Trigonometric multiplicative chaos and applications to random distributions

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Abstract The random trigonometric series $\sum_{n=1}^{\infty} \rho_n \cos(nt + \omega_n)$ on the circle $T$ are studied under the conditions $\sum |\rho_n|^2 = \infty$ and $\rho_n \to 0$, where $\{\omega_n\}$ are independent and uniformly distributed random variables on $T$. They are almost surely not Fourier-Stieltjes series but determine pseudo-functions. This leads us to develop the theory of trigonometric multiplicative chaos, which produces a class of random measures. The kernel and the image of chaotic operators are fully studied and the dimensions of chaotic measures are exactly computed. The behavior of the partial sums of the above series is proved to be multifractal. Our theory holds on the torus $T^d$ of dimension $d \geq 1$.

Keywords multiplicative chaos, random Fourier series, Hausdorff dimension, Riesz potential

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1 Introduction

Let us consider a random trigonometric series

$$\sum_{n=1}^{\infty} X_n \cos(nt + \Phi_n),$$

(1.1)

where the random complex variables $A_n := X_n e^{i\Phi_n}$ with $X_n$ and $\Phi_n$ being real are supposed to be independent and symmetric. It is the real part of the random Taylor trigonometric series

$$\sum_{n=1}^{\infty} A_n e^{int}.$$

(1.2)

These series are well studied in [33,50–52] (see also [35]) under the condition $E X_n^4 \leq C (EX_n^2)^2$. If $\sum E(X_n^2)$ $= +\infty$, the series (1.1) diverges almost surely almost everywhere and is almost surely not a Fourier-Stieltjes series (see [35, p.54]). Regular and irregular properties of the function defined by the series (1.1) are studied under the condition $\sum E(X_n^2) < +\infty$ (see [35, 49]).

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We study the series (1.1) under the following assumptions:

$$\sum_{n=1}^{\infty} E X_n^2 = \infty, \quad \sum_{n=1}^{\infty} E X_n^4 < \infty.$$  \hspace{1cm} (1.3)

One of our objects of study is the behavior of the partial sums of the pseudo function defined by the series (1.1):

$$S_N(t) := S_N(t, \omega) := \sum_{n=1}^{N} X_n \cos(nt + \Phi_n).$$  \hspace{1cm} (1.4)

We also assume that \(\{\Phi_n\}\) are independent and uniformly distributed on the circle \(T := \mathbb{R}/2\pi\mathbb{R}\) which is identified with the interval \([0, 2\pi)\) and that \(\{X_n\}\) are independent quasi-gaussian and independent of \(\{\Phi_n\}\). A real random variable \(X\) is said to be quasi-gaussian if \(E(X^{2m}) = O(K^m m!)\) for some \(K \geq 0\). Recall that \(X\) is said to be subgaussian if \(E(e^{\lambda X}) \leq e^{\frac{1}{2}\tau^2 \lambda^2}\) for some \(\tau > 0\) and all \(\lambda \in \mathbb{R}\). We know that \(X\) is subgaussian if and only if it is centered and quasi-gaussian (see [35, p. 82]).

As we shall prove, the behavior of \(S_N(t)\) is very multifractal, meaning that for uncountably many functions \(L : \mathbb{N} \rightarrow \mathbb{R}\) of different orders, the sets \(\{t : S_N(t) \sim L(N)\}\) have positive Hausdorff dimensions.

The partial sum \(S_N\) is a stationary process on \(T\) with its correlation function equal to

$$E S_N(t) S_N(s) = H_N(t - s),$$

where

$$H_N(t) = \frac{1}{2} \sum_{n=1}^{N} E X_n^2 \cos nt.$$  \hspace{1cm}

The limit

$$H(t) := \lim_{N \rightarrow \infty} H_N(t),$$

if it exists, is defined to be the correlation function of the series (1.1). In many cases, the correlation function

$$H(t) = \frac{1}{2} \sum_{n=1}^{\infty} E X_n^2 \cos nt$$

is well defined, integrable on \(T\) and continuous everywhere except \(t = 0\) (see Section 3). The correlation function \(H\) will play an important role in the study of the series (1.1).

When \(E X_n^2 = \frac{\alpha^2}{n}\), we obtain a special correlation function

$$H_\alpha(t) = \frac{\alpha^2}{2} \sum_{n=1}^{\infty} \frac{\cos nt}{n} = -\frac{\alpha^2}{2} \log \left( \frac{1}{2} \left| \sin \frac{t}{2} \right| \right).$$

Its exponential is the \(\frac{\alpha^2}{4}\text{-order\ Riesz kernel}\)

$$e^{H_\alpha(t)} \simeq \frac{1}{\left| \sin \frac{t}{2} \right|^2}. $$

An effective tool that we shall use to study the series (1.1) is a class of multiplicative chaotic measures, which are formally defined by

$$\mu_\omega = \prod_{n=1}^{\infty} \exp(X_n \cos(nt + \Phi_n) - \log E I_0(X_n))dt,$$  \hspace{1cm} (1.5)

where \(I_0\) is the modified Bessel function of the first kind:

$$I_0(\alpha) = \int_{0}^{2\pi} e^{\alpha \sin x} \frac{dx}{2\pi}, \quad \alpha \in \mathbb{R}. $$
More precisely, $\mu_\omega$ is the weak limit of the partial products of the infinite product in (1.5). These partial products form a non-negative martingale for each fixed $t$, which ensures the existence of the limit. We shall find conditions ensuring the non-nullity of the measure $\mu_\omega$. A simple condition like this is

$$
\int \int e^{H(t-s)}dtds < \infty
$$

(1.6)

(see Proposition 4.1). Under the energy condition (1.6), we can define a probability measure $Q$ on the product space $\mathbb{T} \times \Omega$ by the equality

$$
E_Q h(t, \omega) = E \int_0^{2\pi} h(t, \omega) d\mu_\omega(t)
$$

holding for all the bounded functions $h$. This measure is called the Peyri`ere measure. An important fact is that “$Q$-almost surely” means “almost surely (with respect to $P$) $\mu_\omega$-almost everywhere”. Another important fact is that the random variables $\{X_n \cos(nt + \Phi_n)\}$ defined on $\mathbb{T} \times \Omega$ are $Q$-independent (see Theorem 2.5). On the other hand, the measure $\mu_\omega$, through the measure $Q$, well captures the points $t$ for which the series (1.1) has a specific property. In the definition of $\mu_\omega$ we can replace $\{X_n\}$ by $\{\alpha X_n\}$ to give a measure $\mu_{\omega,\alpha}$. So we possess many measures as tools.

For the typical case $X_n = \frac{1}{\sqrt{n}}$, we shall prove that $\mu_{\omega,\alpha} = 0$ almost surely if $|\alpha| > 2$ (see Theorem 5.2). However, for $|\alpha| < 2$ the Hausdorff dimension of the measure $\mu_{\omega,\alpha}$ is almost surely equal to

$$
dim \mu_{\omega,\alpha} = 1 - \frac{\alpha^2}{4}, \quad |\alpha| < 2
$$

(see Theorem 6.3). The Peyri`ere measure is denoted by $Q_\alpha$ when $|\alpha| < 2$.

Let us state the following two representative results obtained in this paper on the series (1.1) (see Theorems 6.1 and 7.10).

**Theorem 1.1.** Let us consider the series (1.1) with $X_n = \frac{1}{\sqrt{n}}$. Let $\alpha \in (-2, 2)$. Almost surely $\mu_{\omega,\alpha}$-almost everywhere we have

$$
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{\cos(nt + \Phi_n)}{\sqrt{n}} = \frac{\alpha}{2}.
$$

Moreover, for any $\eta > 0$, we have the following large deviation:

$$
\lim_{N \to \infty} \frac{1}{\log N} \log Q_\alpha \left\{ (t, \omega) : \frac{1}{\log N} \sum_{n=1}^N \frac{\cos(nt + \Phi_n)}{\sqrt{n}} \not\in \left[ \frac{\alpha}{2} + [-\eta, \eta] \right] \right\} = -\eta^2.
$$

On the torus $\mathbb{T}_d$, a standard trigonometric multiplicative chaotic measure is defined by

$$
Q_\alpha \sigma = \prod_n \exp[\alpha X_n \cos(nx + \Phi_n)]d\sigma(x),
$$

where $\alpha$ is a real parameter, $\sigma$ is a finite Borel measure on the torus $\mathbb{T}_d$, $\{\Phi_n\}$ are i.i.d. (independent and identically distributed) and uniformly distributed on the torus $\mathbb{T}_d$, $EX_n^2 = \frac{1}{|m|}$ and the product is taken over $n \in \mathbb{Z}_d \setminus \{0\}$ but only one of $n$ and $-n$ is taken into account.

Let us quote the following result in the case $X_n = \frac{1}{\sqrt{|m|}}$ (see Theorems 6.3 and 7.10).

**Theorem 1.2.** Let $\tau(d) = \frac{d^{d/2}}{\Gamma(d/2)}$, which is the half area of the unit sphere in $\mathbb{R}_d$. For any unidimensional measure $\sigma$ on $\mathbb{T}_d$, we have

1. $Q_\alpha \sigma = 0$ if $\dim \sigma < \frac{\alpha^2}{4} \tau(d)$;
2. $\dim Q_\alpha \sigma = \dim \sigma - \frac{\alpha^2}{4} \tau(d)$ if $\dim \sigma > \frac{\alpha^2}{4} \tau(d)$.

Our main effort is to determine the kernel and the image of the chaotic operator $EQ_\alpha$ for which we obtain the following result (see Theorems 6.2 and 7.9).
Theorem 1.3. Let \( \tau(d) = \frac{d^{d/2}}{\Gamma(d/2)} \). We have
\[
S_{\frac{\log a}{a}} \subset \text{Ker}EQ_{a} \subset S_{\frac{\log a}{a} - 1}, \quad \mathcal{R}_{\frac{\log a}{a}} \subset \text{Im}EQ_{a} \subset \mathcal{R}_{\frac{\log a}{a} - 1}.
\]

The spaces \( S_{\frac{\log a}{a}}, \mathcal{R}_{a}, \) etc. will be discussed in Section 3. Let us point out that Theorem 1.2 is actually a consequence of Theorem 1.3 and the decomposition principle stated in Theorem 2.6. It is a very interesting problem to completely determine the image and the kernel of \( EQ_{a} \). Associated with the percolation problem on a tree, there is a multiplicative chaotic operator for which the image and the kernel are completely determined (see [16] and [17] for the detailed proof).

Our work will actually concentrate on the study of trigonometric multiplicative chaos. This enters into the general theory of multiplicative chaos of Kahane [37]. Other multiplicative chaos has already been studied, including gaussian chaos [14,34], Lévy stable chaos [23], Dvoretsky covering chaos [36], and tree percolation chaos [17]. General infinitely divisible chaos, which includes the Lévy stable chaos, is studied in [1]. There is recently an active study, which produces a great amount of literature, on gaussian chaos because of its link to physics, especially to Liouville quantum fields. We just cite here some of these works [6,7,12,31,53,56] and invite the reader to refer to the references therein and the survey papers [54,55]. We point out that for both Dvoretsky covering chaos and tree percolation chaos and these works [6,7,12,31,53,56] and invite the reader to refer to the references therein and the survey papers [54,55]. We point out that for both Dvoretsky covering chaos and tree percolation chaos and these works [6,7,12,31,53,56] and invite the reader to refer to the references therein and the survey papers [54,55]. We point out that for both Dvoretsky covering chaos and tree percolation chaos and these works [6,7,12,31,53,56] and invite the reader to refer to the references therein and the survey papers [54,55]. We point out that for both Dvoretsky covering chaos and tree percolation chaos and these works [6,7,12,31,53,56] and invite the reader to refer to the references therein and the survey papers [54,55].

In Section 2, we give a brief review of the general theory of multiplicative chaos and state all the known basic results that we shall use. Our trigonometric chaos on \( T \) is constructed in Section 3, where the correlation functions and the associated kernels are examined, and the capacity and the dimension of measures as well as their relations are recalled. Section 4 is devoted to \( L^{2} \)-theory and its consequences, and in this section, we also discuss the decomposition and the composition of our typical chaotic operators. The degeneracy of chaotic operators is investigated in Section 5, while the kernel and the image of the projection \( EQ_{a} \) are studied in Section 6. All these can be generalized to the torus \( T^{d} \) with \( d \geq 1 \) as we show in Section 7. Our chaotic measures are used in Section 8 to study the divergence of the series (1.1).

Notation. We adopt the following notation. Let \( u(x) \) and \( v(x) \) be two functions. We write \( u(x) \ll v(x) \) if there exists a positive constant \( C > 0 \) such that \( u(x) \leq Cv(x) \) for \( x \) in the domain of definition of \( u \) and \( v \). If \( u(x) \ll v(x) \) and \( v(x) \ll u(x) \), we write \( u(x) \asymp v(x) \). When \( \lim_{x \to x_{0}} \frac{u(x)}{v(x)} = 1 \), we write \( u(x) \sim v(x) \) \( (x \to x_{0}) \) or simply \( u(x) \sim v(x) \).

2 General multiplicative chaos

In this section, we recall the basic results in the theory of \( T \)-martingales, or of multiplicative chaos. The origin goes back to [42,45–48], where understanding turbulence was the motivation. The general theory was developed by Kahane [37]. The first seminal works on the subject are [34] and [40].

In the next subsection, we introduce our \( T \)-martingales that we study in this paper. Let us point out that the theory in [37] is generalized in [59] to a case including some dependent multiplicative cascades. The materials presented below come from [18,29,37].

2.1 The construction of \( T \)-martingales

Let \( (T,d) \) be a compact (or locally compact) metric space and \((\Omega,\mathcal{A},P)\) be a probability space. We are given an increasing sequence \( \{A_{n}\}_{n \geq 1} \) of sub-\( \sigma \)-fields of \( \mathcal{A} \) and a sequence of random functions \( \{P_{n}(t,\omega)\}_{n \geq 1} \) \( t \in T, \omega \in \Omega \). We make the following assumptions:

(H1) \( P_{n}(t) = P_{n}(t,\omega) \) are non-negative and independent processes; \( P_{n}(\cdot,\omega) \) is Borel measurable for almost all \( \omega \); \( P_{n}(\cdot,\omega) \) is \( A_{n} \)-measurable for each \( t \).

(H2) \( EP_{n}(t) = 1 \) for all \( t \in T \).
Such a sequence \( \{P_n\} \) is called a sequence of weights adapted to \( \{A_n\} \).

Let
\[
Q_n(t) = Q_n(t, \omega) = \prod_{j=1}^{n} P_j(t, \omega).
\]

For any \( t \in T \), \( \{Q_n(t)\} \) is a martingale. We call \( \{Q_n(t)\} \) a \( T \)-martingale, or a martingale indexed by \( T \). For any \( n \geq 1 \) and any positive Radon measure \( \sigma \) on \( T \) (we write \( \sigma \in \mathcal{M}^+(T) \)), we consider the random measures \( Q_n \sigma \)'s defined by
\[
Q_n \sigma(A) = \int_A Q_n(t) d\sigma(t), \quad A \in B(T),
\]
where \( B(T) \) is the Borel field of \( T \). It is clear that for any \( A \in B(T) \), \( Q_n \sigma(A) \) is a positive martingale, so it converges almost surely (a.s. for short). So does \( \int \phi(t) dQ_n \sigma(t) \) for any bounded Borel function \( \phi \).

