Derivation of the Maxwell-Schrödinger Equations from the Pauli-Fierz Hamiltonian

Nikolai Leopold* and Peter Pickl†‡

July 2, 2018

Abstract

We consider the spinless Pauli-Fierz Hamiltonian which describes a quantum system of non-relativistic identical particles coupled to the quantized electromagnetic field. We study the time evolution in a mean-field limit where the number $N$ of charged particles gets large while the coupling to the radiation field is rescaled by $1/\sqrt{N}$. At time zero we assume that almost all charged particles are in the same one-body state (a Bose-Einstein condensate) and we assume also the photons to be close to a coherent state. We show that at later times and in the limit $N \to \infty$ the charged particles as well as the photons exhibit condensation, with the time evolution approximately described by the Maxwell-Schrödinger system, which models the coupling of a non-relativistic particle to the classical electromagnetic field. Our result is obtained by an extension of the "method of counting", introduced in [13], to condensates of charged particles in interaction with their radiation field.

MSC class: 35Q40, 81Q05, 81V10, 82C10

Keywords: mean-field limit, Pauli-Fierz Hamiltonian, Maxwell-Schrödinger equations

I Setting of the problem

The existence of light quanta, later named photons, was first postulated by Albert Einstein in his renowned paper "On a heuristic point of view about the creation and conversion of light" [3]. This led to the invention of Quantum Electrodynamics and supplemented the nature of light, which was formerly described as a wave in classical electromagnetism, with a particle interpretation. During the last decades the predictions of Quantum Electrodynamics has been tested up to highest accuracy. Nevertheless, in a lot of situations the corpuscular character of light is subordinate and the second-quantized electromagnetic field can be approximated by a classical field satisfying Maxwell’s equations. In this paper, the validity of such an approximation is justified in the mean-field regime. More explicitly, we derive the Maxwell-Schrödinger equations from the spinless Pauli-Fierz Hamiltonian. Such a derivation is of great interest to fundamental physics. Moreover, since the applied mean-field approximation reduces the degrees of freedom of the original system tremendously explicit error bounds might also be of interest for numerical simulations. We consider a system, described by a
wave function $\Psi_{N,t} \in \mathcal{H}^{(N)}$, of $N$ identical charged bosons in interaction with a photon field. Here,

$$\mathcal{H}^{(N)} := L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}_p,$$

where the photon field is represented by elements of the Fock space

$$\mathcal{F}_p := \bigoplus_{n \geq 0} \left[ L^2(\mathbb{R}^{3}) \otimes \mathbb{C}^2 \right]^\otimes_n.$$

The subscript $s$ indicates symmetry under interchange of variables. The Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^{3}) \otimes \mathbb{C}^2$ consists of wave functions $f(k, \lambda)$, with wave number $k \in \mathbb{R}^{3}$ and helicity $\lambda = 1, 2$. It is equipped with the inner product

$$\langle f, g \rangle_{\mathfrak{h}} := \sum_{\lambda=1,2} \int d^3k f^*(k, \lambda) g(k, \lambda).$$

The time evolution of $\Psi_{N,t}$ is governed by the Schrödinger equation

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t},$$

denotes the Pauli-Fierz Hamiltonian and

$$\hat{A}_\kappa(x) = \sum_{\lambda=1,2} \int d^3 k \tilde{\kappa}(k) \frac{1}{\sqrt{|k|}} \epsilon_\lambda(k) \left( e^{ikx}a(k, \lambda) + e^{-ikx}a^*(k, \lambda) \right)$$

the quantized transverse vector potential. The function

$$\tilde{\kappa}(k) = (2\pi)^{-3/2} \mathbb{1}_{|k| \leq \Lambda}(k), \quad \text{with } \mathbb{1}_{|k| \leq \Lambda}(k) = \begin{cases} 1 & \text{if } |k| \leq \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

cuts off the high frequency modes of the radiation field. There are two real polarization vectors $\epsilon_1(k)$ and $\epsilon_2(k)$ with

$$|\epsilon_1(k)| = |\epsilon_2(k)| = 1, \quad \epsilon_1(k) \cdot k = \epsilon_2(k) \cdot k = \epsilon_1(k) \cdot \epsilon_2(k) = 0.$$

The operator valued distributions $a(k, \lambda)$ and $a^*(k, \lambda)$ ($k \in \mathbb{R}^{3}, \lambda \in \{1,2\}$) are the usual pointwise annihilation and creation operators in $\mathcal{F}_p$, satisfying

$$[a(k, \lambda), a^*(l, \mu)] = \delta_{\lambda,\mu} \delta(k-l), \quad [a(k, \lambda), a(l, \mu)] = [a^*(k, \lambda), a^*(l, \mu)] = 0.$$

The energy of the photon field is given by

$$H_f = \sum_{\lambda=1,2} \int d^3k |k| a^*(k, \lambda) a(k, \lambda)$$

and the potential $v$ describes a direct interaction between the charged particles.
We assume:

(A1) The (repulsive) interaction potential $v$ is a positive, real, and even function satisfying

$$||v||_{L^2 + L^\infty} = \inf_{v = v_1 + v_2} \{||v_1||_{L^2(\mathbb{R}^3)} + ||v_2||_{L^\infty(\mathbb{R}^3)}\} < \infty$$

(11)

such that the Pauli-Fierz Hamiltonian $H_N$ is self-adjoint on the domain $\mathcal{D}(H_N) := \mathcal{D}(\sum_{i=1}^N -\Delta_i + H_f)$ (see [8] and [22, p.164]).

The mean-field scaling $1/N$ in front of the interaction potential and the scaling $1/\sqrt{N}$ in front of the vector potential ensure that the kinetic and potential energy of $H_N$ are of the same order. For simplicity, we are first interested in the evolution of initial states of the product form

$$\varphi_0^\otimes N \otimes W(\sqrt{N} \alpha_0)\Omega.$$ 

(12)

Here, $\Omega$ denotes the vacuum in $\mathcal{F}_p$ and $W(f)$ (with $f \in \mathfrak{h}$) is the unitary Weyl operator

$$W(f) := \exp \left( \sum_{\lambda=1,2} \int d^3k f(k, \lambda) a^*(k, \lambda) - f^*(k, \lambda) a(k, \lambda) \right).$$

(13)

This choice of initial data corresponds to situations in which both the charged particles and the photons exhibit condensation. Due to different types of interactions, correlations take place and the time evolved state will no longer have an exact product structure. However, for large $N$ and times of order one it can be approximated, in a sense specified below, by a state of the product form $\varphi^\otimes N \otimes W(\sqrt{N} \alpha_t)\Omega$, where

$$|k|^{1/2} \alpha_t(k, \lambda) := \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{F}T[A](k, t) - i \mathcal{F}T[E](k, t))$$

(14)

and $(\varphi_t, A(t), E(t))$ solve the Maxwell-Schrödinger system

$$\begin{cases}
  i \partial_t \varphi_t(x) = \left(-i \nabla - (\kappa \ast A)(x, t)\right)^2 + (v \ast |\varphi_t|^2)(x) \varphi_t(x), \\
  \nabla \cdot A(x, t) = 0, \\
  \partial_t A(x, t) = -E(x, t), \\
  \partial_t E(x, t) = (-\Delta A)(x, t) - (1 - \nabla \text{div} \Delta^{-1})(\kappa \ast j_i)(x), \\
  j_i(x) = 2 \left( \text{Im}(\varphi_t^* \nabla \varphi_t)(x) - |\varphi_t|^2(x)(\kappa \ast A)(x, t) \right)
\end{cases}$$

(15)

with initial datum

$$\begin{cases}
  \varphi_0, \\
  A(x, 0) = (2\pi)^{-3/2} \sum_{\lambda=1,2} \int d^3k \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) \left( e^{ikx} \alpha_0(k, \lambda) + e^{-ikx} \alpha_0^*(k, \lambda) \right), \\
  E(x, 0) = (2\pi)^{-3/2} \sum_{\lambda=1,2} \int d^3k \frac{k}{\sqrt{2|k|}} \epsilon_\lambda(k) i \left( e^{ikx} \alpha_0(k, \lambda) - e^{-ikx} \alpha_0^*(k, \lambda) \right).
\end{cases}$$

(16)

These equations determine the time evolution of a single quantum particle interacting with the classical electromagnetic field it generates. The solution theory of this system is well studied, see [17] and references therein.

---

1 Hereby, $(\kappa \ast A)(x, t) = \int d^3k e^{ikx} \tilde{\kappa}(k) A(k, t)$ and $\tilde{f}$ as well as $\mathcal{F}T[f]$ are used to denote the Fourier transform of a function $f$. 

---
II Main result

The physical situation we are interested in is the dynamical description of a Bose-Einstein condensate of charged particles. We start with an initial wave function of product form \( |\Psi_{N,t}\rangle = \prod_{i=1}^{N} |\psi_i(t)\rangle |\gamma_i(t)\rangle \) (a condition that will be relaxed later) and show that the condensate is stable over time, i.e. correlations are small at later times. Let \( \Psi_{N,t} \in (L^2(\mathbb{R}^3))^N \cap \mathcal{H}^{(N)}\) with \( ||\Psi_{N,t}|| = 1 \).

