AMERICAN OPTIONS IN THE HOBSON-ROGERS MODEL

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Abstract. In this article, we consider a risky asset $X$ for which evolution follows a model proposed by D.G. Hobson and L.C.G. Rogers\cite{7}. We assume that the volatility of $X$ depends on the ratio of the present value and the exponentially weighted average of the past value. Using the Markovian modelling of the enlarged two-dimensional process, we show that, for the American put option with $X$ as the underlying asset, the continuation region and the stopped region are separated a striking curve. This striking curve lies between the two striking curves from the basic BSM model, yet is not monotone.

Keywords: Hobson-Rogers model; volatility smile; delay stochastic differential equation; delay geometric Brownian motion; Itô formula for delay; American options.

1. Introduction

In this article, we propose the optimal stopping problem for a diffusion-type process $X(t)$ in $\mathbb{R}$, which is governed by a certain stochastic delay differential equation (SDDE). The main result is to present the optimal striking for the American put option for which the underlying risky asset $X(t)$ obeys a delay geometric Brownian motion (DGBM) which was proposed by Hobson and Rogers\cite{7}; we use their offset function of order 1 in the below. The model in \cite{7} is also regarded with the stochastic volatility and with the volatility smile; yet it has the advantage that the model preserves the market completeness of which the usual SV model is lack.

We remark that the SDDE has been under active research for years, useful references are \cite{9} and \cite{4}. The applications to European options for which underlying stock is DGBM are well studied; we mention \cite{1,8,10}, among others. Optimal stopping problem related to DGBM appeared in \cite{5,6}. The feature for the delay equation is that the solution brings the memory from the past, and thus the vast literature on the Markovian solution of an SDE is not readily applicable.

This article is organized as follows. In Section 2, we formulate our SDDE and propose an enlargement of the dimension to fit the Markovian setting in the 1+1 dimension. In Section 3, we present our main result, namely to consider the underlying
risky asset following a certain DGBM, which was proposed in [4], and to present the optimal striking curve for the associated American put option. The proofs of our results are given in the Section 4. The final Section 5 is the conclusion, in which we discuss the novelty of the result in the article and some related direction in financial economics.

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2. The asset model

We consider the following SDDE:

\[ \text{d}X(t) = b(X(t), Y(t)) \, \text{d}t + \sigma(X(t), Y(t)) \, \text{d}B(t), \quad t \geq 0, \]

in which \( B(t) \) is the standard Brownian motion. The process \( Y \) defined by

\[ Y(t) = \frac{\int_{-\infty}^{0} e^{\lambda s} X(t+s) \, \text{d}s}{\int_{-\infty}^{0} e^{\lambda s} \, \text{d}s} = \lambda \int_{-\infty}^{0} e^{\lambda s} X(t+s) \, \text{d}s. \]

The parameter \( \lambda > 0 \) for the exponential averaging, and the continuous deterministic past-memory

\[ X(s) = \xi(s), \quad s \in (-\infty, 0], \]

are pre-given.

We notice that the \( Y \) has the differential

\[ \text{d}Y(t) = \lambda (X(t) - Y(t)) \, \text{d}t. \]

Under suitable Lipschitz and growth conditions, there exits a unique strong solution for (1); see [9] for the detail. In view of (1) and (2), we see that the two-dimensional process \((X(t), Y(t))\) constitutes a Markovian process in \( \mathbb{R}^2 \) with continuous paths. We should remark that, however, the one-dimensional \( X(t) \) is not Markovian. The following \textit{Itô formula for delay} is adapted from [4]. Let \( F(x, y) \) be a given twice differential function in \((x, y)\).

\[ \text{d}F(X(t), Y(t)) = \left( b(X(t), Y(t)) \frac{\partial F}{\partial x} + \lambda (X(t) - Y(t)) \frac{\partial F}{\partial y} + \frac{1}{2} \sigma^2(X(t), Y(t)) \frac{\partial^2 F}{\partial x^2} \right) \, \text{d}t \]
\[ + \left( \sigma(X(t), Y(t)) \frac{\partial F}{\partial x} \right) \, \text{d}B(t). \]

