Imaginary Powers of \((k, 1)\)-Generalized Harmonic Oscillator

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Received: 14 April 2022 / Accepted: 23 May 2022
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Abstract
In this paper we will define and investigate the imaginary powers \((-\triangle_{k,1})^{-i\sigma}, \sigma \in \mathbb{R}\) of the \((k, 1)\)-generalized harmonic oscillator \(-\triangle_{k,1} = -\|x\| \triangle_k + \|x\|\) and prove the \(L^p\)-boundedness \((1 < p < \infty)\) and weak \(L^1\)-boundedness of such operators. It is a parallel result to the \(L^p\)-boundedness \((1 < p < \infty)\) and weak \(L^1\)-boundedness of the imaginary powers of the Dunkl harmonic oscillator \(-\triangle_{k} + \|x\|^2\). To prove this result, we develop the Calderón–Zygmund theory adapted to the \((k, 1)\)-generalized setting by constructing the metric space of homogeneous type corresponding to the \((k, 1)\)-generalized setting, and show that \((-\triangle_{k,1})^{-i\sigma}\) are singular integral operators satisfying the corresponding Hörmander type condition.

Keywords Imaginary powers \cdot (k , 1)-generalized harmonic oscillator \cdot Hörmander type condition

Mathematics Subject Classification 43A85 \cdot 31B10 \cdot 42B20 \cdot 53C23

1 Introduction

Dunkl theory is a far-reaching generalization of classical Fourier analysis related to root system initiated by Dunkl [9]. In the past twenty years, there have been many studies on Dunkl theory, including maximal functions, Bochner–Riesz means, multipliers, Riesz transforms and Calderón–Zygmund theory. In [16], A. Nowak and K. Stempak studied the Riesz transforms related to the Dunkl harmonic oscillator \(L_k\) and proved...
when the finite reflection group $G$ is isomorphic to $\mathbb{Z}_2^N$ such operators are $L^p$-bounded ($1 < p < \infty$) and weakly $L^1$-bounded. They then continued the study of the Dunkl harmonic oscillator $L_k$ in [17] by considering its imaginary powers $L_k^{-i\sigma}$, $\sigma \in \mathbb{R}$ and proved the $L^p$-boundedness ($1 < p < \infty$) and weak $L^1$-boundedness of such operators when $G$ is isomorphic to $\mathbb{Z}_2^N$. It is a generalization of the result obtained by Stempak and Torrea [20] on the imaginary powers of classical harmonic oscillator corresponding to the case when the multiplicity function $k \equiv 0$. In [1], B. Amri extended the result in [16] to general finite reflection groups $G$ in arbitrary dimensions. And the result in [17] can also be extended to general finite reflection groups using the same techniques as in [1] with slight modifications. The principal tool of the above results is the adaptation of the classical Calderón–Zygmund theory to Dunkl setting due to Amri and Sifi [2].

The framework of Dunkl theory is as follows: Given a root system $R$ in the Euclidean space $\mathbb{R}^N$, denote by $\sigma_\alpha$ the reflection in the hyperplane orthogonal to $\alpha$ and $G$ the finite subgroup of $O(N)$ generated by the reflections $\sigma_\alpha$ associated to the root system. Define a multiplicity function $k : R \rightarrow \mathbb{C}$ such that $k$ is $G$-invariant, that is, $k(\alpha) = k(\beta)$ if $\sigma_\alpha$ and $\sigma_\beta$ are conjugate. The Dunkl operators $T_j$, $1 \leq j \leq N$, which were introduced in [9], are defined by the following deformations by difference operators of directional derivatives $\partial_j$:

$$T_j f(x) = \partial_j f(x) + \sum_{\alpha \in R^+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product and $R^+$ is any fixed positive subsystem of $R$. They commute pairwise and are skew-symmetric with respect to the $G$-invariant measure $dm_k(x) = h_k(x)dx$, where $h_k(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k(\alpha)}$.

The Dunkl harmonic oscillator is defined as $L_k = -\Delta_k + \|x\|^2$, where $\Delta_k$ denotes the Dunkl Laplacian $\Delta_k = \sum_{j=1}^N T_j^2$. When $k \equiv 0$, the operator $L_k$ recedes to the classical harmonic oscillator $-\Delta + \|x\|^2$, where $\Delta$ stands for the classical Euclidean Laplacian. The eigenfunction of $\Delta_k$ for fixed $y$ is the integral kernel of the generalized Fourier transform called Dunkl transform. It takes the place of the exponential function $e^{-i(x,y)}$ in classical Fourier transform.

The operators $\partial_j$ and $T_j$ are intertwined by a Laplace-type operator (see [10])

$$V_k f(x) = \int_{\mathbb{R}^N} f(y) d\mu_x(y)$$  \hspace{1cm} (1.1)

associated to a family of probability measures $\{\mu_x | x \in \mathbb{R}^N\}$ with compact support (see [19]), that is, $T_j \circ V_k = V_k \circ \partial_j$. Specifically, the support of $\mu_x$ is contained in the convex hull $co(G \cdot x)$, where $G \cdot x = \{ g \cdot x | g \in G \}$ is the orbit of $x$. For any Borel set $B$ and any $r > 0$, $g \in G$, the probability measures satisfy

$$\mu_{rx}(B) = \mu_x(r^{-1}B), \quad \mu_{gx}(B) = \mu_x(g^{-1}B).$$

The intertwining operator $V_k$ is one of the most important operators in Dunkl theory.
More recently, S. Ben Saïd, T. Kobayashi and B. Ørsted [5] gave a further far-reaching generalization of Dunkl theory by introducing a parameter $a > 0$ arisen from the “interpolation” of the two $sl(2, \mathbb{R})$ actions on the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$ and the minimal unitary representation of the conformal group $O(N + 1, 2)$. They defined the $a$-deformed Dunkl harmonic oscillator as $\Delta_{k,a} := \|x\|^{2-a} \Delta_k - \|x\|^a$. The operator is an essentially self-adjoint operator on $L^2(\mathbb{R}^N, \vartheta_{k,a} (x) \, dx)$ with only negative discrete spectrum, where $\vartheta_{k,a} (x) = \|x\|^{a-2} h_k(x)$. They then proved the existence of a $(k, a)$-generalized holomorphic semigroup $I_{k,a}(z) := \exp \left( \frac{z}{a} \Delta_{k,a} \right)$, $Re z \geq 0$ with infinitesimal generator $\frac{1}{a} \Delta_{k,a}$. This holomorphic semigroup recedes to the Hermite semigroup studied by Howe [13] when $k \equiv 0$ and $a = 2$; to the Laguerre semigroup studied by Kobayashi and Mano [14, 15] when $k \equiv 0$ and $a = 1$; to the Dunkl Hermite semigroup studied by Rösler [18] when $k \geq 0$, $a = 2$ and $z = 2t$, $t > 0$. The $(k, a)$-generalized Fourier transform is then defined as the boundary value $z = \frac{\pi i}{2}$ of the semigroup, i.e., $F_{k,a} = e^{i\pi(\frac{k}{2} + N + a - 2)} I_{k,a}(\frac{\pi i}{2})$, where $(k) := \sum_{\alpha \in R^+} k(\alpha)$.

The $(k, a)$-generalized Fourier transform defined on $L^2(\mathbb{R}^N, \vartheta_{k,a} (x) \, dx)$ has its integral representation as (see [5, (5.8)])

$$F_{k,a} f(\xi) = c_{k,a} \int_{\mathbb{R}^N} f(y) B_{k,a}(\xi, y) \vartheta_{k,a}(y) \, dy, \quad \xi \in \mathbb{R}^N,$$

where $c_{k,a}$ is a constant and $B_{k,a}(x, y)$ is a symmetric kernel. It recedes to the Dunkl transform when $a = 2$ for $f \in (L^1 \cap L^2)(\mathbb{R}^N, \vartheta_{k,a} (x) \, dx)$. In [5, Theorem 5.11], the authors showed that the integral kernel $B_{k,a}(x, y)$ satisfies the condition

$$|B_{k,a}(x, y)| \leq |B_{k,a}(0, y)| \leq 1 \quad (1.2)$$

if $a = 1$ or $2$ assuming that $2(k) + N + a - 3 \geq 0$. In this case one can define the $(k, a)$-generalized translation operator via an integral combining the inversion formula of the $(k, a)$-generalized Fourier transform for $a = \frac{2}{n}$, $n \in \mathbb{N}$ (see [5, Theorem 5.3]). For the general case of $2(k) + N + a - 3 \geq 0$, the condition of boundedness (1.2) is not necessarily true and it remains an open problem whether it holds (see [12, Section 6]). In [7], the authors proved such boundedness for $k \equiv 0$ and $a = \frac{2}{n}$, $n \in \mathbb{N}$ only.

