ON DISCRETELY SELF-SIMILAR SOLUTIONS OF THE EULER EQUATIONS

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Abstract. This notes gives several criteria which exclude the existence of discretely self-similar solutions of the three dimensional incompressible Euler equations.

1. Introduction

Let $I = (-\infty, 0)$ or $I = (0, \infty)$ be a time interval. We are concerned with the Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3 \times I$,

\[
\begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, \\
\text{div} v = 0,
\end{cases}
\]

where $v = (v_1, v_2, v_3)$, $v_j = v_j(x, t)$, $j = 1, 2, 3$, is the velocity of the flow, and $p = p(x, t)$ is the scalar pressure. It is called backward or forward depending on whether $I = (-\infty, 0)$ or $I = (0, \infty)$. Thanks to the time reversal symmetry of the Euler equations there is a one to one correspondence between backward and forward solutions, and we may only consider the backward case $I = (-\infty, 0)$.

Recall that the system (E) has the scaling property that if $(v, p)$ is a solution of the system (E), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

\[
v^{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda, \alpha}(x, t) = \lambda^2 \alpha p(\lambda x, \lambda^{\alpha+1} t)
\]

are also solutions of (E). One can also include space-time translation in (1.1), but we omit it for simplicity. We say that a solution $(v, p)$ of (E) is self-similar (SS) with respect to the space-time origin $(0, 0)$ if there exists $\alpha \in (-1, \infty)$ such that, for all $\lambda > 0$,\n
\[
v^{\lambda, \alpha}(x, t) = v(x, t), \quad p^{\lambda, \alpha}(x, t) = p(x, t), \quad (x, t) \in \mathbb{R}^3 \times I.
\]

It follows that $v(x, t) = \frac{1}{|x|^{\alpha}} V(\frac{x}{|x|^\alpha})$ for $V(y) = v(y, \text{sign} t)$ and

\[
a = \frac{\alpha}{\alpha + 1}, \quad b = \frac{1}{\alpha + 1}, \quad \alpha > -1.
\]

The condition $\alpha > -1$ ensures that the solution concentrates at the origin as $t \to 0$. If a solution satisfies (1.2) for one single $\lambda > 1$, we say it is discretely self-similar (DSS) with factor $\lambda$. It does not need to satisfy (1.2) for every $\lambda$, and a self-similar

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solution is considered as a special case. The possibilities of self-similar singularities in the Euler equations are studied in [3, 4, 6, 7, 8, 9, 10]. The existence of DSS solutions of (E) has not been studied, and is the main concern of this note.

Our analysis is based on the self-similar transform. The self-similar transform with respect to \((0, 0)\) is the map \((v, p) \mapsto (V, P)\) given by

\[
\begin{aligned}
&v(x, t) = \frac{1}{(-t)^a} V(y, s), \quad p(x, t) = \frac{1}{b(-t)^{2a}} P(y, s), \\
&y = (-t)^{-b} x, \quad s = -\log(-t).
\end{aligned}
\]

where \(a \in \mathbb{R}\) and \(b > 0\) are given by (1.3), and

\[
y = (-t)^{-b} x, \quad s = -\log(-t).
\]

Substituting (1.4)–(1.5) into (E), we obtain the following system for \((V, P)\):

\[
\begin{aligned}
&\frac{\partial V}{\partial s} + \frac{\alpha}{\alpha + 1} V + \frac{1}{\alpha + 1} (y \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P, \\
&\text{div } V = 0.
\end{aligned}
\]

A solution \((v, p)\) of (E) is self-similar if and only if \((V, P)\) is independent of \(s\). A solution \((v, p)\) of (E) is discretely self-similar with factor \(\lambda > 1\) if and only if

\[
V(y, s) = V(y, s + S_0), \quad \forall (y, s) \in \mathbb{R}^{3+1}
\]

where \(S_0 = (\alpha + 1) \log \lambda > 0\). In other words, \((V, P)\) is a time periodic solution of (1.6) with period \(S_0\).

Discretely self-similar solutions of partial differential equations have been considered in many other contexts such as the cosmology. See the review [11, Section 5] and the references therein.

