UTILITY MAXIMIZATION WITH ADDICTIVE CONSUMPTION HABIT FORMATION IN INCOMPLETE SEMIMARTINGALE MARKETS

BY XIANG YU

University of Texas at Austin

This paper studies the problem of continuous time expected utility maximization of consumption together with addictive habit formation in general incomplete semimartingale markets. Introducing the set of auxiliary state processes and the modified dual space, we embed our original problem into an abstract time-separable utility maximization problem with a shadow random endowment on the product space \( \mathbb{L}_0^0 (\Omega \times [0, T], \mathcal{O}, \bar{\mathbb{P}}) \). We establish existence and uniqueness of the optimal solution using convex duality by defining the primal value function as depending on two variables, i.e., the initial wealth and the initial standard of living. We also provide market independent sufficient conditions both on the stochastic discounting processes and on the utility function for the well-posedness of our original optimization problem. Under the same assumptions, we can carefully modify the classical proofs in the approach of convex duality analysis when the auxiliary dual process is not necessarily integrable.

1. Introduction. During the past decades, the assumption of time separability of von Neumann-Morgenstern preferences on consumption plan has been challenged due to its lack of consistency with many observed empirical evidences. For instance, the celebrated magnitude of the equity premium (Mehra and Prescott [23]) can not be reconciled with the preference \( \mathbb{E}[\int_0^T U(t, c_t)dt] \) when the instantaneous utility function \( U \) is only derived from the consumption rate. As an alternative modeling tool, linear addictive habit formation preference has attracted a lot of attention and has been actively investigated in recent years. This new way to compare consumption streams is defined by \( \mathbb{E}[\int_0^T U(t, c_t - Z_t)dt] \), where \( U : [0, T] \times (0, \infty) \rightarrow \mathbb{R} \) and the additional accumulative process \( Z_t \), called the habit formation or the standard of living process, describes the consumption history impact. In particular, \( Z \triangleq Z(\cdot; c) \) is defined in the following way:

\[
\begin{align*}
dZ_t &= (\delta_t c_t - \alpha_t Z_t)dt, \\
Z_0 &= z,
\end{align*}
\]

where the discounting factors \( \alpha_t \) and \( \delta_t \) are generally assumed to be nonnegative optional processes and the given real number \( z \geq 0 \) is called the “initial habit” or the “initial standard of living”. Moreover, the consumption habits are assumed to be addictive in the sense that we impose the constraint that \( c_t \geq Z_t \) for all time \( t \in [0, T] \).

Compared to the utility from consumption itself, a small drop in consumption may cause a large fluctuation in consumption net of the subsistence level due to the standard of living constraint and hence can possibly explain sizable excess returns on risky assets in equilibrium.
models even for moderate values of the degree of risk aversion. Based on this, there is a vast literature that recommends the habit formation preference as the new economic paradigm which can resolve the equity premium puzzle as well as many other empirical observations, we refer the readers to, for instance, Constantinides [4], Samuelson [28] and Campbell and Cochrane [3].

At the intuitive level, the other remarkable feature of the habit formation preference is its reflection of consumers’ rationality from the psychological perspective. The concept of habit formation characterizes the non-negligible effect of past consumption patterns on current and future economic decisions. Consumption behaviors in daily life often are repetitive and performed in customary places, leading consumers to develop habits. And high consumption history will generically lift up the investor’s desired consumption plan for the future. On the other hand, an increase in consumption today increases his current utility but will depress all future utilities through the induced growth in future standards of living.

The study of habit formation in modern economics dates back to Hicks [13] in 1965 and Ryder and Heal [27] in 1973. More recently, in complete Itô processes markets, Detemple and Zapatero [10, 11] employed martingale methods to study the general nonlinear habit formation utility optimization problem \( \mathbb{E}\left[\int_0^T U(t, c_t, Z_t)dt\right] \) and established some recursive stochastic differential equations for the consumption rate process \( c_t \). Later, Schroder and Skiadas [30] made an insightful observation that to solve the optimal portfolio selection with utilities incorporating linear addictive habit formation \( \mathbb{E}\left[\int_0^T U(t, c_t - Z_t)dt\right] \) in the complete market is equivalent to solving the time separable utility maximization \( \mathbb{E}\left[\int_0^T U(t, c'_t)dt\right] \) in the isomorphic complete market without habit formation. They also gave the construction of the isomorphic market based on the original market under some appropriate assumptions. Munk [24] applied the Market Isomorphism result in the complete market model with mean reverting drift process and stochastic interests rates process, and provides closed form optimal strategies in several special cases. Detemple and Karatzas [9] further considered the linear non-addictive habits \( \mathbb{E}\left[\int_0^T U(t, c_t - Z_t)dt\right] \), where instead they define \( U : [0, \infty) \times (-\infty, \infty) \rightarrow \mathbb{R} \). Their consumption \( c_t \) is required to be non-negative but is allowed to fall below the “the standard of living” index \( Z_t \) that aggregates past consumption. They provided a constructive proof for the existence of an optimal consumption plan, however, the market completeness is still a key assumption for their analysis. Englezos and Karatzas [12] exploited stochastic partial differential equations and the first order condition in the non-Markovian complete market, and obtained stochastic feedback formulae for the optimal portfolio and consumption policies.

Although significant progress has been made in the complete market setting, in the words by Englezos and Karatzas [12], “The existence of an optimal portfolio/consumption pair in an incomplete market is an open question. . . ., and new methodologies are needed to handle the problem.” Therefore, in this paper, we are interested in the general incomplete semimartingale framework and aim to prove the existence and uniqueness of the optimal solution of this path dependent optimization problem by using convex duality analysis. To the best of our knowledge, our work is the first one which aims to solve the continuous time expected utility maximization problem with consumption habit formation in general incomplete financial markets. However, we also refer the readers to the very recent work by Muraviev [25], which treats the additive habit formation and random endowment in the discrete time incomplete markets using a very different analysis.

The convex duality approach plays an important role in the treatment of general utility
maximization problems in the framework of incomplete markets. To list a very small subset of the existing literature in optimal investment and consumption problems, we refer to Karatzas, Lehoczky, Shreve, and Xu [16], Kramkov and Schachermayer [20], [21], Ćvitanic, Schachermayer and Wang [5], Karatzas and Žitković [17], Hugonnier and Kramkov [15], Žitković [31], [32], Kauppila [18] and Larsen and Žitković [22].

Typically, the critical step to build conjugate duality results for utility maximization problems is to define the dual space as the proper extension of space $\mathcal{M}$, the set of density processes of equivalent local martingale measures. Due to the presence of the habit formation process, the choice of the dual space and the formulation of the associated dual problem will become more elegant. By working in the product space $L^0_+ (\Omega \times [0, T], \mathcal{O}, \mathbb{P})$, the first natural choice is the bipolar set of the space $\mathcal{M}$, which is the smallest convex, closed and solid set containing space $\mathcal{M}$. Kramkov and Schachermayer [20], [21], or, more precisely in the filtered version by Žitković [31], proved that this bipolar set can be characterized as the solid hull of the set $\mathcal{Y}(y)$, which is defined as the set of supermartingales deflators

$$\mathcal{Y}(y) = \left\{ Y \geq 0 \mid Y_0 = y \text{ and } XY = (X_t Y_t)_{0 \leq t \leq T} \text{ is a supermartingale} \right\},$$

where $\mathcal{X}(x)$ denotes the set of accumulated gains/losses process under some admissible investment strategies with initial endowment less than or equal to $x$. However, according to the definition of the habit formation process $Z_t$, the dual functional is no longer necessarily lower semicontinuous with respect to process $Y_t \in \mathcal{Y}(y)$. As a matter of fact, if we formally derive the naive dual problem by using the Legendre-Fenchel transform and the first order condition, we arrive at

$$\inf_{y>0,Y \in \mathcal{Y}(y)} \mathbb{E} \left[ \int_0^T V \left( Y_t + \delta_t \mathbb{E} \left[ \int_t^T e^{\int_s^t (\delta_v - \alpha_v) dv} Y_s ds \bigg| \mathcal{F}_t \right] \right) dt \right] - \mathbb{E} \left[ \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} Y_t dt \right].$$

The first mathematical difficulty is the extra integral $\mathbb{E} \left[ \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} y_t dt \right]$, for which $\mathcal{Y}(y)$ is not the appropriate dual space to show the existence of the optimal solution for the above optimization problem. However, it can still remind us to invoke the general treatment of random endowment developed by Ćvitanic, Schachermayer and Wang [5], Karatzas and Žitković [17] and Žitković [32]. Their work requires another extension of the set $\mathcal{M}$, which is now considered as the set of equivalent local martingale measures, to the set $\mathcal{D}$ of bounded finitely additive measures. Nevertheless, their approach is inadequate to deal with the first term of the dual problem, when the conditional integral part $\mathbb{E} \left[ \int_t^T e^{\int_s^t (\delta_v - \alpha_v) dv} y_s ds \bigg| \mathcal{F}_t \right]$ in the conjugate function $V$ is taken into account. The analysis becomes more complicated since the conditional expectation is not well defined under finitely additive measures and the primal optimizer will possibly depends on the singular part of some finitely additive measures.

In order to avoid the complexity of the path-dependence and difficulties stated above, we propose the novel transformation from the consumption rate process $c_t$ to its auxiliary process $\tilde{c} = c_t - Z_t$, so that the primal utility maximization problem becomes time separable with respect to the process $\tilde{c}_t$. This substitution idea from $c_t$ to $\tilde{c}_t$ appeared firstly in the Market
Isomorphism result for complete markets by Schroder and Skiadas [30]. And meanwhile, for each equivalent local martingale measure density process \( Y \in \mathcal{M} \), we define the auxiliary dual process \( \Gamma_t \) exactly by

\[
\Gamma_t \triangleq Y_t + \delta_t \mathbb{E} \left[ \int_t^T e^{\int_s^t (\delta_v - \alpha_v) dv} Y_s ds \bigg| \mathcal{F}_t \right], \quad \forall t \in [0, T].
\]

We naturally intend to rewrite the dual problem given above in terms of auxiliary process \( \Gamma_t \) instead of \( Y_t \) so that the path dependence of \( Y_t \) can be also hidden in the definition of process \( \Gamma_t \).

However, the integral \( \mathbb{E} \left[ \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} Y_t dt \right] \) remains in the formulation of the dual problem. By introducing the stochastic process \( \tilde{w}_t = \exp(\int_0^t (-\alpha_v) dv) \), which itself is fully determined by the discounting processes \( \delta_t \) and \( \alpha_t \), one can shift the integral \( \mathbb{E} \left[ \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} Y_t dt \right] \) involving process \( Y_t \in \mathcal{M} \) to the integral \( \mathbb{E} \left[ \int_0^T \tilde{w}_t \Gamma_t dt \right] \) with respect to its auxiliary process \( \Gamma_t \). With the aid of this equality, we can naturally treat the extra exogenous random term \( \tilde{w}_t \) as some shadow random endowment density process in the abstract product space, and define the dual functional on the properly modified space of \( \Gamma_t \) instead of \( Y_t \). Now, as long as the initial standard of living value \( z \) is regarded as the variable of the value function, we can add one more dimension to the conjugate duality results and hide the extra integral term \( \mathbb{E} \left[ \int_0^T \tilde{w}_t \Gamma_t dt \right] \) by controlling its values. In essence, by enlarging the effective domain of values for \( x \) and \( z \), we arrange to embed our original utility maximization problem with consumption habit formation into the framework of Hugonnier and Kramkov [15] for an abstract time separable utility maximization on the product space.

On the other hand, we are facing issues in trying to apply the classical convex duality results when the auxiliary dual space may not be a subset of \( L^1(\Omega \times [0, T], \mathcal{O}, \mathbb{P}) \). Therefore, we impose the additional market independent sufficient conditions on habit formation discounting factors \( \alpha_t \) and \( \delta_t \), see Assumption (3.3) and (3.4), to guarantee the well-posedness of the Primal optimization problem. We also ask for Reasonable Asymptotic Elasticity conditions on utility functions \( U \) both at \( x \to 0 \) and \( x \to \infty \), i.e., \( AE_0[U] < \infty \) and \( AE_\infty[U] < 1 \) (see Assumption (2.9) and (2.10)), for the validity of several key assertions of our main results to hold true.

The rest of this paper is organized in the following way: Section 2 first describes the financial market. For the purpose to ensure the original utility optimization problem is well defined and to assist future proofs of the main results, we impose the Reasonable Asymptotic Elasticity Condition of the Utility function both for \( x = \infty \) and \( x = 0 \). In Section 3, we introduce some functional set-up on the product space \( L^1_\mathbb{P}(\tilde{\Omega} \times [0, T], \tilde{\mathcal{O}}, \tilde{\mathbb{P}}) \), and define the auxiliary process domain \( \mathcal{A}(x, z) \) and the auxiliary dual space \( \tilde{\mathcal{M}} \). We embed our original problem into an abstract time separable utility maximization problem over the enlarged abstract admissible space \( \tilde{\mathcal{A}}(x, z) \), however, with the shadow random endowment. We first assume that the extra exogenous term \( \mathcal{E} = \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} dt \) is not replicable under the original market in Section 4, and this section is devoted to the definition of the two dimensional dual problem over the properly enlarged dual space \( \tilde{\mathcal{Y}}(y, r) \) for the auxiliary primal optimization problem and our main results are stated in the end. Section 5 contains the proofs of our main results. Section 6 complements our main results by concerning the special case of replicable extra
exogenous term $\mathcal{E}$. Some important and interesting features of the abstract dual space are discussed and one explicit example is presented at the end by assuming that the discounting factors are deterministic functions.

### 2. Market Model.

#### 2.1. The Financial Market Model.

We consider a financial market with $d \in \mathbb{N}$ risky assets modeled by a $d$-dimensional semimartingale

$$S = (S^{(1)}_t, \ldots, S^{(d)}_t)_{t \in [0,T]}$$

on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where the filtration $\mathcal{F}$ satisfies the usual conditions and the maturity time is given by $T$. To simplify our notation, we take $\mathcal{F} = \mathcal{F}_T$.

We make the standard assumption that there exists one riskless bond $S^{(0)}_t \equiv 1, \forall t \in [0, T]$, which amounts to consider $S^{(0)}_t$ as the numéraire asset.

The portfolio process $H = (H^{(1)}_t, \ldots, H^{(d)}_t)_{t \in [0,T]}$ is a predictable $S$-integrable process representing the number of shares of each risky asset held by the investor at time $t \in [0,T]$.

The accumulated gains/losses process of the investor under his trading strategy $H$ by time $t$ is given by:

$$X^H_t = (H \cdot S)_t = \sum_{k=1}^{d} \int_0^t H^{(k)}_u dS^{(k)}_u, \quad t \in [0, T].$$

#### 2.2. Admissible Portfolios and Consumption Habit Formation.

The portfolio process $(H_t)_{t \in [0,T]}$ is called admissible if the gains/losses process $X^H_t$ is bounded below, which is to say, there exists a constant bound $a \in \mathbb{R}$ such that $X^H_t \geq a$, a.s. for all $t \in [0, T]$.

Now, given the initial wealth $x > 0$, the agent will also choose an intermediate consumption plan during the whole investment horizon, and we denote the consumption rate process by $c_t$. The resulting self-financing wealth process $(W^{x,H,c}_t)_{t \in [0,T]}$ is given by

$$W^{x,H,c}_t = x + (H \cdot S)_t - \int_0^t c_u ds, \quad t \in [0, T].$$

Besides of the wealth process, as we already defined in the Introduction, the associated consumption habit formation or the standard of living process $Z_t \triangleq Z(\cdot; c)$ is given equivalently by the following exponentially weighted average of agent’s past consumption integral and the initial habit

$$Z_t = ze^{-\int_0^t \alpha_v dv} + \int_0^t \delta_s e^{-\int_s^t \alpha_v dv} c_s ds,$$

where discounting factors $\alpha_t$ and $\delta_t$ measure, respectively, the persistence of the initial habits level and the intensity of consumption history. In this paper, we shall be mostly interested
in the general case when the discounting factors $\alpha_t$ and $\delta_t$ are stochastic processes which are allowed to be unbounded. The stochastic nature of the discounting factors corresponds to various market features. For instance, the investor may randomly change his weights on the consumption habits impact due to his risk preference change, time-varying impatience or other time inconsistent factors from the financial market.

Throughout this paper, we make the assumption that the consumption habit is addictive, i.e., $c_t \geq Z_t$, $\forall t \in [0, T]$, which is to say, the investor’s current consumption rate shall never fall below “the standard of living” process.

A consumption process $(c_t)_{t \in [0,T]}$ is defined to be $(x, z)$-financeable if there exists an admissible portfolio process $(H_t)_{t \in [0,T]}$ such that $W^{x,H,c}_t \geq 0$, $\forall t \in [0, T]$, a.s. and the addictive habit formation constraint $c_t \geq Z_t$, $\forall t \in [0, T]$ a.s. holds. The class of all $(x, z)$-financeable consumption rate processes will be denoted by $A(x, z)$, for $x > 0, z \geq 0$.

2.3. Absence of Arbitrage. A probability measure $Q$ is called an equivalent local martingale measure if it is equivalent to $P$ and if $X^H_t$ is a local martingale under $Q$. We denote by $\mathcal{M}$ the family of equivalent local martingale measures and in order to rule out the arbitrage opportunities in the market, we assume that

\[(2.5)\quad \mathcal{M} \neq \emptyset.\]

We refer the readers to Delbaen and Schachermayer [6] and [7] for a comprehensive discussion and treatment on the topic of no arbitrage.

Define the RCLL process $Y^Q_t$ by

$$ Y^Q_t = \mathbb{E}\left[\frac{dQ}{dP} \middle| \mathcal{F}_t\right] $$

for the $Q \in \mathcal{M}$, then $Y^Q_t$ is called an equivalent local martingale measure density and we shall always identify the equivalent local martingale measure $Q$ with its density process $Y^Q_t$, and with a slight abuse of notation, we denote $\mathcal{M}$ also as the set of all equivalent local martingale density processes.

The celebrated Optional Decomposition Theorem, see Kramkov [19], enables us to characterize the $(x, z)$-financeable consumption process in terms of linear inequalities with respect to $Y_t \in \mathcal{M}$, called Budget Constraint, and this serves as an important ingredient in the treatment of our utility maximization problem via convex duality approach.

**Proposition 2.1.** The process $(c_t)_{t \in [0,T]}$ is $(x, z)$-financeable if and only if $c_t \geq Z_t$, $\forall t \in [0, T]$ and

\[(2.6)\quad \mathbb{E}\left[\int_0^T c_tY_tdt\right] \leq x, \quad \forall Y_t \in \mathcal{M}.\]

**Remark 2.1.** It is easy to see the set of $(x, z)$-financeable consumption processes is in general not solid. This means it is not allowed to consume as less as we want since the habit formation constraint may not be retained. Some technical questions in convex duality arise as we expect the solid hull of the financeable set is now generically complicated to describe.
2.4. The Utility Function. The individual investor’s preference is represented by a utility function $U : [0, T] \times (0, \infty) \to \mathbb{R}$, such that, for every $x > 0$, $U(\cdot, x)$ is continuous on $[0, T]$, and for every $t \in [0, T]$, the function $U(t, \cdot)$ is strictly concave, strictly increasing, continuously differentiable and satisfies the Inada conditions:

\begin{equation}
U'(t, 0) \triangleq \lim_{x \to 0} U'(t, x) = \infty, \quad U'(t, \infty) \triangleq \lim_{x \to \infty} U'(t, x) = 0.
\end{equation}

where $U'(t, x) \triangleq \frac{\partial}{\partial x} U(t, x)$. For each $t \in [0, T]$, we extend the definition of the utility function by $U(t, x) = -\infty$ for all $x < 0$, which is equivalent to the addictive habit formation constraint $c_t \geq Z_t$ when the utility function is defined on the difference between the consumption rate process $c_t$ and habit formation process $Z_t$.

According to these assumptions, the inverse $I(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ of the function $U'(t, \cdot)$ exists for every $t \in [0, T]$, and is continuous and strictly decreasing with:

\begin{equation}
I(t, 0) \triangleq \lim_{x \to 0} I(t, x) = \infty, \quad I(t, \infty) \triangleq \lim_{x \to \infty} I(t, x) = 0.
\end{equation}

The convex conjugate of the agents’ utility function, also known as the Legendre-Fenchel transform, is defined as follows:

\begin{equation}
V(t, y) \triangleq \sup_{x > 0} \{ U(t, x) - xy \}, \quad y > 0.
\end{equation}

Under the Inada conditions (2.7), the conjugate of $V(t, \cdot)$ is a continuously differentiable, strictly decreasing and strictly convex function satisfying $V'(t, 0) = -\infty$, $V'(t, \infty) = 0$ and $V(t, 0) = U(t, \infty)$, $V(t, \infty) = U(t, 0)$, see, for example, Karatzas, Lehoczky, Shreve, and Xu [16] for reference.