### 2.2 Basic results in the theory of multiplicative chaos

The following fundamental theorem is proved based on the last fact stated above.

**Theorem 2.1** (See [37]). For any Radon measure \( \sigma \in \mathcal{M}^+(T) \), almost surely the random measures \( Q_n \sigma \)'s converge weakly to a random measure \( S \), which will be denoted by \( Q \sigma \).

We may consider \( Q \) as an operator which maps measures into random measures. We call \( Q \) a multiplicative chaotic operator or simply a chaotic operator, and \( Q \sigma \) a multiplicative chaotic measure or simply a chaotic measure which, in some special cases, describes the limit energy state of turbulence [15, 35, 47, 48]. There are two possible extreme cases. The first one is that \( Q \sigma = 0 \) a.s. (the energy is totally dissipated). The second one is that \( Q \sigma(T) \)'s converge in \( L^1 \) or equivalently \( EQ \sigma(T) = \sigma(T) \) (the energy is conserved). If the first case occurs, we say that \( Q \) degenerates on \( \sigma \) or \( \sigma \) is \( Q \)-singular. If the second case occurs, we say that \( Q \) fully acts on \( \sigma \) or \( \sigma \) is \( Q \)-regular.

We define a map \( EQ : \mathcal{M}^+(T) \to \mathcal{M}^+(T) \) by
\[
(EQ \sigma)(A) = E(Q\sigma(A)), \quad A \in B(T).
\]
That \( \sigma \) is \( Q \)-singular (resp. \( Q \)-regular) is equivalent to \( EQ \sigma = 0 \) (resp. \( EQ \sigma = \sigma \)).

**Theorem 2.2** (See [37]). Any Radon measure \( \sigma \in \mathcal{M}^+(T) \) can be uniquely decomposed into \( \sigma = \sigma_r + \sigma_s \), where \( \sigma_r \) is a \( Q \)-regular measure and \( \sigma_s \) is a \( Q \)-singular measure. Both \( \sigma_r \) and \( \sigma_s \) are restrictions of \( \sigma \), that is to say \( \sigma_r = \sigma_{1_B} \) and \( \sigma_s = \sigma_{1_{B^c}} \) for some Borel set \( B \).

The operator \( EQ \) extended to the space \( \mathcal{M}(T) \) is thus a projection whose image (resp. kernel) consists of \( Q \)-regular (resp. \( Q \)-singular) measures. We call \( EQ \) the chaotic projection operator.

We are concerned with properties of the random measure \( Q \sigma \), of the operator \( Q \) or of the projection \( EQ \). Here are some fundamental questions:

**Question 1.** Does \( Q \) degenerate on \( \sigma \)?

**Question 2.** Does \( Q \) fully act on \( \sigma \)?

**Question 3.** What is the dimension of the measure \( Q \sigma \)?

**Question 4.** What are the possible relations between two measures \( Q' \sigma' \) and \( Q'' \sigma'' \) for two different operators \( Q' \) and \( Q'' \) defined in the same way as \( Q \)? For example, when are \( Q' \sigma' \) and \( Q'' \sigma'' \) mutually singular or absolutely continuous?

In the following, we state some results in the general case. Either they are partial answers to one of these questions, or they provide some useful tools.

**Theorem 2.3** (See [37]). Suppose that \( H^n(T) < \infty \) where \( H^n \) denotes the \( \alpha \)-dimensional Hausdorff measure and that there exist constants \( h \) (\( 0 < h < 1 \)) and \( C > 0 \) with the property that for any ball \( B \)
with radius $r$, there exists an integer $n = n(B)$ such that
\[
E \left( \sup_{y \in B} Q_n(y) \right)^h \leq C r^\alpha (1-h). \tag{2.2}
\]

Then all the Radon measures on $T$ are $Q$-singular.

This provides a good tool to verify the $Q$-singularity of $\sigma$. In fact, the condition $\dim \sigma < \alpha$ together with (2.2) implies the $Q$-singularity of $\sigma$.

On the other hand, the following theorem gives a simple condition of $Q$-regularity, which is the condition for the $L^2$-convergence of the martingale $Q_\omega \sigma(T)$.

**Theorem 2.4** (See [37]). The operator $Q$ fully acts on $\sigma$ and $E(Q\sigma(T))^2 < \infty$ if and only if
\[
\lim_{N \to \infty} \int \prod_{n=1}^N EP_n(t) P_n(s) d\sigma(t) d\sigma(s) < \infty. \tag{2.3}
\]

Suppose that $\sigma$ is a $Q$-regular probability measure. A useful tool for studying the measure $Q\sigma$ is the Peyrière measure $Q$ on the product space $T \times \Omega$ defined by
\[
\int_{T \times \Omega} \varphi(t,\omega) dQ(t,\omega) = E \int_{T} \varphi(t,\omega) dQ\sigma(t) \tag{2.4}
\]
for the non-negative measurable function $\varphi$.

**Theorem 2.5** (See [37]). Suppose that $\sigma$ is a $Q$-regular probability measure and that the probability law of the variable $P_n(t)$ is independent of $t$. Then $P_n$’s, considered as random variables with respect to $Q$, are $Q$-independent.

If the conditions in Theorem 2.5 are satisfied, for any bounded function or positive function $\Phi$, we have
\[
E_Q \Phi(P_n(t)) = E\Phi(P_n(t)) P_n(t), \tag{2.5}
\]
where the term on the right-hand side is independent of $t$.

A direct application of the Peyrière measure is that almost surely for $Q\sigma$-almost every $t \in T$, we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log P_k(t,\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} EP_k \log P_k. \tag{2.6}
\]
This result may be used to study the dimension and the multifractality of the measure $Q\sigma$.

Now suppose that we are given two sequences of weights $\{P'_n(t)\}_{n \geq 1}$ and $\{P''_n(t)\}_{n \geq 1}$ adapted to $\{A'_n\}$ and $\{A''_n\}$, respectively defined on probability spaces $(\Omega', A', P')$ and $(\Omega'', A'', P'')$. It is obvious that $\{P_n\}$ defined by $P_n(t) = P'_n(t) P''_n(t)$ is a sequence of weights adapted to $\{A'_n \otimes A''_n\}$, defined on the product space $(\Omega, A, P) = (\Omega' \times \Omega'', A' \otimes A'', P' \otimes P'')$. We denote by $Q'$, $Q''$ and $Q$ the three operators corresponding to the above three sequences of weights. The following decomposition principle establishes a relationship between $Q'$, $Q''$ and $Q$.

**Theorem 2.6** (See [29]). Under the above condition, we have
(a) a.s. $Q\sigma = Q'(Q'\sigma)$ for any measure $\sigma \in M^+(T)$;
(b) $\sigma \in \text{Im} EQ \Rightarrow Q'\sigma \in \text{Im} EQ''$ for almost all $\omega' \in \Omega'$;
(c) $\sigma \in \text{Ker} EQ \Rightarrow Q'\sigma \in \text{Ker} EQ''$ for almost all $\omega' \in \Omega'$;
(d) $EQ = EQ' EQ''$.

Let $Q'$ and $Q''$ be two operators associated, respectively, with $\{P'_n\}$ and $\{P''_n\}$. Now we do not suppose the independence of $\{P'_n\}$ and $\{P''_n\}$. But we suppose that the law of the vector $(P'_n(t), P''_n(t))$ is independent of $t$. We have a Kakutani type criterion for the dichotomy of the mutual absolute continuity of $Q'\sigma$ and $Q''\sigma$.

**Theorem 2.7** (See [18]). Assume the above assumptions. Suppose furthermore that $\sigma$ is a $Q'$-regular probability measure. We have
(a) $\prod_{n=1}^{\infty} E \sqrt{P'_n P''_n} > 0 \Rightarrow a.s. \ Q'\sigma \ll Q'\sigma$ and $\sigma$ is $Q''$-regular;
(b) $\prod_{n=1}^{\infty} E \sqrt{P'_n P''_n} = 0 \Rightarrow a.s. \ Q'\sigma \perp Q'\sigma$. 

3 Trigonometric multiplicative chaos on $\mathbb{T}$

Now, in this section, we construct a class of $\mathbb{T}$-martingales, which are used to define our trigonometric multiplicative chaos. We also discuss the associated correlation functions and their relations to the potential theory and the dimension theory.

3.1 The definition of $\mathbb{T}$-martingales

We always make the following assumptions:

(H1) $X_n = \alpha_n Y_n$, where $Y_n$’s are normalized independent quasi-gaussian real random variables. That means

$$\mathbb{E}Y_n^2 = 1, \quad \mathbb{E}Y_n^{2m} = O(K^m m!)$$

for some $K > 0$ and for all $m \geq 2$ and $n \geq 1$;

(H2) $\Phi_n$’s, which will be denoted by $\omega_n$, are independent random variables which are uniformly distributed on $\mathbb{T}$;

(H3) the real coefficients $\alpha_n$’s satisfy

$$\sum_{n=1}^{\infty} |\alpha_n|^2 = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n|^4 < \infty.$$

We define the weights

$$P_n(t) = \frac{e^{\alpha_n Y_n \cos(nt + \omega_n)}}{e^{\alpha_n Y_n \cos(nt + \omega_n)}} = \frac{e^{\alpha_n Y_n \cos(nt + \omega_n)}}{I_0(\alpha_n Y_n)}, \quad (3.1)$$

where $I_0$ denotes the modified Bessel function of the first kind.

Then we define the $\mathbb{T}$-martingales $\{Q_n(t)\}$ by

$$Q_n(t) = \exp \left[ \sum_{k=1}^{n} (\alpha_k Y_k \cos(nt + \omega_k) - \log I_0(\alpha_k Y_k)) \right]. \quad (3.2)$$

We denote by $Q$ the multiplicative operator defined by these $\mathbb{T}$-martingales $\{Q_n(t)\}$.

For this multiplicative chaotic operator $Q$, we define its correlation function by

$$H(t) = \frac{1}{2} \sum_{n=1}^{\infty} \alpha_n^2 \cos nt \quad (3.3)$$

and its associated kernel

$$\Phi(t) = e^{H(t)}. \quad (3.4)$$

The following simple properties of the Bessel function $I_0$ are frequently used in the sequel.

**Lemma 3.1.** Suppose that $X$ is quasi-gaussian and $EX^2 = 1$. As $\alpha \to 0$, we have

$$\mathbb{E}I_0(\alpha X) = 1 + \frac{\alpha^2}{4} + O(\alpha^4) = e^{\frac{\alpha^2}{4}} + O(\alpha^4),$$

$$\frac{d}{d\alpha} I_0(\alpha X) = \frac{\alpha}{2} + O(\alpha^2),$$

$$\frac{d^2}{d\alpha^2} I_0(\alpha X) = \frac{1}{2} + O(\alpha^2).$$

**Proof.** From the well-known fact $I_0(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{x}{2} \right)^{2m}$, we obtain

$$\mathbb{E}I_0(\alpha X) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\alpha^{2m}}{2^{2m}} \mathbb{E}(X^{2m}) = 1 + \frac{\alpha^2}{4} + \sum_{m=2}^{\infty} \frac{1}{m!} \frac{\alpha^{2m}}{2^{2m}} \mathbb{E}(X^{2m}).$$

Since $X$ is quasi-gaussian, we have $\mathbb{E}(X^{2m}) \leq MK^{m} m!$ for some constants $M > 0$ and $K > 0$. It follows that the last sum is bounded by $\frac{MK^2}{4} \alpha^4 e^{\frac{\alpha^2}{4}}$. The other two estimates can be similarly obtained. \qed
The next lemma states a basic relation between the chaotic operator and its correlation function.

**Lemma 3.2.** Under the assumptions (H1)–(H3), we have

\[ \mathbb{E}[Q_n(t)Q_n(s)] \approx \exp \left( \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \cos k(t - s) \right). \]

**Proof.** For any real number \( \lambda \), we have the equality

\[ \mathbb{E} \exp (\lambda [\cos (kt + \omega_k) + \cos (ks + \omega_k)]) = I_0(2\lambda \cos \pi k(t - s)). \]

Indeed, using the formula \( \cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \), we obtain

\[ \cos (kt + \omega_k) + \cos (ks + \omega_k) = 2 \cos (k(t + s)/2 + \omega_k) \cos \pi k(t - s). \]

Then the equality follows from the definition of \( I_0 \) and the translation invariance of the Lebesgue measure. It follows that

\[ \mathbb{E}[Q_n(t)Q_n(s)] = \prod_{k=1}^{n} \frac{\mathbb{E}I_0(2\alpha_k Y_k \cos \pi k(t - s))}{[\mathbb{E}I_0(\alpha_k Y_k)]^2}. \]

However, according to Lemma 3.1, we have

\[ \frac{\mathbb{E}I_0(2\alpha_k Y_k \cos \pi k(t - s))}{[\mathbb{E}I_0(\alpha_k Y_k)]^2} = \exp \left( \alpha_k^2 \left( \cos^2 \pi k(t - s) - \frac{1}{2} \right) + O(\alpha_k^4) \right). \]

Now we can conclude because \( \cos^2 x - \frac{1}{2} = \frac{1}{4} \cos 2x \) and \( \sum \alpha_k^4 < \infty \).

Lemma 3.2 is fundamental for all the computations in the sequel. Remark that the estimate in Lemma 3.2 only depends on \( \alpha_k^2 \) but not on the distributions of \( Y_k \). So, without loss of generality, we can assume that \( Y_k = 1 \) for all \( k \). In other words, we can treat all the sequences \( \{Y_k\} \) satisfying the assumption (H1) just as in the special case of \( Y_k = 1 \). From now on, we assume that \( Y_k = 1 \).

A typical case is \( \alpha_n = \frac{\alpha}{\sqrt{n}} \), for which the correlation function is equal to

\[ H_{\alpha}(t) = -\frac{\alpha^2}{2} \log |\sin \pi t| + O(1). \]

We finish our construction by pointing out that instead of \( \cos x \) we can consider any \( 2\pi \)-periodic function \( f(x) \) such that

\[ J(\alpha) := \int_0^{2\pi} e^{\alpha f(x)} \frac{dx}{2\pi} < \infty \quad \text{for} \quad \alpha \in (-\delta, \delta) \]

for some \( \delta > 0 \). This function \( J \) plays the role of the Bessel function \( I_0 \). Let

\[ m_n = \int_0^{2\pi} f(x) \frac{dx}{2\pi} \]

be the moments \( (n \geq 0) \). We have

\[ J(\alpha) = \sum_{n=0}^{\infty} \frac{m_n}{n!} \alpha^n. \]

Some conditions on the moments of \( f \) are needed.

### 3.2 The correlation function \( H \) and the kernel \( \Phi \)

Here, we first present some conditions for \( H \) to be pointwise well defined and for \( H \) and even \( \Phi \) to be integrable. The function \( \Phi \) will play the role of the kernel in the sense of potential theory. We refer to [61, Volume 1, Chapter V].
It is well known that if $\alpha_n^2 \downarrow 0$, then the cosine series (3.3) defining $H$ converges uniformly on every interval $[\delta, 2\pi - \delta]$ ($\delta > 0$) and consequently the function $H$ is continuous in the open interval $(0, 2\pi)$.

A sequence $\{a_n\}$ is said to be convex if $a_{n+2} - 2a_{n+1} + a_n \geq 0$ for all $n$. A convex sequence tending to 0 must be decreasing. It is also well known that if $\{\alpha_n^2\}$ is convex and $\alpha_n^2 \downarrow 0$, then the function $H$ defined by the series (3.3) is bounded from below and Lebesgue-integrable. In the following lemma, we give a condition ensuring that the partial sums of (3.3) are bounded by the sum of (3.3).

