Attractivity, Degeneracy and Codimension of a Typical Singularity in 3D Piecewise Smooth Vector Fields

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Abstract. We address the problem of understanding the dynamics around typical singular points of 3D piecewise smooth vector fields. A model $Z_0$ in 3D presenting a T-singularity is considered and a complete picture of its dynamics is obtained in the following way: (i) $Z_0$ has an invariant plane $\pi_0$ filled up with periodic orbits (this means that the restriction $Z_0|_{\pi_0}$ is a center around the singularity); (ii) All trajectories of $Z_0$ converge to the surface $\pi_0$; (iii) given an arbitrary integer $k \geq 0$ then $Z_0$ can be approximated by $\pi_0$-invariant piecewise smooth vector fields $Z_\varepsilon$ such that the restriction $Z_\varepsilon|_{\pi_0}$ has exactly $k$-hyperbolic limit cycles; (iv) the origin can be chosen as an asymptotic stable equilibrium of $Z_\varepsilon$ when $k = 0$; and finally, (v) $Z_0$ has infinite codimension in the set of all 3D piecewise smooth vector fields.

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1. Introduction

1.1. The goal

Our purpose in this paper is to present a detailed qualitative analysis of the phase portrait of perturbations of a distinguished piecewise smooth vector field (PSVF for short) around typical singularities. These singularities, known in the literature as two-fold singularities (or T-singularities) have been, lately, object of attention of many researchers due to their high complexity and applications.

1.2. Historical Facts and Motivations

Mechanics Engineering (see [4, 9, 14]), Electric Engineering (see [2, 13]), Biologic/Control Theory/Economics (with sudden external influences, see [8]), among others, are natural source of mathematical models of PSVFs. In fact, every system susceptible to on-off operations are modelled by PSVFs which imposes an interdisciplinary aspect on this theory.
It is worth mentioning that Anosov in [1] studied the asymptotic stability in the class of relay systems in $\mathbb{R}^n$ of the form
\[ \dot{u} = Au + \text{sign}(u_1)k \]
where the variable $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $A$ is an $n \times n$ real valued matrix and $k = (k_1, \ldots, k_n)$ is a constant vector in $\mathbb{R}^n$.

In the 2-dimensional case a formal study of systems presenting a two-fold singularity is performed in [5]. The conclusion is that infinitely many topological types (see Definition 2.3 below) nearby the system are observed. So a natural question is about an equivalent problem in higher dimensions.

### 1.3. The two-fold singularity

As stated in [5], a PSVF $Z$ is a concatenation of two $C^r$-vector fields $X$ and $Y$, in such a way that just their restrictions to some regions (half-spaces) separated by a codimension one surface $\Sigma$ (called switching manifold) are considered (see Figure 1). So, the notation $Z = (X,Y)$ is adopted.

![Figure 1](image)

**Figure 1.** At the left we get two vector fields $X$ and $Y$ and the domain of both is separated by $\Sigma$. At the middle these vector fields are defined just in a half-space. At the right it appears the concatenation of the flows.

In this context, points where $X$ and/or $Y$ are tangent to $\Sigma$ must be distinguished. The most known tangential singularity of a smooth system $X$ with $\Sigma$ is the fold singularity (also called fold point), which is characterized by the quadratic contact of an orbit of $X$ with $\Sigma$. A fold point $p$ can be visible or invisible. It is visible for $X$ if, locally, the $X$-trajectory (i.e., the trajectory of $X$) passing through $p$ remains in the same side where $X$ is defined, otherwise it is invisible. In $3D$, generically there exists a curve of tangential singularities $S_X \subset \Sigma$ passing through $p$ (for a rigorous definition, see Section 2). See Figure 2.

If $p \in \Sigma$ is a fold singularity of both systems $X$ and $Y$ then it is called a two-fold singularity. As pointed out in [7], a two-fold singularity is an important organizing center because it brings together all of the basic forms of dynamics possible in a 3D
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Figure 2. 3D fold singularities.

PSVF. We say that a two-fold singularity $p$ is a $T$-singularity (or Teixeira-singularity – due to the pioneering work [15] – or invisible two-fold singularity) for $Z = (X, Y)$ if $p$ is an invisible fold point of both $X$ and $Y$ and $S_X$ meets $S_Y$ transversally at $p$ (see Figure 3). There are a lot of papers studying specifically this singularity and the interested reader can see more details about the T-singularity in [2, 3, 6, 7, 11, 12, 15].

Some of the (distinct) T-singularity models presented in the literature reveal behaviors like stability ([11]), chaos ([7]), among others. In this paper we clarify the existence of such rich behavior since the T-singularity has infinite codimension, or even more, it is possible to obtain PSVFs of infinitely many topological types nearby a PSVF presenting a T-singularity.

Figure 3. T-singularity.

1.4. Setting the problem and Statement of the Main Results

We study piecewise smooth nonlinear perturbations of the following 3D-model:

$$Z_0(x, y, z) = (\dot{x}, \dot{y}, \dot{z}) = F(x, y, z) + \text{sign}(z)K,$$

where

$$F(x, y, z) = \frac{1}{2} \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2} \text{sign}(z) \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and

$$K = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$
or equivalently:

\[ Z_0(x, y, z) = \begin{cases} 
X(x, y, z) = (-1 - (x + y), 1 - (x + y), -y) & \text{if } z \geq 0, \\
Y_0(x, y, z) = (1, -1, x) & \text{if } z \leq 0. \end{cases} \quad (1.1) \]

Figure 8 illustrates the phase portrait of \( Z_0 \). The richness of our model is revealed after suitable perturbations of it. We are able to produce an arbitrary number of distinct topological types of PSVF$s$ very close to \( Z_0 \). In fact, these perturbations will involve just the vector field \( Y_0 \), this is the reason why \( X \) does not have a subscript.

