A distance formula related to a family of projections orthogonal to their symmetries

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Abstract
Let \( u \) be a hermitian involution, and \( e \) an orthogonal projection, acting on the same Hilbert space \( H \). We establish the exact formula, in terms of \( \|eue\| \), for the distance from \( e \) to the set of all orthogonal projections \( q \) from the algebra generated by \( e, u \), and such that \( quq = 0 \).

Keywords: orthogonal projection, involution, \( C^* \)-algebra, \( W^* \)-algebra
2010 MSC: 47A05, 47A30

1. Introduction
Let \( H \) be a Hilbert space and let \( B(H) \) stand for the \( C^* \)-algebra of all bounded linear operators acting on \( H \). Given a hermitian involution \( u \in B(H) \), denote by \( Q_u \) the set of all orthogonal projections \( q \in B(H) \) for which \( quq = 0 \).

Theorem 1.2 of [1] can be stated as follows:

**Theorem 1.** Let \( e \in B(H) \) be an orthogonal projection such that
\[
\|eue\| < \xi (\approx 0.455).
\]
Then there exists \( q \in Q_u \) for which
\[
\|e - q\| \leq \frac{1}{2} \|eue\| + 4 \|eue\|^2.
\]
Further, \( q \) is in the \( C^* \)-subalgebra of \( B(H) \) generated by \( e, ueu^* \).

Note that the distance between any two orthogonal projections does not exceed one. So, estimate [1] is useful only when \( \|eue\| \) is smaller than the positive root of \( 8x^2 + x - 2 \), that is, approximately 0.441.

We will provide an explicit formula for the distance from \( e \) to the intersection of \( Q_u \) with the \( W^* \)-algebra \( W(e, u) \) generated by \( e, u \), as well as for the element on which this distance is attained. No a priori restriction on \( \|eue\| \) is needed, and the respective \( q \) indeed lies in the \( C^* \)-algebra \( C(e, ueu^*) \) generated by \( e, ueu^* \) whenever \( \|eue\| < 1 \).
Theorem 2. Let \( e, u \in \mathcal{B}(\mathcal{H}) \) be, respectively, an orthogonal projection and a hermitian involution. Denote by \( \mathcal{H}_\pm \) the eigenspace of \( u \) corresponding to its eigenvalue \( \pm 1 \). Then the distance \( d \) from \( e \) to \( Q_u \cap \mathcal{W}(e, u) \) is one if the range of \( e \) has a non-trivial intersection with \( \mathcal{H}_+ \) or \( \mathcal{H}_- \), and is given by the formula

\[
d = \sqrt{\frac{1}{2} \left( 1 - \sqrt{1 - \|ueu\|^2} \right)}
\]  

otherwise.

For small values of \( \|ueu\| \), it is instructive to compare (1) with the Taylor expansion of (2):

\[
d = \frac{1}{2} \|ueu\| + \frac{1}{16} \|ueu\|^3 + \cdots
\]

Figure 1: Estimate (1) versus formula (2) as functions of \( \|ueu\| \)

2. Proof of the main result

Using the canonical representation [2] (see also [3] or a more recent survey [4]) of the pair \( e, (u + I)/2 \) of orthogonal projections, we can find an orthogonal decomposition of \( \mathcal{H} \) into six summands,

\[
\mathcal{H} = \mathcal{M}_{00} \oplus \mathcal{M}_{01} \oplus \mathcal{M}_{10} \oplus \mathcal{M}_{11} \oplus \mathcal{M} \oplus \mathcal{M},
\]  

(3)
with respect to which
\[ u = I \oplus I \oplus (-I) \oplus (-I) \oplus \text{diag}[I, -I], \]
\[ e = I \oplus 0 \oplus 0 \oplus I \oplus \begin{bmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I - H \end{bmatrix}. \]  \hspace{1cm} (4)

(Here and in what follows we use the notation \(\text{diag}[X_1, \ldots, X_k]\) for block diagonal matrices with \(X_1, \ldots, X_k\) as their diagonal blocks.) Note that in (3) the subspaces \(\mathcal{M}_{00}\) and \(\mathcal{M}_{11}\) (resp, \(\mathcal{M}_{01}\) and \(\mathcal{M}_{10}\)) are the intersections of the range (resp, the kernel) of \(e\) with \(\mathcal{H}_+\) and \(\mathcal{H}_-\). The (hermitian) operator \(H\) is the compression of \(e\) onto \(\mathcal{M} := \mathcal{H}_+ \oplus (\mathcal{M}_{00} \oplus \mathcal{M}_{01})\). By construction, \(H\) has its spectrum \(\Delta\) lying in \([0, 1]\) and \(0, 1\) are not its eigenvalues.

