Collinear Factorization and Splitting Functions for Next-to-next-to-leading Order QCD Calculations

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Abstract

We consider the singular behaviour of tree-level QCD amplitudes when the momenta of three partons become simultaneously parallel. We discuss the universal factorization formula that controls the singularities of the multiparton matrix elements in this collinear limit and present the explicit expressions of the corresponding splitting functions. The results fully include spin (azimuthal) correlations up to $O(\alpha_s^2)$. In the case of spin-averaged splitting functions we confirm similar results obtained by Campbell and Glover.
1 Introduction

The universal properties of multiparton matrix elements in the infrared (soft and collinear) limit play a relevant role in our capability to make reliable QCD predictions for hard-scattering processes [1, 2].

At leading order in the QCD coupling $\alpha_S$, these properties are embodied in process-independent factorization formulae of tree-level [3–5] and one-loop [6–8] amplitudes that are at the basis of (at least) three important tools in perturbative QCD. The leading-logarithmic (LL) parton showers, which are implemented in Monte Carlo event generators [1] to describe the exclusive structure of hadronic final states, are based on these factorization formulae supplemented with ‘jet calculus’ techniques [9] and colour-coherence properties [10, 11]. Analytical techniques to perform all-order resummation of logarithmically enhanced contributions at next-to-leading logarithmic (NLL) accuracy [12] rely on the factorization properties of soft and collinear emission. More recently, the leading-order factorization formulae have been fully exploited to set up completely general algorithms [7, 13–15] to handle and cancel infrared singularities when combining tree-level and one-loop contributions in the evaluation of jet cross sections at the next-to-leading order (NLO) in perturbation theory.

Any further (although challenging) theoretical improvement of these tools requires the understanding of infrared-factorization properties at the next order in $\alpha_S$.

This topic has recently received considerable attention. The infrared singular behaviour of two-loop QCD amplitudes has been discussed in Ref. [16]. The collinear limit of one-loop amplitudes is known [17]. In the case of tree-level amplitudes, the factorization structure in the double-soft [18] and triple-collinear [19] limits has been studied.

In this letter, we reconsider the triple-collinear limit examined by Campbell and Glover [19]. In particular, we extend their results by fully taking into account azimuthal (spin) correlations. Besides improving the general understanding of the collinear behaviour of tree amplitudes, this extension is essential to apply some general methods to perform exact fixed-order calculations at the next-to-next-to-leading order (NNLO) in perturbation theory. For instance, the subtraction method [14, 15] works by regularizing the infrared singularities of the tree-level matrix element by identifying and subtracting a proper local counterterm. Thus, the knowledge of the azimuthally averaged collinear limit studied in Ref. [19] is not sufficient for this purpose.

The outline of the paper is as follows. In Sect. 2, we define our notation and the collinear kinematics and, after reviewing the known collinear-factorization formulae at $\mathcal{O}(\alpha_S)$, we present their generalization at the next perturbative order. Our explicit results for the collinear-splitting functions at $\mathcal{O}(\alpha_S^2)$ are given in Sect. 3, where the comparison with the azimuthally-averaged case considered in Ref. [19] is also discussed. Details on our calculation of the splitting functions are not discussed here and will be presented elsewhere [20]. Some final remarks are left to Sect. 4.
2 Notation and kinematics

We consider a generic scattering process involving final-state QCD partons (massless quarks and gluons) with momenta $p_1, p_2, \ldots$. Non-QCD partons ($\gamma^*, Z^0, W^\pm, \ldots$), carrying a total momentum $Q$, are always understood. The corresponding tree-level matrix element is denoted by

$$
M_{c_1, c_2, \ldots}^{s_1, s_2, \ldots}(p_1, p_2, \ldots)
$$

where $\{c_1, c_2, \ldots\}$, $\{s_1, s_2, \ldots\}$ and $\{a_1, a_2, \ldots\}$ are respectively colour, spin and flavour indices. The matrix element squared summed over final-state colour $s$ and spins will be denoted by $|M_{a_1, a_2, \ldots}(p_1, p_2, \ldots)|^2$. To the purpose of the present paper, it is also useful to consider the sum over the spins of all the final-state partons but one. For instance, if the sum over the spin polarizations of the parton $a_1$ is not carried out, we define the following ‘spin-polarization tensor’

$$
T_{s_1, s_1'}(p_1, \ldots) \equiv \sum_{\text{spins} \neq s_1} \sum_{\text{colours}} M_{a_1, a_2, \ldots}^{c_1, c_2, \ldots, s_1, s_1', \ldots}(p_1, p_2, \ldots) \left[M_{a_1, a_2, \ldots}^{c_1, c_2, \ldots, s_1', s_1, \ldots}(p_1, p_2, \ldots)\right]^\dagger.
$$

