FILTRATION OF HEEGAARD FLOER HOMOLOGY AND GLUING FORMULAS

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ABSTRACT. We introduce an extra filtration on the complex \( \widehat{CFK}(Y, K) \) associated with a null-homologous knot \( K \) inside the three-manifold \( Y \), denoted by \( \widehat{CFK}_\bullet(Y, K) \), with \( \bullet \in \{-, 0, +\} \). This filtration will present the longitude theory \( \widehat{CFL}(Y, K) \) as a subcomplex of \( \widehat{CFK}(Y, K) \). The surgery exact sequences respect this filtration. Besides some basic properties of these filtered complexes, we derive a formula for \( \widehat{CFK}_\bullet \) of the knot \((Y, K)\) obtained by gluing the knot complements \( Y_1 \setminus \text{nd}(K_1) \) and \( Y_2 \setminus \text{nd}(K_2) \). We will also compute the filtered complex \( \widehat{CFK}_\bullet(S^3, K) \) for an alternating knot \( K \).

1. Introduction

The introduction of the knot Floer homology by Ozsváth and Szabó (8, 9) and independently by Rasmussen (15) has brought an amazingly powerful technology package to classical knot theory. These are invariants of the null homologous knots in three-manifolds, which are constructed following the general procedure introduced by Ozsváth and Szabó in the context of three-manifolds (8, 9).

The knot Floer homology assigned to a knot \((Y, K)\) comes as a package of filtered chain complexes, the main object being \( CFK^\infty(Y, K) \). There are easier filtered packages which may be realized as the subcomplexes, quotient complexes, etc., of \( CFK^\infty(Y, K) \) with the help of \( \mathbb{Z} \oplus \mathbb{Z} \) filtration induced by the knot \( K \).

In this paper we will be dealing with the filtered chain complex \( \widehat{CFK}(Y, K) \), which is somewhat easier to compute in general. Suppose that the Heegaard diagram

\[(\Sigma, \alpha, \beta_0 \cup \{m\}, z)\]

is chosen so that \((\Sigma, \alpha, \beta_0)\) represents a Heegaard diagram for the knot complement \( Y \setminus \text{nd}(K) \), which is completed to a Heegaard diagram for \( Y \) by adding the special \( \beta \)-curve \( m \). Here \( \alpha = \{\alpha_1, ..., \alpha_g\} \) and \( \beta = \beta_0 \cup \{m\} = \{m, \beta_2, ..., \beta_g\} \) are two \( g \)-tuples of linearly independent simple closed curves on the genus \( g \) surface \( \Sigma \). Furthermore, assume that \( z \) is a marked point on \( m \setminus \alpha \). The points \( u \) and \( w \) are chosen on the two sides of the curve \( m \) and very close to the marked point \( z \). The diagram should be chosen
to be admissible in the sense of [8, 6], a condition that we will drop from this exposition. Let $\widehat{CFK}(Y, K)$ be the complex freely generated by the intersections of the two tori

$$T_\alpha = \alpha_1 \times \ldots \times \alpha_g, \text{ and } T_\beta = m \times \beta_2 \times \ldots \times \beta_g,$$

in the symplectic manifold $\text{Sym}^g(\Sigma) = \frac{\Sigma \times \ldots \times \Sigma}{\mathbb{Z}_g}$, which is the $g$-th symmetric product of the surface $\Sigma$.

The differential map of the complex $\widehat{CFK}(Y, K)$ is defined as follows. For any two generators $x, y \in T_\alpha \cap T_\beta$ let $\pi_2(x, y)$ denote the set of homotopy classes of the disks $\phi : [0, 1] \times \mathbb{R} \to \text{Sym}^g(\Sigma)$ such that $\phi(t, s)$ converges to $x$ as $s$ goes to $\infty$ and converges to $y$ as $s$ goes to $-\infty$, and such that $\phi(0, s) \in T_\alpha$ and $\phi(1, s) \in T_\beta$. For any element $\phi \in \pi_2(x, y)$ let $M(\phi)$ denote the moduli space of holomorphic representatives of $\phi$ with respect to the standard complex structure on the complex plane, and a generic one parameter family of complex structures $\{J_t\}_{t \in [0, 1]}$ on $\text{Sym}^g(\Sigma)$. Let $n_u(\phi)$ and $n_w(\phi)$ denote the algebraic intersection numbers of the homotopy class $\phi$ with the codimension 2 submanifolds $\{u\} \times \text{Sym}^{g-1}(\Sigma)$ and $\{w\} \times \text{Sym}^{g-1}(\Sigma)$ respectively. Then define

$$\partial [x] = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y)} \sum_{\mu(\phi) = 1}^{\#(\frac{M(\phi)}{\sim_\mathbb{R}})} [y],$$

where $\sim_\mathbb{R}$ represents the equivalence relation induced by the $\mathbb{R}$-action on the moduli space $M(\phi)$, and $\mu(\phi)$ is the Maslov index associated with the homotopy class $\phi$.

The complex $\widehat{CFK}(Y, K)$ is filtered using the marked point $w$. In fact, we may assign a Spin$^c$ structure $s_u(x) = s_w(x)$ to any generator $x$ of the complex using either of the marked points $u$ or $w$, which is an element of $\text{Spin}^c(Y, K) = \text{Spin}^c(Y_0(K)) \simeq \mathbb{Z} \oplus \text{Spin}^c(Y)$, such that the boundary map $\partial$ does not increase the $\mathbb{Z}$-factor of the generators.

In this paper, very often we may refer by $\widehat{CFK}(Y, K)$ to the above complex with a different boundary map, which consists only of the part of $\partial$ which does not change the $\mathbb{Z}$-component of the associated Spin$^c$ structure of the generators in $\widehat{CFK}(Y, K)$. Usually, this is the case unless it is clear from the context that we are considering the whole boundary map.

The complex $\widehat{CFK}(Y, K)$ behaves quite well under the connected sum ([6]). However, there are other ways of obtaining a new knot $(Y, K)$ from two given knots $(Y_1, K_1)$ and $(Y_2, K_2)$, where there is not much known on the behavior of the Floer homology group $\widehat{CFK}(Y, K)$ in terms of $\widehat{CFK}(Y_i, K_i)$,
We describe here an extra filtration of the complex $\hat{CFK}(Y,K)$ by the elements of the set $\{-,0,\}$, where the trivial order $- < 0 < +$ is assumed on this set. Correspondingly, for any relative Spin$^c$ structure $\mathfrak{g} \in \text{Spin}^c(Y,K)$ we obtain a sequence of subcomplexes

$$
\hat{CFK}_-(Y,K,\mathfrak{g}) \subset \hat{CFK}_0(Y,K,\mathfrak{g}) \subset \hat{CFK}_+(Y,K,\mathfrak{g}) = \hat{CFK}(Y,\mathfrak{g}).
$$

We show furthermore that there is a short exact sequence

$$
0 \to \hat{CFK}_0(Y,K,\mathfrak{g}) \to \hat{CFK}_+(Y,K,\mathfrak{g}) \xrightarrow{\partial'} \hat{CFK}_-(Y,K,\mathfrak{g} - \text{PD}[m]) \to 0,
$$

where PD$[m]$ denotes the Poincaré dual of the homology class represented by the meridian $m$ of the knot $K$ in the three-manifold $Y$. The complex $\hat{CFK}_*(Y,K)$ and the map $\partial'$ may be used to construct a complex which in fact has the homotopy type of $\hat{CF}(Y)$.

Our claim in this paper is that this whole package is a topological invariant of the knot $(Y,K)$ and the relative Spin$^c$ structure $\mathfrak{g} \in \text{Spin}^c(Y,K)$.

**Theorem 1.1.** The chain homotopy type of the flag of complexes

$$
\hat{CFK}_-(Y,K,\mathfrak{g}) \xrightarrow{i} \hat{CFK}_0(Y,K,\mathfrak{g}) \xrightarrow{i_0} \hat{CFK}_+(Y,K,\mathfrak{g})
$$

is a topological invariant of the knot $K$ in the three-manifold $Y$, and the relative Spin$^c$ structure $\mathfrak{g} \in \text{Spin}^c(Y,K) = \text{Spin}^c(Y_0(K))$. Moreover the map $\partial'$ gives a chain map which makes the following sequence exact:

$$
0 \to \hat{CFK}_0(Y,K,\mathfrak{g}) \xrightarrow{i_0} \hat{CFK}_+(Y,K,\mathfrak{g}) \xrightarrow{\partial'} \hat{CFK}_-(Y,K,\mathfrak{g} - \text{PD}[m]) \to 0.
$$

The chain homotopy type of this exact sequence is also a topological invariant of the knot $K$ and the relative Spin$^c$ structure $\mathfrak{g}$.

This extra filtered complex enjoys some of the properties of the non-filtered complex $\hat{CFK}(Y,K)$. For example it does not depend on the orientation of the knot $K$, in the following sense:

**Proposition 1.2.** If $(Y,K)$ is a null-homologous knot and if $(Y,-K)$ denotes the same knot with the opposite orientation, then for any relative Spin$^c$ structure $\mathfrak{g} \in \text{Spin}^c(Y,K)$, the filtered complexes $\hat{CFK}(Y,K,\mathfrak{g})$ and $\hat{CFK}(Y,-K,\mathfrak{g})$ have the same homotopy type. Moreover if $\mathfrak{g}$ extends a torsion Spin$^c$ structure $\mathfrak{s} \in \text{Spin}^c(Y)$, then the homotopy equivalence shifts the absolute Maslov grading by $-2k$, where

$$
k = \frac{1}{2}\langle c_1(\mathfrak{g}), [\hat{F}] \rangle,$$

where $\langle a, b \rangle$ denotes the intersection number of $a$ and $b$.
for a capping \( \hat{F} \) of a Seifert surface \( F \) for the oriented knot \( K \). In particular
\[
H_d(\hat{CF}K_\bullet(Y,K,\hat{\mathfrak{g}})) \cong H_{d-2k}(\hat{CF}K_\bullet(Y,-K,\hat{\mathfrak{g}})),
\]
for any of the filtration levels \( \bullet \in \{-,0,+,\} \).

Here we are denoting the homology of a complex \( C \) by \( H_\bullet(C) \), where \( * \) denotes the degree of the elements in \( C \), when there is such a well-defined degree, respected by the differential (boundary map) of the complex \( C \).

Although the definition of these complexes uses special types of Heegaard diagrams for the knots \( (Y,K) \), which are in general considerably more complicated compared to the Heegaard diagrams used for the computation of non-filtered complex, it is still possible to do certain explicit computations, as is illustrated in this paper. As usual, the alternating knots are the first target of the Floer homology computations. We prove:

**Theorem 1.3.** Suppose that \((S^3,K)\) is an alternating knot in \( S^3 \), and let \( \Delta_K(t) \) and \( \sigma(K) \) denote the Alexander polynomial and the signature of the knot \( K \) respectively. Then the homotopy type of the filtered chain complex \( \hat{CF}K_\bullet(S^3,K) \) is completely determined by \( \Delta_K(t) \) and \( \sigma(K) \).

In a similar way, we may introduce a filtered version \( \hat{CFL}_\bullet(Y,K) \) of the longitude Floer homology \( \hat{CFL}(Y,K) \). This complex was introduced in \([2]\) as a variant of the complex \( \hat{CFK}(Y,K) \) obtained by interchanging the role of the meridian and the longitude of \( K \) in the construction of the Floer homology groups. There are many similar properties that the filtered complexes \( \hat{CFL}_\bullet(Y,K) \) share with \( \hat{CFK}_\bullet(Y,K) \). The computations are also very much related. In particular we will also have:

**Theorem 1.4.** Suppose that \((S^3,K)\) is an alternating knot, and assume that \( \Delta_K(t) \) and \( \sigma(K) \) are as in the above theorem. Then the homotopy type of the filtered chain complex \( \hat{CFL}_\bullet(S^3,K) \) is also completely determined by \( \Delta_K(t) \) and \( \sigma(K) \).

These filtered complexes are all graded by the Spin\(^c\) structures \( \text{Spin}^c(Y_0(K)) \). In fact, in \([2]\) we mentioned that in the computation of \( \hat{CFL}(Y,K) \) the two maps
\[
\hat{s}_u, \hat{s}_w : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \text{Spin}^c(Y,K)
\]
differ from each other by a factor \( \text{PD}[m] \), where \( m \) is a meridian for \( K \) as before, and that one may either choose to filter \( \hat{CFL} \) using the map \( \hat{s}_u \), or more invariantly using the average \( \text{Spin}^c \) structure assignment
\[
x \mapsto \hat{s}(x) = \frac{\hat{s}_u(x) + \hat{s}_w(x)}{2} \in \frac{1}{2} \text{PD}[m] + \text{Spin}^c(Y,K).
\]
For most of the paper \([2]\) this later choice was the convention. Here in the exposition we choose to filter the complex using the map \( \hat{s}_u \). Then the
two filtered chain complexes $\widehat{CFK}_\bullet(Y, K)$ and $\widehat{CFL}_\bullet(Y, K)$ are very closely related to each other. Namely,

**Proposition 1.5.** Suppose that $(Y, K)$ is a null homologous knot. Then for any Spin$^c$ structure $\mathfrak{s} \in \text{Spin}^c(Y, K)$ we have isomorphism of (non-filtered) chain complexes

$$\widehat{CFK}_{-}(Y, K, \mathfrak{s} - \text{PD}[m]) \simeq \frac{\widehat{CFK}_{+}(Y, K, \mathfrak{s})}{\widehat{CFK}_{0}(Y, K, \mathfrak{s})} \simeq \widehat{CFL}(Y, K, \mathfrak{s}),$$

and

$$\widehat{CFL}_{-}(Y, K, \mathfrak{s}) \simeq \frac{\widehat{CFL}_{+}(Y, K, \mathfrak{s})}{\widehat{CFL}_{0}(Y, K, \mathfrak{s})} \simeq \widehat{CFK}(Y, K, \mathfrak{s}),$$

where all the chain complexes are assumed to be graded using the Spin$^c$ structures in $\text{Spin}^c(Y, K)$.

The techniques used in [2] may be used to derive the following genus formula for the knots in $S^3$. Suppose that $(S^3, K)$ is a knot, and identify the set of Spin$^c$ structures Spin$^c(S^3, K) = \text{Spin}^c(S^3_0(K))$ with $\mathbb{Z}$ by evaluating the Chern class of a Spin$^c$ structure on a capped Seifert surface for $K$. For a complex $C$ which is graded by $\mathbb{Z}$ (in the sense that the differential of $C$ takes elements in a grading level $s$ to elements in grading level $s$), define the degree $d_+(C)$ to be given by

$$d_+(C) = \max \{s \in \mathbb{Z} \mid C(s) \text{ does not have trivial chain homotopy type} \},$$

and similarly define

$$d_-(C) = \min \{s \in \mathbb{Z} \mid C(s) \text{ does not have trivial chain homotopy type} \}.$$

Here we are denoting the part of complex $C$ in grading level $s$ by $C(s)$. Then we have the following theorem.

