Fixed-time Observer Design for LTI Systems by Time-varying Coordinate Transformation

Gilberto Pin, Guitao Yang, Andrea Serrani, and Thomas Parisini

Abstract—In this work, we present a novel fixed-time state observer for LTI systems based on a time-varying coordinate transformation yielding the cancellation of the effect of the unknown initial conditions from the state estimates. This coordinate transformation allows one to map the state of the original system into that of an auxiliary system that evolves from initial conditions that are known by definition. After a stable observer is designed in the transformed coordinates, an estimate for the state of the original system can be obtained by inverting the above-mentioned map. The invertibility of the map is guaranteed for any time strictly greater than zero, so that the convergence time can be made arbitrarily small in nominal conditions. The robustness of the observer with respect to bounded measurement disturbances is characterized in terms of both transient and norm bounds on the asymptotic state-estimation error. Compared to existing finite- and fixed-time approaches, the proposed method does not require high-gain output-error injection, state augmentation, delay operators, or moving-windows. The dimensionality of the observer matches that of the observed system, and its realization takes the form of an LTV system.

I. INTRODUCTION

In many applications such as fault isolation, change-point detection, or in the case of processes that can be observed in finite time-windows, estimates of hidden states are often required to converge with fast dynamics, independently from unknown initial conditions. This task can – in principle – be accomplished by “finite-time” and “fixed-time” observers. In the case of finite-time algorithms, the convergence time depends on the initial estimation error, while in the case of fixed-time observers the convergence time can be chosen by design, being independent from the initial conditions. Since the convergence time of fixed-time observers can be chosen a priori by the designer, the convergence is also termed “prescribed-time” or “free-will” in the literature.

A. A Glimpse on the State of the Art

While for linear discrete-time systems the design of finite-time (also known as “deadbeat”) observers is trivial, the derivation of a continuous-time counterpart poses technical challenges that have been tackled from different directions.

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The first solution to this problem has been proposed by Kalman and his co-workers in the book [1], where it has been shown that the initial state of a system could be “observed” (according to the lexicon used there) by processing the input/output measurements of a system by causal integration; the same source also offers the first fixed-time state “reconstruction” formula, that permits to estimate exactly the current state of a linear system with unknown initial conditions, again by processing input/output measurements by causal integration. However, said formula is not well suited for implementation due to the presence of nested time-integrals. After the seminal work [1], most of the theory on finite-time state reconstruction was rooted into the use of sliding mode (SM) observers and homogeneity tools [2]–[10]. Conventional SM observers, however, can only guarantee semi-global stabilizability of the estimation-error dynamics, that is, the convergence can be proven for initial states restricted to a given bounded region. Higher-order SM update laws can be shown to achieve global convergence, at the cost of increased complexity in the implementation. Notably, the SM methodology achieves finite-time convergence by discontinuous high-gain output injection, such that measurement noise may prevent its applicability. The convergence time of SM observers can be shortened at the cost of boosting the observer gains. The higher-order SM formulation offers a solution to the chattering issue, but still requires high-gain output injection.

Among methodologies alternative to sliding mode, one finds the moving-horizon observer described in [11] and the delay-based filters proposed by [12] and [13]. Practical implementation of the aforementioned observers is, however, computationally demanding. Indeed, moving-horizon observers require time discretization and the need to solve repeated dynamic optimization problems on-line, whose complexity depends on the system dimension and on the length of the observation window. Moreover, the implementation of the delay operators required by [12] and [13] poses additional challenges in the continuous-time framework. Further alternatives to sliding mode observers are provided by Fliess’s algebraic state reconstruction method (see [14]–[16]), or the integral-based modulating functions (see [17]–[20]), that exploit integral operators over compact domains able to annihilate the effect of initial conditions. Modulation function methods also found application to signal differentiation [21] and state estimation for fractional order systems [22]. To avoid error accumulation, the implementation of modulating function observers calls for the use of sliding windows (integration over compact domains) or require periodic re-scaling as discussed in [17].

A different finite-time convergent observer, based on impulsive innovation updates, has been proposed in [23], [24].
This approach does not need sliding integration windows, but dynamic augmentation, where the dimension of the observer becomes twice that of the observed system. However, compared to SM observers, this type of observer does not make use of high-gain output injection.

