Gravitational collapse with equation of state

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We investigate here gravitational collapse of a perfect fluid with a linear isentropic equation of state \( p = k\rho \). A class of collapse models is given which is a family of solutions to Einstein equations and the final fate of collapse is analyzed in terms of the formation of black holes and naked singularities. The collapse evolves from a regular initial data and the positivity of energy conditions and other physical regularity conditions are satisfied. As we provide here an explicit class, this gives useful insights into the endstates of collapse with a physically reasonable and relevant equation of state and for the cosmic censorship hypothesis.

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I. INTRODUCTION

The cosmic censorship conjecture (CCC) articulated by Penrose \textsuperscript{1} is fundamental to many aspects of theory and astrophysical applications of black hole physics today. Despite many attempts over past decades no theoretical proof or even any satisfactory mathematical formulation of CCC is available as of today in the case of dynamical gravitational collapse. In the mean time, many authors have studied mainly spherical gravitational collapse of a massive matter cloud within the framework of general relativity. As the nuclear fuel of a massive star exhausts, it loses its equilibrium and gravity becomes the central dominant force which lends the star to its perpetual collapse. The gravitational collapse studies then show that the collapse end state is either a black hole (BH) or a naked singularity (NS), depending on the nature of the initial data from which the collapse evolves, arising from a regular initial state to the final super dense state \textsuperscript{2–9}.

The works such as above analyze physically relevant general matter fields that include most of the known physical forms of matter like dust, perfect fluids, and such others. Wide classes of solutions to Einstein equations are shown in these cases to exist where the collapse end state is a black hole or naked singularity depending on the nature of initial data and the collapse evolutions allowed by the Einstein equations subject to regularity and physical reasonability conditions. An equation of state for the matter is, however, not always assumed in many of these investigations. As such, the physical characteristics of the matter field constituting the star are described by an equation of state relating the density and pressures, and therefore it is important to know if the naked singularities would still form when a suitable equation of state is assumed for the collapsing cloud. But it is to be noted that such an equation of state is not precisely known for a collapsing massive star which will be assuming super dense states of matter closer to the final later stages of the collapse, where the physical region has ultra-high densities, energies and pressures. It is far from clear as of now what would be a physically realistic equation of state that would describe such an ultra-high density region of collapse, and also whether the equation of state would remain unchanged or it would actually evolve and change as the collapse develops.

Nevertheless it is important to understand the final fate of a gravitational collapse when a suitable equation of state is assumed, and to know if the naked singularities still form as collapse final states. One can thus choose the equation of state to be linear isentropic or polytropic to describe the collapse of a massive star, right after its departure from the equilibrium configuration where gravity was balanced by the nuclear reactions taking place at the core region of the star.

The gravitational collapse of a perfect fluid with a linear equation of state is of interest from both theoretical as well as numerical relativity perspectives. Our motivation also comes from certain other questions such as, what if the value of \( k \) increases in the range \( 0 \leq k \leq 1 \) when a naked singularity appears as collapse final state for a given spacetime dimension and for a critical positive tangent to the singularity curve as to be discussed below. In such a case will the singularity sustain its nature, or a formation of event horizon shall precede the formation of singularity to preserve the CCC? We believe as answer to these and similar issues would be important to understand better the physical conditions for collapse and the role of an equation of state towards providing a suitable mathematical and physical formulation of CCC.