Follow the asymptotic growth control of the utility functions coined by Kramkov and Schachermayer [20], see also Karatzas and Žitković [17], we shall make additional assumptions on the asymptotic behavior of $U$ at both $x = 0$ and $x = \infty$ for future purposes:

**Assumption 2.1.**
The utility functions $U$ satisfies the Reasonable Asymptotic Elasticity condition both at $x = \infty$ and $x = 0$, i.e.,

\begin{equation}
AE_{\infty}[U] = \limsup_{x \to \infty} \left( \sup_{t \in [0, T]} \frac{x U'(t, x)}{U(t, x)} \right) < 1,
\end{equation}

and

\begin{equation}
AE_{0}[U] = \limsup_{x \to 0} \left( \sup_{t \in [0, T]} \frac{x U'(t, x)}{|U(t, x)|} \right) < \infty.
\end{equation}

Moreover, in order to get some inequalities uniformly in time $t$, we shall assume

\begin{equation}
\lim_{x \to \infty} \left( \inf_{t \in [0, T]} U(t, x) \right) > 0,
\end{equation}

and

\begin{equation}
\lim_{x \to 0} \left( \sup_{t \in [0, T]} U(t, x) \right) < 0.
\end{equation}
Remark 2.2. Many well known utility functions satisfy Reasonable Asymptotic Elasticity Assumptions (2.9) and (2.10), for example, the discounted log utility function \( U(t, x) = e^{-\beta t} \log(x) \) or discounted power utility function \( U(t, x) = e^{-\beta t} \frac{x^p}{p} \) (\( p < 1 \) and \( p \neq 0 \)), for a constant \( \beta > 0 \). However, it is also easy to check that the utility function \( U(t, x) = -e^{x} \) does not satisfy the Assumption (2.10) and the utility function \( U(t, x) = \frac{x}{\log x} \) does not satisfy the Assumption (2.9).

Remark 2.3. If the utility function satisfies the lower bound assumption \( \inf_{t \in [0, T]} U(t, 0) > -\infty \), then our Assumption (2.10) is automatically verified. And if the utility function satisfies the upper bound assumption \( \sup_{t \in [0, T]} U(t, \infty) < \infty \), the Assumption (2.9) holds true.

Remark 2.4. The utility function \( U(t, x) \) satisfies Reasonable Asymptotic Elasticity Assumptions (2.9) and (2.10) if and only if its affine transform \( a + bU(t, x) \) satisfies Reasonable Asymptotic Elasticity Assumptions (2.9) and (2.10) for arbitrary constants \( a, b > 0 \). Hence, the adjoint Assumption (2.11) and Assumption (2.12) are not restrictive.

The next technical result gives the equivalent characterization of the Reasonable Asymptotic Elasticity condition \( AE_{\infty}[U] \), which follows the identical proof of Lemma 6.3 of Kramkov and Schachermayer [20], see also Proposition 3.7 of Karatzas and Žitković [17].

Lemma 2.1. Let \( U(t, x) \) be a utility function satisfying (2.9) and (2.11). In each of the subsequent assertions, the infimum of \( \gamma > 0 \) for which these assertions hold true equals the Reasonable Asymptotic Elasticity \( AE_{\infty}[U] \).

(i) There is \( x_0 > 0 \) for all \( t \in [0, T] \) s.t.
\[
U(t, \lambda x) < \lambda^\gamma U(t, x) \quad \text{for} \quad \lambda > 1, x \geq x_0.
\]

(ii) There is \( x_0 > 0 \) for all \( t \in [0, T] \) s.t.
\[
U'(t, x) < \gamma \frac{U(t, x)}{x} \quad \text{for} \quad x \geq x_0.
\]

(iii) There is \( y_0 > 0 \) for all \( t \in [0, T] \) s.t.
\[
V(t, \mu y) < \mu^{\gamma} V(t, y) \quad \text{for} \quad 0 < \mu < 1, 0 < y \leq y_0.
\]

(iv) There is \( y_0 > 0 \) for all \( t \in [0, T] \) s.t.
\[
-V'(t, y) < \left( \frac{\gamma}{1 - \gamma} \right) \frac{V(t, y)}{y} \quad \text{for} \quad 0 < y \leq y_0.
\]

Corollary 2.1. Under Assumptions (2.10) and (2.12), we have \( AE_0[U] < \infty \) if and only if \( AE_{\infty}[V] < 1 \), where we define
\[
AE_{\infty}[V] = \lim_{y \to \infty} \left( \sup_{t \in [0, T]} \frac{y V'(t, y)}{V(t, y)} \right) < 1,
\]
and hence similarly, we have each of the following assertions, the infimum of \( \gamma > 0 \) for which these assertions hold true equals the Reasonable Asymptotic Elasticity \( AE_{\infty}[V] \).
(i) There is \( y_0 > 0 \) for all \( t \in [0, T] \) s.t.
\[
V(t, \lambda y) > \lambda \gamma V(t, y) \quad \text{for } \lambda > 1, y \geq y_0.
\]

(ii) There is \( y_0 > 0 \) for all \( t \in [0, T] \) s.t.
\[
V'(t, y) > \gamma \frac{V(t, y)}{y} \quad \text{for } y \geq y_0.
\]

(iii) There is \( x_0 > 0 \) for all \( t \in [0, T] \) s.t.
\[
U(t, \mu x) > \mu^{-\gamma} \frac{1}{1-\gamma} U(t, x) \quad \text{for } 0 < \mu < 1, 0 < x \leq x_0.
\]

(iv) There is \( x_0 > 0 \) for all \( t \in [0, T] \) s.t.
\[
- U'(t, x) > \gamma \left( 1 - \frac{x}{1-\gamma} \right) \frac{U(t, x)}{x} \quad \text{for } 0 < x \leq x_0.
\]

3. A New Characterization of Financeable Consumption Processes.

3.1. Some Functional Set Up. In the spirit of Bouchard and Pham [1] which treated the wealth dependent problem (see also Žitković [32] on consumption and endowment with stochastic clock), let \( \mathcal{O} \) denotes the \( \sigma \)-algebra of optional sets relative to the filtration \( (\mathcal{F}_t)_{t \in [0,T]} \) and we define the product measure \( d\bar{\mathbb{P}} = dt \times d\mathbb{P} \) be the finite measure on the product space \((\Omega \times [0,T], \mathcal{O})\):

\[
\bar{\mathbb{P}}[A] = \mathbb{E}[\int_0^T 1_A(t, \omega) dt], \quad \text{for } A \in \mathcal{O}.
\] (3.1)

We denote by \( \mathbb{L}^0(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}}) \) \( (\mathbb{L}^0 \) for short) the set of all random variables on the product space \( \Omega \times [0,T] \) under the product measure \( \bar{\mathbb{P}} \) with respect to the optional \( \sigma \)-algebra \( \mathcal{O} \) endowed with the topology of convergence in measure \( \bar{\mathbb{P}} \). And from now on, we shall always identify the optional stochastic process \((Y_t)_{t \in [0,T]}\) with the random variable \( Y \in \mathbb{L}^0(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}}) \). We also define the positive orthant \( \mathbb{L}^0_+ (\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}}) \) \( (\mathbb{L}_+^0 \) for short) the set of elements \( Y = Y(t, \omega) \) of \( \mathbb{L}^0 \) such that:

\[
Y \geq 0, \quad \bar{\mathbb{P}} \text{ a.s.}
\]

For any \( Y^1, Y^2 \in \mathbb{L}_+^0 \), we shall say that

\[
Y^1 \equiv Y^2 \quad \text{if } Y^1 = Y^2, \quad \bar{\mathbb{P}} \text{ a.s.}
\]

Endow \( \mathbb{L}_+^0 \) with the bilinear form valued in \([0, \infty]\) as:

\[
\langle X, Y \rangle = \mathbb{E}\left[ \int_0^T X_t Y_t dt \right], \quad \text{for all } X, Y \in \mathbb{L}_+^0.
\]

We also define a partial ordering on \( \mathbb{L}_+^0 \) for convenience:

\[
Y^1 \preceq (\prec) Y^2 \iff Y^1 \leq (\prec) Y^2, \quad \bar{\mathbb{P}} \text{ a.s.}
\]
3.2. Path-dependence Reduction by Auxiliary Processes. At this point, we are able to define the set of all \((x, z)\)-financeable consumption rate processes as a set of random variables on the product space \((\Omega \times [0, T], \mathcal{O}, \mathbb{P})\) and the Budget Constraint Proposition 2.1 states that:

\[
\mathcal{A}(x, z) \triangleq \left\{ c \in L^0_+ : c_t \geq Z_t \text{ and } W_t = x + (H : S)_t - \int_0^t c_s ds \geq 0, \forall t \in [0, T] \text{ and } H \text{ is admissible} \right\} = \left\{ c \in L^0_+ : c_t \geq Z_t, \forall t \in [0, T] \text{ and } \langle c, Y \rangle \leq x, \forall Y \in \mathcal{M} \right\}.
\]

where process \(Z_t\) is defined by (2.4). However, the family \(\mathcal{A}(x, z)\) may be empty for some values \(x > 0, z \geq 0\). We shall restrict ourselves to the effective domain \(\bar{H}\) which is defined as the union of the interior of set such that \(\mathcal{A}(x, z)\) is not empty and the one side boundary \(\{x > 0, z = 0\}\):

\[
(3.2) \quad \bar{H} \triangleq \text{int}\left\{(x, z) \in (0, \infty) \times [0, \infty) : \mathcal{A}(x, z) \neq \emptyset \right\} \cup (0, \infty) \times \{0\}.
\]

We want the effective domain \(\bar{H}\) to include the special case of zero initial habit by \(z = 0\).

Before we state the next result, we shall first impose some additional conditions on the stochastic discounting factors \(\alpha_t\) and \(\delta_t\), which are essential for the well-posedness of our primal utility optimization problem:

**Assumption 3.1.**

\[\text{We assume the nonnegative optional processes } \alpha_t \text{ and } \delta_t \text{ satisfy:} \]

\[
\sup_{Y \in \mathcal{M}} \mathbb{E}\left[\int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_{t} dt \right] < \infty.
\]

and there exists a constant \(\bar{x} > 0\) such that

\[
(3.4) \quad \mathbb{E}\left[\int_0^T U(t, \bar{x} e^{-\int_0^t \alpha_v dv}) dt \right] > -\infty.
\]

**Remark 3.1.** If stochastic discounting processes \(\alpha_t\) and \(\delta_t\) are assumed to be bounded, Assumptions (3.3) and (3.4) will be satisfied, and are redundant. Assumption (3.3) is the well known super-hedging property of the random variable \(\int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} dt \) in our original financial market. We make Assumption (3.4) here to guarantee the existence of some \((x, z) \in \bar{H}\) such that the value function \(u(x, z) > -\infty\). It is interesting to note that the process \(\tilde{w}_t \triangleq e^{-\int_0^t \alpha_v dv}\) somehow plays the same role as the constant 1 to be a universal strictly positive element in the corresponding admissible space by rescaling. And we remark here that one can also take \(\tilde{w}_t \triangleq e^{-\int_0^t \alpha_v dv}\) as the abstract numéraire.

**Lemma 3.1.** Under Assumption (3.3), the effective domain \(\bar{H}\) can be rewritten explicitly as:

\[
(3.5) \quad \bar{H} = \left\{(x, z) \in (0, \infty) \times [0, \infty) : x > z \sup_{Y \in \mathcal{M}} \mathbb{E}\left[\int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_{t} dt \right] \right\}.
\]
PROOF. It is enough to show for all \((x, z) \in (0, \infty) \times [0, \infty)\),

\[
x \geq z \sup_{Y \in \mathcal{M}} \mathbb{E} \left[ \int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_t dt \right]
\]

if and only if \(A(x, z) \neq \emptyset\).

On one hand, if \((x, z) \in (0, \infty) \times [0, \infty)\) and \(A(x, z) \neq \emptyset\), by definition, there exists \(c \in \mathbb{L}^0_{+}\) such that \(c_t \geq Z_t\) for all \(t \in [0, T]\) and \(\langle c, Y \rangle \leq x, \forall Y \in \mathcal{M}\). We now claim that we should always have \(c_t \geq \bar{c}_t\) for all \(t \in [0, T]\) where \(\bar{c}_t \equiv Z(\bar{c})_t\) is the subsistent consumption plan which equals its standard of living process. To this end, we first recall by the definition of \(A\)

\[
x = \sup_{Y \in \mathcal{M}} \mathbb{E} \left[ \int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_t dt \right]
\]

which equals its standard of living process. To this end, we first recall by the definition of \(Z\) that \(\frac{dZ_t}{dt} = (\delta_t c_t - \alpha_t Z_t)dt\) with \(Z_0 = z \geq 0\), and the constraint that \(c_t \geq Z_t\) implies

\[
d\bar{c}_t = (\delta_t \bar{c}_t - \alpha_t \bar{c}_t)dt, \quad \bar{c}_0 = z.
\]

and we can solve \(\bar{c}_t = z e^{\int_0^t (\delta_s - \alpha_s)ds} dt\) for \(t \in [0, T]\).

By the simple subtraction of (3.7) and (3.8), one can get

\[
d(Z_t - \bar{c}_t) \geq (\delta_t - \alpha_t)(Z_t - \bar{c}_t)dt, \quad Z_0 - \bar{c}_0 = 0,
\]

from which we can derive that

\[
e^{\int_0^t (\delta_s - \alpha_s)ds} (Z_t - \bar{c}_t) \geq 0, \quad \forall t \in [0, T].
\]

Hence, we will conclude that \(c_t \geq z e^{\int_0^t (\delta_s - \alpha_s)ds} dt\) for all \(t \in [0, T]\), which gives

\[
x \geq z \sup_{Y \in \mathcal{M}} \mathbb{E} \left[ \int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_t dt \right]
\]

by the consumption Budget Constraint condition (2.6).

One the other hand, if \((x, z) \in (0, \infty) \times [0, \infty)\) and (3.6) holds, we can obviously always construct \(\bar{c}_t = z e^{\int_0^t (\delta_s - \alpha_s)ds} dt\) such that \(\bar{c}_t \equiv Z(\bar{c})_t\) for all \(t \in [0, T]\) and \(\langle c, Y \rangle \leq x, \forall Y \in \mathcal{M}\), and hence \(A(x, z) \neq \emptyset\). The proof is complete. 

By choosing \((x, z) \in \mathcal{H}\), we can now define the preliminary version of our Primal Utility Maximization Problem as:

\[
u(x, z) \triangleq \sup_{c \in \mathcal{A}(x, z)} \mathbb{E} \left[ \int_0^T U(t, c_t - Z_t) dt \right], \quad (x, z) \in \mathcal{H}.
\]

Now, for fixed \((x, z) \in \mathcal{H}\), and each \((x, z)\)-financeable consumption rate process, we want to generalize the Market Isomorphism idea by Schroder and Skiadas [30] in order to reduce the path dependency. We are ready to introduce the auxiliary process \(\tilde{c}_t = c_t - Z_t\), and define the auxiliary set of \(A(x, z)\) as:

\[
\tilde{A}(x, z) \triangleq \left\{ \tilde{c} \in \mathbb{L}^0_{+} : \tilde{c}_t = c_t - Z_t, \forall t \in [0, T], \quad c \in A(x, z) \right\}.
\]
Lemma 3.2. For each fixed \((x, z) \in \mathcal{H}\), there is a one to one correspondence between sets \(\mathcal{A}(x, z)\) and \(\bar{\mathcal{A}}(x, z)\), and hence we have \(\bar{\mathcal{A}}(x, z) \neq \emptyset\) for \((x, z) \in \mathcal{H}\).

**Proof.** Fix each pair \((x, z) \in \mathcal{H}\) so that \(\mathcal{A}(x, z) \neq \emptyset\), it is clear by the definition that for each \(c \in \mathcal{A}(x, z)\), there exists a unique \(\tilde{c}_t = c_t - Z_t\) such that \(\tilde{c} \in \bar{\mathcal{A}}(x, z)\).

Now for each fixed \((x, z) \in \mathcal{H}\) and \(\tilde{c} \in \bar{\mathcal{A}}\), denote the process
\[
c_t \triangleq \tilde{c}_t + \tilde{Z}_t,
\]
where the process \(\tilde{Z}_t\) is uniquely determined by the process \(\tilde{c}\) as
\[
\tilde{Z}_t = z e^{f_t^0(\delta_v - \alpha_v)dv} + \int_0^t \delta_s e^{f_s^0(\delta_v - \alpha_v)dv} \tilde{c}_s ds.
\]
It is easy to check by definition that \(c_t - Z_t = \tilde{c}_t \geq 0\), where we know
\[
Z_t = z e^{-f_t^0 \alpha_v dv} + \int_0^t \delta_s e^{-f_s^0 \alpha_v dv} c_s ds.
\]

Now by the definition of set \(\bar{\mathcal{A}}(x, z)\) and the uniqueness of process \(c_t\) such that \(c_t - Z_t = \tilde{c}_t\), we can therefore conclude there exists a unique \(c_t \in \mathcal{A}(x, z)\) for each \(\tilde{c} \in \bar{\mathcal{A}}(x, z)\). \(\square\)

Let’s turn our attention to the set \(\mathcal{M}\) of equivalent local martingale measures, and for each \(Y \in \mathcal{M}\), according to Assumption (3.3) we can define the auxiliary optional process with respect to \(Y_t\) as:
\[
(3.12) \quad \Gamma_t \triangleq Y_t + \delta_t \mathbb{E} \left[ \int_t^T e^{f_s^0(\delta_v - \alpha_v)dv} Y_s ds \middle| \mathcal{F}_t \right], \quad \forall t \in [0, T].
\]

Let’s denote the set of all these auxiliary optional processes as:
\[
(3.13) \quad \tilde{\mathcal{M}} = \left\{ \Gamma \in \mathbb{L}^0_+ : \Gamma_t = Y_t + \delta_t \mathbb{E} \left[ \int_t^T e^{f_s^0(\delta_v - \alpha_v)dv} Y_s ds \middle| \mathcal{F}_t \right], \quad \forall t \in [0, T], \quad Y \in \mathcal{M} \right\}.
\]
We remark again here that since stochastic discounting processes \(\delta_t\) and \(\alpha_t\) are unbounded, under Assumption (3.3), the auxiliary dual process \(\Gamma\) is well defined, but it is not necessarily in \(\mathbb{L}^1\).

The following important equalities serve as critical ingredients in embedding our original utility maximization problem into an auxiliary abstract optimization problem on the product space, for which we are able to apply the convex duality approach:

**Proposition 3.1.** Under Assumption (3.3), for each nonnegative optional process \(c_t\) such that \(c_t \geq Z_t\) with \(Z_t\) defined by (2.4) for fixed initial standard of living \(z \geq 0\) and the nonnegative optional process \(\tilde{Y}_t\), we have the following equalities with respect to their corresponding auxiliary processes \(\tilde{c}_t = c_t - Z_t\) and \(\tilde{\Gamma}_t\) which is defined by (3.12), that:
\[
(3.14) \quad \langle c, Y \rangle = \langle \tilde{c}, \Gamma \rangle + z \langle w, Y \rangle
\]
where we define these extra exogenous random processes \(w, \tilde{w} \in \mathbb{L}^0_+\) as
\[
(3.15) \quad w_t \triangleq e^{f_t^0(\delta_v - \alpha_v)dv} \quad \text{and} \quad \tilde{w}_t \triangleq e^{f_t^0(-\alpha_v)dv} \quad \text{for all} \ t \in [0, T].
\]
Proof. By the definition, $Z_t$ solves the ODE: $dZ_t = (\delta_t c_t - \alpha_t Z_t)dt$ with $Z_0 = z$, for each $t \in [0, T]$. If we set $\tilde{c}_t = c_t - Z_t$, we can rewrite $c_t$ in terms of $\tilde{c}_t$ as:

$$c_t = ze^{\int_0^t (\delta_s - \alpha_s)ds} + \tilde{c}_t + \int_0^t \delta_s e^{\int_s^t (\delta_r - \alpha_r)dr} \tilde{c}_r ds,$$

and hence we will have the following chain equivalence by Fubini-Tonelli’s theorem:

$$\langle c, Y \rangle = zE \left[ \int_0^T e^{\int_0^t (\delta_s - \alpha_s)ds} Y_t dt \right] + E \left[ \int_0^T (\tilde{c}_t + \int_0^t \delta_s e^{\int_s^t (\delta_r - \alpha_r)dr} \tilde{c}_s ds) Y_t dt \right]$$

$$= z\langle w, Y \rangle + E \left[ \int_0^T \tilde{c}_t Y_t dt + \int_0^T \delta_s \tilde{c}_s \left( \int_s^T e^{\int_s^t (\delta_r - \alpha_r)dr} Y_r ds \right) dt \right]$$

$$= z\langle w, Y \rangle + E \left[ \int_0^T \tilde{c}_t Y_t dt + \int_0^T \delta_t \tilde{c}_t E \left[ \int_t^T e^{\int_t^s (\delta_r - \alpha_r)dr} Y_s ds \right] \mathcal{F}_t dt \right]$$

$$= z\langle w, Y \rangle + \langle \tilde{c}, \Gamma \rangle,$$

which gives the first equality. Similarly, we just observe that:

$$\langle \tilde{w}, \Gamma \rangle = E \left[ \int_0^T e^{\int_0^s (-\alpha_r)dr} Y_t dt \right] + E \left[ \int_0^T e^{\int_0^s (\delta_r - \alpha_r)dr} \delta_t \left( \int_t^s e^{\int_t^r (\delta_s - \alpha_s)ds} Y_r dr \right) \mathcal{F}_t dt \right]$$

$$= E \left[ \int_0^T e^{\int_0^s (-\alpha_r)dr} Y_t dt \right] + E \left[ \int_0^T e^{\int_0^s (\delta_r - \alpha_r)dr} Y_s \left( \int_0^s \delta_t e^{\int_t^0 (\delta_s - \alpha_s)ds} dt \right) ds \right]$$

$$= E \left[ \int_0^T e^{\int_0^s (-\alpha_r)dr} Y_t dt \right] - E \left[ \int_0^T e^{\int_0^s (\delta_r - \alpha_r)dr} Y_s \left( e^{\int_0^s \delta_s ds} - 1 \right) ds \right]$$

$$= E \left[ \int_0^T e^{\int_0^s (-\alpha_r)dr} Y_t dt \right] - E \left[ \int_0^T e^{\int_0^s (\delta_r - \alpha_r)dr} Y_t dt \right] + E \left[ \int_0^T e^{\int_0^s (\delta_r - \alpha_r)dr} Y_t dt \right]$$

$$= \langle w, Y \rangle,$$

which gives the second equality. ∎

Remark 3.2. These extra random processes $w_t$ and $\tilde{w}_t$ in (3.15) defined by stochastic discounting factors $\alpha_t$ and $\delta_t$ will play the role of shadow random endowment rate processes in the future formulation of the dual optimization problem. In an attempt to analyze this special structure, we will naturally adopt some classical convex duality analysis with respect to market random endowment source, and try to prove some similar results.

