STRONGLY PRODUCTIVE ULTRAFILTERS ON SEMIGROUPS

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ABSTRACT. We prove that if $S$ is a commutative semigroup with well founded universal semilattice or a solvable inverse semigroup with well founded semilattice of idempotents, then every strongly productive ultrafilter on $S$ is idempotent. Moreover we show that any very strongly productive ultrafilter on the free semigroup with countably many generators is sparse, answering a question of Hindman and Legette Jones.

1. INTRODUCTION

Let $S$ be a multiplicatively denoted semigroup. If $\vec{x} = (x_n)_{n \in \omega}$ is a sequence of elements of $S$, then the finite products set associated with $\vec{x}$, denoted by $FP(\vec{x})$, is the set of products (taken in increasing order of indices)

$$\prod_{i \in a} x_i \in S$$

where $a$ ranges among the finite subsets of $\omega$. If $k \in \omega$ then $FP_k(\vec{x})$ stands for the FP-set associated with the sequence $(x_{n+k})_{n \in \omega}$. A subset of $S$ is called an FP-set if it is of the form $FP(\vec{x})$ for some sequence $\vec{x}$ in $S$, and an IP-set if it contains an FP-set (see [10, Definition 16.3]). An ultrafilter $p$ on $S$ is strongly productive (see [7, Section 1]) if it has a basis of FP-sets. This means that for every $A \in p$ there is an FP-set contained in $A$ that is a member of $p$. When $S$ is an additively denoted abelian semigroup, then the finite products sets are called finite sums sets or FS-sets and denoted by $FS(\vec{x})$. Moreover strongly productive ultrafilters are called in this context strongly summable (see [8, Definition 1.1]).

The concept of strongly summable ultrafilter was first considered in the case of the semigroup of positive integers in [6] by Hindman upon suggestion of van Douwen (see also the notes at the end of [10, Chapter 12]). Later Hindman, Protasov, and Strauss studied in [8] strongly summable ultrafilters on arbitrary abelian groups. Theorem 2.3 in [8] asserts that any strongly summable ultrafilter on an abelian group $G$ is idempotent, i.e. an idempotent element of the semigroup compactification $\beta G$ of $G$, which can be seen as the collection of all ultrafilters on $G$. Similarly, strongly productive ultrafilters on a free semigroup are also idempotent by [7, Lemma 2.3].

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In this paper we provide a common generalization of these results to a class of semigroups containing in particular all abelian semigroups with well founded universal semilattice, and solvable inverse semigroups with well founded semilattice of idempotents. (The notion of universal semilattice of a semigroup is presented in [4, Section III.2]. Inverse semigroups are introduced in [4, Section I I.2], and within these the class of solvable semigroups is defined in [13, Definition 3.2]. Solvable groups are the solvable inverse semigroups with exactly one idempotent element by [13, Theorem 3.4].)

**Theorem 1.1.** If $S$ is either an abelian semigroup with well founded universal semilattice, or an inverse semigroup with well founded semilattice of idempotents, then every strongly summable ultrafilter on $S$ is idempotent.

In order to prove Theorem 1.1 we find it convenient to consider the following strengthening of the notion of strongly summable ultrafilter:

**Definition 1.2.** A nonprincipal strongly productive ultrafilter $p$ on a semigroup $S$ is regular if it contains an element $B$ with the following property: Whenever $\vec{x}$ is a sequence in $S$ such that $\text{FP}(\vec{x}) \subset B$, the set $x_0 \text{FP}_1(\vec{x})$ does not belong to $p$.

**Remark 1.3.** Suppose that $S, T$ are semigroups, $f : S \to T$ is a semigroup homomorphism, and $p$ is a strongly productive ultrafilter on $S$. Denote by $q$ the ultrafilter on $T$ defined by $B \in q$ if and only if $f^{-1}[B] \in p$ (note that $q$ is the image of $p$ under the unique extension of $f$ to a continuous function from the Stone-Cech compactification of $S$ to the Stone-Cech compactification of $T$). It is easy to see that $q$ is a strongly summable ultrafilter on $T$; Moreover if $q$ is regular then $p$ is regular.

We will show that the notions of strongly productive and regular strongly productive ultrafilter coincide for the classes of semigroups considered in Theorem 1.1.

**Theorem 1.4.** If $S$ is either an abelian semigroup with well founded universal semilattice, or a solvable inverse semigroup with well founded semilattice of idempotents, then every nonprincipal strongly productive ultrafilter on $S$ is regular.

The universal semilattice of a semigroup $S$ is the quotient of $S$ by the smallest semilattice congruence $\mathcal{N}$ on $S$, see [4, Section III.2]. When $S$ is an inverse semigroup the set $E(S)$ of idempotent elements of $S$ is a semilattice, and the restriction to $E(S)$ of the quotient map from $S$ to $S/\mathcal{N}$, cf. [4, Exercise III.2.2, Proposition II.2.6, and Section VII.1].

Although not using this terminology, there is a well-known argument showing that any regular strongly productive ultrafilter is idempotent, see for example [10, Theorem 12.19], or [7, Lemma 2.3]. We reproduce the argument in Lemma 1.5 below for convenience of the reader. Using this fact, Theorem 1.1 will be a direct consequence of Theorem 1.4.

**Lemma 1.5.** Suppose that $S$ is a semigroup, and $p$ is an ultrafilter on $S$. If $p$ is regular strongly productive, then $p$ is idempotent.