In [3], S. Ben Saïd and L. Deleaval studied the particular case when $k > 0$ and $a = 1$. They defined the generalized translation operator via an integral in this setting and derived a positive radial formula of the $(k, 1)$-generalized translation from the product of the integral kernel of the generalized Fourier transform. They then found many parallel results to Dunkl’s analysis (the case when $k \geq 0$ and $a = 2$) from such definition of the generalized translation operator. We continue the study of the $(k, 1)$-generalized Fourier analysis by S. Ben Saïd and L. Deleaval [3] in this paper.

We will define and investigate the imaginary powers $(-\Delta_{k,1})^{-i\sigma}$, $\sigma \in \mathbb{R}$ of the $(k, 1)$-generalized harmonic oscillator $-\Delta_{k,1} = -\|x\| \Delta_k + \|x\|$ and prove a parallel result for general finite reflection groups $G$ to that in [17]. According to the radial formula of $(k, 1)$-generalized translation operator given in [3], we develop the Calderón–Zygmund theory adapted to the $(k, 1)$-generalized setting by constructing
the metric space of homogeneous type corresponding to the \((k, 1)\)-generalized setting and giving the corresponding Hörmander type condition to prove the \(L^p\)-boundedness \((1 < p < \infty)\) and weak \(L^1\)-boundedness of \((-\triangle_{k,1})^{-i\sigma}\).

For a general metric space, a well-known definition of differentiation by Cheeger [6] is given via integration on continuous rectifiable curves. Unfortunately, rectifiable curves between two distinct points do not necessarily exist (or in other words, the induced length metric could be infinite) with respect to the metric corresponding to \((k, 1)\)-generalized analysis and derivatives on the metric space cannot be defined. We will make use of an estimate of difference quotient analogue in substitute of estimate of derivative. In the \((k, 1)\)-generalized setting, it is reasonable to consider the operator \(\|x\| \triangle_k\) as the \((k, 1)\)-generalized Laplacian because the distribution kernel of the \((k, 1)\)-generalized Fourier transform is the eigenfunction of the operator \(\|x\| \triangle_k\) (see [5, Theorem 5.7]). The imaginary powers of \((k, 1)\)-generalized harmonic oscillators motivates us to develop the Calderón–Zygmund theory in \((k, 1)\)-generalized setting. The development of \((k, 1)\)-generalized Fourier analysis is still at its infancy and there have only been [3] and [4] on this field.

The paper is organized as follows. In Sect. 2 we recall some results in \((k, 1)\)-generalized Fourier analysis and the translation operator in this setting. In Sect. 3, we will study the corresponding metric space of homogeneous type and develop the Calderón–Zygmund theory adapted to the \((k, 1)\)-generalized setting. In Sect. 4, we define and investigate the imaginary powers \((-\triangle_{k,1})^{-i\sigma}\) of the \((k, 1)\)-generalized harmonic oscillator and state the main theorem. In the last section we will show that such operators satisfy the corresponding Hörmander type condition given in Sect. 3 to prove the main theorem. We assume \(k > 0\) in this paper and most of the results will be under the condition \(2(k + N - 2) > 0\). Throughout the paper we denote \(C, C_1, C_2\) to be constants varying from line to line and \(b, c, b_1, b_2\) to be some positive absolute constants. The root system we are concerned with is not necessarily crystallographic.

## 2 Preliminaries

The study of Dunkl theory originates from a generalization of spherical harmonics with the Dunkl weight measure \(dm_k(x) = h_k(x)dx\), where \(h_k(x) = \prod_{\alpha \in \mathbb{R}^+} |\langle \alpha, x \rangle|^{k(\alpha)}\), playing the role of Lebesgue measure \(dx\) in the classical theory of spherical harmonics. Let \(P_m\) be the space of homogeneous polynomials on \(\mathbb{R}^N\) of degree \(m\). The so called Dunkl Laplacian \(\triangle_k\) was constructed in such a way that \(P_m \cap \ker \triangle_k\) are orthogonal to each other for \(m = 0, 1, \cdots\) with respect to Dunkl weight measure \(m_k\). It has the following explicit expression,

\[
\triangle_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathbb{R}^+} k(\alpha) \left( \frac{\langle \nabla f, \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right).
\]

Denote \(\mathcal{H}_k^m(\mathbb{R}^N) := P_m \cap \ker \Delta_k\) to be the space of \(h\)-harmonic polynomials of degree \(m\). Then the elements in the restriction \(\mathcal{H}_k^m(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}\) of \(\mathcal{H}_k^m(\mathbb{R}^N)\) to the unit sphere
The spaces $\mathcal{H}_k^m (\mathbb{R}^N) |_{S^{N-1}}$, $m = 0, 1, \ldots$ are finite dimensional and there is the spherical harmonics decomposition

$$L^2 \left( S^{N-1}, h_k (x') \, d\sigma (x') \right) = \bigoplus_{m \in \mathbb{N}} \mathcal{H}_k^m \left( \mathbb{R}^N \right) |_{S^{N-1}} ,$$

(2.1)

where $d\sigma$ denotes the spherical measure. For each fixed $m \in \mathbb{N}$, denote by $d(m) = \dim (\mathcal{H}_k^m (\mathbb{R}^N) |_{S^{N-1}})$. Let $\{ Y_i^m : i = 1, 2, \ldots, d(m) \}$ be an orthonormal basis of $\mathcal{H}_k^m (\mathbb{R}^N) |_{S^{N-1}}$. They are the eigenvectors of the generalized Laplace–Beltrami operator $\Delta_k |_{S^{N-1}}$.

From the spherical harmonic decomposition (2.1) of $L^2 \left( S^{N-1}, h_k (x') \, d\sigma (x') \right)$, there is a unitary isomorphism (see [5, (3.25)])

$$\sum_{m \in \mathbb{N}} (\mathcal{H}_k^m (\mathbb{R}^N) |_{S^{N-1}}) \otimes L^2 \left( \mathbb{R}_+, r^{2(k)+N-2} \, dr \right) \sim L^2 \left( \mathbb{R}^N, \vartheta_{k,1} (x) \, dx \right) ,$$

where $\vartheta_{k,1} (x) = \| x \|^{-1} h_k (x)$. Let $\lambda_{k,m} := 2m + 2 \langle k \rangle + N - 2$ and define the Laguerre polynomials as

$$L^2_\mu (t) := \sum_{j=0}^{l} \frac{(-1)^j \Gamma (\mu + l + 1)}{(l - j)! \Gamma (\mu + j + 1) \, j!} t^j , \quad \text{Re} \mu > -1 .$$

In [5] the authors constructed an orthonormal basis $\{ \Phi_{l,m,j} | l \in \mathbb{N}, m \in \mathbb{N}, j = 1, 2, \ldots, d(m) \}$ of $L^2 (\mathbb{R}^N, \vartheta_{k,1} (x) \, dx)$, where

$$\Phi_{l,m,j} (x) := \left( \frac{2^{\lambda_{k,m}+1} \Gamma (l+1)}{\Gamma (\lambda_{k,m} + l + 1)} \right)^{1/2} Y_j^m (x) L^2_{l} (\| x \| \exp (-\| x \|)) .$$

They are eigenfunctions for the $(k, 1)$-generalized harmonic oscillator $-\Delta_{k,1} = -\| x \| \Delta_k + \| x \|$, i.e.,

$$-\Delta_{k,1} \Phi_{l,m,j} (x) = (2l + \lambda_{k,m} + 1) \Phi_{l,m,j} (x) .$$

(2.2)

The $(k, 1)$-generalized Laguerre holomorphic semigroup $e^{z \Delta_{k,1}} (\Re z \geq 0)$ on $L^2 (\mathbb{R}^N, \vartheta_{k,1} (x) \, dx)$ has its spectral decomposition (see [22, (4.3)])

$$e^{z \Delta_{k,1}} (f) (x) = \sum_{l,m,j} e^{-z(2l+\lambda_{k,m}+1)} \langle f, \Phi_{l,m,j} \rangle_{k,1} \Phi_{l,m,j} (x) , \quad f \in L^2 (\mathbb{R}^N, \vartheta_{k,1} (x) \, dx) ,$$

