Abstract

Once the action for Einstein’s equations is rewritten as a functional of an SO(3, C) connection and a conformal factor of the metric, it admits a family of “neighbours” having the same number of degrees of freedom and a precisely defined metric tensor. This paper analyzes the relation between the Riemann tensor of that metric and the curvature tensor of the SO(3) connection. The relation is in general very complicated. The Einstein case is distinguished by the fact that two natural SO(3) metrics on the GL(3) fibers coincide. In the general case the theory is bimetric on the fibers.
1. INTRODUCTION.

The principles of general relativity include requirements such as diffeomorphism invariance and a dynamically determined space-time metric. It is of philosophical interest to ask to what extent the form of Einstein’s equations are enforced by these principles. We want to know how tightly they constrain the world. It is also well to keep in mind a more practical reason for trying to modify Einstein’s equations; indeed there is observational evidence (notably galactic rotation curves) which, if taken at face value, suggests that the equations fail at large distances. Of course, the most likely explanation of this evidence is that we have a “dark matter” problem on our hands, rather than a problem with Einstein’s equations, but nevertheless these observations do add further interest to the question of the degree of uniqueness of the latter.

In 1915, Einstein was under the impression that he had found the unique field equations embodying the principles of general relativity. This first impression has stood the test of time rather well. Although a one parameter ambiguity was found early on (with the parameter christened “the cosmological constant”), most later attempts at generalization have involved rather drastic changes in the theory. At the very least, new degrees of freedom are added to the theory, as in scalar-tensor theories and theories based on higher derivative actions. The latter have some manifestly unphysical features. A more radical, and as yet unfinished, construction is provided by string theory. This involves a very large number of additional degrees of freedom, and presumably also some changes in the underlying principles - as should be true for all “unified” theories. The only attempt known to the author in which Einstein’s equations are generalized with no additional degrees of freedom - except for the well known family of theories parametrized by the cosmological constant - is a class of possible generalizations of Einstein’s equations to which we refer as “neighbours of Einstein’s equations”. The field equations can be derived from an action which is a functional of an SO(3,C) connection $A_{\alpha i}$ and a scalar field $\eta$ with tensor density weight minus one, and the action is of the form

$$S[A, \eta] = \int \mathcal{L}(\eta, Tr\Omega, Tr\Omega^2, Tr\Omega^3),$$

where the function $\mathcal{L}$ is chosen so that the integrand has density weight one, “$Tr$” denotes an SO(3) trace, and

$$F^{\alpha i}_{\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha + i\gamma^{im}\epsilon_{mjk}A^j_\alpha A^k_\beta$$

$$\Omega^{ij} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F^{i}_{\alpha\beta}F^{j}_{\gamma\delta}.$$
It will then be seen that for generic choices of $\mathcal{L}$ there are two degrees of freedom per space-time point. Moreover, since the space-time metric can be constructed from the structure functions of the constraint algebra, one can calculate a metric tensor which, when rewritten in manifestly four-dimensional notation, is

$$g_{\alpha\beta} = \frac{4}{3} \eta^{ijkl} \epsilon_{\mu\nu\rho\sigma} F_{\alpha i}^{\mu} F_{\nu j}^{\nu} F_{\rho k}^{\sigma}.$$  \hspace{1cm} (4)

This formula holds whenever the field equations hold, and the expression is unique up to conformal transformations, so that the conformal structure is uniquely determined in a space-time which solves the field equations. We stress that the equation is not a definition of the metric tensor, it is the statement of a theorem \[4\] \[5\]. A further theorem then follows immediately, namely that the SO(3) field strengths are self-dual with respect to this metric \[6\].

It is known that the original Einstein equations are equivalent to the field equations following from the above action, provided that $\mathcal{L}$ is suitably specified \[1\]:

$$S[A, \eta] = \frac{1}{2} \int \frac{\eta}{\sqrt{-\gamma}} (Tr \Omega^2 - \frac{1}{2}(Tr \Omega)^2).$$  \hspace{1cm} (5)

(Actually, this is not quite true, since the formalism breaks down for Petrov types \{3, 1\}, \{4\} and \{-\}, but this is the least of our problems, and we will ignore this point in the sequel.) Another highly intricate choice of $\mathcal{L}$ gives equations equivalent to Einstein’s with a non-vanishing cosmological constant \[4\]. The Einstein case is distinguished in that the two fibre metrics ($\gamma_{ij}$ and $\mu_{ij}$) coincide, whereas in general they do not. We will return to this point in section 5.

What we have just described is very much an unfinished construction, for several reasons:

1) While we know how to characterize solutions with real Lorentzian metrics, and some exact solutions with a real Lorentzian metric are known, we have no useful characterization of the set of such solutions, which makes comparison to the special Einstein case difficult.

2) It is not known whether propagation is causal with respect to the metric that we have identified.

3) It is technically difficult to add matter degrees of freedom to the theory, and only some very special cases have been worked out.

