Abstract

We study infinitely divisible (ID) distributions on the nonnegative half-line \( \mathbb{R}_+ \). The Lévy-Khintchine representation of such distributions is well-known. Our primary contribution is to cast the probabilistic objects and the relations amongst them in a unified visual form that we refer to as the Lévy-Khintchine commutative diagram (LKCD). While it is introduced as a representational tool, the LKCD facilitates the exploration of new ID distributions and may thus also be looked upon, at least in part, as a discovery tool. The basic object of the study is the gamma distribution. Closely allied to this is the \( \alpha \)-stable distribution on \( \mathbb{R}_+ \) for \( 0 < \alpha < 1 \), which we regard as arising from the gamma distribution rather than as a separate object. It is indeed often characterised as an instance of a class of ID distributions known as generalised gamma convolutions (GGCs). We make use of convolutions and mixtures of gamma and stable densities to generate densities of other GGC distributions, with particular cases involving Bessel, confluent hypergeometric, Mittag-Leffler and parabolic cylinder functions. We present all instances as LKCD representations.

Keywords— infinite divisibility, Lévy-Khintchine representation, commutative diagrams; generalised gamma convolutions; gamma, stable, beta, fractional gamma distributions; Bessel, confluent hypergeometric, Mittag-Leffler, parabolic cylinder functions.

1 Introduction

We are interested in infinitely divisible (ID) distributions on the nonnegative half-line \( \mathbb{R}_+ \), that we shall occasionally refer to as ‘positive ID distributions’ for short, even though the term is a bit imprecise. Such distributions find application in numerous settings involving positive (non-negative) random phenomena that are additive over partitions of their domain of definition, the domain being one-dimensional (notably time) or multidimensional (physical or more abstract space). Given that the theory of positive ID distributions is well-established (as briefly reviewed
ID distributions feature prominently in the book by Feller [8], while the one by Steutel and van Harn [25] is exclusively dedicated to the topic of infinite divisibility. Well-known examples of ID distributions on real variables are the Gaussian and Cauchy distributions (they also happen to be the two instances of stable distributions on \( \mathbb{R} \) with known closed form densities). In the case of variables on \( \mathbb{R}_+ \), the gamma distribution is the classical example, along with the family of stable distributions on \( \mathbb{R}_+ \) ("positive stable distributions"). The latter two examples are related in the sense that the gamma distribution might be described as a “founding member” of a family of distributions known as the generalised gamma convolutions (GGCs), to which the positive stable distributions also belong.

Thorin [26, 27] introduced the GGC class (which belongs to the broader class of positive ID distributions) as he sought to prove the infinite divisibility of the Pareto and the log-normal distributions. GGC theory was subsequently studied in depth in the text by Bondesson [5], exploring the powerful ramifications of the GGC concept. The more recent survey of the GGC class by James et al. [15] includes theory and examples.

All ID distributions are characterised by the Lévy-Khintchine representation. The primary purpose of this paper is to introduce a visualisation of infinite divisibility by casting the Lévy-Khintchine representation as a commutative diagram, a construct borrowed from category theory. In our view, the commutative diagram lends a welcome perspective to what can often be a bewildering morass of equations in the study of ID distributions.

The Lévy-Khintchine commutative diagram (LKCD) might be said to be an organising principle that displays the objects of an infinitely divisible probabilistic structure as vertices connected by arrows denoting relationships between objects. The passage from one object to another is path-independent. Our experience has been that the assignment of arrow mappings such that path-independence holds can trigger a thought process about ID/GGC structure that does not readily arise in the absence of the commutative diagram setting. In some instances, this has prompted novel ideas on the representation of known densities. This will be especially apparent when we describe the fractional gamma distribution in terms of the parabolic cylinder function alongside the commonly used Mittag-Leffler function and its three-parameter generalisation known as the Prabhakar function.

Hence the LKCD can facilitate the discovery of novel probabilistic representations. It may not be a discovery tool in its own right, but we believe it to be a worthy addition to the study of infinite divisibility.

1.1 Structure of Paper

Along with the well-known gamma distribution, the stable distribution is central to the paper. We start by introducing the stable distribution in the conventional way as a standalone object. We summarise infinite divisibility on \( \mathbb{R}_+ \) in Section 2. We then introduce the Lévy-Khintchine
commutative diagram (LKCD) in Section 3, with gamma and stable LKCD examples as the objects of primary interest. Section 4 discusses the ID class known as generalised gamma convolutions (GGCs), followed by the stable GGC LKCD in Section 5. With the preparatory background in place, Section 6 moves to the convolution of two gamma densities with several associated LKCD examples. This is the first part of the core message of the paper. The second part in Section 7 introduces mixtures of stable densities, with both stable and gamma mixing densities as examples. The former allows the generation of new stable densities from given instances. The latter gives the fractional gamma density, which is discussed at length. We discover a novel integral representation of the fractional gamma density for $\alpha = 1/2$ in terms of the parabolic cylinder function. All examples are presented as LKCDs. This is followed by a discussion in Section 10, and concluding remarks and pointers to future work in Section 11.

### 1.2 Stable distribution

The $\alpha$-stable distribution for $0 < \alpha < 1$, defined on the positive half-line, has density $f_\alpha(x)$, $x \geq 0$ with Laplace transform

$$\tilde{f}_\alpha(s) = \exp(-s^\alpha) \quad (1)$$

The $\alpha$-stable distribution is of interest in probability theory and various applications. In physics, $\tilde{f}_\alpha(s)$ is often referred to as the stretched exponential or the Kohlrausch function (Berberan-Santos et al. [4], Penson and Górska [18]). It is intimately associated with relaxation and diffusion phenomena. To paraphrase [18], $f_\alpha(x)$ arises in condensed and soft matter physics, geophysics, meteorology, economics, fractional kinetics: “For instance, the value $\alpha = 1/4$ is thought to describe mechanical and dielectric properties of glassy polymers. It is also confirmed that the same value of $\alpha$ is relevant for a statistical description of subrecoil laser cooling”.

Yet the functional form of $f_\alpha(x)$ is elusive. Pollard [21] showed that Laplace inversion gives the rather forbidding infinite series

$$f_\alpha(x) = -\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\sin(\pi k \alpha)}{\Gamma(k \alpha + 1)} \frac{\Gamma(k \alpha + 1)}{x^{k \alpha + 1}} \quad (2)$$

Feller [8] (p581) derived the expression using the Fourier transform, along with another expression for $1 < \alpha < 2$. (We shall not discuss here stable distributions on the real line indexed by $1 \leq \alpha \leq 2$, with known closed form only for the Cauchy distribution for $\alpha = 1$ and the Gaussian distribution for $\alpha = 2$.)

For $\alpha = 1/2$, the series representation (2) reduces to the simple form

$$f_{1/2}(x) = \frac{1}{2\sqrt{\pi}} e^{-3/2} e^{-1/4x} \quad (3)$$

Other forms can be inferred from (2), such as $f_{1/3}$ in terms of $K_{1/3}$, the modified Bessel function of the second kind of order 1/3. Forms for rational $\alpha$ are typically cast in terms of hypergeometric functions or the allied Whittaker functions, e.g. $\alpha = \{2/3, 1/4, 3/4\}$ (Barkai [2], Penson and Górska [18], Scher and Montroll [22]). Indeed, [18, 22] state that, for any rational $\alpha = l/k$ ($0 < l < k$), $f_\alpha$ may be expressed as a finite sum of generalised hypergeometric functions. But generality often comes at the cost of simplicity. Hypergeometric functions are flexible series
representations that are not routinely encountered mathematical objects, even though they yield many common functions as particular cases.

Amongst other things, we will discuss a simple and known integral representation of \( f_{\alpha\beta} \) in terms of \( f_\alpha \) and \( f_\beta \). In particular, we shall infer \( f_{1/4} \) from \( f_{1/2} \) as an integral representation instead of the hypergeometric representation of \( f_{1/4} \) given in [2, 18].

In Section 3 we shall motivate the stable distribution as an intimate relative of the gamma distribution rather than as a standalone object. To that end and beyond, we discuss next the concept of infinitely divisible (ID) distributions.

## 2 Infinitely Divisible Distributions

The theory of infinitely divisible distributions summarised here is well-known and can be found in several probabilistic texts such as Feller [8], Kingman [17], Steutel and van Harn [25]. Our contribution is a commutative diagram representation that, in our view, offers a helpful visual summary of the theoretical framework.

By way of basic motivation, consider a set of points that are randomly scattered over some domain. In practical application, the domain might be an interval in time or a region in space. A point might be an event in time like a vehicle crossing a bridge in sparse traffic, an isolated day of rain or, in a spatial context, a point source at some location in the sky. In addition, each point carries a random positive additive attribute, such as the mass of the vehicle, the amount of rainfall on the given day or the brightness of the point source. In each case, we may meaningfully speak of the total vehicle mass that the bridge bears in a day, the rainfall in a month or the brightness of the patch of sky by simply adding up the respective attributes over all point occurrences within the specified domain.

More abstractly, let \( n \), the number of point occurrences in a specified domain, be governed by a Poisson distribution with mean rate \( \mu \) (typically the size of the domain). Let each point \( i \) have an associated attribute \( X_i \) where the \( \{X_i : i = 1 \ldots n\} \) are independent, identically distributed positive (nonnegative) variables governed by a common distribution with density \( \ell(x) \). Then, as is well-known, the sum \( X = X_1 + X_2 + \cdots + X_n \) is governed by the density \( \Pr(x|n) = \ell^{n*}(x) \), where \( \ell^{n*} \) is the \( n \)-fold Laplace convolution of \( \ell \) (where \( \ell^{1*} \equiv \ell \)). For \( n = 0 \), \( X \equiv 0 \), so that \( \Pr(x|n = 0) \equiv \ell^0(x) = \delta(x) \) is an atom at \( x = 0 \).