Another class of finite-time state estimators is described in [25], [26], where Volterra integral operators with suitably designed kernels are used to cancel the effect of unknown initial conditions. Volterra integral operators with tailored kernel functions have been also used to provide combined state-parameter estimates [27], [28]. The use of suitable bivariate kernel functions in place of univariate modulating functions is a key ingredient to enforce internal stability. Due to this desirable feature, these observers do not require sliding-windows, resetting or re-scaling. Indeed, the observers described in [25]–[27] can be implemented as LTV systems whose dimension matches that of the original system. As a drawback, their applicability is limited to SISO systems.

Recently, in [29] an original approach has been proposed towards the design of a finite-time observer for SISO linear systems that uses time-varying output-injection gains. The design of the observer gains stems from a state-mapping perspective, based on the rationale that, in transformed coordinates, the observer-error can be made to converge in finite-time, independently from unknown initial conditions. As a drawback, gain functions in [29] becomes unbounded as time approaches the prescribed convergence time.

B. Contributions

The present paper describes a new fixed-time observer design technique for linear systems that overcomes the weak points of all the aforementioned approaches. More specifically:

- The dimension of the observer matches that of the observed system (no state augmentation is required)
- There is no need for high-gain output-error injection
- No sliding-window or periodic re-scaling is needed
- The proposed fixed-time observer is directly applicable to MIMO systems.

Indeed, the method proposed in this paper – rooted in the idea presented in [30] – is based on the use of a state-dependent mapping such that the system, in the transformed coordinates, has initial conditions at the origin by construction.

The paper is organized as follows. The next section is focused on the problem formulation whereas in Section III the design of the proposed observer is addressed. Section IV deals with the robustness analysis and Section V presents numerical results showing the effectiveness of the proposed methodology. Concluding remarks are reported in Section VI.

II. PROBLEM SETUP

Consider the following LTI system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t),
\end{align*}
\]

with unavailable state \( x \in \mathbb{R}^n \) of known dimension, available input and output signals \( u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p \), known system matrices \( C, A, B \) and unknown initial state \( x(0) = x_0 \). It is assumed that the pair \( (C,A) \) is observable. Our objective is to design a fixed-time observer for system (1), that is, an observer with a positive convergence time smaller or equal than a prescribed arbitrarily time \( T > 0 \).

The key idea behind the proposed methodology consists in determining a time-varying coordinate transformation matrix \( T(t) \in \mathbb{R}^{n \times n} \) to map the state \( x(t) \) into new coordinates

\[
z(t) = T(t)x(t),
\]

such that

1) \( T(0) = 0_{n \times n} \)
2) \( T(t) \) is invertible for any \( t > 0 \).

Note that the first point implies that the initial condition for the transformed state is structurally constrained to

\[z(0) = z_0 = 0_{n \times 1},\]

while the second condition is needed to guarantee the existence of an inverse mapping from \( z(t) \) to \( x(t) \). Therefore, the design of a fixed-time estimator for \( x(t) \) is converted into the problem of designing a stable observer in the transformed coordinate \( z(t) \), yielding an estimate \( \hat{z}(t) \) that is then mapped back to the original coordinates by

\[
\hat{z}(t) = T(t)^{-1}z(t), \quad t > 0.
\]

Next, we describe the observer design methodology, leading to the map \( T(t) \) and the dynamics of the transformed-coordinate system.

III. FIXED-TIME OBSERVER DESIGN

The first step of the design consists in adding and subtracting the term \( LCx(t) \) to the r.h.s. of the first equation of (1), with \( L \) chosen such that the matrix \( AT := -A + LC \) is Hurwitz. The existence of such an \( L \) is guaranteed by observability of the pair \( (C,A) \):

\[
\dot{x}(t) = (A - LC)x(t) + Bu(t) + Ly(t) = -ATx(t) + Bu(t) + Ly(t).
\]

In the light of (2) and (3), the time derivative of \( z(t) \) can be written as

\[
\dot{z}(t) = \dot{T}(t)x(t) + T(t)[-ATx(t) + Bu(t) + Ly(t)]
\]

\[
= \dot{T}(t) - T(t)AT\dot{x}(t) + T(t)Bu(t) + T(t)Ly(t).
\]

