A NOVEL ALGORITHM FOR APPROXIMATING COMMON SOLUTION OF A SYSTEM OF MONOTONE INCLUSION PROBLEMS AND COMMON FIXED POINT PROBLEM

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(Communicated by Yongzhong Song)

Abstract. In this paper, we study the problem of finding a common element of the set of solutions of a system of monotone inclusion problems and the set of common fixed points of a finite family of generalized demimetric mappings in Hilbert spaces. We propose a new and efficient algorithm for solving this problem. Our method relies on the inertial algorithm, Tseng’s splitting algorithm and the viscosity algorithm. Strong convergence analysis of the proposed method is established under standard and mild conditions. As applications we use our algorithm for finding the common solutions to variational inequality problems, the constrained multiple-set split convex feasibility problem, the convex minimization problem and the common minimizer problem. Finally, we give some numerical results to show that our proposed algorithm is efficient and implementable from the numerical point of view.

1. Introduction. The concept of inclusion problems and fixed point problems has been interesting to many mathematical researchers. The reason is that these problems can be applied to several other problems. For instance, these problems are applicable to solving convexly constrained linear inverse problem, split feasibility problem, convexly constrained minimization problem, variational inequalities and many more. As a result, some applications of such problems are able to be taken into consideration, such as signal processing, computer vision, machine learning, the image restoration problem, sensor networks in computerized tomography and data compression, and intensity modulated radiation therapy treatment planning, see [7, 10, 11, 13, 19, 20, 22, 30, 43].

Let us recall the definition of the monotone inclusion problem. Let $\mathcal{H}$ be a real Hilbert space. Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $A : \mathcal{H} \to \mathcal{H}$

2020 Mathematics Subject Classification. Primary: 47H05, 47H10; Secondary: 65Y05.

Key words and phrases. Inclusion problem, generalized demimetric mappings, Tseng’s splitting algorithm, inertial algorithm.

This research was in part supported by a grant from IPM (No.1400470032).

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be a Lipschitz continuous monotone operator. The monotone inclusion problem is formulated as follows:

\[
\text{Find an element } x^* \in \mathcal{H} \text{ such that } 0 \in (A + B)x^*. \tag{1}
\]

A popular method for solving problem (1) in real Hilbert spaces is the well-known forward-backward splitting method introduced by Passty [41] and Lions and Mercier [31]. The method is formulated as:

\[
x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n, \quad \lambda_n > 0,
\]

under the condition that \( \text{Dom}(B) \subseteq \text{Dom}(A) \). In this setting (2), \( A \) and \( B \) are usually called the forward operator and the backward operator, respectively.

It was shown, see for example [17], that weak convergence of (2) requires quite restrictive assumptions on \( A \) and \( B \), such that the inverse of \( A \) is strongly monotone or \( B \) is Lipschitz continuous and monotone and the operator \( A + B \) is strongly monotone on \( \text{Dom}(B) \). In a direction to weaken these assumptions, Tseng in [54] included an extra step \( y_n \) per each step of (2) and introduced the following modified forward-backward splitting method, also known as Tseng’s splitting algorithm. Let \( C \subseteq \mathcal{H} \) be closed and convex set which intersects the solution set of equation (1). Choose an arbitrary starting point \( x_0 \in C \); given the current iterate \( x_n \), generate the next iterate as via the update rule

\[
y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n, \tag{3}
\]

where \( \lambda_n \) is chosen to be the largest \( \lambda \in \{\delta, \delta l, \delta l^2, \ldots\} \) satisfying

\[
\lambda \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|,
\]

where \( \delta > 0 \), \( l \in (0, 1) \) and \( \mu \in (0, 1) \) are constraints. Set

\[
x_{n+1} = P_C(y_n - \lambda_n(AY_n - Ax_n)). \tag{4}
\]

Tseng’s splitting algorithm (3)-(4) only obtain weak convergence in real Hilbert spaces. Quite recently, Gibali and Thong [25] have obtained strong convergence result by modifying Tseng’s splitting algorithm and Moudafi viscosity algorithm [34] in real Hilbert spaces. They present the following algorithm:

**Algorithm 1**

**Initialization** Choose \( \lambda_1 > 0, \mu \in (0, 1), x_1 \in \mathcal{H} \) and a sequence \( \{\xi_n\} \subset (0, 1) \) such that \( \lim_{n \to \infty} \xi_n = 0 \) and \( \sum_{n=1}^{\infty} \xi_n = \infty \). Assume that \( f \) is a contraction of \( \mathcal{H} \) into itself.

**Step 1:** Compute

\[
y_n = (I + \lambda_n B)^{-1}(x_n - \lambda_n A(x_n))
\]

If \( x_n = y_n \) then stop and return \( y_n \) as a solution. Otherwise go to step 2.

**Step 2:** Compute

\[
z_n = y_n + \lambda_n(A(x_n) - A(y_n)),
\]

and

\[
x_{n+1} = \xi_n f(x_n) + (1 - \xi_n)z_n
\]

and update

\[
\lambda_{n+1} = \begin{cases} 
\min \{ \frac{\mu \|x_n - y_n\|}{\|A(x_n) - A(y_n)\|}, \lambda_n \}, & A(x_n) - A(y_n) \neq 0 \\
\lambda_n, & \text{otherwise}
\end{cases}
\]

\[
(5)
\]
Set $n := n + 1$ and go to step 1.

To accomplish a better algorithm, one can consider the algorithm with the higher rate of convergence. In 1964, Polyak [42] solved the smooth convex minimization problem by the heavy ball method. Next, the modification of this method for solving the smooth convex minimization problem was used to increase the rate of convergence by Nesterov [35] as the following

$$
\begin{align*}
x_0, x_1 &\in \mathcal{H}, \\
y_n &= x_n + \theta_n(x_n - x_{n-1}), \\
x_{n+1} &= y_n - \lambda_n \nabla F(y_n), \quad \forall n \in \mathbb{N},
\end{align*}
$$

(6)

where $\theta_n \in [0, 1)$ is an extrapolation factor and $\lambda_n$ is a step-size parameter (sufficiently small) and $\nabla F$ is the gradient of a smooth convex function $F$. The algorithm is more effective and converges faster because of the term $\theta_n(x_n - x_{n-1})$ in (6) as the inertial step (see for instance [1, 2, 8, 9, 12, 21, 32, 35, 37, 38, 39, 42, 50, 51, 52, 56] and the references therein).

In [32], Lorenz and Pock, proposed a modification of the forward backward splitting algorithm. The scheme is as follows

$$
\begin{align*}
x_0, x_1 &\in \mathcal{H}, \\
y_n &= x_n + \theta_n(x_n - x_{n-1}), \\
x_{n+1} &= (I + \lambda_n B)^{-1}(y_n - \lambda_n A(y_n)), \quad \forall n \in \mathbb{N},
\end{align*}
$$

(7)

where $\theta_n \in [0, 1)$ is an extrapolation factor, $\lambda_n$ is a step-size parameter. They proved that the $\{x_n\}$ generated by (7) converges weakly to a zero of $A + B$.

Let $T : \mathcal{H} \to \mathcal{H}$ be a mapping. Then the set of fixed points of $T$, denoted by $\text{Fix}(T)$, is defined by $\text{Fix}(T) := \{x \in \mathcal{H} : T(x) = x\}$. In 2018, Kawasaki and Takahashi [28] introduced a new general class of mappings, called generalized demimetric mappings, as follows:

**Definition 1.1.** Let $\zeta$ be a real number with $\zeta \neq 0$. A mapping $T : \mathcal{H} \to \mathcal{H}$ with $\text{Fix}(T) \neq \emptyset$ is called generalized demimetric, if

$$
\zeta \langle x - x^*, x - Tx \rangle \geq \|x - Tx\|^2,
$$

for all $x \in \mathcal{H}$ and $x^* \in \text{Fix}(T)$. This mapping $T$ is called $\zeta$-generalized demimetric.

