On the role of injection in kinetic approaches to non-linear particle acceleration at non-relativistic shock waves

P. Blasi,1⋆ S. Gabici2 and G. Vannoni2
1INAF/Osservatorio Astrofisico di Arcetri, Largo E. Fermi, 5 I-50125 Firenze, Italy
2Max-Planck-Institut für Kernphysik, Saupfercheckweg, 1 D-69117 Heidelberg, Germany

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ABSTRACT
The dynamical reaction of the particles accelerated at a shock front by the first-order Fermi process can be determined within kinetic models that account for both the hydrodynamics of the shocked fluid and the transport of the accelerated particles. These models predict the appearance of multiple solutions, all physically allowed. We discuss here the role of injection in selecting the real solution, in the framework of a simple phenomenological recipe, which is a variation of what is sometimes referred to as thermal leakage. In this context we show that multiple solutions basically disappear and when they are present they are limited to rather peculiar values of the parameters. We also provide a quantitative calculation of the efficiency of particle acceleration at cosmic ray modified shocks and we identify the fraction of energy which is advected downstream and that of particles escaping the system from upstream infinity at the maximum momentum. The consequences of efficient particle acceleration for shock heating are also discussed.

Key words: acceleration of particles – cosmic rays.

1 INTRODUCTION
Diffusive particle acceleration at non-relativistic shock fronts is an extensively studied phenomenon. Detailed discussions of the current status of the investigations can be found in some excellent reviews (Drury 1983; Blandford & Eichler 1987; Berezhko & Krimsky 1988; Jones & Ellison 1991; Malkov & Drury 2001). While much is by now well understood, some issues are still subjects of much debate, for the theoretical and phenomenological implications that they may have. One of the most important of these is the reaction of the accelerated particles on the shock: the violation of the test particle approximation occurs when the acceleration process becomes sufficiently efficient that the pressure of the accelerated particles is comparable with the incoming gas kinetic pressure. Both the spectrum of the particles and the structure of the shock are changed by this phenomenon, which is therefore intrinsically non-linear.

At present, there are three viable approaches to determine the reaction of the particles upon the shock. The first is based on the ever-improving numerical simulations (Jones & Ellison 1991; Bell 1987; Ellison, Möbius & Paschmann 1990; Ellison, Baring & Jones 1995, 1996; Kang & Jones 1997, 2005; Kang, Jones & Gieseler 2002) that allow one to achieve a self-consistent treatment of several effects.

The second approach is based on the so-called two-fluid model, and treats cosmic rays as a relativistic second fluid. This class of models was proposed and discussed in Drury & Völk (1980, 1981), Drury, Axford & Summers (1982), Axford, Leer & McKenzie (1982) and Duffy, Drury & Völk (1994). These models allow one to obtain the thermodynamics of the modified shocks, but do not provide information about the spectrum of accelerated particles.

The third approach is semi-analytical and may be very helpful to understand the physics of the non-linear effects in a way that sometimes is difficult to achieve through simulations, due to their intrinsic complexity and limitations in including very different spatial scales. Blandford (1980) proposed a perturbative approach in which the pressure of accelerated particles was treated as a small perturbation. By construction, this method provides the description of the reaction only for weakly modified shocks.

Alternative approaches were proposed by Eichler (1984, 1985) and Ellison & Eichler (1984, 1985), based on the assumption that the diffusion of the particles is sufficiently energy-dependent that different parts of the fluid are affected by particles with different energies. The way the calculations are carried out implies a sort of separate solution of the transport equation for subrelativistic and relativistic particles, so that the two spectra must be somehow connected at $p \sim mc$ a posteriori.

In Berezhko, Yelshin & Ksenofontov (1994), Berezhko, Ksenofontov & Yelshin (1995) and Berezhko (1996), the effects of the non-linear reaction of accelerated particles on the maximum
achieved for collisionless shocks, does not allow us to establish a clear and universal connection between the injection efficiency and the macroscopic shock properties. Putting aside the possibility to have a fully self-consistent picture of this phenomenon, one can try to achieve a phenomenological description of it. Kang et al. (2002) introduced a sort of weight function to determine a return probability of particles in the downstream fluid to the upstream fluid, as a function of particle momentum. Only sufficiently suprathermal particles can jump back to the upstream region and therefore take part in the acceleration process. Here we adopt an injection recipe which is similar to the thermal leakage model of Kang et al. (2002) (see also previous papers by Malkov 1998 and by Gieseler, Jones & Kang 2000) and implement it in the semi-analytical approach of Blasi (2002, 2004). We investigate then the phenomenon of multiple solutions and show that the injection recipe dramatically reduces the appearance of these situations. We also study in some detail the efficiency for particle acceleration as a function of the Mach number of the shock and the maximum momentum of the accelerated particles.

The paper is structured as follows. In Section 2 we briefly describe the method proposed by Blasi (2002) for the calculation of the spectrum and pressure of particles accelerated at a modified shock. We describe the appearance of multiple solutions in Section 3, and the comparison with the method of Malkov (1997) in Section 4. In Section 5 we introduce a recipe for the injection of particles from the thermal pool. This recipe is then used in Section 6 to show how the regions of parameter space where multiple solutions appear are drastically reduced by the self-regulated injection. In Section 7 we discuss the efficiency of particle acceleration at modified shocks, and stress the role of escape of particles from upstream infinity. The consequences of the cosmic ray modification on the shock heating are investigated in Section 8. We conclude in Section 9.

2 A KINETIC MODEL FOR PARTICLE ACCELERATION AT MODIFIED SHOCKS

In this section we describe the method proposed by Blasi (2002, 2004) for the calculation of the spectrum and pressure of the particles accelerated at a shock surface, when the reaction of the particles is taken into account. No seed particles are included here.

