrary $l$-pole moment is calculated. It turns out, that $\kappa_l \sim r_0^{2l+1}$, where $r_0 := \sqrt{|e|/4\pi b}$ and $b$ is the Born-Infeld non-linearity constant. Physical implications are considered.

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*On leave from Institute of Physics, Nicholas Copernicus University, ul. Grudziądzka 5/7, 87-100 Toruń, Poland.
1. Introduction

Recently, one of us has proposed a consistent, relativistic theory of the classical Maxwell field interacting with classical, charged, point-like particles (cf. [1]). For this purpose an “already renormalized” formula for the total four-momentum of a system composed of both the moving particles and the surrounding electromagnetic field was used. It was proved, that the conservation of the total four-momentum defined by this formula is equivalent to a certain boundary condition for the behaviour of the Maxwell field in the vicinity of the particle trajectories. Without this condition, Maxwell theory with point-like sources is not dynamically closed: initial conditions for particles and fields do not imply uniquely the future and the past of the system. Indeed, the particles trajectories fulfilling initial conditions can be arbitrarily chosen and then the initial-value-problem for the field alone can be solved uniquely. The boundary condition derived this way was called fundamental equation. When added to Maxwell equations, it provides the missing dynamical equation: now, particles trajectories cannot be chosen arbitrarily and initial data uniquely imply the future and the past evolution of the composed “particles + fields” system.

Physically, the “already renormalized” formula for the total four-momentum was suggested by a suitable approximation procedure applied to an extended-particle model. In such a model we suppose that the particle is a stable, soliton-like solution of a hypothetical fundamental theory of interacting electromagnetic and matter fields. We assume that this hypothetical theory tends asymptotically to the linear Maxwell electrodynamics, in the weak field regime (i. e. for weak electromagnetic fields and “almost vanishing” matter fields). This means, that “outside of the particles” the entire theory reduces to Maxwell electrodynamics. Starting from this model, a formula was found, which gives in a good approximation the total four-momentum of the system composed of both the moving (extended) particles and the surrounding electromagnetic field. This formula uses only the “mechanical” information about the particle (position, velocity, mass $m$ and the electric charge $e$) and the free electromagnetic field outside of the particle. It turns out, that this formula does not produce any infinities when applied to the case of point particles, i. e. it is “already renormalized”. Using this philosophy, this formula was taken as a starting point for a mathematically self-consistent theory of point-like particles interacting with the linear Maxwell field. The “fundamental equation” of the theory is precisely the conservation of the total four momentum of the system “particles + fields”, defined by this formula.

At this point a natural idea arises, to construct a “second-generation” theory, which approximates better the real properties of an extended particle, and takes into account also possible deformations of its interior, due to strong external field. In [2] a simple mechanism of generation of a particle’s electric dipole moment was proposed. As a specific particle’s model we have used the Born-Infeld particle described by the Born-Infeld nonlinear electrodynamics.

In the present paper we prove that a similar mechanism is responsible for the generation of higher multipole moments.
Mathematically, such a polarizability is due to the elliptic properties of the field equations describing the statics of the physical system under consideration. Given a particular model of the matter fields interacting with electromagnetism, the “particle at rest”-solution corresponds to a minimum of the total field energy. It is, therefore, described by a solution of a system of elliptic equations (Euler-Lagrange equations derived from the total Hamiltonian of the hypothetical fundamental theory of interacting matter fields and electromagnetic field). Far away from the particle, these equations reduce to the free Maxwell equations.

This solution corresponds to the vanishing boundary conditions at infinity. Physical situation “particle in a non-vanishing external field” corresponds to the solution of the same elliptic equations but with non-vanishing boundary conditions $E_\infty$ at infinity. More precisely, we assume that the electric field at infinity behaves like

$$E^k(x) = E^k_{reg}(x) + E^k_\infty(x), \tag{1}$$

where

$$E^k_\infty(x) := Q^k_{i_2...i_l} x^{i_2}...x^{i_l}, \tag{2}$$

(the $l$-pole tensor $Q_{i_1i_2...i_l}$ is completely symmetric and traceless) and the regular part $E^k_{reg}(x)$ vanishes at infinity.