The construction will remain an unfinished one also at the end of the present paper. The mathematical problem that we will solve here is how to relate the Riemann tensor of the metric to the SO(3) field strength that occurs in the action. Actually, \"study\" may be more appropriate than \"solve\", since no computationally useful formula will emerge. In the Einstein case, the SO(3) field strength is simply the self dual part of the Riemann tensor, but in the general case the relation is decidedly complicated, as we will see. The problem will be formulated more precisely in section 2, where some further background information can be found as well. In section 3 we extend the SO(3) covariant derivative to a derivative which is covariant under local GL(3) transformations of the fibres as well as under space-time diffeomorphisms. We also demand that the extended derivative shall be...
“compatible” with a triad of two-forms and a scalar density. The formalism that we use is based on a peculiarly four dimensional variation of Riemannian geometry which uses triads of two-forms rather than tetrads of vector fields, and which is due to ’tHooft ([8] [9].) In section 4 it will be shown how to use this formalism to establish a relation between the metric Riemann tensor and the SO(3) curvature. The establishment of this formula was one of the main goals of the investigation that is reported here - naturally I hoped that a fairly simple formula would emerge, but this hope was crossed by the facts. In section 5 we discuss the dynamical equations derived for a one parameter family of neighbours of Einstein’s equations; in particular we will see how the two fibre metrics are related to each other. Then we give some examples of exact solutions of the same equations: “Kasner-like” solutions in section 6 and “Schwarzschild-like” in section 7. In the concluding section 8 we summarize the argument (for the benefit of those readers who do not want to lose themselves among calculational details) and make some further comments.

2. NEIGHBOURS AND NOTATION.

Here we will develop the “neighbours of Einstein’s equations” a little bit further, in order to make our notation clear. There will be not only an SO(3) fiber over every space-time point, but a fiber equipped with two different metrics \( \gamma_{ij} \) and \( m_{ij} \), which define two different SO(3) subspaces of GL(3). This feature will put a strain on our notation. First we make clear that “Tr” in eq. (1) means

\[ \text{Tr } \Omega \equiv \gamma_{ij} \Omega^{ij}, \]  \( (6) \)

and we stress that \( \gamma_{ij} \) is some metric that we are allowed to specify in any way we wish (in most cases, one would set it equal to Kronecker’s delta).

Before we derive the field equations from the action (1), we make two definitions:

\[ \Psi_{ij} \equiv \frac{\partial L}{\partial \Omega^{ij}}, \]  \( (7) \)

\[ \Sigma_{\alpha\beta i} \equiv \Psi_{ij} F_{\alpha\beta}^j. \]  \( (8) \)

Then the field equations take the form

\[ \epsilon^{\alpha\beta\gamma\delta} D_\beta \Sigma_{\gamma\delta i} = 0 \]  \( (9) \)

\[ \frac{\partial L}{\partial \eta} = 0. \]  \( (10) \)

(The second equation here is equivalent to the Hamiltonian constraint in the canonical formulation.)
We will analyze these equations further later on. Eq. (9), which is shared by all the neighbours (and also turns up in other contexts), will be dealt with in the following two sections, while eqs. (8) and (10) will be discussed for a special one parameter family of neighbours in section 5. For the moment, we use the triad $\Sigma_i$ of two-forms to introduce a second metric on the fibers, namely

$$m_{ij} \equiv \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \Sigma_{\alpha\beta i} \Sigma_{\gamma\delta j}. \quad (11)$$

We will now adopt the convention that we use $\gamma_{ij}$ to raise and lower the latin indices on the objects

$$A^i_\alpha, \quad F^i_{\alpha\beta}, \quad \Omega^{ij}, \quad (12)$$

and $m_{ij}$ to raise and lower the Latin indices on all other objects except the $\epsilon$-tensors. Greek indices will be raised and lowered with the space-time metric, which is unique. The inverse and determinant of $\gamma_{ij}$ are denoted respectively by $\gamma^{ij}$ and $\gamma$, and similarly for the other metrics. An over-riding convention is that all our $\epsilon$-tensors take the values $\pm 1$ in every coordinate system, and hence their indices are never raised or lowered with any metric. A further useful piece of notation is

$$\tilde{\Sigma}_{\alpha\beta} \equiv \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \Sigma_{\gamma\delta}. \quad (13)$$

As shown in refs. [4], [5], in any solution of the equations the space-time metric is

$$g_{\alpha\beta} = \frac{8}{3} \eta_{ijk} F^i_{\alpha\gamma} \tilde{F}^{\gamma\delta j} F^{k}_{\delta\beta} = \frac{8}{3} \sigma \epsilon^{ijk} \Sigma_{\alpha\gamma i} \tilde{\Sigma}_{\gamma\delta j} \Sigma_{\delta\beta k}, \quad (14)$$

where

$$\sigma = \frac{\eta}{\det \Psi}. \quad (15)$$

This metric has a very special geometrical significance [8]: It can be defined (uniquely up to a conformal factor) by the requirement that the $F_i$’s, as well as the $\Sigma_i$’s, span the subspace of self-dual two-forms. Or stated the other way around, in any solution of the equations these objects are self-dual two-forms. For our purposes it is of course essential to know under which conditions on the (complex valued) two-forms the metric is real and Lorentzian. This condition turns out to be

$$\epsilon^{\alpha\beta\gamma\delta} F^i_{\alpha\beta} \tilde{F}^{\gamma\delta j}_{\gamma\delta} = 0, \quad (16)$$

where the bar denotes complex conjugation. An extra condition is needed to ensure the reality of the conformal factor.