In the evaluation of the matrix element, we use dimensional regularization in $d = 4 - 2\epsilon$ space-time dimensions and consider two helicity states for massless quarks and $d-2$ helicity states for gluons. This defines the usual dimensional-regularization scheme. Thus, the fermion spin indices are $s = 1, 2$ while to label the gluon spin it is convenient to use the corresponding Lorentz index $\mu = 1, \ldots, d$. The $d$-dimensional average of the matrix element over the polarizations of a parton $a$ is obtained by means of the factors

$$
\frac{1}{2}\delta_{ss'}
$$

for a fermion, and (the gauge terms are proportional either to $p^\mu$ or to $p'^\nu$)

$$
\frac{1}{d-2}d_{\mu\nu}(p) = \frac{1}{2(1-\epsilon)}(-g_{\mu\nu} + \text{gauge terms})
$$

with

$$
-g_{\mu\nu}d_{\mu\nu}(p) = d-2, \quad p^\mu d_{\mu\nu}(p) = d_{\mu\nu}(p) p'^\nu = 0,
$$

for a gluon with on-shell momentum $p$.

The relevant collinear limit at $\mathcal{O}(\alpha_s)$ is approached when the momenta of two partons, say $p_1$ and $p_2$, become parallel. Usually, this limit is precisely defined as follows:

$$
p_1^\mu = z p^\mu + k_\perp^\mu - k_\perp^2 z n_\mu \over 2p \cdot n,
\quad
p_2^\mu = (1-z) p^\mu - k_\perp^2 \over 1-z 2p \cdot n,
\quad
s_{12} \equiv 2p_1 \cdot p_2 = -k_\perp^2 \over z(1-z), \quad k_\perp \to 0.
$$

In Eq. (6) the light-like ($p^2 = 0$) vector $p^\mu$ denotes the collinear direction, while $n_\mu$ is an auxiliary light-like vector, which is necessary to specify the transverse component $k_\perp$.\footnote{The case of incoming partons can be recovered by simply crossing the parton flavours and momenta.}
In the small-$k_\perp$ limit (i.e. neglecting terms that are less singular than $1/k_\perp^2$), the square of the matrix element in Eq. (1) fulfils the following factorization formula [1]

$$|\mathcal{M}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)|^2 \simeq \frac{2}{s_{12}} 4\pi \mu^2 \alpha_S \mathcal{T}_{ss'}^{a\ldots}(p,\ldots) \hat{P}_{a_1a_2}^{ss'}(z, k_\perp; \epsilon) ,$$

where $\mu$ is the dimensional-regularization scale. The spin-polarization tensor $\mathcal{T}_{ss'}^{a\ldots}(p,\ldots)$ is obtained by replacing the partons $a_1$ and $a_2$ on the right-hand side of Eq. (2) with a single parton denoted by $a$. This parton carries the quantum numbers of the pair $a_1 + a_2$ in the collinear limit. In other words, its momentum is $p^a$ and its other quantum numbers (flavour, colour) are obtained according to the following rule: anything + gluon gives anything and quark + antiquark gives gluon.

The kernel $\hat{P}_{a_1a_2}$ in Eq. (7) is the $d$-dimensional Altarelli–Parisi splitting function [21]. It depends not only on the momentum fraction $z$ involved in the collinear splitting $a \to a_1 + a_2$, but also on the transverse momentum $k_\perp$ and on the helicity of the parton $a$ in the matrix element $\mathcal{M}_{a_1,a_2,\ldots}^{ss'}(p,\ldots)$. More precisely, $\hat{P}_{a_1a_2}$ is in general a matrix acting on the spin indices $s, s'$ of the parton $a$ in the spin-polarization tensor $\mathcal{T}_{ss'}^{a\ldots}(p,\ldots)$. Because of these spin correlations, the spin-average square of the matrix element $\mathcal{M}_{a_1,a_2,\ldots}^{ss'}(p,\ldots)$ cannot be simply factorized on the right-hand side of Eq. (7).

