On the fusion procedure for the symmetric group

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Abstract

We give a new version of the fusion procedure for the symmetric group which originated in the work of Jucys and was developed by Cherednik. We derive it from the Jucys–Murphy formulas for the diagonal matrix units for the symmetric group.

1 Introduction

A key role in the quantum inverse scattering method is played by solutions of the Yang–Baxter equation. The fusion procedure is commonly understood as a certain way to obtain new solutions of this equation out of the old ones. Consider the Yang–Baxter equation with the spectral parameter

\[ R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u), \]

where \( R(u) \) is a function of \( u \) with values in the endomorphism algebra of the tensor square \( V \otimes V \) of a vector space \( V \). Both sides of (1) are endomorphisms of \( V \otimes V \otimes V \) and the subscripts of \( R(u) \) indicate the copies of \( V \) so that \( R_{12}(u) = R(u) \otimes 1 \), etc. The Yang \( R \)-matrix is a simplest solution of (1), which is given by

\[ R(u) = 1 - Pu^{-1}, \]

where \( P \) is the permutation operator

\[ P : \xi \otimes \eta \mapsto \eta \otimes \xi, \quad \xi, \eta \in V. \]
Note that for the values $u = -1$ or $u = 1$ the endomorphism (2) maps the space $V \otimes V$ into the subspace of symmetric or anti-symmetric tensors, respectively. For instance, take $W$ to be the subspace of symmetric tensors. Due to (1), the subspace $V \otimes W$ of $V \otimes V \otimes V$ is preserved by the operator $R_{12}(u)R_{13}(u-1)$. Similarly, introducing an extra copy of $V$ labeled by 0, we can define the operator

$$R_W(u) = R_{12}(u+1)R_{13}(u)R_{02}(u)R_{03}(u-1).$$

(3)

Then the restriction of $R_W(u)$ to the subspace $W \otimes W$ is a “fused $R$-matrix” which is again a solution of the Yang–Baxter equation.

More generally, for any different complex numbers $c_1, \ldots, c_n$ the operator

$$R_{01}(u-c_1)R_{02}(u-c_2)\ldots R_{0n}(u-c_n)$$

preserves the subspace $V \otimes W \subseteq V \otimes V \otimes n$, where

$$W = \left( \prod_{1 \leq i < j \leq n} R_{ij}(c_i-c_j) \right) V^\otimes n,$$

(4)

and the factors are taken in the lexicographical ordering on the pairs $(i, j)$.

The symmetric group $\mathfrak{S}_n$ acts naturally on $V^\otimes n$ by permutations of the tensor factors. When the parameters $c_i$ are chosen in a certain particular way, the product of the $R$-matrices in (4) turns out to coincide with the image of a diagonal matrix element of an irreducible representation of $\mathfrak{S}_n$. Hence, due to the Schur–Weyl duality, this leads, by analogy with the above example, to the construction of solutions of the Yang–Baxter equation of the type $R_W(u)$, where $W$ is an arbitrary polynomial representation of $GL_N$. More precisely, the $c_i$ should be taken to be equal to the respective contents of a standard tableau $T$ associated with a partition of $n$. It may happen, however, that $c_i = c_j$ for some $i < j$ so that the corresponding factor $R_{ij}(c_i-c_j)$ in (4) is not defined. Nevertheless, it turns out that the product can be interpreted as a well-defined operator via a certain limiting procedure. Such a procedure providing an expression for the matrix elements of irreducible representations of $\mathfrak{S}_n$ originates in the work of Jucys [8]. A similar approach was developed by Cherednik [1] in greater generality for representations of Hecke algebras. Cherednik’s paper does not contain complete proofs, however. More details were given by Jimbo, Kuniba, Miwa and Okado [7, Lemmas 3.2 and A.1] while a complete proof of a version of the fusion theorem was given by Nazarov [11, Theorem 2.2] with simpler arguments than in [8]; see also Guizzi and Papi [5]. Nazarov’s theorem establishes a continuity property of the restriction of the product of the $R$-matrices in (4) on a certain subset of the parameters $c_i$. A hook version of the fusion procedure was developed in recent work of Grime [3].
In this paper we give a new version of the fusion procedure which is similar to [8] but with a different definition of the limits of rational functions: we take them \textit{consecutively} for each single variable. We show that the procedure is essentially equivalent to another construction of Jucys [9] (which was re-discovered by Murphy [10]) providing explicit formulas for the diagonal matrix elements of irreducible representations of $\mathfrak{S}_n$ in terms of certain elements of the group algebra $\mathbb{C}[\mathfrak{S}_n]$, known as the \textit{Jucys–Murphy elements}. The proof of these formulas is rather simple (its version is reproduced below) thus leading to a short derivation of the fusion procedure. Comparing this derivation with the other proofs, note that the approaches of Jucys [8] and Nazarov [11] rely on the formulas for the diagonal matrix elements involving the Young symmetrizers and do not establish a direct relationship with the Jucys–Murphy construction. Some versions of the procedure were given by Nazarov [12] and Grime [4] for the Hecke algebra, and Nazarov [13] developed a “skew fusion procedure”; see also earlier results of Cherednik [1, 2].