**Proof.** Fix an element $B$ of $p$ witnessing the fact that $p$ is regular. Suppose that $A$ is an element of $p$ and $\vec{x}$ is a sequence in $S$ such that $\text{FP}(\vec{x}) \subset A \cap B$. Since
\[ \text{FP}(\vec{x}) = \{x_0\} \cup x_0 \text{FP}_1(\vec{x}) \cup \text{FP}_1(\vec{x}), \]

and $p$ is nonprincipal and regular, it follows that $\text{FP}_1(\vec{x}) \in p$. Using this argument, one can show by induction that $\text{FP}_n(\vec{x}) \in p$ for every $n \in \omega$. Now notice that, if $x = \prod_{i \in a} x_i \in \text{FP}(\vec{x})$ and $n = \max(a) + 1$, then $x \text{FP}_n(\vec{x}) \subseteq \text{FP}(\vec{x}) \subseteq A$. Therefore $\text{FP}_n(\vec{x}) \subseteq x^{-1}A$ and since the former set is an element of $p$, so is the latter. Hence

$$\text{FP}(\vec{x}) \subset \{ x \in S : x^{-1}A \in p \}$$

and so the latter set belongs to $p$. This shows that $p$ is idempotent. □

To our knowledge it is currently not known if the existence of a semigroup $S$ and a nonprincipal strongly productive ultrafilter on $S$ that is not idempotent is consistent with the usual axioms of set theory. We think that Theorem 1.1 as well as [8, Theorem 2.3] provide evidence that for all semigroups $S$, every strongly productive ultrafilter on $S$ should be idempotent.

**Conjecture 1.6.** A strongly productive ultrafilter on an arbitrary semigroup is idempotent.

This paper is organized as follows: In Section 2 we introduce the notion of IP-regular (partial) semigroup and observe that IP-regular semigroups satisfy the conclusion of Theorem 1.4. In Section 3 we show that commutative cancellative semigroups are IP-regular; In Section 4 we record some closure properties of the class of IP-regular semigroups, implying in particular that all (virtually) solvable groups are IP-regular; In Section 5 we present the proof of Theorem 1.4. Finally in Section 6 we discuss sparseness of strongly productive ultrafilters, and show that a very strongly productive on the free semigroup with countably many generators is sparse, answering Question 3.8 from [7].

2. IP-regularity

A **partial semigroup** as defined in [5, Section I.3] is a set $P$ endowed with a partially defined (multiplicatively denoted) operation such that for every $a, b, c \in P$

$$(ab)c = a(bc)$$

whenever both $(ab)c$ and $a(bc)$ are defined. Observe that in particular every semigroup is a partial semigroup. Moreover any subset of a partial semigroup is naturally endowed with a partial semigroup structure when one considers the restriction of the operation. An element $a$ of a partial semigroup $P$ is idempotent if $a \cdot a$ is defined and equal to $a$. The set of idempotent elements of $P$ is denoted by $E(P)$. The notion of FP-set and IP-set admit straightforward generalizations to the framework of partial semigroups. If $\vec{x}$ is a sequence of elements of a partial semigroup $P$ such that all the products (taken in increasing order of indices)

$$\prod_{i \in a} x_i$$

where $a$ is a finite subset of $\omega$ are defined, then the FP-set $\text{FP}(\vec{x})$ is the set of all such products. A subset of $P$ is an IP-set if it contains an FP-set. Analogous to the case of semigroups, an ultrafilter $p$ on a partial semigroup $P$ is **strongly productive** if it is has a basis of FP-sets. A strongly productive ultrafilter is **regular** if it contains a set $B$ with the following property: If $\vec{x}$ is a sequence in $P$ such that all finite products from $\vec{x}$ are defined and belong to $B$, then $x_0 \text{FP}_1(\vec{x})$ does not belong to $p$. 
Definition 2.1. A partial semigroup $P$ is strongly IP-regular if for every sequence $\vec{x}$ in $P$ such that all the finite products from $\vec{x}$ are defined the set

$$x_0 FP_1(\vec{x})$$

is not an IP-set.

Since a subset of a partial semigroup is still a partial semigroup, we can speak about strongly IP-regular subsets of a partial semigroups, which are just strongly IP-regular partial semigroups with respect to the induced partial semigroup structure.

Definition 2.2. A partial semigroup $P$ is IP-regular if $P \setminus E(P)$ is the union of finitely many strongly IP-regular sets.

It is immediate from the definition that a partial semigroup is IP-regular whenever it is the union of finitely many IP-regular subsets (i.e. subsets which are IP-regular partial semigroups with respect to the induced partial semigroup structure).

Remark 2.3. Any subset of a (strongly) IP-regular partial semigroup is (strongly) IP-regular. Any finite partial semigroup is strongly IP-regular.

We will show in Section 3 that every cancellative commutative semigroup is IP-regular. The relevance of the notion of IP-regularity stems from the fact that any IP-regular group satisfies the conclusion of Theorem 1.4. The same is in fact true for any IP-regular partial semigroup with finitely many idempotent elements.

Lemma 2.4. If $S$ is an IP-regular partial semigroup with finitely many idempotent elements, then every nonprincipal strongly productive ultrafilter on $S$ is regular.

Proof. Suppose that $p$ is a strongly summable ultrafilter on $S$. Since $p$ is nonprincipal, $S \setminus E(S) \in p$. Given that $S$ is IP-regular, the ultrafilter $p$ must have a strongly IP-regular member $A$. It is clear that $A$ witnesses the fact that $p$ is regular. \qed

The class of IP-regular partial semigroups has interesting closure properties. We have already observed that a subset of an IP-regular partial semigroup is IP-regular. Moreover the inverse image of an IP-regular partial semigroup under a partial homomorphism is IP-regular. Recall that a partial homomorphism from a partial semigroup $P$ to a partial semigroup $Q$ is a function $f : P \to Q$ such that for every $a, b \in P$ such that $ab$ is defined, $f(a)f(b)$ is defined and $f(ab) = f(a)f(b)$

Lemma 2.5. Suppose that $f : P \to Q$ is a partial homomorphism. If $Q$ is IP-regular and $f^{-1}[E(Q)]$ is IP-regular, then $P$ is IP-regular.