We now sketch the structure of the rest of the paper. In Subsection 1.1 we review related results for Navier-Stokes equations. In Section 2 we give nonexistence criteria based on vorticity integrability, and we will state and prove Theorems 2.1 and 2.2. In Section 3 we give nonexistence criteria based on velocity integrability, and we will state and prove Theorems 3.1 and 3.2.

1.1. Related results for Navier-Stokes equations. For comparison, we review related results for Navier-Stokes equations (NS) for which we add \(\Delta v\) to the right side of (E). For (NS), the backward and forward cases are very different. Introduce the similarity variables: We take parameter \(a < 0\) for backward case and \(a > 0\) for forward case, and let

\[
\begin{aligned}
&v(x, t) = \frac{1}{\sqrt{2at}} V(y, s), \quad p(x, t) = \frac{1}{2at} P(y, s), \\
&y = \frac{x}{\sqrt{2at}}, \quad s = \frac{1}{2a} \log(2at).
\end{aligned}
\]

The corresponding time-dependent Leray’s equations for \((V, P)\) read

\[
\begin{aligned}
&\frac{\partial V}{\partial s} - \Delta U - aV - ay \cdot \nabla V + (V \cdot \nabla) V = -\nabla P, \\
&\text{div } V = 0.
\end{aligned}
\]

For the forward case \(a > 0\), one can consider the Cauchy problem for (NS) with initial data \(v_0(x)\) which is also SS or DSS, i.e., it satisfies (1.2) with no time dependence.
For small data, the unique existence of small mild solutions by \cite{13,1,2} implies those with SS or DSS data are also SS or DSS. For large SS data, the corresponding SS solution has recently been constructed by \cite{14}, and extended to large DSS data by \cite{19} if the DSS-factor $\lambda$ is sufficiently close to 1 according to the size of the data.

For the backward case $a < 0$, the existence question of self-similar solutions was raised by Leray \cite{15}. It was excluded if $V \in L^3(\mathbb{R}^3)$ by \cite{16}, and if $V \in L^q(\mathbb{R}^3)$, $3 < q \leq \infty$, by \cite{18}. Further extensions were given in \cite{4,5,7,9}. The existence of backward DSS solutions has not been addressed in literature, except that if $V \in L^\infty(\mathbb{R}, L^3(\mathbb{R}^3))$, which is equivalent to $v \in L^\infty((\infty,0), L^3(\mathbb{R}^3))$, it must be zero by the result of \cite{12}. Thus one is concerned, e.g., if $V$ only has the bound

$$|V(y, s)| \leq \frac{C_*}{1 + |y|}, \quad \forall (y, s) \in \mathbb{R}^{3+1},$$

for some large constant $C_*$. A special case that $V(y, s) = R(s\vec{k})\tilde{V}(y)$, with $R(s\vec{k})$ being the rotation about a fixed axis $\vec{k}$ by angle $s|\vec{k}|$, was proposed by G. Perelman. Then $\tilde{V}$ satisfies a time-independent system. As pointed out to one of us by R. Kohn, the examples of Scheffer \cite{17} are DSS solutions with singular DSS forces. In view of the forward case, one may hope that the case with the DSS-factor $\lambda$ sufficiently close to 1 might be easier to exclude.

2. Criteria based on vorticity

In this section we give nonexistence criteria based on vorticity integrability.

**Theorem 2.1.** Let $V(y, s) \in C_x^2C_t^1(\mathbb{R}^{3+1})$ be a time periodic solution of (1.6) with period $S_0 > 0$ that has bounded first derivatives and satisfies

$$\lim_{|y| \to \infty} V(y, s) = 0, \quad \forall s \in [0, S_0),$$

and for some $r > 0$,

$$\Omega := \text{curl} V \in \bigcap_{0 < q < r} L^q(\mathbb{R}^3 \times [0, S_0]).$$

Then $V = 0$ on $\mathbb{R}^{3+1}$.

