EXACT ASYMPTOTICS OF POSITIVE SOLUTIONS TO DICKMAN EQUATION

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Abstract. The paper considers the Dickman equation
\[ \dot{x}(t) = \frac{1}{t} x(t-1), \]
for \( t \to \infty \). The number theory uses what is called a Dickman (or Dickman-de Bruijn) function, which is the solution to this equation defined by an initial function \( x(t) = 1 \) if \( 0 \leq t \leq 1 \). The Dickman equation has two classes of asymptotically different positive solutions. The paper investigates their asymptotic behaviors in detail. A structure formula describing the asymptotic behavior of all solutions to the Dickman equation is given, an improvement of the well-known asymptotic behavior of the Dickman function, important in number theory, is derived and the problem of whether a given initial function defines dominant or subdominant solution is dealt with.

1. Introduction and preliminaries. The paper investigates the properties of solutions to the Dickman equation
\[ \dot{x}(t) = -\frac{1}{t} x(t-1) \tag{1} \]
for \( t \to \infty \) where \( t \geq t_0 > 0 \). Throughout the paper, the value \( t_0 \) may differ as different results are formulated and, in general, it is assumed to be sufficiently large in order to guarantee all the computations performed being well defined. This is mentioned in each particular case. A continuous function \( x: [t_0 - 1, \infty) \to \mathbb{R} \) is called a solution of (1) on \( [t_0 - 1, \infty) \) if it is continuously differentiable on \( [t_0, \infty) \) and satisfies (1) for every \( t \in [t_0, \infty) \) (at \( t = t_0 \), the derivative is regarded as the derivative on the right). The initial problem \( x = \varphi(t), t \in [t_0 - 1, t_0) \), where \( \varphi \) is a continuous function, defines a unique solution \( x = x(t_0, \varphi)(t), t \geq t_0 - 1 \) of (1) such that \( x(t_0, \varphi)(t) \equiv \varphi(t) \) if \( t \in [t_0 - 1, t_0] \).

A solution \( x \) of (1) on \( [t_0 - 1, \infty) \) is called positive if \( x(t) > 0 \) for every \( t \in [t_0 - 1, \infty) \), negative if \( x(t) < 0 \) for every \( t \in [t_0 - 1, \infty) \), and oscillating if it has arbitrarily large zeros on \( [t_0 - 1, \infty) \).

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The equation (1) keeps attracting much attention of experts in number theory and differential and difference equations, (see classic and recent sources, e.g., [1, 5, 6, 7, 8, 12, 18, 20, 21, 23, 24, 26, 27, 28], the www pages [25] and the references therein). The reason is that the investigation of the properties of solutions of (1) is closely connected with a problem in number theory, which we now shortly describe (see [1, 2, 6]).

Let $\Psi(y_1, y_2)$ be the number of positive integers not exceeding $y_1$ having no prime divisors exceeding $y_2$. Then,

$$\lim_{y \to \infty} \Psi(y^t, y) y^{-t} = \rho(t), \quad t > 0$$

where $\rho$ is what is called the Dickman function (or Dickman - de Bruin function because the latter author intensively studied it), defined for real $t \geq 0$ by the relation

$$\rho(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ \frac{1}{t} \int_{t-1}^{t} \rho(s) \, ds & \text{if } t > 1. \end{cases} \quad (2)$$

As noted, e.g., in [20], the Dickman function was first studied by Dickman [12] and later by de Bruijn [6, 7]. Differentiating (2), we can see that, assuming $t_0 = 1$, $x = \rho(t)$ is a solution of equation (1) satisfying the unit initial condition on $[0, 1]$. Moreover, $0 < \rho(t) \leq 1$, $|\rho'(t)| \leq 1$, $\rho(t)$ is nonincreasing, $t \in [0, \infty)$, and $\rho(t) \leq 1/\lfloor t \rfloor$ where $\lfloor \cdot \rfloor$ is the floor integer function. It is also known ([6], see also [2]) that

$$\rho(t) = \exp\left(-t \ln t - t \ln(\ln t) + O(t)\right), \quad t > 3 \quad (3)$$

(throughout the paper, we use the well-known Landau order symbols $O$ (“big” $O$) and $o$ (“small” $o$) in computations with $t \to \infty$). In [8] (see also [18]) an improved version of formula (3) is given:

$$\rho(t) = \exp\left(-t \ln t - t \ln(\ln t) + t \ln(\ln t) + \frac{t}{\ln t} + O\left(\frac{t(\ln(\ln t))^2}{(\ln t)^2}\right)\right) \quad (4)$$

for all sufficiently large $t$ and [24, p. 508] (formula for $j_\kappa^{(n+1)}(u)$ where $n = 0$ and $\kappa = 1$) includes an improvement of formula (4)

$$\rho(t) = \exp\left(-t \ln t - t \ln(\ln t) + t \ln(\ln t) + \frac{t}{\ln t} + \frac{t(\ln(\ln t))^2}{2(\ln t)^2} + O\left(\frac{t}{(\ln t)^2}\right)\right). \quad (5)$$

To the best of our knowledge, the formula (5) gives the best-possible asymptotic behavior of the function $\rho$ published in available sources. For an overview of properties of the function $\rho$ see also [18].

In the paper, we perform a qualitative analysis of the asymptotic behavior of the family of all solutions of (1) in terms of the theory of dominant and subdominant solutions to (1). We give the exact asymptotic behavior of the dominant solutions and sharp asymptotic behavior of the subdominant solutions. As a special case, we significantly improve the asymptotic behavior of the function $\rho$ (being a subdominant solution using the below terminology) given by formula (4).

The paper is organized as follows. In part 1.1, the theory of dominant and subdominant solutions is shortly described. Then, the main results of the paper in part 2 are formulated where the existence of dominant and subdominant solutions
to (1) and their asymptotic behaviors are discussed. Part 3 is devoted to important consequences of the main results. Namely, the structure formula describing the behavior of the family of all solutions to (1) is derived, the asymptotic behavior of Dickman function given by formula (5) is improved, and classes of initial functions defining either dominant or subdominant solutions are described. Proofs of the statements with the necessary auxiliary information are brought together in part 4.

1.1. Dominant and subdominant solutions. We will shortly overview the representation theory of solutions of equation (1) by what is called dominant and subdominant solutions. To this end, we adapt Theorems 8–10 and Definition 2 from [11] where more general equations than equation (1) are treated and formulate the relevant Theorems 1.1–1.3 and Definition 1.4. For this type of investigation, see also [15, 16, 17, 19].

Theorem 1.1. Let a positive solution of (1) on $[t_0 - 1, \infty)$ exist. Then, there exist two positive solutions $x_1$ and $x_2$ of (1) on $[t_0 - 1, \infty)$ satisfying

$$\lim_{t \to \infty} \frac{x_2(t)}{x_1(t)} = 0.$$ (6)

Moreover, every solution $x = x(t)$ of (1) on $[t_0 - 1, \infty)$ can be uniquely represented by the formula

$$x(t) = Kx_1(t) + O(x_2(t)),$$ (7)

where the constant $K$ depends on $x$.

Theorem 1.2. Let $\tilde{x}_1$ and $\tilde{x}_2$ be positive solutions of (1) on $[t_0 - 1, \infty)$ such that

$$\lim_{t \to \infty} \frac{\tilde{x}_2(t)}{\tilde{x}_1(t)} = 0.$$ (8)

Then, the formula (7) remains valid if $x_1$ is replaced by $\tilde{x}_1$, the constant $K$ is replaced by a constant $\tilde{K}$ and $x_2$ is replaced by $\tilde{x}_2$.

Theorem 1.3. The equation (1) cannot have three positive solutions $x_1$, $x_2$ and $x_3$ on $[t_0 - 1, \infty)$ satisfying the relations (6) and

$$\lim_{t \to \infty} \frac{x_3(t)}{x_2(t)} = 0.$$ (9)

The above theorems make possible the following definition.

