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We study a revised version of Witten’s 2d black hole, in which the matter and \((b, c)\) ghosts are mixed. The level of the coset model is still \(9/4\). We show that this model is equivalent to that of Mukhi and Vafa, in which the level of the coset model is taken as 3, and the stress tensor is improved. We argue that the exact metric in such a model is just the semi-classical one, quite different from the exact metric in Witten’s black hole, being studied by Dijkgraaf, Verlinde and Verlinde. In addition, there appear ghost-related terms as a part of the background in the world sheet action.
There has been intensive interest in studying quantum aspects of black holes recently, partly inspired by Witten’s construction of a classically exact black hole in the two dimensional string theory. The model is provided by a conformal field theory associated to the coset model $SL(2, R)_k/U(1)$, in case of the Euclidean black hole, and by the one associated to $SL(2, R)_k/R$, in case of the minkowskian black hole. To have a critical central charge 26, the level $k$ must be $9/4$. With such a conformal field theory in hand, one hopes to tackle some problems which may shed light to the role of string theory in quantum gravity. Though much work has been done along this line, we are still far from any deep understanding of this black hole background. Despite many interesting problems pertaining to the black hole itself, any further understanding of it will help us in understanding the 2d string theory, as the black hole is just one of many possible time-dependent backgrounds in the 2d string theory.

It was suggested recently by Mukhi and Vafa, that one should not take Witten’s original model as the right black hole background. Instead, one should consider a modified version in which the matter and the diffeomorphism $(b, c)$ ghosts are mixed. The recipe is necessary, as argued by these two authors, in order to have the no-ghost theorem hold in this particular time-dependent background. The precise construction of this modified model is the following. One starts with a WZW model $SL(2, R)_3$, with a central charge 9, one then improves the stress tensor by adding a term $\partial J_3$. The resulting theory has a central charge 27, together with $(b, c)$, the sum of central charges is 1. Finally, one takes the $U(1)$ quotient with the current $J_3 + bc$. This theory has zero central charge too. It is in the last step that the matter and ghost get mixed. We will adopt a different construction. Our model will be

$$\frac{SL(2, R)_{9/4} \otimes [b, c]}{U(1)},$$

the $U(1)$ current is $J'_3 + bc$. Here $J'_3$ is a current different from $J_3$ in the $SL(2, R)$ current algebra, in order to have $J'_3 + bc$ be primary. Note that, here we still start with the WZW model at level $9/4$, with a central charge 27. We shall show however that this model is equivalent to the Mukhi-Vafa model.

One of our main results is that the semi-classical metric, as found in [4] [5], is the exact metric in the sense that there is no more $\alpha'$ or $1/k$ corrections. This fact should be contrasted to the exact metric found in [3], where there are $\alpha'$ corrections. What does this imply to the 2d string theory? This seems to indicate that the low energy action for the dilaton and metric in fact is the exact action. A low energy topological model for all
topological modes in the 2d string theory, constructed in [3], seems also to support this result. If one attempts to add \( \alpha' \) corrections to that model, there could be no way to preserve topological invariance.

Now we set off to construct the model. We will keep the level \( k \) arbitrary. Let \( k' \) denote \( k - 2 \), for \( k = 9/4 \), \( k' = 1/4 \). The current algebra \((J^\pm, J_3)\) of the \( SL(2,R) \) WZW model satisfies

\[
J^+(z)J^-(w) = \frac{k}{(z-w)^2} - \frac{2}{z-w} J_3(w) + \ldots,
\]

\[
J_3(z)J^\pm(w) = \pm \frac{1}{z-w} J^\pm(w) + \ldots,
\]

\[
J_3(z)J_3(w) = - \frac{k/2}{(z-w)^2} + \ldots.
\]

