Totally real bi-quadratic fields with large Pólya groups

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Abstract

For an algebraic number field $K$ with ring of integers $\mathcal{O}_K$, an important subgroup of the ideal class group $\text{Cl}_K$ is the Pólya group, denoted by $\text{Po}(K)$, which measures the failure of the $\mathcal{O}_K$-module $\text{Int}(\mathcal{O}_K)$ of integer-valued polynomials on $\mathcal{O}_K$ from admitting a regular basis. In this paper, we prove that for any integer $n \geq 2$, there are infinitely many totally real bi-quadratic fields $K$ with $\text{Po}(K) \cong (\mathbb{Z}/2\mathbb{Z})^n$. In fact, we explicitly construct such an infinite family of number fields. This also provides an infinite family of bi-quadratic fields with ideal class groups of 2-ranks at least $n$.

Keywords: Pólya field, Pólyagroup, Real bi-quadratic field, Galois cohomology

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1 Introduction

Let $K$ be an algebraic number field with ring of integers $\mathcal{O}_K$ and ideal class group $\text{Cl}_K$. Let

$$\text{Int}(\mathcal{O}_K) = \{ f(X) \in K[X] : f(\alpha) \in \mathcal{O}_K \text{ for all } \alpha \in \mathcal{O}_K \}.$$ 

It is a well-known fact (cf. [2, Remark II.3.7]) that $\text{Int}(\mathcal{O}_K)$ is a free $\mathcal{O}_K$-module. An $\mathcal{O}_K$-basis $\{ f_i \}_{i \geq 0}$ of $\text{Int}(\mathcal{O}_K)$ is said to be a regular basis if degree$(f_i) = i$ for each $i \geq 0$. The existence of a regular basis leads to the definition of a Pólya field as follows.

Definition 1 (cf. [2, Definition II.4.1]) An algebraic number field $K$ is said to be a Pólya field if the $\mathcal{O}_K$-module $\text{Int}(\mathcal{O}_K)$ has a regular basis.

It is quite interesting that the existence of a regular basis for $K$ is closely governed by the structure of $\text{Cl}_K$. To describe the interplay between $\text{Cl}_K$ and $\text{Int}(\mathcal{O}_K)$, we first introduce the Ostrowski ideals $\Pi_q(K)$ [14, 15] as follows.

$$\Pi_q(K) = \begin{cases} \prod_{p \in \text{Spec}(\mathcal{O}_K) \cap \mathbb{Q} \mid p \neq q} p ; & \text{if } \mathcal{O}_K \text{ has some primes } p \text{ of norm } q, \\ \mathcal{O}_K; & \text{otherwise}. \end{cases}$$  

(1)

Definition 2 (cf. [2], §II.4) The Pólya group $\text{Po}(K)$ of $K$ is defined to be the subgroup generated by the ideal classes $[\Pi_q(K)]$ in $\text{Cl}_K$. 

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It is known (cf. [18, Section 1]) that $K$ is a Pólya field if and only if $\text{Po}(K)$ is trivial. Thus the study of $\text{Int}(O_K)$ is translated into the study of the subgroup $\text{Po}(K)$ of $\text{Cl}_K$. Moreover, when $K/\mathbb{Q}$ is a finite Galois extension, it immediately follows from Definition 2 that $\text{Po}(K)$ is generated precisely by the ideal classes $[\Pi_q(K)]$, where $q$ runs over the norms of ramified primes [2, Proposition II.4.2]. Therefore, there is a close connection between the ramified primes in $K$ and the Pólya group $\text{Po}(K)$. In particular, for quadratic fields, the following result is known due to the work of Hilbert.