**Proof.** We first observe that from the calculus identity

$$V(y, s) = V(0, s) + \int_0^1 \partial_\tau V(\tau y, s)d\tau = V(0, s) + \int_0^1 y \cdot \nabla V(\tau y, s)d\tau,$$

we have $|V(y, s)| \leq |V(0, s)| + |y|\|\nabla V(s)\|_{L^\infty}$, and hence

$$\sup_{(y,s)\in\mathbb{R}^{3+1}} \frac{|V(y,s)|}{1 + |y|} \leq C_1 := \max_s |V(0, s)| + \|\nabla V\|_{L^\infty(\mathbb{R}^{3+1})}.$$

Let us consider the radial cut-off function $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases}$$

\footnote{private communication of G. Seregin.}
and \(0 \leq \sigma(x) \leq 1\) for \(1 < |x| < 2\). Then, for each \(R > 0\), we define \(\sigma_R(y) := \sigma\left(\frac{|y|}{R}\right) \in C_0^\infty(\mathbb{R}^N)\). We operate curl on \((1.6)\),

\[
(2.5) \quad \frac{\partial}{\partial s} \Omega + \Omega + \frac{1}{\alpha + 1}(y \cdot \nabla)\Omega + (V \cdot \nabla)\Omega = (\Omega \cdot \nabla)V.
\]

We multiply \((2.5)\) by \(|\Omega|^{q-2}\Omega\sigma_R\), and integrate over \(\mathbb{R}^3 \times (0, S_0)\). The first term vanishes by periodicity. After integration by parts we get

\[
(2.8) \quad \lim_{S \to \infty} \int_0^{S_0} |\Omega|^q \sigma_R dy ds = \int_0^{S_0} \frac{\xi \cdot \nabla V \cdot \xi |\Omega(s)|^q \sigma_R dy ds.
\]

Remark after the proof: The above proof works for more general system

\[
(2.9) \quad \begin{cases} V_s + aV + b(y \cdot \nabla)V + (V \cdot \nabla)V = -\nabla P, \\ \text{div} V = 0, \end{cases}
\]

where \(a, b\) are arbitrary real constants with \(b \neq 0\).

**Theorem 2.2.** Let \(V(y, s) \in C_2^2 C^1_t(\mathbb{R}^{3+1})\) be a time periodic solution of \((1.6)\) with period \(S_0 > 0\) that has bounded first derivatives satisfying

\[
(2.10) \quad \lim_{|y| \to \infty} \sup_{0 < s < S_0} |V(y, s)| + |\nabla V(y, s)| = 0,
\]

and there exists \(q \in (0, \frac{3}{1+\alpha})\) such that

\[
(2.11) \quad \Omega \in L^q(\mathbb{R}^3 \times [0, S_0]).
\]

Then, \(V = 0\) on \(\mathbb{R}^{3+1}\).

**Proof.** Writing \((2.11)\) in terms of spherical coordinates,

\[
\int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q dy ds = \int_0^{\infty} \int_{|y|=r} |\Omega|^q dS_r dr < \infty,
\]
one finds that there exists a sequence $R_j \uparrow \infty$ such that
\begin{equation}
(2.12) \quad R_j \int_0^{S_0} \int_{|y|=R_j} |\Omega|^q dS_R ds \to 0 \quad \text{as } j \to \infty.
\end{equation}