To simply life, we shall use a free field realization of the current algebra. We choose the Wakimoto representation. Three basic free fields are needed. A pair of bosonic \((\beta, \gamma)\) with spins \((1,0)\), and a scalar \(\phi\). The relevant propagators are \(\gamma(z)\beta(w) \sim 1/(z-w)\) and \(\phi(z)\phi(w) \sim -\ln(z-w)\). \(SL(2,R)_k\) currents are realized as

\[
J^-(z) = \beta \gamma^2 + \sqrt{2k'} \gamma \partial \phi - k \partial \gamma,
\]

\[
J^+(z) = \beta, \quad J_3(z) = \beta \gamma + \sqrt{\frac{k'}{2}} \partial \phi.
\]

To construct the coset, we need to take a quotient with respect to the \(U(1)\) group with current \(J'_3 + bc\). Since \(bc\) is not primary, \(J'_3\) can not be \(J_3\). We use \(J'_3 = \beta \gamma - \sqrt{2k'} \partial \phi\), then \(J'_3 + bc\) becomes primary. We introduce the gauge scalar \(X\), and the ghosts \((\eta, \xi)\) for gauge fixing \(U(1)\). These ghosts have spins \((1,0)\). Associated to current \(J'_3 + bc\), the anomaly free current is \(J'_3 + bc - i\sqrt{2k'} \partial X\). The radius of \(X\) is \(\sqrt{2k'}\), and becomes self-dual when \(k' = 1/4\). The \(U(1)\) BRST charge is then

\[
Q_{U(1)} = \oint \xi (J_3 + bc - i\sqrt{2k'} \partial X).
\]

Throughout this letter, \(\oint\) stands for \(1/(2\pi i) \oint dz\). This BRST charge is nilpotent.

Now, the total stress tensor including the \((b,c)\) ghosts is

\[
T(z) = -\beta \partial \gamma - \eta \partial \xi - \frac{1}{2} (\partial X)^2 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{\sqrt{2k'}} \partial^2 \phi - 2b \partial c + c \partial b
\]

The central charge of \((\beta, \gamma)\) and \((\eta, \xi)\) sums to zero, it is reasonable to expect that one can eliminate these fields effectively. We adopt a similar transformation on the stress tensor as found by Eguchi et al. in [3]. Consider the BRST anti-commutator

\[
\{Q_{U(1)}, \eta \partial \ln \gamma\} = \beta \partial \gamma + \eta \partial \xi + (bc - \sqrt{2k'} (\partial \phi + i \partial X)) \partial \ln \gamma + \frac{1}{2} (\partial \ln \gamma)^2 - \frac{1}{2} \partial^2 \ln \gamma.
\]
Note that $\partial \ln \gamma$ is a conformal operator. The above formula suggests us to add it to the stress tensor to cancel out the kinetic part for $(\beta, \gamma)$ and $(\eta, \xi)$. We arrive at a new stress tensor $T'$, equivalent to $T$:

$$T' = T + \{Q_{U(1)}, \eta \partial \ln \gamma\}$$

$$= -\frac{1}{2} (\partial X')^2 - \frac{1}{2} (\partial \phi')^2 + \frac{1}{\sqrt{2k'}} \partial^2 \phi' - 2b' \partial c' + c' \partial b' + \partial^2 \ln \gamma,$$

where

$$X' = X + i\sqrt{2k'} \ln \gamma, \quad \phi' = \phi + \sqrt{2k'} \ln \gamma,$$

$$b' = b \gamma, \quad c' = c \gamma^{-1}.$$

We note that with the new fields defined above, the new stress tensor in (2) is almost the same as the one for the $c=1$ Liouville theory, when $k' = 1/4$. The only difference is the last term associated to $\gamma$. Since there is no kinetic term for $\gamma$ in the new stress tensor, $\gamma$ is no longer dynamic. It is then consistent to take $\gamma$ being constant, if one works with new fields and new stress tensor. Thus, the total derivative term associated to $\gamma$ can be dropped out. This phenomenon is slightly different from that in the work of Eguchi et al, where no such a total derivative term is left after do a similar BRST transformation. It is easy to check that all new fields are neutral with respect to $Q_{U(1)}$, so operators constructed from them are automatically BRST invariant. Note that, if we set $\gamma$ be constant, then one can replace $c$ in the diffeomorphism BRST operator by $c'$.