Now we state the main results of the paper.

**Theorem A.** Let \( Z_0 \) be given by (1.1) and \( \pi_0 \) the plane \( \{y + x = 0\} \). For each integer \( k \geq 0 \), there exists a one-parameter family of \( \pi_0 \)-invariant PSVF$s$ \( Z_\varepsilon \) satisfying:

(a) \( Z_\varepsilon \to Z_0 \) when \( \varepsilon \to 0 \);

(b) \( Z_\varepsilon \) has exactly \( k \) hyperbolic limit cycles in a neighborhood of the origin. The same holds for \( k = \infty \) and,

(c) All trajectories of \( Z_0 \) and \( Z_\varepsilon \) converge to \( \pi_0 \).

(d) When \( \varepsilon > 0 \) and \( k = 0 \), the origin is an asymptotically stable equilibrium point for \( Z_\varepsilon \).

**Theorem B.** The PSVF \( Z_0 \), given by (1.1), has infinite codimension over the set of all 3D PSVF$s$.

The paper is organized as follows. In Section 2 we introduce the terminology, some definitions and basic theory about PSVF$s$. In Section 3 we present properties towards the understanding of the phase portrait of (1.1). In Section 4 suitable perturbations of (1.1) are considered and the birth of limit cycles are explicitly exhibited. In Section 5 we prove the main results. In Section 6 we picture some numerical simulations and the phase portrait of (1.1) around the origin.

## 2. Preliminaries

Let \( V \) be an arbitrarily small neighborhood of \( 0 \in \mathbb{R}^3 \). We consider a codimension one manifold \( \Sigma \) of \( \mathbb{R}^3 \) given by \( \Sigma = f^{-1}(0) \), where \( f : V \to \mathbb{R} \) is a smooth function having \( 0 \in \mathbb{R} \) as a regular value (i.e. \( \nabla f(p) \neq 0 \), for any \( p \in f^{-1}(0) \)). We call \( \Sigma \) the switching manifold that is the separating boundary of the regions \( \Sigma^+ = \{q \in V \mid f(q) \geq 0\} \) and \( \Sigma^- = \{q \in V \mid f(q) \leq 0\} \). Throughout the paper we assume that \( \Sigma = f^{-1}(0) \), where \( f(x, y, z) = z \).

Designate by \( \chi \) the space of \( C^r \)-vector fields on \( V \subset \mathbb{R}^3 \) endowed with the \( C^r \)-topology, with \( r \geq 1 \) large enough for our purposes. Call \( \Omega^r \) the space of vector fields \( Z : V \to \mathbb{R}^3 \) such that

\[
Z(x, y, z) = \begin{cases} 
X(x, y, z), & \text{for } (x, y, z) \in \Sigma^+, \\
Y(x, y, z), & \text{for } (x, y, z) \in \Sigma^-,
\end{cases} \quad (2.1)
\]

where \( X = (X_1, X_2, X_3), Y = (Y_1, Y_2, Y_3) \in \chi \). We endow \( \Omega^r \) with the product topology. The trajectories of \( Z \) are solutions of \( \dot{q} = Z(q) \) and we will accept it to
be multi-valued in points of $\Sigma$. The basic results of differential equations, in this context, were first stated in [10].

On $\Sigma$ we distinguish the following regions:

- Crossing Region: $\Sigma^c = \{ p \in \Sigma \mid X_3(p) \cdot Y_3(p) > 0 \}$. Moreover, we denote $\Sigma^{c+} = \{ p \in \Sigma \mid X_3(p) > 0, Y_3(p) > 0 \}$ and $\Sigma^{c-} = \{ p \in \Sigma \mid X_3(p) < 0, Y_3(p) < 0 \}$.
- Sliding Region: $\Sigma^s = \{ p \in \Sigma \mid X_3(p) < 0, Y_3(p) > 0 \}$.
- Escaping Region: $\Sigma^e = \{ p \in \Sigma \mid X_3(p) > 0, Y_3(p) < 0 \}$.

When $q \in \Sigma^s \cup \Sigma^e$, following the Filippov’s convention, the sliding vector field associated to $Z \in \Omega^r$ is the vector field $\tilde{Z}^s$ tangent to $\Sigma^s \cup \Sigma^e$ and expressed in coordinates as

$$\tilde{Z}^s(q) = \frac{1}{(Y_3 - X_3)(q)}((X_1Y_3 - Y_1X_3)(q), (X_2Y_3 - Y_2X_3)(q), 0).$$

Associated to (2.2) there exists the planar normalized sliding vector field

$$Z^s(q) = ((X_1Y_3 - Y_1X_3)(q), (X_2Y_3 - Y_2X_3)(q)).$$

**Remark 2.1.** Note that, if $q \in \Sigma^s$ then $X_3(q) < 0$ and $Y_3(q) > 0$. So, $(Y_3 - X_3)(q) > 0$ and therefore, $\tilde{Z}^s$ and $Z^s$ are topologically equivalent since both have the same orientation and they can be $C^r$-extended to the closure $\Sigma^s$ of $\Sigma^s$. If $q \in \Sigma^e$ then $\tilde{Z}^s$ and $Z^s$ have opposite orientation.

**Definition 2.2.** The orbit (trajectory) of a point $p \in V$ is the set $\gamma(p) = \{ \phi_Z(t, p) : t \in \mathbb{R} \}$ obtained by the concatenation of the flows of $X$, $Y$, and $\tilde{Z}^s$.