We should remark that, in generality as it is presented in [9], an SDDE \( X(t) \) is regarded as an \textit{infinite-dimensional} Markov process. Here we have a two-dimensional
Markovian enlargement \((X(t), Y(t))\) is due to the choice of the memory \(Y(t)\). It is seen that \((X(t), Y(t))\) is \textit{not} Markovian if we choose \(Y(t) = X(t - t_0)\) for some time-instant \(t_0\); this type of delay indeed appears in the main literature of SDDE, see the final Section 5 for some discussions. We also remark that, the above Itô formula for delay is so-named is mainly to remind the reader for such formula existing in the general SDDE context; the above one can indeed be induced from the two-variate Itô formula for the two-dimensional Markovian diffusion \((X, Y)\), as shown in §6.6 of [15]

3. Main Result

In this main section, we consider the American put option for which the underlying risky asset \(X(t)\) is with the constant risk-less interest rate \(r > 0\), and is with the volatility \(\sigma(s)\) depending solely on the ratio of the present-value \(X(t)\) and the past-value \(Y(t)\), where \(Y\) is given by (2). Thus, \(X(t)\) obeys the following DGBM
\[
dX(t) = rX(t)\,dt + \sigma(Z(t))X(t)\,dB(t), \quad t \geq 0,
\]
in which \(Z\) is the ratio process
\[
Z(t) = \frac{X(t)}{Y(t)}.
\]
By Itô formula for delay in Section 2, we have the following differential for \(Z\),
\[
dZ(t) = (r + \lambda - \lambda Z(t))Z(t)\,dt + \sigma(Z(t))Z(t)\,dB(t).
\]
We remark that, from [3] and [4], we have a certain stochastic volatility model for the risky asset \(X\). However, since we have only one source of randomness, namely \(B(t)\), the market is still complete; see the Remark 3.2 in [7] for this aspect.

The \((X(t), Z(t))\) is a strong Markovian process in \(\mathbb{R}^2\) with continuous paths; we also remark that the log-process \(\ln Z\) has a certain mean reverting property, indeed by applying Itô formula to \(Z\) and \(\ln z\) we have
\[
Z(t) = Z(0) \exp \left\{ \int_0^t \left[ -\lambda Z(s) - (1 + \frac{r}{\lambda} - \frac{\sigma^2(Z(s))}{2\lambda}) \right] ds + \int_0^t \sigma(Z(s)) dB(s) \right\}.
\]
By the Assumption 1 in the below, we may see that \(\ln Z(t)\) is mean-reverting to a zone \([1 + \frac{r}{\lambda} - \frac{\sigma^2(Z(s))}{2\lambda}, 1 + \frac{r}{\lambda} - \frac{\sigma^2(Z(s))}{2\lambda}]\). We remark that, a diffusion which exhibits such mean-reverting to a zone, rather than to a constant or to a time-curve, seems to be a new class, if we compare with the usual mean-reverting Ornstein-Uhlenbeck process; a discussion such “zone-reverting” diffusions will appear in [16].
The fair price of the American put option in the time-horizon $[0, T]$, associated with the two-dimensional Markovian process $(X(t), Z(t))$, is defined to be

$$V(x, z) = \text{esssup}_\tau \mathbb{E}_{(x,z)}[e^{-r\tau}(K - X(\tau))^+]$$

where the notation $\mathbb{E}_{(x,z)}[\cdot]$ denotes the expectation w.r.t. the process starting at $(x, z)$. The $\tau$ is ranging over the class of all stopping times over the time-horizon $[0, T]$, w.r.t. the Brownian filtration $\{\mathcal{F}(t), t \geq 0\}$; see, for example, Section 25.1 of [13].

Our standing assumption is that, besides the continuity of $\sigma(z)$,

**Assumption 1.** $0 < \sigma_2 = \inf \sigma(z) < \sup \sigma(z) = \sigma_1 < \infty$.

This is a reasonable assumption for the non-constant volatility function; see [7, 2] for more detailed discussions.

We state two lemmas on the fair price $V(x, z)$.

**Lemma 3.1.** Under Assumption 1, it has

$$V_2(x) \leq V(x, z) \leq V_1(x), \quad \forall (x, z) \in \mathbb{R}_+^2,$$

where $V_i(x)$ is the fair price of the American put option on the time-horizon $[0, T]$ associated with the standard GBM,

$$dX_i(t) = rX_i(t) \, dt + \sigma_i X_i(t) \, dB(t), \quad X_i(0) = x, \quad i = 1, 2.$$

**Lemma 3.2.** We have

$$(K - x)^+ \leq V(x, z) \leq K, \quad \forall (x, z);$$

moreover, for each $z > 0$, the map

$$x \mapsto V(x, z)$$

is convex, continuous, and decreasing in $x \in [0, \infty)$.