The equation that describes the diffusive transport of particles in one dimension is

$$\frac{\partial}{\partial x} \left[ D \frac{\partial f(x, p)}{\partial x} \right] - \frac{\partial f(x, p)}{\partial x} - \frac{1}{3} \frac{\partial f(x, p)}{\partial p} + Q(x, p) = 0, \quad (1)$$

where we have assumed stationarity ($\partial f/\partial t = 0$). The $x$-axis is oriented from upstream to downstream, as in Fig. 1. The pressure of the accelerated particles slows down the fluid upstream before it crosses the shock surface; therefore, the gas velocity at upstream...
infinity, \( u_0 \), is different from \( u_1 \), the fluid speed just upstream of the shock. The injection term is taken in the form \( Q(x, p) = Q_0(p) \delta(x) \).

As a first step, we integrate equation (1) around \( x = 0 \), from \( x = 0^{-} \) to \( x = 0^{+} \), denoted in Fig. 1 as points ‘1’ and ‘2’, respectively, so that the following equation can be written

\[
\left( \frac{D}{\partial x} \right) \left[ \frac{3}{2} p \frac{df_0}{dp} (u_2 - u_1) + Q_0(p) \right] = 0, \tag{2}
\]

where \( u_1(u_2) \) is the fluid speed immediately upstream (downstream) of the shock and \( f_0 \) is the particle distribution function at the shock location. By requiring that the distribution function downstream is independent of the spatial coordinate (homogeneity), we obtain \( \{D(\bar{\partial} f/\partial x)\}_2 = 0 \), so that the boundary condition at the shock can be re-written as

\[
\left( \frac{D}{\partial x} \right) \left[ \frac{3}{2} p \frac{df_0}{dp} (u_2 - u_1) + Q_0(p) \right] = 0. \tag{3}
\]

We can now perform the integration of equation (1) from \( x = -\infty \) to \( x = 0^{+} \) (point ‘1’), in order to take into account the boundary condition at upstream infinity. Using equation (3) we obtain

\[
\frac{1}{3} p \frac{df_0}{dp} (u_2 - u_1) - u_1 f_0 + Q_0(p) + \int_{-\infty}^{0^{+}} dx f(x, p) \frac{df}{dx} = 0.
\tag{4}
\]

We introduce the quantity \( u_p \), defined as

\[
u_p = u_1 - \frac{1}{f_0} \int_{-\infty}^{0^{+}} dx \frac{df}{dx} f(x, p), \tag{5}\]

whose physical meaning is instrumental to understanding the non-linear reaction of particles. The function \( u_p \) is the average fluid velocity experienced by particles with momentum \( p \) while diffusing upstream away from the shock surface. In other words, the effect of the average is that, instead of a constant speed \( u_1 \) upstream, a particle with momentum \( p \) experiences a spatially variable speed, due to the pressure of the accelerated particles. Because the diffusion coefficient is in general \( p \)-dependent, particles with different energies feel a different compression coefficient, higher at higher energies if, as expected, the diffusion coefficient is an increasing function of momentum (see Blasi 2004 for further details on the meaning of the quantity \( u_p \)).

With the introduction of \( u_p \), equation (4) becomes

\[
\frac{1}{3} p \frac{df_0}{dp} (u_2 - u_1) - u_1 f_0 + Q_0(p) + \int_{-\infty}^{0^{+}} dx f(x, p) \frac{df}{dx} = 0. \tag{6}
\]

where we used the fact that

\[
\frac{d}{dp} \int_{-\infty}^{0^{+}} dx \frac{df}{dx} f = \left[ \frac{df_0}{dp} (u_1 - u_p) - f_0 \frac{du_p}{dp} \right].
\]

The solution of equation (6) for a monochromatic injection at momentum \( p_{0,ij} \) is

\[
f_0(p) = \int_{p_{0,ij}}^{p} \frac{3}{p} \frac{Q_0(p)}{u_p - u_2} \exp \left[ -\int_{p_{0,ij}}^{p} \frac{dp'}{p'} \frac{3}{p'} (u_p + \frac{1}{3} p \frac{du_p}{dp'}) \right] = \frac{3 R_{sub}}{R_{sub} - 4 \pi p_{0,ij}^2} \eta_n \frac{d}{dp} \left[ \int_{p_{0,ij}}^{p} \frac{dp'}{p'} \frac{3}{p'} (u_p + \frac{1}{3} p \frac{du_p}{dp'}) \right]. \tag{7}
\]

Here we have used

\[
Q_0(p) = \frac{\eta_n g_3 u_1}{4\pi p_{0,ij}^2} \delta(p - p_{0}),
\]

where \( n_{g3,1} \) is the gas density immediately upstream \( (x = 0^{-}) \) and \( \eta \) is the fraction of particles crossing the shock which take part in the acceleration process.

Here we have introduced the two quantities \( R_{sub} = u_1/u_2 \) and \( R_{tot} = u_0/u_2 \), which are respectively the compression factor at the gas subshock and the total compression factor between upstream infinity and downstream. For a modified shock, \( R_{sub} \) can attain values much larger than \( R_{sub} \) and more in general, much larger than 4, which is the maximum value achievable for an ordinary strong non-relativistic shock. The increase of the total compression factor compared with the prediction for an ordinary shock is responsible for the peculiar flattening of the spectra of accelerated particles that represents a feature of non-linear effects in shock acceleration. In terms of \( R_{sub} \) and \( R_{tot} \), the density immediately upstream is \( n_{gas,1} = (\rho_0/\eta_0) R_{sub}/R_{tot} \).

In equation (7) we can introduce a dimensionless quantity \( U(p) = u_p/\eta_0 \) so that

\[
f_0(p) = \left( \frac{3 R_{sub}}{R_{tot} - 1} \right) \frac{\eta_n g_3 u_1}{4 \pi p_{0,ij}^2} \exp \left[ -\int_{p_{0,ij}}^{p} \frac{dp'}{p'} \frac{3}{p'} R_{tot} U(p') \right] \tag{8}
\]

The non-linearity of the problem reflects in the fact that \( U(p) \) is in turn a function of \( f_0 \) as it is clear from the definition of \( u_p \). In order to solve the problem, we need to write the equations for the thermodynamics of the system including the gas, the cosmic rays accelerated from the thermal pool and the shock itself.

