On dispersionless BKP hierarchy and its reductions

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Abstract

Integrable dispersionless Kadomtsev-Petviashvili (KP) hierarchy of B type is considered. Addition formula for the τ-function and conformally invariant equations for the dispersionless BKP (dBKP) hierarchy are derived. Symmetry constraints for the dBKP hierarchy are studied.

1 Introduction

Dispersionless integrable hierarchies play an important role in the study of different nonlinear phenomena in various fields of physics and mathematics (see e.g. [1]-[12]). They have attracted recently a considerable interest (see e.g. [13]-[17]).

Quasi-classical ∂-dressing method, proposed in [18, 19, 20, 21], gives a new approach to study the properties of dispersionless integrable hierarchies, including various addition formulae, symmetry constraints etc. [22, 23, 24]. Most of the results obtained using this approach are connected with dispersionless Kadomtsev-Petviashvili (dKP), modified dispersionless Kadomtsev-Petviashvili (dmKP) and dispersionless 2-dimensional Toda lattice (d2DTL) hierarchies.

In the full ‘dispersive’ case not only the standard KP hierarchy (the hierarchy of A type), but also BKP, CKP and DKP hierarchies play an important role [25, 26, 27]. In the dispersionless case the study of hierarchies of B type is just in the very beginning [28, 20].

In the present paper we analyze the dBKP hierarchy in detail. We formulate the ∂-dressing approach to this hierarchy, derive an integral formula

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for the $\tau$-function, obtain the fundamental equation for the basic homeomorphism which represents a generating equation for the whole hierarchy. We derive also the dispersionless addition formula for the $\tau$-function and obtain conformally invariant equation for its symmetries. We consider symmetry constraints for the dBKP hierarchy and find generating equations and Sato functions for the constrained hierarchies.

2 dKP and dBKP hierarchies

The quasiclassical $\bar{\partial}$-dressing scheme for dispersionless KP hierarchy \cite{18, 19, 20, 21} is based on nonlinear Beltrami equation

$$S_{\bar{z}} = W(z, \bar{z}, S_z),$$

where $\bar{\partial}$-data $W$ are localized in the unit disc, $S(z, \bar{z}, t) = S_0 + \bar{S}$, $S_0(z, t) = \sum_{n=1}^{\infty} t_n z^n$, $\bar{S}$ is analytic outside the unit disc, and at infinity it has an expansion $\bar{S} = \sum_{i=1}^{\infty} \bar{S}_i(t) z^{-i}$. The quantity $p = \frac{\partial S}{\partial t_1}$ is a basic homeomorphism \cite{19}. Important role in the theory of dKP hierarchy is played by the equation

$$p(z) - p(z_1) + z_1 \exp(-D(z_1)S(z)) = 0, \quad z, z_1 \in \mathbb{C},$$

which generates Hamilton-Jacobi equations of the hierarchy by expansion into the powers of $z^{-1}$ at infinity (see, e.g., \cite{22}). This equation also implies existence of the $\tau$-function, characterized by the relation

$$\bar{S}(z, t) = -D(z)F(t),$$

and provides the dispersionless addition formula.

Dispersionless BKP hierarchy is characterized by the symmetry condition

$$S(-z, t) = -S(z, t),$$

which is preserved only by odd flows of the hierarchy ($t_n$ with odd $n$), $S_0(z, t) = \sum_{n=0}^{\infty} t_{2n+1} z^{2n+1}$. Thus dBKP hierarchy is dKP hierarchy with even times frozen at zero plus symmetry \cite{14, 28}. In terms of $\bar{\partial}$-data this symmetry is provided by the condition

$$W(-z, S_z) = W(z, S_z).$$

To obtain the analogue of relation \cite{24} for dBKP hierarchy, we introduce B-type quasiclassical vertex operator $D(z) = 2 \sum_{n=0}^{\infty} \frac{z^{-(2n+1)}}{2n+1} \frac{\partial}{\partial z^{2n+1}}, |z| > 1$, where

$$D(z) = 2 \sum_{n=0}^{\infty} \frac{z^{-(2n+1)}}{2n+1} \frac{\partial}{\partial z^{2n+1}}.$$
characterized by the property \( \mathcal{D}(z_1)S_0(z) = \ln \frac{z_1 + z}{z_1 - z} \). Then, similar to dKP case \([22]\), we get

\[
\frac{p(z) - p(z_1)}{p(z) + p(z_1)} = \exp(-\mathcal{D}(z_1)S(z)),
\]