**Proposition 1** (cf. [9], [4, Proposition 1.4]) Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field. Let $\varepsilon_K$ be the fundamental unit of $K$ and let $r_K$ denote the number of ramified primes in $K/\mathbb{Q}$. Then

$$|\text{Po}(K)| = \begin{cases} 2^{r_K} - 2 & \text{if } d > 0 \text{ and } N(\varepsilon_K) = 1, \\ 2^{r_K} - 1 & \text{otherwise.} \end{cases}$$

From Proposition 1, it is clear that the order of the Pólya group for a quadratic field is necessarily 1 or a power of 2. In [18], Zantema generalized this to finite Galois extensions $K/\mathbb{Q}$ (see Proposition 2 below). Recently, Chabert [4] and Maarefparvar-Rajaei [13] independently extended the result for finite Galois extensions of number fields (whereas the first approach for relativization of Pólya group appeared in Brumer-Rosen paper [1]).

There has been a lot of studies on the Pólya group of bi-quadratic fields in recent times (cf. [3,6,7,10–12]). In [6], the authors determined the Pólya groups of bi-quadratic fields that are known to have an “Euclidean ideal class”. The interested reader is encouraged to look into [5] and the references listed therein to know about Euclidean ideal class.

In this paper, using Zantema’s result, we explicitly construct an infinite family of totally real bi-quadratic fields $K$ with Pólya group $\text{Po}(K) \simeq (\mathbb{Z}/2\mathbb{Z})^n$. Our result extends the previously known family of bi-quadratic fields with Pólya group $\mathbb{Z}/2\mathbb{Z}$ (cf. [6, Theorem 3]). More precisely, we prove the following theorem.

**Theorem 1** Let $n \geq 1$ be an integer. Then there exist infinitely many totally real bi-quadratic fields $K$ with Pólya group $\text{Po}(K) \simeq (\mathbb{Z}/2\mathbb{Z})^n$.

Since the Pólya group $\text{Po}(K)$ is a subgroup of the ideal class group $\text{Cl}_K$, we immediately have the following corollary.

**Corollary 1** For an integer $n \geq 1$, there exist infinitely many totally real bi-quadratic fields $K$ with 2-rank of $\text{Cl}_K$ being at least $n$.

2 Preliminaries

In [18, Section 3], Zantema made use of Galois cohomology to study $\text{Po}(K)$ and its relation with the ramified primes in $K/\mathbb{Q}$, when $K$ is a finite Galois extension of $\mathbb{Q}$. The Galois group $G = \text{Gal}(K/\mathbb{Q})$ acts on the multiplicative group of units $O_K^\times$ via the action $(\sigma, \alpha) \mapsto \sigma(\alpha)$, thus endowing $O_K^\times$ with a $G$-module structure. Zantema’s result can now be stated as follows (for a generalization in relative case, see [1, Proposition 2.2] and [13, Theorem 2.2]).

**Proposition 2** [18, Proposition 3.1] Let $K$ be a finite Galois extension of $\mathbb{Q}$ with Galois group $G = \text{Gal}(K/\mathbb{Q})$. Let $p_1, \ldots, p_m$ be the rational primes that ramify in $K$. Let the
ramification index of \( p_i \) be \( e_i \) for each \( i \in \{1, \ldots, m\} \). Then there is an exact sequence

\[
0 \rightarrow H^1(G, \mathcal{O}_K^*) \rightarrow \bigoplus_{i=1}^{m} \mathbb{Z}/e_i\mathbb{Z} \rightarrow \text{Po}(K) \rightarrow 0
\]

of abelian groups.

The next lemma helps us understand the structure of the group \( H^1(G, \mathcal{O}_K^*) \) via its subgroup of 2-torsion elements.

**Lemma 1** [16, Theorem 4] Let \( K \) be a totally real bi-quadratic field with quadratic subfields \( K_1, K_2 \) and \( K_3 \). Let \( H[2] \) be the 2-torsion subgroup of \( H^1(G, \mathcal{O}_K^*) \). Then the following statements hold.