Proof. Observe that the image of an IP-set under a partial homomorphism is an IP-set. It follows that $f^{-1}[B]$ is strongly IP-regular whenever $B \subset Q \setminus E(Q)$ is strongly IP-regular. The fact that $P$ is IP-regular follows easily from these observations. \qed

In particular Lemma 2.5 guarantees that the extension of an IP-regular group by an IP-regular group is IP-regular. More generally one can consider groups admitting a subnormal series with IP-regular factor groups. Recall that a subnormal series of a group $G$ is a finite sequence $(a)$ of subgroups of $G$ such that $A_0 = \{1_G\}$, $A_n = G$, and $A_i \subset A_{i+1}$ for every $i \in n$. The quotients $A_{i+1}/A_i$ for $i \in n$ are called factor groups of the series. Proposition 2.4 can be easily obtained from Lemma 2.5 by induction on the length of the subnormal series.
Proposition 2.6. Suppose that $G$ is a group, and $(a)$ is a subnormal series of $G$. If the factor groups $A_{i+1}/A_i$ are IP-regular for all $i \in n$, then $G$ is IP-regular.

A consequence of Proposition 2.6 is that solvable groups are IP-regular. This follows from the facts that solvable groups are exactly those that admit a subnormal series where all the factor groups are abelian, and that all abelian groups are IP-regular. The latter fact will be proved in the following section.

3. Cancellative commutative semigroups

Throughout this section all (partial) semigroups are additively denoted and assumed to be cancellative and commutative.

Proposition 3.1. Every cancellative commutative semigroup is IP-regular (as in Definition 2.2).

A key role in the proof of Proposition 3.1 is played by the notion of rank function.

Definition 3.2. A rank function on a cancellative commutative partial semigroup $P$ is a function $\rho$ from $P$ to a well ordered set with the property that if $\vec{x}$ is a sequence in $P$ such that all finite sums from $\vec{x}$ are defined, then the two following conditions are satisfied:

1. The restriction of $\rho$ to the range $\{x_n | n \in \omega\}$ of the sequence $\vec{x}$ is a finite-to-one function, and
2. if $\rho(x_n) \geq \rho(x_0)$ for every $n \in \omega$ then $x_0 + FS_1(\vec{x})$ is not an IP-set.

Remark 3.3 provides an example of a rank function.

Remark 3.3. If $\rho$ is a function from $P$ to a well order such that $\rho(x + y) = \min\{\rho(x), \rho(y)\}$ and $\rho(x) \neq \rho(y)$ whenever $x, y \in P$ and $x + y$ is defined, then $\rho$ is a rank function on $A$.

The relevance of rank functions for the proof of Proposition 3.1 is stated in Lemma 3.4. Recall that the family of IP-sets of a (partial) semigroup $P$ is partition regular (see [10, Corollary 5.15]). This means that if $F$ is a finite family of subsets of $P$ and $\bigcup F$ is an IP-set, then $F$ contains an IP-set. This fact will be used in the proof of Lemmas 3.4 and 3.6.

Lemma 3.4. If there is a rank function on a partial semigroup $P$, then $P$ is strongly IP-regular.

Proof. Fix a rank function $\rho$ from $P$ to a well ordered set. Suppose that $\vec{x}$ is a sequence in $P$ such that all the finite sums are defined. We claim that $x_0 + FS_1(\vec{x})$ is not an IP-set. Since $\rho$ restricted to the range of $\vec{x}$ is finite-to-one, it is possible to pick a permutation $\sigma$ of $\omega$ such that

$$\rho(x_{\sigma(n)}) \leq \rho(x_{\sigma(m)})$$

for every $n, m \in \omega$. Define $y_n = x_{\sigma(n)}$ and $\vec{y} = (y_n)_{n \in \omega}$. Observe that if $x_0 = y_{n_0}$ then

$$x_0 + FS_1(\vec{x}) = (y_{n_0} + FS_{n_0+1}(\vec{y})) \cup (y_{n_0} + FS\left((y_i)_{i=0}^{n_0-1}\right)) \cup$$

$$\cup (y_{n_0} + FS\left((y_i)_{i=0}^{n_0-1}\right) + FS_{n_0+1}(\vec{y})).$$
Applying the hypothesis that $\rho$ is a rank function to the sequence $(y_{n_0+k})_{k \in \omega}$, it follows that

$$y_{n_0} + \text{FS}_{n_0+1}(\vec{y})$$

is not an IP-set. Now for every $y = \sum_{i \in a} y_i \in \text{FS}((y_i)_{i \in n_0})$, if $m = \min(a)$ then we have that

$$y_{n_0} + y + \text{FS}_{n_0+1}(\vec{y}) \subseteq y_m + \text{FS}_{m+1}(\vec{y})$$

and since the latter is not an IP-set (by applying the fact that $\rho$ is a rank function to the sequence $(y_{m+k})_{k \in \omega}$), neither is the former. Finally

$$y_{n_0} + \text{FS}((y_i)_{i \in n_0})$$

is finite and hence not an IP-set. This allows one to conclude that

$$x_0 + \text{FS}_1(\vec{x})$$

is not an IP-set, as claimed. \qed

Denote in the following by $\mathbb{R}/\mathbb{Z}$ the quotient of $\mathbb{R}$ by the subgroup $\mathbb{Z}$. It is a well known fact that any commutative cancellative semigroup embeds into an abelian group, see [5, Proposition II.3.2]. Moreover any abelian group can be embedded in a divisible abelian group, and a divisible abelian group in turn can be embedded into a direct sum of copies of $\mathbb{R}/\mathbb{Z}$ (see for example [3, Theorems 24.1 and 23.1]). It follows that every cancellative commutative semigroup is a subsemigroup of a direct sum $(\mathbb{R}/\mathbb{Z})^{\oplus \kappa}$ of $\kappa$ copies of $\mathbb{R}/\mathbb{Z}$ for some cardinal $\kappa$. Therefore it is enough to prove Proposition 3.1 for $(\mathbb{R}/\mathbb{Z})^{\oplus \kappa}$. The proof of this fact will occupy the rest of this section.