Elements of \(\mathcal{W}(e, u)\) with respect to the same decomposition \(3\) look as
\[ q = a_{00}I \oplus a_{01}I \oplus a_{10}I \oplus a_{11}I \oplus \Phi(H), \] \hspace{1cm} (5)
where \(\Phi = \begin{bmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{bmatrix}\), \(a_{ij} \in \mathbb{C}\), and the functions \(\phi_{ij}\) are Borel-measurable and essentially bounded on \(\Delta\), in the sense of the spectral measure of \(H\) (\[5\], see also \([3, 4]\)). Consequently, \(q \in \mathcal{W}(e, u)\) is an orthogonal projection if and only if \(a_{ij} \in \{0, 1\}\), the functions \(\phi_{00}, \phi_{11}\) are real-valued, while \(\phi_{01}, \phi_{10}\) are complex conjugate, and
\[ \phi_{00} - \phi_{00}^2 = \phi_{11} - \phi_{11}^2 = |\phi_{01}|^2, \quad (\phi_{00} + \phi_{11} - 1)\phi_{01} = 0. \]  \hspace{1cm} (6)

On the other hand, direct computations immediately reveal that condition \(quq = 0\) is equivalent to
\[ \phi_{00}^2 = \phi_{11}^2 = \phi_{01}\phi_{10}, \quad (\phi_{00} - \phi_{11})\phi_{01} = (\phi_{00} - \phi_{11})\phi_{10} = 0. \]  \hspace{1cm} (7)
Solving the system of equations (6)–(7) yields
\[ \phi_{00} = \phi_{01} = \frac{1}{2}\chi, \quad \phi_{01} = \frac{1}{2}\chi\omega, \quad \phi_{10} = \frac{1}{2}\chi\overline{\omega} \]
with \(\chi\) being a characteristic function of some subset of \(\Delta\) and unimodular \(\omega\).

So, elements of \(\mathcal{Q}_u \cap \mathcal{W}(e, u)\) have the form
\[ q = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{bmatrix} \chi & \chi\omega \\ \chi\overline{\omega} & \chi \end{bmatrix}(H). \]  \hspace{1cm} (8)

The rest of the reasoning depends on whether or not the subspaces \(\mathcal{M}_{00}, \mathcal{M}_{11}\) are actually present in the decomposition \(3\).

Case 1. At least one of the subspaces \(\mathcal{M}_{00}, \mathcal{M}_{11}\) is different from zero, that is, the range of \(e\) contains some eigenvectors of \(u\).

Since for any \(q\) of the form (5) the restriction of \(e - q\) on \(\mathcal{M}_{00} \oplus \mathcal{M}_{11}\) is the identity, we then have \(\|e - q\| = 1\). Consequently, \(d = 1\). Note that in this case also \(\|eue\| = 1\).
Case 2. \(M_{00} = M_{11} = \{0\}\). Since both \(e\) given by (4) and \(q\) given by (8) have zero restrictions onto \(M_{01} \oplus M_{11}\), we may without loss of generality suppose that in place of (3) simply \(H = M \oplus M\), and respectively
\[
e = \begin{bmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & I - H \end{bmatrix}, \quad q = \frac{1}{2} \begin{bmatrix} \chi & \chi \omega \\ \chi \omega & \chi \end{bmatrix} (H).
\] (9)

So, \(e - q = \Phi_{\chi,\omega}(H)\), where
\[
\Phi_{\chi,\omega}(t) = \begin{bmatrix} t - \frac{1}{2} \chi(t) & \sqrt{t(1-t)} - \frac{1}{2} \chi(t) \omega(t) \\ \sqrt{t(1-t)} - \frac{1}{2} \chi(t) \omega(t) & 1 - t - \frac{1}{2} \chi(t) \end{bmatrix}.
\]

Consequently,
\[
\|e - q\| = \text{ess sup}_{t \in \Delta} \lambda_{\chi,\omega}(t),
\]
where \(\lambda_{\chi,\omega}(t)\) is the positive eigenvalue of \(\Phi_{\chi,\omega}(t)\), and \(\text{ess}\) is understood in the sense of the spectral measure of \(H\).

