La interacción de apareamiento es de gran interés debido a que es una de las componentes principales de la interacción residual en sistemas de muchos cuerpos. Las aproximaciones de Bardeen-Cooper-Schrieffer (BCS) y Lipkin-Nogami (LN) dan soluciones aproximadas al Hamiltoniano de apareamiento. Por otro lado, el apareamiento constante, admite solución exacta, llamada solución de Richardson. Los núcleos alejados de la línea de estabilidad tienen importantes correlaciones con los estados de dispersión y por ende, los estados del continuo deben ser tenidos en cuenta en forma explícita en la descripción de tales sistemas. Una forma de incluir el continuo es a través de los estados de Gamow, estos es, soluciones de la ecuación de Schrödinger con energía compleja. En este trabajo comparamos las soluciones aproximadas de BCS y LN con la solución exacta en el modelo simétrico de Lipkin con energía compleja. Encontramos que la solución de LN resulta muy parecida a la exacta y que la extensión de la solución de la BCS al plano de energía complejo produce soluciones puramente imaginarias para el gap cuando \( G < G_c \).

Palabras claves: Lipkin-Nogami, Richardson, energía compleja, apareamiento.

I. INTRODUCTION

Many-body configurations in the continuum is an important issue to understand the properties of loosely bound systems, as for example, nuclei close to the drip line [1]. The complex energy poles of the scattering matrix correspond to complex energy eigenvalue of a single particle Hamiltonian with purely outgoing boundary condition. They are called Gamow states and they represent decay states in the continuum [2]. They have information of the structure of the real energy continuum spectrum: the real part of the complex pole gives the resonant energy while the reciprocal of its imaginary part is proportional to half-live of the unstable state [3].

The constant pairing interaction, although simple, is an important component of the particle-particle interaction [4, 5]. The pairing Hamiltonian can be solved using the Bardeen-Cooper-Schrieffer (BCS) approximation [6]; but, for small number of particles, this approximation is not a satisfactory solution. An improve solution was given by Nogami [7, 8] using the technique developed by Lipkin ir Ref. [9], now known as Lipkin-Nogami (LN) approximation. The solutions in the LN approximation including the continuum single particle density was work out recently in Ref. [10]. But, the constant pairing has exact solution worked out by Richardson [11, 12]. The eigenfunctions of the BCS and LN solutions do not conserve the number of particles of the system, while the Richardson solution does.

A test system, which is non trivial but it is simple enough to be exactly solvable was given by Lipkin, Meshikov and Glick [13], now know as Lipkin model or symmetric model. It is used to test the validity of new formalisms and techniques as well as to illustrate more complicated models in many-body systems [14, 15].
In this work we calculated the many-body energy in the BCS, LN and Richardson frameworks with complex energy in the Lipkin model. The Lipkin formalism is described in section II. In section III the BCS, LN and Richardson solutions are given. The application to the Lipkin model is presented in section IV. Finally, in the last section V we draw some conclusions.

II. LIPKIN FORMALISM

Let us assume that |Φ⟩ is an approximate solution of the unknown exact ground state of a many-body system which is described by a Hamiltonian H. The wave function (w.f.) |Φ⟩ describes nicely some properties of the system but, at the same time, it violates some other property of the Hamiltonian H. For example, the BCS w.f. |Φ_{BCS}⟩ = \prod_{k>0}(1 + \phi_k a_k^\dagger |0⟩) describes nicely the pairing property of the many-body system but it is not an eigenfunction of the particle number operator N. We may expand |Φ⟩ in a basis of eigenfunctions of \hat{S}_z, i.e., |Φ⟩ = \sum_S c_S |Φ_S⟩, with ⟨Φ|Φ⟩ = 1 \quad (\sum_S c_S^2 = 1 \text{ and } c_S \text{ real numbers})

In our example |Φ_{BCS}⟩ = \sum_{N=0}^{N_{\text{max}}} c_N |Φ_N⟩ were N is even, N_{\text{max}} is the maximum particle number allowed by the representation and \langle Φ|N⟩ = N(|Φ⟩).