The explicit expressions of $\hat{P}_{a_1a_2}$, for the splitting processes

$$a(p) \to a_1(zp + k_\perp + \mathcal{O}(k_\perp^2)) + a_2((1-z)p - k_\perp + \mathcal{O}(k_\perp^2)) ,$$

depend on the flavour of the partons $a_1, a_2$ and are given by

$$\hat{P}_{qq}^{ss'}(z, k_\perp; \epsilon) = \hat{P}_{qq}^{ss'}(z, k_\perp; \epsilon) = \delta_{ss'} C_F \left[ \frac{1 + z^2}{1 - z} - \epsilon(1 - z) \right] ,$$

$$\hat{P}_{gq}^{ss'}(z, k_\perp; \epsilon) = \hat{P}_{gq}^{ss'}(z, k_\perp; \epsilon) = \delta_{ss'} C_F \left[ \frac{1 + (1-z)^2}{z} - \epsilon z \right] ,$$

$$\hat{P}_{gg}^{ss'}(z, k_\perp; \epsilon) = \hat{P}_{gg}^{ss'}(z, k_\perp; \epsilon) = T_R \left[ -g^{\mu\nu} + 4z(1-z)\frac{k^\mu_\perp k^\nu_\perp}{k_\perp^2} \right] ,$$

$$\hat{P}_{gg}^{ss'}(z, k_\perp; \epsilon) = 2C_A \left[ -g^{\mu\nu} \left( \frac{z}{1 - z} + \frac{1 - z}{z} \right) - 2(1 - \epsilon) z (1 - z) \frac{k^\mu_\perp k^\nu_\perp}{k_\perp^2} \right] ,$$

where the $SU(N_c)$ QCD colour factors are

$$C_F = \frac{N_c^2 - 1}{2N_c} , \quad C_A = N_c , \quad T_R = \frac{1}{2} ,$$

and the spin indices of the parent parton $a$ have been denoted by $s, s'$ if $a$ is a fermion and $\mu, \nu$ if $a$ is a gluon.

Note that when the parent parton is a fermion (cf. Eqs. (8) and (9)) the splitting function is proportional to the unity matrix in the spin indices. Thus, in the factorization
formula (4), spin correlations are effective only in the case of the collinear splitting of a gluon. Owing to the $k_\perp$-dependence of the gluon splitting functions in Eqs. (11) and (12), these spin correlations produce a non-trivial azimuthal dependence with respect to the directions of the other momenta in the factorized matrix element.

Equations (1)–(12) lead to the more familiar form of the $d$-dimensional splitting functions only after average over the polarizations of the parton $a$. The $d$-dimensional average is obtained by means of the factors in Eqs. (3) and (4). Denoting by $\langle \hat{P}_{a_1a_2} \rangle$ the average of $\hat{P}_{a_1a_2}$ over the polarizations of the parent parton $a$, we have:

$$\langle \hat{P}_{qq}(z; \epsilon) \rangle = \langle \hat{P}_{qg}(z; \epsilon) \rangle = C_F \left[ \frac{1 + z^2}{1 - z} - \epsilon(1 - z) \right], \quad (14)$$

$$\langle \hat{P}_{gq}(z; \epsilon) \rangle = \langle \hat{P}_{gq}(z; \epsilon) \rangle = C_F \left[ \frac{1 + (1 - z)^2}{z} - \epsilon z \right], \quad (15)$$

$$\langle \hat{P}_{qq}(z; \epsilon) \rangle = \langle \hat{P}_{qg}(z; \epsilon) \rangle = T_R \left[ 1 - \frac{2z(1 - z)}{1 - \epsilon} \right], \quad (16)$$

$$\langle \hat{P}_{gg}(z; \epsilon) \rangle = 2C_A \left[ \frac{z}{1 - z} + \frac{1 - z}{z} + z(1 - z) \right]. \quad (17)$$