On the Hilbert space \( L^2(\mathbb{R}^3) \), define the "one-particle reduced density matrix of the charged particles" by

\[
\gamma_{N,t}^{(1,0)} := \text{Tr}_{2,...,N} \otimes \text{Tr}_F |\Psi_{N,t}\rangle \langle \Psi_{N,t}|,
\]

where \( \text{Tr}_{2,...,N} \) denotes the partial trace over the coordinates \( x_2, \ldots, x_N \) and \( \text{Tr}_F \) the trace over Fock space. The charged particles of the many-body state \( \Psi_{N,t} \) are said to exhibit complete asymptotic Bose-Einstein condensation at time \( t \), if there exists \( \varphi_t \in L^2(\mathbb{R}^3) \) with \( ||\varphi_t|| = 1 \), such that

\[
\text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)}| - ||\varphi_t|| \langle \varphi_t | \rightarrow 0,
\]

as \( N \to \infty \). Such \( \varphi_t \) is called the condensate wave function. For other indicators of condensation and their relation we refer to \([15]\). Given \( \Psi_{N,t} \in \mathcal{D}(H_f) \) with \( ||\Psi_{N,t}|| = 1 \), we introduce the "one-particle reduced energy matrix of the photons" with kernel

\[
\gamma_{N,t}^{(0,1)}(k, \lambda; k', \lambda') := N^{-1} |k|^{1/2} |k'|^{1/2} \langle \Psi_{N,t}, a^*(k', \lambda') a(k, \lambda) \Psi_{N,t} \rangle_{H^{(N)}}.
\]

\( \gamma_{N,t}^{(0,1)} \) is a positive trace class operator on \( \mathfrak{h} \) with \( \text{Tr}_h(\gamma_{N,t}^{(0,1)}) = N^{-1} \langle \Psi_{N,t}, H_f \Psi_{N,t} \rangle_{\mathcal{H}^{(N)}} \). It is important to note, that \((19)\) differs from the usual definition (e.g. \([21\) p.8]) by the weight factor \( |k|^{1/2} |k'|^{1/2} \langle \Psi_{N,t}, N \Psi_{N,t} \rangle_{\mathcal{H}^{(N)}} / N \) with \( N \) being the number of photons operator. Our choice ensures that we neglect photons with small energies and measure only deviations from the photon field that are at least of order \( N \). This is reasonable because due to the scaled coupling many photon states with a mean particle number smaller than of order \( N \) only have a subleading effect on the dynamics of the charged particles. We say the photons exhibit asymptotic Bose-Einstein condensation, if there exists a state \( u_t \in \mathfrak{h} \), such that

\[
\text{Tr}_h |\gamma_{N,t}^{(0,1)}| - ||u_t|| \langle u_t || \rightarrow 0,
\]

as \( N \to \infty \).

In order to prove our main theorem, we have to require regularity assumptions on the solutions of the Maxwell-Schrödinger system.

**Definition II.1.** Let \( m \in \mathbb{N} \), and \( H^m(\mathbb{R}^3, \mathbb{C}^k) \) denote the Sobolev space of order \( m \) with norm \( ||f||_{H^m(\mathbb{R}^3, \mathbb{C}^k)} = \sum_{i=1}^k \int d^3k (1 + |k|^2)^m |\mathcal{F}(f^i)(k)|^2 \). We define the following set of solutions of the Maxwell-Schrödinger equations:

\[
(\varphi_t, A(t), E(t)) \in \mathcal{G} \Leftrightarrow (a) \quad (\varphi_0, A(0), E(0)) \text{ is given by } (16) \text{ with } \alpha_0 \in \mathfrak{h} \text{ and } \varphi_0 \in L^2(\mathbb{R}^3) \\
(b) \quad (\varphi_t, A(t), E(t)) \text{ is a } L^2 \oplus L^2 \oplus L^2 \text{ solution of } (13) \\
(c) \quad (\varphi_t, A(t), E(t)) \in H^2(\mathbb{R}^3, \mathbb{C}) \oplus H^2(\mathbb{R}^3, \mathbb{C}^3) \oplus H^1(\mathbb{R}^3, \mathbb{C}^3) \\
(d) \quad ||\varphi_t||_{L^2(\mathbb{R}^3)} = 1 \quad \text{for all } t \geq 0.
\]

We expect these assumptions to follow from appropriately chosen initial data. In the absence of a cutoff function and \( v \) being the Coulomb potential it has for example been shown in \([17]\)
that the Maxwell-Schrödinger system is globally well-posed in the space $C(\mathbb{R}_t, H^2(\mathbb{R}^3, \mathbb{C}) \oplus H^2(\mathbb{R}^3, \mathbb{C}^3) \oplus H^1(\mathbb{R}^3, \mathbb{C}^3))$.

Moreover, we have to introduce the energy mode function of the electromagnetic field:

$$u_t(k, \lambda) := |k|^{1/2} \alpha(k, \lambda) := \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{F}T[A](k, t) - i \mathcal{F}T[E](k, t)).$$

(22)

and the energy functional of the Maxwell-Schrödinger system

$$\mathcal{E}_M[\varphi_t, u_t] := |\langle (\nabla - A_\lambda(t)) \varphi_t \rangle|^2 + 1/2 \langle \varphi_t, (v * |\varphi_t|^2) \varphi_t \rangle + ||u_t||^2_\mathcal{H}.$$  

(23)

**Theorem II.2.** Let $v$ satisfy (A1), $\varphi_0 \in L^2(\mathbb{R}^3)$ with $||\varphi_0|| = 1$, $u_0 \in \mathfrak{h}$ so that $\alpha_0 = |k|^{-1/2}u_0 \in \mathfrak{h}$ and $\Psi_{N,0} \in \mathcal{D}(H_N) \cap (L^2(\mathbb{R}^3)^N \otimes \mathcal{F}_p)$ such that

$$a_N := Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,0}^{(1,0)} - |\varphi_0|| \rightarrow 0,$$

(24)

$$b_N := N^{-1}\langle W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}, H_f W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0} \rangle_{H(N)} \rightarrow 0$$

(25)

$$c_N := \left(\langle (N^{-1}H_N - \mathcal{E}_M[\varphi_0, u_0]) \Psi_{N,0} \rangle \right)^2 \rightarrow 0$$

(26)

as $N \to \infty$. Let $\Psi_{N,t}$ be the unique solution of $\mathbf{1}$, $(\varphi_t, A(t), E(t)) \in \mathcal{G}$ and $u_t$ be defined by $\mathbf{22}$. Then, there exists a monotone increasing function $C(s)$ of the norms $||\varphi_s||_{H^2(\mathbb{R}^3, \mathbb{C})}$, $||A(s)||_{H^2(\mathbb{R}^3, \mathbb{C})}$ and $||E(s)||_{L^2(\mathbb{R}^3, \mathbb{C})}$ such that

$$Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t|| \leq a_N + b_N + c_N + N^{-1} A e^{A^4 \int_0^t ds C(s)},$$

(27)

$$Tr_{\mathfrak{h}}|\gamma_{N,t}^{(0,1)} - |u_t|| \leq a_N + b_N + c_N + N^{-1} A e^{A^4 \int_0^t ds C(s)}.$$  

(28)

for any $t \geq 0$. In particular, for $\Psi_{N,0} = \varphi_0^N \otimes W(\sqrt{N}\alpha_0)\Omega$ one obtains

$$Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t|| \leq N^{-1/2} A e^{A^4 \int_0^t ds C(s)}$$

(29)

$$Tr_{\mathfrak{h}}|\gamma_{N,t}^{(0,1)} - |u_t|| \leq N^{-1/2} A e^{A^4 \int_0^t ds C(s)}.$$  

(30)

**Remark II.3.** Assumption (A1) allows to consider the Coulomb potential $v(x) = |x|^{-1}$. The requirements on the interaction potential can easily be relaxed because our estimates only rely on the finiteness of $||v * |\varphi_t|^2||_{L^\infty(\mathbb{R}^3)}$ and $||v^2 * |\varphi_t|^2||_{L^\infty(\mathbb{R}^3)}$. This is captured by (A1) and $\varphi_t \in H^2(\mathbb{R}^3)$ but also by other means.

**Remark II.4.** For simplicity we apply the mean-field scaling $1/N$ in front of the direct interaction. Using techniques from [19] and [20] it seems possible to treat the direct interaction also in the NLS or Gross-Pitaevskii regime.

**Remark II.5.** The minimal coupling term in the Pauli-Fierz Hamiltonian leads to an interaction between the charges and the radiation field which is more singular than for example in the Nelson model. This makes it difficult to control the number of photons with small energies during the time evolution and causes technical problems in the estimates. To overcome these difficulties we neglect contributions from photons with small energies and restrict our initial data to a subspace of many-body states whose energy per particle only fluctuates little around the energy of the effective system. This is the reason why we consider the one-particle reduced energy matrix instead of the one-particle reduced density matrix of the photons. Moreover, it explains the appearance of condition [20].

**Remark II.6.** The ultraviolet cutoff is essential in our derivation but can be chosen $N$-dependent.

---

2 The direct sum of the Sobolev spaces refers to $(\varphi_t, A(t), E(t))$.

3 The name energy mode function is motivated from the fact that $||u_t||^2_\mathcal{H} = \langle u_t, u_t \rangle_\mathcal{H}$ is the energy of the electromagnetic field. It should be noted that $u_t \in \mathfrak{h}$ and $\mathcal{E}_M[\varphi_t, u_t] < \infty$ follow from $(\varphi_t, A(t), E(t)) \in \mathcal{G}$ (see [31] and [32]).
III Comparison with the literature

Derivations of classical field equations from Many-body Quantum Dynamics has been established in a series of works: In [7], Ginibre, Nironi and Velo derived the Schrödinger-Klein-Gordon system of equations from the Nelson model with cutoff. They considered a mean-field limit where a finite number of charged particles interacts with a coherent state of gauge bosons whose particle number goes to infinity. Falconi [4] derived the Schrödinger-Klein-Gordon system of equations in a mean-field limit where both the number of the charged particles and the gauge bosons go to infinity. Making use of a Wigner measure approach Ammari and Falconi [1] were able to establish the classical limit of the renormalized Nelson model without cutoff. The replacement of quantized radiation fields by classical interactions has also been justified in other limits. Teufel [24] considered the adiabatic limit of the Nelson model and showed that the interaction mediated by the quantized radiation field is well approximated by a direct Coulomb interaction. In [10], Frank, Gang and Schlein showed that in the strong coupling limit the dynamics of a polaron is described by an effective equation, in which the phonon field is treated as a classical field. Knowles [18] analyzed a finite number of heavy particles in a strong radiation field and derived the Newton-Maxwell equations from the Pauli Fierz Hamiltonian. In [22] it is shown that the semiclassical set of coupled Maxwell-Schrödinger equations is obtained by neglecting certain terms of the Pauli-Fierz Hamiltonian. To our best knowledge, this is the first rigorous result concerning a mean-field limit of the Pauli-Fierz Hamiltonian. This work continues the master thesis [14].

