SU(N) Gauge Theories Near $T_c$*

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We study the deconfinement phase transition in SU(N) gauge theories for $N=2,3,4,6,8$. The transition is first order for $N \geq 3$, with the strength increasing as $N$ increases. We extrapolate $T_c/\sqrt{\sigma}$ to the continuum limit for each $N$, and observe a rapid approach to the large $N$ limit. As $N$ increases the phase transition becomes clear-cut on smaller spatial volumes, indicating the absence of (non-singular) finite volume corrections at $N=\infty$ – reminiscent of large $N$ reduction. The observed rapid increase of the inter-phase surface tension with $N$ may indicate that for $N=\infty$ the deconfinement transition cannot, in practice, occur.

1. INTRODUCTION

Originally proposed as a possible way of solving QCD \cite{1}, the large $N$ limit of SU(N) gauge theories plays an important role for our understanding of this class of theories. It has been shown numerically \cite{2} that several quantities at zero temperature have a well-defined large $N$ limit and that deviations from that limit at finite $N$ are small and accounted for by a $O(1/N^2)$ correction (the first expected correction by perturbative arguments) all the way down to $N=2$. In \cite{3} similar features were observed at finite temperature. In particular, for the deconfinement temperature $T_c$ it was found

$$T_c/\sqrt{\sigma} = 0.582(15) + 0.43(13)/N^2.$$  (1)

Here we extend the investigation of \cite{3}. We study SU(N) groups for $N = 2, 3, 4, 6, 8$. New improved results for both $T_c$ and $\sqrt{\sigma}$ help us to keep under better control the extrapolation to infinite $N$. In the following we will discuss our numerical results for the critical temperature and the latent heat $L_h$ as a function of $N$ and what the observed behaviour at finite $N$ may imply for the physics at finite temperature in the large $N$ limit. This work is based on \cite{4}, to which we refer the reader for a more detailed discussion.

2. LATTICE SETUP

In our numerical simulations we have used the Wilson action

$$S = \beta \sum_P \left(1 - \frac{1}{N} \text{ReTr} U_P \right),$$  (2)

where $\beta = 2N/g^2$, with $g$ the coupling of the theory, $U_P$ is the path ordered product of the links around the plaquette $P$ and the links are SU(N) matrices. The sum is performed over all the plaquettes $P$. The lattice size is $L$ in three directions and $N_t$ (with $N_t \ll L$) in the fourth one. We call spatial directions the three of equal size and temporal direction the remaining one.

The system can be regarded either as a statistical system in four dimensions with the role of temperature played by $\beta$ or as a three-dimensional quantum field theory at finite temperature, with the physical temperature given by

$$T_c = 1/N_t a(\beta),$$  (3)

$a$ being the lattice spacing. Thermodynamical quantities have been obtained via a finite size study in the critical region of susceptibilities like the specific heat and the susceptibility of the Polyakov loop.

3. CRITICAL TEMPERATURE

The critical temperature is unambiguously defined only in the thermodynamic limit. On a fi-
finite lattice, one usually takes as a definition the value at which the susceptibility of some observable relevant for the transition has a peak. More specifically, one defines the pseudocritical coupling \( \beta_c(N_t, L) \) as the \( \beta \) at which the chosen observable displays a peak at fixed \( L, N_t \). Then the value of the critical coupling is given by a fit to the finite size scaling relation

\[
\beta_c(N_t, L = \infty) = \beta_c(N_t, L) + hL^{-1/\nu},
\]

where, if \( d \) is the number of spatial dimensions, we get \( \nu = 1/d \) for a first order phase transition and \( \nu < 1/d \) for a second order one. The value of \( \beta_c(N_t, L = \infty) \) does not depend on the chosen observable.

In principle the order of the phase transition can be extracted from the very same procedure used for determining \( \beta_c(L = \infty) \). However, in practise that proves to be hard: using (3) as a fit ansatz gives large errors on \( \nu \). For this reason, we have determined this exponent from the scaling of the maximum of a susceptibility. In fact, from general arguments

\[
\chi_{\text{max}}(N_t, L = \infty) \propto L^{\gamma/\nu},
\]

where \( \gamma \) is the critical exponent governing the scaling of the considered observable. The fitted value must satisfy the constraint \( 0 < \gamma/\nu \leq d \), with the upper bound saturated by first order phase transitions.