(i) The index of \( H[2] \) in \( H^1(G, \mathcal{O}_K^*) \) is \( \leq 2 \). The index is 2 if and only if the rational prime 2 is totally ramified in \( K/\mathbb{Q} \) and there exist \( a_i \in \mathcal{O}_{K_i} \) for each \( i = 1, 2, 3 \) with

\[
N_{K_i/\mathbb{Q}}(a_i) = 2 \text{ or } -2.
\]

(ii) For \( i \in \{1, 2, 3\} \), let \( \Delta_i \) be the square-free part of the discriminant of \( K_i \) and let \( u_i = z_i + t_i\sqrt{\Delta_i} \) be a fundamental unit of \( \mathcal{O}_{K_i} \) with \( z_i > 0 \). Let us define

\[
a_i = \begin{cases} 
N_{K_i/\mathbb{Q}}(u_i + 1) & \text{if } N_{K_i/\mathbb{Q}}(u_i) = 1, \\
1 & \text{if } N_{K_i/\mathbb{Q}}(u_i) = -1.
\end{cases}
\]

Then \( H[2] \) is isomorphic to the subgroup of \( \mathbb{Q}^*/(\mathbb{Q}^*)^2 \) generated by the images of \( \Delta_1, \Delta_2, \Delta_3, a_1, a_2 \) and \( a_3 \) in \( \mathbb{Q}^*/(\mathbb{Q}^*)^2 \).

The next lemma is very crucial for proving Theorem 1 and we furnish a detailed proof here.

**Lemma 2** Let \( t \geq 2 \) be an integer. For every unordered pair \((i, j)\) with \( 1 \leq i \neq j \leq t \), let \( \varepsilon_{ij} \in \{\pm 1\} \) be given. Then there exist infinitely many \( t \)-tuples of prime numbers \( p_1, \ldots, p_t \) with \( p_i \equiv 1 \) (mod 8) for each \( i \), and the Legendre symbol \( \left( \frac{p_i}{p_j} \right) = \varepsilon_{ij} \) for all \( i \neq j \).

**Proof** We prove by induction on \( t \). We begin with \( t = 2 \). We choose \( p_1 = 73 \). If \( \varepsilon_{12} = 1 \), then we take a prime \( p_2 \equiv 1 \) (mod 8p_1). On the other hand, if \( \varepsilon_{12} = -1 \), we take \( p_2 \equiv 17 \) (mod 8p_1). By Dirichlet’s theorem for primes in arithmetic progression, we obtain infinitely many choices for \( p_2 \) in both the cases. Thus, we obtain infinitely many 2-tuples \( \{p_1, p_2\} \) satisfying the requirements of the lemma. Consequently, the lemma holds for \( t = 2 \).

Now, suppose that we have a \((t - 1)\)-tuple of prime numbers \( \{p_1, \ldots, p_{t-1}\} \) such that \( p_i \equiv 1 \) (mod 8) for each \( i \in \{1, \ldots, t-1\} \) and \( \left( \frac{p_i}{p_j} \right) = \varepsilon_{ij} \) for all \( 1 \leq i \neq j \leq t-1 \). We wish to find infinitely many prime numbers \( p_t \) with \( \left( \frac{p_t}{p_i} \right) = \varepsilon_{it} \) for all \( 1 \leq i \leq t - 1 \). For each integer \( i \in \{1, \ldots, t-1\} \), we choose integers \( n_i \) such that \( 1 \leq n_i \leq p_i - 1 \) and \( \left( \frac{n_i}{p_i} \right) = \varepsilon_{it} \). We consider the following system of congruences.

\[
X \equiv 1 \pmod{8} \\
X \equiv n_1 \pmod{p_1} \\
\vdots \\
X \equiv n_{t-1} \pmod{p_{t-1}}.
\]
By the Chinese remainder theorem, there exists a unique integer, say \( x_0 \), modulo \( 8p_1 \cdots p_{t-1} \) satisfying the above system of congruences. Since \( \gcd(x_0, 8p_1 \cdots p_{t-1}) = 1 \), there exist infinitely many prime numbers \( \ell \) such that \( \ell \equiv x_0 \pmod{8p_1 \cdots p_{t-1}} \), by Dirichlet’s theorem for primes in arithmetic progression. Then \( \ell \equiv 1 \pmod{8} \) and \( \left( \frac{\ell}{p_1} \right) = \left( \frac{\ell}{p_2} \right) = \cdots = \left( \frac{\ell}{p_t} \right) = \varepsilon_\ell \) for each \( i \in \{1, \ldots, t-1\} \). Taking \( p_t = \ell \), we conclude that the \( t \)-tuples \( \{p_1, \ldots, p_t\} \) fulfill the desired requirements of the lemma.

