An Eigenvector-Based Method of Radio Array Calibration and Its Application to the Tianlai Cylinder Pathfinder

Shifan Zuo1,2, Ue-Li Pen3,4, Fengquan Wu1, Jixia Li1,2, Albert Stebbins5, Yougang Wang1, and Xuelei Chen1,2,6

1 Key Laboratory for Computational Astrophysics, National Astronomical Observatories, Chinese Academy of Sciences, Beijing 100101, People’s Republic of China
2 School of Astronomy and Space Science, University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
3 Canadian Institute for Theoretical Astrophysics, 60 St. George Street, Toronto, Ontario MSS 3H8, Canada
4 Canadian Institute for Advanced Research, CIFAR Program in Gravitation and Cosmology, Toronto, Ontario MSG 1Z8, Canada
5 Fermi National Accelerator Laboratory, Batavia, IL 60510, USA
6 Center of High Energy Physics, Peking University, Beijing 100871, People’s Republic of China

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Abstract

We propose an eigenvector-based formalism for the calibration of radio interferometer arrays. In the presence of a strong dominant point source, the complex gains of the array can be obtained by taking the first eigenvector of the visibility matrix. We use the stable principle component analysis method to help separate outliers and noise from the calibrator signal to improve the performance of the method. This method can be applied with poorly known beam model of the antenna, is insensitive to outliers or imperfections in the data, and has low computational complexity. It thus is particularly suitable for the initial calibration of the array, which can serve as the initial point for more accurate calibrations. We demonstrate this method by applying it to the cylinder pathfinder of the Tianlai experiment, which aims to measure the dark energy equation of state using the baryon acoustic oscillation features in the large-scale structure by making intensity mapping observation of the redshifted 21 cm emission of the neutral hydrogen (HI). The complex gain of the array elements and the beam profile in the east–west direction (short axis of the cylinder) are successfully obtained by applying this method to the transit data of bright radio sources.

Key words: instrumentation: interferometers – methods: data analysis – techniques: interferometric

1. Introduction

Calibration of a telescope is performed to determine the various parameters that characterize the telescope model by solving equations linking the observational data to these parameters. In the case of a radio interferometer array, the model typically includes the beam and polarization response, the bandpass, and the complex gain of the receiving elements. In most cases, even if the beam response of the telescope is relatively stable, the amplitudes and phases of the receivers (complex gain) still vary significantly and must be calibrated during observation. Many interferometer array calibration methods have been developed and are in wide use (see e.g., SAult et al. 1996; Thompson et al. 1986; Perley et al. 1999; Hamaker 2000; Smirnov 2011a, 2011b). In recent years, with the need to achieve high precision for arrays with very large numbers of elements, and especially the low-frequency arrays with large fields of view, where direction-dependent beam response must be taken into account, the calibration methods are further developed and refined, such as the SAGECal algorithm (Kazemi et al. 2011), the Wirtinger derivative method (Tasse 2014), the Statistically Efficient and Fast Calibration (Salvini & Wijholds 2014), the complex optimization method (Smirnov & Tasse 2015), and the Facet calibration method (van Weeren et al. 2016) among others.

Calibration is usually a multistep and iterative process. After a reasonably good initial model of the telescope is achieved, the model is refined to take into account smaller effects. Although the initial calibration is a coarse one, it also has the challenge that the model of the telescope is largely unknown, so it needs to be blind and robust. In this article, we present a method of calibrating the complex gains of the interferometer array on the basis of eigenvector decomposition,7 which is accurate and computationally efficient. To make it more robust in the presence of missing data or occasional outliers, we also improve the method by using a technique called the stable principal component analysis (SPCA) to separate the dominant calibrator signal, the noise, and the occasional outlier components by exploiting their different properties in the covariance matrix. As a concrete example, the method is applied to the calibration of the Tianlai cylinder array pathfinder.

The Tianlai8 (Chinese for “heavenly sound”) project (Chen 2012; Xu et al. 2015) is an experimental effort to make intensity mapping (Chang et al. 2008) observations of the redshifted 21 cm line from the neutral hydrogen, to measure the baryon acoustic oscillation signal of large-scale structure and measure the dark energy equation of state. The Tianlai pathfinder includes both a dish array with 16 dishes, compactly arranged in two concentric rings (Zhang et al. 2016a), and a cylinder array with three north–south oriented cylinders (Zhang et al. 2016b) containing 31, 32, and 33 feed elements, respectively. The construction of the two arrays was completed in 2015, and the first trial observations were done in 2016 September. We have developed a data-processing pipeline for the arrays, and here we present the method of its initial calibration.

This article is organized as follows: in Section 2 we introduce the basic principle of the complex gain determination

7 This method was previously used by K. Bandura in the calibration of the Pittsburgh cylinder in his PhD thesis (Bandura 2011).
8 http://tianlai.bao.ac.cn
using eigenvector analysis method and its generalization to the SPCA method. In Section 3 we apply the method to the Tianlai array. We summarize the results in Section 4.

The notation used in this article is as follows: the vectors and matrices as a whole are denoted by bold letters. The \( l_0 \)-norm of a vector \( z \), denoted as \( |z|_{l_0} \), is defined as the number of nonzero elements of \( z \); the \( l_1 \)-norm of \( z \) is defined as \( |z|_1 = \sum_i |z_i| \). The \( l_0 \)- and \( l_1 \)- norms for a matrix \( X \) are defined by taking it as an vector. The Frobenius norm of a matrix \( X \) is defined as \( |X|_F = \sqrt{\text{Tr}(XX^T)} \), where \( \text{Tr} M \) denotes the trace of the matrix \( M \). Finally, the vector hard-thresholding operator \( \Theta_{\lambda}(z) \) is defined component-wise as

\[
\Theta_{\lambda}(z_i) = \begin{cases} z_i & \text{if } |z_i| > \lambda; \\ 0 & \text{otherwise.} \end{cases}
\]  