We therefore consider here a linear isentropic equation of state, \( p = kp \), \( 0 \leq k \leq 1 \) in the study of spherical gravitational collapse of a perfect fluid. Self-similar perfect fluid collapse models with a linear equation of state were considered through numerical simulations by Ori and Piran \textsuperscript{10} and analytically by Joshi and Dwivedi \textsuperscript{11} to show how black holes and naked singularities de-
velop as collapse final states in this scenario. Further, Goswami and Joshi\cite{goswami2015} studied the case of an isentropic perfect fluid with a linear equation of state in four dimensional spacetime without the self-similarity assumption, wherein they showed that the occurrence of BH and NS evolving from regular initial data depends on the choice of rest of the free functions available. It was also proved that in a general $N$-dimensional dust collapse, the occurrence of NS can be removed when one goes to a higher dimensional spacetime, thus restoring CCC\cite{carr1974}. Dadhich et al\cite{dadhich2006} studied spherically symmetric collapse of a fluid with a non-vanishing radial pressure in higher dimensional spacetimes with an equation of state $p_r = k \rho$ using the construction of the Joshi-Dwivedi root equation that governs the nature (BH versus NS) of the central singularity. Also other cases of collapse with pressure have been studied by various authors to understand the collapse final fate\cite{goswami2017}.

Our aim here is to give an explicit class of collapse models for the perfect fluid case with a linear equation of state, and to analyze the same for the collapse end states in terms of formation of black holes and naked singularities. We analyze here the marginally bound case for the sake of simplicity and transparency and the results show that BH and NS form for a wide range of values of the parameter $k$ in the equation of state $p = k \rho$.

The plan of work is as follows: In section II, we study Einstein field equations for spherically symmetric metric. We obtain the function $M(r, v)$ as a class of solutions to the constraint equation that enters through dynamical equations for $p = k \rho$. The dynamics of collapse for occurrence of NS/BH phases due to the effect of regular initial data is studied in the subsection II.1. The implication of the function $M(r, v)$ on singularity curve is investigated, which corroborate the study of apparent horizon in section III. The occurrence of NS/BH phases is investigated for increasing value of the parameter $k$, providing illustration of its behavior in subsections III.1 and III.2. The conclusions are specified in section IV.

II. COLLAPSE WITH A LINEAR EQUATION OF STATE

The general spherically symmetric metric describing spacetime geometry of a collapsing cloud can be described in comoving coordinates $(t, r, \theta, \phi)$ by the metric,

$$ds^2 = -e^{2\nu(t, r)}dt^2 + e^{2\nu(t, r)}dr^2 + R^2(t, r)d\Omega^2$$

where

$$d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$$

is the metric on a two-sphere. We aim here at the question, whether the choice of an equation of state $p = k \rho$, $k \in [0, 1]$ contributes through the parameter $k$ in the development of BH/NS phases. The stress-energy tensor for the type I matter fields, and in particular for perfect fluids, is in a diagonal form in a comoving coordinate system and is given by\cite{weinberg1972},

$$T^t_t = -\rho, \quad T^r_r = T^\theta_\theta = T^\phi_\phi = p.$$ 

The quantities $\rho$ and $p$ are the energy density and pressure respectively. We take the matter field to satisfy the weak energy condition which implies $\rho \geq 0; \rho + p \geq 0$. The linear equation of state for a perfect fluid is,

$$p(t, r) = k \rho(t, r) \text{ where } k \in [0, 1].$$

The Einstein field equations for the metric are derived as\cite{weinberg1972}

$$\rho = \frac{F'}{R^2 R'} = -\frac{1}{k} \frac{\dot{F}}{k R^2 R}$$

$$\nu' = -\frac{k}{k+1}[\ln(\rho)]'$$

$$R'G - 2\dot{R}G \nu' = 0$$

$$G - H = 1 - \frac{F}{R}.$$ 

Here $F = F(t, r)$ is an arbitrary function, and has an interpretation of the mass function for the cloud. It gives the total mass in a shell of comoving radius $r$ on any spacelike hypersurface $t = \text{const}$. The energy conditions impose the restriction on $F$, namely that $F(t, r) \geq 0$. In order to preserve the regularity at the initial epoch, we have $F(t, 0) = 0$, that is, the mass function has to vanish at the center of the cloud. Since we are considering collapse, we have $R < 0$, i.e. the physical radius $R$ of the collapsing cloud keeps decreasing in time and ultimately it reaches $R = 0$, which denotes the spacetime singularity where all the matter shells collapse to a zero physical radius. The functions $G$ and $H$ are defined as $G(t, r) = e^{-2\nu}R^2$ and $H(t, r) = e^{-2\nu}\dot{R}^2$.