In the case that $\zeta > 0$, the mapping $T$ is called demimetric mapping ([48]). This class of mappings is fundamental because it includes many types of nonlinear mappings arising in applied mathematics and optimization, see [28, 49] for details. In [47], Takahashi proved a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Very recently, Eslamian [23] present an algorithm for finding a common element of the set of common fixed points of a finite family of generalized demimetric mappings and the set of common zero points of a finite family of inverse strongly monotone mappings in Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space. Let for each $i \in \{1, 2, ..., m\}$, $B^{(i)} : \mathcal{H} \to 2^\mathcal{H}$ be a maximal monotone operator, let $A^{(i)} : \mathcal{H} \to \mathcal{H}$ be a monotone and $L^{(i)}$-Lipschitz
continuous operator and let $T^{(i)} : H \to H$ be a $\zeta^{(i)}$-generalized demimetric mapping. In this paper we study the following problem:

Find an element $x^* \in \bigcap_{i=1}^{m} Fix(T^{(i)})$ such that $0 \in \bigcap_{i=1}^{m} (A^{(i)} + B^{(i)})x^*$. \hfill (8)

Inspired by the inertial algorithm, Tseng’s splitting algorithm and the viscosity algorithm, we introduce a new and efficient iterative method for solving the problem (8). Strong convergence theorem of the proposed method is established under standard and mild conditions. As applications, we use our algorithm for finding the common solutions to variational inequality problems, the constrained multiple-set split convex feasibility problem, the convex minimization problem and the common minimizer problem. Finally, we give some numerical results to show that our proposed algorithm is efficient and implementable from the numerical point of view.

2. Preliminaries. We use the following notations in the sequel:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

Given a nonempty closed convex set $C \subset H$, the mapping that assigns every point $x \in H$ to its unique nearest point in $C$ is called the metric projection onto $C$ and is denoted by $P_C$; i.e., $P_C(x) \in C$ and $\|x - P_C(x)\| = \inf_{y \in C}\|x - y\|$. The metric projection $P_C$ is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x) \rangle \leq 0, \quad \forall x \in H, y \in C.$$  

It is known that $P_C$ is 1-generalized demimetric mapping, (see [49]).

We recall the following definitions concerning operator $F : H \to H$.

**Definition 2.1.** The operator $F : H \to H$ is called

- Lipschitz continuous if there exists a constant $L > 0$ such that
  $$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in H.$$
- Contraction if there exists a constant $0 \leq k < 1$ such that
  $$\|F(x) - F(y)\| \leq k\|x - y\|, \quad \forall x, y \in H.$$
- Monotone if
  $$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in H.$$
- Nonexpansive if
  $$\|Fx - Fy\| \leq \|x - y\|, \quad \forall x, y \in H.$$
- Firmly nonexpansive if
  $$\|Fx - Fy\|^2 \leq \|x - y\|^2 - \|(x - Fx) - (y - Fy)\|^2, \quad \forall x, y \in H.$$
- Demicontractive if $Fix(T) \neq \emptyset$ and there exists $\beta \in [0, 1)$ such that
  $$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \beta\|x - Tx\|^2, \quad \forall x \in H, \quad \forall x^* \in Fix(T).$$

This is equivalent to

$$\langle x - x^*, x - Tx \rangle \geq \frac{1 - \beta}{2}\|x - Tx\|^2, \quad \forall x \in H, \quad \forall x^* \in Fix(T).$$

- Strict pseudo-contraction if there exists a constant $\beta \in [0, 1)$ such that
  $$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in H.$$
Note that every $\beta$-strict pseudo-contraction mapping with nonempty fixed point set is $\beta$-demicontractive. Also a $\beta$-demicontractive mapping is $\frac{2}{1+\beta}$-generalized demimetric. We give an example of a generalized demimetric mapping which is not demicontractive.

**Example 1.** Let $H = \mathbb{R}$ be the real line. Define $T$ on $\mathbb{R}$ by $T(x) = 3x - 1$. Clearly, $x^* = \frac{1}{3}$ is the only fixed point of $T$. Then $T$ is $(-2)$-generalized demimetric mapping. Indeed, for each $x \in \mathbb{R}$ we have

$$(-2)(x - \frac{1}{3})(1 - 2x) = \zeta (x - x^*, x - Tx) = \|x - Tx\|^2 = (1 - 2x)^2.$$ 

Putting $x^* = \frac{1}{2}$ and $x = 1$, we see that $T$ is not demicontractive mapping.

**Definition 2.2.** Assume that $T : H \to H$ is a nonlinear mapping with $Fix(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in $H$, the following implication holds:

$$x_n \to x \quad \text{and} \quad (I - T)x_n \to 0 \Rightarrow x \in Fix(T).$$

Let $T : H \to H$ be a $k$-strict pseudo-contraction mapping. Then $I - T$ is demiclosed at zero. Also we know that $T$ is a $\frac{2}{1+k}$-generalized demimetric mapping when $Fix(T) \neq \emptyset$. Note that every nonexpansive mapping is $0$-strict pseudo-contraction. Let $T : H \to H$ be a nonexpansive mapping. Then $T$ is $2$-generalized demimetric and $I - T$ is demiclosed at zero. See [48, 49] for details.

**Lemma 2.3.** [49] Let $H$ be a real Hilbert space and let $\theta$ be a real number with $\theta \neq 0$. Let $T : H \to H$ be a $\theta$-generalized demimetric mapping. Then the fixed point set $Fix(T)$ of $T$ is closed and convex.

Let $B$ be a mapping of $H$ into $2^H$. The effective domain of $B$ is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping $B$ on $H$ is said to be monotone if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$ and $v \in By$. A monotone mapping $B$ on $H$ is said to be maximal if its graph is not properly contained in the graph of any other monotone mapping on $H$. For a maximal monotone mapping $B$ on $H$ and $r > 0$, we may define a single-valued mapping

$$J_r = (I + rB)^{-1} : H \to D(B),$$

which is called the resolvent of $B$ for $r$. Let $B$ be a maximal monotone mapping on $H$ and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent $J_r$ is $1$-generalized demimetric mapping and $B^{-1}0 = Fix(J_r)$ for all $r > 0$; see [46, 49] for more details.

**Lemma 2.4.** [4] Let $B : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous and monotone mapping. Then the mapping $A + B$ is a maximal monotone mapping.

**Lemma 2.5.** [46] Let $B : H \to 2^H$ be a maximal monotone operator and $A : H \to H$ be a mapping on $H$. Define $T_\lambda := (I + \lambda B)^{-1}(I - \lambda A), (\lambda > 0)$, Then we have

$$Fix(T_\lambda) = (A + B)^{-1}(0).$$

**Lemma 2.6.** ([27]) Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that

$$\begin{align*}
 s_{n+1} &\leq (1 - \eta_n)s_n + \eta_n \delta_n, \quad n \geq 0, \\
 s_{n+1} &\leq s_n - \varrho_n + \zeta_n, \quad n \geq 0,
\end{align*}$$

where $\{\eta_n\}$ is a sequence in $(0, 1)$, $\{\varrho_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\zeta_n\}$ are two sequences in $\mathbb{R}$ such that
Proof. Then, the sequence

\[
\{n_k\} \subset \mathbb{N},
\]

implies \(\limsup_{k \to \infty} \delta_{n_k} \leq 0\) for any subsequence \(\{n_k\} \subset \{n\}\).