Definition 1.4. Let the positive solutions $x_1$ and $x_2$ of (1) on $[t_0 - 1, \infty)$ satisfy relation (6). Then, we call the solution $x_1$ a dominant solution and the solution $x_2$ a subdominant solution.

2. Main results. In this part, we formulate the main results of the paper. First, in part 2.1, we investigate the exact asymptotic behavior of a class of dominant solutions. Next, in part 2.2, we describe the asymptotic behavior of a subdominant solution. The fact that both solutions represent a pair of dominant and subdominant solutions within the meaning of Definition 1.4 follows immediately from their asymptotic behavior (formulas (10) and (20) below) since it is a trivial matter to verify that (6) holds. The proofs of the statements in this part are given in part 4.

In what follows, we will work with what is called iterated logarithms. We define them as follows. The $n$th iterated logarithm $\ln_n t$ ($n \geq 0$) is defined as

$$\ln_n t := \ln(\ln_{n-1} t), \quad n \geq 1, \quad \ln_0 t := t$$
where we assume $t > e^{n-1}$ for this definition to be correct.

In parts 2.1, 2.2 below, we use the terms “dominant” and “subdominant” solution to (1) in advance. When the existence of both types of solutions is proved, a verification of Definition 1.4 is simple and is given in part 2.3.

2.1. Dominant solutions to (1). Let us look for a formal solution $x = x(t)$ to (1) in the form of a power series with negative powers in a neighborhood of infinity

$$x(t) = S(t) := \sum_{n=1}^{\infty} \frac{C_n}{t^n},$$

with real coefficients $C_n$ defined by the following lemma.

**Lemma 2.1.** Coefficients $C_n$ of the formal solution $x(t) = S(t)$ to (1) are defined by the formula

$$C_n = \frac{1}{n-1} \sum_{s=1}^{n-1} \left( \frac{n-1}{s-1} \right) C_s$$

where $n \geq 2$ and the coefficient $C_1$ is chosen arbitrarily.

The proof of Lemma 2.1 is given in part 4.1.

**Remark 1.** From (9) it follows that, if $C_1 > 0$, then $C_i > 0$ for all $i = 2, 3, \ldots$. Similarly, $C_i < 0$ for all $i = 2, 3, \ldots$ if $C_1 < 0$. To establish the convergence or divergence of the formal series $S(t)$ defined by (8) is an open problem since the well-known criteria for the convergence or divergence of power series are not directly applicable. An attempt to utilize formula (9) to get estimates of the coefficients for the convergence/divergence tests to be applicable does not lead to applicable estimates.

The convergence/divergence problem explained in Remark 1 is the reason why we derive the following result on the existence of solutions to equation (1) asymptotically described by the formal series $S(t)$ defined by (8).

**Theorem 2.2.** Let $p \in \mathbb{N}$ be fixed, let $C_1 > 0$ be fixed and let $\varepsilon$ be a positive number such that $\varepsilon > C_{p+1}$. Then, there exists a dominant solution $x(t)$ of (1) satisfying the inequality

$$\left| x(t) - \sum_{n=1}^{p} \frac{C_n}{t^n} \right| < \frac{\varepsilon}{t^{p+1}}$$

for all $t \in [t_0, \infty)$ if $t_0 = t_0(p)$ is sufficiently large.

The proof of Theorem 2.2 is given in part 4.2. Let us remark that the asymptotic relation (10) is often written as

$$x(t) \sim \sum_{n=1}^{\infty} \frac{C_n}{t^n}$$

where $t \to \infty$.

2.2. Subdominant solutions to (1). The statement of Theorem 2.2 implies the existence of positive solutions to (1) decreasing to zero as polynomials with negative powers do. In this part, we show that there exist positive solutions decreasing to zero even faster. Using the above terms, such solutions are called subdominant. We will describe them using a class of functions $M_\beta$ with specific asymptotic properties defined below.
Definition 2.3. By $\mathcal{M}_\beta$, we denote all continuously differentiable functions $\beta: [t_0, \infty) \to (0, \infty)$ satisfying

$$\lim_{t \to \infty} \frac{\beta(t) \ln^2 t}{t \ln_2 t} = 0, \quad (11)$$

$$\lim_{t \to \infty} \beta'(t) \ln t = 0, \quad (12)$$

$$\text{sign}(\beta(t) - 1) - \beta(t)) \neq 0, \quad (13)$$

$$\lim_{t \to \infty} (\beta(t) - 1 - \beta(t)) = 0, \quad (14)$$

$$\lim_{t \to \infty} \frac{1}{(\beta(t) - 1 - \beta(t)) \ln^2 t} = 0. \quad (15)$$

The following lemma says that $\mathcal{M}_\beta \neq \emptyset$ by defining a class of functions satisfying all afore-mentioned properties. In addition, a sign property, necessary in the following investigation, is emphasized.

Lemma 2.4. Let

$$\beta(t) = \mu t \cdot \frac{\ln^\nu t}{\ln^2 t \ln_3^\alpha t \ln_4^\gamma t \cdots \ln_s^\nu t} \quad (16)$$

where $t \geq t_0$, $\mu > 0$, $s \geq 3$, $0 < \nu < 1$ and $\alpha_i$, $i = 3, \ldots, s$ are arbitrary reals. Then, $\beta \in \mathcal{M}_\beta$. Moreover,

$$\text{sign}(\beta(t) - \beta(t)) = -1 \quad (17)$$

for all $t \in [t_0, \infty)$ assuming $t_0$ sufficiently large.

The proof of Lemma 2.4 is given in part 4.3.

The following result describes the asymptotic behavior of a subdominant solution.

Theorem 2.5. Let $\beta \in \mathcal{M}_\beta$ and

$$\text{sign}(\beta(t) - 1 - \beta(t)) = -1, \quad t \in [t_0, \infty). \quad (18)$$

Then, there exists a subdominant solution $x = x(t)$, $t \in [t_0 - 1, \infty)$ of $(1)$ such that the inequality

$$e^{-\beta(t)} < x(t) \cdot \exp \left( t \ln t + t \ln_2 t - t + t \ln_2^\alpha t + \frac{t \ln_2^2 t}{2 \ln_2 t} + 2 \frac{t \ln_2 t}{\ln_2 t} \right) < e^{\beta(t)} \quad (19)$$

holds for all $t \in [t_0 - 1, \infty)$ assuming $t_0$ sufficiently large.

The proof of Theorem 2.5 is given in part 4.4.

Remark 2. Due to (11), we have $\beta(t) = o(t \ln_2 t / \ln^2 t)$ when $t \to \infty$. Therefore, from inequality (19), we get

$$x(t) = \exp \left( -t \ln t - t \ln_2 t + t - t \frac{\ln_2 t}{\ln t} + t \frac{\ln_2^2 t}{2 \ln_2 t} - 2 \frac{t \ln_2 t}{\ln_2 t} \right) \quad (20)$$

for $t \to \infty$. Note that there are always functions from $\mathcal{M}_\beta$ satisfying (18) (e.g., functions of the set $\mathcal{M}_\beta$ described by formula (16) in Lemma 2.4). If $\beta \in \mathcal{M}_\beta$ and functions $\beta$ are defined by formula (16), from inequality (19), we get

$$x(t) = \exp \left( -t \ln t - t \ln_2 t + t - t \frac{\ln_2 t}{\ln t} + t \frac{\ln_2^2 t}{2 \ln_2 t} - 2 \frac{t \ln_2 t}{\ln_2 t} \right) \quad (21)$$
where the function $v$ satisfies $v(\infty) = 0$ and
\[
v(t) = O\left(\frac{\ln_{\nu - 1} t}{\ln_{\nu - 3}^3 t \ln_{\nu - 4}^4 t \cdots \ln_{\nu - s}^s t}\right)
\] as $t \to \infty$.

