A secure multi-party batch matrix multiplication problem (SMBMM) is considered, where the goal is to allow a master node to efficiently compute the pairwise products of two batches of massive matrices that originate at external source nodes, by distributing the computation across $S$ honest but curious servers. Any group of up to $X$ colluding servers should gain no information about the input matrices, and the master should gain no additional information about the input matrices beyond the product. A solution called Generalized Cross Subspace Alignment codes with Noise Alignment (GCSA-NA in short) is proposed in this work, based on cross-subspace alignment codes. These codes originated in secure private information retrieval, and have recently been applied to distributed batch computation problems where they generalize and improve upon the state of art schemes such as Entangled Polynomial Codes and Lagrange Coded Computing. The prior state of art solution to SMBMM is a coding scheme called polynomial sharing (PS) that was proposed by Nodehi and Maddah-Ali. GCSA-NA outperforms PS codes in several key aspects – more efficient and secure inter-server communication (which can entirely take place beforehand, i.e., even before the input matrices are determined), flexible inter-server network topology, efficient batch processing, and tolerance to stragglers. The idea of noise-alignment can also be applied to construct schemes for $N$ sources based on $N$-CSA codes, and to construct schemes for symmetric secure private information retrieval to achieve the asymptotic capacity.

I. INTRODUCTION

Recent interest in coding for secure, private, and distributed computing combines a variety of elements such as coded distributed massive matrix multiplication, straggler tolerance, batch computing, and private information retrieval [1]–[39]. Coded distributed massive matrix multiplication [1]–[30] is currently an active research area in information theory where the goal is to perform matrix multiplications that are too massive to be feasible for one node, by partitioning the task into smaller subtasks and outsourcing these subtasks to a set of servers. Straggler tolerance is important for massive computations because of the significant risk of a server failing during the course of such a computation. Batch computing codes [6], [30] seek to combine multiple computation tasks in a way that the efficiency of computation is improved over separate computations. Private information retrieval [31]–[39] seeks ways for a user to retrieve desired information from a set of distributed servers without disclosing what specifically is desired. These related ideas converged recently in Generalized Cross Subspace Alignment (GCSA) codes presented in [30]. These codes originated in the setting of secure private information retrieval [57] and have recently been developed further in [30] for applications to coded distributed batch computation problems where they generalize and improve upon the state of art schemes such as Polynomials codes [2], MatDot codes and PolyDot codes [3], Generalized PolyDot codes [4] and Entangled Polynomial Codes [5] (all based on matrix partitioning) and Lagrange Coded Computing [6] (based on batch processing).

As the next step in the expanding scope of coding for distributed computing, Nodehi and Maddah-Ali recently explored its application to secure multiparty computation in [40]. Introduced by Andrew Yao in the 1980s [41], secure multiparty computation is an active research topic in theoretical computer science and cryptography that seeks efficient ways to compute a function on a set of inputs that must be kept private [42]–[44]. Specifically, Nodehi et al. consider a system including $N$ source nodes, $S$ honest but curious servers (worker nodes) and one master node. Each source sends a coded function of its data (called a share) to each server. The servers process their inputs and while doing so, may communicate with each other. After that each server sends a message to the master, such that the master can recover the required function of the source inputs. The input data must be kept perfectly secure from the servers even if any group of up to $X$ servers collude among themselves to share everything they know. Moreover, the master node must not gain any information about the input data beyond the result of the computation. Nodehi et al. propose a scheme called polynomial sharing (PS), which admits basic matrix operations such as addition, multiplication, and transposition. By concatenating the procedures for basic operations, arbitrary polynomial functions of the input matrices can be calculated. However, the PS scheme has a few key limitations. It needs multiple rounds of communication among servers where every server needs to send messages to every
other server. This carries a high communication cost and requires the network topology among servers to be a complete graph (otherwise data security would be compromised), does not tolerate stragglers, and does not lend itself to batch processing. These aspects (batch processing, improved inter-server communication efficiency, various network topologies) are highlighted as open problems by Nodehi et al. in [40].

Since GCSA codes are particularly efficient at batch processing and already encompass prior approaches to coded distributed computing, in this work we explore whether GCSA codes can also be applied to the problems identified by Nodehi et al. In particular, we will focus on the problem of multiplication of two matrices. As it turns out, in this context the answer is in the affirmative. Securing the data against any communication is leaked, it can reveal nothing about the data inputs. In fact, the inter-server communication can be chosen to be arbitrarily large by the coding algorithm.

Let us refer to the additional terms that are contained in the answers sent by the servers to the master node, which may collectively reveal information about anything about the inputs besides the result of the computation. Notably, the idea of Noise Alignment (NA) – the workers communicate among themselves to share noise terms (unknown to the master node) that are structured in the same manner as the interfering terms. Because these terms do not depend on the data inputs, the inter-server communication can take place beforehand, before the input data is determined, say during off-peak hours. This directly leads to another advantage. The GCSA-NA scheme allows the inter-server communication network graph to be any connected graph unlike PS schemes which require a complete graph.

The rest of the paper is organized as follows. Section II presents the problem statement. In Section III we state the results and proof of GCSA-NA are shown in Section V. Section VI concludes the paper.

Notation: For positive integers $M, N$ ($M < N$), $[N]$ stands for the set $\{1, 2, \ldots, N\}$ and $[M : N]$ stands for the set $\{M, M+1, \ldots, N\}$. The notation $X_{[N]}$ denotes the set $\{X_1, X_2, \ldots, X_N\}$. For $\mathcal{I} = \{i_1, i_2, \ldots, i_N\}$, $X_{\mathcal{I}}$ denotes the set $\{X_{i_1}, X_{i_2}, \ldots, X_{i_N}\}$. The notation $\otimes$ is used to denote the Kronecker product of two matrices, i.e., for two matrices $A$ and $B$, where $(A)_{r,s} = a_{rs}$ and $(B)_{v,w} = b_{vw}$, $(A \otimes B)_{p(r-1)+q,s+w} = a_{rs}b_{vw}$. $I_N$ denotes the $N \times N$ identity matrix. $T(X_1, X_2, \cdots, X_N)$ denotes the $N \times N$ lower triangular Toeplitz matrix, i.e.,

$$T(X_1, X_2, \cdots, X_N) = \begin{bmatrix} X_1 & X_2 & \cdots & X_1 \\ X_2 & X_1 & \cdots & X_2 \\ \vdots & \vdots & \ddots & \vdots \\ X_N & \cdots & \cdots & X_1 \end{bmatrix}.$$ 

For a matrix $M$, $|M|$ denotes the number of elements in $M$. For a polynomial $P$, $\text{deg}_\alpha(P)$ denotes the degree of $P$ with respect to a variable $\alpha$. Define the degree of the zero polynomial as $-1$. The notation $\mathcal{O}(a \log^2 b)$ suppresses polylog terms. It may be replaced with $\mathcal{O}(a \log^2 b)$ if the field $\mathbb{F}$ supports the Fast Fourier Transform (FFT), and with $\mathcal{O}(a \log^2 b \log \log(b))$ if it does not.

II. Problem Statement

Consider a system including 2 sources nodes (Source $A$ and Source $B$), $S$ servers nodes (workers nodes) and one master node, as illustrated in Fig. 1. Each source is connected to (has a direct communication link to) every single server. Servers are connected to each other, and all of the servers are connected to the master. All of these links are secure and error free.