Suppose that the free particle (the unperturbed solution) displays no internal structure. This means that for vanishing external field $Q = 0$ the regular part $E^k_{reg}$ reduces to the simplest Coulomb field describing the monopole with a given electric charge $e$. But for nontrivial perturbation $Q$ the regular field may contain an extra multipole term at infinity, of the type:

$$E^k_M := \frac{r^2 M^k_{i_2...i_l} x^{i_2}...x^{i_l} - \frac{2l+1}{l} r^k M_{i_1i_2...i_l} x^{i_1}x^{i_2}...x^{i_l}}{r^{3+2l}}. \tag{3}$$

The reason for creation of this extra multipole moments is the field nonlinearity in the strong field region. For weak perturbations, the relation between the particle’s $l$-pole moment $M_{i_1i_2...i_l}$ created this way and the $l$-pole moment $Q_{i_1i_2...i_l}$ of the external field is expected to be linear in the first approximation:

$$M_{i_1i_2...i_l} = \kappa_l Q_{i_1i_2...i_l} \tag{4}$$

and the coefficient $\kappa_l$ describes the “deformability” of the particle, due to non-linear character of the interaction of the matter fields (constituents of the particle) with the electromagnetic field.

The coefficients $\kappa_l$ arise, therefore, similarly as “reflection” or “transmission” coefficients in the scattering theory. Formulae (2) and (3) describe two independent solutions of the second order, linear, elliptic equation describing the free, statical Maxwell field surrounding the particle. Outside of the particle they may be mixed in an arbitrary proportion. Such an arbitrary mixture is no longer possible if it has to match an exact solution of non-linear equations describing the interior of the particle. The relation (4) arises, therefore, as the matching condition between these two solutions.
In the present paper we assume that the unperturbed particle is described by the spherically symmetric, static solution of non-linear Born-Infeld electrodynamics with a $\delta$-like source. We find explicitly the two-dimensional family of all the $l$-pole perturbations of the above solution. They all behave correctly at $r \to \infty$. For $r \to 0$, however, there is one perturbation which remains regular, and another one which increases faster than the unperturbed solution. The variation of the total field energy due to the latter perturbation is divergent in the vicinity of the particle, which we consider to be an unphysical feature. We conclude that all the physically admissible perturbations must be proportional to the one which is regular at $0$. At $r \to \infty$ this solution behaves as a mixture of solutions (2) and (3). We calculate the ratio between these two ingredients and we interpret it as the $l$-th polarizability coefficient of the Born-Infeld particle.

2. Perturbations of the Born-Infeld particle

The Born-Infeld electrodynamics \cite{3} (see also \cite{4}) is defined by the following lagrangian

$$\mathcal{L}_{BI} := b^2 \left[ 1 - \sqrt{1 - 2b^{-2}S - b^{-4}P^2} \right],$$

where $S$ and $P$ are the following Lorentz invariants:

$$S := -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} = \frac{1}{2} (E^2 - B^2),$$

$$P := -\frac{1}{4} \epsilon_{\mu\nu\lambda\kappa} f^{\mu\nu} f^{\lambda\kappa} = EB,$$

and $f_{\mu\nu}$ is a tensor of the electromagnetic field defined in a standard way via a four-potential vector. The parameter "$b$" has a dimension of a field strength (Born and Infeld called it the absolute field, cf. \cite{3}) and it measures the nonlinearity of the theory. In the limit $b \to \infty$ the lagrangian $\mathcal{L}_{BI}$ tends to the standard Maxwell lagrangian

$$\mathcal{L}_{Maxwell} = S.$$ 

Note, that field equations derived from (3) have the same form as Maxwell equations derived from (8), i.e.

$$\nabla \times E + \dot{B} = 0, \quad \nabla \cdot B = 0,$$

$$\nabla \times H - \dot{D} = j, \quad \nabla \cdot D = \rho,$$

(to obtain the Born-Infeld equations with sources $\rho$ and $j$ one has to add to $\mathcal{L}_{BI}$ the standard interaction lagrangian "$j^{\mu} A_{\mu}$"). However, the relation between fields ($E, B$) and ($D, H$) is now highly nonlinear:

$$D := \frac{\partial \mathcal{L}_{BI}}{\partial E} = \frac{E + b^{-2}(EB)B}{\sqrt{1 - b^{-2}(E^2 - B^2) - b^{-4}(EB)^2}},$$

$$H := -\frac{\partial \mathcal{L}_{BI}}{\partial B} = \frac{B - b^{-2}(EB)E}{\sqrt{1 - b^{-2}(E^2 - B^2) - b^{-4}(EB)^2}}.$$
The above formulae are responsible for the nonlinear character of the Born-Infeld theory. In the limit $b \to \infty$ we obtain linear Maxwell relations: $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$ (we use the Heaviside-Lorentz system of units).