(2.3)

where $\langle f, g \rangle_{k,1} = \int_{\mathbb{R}^N} f (x) g (x) \vartheta_{k,1} (x) \, dx$. It is a Hilbert–Schmidt operator for $\Re z > 0$ and a unitary operator on $\Re z = 0$ (see [5, Theorem 3.39]). By Schwartz kernel
theorem, the operator \( e^{z \Delta_{k,1}} \) (\( \Re z \geq 0 \)) has the following integral representation (see [5, (4.56)])

\[
e^{z \Delta_{k,1}} (f) (x) = c_{k,1} \int_{\mathbb{R}^N} f (y) \Lambda_{k,1} (x, y; z) \vartheta_{k,1} (y) \, dy,
\]

(2.4)

where \( c_{k,1} = (\int_{\mathbb{R}^N} \exp (-\| x \|) \vartheta_{k,1} (x) \, dx)^{-1} \) and

\[
\Lambda_{k,1} (r \omega, s \eta; z) := (\tilde{V}_k h_{k,1} (r, s; z; \cdot)) (\omega, \eta)
\]

(2.5)

for \( x = r \omega, y = s \eta, r, s > 0 \) and \( \omega, \eta \in S^{N-1} \). Here \( \tilde{V}_k \) is defined by \((\tilde{V}_k h) (x, y) := (V_k h_y) (x)\), where \( h_y (\cdot) := h (\langle \cdot, y \rangle) \) for a continuous function \( h(t) \) of one variable. And \( h_{k,1} (r, s; z; w) \) has its closed formula

\[
h_{k,1} (r, s; z; w) = \frac{\exp (- (r + s) \coth (z))}{\sinh (z)^{2(k)+N-1}} \Gamma \left( (k) + \frac{N - 1}{2} \right) \tilde{I}_{(k)+\frac{N-3}{2}} \left( \frac{\sqrt{2} (r s)^{\frac{1}{2}}}{\sinh z} (1 + w)^{\frac{1}{2}} \right).
\]

(2.6)

where \( \tilde{I}_v \) is the normalized \( I \)-Bessel function and has the following integral formula (see, e.g., [23, 6.15 (2)])

\[
\tilde{I}_v (w) = \frac{1}{\sqrt{\pi} \Gamma \left( v + \frac{1}{2} \right)} \int_{-1}^{1} e^{wu} \left( 1 - u^2 \right)^{v - \frac{1}{2}} du, \quad v > -1/2, \ w \in \mathbb{C}.
\]

The integral on the right hand side of (2.4) converges absolutely for all \( f \in L^2 (\mathbb{R}^N, \vartheta_{k,1} (x) \, dx) \) if \( \Re z > 0 \) and for all \( f \in (L^1 \cap L^2) (\mathbb{R}^N, \vartheta_{k,1} (x) \, dx) \) if \( \Re z = 0 \) (see [5, Corollary 4.28]). From (2.5) and (2.6) we get an expression of \( \Lambda_{k,1} (x, y; z) \) (a slight modification of Proposition 5.10 in [5])

\[
\Lambda_{k,1} (x, y; z) = \frac{\exp (- (\| x \| + \| y \|) \coth (z)))}{\sinh (z)^{2(k)+N-1}} \Gamma \left( (k) + \frac{N - 1}{2} \right) \\
\times V_k \left( \tilde{I}_{(k)+\frac{N-3}{2}} \left( \frac{1}{\sinh z} \sqrt{2 (\| x \| \| y \| + \langle x, \cdot \rangle)} \right) \right) (y).
\]

(2.7)

Let

\[
B_{k,1} (x, y) := e^{i z (\frac{N}{2})} \Lambda_{k,1} \left( x, y; i \frac{\pi}{2} \right).
\]
Then the $(k, 1)$-generalized Fourier transform on $L^2(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx)$ can be expressed as

$$F_{k,1}f(\xi) = c_{k,1} \int_{\mathbb{R}^N} f(y) B_{k,1}(\xi,y) \vartheta_{k,1}(y)\,dy, \quad \xi \in \mathbb{R}^N$$

because $F_{k,1} := e^{i \frac{1}{2}(2(k)+N-1)\mathcal{I}_{k,1}(\frac{\pi i}{2})}$. It has the property (see [5, Theorem 5.3])

$$F_{k,1}^{-1}(f) = F_{k,1}(f).$$

The $(k, 1)$-generalized translation $\tau_y$ is defined on $L^2(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx)$ by (see [3])

$$F_{k,1}(\tau_y f)(\xi) := B_{k,1}(y,\xi) F_{k,1}(f)(\xi), \quad \xi \in \mathbb{R}^N.$$

It is analogous to the translation operator $\tau_y f(x) = f(x-y)$ in classical Fourier analysis. The above definition makes sense as $F_{k,1}$ is an isometry from $L^2(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx)$ onto itself. Assume $\langle k \rangle + \frac{N-2}{2} > 0$. Then $|B_{k,1}(x,y)| \leq 1$ (see [5, Theorem 5.11]) and so $\tau_y$ can also be defined as

$$\tau_y f(x) = c_{k,1} \int_{\mathbb{R}^N} B_{k,1}(x,\xi) B_{k,1}(y,\xi) F_{k,1}(f)(\xi) \vartheta_{k,1}(\xi)\,d\xi$$

for $f \in L^1_k(\mathbb{R}^N)$, where $L^1_k(\mathbb{R}^N) := \{ f \in L^1(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx) : F_{k,1}(f) \in L^1(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx) \}$. This formula holds true on Schwartz space $S(\mathbb{R}^N)$ since $S(\mathbb{R}^N)$ is a subspace of $L^1_k(\mathbb{R}^N)$. The operator $\tau_y$ satisfies the following properties:

1. For every $x, y \in \mathbb{R}^N$,

$$\tau_y f(x) = \tau_x f(y), \quad f \in S(\mathbb{R}^N). \quad (2.8)$$

2. For every $y \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} \tau_y f(x) g(x) \vartheta_{k,1}(x)\,dx = \int_{\mathbb{R}^N} f(x) \tau_y g(x) \vartheta_{k,1}(x)\,dx, \quad f, g \in S(\mathbb{R}^N). \quad (2.9)$$

Here the property (1) corresponds to $\tau_y f(x) = \tau_{-x} f(-y)$ and (2) corresponds to the skew-symmetry in classical Fourier analysis and Dunkl analysis.

For any radial function $f \in S(\mathbb{R}^N)$, i.e., $f(x) = f_0(\|x\|), \langle k \rangle + \frac{N-2}{2} > 0$, $\tau_y$ can be expressed as follows (see [3])
\[ \tau_y f(x) = \frac{\Gamma \left( \frac{N-1}{2} + \langle k \rangle \right)}{\sqrt{\pi} \Gamma \left( \frac{N-2}{2} + \langle k \rangle \right)} \times V_k \left( \int_{-1}^{1} f_0 \left( \| x \| + \| y \| - \sqrt{2 (\| x \| \| y \| + \langle \cdot, y \rangle)} u \right) \left( 1 - u^2 \right)^{\frac{N}{2} + \langle k \rangle - 2} \, du \right)(x). \]

(2.10)

And so \( \tau_y \) is positive on radial functions and can be extended as a bounded operator to the space of all radial functions on \( L^p \left( \mathbb{R}^N, \vartheta_{k,1} (x) \, dx \right), 1 \leq p \leq 2 \). Further, if \( f \) is a nonnegative radial function on \( L^1 \left( \mathbb{R}^N, \vartheta_{k,1} (x) \, dx \right) \), then

\[ \int_{\mathbb{R}^N} \tau_y f(x) \vartheta_{k,1} (x) \, dx = \int_{\mathbb{R}^N} f(x) \vartheta_{k,1} (x) \, dx. \]

(2.11)

The authors in [3] also gave a special case of the formula for radial functions

\[ \tau_y \left( e^{-\lambda \| \cdot \|} \right)(x) = \frac{\Gamma \left( \langle k \rangle + \frac{N-1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{N-2}{2} + \langle k \rangle \right)} e^{-\lambda (\| x \| + \| y \|)} V_k \left( \widetilde{I}_{\langle k \rangle + \frac{N-3}{2}} \left( \lambda \sqrt{2 (\| x \| \| y \| + \langle x, \cdot \rangle}) \right) \right)(y). \]

(2.12)