2.3. Specific dominant and subdominant solutions to (1). Let us show that Definition 1.4 is valid for dominant solutions described in part 2.1 and subdominant solutions defined in part 2.2. By $x_1(t)$, denote a solution $x = x(t)$ to equation (1) satisfying inequality (10) and, by $x_2(t)$, denote a solution $x = x(t)$ to equation (1) satisfying (20). We get (after simplification, l’Hospital’s rule can be applied, omitting tedious computations)
\[
\lim_{t \to \infty} \frac{x_2(t)}{x_1(t)} = \lim_{t \to \infty} \exp\left(-t \ln t - t \ln 2 + t \frac{\ln_2 t}{\ln t} + t - 2 \frac{t \ln_2 t}{2 \ln^2 t} (1 + o(1))\right) = 0
\] (23)
and Definition 1.4 holds.

3. Some consequences. The asymptotic behavior of both dominant and subdominant solutions to (1) together with representation formula (7) make it possible to formulate important properties of Dickman equation. Below, formula (7) is shown for the case in question, the asymptotic behavior of Dickman function is improved and a discussion on the sets of initial functions defining either dominant or subdominant solutions follows.

3.1. Structure formula describing the family of all solutions to (1). It is easy to write, utilizing formula (7) in Theorem 1.1, a structure formula describing the asymptotic behavior of all solutions to (1). As a solution $x_1$ we can take the solution described by formula (10) in Theorem 2.2, assuming that $C_1 > 0$ and $\varepsilon > 0$ are fixed, i.e.,
\[
x_1(t) := \sum_{n=1}^{p} \frac{C_n}{t^n} + O\left(\frac{\varepsilon}{t^{p+1}}\right).
\] (24)
As a solution $x_2$ we can take the solution described in Theorem 2.5 and given by formula (20) in Remark 2, i.e.
\[
x_2(t) := \exp\left(-t \ln t - t \ln 2 + t \frac{\ln_2 t}{\ln t} + t - 2 \frac{t \ln_2 t}{2 \ln^2 t} (1 + o(1))\right)
\] (25)
Since, by (23), $\lim_{t \to \infty} x_2(t)/x_1(t) = 0$, Theorem 1.1 holds and every solution $x = x(t)$ of (1) on $[t_0 - 1, \infty)$ can be represented by the formula
\[
x(t) = K x_1(t) + O(x_2(t)),
\] (26)
where the constant $K$ depends on $x$ and $x_1, x_2$ are defined by formulas (24), (25). From (26), we can, e.g., conclude that every oscillating solution $x = x(t)$ of (1) must satisfy (in such a case $K = 0$)
Theorem 3.1. For the Dickman function

\[ x(t) = O \left( \exp \left( -t \ln t - t \ln_2 t + t - \frac{\ln_2 t}{\ln t} + \frac{t \ln^2 t}{2 \ln^2 t} - 2 \frac{t \ln_2 t}{\ln^2 t} (1 + o(1)) \right) \right). \]

3.2. Improved asymptotic behavior of the Dickman function. The above results make an important contribution to the investigation of the Dickman equation by making it possible to improve the asymptotic behavior of Dickman function, given by formulas (3)–(5).

The following theorem provides the relevant statement and is proved in part 4.5.

**Theorem 3.1.** For the Dickman function \( \rho(t) \) defined by (2), which is the solution of equation (1) satisfying the unit initial condition on \([0, 1]\), we have

\[ \rho(t) = \exp \left( -t \ln t - t \ln_2 t + t - \frac{\ln_2 t}{\ln t} + \frac{t \ln^2 t}{2 \ln^2 t} - 2 \frac{t \ln_2 t}{\ln^2 t} + o \left( \frac{t \ln^2 t}{\ln^2 t} \right) \right), \]

for \( t \to \infty \) where \( \nu \in (0, 1) \).

3.3. On initial functions defining dominant and subdominant solutions.

Let \( x(t) = x(t_0, \varphi)(t) \) be the unique solution of (1) with the initial data

\[ x(t) = \varphi(t), \quad \text{for every} \quad t \in [t_0 - 1, t_0] \quad (28) \]

where \( \varphi: [t_0 - 1, t_0] \to \mathbb{R} \) is a continuous function. The initial-value problem (1), (28) is equivalent to the following one

\[ x(t) = \begin{cases} 
\varphi(t) & \text{if } t_0 - 1 \leq t \leq t_0, \\
\frac{1}{t} \int_{t_0}^{t} x(s) ds + \frac{1}{t} C(t_0, \varphi) & \text{if } t > t_0, 
\end{cases} \quad (29) \]

where

\[ C(t_0, \varphi) = t_0 \varphi(t_0) - \int_{t_0}^{t_0} \varphi(s) ds. \]

For \( t_0 = 1 \), the limit value

\[ \lim_{t \to \infty} t \cdot x(1, \varphi)(t) = C(1, \varphi) \]

is discussed, e.g., in [11, 14, 20, 28]. If the limit exists and is finite, then

\[ C(1, \varphi) = \varphi(1) - \int_{0}^{1} \varphi(t) dt. \quad (31) \]

Formula (31) is derived in [28] but without proving the existence of the limit (30). The existence of the limit can be deduced from the results in [11] and [14], or from formula (26), but these results cannot be used to derive formula (31). In [20], the authors gave an alternative proof of the limit equation (31) including the existence of the limit (30) (in connection with a discussion on the asymptotic convergence of solutions, we also refer to [3, 4]). Let us also mention a recent paper [13], which describes a method for studying the asymptotic behavior of the dominant positive solutions to a similar class of scalar delay differential equations.

Analyzing the initial-value problems (1), (28) and (29) with \( t_0 = 1 \), we conclude that the following theorem is valid.

**Theorem 3.2.** Let \( \varphi(t) > 0, \ t \in [0, 1] \) and \( C(1, \varphi) \geq 0 \). Then, the solution \( x(1, \varphi)(t) \) of the initial-value problem (1), (28) is positive for every \( t \in [0, \infty) \).

If, moreover, \( C(1, \varphi) > 0 \), then \( x(1, \varphi)(t) \) is a dominant solution to (1) and, if \( C(1, \varphi) = 0 \), then \( x(1, \varphi)(t) \) is a subdominant solution to (1).
The exact asymptotic behavior of the dominant solution \( x(1, \varphi)(t) \) is specified in following theorem.

**Theorem 3.3.** Let \( \varphi(t) > 0, t \in [0,1] \) and \( C(1, \varphi) > 0 \). Then, the dominant solution \( x(1, \varphi)(t) \) of the initial-value problem (1), (28) for \( t \to \infty \) is asymptotically described as

\[
x(1, \varphi)(t) \sim \sum_{n=1}^{\infty} \frac{C_n(\varphi)}{t^n} + O(x_2(t))
\]

where the coefficients of the series in (32) are computed by formulas

\[
C_n(\varphi) = \frac{1}{n-1} \sum_{s=1}^{n-1} \binom{n-1}{s-1} C_s(\varphi), \quad n \geq 2, \quad C_1(\varphi) = C(1, \varphi)
\]

and \( x_2(t) \) is arbitrary subdominant solution to (1).

The theorem is proved in part 4.6.

**Remark 3.** Compare the asymptotic behavior of \( x(1, \varphi)(t) \) given by formulas (30) and (32). By (30) we get

\[
x(1, \varphi)(t) = \frac{C_1(\varphi)}{t} (1 + o(1))
\]

for \( t \to \infty \). Describing exactly the asymptotic behavior by a power series with negative powers, formula (32) substantially improves formula (34). The order of asymptotic accuracy is improved as well. A subdominant solution \( x_2(t) \) can be described, e.g., by formula (25) and, therefore, \( \lim_{t \to \infty} t^n x_2(t) = 0 \) for arbitrary \( n \in \mathbb{N} \).

4. **Proofs and additional material.** This part contains proofs of the statements formulated above and the necessary auxiliary results and material.