The next lemma, due to Trotter [17], provides a useful criterion for a real quadratic field of a particular type to have a fundamental unit of negative norm.

**Lemma 3** [17, Page 3] Let \( p_1, \ldots, p_t \) be prime numbers with \( p_i \equiv 1 \pmod{4} \) for each \( i \in \{1, \ldots, t\} \). Let \( \varepsilon_{ij} = \left( \frac{p_i}{p_j} \right) \) for \( 1 \leq i \neq j \leq t \). Then each of the following two conditions is sufficient to ensure that the fundamental unit of the real quadratic field \( \mathbb{Q}(\sqrt{p_1 \cdots p_t}) \) to have norm \(-1\).

(i) \( t \) is odd and \( \varepsilon_{ij} = -1 \) for all \( 1 \leq i \neq j \leq t \).

(ii) \( t \) is even, \( \varepsilon_{12} = -1, \varepsilon_{11} = 1 \) for all \( i > 2 \) and \( \varepsilon_{ij} = -1 \) for all \( 2 \leq i \neq j \leq t \).

### 3 Proof of Theorem 1

Let \( t \geq 3 \) be an integer and let \( p_1, \ldots, p_t \) be prime numbers with \( p_i \equiv 1 \pmod{8} \). Lemma 2 allows us to choose infinitely many \( t \)-tuples \( \{p_1, \ldots, p_t\} \) of prime numbers satisfying any condition on the Legendre symbols \( \left( \frac{p_i}{p_j} \right) \). Let \( K = \mathbb{Q}(\sqrt{\ell}, \sqrt{p_1 \cdots p_t}) \) with three quadratic subfields \( K_1 = \mathbb{Q}(\sqrt{\ell}), K_2 = \mathbb{Q}(\sqrt{p_1 \cdots p_t}) \) and \( K_3 = \mathbb{Q}(\sqrt{2p_1 \cdots p_t}) \). The only ramified primes in \( K/\mathbb{Q} \) are 2 and \( p_i \)'s. Moreover, since \( K = K_1K_2 \) and \( K_1 \cap K_2 = \mathbb{Q} \), we conclude that the ramification indices of all these primes are 2. Therefore, by Proposition 2, \( H^1(G, O^n_K) \) embeds into \( \bigoplus_{i=1}^{t+1} \mathbb{Z}/2\mathbb{Z} \), where \( G \) is the Galois group \( \text{Gal}(K/\mathbb{Q}) \). Now, we prove that \( |H^1(G, O^n_K)| = 4 \), which, by Proposition 2, implies that

\[
|\text{Po}(K)| = \frac{|H^1(G, O^n_K)|}{\#(\mathbb{Z}/2\mathbb{Z})} = 2^{t+1} \cdot 2 = 2^{t-1}.
\]

We note that the rational prime 2 is not totally ramified in \( K/\mathbb{Q} \). Therefore, by Lemma 1, we get that \( H^2[2] = H^2(G, O^n_K) \). Following the notations of Lemma 1, let \( \Delta_i \) be the square-free part of the discriminant of \( K_i \), namely, \( \Delta_1 = 2, \Delta_2 = p_1 \cdots p_t \) and \( \Delta_3 = 2p_1 \cdots p_t \). Using the notation \([x]\) to denote the image of a non-zero element \( x \in \mathbb{Q} \) in \( \mathbb{Q}^+/(\mathbb{Q}^+)^2 \), we see that \([\Delta_1],[\Delta_2],[\Delta_3]\) \( \in \langle [2], [p_1 \cdots p_t] \rangle \) in \( \mathbb{Q}^+/(\mathbb{Q}^+)^2 \). We note that 1 + \( \sqrt{2} \) is a fundamental unit of \( K_1 \) with negative norm. Also, by Lemma 3, the fundamental unit of \( K_2 \) has negative norm and consequently, \([a_1]\) and \([a_2]\) are trivial in \( \mathbb{Q}^+/(\mathbb{Q}^+)^2 \), where \( a_i \)'s are as defined in Lemma 1.