**Theorem 1.6.** Suppose that $K$ is a knot in $S^3$ of genus $g$. Then for the complex $\widehat{CFK}(K) = \widehat{CFK}(S^3, K)$ we have

$$d_+(\widehat{CFK}_-(K)) + 1 = -d_-(\widehat{CFK}_-(K)) = g,$$

$$d_+(\widehat{CFK}_0(K)) = -d_-(\widehat{CFK}_0(K)) = g,$$

and

$$d_+(\widehat{CFK}_+(K)) = -d_-(\widehat{CFK}_+(K)) = g.$$

Furthermore, for the complex $\widehat{CFL}(K) = \widehat{CFL}(S^3, K)$ we have

$$d_+(\widehat{CFL}_-(K)) = -d_-(\widehat{CFL}_-(K)) = g,$$

$$d_+(\widehat{CFL}_0(K)) = -d_-(\widehat{CFL}_0(K)) = g,$$

and

$$d_+(\widehat{CFL}_+(K)) = 1 - d_-(\widehat{CFL}_+(K)) = g.$$
As we mentioned before, for many gluing constructions, it is not known how to relate the Floer homology of the final object to the Floer homology of the building blocks of the construction. The main type of construction we have in mind is gluing of three manifold along certain null-homologous knots inside them.

Suppose that $(Y_1, K_1)$ and $(Y_2, K_2)$ are two null-homologous knots and consider the knot complements $W_i = Y_i \setminus \text{nd}(K_i)$. There are two distinguished curves on the torus boundary of each $W_i$, $i = 1, 2$. One of them is the meridian $m_i$ of the knot $K_i$ in $Y_i$, and the other one is a longitude $l_i$ for $K_i$, so that it gives the three manifold $(Y_i)_0(K_i)$, obtained from $Y_i$ by a zero surgery on $K_i$. The two curves $(m_i, l_i)$, $i = 1, 2$, determine a framing of the boundary of the three-manifold $W_i$. One may glue $W_1$ to $W_2$ along their torus boundary in many ways. Of special interest to us are the following two special cases. The first case is when the curve $m_1$ is glued to $m_2$ and the curve $l_1$ is glued to $l_2$ under the above identification of the boundaries of $W_1$ and $W_2$. The identification of $l_1$ with $l_2$, in fact, identifies the two knots $K_1$ and $K_2$ to give a knot $K$ in the resulting three-manifold $Y$. The result of this construction will be denoted by $(Y, K) = (Y_1, K_1) \parallel (Y_2, K_2)$. We may also choose the identification of the boundaries so that $m_1$ is identified with $l_2$ and $m_2$ is identified with $l_1$. A parallel copy of $K_1$ gives a knot $\hat{K}$ in the resulting three-manifold $Y$, and we will denote the result of this construction by $(Y, \hat{K}) = (Y_1, K_1) \perp (Y_2, K_2)$. The Heegaard Floer homology groups (non-filtered versions) $CFK(Y, K)$ and $\overline{CFL}(Y, K)$ may be computed in terms of filtered Floer homologies of $(Y_i, K_i)$ for these constructions. In particular, we prove the following two parallel theorems:

**Theorem 1.7.** Suppose that $(Y_1, K_1)$ and $(Y_2, K_2)$ are two null-homologous knots and that $(Y, K) = (Y_1, K_1) \parallel (Y_2, K_2)$ as above. Then for any relative Spin$^c$ structure

$$\mathfrak{g}_1 \# \mathfrak{g}_2 \in \text{Spin}^c(Y, K) = \text{Spin}^c(Y_1, K_1) \oplus \text{Spin}^c(Y_2, K_2),$$

the Heegaard Floer homology $\overline{CFK}(Y, K)$ in the Spin$^c$ structure $\mathfrak{g}_1 \# \mathfrak{g}_2$ will be given as the quotient complex

$$\mathcal{C}(\mathfrak{g}_1 \# \mathfrak{g}_2) = \frac{[\overline{CFL}(Y_1, K_1, \mathfrak{g}_1) \otimes \overline{CFL}(Y_2, K_2, \mathfrak{g}_2)]}{\overline{CFL}_0(Y_1, K_1, \mathfrak{g}_1) \otimes \overline{CFL}_0(Y_2, K_2, \mathfrak{g}_2)},$$

where the equivalence relation $\sim$ is induced by the isomorphism of the subcomplexes

$$\rho : \overline{CFL}_-(Y_1, K_1, \mathfrak{g}_1) \otimes \overline{CFL}_0(Y_2, K_2, \mathfrak{g}_2) \longrightarrow \overline{CFL}_0(Y_1, K_1, \mathfrak{g}_1) \otimes \overline{CFL}_-(Y_2, K_2, \mathfrak{g}_2).$$
Here $\rho$ is induced by the isomorphisms $\widehat{CFL}(\cdot) \cong \widehat{CFL}_0$. Moreover, the groups $\widehat{CFL}(Y, K)$ are given by

$$\widehat{CFL}(Y, K, \mathfrak{s}_1 \# \mathfrak{s}_2) \cong \widehat{CFL}(Y, K_1, \mathfrak{s}_1) \otimes \widehat{CFL}(Y, K_2, \mathfrak{s}_2).$$

\textbf{Theorem 1.8.} Suppose that $(Y, K) = (Y_1, K_1) \perp (Y_2, K_2)$ as above. Then

$$\text{Spin}^c(Y, K) \oplus \mathbb{Z} = \text{Spin}^c(Y_1, K_1) \oplus \text{Spin}^c(Y_2, K_2),$$

with $\mathbb{Z}$ being generated by $PD[m_2]$. Furthermore, for any relative Spin$^c$ structure $\mathfrak{s} \in \text{Spin}^c(Y, K)$ we have

$$\widehat{CFL}(Y, K, \mathfrak{s}) = \bigoplus_{\mathfrak{s}_1 \in \text{Spin}^c(Y_1, K_1), \mathfrak{s}_2 \in \text{Spin}^c(Y_2, K_2)} \widehat{CFL}(Y_1, K_1, \mathfrak{s}_1) \otimes \widehat{CFL}(Y_2, K_2, \mathfrak{s}_2),$$

$$\widehat{CFK}(Y, K, \mathfrak{s}) = \left[ \bigoplus_{\mathfrak{s}_1 \in \text{Spin}^c(Y_1, K_1), \mathfrak{s}_2 \in \text{Spin}^c(Y_2, K_2)} \mathcal{C}(\mathfrak{s}_1, \mathfrak{s}_2) \right] \cong$$

$$\mathcal{C}(\mathfrak{s}_1, \mathfrak{s}_2) = \frac{\widehat{CFL}(Y_1, K_1, \mathfrak{s}_1) \otimes \widehat{CFK}(Y_2, K_2, \mathfrak{s}_2)}{\widehat{CFL}_0(Y_1, K_1, \mathfrak{s}_1) \otimes \widehat{CFK}_0(Y_2, K_2, \mathfrak{s}_2)},$$

$$\widehat{CFL}_{-}(Y_1, K_1, \mathfrak{s}_1) \otimes \widehat{CFK}_{-}(Y_2, K_2, \mathfrak{s}_2) \cong$$

$$\frac{\widehat{CFL}(Y_1, K_1, \mathfrak{s}_1)}{\widehat{CFL}_0(Y_1, K_1, \mathfrak{s}_1)} \otimes \frac{\widehat{CFK}(Y_2, K_2, \mathfrak{s}_2)}{\widehat{CFK}_0(Y_2, K_2, \mathfrak{s}_2)} \cong \frac{\widehat{CFL}(Y_1, K_1, \mathfrak{s}_1)}{\widehat{CFL}_0(Y_1, K_1, \mathfrak{s}_1)} \otimes \frac{\widehat{CFK}(Y_2, K_2, \mathfrak{s}_2)}{\widehat{CFK}_0(Y_2, K_2, \mathfrak{s}_2)}.$$

As before the equivalence relation $\sim$ is induced by the connecting isomorphisms $\widehat{CFK} \cong \widehat{CFK}_0$ and $\widehat{CFL} \cong \widehat{CFL}_0$.

These gluing formulas are in fact our main justification, besides their nice properties, for definition of these extra filtered complexes. Some of the computation technology introduced in \cite{[6, 7]} naturally generalizes to these extra filtered versions of our complexes. One thing we would like to emphasize here, is the surgery long exact sequence. If $(Y, K)$ denotes a knot in a three-manifold $Y$, and if $\gamma$ is another knot which has zero linking number with $K$, we may extend the surgery short exact sequence of Ozsváth and Szabó to the context of filtered chain complexes to obtain the following long exact sequence formula in the level of homology.

\textbf{Theorem 1.9.} Let $(Y, K)$ be a knot and let $\gamma$ be a framed knot in $Y$ which is disjoint from $K$ and has zero linking number with it. Then for a correct choice of the Seifert surface, and for each integer $m \in \mathbb{Z}$ we obtain the exact sequences:

$$\ldots \to H_*(\widehat{CFK}_*(Y_{-1}(\gamma), K, m)) \xrightarrow{f_1} H_*(\widehat{CFK}_*(Y_{0}(\gamma), K, m)) \xrightarrow{f_2} H_*(\widehat{CFK}_*(Y, K, m)) \xrightarrow{f_3} \ldots,$$
where $\bullet \in \{-, 0, +\}$. The maps $f_1^\bullet$ and $f_2^\bullet$, when the groups are graded, each will lower the absolute grading by $\frac{1}{2}$, and $f_3^\bullet$ will not increase the absolute grading.

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2. **Construction of the filtration**

We begin by considering a special Heegaard diagram for the null-homologous knot $K$ in the three-manifold $Y$. To obtain such a diagram, start with a weakly admissible marked Heegaard diagram $(\Sigma, \alpha, \beta_0 \cup \{m\}, z)$ for $K \subset Y$, as defined by Ozsváth and Szabó in [6]. As usual $\alpha = \{\alpha_1, ..., \alpha_g\}$ and $\beta_0 = \{\beta_2, ..., \beta_g\}$ give the Heegaard diagram for the knot complement $Y \setminus nd(K)$ and $m$ represents a meridian of $K$ in $Y$. Furthermore, the marked point $z$ is located on $m$.

We may consider a longitude $l$ for the knot $K$ on this surface $\Sigma$ with the property that it is disjoint from the curves in $\beta_0$, cuts $m$ transversely in a single point and such that the Heegaard diagram $(\Sigma, \alpha, \beta_0 \cup \{l\})$ represents the three manifold $Y_0(K)$ obtained from $Y$ by a zero surgery on $K$. Such a curve $l$ is unique up to handle slides along the curves in $\beta_0$ (and also isotopies which are disjoint from $\beta$).

We may assume that $l$ cuts the meridian $m$ in the marked point $z$, without losing the generality.

To get the desired Heegaard diagram, modify this data as follows: Let $\delta$ be a small oriented arc on $l$, forming a neighborhood of $z$ in $l$ and disjoint from all the other curves in $\alpha$ and $\beta_0$. Let $P$ and $Q$ be the start point and the end point of this arc. Attach a handle to the surface $\Sigma$ which connects $P$ to $Q$. Using this handle we may replace the arc $\delta$ on $l$ by a path which avoids $m$, to get a new curve $\lambda$. The loop $\mu$ which goes once around the handle cuts $\lambda$ in a single point. Moving $\mu$ by an isotopy, we may also create a pair of cancelling intersection points with $m$.

Let us denote the new surface by $\Sigma'$. If we connect the end points of $\delta$ to each other using the handle, so that the resulting closed curve $l'$ only has a single intersection point with $m$ and a single intersection point with $\mu$, then the Heegaard diagram

$$(\Sigma', \alpha \cup \{\mu\}, \beta_0 \cup \{\lambda\} \cup \{m\}, z)$$
also represents the same knot \( K \subset Y \), and \( l' \) represents a longitude for \( K \) in this new diagram (in the above sense).

Note that given the orientation on \( m \) and on \( l \), the above construction may be done in a unique way, say as is suggested by figure 1.

There are three intersection points on the curve \( \mu \) which are named by the letters \( A, B \) and \( C \) in figure 2. Any generator corresponding to the Heegaard diagram
\[(\Sigma', \alpha \cup \{\mu\}, \beta_0 \cup \{\lambda\} \cup \{m\}, z)\]
will contain exactly one of these three intersection points, and accordingly we may partition the generators into three types: type A, type B and type C.

Put the marked points \( u \) and \( w \) on the left and on the right hand side of the point \( z \in m \) in order to get a doubly pointed Heegaard diagram, as in [6].

Note that any boundary map starting from a generator of type C, will go to a generator of type C. The reason is that if a homotopy type \( \phi \in \pi_2(x, y) \) of disks connecting the generators \( x \) and \( y \) contains a holomorphic representative, then the associated domain \( D(\phi) \) will only have positive coefficients. Since three of the four quadrants around the intersection point \( C \) are forced to have multiplicity zero, the fourth domain will have a coefficient of \( \pm 1 \), if \( \phi \) connects a generator of type C to a generator of a different type. Having

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{diag.png}
\caption{From the given Heegaard diagram on \( \Sigma \), where the \( \alpha \) curves are represented by the bold curves, the elements of \( \beta_0 \) and the meridian \( m \) are the normal lines, and \( l \) is the dotted curve, we may obtain a new Heegaard diagram for the same knot, by adding a handle and the two curves \( \lambda \) and \( \mu \). The new longitude \( l' \) is the dotted curve on the right hand side.}
\end{figure}
a holomorphic representative means that this coefficient is in fact +1.

But then the differential map associated with this disk will "end up" within the generator of type C, rather than "start from" such a state.

A similar argument shows that the boundary of generators of type B consists of generators of types B and C. As a result we have a filtration associated with the ordering

\[
\text{type } C < \text{type } B < \text{type } A.
\]

According to the general properties of filtered chain complexes we get a filtration of the complex \( \widehat{CFK}(Y,K) \) by a flag

\[
\widehat{CFK}_-(Y,K) \xrightarrow{i_-} \widehat{CFK}_0(Y,K) \xrightarrow{i_0} \widehat{CFK}_+(Y,K) = \widehat{CFK}(Y,K).
\]

Here \( \widehat{CFK}_-(Y,K) \) denotes the subcomplex generated by the generators of type C, \( \widehat{CFK}_0(Y,K) \) consists of generators of types B and C, and \( \widehat{CFK}_+(Y,K) \) consists of all the generators (of types A,B and C). The maps \( i_- \) and \( i_0 \) are embeddings of the subcomplexes. The boundary maps are the restrictions of the standard boundary map \( \partial_+ \) of \( \widehat{CFK}(Y,K) \) to the corresponding subcomplex.