### 3 Hörmander Type Condition

Let \( (X, d) \) be a metric space. Denote \( B(x, r) \) to be the ball \( B(x, r) := \{ y \in X : d(x, y) \leq r \} \) for \( x \in X \). If there exists a doubling measure \( m \), i.e., there exists a measure \( m \) such that for some absolute constant \( C \),

\[ m \left( B(x, 2r) \right) \leq C m \left( B(x, r) \right), \forall x \in \mathbb{R}^N, \ r > 0, \]

(3.1)

then \( (X, d) \) is a space of homogeneous type. The Calderón–Zygmund theory on a space of homogeneous type \( (X, d, m) \) says that for \( f \in L^1(X, m) \cap L^2(X, m) \) and \( \lambda > \frac{\| f \|_1}{m(X)} \), there exists the Calderón–Zygmund decomposition \( f = h + b \) with \( b = \sum_j b_j \) and a sequence of balls \( (B(y_j, r_j)) \) such that for some absolute constant \( C \),

1. \( \| h \|_\infty \leq C \lambda; \)
2. \( \text{supp}(b_j) \subset B_j; \)
3. \( \int_{B_j} b_j(x) dm(x) = 0; \)
4. \( \| b_j \|_{L^1(X,m)} \leq C \lambda m(B_j); \)
5. \( \sum_j m(B_j) \leq C \frac{\| f \|_{L^1(X,m)}}{\lambda}. \)

From the Calderón–Zygmund decomposition one can deduce that for a bounded operator \( S \) on \( L^2(X, m) \) associated with kernel \( K(x, y) \), if \( K(x, y) \) satisfies a Hörmander
type condition, then the operator $S$ can be extended to a bounded operator on $L^p(X, m)$ $(1 < p \leq 2)$ and a weakly bounded operator on $L^1(X, m)$. We refer to [8, Chapter III] for this theory.

Now we adapt Calderón–Zygmund theory to the $(k, 1)$-generalized setting by constructing the metric space corresponding to this setting first. For $x, y \in \mathbb{R}^N$, define a function $d$ from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}$ as

$$d(x, y) := \sqrt{\|x\| + \|y\| - \sqrt{2 (\|x\| \|y\| + \langle x, y \rangle)}}$$

$$= \sqrt{\|x\| + \|y\| - 2\sqrt{\|x\| \|y\|} \cos \frac{\theta}{2}} \geq \sqrt{\|x\| - \|y\|},$$

where $\theta = \arccos \frac{\langle x, y \rangle}{\|x\| \|y\|}$, $0 \leq \theta \leq \pi$. We need to equip $\mathbb{R}^N$ with this function as the metric in $(k, 1)$-generalized analysis in view of the expression (2.10) of $(k, 1)$-generalized translation operators.

**Proposition 3.1** The function $d(x, y)$ is a metric.

**Proof** The symmetry property is obvious. For the positivity property, if $d(x, y) = 0$, then $\|x\| = \|y\|$ and $\sqrt{\|x\| - \|x\| \cos \frac{\theta}{2}} = 0$ leading to $\theta = 0$. Hence $x = y$.

Then we turn to prove the triangle inequality. Let $\alpha = \arccos \frac{\langle x, y \rangle}{\|x\| \|y\|}$, $\beta = \arccos \frac{\langle x, z \rangle}{\|x\| \|z\|}$, $\gamma = \arccos \frac{\langle z, y \rangle}{\|z\| \|y\|}$, $0 \leq \alpha, \beta, \gamma \leq \pi$.

Then we have $\beta + \gamma \geq \alpha$ from the triangle inequality of the spherical distance. Therefore,

$$d(x, y) \leq \sqrt{\|x\| + \|y\| - 2\sqrt{\|x\| \|y\|} \cos \frac{\beta + \gamma}{2}}.$$

It suffices to show that

$$\sqrt{\|x\| + \|y\| - 2\sqrt{\|x\| \|y\|} \cos \frac{\beta + \gamma}{2}} \leq d(x, z) + d(z, y).$$

Take the square of the above inequality and eliminate some items. It suffices to show the following inequality,

$$\|x\| \|y\| \sin^2 \frac{\beta + \gamma}{2} + \|x\| \|z\| \sin^2 \frac{\beta}{2} + \|z\| \|y\| \sin^2 \frac{\gamma}{2}$$

$$+ 2 \|z\| \sqrt{\|x\| \|y\|} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$+ 2 \|x\| \sqrt{\|z\| \|y\|} \cos \frac{\beta + \gamma}{2} + 2 \|y\| \sqrt{\|z\| \|x\|} \cos \frac{\gamma}{2} \cos \frac{\beta + \gamma}{2}$$

$$\geq 2 \|x\| \sqrt{\|z\| \|y\|} \cos \frac{\gamma}{2} + 2 \|y\| \sqrt{\|z\| \|x\|} \cos \frac{\beta}{2} + 2 \|z\| \sqrt{\|x\| \|y\|} \cos \frac{\beta + \gamma}{2}.$$
And the inequality is equivalent to

\[
\left( \sqrt{\|x\| \|y\|} \sin \frac{\beta + \gamma}{2} - \sqrt{\|x\| \|z\|} \sin \frac{\beta}{2} - \sqrt{\|z\| \|y\|} \sin \frac{\gamma}{2} \right)^2 \geq 0.
\]

Proposition 3.1 is therefore proved.

\[\Box\]

**Remark 3.2** i). For the one dimensional case, the metric \(d(x, y)\) recedes to

\[
d(x, y) = \begin{cases} \sqrt{|x - y|} , & xy \leq 0 \\ |\sqrt{|x|} - \sqrt{|y|}| , & xy > 0 \end{cases}.
\]

The ball with respect to this metric in this case was already used in [4] to define the generalized Hardy–Littlewood maximal operator.

ii). A continuous rectifiable curve between two distinct points does not necessarily exist with respect to this metric. For example, if we take \(x = -1\) and \(y = 1\) for the one dimensional case, then distance between \(x\) and \(y\) with respect to the induced length metric is no less than \(\sup_n \sum_{i=1}^n \sqrt{\frac{2}{n}} = \infty\).

**Proposition 3.3** \((\mathbb{R}^N, d)\) is a complete metric space.

**Proof** We will show that \(d(x, y)\) is equivalent to the Euclidean metric. If \(y_n \to y\) with respect to the Euclidean metric, then \(d(y_n, y) \to 0\) obviously. If \(d(y_n, y) \to 0\), then \(\|y_n\| \to \|y\|\). Denote \(\theta_n = \arccos \frac{\langle y_n, y \rangle}{\|y_n\| \|y\|}\). Then

\[
\sqrt{2 \|y\| - 2 \|y\|} \lim_{n \to \infty} \cos \frac{\theta_n}{2} = 0.
\]

So, \(\lim_{n \to \infty} \cos \theta_n = 1\) and \(\lim_{n \to \infty} \langle y_n, y \rangle = \|y\|^2\). Hence

\[
\lim_{n \to \infty} \|y_n - y\| = \lim_{n \to \infty} \sqrt{\|y_n\|^2 + \|y\|^2 - 2 \langle y_n, y \rangle} = 0.
\]

\[\Box\]

The closure of an open ball in a metric space is not necessarily the closed ball. In [24] the authors gave a sufficient but not necessary condition such that the closure of the open ball is the closed ball. They showed that if the metric is weakly convex, i.e., for any two different points \(x\) and \(y\), there exists \(z \neq x, y\), such that \(d(x, y) = d(x, z) + d(z, y)\). The metric \(d\) we are concerned with is not weakly convex obviously but the closure of the open ball with respect to this metric is still the closed ball.