In the rest of the paper we are interested in the collinear limit at $\mathcal{O}(\alpha_S^2)$. In this case three parton momenta can simultaneously become parallel. Denoting these momenta by $p_1, p_2$ and $p_3$, their most general parametrization is

$$p_i^\mu = x_i p^\mu + \frac{k_{1i}^2}{2p \cdot n} n^\mu, \quad i = 1, 2, 3, \quad (18)$$

where, as in Eq. (3), the light-like vector $p^\mu$ denotes the collinear direction and the auxiliary light-like vector $n^\mu$ specifies how the collinear direction is approached ($k_{1i} \cdot p = k_{1i} \cdot n = 0$). Note that no other constraint (e.g. $\sum_i x_i = 1$ or $\sum_i k_{1i} = 0$) is imposed on the longitudinal and transverse variables $x_i$ and $k_{1i}$. Thus, we can easily consider any (asymmetric) collinear limit at once.

In the triple-collinear limit, the matrix element squared $|\mathcal{M}_{a_1,a_2,a_3 \ldots}(p_1, p_2, p_3, \ldots)|^2$ has the singular behaviour $|\mathcal{M}_{a_1,a_2,a_3 \ldots}(p_1, p_2, p_3, \ldots)|^2 \sim 1/(ss')$, where $s$ and $s'$ can be either two-particle ($s_{ij} = (p_i + p_j)^2$) or three-particle ($s_{123} = (p_1 + p_2 + p_3)^2$) sub-energies. More precisely, it can be shown [18, 20] that the matrix element squared still fulfils a factorization formula analogous to Eq. (4), namely

$$|\mathcal{M}_{a_1,a_2,a_3 \ldots}(p_1, p_2, p_3, \ldots)|^2 \simeq \frac{4}{s_{123}} \left( 4\pi \mu^{2\epsilon} \alpha_S \right)^2 T_{a \ldots}^{ss'}(p, \ldots) \hat{P}_{a_1a_2a_3}^{ss'}. \quad (19)$$

Likewise in Eq. (4), the spin-polarization tensor $T_{a \ldots}^{ss'}(p, \ldots)$ is obtained by replacing the partons $a_1, a_2$ and $a_3$ with a single parent parton, whose flavour $a$ is determined (see Sect. 3) by flavour conservation in the splitting process $a \to a_1 + a_2 + a_3$. 

4
The three-parton splitting functions $\hat{P}_{a_1 a_2 a_3}$ generalize the Altarelli–Parisi splitting functions in Eq. (7). The spin correlations produced by the collinear splitting are taken into account by the splitting functions in a universal way, i.e. independently of the specific matrix element on the right-hand side of Eq. (15). Besides depending on the spin of the parent parton, the functions $\hat{P}_{a_1 a_2 a_3}$ depend on the momenta $p_1, p_2, p_3$. However, due to their invariance under longitudinal boosts along the collinear direction, the splitting functions can depend in a non-trivial way only on the sub-energy ratios $s_{ij}/s_{123}$ and on the following longitudinal and transverse variables:

$$z_i = \frac{x_i}{\sum_{j=1}^{3} x_j},$$

$$\tilde{k}_i^\mu = k_{1 i}^\mu - \frac{x_i}{\sum_{k=1}^{3} x_k} \sum_{j=1}^{3} k_{1 j}^\mu,$$

which automatically satisfy the constraints $\sum_{i=1}^{3} z_i = 1$ and $\sum_{i=1}^{3} \tilde{k}_i = 0$. To simplify the explicit expressions of the splitting functions, we find it convenient to introduce also the variables

$$t_{ij,k} \equiv 2 \frac{z_i s_{jk} - z_j s_{ik}}{z_i + z_j} + \frac{z_i - z_j}{z_i + z_j} s_{ij}.$$ 

The results of our calculation are presented in the next section.

### 3 Collinear splitting functions at $O(\alpha_S^2)$

To evaluate the three-parton splitting functions, we use power-counting arguments and the universal factorization properties of collinear singularities. The method [22] consists in directly computing process-independent Feynman subgraphs in a physical gauge. Details on the method and on our calculation are given in Ref. [20]. In the following we present the complete results for the spin-dependent splitting functions.