Let us fix a cardinal $\kappa$. If $x \in (\mathbb{R}/\mathbb{Z})^{\oplus \kappa}$ and $\alpha \in \kappa$ then $\pi_\alpha(x)$ denotes the $\alpha$-th coordinate of $x$. Elements of $\mathbb{R}/\mathbb{Z}$ will be freely identified with their representatives in $\mathbb{R}$ (thus we might write something like $t \neq 0$, and this really means $t \notin \mathbb{Z}$), and if we need to specify a particular representative, we will choose the unique such in $[0, 1)$. Consider the partition

$$(\mathbb{R}/\mathbb{Z})^{\oplus \kappa} = C \cup B \cup \{0\}$$

of $(\mathbb{R}/\mathbb{Z})^{\oplus \kappa}$, where $B$ is the set of elements of $(\mathbb{R}/\mathbb{Z})^{\oplus \kappa}$ of order 2.

**Lemma 3.5.** The subset $B$ of $(\mathbb{R}/\mathbb{Z})^{\oplus \kappa}$ is strongly IP-regular.

**Proof.** Observe that $B \cup \{0\}$ is a subgroup of $(\mathbb{R}/\mathbb{Z})^{\oplus \kappa}$ isomorphic to the direct sum of $\kappa$ copies of $\mathbb{Z}/2\mathbb{Z}$. Thus $B \cup \{0\}$ has the structure of $\kappa$-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. If $\vec{x}$ is a sequence in $B$ then $\text{FS}(\vec{x}) \cup \{0\}$ is the vector space generated by $\vec{x}$. Moreover the sequence $\vec{x}$ is linearly independent if and only if $0 \notin \text{FS}(\vec{x})$ (see [2, Proposition 4.1]). Thus if $\vec{x}$ is a sequence in $B$ such that $\text{FS}(\vec{x}) \subset B$ then $\vec{x}$ is a linearly independent sequence, and hence any element of $\text{FS}(\vec{x})$ can be written in a unique way as a sum of elements of the sequence $\vec{x}$. In particular $x_0 + \text{FS}_1(\vec{x})$ consists of those finite sums $\sum_{i \notin a} x_i$ such that $0 \in a$. Thus if $a, b$ are finite subsets of $\omega \setminus 1$, then so is $a \Delta b$, hence

$$x_0 + \sum_{i \in a} x_i + x_0 + \sum_{i \in b} x_i = \sum_{i \in a \Delta b} x_i \notin x_0 + \text{FS}_1(\vec{x}).$$

This shows that whenever $x, y \in x_0 + \text{FS}_1(\vec{x})$ then $x + y \notin x_0 + \text{FS}_1(\vec{x})$, which implies that $x_0 + \text{FS}_1(\vec{x})$ is not an IP-set and $B$ is IP-regular. \qed
It remains to show now that $C$ is IP-regular. Elements $x \in C$ have order strictly greater than 2, thus there is at least one $\alpha < \kappa$ such that $\pi_\alpha(x) \notin \{0, \frac{1}{2}\}$, hence it is possible to define the function $\mu : C \to \kappa$ by

$$\mu(x) = \min \left\{ \alpha \in \kappa : \pi_\alpha(x) \notin \left\{ 0, \frac{1}{2} \right\} \right\}.$$

Consider

$$C_1 = \left\{ x \in C : \pi_{\mu(x)}(x) = \frac{1}{4} \right\};$$

$$C_3 = \left\{ x \in C : \pi_{\mu(x)}(x) = \frac{3}{4} \right\};$$

$$C_2 = \left\{ x \in C : \pi_{\mu(x)}(x) \notin \left\{ \frac{1}{4}, \frac{3}{4} \right\} \right\}.$$

Observe that

$$C = C_1 \cup C_2 \cup C_3$$

is a partition of $C$.

**Lemma 3.6.** The function $\mu$ restricted to $C_1$ is a rank function on $C_1$ as in Definition \[\text{[3.2]}\].

**Proof.** Suppose that $\vec{x}$ is a sequence in $(\mathbb{R} / \mathbb{Z})^{\omega}$ such that $\text{FS}(\vec{x}) \subset C_1$. We will show that the function $\mu$ restricted to $\{x_n | n \in \omega\}$ is at most two-to-one, in particular finite-to-one. This is because if $n, m, k \in \omega$ are three distinct numbers such that $\mu(x_n) = \mu(x_m) = \mu(x_k) = \alpha$, then for $\beta < \alpha$ we get that $\pi_\beta(x_n + x_m + x_k)$ is an element of $\{0, \frac{1}{2}\}$ because so are $\pi_\beta(x_n), \pi_\beta(x_m), \pi_\beta(x_k)$. On the other hand, $\pi_\alpha(x_n + x_m + x_k) = \frac{3}{4}$, which shows that $\mu(x_n + x_m + x_k) = \alpha$ but $x_n + x_m + x_k \in C_3$, a contradiction.

Now assume also that $\alpha = \mu(x_0) \leq \mu(x_i)$ for every $i \in \omega$. By the previous paragraph, there is at most one $n \in \omega \setminus \{1\}$ such that $\mu(x_n) = \mu(x_0) = \alpha$. Thus the first case is when there is such $n$. The first thing to notice is that for each $k \in \omega \setminus \{0, n\}$, we have that $\pi_\alpha(x_k) = 0$. This is because otherwise, since $\mu(x_k) > \alpha$ we would have that $\pi_\alpha(x_k) = \frac{1}{2}$ and so $\pi_\alpha(x_0 + x_k) = \frac{3}{4}$, so by an argument similar to that in the previous paragraph, $\mu(x_0 + x_k) = \alpha$, which would imply that $x_0 + x_k \in C_3$, a contradiction. Now write

$$x_0 + \text{FS}_1(\vec{x}) = \{x_0 + x_n \cup (x_0 + \text{FS}((x_k)_{k \in \omega \setminus \{0, n\}})) \cup (x_0 + x_n + \text{FS}((x_k)_{k \in \omega \setminus \{0, n\}})) \}
$$

Clearly $\{x_0 + x_1\}$ is not an IP-set, as it is finite. Now since $\pi_\alpha(x_k) = 0$ for $i \notin \omega \setminus \{0, n\}$, it follows that every element $x \in x_0 + \text{FS}((x_k)_{k \in \omega \setminus \{0, n\}})$ must satisfy $\pi_\alpha(x) = \frac{1}{4}$, which implies that $x_0 + \text{FS}((x_k)_{k \in \omega \setminus \{0, n\}})$ cannot contain the sum of any two of its elements and consequently it is not an IP-set. Similarly, every element $x \in x_0 + x_n + \text{FS}((x_k)_{k \in \omega \setminus \{0, n\}})$ satisfies $\pi_\alpha(x) = \frac{1}{2}$, so this set is, by the same argument, not an IP-set. Hence $x_0 + \text{FS}_1(\vec{x})$ is not an IP-set.