Let us define a new function $v(t, r)$ by $v(t, r) = R/r$, and now we use the scaling independence of the comoving coordinate $r$ to write

$$R(t, r) = r \ v(t, r)$$

This gives $v(t_0, r) = 1, v(t_s(r), r) = 0$ and the collapse condition is now written as $\dot{v} < 0$. The time $t = t_s(r)$ corresponds to the shell-focusing singularity at $R = 0$, where all the matter shells collapse to a vanishing physical radius. The introduction of the parameter $v$ as above allows us to distinguish the genuine spacetime singularity from the regular center at $r = 0$, with $v = 1$ at the initial epoch including the center $r = 0$, and $v$ then decreases monotonically with time as collapse progresses to the value $v = 0$ at the singularity $R = 0$. At the regular center the mass function $F(t, r)$ behaves suitably so that the density remains finite and regular there at all times till the occurrence of singular epoch. From the Einstein
equation for density, we see that the mass function for the cloud can be written in general as,

$$F(t, r) = r^3 M(r, v)$$  \(8\)

where \(M(r, v)\) is regular and continuously twice differentiable. Using equation (8) in equations (3), we obtain

$$\rho = \frac{3M + r[M_M + M_v v']}{v^2(v + rv')} = -\frac{M_v}{k v^2}.$$  \(9\)

Then as \(v \to 0, \rho \to \infty\) and \(p \to \infty, i.e.\) both the density and pressure blow up at the singularity. We rearrange equation (9) as follows,

$$kr M_{,v} + [(k + 1)rv' + v] M_v = -3kM$$  \(10\)

The general solution of equation (10) represents many classes of solutions but only those classes are to be considered which satisfy the energy conditions, which are regular and those which give \(\rho \to \infty\) as \(v \to 0\). This means that the energy conditions and equation of state \(p = kp\) isolate the class of functions \(M(r, v)\) so that the mass function \(F(t, r)\), the metric function \(\nu(t, r)\) and other concerned functions of \(v\) evolve as the collapse begins according to the Type I field equations.

To obtain such a class of solutions, we consider here the ansatz,

$$\frac{rv'}{v} = g(v)$$  \(11\)

due to which the equation (10) has a general solution of the form,

$$M(r, v) = f(re^{-Z(v)}) e^{-3Z(v)}$$  \(12\)

where \(Z(v) = \int_1^v \frac{k}{v[(k + 1)g(v) + 1]} dv$$  \(13\)

and \(f\) is an arbitrary function. Therefore, we can choose a physically relevant \(M\) as given by,

$$M(r, v) = M_0 \text{sech}(re^{-Z(v)}) e^{-3Z(v)}$$  \(14\)

as a solution of equation (10), where \(M_0\) is a positive constant and the choice of the function \(g(v)\) should be such that integrand of equation (13) should be at least piecewise continuous on \([0, 1]\) and also we note that \(g(1) = 0\) and \(g(0) = v'\) for \(v \to 0, r \to 0\). Also at \(r = 0\), \(g(v) = 0\) i.e. \(g(v)\) vanishes at the central shell at all regular points of the spacetime. The introduction of the equation (11) demands its compatibility with other field equations or their subsequent equation (27), which is discussed in the Appendix.

From equation (14), we can write

$$M_0(v) = M(0, v) = M_0 e^{-3Z}, M_1(v) = \frac{M_{,r}(0, v)}{1!} = 0$$

and

$$M_2(v) = \frac{M_{,rr}(0, v)}{2!} = -\frac{M_0 e^{-5Z}}{2}.$$  

Here \(M_1(v) = 0\) is in accordance with the requirement that the energy density has no cusps at the center and that its first derivative vanishes at the center. The density profile for this class of models then takes the form,

$$\rho(r, v) = \frac{M_0 e^{-3Z(v)} \text{sech}(re^{-Z(v)})}{v^3[(k + 1)g(v) + 1]} \times [3 - re^{-Z(v)} \tanh(re^{-Z(v)})].$$  \(15\)