In what follows, we will work with the group algebra $\mathbb{C}[\mathfrak{S}_n]$, since the product of the $R$-matrices in (4) obviously coincides with the image of the ordered product

$$\prod_{1 \leq i < j \leq n} \varphi_{ij}(c_i, c_j), \quad \varphi_{ij}(u, v) = 1 - \frac{(i, j)}{u - v}.$$ 

\section{Young basis}

Let us fix some notation and recall some well known facts about the representations of the symmetric group $\mathfrak{S}_n$; see e.g. [6]. We write a partition $\lambda$ as a sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ of integers such that $\lambda_1 \geq \cdots \geq \lambda_l \geq 0$. We shall identify a partition $\lambda$ with its diagram which is a left-justified array of rows of cells such that the top row contains $\lambda_1$ cells, the next row contains $\lambda_2$ cells, etc. Let us fix a positive integer $n$. If $\lambda_1 + \cdots + \lambda_l = n$ then $\lambda$ is a partition of $n$, written $\lambda \vdash n$. A cell of $\lambda$ is called \textit{removable} if its removal leaves a diagram. Similarly, a cell is \textit{addable} to $\lambda$ if the union of $\lambda$ and the cell is a diagram. We shall write $\mu \rightarrow \lambda$ if $\lambda$ is obtained from $\mu$ by adding one cell. A tableau $T$ of shape $\lambda$ (or a $\lambda$-tableau $T$) is obtained by filling in the cells of the diagram bijectively with the numbers $\{1, \ldots, n\}$. We write $\text{sh}(T) = \lambda$ if the shape of $T$ is $\lambda$. A tableau $T$ is called standard if its entries strictly increase along the rows and down the columns.

The irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$ are parameterized by partitions of $n$. Given a partition $\lambda$ of $n$ denote the corresponding irreducible representation of $\mathfrak{S}_n$ by $V_\lambda$. The vector space $V_\lambda$ is equipped with an $\mathfrak{S}_n$-invariant inner product $(\cdot, \cdot)$. The orthonormal Young basis $\{v_T\}$ of $V_\lambda$ is parameterized by the set of standard $\lambda$-tableaux $T$. The action of the standard generators $s_i = (i, i+1)$ of $\mathfrak{S}_n$ in the Young
basis is described as follows. If $\alpha$ is a cell of $\lambda$ which occurs in row $i$ and column $j$ then the content of $\alpha$ is the number $j - i$. Now let a standard tableau $T$ be given. We denote by $c_k = c_k(T)$ the content of the cell occupied by the number $k$. Then for any $i \in \{1, \ldots, n-1\}$ we have

$$s_i \cdot v_T = dv_T + \sqrt{1 - d^2} v_{s_i T},$$

where $d = (c_{i+1} - c_i)^{-1}$, the tableau $s_i T$ is obtained from $T$ by swapping the entries $i$ and $i+1$, and we assume $v_{s_i T} = 0$ if the tableau $s_i T$ is not standard.

The group algebra $\mathbb{C}[\mathfrak{S}_n]$ is isomorphic to the direct sum of matrix algebras

$$\mathbb{C}[\mathfrak{S}_n] \cong \bigoplus_{\lambda \vdash n} \text{Mat}_{f_\lambda}(\mathbb{C}),$$

where $f_\lambda = \dim V_\lambda$. The matrix units $E_{TT'} \in \text{Mat}_{f_\lambda}(\mathbb{C})$ are parameterized by pairs of standard $\lambda$-tableaux $T$ and $T'$. An isomorphism between the algebras is provided by the formulas

$$E_{TT'} = \frac{f_\lambda}{n!} \Phi_{TT'},$$

where $\Phi_{TT'}$ is the matrix element corresponding to the basis vectors $v_T$ and $v_{T'}$ of the representation $V_\lambda$,

$$\Phi_{TT'} = \sum_{s \in \mathfrak{S}_n} (s \cdot v_T, v_{T'}) \cdot s^{-1} \in \mathbb{C}[\mathfrak{S}_n].$$