(5)

This equation generates Hamilton-Jacobi equations of the dBKP hierarchy by expansion into the powers of \( z^{-1} \) at infinity. The first two Hamilton-Jacobi equations are

\[
S_y = p^3 + 3up, \quad S_t = p^5 + 5up^3 + vp,
\]

(6) \( \quad \) (7)

where \( x = t_1, \ y = t_3, \ t = t_5, \ u = -\partial_x \tilde{S}_1, \ v_x = \frac{5}{3}u_y + 5(u^2)_x \). Compatibility condition for these equations gives dispersionless BKP equation \([20]\)

\[
\frac{1}{5}u_t + u^2u_x - \frac{1}{3}u_{yy} - \frac{1}{3}u_x \partial_x^{-1} u_y - \frac{1}{9} \partial_x^{-1} u_{yy} = 0.
\]

(8)

Relation (5) for dBKP hierarchy can be also obtained starting with (2). Indeed, using (2), in the framework of dKP hierarchy we get a relation

\[
\frac{p(z) - p(z_1)}{p(z) - p(-z_1)} = -\exp(-(D(z_1) - D(-z_1))S(z)),
\]

(9)

Then, freezing even times at zero and using symmetry \([4]\), we get (5) \( (D(z) = D(z) - D(-z)) \).

Equation (5) also implies existence of the \( \tau \)-function, characterized by the relation

\[
\tilde{S}(z, t) = -\mathcal{D}(z)F_{dBKP}(t).
\]

Comparing this relation with (3), we come to the conclusion that

\[
2F_{dBKP} = F_{dKP},
\]

if \( F_{dKP} \) is taken at zero even times and symmetry condition \([11]\) is satisfied \([28]\). This symmetry condition is equivalent to a simple condition for the function \( F_{dKP} \) itself, namely that its derivatives \( \partial F_{dKP}/\partial t_{2(n+1)} \) taken at zero even times are equal to zero.

In complete analogy with dKP hierarchy, it is possible to find explicit representation of dBKP \( \tau \)-function as an action for nonlinear Beltrami equation \([11]\) evaluated on its solution.
Proposition 1 The function

\[ F(t) = -\frac{1}{2\pi i} \int_D \left( \frac{1}{2} \tilde{S}_t(t) \tilde{S}_z(t) - W_{-1}(z, \bar{z}, S_z(t)) \right) dz \wedge d\bar{z}, \]  

(10)
i.e., the action for the problem (11) evaluated on its solution, where \( \partial_\eta W_{-1}(z, \bar{z}, \eta) = W(z, \bar{z}, \eta) \), and \( W \) satisfies a symmetry condition

\[ W(-z, -\bar{z}, S_z) = W(z, \bar{z}, S_z), \]
is a \( \tau \)-function of dBKP hierarchy.

Variations of the \( \partial \)-data define infinitesimal symmetries of the \( \tau \)-function \([23, 24]\). One should take into account that these variations should satisfy the symmetry condition. Considering variations localized in the pair of points \( z_0, -z_0 \), we get a symmetry

\[ \delta F = f(S_z(z_0)), \]
where \( f \) is an arbitrary analytic function. Variations localized on the set of curves lead to infinitesimal symmetry

\[ \delta F = \sum_{i=1}^{N} c_i (S_i - \tilde{S}_i), \]

(12)
where \( S_i = S(z_i), \tilde{S}_i = S(z_i), z_i, \tilde{z}_i \) are some sets of points, and \( c_i \) are arbitrary constants. Due to the symmetry \( S(-z) = -S(z) \) in dBKP case, it is possible to take \( \tilde{z}_i = -z_i \) and consider symmetries of the form

\[ \delta F = 2 \sum_{i=1}^{N} c_i S_i. \]

(13)

