The Semi-Chiral Quotient, Hyperkähler Manifolds and T-duality

P. Marcos Crichigno†

†C.N. Yang Institute for Theoretical Physics
State University of New York at Stony Brook, NY 11790, USA

Abstract

We study the construction of generalized Kähler manifolds, described purely in terms of $\mathcal{N} = (2, 2)$ semichiral superfields, by a quotient using the semichiral vector multiplet. Despite the presence of a $b$-field in these models, we show that the quotient of a hyperkähler manifold is hyperkähler, as in the usual hyperkähler quotient. Thus, quotient manifolds with torsion cannot be constructed by this method. Nonetheless, this method does give a new description of hyperkähler manifolds in terms of two-dimensional $\mathcal{N} = (2, 2)$ gauged non-linear sigma models involving semichiral superfields and the semichiral vector multiplet. We give two examples: Eguchi-Hanson and Taub-NUT. By T-duality, this gives new gauged linear sigma models describing the T-dual of Eguchi-Hanson and NS5-branes. We also clarify some aspects of T-duality relating these models to $\mathcal{N} = (4, 4)$ models for chiral/twisted-chiral fields and comment briefly on more general quotients that can give rise to torsion and give an example.

†crichigno@max2.physics.sunysb.edu
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1 Introduction

Recent developments in both physics and mathematics are renewing the interest in general $d = 2$, $\mathcal{N} = (2, 2)$ sigma models. From the physics perspective, these models describe string compactifications with NS-NS fluxes and, from the mathematics perspective, they provide a useful tool in exploring aspects of Generalized Complex Geometry. This is an example of the interesting interplay between geometry and supersymmetry, initiated by Zumino in the classic work [1]. It is well known by now that the conditions under which $d = 2$, $\mathcal{N} = (1, 1)$ sigma models (with no Wess-Zumino term) admit an extended supersymmetry, can be solved by requiring the target space to be Kähler, for the case of $\mathcal{N} = (2, 2)$, and hyperkähler for $\mathcal{N} = (4, 4)$ [2]. The action for the sigma model is then simply given by the Kähler potential $K(\Phi^a, \bar{\Phi}^a)$ of the target space, with the complex coordinates $\Phi^a$ identified with $\mathcal{N} = (2, 2)$ chiral superfields satisfying $\bar{D}_+ \Phi^a = D_- \Phi^a = 0$. These ideas lead to a variety of applications of supersymmetric methods to Kähler geometry. An example of this is the hyperkähler quotient [3, 4]. This method is based on the gauging of isometries along directions parametrized by chiral superfields and provides a powerful method for constructing potentials describing hyperkähler manifolds.

The introduction of a Wess-Zumino term generalizes these models in an interesting way by introducing torsion (i.e., H-flux), leading to what is known as bihermitean geometry [5]. To describe general bihermitean models in $\mathcal{N} = (2, 2)$ superspace, it is necessary to include directions parametrized by twisted-chiral and semichiral superfields. Since the $\mathcal{N} = (2, 2)$ vector multiplets introduced in [6, 7] allow the gauging of isometries in general bihermitean manifolds, it has led us in the present paper to consider more general quotients. This can be used to construct explicit generalized potentials, few of which are known. It would be particularly interesting if one could find potentials describing non-Kählerian manifolds (see, e.g., [8] for a discussion of related issues), but it would also be useful to have explicit generalized descriptions of usual hyperkähler manifolds.

The main goal of this paper is therefore to study certain quotients in a bihermitean setting. We focus on $d = 2$ sigma models involving only semichiral superfields, with a $U(1)$ isometry, gauged by the action of the semichiral vector multiplet. We show that the quotient of a hyperkähler manifold is hyperkähler. Thus, despite what could have been naively expected, the resulting quotient manifold has no torsion. We give two explicit examples in four dimensions: Eguchi-Hanson and Taub-NUT. We also perform a T-duality on the latter, which gives us a new $\mathcal{N} = (2, 2)$ gauged linear sigma model describing NS5-branes and we briefly discuss a type of quotient that does lead to an H-flux, by incorporating coordinates other than semichiral.

The paper is organized as follows. The remainder of this Section contains no new results, but simply reviews some basic elements of general $\mathcal{N} = (2, 2)$ models. We focus on the geometry of general semichiral sigma models and the semichiral vector multiplet,
which gauges isometries of these sigma models. In Section 2 we describe the semichiral quotient and state one of the main results of the paper. In Section 3 we clarify and extend a duality relation of these semichiral models with $\mathcal{N} = (4, 4)$ models for chiral and twisted-chiral fields. In Sections 4 and 5 we give the explicit construction of two well-known gravitational instanton solutions (Eguchi-Hanson and Taub-NUT) and in Section 6 we describe NS5-branes as a T-dual of Taub-NUT and comment on instanton corrections. In Section 7 we present the T-dual of Eguchi-Hanson. We conclude with a summary and discussion of open problems.

1.1 General $\mathcal{N} = (2, 2)$ sigma models

The models originally studied in [11] are not the most general since they don’t have a Wess-Zumino term. This motivated the study of general $\mathcal{N} = (1, 1)$ models [5] (see also [9, 10]), containing both a metric and a $b$-field, the latter corresponding to a Wess-Zumino term in the action. A general $\mathcal{N} = (1, 1)$ sigma model is described by

$$\mathcal{L} = -\frac{1}{4} \int d^2\theta (D_+ \Phi^\mu)(D_- \Phi^\nu) \left( g_{\mu\nu}(\Phi) + b_{\mu\nu}(\Phi) \right),$$

(1)

where the $\mathcal{N} = (1, 1)$ superfields $\Phi^\mu$ are target space coordinates, $g_{\mu\nu}$ is the target space metric, and $b_{\mu\nu} = -b_{\nu\mu}$ is the NS-NS 2-form. In the case $b = 0$, it reduces to the original case studied by Zumino.

Studying the conditions under which such models admit an extended supersymmetry led to the discovery [5] of a rich geometrical structure: generalized Kähler geometry. It was found that, associated to the $\mathcal{N} = (2, 2)$ supersymmetry, there are two complex structures $J_\pm$ and the metric is hermitean with respect to both. Furthermore, the presence of the $b$-field induces a connection with torsion (proportional to $H = db$) and the complex structures are covariantly constant with respect to this connection. This is what is known as bihermitean geometry. The framework of Generalized Complex Geometry, recently developed by Hitchin [11] and Gualtieri [12], describes this geometry as a generalized Kähler geometry and we will use these terms interchangeably.

Since the class of models studied by Zumino admits an explicit $\mathcal{N} = (2, 2)$ formulation, it is natural to wonder if these general models admit such a description and, indeed, they do. This was made possible by the introduction of $\mathcal{N} = (2, 2)$ twisted-chiral superfields satisfying $\bar{D}_+ \chi = D_- \chi = 0$. One considers a scalar function, depending both on chiral and twisted-chiral superfields, i.e., $K(\Phi, \bar{\Phi}, \chi, \bar{\chi})$, as a potential for the bihermitean geometry. Due to the twisted nature of the constraints on $\chi$ (relative to $\Phi$), upon reduction to $\mathcal{N} = (1, 1)$, one finds an action of the type (1) with a non-zero $b_{\mu\nu}$, provided by cross-terms like $K_{\Phi \chi}$. Interestingly, these models fall outside the classification of [2]. Indeed, when the condition $K_{\Phi \Phi} + K_{\bar{\Phi} \bar{\Phi}} = 0$ is satisfied, the model has $\mathcal{N} = (4, 4)$ supersymmetry without being hyperkähler [5]. An example of this is the $S^3 \times S^1$ WZW model [13].

For some time, however, it remained unclear what set of $\mathcal{N} = (2, 2)$ fields provides a complete description of bihermitean geometry. In $d = 2$, $\mathcal{N} = (2, 2)$ superspace has four
fermionic coordinates $\theta^\pm, \bar{\theta}^\pm$, where the $\pm$ index stands for chirality under Lorentz transformations, and $\bar{\theta}^\pm = (\theta^\pm)^*$. Thus, the most general, linear, SUSY-invariant, constraints one can impose are \[14\]

\[
\begin{align*}
\bar{D}_+ \Phi &= \bar{D}_- \Phi = 0 & \text{Chiral} \\
\bar{D}_+ \chi &= \bar{D}_- \chi = 0 & \text{Twisted chiral} \\
\bar{D}_+ X_L &= \bar{D}_- X_R = 0 & \text{Left and Right semichiral}.
\end{align*}
\] (2)

It was believed at the time, and explicitly conjectured in \[14\], that this is the set of fields which gives the most general description of an $\mathcal{N} = (2,2)$ sigma model. This was finally proven\footnote{To be able to integrate out the auxiliary $\mathcal{N} = (1,1)$ spinor superfields, it is necessary to have the same number of left and right semichiral fields.} in \[16\]. The action is given in terms of a generalized Kähler potential

\[\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} K(\Phi, \bar{\Phi}, \chi, \bar{\chi}, X_L, \bar{X}_L, X_R, \bar{X}_R).\] (3)

From the constraints (2) one sees that the generalized potential $K$ is defined up to generalized Kähler transformations $f(\phi, \chi, X_L) + g(\bar{\phi}, \bar{\chi}, \bar{X}_L) + \bar{f}(\bar{\phi}, \chi, \bar{X}_L) + \bar{g}(\phi, \bar{\chi}, X_L)$, since these vanish upon integration in superspace. Upon reduction to $\mathcal{N} = (1,1)$ fields, the action has the form (1) and one can read off the metric and $b$-field completely in terms of second derivatives of $K$.

A comment on notation: $\mathcal{N} = (2,2)$ spinor derivatives are denoted by $\mathbb{D}_\pm$ to distinguish them from the $\mathcal{N} = (1,1)$ derivatives $D_\pm$. We usually denote the lowest $\mathcal{N} = (1,1)$ components of chiral and twisted-chiral fields by the same letters as the $\mathcal{N} = (2,2)$ fields, whereas for semichiral fields we write $X_{L,R}\mid = X_{L,R}$. When writing the metric and $b$-field, it should be understood that we are referring to the $\mathcal{N} = (1,1)$ components.