Based on previous Propositions 2.1 and 3.1, under Assumptions (3.3) and (3.4), clearly we will have the alternative budget constraint characterization of the consumption rate process $c_t$ as:

**Proposition 3.2.** For any given pair $(x, z) \in \tilde{\mathcal{H}}$, we call the consumption rate process $c$ is $(x, z)$-financeable if and only if $c_t \geq Z_t$, $\forall t \in [0, T]$ and

$$\langle c - Z, \Gamma \rangle \leq x - z \langle \tilde{w}, \Gamma \rangle,$$

for all $\Gamma \in \tilde{M}$. 

Proposition 3.2 provides us the alternative definition of set $\tilde{A}(x, z)$ for $(x, z) \in \tilde{H}$ as:

\[(3.16) \quad \tilde{A}(x, z) = \{ \tilde{c} \in \mathbb{L}_+^0 : \langle \tilde{c}, \Gamma \rangle \leq x - z \langle \tilde{w}, \Gamma \rangle, \forall \Gamma \in \tilde{M} \} .\]

We see that the path-dependent addictive habits constraint on $c_t$ such that $c_t \geq Z_t$ eventually turns to be a natural constraint that $\tilde{c} \in \mathbb{L}_+^0$, and (3.16) states that the auxiliary set $\tilde{A}(x, z)$ is solid, convex and closed in measure $\tilde{P}$ although $A(x, z)$ does not hold all these properties. Hence this path-dependence reduction from $c_t$ to $\tilde{c}_t$ is crucial to enable us to work with convex duality approach.

3.3. Embedding into an Abstract Utility Maximization Problem with the Shadow Random Endowment. In order to apply the classical convex duality analysis for the random endowment and build conjugate duality relations between value functions in the next section, due to some technical reasons, we need to first enlarge the domain of the set $\tilde{H}$ to $H$ and enlarge the corresponding auxiliary set $\tilde{A}(x, z)$ to $\tilde{A}(x, z)$ defined as:

\[(3.17) \quad \tilde{A}(x, z) \triangleq \{ \tilde{c} \in \mathbb{L}_+^0 : \langle \tilde{c}, \Gamma \rangle \leq x - z \langle \tilde{w}, \Gamma \rangle, \forall \Gamma \in \tilde{M} \},\]

where now $(x, z) \in \mathbb{R}^2$, and is restricted in the enlarged domain $\tilde{H}$:

\[
\tilde{H} \triangleq \text{int}\{ (x, z) \in \mathbb{R}^2 : \tilde{A}(x, z) \neq \emptyset \} .
\]

Under Assumption (3.3) and Proposition 3.1, we have the following equivalent characterization of $\tilde{A}(x, z)$:

**Lemma 3.3.**

\[
\begin{align*}
\tilde{H} &= \{(x, z) \in \mathbb{R}^2 : x > z \langle \tilde{w}, \Gamma \rangle, \text{ for all } \Gamma \in \tilde{M}\} \\
&= \{(x, z) \in \mathbb{R}^2 : x > \bar{p} z, z \geq 0\} \cup \{(x, z) \in \mathbb{R}^2 : x > \underline{p} z, z < 0\} ,
\end{align*}
\]

where

\[
\bar{p} \triangleq \sup_{Y \in \mathcal{M}} \langle w, Y \rangle = \sup_{\Gamma \in \tilde{M}} \langle \tilde{w}, \Gamma \rangle ,
\]

and

\[
\underline{p} \triangleq \inf_{Y \in \mathcal{M}} \langle w, Y \rangle = \inf_{\Gamma \in \tilde{M}} \langle \tilde{w}, \Gamma \rangle .
\]

where $\bar{p}, \underline{p} < \infty$ and $\tilde{H}$ is a well defined convex cone in $\mathbb{R}^2$. Moreover

\[
\text{cl}\tilde{H} = \{(x, z) \in \mathbb{R}^2 : \tilde{A}(x, z) \neq \emptyset\} \\
= \{(x, z) \in \mathbb{R}^2 : x \geq z \langle \tilde{w}, \Gamma \rangle, \text{ for all } \Gamma \in \tilde{M}\} ,
\]

where $\text{cl}\tilde{H}$ denotes the closure of the set $\tilde{H}$ in $\mathbb{R}^2$. 

PROOF. Again, it is just enough to show \( \{ (x, z) \in \mathbb{R}^2 : x \geq z \langle \tilde{w}, \Gamma \rangle, \text{ for all } \Gamma_t \in \tilde{M} \} \) is equivalent to \( \{ (x, z) \in \mathbb{R}^2 : \tilde{A}(x, z) \neq \emptyset \} \).

On one hand, if \( (x, z) \in \{ (x, z) \in \mathbb{R}^2 : \tilde{A}(x, z) \neq \emptyset \} \), there exists \( \tilde{c} \in \mathbb{L}_{L_0}^\Gamma \) such that \( \langle \tilde{c}, \Gamma \rangle \leq x - z \langle \tilde{w}, \Gamma \rangle \) for all \( \Gamma \in \tilde{M} \), clearly, we get \( x \geq z \langle \tilde{w}, \Gamma \rangle \), for all \( \Gamma \in \tilde{M} \).

On the other hand, if \( (x, z) \in \{ (x, z) \in \mathbb{R}^2 : x \geq z \langle \tilde{w}, \Gamma \rangle, \text{ for all } \Gamma_t \in \tilde{M} \} \), it is trivial to construct \( \tilde{c}_t \equiv 0 \in \tilde{A}(x, z) \) for all \( t \in [0, T] \), therefore, we have \( (x, z) \in \{ (x, z) \in \mathbb{R}^2 : \tilde{A}(x, z) \neq \emptyset \} \), which completes the proof.

We will now define the **Auxiliary Primal Utility Maximization Problem** based on the abstract auxiliary domain \( \tilde{A}(x, z) \) as:

\[
\tilde{u}(x, z) \triangleq \sup_{\tilde{c} \in \tilde{A}(x, z)} \mathbb{E} \left[ \int_0^T U(t, \tilde{c}_t) dt \right], \quad (x, z) \in \mathcal{H}.
\]

By definitions of \( \bar{A}(x, z) \) for \( (x, z) \in \bar{H} \) and \( \tilde{A}(x, z) \) for \( (x, z) \in \mathcal{H} \), we successfully embedded our original utility maximization problem (3.10) with consumption habit formation into the auxiliary abstract utility maximization problem (3.22) without habit formation, however, with some shadow random endowments. More precisely, the following equivalence can be guaranteed that for any \( (x, z) \in \bar{H} \subset \mathcal{H} \):

\[
\bar{A}(x, z) = \tilde{A}(x, z),
\]

and the two value functions coincide

\[
u(x, z) = \tilde{u}(x, z),
\]
in addition, the immediate byproduct consequence states that \( c_t^* \) is the optimal solution for \( u(x, z) \) if and only if \( \tilde{c}_t^* = c_t^* - Z_t^* \geq 0 \) for all \( t \in [0, T] \) is the optimal solution for \( \tilde{u}(x, z) \), when \( (x, z) \in \mathcal{H} \).

4. The Dual Optimization Problem and Main Results. Inspired by the idea in Hugonnier and Kramkov [15] for optimal investment with random endowment, we concentrate now on the construction of the dual problem by first introducing the set \( \mathcal{R} \), which is the relative interior of the polar cone of \(-\mathcal{H}\):

\[
\mathcal{R} \triangleq ri\left\{ (y, r) \in \mathbb{R}^2 : xy - zr \geq 0 \text{ for all } (x, z) \in \mathcal{H} \right\}.
\]

Let’s make the following assumption on stochastic discounting processes \( \alpha_t \) and \( \delta_t \):

**Assumption 4.1.**

The random variable defined by

\[
\mathcal{E} \triangleq \int_0^T w_t dt = \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} dt
\]

is not replicable under our original financial market.
Remark 4.1. Under our Assumption (4.2), the existence of Market Isomorphism by Schroder and Skiadas [30] may no longer hold and our work generally extends their conclusion and provides the existence and uniqueness of the optimal solution in incomplete markets using convex duality.

Remark 4.2. We remark here that even if \( E = \int_0^T w_t dt \) is replicable in the original incomplete market such that \( \bar{p} = \underline{p} \), the market isomorphism relation by Schroder and Skiadas [30] may still not hold. In this case, however, the original utility maximization problem becomes easier since we do not need to take care of the exogenous term \( \tilde{w}_t \) and the primal value function \( \tilde{u} \) becomes one dimensional function. The special case is discussed in detail in the final section 6.

Lemma 4.1. By Assumption (4.2), we know that \( R \) is an open convex cone in \( \mathbb{R}^2 \), and can be rewritten as:

\[
R = \left\{ (y, r) \in \mathbb{R}^2 : y > 0, \text{ and } py < r < \bar{p}y \right\},
\]

where \( \bar{p} \) and \( \underline{p} \) are defined by (3.19) and (3.20), and \( \bar{p} < \underline{p} \).

Proof. Since \( p < \bar{p} \) by Assumption (4.2), and by Lemma 3.3 the set \( \text{cl} \mathcal{H} = \{(x, z) \in \mathbb{R}^2 : x \geq \bar{p}z, z \geq 0\} \cup \{(x, z) \in \mathbb{R}^2 : x \geq \underline{p}z, z < 0\} \) does not contain any lines passing through the origin. By the properties of polars of convex sets (See Rockafellar [26], Corollary 14.6.1), \( R \) is an open convex cone in the first orthant of \( \mathbb{R}^2 \). Moreover, by the inequality constraint \( xy - zr \geq 0 \) for all \( (x, z) \in \mathcal{H} \) and the definition of \( \mathcal{H} \), it is obvious that (4.3) holds.

For an arbitrary pair \((y, r) \in \mathcal{R}\), we denote by \( \tilde{\mathcal{Y}}(y, r) \) the set of nonnegative processes as a proper extension of the auxiliary set \( \tilde{M} \) in the way that:

\[
\tilde{\mathcal{Y}}(y, r) \triangleq \left\{ \Gamma \in \mathbb{L}^0_+ : \langle \tilde{c}, \Gamma \rangle \leq xy - zr, \text{ for all } \tilde{c} \in \tilde{A}(x, z), \text{ and } (x, z) \in \mathcal{H} \right\}.
\]

Based on previous efforts, we are ready to establish the Auxiliary Dual Utility Maximization Problem to (3.22) defined as:

\[
\tilde{v}(y, r) \triangleq \inf_{\Gamma \in \tilde{Y}(y, r)} \mathbb{E} \left[ \int_0^T V(t, \Gamma_t) dt \right], \quad (y, r) \in \mathcal{R}.
\]

The following theorems constitute our main results. And we provide their proofs through a number of auxiliary results in the next section.

Theorem 4.1. Assume conditions (2.5), (2.7), (3.3), (3.4), (4.2) hold true. Assume also that (2.11), (2.12) and (2.10) (i.e., \( AE_0[U] < \infty \)) together with

\[
\tilde{u}(x, z) < \infty \quad \text{for some } (x, z) \in \mathcal{H}.
\]

we will have:
(i) The function $\tilde{u}$ is $(-\infty, \infty)$-valued on $\mathcal{H}$ and $\tilde{v}(y, r)$ is $(-\infty, \infty]$-valued on $\mathcal{R}$. And for each $(y, r) \in \mathcal{R}$ there exists a constant $s = s(y, r) > 0$ such that $\tilde{v}(sy, sr) < \infty$. Moreover, we have the conjugate duality of value functions $\tilde{u}$ and $\tilde{v}$:

$$\tilde{u}(x, z) = \inf_{(y, r) \in \mathcal{R}} \{ \tilde{v}(y, r) + xy - zr \}, \quad (x, z) \in \mathcal{H},$$

$$\tilde{v}(y, r) = \sup_{(x, z) \in \mathcal{H}} \{ \tilde{u}(x, z) - xy + zr \}, \quad (y, r) \in \mathcal{R}.$$

(ii) The solution $\Gamma^*(y, r)$ to the optimization problem (4.5) exists and is unique (in the sense of $\equiv$ in $L^0_+$) for all $(y, r) \in \mathcal{R}$ such that $\tilde{v}(y, r) < \infty$.

**Theorem 4.2.** We now assume in addition to conditions of Theorem 4.1 that Assumption (2.9) (i.e., $AE_\infty[U] < 1$) holds. Then in addition to assertions of Theorem 4.1, we also have:

(i) The value function $\tilde{v}(y, r)$ is $(-\infty, \infty)$-valued on $(y, r) \in \mathcal{R}$ and $\tilde{v}$ is continuously differentiable on $\mathcal{L}$.

(ii) The solution $\tilde{c}^*(x, z)$ to optimization problem (3.22) exists and is unique (in the sense of $\equiv$ in $L^0_+$) for any $(x, z) \in \mathcal{H}$, and there exists a representation of the optimal solution such that $\tilde{c}^*(x, z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$.

(iii) The superdifferential of $\tilde{u}$ maps $\mathcal{H}$ into $\mathcal{R}$, i.e.,

$$\partial \tilde{u}(x, z) \subset \mathcal{R}, \quad (x, z) \in \mathcal{H}. \tag{4.7}$$

Moreover, if $(y, r) \in \partial \tilde{u}(x, z)$, then there exists a representation of the optimal solution such that $\Gamma^*_t(y, r) > 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$ and $\tilde{c}^*(x, z)$ and $\Gamma^*(y, r)$ are related by:

$$\Gamma^*_t(y, r) = U'(t, \tilde{c}^*_t(x, z)) \quad \text{or} \quad \tilde{c}^*_t(x, z) = I(t, \Gamma^*_t(y, r)), \tag{4.8}$$

$$\langle \Gamma^*(y, r), \tilde{c}^*(x, z) \rangle = xy - zr.$$

(iv) If we restrict the choice of initial wealth $x$ and initial standard of living $z$ such that $(x, z) \in \tilde{\mathcal{H}} \subset \mathcal{H}$, the solution $\tilde{c}^*_t(x, z)$ to our primal utility optimization problem (3.10) exists and is unique, moreover,

$$\tilde{c}^*_t(x, z) = c^*_t(x, z) - Z^*_t(x, z).$$

5. Proofs of Main Results.

5.1. The Proof of Theorem 4.1. The following Proposition will serve as the key step to build some future Bipolar relationships:

**Proposition 5.1.** Assume all assumptions of Theorem 4.1 hold true. Then the families $(\tilde{A}(x, z))_{(x, z) \in \mathcal{H}}$ and $(\tilde{Y}(y, r))_{(y, r) \in \mathcal{R}}$ have the following properties:
For any \((x, z) \in \mathcal{H}\), the set \(\tilde{\mathcal{A}}(x, z)\) contains a strictly positive random variable on the product space. A nonnegative random variable \(\tilde{c}\) belongs to \(\tilde{\mathcal{A}}(x, z)\) if and only if
\[
\langle \tilde{c}, \Gamma \rangle \leq xy - zr \quad \text{for all} \quad (y, r) \in \mathcal{R} \quad \text{and} \quad \Gamma \in \tilde{\mathcal{Y}}(y, r).
\]

(ii) For any \((y, r) \in \mathcal{R}\), the set \(\tilde{\mathcal{Y}}(y, r)\) contains a strictly positive random variable on the product space. A nonnegative random variable \(\Gamma\) belongs to \(\tilde{\mathcal{Y}}(y, r)\) if and only if
\[
\langle \tilde{c}, \Gamma \rangle \leq xy - zr \quad \text{for all} \quad (x, z) \in \mathcal{H} \quad \text{and} \quad \tilde{c} \in \tilde{\mathcal{A}}(x, z).
\]

In order to prove Proposition 5.1, for any \(p > 0\), we denote by \(\mathcal{M}(p)\) the subset of \(\mathcal{M}\) such that \(\langle w, Y \rangle = p\). Then for any density process \(Y \in \mathcal{M}(p)\), define the auxiliary set as
\[
\tilde{\mathcal{M}}(p) \triangleq \left\{ \Gamma \in \mathbb{C}_+^0 : \Gamma_t = Y_t + \delta_t \mathbb{E} \left[ \int_t^T e^{f_s'(\delta_v - \alpha_v)dv} Y_s \bigg| \mathcal{F}_t \right], \forall t \in [0, T], Y \in \mathcal{M}(p) \right\}.
\]
We have \(\langle \tilde{w}, \Gamma \rangle = \langle w, Y \rangle = p\).

Define \(\mathcal{P}\) as the open interval \(\mathcal{P} = (\underline{p}, \bar{p})\), where \(\underline{p}, \bar{p}\) are defined in (3.19) and (3.20). We have the following result.

**Lemma 5.1.** Assume that conditions of Proposition 5.1 hold true and let \(p > 0\). Then the set \(\tilde{\mathcal{M}}(p)\) is not empty if and only if \(p \in \mathcal{P} = (\underline{p}, \bar{p})\), where \(\underline{p}, \bar{p}\) are defined in (3.19) and (3.20). In particular,
\[
\bigcup_{p \in \mathcal{P}} \tilde{\mathcal{M}}(p) = \tilde{\mathcal{M}}.
\]
where the set \(\tilde{\mathcal{M}}\) is defined by (3.13).

**Proof.** The proof reduces to verifying that \(\mathcal{P} = \mathcal{P}'\), where we define
\[
\mathcal{P}' \triangleq \{ p > 0 : \tilde{\mathcal{M}}(p) \neq \emptyset \}.
\]

Similar to the proof of Lemma 8 of Hugonnier and Kramkov [15], one direction inclusion that \(\mathcal{P} \subseteq \mathcal{P}'\) is obvious.

For the inverse, let \(p \in \mathcal{P}'\), \((x, z) \in \partial \mathcal{H}\), \(\Gamma \in \tilde{\mathcal{M}}(p)\), and we first claim there exists a \(\tilde{c} \in \tilde{\mathcal{A}}(x, z)\) such that
\[
\bar{p}[\tilde{c} \succ 0] > 0,
\]
so we get
\[
0 < \langle \tilde{c}, \Gamma \rangle \leq x - zp.
\]
As \((x, z)\) is an arbitrary element of \(\partial \mathcal{H}\), we have \(p \in \mathcal{P}\).

As for the above claim, according to Theorem 2.11 of Schachermayer [29], Assumption (4.2) guarantees that for all \(Y \in \mathcal{M}\), we have
\[
\underline{p} < \langle w, Y \rangle < \bar{p},
\]
which is
\[ p < \langle \tilde{w}, \Gamma \rangle < \bar{p}, \]
for all the \( \Gamma \in \tilde{M} \). Then by the definition of \( clH \) in Lemma 3.3, we observe that for any \( (x, z) \in clH \), we will have
\[ x - z\langle \tilde{w}, \Gamma \rangle > 0, \]
for all the \( \Gamma \in \tilde{M} \), and the claim holds by the definition of \( \tilde{A}(x, z) \).

**Lemma 5.2.** Assume that conditions of Proposition 5.1 hold true and let \( p \in \mathcal{P} = (p, \bar{p}) \), we have then \( \tilde{M}(p) \subseteq \tilde{Y}(1, p) \).

**Proof.** The conclusion can be directly derived in light of the definition of \( \tilde{A}(x, z) \) and \( \tilde{Y}(1, p) \).

**Lemma 5.3.** Assume that conditions of Proposition 5.1 hold true. For any \((x, z) \in H\), a nonnegative random variable \( \tilde{c} \) belongs to \( \tilde{A}(x, z) \) if and only if
\[(5.5) \quad \langle \tilde{c}, \Gamma \rangle \leq x - zp \quad \text{for all } p \in \mathcal{P} \text{ and } \Gamma \in \tilde{M}(p). \]

**Proof.** If \( \tilde{c} \in \tilde{A}(x, z) \), the definition of \( \tilde{A}(x, z) \) and the fact \( \tilde{M}(p) \subset \tilde{M} \) guarantee the validity of (5.5).