### 2. Basic Principle

In radio interferometry a visibility \( V_i \) is the instantaneous correlation between the voltages from two receiver feed elements \( F_i \) and \( F_j \). Without losing generality, we may assume that there are two orthogonal polarizations, \( X \) and \( Y \), in each feed. In the Tianlai cylinder case, the feeds are dipoles with linear polarization, and we shall call the east–west polarization \( X \) and north–south polarization \( Y \). The interferometer takes four combinations of the measurements \( V_{ij}^X, V_{ij}^Y, V_{ij}^{XX} \) and \( V_{ij}^{YY} \) for each baseline \((i, j)\). In this article, we deal with only the nonpolarized calibration (i.e., we do the calibration for only \( V_{ij}^X \) and \( V_{ij}^{YY} \) independently). For symbolic simplicity, we omit the \( XX \) and \( YY \) superscript in the following discussion. With noise, the voltage of element \( i \) is

\[
F_i = g_i \int d^2 \hat{n} A_i(\hat{n}) \mathcal{E}(\hat{n}) e^{-2\pi i \hat{n} \cdot u_i} + n_i,
\]

where \( \mathcal{E}(\hat{n}) \) is the electric field of the radio wave coming from direction \( \hat{n} \) on the celestial sphere. \( A_i(\hat{n}) \) is the primary beam of feed \( i \), and \( g_i \) is a direction-independent complex gain factor that calibration seeks to solve, and \( n_i \) is the noise in receiver \( i \). Assuming that the signal and noise are uncorrelated, and neglecting the couplings between the feeds, the visibility is ideally given by

\[
V_{ij} = \langle F_i F_j^* \rangle = g_i g_j^* \int d^2 \hat{n} A_i(\hat{n}) A_j^*(\hat{n}) e^{-2\pi i \hat{n} \cdot u_{ij}} I(\hat{n}) + \langle n_i n_j^* \rangle,
\]

where \( u_{ij} = (r_i - r_j)/\lambda \) is the baseline vector between the two feeds in units of wavelength, and \( I(\hat{n}) \) is the sky intensity distribution. One can substitute the sky model and telescope model into these equations to solve for the complex gains.

#### 2.1. Complex Gain as Eigenvectors

If we have a good knowledge on the primary beam responses \( A_i(\hat{n}) \), the positions \( u_i \), and a sky model \( I(\hat{n}) \), neglecting noise, we can compute the visibilities induced by the sky model \( V_{ij}^{\text{model}} \). The ratio between the observation and data is

\[
R_{ij} = V_{ij}^{\text{obs}} / V_{ij}^{\text{model}} = g_i g_j^*,
\]

or in matrix form,

\[
R = gg^T,
\]

where \( g \) is a vector with its \( i \)-th element being the gain \( g_i \).

One can simply go for numerical solution of Equation (4) by putting in the model and observed values. However, taking note of the form of Equation (4), an eigen-analysis method presents itself for solution. Specifically, because \( R \) is a rank-one matrix, it has only one nonzero eigenvalue in the absence of noise. Note that \( R g = (gg^T) g = g (g^T \cdot g) \) and \( g \cdot g = ||g||^2 = \sum_i |g_i|^2 \) is a real number, so the (unnormalized) eigenvector of \( R \) is \( g \), with eigenvalue \( ||g||^2 \). Thus, in principle, the complex gains of the array could be obtained by solving the eigenvalue problem for the matrix \( R \).

However, noise is present in actual measurement, and the beam response is not precisely known so the computation of the model visibility is inaccurate or even impossible, making the solution with Equations (3) and (4) impractical in the general case. However, if there is a strong radio point source with flux \( S_r \) at direction \( \hat{n}_r \) that dominates over the noise (we treat all subdominant signal except outliers as noise), then

\[
V_{ij} = V_{ij}^0 + n_i n_j^*,
\]

where \( n_i, n_j \) are the noise from the receivers \( i, j \) respectively, and

\[
V_{ij}^0 = S_r G_i G_j^*,
\]

with

\[
G_i = g_i A_i(\hat{n}_r) e^{-2\pi i \hat{n}_r \cdot u_{ij}};
\]

in matrix form,

\[
V_0 = S_r G G^T.
\]

The vector \( G \), which includes complex gain and beam response, is an eigenvector of \( V_0 \).

If noise is present but small compared with the calibrator source and statistically equal in all elements (i.e., \( \mathbf{V} = V_0 + \mathbf{N} \), where \( \mathbf{N} = \langle \mathbf{m} \mathbf{m}^T \rangle \)), the vector \( G \) could be obtained by principal component analysis (PCA): solving the eigenvector of the matrix \( V \), with the eigenvector associated with the largest eigenvalue identified as \( G \). This is also the least square solution of the form \( V = gg^T \). To prove this, introduce a Lagrangian multiplier \( \lambda \), and normalize the solution to satisfy

\[
g_i = \sqrt{\lambda} v_i, \quad \sum v_i^2 = 1,
\]

define the residual error

\[
\epsilon \equiv \sum_{ij} (V_{ij} - \lambda v_i v_j)^2.
\]

The least square solution is obtained by \( \partial \epsilon / \partial v_i = 0 \), that is,

\[
\sum_j V_{ij} v_i = \lambda v_i,
\]

which is the eigenvector equation. Note also that adding a constant along the diagonal of the matrix does not change the solution, and for a unit normalized covariance matrix, setting the diagonals to zero does not affect the solution either.

This is the basic idea of calibration with eigenvector analysis. The solution obtained as an eigenvector automatically satisfies both the phase and the amplitude closure relations. This is because the quantity \( g_i \) is of the form \( g_i = |g_i| e^{i \phi_i} \), from the algebraic identity \( (\phi_i - \phi_j) + (\phi_j - \phi_k) + (\phi_k - \phi_i) = 0 \) and \( |g_i||g_j||g_k| = |g_i||g_j||g_k| \) we always have

\[
\text{Arg}(g_i g_j^*) + \text{Arg}(g_j g_k^*) + \text{Arg}(g_k g_i^*) = 0,
\]
and
\[ |g_s g_s^\dagger| = |g_s g_s^*| = |g_s g_s^*|].

In the above analysis, we have assumed a single dominate point source that is much stronger than the rest of the sky to serve as the calibrator. In reality, there are diffuse foreground and other discrete point sources in the sky that may affect the calibration process. Here we consider the effects of these.