1.2 Geometry of semichiral sigma models

Consider a non-linear sigma model of a set of semichiral superfields $X_{L,R}^a, a, a' = 1, \ldots, d_s$ with an action given by

\[\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} K(X_L, \bar{X}_L, X_R, \bar{X}_R).\] (4)

These models were first studied in \[17\], showing that upon reduction to $\mathcal{N} = (1,1)$, they lead to a general non-linear sigma model of the type of (1). However, semichiral superfields are less constrained than chiral and twisted-chiral fields and contain auxiliary superfields which, when integrated out, induce non-linearities in the $\mathcal{N} = (1,1)$ action. As a consequence, the metric and $b$-field are non-linear functions of second derivatives of

\footnote{To avoid the reader’s confusion, it is worth mentioning that the conclusion in Ref. [15] (which includes other important results), that this is not the case, is erroneous. See [16] for an explanation.}
These can be written compactly \cite{15, 16} in terms of the complex structures $J_\pm$ and a closed 2-form $\Omega$ as

$$g = \Omega [J_+, J_-], \quad b = \Omega \{J_+, J_-\}. \quad (5)$$

The complex structures and $\Omega$ are completely determined by the generalized potential by

$$J_+ = \left( \begin{array}{ccc} J_s & 0 \\ \mathcal{K}_{RL}^{-1} C_{LL} & \mathcal{K}_{RL}^{-1} \mathcal{J}_s C_{LR} \end{array} \right), \quad J_- = \left( \begin{array}{ccc} \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} & \mathcal{K}_{LR}^{-1} C_{RR} \\ 0 & J_s \end{array} \right), \quad (6)$$

where $J_s$ is a 2$d_s$-dimensional matrix of the form $\text{diag}(i, -i)$ and

$$\Omega = \left( \begin{array}{ccc} 0 & \mathcal{K}_{LR} \\ -(\mathcal{K}_{LR})^t & 0 \end{array} \right), \quad (7)$$

with

$$\mathcal{K}_{LL} = \left( \begin{array}{ccc} K_{LL} & K_{LL} \\ K_{LL} & K_{LL} \end{array} \right), \quad \mathcal{K}_{LR} = \left( \begin{array}{ccc} K_{LR} & K_{LR} \\ K_{LR} & K_{LR} \end{array} \right), \quad (8)$$

where $K_{LR} \equiv \frac{\partial^2 K}{\partial \bar{X}_L \partial X_R}$, etc. and $\mathcal{K}_{LR}^{-1} \equiv (\mathcal{K}_{RL})^{-1}$.

In four dimensions \textit{(i.e.}, $d_s = 1$) there’s an additional structure, leading to the anti-commutator of the complex structures to being proportional to the identity, namely

$$\{J_+, J_-\} = c \mathbb{I}, \quad (9)$$

where $c$ is a scalar function given by

$$c = -2 \frac{|K_{LR}|^2 + |K_{LR}|^2 - 2K_{LL}K_{RR}}{|K_{LR}|^2 - |K_{LR}|^2}. \quad (10)$$

As we shall review next, it contains important information about the geometry; when $c$ is a constant and $|c| < 2$, the manifold is hyperkähler.

### 1.3 Hyperkähler case

As shown in \cite{16}, a generalized Kähler manifold of $4N$ real dimensions, described in terms of semichiral superfields, is hyperkähler if $\{J_+, J_-\} = c \mathbb{I}$ with $c$ a constant and $|c| < 2$ (see also \cite{15} for the particular case $c = 0$). This is easy to see from the expression for the $b$-field in \cite{5}, since $\Omega$ is a closed 2-form, the torsion, $H = db = \Omega dc$, vanishes for constant $c$. If the manifold is hyperkähler, there must be three complex structures and, indeed, a third complex structure $J_3$ can be constructed from $J_\pm$ by

$$J_3 = \frac{1}{\sqrt{\left(\frac{2}{c}\right)^2 - 1}} \left( \mathbb{I} - \frac{2}{c} J_+ J_- \right). \quad (11)$$
A trivial example of a hyperkähler manifold (and one which will be used in what follows) is flat $\mathbb{R}^{4n}$ with a constant $b$-field. This is described by the generalized potential

$$K_{\mathbb{R}^{4n}} = \sum_{i=1}^{n} (\bar{X}^{i}_L X^{i}_L + \bar{X}^{i}_R X^{i}_R + \alpha (\bar{X}^{i}_R X^{i}_L + \bar{X}^{i}_L X^{i}_R)) \quad (12)$$

From equations (5-8), one finds the (constant) metric, $b$-field, and complex structures satisfying

$$\{J_{+}, J_{-}\} = 2(1 - \frac{2}{\alpha^2})\mathbb{I}. \quad (13)$$

For the metric to be positive definite, $\alpha^2 > 1$ is required, which also ensures $|c| < 2$. For the special value $\alpha^2 = 2$, the $b$-field vanishes.

### 1.4 Semichiral vector multiplet

The semichiral vector multiplet [6, 7] was introduced to gauge isometries along semichiral directions, e.g.,

$$\delta X_L = i\lambda, \quad \delta X_R = i\lambda. \quad (14)$$

It is described in terms of three real supervector fields $V^\alpha = (V_L, V_R, V')$, with gauge transformations

$$\delta V_L = i(\bar{\Lambda}_L - \Lambda_L), \quad \delta V_R = i(\bar{\Lambda}_R - \Lambda_R), \quad \delta V' = (\Lambda_R + \bar{\Lambda}_R - \Lambda_L - \bar{\Lambda}_L). \quad (15)$$

It’s convenient to introduce the complex combinations

$$\mathbb{V} = \frac{1}{2}(-V' + i(V_L - V_R)), \quad \bar{\mathbb{V}} = \frac{1}{2}(-V' + i(V_L + V_R)), \quad (16)$$

with gauge transformations

$$\delta \mathbb{V} = \Lambda_L - \Lambda_R, \quad \delta \bar{\mathbb{V}} = \Lambda_L - \bar{\Lambda}_R. \quad (17)$$

The corresponding chiral and twisted-chiral field strengths are

$$F = \bar{D}_+ D_- \mathbb{V}, \quad \bar{F} = \bar{D}_+ D_- \bar{\mathbb{V}}. \quad (18)$$

Thus, the nonvanishing commutation relations are [7]

$$\{\bar{\nabla}_+, \nabla_\pm\} = iD_\pm, \quad i\{\bar{\nabla}_+, \bar{\nabla}_\pm\} = F, \quad i\{\nabla_+, \nabla_-\} = \bar{F},$$

where $\nabla_\pm$ are gauge-covariant superderivatives. The kinetic terms for the gauge fields are given by

$$L_{\text{gauge}} = \int d^4\theta \frac{1}{e^2}(FF - \bar{F}\bar{F}). \quad (19)$$
It’s also possible to add Fayet-Iliopoulos (FI) terms of the form

\[
\mathcal{L}_{FI} = - \int d^4 \theta \left( t \bar{V} + s \delta \bar{V} + \text{c.c.} \right) = - \int d^4 \theta t_\alpha V^\alpha,
\]

where we defined \( t_\alpha \equiv -(\text{Im}(s + t), \text{Im}(s - t), \text{Re}(s + t)) \). These will play an important role in what follows. Upon reduction to \( \mathcal{N} = (1, 1) \), \( (\ref{eq:kin}) \) gives the usual kinetic terms. The only dimensionful scale is \( |e| = 1 \) and the low energy limit corresponds to taking \( e \to \infty \). Therefore, the kinetic terms are irrelevant in the IR limit and the gauge fields \( V', V_L, V_R \) become non-dynamical and are integrated out. Thus, the gauged linear sigma model will flow in the IR to a non-linear sigma model given by a semichiral quotient, which we now describe.

2 The Semichiral Quotient

Here we describe what we refer to as the semichiral quotient. We consider a bihermitean manifold \( \mathcal{M} \) of \( d = 4(N + 1) \) real dimensions, parameterized by semichiral coordinates \((X^a_L, X^{a'}_R)\) with \( a, a' = 1, ..., N + 1 \) and generalized potential \( K(X^a_L, X^{a'}_R) \). We assume the existence of a \( U(1) \) Killing vector

\[
k = k^a \partial_a + k^{a'} \partial_{a'} + k^{a'} \partial_{a'},
\]

(21)

generating the isometry

\[
\delta X = [\lambda k, X],
\]

(22)

where \( \lambda \) is the parameter of the transformation and \( X \) is any of the coordinates. We now choose coordinates \((X^i_L, X^{i'}_R) = (X^i_L, X^{i'}_R, X_L, X_R)\), with \( i, i' = 1, ..., N \), which are adapted to the isometry and the Killing vector takes the form

\[
k = i (\partial_L - \bar{\partial}_L + \partial_R - \bar{\partial}_R).
\]

(23)

In these adapted coordinates, the generalized potential depends explicitly on the \( 4N \) neutral coordinates \((X^i_L, X^{i'}_R)\) and the 3 invariant combinations \( X^\alpha = (X_L + \bar{X}_L, X_R + \bar{X}_R, i(X_R - \bar{X}_R - X_L + \bar{X}_L)) \). Now we proceed to gauge this isometry by promoting the parameter \( \lambda \) to a corresponding semichiral field and introducing a semichiral vector multiplet. Then, the function \( \hat{K} \) is defined by

\[
\hat{K}(X^i_L, X^{i'}_R) = K(X^i_L, X^{i'}_R, X^\alpha + V^\alpha) - t_\alpha V^\alpha,
\]

(24)

where \( V^\alpha = V^\alpha(X^i_L, X^{i'}_R) \) is given by solving its equations of motion

\[
\frac{\partial K(X^i_L, X^{i'}_R; X^\alpha + V^\alpha)}{\partial V^\alpha} = t_\alpha
\]

(25)

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and choosing the gauge $X^\alpha = 0$. The new potential $\hat{K}$ depends on $4N$ coordinates (and three FI parameters $t_\alpha$), and describes the quotient manifold $\hat{\mathcal{M}}$ of real dimension $4N$.

Now we state one of our main results. Assume that $\mathcal{M}$ is a hyperkähler manifold and therefore

$$\{J_+, J_-\} = c \mathbb{I}, \quad \text{(26)}$$

with $c$ a constant, as discussed in Section 1.3. Then, the anticommutator of the complex structures on the quotient manifold $\hat{\mathcal{M}}$ is given by

$$\{\hat{J}_+, \hat{J}_-\} = c \mathbb{I} \quad \text{(27)}$$

(with the same $c$ on the right-hand side). In particular, this implies that the quotient manifold is also hyperkähler. In the current setting, the proof of (27) requires some rather tedious algebra, but is straightforward. Imposing (26) leads to the set of equations

$$\{\mathcal{K}_{LR}^{-1} C_{RR} \mathcal{K}_{RL}^{-1}, J_s\} = 0, \quad \text{(28)}$$

$$J_s \mathcal{K}_{LR}^{-1} I_s \mathcal{K}_{RL} + \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} I_s + \mathcal{K}_{LR}^{-1} C_{RR} \mathcal{K}_{RL}^{-1} C_{LL} = c \mathbb{I}, \quad \text{(29)}$$

and those which follow from these exchanging ($L \leftrightarrow R$). Using standard relations between second derivatives of Legendre-transformed functions, and identities for matrix inverses, we show that these equations also hold for $\hat{K}$, proving the assertion (27) (see Appendix A for more details).

A brief comment is in order. In showing that the structure (26) is preserved by the quotient, we have actually not made use of the fact that $c$ is a constant. Thus, one could in principle extended our results to bihermitean geometries satisfying (26), other than hyperkähler (with $c$ an arbitrary function), if there are any such manifolds. This, however, is not the case due to the following result [18]. Although the set of equations (28, 29) are satisfied identically in four dimensions, they highly restrict the geometry in higher dimensions. So much indeed, that the only manifolds satisfying (26) in $d \geq 8$ are those with a constant $c$, i.e., hyperkähler manifolds.