We note that at the initial epoch, \(g(1) = 0, Z(1) = 0\), and the regular density distribution takes the form

$$\rho_0(r) = M_0 \text{sech}(r)[3 - r \tanh(r)].$$  \(16\)

Such a density profile is decreasing away from the center \(r = 0\), which is a physically reasonable feature for the collapsing matter cloud. For example, at the epoch \(v = 1\), \(\rho_0(r) = 0\) at (an approximate) value \(r = 3.014482\). Hence we would take such a value of \(r\) as the boundary of the cloud where the energy density is zero, and where the interior is matched to a suitable exterior metric.

At \(r = 0\) we have \(g(v) = 0\) and therefore \(e^{-3Z(v)} = v^{-3k}\), and the energy density at the center takes the form,

$$\rho(0, v) = \frac{3M_0}{v^3(k + 1)},$$

Therefore \(\rho\) satisfies the energy condition \(\rho \geq 0\) at all the epochs. The gradient of \(\rho(r, v)\) is obtained as

$$\rho_r(r, v) = -\frac{M_0 e^{-4Z} \text{sech}(re^{-Z})}{v^3[(k + 1)g(v) + 1]} \times \{4 \tanh(re^{-Z}) - re^{-Z}[2 \tanh^2(re^{-Z}) - 1]\}.$$  

At the initial epoch \(v = 1\), we get \(\rho_r = 0\) at the center \(r = 0\). The radial coordinate \(r\) has the range \(0 \leq r \leq r_b\) which defines the interior of the compact collapsing object, and wherein the density profile is smooth and monotone decreasing away from the center of the cloud as one moves towards the boundary of the cloud.

II.1. Dynamics of collapse and the BH/NS phases

Based on the above, we would now like to determine the final states for such a perfect fluid collapsing cloud in terms of the black holes and naked singularities. As we shall point out, the key factor that determines this final outcome is the geometry of the trapped surfaces. If the trapped surfaces developed early enough in collapse, well before the formation of the singularity, then a black hole is the collapse final state. On the other hand, when the trapped surface formation is delayed as the collapse evolves, due to the internal dynamics of the collapsing cloud, in that case the singularity is no longer fully covered by an event horizon, and we can have light rays and material particles escaping away from an arbitrary vicinity of the same, thus giving rise to a naked singularity.

To decide and understand this trapped surface geometry, we need to determine the behavior of the singularity curve in the spacetime, which describes how and when different collapsing shells arrive at the zero physical radius, thus forming the spacetime singularity.
To determine this, firstly on integrating equation (4), we obtain the general metric function,
\[ \nu(r, t) = -l \ln(\rho) + a(t). \]  
where \( l = \frac{1}{\rho R'}. \) We can set \( a(t) = 0. \) Let \( A(r, v) \) be a suitably differentiable function defined by \( A(r, v) = \nu' / R', \) and on using equations (11) and (17), it takes the form
\[ A(r, v) = \frac{l \frac{\rho'}{\rho R'}}{v[1 + g(v)] [3 - re^{-Z} T]}, \]  
where \( T(r, v) = \tanh(re^{-Z}) \) and \( T(r, 1) = \tanh(r). \)

Then on integrating equation (19), we obtain
\[ A(r, v) = \int_{v}^{1} \frac{l \{4T(r, 1) - r[2T^{2}(r, 1) - 1]\}}{[3 - rT'(r, 1)]} \]  
\[ [A(r, v), v = 1 \]  
where \( T(r, v) = \tanh(re^{-Z}) \) and \( T(r, 1) = \tanh(r). \)

Our aim below is to find the BH/NS phases for collapse at \( R = 0 \) or \( v = 0. \) The field equation (10) can be written in the form
\[ \dot{R}^{2} = e^{2\nu} \left[ \frac{F}{R} + G - 1 \right] \]  
where the function \( b(r) \) basically characterizes the energy distribution for the collapsing shells. For the sake of simplicity and transparency of consideration and for a better understanding of the model, we consider here only the ‘marginally bound case’, which corresponds to \( b(r) = 0. \)