**Lemma 3.3.** Suppose that $\{\alpha_n^2\}$ is convex and
\[
\alpha_n^2 \downarrow 0, \quad \alpha_n^2 = O(n^{-1}), \quad \alpha_n^2 - \alpha_{n+1}^2 = O(n^{-2}).
\] (3.5)
Then the partial sums of the series (3.3) are uniformly bounded from above by $H(x) + C$:
\[
\frac{1}{2} \sum_{n=1}^{N} \alpha_n^2 \cos nt \leq H(t) + C, \quad \forall N \geq 1, \quad \forall x \in T,
\]
where $C$ is a constant.

**Proof.** We repeat the classical proof (see [8, p.92]) and make the observation for the boundedness. For simplicity, let $a_n = \frac{\alpha_n^2}{2}$. Choose a positive $a_0$ such that $\{a_n\}_{n \geq 0}$ is convex. Let
\[
S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nt.
\]
By a double application of Abel’s transformation, we obtain
\[
S_N(x) = \sum_{n=0}^{N-2} (n+1)\Delta^2 a_n K_n(x) + N\Delta a_{N-1} K_{N-1}(x) + a_N D_N(x),
\]
where $D_n$ is the Dirichlet kernel, $K_n$ is the Fejér kernel, and both of them are uniformly bounded by $n$.

By the assumption (3.5), we have
\[
|N\Delta a_{N-1} K_{N-1}(x) + a_N D_N(x)| \leq C
\]
for some constant $C > 0$. Now we can conclude the statement because $K_n$’s are non-negative and
\[
\sum_{n=0}^{\infty} (n+1)\Delta^2 a_n K_n(x) - \frac{a_0}{2} = H(t).
\]

This completes the proof. \(\Box\)

The conditions in Lemma 3.3 are satisfied by
\[
\alpha_n^2 = \frac{1}{n \log^\beta n}, \quad n \geq 3, \quad \beta \geq 0
\]
($\alpha_n^2$ and $\alpha_n^2$ are conveniently chosen).

If $\alpha_n^2 \downarrow 0$, the function $H$ is continuous for $x \neq 0$. Also observe that $H$ is even. So, in this case, for any Borel probability measure $\sigma$ on $T$ we have the equivalence
\[
\int \int e^{H(t-s)} d\sigma(t) d\sigma(s) < \infty \iff \int_{|t-s| \leq \delta} e^{H(t-s)} d\sigma(t) d\sigma(s) < \infty
\]
for some or all $\delta > 0$. If $\sigma$ is the Lebesgue measure, after a change of variables we obtain
\[
\int \int e^{H(t-s)} dt ds < \infty \iff \int e^{H(t)} dt < \infty \iff \int_0^\delta e^{H(t)} dt < \infty
\]
for some or all $\delta > 0$.

Thus, in order to obtain the integrability of the kernel $\Phi$, we only need to know the behavior of $H$ at $t = 0$. Let us quote the following result for a family of functions $H$. Recall that a positive function $b(u)$ for $u \geq u_0$ is said to be slowly varying if for any $\delta > 0$, $b(u)u^{-\delta}$ is increasing and $b(u)u^{-\delta}$ is decreasing for $u$ large enough. For any real number $\beta$, the function $\log^\beta u$ ($u \geq 3$) is slowly varying.
Lemma 3.4 (See [61, pp.187–188]). Let $f_\tau(t) = \sum_{n=1}^{\infty} \frac{b_n}{n^\tau} \cos nt$, where $0 < \tau \leq 1$ and $b_n = b(n)$ for some slowly varying function $b(.)$. When $0 < \tau < 1$, we have

$$f_\tau(t) \sim \Gamma(1 - \tau) \sin \frac{\pi \tau}{2} \frac{b(t^{-1})}{t^{1-\tau}}, \quad t \to +0.$$ 

When $\tau = 1$ and $\sum \frac{b_n}{n} = \infty$, we have

$$f_1(t) \sim \int_1^{t^{-1}} \frac{b(u)}{u} du, \quad t \to +0.$$ 

Let us consider the functions

$h_{\alpha,\tau,\beta}(t) = \sum_{n=3}^{\infty} \frac{\alpha}{n^{\tau} \log^{\beta} n} \cos nt, \quad \alpha > 0, \quad 0 < \tau \leq 1, \quad \beta \in \mathbb{R}.$

Corollary 3.5. If $0 < \tau < 1$, we have

$$h_{\alpha,\tau,\beta}(t) \sim \frac{C_{\alpha,\tau}}{t^{1-\tau} \log^{\beta} \frac{1}{t}}, \quad t \to +0$$

with $C_{\alpha,\tau} = \alpha \Gamma(1 - \tau) \sin \frac{\pi \tau}{2}$, and $e^{h_{\alpha,\tau,\beta}(t)}$ is not integrable on $\mathbb{T}$.

Corollary 3.6. If $\tau = 1$ and $0 < \beta < 1$, we have

$$h_{\alpha,1,\beta}(t) \sim \alpha \int_3^{t^{-1}} \frac{du}{u \log^{\beta} u} \sim \frac{\alpha}{1 - \beta} \log^{1-\beta} \frac{1}{t}, \quad t \to +0,$$

and $e^{h_{\alpha,1,\beta}(t)}$ is integrable on $\mathbb{T}$. If $\tau = 1$ and $\beta > 1$, $h_{\alpha,1,\beta}$ is bounded.

Corollary 3.7. If $\tau = 1$ and $\beta = 1$, we have

$$h_{\alpha,1,1}(t) \sim \alpha \int_3^{t^{-1}} \frac{du}{u \log u} \sim \alpha \log \log \frac{1}{t}, \quad t \to +0,$$

and $e^{h_{\alpha,1,1}(t)}$ is integrable on $\mathbb{T}$.

Corollary 3.8. If $\tau = 1$ and $\beta = 0$, we have

$$h_{\alpha,1,0}(t) \sim \alpha \int_3^{t^{-1}} \frac{du}{u} \sim \alpha \log \frac{1}{t}, \quad t \to +0,$$

and $e^{h_{\alpha,1,0}(t)}$ is integrable on $\mathbb{T}$ if and only if $\alpha < 1$.

3.3 The capacity, the dimension and the Kahane decomposition

We present here some notions from the potential theory and their relations to the dimensions of measures. Consider the Riesz kernel

$$\Phi_\alpha(t) = e^{H_\alpha(t)} \sim \frac{1}{| \sin \frac{\pi}{2} |^{\alpha}}.$$ 

The $\alpha$-energy of a finite Borel measure $\sigma$ is defined by

$$I_\alpha^\sigma = \iint \frac{d\sigma(t)d\sigma(s)}{| \sin \frac{\pi}{2} |^{\alpha}}.$$ 

We have forgotten the constant $\frac{1}{| \sin \frac{\pi}{2} |^{\alpha}}$, because it is usually the finiteness of the energy which plays the role.

Generally, for a given kernel $\Phi$, the $\Phi$-energy of $\sigma$ is similarly defined by

$$I_\Phi^\sigma = \iint \Phi(t-s) d\sigma(t)d\sigma(s).$$
and the Φ-potential of σ is defined by
\[ U_\sigma^\Phi(t) = \int \Phi(t-s) d\sigma(s). \]
If Φ ∈ L^1(T) is non-negative, even and convex on (0, 2π), the Φ-potential theory is well developed (see [41]). The Fourier coefficients \( \Phi(n) \)'s are non-negative and we have the following formula for the energy in terms of Fourier coefficients:
\[ I_\sigma^\Phi = 4\pi^2 \sum_{n=-\infty}^{\infty} |\hat{\Phi}(n)|^2 |\hat{\sigma}(n)|^2. \]

The Φ-capacity of a set \( E \subset T \), which is an important notion, is defined by
\[ \text{cap}_\Phi(E) = \inf I_\sigma^\Phi, \]
where the infimum is taken over all the probability measures concentrated in \( E \). When \( \Phi = \Phi_\alpha \), we write \( I_\sigma^\Phi \) for \( I_\sigma^{\Phi_\alpha} \). \( U_\sigma^\Phi \) and \( \text{cap}_\Phi(E) \) are similarly understood.

The potential \( U_\sigma^\Phi(x) \) describes the local behavior of the measure \( \sigma \) at \( x \) well:
\[ A \sup_{r>0} \frac{\sigma(B(x,r))}{r^\alpha} \leq U_\sigma^\Phi(x) \leq B \sup_{r>0} \frac{\sigma(B(x,r))}{r^{\alpha+\epsilon}}, \]
where \( B(x,r) \) denotes the ball centered at \( x \) of radius \( r \) (an interval of length 2r), and \( A \) and \( B \) are constants depending only on \( \alpha \) and \( \epsilon > 0 \) (see [5, p.103]). So it is easy to see that when \( I_\sigma^\Phi < \infty \), we have \( \dim_\sigma \sigma \geq \alpha \), where \( \dim_\sigma \sigma \) denotes the lower Hausdorff dimension of the measure \( \sigma \):
\[ \dim_\sigma \sigma = \inf \{ \dim E : \sigma(E) > 0 \}, \]
where \( \dim E \) denotes the Hausdorff dimension of the set \( E \) (see [20] for the definition). In [20], the upper Hausdorff dimension of the probability measure \( \sigma \) is defined by
\[ \dim^* \sigma = \inf \{ \dim E : \sigma(E) = 1 \}. \]

If \( \dim_\sigma \sigma = \dim^* \sigma \), we write \( \dim \sigma \) for the common value, called the Hausdorff dimension of \( \sigma \). In this case, we say that \( \sigma \) is unidimensional. Also recall that the lower local dimension of \( \sigma \) at \( x \) is defined by
\[ \frac{\log \sigma(B(x,r))}{\log r}. \]
Here, we can discretize \( r \) by taking a sequence \( \{r_n\} \) such that \( r_n \downarrow 0 \) and \( \log r_n \sim \log r_{n+1} \). It is proved that (see [20])
\[ \dim_\sigma \sigma = \text{ess inf} D(\sigma, x), \quad \dim^* \sigma = \text{ess sup} D(\sigma, x). \]

A finite Borel measure \( \sigma \) is said to be \( \alpha \)-regular, if \( \sigma \) is a countable sum \( \sum \sigma_i \) (convergent in the norm of total variation with \( \sigma_i \) having disjoint Borel supports) such that \( I_\sigma^{\Phi_\alpha} < \infty \) for all \( i \); it is said to be \( \alpha \)-singular if \( \sigma \) is supported by a set of zero \( \alpha \)-capacity. We denote by \( \mathcal{R}_\alpha \) (resp. \( \mathcal{S}_\alpha \)) the set of all the \( \alpha \)-regular (resp. \( \alpha \)-singular) measures. We have the relations
\[ \sigma \in \mathcal{S}_\alpha \Rightarrow \dim^* \sigma \leq \alpha, \quad \sigma \in \mathcal{R}_\alpha \Rightarrow \dim_\sigma \sigma \geq \alpha. \]

Any finite Borel measure \( \sigma \in \mathcal{M}^+(\mathbb{T}) \) has the following Kahane decomposition.

**Theorem 3.9** (See [38]). *Every measure in \( \mathcal{M}^+(\mathbb{T}) \) is uniquely decomposed into \( \sigma_r + \sigma_s \) with \( \sigma_r \in \mathcal{R}_\alpha \) and \( \sigma_s \in \mathcal{S}_\alpha \).*
Actually, the singular measure $\sigma_s$ is supported by the singular part of $\sigma$ defined by

$$S_\sigma := \{ x : U_\sigma^s(x) = \infty \}.$$  

The key step for proving Theorem 3.9 is to prove $\text{cap}_\alpha(S_\sigma) = 0$. The regular measure $\sigma_r$ can then be decomposed as the sum of restrictions of $\sigma$ on $\{ x : i - 1 \leq U_\sigma^s(x) < i \}$ for $i \geq 1$.

It is clear that $S_\alpha$ increases with $\alpha$, while $R_\alpha$ decreases. Let

$$S_{\alpha+} = \bigcap_{\beta > \alpha} S_{\beta}, \quad R_{\alpha-} = \bigcap_{\beta < \alpha} R_{\beta}.$$  

We can similarly define $S_{\alpha-}$ and $R_{\alpha+}$:

$$S_{\alpha-} = \bigcup_{\beta < \alpha} S_{\beta}, \quad R_{\alpha+} = \bigcup_{\beta > \alpha} R_{\beta}.$$  

The following proposition, which is obvious, shows how the potential theory can be used to compute the dimension of a measure.

**Proposition 3.10.** We have $\sigma \in S_{\alpha+} \cap R_{\alpha-} \Rightarrow \dim \sigma = \alpha$.

We also introduce the class $R^*_\alpha$, which describes the image of our operator $EQ_\alpha$ well, as we shall prove. A finite measure $\sigma \in R^*_\alpha$ is defined by the following property: for any $\epsilon > 0$, there exist a number $\beta > \alpha$ and a compact set $K$ such that

$$\sigma(K^c) < \epsilon, \quad \forall x \in K, \quad D(\sigma 1_K, x) \geq \beta,$$  

(3.7)

where $\sigma 1_K$ means the restriction to $K$ of $\sigma$. This class is not far from $R_\alpha$.

**Proposition 3.11.** We have the relation $R_{\alpha+} \subset R^*_\alpha \subset R_\alpha$.

**Proof.** Assume $\sigma \in R_{\alpha+}$. Then $\sigma \in R_\beta$ for some $\beta > \alpha$, so we can write $\sigma = \sum_{i=1}^{\infty} \sigma_i$ with $I_{\sigma_i}^\alpha < \infty$. Rewrite $\sigma = \sigma' + \sigma''$ with $\sigma' = \sum_{i=1}^{N} \sigma_i$ and $\sigma'' = \sum_{i=N+1}^{\infty} \sigma_i$. For any $\epsilon > 0$, take $N$ large enough such that $\sigma''(\mathbb{T}) < \frac{\epsilon}{2}$. Since $I_{\sigma''}^\alpha < \infty$, by Fubini’s theorem we have $U_{\sigma'}^\alpha(x) < \infty \sigma'$-a.e. By the first estimate in (3.6), we obtain $D(\sigma', x) \geq \beta$ $\beta$-a.e. Recall that $\sigma'$ is a restriction of $\sigma$ on some set $B$. We can find a compact set $K$ in $B$ such that $\sigma(B \setminus K) < \frac{\epsilon}{2}$ so that $\sigma(K) < \epsilon$, and $D(\sigma 1_K, x) \geq \beta$ for all $x \in K$. Thus we have proved that $\sigma \in R^*_\alpha$.

Now suppose that $\sigma \in R^*_\alpha$. Take $\epsilon_j = 2^{-j}$, and there exist a sequence $\beta_j > \alpha$ and a compact set $K_j$ such that

$$\sigma(K_j^c) < \epsilon_j, \quad \forall x \in K_j, \quad D(\sigma 1_{K_j}, x) \geq \beta_j > \alpha.$$  

Notice that $K = \bigcup_j K_j$ is large, i.e., $\sigma(K^c) < \epsilon$. On the other hand, by using the second estimate in (3.6), we can prove that $\sigma 1_{K_j} \in R_\alpha$ so that $\sigma 1_K \in R_\alpha$. By letting $\epsilon \to 0$, we can obtain other components of $\sigma$ which are of finite $\alpha$-energy.