The proofs of the main results utilize the Ważewski method in a setting suitable to be applied to delayed differential equations [22]. This method is used in [9] to prove a theorem on the existence of solutions of delayed functional differential equations with graphs embedded in a previously defined domain. We employ the following particular case (but sufficient to determine the asymptotic behavior of dominant and subdominant solutions to (1)) of Theorem 1 in [9] adapted for equation (1).

For continuous functions \( \pi, \delta: [t_0 - 1, \infty) \to \mathbb{R} \) with \( \pi(t) < \delta(t) \) for \( t \in [t_0 - 1, \infty) \), continuously differentiable on \([t_0, \infty)\), we define \( \omega := \{(t, y): t \in [t_0 - 1, \infty), \pi(t) < y < \delta(t)\} \).

**Theorem 4.1.** Assume that \( \forall t \geq t_0 \) and for every continuous function \( \varphi: [t-1, t] \to \mathbb{R} \) such that \((t + \theta, \varphi(t + \theta)) \in \omega, \forall \theta \in [-1, 0)\), inequalities

\[
\begin{align*}
\delta'(t) &< -\frac{1}{t} \varphi(t - 1), \\
\pi'(t) &> -\frac{1}{t} \varphi(t - 1)
\end{align*}
\]

hold. Then, there exists a solution \( x \) of (1) on \([t_0 - 1, \infty)\) such that

\[
\pi(t) < x(t) < \delta(t)
\]

for every \( t \in [t_0 - 1, \infty) \).
For further computations, we need auxiliary formulas on asymptotic decompositions given in Lemma 4.1 and Lemma 4.2 in [10]. The following lemma summarizes the necessary formulas.

**Lemma 4.2.** Let reals \( \sigma \) and \( \tau \) be fixed. Then, the following asymptotic representation

\[
(t - \tau)^\sigma = t^\sigma \left[ 1 - \frac{\sigma \tau}{t} + \frac{\sigma(\sigma - 1)\tau^2}{2t^2} - \frac{\sigma(\sigma - 1)(\sigma - 2)\tau^3}{6t^3} + o\left(\frac{1}{t^3}\right) \right] \quad (38)
\]

holds for \( t \to \infty \). Let \( k \in \mathbb{N} \) and reals \( \sigma, \tau \) be fixed. Then, the following asymptotic representation

\[
\ln_k^\sigma (t - \tau) = (\ln_k t)^\sigma \left[ 1 - \frac{\sigma \tau}{t \ln t \ln_2 t \cdots \ln_{k-1} t \ln_k t} - \frac{\sigma \tau^2}{2t \ln t \ln_2 t \cdots \ln_{k-1} t \ln_k t} t \ln_k t - \cdots \right]
\]

\[
- \frac{\sigma \tau^2}{2t \ln t \ln_2 t \cdots \ln_{k-1} t \ln_k t} t \ln_k t \ln_2 t \cdots \ln_{k-1} t \ln_k t + \frac{(\sigma \tau)^2}{2t \ln t \ln_2 t \cdots \ln_{k-1} t \ln_k t} t \ln_k t + o\left(\frac{1}{t^3}\right) \quad (39)
\]

holds for \( t \to \infty \).

4.1. **Proof of Lemma 2.1.** In the below computations, for arbitrary non-negative integers \( n \) and \( \lambda \), we define binomial number

\[
\binom{-n}{\lambda} = \begin{cases} 
-n(-n-1)\cdots(-n-(\lambda-1)) & \text{if } \lambda > 0, \\
1 & \text{if } \lambda = 0.
\end{cases}
\]

Substituting \( x(t) = \sum_{n=1}^{\infty} C_n/t^n \) in (1), we get

\[
- \sum_{n=1}^{\infty} \frac{nC_n}{t^{n+1}} = - \frac{1}{t} \sum_{n=1}^{\infty} \frac{C_n}{(t-1)^n}
\]

or, assuming \( t \geq t_0 > 1 \),

\[
\sum_{n=1}^{\infty} \frac{nC_n}{t^n} = \sum_{n=1}^{\infty} \frac{C_n}{(t-1)^n} = \sum_{n=1}^{\infty} \frac{C_n}{t^n} \left(1 - \frac{1}{t}\right)^{-n} = \sum_{n=1}^{\infty} \frac{C_n}{t^n} \sum_{\lambda=0}^{\infty} \binom{-n}{\lambda} \left(\frac{1}{t}\right)^{\lambda} \quad (40)
\]

\[
= \sum_{n=1}^{\infty} C_n \sum_{\lambda=0}^{\infty} (-1)^\lambda \binom{-n}{\lambda} \left(\frac{1}{t}\right)^{\lambda + \lambda} = [\text{set } \lambda = s - n] 
\]

\[
= \sum_{s=n}^{\infty} C_n \sum_{s=n}^{\infty} (-1)^{s-n} \binom{-n}{s-n} \left(\frac{1}{t}\right)^s = [\text{after re-arranging powers}] 
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{t^n} \sum_{\ell=1}^{n} (-1)^{n-\ell} C_\ell \binom{-\ell}{n-\ell}, \quad (41)
\]

Matching the multipliers of the identical powers in (40) and (41), we obtain

\[
nC_n = \sum_{\ell=1}^{n} (-1)^{n-\ell} C_\ell \binom{-\ell}{n-\ell} = \sum_{\ell=1}^{n-1} (-1)^{n-\ell} C_\ell \binom{-\ell}{n-\ell} + (-1)^n C_n \binom{-n}{0}
\]

or
\[(n - 1)C_n = \sum_{\ell=1}^{n-1} (-1)^{n-\ell} C_{\ell} \binom{n-\ell}{n} = \text{[re-arrange and then set } \ell = s] \]
\[
= \sum_{s=1}^{n-1} \binom{n-1}{s-1} C_s. \tag{42}
\]

The last formula is equivalent to (9), i.e., a formal solution \(S(t)\) to (1) is of form (8) with coefficients defined by (9).

**4.2. Proof of Theorem 2.2.** In the proof, we apply Theorem 4.1 with

\[
\pi(t) := \sum_{n=1}^{p} \frac{C_n}{t^n} - \frac{\varepsilon}{t^{p+1}}, \quad \delta(t) := \sum_{n=1}^{p} \frac{C_n}{t^{n+1}} + \frac{\varepsilon}{t^{p+1}}.
\]

Verify inequality (35) first. Due to the assumptions of Theorem 4.1, we have \(\varphi(t + \theta) < \delta(t + \theta)\) for every \(\theta \in [-1, 0]\) and, therefore, \(\varphi(t - 1) < \delta(t - 1)\). Since \(-\varphi(t - 1) > -\delta(t - 1)\), inequality (35) will be valid if

\[
\delta'(t) = -\frac{1}{t}\delta(t - 1). \tag{43}
\]

So, it is sufficient to show that (43) holds. For the left-hand side of (43), we get

\[
\delta'(t) = \left(\sum_{n=1}^{p} \frac{C_n}{t^n} + \frac{\varepsilon}{t^{p+1}}\right)' = -\sum_{n=1}^{p} nC_n \frac{1}{t^{n+1}} - \frac{\varepsilon(p + 1)}{t^{p+2}}
\]

and the right-hand side of (43) turns into (we use formula (9) with some technicalities being the same as in the proof of Lemma 2.1)

\[
-\frac{1}{t}\delta(t - 1) = -\frac{1}{t} \left( \sum_{n=1}^{p} \frac{C_n}{(t-1)^n} + \frac{\varepsilon}{(t-1)^{p+1}} \right)
\]

\[
= -\frac{1}{t} \left( \sum_{n=1}^{p} \frac{C_n}{t^n} \left(1 - \frac{1}{t}\right)^{-n} + \frac{\varepsilon}{t^{p+1}} \left(1 - \frac{1}{t}\right)^{-(p+1)} \right)
\]