The velocity, density and thermodynamic properties of the fluid can be determined by the mass and momentum conservation equations, with the inclusion of the pressure of the accelerated particles. We write these equations between a point far upstream \( (x = -\infty) \), where the fluid velocity is \( u_0 \) and the density is \( \rho_0 = \eta_0 n_{gas,0} \), and the point where the fluid velocity is \( u_{ps} \) (density \( \rho_{ps} \)). The index \( p \) denotes quantities measured at the point where the fluid velocity is \( u_{ps} \) (density \( \rho_{ps} \)). That is clearly an approximation, but as shown in Section 4 it provides a good agreement with other calculations where this approximation is not used).

The mass conservation implies

\[
\rho_{ps} u_{ps} = \rho_p u_p, \tag{9}
\]

Conservation of momentum reads

\[
\rho_{ps} u_{ps}^2 + P_{ps} = \rho_p u_p^2 + P_p + P_{CR,p}, \tag{10}
\]

where \( P_{ps} \) is the gas pressure at the point \( x = x_p \) and \( P_{CR,p} \) is the pressure of accelerated particles at the same point (we use the subscript ‘CR’ to mean cosmic rays, in the sense of accelerated particles). The mass and momentum escaping fluxes in the form of accelerated particles have reasonably been neglected. Note that at this point the equation for energy conservation has not been used.

Our basic assumption, similar to that used in Eichler (1984), is that the diffusion is \( p \)-dependent and more specifically that the diffusion coefficient \( D(p) \) is an increasing function of \( p \). Therefore, the typical distance that a particle with momentum \( p \) travels away from the shock is approximately \( \Delta x \sim D(p)/u_p \), larger for
high-energy particles than for lower-energy particles.¹ As a consequence, at each given point x, only particles with momentum larger than p are able to affect appreciably the fluid. Strictly speaking, the validity of the assumption depends on how strongly the diffusion coefficient depends on the momentum p (see Section 4).

Because only particles with momentum \( p \geq p_{\text{max}} \) can reach the point \( x = x_p \), we can write

\[
P_{\text{CR}, p} \approx \frac{4\pi}{3} \int_{p_{\text{max}}}^{\infty} dp \rho v(p) f_0(p),
\]

where \( v(p) \) is the velocity of particles with momentum \( p \), and \( p_{\text{max}} \) is the maximum momentum achievable in the specific situation under investigation.

From equation (10) we can see that there is a maximum distance, corresponding to the propagation of particles with momentum \( p_{\text{max}} \) such that at larger distances the fluid is unaffected by the accelerated particles and \( u_p = u_0 \).

The equation for momentum conservation is

\[
\frac{dU}{dp} \left[ 1 - \frac{1}{M_0^2} U^{-(\gamma_1+1)} \right] + \frac{1}{\rho_0 u_0^2} \frac{dP_{\text{CR}}}{dp} = 0.
\]

where \( \gamma_1 \) is the ratio of specific heats for the gas. Using the definition of \( P_{\text{CR}} \) and multiplying by \( p \), this equation becomes

\[
\frac{dU}{dp} \left[ 1 - \frac{1}{M_0^2} U^{-(\gamma_1+1)} \right] = \frac{4\pi}{3\rho_0 u_0^2} p^2 v(p) f_0(p),
\]

where \( f_0(p) \) is known once \( U(p) \) is known. Equation (13) is therefore an integral-differential non-linear equation for \( U(p) \). The solution of this equation also provides the spectrum of the accelerated particles.

The last missing piece is the connection between \( R_{\text{up}} \) and \( R_{\text{tot}} \), the two compression factors appearing in equation (8). The compression factor at the gas shock around \( x = 0 \) can be written in terms of the Mach number \( M_1 \) of the gas immediately upstream through the well-known expression

\[
R_{\text{up}} = \frac{(\gamma_1 + 1)M_1^2}{(\gamma_1 - 1)M_1^2 + 2}.
\]

On the other hand, if the upstream gas evolution is adiabatic, then the Mach number at \( x = 0^- \) can be written in terms of the Mach number of the fluid at upstream infinity \( M_2 \) as

\[
M_2 = M_0^2 \left( \frac{u_1}{u_0} \right)^{\gamma_1+1} = M_0^2 \left( \frac{R_{\text{up}}}{R_{\text{tot}}} \right)^{\gamma_1+1},
\]

so that from the expression for \( R_{\text{tot}} \) we obtain

\[
R_{\text{tot}} = M_0^{[2/(\gamma_1+1)]} \times \left[ (\gamma_1 + 1)R_{\text{up}}^\gamma - (\gamma_1 - 1)R_{\text{up}}^{\gamma+1} \right]^{[1/(\gamma_1+1)]}. \tag{15}
\]

Now that an expression between \( R_{\text{up}} \) and \( R_{\text{tot}} \) has been found, equation (13) basically is an equation for \( R_{\text{up}} \), with the boundary condition that \( U(p_{\text{max}}) = 1 \). Finding the value of \( R_{\text{up}} \) (and the corresponding value for \( R_{\text{tot}} \)) such that \( U(p_{\text{max}}) = 1 \) also provides the whole function \( U(p) \) and, through equation (8), the distribution function \( f_0(p) \). If the reaction of the accelerated particles is small, the test particle solution is recovered.