In what follows we only use the diagonal matrix units so we shall write $E_T = E_{TT}$ and $\Phi_T = \Phi_{TT}$. Now recall the construction of the matrix units $E_T$ which is due to Jucys [9] and Murphy [10]. Consider the Jucys–Murphy elements $X_1, \ldots, X_n \in \mathbb{C}[\mathfrak{S}_n]$ given by

$$X_1 = 0, \quad X_i = (1 i) + (2 i) + \cdots + (i - 1 i), \quad i = 2, \ldots, n.$$

The vectors of the Young basis are eigenvectors for the action of $X_i$ on $V_\lambda$: for any standard $\lambda$-tableau $T$ we have

$$X_i \cdot v_T = c_i(T) v_T, \quad i = 2, \ldots, n. \tag{5}$$

For any $n \geq 2$ we regard $\mathfrak{S}_{n-1}$ as the natural subgroup of $\mathfrak{S}_n$. The branching properties of the Young basis imply the following properties of the matrix units. If $U$ is a given standard tableau with the entries $1, \ldots, n - 1$ then

$$E_U = \sum_{U \rightarrow T} E_T, \tag{6}$$
where $U \rightarrow T$ means that the standard tableau $T$ is obtained from $U$ by adding one cell with the entry $n$. Relations (5) imply

$$X_i E_T = E_T X_i = c_i(T) E_T, \quad i = 2, \ldots, n$$

for any standard $\lambda$-tableau $T$. In particular, we have the identity in $\mathbb{C}[\mathfrak{S}_n]$,

$$X_n = \sum_{\lambda \vdash n} \sum_{sh(T) = \lambda} c_n(T) E_T. \quad (8)$$

Obviously, $E_{T_0} = 1$ if $T_0$ is the (1)-tableau with the entry 1. The other matrix units are given by the following recurrence relation which yields an explicit expression of $E_T$ in terms of the Jucys–Murphy elements $X_2, \ldots, X_n$. Let $\lambda \vdash n$ for $n \geq 2$ and let $T$ be a standard $\lambda$-tableau. Let $U$ be the standard tableau obtained from $T$ by removing the cell $\alpha$ occupied by $n$ and let $\mu$ be the shape of $U$. Then

$$E_T = E_U \frac{(X_n - a_1) \ldots (X_n - a_k)}{(c - a_1) \ldots (c - a_k)}, \quad (9)$$

where $a_1, \ldots, a_k$ are the contents of all addable cells of $\mu$ except for $\alpha$, while $c$ is the content of the latter. The relation follows from (6) and (7). It admits the following interpretation. Let $u$ be a complex variable. Due to (8), the following is a well-defined rational function in $u$ with values in $\mathbb{C}[\mathfrak{S}_n]$,

$$E_T(u) = E_U \frac{u - c}{u - X_n}. \quad (10)$$

Then $E_T(u)$ is regular at $u = c$ and $E_T(c) = E_T$. Indeed, by (6) and (7) we have

$$E_U \frac{u - c}{u - X_n} = \sum_{U \rightarrow T'} E_{T'} \frac{u - c}{u - c_n(T')} = E_T + \sum_{U \rightarrow T', T' \neq T} E_{T'} \frac{u - c}{u - c_n(T')}.$$

Since $c_n(T') \neq c$ for all standard tableaux $T'$ distinct from $T$, the value of this rational function at $u = c$ is $E_T$. Thus, the Jucys–Murphy formula (9) can also be written as

$$E_T = E_U \frac{u - c}{u - X_n} \bigg|_{u = c}. \quad (11)$$

We shall need the corresponding relation for the matrix elements $\Phi_U$ and $\Phi_T$. Recalling that the ratio $n! / f_\lambda$ equals the product of the hooks of $\lambda$, we get

$$\Phi_T = H_{\lambda, \mu} \Phi_U \frac{u - c}{u - X_n} \bigg|_{u = c}, \quad (10)$$

where the coefficient $H_{\lambda, \mu}$ is the ratio of the product of hooks of $\lambda$ and the product of hooks of $\mu$. It can be given by

$$H_{\lambda, \mu} = \frac{(a_1 - c) \ldots (a_p - c)(c - a_{p+1}) \ldots (c - a_k)}{(b_1 - c) \ldots (b_q - c)(c - b_{q+1}) \ldots (c - b_r)}, \quad (11)$$
where the numbers \(a_1, \ldots, a_p, c, a_{p+1}, \ldots, a_k\) are the contents of all addable cells of \(\mu\) and \(b_1, \ldots, b_q, c, b_{q+1}, \ldots, b_r\) are the contents of all removable cells of \(\lambda\) with both sequences written in the decreasing order.