**Definition 2.3.** Two PSVFs $Z = (X, Y)$, $\tilde{Z} = (\tilde{X}, \tilde{X}) \in \Omega^r$ have the same topological type if they are $\Sigma$-equivalent, i.e., there exists an orientation preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ that sends $\Sigma$ to $\Sigma$, the orbits of $X$ restricted to $\Sigma^+$ to the orbits of $\tilde{X}$ restricted to $\Sigma^+$, and the orbits of $Y$ restricted to $\Sigma^-$ to the orbits of $\tilde{Y}$ restricted to $\Sigma^-$.

Points $p \in \Sigma$ such that $X_3(p) \cdot Y_3(p) = 0$ are called tangential singularities of $Z$ (i.e., the trajectory through $p$ is tangent to $\Sigma$).

For practical purposes, the contact between the smooth vector field $X$ and the switching manifold $\Sigma = f^{-1}(0)$ is characterized by the expression $X.f(p) = \langle \nabla f(p), X(p) \rangle$ where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^3$. In this way, we say that a point $p \in \Sigma$ is a fold point of $X$ if $X.f(p) = 0$ but $X^2.f(p) \neq 0$, where $X^i.f(p) = \langle \nabla X^{i-1}.f(p), X(p) \rangle$ for $i \geq 2$. Moreover, $p \in \Sigma$ is a visible (respectively invisible) fold point of $X$ if $X.f(p) = 0$ and $X^2.f(p) > 0$ (respectively $X^2.f(p) < 0$). In addition, a tangential singularity $q$ is singular if $q$ is an invisible tangency for both $X$ and $Y$. On the other hand, a tangential singularity $q$ is regular if it is not singular.

Call $S_X$ (resp. $S_Y$) the set of all tangential singularities of $X$ (resp. $Y$).

Consider $0 \neq p \in \Sigma^{c+}$ and that there exists a time $t_1(p) > 0$, called $X$-fly time, such that the forward trajectory of $X$ passing through $p$ at $t = 0$ return to $\Sigma$ after $t_1(p)$. We define the half return map associated to $X$ by $\varphi_X(p) = \phi_X(t_1(p), p) = p_1 \in \Sigma$. When $p_1 \in \Sigma^{c-}$, suppose that there exists $t_2(p_1) > 0$, the $Y$-fly time of the trajectory of $Y$ passing through $p_1$. Define the half return map associated to
Y by \( \varphi_Y(p_1) = \phi_Y(t_2(p_1), p_1) \in \Sigma \). The \( C^r \) involution \( \varphi_X \) (resp. \( \varphi_Y \)) is such that \( \text{Fix}(\varphi_X) = S_X \) (resp. \( \text{Fix}(\varphi_Y) = S_Y \)), where Fix represents the set of fixed points. The first return map associated to \( Z = (X, Y) \) is defined by the composition of these involutions, i.e.,
\[
\varphi_Z(p) = \varphi_Y \circ \varphi_X(p) = \phi_Y(t_2(p_1), \phi_X(t_1(p), p))
\] (2.4)
or the reverse, applying first the flow of \( Y \) and after the flow of \( X \). See Figure 4 and details in [16].

![Figure 4. First Return Map.](image)

The mapping \( \varphi_Z \) is an important object in order to study the behavior of \( Z \) around a T-singularity. The proofs of the main results require a detailed analysis of the return map and different domains (departure regions), according to the Filippov’s decomposition of \( \Sigma \), must be considered. For example, if \( q \in \Sigma_c \) then the iteration of both mappings \( \varphi_Y \circ \varphi_X(q) \) and \( \varphi_X \circ \varphi_Y(q) \) must be considered.

### 3. Properties of System \( Z_0 \) given by (1.1)

In this section we describe some important features about the PSVF given by (1.1). In Subsection 3.1 we describe the sliding vector field associated to (1.1). In Subsection 3.2 we analyze the first return map associated to it.

#### 3.1. The sliding vector field associated to (1.1)

First of all, let us determine the distinguished regions. We get:
\[
\Sigma^c_+ = \{(x, y, 0) \in \mathbb{R}^3 \mid y < 0 \text{ and } x > 0\}, \\
\Sigma^c_- = \{(x, y, 0) \in \mathbb{R}^3 \mid y > 0 \text{ and } x < 0\}, \\
\Sigma^s = \{(x, y, 0) \in \mathbb{R}^3 \mid y > 0 \text{ and } x > 0\}, \\
\Sigma^e = \{(x, y, 0) \in \mathbb{R}^3 \mid y < 0 \text{ and } x < 0\}.
\]
The tangential sets are
\[
S_X = \{(x, 0, 0) \in \mathbb{R}^3\}, \\
S_Y = \{(0, y, 0) \in \mathbb{R}^3\}.
\]
Moreover,
\[
X^2.f(x, 0, 0) = -X^2(x, 0, 0) = 1 - x \Rightarrow X^2.f(x, 0, 0) > 0 \text{ when } x < 1, \\
Y^2.f(0, y, 0) = Y^1(0, y, 0) = 1 > 0.
\]
As consequence, $S_X \cap S_Y = \{(0,0,0)\}$ is a T-singularity of $Z = (X,Y)$.

Using Equation (2.3), associated to (1.1) we have the normalized sliding vector field

$$Z_0^s(x,y,z) = \left((1 - (x+y))x + y, (1 - (x+y))x - y\right). \quad (3.1)$$

Let us understand the phase portrait of (3.1). In order to do it, we will consider $Z_0^s$ defined in $\mathbb{R}^2$.