**Theorem 3.4** The closure \(\overline{B_0(x, r)}\) of the open ball \(B_0(x, r) = \{ y : d(y, x) < r \}\), \(r > 0\) is the closed ball \(B(x, r)\).
Let $y$ be a point $\mathbb{R}^N$ distinct from $x$ such that $d(x, y) = r$. We show that for any $\varepsilon > 0$, there exists $z \in B(y, \varepsilon)$, such that $d(x, z) < r = d(x, y)$. Let $M_x$, $x \neq 0$ be the mapping $M_x : \mathbb{R}^N \to [0, +\infty)$, $y \mapsto d(x, y)$ and $L_x(y) := M_x(y)^2$. It suffices to show that the function $L_x$ takes no minimum point on $\mathbb{R}^N$ except at $y = x$. Notice that $L_x$ is differentiable on $\mathbb{R}^N \setminus \{0\}$. We calculate the points such that

$$0 = \frac{\partial L_x}{\partial y_i} = \frac{y_i}{\|y\|} - \frac{\|x\| y_i + x_i}{\sqrt{2} (\|x\| \|y\| + \langle x, y \rangle)}, \quad i = 1, 2, \ldots, N.$$ 

By summing up the square, we get

$$\sqrt{2 (\|x\| \|y\| + \langle x, y \rangle)} = 2 \|y\| \text{ and } x_i = ty_i, \quad \text{where } t = 2 - \frac{\|x\|}{\|y\|}.$$ 

Thus $2y_i = |t| y_i + ty_i$ and $y = x$. For the point $y = 0$, consider the function

$$L(y_1) := L_x(y_1, 0, \ldots, 0) = \|x\| + |y_1| - \sqrt{2 (\|x\| |y_1| + x_1y_1)}$$

$$= \begin{cases} \|x\| + y_1 - \sqrt{2 (\|x\| + x_1)y_1}, & y_1 \geq 0 \\ \|x\| - y_1 - \sqrt{-2 (\|x\| - x_1)y_1}, & y_1 < 0. \end{cases}$$

It does not take minimum at $y_1 = 0$ obviously. Therefore, $L_x$ takes no minimum point on $\mathbb{R}^N$ except at $y = x$. $\square$

The metric space $(\mathbb{R}^N, d)$, rather than the standard Euclidean metric space, is the natural metric space corresponding to the $(k, 1)$-generalized setting when metric is involved due to the expression of the $(k, 1)$-generalized translation operators. In the following theorem we give a characterization of support of the $(k, 1)$-generalized translation of a function supported in $B(0, r) = \{y \in \mathbb{R}^N : \sqrt{\|y\|} \leq r\}$.

**Theorem 3.5** Let $f = f_0 (\|\cdot\|)$ be a nonnegative radial function on $L^2 (\mathbb{R}^N, \varphi_{k,1}(x) \, dx)$, supp $f = B(0, r)$, then

$$\text{supp} \tau_x f = \bigcup_{g \in G} B(gx, r).$$

**Proof** We extend the formula of $(k, 1)$-generalized translations on radial Schwartz functions (2.10) to all continuous radial functions on $L^2 (\mathbb{R}^N, \varphi_{k,1}(x) \, dx)$ first. The proof goes similar as the Lemma 3.4 in [1]. The only difference is to take the set $A_n$ in the proof as

$$A_n \equiv A_n(y) := \{x \in \mathbb{R}^N : 2^{-n} \leq \sqrt{\|x\|} - \sqrt{\|y\|} \leq \sqrt{\|x\|} + \sqrt{\|y\|} \leq 2^n\}$$
for $n \in \mathbb{N}$ and $n \geq \frac{1}{2} \left[ \log \frac{\|y\|}{\log 2} \right] + 1$, since

$$\left| \sqrt{\|x\|} - \sqrt{\|y\|} \right| \leq \sqrt{\|x\| + \|y\|} - \sqrt{\frac{2}{2}(\|x\| \|y\| + \langle \eta, y \rangle)}u \leq \sqrt{\|x\| + \|y\|}$$

for $\eta \in \co(G.x)$ and $u \in [-1, 1]$.

Then we prove the theorem for continuous nonnegative radial functions. For the proof of $\text{supp} \tau_x f = \bigcup_{g \in G} B(gx, r)$, from the radial formula (2.10) of $(k, 1)$-generalized translations and notice that for any $\eta \in \co(G.x)$ and $u \in [-1, 1]$,

$$\sqrt{\|x\| + \|y\|} - \sqrt{\frac{2}{2}(\|x\| \|y\| + \langle \eta, y \rangle)}u \geq \min_{g \in G} d(gx, y), \quad (3.2)$$

we have $\tau_x f(y) = 0$ for $y \in \left( \bigcup_{g \in G} B(gx, r) \right)^c$ if $\text{supp} f \subseteq B(0, r)$. For the converse part $\bigcup_{g \in G} B(gx, r) \subseteq \text{supp} \tau_x f$, we will show that $\bigcup_{g \in G} B_0(gx, r) \subseteq \text{supp} \tau_x f$ first. Suppose there exists a $y \in \bigcup_{g \in G} B_0(gx, r)$ for which $y \notin \text{supp} \tau_x f$. Then there exists $\varepsilon > 0$, such that for any $z \in B(y, \varepsilon)$, we have $z \in \bigcup_{g \in G} B_0(gx, r)$ (that is, there also exists a $g \in G$ such that $d(z, gx) < r$) and

$$0 = \tau_x f(z) - \frac{\Gamma \left( \frac{N-1}{2} + \langle k \rangle \right)}{\sqrt{\pi} \Gamma \left( \frac{N-2}{2} + \langle k \rangle \right)} \times \int_{\mathbb{R}^N} \int_{-1}^{1} f_0 \left( \|x\| + \|z\| - \sqrt{\frac{2}{2}(\|x\| \|z\| + \langle \eta, z \rangle)}u \right) \times \left( 1 - u^2 \right)^{\frac{N}{2} + \langle k \rangle - 2} dud\mu_x(\eta).$$

Thus

$$f_0 \left( \|x\| + \|z\| - \sqrt{\frac{2}{2}(\|x\| \|z\| + \langle \eta, z \rangle)}u \right) = 0$$

for any $\eta \in \text{supp} \mu_x$ and $u \in [-1, 1]$. Then from a result of Gallardo and Rejeb (see [11]), that the orbit of $x, G.x$, is contained in $\text{supp} \mu_x$, we can select $u = 1$ and $\eta = gx$ for the above $g$. Then we get $f_0 \left( d(gx, z) \right) = 0$ for all $z \in B(y, \varepsilon)$. But $d(z, gx) < r$, which contradicts to that supp $f_0 = [0, r^2]$. Then from Theorem 3.4, we get $\bigcup_{g \in G} B(gx, r) \subseteq \text{supp} \tau_x f$.

The conclusion for all nonnegative radial function on $L^2 \left( \mathbb{R}^N, \vartheta_{k,1}(x) dx \right)$ can then be derived from the density of continuous functions with compact support $B(0, r)$ in $L^2 \left( B(0, r), \vartheta_{k,1}(x) dx \right)$ and the positivity of the $(k, 1)$-generalized translations on radial functions as in [21, Theorem 1.2].

Denote by $dm_{k,1}(x) = \vartheta_{k,1}(x) dx$. The measure $m_{k,1}$ satisfies the scaling property

$$m_{k,1} \left( B(tx, \sqrt{tr}) \right) = t^{2(k)+N-1} m_{k,1} \left( B(x, r) \right), \quad t > 0. \quad (3.3)$$
From polar coordinate transformation we have

\[ m_{k,1}(B(x, r)) = \int_{S^{N-1}} \int_{(0, +\infty)} \rho^{2(k)+N-2} d\rho h_k(\omega) d\omega \]

\[ u=\sqrt{\rho}, \quad \int_{S^{N-1}} \int_{(0, +\infty)} u^{2(2(k)+N)-3} du h_k(\omega) d\omega \]

\[ z=\mu \omega, \quad \int_{E(x_0, r)} \|z\|^{2(k)+N-2} h_k(z) dz, \]

where

\[ \theta = \arccos \frac{\langle x, \omega \rangle}{\|x\|} \quad x_0 = \sqrt{\|x\|} \frac{x + \|x\| \omega}{\|x + \|x\| \omega\|}, \]

and \( E(x_0, r) \) denotes the Euclidean ball centered at \( x_0 \) with radius \( r \). For the one dimensional case, this expression coincides that of the measure of the ball in the proof of Lemma 2.2 in [4]. So if \( 2 \langle k \rangle + N - 2 > 0 \), then for any \( x \in \mathbb{R}^N \) and \( r > 0 \), \( m_{k,1}(B(x, r)) \) is finite and \( m_{k,1}(B(t x, r)) \) is nondecreasing as \( t \) grows. It is then easy to check that \( m_{k,1} \) is a doubling measure when \( 2 \langle k \rangle + N - 2 > 0 \) combining (3.3). Therefore, \( (\mathbb{R}^N, d, m_{k,1}) \) is a space of homogeneous type and for all \( f \in L^1(\mathbb{R}^N, \vartheta_{k,1}(x) dx) \cap L^2(\mathbb{R}^N, \vartheta_{k,1}(x) dx) \) and \( \lambda > 0 \), there exists the corresponding Calderón–Zygmund decomposition of \( f \) satisfying (i)–(v).