Now, consider a point-like, Born-Infeld charged particle at rest. It is described by the static solution of the Born-Infeld field equations (9)–(12) with $\rho = e\delta(r)$ and $\mathbf{j} = 0$, where $e$ denotes the particle’s electric charge. Obviously $\mathbf{B} = \mathbf{H} = 0$ (Born-Infeld electrostatics). Moreover, the spherically symmetric solution of $\nabla \cdot \mathbf{D} = e\delta(r)$ is given by the Coulomb formula

$$D_0 = \frac{e}{4\pi r^3} r .$$

(13)

Using (11) one easily finds the corresponding $\mathbf{E}_0$ field

$$\mathbf{E}_0 = \frac{\mathbf{D}_0}{\sqrt{1 + b^{-2}D_0^2}} = \frac{e}{4\pi r} \frac{r}{\sqrt{r^4 + r_0^4}} ,$$

(14)

where we introduced $r_0 := \sqrt{\frac{|e|}{4\pi b}}$. Note, that the field $\mathbf{E}_0$, contrary to $\mathbf{D}_0$, is bounded in the vicinity of a particle: $|\mathbf{E}_0| \leq \frac{|e|}{4\pi r_0} = b$. It implies that the energy of a point charge is already finite.

Now, let us perturb the static Born-Infeld solution $\mathbf{E}_0$:

$$\mathbf{E} = \mathbf{E}_0 + \tilde{\mathbf{E}} ,$$

(15)

where $\tilde{\mathbf{E}}$ denotes a weak perturbation, i.e. $|\tilde{\mathbf{E}}| \ll |\mathbf{E}_0|$. The corresponding $\mathbf{D}$ field may be obtained from (11). In the electrostatic case, i.e. $\mathbf{B} = \mathbf{H} = 0$, formula (11) reads:

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{1 - b^{-2}E^2}} , \quad \mathbf{E} = \frac{\mathbf{D}}{\sqrt{1 + b^{-2}D^2}} .$$

(16)

Therefore, using (13) we get

$$\mathbf{D} = \mathbf{D}_0 + \frac{1}{\sqrt{1 - b^{-2}E_0^2}} \left( \tilde{\mathbf{E}} + b^{-2}(\mathbf{D}_0 \tilde{\mathbf{E}})\mathbf{D}_0 \right) + O(\tilde{\mathbf{E}}^2) ,$$

(17)

where $O(\tilde{\mathbf{E}}^2)$ denotes terms vanishing for $|\tilde{\mathbf{E}}| \to 0$ like $\tilde{\mathbf{E}}^2$ or faster. In the present paper we study only the linear perturbation, i.e. we keep in (17) terms linear in $\tilde{\mathbf{E}}$ and neglect $O(\tilde{\mathbf{E}}^2)$. Therefore, in this approximation the perturbation $\tilde{\mathbf{D}} = \mathbf{D} - \mathbf{D}_0$ of the field $\mathbf{D}_0$ equals:

$$\tilde{\mathbf{D}} = \frac{1}{\sqrt{1 - b^{-2}E_0^2}} \left( b^{-2}(\mathbf{D}_0 \tilde{\mathbf{E}})\mathbf{D}_0 + \tilde{\mathbf{E}} \right) = \sqrt{1 + b^{-2}D_0^2} \left( b^{-2}(\mathbf{D}_0 \tilde{\mathbf{E}})\mathbf{D}_0 + \tilde{\mathbf{E}} \right) .$$

(18)

The fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{D}}$ fulfill the following equations:

$$\nabla \times \tilde{\mathbf{E}} = 0 , \quad \nabla \cdot \tilde{\mathbf{D}} = 0 .$$

(19)
The first one implies that $\vec{E} = -\nabla \vec{\phi}$. Therefore, using (18), $\nabla \cdot \vec{D} = 0$ leads to the following equation for the potential $\vec{\phi}$:

$$\Delta \vec{\phi} + \left( \frac{r_0}{r} \right)^4 \left[ \frac{\partial^2 \vec{\phi}}{\partial r^2} - \frac{4}{r} \frac{\partial \vec{\phi}}{\partial r} \right] = 0,$$

(20)

where $\Delta$ stands for a 3-dimensional Laplace operator in $\mathbb{R}^3$. Observe, that in the limit $r_0 \to 0$ (i. e. Maxwell theory) we obtain simply the Laplace equation $\Delta \vec{\phi} = 0$ for the electrostatic potential $\vec{\phi}$. Using spherical coordinates in $\mathbb{R}^3$ the Laplace operator $\Delta$ reads:

$$\Delta \vec{\phi} = \frac{1}{r \partial^2} \left( r \vec{\phi} \right) + \frac{1}{r^2} L^2 \vec{\phi},$$

(21)

where $L^2$ denotes the Laplace-Beltrami operator on the unit sphere (it is equal to the square of the quantum-mechanical angular momentum).

