The Complexity of Maximum $k$-Order Bounded Component Set Problem

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Abstract
Given a graph $G = (V, E)$ and a positive integer $k$, in MAXIMUM $k$-ORDER BOUNDED COMPONENT SET problem (MAX-$k$-OBCS), it is required to find a vertex set $S \subseteq V$ of maximum size such that each component in the induced graph $G[S]$ has at most $k$ vertices. We prove that for constant $k$, MAX-$k$-OBCS is hard to approximate within a factor of $n^{1-\epsilon}$, for any $\epsilon > 0$, unless $P = NP$. We provide lower bounds on the approximability when $k$ is not a constant as well. MAX-$k$-OBCS can be seen as a generalization of MAXIMUM INDEPENDENT SET problem (MAX-IS). We generalize Turán’s greedy algorithm for MAX-IS and prove that it approximates MAX-$k$-OBCS within a factor of $(2k - 1)\bar{d} + k$, where $\bar{d}$ is the average degree of the input graph $G$. This approximation factor is a generalization of Turán’s approximation factor for MAX-IS.

Keywords: Approximation algorithms, Graph algorithm, Dissociation set, $k$-order bounded component

1. Introduction
In this paper, we consider the computational complexity of an optimization problem called the MAXIMUM $k$-ORDER BOUNDED COMPONENT SET problem (MAX-$k$-OBCS). In MAX-$k$-OBCS, given a graph $G = (V, E)$ and a positive integer $k$, it is required to find a vertex set $S \subseteq V$ of maximum size such that order of each component in $G[S]$ is at most $k$.

This is a generalization of several important problems in combinatorics. For $k = 1$, this problem is the same as the MAXIMUM INDEPENDENT SET problem (MAX-IS), where given a graph $G = (V, E)$, the objective is to find a subset of vertices $S \subseteq V$ of maximum size such that there is no edge between any pair of vertices in $G[S]$. The case of $k = 2$ gives rise to the MAXIMUM DISSOCIATION SET problem (MAX-DISSO-SET),
where given a graph $G$, the objective is to find a vertex set $S \subseteq V$ of maximum size such that the degree of each vertex in $G[S]$ is at most 1. Both these problems are $\text{NP}$-hard and thus $\text{MAX-k-OBCS}$ is $\text{NP}$-hard as well.

Finding a maximum sized independent set is one of the widely studied problems in combinatorics and its hardness to compute has prompted several attempts at developing approximate solutions. It has been established that, for any $\epsilon > 0$, $\text{MAX-IS}$ cannot be approximated within a factor of $n^{1-\epsilon}$, unless $P = \text{NP}$ [1], and this lower bound result rules out the existence of a polynomial time efficient algorithm on general graphs. However, the situation is not quite as grim when restricted to special graph classes. It is $P$-time solvable to find a maximum independent set in claw-free graphs [2], $P_5$-free graphs [3], and perfect graphs [4]. For other classes, although finding a polynomial time optimal solution might be out of the question unless $P = \text{NP}$, we nevertheless provide good approximation algorithms. In planar graphs, there exists a PTAS for $\text{MAX-IS}$ [5] whereas for graphs with bounded degree, the problem can approximated within a constant factor [6]. One such algorithm greedily includes a vertex of minimum degree into the solution set, deletes that vertex along with its neighbors from the graph and then repeats this selection process until there are no more vertices to process. This simple greedy algorithm returns an independent set which is within a factor of $(\frac{d}{2}+1)$ from the optimal solution (Turán’s Theorem) [7]. A tighter analysis has improved the approximation factor to $\frac{1}{2}(2d+3)$ [8].

In [9], Yannakakis defined dissociation set and proved that it is $\text{NP}$-complete to find a vertex set $S$ of minimum size in a given bipartite graph $G$ such that $G[V \setminus S]$ has maximum degree 1. From this result it follows that $\text{MAX-DISSO-SET}$ is $\text{NP}$-complete even when restricted to bipartite graphs. It is also known to be $\text{NP}$-complete for $K_{1,4}$-free bipartite graphs, $C_4$-free bipartite graphs of maximum degree 3 [10], planar graphs with maximum degree 4 [11] and planar line graphs of planar bipartite graphs [12].

Approximation algorithms for the complementary problem of $\text{MAX-k-OBCS}$ have been designed by various researchers. In this minimization problem, we are asked to find a vertex set $S$ of minimum size in a given graph $G$ such that each component in $G[V \setminus S]$ has at most $k$ vertices. For $k = 1, 2$, these problems are the $\text{MINIMUM VERTEX COVER}$ problem and the $\text{MINIMUM DISSOCIATION SET}$ problem, both of which have a factor 2 approximation algorithm [13]. For larger values of $k$, the deletion problem is known to be approximable within a factor of $k$ and approximable within a factor of $(k-1)$ when the input graph has girth at least $k$ [14]. However, to the best of our knowledge, there is no known result about the approximability of $\text{MAX-k-OBCS}$ for arbitrary graphs.