Now if there is no such $n$, i.e. if $\mu(x_k) > \mu(x_0) = \alpha$ for all $k > 0$, then arguing as in the previous paragraph we get that $\pi_\alpha(x_k) = 0$ for all $k > 0$. Hence every element $x \in x_0 + \text{FS}_1(\vec{x})$ satisfies that $\pi_\alpha(x) = \frac{1}{4}$, therefore the set $x_0 + \text{FS}_1(\vec{x})$ cannot be an IP-set. This concludes the proof that $\mu$ is a rank function on $C_1$. \[\square\]
Considering the fact that the function \( t \mapsto -t \) is an automorphism of \( (\mathbb{R}/\mathbb{Z})^{\times \kappa} \) mapping \( C_3 \) onto \( C_5 \) and preserving \( \mu \) allows one to deduce from Lemma 3.6 that \( \mu \) is a rank function on \( C_3 \) as well. Thus it only remains to show that \( C_2 \) is IP-regular.

Define
\[
Q_{i,j} = \left\{ x \in C_2 : \pi_{\mu(x)}(x) \in \bigcup_{m \in \omega} \left[ \frac{i}{4} + \frac{1}{2^{3m+j+3}}, \frac{i}{4} + \frac{1}{2^{3m+j+2}} \right) \right\}
\]
for \( i \in \{0, 1, 2, 3\} \) and \( j \in \{0, 1, 2\} \). Observe that
\[
C_2 = \bigcup_{i \in 4} \bigcup_{j \in 3} Q_{i,j}
\]
is a partition of \( C_2 \). In order to conclude the proof of Proposition 3.1 it is now enough to show that for every \( i \in 4 \) and \( j \in 3 \) the set \( Q_{i,j} \) is IP-regular. This will follow from Lemma 3.7 by Lemma 3.4.

**Lemma 3.7.** Consider \( \kappa \times \omega \) well ordered by the lexicographic order. The function \( \rho : Q_{i,j} \to \kappa \times \omega \) defined by \( \rho(x) = (\mu(x), m) \) where \( m \) is the unique element of \( \omega \) such that
\[
\pi_{\mu(x)}(x) \in \left[ \frac{i}{4} + \frac{1}{2^{3m+j+3}}, \frac{i}{4} + \frac{1}{2^{3m+j+2}} \right)
\]
is a rank function on \( Q_{i,j} \).

**Proof.** To simplify the notation let us run the proof in the case when \( i = j = 0 \). The proof in the other cases is analogous. By Remark 3.3 it is enough to show that if \( x \) and \( y \) are such that \( x, y, x + y \in Q_{0,0} \) then \( \rho(x) \neq \rho(y) \) and \( \rho(x + y) = \min \{\rho(x), \rho(y)\} \), so suppose that \( x, y \) are elements of \( Q_{0,0} \) such that \( x + y \in Q_{0,0} \) and assume by contradiction that \( \rho(x) = \rho(y) = (\alpha, m) \). Thus
\[
\pi_{\alpha}(x), \pi_{\alpha}(y) \in \left[ \frac{1}{2^{3m+3}}, \frac{1}{2^{3m+2}} \right)
\]
and hence
\[
\pi_{\alpha}(x + y) \in \left[ \frac{1}{2^{3m+2}}, \frac{1}{2^{3m+1}} \right).
\]
If \( m = 0 \) then
\[
\pi_{\alpha}(x + y) \in \left[ \frac{1}{4}, \frac{1}{2} \right)
\]
thus
\[
x + y \in C_1 \cup Q_{1,0} \cup Q_{1,1} \cup Q_{1,3}.
\]
If \( m > 0 \) then
\[
\pi_{\alpha}(x + y) \in \left[ \frac{1}{2^{3(m-1)+2+3}}, \frac{1}{2^{3(m-1)+2+2}} \right)
\]
and therefore
\[
x + y \in Q_{0,2}.
\]
In either case one obtains a contradiction from the assumption that \( x + y \in Q_{0,0} \). This concludes the proof that \( \rho(x) \neq \rho(y) \). We now claim that \( \rho(x + y) = \min \{\rho(x), \rho(y)\} \). Define \( \rho(x) = (\alpha, m) \) and \( \rho(y) = (\beta, n) \). Let us first consider the case when \( \alpha = \beta \) and without loss of generality \( m > n \). In this case
\[
\pi_{\xi}(x + y) \in \left\{ 0, \frac{1}{2} \right\}
\]
for $\xi < \alpha$, while

$$
\pi_\alpha (x + y) \in \left[ \frac{1}{2^n+3}, \frac{1}{2^{n+3}}, \frac{1}{2^{n+2}}, \frac{1}{2^{n+3}} \right]
$$

where

$$
\frac{1}{2^{n+2}} + \frac{1}{2^{n+2}} < \frac{1}{2^{n+1}} < \frac{1}{2^{n(n-1)+3}}.
$$

