SOME ARITHMETIC PROPERTIES OF NUMBERS OF THE FORM $\lfloor p^c \rfloor$

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Abstract. Let

$$\mathbb{P}^c = (\lfloor p^c \rfloor)_{p \in \mathbb{P}} \quad (c > 1, c \not\in \mathbb{N}),$$

where $\mathbb{P}$ is the set of prime numbers, and $\lfloor \cdot \rfloor$ is the floor function. We show that for every such $c$ there are infinitely many members of $\mathbb{P}^c$ having at most $R(c)$ prime factors, giving explicit estimates for $R(c)$ when $c$ is near one and also when $c$ is large.

1. Introduction

1.1. Motivation. Piatetski-Shapiro sequences are those sequences of the form

$$\mathbb{N}^c = ([n^c])_{n \in \mathbb{N}} \quad (c > 1, c \not\in \mathbb{N}),$$

where $[t]$ denotes the integer part of any real number $t$. Such sequences are named in honor of Piatetski-Shapiro, who showed (cf. [12]) that for any fixed $c \in (1, \frac{12}{11})$ there are infinitely many primes in $\mathbb{N}^c$. The admissible range of $c$ for this result has been extended many times over the years, and currently it is known to hold for all $c \in (1, \frac{243}{205})$ thanks to Rivat and Wu [14].

Many authors have studied arithmetic properties of Piatetski-Shapiro sequences (see Baker et al [3] and the references contained therein), and it is natural to ask whether certain properties also hold on special subsequences of the Piatetski-Shapiro sequences. Perhaps the most important of these are the subsequences of the form

$$\mathbb{P}^c = (\lfloor p^c \rfloor)_{p \in \mathbb{P}} \quad (c > 1, c \not\in \mathbb{N}),$$

where $\mathbb{P} = \{2, 3, 5, \ldots\}$ is the set of prime numbers; however, up to now very little has been established about the arithmetic structure of $\mathbb{P}^c$ for fixed $c > 1$. Balog [5] has shown that for almost all $c > 1$, the counting function

$$\Pi_c(x) = |\{\text{prime } p \leq x : \lfloor p^c \rfloor \text{ is prime}\}|$$

satisfies

$$\limsup_{x \to \infty} \frac{\Pi_c(x)}{x/(c \log^2 x)} \geq 1,$$
but this result gives no information for any specific choice of $c$.

Thanks to the work of Cao and Zhai [7] it is known that the set $\mathbb{P}^c$ contains infinitely many squarefree natural numbers provided that $c$ is not too large. More precisely, as a special case of the main result in [7], one knows that for any $c \in (1, \frac{149}{87})$ there exists $\varepsilon > 0$ (depending only on $c$) such that the estimate

$$\left| \{ \text{prime } p \leq x : \lfloor p^c \rfloor \text{ is squarefree} \} \right| = \frac{6}{\pi^2} \cdot \pi(x) + O(x^{1-\varepsilon})$$

holds, where $\pi(x)$ denotes the number of primes not exceeding $x$.

In the present paper, as a step towards better understanding the arithmetic properties of $\mathbb{P}^c$, we consider the related question of whether or not $\mathbb{P}^c$ contains infinitely many almost primes.

1.2. Main results. For every $R \geq 1$, we say that a natural number is an $R$-almost prime if it has at most $R$ prime factors, counted with multiplicity.

We study almost prime values of $\lfloor p^c \rfloor$ in two different regimes in order to demonstrate the underlying ideas: (i) values of $c$ close to one, and (ii) large values of $c$.

In the first regime, our result is stated in terms of the following set of admissible pairs $(R, c_R)$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$R$ & $c_R$ & $R$ & $c_R$ & $R$ & $c_R$ \\
\hline
8  & 1.0521 & 12 & 1.1649 & 16 & 1.2073 \\
9  & 1.1056 & 13 & 1.1780 & 17 & 1.2148 \\
10 & 1.1308 & 14 & 1.1891 & 18 & 1.2214 \\
11 & 1.1494 & 15 & 1.1988 & 19 & 1.2273 \\
\hline
\end{tabular}
\caption{Admissible pairs $(R, c_R)$}
\end{table}

**Theorem 1.1.** Let $(R, c_R)$, $R = 8, \ldots, 19$, be a pair from Table 1.1. Then for any fixed $c \in (1, c_R]$ there is a real number $\eta > 0$ such that the lower bound

$$\left| \{ \text{prime } p \leq x : \lfloor p^c \rfloor \text{ is an } R\text{-almost prime} \} \right| \geq \eta \frac{x}{\log^2 x}$$

holds for all sufficiently large $x$.