### 2.1 Geometrical interpretation

It might seem surprising at first that the semichiral quotient coincides with the hyperkähler quotient. However, this is clarified by the following geometrical interpretation [19]. The hyperkähler quotient [3, 4] is based on assuming the existence of three symplectic 2-forms $\omega^p$, $p = 1, 2, 3$, and a triholomorphic Killing vector $k$, i.e.,

$$\mathcal{L}_k \omega^p = i_k d\omega^p + d(i_k \omega^p) = 0. \quad \text{(30)}$$

Since $d\omega^p = 0$, this implies the existence (locally) of the three moment maps, $\mu^p$, such that

$$i_k \omega^p = d\mu^p. \quad \text{(31)}$$
Setting the moment maps to zero (and dividing by the isometry), leads to the hyperkähler quotient. The relation with the semichiral quotient is based on the observation that if $[J_+, J_-]$ is invertible (which requires the presence of only semichiral fields), the closed 2-form

$$\Omega = g^{-1} [J_+, J_-]$$

is well defined. This symplectic form can be decomposed into its holomorphic and anti-holomorphic part, with respect to both complex structures $J_{\pm}$, i.e.,

$$\Omega = \Omega_{-}^{(2,0)} + \bar{\Omega}_{-}^{(0,2)} = \Omega_{+}^{(2,0)} + \bar{\Omega}_{+}^{(0,2)}$$

and $d\Omega = 0$ implies

$$\partial \Omega_{\pm}^{(2,0)} = \bar{\partial} \Omega_{\pm}^{(2,0)} = 0,$$

and the complex conjugates. This implies the existence of four moment maps $\mu_{\pm}, \bar{\mu}_{\pm}$, subject to the reality condition

$$\mu_- + \mu_- = \mu_+ + \mu_+,$$

which follows from (33). Thus, there are three independent moment maps and the semichiral quotient coincides with the hyperkähler quotient.

It can also be understood in these geometrical terms why only hyperkähler manifolds satisfy (26). In a generalized Kähler manifold, the 3-form $H = db$ has no $(3,0)$ or $(0,3)$ part (see, e.g., (12)) with respect to both $J_{\pm}$, i.e.,

$$H = H_{\pm}^{(1,2)} + H_{\pm}^{(2,1)}.$$

Assuming (26), one has $H = \Omega dc$. Using (33) and $dc = \partial c + \bar{\partial} c$, one sees that $(3,0)$ and $(0,3)$ parts appear. The requirement that they vanish implies

$$\partial c = \bar{\partial} c = 0.$$

Thus, $c$ is a constant and $H$ vanishes completely.

2.2 Comment on more General Quotients

As we have just seen, quotients involving only semichiral fields will not lead to a non-trivial $b$-field. However, considering several types of fields typically does. Here we give a simple

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4Even in the presence of only semichiral fields, $[J_+, J_-]$ can fail to be invertible at some points or loci in the manifold, leading to type change. We shall not consider this case here.

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example. Consider a set of semichiral fields and a single chiral field $\Phi$, gauged by the usual vector multiplet $V$, i.e.,

$$K = \bar{X}_L e^V X_L + \bar{X}_R e^V X_R + \alpha \left( \bar{X}_R e^V X_L + \bar{X}_L e^V X_R \right) + t\bar{\Phi} e^V \Phi - rV. \quad (38)$$

Integrating out $V$ (and choosing the gauge $\Phi = 1$) leads to

$$K = r \log \left( \bar{X}_L X_L + \bar{X}_R X_R + \alpha \left( \bar{X}_R X_L + \bar{X}_L X_R \right) + t \right). \quad (39)$$

From (11) we find

$$c = -2 + \frac{4t(\alpha^2 - 1)}{\alpha(\alpha t - R)}, \quad (40)$$

where $R \equiv \bar{X}_R X_L + \bar{X}_L X_R + \alpha(\bar{X}_L X_L + \bar{X}_R X_R)$. Thus $c$ is not a constant and there’s a non-trivial $b$-field. Although we will not analyze this model in full detail here, we can already study some features. From (39), one sees that the limit $t \to \infty$ corresponds to flat space, while $t \to 0$ gives a singular metric. For finite $t$, the metric becomes singular ($c = \pm 2$) for $R = t/\alpha$ and $R \to \infty$.

## 3 T-Duality

A duality relation between hyperkähler manifolds, described in terms of semichiral superfields with $c = 0$, and $\mathcal{N} = (4,4)$ models for chiral/twisted-chiral fields was described in [15]. Actually, understanding this relation was one of the motivations for introducing the new vector multiplets and studying T-duality [20]. In this Section we would like clarify the exact relation of the duality in [15] to T-duality and offer a geometrical interpretation, which also allows us to consider Kähler manifolds with $c \neq 0$ and even non-Kählerian manifolds (that may still have $\mathcal{N} = (4,4)$). As we shall see, this depends on the character of the isometry along which the duality is performed. We first discuss T-duality along a translational isometry, which leads to a hyperkähler manifold. Then, we discuss T-duality along a general isometry.

### 3.1 Translational isometry

The duality described in [15] involves two steps. Given a potential $\hat{F}(\Phi, \bar{\Phi}, \chi, \bar{\chi})$ satisfying the Laplace equation, one first constructs a potential $F(\Phi, \bar{\Phi}, \chi, \bar{\chi})$. Then, one performs a Legendre transformation to semichiral superfields. It is the first step which we reinterpret as a rotation of the $\mathcal{N} = (1,1)$ components by a fixed angle. As we shall see below, considering an arbitrary (constant) rotation by an angle $\nu$ leads to a non-zero (constant) $c$.  

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5The author wishes to thank Martin Rocek for this suggestion.
Consider a potential $\hat{F}(\phi, \bar{\phi}, \chi, \bar{\chi})$ and assume that there’s a translational isometry, generated by the Killing vector
\begin{equation}
k = i(\partial_\phi - \partial_{\bar{\phi}} - \partial_\chi + \partial_{\bar{\chi}}). \tag{41}\end{equation}
Thus, in adapted coordinates
\begin{equation}\hat{F} = \hat{F}(\phi + \bar{\phi}, \chi + \bar{\chi}, i(\phi - \bar{\phi} + \chi - \bar{\chi})). \tag{42}\end{equation}
Assume now that the potential describes an $\mathcal{N} = (4, 4)$ model and, therefore, satisfies the Laplace equation
\begin{equation}\hat{F}_{\phi\bar{\phi}} + \hat{F}_{\chi\bar{\chi}} = 0. \tag{43}\end{equation}
The important observation now is that a rotation among the $\mathcal{N} = (1, 1)$ fields, $(\phi, \chi)$, is allowed and preserves the Laplace equation. Then, when integrating up to the $\mathcal{N} = (2, 2)$ potential, one must choose what to call a chiral or twisted-chiral field and we choose to take the rotated fields. That is, we consider the transformation
\begin{equation}\phi \rightarrow \cos(\theta)\phi + \sin(\theta)\chi, \quad \chi \rightarrow \cos(\theta)\chi - \sin(\theta)\phi. \tag{44}\end{equation}
For convenience, we introduce $\theta = \nu + \frac{\pi}{4}$ and define the potential $F(\Phi, \bar{\Phi}, \chi, \bar{\chi})$ by
\begin{equation}F = \hat{F}(\Phi + \bar{\Phi}, \chi + \bar{\chi}, i(c(\Phi - \bar{\Phi}) + s(\chi - \bar{\chi}))), \tag{45}\end{equation}
where we have abbreviated $\cos(\theta) = c, \sin(\theta) = s$. The Killing vector is now given by
\begin{equation}k = i[(s(\partial_\phi - \partial_{\bar{\phi}}) - c(\partial_\chi - \partial_{\bar{\chi}})], \tag{46}\end{equation}
which implies the transformations for the matter fields
\begin{equation}\delta\Phi = is\lambda, \quad \delta\chi = -ic\lambda. \tag{47}\end{equation}
This isometry can be gauged by the Large Vector Multiplet (LVM) \[ \text{LVM} \] defined similarly to the SVM by
\begin{equation}V_L = \frac{1}{2}(-V' + i(V^\phi - V^\chi)), \quad V_R = \frac{1}{2}(-V' + i(V^\phi + V^\chi)), \tag{48}\end{equation}
where the real vector fields $V^\alpha = (V^\phi, V^\chi, V')$ transform as
\begin{equation}\delta V^\phi = i(\bar{\Lambda} - \Lambda), \quad \delta V^\chi = i(\bar{\Lambda} - \Lambda), \quad \delta V' = -(\Lambda + \bar{\Lambda}) + \bar{\Lambda} + \bar{\Lambda}. \tag{49}\end{equation}
Following [20], we perform a T-duality to semichiral fields by defining
\begin{equation}K(X_L, X_R) = F(\Phi + \bar{\Phi} + sV^\phi, \chi + \bar{\chi} + cV^\chi, i(c(\Phi - \bar{\Phi}) + s(\chi - \bar{\chi}))) - csV' - [X_L V_L + X_R V_R + c.c.]. \tag{50}\end{equation}
In the gauge $\Phi = \chi = 0$, we have
\[ K(\bar{X}_L, \bar{X}_R) = F(sV^\phi, cV^\chi, -csV') - \frac{1}{2} \left[ iV^\phi(\bar{X}_L - \bar{X}_L + \bar{X}_R - \bar{X}_R) - iV^\chi(\bar{X}_L - \bar{X}_L - \bar{X}_R + \bar{X}_R) - V'(\bar{X}_L + \bar{X}_L + \bar{X}_R + \bar{X}_R) \right]. \] (51)

Integrating out the LVM, i.e., solving
\[ \frac{\partial K}{\partial V^\alpha} = 0 \] (52)
for the vector fields $V^\alpha$ leads to the semichiral potential. From the definition (10), and using standard implicit differentiation relations (see Appendix B for more details), we find a non-zero $c$ given by
\[ c = -2 \cos(2\theta). \] (53)
For the particular case $\theta = \pi/4$, this reduces to the duality described in [15].

A short observation that will be useful later is that one may alternatively rescale the fields $\phi, \chi$ in (46) to bring the Killing vector to its usual form. Then, the potential $F$ will satisfy a scaled Laplace equation: If $K$ describes a hyperkähler manifold with a constant $c = 2(1 - \frac{\alpha^2}{2})$, the dual potential satisfies
\[ F_{\phi\bar{\phi}} + (\alpha^2 - 1)F_{\chi\bar{\chi}} = 0. \] (54)

### 3.2 General isometry

As we have just discussed, T-dualizing an $\mathcal{N} = (4,4)$ model along a translational isometry using the LVM leads to a hyperkähler manifold, described in terms of semichiral fields. In showing this, the form of the Killing vector was crucial. Indeed, if it acts by translation on $\Phi$ and $\chi$ by equal amounts, then $c = 0$, while if it acts by different amounts, it leads to a non-zero (but constant) $c$. We wish to investigate now what happens for a general isometry of the form $k = k^\Phi(\Phi)\partial_\Phi + k^\chi(\chi)\partial_{\bar{\chi}} + c.c.$ If $K$ is invariant under the isometry, the gauging along a general Killing vector is given by [20]
\[ K^{(g)} = \exp \left( -\frac{1}{4}V^\phi L_{(J_+J_-)k} - \frac{1}{4}V^\chi L_{(J_+J_-)k} - \frac{1}{4}V' L_{J_+J_-k} \right) K. \] (55)

By implicit differentiation (again, see Appendix B for details), we find
\[ c = 2 \left( \frac{|k^\Phi|^2 - |k^\chi|^2}{|k^\Phi|^2 + |k^\chi|^2} \right) \left. \frac{\partial K}{\partial V^\alpha} \right|_{\partial K/\partial V=0}. \] (56)

\[ ^6 \text{In coming Sections we will perform T-duality transformations in the other direction, namely from semichiral fields to chiral/twisted-chiral by the use of the semichiral vector multiplet. We expect, however, the same relations to hold.} \]
Note that although this expression does not depend on the potential explicitly, it does depend on it implicitly; to write the right-hand side in terms of semichiral coordinates, the relation of chiral/twisted-chiral fields to semichiral fields given by the Legendre transform is needed. In the case 
\[ k^\Phi = -k^\bar{\Phi} = i \cos(\theta) \text{ and } k^{\chi} = -k^{\bar{\chi}} = i \sin(\theta), \]
we recover (53). We conclude from (56) that for a general isometry \( c \) will not be a constant and the dual geometry will not be hyperkähler, even if the T-duality preserves the supersymmetry (the isometries preserving \( \mathcal{N} = (4, 4) \) in this context are translational and rescaling [21]).