As we see now, having supplied an explicit function \( M(r, v) \) which is a solution to the first two Einstein equations, that determines \( \rho(r, v), \) and therefore the metric function \( \nu(r, v) \) is fully determined as above. The function \( A(r, v) \) is also determined as above and is now a known function, and therefore \( G(r, v) \) is also determined. Finally, the metric function \( R \) is determined as we shall mention below. We can write the metric in the neighborhood of the center of the cloud as,
\[ ds^{2} = -\rho^{-2} dt^{2} + \frac{R^{2}}{e^{2\nu}} dr^{2} + R^{2}(t, r) d\Omega^{2} \]  
In order to determine the nature of the singularity curve \( t_{s}(r) \) which corresponds to the physical singularity at \( R = 0 \) or \( v = 0, \) we consider here only the ‘marginally bound case’, which corresponds to \( b(r) = 0. \)

The time for other collapsing shells to arrive at the singularity and therefore can be expanded around the center as,
\[ t(r, v) = t_{i} + \int_{v}^{1} \frac{e^{-\nu} dv}{\sqrt{\frac{M(r, v)}{v} + \frac{e^{2\nu} - 1}{r^{2}}}}. \]  
where \( \chi_{1}(v) = \frac{dt}{dr} \bigg|_{r=0} \) and \( \chi_{2}(v) = \frac{d^{2}t}{dr^{2}} \bigg|_{r=0} \) and using equations (17) and (21), we have
\[ t_{s}(r) = t_{i} + \int_{0}^{1} \frac{e^{-\nu} dv}{\sqrt{\frac{M(r, v)}{v} + \frac{e^{2\nu} - 1}{r^{2}}}}. \]  

In above equation, the variable \( r \) is treated as a constant. Regularity ensures that, in general, \( t(r, v) \) is at least \( C^{2} \) near the singularity and therefore can be expanded around the center as,
\[ t(r, v) = t(0, v) + r\chi_{1}(v) + \frac{r^{2}}{2!}\chi_{2}(v) + O(r^{3}) \]  
where
\[ \chi_{1}(v) = \frac{dt}{dr} \bigg|_{r=0}, \chi_{2}(v) = \frac{d^{2}t}{dr^{2}} \bigg|_{r=0} \]  
and using equations (17) and (21), we have
\[ t(0, v) = t_{i} + (3M_{0})^{\frac{1}{4}} \int_{v}^{1} \frac{e^{-\frac{3}{4}Z} dv}{\sqrt{\frac{M(0, v)}{v} + 2A_{1}(v)}}. \]  

The time for other collapsing shells to arrive at the singularity can be expressed by
\[ t_{s}(r) = t(r, 0) = t_{i} + \int_{0}^{1} \frac{e^{-\nu} dv}{\sqrt{\frac{M(r, v)}{v} + \frac{e^{2\nu} - 1}{r^{2}}}}. \]
Therefore the positive or negative sign of $\chi$ is crucial for the determination of $k$ and $M$, which are the key functions of the model. Moreover, the absolute constant $c$ is related with the behavior of the term $M$ at the center taken on a cross section of an initial surface $v = 1$ at the beginning of the collapse at the instant $t = t_i$.

### III. THE APPARENT HORIZON

The significance of the term $\chi_2(0)$ towards deciding the visibility or otherwise of the final spacetime singularity is as follows. As seen by the equation for $t_s(r)$ above for the singularity curve, this term and its sign basically decides the behavior of the singularity curve away from the center. In particular, when the singularity curve is increasing, it is shown that the apparent horizon which becomes singular, and which is obtained as

$$t(0, 0) = t_i + 3I(M_o)^{3/2} \times$$

$$\left\{\frac{2 M_o}{3(1 - k)} - \frac{5}{3(k + 1)} \left[\frac{1}{5 - k} - \frac{1}{5 + 3k}\right]\right\}$$

The time taken by the central shell to reach the singularity should be positive and finite, hence we have the model realistic condition (MRC), namely that,

$$\left\{\frac{2 M_o}{3(1 - k)} - \frac{5}{3(k + 1)} \left[\frac{1}{5 - k} - \frac{1}{5 + 3k}\right]\right\} > 0$$

and it must be finite for any $k \in [0, 1)$.