In the second regime, we prove the following result.

**Theorem 1.2.** For fixed $c \geq \frac{11}{5}$ there is a positive integer

$$R \leq \begin{cases} 
16c^3 + 179c^2 & \text{if } c \in \left[\frac{11}{5}, 3\right), \\
16c^3 + 88c^2 & \text{if } c \geq 3,
\end{cases}$$

if $c \in \left[\frac{11}{5}, 3\right)$,
and a real number $\eta > 0$ such that the lower bound

$$\left| \{ \text{prime } p \leq x : \lfloor p^c \rfloor \text{ is an } R\text{-almost prime} \} \right| \geq \eta \frac{x}{\log^2 x}$$

holds for all sufficiently large $x$.

These results are based on bounds of bilinear exponential sums and estimates on the uniformity of distribution of fractional parts $\{p^c d^{-1}\}$. We use the notion of level of distribution from sieve theory in a precise form stated in §2.1; see Friedlander and Iwaniec [8] and Greaves [10].

We remark that although the ranges of Theorems 1.1 and 1.2 do not overlap, using the same methods and sacrificing on the explicitness of the bounds for $R$, one can cover the gap as well.

### 1.3. Notation

Throughout the paper, we use the symbols $O$, $\ll$, $\gg$ and $\asymp$ along with their standard meanings; any constants or functions implied by these symbols may depend on $c$ and (where obvious) on the parameters $\varepsilon$ and $\nu$ but are absolute otherwise. We use the notation $m \sim M$ as an abbreviation for $M < m \leq 2M$.

The letter $p$ always denotes a prime number. As usual, $\mu(\cdot)$ is the Möbius function, and $\Lambda(\cdot)$ is the von Mangoldt function.

We write $e(t) = \exp(2\pi it)$ for all $t \in \mathbb{R}$.

### 2. Proof of Theorem 1.1

#### 2.1. Preliminaries

As we have mentioned the following notion plays a crucial rôle in our arguments. We specify it to the form that is suited to our applications; it is based on a result of Greaves [10] that relates level of distribution to $R$-almost primality. More precisely, we say that an $N$-element set of integers $A$ has a level of distribution $D$ if for a given multiplicative function $f(d)$ we have

$$\sum_{d \leq D} \max_{\gcd(s,d)=1} \left| \{a \in A, \ a \equiv s \mod d\} - \frac{f(d)}{d} N \right| \leq \frac{N}{\log^2 N}.$$  

As in [10, pp. 174–175] we define the constants

$$\delta_2 = 0.044560, \quad \delta_3 = 0.074267, \quad \delta_4 = 0.103974$$

and

$$\delta_R = 0.124820, \quad R \geq 5.$$  

We have the following result, which is [10, Chapter 5, Proposition 1].

**Lemma 2.1.** Suppose $A$ is an $N$-element set of positive integers with a level of distribution $D$ and degree $\rho$ in the sense that

$$a < D^\rho \quad (a \in A)$$
holds with some real number $\rho < R - \delta_R$. Then

$$\left| \{a \in A : a \text{ is an } R\text{-almost prime}\} \right| \gg \rho \frac{N}{\log^2 N}.$$  

Note that we always have $R \geq 5$ in what follows.

Using Baker and Pollack [4, Lemma 1] together with Lemma 2.1, it is easily seen that the proof of Theorem 1.1 reduces to showing that, for a fixed pair $(R, c_R)$ as in Table 1.1, for any fixed numbers $c \in (1, c_R]$ and $\vartheta \in (0, 1/R)$ the uniform bound

$$\sum_{1 \leq h \leq H} \sum_{d \sim D} \sum_{n \sim x} \Lambda(n) e(hd^{-1}n^c) \ll \vartheta D x \frac{\log^3 x}{\log^{3/2} x}$$  

holds with any $D \leq x^{\alpha}$ and $H = D \log^{3/2} x$. To estimate the triple sums in (2.1) we treat the summation over $h$ with straightforward estimates after estimating the inner sums over $d$ and $n$. Choosing a sufficiently small $\kappa > 0$ and applying Rivat and Sargos [13, Lemma 2] with

$$\alpha = \max\{1/20, \vartheta + \kappa\} < 1/6,$$

it suffices to show that

$$\sum_{d \sim D} c_d \sum_{m \sim M} a_m \sum_{\ell \sim x/m} b_\ell e(hd^{-1}\ell^c m^c) \ll \vartheta, \kappa, \xi x^{1-\xi}$$

with some fixed $\xi > 0$ (depending on $\vartheta$), arbitrary weights $c_d, a_m, b_\ell$ of size $O(1)$, and in three ranges of $M$ that correspond to two Type I sums and one Type II sum. More precisely, denoting

$$u_0 = x^\alpha,$$

these ranges are the following:

(i) Type II sums: $u_0 \ll x/M \ll u_0^2$;

(ii) Type I sums: $u_0^2 \ll x/M \ll x^{1/3}$ with $b_\ell$ being the characteristic function of an interval;

(iii) Type I sums: $M \ll x^{1/2} u_0^{1/2}$ with $b_\ell$ being the characteristic function of an interval.

By a standard application of the Fourier analysis (see, e.g., Garaev [9] or Banks et al [6]) the hyperbolic region of summation in (2.2) can be replaced with a rectangular region; in other words, it is enough to derive the bound

$$\sum_{d \sim D} c_d \sum_{m \sim M} a_m \sum_{\ell \sim L} b_\ell e(hd^{-1}\ell^c m^c) \ll \vartheta, \kappa, \xi x^{1-\xi}$$

for some $L$ and $M$ with $LM \asymp x$ in the following three ranges:

(i) Multilinear Type II sums: $u_0 \ll L \ll u_0^2$;
(ii) Multilinear Type I sums: $u_0^2 \ll L \ll x^{1/3}$ with $b_\ell$ being the characteristic function of an interval;

(iii) Multilinear Type I sums: $M \ll x^{1/2}u_0^{1/2}$ with $b_\ell$ being the characteristic function of an interval.

Before proceeding, we record the following technical result which simplifies the exposition below.

**Lemma 2.2.** Fix an admissible pair $(R, c_R)$ from Table 1.1. For any fixed numbers $c \in (1, c_R]$ and $\vartheta \in (0, 1/R)$, there is a positive number $\kappa$ such that if we define

$$\alpha = \max\{1/20, \vartheta + \kappa\},$$

then all of the following inequalities hold:

1. $2\vartheta + 2\alpha < c$
2. $c + 5\vartheta + 2\alpha < 2$
3. $365/3 + 32c + 147\vartheta < 174$
4. $8/3 + c + 2\vartheta < 4$
5. $2\vartheta + 4\vartheta < 4$
6. $1 + \vartheta - 2\alpha < 1$
7. $1 + \vartheta/2 - \alpha < 1$
8. $2/3 + \vartheta < 1$
9. $1 - c/2 + 3\vartheta/2 < 1$
10. $2\vartheta + (1 + \alpha)/2 < c$
11. $2c + 6\vartheta + \alpha < 3$

**Remark 2.3.** These inequalities are listed for convenience only and in some cases are redundant (for instance, (vi) and (vii) are equivalent). The proof of Lemma 2.2 is straightforward.

2.2. **General multilinear sums.** First, we need an adaptation of a result of Baker [1, Theorem 2], which is given here only for the specific exponent pair $(\kappa, \lambda) = (1/2, 1/2)$. Note that we use $D$ and $L$ instead of $M_1$ and $M_2$, respectively in the notation of [1, Theorem 2], and thus we use $d$ and $\ell$ instead of $m_1$ and $m_2$. However, $M$ and $m$ retain the same meaning.

**Lemma 2.4.** Let $\alpha_1, \alpha_2, \beta$ be nonzero real numbers such that $\beta < 1$, let $h, D, L, M$ be positive integers, and let $g$ be a real function on the interval $[M, 2M]$ such that

$$g'(x) \asymp hM^{\beta - j} \quad (x \sim M).$$

Let

$$S = \sum_{m \sim M} \sum_{d \sim D} \sum_{\ell \sim L} a_{m, d, \ell} e(g(m)d^{\alpha_1} \ell^{\alpha_2})$$
where \( a_m, c_d, \ell \) are complex numbers with \( a_m, c_d, \ell \ll 1 \). If the number \( X = hD^{\alpha_1}L^{\alpha_2}M^\beta \) is such that \( X \geq DL \), then
\[
S \ll DLM \left( (DL)^{-1/2} + \left( X/(DLM^2) \right)^{1/6} \right) \log 2DL.
\]

**Proof.** As this is a straightforward variant of [1, Theorem 2] we indicate mainly the changes that are needed in the proof.