Now, let \( u_3 = z + w\sqrt{2p_1 \cdots p_t} \) be a fundamental unit of \( K_3 \). If \( N_{K_3/\mathbb{Q}}(u_3) = -1 \), then \([a_3]\) is also trivial in \( \mathbb{Q}^+/(\mathbb{Q}^+)^2 \) and in this case, Lemma 1 implies that \( |H^1(G, O^n_K)| = 4 \). Otherwise, we assume that \( N_{K_3/\mathbb{Q}}(u_3) = 1 \). Then by Lemma 1, we have \( a_3 = N_{K_3/\mathbb{Q}}(u_3 + 1) = 2(z+1) \). Since \( z^2 - 1 = 2w^2p_1 \cdots p_t \) is even, we conclude that \( z \) is odd and therefore \( \gcd(z-1, z+1) = 2 \). Therefore, \( w = 2w_1 \) for some integer \( w_1 \) and hence

\[
\frac{z-1}{2} \cdot \frac{z+1}{2} = 2w_1^2p_1 \cdots p_t.
\]
Since \( \gcd \left( \frac{z - 1}{2}, \frac{z + 1}{2} \right) = 1 \), from Eq. (2), we get that \( \frac{z - 1}{2} = a^2 \alpha \) and \( \frac{z + 1}{2} = b^2 \beta \), where \( a, b \) are relatively prime integers with \( ab = w_1 \) and \( \alpha, \beta \) are relatively prime integers with \( \alpha \beta = 2p_1 \cdots p_t \). Thus we have \([2(z + 1)] = [4b^2 \beta] = [\beta] \in \mathbb{Q}^*/(\mathbb{Q}^*)^2\). If \( \beta = 1 \) or 2 or \( p_1 \cdots p_t \) or \( 2p_1 \cdots p_t \), then \([\beta] \notin \{2, [p_1 \cdots p_t]\}\). We now prove that other choices cannot occur for \( \beta \).

**Case I.** Let \( t \) be odd. By Lemma 2, we can choose prime numbers \( p_1, \ldots, p_t \) such that \( p_i \equiv 1 \pmod{8} \) for each \( i \) and they satisfy condition (1) of Lemma 3.

- **Case I-1.** \( \beta = 2B \), where \( 1 < B \) is a strict integer divisor of \( p_1 \cdots p_t \) and consisting of an odd number of prime divisors. Then for a prime divisor \( \ell \) of \( \alpha \), from the equation
  \[
  1 = \frac{z + 1}{2} - \frac{z - 1}{2} = b^2 \beta - a^2 \alpha, 
  \]
  \( \text{(3)} \)
  we have
  \[
  1 = \left( \frac{1}{\ell} \right) = \left( \frac{2b^2 \beta - a^2 \alpha}{\ell} \right) = \left( \frac{2}{\ell} \right) \left( \frac{B}{\ell} \right) = \left( \frac{B}{\ell} \right) = -1, 
  \]
  which is a contradiction.

- **Case I-2.** \( \beta = 2B \), where \( 1 < B \) is a strict integer divisor of \( p_1 \cdots p_t \) and consisting of an even number of prime divisors. Then \( \alpha \) consists of an odd number of prime divisors, since \( t \) is odd. Therefore, for an odd prime divisor \( \ell \) of \( \beta \), we obtain
  \[
  1 = \left( \frac{1}{\ell} \right) = \left( \frac{2b^2 \beta - a^2 \alpha}{\ell} \right) = \left( \frac{-\alpha}{\ell} \right) = \left( \frac{\alpha}{\ell} \right) = -1, 
  \]
  which is a contradiction.