An approximation of the ground state energy would be ⟨Φ|H|Φ⟩ but since |Φ⟩ does not conserve the property \hat{S}_z we hope that ⟨Φ_S|H|Φ_S⟩ will be a better approximation for some specific value of S of the observable \hat{S}. The trick consist to use the approximate w.f. |Φ⟩ (called model w.f.), which we assume it is easier to handle, to obtain ⟨Φ_S|H|Φ_S⟩, together with a model Hamiltonian \hat{H} (to be build) for which our model w.f. is an eigenvector.

We defined our model Hamiltonian as

\hat{H} = H - f(\hat{S}), \quad (1)

with f(\hat{S})|Φ_S⟩ = f(S)|Φ_S⟩, then

⟨Φ|\hat{H}|Φ⟩ = ⟨Φ_S|H - f(S)|Φ_S⟩ \quad (2)

If f(\hat{S}) is in chosen in such a way that |Φ_S⟩ are all degenerate eigenfunctions of \hat{H}, i.e. \hat{H}|Φ_S⟩ = \mathcal{E}|Φ_S⟩ then

⟨Φ|\hat{H}|Φ⟩ = \mathcal{E}. \quad (3)

By combining Eqs. (2) and (3) we get,

\mathcal{E} = ⟨Φ_S|H - f(S)|Φ_S⟩, \quad (4)

notice that \mathcal{E} is not the eigenvalue of our system but,

⟨Φ_S|H|Φ_S⟩ = \mathcal{E} + f(S), \quad (5)

then, the above discussion assume that \mathcal{E} = ⟨Φ|\hat{H}|Φ⟩ is easier to calculate than ⟨Φ_S|H|Φ_S⟩, i.e., it is easier to solve the eigenvalue problem for the model Hamiltonian \hat{H}|Φ⟩ than the original one H|Φ⟩.

Probably, the only exactly known f(\hat{S}) is the momentum operator, in all the other cases this function has to be approximated by a Taylor’s series, with the hope that a few terms will be enough to reproduce the truly ground state energy. So, let us assume that

\begin{equation}
\mathcal{E} = f_1 \hat{S} + f_2 \hat{S}^2 + \ldots. \quad (6)
\end{equation}

In the case that one truncates the series, the condition that ⟨Φ_S|\hat{H}|Φ_S⟩ may be degenerated for all S is not fulfill. In such a case one must complement the problem with some other subsidiary condition.

If \hat{N} represents the particle number operator \hat{N}, the simplest approximation of f(\hat{N}) is when we keep the first term of the series, f(\hat{N}) = \lambda \hat{N}, then \langle Φ_{BCS}|H|Φ_{BCS}\rangle = \mathcal{E}_{BCS} and the subsidiary condition is that the mean value of the particle number operator is fixed, hence ⟨Φ_S|\hat{N}|Φ_S⟩ = \mathcal{E}_{BCS} + \lambda \hat{N}.

The condition ⟨Φ_{BCS}|\hat{N}|Φ_{BCS}⟩ = N determines the value of the parameter \lambda.

The equation

\begin{equation}
\langle Φ_S|\hat{H}|Φ_S⟩ = \mathcal{E} + f(S) = ⟨Φ|\hat{H}|Φ⟩ + f(S) \quad (7)
\end{equation}

can be rearranged to be written,

\begin{equation}
\langle Φ_S|\hat{H}|Φ_S⟩ = (f_1 S - \langle Φ_s|\hat{S}|Φ_s⟩) + f_2 (\hat{S}^2 - \langle Φ_s|\hat{S}^2|Φ_s⟩) + \ldots. \quad (8)
\end{equation}

where each term can be interpreted as a correction term. This way of writing the mean value ⟨Φ_S|H|Φ_S⟩ is a bit tricky. For example, in the BCS example