Before we state the invariance of this filtration, let us introduce one extra map \( \partial' \) on the complex.

\[
\text{Figure 2. When a handle slide happens between the } \beta \text{ circles in the complement of the marked point } z, \text{ we may consider a Hamiltonian isotope of the curves } \lambda \text{ and } m \text{ which are denoted by the dotted curve. We assume that there are a pair of cancelling intersection points between each curve and its Hamiltonian translate, denoted by } \Theta \text{ and } \Theta_1.
\]
If we allow the connecting disks to have a nonzero coefficient at the point \( u \), i.e. if we allow \( n_u(\phi) \neq 0 \), then the only possible disks between two generators with \( n_u(\phi) \neq 0 \) which also have holomorphic representatives will go from a generator of type A to a generator of type C. Furthermore, for such a disk, \( n_u(\phi) = 1 \) and the other regions which have A or C as a corner will get a zero coefficient in the domain \( D(\phi) \) of \( \phi \).

If the two generators agree, except for the intersection points A and C, then there is a unique holomorphic disk connecting them.

As a result of this discussion we see that the map \( \partial_1 \), which counts such disks, is trivial except on the generators of type A, and that the image of \( \partial_1 \) lies in \( \hat{\text{CFK}}^- (Y,K) \).

It is important to note that the complex \( \hat{\text{CFK}}(Y,K) \) equipped with \( \partial = \partial_1 + \partial_+ \) gives the chain complex which evaluates the Floer homology of the three-manifold \( Y \) (the hat theory), where \( \partial_+ \) is the differential of \( \hat{\text{CFK}}^+(Y,K) \). We may combine the map \( \partial_1 \) with the sign map, assigning the values \( \pm 1 \) to the generators of the complex according to their absolute \( \mathbb{Z}/2\mathbb{Z} \) grading to get a chain map \( \partial' = \epsilon \partial_1 \).

We also make the remark that as usual the Spin\(^c\) structures of \( Y_0(K) \) will filter the complexes:

\[
\text{CFK}_-(Y,K,\mathfrak{g}) \overset{i}{\longrightarrow} \text{CFK}_0(Y,K,\mathfrak{g}) \overset{i_0}{\longrightarrow} \text{CFK}_+(Y,K,\mathfrak{g}) = \text{CFK}(Y,K,\mathfrak{g}).
\]

Here \( \text{CFK}(Y,K) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y_0(K))} \text{CFK}(Y,K,\mathfrak{s}) \), etc..

The first result of this paper is the following:

**Theorem 2.1.** The chain homotopy type of the flag of complexes

\[
\text{CFK}_-(Y,K,\mathfrak{g}) \overset{i}{\longrightarrow} \text{CFK}_0(Y,K,\mathfrak{g}) \overset{i_0}{\longrightarrow} \text{CFK}_+(Y,K,\mathfrak{g})
\]

is a topological invariant of the knot \( K \) in the three-manifold \( Y \), and the relative Spin\(^c\) structure \( \mathfrak{g} \in \text{Spin}^c(Y) = \text{Spin}^c(Y_0(K)) \). Moreover the map \( \partial' \) satisfies \( \partial' \circ \partial_+ - \partial_+ \circ \partial' = 0 \) which gives the following exact sequence:

\[
0 \rightarrow \text{CFK}_0(Y,K,\mathfrak{g}) \overset{i_0}{\longrightarrow} \text{CFK}_+(Y,K,\mathfrak{g}) \overset{\partial'}{\longrightarrow} \text{CFK}_-(Y,K,\mathfrak{g} - PD[m]) \rightarrow 0.
\]

The chain homotopy type of this exact sequence is also a topological invariant of the knot \( K \) and the relative Spin\(^c\) structure \( \mathfrak{g} \).

**Proof.** The independence from the almost complex structure is easy and standard. We have to show that different choices of the Heegaard diagram do not change the homotopy types of the the above sequences.
To do so, we should show that:

1) The final choice of \( l \), up to a handle slide does not change the homotopy types.
2) Handle slides and isotopies of \( \alpha \), away from the base point \( z \), preserve the homotopy types.
3) Handle slides of \( \beta_0 \) and \( m \) along \( \beta_0 \) keep the homotopy type invariant.
4) The same is true for the isotopies of \( \beta_0 \) and \( m \).
5) The homotopy type is not changed in the process of handle addition.

For the first claim, assume that \( l^1 \) is obtained from \( l \) by a handle slide along one of the curves in \( \beta_0 \), say \( \beta_2 \). We implicitly assume that \( l^1 \) is moved by a hamiltonian isotopy so that a pair of cancelling transverse intersections are created between \( l \) and \( l^1 \). Correspondingly introduce \( \lambda^1 \), which is constructed from \( l^1 \) in the same way that \( \lambda \) is constructed from \( l \).

Let \( m^1, \gamma_2, \gamma_3, ..., \gamma_g \) be Hamiltonian isotopes of \( m, \beta_2, ..., \beta_g \), so that there are pairs of cancelling intersection points between \( \beta_i \) and \( \gamma_i \), and between \( m \) and \( m^1 \). Then the three sets of curves:

\[
\alpha' = \alpha \cup \{ \mu \} = \{ \alpha_1, ..., \alpha_g, \mu \},
\beta' = \beta \cup \{ \lambda \} = \{ m, \beta_2, ..., \beta_g, \lambda \}, \text{ and }
\gamma' = \gamma \cup \{ \lambda^1 \} = \{ m^1, \gamma_2, ..., \gamma_g, \lambda^1 \}
\]

will form a Heegaard triple. \( \alpha' \) and \( \beta' \) will give the initial diagram \( H_1 \), the pair \( (\beta', \gamma') \) will give a Heegaard diagram \( H_2 \) for the connected sum of \( g \) copies of \( S^1 \times S^2 \), and finally the pair \( (\alpha', \gamma') \) will give the diagram \( H_3 \) obtained by a handle slide on \( H_1 \). The marked points \( u \) and \( w \) will be fixed for each of the three Heegaard diagram obtained from the three pairs of \( \alpha, \beta \) and \( \gamma \). However, on \( H_2 \), they are both located in the same region.

The Floer homology of the Heegaard diagram

\[ H_2 = (\Sigma', \beta', \gamma'; u = w) \]

gives \( \widehat{HF}(\#^g(S^1 \times S^2)) \). There is a top generator of this Floer homology group in the Spin\(^c\) structure with trivial first Chern class, which will be denoted by \( \Theta \). The map

\[ G : \widehat{CFK}(Y, K, g; H_1) \rightarrow \widehat{CFK}(Y, K, g; H_3) \]

is defined by \( G(x) = F(x \otimes \Theta) \) for any generator \( x \in T_\alpha \cap T_\beta \). Here

\[ T_\alpha = \mu \times \alpha_1 \times ... \times \alpha_g, \]
\[ T_\beta = \lambda \times m \times \beta_2 \times ... \times \beta_g, \text{ and } \]
\[ T_\gamma = \lambda^1 \times m^1 \times \gamma_2 \times ... \times \gamma_g, \]
and the map

\[ \mathcal{F} : \hat{CFK}(H_1) \otimes \hat{CFK}(H_2) \to \hat{CFK}(H_3) \]

is defined for any \( x \otimes y \) with \( x \in T_\alpha \cap T_\beta \) and \( y \in T_\beta \cap T_\gamma \) to be

\[ \mathcal{F}(x \otimes y) = \sum_{w \in T_\alpha \cap T_\gamma} \sum_{\phi \in \pi_2(x, y, w) \atop \mu(\phi) = 0 \atop n_u(\phi) = n_w(\phi) = 0} \#(M(\phi)).w. \]

The similar arguments in [6, 8, 9] may be followed to show that this map is in fact a chain homotopy equivalence. However, we need to show furthermore, that this map in fact respects the filtrations of the complexes \( \hat{CFK}(Y, K; H_1) \) and \( \hat{CFK}(Y, K; H_3) \).

It suffices to show that when \( x \) is a generator of type C, then only generators of type C can appear in the formal sum \( G(x) \), and that if \( x \) is of type B, then every generator that appears in the formal expression \( G(x) \) is either of type B or of type C.

Note that if the coefficient of \( w \in T_\alpha \cup T_\gamma \) is positive in \( G(x) \), then we should have a holomorphic representative for a homotopy class of disks \( \phi \in \pi_2(x, \Theta, w) \) with \( n_u(\phi) = n_w(\phi) = 0 \). If \( x \) is of type C, as we travel counterclockwise on \( \mu \) starting at \( C \), before getting to the intersection point corresponding to \( w \), the coefficient on the right hand side, is always strictly lower than the one on the left hand side. Here we are referring to a picture which is locally illustrated in figure 2. If \( w \) is not of type C, then at some point, we will have a zero coefficient on the left, which implies that the coefficient on the right is forced to be negative, a contradiction.

A similar argument shows that for any \( x \in T_\alpha \cap T_\beta \) of type B, \( G(x) \) will only consist of nonzero multiples of generators in \( T_\alpha \cap T_\gamma \) of types B and C.

This proves that \( G \) respects the filtration and that the restriction to each of the subcomplexes is a homotopy equivalence as well.

To show that the homotopy type of the exact sequences are preserved, we have to show that for the vertical maps \( G \) in the diagram

\[
\begin{array}{ccl}
0 & \to & \hat{CFK}_0(H_1) \\
& \downarrow G & \downarrow G \\
0 & \to & \hat{CFK}_0(H_3)
\end{array}
\]

(1)

\[
\begin{array}{ccl}
\hat{CFK}_+(H_1) & \to & \hat{CFK}_-(H_1) \\
& \downarrow G & \downarrow G \\
\hat{CFK}_+(H_3) & \to & \hat{CFK}_-(H_3)
\end{array}
\]

\[
\begin{array}{ccl}
0 & \to & 0
\end{array}
\]
the expressions $G \circ \partial' - \partial' \circ G$ and $G \circ i_B - i_B \circ G$ are null homotopic. This is also a typical application of techniques in [9] and [6].

For the other cases, the argument continues exactly in the same way. We will follow the steps showing that the moves of type 2, 3, 4 and 5 do not change the homotopy type of the complex $\widehat{CFK}(Y, K)$ as in [9], and at each step, we will also check that the filtration is also preserved by the homotopy equivalence suggested by the argument of Ozsváth and Szabó in [6]. This completes the proof.

It is interesting to note that the homology of the subcomplex $\widehat{CFK}_-(Y, K)$, gives the longitude Floer homology of the knot $K$ introduced in [2]:

**Theorem 2.2.** The homology of the subcomplex $\widehat{CFK}_-(Y, K)$ and the quotient complex

$$\frac{\widehat{CFK}_+(Y, K)}{i_0(\widehat{CFK}_0(Y, K))}$$

give the longitude Floer homology $\widehat{HFL}(Y, K)$. More precisely:

$$H_*(\widehat{CFK}_-(Y, K, \mathfrak{s})) \simeq \widehat{HFL}(Y, K, \mathfrak{s} + \frac{1}{2}PD[m]),$$

and

$$H_*(\frac{\widehat{CFK}_+(Y, K, \mathfrak{s})}{i_0(\widehat{CFK}_0(Y, K, \mathfrak{s}))}) \simeq \widehat{HFL}(Y, K, \mathfrak{s} - \frac{1}{2}PD[m]),$$

where $PD[m]$ denotes the Poincaré dual of the meridian $m$ of the knot $K$ in the three-manifold $Y$.

**Proof.** Note that if we replace the meridian $m$ of the diagram shown in figure [1] by the dotted curve $l'$, we obtain a weakly admissible diagram for the zero surgery $Y_0(K)$ on the knot $K$. This diagram may be used for the longitude Floer homology (the hat theory), as any periodic domain has both positive and negative coefficients.

The only curve which cuts $l'$ is the curve $\mu$. It cuts $l'$ in a single point $z$ and any intersection of the tori associated with the tuples $\alpha \cup \{\mu\}$ and $\{l', \lambda\} \cup \beta_0$ will consist of a $(g + 1)$-tuple of points, where one of the points is forced to be $z$. These $g$ tuples are in one to one correspondence with the intersection points of $T_\alpha$ and $\mathbb{T}_\beta$ of type $C$. Namely, any $(g + 1)$-tuple $\{z, \bullet\}$ is in correspondence with $\{C, \bullet\}$.

Since the disks are forced to have zero coefficients at $u$ and $w$, we conclude that the domains of the disks between the generators in the longitude theory, are in fact exactly the same as the domains of the disks between the
corresponding generators of type C.

This gives the first isomorphism claimed in the above theorem. The second isomorphism is quite similar.

The claim about the \( \frac{1}{2} \text{PD}[m] \) shifts in the Spin\(^c\) structure, is because of our special convention in \([2]\), on averaging the Spin\(^c\) structures assigned to a generator \( x \) by the maps \( s_u \) and \( s_w \).

\[ \square \]

There is a second way of getting a filtration, this time on \( \widehat{CFL}(Y,K) \) rather than on \( \widehat{CFK}(Y,K) \). Namely, in the initial diagram of figure 1, we replace the role of the meridian \( m \) and the longitude \( l \) and will continue with the construction, as is suggested in figure 3. The points \( u \) and \( w \) will give us two maps

\[ s_u, s_w : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \text{Spin}^c(Y,K) = \text{Spin}^c(Y_0(K)), \]

which are related by the formula

\[ s_u(x) = s_w(x) + \text{PD}[m]. \]

This choice of the order of \( u \) and \( w \) is the one that has been implicit in this paper.

We may choose to assign the element

\[ s_u(x) - \frac{1}{2} \text{PD}[m] = s_w(x) + \frac{1}{2} \text{PD}[m] \in \frac{1}{2} \text{PD}[m] + \text{Spin}^c(Y_0(K)), \]

\[ \text{Figure 3.} \] We may obtain a new Heegaard diagram for the longitude theory of the knot \( K \), representing the three manifold \( Y_0(K) \), by adding a handle and the two curves \( \lambda \) and \( \mu \) to the collections \( \beta_0 \cup \{ l \} \) and \( \alpha \) respectively.
to the generators $x$, which will be for the sake of symmetry as in [2], or just choose to assign the “higher” Spin$^c$ structure $s_u(x)$ to any $x \in T_\alpha \cap T_\beta$. Unlike [2] where for most of the paper we averaged the Spin$^c$ structures, here we will usually use $s_u(x)$ as the Spin$^c$ structure assigned to $x$. Note that by abuse of notation, we are using the notations $T_\alpha$ and $T_\beta$ for the tori $\alpha_1 \times ... \times \alpha_g \times \mu$, and $l \times \beta_2 \times ... \times \beta_g \times \lambda$.