First, without losing generality, assume in addition to the strongest source \( s_0 \) with flux \( S_{s0} \) at direction \( \mathbf{n}_0 \), there is another source \( s_1 \) with flux \( S_{s1} \) at direction \( \mathbf{n}_1 \) also in the field view. The observed visibilities would then be (ignore the noises)
\[ V = S_{s0} G_0 G_0^\dagger + S_{s1} G_1 G_1^\dagger, \]
where the \( i \)th elements of \( G_0 \) is \( g_i A_i(\mathbf{n}_0) e^{-2\pi i \mathbf{n}_0 \cdot u} \), for \( s = 0, 1 \). Here we consider direction-independent calibration, so the gain \( g_i \) is the same in direction \( \mathbf{n}_0 \) and \( \mathbf{n}_1 \). By absorbing the amplitude of the vector \( G_0 \) into the flux \( S_{s0}/|G_0|^2 \), we can write the above equation as
\[ V = S_{s0}' \tilde{G}_0 \tilde{G}_0^\dagger + S_{s1}' \tilde{G}_1 \tilde{G}_1^\dagger, \]
where \( \tilde{G}_0 = G_0/|G_0| \), the normalized unit vector of \( G_0 \). We assume the voltage induced by \( s_0 \) is stronger by \( s_0, S_{s0}' > S_{s1}' \). Note that \( s_1 \) may have smaller flux than \( s_0 \), or it may have greater flux than \( s_0 \) but away from the beam center such that its contribution is smaller. We may decompose \( \tilde{G}_1 \) into two parts, one parallel to \( \tilde{G} \) and the other perpendicular to \( \tilde{G} \),
\[ \tilde{G}_1 = \alpha \tilde{G} + \mathbf{g}, \]
where \( \alpha = \tilde{G}^\dagger \cdot \tilde{G}_1 \) and \( |\alpha| \leq 1, \tilde{G}^\dagger \cdot \tilde{G}_1 = \tilde{G}^\dagger \cdot \tilde{G} = 0 \), then we have
\[
V = (S_{s0} + |\alpha|^2 S_{s1} ) \tilde{G} \tilde{G}^\dagger + S_{s1}' G_1 G_1^\dagger
+ \alpha S_{s1}' \tilde{G}_1 \tilde{G}_1^\dagger + \alpha^* S_{s1}' G_1 \tilde{G}_1^\dagger
= \lambda_1 u_1 u_1^\dagger + \lambda_2 u_2 u_2^\dagger, \tag{10}
\]
where \( \lambda_1 > \lambda_2 \) are the eigenvalues of \( V \) and \( u_1 \) and \( u_2 \) are the corresponding eigenvectors.

First we consider the extreme case \( G_1 = 0 \), in this case, \( |\alpha| = 1, \lambda_1 = (S_{s0} + S_{s1}'), u_1 = \tilde{G} \) and \( \lambda_2 = 0 \) (i.e., \( V \) is still rank one), in which the gain solution will be a factor larger in amplitude, but exactly the same in phase. Note that for this to happen, we must have for each baseline \( u_i \), the direction of the two sources \( \mathbf{n}_0, \mathbf{n}_1 \) must satisfy
\[ A_i(\mathbf{n}_0) e^{-2\pi i \mathbf{n}_0 \cdot u_i} = A_i(\mathbf{n}_1) e^{-2\pi i \mathbf{n}_1 \cdot u_i}. \]
For regularly spaced arrays with spacing greater than half wavelength, such a situation could arise if \( \mathbf{n}_1 \cdot u_i = \mathbf{n}_0 \cdot u_i + m_i \) where \( m_i \) is an integer for each \( i \), and \( A_i(\mathbf{n}_0) \approx A_i(\mathbf{n}_1) \). This is the case when the second source happens to be located at a grating lobe position conjugate to the position of the first source. Given that the spacing of the Tianlai cylinder array is not completely regular (each cylinder has different spacing), this is very unlikely in our case.

Second, for the case \( \alpha = 0, \lambda_1 = S_{s0}', u_1 = \tilde{G}, \lambda_2 = S_{s1}'/|G_1|^2, \) and \( u_2 = G_1/|G_1| \) (i.e., \( V \) has a rank of two, and its largest eigenvalue and eigenvector gives exactly the gain solution exactly).

More generally, we have \( 0 < |\alpha| < 1 \) and \( G_1 \neq 0, V \) will be a rank-two matrix. The last two terms in Equation (10) add small perturbations to the first two terms. The largest eigenvalue is \( \lambda_1 \approx (S_{s0}' + |\alpha|^2 S_{s1}'), \) and its corresponding eigenvector is \( u_1 \approx \tilde{G} \). Now if we truncated to the first principal component of \( V \), we will get
\[ G' \approx \sqrt{1 + |\alpha|^2} \frac{S_{s0}'|G_j|^2}{S_{s1}'|G_j|^2} G. \tag{11} \]

From this we see the existence of the additional source \( s_1 \) will generate a bias in the amplitude of the gain by a factor of about \( \sqrt{1 + |\alpha|^2}, \) and also a small bias in the phase.

The fainter sources and diffuse background in most cases will behave as noise. As will be shown below in the case of Tianlai cylinder array, the calibration could be performed by the PCA with only small errors.

### 2.2. SPCA

In the real world, in addition to the calibrator source and noise, there may be radio frequency interferences (RFIs), or some data might be missing for various reasons, such as receiver malfunction. Even though some of the RFIs and missing data might be removed in preprocessing, some large residues may still be present and wreck the PCA. In such a case, the observed visibilities can be modeled as
\[ V = V_0 + S + N, \tag{12} \]
where \( V_0 = S, GG^\dagger \) is a rank-one matrix from the calibrator (strong point source), \( S \) is a sparse matrix whose elements are outliers (the unflagged RFI, abnormal value, and so forth), which may have large magnitude, and \( N \) is a matrix with dense small elements, which represents the noise, signal of fainter objects in the field of view, cross-talks (i.e., signals from an adjacent antenna/feed that are picked up by an adjacent antenna/feed, causing an erroneous correlation), and so on, and we assume it has a magnitude smaller than the nonzero elements of the outliers \( S \). The SPCA method (Zhou et al. 2010) may be applied to solve the problem in this case. In this approach, the observed data matrix \( X \) is decomposed as \( X = L + S + N \) where \( L \) is a matrix of low rank, \( S \) a sparse matrix (i.e., only a small fraction of its elements are nonzero), and \( N \) a dense noise matrix. In our case, the SPCA would yield \( L = S, GG^\dagger = V_0 \).