Our Contribution. In this paper, we explore the $\text{MAX-k-OBCS}$ problem and provide both strong lower bounds and approximation algorithms for solving the hard cases of $k$. In Section 3, we prove inapproximability results by establishing an approximation preserving reduction from $\text{MAX-IS}$, which shows that for constant $k$, this problem has no efficient approximation algorithms for general graphs. The reduction used can be composed to prove hardness results even for non-constant $k$. In particular, we show that for $k = O(\frac{n}{\log n})$, it is not possible to develop an algorithm with a better approximation guarantee than $O(\log n)$, unless $P = \text{NP}$. Having established the hardness of computing efficient solutions to this problem, in the subsequent sections we concentrate
on developing upper bounds. In Section 4, we prove that MAX-WEIGHTED-$k$-OBCS can be approximated within a factor of $(\Delta + 1)$, where $\Delta$ is the maximum degree of the input graph $G$. This also proves that the problem can be approximated within a constant factor for degree bounded graphs. It is important to note that this bound holds not just for the unweighted case but also for the weighted version of the problem, where there is a weight function $w : V \rightarrow \mathbb{R}^+$ as an additional input and the objective is to find a vertex set $S$ of maximum weight such that $G[S]$ has no component whose order exceeds $k$. The technique used in the algorithm is the Local Ratio Method [13]. In Section 5, we build upon the algorithm for MAX-IS to develop a greedy algorithm for MAX-DISSO-SET, with a performance guarantee of $(3\overline{d} + 2)$, where $\overline{d}$ is the average degree of the graph. We then generalize the algorithm and the analysis to the larger problem with arbitrary values of $k$ and prove that the extension of the algorithm yields a solution which is within a factor of $(2k - 1)\overline{d} + k$ from the optimal solution. This can be observed as a generalization of Turán’s bound mentioned earlier for MAX-IS. Finally, in Section 6, we establish a reduction to MAX-IS and use existing upper bounds for MAX-IS [16, 17] to prove that MAX-$k$-OBCS can be approximated within a factor of $O\left(\frac{k\Delta \log \log \Delta}{\log \Delta}\right)$, for sufficiently large values of $\Delta$.

2. Notations

Throughout this paper, we assume that any graph $G = (V, E)$ mentioned is simple and undirected, with $|V| = n$ vertices and $|E| = m$ edges. Given such a graph $G = (V, E)$, the degree of a vertex $v$ in the graph is the number of edges from $v$ to the other vertices in the graph, and is denoted by $d_G(v)$, the maximum degree of the graph is $\Delta_G = \max_{v \in V} d_G(v)$. When the underlying graph $G$ is unambiguous, we use just $\Delta$ to denote the maximum degree. The average degree of the graph $G$ is $\overline{d} = \frac{1}{n} \sum_{v \in V} d(v)$. There is an interesting relationship between the average degree of a graph and the total number of edges in the graph, which we use in this paper, viz. the sum of the degrees of all the vertices in a graph is twice the number of edges. So $\sum_{v \in V} d_G(v) = 2|E|$, which gives us $|V|\overline{d} = 2|E|$.

Given a subset of vertices, $S \subseteq V$, the graph induced on this set, $G[S]$, is the subgraph whose vertex set is $S$ and the edge set is the subset of edges $E$ such that both endpoints are in $S$, i.e., $G[S] = (V', E')$ where $V' = S$ and $E' = \{(u, v) \in E | u \in S$ and $v \in S\}$.

A vertex set $S \subseteq V$ is called an independent set if $G[S]$ has no edges. We shall denote $\alpha(G)$ as the size of a maximum independent set in $G$. An independent set $S$ is maximal if it is not a proper subset of another independent set of $G$. A set of vertices $S$ of $G$ is called a dissociation set if degree of each vertex in $G[S]$ is at most 1 (equivalently, each component in $G[S]$ has at most two vertices). In the maximum dissociation set problem (MAX-DISSO-SET), the objective is to find a dissociation set of maximum size in a given graph $G$. We denote the cardinality of a maximum dissociation set in a graph $G$ by $\text{diss}(G)$. A set $S \subseteq V$ is called a $k$-component set in $G$ if each component in $G[S]$ has at most $k$ vertices. We shall denote the size of a maximum $k$-component set in $G$ by $\text{comp}_k(G)$. It is easy to observe that $\alpha(G) = \text{comp}_1(G)$ and $\text{diss}(G) = \text{comp}_2(G)$.
3. Hardness of Approximation

In this section, we shall prove inapproximability results for MAX-$k$-OBCS by establishing an AP-reduction \textsuperscript{18} from MAX-IS. The following lower bound result for MAX-IS makes this reduction useful.

**Theorem 3.1.** \textsuperscript{1} Unless $P = NP$, for any $\epsilon > 0$, MAX-IS is not approximable within a factor of $n^{1-\epsilon}$.