Define the distance between the two orbits \( G.x \) and \( G.y \) as \( d_G(x, y) = \min_{g \in G} d(g x, y) \). Now we are ready to give the Hörmander type condition adapted to \( (k, 1) \)-generalized setting. It is a modification of the Hörmander type condition on a homogeneous space in [8, Chapter III, Theorem 2.4] because the Calderón–Zygmund theory on homogeneous spaces cannot be applied to this setting. We omit the proof because it is similar to that of Theorem 3.1 in [2].

**Theorem 3.6** For \( 2 \langle k \rangle + N - 2 > 0 \), let \( K \) be a measurable function on \( \mathbb{R}^N \times \mathbb{R}^N \setminus \{(x, g x) : x \in \mathbb{R}^N, \ g \in G\} \) and \( S \) be a bounded operator on \( L^2(\mathbb{R}^N, \vartheta_{k,1}(x) dx) \) associated with the kernel \( K \) such that for any compactly supported function \( f \in L^2(\mathbb{R}^N, \vartheta_{k,1}(x) dx) \),

\[
S(f)(x) = \int_{\mathbb{R}^N} K(x, y) f(y) \vartheta_{k,1}(y) dy, \quad G.x \cap \text{supp } f = \emptyset.
\]

If \( K \) satisfies

\[
\int_{d_G(x, y) > 2d(y, y_0)} |K(x, y) - K(x, y_0)| \vartheta_{k,1}(x) dx \leq C, \quad y, y_0 \in \mathbb{R}^N,
\]

then \( S \) extends to a bounded operator on \( L^p(\mathbb{R}^N, \vartheta_{k,1}(x) dx) \) for \( 1 < p \leq 2 \) and a weakly bounded operator on \( L^1(\mathbb{R}^N, \vartheta_{k,1}(x) dx) \).
4 Main Results

We reformulate the reproducing kernel $\Lambda_{k,1} (x, y; z)$ of $e^{z\Delta_{k,1}}$ from (2.7), (2.12) and (2.10) as

$$\Lambda_{k,1} (x, y; z) = \frac{1}{(\sinh z)^{2(k)+N-1}} e^{-\tanh \frac{z}{2} (\|x\|+\|y\|)} \tau_y \left( e^{-\frac{1}{2} \|\cdot\|} \right)(x)$$

$$= \frac{\Gamma \left( \frac{N-1}{2} + \langle k \rangle \right)}{\sqrt{\pi} \Gamma \left( \frac{N-2}{2} + \langle k \rangle \right) (\sinh z)^{2(k)+N-1}} e^{-\tanh \frac{z}{2} (\|x\|+\|y\|)}$$

$$\cdot V_k \left( \int_{-1}^{1} e^{-\frac{1}{2} \sinh z (\|x\|+\|y\|-\sqrt{2(\|x\|\|y\|+\langle \cdot, \cdot \rangle)u})} \left( 1 - u^2 \right)^{\frac{N}{2} + \langle k \rangle - 2} du \right)(x).$$

(4.1)

Let $z = t, t > 0$. For $0 < t \leq 1$, sinh $t$ behaves like $t$. So

$$|\Lambda_{k,1} (x, y; t)| \leq C \frac{1}{t^{2(k)+N-1}} \tau_y \left( e^{-\frac{z}{2} \|\cdot\|} \right)(x).$$

(4.2)

For $t > 1$, sinh $t$ behaves like $e^t$. So

$$|\Lambda_{k,1} (x, y; t)| \leq C e^{-(2(k)+N-1)t} \tau_y \left( e^{-b \|\cdot\|} \right)(x).$$

(4.3)

From (2.2) we can define the imaginary powers $(-\Delta_{k,1})^{-i\sigma}, \sigma \in \mathbb{R}$ for $f \in L^2 (\mathbb{R}^N, \partial_{k,1} (x) dx)$ of the $(k, 1)$-generalized harmonic oscillator $-\Delta_{k,1}$ naturally as

$$(-\Delta_{k,1})^{-i\sigma} (f) (x) = \sum_{l,m,j} \left( 2l + \lambda_{k,m} + 1 \right)^{-i\sigma} \left( f, \Phi_{l,m,j} \right) \Phi_{l,m,j} (x).$$

(4.4)

It is obviously a bounded operator on $L^2 (\mathbb{R}^N, \partial_{k,1} (x) dx)$ from its spectrum.

In what follow we put

$$K (x, y) = \int_{0}^{\infty} \Lambda_{k,1} (x, y; t) t^{i\sigma-1} dt.$$  

(4.5)

It is then easy to verify the integral (4.5) converges absolutely and that $|K (x, y)| \leq C d_G(x,y)^{2(k)+N-1}$ for all $x, y \in \mathbb{R}^N, y \notin G, x, 2 \langle k \rangle + N - 2 > 0$.

Based on the formula

$$\lambda^{-i\sigma} = \frac{1}{\Gamma (i\sigma)} \int_{0}^{\infty} e^{-\lambda t^{i\sigma-1}} dt, \quad \lambda > 0$$
and (4.4), (2.3), (2.4), we can write \((-\Delta_{k,1})^{-i\sigma}\) in the following way (such definition goes back to [17] and [20])

\[
(-\Delta_{k,1})^{-i\sigma} (f)(x) = \frac{1}{\Gamma(i\sigma)} \int_0^\infty e^{i\Delta_{k,1}} (f)(x) t^{i\sigma - 1} dt
= \frac{c_{k,1}}{\Gamma(i\sigma)} \int_0^\infty t^{i\sigma - 1} dt \int_{\mathbb{R}^N} f(y) \Lambda_{k,1}(x, y; t) \vartheta_{k,1}(y) dy.
\]

We can observe that this integral converges absolutely for all compactly supported functions \(f \in L^2(\mathbb{R}^N, \vartheta_{k,1}(x) \, dx)\) with \(\text{supp} \, f \cap G.x = \emptyset\). And for compactly supported functions \(f \in L^2(\mathbb{R}^N, \vartheta_{k,1}(x) \, dx)\), \(G.x \cap \text{supp} \, f = \emptyset\), \((-\Delta_{k,1})^{-i\sigma}\) satisfies

\[
(-\Delta_{k,1})^{-i\sigma} (f)(x) = \frac{c_{k,1}}{\Gamma(i\sigma)} \int_{\mathbb{R}^N} K(x, y) f(y) \vartheta_{k,1}(y) dy
\]

by changing the order of integration. We will show that the kernel \(K(x, y)\) of \((-\Delta_{k,1})^{-i\sigma}\) satisfies the condition in Theorem 3.6 to prove the following main theorem.

**Theorem 4.1** For \(2 \langle k \rangle + N - 2 > 0\), the imaginary powers \((-\Delta_{k,1})^{-i\sigma}, \sigma \in \mathbb{R}\) of the \((k, 1)\)-generalized harmonic oscillator \(-\Delta_{k,1}\) are bounded operators on \(L^p(\mathbb{R}^N, \vartheta_{k,1}(x) \, dx)\), \(1 < p < \infty\) and weakly bounded on \(L^1(\mathbb{R}^N, \vartheta_{k,1}(x) \, dx)\).

**5 Proof of Theorem 4.1**

We start the proof with two lemmas. The first one is an enhancement of the triangle inequality of the metric \(d(x, y)\).

**Lemma 5.1** For \(u \in [-1, 1]\), \(\eta \in co(G.x)\), and \(x, y \in \mathbb{R}^N\),

\[
\sqrt{\|x\| + \|y\|} - \sqrt{2 \left(\|x\| \|y\| + \langle \eta, y \rangle \right)} u - \sqrt{\|x\| + \|z\|} - \sqrt{2 \left(\|x\| \|z\| + \langle \eta, z \rangle \right)} u \leq d(y, z).
\]

**Proof** If \(\eta \in co(G.x)\), then there exists a rotation transformation \(T\) such that \(\eta = kT(x), 0 \leq k \leq 1\). So we assume \(\eta = kx\) in the proof since \(\|T(x)\| = \|x\|\). Then

\[
\sqrt{2 \left(\|x\| \|y\| + \langle \eta, y \rangle \right)} u = 2\sqrt{\|x\| \|y\|} \sqrt{\frac{1}{2} (1 + k \cos \alpha) u};
\]

\[
\sqrt{2 \left(\|x\| \|z\| + \langle \eta, z \rangle \right)} u = 2\sqrt{\|x\| \|z\|} \sqrt{\frac{1}{2} (1 + k \cos \beta) u}.
\]
Denote by
\[ \alpha_k = 2 \arccos \sqrt{\frac{1}{2} (1 + k \cos \alpha)}, \quad \beta_k = 2 \arccos \sqrt{\frac{1}{2} (1 + k \cos \beta)}. \]

We assert that
\[ |\alpha_k - \beta_k| \leq \gamma. \tag{5.1} \]

Here \( \alpha, \beta \) and \( \gamma \) are given as in the proof of Proposition 3.1.