\[
= -\sum_{n=1}^{p} \frac{C_n}{t^{n+1}} \sum_{\lambda=0}^{\infty} \binom{-n}{\lambda} \left(\frac{-1}{t}\right)^{\lambda} \frac{\varepsilon}{t^{p+2}} \sum_{\lambda=0}^{\infty} \binom{-p-1}{\lambda} \left(\frac{-1}{t}\right)^{\lambda}
\]

\[
= -\sum_{n=1}^{p} C_n \sum_{s=n}^{\infty} \frac{(-1)^s}{s - n} \left(\frac{1}{7}\right)^{s+1} - \frac{\varepsilon}{t^{p+2}} \left(1 + O\left(\frac{1}{7}\right)\right)
\]

\[
= -\sum_{n=1}^{p} \frac{1}{t^{n+1}} \sum_{\ell=1}^{n} \frac{(-1)^{n-\ell} C_{\ell}}{t^{n-\ell}}
\]

\[
- \frac{1}{t^{p+2}} \sum_{\ell=1}^{p+1} \frac{(-1)^{p+1-\ell} C_{\ell}}{t^{p+1-\ell}} + O\left(\frac{1}{t^{p+3}}\right) - \frac{\varepsilon}{t^{p+2}} \left(1 + O\left(\frac{1}{7}\right)\right)
\]

\[
= -\sum_{n=1}^{p} \frac{1}{t^{n+1}} \sum_{\ell=1}^{n} \frac{(-1)^{n-\ell} C_{\ell}}{t^{n-\ell}} - \frac{1}{t^{p+2}} \sum_{\ell=1}^{p+1} \frac{(-1)^{p+1-\ell} C_{\ell}}{t^{p+1-\ell}}
\]

\[
+ \frac{1}{t^{p+2}} (-1)^p C_{p+1} \left(\frac{-\ell}{0}\right) - \frac{\varepsilon}{t^{p+2}} + O\left(\frac{1}{t^{p+3}}\right)
\]
\[= -\sum_{n=1}^{p} \frac{nC_n}{t^{n+1}} - \frac{(p+1)C_{p+1}}{t^{p+2}} + \frac{C_{p+1}}{t^{p+2}} - \varepsilon \frac{1}{t^{p+2}} + O\left(\frac{1}{t^{p+3}}\right)\]

Now it is easy to see that (43) will hold if
\[\sum_{n=1}^{p} \frac{nC_n}{t^{n+1}} + \varepsilon(p+1) < \sum_{n=1}^{p} \frac{nC_n}{t^{n+1}} + \frac{pC_{p+1}}{t^{p+2}} + \varepsilon \frac{1}{t^{p+2}} + O\left(\frac{1}{t^{p+3}}\right)\]

or
\[\varepsilon > C_{p+1} + O\left(\frac{1}{t}\right)\]

This inequality is valid for all sufficiently large \( t \) and, therefore, inequality (43) holds.

Now we show that inequality (36) holds as well. Since \( \varphi(t-1) > \pi(t-1) \), inequality (36) will be valid if
\[\pi'(t) > -\frac{1}{t} \pi(t-1)\]

Proceeding as above, we obtain inequality (44) where the symbol of inequality “\( > \)” is replaced by the opposite symbol “\( < \)” and \( \varepsilon \) replaced by \( -\varepsilon \). Consequently, we conclude that inequality (36) holds if an inequality relevant to (45), i.e.,
\[-\varepsilon < C_{p+1} + O\left(\frac{1}{t}\right)\]

is valid. Since the last inequality holds for all sufficiently large \( t \), inequalities (46) and (36) hold as well.

4.3. Proof of Lemma 2.4. By L’ Hospital’s rule we can verify that
\[\lim_{t \to \infty} \frac{\beta(t) \ln^2 t}{t \ln t} = \lim_{t \to \infty} \mu t \cdot \frac{\ln^2 t}{t \ln \alpha_3 t \ln \alpha_4 t \cdots \ln \alpha_s t} \cdot \frac{\ln^2 t}{t \ln \alpha_3 t} = \lim_{t \to \infty} \mu \cdot \frac{\ln^2 t}{t \ln \alpha_3 t \ln \alpha_4 t \cdots \ln \alpha_s t} = 0.\]

This means that the first assumption (11) holds. Compute
\[\beta'(t) = \mu \cdot \frac{\ln^2 t}{\ln^2 t \ln \alpha_3 t \ln \alpha_4 t \cdots \ln \alpha_s t} + \mu^2 \cdot \frac{\ln^2 t}{\ln^3 t \ln \alpha_3 t \ln \alpha_4 t \cdots \ln \alpha_s t} - 2\mu \cdot \frac{\ln^2 t}{\ln^3 t \ln \alpha_3 t \ln \alpha_4 t \cdots \ln \alpha_s t} - \alpha_3 \mu \cdot \frac{\ln^2 t}{\ln^3 t \ln \alpha_3 t \ln \alpha_4 t \cdots \ln \alpha_s t} - \cdots - \alpha_s \mu \cdot \frac{\ln^2 t}{\ln^3 t \ln \alpha_3 t \ln \alpha_4 t \cdots \ln \alpha_s t}.
\]

This expression clearly implies
\[\beta'(t) = \mu \cdot \frac{\ln^2 t}{\ln^2 t \ln \alpha_3 t \ln \alpha_4 t \cdots \ln \alpha_s t} \left(1 + o(1)\right)\]
for \( t \to \infty \). Then,
\[
\lim_{t \to \infty} \beta'(t) \ln t = \lim_{t \to \infty} \mu \cdot \frac{\ln^\nu t}{\ln t \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t} (1 + o(1)) = 0
\]
and (12) holds.

Now consider the difference (whenever it is necessary we use formula (39))
\[
\beta(t-1) - \beta(t)
\]
\[
= \mu(t-1) \cdot \frac{\ln^\nu t}{\ln^2 t \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t}
\]
\[
- \mu t \cdot \frac{\ln^\nu t}{\ln^2 t \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t}
\]
\[
= \mu t \cdot \frac{\ln^\nu t}{\ln^2 t \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t}
\]
\[
	imes \left[ \frac{(t-1) \ln^\nu t}{t} \left(1 - \frac{\nu}{t \ln t} + O \left(\frac{1}{t^2}\right)\right) \left(1 + \frac{2}{t \ln t} + O \left(\frac{1}{t^2}\right)\right) \right.
\]
\[
	imes \left(1 + \frac{\alpha_3}{t \ln t} \ln^\alpha_3 t + O \left(\frac{1}{t^2}\right)\right) \left(1 + \frac{\alpha_4}{t \ln t} \ln^\alpha_4 t + O \left(\frac{1}{t^2}\right)\right) \cdots
\]
\[
\left. \cdots \left(1 + \frac{\alpha_s}{t \ln t} \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t + O \left(\frac{1}{t^2}\right)\right) - 1 \right]
\]
\[
= \mu t \cdot \frac{\ln^\nu t}{\ln^2 t \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t}
\]
\[
	imes \left[ 1 - \frac{1}{t} - \frac{\nu}{t \ln t} + O \left(\frac{1}{t^2}\right) + \frac{2}{t \ln t} + \frac{\alpha_3}{t \ln t} \ln^\alpha_3 t + \cdots
\]
\[
+ \frac{\alpha_s}{t \ln t} \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t - 1 + O \left(\frac{1}{t^2}\right) \right]
\]
\[
= - \mu \cdot \frac{\ln^\nu t}{\ln^2 t \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t} (1 + o(1)).
\]
(47)

Obviously, (13) is valid. Moreover, (14) is valid as well since
\[
\lim_{t \to \infty} (\beta(t-1) - \beta(t)) = 0.
\]

The last property (15) holds because
\[
\lim_{t \to \infty} \frac{1}{(\beta(t-1) - \beta(t)) \ln^2 t} = - \lim_{t \to \infty} \frac{1}{\mu \cdot \ln^2 t \ln^\alpha_3 t \ln^\alpha_4 t \cdots \ln^\alpha_s t} \frac{1}{\ln^2 t}.
\]
Lemma 4.3. Assuming $t \to \infty$, we utilize the following auxiliary result.

