Quantum properties of two-dimensional linear harmonic oscillator in polar coordinate system

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Abstract: Using quantum theory and MATLAB software, the basic properties of two-dimensional linear harmonic oscillators in quantum mechanics are systematically studied in polar coordinate system, and obtain the visualized results. The results show that, in polar coordinate system, with the exception of special case \( n_r = 0, |m| = 0 \), the degeneracy of two-dimensional linear harmonic oscillator is \( 2n_r + |m| + 1 \), and the corresponding energy eigenvalues is \( \hbar \omega (2n_r + |m| + 1) \). The number of intersection line between wave function and the plane with \( \Psi = 0 \) is \( 2n_r + |m| \). In the case of \( n_r = 0 \), the maximum number of probability density distributions is \( 2|m| \). The results of this visualization are in complete agreement with the theoretical results. The visualization results in different coordinate systems can be verified with each other, which opens up a new research idea and also provides an idea for other quantum theoretical models to be studied.

1. Introduction

As the basic and core theoretical course of modern physics, quantum mechanics is a subject that studies the motion laws of microscopic particles [1,2]. It elaborates the wave-particle duality of matter, as well as the interaction between energy and matter, and is the basis of modern disciplinary fields, such as big data, biological science, communication engineering, power electronics, automation and quantum communication, etc.[3-8]. As one of the most important models in quantum mechanics, the study of harmonic oscillator is very important for the exploration of quantum theory and even the microscopic world. The study of harmonic oscillator motion and its related characteristics is of great significance both in theory and in application. In recent years, the research on harmonic oscillators and their characteristics has been heated up [9-15], and some achievements have been made, especially one-dimensional linear harmonic oscillators. The two-dimensional isotropic harmonic oscillator is one of the most representative harmonic oscillator models, and is also one of the important models in quantum mechanics. To study this model correctly, we must analyze its ground state energy and wave function.

This article discusses issue of two-dimensional isotropic harmonic oscillator in polar coordinates. Using theoretical derivation and simulation method, we calculate ground-state energy of two-dimensional isotropic harmonic oscillator, then obtain the visual results of characteristics, including 3-d graphics, contour, pseudo panel of wave function and probability density. This kind of research from different angles and different ideas is more helpful to the understanding and mastering of abstract concepts.
2. Theoretical model and derivation

After studying wave functions in cartesian coordinates, we shall proceed in terms of polar coordinates. According to the principle of one-dimensional linear harmonic oscillator and the theory of two-dimensional linear harmonic oscillator, this paper discusses in polar coordinates.

In polar coordinates, the stationary Schrödinger equation of the system is

$$\left[ -\frac{\hbar^2}{2\mu} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) + \frac{1}{2} \mu \omega^2 r^2 \right] \psi = E \psi, \quad (1)$$

With the natural unit ($\hbar = \mu = \omega = 1$), use the separation of variables method. By studying the two special cases of $r \to 0$ and $r \to \infty$ [16-18], combining with the normalization, the radial wave function is obtained

$$R_{n,m}(r) = N_{n,m} r^{|m|+1/2} F(-n,|m|+1, r^2). \quad (2)$$

Where $N_{n,m}$ is the normalized coefficient, and its expression is

$$N_{n,m} = \left[ \frac{2(n_r + |m|)!}{n_r! (|m|)!^2} \right]^\frac{1}{2} \quad (3)$$

$F(-n,|m|+1, r^2)$ is called a confluence hypergeometric function, namely

$$F(-n,|m|+1, r^2) = 1 + \frac{n_r}{1(|m|+1)} r^2 + \frac{n_r(n_r + 1)}{2(|m|+1)(|m|+1+1)} r^2 + \frac{n_r(n_r + 1)(n_r + 2)}{3(|m|+1)(|m|+1+1)(|m|+1+2)} r^2 + \cdots \quad (4)$$

Then the expression of the wave function is

$$\psi(r, \phi) = \frac{1}{\sqrt{2\pi}} N_{n,m} r^{|m|+1/2} F(-n,|m|+1, r^2) \left[ \sin(m\phi) + \cos(m\phi) \right]. \quad (5)$$

3. Simulation results and discussion

3.1. The energy levels

According to the relationship between energy levels, the energy levels of one-, two- and three-dimensional linear harmonic oscillators are obtained by numerical simulation, as shown in Fig. 1.

It can be seen from Fig. 1 that the energy level of the linear harmonic oscillator is of a discrete type and can only take discrete values, and the energy is quantized. The energy of a harmonic oscillator is evenly distributed, the adjacent two level spacing $\Delta E = h \omega$ omega. This result conforms to Planck's hypothesis and is in complete agreement with theoretical results.
3.2. The wave function and its probability density

According to Eqs. (2) ~ (4), in polar coordinate system, MATLAB software is used to draw the wave function simulation diagram of two-dimensional linear harmonic oscillator with different particle numbers, and the wave function grid diagram of 9 cases is obtained, as shown in Fig. 2. At the same time, combining with the theoretical derivation of two-dimensional linear harmonic oscillator, MATLAB software is used to draw the probability density of the two-dimensional linear harmonic oscillator under different energy levels, and 9 kinds of wave function diagrams corresponding to 9 wave function probability density distribution diagrams are obtained, as shown in Fig. 3.

Fig. 1 The distributions of energy levels of one-, two- and three-linear harmonic oscillator.

Fig. 2 Simulation diagram of wave function in polar coordinate system.
Fig. 3 The distributions of the wave function probability density in polar coordinate system.

It can be from graph that, in polar coordinates, with the exception of special case $n_r=0$, $|m|=0$, the degeneracy of two-dimensional linear harmonic oscillator is $2n_r+|m|+1$, the corresponding energy eigenvalues is $\hbar\omega(2n_r+|m|+1)$. Wave function and the intersection of a plane $\Psi=0$ is $2n_r+|m|$. In the case of $n_r=0$, the probability density distribution of the maximum number is $2|m|$. The results shows that are in complete agreement with the theoretical results in the two figures, which well reflect the characteristics of the wave function and its probability density.

4. Conclusion

Based on quantum theory, we systematically studies the basic characteristics of two-dimensional linear harmonic oscillator in quantum mechanics in polar coordinate system, and obtains the visualization results using MATLAB software. We find that, in polar coordinates, with the exception of special case $n_r=0$, $|m|=0$, the degeneracy of two-dimensional linear harmonic oscillator is $2n_r+|m|+1$, the corresponding energy eigenvalues is $\hbar\omega(2n_r+|m|+1)$. The wave function and the intersection of a plane $\Psi=0$ to $2n_r+|m|$. In the case of $n_r=0$, the probability density distribution of the maximum number is $2|m|$. The simulation results reflect the characteristics of the wave function and probability density in polar coordinate system, and are in complete agreement with the theoretical results. The visualization results in different coordinate systems can be verified with each other, which opens up a new research idea, and also provides ideas for the study of other quantum theoretical models.

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