On the other hand, for any \( \tilde{c} \in L^0_+ \) such that (5.5) holds true, we will have:
\[
\sup_{\Gamma \in \tilde{M}} \langle \tilde{c} + z\tilde{w}, \Gamma \rangle = \sup_{p \in \mathcal{P}} \sup_{\Gamma \in \tilde{M}(p)} \langle \tilde{c} + z\tilde{w}, \Gamma \rangle \\
= \sup_{p \in \mathcal{P}} \sup_{\Gamma \in \tilde{M}(p)} \left( \langle \tilde{c}, \Gamma \rangle + zp \right) \leq x.
\]
The claim holds according to the definition of \( \tilde{A}(x, z) \).

**Proof of Proposition 5.1.**
For the validity of assertion (i), consider \((x, z) \in H\), there exists a \( \lambda > 0 \) such that \((x - \lambda, z) \in H\) since \( H \) is an open set.

Let \( \tilde{c} \in \tilde{A}(x - \lambda, z) \), we will have for any \( \Gamma \in \tilde{M} \), and \( \tilde{w}_t = e^{-\int_0^t \alpha_v dv} > 0 \),
\[(5.6) \quad \langle \tilde{c}, \Gamma \rangle \leq x - \lambda - z\langle \tilde{w}, \Gamma \rangle. \]

By Assumption (3.3) and Proposition 3.1, we define \( \rho_t \triangleq \frac{1}{\bar{p}} \tilde{w}_t > 0 \) for all \( t \in [0, T] \), then for all \( \Gamma \in \tilde{M} \):
\[
\langle \rho, \Gamma \rangle \leq \langle \tilde{c} + \rho, \Gamma \rangle \leq x - \lambda - z\langle \tilde{w}, \Gamma \rangle + \frac{\lambda}{\bar{p}}\langle \tilde{w}, \Gamma \rangle \\
\leq x - \lambda - z\langle \tilde{w}, \Gamma \rangle + \lambda \leq x - z\langle \tilde{w}, \Gamma \rangle.
\]
Hence, we have shown the existence of a strictly positive element \( \rho_t > 0 \in \tilde{A}(x, z) \) by the definition of \( \tilde{A}(x, z) \).
If (5.1) holds for some $\tilde{c} \in \mathbb{L}_0^p$. The density process $\Gamma \in \tilde{\mathcal{M}}(p)$ belongs to $\tilde{\mathcal{Y}}(1, p)$ for all $p \in P$ by Lemma 5.2, and hence (5.5) holds. Lemma 5.3 then implies that $\tilde{c} \in \tilde{\mathcal{A}}(x, z)$. Conversely, suppose now $\tilde{c} \in \tilde{\mathcal{A}}(x, z)$, the definition of sets $\tilde{\mathcal{Y}}(y, r)$, $(y, r) \in R$ implies (5.1) and we complete the proof of assertion (i).

For the proof the assertion (ii), notice
\[ k\tilde{Y}(y, r) = \tilde{Y}(ky, kr) \text{ for all } k > 0, (y, r) \in R. \]

Therefore we just need to consider $(y, r) = (1, p)$ for some $p \in P$. Lemma 5.2 implies $\Gamma \in \tilde{\mathcal{M}}(p) \subseteq \tilde{\mathcal{Y}}(1, p)$, and the existence of $Y > 0 \in \mathcal{M}(p)$ takes care of the existence $\Gamma > 0 \in \tilde{\mathcal{M}}(p)$, $\bar{P}$-a.s.

The second part is a direct consequence of the definition of $\tilde{\mathcal{Y}}(y, r)$.

For the proof of Theorem 4.1, we will also need the following lemmas:

**Lemma 5.4.** Under assumptions of Theorem 4.1, the value function $\tilde{u}$ is $(-\infty, \infty)$-valued on $\mathcal{H}$.

**Proof.** First, by Lemma 2.1, the assumption $AE_0[U] < \infty$ implies that for any positive constant $s > 0$, the existence of $s_1 > 0$ and $s_2 > 0$ such that for all $t \in [0, T]$:
\begin{equation}
U(t, x/s) \geq s_1 U(t, x) + s_2, \quad x > 0,
\end{equation}

According to Assumption (3.4) and the proof of Proposition 5.1, for each fixed pair $(x, z) \in \mathcal{H}$, there exists $\lambda = \lambda(x, z) > 0$ such that $\frac{1}{p} \tilde{w}_t \in \tilde{\mathcal{A}}(x, z)$, therefore we deduce that $\tilde{w}_t \in \tilde{\mathcal{A}}(\frac{p}{\lambda} x, \frac{p}{\lambda} z)$, and
\[ \tilde{u}((\frac{p}{\lambda} x, \frac{p}{\lambda} z)) = \sup_{\tilde{c} \in \tilde{\mathcal{A}}(\frac{p}{\lambda} x, \frac{p}{\lambda} z)} \mathbb{E}
\left[ \int_0^T U(t, \tilde{c}_t)dt \right] \geq \mathbb{E}
\left[ \int_0^T U(t, \tilde{w}_t)dt \right] > -\infty, \]

hence, for any $(x, z) \in \mathcal{H}$, we get the existence of a constant $s(x, z) > 0$, such that $\tilde{u}(sx, sz) > -\infty$, with $s(x, z) = \frac{p}{\lambda}$.

Since, for any constant $s > 0$,
\[ \tilde{\mathcal{A}}(x, z) = \tilde{\mathcal{A}}(sx, sz)/s, \]
we derive $\tilde{u}(x, z) > -\infty$ if $\tilde{u}(sx, sz) > -\infty$ holds for a constant $s = s(x, z) > 0$, follow the result above, we conclude that $\tilde{u}(x, z) > -\infty$ in the whole domain $\mathcal{H}$.

Now, since the set $\mathcal{H}$ is open and $\tilde{u}(x, z) < \infty$ for some $(x, z) \in \mathcal{H}$ by assumption (4.6), we deduce that $\tilde{u}$ is finitely valued on $\mathcal{H}$ by the concavity of $\tilde{u}$ on $\mathcal{H}$. And the proof is complete. \hfill \qed

Before we state the next lemma, let’s introduce a special concept of compactness which was originally defined in Žitković [33].
**Definition 5.1.** A convex subset $C$ of a topological vector space $X$ is said to be convexly compact if for any non-empty set $A$ and any family $\{F_a\}_{a \in A}$ of closed, convex subsets of $C$, the condition

$$\forall D \in \text{Fin}(A), \bigcap_{a \in D} F_a \neq \emptyset \implies \bigcap_{a \in A} F_a \neq \emptyset$$

where the set $\text{Fin}(A)$ consists of all non-empty finite subsets of $A$ for an arbitrary non-empty set $A$.

Without the restriction that the sets $\{F_a\}_{a \in A}$ must be convex, this definition would be equivalent to compactness in the original sense. Thus any convex and compact set is convexly compact and Definition 5.1 extends the concept of compactness.

Žitković [33] furthermore derived an easy characterization on the space of non-negative, measurable functions, see Theorem 3 of Žitković [33] which states that

**Theorem 5.1.** A closed and convex subset $C$ of $L^0_+$ is convexly compact if and only if it is bounded in finite measure.

Based on the above theorem, we have the following lemma on the convex compactness of sets $\tilde{A}(x, z)$ and $\tilde{Y}(y, r)$:

**Lemma 5.5.** For each pair $(x, z) \in H$ and $(y, r) \in R$, the sets $\tilde{A}(x, z)$ and $\tilde{Y}(y, r)$ are convex, solid and closed in the topology of convergence in measure $\bar{P}$. Moreover, they are both bounded in $L^0_+(\Omega \times [0, T], \mathcal{O}, \bar{P})$, hence they are both convexly compact.

**Proof.** For $(y, r) \in R$, we now define two auxiliary sets as

$$H(y, r) \triangleq \left\{(x, z) \in H : xy - rz \leq 1\right\}$$

$$\mathcal{A}(k) \triangleq \bigcup_{(x, z) \in K \cap H(y, r)} \tilde{A}(x, z),$$

and denote by $\tilde{\mathcal{A}}(k)$ the closure of $\mathcal{A}(k)$ with respect to convergence in measure $\bar{P}$.

From Proposition 5.1, we deduce that

$$\Gamma \in \tilde{Y}(y, r) \iff \langle \tilde{c}, \Gamma \rangle \leq 1, \quad \forall \tilde{c} \in \tilde{\mathcal{A}}(1)$$

Hence, sets $\tilde{Y}(y, r)$ and $\tilde{\mathcal{A}}(1)$ satisfy

$$\tilde{Y}(y, r) = \tilde{\mathcal{A}}(1)^\circ.$$

At the same time, by its definition, we have $\tilde{\mathcal{A}}(1)$ itself is closed, convex and solid, by the Bipolar theorem in Brannath and Schachermayer [2], we have $\tilde{\mathcal{A}}(1) = \tilde{\mathcal{A}}(1)^{\circ\circ}$, and hence we have the following Bipolar relationship:

$$\tilde{\mathcal{A}}(1) = \tilde{Y}(y, r)^\circ$$

(5.9)

$$\tilde{Y}(y, r) = \tilde{\mathcal{A}}(1)^\circ.$$

(HABIT FORMATION)
The Bipolar theorem on $L^0_+$ gives the convexity, solidness and closure in measure $\bar{P}$.
Similarly, for $(x, z) \in \mathcal{H}$, now define the set:

$$\mathfrak{R}(x, z) \triangleq \{(y, r) \in \mathcal{R} : xy - zr \leq 1\},$$

and denote by $\mathfrak{Y}(k)$ the closure of $\mathfrak{Y}(k)$ with respect to convergence in measure $\bar{P}$.

Now, again Proposition 5.1 implies

$$\check{c} \in \check{A}(x, z) \iff \langle \check{c}, \Gamma \rangle \leq 1, \forall \Gamma \in \mathfrak{Y},$$

and the Bipolar relationship:

$$\mathfrak{Y}(1) = \check{A}(x, z)^{\circ}, \quad \check{A}(x, z) = \mathfrak{Y}(1)^{\circ}.$$

Hence, we also have $\check{A}(x, z)$ is convex, solid and closed in the topology of convergence in measure $\bar{P}$.

Moreover, thanks to the existence of $0 \prec \Gamma \in \mathfrak{M}(p)$ which is also in $\mathfrak{Y}(1, p)$, we deduce the set $\check{A}(x, z)$ is bounded in measure $\bar{P}$ by Proposition 5.1 part (i).

Similarly, as in the proof of Proposition 5.1, we have derived the existence of $\lambda = \lambda(x, z)$ such that $0 \prec \rho_t = \frac{\lambda}{\bar{p}} \tilde{w}_t \in \check{A}(x, z)$, due to Proposition 5.1 part (ii), we get the set $\mathfrak{Y}(y, r)$ is also bounded in measure $\bar{P}$ and therefore both of them are convexly compact in $L^0_+$. 

A major difficulty arises in the proof of the existence of the dual optimizer in our setting due to the lack of integrability of the dual process $\Gamma \in \mathfrak{Y}(y, r)$ for $(y, r) \in \mathcal{R}$. In fact, the trick of applying de la Vallée-Poussin theorem in the proof of Lemma 3.2 in Kramkov and Schachermayer [20] and Lemma A.1 in Karatzas and Žitković [17] does not work. And the argument of contradiction mimicking the proof of Lemma 1 in Kramkov and Schachermayer [21] using the subsequence splitting lemma will also fail by observing the constant may not be contained in the dual space. Contrary to the results in the literature, much effort has to be made to modify the classic analysis, where the Assumptions of $\text{AE}[U]_0 < \infty$ and $\mathbb{E}\left[\int_0^T U(t, \tilde{x} \tilde{w}_t)dt\right] > -\infty$ are critical for the procedure of our proof of the following lemmas.

**Lemma 5.6.** Under assumptions of theorem 4.1, we have for each fixed $(y, r) \in \mathcal{R}$

$$\sup_{\Gamma \in \mathfrak{Y}(y, r)} \mathbb{E}\left[\int_0^T V^-(t, \Gamma_t)dt\right] < \infty.$$

**Proof.** Assumption (3.4) admits the existence of $\tilde{x} \tilde{w}_t \in L^0_+$ such that

$$\mathbb{E}\left[\int_0^T U(t, \tilde{x} \tilde{w}_t)dt\right] > -\infty,$$

and moreover, by the proof of Proposition 5.1, we also know for each fixed $(y, r) \in \mathcal{R}$, find the fixed pair $(x, z) \in \tilde{\mathfrak{Y}}(y, r)$, there exists a constant $\lambda(x, z) > 0$
such that \( \tilde{w} \in \tilde{\mathcal{A}}(\frac{t}{\lambda}) \), where \( \tilde{p} \) is defined by (3.19). Taking into account the inequality \( U(t, x) \leq V(t, y) + xy \), we have for any \( \Gamma \in \tilde{\mathcal{Y}}(y, r) \) and \( y_0(t) \triangleq \inf\{y > 0 : V(t, y) < 0\} \)

\[
\mathbb{E} \left[ \int_0^T V^-(t, \Gamma_t) dt \right] \leq -\mathbb{E} \left[ \int_0^T V(t, \Gamma_t 1_{\{\Gamma_t \geq y_0(t)\}} + y_0(t) 1_{\{\Gamma_t < y_0(t)\}}) dt \right]
\]

\[
\leq -\mathbb{E} \left[ \int_0^T U(t, \tilde{w} t) dt \right] + \tilde{x} \mathbb{E} \left[ \int_0^T \tilde{w} t \Gamma_t dt \right] + \tilde{x} \mathbb{E} \left[ \int_0^T \tilde{w} t (y_0(t) - \Gamma_t) 1_{\{\Gamma_t < y_0(t)\}} dt \right]
\]

\[
\leq -\mathbb{E} \left[ \int_0^T U(t, \tilde{w} t) dt \right] + \tilde{x} \tilde{p} + \tilde{x} \int_0^T y_0(t) dt,
\]

which is finitely valued and independent of the initial choice of \( \Gamma \) since we have \( \tilde{w} t \triangleq e^{\int_0^t (-\alpha_\gamma) ds} \leq 1 \) for \( t \in [0, T] \) and \( \sup_{t \in [0, T]} y_0(t) < \infty \) by Assumption (2.12), and thus our conclusion holds true.

**Lemma 5.7.** Under assumptions of Theorem 4.1, we have for any \((y, r) \in \mathcal{R}, \left(V^-(\cdot, \Gamma)\right)\) is uniformly integrable for all \( \Gamma \in \tilde{\mathcal{Y}}(y, r) \).

**Proof.** By Corollary 2.1, the assumption \( AE_0[U] < \infty \) is equivalent to the following assertions:

\[
\exists y_0 > 0, \text{ and } \mu \in (1, 2), \quad \forall y \geq y_0, \quad V(t, 2y) \geq \mu V(t, y).
\]

Let \( y_0 > 0 \) and \( \mu \in (1, 2) \) be the constants in the above (5.12). Take \( \gamma = \log_2 \mu \in (0, 1) \), we define the auxiliary function \( \tilde{V}(t, y) : [0, \infty) \times (0, \infty) \rightarrow \mathbb{R} \) by

\[
\tilde{V}(t, y) \triangleq \begin{cases} 
-\frac{2y_0}{\gamma} V'(t, 2y_0) - V(t, y), & y \geq 2y_0, \\
- V(t, 2y_0) - \frac{2y_0}{\gamma} V'(t, 2y_0)(\frac{y_0}{2y_0})^\gamma, & y < 2y_0.
\end{cases}
\]

For each fixed \( t > 0 \), \( \tilde{V}(t, y) \) is a nonnegative, concave, and nondecreasing function which agrees with \( -V(t, y) \) up to a constant for large enough values of \( y \) and satisfies

\[
\tilde{V}(t, 2y) \leq \mu \tilde{V}(t, y), \quad \text{for all } y > 0.
\]

Lemma 5.6 asserts

\[
\sup_{\Gamma \in \tilde{\mathcal{Y}}(y, r)} \mathbb{E} \left[ \int_0^T V^-(t, \Gamma_t) dt \right] < \infty,
\]

and hence in light of the fact that \( V^- \) and \( \tilde{V} \) differ only by a constant in a neighborhood of \( \infty \), we will get

\[
\sup_{\Gamma \in \tilde{\mathcal{Y}}(y, r)} \mathbb{E} \left[ \int_0^T \tilde{V}(t, \Gamma_t) dt \right] < \infty.
\]

The validity of uniform integrability of the sequence \((V^-(\cdot, \Gamma^n))\) \( n \geq 1 \) for \( \Gamma^n \in \tilde{\mathcal{Y}}(y, r) \), is therefore equivalent to the uniform integrability of \((\tilde{V}(\cdot, \Gamma^n))\) \( n \geq 1 \).
To this end, we argue by contradiction. Suppose this sequence is not uniformly integrable, then by Rosenthal’s subsequence splitting lemma, we can find a subsequence \((f^n)_{n \geq 1}\), a constant \(\varepsilon > 0\) and a disjoint sequence \((A^n)_{n \geq 1}\) of \((\Omega \times [0, T], \mathcal{O})\) with

\[ A^n \in \mathcal{O}, \quad A^i \cap A^j = \emptyset \quad \text{if} \quad i \neq j, \]

such that

\[ \mathbb{E} \left[ \int_0^T \bar{V}(t, f^n_t)1_{A^n} dt \right] \geq \varepsilon, \quad \text{for} \quad n \geq 1 \]

We define the sequence of random variables \((h^n)_{n \geq 1}\)

\[ h^n_t = \sum_{k=1}^{n} f^k_t 1_{A^k}. \]

For any \(\tilde{c} \in \tilde{\mathfrak{A}}(1)\),

\[ \langle \tilde{c}, h^n \rangle \leq \sum_{k=1}^{n} \langle \tilde{c}, f^k \rangle \leq n. \]

Hence \(h^n_n \in \tilde{\mathcal{Y}}(y,r)\).

One the other hand,

\[ \mathbb{E} \left[ \int_0^T \bar{V}(t, h^n_n) dt \right] \geq \sum_{k=1}^{n} \mathbb{E} \left[ \int_0^T \bar{V}(t, f^k_t)1_{A^k} dt \right] \geq \varepsilon n, \]

and therefore by taking \(n = 2^m\), via iteration, it produces

\[ \mu^m \sup_{\Gamma \in \tilde{\mathcal{Y}}(y,r)} \mathbb{E} \left[ \int_0^T \bar{V}(t, \Gamma_t) dt \right] \geq \mu^m \mathbb{E} \left[ \int_0^T \bar{V}(t, \frac{h^n_n}{2^m}) dt \right] \geq \mathbb{E} \left[ \int_0^T \bar{V}(t, h^n_n) dt \right] \geq 2^m \varepsilon, \]

since \(\mu \in (1,2)\), this contradicts (5.15) for \(m\) large enough, therefore the conclusion holds true.

\textbf{Lemma 5.8.} For any pair \((y,r) \in \mathcal{R}\) such that \(\bar{v}(y,r) < \infty\), the optimal solution \(\Gamma^*\) to the optimization problem (4.5) exists and is unique.

\textbf{Proof.} Now fix \((y,r) \in \mathcal{R}\), let \((\Gamma^n)_{n \geq 1}\) be a sequence in \(\tilde{\mathcal{Y}}(y,r)\) such that

\[ \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T V(t, \Gamma^n_t) dt \right] = \bar{v}(y,r). \]

There exists a sequence of forward convex combinations \(f^n \in \text{conv}(\Gamma^n, \Gamma^{n+1}, \ldots)\) which converges almost surely to a random variable \(\Gamma^*\) with values in \([0, \infty]\). Since the set \(\tilde{\mathcal{Y}}(y,r)\) is closed and bounded in measure \(\mathbb{P}\) in \(L^0_+\) by Lemma 5.5, we
deduce that $\Gamma^*$ is almost surely finitely valued, moreover, $\Gamma^*$ belongs to $\mathcal{Y}(y, r)$. We claim that $\Gamma^*$ is the optimal solution to (4.5), that is

$$
\mathbb{E}\left[ \int_0^T V(t, \Gamma^*_t)dt \right] = \tilde{v}(y, r).
$$

The concavity of $V$ produces

$$
\lim_{n \to \infty} \mathbb{E}\left[ \int_0^T V(t, f^n_t)dt \right] \leq \tilde{v}(y, r),
$$

and Fatou's lemma implies

$$
\liminf_{n \to \infty} \mathbb{E}\left[ \int_0^T V^+(t, f^n_t)dt \right] \geq \mathbb{E}\left[ \int_0^T V^+(t, \Gamma^*_t)dt \right].
$$

The optimality of $\Gamma^*_t$ will follow if we can show

$$
\lim_{n \to \infty} \mathbb{E}\left[ \int_0^T V^-(t, f^n_t)dt \right] = \mathbb{E}\left[ \int_0^T V^-(t, \Gamma^*_t)dt \right],
$$

but the validity of (5.16) is a consequence of Lemma 5.7. \hfill \Box

For the proof of conjugate duality relations between value functions $\tilde{u}(x, z)$ and $\tilde{v}(y, r)$, similar to the proof of Lemma 11 of Hugonnier and Kramkov [15], we have the following general result:

**Lemma 5.9.** If $\mathcal{G} \subseteq L^0_+$ is convex and contains a strictly positive random variable. Then

$$
\sup_{g \in \mathcal{G}} \mathbb{E}\left[ \int_0^T U(t, xg_t)dt \right] = \sup_{g \in \text{cl}\mathcal{G}} \mathbb{E}\left[ \int_0^T U(t, xg_t)dt \right], \quad x > 0
$$

where $\text{cl}\mathcal{G}$ denotes the closure of $\mathcal{G}$ with respect to convergence in measure $\bar{\mathbb{P}}$.