The basic idea is that the $\Sigma_i$’s, or alternatively the $F_i$’s, by construction form a basis for the three dimensional space of self-dual two-forms. (The CDJ formalism then fails for certain algebraically degenerate field configurations, for which the $F_i$’s are not linearly independent.) I reviewed these matters recently [9], and do not propose to do so again.

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2I have to add that the preprint version of ref. [9] contains an embarrassing sign error.
here, but two useful formulæ must be recorded:

\[ \tilde{\Sigma}_{\alpha\beta} = 4\sigma^2 m\Sigma_{\alpha\beta} \] (17)

\[ \Sigma_{\alpha\gamma\delta} \Sigma_{\beta j}^{\gamma} = -\frac{1}{4} m_{ij} g_{\alpha\beta} - \frac{1}{4\sigma} \epsilon_{ijk} \tilde{\Sigma}_{\alpha\beta}^k . \] (18)

These formulæ were used liberally in the calculations to be reported below.

3. COVARIANT DERIVATIVES.

Let us recall the starting point, which is that we have available an SO(3) covariant derivative defined by

\[ D_\alpha V_i = \partial_\alpha V_i + i\epsilon_{ijk} A_j^\gamma \gamma^{km} V_m . \] (19)

Note the presence of the imaginary unit; actually we are dealing with an SO(3, C) connection here. The group metric is denoted by \( \gamma_{ij} \); we do not use it to raise and lower indices because we will shortly introduce another metric which we will use for such a purpose.

It is assumed that

\[ D_{[\alpha} \Sigma_{\beta\gamma]} i = 0 . \] (20)

And the problem is to use this equation to set up a relation between the curvature tensor \( F_{\alpha\beta i} \) and the Riemann tensor of the metric

\[ g_{\alpha\beta} = \frac{8}{3} \sigma \epsilon_{ijk} \Sigma_{\alpha\gamma i} \Sigma_{\gamma j}^{\delta k} . \] (21)

Note that no special relation between \( F_{\alpha\beta i} \) and \( \Sigma_{\alpha\beta i} \) is assumed in this or in the following section. Conversely, any special relation is allowed.

To solve the problem, it will be convenient to introduce a number of new covariant derivatives, which extend the SO(3) covariant derivative which is presented to us at the outset. First we extend \( D \) to a GL(3) covariant derivative

\[ D_\alpha V_{\beta i} = D_\alpha V_{\beta i} + \beta_{\alpha j}^i V_{\beta j} . \] (22)

Then we introduce no less than three different GL(4) covariant derivatives, viz.

\[ \nabla_\alpha^{(g)} V_{\beta i} = \partial_\alpha V_{\beta i} - \left\{ \gamma_{\alpha\beta} \right\} V_{\gamma i} \] (23)

(this makes use of the Christoffel symbol, i.e. of an affine connection that is a function of the metric),

\[ \nabla_\alpha^{(T)} V_{\beta i} = \partial_\alpha V_{\beta i} - \Gamma_{\alpha\beta}^{\gamma} V_{\gamma i} \] (24)

(defined using a symmetric affine connection), and
\[
\n\nabla^{(T)}_{\alpha} V_{\beta i} = \nabla^{(T)}_{\alpha} V_{\beta i} + \frac{1}{2} T_{\alpha \beta} \gamma^\gamma V_{\gamma i} ,
\]

where the torsion tensor is self-dual, which means that it is of the form

\[
T_{\alpha \beta} \gamma^\gamma = \Sigma_{\alpha \beta i} \gamma^\gamma i .
\]

Finally, we extend \( D \) to a derivative which is both GL(3) and GL(4) covariant:

\[
\nabla_{\alpha} V_{\beta i} = D_{\alpha} V_{\beta i} + \beta_{\alpha i} V_{\beta j} - \Gamma_{\alpha \gamma} V_{\gamma i} + \frac{1}{2} T_{\alpha \beta} \gamma^\gamma V_{\gamma i} .
\]

The idea is that the various connections that we have introduced - altogether \( 12 + 24 + 40 + 12 = 88 \) unknown components - will be determined through the \( 12 + 72 + 4 = 88 \) conditions

\[
D_{[\alpha \Sigma_{\beta \gamma j i]} = 0
\]

\[
\nabla_{\alpha} \Sigma_{\beta \gamma i} = D_{\alpha} \Sigma_{\beta \gamma i} + \beta_{\alpha i} \Sigma_{\beta j} - \Gamma_{\alpha \gamma} \Sigma_{\beta \delta i} + \frac{1}{2} T_{\alpha \beta} \Sigma_{\delta \gamma i} + \frac{1}{2} T_{\alpha \gamma} \Sigma_{\beta \delta i} = 0
\]

\[
\nabla_{\alpha} \sigma = \partial_{\alpha} \sigma - \beta_{\alpha \gamma} \sigma + \Gamma_{\alpha \gamma} \sigma - \frac{1}{2} T_{\alpha \gamma} \gamma \sigma = 0
\]

The status of these conditions is the same as that of the “metric postulate” in the customary presentation of Riemannian geometry. The task at hand is to solve these equations for the connections that we have introduced. This is easily done in a step by step analysis.