IV Notations

We set Planck’s constant $\hbar$, the speed of light $c$, the charge $e$, and twice the mass of the particles $2m$ equal to one. Except in definitions, results, and where confusion might be possible, we refrain from indicating the explicit dependence of a quantity on the time $t$. We use the notations $\varphi(t)$ and $\varphi_t$ interchangeably to denote a quantity $\varphi$ at time $t$. The symbol $C$ is used as a generic positive constant independent of $t$. We use expressions like $C(||\varphi||_{H^2(\mathbb{R}^3)}, ||A||_{L^2(\mathbb{R}^3, \mathbb{C})})$ to denote positive monotone increasing function of the norms indicated. Both $\tilde{f}$ and $\mathcal{F}f$ stand for the Fourier transform of $f$. With a slight abuse of notation $A$ and $E$ denote the vector potential and the electric field, but also their respective Fourier transforms. If we write $A(t)$ or $E(t)$, we always refer to the coordinate representation of the electromagnetic fields. Furthermore, we apply the shorthand notation $A_\kappa(x, t) = (\kappa \ast A)(x, t)$. $H^m(\mathbb{R}^3, \mathbb{C}^k)$ stand for the Sobolev space of order $m$ with norm $\|f\|^2_{H^m(\mathbb{R}^3, \mathbb{C}^k)} = \sum_{i=1}^k \int d^3k (1 + |k|^2)^m |\mathcal{F}f| |k|^2$. To simplify the notation we use $H^m(\mathbb{R}^3)$ for $H^m(\mathbb{R}^3, \mathbb{C}^1)$ with $k \in \{1, 2, 3\}$. $\|A\|_{HS} = \sqrt{TrA^\dagger A}$ stands for the Hilbert-Schmidt norm and $\langle \cdot, \cdot \rangle$ denotes the scalar products on $\mathcal{H}(N)$. $L^2(\mathbb{R}^3)$ and $\mathfrak{h}$. Furthermore, we use the shorthand notation $\langle \cdot, \cdot \rangle_2 = \int d^3y \langle \cdot, \cdot \rangle$ and $||.||_2 = \int d^3y ||\cdot||$. In order to stress the connection between the annihilation operator and the mode function of the electromagnetic field we mostly write $|k|^{1/2} \alpha_t(k, \lambda)$ instead of $u_t(k, \lambda)$.

V Organization of the proof

Our result is obtained by an extension of the "method of counting", introduced in [18], to condensates of charged particles in interaction with their radiation field [18]. The key idea is not to prove condensation in terms of reduced density matrices but to consider a different indicator of condensation. More specific, we introduce a functional $\beta(t) : \mathcal{D}(H_\mathcal{N}) \times H^2(\mathbb{R}^3) \times$
\( \mathfrak{h} \to \mathbb{R}_0^+ \) with the properties:

(a) \( \beta(0) := \beta [\Psi_0, \varphi_0, u_0] \to 0 \) as \( N \to 0 \) for appropriately chosen initial data.

(b) For each \( t \in \mathbb{R}_0^+ \) we are able to control the time-derivative of the functional by \( |d_t \beta(t)| \leq C(\beta(t) + N^{-1}) \). Then, \( \beta(t) \leq e^{C t}(\beta(t) + N^{-1}) \) follows by Gronwall’s Lemma.

(c) \( \beta(t) \to 0 \) as \( N \to \infty \) implies \([18]\) and \([20]\).

The proof is organized as follows:

(a) In Section \([\text{VIII}]\) we define the counting functional. Afterwards, we show that convergence of the functional to zero in the limit \( N \to \infty \) implies condensation in terms of reduced density matrices.

(b) In section \([\text{VIII}]\) we control the growth of \( \beta \) by means of a Gronwall estimate. To this end, we provide preliminary estimates and control the time derivative of \( \beta \).

(c) Then, we relate the value of the functional at time zero to the initial data of Theorem \([\text{II}2]\).

Subsequently, we require \((\varphi_t, A(t), E(t)) \in \mathcal{G} \). This implies

\[
|\nabla \varphi_t| < \infty, \quad |\Delta \varphi_t| < \infty, \quad |||\varphi|||_\infty < \infty, \quad |||A_k(t)|||_\infty < \infty,
\]

and continue with

\[
||u_t||_\mathfrak{h} < \infty, \quad |||\cdot|||^{1/2} u_t|||_\mathfrak{h} = \left( \sum_{\lambda=1,2} d^3 k |k|^2 |\alpha_t(k, \lambda)|^2 \right)^{1/2} < \infty.
\]

Proof. Assuming that \((\varphi_t, A(t), E(t)) \in \mathcal{G} \) we have \( \varphi_t \in H^2(\mathbb{R}^3) \) and the first three relations follow from Sobolev inequalities. In order to show the finiteness of \(||A_k(t)||_\infty \) we define the functions

\[
\kappa_<(k) := (2\pi)^{-3/2} 1_{|k| \leq 1}(k), \quad \kappa_>(k) := (2\pi)^{-3/2} |k|^{-2} 1_{1 \leq |k| \leq \Lambda}(k)
\]

and with

\[
A_k(x, t) = (2\pi)^{-3/2} \int d^3 k e^{ikx} 1_{|k| \leq \Lambda}(k) A(k, t) = (2\pi)^{-3/2} \int d^3 k e^{ikx} 1_{|k| \leq 1}(k) A(k, t)
\]

\[
+ (2\pi)^{-3/2} \int d^3 k e^{ikx} |k|^{-2} 1_{1 \leq |k| \leq \Lambda}(k) |k|^2 A(k, t)
\]

\[
= (\kappa_< \ast A)(x, t) - (\kappa_\ast \Delta A)(x, t).
\]

So if we use Young’s inequality, \( ||\kappa_\ast||_2^2 = (6\pi^2)^{-1} \) and \( ||\kappa_\ast||^2_2 \leq (4\pi^2)^{-1} \), we get

\[
||A_k(t)||_\infty \leq ||(\kappa_\ast \ast A)(t)||_\infty + ||(\kappa_\ast \ast \Delta A)(t)||_\infty
\]

\[
\leq ||\kappa_\ast|| ||A(t)|| + ||\kappa_\ast|| ||\Delta A(t)|| < ||A(t)||_{H^2(\mathbb{R}^3)}.
\]

By means of

\[
\sum_{\lambda=1,2} e^\lambda(k) e^\lambda(k) = \delta_{ij} - \frac{k_i k_j}{|k|^2}
\]

one easily shows

\[
||u_t||_2^2 = 1/2 \int d^3 k \left( |k|^2 A^2(k, t) + E^2(k, t) \right) \leq ||A(t)||_{H^1(\mathbb{R}^3)}^2 + ||E(t)||^2
\]

\[
|||\cdot|||^{1/2} u_t|||_\mathfrak{h}^2 = 1/2 \int d^3 k \left( |k|^3 A^2(k, t) + |k| E^2(k, t) \right)
\]

\[
\leq ||A(t)||_{H^1(\mathbb{R}^3)} ||A(t)||_{H^2(\mathbb{R}^3)} + ||E(t)|| ||E(t)||_{H^1(\mathbb{R}^3)}.
\]

\( \square \)
VI The counting functional

In this section, we introduce a new indicator of condensation referred to as the "counting functional. Our system under consideration describes the interaction between charged particles and the quantized electromagnetic field. Initially, we assume the charges and photons to exhibit condensation and we would like to show that both condensates are stable over time.

In case of the charges, this is done by means of a functional, denoted by \( \beta^a \), which counts for each time \( t \) the relative number of charges which are not in the state \( \varphi_t \). Under suitable conditions on the photon field it is then possible to show that the rate of particles which leave the condensate is small, if initially almost all particles are in the state \( \varphi_0 \). The situation is different for the radiation field because the number of photons is not a conserved quantity.

On that account not only existing photons gets correlated but also new photons are created or destroyed. One should note that the high frequency modes of the radiation field do not interact with the charges due to the ultraviolet cutoff and evolve according to the free evolution. This is why neither the number of photons changes nor the photon state shows correlations for wave-numbers \( |k| \geq \Lambda \). However in the long wave-length sector of \( \mathcal{F}_b \) correlations take place and the number of photons varies.

To show that the photon field remains coherent we introduce the functional \( \beta^b \) measuring for each time \( t \) the fluctuations of the photon field around the classical mode function. An additional factor of \( |k| \) in the integral implies that we neglect contributions from photons with small energies. The main difficulties in our derivation arise from the minimal coupling term in the Pauli-Fierz Hamiltonian. On that account we have to control expectation values of certain unbounded operators, see Subsection VIII.1.

This is established by \( \beta^c \) which restricts our consideration to a subspace of many-body states whose energy per particle only fluctuates little around the energy functional of the effective system.

In order to define the counting functional we introduce the projectors \( p^c_j \) and \( q^c_j \).