An additional structural constraint is imposed on \( T(t) \), aimed at simplifying the design of a transformation ensuring that the requirements 1) and 2) are verified. Specifically, let \( T(t) \) be generated by the linear system

\[
\begin{align*}
\dot{T}(t) &= AT^2T(t) + T(t)AT + C^TC \quad (5)
\end{align*}
\]

whose dynamics obeys a Differential Lyapunov Equation (DLE). By exploiting the differential constraint in (5), (4) can be recast in the following form:

\[
\dot{z}(t) = [AT^2T(t) + C^TC]x(t) + T(t)[Bu(t) + Ly(t)].
\]

Setting \( A_z := A_T^T \), \( z(t) \) evolves according to the following dynamics:

\[
\dot{z}(t) = A_z\hat{z}(t) + T(t)Bu(t) + (C^T + T(t)L)y(t) \quad (7)
\]

\[
z(0) = 0_{n \times 1}.
\]
Next, an observer for (7) is designed as
\[ \dot{\hat{z}}(t) = A\hat{z}(t) + T(t)Bu(t) + [C^\top + T(t)L]y(t) \] 
(\text{8})
where we have exploited the fact that the initial condition in the transformed coordinates is known to be at the origin. Then, the error \( \hat{z}(t) = \hat{z}(t) - z(t) \) evolves according to
\[ \dot{\hat{z}}(t) = A\hat{z}(t), \quad \hat{z}(0) = 0_{n \times 1}, \] 
(\text{9})
The above system is readily seen to be exponentially stable, as \( A = A^\top \) is Hurwitz by design. Furthermore, since the initial condition is at the origin \( \hat{z} = 0 \), the observation error vanishes for all \( t \geq 0 \).

Next, we proceed by determining a suitable expression for the map \( T(t) \) by solving (5) and characterizing its invertibility. A solution for the linear Lyapunov matrix differential equation (5) is given by (see Appendix AI)
\[ T(t) = e^{A\frac{\tau}{2}}T_0e^{A\frac{\tau}{2}} + \int_0^t e^{A\frac{\tau}{2}}Ce^{A\frac{\tau}{2}}d\tau. \]
(\text{10})
Note that for the initial condition \( T_0 = 0_{n \times n} \), the Gram matrix \( T(t) \) corresponds to the Observability Gramian of \( (C, A_T) \). Under the assumption that \( (C, A_T) \) is observable and since \( A_T \) Hurwitz, it follows that \( T(t) \) is unique, symmetric and positive-definite \( \forall t > 0 \). Therefore, \( T(t) \) is invertible for all \( t > 0 \). Moreover, \( \lim_{\tau \to +\infty} T(t) = W_o \), where \( W_o \) is the asymptotic Observability Gramian of \( (C, A_T) \) given by the solution of the algebraic Lyapunov equation
\[ A_T^\top W_o + W_oA_T = -C^\top C. \]
The dimension of the observer, which is comprised of (8) and the linear matrix differential equation (5), may appear at a first glance to equal \( n + n^2 \). Nonetheless, one can exploit the explicit solution of (5) given in (10), so that \( T(t) \) becomes a known time-dependent matrix. This is to say that the dynamic part of the observer can be restricted to (8), which is a linear observer in the transformed coordinates with a time-varying input/output injection gain matrix.

IV. ROBUSTNESS TO OUTPUT DISTURBANCE

In the previous section, it has been shown that the proposed observer allows one to estimate the state of a linear system in an arbitrary fixed-time in nominal conditions. The objective of this section is to characterize the behavior of the observer in case the output measurement is perturbed by a bounded disturbance. In this scenario, transient and asymptotic worst-case bounds for the state-estimation error will be derived. In what follows, we will denote by \( | \cdot | \) the Euclidean norm for vectors and by \( \| \cdot \| \) the corresponding induced matrix norm: \( \| M \| := \sup_{x \in \mathbb{R}^n, x \neq 0} |Mx|/\|x\| \), where \( M \in \mathbb{R}^{m \times n} \). The notation \( \lambda(P) \) and \( \lambda(P) \) stands respectively for the smallest and largest eigenvalues of the symmetric positive semi-definite matrix \( P \in \mathbb{R}^{n \times n} \).