Source $A$ and Source $B$ independently generate sequences of $L$ matrices, denoted as $A = (A^{(1)}, A^{(2)}, \ldots, A^{(L)})$, and $B = (B^{(1)}, B^{(2)}, \ldots, B^{(L)})$, respectively, such that for all $l \in [L]$, $A^{(l)} \in \mathbb{F}^{\lambda \times k}$ and $B^{(l)} \in \mathbb{F}^{\kappa \times \mu}$, i.e., $A^{(l)}$ and $B^{(l)}$
\[ A = \{A^{(1)}, \ldots, A^{(L)}\}, Z^A \quad \text{and} \quad B = \{B^{(1)}, \ldots, B^{(L)}\}, Z^B \]

\[ \begin{array}{c}
\text{Source } A \\
A^1 \\
A^2 \\
\vdots \\
A^L \\
\text{Server } S \\
\text{Server } 1 \\
\mathbf{\cdots} \\
\text{Server } i_X \\
\text{Master} \\
\end{array} \quad \begin{array}{c}
\text{Source } B \\
B^1 \\
B^2 \\
\vdots \\
B^L \\
\text{Server } S \\
\text{Server } 1 \\
\mathbf{\cdots} \\
\text{Server } i_X \\
\text{Master} \\
\end{array} \]

\[ Y_1, Y_2, \ldots, Y_S \]

A total of \( R \) answers downloaded

\[ AB = \{A^{(1)}B^{(1)}, \ldots, A^{(L)}B^{(L)}\} \]

\[ I(A, B; Y_1, Y_2, \ldots, Y_S | AB) = 0 \]

Fig. 1: The SMBMM problem. Source nodes generate matrices \( A = \{A^{(1)}, \ldots, A^{(L)}\} \) with separate noise \( Z^A \) and \( B = \{B^{(1)}, B^{(2)}, \ldots, B^{(L)}\} \) with separate noise \( Z^B \), and upload information to \( S \) distributed servers in coded form \( \bar{A}^{[s]}, \bar{B}^{[s]} \), respectively. For security, any \( X \) colluding servers (e.g., \( s \) to \( i_X \) in the figure) gain nothing about \( A, B \). Servers could send to each other some messages which are independent of \( A, B \). The \( s^{th} \) server computes the answer \( Y_s \), which is a function of all information available to it. For effective straggler (e.g., Server \( S \) in the figure) mitigation, upon downloading answers from any \( R \) servers, where \( R < S \), the master must be able to recover the product \( AB = \{A^{(1)}B^{(1)}, A^{(2)}B^{(2)}, \ldots, A^{(L)}B^{(L)}\} \).

For privacy, the master must not gain any additional information about \( A, B \) beyond the desired product \( AB \).

1) Sharing: In this phase, each source independently and privately generates a set of random matrices, denoted as \( Z^A \) and \( Z^B \), and encodes its matrices using its private randomness according to the functions \( f = (f_1, f_2, \ldots, f_S) \) and \( g = (g_1, g_2, \ldots, g_S) \), where \( f_s \) and \( g_s \) correspond to the \( s^{th} \) server, \( s \in [S] \). Specifically, let us denote the encoded matrices for the \( s^{th} \) server as \( \bar{A}^s \) and \( \bar{B}^s \), so we have

\[ \bar{A}^s = f_s(A, Z^A) \quad \text{(1)} \]

\[ \bar{B}^s = g_s(B, Z^B). \quad \text{(2)} \]

The encoded matrices, \( \bar{A}^s, \bar{B}^s \), are sent to the \( s^{th} \) server.

2) Computation and Communication: In this phase, servers may send some messages to other servers, and process what they received from both the sources and other servers. Denote \( M_{s \rightarrow s'} \) as the messages that server \( s \) sends to server \( s' \). Define \( M_s = \{M_{s' \rightarrow s}, s' \in [S] \setminus \{s\}\} \) as the messages that server \( s \) receives from other servers, and \( M = \{M_s, s \in [S]\} \) as the total messages that all servers receive. Upon receiving the encoded matrices from sources and messages from other servers, each server \( s \), prepares (computes) a response \( Y_s \) and sends it to the master. \( Y_s \) is a function of \( \bar{A}^s, \bar{B}^s \) and \( M_s \), i.e.,

\[ Y_s = h_s(\bar{A}^s, \bar{B}^s, M_s) \quad \text{(3)} \]

where \( h_s, s \in [S] \) are the functions used to produce the answer, and we denote them collectively as \( h = (h_1, h_2, \ldots, h_S) \).

3) Reconstruction: In this phase, the master downloads information from servers. Some servers may fail to respond (or respond after the master executes the reconstruction), such servers are called stragglers. The master decodes the sequence of product matrices \( AB \) based on the information from the responsive servers, using a class of decoding functions (denoted \( d \)). Define \( d = \{d_R : R \subset [S]\} \) where \( d_R \) is the decoding function used when the set of responsive servers is \( R \).

This scheme must satisfy three constraints.

**Correctness:** The master must be able to recover the desired products \( AB \), i.e.,

\[ H(AB | Y_R) = 0, \quad \text{(4)} \]
Define the recovery threshold $R$ among the servers, and download cost of the master. The (normalized)

$$R \subset \text{for any}$$

products from the answers obtained from any

$$r$$

these three constraints. An SMBMM code is said to be

$$\text{-recoverable if the master is able to recover the desired}$$

$$\text{products}$$

$$\text{matrix-partitioning solutions per matrix multiplication.}$$

III. MAIN RESULT

Our main result appears in the following theorem.

3We normalize source upload cost with the number of elements contained in the constituent matrices $A, B$. The server communication cost and master download cost are normalized by the number of elements contained in the desired product $AB$. 

or equivalently $AB = d_R(Y_R)$, for some $\mathcal{R}$.

Security & Strong Security: The servers must remain oblivious to the content of the data $A, B$, even if up to $X(\geq 1)$ of them collude. Formally, $\forall X \subset [S], |X| \leq X,$

$$I(A, B; A^X, B^X, M_X) = 0,$$  \hspace{1cm} (5)

where $A^X = \{A^s, s \in X\}$, $B^X = \{B^s, s \in X\}$. This is the privacy for workers in \[40\].

In this paper, strong security is also considered. It requires that the information transmitted among servers is independent of data $A, B$ and all the shares $\bar{A}^{|S|}, \bar{B}^{|S|}$, i.e.,

$$I(A, B, \bar{A}^{|S|}, \bar{B}^{|S|}; M) = 0.$$  \hspace{1cm} (6)

This property makes it possible that inter-server communications happen before receiving data from sources, and makes the server communication network topology more flexible. Note that PS does not satisfy strong security because in the PS scheme, $H(AB \mid M) = 0$.

Privacy: The master must not gain any additional information about $A, B$, beyond the required product. Precisely,

$$I(A, B; Y_1, Y_2, \cdots, Y_S \mid AB) = 0.$$  \hspace{1cm} (7)

This is the privacy for the master in \[40\].

We say that $(f, g, h, d)$ form an SMBMM (Secure coded Multi-party Batch Matrix Multiplication) code if it satisfies these three constraints. An SMBMM code is said to be $r$-recoverable if the master is able to recover the desired products from the answers obtained from any $r$ servers. In particular, an SMBMM code $(f, g, h, d)$ is $r$-recoverable if for any $\mathcal{R} \subset [S], |\mathcal{R}| = r$, and for any realization of $A, B$, we have

$$AB = d_R(Y_R).$$  \hspace{1cm} (8)

Define the recovery threshold $R$ of an SMBMM code $(f, g, h, d)$ to be the minimum integer $r$ such that the SMBMM code is $r$-recoverable.

The communication cost of an SMBMM code is comprised of these parts: upload cost of the sources, communication cost among the servers, and download cost of the master. The (normalized)$^4$ upload costs $U_A$ and $U_B$ are defined as follows.

$$U_A = \frac{\sum_{s \in [S]} |\bar{A}^s|}{L\lambda \kappa}, \hspace{0.5cm} U_B = \frac{\sum_{s \in [S]} |\bar{B}^s|}{L\kappa \mu}.$$  \hspace{1cm} (9)

Similarly, the (normalized) server communication cost $CC$ and download cost $D$ are defined as follows.

$$CC = \frac{|M|}{L\lambda \mu}, \hspace{0.5cm} D = \max_{\mathcal{R}, \mathcal{R} \subset [S], |\mathcal{R}| = r} \frac{\sum_{s \in \mathcal{R}} |Y_s|}{L\lambda \mu}.$$  \hspace{1cm} (10)

Next let us consider the complexity of encoding, decoding and server computation. Define the (normalized) computational complexity at each server, $C_s$, to be the order of the number of arithmetic operations required to compute the function $h_s$ at each server, normalized by $L$. Similarly, define the (normalized) encoding computational complexity $C_{eA}$ for $\bar{A}^{|S|}$ and $C_{eB}$ for $\bar{B}^{|S|}$ as the order of the number of arithmetic operations required to compute the functions $f$ and $g$, respectively, each normalized by $L$. Finally, define the (normalized) decoding computational complexity $C_d$ to be the order of the number of arithmetic operations required to compute $d_R(Y_R)$, maximized over $\mathcal{R}, \mathcal{R} \subset [S], |\mathcal{R}| = R$, and normalized by $L$. Note that normalization by batch-size $L$ is needed to have fair comparisons between batch processing approaches and individual matrix-partitioning solutions per matrix multiplication.
Theorem 1. For SMBMM over a field $\mathbb{F}$ with $S$ servers, $X$-security, and positive integers $(\ell, K_c, p, m, n)$ such that $m | \lambda$, $p | n$, $\mu$ and $L = \ell K_c \leq |\mathbb{F}| - S$, the GCSA-NA scheme presented in Section V is a solution, and its recovery threshold, cost, and complexity are listed as follows.