The joint distribution of \( X \) and \( n \) is

\[
\Pr(x,n|\mu) = \Pr(x|n) \Pr(n|\mu) = \ell^{n*}(x) e^{-\mu} \frac{\mu^n}{n!}
\]  

Hence the unconditional distribution of \( X \) is

\[
\Pr(x|\mu) = \sum_{n=0}^{\infty} \Pr(x|n) \Pr(n|\mu) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \ell^{n*}(x)
\]  

We shall also write this as \( f(x|\mu) \), which is the density of what is known as the compound Poisson distribution that we shall denote by \( \mathcal{CP}(\mu,\ell) \). The Laplace transform of \( f(x|\mu) \) is

\[
\mathcal{L}\{f\}(s) \equiv \tilde{f}(s|\mu) = \int_0^\infty e^{-sx} f(x|\mu) dx
\]
Similarly, $\tilde{\ell}(s)$ is the Laplace transform of $\ell(x)$. Since, by the convolution theorem, the Laplace transform of a convolution of functions is a product of their respective Laplace transforms, $\mathcal{L}\{\ell^n\}(s) = \tilde{\ell}^n(s)$. Hence the Laplace transform of (5) is

$$\tilde{f}(s|\mu) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \tilde{e}^n(s) = \exp\{-\mu(1 - \tilde{\ell}(s))\}$$

$$= \exp\{-\mu \int_0^\infty (1 - e^{-sx}) \ell(x)dx\}$$

where (8) follows from $\ell(x)$ being a density that is normalised (at least at this stage of the discussion). A distribution with Laplace transform (8) is said to be infinitely divisible because any $n^{th}$ root of (8) is the same expression with $\mu$ replaced by $\mu/n$, i.e. the $n^{th}$ root is the Laplace transform of the probability distribution with density $f(x|\frac{\mu}{n})$. The form (8) is the celebrated Lévy-Khintchine representation of an infinitely divisible distribution on positive additive variables, with the definition of $\ell(x)$, known as the Lévy density, broadened beyond a normalised density. We may write the Laplace exponent as

$$\psi(s) \equiv \int_0^\infty (1 - e^{-sx}) \ell(x)dx = \int_0^\infty \int_0^s e^{-xt}dt x\ell(x)dx$$

$$= \int_0^s \int_0^\infty e^{-xt} \rho(x)dx dt \quad \tag{10}$$

$$= \int_0^s \tilde{\rho}(t)dt \quad \tag{11}$$

where $\tilde{\rho}(s)$ is the Laplace transform of $\rho(x) \equiv x\ell(x)$. In light of (11), (8) becomes

$$\tilde{f}(s|\mu) = \exp(-\mu \psi(s)) \quad \tag{12}$$

Hence it is $\tilde{\rho}(s)$ that actually needs to exist rather than $\tilde{\ell}(s)$. The compound Poisson representation $\mathcal{CP}(\mu, \ell)$ need not strictly exist for infinite divisibility to hold. Differentiating (12) gives

$$\tilde{f}'(s|\mu) = -\mu \tilde{\rho}(s)\tilde{f}(s|\mu) \quad \tag{13}$$

$$\Rightarrow \quad \mu \tilde{\rho}(s) = -\frac{\tilde{f}'(s|\mu)}{\tilde{f}(s|\mu)} \quad \tag{14}$$

which is invariant under scaling $f(x|\mu) \to Cf(x|\mu)$ for any constant $C > 0$. An equivalent expression arises from the limiting process

$$\lim_{n \to \infty} n\tilde{f}'(s|\frac{\mu}{n}) = -\lim_{n \to \infty} \mu \tilde{\rho}(s)\tilde{f}(s|\frac{\mu}{n}) = -\mu \tilde{\rho}(s)\tilde{f}(s|0)$$

$$\Rightarrow \quad \mu \tilde{\rho}(s) = -\lim_{n \to \infty} n\tilde{f}'(s|\frac{\mu}{n})/\tilde{f}(s|0) = -\lim_{n \to \infty} n\tilde{f}'(s|\frac{\mu}{n}) \quad \tag{16}$$

given that, by (12), $\lim_{n \to \infty} \tilde{f}(s|\frac{\mu}{n}) = \tilde{f}(s|0) = 1$. Invariance under $f(x|\mu) \to Cf(x|\mu)$ is preserved because, correspondingly, $\tilde{f}(s|0) = 1 \to \tilde{f}(s|0) = C$. Since $-\tilde{f}'(s|\mu)$ is the Laplace transform of $xf(x|\mu)$, it follows that

$$\mu \rho(x) = \lim_{n \to \infty} nx f(x|\frac{\mu}{n})/\tilde{f}(s|0) = \lim_{n \to \infty} nxf(x|\frac{\mu}{n})/\tilde{f}(s|0) = 1$$

An alternative approach to the foregoing starts from Bernstein’s theorem [8] (p439), which states that a function $f(x)$ is a density if and only if its Laplace transform $\tilde{f}(s)$ is completely monotone,
\(i.e. \ (-1)^n \tilde{f}^{(n)}(s) \geq 0, \ n \geq 0 \) \(f(x)\) is a probability density if, in addition, \(\tilde{f}(0) = 1\). Infinite divisibility of a density \(f(x)\) is the case where \(-\tilde{f}'(s)/\tilde{f}(s)\) is also the Laplace transform of a density. Hence an alternative definition of infinite divisibility is that \(f(x)\) is the density of an infinitely divisible distribution if and only if both \(\tilde{f}(s)\) and \(-\tilde{f}'(s)/\tilde{f}(s)\) are completely monotone. This more abstract definition is consistent with the compound Poisson approach although it does not directly assume it. In the compound Poisson construction, \(\ell(x)\) and therefore \(\rho(x) = x\ell(x)\) is an assigned density from the outset so that \(\tilde{\rho}(s)\) is necessarily completely monotone.

The Laplace convolution of two or more densities will arise repeatedly in the discussion that follows. Invoking the convolution theorem once more, it is straightforward to see that the convolution of ID densities is also an ID density whose Lévy density is the sum of Lévy densities of the convolution components.

### 3 Lévy-Khintchine Commutative Diagram

We summarise the objects and relationships amongst them in a graphic that we refer to as the Lévy-Khintchine commutative diagram (LKCD), shown in Figure 1. The compound Poisson relation is dotted to accommodate the observation above that it may formally be undefined despite the existence of all nodes of the LKCD (this may differ from the conventional interpretation of a dotted arrow in category theory).

![Figure 1: Lévy-Khintchine Commutative Diagram (LKCD).](image)

\(\mathcal{L}\) is the Laplace transform and \(\mathcal{CP}(\mu, \ell)\) \((x\ell(x) = \rho(x))\) is the compound Poisson construction (dotted because it may formally be undefined), \(\psi(s)\) is the (definite or indefinite) integral of \(\tilde{\rho}(s)\). The direct Lévy-Khintchine relation is the diagonal from bottom left to top right. It is equivalent to a composition of transitions along the axes: “east then north” or (if \(\mathcal{CP}(\mu, \ell)\) exists) “north then east”.

The commutative diagram illustrates, at a glance, the concept of infinite divisibility, the objects involved and the relationships amongst them. Having assigned or constructed one of the four possible nodes, we may then seek to populate the other nodes by following a path of relationships best suited to the task. Although the LKCD is primarily an organising principle rather than a discovery tool \((i.e.\ a\ mechanism\ to\ construct\ new\ ID\ distributions)\), in our view the visual representation facilitates both the description and construction of ID distributions.
We shall often speak of the upper level of the LKCD as the ID density level and the lower level as the Lévy density level. As noted earlier, any multiplicative constant at the upper level is ‘forgotten’ upon descent to the lower level. The density $\rho(x)$ in the lower level may itself be ID, in which case the LKCD can be extended downward to form a two-level ‘ladder’ where the bottom rung is the Lévy density level of the middle layer, which is in turn the Lévy density level of the top layer. Furthermore, since $f(x|\mu)$ is a density, of necessity, the layer $(f(x|\mu), \tilde{f}(s|\mu))$ can be treated as a Lévy density level for a higher level ID density layer, thereby extending the LKCD ladder upward by another rung. In principle, such upward growth of the LKCD can be repeated indefinitely, although analytic expressions for the ID densities thus generated may become increasingly elusive.

We give LKCD examples for two densities that are central to the rest of our discussion.

### 3.1 Gamma LKCD

The gamma density $g_{\mu,\lambda}(x) \equiv f_\lambda(x|\mu)$ and its LKCD are, respectively (18) and (19)

\[
\begin{align*}
g_{\mu,\lambda}(x) &= \frac{\lambda^\mu}{\Gamma(\mu)} x^{\mu-1} e^{-\lambda x} = \frac{\mu \lambda^\mu}{\Gamma(1+\mu)} x^{\mu-1} e^{-\lambda x} \quad \mu, \lambda > 0 \tag{18} \\
\] \[
\begin{array}{c}
\frac{\lambda^\mu}{\Gamma(\mu)} x^{\mu-1} e^{-\lambda x} \quad \mu e^{-\lambda x} \\
\downarrow \quad \downarrow \\
\lambda^\mu \quad \frac{\lambda^\mu}{(\lambda + s)^\mu} \\
\downarrow \quad \downarrow \\
\mu e^{-\lambda x} \quad \lambda + s \\
\downarrow \quad \downarrow \\
1 \quad 1/s^\mu \\
\downarrow \quad \downarrow \\
\lambda = 0 \quad s = 0 \\
\downarrow \quad \downarrow \\
\mu \quad s \\
\end{array}
\]

The compound Poisson construction is not defined but all other mappings are well-defined. In particular, the second form of (18) makes it clear that the limit (17) is

\[
\lim_{n \to \infty} n x f_\lambda(x|\mu^n) = \mu \rho_\lambda(x) = \mu e^{-\lambda x} \tag{20}
\]

We may omit the normalising factor $\lambda^\mu$, thereby allowing the case $\lambda = 0$ to be well-defined, as shown in the second frame of (19). The density is no longer finite ($s^{-\mu}$ is not defined at $s = 0$), but the LKCD representation remains valid. Henceforth we shall routinely omit $\lambda^\mu$ in the definition of the density $g_{\mu,\lambda}(x)$, thereby making it valid for $\mu > 0$ and $\lambda \geq 0$.

### 3.2 Stable LKCD

We now consider upward extension of the gamma LKCD (19) for $\mu = 1 - \alpha$ ( $0 < \alpha < 1$) and $\lambda = 0$. We reuse $\mu > 0$ as a multiplicative parameter that will be a scale parameter for the layer above. We further multiply by $\alpha$ to generate $\rho_\alpha(x) = \alpha x^{-\alpha}/\Gamma(1-\alpha) = \alpha g_{1-\alpha,0}(x)$. As a gamma density, $\rho_\alpha(x)$ is ID, exactly as discussed above and shown in the unshaded LKCD below. But $\rho_\alpha(x)/x$ is itself the Lévy density of a higher ID density $f_\alpha(x)$, as shown in the
upper shaded extension of the LKCD.

\begin{align*}
\frac{\mu \alpha}{\Gamma(1 - \alpha)} x^{-\alpha} & \quad \mu \alpha s^{\alpha - 1} \\
1 - \alpha & \quad 1 - \alpha
\end{align*}

The shaded top layer, induced by the middle layer as its Lévy density level, is the stable density $f_\alpha(x|\mu)$ with Laplace transform $\tilde{f}_\alpha(s|\mu) = \exp(-\mu s^\alpha)$. All nodes are filled from knowledge of the Laplace transform except for $f_\alpha(x|\mu)$ itself, for which there is no known simple simple closed form expression for general $\alpha$. The LKCD for $\alpha = 1/2$, for which $f_{1/2}(x|\mu)$ is known, is shown on the right of (21).