Assume that the measurement of the output is corrupted by a bounded additive perturbation \( d_y(t) \), that is, let the available measurement be given by
\[ \tilde{y}(t) = y(t) + d_y(t), \]
where \( |d_y(t)| \leq \bar{d}_y, \forall t \geq 0 \). When \( \tilde{y}(t) \) is used in place of \( y(t) \), the observer dynamics read as
\[ \dot{\tilde{z}}(t) = A\tilde{z}(t) + [C^\top + T(t)L] \tilde{y}(t) \]
\[ = A\tilde{z}(t) + T(t)Bu(t) + [C^\top + T(t)L] y(t) + d_y(t) \] 
(\text{11})
\[ \tilde{z}(0) = 0_{n \times 1}. \]
Consequently, the estimation error in the transformed coordinates evolves according to the dynamics
\[ \dot{\tilde{z}}(t) = A\tilde{z}(t) + [C^\top + T(t)L] d_y(t), \]
\[ \tilde{z}(0) = 0_{n \times 1}, \] 
(\text{12})
The effect of the disturbance on the transient and asymptotic behavior of the observation error, is evaluated by means of the quadratic function \( V = \tilde{z}^\top P\tilde{z} \), where \( P \) is the unique solution of the Lyapunov equation \( A_T^\top P + PA_T = -Q \) for a fixed arbitrary matrix \( Q > 0 \). Along trajectories of (12), the derivative of \( V(t) \) reads as
\[ \dot{V}(t) = -\tilde{z}(t)^\top Q\tilde{z}(t) \\
+ 2\tilde{z}(t)^\top P[C^\top + T(t)L]d_y(t) \\
+ \tilde{z}(t)^\top PT(t)Ld_y(t) \]
(\text{13})
where \( L_P := \text{cho}(P) \) denotes the Cholesky factor of \( P \). By using Young’s inequality with a positive scalar \( \epsilon = 2\lambda(P)/(\lambda(Q) - q) \), for some arbitrary scalar \( q \) satisfying \( 0 < q < \lambda(Q) \), the following upper bound is obtained:
\[ \dot{V}(t) \leq -\tilde{z}(t)^\top Q\tilde{z}(t) + \frac{2}{\epsilon} \tilde{z}(t)^\top L_P\tilde{z}(t) \]
\[ + \frac{\epsilon}{\lambda(P)} \| CPT^\top \tilde{z}(t) \| \| L_P \| \| T(t)L \| \] 
\[ \leq -\tilde{z}(t)^\top Q\tilde{z}(t) + \frac{2}{\epsilon} \tilde{z}(t)^\top L_P\tilde{z}(t) \]
\[ + \frac{\epsilon}{\lambda(P)} \| CPT^\top \tilde{z}(t) \| \| L_P \| \| T(t)L \| \] 
\[ \leq -\frac{q}{\lambda(P)} \tilde{z}(t)^\top P\tilde{z}(t) \]
\[ + \frac{\epsilon}{\lambda(P)} \| CPT^\top \tilde{z}(t) \| \| L_P \| \| T(t)L \| \] 
(\text{14})
By invoking the comparison theorem for scalar ODEs (see, for instance, [31]), one obtains \( V(t) \leq v(t) \) for all \( t \geq 0 \), where \( v(t) \) is the unique solution of
\[ \dot{v}(t) = -\frac{q}{\lambda(P)} v(t) + \frac{\epsilon}{\lambda(P)} \| CPT^\top \tilde{z}(t) \| \| L_P \| \| T(t)L \| \] 
\[ v(0) = V(0). \]
Note that the solution of the Cauchy problem for (14) admits the closed form
\[ v(t) = \epsilon \int_0^t e^{-\frac{q}{\lambda(P)}(t-\tau)} \| CPT^\top \tilde{z}(t) \| \| L_P \| \| T(t)L \| d\tau. \]
Noticing that \( |\tilde{z}|^2 \leq \lambda(P)^{-1} V(t) \), the error due to the mea-
measurement disturbances satisfies the following time-dependent upper bound
\[
|\tilde{z}(t)|^2 \leq \frac{1}{\Lambda(P)} V(t) \leq \frac{1}{\Lambda(P)} v(t)
\]
\[
\leq \epsilon \frac{1}{\Lambda(P)} \int_0^t e^{-\frac{\epsilon}{2\Lambda(P)}} (t-\tau) \left\| CPC^T + L^T T(\tau)^T PT(\tau)L \right\| \, d\tau.
\]