3 THE APPEARANCE OF MULTIPLE SOLUTIONS

In the problem described in the previous section there are several independent parameters. While the Mach number of the shock and the maximum momentum of the particles are fixed by the physical conditions in the environment, the injection momentum and the acceleration efficiency are free parameters. The procedure to be followed to determine the solution was defined in Blasi (2002): the basic problem is to find the value of \( R_{\text{up}} \) (and therefore of \( R_{\text{tot}} \)) for which \( U(p_{\text{max}}) = 1 \). In Fig. 2 we plot \( U(p_{\text{max}}) \) as a function of \( R_{\text{tot}} \), for \( T_0 = 10^9 K, \ p_{\text{max}} = 10^5 mc \) and \( p_{\text{inj}} = 10^{-2} mc \) in the left-hand panel and \( p_{\text{inj}} = 10^{-3} mc \) in the right-hand panel (\( m \) here is the mass of protons). The parameter \( \eta \) was taken to be \( 10^{-4} \) in the left-hand panel and \( 10^{-3} \) in the right-hand panel. The different curves refer to different choices of the Mach number at upstream infinity. The physical solutions are those corresponding to the intersection points with the horizontal line \( U(p_{\text{max}}) = 1 \), so that multiple solutions occur for those values of the parameters for which there is more than one intersection with \( U(p_{\text{max}}) = 1 \). These solutions are all physically acceptable, as far as the conservation of mass, momentum and energy are concerned.

It can be seen from both panels in Fig. 2 that for low values of the Mach number, only one solution is found. This solution may be significantly far from the quasi-linear solution. Indeed, for \( M_0 = 10 \)

¹ For the cases of interest, \( D(p) \) increases with \( p \) faster than \( u_p \), does; therefore, \( \Delta x \) is a monotonically increasing function of \( p \).
Injection in non-linear particle acceleration

4 COMPARISON WITH AN ALTERNATIVE APPROACH

Multiple solutions were also found by Malkov (1997) and Malkov et al. (2000), in the context of a semi-analytical kinetic approach. Aside from the technical differences between that method and that proposed by Blasi (2002, 2004), the main difference is in the fact that the former requires the knowledge of the exact expression for the diffusion coefficient as a function of the momentum of the particles, while the latter only requires that such a diffusion coefficient is an increasing function of the particle momentum. While the first approach may provide us with an exact solution to the problem, the second is in fact more practical, in the sense of providing an approximate solution even in those cases, the majority, in which no detailed information on the diffusion properties of the fluid is available. The solution provided in Blasi (2002, 2004) is particularly accurate when the diffusion coefficient is Bohm-like, $D(p) = (1/3) r_L c$, expected in the case of saturated self-generation of waves in the vicinity of the shock surface (Lagage & Cesarsky 1983).

We will now discuss the quantitative comparison between the results of Malkov (1997) and those of Blasi (2002, 2004), by considering a single situation in which multiple solutions are predicted (in both approaches), and determining the spectra and compression factors in both methods. We start with briefly summarizing the approach of Malkov (1997). The following flow potential is introduced there

$$\Psi = \int_0^x dx' u(x'),$$

which is used as a new independent spatial variable to replace $x$. Using the flow potential, it is possible to define an integral transformation of the flow profile as follows

$$\hat{U}(p) = \frac{1}{u_0} \int_0^\infty \exp \left[ -q(p) \frac{\Psi}{D(p)} \right] \frac{du}{dx},$$

where $q(p) = -d \ln f_0 / d \ln p$ is the spectral index of the particle distribution function and $D(p)$ is the diffusion coefficient, which is assumed to be independent of the position. An integral equation for $\hat{U}(p)$ can be derived by applying equation (17) to the $x$-derivative of the Euler equation (Malkov 1997; Malkov et al. 2000):

$$\hat{U}(p) = \frac{R_{\text{ab}} - 1}{R_{\text{tot}}} + \frac{v}{p_{\text{maj}}} \int_{p_{\text{maj}}}^{p_{\text{max}}} \hat{U}(p_{\text{maj}}) \left[ 1 + \frac{q(p) D(p)}{q(p) D(p)} \right]^{\gamma - 1} \frac{d \ln p}{\hat{U}(p)} \exp \left[ -\frac{3}{R R_{\text{ab}}} \int_{p_{\text{maj}}}^{p_{\text{max}}} d \ln p' \frac{\hat{U}(p')}{\hat{U}(p)} \right].$$

Here $v$ is an injection parameter defined as

$$v = \frac{4 \pi}{3} \frac{c}{\eta_0} \frac{p_{\text{maj}}^4}{\beta_0^2} f_0(p_{\text{maj}}),$$

and related to the compression factor by the following equation (Malkov 1997):

$$v = p_{\text{maj}} \left( 1 - \frac{1}{R} \right) \left\{ \int_{p_{\text{maj}}}^{p_{\text{max}}} dp \frac{p}{\sqrt{p^2 + (mc)^2}} \frac{\hat{U}(p_{\text{maj}})}{\hat{U}(p)} \exp \left[ -\frac{3}{R R_{\text{ab}}} \int_{p_{\text{maj}}}^{p_{\text{max}}} d \ln p' \frac{\hat{U}(p')}{\hat{U}(p)} \right] \right\}^{-1}. $$

Note however that even the approach of Malkov (1997) is based on several approximations: the solution is expanded to the first order and the contributions from gas pressure are ignored.
Here \( R = R_{\text{tot}}/R_{\text{sub}} \) is the compression factor in the shock precursor. Equations (15), (18) and (20) form a closed system that can be solved numerically.

Before showing the results, it is worth noticing that the injection parameters \( \eta \) and \( \nu \) adopted in the two approaches are defined in two non-equivalent ways. However, the relation between \( \nu = \frac{\hat{p}_{\text{inj}} c}{m_{\gamma} u_{\gamma}} \) can easily be found by using equations (8) and (19) and can be written as

\[
\nu = \eta \left( \frac{p_{\text{inj}} c}{m_{\gamma} u_{\gamma}} \right) \frac{R_{\text{tot}}}{R_{\text{sub}} - 1}.
\]

One may notice that this relation contains the compression factors \( R_{\text{tot}} \) and \( R_{\text{sub}} \), which are what we are searching for. This fact implies that three solutions characterized by the same value of \( \nu \) may correspond to three distinct values of \( \eta \).