The hardness result mentioned in Theorem \textsuperscript{3.1} establishes the hardness of approximation for MAX-$1$-OBCS. Next, we establish the hardness of approximation for MAX-$k$-OBCS for a general value of $k$. We make use of the following reduction for this purpose.

**Lemma 3.2.** For any $k > 0$, MAX-$k$-OBCS $\leq_{AP}$ MAX-$2k$-OBCS.

**Proof.** Given an instance $G = (V, E)$ of MAX-$k$-OBCS, we construct an instance $G' = (V', E')$ of MAX-$2k$-OBCS as follows. First we make two copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of the graph $G$ then, for each edge $(u, v) \in E$, we add two edges $(u_1, v_2)$ and $(u_2, v_1)$. Thus we have $V' = V_1 \cup V_2$ and $E' = E_1 \cup E_2 \cup \{(u_1, v_2) | (u, v) \in E\} \cup \{(u_2, v_1) | (u, v) \in E\}$. It is easy to observe that $|V'| = 2|V|$.

Let $S^*$ be a maximum $k$-component set in $G$ and $S'^*$ be a maximum $2k$-component set in $G'$. It is easy to observe that, for any $k$-component set $S$ in $G$, the set $S' = \{v_1 | v \in S\} \cup \{v_2 | v \in S\}$ is a $2k$-component set in $G'$, and $2|S| = |S'|$. From this observation it follows that $2|S'^*| \leq |S'^*|$.

Let $S'$ be a $2k$-component set in $G'$, consisting of $t$ components. From $S'$ we will construct a $k$-component set $S$ in $G$ with $|S| \geq \frac{1}{2}|S'|$.

First, we prove a property of the components in $G'[S']$ which is as follows. For any two components $T_1$ and $T_2$, if we look at the vertices in $G$ induced by the components then there are no common vertices. In other words, the sets $\{v \in V | v_1 \in T_1 \text{ or } v_2 \in T_1\}$ and $\{v \in V | v_1 \in T_2 \text{ or } v_2 \in T_2\}$ are disjoint. Suppose, the sets $\{v \in V | v_1 \in T_1 \text{ or } v_2 \in T_1\}$ and $\{v \in V | v_1 \in T_2 \text{ or } v_2 \in T_2\}$ have a common vertex, say $u$. Without loss of generality, we assume that $u_1 \in T_1$ and $u_2 \in T_2$. By the construction, $(u_1, u_2) \in E'$ which implies that $T_1$ and $T_2$ are connected, which is a contradiction.

Now, we construct a set $S \subseteq V$ from $S'$ as follows. Let $T_1, T_2, \ldots, T_t$ be the components in $G'[S']$. For each component $T_i$ in $G'[S']$, if $|T_i \cap V_1| \geq |T_i \cap V_2|$ and $|T_i \cap V_1| \geq k$ choose a set $J_i$ of any $k$ vertices from $T_i \cap V_1$; otherwise if $|T_i \cap V_1| < k$, take $J_i = T_i \cap V_1$. Finally, we update $S = S \cup \{u \in V | u_1 \in J_i\}$. If $|T_i \cap V_2| > |T_i \cap V_1|$, then we take vertices from $V_2$ into the set $J_i$ and update $S = S \cup \{u \in V | u_2 \in J_i\}$.

Let $S = J_1 \cup J_2 \cup \ldots J_t$, where $J_i$ is the vertex set selected from the component $T_i$ of $G'[S'^*]$. We claim that $S$ is a $k$-component set in $G$. It is easy to observe that $G[J_i]$ and can have at most $k$ vertices. Next, we need to prove that there is no edge between a vertex $u$ in $J_i$ and a vertex $v$ in $J_j$. Existence of such an edge would imply that the corresponding components $T_i$ and $T_j$ are not distinct (because of the edges $(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2)$).

From the construction of the set $S$ it follows that $|S| \geq \frac{1}{2}|S'|$.

Thus it follows that, for any given $2k$-component set $S'$ in $G'$, one can construct a $k$-component set $S$ in $G$ in polynomial time, such that $\frac{|S'|}{|S|} \leq \frac{|S'^*|}{|S'|}$.
Corollary 3.3. For \( k > 0 \), MAX-IS is AP-reducible to MAX-\( k \)-OBCS with size amplification of \( kn \), where \( n \) is the number of vertices in the input instance of MAX-IS.

Proof. We reduce MAX-IS to MAX-\( k \)-OBCS by composing the reduction in Lemma 3.2 \( \lceil \log k \rceil \) number times to get an instance of MAX-\( k \)-OBCS. Since the size of the new instance in the reduction of Lemma 3.2 doubles, on applying the reduction \( \lceil \log k \rceil \) times, the size of the MAX-\( k \)-OBCS instance is \( 2^{\lceil \log k \rceil} n \approx kn \). Since the reduction in Lemma 3.2 is a ratio preserving reduction, the composite reduction from MAX-IS to MAX-\( k \)-OBCS is a ratio preserving reduction. Therefore, MAX-IS \( \leq_{AP} \) MAX-\( k \)-OBCS.