3 Addition formula for dBKP hierarchy

Expressing \( S \) in terms of \( F \), from equation (15) we get

\[ \frac{p(z_1) - p(z_2)}{p(z_1) + p(z_2)} = \frac{z_1 - z_2}{z_1 + z_2} e^{\mathcal{D}(z_1) \mathcal{D}(z_2) F}. \]

(14)
Using this equation, we obtain a system of linear equations for \( p(z_i) \),

\[ p(z_i)(f_{ij} - 1) - p(z_j)(f_{ij} + 1) = 0 \]

(15)
where
\[ f_{ij} = \frac{z_i - z_j}{z_i + z_j} e^{D(z_i)D(z_j)F}, \quad 1 \leq i, j \leq 3. \] (16)

To possess nontrivial solutions, this system should have zero determinant,
\[
\det \begin{pmatrix}
 f_{12} - 1 & f_{12} + 1 & 0 \\
 0 & f_{23} - 1 & f_{23} + 1 \\
 f_{13} + 1 & 0 & f_{13} - 1
\end{pmatrix} = 0.
\]

Thus
\[(f_{23} + 1)(f_{31} + 1)(f_{12} + 1) = (f_{23} - 1)(f_{31} - 1)(f_{12} - 1),\] (17)
or, equivalently,
\[f_{23}f_{31}f_{12} + f_{23} + f_{31} + f_{12} = 0.\] (18)

This condition gives addition formula for dispersionless BKP hierarchy,
\[1 + c_2 c_3 e^{-(D_3 D_1 + D_1 D_2)F} + c_1 c_3 e^{-(D_2 D_3 + D_3 D_2)F} + c_2 c_1 e^{-(D_3 D_1 + D_1 D_2)F} = 0,\] (19)

where \(D_i = D(z_i), \ c_i = \frac{z_j + z_k}{z_j - z_k} (i, j, k)\) is a cyclic permutation of \(1, 2, 3\).

4 Conformally invariant equations of dBKP hierarchy

An important object of dispersive integrable hierarchies are discrete Schwarzian equations, which possess Möbius symmetry and have a deep connection with geometry \[29, 30\]. It was demonstrated in \[23\] for KP and 2DTL hierarchies that dispersionless analogues of these equations are given by conformally invariant equations of dispersionless hierarchies, which arise also an a naive continuum limit of discrete Schwarzian equations \[29\].

In the case of dBKP hierarchy we start with the evident analogy of \[15\], \[17\] and the formulae in \[30\] connected with continuum limit of discrete SBKP equation. Using this analogy, we introduce the function \(\Phi, p(z_i) = D_i \Phi\). To demonstrate existence of \(\Phi\), let us rewrite relations \[15\] in the form
\[D_2 \Phi = \frac{f_{12} - 1}{f_{12} + 1} D_1 \Phi, \quad D_3 \Phi = \frac{f_{13} - 1}{f_{13} + 1} D_1 \Phi.\]

Compatibility of this linear system is implied by addition formula \[18\]. Thus the function \(\Phi, p(z_i) = D_i \Phi\) exists.
Relations (14) imply equation for Φ,

$$D_1 \ln \frac{D_2 \Phi + D_3 \Phi}{D_2 \Phi - D_3 \Phi} = D_2 \ln \frac{D_1 \Phi + D_3 \Phi}{D_1 \Phi - D_3 \Phi},$$

(20)

and this equation coincides with continuum limit of discrete SBKP equation introduced in [30].

Equation (20) can be written in symmetric form,

$$\Phi_{23} \Phi_1 (\Phi_2^2 - \Phi_3^2) + \Phi_{31} \Phi_2 (\Phi_3^2 - \Phi_1^2) + \Phi_{12} \Phi_3 (\Phi_1^2 - \Phi_2^2) = 0,$$

(21)

where subscripts denote vertex derivatives.