The SPCA decomposition is achieved by solving the following optimization problem
\[
\min_{L,S} \frac{1}{2} ||X - L - S||_F^2 + \lambda ||S||_0
\]
subject to \( \text{rank}(L) \leq r \). This is done with a block coordinate descent strategy: first take an estimate of outliers \( S \) and subtract it out to get the "cleaned" data \( C = X - S \), and fit \( L \) according to \( C \). Then, we update the outliers \( S \) by hard thresholding on the error \( E = X - L \). That is, iterate the following steps until it converges:
1. \( L = \text{SVD}(X - S); \)
2. \( \lambda = \sqrt{2 \log (mn)} \text{MAD}(X - L)/0.6745; \)
3. \( S = \Theta_{\sqrt{2} \lambda}(X - L). \)
Here, SVD, \( M \) is the rank-\( r \) truncated singular value decomposition (SVD) of the matrix \( M \) (i.e., the SVD with all small singular values being truncated to zero except the largest \( r \) ones), \( \Theta_s(z_c) \) is the hard-thresholding operator defined in Equation (1), and MAD is the median absolute deviation, \( \text{MAD}(E) = \text{med}(|E - \text{med}(E)|) \) for a real matrix \( E \), where \( \text{med} \) denotes the median of the sample. The MAD provides a robust estimate for the standard error, in the case of independent and identically distributed real Gaussian variable \( E \in \mathbb{R}^{m \times n} \),

\[
\hat{\sigma} = \text{MAD}(E)/0.6745.
\]  

(13)

For the complex case,

\[
\text{MAD}(E) = \sqrt{\text{MAD}(\Re(E))^2 + \text{MAD}(\Im(E))^2}
\]  

(14)

Some entries of \( X \) may not be available, such as the data flagged as RFI. Let \( \Omega \subset \{(i, j); i = 1, \ldots, m, j = 1, \ldots, n\} \) be the index set of the available entries of \( X \), and \( \Omega_c \) be its complement. To deal with the missing data, we first set \( X_{ij} = 0 \) for \( (i, j) \in \Omega_c \), while keeping other values unchanged, then solve the optimization problem as before but with the additional constraint,

\[
(L + S + N)_{ij} = 0, \text{ for } (i, j) \in \Omega_c.
\]  

(15)

If in the solution the values of these elements in the low-rank component \( L \) are close to zero, in \( \Omega_c \) they will introduce only small perturbations to the data, which would be separated out as small noises and assigned to the matrix \( N \). On the other hand, if the corresponding values of \( L \) are large, they will be treated as outliers, as long as the support of the true outliers and the identified outliers is not so large that it causes the algorithm to fail. This would have little effect for the recovery of the low-rank component \( L \), which is usually the one in which we are most interested in practice.

Applying the SPCA to Equation (12), we solve for

\[
\min_{V, S} \left[ \frac{1}{2} \| V - V_0 - S \|^2_F + \lambda \| S \|_0 \right] \text{ s.t. rank}(V_0) \leq 1.
\]  

(16)

To solve Equation (16), we need to initialize the outliers \( S \). The simplest choice would be \( S = 0 \), which works in most cases. Alternatively, we may set

\[
S = \begin{cases} 
V - \text{med}(V); & \text{if } |V - \text{med}(V)| \geq \tau \text{MAD}(V); \\
0; & \text{otherwise},
\end{cases}
\]  

(17)

where \( \text{med}(V) = \text{med}(\Re(V)) + i \text{med}(\Im(V)), \) \( \tau \) is a chosen threshold, usually between 3 and 5. The motivation for this initialization is that we expect elements of \( V_0 \) are of similar magnitude, so values that are well above the median would likely be outliers. This initialization helps to make the algorithm converge faster.

The SPCA decomposition and eigen-analysis calibration method only assumed very simple telescope and sky models. It is fairly robust, and the computation complexity \( \propto N^2 \), where \( N \) is the number of elements in the array, which make it scalable to arrays with a very large number of elements.

2.3. Extension to Polarization

The method described above can also be extended to case of full-polarization response calibration with polarized points sources. To characterize the full polarization states, in addition to the same polarization correlations, we should also include the cross-polarization correlations—that is, \( \nu_{ij}^{XX}, \nu_{ij}^{YY}, \nu_{ij}^{XY}, \nu_{ij}^{YX} \) for linear polarization feeds, or \( \nu_{ij}^{LL}, \nu_{ij}^{LR}, \nu_{ij}^{RL}, \) and \( \nu_{ij}^{RR} \) for circular polarization feeds.

Denote the electric field of the incoming wave in orthogonal polarization components \( p, p = (v^X, v^Y) \) for linear polarizations or \( p = (v^L, v^R) \) for circular polarization. For a system of \( N \) antennas/feeds, the \( 2N \) component array voltage response is given by

\[
q = Gp,
\]  

(18)

where \( G \) is an \( N \times 2 \) gain matrix. The observed visibilities are (neglecting noise)

\[
V = \langle qq^* \rangle = G \langle pp^* \rangle G^T.
\]  

(19)

If there is one dominating point source, the brightness matrix is \( B = \langle pp^* \rangle \), and

\[
B = \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix}.
\]  

(20)

for the linear polarization,

\[
B = \begin{pmatrix} I + V & Q + iU \\ Q - iU & I - V \end{pmatrix}.
\]  

(21)

for circular polarization. Substituting Equation (20) or Equation (21) into (19), we see that \( V \) is a rank-two matrix. This is valid for a single dominating source, regardless of whether it is polarized. As in the unpolarized case, we model the imperfections as noise and outliers,

\[
V = V_0 + S + N,
\]  

(22)

where we define

\[
V_0 = GEB^T,
\]  

(23)

which is the visibility generated by the point source, and it is a rank-two matrix. As in the unpolarized case, \( S \) is a sparse matrix whose nonzero elements are outliers, and \( N \) is a small dense noise matrix.

The decomposition in Equation (22) could also be done by the same SPCA algorithm, except that the rank of \( V_0 \) is two instead of one, but now the problem is how to solve for the gain \( G \) from Equation (23). It is not possible to uniquely determine \( G \) by observing a single-calibrator source. To obtain the full solution of the system gain \( G \), three calibrators with different polarizations are needed.

Below we shall solve the \( XX \) and \( YY \) polarizations separately without considering the cross-polarization correlations; this is equivalent to the solution of two unpolarized cases. The calibration of the Tianlai array with full-polarization response will be deferred to future studies.

3. Application to the Tianlai Array

For illustration, we apply the calibration method described above to the Tianlai Cylinder Array, which consists of three adjacent north–south oriented parabolic cylinders. A total of 96 dual-polarization feeds are installed on them, with 31, 32, and
33 units on the three cylinders, which all span the same length of 12.4 m, so that the distances between the feeds are 41.33 cm, 40.00 cm, and 38.75 cm, respectively. This arrangement forms slightly unequal baselines in order to reduce the grating lobes (Zhang et al. 2016b). The data set used here was collected during the first light drift scan observation on 2016 September 27. We choose a period of the data when the sky calibration source Cygnus A (Cyg A) transits over the array. The transit time is 13:25:46 (UT+0h). Before calibration, the data are first preprocessed to remove known bad channels and strong RFI.