**Recovery Threshold:** $R = pmn(\ell + 1)K_c + 2X - 1,$

**Source Upload Cost of $\tilde{A}^{[S]}$, $\tilde{B}^{[S]}$:**

$$(U_A, U_B) = \left( \frac{S}{K_cpn}, \frac{S}{K_cpn} \right),$$

**Server Communication Cost:**

$$CC = \frac{S - 1}{\ell K_c mn},$$

**Master Download Cost:**

$$D = \frac{R}{\ell K_c mn},$$

**Source Encoding Complexity for $\tilde{A}^{[S]}$, $\tilde{B}^{[S]}$:**

$$C_{cA}, C_{cB} = \left( \frac{\lambda \mu S \log^2 S}{K_c pm}, \frac{\lambda \mu S \log^2 S}{K_c pm} \right),$$

**Server Computation Complexity:**

$$C_s = \mathcal{O}\left( \frac{\lambda \mu S \log^2 S}{K_c pm} \right),$$

**Master Decoding Complexity:**

$$C_d = \mathcal{O}\left( \lambda \mu p \log^2 R \right).$$

The following observations place the result of Theorem 1 in perspective.

1) GCSA-NA codes are based on the construction of GCSA codes from [30], combined with the idea of noise alignment (e.g., [45]). In turn, GCSA codes are based on a combination of CSA codes for batch processing [30] and EP codes for matrix partitioning [5]. CSA codes are themselves based on the idea of cross-subspace alignment (CSA) that was introduced in the context of secure PIR [37]. It is a remarkable coincidence that while the idea of CSA originated in the context of PIR [37], and Lagrange Coded Computing (also a batch processing approach) was introduced in parallel independently in [6] for the context of coded computing, the two approaches are essentially identical, with CSA codes being slightly more powerful in the context of coded distributed matrix multiplication (CSA codes offer additional improvements over LCC codes in terms of download cost [30]). Indeed, LCC codes for batch matrix multiplication are recovered as a special case of CSA codes.

2) The idea of noise alignment can also be similarly applied to the $N$-CSA codes that are introduced in Section 6 in [30], for $N$-source secure coded multi-party batch matrix computation.

3) By setting $K_c = 1$, $\ell = L$ and $S = R$, the construction of GCSA-NA codes, with a straightforward generalization, can be further modified to settle the asymptotic (the number of messages goes to infinity) capacity of symmetric $X$-secure $T$-private computation (and also the corresponding private information retrieval setting) [37]. However, the amount of randomness required by the construction is not necessarily optimal. For example, it is shown in [37] that by the achievable scheme for XSTPIR, symmetric security (privacy) is automatically satisfied when $T = 1$, i.e., no randomness among servers is required.

4) A side-by-side comparison of the GCSA-NA solution with polynomial sharing (PS) is presented in Table II. Evidently, GCSA-NA codes have significant advantages over PS codes. These advantages are discussed next. Remarkably, the next six listed advantages of GCSA-NA over PS hold even without batch processing, i.e., even if we set batch size $L = 1$ (which implies $\ell = K_c = 1$ in Table II).

| Polynomial Sharing (PS [40]) | GCSA-NA |
|-----------------------------|---------|
| **Strong Security**         | No      | Yes |
| **Recovery Threshold (R)**  | $2pmn + 2X - 1$ | $pmn(\ell + 1)K_c + 2X - 1$ |
| **Straggler Tolerance**     | Complete Graph | Yes. Tolerates $S - \ell$ stragglers |
| **Server Network Topology** | Complete Graph | Any Connected Graph |
| **Source Encoding Complexity** | $(\frac{\lambda \mu S \log^2 S}{pm}, \frac{\lambda \mu S \log^2 S}{pm})$ | $(\frac{\lambda \mu S \log^2 S}{K_c pm}, \frac{\lambda \mu S \log^2 S}{K_c pm})$ |
| **Source Upload Cost (U_A, U_B)** | $\frac{S}{pm}, \frac{S}{pm}$ | $\frac{S}{K_c pm}, \frac{S}{K_c pm}$ |
| **Server Communication Cost (CC)** | $\frac{S(\ell - 1)}{mn}$ | $\frac{S - 1}{\ell K_c mn}$ |
| **Server Computation Complexity** | $\mathcal{O}\left( \frac{\lambda \mu}{pm} \right) + \mathcal{O}\left( \frac{\lambda \mu}{pm} \right)$ | $\mathcal{O}\left( \frac{\lambda \mu}{K_c pm} \right)$ |
|                             | $\mathcal{O}\left( \frac{\lambda \mu}{pm} \right) + \mathcal{O}\left( \frac{\lambda \mu}{pm} \right)$ | $\mathcal{O}\left( \frac{\lambda \mu}{K_c pm} \right) + \mathcal{O}\left( \frac{\lambda \mu}{K_c pm} \right) + \mathcal{O}\left( \frac{\lambda \mu S \log^2 S}{K_c pm} \right)$ |
| **Master Download Cost (D)** | $\frac{mn + X}{mn}$ | $\frac{R}{\ell K_c mn}$ |
| **Master Decoding Complexity** | $\mathcal{O}\left( \frac{\lambda \mu S \log^2 (mn + X)}{mn} \right)$ | $\mathcal{O}\left( \lambda \mu p \log^2 R \right)$ |

**TABLE I:** Performance Comparison of Polynomial Sharing (PS) and GCSA with Noise Alignment (GCSA-NA).
5) Because all inter-server communication is independent of input data, GCSA-NA schemes are strongly secure, i.e., even if all inter-server communication is leaked it does not compromise the security of input data.

6) Because inter-server communication leaks no information about the input data, GCSA-NA codes allow a much larger class of inter-server network topologies compared to PS codes. In GCSA-NA the inter-server network graph can be any connected graph. This is not possible with PS. For example, if the inter-server network graph is a star graph, i.e., one of the servers acts as a hub through which all inter-server communication must pass, then the hub server can decode $AB$ by monitoring all the inter-server communication in a PS scheme, violating the security constraint. However, even a star network poses no security threat in a GCSA-NA scheme, because the inter-server communication does not depend on the input data.

7) Because inter-server communication does not depend on input data in a GCSA-NA scheme, all inter-server communication can take place during off-peak hours, even before the input data is generated. In contrast, note that inter-server communication in the PS scheme depends on the input data, so it cannot be done beforehand. This allows the GCSA-NA scheme a significant latency advantage over the PS scheme.

8) The high cost of inter-server communication is one of the main limitations of PS because every server must communicate with every server, i.e., $S(S-1)$ such inter-server communications must take place. However, GCSA-NA only requires $S-1$ inter-server communications to propagate structured noise terms across all servers. Thus, the server communication cost of GCSA-NA shows an order-wise (linear in $S$ vs quadratic in $S$) improvement over PS. This improvement is shown numerically in Fig. 2a.

9) Besides the lower server-communication cost, it is notable that the server computation complexity is also lower for the GCSA-NA scheme than the PS scheme. This is because in PS, each server needs to multiply the two shares received from the sources, calculate the shares for every other server and sum up all the shares from every other server. All of this is done in real (peak) time. However, in GCSA-NA, each server only needs to multiply the two shares received from the sources and add noise (which can be precomputed during off-peak hours). This advantage is particularly significant for large number of servers.

10) The GCSA-NA scheme naturally allows robustness to stragglers, which is particularly important for massive matrix multiplications. Stragglers can be an especially significant concern for PS because of the strongly sequential nature of multi-round computation that is central to PS. This is because server failures between computation rounds disrupt the computation sequence. Remarkably, Fig. 2a shows that the inter-server communication cost of GCSA-NA is significantly better than PS even when GCSA-NA accommodates stragglers (while PS does not).

11) Now, let us consider batch processing, i.e., batch size $L > 1$. For these comparisons, let us set $L = K_c, \ell = 1$. PS can be applied to batch processing by repeating the scheme $L$ times. Fig. 2b shows that the normalized server communication cost of GCSA-NA decreases as $L$ increases and is significantly less than that in PS.