The list of (non-vanishing) splitting processes that we have to consider is as follows:

$$q \to q_1' + q_2' + q_3, \quad (\bar{q} \to \bar{q}_1' + q_2' + \bar{q}_3),$$

$$q \to \bar{q}_1 + q_2 + q_3, \quad (\bar{q} \to \bar{q}_1 + q_2 + \bar{q}_3),$$

$$q \to g_1 + g_2 + q_3, \quad (\bar{q} \to g_1 + g_2 + \bar{q}_3),$$

$$g \to g_1 + g_2 + g_3,$$ 

$$g \to g_1 + g_2 + g_3. (27)$$

The superscripts in $q', \bar{q}'$ denote fermions with different flavour with respect to $q, \bar{q}$. The splitting functions for the processes in parenthesis in Eqs. (23) and (24) can be simply obtained by charge-conjugation invariance, i.e. $\hat{P}_{q_1' q_2' q_3} = \hat{P}_{\bar{q}_1' q_2' \bar{q}_3}$ and $\hat{P}_{\bar{q}_1 q_2 q_3} = \hat{P}_{\bar{q}_1 q_2 q_3}$. In summary, we have to compute five independent splitting functions.

In the case of the splitting processes that involve a fermion as parent parton (see Eqs. (23)–(25)), we find that spin correlations are absent. We can thus write

$$\hat{P}_{q_1' q_2' q_3} = \delta^{ss'} (\hat{P}_{q_1' q_2' q_3}),$$

$$\hat{P}_{q_1 q_2 q_3} = \delta^{ss'} (\hat{P}_{q_1 q_2 q_3}).$$

(28)
and likewise for $\hat{P}^{ss'}_{g_1q_3q_3}$ and $\hat{P}^{ss'}_{g_1g_2q_3}$. This feature is completely analogous to the $O(\alpha_S)$ case and follows from helicity conservation in the quark–gluon vector coupling.

The spin-averaged splitting function for non-identical fermions in the final state is

$$
\langle \hat{P}_{q'q_3q_3} \rangle = \frac{1}{2} C_F T_R \frac{s_{12}}{s_{12}s_{123}} \left[ -\frac{t^2_{12,3}}{s_{12}s_{123}} + \frac{4z_3 + (z_1 - z_2)^2}{z_1 + z_2} + (1 - 2\epsilon) \left( z_1 + z_2 - \frac{s_{12}}{s_{123}} \right) \right].
$$

(29)

The analogous splitting function in the case of final-state fermions with identical flavour can be written in terms of that in Eq. (29), as follows

$$
\langle \hat{P}_{q'q_3q_3} \rangle = \left[ \langle \hat{P}_{q'q_3q_3} \rangle + (2 \leftrightarrow 3) \right] + \langle \hat{P}_{(id)}^{(id)} \rangle,
$$

(30)

where

$$
\langle \hat{P}_{q'q_3q_3}^{(id)} \rangle = C_F \left( C_F - \frac{1}{2} C_A \right) \left\{ (1 - \epsilon) \left( \frac{2s_{23}}{s_{12}} - \epsilon \right) + \frac{s_{123}}{s_{12}} \left[ \frac{1 + z_2^2}{1 - z_2} - \frac{2z_2}{1 - z_3} - \epsilon \left( \frac{1 - z_3}{1 - z_2} + 1 + z_1 - \frac{2z_2}{1 - z_3} \right) - \epsilon^2 (1 - z_3) \right] \right.
$$

$$
- \frac{s_{12}^2 z_1}{s_{12}s_{13}} \left[ \frac{1 + z_2^2}{(1 - z_2)(1 - z_3)} - \epsilon \left( 1 + \frac{2(1 - z_2)}{1 - z_3} \right) - \epsilon^2 \right] \right\} + (2 \leftrightarrow 3).
$$

(31)

The splitting function of the remaining quark-decay subprocess can be decomposed according to the different colour coefficients:

$$
\langle \hat{P}_{g_1g_2q_3} \rangle = C_F^2 \langle \hat{P}_{g_1g_2q_3}^{(ab)} \rangle + C_F C_A \langle \hat{P}_{g_1g_2q_3}^{(nab)} \rangle,
$$