Let
\[
Q \leq DL
\]
be a natural number to be determined later. Following [1] we see that either (cf. [1, Equation (3.8)])
\[
S^2 \ll DLM^2 Q \mathcal{L}^2
\]
holds with \( \mathcal{L} = \log 2DL \) (which corresponds to the value \( h = 0 \) in [1, Equation (3.6)]), or else we have (cf. [1, Equation (3.9)])
\[
S^2 \ll D^2 L^2 MQ \Delta \mathcal{L}^2 \left| \sum_{m \sim M} e(f(m)) \right|,
\]
where
\[
f(x) = g(x)(d_1^{\alpha_1} \ell_1^{\alpha_2} - d_2^{\alpha_1} \ell_2^{\alpha_2})
\]
with some quadruple \((d_1, d_2, \ell_1, \ell_2)\) that satisfies
\[
d_1, d_2 \sim D, \quad \ell_1, \ell_2 \sim L, \quad \Delta - \frac{1}{DL} \leq \left| \left( \frac{d_1}{d_2} \right)^{\alpha_1} - \left( \frac{\ell_2}{\ell_1} \right)^{\alpha_2} \right| < 2\Delta
\]
where \( \Delta \) is a number of the form \( \Delta = 2^h(DL)^{-1} \) with some fixed integer \( h \geq 1 \), which satisfies the bound
\[
\Delta \ll Q^{-1}
\]
(recall also the condition (2.4)). Note that
\[
f'(m) \asymp X \Delta M^{-1} \quad (x \sim M)
\]
as in [1].

Now, if the inequality \( X \Delta M^{-1} \leq \varepsilon \) holds with for some sufficiently small (but fixed) \( \varepsilon > 0 \), we can proceed as in Case (i) in the proof of [1, Theorem 2] (making use of [16, Lemma 4.19]) to obtain the bound
\[
\sum_{m \sim M} e(f(m)) \ll X^{-1} \Delta^{-1} M.
\]
Since \( X \geq DL \), upon combining this with (2.6) we again obtain (2.5).

On the other hand, if the inequality \( X \Delta M^{-1} > \varepsilon \) holds, then we can proceed as in Case (ii) in the proof of [1, Theorem 2] (with \( \kappa = \lambda = \frac{1}{2} \)) to derive that
\[
\sum_{m \sim M} e(f(m)) \ll (X \Delta)^{1/2}.
\]
Combining this with (2.6) and (2.7) we have

\[(2.8) \quad S^2 \ll D^2 L^2 M L^2 (X/Q)^{1/2}.\]

Putting (2.5) and (2.8) together, we deduce that

\[S \ll D L M L \left( (Q/(DL))^{1/2} + (X/(M^2 Q))^{1/4} \right).\]

The optimal choice for the natural number \(Q\) is

\[Q = \left\lceil (D^2 L^2 X/M^2)^{1/3} \right\rceil.\]

We note that if for the above choice of \(Q\) condition (2.4) is not satisfied then \(X/M^2 \gg DL\) and the result is trivial. Now, simple calculations lead to the desired bound. \(\square\)

2.3. Multilinear sums: Region (i). In this region, we can apply Lemma 2.4 to bound the sum in (2.3), making the choices \(\alpha_1 = -1\), \(\alpha_2 = \beta = c\), \(c_{d,\ell} = c_d b_\ell\) and \(g(x) = hx^c\). Since \(LM \asymp x\) and \(2\theta + 2\alpha < c\) by Lemma 2.2 (i) we see that

\[(2.9) \quad X = hD^{-1} L^c M^c \gg DL\]

if \(x\) is large, and recalling that \(H = DL^3\) with \(L = \log x\) we also have

\[X \ll HD^{-1} x^c = x^c L^3;\]

hence, for the sum

\[S = \sum_{d \sim D} c_d \sum_{m \sim M} a_m \sum_{\ell \sim L} b_\ell e(hd^{-1} \ell^c m^c)\]

Lemma 2.4 yields

\[S \ll DL M L \left( (DL)^{-1/2} + (x^c L^3/(DL))^{1/6} M^{-1/3} \right) \ll x \left( (D/L)^{1/2} L^3 + (D^5 x^c/(LM^2))^{1/6} L^{3/2} \right).\]

In Region (i) we have \(LM^2 \gg x^2/L \gg x^2 u_0^{-2}\), and therefore

\[(2.10) \quad S \ll x \left( (D/L)^{1/2} L^3 + (D^5 x^{c-2} u_0^2)^{1/6} L^{3/2} \right).\]