The Stable LKCD can be further extended, rather naturally, to the left, to form what is known as a generalised gamma convolution.

### 4 Generalised Gamma Convolution

In keeping with the initial definition, the minimal LKCD for an ID density $f(x|\mu)$ is the rectangle on the right of (22).

\begin{align*}
\mu u(t) & \quad \mu \rho(x) \\
\mu \rho(s) & \quad f(x|\mu) \quad \tilde{f}(s|\mu)
\end{align*}

If, in addition, there exists a density $u(t)$ with Laplace transform $\rho(x)$ (equivalently, if $\rho(x)$ is completely monotone) then $f(x|\mu)$ is said to be a generalised gamma convolution (GGC). We refer to the extended LKCD (22), including $u(t)$, as a GGC LKCD. The GGC name derives from the fact that an arbitrary sum of delta functions $u(t) = \sum_i u_i \delta(t - \lambda_i)$ has Laplace transform $\tilde{u}(x) \equiv \rho(x) = \sum_i u_i \exp(-\lambda_i x)$, whose $f(x|\mu)$ is the convolution of as many gamma densities corresponding to each term in the $\rho(x)$ sum of exponentials. In general, $\rho(x)$ need not be a sum of exponentials, any density $\rho(x)$ that is itself the Laplace transform of another density $u(t)$ generates an $f(x|\mu)$ known as a GGC. For example, $\rho(x)$ might itself be a gamma density, as is the case for the stable density, which is thus an instance of a GGC.

The GGC class was introduced by Thorin [26, 27] as he sought to prove the infinite divisibility of the Pareto and the log-normal distributions. The general density $u(t)$, if it exists, is known as the Thorin density. GGCs were subsequently studied in depth in the book by Bondesson [5].

An alternative but equivalent motivation, as given in the survey of GGCs by James et al. [15],
proceeds as follows. First introduce the concept of self-decomposability, which may be compactly defined as follows (Bondesson [5], p18):

An ID distribution \( f(x) \) on \( \mathbb{R}_+ \) is self-decomposable if and only if it has a Lévy density \( \ell(x) \) such that \( \rho(x) = x\ell(x) \) is decreasing.

If, in addition to \( \rho(x) \) decreasing, there also exists a density \( u(t) \) on \( \mathbb{R}_+ \) such that \( \rho(x) \) is the Laplace transform of \( u(t) \), then \( f(x) \) is GGC with \( u(t) \) as the Thorin density (equivalently, \( \rho(x) \) has to be decreasing and completely monotone for \( f(x) \) to be GGC). The following class hierarchy holds:

1. Let \( I \) be the class of ID distributions on \( \mathbb{R}_+ \)
2. Let \( S \) be the class of self-decomposable distributions on \( \mathbb{R}_+ \)
3. Let \( G \) be the class of generalised gamma convolutions (necessarily defined on \( \mathbb{R}_+ \))

Then \( G \subset S \subset I \), as illustrated in Bondesson [5] p4 (with additional subclasses of \( G \) that we shall not explicitly discuss here, such as the hyperbolically completely monotone distributions). The gamma distribution is, of course, GGC and so is the closely allied stable distribution, as discussed in the next section. Outside GGC, we shall not explore any other particular cases of ID distributions on \( \mathbb{R}_+ \), such as, say, distributions that may be self-decomposable but not GGC.

There is a fundamental relationship between the \( \tilde{\rho}(s) \) and \( u(t) \). Given \( \tilde{\rho}(s) \), Bondesson [5] (p33, Inversion Theorem) used contour integration to show that

\[
\mu u(t) = \frac{1}{\pi} \Im \tilde{\rho}(-t) = \frac{1}{\pi} \Im \frac{\tilde{f}'(-t|\mu)}{\tilde{f}(-t|\mu)}
\]

which, by the limiting rule (16) above, is equivalent to

\[
\mu u(t) = \frac{1}{\pi} \Im \tilde{\rho}(-t) = \frac{1}{\pi} \Im \lim_{n \to \infty} n \tilde{f}'(-t|\frac{\mu}{n})
\]

Bondesson’s proof relied on the theory of Pick functions, defined as functions that are analytic on the upper complex half-plane. We shall give a simple derivation (i.e. without invoking contour integration) of the Inversion Theorem for the stable distribution. We shall use Pollard’s infinite series representation of \( f_\alpha(x|\mu) \) as the basis for discussion.

## 5 The Stable GGC Commutative Diagram

Pollard [21] used contour integration to derive the following integral representation

\[
f_\alpha(x|\mu = 1) \equiv f_\alpha(x) = \frac{1}{\pi} \Im \int_0^\infty e^{-xt} e^{-(te^{-i\pi})^\alpha} dt
\]

\[
= \frac{1}{\pi} \int_0^\infty e^{-xt} e^{-t^\alpha \cos \pi \alpha \sin(t^\alpha \sin \pi \alpha)} dt
\]

The change of variable \( x \to \mu^{-1/\alpha}x \) takes \( f_\alpha(x)dx \) to \( f_\alpha(x|\mu)dx \equiv f_\alpha(\mu^{-1/\alpha}x) \mu^{-1/\alpha}dx \) and the Laplace transform \( \tilde{f}_\alpha(s) = \exp(-s^\alpha) \) to \( \tilde{f}_\alpha(\mu^{1/\alpha}s) = \exp(-\mu s^\alpha) \). Thus, starting with the infinite
series (2), we readily arrive at the equivalent integral representation for $f_\alpha(x|\mu)$:

$$f_\alpha(x|\mu) = -\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \mu^k}{k!} \sin(\pi k\alpha) \frac{\Gamma(k\alpha + 1)}{x^{k\alpha+1}}$$  \hspace{1cm} (27)$$

$$= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \mu^k}{k!} \sin(-\pi k\alpha) \int_{0}^{\infty} e^{-xt} t^{k\alpha} dt$$  \hspace{1cm} (28)$$

$$= \frac{1}{\pi} \text{Im} \int_{0}^{\infty} e^{-xt} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\mu e^{-i\pi\alpha} t^{\alpha})^k dt$$  \hspace{1cm} (29)$$

$$= \frac{1}{\pi} \text{Im} \int_{0}^{\infty} e^{-xt} e^{-\mu(e^{-i\pi\alpha})^\alpha} dt$$  \hspace{1cm} (30)$$

$$= \frac{1}{\pi} \text{Im} \int_{0}^{\infty} e^{-xt} \tilde{f}_\alpha(e^{-i\pi\alpha} t|\mu) dt$$  \hspace{1cm} (31)$$

Hence $f_\alpha(x|\mu)$ is the Laplace transform of

$$\frac{1}{\pi} \text{Im} \tilde{f}_\alpha(e^{-i\pi\alpha} t|\mu) = \frac{1}{\pi} \text{Im} \left\{ e^{-\mu(e^{-i\pi\alpha} t^{\alpha})} \right\}$$  \hspace{1cm} (32)$$

$$= \frac{1}{\pi} e^{-\mu t^{\alpha} \cos \pi \alpha}$$  \hspace{1cm} (33)$$

$$\frac{1}{\pi} \text{Im} \tilde{f}_\alpha'(e^{-i\pi\alpha} t|\mu) = -\frac{\mu \alpha}{\pi} \text{Im} \left\{ t^{\alpha-1} e^{-i\pi\alpha} - e^{-i\pi\alpha} \right\}$$  \hspace{1cm} (34)$$

$$\lim_{n \to \infty} \frac{1}{\pi} \text{Im} \tilde{f}_\alpha'(e^{-i\pi\alpha} n\alpha|\mu) = -\frac{\mu \alpha}{\pi} \text{Im} \left\{ t^{\alpha-1} e^{-i\pi\alpha} \right\} = \frac{\mu \alpha}{\pi} t^{\alpha-1} \sin \pi \alpha$$  \hspace{1cm} (35)$$

(s \to e^{-i\pi\alpha} t = -t so that $ds \to -dt$).

We may thus extend the stable GGC LKCD of (21) to include an additional column on the left, as shown in (36).

\[
\begin{array}{cccc}
\frac{1}{\pi} \text{Im} \left\{ e^{-\mu e^{-i\pi\alpha} t^{\alpha}} \right\} & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad \\
\frac{\mu \alpha}{\pi} \sin(\pi \alpha) t^{\alpha-1} & \quad & \quad & \quad \\
(1 - \alpha) \delta(t) & \quad & \quad & \quad \\
\end{array}
\]

All horizontal arrows represent the Laplace transform or its inverse. All down arrows represent a limiting process from the top layer of densities to the bottom layer, or equivalently involve the logarithmic derivative for the outer down arrows.

We note that $x^{-\alpha}$ is the Laplace transform $\tilde{g}_{\alpha,0}(x)$ of the density $g_{\alpha,0}(t) = t^{\alpha-1}/\Gamma(\alpha)$. That together with the Euler identity $\Gamma(1 - \alpha)\Gamma(\alpha) = \pi/\sin(\pi \alpha)$ ensures consistency between the first two nodes in the middle layer.

The Dirac delta density in the bottom left node arises from the Laplace relation

$$\int_{0}^{\infty} e^{-xt} \delta(t-a) dt = e^{-xa}$$  \hspace{1cm} (37)$$
We do not attempt to define a downarrow to the delta function from the node above.