Then \(\lim_{n \to \infty} s_n = 0\).

3. Algorithm and convergence analysis. Inspired by the discussion proposed in Section 1, in this section we propose a new algorithm for solving the problem (8).

**Theorem 3.1.** Let \(H\) be a Hilbert space. Let for each \(i \in \{1, 2, ..., m\}\), \(B^{(i)} : H \to 2^H\) be a maximal monotone operator and let \(A^{(i)} : H \to H\) be a monotone and \(L^{(i)}\)-Lipschitz continuous operator. Let for each \(i \in \{1, 2, ..., m\}\), \(\zeta^{(i)} \neq 0\) and \(T^{(i)} : H \to H\) be a \(\zeta^{(i)}\)-generalized demimetric mapping such that \(I - T^{(i)}\) is demiclosed at 0. Suppose that \(\Omega = \bigcap_{i=1}^{m} \text{Fix}(T^{(i)}) \cap \bigcap_{i=1}^{m} (A^{(i)} + B^{(i)})^{-1}(0) \neq \emptyset\). Assume that \(\lambda > 0, \gamma^{(i)} \in (0, 1), \lambda_1^{(i)} > 0\) and \(x_1, x_2 \in H\) is two initial points. Let \(\{x_n\}\) be a sequence defined by:

\[
\begin{align*}
\begin{cases}
 w_n = x_n + \alpha_n (x_n - x_{n-1}), \\
 u_n^{(i)} = (I + \lambda_n^{(i)} B^{(i)})^{-1} (w_n - \lambda_n^{(i)} A^{(i)}(w_n)), \\
 v_n^{(i)} = u_n^{(i)} + \lambda_n^{(i)} (A^{(i)}(w_n) - A^{(i)}(u_n^{(i)})), \\
 y_n^{(i)} = v_n^{(i)} + \lambda_n^{(i)} \theta_n^{(i)} (T^{(i)} - I)v_n^{(i)}, \\
 z_n = \sum_{i=1}^{m} d^{(i)} y_n^{(i)}, \\
 x_{n+1} = \xi_n \{z_n\} + (1 - \xi_n) z_n, 
\end{cases}
\end{align*}
\]

where \(d^{(i)} = \frac{\zeta^{(i)}}{[\gamma^{(i)}]}\),

\[
\lambda_{n+1}^{(i)} = \begin{cases}
\min \left\{ \frac{\zeta^{(i)} \left\| w_n - u_n^{(i)} \right\|}{\left\| A^{(i)}(w_n) - A^{(i)}(u_n^{(i)}) \right\|}, \lambda_n^{(i)} \right\}, & \text{if } A^{(i)}(w_n) - A^{(i)}(u_n^{(i)}) \neq 0 \\
\lambda_n^{(i)}, & \text{otherwise},
\end{cases}
\]

and \(0 \leq \alpha_n \leq \overline{\alpha}_n\) such that

\[
\overline{\alpha}_n = \begin{cases}
\min \left\{ \frac{\beta_n}{\left\| x_n - x_{n-1} \right\|}, \alpha \right\}, & x_n - x_{n-1} \neq 0 \\
\alpha, & \text{otherwise}.
\end{cases}
\]

Assume that the sequences \(\{\theta_n^{(i)}\}, \{d^{(i)}\}, \{\xi_n\}\) and \(\{\beta_n\}\) satisfy the following conditions:

(i) \(\{\xi_n\} \subset (0, 1), \lim_{n \to \infty} \xi_n = 0\) and \(\sum_{n=0}^{\infty} \xi_n = \infty\).

(ii) \(d^{(i)} > 0, \sum_{i=1}^{m} d^{(i)} = 1\).

(iii) \(\{\theta_n^{(i)}\} \subset [d^{(i)}, e^{(i)}] \subset (0, 2 \zeta^{(i)})\).

(iv) \(\beta_n > 0\) and \(\lim_{n \to \infty} \frac{\beta_n}{\xi_n} = 0\).

Then, the sequence \(\{x_n\}\) converges strongly to a point \(x^* \in \Omega\).

**Proof.** First we show that \(\{x_n\}\) is bounded. Note that \(P_{\Omega}(f)\) is a contraction of \(H\) into itself. By the Banach contraction principle there exists a unique element \(x^* \in H\) such that \(x^* = P_{\Omega}(f)x^*\).
Since \( T^{(i)} : \mathcal{H} \to \mathcal{H} \) is a \( \zeta^{(i)} \)-demimetric mapping, we get that
\[
\| y_n^{(i)} - x^* \|^2 = \| T^{(i)} \alpha_n \| T^{(i)} v_n^{(i)} - v_n^{(i)} - x^* \|^2 \\
= \| T^{(i)} v_n^{(i)} - x^* \|^2 + 2\langle w_n^{(i)} - x^* , T^{(i)} \alpha_n \rangle T^{(i)} v_n^{(i)} - v_n^{(i)} \rangle \\
+ \| T^{(i)} \alpha_n \| T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2 \\
\leq \| T^{(i)} v_n^{(i)} - x^* \|^2 + 2\langle T^{(i)} \alpha_n \rangle \left( \frac{1}{\zeta^{(i)}} \right) T^{(i)} v_n^{(i)} - v_n^{(i)} \rangle T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2 \\
+ \| T^{(i)} \alpha_n \| T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2 \\
= \| T^{(i)} v_n^{(i)} - x^* \|^2 - \| T^{(i)} \alpha_n \| T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2. \tag{12}
\]

By our assumption that \( \{ \theta_n^{(i)} \} \subset [d^{(i)}, e^{(i)}] \subset (0, 2\varepsilon^{(i)}) \), we have
\[
\| y_n^{(i)} - x^* \| \leq \| v_n^{(i)} - x^* \|.
\]

From [25] we know that the limit of \( \{ \lambda_n^{(i)} \} \) exists, and \( \lim_{n \to \infty} \lambda_n^{(i)} = \lambda^{(i)} > 0 \).
For each \( i \in \{ 1, 2, ..., m \} \), we have (see [25]):
\[
\| v_n^{(i)} - x^* \|^2 \leq \| w_n - x^* \|^2 - (1 - \left( \frac{\gamma^{(i)} \lambda_n^{(i)}}{\lambda^{(i)}_{n+1}} \right) \| w_n - u_n^{(i)} \|^2, \tag{13}
\]
and
\[
\| v_n^{(i)} - u_n^{(i)} \| \leq \frac{\gamma^{(i)} \lambda_n^{(i)}}{\lambda^{(i)}_{n+1}} \| w_n - u_n^{(i)} \|.
\]

From the inequalities (13) and (12) and convexity of \( \| \cdot \|^2 \) we get
\[
\| z_n - x^* \|^2 = \| \sum_{i=1}^m a^{(i)} y_n^{(i)} - x^* \|^2 \leq \sum_{i=1}^m a^{(i)} \| y_n^{(i)} - x^* \|^2 \\
\leq \sum_{i=1}^m a^{(i)} \| v_n^{(i)} - x^* \|^2 - \sum_{i=1}^m a^{(i)} \| T^{(i)} \alpha_n \| T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2 \\
\leq \sum_{i=1}^m a^{(i)} \| w_n - x^* \|^2 - \sum_{i=1}^m a^{(i)} \| T^{(i)} \alpha_n \| T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2 \\
- \sum_{i=1}^m a^{(i)} \| T^{(i)} \alpha_n \| T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2 \\
= \| w_n - x^* \|^2 - \sum_{i=1}^m a^{(i)} \| T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2 \\
\leq \| w_n - x^* \|^2 - \sum_{i=1}^m a^{(i)} \| T^{(i)} v_n^{(i)} - v_n^{(i)} \|^2. \tag{14}
\]