In order to compare the results of the two different approaches, we consider a shock having Mach number \( M_0 = 150 \) and temperature at upstream infinity \( T_0 = 10^4 \) K. We set the value of the injection and maximum momenta equal to \( 10^{-3} mc \) and \( 10^5 mc \), respectively, and we adopt an efficiency \( \eta = 10^{-4} \). Using the approach proposed by Blasi (2002), we find three solutions, characterized by the values of the compression factor \( R \sim 15.3, 3.94 \) and \( 1.05 \). The last solution is the quasi-linear one, in which the precursor is very weak. We adopt now these three values for the precursor compression factor to solve the system of equations given by equations (15), (18) and (20) for different choices of the diffusion coefficient.

In Fig. 4 we plot the velocity profiles for the three solutions, as derived with the method of Blasi (2002, 2004) and detailed above (solid line). In the figure \( U(p) = u_{\gamma}/u_0 \) with \( u_0 \) defined through equation (5) for the method of Blasi (2002, 2004). It is easy to show that \( U(p) \) is related to \( \hat{U}(p) \) through the relation \( \hat{U}(p) + 1/R_{\text{tot}} = U(p) \). The dotted and dashed lines are the results obtained with the calculation of Malkov (1997) with a Bohm and Kolmogorov diffusion, respectively. For Bohm diffusion, the two approaches give very similar results. For Kolmogorov diffusion, the differences are larger, as expected.

\[ \text{Figure 4. Velocity profile upstream of the shock as derived in this paper (solid line) and with the approach of Malkov (1997) with a Bohm diffusion (dotted line) and for a Kolmogorov diffusion (dashed line).} \]

In Fig. 5 we plot the spectra of the accelerated particles, as obtained in this paper (solid line) and with the approach of Malkov (1997) with a Bohm diffusion (dotted line) and for a Kolmogorov diffusion (dashed line).

\[ \text{Figure 5. Spectrum of the accelerated particles as derived in this paper (solid line) and with the approach of Malkov (1997) with a Bohm diffusion (dotted line) and for a Kolmogorov diffusion (dashed line).} \]

The presence of multiple solutions is typical of many non-linear problems and should not be surprising from the mathematical point of view. In terms of physical understanding, however, multiple solutions may be disturbing. The typical situation that takes place in nature when multiple solutions appear in the description of other non-linear systems is that (at least) one of the solutions is unstable and the system falls in a stable solution when perturbed. The stable solutions are the only ones that are physically meaningful. Some attempts to investigate the stability of cosmic ray modified shock waves have been made by Mond & Drury (1998) and Toptygin (1999), but all of them refer to the two-fluid models. A step forward is being carried out by Blasi & Vietri (in preparation) in the context of kinetic models.

In addition to the stability, another issue that enters the physical description of our problem is the identification of possible processes that determine some type of backreaction on the system. It may be expected that when some types of processes of self-regulation are included, the phenomenon of multiple solutions is reduced. In this
section we investigate the type of reaction that takes place when a self-consistent, although simple, recipe for the injection of particles from the thermal pool is adopted. This recipe is similar to that proposed by Kang et al. (2002) in terms of the underlying physical interpretation of the injection, but probably simpler in its implementation.

For non-relativistic shocks, the distribution of particles downstream is quasi-isotropic, so that the flux of particles crossing the shock surface from downstream to upstream can be written as

$$\Phi = -2\pi \int_{p_{\text{min}}}^{\infty} dp \int_{-u_d/v(p)}^{u_d(v(p))} d\mu \frac{f_n(p)}{4\pi} 4\pi p^2 [u_d + v(p)\mu], \quad (22)$$

where $v(p)$ is the velocity of particles with momentum $p$ and $u_d$ is the shock speed in the frame comoving with the downstream fluid. The term $u_d + v(p)\mu$ is the component along the direction perpendicular to the shock surface of the velocity of particles with momentum $p$ moving in the direction $\mu$. It follows that the flux of particles moving tangent to the shock surface [namely with $\mu = -u_d/v(p)$] is zero. We recall that, having in mind collisionless shocks, the typical thickness of the shock, $\lambda$, is the collision length associated with the magnetic interactions that give rise to the formation of the discontinuity. It is useless to say that these interactions are all but well known, and at present the best we can do is to attempt a phenomenological approach to take them into account, without having to deal with their detailed physical understanding. It is however worth recalling that many attempts have been made to tackle the problem of injection at a more fundamental level (Malkov & Völk 1995, 1998; Malkov 1998). Here, we consider the reasonable situation in which $\lambda \propto r_{\text{L}}^3$, where $r_{\text{L}}^3 \propto p_0$ is the Larmor radius of the particles in the downstream fluid that carry most of the thermal energy, namely those with momentum $1.5p_0$ [$p_0 = (2m_{\text{p}}T_0)^{1/2}$] here is the momentum of the particles in the thermal peak of the Maxwellian distribution in the downstream plasma, having temperature $T_0$]. We stress here the important point that the temperature of the downstream gas (and therefore $p_0$) is determined by the shock strength, which in the presence of accelerated particles, is affected by the pressure of the non-thermal component. In particular, the higher the efficiency of the shock as a particle accelerator, the weaker its efficiency in terms of heating of the background plasma (see Section 8).

For collisionless shocks, it is not clear whether the downstream plasma can actually be thermalized and the distribution function be a Maxwellian. On the other hand, it is generally assumed that this is the case, so that in the following we consider the case in which the bulk of the background plasma is thermal and has a Maxwellian spectrum at temperature $T$ given by the generalized Rankine–Hugoniot relations in the presence of accelerated particles (see Section 2). For modified shocks, the points discussed above apply to the so-called subshock, where the injection of particles from the thermal pool is expected to take place. We recall that for strongly modified shocks the subshock is weak, and rather inefficient in the heating of the background plasma.