**Definition VI.1.** For any \( N \in \mathbb{N} \), \( \varphi_t \in L^2(\mathbb{R}^3) \) with \( \| \varphi_t \| = 1 \) and \( 1 \leq j \leq N \) we define the time-dependent projectors \( p^c_j : L^2(\mathbb{R}^3 N) \to L^2(\mathbb{R}^3 N) \) and \( q^c_j : L^2(\mathbb{R}^3 N) \to L^2(\mathbb{R}^3 N) \) by

\[
p^c_j f(x_1, \ldots, x_N) := \varphi_t(x_j) \int d^3x_j \varphi_t^*(x_j) f(x_1, \ldots, x_N) \quad \text{for all } f \in L^2(\mathbb{R}^3 N)
\]

and \( q^c_j := 1 - p^c_j \).

The counting functional is defined by

**Definition VI.2.** Let \( \Psi_{N,t} \in \mathcal{D}(H_N) \), \( \varphi_t \in H^2(\mathbb{R}^3) \), \( u_t \in \mathfrak{h} \) and \( \mathcal{E}_M[\varphi_t, u_t] \) defined by \( \psi \) and \( \kappa \).

Then

\[
\beta^a(\Psi_{N,t}, \varphi_t) := \langle \Psi_{N,t}, q_1 \otimes 1_{\mathcal{F}_b} \Psi_{N,t} \rangle,
\]

\[
\beta^b(\Psi_{N,t}, u_t) := \sum_{\lambda=1,2} \int d^3 k \left| \left| k \right|^{1/2} \frac{a(k, \lambda)}{\sqrt{N}} - u_t(k, \lambda) \right| \Psi_{N,t} \|^{2}
\]

\[
= \sum_{\lambda=1,2} \int d^3 k |k| \left( \frac{a(k, \lambda)}{\sqrt{N}} - \alpha_t(k, \lambda) \right) \Psi_{N,t} \left( \frac{a(k, \lambda)}{\sqrt{N}} - \alpha_t(k, \lambda) \right) \Psi_{N,t},
\]

\[
\beta^c(\Psi_{N,t}, \varphi_t, u_t) := \langle \frac{H_N}{N} - \mathcal{E}_M[\varphi_t, u_t] \rangle \Psi_{N,t} \left( \frac{H_N}{N} - \mathcal{E}_M[\varphi_t, u_t] \right) \Psi_{N,t} \rangle .
\]

The functional \( \beta : \mathcal{D}(H_N) \times H^2(\mathbb{R}^3) \times \mathfrak{h} \to \mathbb{R}_+^a \) is given by \( \beta := \beta^a + \beta^b + \beta^c. \)

\footnote{For ease of notation we mostly omit the superscript \( \varphi_t \) in the following. Additionally, we use the bra-ket notation \( p^c_j = |\varphi_t(x_j)> <\varphi_t(x_j)| \).}
The functional $\beta^a$ was already used in [2, 9, 11, 16, 18, 19, 20] and others to derive the Hartree and Gross-Pitaevskii equation, while $\beta^b$ and $\beta^c$ are introduced to control the interaction with the radiation field.

VII Relation to reduced density matrices

The aim of this section is to show that condensation indicated by the counting functional, $\beta \to 0$ as $N \to \infty$, implies condensation in terms of reduced density matrices.

Lemma VII.1. Let $\Psi_{N,t} \in \mathcal{D}(H_N)$, $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $u_t \in \mathfrak{h}$. Then

$$\beta^a(\Psi_{N,t}, \varphi_t) \leq \text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)}| - |\varphi_t\rangle \langle \varphi_t| \leq \sqrt{8\beta^a(\Psi_{N,t}, \varphi_t)}, \quad (40)$$

$$\text{Tr}_b|\gamma_{N,t}^{(0,1)} - |u_t\rangle \langle u_t| \leq 3\beta^b(\Psi_{N,t}, u_t) + 6||u_t||_b \sqrt{\beta^b(\Psi_{N,t}, u_t)}. \quad (41)$$

Proof. The first inequality follows from [3]:

$$\beta^a = 1 - \langle \Psi_N, p_1 \Psi_N \rangle = 1 - \langle \varphi, \gamma_N^{(1,0)} \rangle = \text{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle \langle \varphi| - |\varphi\rangle \langle \varphi| \gamma_N^{(1,0)})$$

$$\leq ||p_1||_{op} \text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)}| - |\varphi\rangle \langle \varphi| = \text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)}| - |\varphi\rangle \langle \varphi| \leq 2 \beta^a. \quad (42)$$

In order to prove the remaining inequalities we use

$$\text{Tr} |\gamma - p| \leq 2 ||\gamma - p||_{HS} + \text{Tr}(\gamma - p), \quad (43)$$

valid for any one-dimensional projector $p$ and non-negative density matrix $\gamma$. The original argument of the proof was first observed by Robert Seiringer, see [2]. We present a version that is found in [2]: Let $(\lambda_n)_{n \in \mathbb{N}}$ be the sequence of eigenvalues of the trace class operator $A := \gamma - p$. Since $p$ is a rank one projection, $A$ has at most one negative eigenvalue. If there is no negative eigenvalue, $\text{Tr}|A| = \text{Tr}(A)$ and (43) holds. If there is one negative eigenvalue $\lambda_1$, we have $\text{Tr}|A| = |\lambda_1| + \sum_{n \geq 2} \lambda_n = 2|\lambda_1| + \text{Tr}(A)$. Because of $|\lambda_1| \leq ||A||_{op} \leq ||A||_{HS}$, inequality (44) follows.

For the upper bound of (41) we notice that $\text{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle \langle \varphi|) = 0$. Then, (43) reduces to

$$\text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)}| - |\varphi\rangle \langle \varphi| \leq 2 \left| \left| |\gamma_N^{(1,0)}| - |\varphi\rangle \langle \varphi| \right| \right|_{HS} \quad (44)$$

and (40) follows from

$$\text{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle \langle \varphi|) = 1 - 2\text{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle \langle \varphi| \gamma_N^{(1,0)}) + \text{Tr}_{L^2(\mathbb{R}^3)}(|\gamma_N^{(1,0)}|^2) \leq 2(1 - \text{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle \langle \varphi| \gamma_N^{(1,0)})) = 2\beta^a. \quad (45)$$

To prove inequality (41) it is useful to write the kernel of $\gamma_N^{(0,1)} - |u\rangle \langle u|$ as

$$\gamma_N^{(0,1)} - |u\rangle \langle u| = |k|^{1/2} |l|^{1/2} \left( N^{-1} \langle \Psi_N, a^*(l,\mu)a(k,\lambda)\Psi_N \rangle - a^*(l,\mu)a(k,\lambda) \right)$$

$$= |k|^{1/2} |l|^{1/2} \left( \left( N^{-1/2} a(l,\mu) - \alpha(l,\mu) \right) \Psi_N, \left( N^{-1/2} a(k,\lambda) - \alpha(k,\lambda) \right) \Psi_N \right)$$

$$+ |k|^{1/2} |l|^{1/2} \left( \left( N^{-1/2} a(l,\mu) - \alpha(l,\mu) \right) \Psi_N, 0 \right) a^*(l,\mu)$$

$$+ |k|^{1/2} |l|^{1/2} \left( 0, \left( N^{-1/2} a(k,\lambda) - \alpha(k,\lambda) \right) \Psi_N \right). \quad (46)$$

\footnote{For ease of notation, we discard the explicit time dependence and write for example $\Psi_N$ instead of $\Psi_{N,t}$.}
Cauchy-Schwarz inequality gives
\[
|\langle \gamma_N^{(0,1)} - |u\rangle \langle u|)(k, \lambda, \mu) |^2
\leq |k| |l| \left| \left( N^{-1/2} a(k, \lambda) - \alpha(k, \lambda) \right) \right| \Psi_N \left| \right|^2 \left| \left( N^{-1/2} a(l, \mu) - \alpha(l, \mu) \right) \right| \Psi_N \left| \right|^2
+ |k| |l| \left| \left( N^{-1/2} a(k, \lambda) - \alpha(k, \lambda) \right) \right| \Psi_N \left| \right|^2 |\alpha(l, \mu)|^2
+ |k| |l| \left| \left( N^{-1/2} a(l, \mu) - \alpha(l, \mu) \right) \right| \Psi_N \left| \right|^2 |\alpha(k, \lambda)|^2
\]
and
\[
\left| \left| \gamma_N^{(0,1)} - |u\rangle \langle u| \right| \right|_{HS}^2 = \sum_{\lambda, \mu \in \{1, 2\}^2} \int \int d^3 k d^3 l \left| \left( N^{-1/2} a(k, \lambda) - \alpha(k, \lambda) \right) \right| \Psi_N \left| \right|^2 \left| \left( N^{-1/2} a(l, \mu) - \alpha(l, \mu) \right) \right| \Psi_N \left| \right|^2 
\leq (\beta^b)^2 + 2 \| u \|_b \beta^b
\]
follows. Similarly,
\[
\text{Tr}_b(\gamma_N^{(0,1)} - |u\rangle \langle u|) \leq \sum_{\lambda=1,2} \int d^3 k \left| \left( N^{-1/2} a(k, \lambda) - \alpha(k, \lambda) \right) \right| \Psi_N \left| \right|^2 
+ 2 \sum_{\lambda=1,2} \int d^3 k |u(k, \lambda)||k||^1/2 \left| \left( N^{-1/2} a(k, \lambda) - \alpha(k, \lambda) \right) \right| \Psi_N \left| \right|.
\]
Applying Schwarz’s inequality with respect to the scalar product of $\hbar$ yields
\[
\text{Tr}_b(\gamma_N^{(0,1)} - |\alpha\rangle \langle \alpha|) \leq \beta^b + 2 \| u \|_b \left( \sum_{\lambda=1,2} \int d^3 k |k||^1/2 \left| \left( N^{-1/2} a(k, \lambda) - \alpha(k, \lambda) \right) \right| \Psi_N \left| \right|^2 \right)^{1/2}
\leq \beta^b + 2 \| u \|_b \sqrt{\beta^b}.
\]
Monotonicity of the square root and (43) give rise to (41).