From (47), we get

$$\text{sign}(\beta(t) - \beta(t)) = -1$$

for all $t \in [t_0, \infty)$ assuming $t_0$ is sufficiently large. Then, formula (17) is true and Lemma 2.4 is proved.

4.4. Proof of Theorem 2.5. Define auxiliary functions

$$x_{\beta}(t) = \exp \left( -t \ln t - t \ln t + t - t \frac{\ln_2 t}{\ln t} + t + t \frac{\ln^2_2 t}{2 \ln^2 t} - 2 \frac{t \ln_2 t}{\ln^2 t} + \beta(t) \right)$$

and

$$x_{-\beta}(t) = \exp \left( -t \ln t - t \ln t + t - t \frac{\ln_2 t}{\ln t} + t + t \frac{\ln^2_2 t}{2 \ln^2 t} - 2 \frac{t \ln_2 t}{\ln^2 t} - \beta(t) \right).$$

In the proof, we utilize the following auxiliary result.

Lemma 4.3. Assuming $t_0$ is sufficiently large, we have

$$\text{sign} \left( \frac{1}{t} x_{\beta}(t) - x_{\beta}'(t) \right) = -\text{sign}(\beta(t) - \beta(t))$$

for all $t \in [t_0, \infty)$.

Proof. Using (38) and (39), we develop an asymptotic decomposition of

$$- \frac{1}{t} x_{\beta}(t) - x_{\beta}'(t)$$

for $t \to \infty$. We will preserve the necessary order of asymptotic accuracy. We get

$$- \frac{1}{t} x_{\beta}(t) - x_{\beta}'(t)$$

$$= - \frac{1}{t} \exp \left( -(t - 1) \ln(t) - (t - 1) \ln_2(t - 1) + (t - 1) - (t - 1) \frac{\ln_2(t - 1)}{\ln(t - 1)} \right)$$

$$+ \frac{t - 1}{\ln(t - 1)} + \frac{t - 1}{\ln^2(t - 1)} - 2 \left( \frac{t - 1}{\ln(t - 1)} \right)$$

$$= - \frac{1}{t} \exp \left( -t \ln t - t \ln_2 t + t - t \frac{\ln_2 t}{\ln t} + t + t \frac{\ln^2_2 t}{2 \ln^2 t} - 2 \frac{t \ln_2 t}{\ln^2 t} + \beta(t) \right)$$

$$\times \exp \left( -t \ln_2 t - t \ln_2(t - 1) \ln t \right) \exp \left( t - 1 - t \frac{\ln_2 t}{\ln t} \right)$$

$$\times \exp \left( t - 1 - t \frac{\ln_2 t}{\ln t} \ln(t - 1) \right) \exp \left( \frac{t - 1}{\ln t} \right)$$

$$\times \exp \left( -t \ln_2 t - t + t \frac{\ln_2 t}{\ln t} \ln(t - 1) \right) \exp \left( -t \ln_2 t - t + t \frac{\ln_2 t}{\ln t} \ln(t - 1) \right)$$

$$\times \exp \left( t - 1 - t \frac{\ln_2 t}{\ln t} \ln(t - 1) \right) \exp \left( \frac{t - 1}{\ln t} \right)$$

$$\times \exp \left( -t \ln t - t \ln_2 t + t - t \frac{\ln_2 t}{\ln t} + t + t \frac{\ln^2_2 t}{2 \ln^2 t} - 2 \frac{t \ln_2 t}{\ln^2 t} + \beta(t) \right)$$

$$- \left( \exp \left( -t \ln t - t \ln_2 t + t - t \frac{\ln_2 t}{\ln t} + t + t \frac{\ln^2_2 t}{2 \ln^2 t} - 2 \frac{t \ln_2 t}{\ln^2 t} + \beta(t) \right) \right)'$$
\[
\begin{align*}
&= -\frac{1}{t} \exp \left( -t \ln t \left( 1 - \frac{1}{t} \right) \left( 1 - \frac{1}{t \ln t} + O \left( \frac{1}{t^2 \ln t} \right) \right) \right) \\
&\quad \times \exp \left( -t \ln t \left( 1 - \frac{1}{t} \right) \left( 1 - \frac{1}{t \ln t \ln t} + O \left( \frac{1}{t^2 \ln t \ln t} \right) \right) \right) \\
&\quad \times \exp \left( t \left( 1 - \frac{1}{t} \right) \right) \\
&\quad \times \exp \left( -t \ln t \left( 1 - \frac{1}{t} \right) \left( 1 - \frac{1}{t \ln t \ln t} + O \left( \frac{1}{t^2 \ln t \ln t} \right) \right) \right) \\
&\quad \times \left( 1 + \frac{1}{t \ln t} + O \left( \frac{1}{t^2 \ln t} \right) \right) \\
&\quad \times \exp \left( \ln t \ln t \left( 1 - \frac{t \ln t}{t} + \ln t + \ln \ln t + O \left( \frac{1}{t} \right) \right) \right) \\
&\quad \times \exp \left( -t \ln t - t \ln t + t - \frac{\ln t}{\ln t} + \ln t + \frac{t \ln t}{t} + \frac{t \ln t}{\ln t} + \frac{t \ln t}{\ln^2 t} + \beta(t) \right) \\
&\quad + \ln t + 1 + \ln t + \frac{1}{\ln t} - 1 + \ln t + \frac{1}{\ln t} - \frac{\ln t}{\ln t} - \frac{\ln t}{\ln t} - \frac{1}{\ln t} + \frac{1}{\ln^2 t} \\
\end{align*}
\]
\[
- \frac{\ln^2 t}{2 \ln^3 t} - \frac{\ln t}{\ln^3 t} + \frac{\ln^2 t}{\ln^3 t} + 2 \frac{\ln t}{\ln^3 t} + 2 \frac{\ln^3 t}{\ln^3 t} - 4 \frac{\ln t}{\ln^3 t} - \beta'(t)
\]

\[
= x_\beta(t) \left[ -\exp \left( \ln t + \frac{1}{\ln t} + O \left( \frac{1}{t} \right) \right) \exp \left( \frac{\ln^2 t}{\ln t} - \frac{\ln t}{\ln^3 t} + O \left( \frac{1}{t} \right) \right) \right.
\]

\[
\times \exp \left( - \frac{1}{\ln t} + \frac{1}{\ln^2 t} + O \left( \frac{1}{t} \right) \right) \exp \left( - \frac{\ln^2 t}{2 \ln^2 t} - \frac{\ln^3 t}{\ln^3 t} + O \left( \frac{1}{t} \right) \right) \exp \left( \frac{2 \ln t}{\ln^2 t} - \frac{2 \ln^2 t}{\ln^3 t} + O \left( \frac{1}{t} \right) \right)
\]

\[
\times \exp \left( 2 \ln t + \frac{1}{\ln^3 t} + 2 \frac{\ln^3 t}{\ln^3 t} - 5 \frac{\ln t}{\ln^3 t} + 2 \frac{\ln t}{\ln^3 t} - 2 \frac{\ln t}{\ln^3 t} \right)
\]

\[
= x_\beta(t) \left[ -(\ln t) \exp \left( \frac{\ln^2 t}{\ln t} + \frac{2}{\ln^3 t} + \frac{\ln^2 t}{\ln^3 t} - \frac{\ln^3 t}{2 \ln^2 t} - 5 \frac{\ln t}{\ln^3 t} + \frac{\ln^2 t}{\ln^3 t} + 2 \frac{\ln t}{\ln^3 t} \right)
\]

\[
+ \left( \beta(t - 1) - \beta(t) \right) + O \left( \frac{1}{t} \right) \ln t + \frac{\ln t}{\ln^3 t}
\]

\[
+ \frac{2}{\ln^2 t} + \frac{\ln t}{\ln^3 t} + \frac{\ln^2 t}{2 \ln^2 t} - 5 \frac{\ln t}{\ln^3 t} + \frac{\ln t}{\ln^3 t} + 2 \frac{\ln t}{\ln^3 t} - \beta'(t) \right] = (*)
\]