We make the definition

\[
t_{\alpha i} \equiv \gamma^{i j k} \Sigma_{\alpha j} \Sigma_{\gamma k} .
\]

Then we have

\[
D_{[\alpha \Sigma_{\beta \gamma j i]} = 0
\]

\[
D_{\alpha} (\sigma m_{ij}) = 0
\]

\[
\beta_{\alpha i} (\sigma, \Sigma) = \frac{1}{2} \sigma m^{mn} D_{\alpha} (\sigma m_{mn}) - m^{ik} D_{\alpha} (\sigma m_{ik})
\]
\[ \nabla \left[ \alpha \Sigma_{\beta \gamma} \right] = 0 \quad (36) \]
\[ T^{\alpha i}(\sigma, \Sigma) = 2(\Sigma^i_{\alpha \beta} \beta_j - \Sigma^i_{\alpha \beta} \beta_j) \quad (37) \]
\[ \nabla_{\alpha} g_{\beta \gamma} = 0 \quad (38) \]
\[ \Gamma^\gamma_{\alpha \beta}(\sigma, \Sigma) = \{ \gamma_{\alpha \beta} \} - \frac{1}{2} T^\gamma_{\alpha \beta} - \frac{1}{2} T^\gamma_{\beta \alpha} \quad , \quad (39) \]

where the Christoffel symbol is
\[ \{ \gamma_{\alpha \beta} \} = \frac{1}{2} g^{\gamma \delta}(\partial_\beta g_{\delta \alpha} + \partial_\alpha g_{\delta \beta} - \partial_\delta g_{\alpha \beta}) \quad . \quad (40) \]

This solves the problem at hand, which was to ensure that the connections are “compatible” with \( \Sigma \) and \( \sigma \) in the sense of eqs. (29)-(30), while at the same time they extend a given SO(3) connection obeying eq. (28).

In refs. [8] and [9] similar ideas were applied to formulate Einstein’s equations. That is a simpler task than the one which occupies us here, because then the GL(4) connection can be chosen to be torsion free.

### 4. CURVATURE TENSORS.

For each of the covariant derivatives that were introduced in the previous section there is a curvature tensor. First there is an SO(3) curvature
\[ [D_\alpha, D_\beta]V_{\gamma i} = i\epsilon_{ijk} F_{\alpha \beta} j^k V_{\gamma m} \quad (41) \]
and a GL(3) curvature
\[ [\mathcal{D}_\alpha, \mathcal{D}_\beta]V_{\gamma i} = F_{\alpha \beta} j^i V_{\gamma j} = \]
\[ = (i\epsilon_{imn} F_{\alpha \beta} m^i \gamma_{nj} + D_\alpha \beta_{i j} - D_\beta \beta_{i j} + [\beta_\alpha, \beta_\beta]_{i j}) V_{\gamma j} \quad . \quad (42) \]

Then the three different affine connections that we introduced each give rise to a Riemann tensor, as follows:
\[ [\nabla_{\alpha}^{(g)}, \nabla_{\beta}^{(g)}]V_{\gamma i} = R_{\alpha \beta \gamma} \delta V_{\delta i} \quad (43) \]
\[ [\nabla^{(T)}, \nabla^{(T)}]V_{\gamma i} = R^{(T)}_{\alpha \beta \gamma} \delta V_{\delta i} \quad (44) \]
\[ [\nabla^{(T)}, \nabla^{(T)}]V_{\gamma i} = R^{(T)}_{\alpha \beta \gamma} \delta V_{\delta i} + T_{\alpha \beta} \delta \nabla_{\delta} V_{\gamma i} \quad . \quad (45) \]
Finally

\[ [\nabla_\alpha, \nabla_\beta] V_{\gamma i} = R_{\alpha\beta\gamma}^{(T)} \delta V_{\delta i} + T_{\alpha\beta}^{\delta} \nabla_\delta V_{\gamma i} + F_{\alpha\beta i}^{j} V_{\gamma j} . \]  

(I hope that the reader symphtizes with the way I have tried to so lve the notational difficulties here.)

We will also define the self-dual parts of the various Riemann tensors according to

\[ R_{\alpha\beta i} \equiv R_{\alpha\beta\gamma\delta} \tilde{\Sigma}^{\gamma\delta}_{i} , \]  

and similarly for the other cases. Given \( R_{\alpha\beta i} \), the full metric Riemann tensor can be reconstructed by means of complex conjugation, provided that condition (16) for real Lorentzian metrics is fullfilled. For the other Riemann tensors this is not true, since (for instance)

\[ R_{\alpha\beta i}^{(T)} \neq -R_{\alpha\beta i}^{(T)} \]  

in general. However, our aim is to set up a relation between \( F_{\alpha\beta i} \) and \( R_{\alpha\beta i} \), and therefore this need not concern us.