Assume \( \|x\| = \|y\| = \|z\| = 1 \). Then (5.1) is equivalent to
\[ 1 - (y, z)^2 - k^2 (x, y)^2 - k^2 (x, z)^2 + 2k^2 (x, y) (y, z) (x, z) \geq 0. \]

It suffices to show that
\[ k^2 \left( 1 - (y, z)^2 \right) - k^2 (x, y)^2 - k^2 (x, z)^2 + 2k^2 (x, y) (y, z) (x, z) \geq 0. \]

And it is equivalent to
\[ \det \begin{bmatrix} 1 & (x, y) & (x, z) \\ (y, x) & 1 & (y, z) \\ (z, x) & (z, y) & 1 \end{bmatrix} \geq 0. \]

It is the determinant of the Gram matrix of the three vectors \( x, y, \) and \( z \). Thus (5.1) is proved.

From assertion (5.1), similar to the proof in Proposition 3.1, it suffices to show that
\[ \sqrt{\|x\| + \|y\|} - 2\sqrt{\|x\| \|y\| u \cos \frac{\alpha_k}{2}} - \sqrt{\|x\| + \|z\|} - 2\sqrt{\|x\| \|z\| u \cos \frac{\beta_k}{2}} \]
\[ \leq \sqrt{\|y\| + \|z\|} - 2\sqrt{\|y\| \|z\| \cos \frac{\alpha_k - \beta_k}{2}}. \]

And it suffices to show
\[
\begin{align*}
\|x\| \|y\| \left( 1 - u^2 \cos^2 \frac{\alpha_k}{2} \right) + \|x\| \|z\| \left( 1 - u^2 \cos^2 \frac{\beta_k}{2} \right) + \|z\| \|y\| \sin^2 \frac{\alpha_k - \beta_k}{2} \\
+ 2 \|z\| \sqrt{\|x\| \|y\|} u \cos \frac{\beta_k}{2} \cos \frac{\alpha_k - \beta_k}{2} \\
+ 2 \|x\| \sqrt{\|z\| \|y\|} u^2 \cos \frac{\beta_k}{2} \cos \frac{\alpha_k - \beta_k}{2} + 2 \|y\| \sqrt{\|z\| \|x\|} u \cos \frac{\alpha_k - \beta_k}{2} \cos \frac{\alpha_k}{2}
\end{align*}
\]
\[
\geq 2 \|x\| \sqrt{\|z\| \|y\|} \cos \frac{\alpha_k - \beta_k}{2} + 2 \|y\| \sqrt{\|z\| \|x\|} u \cos \frac{\beta_k}{2} \\
+ 2 \|z\| \sqrt{\|x\| \|y\|} u \cos \frac{\alpha_k}{2}.
\]
The above is equivalent to
\[
\left( \sqrt{\|x\| \|y\|} u \sin \frac{\alpha_k}{2} - \sqrt{\|x\| \|z\|} u \sin \frac{\beta_k}{2} - \sqrt{\|z\| \|y\|} \sin \frac{\alpha_k - \beta_k}{2} \right)^2 + \|x\| (\|y\| + \|z\|) (1 - u^2) \geq 2 \|x\| \sqrt{\|z\| \|y\|} (1 - u^2) \cos \frac{\alpha_k - \beta_k}{2}.
\]

The Lemma is therefore proved. \(\square\)

The next lemma is an estimate of the difference quotient analogue. We can no longer make use of estimates of partial derivatives because we cannot define differentiation on the metric space corresponding to \((k, 1)\)-generalized analysis for the failure of the existence of continuous rectifiable curves between two distinct points (see Remark 3.2. ii).

**Lemma 5.2** For \(0 < t < 1, y \neq y_0\),
\[
\left| \frac{\Lambda_{k,1} (x, y; t) - \Lambda_{k,1} (x, y_0; t)}{d (y, y_0)} \right| \leq \frac{C}{t^{2(k) + N - \frac{1}{2}}} \left( \tau_{y_0} \left( e^{-\frac{\tau}{\sqrt{t}} \|\cdot\|} \right) (x) + \tau_y \left( e^{-\frac{\tau}{\sqrt{t}} \|\cdot\|} \right) (x) \right).
\]

**Proof** From (4.1), we write
\[
\frac{\Lambda_{k,1} (x, y; t) - \Lambda_{k,1} (x, y_0; t)}{d (y, y_0)} = \frac{1}{(\sinh t)^{2(k) + N - 1}} \times \left( e^{-\tanh \frac{t}{2} (\|x\| + \|y\|)} \tau_x \left( e^{-\frac{1}{\sqrt{\sinh t}} \|\cdot\|} \right) (y) - e^{-\tanh \frac{t}{2} (\|x\| + \|y_0\|)} \tau_x \left( e^{-\frac{1}{\sqrt{\sinh t}} \|\cdot\|} \right) (y_0) \right) \frac{d (y, y_0)}{d (y, y_0)} + \left( e^{-\tanh \frac{t}{2} (\|x\| + \|y\|)} \tau_x \left( e^{-\frac{1}{\sqrt{\sinh t}} \|\cdot\|} \right) (y_0) - e^{-\tanh \frac{t}{2} (\|x\| + \|y_0\|)} \tau_x \left( e^{-\frac{1}{\sqrt{\sinh t}} \|\cdot\|} \right) (y_0) \right) \frac{d (y, y_0)}{d (y, y_0)} \right) + \frac{1}{(\sinh t)^{2(k) + N - 1}} (I_1 + I_2).
\]

Notice that \(\sinh t\) behaves like \(t\) for \(0 < t \leq 1\). For the second part \(I_2\). If \(||y|| = ||y_0||\), then \(I_2 = 0\). If \(||y|| \neq ||y_0||\), then from the inequality
\[
\left| e^{-\tanh \frac{t}{2} \cdot \frac{x^2}{2}} - e^{-\tanh \frac{t}{2} \cdot \frac{x_0^2}{2}} \right|_{x_1 - x_2} \leq \max_x \left| 2 \tanh \frac{t}{2} \cdot e^{-\tanh \frac{t}{2} \cdot x^2} \right| \leq C_2 \sqrt{t},
\]

Therefore, the second part \(I_2\) is also bounded by \(C_2 \sqrt{t}\).
we have

\[ |I_2| = \left| \tau_x \left( e^{-\frac{1}{\sinh \tau} \cdot \cdot \cdot} \right) (y_0) e^{-\tanh \frac{\tau}{2} (||x||+||y||)} - e^{-\tanh \frac{\tau}{2} (||x||+||y||)} \right| \frac{d(y, y_0)}{\sqrt{||y|| - \sqrt{||y_0||}}}
\]

\[ \leq \tau_x \left( e^{-\frac{1}{\sinh \tau} \cdot \cdot \cdot} \right) (y_0) \cdot \left| e^{-\tanh \frac{\tau}{2} ||y||} - e^{-\tanh \frac{\tau}{2} ||y_0||} \right| \frac{1}{\sqrt{||y|| - \sqrt{||y_0||}}}
\]

\[ \leq C_2 \sqrt{\tau \tau_x} \left( e^{-\frac{b_2}{\tau} \cdot \cdot \cdot} \right) (y_0).
\]

For the first part \( I_1 \), from the inequality

\[ \left| e^{-\frac{1}{\sinh \tau} \cdot \cdot \cdot} \right| \frac{2}{x_2 \leq x \leq x_1} \leq \frac{2}{\sqrt{\sinh \tau}} e^{-\frac{1}{\sinh \tau} \cdot \cdot \cdot} \max_{x_2 \leq x \leq x_1} \frac{1}{\sqrt{\sinh \tau}} e^{-\frac{1}{\sinh \tau} \cdot \cdot \cdot} \right|
\]

\[ \leq C_1 \sqrt{\tau} e^{-\frac{b_2}{\tau} \cdot \cdot \cdot}, \quad x_1 > x_2,
\]

along with Lemma 5.1 and (2.10).