The stress tensor is not the whole story of the coset model. To calculate correlation functions, a screening operator is needed. Or equivalently, one adds a (1,1) operator constructed from the screening operator to the world sheet lagrangian. This operator in the present model is

$$(\beta \gamma^3 - 3 \gamma \partial \gamma)(\overline{\beta} \overline{\gamma}^3 - 3 \overline{\gamma} \overline{\partial \gamma}) e^{\sqrt{2/k'} \phi}.$$

This operator is invariant with respect to $Q_{U(1)}$ (in the ordinary model where there is no matter-ghost mixing, the screening operator is $\beta \overline{\beta} \exp(\sqrt{2/k'} \phi)$). To find the corresponding operator depending only on new fields, we use

$$\{Q_{U(1)}, \eta \gamma^2 e^{\sqrt{2/k'} \phi} \} = (\beta \gamma^3 - 3 \gamma \partial \gamma) e^{\sqrt{2/k'} \phi} + \sqrt{2k'} \left(-\partial(\phi' + iX') + \frac{1}{\sqrt{2k'}} b' c'\right) e^{\sqrt{2/k'} \phi'}. $$

So the first operator on the r.h.s. of the above equation is equivalent to the second operator on the r.h.s., up to a BRST anti-commutator. One can go a step further to show that the operator

$$V = \left(\partial(\phi' + iX') - \frac{1}{\sqrt{2k'}} b' c'\right) \left(\overline{\partial}(\phi' + iX') - \frac{1}{\sqrt{2k'}} \overline{b}' \overline{c}'\right) e^{\sqrt{2/k'} \phi'}$$

(5)
is equivalent to (4), up to a BRST anti-commutator in terms of the total BRST charge \( Q_{U(1)} + \overline{Q}_{U(1)} \). \( V \) is constructed only from new fields, therefore should be used when one works with new stress tensor and new fields. It is remarkable that this operator contains ghosts explicitly, this is how the matter and ghost get mixed in terms of new fields. We have cheated a little in the above discussion. Both operators in (4) and in (5) are not primary operators. It is easy to see that \( V \) is not primary, since \( b'c' \) is not. We need certain prescription for calculating correlation functions in order to circumvent this problem. There is one, that is, to use the twisted \( N=2 \) CFT, as the present model can be thought of as a twisted \( N=2 \) CFT [4].

We now discuss the world sheet action in terms of new fields. The purely kinetic part is

\[
S_0 = \frac{1}{2\pi} \int dz^2 (\partial X \partial X' + \partial \phi \partial \phi' + \sqrt{2k'} R \phi'),
\]

where we omitted the kinetic term for ghosts \( (b', c') \). The dilaton field is \( \Phi = \sqrt{1/2k'} \phi \). We add a term \( \mu/(2\pi)V \) to the lagrangian, the action for bosonic fields now reads

\[
S = \frac{1}{2\pi} \int dz^2 (g_{X'X'} \partial X \partial X' + g_{\phi'\phi'} \partial \phi \partial \phi' + 2g_{X'\phi'} \partial X \partial \phi' + \sqrt{2k'} R \phi'),
\]

with metric components

\[
g_{X'X'} = 1 - \mu e^{\sqrt{2/2k'} \phi'}, \quad g_{\phi'\phi'} = 1 + \mu e^{\sqrt{2/2k'} \phi'},
\]

\[
g_{X'\phi'} = i\mu e^{\sqrt{2/2k'} \phi'}.
\]