(32)

and the abelian and non-abelian contributions are

$$
\langle \hat{P}_{g_1g_2q_3}^{(ab)} \rangle = \left\{ \frac{s_{12}}{s_{13}^2 s_{23}} z_3 \left[ \frac{1 + z_2^2}{z_1 z_2} - \epsilon \frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon (1 + \epsilon) \right] + \frac{s_{123}}{s_{13}} \left[ \frac{z_3 (1 - z_1) + (1 - z_2)^3}{z_1 z_2} + \epsilon^2 (1 + z_3) - \epsilon (z_1^2 + z_1 z_2 + z_2^2) \frac{1 - z_2}{z_1 z_2} \right] + (1 - \epsilon) \left[ \epsilon - (1 - \epsilon) \frac{s_{23}}{s_{13}} \right] \right\} + (1 \leftrightarrow 2),
$$

(33)

$$
\langle \hat{P}_{g_1g_2q_3}^{(nab)} \rangle = \left\{ (1 - \epsilon) \left( \frac{t^2_{12,3}}{4 s_{12}} + \frac{1}{4} - \epsilon \right) + \frac{s_{123}}{2 s_{12} s_{13}} \left[ \frac{(1 - z_3)^2 (1 - \epsilon) + 2 z_3}{z_2} \right] + \frac{s_{123}}{4 s_{13} s_{23}} \left[ \frac{(1 - z_3)^2 (1 - \epsilon) + 2 z_3}{z_1 z_2} \right] + (1 - \epsilon) \right\} + (1 \leftrightarrow 2).
$$

(34)
In the case of collinear decays of a gluon (see Eqs. (26, 27)), spin correlations are highly non-trivial.

The colour-factor decomposition of the splitting function for the decay into a q̄q pair plus a gluon is
\[ \hat{P}_{g_1 g_2 g_3}^{\mu \nu} = C_F T_R \hat{P}_{g_1 g_2 g_3}^{(ab)} + C_A T_R \hat{P}_{g_1 g_2 g_3}^{(nah)} , \] (35)
where the abelian and non-abelian terms are given by
\[ \hat{P}_{g_1 g_2 g_3}^{(ab)} = -g^{\mu \nu} \left[ -2 + \frac{2 s_{123} s_{23} + (1 - \epsilon)(s_{123} - s_{23})^2}{s_{12} s_{13}} \right] \]
\[ + \frac{4 s_{123}}{s_{12} s_{13}} \left( \tilde{k}_2^\mu \tilde{k}_2^\nu + \tilde{k}_3^\mu \tilde{k}_3^\nu - (1 - \epsilon) \tilde{k}_1^\mu \tilde{k}_1^\nu \right) , \] (36)
\[ \hat{P}_{g_1 g_2 g_3}^{(nah)} = \frac{1}{4} \left\{ \frac{s_{123}}{s_{23}} \left[ g^{\mu \nu} t_{23,1}^2 - 16 \frac{z_2^2 z_3^2}{z_1 (1 - z_1)} \left( \frac{\tilde{k}_2}{z_2} - \frac{\tilde{k}_3}{z_3} \right)^\mu \left( \frac{\tilde{k}_2}{z_2} - \frac{\tilde{k}_3}{z_3} \right)^\nu \right] \right. \]
\[ + \left. \frac{s_{123}}{s_{12} s_{23}} \left[ 2 s_{123} g^{\mu \nu} \frac{z_2 (1 - z_1)}{1 - z_1} - 16 \tilde{k}_3^{\mu} \tilde{k}_3^{\nu} \frac{z_2}{z_1 (1 - z_1)} + 8 (1 - \epsilon) \tilde{k}_2^{\mu} \tilde{k}_2^{\nu} \right. \right. \]
\[ + 4 (\tilde{k}_2^{\mu} \tilde{k}_2^{\nu} + \tilde{k}_3^{\mu} \tilde{k}_3^{\nu}) \left( \frac{2 z_2 (z_3 - 1)}{z_2 (1 - z_1)} + (1 - \epsilon) \right) \left) \right] + (2 \leftrightarrow 3) \} . \] (37)