This shows that $\rho (x + y) = (\alpha, n) = \min \{\rho (x), \rho (y)\}$. Let us now consider the case when $\alpha \neq \beta$ and without loss of generality $\alpha > \beta$. In this case

$$
\pi_\xi (x + y) \in \left\{0, \frac{1}{2}\right\}
$$

for $\xi < \beta$ while

$$
\pi_\beta (x) = 0
$$

(because if not then $\pi_\beta (x) = \frac{1}{2}$ and that would imply that $x + y \in \bigcup_{j \in 3} Q_{1,j}$, and hence

$$
\pi_\beta (x + y) = \pi_\beta (y).
$$

This shows that $\rho (x + y) = (\beta, n) = \min \{\rho (x), \rho (y)\}$. This concludes the proof of the fact that $\rho$ satisfies the hypothesis of Remark 3.3 and, hence, it is a rank function on $Q_{0,0}$. □

4. The class of IP-regular semigroups

In this section all semigroups will be denoted multiplicatively. Let us define $\mathcal{R}$ to be the class of all IP-regular semigroups. Observe that by Proposition 3.1 $\mathcal{R}$ contains all commutative cancellative semigroups. We will now show that $\mathcal{R}$ contains all Archimedean commutative semigroups. Recall that a commutative semigroup $S$ is Archimedean if for every $a, b \in S$ there is a natural number $n$ and an element $t$ of $S$ such that $a^n = bt$, see [5, Section III.1]. By [5, Proposition III.1.3] an Archimedean commutative semigroup contains at most one idempotent.

**Proposition 4.1.** Archimedean commutative semigroups are IP-regular.

*Proof.* Suppose that $S$ is a commutative Archimedean semigroup. Let us first assume that $S$ has no idempotent elements: In this case by [4, Proposition IV.4.1] there is a congruence $\mathcal{C}$ on $S$ such that the quotient $S / \mathcal{C}$ is a commutative cancellative semigroup with no idempotent elements. It follows from Proposition 3.1 that $S / \mathcal{C}$ is IP-regular, and therefore $S$ is IP-regular by Lemma 2.5. Let us consider now the case when $S$ has a (necessarily unique) idempotent element $e$. Denote by $H_e$ the maximal subgroup of $S$ containing $e$. By [4, Proposition IV.2.3] $H_e$ is an ideal of $S$ and the quotient $S / H_e$ is a commutative nilsemigroup, i.e. a commutative semigroup with a zero element such that every element is nilpotent. By Lemma 2.5 and Proposition 3.1 it is therefore enough to show that a commutative nilsemigroup $T$ is IP-regular. Denote by $0$ the zero element of $T$. We claim that $T \setminus \{0\}$ is strongly IP-regular (as in Definition 2.1). In fact if $\bar{x}$ is a sequence in $T$ such that $\text{FP}(\bar{x})$ does not contain $0$, then $x_0 \text{FP}(\bar{x})$ is not an IP-set since $x_0$ is nilpotent. This concludes the proof that $T$ is IP-regular, and hence also the proof of Proposition 4.1. □
Let us now comment on the closure properties of the class $\mathcal{R}$. By Remark 2.3, $\mathcal{R}$ is closed with respect to taking subsemigroups, and contains all finite semigroups. Moreover by Proposition 2.6 if a group $G$ has a subnormal series with factor groups in $\mathcal{R}$, then $G$ belongs to $\mathcal{R}$. In particular $\mathcal{R}$ contains all virtually solvable groups and their subgroups. Proposition 4.2 shows that free products of elements of $\mathcal{R}$ with no idempotent elements are still in $\mathcal{R}$.

**Proposition 4.2.** Suppose that $S, T$ are semigroups. If both $S$ and $T$ are IP-regular, and $T$ has no idempotent elements, then the free product $S \ast T$ is IP-regular.

**Proof.** Denote by $T_1$ the semigroup obtained from $T$ adding an identity element 1. Consider the semigroup homomorphism from $S \ast T$ to $T_1$ sending a word $w$ to 1 if $w$ does not contain any letters from $T$, and otherwise sending $w$ to the element of $T$ obtained from $w$ by erasing the letters from $S$ and then taking the product in $T$ of the remaining letters of $w$. Observe that $f^{-1}(\{1\})$ is isomorphic to $S$ and therefore IP-regular. The conclusion now follows from Lemma 2.5. □

The particular case of Proposition 4.2 when $S = T = \mathbb{N}$ lets us obtain that the free semigroup on 2 generators is IP-regular. Considering the function assigning to a word its length, which is a semigroup homomorphism onto $\mathbb{N}$, one can see that a free semigroup in any number of generators is IP-regular, since so is $\mathbb{N}$. Via Lemma 2.4 this observation gives a short proof of [7, Lemma 2.3].

5. The main theorem

We will now present the proof of Theorem 1.4. Suppose that $S$ is a commutative semigroup with well founded universal semilattice. Denote by $\mathcal{N}$ the smallest semilattice congruence on $S$ as in [4, Proposition III.2.1]. Recall that the universal semilattice of $S$ is the quotient $S/\mathcal{N}$ by [4, Proposition III.2.2]. Moreover by [5, Theorem III.1.2] every $\mathcal{N}$-equivalence class is an Archimedean subsemigroup of $S$ known as an archimedean component of $S$. Pick a nonprincipal strongly summable ultrafilter $p$ on $S$. If $p$ contains some Archimedean component of $S$, then $p$ is regular by Lemma 2.4 and Proposition 4.1. Let us then assume that $p$ does not contain any Archimedean component. Denote by $f : S \rightarrow S/\mathcal{N}$ the canonical quotient map, and by $q$ the ultrafilter on $S/\mathcal{N}$ defined by $B \in q$ if and only if $f^{-1}(B) \in p$. By Remark 1.3 $q$ is a nonprincipal strongly productive ultrafilter on $S/\mathcal{N}$, and moreover in order to conclude that $p$ is regular it is enough to show that $q$ is regular. This will follow from Lemma 5.1.

**Lemma 5.1.** If $\Lambda$ is a well-founded semilattice, then any nonprincipal strongly summable ultrafilter $q$ on $\Lambda$ is regular.