**Remark.** Consider the character \(\chi_\lambda\) of \(V_\lambda\),

\[
\chi_\lambda = \sum_{s \in \mathfrak{S}_n} \chi_\lambda(s) s \in \mathbb{C}[\mathfrak{S}_n].
\]

We have

\[
\chi_\lambda = \sum_T \Phi_T,
\]

summed over all standard \(\lambda\)-tableaux \(T\). Formula (9) implies a recurrence relation for the normalized characters \(\hat{\chi}_\lambda = f_\lambda \chi_\lambda / n!\),

\[
\hat{\chi}_\lambda = \sum_{\mu \to \lambda} \hat{\chi}_\mu \frac{(X_n - a_1) \cdots (X_n - a_k)}{(c - a_1) \cdots (c - a_k)}.
\]

Equivalently,

\[
\chi_\lambda = \sum_{\mu \to \lambda} \chi_\mu \frac{(a_1 - X_n) \cdots (a_p - X_n)(X_n - a_{p+1}) \cdots (X_n - a_k)}{(b_1 - c) \cdots (b_q - c)(c - b_{q+1}) \cdots (c - b_r)},
\]

with the notation used in (11).

### 3 Fusion procedure

For any distinct indices \(i, j \in \{1, \ldots, n\}\) introduce the rational function in two variables \(u, v\) with values in the group algebra \(\mathbb{C}[\mathfrak{S}_n]\) by

\[
\varphi_{ij}(u, v) = 1 - \frac{(i \ j)}{u - v}.
\]

Equip the set of all pairs \((i, j)\) with \(1 \leq i < j \leq n\) with the reverse lexicographical ordering so that \((i_1, j_1)\) precedes \((i_2, j_2)\) if \(j_1 < j_2\) or \(j_1 = j_2\) and \(i_1 < i_2\). Take \(n\) complex variables \(u_1, \ldots, u_n\) and consider the ordered product

\[
\Phi(u_1, \ldots, u_n) = \prod_{1 \leq i < j \leq n} \varphi_{ij}(u_i, u_j).
\]

Note that the product taken in the (direct) lexicographical ordering on the pairs \((i, j)\) defines the same rational function. Now let \(\lambda \vdash n\) and fix a standard \(\lambda\)-tableau \(T\). Set \(c_i = c_i(T)\) for \(i = 1, \ldots, n\).
Theorem. The consecutive evaluations
\[ \Phi(u_1, \ldots, u_n) \mid_{u_1 = c_1} \mid_{u_2 = c_2} \cdots \mid_{u_n = c_n} \]
of the rational function \( \Phi(u_1, \ldots, u_n) \) are well-defined. The corresponding value coincides with the matrix element \( \Phi_T \).