**Proposition 3.1.** The normalized sliding vector field $Z_0^s$, given by (3.1), has a saddle-node at the origin.

**Proof.** Identify $\Sigma$ with the $xy$-plane. Consider the change of variables $(u,v) = (x + y, x - y)$. Then (3.1) can be re-written in the form $(\dot{u}, \dot{v}) = -(u+v)u, -2v)$. This last system has the origin as a unique equilibrium with eigenvectors $v_1 = (1,0), v_2 = (0,1)$ associated to the eigenvalues $\lambda_1 = 0, \lambda_2 = -2$ respectively. The phase portrait is pictured at Figure 5. □

**Figure 5.** Phase Portrait of the normalized sliding vector field $Z_0^s$.

**Remark 3.2.** Since (3.1) has a saddle-node at the origin, from Remark 2.1 we must reverse the orientation in $\Sigma^e$. So, we conclude that the sliding vector field associated to $Z_0$ has the phase portrait as shown at Figure 6. Note that the straight line $y + x = 0$ in $\Sigma$, where $Y_3 - X_3 = 0$ in (2.2), is composed only by equilibrium points of the sliding vector field associated to $Z_0$. Moreover, except from the stable invariant manifold, the trajectories of the sliding vector field associated to $Z_0$ depart from $\Sigma^e$.

### 3.2. The first return map associated to (1.1)

In order to exhibit the first return map associated to $Z_0 = (X,Y_0)$, given by (1.1), we have to write the expressions of the half return maps $\varphi_X$ and $\varphi_{Y_0}$. A straightforward calculation shows that the trajectories $\phi_X$ and $\phi_{Y_0}$ are parameterized, respectively, by

$$\phi_X(t) = \left(-t + \frac{1}{2}(e^{-2t}(k_1 + k_2) + k_1 - k_2), t + \frac{1}{2}(e^{-2t}(k_1 + k_2) - k_1 + k_2), \frac{1}{4}e^{-2t}(-2k_1t + k_1 + 2t(k_2 + t) + k_2) + k_1 + k_2)\right),$$

(3.2)
Proposition 3.3. Given an arbitrary point \( (x_0, y_0, 0) \) in \( \Sigma^c_+ \), then
\[
\phi_X(x_0, y_0, 0) = (-t_2 + ((x_0 + y_0)/2)e^{-2t_2} + (x_0 - y_0)/2, t_2 + ((x_0 + y_0)/2)e^{-2t_2} - (x_0 - y_0)/2, 0),
\]
where the X-fly time \( t_2 > 0 \) is given implicitly by
\[
-e^{2t_2}(-2x_0t_2 + x_0 + 2t_2(y_0 + t_2) + y_0) + x_0 + y_0 = 0.
\]
In particular, \( \phi_X(\pi_0 \cap \Sigma^+) \subset (\pi_0 \cap \Sigma^+) \) and \( \phi_X(x_0, -x_0, 0) = (-x_0, x_0, 0) \).

Proof. Considering the initial condition \( (x_0, y_0, 0) \) and (3.2), let \( t_2 > 0 \) be the first time such that
\[
-e^{2t_2}(-2x_0t_2 + x_0 + 2t_2(y_0 + t_2) + y_0) + x_0 + y_0 = 0
\]
or, equivalently,
\[
-t_2^2/2 + ((x_0 + y_0)/2)e^{-2t_2} - ((x_0 - y_0)/2)t_2 - (x_0 + y_0)/4 = 0.
\]
Using (3.2) it is easy to see that
\[
\phi_X(x_0, y_0, 0) = (-t_2 + ((x_0 + y_0)/2)e^{-2t_2} + (x_0 - y_0)/2, t_2 + ((x_0 + y_0)/2)e^{-2t_2} - (x_0 - y_0)/2, 0).
\]
In particular, for \( p_0 = (x_0, y_0, 0) = (x_0, -x_0, 0) \in \pi_0 \cap \Sigma \), we obtain that the first two coordinates of \( \phi_X(t, p_0) \) are \( x(t) = -t + x_0 \) and \( y(t) = t - x_0 = -x(t) \). Moreover, in this case we can solve explicitly the equation \( -t_2^2/2 + ((x_0 + y_0)/2)e^{-2t_2} + ((x_0 - y_0)/2)t_2 - (x_0 + y_0)/4 = 0 \) and we obtain \( t_2 = 2x_0 \). So, \( \phi_X(x_0, -x_0, 0) = (-x_0, x_0, 0) \). This concludes the proof.

Proposition 3.4. Given an arbitrary point \( (x_0, y_0, 0) \in \Sigma^c_- \), then \( \phi_Y(x_0, y_0, 0) = (-x_0, y_0 + 2x_0, 0) \). In particular, \( \phi_Y(\pi_k \cap \Sigma^-) \subset (\pi_k \cap \Sigma^-) \).
**Proof.** Considering the initial condition \((x_0, y_0, 0)\) and (3.3), in order to determine \(\varphi_{Y_0}\) it is enough to obtain the first strictly positive time \(t_1\) such that \(t_1^2/2 + x_0 t_1 = 0\). So \(t_1 = -2x_0\) and \(\varphi_{Y_0}(x_0, y_0, 0) = (-x_0, y_0 + 2x_0, 0)\). In particular, for \(p_0 = (x_0, y_0, 0) = (x_0, -x_0 + k, 0) \in \pi_k \cap \Sigma^r\), the first two coordinates of \(\phi_Y(t, p_0)\) are \(x(t) = t + x_0\) and \(y(t) = -t - x_0 + k = -(t + x_0) + k = -x(t) + k\). This concludes the proof. \(\square\)

**Proposition 3.5.** The plane \(\pi_0 = \{(x, y, z) \in V \mid x+y = 0\}\) is \(Z_0\)-invariant. Moreover, \(Z_0|_{\pi_0}\) is a center.