Using this, we have the following result

Theorem 3.4. For a positive integer \( k \) and for any \( \epsilon > 0 \), MAX-\( k \)-OBCS is hard to approximate within a factor of \( n^{1-\epsilon} \), unless \( P = NP \).

Next, we consider the inapproximability of MAX-\( k \)-OBCS when \( k \) is not a constant and our result is as follows.

Theorem 3.5. If \( k \) is not a constant, then for any \( \epsilon > 0 \), MAX-\( k \)-OBCS cannot be approximated in polynomial time with a better factor than \( n^{\log(n'/k) - \epsilon} \) unless \( P = NP \).

Proof. Note that in the previous theorem, the graph \( G' \) constructed in the reduction of MAX-\( k \)-OBCS from MAX-IS has size \( n' = n2^{\log k} = nk \), where \( n \) is the number of vertices in an instance \( G \) of MAX-IS.

Now suppose that MAX-\( k \)-OBCS can be approximated within a factor of \( n^{\log(n'/k) - \epsilon} \) in polynomial time, then

\[
\frac{\alpha(G)}{|S|} \leq \frac{comp_k(G')}{|S'|} \leq (nk)^{\log(n)/\log(n') - \epsilon}
\]

\[
= n \cdot (nk)^{-\epsilon} \cdot n^{\log n/\log n' - 1} \cdot n^{\log n/\log n'}
\]

\[
= \frac{n}{(nk)^{\epsilon}} \cdot n^{\log k/\log n'}
\]

\[
= \frac{n}{(nk)^{\epsilon}} \cdot \left( n^{-\log k/\log n'} \right)^{\frac{1}{\log n'}}
\]

\[
= \frac{n}{(n')^{\epsilon'}} = n^{1-\epsilon'}
\]

where \( \epsilon' = \epsilon \left[ 1 + \frac{\log k}{\log n} \right] \). This contradicts the lower bound for MAX-IS available in Theorem 3.1.

From Theorem 3.5, we have the following corollary,

Corollary 3.6. If \( k = O\left( \frac{n}{\log n} \right) \), then MAX-\( k \)-OBCS is hard to approximate within a factor of \( O(\log n) \).
4. Approximation Algorithms for MAX-WEIGHTED-k-OBCS

We now turn to the weighted version of the problem, MAX-WEIGHTED-k-OBCS. The input, in addition to a graph $G = (V, E)$, consists of a weight function on the vertices of the graph, $w : V \rightarrow \mathbb{R}^+$. Given a subset of vertices $A \subseteq V$, the weight of the subset is defined as $w(A) = \sum_{v \in A} w(v)$. In MAX-WEIGHTED-k-OBCS, the objective is to find a subset of vertices $S \subseteq V$, of minimum weight $w(S)$ such that the size of the largest component in $G[S]$ is at most $k$. Since the unweighted case looked at thus far is a special case of the weighted case (with the weight function being a constant), the lower bounds derived in the previous section hold for the weighted version as well.

In this section, we look at an approximation algorithm for this problem and prove an approximation guarantee of $(\Delta + 1)$, where $\Delta$ is the maximum degree of the input graph. To this end, we use the local ratio technique [15].

Algorithm 4.1: $k$-Weighted-CS($G = (V, E), w$)

| Input: An undirected simple graph $G = (V, E)$ and $w : V \rightarrow \mathbb{R}^+$ |
| Output: $S \subseteq V$ such that each component in $G[S]$ has at most $k$ vertices |
| 1. if $G = \emptyset$ then |
| 2. return $\emptyset$ |
| 3. end |
| 4. if there exists a vertex $u$ such that $w(u) \leq 0$ then |
| 5. return $k$-Weighted-CS($G \setminus \{u\}, w$); |
| 6. else |
| 7. Let $v$ be the vertex with minimum degree and let $N[v]$ be its closed neighborhood; |
| 8. Define the weight functions |
| $w_1(u) = w(v) \cdot \begin{cases} 1 & u \in N[v] \\ 0 & \text{o.w} \end{cases}$ and $w_2 = w - w_1$ |
| 9. $S' \leftarrow k$-Weighted-CS($G, w_2$); |
| 10. if $S' \cup \{v\}$ is feasible then |
| 11. return $S' \cup \{v\}$; |
| 12. else |
| 13. return $S'$; |
| 14. end |
| 15. end |

The algorithm is described in Algorithm 4.1 and the analysis makes use of the local ratio theorem.

**Theorem 4.1.** [15] Let $(G = (V, E), w)$ be an instance of MAX-WEIGHTED-k-OBCS having $n$ vertices. Let $w_1, w_2 \in \mathbb{R}^n$ such that $w = w_1 + w_2$. Let $S \subseteq V$ be an $r$-approximate feasible solution of MAX-WEIGHTED-k-OBCS for $G$ with respect to $w_1$ and $w_2$. Then $S$ is an $r$-approximate solution of MAX-WEIGHTED-k-OBCS for $G$ with respect to $w$.