**Lemma 5.10.** For $\tilde{w}_t \triangleq e^{\int_0^t (-\alpha_v)dv}$, we have the following result:

$$
\mathbb{E}\left[ \int_0^T V^-(t, U''(t, \tilde{w}_t))dt \right] < \infty.
$$

**Proof.** Similar to the proof of Lemma 5.6, recall the Assumption that $\mathbb{E}\left[ \int_0^T U(t, x\tilde{w}_t)dt \right] > -\infty$, taking into account the inequality $U(t, x) < V(t, y) + xy$, we have
for any $y_0(t) \triangleq \inf\{y > 0 : V(t, y) < 0\}$,

$$
\mathbb{E}\left[ \int_0^T V^-(t, U'(t, \bar{w}_t))dt \right] \\
\leq -\mathbb{E}\left[ \int_0^T V(t, U'(t, \bar{w}_t))1\{U'(t, \bar{w}_t) \geq y_0(t)\} + y_0(t)1\{U'(t, \bar{w}_t) < y_0(t)\}dt \right] \\
\leq -\mathbb{E}\left[ \int_0^T U(t, \bar{w}_t)dt \right] + \bar{x}\mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)dt \right] \\
+ \bar{x}\mathbb{E}\left[ \int_0^T \bar{w}_t(y_0(t) - U'(t, \bar{w}_t))1\{U'(t, \bar{w}_t) < y_0(t)\}dt \right] \\
\leq -\mathbb{E}\left[ \int_0^T U(t, \bar{w}_t)dt \right] + \bar{x}\mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)dt \right] + \bar{x}\int_0^T y_0(t)dt.
$$

(5.18)

We already know the first term and the third term are bounded, as for the second term, we have two different cases:

1. If we have $\bar{x} \leq 1$, then we can rewrite the second term as

$$
\mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)dt \right] = \mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)1\{\bar{w}_t \leq x_0\}dt \right] \\
+ \mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)1\{\bar{w}_t > x_0\}dt \right],
$$

where $x_0$ is the uniform constant in Corollary 2.1 such that for all $t \in [0, T]$,

$$
x U'(t, x) < \left( \frac{\gamma}{1 - \gamma} \right) \left( -U(t, x) \right) \text{ for } 0 < x \leq x_0.
$$

(5.19)

Again, use the fact that $\bar{w} \leq 1$, we have

$$
\mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)1\{\bar{w}_t > x_0\}dt \right] < \infty,
$$

and we also have

$$
\mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)1\{\bar{w}_t \leq x_0\}dt \right] \\
\leq -\left( \frac{\gamma}{1 - \gamma} \right) \mathbb{E}\left[ \int_0^T U(t, \bar{w}_t)dt \right] \\
\leq -\left( \frac{\gamma}{1 - \gamma} \right) \mathbb{E}\left[ \int_0^T U(t, \bar{w}_t)dt \right] < \infty,
$$

by using the inequality (5.19), the increasing property of $U(t, x)$ with respect to $x$ and the Assumption (3.4).

2. If we have $\bar{x} > 1$, then we rewrite the second term as:

$$
\mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)dt \right] = \mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)1\{\bar{w}_t \leq x_0\}dt \right] \\
+ \mathbb{E}\left[ \int_0^T \bar{w}_t U'(t, \bar{w}_t)1\{\bar{w}_t > x_0\}dt \right].
$$
where $x_0$ is the uniform constant in Corollary 2.1 such that for all $t \in [0, T]$, the inequality (5.19) holds and moreover,

\begin{equation}
U(t, \frac{1}{x}x) > \left(\frac{1}{x}\right)^{-\frac{1}{1-\gamma}} U(t, x) \quad \text{for } 0 < x \leq x_0,
\end{equation}

holds for all $t \in [0, T]$.

Then, again, the second term is bounded since $\tilde{x}\tilde{w} \preceq \bar{x}$, and for the first term, we have

\[
\mathbb{E}\left[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t) \mathbb{1}_{\{\tilde{x}\tilde{w}_t \leq x_0\}} \, dt\right] \leq -\left(\frac{\gamma}{1-\gamma}\right) \mathbb{E}\left[\int_0^T U(t, \tilde{w}_t) \mathbb{1}_{\{\tilde{x}\tilde{w}_t \leq x_0\}} \, dt\right]
\leq -\left(\frac{\gamma}{1-\gamma}\right) \left(\frac{1}{x}\right)^{-\frac{1}{1-\gamma}} \mathbb{E}\left[\int_0^T U(t, \tilde{x}\tilde{w}_t) \, dt\right] < \infty,
\]

by the inequality (5.19) and (5.20) and the Assumption (3.4).

Hence we proved the second term in (5.18) is also finite, and we can therefore conclude that result (5.17) holds true.

We should again emphasize the fact that the auxiliary dual domain $\tilde{\mathcal{Y}}(y, r)$ is not necessary a subset of $L^1$, and hence we have to revise the usual Minimax theorem based on $L^1$ to derive the important conjugate duality relationship. Fortunately, the following Minimax theorem proved by Kauppila [18] can serve as a substitute tool on the space $\mathbb{L}^0_+$.

**Theorem 5.2 (Minimax Theorem).** Let $A$ be a nonempty convex subset of a topological space, and $B$ a nonempty, closed, convex, and convexly compact subset of a topological vector space. Let $H : A \times B \to \mathbb{R}$ be convex on $A$, and concave and upper-semicontinuous on $B$. Then

\[
\sup_B \inf_A H = \inf_A \sup_B H.
\]

See the detail proof in Theorem A.1 in Appendix A by Kauppila [18]. We remark this Minimax Theorem is a relaxed version of Theorem 4.9 by Žitković [33]. Contrary to the assumption of Žitković [33] that the target functional needs to be semi-continuous with respect to both vector spaces, Kauppila [18] only requires the functional has semi-continuity property on one of the vector spaces, which can be applied to our case.

**Lemma 5.11.** Under assumptions of Theorem 4.1, the conjugate duality relations hold:

\begin{align}
\hat{u}(x, z) &= \inf_{(y, r) \in \mathcal{R}} \{\hat{v}(y, r) + xy - zr\}, \quad (x, z) \in \mathcal{H}, \\
\hat{v}(y, r) &= \sup_{(x, z) \in \mathcal{H}} \{\hat{u}(x, z) - xy + zr\}, \quad (y, r) \in \mathcal{R}.
\end{align}

**Proof.** For $n > 0$, we define $\mathcal{S}_n$ as a subset in $\mathbb{L}^0_+(\Omega \times [0, T], \sigma, \mathbb{P})$ as

\[
\mathcal{S}_n = \{\tilde{c} \in \mathbb{L}^0_+ : 0 \preceq \tilde{c} \preceq n\tilde{w}\}.
\]
It is clear that sets $S_n$ are closed, convex, and bounded in probability, and hence convexly compact in $L^0_+$. We will first show that the functional
\[
\tilde{c} \mapsto E \left[ \int_0^T \left( U(t, \tilde{c}_t) - \tilde{c}_t \Gamma_t \right) dt \right]
\]
is upper-semicontinuous on $S_n$ in the topology of convergence in measure $\bar{P}$, for all $\Gamma \in \tilde{Y}(y, r)$ and $(y, r) \in R$.

In fact, by passing if necessary to a subsequence denoted by $(\tilde{c}_m)_{m \geq 1}$ converges almost surely to $\tilde{c} \in S_n$, Fatou’s lemma implies both

(5.22) \[ \liminf_{m \to \infty} E \left[ \int_0^T U(t, \tilde{c}_m^m) dt \right] \geq E \left[ \int_0^T U(t, \tilde{c}_t)^- dt \right] \]
and

(5.23) \[ \liminf_{m \to \infty} E \left[ \int_0^T \tilde{c}_m^m \Gamma_t dt \right] \geq E \left[ \int_0^T \tilde{c}_t \Gamma_t dt \right]. \]

Moreover, on $S_n$, it is clear that $E \left[ \int_0^T U(t, \tilde{c}_m^m)^- dt \right]$ is uniformly integrable, and hence

(5.24) \[ \lim_{m \to \infty} E \left[ \int_0^T U(t, \tilde{c}_m^m)^+ dt \right] = E \left[ \int_0^T U(t, \tilde{c}_t)^+ dt \right]. \]

Now, together with (5.22) and (5.23), we have

\[ \limsup_{m \to \infty} E \left[ \int_0^T \left( U(t, \tilde{c}_m^m) - \tilde{c}_m^m \Gamma_t \right) dt \right] \leq E \left[ \int_0^T \left( U(t, \tilde{c}_t) - \tilde{c}_t \Gamma_t \right) dt \right]. \]

Noting that, by Lemma 5.5, $\tilde{Y}(y, r)$ is a closed convex subset of $L^0_+$, we may use the above Minimax Theorem 5.2 to get the following equality, for $n$ fixed:

\[ \sup_{\tilde{c} \in S_n, \Gamma \in \tilde{Y}(y, r)} \inf E \left[ \int_0^T \left( U(t, \tilde{c}_t) - \tilde{c}_t \Gamma_t \right) dt \right] = \inf_{\Gamma \in \tilde{Y}(y, r)} \sup_{\tilde{c} \in S_n} E \left[ \int_0^T \left( U(t, \tilde{c}_t) - \tilde{c}_t \Gamma_t \right) dt \right]. \]

Recall now the Bipolar relationship (5.9), and from the definition, we have

(5.25) \[ \bigcup_{(x, z) \in H} \tilde{A}(x, z) = \bigcup_{k > 0} \tilde{A}(k). \]

As a preparation of the following proof, we define the auxiliary set

\[ \mathfrak{A}'(k) \triangleq \left\{ \tilde{c} \in \mathfrak{A}(k) : \sup_{\Gamma \in \tilde{Y}(y, r)} \langle \tilde{c}, \Gamma \rangle = k \right\} \]
and clearly, we also have

(5.26) \[ \bigcup_{k > 0} \tilde{A}(k) = \bigcup_{(x, z) \in H} \tilde{A}(x, z) = \bigcup_{k > 0} \mathfrak{A}'(k). \]
We show first that
\[
\limsup_{n \to \infty} \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \inf_{\tilde{r} \in \tilde{\mathcal{B}}(k)} \mathbb{E} \left[ \int_0^T \left( U(t, \tilde{c}_i) - \tilde{c}_i \Gamma_t \right) dt \right]
\]
(5.27)
\[
= \sup_{k > 0} \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \inf_{\tilde{r} \in \tilde{\mathcal{B}}(k)} \mathbb{E} \left[ \int_0^T \left( U(t, \tilde{c}_i) - \tilde{c}_i \Gamma_t \right) dt \right].
\]

The direction of inequality “\( \geq \)” holds by
\[
\limsup_{n \to \infty} \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \inf_{\tilde{r} \in \tilde{\mathcal{B}}(k)} \mathbb{E} \left[ \int_0^T \left( U(t, \tilde{c}_i) - \tilde{c}_i \Gamma_t \right) dt \right]
\]
(5.28)
\[
\geq \lim_{n \to \infty} \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \inf_{\tilde{r} \in \tilde{\mathcal{B}}(k)} \mathbb{E} \left[ \int_0^T \left( U(t, \tilde{c}_i) - \tilde{c}_i \Gamma_t \right) dt \right]
\]
\[
= \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \inf_{\tilde{r} \in \tilde{\mathcal{B}}(k)} \mathbb{E} \left[ \int_0^T \left( U(t, \tilde{c}_i) - \tilde{c}_i \Gamma_t \right) dt \right], \quad \forall k > 0,
\]
while the other direction “\( \leq \)” is obvious since for any \((x, z) \in \mathcal{H}\), we have \(n \tilde{w} \in \mathcal{A}'(n\bar{p})\), and hence \(S_n \subset \mathcal{A}'(n\bar{p})\).

To show the next step, we need to prepare some finiteness results as below:

From definitions in Lemma 5.5 and by Lemma 5.9, we know
\[
(5.28) \quad \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \mathbb{E} \left[ \int_0^T U(t, \tilde{c}_i) dt \right] = \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \mathbb{E} \left[ \int_0^T U(t, \tilde{c}_i) dt \right] = \sup_{(x, z) \in \tilde{\mathcal{H}}(y, r)} \tilde{u}(x, z), \quad k > 0,
\]
and we claim that
\[
(5.29) \quad \sup_{(x, z) \in \tilde{\mathcal{H}}(y, r)} \tilde{u}(x, z) < \infty, \quad k > 0.
\]

To prove (5.29), recall that the set \(\mathcal{R}\) is open, the set \(\tilde{\mathcal{H}}(y, r)\) is bounded and (5.29) follows from the concavity of \(\tilde{u}\) and \(\tilde{u}(x, z) < \infty\) for all \((x, z) \in \mathcal{H}\).

Now, by (5.26), (5.29) and the definition of domain \(\mathcal{H}\), we have further equalities:
\[
\sup_{k > 0} \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \inf_{\tilde{r} \in \tilde{\mathcal{B}}(k)} \mathbb{E} \left[ \int_0^T \left( U(t, \tilde{c}_i) - \tilde{c}_i \Gamma_t \right) dt \right]
\]
\[
= \sup_{k > 0} \left\{ \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \mathbb{E} \left[ \int_0^T U(t, \tilde{c}_i) dt \right] - k \right\}
\]
\[
= \sup_{k > 0} \left\{ \sup_{\tilde{c} \in \tilde{\mathcal{A}}(k)} \mathbb{E} \left[ \int_0^T U(t, \tilde{c}_i) dt \right] - k \right\}
\]
\[
= \sup_{k > 0} \left\{ \sup_{(x, z) \in \tilde{\mathcal{H}}(y, r)} \tilde{u}(x, z) - k \right\}
\]
\[
= \sup_{(x, z) \in \mathcal{H}} \{ \tilde{u}(x, z) - xy + zr \}.
\]

On the other hand,
\[
\inf_{\Gamma \in \tilde{\mathcal{Y}}(y, r)} \sup_{\tilde{c} \in S_n} \mathbb{E} \left[ \int_0^T \left( U(t, \tilde{c}_i) - \tilde{c}_i \Gamma_t \right) dt \right] = \inf_{\Gamma \in \tilde{\mathcal{Y}}(y, r)} \mathbb{E} \left[ \int_0^T V^n(t, \Gamma_t, \omega) dt \right] = \tilde{v}^n(y, r),
\]
where we define $V^n(t, y, \omega)$ according to the definition of set $S_n$ as
\[
V^n(t, y, \omega) = \sup_{0 < x \leq n\bar{v}_t} [U(t, x) - xy].
\]
Consequently, it is sufficient to show that
\[
\lim_{n \to \infty} \tilde{v}^n(y, r) = \lim_{n \to \infty} \inf_{\Gamma \in \mathcal{Y}(y, r)} \mathbb{E}\left[\int_0^T V^n(t, \Gamma_t, \omega) dt\right] = \tilde{v}(y, r), \quad (y, r) \in \mathcal{R}.
\]
Evidently, $\tilde{v}^n(y, r) \leq \tilde{v}(y, r)$, for $n \geq 1$. Let $(\Gamma^n)_{n \geq 1}$ be a sequence in $\mathcal{Y}(y, r)$ such that
\[
\lim_{n \to \infty} \mathbb{E}\left[\int_0^T V^n(t, \Gamma^n_t, \omega) dt\right] = \lim_{n \to \infty} \tilde{v}^n(y, r).
\]
Then we can find a sequence $h^n \in conv(\Gamma^n, \Gamma^{n+1}, \ldots)$, $n \geq 1$, converging almost surely to a variable $\Gamma$. We have $\Gamma \in \mathcal{Y}(y, r)$, because the set $\mathcal{Y}(y, r)$ is closed under convergence in probability.

Now, we claim the sequence of processes $(V^n(\cdot, h^n, \omega))$, $n \geq 1$ is uniformly integrable, and in fact, we can rewrite
\[
(V^n(t, h^n_t, \omega))^{-1} + (V^n(t, h^n_t, \omega))^{-1} 1_{\{h^n_t > U'(t, \bar{v}_t)\}},
\]
and since $V^n(t, y, \omega) = V(t, y)$ for $y \geq U'(t, \bar{v}_t) \geq U'(t, n\bar{v}_t)$ by the definition. The argument from Lemma 5.7 asserts the uniform integrability of the sequence of processes
\[
(V^n(\cdot, h^n, \omega))^{-1} 1_{\{h^n > U'(\cdot, \bar{v})\}}, n \geq 1.
\]

On the other hand, by the monotonicity of $(V^n)^-$, we have for all $n > 1$,
\[
(V^n(t, h^n_t, \omega))^{-1} 1_{\{h^n_t \leq U'(t, \bar{v}_t)\}} \leq (V^1(t, h^n_t, \omega))^{-1} 1_{\{h^n_t \leq U'(t, \bar{v}_t)\}} \leq (V(t, U'(t, \bar{v}_t)))^{-1},
\]
and by Lemma 5.10 the right-hand side is integrable in the product space, and hence we conclude the sequence $(V^n(\cdot, h^n, \omega))^{-1} 1_{\{h^n \leq U'(\cdot, \bar{v})\}}, n \geq 1$ is also uniformly integrable, and hence our claim holds true. Moreover, we will have the following inequalities:
\[
\lim_{n \to \infty} \mathbb{E}\left[\int_0^T V^n(t, \Gamma^n_t, \omega) dt\right] \geq \lim_{n \to \infty} \mathbb{E}\left[\int_0^T V^n(t, h^n_t, \omega) dt\right] \geq \mathbb{E}\left[\int_0^T V(t, \Gamma_t) dt\right] \geq \tilde{v}(y, r),
\]
which proves:
\[
\tilde{v}(y, r) = \sup_{(x, z) \in \mathcal{R}} \{\tilde{u}(x, z) - xy + zr\}.
\]

For the other equality (5.21), define the function $f(x, z)$ from $\mathbb{R}^2$ to $\mathbb{R}$ as
\[
f(x, z) \triangleq \begin{cases} cl(-\tilde{u}(x, z)) & (x, z) \in cl\mathcal{H}, \\ \infty & \text{otherwise.} \end{cases}
\]
where $\text{cl}(-\tilde{u}(x, z))$ is the lower semicontinuous hull of function $-u(x, z)$. Then $f$ is a proper, convex and lower-semicontinuous function on $\mathbb{R}$ and notice $\text{int}(\text{dom}(f)) = \mathcal{H}$. By Corollary 12.2.2 in Rockafella [26], its Legendre-Fenchel transform is defined by

$$
\hat{f}(y, r) = \sup_{(x, z) \in \mathbb{R}^2} (-xy + zr - f(x, z)) = \sup_{(x, z) \in \mathcal{H}} (-xy + zr + \tilde{u}(x, z)), \quad (y, r) \in \mathbb{R}^2.
$$

Observe that if $(y, r) \in \mathcal{R}$, we have $\hat{f}(y, r) = \tilde{v}(y, r)$ by (5.32), and if $(y, r) \notin \text{cl} \mathcal{R}$, we have by the increasing property of $\tilde{u}(x, z)$ that

$$
\hat{f}(y, r) \geq s(-x_0y + z_0r) + \tilde{u}(x_0, z_0)
$$

for any $s > 1$ and fixed $(x_0, z_0) \in \mathcal{H}$. We can therefore conclude that $\hat{f}(y, r) = \infty$ for $(y, r) \notin \text{cl} \mathcal{R}$ since $-x_0y + z_0r > 0$ by the definition of $\mathcal{R}$. We can thus apply Theorem 12.2 in Rockafella [26] to derive that

$$
f(x, z) = \sup_{(y, r) \in \mathbb{R}^2} (-xy + zr - \hat{f}(y, r)), \quad \forall (x, z) \in \mathbb{R}^2.
$$

Again, by Corollary 12.2.2 in Rockafella [26] and the fact that $\text{int}(\text{dom}(\hat{f})) = \text{int}(\text{dom}(\tilde{v})) \subseteq \mathcal{R}$, we further have

$$
f(x, z) = \sup_{(y, r) \in \mathcal{R}} (-xy + zr - \tilde{v}(y, r)) = -\inf_{(y, r) \in \mathcal{R}} (\tilde{v}(y, r) + xy - zr), \quad \forall (x, z) \in \mathbb{R}^2.
$$

In particular, we deduce that relation

$$
\tilde{u}(x, z) = \inf_{(y, r) \in \mathcal{R}} \{\tilde{v}(y, r) + xy - zr\}, \quad \forall (x, z) \in \mathcal{H},.
$$

\[\square\]

**PROOF OF THEOREM 4.1.**

It is now sufficient to show the conjugate value function $\tilde{v}$ is $(-\infty, \infty]$-valued on $\mathcal{R}$.