- **Case I-3.** \( \alpha = 2A \), where \( 1 < A \) is a strict integer divisor of \( p_1 \cdots p_t \) and consisting of an even number of prime divisors. Then \( \beta \) consists of an odd number of prime factors. Therefore, for an odd prime divisor \( \ell \) of \( \alpha \), we get
  \[
  1 = \left( \frac{1}{\ell} \right) = \left( \frac{b^2 \beta - 2a^2 A}{\ell} \right) = \left( \frac{\beta}{\ell} \right) = -1, 
  \]
  which is a contradiction.

- **Case I-4.** \( \alpha = 2A \), where \( 1 < A \) is a strict integer divisor of \( p_1 \cdots p_t \) and consisting of an odd number of prime divisors. Then for a prime divisor \( \ell \) of \( \beta \), we obtain
  \[
  1 = \left( \frac{1}{\ell} \right) = \left( \frac{b^2 \beta - 2a^2 A}{\ell} \right) = \left( \frac{-1}{\ell} \right) \left( \frac{2}{\ell} \right) \left( \frac{A}{\ell} \right) = \left( \frac{A}{\ell} \right) = -1, 
  \]
  which is a contradiction.

**Case II.** Now, assume that \( t \) is even. By Lemma 2, we can choose prime numbers \( p_1, \ldots, p_t \) such that \( p_i \equiv 1 \pmod{8} \) for each \( i \) and they satisfy condition (2) of Lemma 3.

- **Case II-1.** \( p_1 \mid \beta \) and \( p_2 \mid \alpha \). Then from Eq. (3), we get
  \[
  1 = \left( \frac{1}{p_1} \right) = \left( \frac{-a^2 \alpha}{p_1} \right) = \left( \frac{\alpha}{p_1} \right) \left( \frac{p_2}{p_1} \right) = -1, 
  \]
  by the conditions on \( p_i \)'s. Therefore, this cannot occur.

- **Case II-2.** \( p_1 \mid \alpha \) and \( p_2 \mid \beta \). Then again from Eq. (3), we get
  \[
  1 = \left( \frac{1}{p_1} \right) = \left( \frac{b^2 \beta}{p_1} \right) = \left( \frac{\beta}{p_1} \right) \left( \frac{p_2}{p_1} \right) = -1, 
  \]
which is a contradiction.

- **Case II-3.** $\beta = p_1p_2B$ or $2p_1p_2B$, where $B \mid p_3 \cdots p_t$. First, suppose that $B$ consists of an even number of prime divisors. Then $\alpha$ also has an even number of odd prime factors and therefore from Eq. (3), we obtain for any odd prime divisor $\ell$ of $\alpha$ that
  \[ 1 = \left( \frac{1}{\ell} \right) = \left( \frac{p_1}{\ell} \right) \left( \frac{p_2}{\ell} \right) \cdot 1 = -1, \]
  which is a contradiction.

  Similarly, if $B$ consists of an odd number of odd prime divisors, then so does $\alpha$. Again, from Eq. (3), we get
  \[ 1 = \left( \frac{1}{p_2} \right) = \left( \frac{-a^2\alpha}{p_2} \right) = \left( \frac{\alpha}{p_2} \right) = -1, \]
  a contradiction.

- **Case II-4.** $p_1 \nmid \beta$ and $p_2 \nmid \beta$. Then $p_1p_2 \mid \alpha$ and considering the equation
  \[ -1 = a^2\alpha - b^2\beta, \]
  we are essentially back to Case 3.

Hence we conclude that for both odd and even values of $t$, we have infinitely many bi-quadratic fields $K$ with $\text{Po}(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-1}$. Putting $n = t - 1$ completes the proof of Theorem 1.  

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