Recalling our choice of \(u_0\), in Region (i) we have

\[L \geq u_0 \geq x^{\theta+\kappa} \geq D x^\kappa;\]

hence the first term in (2.10) is of size \(O(x^{1-\kappa/2})\). For the second term in (2.10), Lemma 2.2 (ii) implies that the inequality

\[5\theta + (c - 2) + 2\alpha < -\kappa\]

holds with a suitably small \(\kappa\), hence the second term in (2.10) is of size \(O(x^{1-\kappa/2})\) as well.
2.4. **Multilinear sums: Region (ii).** In this region, to estimate the sum

\[ S = \sum_{d \sim D} c_d \sum_{m \sim M} a_m \sum_{\ell \sim L} b_\ell e(hd^{-1} \ell^c m^c) \]

in (2.3) we apply a result of Wu [19]. Note that \( b_\ell \) is a characteristic function of an interval. The correspondence between the parameters \((H, M, N, X, \alpha, \beta, \gamma)\) given in [19, Theorem 2] and our parameters is

\[ (H, M, N, X, \alpha, \beta, \gamma) \longleftrightarrow (M, D, L, c, -1, c) \]

(where \( X = hD^{-1}L^cM^c \) as before) and we take \( k = 5 \) in the statement of [19, Theorem 2]; this gives

\[ SL^{-1} \ll (X^{32}M^{114}D^{47}L^{137})^{1/174} + (XM^2D^2L^4)^{1/4} + (XM^2D^4L^2)^{1/4} + MD + M(DL)^{1/2} + M^{1/2}DL + X^{-1/2}MDL. \]

Using the bounds

\[ D^{1-x} \leq X \ll x^cL^3, \quad LM \asymp x, \quad x^{2\alpha} \ll L \ll x^{1/3} \quad \text{and} \quad D \ll x^\theta, \]

it follows that

\[ SL^{-2} \ll (x^{365/3+32\varepsilon+147\theta})^{1/174} + (x^{8/3+\varepsilon+2\theta})^{1/4} + (x^{2+\varepsilon+4\theta})^{1/4} + x^{1+\theta-2\alpha} + x^{1+\theta/2-\alpha} + x^{2/3+\theta} + x^{1-c/2+3\theta/2}. \]

Taking into account the inequalities of Lemma 2.2 (iii)–(ix) we see that \( S = O(x^{1-\kappa}) \) if \( \kappa > 0 \) is small enough.

2.5. **Multilinear sums: Region (iii).** In this region, to estimate the sums in (2.3) we apply a result of Robert and Sargos [15]. Note that \( b_\ell \) is a characteristic function of an interval. The correspondence between the parameters \((H, M, N, X, \alpha, \beta, \gamma)\) given in [15, Theorem 3] and our parameters is

\[ (H, M, N, X, \alpha, \beta, \gamma) \longleftrightarrow (D, L, M, c, -1, c), \]

where

\[ X = hD^{-1}L^cM^c. \]

Applying [15, Theorem 3], for the sum

\[ S = \sum_{d \sim D} c_d \sum_{m \sim M} a_m \sum_{\ell \sim L} b_\ell e(hd^{-1} \ell^c m^c) \]

we have the bound

\[ S \leq (DLM)^{1+o(1)} \left( \left( \frac{X}{DL^2M} \right)^{1/4} + \frac{1}{L^{1/2}} + \frac{1}{X} \right). \]

The third term in this estimate is dominated by the second term since \( X \geq DL \) (cf. (2.9)), and the second term is dominated by the first term.
since \( X \geq DM \), the latter bound holding in Region (iii) in view of the inequality \( c \geq 2\vartheta + (1 + \alpha)/2 \) in Lemma 2.2 (x). Therefore,

\[
S \leq (DLM)^{1+o(1)} \left( \frac{hD^{-1}L^cM^c}{DL^2M} \right)^{1/4}.
\]

Since \( h \leq H = DL^3 \), \( LM \approx x \), \( D \leq x^{\vartheta} \) and \( L \gg x^{1/2}u_0^{-1/2} \), we have

\[
S \leq x^{5/8+c/4+3\vartheta/4+\alpha/8+o(1)}.
\]

To prove (2.3) in this case it is enough to show that

\[
5/8 + c/4 + 3\vartheta/4 + \alpha/8 < 1,
\]

This follows from the inequality

\[
2c + 6\vartheta + \alpha < 3,
\]

which is given in Lemma 2.2 (xi).

