A special class of triple starlike trees characterized by Laplacian spectrum

Muhammad Ajmal¹, Xiwang Cao¹, Muhammad Salman², Jia-Bao Liu³ and Masood Ur Rehman⁴,*

¹ Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 211100, P. R. China
² Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan
³ School of Mathematics and Physics, Anhui Jianzhu University, Hefei, Anhui 230601, P. R. China
⁴ Department of Basic Sciences, Balochistan University of Engineering and Technology Khuzdar, Khuzdar 89100, Pakistan

* Correspondence: Email: masoodqau27@gmail.com; Tel: +923348266836.

Abstract: Two graphs are said to be cospectral with respect to the Laplacian matrix if they have the same Laplacian spectrum. A graph is said to be determined by the Laplacian spectrum if there is no other non-isomorphic graph with the same Laplacian spectrum. In this paper, we prove that one special class of triple starlike tree is determined by its Laplacian spectrum.

Keywords: Laplacian spectrum; tree; triple starlike tree

Mathematics Subject Classification: 05C50

1. Introduction

All graphs mentioned in this paper are finite, undirected and simple. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph of order $n$ with vertex set $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(\Gamma) \subseteq \binom{V(\Gamma)}{2}$. The adjacency matrix of a graph $\Gamma$, denoted by $A(\Gamma) = (a_{i,j})$ is a square matrix of order $n$ such that $a_{i,j} = 1$ if two vertices $v_i$ and $v_j$ are adjacent and 0 otherwise. Let $d_i = d_{v_i}$ be the degree of a vertex $v_i$ in $\Gamma$. The Laplacian matrix and the signless Laplacian matrix are defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$ and $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ respectively, where $D(\Gamma)$ is the diagonal matrix with diagonal entries $\{d_1, d_2, \ldots, d_n\}$ and all others entries are zeros [9]. We know that the matrices $A(\Gamma)$, $L(\Gamma)$ and $Q(\Gamma)$ are real symmetric, their eigenvalues are real. So, we assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are the adjacency, Laplacian, and signless Laplacian eigenvalues of $\Gamma$, respectively. The multiset of eigenvalues of $A(\Gamma)$ (or $L(\Gamma)$, $Q(\Gamma)$) is called the adjacency (or Laplacian, signless Laplacian) spectrum of the graph $\Gamma$. If two non-isomorphic graphs share the same adjacency (or
Laplacian, signless Laplacian) spectrum then we call graphs are cospectral. A graph \( \Gamma \) is said to be determined by the adjacency spectrum (abbreviated to DAS ) if there is no non-isomorphic graph which have same adjacency spectrum. Similarly, we can define DLS graph for the Laplacian matrix \( L(\Gamma) \) and DQS graph for the singless Laplacian matrix \( Q(\Gamma) \).

In 1956, Günthard and Primas [6] raised a question “which graphs are determined by their spectrum” in context of Hückels theory. Basically this problem originates from Chemistry and generally seem to be very difficult. On this problem, Dam and Haemers [24] presented a survey and proposed the modest shape of this problem that is “which trees are determined by their spectrum”. Many researcher have been established results on the graphs DAS, DLS and DQS and some of these results can be found in [1, 2, 21, 23–25], in [1, 5, 11–14, 21, 22, 26, 27, 29] and in [3, 4, 13, 16, 17, 20, 28] respectively.

A tree \( T_{n'} \) of order \( n' \) is a connected graph without cycle. A vertex \( v \in V(T_{n'}) \) is called large if \( d_v \geq 3 \). A tree having one, two or three large vertices is called starlike, double starlike or triple starlike tree, respectively.

We denote by \( T_{n'}(p, q) \), \( n' \geq 2 \) and \( p, q \geq 1 \), one special double starlike tree (of order \( n' \)) obtained by joining \( p \) pendent vertices to an end vertex of a path of length \( n \) and joining \( q \) pendent vertices to another one. In 2009 Liu et al. [11] studied the Laplacian spectrum of \( T_{n'}(p, q) \), for \( q = p \) they showed that \( T_{n'}(p, q) \) can be determined by its Laplacian spectrum. In 2010 Lu et al. [14] showed that for \( q = p - 1 \), \( T_{n'}(p, q) \) can be determined by its Laplacian spectrum. In [15], the authors proved that \( T_{n'}(p, q) \) can be completely determined by its Laplacian spectrum.