In the case of gluon decay into three collinear gluons we find
\[ \hat{P}_{g_1 g_2 g_3}^{\mu \nu} = C_A^2 \left\{ \frac{(1 - \epsilon)}{4 s_{12}^2} \left[ -g^{\mu \nu} t_{12,3}^2 + 16 s_{123} \frac{z_2^2 z_3^2}{z_3 (1 - z_3)} \left( \frac{\tilde{k}_2}{z_2} - \frac{\tilde{k}_1}{z_1} \right)^\mu \left( \frac{\tilde{k}_2}{z_2} - \frac{\tilde{k}_1}{z_1} \right)^\nu \right] \right. \]
\[ - \left. \frac{3}{4} (1 - \epsilon) g^{\mu \nu} + \frac{s_{123}}{s_{12} s_{13}} g^{\mu \nu} \frac{1}{z_3} \left[ 2 (1 - z_3) + 4 z_3^2 \right] - \frac{1 - 2 z_3 (1 - z_3)}{z_1 (1 - z_1)} \right] \]
\[ + \frac{3}{2 (1 - \epsilon)} g^{\mu \nu} \left( \frac{4 z_2 z_3 + 2 z_1 (1 - z_1) - 1}{(1 - z_2) (1 - z_3)} - \frac{1 - 2 z_1 (1 - z_1)}{z_2 z_3} \right) \]
\[ + \left. (\tilde{k}_2^{\mu} \tilde{k}_2^{\nu} + \tilde{k}_3^{\mu} \tilde{k}_3^{\nu}) \left( \frac{2 z_2 (1 - z_1)}{z_3 (1 - z_3)} - 3 \right) \right] \} + (5 \text{ permutations}) \} . \] (38)

The splitting functions in Eqs. (36)–(38) can be averaged over the spin polarizations of the parent gluon according to Eq. (1):
\[ \langle \hat{P}_{a_1 a_2 a_3} \rangle = \frac{1}{2 (1 - \epsilon)} d_{\mu \nu} (p) \hat{P}_{a_1 a_2 a_3}^{\mu \nu} . \] (39)
Performing the average we obtain

\[
\langle \hat{P}^{(ab)}_{g_1g_2q_3} \rangle = -2 - (1 - \epsilon) s_{23} \left( \frac{1}{s_{12}} + \frac{1}{s_{13}} \right) + 2 \frac{s_{123}^2}{s_{12}s_{13}} \left( 1 + z_1^2 - \frac{z_1 + 2z_2z_3}{1 - \epsilon} \right) \\
- \frac{s_{123}}{s_{12}} \left( 1 + 2z_1 + \epsilon - 2\frac{z_1 + z_2}{1 - \epsilon} \right) - \frac{s_{123}}{s_{13}} \left( 1 + 2z_1 + \epsilon - 2\frac{z_1 + z_3}{1 - \epsilon} \right), \tag{40}
\]

\[
\langle \hat{P}^{(nab)}_{g_1g_2q_3} \rangle = -\frac{t_{23}^2}{4s_{23}^2} + \frac{s_{123}^2}{2s_{12}s_{23}} z_3 \left[ (1 - z_1)^2 - \frac{1}{z_1(1 - z_1)} - 2 \frac{z_2}{1 - \epsilon} \right] \\
+ \frac{s_{123}}{2s_{13}} (1 - z_2) \left[ 1 + \frac{1}{z_1(1 - z_1)} - \frac{2z_2}{(1 - \epsilon)z_1(1 - z_1)} \right] \\
+ \frac{s_{123}}{2s_{23}} \left( \frac{1 + z_1^2}{z_1(1 - z_1)} + \frac{z_2}{(1 - \epsilon)z_1(1 - z_1)} \right) \\
- \frac{1}{4} + \frac{\epsilon}{2} - \frac{s_{123}^2}{2s_{12}s_{13}} \left( 1 + z_1^2 - \frac{z_1 + 2z_2z_3}{1 - \epsilon} \right) \right\} + (2 \leftrightarrow 3), \tag{41}
\]

\[
\langle \hat{P}^{(g)}_{g_1g_2q_3} \rangle = C_A^2 \left\{ \frac{(1 - \epsilon)^2}{4s_{12}^2} + \frac{3}{4} (1 - \epsilon) + \frac{s_{123}^2}{s_{12}} \left[ \frac{z_2(z_1^2 - 1 - 2z_2 - 2z_3)}{z_3} + \frac{3}{2} + \frac{5}{2} \right] \\
+ \frac{(1 - z_3(1 - z_3))^2}{z_3z_1(1 - z_1)} \right\} + \frac{s_{123}^2}{s_{12}s_{13}} \left[ \frac{z_2}{z_3(1 - z_3)} + \frac{z_2z_3 + 2 + \frac{1}{z_1(1 + z_1)}}{2} \right] \\
+ \frac{1 + z_1z_2}{2(1 - z_2)(1 - z_3)} \right\} + (5 \text{ permutations}) . \tag{42}
\]