### 4 Full action and $L^2$-theory

Recall that our $T$-martingale is defined by (3.2) and the associated correlation function is

$$H(t) = \frac{1}{2} \sum_{n=1}^{\infty} \alpha_n^2 \cos nt.$$  

In this section, we study the full action of the operator $Q$ defined by this $T$-martingale with the second moment method. We shall come back to the investigation of full action in Section 6 (see Theorem 6.2). The decomposition and composition discussed below are also prepared for this investigation.
4.1 The $L^2$-convergence

For a large class of $T$-martingales, there is a necessary and sufficient condition in terms of energy integral for the $L^2$-convergence of the total mass martingale $\int Q_n(t)d\sigma$, which implies that $Q$ fully acts on $\sigma$.

**Proposition 4.1.** Suppose that $\alpha_n^2 \downarrow 0$ and there exists a constant $C$ such that

$$\frac{1}{2} \sum_{n=1}^{N} \alpha_n^2 \cos nx \leq H(x) + C, \quad \forall N \geq 1, \quad \forall x \in \mathbb{T}. \quad (4.1)$$

Let $\sigma$ be a probability measure on $\mathbb{T}$. Then the martingale $\int Q_n(t)d\sigma(t)$ converges in the $L^2$-norm if and only if the following energy is finite:

$$\iint e^{H(t-s)}d\sigma(t)d\sigma(s) < \infty.$$ 

In this case, $\sigma$ is $Q$-regular.

**Proof.** The martingale $\int Q_n d\sigma$ converges in the $L^2$-norm if and only if

$$\mathbb{E}\left(\int Q_n d\sigma\right)^2 = O(1).$$

But

$$\mathbb{E}\left(\int Q_n d\sigma\right)^2 = \iint \mathbb{E}[Q_n(t)Q_n(s)]d\sigma(t)d\sigma(s).$$

By Lemma 3.2, we have

$$\mathbb{E}\left(\int Q_n d\sigma\right)^2 \approx \iint \exp\left(\frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \cos k(t-s)\right)d\sigma(t)d\sigma(s).$$

Since the series defining $H$ converges everywhere, we can conclude by using the hypothesis (4.1) and the Lebesgue’s dominated convergence theorem.

We should point out that there is a difference between our trigonometric chaos and other well-studied chaos like the gaussian chaos [35], the Lévy chaos [23], and the random covering chaos [16]. The difference is that for these well-studied chaos, $\mathbb{E}Q_n(t)Q_n(s)$ increases with $n$, but it is not the case for the trigonometric chaos. That is why we need the technical condition (4.1).

4.2 Energy of $Q\sigma$

When the random measure $Q\sigma$ is non-null, we know whether its $\alpha$-energy is finite. We have the following result. Recall that $\Phi(t) = e^{H(t)}$.

**Proposition 4.2.** Suppose that the kernel $\Phi$ satisfies the boundedness condition (4.1), and $I_{\Phi,\Phi_n}^\sigma < \infty$. Then we have $I_{\alpha}^{Q\sigma} < \infty$ a.s. Moreover, we have

$$\mathbb{E}I_{\alpha}^{Q\sigma} = \iint \Phi(t-s)\Phi_{\alpha}(t-s)d\sigma(t)d\sigma(s) < \infty.$$ 

**Proof.** Take an arbitrary number $L > 0$. Define

$$\Phi_{\alpha}^{(L)}(t) = L \wedge \Phi_{\alpha}(t),$$

which is continuous and increases to $\Phi_{\alpha}(t)$ as $L \uparrow \infty$. By the definition of $Q\sigma$ as the weak limit and the continuity of $\Phi_{\alpha}^{(L)}$, we have

$$\iint \Phi_{\alpha}^{(L)}(t-s)dQ\sigma(t)dQ\sigma(s) = \lim_{n \to \infty} \iint \Phi_{\alpha}^{(L)}(t-s)Q_n(t)Q_n(s)d\sigma(t)d\sigma(s).$$
By the Fatou lemma, we obtain
\[
E \int \int \Phi_{\alpha}^{(L)}(t-s) dQ\sigma(t) dQ\sigma(s) \leq \liminf_{n \to \infty} \int \int \Phi_{\alpha}^{(L)}(t-s) EQ_n(t) Q_n(s) d\sigma(t) d\sigma(s).
\]
Replace \( \Phi_{\alpha}^{(L)} \) on the right-hand side by \( \Phi_{\alpha} \). Then applying the monotone convergence theorem on the left-hand side, we obtain
\[
EI_{\alpha}^{Q_\sigma} \leq \liminf_{n \to \infty} \int \int \Phi_{\alpha}(t-s) EQ_n(t) Q_n(s) d\sigma(t) d\sigma(s).
\]
The estimate on \( EQ_n(t) Q_n(s) \) in Lemma 3.2 and the boundedness condition (4.1) allow us to apply the Lebesgue’s dominated convergence theorem to obtain
\[
EI_{\alpha}^{Q_\sigma} \leq \int \int \Phi_{\alpha}(t-s) \Phi(t-s) d\sigma(t) d\sigma(s).
\]
To prove the inverse inequality, we start with
\[
\int \int \Phi_{\alpha}(t-s) dQ\sigma(t) dQ\sigma(s) \geq \lim inf_{n \to \infty} \int \int \Phi_{\alpha}^{(L)}(t-s) EQ_n(t) Q_n(s) d\sigma(t) d\sigma(s).
\]
The double integral on the right converges almost surely. It is bounded by \( \int Q_n(t) d\sigma(t) \), which is uniformly integrable by Proposition 4.1. Therefore, it converges in \( L^1(\Omega) \) so that
\[
EI_{\alpha}^{Q_\sigma} \geq \lim inf_{n \to \infty} \int \int \Phi_{\alpha}^{(L)}(t-s) EQ_n(t) Q_n(s) d\sigma(t) d\sigma(s).
\]
Again, by Lemma 3.2 and the boundedness condition (4.1), we obtain
\[
EI_{\alpha}^{Q_\sigma} \geq \int \int \Phi_{\alpha}^{(L)}(t-s) \Phi(t-s) d\sigma(t) d\sigma(s).
\]
We finish the proof by letting \( L \uparrow \infty \).

4.3 The decomposition and the composition of \( Q_\alpha \)

The following decomposition of \( Q_\alpha \) is useful when we study the regularity of \( Q_{\alpha/\sqrt{2}} \). Here, the decomposition is in the sense of Theorem 2.6(a).

Define two sequences of weights as follows:
\[
P'_n(t) := P_{2n-1}(t), \quad P''_n(t) := P_{2n}(t),
\]
where \( \{P_n(t)\} \) are the weights defined by (3.1) with \( \alpha_n = \frac{\alpha}{\sqrt{2}} \). The corresponding multiplicative chaotic operators of these two sequences of weights are denoted by \( Q'_{\alpha/\sqrt{2}} \) and \( Q''_{\alpha/\sqrt{2}} \). The reason for this parametrization \( \alpha/\sqrt{2} \) is that the kernels associated with both \( Q'_{\alpha/\sqrt{2}} \) and \( Q''_{\alpha/\sqrt{2}} \) are the same as that of \( Q_{\alpha/\sqrt{2}} \).

Recall that the correlation function of \( Q_\alpha \) is equal to
\[
H_\alpha(t) = -\frac{\alpha^2}{2} \log \left| \sin \frac{t}{2} \right| + O(1).
\]
Notice that the correlation function of \( Q''_\alpha \) is equal to
\[
H^{(2)}_\alpha(t) = \frac{\alpha^2}{2} \sum_{n=1}^{\infty} \frac{\cos 2nt}{2n} = -\frac{\alpha^2}{4} \log |\sin t| - O(1), \tag{4.2}
\]
and that of \( Q'_{\alpha} \) is equal to
\[
H^{(1)}_\alpha(t) = H_\alpha(t) - H^{(2)}_\alpha(t) = -\frac{\alpha^2}{4} \log \left| \frac{\sin t/2}{2 \cos t/2} \right| + O(1). \tag{4.3}
\]
Notice that $H^{(1)}$ and $H^{(2)}$ have the same size as $H_{\alpha/\sqrt{2}}$ at $t = 0$. After exponentiation, all three determine the same Riesz kernel up to a multiplicative constant.

We are cheating on one point, but with no harm. The kernel $e^{H^{(2)}}$ has two singularities $0$ and $\pi$ in the interval $[0, 2\pi)$. We should consider that the operator $Q_{\alpha/\sqrt{2}}''$ is defined on $\mathbb{R}/\pi\mathbb{Z}$. Actually, the weights are $\pi$-periodic, not only $2\pi$-periodic. The kernel $e^{H^{(1)}}$ has only one singularity at $0$. The figures of the kernels $e^{H^{(1)}}$ and $e^{H^{(2)}}$ are shown in Figure 1.

The decomposition principle stated in Theorem 2.6(a) gives rise to the decomposition

$$Q_{\alpha} = Q_{\alpha/\sqrt{2}}'' Q_{\alpha/\sqrt{2}}.$$

We can continue in the same way to decompose

$$Q_{\alpha/\sqrt{2}}' = Q_{\alpha/\sqrt{2}}'' Q_{\alpha/\sqrt{2}}'$$

so that

$$Q_{\alpha} = Q_{\alpha/\sqrt{2}}'' Q_{\alpha/\sqrt{2}}' Q_{\alpha/\sqrt{2}}'.$$

The procedure can continue with $Q_{\alpha/\sqrt{2}}''$ and so on.

Now let us talk about compositions. Let $Q_{\alpha}$ be the operator by the exponentiation of $\frac{\alpha}{\sqrt{n}} \cos(nt + \omega_n')$ and $Q_{\beta}$ be the operator by the exponentiation of $\frac{\beta}{\sqrt{n}} \cos(nt + \omega_n'')$. We suppose that all $\omega_n'$ and $\omega_n''$ (for all $n \geq 1$) are independent. By the composition of $Q_{\alpha}$ and $Q_{\beta}$, we mean the operator by the exponentiation of both $\frac{\alpha}{\sqrt{n}} \cos(nt + \omega_n')$ and $\frac{\beta}{\sqrt{n}} \cos(nt + \omega_n'')$. In other words, it is the operator defined by the weights

$$P_{2n}(t) = \frac{e^{\frac{\alpha}{\sqrt{n}} \cos(nt + \omega_n')}}{I_0\left(\frac{\alpha}{\sqrt{n}}\right)}, \quad P_{2n-1}(t) = \frac{e^{\frac{\beta}{\sqrt{n}} \cos(nt + \omega_n'')}}{I_0\left(\frac{\beta}{\sqrt{n}}\right)}.$$  

It is clear that the correlation function of the composition operator is equal to

$$-\frac{\alpha^2 + \beta^2}{2} \log \left| \sin \frac{t}{2} \right| + O(1).$$

This composition operator has the same potential properties as $Q_{\gamma}$ where $\gamma$ verifies

$$\gamma^2 = \alpha^2 + \beta^2.$$

4.4 Moments with respect to the Peyri`ere measure

Let $\sigma$ be a probability measure on $\mathbb{T}$. If $Q$ acts fully on $\sigma$, we define the Peyri`ere measure $\mathcal{Q}$ by the following relation:

$$E_{\mathcal{Q}} \phi(t, \omega) = E \int \phi(t, \omega) dQ\sigma(t).$$
It can be proved that \(\{nt + \omega_n \pmod{2\pi}\}\) are \(\mathcal{Q}\)-independent random variables. This is a little more than what is stated in Theorem 2.5, but the proof is the same. The \(\mathcal{Q}\)-moment of \(\varphi(nt + \omega_n)\) is easily computed, as stated in the following proposition.

**Proposition 4.3.** Let \(\mathcal{Q}\) be the chaotic operator defined by \(\{\alpha_n\}\) with \(\alpha_n \to 0\). Suppose that \(\sigma\) is a \(\mathcal{Q}\)-regular probability measure. Let \(\mathcal{Q}\) be the Peyri\`ere measure associated with \(\mathcal{Q}\sigma\). Then for any bounded or non-negative function \(\varphi\), we have

\[
E_{\mathcal{Q}}\varphi(nt + \omega_n) = \frac{1}{I_0(\alpha_n)} \int_0^{2\pi} \varphi(x)e^{\alpha_n \sin x} \frac{dx}{2\pi}.
\]

(4.4)

In particular,

\[
E_{\mathcal{Q}} \cos(nt + \omega_n) = \frac{I_0(\alpha_n)}{I_0(\alpha_n)} = \frac{1}{2} \alpha_n + O(\alpha_n^2),
\]

\[
E_{\mathcal{Q}} \cos^2(nt + \omega_n) = \frac{I_0''(\alpha_n)}{I_0(\alpha_n)} = \frac{1}{2} + O(\alpha_n^2).
\]

**Proof.** The formula (4.4) is a special case of (2.5). From (4.4) and Lemma 3.1, we obtain the other two estimates immediately.

If we consider \(Q_\alpha\) defined by \(\alpha_n = \frac{\alpha}{\sqrt{n}}\), we have

\[
E_{Q_\alpha} \cos(nt + \omega_n) = \frac{\alpha}{\sqrt{n}} + O(n^{-3/2}),
\]

\[
E_{Q_\alpha} \cos^2(nt + \omega_n) = \frac{1}{2} + O(n^{-1}).
\]

Consequently, we have the following proposition.

**Proposition 4.4.** Make the same assumption as in Proposition 4.3. Assume further that \(\sum \alpha_n^2 = \infty\). Let \(P_n(t)\) be the weights defining \(Q\). Then for any \(\delta > 0\), almost surely for \(Q\alpha\)-almost every \(t\) we have

\[
\sum_{n=1}^{N} \log P_n(t) = \frac{1}{4} \sum_{n=1}^{N} \alpha_n^2 + o \left( \left( \sum_{n=1}^{N} \alpha_n^2 \right)^{1/2+\delta} \right).
\]

(4.5)

**Proof.** Using \(I_0(\alpha_n) = \frac{1}{2} \alpha_n^2 + O(\alpha_n^4)\) and Proposition 4.3, we obtain

\[
E_{\mathcal{Q}} \log P_n(t) = \alpha_n E_{\mathcal{Q}} \cos(nt + \omega_n) - \log I_0(\alpha_n) = \frac{1}{4} \alpha_n^2 + O(\alpha_n^4)
\]

and

\[
E_{\mathcal{Q}}(\log P_n(t))^2 = \alpha_n^2 E_{\mathcal{Q}} \cos^2(nt + \omega_n) - 2 \alpha_n \log I_0(\alpha_n) E_{\mathcal{Q}} \cos(nt + \omega_n) + \log^2 I_0(\alpha_n)
\]

\[
= \frac{1}{4} \alpha_n^2 + O(\alpha_n^4).
\]

So we obtain the variance

\[
\text{Var}_{\mathcal{Q}}(\log P_n(t)) = \frac{1}{4} \alpha_n^2 + O(\alpha_n^4).
\]

Let \(Y_n = \log P_n(t) - E_{\mathcal{Q}} \log P_n(t)\). Then the series

\[
\sum_{n=1}^{\infty} \text{Var}_{\mathcal{Q}}(Y_1 + \cdots + Y_n)^{1/2+\delta}
\]

converges \(\mathcal{Q}\)-a.e. Indeed, since \(\{P_n\}\) are \(\mathcal{Q}\)-independent, the partial sums of the series form an \(L^2\)-bounded martingale and Doob’s convergence theorem applies. For the boundedness, we use the fact that

\[
\sum (a_1 + \cdots + a_n)^c < \infty
\]

for any positive numbers \(a_n\) and \(c > 1\). We conclude with the help of the Kronecker lemma.
If we consider $Q_\alpha$ defined by $\alpha_n = \frac{\alpha}{\sqrt{n}}$, we have

$$
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \log P_n(t) = \frac{\alpha^2}{4}.
$$

(4.6)

Intuitively, if $I$ is an interval containing $t$ of length $N^{-1}$, we have

$$
Q_\alpha \sigma(I) = P_1(t)P_2(t) \cdots P_N(t)\sigma(I)
$$

so that

$$
\lim_{|I| \to 0} \frac{\log Q_\alpha \sigma(I)}{\log |I|} = - \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \log P_n(t) + \lim_{|I| \to 0} \frac{\log \sigma(I)}{\log |I|}.
$$

Therefore, $\frac{\alpha^2}{4}$ is the difference of the dimensions of the measure $\sigma$ and its image $Q_\alpha \sigma$:

$$
\dim Q_\alpha \sigma = \dim \sigma - \frac{\alpha^2}{4}.
$$

This will be rigorously proved (see Theorem 6.3).