Similarly we will obtain the following theorem:

**Theorem 2.3.** There is a natural filtration of the complex $\hat{CF}(Y,K)$ respecting the grading by the relative Spin$^c$ structures $\tilde{g} \in \tilde{Spin}^c(Y,K) = Spin^c(Y_0(K))$ given as

$$\hat{CF}_-(Y,K,\tilde{g}) \xrightarrow{l_0} \hat{CF}_0(Y,K,\tilde{g}) \xrightarrow{l_0} \hat{CF}_+ (Y,K,\tilde{g}),$$

whose homotopy type is a topological invariant of the knot $(Y,K)$ and the relative Spin$^c$ structure $\tilde{g}$. Furthermore, there is an exact sequence

$$0 \rightarrow \hat{CF}_0(Y,K,\tilde{g}) \xrightarrow{l_0} \hat{CF}_+ (Y,K,\tilde{g}) \xrightarrow{\partial_l} \hat{CF}_-(Y,K,\tilde{g}) \rightarrow 0,$$

which is also an invariant of the knot $K$. Moreover, if we let $\partial_l$ be the sum $\partial_l^* + \epsilon \partial^*_l$, where $\partial_l^*$ is the boundary map of the complex $\hat{CF}(Y,K)$, then the complex $\hat{CF}(Y,K)$, equipped with $\partial_l$, will give the Floer homology of the three-manifold $Y_0(K)$. Finally the subcomplex $\hat{CF}_-(Y,K)$ has the same chain homotopy type as the (non-filtered) complex $CFK(Y,K)$.

### 3. Basic properties

We start with an investigation of the effect of a change in the orientation of a knot $K$ in the above filtration of $\hat{CF}(Y,K)$. In contrast with the usual hat theory, where the two points $u$ and $w$ basically played the same role and the symmetry was relatively easy to show, here we are giving an essential order to these two marked points, and an essential role to the orientation of the knot $K$.

Let us start with a standard diagram, not necessarily of the type discussed in the previous section, for the null-homologous knot $(Y,K)$. We may run the process of adding a handle in order to get the appropriate diagram for defining the filtration for $(Y,K)$. This process will be different if we were going to add the handle associated with the knot $K$ with the reverse orientation. However, we may continue the process with this new Heegaard diagram, as is shown in figure 4 to add a second handle, and further modify the longitude of $(Y,-K)$ and get the appropriate Heegaard diagram.

We will compare the Floer homology of the Heegaard diagram, after the first step, which gives the filtration of $\hat{CF}(Y,K)$, with that of the final
Figure 4. We may start from a typical Heegaard diagram for the knot \((Y, K)\), change it to an allowed diagram for the pair \((Y, K)\) and then modify it more to get an allowed diagram for \((Y, -K)\).

Heegaard diagram, which gives the filtration on \(\hat{CFK}(Y, -K)\). Let us denote the \(\alpha\) curves added in the first and in the second step by \(\mu_1\) and \(\mu_2\) respectively, and similarly denote the new \(\beta\) curves by \(\lambda_1\) and \(\lambda_2\). Moreover, denote the allowed Heegaard diagram of \((Y, K)\) obtained after the first step, by \(\mathcal{H}_1\) and denote the allowed Heegaard diagram of \((Y, -K)\) obtained after the second step by \(\mathcal{H}_2\).

Some of the special intersection points of these new curves are named in the diagram of figure 4 by the letters \(n, p, q, r, s\) and \(t\). There is a special rectangular region with vertices \(p, q, t\) and \(n\) which we denote by \(\Delta_L\), and another rectangular region \(\Delta_R\) with the vertices \(r, s, n\) and \(t\).

Let us partition the set of generators of the complex associated with \(\mathcal{H}_1\) into sets \(\mathcal{A}, \mathcal{B},\) and \(\mathcal{C}\). Here \(\mathcal{A}\) consists of those generators which are of type \(A\), \(\mathcal{B}\) consists of generators of type \(B\), and \(\mathcal{C}\) consists of type \(C\) generators.

Any generator of type \(A\) in \(\mathcal{H}_2\) will contain the intersection point \(r\). Since we always have to choose one of \(t\) and \(n\), and \(t\) lies on the same \(\alpha\) circle as \(r\), the generators of type \(A\) are forced to contain the pair of points \(\{r, n\}\). Let \(\mathcal{A}_2\) be set of generators of \(\mathcal{H}_2\) of type \(A\). Then the above discussion shows that there is a one-to-one and surjective map

\[
i_A : \mathcal{C} \rightarrow \mathcal{A}_2,
\]
defined by $i_A(\{s, \bullet\}) = \{r, n, \bullet\}$. Similarly, if $C_2$ denotes the set of generators of $H_2$ of type $C$, then any of its elements will contain $q$ and $n$, and there is a correspondence

$$i_C : A \rightarrow C_2,$$

$$i_C(\{p, \bullet\}) = \{q, n, \bullet\}.$$

The generators of type $B$ are more interesting, since they all share the intersection point $t$. This means that any generator of $H_1$ may be completed to a generator of $H_2$ by just adding $t$. As a result we get a correspondence

$$i_B : A \cup B \cup C \rightarrow B_2,$$

$$i_B(\{\bullet\}) = \{t, \bullet\}.$$

If one investigates the effect of the boundary map (differential map) on the generators of the same type, then the above maps will further respect the boundary maps, since for example the domain of a disk connecting $x$ to $y$ in $H_1$, is isomorphic to the domain of the disk connecting $i_B(x)$ to $i_B(y)$ in $H_2$.

Now consider the disks connecting an element $x$ of $A_2$ to an element $y$ in $B_2$ or $C_2$. Any such disk, if it contributes to the boundary map, will have non-negative coefficients in all of the domains.

The intersection point $r$ appears in $x$ and not in $y$, and three of the regions which have $r$ as a corner, get a zero coefficient. As a result the one on the lower right side of $r$ in figure 4 will get a coefficient equal to 1. If $s$ does not appear in $y$ then a similar comparison of coefficients around $s$ shows that the region on the upper right side of $s$ is forced to have a coefficient of $-1$, which is a contradiction. So $y$ has to be of type $B$ and in fact the image of an element of $C$. Let us assume that $x = i_A(z)$ and $y = i_B(w)$, with $z, w \in C$. Then the disk between $x$ and $y$ will have a domain which is the the disjoint union of the rectangle $\Delta_R$ with another domain. An argument similar to the arguments used by Ozsváth and Szabó in proving the connected sum formulas of [9] and [8, 9], may be used to show that the boundary maps going from $A_2$ to $B_2$ is a perturbation of the map $f_A = i_B \circ i_A^{-1}$. The total boundary map on $A_2$ will be chain homotopic to

$$\partial_A = i_A \circ \partial \circ i_A^{-1} + f_A = i_A \circ \partial \circ i_A^{-1} + i_B \circ i_A^{-1},$$

where $\partial$ denotes the boundary map of $\widehat{CFK}(Y, K)$.

Similarly the differentials on type $B$ differ from that of $\widehat{CFK}(Y, K)$ by a factor

$$f_B = i_C \circ i_B^{-1} |_{i_B(A)}.$$
The above discussion may be summarized in the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{d_1} & B \\
\downarrow I_A & & \downarrow I_C \\
C & \xleftarrow{d_2} & B \\
\downarrow I_C & & \downarrow I_A \\
C & \xleftarrow{d_1} & A \\
\end{array}
\]

The maps \(I_A\) and \(I_C\) are isomorphisms and \(d_1\) and \(d_2\) denote just some of the boundary maps. Potentially there can be maps within each of the sets \(A, B\) or \(C\), or we can have a boundary map going from \(A\) to \(C\) on the left hand side, or in the middle row of the right hand side. These potential maps are dropped from diagram (2) for simplicity.

Since the restriction of the boundary map of \(H_1\) to intersection points of type \(A\) (i.e. \(A\)) gives a complex isomorphic to \(\hat{CFK}_-(Y,K)\), We may deduce that up to a possible total shift in the absolute grading by the Maslov index, there is a homotopy equivalence

\[
\hat{CFK}_-(Y,K) \simeq \hat{CFK}_-(Y,-K).
\]

The isomorphism in zero level of the filtration (i.e. \(\hat{CFK}_0(Y,\pm K)\)) is just an algebraic fact about the two complexes above. And of course we already know the equivalence of the chain homotopy types of the full complexes \(\hat{CFK}_+(Y,K)\) and \(\hat{CFK}_+(Y,-K)\).

When the relative Spin\(^c\) structure \(\mathfrak{g} \in \text{Spin}^c(Y,K)\) extends a torsion Spin\(^c\) structure \(\mathfrak{g} \in \text{Spin}^c(Y)\), besides the above isomorphism of the homotopy types of the filtered complexes \(\hat{CFK}(Y,K,\mathfrak{g})\) and \(\hat{CFK}(Y,-K,\mathfrak{g})\), we may compare the absolute Maslov gradings of the two complexes. Comparing the Maslov indices of the intersection points with the definition of Ozsváth and Szabó of the absolute Maslov grading in [6], we derive the following:

**Proposition 3.1.** If \((Y,K)\) is a null-homologous knot and if \((Y,-K)\) denote the same knot with the opposite orientation, then for any relative Spin\(^c\) structure \(\mathfrak{g} \in \text{Spin}^c(Y,K)\), the filtered complexes \(\hat{CFK}(Y,K,\mathfrak{g})\) and \(\hat{CFK}(Y,-K,\mathfrak{g})\) have the same homotopy type. Moreover if \(\mathfrak{g}\) extends a torsion Spin\(^c\) structure \(\mathfrak{g} \in \text{Spin}^c(Y)\), then the homotopy equivalence shifts the absolute Maslov grading by \(-2k\), where

\[
k = \frac{1}{2}\langle c_1(\mathfrak{g}), [\hat{F}] \rangle,
\]

for a capping \(\hat{F}\) of a Seifert surface \(F\) for the oriented knot \(K\). In particular

\[
H_d(\hat{CFK}_\bullet(Y,K,\mathfrak{g})) \simeq H_{d-2m}(\hat{CFK}_\bullet(Y,-K,\mathfrak{g})),
\]

where \(\bullet \in \{-, 0, +\}\).
Note that the whole complex $\widehat{CFK}(Y, K)$ is precisely the complex $\widehat{CFK}(Y, K)$ in [6] and the absolute Maslov grading of generators is induced using this identification. Using this piece of information and the corresponding isomorphism

$$\widehat{CFK}(Y, K) \simeq \widehat{CFK}(Y, -K)$$

in proposition 3.8 of [6], the proof is just to verify the statement via an algebraic comparison. \(\square\)

The next thing we want to consider in this section is the effect of the maps in the long exact sequences on the above filtration.

We remind the reader of the basic setup of the long exact sequences in [6] and [9]. Recall that for any framed knot $\gamma$ in the three manifold $Y$, and for any fixed Spin$^c$ structure $t \in \text{Spin}^c(W_\gamma(Y))$ on the four-manifold $W = W_\gamma(Y)$, counting holomorphic triangles gives a map

$$\hat{f}_{W, t}: \widehat{CF}(Y) \rightarrow \widehat{CF}(Y_\gamma),$$

where $Y_\gamma$ is obtained from $Y$ by a surgery on the framed knot $\gamma$. For the definition of the 4-manifold $W_\gamma(Y)$ we refer the reader to [10]. When we have a knot $(Y, K)$ whose linking number with the framed knot $\gamma$ is zero, we may choose a Seifert surface $F$ for $K$ which is disjoint from $\gamma$. Correspondingly, we will get a filtration induced by $F$ on both $\widehat{CF}(Y)$ and $\widehat{CF}(Y_\gamma)$.

Ozsváth and Szabó showed in [6] that in fact the map $\hat{f}_{W, t}$ respects this filtration. As a result, from the long exact sequence for the three-manifolds, they obtain the following exact sequence for knots:

$$\ldots \rightarrow \widehat{HFK}(Y_{-1}(\gamma), K, k) \xrightarrow{f_1} \widehat{HFK}(Y_0(\gamma), K, k) \xrightarrow{f_2} \widehat{HFK}(Y, K, k) \xrightarrow{f_3} \ldots,$$

for any integer $k \in \mathbb{Z}$, where we use their notation:

$$\widehat{HFK}(Y, K, k) = \bigoplus_{\gamma \in \text{Spin}^c(Y, K)} \widehat{HFK}(Y, K, s).$$

We claim that the following is also true:

**Theorem 3.2.** Let $(Y, K)$ be a knot and let $\gamma$ be a framed knot in $Y$ which is disjoint from $K$ and has zero linking number with it. Then for a correct choice of the Seifert surface, the counts of holomorphic triangles above respects the $\{-, 0, +\}$ filtration, and for each integer $k \in \mathbb{Z}$ we obtain the exact sequences:

$$\ldots \rightarrow H_*(\widehat{CFK}_*(Y_{-1}(\gamma), K, k)) \xrightarrow{f'_1} H_*(\widehat{CFK}_*(Y_0(\gamma), K, k)) \xrightarrow{f'_2} \ldots$$

(3)
Filtration of Heegaard Floer homology and gluing formulas

for any of the filtration levels \( \bullet \in \{-, 0, +\} \). The maps \( f_1^\bullet \) and \( f_2^\bullet \), when the groups are graded, each will lower the absolute grading by \( \frac{1}{2} \), and \( f_3^\bullet \) will not increase the absolute grading.

Proof. The proof is quite straightforward. In the triple Heegaard diagrams used to connect the knot \((Y, K)\) to the knot \((Y_\gamma, K)\), the argument of the second section shows, by examining the positivity of the coefficients of potential holomorphic disks in the domains around the two marked points, that the image of a generator of type C, can only have type C, and that the image of a generator of type B is either of type B, or of type C. As a result the filtration of \( \hat{CFK}(Y, K, k) \) as \( \hat{CFK}_-(Y, K, k) \subset \hat{CFK}_0(Y, K, k) \subset \hat{CFK}_+(Y, K, k) \) is preserved under the connecting homomorphisms. This completes the proof.

4. Gluing along the knots I

Suppose that \((Y_1, K_1)\) and \((Y_2, K_2)\) are two oriented knots. Then there is a unique framing on the boundary of each of the three-manifolds with boundary \( Y_1 \setminus \text{nd}(K_1) \) and \( Y_2 \setminus \text{nd}(K_2) \), determined by a meridian \( m_i \) of each of the knots \( K_i \), and a longitude \( l_i \), such that each \( l_i \) corresponds to a zero surgery on \( K_i \), \( i = 1, 2 \) (i.e. \( l_i \) has zero linking number with \( K_i \)).