\begin{equation}
\langle Φ_N|\hat{N}|Φ_N⟩ = ⟨Φ_{BCS}|\hat{N}|Φ_{BCS}⟩ + \lambda N (⟨Φ_{BCS}|H|Φ_{BCS}⟩ - ⟨Φ_{BCS}|\hat{N}|Φ_{BCS}⟩), \quad (9)
\end{equation}

due the subsidiary condition ⟨Φ_{BCS}|\hat{N}|Φ_{BCS}⟩ = N the correction would be zero. The point we must remember is that we built the model Hamiltonian because is was easy to manipulate with our model wave function, i.e. we don’t solve the eigenvalue problem ⟨Φ|\hat{H}|Φ⟩, instead we solve the eigenvalue problem ⟨Φ|\hat{H}|Φ⟩. In this way we never face terms of the form \langle Φ_{i=1}^i S_i - \langle Φ_{i=1}^i |\hat{S}|Φ⟩⟩ which could be zero. In our BCS example it means that we don’t solve ⟨Φ_{BCS}|H|Φ_{BCS}⟩ but ⟨Φ_{BCS}|H - \lambda \hat{N}|Φ_{BCS}⟩.

The next step is to find a systematic way to obtain the parameters f_i for i ≥ 1 which does not involve the states |Φ_S⟩ but instead involves the model w.f. |Φ⟩. The model w.f. and the model Hamiltonian satisfies the relation

\begin{equation}
⟨Φ|\hat{H}|g(\hat{S})⟩|Φ⟩ = ⟨Φ|\hat{H}|Φ⟩ ⟨Φ|g(\hat{S})⟩|Φ⟩ \quad (10)
\end{equation}

for any function g(\hat{S}). We can choose a set of functions g_i(\hat{S}) = \hat{S}_j with i = 1, 2, \ldots in order to evaluate the coefficients f_i. Then, a self-consistency conditions (independent of |Φ_S⟩) is obtained by rearranged Eq. (10) with g(\hat{S}) replaced by \hat{S}_j,

\begin{equation}
\langle Φ|\hat{H} (\hat{S}_j - ⟨Φ|\hat{S}_j|Φ⟩)|Φ⟩ = 0 \quad (11)
\end{equation}
The ground state energy in this approximation is
\[
\langle \Phi_{BCS} | \mathcal{H} (\hat{N} - \langle \Phi_{BCS} | \hat{N} | \Phi_{BCS} \rangle) | \Phi_{BCS} \rangle = 0
\] (12)
and then
\[
\langle \Phi_{BCS} | \mathcal{H} \hat{N} | \Phi_{BCS} \rangle - \mathcal{E}_{BCS} \langle \Phi_{BCS} | \hat{N} | \Phi_{BCS} \rangle = 0
\] (13)
By inserting \(|\Phi_{BCS} \rangle \langle \Phi_{BCS} |\) between \(\mathcal{H} \hat{N}\) and using the condition \(\mathcal{H}_20 = 0\) we get
\[
\mathcal{E}_{BCS} \langle \Phi_{BCS} | \hat{N} | \Phi_{BCS} \rangle - \mathcal{E}_{BCS} \hat{N} = 0
\] (14)
which gives the standard condition used in BCS, \(\langle \Phi_{BCS} | \hat{N} | \Phi_{BCS} \rangle = N\).

III. MODEL SOLUTIONS

The constant pairing Hamiltonian reads,
\[
H = H_{sp} + V,
\]
where
\[
H_{sp} = \sum_j \epsilon_j \hat{n}_j \quad \hat{n}_j = \sum_m a_{jm}^\dagger a_{jm}
\] (15)
\[
V = -g P^\dagger P = \sum_{jm>0} a_{jm}^\dagger a_{jm}^\dagger
\]
with \(a_{jm}^\dagger \equiv (-)^{j-m} a_{j-m}^\dagger\), and \(g\) the strength of the interaction. The particle number operator is \(\hat{N} = \sum_j \hat{n}_j\).

A. Non-conserving particle number solutions

In this section we will applied the Lipkin method of section III to obtain approximate solutions of the pairing Hamiltonian.