We have \( \alpha_n \| x_n - x_{n-1} \| \leq \beta_n \) for all \( n \), which together with \( \lim_{n \to \infty} \beta_n = 0 \) implies that
\[
\lim_{n \to \infty} \frac{\alpha_n}{\xi_n} \| x_n - x_{n-1} \| = 0. \tag{15}
\]

It follows that there exists a constant \( M_1 > 0 \) such that
\[
\frac{\alpha_n}{\xi_n} \| x_n - x_{n-1} \| \leq M_1.
\]
From the definition of $w_n$ we have
\[ \|w_n - x^*\| = \|x_n + \alpha_n(x_n - x_{n-1}) - x^*\| \]
\[ \leq \|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\| \]
\[ = \|x_n - x^*\| + \xi_n\alpha_n\|x_n - x_{n-1}\| \]
\[ = \|x_n - x^*\| + \xi_nM_1. \]
This implies that
\[ \|z_n - x^*\| \leq \|w_n - x^*\| \leq \|x_n - x^*\| + \xi_nM_1. \tag{16} \]
From the definition of $x_{n+1}$ we have
\[ \|x_{n+1} - x^*\| = \|\xi_n(f(z_n) - x^*) + (1 - \xi_n)(z_n - x^*)\| \]
\[ \leq \xi_n\|f(z_n) - x^*\| + (1 - \xi_n)||z_n - x^*\| \]
\[ \leq \xi_n\|f(z_n) - f(x^*)\| + \xi_n\|f(x^*) - x^*\| + (1 - \xi_n)||z_n - x^*\| \]
\[ \leq \xi_n\|z_n - x^*\| + \xi_n\|f(x^*) - x^*\| + (1 - \xi_n)||z_n - x^*\| \]
\[ \leq (1 - \xi_n(1 - b))\|z_n - x^*\| + \xi_n\|f(x^*) - x^*\|. \]
Hence
\[ \|x_{n+1} - x^*\| \leq (1 - \xi_n(1 - b))\|x_n - x^*\| + \xi_nM_1 + \xi_n\|f(x^*) - x^*\| \]
\[ = (1 - \xi_n(1 - b))\|x_n - x^*\| + (1 - b)\xi_n\frac{M_1 + \|f(x^*) - x^*\|}{(1-b)} \]
\[ \leq \max\{\|x_n - x^*\|, \frac{M_1 + \|f(x^*) - x^*\|}{(1-b)}\}, \]
\[ \leq \cdots \]
\[ \leq \max\{\|x_1 - x^*\|, \frac{M_1 + \|f(x^*) - x^*\|}{(1-b)}\}. \]
This implies that $\{x_n\}$ is bounded. We also get $\{w_n\}$, $\{z_n\}$ and $\{f(z_n)\}$ are bounded.

\[ \|w_n - x^*\|^2 = \|x_n + \alpha_n(x_n - x_{n-1}) - x^*\|^2 \]
\[ \leq \|x_n - x^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - x^*, x_n - x_{n-1}\| \]
\[ \leq \|x_n - x^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - x^*\|\|x_n - x_{n-1}\|. \]
Utilizing inequality $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H$, we arrive at
\[ \|x_{n+1} - x^*\|^2 \]
\[ = \|\xi_n(f(z_n) - x^*) + (1 - \xi_n)(z_n - x^*)\|^2 \]
\[ = \|\xi_n(f(z_n) - f(x^*)) + \xi_n(f(x^*) - x^*) + (1 - \xi_n)(z_n - x^*)\|^2 \]
\[ \leq \|\xi_n(f(z_n) - f(x^*)) + (1 - \xi_n)(z_n - x^*)\|^2 \]
\[ \leq \xi_n\|f(z_n) - f(x^*)\|^2 + (1 - \xi_n)||z_n - x^*\|^2 + 2\xi_n\|f(x^*) - x^*, x_{n+1} - x^*\| \]
\[ \leq \xi_n\|z_n - x^*\|^2 + (1 - \xi_n)||z_n - x^*\|^2 + 2\xi_n\|f(x^*) - x^*, x_{n+1} - x^*\| \]
\[ \leq (1 - (1 - b)\xi_n)||z_n - x^*\|^2 + 2\xi_n\|f(x^*) - x^*, x_{n+1} - x^*\| \]
\[ \leq (1 - (1 - b)\xi_n)||w_n - x^*\|^2 + 2\xi_n\|f(x^*) - x^*, x_{n+1} - x^*\|. \]
It yields
\[
\|x_{n+1} - x^*\|^2 \leq (1 - (1 - b)\xi_n)\|x_n - x^*\|^2 + 2\xi_n (f(x^*) - x^*, x_{n+1} - x^*) \\
\leq \alpha_n \|x_n - x_{n-1}\|^2 + 2\alpha_n \|x_n - x^*\| \|x_n - x_{n-1}\| \\
= (1 - (1 - b)\xi_n)\|x_n - x^*\|^2 + (1 - b)\xi_n \frac{2}{(1 - b)} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
+ \alpha_n \|x_n - x_{n-1}\| \left( \alpha_n \|x_n - x_{n-1}\| + 2\|x_n - x^*\| \right) \\
\leq (1 - (1 - b)\xi_n)\|x_n - x^*\|^2 + (1 - b)\xi_n \frac{2}{(1 - b)} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
+ 3M_2\alpha_n \|x_n - x_{n-1}\| \\
= (1 - (1 - b)\xi_n)\|x_n - x^*\|^2 \\
+ (1 - b)\xi_n \left[ \frac{2}{(1 - b)} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle + \frac{3M_2\alpha_n}{1 - b} \|x_n - x_{n-1}\| \right] \\
= (1 - \eta_n)\|x_n - x^*\|^2 + \eta_n \delta_n,
\]

where
\[
\delta_n = \frac{2}{(1 - b)} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle + \frac{3M_2\alpha_n}{1 - b} \xi_n \|x_n - x_{n-1}\|,
\]
\[
M_2 = \sup_{n \in \mathbb{N}} \{\|x_n - x^*\|, \alpha_n \|x_n - x_{n-1}\|\}, \quad \eta_n = (1 - b)\xi_n.
\]
We observe that
\[
\eta_n \to 0, \quad \sum_{n=1}^{\infty} \eta_n = \infty.
\]

Since \(\{z_n\}\) is bounded, there exists a constant \(M_3 > 0\) such that
\[
2\|z_n - x^*\|.\|f(x^*) - x^*\| + \|f(x^*) - x^*\|^2 \leq M_3.
\]

This implies that
\[
\|x_{n+1} - x^*\|^2 = \|\xi_n (f(z_n) - x^*) + (1 - \xi_n)(z_n - x^*)\|^2 \\
\leq \xi_n \|f(z_n) - f(x^*)\|^2 + \|f(x^*) - x^*\|^2 + (1 - \xi_n)\|z_n - x^*\|^2 \\
\leq \xi_n \|z_n - x^*\|^2 + \|f(x^*) - x^*\|^2 + (1 - \xi_n)\|z_n - x^*\|^2 \\
= \xi_n\|z_n - x^*\|^2 + \|f(x^*) - x^*\|^2 \\
+ \xi_n \|2\|z_n - x^*\|.\|f(x^*) - x^*\| + \|f(x^*) - x^*\|^2 \rangle \\
= \xi_n\|z_n - x^*\|^2 + \|f(x^*) - x^*\|^2 + \xi_n M_3 \\
= \|z_n - x^*\|^2 + \xi_n M_3.
\]