We may glue these two three-manifolds with boundary, by identifying \( l_1 \) with \( l_2 \) and \( m_1 \) with \(-m_2\) (as oriented curves), along their torus boundaries. In fact, the result will be

\[
(Y_1 \setminus \text{nd}(K_1)) \cup_T (Y_2 \setminus \text{nd}(K_2)),
\]

where \( T \) is the identified boundary of \( \text{nd}(K_1) \) and \( \text{nd}(K_2) \). This will give us a new three-manifold \( Y \) and inside this three-manifold, the knots \( K_1 \) and \( K_2 \) will be identified to give a knot \((Y, K)\). Sure enough, \( K \) is null-homologous in \( Y \). We will denote this operation by writing

\[
(Y, K) = (Y_1, K_1) \parallel (Y_2, K_2).
\]

In this section we will find out how the (non-filtered) Floer homology of \((Y, K)\) is related to filtered Floer homologies associated with \((Y_i, K_i)\), \( i = 1, 2 \). In fact, we will prove the following:

**Theorem 4.1.** Suppose that \((Y_1, K_1)\) and \((Y_2, K_2)\) are two null-homologous knots and that \((Y, K) = (Y_1, K_1) \parallel (Y_2, K_2)\) is obtained by gluing \((Y_1, K_1)\) to \((Y_2, K_2)\) as above. Then for any relative Spin\(^c\) structure

\[
\#_{s_1} \, \#_{s_2} \in \text{Spin}^c(Y, K) = \text{Spin}^c(Y_1, K_1) \oplus \text{Spin}^c(Y_2, K_2),
\]

\[
\text{Spin}^c(Y, K) = \text{Spin}^c(Y_1, K_1) \oplus \text{Spin}^c(Y_2, K_2),
\]

where \( s_1 \) and \( s_2 \) are relative Spin\(^c\) structures on \((Y, K)\) and \((Y_1, K_1)\) and \((Y_2, K_2)\), respectively.
the Heegaard Floer homology of \((Y, K)\) in the \(\text{Spin}^c\) structure \(\mathfrak{s}_1 \# \mathfrak{s}_2\) will be given by the quotient complex

\[
C = \left[ \hat{\text{CF}L}(Y_1, K_1, \mathfrak{s}_1) \otimes \hat{\text{CF}L}(Y_2, K_2, \mathfrak{s}_2) \right] \sim \text{CFL}_0(Y_1, K_1, \mathfrak{s}_1) \otimes \text{CFL}_0(Y_2, K_2, \mathfrak{s}_2).
\]

Here the equivalence relation \(\sim\) is induced by the isomorphism of the subcomplexes

\[
\rho : \hat{\text{CF}L}_{-}(Y_1, K_1, \mathfrak{s}_1) \otimes \hat{\text{CF}L}(Y_2, K_2, \mathfrak{s}_2)
\]

\[
\rightarrow \hat{\text{CF}L}(Y_1, K_1, \mathfrak{s}_1) \otimes \text{CFL}_0(Y_1, K_1, \mathfrak{s}_1)
\]

\[
\sim \hat{\text{CF}L}_0(Y_2, K_2, \mathfrak{s}_2),
\]

where \(\rho\) is induced by the isomorphisms \(\hat{\text{CF}L}_{-} \simeq \hat{\text{CF}L}\).

The rest of this section is devoted to a proof of this theorem.

**Proof.** Suppose that \(H_i\) is a Heegaard diagram consisting of a surface \(\Sigma_i\) of genus \(g_i\) and two sets of curves

\[
\alpha^i = \{\alpha^i_1, ..., \alpha^i_{g_i}\}, \quad \text{and}
\]

\[
\beta^i_0 = \{\beta^i_{2g_i}, ..., \beta^i_{2g_i}\},
\]

together with two special curves \(m_i\) and \(l_i\) representing the meridian and the longitude of \(K_i\), for \(i = 1, 2\). Moreover assume that \(l_i\) and \(m_i\) meet each other transversely in a single point \(z_i\).

In order to obtain a Heegaard diagram for \((Y, K)\), connect the surfaces \(\Sigma_1\) and \(\Sigma_2\) by a handle which connects \(z_1\) to \(z_2\). Connect \(l_1\) to \(l_2\) using this handle so that it represents the trivial knot \(K_1 - K_2\). Also, connect \(m_1\) to \(m_2\) so that the resulting closed curve represents the trivial curve \(m_1 + m_2\). Then the resulting surface \(\Sigma\) together with the \(g_1 + g_2\) tuples of curves:

\[
\alpha = \alpha^1 \cup \alpha^2, \quad \text{and}
\]

\[
\beta = \beta^1_0 \cup \{m_1 \# m_2\} = \beta^2_0 \cup \{l_1 \# l_2\} \cup \{m_1 \# m_2\}
\]

will form a Heegaard diagram for \((Y, K)\) with the meridian \(m = m_1 \# m_2\). Denote this Heegaard diagram by \(H_1 \ast H_2\).

We may assume that \(H_1\) and \(H_2\) are any specific Heegaard diagrams for the the knots \((Y_1, K_1)\) and \((Y_2, K_2)\), respectively. In particular, we may assume that \(H_1\) is already modified, so that it is a suitable (allowed) Heegaard diagrams for defining the filtered version of \(\hat{\text{CF}L}(Y_1, K_1)\). We may also assume that \(H_2\) is a weakly admissible diagram, which may be used for a computation of \(\hat{\text{CF}L}(Y_2, K_2)\). It is then implied that \(H = H_1 \ast H_2\) is weakly
admissible. The Heegaard diagram $H$ may still not be an allowed Heegaard diagram for computing $\hat{CFK}(Y,K)$. However, this Heegaard diagram may be used to compute the non-filtered version of $\hat{CFK}(Y,K)$.

The final Heegaard diagram $H$ which defines the complex $\hat{CFK}(Y,K)$ will look like figure 5 around the attached handle. In this figure, the attaching circles of the connecting handle of $\Sigma_1$ and $\Sigma_2$ are denoted by the black circles on the two sides, and each other pair of circles of the same color represents another handle. Moreover, the regular curves denote the elements of $\beta$, while the bold curves represent the elements of $\alpha$.

One may consider another set of $g_1+g_2$ curves, denoted by $\theta$, on the surface $\Sigma$. The set $\theta$ will formally be the union

$$\theta = \beta_0^1 \cup \alpha^2 \cup \{\mu\} = \delta_0 \cup \gamma \cup \{\mu\}$$

$$= \{\delta_2, \ldots, \delta_{g_1}\} \cup \{\gamma_1, \ldots, \gamma_{g_2}\} \cup \{\mu\},$$

where $\gamma$ will consist of Hamiltonian isotopes of the curves in $\alpha^2$ so that the intersections of all pairs of curves are transverse, and consisting of cancelling intersections. Similarly, we assume that the curves in $\delta_0$ are Hamiltonian isotopes of those in $\beta_0^1$. The special curve $\mu$ will be a Hamiltonian isotope of the longitude $l_1$ on $\Sigma_1$ side of the picture, such that it cuts the curve $l_1 \# l_2$ in a pair of cancelling intersection points $z_+$ and $z_-$, and cuts the meridian $m_1 \# m_2$ in a single point $z_0$, and such that all the intersection points are
very close to the attaching circle of the handle connecting \( \Sigma_1 \) to \( \Sigma_2 \). We may choose all the above curves so that the picture is as illustrated in figure 5 with the dotted curves representing the curves in \( \theta \). We choose \( \mu \) so that it stays very close to \( l_1 \) after leaving \( z_+ \) for \( z_- \). Then from \( z_- \) to \( z_0 \) and from \( z_0 \) to \( z_+ \) it stays very close to the \( \alpha \)-curve of figure 5 which we assume to be \( \alpha_1 \). In particular we assume that \( z_- \) and \( z_+ \) are very close to the intersection points in \( \alpha_1 \cap (l_1 \# l_2) \) and that \( z_0 \) is very close to \( \alpha_1 \cap (m_1 \# m_2) \).

Correspondingly, we may consider the triple Heegaard diagram

\[ \overline{\mathcal{H}} = (\Sigma, \alpha, \theta; u, w, v), \]

consisting of the above three sets of curves on \( \Sigma \), together with three marked points \( u, w \) and \( v \). The triple Heegaard diagram \( \overline{\mathcal{H}} \) gives a map \( \mathcal{F} \), through a count of holomorphic triangles associated with this Heegaard diagram \( \overline{\mathcal{H}} \). In order to obtain \( \mathcal{F} \), we will count only those triangles such that their domain has coefficient zero at \( u \) and \( w \), and \( v \). The map \( \mathcal{F} \) goes from the complex

\[ \widehat{CF}(\alpha, \theta) \otimes \widehat{CF}(\theta, \beta) \]

to the complex

\[ \widehat{CF}(\alpha, \beta) = \widehat{CFK}(Y, K). \]

Here, \( \widehat{CF}(\alpha, \theta) \) denotes the chain complex associated with the Heegaard diagram

\[ (\Sigma, \alpha, \theta; u = w, v), \]

where the "\( = \)" signs between \( u \) and \( w \) denotes that they are in the same domain of the Heegaard diagram, when we get rid of \( \beta \).

Similarly, for \( \widehat{CF}(\theta, \beta) \) we will use the Heegaard diagram

\[ (\Sigma, \theta, \beta; u = w, v). \]

The "equality " \( w = u \) is less trivial in this case, compared to the previous equality. Note the difference between the situation here compared to that of the holomorphic triangle map which appeared in the proof of connected sum formulas (9, 10).

For any pair of relative Spin\(^c\) structures \( \mathfrak{s}_1 \in \text{Spin}^c(Y_1, K_1) \) and \( \mathfrak{s}_2 \in \text{Spin}^c(Y_2, K_2) \), there is a unique relative Spin\(^c\) structure

\[ \mathfrak{s} = \mathfrak{s}_1 \# \mathfrak{s}_2 \in \text{Spin}^c(Y, K) \]

which extends \( \mathfrak{s}_1 \) and \( \mathfrak{s}_2 \). This should be compared with the \( \mathbb{Z} \)-many of pairs of relative Spin\(^c\) structures resulting in the same total relative Spin\(^c\) structure, when forming the connected sum of the two knots \( (Y_1, K_1) \) and \( (Y_2, K_2) \).

In order to understand the chain complex \( \widehat{CFK}(Y, K) = \widehat{CF}(\alpha, \beta) \), we will perform a study of the two chain complexes \( \widehat{CF}(\alpha, \theta) \) and \( \widehat{CF}(\theta, \beta) \).
followed by analyzing the effect of the map $F$. These pieces of information will be enough to prove the above theorem.

First of all note that the Heegaard diagram $H^1 = (\Sigma, \alpha, \theta, w, v)$ represents the connected sum $V_1$ of the three-manifold $(Y_1)_0(K_1)$, obtained from $Y_1$ by a zero surgery on the knot $K_1$, with the three manifold $\# g_2 S^1 \times S^2$. For any relative Spin$^c$ structure $s_1 \in \text{Spin}^c(Y_1, K_1)$ we may consider the Spin$^c$ structure $s_1 \# s_0$ on $V_1$, where $s_0$ is the Spin$^c$ structure on $\# g_2 S^1 \times S^2$ with the property that $c_1(s_0)$ represents the trivial homology class. Similar to the methods used in [9, 6], we may couple any generator $x$ of the Heegaard diagram $H_1 = (\Sigma_1, \alpha_1, \beta_1, \gamma, u = w, v)$, corresponding to the relative Spin$^c$ structure $s_1$ (i.e. $s_w(x) = s_1$), with the generator $\Theta_2$ of the complex associated with the Heegaard diagram $(\Sigma_2, \alpha_2, \beta_2, \gamma, u = w)$ corresponding to the top generator of the homology

$$\widehat{HF}(\alpha_2, \gamma) = \widehat{HF}(\# g_2 S^1 \times S^2, s_0),$$

to obtain a corresponding generator of the Heegaard diagram $H_1$ in Spin$^c$ class $s_1 \# s_2$.

As a result, the chain complex $C_1$ associated with the Heegaard diagram $\Pi_1$ may be embedded via the above map into the chain complex $\widehat{CF}(H^1)$:

$$p_1 : C_1(^1) \times_{\Theta_2} \widehat{CF}(H^1, s_1 \# s_0).$$

We are introducing the notation $C(^1)$ for the part of a chain complex $C$ corresponding to the Spin$^c$ class $^1$.

The complex $C_1$ may be identified with the filtered chain complex $\widehat{CFL}_*(Y_1, K_1)$, and under this identification $C_1(^1) = \widehat{CFL}_*(Y_1, K_1, s_1)$.

On the other side, the story is slightly more complicated. Here, in the Heegaard diagram $H^2 = (\Sigma, \theta, \beta, u = w, v)$, we may consider the two sets of curves $\delta_0$ and $\beta^0$, which are Hamiltonian isotopes of one another. As a result, there will be a map to the chain complex associated with $H^2$, from the chain complex associated with the Heegaard diagram

$$\Pi_2 = (\Sigma'_2, \{\mu\} \cup \gamma, \beta_0^2 \cup \{l_1 \# l_2\} \cup \{m_1 \# m_2\}, u = w, v).$$

Here $\Sigma'_2$ is obtained from $\Sigma_2$ by making a connected sum with a torus $T$, in the same way that the surface $\Sigma$ is obtained from $\Sigma_2$ by taking a connected sum of it with $\Sigma_1$. The last two curves $l_1 \# l_2$ and $m_1 \# m_2$ are in fact the curves induced from $\Sigma_1 \# \Sigma_2$ to $T \# \Sigma_2$. The above map will be given by taking a product with the top generator of the Floer homology of $\# g_1^{-1} S^1 \times S^2$ obtained from the $(g_1 - 1)$ tuples of curves $\delta_0$ and $\beta^0$. We will denote the chain complex associated with $\Pi_2$ by $C_2$. 

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**Filtration of Heegaard Floer Homology and Gluing Formulas**

"..."
If we drop the marked points \( u, w \) and \( v \) from the Heegaard diagram \( \mathcal{H}_2 \), we obtain a Heegaard diagram for \((Y_2)_0(K_2)\), and the maps \( s_u \) and \( s_w \) from the generators of \( C_2 \) to \( \text{Spin}^c(Y_2, K_2) \) will differ from each other by a factor \( \text{PD}[m_2] \). In fact for our choice of \( u \) and \( w \) we will have \( s_u(x) = s_w(x) + \text{PD}[m_2] \). We may assign a Spin\(^c\) structure \( s_2 \) to any such generator, using the map \( s_u \), and decompose \( C_2 \) into summands \( C_2(s_2) \) accordingly.

We may re-draw the local configuration of the curves in the Heegaard diagram \( \mathcal{H}_2 \), which results in the picture shown in figure 6. The above discussion shows that there is a second embedding:

\[
p_2 : C_2(s_2) \xrightarrow{\Theta_1 \times} \widehat{CF}(\mathcal{H}_2, \mathfrak{g}_1 \# \mathfrak{g}_2).
\]

We are of course abusing the notation by denoting the special Spin\(^c\) structure of \( #g_1 - 1S^1 \times S^2 \) by the same symbol \( \mathfrak{g}_1 \), which was used for the special Spin\(^c\) structure on \( #g_2 S^1 \times S^2 \).