Now, according to the definition of Legendre-Fenchel transform, we have

$$
U(t, x) \leq V(t, y) + xy
$$

by integration, it is easy to see for any $\tilde{c} \in \tilde{A}(x, z)$ and $\Gamma \in \tilde{Y}(y, r)$, we have

$$
\mathbb{E} \left[ \int_0^T U(t, \tilde{c}_t)dt \right] \leq \mathbb{E} \left[ \int_0^T V(t, \Gamma_t)dt \right] + \mathbb{E} \left[ \int_0^T \tilde{c}_t \Gamma_t dt \right],
$$

from which Proposition 5.1 deduces that

$$
\tilde{u}(x, z) \leq \tilde{v}(y, r) + xy - zr,
$$

and hence we obtain for all $(y, r) \in \mathcal{R}$, we have $\tilde{v}(y, r) > -\infty$ by Lemma 5.4.

On the other hand, thanks to conjugate duality (5.21) and Bipolar relationship (5.9), follow the proofs in Lemma 5.5 and Lemma 5.11, we also have for each fixed $(y, r) \in \mathcal{R}$

$$
\sup_{(x, z) \in \mathcal{K}(y, r)} \tilde{u}(x, z) = \inf_{s > 0} \{\tilde{v}(sy, sr) + ks\}.
$$

The finiteness result (5.29) for all $k > 0$ in the proof of Lemma 5.11 guarantees the existence of a constant $s(y, r) > 0$, such that $\tilde{v}(sy, sr) < \infty$. \[\square\]
5.2. The Proof of Theorem 4.2. Let’s move on to the proof of Theorem 4.2, to this end, we will need some further lemmas and priori results.

**Lemma 5.12.** Under assumptions of Theorem 4.2, we have $\tilde{v}(y, r)$ is $(-\infty, \infty)$-valued on $\mathcal{R}$.

**Proof.** Similar to the proof of Lemma 5.4, under the additional Assumption (2.9), we can show $\tilde{v}(y, r) < \infty$ if $\tilde{v}(sy, sr) < \infty$ for a constant $s = s(y, r) > 0$. And we have shown that Theorem 4.1 asserts the existence of $s = s(y, r) > 0$. \hfill $\square$

We wish to draw the reader’s attention that we can not simply mimic the proofs of Lemma 5.6, 5.7 and 5.8 to obtain the existence and uniqueness of our auxiliary primal Utility Maximization Problem (3.22). In fact, our successful arguments for the dual problem are hinged on the existence of a bounded process $\tilde{w} \in \tilde{A}(\tilde{\lambda})$, which is missing in the dual space. Nevertheless, the prescribed assumptions on the Reasonable Asymptotic Elasticity permits us to interplay the primal optimizer to the optimal solution to some dual problems. To this end, we resort to a further auxiliary optimization problem of the Auxiliary Dual Utility Minimization Problem (4.5), and make advantage of the Bipolar results built in Lemma 5.5.

**Lemma 5.13.** Define the auxiliary optimization problem to the Auxiliary Dual Utility Minimization Problem (4.5) as:

\begin{equation}
\hat{v}(k) = \inf_{\Gamma \in \tilde{\mathcal{A}}(k)} \mathbb{E} \left[ \int_0^T V(t, \Gamma_t)dt \right],
\end{equation}

where $\tilde{\mathcal{A}}(k)$ is defined in Lemma 5.5 as the bipolar set of $\tilde{A}(x, z)$ on the product space for any $(x, z) \in \mathcal{H}$.

Then, for all $k > 0$, under hypothesis of Theorem 4.2, the value function $\hat{v}(k) < \infty$ for all $k > 0$, and the optimal solution $\hat{\Gamma}(k)$ exists and is unique and $\hat{\Gamma}_t(k) > 0$ for all $t \in [0, T]$. Moreover, for each $k > 0$, and any $\Gamma \in \tilde{\mathcal{A}}(k)$, we have

\[ \mathbb{E} \left[ \int_0^T (\Gamma_t - \hat{\Gamma}_t(k))I(t, \hat{\Gamma}_t(k))dt \right] \leq 0. \]

**Proof.** According to the definition in Lemma 5.5, it is easy to see

\[ \hat{v}(k) = \inf_{\Gamma \in \tilde{\mathcal{A}}(k)} \mathbb{E} \left[ \int_0^T V(t, \Gamma_t)dt \right] \leq \inf_{\Gamma \in \tilde{\mathcal{A}}(k)} \mathbb{E} \left[ \int_0^T V(t, \Gamma_t)dt \right] \leq \tilde{v}(y, r) < \infty, \quad k > 0. \]

by Lemma 5.12.

Taking into account the Bipolar relationship (5.11), we have $\tilde{\mathcal{A}}(k)$ is convexly compact in $L_0^0$, the existence and uniqueness of optimal solution $\hat{\Gamma}(k)$ will follow the similar proof of Theorem 4.1.
Now, for $k > 0$, $\epsilon \in (0, 1)$ and define $\Gamma^\epsilon_t = (1 - \epsilon)\hat{\Gamma}_t(k) + \epsilon \Gamma_t$, for all $t \in [0, T]$, the optimality of $\hat{\Gamma}(k)$ implies

$$0 \leq \frac{1}{\epsilon} \mathbb{E} \left[ \int_0^T \left( V(t, \Gamma^\epsilon_t) - V(t, \hat{\Gamma}_t(k)) \right) dt \right] \leq \frac{1}{\epsilon} \mathbb{E} \left[ \int_0^T \left( \hat{\Gamma}_t(k) - \Gamma^\epsilon_t \right) I(t, \Gamma^\epsilon_t) dt \right]$$

$$= \mathbb{E} \left[ \int_0^T \left( \hat{\Gamma}_t(k) - \Gamma_t \right) I(t, \Gamma_t) dt \right]$$

(5.30)

We claim the family $\left\{ \left( (\Gamma_t - \hat{\Gamma}_t(k)) I(t, \Gamma_t) \right)^-, \epsilon \in (0, 1) \right\}$ is uniformly integrable with respect to $\mathbb{P}$, since first

$$\left( (\Gamma_t - \hat{\Gamma}_t(k)) I(t, \Gamma_t) \right)^- \leq \hat{\Gamma}_t(k) I(t, \Gamma_t) \leq \hat{\Gamma}_t(k) I(t, (1 - \epsilon)\hat{\Gamma}_t(k)) \quad \forall t \in [0, T].$$

We fix $\epsilon_0 < 1$ and observe that for $\epsilon < \epsilon_0$, we have for each $t \in [0, T],$

$$\left| \hat{\Gamma}_t(k) I(t, (1 - \epsilon)\hat{\Gamma}_t(k)) \right| \leq \left| \hat{\Gamma}_t(k) I(t, (1 - \epsilon)\hat{\Gamma}_t(k)) \right| 1_{\{\hat{\Gamma}_t(k) \leq y_1\}} + \left| \hat{\Gamma}_t(k) I(t, (1 - \epsilon)\hat{\Gamma}_t(k)) \right| 1_{\{y_1 < \hat{\Gamma}_t(k) < \frac{y_2}{1 - \epsilon_0}\}}.$$

Now fix $\epsilon_0 < 1$ and observe that for $\epsilon < \epsilon_0$, recall by Lemma 2.1 and Corollary 2.1, assumptions on Reasonable Asymptotic Elasticity $AE_0[U] < \infty$ and $AE_\infty[U] < 1$ imply for fixed $\mu > 0$, the existence of constants $C_1 > 0$, $C_2 > 0$, $y_1 > 0$ and $y_2 > 0$ such that

$$-V'(t, \mu y) < C_1 \frac{V(t, y)}{y} \quad \text{for } 0 < y \leq y_1,$$

$$-V'(t, y) < C_2 \frac{V(t, y)}{y} \quad \text{for } y_2 \leq y.$$

(5.31)

Hence, the first term is dominated by

$$\left| \hat{\Gamma}_t(k) I(t, (1 - \epsilon)\hat{\Gamma}_t(k)) \right| 1_{\{\hat{\Gamma}_t(k) \leq y_1\}} \leq \frac{1}{1 - \epsilon_0} C_1 V(t, \hat{\Gamma}_t(k)),$$

and the send term is dominated by

$$\left| \hat{\Gamma}_t(k) I(t, (1 - \epsilon)\hat{\Gamma}_t(k)) \right| 1_{\{y_1 < \hat{\Gamma}_t(k) < \frac{y_2}{1 - \epsilon_0}\}} \leq \frac{1}{1 - \epsilon_0} C_2 V(t, (1 - \epsilon)\hat{\Gamma}_t(k))$$

$$\leq \frac{1}{1 - \epsilon_0} C_2 V(t, \hat{\Gamma}_t(k)).$$

These two terms are both in $L^1$ by the finiteness of $\hat{v}(k)$. On the other hand, the third remaining term $\left| \hat{\Gamma}_t(k) I(t, (1 - \epsilon)\hat{\Gamma}_t(k)) \right| 1_{\{y_1 < \hat{\Gamma}_t(k) < \frac{y_2}{1 - \epsilon_0}\}}$ is dominated by $k\hat{\Gamma}_t(k) 1_{\{y_1 < \hat{\Gamma}_t(k) < \frac{y_2}{1 - \epsilon_0}\}}$ for a constant $k > 0$, and it is obviously integrable as well.

Now we can let $\epsilon \to 0$ and apply Dominated Convergence Theorem and Fatou’s Lemma to obtain the stated inequality.
To show the optimal solution $\tilde{\Gamma}_t(k) > 0$ for all $t \in [0, T]$, choose an element $\Gamma_t > 0 \in \mathcal{F}(k)$ for all $t \in [0, T]$, it is enough to rewrite the inequality (5.30) as

(5.32)
$$
0 \geq \mathbb{E}\left[\int_0^T \left(\Gamma_t - \tilde{\Gamma}_t(k)\right) I(t, \Gamma_t^\dagger) \mathbf{1}_{\{\tilde{\Gamma}_t(k) > 0\}} dt\right] + \mathbb{E}\left[\int_0^T \left(\Gamma_t - \tilde{\Gamma}_t(k)\right) I(t, \Gamma_t^\dagger) \mathbf{1}_{\{\tilde{\Gamma}_t(k) = 0\}} dt\right].
$$

Now suppose $\mathbb{P}\{\tilde{\Gamma}_t(k) = 0\} > 0$, then by the uniform integrability of $\left\{\left(\Gamma_t - \tilde{\Gamma}_t(k)\right) I(t, \Gamma_t^\dagger)\right\}^{-}, \epsilon \in (0, 1)$, let $\epsilon$ converges to 0, the second term of (5.32) goes to $\infty$, since $I(t, 0) = \infty$ and $\Gamma_t > 0$ for all $t \in [0, T]$, and we obtain the contradiction. Hence the conclusion holds.

**Lemma 5.14.** Under Assumptions of Theorem 4.2, the auxiliary dual value function $\hat{v}(k)$ is continuously differentiable on $(0, \infty)$, and

(5.33)
$$
-k\hat{v}'(k) = \mathbb{E}\left[\int_0^T \tilde{\Gamma}_t(k) I(t, \tilde{\Gamma}_t(k)) dt\right].
$$

**Proof.** In order to show $\hat{v}(k)$ is continuously differentiable, notice the convexity property, it is enough to justify that its derivative exists on $(0, \infty)$. Now fix $k > 0$, and define the function

$$
h(s) \equiv \mathbb{E}\left[\int_0^T V(t, \frac{s}{k} \tilde{\Gamma}_t(k)) dt\right].
$$

This function is convex and by optimality of $\hat{\Gamma}(k)$ of problem (5.29), we have $h(s) \geq \hat{v}(s)$ for all $s > 0$ and $h(k) = \hat{v}(k)$. Again, by convexity, we obtain

$$
\Delta^- h(k) \leq \Delta^- \hat{v}(k) \leq \Delta^+ \hat{v}(k) \leq \Delta^+ h(k),
$$

where $\Delta^+$ and $\Delta^-$ denote right- and left-derivatives, respectively. Now

$$
\Delta^+ h(k) = \lim_{\epsilon \to 0} \frac{h(k + \epsilon) - h(k)}{\epsilon}
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}\left[\int_0^T \left( V(t, \frac{k + \epsilon}{k} \tilde{\Gamma}_t(k)) - V(t, \tilde{\Gamma}_t(k)) \right) dt\right]
\leq \liminf_{\epsilon \to 0} \left( -\frac{1}{\epsilon k} \right) \mathbb{E}\left[\int_0^T \epsilon \tilde{\Gamma}_t(k) I(t, \frac{k + \epsilon}{k} \tilde{\Gamma}_t(k)) dt\right]
= -\frac{1}{k} \mathbb{E}\left[\int_0^T \tilde{\Gamma}_t(k) I(t, \tilde{\Gamma}_t(k)) dt\right],
$$

by the Monotone Convergence Theorem.

Similarly, we get

$$
\Delta^- h(k) \geq \limsup_{\epsilon \to 0} \mathbb{E}\left[ -\int_0^T \tilde{\Gamma}_t(k) I(t, \frac{k - \epsilon}{k} \tilde{\Gamma}_t(k)) dt\right].
$$

We can follow the same reasoning as in Lemma 5.13 to show the family

$\left\{\left(\tilde{\Gamma}_t(k) I(t, \frac{k - \epsilon}{k} \tilde{\Gamma}_t(k))\right)^-, \epsilon \in (0, 1 \wedge k)\right\}$ is uniformly integrable, and Dominated Convergence Theorem and Fatou’s Lemma produce that

$$
\Delta^- h(k) \geq -\frac{1}{k} \mathbb{E}\left[\int_0^T \tilde{\Gamma}_t(k) I(t, \tilde{\Gamma}_t(k)) dt\right],
$$
which completes the proof.

**Lemma 5.15.** The auxiliary dual value function \( \hat{v}(\cdot) \) has the asymptotic property:

\[
-\hat{v}'(0) = \infty, \quad -\hat{v}'(\infty) = 0. \tag{5.34}
\]

**Proof.** We first show \(-\hat{v}'(0) = \infty\), and to this end, we can first derive the result that

\[
\hat{v}(0+) \geq \int_0^T V(t, 0+) dt. \tag{5.35}
\]

To prove the validity of (5.35), we observe that for any \( k > 0 \), by the definition we have

\[
\hat{v}(k) = \mathbb{E}\left[ \int_0^T V(t, \hat{\Gamma}_t(k)) dt \right] = \mathbb{E}\left[ \int_0^T V^+(t, \hat{\Gamma}_t(k)) dt \right] - \mathbb{E}\left[ \int_0^T V^-(t, \hat{\Gamma}_t(k)) dt \right],
\]

hence, by Fatou’s Lemma, firstly, we have

\[
\lim_{k \to 0} \mathbb{E}\left[ \int_0^T V^+(t, \hat{\Gamma}_t(k)) dt \right] \geq \mathbb{E}\left[ \int_0^T V^+(t, 0+) dt \right]. \tag{5.36}
\]

On the other hand, similar to the proof of Lemma 5.6, we can show that

\[
\mathbb{E}\left[ \int_0^T V^-(t, \hat{\Gamma}_t(1)) dt \right] < \infty,
\]

and therefore, by the Monotonicity of function \( V^-(t, \cdot) \) and Dominated Convergence Theorem, we can easily derive that

\[
\lim_{k \to 0} \mathbb{E}\left[ \int_0^T V^-(t, \hat{\Gamma}_t(k)) dt \right] = \mathbb{E}\left[ \int_0^T V^-(t, 0+) dt \right],
\]

which together with (5.36) implies that (5.35) holds true.

Therefore, if \( \int_0^T V(t, 0+) dt = \infty \), then we have \( \hat{v}(0+) = \infty \), and by convexity, we have \( \hat{v}'(0+) = -\infty \).

In the case \( \int_0^T V(t, 0+) dt < \infty \), we then have

\[
-\hat{v}(0+) \geq \lim_{k \to 0} \frac{\hat{v}(0) - \hat{v}(k)}{k} \geq \lim_{k \to 0} \frac{\int_0^T V(t, 0+) dt - \mathbb{E}\left[ \int_0^T V(t, \hat{\Gamma}_t(k)) dt \right]}{k},
\]

and hence we have

\[
-\hat{v}(0+) \geq \lim_{k \to 0} \frac{\mathbb{E}\left[ \int_0^T V(t, 0+) dt \right] - \mathbb{E}\left[ \int_0^T V(t, \hat{\Gamma}_t(k)) dt \right]}{k}
\]

\[
\geq \lim_{k \to 0} \mathbb{E}\left[ \int_0^T \hat{\Gamma}_t(1) I(t, k\hat{\Gamma}_t(1)) dt \right] = \infty,
\]

by the Monotone Convergence Theorem.

We can now turn to show that \(-\hat{v}'(\infty) = 0\), and since the function \(-\hat{v}\) is concave and increasing, there is a finite positive limit

\[
-\hat{v}'(\infty) \triangleq \lim_{k \to \infty} -\hat{v}'(y).
\]
By the definition of Legendre Transform, we clearly have for any \( y > 0 \),
\[-V(t, y) \leq -U(t, x) + xy, \text{ for all } x > 0,\]
and then for any \( \epsilon > 0 \), we always have:
\[
0 \leq -\dot{\hat{v}}'(\infty) = \lim_{k \to \infty} \frac{-\dot{\hat{v}}(k)}{k} = \lim_{k \to \infty} \frac{\mathbb{E}\left[ \int_0^T -V(t, \hat{\Gamma}_t(k)) dt \right]}{k} \leq \lim_{k \to \infty} \frac{\mathbb{E}\left[ \int_0^T -U(t, \epsilon \tilde{w}_t) dt \right]}{k} + \lim_{k \to \infty} \frac{\langle \epsilon \tilde{w}, \hat{\Gamma}(k) \rangle}{k}.
\]

Now, recall that for each fixed \((x, z) \in \mathcal{H}\), there exists a constant \( \lambda(x, z) > 0 \) such that we have \( \tilde{w}_t \in \tilde{A}(\tilde{\lambda} x, \frac{\lambda}{\tilde{\lambda}} z) \), and by the definition of \( \mathcal{G}(k) \), we can see the second term above has
\[
\lim_{k \to \infty} \frac{\langle \epsilon \tilde{w}, \hat{\Gamma}(k) \rangle}{k} \leq \lim_{k \to \infty} \frac{\epsilon \lambda}{\tilde{\lambda}} = \epsilon \frac{\tilde{\lambda}}{\lambda}.
\]

As for the first term, we claim that \( \mathbb{E}\left[ \int_0^T -U(t, \epsilon \tilde{w}_t) dt \right] < \infty \) for each fixed \( \epsilon \) small enough, without loss of generality, we just need to consider that \( \epsilon < \bar{x} \), and then we will apply Corollary 2.1 again, and since there exists a constant \( x_0 \) such that for all \( t \in [0, T] \),
\[
U(t, \frac{\epsilon}{x} x) > (\frac{\epsilon}{x})^{-\frac{\gamma}{\mu}} U(t, x) \text{ for } 0 < x \leq x_0,
\]
we will have
\[
\mathbb{E}\left[ \int_0^T -U(t, \epsilon \tilde{w}_t) dt \right] = \mathbb{E}\left[ \int_0^T -U(t, \epsilon \tilde{w}_t) 1_{\{\tilde{w}_t > x_0\}} dt \right] + \mathbb{E}\left[ \int_0^T -U(t, \epsilon \tilde{w}_t) 1_{\{\tilde{w}_t \leq x_0\}} dt \right] \leq \mathbb{E}\left[ \int_0^T -U(t, \tilde{w}_t) 1_{\{\tilde{w}_t > x_0\}} dt \right] + (\frac{\epsilon}{x})^{-\frac{\gamma}{\mu}} \mathbb{E}\left[ \int_0^T -U(t, \tilde{w}_t) dt \right] < \infty
\]
by the fact that \( \tilde{w} \preceq 1 \) and the Assumption (3.4).

Hence, we conclude that
\[
0 \leq -\dot{\hat{v}}'(\infty) = \lim_{k \to \infty} \frac{-\dot{\hat{v}}(k)}{k} \leq \epsilon \frac{\tilde{\lambda}}{\lambda},
\]
and consequently, we have \(-\dot{\hat{v}}'(\infty) = 0\) by letting \( \epsilon \) goes to 0. \( \square \)

**Lemma 5.16.** Under assumptions of Theorem 4.2, for any \((x, z) \in \mathcal{H}\), suppose \( k \) satisfies
\( 1 = -\dot{\hat{v}}(k) \) where \( \hat{v}(k) \) is the value function of the auxiliary dual optimization problem (5.29),
then \( \tilde{c}^*(x, z) \overset{\triangle}{=} I(t, \hat{\Gamma}_t(k)) \) is the unique (in the sense of \( \equiv \) in \( L^0_\mathbb{P} \)) optimal solution to problem (3.22), moreover we have \( \tilde{c}^*(x, z) > 0 \), \( \mathbb{P} \)-a.s. for all \( t \in [0, T] \).
HABIT FORMATION

Proof. Lemma 5.14 asserts
\[ \langle \tilde{c}^*(x, z), \hat{\Gamma}(k) \rangle = -k\tilde{v}'(k) = k. \]

And for any \( \Gamma \in \tilde{\mathcal{Y}}(k) \), by Lemma 5.13, we have
\[ \langle \tilde{c}^*(x, z), \Gamma(k) \rangle \leq \langle \tilde{c}^*(x, z), \hat{\Gamma}(k) \rangle = k. \]

Hence, we get first \( \tilde{c}^*_t(x, z) \in \tilde{A}(x, z) \) by the Bipolar relationship (5.11).