From above inequality and the inequality (14) we have
\[
\|x_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 - \sum_{i=1}^{m} a^{(i)} (1 - \frac{\gamma^{(i)} \lambda^{(i)}}{\lambda^{(i)} + 1}) \|w_n - u_n^{(i)}\|^2 \\
- \sum_{i=1}^{m} a^{(i)} \theta_n^{(i)} \left( \frac{2\theta_n^{(i)}}{\zeta_n^{(i)}} - \theta_n^{(i)} \right) \|T^{(i)} v_n^{(i)} - v_n^{(i)}\|^2 + \xi_n M_3.
\]

Since \(\{x_n\}\) is bounded, there exists a constant \(M_4 > 0\) such that \(2M_1\|x_n - x^*\| + \xi_n M_2^2 \leq M_4\). Therefore
\[\|w_n - x^*\|^2 = (\|x_n - x^*\| + \xi_n M_1)^2 = \|x_n - x^*\|^2 + \xi_n (2M_1 \|x_n - x^*\| + \xi_n M_1^2) \leq \|x_n - x^*\|^2 + \xi_n M_4.\]

Hence from inequality (18) we get
\[\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \xi_n M_4 - \sum_{i=1}^{m} a(i) (1 - \frac{\lambda(i) \lambda_n(i)}{\lambda_n^{(i)}(i)}) \|w_n - u_n^{(i)}\|^2 \]
\[\quad - \sum_{i=1}^{m} a(i) \theta_n(i) \left( \frac{2\lambda(i)}{\Theta(i)} - \theta_n(i) \right) \|T(i) v_n^{(i)} - v_n^{(i)}\|^2 + \xi_n M_3. \quad (19)\]

Now by setting
\[\theta_n = \sum_{i=1}^{m} a(i) (1 - \frac{\lambda(i) \lambda_n(i)}{\lambda_n^{(i)}(i)}) \|w_n - u_n^{(i)}\|^2 \]
\[\quad + \sum_{i=1}^{m} a(i) \theta_n(i) \left( \frac{2\lambda(i)}{\Theta(i)} - \theta_n(i) \right) \|T(i) v_n^{(i)} - v_n^{(i)}\|^2,\]
and
\[\zeta_n = \beta_n (M_3 + M_4), \quad s_n = \|x_n - x^*\|^2, \quad (20)\]

the inequality (19) can be rewritten in the following form:
\[s_{n+1} \leq s_n - \theta_n + \zeta_n. \quad (21)\]

To use Lemma 2.6, (considering inequalities (17) and (21)), it suffices to verify that, for all subsequences \(\{n_k\} \subset \{n\}, \lim_{k \to \infty} \theta_{n_k} = 0\) implies
\[\lim_{k \to \infty} \delta_{n_k} \leq 0.\]

We assume that \(\lim_{k \to \infty} \theta_{n_k} = 0\). By our assumption we get
\[\lim_{k \to \infty} \|T(i) v_{n_k}^{(i)} - v_{n_k}^{(i)}\| = \lim_{k \to \infty} \|u_{n_k}^{(i)} - w_{n_k}\| = 0, \quad i = 1, 2, ..., m. \quad (22)\]

From inequality
\[\|v_{n_k}^{(i)} - u_{n_k}^{(i)}\| \leq \frac{\theta(i) \lambda_n(i)}{\lambda_n^{(i)}} \|w_n - u_n^{(i)}\|,\]
we get
\[\lim_{k \to \infty} \|v_{n_k}^{(i)} - u_{n_k}^{(i)}\| = 0.\]

Note that
\[\|x_n - w_n\| = \alpha_n \|x_n - x_{n-1}\| = \xi_n \frac{\alpha_n}{\xi_n} \|x_n - x_{n-1}\| \to 0.\]

From inequality
\[\|v_{n_k}^{(i)} - x_{n_k}\| \leq \|v_{n_k}^{(i)} - u_{n_k}^{(i)}\| + \|u_{n_k}^{(i)} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\|,\]
we arrive at
\[\lim_{k \to \infty} \|v_{n_k}^{(i)} - x_{n_k}\| = 0, \quad i = 1, 2, ..., m.\]

Since \(\{x_{n_k}\}\) is bounded, there exists a subsequence \(\{x_{n_{k_j}}\}\) of \(\{x_{n_k}\}\) which converges weakly to \(\tilde{x}\). Without loss of generality, we can assume that \(x_{n_k} \rightharpoonup \tilde{x}\). Since \(\lim_{k \to \infty} \|v_{n_k}^{(i)} - x_{n_k}\| = 0\), we have \(v_{n_k}^{(i)} \rightharpoonup \tilde{x}\). Now by similar proof as Lemma 7 in
we obtain that ̂x ∈ ∩_{i=1}^{m}(A(i) + B(i))^{-1}(0). From (22) and the demiclosedness of I - T(i) we have ̂x ∈ ∩_{i=1}^{m}Fix(T(i)). Thus ̂x ∈ Ω. Now we show that
\[ \limsup_{k \to \infty} \langle \hat{f}(x^*) - x^*, x_{nk} - x^* \rangle \leq 0. \] (23)

To show this inequality, we choose a subsequence \{x_{nk}\} of \{x_n\} such that
\[ \lim_{j \to \infty} \langle \hat{f}(x^*) - x^*, x_{nk} - x^* \rangle = \limsup_{k \to \infty} \langle \hat{f}(x^*) - x^*, x_{nk} - x^* \rangle. \]

Since \{x_{nk}\} converges weakly to ̂x ∈ Ω and x* = PΩ(\hat{f}(x*)), by the variational characterization of the metric projection, we have
\[ \limsup_{k \to \infty} \langle \hat{f}(x^*) - x^*, x_{nk} - x^* \rangle = \lim_{j \to \infty} \langle \hat{f}(x^*) - x^*, x_{nk} - x^* \rangle = \langle \hat{f}(x^*) - x^*, ̂x - x^* \rangle \leq 0. \] (24)

From relations (15) and (24) we get
\[ \limsup_{k \to \infty} \delta_{nk} \leq 0. \]

Hence, all conditions of Lemma 2.6 are satisfied. Therefore, we immediately deduce that \(\lim_{n \to \infty} s_n = \lim_{n \to \infty} \|x_n - x^*\|^2 = 0\), that is \{x_n\} converges strongly to \(x^* = PΩ(\hat{f}(x*))\), which completes the proof. □

Remark 1. Our results extend and improve the results of Gibali and Thong [25] from finding a solution of a monotone inclusion problem to a solution of a system of monotone inclusion problem.

4. Applications. In this section we present some applications of our main result.

4.1. Common solutions to variational inequality problems. Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\) and \(A : H \to H\) be an operator. The classical variational inequality problem (VIP) is formulated as follows:

Find an element \(x^* \in C\) such that \(\langle A(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C.\) (25)

The set of solutions of this problem is denoted by \(VI(C, A).\) It is well known that the VIP is a fundamental problem in optimization theory and nonlinear analysis. It is a useful mathematical model which unifies many important concepts in applied mathematics, such as necessary optimality conditions, network equilibrium problems, complementarity problems and systems of nonlinear equations, for instance [3, 26, 29, 55].