We multiply (2.5) by $\Omega |\Omega|^{q-2}$ and rewrite it
\begin{align}
\frac{1}{q} \frac{\partial}{\partial s} |\Omega|^q + |\Omega|^q + \frac{1}{q(\alpha + 1)} \text{div} (y|\Omega|^q) - \frac{3}{q(\alpha + 1)} |\Omega|^q \\
(2.13) \quad = \hat{\alpha} |\Omega|^q - \frac{1}{q} \text{div} (V|\Omega|^q),
\end{align}
where $\hat{\alpha} = \xi \cdot \nabla V \cdot \xi$ with $\xi = \Omega/|\Omega|$. Note that $|\hat{\alpha}| \leq |\nabla V|$. Let us fix an $R > 0$ and integrate (2.13) over the domain $(y, s) \in \{R < |y| < R_j\} \times (0, S_0)$. Applying the divergence theorem, we have
\begin{align}
&\left( \frac{3}{q(\alpha + 1)} - 1 \right) \int_0^{S_0} \int_{R<|y|<R_j} |\Omega|^q dy ds + \frac{R}{q(\alpha + 1)} \int_0^{S_0} \int_{|y|=R} |\Omega|^q dS_R ds \\
&- \frac{R_j}{q(\alpha + 1)} \int_{|y|=R_j} |\Omega|^q dS_R ds \\
&= -\int_0^{S_0} \int_{R<|y|<R_j} \hat{\alpha} |\Omega|^q dy ds - \frac{1}{q} \int_0^{S_0} \int_{|y|=R} V_r |\Omega|^q dS_R ds + \frac{1}{q} \int_0^{S_0} \int_{|y|=R_j} V_r |\Omega|^q dS_R ds,
\end{align}
where $V_r = V \cdot \frac{y}{|y|}$ and we used the fact $\int_0^{S_0} (||\Omega(s)||_{L^q}) ds = 0$, following from the periodicity hypothesis. Passing $j \to \infty$, one obtains
\begin{align}
(2.14) \quad &= -\int_0^{S_0} \int_{|y|>R} \hat{\alpha} |\Omega|^q dy ds - \frac{1}{q} \int_0^{S_0} \int_{|y|=R} V_r |\Omega|^q dS_R ds.
\end{align}

By using (2.10) and choosing $R$ sufficiently large, we have
\begin{align}
|\hat{\alpha}| &\leq \frac{1}{2} \left( \frac{3}{q(\alpha + 1)} - 1 \right), \quad |V_r| \leq \frac{R}{2(\alpha + 1)},
\end{align}
on the right side of (2.14). Consequently,
\begin{equation}
\int_0^{S_0} \int_{|y|>R} |\Omega|^q dy ds = \int_0^{S_0} \int_{|y|=R} |\Omega|^q dS_R ds = 0,
\end{equation}
and hence, $\Omega = 0$ on $\{y \in \mathbb{R}^3 \mid |y| > R\} \times (0, S_0)$. Thus our vorticity $\Omega$ satisfies the condition (2.2) of Theorem 2.1. Applying Theorem 2.1 we obtain $V = 0$ on $\mathbb{R}^{3+1}$.

### 3. Criteria based on velocity

In this section we give nonexistence criteria based on velocity integrability.

We will need to estimate the pressure $P(y, s)$, which satisfies
\begin{equation}
(3.1) \quad - \Delta_y P(\cdot, s) = \sum_{i,j} \partial_i \partial_j (V_i V_j(\cdot, s))
\end{equation}
by taking the divergence of (1.6). One solution of (3.1) is given by

\[
\tilde{P}(y, s) = -\frac{|v(y, s)|^2}{3} + \sum_{i,j} p.v. \int_{\mathbb{R}^3} K_{ij}(y - z)V_iV_j(z)dz,
\]

where the kernel is

\[
K_{ij}(y) = \frac{3y_iy_j - \delta_{i,j}|y|^2}{4\pi|y|^5}.
\]

In general, for each fixed \( t \), the difference \( P - \tilde{P} \) is a harmonic function in \( x \) and may not be constant. We will assume \( P = \tilde{P} \).

**Theorem 3.1.** Suppose that \((V, P) \in C^1_{\text{loc}}(\mathbb{R}^{3+1})\) is a time periodic solution of (1.6) with period \( S_0 > 0 \), that the pressure \( P \) is given by (3.2), and that \( V \) satisfies for some \( 3 \leq r \leq 9/2 \) one of the following conditions,

(i) \( \alpha > 3/2 \) and \( V \in L^3(0, S_0; L^r(\mathbb{R}^3)) \), or

(ii) \(-1 < \alpha < 3/2 \) and \( V \in L^2(0, S_0; L^2(\mathbb{R}^3)) \cap L^3(0, S_0; L^r(\mathbb{R}^3)) \).