Now we present two results. The first one is the existence of the parametric boundary $z \rightarrow b(z)$ of the option’s continuation region in $(z, x)$. The second one is the skewness of the time-parameter curve $t \rightarrow b(z = z(t))$ under the monotone assumption of the volatility function $\sigma(\cdot)$.

**Proposition 3.3.** Under Assumption 1, there exists a continuous curve $x = b(z)$ such that the continuation region

$$C = \{(x, z) \in \mathbb{R}_+^2 : V(x, z) > (K - x)^+\}$$
has the parametric boundary

\[ \partial C = \{(x, z) \in \mathbb{R}^2_+ : x = b(z)\}, \]

and the optimal stopping time over the time-horizon \([0, T]\) is

\[ (5) \quad \tau^* (\omega) = \inf \{ t \in [0, T] : X(t, \omega) \leq b(Z(t, \omega)) \} = \inf \{ t \in [0, T] : X(t, \omega) = b(Z(t, \omega)) \}. \]

**Proposition 3.4.** If we assume that the volatility function \( z \to \sigma(z) \) is monotone increasing in \( z \), besides the Assumption 1, then in Proposition 3.3, the striking curve parametrized in the time, \( t \to b(z = z(t)) \), \( t \in [0, T] \), is not monotone increasing, though it is always squeezed by two increasing convex curves with the same end point \((T, K)\).

**Remark:** In §4.2 of [7], the volatility function is supposed to be

\[ \sigma(z) = \eta \sqrt{1 + \varepsilon z^2} \wedge N, \]

for the simulation of the volatility smile under their model. Such a volatility function satisfies the condition of Proposition 3.4.

4. **Proofs**

**Proof of Lemma 3.1.** We use the time-change technique; see, for example, §5.1 of \([13]\). Define

\[ T(t, \omega) = \left( \frac{1}{\sigma^2} \int_0^t \sigma^2(Z(u, \omega)) \, du \right) \wedge T, \quad t \in [0, T]. \]

which is strictly increasing in \( t \in [0, T] \), and \( T(t) \uparrow T \), a.s. as \( t \uparrow T \), by our uniform lower bound assumption on \( \sigma \), namely Assumption [11]. The inverse

\[ \hat{T}(\theta, \omega) = \inf \{ 0 \leq t \leq T : T(t, \omega) = \theta \}, \quad \theta : \theta \in [0, (\frac{\sigma^2}{\sigma^2}) T], \]

is well-defined, and

\[ \int_0^{\hat{T}(\theta, \omega)} \sigma^2(Z(u, \omega)) \, du = \theta, \quad \theta \geq 0; \]

 moreover, \( \hat{T}(\theta, \omega) \) is also strictly increasing in \( \theta \), and \( \hat{T}(\theta) \uparrow T \), a.s. as \( \theta \uparrow (\frac{\sigma^2}{\sigma^2}) T \). Define the time-changed motion

\[ \hat{B}(\theta, \omega) = \int_0^{\hat{T}(\theta, \omega)} \sigma(Z(u, \omega)) \, dB(u, \omega), \quad \theta \geq 0. \]
Then, as §5.2 [13] shows, the process $\theta \mapsto \hat{B}(\theta, \omega)$ is a standard Brownian motion w.r.t. the filtration $\hat{\mathcal{F}}(\theta) = \mathcal{F}(\hat{T}(\theta))$, $\theta \geq 0$.

Writing in them of $\theta$, we have

$$X(\theta) = X(0)e^{\hat{T}(\theta)\tau - \frac{1}{2} \theta + \hat{B}(\theta)}.$$  

While for $X_i(\theta)$, $i = 1, 2$, we have

$$X_i(\theta) = X(0)e^{\theta \tau - \frac{1}{2} \theta + B_i(\theta)},$$

in which $B_i(\theta)$ is the standard Brownian motion obtained from the scaling $\theta \rightarrow \sigma_i B(\frac{\theta}{\sigma_i})$. We notice that

$$\frac{\theta}{\sigma_1^2} \leq \hat{T}(\theta, \omega) \leq \frac{\theta}{\sigma_2^2}.$$  

Since $t \mapsto \theta$ is a one-to-one transformation, we have

$$V(x, z) = \operatorname{esssup}_{\tau'} \mathbb{E}_{(x, z)}[e^{-r\tau'}(K - X(\tau'))^+],$$

where $\tau'$ is ranging over the class of all stopping times over the scaled time-horizon w.r.t. the time-changed Brownian filtration $\hat{\mathcal{F}}(\theta)$; so are for the $V_i(x)$, $i = 1, 2$.