12) Next, consider source upload cost with batch processing. For the same number of servers $S$, the upload cost
of GCSA-NA is smaller by a factor of $1/K_c$ compared to PS.

13) Finally, let us consider the master download cost and decoding complexity. Essentially, GCSA-NA has higher download cost and decoding complexity than PS by approximately a factor of $p$, which depends on how the matrices are partitioned. If $p$ is a small value, e.g., $p = 1$, then the costs are quite similar. The improvement in download cost and decoding complexity of PS by a factor of $1/p$ comes at the penalty of increased inter-server communication cost by a factor of $S$. But since $S \geq R \geq 2pmn + 2X - 1 \geq p$, and typically $S \gg p$, the improvement is dominated by the penalty, so that overall the communication cost of PS is still significantly larger than GCSA-NA. For example, let us consider the total communication cost of PS versus GCSA-NA with $m = n, \lambda = \kappa = \mu, L = K_c, \ell = 1$. For PS the total communication cost is at least $\frac{2SN\lambda^2}{pm} + \frac{(S-1)\lambda^2}{m^2} + \frac{(m^2+X)\lambda^2}{m^2}$ which is dominated by $\frac{(S-1)\lambda^2}{m^2}$, the inter-server communication cost. For GCSA-NA, the total communication cost is no more than $\frac{2SN\lambda^2}{L_{pm}} + \frac{(S-1)\lambda^2}{L_{m^2}} \leq \frac{2SN\lambda^2}{L_{pm}} + \frac{(m^2+X)\lambda^2}{m^2}$, which is significantly lower for typical values of $S \gg p$.

IV. TOY EXAMPLE

Let us consider a toy example with parameters $\lambda = \kappa = \mu, m = n = 1, p = 2, l = 1, K_c = 2, X = 1$ and $S = R$. Suppose matrices $A, B \in \mathbb{F}^{\lambda \times \lambda}$, and we wish to multiply matrix $A = [A_1 \ A_2]$ with matrix $B = [B_1 \ B_2]$ to compute the product $AB = A_1B_1 + A_2B_2$, where $A_1, A_2 \in \mathbb{F}^{\lambda \times \frac{\lambda}{2}}, B_1, B_2 \in \mathbb{F}^{\frac{\lambda}{2} \times \lambda}$. For this toy example we summarize both the Polynomial Sharing approach [40, 46, 47], and our GCSA-NA approach.

A. POLYNOMIAL SHARING SOLUTION

Polynomial sharing is based on EP code [5]. The given partitioning corresponds to EP code construction for $m = n = 1, p = 2$, and we have

$$P = A_1 + \alpha A_2 \quad (11)$$
$$Q = \alpha B_1 + B_2 \quad (12)$$
$$\implies PQ = A_1B_2 + \alpha (A_1B_1 + A_2B_2) + \alpha^2 A_2B_1. \quad (13)$$

1) Guaranteeing $X$ Security for Inputs against Servers: To satisfy $X = 1$ security, PS includes noise with each share, i.e.,

$$\tilde{A} = P + \alpha^2 Z^A \quad (14)$$
$$\tilde{B} = Q + \alpha^2 Z^B, \quad (15)$$

where $\alpha, \tilde{A}, \tilde{B}$ are generic variables that should be replaced with $\alpha_s, \tilde{A}^s, \tilde{B}^s$ for Server $s$, and $\alpha_1, \alpha_2, \ldots, \alpha_S$ are distinct elements of $\mathbb{F}$. Each server computes the product of the shares that it receives, i.e.,

$$\tilde{A}\tilde{B} = PQ + \alpha^2 PZ^B + \alpha^2 Z^AQ + \alpha^4 Z^A Z^B$$
$$= A_1B_2 + \alpha (A_1B_1 + A_2B_2) + \alpha^2 (A_2B_1 + A_1B_2 + Z^B Z^2B_2) + \alpha^3 (A_2Z^B + Z^A B_1) + \alpha^4 Z^A Z^B. \quad (16)$$

2) Securing Inputs from the Master Node: To secure inputs from the master, PS makes every server send to the master only the desired term $A_1B_1 + A_2B_2$ by using secret sharing scheme among servers. Since $deg_s(\tilde{A}\tilde{B}) = 4$, $A_1B_1 + A_2B_2$ can be calculated from 5 distinct $\tilde{A}\tilde{B}$ according to the Lagrange interpolation rules. In particular, there exist 5 constants $r_1, \ldots, r_5$, such that

$$A_1B_1 + A_2B_2 = \sum_{s \in [5]} r_s \tilde{A}^s \tilde{B}^s. \quad (18)$$

Consider Server $s$, it sends $M_{s \rightarrow j} = r_s A_j^s \tilde{B}^s + \alpha_s Z_s$ to server $j$, for all $j \in [5] \setminus s$ and keeps $M_{s \rightarrow s} = r_s A_s^s \tilde{B}^s + \alpha_s Z_s$ by itself, where $Z_1, \ldots, Z_5$ are i.i.d. uniform noise matrices over $\mathbb{F}^{\lambda \times \lambda}$. After Server $s$ collects all the shares $M_{j \rightarrow s}$, for all $j \in [5]$, it sums up these shares

$$Y_s = \sum_{j \in [5]} M_{j \rightarrow s} = \sum_{j \in [5]} r_j A^j B^j + \alpha_s \sum_{j \in [5]} Z_j = A_1B_1 + A_2B_2 + \alpha_s \sum_{j \in [5]} Z_j, \quad (19)$$

and sends it to the master. Note that after receiving $M_{j \rightarrow s}$ for all $j \in [5]$, Server $s$ still gains no information about the input data, which guarantees the security. However, it does not satisfy the strong security constraint, because $AB$ can be decoded based on $M_{j \rightarrow s}$ for all $j, s \in [5]$. 


The master can decode the desired $\mathbf{AB}$ after collecting 2 responses from servers.\footnote{In \cite{47}, for arbitrary polynomials, $M_{\alpha\rightarrow j} = r_{\alpha} \mathbf{A}^{\alpha} + \alpha^2 \mathbf{Z}$, because $Y_s$ is forced to be casted in the form of entangled polynomial sharing. Therefore, the master needs 3 responses.} Note that in this scheme, at least 5 servers are needed, since it takes 5 distinct $\mathbf{AB}$ to obtain the desired term $Y_s$.

B. GCSA-NA Solution

Let us also start with EP code constructions \cite{11} and \cite{12}. GCSA codes \cite{30} can handle batch processing, therefore let us consider two instances of each ($\ell = 1, K_c = 2$).

$$P = A_1 + (f - \alpha)A_2, \quad Q = (f - \alpha)B_1 + B_2. \quad (20)$$

$$P' = A'_1 + (f' - \alpha)A'_2, \quad Q' = (f' - \alpha)B'_1 + B'_2. \quad (21)$$

Denote the two batches (instances) without and with primes in the superscript, respectively. Using CSA code principles, these instances are coded as follows.

$$P = A_1' + (f - \alpha)A_2', \quad Q = (f - \alpha)B_1' + B_2'. \quad (22)$$

$$P' = A'_1' + (f' - \alpha)A'_2', \quad Q' = (f' - \alpha)B'_1' + B'_2'. \quad (23)$$

and the shares are constructed as follows,

$$\mathbf{A} = \Delta \left( \frac{P}{(f - \alpha)^2} + \frac{P'}{(f' - \alpha)^2} \right) \quad (24)$$

$$\mathbf{B} = \left( \frac{Q}{(f - \alpha)^2} + \frac{Q'}{(f' - \alpha)^2} \right) \quad (25)$$

where

$$\Delta = (f - \alpha)^2(f' - \alpha)^2, \quad (26)$$

and $\alpha, \mathbf{A}, \mathbf{B}$ are generic variables that should be replaced with $\alpha_s, \mathbf{A}^s, \mathbf{B}^s$ for server $s$. Furthermore, $f, f', \alpha_1, \alpha_2, \ldots, \alpha_S$ are distinct elements of $\mathbb{F}$. Each server computes the product of the shares that it receives, i.e.,