The $O(\alpha_S^3)$-collinear behaviour of tree-level QCD matrix elements has been independently examined by Campbell and Glover [19]. Their study differs in many respects from our analysis. Taken for granted the universal factorization formula [19], they compute the three-parton splitting functions by directly performing the collinear limit of the explicit expressions of the $\gamma^* \to$ four- and five-parton squared matrix elements. Moreover, they treat the colour structure in a different way and consider the collinear limit of the colour-ordered sub-amplitudes [3]. Finally, they neglect spin correlations and present only the explicit expressions of the polarization-averaged splitting functions.

By properly taking into account the differences in the colour treatment, we have compared our results with those of Ref. [19] and found complete agreement* for the spin-averaged splitting functions. To be precise, the colour-connected splitting functions $P_{a_1a_2a_3 \to a}$ of Ref. [19] are related to our spin-averaged splitting functions as follows:

\[
\langle \hat{P}^{(g)}_{q_1q_2q_3} \rangle = \frac{s_{123}^2}{4} T_R C_F P_{q_1q_2q_3 \to q}(\text{non-ident}),
\]

\[
\langle \hat{P}^{(id)}_{q_1q_2q_3} \rangle = \frac{s_{123}^2}{4} C_F \left( C_F - \frac{1}{2} C_A \right) P_{q_1q_2q_3 \to q}(\text{ident}),
\]

\[
\langle \hat{P}^{(ab)}_{g_1g_2q_3} \rangle = \frac{s_{123}^2}{4} P_{g_1g_2q_3 \to q},
\]

*Note that the published version of Ref. [19] contains two misprints (in Eqs. (5.8) and (5.19)) that have been corrected in the archive version [hep-ph/9710255] v3.
\[ \langle \hat{P}_{g_1 g_2 g_3}^{(nab)} \rangle = \frac{s_{123}^2}{4} \frac{1}{2} (P_{g_3 g_1 g_2 \rightarrow q} + P_{g_3 g_2 g_1 \rightarrow q} - P_{g_3 g_1 g_2 \rightarrow q}) , \]  
\[ \langle \hat{P}_{g_1 g_2 g_3}^{(ab)} \rangle = \frac{s_{123}^2}{4} P_{g_2 g_1 g_3 \rightarrow \bar{q}} , \]
\[ \langle \hat{P}_{g_1 g_2 g_3}^{(nah)} \rangle = \frac{s_{123}^2}{4} \frac{1}{2} (P_{g_1 g_3 g_2 \rightarrow \bar{q}} + P_{g_3 g_2 g_1 \rightarrow \bar{q}} - P_{g_2 g_1 g_3 \rightarrow \bar{q}}) , \]
\[ \langle \hat{P}_{g_1 g_2 g_3} \rangle = \frac{s_{123}^2}{4} \left( \frac{C_A}{2} \right)^2 [P_{g_1 g_2 g_3 \rightarrow \bar{q}} + (5 \text{ permutations})] . \]

Owing to the completely different methods used by the two groups, this agreement can be regarded as an important cross-check of the calculations.

4 Summary

We have considered the three-parton collinear limit of tree-level QCD amplitudes. In this limit the singular behaviour of the matrix element squared is given by the universal factorization formula (19) and is controlled by process-independent splitting functions, which are analogous to the Altarelli–Parisi splitting functions. In Sect. 3 we have presented the explicit expressions of the splitting functions at \( \mathcal{O} (\alpha_S^2) \), taking fully into account spin correlations.

These splitting functions are one of the necessary ingredients needed to extend QCD predictions at higher perturbative orders. In particular, they are relevant to perform analytic resummed calculations beyond NLL accuracy and to set up general methods to compute jet cross sections at NNLO. The knowledge of the collinear splitting functions, when combined with a consistent analysis of soft-gluon coherence, can also give prospects of improving the logarithmic accuracy of parton showers available at present for Monte Carlo event generators.

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