It is also easy to find a determinant representation for this equation,

$$\det \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \Phi_2 & \Phi_3 & \Phi_1 \\ \Phi_{23} & \Phi_{31} & \Phi_{12} \end{pmatrix} = 0.$$

(22)

We have defined the function Φ implicitly through the relation

$$p_i = D_i \Phi.$$

However, it is possible to construct this function explicitly, using the potential $u$ of dispersionless hierarchy. Indeed, $u = -F_x$, $p(z) = -\partial_x D(z)F + z$, then $p(z) = z + D(z)u$. Thus Φ is ‘almost’ u. For u, instead of (20), we get a generating equation

$$D_1 \ln \frac{u_2 + u_3 + z_2 - z_3}{u_2 - u_3 + z_2 - z_3} = D_2 \ln \frac{u_1 + u_3 + z_1 + z_3}{u_1 - u_3 + z_1 - z_3}.$$

(23)

This equation gives dBKP hierarchy equations for potential u by expansion into parameters $z_1, z_2, z_3$.

To transform relation $p(z) = z + D(z)u$ to the relation $p_i = D_i \Phi$, we define Φ as $u - \sum_{k=0}^{N} c_{2k+1} t^{2k+1}$. It is easy to see that it is possible to satisfy relations $p_i = D_i \Phi$ at some set of points $z_i$ by the choice of constants $c_{2k+1}$ (taking sufficiently large N). Indeed,

$$p(z) = D(z)\Phi = z + \sum_{k=0}^{N} \frac{c_{2k+1}}{2k+1} z^{-(2k+1)},$$

and the relation $p_i = D_i \Phi$ is satisfied if $z_i$ is a zero of polynomial

$$P(z) = (z^2)^{N+1} + \sum_{k=0}^{N} \frac{c_{2k+1}}{2k+1} (z^2)^{N-k}.$$

Taking, e.g., $P(z) = \prod_{i=1}^{l} (z^2 - \alpha_i^2)$, it is possible to express the constants $c_{2k+1}$ through $z_i$ explicitly. Thus solution to equation (20) can be expressed in terms of potential $u$. 
A general conformally invariant equation of dBKP hierarchy

It is also possible, using the approach developed in \cite{23}, to derive a general equation for the symmetry $\phi$ of the function $F$ which is invariant under conformal transformation (we mean that $f(\phi)$ is also a symmetry for arbitrary analytic $f$). We start with addition formula (19). Considering a symmetry $\delta F = e^{\Theta \phi}$, where $\Theta$ is an arbitrary parameter, we get a system of equations

\[
\begin{align*}
\{ & x + y + z = -1, \\
(\phi_{12} + \phi_{13})x + (\phi_{23} + \phi_{12})y + (\phi_{31} + \phi_{23})z = 0, \\
(\phi_1 \phi_2 + \phi_1 \phi_3)x + (\phi_2 \phi_3 + \phi_1 \phi_2)y + (\phi_3 \phi_1 + \phi_2 \phi_3)z = 0,
\}
\tag{24}
\]

where $x = c_2 c_3 e^{-(F_{12} + F_{13})}$, $y = c_1 c_3 e^{-(F_{23} + F_{12})}$, $z = c_1 c_2 e^{-(F_{31} + F_{23})}$. The first line of system (24) (zero order in $\Theta$) is addition formula (19), the second (first order in $\Theta$) defines its symmetry, and the third (second order in $\Theta$) follows from conformal invariance.

Using this system, we express $x, y, z$ through $\phi$ and get a compatibility condition

\[
D_1 \ln \frac{(f^2_{23} + f^2_{31} + f^2_{12})(f^1_{32} + f^1_{31} + f^1_{12})}{(f^2_{23} + f^2_{13} + f^2_{12})(f^1_{23} + f^1_{31} + f^1_{21})} =
D_2 \ln \frac{(f^2_{23} + f^2_{31} + f^2_{12})(f^1_{32} + f^1_{31} + f^1_{12})}{(f^2_{23} + f^2_{31} + f^2_{12})(f^1_{32} + f^1_{31} + f^1_{12})},
\tag{25}
\]

where $f^i_{jk} = D_i \ln \frac{\phi^i}{\phi_k}$. Equation (25) is a general equation for conformally-invariant symmetry of the $\tau$-function of dBKP hierarchy.