It is also worth noting that the right-hand side of the above inequality vanishes for \( t = 0 \) (this is a consequence of the fact that the initial condition is exactly known in the transformed coordinates), and the error builds up due to the injection of the noisy output \( \hat{y}(t) \).

Finally, applying the inverse-mapping, the following upper bound for the state estimation error in the original coordinates can be readily established:
\[
|\hat{\tilde{x}}(t)|^2 \leq \epsilon \frac{1}{\Lambda(P)} \int_0^t e^{-\frac{\epsilon}{2\Lambda(P)}} (t-\tau) \left\| CPC^T + L^T W_o^T PW_o L \right\| \, d\tau.
\]

Owing to the following fact (see Appendix B)
\[
\| W_o \| \geq \| T(t) \|, \forall t \geq 0,
\]
one obtains
\[
|\hat{\tilde{x}}(t)|^2 \leq \epsilon \frac{1}{\Lambda(P)} \int_0^t e^{-\frac{\epsilon}{2\Lambda(P)}} (t-\tau) \left\| CPC^T + L^T W_o^T PW_o L \right\| \, d\tau.
\]

which finally yields
\[
|\hat{\tilde{x}}(t)| \leq \epsilon \frac{1}{\Lambda(P)} \int_0^t e^{-\frac{\epsilon}{2\Lambda(P)}} (t-\tau) \left\| CPC^T + L^T W_o^T PW_o L \right\| \, d\tau \times \| T(t)^{-1} \| \bar{d}_y.
\]

Consequently, the right-hand side of (17) is finite for any \( t > 0 \) and satisfies the asymptotic bound
\[
\lim_{t \to +\infty} |\hat{\tilde{x}}(t)| \leq \epsilon \frac{1}{\Lambda(P)} \left\| CPC^T + L^T W_o^T PW_o L \right\| \times \| W_o^{-1} \| \bar{d}_y.
\]

V. NUMERICAL EXAMPLE
In this section, the effectiveness of the proposed fixed-time observer is shown through a numerical example, in which exact convergence is attained in a prescribed arbitrarily small fixed time-interval in absence of measurement perturbations.

Simulations are carried out in Matlab/Simulink, using the ode45 (Dormand-Prince) variable-step solver.

The fixed-time observer is employed to estimate the state of the system
\[
A = \text{diag} \left( \begin{array}{ccc} -\pi & 10\pi & -\pi \\ -10\pi & -2\pi & -\pi \\ -2\pi & -\pi & -\pi \end{array} \right),
\]
\[
B = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T,
\]
\[
C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}
\]
with \( x(t) \in \mathbb{R}^4 \) and initial condition \( x_0 = [1 \ 1 \ -1 \ 1]^T \).

The input \( u(t) \in \mathbb{R} \) is chosen as \( u(t) = 10 \sin(20\pi t) \), \( t \geq 0 \).

The matrix-gain \( L \) is designed based on (8) as
\[
L = [ -90.967 \ 0.3351 \ -216.91 \ 128.58 ]^T
\]
such that \( A_z := (-A + LC)^T \) is Hurwitz. \( T(t) \) is provided by the analytical solution (10) with \( T_0 = 0_{3 \times 3} \). For \( t < \bar{t} = 0.01 \), with \( \bar{t} \) denoting the prescribed fixed-time after which the matrix \( T(t) \) is inverted, the state estimation is set to zero by design, \( \tilde{x}(t) = 0_{4 \times 1} \), whereas for \( t \geq \bar{t} \), the inverse map is used to obtain the estimate in the original coordinates, \( \tilde{x}(t) = T(t)^{-1} \tilde{z}(t) \). The state estimation error based on the proposed fixed-time observer design is shown in Figure 1, which confirms that convergence of the estimation error is established in the prescribed time interval \([0, \bar{t}]\).