Let \(H\) be a Hilbert space, and let \(h\) be a proper lower semicontinuous convex function of \(H\) into \(R.\) Then the subdifferential \(\partial h\) of \(h\) is defined as follows:
\[ \partial h(x) = \{ z \in H : h(x) + \langle z, u - x \rangle \leq h(u), \quad \forall u \in H \} \]

for all \(x \in H.\) From Rockafellar [44], we know that \(\partial h\) is a maximal monotone operator. Let \(C\) be a nonempty closed convex subset of \(H\), then the indicator function \(i_C\) of \(C,\) is defined as
\[ i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases} \] (26)

Further, the normal cone \(N_Cu\) of \(C\) at \(u \in C\) is defined as
\[ N_Cu = \{ z \in H : \langle z, v - u \rangle \leq 0 : \quad \forall v \in C \}. \]
It is known that $i_C$ is a proper lower semicontinuous convex function on $H$. So, we can define the resolvent operator $J_r^{i_C}$ of $i_C$ for $r > 0$, i.e.,

$$J_r^{i_C}(x) = (I + r i_C)^{-1}(x), \quad x \in H.$$ 

It is easily seen, that for each $x \in C$, $\partial i_C x = N_C x$. We know that $J_r^{i_C}(x) = P_C x$ for all $x \in H$ and $r > 0$. Moreover, for the single valued operator $A : H \to H$ we have (see [45])

$$x \in (\partial i_C + A)^{-1}(0) \Leftrightarrow x \in VI(C, A).$$

Now as application of our main result, we obtain the following theorem for finding a common element of the set of solutions of a system of variational inequality problems and the set of common fixed points of a finite family of generalized demimetric mappings in Hilbert spaces.

**Theorem 4.1.** Let $H$ be a Hilbert space. Let for each $i = 1, 2, ..., m$, $C^{(i)}$ be a nonempty closed convex subset of $H$ and let $A^{(i)} : H \to H$ be a monotone and $L^{(i)}$-Lipschitz continuous operator. Let for each $i \in \{1, 2, ..., m\}$, $\zeta^{(i)} \neq 0$ and $T^{(i)} : H \to H$ be a $\zeta^{(i)}$-generalized demimetric mapping such that $I - T^{(i)}$ is demiclosed at 0. Suppose that $\Omega = \bigcap_{i=1}^m Fix(T^{(i)}) \cap \bigcap_{i=1}^m VI(C^{(i)}, A^{(i)}) \neq \emptyset$. Assume that $f$ is a contraction of $H$ into itself with constant $b \in [0, 1)$. Let $\{x_n\}$ be a sequence defined by:

$$
\begin{align*}
\alpha > 0, \quad \gamma^{(i)} \in (0, 1), \lambda_{n+1}^{(i)} > 0,
\alpha_n, x_0 & \in H \text{ is chosen arbitrarily}, \\
w_n &= x_n + \alpha_n(x_n - x_{n-1}), \\
u_n^{(i)} &= P_{C^{(i)}}(w_n - \lambda_n^{(i)} A^{(i)}(w_n)), \\
v_n^{(i)} &= u_n^{(i)} + \lambda_n^{(i)}(A^{(i)}(w_n) - A^{(i)}(u_n^{(i)})), \\
y_n^{(i)} &= v_n^{(i)} + \bar{f}^{(i)}(T^{(i)} - I)v_n^{(i)} \quad i \in \{1, 2, ..., m\}, \\
z_n &= \sum_{i=1}^m \alpha_n y_n^{(i)}, \\
x_{n+1} &= \xi_n(z_n) + (1 - \xi_n)z_n, \quad n \geq 1,
\end{align*}
$$

where $\bar{f}^{(i)} = \frac{\zeta^{(i)}}{|C^{(i)}|}$,

$$\lambda_{n+1}^{(i)} = \begin{cases} 
\min \left\{ \gamma^{(i)} \frac{\|w_n - v_n^{(i)}\|}{\|A^{(i)}(w_n) - A^{(i)}(u_n^{(i)})\|}, \lambda_n^{(i)} \right\}, & A^{(i)}(w_n) - A^{(i)}(u_n^{(i)}) \neq 0 \\
\lambda_n^{(i)}, & \text{otherwise},
\end{cases}$$

(27)

and $0 \leq \alpha_n \leq \overline{\alpha}_n$ such that

$$\overline{\alpha}_n = \begin{cases} 
\min \left\{ \frac{\beta_n}{\|x_n - x_{n-1}\|}, \alpha \right\}, & x_n \neq x_{n-1} \\
\alpha, & \text{otherwise}.
\end{cases}$$

(28)

Assume that the sequences $\{\theta_n^{(i)}\}$, $\{a^{(i)}\}$, $\{\xi_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\xi_n\} \subset (0, 1)$, $\lim_{n \to \infty} \xi_n = 0$ and $\sum_{n=0}^{\infty} \xi_n = \infty$.

(ii) $a^{(i)} > 0$, $\sum_{i=1}^m a^{(i)} = 1$.

(iii) $\{\theta_n^{(i)}\} \subset [d^{(i)}, e^{(i)}] \subset (0, \frac{2\bar{f}^{(i)}}{\zeta^{(i)}})$.

(iv) $\beta_n > 0$ and $\lim_{n \to \infty} \frac{\beta_n}{\xi_n} = 0$. 

Then, the sequence \( \{x_n\} \) converges strongly to a point \( x^* \in \Omega \).

**Remark 2.** Putting \( T^{(i)} = I \) in Theorem 4.1, we obtain a new method for solving common solutions to variational inequality problems involving monotone and Lipschitz continuous operators.

**Remark 3.** Our results extend and improve the results of Takahashi [47] in the following ways:

(i) From solving variational inequality problem for inverse strongly monotone mappings to monotone Lipschitz continuous mappings.

(ii) From approximating common fixed points of a finite family of demimetric mappings to generalized demimetric mappings.

### 4.2. Convex minimization problem.

For a convex differentiable function \( \Phi : H \to \mathbb{R} \) and a proper convex lower semi-continuous function \( \Psi : H \to (-\infty, +\infty] \), the convex minimization problem is to seek a point \( x^* \in H \) such that

\[
\Phi(x^*) + \Psi(x^*) = \min_{x \in H} \{\Phi(x) + \Psi(x)\}.
\]  

(30)

If \( \nabla \Phi \) and \( \partial \Psi \) represent gradient of \( \Phi \) and subdifferential of \( \Psi \), respectively, then Fermat’s rule ensures the equivalence of problem (30) to the problem of finding a point \( x^* \in H \) such that

\[
0 \in \nabla \Phi(x^*) + \partial \Psi(x^*).
\]

One particular example of this case is the so-called basis pursuit denoising problem [18]:

\[
\min_{x \in H} \frac{1}{2} \|Tx - b\|^2 + \lambda \|x\|_1,
\]

where \( T : H_1 \to H_2 \) is a bounded linear operator, \( b \in H_2 \) and \( \lambda > 0 \). This problem with the \( \ell_1 \)-penalty is relative to a technique, called the LASSO, for least absolute shrinkage and selection operator [53].

Now as application of our main result we obtain the following theorem.

**Theorem 4.2.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( \Phi : H \to \mathbb{R} \) be a convex differentiable function such that its gradient \( \nabla \Phi \) Lipschitz continuous and let \( \Psi : H \to (-\infty, +\infty] \) be a proper function with convexity and lower semi-continuity. Suppose that \( \Omega = \{x \in C : x = \text{argmin}_{z \in H} \Phi(z) + \Psi(z)\} \neq \emptyset \). Assume that \( f \) is a contraction of \( H \) into itself with constant \( b \in [0, 1) \). Let \( \{x_n\} \) be a sequence defined by:

\[
\alpha > 0, \quad \gamma \in (0, 1), \lambda_1 > 0,
\]

\[
x_1, x_0 \in H \text{ is chosen arbitrarily},
\]

\[
w_n = x_n + \alpha_n(x_n - x_{n-1}),
\]

\[
u_n = u_n - \lambda_n \nabla \Phi(u_n),
\]

\[
s_n = \text{argmin}_{z \in H} \{\Psi(z) + \frac{1}{2\lambda_n} \|u_n - z\|^2\},
\]

\[
v_n = s_n + \lambda_n (\nabla \Phi(u_n) - \nabla \Phi(s_n)),
\]

\[
y_n = P_C(v_n),
\]

\[
x_{n+1} = \xi_n f(y_n) + (1 - \xi_n)y_n, \quad n \geq 1,
\]

(31)
Assume that the sequences \( \{\xi_n\} \) and \( \{\beta_n\} \) satisfy the following conditions:

1. \( \{\xi_n\} \subset (0, 1) \), \( \lim_{n \to \infty} \xi_n = 0 \) and \( \sum_{n=0}^{\infty} \xi_n = \infty \).
2. \( \beta_n > 0 \) and \( \lim_{n \to \infty} \frac{\beta_n}{\xi_n} = 0 \).