Let \( P_n \) be a path of length \( n \), where \( n \geq 5 \), attach \( p \) pendent vertices to an end vertex of \( P_n \), \( p \) pendent vertices to another end vertex and \( p \) pendent vertices to any vertex \( v \in V(P_n) \) which is at distance at least two from the end vertices of \( P_n \), where \( p \geq 2 \), by this way we obtain a special triple starlike tree as shown in Figure 1 (or Figures 2 and 3). We denote the aforementioned tree by \( T_{n'}(p, p, p) \). Note that the order \( n' \) of \( T_{n'}(p, p, p) \) is \( n + 3p \), where \( l_1 + l_2 = n - 3 \) see Figure 1. In this paper we show that \( T_{n'}(p, p, p) \) can be determined by its Laplacian spectrum or more precisely, any graph that is determined by its degree sequence is determined by its Laplacian spectrum.

![Figure 1. Triple starlike tree \( T_{n'}(p, p, p) \).](image)

![Figure 2. \( T_{n'}(p, p, p) \) with \( n' = 6 + 3p \).](image)
2. Preliminaries

In this section, we present some known results that play an important role in the results of next section.

Lemma 1. [19, 24] Let $\Gamma$ be a graph. Then the following items are determined by spectrum of adjacency or Laplacian matrix.

1. The number of vertices in $\Gamma$.
2. The number of edges in $\Gamma$.
3. Whether $\Gamma$ is regular.
4. Whether $\Gamma$ is regular with any fixed girth.

For adjacency matrix, the following quantities are determined by its spectrum.

5. Whether $\Gamma$ is bipartite or not.
6. The number of closed walks of any length.

For Laplacian matrix, the following quantities are determined by its spectrum.

7. The number of components.
8. The number of spanning trees.
9. The sum of the squares of degrees of vertices.

Lemma 2. [7] Let $T_n$ be a tree of order $n$ and $\mathcal{L}(T)$ be its line graph. Then $\lambda_i(T) = \mu_i(\mathcal{L}(T)) + 2$, where $1 \leq i \leq n$.

Lemma 3. [8, 10] For a graph $\Gamma$ with a non-empty vertex set $V(\Gamma)$ and non-empty edge set $E(\Gamma)$, let $\Delta(\Gamma)$ be the maximum vertex degree of $\Gamma$. Then we have the following inequality.

$$\Delta(\Gamma) + 1 \leq \mu_1(\Gamma) \leq \max \left\{ \frac{d_{v_i}(d_{v_i} + m_{v_i}) + d_{v_j}(d_{v_j} + m_{v_j})}{d_{v_i} + d_{v_j}} : v_i, v_j \in E(\Gamma) \right\}$$

where $m_{v_i}$ denotes the average of the degrees of the vertices adjacent to a vertex $v_i$ in $\Gamma$.

3. Determination of $T_{n'}(p, p, p)$ by Laplacian spectrum

In this section, first we establish the following two lemmas which will be used in our main result.

Lemma 4. Let $\Gamma'$ and $\Gamma = T_{n'}(p, p, p)$, where $n' = n + 3p$ with $n \geq 5$ and $p \geq 2$ are cospectral graphs with respect to the Laplacian matrix. Then $\Gamma'$ has $t_{p+2} = 1$, $t_{p+1} = 2$, $t_2 = n - 3$, and $t_1 = 3p$, where $t_i$ is number of the vertices of degree $i$ in $\Gamma'$.