Since (14) holds (\(\lim_{t \to \infty} (\beta(t - 1) - \beta(t)) = 0\)), we can asymptotically decompose the exponential function omitting the asymptotically superfluous expressions in (*) to obtain

\[
(*) = x_\beta(t) \left[ -(\ln t) \left( 1 + \frac{\ln^3 t}{\ln^3 t} + \frac{2}{\ln^3 t} + \frac{\ln^2 t}{\ln^3 t} - \frac{\ln^3 t}{2 \ln^2 t} - 5 \frac{\ln^2 t}{\ln^3 t} + \frac{\ln^2 t}{\ln^3 t} + 2 \frac{\ln^2 t}{\ln^3 t} \right)
\]

\[
+ \left( \beta(t - 1) - \beta(t) \right) + O \left( \frac{1}{t} \right) \right)^2
\]

\[
+ O \left( \frac{\ln^3 t}{\ln^3 t} \right) + O \left( \frac{\ln^3 t}{\ln^2 t} (\beta(t - 1) - \beta(t)) \right)
\]

\[
+ O \left( \frac{\ln^2 t}{\ln t} (\beta(t - 1) - \beta(t))^2 \right) + O \left( (\beta(t - 1) - \beta(t))^3 \right)
\]

\[
+ \ln t + \ln^2 t + \frac{2}{\ln^3 t} + \frac{\ln t}{\ln^3 t} - \frac{\ln^2 t}{2 \ln^2 t}
\]
−5 \ln^2 t \left( \frac{1}{\ln^2 t} + \frac{1}{\ln^2 t} + \frac{2}{\ln^2 t} - \beta'(t) \right)

= x_β(t) \left[ -\ln t - \ln^2 t - \frac{2}{\ln t} - \frac{\ln^2 t}{\ln t} - \frac{\ln^2 t}{2 \ln^2 t} - \frac{\ln^2 t}{2 \ln^2 t} - \beta'(t) \right]

− (\ln t) (\beta(t - 1) - \beta(t)) + O \left( \frac{\ln t}{t} \right)

− \frac{1}{2} \ln t \left( \ln^2 t + (\beta(t - 1) - \beta(t))^2 + 2 \ln^2 t \right) (\beta(t - 1) - \beta(t))

+ O \left( \ln^3 t \right) + O \left( \frac{\ln^2 t}{\ln^2 t} (\beta(t - 1) - \beta(t)) \right)

+ O \left( \ln^2 t (\beta(t) - \beta(t))^2 \right) + O \left( (\beta(t - 1) - \beta(t))^3 \right)

+ \ln t + \ln^2 t + \frac{\ln^2 t}{\ln t} + \frac{\ln^2 t}{\ln^2 t} - \frac{\ln^2 t}{2 \ln^2 t}

− 5 \ln^2 t \left( \frac{1}{\ln^3 t} + \frac{1}{\ln^3 t} + \frac{2}{\ln^3 t} - \beta'(t) \right) = [ \text{due to (12)} ]

= x_β(t) \left[ -\frac{2}{\ln t} - (\ln t) (\beta(t - 1) - \beta(t)) - \frac{1}{2} \ln t (\beta(t - 1) - \beta(t))^2

− (\ln^2 t) (\beta(t - 1) - \beta(t)) - \beta'(t)

+ O \left( \ln^3 t \right) + O \left( \frac{\ln^2 t}{\ln t} (\beta(t - 1) - \beta(t)) \right)

+ O \left( \ln^2 t (\beta(t - 1) - \beta(t))^2 \right) + O \left( (\ln t) (\beta(t - 1) - \beta(t))^3 \right) \right]

= (**) \quad (50)

Now we determine the asymptotically leading term in (**) . Obviously, only two terms

− \frac{2}{\ln t} \text{ and } - (\ln t) (\beta(t - 1) - \beta(t))

might determine the asymptotic behavior of (**) because the impact of \beta' is eliminated by (12) . We show that this is really true. Assuming

(****) \sim x_β(t) \left[ -\frac{2}{\ln t} - (\ln t) (\beta(t - 1) - \beta(t)) \right]

we get

(**) \sim -x_β(t)(\ln t) (\beta(t - 1) - \beta(t)) \left[ 1 + \frac{2}{\ln^2 t (\beta(t - 1) - \beta(t))} \right].

Since (15) holds, the limit of the second term in square brackets equals zero. Therefore, our assumption is true and

(**) \sim x_β(t) \left[ - (\ln t) (\beta(t - 1) - \beta(t)) \right]. \quad (51)
Now it can be seen that
\[ \text{sgn} \left( -\frac{1}{t} x_\beta(t) - x'_\beta(t) \right) = -\text{sgn} (\beta(t) - 1) \]
for all \( t \in [t_0, \infty) \) assuming \( t_0 \) is sufficiently large and (48) holds.

Now we are able to finish the proof of Theorem 2.2. We utilize Theorem 4.1 with 
\[ \delta(t) := x_\beta(t), \quad \pi(t) := x_{-\beta}(t). \]
First, verify inequality (35). Due to the assumptions of Theorem 4.1, we have
\[ -\varphi(t + \theta) < \delta(t + \theta) \]
for every \( \theta \in [-1, 0] \) and, therefore, \( \varphi(t) < \delta(t) \). Since
\[ -\varphi(t - 1) > -\delta(t - 1), \]
inequality (35) will be valid if
\[ \delta'(t) < -\frac{1}{t} \delta(t - 1), \]
i.e., if
\[ -\frac{1}{t} x_\beta(t - 1) - x'_\beta(t) > 0. \]
Since, by Lemma 4.3 (formula (48)) and by (18),
\[ \text{sign} \left( -\frac{1}{t} x_\beta(t - 1) - x'_\beta(t) \right) = -\text{sign}(\beta(t) - 1) = 1, \]
inequality (35) holds.

Now we will show that inequality (36) holds as well. It is obvious that, if the function \( \beta \) satisfies all properties (11)–(15), then the function \((-\beta)\) satisfies them as well. Since \( \varphi(t - 1) > \pi(t - 1) \), inequality (36) will be valid if
\[ \pi'(t) > -\frac{1}{t} \pi(t - 1) \]
i.e., if
\[ -\frac{1}{t} x_{-\beta}(t - 1) - x'_{-\beta}(t) < 0. \]
Replacing \( \beta \) by \((-\beta)\) in (49) and carefully tracing the proof of Lemma 4.3, we see that the substitution of the term \((-\beta)'/(+\beta)'\) in (50) has no impact on the leading asymptotical term. Then, in accordance with (51) (where \( x_\beta \) is replaced by \( x_{-\beta} \) and \( \beta \) by \((-\beta)\) ), we derive
\[ -\frac{1}{t} x_{-\beta}(t - 1) - x'_{-\beta}(t) \sim x_{-\beta}(t) \left[ -\left( \ln t \right) \left( -\beta(t - 1) + \beta(t) \right) \right] \]
and, finally,
\[ -\frac{1}{t} x_{-\beta}(t - 1) - x'_{-\beta}(t) < 0. \]
This inequality is valid for all sufficiently large \( t \) and, therefore, inequality (36) holds. From (37), we see that there exists a solution \( x = x(t) \) of (1) satisfying
\[ \exp \left( -t \ln t - t \ln_2 t + t \frac{\ln^2 t}{\ln t} + \frac{t \ln_2 t}{2\ln^2 t} - \frac{2t \ln_2 t}{\ln^2 t} - \beta(t) \right) = x_\beta(t) \]
\[ = \pi(t) < x(t) < \delta(t) \]
and, finally,
\[ = x_{-\beta}(t) = \exp \left( -t \ln t - t \ln_2 t + t \frac{\ln_2 t}{\ln t} + \frac{t \ln_2 t}{2\ln^2 t} - \frac{2t \ln_2 t}{\ln^2 t} + \beta(t) \right). \]
The above inequalities are equivalent with (19).
4.5. Proof of Theorem 3.1. Let us denote by \( \rho_{cl} \) the classical representation of the Dickman \( \rho \) function given by formula (4)

