A the one-dimensional J.J. Thomsons model of radio frequency induction discharge

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Abstract. In this paper, the Thomson’s one-dimensional model of radio-frequency induction discharge is considered and some important relations for investigation of its structure are obtained.

1. Introduction
At present, a great number of papers are devoted to the theory of high-frequency induction (RFI) discharge in atmospheric pressure. All these papers use one-dimensional approximation of so-called “ideal” (in the limit of the infinitely long) inductor for description of the electromagnetic field in a discharge.

The model of constant conductivity, also known as the channel model of HFI discharge, was generally developed by J.J. Thomson in 1926 and published a year later as a part of his famous work [1]. Thomson himself considered only the one-dimensional theory.

Moreover, one should take into account that Thomson himself as well as his successors (e.g., refs. [2-4]) were little interested in the structure of the electromagnetic field of RFI discharge.

They considered mechanisms of the latter only in the context of possible physical and technical applications, concerning basically the issues of heat exchange and optimization of gas heating in the channel of induction RF plasmatron. Therefore, we are going to advance further along the path suggested by J.J. Thomson and, using his main idea, to examine in detail the structure of quasi-stationary electromagnetic field of RFI discharge in one-dimensional cases, into account the cylindrical geometry of the latter.

2. Theoretical part
For methodological consistency of description, let us briefly review the Thomson’s one-dimensional model. A set of Maxwell equations, describing the electromagnetic field of RFI discharge, take the form

\[ \text{rot} \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}; \]
\[
\text{rot}\vec{H} = \frac{4\pi}{c} \sigma \vec{E} ; \\
\text{div}\vec{H} = 0 ; \\
\text{div}\vec{E} = 0 ,
\]

where \( \vec{E} \) and \( \vec{H} \) are the electric and magnetic field vectors, respectively, \( \sigma \) is the conductivity in a discharge, and \( c \) is the velocity of light in vacuum (in Gaussian units).

By taking into account the cylindrical symmetry for the case of infinitely long plasmoid, the set of equations (1) rearrange in the form

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r E_\phi \right) = -\frac{1}{c} \frac{\partial H_z}{\partial t} ; \quad \frac{\partial H_z}{\partial r} = -\frac{4\pi}{c} \sigma E_\phi .
\]

Here \( H_z = H_z(r,t) \) and \( E_\phi = E_\phi(r,t) \) are instantaneous values of longitudinal magnetic and azimuthal electric fields, respectively.

From mathematical point of view, the main idea of the channel model of RFI discharge developed by Thomson is introduction of complex records for electric and magnetic quantities, characterizing the electromagnetic field. In this connection, the electromagnetic field equations transform into the linear differential equations for complex values, which can be solved exactly and can be written using special Bessel and Lord Kelvin functions.

Following J.J. Thomson, let us write the set of Maxwell equations, describing the cylindrically symmetric electromagnetic field of RFI discharge (2) for complex magnetic \( H_z(r,t) = H_z(r) \exp(i\omega t) \) and electric \( E_\phi(r,t) = E_\phi(r) \exp(i\omega t) \) fields in a discharge. As a result, we obtain the following set of equations

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r E_\phi \right) = -\frac{i\omega H_z}{c} ; \\
\frac{\partial H_z}{\partial r} = -\frac{4\pi}{c} \sigma E_\phi ,
\]

where \( \sigma = \text{const} \), and \( H(r) \) and \( E(r) \) are complex amplitudes of longitudinal magnetic and azimuthal electric fields, respectively.

Denoting \( \omega/c \) by \( \alpha \) and \( 4\pi\sigma/c \) by \( \beta \), we have

\[
\frac{\partial^2 E_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial E_\phi}{\partial r} - \frac{E_\phi}{r} = i\alpha \beta E_\phi ; \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial H_z}{\partial r} \right) = i\beta \alpha H_z
\]
or

\[
\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} - i\alpha \beta H_z = 0 ; \\
\frac{\partial^2 E_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial E_\phi}{\partial r} - \left( i\alpha \beta + \frac{1}{r^2} \right) E_\phi = 0 .
\]

3. Results and conclusion

The solution of these equations, satisfying the finiteness condition for \( H \) and \( E \) values on the discharge axis at \( r = 0 \), apparently, is as follows

\[
H_z(r) = H_0 J_0 \left( i\sqrt{\alpha \beta} \ r \right)
\]
and

\[ E_\phi(r) = E_0 J_1(i \sqrt{i \alpha \beta} r) , \]  

(5)

where \( J_0(i \sqrt{i \alpha \beta} r) \) and \( J_1(i \sqrt{i \alpha \beta} r) \) are the Bessel functions of complex argument of the zero and first orders, respectively, \( H = H + i\tilde{H} \), \( E = E + i\tilde{E} \) - complex integration constants, the values of which must be determined on the basis of boundary conditions. At first, let us consider the expression (4) for \( H(r) \). Given the fact that in the point \( r = 0 \) we have \( J_0(0) = 1 \) here follows.