For completeness, we note that the objects in the first and last columns are related by two Laplace transforms via the middle column. The relationship can be formulated as composition of the two Laplace transforms. Let the leftmost object be \( u(t) \), so that \( f(x) = \tilde{u}(x) \) and, in turn, \( \tilde{f}(s) \) is

\[
\tilde{f}(s) = \int_0^\infty dt \ e^{-sx} \int_0^\infty e^{-xt} u(t) dx = \int_0^\infty u(t) \int_0^\infty e^{-(s+t)x} dx \ dt
\]

(38)

Given \( u(t) \), the Stieltjes transform, as (38) is known, bypasses \( f(x) \) to give \( \tilde{f}(s) \) directly. In particular, the Laplace transform of the representation (31) of \( f_\alpha(x|\mu) \) is

\[
\tilde{f}_\alpha(s|\mu) = \frac{1}{\pi} \text{Im} \int_0^\infty \frac{\tilde{f}_\alpha(e^{-i\pi t} | \mu)}{s + t} dt
\]

(39)

But our prime objective is to study the ID density \( f_\alpha(x|\mu) \) in the middle column. It thus seems counter to that objective to seek to bypass \( f_\alpha(x|\mu) \) as the Stieltjes transform does.

We turn next to the convolution of two gamma densities to generate new ID densities. The Lévy density of the convolution is the sum of the Lévy densities of the individual ID densities. Sums are simple, but explicit convolutions can be complicated as the next example illustrates.

### 6 Convolution of Two Gamma Densities

The Laplace convolution \( \{ f \ast g \}(x) \equiv \{ g \ast f \}(x) \) of two functions \( f(x) \) and \( g(x) \) is defined by

\[
\{ f \ast g \}(x) = \int_0^x f(x - y)g(y)dy = x \int_0^1 f(x(1-t))g(xt)dt
\]

(40)

If \( f \) and \( g \) are ID densities, so is \( f \ast g \). The Lévy density of \( f \ast g \) is the sum of the Lévy densities of \( f \) and \( g \). The convolution of two gamma densities \( g_{\mu_1,\lambda_1} \) and \( g_{\mu_2,\lambda_2} \) has the following LKCD

\[
\{g_{\mu_1,\lambda_1} \ast g_{\mu_2,\lambda_2}\}(x) \quad \leftrightarrow \quad \left( \frac{1}{\lambda_1 + s} \right)^{\mu_1} \left( \frac{1}{\lambda_2 + s} \right)^{\mu_2}
\]

\[
\mu_1 e^{-\lambda_1 x} + \mu_2 e^{-\lambda_2 x} \quad \leftrightarrow \quad \frac{\mu_1}{\lambda_1 + s} + \frac{\mu_2}{\lambda_2 + s}
\]

(41)

For \( \lambda_1 = \lambda_2 = \lambda \), the gamma density is closed under convolution, i.e. \( g_{\mu_1,\lambda} \ast g_{\mu_2,\lambda} = g_{\mu_1+\mu_2,\lambda} \), as evident from (41). However, for \( \lambda_1 \neq \lambda_2 \), convolution closure no longer holds, the resultant
density is no longer gamma, although it remains ID. The explicit form of \( \{g_{\mu_1, \lambda_1} \ast g_{\mu_2, \lambda_2}\}(x) \) is

\[
\{g_{\mu_1, \lambda_1} \ast g_{\mu_2, \lambda_2}\}(x) = x \int_0^1 g_{\mu_1, \lambda_1}(x(1-t))g_{\mu_2, \lambda_2}(xt)dt
\]

\[
= \frac{x^{\mu_1+\mu_2-1}}{\Gamma(\mu_1 + \mu_2)} e^{-\lambda_1 x} \int_0^1 (1-t)^{\mu_1-1} t^{\mu_2-1} e^{-(\lambda_2-\lambda_1)xt} dt
\]

\[
= \frac{x^{\mu_1+\mu_2-1}}{\Gamma(\mu_1 + \mu_2)} e^{-\lambda_2 x} \int_0^1 (1-t)^{\mu_2-1} t^{\mu_1-1} e^{-(\lambda_1-\lambda_2)xt} dt
\]

We may interpret the integral form (44) in more than one way, as discussed next.

### 6.1 Hypergeometric Function Interpretation

Here and elsewhere, we draw extensively on Abramowitz and Stegun [1] for Laplace transform pairs and integrals such as the one in (44), which may be expressed as

\[
\{g_{\mu_1, \lambda_1} \ast g_{\mu_2, \lambda_2}\}(x) = \frac{x^{\mu_1+\mu_2-1}}{\Gamma(\mu_1 + \mu_2)} e^{-\lambda_2 x} M(\mu_2, \mu_1 + \mu_2, (\lambda_2 - \lambda_1) x)
\]

\[
= g_{\mu_1+\mu_2, \lambda_3}(x) M(\mu_2, \mu_1 + \mu_2, (\lambda_2 - \lambda_1) x)
\]

where \( M(a, a + b, x) = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \int_0^1 (1-t)^{a-1} t^{b-1} e^{xt} dt \)

is the confluent hypergeometric function ([1] p505, 13.2.1). Since \( M(\cdot, 0, 0) = 1 \), \( \lambda_1 = \lambda_2 = \lambda \) reproduces gamma closure \( \{g_{\mu_1, \lambda} \ast g_{\mu_2, \lambda}\}(x) = g_{\mu_1+\mu_2, \lambda}(x) \). Hypergeometric functions are flexible infinite series representations of a variety of functions for different choices of arguments. In particular, with the aid of [1] p509, 13.6.3 and the Legendre duplication formula \( \sqrt{\pi} \Gamma(2\mu) = 2^{2\mu-1} \Gamma(\mu) \Gamma(\mu + 1/2) \), the case \( \mu_1 = \mu_2 = \mu (\lambda_1 \neq \lambda_2) \) can be shown to be

\[
\{g_{\mu, \lambda_1} \ast g_{\mu, \lambda_2}\}(x) = \frac{\sqrt{\pi}}{\Gamma(\mu)} e^{-(\lambda_1+\lambda_2)x/2} \left( \frac{x}{\lambda_2 - \lambda_1} \right)^{\mu-\frac{1}{2}} I_{\mu-\frac{1}{2}} \left( \frac{\lambda_2 - \lambda_1}{2} x \right)
\]

\( I_{\mu}(x) \) is the modified Bessel function of the first kind of order \( \mu \). Alternatively, (48) is given by the Laplace transform pair [1] p1024 (29.3.50). It leads directly to the sum of two exponentials under the limit (17), noting that \( \sqrt{\pi x/2} I_{-1/2}(x) = \cosh(x) \):

\[
\lim_{n \to \infty} n x \{g_{\mu/n, \lambda_1} \ast g_{\mu/n, \lambda_2}\}(x) = \mu e^{-(\lambda_1+\lambda_2)x/2} \sqrt{\pi(\lambda_2-\lambda_1)x} I_{\mu-\frac{1}{2}} \left( \frac{\lambda_2 - \lambda_1}{2} x \right)
\]

(49)

Let \( (\lambda_1, \lambda_2) = (0, 1) \), for which the convolution (48) is

\( \mu = 1 \): \( e^{-x/2} \sqrt{\pi x} I_{1/2}(x/2) = \{g_{1,0} \ast g_{1,1}\}(x) = 1 - e^{-x} \), where the latter expression follows from \( \sqrt{\pi x/2} I_{1/2}(x) = \sinh(x) \). It is the convolution of a constant and an exponential. It also follows from following the Laplace transform route from the bottom left node of (41).

\( \mu = 1/2 \): so that (48) becomes \( e^{-x/2} I_{0}(x/2) = \{g_{1/2,0} \ast g_{1/2,1}\}(x) \).
The LKCD of these two cases is shown below, with the leftmost limiting downarrow omitted since $\mu$ is fixed. $1 \pm e^{-x}$ may also be written as $e^{-x/2} \sqrt{\pi x} I_{\pm 1/2}(x/2)$.

\[
\begin{align*}
1 - e^{-x} & \quad \frac{1}{s} - \frac{1}{1 + s} \\
1 + e^{-x} & \quad \frac{1}{s} + \frac{1}{1 + s}
\end{align*}
\]

\[
e^{-x/2} I_0\left(\frac{x}{2}\right) \quad \frac{1}{\sqrt{s(1 + s)}}
\]

\[
\begin{align*}
1 - e^{-x} & \quad \frac{1}{s} - \frac{1}{1 + s} \\
1 + e^{-x} & \quad \frac{1}{s} + \frac{1}{1 + s}
\end{align*}
\]

\[
\frac{1}{2} (1 + e^{-x}) \quad \frac{1}{2} \left(\frac{1}{s} + \frac{1}{1 + s}\right)
\]

The mathematical generality of the confluent hypergeometric function can hide probabilistic insight. We can, in fact, interpret it as the Laplace transform of a familiar density, as we do next in the second of our two interpretations of the convolution of two gamma densities.

### 6.2 Beta Density Interpretation

The beta distribution with two shape parameters $a, b$ has the density

\[
\text{Beta}(x|a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad 0 \leq x \leq 1
\]  

with $\text{Beta}(x|1-a, a) = \frac{\sin(\pi a)}{\pi} x^{-a}(1-x)^{a-1}$

We may extend the domain to $\mathbb{R}_+$ by defining

\[
\beta_{a,b}(x) = \begin{cases} 
\text{Beta}(x|a, b) & 0 \leq x \leq 1 \\
0 & x > 1 
\end{cases}
\]

The Laplace transform of (54) is

\[
\tilde{\beta}_{a,b}(x) = \int_0^\infty e^{-xt} \beta_{a,b}(t) dt = M(b, a + b, -x)
\]

\[
\Rightarrow \{g_{\mu_1, \lambda_1} * g_{\mu_2, \lambda_2}\}(x) = g_{\mu_1 + \mu_2, \lambda_1}(x) \tilde{\beta}_{\mu_1, 2}(\lambda_1 - \lambda_2, x)
\]

\[
\equiv g_{\mu_1 + \mu_2, \lambda_1}(x) \tilde{\beta}_{\mu_2, \mu_1}(\lambda_2 - \lambda_1, x)
\]

For $\mu_1 + \mu_2 = 1$, let $\mu_1 \equiv 1 - \alpha, \mu_2 \equiv \alpha$ ($0 < \alpha < 1$). Further, set $\lambda_1 = 0, \lambda_2 = 1$, to get

\[
\{g_{1-\alpha, 0} * g_{\alpha, 1}\}(x) = \tilde{\beta}_{\alpha, 1-\alpha}(x).
\]

We recall that, for the stable case represented in (36), $\rho_\alpha(x) = \alpha g_{1-\alpha, 0}(x)$. Hence we can take the convolution with $g_{\alpha, 1}(x)$ to be at the centre of (36), so that we expand from that point in all four directions. The LKCD thus has the following form (explicit forms for the top-level densities

\[
\text{(58)}
\]
are not readily available).