\[
\rho_{cl}(t) := \exp \left( -t \ln t - t \ln^2 t + t - \frac{\ln^2 t}{\ln t} + \frac{t \ln^2 t}{2 \ln^2 t} + z(t) \right) \tag{52}
\]

where the function \( z \) satisfies \( z(t) = O\left( (\ln^2 t/\ln t) \right) \). The proposed improvement of the asymptotic behavior of the Dickman \( \rho \) function deduced from formula (21) where we set \( \alpha_3 = \alpha_4 = \cdots = \alpha_s = 0 \) in (22), given by formula (27), is denoted by \( \rho_{im} \), i.e.,

\[
\rho_{im}(t) := \exp \left( -t \ln t - t \ln^2 t + t - \frac{\ln^2 t}{\ln t} + \frac{t \ln^2 t}{2 \ln^2 t} - \frac{2 t \ln^2 t}{\ln^2 t} + o \left( \frac{t \ln^2 t}{\ln^2 t} \right) \right). \tag{53}
\]

Note that both functions \( \rho_{cl} \) and \( \rho_{im} \) are positive solutions of equation (1) and, using the terminology of dominant and subdominant solutions, both are subdominant solutions. Therefore, we can use them (based on the statements of Theorems 1.1, 1.2) in the representation structure formula (7) (see also formula (26)). Using the solution \( x_1 \) given by formula (24) as a dominant solution we get, for every solution \( x \) of (1),

\[
x(t) = K_1 x_1(t) + O(\rho_{cl}(t)), \tag{54}
\]

where the constant \( K_1 \) depends on \( x \) and

\[
x(t) = K_2 x_1(t) + O(\rho_{im}(t)), \tag{55}
\]

where the constant \( K_2 \) depends on \( x \). From (54) for \( x(t) = \rho_{im}(t) \) and (55) for \( x(t) = \rho_{cl}(t) \), we get

\[
\rho_{im}(t) = K_1 x_1(t) + O(\rho_{cl}(t)), \tag{56}
\]

and

\[
\rho_{cl}(t) = K_2 x_1(t) + O(\rho_{im}(t)). \tag{57}
\]

Obviously, (56) and (57) can be valid only if \( K_1 = K_2 = 0 \) and, therefore,

\[
\rho_{im}(t) = O(\rho_{cl}(t)),
\]

and

\[
\rho_{cl}(t) = O(\rho_{im}(t)).
\]

By the definition of the Landau order symbol \( O \), there exists a constant \( M > 0 \) such that

\[
\frac{\rho_{im}(t)}{\rho_{cl}(t)} \leq M \quad \text{and} \quad \frac{\rho_{cl}(t)}{\rho_{im}(t)} \leq M
\]

or, using (52) and (53),

\[
\exp \left( -z(t) - \frac{2 t \ln^2 t}{\ln^2 t} + o \left( \frac{t \ln^2 t}{\ln^2 t} \right) \right) \leq M
\]

and

\[
\exp \left( z(t) + \frac{2 t \ln^2 t}{\ln^2 t} + o \left( \frac{t \ln^2 t}{\ln^2 t} \right) \right) \leq M.
\]

A direct consequence of the last two inequalities is the existence of a constant \( M_1 > 0 \) such that

\[
\left| z(t) + \frac{2 t \ln^2 t}{\ln^2 t} + o \left( \frac{t \ln^2 t}{\ln^2 t} \right) \right| \leq M_1.
\]
and, taking into account that \( \lim_{t \to \infty} t \ln^2 t / \ln^2 t = \infty \), it is easy to conclude that the unique possibility of the asymptotic behavior of the function \( z \) is
\[
z(t) = -2t \ln t + o\left(\frac{t \ln^2 t}{\ln^2 t}\right) + O(1) = -2t \ln t + o\left(\frac{t \ln^2 t}{\ln^2 t}\right),
\]
which means that, in such a setting, the behavior of \( \rho_d \) is improved to \( \rho_{im} \).

4.6. **Proof of Theorem 3.3.** In the proof, we utilize following lemma.

**Lemma 4.4.** Let \( x_1^*(t) \), \( x_1^{**}(t) \) be dominant solutions to (1). Then, the limit
\[
K := \lim_{t \to \infty} \frac{x_1^*(t)}{x_1^{**}(t)} \tag{58}
\]
exists, is finite and positive. If, moreover, \( K = 1 \) then
\[
x_1^*(t) - x_1^{**}(t) = O(x_2(t)) \tag{59}
\]
where \( x_2(t) \) is a subdominant solution.

**Proof.** Let \( x_1(t) \) be a dominant solution to (1) and let \( x_2(t) \) be a subdominant solution to (1). Then, by formula (7) in Theorem 1.1 and by Theorem 1.2, every solution \( x = x(t) \) of (1) on \( [t_0 - 1, \infty) \) can be uniquely represented as
\[
x(t) = K x_1(t) + O(x_2(t)), \tag{60}
\]
where the constant \( K \) depends on \( x \). Utilizing (60) for \( x = x_1^* \) and for \( x = x_1^{**} \), we get
\[
x_1^*(t) = K^* x_1(t) + O(x_2(t)) \tag{61}
\]
and
\[
x_1^{**}(t) = K^{**} x_1(t) + O(x_2(t)) \tag{62}
\]
where the constants \( K^* \) and \( K^{**} \) are uniquely defined and, obviously, \( K^* > 0 \), \( K^{**} > 0 \). Then, (using (6))
\[
K = \lim_{t \to \infty} \frac{x_1^*(t)}{x_1^{**}(t)} = \lim_{t \to \infty} \frac{K^* x_1(t) + O(x_2(t))}{K^{**} x_1(t) + O(x_2(t))}
\]
\[
= \lim_{t \to \infty} \frac{K^* + O(x_2(t) / x_1(t))}{K^{**} + O(x_2(t) / x_1(t))} = \frac{K^*}{K^{**}} > 0
\]
and (58) is proved. Prove (59). Let \( K = 1 \), i.e., \( K^* = K^{**} \). Subtracting (62) from (61), we get
\[
x_1^*(t) - x_1^{**}(t) = K^* x_1(t) + O(x_2(t)) - (K^{**} x_1(t) + O(x_2(t))) = O(x_2(t)).
\]

Using Lemma 4.4, the proof of Theorem 3.3 is simple. By (30), the dominant solution is represented by formula (34), i.e.,
\[
x(1, \varphi)(t) = \frac{C_1(\varphi)}{t} \left(1 + o(1)\right), \quad t \to \infty.
\]
A dominant solution \( x^*(1, \varphi)(t) \) expressed by the series on the right-hand side of (32), i.e.,
\[
x^*(1, \varphi)(t) \sim \sum_{n=1}^{\infty} \frac{C_n(\varphi)}{t^n}
\]
with coefficients defined by (33), obviously satisfies
\[
\lim_{t \to \infty} \frac{x(1, \varphi)(t)}{x^*(1, \varphi)(t)} = \frac{C_1(\varphi)}{C_1(\varphi)} = 1.
\]
Therefore, with \(x_1^*(t) := x(1, \varphi)(t)\) and \(x_1^{**}(t) := x^*(1, \varphi)(t)\) formula (58) holds, \(K = 1\) and, by formula (59) of Lemma 4.4, we get
\[
x_1^*(t) = x_1^{**}(t) + O(x_2(t)),
\]
i.e.,
\[
x(1, \varphi)(t) = x^*(1, \varphi)(t) + O(x_2(t)) \sim \sum_{n=1}^{\infty} \frac{C_n(\varphi)}{t^n} + O(x_2(t)).
\]

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