We see, that due to the spherical symmetry of the unperturbed solution $D_0$, different multipole modes decouple in the above equation. Therefore, any solution of (20) may be written as follows:

$$\vec{\phi}(x) = \sum_{l=1}^{\infty} a_l \vec{\phi}_l(x),$$

(22)

where

$$\vec{\phi}_l(r, \text{angles}) := \frac{\Psi_l(r)}{r} Y_l(\text{angles}),$$

(23)

and $Y_l$ denotes the $l$-pole eigenfunction of $L^2$, i. e. $L^2 Y_l = -l(l+1)Y_l$. Obviously, an eigenfuction $Y_l$ is related to the $l$-pole moment tensor by

$$Y_l = r^{-l} x^{i_1} ... x^{i_l} Q_{i_1 i_2 ... i_l}.$$

(24)

Observe, that we do not consider the monopole term (i. e. $a_0 \vec{\phi}_0$) in (22). This term corresponds to the gauge transformation of $\vec{\phi}$ and due to the gauge invariance of (20) it is unessential (we may fix the gauge by putting e. g. $a_0 = 0$).

Let us look for the $l$-pole-like deformation, i. e. for a function $\vec{\phi}_l$. Inserting the ansatz (23) to (20) we obtain the following equation for $\Psi_l(r)$:

$$\left( \Psi''_l - \frac{l(l+1)}{r^2} \Psi_l \right) + \left( \frac{r_0}{r} \right)^4 \left( \Psi''_l - \frac{6}{r} \Psi'_l + \frac{6}{r^2} \Psi_l \right) = 0,$$

(25)

where $\Psi'_l$ stands for $\partial \Psi_l / \partial r$. In the next section we find 2-dimensional space of solution of (25).
3. Exact solution of the “deformed” Laplace equation

Let us note that for \( r \gg r_0 \), equation (20) reduces to the standard Laplace equation with 2 independent \( l \)-pole-like solutions: the one corresponding to the constant \( l \)-pole moment (it behaves like \( r^{l+1} \)) and the external \( l \)-pole solution (behaving like \( r^{-l} \)).

On the other hand, a basis may be chosen corresponding to the behaviour of \( \Psi_l \) at \( r \to 0 \). From the asymptotic analysis of (25) it follows that there is a solution which behaves like \( r^6 \) and another one which behaves like \( r \). Let us define \( \Phi_l := r^{-6} \Psi_l \) and introduce the following variable:

\[
z := -\left( \frac{r}{r_0} \right)^4 .
\]

Using (25) one gets the following equation for \( \Phi_l \):

\[
z (1 - z) \frac{d^2 \Phi_l}{dz^2} + \left( \frac{9}{4} - \frac{15}{4} z \right) \frac{d \Phi_l}{dz} - \frac{30 - l(l + 1)}{16} \Phi_l = 0
\]

(27)

This is a hypergeometric equation (cf. [5]). A hypergeometric equation for a function \( u = u(z) \)

\[
z (1 - z) u'' + \left[ c - (a + b + 1) \right] u' - ab u = 0,
\]

(28)

has two independent solutions. One of them is given by the hypergeometric function \( _2F_1(a, b, c, z) \). The other one has the following form (for \( c \neq 1 \)):

\[
z^{1-c} _2F_1(b - c + 1, a - c + 1, 2 - c, z).
\]

(29)

Therefore, the general solution of (25) reads:

\[
\Psi_l(r) = A_l r^6 _2F_1 \left( \frac{l + 6}{4}, \frac{5 - l}{4}, \frac{9}{4}, -\left( \frac{r}{r_0} \right)^4 \right) + B_l r _2F_1 \left( \frac{l + 1}{4}, -\frac{l}{4}, -\frac{1}{4}, -\left( \frac{r}{r_0} \right)^4 \right).
\]

(30)

The first term on the r.h.s. of (30) behaves at \( r \to 0 \) like \( r^6 \), the other one like \( r \).