3. Proof of Theorem 1.2

3.1. Preliminaries. Let \( c \) be fixed, and put

\[
\sigma = \frac{1}{16c^2 + 179c - 1.15c^{-1}} \quad \text{and} \quad \beta = 47\sigma.
\]

For our purposes below, we record that the inequality

\[
\frac{c_1(1/2 - \beta)^3 - (1/2 - \beta)^4}{(c_1 + \frac{1}{2} - \beta)(c_1 + 1 - 2\beta)(2c_1 + \frac{1}{2} - \beta)} > \sigma
\]

holds with \( c_1 = c + \sigma \) for all \( c \geq 2.081 \), and the inequalities

\[
\frac{\frac{1}{27}c_2 - \frac{16}{31}}{(c_2 + \frac{1}{3})(2c_2 + 2)} > 2\sigma
\]

and

\[
\frac{c_2(1 - 2\beta)^3 - (1 - 2\beta)^4}{(c_2 + 2 - 4\beta)(c_2 + 3 - 6\beta)(2c_2 + 3 - 6\beta)} > 2\sigma
\]

both hold with \( c_2 = c - 1 + 3\sigma \) for all \( c \geq 2.198 \).

Suppose that we have the uniform bound

\[
\sum_{p \leq x} e(hd^{-1}p^c) \ll x^{1-\sigma} \quad (d, h \leq x^\sigma).
\]

Let \( A \) be the sieving set given by

\[
A = \{ n : n = \lfloor p^c \rfloor \text{ for some prime } p \leq x \},
\]
If (3.5) holds, then (as in the proof of Theorem 1.1) for any fixed \( \varepsilon > 0 \) we obtain a level of distribution \( D = x^{\sigma - \varepsilon} \) for \( A \). Thus, we can apply Lemma 2.1 with \( g = c/\sigma + \varepsilon \) (since \( a \leq x^c \) for all \( a \in A \)) and with
\[
(3.6) \quad R \leq \frac{c}{\sigma} + 1.15 = 16c^3 + 179c^2,
\]
which implies the stated result for \( c \in \left[ \frac{11}{5}, 3 \right) \).

For \( c \geq 3 \) we replace 179 with 88 in the definition of \( \sigma \) and take \( \beta = 20\sigma \) in (3.1), and the estimates (3.4)–(3.6) continue to hold (as well as the bound \( \beta < 0.1 \); see §3.3 below). Hence, we can also replace 179 with 88 in (3.6) as well.

3.2. Bounds on some auxiliary sums. Here, it is convenient to introduce the notations \( A \lesssim B \) and \( B \gtrsim A \), which are equivalents of an inequality of the form \( A \leq B + O(\mathcal{L}^{-1}) \), where \( \mathcal{L} = \log N \).

To prove that (3.5) holds, we need the following bound of exponential sums; it is used to establish (3.14) and (3.17) below.

**Lemma 3.1.** Let \( c, \Theta, \Delta, \varepsilon > 0 \) be fixed, and put
\[
k = \left\lfloor \frac{c+\Delta}{\Theta} \right\rfloor + 1.
\]
If \( k \geq 3 \), then the exponential sum
\[
S(N) = \sum_{z \sim N^\Theta} \exp(z^c N^\Delta)
\]
satisfies the bound
\[
(3.8) \quad S(N) \ll N^{\Theta(1 - \vartheta)},
\]
where the implied constant depends only on \( c \) and \( \varepsilon \), and
\[
(3.9) \quad \vartheta = \frac{k - 2 - \varepsilon}{k(k + 1)(2k - 1)}.
\]

**Proof.** Let \( s = k^2 - 1 \). Applying the result of Vinogradov [17, Chapter VI, Lemma 7] with the function \( F(z) = z^c N^\Delta \) and \( n = k \), for any fixed \( \varrho \in (0, 1) \) we have the bound
\[
(3.10) \quad S(N)^{2s} \ll P^{-2s + \frac{1}{2}k(k+1)}(N^\Theta)^{2s - 1 + 2/(k+1)\varrho} I + (N^\Theta)^{2s(1 - \vartheta)},
\]
where
\[
I = \int_0^1 \cdots \int_0^1 \left| \sum_{z=1}^P \exp(\alpha_1 z + \cdots + \alpha_k z^k) \right|^{2s} d\alpha_1 \cdots d\alpha_k
\]
and \( P \) is the integer given by
\[
P = \left\lfloor A_0^{(1 - \vartheta)/(k+1)} \right\rfloor, \quad \text{where} \quad A_0 = \left| \frac{(k + 1)!}{F(k+1)(N^\Theta)} \right|.
\]
Noting that
\[ A_0 \asymp N^{\Theta(k + 1 - c) - \Delta}, \]
in order to apply [17, Chapter VI, Lemma 7] it must be the case that
\[ \Theta \ll \Theta(k + 1 - c) - \Delta \ll \Theta(2 + 2/k), \]
or in other words,
\[ c + \Delta/\Theta \ll k \ll 1 + 2/k + c + \Delta/\Theta. \]
However, this condition is guaranteed by (3.7).