**Proof.** We can assume without loss of generality that $\Lambda$ has a maximum element $x_{\text{max}}$. For $x \in \Lambda$, denote by pred$(x)$ the set
\[
\{y \in \Lambda : y \leq x \text{ and } y \neq x\}
\]
of strict predecessors of $x$. We will show by well founded induction that, for every $x \in \Lambda$, if pred$(x) \in q$ then $q$ is regular. The conclusion will follow from the observation that $p(x_{\text{max}}) \in q$. Suppose that $x$ is an element of $\Lambda$ such that, for every $y \in \text{pred}(x)$, if $\text{pred}(y) \in q$ then $q$ is regular. Suppose that pred$(x) \in q$. If pred$(x)$ witnesses the fact that $q$ is regular, then this concludes the proof. Otherwise there is a sequence $\vec{y}$ in $\Lambda$ such that $FP(\vec{y}) \subset \text{pred}(x)$ and $y_0 \text{FP}_1(\vec{y}) \in p$. Observing
that \( y_0 \text{FP}(\vec{y}) \subset \text{pred}(y_0) \cup \{y_0\} \) allows one to conclude that \( \text{pred}(y_0) \in q \), where \( y_0 \in \text{pred}(x) \). Thus by inductive hypothesis \( q \) is regular. \( \square \)

This concludes the proof of the fact that a nonprincipal strongly summable ultrafilter on a commutative semigroup with well founded universal semilattice is regular. We will now show that the same fact holds for solvable inverse semigroups with well founded semilattice of idempotents. An introduction to inverse semigroups can be found in [4, Chapter VII] or in the monograph [12]. Recall that the semilattice of idempotents of an inverse semigroup is isomorphic to its universal semilattice. The notion of solvable inverse semigroup has been introduced by Piochi in [13] as a generalization of the notion of solvable group to the context of inverse semigroups (solvable groups are thus exactly the solvable inverse semigroups with only one idempotent, see [13, Theorem 3.4]). Observe that by definition a solvable inverse semigroup \( S \) of class \( n + 1 \) has a commutative congruence \( \gamma_S \) such that, if \( f : S \to S/\gamma_S \) is the canonical quotient map, then

\[
f^{-1}[E(S/\gamma_S)]
\]

is an inverse subsemigroup of \( S \) of solvability class \( n \). Moreover the solvable inverse semigroups of solvability class 1 are exactly the commutative semigroups. The fact that solvable inverse semigroups with well-founded semilattice of idempotents satisfy the conclusion of Theorem [14] will then follow from Remark [14] by induction on the solvability class, after we observe that a commutative homomorphic image of an inverse semigroup with well founded semilattice of idempotents has well founded universal semilattice. This is the content of Lemma [5.2].

**Lemma 5.2.** Suppose that \( S, T \) are semigroups, and \( f : S \to T \) is a surjective semigroup homomorphism. If \( S \) is an inverse semigroup with well founded semilattice of idempotents and \( T \) is commutative, then \( T \) is has well founded universal semilattice.

**Proof.** Observe that \( T \) is a von Neumann regular semigroup, as in [4, Section II.2]. In fact if \( f(x) \) is an element of \( T \) then \( f(x^{-1}) \) is an inverse of \( f(x) \), as in [4, Section II.2]. Moreover, since \( T \) is commutative, it is an inverse semigroup by [4, Proposition II.2.6]. Therefore the universal semilattice of \( T \) is isomorphic to the semilattice \( E(T) \) of idempotent elements of \( T \). Suppose that \( B \) is a nonempty subset of \( E(T) \); By [12, Chapter 1, Proposition 21] there is a set \( A \) of idempotent elements of \( S \) such that \( B = f[A] \). Pick a minimal element \( e_0 \) of \( A \). If \( e \in A \) is such that \( f(e) \leq f(e_0) \) then

\[
f(e_0) = f(ee_0) = f(e)f(e_0) = f(e).
\]

This shows that \( f(e_0) \) is a minimal element of \( B \), and hence \( E(T) \) is well founded. \( \square \)

### 6. Sparseness

A strongly productive ultrafilter \( p \) on a (multiplicatively denoted) semigroup \( S \) is **sparse** (see [7, Definition 3.9]) if for every \( A \in p \) there are a sequence \( \vec{x} = (x_n)_{n \in \omega} \) in \( S \) and a subsequence \( \vec{y} = (y_n)_{n \in \omega} \) of \( \vec{x} \) such that:

- \( \text{FP}(\vec{y}) \in p \);
- \( \text{FP}(\vec{x}) \subset A \);
- \( \{k_n : n \in \omega\} \) is coinfinitely in \( \omega \).
Suppose that $\mathbb{F}$ is the partial semigroup of finite nonempty subsets of $\omega$, where, for $a, b \in \mathbb{F}$ the product $ab$ is defined and equal to $a \cup b$ if and only if $\max(a) < \min(b)$. A strongly productive ultrafilter on the partial semigroup $\mathbb{F}$ is an \textit{ordered union ultrafilter} as defined in [1, page 92]. A strongly productive ultrafilter $p$ on a multiplicatively denoted semigroup $S$ is \textit{multiplicatively isomorphic to an ordered union ultrafilter} if there is a sequence $\vec{x}$ such that the function

$$ f: \mathbb{F} \to \text{FP}(\vec{x}) $$

$$ a \mapsto \prod_{i \in a} x_i $$

is injective, and furthermore

$$ \{f^{-1}[A] : A \in p\} $$

is an ordered union ultrafilter.

\textbf{Lemma 6.1.} If $p$ is multiplicatively isomorphic to an ordered union ultrafilter, then $p$ is sparse strongly productive. In particular every ordered union ultrafilter is sparse.