Noting that $\nu'(0, v) = 0$, we have

$$\chi_1(v) = -\frac{1}{2} \int_v^1 \frac{2e^{-\nu}A_2}{\left[\frac{M(0, v)}{v} + 2A_1\right]^{3/2}} dv$$

and since $A_2(v) = 0$, we have $\chi_1(0) = 0$ for all $k$. Also, at $r = 0$, $g(v) = 0$ and therefore $Z = k \ln(v)$ and $e^{-3Z} = v^{-3k}$, so we have

$$A_1(v) = \frac{5}{6(k + 1)}[v^{-2k} - 1], A_3(v) = \frac{5}{36(k + 1)}(1 - v^{-4k})$$

Since $\chi_1(0) = 0$, using expressions for $M_0(v)$, $A_1(v)$ and $A_3(v)$, we compute $\chi_2(0)$ for an analysis of the nature of the central singularity in terms of its visibility or otherwise. We get,

$$\chi_2 = \left(\frac{3!M_o}{M_o^{3/2}}\right) \int_v^1 \left\{\frac{5l}{3} \left[\frac{M(0, v)}{v} + 2A_1\right]^{1/2} + \frac{\frac{M_o + 2A_3 + 2A_2}{M(0, v)}}{\left[\frac{M(0, v)}{v} + 2A_1\right]^{3/2}} \right\} dv$$

where $c_1 = \left[\frac{3!k^2 + 136k + 25}{(5 + 3k)(5 - k)}\right]$ and $c_2 = \left[\frac{3k^2 - 20k + 7}{(r + k)(1 - 3k)(7 + 3k)(7 + 9k)}\right]$ are functions of $k.$
Such a singularity can also be globally visible depending on the nature of the mass function away from the center. On the other hand, when the singularity curve is constant ($\chi_2$ and other higher order terms are all vanishing), or would be decreasing, then a black hole will necessarily form as the collapse final state.

Basically, a naked singularity could occur when a co-moving observer (at a fixed radius $r$) does not encounter any trapped surfaces until the time of singularity formation. On the other hand, for a black hole the trapped surfaces form before the singularity. Thus for a black hole to form we require,

$$t_{ah}(r) \leq t_0 \text{ for } r > 0, \text{ near } r = 0.$$  \hspace{1cm} (36)

where $t_0$ is the epoch at which the central shell hits the singularity. In the general case this condition is violated when $\chi_2 > 0$ as seen from the equation for $t_s(r)$ above. The apparent horizon curve that initiates at the singularity $r = 0$ at the epoch $t_0$, then increases with increasing $r$, moving to the future and we have $t_{ah} > t_0$ for $r > 0$ near the center. The behavior of the outgoing families of null geodesics has been analyzed in detail in these cases, and it is known that the geodesics terminate at the singularity in the past, which results in a naked singularity. In such cases then the extreme strong gravity regions can communicate with external observers.

As these aspects are already discussed earlier as cited above, we therefore mainly focus here on determining the behavior of the quantity $\chi_2$ for the perfect fluid collapse with an equation of state $p = k\rho$, and we examine the impact of changing the values of the parameter $k$ on the visibility or otherwise of the singularity.

### III.1. Dust collapse

We note that for the well-studied case of the dust collapse, we have $p = 0$ at $k = 0$ and the MRC reduces to the form $M_o > 0$. We have

$$\chi_1(0) = 0 \text{ and } \chi_2^{k=0}(0) = \frac{1}{3M_o^{1/2}} > 0.$$ \hspace{1cm} (37)

In this case a naked singularity exists for the given initial data of mass function.