Let us note, however, that the second term (behaving like \( r \)) corresponds to the unphysical solution. To see this let us look for the behaviour of the electric field \( \vec{E} \) “produced” by it in the vicinity of the Born-Infeld particle, i. e. for \( r \to 0 \). From (23) and (24) we get

\[
\vec{E}_k = -\partial_k \vec{\phi} = -\frac{1}{r} [x \cdot \mathcal{Q}]_m \left( \delta^m_k - \frac{x^m x_k}{r^2} \right) + l(l+1) \frac{r^3}{4r_0^4} \left( 4 - \frac{1}{r} [x \cdot \mathcal{Q}] x_k + l [x \cdot \mathcal{Q}]_k \right) + O(r^7),
\]

(31)
where

\[ [x \cdot Q] := r^{-l} x^{i_1} \cdots x^{i_l} Q_{i_1 \cdots i_l}, \]

\[ [x \cdot Q]_k := r^{-l+1} x^{i_1} \cdots x^{i_{l+1}} Q_{i_1 \cdots i_{l+1}k}. \]

Therefore, due to the first term in (31), \( \tilde{E} \) exceeds the unperturbed field \( E_0 \) itself.

Moreover, the “perturbation” \( \tilde{E} \) leads to infinite variation of the total field energy. The “electrostatic” energy \( H \) corresponding to electric induction \( D \) is given by (cf. [3], [4]):

\[ H = \int b^2 \left( \sqrt{1 + b^{-2}D^2} - 1 \right) d^3x. \]  

Therefore, its variation reads

\[ \delta H|_{D_0} = \int \frac{D_k^0 \delta D_k}{\sqrt{1 + b^{-2}D_0^2}} d^3x = \int E_k^0 \delta D_k d^3x = \int E_k^0 \tilde{D}_k d^3x. \]  

Now, let us investigate the behaviour of \( E_0 \tilde{D} \) for \( r \to 0 \). From (18) it follows that this expression contains highly singular term which behaves like \( r^{-6} \times \) radial component of \( \tilde{E} \). Using the expansion (31) we see that the first term is purely tangential, i.e. it is orthogonal to any radial vector. However, the second one does contain a radial part \((e/4\pi)r^3l(l + 1)[x \cdot Q]\) and this way \( E_0 \tilde{D} \) produces nonintegrable singularity \( \sim r^{-3}[x \cdot Q] \). Therefore, we conclude that the solution behaving like \( r \) for \( r \to 0 \) has to be excluded from our consideration, i.e. \( B_l = 0 \). This way the solution of (25) is given by

\[ \Psi_l(r) = A_l r^6 2F_1 \left( \frac{l + 6}{4}, \frac{5 - l}{4}, \frac{9}{4}, -\left( \frac{r}{r_0} \right)^4 \right). \]

4. **Multipole moments of the particle**

Knowing the exact solution of (25) we are ready to calculate the \( l \)-th deformability coefficient of the Born-Infeld particle. We know that at infinity, i.e. for \( r \gg r_0 \), \( \Psi_l \) is a combination of 2 solutions: one behaving like \( r^{l+1} \) (it corresponds to the constant \( l \)-pole field) and the other one behaving like \( r^{-l} \) (corresponding to the external \( l \)-pole solution).

Due to the fact that the space of physically admissible solution is only 1-dimensional (see (33)) the proportion between these two solutions is no longer arbitrary. The ratio between them has therefore physical meaning. To find this ratio we shall use the following property of a hypergeometric function \( 2F_1(a, b, c, z) \) (cf. [3]):

\[ 2F_1(a, b, c, z) = \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)}(-z)^{-a} 2F_1 \left( a, a + 1 - c, a + 1 - b, \frac{1}{z} \right) + \]

\[ + \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)}(-z)^{-b} 2F_1 \left( b, b + 1 - c, b + 1 - a, \frac{1}{z} \right), \]  

(37)
where \( \Gamma \) denotes the Euler \( \Gamma \)-function. This identity allows us to find the asymptotic behaviour of \( \Psi_l \) at infinity. Using (36) and (37) one immediately gets:

\[
\Psi_l(r) = X_l r^{l+1} _2 F_1 \left( \begin{array}{c} -\frac{l}{4}, \frac{5-l}{4}, \frac{3-2l}{4}, \left( \frac{r_0}{r} \right)^4 \end{array} \right) + \\
+ Y_l r^{-l} _2 F_1 \left( \begin{array}{c} \frac{l+6}{4}, \frac{l+1}{4}, \frac{5+2l}{4}, \left( \frac{r_0}{r} \right)^4 \end{array} \right),
\]