Then, the sequence \( \{x_n\} \) converges strongly to a point \( x^* \in \Omega \).

Proof. We know that the subdifferential mapping \( \partial \Psi \) of a proper, convex and lower semi-continuous function \( \Psi \) is maximal monotone. Also, we have

\[
\begin{align*}
    s_n &= (I + \lambda_n \partial \Psi)^{-1} u_n = \arg\min_{z \in \mathcal{H}} \{ \Psi(z) + \frac{1}{2\lambda_n} \|u_n - z\|^2 \}.
\end{align*}
\]

Putting \( T := P_C \), we have that \( T \) is 1-generalized demimetric mapping. Now setting \( \theta_n = 1 \), we obtain the desired result from Theorem 3.1. \( \square \)

4.3. The common minimizer problem. Another problem which can be seen as a special case of the problem (8) is the Common Minimizer Problem (CMP).

Given \( m \) nonempty, closed and convex subsets \( C^{(i)} \subset \mathcal{H} \), with \( \bigcap_{i=1}^{m} C^{(i)} \neq \emptyset \), and functions \( \Phi^{(i)}, i = 1, 2, \ldots, m \), that are continuously differentiable and convex on \( C^{(i)} \), respectively, the CMP [16] is to find a point \( x^* \) so that

\[
    x^* = \bigcap_{i=1}^{m} C^{(i)} \quad \text{and} \quad x^* = \arg\min_{x \in C^{(i)}} \{ \Phi^{(i)}(x) : x \in C^{(i)} \}, \quad \forall i = 1, 2, \ldots, m.
\]

By choosing \( A^{(i)} = \nabla \Phi^{(i)} \), \( B^{(i)} = \partial i_{C^{(i)}} \), \( T^{(i)} = P_{C^{(i)}} \), we see the (CMP) is a special case of the problem (8).

4.4. The multiple-set split feasibility problem. Let \( \mathcal{H} \) and \( \mathcal{K} \) be real Hilbert spaces, \( S : \mathcal{H} \to \mathcal{K} \) be a bounded linear operator and let \( \{C^{(i)}\}_{i=1}^{p} \) be a family of nonempty closed convex subsets in \( \mathcal{H} \) and \( \{Q^{(j)}\}_{j=1}^{r} \) be a family of nonempty closed convex subsets in \( \mathcal{K} \). The multiple-set split feasibility problem (MSSFP) was introduced by Censor et al. (2005) [15] and is formulated as finding a point \( x^* \) with the property:

\[
    x^* \in \bigcap_{i=1}^{p} C^{(i)} \quad \text{and} \quad Sx^* \in \bigcap_{j=1}^{r} Q^{(j)}.
\]

Masad and Reich [33] studied the constrained multiple-set split convex feasibility problem (CMSSCFP). Let \( S^{(j)} : \mathcal{H} \to \mathcal{K}, j = 1, 2, \ldots, r \), be \( r \) bounded linear operators and let \( \Omega \) be another closed and convex subset of \( \mathcal{H} \). The CMSSCFP is formulated as follows:

find a point \( x^* \in \Omega \) such that

\[
    x^* \in \bigcap_{i=1}^{p} C^{(i)} \quad \text{and} \quad S^{(j)}(x^*) \in Q^{(j)} \quad \text{for each} \quad j = 1, 2, \ldots, r.
\]
The multiple-set split feasibility problem with \( p = r = 1 \) is known as the split feasibility problem (SFP) which is formulated as finding a point \( x^\ast \) with the property:

\[
x^\ast \in C \text{ and } Sx^\ast \in Q,
\]

where \( C \) and \( Q \) are nonempty closed convex subsets of \( H \) and \( K \), respectively. The split feasibility problem was introduced by Censor and Elfving (1994) ([14]). It has attracted many authors attention due to its application in optimization problem and signal processing, ( see [5, 6, 24]).

Take

\[
A(j)x := \nabla \left( \frac{1}{2} \| S(j)x - P_{Q(j)}S(j)x \|^2 \right) = (S(j))^*(I - P_{Q(j)})S(j)x, \quad (S(j))^* \text{ is the adjoint of } S(j) \quad \text{ and } \quad B(j) = \partial i_{C(j)}, \quad j = 1, 2, ..., r \quad (\text{the indicator function }) .
\]

Then

\[
A(j) \text{ is monotone and Lipschitz continues with constant } L(j) = \| S(j) \|^2 \text{ and } B(j) \text{ is maximal monotone.}
\]

So the constrained multiple-set split convex feasibility problem is a especial case of the problem (8).

5. Numerical experiments. In this section, we provide three computational experiments. Example 5.1 is a multiple-set split feasibility problem, Example 5.2 is a convex minimization problem and Example 5.3 is a signal processing problem. In order to illustrate the efficiency of the new method proposed in Section 3, we compare it with Algorithm 1 presented by Gibali and Thong [25]. All codes were written in MATLAB R2014a and performed on a laptop with Core i7 Processor and 12GB RAM.

Example 5.1. Let \( \mathcal{H} := L^2([0,1]) \) with norm \( \| x \| := \left( \int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}} \) and inner product \( \langle x, y \rangle := \int_0^1 x(t)y(t)dt, \ x, y \in L^2([0,1]). \) Furthermore, let us take

\[
C := \{ x \in L^2([0,1]) : \langle a, x \rangle \leq b \},
\]

where \( 0 \neq a \in L^2([0,1]) \) and \( b \in \mathbb{R} \), then (see, for example, [7])

\[
P_C(x) = \begin{cases} \frac{b-\langle a, x \rangle}{\| a \|_{L^2}} a + x, & \langle a, x \rangle > b \\ x, & \langle a, x \rangle \leq b. \end{cases}
\]

Let

\[
Q = \{ x \in L^2([0,1]) : \| x - d \|_{L^2} \leq r \}
\]

be a closed ball centered at \( d \in L^2([0,1]) \) with radius \( r > 0 \). Then

\[
P_Q(x) = \begin{cases} d + r \frac{x - d}{\| x - d \|}, & x \notin Q \\ x, & x \in Q. \end{cases}
\]

Suppose that \( S : L^2([0,1]) \to L^2([0,1]) \) is defined by

\[
(Sx)(t) = \int_0^t x(s)ds, \quad \forall x \in L^2([0,1]).
\]

It can be easily shown that \( S \) is a bounded linear operator with \( \| S \| = \frac{2}{\pi} \) and the adjoint \( S^* \) of \( S \) is defined by

\[
(S^*x)(t) = \int_t^1 x(s)ds, \quad \forall x \in L^2([0,1]).
\]
We take $m=2$ in Theorem 3.1. Now, suppose

$$C^{(1)} := \{ x \in L^2([0,1]) : \int_0^1 (t^2 + 1)x(t)dt \leq 1 \}$$
$$C^{(2)} := \{ x \in L^2([0,1]) : \int_0^1 |x(t) - \sin t|^2 dt \leq 16 \}$$
$$Q^{(1)} := \{ x \in L^2([0,1]) : ||x||_{L^2} \leq 2 \}$$
$$Q^{(2)} := \{ x \in L^2([0,1]) : ||x||_{L^2} \leq 1 \}$$
$$C := \{ x \in L^2([0,1]) : \int_0^1 tx(t)dt \leq 1 \}.$$ 