$$\mathbf{A} \mathbf{B} = \frac{c_0}{(f - \alpha)^2} P Q + \frac{c_1}{(f - \alpha)^2} P Q' + \frac{c_0'}{(f' - \alpha)^2} P' Q' + \frac{c_1'}{(f' - \alpha)^2} P' Q + I_0 + \alpha I_1 + \alpha^2 I_2, \quad (27)$$

where $c_0, c_1, c'_0, c'_1$ are constants. Note\footnote{We use the notation $I_0 + \alpha I_1 + \alpha^2 I_2$, etc., only to highlight the dependence on $\alpha$ of the unnecessary (interference) terms. The precise values for $I_0, I_1, I_2$ are not important, and may change from one step to another as we manipulate various terms. This is done to suppress irrelevant details that could otherwise make the derivation extremely tedious. In other words, the notation $I_0 + \alpha I_1 + \alpha^2 I_2$ should be interpreted as 'some terms that do not depend on $\alpha$'+$\alpha$(some other terms that do not depend on $\alpha$)+$\alpha^2$(some more terms that do not depend on $\alpha$).} that $I_0$ contains contributions from all of $PQ, P'Q', PQ', P'Q$, but $I_1, I_2$ contain only contributions from $PQ', P'Q$. Expanding, we have

$$\mathbf{A} \mathbf{B} = \frac{c_0 A_1 B_2}{(f - \alpha)^2} + \frac{c_0 A_1 B_1 + c_0 A_2 B_2 + c_1 A_1 B_2}{f - \alpha} + \frac{c_0' A_1' B'_2}{(f' - \alpha)^2} + \frac{c_0' A_1' B'_1 + c_0' A_2' B'_2 + c_1' A_1' B'_2}{f' - \alpha} + I_0 + \alpha I_1 + \alpha^2 I_2 \quad (28)$$

Note that for this construction we need $R = pmn((\ell + 1)K_c - 1) + p - 1 = 7$ responses to recover each coefficient. Then from $A_1 B_2$ and $c_0 A_1 B_1 + c_0 A_2 B_2 + c_1 A_1 B_2$, we obtain $A_1 B_1 + A_2 B_2$. This is the original GCSA scheme of \cite{30}.

1) Guaranteeing $X$ Security for Inputs against Servers: Next, let us modify the scheme to make it $X = 1$ secure by including noise with each share, i.e.,

$$\mathbf{A} = \Delta \left( \frac{P}{(f - \alpha)^2} + \frac{P'}{(f' - \alpha)^2} + \mathbf{Z}^A \right), \quad (29)$$

$$\mathbf{B} = \left( \frac{Q}{(f - \alpha)^2} + \frac{Q'}{(f' - \alpha)^2} + \mathbf{Z}^B \right). \quad (30)$$

And we have

$$\mathbf{A} \mathbf{B} = \left( \frac{(f' - \alpha)^2}{(f - \alpha)^2} P Q + \frac{(f - \alpha)^2}{(f' - \alpha)^2} P' Q' + I_0 + \alpha I_1 + \alpha^2 I_2 + \alpha^3 I_3 + \alpha^4 I_4 \right) \quad (32)$$

$$= \frac{c_0 P Q}{(f - \alpha)^2} + \frac{c_1 P Q'}{f - \alpha} + \frac{c_0' P' Q'}{(f' - \alpha)^2} + \frac{c_1' P' Q'}{f' - \alpha} + I_0 + \alpha I_1 + \alpha^2 I_2 + \alpha^3 I_3 + \alpha^4 I_4. \quad (33)$$

Note that as a result of the added noise terms, the recovery threshold is now increased to $R = 9$. Also note that the term $I_4$ contains only contributions from $\Delta \mathbf{Z}^A \mathbf{Z}^B$, i.e., this term leaks no information about $\mathbf{A}, \mathbf{B}$ matrices.
2) Securing Inputs from the Master Node: Note that if the servers directly return their computed values of $\tilde{A}\hat{B}$ to the master node, then besides the result of the computation some additional information about the input matrices $A, B$ may be leaked by the terms

$$
\left(\frac{c_0}{(f-\alpha)^2} + \frac{c_1}{f-\alpha}\right) A_1 B_2 + \left(\frac{c_0'}{(f'-\alpha)^2} + \frac{c_1'}{f'-\alpha}\right) A'_1 B_2' + I_0 + \alpha I_1 + \alpha^2 I_2 + \alpha^3 I_3
$$

which can be secured by the addition of aligned noise terms

$$
\tilde{Z} = \left(\frac{c_0}{(f-\alpha)^2} + \frac{c_1}{f-\alpha}\right) Z + \left(\frac{c_0'}{(f'-\alpha)^2} + \frac{c_1'}{f'-\alpha}\right) Z' + Z_0 + \alpha Z_1 + \alpha^2 Z_2 + \alpha^3 Z_3
$$

at each server so that the answer returned by each server to the master node is $\tilde{A}\hat{B} + \tilde{Z}$. Here $Z, Z', Z_0, Z_1, Z_2, Z_3$ are i.i.d. uniform noise matrices over $\mathbb{F}^{\times \times}$, that can all be privately generated by one server, who can then share their aligned form $\tilde{Z}$ with all other servers. This sharing of $\tilde{Z}$ is the only inter-server communication needed in the GCSA-NA scheme, and since it does not depend on the data inputs, it can be done beforehand, during off-peak hours, thereby reducing the latency of server computation. The strong security is also automatically satisfied.

V. CONSTRUCTION OF GCSA-NA

Now let us present the general construction. Let $L = \ell K_c$ instances of $A$ and $B$ matrices are split into $\ell$ groups. $\forall l \in [\ell], \forall k \in [K_c]$, denote

$$
A^{l,k} = A^{(K_c(l-1)+k)}, \quad B^{l,k} = B^{(K_c(l-1)+k)}.
$$

Further, each matrix $A^{l,k}$ is partitioned into $m \times p$ blocks and each matrix $B^{l,k}$ is partitioned into $p \times n$ blocks, i.e.,

$$
A^{l,k} = 
\begin{bmatrix}
A_{l,k}^{1,1} & A_{l,k}^{1,2} & \cdots & A_{l,k}^{1,p}

A_{l,k}^{2,1} & A_{l,k}^{2,2} & \cdots & A_{l,k}^{2,p}

: & : & \cdots & :

A_{l,k}^{m,1} & A_{l,k}^{m,2} & \cdots & A_{l,k}^{m,p}
\end{bmatrix},

B^{l,k} = 
\begin{bmatrix}
B_{l,k}^{1,1} & B_{l,k}^{1,2} & \cdots & B_{l,k}^{1,n}

B_{l,k}^{2,1} & B_{l,k}^{2,2} & \cdots & B_{l,k}^{2,n}

: & : & \cdots & :

B_{l,k}^{p,1} & B_{l,k}^{p,2} & \cdots & B_{l,k}^{p,n}
\end{bmatrix},
$$

where $(A^{l,k}_{i,j})_{i \in [m], j \in [p]} \in \mathbb{F}^{m \times p}$ and $(B^{l,k}_{i,j})_{i \in [m], j \in [p]} \in \mathbb{F}^{n \times n}$.

Let $f_{1,1}, f_{1,2}, \cdots, f_{\ell, K_c}, \alpha_1, \alpha_2, \cdots, \alpha_S$ be $(S + L)$ distinct elements from the field $\mathbb{F}$. For convenience, define

$$
R' = mn,
$$

$$
D_E = \max(pm, pmn - pm + p) - 1,
$$

$$
\mathcal{E} = \{p + p(m'-1) + pm(n'-1) \mid m' \in [m], n' \in [n]\},
$$

$$
\Delta_{K_c} = \prod_{k \in [K_c]} (f_{1,k} - \alpha_s)^{R'}, \forall l \in [\ell], \forall s \in [S].
$$

Define $c_{l,k,i} \in \{0, 1, \cdots, R'(K_c - 1)\}$ to be the coefficients satisfying

$$
\Psi_{l,k}(\alpha) = \prod_{k \in [K_c] \setminus \{k\}} (\alpha + (f_{1,k} - f_{1,l}))^{R'} = \sum_{i=0}^{R'(K_c-1)} c_{l,k,i} \alpha^i, \forall l \in [\ell], \forall k \in [K_c]
$$

i.e., they are the coefficients of the polynomial $\Psi_{l,k}(\alpha) = \prod_{k \in [K_c] \setminus \{k\}} (\alpha + (f_{1,k} - f_{1,l}))^{R'}$, which is defined by its roots. Note that all the coefficients $\alpha_{[1:S], \{f_{1,k} \in [L], k \in [K]\}, \{c_{l,k,i} \in [L], k \in [K_c], i \in [0,1, \cdots, R'(K_c-1)\}}$ are globally known.