1. BCS solution

The Taylor’s expansion Eq. (15) in the particle number operator \(\hat{N}\) up to the second order defines the Lipkin-Nogami (LN) model Hamiltonian \(\mathcal{H}_{LN}\) as Eq. (15). The model w.f. \(|\Psi_{LN}\rangle\) is like Eq. (18) but with different coefficients \(u_j\) and \(v_j\). They are determined in terms of the parameters \(\Delta, \lambda_1\) and \(\lambda_2\) by solving the following system of three equations
\[
\frac{4}{g} = \sum_j \frac{(2j+1)}{E_j}
\] (25)
\[
N = \sum_j (2j+1)v_j^2
\] (26)
\[
\frac{4\lambda_2}{g} = \frac{(\sum_{jm} u_j^2 v_j^2)(\sum_{jm} u_j v_j^2) - 2 \sum_{jm} (u_j v_j)^4}{(\sum_{jm} (u_j v_j)^2)^2} - 2 \sum_{jm} (u_j v_j)^4
\] (27)
with \(\sum_m = 2j + 1\) and
\[
v_j^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_j}{E_j} \right)
\]
\[
E_j = \sqrt{\epsilon_j^2 + \Delta^2}
\]
\[
\epsilon_j = \epsilon_j - \lambda - g v_j^2
\]
\[
\lambda = \lambda_1 + 2\lambda_2(N + 1)
\]
The ground state energy is
\[
E_{LN} = \langle \Phi_{LN} | \mathcal{H}_{LN} | \Phi_{LN} \rangle + \lambda_1 N + \lambda_2 N^2
\]
\[
= \sum_{jm} (\epsilon_j - g v_j^2) v_j^2 - \frac{\Delta^2}{g} - g \sum_{jm} 2u_j^2 v_j^2
\] (28)

B. Conserving particle number solution

The conserving particle number solution for a system of even \(N\) fermions is given in terms \(N_{pairs} = N/2\).
parameter $E_n$ called pair energies. These parameters are obtained by solving a system of $N_{pairs}$ equations, called Richardson’s equations \[11, 12\]

\[
\frac{1}{g} - \frac{1}{2} \sum_j 2\epsilon_j - E_n + 2 \sum_{n'=1, n' \neq n}^{N_{pairs}} \frac{1}{E_{n'} - E_n} = 0 \quad (29)
\]

The many-body ground state energy $E_{\text{Rich}}$ is defined by taking the lowest $N_{pairs}$ pair-energy $E_n$

\[
E_{\text{Rich}} = \sum_{n} E_n \quad (30)
\]

### IV. APPLICATION: LIPKIN MODEL

The Lipkin model, also called symmetric model, consists of two equally degenerate levels with energies $\epsilon_u$ and $\epsilon_d$ at half filling. The notation of the previous section reduces to $\sum_{jm} = \sum_{u, d} \sum_{\Omega} m = -\Omega$. Hence, the degeneracy for each level is $2\Omega$ and $N = 2\Omega$. Let us introduce the parameter $\epsilon > 0$

\[
\epsilon = \epsilon_u - \epsilon_d, \quad (31)
\]

which defines the energy separation between the two levels. The following values are used for the applications:

\[
\begin{align*}
N &= 10 \\
\epsilon_d &= -0.5 \text{ MeV} \\
\epsilon_u &= (0.5 - i \gamma) \text{ MeV}
\end{align*}
\]

By using the relations of the previous section we found the following algebraic solutions for the BCS and LN approximations:

a. **BCS solution:**

\[
\lambda = \frac{\epsilon_u + \epsilon_d}{2} + \frac{g}{2} \\
E = E_u = E_d = g \Omega \\
\Delta = g \sqrt{1 - \frac{\epsilon^2}{(2g\Omega + \alpha)^2}}
\]

with

\[
g > g_c = \frac{\epsilon - \alpha}{2\Omega} \quad (32)
\]