The SumThreshold method (Offringa et al. 2010) is used for RFI flagging. Currently, the Tianlai Cylinder Array has an observational frequency band of 100 MHz (700—800 MHz), with a frequency resolution of 99.765 kHz (1024 equally spaced frequency bins). The calibration method can be applied to each frequency bin. As an example, here we show the result for the frequency channel of $\nu = 750$ MHz.

For logistic reasons, the Tianlai array antenna is located a relatively long distance away from the station house, where the electronic systems sit. The radio frequency signal from the antenna feed, after being amplified by a low noise amplifier, is converted to an analog optical signal and transmitted via optical fiber (radio frequency over fiber) to the station house. The cable length is about 7 km and varies slightly as the environment temperature changes. This necessitates a two-step calibration procedure: in the first step, we use a periodically broadcasted artificial noise source signal to do a relative phase calibration, so as to compensate the phase variations over time induced by the cable delay; then we perform an absolute calibration by using a strong radio source on the sky.

### 3.1. Noise Source Calibration

For the purpose of calibrating relative phase change, we have installed an artificial noise calibrator, which is a disk-cone antenna located on a wooden post on top of a small hill about 250 m away from the center of the cylinder array. After a field test around the antenna array, this position is chosen where all cylinders could receive the signal and the signal strength is comparable. The on-off period is adjustable; during the observation presented here, it broadcasts a broadband noise lasting for 20 s at a period of 4 min. During the on time (20 s), the sky rotates only about 5 arcmin at the equator, which is only a small fraction of the primary beamwidth, so that the sky signal measured during this time does not change much. The signal from the artificial noise calibrator is much stronger than the signal from the sky, and its broadcasting time is known, so it is easily recognized in the data. The noise source can be viewed as a near-field source. Approximately, its visibility can be approximated as

$$V_{ij}^{\text{ns}} \approx S_n \frac{A_i(\hat{n}_i)A_j(\hat{n}_j)}{\sqrt{\Omega_i \Omega_j}} \frac{r^2}{r_ir_j} e^{-i\Delta_{i-j} \nu + \phi_{ij}}$$  \hspace{1cm} (24)

where $r_i, r_j$ are the distance between the noise source and the receiving feed $i, j$ respectively. However, even if the prefactor in Equation (24) is not exact, the phase factor $e^{-i\Delta_{i-j} \nu + \phi_{ij}}$ would still be right. The noise source is switched on and off periodically, and the time averaged (20 s on and 20 s off just before the on signal) visibility data obtained are

$$V_{ij}^{\text{on}} = G_{ij} (V_{ij}^{\text{sk}} + V_{ij}^{\text{ns}} + n_{ij})$$

$$V_{ij}^{\text{off}} = G_{ij} (V_{ij}^{\text{sk}} + n_{ij}),$$

so we have

$$V_{ij}^{\text{on}} - V_{ij}^{\text{off}} = G_{ij} V_{ij}^{\text{ns}} + \delta n_{ij}$$

$$\approx C |G_{ij}| e^{-i\Delta_{i-j} \nu + \phi_{ij}}$$

where $\delta n_{ij}$ is the difference of the random noise. Because the phase factor $e^{-i\Delta_{i-j} \nu + \phi_{ij}}$ is the equivalent instrument delay difference between the channels $i$ and $j$, which is mostly due to the variation in the cable length for the two channels. The phase of it is $\phi_{ij} = \text{Arg}(V_{ij}^{\text{on}} - V_{ij}^{\text{off}}) = k \Delta L_{ij} + \text{const.}$, (25)

where we have used the fact that the relative position of the noise source and the receiving feeds are fixed and the coefficient $C$ is stable.

For most receivers, the calibrator is in the side lobe of the beam, so the observed amplitude is not very stable. We therefore only use it to correct for the phase change. An example of this phase change is shown in Figure 1 for the XX linear polarization of several baselines (the baselines are marked by the pair of receiver number), where the points mark the values measured when the noise source is turned on, whereas the curves are cubic spline interpolations. The top panel shows the changes of phase during a full day, and the middle panel shows an interval of 2 hr during the night. We see the phases change smoothly for a small amount during the night, varying a few degrees for most baselines. However, the changes are more significant and rapid during daytime, probably because the temperature varies more significantly during daylight, which affects the length of the optical fibers. To check this point, we note that if the phase variation is mainly due to the change in optical fiber length, then $\Delta \phi_{ij} = 2\pi \nu \Delta L_{ij}/c = 2\pi \nu \Delta L_{ij}/c$, so the large part of the phase change should be linear in frequency. In the bottom panel of Figure 1 we plot $\Delta L_{ij} = \Delta \phi_{ij}/2\pi \nu$ for a typical feed pair. The horizontal color stripes show that the linear term indeed dominates the phase change, though of course there are RFI s and also small nonlinear variations over the frequency. The time difference is at the level of a few to a few tens of picoseconds, corresponding to millimeter to centimeter changes in the optical cable over a distance of a few kilometers.

We compensate for the relative phase change due to $\Delta L$ in the relative phase calibrated visibility:

$$V_{ij}^{\text{rel-cal}} = e^{-i\nu_{\phi}} V_{ij},$$

(26)

After this step, the phase variation over time in the observed visibility is removed, but there is still an unknown constant phase factor to be determined in $V_{ij}^{\text{rel-cal}}$, which must be determined from absolute calibration using a sky source.

### 3.2. Sky Source Calibration

After compensating for the relative phase changes, we use the noise source calibrator off data for sky calibration. In this first-stage calibration, we use the strongest radio point sources during its meridian transit as the calibrator. Cyg A is an
excellent source for such purpose, because its position is very close to the zenith of the array. We also used several other strong sources, such as the Cassiopeia A (Cas A), Taurus A (Tau A), and Virgo A (Vir A). As discussed in Section 2.1, to get an accurate gain by using the eigenvector-based calibration method, we need the signal of a bright point source dominating over the noise (including the signal from the rest of the sky). This is true for the four sources above, as can be seen from the observed visibilities; one example is shown in Figure 2, where the very strong signal of Cyg A (cyg), Cas A (cas), and Tau A (crab) can be clearly seen in the data observed during the night. In this first-stage calibration, we calibrate the XX and YY linear polarizations separately; the full-polarization calibration will be deferred to future works.