A. Sharing

Firstly, each source encodes each constituent matrix blocks $A^{l,k}$ and $B^{l,k}$ with Entangled Polynomial code. For all $l \in [\ell], k \in [K_c]$, define

$$
P_{s}^{l,k} = \sum_{m' \in [m]} \sum_{p' \in [p]} A_{m',p'}^{l,k}(f_{1,k} - \alpha_s)^{p'-1+p(m'-1)},
$$

$$
Q_{s}^{l,k} = \sum_{p' \in [p]} \sum_{n' \in [n]} B_{p',n'}^{l,k}(f_{1,k} - \alpha_s)^{p-p'n'+pm(n'-1)}.
$$

Note that the original Entangled Polynomial code can be regarded as polynomials of $\alpha_s$, and here for each $(l, k)$, Entangled Polynomial code is constructed as polynomials of $(f_{1,k} - \alpha_s)$. 

Each source generates $\ell X$ independent random matrices, i.e., $Z^A = \{Z_{1,1}^A, \ldots, Z_{1,X}^A, Z_{2,1}^A, \ldots, Z_{2,X}^A\}$ and $Z^B = \{Z_{1,1}^B, \ldots, Z_{2,1}^B, \ldots, Z_{2,X}^B\}$, where $\forall l \in [\ell], \forall x \in [X], Z_{l,x}^A$ is uniformly distributed over $\mathbb{F}^{m \times \frac{n}{2}}$ and $Z_{l,x}^B$ is uniformly distributed over $\mathbb{F}^{\frac{m}{2} \times \frac{n}{2}}$. The independence is established as follows.

$$H(Z^A, Z^B, A, B) = H(A) + H(B) + \sum_{l \in [\ell], x \in [X]} H(Z_{l,x}^A) + \sum_{l \in [\ell], x \in [X]} H(Z_{l,x}^B). \tag{44}$$

For all $s \in [S]$, the shares of matrices $A$ and $B$ at the $s^{th}$ server are constructed as follows.

$$\ddot{A}_s = (\ddot{A}_1^s, \ddot{A}_2^s, \ldots, \ddot{A}_\ell^s), \tag{45}$$
$$\ddot{B}_s = (\ddot{B}_1^s, \ddot{B}_2^s, \ldots, \ddot{B}_\ell^s), \tag{46}$$

where for all $l \in [\ell]$,

$$\ddot{A}_l^s = \Delta^{s}_{l,K_c} \left( \sum_{k \in [K_c]} \frac{P^{l,k}_\ell}{(f_l k - \alpha_s)^R} + \sum_{x \in [X]} \alpha_s^{-1} Z_{l,x}^A \right), \tag{47}$$
$$\ddot{B}_l^s = \sum_{k \in [K_c]} \frac{Q^{l,k}_\ell}{(f_l k - \alpha_s)^R} + \sum_{x \in [X]} \alpha_s^{-1} Z_{l,x}^B. \tag{48}$$

Then each pair of shares $\ddot{A}_s, \ddot{B}_s$ is sent to the corresponding server.

B. Computation and Communication

One of the servers generates a set of $\frac{m}{\ell} \times \frac{n}{\ell}$ matrices $Z^{server}_i$, which contains $R'(K_c - 1) + X + D_E + \ell K_c(p - 1)mn$ random matrices and $\ell K_c mn$ zero matrices. In particular, $Z^{server}_0 = \{Z_i^0 | i \in [R'(K_c - 1) + X + D_E]\}$, and $Z^{server}_2 = \{Z_i^{l,k,i} | l \in [\ell], k \in [K_c], i \in [R']\}$. Here,

$$Z_{l,k,i}^{s} = \begin{cases} 0, & \text{if } i \in \mathcal{E} \\ Z_{l,k,i}, & \text{otherwise}, \forall l \in [\ell], \forall k \in [K_c]. \end{cases}$$

$\forall i \in [R'(K_c - 1) + X + D_E], Z_i^s$ is uniformly distributed over $\mathbb{F}^{\frac{m}{\ell} \times \frac{n}{\ell}}$ and $\forall l \in [\ell], \forall k \in [K_c], \forall i \in [R'] \setminus \mathcal{E}, Z_{l,k,i}^{s}$ is uniformly distributed over $\mathbb{F}^{\frac{m}{\ell} \times \frac{n}{\ell}}$, $0$ is a zero matrix. The independence is established as follows.

$$H(Z^{server}, A, B) = H(A) + H(B) + \sum_{i \in [R'(K_c - 1) + X + D_E]} H(Z_i^s) + \sum_{l \in [\ell], k \in [K_c], i \in [R']} H(Z_{l,k,i}^s). \tag{49}$$

Without loss of generality, assume the first server generates $Z^{server}_0$, encodes them into

$$\ddot{M}_s = \sum_{i \in [R'(K_c - 1) + X + D_E]} \alpha_s^{-1} Z_i^s + \sum_{l \in [\ell]} \sum_{k \in [K_c]} \sum_{i = 0}^{R'-1} \sum_{i' = 0}^{R'-1} \frac{\alpha_s^{i} \alpha_{s}^{-i'} Z_{l,k,i+1}^{s}}{(f_l k - \alpha_s)^R} \tag{50}$$

and sends $\ddot{M}_s$ to server $s, s \in [S] \setminus \{1\}$, where $\{(c_{l,k,i})_{l \in [\ell], k \in [K_c], i \in [0, 1, \ldots, R'(K_c - 1)]}\}$ is defined in $[41]$. The answer returned by the $s^{th}$ server to the master is constructed as

$$Y_s = \sum_{l \in [\ell]} A_l^s B_l^s + M_s.$$

C. Reconstruction

After the master collects any $R$ answers, it decodes the desired products $AB$. The decodability is proved in the next section.

D. Proof

To begin, let us recall the standard result for Confluent Cauchy-Vandermonde matrices $[48]$, replicated here for the sake of completeness.

Lemma 1. If $f_1, f_2, \ldots, f_\ell, K_c, \alpha_1, \alpha_2, \ldots, \alpha_R$ are $R + L$ distinct elements of $\mathbb{F}$, with $|\mathbb{F}| \geq R + L$, $L = \ell K_c$ and $R = R'(\ell + 1)K_c + 2X - 1$, then the $R \times R$ Confluent Cauchy-Vandermonde matrix $[51]$ is invertible over $\mathbb{F}$.

$$\tilde{V}_{\ell, K_c, R', X, R} \triangleq \begin{bmatrix} \frac{1}{(f_1 - \alpha_1)^R} & \cdots & \frac{1}{f_1 - \alpha_1} & \cdots & \frac{1}{f_1 - \alpha_1} & \cdots & \frac{1}{f_1 - \alpha_1} & \cdots & 1 & \cdots & \alpha_1^{R'(K_c+2X-2)} \\ \frac{1}{(f_1 - \alpha_2)^R} & \cdots & \frac{1}{(f_1 - \alpha_2)^R} & \cdots & \frac{1}{(f_1 - \alpha_2)^R} & \cdots & \frac{1}{(f_1 - \alpha_2)^R} & \cdots & 1 & \cdots & \alpha_2^{R'(K_c+2X-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(f_1 - \alpha_R)^R} & \cdots & \frac{1}{(f_1 - \alpha_R)^R} & \cdots & \frac{1}{(f_1 - \alpha_R)^R} & \cdots & \frac{1}{(f_1 - \alpha_R)^R} & \cdots & 1 & \cdots & \alpha_R^{R'(K_c+2X-2)} \end{bmatrix} \tag{51}$$
Firstly, let us prove that the GCSA-NA codes are $R = pmn(\ell + 1)K_c + 2T - 1$ recoverable. Rewrite $Y_s$ as follows.

$$Y_s = \tilde{A}_1^s \tilde{B}_1^s + \tilde{A}_2^s \tilde{B}_2^s + \cdots + \tilde{A}_l^s \tilde{B}_l^s + \tilde{M}_s$$

(52)

$$= \sum_{l \in [\ell]} \Delta_s^{l,K_c} \left( \sum_{k \in [K_c]} \frac{p_{l,k}}{(f_{l,k} - \alpha_s)^R} \right) \left( \sum_{x \in [X]} \sum_{k \in [K_c]} \frac{\alpha_s x_z^1 A_x^s}{(f_{l,k} - \alpha_s)^R} \right) + \tilde{M}_s$$

(53)

$$= \sum_{l \in [\ell]} \Delta_s^{l,K_c} \left( \sum_{k \in [K_c]} \frac{p_{l,k}}{(f_{l,k} - \alpha_s)^R} \right) \left( \sum_{x \in [X]} \sum_{k \in [K_c]} \frac{Q_s^{l,k}}{(f_{l,k} - \alpha_s)^R} \right) + \tilde{M}_s$$

(54)

In (55), we split the first term in (54) into two parts depending on whether $k = k'$. To be convenient, let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ represent the second to fifth terms in (55).