Applying Wooley [18, Theorem 1.1] with \( \varepsilon/k \) in place of \( \varepsilon \), we see that the integral \( \mathcal{I} \) is bounded by
\[
(3.11) \quad \mathcal{I} \ll P^{2s-1/2(k+1)+\varepsilon/k}.
\]
Taking into account that
\[ P \ll A_0^{1/(k+1)} \ll N^{(\Theta(k+1-c)-\Delta)/(k+1)} \ll N^\Theta, \]
after combining (3.10) and (3.11) we derive the bound
\[
S(N)^{2s} \ll (N^\Theta)^{2s-1+2s+\varepsilon/k} \mathcal{I} + (N^\Theta)^{2s(1-\varepsilon)}. 
\]
To optimize, we choose \( \rho \) so that
\[ 2s - 1 + (2 + \varepsilon)/k + (k + 1)\rho = 2s(1 - \rho); \]
recalling that \( s = k^2 - 1 \) this leads to (3.9), and (3.8) follows. \( \square \)

3.3. Concluding the proof. We now turn our attention to (3.5). We use the Heath-Brown decomposition (cf. Heath-Brown [11]) to reduce the problem to that of bounding Type I and Type II sums. In the present situation, to prove (3.5) it suffices to show, for some sufficiently small \( \varepsilon > 0 \) which depends only on \( c \), that \( B = N^{1-\sigma-\varepsilon} \) is an upper bound on all Type I sums
\[
(3.12) \quad S_I(X, Y) = \sum_{x \sim X} \sum_{y \sim Y} a_x e(hd^{-1}x c y^c) \quad (Y \gg N^{1/2-\beta}),
\]
and an upper bound on all Type II sums
\[
(3.13) \quad S_{II}(X, Y) = \sum_{x \sim X} \sum_{y \sim Y} a_x b_y e(hd^{-1}x c y^c) \quad (N^{2\beta} \ll Y \ll N^{1/3}),
\]
where \( |a_x| \leq 1 \) and \( |b_y| \leq 1 \), and \( XY \asymp N \); we refer the reader to the discussion on [11, pp. 1367-1368]. We specify \( \varepsilon > 0 \) below.
Let $\mathcal{L} = \log N$ as before. Using van der Corput’s inequality with $Q = N^{2\sigma+2\varepsilon}$ and following the proof of Baker [2, Theorem 5], we are lead to the bound [2, Equation (4.18)] with some $q \in [1, Q]$:

$$S_H(X, Y)^2 \mathcal{L}^{-2} \ll \frac{N^2}{Q} + \frac{N \mathcal{L}q}{Q} \left| \sum_{y \sim Y} \sum_{x \sim X} b_{y+q} y e \left( h d^{-1} x^c ((y + q)^c - y^c) \right) \right|$$

$$\ll N^{2-2\sigma-2\varepsilon} + N \mathcal{L} \left| \sum_{y \sim Y} \sum_{x \sim X} e \left( h d^{-1} x^c ((y + q)^c - y^c) \right) \right|. $$

For the moment, put $\Theta = (\log X) / \mathcal{L}$, so that $X = N^\Theta$. Noting that $qY^{c-1}N^{-\sigma} \ll hd^{-1}((y + q)^c - y^c) \ll qY^{c-1}N^\sigma$ \quad (y \sim Y),

we see that in the Type II case it suffices to show that

$$\sum_{z \sim N^\Theta} e(z^c N^\Delta) \ll N^{\Theta-2\sigma-3\varepsilon}$$

holds uniformly for

$$2/3 \lesssim \Theta \lesssim 1 - 2\beta$$

and

$$(1 - \Theta)(c - 1) - \sigma \lesssim \Delta \lesssim (1 - \Theta)(c - 1) + 3\sigma + 2\varepsilon,$$

where continue to use the notation $A \ll B$ from §3.2.