**Proof.** By Propositions 3.3 and 3.4 we get that \(\pi_0\) is invariant by the flow of \(Z_0\). In order to see that \(Z_0\) has a center at \(\pi_0\) it is enough to see that

\[\varphi_{Z_0}(x_0, -x_0, 0) = \varphi_{Y_0} \circ \varphi_X(x_0, -x_0, 0) = \varphi_{Y_0}(-x_0, x_0, 0) = (x_0, -x_0, 0).\]

Now we will prove that \(\pi_0\) is a hyperbolic global attractor for the trajectories of \(Z_0\).

**Proposition 3.6.** Let \(r_0\) be the straight line given by \(r_0 = \pi_0 \cap \Sigma\). Given \((x_0, y_0, 0) \in \Sigma^c\), then

\[d(\varphi_{Z_0}(x_0, y_0, 0), r_0) < d((x_0, y_0, 0), r_0).\]

Moreover,

\[\varphi_{Z_0}^n(x_0, y_0, 0) = (x_n, -x_n + (x_0 + y_0)e^{-2(t_2^{(1)} + ... + t_2^{(n)})}, 0)\]

where \(t_2^{(i)}\) is the fly time necessary to the \(X\)-trajectory by \(\varphi_{Z_0}^{i}(x_0, y_0, 0)\) returns to \(\Sigma\) and \(x_n = t_2^{(n)} - ((x_0 + y_0)/2)e^{-2(t_2^{(1)} + ... + t_2^{(n)})} - x_{n-1} + ((x_0 + y_0)/2)e^{-2(t_2^{(1)} + ... + t_2^{(n-1)})}\).

**Proof.** By Proposition 3.3 we obtain

\[\varphi_X(x_0, y_0, 0) = (-t_2 + ((x_0 + y_0)/2)e^{-2t_2} + (x_0 - y_0)/2, t_2 + ((x_0 + y_0)/2)e^{-2t_2} - (x_0 - y_0)/2, 0),\]

where \(t_2 > 0\) is given implicitly by

\[-e^{2t_2}(-2x_0t_2 + x_0 + 2t_2(y_0 + t_2) + y_0) + x_0 + y_0 = 0.\]

By Proposition 3.4,

\[\varphi_{Z_0}(x_0, y_0, 0) = (\varphi_{Y_0} \circ \varphi_X)(x_0, y_0, 0) = (t_2 - \frac{(x_0 + y_0)}{2}e^{-2t_2} - \frac{(x_0 - y_0)}{2}, -t_2 + 3\frac{(x_0 + y_0)}{2}e^{-2t_2} + \frac{(x_0 - y_0)}{2}, 0)\]  \(\text{(3.5)}\)

Then we get,

\[d(\varphi_{Z_0}(x_0, y_0, 0), r_0) = \frac{\sqrt{2}}{2}(x_0 + y_0)e^{-2t_2} < \frac{\sqrt{2}}{2}(x_0 + y_0) = d((x_0, y_0, 0), r_0).\]

In order to obtain that

\[\varphi_{Z_0}^n(x_0, y_0, 0) = (x_n, -x_n + (x_0 + y_0)e^{-2(t_2^{(1)} + ... + t_2^{(n)})}, 0),\]
with
\[ x_n = -t_2^{(n)} + \frac{(x_0 + y_0)}{2} e^{-2t_2^{(1)} + \cdots + t_2^{(n)}} - x_{n-1} + \frac{(x_0 + y_0)}{2} (e^{-2t_2^{(1)} + \cdots + t_2^{(n-1)}}), \]
it is enough to use \( n \) times Propositions 3.3 and 3.4.

\[ \square \]

**Proposition 3.7.** Let \( p_0 = (x_0, -x_0, 0) \in \pi_0 \). The first return map \( \varphi_{Z_0}(p_0) \) has eigenvalues \( \mu_1 = 1 \) and \( \mu_2 = e^{-4x_0} \). Moreover, since \( x_0 > 0 \) then \( e^{-4x_0} < 1 \) and \( \pi_0 \) behaves like an attractive center manifold.

**Proof.** First of all, note that it is enough to consider \( \varphi_{Z_0} : \mathbb{R}^2 \to \mathbb{R}^2 \). From Equation 3.4 we get

\[ -\frac{e^{-2t_2(x_0 + y_0)}}{2} = -t_2^2 + t_2(x_0 - y_0) - \frac{x_0 + y_0}{2}. \]

Putting this in Equation (3.5) we obtain the jacobian matrix

\[ J\varphi_{Z_0}(x_0, y_0) = \begin{pmatrix} (\frac{-2t_2 + x_0 - y_0 + 1}{\partial x_0}) + t_2 - 1 & (-2t_2 + x_0 - y_0 + 1)\frac{\partial t_2}{\partial y_0} - t_2 \\ (6t_2 - 3(x_0 - y_0) - 1)\frac{\partial t_2}{\partial x_0} - 3t_2 + 2 & (6t_2 - 3(x_0 - y_0) - 1)\frac{\partial t_2}{\partial y_0} + 3t_2 + 1 \end{pmatrix} \]

where \( t_2 = t_2(x_0, y_0) \).