As an example, consider the gauging of the isometry along the \( S^1 \) in the \( SU(2) \times U(1) \) WZW model, described in terms of chiral/twisted-chiral superfields [13], recently studied in [22]. The isometry in this case acts by a rescaling of the fields, i.e.,
\[ k = \Phi \partial_\Phi + \bar{\Phi} \partial_{\bar{\Phi}} + \chi \partial_\chi + \bar{\chi} \partial_{\bar{\chi}}. \]

\[ \text{T-dualizing along this direction, the dual potential again describes an } SU(2) \times U(1) \text{ WZW model, which is not hyperkähler. Indeed, from (56), one finds} \]
\[ c = \frac{2}{\sqrt{1 - 4e^{-X'}}}, \]
where \( X' = X_L + X_R + X_R + X_R \). Since the isometry in the \( SU(2) \times U(1) \) WZW model corresponds to a rescaling, the semi-chiral description of this space allows for \( \mathcal{N} = (4, 4) \).

4 Eguchi-Hanson

Here we give the first example of the semichiral quotient. We consider \( \mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4 \), described by two copies of a left and right semichiral field, \( (\bar{X}^{(1)}_L, \bar{X}^{(1)}_R) \) and \( (\bar{X}^{(2)}_L, \bar{X}^{(2)}_R) \), as discussed in Section 1.3. We assign equal \( U(1) \) charges \( q_1 = q_2 = 1 \) to both and proceed as described, defining
\[ \hat{K} = \sum_{i=1,2} \left[ \bar{X}^{(i)}_L e^{V_i} X^{(i)}_L + \bar{X}^{(i)}_R e^{V_i} X^{(i)}_R + \alpha(\bar{X}^{(i)}_R e^{-i\bar{\psi}} X^{(i)}_L + \bar{X}^{(i)}_L e^{i\bar{\psi}} X^{(i)}_R) \right] - \alpha V^\alpha. \]

Based on our results of Section 2, we know the resulting quotient manifold will be hyperkähler, with \( c = 2(1 - \frac{2}{\alpha}) \). We show below that this is actually the well-known Eguchi-Hanson manifold. Before showing this explicitly, by computation of the quotient potential and metric, we show that this quotient construction actually reduces to the usual hyperkähler quotient construction of Eguchi-Hanson in terms of \( \mathcal{N} = 1 \) fields.

\[ \text{It is worth mentioning that one can invert the charge of one of the pairs, say } (\bar{X}^{(2)}_L, X^{(2)}_R), \text{ by dualizing to fields } \bar{X}_L, \bar{X}_R \text{ that impose the semichiral constraints on the original pair [22]. This duality is not based on an isometry and does not change the geometry. Hence, we expect the quotient involving two pairs of semis, either with charges } (+, +) \text{ or } (+, -), \text{ to lead to the same geometry.} \]
4.1 Reduction to $\mathcal{N} = (1,1)$: Comparison to the hyperkähler quotient

The procedure to reduce to $\mathcal{N} = (1,1)$ is well known (see, e.g., [16] for a review). One decomposes the $\mathcal{N} = (2,2)$ gauge-covariant (super)derivatives into their real and imaginary part, namely

$$\nabla_\pm = \frac{1}{2} (D_\pm - iQ_\pm), \quad \bar{\nabla}_\pm = \frac{1}{2} (D_\pm + iQ_\pm).$$  \hfill (60)

We perform the reduction of the matter fields $X_L, X_R$ in the covariant approach (see Appendix C for more details), defining

$$\hat{X}_L = \bar{X}_L e^V, \quad \check{X}_L = X_L, \quad \check{X}_R = \bar{X}_R e^{-i\bar{V}},$$

in terms of which the Lagrangian (59) reads (relabeling the fields $\hat{X}_{L,R} \rightarrow X_{L,R}$)

$$L = \int d^2\theta Q_+ Q_- \left[ \bar{X}_L X_L + \bar{X}_R X_R + \alpha (\bar{X}_L X_R + \bar{X}_R X_L) \right]$$

$$+ \int d^2\theta \left[ t\bar{F} - s\bar{\tilde{F}} - c.c. \right],$$ \hfill (61)

where $d^2\theta$ is the $\mathcal{N} = (1,1)$ measure and the relative minus sign between $s$ and $t$ comes from the ordering in the measure. Next, one imposes the fields to be gauge-covariantly semichiral and defines components with gauge-covariant $Q_\pm$’s, i.e.,

$$X_L = X_L \bigg|, \quad Q_+ X_L = iD_+ X_L, \quad Q_- X_L = \Psi_-, \quad X_R = X_R \bigg|, \quad Q_- X_R = iD_- X_R, \quad Q_+ X_R = \Psi_+. \hfill (62)$$

The reduction of the semichiral vector multiplet is given by [6, 7]

$$d^1 = (F + \bar{F}) \bigg|, \quad d^2 = (\tilde{F} + \bar{\tilde{F}}) \bigg|, \quad d^3 = i \left( F - \bar{F} - \tilde{F} + \bar{\tilde{F}} \right) \bigg|, \quad f = -i \left( F - \bar{F} - \tilde{F} + \bar{\tilde{F}} \right) \bigg|. \hfill (63)$$

Rescaling $X_L \rightarrow \alpha/(\sqrt{4 - \alpha^2})X_L$ and writing

$$X_L = \frac{1}{4} \left( \frac{\phi_-}{\sqrt{\alpha + 1}} - \frac{\bar{\phi}_+}{\sqrt{\alpha - 1}} \right), \quad X_R = \frac{1}{4} \left( \frac{\phi_-}{\sqrt{\alpha + 1}} + \frac{\bar{\phi}_+}{\sqrt{\alpha - 1}} \right),$$

the kinetic terms are diagonalized, i.e.,

$$L_{\text{kin.}} \sim \int d^2\theta \left[ D_+ \bar{\phi}_+ D_- \phi_+ + D_+ \bar{\phi}_- D_- \phi_- \right]$$ \hfill (64)

and the constraints read

$$\bar{\phi}_+ \phi_+ - \bar{\phi}_- \phi_- = p, \quad \phi_+ \phi_- + ib = 0,$$ \hfill (65)
where we have defined
\[ r \equiv -2\text{Re}[s + t], \quad q \equiv -2\text{Im}[t], \quad p \equiv -2\text{Im}[s], \] (67)
and \(2b \equiv (r + iq)\sqrt{\alpha^2 - 1}\). The free action (65), subject to the constraints (66), is the usual hyperkähler quotient construction for Eguchi-Hanson \cite{3} (see also, e.g., \cite{24, 25}). This is a specific example of our discussion in Section 2.1 of the semichiral quotient reducing to the hyperkähler quotient. Thus, performing the quotient at the \(\mathcal{N} = (2, 2)\) level will give the generalized potential for this manifold.

### 4.2 Generalized Potential

We have learned that the semichiral quotient \cite{59} coincides, in \(\mathcal{N} = (1, 1)\) language, to the hyperkähler construction of Eguchi-Hanson. Therefore, performing the quotient in terms of \(\mathcal{N} = (2, 2)\) superfields will lead us to the generalized description of this manifold. From \cite{59}, the equations of motion for the vector multiplet read

\[
\begin{align*}
 e^{V_L} \left(1 + |X_L|^2\right) + \frac{\alpha}{2} \left[e^{-i\tilde{V}} \left(1 + \bar{X}_R X_L\right) + e^{i\tilde{V}} \left(1 + \bar{X}_L X_R\right)\right] - \frac{p + q}{2} &= 0, \\
 e^{V_R} \left(1 + |X_R|^2\right) + \frac{\alpha}{2} \left[e^{-i\tilde{V}} \left(1 + \bar{X}_R X_L\right) + e^{i\tilde{V}} \left(1 + \bar{X}_L X_R\right)\right] - \frac{p - q}{2} &= 0, \\
 i\alpha \left[e^{-i\tilde{V}} (\bar{X}_R X_L + 1) - e^{i\tilde{V}} (\bar{X}_L X_R + 1)\right] - \frac{r}{2} &= 0,
\end{align*}
\] (68)

where we have chosen the gauge \(X_L(2) = X_R(2) = 1\) (and relabeled the remaining fields). These can be easily solved for \(V_L, V_R, V'\), leading to the quotient potential

\[
\begin{align*}
 \hat{K}_{EH} &= -\frac{p}{2} \log \left(\frac{-(q^2 + r^2) \left(S^2 - \alpha^2 T^2\right) + p^2 \left(S^2 + T^2 \alpha^2\right) - 2ipQ}{\left(S^2 - \alpha^2 T^2\right)^2}\right) \\
 &\quad -\frac{q}{2} \log \left(\frac{(1 + |X_R|^2)^2 \left(p^2 S^2 + r^2 S^2 - q^2 \left(S^2 - 2T^2 \alpha^2\right) + 2iqQ\right)}{(p - q)^2 + r^2 S^4}\right) \\
 &\quad -\frac{ir}{2} \log \left(\frac{(1 + \bar{X}_L X_R)^2 \left(-2r^2 S^2 + (p^2 - q^2 + r^2) T^2 \alpha^2 - 2rQ\right)}{T^4 \alpha^2}\right),
\end{align*}
\] (69)

where we have defined

\[
S^2 = (1 + |X_L|^2)(1 + |X_R|^2), \quad T^2 = (1 + \bar{X}_R X_L)(1 + \bar{X}_L X_R),
\] (70)

and

\[
Q^2 = r^2 S^4 - (p^2 - q^2 + r^2) S^2 T^2 \alpha^2 - q^2 T^4 \alpha^4.
\] (71)

This quotient construction has been discussed to some extent in \cite{26}, where the authors suggest that this will lead to a non-trivial H-flux. Based on our result of Section 2, we know
this is not the case. Instead, it must describe a hyperkähler manifold; in this case, Eguchi-
Hanson. By explicit calculation, one can also verify that (69) satisfies the Monge-Ampere 
equation (10) with
$$c = 2(1 - \frac{2}{\alpha^2}),$$
i.e.,
$$\{J_+, J_-\} = 2(1 - \frac{2}{\alpha^2})I. \tag{72}$$
To show explicitly that one can derive the standard metric for Eguchi-Hanson from this 
potential, we set the FI parameters to some convenient value for which the potential is 
simplified. The choice $r = q = 0$, for instance, leads to the left-right symmetric potential
$$K = p \log [S + \alpha T], \tag{73}$$
while the choice $r = 0, p = -q$ leads to
$$K = p \log \left[ \frac{S^2 - \alpha^2 T^2}{1 + |X|^2} \right]. \tag{74}$$
This form of the potential also coincides with that of [27], constructed by twistor methods. 
Note how these potentials are more compact than the usual Kähler potential and contain 
no term with a square root outside the log. Working with the potential (74), we will 
explicitly construct the Eguchi-Hanson metric, but first we will study the $SU(2)$ symmetry 
of the problem.