**VIII  Estimates on the time derivative**

In this section, we control the change of $\beta$ in time. To this end we separately estimate the time derivative of $\beta^a$ and $\beta^b$. The value of $\beta^a$ is constant in time because the energies of the many-body and the Maxwell-Schrödinger system are conserved quantities. To control the difference between the quantized and classical vector potential by the functional $\beta^b$ it is convenient to introduce their positive and negative frequency parts.

\[
\hat{A}_\kappa^+(x) := \sum_{\lambda=1,2} \int d^3 k \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) e^{ikx} a(k, \lambda),
\]

\[
\hat{A}_\kappa^-(x) := \sum_{\lambda=1,2} \int d^3 k \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) e^{-ikx} a^*(k, \lambda),
\]

\[
A_\kappa^+(x, t) := \sum_{\lambda=1,2} \int d^3 k \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) e^{ikx} a_t(k, \lambda),
\]

\[
A_\kappa^-(x, t) := \sum_{\lambda=1,2} \int d^3 k \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) e^{-ikx} a_t^*(k, \lambda).
\]
Moreover, it is helpful to define the positive and negative frequency parts of the quantum mechanical and classical electric field.

\[
\tilde{E}_\kappa^+(x) := \sum_{\lambda=1,2} \int d^3k \tilde{\kappa}(k)\sqrt{\frac{|k|}{2}}\epsilon_\lambda(k)ie^{ikx}a(k, \lambda),
\]

\[
\tilde{E}_\kappa^-(x) := \sum_{\lambda=1,2} \int d^3k \tilde{\kappa}(k)\sqrt{\frac{|k|}{2}}\epsilon_\lambda(k)(-i)e^{-ikx}a^*(k, \lambda),
\]

\[
E_\kappa^+(x, t) := \sum_{\lambda=1,2} \int d^3k \tilde{\kappa}(k)\sqrt{\frac{|k|}{2}}\epsilon_\lambda(k)ie^{ikx}\alpha(k, \lambda),
\]

\[
E_\kappa^-(x, t) := \sum_{\lambda=1,2} \int d^3k \tilde{\kappa}(k)\sqrt{\frac{|k|}{2}}\epsilon_\lambda(k)(-i)e^{-ikx}\alpha^*(k, \lambda).
\]

(52)

For \(\tilde{\kappa} \in \{\, , +, -\, \},\) we introduce the shorthand notations

\[
E^\tilde{\kappa}(x, t) := \frac{\tilde{E}_\kappa^\tilde{\kappa}(x)}{\sqrt{N}} - E_\kappa^\mp(x, t), \quad A^\tilde{\kappa}(x, t) := \frac{\tilde{A}_\kappa^\tilde{\kappa}(x)}{\sqrt{N}} - A_\kappa^\mp(x, t).
\]

(53)

By means of the cutoff function

\[
\tilde{\eta}(k) := |k|^{-1}\tilde{\kappa}(k) = (2\pi)^{-\frac{d}{2}}|k|^{-1}1_{|k| \leq \Lambda}(k)
\]

(54)

we are able to express the vector potential in terms of the electric field.

**Lemma VIII.1.** Let \(\eta\) be the Fourier transform of \(\tilde{\kappa}\), then

\[
\tilde{A}_\kappa^+(x) = -i(\eta * \tilde{E}_\kappa^+(x)), \quad \tilde{A}_\kappa^-(x) = i(\eta * \tilde{E}_\kappa^-)(x),
\]

\[
A_\kappa^+(x, t) = -i(\eta * E_\kappa^+(x, t)), \quad A_\kappa^-(x, t) = i(\eta * E_\kappa^-)(x, t).
\]

(55)

**Proof.** The proof is a simple application of the convolution theorem. \(\square\)

At various points in our estimates, we replace the vector potential by the electric field and make use of (see Lemma [11])

\[
\int d^3y \langle \Psi_{N, t} | N^{-1/2}\tilde{E}_\kappa^-(y) - E_\kappa^-(y, t) \rangle \left( N^{-1/2}\tilde{E}_\kappa^+(y) - E_\kappa^+(y, t) \right) \Psi_{N, t} \leq \beta(t).
\]

(56)

To obtain proper bounds it is crucial that the \(L^2\)-norm of the cutoff functions

\[
||\kappa||^2_2 = \Lambda^3/(6\pi^2) \quad \text{and} \quad ||\eta||^2_2 = \Lambda/(2\pi^2)
\]

(57)

is finite.

**VIII.1 Preliminary estimates**

The minimal coupling term in the Pauli-Fierz Hamiltonian

\[
\sum_{j=1}^{N}(-i\nabla_j - N^{-1/2}\tilde{A}_\kappa(x_j))^2 = \sum_{j=1}^{N}\left(-\Delta_1 + 2iN^{-1/2}\tilde{A}_\kappa(x_1) \cdot \nabla_1 + N^{-1}\tilde{A}_\kappa^2(x_1)\right)
\]

(58)

contains an interaction that is quadratic in the vector potential. If we want to control the growth of \(\beta(t)\) in time this quadratic part (see for example the estimate of [11]) requires that
quantities like $||a\nabla q||^2$ and $N^{-1}||\hat{A}_k(x_1)q_1\Psi_{N,t}||^2$ are not only finite but bounded by $\beta(t)$. This holds for every bounded operator $B$ because of

$$\langle \Psi_{N,t}, q_1Bq_1\Psi_{N,t}\rangle \leq C ||q_1\Psi_{N,t}||^2 \leq C \beta^a(t)$$

(59)

but must not be true in general. In case of unbounded operators smallness can sometimes be inferred on a subclass of states which have sufficient decay in the occupation of eigenstates with large eigenvalues. For a self-adjoint operator $O$ with $[O,q_1] \approx 0$ and $c \in \mathbb{R}$ one has

$$\langle \Psi_{N,t}, q_1Oq_1\Psi_{N,t}\rangle \approx \langle \Psi_{N,t}, q_1O\Psi_{N,t}\rangle = \langle \Psi_{N,t}, q_1(O-c)\Psi_{N,t}\rangle + c\langle \Psi_{N,t}, q_1\Psi_{N,t}\rangle$$

$$\leq (c+1)\langle \Psi_{N,t}, q_1\Psi_{N,t}\rangle + \langle \Psi_{N,t}, (O-c)^2\Psi_{N,t}\rangle.$$  

(60)

Thus, $\langle \Psi_{N,t}, q_1Oq_1\Psi_{N,t}\rangle$ can be bounded by $\beta(t)$ plus a small error if $\Psi_{N,t}$ occupies eigenstates of $O$ with eigenvalues $\lambda \neq c$ only with small probability. This is in the spirit of Chebyshev’s inequality from probability theory. Requiring $\langle \Psi_{N,0}, (O-c)^2\Psi_{N,0}\rangle \approx 0$ initially does not imply smallness at later times. However, if $O$ is a conserved quantity its variance is constant during the time evolution and we only have to restrict the class of initial states. In the following, we consider the variance of the energy per particles of the many-body system (see $\beta^a$). Then, we estimate the vector potential and the Laplacian by $H_N/N$ and bound expression like $N^{-1}\langle \Psi_{N,t}, q_1\hat{A}_k^2(x_1)q_1\Psi_{N,t}\rangle$ by the counting functional.

**Lemma VIII.2.** Let $y \in \mathbb{R}^3$ or $y \in \{x_1, \ldots, x_N\}$ and $\Psi_N \in \mathcal{D}(H_N)$. Then

$$\left| \left| N^{-1/2} \hat{A}_k(y)\Psi_N \right| \right|^2 \leq \frac{\Lambda}{(2\pi^2)} \langle \Psi_N, N^{-1}H_f\Psi_N \rangle,$$

$$\left| \left| N^{-1/2} \hat{A}_k(y)\Psi_N \right| \right|^2 \leq \frac{\Lambda}{(2\pi^2)} \langle \Psi_N, N^{-1}H_f\Psi_N \rangle + \Lambda^2/(4\pi^2N) \left| \left| \Psi_N \right| \right|^2,$$

$$\left| \left| N^{-1/2} \hat{A}_k(y)\Psi_N \right| \right|^2 \leq \frac{2\Lambda^2}{(2\pi^2N)} \langle \Psi_N, N^{-1}H_f\Psi_N \rangle + \Lambda^2/(2\pi^2N) \left| \left| \Psi_N \right| \right|^2.\quad (61)$$

**Proof.** To ease notation, we define the vector-valued function $f(k,\lambda) := \frac{\hat{k}(k)}{\sqrt{2|k|}}e_\lambda(k)$. The first estimate follows from Cauchy-Schwarz inequality

$$\left| \left| \sum_{\lambda=1,2} \int d^3k f(k,\lambda)e^{\pm iky}a(k,\lambda)\Psi_N \right| \right|^2$$

$$\leq \left( \sum_{\lambda=1,2} \int d^3k |f(k,\lambda)||k|^{-1/2} \left| \left| k^{-1/2}a(k,\lambda)\Psi_N \right| \right|^2 \right)^2$$

$$\leq \left( \sum_{\lambda=1,2} \int d^3k |f(k,\lambda)|^2|k|^{-1} \right) \left( \sum_{\lambda=1,2} \int d^3k |k| |a(k,\lambda)\Psi_N| \right)^2$$

$$= \frac{\Lambda}{(2\pi^2)} \langle \Psi_N, H_f\Psi_N \rangle.\quad (62)$$

By use of the canonical commutation relations (I), the second bound is obtained via

$$\left| \left| \sum_{\lambda=1,2} \int d^3k f(k,\lambda)e^{\pm iky}a^*(k,\lambda)\Psi_N \right| \right|^2 = \left| \left| \sum_{\lambda=1,2} \int d^3k f(k,\lambda)e^{\mp iky}a(k,\lambda)\Psi_N \right| \right|^2$$

$$+ \|f\|_0^2 \langle \Psi_N \rangle^2 \leq \Lambda^2/(4\pi^2) \left| \left| \Psi_N \right| \right|^2 + \Lambda/(2\pi^2) \langle \Psi_N, H_f\Psi_N \rangle.\quad (63)$$

The last estimate follows by triangular inequality.