5 Symmetry constraints for dBKP hierarchy

Using the symmetry (13), we define a symmetry constraint

\[
F_x = \sum_{i=1}^{N} c_i S_i,
\]

or, in terms of potential $u$,

\[
u = 2 \sum_{i=1}^{N} c_i p_i, \quad p_i = p(z_i).
\]

Evaluating first dBKP Hamilton-Jacobi equation

\[
S_y = p^3 + 3up
\]
where \( y = t_3 \), at \( z \) equal to \( z_i \), we get a system of hydrodynamic type

\[
\partial_y p_k = \partial_x (p_k^3 + 6p_k \sum_i c_i p_i).
\] (26)

Higher Hamilton-Jacobi equations will give higher systems of constrained dBK hierarchy. The Sato function \( z(p) \) for this hierarchy is constructed similar to constrained dKP case [24],

\[ z = p - \sum_{i=1}^{N} c_i \ln \frac{p - p_i}{p + p_i}, \] (27)

Its expansion at infinity is

\[ z \to p + \sum_{n=0}^{\infty} v_{2n+1} p^{-(2n+1)}, \quad v_{2n+1} = \frac{2}{2n + 1} \sum_{i=1}^{N} c_i p_i^{2n+1}. \] (28)

From (5) we obtain a generating system for the constrained hierarchy,

\[
D(z) p_k = -\partial_x \ln \frac{p - p_k}{p + p_k},
\] (29)

where \( p \) is a function of \( z \), \((p_1, \ldots, p_N)\), defined by the relation (27). Expanding both sides of this system into the powers of \( z^{-1} \), one gets the systems (26) and its higher counterparts. Expansion of \( p(z) \) at infinity is given by the formula

\[
p(z) = z + \sum_{n=0}^{\infty} \frac{1}{2n + 1} \text{Res}_{p=\infty} (z(p)^{2n+1}) z^{-(2n+1)}.
\]

In the same manner, it is possible to define constrained hierarchy using symmetry (12) (which can be considered as a special case of (13)) and (11). We will give the basic formulae for constrained hierarchy connected with (12) and obtain constrained hierarchy for the symmetry of the type (11) as a limit.

Using the symmetry (12), we define a symmetry constraint

\[
F_x = \frac{1}{2} \sum_{i=1}^{N} c_i (S_i - \tilde{S}_i),
\]

or, in terms of \( u \),

\[
u = \sum_{i=1}^{N} c_i (p_i - \tilde{p}_i).\]
The first hydrodynamic type system of constrained hierarchy is

$$\begin{aligned}
\partial_y p_k &= \partial_x \left((p_k^3) + 3p_k \sum_i c_i (p_i - \tilde{p}_i)\right) \\
\partial_y \tilde{p}_k &= \partial_x \left((\tilde{p}_k^3) + 3\tilde{p}_k \sum_i c_i (p_i - \tilde{p}_i)\right)
\end{aligned} \quad (30)$$

The Sato function for constrained hierarchy is given by

$$z = p - \frac{1}{2} \sum_{i=1}^N c_i \ln \frac{p - p_i p + \tilde{p}_i}{p + p_i p - \tilde{p}_i}.$$ 

The generating equation for the constrained hierarchy is

$$D(z)p_k = -\frac{1}{2} \partial_x \ln \frac{p - p_k p + \tilde{p}_i}{p + p_k p - \tilde{p}_i}.$$ 

Finally, we will consider symmetry constraint connected with the symmetry (13),

$$F_x = \sum_{i=1}^N c_i S_z(z_i), \quad u = \sum_{i=1}^N c_i \phi_i, \quad \phi_i = \partial_x S_z(z_i).$$

Though it is possible to consider this constrained hierarchy directly, we will obtain it as a limit of the previous case, when $p_i \to \tilde{p}_i$. Then from (30) we obtain a first system of constrained hierarchy,

$$\begin{aligned}
\partial_y p_k &= \partial_x \left((p_k^3) + 3p_k \sum_i c_i \phi_i\right) \\
\partial_y \phi_k &= \partial_x \left(3p_k^2 \phi_k + 3\phi_k \sum_i c_i \phi_i\right)
\end{aligned} \quad (31)$$

The Sato function for the constrained hierarchy is

$$z = p + \sum_{i=1}^N c_i \frac{p \phi_i}{p^2 - p_i^2}.$$ 

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