**4.3 $SU(2)$ symmetry**

The action (59) is invariant under the global $SU(2)$ transformations which rotate $(\mathbb{X}^{(1)}, \mathbb{X}^{(2)})$, 
as well as under $U(1)$ gauge transformations. Recall that we have chosen the $U(1)$ gauge
$$\mathbb{X}_L^{(2)} = \mathbb{X}_R^{(2)} = 1, \tag{75}$$
which is not preserved by the $SU(2)$. Nevertheless, the $SU(2)$ symmetry can be realized 
non-linearly in the gauged action by introducing a compensating $U(1)$ transformation with 
parameter $\Lambda_C$, namely
$$\begin{pmatrix}
\delta \mathbb{X}_L^{(1)} \\
\delta \mathbb{X}_L^{(2)}
\end{pmatrix} = i \begin{pmatrix} \alpha & -i \beta \\
\beta & \alpha
\end{pmatrix} \begin{pmatrix}
\mathbb{X}_L^{(1)} \\
\mathbb{X}_L^{(2)}
\end{pmatrix} + i \begin{pmatrix} \Lambda_C \mathbb{X}_L^{(1)} \\
\Lambda_C \mathbb{X}_L^{(2)}
\end{pmatrix}, \tag{76}
$$
and similarly for $\mathbb{X}_R$. Imposing that the transformation preserves the gauge (75), and 
relabelling $\mathbb{X}_{L,R}^{(1)} = \mathbb{X}_{L,R}$ henceforth, one finds
$$\delta \mathbb{X}_L = 2i\alpha \mathbb{X}_L + \bar{\beta}(\mathbb{X}_L)^2 + \beta, \quad \delta \mathbb{X}_R = 2i\alpha \mathbb{X}_R + \bar{\beta}(\mathbb{X}_R)^2 + \beta. \tag{77}$$
The infinitesimal transformations are generated by the vector field
$$\xi = \delta \mathbb{X}_L \partial_L + \delta \mathbb{X}_R \partial_R + c.c. \tag{78}$$
16
and the finite transformations are given by the Möbius transformations
\[ X_L \rightarrow \frac{aX_L + b}{\bar{a} - b\bar{X}_L}, \quad X_R \rightarrow \frac{aX_R + b}{\bar{a} - b\bar{X}_R}, \] (79)
with \(|a|^2 + |b|^2 = 1\). Given the \(SU(2)\) invariance of the potential (and therefore the metric), it will be convenient to find coordinates in which this symmetry is manifest. The first step in doing this is to note that a natural radial coordinate \(R\) is defined by the invariant cross-ratio
\[ R^2 \equiv \frac{Z_{13}Z_{24}}{Z_{23}Z_{14}}, \] (80)
where \(Z_{ij} = Z_i - Z_j\). Since we have only two complex variables, namely \(X_L, X_R\), there is only one, non-zero, independent cross ratio we can form. Taking \(Z_1 = X_L, Z_2 = X_R, Z_3 = -1/\bar{X}_L\) and \(Z_4 = -1/\bar{X}_R\) we have
\[ R^2 = \frac{(1 + |X_L|^2)(1 + |X_R|^2)}{(1 + X_LX_R)(1 + \bar{X}_RX_L)} = \frac{S^2}{T^2}. \] (81)
One can easily verify that \(\mathcal{L}_\xi R = \xi R = 0\). Therefore, one can reach every point \((X_L, X_R)\), at a certain radius \(R\), by choosing a point \((X'_L, X'_R)\) on the sphere of that radius and acting by a \(SU(2)\) transformation with parameters \((a, b)\). Thus, we can parameterize any point \((X_L, X_R)\) by \(a, b\) (subject to \(|a|^2 + |b|^2 = 1\)), and the radial coordinate \(R\). Then, the natural remaining invariants are the Cartan 1-forms \(\sigma^i\) on the group manifold. As shown in Appendix \[D\] this parameterization of the \(X_L, X_R\) coordinates leads to
\[ dX_L = \frac{1}{\bar{a}^2} (i\sigma^1 - \sigma^2), \]
\[ dX_R = \frac{1}{(\bar{a} - \rho\bar{b})^2} \left[ 2i\rho \sigma^3 + i(1 - \rho^2)\sigma^1 - (1 + \rho^2)\sigma^2 + d\rho \right], \] (82)
where \(\rho^2 \equiv R^2 - 1\). As we shall see below, when writing the line element in this \(SU(2)\) parameterization, all the dependence in \(a, b\) drops out as a consequence of the invariance of the metric. Also, one can see by explicit calculations of \(J_\pm\) from the potential (73) that
\[ \mathcal{L}_\xi J_\pm = 0. \] (83)
That is, both complex structures, \(J_\pm\) (and therefore the third one), are preserved by the \(SU(2)\), which is an important property of Eguchi-Hanson (see Appendix \[D\] for more details). To show explicitly that the potential (74) indeed describes this manifold, we compute the metric.

### 4.4 Metric

From the potential (74), and Eqs. (5)–(8), one finds the metric
\[ g_{\mu\nu} = \alpha^2 \delta_{\mu\nu}, \] (84)
for simplicity, we have taken \(\alpha = \sqrt{2}\) here, but the final result (87) holds for any \(\alpha\), with appropriate redefinitions.
\[
ds^2 = \frac{F(R)(\bar{X}_L - \bar{X}_R)^2}{(1 + \bar{X}_R X_L)^2(1 + |X_L|^2)}dX_L dX_L + \frac{F(R)(\bar{X}_R - \bar{X}_L)^2}{(1 + \bar{X}_L X_R)^2(1 + |X_R|^2)}dX_R dX_R + \frac{G(R)}{(1 + |X_L|^2)}dX_L dX_L + \frac{G(R)}{(1 + |X_R|^2)}dX_R dX_R + \frac{H(R)(\bar{X}_L - \bar{X}_R)^2}{(1 + |X_L|^2)(1 + |X_L|^2)}dX_L dX_R + \frac{I(R)(1 + \bar{X}_L X_R)^2}{(1 + |X_L|^2)(1 + |X_R|^2)}dX_L d\bar{X}_R + \text{c.c.}
\]

where

\[
F(R) = -\frac{16 (2 - 2 R^2 + R^4)}{(-2 + R^2)^3 R^2}, \quad G(R) = -\frac{8(2 - 2 R^2 + R^4)^2}{(-2 + R^2)^3 R^2},
\]

\[
H(R) = \frac{4 R^2 (4 - 2 R^2 + R^4)}{(-2 + R^2)^3}, \quad I(R) = \frac{4 R^2 (4 - 6 R^2 + 3 R^4)}{(-2 + R^2)^3}.
\]

Defining \( r \) through

\[
R^2 = \frac{2r^2}{r^2 - 1},
\]

and using (82), after some algebra the line element reads

\[
\frac{1}{8} \, ds^2 = \frac{1}{1 - \frac{1}{r^2}} dr^2 + r^2 \left( \sigma_1^2 + \sigma_2^2 + (1 - \frac{1}{r^2}) \sigma_3^2 \right),
\]

which is the usual Eguchi-Hanson metric (see, e.g., [28] for a review).

5 Taub-NUT

5.1 A gauged linear sigma model

Here we present a gauged linear sigma model in terms of semichiral superfields whose low-energy limit target space is Taub-NUT. Consider a gauged linear sigma model with two copies of semichiral superfields, just as the Eguchi-Hanson case, but with the difference that the isometry acts by translations on one of the pairs, i.e.,

\[
K = \bar{x}_L^{(1)} e^{V_L} x_L^{(1)} + \bar{x}_R^{(1)} e^{V_R} x_R^{(1)} + \alpha (\bar{x}_R^{(1)} e^{-i\bar{\nu}} x_L^{(1)} + \bar{x}_L^{(1)} e^{i\bar{\nu}} x_R^{(1)}) + \frac{1}{2} (x_L^{(2)} + \bar{x}_L^{(2)} + V_L)^2 + \frac{1}{2} (x_R^{(2)} + \bar{x}_R^{(2)} + V_R)^2 + \frac{\alpha}{2} (x_L^{(2)} + \bar{x}_L^{(2)} - i\bar{\nu})^2 + (x_R^{(2)} + \bar{x}_R^{(2)} + i\nu)^2 - (i\nu + s\bar{\nu} + \text{c.c.}).
\]
It is known in general that such constructions (where the isometry acts transitively on some fields) lead to ALF (as opposed to ALE) spaces and we claim that performing the semichiral quotient in this way leads to the semichiral description of Taub-NUT. Although integrating out the vector field cannot be done explicitly, by implicit differentiation we could still compute the metric. Instead, we shall study the geometry of the T-dual theory.

5.2 T-dual

To perform a T-duality from the worldsheet perspective, one proceeds as according to [20, 29]. We introduce an additional vector multiplet $U^\alpha$, which acts on the second pair and constrain its field strengths to be trivial by Lagrange multipliers $\Phi, \chi$, i.e.,

$$\tilde{K} = \tilde{X}_L^{(1)} e^{V_L} \tilde{X}_L^{(1)} + \tilde{X}_R^{(1)} e^{V_R} \tilde{X}_R^{(1)} + \alpha \left( \tilde{X}_R^{(1)} e^{-i\tilde{V}_L} \tilde{X}_R^{(1)} + \tilde{X}_L^{(1)} e^{i\tilde{V}_R} \tilde{X}_R^{(1)} \right)$$

$$+ \frac{1}{2} \left( \tilde{X}_L^{(2)} + \tilde{X}_R^{(2)} + U_L \right)^2 + \frac{1}{2} \left( \tilde{X}_R^{(2)} + \tilde{X}_R^{(2)} + U_R \right)^2$$

$$+ \frac{\alpha}{2} \left( (\tilde{X}_L^{(2)} + \tilde{X}_R^{(2)} - i\tilde{U})^2 + (\tilde{X}_R^{(2)} + \tilde{X}_L^{(2)} + i\tilde{U})^2 \right)$$

$$- ((t + \Phi)V + (s + \chi)\tilde{V} + c.c.) + (\Phi \tilde{U} + \chi \tilde{\tilde{U}} + c.c.),$$

where we have shifted $U^\alpha \rightarrow U^\alpha - V^\alpha$. Integrating out $U^\alpha$ yields the T-dual gauged linear sigma model

$$\tilde{K} = \frac{1}{g^2} \left( - \frac{\tilde{X}_L \chi}{\alpha^2 - 1} + \tilde{X}_L e^{V_L} \tilde{X}_L + \tilde{X}_R e^{V_R} \tilde{X}_R + \alpha (\tilde{X}_L e^{i\tilde{V}_L} \tilde{X}_R^{(1)} + \tilde{X}_L e^{i\tilde{V}_R} \tilde{X}_R^{(1)}) \right)$$

$$- (\Phi \tilde{V} + \chi \tilde{\tilde{V}} + c.c.),$$

where we have shifted $\chi, \phi$ to get rid of the FI parameters $s$ and $t$, dropped terms that vanish upon integration in superspace (i.e., generalized Kähler transformations) and rescaled the fields appropriately. As we will see in Section 6, this gauged linear sigma model describes a smeared NS5-brane and, therefore, the original theory (88) is a gauged sigma model for Taub-NUT.