To verify the potential impact of the fainter sources, we performed a simulation by generating simulated visibilities using only the calibrator (Cygnus A) and the calibrator plus the radio background given by the Global Sky Model (de Oliveira-Costa et al. 2008), and compare the solution of complex gains. In the simulation, the amplitude of the complex gains for each element are the same, whereas the phases are randomly distributed. The result is shown in Figure 3. The top panel shows the amplitude, the middle panel shows the solved phase, and the bottom panel shows the difference in the phases for these two cases. We see that the differences in both phase and amplitude are small; indeed, for the phases the points coincide with each other so that in the middle panel, one can not even see the difference. This demonstrates that our proposed method could work in the presence of realistic point sources and diffuse backgrounds.

As shown by the discussions above, in the ideal case of a single dominating point source, we should have only one nonzero eigenvalue in the visibility $V$. In Figure 4, we show the eigenvalues for the observed visibility data $V$ (blue points). Actually, besides the largest one, there are also a few other sizable eigenvalues. In contrast, we also show the eigenvalues of the matrix $V_0$ (green points), $S$ (red points), and $N$ (cyan points) obtained by the SPCA decomposition on the same plot. We see there is only a single large eigenvalue of $V_0$, and the remaining eigenvalues are all very small; the eigenvalues (except the largest two) of the outlier matrix $S$ almost overlaps with the ones of $V$, and the eigenvalues of the noise matrix $N$ are all very small. This clearly show that the other sizable
The eigenvalues of $V$ are mainly due to the contribution of the outliers $S$. In other words, outliers in the data may severely bias the gain solution if one simply does a rank-one PCA of the matrix $V$, and the SPCA decomposition helps to separate the components and get better calibration precision.

In Figure 5 we plot the magnitude of the eigenvector corresponding to the largest eigenvalue of $V$ (blue) and $V_0$ (green). The top and bottom panels are for the $XX$ (east–west) polarization and $YY$ (north–south) polarization, respectively.

We show the SPCA decomposition of the observed visibility in Figure 6 for the Cyg A transit. The three components are successfully separated. Although the autocorrelation (the main diagonal of the $V$ matrix) of each feed is high, it does not appear in the recovered rank-one matrix $V_0$. The autocorrelation is dominated by noise, but amazingly, the much smaller visibility induced by the sky source is extracted from the data.
Also we note that, in the recovered $V_0$, there are several apparently symmetric horizontal/vertical strips that have a value of zero. We have checked that they correspond to the bad feeds, which are automatically detected in this decomposition. The outliers are picked out and put in the sparse $S$ as expected. Though we call $S$ the outliers matrix, not all of its nonzero elements are outliers. The high noise in the autocorrelations and short baselines also come under $S$ in this classification. For the same reason, elements in $N$ are not all pure random noise, and we can see some obvious patterns in it. Three squares along the main diagonal are formed by the correlations/cross-talks between feeds along the same cylinder.

In Figure 7 we show a snapshot of solved $G$ at the transit time of Cyg A obtained from the SPCA analysis. Both the real and imaginary parts as well as the amplitude of $G_i$ are shown. We see that the phase of $G$ are randomly distributed, but most feeds has a typical $|G_i|$ value of 300 ~ 400 (digital output, arbitrary units). However, a few feeds have small gain amplitude, $|G_i| \approx 0$; these are the malfunction ones.

If the beam response $A_i(\hat{\theta})$ and the positions of the antenna/feed $u_i$ are accurately known, we can solve the gain $g_i$ for each feed $i$ from $G_i$. However, the beam response of the Tianlai cylinder array has not been measured before. Although a beam model was computed with electromagnetic field simulation (Cianciara et al. 2017), it is based on the ideal model, whereas the actual construction could be different. Here we fit the beam profile from the observed data.

From Equation (7), the normalized $G_i$ is given by

$$\hat{G}_i \equiv \frac{G_i}{|G_i|} = \hat{g}_i E_i,$$

where $\hat{g}_i \equiv g_i/|g_i|$ and $E_i = e^{-2\pi i h_i u_i}$. We used the fact that the beam profile $A_i(\hat{\theta}_0)$ is real. $E_i$ varies as the calibration source transits over the beam. Assuming that the receiver is
stable and its complex gain \( g_i \), a constant during this period, we may fit \( \delta g_i \) with the observational data. This determines the phase of the gain \( g_i \). For the amplitude \( |g_i| \),

\[
|G_i| = |g_i| A_i(\hat{n}_0).
\]

(28)

We see \(|g_i|\) is degenerate with the normalization of beam response \( A_i(\hat{n}_0) \). We may choose a normalization (e.g., take \( A_i(\hat{\chi}) = 1 \) in the direction of zenith). In fact, it happens that the decl. of Cyg A (40°44′02") is close to the latitude of the Tianlai site (44°09′08"), so it crosses near the zenith during its transit. The cylinder array beam response is a narrow strip along the north–south direction and it varies slowly near the zenith (Cianciara et al. 2017), so as a first approximation we can normalize \( A_i \approx 1 \) at the direction of Cyg A when it transits over the array.

In Figure 8, we show the total sky visibility and the various components (the Cyg A, the outlier and the noise) obtained by decomposition for two baselines during a period of 40 minutes centered at Cyg A’s transit time. The two baselines shown are for the element pair (1, 13) (short, north–south direction) and the element pair (15, 80) (long, nearly east–west) XX polarization. We have removed the part when the artificial noise calibrator was on, so the curves are broken at the time of their broadcasting. As expected, for the Cyg A (marked “cyg” in the figure) component \( V_0 \), the NS baseline show a general profile of the primary beam, whereas the east–west baseline shows interferometer fringes with primary beam as the envelope. The outlier and noise components are small for these two baselines during the Cyg A transit.

The outlier matrix is a sparse one, and for most baselines it is small, as in the last figure, but occasionally the decomposition yields nonzero outlier components; two examples are shown in Figure 9. These are more frequently seen in the visibility of the short baselines, which perhaps have higher noise levels due to cross-talk. In the top panel of Figure 9, the outlier component is much greater than the threshold and so varies smoothly during the observation. In the bottom panel, because the level of the outlier component is close to the threshold \( \lambda = \sqrt{2 \log(mn) \sigma} \), there is some degeneracy of the two components and we can see the mixing or rapid switching of the two during the observation. This, however, does not affect the calibration, which uses only the point-source component that is still stable.