Proof. Given that the graphs $\Gamma'$ and $\Gamma$ are cospectral with respect to the Laplacian matrix. Then by 1, 2, 7, and 8 of Lemma 1, the graph $\Gamma'$ is a tree with $|V(\Gamma')| = n + 3p$ and $|E(\Gamma')| = n + 3p - 1$. From

![Figure 3. $T_{n'}(p, p, p)$ with $n' = 10 + 3p$.](image-url)
Lemma 3, we have $p + 3 \leq \mu_1 \leq p + 4 - \frac{2}{p+3}$, which implies the maximum degree in the graph $\Gamma'$ is at most $p + 2$. Now, assume that the graph $\Gamma'$ has $t_i$ vertices of degree $i$, for $i = 1, 2, \ldots, \Delta'$, where $\Delta' \leq p + 2$ is the maximum degree of $\Gamma'$. The following equations follow from 1, 2, and 9 of Lemma 1.

\[
\sum_{i=1}^{\Delta'} t_i = n + 3p \tag{3.1}
\]

\[
\sum_{i=1}^{\Delta'} it_i = 2(n + 3p - 1) \tag{3.2}
\]

\[
\sum_{i=1}^{\Delta'} i^2t_i = 3p^2 + 11p + 4n - 6 \tag{3.3}
\]

Then
\[
\sum_{i=1}^{\Delta'} (i^2 - 3i + 2)t_i = 3p^2 - p \tag{3.4}
\]

The line graphs of $\Gamma$ and $\Gamma'$ have same spectrum with respect to the adjacency matrix, by Lemma 2. Hence, from 6 of Lemma 1, they have same number of triangles (closed walk of length three). Therefore,

\[
\binom{p + 2}{3} + 2 \binom{p + 1}{3} = \sum_{i=1}^{\Delta'} \binom{i}{3} t_i
\]

First we show that there is only one vertex of degree $p + 2$, i.e., $t_{p+2} = 1$. If $t_{p+2} = 0$, i.e., $\Delta' < p + 2$

\[
\binom{p + 2}{3} + 2 \binom{p + 1}{3} = \sum_{i=1}^{\Delta'} \binom{i}{3} t_i \leq \frac{p + 1}{6} \sum_{i=1}^{\Delta'} (i - 1)(i - 2)t_i
\]

i.e.,

\[3p^2 \leq \sum_{i=1}^{\Delta'} (i^2 - 3i - 2)t_i\]

By Eq (3.4), $p \leq 0$ which is a contradiction.

If $t_{p+2} \geq 2$, i.e., there are at least two vertices of degree $\Delta = p + 2$, then we have

\[
\binom{p + 2}{3} + 2 \binom{p + 1}{3} = \sum_{i=1}^{\Delta'} \binom{i}{3} t_i \geq 2 \binom{p + 2}{3} + \sum_{i=1}^{p+1} \binom{i}{3} t_i
\]

i.e.,

\[\binom{p + 2}{3} + 2 \binom{p + 1}{3} \geq \sum_{i=1}^{p+1} \binom{i}{3} t_i\]

This is a contradiction, hence $t_{p+2} = 1$. 
Second we prove that $t_{p+1} = 2$. If $t_{p+1} \geq 3$, i.e., there are at least three vertices of degree $p + 1$, then we have

\[
\left( \binom{p+2}{3} \right) + 2 \left( \binom{p+1}{3} \right) = \sum_{i=1}^{\Delta'} \left( \binom{i}{3} \right) t_i \geq \left( \binom{p+2}{3} \right) + 3 \left( \binom{p+1}{3} \right) + \sum_{i=1}^{p} \left( \binom{i}{3} \right) t_i
\]

i.e.,

\[- \left( \binom{p+1}{3} \right) \geq \sum_{i=1}^{p} \left( \binom{i}{3} \right) t_i\]

This is a contradiction, so $t_{p+1} \leq 2$.

If $t_{p+1} = 0$, i.e., there is no vertex of degree $p + 1$, then we have

\[
\left( \binom{p+2}{3} \right) + 2 \left( \binom{p+1}{3} \right) = \sum_{i=1}^{\Delta'} \left( \binom{i}{3} \right) t_i \leq \left( \binom{p+2}{3} \right) + \sum_{i=1}^{p} \left( \binom{i}{3} \right) t_i
\]

i.e.,

\[
2 \left( \binom{p+1}{3} \right) \leq \frac{p}{6} \sum_{i=1}^{p} (i-1)(i-2)t_i
\]

i.e.,

\[2(p+1)(p-1) \leq \sum_{i=1}^{p} (i-1)(i-2)t_i\]

i.e.,

\[2(p+1)(p-1) + p(p+1) \leq \sum_{i=1}^{p} (i-1)(i-2)t_i + p(p+1)\]

i.e.,

\[3p^2 + p - 2 \leq \sum_{i=1}^{p+2} (i-1)(i-2)t_i\]

By Eq (3.4), $3p^2 + p - 2 \leq 3p^2 - p$, i.e., $2p - 2 \leq 0$, a contradiction.