From equation (22) we obtain

$$\Phi = \frac{1}{4} \int_{p_{\text{min}}}^{\infty} dp 4\pi p^2 f_n(p) \frac{(v(p) - u_d)^2}{v(p)}, \quad (23)$$

where we have assumed that the temperature downstream implies non-relativistic motion of the quasi-thermal particles [$v \approx mv(p)$]. In equation (23) we write the minimum momentum in terms of a parameter $\alpha$, such that $\lambda = \alpha r_{\text{L}}^3$. With this formalism, the particles that can cross the shock surface are those that satisfy the condition:

$$p > p_{\text{min}} = 1.5\alpha p_0. \quad (24)$$

The parameter $\alpha$ defines the thickness of the shock in units of the gyration radius of the bulk of the thermal particles. Our recipe for injection is pictorially illustrated in Fig. 6: thermal particles have a path-length smaller than the shock thickness and cannot cross the shock surface, being advected downstream before the crossing occurs. Only particles with momentum sufficiently larger than the thermal momentum of the downstream particles can actually return upstream and be accelerated.

In the following we will neglect the fluid speed $u_d$ compared with $v(p)$, which is a good approximation if the injected particles are sufficiently more energetic than the thermal particles. This is done only to make the interpretation of the result simpler, but there is no technical difficulty in keeping the dependence of the results on $u_d$.

We introduce an effective injection momentum $p_{\text{inj}} = \xi p_0$ defined by the equation

$$\Phi = \int_{p_{\text{inj}}}^{\infty} dp 4\pi p^2 f_n(p)v(p), \quad (25)$$

which in terms of dimensionless quantities, with $f_n(p) = e^{-\beta p}/p_0^2$ reads

$$\int_{1.5\xi}^{\infty} dx x^3 e^{-x^2} = 4 \int_{1.5\xi}^{\infty} dx x^3 e^{-x^2}. \quad (26)$$

It is easy to show that $\xi \approx 2$ for $\alpha = 1$ (half a Larmor rotation of the particles with momentum $1.5p_0$ inside the thickness of the shock) and $\xi \approx 3.25$ for $\alpha = 2$ (one full Larmor rotation of the particles with momentum $1.5p_0$ inside the thickness of the shock). The fraction of particles at momentum $\xi$ times larger than the thermal one is $\sim 5$ per cent for $\xi = 2$ and $\sim 10^{-4}$ for $\xi = 3.25$. The actual values of $\xi$ are expected to be somewhat higher if the effect of advection with the downstream fluid is not neglected. The sharp decrease in the fraction of leaking particles that may take part in the acceleration process is due to the exponential behaviour of the Maxwellian at large momenta. Although the fraction of particles in the Maxwellian that become accelerated only depends on the parameter $\xi$, which in turn is expected to keep the information about the microscopic structure of the shock, the absolute number of and energy carried by these

![Figure 6. Graphic illustration of the structure of a collisionless shock and of the basic idea underlying our injection recipe.](https://academic.oup.com/mnras/article-abstract/361/3/907/972633)
particles depend on the temperature of the downstream gas, which is an output of our calculation. This simple argument serves as an explanation of the physical reason why there is a non-linear reaction on the system due to injection. If the parameter $\xi$ is assumed to be determined by the microphysics of the shock, and if we adopt our simple recipe to describe such microphysics, then the shock thickness is easily estimated once the temperature of the downstream gas is known, and the latter can be calculated from the modified Rankine–Hugoniot relations. The parameter $\eta$ in equation (8) is no longer a free parameter, being related in a unique way to the parameter $\xi$ and to the physical conditions at the subshock. The condition that fixes $\eta$ is that the total number of particles in the non-thermal spectrum equals the number of particles in the Maxwellian at momenta larger than $p_{\text{inj}}$. Due to the very strong dependence of the spectrum on the momentum for both the Maxwellian and the power law at low momenta, the condition described above is very close to require the continuity of the distribution function, namely that $f_{\text{inj}}(p_{\text{inj}}) = f_{\text{max}}(p_{\text{inj}})$. In the following we adopt this condition for the calculations. This can be shown to imply the following expression for $\eta$:

$$\eta = \frac{4}{3\pi^{1/2}}(R_{\text{sub}} - 1)\xi^{-1} e^{-\xi^2}.$$  \hfill (27)

We recall that the compression factor at the subshock, $R_{\text{sub}}$, approaches unity when the shock becomes cosmic ray dominated. This makes evident how the backreaction discussed above works: when the shock becomes increasingly more modified, the efficiency $\eta$ tends to decrease, limiting the amount of energy that can be channelled in the non-thermal component. Although the recipe provided here is certainly far from representing the complexity of the reality of injection of particles from the thermal pool, it may be considered as a useful attempt to include the main physical aspects of this phenomenon.

6 SELF-CONSISTENT INJECTION AND MULTIPLE SOLUTIONS

In this section we describe the role played by the injection recipe discussed above for the appearance of multiple solutions. It can be expected that the phenomenon is somewhat reduced because, as discussed in the previous section, the injection provides an efficient backreaction mechanism on the shock as a particle accelerator. Indeed we find that the appearance of multiple solutions is drastically reduced, and that the phenomenon still exists only in regions of the parameter space which are very narrow and of limited physical interest. In the quantitative calculations, we use the value $\xi = 3.5$ for the injection parameter, as suggested by the simple estimate in Section 5 and as suggested also in the numerical work of Kang & Jones (1995). The dependence of the effect on the value of $\xi$ is discussed below. In Fig. 7 we illustrate the dramatic change in the physical picture by plotting $U(p_{\text{max}})$ as a function of $R_{\text{tot}}$ for $\xi = 3.5$ and adopting the same values for the parameters as those used in obtaining Fig. 2. The efficiency $\eta$ is now calculated according to the recipe described in the previous section. It can be seen very clearly that when the Mach number of the shock is changed, there is a single solution (compare with Fig. 2 where multiple solutions where found for the same values of the parameters, but without thermal leakage).