6 NS5-branes

It is well known that under type II string theory T-duality, Taub-NUT is mapped to an NS5-brane. A worldsheet discussion of such relation is given in [30]. There, a gauge theory description of NS5-branes involving a hypermultiplet, a twisted hypermultiplet, and a vector multiplet acting on the former is given and instanton corrections are discussed.

---

9We have chosen to keep the kinetic terms of $\Phi$ with the usual normalization, leading to the $1/(\alpha^2 - 1)$ factor for $\chi$. This relative coefficient, as we will see, is important to ensure the $N = (4, 4)$ symmetry of the quotient model, as is expected of a model which is dual in this manner to a model describing a hyperkähler manifold in terms of semichiral fields, as discussed in Section 3.
We shall first show that the gauge theory (90), involving semichiral fields, also describes NS5-branes and we shall comment in Section 6.2 on instanton effects.

### 6.1 A gauged linear sigma model

Consider the action

\[
\mathcal{L} = \int d^4\theta \left[ \frac{1}{e^2} (\dddot{F} \dddot{F} - \dddot{\Phi} \dddot{\Phi}) + \frac{1}{g^2} \left( -\frac{\dddot{\chi}}{\alpha^2 - 1} + \dddot{\Phi} \dddot{\Phi} \right) \right. \\
+ \dddot{X}_L e^{V_L} \dddot{X}_L + \dddot{X}_R e^{V_R} \dddot{X}_R + \alpha (\dddot{X}_R e^{-i\bar{V}} \dddot{X}_L + \dddot{X}_L e^{i\bar{V}} \dddot{X}_R) \\
\left. - (\dddot{\Phi} \dddot{V} + \dddot{\chi} \dddot{\bar{V}} + c.c.) \right].
\]

In the IR limit \((e^2 \to \infty)\), the equations of motion for the semichiral vector field are

\[
\dddot{X}_L e^{V_L} \dddot{X}_L + \alpha \left( \dddot{X}_R e^{-i\bar{V}} \dddot{X}_L + \dddot{X}_L e^{i\bar{V}} \dddot{X}_R \right) - \frac{i}{2} (\dddot{\Phi} - \dddot{\Phi} + \dddot{\chi} - \dddot{\bar{\chi}}) = 0, \\
\dddot{X}_R e^{V_R} \dddot{X}_R + \alpha \left( \dddot{X}_R e^{-i\bar{V}} \dddot{X}_L + \dddot{X}_L e^{i\bar{V}} \dddot{X}_R \right) - \frac{i}{2} (\dddot{\Phi} + \dddot{\bar{\Phi}} + \dddot{\chi} + \dddot{\bar{\chi}}) = 0, \\
\alpha \left( \dddot{X}_R e^{-i\bar{V}} \dddot{X}_L - \dddot{X}_L e^{i\bar{V}} \dddot{X}_R \right) + \frac{1}{2} \left( \dddot{\chi} + \dddot{\bar{\chi}} + \dddot{\Phi} + \dddot{\bar{\Phi}} \right) = 0.
\]

For simplicity, we gauge the semis to \(\dddot{X}_L = \dddot{X}_R = 1\). Solving these equations leads to

\[
K(\Phi, \chi) = \frac{1}{g^2} \left( -\frac{\dddot{\chi}}{\alpha^2 - 1} + \dddot{\Phi} \dddot{\Phi} \right) + \Delta K(\Phi, \chi)
\]

with

\[
\Delta K(\Phi, \chi) \equiv -i\chi \log \left[ i(\chi - \bar{\chi})\alpha^2 + i(\chi + \bar{\chi} + \Phi + \bar{\Phi})(\alpha^2 - 1) - 2R \right] \\
- i\Phi \log \left[ - \frac{i(\chi + \bar{\chi} + \Phi + \bar{\Phi}) + i(\Phi - \bar{\Phi})\alpha^2 + 2R}{2i(\Phi + \bar{\chi})} \right] + c.c.,
\]

where we have defined

\[
R \equiv \frac{1}{2} \sqrt{(\chi + \bar{\chi} + \Phi + \bar{\Phi})^2(\alpha^2 - 1) - (\chi - \bar{\chi})^2\alpha^2 - (\Phi - \bar{\Phi})^2\alpha^2(\alpha^2 - 1)}.
\]

Note that \(\alpha^2 \geq 1\) ensures the reality of \(R\). From here we find

\[
K_{\chi \bar{\chi}} = -\frac{1}{\alpha^2 - 1} \left( \frac{1}{g^2} + \frac{\alpha^2 - 1}{2R} \right), \quad K_{\Phi \bar{\Phi}} = \frac{1}{g^2} + \frac{\alpha^2 - 1}{2R}, \\
K_{\chi \Phi} = -\frac{1}{2R} \left( \frac{\alpha^2 - 1(\Phi + \bar{\chi})}{2iR - (\chi - \bar{\chi}) - (\alpha^2 - 1)(\Phi - \bar{\Phi})} \right).
\]
Note that the $1/(\alpha^2 - 1)$ factor for $\bar{\chi}\chi$ in (91) is crucial for the potential to satisfy the scaled Laplace equation (54) (although in Section 3 we performed the duality in the other direction, one would expect the same relations to hold). After a trivial rescaling of the fields, the line element is given by

$$ds^2 = 2(K\Phi d\Phi d\bar{\Phi} - K\chi d\chi d\bar{\chi}) = 2H(r)(d\Phi d\bar{\Phi} + d\chi d\bar{\chi})$$

(97)

with

$$H(r) \equiv \left(\frac{1}{g^2} + \frac{1}{2r}\right).$$

(98)

Defining

$$\chi = \frac{(r_1 + \theta)}{2} + i\frac{r_2}{\sqrt{2}}, \quad \Phi = \frac{(r_1 - \theta)}{2} + i\frac{r_3}{\sqrt{2}},$$

(99)

we finally have

$$ds^2 = H(r)(d\vec{r} \cdot d\vec{r} + d\theta^2),$$

(100)

which is the metric for an NS5-brane, smeared along the $\theta$ direction.

### 6.2 Comment on instanton corrections

In [30] a gauge theory description of smeared NS5-branes and a worldsheet T-dual description of Taub-NUT was also given. It was argued that worldsheet instanton corrections to the effective action un-smear the NS5-brane, localizing it in the $\theta$ direction. (For a recent discussion of this phenomenon in the context of double field theory [31], see [32].)

In two dimensions, instantons are Nielsen-Olesen vortices, which arise as BPS solutions to an abelian Higgs model contained in the gauge theory. Although our gauge theory construction is quite different (from the $\mathcal{N} = (2,2)$ point of view), the same arguments hold so we expect the same mechanism to be at work. Our construction does not add to the results of [30], but is consistent with it. This is more easily seen upon reduction of the gauge theory (91) to $\mathcal{N} = (1,1)$. Following [6], we get (see Appendix C for details)

$$\mathcal{L} = \int d^2\theta \left[ \frac{1}{4g^2} (D_+d^a) (D_-d^b) g_{ab} + \frac{1}{g^2} (D_+\bar{\phi} D_-\phi + D_+\bar{\chi} D_-\chi) + (D_+X^i)(D_-X^j)E_{ij} ight.$$

$$+ 2id^1(\bar{X}_L X_L - \bar{X}_R X_R - \frac{i}{8}(\phi - \bar{\phi})) + d^3(\alpha(\bar{X}_R X_L - \bar{X}_L X_R) - \frac{i}{8}(\phi + \bar{\phi} + \chi + \bar{\chi}))$$

$$- 2id^2(\bar{X}_L X_L + \bar{X}_R X_R + \alpha(\bar{X}_R X_L + \bar{X}_L X_R) - \frac{i}{8}(\chi - \bar{\chi})) + if(\phi + \bar{\phi} - \chi - \bar{\chi}) \left],

where $d^a = (f, d^1, d^2, d^3)$, $X^i = (X_L, \bar{X}_L, X_R, \bar{X}_R)$ and $g_{ab} = \text{diag}(1, 2, 2, 1)$. One can rewrite this in terms of the fields $\phi_{\pm}$ from Section 4.1 which diagonalize the kinetic terms.
for the semis. Then, following Tong, we allow only the lowest component of, say, \( \phi_+ \) to vary over space and set all other fields to their classical expectation values. This results in an abelian Higgs model with a \( \theta \) term for the gauge field, whose instanton solutions (in the limit \( g^2 \to 0 \)) are conjectured to contribute to the low-energy effective action, effectively replacing

\[
H(r) \to H(r, \theta) = \frac{1}{g^2} + \frac{1}{2r \sinh r} \cosh r - \cos \theta,
\]

therefore unsmearing the NS5.

### 7 T-dual of Eguchi-Hanson

For completeness, we finally discuss the T-dual of Eguchi-Hanson. We can perform a T-duality before taking the quotient. As before, we introduce an additional semichiral vector multiplet \( U^\alpha \) which acts only on the second pair \( X_L^{(2)} \), and defines

\[
K = (\bar{X}_L^{(1)} X_L^{(1)} + \bar{X}_L^{(2)} X_L^{(2)} e^{U_L}) e^{V_L} + (\bar{X}_R^{(1)} X_R^{(1)} + \bar{X}_R^{(2)} X_R^{(2)} e^{U_R}) e^{V_R}
\]

\[
+ \alpha (\bar{X}_L^{(1)} \bar{X}_R^{(1)} + \bar{X}_L^{(2)} \bar{X}_R^{(2)} e^{-i\bar{U}}) e^{-i\bar{V}} + \alpha (\bar{X}_L^{(1)} X_R^{(1)} + \bar{X}_L^{(2)} X_R^{(2)} e^{i\bar{U}}) e^{i\bar{V}}
\]

\[
- \left[ \Phi U + \chi \bar{U} + c.c. \right].
\]

Shifting \( U^\alpha \to U^\alpha - V^\alpha \), the Lagrangian decouples and, gauging all the semis to 1, we have

\[
K = K_1 + K_2,
\]

where

\[
K_1 = e^{U_L} + e^{U_R} + \alpha (e^{-i\bar{U}} + e^{i\bar{U}}) + \left( \Phi U + \chi \bar{U} + c.c. \right),
\]

\[
K_2 = e^{V_L} + e^{V_R} + \alpha (e^{-i\bar{V}} + e^{i\bar{V}}) - \left( (\Phi + t)V + (\chi + s)\bar{V} + c.c. \right).
\]