By using this theorem we have the following result.

**Theorem 4.2.** MAX-WEIGHTED-k-OBCS can be approximated within a factor of $(\Delta + 1)$, where $\Delta$ is the maximum degree of the input graph $G$. 


Proof. We prove correctness by induction. In the base case, Line 2 is executed, and it is \((\Delta + 1)\)-approximate. In the inductive step, when Line 9 is executed, \(S'\) is \((\Delta + 1)\)-approximate with respect to \(w_2\). Now, if none of the vertices in \(N(v)\) are part of \(S'\), then \(S' \cup \{v\}\) is a feasible solution and thus any maximal solution with respect to \(w_1\) always has cost at least \(w(v)\) and since the maximum possible weight is \((\Delta + 1)w(v)\), we get that the solution is \((\Delta + 1)\)-approximate with respect to \(w_1\) as well.

Thus using the local ratio theorem, we get a \((\Delta + 1)\)-factor approximation algorithm for MAX-WEIGHTED\(-k\)-OBCS.

It should be noted that due to the relation between the weighted and unweighted cases, the upper bound derived here holds even for the unweighted case.

5. Approximation Algorithms for MAX-DISSOCIATION-SET

A subset of vertices \(S \subseteq V\) in a graph \(G = (V, E)\) is called a dissociation set if it induces a subgraph with vertex degree of at most 1. Since MAX-DISSOCIATION-SET is equivalent to MAX-2-OBCS, it is hard to approximate within a factor of \(n^{1-\epsilon}\), unless \(P = NP\), by Theorem 3.4. In this section, we prove that a greedy algorithm approximates MAX-DISSOCIATION-SET within a factor of \(3\Delta + 2\).

In this algorithm, we keep track of the components of size 1 and 2 in \(G[S]\). \(S_1\) is the set of vertices in \(S\) which form components of size 1 and \(S_2\) is the set of vertices which form components of size 2 in \(G[S]\). Each time we include a vertex into \(S\), we update the set \(X\) such that each component in \(G[S \cup X]\) has at least 3 vertices.

| Algorithm 5.1: MAX-DISSOCIATION-SET |
|-------------------------------------|
| **Input:** An undirected simple graph \(G = (V, E)\) |
| **Output:** \(S \subseteq V\) such that each component in \(G[S]\) has at most 2 vertices |
| 1 \(S_1 = \emptyset; S_2 = \emptyset; X = \emptyset; S = \emptyset; V' = V \setminus (S \cup X); i = 1;\) |
| 2 while \([V' \neq \emptyset]\) do |
| 3 \(v \in \arg \min\ \{d_{G[V']} (v)\} ;\) |
| 4 \(d_i = d_{G[V']} (v) ;\) |
| 5 \(S = S \cup \{v\} ;\) |
| 6 if there exists a vertex \(p \in S_1\) such that \((p, v) \in E\) then |
| 7 \(X = X \cup N(p) \cup N(v) ;\) |
| 8 \(S_2 = S_2 \cup \{v\} ;\) |
| 9 \(V' = V \setminus (S \cup X) ;\) |
| 10 else |
| 11 \(X = X \cup [N(v) \cap_{t \in S_1} N(t)] ;\) |
| 12 \(S_1 = S_1 \cup \{v\} ;\) |
| 13 \(V' = V \setminus (S \cup X) ;\) |
| 14 end |
| 15 \(i = i+1 ;\) |
| 16 end |
| 17 Return \(S\); |

We first prove the correctness of the above algorithm and then turn to its approximation guarantee. Assume that the algorithm runs for \(q\) iterations. Since we include
Lemma 5.1. At any iteration of the algorithm, for each vertex \( v \in S_2 \), \( N(v) \subseteq X \).

Proof. We prove this statement by induction on the iteration number. For the base case \( S_2 = \emptyset \) and the statement trivially holds. Assume that the statement is true for the \( i \)th iteration. Let \( v_{i+1} \) be the vertex added to \( S \) in \((i+1)\)th iteration. If \( v_{i+1} \) has no neighbor in \( S_1 \) then \( S_2 \) remain unchanged and the statement holds. Otherwise, a new pair of vertices is added to \( S_2 \) and line 7 ensures that the neighbors of these two vertices are included in \( X \). Thus the lemma holds in this case as well. \( \square \)

The correctness of the algorithm can be observed easily using the above lemma. Consider a solution set \( S = S_1 \cup S_2 \) returned by this algorithm. From Lemma 5.1 we can see that each of the vertices in \( S_2 \) are in a component of size exactly 2. Now consider a vertex \( v \in S_1 \). It cannot be a neighbor to a vertex in \( S_2 \), since if it did, line 8 would have moved these two vertices to the set \( S_2 \) during the run of the algorithm. Thus we see that the set \( S_1 \) forms an independent set in \( G[S] \) and thus the maximum size of a component in \( G[S] \) is 2, implying that \( S \) is a feasible solution.