### III.2. The Radiation collapse and other values of $k$

The radiation collapse can be considered through the case $k = 1/3$ for which the MRC takes the form $M_o > 5/84$ and we have $\chi_1(0) = 0$ and

$$\chi_2^{k=1/3}(0) = -\frac{(3)^{1/4}}{M_o^{3/4}} \left[ \frac{1}{4} M_o^2 + 121 \frac{M_o}{1008} + \frac{5}{6864} \right].$$ \hspace{1cm} (38)

Since the bracketed term has varying sign, we find that $\chi_2(0) < 0$ for small positive values of $M_o$, and for higher values of $M_o$, $\chi_2(0)$ changes its sign.

In general, to understand the nature of $\chi_2(0)$ in the interval $k \in (0, 1)$, we can obtain expressions for $\chi_2(0)$

![Graph](image1)

FIG. 2. For a fixed value of the central density corresponding to $M_o = 1$, the quantity $\chi_2(0)$ takes positive values for $k = 0$ and also for other small values of $k$. Then it crosses the $k$-axis and takes negative values as $k \rightarrow 1$.

![Graph](image2)

FIG. 3. The nature of $\chi_2(0)$ is illustrated through the above graphs. We note that if the model begins with a higher value of $M_o$, higher values of $k$ would satisfy the condition $\chi_2(0) > 0$. Thus, it is the initial central density that can specify and would have ramification on the nature of singularity in terms of its visibility or otherwise.
for different values of $k$ in order to see how it changes with a change in the parameter values $k$. The graph of $\chi_2(0)$ for various specific $k$ values is given in Fig.3.

![Graph of $\chi_2(0)$ for various values of $k$ and $M_0$, $b(r)=0$](Image)

FIG. 4. An implicit graph of $\chi_2(0)$ specifies the regions for formation of naked singularity and black hole, as determined by the variation in parameters $M_0$ and $k$.

The equation of apparent horizon in a spherically symmetric spacetime is given by $F/R = 1$. Use of equations 8 and 14 give us the expression

$$r^2M_0 \text{ sech}(re^{-Z(v)})e^{-3Z(v)} = v$$

It is clear that as $v \to 0$, we must have $r \to 0$ on the apparent horizon. Therefore, the only singularity that can eventually be visible is that at the center of the collapsing cloud. Certainly, it is not possible to satisfy $1 - F/R > 0$, i.e.

$$1 - \frac{r^2M_0\text{ sech}(re^{-Z(v)})e^{-3Z(v)}}{v} < 0 \text{ as } v \to 0 \text{ for } r > 0$$

near the singularity and away from center, that is with $r > 0$. It follows that the region surrounding the singularity cannot be timelike, and therefore any singularity that might eventually form near the center with $r > 0$ must not be visible.

IV. CONCLUSIONS

The investigation of gravitational collapse with a linear equation of state has revealed role of the parameter $k$ in deciding the nature of central singularity for the given initial data set of the mass function $M(r,v)$. We examined here the marginally bound case with $b(r) = 0$. In particular, we gave here an explicit class of mass functions that satisfies the Einstein equations and the metric obtained represents a class of solutions for a unique choice of functions $g(v)$, representing the exact solution for a perfect fluid sphere collapse with an isentropic equation of state.

The occurrence of a locally naked singularity as collapse end state is shown for these collapse models for a wide variety of values of $k \in [0,1)$, and also formation of black hole is seen for a range of $k$ values for this class of mass functions which are an appropriate choice of $M(r,v)$ that are solutions to the Einstein field equations.

We thus see, through an explicit demonstration, that both black holes and naked singularities arise naturally as collapse final states for perfect fluid collapse with a linear isentropic equation of state. The class of solutions we gave here have several intriguing properties some of which we discussed here. It is thus seen that the choice of $k \in [0,1)$ influences and contributes to decide in the formation of NS/BH phases of gravitational collapse. The model realistic condition discussed here emphasizes the role of pressure in the gravitational collapse due to which the collapse could stop momentarily, and a bounce may take place. In other cases the cloud would collapse continually without a bounce.