Thus, integrating out both \( U^\alpha \) and \( V^\alpha \) reduces to the case studied for NS5-branes with \( K_1 = \Delta K(-\Phi, -\chi) \), \( K_2 = \Delta K(\Phi + t, \chi + s) \) and therefore

\[
\tilde{K} = \Delta K(-\Phi, -\chi) + \Delta K(\Phi + t, \chi + s),
\]

with \( \Delta K \) given in (94). Since the metric is linear in second derivatives of the potentials, we have

\[
\tilde{K}_{\chi \bar{\chi}} = -\frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right), \quad \tilde{K}_{\Phi \Phi} = \frac{\alpha^2 - 1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right).
\]

and similarly for the torsion terms. Again, this potential satisfies the scaled Laplace equation

\[\tilde{K}_{\Phi \Phi} + (\alpha^2 - 1)\tilde{K}_{\chi \bar{\chi}} = 0,\]

in accordance with our results of Section 3. Note that changing the relative position of the mass-points corresponds to rotating the complex structures.
8 Summary and Conclusions

We have studied a supersymmetric quotient construction by the use of general $\mathcal{N} = (2, 2)$ sigma models and the semichiral vector multiplet. We first restricted ourselves to the case in which only semichiral fields are involved. Due to the presence of a $b$-field in these models, one may naively think that a non-zero H-flux could be induced on the quotient manifold $\hat{\mathcal{M}}$, even if the original manifold $\mathcal{M}$ is hyperkähler. This, however, is prevented by our result of Section 2 asserting that the quotient of a hyperkähler manifold is hyperkähler, as in the usual hyperkähler quotient. Furthermore, the value of the anticommutator of the complex structures is preserved under the studied quotient. Thus, although the quotient manifold in general does have a $b$-field, its field strength $H = db$ vanishes. Nonetheless, the quotient provides a powerful method for constructing generalized potentials for hyperkähler manifolds, of which few explicit examples are known. We gave two examples of well-known hyperkähler manifolds, namely Eguchi-Hanson and Taub-NUT. We also used the SVM to perform T-duality transformations, giving a new $\mathcal{N} = (2, 2)$ gauged linear sigma model description of (smeared) NS5-branes involving semichiral, chiral, and twisted-chiral superfields. This description is consistent with previous ones in that it contains an abelian Higgs model whose instanton solutions unsmear the NS5.

We have also clarified and extended some previous results on the duality relation of these semichiral models with $\mathcal{N} = (4, 4)$ models for chiral/twisted-chiral fields. We showed that the T-dual of an $\mathcal{N} = (4, 4)$ model for chiral/twisted-chiral fields, may or may not describe a hyperkähler manifold, depending on the character of the isometry along which the duality is performed. If the isometry is translational, the dual manifold is hyperkähler. For a general isometry, however, the dual manifold is in general not hyperkähler, even if the $\mathcal{N} = (4, 4)$ SUSY is preserved. This, for instance, is the case of the $SU(2) \times U(1)$ WZW model which was briefly discussed.

We also commented on more general quotients that can lead to manifolds with torsion, noting that this requires the presence of more than one type of $\mathcal{N} = (2, 2)$ field and gave an example involving a chiral and a pair of semichiral fields. A more thorough analysis of such general quotients remains open.

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A Semichiral Quotient

Here we give the necessary elements and sketch the proof of (27). As mentioned in the text, the requirement \( \{ J_+, J_- \} = c I \) implies the set of equations

\[
\{ \mathcal{K}_{LR}^{-1} C_{RR} \mathcal{K}_{RL}^{-1} , J_s \} = 0 ,
\]

\( J_s \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} + \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} J_s + \mathcal{K}_{LR}^{-1} C_{RR} \mathcal{K}_{RL} C_{LL} = c I .
\]

We define the potential \( \hat{K} \) by

\[
\hat{K}(X_i, X_r) = K(X_i, X_r; X^\alpha + V^\alpha) - t_\alpha V^\alpha ,
\]

from where the standard relation of second derivatives

\[
\hat{K}_{\mu\nu} = K_{\mu\nu} - K_{\mu\alpha} K_{\nu}^{-1} K_{\beta\nu}
\]

follows, where \( \mu = (i, i', \bar{i}, \bar{i}') \) labels the \( 4N \) coordinates. From now on we suppress obvious indices, writing \( (L, R) = (l, r, \alpha) \). Capital letters refer to the manifold \( \mathcal{M} \), while lower-case are coordinates on \( \hat{\mathcal{M}} \) and \( \alpha \) labels coordinates which are gauged away. We decompose the relevant matrices as

\[
\mathcal{K}_{LR} = \begin{pmatrix} K_{lr} & K_{l\alpha} \\ K_{\beta r} & K_{\beta\alpha} \end{pmatrix} , \quad \mathcal{K}_{LL} = \begin{pmatrix} C_{ll} & C_{l\alpha} \\ C_{\beta l} & C_{\beta\alpha} \end{pmatrix} , \quad J_s = \begin{pmatrix} \hat{J} & 0 \\ 0 & j \end{pmatrix} ,
\]

with \( \hat{J}^2 = -1 \) and \( j^2 = -1 \) and

\[
C_{ll} = [\hat{J}, K_{ll}] , \quad C_{l\alpha} = [j, K_{l\alpha}] , \quad C_{\beta l} = j K_{\beta l} - K_{\beta l} \hat{J} , \quad C_{\beta\alpha} = [\hat{J}, K_{\beta\alpha}]
\]

(111)

and similarly for \( C_{RR} \). The inverse matrices are given by

\[
\mathcal{K}_{RL}^{-1} \equiv (\mathcal{K}_{LR})^{-1} = \begin{pmatrix} \hat{K}_{lr}^{-1} & -\hat{K}_{lr}^{-1} K_{l\alpha} K_{\beta\alpha}^{-1} \\ K_{\beta\alpha} K_{lr} K_{\beta\alpha}^{-1} & T_{\alpha\beta} \end{pmatrix}
\]

(112)

and \( \mathcal{K}_{LR}^{-1} = (\mathcal{K}_{RL})^t \) and where

\[
\hat{K}_{lr} = K_{lr} - K_{l\alpha} K_{\beta\alpha}^{-1} K_{\beta r} ,
\]

\[
T_{\alpha\beta} = K_{\beta\alpha}^{-1} + K_{\gamma\alpha}^{-1} K_{\beta r} K_{lr}^{-1} K_{\gamma l} K_{\beta\gamma}^{-1} .
\]

(113)

(114)

(Here we have changed the notation slightly to mean \( \hat{K}_{lr}^{-1} = (\hat{K}_{lr})^{-1} , \quad K_{\alpha\beta}^{-1} = (K^{-1})_{\alpha\beta} , \quad etc. \)). Similarly, we also have

\[
\hat{C}_{rr} = C_{rr} - [\hat{J}, K_{r\beta} K_{\alpha\beta}^{-1} K_{\alpha r}] , \quad \hat{C}_{ll} = C_{ll} - [\hat{J}, K_{l\beta} K_{\alpha\beta}^{-1} K_{\alpha l}] .
\]

(115)
By rather straightforward (albeit tedious) algebraic manipulations, one can show that (107) and (108) lead to
\[
\{ \hat{K}^{-1} \hat{C}_{t \tau} \hat{K}^{-1}_{t \tau}, \hat{J} \} = 0, \tag{116}
\]
\[
\hat{J} \hat{K}^{-1}_{t \tau} \hat{J} \hat{K}_{t \tau} + \hat{K}^{-1}_{t \tau} \hat{J} \hat{K}_{t \tau} \hat{J} + \hat{K}^{-1}_{t \tau} \hat{C}_{t \tau} \hat{K}^{-1}_{t \tau} \hat{C}_{t \tau} = c \mathbb{I}, \tag{117}
\]
which is equivalent to the statement that
\[
\{ \hat{J}_+, \hat{J}_- \} = c \mathbb{I}, \tag{118}
\]
as we wanted to prove.

\section{B. T-duality}

Here we give some of the details leading to (53) and (56). Writing the Legendre transform (51) as
\[
K(X^i) = F(V^\alpha) - \frac{1}{2} V^\alpha \delta_{\alpha \beta} X^i, \tag{119}
\]
where we defined
\[
X^i \equiv (i(X_L - \bar{X}_L + X_R - \bar{X}_R), -i(X_L - \bar{X}_L - X_R + \bar{X}_R), -(X_L + \bar{X}_L + X_R + \bar{X}_R)),
\]
we find the standard relation of second derivatives
\[
K_{ij} = -\frac{1}{2} \delta_{\alpha \beta} (F^{-1})^{\alpha \beta} \delta_{\beta j}. \tag{120}
\]
Explicitly inverting the general $3 \times 3$ matrix $F_{\alpha \beta}$ and using these relations in the definition (10), one finds
\[
c = 2 \left( F_{\phi \phi} + F_{\chi \chi} + 2F_{\phi \chi} \right) \left( F_{\phi \phi} + F_{\chi \chi} \right). \tag{121}
\]
The important point now is that the Laplace equation $(F_{\phi \phi} + F_{\chi \chi} = 0)$ translates into
\[
\cos^2(\theta) F_{\phi \phi} + \sin^2(\theta) F_{\chi \chi} + F_{\phi \chi} = 0, \tag{122}
\]
which is a direct consequence of how the gauging was performed in (50) (i.e., the charges of the fields). Using (122) in (121) finally leads to
\[
c = -2 \cos(2\theta). \tag{123}
\]
To prove (56) it is more convenient to redefine the fields so that the Killing vector acts by translations. Note that this is allowed due to the chirality properties of the components of the Killing vector. This, of course, does not preserve the form of the Laplace equation, but instead turns into \( \frac{1}{|X_L|^2} F_{\phi \phi} + \frac{1}{|X_R|^2} F_{\chi \chi} = 0 \). Using this in (121) leads to (56).
C Reduction to $\mathcal{N} = (1, 1)$

To reduce to $\mathcal{N} = (1, 1)$ (here we follow mostly [6, 7, 16]), one decomposes the $\mathcal{N} = (2, 2)$ gauge covariant superderivatives into their real and imaginary part, namely

$$\nabla_\pm = \frac{1}{2}(\mathcal{D}_\pm - iQ_\pm), \quad \bar{\nabla}_\pm = \frac{1}{2}(\mathcal{D}_\pm + iQ_\pm).$$

(124)

Here $\mathcal{D}_\pm$ are $\mathcal{N} = (1, 1)$ derivatives, which satisfy the algebra

$$\{\mathcal{D}_\pm, \mathcal{D}_\mp\} = i\mathcal{D}_{\pm\mp},$$

(125)

where $\mathcal{D}_{\pm\pm}$ is the gauge-covariant space derivative and $Q_\pm$ generate the non-manifest supersymmetries. We perform the reduction of the matter fields $\chi_L, \chi_R$ in the covariant approach. That is, we define

$$\hat{\chi}_R = e^{-\bar{\nabla}}e^{\nabla}\chi_R, \quad \bar{\hat{\chi}}_R = \chi_R e^{\nabla}e^{-\bar{\nabla}},$$

(126)

so that there are no factors $e^V$ anywhere. For instance, $\bar{\hat{\chi}}_R \hat{\chi}_R = \chi_R e^{\nabla}e^{-\bar{\nabla}}\chi_R = \bar{\chi}_R e^{\bar{\nabla}}\chi_R$ and the Lagrangian is simply (dropping the hats)

$$K = \bar{\chi}_L \chi_L + \bar{\chi}_R \chi_R + \alpha (\bar{\chi}_L \chi_R + \bar{\chi}_R \chi_L).$$