Composing these embeddings with the count \( F \) of the holomorphic triangles, we obtain a chain map, still denoted by \( F \):

\[
F : (C_1, \mathfrak{g}_1) \otimes (C_2, \mathfrak{g}_2) \longrightarrow \widehat{CFK}(Y, K, \mathfrak{g}_1 \# \mathfrak{g}_2),
\]

\( \mathfrak{g}_1 \in \text{Spin}^c(Y_1, K_1), \mathfrak{g}_2 \in \text{Spin}^c(Y_2, K_2) \).

Any generator of the complex \( C_2 \) will contain an intersection point on the image of the special curve \( \mu \). There are three intersection points between \( \mu \) and the two curves \( \ell = l_1 \# l_2 \) and \( m = m_1 \# m_2 \). The ones on \( \ell \) are denoted by \( z^+ \) and \( z^- \), and the intersection with \( m \) is denoted by \( z_0 \). We choose them so that the trivial disk between the generators \( \{z^+, \bullet\} \) and \( \{z^-, \bullet\} \) goes from the first one to the second. Correspondingly, we may divide the generators of \( \widehat{CF}(\mathcal{H}_2) \) into the groups \( \mathcal{Z}^+, \mathcal{Z}^- \) and \( \mathcal{Z}_0 \).

It is clear that the complex \( C_2 \) may be identified with the filtered complex \( \widehat{CFL}(Y_2, K_2) \) and that there is a natural bijection

\[
j : \widehat{CF}(\mathcal{H}_2) \longrightarrow \widehat{CFL}(Y_2, K_2),
\]

which reduces to maps

\[
j_- : \mathcal{Z}^- \longrightarrow \widehat{CFL}_-(Y_2, K_2),
\]

\[
j_0 : \mathcal{Z}^- \cup \mathcal{Z}_0 \longrightarrow \widehat{CFL}_0(Y_2, K_2) \quad \text{and}
\]

\[
j_+ : \mathcal{Z}^- \cup \mathcal{Z}_0 \cup \mathcal{Z}_+ \longrightarrow \widehat{CFL}_+(Y_2, K_2).
\]

Under the above identifications

\[
C_1 \simeq \widehat{CFL}(Y_1, K_1), \quad \text{and}
\]

\[
C_2 \simeq \widehat{CFL}(Y_2, K_2),
\]

suppose that \( x \otimes y \in C_1 \otimes C_2 \) belongs to \( \widehat{CFL}(Y_1, K_1) \otimes \widehat{CFL}(Y_2, K_2) \).

This means that the generator \( y \) contains one of the intersection points \( z^- \).
or $z_0$, and that the intersection point $A$ of figure 5 is not included in the generator $x$. It is not hard to check, by examining the possible coefficients of the formal domain associated with any holomorphic triangle, that the image $\mathcal{F}(x \otimes y)$ is trivial. In fact, if $\phi$ is a triangle connecting $x \times \Theta_2$, $\Theta_1 \times y$ and $w$, with $w \in \widehat{CFK}(Y, K) = \widehat{CF}(H)$, then $\phi$ can not have non-negative coefficients in all the domains appearing in figure 5 and zero coefficients at $u, w$ and $v$.

As a result of the above discussion we see that $\widehat{CF}_{L_0}(Y_1, K_1) \otimes \widehat{CF}_{L_0}(Y_2, K_2)$ is included in the kernel of the map $\mathcal{F}$. So we may think of $\mathcal{F}$ as a map

$$\mathcal{F} : \frac{\widehat{CF}_{L}(Y_1, K_1) \otimes \widehat{CF}_{L}(Y_2, K_2)}{\widehat{CF}_{L_0}(Y_1, K_1) \otimes \widehat{CF}_{L_0}(Y_2, K_2)} \rightarrow \widehat{CFK}(Y, K).$$

Now suppose that $x_- \otimes y_+$ is a generator in

$$\widehat{CF}_{L_-}(Y_1, K_1) \otimes \widehat{CF}_{L}(Y_2, K_2) \otimes \widehat{CF}_{L_0}(Y_2, K_2)$$

which corresponds to a coupling of a generator $x_-$ of type $C$ and a generator $y_+$ of type $A$. Denote the type $A$ generator of the complex $C_1$ which corresponds to $x_-$, by $x_+$. Similarly, the type $C$ generator of $C_2$ which is associated with $y_+$ will be denoted by $y_-$. We will obtain a generator

$$x_+ \otimes y_- \in \frac{\widehat{CF}_{L_1}(Y_1, K_1)}{\widehat{CF}_{L_0}(Y_1, K_1)} \otimes \widehat{CF}_{L_-}(Y_2, K_2).$$

There is a unique generator $w \in \widehat{CF}(H) \simeq \widehat{CFK}(Y, K)$ with the property that there is triangle connecting $x_- \times \Theta_2$, $\Theta_1 \times y_+$ and $w$, whose domain
consists of $g_1 + g_2$ disjoint small triangles with a total area that may be assumed to be arbitrarily small, if we choose the curves in $\theta$ sufficiently close to the corresponding curves in $\alpha$ and $\beta$. This triangle supports a unique holomorphic representative. Moreover, the same generator $w$ has the extra property that there is another unique holomorphic triangle $\psi$ with small area (energy) that connects $x_+ \times \Theta_2$, $\Theta_1 \times y_-$ and $w$. As a result

$$ \mathcal{F}(x_- \otimes y_+) = w + \text{significantly lower energy terms}, $$

and

$$ \mathcal{F}(x_+ \otimes y_-) = w + \text{significantly lower energy terms}. $$

Using the arguments of Ozsváth and Szabó on the energy filtration ([2], section 6), we may use this correspondence to modify the isomorphism

$$ \rho : \widehat{CFL}_-(Y_1, K_1) \otimes \widehat{CFL}(Y_2, K_2) \rightarrow \frac{\widehat{CFL}(Y_1, K_1)}{\widehat{CFL}_0(Y_1, K_1)} \otimes \frac{\widehat{CFL}_-(Y_2, K_2)}{\widehat{CFL}_0(Y_2, K_2)} $$

by a chain homotopy, so that for any

$$ z = x_- \otimes y_+ \in \frac{\widehat{CFL}_-(Y_1, K_1)}{\widehat{CFL}_0(Y_1, K_1)} \otimes \frac{\widehat{CFL}_-(Y_2, K_2)}{\widehat{CFL}_0(Y_2, K_2)}, $$

we have $\mathcal{F}(z - \rho(z)) = 0$.

With this observation in hand, it is now enough to show that the map $\mathcal{F}$, induced on the quotient complex

$$ \mathcal{C} = \left[ \frac{\widehat{CFL}(Y_1, K_1) \otimes \widehat{CFL}(Y_2, K_2)}{\widehat{CFL}_0(Y_1, K_1) \otimes \widehat{CFL}_0(Y_2, K_2)} \right] / \sim $$

to $\widehat{CFK}(Y, K)$ is a bijection, where $\sim$ is induced by the modified isomorphism $\rho$.

But this is quite easy. The map $\mathcal{F}$ is of the form

$$ \mathcal{F} = \mathcal{G} + \text{significantly lower energy terms}, $$

where the map $\mathcal{G}$ is defined from

$$ \left[ \frac{\widehat{CFL}(Y_1, K_1) \otimes \widehat{CFL}(Y_2, K_2)}{\widehat{CFL}_0(Y_1, K_1) \otimes \widehat{CFL}_0(Y_2, K_2)} \right] / \sim $$

$$ \left[ \frac{\widehat{CFL}(Y_1, K_1) \otimes \widehat{CFL}(Y_2, K_2)}{\widehat{CFL}_0(Y_1, K_1) \otimes \widehat{CFL}_0(Y_2, K_2)} \right] \oplus \left[ \frac{\widehat{CFL}(Y_1, K_1) \otimes \widehat{CFL}_0(Y_2, K_2)}{\widehat{CFL}_0(Y_1, K_1) \otimes \widehat{CFL}_0(Y_2, K_2)} \right] $$

to $\widehat{CFK}(Y, K)$ as follows. The first summand corresponds to a coupling of an arbitrary element $x$ of $\widehat{CFL}(Y_1, K_1)$, with an element $y$ of $\mathcal{Z}_+$. There is a unique generator $w = \mathcal{G}(x \otimes y)$ of $\widehat{CFK}(Y, K)$ such that the three generators $x \otimes \Theta_2, \Theta_1 \times y$ and $w$ are connected by a unique triangle $\phi$, whose domain $\mathcal{D}(\phi)$ consists of small triangles of arbitrarily small area, and such
that \( \phi \) supports a holomorphic disk.

The definition of \( G \) on the second summand is done in a quite similar fashion. The argument which finds the multiplicities of the connecting disk is slightly more complicated. An examination of possible multiplicities for the domain of such a triangle, in the part of surface that appears in figure and just another game of trying to keep the multiplicities positive, will find two candidates for \( w = G(x \otimes y) \), where

\[
x \otimes y \in \frac{\text{CFL}(Y_1, K_1)}{\text{CFL}_0(Y_1, K_1)} \otimes \frac{\text{CFL}_0(Y_2, K_2)}{\text{CFL}_-(Y_2, K_2)}.
\]

However, only one of them has the property that the triangle associated with it has an arbitrarily small area, once \( \mu \) is chosen as it was described before.

It is not hard to check that the map \( G \) is surjective. From the construction it is clear that our claim is satisfied, i.e.

\[
\mathcal{F} = G + \text{significantly lower energy terms}.
\]

Now the argument of Ozsváth and Szabó in \cite{9, 6} may be followed to conclude that \( \mathcal{F} \) is surjective and injective from \( \mathcal{C} \) to \( \hat{\text{CFK}}(Y, K) \). This completes the proof of the theorem.

5. Gluing along the knots II

We may consider several other gluing constructions along the knots, as well as the operation of taking the connected sum of \( (Y_1, K_1) \) with \( (Y_2, K_2) \), and look into the complexes \( \hat{\text{CFL}}(Y, K) \) and \( \hat{\text{CFK}}(Y, K) \) of the resulting knot. Now that we have treated the above case carefully, it is not hard to state an prove some similar results.

The first of these results, is the effect of connected sum on \( \hat{\text{CFL}} \). We already know that (see \cite{6}) if \( (Y_1, K_1) \) and \( (Y_2, K_2) \) are null homologous knots and if \( (Y, K) \) is obtained by taking the connected sum of them, then for any relative \( \text{Spin}^c \) structure \( \mathfrak{g} \in \text{Spin}^c(Y, K) \) there are \( \mathbb{Z} \)-many of pairs

\[
\mathfrak{g}_1 \# \mathfrak{g}_2 \in \text{Spin}^c(Y_1, K_1) \oplus \text{Spin}^c(Y_2, K_2)
\]

which extend \( \mathfrak{g} \). The complex \( \hat{\text{CFK}}(Y, K) \) may be described as

\[
\hat{\text{CFK}}(Y, K, \mathfrak{g}) = \bigoplus_{\mathfrak{g} \in \text{Spin}^c(Y, K), \mathfrak{g}_1 \# \mathfrak{g}_2 \text{ extends } \mathfrak{g}} \hat{\text{CFK}}(Y_1, K_1, \mathfrak{g}_1) \otimes \hat{\text{CFK}}(Y_2, K_2, \mathfrak{g}_2).
\]

We claim that \( \hat{\text{CFL}}(Y, K) \) is given by the following theorem.
**Theorem 5.1.** If \((Y, K)\) is obtained as the connected sum of the two knots \((Y_1, K_1)\) and \((Y_2, K_2)\), and if \(\mathfrak{s} \in \text{Spin}^c(Y, K)\) is a relative Spin\(^c\) structure on \((Y, K)\), then

\[
\widehat{\text{CFL}}(Y, K, \mathfrak{s}) = \left[ \bigoplus_{\mathfrak{s}_1 \in \text{Spin}^c(Y_1, K_1)} \mathcal{C}(\mathfrak{s}_1), \mathfrak{s}_2 \right]_{\mathfrak{s}_1 \# \mathfrak{s}_2 \text{ extends } \mathfrak{s}} \]

\[
\mathcal{C}(\mathfrak{s}_1, \mathfrak{s}_2) = \frac{\widehat{CFK}(Y_1, K_1, \mathfrak{s}_1) \otimes \widehat{CFK}(Y_2, K_2, \mathfrak{s}_2)}{\widehat{CFK}_0(Y_1, K_1, \mathfrak{s}_1) \otimes \widehat{CFK}_0(Y_2, K_2, \mathfrak{s}_2)}
\]

\[
\widehat{CFK}^{-}(Y_1, K_1, \mathfrak{s}_1) \otimes \frac{\widehat{CFK}(Y_2, K_2, \mathfrak{s}_2)}{\widehat{CFK}_0(Y_2, K_2, \mathfrak{s}_2)} \sim \frac{\widehat{CFK}(Y_1, K_1, \mathfrak{s}_1 + PD[m_1])}{\widehat{CFK}_0(Y_1, K_1, \mathfrak{s}_1 + PD[m_1])} \otimes \frac{\widehat{CFK}^{-}(Y_2, K_2, \mathfrak{s}_2 - PD[m_2])}{\widehat{CFK}^{-}(Y_2, K_2, \mathfrak{s}_2 - PD[m_2])}.
\]

Here the isomorphisms inducing \(\sim\) are coming from the connecting isomorphisms \(\partial' : \frac{\widehat{CFK}}{\widehat{CFK}_0} \to \widehat{CFK}^{-}\).

**Proof.** The proof is almost identical to the proof of the above theorem. In fact, if we change the role of the meridians \(m_1\) and \(m_2\) with the longitudes \(l_1\) and \(l_2\), then the Heegaard diagram \(H = H_1 \ast H_2\) which was used in the proof of theorem 4.1 will compute the longitude Floer homology \(\widehat{CFL}(Y, K)\), where \((Y, K) = (Y_1, K_1) \# (Y_2, K_2)\). The allowed Heegaard diagrams \(H_1\) and \(H_2\), and also \(H_2\) will compute the filtered chain complexes \(\widehat{CFK}(Y_1, K_1)\) and \(\widehat{CFK}(Y_2, K_2)\). The equivalence relation \(\sim\) will come from, instead of the isomorphism between the type A and type C generators of \(\widehat{CFL}\), the isomorphism between type A and type C generators of \(\widehat{CFK}(Y_i, K_i)\). This last isomorphism changes the Spin\(^c\) structure within each of the complexes \(\widehat{CFK}(Y_i, K_i)\). That is the reason for the appearance of the shifts by \(+PD[m_1]\) and \(-PD[m_2]\), which does not change the total relative Spin\(^c\) structure \(\mathfrak{s} \in \text{Spin}^c(Y, K)\), since the total change is by the Poincaré dual of \([m_1 - m_2]\).

Since the effect of a connected sum on the longitude theory is so closely related to the effect of the gluing along the knots on the standard complex \(\widehat{CFK}\), we may expect that the converse is also true. In fact, if we drop our filtration of the complex \(\widehat{CFL}\), introduced in this paper, we get the following proposition.