The derivation of Equation (3.4) with respect to \( x_0 \) gives

\[ \frac{\partial t_2}{\partial x_0}(x_0, -x_0) = 1 - \frac{1}{4x_0} + \frac{e^{-4x_0}}{4x_0}, \]

and the derivation of Equation (3.4) with respect to \( y_0 \) gives

\[ \frac{\partial t_2}{\partial y_0}(x_0, -x_0) = -1 - \frac{1}{4x_0} + \frac{e^{-4x_0}}{4x_0}. \]

So, we are able to evaluate (3.6) when \( y_0 = -x_0 \). A consequence, the eigenvalues of \( J\varphi_{Z_0}(x_0, -x_0, 0) \) are \( \mu_1 = 1 \) and \( \mu_2 = e^{-4x_0} \).

\[ \square \]

**Corollary 3.8.** All trajectories of \( Z_0 \), given by (1.1), converge to the plane \( \pi_0 \).

**Proof.** First note that each \( q \in V \) hits \( \Sigma \) for some positive time. Using Remark 3.2, the position of the eigenspaces of (3.1) and Propositions 3.6 and 3.7 it is easy to see that given \( p \in \Sigma = \Sigma^s \cup \Sigma^d \cup \Sigma^c \) the trajectories of \( Z_0 \) by \( p \) converge to \( \pi_0 \). \[ \square \]

### 4. Properties of an oriented perturbation of System \( Z_0 \)

#### 4.1. Auxiliary results

In what follows, \( h : \mathbb{R} \to \mathbb{R} \) will denote the \( C^\infty \)-function given by

\[ h(w) = \begin{cases} 0, & \text{if } w \leq 0; \\ e^{-1/w}, & \text{if } w > 0. \end{cases} \]

Consider the function \( F^0_\varepsilon(x, y) : \mathbb{R}^2 \times \mathbb{R} \), where either \( \rho = f \) or \( \rho = i \), such that

\[ F^f_\varepsilon(x, y) = -\varepsilon h(x)h(-y)(\varepsilon - x)(2\varepsilon - x)\ldots(k\varepsilon - x) \quad (4.1) \]
with $k \in \mathbb{N}$,

$$F^{i} \xi(x, y) = -\varepsilon h(x)h(-y)$$

(4.2)

with $k = 0$ and

$$F^{i} \xi(x, y) = h(x)h(-y)\sin(\pi \varepsilon^2 / x).$$

(4.3)

**Lemma 4.1.** Consider the function $F^{i} \xi(x, y)$ given by (4.1).

(i) If $\varepsilon < 0$, then $F^{i} \xi$ does not have roots in $(0, +\infty) \times \{y\}$.

(ii) If $\varepsilon > 0$, then $F^{i} \xi$ has exactly $k$ roots in $(0, +\infty) \times \{y\}$, these roots are

$$\{(\varepsilon, y), (2\varepsilon, y), \ldots, (k\varepsilon, y)\}.$$

(iii) $\partial F^{i} \xi \frac{\partial}{\partial x}(j\varepsilon, y) = -eh(-y)(-1)^{j}(j\varepsilon)(k - j)!(j - 1)!$ for $j \in \{1, 2, \ldots, k\}$. It means that such partial derivative at $(j\varepsilon, y)$ is positive for $j$ odd and negative for $j$ even.

**Proof.** When $x > 0$, by a straightforward calculation $F^{i} \xi(x, y) = 0$ if, and only if, $(\varepsilon - x)(2\varepsilon - x)\ldots(k\varepsilon - x) = 0$. So, the roots of $F^{i} \xi(x, y)$ in $(0, +\infty)$ are $\varepsilon, 2\varepsilon, \ldots, k\varepsilon$. Moreover,

$$\frac{\partial F^{i} \xi}{\partial x}(x, y) = -eh(-y)\frac{\partial}{\partial x}((j\varepsilon - x)H(x)) = -eh(-y)((j\varepsilon - x)\frac{\partial H}{\partial x}(x) - H(x)),$$

where $H(x) = F^{i} \xi(x, y)/(eh(-y)(j\varepsilon - x))$. So,

$$\frac{\partial F^{i} \xi}{\partial x}(j\varepsilon, y) = eh(-y)H(j\varepsilon)$$

$$= eh(-y)e^{k}h(j\varepsilon)(1 - j)\ldots((j - 1) - j)((j + 1) - j)\ldots(k - j)$$

$$= -eh(-y)e^{k}h(j\varepsilon)(-1)^{j}((j - 1)\ldots(j - (j - 1))((j + 1) - j)\ldots(k - j))$$

$$= -eh(-y)(-1)^{j}e^{k}h(j\varepsilon)(k - j)!(j - 1)!$$

This proves items (ii) and (iii). Item (i) follows immediately. \qed

**Lemma 4.2.** Consider the function $F^{i} \xi(x, y)$ given by (4.3). For $\varepsilon \neq 0$ the function $F^{i} \xi$ has infinitely many roots in $(0, \varepsilon^2) \times \{y\}$, these roots are

$$\{(\varepsilon^2, y), (\varepsilon^2/2, y), (\varepsilon^2/3, y), \ldots\}$$

and

$$\frac{\partial F^{i} \xi}{\partial x}(\varepsilon^2/j, y) = h(-y)(-1)^{j}(-\pi j^2/\varepsilon^2)h(\varepsilon^2/j)$$

for $j \in \{1, 2, 3, \ldots\}$. It means that such derivative at $(\varepsilon^2/j, y)$ is positive for $j$ odd and negative for $j$ even.