Since the screening operator is believed to be exactly marginal, the metric we get should be taken as exact. The above metric can be diagonalized. If we define new coordinates via \( \tanh^2 r = 1 - \mu \exp(\sqrt{2/2k'} \phi') \), and \( \theta = X' - i\sqrt{2k'} \ln(\tanh r) \), the diagonalized metric is

\[
g_{rr} = 2k', \quad g_{\theta\theta} = \tanh^2 r,
\]

the standard euclidean black hole metric, found as a solution to the low energy effective action. The dilaton field in term of new coordinates is \( 2\Phi = -\ln(\mu \cosh^2 r) \), also the standard one. The radius of \( \theta \) is \( \sqrt{2k'} \), the same as that of \( X' \). If \( X' \) were real, \( \theta \) would be complex. To get rid of this embarrassing situation, one can Wick-rotate \( X' \) and \( \theta \), thus gets the minkowskian black hole. Why is the exact metric the same as the solution to the
low energy effective action? The answer is that there is another part in the world sheet action, related to \((b', c')\) ghosts. This part is written, in terms of new coordinates, as

\[
\frac{1}{2\pi} \int dz^2 \frac{1}{\sqrt{2k'}} \frac{1}{\cosh^2 r} \left[ (-i\partial\theta + \sqrt{2k'} \coth r \partial r) b' c' + (-i\overline{\partial}\theta + \sqrt{2k'} \coth r \overline{\partial} r) b' c' - \frac{1}{\sqrt{2k'}} b' \overline{b} c' \overline{c}' \right].
\]

Indeed this part together with that for bosonic fields gives us almost a N=2 system, except that the spins of \((b', c')\) are not \((1/2, 1/2)\) but \((2, -1)\). The model can be treated as a twisted N=2 system, with fermionic fields \((b', c')\). The hidden N=2 symmetry is the reason why the beta functions for the dilaton and metric calculated at semi-classical level are exact. The above term involving ghosts can be compared to that discussed in [7], where a supersymmetric gauged WZW model based on group \(SL(2, R)\) is considered. If one replaces \(k\) by \(k' = k - 2\) in those semiclassical formulas in [7], one gets the same metric and the ghost-related terms. So the only renormalization effect is the shift of the level.

In the Witten’s version of the 2d euclidean black hole, it was shown in the first paper in [3] that the screening operator corresponds to the wrongly dressed operator \(W_{1,0}^-\) in the c=1 model. What does our screening operator in the revised model correspond to? In the first sight it seems impossible to find any corresponding operator in the c=1 model, since our screening operator contains ghosts. The right way is to consider the conformal dimension zero, BRST invariant operator \(c' \overline{c'} V\). It is the factor \(c' \overline{c'}\) that annihilates the ghost terms in \(V\). Still, this operator is different from that in the Witten’s model. To see that the screening operator is indeed equivalent to \(c' \overline{c'} W_{1,0}^-\), we note that \(W_{1,0}^- \sim \partial X \overline{\partial} X \exp(\sqrt{2/k'} \phi')\). In addition to this term, there are terms in \(c' \overline{c'} V\) proportional to \(\partial \phi' \exp(\sqrt{2/k'} \phi')\) or its anti-holomorphic counterpart. Let \(Q'\) be the diffeomorphism BRST operator constructed from \((b', c')\) and the new stress tensor \((2)\), then one can show that

\[
c' \partial \phi' e^{\sqrt{2/k'} \phi'} = [Q', \sqrt{\frac{k'}{2}} e^{\sqrt{2/k'} \phi'}].
\]

Thus, the operator \(c' \overline{c'} V\) is equivalent to \(c' \overline{c'} W_{1,0}^-\). It is remarkable that both screening operators in the matter-ghost mixing model and the non-mixing model correspond to the same BRST invariant operator in the c=1 model, while they are markedly different as \((1, 1)\) operators.

As we showed before, when \(\gamma\) is effectively dropped out after doing transformation in \((2)\), one can use \(Q'\) to calculate the BRST cohomology. This BRST cohomology is just
the same as in the c=1 Liouville model \([8]\). One can use the old fields \(\beta, \gamma, \phi\) and \(X\), and \(b, c\) to express those operators. However, when one considers (1, 1) operators which can be used to perturb the string background, one should be more careful. The simplest example is just \(V\) in \([5]\). This operator provides a time-dependent background, and simultaneously matter-ghost mixing. It is the latter that is missing in the corresponding BRST invariant operator, as we showed in the last paragraph. It is generally true that any (1, 1) discrete operator gives rise to a time-dependent background, therefore one should carefully find out the ghost-dependent piece.