If \(\chi(t) = 0\) for some \(t \in \Delta\), then the respective \(\lambda_{\chi,\omega}(t)\) equals one, guaranteeing \(\|e - q\| = 1\). We should concentrate therefore on elements \(q\) with \(\chi(t) \equiv 1\). Then we have
\[
\Phi_{1,\omega}(t) = \begin{bmatrix} t - \frac{1}{2} & \sqrt{t(1-t)} - \frac{1}{2} \omega(t) \\ \sqrt{t(1-t)} - \frac{1}{2} \omega(t) & 1 - t - \frac{1}{2} \end{bmatrix},
\]
and
\[
\lambda_{1,\omega}(t) = \sqrt{\frac{1}{2} - \sqrt{t(1-t)}} \text{ Re } \omega(t).
\]

Since \(\omega\) is unimodular, to minimize \(\lambda_{1,\omega}(t)\) for any given \(t\) we should take \(\omega(t) = 1\). The respective element \(q\) is simply
\[
q_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\] (10)
\[
\lambda_{1,1}(t) = \sqrt{\frac{1}{2} - \sqrt{t(1-t)}}, \quad \text{and}
\]
\[
\|e - q_0\| = \sqrt{\frac{1}{2} - \min_{t \in \Delta} \sqrt{t(1-t)}} = \sqrt{\frac{1}{2} \left( 1 - \sqrt{1 - \max_{t \in \Delta} |2t - 1|^2} \right)}.
\]

In order to justify (2), it remains only to observe that
\[
\max_{t \in \Delta} |2t - 1| = \|eue\|.
\] (11)

But this is indeed the case, since \(eue = \Phi(H)\) with the matrix
\[
\Phi(t) = (2t - 1) \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1 - t \end{bmatrix},
\]
the eigenvalues of which are zero and \(2t - 1\).
3. Additional comments

1. Recall \([6]\) that elements of \(C^*\)-algebra \(C(e, u)\) generated by \(e\) and \(u\) are those of the form \([6]\) for which the functions \(\phi_{ij}\) are continuous on \(\Delta\) and such that \(\phi_{01}(j) = \phi_{10}(j) = 0, a_{ij} = \phi_{ii}(j)\) if \(j \in \Delta\). From \([10]\) we therefore conclude that the element \(q_0 \in W(e, u)\) on which the distance from \(e\) to \(Q_u\) is attained does not lie in \(C(e, u)\) if the spectrum of \(H\) contains 0 or 1.

On the other hand, due to \([11]\) condition \(\|eue\| < 1\) guarantees that \(0, 1 \notin \Delta\), and thus the invertibility of the operator \(H(I - H)\). Moreover, \(M_{00} = M_{11} = \{0\}\), as was observed in Section 2. So, without loss of generality \(e\) is given by the first formula in \([9]\), while \(u = \text{diag}[I, -I]\). From here:

\[
z := \frac{1}{2}(e - ueu^*) = \begin{bmatrix} 0 & \sqrt{H(I - H)} \\ \sqrt{H(I - H)} & 0 \end{bmatrix} \in C(e, ueu^*),
\]

\[
z^2 = \text{diag}[H(I - H), H(I - H)]\]

is positive definite and also lies in \(C(e, ueu^*)\), and therefore so does \((z^2)^{-1/2} = \text{diag}[(H(I - H))^{-1/2}, (H(I - H))^{-1/2}]\). Along with \(z\) and \((z^2)^{-1/2}\), the algebra \(C(e, ueu^*)\) contains their product

\[
z(z^2)^{-1/2} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.
\]

We conclude from \([10]\) that \(q_0 = \frac{1}{2}(I + z(z^2)^{-1/2}) \in C(e, ueu^*)\).

2. The distances from \(e\) to the sets \(Q_u\) and \(Q_u \cap W(e, u)\) may not coincide. To illustrate, consider \(H = \mathbb{C}^3\), \(u = \text{diag}[1, 1, -1]\) and \(e = \text{diag}[1, 0, 0]\). Then \(Q_u\) consists of zero and all matrices of the form

\[
q_{x,y} = \frac{1}{2} \begin{bmatrix} |x|^2 & x\bar{y} & x \\ x\bar{y} & |y|^2 & y \\ x & y & 1 \end{bmatrix},
\]

with the parameters \(x, y \in \mathbb{C}\) satisfying \(|x|^2 + |y|^2 = 1\). An easy computation shows that

\[
\|e - q_{x,y}\| = \sqrt{(1 + |y|^2)/2}.
\]

So, the distance from \(e\) to \(Q_u\) equals \(1/\sqrt{2}\) and is attained on all the matrices \(q_{\omega,0}\) with \(|\omega| = 1\), that is, having the form

\[
\frac{1}{2} \begin{bmatrix} 1 & 0 & \omega \\ 0 & 0 & 0 \\ \bar{\omega} & 0 & 1 \end{bmatrix}.
\]

On the other hand, the algebra generated by \(e\) and \(u\) consists simply of all 3-by-3 diagonal matrices. The only diagonal matrix lying in \(Q_u\) is 0, and \(d = 1\) in full agreement with Theorem \(2\).
4. Acknowledgments

The author was supported in part by Faculty Research funding from the Division of Science and Mathematics, New York University Abu Dhabi.

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