\[ H_z(r = 0) = \sqrt{H_0^2 + \tilde{H}_0^2} , \]

so the physical amplitude \( H_{\alpha}^0(0) \) of the magnetic field on the discharge axis are respectively expressed as

\[ H_{\alpha}^0(r = 0) = \sqrt{H_0^2 + \tilde{H}_0^2} . \]  

(6)

By writing this formula through its real and imaginary part, we have

\[ H_z(r) = H_z^0(0) \left[ \text{ber}_0(i \sqrt{\alpha \beta} r) + i \text{bei}_0(i \sqrt{\alpha \beta} r) \right] , \]

(7)

where \text{ber}_0 \ and \ \text{bei}_0 \ are the Thomson (Lord Kelvin) functions of the first kind of zero order [5, 6]. For the amplitude and phase of the physical magnetic field in the discharge, by definition, from here we have

\[ H_{\alpha}^0(r) = H_z^0(0) \left[ \text{ber}_0(i \sqrt{\alpha \beta} r) + i \text{bei}_0(i \sqrt{\alpha \beta} r) \right] ; \]

\[ \varphi_{H_{\alpha}}(r) = \arctg \left[ \frac{\text{bei}_0(i \sqrt{\alpha \beta} r)}{\text{ber}_0(i \sqrt{\alpha \beta} r)} \right] , \]

(8)

(9)

Let us now consider the behavior of the azimuthal electric field in RFI discharge within the framework of the Thomson’s model. In general form, the radial dependence of the complex value \( E_\phi(r) \) is determined by formula (5). However, the simplest way is to derive the expression for \( E_\phi(r) \) directly using the initial Maxwell equations (3). Then, on substitution of the relation (7) for \( E_\phi(r) \) into the first of the equations of this system, and using standard table integrals [7], we get

\[ E_\phi(r) = \frac{\alpha}{r} H_z(r) dr = \frac{i \alpha}{r} H_z^0(0) \left[ J_0(i \sqrt{\alpha \beta} r) \right] dr = \]

\[ = - \sqrt{\frac{\alpha}{2 \beta}} H_z^0(0) (1-i) J_1(i \sqrt{\alpha \beta} r) = \]

\[ = - \sqrt{\frac{\alpha}{2 \beta}} H_z^0(0) \left[ \text{ber}_1(i \sqrt{\alpha \beta} r) + i \text{bei}_1(i \sqrt{\alpha \beta} r) \right] = \]

\[ = - \sqrt{\frac{\alpha}{2 \beta}} H_z^0(0) \left[ \text{bei}_1(i \sqrt{\alpha \beta} r) - \text{ber}_1(i \sqrt{\alpha \beta} r) \right] , \]

(10)

whence it follows

\[ E_\phi(r) = \sqrt{\frac{\alpha}{\beta}} H_z^0(0) \left[ \text{ber}_1(i \sqrt{\alpha \beta} r) + i \text{bei}_1(i \sqrt{\alpha \beta} r) \right] ; \]

\[ \varphi_{E_{\phi}}(r) = \arctg \left[ \frac{\text{be}_1(i \sqrt{\alpha \beta} r) - \text{ber}_1(i \sqrt{\alpha \beta} r)}{\text{be}_1(i \sqrt{\alpha \beta} r) + \text{ber}_1(i \sqrt{\alpha \beta} r)} \right] . \]

(11)

Herewith for small values of the argument in accordance with [8] from here we have

\[ \text{ber}_1(x) \approx - \frac{\sqrt{2}}{4} x - \frac{\sqrt{2}}{32} x^3 , \quad \text{bei}_1(x) \approx \frac{\sqrt{2}}{4} x - \frac{\sqrt{2}}{32} x^3 . \]
Substituting these expansions in the dependence (11), we obtain, in the limit \( r \to 0 \), the values \( E_0''(0) = 0 \) and \( \varphi_{E_0}(r) = -\pi/2 \). As you can see, on the axis of the plasmoid, the phase difference of the longitudinal magnetic and azimuthal electric fields \( \Delta \varphi(0) = \varphi_{H_z}(0) - \varphi_{E_\varphi}(0) \) is equal to the value \( \pi/2 \).

It should be emphasized that the latter result does not depend on the choice made above of specific values of the integration constants and, \( \overline{H}_0 \) and \( \overline{H}_z \). It can easily be shown that exactly the same dependence \( \Delta \varphi(0) = \pi/2 \) will be obtained, for example, when choosing the integration constants \( \overline{H}_0 = 0 \), \( \overline{H}_z = H''(0) \), or \( \overline{H} = \overline{H} = \frac{\sqrt{2}}{2} H(0) \), so on.

4. Conclusion

From the point of view of physics, the result obtained corresponds to the obvious fact that the specific numerical values of the phase angles of certain components of the electromagnetic field in the discharge are veryconditional values, since the initial values of these angles can be chosen by any. At the same time, the phase difference between the components of the electro-magnetic field in the discharge is already the specific physical quantity that can be subjected to real experimental measurement in practice.

References

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