We proceed with the calculation; we use the equation

\[ [\nabla_\alpha, \nabla_\beta] \Sigma_{\gamma i} = R_{\alpha\beta\gamma}^{(T)} \sigma \Sigma_{\delta i} - R_{\alpha\beta\delta}^{(T)} \sigma \Sigma_{\gamma i} + F_{\alpha\beta i}^{j} \Sigma_{\gamma j} = 0 \]  

(49)

to relate the self-dual part of the Riemann tensor, for that connection which includes torsion, to the GL(3) curvature tensor. The result is

\[ F_{\alpha\beta i j} = -\frac{1}{2} R_{\alpha\beta\gamma}^{(T)} \gamma m_{ij} + \frac{1}{2} \epsilon_{i j k} R_{\alpha\beta}^{(T)k} \]  

\[ R_{\alpha\beta}^{(T)i} = \sigma \epsilon_{i j k} F_{\alpha\beta j k} . \]  

(51)

It is now straightforward but tedious to extract the desired relation. We will quote some intermediate steps for reference; round and square brackets denote symmetrization and anti-symmetrization, respectively, both with weight one. Thus:

\[ R_{\alpha\beta\gamma}^{(T)} \delta = R_{\alpha\beta\gamma}^{(T)} \delta + \nabla_{[a T_{\beta]} \gamma} \delta - \frac{1}{2} T_{\alpha[a} \delta T_{\beta] \gamma} \sigma - \frac{1}{2} T_{\alpha\beta \sigma} T_{\delta \gamma} \]  

\[ R_{\alpha\beta\gamma}^{(T)} \delta = R_{\alpha\beta\gamma}^{(T)} \delta + 2 \nabla_{[a T_{\beta]} \gamma} \delta - T_{\alpha\beta} \sigma T_{\delta (\gamma \sigma)} + T_{\gamma[a} \sigma T_{\delta (\beta \sigma)} - T_{\sigma[a} \delta T_{(\beta \gamma)} + 2 T_{\delta (\sigma[a)} T_{(\beta \gamma)} \]  

\[ R_{\alpha\beta i}^{(T)} = R_{\alpha\beta i} + \frac{2}{\sigma} \sum_{[a} \gamma^{j} \nabla_\beta D_{\gamma i} (\sigma m_{ij}) + \]
\[ \begin{align*}
+ \ &\frac{1}{4} \sum_{[\alpha} T_{\beta\gamma j] T_{\gamma i} + \frac{1}{4\sigma} \epsilon_{ijk} \sum_{[\alpha} m \sum_{[\beta} T_{\gamma k} T_{m]} + \\
+ \ &\frac{1}{2} \sum_{[\alpha} T_{\beta\gamma j} T_{\gamma j} + \frac{1}{4\sigma} \epsilon_{ijk} T_{\alpha} T_{\beta} + \\
+ \ &\frac{1}{8} \sum_{[\alpha} T_{\alpha} T_{\beta} + \frac{2}{\sigma} \epsilon_{imn} T_{\gamma j} \sum_{[\gamma \delta m} T_{\delta n] + \frac{1}{\sigma} \epsilon_{jmn} T_{\gamma m} \sum_{\gamma \delta i} T_{\delta n} ) - \frac{1}{8} \sum_{[\alpha} T_{\gamma j} T_{\gamma j} .
\end{align*} \]

Unfortunately, although we have manipulated this expression further, we have been unable to simplify it much. Therefore the best summary we can give at this point is to say that an explicit formula for the Riemann tensor as a function of the \( \Sigma \)'s and \( \sigma \) can be obtained by using eqs. (51) together with (54). This solves our problem (in principle).

5. THE DYNAMICAL EQUATIONS.

The problem that was discussed in the two previous sections is a geometrical one, with no dynamical restrictions on the two-forms imposed. The precise form of the action (1) enters when we turn to the dynamical field equations, that is to say when we assume that

\[ \Sigma_{[\alpha} = \Psi_{ij} F_{\alpha\beta}^{\ j}, \]

with the matrix \( \Psi_{ij} \) given by eq. (7). The field equation (9), together with the Bianchi identity for the field strength, can be rewritten in the form

\[ \tilde{F}^{\alpha\beta j} D_\beta \Psi_{ij} = 0 . \]

We also add the constraint (10) that ensures that the action is stationary with respect to variations in the field \( \eta \). It is out of the question to discuss the general case here, so we will concentrate on a one parameter family of neighbours of Einstein’s equations given by the action (3)

\[ S[A, \eta] = \frac{1}{2} \int \frac{\eta}{\sqrt{g}} (Tr \Omega^2 + \alpha (Tr \Omega)^2) . \]

Note that the parameter \( \alpha \) is dimensionless, and that \( \alpha = -1/2 \) gives Einstein’s vacuum equations (1). As far as I know, the results that follow are typical - at least I did not choose this special action with any deliberate intention of obtaining especially simple results. Anyway, we will not go very far into the subject - essentially only one conclusion will be drawn.

To begin, we obtain from the definition in eq. (7) that

\[ \Psi_{ij} = \frac{\eta}{\sqrt{g}} (\Omega_{ij} + \alpha \gamma_{ij} Tr \Omega) . \]
(Recall that the fixed fibre metric $\gamma_{ij}$ is used to raise and lower indices on $\Omega_{ij}$.)