(127)

Next, one imposes the fields to be gauge-covariantly semichiral and defines components with gauge-covariant $Q_\pm$'s, i.e.,

$$X_L = \chi_L |, \quad Q_+ \chi_L = i\mathcal{D}_+ \chi_L, \quad Q_- \chi_L = \Psi_-, \quad X_R = \chi_R |, \quad Q_- \chi_R = i\mathcal{D}_- \chi_R, \quad Q_+ \chi_R = \Psi_+.$$

(128)

(129)

The reduction of the semichiral vector multiplet is given by

$$d^1 = (\mathcal{F} + \bar{\mathcal{F}}) |, \quad d^2 = (\bar{\mathcal{F}} + \mathcal{F}) |, \quad d^3 = i \left(\mathcal{F} - \bar{\mathcal{F}} - \bar{\mathcal{F}} + \mathcal{F}\right) |, \quad f = -i \left(\mathcal{F} - \bar{\mathcal{F}} + \bar{\mathcal{F}} - \mathcal{F}\right) |,$$

(130)

from where

$$\mathcal{F} = \frac{1}{2} \left( d^1 + \frac{i}{2} (f - d^3) \right), \quad \bar{\mathcal{F}} = \frac{1}{2} \left( d^2 + \frac{i}{2} (f + d^3) \right).$$

(131)

From the definitions $\mathcal{F} = i\{\nabla_+, \nabla_-\}$ and $\bar{\mathcal{F}} = i\{\nabla_+, \nabla_-\}$, one can solve for the commutation relations

$$\{Q_+, \mathcal{D}_-\} = \mp (d_1 + d_2), \quad \{\mathcal{D}_+, Q_-\} = \mp (d_1 - d_2),$$

$$\{Q_+, Q_-\} = \pm d_3, \quad \{\mathcal{D}_+, \mathcal{D}_-\} = f,$$

(132)

where the upper(lower) sign is chosen for positive(negative) charge. These are used repeatedly when reducing the matter fields, and the appropriate sign must be chosen depending on the charge of the field it acts on. Note that $f$ is the usual field strength which, in two dimensions, is a total derivative giving the topological charge.
SU(2) symmetry

As described in the text, the action (59) is invariant under the global SU(2) transformations which rotate \((X^{(1)}, X^{(2)})\) and the cross-ratio (81) is a natural radial coordinate. At a fixed radius \(R\), we can reach any point by a finite SU(2) transformation from a single point \(X^0_L, X^0_R\). We take \(X^0_L = 0\) and \(X^0_R = \sqrt{R^2 - 1}\). Thus, by acting with a finite SU(2) transformation, an arbitrary point is parameterized as

\[
X_L = \frac{b}{a}, \quad X_R = \frac{a \rho + b}{a - b \rho},
\]

(133)

where we have defined \(\rho^2 \equiv R^2 - 1\). By means of this identification, the natural remaining invariants are given by the Cartan 1-forms on the group manifold. Consider a group element \(g\) of SU(2),

\[
g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.
\]

(134)

The (real) invariant 1-forms \(\sigma^i\) are defined by

\[
g^{-1} dg = i \begin{pmatrix} \sigma^3 \\ \sigma^1 - i \sigma^2 \\ -\sigma^3 \end{pmatrix}.
\]

(135)

In the parameterization (134), we have

\[
\sigma^1 = \text{Im}(\bar{a}db - bd\bar{a}), \quad \sigma^2 = -\text{Re}(\bar{a}db - bd\bar{a}), \quad \sigma^3 = -i(\bar{a}da + bd\bar{b}).
\]

(136)

The constraint \(|a|^2 + |b|^2 = 1\) ensures the reality of \(\sigma^3\). From (133) and (136) we find

\[
dX_L = \frac{1}{a^2} (i \sigma^1 - \sigma^2),
\]

\[
dX_R = \frac{1}{(a - \rho b)^2} \left[ 2i \rho \sigma^3 + i(1 - \rho^2) \sigma^1 - (1 + \rho^2) \sigma^2 + d\rho \right].
\]

(137)

These are the expressions which allow us to rewrite the Eguchi-Hanson metric in an explicitly SU(2)-invariant form. Another well-known property of Eguchi-Hanson is that its complex structures are preserved by the SU(2) (in the Taub-NUT case they form a triplet). The Lie derivative along \(\xi\) of a (1, 1) tensor such as a complex structure is given by

\[
\mathcal{L}_\xi J_{\pm} = \xi J_{\pm} - [\partial \cdot \xi, J_{\pm}], \quad \partial \cdot \xi \equiv \begin{pmatrix} \partial_L \xi^L & 0 \\ 0 & \partial_R \xi^R \end{pmatrix},
\]

(138)

where

\[
\partial_L \xi^L \equiv \begin{pmatrix} \partial_l \xi^l & 0 \\ 0 & \partial_l \bar{\xi}^l \end{pmatrix}, \quad \partial_R \xi^R \equiv \begin{pmatrix} \partial_r \xi^r & 0 \\ 0 & \partial_r \bar{\xi}^r \end{pmatrix}.
\]

(139)
The equations from $L_\xi J_+ = 0$ read

$$\xi^\mu \partial_\mu (\mathcal{K}_{RL}^{-1} C_{LL}) - (\partial_R \xi^R \mathcal{K}_{RL}^{-1} C_{RR} - \mathcal{K}_{RL}^{-1} C_{LL} \partial_L \xi^L) = 0,$$

$$\xi^\mu \partial_\mu (\mathcal{K}_{RL}^{-1} J_s \mathcal{K}_{LR}) - [\partial_R \xi^R, \mathcal{K}_{RL}^{-1} J_s \mathcal{K}_{LR}] = 0,$$

(140)

and similarly for $J_-$, exchanging $R$ by $L$. We verified that these equations are satisfied by explicit calculations from the potential (73).

References

[1] B. Zumino, “Supersymmetry and Kähler Manifolds,” Phys. Lett. B87, 203 (1979).

[2] L. Alvarez-Gaume, D. Z. Freedman, “Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model,” Commun. Math. Phys. 80, 443 (1981).

[3] U. Lindstrom, M. Rocek, “Scalar Tensor Duality and N=1, N=2 Nonlinear Sigma Models,” Nucl. Phys. B222, 285-308 (1983).

[4] N. J. Hitchin, A. Karlhede, U. Lindstrom, M. Rocek, “Hyperkähler Metrics and Supersymmetry,” Commun. Math. Phys. 108, 535 (1987).

[5] S. J. Gates, Jr., C. M. Hull, M. Rocek, “Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models,” Nucl. Phys. B248, 157 (1984).

[6] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge and M. Zabzine, “New N = (2,2) vector multiplets,” JHEP 0708, 008 (2007) [arXiv:0705.3201 [hep-th]].

[7] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge and M. Zabzine, “Nonabelian Generalized Gauge Multiplets,” JHEP 0902, 020 (2009) [arXiv:0808.1535 [hep-th]].

[8] A. Kapustin, A. Tomasiello, “The General (2,2) gauged sigma model with three-form flux,” JHEP 0711, 053 (2007). [hep-th/0610210].

[9] T. L. Curtright, C. K. Zachos, “Geometry, Topology and Supersymmetry in Nonlinear Models,” Phys. Rev. Lett. 53, 1799 (1984).

[10] P. S. Howe, G. Sierra, “Two-dimensional Supersymmetric Nonlinear Sigma Models With Torsion,” Phys. Lett. B148, 451-455 (1984).

[11] N. Hitchin, “Generalized Calabi-Yau manifolds,” Quart. J. Math. Oxford Ser. 54, 281-308 (2003). [math/0209099 [math-dg]].

[12] M. Gualtieri, “Generalized complex geometry,” [math/0401221 [math-dg]].

[13] M. Rocek, K. Schoutens, A. Sevrin, “Off-shell WZW models in extended superspace,” Phys. Lett. B265, 303-306 (1991).
[14] A. Sevrin, J. Troost, “Off-shell formulation of N=2 nonlinear sigma models,” Nucl. Phys. B492, 623-646 (1997). [hep-th/9610102].

[15] J. Bogaerts, A. Sevrin, S. van der Loo, S. Van Gils, “Properties of semichiral superfields,” Nucl. Phys. B562, 277-290 (1999). [hep-th/9905141].

[16] U. Lindstrom, M. Rocek, R. von Unge, M. Zabzine, “Generalized Kähler manifolds and off-shell supersymmetry,” Commun. Math. Phys. 269, 833-849 (2007). [hep-th/0512164].

[17] T. Buscher, U. Lindstrom, M. Rocek, “NEW SUPERSYMMETRIC sigma MODELS WITH WESS-ZUMINO TERMS,” Phys. Lett. B202, 94 (1988).

[18] N. Hitchin and M. Rocek. Private communication.

[19] U. Lindström, M. Rocek, R. von Unge, M. Zabzine. Unpublished. (M. Rocek, private communication)

[20] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge, M. Zabzine, “T-duality and Generalized Kähler Geometry,” JHEP 0802, 056 (2008). [arXiv:0707.1696 [hep-th]].

[21] P. M. Crichigno, M. Göteman. In preparation

[22] A. Sevrin, W. Staessens, D. Terryn, “The generalized Kaehler geometry of N=(2,2) WZW-models,” [arXiv:1111.0551 [hep-th]].

[23] M. T. Grisaru, M. Massar, A. Sevrin and J. Troost, “Some aspects of N=(2,2), D = 2 supersymmetry,” Fortsch. Phys. 47, 301 (1999) [hep-th/9801080].

[24] T. L. Curtright, D. Z. Freedman, “NONLINEAR sigma MODELS WITH EXTENDED SUPERSYMMETRY IN FOUR-DIMENSIONS,” Phys. Lett. B90, 71 (1980).

[25] M. Rocek, P. K. Townsend, “Three Loop Finiteness Of The N=4 Supersymmetric Nonlinear Sigma Model,” Phys. Lett. B96, 72 (1980).

[26] W. Merrell, D. Vaman, “T-duality, quotients and generalized Kähler geometry,” Phys. Lett. B665, 401-408 (2008). [arXiv:0707.1697 [hep-th]].

[27] M. Dyckmanns, “A twistor sphere of generalized Kahler potentials on hyperkahler manifolds,” [arXiv:1111.3893 [hep-th]].

[28] T. Eguchi, P. B. Gilkey, A. J. Hanson, “Gravitation, Gauge Theories and Differential Geometry,” Phys. Rept. 66, 213 (1980).

[29] M. Rocek, E. P. Verlinde, “Duality, quotients, and currents,” Nucl. Phys. B373, 630-646 (1992). [hep-th/9110053].
[30] D. Tong, “NS5-branes, T duality and world sheet instantons,” JHEP 0207, 013 (2002). [hep-th/0204186].

[31] C. Hull, B. Zwiebach, “Double Field Theory,” JHEP 0909, 099 (2009). [arXiv:0904.4664 [hep-th]].

[32] S. Jensen, “The KK-Monopole/NS5-Brane in Doubled Geometry,” JHEP 1107, 088 (2011). [arXiv:1106.1174 [hep-th]].