**Proof.** When $x > 0$, by a straightforward calculation $F^{i} \xi(x, y) = 0$ if, and only if, $\sin(\pi \varepsilon^2 / x) = 0$. So, the roots of $F^{i} \xi(x, y)$ in $(0, \varepsilon^2) \times \{y\}$ are $(\varepsilon^2, y), (\varepsilon^2/2, y), (\varepsilon^2/3, y), \ldots$. Moreover,

$$\frac{\partial F^{i} \xi}{\partial x}(x, y) = h(-y)[h'(x)\sin(\pi \varepsilon^2 / x) + h(x)\cos(\pi \varepsilon^2 / x)(-\pi \varepsilon^2 / x^2)].$$
\[
\frac{\partial F_\varepsilon^i}{\partial x}(\varepsilon^2/j, y) = h(-y)[h'(\varepsilon^2/j) \sin(\pi \varepsilon^2/j) + h(\varepsilon^2/j) \cos(\pi \varepsilon^2/j)(-\pi j^2/\varepsilon^2)] = h(-y)(-1)^j(-\pi j^2/\varepsilon^2)h(\varepsilon^2/j). \]

So,

Consider \( Z_0 \) given by (1.1) and

\[
Z_\rho^0(x, y, z) = \begin{cases} 
X(x, y, z) = (-1 - (x + y), 1 - (x + y), -y) & \text{if } z \geq 0, \\
\Y_\rho^0(x, y, z) = (1, -1, x + \frac{\partial F^0_\varepsilon}{\partial x}(x, y)) & \text{if } z \leq 0.
\end{cases} \tag{4.4}
\]

**Remark 4.3.** Take \( Z_\varepsilon = Z_\rho^0 \), with \( \rho = i, f \). It is easy to see that \( Z_\varepsilon \to Z_0 \) when \( \varepsilon \to 0 \).

Associated to (4.4) we have the normalized sliding vector field given by

\[
Z_{\rho, \varepsilon}^0(x, y, z) = \left( (1 - (x + y))(x + \frac{\partial F^0_\varepsilon}{\partial x}(x, y)) + y, (1 - (x + y))(x + \frac{\partial F^0_\varepsilon}{\partial x}(x, y)) - y, 0 \right).
\]

A straightforward calculation shows that the trajectory \( \phi_{Y_\rho^0} \) of \( Y_\rho^0 \) given in (4.4) are parameterized by

\[
\phi_{Y_\rho^0}(t) = (t + l_1, -t + l_2, t^2/2 + l_1 t + F^0_\varepsilon(t + l_1, -t + l_2) + l_3). \tag{4.5}
\]

**Proposition 4.4.** Given an arbitrary point \((x_0, y_0, 0) \in \Sigma^c\), then \( \varphi_{Y_\rho^0}(x_0, y_0, 0) = (t_1 + x_0, -t_1 + y_0, 0) \), where the \( Y_\rho^0 \)-fly time \( t_1 > 0 \) is given implicitly by \( t_1^2/2 + x_0 t_1 + F^0_\varepsilon(t_1 + x_0, -t_1 + y_0) = 0 \). In particular, \( \varphi_{Y_\rho^0}(\pi_k \cap \Sigma^-) \subset (\pi_k \cap \Sigma^-) \).

**Proof.** Let \( t_1 > 0 \) the first time such that \( t_1^2/2 + x_0 t_1 + F^0_\varepsilon(t_1 + x_0, -t_1 + y_0) = 0 \). Using (4.5) it is easy to see that \( \varphi_{Y_\rho^0}(x_0, y_0, 0) = (t_1 + x_0, -t_1 + y_0, 0) \). In particular, for \( p_0 = (x_0, y_0, 0) = (x_0, -x_0 + k, 0) \in \pi_k \cap \Sigma^- \), the first two coordinates of \( \phi_{Y_\rho^0}(t, p_0) \) are \( x(t) = t + x_0 \) and \( y(t) = -t - x_0 + k = -(t + x_0) + k = -x(t) + k \). This concludes the proof. \( \square \)

**Proposition 4.5.** The plane \( \pi_0 \) is invariant by the flow of \( Z^0_\varepsilon \), given by (4.4).

**Proof.** By Proposition 4.4 and Proposition 3.3 we get that \( \pi_0 \) is invariant by the flow of \( Z^0_\varepsilon \).

**Remark 4.6.** Since, by Proposition 3.6, we get that all trajectories of \( Z_0 \) converge to \( \pi_0 \) we obtain that, for \( \varepsilon \) sufficiently small, the same holds for the trajectories of \( Z^0_\varepsilon \).

**Proposition 4.7.** Consider an integer \( k \geq 0 \) and \( Z^0_\varepsilon \) given by (4.4), where either \( \rho = f \) or \( \rho = i \). Then \( Z^1_\varepsilon \) has exactly \( k \) limit cycles and \( Z^i_\varepsilon \) has infinite many limit cycles, all of then situated in \( \pi_0 \).

**Proof.** According to Remark 4.6, \( \pi_0 \) is a global attractor for \( Z^0_\varepsilon \). Also, according to Proposition 4.5, \( \pi_0 \) is invariant by the flow of \( Z^0_\varepsilon \). So, if there exists limit cycles, then they are situated at \( \pi_0 \). Moreover when we restrict the flow of \( Z^0_\varepsilon \) to \( \pi_0 \), by
Propositions 4.4 and 3.3, the fixed points of the first return map $\varphi_{Z_0^\varepsilon} = \varphi_{Y_0^\varepsilon} \circ \varphi_X$ occurs when $t = t_3 = 4x_0$. So, take $p_0 = (x_0, -x_0, 0)$ and we get

$$\varphi_{Z_0^\varepsilon}(p_0) = \varphi_{Y_0^\varepsilon}(2x_0, -p_0) = (x_0, -x_0, F_\varepsilon^\rho(x_0, -x_0)). \quad (4.6)$$