4.5 The mutual singularity and the continuity

Let $\{\alpha_n\}$ and $\{\alpha'_n\}$ be two sequences, which are used to define two operators $Q$ and $Q'$, respectively, by using the same random variables $\{\omega_n\}$. There is a simple criterion for the mutual singularity and the continuity of $Q\sigma$ and $Q'\sigma$.

**Proposition 4.5.** Suppose $\sum |\alpha_n|^4 < \infty$ and $\sum |\alpha'_n|^4 < \infty$. Suppose that $\sigma$ is a $Q$-regular measure. We have

(a) $\sum_{n=1}^{\infty} |\alpha_n - \alpha'_n|^2 < \infty \Rightarrow Q'\sigma \ll Q\sigma$ and $\sigma$ is $Q'$-regular;

(b) $\sum_{n=1}^{\infty} |\alpha_n - \alpha'_n|^2 = \infty \Rightarrow Q'\sigma \perp Q\sigma$.

**Proof.** This is a special case of Theorem 2.7. Let us compute

$$
E \sqrt{P_n(t)}P'_n(t) = \frac{E_0^{\frac{\alpha_n + \alpha'_n}{2} \cos(\alpha t + \omega)}}{\sqrt{I_0(\alpha_n)I_0(\alpha'_n)}} = \frac{I_0(\alpha_n + \alpha'_n)}{\sqrt{I_0(\alpha_n)I_0(\alpha'_n)}}.
$$

Observe that

$$
I_0\left(\frac{\alpha_n + \alpha'_n}{2}\right) = 1 + \frac{(\alpha_n + \alpha'_n)^2}{16} + O(\alpha_n^4 + \alpha'_n^4),
$$

$$
I_0(\alpha_n)I_0(\alpha'_n) = \left(1 + \frac{\alpha_n^2}{4} + O(\alpha_n^4)\right)\left(1 + \frac{\alpha'_n^2}{4} + O(\alpha'_n^4)\right)
$$

\[= 1 + \frac{\alpha_n^2 + \alpha'_n^2}{4} + O(\alpha_n^4 + \alpha'_n^4).\]

It follows that

$$
E \sqrt{P_n(t)}P'_n(t) = \exp\left(\frac{(\alpha_n + \alpha'_n)^2}{16} - \frac{\alpha_n^2 + \alpha'_n^2}{8} + O(\alpha_n^4 + \alpha'_n^4)\right)
$$

\[= \exp\left(-\frac{(\alpha_n - \alpha'_n)^2}{16} + O(\alpha_n^4 + \alpha'_n^4)\right).
$$

Finally we conclude by applying Theorem 2.7. \(\square\)
5 Degeneracy

In this section, we see that the chaotic operator $Q$ can be completely degenerate in the sense that $Q\sigma = 0$ for all the Borel probability measures $\sigma$. Recall that $H$, defined by (3.3), is the correlation function associated with $Q$. We denote its partial sums by

$$H_n(t) = \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \cos kt.$$  

### 5.1 A general result

Let

$$S_n(t) = \sum_{k=1}^{n} \alpha_k \cos(kt + \omega_k).$$

The following estimates allow us to draw conditions for the complete degeneracy in different cases.

#### Proposition 5.1

Let $h$ (0 < $h$ < 1) be a positive number, $n \geq 1$ be an integer, $I \subset \mathbb{T}$ be an interval and $s \in I$ be a fixed point. Let $(p, q)$ be a conjugate pair such that $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$E\sup_{t \in I} Q_n(t)^h \leq (E Q_n(s)^h)^{1/p} (E e^{h|I||S_n'|_\infty})^{1/q},$$  \hspace{1cm} (5.1)

$$E Q_n(s)^h)^{1/p} \leq \left[ e^{-\frac{h(1-h)p}{4} \sum_{k=1}^{n} \alpha_k^2} \right]^{1-h},$$  \hspace{1cm} (5.2)

$$E e^{h|I|q||S_n'|_\infty})^{1/q} \leq n^{1/q} \exp \left( \frac{h^2 q I^2}{4} \sum_{k=1}^{n} k^2 \alpha_k^2 \right).$$  \hspace{1cm} (5.3)

**Proof.** We write

$$Q_n(t) = Q_n(s) e^{S_n(t) - S_n(s)} = Q_n(s) e^{(t-s)S_n'(r)},$$

where $r$ is a point between $s$ and $t$, and

$$S_n'(r) = \sum_{k=1}^{n} k \alpha_k \sin(kr + \omega_k).$$

It follows that

$$\sup_{t \in I} Q_n(t)^h \leq Q_n(s)^h e^{h|I||S_n'|_\infty}.$$  

Thus, by Hölder’s inequality, we obtain

$$E\sup_{t \in I} Q_n(t)^h \leq (E Q_n(s)^h)^{1/p} (E e^{h|I||S_n'|_\infty})^{1/q},$$

which is (5.1).

The first expectation on the right-hand side in the above inequality is easy to estimate by the independence and the local behavior of the Bessel function $I_0(\cdot)$ (see Lemma 3.1). Indeed,

$$E Q_n(s)^h = \prod_{k=1}^{n} \frac{I_0(h \alpha_k \omega_k)}{I_0(\alpha_k)^h} \ll e^{-\frac{h(1-h)p}{4} \sum_{k=1}^{n} \alpha_k^2}.$$  

Here, the assumption $\sum |\alpha_k|^4 < \infty$ is also used. Thus

$$E Q_n(s)^h)^{1/p} \ll e^{-\frac{h(1-h)p}{4} \sum_{k=1}^{n} \alpha_k^2} = \left[ e^{-\frac{h^2 q I^2}{4} \sum_{k=1}^{n} k^2 \alpha_k^2} \right]^{1-h},$$

which is (5.2).

Now let us prove the last desired estimate. The argument used below is inspired from [35, pp.68–69]. Assume $\|S_n'|_\infty = |S_n'(r)|$, where $r$ is a random point. By Bernstein’s inequality, there exists an interval $J$ of length $n^{-1}$ such that $|S'(\cdot)| \geq \frac{1}{2} |S'(r)|$. Thus,

$$e^{h|I|q||S_n'|_\infty} \leq \frac{1}{2 |J|} \int_J e^{h|I|q||S_n'(x)|} dx \leq \frac{n}{2} \int_T e^{h|I|q||S_n'(x)|} dx.$$
Take the expectation to obtain
\[ E\|I\|\|S_n\|_* \leq \frac{n}{2} \int_T E\|I\|\|S_n(x)\|_* dx \]
\[ \leq \frac{n}{2} \int_T (E\|I\|\|S_n(x)\|_* + E\|I\|\|S_n(x)\|_*^2) dx. \]

The last two expectations do not depend on \( x \), which can be easily and exactly computed. Thus
\[ E\|I\|\|S_n\|_* \leq n \prod_{k=1}^{n} I_0(\|I\|\|\alpha_k\|) \ll n \exp \left( \frac{\|I\|}{4} \sum_{k=1}^{n} k^2 \alpha_k^2 \right). \]

Take the \( q \)-th roots of both sides to obtain (5.3).

\section*{5.2 Degeneracy of \( Q_\alpha \)}

Let us apply Proposition 5.1 to treat the special case of \( Q_\alpha \).

\textbf{Theorem 5.2.} If \( \dim^* \sigma < \frac{\alpha^2}{4} \), then \( Q_\alpha \sigma = 0 \) a.s. In particular, the operator \( Q_\alpha \) is completely degenerate when \( |\alpha| > 2 \).

\textbf{Proof.} Since \( \alpha_n = \frac{\alpha}{\sqrt{n}} \), we have
\[ \sum_{k=1}^{n} \alpha_k^2 \sim \alpha^2 \log n, \quad \sum_{k=1}^{n} k^2 \alpha_k^2 \sim \frac{\alpha^2}{2} n^2. \]

Take a very small \( \epsilon > 0 \). For a given interval \( I \), choose \( n \) such that \( |I| \sim \frac{1}{\sqrt{n}+\epsilon} \). Then choose \( q = n^\epsilon \) so that \( |I|^2 n^q q = O(1) \). Also notice that \( n^{1/q} = n^{1/n^\epsilon} = O(1) \). So by Proposition 5.1, we obtain
\[ \mathbb{E} \sup_{t \in I} Q_n(t)^h \ll \left( \frac{1}{n} \right)^{\frac{\alpha^2}{4} \frac{h(1-h)p}{(1-h)(1+\epsilon)}} \ll |I|^{\frac{\alpha^2}{4} \frac{h(1-h)p}{(1-h)(1+\epsilon)}}(1-h). \]

Since \( \dim^* \sigma < \frac{\alpha^2}{4} \), for \( \epsilon > 0 \) small enough we have
\[ \dim^* \sigma < (1-\epsilon) \frac{\alpha^2}{4}. \]

Take \( 1 > h > 1 - \epsilon^2 \). Take \( p \) close to 1 (i.e., \( |I| \) is sufficiently small) so that
\[ \frac{h(1-h)p}{(1-h)(1+\epsilon)} \geq 1 - \epsilon. \]

We have proved that for any small interval \( I \) (i.e., \( |I| < \delta \) where \( \delta \) depends on \( \epsilon \) and \( p \)), there exists an integer \( n \) such that
\[ \mathbb{E} \sup_{t \in I} Q_n(t)^h \ll |I|^{\frac{\alpha^2}{4} (1-\epsilon)(1-h)}. \]

We conclude the first assertion by Theorem 2.3. The second assertion is a trivial consequence of the first one.

If \( \alpha_n = \frac{\alpha}{\sqrt{n}} \) with \( a_n \to +\infty \), the operator is completely degenerate.

\section{6 The image and the kernel of the projection \( EQ_\alpha \)}

Now we are going to show that if \( \sigma \) is rather regular in the sense of Riesz potential theory, then it is \( Q_\alpha \)-regular. The following theorem gives an exact statement. This result and Theorem 5.2 on the degeneracy will give us satisfactory descriptions of the image and the kernel of the projection \( EQ_\alpha \). It is then possible to compute the dimension of \( Q_\alpha \sigma \).
6.1 $Q_{\alpha}$-regular measures

**Theorem 6.1.** Suppose that $D > \frac{\alpha^2}{4}$. If $\sigma \in \mathcal{R}_D^*$, then $Q_\alpha \sigma \in \mathcal{R}_{D-\frac{\alpha^2}{4}}$.

**Proof.** We follow [34]. The method was also used in [23]. By the hypothesis on $\sigma$, for any $\epsilon > 0$ there exist a number $\eta > D$ and a compact set $K$ such that $\sigma(K^\eta) < \epsilon$ and

$$I_\eta(\sigma 1_K) = \int_{K^\eta} \int_K \frac{d\sigma(t)d\sigma(s)}{|t-s|^\eta} < \infty.$$

**Step I.** We first suppose that $D > \frac{\alpha^2}{2}$. By the $L^2$-theory (see Proposition 4.1), $E_Q \sigma_K = \sigma_K$, where $\sigma_K$ is the restriction $\sigma 1_K$ of $\sigma$ on $K$. The associated Peyri`ere measure $Q_{\alpha}$ is well defined. Denote by $R_n$ the chaotic operator defined by the weights $P_m$ with $m \geq e^n$ and consider the mass of $R_n \sigma_K$ concentrated in the ball $B(t, e^{-n})$ centered at $t$ of radius $e^{-n}$:

$$\rho_n(t) = R_n \sigma_K(B(t, e^{-n})),$$

which can be rewritten as

$$\rho_n(t) = \int \chi_n(t,s)dR_n\sigma(s),$$

where $\chi_n(t,s)$ is the characteristic function of the set of the points $(t,s) \in K \times K$ such that $|t-s| \leq e^{-n}$. We can then compute the $Q_{\alpha}$-expectation of $\rho_n$. Indeed, by the definition of $Q_{\alpha}$ and Proposition 4.1, we have

$$E_{Q_{\alpha}} \rho_n = \int \int \chi_n(t,s) \prod_{m=|e^n|}^{\infty} E_P(m)P_m(s)d\sigma_K(t)d\sigma_K(s).$$

Notice that

$$\sum_{m=1}^{[e^n]} E_P(m)^2 = \exp \left( \frac{\alpha^2}{2} \sum_{m=1}^{[e^n]} \frac{\cos m(t-s)}{m} + O(1) \right).$$

If $|t-s| \leq e^{-n}$, then

$$\sum_{m=1}^{[e^n]} E_P(m)^2 = \exp \left( \frac{\alpha^2}{2} n + O(1) \right).$$

Therefore,

$$E_{Q_{\alpha}} \rho_n \ll \int e^{-\frac{\alpha^2 n}{2} \frac{\sin \frac{t-s}{2}}{\alpha^2}} \chi_n(t,s) d\sigma_K(t)d\sigma_K(s).$$

If $\chi_n(t,s) = 1$, we have $e^n \leq 1/|t-s|$ so that

$$\sum_{n=1}^{\infty} e^{(\eta - \frac{\alpha^2}{2})n} \frac{\chi_n(t,s)}{|\sin \frac{t-s}{2}|^{\alpha^2}} \ll \frac{1}{|t-s|^\eta}.$$

It follows that

$$E_{Q_{\alpha}} \left( \sum_{n=1}^{\infty} e^{\eta n} \rho_n \right) \ll \frac{1}{\epsilon}.$$

and consequently $\rho_n = O(e^{-\eta n})$ $Q_{\alpha}$-a.s. So almost surely

$$\rho_n(t) = O(e^{-\eta n}) \quad Q_{\alpha}$-$\text{a.e.}$

Then for any $\epsilon > 0$, almost surely there exists a compact subset $K_\epsilon^1 \subset K$ such that

$$Q_\alpha \sigma(K \setminus K_\epsilon^1) < \epsilon.$$
and
\[
\limsup_{n \to \infty} \frac{\log \rho_n(t)}{n} \leq -\eta \quad \text{uniformly on } K^1_t.
\]

On the other hand, from Proposition 4.4 we can deduce that almost surely for every \(\epsilon > 0\), there exists a compact set \(K^2\) contained in \(K\) such that
\[
Q_\alpha \sigma(K \setminus K^2) < \epsilon
\]
and
\[
\frac{\log Q_n(t)}{\log n} \to \frac{\alpha^2}{4} \quad \text{uniformly on } K^2.
\]

Now let
\[
S_\epsilon = 1_{K^1 \cap K^2} Q \sigma.
\]
Then
\[
Q \sigma(K \setminus (K^1 \cap K^2)) < 2\epsilon
\]
and
\[
\limsup_{n \to \infty} \frac{\log S_\epsilon(B(t, e^{-n}))}{n} \leq \limsup_{n \to \infty} \frac{\log Q_{\epsilon n^\alpha}(t)}{n} + \limsup_{n \to \infty} \frac{\log \rho_n(t)}{n} \leq \frac{\alpha^2}{4} - \eta
\]
uniformly on \(K^1 \cap K^2\). This implies \(Q_\alpha \sigma \in M^+_{(D - \frac{\alpha^2}{4})^+}(T)\). Let us summarize as follows:
\[
D > \frac{\alpha^2}{4}, \quad \sigma \in R^*_D \Rightarrow Q_\alpha \sigma K \in R^*_{D - \frac{\alpha^2}{4}},
\]
where \(\sigma(K^c)\) can be made as small as we like.