If $t_{p+1} = 1$, i.e., one vertex of degree $p + 1$, then

\[
\left( \binom{p+2}{3} \right) + 2 \left( \binom{p+1}{3} \right) = \sum_{i=1}^{\Delta'} \left( \binom{i}{3} \right) t_i \leq \left( \binom{p+2}{3} \right) + \left( \binom{p+1}{3} \right) + \sum_{i=1}^{p} \left( \binom{i}{3} \right) t_i
\]

i.e.,

\[
\left( \binom{p+1}{3} \right) \leq \frac{p}{6} \sum_{i=1}^{p} (i-1)(i-2)t_i
\]

i.e.,

\[(p+1)(p-1) \leq \sum_{i=1}^{p} (i-1)(i-2)t_i\]
\( (p + 1)(p - 1) + p(p - 1) + p(p + 1) \leq \sum_{i=1}^{p} (i - 1)(i - 2) t_i + p(p - 1) + p(p + 1) \)

i.e.,
\[ 3p^2 - 1 \leq \sum_{i=1}^{p+2} (i^2 - 3i + 2)t_i \]

By Eq (3.4), \( 3p^2 - 1 \leq 3p^2 - p \), i.e., \( p - 1 \leq 0 \), a contradiction. Thus \( t_{p+1} = 2 \).

For each \( i = 3, 4, \ldots, p \), \( t_i = 0 \) from Eq (3.4). Finally, Eqs (3.1) and (3.2), immediately yield \( t_1 = 3p \) and \( t_2 = n - 3 \). This finishes the proof.

Lemma 5. Let \( \Gamma \) be any tree of order \( n + 3p \), where \( n \geq 5 \) and \( p \geq 2 \), such that \( t_{p+2} = 1 \), \( t_{p+1} = 2 \), \( t_2 = n - 3 \), and \( t_1 = 3p \). Then \( \Gamma \) is isomorphic to \( T_w(p, p, p) \).

Proof. It is clear that there exists 2, \( p + 2 \), \( 2p \), and \( n - 5 \) vertices of degree \( p + 2 \), \( p + 1 \), \( p \), and 2, respectively in the line graph \( L \). Here, we divide the proof into two main cases.

Case 1. When two vertices of degree \( p + 1 \) are joined by a path of length \( l_1 + 1 \), \( (l_1 \geq 0) \) and one vertex of them joined by a path of length \( l_2 + 1 \), \( (l_2 \geq 0) \) with a vertex of degree \( p + 2 \). Without loss of generality, suppose that there exist \( p - x \), \( p - y - 1 \), and \( p + 1 - z \) vertices of degree 1 which are adjacent to the vertex of degree \( p + 1 \), \( p + 1 \), and \( p + 2 \), respectively in the graph \( \Gamma \) as shown in Figure 4. Where \( 0 \leq x \leq p \), \( 0 \leq y \leq p - 1 \), and \( 0 \leq z \leq p + 1 \). Therefore, we have

\[ l_1 + l_2 + \sum_{i=0}^{x} l_i' + \sum_{j=0}^{y} l_j' + \sum_{k=0}^{z} l_k'' + 3p + 3 = n + 3p \]  \hspace{1cm} (3.5)

where \( l'_0 = l''_0 = l''_0 = 0 \). That is

\[ l_1 + l_2 + \sum_{i=0}^{x} l_i' + \sum_{j=0}^{y} l_j' + \sum_{k=0}^{z} l_k'' = n - 3 \]  \hspace{1cm} (3.6)