\[
\frac{1}{\pi} \text{Im} \left\{ \tilde{h}_\alpha (e^{-i\pi t} | \mu) \right\} \quad \mapsto \quad h_\alpha (x | \mu) \quad \mapsto \quad \tilde{h}_\alpha (s | \mu)
\]

\[
\mu \beta_{\alpha,1-\alpha}(t) \quad \mapsto \quad \mu \{ g_{1-\alpha,0} * g_{\alpha,1} \}(x) \quad \mapsto \quad \mu \frac{s^\alpha - 1}{(1 + s)^\alpha}
\]

\[
(1 - \alpha) \delta(t) + \alpha \delta(t - 1) \quad \mapsto \quad 1 - \alpha + \alpha e^{-x} \quad \mapsto \quad \frac{1 - \alpha}{s} + \frac{\alpha}{1 + s}
\]

In keeping with the foregoing discussion, we may also write \( \beta_{\alpha,1-\alpha}(t) \) as

\[
\beta_{\alpha,1-\alpha}(t) = \frac{1}{\pi} \text{Im} \left\{ \left( e^{-i\pi t} \right)^{\alpha - 1} \right\} = \begin{cases} 
\frac{\sin \frac{\pi \alpha}{2}}{\pi} \frac{\alpha - 1}{(1 - t)^{-\alpha}} & 0 \leq t \leq 1 \\
0 & t > 1
\end{cases}
\]

(60)

Setting \((\lambda_1, \lambda_2) = (1, 0)\) gives (where \(h_\alpha (x | \mu)\) denotes a correspondingly different density)

\[
\frac{1}{\pi} \text{Im} \left\{ \tilde{h}_\alpha (e^{-i\pi t} | \mu) \right\} \quad \mapsto \quad h_\alpha (x | \mu) \quad \mapsto \quad \tilde{h}_\alpha (s | \mu)
\]

\[
\mu \beta_{1-\alpha,\alpha}(t) \quad \mapsto \quad \mu \{ g_{1-\alpha,1} * g_{\alpha,0} \}(x) \quad \mapsto \quad \mu \frac{(1 + s)^{\alpha - 1}}{s^\alpha}
\]

\[
\alpha \delta(t) + (1 - \alpha) \delta(t - 1) \quad \mapsto \quad \alpha + (1 - \alpha) e^{-x} \quad \mapsto \quad \frac{\alpha}{s} + \frac{1 - \alpha}{1 + s}
\]

For \(\alpha = 1/2\), we have explicit forms, recalling that \(e^{-x/2} I_0(x/2) \equiv \{ g_{1/2,0} * g_{1/2,1} \}(x)\):

\[
\frac{1}{\pi} \text{Im} \left( \sqrt{1 - t} + i \sqrt{t} \right)^{2\mu} \quad \mapsto \quad \mu e^{-x/2} I_{\mu} \left( \frac{x}{2} \right) \quad \mapsto \quad \left( \sqrt{1 + s} - \sqrt{s} \right)^{2\mu}
\]

\[
\mu \beta_{1/2,1/2}(t) \quad \mapsto \quad \mu e^{-x/2} I_0 \left( \frac{x}{2} \right) \quad \mapsto \quad \mu \frac{1}{\sqrt{s}(1 + s)}
\]

\[
\frac{1}{2} \left( \delta(t) + \delta(t - 1) \right) \quad \mapsto \quad \frac{1}{2} \left( 1 + e^{-x} \right) \quad \mapsto \quad \frac{1}{2} \left( \frac{1}{s} + \frac{1}{1 + s} \right)
\]

(62)

We have used the Laplace transform pair [1] p1024 (29.3.53) for the density \(\mu e^{-x/2} I_\mu(x/2)/x\) and its Laplace transform in the top right node, which may be written in several forms

\[
\left( \sqrt{1 + s} + \sqrt{s} \right)^{-2\mu} = \left( \sqrt{1 + s} - \sqrt{s} \right)^{2\mu} = \left( 1 + 2s + 2\sqrt{s(1 + s)} \right)^{-\mu} = \left( 1 + 2s - 2\sqrt{s(1 + s)} \right)^{\mu}
\]
Bessel functions play a prominent role in random walks and Brownian motion, with numerous applications to random phenomena in physics, chemistry, biology, finance etc. For example, in a section titled “Bessel Functions and Random Walks” ([8] p58-61), Feller showed that the distribution of the first passage through \( \mu > 0 \) (i.e. the time it takes to reach the point \( \mu \) for the first time in a random walk in one dimension, starting at the point 0) has the density \( \mu e^{-\xi}I_{\mu}(x)/x \) (where \( x \) is time in this context). Feller proceeded to calculate the Laplace transform of this density ([8] p437) and to demonstrate its infinite divisibility, with \( \rho(x) = e^{-\xi}I_{\mu}(x) \) ([8] p451).

Without explicit reference to the Bessel function form, Bondesson ([5] p37) showed that the first passage distribution is GGC with Thorin density \( \beta_{1/2,1/2}(t) \). The latter is often referred to as the density of the arc-sine distribution in the literature on random walks and Brownian motion.

Lastly, let \( \mu_1 + \mu_2 = 1 - \theta \) for some parameter \( \theta \). Once again set \( \mu_1 = 1 - \alpha \) so that \( \mu_2 = \alpha - \theta \). Since the gamma shape parameters \( \mu_1, \mu_2 \) are always positive, we must have \( 0 \leq \theta < \alpha \) where \( 0 < \alpha < 1 \) as before. Then (56) becomes

\[
\{g_{1-\alpha,0} \ast g_{\alpha-\theta,1}\}(x) = g_{1-\theta,0}(x) \tilde{\beta}_{\alpha-\theta,1-\alpha}(x) \tag{63}
\]

which reduces to (58) for \( \theta = 0 \). For \( 0 < \theta < \alpha \), (63) is, in turn, the Laplace transform of a convolution of a gamma and a beta density

\[
\frac{\sin \pi \theta}{\pi} t^{\theta-1} \ast \beta_{\alpha-\theta,1-\alpha}(t) \longrightarrow g_{1-\theta,0}(x) \tilde{\beta}_{\alpha-\theta,1-\alpha}(x) \tag{64}
\]

This generalises (59) (i.e. the case \( \theta = 0 \)) to the following:

\[
\frac{1}{\pi} \text{Im} \left\{ \tilde{h}_{\alpha,\theta}(e^{-i\pi t}|\mu) \right\} \quad \text{h}_{\alpha,\theta}(x|\mu) \quad \text{h}_{\alpha,\theta}(s|\mu)
\]

\[
\mu \frac{\sin \pi \theta}{\pi} t^{\theta-1} \ast \beta_{\alpha-\theta,1-\alpha}(t) \quad \mu \{g_{1-\alpha,0} \ast g_{\alpha-\theta,1}\}(x) \quad \mu \frac{s^{\alpha-1}}{(1+s)^{\alpha-\theta}}
\]

\[
(1-\alpha)\delta(t) + (\alpha-\theta)\delta(t-1) \quad 1-\alpha + (\alpha-\theta)e^{-x} \quad \frac{1-\alpha}{s} + \frac{\alpha-\theta}{1+s}
\]

In this case, \( \alpha = 1/2 \) gives

\[
\frac{1}{\pi} \text{Im} \left\{ \tilde{h}_{1/2,\theta}(e^{-i\pi t}|\mu) \right\} \quad \text{h}_{1/2,\theta}(x|\mu) \quad \text{h}_{1/2,\theta}(s|\mu)
\]

\[
\mu \frac{\sin \pi \theta}{\pi} t^{\theta-1} \ast \beta_{1/2-\theta,1/2}(t) \quad \mu \{g_{1/2,0} \ast g_{1/2-\theta,1}\}(x) \quad \mu \frac{(1+s)^{\theta}}{\sqrt{s(1+s)}}
\]

\[
\frac{1}{2} \delta(t) + \left(\frac{1}{2} - \theta \right)\delta(t-1) \quad \frac{1}{2} + \left(\frac{1}{2} - \theta \right)e^{-x} \quad \frac{1}{2s} + \frac{1/2 - \theta}{1+s}
\]

\( h_{\alpha,\theta}(x|\mu) \) is a three-parameter ID density generated by \( \{g_{1-\alpha,0} \ast g_{\alpha-\theta,1}\}(x) \) which, in turn, is a two-parameter ID density. Aside from \( h_{1/2,0}(x|\mu) = \mu e^{-x/2}I_{\mu}(x/2)/x \) of the LKCD given
in (62), we shall not pursue further here the explicit form for the general case \( h_{\alpha,\theta}(x|\mu) \). We note only that, as shown in (65), it satisfies

\[
\frac{\tilde{h}'_{\alpha,\theta}(s|\mu)}{\tilde{h}_{\alpha,\theta}(s|\mu)} = \mu \frac{s^{\alpha-1}}{(1+s)^{\alpha-\theta}} \equiv \mu \rho_{\alpha,\theta}(s)
\]

\[
\implies \tilde{h}_{\alpha,\theta}(s|\mu) = \exp \left\{ -\mu \int_0^s \rho_{\alpha,\theta}(t) dt \right\}
\]

(67)

(68)

Setting \((\lambda_1, \lambda_2) = (1, 0)\) gives

\[
\frac{1}{\pi} \text{Im} \left\{ \tilde{h}_{\alpha,\theta}(e^{-i\pi t}|\mu) \right\} \quad \longleftrightarrow \quad h_{\alpha,\theta}(x|\mu) \quad \longleftrightarrow \quad \tilde{h}_{\alpha,\theta}(s|\mu)
\]

\[
\mu \frac{\sin \pi \theta}{\pi} t^{\theta-1} * \beta_{1-\alpha,\alpha-\theta}(t) \quad \longleftrightarrow \quad \mu \{ g_{1-\alpha,1} * g_{\alpha-\theta,0} \}(x) \quad \longleftrightarrow \quad \mu \frac{(1+s)^{\alpha-1}}{s^{\alpha-\theta}}
\]

(69)

\[
(\alpha - \theta) \delta(t) + (1-\alpha) \delta(t-1) \quad \longleftrightarrow \quad \alpha - \theta + (1-\alpha) e^{-x} \quad \Longleftrightarrow \quad \frac{\alpha - \theta}{s} + \frac{1 - \alpha}{1+s}
\]