**Step II.** The general case. We use the decomposition
\[
Q_\alpha = Q^{(m)}_{\sqrt{\frac{\alpha^2}{4}}} Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} \cdots Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} Q^\prime_{\frac{\alpha^2}{4}},
\]
which is presented in Subsection 4.3. First, apply the principle (6.1) to \(Q^{(m)}_{\sqrt{\frac{\alpha^2}{4}}}\) and \(\sigma\). We check that
\[
D > \frac{\alpha^2}{4} = \frac{(\alpha/\sqrt{2})^2}{2}
\]
and \(\sigma \in R^*_D\), and we obtain \(Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} \sigma \in R^*_{D - \frac{\alpha^2}{4}}\). Now observe that
\[
D - \frac{\alpha^2}{4} - \frac{\alpha^2}{2} > \frac{(\alpha / \sqrt{2})^2}{2}.
\]
So the principle (6.1) applies to \(Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}}\) and \(Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} \sigma\). Thus, we obtain
\[
Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} \sigma \in R^*_{D - \frac{\alpha^2}{4} - \frac{\alpha^2}{4}}.
\]
Inductively, we obtain
\[
Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} \cdots Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} Q^\prime_{\sqrt{\frac{\alpha^2}{2}}} \sigma \in R^*_{D - \frac{\alpha^2}{4} - \frac{\alpha^2}{4} - \cdots - \frac{\alpha^2}{4}},
\]
since \(D - \frac{\alpha^2}{4} > 0\). When \(m\) is sufficiently large, we have
\[
D - \frac{\alpha^2}{4} - \frac{\alpha^2}{4} - \cdots - \frac{\alpha^2}{4} - \frac{(\alpha/\sqrt{2})^2}{2} = \frac{\alpha^2}{2^{m+1}}.
\]
So we can apply once more the principle (6.1) to \(Q^{(m)}_{\sqrt{\frac{\alpha^2}{4}}}\). Finally, we obtain
\[
Q^{(m)}_{\sqrt{\frac{\alpha^2}{4}}} Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} \cdots Q^{\prime}_{\sqrt{\frac{\alpha^2}{2}}} Q^\prime_{\sqrt{\frac{\alpha^2}{2}}} \sigma \in R^*_{D - \frac{\alpha^2}{4} - \frac{\alpha^2}{4} - \cdots - \frac{\alpha^2}{4} - \frac{\alpha^2}{4}} = R^*_{D - \frac{\alpha^2}{4}}.
\]
This completes the proof.
6.2 Ker $EQ_\alpha$ and Im $EQ_\alpha$

We have the following descriptions for the kernel and the image of $EQ_\alpha$.

**Theorem 6.2.** We have

\[ S_{a^2/4} \subset \text{Ker} EQ_\alpha \subset S_{a^2/4}^+ \]

and

\[ R_{a^2/4} \subset \text{Im} EQ_\alpha \subset R_{a^2/4}^- . \]

**Proof.** The first inclusion follows immediately from Theorem 5.2. Now we prove the second inclusion by contradiction. Suppose $\sigma \in \text{Ker} EQ_\alpha$ but $\sigma \not\in S_{a^2/4}^-$. Then $\sigma \in S_\beta^-$ for some $\beta > a^2/4$. That means $\sigma$ has its $\beta$-regular component $\sigma_r \neq 0$ according to the Kahane decomposition. Hence, by Theorem 6.1,

\[ \dim_* Q_\sigma \sigma = \dim_* Q_\sigma \sigma_r \geq \beta - \frac{a^2}{4} > 0 \text{ a.s.}, \]

which contradicts $\sigma \in \text{Ker} EQ_\alpha$.

Assume $\sigma \in R_{a^2/4}^\alpha$. That means $\sigma \in R_\beta^\alpha$ for some $\beta > a^2/4$. We claim that $\sigma \in \text{Im} EQ_\alpha$. Otherwise, there is a non-null component of $\sigma$, i.e., $\sigma_1 \in \text{Ker}(K) > 0$, which is killed by $EQ_\alpha$. That is to say $\sigma_1 \in \text{Ker} EQ_\alpha$. So $\sigma_1 \in S_{a^2/4}^+$, by what we have just proved above. Hence $\text{Cap}_\beta(K) = 0$. This contradicts $\sigma_1 \in \text{R}_\beta$.

The last inclusion is also proved by contradiction. Assume $\sigma \in EQ_\alpha$ but $\sigma \not\in R_{a^2/4}^-$. Then $\sigma \not\in R_\beta$ for some $\beta < a^2/4$. By the Kahane decomposition, $\sigma$ has a non-null $\beta$-singular component which must be killed by $EQ_\alpha$ because $\beta < a^2/4$ (see Theorem 5.2). This contradicts $\sigma \in EQ_\alpha$. \qed

Recall that $\dim \sigma = D$ means $\dim_* \sigma = \dim^* \sigma = D$. In this case, we say that $\sigma$ is unidimensional. The dimension of $Q_\sigma$ is given by the following formula.

**Theorem 6.3.** Suppose that $\sigma$ is a unidimensional measure and $\dim \sigma > a^2/4$. We have

\[ \dim Q_\sigma = \dim \sigma - \frac{a^2}{4}. \]

**Proof.** Let $D = \dim \sigma$. For any $\epsilon > 0$, the fact $\dim_* \sigma = D$ implies $\sigma \in R_{D-\epsilon}^\alpha$. Then by Theorem 6.1, we obtain $\dim_* Q_\sigma \sigma \geq D - \epsilon - \frac{a^2}{4}$ a.s. This proves $\dim_* Q_\sigma \sigma \geq D - \frac{a^2}{4}$ a.s.

Take $\beta$ such that $\frac{\beta^2}{4} + \frac{\alpha^2}{4} > D$. Let $\gamma$ be such that $\gamma^2 = \beta^2 + \alpha^2$. We consider an operator $Q_\beta$, which is independent of $Q_\alpha$. The product $Q_\beta Q_\alpha$ has the same law as $Q_\gamma$. The fact $\dim^* \sigma = D$, together with $\frac{\beta^2}{4} + \frac{\alpha^2}{4} > D$, implies $\sigma \in \text{Ker} EQ_\gamma$. Then by Theorem 2.6(c), we obtain $Q_\sigma \sigma \in \text{Ker} EQ_\beta$ a.s. Now we apply Theorem 6.2 to obtain $\dim^* Q_\sigma \sigma \leq \beta$ a.s. Optimizing $\beta$ leads to $\dim^* Q_\sigma \sigma \leq D - \frac{a^2}{4}$ a.s. \qed

7 Trigonometric multiplicative chaos on $\mathbb{T}^d$

Our theory can be extended to torus $\mathbb{T}^d$ of all dimensions with few points to be checked in the higher-dimensional case. On $\mathbb{T}^d$ we propose studying the random series

\[ \sum_{n \in \mathbb{Z}_d^d} \rho_n \cos(n \cdot x + \omega_n) , \]  

where $n \cdot x = \sum_{j=1}^d n_j x_j$ denotes the inner product, $\mathbb{Z}_d^d$ is the set $\mathbb{Z}^d \setminus \{0\}$ but with $n$ and $-n$ identified, and $\{\omega_n\}$ are i.i.d. random variables uniformly distributed on $\mathbb{T}$ (not on $\mathbb{T}^d$). It can be rewritten as

\[ \frac{1}{2} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \rho_n e^{i(n \cdot x + \omega_n)} \]
with \( \rho_n = \rho_{-n} \) and \( \omega_n = \omega_{-n} \). When \( d = 1 \), \( \mathbb{Z}^d_+ \) is the set of natural numbers. The correlation function of (7.1) is defined by

\[
H(t) = \frac{1}{2} \sum_{n \in \mathbb{Z}^d_+} \rho_n^2 \cos(n \cdot t), \quad t \in \mathbb{T}^d.
\]

Let \( |x|_\infty = \max_{1 \leq i \leq d} |x_i| \) for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \). Let \( |x| \) denote the Euclidean norm \( \sqrt{\sum_{i=1}^{d} x_i^2} \).

### 7.1 The definition of \( T^d \)-martingales

Let \( \{\alpha_k\}_{k \in \mathbb{Z}^d_+} \) be a family of real numbers such that \( \sum_{k \in \mathbb{Z}^d_+} |\alpha_k|^4 < \infty \). Let \( \{A_n\} \) be an increasing sequence of finite subsets of \( \mathbb{Z}^d_+ \) such that \( A_0 = \emptyset \). We define the weights

\[
P_n(t) = \frac{\exp \sum_{k \in A_n \setminus A_{n-1}} \alpha_k \cos(k \cdot t + \omega_k)}{\prod_{k \in A_n \setminus A_{n-1}} I_0(\alpha_k)}. \tag{7.2}
\]

Here is a choice for \( \{A_n\} \). For \( n \geq 1 \), we consider the hypercube \( C_n = \{k \in \mathbb{Z}^d : |k|_\infty \leq n\} \) and choose \( A_n \) to be the positive part \( C_n^+ = C_n \cap \mathbb{Z}^d_+ \). In this case, \( C_n^+ \setminus C_{n-1}^+ \) is the boundary \( \partial C_n^+ \). We have \( \#C_n = (2n + 1)^d - 1 \). It follows that

\[
\#C_n^+ = \frac{(2n + 1)^d - 1}{2}, \quad \#\partial C_n^+ \sim d2^{d-1}n^{d-1}.
\]

Another choice for \( A_n \) is the ball \( \{k \in \mathbb{Z}^d_+ : |k| \leq n\} \).

The \( T^d \)-martingale defined by the weight (7.2) produces a chaotic operator \( Q \). The same computation as in the case of dimension 1 leads to the basic relation

\[
EQ_\alpha(t)Q_\alpha(s) = \exp \left( \frac{1}{2} \sum_{k \in A_n} \alpha_k^2 \cos k \cdot (t - s) + O(1) \right). \tag{7.3}
\]

For a given real number \( \alpha \), we have the typical sequence of coefficients

\[
\alpha_k = \frac{\alpha}{|k|^{d/2}}.
\]

As we see below (see Theorem 7.1), the corresponding correlation function \( H_\alpha(t) \) is smooth in \( T^d \setminus \{0\} \) and around 0 it behaves as

\[
H_\alpha(t) = \frac{\alpha^2}{2} \sum_{k \in \mathbb{Z}^d_+} \frac{\cos k \cdot t}{|k|^d} \sim \frac{\alpha^2}{2} \cdot \frac{\pi^{d/2}}{\Gamma(d/2)} \log \frac{1}{|t|}, \quad t \to 0.
\]

Its exponentiation \( \Phi_\alpha = e^{H_\alpha(t)} \) is then a Riesz kernel

\[
\Phi_\alpha(t) = \frac{1}{|t|^{\tau(\alpha)\pi/d/2}} \quad \text{with} \quad \tau(\alpha) = \frac{\pi^{d/2}}{\Gamma(d/2)}. \tag{7.4}
\]

### 7.2 The correlation function \( H_\alpha \)

We consider the Jacobi function \( G \) defined by the trigonometric series

\[
G(x) = \sum_{k \in \mathbb{Z}^d, k \neq 0} |k|^{-d} \exp(ik \cdot x). \tag{7.5}
\]

For \( m \geq 1 \), we denote by \( S_m \) its partial sum:

\[
S_m(G) = \sum_{|k| \leq m, k \neq 0} |k|^{-d} \exp(ik \cdot x).
\]

We set \( s_d = 2 \pi^{d/2} / \Gamma(d/2) \), which is the area of the unit sphere in \( \mathbb{R}^d \).
Theorem 7.1. The Jacobi function $G$ has the following properties:

(a) The function $G(x)$ is $C^\infty$ in $\mathbb{T}^d \setminus \{0\}$.

(b) There is a bounded function $E(x)$ on $\mathbb{T}^d$ such that

\[ \forall x \in \mathbb{T}^d, \quad G(x) = s_d \log \frac{1}{|x|} + E(x). \]

(c) There exists a constant $C$ such that for any integer $m \geq 1$, we have

\[ S_m(G)(x) \leq s_d \log \frac{1}{|x|} + C. \]

Proof. The proof mimics the argument given by Titchmarsh and Riemann himself (see [58]) of the functional equation of the zeta function. One considers the Jacobi function $\theta(u,x)$ defined by

\[ \theta(u,x) = \sum_{n \in \mathbb{Z}^d} \exp(-|n|^2 u) \exp(i n \cdot x). \]

Here, $u > 0$ and $x \in \mathbb{T}^d$. The notations used here slightly differ from the standard ones. Let us consider $\theta_0 = \theta - 1$. Then the function $\theta_0(u,x)$ has an exponential decay as $u$ tends to $+\infty$ and this decay is uniform in $x$. Assume $s > d$. We introduce the auxiliary Jacobi function $G(s,x)$ defined by

\[ G(s,x) = \sum_{n \in \mathbb{Z}^d, n \neq 0} |n|^{-s} \exp(i n \cdot x), \]

where the series converges absolutely. Fubini’s theorem implies

\[ \int_0^{\infty} \theta_0(u,x) u^{s/2} \frac{du}{u} = \Gamma(s/2)G(s,x). \]

One splits this integral into $\int_0^1 + \int_1^{\infty}$ which yields

\[ \Gamma(s/2)G(s,x) = A(s,x) + B(s,x). \]

The function $B(s,\cdot)$ is obviously a $C^\infty$ function for every $s \geq 0$, especially $B(d,x)$. To study $A(s,x)$, one rewrites $\theta_0 = \theta - 1$ and uses the following functional equation for the Jacobi function:

\[ \theta(u,x) = \frac{\pi d/2}{u^{d/2}} \sum_{k \in \mathbb{Z}^d} \exp \left( - \frac{|x - 2\pi k|^2}{4u} \right), \]

which follows from the Poisson summation formula. One can ignore $1$ in $\theta_0 = \theta - 1$ since its contribution to $A(s,x)$ yields a constant. When $x$ belongs to the ball centered at $0$ with radius $1$, all the terms together in the above sum but the term $k = 0$ yield a $C^\infty$ contribution. Therefore, one is led to

\[ J(s,x) := \frac{\pi d/2}{u^{d/2}} \int_0^1 \exp \left( - \frac{|x|^2}{4u} \right) u^{s-d} \frac{du}{u}. \]

We have thus proved the following lemma.

Lemma 7.2. The difference

\[ \Gamma(s/2)G(s,x) - J(s,x) = R(s,x) \]  \hspace{1cm} (7.6)

is a $C^\infty$ function for every $s \geq 0$.