Lemma 5.2. \( n \leq \sum_{i=1}^{d_i} (d_i + 1) \leq 2n \), where \( n \) is the number of vertices in the input graph.

Proof. When we choose a vertex \( v \in G[V'] \) of minimum degree, we denote it as a chosen vertex and denote its neighbors in \( G[V'] \) as looked-at vertices. From the algorithm it is clear that a chosen vertex is always included in \( S \). Every time a vertex is looked-at, exactly one of its neighbors is included in \( S \). Once a vertex is looked-at twice, the algorithm includes it in the set \( X \). Therefore, a vertex can be looked-at atmost twice and can be chosen atmost once.

Based on these notions, it follows that \( X \) is the set of vertices which are marked as looked-at twice by the algorithm. In the \( i \)th step of the algorithm a minimum degree vertex can be a looked-at vertex in \( V' \). Therefore, \( \sum_{i=1}^{d} d_i \geq |X| \). From this observation it follows that \( \sum_{i=1}^{d} (d_i + 1) \geq |X| + |S| = n \).

The greedy algorithm finally partitions the vertex set \( V \) as \( S \cup X \). Each vertex in \( V \) is probed atmost twice (as chosen, or looked-at and chosen, or looked-at and looked-at) before it is put in \( S \) or \( X \). At the \( i \)th step \( (d_i + 1) \) represents the number of looked-at vertices plus the chosen vertex. Therefore, \( \sum_{i=1}^{d} (d_i + 1) \leq 2n \).

Lemma 5.3. \( \sum_{i=1}^{d} (d_i + 1)d_i \leq 3n\overline{d} \).

Proof. Every time we choose a vertex \( v \in V' = [V \setminus (S \cup X)] \), we design a token to each edge \( e \) incident on a vertex of \( N_{V'[v]} \).

Since degree of vertex \( v \) in the \( i \)th step is \( d_i \) and it is of minimum degree, we assign at least \( \frac{1}{2} d_i (d_i + 1) \) tokens in \( i \)th step. At the end of the algorithm, an edge can have atmost 3 tokens as the end vertices of an edge can be marked as looked-at atmost 3 times by their neighbors.

Therefore, we have \( \sum_{i=1}^{d} \frac{1}{2} (d_i + 1)d_i \leq 3|E| = \frac{3}{2}n\overline{d} \), and thus \( \sum_{i=1}^{d} (d_i + 1)d_i \leq 3n\overline{d} \). \( \square \)
Lemma 5.4. The solution set returned by Algorithm 5.1 has size at least $\frac{n}{3d + 2}$.

Proof. By using the inequalities in Lemma 5.2 and Lemma 5.3 we have

$$\sum_{i=1}^{q} (d_i + 1)^2 = \sum_{i=1}^{q} d_i (d_i + 1) + \sum_{i=1}^{q} (d_i + 1) \leq 3nd + 2n.$$  

From the Cauchy-Schwarz inequality, we get

$$\sum_{i=1}^{q} (d_i + 1)^2 \geq \frac{\left(\sum_{i=1}^{q} (d_i + 1)\right)^2}{\sum_{i=1}^{q} 1} \geq \frac{n^2}{q}.$$  

Combining these two inequalities yields $q \geq \frac{n}{3d + 2}$. \square

From the above lemma, we have the following result.

Theorem 5.5. MAX-DISSOCIATION-SET can be approximated within a factor of $(3d + 2)$, where $d$ is the average degree of the input graph $G$.

5.1. Approximation Algorithm for MAX-k-OBCS

In this section, we will generalize the $(3d + 2)$ factor approximation algorithm for MAX-DISSOCIATION-SET to get an algorithm for MAX-k-OBCS which is a $(2k - 1)d + k$ factor approximation algorithm. The following is a generalization of Algorithm 5.1.

**Algorithm 5.2: Algorithm for MAX-k-OBCS**

```plaintext
Input: An undirected simple graph $G = (V, E)$  
Output: $S \subseteq V$ such that each component in $G[S]$ has at most $k$ vertices
1 $S = \emptyset; V' = V \setminus (S \cup X); i=1;$
2 while $[V' \neq \emptyset]$ do
3 choose a vertex $v \in V'$ of minimum degree in $G[V']$;
4 $d_i = d_{G[V']}(v);$  
5 $S = S \cup \{v\};$  
6 for each vertex $p \in V'$ and $p \neq v$ do
7 $\text{if } G[S \cup \{p\}] \text{ has a component of size at least } k + 1$ then
8 $X = X \cup \{p\};$
9 end
10 $V' = V' \setminus (S \cup X);$  
11 $i = i + 1;$
12 end
13 Return $S;$
```

By using a similar argument we have the following results.
Lemma 5.6. Let $q$ be the number of iterations made by the while loop in the Algorithm 5.2.