Now put $\Theta = (\log Y) / \mathcal{L}$. Noting that

$$X^c N^{-\sigma} \ll hd^{-1} x^c \ll X^c N^\sigma \quad \text{\quad (x \sim X)}$$

and $X \sim N^{1-\Theta}$, in the Type I case we only need to show that

$$\sum_{z \sim N^\Theta} e(z^c N^\Delta) \ll N^{\Theta-\sigma-\varepsilon}$$

holds uniformly for

$$1/2 - \beta \lesssim \Theta \lesssim 1$$

and

$$(1 - \Theta)c - \sigma \lesssim \Delta \lesssim (1 - \Theta)c + \sigma.$$

Suppose first that $\Theta, \Delta$ are such that (3.18) and (3.19) hold, and fix $\varepsilon > 0$. Define $k$ by (3.7) and $\rho$ by (3.9). Note that $\rho = f(k)$, where

$$f(t) = \frac{t - 2 - \varepsilon}{t(t + 1)(2t - 1)}.$$
Since $f$ is decreasing on $[3, \infty)$, and noting that the bounds

$$3 \leq k \leq c + \Delta/\Theta + 1 \leq (c + \sigma)/\Theta + 1$$

hold in view of (3.18) and (3.19), it follows that

$$\Theta \varrho \geq f_1(\Theta),$$

where

$$f_1(t) = \frac{c_1 t^3 - (1 + \varepsilon) t^4}{(c_1 + t)(c_1 + 2t)(2c_1 + t)}$$

with $c_1 = c + \sigma$. Since $c \geq 1.6$ and $\varepsilon \leq 0.01$ (say), the function $f_1$ is increasing on $[0, 1]$; consequently, as $\Theta \geq 1/2 - \beta$ we have

$$\Theta \varrho \geq f_1(1/2 - \beta) = \frac{c_1 (1/2 - \beta)^3 - (1 + \varepsilon)(1/2 - \beta)^4}{(c_1 + 1/2 - \beta)(c_1 + 1 - 2\beta)(2c_1 + 1/2 - \beta)}.$$

In view of (3.2) we can choose $\varepsilon > 0$ sufficiently small, depending only on $c$, such that

$$\Theta \varrho \geq \sigma + \varepsilon.$$

Then, using the equation (3.8) of Lemma 3.1, we derive the required bound (3.17) for the Type I sums (3.12).

Next, suppose that $\Theta, \Delta$ are such that (3.15) and (3.16) hold, and let $\varrho > 0$ be chosen as above. We again define $k$ by (3.7) and put $\varrho = f(k)$. Since $f$ is decreasing on $[3, \infty)$, and noting that the bounds

$$3 \leq k \leq c + \Delta/\Theta + 1 \leq (c - 1 + 3\sigma + 2\varepsilon)/\Theta + 2$$

hold in view of (3.15) and (3.16), it follows that

$$\Theta \varrho \geq f_2(\Theta),$$

where

$$f_2(t) = \frac{(c_2 + 2\varepsilon)t^3 - (1 + \varepsilon)t^4}{(c_2 + 2t + 2\varepsilon)(c_2 + 3t + 2\varepsilon)(2c_2 + 3t + 4\varepsilon)}$$

with $c_2 = c - 1 + 3\sigma$. Since $c \geq 11/6$ and $\varepsilon \leq 0.01$, one verifies that $f_2$ attains a unique maximum on $[0, 1]$; therefore, as $2/3 \leq \Theta \leq 1 - 2\beta$ we have either

$$\Theta \varrho \geq f_2(2/3) = \frac{8/27(c_2 + 2\varepsilon) - 16/21(1 + \varepsilon)}{(c_2 + 4/3 + 2\varepsilon)(c_2 + 2 + 2\varepsilon)(2c_2 + 2 + 4\varepsilon)}$$

or else

$$\Theta \varrho \geq f_2(1 - 2\beta)$$

$$= \frac{(c_2 + 2\varepsilon)(1 - 2\beta)^3 - (1 + \varepsilon)(1 - 2\beta)^4}{(c_2 + 2 - 4\beta + 2\varepsilon)(c_2 + 3 - 6\beta + 2\varepsilon)(2c_2 + 3 - 6\beta + 4\varepsilon)}.$$
In view of the inequalities (3.3) and (3.4), we can take \( \varepsilon > 0 \) sufficiently small to guarantee that
\[
\Theta \varphi \gtrsim 2\sigma + 3\varepsilon.
\]
Using the equation (3.8) of Lemma 3.1 once again, we derive the required bound (3.14) for the Type II sums (3.13).

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