In term of $\theta$, $V(x, z)$ and $V_i(x)$ are all driven by the standard Brownian motion. We may compare the first term of the three exponentials in (6) and (7), together with (8), and conclude that, for all $\theta' > 0,$

$$\mathbb{E}_x[e^{-r\theta'}(K - X_2(\theta'))^+] \leq \mathbb{E}_{(x, z)}[e^{-r\theta'}(K - X(\theta'))^+] \leq \mathbb{E}_x[e^{-r\theta'}(K - X_1(\theta'))^+].$$

Substituting $\theta'$ by $\tau'(\theta)$, we have the desired bound. □

**Proof of Lemma 3.2.** Taking $\tau = 0$ in the defining equality of $V(x, z)$, we have $V(x, z) \geq (K - x)^+$; that $V(x, z) \leq K$ is obvious. Using the (3), we can write $V(x, z)$ explicitly as

$$V(x, z) = \operatorname{esssup}_\tau \mathbb{E}[e^{-r\tau'}(K - x \exp\{r\tau - \frac{1}{2} \int_0^\tau \sigma^2(u) \, du + \int_0^\tau \sigma(u) \, dB(u)\})^+],$$

in which $\sigma(u) = \sigma(Z(u))$, $X(0) = x$, $Z(0) = z$. From this display, it is seen that, for each $z$, $x \mapsto V(x, z)$ is convex, continuous, and decreasing in $x$. □

**Proof of Proposition 3.3.** Define, for each $z$,

$$b(z) = \sup\{x \leq K : \, V(x, z) = (K - x)^+\}.$$
By Lemma 3.1, \(0 < V_2(x) \leq V(x, z) \leq V_1(x) \leq K, \forall (x, z)\), and the striking line for \(V_1(x)\) is known to be \(x = b_i\); see, for example, §25.1 of [13]. Therefore, for each \(z\),

\[
0 < b_1 \leq b(z) \leq b_2 < K.
\]

Since \(V(x, z)\) is continuous in \((x, z)\) (recall that \((X(t), Z(t))\) is a strong Markovian process with continuous paths, and hence it has the Fell property), the curve \(z \mapsto b(z)\) is continuous. We claim that

\[
(9) \quad V(x, z) = (K - x)^+, \quad \forall x : x \leq b(z);
\]

so that the curve \(z \mapsto b(z)\) is indeed defining the boundary of the continuation region \(C\).

Suppose, on the contrary, that, for some \(0 < b_0(z) < b(z)\),

\[
V(b_0(z), z) > K - b_0(z).
\]

Since \(V(b(z), z) = K - b(z)\), we must have, for some \(\beta > 1\),

\[
\frac{V(b(z), z) - V(b_0(z), z)}{b(z) - b_0(z)} = -\beta < -1.
\]

We recall that, by Lemma 3.2, \(x \mapsto V(x, z)\) is decreasing. By Lemma 3.2 again, \(x \mapsto V(x, z)\) is convex, and thus we have,

\[
\frac{V(b(z), z) - V(x, z)}{b(z) - x} \leq -\beta, \quad \forall x \leq b_0(z).
\]

This will imply that

\[
V(x, z) \geq V(b(z), z) + \beta(b(z) - x) = (K - b(z)) + \beta(b(z) - x) = K + (\beta - 1)b(z) - \beta x,
\]

which implies that \(V(x, z) > K\), whenever \(x < (\beta - 1)b(z)/\beta\). This is a contraction to the fact that \(V(x, z) \leq K\) (Lemma 3.2). Therefore, the supposition must be false. That the curve \(z \mapsto b(z)\) lies between two parallels \(x = b_i, i = 1, 2\), is a consequence of Lemma 3.1.