Assume that $T^{(1)}x(t) := tx(t)$, $T^{(2)} := P_C$ and $f(x) := \frac{x}{2}$ for $x \in L^2([0,1])$. Then $T^{(1)}$ is 2-generalized demimetric mapping, $T^{(2)}$ is 1-generalized demimetric mapping and $f$ is a contraction. Put $(S^{(1)}x)(t) = (S^{(2)}x)(t) = \int_0^t x(s)ds$, $\forall x \in L^2([0,1])$, $A^{(j)}x := \nabla \left( \frac{1}{2} ||S^{(j)}x - P_{Q^{(j)}}S^{(j)}x||^2 \right) = (S^{(j)})^* (I - P_{Q^{(j)}})S^{(j)}x$ and $B^{(j)} = \partial_i C^{(j)}$ $j = 1,2$. Take $\theta^{(1)}_n = \frac{n+1}{2(n+2)}$, $\theta^{(2)}_n = 1$, $\xi_n = \frac{1}{n+5}$, $\beta_n = \frac{1}{(n+1)(n+2)}$, $\alpha = 0.55$. Also for $i = 1,2$, we take $a^{(i)} = \frac{1}{2}$, $\lambda_1^{(i)} = 0.01$, $\gamma^{(i)} = \frac{1}{2}$. We use stopping criterion $||e_n||_2 := ||x_{n+1} - x_n||_2 \leq \varepsilon$, where the tolerance is given as $\varepsilon = 10^{-5}$.

**Example 5.2.** Finding the solution of convex minimization problem:

$$\min_{x \in \mathbb{R}^5} \frac{||x||_2^2}{10} + \frac{1}{2} \langle Ux, x \rangle + 0.5.$$  \hspace{1cm} (35)

We consider $\mathcal{H} = \mathbb{R}^5$. We set $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ and $\Psi : \mathcal{H} \rightarrow (-\infty, +\infty]$ by

$$\Phi(x) = \frac{||x||_2^2}{10} + 0.5, \quad \Psi(x) = \frac{1}{2} \langle Ux, x \rangle,$$

where

$$U = \begin{pmatrix} 2 & -2 & -2 & -2 & -2 \\ -2 & 2 & 2 & 2 & 2 \\ -2 & 2 & 2 & 2 & 2 \\ -2 & 2 & 2 & 2 & 2 \\ -2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$ 

The convexity and differentiability of $\Phi$ can be easily shown. Also for all $x \in \mathcal{H}$, $\nabla \Phi(x) = \xi$ is $\frac{1}{2}$-Lipschitz continuous and monotone. Further, $\Psi$ can be easily proved proper, convex and continuous function. From [40] we know that, (for $r > 0$)

$$\arg \min_{z \in \mathcal{H}} \{ \Psi(z) + \frac{1}{2r} ||z - x||^2 \} = (1 + rU)^{-1}(x), \quad x \in \mathcal{H}.$$ 

For this example, the parameters and the algorithms of the problems is considered as: $\alpha = 0.35$, $\gamma = 0.5$, $x_1 = x_0$, $\xi_n = \frac{1}{n+1}$ and $\beta_n = \frac{10}{(n+1)(n+2)}$, $\lambda_1 = 2$ and the tolerance is $10^{-8}$.

**Example 5.3.** The signal recovery problems can be represented as solving the following systems of linear equations

$$b = Tx + \mathcal{N},$$  \hspace{1cm} (36)
Table 2. Numerical results of comparison of the new algorithm and the Algorithm 1 for Example 5.2

| Starting point(s) | Algorithm 1 | The new algorithm |
|-------------------|-------------|-------------------|
|                   | No. iterations | CPU time | No. iterations | CPU time |
| $[-5, 4, -3, 2, -1]$ | 63 | 0.0133 | 31 | 0.0025 |
| $[-50, 40, -30, 20, -1]$ | 72 | 0.0263 | 35 | 0.0093 |
| $[5, 4, 3, 2, 1]$ | 64 | 0.0146 | 31 | 0.0120 |
| $[50, 40, 30, 20, 10]$ | 72 | 0.0150 | 29 | 0.0095 |

Figure 1. The graph of the error $\|x_n - x_{n-1}\|_2$

| n | $x_n$ | $\|x_n - x_{n-1}\|_2$ |
|---|------|-------------------|
| 0 | [-5.000, 4.0000, -3.0000, 2.0000, -1.0000] | – |
| 1 | [-5.000, 4.0000, -3.0000, 2.0000, -1.0000] | – |
| 2 | [-6.3896, 5.0620, -4.2310, 2.4069, -1.5759] | 2.2520e+00 |
| 8 | [-0.0295, 0.0190, -0.0547, -0.0021, -0.0336] | 1.8203e-01 |
| 14 | [0.0019, -0.0014, 0.0024, -0.0003, 0.0013] | 2.6378e-03 |
| 20 | [0.0000, -0.0000, 0.0000, -0.0000, 0.0000] | 5.5509e-05 |
| 22 | [0.0000, -0.0000, 0.0000, -0.0000, 0.0000] | 8.9023e-06 |

Table 3. The details of iterations of the new algorithm for Example 5.2
where \( x \in \mathbb{R}^n \) is the true signal that need to be recovered, \( b \in \mathbb{R}^m \) is the observed signal with noise \( \mathcal{N} \in \mathbb{R}^m \) and \( T \in \mathbb{R}^{mn} \) is a linear operator. If the modeling of the phenomenon assumes that \( x \) vector is sparse (i.e., the vector has many zero components) the problem (36) can be formulated as:

\[
\min_{x \in \mathbb{R}^n} \frac{1}{2} \| b - Tx \|_2^2 + r \| x \|_1,
\]

where \( r \) is a positive parameter that sets a tradeoff between error and sparsity and is chosen based on empirical considerations.

Therefore, we use our Theorem 4.3 to solve (37). We set \( \Phi(x) = \frac{1}{2} \| b - Tx \|_2^2 \), \( \Psi(x) = r \| x \|_1 \). In our experiment, the size of signal is selected to be \( n = 512 \), where the original signal is a random sparse signal with \( p\% \) nonzero elements. Let \( m = 256 \) and the matrix \( T \) be the Gaussian matrix generated by command \( \text{randn}(m, n) \). The parameters and sequences of the algorithms is considered as: \( \alpha = 0.9 \), \( \gamma = 0.5 \), \( x_1 = x_0 = \text{zeros}(512, 1) \), \( \xi_n = \frac{0.001}{n+1} \) and \( \beta_n = \frac{10}{(n+1)^{1.5}} \), \( \lambda_1 = 7.55 \) and the tolerance \( \| x - y \| < 10^{-5} \) is considered to the equality \( x = y \). The regularization parameter is also estimated by \( r = \frac{5 \| b \|_1}{\| T \|_2} \).
Remark 4. The reported numerical results indicate the efficiency and robustness of the new algorithm for finding common element of the set of solutions of a system of monotone inclusion problems and the set of common fixed points of a finite family of generalized demimetric mappings.

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Figure 4. Comparison of the new algorithm and the Algorithm 1 to recovery of a sparse signal with 5% nonzero elements.

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Received July 2021; 1st and 2nd revision October 2021; early access December 2021.

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