In this case, \( \alpha = 1/2 \) gives

\[
\frac{1}{\pi} \text{Im} \left\{ \tilde{h}_{1/2,\theta}(e^{-i\pi t}|\mu) \right\} \quad \longleftrightarrow \quad h_{1/2,\theta}(x|\mu) \quad \longleftrightarrow \quad \tilde{h}_{1/2,\theta}(s|\mu)
\]

\[
\mu \frac{\sin \pi \theta}{\pi} t^{\theta-1} * \beta_{1/2,1/2-\theta}(t) \quad \longleftrightarrow \quad \mu \{ g_{1/2,1} * g_{1/2-\theta,0} \}(x) \quad \longleftrightarrow \quad \mu \frac{s^{\theta}}{\sqrt{s(1+s)}}
\]

(70)

\[
(\frac{1}{2} - \theta) \delta(t) + \frac{1}{2} \delta(t-1) \quad \longleftrightarrow \quad (\frac{1}{2} - \theta) + \frac{1}{2} e^{-x} \quad \Longleftrightarrow \quad \frac{1/2 - \theta}{s} + \frac{1/2}{1+s}
\]

In the latter discussion

\[
\frac{\tilde{h}'_{\alpha,\theta}(s|\mu)}{\tilde{h}_{\alpha,\theta}(s|\mu)} = \mu \frac{(1+s)^{\alpha-1}}{s^{\alpha-\theta}} \equiv \mu \rho_{\alpha,\theta}(s)
\]

\[
\implies \tilde{h}_{\alpha,\theta}(s|\mu) = \exp \left\{ -\mu \int_0^s \rho_{\alpha,\theta}(t) dt \right\}
\]

(71)

(72)

We note that many densities that have arisen in our investigation of the convolution of two gamma densities can be linked to a variety of other probabilistic studies. For example, \( \beta_{1-\alpha,n\alpha+\theta} \) for \( \theta > -\alpha \) and integer \( n > 0 \) arises in the construction of the Pitman-Yor or two-parameter Poisson-Dirichlet distribution PD(\( \alpha, \theta \)) [20], which extends the original one-parameter formulation PD(\( \alpha \)) due to Kingman [16, 17].

It is only natural to explore the generation of further densities through higher convolutions of known densities, or sums of such densities treated as Levy densities of some higher ID densities. We defer such further investigation to a separate study. Instead, we turn next to the study of mixtures of stable densities and explore associated LKCD representations.
7 Mixtures of Stable Densities

As discussed above, the convolution of ID densities is also ID. It turns out that the sum of ID densities can also be ID, although this is by no means obvious from the ID representation by itself. An example is a mixture (weighted sum or integral) of exponentials, which was shown to be ID by Steutel [24]. We now broaden the foregoing discussion by allowing the scale parameter $y > 0$ of the stable density $f_\alpha(x|y)$ ($0 < \alpha < 1$) to be a variable governed by a distribution with density $f(y|\mu)$. In principle, $f$ might be any density, possibly involving several parameters. We retain explicit dependence on at least one parameter $\mu$ in anticipation of choosing an infinitely divisible $f(y|\mu)$ of the generic form illustrated in the LKCD of Figure 1.

The two-dimensional joint density of $x$ and $y$ is $Pr(x,y|\mu) = Pr(x|y)Pr(y|\mu) = f_\alpha(x|y)f(y|\mu)$. Hence the one-dimensional marginal density of $x$ is

$$m_\alpha(x|\mu) \equiv Pr(x|\mu) = \int_0^\infty Pr(x,y|\mu)dy = \int_0^\infty f_\alpha(x|y)f(y|\mu)dy$$

(73)

This may be regarded as a weighted mixture of stable densities at different scales, with mixing density $f(y|\mu)$. Since $f_\alpha(x|y)$ has $f_\alpha(s|y) = \exp(-ys^\alpha)$, $m_\alpha(x|\mu)$ has Laplace transform

$$\tilde{m}_\alpha(s|\mu) = \int_0^\infty f_\alpha(s|y)f(y|\mu)dy = \int_0^\infty e^{-ys^\alpha}f(y|\mu)dy = \tilde{f}(s^\alpha|\mu)$$

(74)

where $\tilde{f}(s|\mu)$ is the Laplace transform of $f(y|\mu)$. Although $0 < \alpha < 1$ for a stable distribution on a positive variable, we can accommodate $\alpha = 1$ by defining $f_{\alpha=1}(x|y) = \delta(x-y)$ with Laplace transform $\exp(-ys)$, so that $m_{\alpha=1}$ reproduces $f$, i.e. $m_{\alpha=1}(x|\mu) = f(x|\mu)$ and $\tilde{m}_{\alpha=1}(s|\mu) = \tilde{f}(s|\mu)$.

An alternative approach that leads to (74) is to consider the density of the product $Y^{1/\alpha}X$ where $X$ and $Y$ are independent variables with densities $f_\alpha(x)$ and $f(y)$ respectively, as discussed by Feller [8] (p463, Problem 10) and Bondesson [5] (p38, Example 3.2.4 for the case where $f$ is the gamma density). The study of distributions of products of independent variables is a recurring theme in James [14] and James et al. [15].

Since $f_\alpha$ is not available in closed form for general $0 < \alpha < 1$, neither is $m_\alpha$. But $\tilde{m}_\alpha$ depends solely on the availability of $\tilde{f}$. If $f$ is ID then so is $m_\alpha$, as discussed under subordination and completely monotone functions in Feller [8] p451. Furthermore, Bondesson [5] (p41, Theorem 3.3.2) proved that if $f$ is GGC then so is $m_\alpha$, i.e. if $\tilde{f}(s|\mu)$ is the Laplace transform of a GGC, so is $\tilde{f}(s^\alpha|\mu)$. In this case, the availability of $\tilde{f}(s^\alpha|\mu)$ enables the generation of Thorin densities that may not otherwise be readily identifiable as legitimate densities. Henceforth we shall confine interest to the GGC case.

We always take $f_\alpha(x|\mu)$ ($0 < \alpha < 1$) to denote the stable density. If we also reserve $\rho_\alpha(x)$ to be the $\rho$-density of $f_\alpha(x|\mu)$, we need a different symbol, $r_\alpha$, say, for the $\rho$-density of $m_\alpha(x|\mu)$ – i.e. $r_\alpha(x)$ is to $m_\alpha(x|\mu)$ what $\rho(x)$ is to the general ID density $f(x|\mu)$ (and what $\rho_\alpha(x)$ is to $f_\alpha(x|\mu)$). Hence $\{\tilde{\rho}(s), \tilde{r}_\alpha(s)\}$, the Laplace transforms of $\{\rho(x), r_\alpha(x)\}$ respectively, are given by

$$\mu\tilde{\rho}(s) = -\frac{\int f(s|\mu)\tilde{f}(s|\mu)}{\int f(s|\mu)}$$

and

$$\mu\tilde{r}_\alpha(s) = -\frac{\int f(s^\alpha|\mu)\tilde{f}(s^\alpha|\mu)}{\int f(s^\alpha|\mu)}$$

(75)

To be explicit

$$\mu\tilde{r}_\alpha(s) = -\frac{1}{\int f(s^\alpha|\mu)} \frac{d}{ds} \tilde{f}(s^\alpha|\mu) = -\frac{\alpha s^{\alpha-1}}{\int f(s^\alpha|\mu)} \frac{d}{ds} \tilde{f}(s^\alpha|\mu) = \mu \alpha s^{\alpha-1} \tilde{\rho}(s^{\alpha})$$

(76)
Accordingly, for GGC \( f(x|\mu) \), \( m_\alpha(x|\mu) \) has the following GGC LKCD:

\[
\frac{1}{\pi} \text{Im} \left\{ \tilde{f}(e^{-i\pi\alpha}t^{\alpha}|\mu) \right\} \quad \overleftrightarrow{\quad m_\alpha(x|\mu) \quad \overleftrightarrow{\quad \tilde{f}(s^{\alpha}|\mu) \quad} \right. \\
\overleftrightarrow{\quad \mu r_\alpha(x) \quad \overleftrightarrow{\quad \mu \alpha s^{\alpha-1}\tilde{\rho}(s^{\alpha}) \quad}}
\]

(81)

The assertion that the bottom left node of (81) is a density, even though the positivity of the expression may not be obvious from mere inspection, restates Bondesson’s Theorem 3.3.2, with an overlay of the GGC LKCD theme of this paper.

8 Stable Mixing Density

The stable-stable mixture density is discussed by Feller at various places in [8] (pp176, 348, 452). Let \( f(y|\mu) = f_\beta(y|\mu) \implies \tilde{f}(s|\mu) = \exp(-\mu s^\beta) \). We relabel \( m_\alpha \) as \( m_{\alpha,\beta} \). Then \( \tilde{m}_{\alpha,\beta}(s|\mu) = \tilde{f}_\beta(s^{\alpha}|\mu) = \exp(-\mu s^{\alpha\beta}) \), as can readily be verified. Hence \( m_{\alpha,\beta} \equiv f_{\alpha\beta} \), the stable density with parameter \( \alpha\beta \). The GGC LKCD of \( f_{\alpha\beta}(x|\mu) \) is given in (36), with \( \alpha \) replaced by \( \alpha\beta \).

The integral representations (73) and (80) take the form

\[
m_{\alpha,\beta}(x|\mu) \equiv f_{\alpha\beta}(x|\mu) = \int_0^\infty f_\alpha(x|y)f_\beta(y|\mu)dy \quad (82)
\]

\[
r_{\alpha,\beta}(x) \equiv \rho_{\alpha\beta}(x) = x \int_0^\infty f_\alpha(x|y)\rho_\beta(y)dy \quad (83)
\]

For \( \alpha = \beta = 1/2 \), the known density \( f_{1/2} \) of (21) induces an integral representation for \( f_{1/4} \)

\[
f_{1/4}(x|\mu) = \int_0^\infty f_{1/2}(x|y)f_{1/2}(y|\mu)dy
\]

\[
= \int_0^\infty \frac{1}{2\sqrt{\pi}} e^{-y^2/4x} \frac{\mu}{2\sqrt{y^3}} e^{-\mu^2/2y} dy
\]

\[
= \frac{\mu}{4\pi\sqrt{x^3}} \int_0^\infty \frac{1}{\sqrt{y}} e^{-\mu^2/4y-y^2/4x} dy \quad (85)
\]

Berberan-Santos [3] derived (85) for \( \mu = 1 \) through Laplace inversion.
The corresponding $\rho_{\alpha\beta}(x)$, with $\alpha = \beta = 1/2$, is

$$\frac{\alpha\beta x^{-\alpha\beta}}{\Gamma(1-\alpha\beta)} = \frac{1}{4} \frac{x^{-1/4}}{\Gamma(\frac{3}{4})}$$

which is reproduced, as it should be, by the limit

$$\lim_{n \to \infty} n x f_{1/4}(x|\mu_n) = \mu \frac{\sqrt{\pi}}{8} \int_0^\infty y^{-3/4} e^{-y/4x} \, dy \quad (y^2 \to y)$$

$$= \mu \frac{x^{-1/4}}{4} \Gamma\left(\frac{1}{4}\right) (4x)^{1/4}$$

since $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi \sqrt{2}$ by the Legendre duplication formula.