(38)

where

\[
X_l = A_l \frac{\Gamma(l+6)\Gamma(-\frac{2l+1}{4})}{\Gamma(l+\frac{1}{4})\Gamma(\frac{l+4}{4})} r_0^{5-l},
\]

\[
Y_l = A_l \frac{\Gamma(\frac{l}{4})\Gamma(-\frac{2l+1}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})} r_0^{6+l}.
\]

(39)

(40)

Note, that the first term on the r.h.s. of (38) behaves at infinity like \( r^{l+1} \) and corresponds to the constant \( l \)-pole field \( E_\infty \) given by the formula (2). The second one behaving like \( r^{-l} \) corresponds to the external \( l \)-pole solution \( E_\mathcal{M} \) given by (3). We interpret the ratio between these two ingredients in formula (38)

\[
\kappa_l := \frac{Y_l}{X_l},
\]

(41)

as the “deformability coefficient” (more precisely, the \( l \)-th deformability coefficient) of the Born-Infeld particle. It measures the particle’s \( l \)-pole moment \( \mathcal{M}_{i_1...i_l} \) generated by the constant \( l \)-pole moment \( Q_{i_1...i_l} \) of the external electric field. Using (39) and (40) we obtain:

\[
\kappa_l = \frac{\Gamma(\frac{l+6}{4})\Gamma(l+\frac{1}{4})\Gamma(-\frac{2l+1}{4})}{\Gamma(l+\frac{1}{4})\Gamma(\frac{3}{4})\Gamma(\frac{2l+1}{4})} r_0^{-2l+1}.
\]

(42)

Obviously, in the limit of Maxwell theory \( (r_0 \to 0) \), the external electric field does not generate any particle’s \( l \)-pole moment. Therefore, this mechanism comes entirely from the non-linearity of the Born-Infeld theory.

5. Physical implications

We used in this paper very specific model of non-linear electrodynamics. There is a natural question: why this particular model? It turns out that among other non-linear theories of electromagnetism, Born-Infeld theory possesses very distinguished physical properties [6]. For example it is the only causal spin-1 theory [7] (aside from the Maxwell theory). The assumption, that the theory is effectively non-linear in the vicinity of a charged particle is very natural from the physical point of view. Actually, we have learned this from quantum electrodynamics. There were attempts to identify non-polynomial Born-Infeld lagrangian as an effective Euler-Heisenberg lagrangian [8]. It was shown [3] that the
effective lagrangian can coincide with the \( (3) \) up to six-photon interaction terms. Recently, there is a new interest in the Born-Infeld electrodynamics due to the investigation in the string theory (see e. g. \([9]\)), where \( (3) \) is not postulated but derived.

Now, let us make some comments concerning physical implications of the obtained results. First of all taking \( l = 1 \) one gets

\[
\kappa_1 = -1.85407 \, r_0^3,
\]

which reproduces result obtained in \([2]\). In \([2]\) it was mentioned that for \( l = 1 \) one might describe, this way, the polarizability coefficient of the proton. According to \([11]\), we have \( \kappa_1 = (12.1 \pm 0.9) \times 10^{-4} \) fm\(^3\). To fit this value one has to take \( r_0 \approx 0.09 \) fm. The total mass of the corresponding Born-Infeld unperturbed field accompanying such a particle is about 32 electron masses. We see, that the main part of the proton total mass cannot be of electromagnetic nature and has to be concentrated in the material core of the particle.

Unfortunately, there are no experimental data concerning particles polarizability for \( l \neq 1 \). Probably, it is highly nontrivial task to measure these quantities experimentally. It would be very interesting to have the possibility to compare such quantities with the formula \( (12) \).

Let us note that \( \kappa_3 \) and \( \kappa_5 \) vanish due to \( \Gamma(0) \) in the denominator of \( (12) \). All other \( \kappa_l \neq 0 \). It does not mean that the particle is not polarizable for \( l = 3 \) and \( l = 5 \). It is true in the linear approximation only. However, this result suggests that in these two sectors it is much more difficult to polarize the particle than in the other ones.

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