We note that the present model can be extended for a more general case with $b(r) \neq 0$, and the results on the same shall be reported elsewhere.

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APPENDIX-

COMPATIBILITY CONDITION

The introduction of the equation (11) demands its compatibility with other field equations or their subsequent equation (27). So, we consider \( v' = v g(v)/r = W(r, v) \) and \( \dot{v} = U(r, v) \). The condition of compatibility for non-linear partial differential equations of order one yields,

\[
W_{U,v} - UW_{v,v} = -U_{r,r}.
\]

where \( U(r, v) = -e^{\nu(r,v)} \sqrt{D(r,v)} \) and

\[
D(r, v) = \frac{M(r, v)}{v} + \frac{e^{2rA(r,v)} - 1}{r^2}.
\]

Equation (39) takes the form

\[
- \frac{e^\nu}{2r \sqrt{D}} \left[ 2D \left( v g(v) \nu, v - g(v) - v \frac{dg}{dv} + r \nu, v \right) + v g(v) D, v + r D, r \right] = 0
\]

Using equation (10), and values of \( \nu_r, \nu_v \) (and subsequently \( \rho, \rho_v \) and \( Z_v \)) above equation becomes

\[
\frac{r^2 v^2 (1 + g(v)) D}{Y} \frac{dg}{dv} - r^3 v^2 A_v D - \frac{r^3 M_r}{2} + r^2 v g(v) D \left\{ \frac{\partial X [X \left[ 1 - \tanh^2(X) \right] + \tanh(X)]}{Y [3 - X \tanh(X)]} \\
+ (1 - 3l) + lv \frac{M_v}{M} - rv A_v \right\}
\]

\[
+ \frac{r^2 (g(v)) [r M - 2 u^2 e^{2rA} A_v - rv M_v]}{2} - r v e^{2rA} (A + r A_v) + v (e^{2rA} - 1) = 0.
\]

where \( X = re^{-Z(v)} \) and \( Y = 1 + (k + 1) g(v) \). Since \( Z(v) \) and \( A_v \) are integration terms, therefore, above equation is an integro-differential equation. The dependent function \( g(v) \) can be obtained as its solution in terms of \( v \)'s and integration constant. In turn, \( v' \) will have a regulated form consistent with the field equations. So, \( v' \) and \( \dot{v} \) given by equations (11) and (27) respectively, together determine \( v \), and hence \( R(t, r) \) to obtain the requisite exact solution of the Einstein’s field equations.

In the case of dust model, for \( k = 0 \), equation (42) gives

\[
v \frac{dg}{dv} + \frac{3}{2} g(v) = -r \tan(r) \frac{v}{2}
\]

The solution of above equation with treating \( r \)-terms as constant and using initial condition \( g(1) = 0 \) is obtained as

\[
g(v) = \frac{r \tan(r)}{3} \left[ \frac{1}{v^{3/2} - 1} \right]
\]

and therefore,

\[
v' = v \tan(r) \frac{1}{3} \left[ \frac{1}{v^{3/2} - 1} \right].
\]

On integrating it with respect to \( r \), we have

\[
v^{3/2}(t, r) = 1 - \sqrt{sech(r)} S(t)^{-1/2}
\]

where \( S(t) \) is due to a constant of integration, and it can be defined aptly to satisfy the requisite boundary conditions. As expected above equation satisfies the equation (27) for \( k = 0 \),

\[
\dot{v} = -\sqrt{M_0 sech(r)} \frac{1}{v} \text{ with } \dot{S}(t) = -3 \sqrt{M_0 S(t)^{3/2}}.
\]

Thus, condition of compatibility has established that definition of \( v' \) given in equation (11) is consistent with field equations. Its use, determines the form of \( g(v) \) and therefore role of \( v' \) is further become more evident in the analysis of existence of naked singularity.