Now, for any \( \tilde{c} \in \tilde{A}(x, z) \), we have
\[ \langle \tilde{c}, \hat{\Gamma}(k) \rangle \leq k, \]
\[ U(t, \tilde{c}_t) \leq V(t, \hat{\Gamma}_t(k)) + \tilde{c}_t\hat{\Gamma}_t(k), \quad \forall t \in [0, T]. \]

It follows that
\[ \mathbb{E}\left[ \int_0^T U(t, \tilde{c}_t)dt \right] \leq \tilde{v}(k) + k = \mathbb{E}\left[ \int_0^T \left( V(t, \hat{\Gamma}_t(k)) + \hat{\Gamma}_t I(t, \hat{\Gamma}_t(k)) \right) dt \right] \]
\[ = \mathbb{E}\left[ \int_0^T U(t, I(\hat{\Gamma}_t(k)))dt \right] = \mathbb{E}\left[ \int_0^T U(t, \tilde{c}^*_t)dt \right], \tag{5.37} \]

which shows the optimality of \( \tilde{c}^* \). The uniqueness of the optimal solution follows from the strict concavity of the function \( U \).

Moreover, under assumptions of Theorem 4.2, for any pair \( (x, z) \in \mathcal{H} \), by the fact that \( \tilde{\mathcal{Y}}(k) \) is convexly compact and \( \hat{\Gamma}_t(k) \) is bounded in probability, we actually have the optimal solution \( \tilde{c}^*_t(x, z) > 0, \mathbb{P}\text{-a.s.} \) for all \( t \in [0, T] \) since \( \hat{\Gamma}_t(k) \) is bounded in probability if and only if \( \hat{\Gamma}_t(k) \) is finite \( \mathbb{P}\text{-a.s.} \) and by definition, we know \( I(t, x) > 0 \) for \( x < \infty \).

For the proof of Theorem 4.2, we shall also need the following lemma.

Lemma 5.17. Assume that the assumptions of Proposition of 5.1 hold true. Let the sequences \( (y^n, r^n) \in \mathcal{R} \) and \( \Gamma^n \in \tilde{\mathcal{Y}}(y^n, r^n), \ n \geq 1, \) converges to \( (y, r) \in \mathcal{R} \) and \( \Gamma \in L^0_1 \), respectively. If \( \Gamma \) is a strictly positive random variable, then \( (y, r) \in \mathcal{R} \) and \( \Gamma \in \tilde{\mathcal{Y}}(y, r) \).

Proof. Let \( (x, z) \in cl\mathcal{H} \) and \( \tilde{c} \in \tilde{A}(x, z) \) Let \( (x, z) \in cl\mathcal{H} \), the proof of Lemma 5.1 states there exists \( \tilde{c} \in \tilde{A}(x, z) \) such that
\[ \mathbb{P}[^c > 0] > 0. \]

According to Proposition 5.1, we can get
\[ 0 < \langle \tilde{c}, \Gamma \rangle \leq xy - zr, \]

by Fatou’s lemma.

As \( (x, z) \) is an arbitrary element in \( cl\mathcal{H} \), it implies that \( (y, r) \in \mathcal{R} \). The conclusion that \( \Gamma \in \tilde{\mathcal{Y}}(y, r) \) holds by applying Fatou’s lemma and Proposition 5.1.
PROOF OF THEOREM 4.2.

We first show the dual value function $\tilde{v}(y, z)$ is continuously differentiable on $\mathcal{R}$. Theorem 4.1.1 and 4.1.2 in Hiriart-Urruty and Lemaréchal [14] gives the equivalence between the above statement and the fact that the value function $\tilde{u}(x, z)$ is strictly concave on $\mathcal{H}$. Since $U$ is a strictly concave function, to show the value function is strictly concave is equivalent to Assumption (4.1) which is equivalent to Assumption (4.2).

As for the remaining piece of the proof, it amounts to show the assertion (ii) hold, and recall $\hat{\Gamma}(k)$ is the optimal solution of the auxiliary dual problem (5.29), such that

\[ \hat{\Gamma}_t(k) = U'(t, \bar{c}^*(x, z)), \quad \forall t \in [0, T], \quad k = \langle \bar{c}^*(x, z), \hat{\Gamma}(k) \rangle. \]

By the definition that $\bar{\mathcal{Y}}(k)$ is closed with respect to convergence in measure $\bar{\mathbb{P}}$, there exists a sequence $(y^n, r^n) \in k\mathcal{R}(x, z)$ such that $\Gamma^n \in \bar{\mathcal{Y}}(y^n, r^n)$ and $\Gamma^n$ converges to $\hat{\Gamma}(k)$ $\mathcal{P}$-a.s. by passing to a subsequence if necessary, and since set $k\mathcal{R}(x, z)$ is bounded, there exists a further subsequence $(y^n, r^n)$ converges to $(y, r) \in \mathbb{R}^2$. By passing to this further subsequence, as we have shown $\bar{\mathbb{P}}[\hat{\Gamma}(k) > 0] = 1$, we will have $(y, r) \in k\mathcal{R}(x, z)$ such that $\hat{\Gamma}(k) \in \bar{\mathcal{Y}}(y, r)$ due to Lemma 5.17. Moreover, for this pair $(y, r) \in \mathcal{R}$, by Fatou’s Lemma and Proposition 5.1, we have the equality that

\[ (5.38) \quad xy - zr = k = \langle \bar{c}^*(x, z), \hat{\Gamma}(k) \rangle. \]

And we have the corresponding optimizer $\Gamma^*_t(y, r)$ of (4.5) verifies

\[ (5.39) \quad \Gamma^*_t(y, r) = \hat{\Gamma}_t(k) = U'(t, \bar{c}^*(x, z)), \]

because on one hand, we have $\hat{\Gamma}(k) \in \bar{\mathcal{Y}}(y, r)$, hence

\[ \mathbb{E} \int_0^T V(t, \Gamma^*_t(y, r)) = \inf_{\Gamma \in \bar{\mathcal{Y}}(y, r)} \mathbb{E} \int_0^T V(t, \Gamma_t(y, r)) \leq \mathbb{E} \int_0^T V(t, \hat{\Gamma}_t(k)), \]

and on the other hand, we have

\[ \mathbb{E} \int_0^T V(t, \hat{\Gamma}_t(y, r)) = \inf_{\Gamma \in \bar{\mathcal{Y}}(y, r)} \mathbb{E} \int_0^T V(t, \Gamma_t(y, r)) \leq \mathbb{E} \int_0^T V(t, \Gamma^*_t(y, r)) = \mathbb{E} \int_0^T V(t, \Gamma^*_t(y, r)). \]

By the equality

\[ U(t, \bar{c}^*_t(x, z)) = V(t, \hat{\Gamma}_t(k)) + \bar{c}^*_t(x, z)\hat{\Gamma}_t(k), \]

we can conclude $(y, r) \in \partial \tilde{u}(x, z)$ by Theorem 23.5 of Rockafellar [26], since we have

\[ (5.40) \quad \tilde{u}(x, z) = \tilde{v}(y, z) + xy - zr \]
In particular, we get
\[ \partial \tilde{u}(x, z) \cap \mathcal{R} \neq \emptyset. \]

Similar to the proof of Theorem 2 in Hugonnier and Kramkov [15], we can actually show
\[ \partial \tilde{u}(x, z) \subset \mathcal{R}. \]

For any \((y, r) \in \partial \tilde{u}(x, z)\), we can find a sequence \((y^n, r^n) \in \partial \tilde{u}(x, z) \cap \mathcal{R}\) converging to \((y, r)\) by (5.41) and the fact that \(\partial \tilde{u}(x, z)\) is closed and convex. Since \(U'(\cdot, \tilde{c}^*(x, z))\) is strictly positive and we know \(U'(\cdot, \tilde{c}^*(x, z)) \in \hat{\mathcal{Y}}(y, r)\). Lemma 5.17 now infers \((y, r) \in \mathcal{R}\).

Conversely, for any \((y, r) \in \partial \tilde{u}(x, z)\), then
\[
\mathbb{E}\left[ \int_0^T \left| V(t, \Gamma^*_t(y, r)) + \tilde{c}^*_t(x, z) \Gamma^*_t(y, r) - U(t, \tilde{c}^*_t(x, z)) \right| dt \right] \\
= \mathbb{E}\left[ \left( \int_0^T \left| V(t, \Gamma^*_t(y, r)) + \tilde{c}^*_t(x, z) \Gamma^*_t(y, r) - U(t, \tilde{c}^*_t(x, z)) \right| dt \right) \right] \\
\leq \tilde{v}(y, r) + xy - zr - \tilde{u}(x, z) = 0,
\]
which infers (5.38) and (5.39).

6. The Special Case When \(\mathcal{E}\) is Replicable.

6.1. The One Dimensional Primal Value Function. We begin this section by describing the special case when the random variable defined by
\[ \mathcal{E} \triangleq \int_0^T w_t dt = \int_0^T e^{\int_0^t (\delta v - \alpha v) dv} dt \]
is replicable, i.e., there exists a constant \(\tilde{r} > 0\) such that
\[ \langle w, Y \rangle = \tilde{r}, \quad \forall Y \in \mathcal{M}, \]
where \(\mathcal{M}\) is the set of all equivalent local martingale measure densities.

In the light of the fact that \(\bar{p} = p = \tilde{r}\), where \(\bar{p}\) and \(p\) are defined in (3.19) and (3.20), we see the closure of domain \(\mathcal{R}\) for the pair \((y, r)\) defined in (4.3) shrinks into a line, and hence the set \(\mathcal{R}\) is not well defined. In order to build the similar conjugate duality, we will need to reconsider our primal utility maximization problem and the corresponding dual optimization problem, and embed our problem into the framework of Kramkov and Schachermayer [20], [21]. In particular, instead of defining the primal value function on two variables of initial wealth \(x > 0\) and initial standard of living \(z \geq 0\), we can reduce its dimension and define the new variable \(\tilde{x} = x - z\tilde{r}\). According to Lemma 3.1, the effective domain \(\mathcal{H}\) for \(x\) and \(z\) mandating the constraint that \(x > z\tilde{r}\) can therefore be transformed to the constraint on the choice of \(\tilde{x}\) as \(\tilde{x} > 0\).

Recall the auxiliary dual set \(\tilde{\mathcal{M}}\) is defined by
\[
\tilde{\mathcal{M}} = \left\{ \Gamma \in \mathbb{L}_+^0 : \Gamma_t = Y_t + \delta_t \mathbb{E}\left[ \int_t^T e^{\int_s^t (\delta v - \alpha v) dv} Y_s ds \bigg| \mathcal{F}_t \right], \quad \forall t \in [0, T], \quad \forall Y \in \mathcal{M} \right\}.
\]
In order to invoke the one dimensional convex duality analysis in Kramkov and Schachermayer [20], [21], we first define the domain of the auxiliary processes \( \tilde{c} \) depending on the value of \( \tilde{x} \) as

\[
\tilde{A}(\tilde{x}) \triangleq \left\{ \tilde{c} \in \mathbb{L}_0^+ : \langle \tilde{c}, \Gamma \rangle \leq \tilde{x}, \forall \Gamma \in \tilde{M} \right\}, \text{ for } \tilde{x} > 0.
\]

Then, it is clear that we have the equivalence that \( \tilde{A}(x, z) = \tilde{A}(\tilde{x}) \) if we take \( \tilde{x} = x - z \bar{r} \).

It is now ready to define the **Auxiliary Primal Utility Maximization Problem** with respect to \( \tilde{c} \) as

\[
\tilde{u}(\tilde{x}) \triangleq \sup_{\tilde{c} \in \tilde{A}(\tilde{x})} \mathbb{E}\left[ \int_0^T U(t, \tilde{c}_t) dt \right], \quad \tilde{x} > 0.
\]

6.2. **The Dual Optimization Problem and Main Results.** We denote the set \( \tilde{Y}(1) \) as the bipolar set of \( \tilde{M} \) in \( \mathbb{L}_0^+ \), i.e.,

\[
\tilde{Y}(1) = \tilde{M}^\circ \circ,
\]

then we know the set \( \tilde{Y}(1) \) is the closure of the solid hull of \( \tilde{M} \), and we are ready to define the **Auxiliary Dual Utility Optimization Problem** as

\[
\tilde{v}(y) = \inf_{\Gamma \in y\tilde{Y}(1)} \mathbb{E}\left[ \int_0^T V(t, \Gamma_t) dt \right].
\]

Then, we have the following theorems which consist of the main results in this section

**Theorem 6.1.** Assume conditions (2.5), (2.7), (3.4) hold. Assume also that (2.11), (2.12) and (2.10), (i.e., \( \mathcal{A}\mathbb{E}_0[U] < \infty \)) hold true together with

\[
\tilde{u}(\tilde{x}) < \infty \quad \text{for some } \tilde{x} > 0,
\]

we will have:

(i) The value function \( \tilde{u}(\tilde{x}) \) takes value \((-\infty, \infty)\) for all \( \tilde{x} > 0 \), \( \tilde{v}(y) \) takes value \((-\infty, \infty)\) for all \( y > 0 \). And there exists a constant \( y_0 > 0 \) such that \( \tilde{v}(y) < \infty \) for \( y > y_0 \). The value functions \( \tilde{u} \) and \( \tilde{v} \) are conjugate,

\[
\tilde{v}(y) = \sup_{\tilde{x} > 0} \left[ \tilde{u}(\tilde{x}) - \tilde{x}y \right], \quad y > 0,
\]

\[
\tilde{u}(\tilde{x}) = \sup_{y > 0} \left[ \tilde{v}(y) + \tilde{x}y \right], \quad \tilde{x} > 0.
\]

(ii) The function \( \tilde{u} \) is continuously differentiable on \((0, \infty)\) and the function \( \tilde{v} \) is strictly convex on \( \{ \tilde{v} < \infty \} \).

(iii) The functions \( \tilde{u}' \) and \( \tilde{v}' \) satisfy

\[
\tilde{u}'(0) = \lim_{\tilde{x} \to 0} \tilde{u}'(\tilde{x}) = \infty, \quad -\tilde{v}'(\infty) = \lim_{y \to \infty} -\tilde{v}'(y) = 0.
\]

(iv) If \( \tilde{v}(y) < \infty \), then the optimal solution \( \Gamma_\alpha(y) \in y\tilde{Y}(1) \) to (6.5) exists and is unique.
Theorem 6.2. We now assume in addition to conditions of Theorem (2.9) holds, \(\text{i.e., } AE_\infty [U] < 1\). Then in addition to assertions of Theorem 6.1, we also have:

(i) \(\tilde{v}(y) < \infty\), for all \(y > 0\). The value functions \(\tilde{u}\) is continuously differentiable on \((0, \infty)\) and \(\tilde{v}\) is continuously differentiable on \((0, \infty)\) and the functions \(\tilde{u}'\) and \(-\tilde{v}'\) are strictly decreasing and satisfy

\[
\tilde{u}'(\infty) = \lim_{\tilde{x} \to \infty} \tilde{u}'(\tilde{x}) = 0, \quad -\tilde{v}'(0) = \lim_{y \to 0} -\tilde{v}'(y) = \infty.
\]

(ii) The optimal solution \(\tilde{c}^* (\tilde{x})\) to (6.3) exists and is unique. If \(\Gamma^*(y)\) is the optimal solution to (6.5), where \(y = \tilde{u}'(\tilde{x})\), we have the dual relation

\[
\tilde{c}^*_t (\tilde{x}) = I(t, \Gamma^*_t (y)), \quad \Gamma^*_t (y) = U'(t, \tilde{c}^*_t (\tilde{x})).
\]

(iii) For the choice of initial wealth \(x\) and initial standard of living \(z\) such that \((x, z) \in \mathcal{H}\), \(\text{i.e., } x > z\tilde{r}\), we have the optimal solution to our primal utility maximization problem (3.10) exists and is unique, moreover,

\[
c^*_t (x, z) - Z^*_t (x, z) = \tilde{c}^*_t (\tilde{x}), \quad \forall t \in [0, T],
\]

where we have \(x - z\tilde{r} = \tilde{x}\).

The proofs of Theorem 6.1 and part (i), (ii) of Theorem (6.2) are very similar to the proofs of Theorem (4.1) and (4.2) in Section 5 for the case when \(\mathcal{E} = \int_0^T w_t dt = \int_0^T e^{\int_0^t (d_{v} - \alpha_v) dv} dt\) is not replicable, and therefore will be omitted here. As for part (iii) of Theorem (6.2), we just recall that if we have \(x - z\tilde{r} = \tilde{x} > 0\), then \(\mathcal{A}(x, z) = \tilde{\mathcal{A}}(\tilde{x})\) and there is a one-to-one correspondence between set \(\mathcal{A}(x, z)\) for the consumption rate process \(c\) and the set \(\tilde{\mathcal{A}}(\tilde{x})\) for the auxiliary process \(\tilde{c}\).

We are now interested in a closer look at the dual domain \(\tilde{\mathcal{Y}}(1) = \tilde{\mathcal{M}}^\infty\). Recall the construction of the auxiliary set \(\tilde{\mathcal{M}}\), we know that each element \(\Gamma \in \tilde{\mathcal{M}}\) is a linear transform of the equivalent local martingale measure density process \(Y \in \mathcal{M}\) such that

\[
\Gamma_t = Y_t + \delta_t E \left[ \int_t^T e^{\int_t^s (d_v - \alpha_v) dv} Y_s ds \middle| \mathcal{F}_t \right], \forall t \in [0, T].
\]

A natural question is whether the dual domain \(\tilde{\mathcal{Y}}(y)\) can be fully characterized by the same linear transform of supermartingale deflators defined in Kramkov and Schachermayer [20], [21], where we define the set \(\mathcal{Y}(y)\) as

\[
\mathcal{Y}(y) \triangleq \left\{ Y \in \mathbb{L}^0_+ : Y_0 = y, YX \text{ is a supermartingale for } X = x + H \cdot S \right\},
\]

where \(H\) is admissible,

and define the corresponding auxiliary set \(\tilde{\mathcal{Y}}(y)\) by

\[
\tilde{\mathcal{Y}}(y) = \text{solid} \left( \left\{ \Gamma \in \mathbb{L}^0_+ : \Gamma_t = Y_t + \delta_t E \left[ \int_t^T e^{\int_t^s (d_v - \alpha_v) dv} Y_s ds \middle| \mathcal{F}_t \right], \forall t \in [0, T] \right. \right. \]

\[
\text{for } Y \in \mathcal{Y}(y) \right\} \right). \quad (6.7)
\]
Then

\begin{equation}
\tag{6.8}
\text{Question: Is it true that } \overline{\mathcal{Y}(1)} = \mathcal{Y}(1) \, ?
\end{equation}

If the answer is TRUE, then we can always redefine the dual problem over the set of supermartingale deflator \( Y_t \) instead of the abstract auxiliary process \( \Gamma_t \), i.e., we can define the dual optimization problem as:

\begin{equation}
\tag{6.9}
v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[ \int_0^T V(t, Y_t + \delta_t) \mathbb{E}\left[ \int_t^T e^{\int_s^t (\delta_v - \alpha_v) dW_s} Y_s ds \big| \mathcal{F}_t \right] \right],
\end{equation}

and if the dual optimizer \( \Gamma^*_t(y) \) in (6.5) exists, we should always have the dual optimizer \( Y^*_t(y) \) to the problem (6.9) exists and we have the equivalence that

\[ \Gamma^*_t(y) = Y^*_t(y) + \delta_t \mathbb{E}\left[ \int_t^T e^{\int_s^t (\delta_v - \alpha_v) dW_s} Y^*_s(y) ds \big| \mathcal{F}_t \right]. \]

Unfortunately, the answer to the question (6.8) is not always yes. It is important to point out that the set \( \mathcal{Y}(y) \) is not necessarily closed due to the conditional expectation of the future path integral with respect to \( Y_t \). Actually, we have the following Lemma:

**Lemma 6.1.**

\begin{equation}
\tag{6.10}
\overline{\mathcal{Y}(y)} = \mathcal{Y}(y).
\end{equation}

where \( \mathcal{Y}(y) \) denote the closure of set \( \mathcal{Y}(y) \) on the product space \( \mathbb{L}^0_+ \).

**Proof.** Without loss of generality, we just need to check the case when \( y = 1 \). First, it is trivial to show that \( \overline{\mathcal{Y}(1)} \subseteq \mathcal{Y}(1) \) holds, since we clearly have \( \mathcal{M} \subseteq \overline{\mathcal{Y}(1)} \), and the set \( \mathcal{Y}(1) \) is closed, solid and convex set containing \( \mathcal{M} \).