Consider the first term in (55). For each $l \in [\ell], k \in [K_c]$, we have

$$\prod_{k' \in [K_c] \setminus \{k\}} \frac{(f_{l,k'} - \alpha_s)^R}{(f_{l,k} - \alpha_s)^R} P_s^{l,k} Q_s^{l,k}$$

(56)

$$= \frac{\Psi_{l,k}(f_{l,k} - \alpha_s) P_s^{l,k} Q_s^{l,k}}{(f_{l,k} - \alpha_s)^R}$$

(57)

$$= \frac{\Psi_{l,k}(f_{l,k} - \alpha_s) P_s^{l,k} Q_s^{l,k}}{(f_{l,k} - \alpha_s)^R}$$

(58)

where (58) results from the definition of $\Psi_{l,k}(\cdot)$ and in (59) the polynomial $\Psi_{l,k}(f_{l,k} - \alpha_s)$ is rewritten in terms of its coefficients. Let $\Gamma_5$ represent the second term in (59).

Recall that by the construction of Entangled Polynomial code (32) (43), the product $P_s^{l,k} Q_s^{l,k}$ can be written as weighted sums of the terms $1, (f_{l,k} - \alpha_s), \cdots, (f_{l,k} - \alpha_s)^{R' + p - 2}$, i.e.,

$$P_s^{l,k} Q_s^{l,k} = \sum_{m' \in [m]} \sum_{p' \in [p]} \sum_{p'' \in [p']} \sum_{n' \in [n']} A^{l,k}_{m',p',p''} B^{l,k}_{p',n'} (f_{l,k} - \alpha_s)^{p + p' + p'' + pm(n'' - 1) + pm(n'' - 1) - 1}$$

(60)

where $C_{i+1}^{l,k}(f_{l,k} - \alpha_s)^i$ are various linear combinations of products of blocks of $A^{l,k}$ and blocks of $B^{l,k}$. Consider the first term in (59).

$$\prod_{i=0}^{\infty} \left( \frac{c_{l,k,0}}{(f_{l,k} - \alpha_s)^R} + \frac{c_{l,k,1}}{(f_{l,k} - \alpha_s)^{R - 1}} + \cdots + \frac{c_{l,k,R' - 1}}{(f_{l,k} - \alpha_s)} \right) P_s^{l,k} Q_s^{l,k}$$

(59)

$$= \sum_{i=0}^{\infty} C_{i+1}^{l,k}(f_{l,k} - \alpha_s)^i$$

(61)
\[
\begin{aligned}
&= \sum_{i=0}^{R'-1} \sum_{l'=0}^{i} c_{l,k,i-l'} C_{l',i+1}^{l,k} + \sum_{i=0}^{p-2} (f_{l,k} - \alpha_s)^i \left( \sum_{i'=i+1}^{R'+i'} c_{l,k,R'-i'+i} C_{i'+1}^{l,k} \right) \\
&\quad + \sum_{i=p-1}^{R'+p-3} (f_{l,k} - \alpha_s)^i \left( \sum_{i'=i+1}^{R'+p-2} c_{l,k,R'-i'+i} C_{i'+1}^{l,k} \right).
\end{aligned}
\]

Let \(\Gamma_0, \Gamma_7\) represent the second and third terms in (63). Note that if \(K_c = 1, \forall i \neq 0, c_{l,k,i} = 0, \Gamma_0\), the second term in (59) and \(\Gamma_7\), the third term in (63) are zero polynomials. Now let us consider the degree with respect to \(\alpha_s\) of \(\Gamma_1, \cdots, \Gamma_7\).

\[
\deg_{\alpha_s} (\Gamma_1) = \begin{cases} 
R'(K_c - 1) + p - 2, & \text{if } K_c > 1 \\
-1, & \text{otherwise}
\end{cases}
\]

\[
\deg_{\alpha_s} (\Gamma_2) = R'(K_c - 1) + pm + X - 2,
\]

\[
\deg_{\alpha_s} (\Gamma_3) = R'(K_c - 1) + pmn - pm + p + X - 2,
\]

\[
\deg_{\alpha_s} (\Gamma_4) = R'K_c + 2X - 2,
\]

\[
\deg_{\alpha_s} (\Gamma_5) = \begin{cases} 
R'(K_c - 1) + p - 2, & \text{if } K_c > 1 \\
-1, & \text{otherwise}
\end{cases}
\]

\[
\deg_{\alpha_s} (\Gamma_6) = p - 2,
\]

\[
\deg_{\alpha_s} (\Gamma_7) = \begin{cases} 
R' + p - 3, & \text{if } K_c > 1 \\
-1, & \text{otherwise}
\end{cases}
\]

Recall \(X, p, m, n, K_c\) are positive integers, if \(K_c > 1\), it is easy to see that \(R'K_c + 2X - 2\) is the largest. If \(K_c = 1, R' = pmn \geq p > p - 2, R'K_c + 2X - 2\) is also the largest. Therefore the sum of \(\Gamma_1, \cdots, \Gamma_7\) can be expanded into weighted sums of the terms \(1, \alpha_s, \cdots, \alpha_s^{R'K_c + 2X - 2}\). Note that the weights of terms \(\alpha_s^{R'(K_c-1)+X+D_E+1}, \cdots, \alpha_s^{R'K_c+2X-2}\) are functions of \(Z^A, Z^B\). \(Y_s\) can be rewritten as

\[
Y_s = \sum_{i \in [\ell]} \sum_{k \in [K_c]} \sum_{i=0}^{R'-1} \sum_{l'=0}^{i} c_{l,k,i-l'} C_{l',i+1}^{l,k} + \sum_{x \in [R'K_c + 2X - 1]} \alpha_s^{-1} I_x + \tilde{M}_s
\]

\[
+ \sum_{i \in [\ell]} \sum_{k \in [K_c]} \sum_{i=0}^{R'-1} \sum_{l'=0}^{i} c_{l,k,i-l'} Z_{l',i+1}'' \left( \sum_{x \in [R'K_c + 2X - 1]} \alpha_s^{-1} I_x + \sum_{x \in [R'(K_c - 1) + X + D_E]} \alpha_s^{-1} I_x + \sum_{x \in [R'(K_c - 1) + X + D_E + 1]} \alpha_s^{-1} I_x \right)
\]

\[
= \sum_{i \in [\ell]} \sum_{k \in [K_c]} \sum_{i=0}^{R'-1} \sum_{l'=0}^{i} c_{l,k,i-l'} \left( C_{l',i+1}^{l,k} + Z''_{l',i+1} \right) + \sum_{x \in [R'(K_c - 1) + X + D_E]} \alpha_s^{-1} I_x + \sum_{x \in [R'(K_c - 1) + X + D_E + 1]} \alpha_s^{-1} I_x
\]

where \(D_{l,k}^{l,k} = C_{l,k}^{l,k} + Z''_{l,k,i+1}, l \in [\ell], k \in [K_c], i \in [R'], J_x = I_x + Z''_x, x \in [R'(K_c - 1) + X + D_E] \) and \(J_x = I_x, x \in [R'(K_c - 1) + X + D_E + 1] \). In the matrix form, answers from any \(R = R'K_c + 2X - 1 + R' = pmn(\ell+1)K_c + 2X - 1\) servers, whose indices are denoted as \(s_1, s_2, \cdots, s_{R'}\), can be written as (75).