For the values of \( l_1, l_2 \), there exists four different shape of the graph \( \Gamma \). Let \( l_i' \) be the number of the vertices of degree \( i \). Clearly, there exists \( l_i' = x + y + z \), \( l_2' = n - i - x - y - z \), \( l_p' = 2p - x - y - 1 \), \( l_{p+1}' = p + j + x + y - z \), \( l_{p+2}' = k + z \), \( l_{2p}' = r \), \( l_{2p+1}' = s \), and \( l_i' = 0 \) for \( l = 3, 4, \ldots, p - 1 \) and \( p + 2 < l < 2p \) in the line graph \( L(\Gamma) \), where the values of \( i, j, k, r \) and \( s \) are listed in Table 1 corresponding to each shape of \( L(\Gamma) \).
Table 1. Parameters for the Eq (3.7).

| Values of $l_1$ and $l_2$ | $i$ | $j$ | $k$ | $r$ | $s$ |
|--------------------------|-----|-----|-----|-----|-----|
| $l_1 = 0, l_2 = 0$       | 3   | 1   | 0   | 1   | 1   |
| $l_1 = 0, l_2 \geq 1$   | 4   | 2   | 1   | 1   | 0   |
| $l_1 \geq 1, l_2 = 0$   | 4   | 3   | 0   | 0   | 1   |
| $l_1 \geq 1, l_2 \geq 1$| 5   | 4   | 1   | 0   | 0   |

From 2 and 5 of Lemma 1, the line graphs $L(\Gamma)$ and $L(\Gamma')$ have the same number of edges and closed walks of length 4, respectively. Clearly, they have the same number of 4-cycles $2(\binom{p+1}{2} + \binom{p+2}{2})$. Hence the line graphs $L(\Gamma)$ and $L(\Gamma')$ have the same number of induced paths of length 2. Then we have

\[
2\left(\frac{p+2}{2}\right) + (p+2)\left(\frac{p+1}{2}\right) + 2p\left(\frac{p}{2}\right) + (n-5)\left(\frac{2}{2}\right) = s\left(\frac{2p+1}{2}\right) + r\left(\frac{2p}{2}\right) \\
+ (k+z)\left(\frac{p+2}{2}\right) + (p+j+x+y-z)\left(\frac{p+1}{2}\right) + (2p-x-y-1)\left(\frac{p}{2}\right) + (n-i-x-y-z)\left(\frac{2}{2}\right)
\]

i.e.,

\[
(4s+4r+j+k-5)p^2 + (2s-2r+j+3k+2x+2y+2z-7)p - 2(i-k+x+y-3) = 0 \tag{3.7}
\]

Since $p \geq 2, 0 \leq x \leq p, 0 \leq y \leq p-1$, and $0 \leq z \leq p+1$. For each case of $l_1$ and $l_2$ that mentioned in Table 1 and take corresponding values of $i, j, k, r, s$. It is easy to check that the Eq (3.7) is not equal to zero, which is a contradiction.

**Case 2.** When a vertex of degree $p+2$ joined with two vertices of degree $p+1$ by paths of length $l_1+1, (l_1 \geq 0)$ and $l_2+1, (l_2 \geq 0)$, respectively. Without loss of generality, suppose that there exits $p-x, p-x$ and $p-z$ vertices of degree 1 which are adjacent to vertex of degree $p+1, p+1,$ and $p+2$ respectively in the graph $\Gamma'$ as shown in Figure 5. Where $0 \leq x, y, z \leq p$. Therefore, we have same two equations as Eqs (3.5) and (3.6).

Figure 5. Graph $\Gamma'$.

For the values of $l_1, l_2$, there exits three different shape of the graph $\Gamma'$. Clearly, there exits $t_1 = x+y+z, t_2 = n-i-x-y-z, t_p = 2p-x-y, t_{p+1} = p+j+x+y-z, t_{p+2} = k+z, t_{2p+1} = r$ and $t_l = 0, \text{for } l = 3, 4, \ldots, p-1 \text{ and } p+2 < l < 2p+1$ in the line graph $L(\Gamma')$. Where values of $i, j, k, r, s$ are listed in Table 2 corresponding to each shape of $L(\Gamma')$.  