(a) $n \leq \sum_{i=1}^{d} (d_i + 1) \leq nk$

(b) $\sum_{i=1}^{d} d_i (d_i + 1) \leq \overline{d}(2k-1)n$.

Proof. (a) It is clear again that $\sum_{i=1}^{d} d_i \geq |X|$, and thus we get $n \leq \sum_{i=1}^{d} (d_i + 1)$. For the other inequality, consider the following argument: In each iteration, when a vertex $v$ is picked, assign one token to that vertex and each of its neighbors in the graph $G[V']$. Thus, in the $i$th iteration, the number of tokens assigned is exactly $d_i + 1$, and the total number of tokens assigned is $\sum_{i=1}^{d} (d_i + 1)$. Now, we look at the maximum number of tokens that any particular vertex can get. Note that a vertex gets a token if it is either chosen or looked-at, and once a vertex is chosen or included in the set $X$, it gets no more tokens. Suppose that at the end of the algorithm, a vertex $u$ has $(k+1)$ or more tokens. This implies that either it was looked-at at least $k+1$ times and then moved to the set $X$ or it was looked-at at least $k$ times and then chosen. Neither of these cases is possible, since when the vertex $u$ is looked-at $k$ times its neighbors have been chosen, and thus the step [6] will remove $u$ after $k$ of its neighbors have been chosen. Thus, any vertex can have almost $km$ tokens, and thus the total number of tokens available is atmost $nk$. This proves the second inequality.

(b) In the $i$th iteration, assume that vertex $v$ is chosen. Assign a token to all the edges incident on $v$ and its neighbors from the set $V'$ of the $i$th iteration. As argued in the case of the dissociation set problem, due to minimality of $d_i$, we assign at least $\sum_{i=1}^{d} \frac{1}{2} d_i (d_i + 1)$ tokens totally. Now, we produce an upper bound on the total number of tokens by seeing the maximum number of tokens that can be assigned to any edge. Consider an edge $e = (u,v)$ - it gets a token whenever at least one of its endpoints is either looked-at or chosen. Now, from the proof of (a), it follows that any vertex is looked-at at most $k$ times and put into the set $X$, or looked-at at most $k-1$ times and then chosen. Thus the edge $e$ can have atmost $(2k-1)$ tokens, since if it receives that amount, then at least one of its endpoints has been moved to either the set $X$ or the set $S$ and the edge no longer receives any tokens. Thus the total number of tokens that any edge can receive is $(2k-1)$ and the total number of tokens given out is atmost $|E|/(2k-1)$. Thus we get $\sum_{i=1}^{d} d_i (d_i + 1) \leq 2|E|/(2k-1) = \overline{d}(2k-1)n$. □

By using the inequalities in Lemma 5.6 and Cauchy-Schwarz inequality, we have

Theorem 5.7. Max-$k$-OBCS can be approximated within a factor of $(2k-1)\overline{d} + k$, where $\overline{d}$ is the average degree of the input graph $G$.

6. Upper bound for Max-$k$-OBCS with large $\Delta$

In this section, we provide a reduction from the Max-Dissociation-Set problem to the Max-IS problem, and use the upper bound result for Max-IS from [16, 17] to provide a $O(\frac{\Delta \log \log \Delta}{\log \Delta})$ approximation factor, for sufficiently large values of $\Delta$.

Lemma 6.1. Max-Dissociation-Set $\leq_{AP}$ Max-IS.

Proof. The reduction is a straightforward one, where we solve the Max-IS problem on the input instance. Formally, the reduction is the identity mapping, and maps the input
to the \textsc{Max-Dissociation-Set}, $G = (V, E)$, to the same instance $G$ for the \textsc{Max-IS} problem.

Consider an approximate solution $S$ to the \textsc{Max-IS} problem. Clearly, the solution set $S' = S$ is a feasible solution to the Max-Dissociation-Set problem as well, since we work on the same graph in both problems. This directly gives us $|S| \leq |S'|$.

Now comparing the optimal solutions to both problems, suppose that the optimal solution to \textsc{Max-Dissociation-Set} is the set $S^*$. The induced graph $G[S^*]$ will have maximum degree of at most 1, and we can partition the set $S^*$ into two sets $S_0 = \{v \in S^* | d_{G[S^*]}(v) = 0\}$ and $S_1 = \{v \in S^* | d_{G[S^*]}(v) = 1\}$, where $G[S_0]$ is independent and $G[S_1]$ forms an induced matching. We construct an approximate solution $S$ to the \textsc{Max-IS} by taking all the vertices in $S_0$ and one vertex from each matching in $S_1$. This set $S$ of vertices is an independent set and $|S| \geq \frac{1}{2}|S^*|$. Thus we have $\text{diss}(G) \leq 2|S| \leq 2\alpha(G)$, yielding

$$\frac{\text{diss}(G)}{|S'|} \leq 2\frac{\alpha(G)}{|S|}.$$

\[\square\]

**Theorem 6.2.** [16, 17] \textsc{Max-IS} can be approximated within a factor of $O\left(\frac{\Delta \log \log \Delta}{\log \Delta}\right)$ for sufficiently large values of $\Delta$, where $\Delta$ is the maximum degree of the input graph.