\textit{Proof.} Suppose that the sequence $\vec{x}$ in $S$ and the function $f: \mathbb{F} \to \text{FP}(\vec{x})$ witness the fact that $p$ is multiplicatively isomorphic to an ordered union ultrafilter. Fix an element $B$ of $p$, and observe that

$$ q = \{f^{-1}[A \cap B] : A \in p\} $$

is an ordered union ultrafilter. Therefore there is a sequence $\vec{b}$ in $\mathbb{F}$ such that all the products from $\vec{b}$ are defined (equivalently, $\max(b_i) < \min(b_{i+1})$ for every $i \in \omega$), and $\text{FP}(\vec{b}) \in q$. Moreover by [11, Theorem 4] (see also [9, Theorem 2.6]) there is an element $W$ of $q$ contained in $\text{FP}(\vec{b})$ such that $\bigcup W$ has infinite complement in $\bigcup_{i \in \omega} b_i$. Denote by $D$ the set of $i \in \omega$ such that $b_i \subset \bigcup W$. Observe that

$$ \bigcup_{i \in D} b_i = \bigcup W $$

and

$$ W \subset \text{FP} \left( (b_i)_{i \in D} \right). $$

In particular $D$ has infinite complement in $\omega$ and $\text{FP} \left( (b_i)_{i \in D} \right)$ belongs to $q$. Therefore the sequence $\vec{x}$ in $S$ such that $x_i = f(b_i)$ for every $i \in \omega$ is such that $\text{FP}(\vec{x}) \subset B$ and $\text{FP} \left( (x_i)_{i \in D} \right) \in p$, witnessing the fact that $p$ is sparse strongly productive. \hfill $\square$

We will now define a condition on sequences that ensures the existence of a multiplicative isomorphism with an ordered union ultrafilter. This can be seen as a noncommutative analogue of the notion of \textit{strong uniqueness of finite sums} introduced in [9, Definition 3.1] in a commutative context.

\textbf{Definition 6.2.} A sequence $\vec{x}$ in a semigroup $S$ satisfies the \textit{ordered uniqueness of finite products} if the function

$$ f: \mathbb{F} \to \text{FP}(\vec{x}) $$

$$ a \mapsto \prod_{i \in a} x_i $$
is an isomorphism of partial semigroups from $\mathbb{F}$ to $FP(\vec{x})$. Equivalently $f$ is injective and if $a, b$ are elements of $\mathbb{F}$ such that $f(a) f(b) \in FP(\vec{x})$, then the maximum element of $a$ is strictly smaller than the minimum element of $b$.

For example suppose that $S$ is the free semigroup on countably many generators $\{s_n : n \in \omega\}$. It is not difficult to see that the sequence $(s_n)_{n \in \omega}$ in $S$ satisfies the ordered uniqueness of finite products.

**Remark 6.3.** If a strongly productive ultrafilter $p$ on $S$ contains $FP(\vec{x})$ for some sequence $\vec{x}$ in $S$ satisfying the ordered uniqueness of finite products, then $p$ is multiplicatively isomorphic to an ordered union ultrafilter.

Remark 6.3 follows immediately from the fact that an ordered union ultrafilter is just a strongly productive ultrafilter on the partial semigroup $\mathbb{F}$.

The following immediate consequence of Remark 6.3 and Lemma 6.1 can be seen as a noncommutative analogue of [9, Theorem 3.2] (see also [2, Corollary 2.9]).

**Corollary 6.4.** Let $p$ be a strongly productive ultrafilter on a semigroup $S$. If $p$ contains $FP(\vec{x})$ for some sequence $\vec{x}$ satisfying the ordered uniqueness of finite products, then $p$ is sparse.

In the remainder of this section, we will present an application of Corollary 6.4 to a question of Neil Hindman and Lakeshia Legette Jones from [7] about very strongly productive ultrafilters on the free semigroup on countably many generators.

Recall that a sequence $\vec{y}$ on a semigroup $S$ is a product subsystem of the sequence $\vec{x}$ in $S$ if there is a sequence $(a_n)_{n \in \omega}$ in $\mathbb{F}$ such that $y_n = \prod_{i \in a_n} x_i$ and the maximum element of $a_n$ is strictly smaller than the minimum element of $a_{n+1}$ for every $n \in \omega$. Suppose that $S$ is the free semigroup on countably many generators, and $\vec{s}$ is an enumeration of its generators. A very strongly productive ultrafilter on $S$ as in [7, Definition 1.2] is an ultrafilter $p$ on $S$ generated by sets of the form $FP(\vec{x})$ where $\vec{x}$ is a product subsystem of $\vec{s}$.

**Theorem 6.5.** Every very strongly productive ultrafilter on the free semigroup $S$ is multiplicatively isomorphic to an ordered union ultrafilter, and hence sparse.

**Proof.** Observe that by [7, Theorem 4.2] very strongly productive ultrafilters on $S$ are exactly the strongly productive ultrafilters containing $FP(\vec{s})$ as an element. In particular, since the sequence $\vec{s}$ satisfies the ordered uniqueness of finite products, all very strongly productive ultrafilters on $S$ are multiplicatively isomorphic to ordered union ultrafilters by Remark 6.3 and hence sparse by Lemma 6.1. □

Theorem 6.5 answers Question 3.26 from [7]. Corollary 3.11 of [7] asserts that a sparse very strongly productive ultrafilter on $S$ can be written only trivially as a product of ultrafilters on the free group on the same generators. Since by Theorem 6.5 any very strongly productive ultrafilter on $S$ is sparse, one can conclude that the conclusion of [7, Corollary 3.11] holds for any very strongly productive ultrafilter on $S$. This is the content of Corollary 6.6.

**Corollary 6.6.** Let $G$ be the free group on the sequence of generators $\vec{s}$, and let $S$ be the free semigroup on the same generators. Suppose that $p$ is a very strongly productive ultrafilter on $S$. If $q, r$ are ultrafilters on $G$ such that $qr = p$, then there is an element $w$ of $G$ such that one of the following statements hold:

1. $r = wp$ and $q = pw^{-1}$;
(2) \( r = w \) and \( q = pw^{-1} \);
(3) \( r = wp \) and \( q = w^{-1} \).

In particular, if \( q, r \in G^* \) are such that \( qr = p \), then \( r = wp \) and \( q = pw^{-1} \) for some \( w \in G \).

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