Therefore, one can pass to the limit $s \to d$ in (7.6) and we are led to the computation of $J(d,x)$. Making a change of variables $t = \frac{|x|^2}{4u}$, we obtain

\[ J(d,x) = \frac{\pi d/2}{u^{d/2}} \int_0^{|x|^2} e^{-t} \frac{dt}{t} = 2\pi^{d/2} \log \frac{1}{|x|} + O(1), \quad x \to 0. \]

We have thus proved (a) and (b).

(c) will follow from (a), (b) and the following lemma.
Lemma 7.3. If $\phi$ denotes a non-negative compactly supported radial function of integral 1 and if $\phi_m(x) = m^d \phi(mx)$, we have
$$\|G * \phi_m - S_m(G)\|_\infty \leq C.$$ 

The proof of this lemma is immediate. We have
$$\left| \hat{\phi} \left( \frac{y}{m} \right) - 1 \right| \leq C \frac{|y|^2}{m^2}, \quad \text{if } |y| \leq m; \quad \left| \hat{\phi} \left( \frac{y}{m} \right) \right| \leq C \frac{m^2}{|y|^2}, \quad \text{if } |y| \geq m.$$ 

This together with the decay of the Fourier coefficients of $G$ yields the result.

Now we prove (c). By (a) and (b), the function $G$ can be forgotten and replaced by $f(x) = s_d \log |x|$, because the question arises around 0. It suffices to check $f * \phi_m \leq f + C$, which is an easy calculation. 

It is easy to deduce that the partial sums over cubes $C_n$ are also bounded by $s_d \log(1/|x|) + C$.

7.3 Riesz potential theory on $\mathbb{T}^d$ and $L^2$-theory

Let $\beta \in (0, d)$ be a fixed number. It is known that (see [57, p. 256])
$$\sum_{m \in \mathbb{Z}^d, m \neq 0} \frac{1}{|m|^{d-\beta}} e^{im \cdot x} \sim \gamma_{\beta} \frac{1}{|x|^\beta} + b(x),$$
where $\gamma_{\beta} = 2^\beta \pi^{\beta-\frac{d}{2}} \frac{\Gamma(d-\beta)}{\Gamma(d)}$ and $b \in C^\infty(\mathbb{T}^d)$. This means that the function on the right, which is Lebesgue-integrable on $\mathbb{T}^d$, admits the series on the left as its Fourier series. The function $b$ is necessarily real-valued. Let $B = \max_x b(x)$. We define the $\beta$-order Riesz kernel by
$$R_{\beta}(x) = \gamma_{\beta} \frac{1}{|x|^\beta} + b(x) + B.$$

It is a positive, lower semicontinuous function having positive Fourier coefficients. We also have the following simple estimate:
$$R_{\beta}(x) \asymp \frac{1}{|x|^\beta}, \quad |x_1| \leq \pi, \ldots, |x_d| \leq \pi.$$

As in the case $d = 1$, the $\beta$-order energy $I_{\beta}^\sigma$ of a measure $\sigma$ on $\mathbb{T}^d$ can be defined. As shown in [41], we have the formula
$$I_{\beta}^\sigma = B|\hat{\sigma}(0)|^2 + \sum_{|m| > 0} |\hat{\sigma}(m)|^2 \frac{|m|^{d-\beta}}{|m|^{d-\beta}}.$$

More generally, we have the results on $\mathbb{T}^d$ as developed in Subsection 3.3 about the potential, the capacity and the dimension (see [20, 38, 41]).

By Theorem 7.1, we have
$$H_{\alpha}(t) = \frac{\alpha}{2} \cdot \frac{\pi^{d/2}}{\Gamma(d/2)} \log \frac{1}{|t|} + O(1), \quad \text{as } t \to 0.$$ 

Theorem 7.1 allows us to develop the $L^2$-theory for our $\mathbb{T}^d$-martingales, which was developed in Section 4 for the case $d = 1$. We do not restate the results here, but we are free to use them.

7.4 Degeneracy

Proposition 7.4. The result of Proposition 5.1 remains true when $d \geq 2$. It suffices to replace $\sum_{k=1}^n \alpha_k^2$ and $\sum_{k=1}^n k^2 \alpha_k^2$, respectively, by
$$\sum_{k \in A_n} \alpha_k^2 \quad \text{and} \quad \sum_{k \in A_n} |k|^2 \alpha_k^2.$$
Lemma 7.5. Let $S_n(t) = \sum_{k \in A_n} \alpha_k \cos(k \cdot t + \omega)$. Recall that $k \cdot t = \sum_{j=1}^d k_j t_j$. Now

$$|S_n(t) - S_n(s)| = \left| \sum_{j=1}^d \frac{\partial S_n}{\partial t_j}(t_j - s_j) \right| \leq |t - s| \sum_{j=1}^d \left| \frac{\partial S_n}{\partial t_j}(r) \right|,$$

where

$$\frac{\partial S_n}{\partial t_j}(r) = - \sum_{k \in A_n} k_j \alpha_k \sin(k \cdot t + \omega).$$

Let $I$ be a cube and $|I|$ be its side length. We have to estimate the following expectation:

$$\mathbb{E}\exp \left( h|I| \sum_{j=1}^d \left| \frac{\partial S_n}{\partial t_j} \right|_{\infty} \right),$$

which, by Hölder’s inequality, is bounded by

$$\left\{ \prod_{j=1}^d \mathbb{E}\exp \left( dh|I| \left| \frac{\partial S_n}{\partial t_j} \right|_{\infty} \right) \right\}^{1/d}.$$

Each of these expectations can be estimated as before. Finally, we obtain

$$\mathbb{E}\exp \left( h|I| \sum_{j=1}^d \left| \frac{\partial S_n}{\partial t_j} \right|_{\infty} \right) \ll n \exp \left( \frac{dh|I|^2}{4}(k^2 + \cdots + k_2^2)\alpha_k^2 \right).$$

This completes the proof.

Let us apply the result to the special case $\alpha_k = \frac{\alpha}{|k|^d}$. Let us take $A_n = C_n^\dagger$. We are led to estimate

$$\sum_{k \in C_n^\dagger} \alpha_k^2 = \sum_{k \in C_n^\dagger} \frac{\alpha^2}{|k|^d}, \quad \sum_{k \in C_n^\dagger} |k|^2 \alpha_k^2 = \sum_{k \in C_n^\dagger} \frac{\alpha^2}{|k|^{d-2}}.$$

The sum over $C_n$ doubles the sum over $C_n^\dagger$. The sum over $C_n$ and the sum over the ball $B(0,n)$ are comparable, because the cube $C_n$ is contained in the ball centered at 0 of radius $\sqrt{d}n$, and contains the ball of radius $n$.

Lemma 7.5. Let $B(0,R)$ be the ball of radius $R$ centered at 0. We have

$$\sum_{1 \leq |k| \leq R} \frac{1}{|k|^d} = s_d \log R + O(1), \quad \sum_{1 \leq |k| \leq R} \frac{1}{|k|^{d-2}} = s_d R^2 + O(R),$$

where $s_d = 2 \frac{\pi^{d/2}}{\Gamma(d/2)}$. Also we have

$$\int_{n \leq |x| \leq \sqrt{d}n} \frac{dx}{|x|^d} = O(1), \quad \int_{n \leq |x| \leq \sqrt{d}n} \frac{dx}{|x|^{d-2}} \approx n^2.$$

Proof. Assume $d \geq 2$. When $x$ is in the unit cube centered at $k$, we have $|k|^{-d} - |x|^{-d} = O(|k|^{-(d+1)})$.

Since $\sum |k|^{-(d+1)} < \infty$, we have

$$\sum_{1 \leq |k| \leq R} \frac{1}{|k|^d} = \int_{1 \leq |x| \leq R} \frac{dx}{|x|^d} + O(1).$$

The integral is equal to $s_d \int_1^R \frac{dx}{x} = s_d \log R$. The second equality can be proved in the same manner. We need the estimate $|k|^{-d+2} - |x|^{-d+2} = O(|k|^{-(d-1)})$ for $x$ in the unit cube centered at $k$. The estimates of two integrals are direct.
This lemma implies
\[
\sum_{k \in C^+} \alpha_k^2 = \sum_{k \in C^+} \frac{\alpha_k^2}{|k|^d} = \frac{\alpha^2}{2} s_d \log n + O(1),
\]
\[
\sum_{k \in C^+} |k|^2 \alpha_k^2 = \sum_{k \in C^+} \frac{\alpha_k^2}{|k|^{d-2}} = \frac{\alpha^2}{2} s_d n^2 + O(n).
\]

Now we are ready to state the following theorem which we can prove using the last two estimates and the same proof as that of Theorem 5.2.

**Theorem 7.6.** Let us consider the operator \( Q_\alpha \) on \( \mathbb{T}^d \) with \( d \geq 2 \). If \( \dim \sigma^* < \frac{\alpha^2}{4} \frac{s_d}{2} = \frac{\alpha^2}{4} \frac{\Gamma(d/2)}{\pi^{d/2}} \), then \( Q_\alpha \sigma = 0 \). Consequently, the operator \( Q_\alpha \) is completely degenerate when \( |\alpha|^2 > 4d \frac{\Gamma(d/2)}{\pi^{d/2}} \).

The above statement coincides with \( d = 1 \) if we define \( s_1 = 2 \). So the degeneracy condition \( |\alpha|^2 > 4d \frac{\Gamma(d/2)}{\pi^{d/2}} \) holds for all \( d \geq 1 \).

### 7.5 Decompositions of \( H_\alpha \) and \( Q_\alpha \), the kernel and the image of \( EQ_\alpha \)

The following result was known for \( d = 1 \) (see Subsection 4.3). It allows us to decompose the operator \( Q_\alpha \) in a suitable way. Recall that the function \( G \) is defined by (7.5).

**Lemma 7.7.** There is a partition of \( \mathbb{Z}^d \setminus \{0\} \) into two parts \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) such that the decomposition \( G = G_1 + G_2 \) with
\[
G_1(x) = \sum_{k \in \mathcal{P}_1} |k|^{-d} \exp(ik \cdot x), \quad G_2(x) = \sum_{k \in \mathcal{P}_2} |k|^{-d} \exp(ik \cdot x)
\]
has the property
\[
G_1(x) = \frac{1}{2} G(x) + O(1), \quad G_2(x) = \frac{1}{2} G(x) + O(1), \quad \text{as } x \to 0.
\]
The partial sums, cubic or spheric, of both \( G_1 \) and \( G_2 \) are, respectively, bounded by \( G_1(x) + C \) or \( G_2(x) + C \) for some constant \( C \).

**Proof.** We first look at the case \( d = 2 \). Let
\[
A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix}.
\]

The linear map defined by \( A \) is a similitude such that \( |Ak| = 2|k| \) and preserves the lattice \( \mathbb{Z}^d \). We choose \( \mathcal{P}_1 = A(\mathbb{Z}^d \setminus \{0\}) \) and \( \mathcal{P}_2 = \mathbb{Z}^d \setminus \{0\} \setminus \mathcal{P}_1 \). Then
\[
G_1(x) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{-d} \exp(ik \cdot x) = \frac{1}{2} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{-d} \exp(ik \cdot A x).
\]

Now, by Theorem 7.1, around 0 we have
\[
G_1(x) = \frac{1}{2} G(A x) = \frac{1}{2} s_d \log \frac{1}{|A x|} + O(1) = \frac{1}{2} G(x) + O(1).
\]

Consequently,
\[
G_2(x) = G(x) - G_1(x) = \frac{1}{2} G(x) + O(1).
\]

When \( d \geq 3 \), we only need to replace the similitude \( A \) by
\[
B = \begin{pmatrix} A \\ I_{d-2} \end{pmatrix},
\]
where \( I_{d-2} \) is the \((d-2) \times (d-2)\) unit matrix. The boundedness of partial sums follows from Lemma 7.3. \( \square \)
Let $G^{(1)}_1 = G_1$ and $G^{(1)}_2 = G_2$. In the same way, we can continue to decompose
\[ G^{(1)}_1(x) = G^{(2)}_1 + G^{(2)}_2 \]
with
\[ G^{(2)}_1(x) = \frac{1}{2}G^{(1)}_1(Ax) = \frac{1}{2^2}G^{(2)}_1(A^2x) = \frac{1}{4} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{-d} \exp(ik \cdot A^2x). \]

In our application later, we do not decompose $G^{(1)}_2$ or $G^{(2)}_2$ further. But we need to decompose the first component several times. All these components $G^{(j)}_i$ have the same properties of $G$ as stated in Lemma 7.7.

What we are really interested in is $H_\alpha$, which is equal to $\frac{d}{2} G(x)$. The corresponding decomposition of $H_\alpha$ allows us to decompose the chaotic operator $Q_\alpha$ as follows:
\[ Q_\alpha = \tilde{Q} \frac{d}{2} \cdot \cdots \cdot Q \frac{d}{2} \cdot Q \frac{d}{2}, \]
where all the operators on the right are independent, the correlation function of $Q \frac{d}{2}$ has the same behavior as $H \frac{d}{2}$, and that of $\tilde{Q} \frac{d}{2}$ has the same behavior as $H \frac{d}{2}$.

The kernel $\Phi_\alpha = e^{H_\alpha}$ is the Riesz kernel of the order $\frac{\tau(d)\alpha^2}{4}$ (see (7.4)). If $\frac{\tau(d)\alpha^2}{2} < d$, the Lebesgue measure $\lambda$ is $Q_\alpha$-regular and the Peyrière measure $Q_\alpha$ associated with $Q_\alpha \lambda$ is well defined. It can be proved that
\[ Q_\alpha\text{-a.s.} \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \log P_n(t) = \frac{\tau(d)\alpha^2}{4}. \]
(7.7)
The proof is the same as that of (4.6).

Now we can state the following description for the measure $Q_\alpha \sigma$ when $\sigma$ is sufficiently regular. The proof is the same as that of Theorem 6.1, but the above preparations are necessary.

**Theorem 7.8.** Suppose that $D > \frac{\tau(d)\alpha^2}{4}$. If $\sigma \in \mathcal{R}_D$, then $Q_\alpha \sigma \in \mathcal{R}_D^{* \frac{\tau(d)\alpha^2}{4}}$.

Hence, following the same proof as that of Theorem 6.2, we can prove the following descriptions of the kernel and the image of the operator $E \lambda$.

**Theorem 7.9.** Let $\tau(d) = \frac{e^{d/2}}{\Gamma(d/2)}$. We have
\[ S_{\frac{\tau(d)\alpha^2}{4}} \subseteq \text{Ker}E \lambda \subseteq S_{\frac{\tau(d)\alpha^2}{4}}, \]
and
\[ \mathcal{R}_{\frac{\tau(d)\alpha^2}{4}} \subseteq \text{Im}E \lambda \subseteq \mathcal{R}_{\frac{\tau(d)\alpha^2}{4}}. \]

Let us finish by stating the counterpart of Theorem 6.3 in the higher-dimensional case.