**Proposition 5.2.** Suppose that the knot \((Y, K)\) is obtained from \((Y_1, K_1)\) and \((Y_2, K_2)\), by gluing the knot complements \(Y_1 \setminus nd(K_1)\) and \(Y_2 \setminus nd(K_2)\) along their boundary and identifying the longitudes (resp. meridians) of \(K_1\) and \(K_2\) (i.e. we have \((Y, K) = (Y_1, K_1) \# (Y_2, K_2)\)). Then for any

\[
\mathfrak{s} = \mathfrak{s}_1 \# \mathfrak{s}_2 \in \text{Spin}^c(Y, K) = \text{Spin}^c(Y_1, K_1) \oplus \text{Spin}^c(Y_2, K_2)
\]
we will have
\[
\widehat{\text{CFL}}(Y,K,\mathfrak{a}) = \widehat{\text{CFL}}(Y_1,K_1,\mathfrak{a}_1) \otimes \widehat{\text{CFL}}(Y_2,K_2,\mathfrak{a}_2).
\]

**Proof.** Again the proof is almost identical to the proof of the connected sum formula for \(\widehat{\text{CFK}}(Y,K)\) in [6]. In the Heegaard diagram \(H = H_1 \ast H_2\) of theorem 4.1, \(m = m_1 \# m_2\) will be the meridian of the knot \(K\), and a parallel copy \(l'_1\) of \(l_1\) may be used as the longitude of \(K\). Thus, a Heegaard diagram for computing \(\widehat{\text{CFL}}(Y,K)\) may be obtained by trading \(m\) for \(l'_1\) in the Heegaard diagram \(H\). Up to a change in the names, this is similar to the Heegaard diagram used in section 7 of [6]. The argument of Ozsváth and Szabó may be followed word by word.

The next thing we may derive without any extra effort is a gluing formula for the following construction. Suppose that \((Y_1,K_1)\) and \((Y_2,K_2)\) are as before. We may glue \(Y_1 \setminus \text{nd}(K_1)\) to \(Y_2 \setminus \text{nd}(K_2)\) along their torus boundary, in such a way that the meridian \(m_1\) of \(K_1\) is identified with the longitude \(l_2\) of \(K_2\) and vice versa. The knot \(K_1\) will induce a knot \(K\) inside the threemanifold \(Y\) obtained as above. We will denote the result of this construction by \((Y_1,K_1) \perp (Y_2,K_2) = (Y,K)\). The following result describes \(\widehat{\text{CFK}}(Y,K)\) and \(\widehat{\text{CFL}}(Y,K)\).

**Theorem 5.3.** Suppose that \((Y,K) = (Y_1,K_1) \perp (Y_2,K_2)\) as above. Then
\[
\text{Spin}^c(Y,K) \oplus \mathbb{Z} = \text{Spin}^c(Y_1,K_1) \oplus \text{Spin}^c(Y_2,K_2),
\]
with \(\mathbb{Z}\) being generated by \(\text{PD}[m_2]\). Furthermore, for any relative \(\text{Spin}^c\) structure \(\mathfrak{a} \in \text{Spin}^c(Y,K)\) we have
\[
\widehat{\text{CFL}}(Y,K,\mathfrak{a}) = \bigoplus_{\mathfrak{a} \in \text{Spin}^c(Y_1,K_1) \atop (\mathfrak{a}_1,\mathfrak{a}_2) \text{ induce } \mathfrak{a}} \widehat{\text{CFL}}(Y_1,K_1,\mathfrak{a}_1) \otimes \widehat{\text{CFK}}(Y_2,K_2,\mathfrak{a}_2),
\]
\[
\widehat{\text{CFK}}(Y,K,\mathfrak{a}) = \left[ \bigoplus_{\mathfrak{a} \in \text{Spin}^c(Y_1,K_1) \atop (\mathfrak{a}_1,\mathfrak{a}_2) \text{ induce } \mathfrak{a}} C(\mathfrak{a}_1,\mathfrak{a}_2) \right],
\]
\[
C(\mathfrak{a}_1,\mathfrak{a}_2) = \frac{\widehat{\text{CFL}}(Y_1,K_1,\mathfrak{a}_1) \otimes \widehat{\text{CFK}}(Y_2,K_2,\mathfrak{a}_2)}{\widehat{\text{CFL}}_0(Y_1,K_1,\mathfrak{a}_1) \otimes \widehat{\text{CFK}}_0(Y_2,K_2,\mathfrak{a}_2)}.
\]

As before the equivalence relation \(\sim\) is induced by the connecting isomorphisms \(\widehat{\text{CFK}} \sim \frac{\widehat{\text{CFK}}}{\text{CFK}_0}\) and \(\widehat{\text{CFL}} \sim \frac{\widehat{\text{CFL}}}{\text{CFL}_0}\).
Proof. If $H_1$ and $H_2$ are Heegaard diagrams for $(Y_1,K_1)$ and $(Y_2,K_2)$ respectively, we may connect them by a one-handle which is attached to the surfaces at the intersection of the meridian and the longitude in each diagram. If we take the two connected sums of curves $m_1 \# l_2$ and $l_1 \# m_2$ using this one-handle, in such a way that they stay disjoint from each other, the result will be a Heegaard diagram for $Y$. The meridian of $K$ in this Heegaard diagram may be thought of as the curve $m_1 \# l_2$, and the longitude may be represented by a parallel copy of $l_1$ that stays on $\Sigma_1$. It is now clear that for a computation of $\hat{CFL}(Y,K)$ one can follow the argument of Ozsváth and Szabó, and that for a computation of $\hat{CFK}(Y,K)$, the proof of theorem 4.1 may be followed almost word by word.

6. Examples; Alternating knots

We have shown in the previous sections how our new $\{-,0,\}$ filtrations of the complexes $\hat{CFK}(Y,K)$ and $\hat{CFL}(Y,K)$ play a major role in gluing formulas. This role seems to be enough of a justification for introducing them. However, the next question is how hard are these filtered complexes to compute? The computations of Ozsváth and Szabó ([7]) and Rasmussen ([14, 15]) illustrated how well-behaved the alternating knots, and more generally the perfect knots, are for the purpose of Floer homology computations. So, it is natural to turn to the computation of the $\{-,0,\}$-filtered complexes associated with an alternating knot in $S^3$ (or more generally, perfect knots) as the first computation.

In this section we will first compute the complexes $\hat{CFL}_*(S^3,K)$ for an alternating knot $(S^3,K)$:

**Theorem 6.1.** Suppose that $(S^3,K)$ is an alternating knot in $S^3$. Then the filtered chain complex $\hat{CFL}_*(S^3,K)$ is completely determined by the Alexander polynomial $\Delta_K(t)$ and the signature $\sigma(K)$.

**Proof.** The proof is quite similar to our computation of $\hat{CFL}(T(2,2n+1))$, where $T(2,2n+1)$ is the $(2,2n+1)$ torus knot (c.f. [2]). Consider a standard Heegaard diagram

\[ H_0 = (\Sigma', \alpha, \beta_0 \cup \{\beta_1\}, z) \]

for $(S^3,K)$, which is obtained from an alternating projection of the knot $K$. The longitude $l$ of $(S^3,K)$ may be realized as a curve which does not cut any of the curves in $\beta_0$, but it cuts the meridian $m = \beta_1$ exactly once at the marked point $z$. We may spin $l$ along the meridian $m$ for a large number $N$ of times to obtain a Heegaard diagram which is more appropriate for our purposes. Now, add a new one-handle to the Heegaard diagram so that an allowed Heegaard diagram for the computation of $\hat{CFL}(K)$ is obtained. There are two new curves added to the Heegaard diagram, which will be
denoted by \( \lambda \) (the new \( \alpha \) curve) and \( \mu \) (the new \( \beta \) curve, obtained from \( m \)). Thus we get the Heegaard diagram

\[
H = (\Sigma, \alpha \cup \{\lambda\}, \beta_0 \cup \{\mu\} \cup \{l\}, z).
\]

The final picture will locally look like figure 7.

Let us denote the set of generators of the alternating projection of \( K \) by \( Z \). The elements of \( Z \) are in one-to-one correspondence with the generators of the complex associated with the initial Heegaard diagram \( H_0 \). In \( H_0 \) there is a unique \( \alpha \) circle which cuts \( m \). We may assume that this curve is named \( \alpha_1 \). Spinning \( l \) along \( m \) creates \( 2N \) new intersections between \( l \) and \( \alpha_1 \). Denote these intersection points by \( x_0, x_1, \ldots, x_{N-1} \) on the right, and by \( y_0, y_1, \ldots, y_{N-1} \) on the left. As usual, denote the intersection points on \( \lambda \) by \( A, B \) and \( C \), and correspondingly divide the generators of the chain complex into groups \( A, B \) and \( C \). Finally, denote the unique intersection point of \( \mu \) and \( \alpha_1 \) by \( x \).

Spinning of \( l \) along \( m \) implies (c.f. [9, 6, 14, 2]) that the generators in the relevant Spin\(^c\) structures should contain one of the intersection points \( x, x_0, \ldots, x_{N-1} \), or one of \( y_0, y_1, \ldots, y_{N-1} \). Let us denote the relevant generators in \( A, B \) and \( C \), by \( A_0, B_0 \) and \( C_0 \) respectively. Then it is easy to check that

\[\begin{array}{ccc}
0 & 1 & 2 \\
\end{array}\]

\[\begin{array}{c}
w \\
A \\
B \\
C \\
u \\
\end{array}\]

\[\begin{array}{cccc}
y_2 & y_1 & y_0 & x_0 \\
x_1 & x_2 & x_0 & x_1 \\
\end{array}\]

**Figure 7.** The longitude of the alternating knot \( K \) may spin along the meridian several times to generate the intersection points \( x_0, x_1, \ldots \) and \( y_0, y_1, \ldots \) with \( \alpha_1 \). Then by adding a handle, we may construct an allowed Heegaard diagram for the computation of \( \hat{CF}L_* (S^3, K) \).
we may make the identifications
\[ A_0 = \left\{ \{A, x, z\} \mid z \in \mathbb{Z} \right\}, \]
\[ C_0 = \left\{ \{C, x, z\} \mid z \in \mathbb{Z} \right\}, \]
and
\[ B_0 = \left\{ \{B, x_i, z\}, \{B, y_i, z\} \mid z \in \mathbb{Z}, \quad i = 0, 1, \ldots, N - 1 \right\}. \]

Suppose that \( F \) is a Seifert surface for the knot \( K \), in \( S^3 \). Such a surface may be obtained by capping off in \( S^3 \setminus \text{nd}(K) \) a periodic domain \( D \) for \( K \) which has a zero coefficient at \( w \) and coefficient 1 at \( u \). We have \( D = D_1 - D_2 \), where \( D_1 \) is the small domain containing \( u \). Then the difference between the Maslov gradings of the two generators \( \{A, x, z\} \) and \( \{C, x, z\} \) may be computed as follows. If we had illegally used the domain \( D_1 \) to connect these two generators, the difference between the Maslov gradings would have been one. In reality, we have to use the more complicated domain \( D_2 \). Thus
\[ \mu(\{A, x, z\}) - \mu(\{C, x, z\}) = \mu(D_2) = 1 - \langle c_1(s(z)), F \rangle. \]

The number \( \langle c_1(s(z)), F \rangle \) is precisely \( 2s(z) \), where \( s(z) \) is the \( \mathbb{Z} \)-grading induced by the knot \( K \) on the complex. Since \( K \) is alternating and the diagram \( H_0 \) is obtained from an alternating projection, \( s(z) = \mu(z) - \frac{\sigma(K)}{2} \), where \( \sigma(K) \) denotes the signature of the knot \( K \) (see [7]). Here we are already using the fact that the the Spin\(^c\) structure
\[ s(\{A, x, z\}) \in \text{Spin}^c(S^3, K) = \text{Spin}^c(S_0^3(K)) \]
is the same as the Spin\(^c\) structure \( s(z) \) assigned to \( z \) via the Heegaard diagram \( H_0 \).

In fact the Spin\(^c\) structure assigned by the map \( s = s_u \) to the above generators may be computed via the formulas
\[ s(\{A, x, z\}) = s(z), \]
\[ s(\{C, x, z\}) = s(z), \quad \text{and} \]
\[ s(\{B, x_i, z\}) = s(\{B, y_i, z\}) = s(z) - i\text{PD}[m], \]
where \( \text{PD}[m] \) denotes the Poincaré dual of the meridian \( m \) of the knot.

A consideration similar to that of section 3 of [2] shows that the Maslov grading levels of \( \{B, x_i, z\} \) and \( \{B, y_i, z\} \) are independent of \( i \). Also, because of the trivial disks between \( \{B, x_0, z\} \) and \( \{A, x, z\} \), and between \( \{B, y_0, z\} \) and \( \{C, x, z\} \) we know that their Maslov gradings differ by one. We may summarize all this information as
\[ \mu(\{A, x, z\}) = 1 + \sigma(K) - \mu(z), \]
\[ \mu(\{C, x, z\}) = \mu(z), \]
\[ \mu(\{B, x_i, z\}) = \sigma(K) - \mu(z), \quad \text{and} \]
\[ \mu(\{B, y_i, z\}) = \mu(z) - 1. \]
The boundary maps between the generators of $A_0$ and between the
generators of $C_0$ are exactly the boundary maps between the
generators in $Z$, considered as the generators of the complex associated with $H_0$. The boundary maps going from $A_0$ to $B_0$ will give a map on the chain complex which is chain homotopic to the map sending $\{A, x, z\}$ to $\{B, x_0, z\}$. Similarly, the map going from $B_0$ to $C_0$ is chain homotopic to the map sending $\{B, y_0, z\}$ to $\{C, x, z\}$. In order to complete the computation, we should study the maps within the generators of type $B$, i.e. elements of $B_0$.

We remind the reader of the symmetry within the non-filtered complex $\hat{CFL}(K)$, which was proved in [2]. If we re-consider the Spin$^c$ class of the generator $x$ as the average $\frac{1}{2}(s_u(x) + s_w(x))$ of the Spin$^c$ structures in Spin$^c(S^3, K)$ associated to $x$ via the maps $s_u$ and $s_w$, then we have shown in [2] that the non-filtered complexes $\hat{CFL}(S^3, K, s)$ and $\hat{CFL}(S^3, K, -s)$ are in fact isomorphic, where $s \in \frac{1}{2}\text{PD}[m] + \text{Spin}^c(S^3, K)$). Here $m$ is the meridian of $K$, and $-s$ denotes the element with the property that formally $c_1(s) + c_1(-s) = 0$.