The constraint (10) becomes

$$Tr\Omega^2 + \alpha(Tr\Omega)^2 = 0 .$$

(59)

It will also be useful to recall the characteristic equation for three-by-three matrices:

$$M^3 - M^2TrM - \frac{1}{2}M(TrM^2 - (TrM)^2) - \det M = 0 .$$

(60)

Now we are in a position to express the fibre metric $m_{ij}$ in terms of the SO(3) field strengths. Using the characteristic equation in conjunction with the constraint (59), we find that

$$m_{ij} \equiv \Sigma_{\alpha\beta} \Sigma_{\alpha\beta} = \Psi_{im} \Omega^{mn} \Psi_{nj} =$$

$$= \eta^2 \gamma_{ij} \det \Omega + (\alpha + \frac{1}{2})(\alpha - 1)(Tr\Omega)^2 \Omega_{ij} + (1 + 2\alpha)Tr\Omega \Omega^2_{ij} =$$

(61)

$$= \sigma^2 m(\gamma_{ij} + (1 + 2\alpha)\frac{Tr\Omega}{\det \Omega}(\Omega^2_{ij} + \frac{1}{2}(\alpha - 1)Tr\Omega \Omega_{ij})).$$

(62)

If we choose $\alpha = -1/2$ this equation simplifies dramatically, and the two fibre metrics are then proportional. It is easy to show that

$$\alpha = -\frac{1}{2} \quad \Rightarrow \quad \sigma m_{ij} = \frac{1}{\sqrt{\gamma}} \gamma_{ij} .$$

(63)

Now the calculations in section 4 collapse:

$$\alpha = -\frac{1}{2} \quad \Rightarrow \quad \beta_{\alpha}^i = 0 \quad \Rightarrow \quad R_{\alpha\beta}^i = -\frac{2i}{\sqrt{\gamma}} F_{\alpha\beta}^i .$$

(64)

So we see that the Einstein case ($\alpha = -\frac{1}{2}$) is indeed very special, in that when the field equations hold the two a priori different metrics that were introduced on the fibers coincide. This brings further simplifications in train; thus the SO(3) field strength that occurs in the action equals the self-dual part of the Riemann tensor of the spacetime metric. In turn, since the SO(3) field strengths span the space of self-dual two-forms, this implies that the self-dual part of the Riemann tensor is self-dual also in its remaining part of indices, which is an exceptional situation which holds if and only if the traceless part of the Ricci tensor vanishes.

In the general case we have a theory that is bimetric on the fibres, even though there is a unique conformal structure in space-time, as determined by the Hamiltonian constraint.
algebra [3]. To our disappointment we have been unable to use the specific choice of the action to simplify eq. (54) for the Riemann tensor in any significant way.

6. KASNER-LIKE SOLUTIONS.

To add concreteness to the above, we will consider some exact solutions of the equations that were considered in the previous section. Throughout, we make the convenient choice that

\[
\gamma_{ij} = \delta_{ij} . \tag{65}
\]

To obtain our first example, we make an Ansatz which is known to give the familiar Kasner solution in the Einstein case, namely [11]

\[
A_{x1} = a_1(t) \quad A_{y2} = a_2(t) \quad A_{z3} = a_3(t) , \tag{66}
\]

all other components vanishing. After some manipulation, one finds that the field equations collapse to

\[
\left(\frac{\dot{a}_1}{a_1}\right)^2 + \left(\frac{\dot{a}_2}{a_2}\right)^2 + \left(\frac{\dot{a}_3}{a_3}\right)^2 + \alpha \left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3}\right)^2 = 0 \tag{67}
\]

\[
\partial_t (\eta a_2 a_3 \dot{a}_1) = \partial_t (\eta a_3 a_1 \dot{a}_2) = \partial_t (\eta a_1 a_2 \dot{a}_3) = 0 . \tag{68}
\]

A convenient choice of gauge is

\[
\eta = -\frac{1}{16\dot{a}_1 \dot{a}_2 \dot{a}_3} . \tag{69}
\]

Then the solution is

\[
a_1 = d_1 t^{\gamma_1 - 1} \quad a_2 = d_2 t^{\gamma_2 - 1} \quad a_3 = d_3 t^{\gamma_3 - 1} , \tag{70}
\]

where the “Kasner exponents” obey

\[
\gamma_1 + \gamma_2 + \gamma_3 = 1 \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = -1 - 4\alpha \tag{71}
\]

We choose the integration constants so that we obtain a simple expression for the spacetime metric - which means that certain other expressions appear to be more complicated than they are, because one can not have everything. Specifically, we choose

\[
d_1 = \sqrt{(\gamma_2 - 1)(\gamma_3 - 1)} \quad d_2 = \sqrt{(\gamma_3 - 1)(\gamma_1 - 1)} \quad d_3 = \sqrt{(\gamma_1 - 1)(\gamma_2 - 1)} . \tag{72}
\]

This ensures that the spacetime metric takes the familiar Kasner form

\[
ds^2 = -dt^2 + t^{2\gamma_1} dx^2 + t^{2\gamma_2} dy^2 + t^{2\gamma_3} dz^2 . \tag{73}
\]
For our other geometrical objects, we obtain

\[ \Sigma_{tx} = \frac{i}{4d_1}(1 + 2\alpha - \gamma_1)t^{\gamma_1} \quad \Sigma_{yz} = \frac{1}{4d_1}(\gamma_1 - 1 - 2\alpha)t^{1-\gamma_1}, \]  

(74)

and similarly for \( \Sigma_{ty}, \Sigma_{tz} \) and \( \Sigma_{zx}, \Sigma_{xy} \) (and so on below). Further,

\[ \sigma = 4i\frac{(\gamma_1 - 1)(\gamma_2 - 1)(\gamma_3 - 1)}{(\gamma_1 - 1 - 2\alpha)(\gamma_2 - 1 - 2\alpha)(\gamma_3 - 1 - 2\alpha)}t^{-1} \]  