**Theorem 7.10.** Suppose that $\sigma$ is a unidimensional measure on $\mathbb{R}^d$ and $\dim \sigma > \frac{\tau(d)\alpha^2}{4}$. We have
\[ \dim Q_\alpha \sigma = \dim \sigma - \frac{\tau(d)\alpha^2}{4}. \]

8 **Random trigonometric series**

8.1 **Study of $\sum_{n=1}^{\infty} \rho_n \cos(nt + \omega_n)$**

Now come back to our series
\[ \sum_{n=1}^{\infty} X_n(t) = \sum_{n=1}^{\infty} \rho_n \cos(nt + \omega_n), \]
(8.1)
which was proposed in Section 1. Suppose that $\sum \rho_n^2 = \infty$. This series diverges almost surely almost everywhere with respect to the Lebesgue measure. Indeed, for any $t$ fixed, the series diverges almost
surely ([35, Theorem 4, p. 31] applies). Then Fubini’s theorem allows us to conclude. The same argument shows that for any given measure \( \sigma \) on \( \mathbb{T} \), the series diverges almost surely \( \sigma \)-almost everywhere. When \( \rho_n = \frac{1}{\sqrt{n}} \), almost surely the series diverges everywhere (see [35, p. 108]). In the following, we investigate how the series diverges. As we shall see, we can obtain speeds of divergence for points in the supports of our chaotic measures \( Q_\alpha \). Similar results also hold for \( Q_\alpha \sigma \) with \( \sigma \neq \lambda \), but we do not state them.

Let \( \{ \alpha_n \} \) be a sequence of real numbers which determines a chaotic operator \( Q \). Suppose that the Lebesgue measure \( \lambda \) is \( Q \)-regular. For example, \( \lambda \) is \( Q_\alpha \)-regular if \( |\alpha| < 2 \) (see Theorem 6.1). Consider the Peyrière measure \( Q \) associated with \( Q \lambda \). By Proposition 4.3, we have

\[
E_Q X_n(t) = \frac{1}{2} \rho_n \alpha_n + O(\rho_n |\alpha_n|^3),
\]
\[
E_Q X_n(t)^2 = \frac{1}{2} \rho_n^2 + O(\rho_n^2 |\alpha_n|^2).
\]

Remark that the variance of \( X_n \) is equivalent to \( \frac{1}{2} \rho_n^2 \) if \( \alpha_n \to 0 \). Notice that we have coefficients \( \rho_n \)'s in the series (8.1), but our chaotic measures are defined by the exponentiation of \( \alpha_n \cos(nt + \omega_n) \), where \( \alpha_n \)'s are not necessarily \( \rho_n \)'s.

**Theorem 8.1.** Suppose that \( \sum \rho_n^2 = \infty \) and \( \lambda \) is \( Q \)-regular, where \( Q \) is the chaotic operator defined by \( \{ \alpha_n \} \) with \( \alpha_n \to 0 \). Then almost surely \( Q \lambda \)-almost everywhere the series (8.1) diverges and

\[
\sum_{n=1}^{N} \rho_n \cos(nt + \omega_n) = \frac{1}{2} \sum_{n=1}^{N} \rho_n \alpha_n + O\left( \varphi\left( \sum_{n=1}^{N} \rho_n^2 \right) \right),
\]

where \( \varphi : [0, \infty) \to [0, \infty) \) is a function satisfying the condition

\[
\sum_{n=1}^{\infty} \frac{\rho_n^2}{\varphi\left( \sum_{n=1}^{N} \rho_n^2 \right)^2} < \infty.
\]

**Proof.** The divergence is already discussed at the beginning of this section, because \( \{ nt + \omega_n \} \) are \( Q \)-independent. The proof for the estimation is the same as that of Proposition 4.4. Instead we consider the series

\[
\sum_{n=1}^{\infty} X_n(t) - E_{Q} X_n(t),
\]

the partial sums of which form an \( L^2 \)-bounded martingale. This boundedness is checked by the fact that the variance of \( X_n(t) \) is of size \( \rho_n^2 \).

We can improve the above result by a law of iterated logarithm as a special case of the law of iterated logarithm obtained by Wittmann [60].

**Theorem 8.2.** Suppose that \( \sum \rho_n^2 = \infty \) and \( \lambda \) is \( Q \)-regular, where \( Q \) is the chaotic operator defined by \( \{ \alpha_n \} \) with \( \alpha_n \to 0 \). Then almost surely \( Q \lambda \)-almost everywhere we have

\[
\limsup_{n \to \infty} \frac{\sum_{n=1}^{N} \rho_n \cos(nt + \omega_n) - \frac{1}{2} \alpha_n}{\sqrt{\sum_{n=1}^{N} \rho_n^2 \log \log \sum_{n=1}^{N} \rho_n^2}} = 1,
\]

if the following condition is satisfied for some \( p \) (\( 2 < p \leq 3 \)):

\[
\sum_{n=1}^{\infty} \frac{\rho_n^p}{\left( \sqrt{\sum_{n=1}^{N} \rho_n^2 \log \log \sum_{n=1}^{N} \rho_n^2} \right)^p} < \infty. \tag{8.2}
\]

**Proof.** It is a special case of the following law of iterated logarithm due to Wittmann [60, Theorem 1.2]. Let \( \{ Z_n \} \) be a sequence of independent real random variables such that \( E Z_n = 0 \) and \( E Z_n^2 < \infty \). Let

\[
S_n = \sum_{k=1}^{n} Z_k, \quad s_n = \sqrt{E S_n^2}, \quad t_n = \sqrt{2 \log \log s_n^2}.
\]
Suppose that
\[
\lim s_n = \infty, \quad s_{n+1} = O(s_n), \quad \sum_{n=1}^{\infty} \frac{|E[Z_n]|^p}{(s_n t_n)^p} < \infty, \quad \exists \ p \ (2 < p \leq 3).
\] (8.3)

Then almost surely we have \( \lim \sup_{n \to \infty} \frac{S_n}{s_n} = 1 \). We can apply this result to \( Z_n = X_n(t) - E_Q X_n(t) \), where \( X_n(t) = \rho_n \cos(nt + \omega_n) \), to obtain the announced result. Indeed, we have

\[
s_n^2 \sim \frac{1}{2} \sum_{k=1}^{n} \rho_k^2.
\]

So, since \( |Z_n| \leq 2\rho_n \), the convergence of the series in (8.3) is ensured by the condition (8.2). The assumption \( \sum \rho_n^2 = \infty \) is nothing but \( \lim_{n \to \infty} s_n = \infty \). The condition \( s_{n+1} = O(s_n) \) in (8.3) is also satisfied because \( \rho_n \to 0 \).

In Theorems 8.1 and 8.2, we can consider the operator \( Q \) as well as \( Q' \) associated with \( \{\eta_n, \alpha_n\} \), where \( \{\eta_n\} \) is a sequence of +1 or −1 and then consider the measures \( Q_\lambda \) and \( Q' \lambda \). Both operators \( Q \) and \( Q' \) have the same correlation function. Both measures \( Q' \lambda \) and \( Q_\lambda \) share the same properties, i.e.,

\[
\dim Q' \lambda = \dim Q_\lambda = 1 - \frac{\alpha^2}{4}, \quad |\alpha| < 2
\]

in the case \( \alpha_n = \frac{\alpha}{\sqrt{n}} \). But they are usually mutually singular (see Proposition 4.5):

\[
a.s. \quad Q' \lambda \perp Q_\lambda \iff \sum_{n=1}^{\infty} |1 - \eta_n|^2 |\alpha_n|^2 = \infty.
\]

By choosing different sequences \( \{\alpha_n\} \) and \( \{\eta_n\} \), we can obtain points with different properties of the series (8.1), even when \( \{\alpha_n\} \) is fixed. Through all these chaotic measures, we see that the partial sums of the series (8.1) are very multifractal.

### 8.2 Special series \( \sum_{n=1}^{\infty} \frac{\cos(nt + \omega_n)}{n^r} \) (0 < \( r \leq \frac{1}{2} \))

Let us look at the special series

\[
\sum_{n=1}^{\infty} \frac{\cos(nt + \omega_n)}{n^r}, \quad 0 < \ r \leq \frac{1}{2}
\] (8.4)

First assume \( r = \frac{1}{2} \). The variables \( nt + \omega_n \) are i.i.d. with respect to \( \lambda \otimes P \) and we have

\[
E_{\lambda \otimes P} \cos(nt + \omega_n) = 0, \quad E_{\lambda \otimes P} \cos^2(nt + \omega_n) = \int \cos^2 x \, dx = \frac{1}{2}.
\]

Then by the classical law of iterated logarithm, a.s. for \( \lambda \)-almost all \( t \) we have

\[
\lim_{N \to \infty} \sup_{n=1}^{N} \frac{\sum_{n=1}^{N} \cos(nt + \omega_n)}{\sqrt{N \log \log N}} = 1.
\]

By the law of iterated logarithm in [60], we have a.s. Lebesgue-almost everywhere

\[
\lim_{N \to \infty} \sup_{n=1}^{N} \frac{\sum_{n=1}^{N} \cos(nt + \omega_n)}{\sqrt{\log N \log \log \log N}} = 1.
\]

More generally, if \( |\alpha| < 2 \), by Theorem 8.2, we have a.s. \( Q_\alpha \lambda \)-almost everywhere

\[
\lim_{N \to \infty} \sup_{n=1}^{N} \frac{\sum_{n=1}^{N} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} \cdot \cos(nt + \omega_n) - \frac{\alpha}{n} \log N}{\sqrt{\log N \log \log \log N}} = 1.
\] (8.5)
In particular, we have a.s. $Q_\alpha\lambda$-almost everywhere

$$\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \cos(n t + \omega_n) = \frac{\alpha}{2}. \quad (8.6)$$

Now assume $0 \leq r < \frac{1}{2}$. Since

$$\sum_{k=1}^{n} k^{-1/2-r} = \frac{n^{1/2-r}}{1/2 - r} + O(1)$$

and

$$\sum_{k=1}^{n} k^{-2r} = \frac{n^{1-2r}}{1 - 2r} + O(1),$$

if $|\alpha| < 2$, by Theorem 8.2, we have a.s. $Q_\alpha\lambda$-almost everywhere

$$\limsup_{N \to \infty} \frac{\sum_{n=1}^{N} \frac{1}{\pi} \cos(n t + \omega_n) - \frac{\alpha}{r} N^{1/2-r}}{\sqrt{N^{1-2r} \log \log N}} = \sqrt{1 - 2r}.$$  

Since $N^{1/2-r} = O(\sqrt{N^{1-2r} \log \log N})$, we have a.s. $Q_\alpha\lambda$-almost everywhere

$$\limsup_{N \to \infty} \frac{\sum_{n=1}^{N} \frac{1}{\pi} \cos(n t + \omega_n)}{\sqrt{N^{1-2r} \log \log N}} = \sqrt{1 - 2r}. \quad (8.7)$$

It is well known that the series $\sum_{n=1}^{\infty} \frac{\sin nt}{\sqrt{n}}$ converges everywhere, but not uniformly, to a Lebesgue integrable function, and the convergence is uniform on any interval $[\delta, 1 - \delta]$ (0 < $\delta$ < 1/2) (see [61, Volume I, Chapter V] and [8, p. 87]). But the randomized series $\sum_{n=1}^{\infty} \frac{\sin(nt + \omega_n)}{\sqrt{n}}$ loses all these properties almost surely.

### 8.3 The large deviation of $\sum_{n=1}^{N} \frac{\cos(nt + \omega_n)}{\sqrt{n}}$

Suppose $|\alpha| < 2$. By (8.5) which is a consequence of Theorem 8.1, we have

$Q_\alpha$-a.s. $\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{\cos(n t + \omega_n)}{\sqrt{n}} = \frac{\alpha}{2}.$

We have the following stronger result, a large deviation result.

**Theorem 8.3.** Suppose $|\alpha| < 2$. For any $\eta > 0$, we have

$$\lim_{N \to \infty} \frac{1}{\log N} \log Q_\alpha \left\{ (t, \omega) : \frac{1}{\log N} \sum_{n=1}^{N} \frac{\cos(nt + \omega_n)}{\sqrt{n}} \notin \frac{\alpha}{2} + [-\eta, \eta] \right\} = -\eta^2.$$

**Proof.** Let $W_n = \sum_{k=1}^{n} \frac{\cos(k t + \omega_k)}{\sqrt{k}}$, the random variable in question and $a_n = \log n$, the chosen normalizer. We shall prove that the following limit exits:

$$c(\beta) := \lim_{n \to \infty} \frac{1}{a_n \log \log Q_\alpha} \log E_{Q_\alpha} e^{\beta W_n} = \frac{(\beta + \alpha)^2 - \alpha^2}{4}, \quad \forall \beta \in \mathbb{R},$$

which is called the free energy function of $(W_n)$ with the weight $(a_n)$ with respect to the probability measure $Q_\alpha$. By the large deviation theorem (see [13, p. 230]), for any nonempty interval $K \subset \mathbb{R}$ we have

$$\lim_{n \to \infty} \frac{1}{a_n} \log Q_\alpha \{ a_n^{-1} W_n \notin K \} = -\inf_{\gamma \in K} c^*(\gamma),$$

where $c^*(\gamma) = \sup_{\beta} (\gamma \beta - c(\beta))$ is the Legendre transform of $c(\cdot)$. Then the announced result follows.
Indeed, the limit \( c(\beta) = \frac{\beta^2 + 2 \alpha \beta}{4} \) is easy to obtain, because
\[
\mathbb{E}_{Q_\alpha} e^{\beta W_n} = \prod_{k=1}^{n} \frac{I_0\left(\frac{\beta \sqrt{k}}{\sqrt{\alpha}}\right)}{I_0\left(\frac{\alpha \sqrt{k}}{\sqrt{k}}\right)} \approx \exp\left(\frac{\beta^2 + 2 \alpha \beta}{4} \log n\right).
\]
It is also direct to obtain the Legendre transform \( c^*(\gamma) = (\gamma - \frac{\alpha}{2})^2 \). The convex function \( c^*(\cdot) \) attains its minimal value at \( \gamma = \frac{\alpha}{2} \). For \( K = \frac{\alpha}{2} + [-\eta, \eta] \), we have clearly
\[
\inf_{\gamma \in K} c^*(\gamma) = c^*(\alpha/2 + \eta) = \eta^2.
\]
This completes the proof. \( \square \)

The same method applies to other cases. Assume \( \rho_n \rightarrow 0 \) and \( \frac{1}{\sqrt{n}} = o(\rho_n) \). That is the case for \( \rho_n = \frac{1}{n^r} \) with \( 0 < r < \frac{1}{2} \). If we consider
\[
W_n = \sum_{k=1}^{n} \rho_n \cos(kt + \omega_k),
\]
we have
\[
\mathbb{E}_{Q_\alpha} e^{\beta W_n} = \prod_{k=1}^{n} \frac{I_0\left(\beta \rho_k + \frac{\alpha \sqrt{k}}{\sqrt{n}}\right)}{I_0\left(\frac{\alpha \sqrt{k}}{\sqrt{n}}\right)} \approx \exp\left(\frac{\beta^2}{4} \sum_{k=1}^{n} \rho_k^2\right).
\]
Take
\[
a_n = \sum_{k=1}^{n} \rho_k^2
\]
as the normalizer. Then we obtain that the free energy function of \((W_n)\) is equal to \( c(\beta) = \frac{\beta^2}{4} \). Its Legendre transform \( c^*(\gamma) = \gamma^2 \). Therefore,
\[
\lim_{N \rightarrow \infty} \frac{1}{\sum_{n=1}^{N} \rho_n^2} \log Q_\alpha\left\{ (t, \omega) : \sum_{n=1}^{N} \rho_n \cos(nt + \omega_n) \notin [-\eta, \eta] \right\} = -\eta^2.
\]

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