To prove the other direction inclusion, it amounts to verify that \( \overline{\mathcal{Y}(1)} \subseteq \overline{\mathcal{A}(1)} \). By the definition, for each \( \Gamma \in \overline{\mathcal{A}(1)} \), we can find a sequence \( \{\Gamma^n\}_{n \geq 1} \) converges to \( \Gamma \) a.s. in \( \overline{\mathbb{P}} \) and \( \Gamma^n \preceq \Gamma \in \mathcal{Y}(1) \) such that there exists \( Y^n \in \mathcal{Y}(1) \) with \( \Gamma^n_t = Y^n_t + \delta_t \mathbb{E}\left[ \int_t^T e^{\int_s^t (\delta_v - \alpha_v) dW_s} Y^n_s ds \big| \mathcal{F}_t \right], \forall t \in [0, T] \). So for each \( \bar{c} \in \overline{\mathcal{A}(1)} \), there exists a pair \( (x, z) \in H \) with \( x - z\bar{r} = \bar{x} = 1 \), and by Proposition 3.1, there exists \( c \in \mathcal{A}(x, z) \) and we have

\[ \langle \bar{c}, \Gamma \rangle = \langle \bar{c}, \lim_{n} \Gamma^n \rangle \leq \langle \bar{c}, \lim_{n} \Gamma^n \rangle = \lim_{n} \langle c, Y^n \rangle - z\bar{r} \leq x - z\bar{r} = \bar{x} = 1, \]

and our claim holds, where we used Fatou’s lemma and the fact that \( \mathcal{Y}(1) = \mathcal{M}^{**} \) and Consumption Budget Constraint to conclude that \( \langle c, Y^n \rangle \leq x - z\bar{r} \) for each \( Y^n \in \mathcal{Y}(1) \). Now, the fact that \( \overline{\mathcal{Y}(1)} = \overline{\mathcal{A}(1)} \) implies that \( \overline{\mathcal{Y}(1)} \subseteq \overline{\mathcal{Y}(1)} \) which completes the proof.

The optimal dual solution for the problem (6.5) is an abstract process in general, which does not provide any explicit financial intuitions with respect to the market. However, one special case we want to discuss in this section is to assume the discounting processes \( \delta_t \) and
\(\alpha_t\) satisfies the condition that \(\delta_t - \alpha_t\) is a deterministic function in time \(t\).

In this special case, it is clear that we can rewrite the auxiliary set \(\hat{\mathcal{M}}\) as

\[
\hat{\mathcal{M}} = \left\{ \Gamma \in \mathbb{L}_+^0 : \Gamma_t = Y_t \left( 1 + \delta_t \int_t^T e^{\int_s^t (\delta_u - \alpha_u)du} ds \right), \forall Y \in \mathcal{M} \right\}.
\]

And we will define another auxiliary dual domain by

\[
(6.11) \quad \hat{\mathcal{Y}}(y) = \left\{ \Gamma \in \mathbb{L}_+^0 : \Gamma_t = Y_t \left( 1 + \delta_t \int_t^T e^{\int_s^t (\delta_u - \alpha_u)du} ds \right), \forall Y \in \text{solid}(\mathcal{Y}(y)) \right\}.
\]

We want to show the following lemma holds,

**Lemma 6.2.** If \(\delta_t - \alpha_t\) is a deterministic function in time \(t\), then

\[
(6.12) \quad \hat{\mathcal{Y}}(y) = \hat{\mathcal{Y}}(y).
\]

**Proof.** Again, it is enough for us to prove the conclusion for \(y = 1\). For one direction, since the set \(\mathcal{Y}(1)\) is closed, convex and solid, from the definition, it is also true that the set \(\hat{\mathcal{Y}}(1)\) is closed, convex and solid on \(\mathbb{L}_+^0\). Notice again that \(\hat{\mathcal{M}} \subseteq \hat{\mathcal{Y}}(1)\), we can conclude that \(\hat{\mathcal{Y}}(1) \subseteq \hat{\mathcal{Y}}(1)\), as \(\hat{\mathcal{Y}}(1)\) is the smallest closed, convex and solid set containing \(\hat{\mathcal{M}}\).

One the other hand, by the fact that \(\text{solid}(\mathcal{Y}(1)) = \mathcal{M}^{\infty}\), for any \(\Gamma_t = Y_t K_t \in \hat{\mathcal{Y}}(1)\) where we denote \(K_t = \left( 1 + \delta_t \int_t^T e^{\int_s^t (\delta_u - \alpha_u)du} ds \right)\), there exists a sequence of processes \(Y^n \in \mathcal{M}\) such that \(Y^n\) converges to \(Y\). Therefore, we have for any \(\bar{c} \in \hat{\mathcal{A}}(1)\)

\[
\langle \bar{c}, \Gamma \rangle = \langle \bar{c}, YK \rangle \leq \lim_n \langle \bar{c}, Y^n K \rangle,
\]

and since \(Y^n\) is a true martingale, we have

\[
Y^n_t K_t = Y^n_t + \delta_t \mathbb{E} \left[ \int_t^T e^{\int_s^t (\delta_u - \alpha_u)du} Y^n_s ds \Big| \mathcal{F}_t \right].
\]

Consequently, we can derive that the existence of a pair \((x, z)\) \(\in \bar{H}\) such that \(x - z\bar{r} = 1\) and \(c \in \mathcal{A}(x, z)\), and

\[
\langle \bar{c}, Y^n K \rangle = \langle c, Y^n \rangle - z\bar{r} \leq x - z\bar{r} = 1.
\]

We proved that \(\hat{\mathcal{Y}}(1) \subseteq \hat{\mathcal{A}}(1)^\circ\), and we complete the proof by the fact that \(\hat{\mathcal{Y}}(y) = \hat{\mathcal{A}}(1)^\circ\). \(\square\)

Under the assumption that \(\delta_t - \alpha_t\) is deterministic, the above Lemma gives a nice characterization of our dual domain that each element \(\Gamma_t(y)\) in \(\hat{\mathcal{Y}}(y)\) is actually the product of the discounted supermartingale \(D_t Y_t(y)\) and the unique discounting stochastic process \(K_t = \left( 1 + \delta_t \int_t^T e^{\int_s^t (\delta_u - \alpha_u)du} ds \right)\), where the optional process \(D_t\) takes values in \([0, 1]\) and \(Y_t(y) \in \mathcal{Y}(y)\), here the set \(\mathcal{Y}(y)\) is conventionally defined as the set of supermartingale deflators with respect to the admissible wealth process as in Kramkov and Schachermayer [20], [21]. Furthermore, it enables us to find some examples of the explicit form of the optimal dual process.
6.3. An Example in the Itô Process Market Model. In this section, we adopt the same incomplete market model driven by Itô processes in the framework of Karatzas, Lehoczky, Shreve and Xu [16], and we still assume \( \delta_t - \alpha_t \) is a deterministic function in time \( t \).

Specifically, we consider the financial market with one risk-less bond \( S^0 \) modeled by
\[
dS^0_t = r_t S^0_t dt, \quad S^0_0 = 1,
\]
and there are \( m \) stocks whose price process evolve as
\[
dS^i_t = S^i_t b^i_t dt + \sum_{j=1}^{d} S^i_t \sigma^i_{tj} dW^j_t, \quad i = 1, \ldots, m.
\]

Here \( W = (W_t)_{0 \leq t \leq T} \) is a \( d \)-dimensional Brownian motion on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), and we denote \( \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \) the \( \mathbb{P} \)-augmentation of the filtration generated by \( W \). We assume generally \( d \geq m \), i.e., the number of sources of uncertainty in the model is at least as large as the number of stocks available for investment.

We assume the interest rate \( r_t \) and the stock drift vector \( b_t \) are progressively measurable with respect to \( \mathcal{F}_t \) and satisfy \( \int_0^T \| b_t \| dt < \infty \) and \( \int_0^T |r_t| dt \leq L \) a.s. for some constant \( L > 0 \). The volatility matrix \( \sigma_t \) is also progressively measurable with respect to \( \mathcal{F}_t \) and we assume the relative risk process
\[
\theta_t \triangleq \sigma_t^T (\sigma_t \sigma_t^T)^{-1} [b_t - r_t 1_m]
\]
is well defined. Moreover, we assume \( \int_0^T \| \theta_t \|^2 dt < \infty \), a.s. under \( \mathbb{P} \).

To be consistent with Theorem 6.1 and 6.2 and Lemma 6.2, we assume our financial market satisfies the NFLVR condition, i.e., \( \mathcal{M} \neq \emptyset \).

In incomplete Itô processes markets, Detemple and Zapatero [11] and Englezos and Karatzas [12] made the suggestion that it is possible to perform the same program brought up by Karatzas, Lehoczky, Shreve and Xu [16], which is to complete the market with some fictitious stocks and invest in a least favorable manner such that the investor does not invest in the additional stocks at all. For the utility maximization problem with consumption habit formation, their dual optimization problem eventually is written in the following form
\[
\inf_{\nu > 0, \nu \in H(\sigma)} \mathbb{E} \left[ \int_0^T V(y Y^\nu_t + \delta_t \mathbb{E} \left[ \int_t^T e^{\int_0^s (\delta_x - \alpha_x) ds} y Y^\nu_s ds | \mathcal{F}_t \right] ) dt \right],
\]
where they define the parameterized exponential local martingale \( Y^\nu_t \) by
\[
Y^\nu_t \triangleq \exp \left\{ - \int_0^t \left( \theta^T_s + \nu_s^T \right) dW_s - \frac{1}{2} \int_0^t \left( \| \theta_s \|^2 + \| \nu_s \|^2 \right) ds \right\}.
\]
Here, the Hilbert space \( H(\sigma) \) is defined as
\[
H(\sigma) \triangleq \left\{ \nu \in K(\sigma) : \mathbb{E} \left[ \int_0^T \| \nu_s \|^2 ds < \infty \right] \right\},
\]
and the appropriate set \( K(\sigma) \) for \( \nu \) is generally given by
\[
K(\sigma) \triangleq \left\{ \nu : \int_0^T \| \nu_t \|^2 dt < \infty, \text{ a.s. and } \sigma_t \nu_t = 0, \forall t \in [0, T], \text{ a.s.} \right\}.
\]
See Karatzas, Lehoczky, Shreve and Xu [16] for the detail definition and arguments.

Contrary to the optimal consumption problem without habit formation, the previous authors acknowledged that the optimization problem (6.16) is generically more difficult since the dual functional becomes non-convex over the parameter process $\nu \in H(\sigma)$, and some new techniques in non-convex optimization is evidently needed.

However, as we discussed in the previous section, the optimal dual solution lies in the closure of the linear transform of a family of supermartingales for general incomplete semi-martingale financial market. Therefore, if we formulate the dual functional in the form of (6.16), then the set of local martingale deflators is generally too small to contain the dual optimizer.

On the other hand, the market completion argument by Karatzas, Lehoczky, Shreve and Xu [16] should work in general. According to Lemma 6.2 and our main results Theorem 6.1 and 6.2, we can instead define the dual problem in the form

$$
\tilde{v}(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[ \int_0^T V(t, yY_t K_t) \right],
$$

where we define

$$
K_t = \left( 1 + \delta_t \int_t^T e^{\int_s^t (\delta_v - \alpha_v) dv} ds \right),
$$

and it is clear we can now apply the convex duality in Kramkov and Schachermayer [20], [21], and the optimal solution $Y^*_t$ happens to be parameterized exponential local martingale if we assume all the market coefficients are bounded, see the proof of maximal elements of set $\mathcal{Y}(1)$ in Karatzas and Žitković [17]. These results can successfully resolve the open problem mentioned by Detemple and Zapatero [11] and Englezos and Karatzas [12].

We end up this section with an explicit example and we consider the utility function given by $U(t, x) = \log(x)$, such that the conjugate utility function is $V(t, y) = -\log(y) - 1$.

We give the same construction of the financial market as in Delbaen and Schachermayer [8], see also example 5.1 in Kramkov and Schachermayer [20]. One the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t \geq 0}$ is generated by two independent Brownian motions $B$ and $W$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

Define the process $L$ by

$$
L_t = \exp(B_t - \frac{1}{2} t), \quad t \geq 0.
$$

Define the stopping time $\tau$ by

$$
\tau = \inf\{ t \geq 0 : L_t = \frac{1}{2} \}.
$$

Clearly, we have $\tau < \infty$ a.s. Similarly, construct a martingale

$$
M_t = \exp(W_t - \frac{1}{2} t).
$$

The stopping time $\iota$ is defined as

$$
\iota = \inf\{ t \geq 0 : M_t = 2 \}.
Define the financial market with the time horizon

\[ T = \tau \wedge \iota, \]

and define the stock price process

\[ S_t = \exp(-B_t + \frac{1}{2}t), \]

such that \( b_t \equiv 1 \) and \( \sigma_t \equiv -1 \) and the Bond price equals constant 1 at any time \( t \) for simplicity.

Theorem 2.1 in Delbaen and Schachermayer [8] showed the process \( Y_t^* \) defined as

\[ Y_t^* = L_{T \wedge \iota \wedge t}, \]

is a strictly local martingale under \( \mathbb{P} \) and corresponds to \( Y_t^\nu \) for \( \nu \equiv 0 \) by the definition of \( Y_t^\nu \) in (6.17). And similar to the argument by Proposition 5.1 in Kramkov and Schachermayer [20], we can show that \( Y_t^* K_t \) is the unique optimal solution of the dual optimization problem (6.18).

To this end, for each \( Y \in \mathcal{Y} \), the process \( \frac{Y_t K_t}{Y_t} = Y S_t \) is a supermartingale starting at \( Y_0 S_0 = 1 \). Jensen’s inequality implies that

\[ \mathbb{E} \left[ \int_0^T \log \left( Y_t K_t \right) dt \right] = \mathbb{E} \left[ \int_0^T \log \left( \frac{Y_t K_t}{Y_t^* K_t} \right) dt \right] + \mathbb{E} \left[ \int_0^T \log \left( Y_t^* K_t \right) dt \right] \leq \int_0^T \log \left( \mathbb{E} \left[ \frac{Y_t K_t}{Y_t^* K_t} \right] \right) dt + \mathbb{E} \left[ \int_0^T \log \left( Y_t^* K_t \right) dt \right] \leq \mathbb{E} \left[ \int_0^T \log \left( Y_t^* K_t \right) dt \right]. \]

Also, following exactly the same proof of Proposition 5.1 in Kramkov and Schachermayer [20], we can show that \( (Y_t^* M_{t \wedge T})_{t \geq 0} \) is the density process of an equivalent martingale measure and hence \( \mathcal{M} \neq \emptyset \) and \( \hat{v}(1) < \infty \). This completes the proof of our claim that \( Y_t^* K_t \) is the unique optimal solution of the dual optimization problem (6.18).

Moreover, we can choose some special discounting processes \( \delta_t \) and \( \alpha_t \) such that there does not exist an exponential local martingale \( Y_t^\nu \) such that

\[ Y_t^* K_t = Y_t^\nu + \delta_t \mathbb{E} \left[ \int_t^T e^{\int_s^t (\delta_v - \alpha_v) dv} Y_s^\nu ds \bigg| \mathcal{F}_t \right]. \]

To this end, we choose \( \delta_t \) and \( \alpha_t \) such that the process

\[ G_t \triangleq Y_t^* K_t - \delta_t \mathbb{E} \left[ \int_t^T e^{-\int_s^t \alpha_v dv} Y_s^* K_s ds \bigg| \mathcal{F}_t \right] \]

is a continuous nonnegative semimartingale, this can be easily achieved by taking \( \delta_t \) and \( K_t \) to be continuous semimartingales since the product of two semimartingales is still a semimartingale. Furthermore, we choose \( \delta_t \) and \( \alpha_t \) such that the finite variation process \( A_t \) appeared in the decomposition of \( G_t = A_t + R_t \) is not identically the constant \( 1 - R_0 \).

Then if (6.20) holds for some exponential local martingale \( Y_t^\nu \), it is easy to verify by Fubini-Tonelli’s theorem that

\[ Y_t^\nu = G_t, \]
which contradicts the condition that the finite variation process is not identically $1 - R_0$, since a continuous local martingale with finite variation is a constant. This eventually provides us an counterexample to show the set of parameterized exponential local martingales is too small to contain the dual optimizer for the dual problem (6.16), however, it is the proper dual space for the modified optimization problem (6.18) and (6.18) is the correct dual problem to the associated utility maximization problem with consumption habit formation in the Itô processes model.

Acknowledgements. I sincerely thank my advisor Mihai Sirbu for numerous helpful discussions and comments on the topic of this paper as well as his generous support and kind encouragement during my learning and research. This work is part of the author’s Ph.D. dissertation and is partially supported by National Science Foundation under the award number DMS-0908441.

References.

[1] B. Bouchard and H. Pham, Wealth-path dependent utility maximization in incomplete markets, Finance Stoch. 8 (2004), no. 4, 579–603.

[2] W. Brannath and W. Schachermayer, A bipolar theorem for $L^p_0(\Omega, \mathcal{F}, \mathcal{P})$, Séminaire de Probabilités, XXXIII, Lecture Notes in Math., vol. 1709, Springer, Berlin, 1999, pp. 349–354.

[3] J. Y. Campbell and J. H. Cochrane, By force of habit: A consumption-based explanation of aggregate stock market behavior, Journal of Political Economy 107 (1999), 205–251.

[4] G. M. Constantinides, Habit formation: a resolution of the equity premium puzzle, J. Political Econ. (1990), 519–543.

[5] J. Cvitanić, W. Schachermayer, and H. Wang, Utility maximization in incomplete markets with random endowment, Finance Stoch. 5 (2001), no. 2, 259–272.

[6] F. Delbaen and W. Schachermayer, A general version of the fundamental theorem of asset pricing, Math. Ann. 300 (1994), no. 3, 463–520.

[7] _____, The fundamental theorem of asset pricing for unbounded stochastic processes, Math. Ann. 312 (1998), no. 2, 215–250.

[8] _____, A simple counterexample to several problems in the theory of asset pricing, Math. Finance 8 (1998), no. 1, 1–11. MR1613358 (99i:90014)

[9] J. B. Detemple and I. Karatzas, Non-addictive habits: optimal consumption-portfolio policies, Journal of Economic Theory 113 (2003), no. 2, 265 – 285.

[10] J. B. Detemple and F. Zapatero, Asset prices in an exchange economy with habit formation, Econometrica 59 (1991), no. 6, 1633–57.

[11] _____, Optimal consumption-portfolio policies with habit formation, Mathematical Finance 2 (1992), no. 4, 251–274.

[12] N. Englezos and I. Karatzas, Utility maximization with habit formation: Dynamic programming and stochastic pdes, Siam Journal on Control and Optimization 48 (2009), 481–520.

[13] J. Hicks, Capital and growth, Oxford University Press, New York, 1965.

[14] J. B. Hiriart-Urruty and C. Lemaréchal, Fundamentals of convex analysis, Grundlehren Text Editions, Springer-Verlag, Berlin, 2001.

[15] J. Hugonnier and D. Kramkov, Optimal investment with random endowments in incomplete markets, Ann. Appl. Probab. 14 (2004), no. 2, 845–864.

[16] I. Karatzas, J. P. Lehoczky, S. E. Shreve, and G. Xu, Martingale and duality methods for utility maximization in an incomplete market, SIAM J. Control Optim. 29 (1991), no. 3, 702–730.

[17] I. Karatzas and G. Žitković, Optimal consumption from investment and random endowment in incomplete semimartingale markets, Annals of Probability 31 (2003), no. 4, 1821–1858.

[18] H. Kauppila, Convex duality in singular control: Optimal consumption with intertemporal substitution and
optimal investment in incomplete markets, Graduate School in Arts and Science, Columbia University (2010).

[19] D. Kramkov, *Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets*, Probab. Theory Related Fields 105 (1996), no. 4, 459–479.

[20] D. Kramkov and W. Schachermayer, *The asymptotic elasticity of utility functions and optimal investment in incomplete markets*, Ann. Appl. Probab. 9 (1999), no. 3, 904–950.

[21] ———, *Necessary and sufficient conditions in the problem of optimal investment in incomplete markets*, Ann. Appl. Probab. 13 (2003), no. 4, 1504–1516.

[22] K. Larsen and G. Žitković, *Utility maximization under convex portfolio constraints*, to appear in Annals of Applied Probability, 2011.

[23] R. Mehra and E. C. Prescott, *The equity premium: A puzzle*, Journal of Monetary Economics 15 (1985), no. 2, 145 – 161.

[24] C. Munk, *Portfolio and consumption choice with stochastic investment opportunities and habit formation in preferences*, Journal of Economic Dynamics and Control 32 (2008), no. 11, 3560–3589.

[25] R. Muraviev, *Additive habit formation: consumption in incomplete markets with random endowments*, Mathematics and Financial Economics 5 (2011), 67–99.

[26] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.

[27] H. E. Ryder and G. M. Heal, *Optimal growth with intertemporally dependent preferences*, review of economic studies 40 (1973), 1–33.

[28] P. A. Samuelson, *Lifetime portfolio selection by dynamic stochastic programming*, The Review of Economics and Statistics 51 (1969), no. 3, 239–46.

[29] W. Schachermayer, *Portfolio optimization in incomplete financial markets*, Cattedra Galileiana. [Galileo Chair], Scuola Normale Superiore, Classe di Scienze, Pisa, 2004. MR2144570 (2005m:91113)

[30] M. Schröder and C. Skiadas, *An isomorphism between asset pricing models with and without linear habit formation*, Review of Financial Studies 15 (2002), no. 4, 1189–1221.

[31] G. Žitković, *A filtered version of the bipolar theorem of Brannath and Schachermayer*, Journal of Theoretical Probability 15 (2002), no. 1, 41–61.

[32] ———, *Utility maximization with a stochastic clock and an unbounded random endowment*, Annals of Applied Probability 15 (2005), no. 1B, 748–777.

[33] G. Žitković, *Convex-compactness and its applications*, Mathematics and Financial Economics 3 (2009), no. 1, 1–12.

Xiang Yu, Department of Mathematics, The University of Texas at Austin, USA.

*Email: xiangyu@math.utexas.edu*