Proof. Clearly, it is sufficient to consider the last evaluation \( u_n = c_n \). We argue by induction on \( n \) and suppose that \( n \geq 2 \). By the induction hypothesis, setting \( u = u_n \) we get
\[ \Phi(u_1, \ldots, u_n) \mid_{u_1 = c_1} \cdots \mid_{u_{n-1} = c_{n-1}} = \Phi_U \varphi_{1n}(c_1, u) \cdots \varphi_{n-1,n}(c_{n-1}, u), \]
where the standard tableau \( U \) is obtained from \( T \) by removing the cell occupied by \( n \). Let us verify that the expression on the right hand side can be given by
\[ \Phi_U \varphi_{1n}(c_1, u) \cdots \varphi_{n-1,n}(c_{n-1}, u) = \prod_{i=1}^{n-1} \left( 1 - \frac{1}{(u - c_i)^2} \right) \Phi_U (1 - X_n u^{-1})^{-1}. \tag{12} \]
Note that due to (8), the expression \( (1 - X_n u^{-1})^{-1} \) is a well-defined rational function in \( u \). Since
\[ \varphi_{1n}(c_1, u)^{-1} \left( 1 - \frac{1}{(u - c_i)^2} \right) = \varphi_{1n}(-c_i, -u), \]
relation (12) is equivalent to
\[ \Phi_U \varphi_{n-1,n}(-c_{n-1}, -u) \cdots \varphi_{1n}(-c_1, -u) = \Phi_U (1 - X_n u^{-1}). \]
We verify by induction on \( n \) a slightly more general identity
\[ \Phi_U \varphi_{n-1,r}(-c_{n-1}, -u) \cdots \varphi_{1r}(-c_1, -u) \]
\[ = \Phi_U \left( 1 - \frac{(1 \cdot r) \cdot (2 \cdot r) \cdots + (n-1 \cdot r)}{u} \right), \tag{13} \]
where \( r \) is a fixed index, \( r \geq n \). By (9) we can write \( \Phi_U \) as the product
\[ \Phi_U = \gamma \cdot \Phi_U \Phi_Y, \]
where \( Y \) is the standard tableau obtained from \( U \) by removing the cell occupied by \( n - 1 \) and \( \gamma \) is a nonzero constant. Hence, using the induction hypothesis we can transform the left hand side of (13) as
\[ \gamma \cdot \Phi_U \Phi_Y \varphi_{n-1,r}(-c_{n-1}, -u) \cdots \varphi_{1r}(-c_1, -u) \]
\[ = \gamma \cdot \Phi_U \varphi_{n-1,r}(-c_{n-1}, -u) \Phi_Y \varphi_{n-2,r}(-c_{n-2}, -u) \cdots \varphi_{1r}(-c_1, -u) \]
\[ = \gamma \cdot \Phi_U \varphi_{n-1,r}(-c_{n-1}, -u) \Phi_Y \left( 1 - \frac{(1 \cdot r) \cdot (2 \cdot r) \cdots + (n-2 \cdot r)}{u} \right). \]
This equals
\[ \Phi_U \left( 1 - \frac{(n - 1)r}{u - c_{n-1}} \right) \left( 1 - \frac{(1)r + (2)r + \cdots + (n-2)r}{u} \right). \]  
\[ (14) \]

Now observe that
\[ (n - 1)r \left( (1)r + (2)r + \cdots + (n-2)r \right) = X_{n-1} (n - 1)r \]
and recall that \( \Phi_U X_{n-1} = c_{n-1} \Phi_U \) by (7). Hence, (14) simplifies to (13) as required.

Now write the right hand side of (12) as
\[ \prod_{i=1}^{n-1} \left( 1 - \frac{1}{(u - c_i)^2} \right) \frac{u}{u - c_n} \cdot \Phi_U \frac{u - c_n}{u - X_n}. \]  
\[ (15) \]

Observe that the product
\[ \prod_{i=1}^{n-1} \left( 1 - \frac{1}{(u - c_i)^2} \right) \frac{u}{u - c_n} \]
only depends on the shape \( \mu \) of \( U \) so we may choose a particular (e.g. row-standard) tableau \( U \) for its evaluation. A short calculation shows that this product is regular at \( u = c_n \) with the value \( H_{\lambda,\mu} \) for \( c = c_n \). Due to (10), the value of (15) at \( u = c_n \) is \( \Phi_T \).

**Example.** Let \( \lambda = (2^2) \) so that \( n = 4 \). Take the standard \( \lambda \)-tableau
\[ T = \begin{array}{ccc}
1 & 2 \\
3 & 4 
\end{array} \]
The contents are \( c_1 = 0, c_2 = 1, c_3 = -1, c_4 = 0 \). Therefore,
\[ \Phi(0,1,-1,u) = \left( 1 + (12) \right) \left( 1 - (13) \right) \left( 1 - \frac{(23)}{2} \right) \times \left( 1 + \frac{(14)}{u} \right) \left( 1 + \frac{(24)}{u-1} \right) \left( 1 + \frac{(34)}{u+1} \right). \]

By the Theorem, this rational function is well-defined at \( u = 0 \). The corresponding value is
\[ \Phi_T = \Phi(0,1,-1,0) = \left( 1 + (12) \right) \left( 1 - (13) \right) \left( 1 - \frac{(23)}{2} \right) \times \left( 2 - (14) - (24) - (34) \right) \left( 2 + (14) + (24) + (34) \right). \]
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