As noted earlier, $f_{1/4}$ is of particular interest in physics. We may then use $\alpha = 1/2$ and $\beta = 1/4$ (or the other way round) in (82) to generate $f_{1/8}$, although the integral representation will inevitably be more complex.

### 9 Gamma Mixing Density

Let $f(y|\mu)$ be the gamma density $g_{\mu,\lambda}(y)$ of (18). Then $m_{\alpha,\lambda}(x|\mu)$ has Laplace transform

$$\tilde{m}_{\alpha,\lambda}(s|\mu) = \tilde{g}_{\mu,\lambda}(s^\alpha) = \frac{\lambda^\mu}{(\lambda + s^\alpha)^\mu}$$

(91)

$\alpha = 1$ reproduces the Laplace transform of the gamma density $\tilde{g}_{\mu,\lambda}(s)$. For $0 < \alpha < 1$ (91) is the Laplace transform of what is known as the fractional gamma distribution, e.g. Di Nardo et al. [6] (Definition 2.1). It is also called the positive Linnik distribution. It also seems reasonable to refer to it as the stable-gamma mixture distribution.

The GGC LKCD of the fractional gamma density $m_{\alpha,\lambda}(x|\mu)$ is

$$\frac{1}{\pi} \Im \left\{ \frac{\lambda^\mu}{(\lambda + e^{-i\pi \alpha} t^\alpha)^\mu} \right\} \quad \longleftrightarrow \quad m_{\alpha,\lambda}(x|\mu) \quad \longleftrightarrow \quad \frac{\lambda^\mu}{(\lambda + s^\alpha)^\mu}$$

(92)

The Thorin density in the bottom left node takes the explicit form

$$\frac{\mu\alpha}{\pi} \Im \left\{ \frac{(e^{-i\pi t})^{\alpha - 1}}{\lambda + e^{-i\pi \alpha} t^\alpha} \right\} = \frac{\mu\alpha}{\pi} \frac{\lambda t^{\alpha - 1} \sin \pi \alpha}{\lambda^2 + 2\lambda t^\alpha \cos \pi \alpha + t^{2\alpha}}$$

(93)

This was derived for $\lambda = 1$ by Bondesson [5] (p38), as a consequence of Theorem 3.3.2 (p41). As previously discussed, the theorem gives assurance that (93) is a valid density.

What form does $m_{\alpha,\lambda}(x|\mu)$ itself take?
9.1 Geometric Series Representation

Let \( m_{\alpha,\lambda}(x) \equiv m_{\alpha,\lambda}(x|\mu = 1) \) with Laplace transform \( \tilde{m}_{\alpha,\lambda}(s) \), which we expand as a geometric series

\[
\tilde{m}_{\alpha,\lambda}(s) = \frac{\lambda}{\lambda + s^\alpha} = \frac{\lambda/s^\alpha}{1 + \lambda/s^\alpha} = - \sum_{k=1}^{\infty} \left( \frac{-\lambda}{s^\alpha} \right)^k
\]

\[
\implies m_{\alpha,\lambda}(x) = - \sum_{k=1}^{\infty} (-\lambda)^k x^{\alpha k-1} \frac{1}{\Gamma(\alpha k)}
\]

Let \( F_{\alpha,\lambda}(x) \) be the distribution with density \( m_{\alpha,\lambda}(x) \), so that

\[
F_{\alpha,\lambda}(x) = \int_0^x m_{\alpha,\lambda}(y) dy
\]

\[
= - \sum_{k=1}^{\infty} \frac{(-\lambda x^\alpha)^k}{\Gamma(\alpha k + 1)} = 1 - E_{\alpha}(-\lambda x^\alpha)
\]

where \( E_{\alpha}(x) \) is the Mittag-Leffler function

\[
E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}
\]

Hence the density \( m_{\alpha,\lambda}(x) \equiv m_{\alpha,\lambda}(x|\mu = 1) \) can be written as

\[
m_{\alpha,\lambda}(x) = F'_{\alpha,\lambda}(x) = -E'_{\alpha}(-\lambda x^\alpha)
\]

Feller ([8] p453) discussed the Mittag-Leffler function in the context of Laplace transforms in two dimensions. Pillai [19] defined the Mittag-Leffler distribution as the case \( F_{\alpha,\lambda=1}(x) \) (in the notation of this paper). Hauboldt et al. [13] gave a review of the Mittag-Leffler function. The general case \( F_{\alpha,\lambda}(x|\mu) \) with density \( m_{\alpha,\lambda}(x|\mu) \) may be expressed in terms of the generalised (three parameter) Mittag-Leffler function, also known as the Prabhakar function, as reviewed by Garra and Garrappa [11].

In light of the foregoing discussion, we may plausibly refer to \( m_{\alpha,\lambda}(x) \) as the Mittag-Leffler density. Commonly used though the Mittag-Leffler geometric series representation may be, it is not a unique representation of the fractional gamma distribution. We demonstrate an alternative perspective next that directly adheres to the integral representation of \( m_{\alpha,\lambda}(x|\mu) \).

9.2 Integral Representation of Fractional Gamma Density

The integral representations (73), (80) are

\[
m_{\alpha,\lambda}(x|\mu) = \frac{\lambda^\mu}{\Gamma(\mu)} \int_0^{\infty} f_{\alpha}(x|y) y^{\mu-1} e^{-\lambda y} dy
\]

\[
r_{\alpha,\lambda}(x) = x \int_0^{\infty} f_{\alpha}(x|y) y^{-1} e^{-\lambda y} dy
\]
Choosing $\alpha = 1/2$ gives

$$m_{1/2,\lambda}(x|\mu) = \frac{\lambda^\mu}{2\Gamma(\mu)\sqrt{\pi}x^{\lambda-1}} \int_0^\infty y^\mu e^{-y^2/4x-\lambda y} dy$$  \hspace{1cm} (102)

$$= \sqrt{\frac{2}{\pi}} \mu^\lambda (2x)^{\mu-1/2} e^{x^2/2} D_{\mu-1}(\lambda\sqrt{2x})$$  \hspace{1cm} (103)

where $D_\mu(x)$ is the parabolic cylinder function [12] (p365, 3.462.1). $D_\mu(x)$ arises in the solution of Laplace’s equation by separation of variables in parabolic cylinder coordinates. Also

$$r_{1/2,\lambda}(x) = \frac{1}{2} \sqrt{\frac{2}{\pi}} e^{x^2/2} D_{\mu-1}(\lambda\sqrt{2x}) = \frac{1}{2} e^{x^2} \text{erfc}(\lambda\sqrt{x})$$  \hspace{1cm} (104)

The rightmost form follows from [12] (p1030, 9.254.1), erfc $(x)$ being the complementary error function. It is compatible with the Laplace transform route, with the aid of the Laplace transform pairs [1] (p1028, 29.3.114 and p1027, 29.3.90). The GGC LKCD of the fractional gamma density $m_{1/2,\lambda}(x|\mu)$ is

$$1 \frac{\text{Im}(\lambda + i\sqrt{t})^\mu}{\pi (\lambda^2 + t)^\mu} \rightarrow m_{1/2,\lambda}(x|\mu) \rightarrow \frac{\lambda^\mu}{\lambda + \sqrt{s}}$$  \hspace{1cm} (105)

$$\frac{\mu}{2\pi \sqrt{t}} \frac{1}{(\lambda^2 + t)^\mu} \rightarrow \lambda \rightarrow \frac{\mu}{2} e^{x^2} \text{erfc}(\lambda\sqrt{x}) \rightarrow \frac{\mu}{2} \frac{1}{\sqrt{s}}$$

With the aid of (82) and (83) in the stable-stable case above we get

$$m_{\alpha\beta,\lambda}(x|\mu) = \frac{\lambda^\mu}{\Gamma(\mu)} \int_0^\infty f_{\alpha\beta}(x|y) y^{\mu-1} e^{-\lambda y} dy$$  \hspace{1cm} (106)

$$= \int_0^\infty f_{\alpha}(x|u) \left\{ \frac{\lambda^\mu}{\Gamma(\mu)} \int_0^\infty f_{\beta}(u|y) y^{\mu-1} e^{-\lambda y} dy \right\} du$$  \hspace{1cm} (107)

$$= \int_0^\infty f_{\alpha}(x|u) m_{\beta,\lambda}(u|\mu) du$$  \hspace{1cm} (108)

$$r_{\alpha\beta,\lambda}(x) = x \int_0^\infty f_{\alpha}(x|u) u^{\mu-1} r_{\beta,\lambda}(u) du$$  \hspace{1cm} (109)

Hence we may use the case $\alpha = 1/2$ to induce the $\alpha = 1/4$ case

$$m_{1/4,\lambda}(x|\mu) = \int_0^\infty f_{1/2}(x|z) m_{1/2,\lambda}(z|\mu) dz$$  \hspace{1cm} (110)

$$= \frac{1}{2\sqrt{\pi x^3}} \int_0^\infty z e^{-z^2/4x} m_{1/2,\lambda}(z|\mu) dz$$  \hspace{1cm} (111)

$$r_{1/4,\lambda}(x) = \frac{1}{2\sqrt{\pi x}} \int_0^\infty e^{-z^2/4x} r_{1/2,\lambda}(z) dz$$  \hspace{1cm} (112)

$$= \frac{1}{4\sqrt{\pi x}} \int_0^\infty e^{-z^2/4x+\lambda^2z} \text{erfc}(\lambda\sqrt{z}) dz$$  \hspace{1cm} (113)
The GGC LKCD of the fractional gamma density $m_{1/4,\lambda}(x|\mu)$ is shown below

\[
\frac{1}{\pi} \text{Im} \left\{ \frac{\lambda^\mu}{(\lambda + e^{-i\pi/4}t^{1/4})^\mu} \right\} \quad m_{1/4,\lambda}(x|\mu) \quad \frac{\lambda^\mu}{(\lambda + s^{1/4})^\mu}
\]

\[
\frac{\mu}{4\sqrt{2\pi}} \frac{\lambda t^{-3/4}}{(\lambda^2 + \sqrt{2}\lambda t^{1/4} + \sqrt{t})} \quad \mu r_{1/4,\lambda}(x) \quad \frac{\mu}{4} \frac{s^{3/4}}{(\lambda + s^{1/4})}
\]

We can proceed to generate an integral representation for $\alpha = 1/8$ with the aid of $\alpha = 1/2, 1/4$. We can similarly start from $\alpha = 1/3$ combined with $\alpha = 1/2$ to generate the sequence for $\alpha = \{1/3, 1/6, 1/9, \ldots \}$.