\[ |I_1| \leq C_1 e^{-\tanh \frac{\tau}{2} (||x||+||y||)} V_k \left( \int_{-1}^{1} \frac{e^{-\frac{1}{\sinh \tau} \cdot \cdot \cdot} \left( \frac{1}{1 - u^2} \right)^{\frac{N}{2} + (k-2)} du \right) (x)
\]

\[ \leq C_1 \sqrt{\tau} e^{-\tanh \frac{\tau}{2} (||x||+||y||)} \int_{\mathbb{R}^N} \left( \int_{\{u \in [-1,1] : ||y|| - \sqrt{2(||x||+||y||)} u > ||y_0|| - \sqrt{2(||x||+||y||)} u \}} e^{-\frac{1}{\sinh \tau} \cdot \cdot \cdot} \left( \frac{1}{1 - u^2} \right)^{\frac{N}{2} + (k-2)} du 
\]

\[ + \int_{\{u \in [-1,1] : ||y|| - \sqrt{2(||x||+||y||)} u < ||y_0|| - \sqrt{2(||x||+||y||)} u \}} e^{-\frac{1}{\sinh \tau} \cdot \cdot \cdot} \left( \frac{1}{1 - u^2} \right)^{\frac{N}{2} + (k-2)} du \right) d\mu_\xi (\eta)
\]

\[ \leq C_1 \left( \tau y \left( e^{-\frac{b_2}{\tau} \cdot \cdot \cdot} \right) (x) + \tau y \left( e^{-\frac{b_2}{\tau} \cdot \cdot \cdot} \right) (x) \right).
\]
Thus
\[
\begin{align*}
\left| \Lambda_{k,1} (x, y; t) - \Lambda_{k,1} (x, y_0; t) \right| \\
\leq \frac{1}{t^{2(k)+N-1}} \left( C_2 \sqrt{t} + C_1 \right) \left( \tau_{y_0} \left( e^{-\frac{t\|x\|}{t}} \right) (x) + \tau_y \left( e^{-\frac{t\|x\|}{t}} \right) (x) \right) \\
\leq \frac{C}{t^{2(k)+N-1}} \left( \tau_{y_0} \left( e^{-\frac{t\|x\|}{t}} \right) (x) + \tau_y \left( e^{-\frac{t\|x\|}{t}} \right) (x) \right).
\end{align*}
\]

**Proof of Theorem 4.1** We only need to show that the operator \((-\Delta_{k,1})^{-i\sigma}\) is \(L^p\)-bounded for \(1 < p \leq 2\) and weakly \(L^1\)-bounded since it is symmetric on \(L^2(\mathbb{R}^N, \vartheta_{k,1}(x) \, dx)\) and its \(L^p\)-boundedness for \(2 < p < \infty\) can be derived from the duality argument. From (4.5), we write
\[
K (x, y) = \int_0^1 \Lambda_{k,1} (x, y; t) t^{i\sigma-1} \, dt + \int_1^\infty \Lambda_{k,1} (x, y; t) t^{i\sigma-1} \, dt = K^{(1)} (x, y) + K^{(2)} (x, y),
\]
where \(x, y \in \mathbb{R}^N, y \notin G.x\). We claim that \(K (x, y)\) satisfies the condition in Theorem 3.6.

For the second part \(K^{(2)} (x, y)\), by (4.2), (2.8) and (2.11),
\[
\begin{align*}
\int_{\mathbb{R}^N} \left| K^{(2)} (x, y) \right| \vartheta_{k,1} (x) \, dx \\
\leq C \int_{\mathbb{R}^N} \int_1^\infty e^{-(2k+N-1)t} \tau_y \left( e^{-b\|x\|} \right) (x) \frac{1}{t} \vartheta_{k,1} (x) \, dt \, dx \\
= C \int_1^\infty \int_{\mathbb{R}^N} e^{-(2k+N-1)t} e^{-b\|x\|} \frac{1}{t} \vartheta_{k,1} (x) \, dx \, dt \\
\leq C \int_1^\infty e^{-(2k+N-1)t} \frac{1}{t} \, dt \leq C.
\end{align*}
\]

Then we have
\[
\int_{d_G(x,y)>2d(y,y_0)} \left| K^{(2)} (x, y) - K^{(2)} (x, y_0) \right| \vartheta_{k,1} (x) \, dx \\
\leq 2 \int_{\mathbb{R}^N} \left| K^{(2)} (x, y) \right| \vartheta_{k,1} (x) \, dx \leq C.
\]

For the first part \(K^{(1)} (x, y)\), from Lemma 5.2,
\[
\begin{align*}
\left| K^{(1)} (x, y) - K^{(1)} (x, y_0) \right| \\
\leq \int_0^1 \left| \Lambda_{k,1} (x, y; t) - \Lambda_{k,1} (x, y_0; t) \right| \frac{1}{t} \, dt \\
\leq C d (y, y_0) \int_0^1 \frac{1}{t^{2(k)+N+\frac{1}{2}}} \left( \tau_{y_0} \left( e^{-\frac{t\|x\|}{t}} \right) (x) + \tau_y \left( e^{-\frac{t\|x\|}{t}} \right) (x) \right) \, dt.
\end{align*}
\]
When $d_G(x, y) > 2d(y, y_0)$, we have

$$d_G(x, y_0) \geq d_G(x, y) - d(y_0, y) > d(y, y_0), \quad d_G(x, y) > d(y, y_0).$$

Then from (3.2), for any $u \in [-1, 1]$ and $\eta \in co(G_x)$, we have

$$\sqrt{\|x\| + \|y_0\|} - \sqrt{2(\|x\| \|y_0\| + \langle \eta, y_0 \rangle)}u \geq d_G(x, y_0) > d(y, y_0),$$

$$\sqrt{\|x\| + \|y\|} - \sqrt{2(\|x\| \|y\| + \langle \eta, y \rangle)}u \geq d_G(x, y) > d(y, y_0).$$

So

$$\tau_x \left( e^{-\frac{x}{\tau} \|\cdot\|} \right)(y_0) \leq \tau_x \left( e^{-\frac{x}{\tau} \left( \sqrt{\|\cdot\| + d(y, y_0)} \right)^2} \right)(y_0), \quad \tau_x \left( e^{-\frac{x}{\tau} \|\cdot\|} \right)(y) \leq \tau_x \left( e^{-\frac{x}{\tau} \left( \sqrt{\|\cdot\| + d(y, y_0)} \right)^2} \right)(y).$$

Therefore, from (2.8) and (2.11),

$$\int_{d_G(x, y) > 2d(y, y_0)} \left| K^{(1)}(x, y) - K^{(1)}(x, y_0) \right| \vartheta_{k, 1}(x) \, dx$$

$$\leq Cd(y, y_0) \int_0^1 \frac{1}{t^{2(k)+N+\frac{1}{2}}} \left( \int_{\mathbb{R}^N} \tau_{y_0} \left( e^{-\frac{x}{\tau} \left( \sqrt{\|\cdot\| + d(y, y_0)} \right)^2} \right)(x) \vartheta_{k, 1}(x) \, dx \right) dt$$

$$= Cd(y, y_0) \int_0^1 \frac{1}{t^{2(k)+N+\frac{1}{2}}} dt \int_{\mathbb{R}^N} 2e^{-\frac{x}{\tau} \left( \sqrt{\|\cdot\| + d(y, y_0)} \right)^2} \vartheta_{k, 1}(x) \, dx$$

$$\leq Cd(y, y_0) \int_0^\infty r^{2(k)+N-2} dr \int_0^1 \frac{2}{t^{2(k)+N+\frac{1}{2}}} e^{-\frac{x}{\tau} \left( \sqrt{\tau + d(y, y_0)} \right)^2} dt$$

$$\leq Cd(y, y_0) \int_0^\infty \frac{r^{2(k)+N-2}}{(\sqrt{\tau + d(y, y_0)})^2} \, dr \int_0^\infty \frac{2}{u^{2(k)+N+\frac{1}{2}}} e^{-\frac{x}{\tau} du}$$

$$\leq Cd(y, y_0) \int_0^\infty \frac{1}{(\sqrt{\tau + d(y, y_0)})^3} dr = C.$$

The proof of Theorem 4.1 is complete. \square

**Acknowledgements** The author would like to thank his adviser Nobukazu Shimeno for valuable comments and advice. All data included in this study are available upon request by contact with the corresponding author.
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