The ground state energy $E_{\text{BCS}}$, Eq. [10], relative to the non-interacting system is

\[
E_{\text{BCS}} - 2\Omega \epsilon_d = \Omega \left(\epsilon - \frac{g}{2}\right) - \Delta^2 \Omega g \left(\epsilon + \frac{g}{2} \chi\right) \quad (33)
\]

where

\[
\chi = \frac{\epsilon}{2\Omega g - g} \quad (34)
\]

b. **LN solution:** The Lipkin-Nogami solution can also be obtained analytically,

\[
\lambda = \frac{\epsilon_u + \epsilon_d + \alpha}{2}; \quad \alpha = 4\lambda_2 - g > 0 \\
E = E_u = E_d = g \Omega \\
\Delta = g \sqrt{1 - \frac{\epsilon^2}{(2g\Omega + \alpha)^2}}
\]

with

\[
g > g_c = \frac{\epsilon - \alpha}{2\Omega} \quad (35)
\]

Since $4\lambda_2 - g > 0$ we are interested in the positive $\alpha$ solution of the following cubic equation (see also Eq. (15) in Ref. [8])

\[
\alpha(2\Omega - 1)[(2g\Omega + \alpha)^2 - \epsilon^2] - 2g\Omega \epsilon^2 = 0 \quad (36)
\]

Figure 1 shows the three possible solutions of Eq. (36) for each value of the strength $g$ in the range [0, 0.5] MeV. As the strength goes to zero, the parameter $\alpha$ goes to zero and to $\pm \epsilon$ (with $\epsilon = 1$ MeV).

![Figure 1: Solutions of the cubic equation](image)
FIG. 2: $g_c$ of Eqs. (32) and (35) (for $\alpha > 0$) versus $g$ for the BCS and LN approximations, respectively.

in the BCS approximation $g$ has to be bigger than a threshold value, while the LN approximation has no trivial solution for any value of the strength.

Figure 4 extends the comparison done in Fig. 1 of Ref. [8] between the BCS and LN gap parameter $\Delta$ as a function of the strength $g$ for $\alpha > 0$ and $\gamma = 0$ to stronger strength. They compare well for strong correlation but they depart each other for small value of the strength. The figure also shows the nonphysical behavior of the pairing gap in the BCS approximation, i.e. $\Delta = 0$ for $g \lesssim 0.1$ MeV.

FIG. 3: Pairing gap versus $g$.

The ground state energy is calculated in the BCS and LN approximations for $\gamma = 0.25$ MeV and shown in Fig. 6. The energy of the BCS approximation diverges for values of the strength for which its gap is purely imaginary. While the imaginary part of the energy are similar in both approximations, the real one differs for the same value of the strength. Figure 6 also compare the real and imaginary parts of the energy with the exact Richardson solution. A good agreement with the LN approximation for all value of $g$ can be observed.

FIG. 4: Real and imaginary part of the pairing parameter calculated in the BCS approximation for different value of $\gamma$ for $g < 0.25 MeV$.

FIG. 5: Binding energy relative to the non-interacting Fermi sea versus $g$ for $\gamma = 0$. We observe a very good agreement between the approximate LN and the exact (Richardson) solutions for all values of the pairing strength.

FIG. 5: Binding energy relative to the non-interacting Fermi sea versus $g$ for $\gamma = 0$. We observe a very good agreement between the approximate LN and the exact (Richardson) solutions for all values of the pairing strength.

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V. DISCUSSION AND CONCLUSIONS

The Lipkin model with complex energy had been solved in the BCS and LN approximations and compared with the exact Richardson solution.

The extension of the pairing solution to the complex energy plane, shows that there is solution for any value of the strength, even in the BCS approximation, but this solution is completely nonphysical since the energy diverges.

It was found the LN approximation agreed very well with the exact Richardson solution for real and complex energy for any value of the strength.

A limitation of the exact Richardson solution is that it can be applied only to constant pairing interaction. But the good agreement with the LN solution seems to indicate that, for more general interactions, the LN method would be a well founded alternative, even in a complex energy representation.

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