Fig. 1. Estimation errors by using the proposed fixed-time observer in the noise-free scenario. The inversion of the map \( T(t) \) is applied from time \( \bar{t} = 0.01 \) to avoid numerical issues. After this time, the estimator yields exact estimates.

Next, we consider the case where the output signal fed into the observer is corrupted by a bounded additive perturbation, as described in Section IV, where \( d_y(t) \) is selected as a random signal with uniform distribution in the interval \([-0.1, 0.1]\) bounded by \( \bar{d}_y = 0.1 \). The matrix \( Q > 0 \) and the corresponding solution of the Lyapunov equation \( P > 0 \) are given respectively as
\[
Q = \begin{bmatrix} 173.96 & 23.462 & 28.273 & 49.852 \\ 23.462 & 71.134 & -3.4691 & 10.623 \\ 28.273 & -3.4691 & 128.49 & 17.568 \\ 49.852 & 10.623 & 17.568 & 138.97 \end{bmatrix}
\]
\[
P = \begin{bmatrix} 4.8736 & -1.8343 & -3.1157 & -2.8863 \\ -1.8343 & 7.0882 & 1.2071 & 0.31947 \\ -3.1157 & 1.2071 & 2.7891 & 1.0901 \\ -2.8863 & 0.31947 & 1.0901 & 5.6087 \end{bmatrix}
\]

The asymptotic Observability Gramian of \( (C, A_T) \) is given by
\[
W_o = 10^{-4} \times \begin{bmatrix} 38.364 & 3.6687 & 9.3429 & 2.1731 \\ 3.6687 & 36.687 & -0.42121 & 2.0199 \\ 9.3429 & -0.42121 & 19.732 & 0.26097 \\ 2.1731 & 2.0199 & 0.26097 & 10.961 \end{bmatrix}
\]
VI. CONCLUSIONS

In this work, we have described a new fixed-time observer for LTI systems based on a time-varying coordinate transformation that is able to suppress the effect of unknown initial conditions on the estimates. The proposed observer does not rely on high-gain output injection and is robust to additive perturbations affecting the output-measurement equation.

In this setup, further analysis is needed to characterize the robustness of the observer in the face of input perturbations and the noises in the system equations, to reduce the conservative nature of the norm-bound on the state-estimation error, and to extend the formulation to specific classes of nonlinear systems.

APPENDIX A: SOLUTION OF THE DLE (5)

We will prove that $T(t)$ given in (10) solves (5). First, it is noted that

$$
\dot{T}(t) = \frac{\partial}{\partial t} T(t) = A_T^T e^{A_T^T t} T_0 e^{A_T t} t + e^{A_T^t t} T_0 e^{A_T t} A_T + e^{A_T^t t} C^T C e^{A_T t} t.
$$

Substituting (10) in the right-hand side of (5), one obtains

$$
\dot{T}(t) = A_T^T \left( e^{A_T^t t} T_0 e^{A_T t} + \int_0^t e^{A_T^t t} C^T C e^{A_T t} d\tau \right) + \left( e^{A_T^t t} T_0 e^{A_T t} t + \int_0^t e^{A_T^t t} C^T C e^{A_T t} d\tau \right) A_T + C^T C.
$$

Then accordingly we have

$$
\dot{T}(t) = \int_0^t A_T^T e^{A_T^t T_0 e^{A_T t} t} t + e^{A_T^t T_0 e^{A_T t} t} A_T + C^T C
t + \int_0^t \frac{\partial}{\partial t} e^{A_T^t T_0 e^{A_T t} t} + e^{A_T^t t} T_0 e^{A_T t} A_T + C^T C
+ A_T^T e^{A_T^t T_0 e^{A_T t} t} t + e^{A_T^t T_0 e^{A_T t} t} A_T + C^T C
= \left[ e^{A_T^t t} C^T C e^{A_T t} t \right]_{t=0}^T + C^T C
+ A_T^T e^{A_T^t T_0 e^{A_T t} t} + e^{A_T^t T_0 e^{A_T t} t} A_T
+ e^{A_T^t t} T_0 e^{A_T t} A_T + C^T C,
$$

which proves that $T(t)$ given in (10) verifies the DLE (5).