The appearance of multiple solutions can be investigated in the whole parameter space, in order to define the regions where the phenomenon appears, when it does. In Fig. 8 we highlight the regions where there are multiple solutions (dark regions) in a plane $\xi-\log(p_{\text{max}})$, for different values of the Mach number of the shock. In most cases, the dark regions are very narrow and cover a region of values of $\xi$, which is rather high (small efficiency). In Fig. 9 we plot the value of $R_{\text{tot}}$ as a function of $\xi$ for $M_0 = 200$, $u_0 = 5 \times 10^8 \, \text{cm s}^{-1}$ and $p_{\text{max}} = 10^3, 10^4, 10^5 \, \text{meV}$ from left to right. The line is continuous when there are no multiple solutions and dashed when multiple solutions appear. The dashed regions are, as stressed above, rather narrow. For instance, for $p_{\text{max}} = 10^4 \, \text{meV}$ there are multiple solutions only for $3.67 \leq \xi \leq 3.7$. Any small perturbation of the system that changes the values of $\xi$ at the per cent level implies that the system shifts to one of the single solutions if it is sitting in the intermediate solution before the perturbation. The sharp transition between the strongly modified solution and the quasi-linear solution when $\xi$ is increased suggests that the intermediate solution may be unstable, although a formal demonstration cannot be provided here. In order to make sure that this is the case, a careful analysis of the stability is required, and will be presented in a forthcoming publication (Blasi & Vietri, in preparation). On the other hand, a previous study, carried out in the context of the two-fluid models, showed that when multiple solutions are present, the solution with intermediate efficiency is in fact unstable to corrugations of the shock surface (Mond & Drury 1998).

7 ESCAPING FLUX OF ACCELERATED PARTICLES

It is rather remarkable that the kinetic model of Blasi (2002, 2004) does not require explicitly the use of the equation for energy flux conservation. However, once the solution of the kinetic problem has been found, the equation for conservation of the energy flux provides very useful information, as we show below. The equation can be written in the following form

$$\frac{1}{2} \rho u_1^2 + \frac{\gamma_k}{\gamma_k - 1} P_{\text{GR}2} + \frac{\gamma_k}{\gamma_k - 1} P_{\text{GR}2} = \frac{1}{2} \rho u_1^2 + \frac{\gamma_k}{\gamma_k - 1} P_{\text{GR}2} u_0 - F_{\text{E}}.$$  \hfill (28)

where $F_{\text{E}}$ is the flux of particles escaping at the maximum momentum from the upstream section of the fluid (Berezhko &
Ellison 1999). Notice that this term is usually neglected in the linear approach to particle acceleration at shock waves because the spectra are steep enough that, in most cases, we can neglect the flux of particles leaving the system at the maximum momentum. The fact that particles leave the system make the upstream fluid behave as a radiative fluid, and makes it more compressible. This is a crucial consequence of particle acceleration at modified shocks, and is shown here to be a natural consequence of energy conservation.

In equation (28) we can divide all terms by \((1/2)\rho_0 u_0^3\) and calculate the normalized escaping flux:

\[
F'_e = 1 - \frac{1}{R_{tot}} + \frac{2}{M_0^2(y_e - 1)} - \frac{2}{R_{tot}} \frac{\gamma_e}{y_e - 1} \frac{P_{e,2}}{\rho_0 u_0^2} - \frac{2}{R_{tot}} \frac{\gamma_e}{y_e - 1} \frac{P_{e,2}}{\rho_0 u_0^2}.
\]

From momentum conservation at the subshock we also have

\[
\frac{P_{e,2}}{\rho_0 u_0^2} = \frac{R_{sub}}{R_{tot}} - \frac{1}{R_{tot}} + \frac{1}{\gamma_e M_0^2} \left( \frac{R_{sub}}{R_{tot}} \right)^{-\gamma_e}.
\]

Note also that the adiabatic index for cosmic rays, \(\gamma_c\), is here calculated self-consistently as

\[
\gamma_c = 1 + \frac{P^c}{E^c} = 1 + \left(1/3\right) \int_{p_{inj}}^{p_{max}} dp\sqrt{\frac{\gamma_c M_0^2}{p_0^2}} f_0(p)\left(\int_{p_{inj}}^{p_{max}} dp\sqrt{\frac{\gamma_c M_0^2}{p_0^2}} f_0(p)\right)^{-1},
\]

where \(E_c\) is the energy density in the form of accelerated particles and \(\epsilon(p)\) is the kinetic energy of a particle with momentum \(p\). It can be easily seen that \(\gamma_c \to 4/3\) when the energy budget is dominated by the particles with \(p \sim p_{max}\) (namely for strongly modified shocks) and \(\gamma_c \to 5/3\) for weakly modified shocks. In equation (29) the term

\[
F'_{adv} = \frac{2}{R_{tot}} \frac{\gamma_c}{y_c - 1} \frac{P_{e,2}}{\rho_0 u_0^2}
\]

is clearly the fraction of flux which is advected downstream with the fluid.

In Fig. 10 we plot the escaping flux \((F'_e)\), the advected flux \((F'_{adv})\), and the sum of the two \((F'_{tot})\) normalized to the incoming flux \((1/2)\rho_0 u_0^3\), as functions of the Mach number at upstream infinity \(M_0\). Here we used \(u_0 = 5 \times 10^8\) cm s\(^{-1}\), and \(\xi = 3.5\), while the maximum momentum has been chosen as \(p_{max} = 10^6 m_c\) in the left-hand panel and \(p_{max} = 10^2 m_c\) in the right-hand panel. Several comments are in order.
Figure 9. Dependence of $R_{\text{tot}}$ as a function of $\xi$ for $M_0 = 200$, $u_0 = 5 \times 10^8$ cm s$^{-1}$ and $p_{\text{max}} = 10^3, 10^5, 10^7 mc$ from left to right. The sharpness of the transition suggests that the small perturbations of the parameters make the solution fall on one of the two sides.

(i) At low Mach numbers, the escaping flux is inessential, as one would expect for a weakly modified shock. We recall that the escaping flux is due to the particles with momentum $p_{\text{max}}$ leaving the system from upstream infinity. For a weakly modified shock at low Mach number, the spectrum is steeper than $E^{-2}$, so that the energy carried by the highest energy particles is a small fraction of the total.