We see that the signal of the point source Cyg A is dominant in about half an hour. When its signal dominates, the SPCA algorithm can successfully extract it from the observed visibility, but the algorithm fails when its strength drops to the level of noise. When the algorithm fails, the solution of the low-rank component is somewhat unstable. The relative strength between the signal and the noise level can be roughly quantified by the ratio between the largest eigenvalues of \( V_0 \) and \( V - V_0 \). In Figure 10, we show the largest and second largest eigenvalues of the matrix \( V \) (marked as \( V_1, V_2 \)) in solid and dashed blue curves, respectively, and the largest eigenvalue of the SPCA component \( V_0 \) in green curve, as well as the eigenvalue of the matrix \( V - V_0 \) in red curve during the transit. The largest eigenvalue of \( V \) is significantly larger than the second from 13:19 UT to 13:32 UT. At the same time, the largest eigenvalue of \( V_0 \) is significantly larger than the largest eigenvalue of the remaining components \( V - V_0 \), showing the dominance of the calibrator signal. Beyond this time interval, the eigenvalues of \( V \) become comparable with each other, and the largest eigenvalue of \( V_0 \) drops below that of \( V - V_0 \). The algorithm begins to fail to extract the low-rank signal component as the signal strength drops. We can truncate the algorithm here. However, in practice we find that the algorithm can go a much longer way until the largest eigenvalue of \( V_0 \) drops below a factor \( c < 1 \) of that of \( V - V_0 \). To make the computation run smoothly, we make the following remedy: when the largest nonzero eigenvalue of the solved low-rank component \( V_0 \) falls below a factor \( c \) (we take \( c = 0.2 \)) of the largest eigenvalue of the residual matrix \( V - V_0 \), we take \( V_0^{\text{tr}} = \text{svd}(V - V_0) \). This makes the algorithm works more smoothly during a run, but note that this solution is no longer the true dominating low-rank components (i.e., the visibility matrix of the point source).

To check the precision of the calibration, we compare the gains obtained for several strong point sources with different transit times, including the Cyg A, the Cas A, and the Tau A. They all transit over the array during the night in that observation, over a time span of about 9 hr. The result is shown in Figure 11. We see the complex gain solutions obtained for the three calibrator sources are highly consistent with each other, especially in their phases. The rms of the difference in phase between pairs of calibrators \( s1 \) and \( s2 \) before and after applying the artificial noise source calibration are given in Table 1, from which we see the relative phase calibration by
using the noise source calibrator can effectively compensate for
the phase variations over time and improve the phase precision
of the eigenvector-based calibration by using strong sky
sources. The amplitudes of the gains have some differences,
but note that in the approach described above, in each case the
beam $A_i(\hat{\theta}_0)$ is normalized to 1 at the peak of the transit, but the
three calibrators are actually located at different declinations, so
part of this difference may come from the north–south beam
profile.

3.3. Redundant Baselines

The redundant baselines of an interferometer array are
baselines with the same direction and length but are formed by
different pairs of receivers. Theoretically, the redundant
baselines should all have identical outputs, so they provide a
good check on the calibration. The difference in their output
reflects the nonuniformity of the system.

In the Tianlai cylinder array, the receivers on the same
cylinder are placed along the due north–south direction with

Figure 8. Samples of SPCA decomposition result with nearly zero outliers
during Cyg A transit. Top: a north–south short baseline (1, 31) XX polarization;
Bottom: an east–west long baseline (15, 80) XX polarization. The three
subpanels in each plot are, from top to bottom, the real part, imaginary part and
the magnitude of the visibility.

Figure 9. Samples of SPCA decomposition result with nonzero outliers during
Cyg A transit. Similar to Figure 8, but for two different baselines that exhibit
the nonzero outliers. Top: baseline (2, 3) XX polarization; Bottom: baseline
(32, 40) XX polarization.

Figure 10. The variation of the largest eigenvalue of $V$, $V_0$ and $V - V_0$. 
equal spacings, though the spacings for the three cylinders are different (the three cylinders have 31, 32, and 33 feeds, respectively, each with a total length of 12.4 m, so that and the average center-to-center spacings are 0.4133 m, 0.40 m, and 0.3875 m, respectively). Thus, for receivers on the same cylinder, except for the longest one, the baselines all have redundancy, with the shorter ones having more redundancy.

Table 1

| Pairs     | XX-pol | YY-pol | XX-pol | YY-pol |
|-----------|--------|--------|--------|--------|
| cyg–cas   | 11°78  | 13°77  | 8°5    | 7°3    |
| cyg–crab  | 10°5   | 14°71  | 8°1    | 6°9    |
| cas–crab  | 8°7    | 7°8    | 7°7    | 10°1   |

In Figure 12 we show all the visibilities of the redundant baselines for a single frequency at the moment of Cyg A meridian transit. If the gains of the receivers are the same, with only differences in the phase, we would expect that the visibilities all have the same magnitude, but with different
phases, so that in the complex plane they should form a circle. As shown in the top panel of the figure, there is a circular distribution of the visibilities, but the magnitudes spread over the whole circle area, because of differences in both the receiver gain amplitude and the noise. The uncalibrated $V_0$ component extracted by the SPCA process is shown in the middle panel of Figure 12, where the ring of data points has less spread in the radius, because the noise is removed. Some points in the origin are from malfunctioning feeds that produce too small output. After calibration (bottom panel), for each redundant baseline, the visibilities from the different pairs are indeed collapsed to a single point, and all the points of the different baselines form a nearly perfect circle, as one would have expected from the theory. This shows that our method of calibration indeed works with high precision. The “redundant baseline” calibration method (Pearson & Readhead 1984; Yang 1988; Wieringa 1992; Liu et al. 2010) assumes that the visibility from redundant baselines should all be the same and uses this to solve for the array gain. However, noise or outliers may affect the precision of such calibration method, as shown by Figure 12. If the redundant baseline calibration is performed when a strong source dominates, the SPCA method may also be applied to extract the signal component for use in the redundant baseline calibration, which may help improve the signal-to-noise ratio.

3.4. Beam Profile

Figure 13 shows the solved $|G|$ for the $XX$ and $YY$ polarization during 40 minutes of the Cyg A transit period. The center of beam profiles is found by fitting a sinc function to the good data.

Figure 14. The fitted beam profile.
feeds. The cylinder reflector surface may also have some error. As shown in Figure 13, the measured beam profiles for the different feeds are not completely aligned; this is especially obvious in the second (middle) and third (top) cylinders. We flag out the abnormal ones and then fit the remaining ones with a Gaussian function or a sinc function along the east–west direction. The center point of the profile for the sinc function fit is plotted in Figure 13 as the blue points in the center, from which the misalignment of the beam is more apparently shown. The maximum deviation of the transit peak is 108 s, corresponding to an angle of 0°45. The median value of the deviation is 28 s, corresponding to an angle of 0°12. Field inspection and experiment is needed to determine the actual cause of the misalignment, which is beyond the scope of the present article and will be investigated in future works on the testing of the Tianlai array.