AIMS Mathematics

Volume 6, Issue 5, 4394–4403.
Table 2. Parameters for the Eq (3.8).

| Values of $l_1$ and $l_2$ | $i$ | $j$ | $k$ | $r$ |
|---------------------------|-----|-----|-----|-----|
| $l_1 = 0$, $l_2 = 0$     | 3   | 0   | 0   | 2   |
| $l_1 = 0$, $l_2 \geq 1$ or $l_1 \geq 1$, $l_2 = 0$ | 4   | 1   | 1   | 1   |
| $l_1 \geq 1$, $l_2 \geq 1$ | 5   | 2   | 2   | 0   |

By same argument that we used in first case, we have

$$2\binom{p+2}{2} + (p+2)\binom{p+1}{2} + 2p\binom{p}{2} + (n-5)\binom{2}{2} = r\binom{2p+1}{2} + (k+z)\binom{p+2}{2}$$

$$+ (p+j+x+y-z)\binom{p+1}{2} + (2p-x-y)\binom{p}{2} + (n-i-x-y-z)\binom{2}{2}$$

i.e.,

$$(j + k + 4r - 4)p^2 + (j + 3k + 2r + 2x + 2y + 2z - 8)p - 2(i - k + x + y - 3) = 0 \quad (3.8)$$

Since $p \geq 2$ and $0 \leq x, y, z \leq p$. For first two cases of $l_1$ and $l_2$ that mentioned in Table 2 and take their corresponding values of $i, j, k, r, s$. It is easy to check that the Eq (3.8) is not equal to zero, a contradiction. For the last case $l_1, l_2 \geq 1$, we obtained $(x+y+z)p-(x+y) = 0$, so $x = y = z = 0$. By Eq (3.6), we have $l_1 + l_2 = n-3$. Thus the vertices of degree 1 adjacent to the two vertices of degree $p+1$ and one vertex of degree $p+2$. Hence, $\Gamma'$ is isomorphic to $\Gamma$. □

Now, we ready to prove our main result.

**Theorem 6.** The tree $T_n'(p, p, p)$, where $n' = n + 3p$ with $n \geq 5$ and $p \geq 2$, is determined by its Laplacian spectrum.

**Proof.** The proof of this result immediately follows from Lemmas 4 and 5. □

**Acknowledgments**

The authors are grateful to the editor and anonymous referees for their comments and suggestions to improve quality of this article. The research of Muhammad Ajmal was supported by Post Doctoral Research Fellowship at Nanjing University of Aeronautics and Astronautics (NUAA), Nanjing, China.

**Conflict of interest**

The authors declare that they have no conflict of interest.

**References**

1. R. Boulet, B. Jouve, The lollipop is determined by its spectrum, *Electron. J. Comb.*, 15 (2008), R74.
2. R. Boulet, Spectral characterizations of sun graphs and broken sun graphs, *Discrete Math. Theor. Comput. Sci.*, 11 (2009), 149–160.

3. C. J. Bu, J. Zhou, Starlike trees whose maximum degree exceed 4 are determined by their Q-spectra, *Linear Algebra Appl.*, 436 (2012), 143–151.

4. C. J. Bu, J. Zhou, Signless Laplacian spectral characterization of the cones over some regular graphs, *Linear Algebra Appl.*, 436 (2012), 3634–3641.

5. N. Ghareghani, G. R. Omidi, B. Tayfeh-Rezaie, Spectral characterization of graphs with index at most $\sqrt{2 + \sqrt{5}}$, *Linear Algebra Appl.*, 420 (2007), 483–489.

6. Hs. H. Günthard, H. Primas, Zusammenhang von Graphentheorie und Mo-Theorie von Molekeln mit Systemen konjugierter Bindungen, *Helv. Chim. Acta.*, 39 (1956), 1645–1653.

7. I. Gutman, V. Gineityte, M. Lepović, M. Petrović, The high-energy band in the photoelectron spectrum of alkaners and its dependence on molecular structure, *J. Serb. Chem. Soc.*, 64 (1999), 673–680.