**Lemma VIII.2** leads to
Corollary VIII.3. For \( y \in \mathbb{R}^3 \) or \( y \in \{ x_1, \ldots, x_N \} \) and \( \Psi_N \in \mathcal{D}(H_N) \) we have
\[
\left\| N^{-1/2} \hat{A}_k(y) q_1 \Psi_N \right\|^2 \leq 2 \lambda / \pi^2 \left\langle \Psi_N, q_1 N^{-1} H_f q_1 \Psi_N \right\rangle + \lambda^2 / (2 \pi^2 N^2) \beta \alpha,
\]
\[
\left\| N^{-1/2} \hat{A}_k(y) p_1 q_2 \Psi_N \right\|^2 \leq 2 \lambda / \pi^2 \left\langle \Psi_N, q_1 N^{-1} H_f q_1 \Psi_N \right\rangle + \lambda^2 / (2 \pi^2 N^2) \beta \alpha.
\]

Lemma VIII.4. Let \( v \) satisfy (A1), \( \Psi_{N,t} \in (L^2_0(\mathbb{R}^3) \otimes F_N) \cap \mathcal{D}(H_N) \) and \( (\varphi_t, A(t), E(t)) \in \mathcal{G} \). Then, there exists a monotone increasing function \( C(t) \) of \( \mathcal{E}_M[\varphi_t, u_t] \), \( ||\varphi_t||_{H^2(\mathbb{R}^3)} \) and \( ||\varphi_t||_{L^\infty(\mathbb{R}^3)} \) such that
\[
\left\langle \Psi_{N,t}, q_1^{v_1} N^{-1} H_N q_1^{v_2} \Psi_{N,t} \right\rangle \leq C(t) (\beta(t) + \Lambda/N).
\]

Proof. We decompose the Pauli-Fierz Hamiltonian into
\[
\left\langle \Psi_N, q_1 N^{-1} H_N q_1 \Psi_N \right\rangle = \left\langle \Psi_N, q_1 N^{-1} \sum_{j=1}^N \left( -i \nabla_j - N^{-1/2} \hat{A}_k(x_j) \right)^2 q_1 \Psi_N \right\rangle
\]
\[
+ \left\langle \Psi_N, q_1 N^{-2} \sum_{1 \leq j < k \leq N} v(x_j - x_k) q_1 \Psi_N \right\rangle
\]
\[
+ \left\langle \Psi_N, q_1 N^{-1} H_f q_1 \Psi_N \right\rangle.
\]

Then, we write the first line as
\[
(65) = \left\langle \Psi_N, q_1 N^{-1} \sum_{j=1}^N \left( -i \nabla_j - N^{-1/2} \hat{A}_k(x_j) \right)^2 \Psi_N \right\rangle
\]
\[
+ N^{-1} \sum_{j=1}^N \left\langle \Psi_N, q_1 \left( -i \nabla_j - N^{-1/2} \hat{A}_k(x_j) \right)^2, q_1 \right\rangle \Psi_N
\]
\[
= \left\langle \Psi_N, q_1 N^{-1} \sum_{j=1}^N \left( -i \nabla_j - N^{-1/2} \hat{A}_k(x_j) \right)^2 \Psi_N \right\rangle
\]
\[
+ N^{-1} \left\langle \Psi_N, q_1 \left( -i \nabla_1 - N^{-1/2} \hat{A}_k(x_1) \right)^2, q_1 \right\rangle \Psi_N \right\rangle.
\]

The second line is given by
\[
(66) = \left\langle \Psi_N, q_1 N^{-2} \sum_{1 \leq j < k \leq N} v(x_j - x_k) \Psi_N \right\rangle + N^{-2} \sum_{1 \leq j < k \leq N} \left\langle \Psi_N, q_1 [v(x_j - x_k), q_1] \Psi_N \right\rangle
\]
\[
= \left\langle \Psi_N, q_1 N^{-2} \sum_{1 \leq j < k \leq N} v(x_j - x_k) \Psi_N \right\rangle + (N - 1) N^{-2} \left\langle \Psi_N, q_1 [v(x_1 - x_2), q_1] \Psi_N \right\rangle.
\]

In line (68) we use that \( H_f \) commutes with operators which only act on the sector of the non-relativistic particles. This leads to
\[
\left\langle \Psi_N, q_1 N^{-1} H_N q_1 \Psi_N \right\rangle = \left\langle \Psi_N, q_1 N^{-1} H_N \Psi_N \right\rangle
\]
\[
+ N^{-1} \left\langle \Psi_N, q_1 \left( -i \nabla_1 - N^{-1/2} \hat{A}_k(x_1) \right)^2, q_1 \right\rangle \Psi_N \right\rangle
\]
\[
+ (N - 1) N^{-2} \left\langle \Psi_N, q_1 [v(x_1 - x_2), q_1] \Psi_N \right\rangle.
\]
The first term is estimated by
\[
|71| = N^{-1} |\langle \Psi_N, q_1 \left( -i \nabla_1 - N^{-1/2} \hat{A}_n(x_1) \right)^2 p_1 \rangle \Psi_N| \\
= N^{-1} |\langle \Psi_N, q_1 \left( -i \nabla_1 - N^{-1/2} \hat{A}_n(x_1) \right)^2 p_1 \Psi_N \rangle| \\
\leq N^{-1} |\langle q_1 \Psi_N, (-\Delta_1) p_1 \Psi_N \rangle| + N^{-1} |\langle N^{-1/2} \hat{A}_n(x_1) q_1 \Psi_N, \nabla_1 p_1 \Psi_N \rangle| \\
+ N^{-1} |\langle N^{-1/2} \hat{A}_n(x_1) q_1 \Psi_N, N^{-1/2} \hat{A}_n(x_1) p_1 \Psi_N \rangle| \\
\leq N^{-1} \left( \beta^3 + ||\Delta p_1 \Psi_N||^2 + ||\nabla p_1 \Psi_N||^2 \right) \\
+ N^{-1} \left( ||N^{-1/2} \hat{A}_n(x_1) q_1 \Psi_N||^2 + ||N^{-1/2} \hat{A}_n(x_1) p_1 \Psi_N||^2 \right). 
\] (73)

Lemma VIII.2 and the positivity of the interaction potential \( v \) let us continue with
\[
|71| \leq N^{-1} \Lambda C(||v||_{H^2}) \left( |\langle \Psi_N, N^{-1} H J \Psi_N \rangle| + \Lambda/N \right) \\
\leq N^{-1} \Lambda C(||v||_{H^2}) \left( |\langle \Psi_N, N^{-1} H \Psi_N \rangle| + \Lambda/N \right) \\
\leq N^{-1} \Lambda C(||\varphi||_{H^2}) \left( |\langle \Psi_N, (N^{-1} H_N - E_M) \Psi_N \rangle| + E_M \right) \\
\leq N^{-1} \Lambda C(||\varphi||_{H^2}) \left( \sqrt{\beta^3 + \epsilon_M} \right) \leq N^{-1} \Lambda C(||\varphi||_{H^2}, E_M). 
\] (74)

The second term is bounded by
\[
|72| \leq N^{-1} |\langle \Psi_N, q_1 \left[ v(x_1 - x_2), p_1 \right] \Psi_N \rangle| = N^{-1} |\langle \Psi_N, q_1 v(x_1 - x_2) p_1 \Psi_N \rangle| \\
\leq 1/2 ||q_1 \Psi_N||^2 + 1/(2N^2) ||v(x_1 - x_2) p_1 \Psi_N||^2 \\
= 1/2 \beta^3 + 1/(2N^2) |\langle \Psi_N, p_1 v(x_2 - x_1) \rangle| \\
= 1/(2 \beta^3) + 1/(2N^2) |\langle \Psi_N, p_1 v^2 \rangle (x_2) \Psi_N \rangle \\
\leq 1/(2 \beta^3) + 1/(2N^2) ||v^2 \varphi||^2_{\infty}. 
\] (75)

We use assumption (A1) and decompose the interaction potential \( v = v_1 + v_2 \) into \( v_1 \in L^2(\mathbb{R}^3) \) and \( v_2 \in L^\infty(\mathbb{R}^3) \). Then, we apply Young’s inequality and obtain
\[
||v^2 \varphi||^2_{\infty} \leq \frac{1}{2} ||v_1||^2_{L^2} ||\varphi||^2_{L^2} + ||v_2||^2_{L^\infty} ||\varphi||^2_{L^2} \leq C(||\varphi||_{L^\infty}). 
\] (76)