For general $\alpha$, we may revert to the Pollard infinite series representation (27) for $f_\alpha$ in the integral representations (100) and (101):

\[
m_{\alpha,\lambda}(x|\mu) = -\frac{\lambda^\mu}{\pi \Gamma(\mu)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sin(\pi k\alpha) \frac{\Gamma(\mu k + 1)}{x^{k\alpha+1}} \int_0^\infty y^{\mu+k-1} e^{-\lambda y} dy \quad (115)
\]

\[
\text{and } r_{\alpha,\lambda}(x) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(\pi k\alpha) \frac{\Gamma(\mu k + 1)}{x^{k\alpha+1}} = -\alpha \sum_{k=1}^{\infty} \frac{(-\lambda x^\alpha)^{-k}}{\Gamma(1-k\alpha)} \quad (117)
\]

The geometric series Mittag-Leffler representation is, of course, fully consistent with the integral representation, including the parabolic cylinder function representation. Explicitly, for $\alpha = 1/2$

\[
2 r_{1/2,\lambda}(x) = \sqrt{\frac{2}{\pi}} e^{\lambda x^2/2} D_{-1}(\lambda \sqrt{2}x) = E_{1/2}(-\lambda \sqrt{x}) = e^{\lambda x^2} \text{erfc}(\lambda \sqrt{x}) \quad (118)
\]

\[
m_{1/2,\lambda}(x) = \frac{\lambda}{\sqrt{\pi} x} e^{\lambda x^2/2} D_{-2}(\lambda \sqrt{2}x) \quad (119)
\]

\[
= -\frac{d}{dx} \left( e^{\lambda x^2} \text{erfc}(\lambda \sqrt{x}) \right) = -E_{1/2}'(-\lambda \sqrt{x}) \quad (120)
\]

where, by [12] (p1030, 9.254.2)

\[
D_{-2}(\lambda \sqrt{2}x) = e^{-\lambda x^2/2} - \lambda \sqrt{\pi} x e^{\lambda x^2/2} \text{erfc}(\lambda \sqrt{x}) \quad (121)
\]

### 9.3 Convolution Revisited

Finally, we note that the fractional gamma mixture may be looked upon as arising from a convolution of two densities: $g_{1-\alpha,0}(x)$ and $m_{\alpha,\lambda}(x|\mu = 1)$. For brevity, we define

\[
\{g \ast m\}_{\alpha,\lambda}(x) \equiv g_{1-\alpha,0}(x) \ast m_{\alpha,\lambda}(x) / \lambda \quad (122)
\]

and

\[
\phi_{\alpha,\lambda}(t) = \frac{\alpha}{\pi} \text{Im} \left\{ \frac{(e^{-i\pi t})^{\alpha-1}}{\lambda + e^{-i\pi t} \lambda^{\alpha}} \right\} = \frac{\alpha}{\pi} \frac{\lambda t^{\alpha-1} \sin \pi \alpha}{\lambda^2 + 2\lambda t^\alpha \cos \pi \alpha + t^{2\alpha}} \quad (123)
\]

$\phi_{\alpha,\lambda}(t)$ is the Thorin density of (93).
Hence, in keeping with the convolution of two gamma densities studied earlier, we have the followed “convolution-centred” GGC LKCD for the fractional gamma density $m_{\alpha,\lambda}(x|\mu)$

\[
\frac{1}{\pi} \text{Im} \left\{ \frac{\lambda^\mu}{(\lambda + e^{-i\pi\alpha} t^{\alpha})^\mu} \right\} \rightarrow m_{\alpha,\lambda}(x|\mu) \rightarrow \frac{\lambda^\mu}{(\lambda + s^{\alpha})^\mu} \\
\mu \phi_{\alpha,\lambda}(t) \rightarrow \mu \alpha (g * m)_{\alpha,\lambda}(x) \rightarrow \mu \alpha \frac{s^{\alpha-1}}{\lambda + s^{\alpha}} \\
(1 - \alpha) \delta(t) + \phi_{\alpha,\lambda}(t) \rightarrow 1 - \alpha + \alpha (g * m)_{\alpha,\lambda}(x) \rightarrow \frac{1 - \alpha}{s} + \alpha \frac{s^{\alpha-1}}{\lambda + s^{\alpha}} \quad (124)
\]

This demonstrates a unifying theme of the convolution of two densities for all objects studied in this paper, despite their apparent diversity.

10 Discussion

It is worth restating the objective of this paper, building upon the introductory remarks. The novelty of the paper is primarily one of perspective and representation rather than discovery of new, previously undocumented, ID distributions. Such novelty lies in representing ID densities, known or novel, as nodes of a commutative diagram, with

- the Laplace transform or its inverse as horizontal connections of the diagram
- a limiting process or logarithmic derivative as downward connections
- compound Poisson sum or integral of logarithmic derivative as upward connections.

Such visual representation has vastly contributed to our own appreciation of the coherence and connection amongst densities that appear disparate at first.

The typical goal of mathematical research is the quest for generality. A case in point is the representation of the stable density in terms of the very general Meijer G-function (Penson and Górska [18]). By contrast, the approach of this paper has been one of conceptual simplicity, such as the convolution of two densities, and then exploring the generality that may flow from such simplicity.

Accordingly, we have introduced only one basic ID object, the gamma density $g_{\mu,\lambda}(x)$, which gave rise to the gamma LKCD (19). Then, rather than introducing the stable density as a new object, we inferred it from the gamma case by growing the gamma LKCD upward from $g_{1-\alpha,0}(x) = x^{-\alpha}/\Gamma(1 - \alpha)$ ($0 < \alpha < 1$). We then also extended the LKCD to the left since $x^{-\alpha}$ is the Laplace transform of $g_{\alpha,0}(t) = t^{\alpha-1}/\Gamma(\alpha)$, to form the LKCD of a generalised gamma convolution, the GGC LKCD. We then turned to the convolution of two gamma densities, thereby entering the world of beta and Bessel densities via the confluent hypergeometric function.

That all these objects can be seen to arise from the gamma density or the convolution of two gamma densities (which is not simply another gamma density if the decays are different), can easily pass unappreciated, leading to a less joined-up conversation about them than might
otherwise be the case. We take conceptual simplicity and the intrinsically joined-up commutative
diagram to be intimately related fundamentals.

Another key perspective that we explored is the mixture of stable densities. If the mixing density
is itself stable, the mixture is also a stable density of index \( \alpha \beta \) generated from stable densities
of separate indices \( \alpha \) and \( \beta \). A gamma mixing density led to the fractional gamma density. The
latter is routinely represented in terms of the Mittag-Leffler function. We also demonstrated an
integral representation that gives an expression involving the parabolic cylinder function for the
stable index \( \alpha = 1/2 \). To our awareness, the parabolic cylinder function representation of the
fractional gamma distribution has not been reported in the literature. As a solution of Laplace’s
equation in cylindrical coordinates, the parabolic cylinder function perspective suggests a link
between the fractional gamma distribution and the study of harmonic functions in potential
theory.

11 Conclusion and Future Work

We have introduced a commutative diagram visualisation of infinitely divisible (ID) distributions
(or their densities, to be precise). We referred to this novel representation as the Lévy-Khintchine
commutative diagram (LKCD).

Much remains to be explored. Notably, the ID densities studied here can form the basis for the
construction of multivariate distributions. The simplest case is the Dirichlet distribution, which
Ferguson [9, 10] constructed as follows. Let \( G(\mu, \lambda) \) denote the gamma distribution with shape
\( \mu \) and decay \( \lambda \) and let \( \{X_i \sim G(\mu_i, \lambda)\} \) be \( N \) independently distributed variables with individual
gamma distributions that may have different shapes \( \{\mu_i\} \) but share a common decay \( \lambda \). The
distribution of the sum \( X = \sum_{i=1}^{N} X_i \) is a convolution of the individual gamma distributions and
thus also a gamma distribution whose shape is a sum of the individual shapes with the same
shared decay. Then the multivariate distribution on the normalised variables \( \{X_i/X\} \) is known
as the Dirichlet distribution. It is defined on the \( (N-1) \)-dimensional probability simplex in the
\( N \)-dimensional space defined by the independent \( \{X_i\} \). Notably, it depends only on the \( \{\mu_i\} \)
and is independent of the shared decay \( \lambda \).

Alternatives to Dirichlet are possible but more complex. For instance, Sibisi and Skilling [23]
suggested the convolution of two gamma distribution \( \{X_i \sim G(\mu_i, \lambda_1) \ast G(\mu_1, \lambda_2)\} \), which they
referred to as the supergamma distribution and the induced normalised distribution the super-
Dirichlet distribution. Di Nardo et al. [6] explored \( \{X_i \sim F\mathcal{G}(\mu_i, \alpha, \lambda)\} \) where \( F\mathcal{G}(\mu, \alpha, \lambda) \)
is the fractional gamma distribution, which they expressed in terms of the three parameter
Mittag-Leffler (Prabhakar) function. They referred to the associated normalised distribution as
the fractional generalisation of the Dirichlet distribution. Favaro et al. [7] discussed the general
case where each \( X_i \) has an arbitrary ID distribution that need not be in the same family as the
other \( X_{j\neq i} \), i.e. the individual distributions need not all be gamma or all fractional gamma with
different parameter choices. They referred to the associated normalised distribution generically
as the class of distributions on the simplex.

There is ample room for further exploration of multivariate alternatives to Dirichlet building
upon the classes of densities explored in this paper. In addition to that, there is still much to
explore in the world of univariate ID and GGC densities and the form of LKCD representation
that they induce which, in turn, can lead to further insights.
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