APPENDIX B: PROOF OF $\|W_0\| \geq \|T(\tau)\|, \forall \tau > 0$

Recalling the definition of vector-induced matrix norm

$$
\|T(t)\| = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{|T(t)\|}{\|x\|} = \sup_{x \in \mathbb{R}^n, x \neq 0} \sqrt{x^T T(t)^T T(t) x / x^T x},
$$

and owing to the fact that

$$
T(t) = T(t') + \int_{t'}^t e^{A_T^t T_0 e^{A_T t} t} C e^{A_T t} t d\tau,
$$

it follows that, for any $t' : 0 \leq t' \leq t$,

$$
\|T(t)\| = \sup_{x \in \mathbb{R}^n, x \neq 0} \sqrt{x^T T(t')^T T(t') x / x^T x} \geq 0,
$$

Since $x^T \left( \int_{t'}^t e^{A_T^t T_0 e^{A_T t} t} C e^{A_T t} t d\tau \right) x \geq 0$, one obtains

$$
\|T(t)\| \geq \sup_{x \in \mathbb{R}^n, x \neq 0} \sqrt{x^T T(t')^T T(t') x / x^T x} = \|T(t')\|,
$$

which implies $\|W_0\| = \| \lim_{t \to \infty} T(t) \| \geq \|T(\tau)\|, \forall \tau > 0$.

REFERENCES

[1] R.E. Kalman and P.L. Falb and M. Arbib, Topics in Mathematical Systems Theory. New York: McGraw-Hill, 1969.
[2] A. Levant, “Robust exact differentiation via sliding mode technique,” Automatica, vol. 34, no. 3, pp. 379 – 384, 1998.
[3] I. Hasakara, U. Özgüner, and V. Utkin, “On sliding mode observers via equivalent control approach,” Int. J. Contr., vol. 71, no. 6, pp. 1051–1067, 1998.
[4] A. Levant, “Homogeneity approach to high-order sliding mode design,” Automatica, vol. 41, no. 5, pp. 823 – 830, 2005.
[5] W. Perruquetti, T. Floquet, and E. Moulay, “Finite-time observers: Application to secure communication,” IEEE Transactions on Automatic Control, vol. 53, no. 1, pp. 356–360, Feb 2008.
[6] A. Levant, “On fixed and finite time stability in sliding mode control,” in 52nd IEEE Conference on Decision and Control, Dec 2013, pp. 4260–4265.
[7] M. T. Angulo, J. A. Moreno, and L. Fridman, “Robust exact uniformly convergent arbitrary order differentiator,” Automatica, vol. 49, no. 8, pp. 2489 – 2495, 2013.
[8] T. Menard, E. Moulay, and W. Perruquetti, “A global high-gain finite-time observer,” IEEE Transactions on Automatic Control, vol. 55, no. 6, pp. 1500–1506, June 2010.
[9] F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti, “Finite-time and fixed-time observer design: Implicit Lyapunov function approach,” Automatica, vol. 87, pp. 52 – 60, 2018.
[10] J. C. L. Chan, C. P. Tan, H. Trinh, and M. A. S. Kamal, “State and fault estimation for a class of non-infinitely observable descriptor systems using two sliding mode observers in cascade,” Journal of the Franklin Institute, vol. 356, no. 5, pp. 3010–3029, 2019.

[11] S. Han, W. Kwon, and P. Kim, “Receding-horizon unbiased FIR filters for continuous-time state-space models without a priori state informations,” IEEE Transactions on Automatic Control, vol. 46, pp. 766–770, 2001.

[12] R. Engel and G. Kresselmeier, “A continuous-time observer which converges in finite-time,” IEEE Trans. Automatic Control, vol. 47, pp. 1202–1204, 2002.

[13] A. Medvedev and T. Toivonen, “Feedback time-delay structures in state estimation: finite memory smoothing and continuous deadbeat observers,” IEE Proceedings of Control Theory and Applications, vol. 141, pp. 121–129, 1994.