Then \( V = 0 \) on \( \mathbb{R}^{3+1} \).

**Proof.** Since \( P \) is given by (3.2), by the Calderon-Zygmund inequality we have

\[
\|P(s)\|_{L^q} \leq C_q\|V(s)\|_{L^r}^2, \quad \forall q \in (1, \infty), \forall s \in \mathbb{R}.
\]

The case (i): Let \( \sigma_R \) be the cut-off function introduced in the proof of Theorem 1.1. We multiply (1.6) by \( V \sigma_R \), and integrate over \( \mathbb{R}^3 \times (0, S_0) \), then from the time periodicity condition and by integration by part we obtain

\[
\frac{1}{\alpha + 1} \left( \alpha - \frac{3}{2} \right) \int_0^{S_0} \int_{\mathbb{R}^3} |V|^2 \sigma_R dy ds - \frac{1}{2(\alpha + 1)} \int_0^{S_0} \int_{\mathbb{R}^3} |V|^2 y \cdot \nabla \sigma_R dy ds
\]

\[
= \frac{1}{2} \int_0^{S_0} \int_{\mathbb{R}^3} |V|^2 y \cdot \nabla \sigma_R dy ds + \int_0^{S_0} \int_{\mathbb{R}^3} PV \cdot \nabla \sigma_R dy ds
\]

(3.4)

Since \( y \cdot \nabla \sigma_R \leq 0 \) for all \( y \in \mathbb{R}^3 \), and the first term of the left hand side of (3.4) is monotonically non-decreasing function of \( R \), we find that

\[
\frac{1}{\alpha + 1} \left( \alpha - \frac{3}{2} \right) \int_0^{S_0} \int_{\mathbb{R}^3} |V|^2 \sigma_R dy ds
\]

\[
\leq \frac{1}{2} \int_0^{S_0} \int_{\mathbb{R}^3} |V|^2 y \cdot \nabla \sigma_R dy ds + \int_0^{S_0} \int_{\mathbb{R}^3} PV \cdot \nabla \sigma_R dy ds
\]

(3.5)

for all \( 0 < R_1 < R_2 < \infty \). Passing \( R_2 \to \infty \), one has

\[
I_1 \leq \frac{\|\nabla \sigma\|_{L^\infty}}{2R_2} \int_0^{S_0} \int_{R_2 < |y| < 2R_2} |V|^3 dy ds
\]

\[
\leq \frac{\|\nabla \sigma\|_{L^\infty}}{2R_2} \int_0^{S_0} \left( \int_{R_2 < |y| < 2R_2} |V|^r dy \right)^{\frac{3}{r}} \left( \int_{R_2 < |y| < 2R_2} dy \right)^{1-\frac{3}{r}} ds
\]

\[
\leq CR_2^{2-\frac{3}{r}} \int_0^{S_0} \|V(s)\|_{L^r(R_2 < |y| < 2R_2)}^3 ds \to 0,
\]
and

\[ I_2 \leq \left\| \nabla \sigma \right\|_{L^\infty} \int_0^{S_0} \int_{R^3} \frac{|V||P|dy}{2R_2} ds \]

\[ \leq \left\| \nabla \sigma \right\|_{L^\infty} \int_0^{S_0} \left( \int_{R^3} \frac{|V||P|^r}{dy} \right)^{\frac{1}{r}} \left( \int_{R^3} |P|^2 dy \right)^{\frac{1}{2}} \left( \int_{R^2} |y|^2 dy \right)^{\frac{1-\frac{3}{2}}{2}} ds \]

\[ \leq CR_2^{\frac{3}{2}} \int_0^{S_0} \left\| V \right\|_{L^r(R_2<|y|<R_2)} \left\| V \right\|_{L^r}^2 ds \]

\[ \leq CR_2^{\frac{3}{2}} \left( \int_0^{S_0} \left| V \right|_{L^r(R_2<|y|<R_2)}^3 ds \right)^{\frac{1}{3}} \left( \int_0^{S_0} \left| V \right|_{L^r}^3 ds \right)^{\frac{2}{3}} \to 0, \]

where we used (3.3). Therefore, we have

\[ \left( \alpha - \frac{3}{2} \right) \int_0^{S_0} \int_{R^3} |V|^2 \sigma R^2 dy ds = 0 \]

for all \( R_1 > 0 \). This shows that \( V = 0 \) on \( R^3 \times (0, S_0) \).