When $\rho = f$, by Item (ii) of Lemma 4.1,

$$\varphi_{Z_0^\varepsilon}(x_0, -x_0, 0) = (x_0, -x_0, 0) \iff x_0 = j\varepsilon \text{ with } \varepsilon > 0 \text{ and } j = 1, 2, \ldots, k.$$

Therefore, $Z_0^f$ has $k$ limit cycles, all of them situated in $\pi_0$. When $\rho = i$, by Lemma 4.2,

$$\varphi_{Z_0^\varepsilon}(x_0, -x_0, 0) = (x_0, -x_0, 0) \iff x_0 = \varepsilon^2/j \text{ with } j = 1, 2, \ldots.$$

Therefore, $Z_0^i$ has infinite many limit cycles, all of them situated in $\pi_0$. □

**Proposition 4.8.** All limit cycles in Proposition 4.7 are hyperbolic. Moreover, for $\varepsilon > 0$, if $j = \text{even}$ then it is an attractor and if $j = \text{odd}$ then it is a repeller. When $k = 0$, the origin is an attractor equilibrium of the system.

**Proof.** In fact, in order to prove this we must consider the expression of the derivatives in Item (iii) of Lemma 4.1 and Lemma 4.2. Observe that when $k = 0$ in Proposition 4.7 there is no limit cycles. In such a case, it is easy to see that Equation (4.2) is decreasing. So, the origin is an attractor equilibrium of the system. □

5. Proof of main results

Now we prove the main results of the paper:

**Proof of Theorem A.** Item (a): It follows from Remark 4.3.

Item (b): It follows from Propositions 4.7 and 4.8.

Item (c): It follows from Corollary 3.8 and Remark 4.6.

Item (d): It follows from Proposition 4.8. □

Before to proceed the prove of Theorem B, we indicate the reading of Appendix A.1 of [5], where the classical notion of codimension of vector fields (using versal and mini-versal unfolding) is exhibited. Rough speaking, after reading [5], we can say that the codimension of $Z_0$ is the minimal number of parameters that can be used in a single unfolding in order to obtain all topological types nearby $Z_0$. Moreover, in Theorem C of [5] it is proved that a PSVF presenting a planar non-smooth center has infinite codimension.

**Proof of Theorem B.** Consider $Z_0$ given in (1.1). Proposition 3.5 ensures that there exists a plane $\pi_0$ (transversal to $\Sigma$), such that $Z_0|_{\pi_0}$ has a planar non-smooth center. From Theorem C of [5] we conclude that $Z_0|_{\pi_0}$ has infinite codimension. So, $Z_0$ has infinite codimension in $\Omega^r$. □
6. Appendix

Now we illustrate the theoretical analysis performed by means of some numerical simulations. In the next illustrations we use software package “Mathematica”.

We use the following line of commands:

\[
solutions = \text{Table}[\text{First}[\text{NDSolve}[[x'[t] == If[z[t] > 0,} \\
-1 - (x[t] + y[t]), 1], y'[t] == If[z[t] > 0, 1 - (x[t] + y[t]), -1],} \\
z'[t] == If[z[t] > 0, -y[t], x[t]], x[0] == \theta, y[0] == -\theta,} \\
z[0] == 0, \{x, y, z\}, \{t, 0, 2.5\}]], \{\theta, 0.1, 2\pi - 0.1, 0.1\}];
\]

\[
solutions2 = \text{Table}[\text{First}[\text{NDSolve}[[x'[t] == If[z[t] > 0,} \\
-1 - (x[t] + y[t]), 1], y'[t] == If[z[t] > 0, 1 - (x[t] + y[t]), -1],} \\
z'[t] == If[z[t] > 0, -y[t], x[t]], x[0] == \cos[\theta]/4, y[0] == \sin[\theta]/4,} \\
z[0] == 0, \{x, y, z\}, \{t, 0, 2.5\}]], \{\theta, 0.1, 2\pi - 0.1, 0.1\}];
\]

\[
c1 = \text{ContourPlot3D}[z, \{x, -0.4, 0.4\}, \{y, -0.4, 0.4\}, \{z, -0.5, 0.5\},} \\
\text{Contours} \to 0, \text{Mesh} \to \text{False}];
\]

\[
c2 = \text{ContourPlot3D}[y, \{x, -0.5, 0.5\}, \{y, -0.4, 0.4\}, \{z, -0.5, 0.5\},} \\
\text{Contours} \to 0, \text{Mesh} \to \text{False}];
\]

\[
c3 = \text{ContourPlot3D}[x + y, \{x, -0.5, 0.5\}, \{y, -0.4, 0.4\}, \{z, -0.5, 0.5\},} \\
\text{Contours} \to 0, \text{Mesh} \to \text{False}];
\]

The command

\[
p3 = \text{ParametricPlot3D}[[\text{Evaluate}[[x[t], y[t], z[t]]] /. \text{solutions}]], \{t, 0, 2.5\},} \\
\text{PlotStyle} \to \{\text{Thickness}[0.0015], \text{Red}\}; \text{Show}[p3, \text{ImageSize} \to \text{Large}]
\]
generates Figure 7.

The command

\[
p4 = \text{ParametricPlot3D}[[\text{Evaluate}[[x[t], y[t], z[t]]] /. \text{solutions2}]], \{t, 0, 2.5\},} \\
\text{PlotStyle} \to \{\text{Thickness}[0.0015], \text{Red}\}; \text{Show}[p4, \text{ImageSize} \to \text{Large}]
\]
generates Figure 8.

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Figure 7. Phase Portrait of the center of $Z_0$.

Figure 8. Phase Portrait of trajectories in a neighborhood of the origin.

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