[14] M. Fliess and H. S. Ramirez, “State Reconstructions: A possible alternative to asymptotic observers and Kalman filtering,” in Multiconf. on Computational Eng. in Systems Appl. (CESA-2003), Lille, France, 2003.

[15] M. Fliess and H. Sira-Ramirez, “Reconstructeurs d’état,” C.R. Acad. Sci., Paris, vol. 338, no. 1, pp. 91 – 96, 2004.

[16] J. Reger and J. Jouffroy, “On algebraic time-derivative estimation and deadbeat state reconstruction,” in Proc. IEEE Conf. on Decision and Control, Shanghai, 2009, pp. 1740–1745.

[17] G. Pin, B. Chen, and T. Parisini, “Robust deadbeat continuous-time observer design based on modulation integrals,” Automatica, vol. 107, pp. 95 – 102, 2019.

[18] S. Djennoune, M. Bettayeb, and U. M. Al-Saggaf, “Modulating function-based fast convergent observer and output feedback control for a class of non-linear systems,” IET Control Theory Applications, vol. 13, no. 16, pp. 2681–2693, 2019.

[19] F. Mazenc, S. Ahmed, and M. Malisoff, “Reduced order finite time observers for time-varying nonlinear systems,” in 2018 IEEE Conference on Decision and Control (CDC), Dec 2018, pp. 2182–2186.

[20] S. Ahmed, M. Malisoff, and F. Mazenc, “Finite time estimation for time-varying systems with delay in the measurements,” Systems & Control Letters, vol. 133, p. 104551, 2019.

[21] D.-Y. Liu and T.-M. Laleg-Kirati, “Robust fractional order differentiation using generalized modulating functions method,” Signal Processing, vol. 107, pp. 395 – 406, 2015, special Issue on ad hoc microphone arrays and wireless acoustic sensor networks Special Issue on Fractional Signal Processing and Applications.

[22] D. Liu, O. Gibaru, W. Perruquetti, and T. Laleg-Kirati, “Fractional order differentiation by integration and error analysis in noisy environment,” IEEE Transactions on Automatic Control, vol. 60, no. 11, pp. 2945–2960, Nov 2015.

[23] T. Raff and F. Allgöwer, “Observers with impulsive dynamical behavior for linear and nonlinear continuous-time systems,” in Proc. IEEE Conf. Decision and Control, New Orleans, 2019, pp. 1–7.

[24] T. Raff and F. Allgöwer, “An observer that converges in finite time due to measurement-based state updates,” in Proc. of the 17th IFAC World Congress, 2008, pp. 2693–2695.

[25] G. Pin, B. Chen, and T. Parisini, “Robust finite-time estimation of biased sinusoidal signals: A volterra operators approach,” Automatica, vol. 77, pp. 120 – 132, 2017.

[26] B. Chen, P. Li, G. Pin, G. Fedele, and T. Parisini, “Finite-time estimation of multiple exponentially-damped sinusoidal signals: A kernel-based approach,” Automatica, vol. 106, pp. 1 – 7, 2019.

[27] P. Li, F. Boem, G. Pin, and T. Parisini, “Kernel-based simultaneous parameter-state estimation for continuous-time systems,” IEEE Transactions on Automatic Control, pp. 1–1, 2019.

[28] P. Li, G. Pin, G. Fedele, and T. Parisini, “Deadbeat source localization from range-only measurements: A robust kernel-based approach,” IEEE Transactions on Control Systems Technology, vol. 27, no. 3, pp. 923–933, May 2019.

[29] J. Holloway and M. Krstic, “Prescribed-time observers for linear systems in observer canonical form,” IEEE Transactions on Automatic Control, vol. 64, no. 9, pp. 3905–3912, Sep. 2019.

[30] G. Pin, P. Li, G. Fedele, and T. Parisini, “A deadbeat observer for lti systems by time/output-dependent state mapping,” in 2017 IEEE 56th Annual Conference on Decision and Control (CDC), Dec 2017, pp. 4795–4800.

[31] I. Karafyllis, “Lyapunov theorems for systems described by retarded functional differential equations,” Nonlinear Analysis: Theory, Methods and Applications, vol. 64, no. 3, pp. 590 – 617, 2006.