**The case (ii):** In this case from (3.4) we have

\[ \frac{1}{\alpha + 1} \left| \alpha - \frac{3}{2} \right| \int_0^{S_0} \int_{R^3} |V|^2 \sigma R^2 dy ds \leq \frac{1}{2(\alpha + 1)} \int_0^{S_0} \int_{R^3} |V|^2 y \cdot \nabla \sigma R^2 dy ds \]

\[ + \frac{1}{2} \int_0^{S_0} \int_{R^3} |V|^2 V \cdot \nabla \sigma R^2 dy ds + \int_0^{S_0} \int_{R^3} |P||V| \cdot \nabla \sigma R^2 dy ds \]

(3.6) \[ := J_1 + J_2 + J_3. \]

From the above computations we know that \( |J_2| + |J_3| \to 0 \) as \( R \to \infty \). For \( J_1 \) we estimate easily

\[ |J_1| \leq C \int_0^{S_0} \int_{R^3} \frac{|V|^2}{y} dy ds \leq C \int_0^{S_0} \left\| V(s) \right\|_{L^2(R<|y|<2R)}^2 ds \to 0 \]

as \( R \to \infty \). Hence \( \int_0^{S_0} \int_{R^3} |V|^2 dy ds = 0 \), and \( V = 0 \) on \( R^{3+1} \). \( \square \)

The next result, Theorem 3.2, is an extension of Chae-Shvydkoy [10] Theorem 3.2 to the case of discretely self-similar solutions. An important role is played by the following lemma, which extends the local energy inequality in [10].

**Lemma 3.1.** Suppose \((V, P) \in C^{1}_{loc}(R^{3+1})\) is a time periodic solution of (1.6) with period \( S_0 > 0 \). Let \( \lambda = e^{\beta s}, \beta > 1 \). For \( -\infty < s_1 < s_2 < \infty \), let \( l_j = e^{\beta s_j} \) and

\[ I_j = \int_0^{S_0} \int_{R^3} |V(y, s_j + \tau)|^2 \sigma (e^{-b(s_j + \tau)} y) dy d\tau, \quad (j = 1, 2). \]

We have

\[ \int_0^{S_0} \int_{|y| \leq \frac{1}{2}l_j} |V(y, \tau)|^2 dy d\tau \leq I_j \leq \int_0^{S_0} \int_{|y| \leq M_j} |V(y, \tau)|^2 dy d\tau \]

(3.7)
for $j = 1, 2$, and for some constant $C = C(S_0)$

$$
|I_1 - I_2| \leq C \int_0^{S_0} \int_{\frac{1}{2}l_j < |y| < \lambda l_2} |y|^{2a-4}(|V|^{3} + |PV|)(y, s)\,dy\,ds.
$$

Note that $\lambda$ is the factor for discrete self-similarity, see (3.7).

**Proof.** Let $\sigma(x)$ be a radial function with $\sigma \geq 0$, $\sigma(r) = 1$ for $r < 1/2$ small and $\sigma(r) = 0$ for $r \geq 1$. Let $t_j = -e^{-b s_j}$, $j = 1, 2$. Testing the Euler equation with $\sigma v$ in $\mathbb{R}^3 \times (t_1, t_2)$ we get

$$
|v(t_2, t_2)|^2 \sigma(x)\,dx - \int |v(t_1, t_1)|^2 \sigma(x)\,dx = \int_{t_1}^{t_2} (|v|^2 + 2p)\nabla \sigma(x)\,dx \,dt.
$$

In self-similar variables (1.4)–(1.5) it becomes

$$
e^{(2a-3b)s_2} \int |V(y, s_2)|^2 \sigma(e^{-b s_2} y)\,dy - e^{(2a-3b)s_1} \int |V(y, s_1)|^2 \sigma(e^{-b s_1} y)\,dy
$$

$$= \int_{s_1}^{s_2} e^{(3a-3b-1)\tau} \int (|V|^2 + 2P)\nabla \sigma(e^{-b s} y)\,dy\,ds.
$$

(3.11)

Assume now that $v$ is DSS, so that $V(y, s)$ is periodic in $s$ with period $S_0 > 0$.