Now we prove that the \(\tau^*\) defined by [3] is indeed the optimal stopping time; that is, \(\tau^*(\omega) = \tau_D(\omega)\), and

\[
\tau_D(\omega) = \inf\{t \in [0, T] : (X(t, \omega), Z(t, \omega)) \in D\},
\]

where \(D\) is the stopping region \(D = \{(x, z) : V(x, z) = (K - x)^+\}\); see §2.2 of [13]. For each \(t > 0\), we observe that, by the definition of \(b(z)\),

\[
(X(t, \omega), Z(t, \omega)) \in D \quad \text{if and only if} \quad X(t, \omega) \leq b(Z(t, \omega)).
\]
Therefore,

\[
\tau_D(\omega) = \inf\{t \in [0, T] : X(t, \omega) \leq b(Z(t, \omega))\} \\
= \inf\{t \in [0, T] : X(t, \omega) = b(Z(t, \omega))\} \\
= \tau^*(\omega)
\]

The second “=” in the above is due to the path-continuity of the process. Indeed, suppose, on the contrary that the “<” held there, then there will exist \( t_2 < t_1 \) such that

\[
X(t_2, \omega) < b(Z(t_2, \omega)); \quad X(t_1, \omega) = b(Z(t_1, \omega)).
\]

This is impossible whenever we start the process \((X, Z)\) at \((x, z) \in C\) which is above the curve \( z \mapsto b(z) \). □

**Proof of Proposition 3.4.** we claim that, for any two \( z \) and \( z' \),

\[
(b(z) - b(z'))(V(x, z) - V(x, z')) \leq 0, \quad \text{for any } x \text{ between } b(z), b(z').
\]

Indeed, suppose that \( b(z) < b(z') \). Then, for any \( x : b(z) < x < b(z') \), by the definition of \( z \mapsto b(z) \), and the proof of Theorem 3.3, \( V(x, z') = (K - x)^+ \), while \( V(x, z) > (K - x)^+ \). Thus, \( V(x, z') > V(x, z) \). On that other hand, if \( b(z) > b(z') \), then the same argument gives \( V(x, z') < V(x, z) \).

Now consider the time-parameter striking curve \( t \to b(z(t)), \quad t \in [0, T] \), and suppose that it is monotone increasing in \( t \). Then, by the above “anti-comonotone” property, since \( b(z) \) in increasing in \( z \), the value \( z \to V(x, z) \) must be monotone decreasing, for each \( x \in (\inf b(\cdot), \sup b(\cdot)) \). This is a contraction. We have assumed that the volatility \( \sigma(z) \) is increasing in \( z \), thus as a consequence the value \( V(\cdot) \) must be monotone increasing too; any option must get higher value when the volatility of the underlying asset gets higher. Therefore the curve \( t \to b(z(t)) \) cannot be monotone increasing in \( t \in [0, T] \). The two increasing convex curves which squeeze our striking curve are those striking curves for the two American options of each the underlying asset follows the standard GBM with constant volatilities \( \sigma_1 \) and \( \sigma_2 \) respectively . □

5. Conclusion

1. It is the contribution of SDDE in financial economics to formulate a risky asset for which its present value brings the memory of its historical values. The choice of the memory \( Y(t) = X(t - t_0) \) mentioned in Section 1 reflects that a past time-instant \( t_0 \) is the source of the “after-effect”, and this can be extended to \( n \) time-instants \( t_0, \cdots, t_n \). The European option pricing based on the SDDE of this type
appeared in [1], [8], and [10]; these papers rely on the martingale aspect of the pricing theory. The American option pricing is certainly at the beginning to be viewed from the martingale aspect, as one may see from [13]. However it is more important then to move to view American options in the Markov process aspect; since only then the striking curve can be discussed, that is, the parametric curve to separate the region in which the owner of the option holds and waits, and the region in which the owner exercises and gets the (positive) reward. In the basic (that is the constant volatility is assumed) BSM theory, the striking curve is a monotone increasing and convex curve across the time horizon $[0,T]$; see Chapter 8 of [15] or Section 25.1 of [13]. The novelty of this article is that, if we assume the volatility is the ratio of the asset’s present value $X(t)$ and historical value $Y(t)$, with the choice of the historical values being exponentially averaged, then a parametric striking curve still appears, yet it is a “anti-comonotone” curve, as shown in Proposition 3.4. This would assert that the striking curve is skewed, due to the historical value of the asset. We mention that, to our knowledge, this situation is firstly observed, and we would compare this result with one main conclusion in [7], in which the authors discuss the volatility smile of European options under the model.

2. Financial economics under uncertainly is one fundamental topic in Microeconomics, and we refer to Chapter 6 of [11]. Option pricing is one aspect, and here we would discuss of the effect of the asset’s historical value to the pricing turnout; the volatility smile for European options in [7], and the striking-curve skewness for American options in this article. Portfolio selection (under uncertainty) is also one classical aspect, it can be traced to the classics [12], and we cite two very recent papers in this aspect [3] and [14]. Study of portfolio selections of risky assets with the memory effects seems to be very promising.

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