(75)

\[ \sigma_{ij} = \delta_{ij} - 2(1 + 2\alpha) \left( \begin{array}{ccc} -\frac{\gamma_1 + \alpha}{(\gamma_2 - 1 - 2\alpha)\gamma_3 - 1 - 2\alpha} & 0 & 0 \\ 0 & \frac{\gamma_2 + \alpha}{\gamma_3 - 1 - 2\alpha}(\gamma_1 - 1 - 2\alpha) & 0 \\ 0 & 0 & \frac{\gamma_3 + \alpha}{(\gamma_1 - 1 - 2\alpha)(\gamma_2 - 1 - 2\alpha)} \end{array} \right) \]  

(76)

\[ R_{tx} = \frac{1}{2d_1}\gamma_1(1 - \gamma_1)(\gamma_1 - 1 - 2\alpha)t^{\gamma_1-1} \quad R_{yz} = \frac{i}{2d_1}\gamma_2\gamma_3(1 + 2\alpha - \gamma_1)t^{-\gamma_1}. \]  

(77)

We observe that there is only one non-vanishing component of the Ricci tensor, viz.

\[ R_{tt} = 2(1 + 2\alpha)t^{-2}. \]  

(78)

7. SCHWARZSCHILD-LIKE SOLUTIONS.

We will give one more explicit example of a geometry, namely static and spherically symmetric solutions. The derivation was given in ref. [4], and here we will confine ourselves to stating the result. In order to make a spherically symmetric Ansatz, it is convenient to use spherical polar coordinates, and to introduce the vectors

\[ U_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad V_i = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \]  

(79)

\[ W_i = (-\sin \phi, \cos \phi, 0). \]

A form of the solution which is the most general one compatible with a spherically symmetric and static line element is then given by

\[ A_{ti} = -\frac{(2M)^{1-c}}{\sqrt{2c}}(1 - \varphi^2)^cU_i \quad A_{ri} = 0 \]  

(80)

\[ A_{\theta i} = i(1 - \varphi)W_i \quad A_{\phi i} = -i\sin \theta(1 - \varphi)V_i \]

\[ \eta = \frac{\sqrt{2c}(2M)^{2c-1}}{16} \frac{(1 - \varphi^2)^{-2c-1}}{\varphi \varphi' \sin \theta} \]  

(81)
where \( \varphi = \varphi(r) \) is an arbitrary function of \( r \), the slash denotes differentiation with respect to \( r \), \( M \) is an integration constant (adjusted to simplify the metric), and

\[
c = \frac{1}{1 + \alpha}(\pm \sqrt{-\frac{1}{2}(1 + 3\alpha) - \alpha}) .
\]  

(82)

The fact that there are two roots of this equation may appear to contradict Birkhoff’s theorem on the uniqueness of the Schwarzschild solution, but appearances deceive. In the Einstein case, \( \alpha = -1/2 \), what happens is that \( c = 2 \) gives the Schwarzschild solution while \( c = 0 \) gives a degenerate metric. In the generic case both roots give rise to non-degenerate metrics, but then there is no Birkhoff’s theorem to evade.

The results are

\[
F_i \equiv \frac{1}{2} dx^\alpha dx^\beta F_{\alpha\beta i} = -\sqrt{2c}(2M)^{1-c} \varphi\varphi'(1 - \varphi^2)^{c-1} dt dr U_i +
\]

\[
+ \frac{(2M)^{1-c}}{\sqrt{2c}} \varphi(1 - \varphi^2)^c (dtd\theta V_i + \sin \theta dt d\phi W_i) -
\]

\[
- i\varphi'(dr d\theta W_i + \sin \theta d\phi dr V_i) + i(1 - \varphi^2) \sin \theta d\theta d\phi U_i
\]

(83)

\[
\Sigma_i = \sqrt{2c} M (c + \alpha c + \alpha) \varphi \varphi'(1 - \varphi^2)^{-2} dt dr U_i -
\]

\[
- \frac{iM}{2\sqrt{2c}}(2\alpha c + 1 + 2\alpha) \varphi(1 - \varphi^2) - 1 (dtd\theta V_i + \sin \theta dt d\phi W_i) -
\]

\[
- \frac{(2M)^c}{4} (2\alpha c + 1 + 2\alpha) \varphi'(1 - \varphi^2)^{-c-1} (dr d\theta W_i + \sin \theta d\phi dr V_i) +
\]

\[
+ \frac{(2M)^c}{2} (c + \alpha c + \alpha)(1 - \varphi^2)^{-c} \sin \theta d\theta d\phi U_i
\]

(84)

\[
\sigma = -\frac{4i(2M)^{-c-1}}{\sqrt{2c}(1 + 3\alpha)(2\alpha c + 1 + 2\alpha)} \varphi \varphi' \sin \theta
\]

(85)

\[
ds^2 = -(2M)^{2-c} \varphi^2 (1 - \varphi^2)^{c-2} dt^2 +
\]

\[
+ 2c(2M)^c (\varphi')^2 (1 - \varphi^2)^{-c-2} dr^2 + (2M)^c (1 - \varphi^2)^{-c} (d\theta^2 + \sin^2 \theta d\phi^2)
\]