Replacing $s_j$ by $s_j + \tau$, dividing by $e^{(2a-3b)\tau}$, and integrating over $\tau \in [0, S_0]$, we get

$$
I_1 - I_2 = I_3
$$

where $I_1$ and $I_2$ are given in (3.7), and

$$
I_3 = \int_0^{S_0} e^{-(2a-3b)\tau} \int_{s_1+\tau}^{s_2+\tau} e^{(3a-3b-1)\tau} \int (|V|^2 + 2P)\nabla \sigma(e^{-b s} y)\,dy\,ds\,d\tau.
$$

(3.13)

The estimate (3.8) for $I_1$ and $I_2$ is because that $\sigma(e^{-b(s_j+\tau)})$ is supported in $\frac{1}{2}l_j \leq |y| \leq \lambda l_j$, and also using the periodicity.

For $I_3$, since $e^{-(2a-3b)\tau} \leq C$ and $\sigma(e^{-b s} y)$ is supported in $\frac{1}{2}e^{bs} \leq |y| \leq e^{bs}$,

$$
|I_3| \leq C \int_E \int_0^{S_0} \int_{s_1+\tau}^{s_2+\tau} e^{3a-3b-1)\tau} Q(y, s)\nabla \sigma(e^{-b s} y)\,dy\,d\tau\,d\tau,
$$

(3.14)

where $E$ denotes the spatial region $E = \{ y : \frac{1}{2}l_1 \leq |y| \leq \lambda l_2 \}$ and $Q = |V|^3 + |P||V|$. If we denote by $f(y, s)$ the integrand, the inner integral

$$
\int_0^{S_0} \int_{s_1+\tau}^{s_2+\tau} f(y, s)\,ds\,d\tau \leq S_0 \int_{s_1}^{s_2+S_0} f(y, s)\,ds \leq S_0 \int_{J_y} f(y, s)\,ds
$$

where we have used $\sigma(e^{-b s} y)$ is supported in the time interval

$$
J_y = \{ s : |y| \leq e^{bs} \leq 2|y| \} = \{ s : \frac{\ln |y|}{b} \leq s \leq \frac{\ln |y|}{b} + \frac{\ln 2}{b} \}.
$$

(3.15)

In $J_y$ we have $e^{(3a-3b-1)s} \leq C|y|^\frac{3a-3b-1}{b} = C|y|^{2a-4}$. Thus

$$
|I_3| \leq C \int_E \int_{J_y} |y|^{2a-4}Q(y, s)\,ds\,dy.
$$

(3.17)
Let $k$ be the positive integer so that $(k - 1)S_0 < \frac{\ln 2}{b} \leq kS_0$. Using the periodicity of $Q(y, s)$ in $s$,
\begin{equation}
|I_3| \leq Ck \int_E \int_0^{S_0} |y|^{2\alpha - 4}Q(y, s)ds dy.
\end{equation}

This shows (3.3). □

**Theorem 3.2.** Suppose $(V, P) \in C_1^1(\mathbb{R}^{3+1})$ is a time periodic solution of (1.6) in $\mathbb{R}^{3+1}$ with period $S_0 > 0$, $V \in L^p(\mathbb{R}^3 \times ([0, S_0])$ for some $3 \leq p \leq \infty$, and $P$ is given by (3.2). If $-1 < \alpha \leq 3/p$ or $3/2 < \alpha < \infty$, then $V = 0$ on $\mathbb{R}^{3+1}$.

**Proof.** The proof of [10, Theorem 3.2] goes through with the help of Lemma 3.1. One adds the temporal integral $\int_0^{S_0} ds$ in front of every spatial integral in its proof. □

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