8. A. K. Kelmans, V. M. Chelnokov, A certain polynomial of a graph and graphs with an extremal number of trees, *J. Comb. Theory B*, 16 (1974), 197–214.

9. D. J. Klein, Graph geometry, graph metrics, & wiener, *MATCH Communi. Math. Compt. Chem.*, 35 (1997), 7–27.

10. J. X. Li, X. D. Zhang, On the Laplacian eigenvalues of a graph, *Linear Algebra Appl.*, 285 (1998), 305–307.

11. X. G. Liu, Y. P. Zhang, P. L. Lu, One special double starlike graph is determined by its Laplacian spectrum, *Appl. Math. Lett.*, 22 (2009), 435–438.

12. F. J. Liu, Q. X. Huang, Laplacian spectral characterization of 3-rosegraphs, *Linear Algebra Appl.*, 439 (2013), 2914–2920.

13. M. L. Liu, Y. L. Zhu, H. Y. Shan, K. C. Das, The spectral characterization of butterfly-like graphs, *Linear Algebra Appl.*, 513 (2017), 55–68.

14. P. L. Lu, X. D. Zhang, Y. P. Zhang, Determination of double quasi-star tree from its Laplacian spectrum, *Journal Shanghai University*, 14 (2010), 163–166.

15. P. L. Lu, X. G. Liu, Laplacian spectral characterization of some double starlike trees, *Journal of Harbin Engineering University*, 37 (2016), 242–247.

16. X. L. Ma, Q. X. Huang, Signless Laplacian spectral characterization of 4-rose graphs, *Linear Multi-linear Algebra*, 64 (2016), 2474–2485.

17. M. Mirzakhah, D. Kiani, The sun graph is determined by its signless Laplacian spectrum, *Electron. J. Linear Algebra*, 20 (2010), 610–620.

18. G. R. Omidi, K. Tajbakhsh, Startlike trees are determined by their Laplacian spectrum, *Linear Algebra Appl.*, 422 (2007), 654–658.

19. C. S. Oliveira, N. M. M. de Abreu, S. Jurkiewicz, The characteristic polynomial of the Laplacian of graphs in $(a, b)$-linear cases, *Linear Algebra Appl.*, 356 (2002), 113–121.

20. G. R. Omidi, E. Vatandoost, Starlike trees with maximum degree 4 are determined by their signless Laplacian spectra, *Electron. J. Linear Algebra*, 20 (2010), 274–290.
21. X. L. Shen, Y. P. Hou, Y. P. Zhang, Graph $Z_n$ and some graphs related to $Z_n$ are determined by their spectrum, *Linear Algebra Appl.*, **404** (2005), 58–68.

22. X. L. Shen, Y. P. Hou, Some trees are determined by their Laplacian spectra, *Journal of Nature Science Hunan Normal University*, **1** (2006), 241–272.

23. S. Sorgun, H. Topcu, On the spectral characterization of kite graphs, *J. Algebra Comb. Discrete Struct. Appl.*, **3** (2016), 81–90.

24. E. R. van Dam, W. H. Haemers, Which graphs are determined by their spectrum? *Linear Algebra Appl.*, **373** (2003), 241–272.

25. E. R. van Dam, W. H. Haemers, Developments on spectral characterizations of graphs, *Discrete Math.*, **309** (2009), 576–586.

26. W. Wang, C. X. Xu, Note: The $T$-shape tree is determined by its Laplacian spectrum, *Linear Algebra Appl.*, **419** (2006), 78–81.

27. F. Wen, Q. X. Huang, X. Y. Huang, F. J. Liu, On the Laplacian spectral characterization of $\prod$-shape trees, *Indian J. Pure Applied Math.*, **49** (2018), 397–411.

28. Y. P. Zhang, X. G. Liu, B. Y. Zhang, X. R. Yong, The lollipop graph is determined by its $Q$-spectrum, *Discrete Math.*, **309** (2009), 3364–3369.

29. J. Zhou, C. J. Bu, Laplacian spectral characterization of some graphs obtained by product operation, *Discrete Math.*, **312** (2012), 1591–1595.

© 2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)