(ii) At large Mach numbers, the shock becomes increasingly more cosmic ray dominated, and for the cases at hand the total efficiency comes very close to unity, meaning that the shock behaves as an extremely efficient accelerator. At Mach numbers around 4 on the other hand, the total efficiency is around 20 per cent for $p_{\text{max}} = 10^6 mc$ and $\sim 10$ per cent for $p_{\text{max}} = 10^7 mc$, dropping fast below Mach number 4. Clearly the efficiency would be higher in this region for lower values of the parameter $\xi$.

(iii) Despite the fact that the total efficiency of the shock as a particle accelerator is close to unity at large Mach numbers, the fraction of the incoming energy which is actually advected toward downstream infinity is only $\sim 20$ per cent at $M_0 \approx 100$ for $p_{\text{max}} = 10^6 mc$. Most of the energy flux in this case is in fact in the form of energy escaping from upstream infinity at the highest momentum $p_{\text{max}}$. For $p_{\text{max}} = 10^7 mc$ the normalized advected flux roughly saturates at $\sim 40$ per cent and is comparable with the escape flux at the same Mach number. For a distant observer, these escaping particles would have a spectrum close to a delta function around $p_{\text{max}}$.

8 SHOCK HEATING IN THE PRESENCE OF EFFICIENT PARTICLE ACCELERATION

Energy conservation has the natural consequence that a smaller fraction of the kinetic energy of the fluid is converted into thermal energy of the downstream plasma in cosmic ray modified shocks, compared with the case of ordinary shocks. The reduction of the heating at non-linear shock waves is fully confirmed by our calculation in the context of the injection recipe introduced in Section 5. In Fig. 11 we plot the temperature jump between downstream infinity (at temperature $T_2$) and upstream infinity (at temperature $T_0$). The thick solid line is the jump as predicted by the standard Rankine–Hugoniot relations without cosmic rays. The other lines represent the temperature jump at cosmic ray modified shocks with $p_{\text{max}}/mc = 10^3$ (thin solid line), $p_{\text{max}}/mc = 10^5$ (dashed line), $p_{\text{max}}/mc = 10^7$ (dotted line) and $p_{\text{max}}/mc = 5 \times 10^{10}$ (dash-dotted line).

Such a drastic reduction of the downstream temperature is expected to reflect directly in the thermal emission of the downstream gas in those environments in which collisions are relevant. Note that for strongly modified shocks the compression factor between upstream infinity and downstream are much larger than for ordinary shocks, so that the downstream turns out to be denser but colder than in the linear case. The missing energy ends up in the form of accelerated particles.

The effect of suppression of the heating in cosmic ray modified shocks also appears in the spectra of the particles (thermal plus non-thermal) in the shock vicinity. In Fig. 12 we show these spectra (including the Maxwellian thermal bump) for $u_0 = 5 \times 10^8$ cm s$^{-1}$,
9 CONCLUSIONS

The efficiency of the first-order Fermi acceleration at shock fronts depends in a crucial way upon details of the mechanism that determines the injection of a small fraction of the particles from the thermal pool to the acceleration box. In reality, the processes of formation of a collisionless shock wave of plasma heating and particle acceleration are expected to be all parts of the same problem, although on different spatial scales. We hide our lack of knowledge of the microphysics of the shock structure in a simple recipe for injection, in which the particles that take part in the acceleration process are those that have momentum larger by a factor $\xi$ than the momentum of the thermal particles in the downstream fluid. This is motivated by the fact that for collisionless shocks thickness of the shock surface is determined by the gyromotion of the bulk of the thermal particles. We estimated that $\xi \sim 2-4$. This recipe implies that the fraction of particles that become accelerated is rather small ($0.01-10^{-5}$).

We implemented this recipe in a calculation of the non-linear reaction of cosmic rays on the shock structure proposed by Blasi (2002, 2004). Similarly to other models, also this approach shows the appearance of multiple solutions, for a wide choice of the parameters of the problem. When the simple model of particle injection is used, this phenomenon is drastically reduced: the multiple solutions disappear for most of the parameter space, and when they appear they look like narrow strips in the parameter space, at the boundary between unmodified and modified shocks. We argued that this result suggests that the narrow regions may signal the transition between two stable solutions, although this needs further confirmation through detailed analyses of the stability of the solutions. This interpretation seems to be supported in part by the calculations of Mond & Drury (1998), which showed that when three solutions are present, the intermediate one is indeed unstable for small corrugations of the shock front. This calculation was however performed in the context of a two-fluid model, while an investigation of the stability for kinetic models is still lacking.

We find that the phenomenology of the particle acceleration at modified shocks is characterized by three main features, as follows.

(i) The modification of the shock increases with the Mach number of the fluid. For low Mach numbers, the quasi-linear solution is recovered, but departures from it are evident already at relatively low Mach numbers. The modification of the spectra manifests itself with a hardening at high momenta and a softening at low momenta. The $p^3 f_\perp(p)$ shows a characteristic dip at intermediate momenta, typically around $p/mC \sim 1-100$ (for very large values of $p_{\max}$, the dip can be found at even larger momenta, which is of interest for the acceleration of ultrahigh-energy cosmic rays).

(ii) The total efficiency for particle acceleration saturates at large Mach numbers at a number of order unity. However, as shown in Fig. 10, the largest fraction of the energy is not advected downstream but rather escapes from upstream infinity at the maximum momentum. This effect was also discussed in the context of the simple model by Berezhko & Ellison (1999).

(iii) The high efficiency for particle acceleration reflects in a reduced ability of cosmic ray modified shocks in the heating of the background plasma. This effect is at the very basis of the backreaction introduced by the injection recipe on the acceleration process, and determines the saturation of the total efficiency for large Mach numbers. The heating suppression is shown in Figs 11 and 12.
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