\[
\begin{bmatrix}
Y_{s_1} \\
Y_{s_2} \\
\vdots \\
Y_{s_R}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{(f_{l,1}-\alpha_{s_1})^R} & \cdots & \frac{1}{(f_{l,1}-\alpha_{s_1})^R} \\
\frac{1}{(f_{l,1}-\alpha_{s_2})^R} & \cdots & \frac{1}{(f_{l,1}-\alpha_{s_2})^R} \\
\vdots & \ddots & \vdots \\
\frac{1}{(f_{l,1}-\alpha_{s_R})^R} & \cdots & \frac{1}{(f_{l,1}-\alpha_{s_R})^R}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{(f_{l,k}-\alpha_s)^R} & \cdots & \frac{1}{(f_{l,k}-\alpha_s)^R} \\
\frac{1}{(f_{l,k}-\alpha_s)^R} & \cdots & \frac{1}{(f_{l,k}-\alpha_s)^R} \\
\vdots & \ddots & \vdots \\
\frac{1}{(f_{l,k}-\alpha_s)^R} & \cdots & \frac{1}{(f_{l,k}-\alpha_s)^R}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\]
where (79) holds because of the fact that the map from $f : \mathbb{T} \rightarrow \mathbb{R}$ is invertible. Guaranteed by Lemma 1 and the fact that the Kronecker product of non-singular matrices is non-singular, the matrix $(\mathfrak{V}_{l,K,c,R},X,R) \otimes I_{l/m}$ is invertible. Therefore, the master is able to recover $(D_{l}^{i,k})_{l \in [\ell], k \in [K], i \in \mathcal{E}}$ by inverting the matrix. Note that $Z_{l,m} = 0, l \in [\ell], k \in [K], i \in \mathcal{E}$, therefore $(C_{l}^{i,k})_{l \in [\ell], k \in [K], i \in \mathcal{E}}$ are recoverable from $(C_{l}^{i,k})_{l \in [\ell], k \in [K], i \in \mathcal{E}}$ guaranteed by the correctness of Entangled Polynomial code. This completes the proof of recovery threshold $R = pmn(\ell + 1)K + 2X - 1$.

Consider the strong security. According to the construction, $M_{1} = 0, M_{s} = \widetilde{M}_{s}, s \in [S] \setminus \{1\}$, and $M = \{\widetilde{M}_{s} \mid s \in [S] \setminus \{1\}\}$. Since $\widetilde{M}_{s}$ is a function of $Z_{\text{server}}$,

$$I(A, B, \bar{A}^{[S]}, \bar{B}^{[S]}, M) \leq I(A, B, \bar{A}^{[S]}, \bar{B}^{[S]}, Z_{\text{server}}) = 0. \quad (76)$$

Strong security is satisfied. Security is guaranteed because $\forall X \subset [S], |X| = X$,

$$I(A, B, \bar{A}^{X}, \bar{B}^{X}, M_X) = I(A, B, M_X) + I(A, B, \bar{A}^{X}, \bar{B}^{X} \mid M_X) \quad (77)$$

$$= I(A, B, M_X) + I(A, B, \bar{A}^{X}, \bar{B}^{X}) = 0, \quad (78)$$

where (78) is due to (44), (49) and the facts that each share is encoded with $(X, S)$ Reed-Solomon code with uniformly and independently distributed noise.

Consider the privacy,

$$I(Y_1, Y_2, \cdots, Y_S; A, B | AB)$$

$$= I\left(\left(D_{i}^{j,k}\right)_{l \in [\ell], k \in [K], i \in [R]} ; (J_{x})_{x \in [R'K_{c}+2X-1]} ; A, B | AB\right) \quad (79)$$

$$= I\left(\left(D_{i}^{j,k}\right)_{l \in [\ell], k \in [K], i \in [R]} ; A, B | AB\right)$$

$$+ I\left((J_{x})_{x \in [R'K_{c}+2X-1]} ; A, B, \left(D_{i}^{j,k}\right)_{l \in [\ell], k \in [K], i \in [R]} \right), \quad (80)$$

where (79) holds because of the fact that the map from $(Y_1, Y_2, \cdots, Y_S)$ to $\left(D_{i}^{j,k}\right)_{l \in [\ell], k \in [K], i \in [R]} (J_{x})_{x \in [R'K_{c}+2X-1]}$ is bijective. The first term of (80),

$$I\left(\left(D_{i}^{j,k}\right)_{l \in [\ell], k \in [K], i \in [R]} ; A, B | AB\right) = 0, \quad (81)$$
due to \([49]\) and the fact that \((C^{(i,k)}_c)_{i \in [\ell], k \in [K_c], i \in \mathcal{E}}\) are functions of \(AB\). Consider the second term of \((80)\).

\[
I \left( J_x \mid x \in [R'_cK_c2X-1]; A, B \mid AB, \left( D^{(i,k)}_i \right)_{i \in [\ell], k \in [K_c], i \in [R']} \right)
\leq I \left( J_x \mid x \in [R'_cK_c2X-1]; A, B, AB, \left( D^{(i,k)}_i \right)_{i \in [\ell], k \in [K_c], i \in [R']} \right)
\]

\[
= I \left( I_x + Z_x \mid x \in [R'_cK_c1X+D_B]; \left( I_x \mid x \in [R'_cK_c1X+D_B] \right), \left( I_x \mid x \in [R'_cK_c1X+D_B] \right); A, B, \left( D^{(i,k)}_i \right)_{i \in [\ell], k \in [K_c], i \in [R']} \right)
\]

\[
\leq I \left( Z^{\text{server}}, Z^A, Z^B; A, B, \left( D^{(i,k)}_i \right)_{i \in [\ell], k \in [K_c], i \in [R']} \right) = 0,
\]

where inequation \((85)\) holds because of the fact that \((I_x)_{x \in [R'_cK_c1X+D_B]}\) are functions of \(Z^A, Z^B\).

Combine \((81)\) and \((85)\), \(I(Y_1, Y_2, \ldots, Y_S; A, B \mid AB) = 0\).

Consider the communication cost. The source upload cost \(U_A = \frac{S}{K_c pm}\) and \(U_B = \frac{S}{K_c pm}\). The server communication cost \(CC = \frac{S-1}{Ibmn}\). Note that the master is able to recover \(lmn\) desired symbols from \(R\) downloaded symbols, the master download cost is \(D = R = \frac{pmn(l+1)K_c2X-1}{Ibmn}\). Thus the desired costs are achievable.

Now let us consider the computation complexity. Note that the source encoding procedure can be regarded as products of confluent Cauchy matrices by vectors. So by fast algorithms \([49]\), the encoding complexity of \((C_{cA}, C_{cB}) = (\tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm}), \tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm}))\) is achievable. For the server computation complexity, each server multiplies the \(\ell\) pairs of shares \(A^\ell_1, B^\ell_1, l \in [\ell]\), and returns the sum of these \(\ell\) products and structured noise \(\tilde{M}_s\). With straightforward matrix multiplication algorithms, each of the \(\ell\) matrix products has a computation complexity of \(\tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm})\) for a total of \(\tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm})\). The complexity of summation over the products and noise is \(\tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm})\). To construct the noise, one server needs to encode the noise, whose complexity is \(\tilde{O}(\frac{\lambda S + \log^2 S}{mn})\) by fast algorithms \([49]\). Normalized by the number of servers, it is \(\tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm})\). Considering these 3 procedures, upon normalization by \(L = \ell K_c\), it yields a complexity of \(\tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm}) + \tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm}) + \tilde{O}(\frac{\lambda S + \log^2 S}{K_c pm})\) per server. The master decoding complexity is inherited from that of GCSA codes \([30]\), which is at most \(\tilde{O}(\lambda S + \log^2 S)\). This completes the proof of Theorem 1.

Remark: When \(L = \ell = K_c = 1, S = R\), by setting \(f_{1,1} = 0\), our construction of shares of \(\tilde{A}^\ell\) and \(\tilde{B}^\ell\) essentially recovers the construction of shares in \([40]\).

VI. Conclusion

The class of GCSA codes is expanded by including noise-alignment, so that the resulting GCSA-NA code is a solution for secure coded multi-party computation of massive matrix multiplication. For two sources and matrix multiplication, GCSA-NA strictly generalizes PS \([40]\) and outperforms it in several key aspects. This construction also settles the asymptotic capacity of symmetric X-secure T-private information retrieval. The idea of noise-alignment can be applied to construct a scheme for \(N\) sources based on \(N\)-CSA codes. As an open problem, exploring the optimal amount of randomness is an interesting direction. Also finding the communication efficient schemes for arbitrary polynomial is worth exploring.

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