We fit a common beam profile by combining the normalized and aligned data (exclude the bad/abnormal ones). The result for the frequency of 750 MHz is shown in Figure 14, where we plot both the Gaussian and the sinc function fitting curves. The different receiver units have almost identical beam profile in the central part. In the side lobes, the profiles of different units vary a lot, but note that in the side lobe there is also large measurement error, because the calibrator signal is no longer dominant over the noise. The Gaussian and sinc fitting curves also coincide with each other in the center part. From the fitted Gaussian function we can obtain the FWHM of the beamwidth. Using \( \theta_{\text{FWHM}} = \Omega_{\text{Earth}} \Delta f_{\text{FWHM}} \cos \delta_{\text{src}} \) where \( \delta_{\text{src}} \) is the decl. of the source we find the FWHM beamwidth is 2°6 for the XX polarization, and 2°4 for the YY polarization.

4. Conclusion

We have developed a method for the initial calibration of the complex gains of a radio interferometer array by taking the observational data of a strong point source, arranging the visibilities (interferometer correlations) as a matrix \( V_i \) with indices denoting the pairs of receiver feeds, and then solving for the eigenvector of the matrix. The eigenvector of the matrix with the largest eigenvalue gives a least square solution to the complex gains of the receivers. To deal with the noise and outliers (e.g., malfunctioning feeds and residual RFIs) that are frequently seen in such data, we improve the method by first using an SPCA algorithm to decompose the visibility matrix into the point calibrator signal (a low-rank matrix), an outlier component (a sparse matrix), and a noise component (a matrix with dense small elements). When the calibrator signal is strong, this decomposition yields unique solution. Although in this article we have applied the method to transit observations, it can also be used for tracking observation. The method can also be extended to treat the calibration of full-polarization responses, though in that case, calibration observation for at least three polarized calibrator sources is need for solving the additional parameters in the measurement equation.

We applied this method to the first light data of the Tianlai cylinder pathfinder array. The calibration is performed using both periodically broadcasted artificial noise calibrator for the relative instrument phases and strong astronomical radio sources for both phase and amplitude of complex gains. We find that the instrument phases are very stable during the night, though during daytime the phases vary as the environment temperature changes. Checking with visibilities of the redundant baselines, we find that, as expected, the calibrated visibilities form a circle on the complex plane, whereas the raw visibilities spread out as an irregular disk. The SPCA algorithm can be used to extract the signal component from the noise and outliers, which may also be useful to help improve the signal-to-noise ratio in the calibrations on the basis of the redundant baselines.

On the basis of the strong source transit data, the cylinder beam profile is measured along the east–west direction for each feed. We find that despite engineering efforts, there is some misalignment in the feed response, the exact cause of which is still to be determined. We also have fitted the beam profile with Gaussian and sinc functions. After adjusting for the misalignment, the central part of the beam for the different feeds agree very well, and the FWHM beamwidth are measured.

Much further analysis with more data is necessary to fully characterize the performance of the Tianlai array and to accurately calibrate its response. The aim of the present work is to present a method of array calibration based on eigenvector analysis and SPCA decomposition. The method is shown to work with a sample of the Tianlai data. We have incorporated this method in the Tianlai data-processing pipeline, and it will be used in our subsequent works on the testing and commissioning of the Tianlai pathfinder arrays.

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References

Bandura, K. 2011, PhD thesis, Carnegie Mellon Univ.
Chang, T.-C., Pen, U.-L., Peterson, J. B., & McDonald, P. 2008, PhRvL., 100, 091303
Chen, X. 2012, JUMPS, 12, 256
Cianciara, A. J., Anderson, C. J., Chen, X., et al. 2017, JAI, 6, 1750003
de Oliveira-Costa, A., Tegmark, M., Gaensler, B. M., et al. 2008, MNRAS, 388, 247
Hamaker, J. P. 2000, A&AS, 143, 515
Kazemi, S., Yatawatta, S., Zaroubi, S., et al. 2011, MNRAS, 414, 1656
Liu, A., Tegmark, M., Morrison, S., Lutomirski, A., & Zaldarriaga, M. 2010, MNRAS, 408, 1029
Offringa, A. R., de Bruyn, A. G., Biehl, M., et al. 2010, MNRAS, 405, 155
Pearson, T. J., & Readhead, A. C. S. 1984, ARA&A, 22, 97
Perley, R. A., Carilli, C. L., & Taylor, G. B. G. 1999, in Synthesis Imaging in Radio Astronomy II: A Collection of lectures from the Sixth NRAO/NMIMT Synthesis Imaging Summer School held at Socorro (San Francisco, CA: ASP)
Salvini, S., & Wijnholds, S. J. 2014, in 2014 XXXIth URSI General Assembly and Scientific Symp. (URSI GASS) (New York: IEEE), doi:10.1109/URSIGASS.2014.6930038

9 https://github.com/TianlaiProject/tlpipe
Sault, R. J., Hamaker, J. P., & Bregman, J. D. 1996, A&AS, 117, 149
Smirnov, O. M. 2011a, A&A, 527, A106
Smirnov, O. M. 2011b, A&A, 527, A107
Smirnov, O. M., & Tasse, C. 2015, MNRAS, 449, 2668
Tasse, C. 2014, arXiv:1410.8706
Thompson, A. R. A. R., Moran, J. M., & Swenson, G. W. 1986, Interferometry 
and Synthesis in Radio Astronomy (New York: Wiley)
van Weeren, R. J., Williams, W. L., Hardcastle, M. J., et al. 2016, ApJS, 223, 2
Wieringa, M. H. 1992, ExA, 2, 203
Xu, Y., Wang, X., & Chen, X. 2015, ApJ, 798, 40
Yang, Y.-P. 1988, A&A, 189, 361
Zhang, J., Ansari, R., Chen, X., et al. 2016a, MNRAS, 461, 1950
Zhang, J., Zuo, S., Ansari, R., et al. 2016b, RAA, 16, 158
Zhou, Z., Li, X., Wright, J., Candès, E., & Ma, Y. 2010, in 2010 IEEE Int. 
Symp. Information Theory 1518 (New York: IEEE), doi:10.1109/ISIT. 
2010.5513535