The tachyon vertex operator \(\exp(i p X' + p \phi')\) is
\[
\gamma \sqrt{2k'} (p \phi - p X) e^{ipX'+ip\phi}
\]
in terms of \(SL(2, R)\) free fields. This is also different from the one in the ordinary model without matter-ghost mixing \([3]\). Given that both the tachyon vertex operator and the screening operator \((\beta \gamma^3 - 3 \gamma \partial \gamma)(\beta \gamma^3 - 3 \gamma \partial \gamma)\exp(\sqrt{2/k'} \phi)\) are different from those in the ordinary model, we expect that tachyon amplitudes will be different in these two models, even at tree level. It would be extremely interesting to calculate just two point amplitudes in both models and compare them.

Next we show that the model proposed by Mukhi and Vafa in \([4]\) is equivalent to ours. Their starting point is a WZW model \(SL(2, R)_3\), with the improved stress tensor \(T + \partial J_3\). The central charge of this model is again 27. We shall replace the level 3 by \(k + 2\), for a general \(k\). Once again we use fields \((\beta, \gamma)\) and \(\phi\) to bosonize the current algebra. We have to replace \(k'\) by \(k\) and \(k\) by \(k + 2\) in formulas in \([4]\). With the improvement, the stress tensor for this WZW model is
\[
T(z) = \gamma \partial \beta - \frac{1}{2} (\partial \phi)^2 + \frac{1}{\sqrt{2}} (\sqrt{k} + 1/\sqrt{k}) \partial^2 \phi.
\] (8)
Now the spins of \((\beta, \gamma)\) after improvement are longer \((1, 0)\) but \((0, 1)\), so the roles of \(\beta\) and \(\gamma\) are exchanged. This implies particularly that current \(J^+\) is of conformal dimension 0. Define \(1/\sqrt{k'} = \sqrt{k} + 1/\sqrt{k}\), then the stress tensor in \((8)\) is the same as that of \(SL(2, R)\) at level \(k' + 2\). Taking \(k = 1\), the value proposed in \([4]\), \(k' = 1/4\), the value in our model studied before. This is the main reason why the two models are the same. The \(U(1)\) current to be modded out is \(-J_3 + bc\). Although both \(J_3\) and \(bc\) are not primary with the improved stress tensor, \(-J_3 + bc\) is. Now the \(U(1)\) BRST charge is
\[
Q_{U(1)} = \oint \xi (-J_3 + bc - i \sqrt{\frac{k}{2}} \partial X).
\]
Again taking $k = 1$, we find that this BRST charge is the same as the one we defined before when $k' = 1/4$, provided we use $\beta \to \gamma$ and $\gamma \to -\beta$. The statement that with the improved stress tensor, one puts constraint $J^+ = \beta = \text{const}$ is equivalent to what we had to do with $\gamma$ in our model, in order to get rid the total derivative term in the stress tensor $T'$ in (2).

To conclude, we have shown that in the revised version of Witten’s black hole, the semi-classical metric and dilaton become exact, due to the appearance of ghost-related terms in the world sheet action. The necessity of the mixing of the matter and ghosts can be most easily seen by examining the one-loop partition function, where appropriate cancellation has to occur in order to have the right amount of degrees of freedom [9]. If the lesson we learned from this example is correct, we have to be careful when we consider any time-dependent background, obtained by adding marginal perturbation to the action. The marginal operator should contain ghost-related terms in an appropriate way. The guide for introducing these terms may be some hidden N=2 supersymmetry, as advocated by the authors of [4]. Indeed the hidden N=2 symmetry is the reason why the semi-classical solution is exact in the model discussed in this letter. Consider a N=2 non-linear sigma model with a dilaton field $\Phi$, the Ricci tensor associated to the rescaled metric $\exp(-2\Phi)g$ should vanish. This condition is exact for the N=2 non-linear sigma model. When the spacetime is two dimensional, this implies that the rescaled metric is flat, a consequence of the semi-classical equation of motion.

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