The cleaner signal for the fit is generally given by the specific heat, from which we get evidence for a first order phase transition for \( N > 2 \). Inserting this information back into eq. (4), we obtain \( \beta_c(N_t, \infty) \). Next, by measuring the string tension at zero temperature at that value of \( \beta \), we get \( T_c/\sqrt{\sigma} \).

We have determined \( \beta_c(L = \infty) \) for at least three values of \( N_t \), chosen in such a way to avoid any bulk phase for \( N \geq 4 \). This allows us to obtain the continuum limit of \( T_c/\sqrt{\sigma} \).

The final step is the extrapolation to \( N = \infty \). We find

\[
T_c/\sqrt{\sigma} = 0.596(4) + 0.453(30)/N^2.
\]

Our finite size analysis has been performed in full only at \( N_t = 5 \). For other values of \( N_t \), we have studied one lattice size in the thermodynamic regime and then we have used the scaling with \( \beta \) of \( h \) in eq. (4) to determine \( \beta_c \).

Our data for \( T_c/\sqrt{\sigma} \) as a function of \( 1/N^2 \) are plotted in fig. 1. Despite the increased precision, which reflects on a noticeable reduction of the error bars with respect to (4), still the leading correction in \( 1/N^2 \) accurately describes the data all the way down to \( SU(2) \).

4. LATENT HEAT

At finite \( N \geq 3 \) the deconfinement phase transition is first order. First order phase transitions are characterised by a latent heat, i.e. a difference in action between the confined and the deconfined vacuum at \( T = T_c \). The latent heat (per site) can be obtained from the infinite volume limit of the maximum of the specific heat in the following way:

\[
L_h(N_t) = \lim_{L \to \infty} \left( \frac{2}{\beta_c(L, N_t)} \sqrt{\frac{C_{\text{max}}(L, N_t)}{6L^3}} \right). \tag{7}
\]

We have measured \( L_h \) for \( N_t = 5 \) and \( N \geq 3 \), and we have found that its value increases with \( N \). Assuming a leading \( O(1/N^2) \) correction, our

\[\text{Figure 1. Extrapolation of } T_c/\sqrt{\sigma} \text{ to infinite } N. \text{ The dashed line is the best fit to the data.}\]

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results at finite $N$ can be extrapolated to $N = \infty$ (cfr. fig. 2). We obtain a finite value for $L_h$ in that limit, which implies that the transition is first order at $N = \infty$. The negative coefficient for the correction indicates that the strength of the transition increases with $N$.

5. SURFACE AND FINITE SIZE EFFECTS

A consequence of the increasing strength of the phase transition is that a stronger interface separates the confined and the deconfined vacua at larger $N$. Hence the life time of a metastable state at fixed spatial volume increases as $N$ is increased, suggesting that it becomes infinite at $N = \infty$ even for $\beta$ close to but different from $\beta_c$. In that case, we would have two stable states in a finite region of $\beta$’s. Interpreted in terms of the Master Field [5], this suggests the possibility that this field is not unique: at infinite $N$ there must be more than one Master Field, separated from each other by infinite energy barriers. The interplay between the different vacua at finite but large $N$ guarantees a rich physics in the limiting case.

When we vary the volume, we observe that as $N$ increases (a) the thermodynamical limit is reached earlier and (b) finite size corrections are smaller. This suggests that at large $N$ the scaling regime is obtained for $L = N_t + \epsilon$, with $\epsilon \to 0$ as $N \to \infty$. If we reduce $L$ below $N_t$, the system undergoes a deconfinement phase transition due to the breaking of a spatial $Z(N)$ symmetry. This has been investigated from a different perspective in [5]. The breaking of a spatial $Z(N)$ symmetry could be prevented by twisting the boundary conditions in all spatial directions. In this way a finite temperature regime can be obtained for $L < N_t$. It is then plausible that at infinite $N$ the spatial volume can be shrunk to a single plaquette. This naturally connects to the twisted Eguchi-Kawai model [7].

6. CONCLUSIONS

Our results for $T_c$ and $L_h$ indicates that a sensible large $N$ limit exists for $SU(N)$ gauge theories also at finite temperature. Deviations from that limit at finite $N$ are small and can be accounted for by a leading $\mathcal{O}(1/N^2)$ correction. Our extended range of $N$ (up to eight) reveals some interesting scenarios for the physics at $N = \infty$. At present we are extending this analysis to other observables.

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