(86)
\[ \sigma_{ij} = \frac{1}{2(c + \alpha c + \alpha)} (\delta_{ij} - (4c + 6\alpha c + 1 + 6\alpha)U_iU_j) \]  
(87)

\[ R_{\alpha\beta i} = 4(2M)^2(c + 2\alpha + \frac{1}{2} + 2\alpha) \varphi^2(\varphi')^2 \frac{(1 - \varphi^2)}{(1 - \varphi^2)^3} \sin \theta dt dr U_i - \]

\[ - \frac{(2M)^2}{4} (2\alpha c + 1 + 2\alpha) \varphi^2 \varphi' \frac{(1 + (1 - c)\varphi^2)}{(1 - \varphi^2)^3} \sin \theta (dt d\theta V_i + \sin \theta dt d\phi W_i) + \]

\[ + \frac{i}{4} \sqrt{2c}(2M)^{c+1} (2\alpha c + 1 + 2\alpha) \varphi(\varphi')^2 \frac{(1 - \varphi^2)}{(1 - \varphi^2)^{c+2}} \sin \theta (dr d\theta W_i + \sin \theta d\phi dr V_i) + \]

\[ + i\sqrt{2c}(2M)^{1+c} (c + \alpha c + \alpha) \varphi \varphi' \frac{(1 - \frac{c}{2}\varphi^2)}{(1 - \varphi^2)^{2+c}} d\theta d\phi U_i . \]

The only non-vanishing components of the Ricci tensor are

\[ R_{tt} = (\frac{2}{c} - 1)(2M)^{2-2c} \varphi^2 (1 - \varphi^2)^{2c-2} \quad R_{rr} = 2(c - 2)(2M)(\varphi')^2 \frac{(1 - c\varphi^2)}{(1 - \varphi^2)^2}. \]  
(89)

Finally, the “Schwarzschild” form of the line element is obtained if we choose the free function \( \varphi(r) \) according to

\[ \varphi^2 = 1 - \frac{2M}{r^{2+c}}. \]  
(90)

8. CONCLUSIONS.

Let us make a brief sketch of the argument. The action (1), with a general choice of the integrand \( \mathcal{L} \), gives field equations of the form

\[ D_{[\alpha} \Sigma_{\beta\gamma]} = 0 \]  
(91)

\[ \frac{\partial \mathcal{L}}{\partial \eta} = 0, \]  
(92)

where the \( \Sigma_i \)'s are definite functions of \( \eta \) and the \( F_i \)'s. From an analysis of the constraint algebra in the Hamiltonian formulation, we conclude that the space-time metric is given by the Schönberg-Urbantke expression, eq. (4), which guarantees that the two-forms are self-dual. We found in this paper that the SO(3) covariant derivative may be extended to a derivative which is covariant under both local GL(3) transformations of the fibers.
as well as space-time diffeomorphisms, and that the extension becomes unique when we require that this derivative annihilates both the triad $\Sigma_i$ of two-forms and the conformal factor $\sigma$ of the metric. It was also shown that one can use these compatibility conditions to express the Riemann tensor of this derivative, as well as the metric Riemann tensor, as a definite function of the two-forms, the conformal factor, and the SO(3) curvature tensor. The expression is unwieldy, however.

We then concentrated our attention to a special one parameter family of action integrands, which includes the special choice which gives rise to Ricci flat metrics, and analyzed the relation between the two natural metrics on the fibers, the inert metric $\gamma_{ij}$ which is part of the definition of the theory, and the dynamical metric $m_{ij}$ that is defined by the triad of two-forms. It was found that they coincide in the Einstein case, and only in the Einstein case. Finally some exact solutions of the equations were examined.

Although no computationally useful formula for the Riemann tensor emerged in the non-Einstein case, the results do give insight into the structure of “neighbour geometry”.

It is legitimate to ask why we regard $m_{ij}$ rather than $\Omega_{ij}$ as a natural fibre metric. The answer is that in the Einstein case the latter matrix is degenerate for certain field configurations, such as flat Minkowski space, for which the field strengths $F_i$’s fail to provide a basis for the space of self-dual two-forms. On the other hand the $\Sigma_i$’s do form such a basis by assumption, since again the space-time metric is non-degenerate by assumption. To this one may object that this is an unnatural restriction within the present framework, and moreover that the situation may be different in the non-Einstein case. I do not have a good retort to this objection available. It is of course possible to replace $\Sigma_i$ and $\sigma$ by $F_i$ and $\eta$ throughout the calculations of sections 3 and 4, but as far as I can see this leads to no significant simplifications.

How does our construction evade Lovelock’s theorem, which is a strong uniqueness theorem for Einstein’s equations in four space-time dimensions \[12\]? Suppose that we rewrite our “vacuum” equations as an equation which has the Einstein tensor of the metric on the left hand side, and something else on the right hand side. Without going into details, we remark - for the benefit of those who more or less remember the theorem - that apart from the fact that the right hand side is not directly given as a functional of the metric, we also escape Lovelock’s theorem because the divergence of the right hand side is zero only as a consequence of the field equations. Indeed the metric takes the Schöenberg-Urbantke form only when the equations of motion hold. It is therefore understandable that only partial results \[11\] \[13\] are known for matter couplings.

Finally, the phenomenological prospect for the neighbours is known to be bleak \[14\]. However, if I am allowed to end on a speculative note, this prospect may change in unpredictable ways if one is able to turn the fibre metric $\gamma_{ij}$ in the action into a dynamical field. Which seems a natural thing to try, anyway.

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