From Lemma [6.1] and Theorem [6.2] we get the following result,

**Theorem 6.3.** \textsc{Max-Dissociation-Set} is approximable within a factor of $O\left(\frac{\Delta \log \log \Delta}{\log \Delta}\right)$ for sufficiently large values of $\Delta$, where $\Delta$ is the maximum degree of the input graph.

We generalize the above result for the case of \textsc{Max-k-OBSC}, and using the inequalities $|S| \leq |S'|$ and $\text{comp}_k(G) \leq k|S| \leq k\alpha(G)$, we get

$$\frac{\text{comp}_k(G)}{|S'|} \leq k\frac{\alpha(G)}{|S|}$$

and thus giving the following upper bound result,

**Theorem 6.4.** For $k > 0$, \textsc{Max-k-OBSC} is approximable within a factor of $O\left(\frac{k\Delta \log \log \Delta}{\log \Delta}\right)$ for sufficiently large values of $\Delta$, where $\Delta$ is the maximum degree of the input graph.

7. Conclusion

We have proved that \textsc{Max-k-OBSC} is hard to approximate like \textsc{Max-IS}. We also show that for $k = O\left(\frac{\Delta}{\log n}\right)$, \textsc{Max-k-OBSC} is hard to approximate within a factor of $O(\log n)$. We generalize the greedy algorithm for \textsc{Max-IS} to obtain an algorithm for \textsc{Max-k-OBSC} with an approximation factor of $(2k - 1)d + k$. This approximation factor is a generalization of Turán’s bound for \textsc{Max-IS}.
References

[1] J. Hastad, Clique is hard to approximate within $n^{1-\epsilon}$, in: Foundations of Computer Science, 1996. Proceedings., 37th Annual Symposium on, IEEE, 1996, pp. 627–636.
[2] D. Nakamura, A. Tamura, A revision of minty’s algorithm for finding a maximum weight stable set of a claw-free graph, Journal of the Operations Research Society of Japan 44 (2) (2001) 194–204.
[3] D. Lokshantov, M. Vatshelle, Y. Villanger, Independent set in $P_5$-free graphs in polynomial time, in: Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, 2014, pp. 570–581.
[4] M. Grötschel, L. Lovász, A. Schrijver, Geometric algorithms and combinatorial optimization, Vol. 2, Springer Science & Business Media, 2012, pp. 296–298.
[5] B. S. Baker, Approximation algorithms for np-complete problems on planar graphs, Journal of the ACM (JACM) 41 (1) (1994) 152–180.
[6] C. H. Papadimitriou, M. Yannakakis, Optimization, approximation, and complexity classes, Journal of computer and system sciences 43 (3) (1991) 425–440.
[7] D. S. Hochbaum, Efficient bounds for the stable set, vertex cover and set packing problems, Discrete Applied Mathematics 6 (3) (1983) 243–254.
[8] M. M. Halldórsson, J. Radhakrishnan, Greed is good: Approximating independent sets in sparse and bounded-degree graphs, Algorithmica 18 (1) (1997) 145–163.
[9] M. Yannakakis, Node-deletion problems on bipartite graphs, SIAM Journal on Computing 10 (2) (1981) 310–327.
[10] R. Bolíac, K. Cameron, V. V. Lozin, On computing the dissociation number and the induced matching number of bipartite graphs, Ars Comb. 72.
[11] C. H. Papadimitriou, M. Yannakakis, The complexity of restricted spanning tree problems, Journal of the ACM (JACM) 29 (2) (1982) 285–309.
[12] Y. Orlovich, A. Dolgui, G. Finke, V. Gordon, F. Werner, The complexity of dissociation set problems in graphs, Discrete Applied Mathematics 159 (13) (2011) 1352–1366.
[13] T. Fujito, A unified approximation algorithm for node-deletion problems, Discrete applied mathematics 86 (2) (1998) 213–231.
[14] Y. Zhang, Y. Shi, Z. Zhang, Approximation algorithm for the minimum weight connected k-subgraph cover problem, Theoretical Computer Science 535 (2014) 54–58.
[15] R. Bar-Yehuda, K. Bendel, A. Freund, D. Rawitz, Local ratio: A unified framework for approximation algorithms. in memoriam: Shimon even 1935-2004, ACM Computing Surveys (CSUR) 36 (4) (2004) 422–463.
[16] D. Karger, R. Motwani, M. Sudan, Approximate graph coloring by semidefinite programming, Journal of the ACM (JACM) 45 (2) (1998) 246–265.
[17] N. Alon, N. Kahale, Approximating the independence number via the $\delta$-function, Mathematical Programming 80 (3) (1998) 253–264.
[18] G. Ausiello, Complexity and Approximability Properties: Combinatorial Optimization Problems and Their Approximability Properties, Springer, 1999.