Basically the same argument may be applied to show the symmetry of the filtered version in the presence of the generators of different types $A, B$, and $C$. It is thus enough, as we did in the computation of $\hat{CFL}(S^3, K)$ for the $(2, 2n + 1)$ torus knot $K = T(2, 2n + 1)$ in $S^3$, to compute the complexes $\hat{CFL}(K, s)$ for those elements $s \in \frac{1}{2}\text{PD}[m] + \text{Spin}^c(S^3, K))$ with $c_1(s) > 0$. Note that here we are taking the formal first Chern class of an element of the form

$$\frac{1}{2}\text{PD}[m] + s \in \frac{1}{2}\text{PD}[m] + \text{Spin}^c(S^3, K) \simeq \frac{1}{2}\text{PD}[m] + \text{Spin}^c(S^3, K))$$

to be the cohomology class with rational coefficients which evaluates on a capped Seifert surface $F$ of $K$ to give $\frac{1}{2} + \langle c_1(s), F \rangle$.

In this regard, it is sufficient to compute the whole filtered complex for the positive Spin$^c$ classes $s$ (i.e. those with $\langle c_1(s), F \rangle > 0$).

When this is the case, there are no boundary maps going from the generators of the form $\{B, x_i, z\}$ to the generators of the form $\{B, y_j, w\}$. This is because the Maslov gradings of the generators of the first type are always less than the Maslov gradings of the generators of the second type, as long as these two generators are in the same Spin$^c$ class, and the associated Spin$^c$ class $s$ is a positive one. We refer the reader to [2] for a more detailed argument.

On the other hand, for orientational reasons, the domain of any possible boundary map going from $\{B, y_j, w\}$ to $\{B, x_i, z\}$ has to have negative coefficients in the cylindrical region of figure 7 if both $i$ and $j$ are small,
compared to the number of twists (which may be chosen to be arbitrarily large). The result of this discussion is that the only boundary maps we have to understand for the positive Spin\(^c\) structures \(s\), are the boundary maps within the generators of the form \(\{B, x_i, z\}\) (which we call the generators of type BX), as well as the boundary maps within the generators of the form \(\{B, y_j, w\}\) (which will be called generators of type BY).

Consider the Heegaard diagram obtained from \(H_0\), by replacing the meridian by the longitude \(l_N = l + N.m\), which is obtained from \(l\) by spinning it \(N\) times around the meridian \(m\). Let us denote this Heegaard diagram by \(H_N\):

\[
H_N = (\Sigma', \alpha, \beta_0 \cup \{l_N\}, u).
\]

Here \(u\) denotes one of the marked points of the previous Heegaard diagram \(H_0\). Note that the Floer homology of this Heegaard diagram computes \(\tilde{CF}(S^3_N(K))\), of the three-manifold obtained from \(S^3\) by a \(N\)-surgery on the knot \(K\).

We may re-draw this Heegaard diagram around the marked point \(u\), so that the picture looks like figure 8. The curve \(l_N\) will be the curve spinning around the center, and the unique \(\alpha\)-circle which cuts \(m\) in a single point, is the bold curve cutting \(l_N\) in several (in fact \(N\)) intersection points \(z_0, z_1, ..., z_{N-1}\). Again we assume that \(z_0\) is the closest to the center of the spiral.

If we consider a sufficiently large number \(N\), then except for a fixed number \(k\) (independent of \(N\)) of the Spin\(^c\) classes \(t \in \text{Spin}^c(S^3_N(K))\), all the

![Figure 8](image-url)

**Figure 8.** If we redraw the diagram \(H_N\), around the marked point \(u\), there will be the spined longitude \(l_N\) cutting \(\alpha_1\) in several points \(z_0, z_1, ..., \), and cutting the meridian \(m\) in \(x\). \(\alpha_1\) is denoted by the bold curve, and \(m\) by the dotted curve.
generators in the Spin\(^c\) class \(t\) will contain an intersection point \(z_i\) on \(l_N\). In fact, one of these Spin\(^c\) classes will be generated by the elements

\[
\{z_{g+s(z)}, y\} \mid z = \{x, y\} \in \mathcal{Z},
\]

where \(g\) is the genus of the alternating knot \(K\).

Note that the generator \(\{z_i, y\}\) is quite similar to the generator corresponding to \(\{B, y_i, z\}\) of type BY. In fact the complex generated by the generators of type BY, and with the boundary map induced from \(\widehat{CFL}(K)\), is isomorphic to the complex generated by the generators of \(\widehat{CF}(S^3_N(K))\) of the above form.

We will first try to understand the part of \(\widehat{CF}(S^3_N(K))\) in the Spin\(^c\) class \(t\) considered above. Then we will use this information to understand the boundary maps within the generators of type BX and within the generators of type BY.

Denote by \(\beta_N\) the set of simple closed curves \(\beta_0 \cup \{l_N\}\), and let \(\gamma\) be another set of simple closed curves obtained as the union of Hamiltonian isotopes \(\gamma_0\) of \(\beta_0\), with the single curve \(\{m\}\). Consider the triple Heegaard diagram

\[
\Pi_N = (\Sigma', \alpha, \gamma, \beta_N; u),
\]

and the corresponding chain map

\[
\mathcal{F} : \widehat{CF}(\alpha, \gamma) \otimes \widehat{CF}(\gamma, \beta_N) \rightarrow \widehat{CF}(\alpha, \beta_N),
\]

obtained by a count of holomorphic triangles associated with the Heegaard diagram \(\Pi_N\). Note that the chain complex \(\widehat{CF}(\alpha, \gamma)\) is the filtered chain complex \(\widehat{CF}(S^3)\), which is filtered using the alternating knot \(K\). Moreover the chain complex \(\widehat{CF}(\gamma, \beta_N)\) computes the Floer homology of the three-manifold \(\#^{g-1}(S^1 \times S^2)\). Let \(\Theta\) denote the top generator of this homology group in the complex \(\widehat{CF}(\gamma, \beta_N)\). Moreover, denote the part of \(\mathcal{F}(x \otimes \Theta)\) in Spin\(^c\) class \(t\) by \(\mathcal{F}_t(x)\). This way, we get a chain map

\[
\mathcal{F}_t : \widehat{CF}(K, S^3) \rightarrow \widehat{CF}(S^3_N(K), t),
\]

where we use \(\widehat{CF}(K, S^3)\) to emphasize that our complex \(\widehat{CF}(S^3)\) admits a filtration coming from the knot \(K\).

It is quite straightforward to check, using the energy filtration of [9], that for any generator \(z = \{x, y\}\) of the complex \(\widehat{CF}(K, S^3)\) we have

\[
\mathcal{F}_t(z) = \{z_{g+s(z)}, y\} + \text{lower energy terms}, \tag{4}
\]

This is enough to show that the map \(\mathcal{F}_t\) induces an isomorphism of the chain complexes. Moreover, note that using the isomorphism \(\mathcal{F}_t\) and the filtration of \(\widehat{CF}(S^3)\) by \(K\), we may induce a \(\mathbb{Z}\)-filtration on the target complex.
\( \widehat{CF}(S^3_N(K), t) \). Using equation \([4]\) it is not hard to show that this filtration, and the filtration given by

\[
\text{filtration level of } \{z_i, y\} = \text{filtration level of } \{x, y\},
\]
give isomorphic filtered chain complexes.

Let us denote by \( G^{-}(K, \ell) \) the subcomplex of \( \widehat{CF}(K, S^3) \) generated by the generators in filtration levels less than \( \ell \in \mathbb{Z} \). Denote the quotient complex by

\[
G^{+}(K, \ell) = \frac{\widehat{CF}(K, S^3)}{G^{-}(K, \ell)}.
\]

For a positive Spin\(^c\) structure \( s \), the generators of type BY which correspond to \( s \) are precisely those \( \{B, y_i, z\} \) such that \( s(z) - i\text{PD}[m] = s \). If we use the isomorphisms

\[
\mathbb{Z} \simeq \text{Spin}^c(S^3_0(K)), \quad \text{and}
\]

\[
\frac{1}{2} + \mathbb{Z} \simeq \frac{1}{2}\text{PD}[m] + \text{Spin}^c(S^3_0(K)),
\]

and denote the images of \( s(z) \) and \( s \) in \( \mathbb{Z} \) and \( \frac{1}{2} + \mathbb{Z} \) under these isomorphism by \( s(z) \) and \( s \) respectively, then the relevant generators of type BY are precisely

\[
\left\{ \{B, y_{s(z)} - s - \frac{1}{2}, z\} \mid z \in \mathbb{Z}, \ s(z) > s \right\}.
\]

The corresponding complex generated using the generators of \( \widehat{CF}(S^3_N(K), t) \) will be \( G^{+}(K, s + \frac{1}{2}) \) and in fact we see that this correspondence is an isomorphism of the chain complexes.

Similarly, we may prove that the generators of type BX in the Spin\(^c\) class corresponding to \( s \in \frac{1}{2} + \mathbb{Z} \) generate a complex isomorphic to \( G^{-}(K, -s + \frac{1}{2}) \). Since we already know the differentials between the generators of different types, and all the complexes involved in this computation may be determined using the filtered chain complex \( \widehat{CFK}(K, S^3) \), we may argue as in \([7, 15]\) that the Alexander polynomial \( \Delta_K(t) \) of \( K \), and its signature \( \sigma(K) \), determine the \( \{-, 0, +\}\)-filtered chain complex \( \widehat{CFL}_\bullet(S^3, K) \).

In fact, if we continue to show the part of the \( \mathbb{Z} \)-filtered complex \( \widehat{CF}(S^3) \) in filtration level less than \( \ell \) by \( G^{-}(K, \ell) \) and its quotient complex by

\[
G^{+}(K, \ell) = \frac{\widehat{CF}(K, S^3)}{G^{-}(K, \ell)},
\]

then we may explicitly determine the complex \( \widehat{CFL}_\bullet \) as is stated in the following proposition.
Proposition 6.2. Suppose that $s \in 1/2 + \mathbb{Z}$ represents a Spin$^c$ structure

$$s \in 1/2 \text{PD}[m] + \text{Spin}^c(S^3_0(K)) \cong 1/2 + \mathbb{Z}$$

for an alternating knot $K$, and suppose that the complexes $C_\bullet(s)$ for $\bullet \in \{-,0,+\}$ are defined as

$$C_-(s) = \frac{G^-(K,-s+1/2)}{G^-(K,-s-1/2)}$$

$$C_0(s) = \left[ \frac{G^-(K,-s+1/2)}{G^+(K,s+1/2)} \right] \bigoplus \left[ G^-(K,-s+1/2) \oplus G^+(K,s+1/2) \right]$$

$$C_+(s) = \left[ \frac{G^-(K,-s+1/2)}{G^+(K,s+1/2)} \right] \bigoplus \left[ G^-(K,-s+1/2) \oplus G^+(K,s+1/2) \right]$$

$$\bigoplus \left[ G^-(K,s+1/2) \bigoplus G^+(K,s+1/2) \right].$$

Then the complex $\widehat{CFL}_\bullet(S^3,K,s)$ is given by $C_\bullet(s)$, with its boundary map modified by adding the maps

$$d_{0-} : C_0 \longrightarrow C_-$$

$$G^-(K,-s+1/2) \xrightarrow{d_{0-}} \frac{G^-(K,-s+1/2)}{G^-(K,-s-1/2)}, \text{ and}$$

$$d_{+0} : C_+ \longrightarrow C_0$$

$$\frac{G^-(K,s+3/2)}{G^-(K,s+1/2)} \xrightarrow{d_{+0}} G^+(K,s+1/2) = \frac{\widehat{CF}(K,S^3)}{G^-(K,s+1/2)}.$$

Here $\bullet \in \{-,0,+\}$.

Proof. The proof is just a simple algebraic argument, once we use our information on the structure of the boundary maps between different types of generators in positive Spin$^c$ structures, which was obtained in the proof of theorem 6.1. The symmetry with respect to the Spin$^c$ structures implies the theorem in general.

We may modify the Heegaard diagram $H_0$ in a different way to obtain an allowed Heegaard diagram for the computation of $\{-,0,+\}$-filtered complex $\widehat{CFK}(S^3,K)$. Namely, we may spin the longitude along the meridian several times, and then by adding a one handle and a pair of curves to the diagram, change it to an allowed Heegaard diagram for the computation of $\widehat{CFK}_\bullet(S^3,K)$.

An argument almost identical to the argument used for proving theorem 6.1 shows the following:
Theorem 6.3. Suppose that \((S^3,K)\) is an alternating knot in \(S^3\), and let \(\Delta_K(t)\) and \(\sigma(K)\) denote the Alexander polynomial and the signature of the knot \(K\). Then the homotopy type of the filtered chain complex \(\hat{CFK}_\bullet(S^3,K)\) is completely determined by \(\Delta_K(t)\) and \(\sigma(K)\).

7. Filtration of Floer homology and genus formulas

This section is a short remark on how the genus of a knot \(K\) is related to the new \(\{-,0,+,\}\)-filtration of the complexes \(\hat{CFK}(K)\) and \(\hat{CFL}(K)\). In [2] we used a technique similar to the one used in the previous section, to obtain a sharp genus bound from \(\hat{CFL}(K)\). In fact, by spinning the longitude along the meridian of a given knot \(K\), we showed that the non-triviality of the group \(\hat{CFK}(K,g)\) implies the non-triviality of \(\hat{CFK}(K,g-\frac{1}{2})\), provided that \(g > 0\) is the genus of \(K\). The same argument may be used, almost word by word, to show the following (slight) generalization, in presence of \(\{-,0,+,\}\) filtration.

For a complex \(C\) which is graded by \(\mathbb{Z}\) (in the sense that the differential of \(C\) takes elements in a grading level \(s\) to elements in grading level \(s\)), we define the degree \(d_+(C)\) to be given by

\[
d_+(C) = \max \left\{ s \in \mathbb{Z} \mid C(s) \text{ does not have trivial chain homotopy type} \right\},
\]

and similarly define

\[
d_-(C) = \min \left\{ s \in \mathbb{Z} \mid C(s) \text{ does not have trivial chain homotopy type} \right\}.
\]

Theorem 7.1. Suppose that \(K\) is a knot in \(S^3\) of genus \(g\). Then for the complex \(\hat{CFK}(K) = \hat{CFK}(S^3,K)\) we have

\[
d_+(\hat{CFK}_{-}(K)) + 1 = -d_- (\hat{CFK}_{-}(K)) = g,
\]

\[
d_+(\hat{CFK}_{0}(K)) = -d_- (\hat{CFK}_{0}(K)) = g, \text{ and}
\]

\[
d_+(\hat{CFK}_{+}(K)) = -d_- (\hat{CFK}_{+}(K)) = g.
\]

Furthermore, for the complex \(\hat{CFL}(K) = \hat{CFL}(S^3,K)\) we have

\[
d_+(\hat{CFL}_{-}(K)) = -d_- (\hat{CFL}_{-}(K)) = g,
\]

\[
d_+(\hat{CFL}_{0}(K)) = -d_- (\hat{CFL}_{0}(K)) = g, \text{ and}
\]

\[
d_+(\hat{CFL}_{+}(K)) = 1 - d_- (\hat{CFL}_{+}(K)) = g.
\]

We skip a more detailed proof of this last theorem, and refer the reader to [2].
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