Thus,
\[
|71| + |72| \leq C(||\varphi||_{H^2}, ||\varphi||_{L^\infty}, E_M) (\beta + \Lambda/N) 
\] (77)

and
\[
\langle \Psi_N, q_1 N^{-1} H_N \Psi_N \rangle \leq |\langle \Psi_N, q_1 N^{-1} H_N \Psi_N \rangle| + |71| + |72| \\
\leq |\langle \Psi_N, q_1 \left( N^{-1} H_N - E_M \right) \Psi_N \rangle| + E_M \beta^3 + C(||\varphi||_{H^2}, ||\varphi||_{L^\infty}, E_M) (\beta + \Lambda/N) \\
\leq |\langle \Psi_N, q_1 \left( N^{-1} H_N - E_M \right) \Psi_N \rangle| + C(||\varphi||_{H^2}, ||\varphi||_{L^\infty}, E_M) (\beta + \Lambda/N) \\
\leq \left( ||N^{-1} H_N - E_M \right) \Psi_N \left| ^2 + \beta^3 + C(||\varphi||_{H^2}, ||\varphi||_{L^\infty}, E_M) (\beta + \Lambda/N) \\
\leq C(||\varphi||_{H^2}, ||\varphi||_{L^\infty}, E_M) (\beta + \Lambda/N). 
\] (78)
Lemma VIII.5. Let \( y \in \mathbb{R}^3 \) or \( y \in \{ x_1, \ldots, x_N \} \), \( v \) satisfy (A1), \( \Psi_{N,t} \in (L_2^2(\mathbb{R}^3) \otimes \mathcal{F}_p) \cap \mathcal{D}(H_N) \) and \((\varphi_t, A(t), E(t)) \in \mathcal{G} \). Then, there exists a monotone increasing function \( C(t) \) of \( \mathcal{E}_M[\varphi_t, u_t] \), \( \|\varphi_t\|_{H^2(\mathbb{R}^3)} \) and \( \|\varphi_t\|_{L^\infty(\mathbb{R}^3)} \) such that

\[
\left\| N^{-1/2} \hat{A}_N(y) q_1 \Psi_{N,t} \right\|^2 \leq \mathcal{C}(t) \left( \beta(t) + \Lambda/N \right),
\]

\[
\left\| N^{-1/2} \hat{A}_N(y) q_2 \Psi_{N,t} \right\|^2 \leq \mathcal{C}(t) \left( \beta(t) + \Lambda/N \right),
\]

\[
\left\| N^{-1/2} \hat{A}_N(y) p_1 q_2 \Psi_{N} \right\|^2 \leq \mathcal{C}(t) \left( \beta(t) + \Lambda/N \right). \quad (79)
\]

Proof. We have

\[
\langle \Psi_N, q_1 N^{-1} H f q_1 \Psi_N \rangle \leq \langle \Psi_N, q_1 N^{-1} H N q_1 \Psi_N \rangle
\]

because \( v \) is positive. Lemma VIII.4 and Corollary VIII.3 then lead to

\[
\langle \Psi_N, q_1 N^{-1} H f q_1 \Psi_N \rangle \leq C(\|\varphi\|_{H^2}, \|\varphi\|_{L^\infty}, \mathcal{E}_M) \left( \beta + \Lambda/N \right) \quad (80)
\]

and

\[
\left\| N^{-1/2} \hat{A}_N(y) q_1 \Psi_N \right\|^2 \leq \Lambda C(\|\varphi\|_{H^2}, \|\varphi\|_{L^\infty}, \mathcal{E}_M) \left( \beta + \Lambda/N \right). \quad (81)
\]

The other inequalities are shown analogously. \( \square \)

Lemma VIII.6. Let \( v \) satisfy (A1), \( \Psi_{N,t} \in (L_2^2(\mathbb{R}^3) \otimes \mathcal{F}_p) \cap \mathcal{D}(H_N) \) and \((\varphi_t, A(t), E(t)) \in \mathcal{G} \). Then, there exists a monotone increasing function \( C(t) \) of \( \mathcal{E}_M[\varphi_t, u_t] \), \( \|\varphi_t\|_{H^2(\mathbb{R}^3)} \) and \( \|\varphi_t\|_{L^\infty(\mathbb{R}^3)} \) such that

\[
\int d^3 y \left\| N^{-1} \sum_{j=1}^N q_j \kappa(x_j - y) \left( -i \nabla_j - N^{-1/2} \hat{A}_N(x_j) \right) \Psi_{N,t} \right\|^2 \leq \Lambda^3 C(t) \left( \beta + \Lambda/N \right). \quad (83)
\]

Proof. In the following, we use \( \langle \cdot, \cdot \rangle_y = \int d^3 y \langle \cdot, \cdot \rangle \) and \( \|\cdot\|_y = \sqrt{\int d^3 y \langle \cdot, \cdot \rangle} \) to ease the notation. We estimate

\[
\int d^3 y \left\| N^{-1} \sum_{j=1}^N q_j \kappa(x_j - y) \left( -i \nabla_j - N^{-1/2} \hat{A}_N(x_j) \right) \Psi_N \right\|^2
\]

\[
= N^{-2} \left\{ \sum_{i=1}^N q_i \kappa(x_i - y) \left( -i \nabla_i - N^{-1/2} \hat{A}_N(x_i) \right) \Psi_N, \sum_{j=1}^N q_j \kappa(x_j - y) \left( -i \nabla_j - N^{-1/2} \hat{A}_N(x_j) \right) \Psi_N \right\}_y
\]

\[
= N^{-1} \langle q_1 \kappa(x_1 - y) \left( -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \right) \Psi_N, q_1 \kappa(x_1 - y) \left( -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \right) \Psi_N \rangle_y
\]

\[
+ (N-1) N^{-1} \langle q_1 \kappa(x_1 - y) \left( -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \right) \Psi_N, q_2 \kappa(x_2 - y) \left( -i \nabla_2 - N^{-1/2} \hat{A}_N(x_2) \right) \Psi_N \rangle_y
\]

\[
\leq N^{-1} \left\| \kappa(x_1 - y) \left( -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \right) \Psi_N \right\|^2_y
\]

\[
+ (N-1) N^{-1} \left\| \kappa(x_1 - y) \left( -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \right) q_2 \Psi_N, \kappa(x_2 - y) \left( -i \nabla_2 - N^{-1/2} \hat{A}_N(x_2) \right) q_1 \Psi_N \right\|^2_y
\]

\[
\leq N^{-1} \left\| \kappa(x_1 - y) \left( -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \right) \Psi_N \right\|^2_y
\]

\[
+ (N-1) N^{-1} \left\| \kappa(x_1 - y) \left( -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \right) q_2 \Psi_N \right\|^2_y
\]

\[
= N^{-1} \langle -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \Psi_N, \int d^3 y |\kappa(x_1 - y)|^2 \rangle \left( -i \nabla_1 - N^{-1/2} \hat{A}_N(x_1) \right) \Psi_N \rangle \]
\[(N - 1)N^{-1}\langle -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1)\rangle q_2\Psi_N, \langle \int d^3y\, |\kappa(x_1 - y)|^2 \rangle \langle -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1) \rangle q_2\Psi_N\rangle
\]
\[= N^{-1}||\kappa||_2^2 \langle \Psi_N, -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1) \rangle^2 \Psi_N\rangle
\]
\[+(N - 1)N^{-1}||\kappa||_2^2 \langle \Psi_N, q_2 \left( -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1) \right)^2 q_2\Psi_N\rangle. \quad (84)
\]
So if we insert the identity \(1 = p_1 + q_1\) and use the symmetry of the wave function, we get
\[
\int d^3y \left| N^{-1} \sum_{j=1}^N q_j\kappa(x_j - y) \left( -i\nabla_j - N^{-1/2}\hat{A}_\kappa(x_j) \right) \Psi_N \right|^2
\]
\[\leq N^{-1}||\kappa||_2^2 \langle \Psi_N, q_1 \left( -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1) \right)^2 q_1\Psi_N\rangle
\]
\[+ 2N^{-1}||\kappa||_2^2 \langle \Psi_N, q_1 \left( -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1) \right)^2 p_1\Psi_N\rangle |
\]
\[+ N^{-1}||\kappa||_2^2 \langle \Psi_N, p_1 \left( -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1) \right)^2 p_1\Psi_N\rangle |
\]
\[+ N^{-1}||\kappa||_2^2 \sum_{j=2}^N \langle \Psi_N, q_1 \left( -i\nabla_j - N^{-1/2}\hat{A}_\kappa(x_j) \right)^2 q_1\Psi_N\rangle. \quad (85)
\]
Adding the lines together this simplifies to
\[
\int d^3y \left| N^{-1} \sum_{j=1}^N q_j\kappa(x_j - y) \left( -i\nabla_j - N^{-1/2}\hat{A}_\kappa(x_j) \right) \Psi_N \right|^2
\]
\[= N^{-1}||\kappa||_2^2 \sum_{j=1}^N \langle \Psi_N, q_1 \left( -i\nabla_j - N^{-1/2}\hat{A}_\kappa(x_j) \right)^2 q_1\Psi_N\rangle
\]
\[+ 2N^{-1}||\kappa||_2^2 \langle \Psi_N, q_1 \left( -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1) \right)^2 p_1\Psi_N\rangle |
\]
\[+ N^{-1}||\kappa||_2^2 \langle \Psi_N, p_1 \left( -i\nabla_1 - N^{-1/2}\hat{A}_\kappa(x_1) \right)^2 p_1\Psi_N\rangle |
\]
\[+ N^{-1}||\kappa||_2^2 \sum_{j=2}^N \langle \Psi_N, q_1 \left( -i\nabla_j - N^{-1/2}\hat{A}_\kappa(x_j) \right)^2 q_1\Psi_N\rangle. \quad (86)
\]
Now we estimate the last two lines analogously to (71) and obtain
\[
\int d^3y \left| N^{-1} \sum_{j=1}^N q_j\kappa(x_j - y) \left( -i\nabla_j - N^{-1/2}\hat{A}_\kappa(x_j) \right) \Psi_N \right|^2 \leq ||\kappa||_2^2 \Lambda/NC(||\varphi||_{H^2}, E_M)
\]
\[+ N^{-1}||\kappa||_2^2 \sum_{j=1}^N \langle \Psi_N, q_1 \left( -i\nabla_j - N^{-1/2}\hat{A}_\kappa(x_j) \right)^2 q_1\Psi_N\rangle. \quad (87)
\]
Because \(H_f\) and \(v\) are positive operators, this is bounded by
\[
||\kappa||_2^2 \langle \Psi_N, q_1N^{-1}H_Nq_1\Psi_N \rangle + ||\kappa||_2^2 \Lambda/NC(||\varphi||_{H^2}, E_M). \quad (88)
\]
Then, we apply Lemma [VIII.4] and obtain
\[
\int d^3y \left| N^{-1} \sum_{j=1}^N q_j\kappa(x_j - y) \left( -i\nabla_j - N^{-1/2}\hat{A}_\kappa(x_j) \right) \Psi_N \right|^2 \leq ||\kappa||_2^2 C(t) (\beta + \Lambda/N)
\]
\[\leq \Lambda^3C(t) (\beta + \Lambda/N), \quad (89)
\]
where \(C(t)\) is a monotone increasing function of \(E_M[\varphi_t, u_t], ||\varphi_t||_{H^2(\mathbb{R}^3)}\) and \(||\varphi_t||_{L^\infty(\mathbb{R}^3)}\). \(\square\)
VIII.2 Bound on $d_t \beta^a$:

Lemma VIII.7. Let $v$ satisfy (A1), $(\varphi, A(t), E(t)) \in \mathcal{G}$ and $\Psi_{N,t}$ be the unique solution of $\Box$ with initial data $\Psi_{N,0} \in (L^2(\mathbb{R}^3)) \otimes \mathcal{F}_p \cap \mathcal{D}(H_N)$. Then, there exists a monotone increasing function $C(t)$ of $\|A_h\|_\infty$, $\mathcal{E}_M[\varphi_t, u_t]$, $\|\varphi_t\|_{H^2(\mathbb{R}^3)}$ and $\|\varphi_t\|_{L^\infty(\mathbb{R}^3)}$ such that

\[ |d_t \beta^a(\Psi_{N,t}, \varphi_t)| \leq A^2 C(t) \left( \beta(\Psi_{N,t}, \varphi_t, u_t) + \Lambda/N \right). \tag{90} \]

Proof. The time derivative of the projector $q_1^{\varphi t}$ is given by

\[ d_t q_1^{\varphi t} = -i \left[ H^H_{1M}, q_1^{\varphi t} \right], \tag{91} \]

where $H^H_{1M}$ denotes the effective Hamiltonian $H^H_{1M} := (-i \nabla - A_h(x, t))^2 + (v \ast |\varphi_t|^2)(x)$. This allows us to compute the derivative of $\beta^a(t)$ by

\[
\begin{align*}
    d_t \beta^a(t) &= d_t \langle \Psi_{N,t}, q_1^{\varphi t} \Psi_{N,t} \rangle = i \langle \Psi_{N,t}, [H_N - H^H_{1M}], q_1^{\varphi t} \rangle \\
    &= -2 \langle \Psi_{N,t}, \left( N^{-1/2} \hat{A}_h(x_1) - A_h(x_1, t) \right) \cdot \nabla, q_1^{\varphi t} \rangle \Psi_{N,t} \\
    &\quad + i \langle \Psi_{N,t}, \left( N^{-1} \hat{A}_h^2(x_1) - A_h^2(x_1, t) \right), q_1^{\varphi t} \rangle \Psi_{N,t} \\
    &\quad + i \langle \Psi_{N,t}, \left( N^{-1} \sum_{1 \leq j < k \leq N} v(x_j - x_k) - (v \ast |\varphi_t|^2)(x_1) \right), q_1^{\varphi t} \rangle \Psi_{N,t} \\
    &= -4 \text{Re} \langle \Psi_{N,t}, \left( N^{-1/2} \hat{A}_h(x_1) - A_h(x_1, t) \right) \cdot \nabla, q_1 \Psi_{N,t} \rangle \\
    &\quad - 2 \text{Im} \langle \Psi_{N,t}, \left( N^{-1} \hat{A}_h^2(x_1) - A_h^2(x_1, t) \right), q_1 \Psi_{N,t} \rangle \\
    &\quad - 2 \text{Im} \langle \Psi_{N,t}, \left( (N - 1)N^{-1} v(x_1 - x_2) - (v \ast |\varphi_t|^2)(x_1) \right), q_1 \Psi_{N,t} \rangle.
\end{align*}
\]

Inserting the identity $1 = p_1 + q_1$ and the relations

\[
\begin{align*}
    \text{Re} \langle \Psi_{N}, q_1 \left( N^{-1/2} \hat{A}_h(x_1) - A_h(x_1, t) \right) \cdot \nabla, q_1 \Psi_N \rangle &= 0, \\
    \text{Im} \langle \Psi_{N}, q_1 \left( N^{-1} \hat{A}_h^2(x_1) - A_h^2(x_1, t) \right), q_1 \Psi_N \rangle &= 0, \\
    \text{Im} \langle \Psi_{N}, q_1 \left( (N - 1)N^{-1} v(x_1 - x_2) - (v \ast |\varphi_t|^2)(x_1) \right), q_1 \Psi_N \rangle &= 0,
\end{align*}
\]

lead to

\[
\begin{align*}
    d_t \beta^a(t) &= -4 \text{Re} \langle \Psi_{N}, p_1 \left( N^{-1/2} \hat{A}_h(x_1) - A_h(x_1, t) \right) \cdot \nabla, q_1 \Psi_N \rangle \tag{93} \\
    &\quad - 2 \text{Im} \langle \Psi_{N}, p_1 \left( N^{-1} \hat{A}_h^2(x_1) - A_h^2(x_1, t) \right), q_1 \Psi_N \rangle \tag{94} \\
    &\quad - 2 \text{Im} \langle \Psi_{N}, p_1 \left( (N - 1)N^{-1} v(x_1 - x_2) - (v \ast |\varphi_t|^2)(x_1) \right), q_1 \Psi_N \rangle. \tag{95}
\end{align*}
\]

In the following, we estimate each line separately. To simplify the presentation we use the shorthand notation $\Box$. 

17
Lemma XI.1 and the symmetry of the wave function lead to

\[
\text{By means of Lemma VIII.1, we bound the first line by}
\]

\[
\text{Integration by parts and triangular inequality let us estimate}
\]

\[
\text{VIII.2.1 Bound on (93)}
\]

\[
|\text{(93)}| \leq 4|\langle \Psi_N, p_1 (A^+(x_1, t) + A^-(x_1, t)) \cdot \nabla_1 q_1 \Psi_N \rangle| \\
\leq 4|\langle \nabla_1 p_1 \Psi_N, A^- (x_1, t) q_1 \Psi_N \rangle| \\
+ 4|\langle \nabla_1 p_1 \Psi_N, A^+ (x_1, t) q_1 \Psi_N \rangle|.
\]

By means of Lemma VIII.1 we bound the first line by

\[
\text{(96)} = 4|\langle \nabla_1 p_1 \Psi_N, (\eta \ast E^-) (x_1, t) q_1 \Psi_N \rangle| \\
= 4|\langle E^+(y, t) \nabla_1 p_1 \Psi_N, \eta (x_1 - y) q_1 \Psi_N \rangle| \\
\leq 4 \|E^+(y, t) \cdot \nabla_1 p_1 \Psi_N\| \|\eta(y - x_1) q_1 \Psi_N\|_y \\
\leq 2 \|E^+(y, t) \cdot \nabla_1 p_1 \Psi_N\|_y^2 + 2 \|\eta\|_2^2 \|q_1 \Psi_N\|^2 \\
\leq \Lambda \beta^{-2} \beta + C(\|\nabla \varphi\|_2) \beta \leq \Lambda C(\|\nabla \varphi\|_2) \beta,
\]

where we made use of Lemma XI.1 and (97).

The second term is bounded by

\[
\text{(97)} = 4|\langle \nabla_1 p_1 \Psi_N, \int d^3 y \eta(x_1 - y) E^+(y, t) q_1 \Psi_N \rangle| \\
= 4|\langle \nabla_1 p_1 \Psi_N, \eta(x_1 - y) E^+(y, t) q_1 \Psi_N \rangle| \\
= 4|\langle q_1 \eta(x_1 - y) \nabla_1 p_1 \Psi_N, E^+(y, t) \Psi_N \rangle| \\
= 4|\langle \sum_{i=1}^N q_i \eta(x_i - y) \nabla_1 p_i \Psi_N, E^+(y, t) \Psi_N \rangle| \\
\leq 2 \|E^+(y, t) \Psi_N\|_y^2 + 2 \left| \sum_{i=1}^N q_i \eta(x_i - y) \nabla_1 p_i \Psi_N \right|_y^2.
\]

Lemma XI.1 and the symmetry of the wave function lead to

\[
\text{(97)} \leq 2 \beta \beta + 2 \beta \beta + 2N^{-2} \left( \sum_{i=1}^N q_i \eta(x_i - y) \nabla_1 p_i \Psi_N, \sum_{j=1}^N q_j \eta(x_j - y) \nabla_1 p_j \Psi_N \right)_y \\
\leq 2 \beta \beta + 2N^{-1} \|q_1 \eta(x_1 - y) \nabla_1 p_1 \Psi_N\|_y^2 \\
+ 2 \left( \sum_{i=1}^N q_i \eta(x_i - y) \nabla_1 p_i \Psi_N, q_2 \eta(x_2 - y) \nabla_2 p_2 \Psi_N \right)_y \\
\leq 2 \beta \beta + 2N^{-1} \|\eta(x_1 - y) \nabla_1 p_1 \Psi_N\|_y^2 \\
+ 2 \left( \eta(x_1 - y) \nabla_1 p_1 \Psi_N, q_2 \eta(x_2 - y) \nabla_2 p_2 q_1 \Psi_N \right)_y \\
\leq 2 \beta \beta + 2N^{-1} \|\eta(x_1 - y) \nabla_1 p_1 \Psi_N\|_y^2 \\
+ 2 \|\eta(x_1 - y) \nabla_1 p_1 q_2 \Psi N\|_y \|\eta(x_2 - y) \nabla_2 p_2 q_1 \Psi N\|_y \\
\leq 2 \beta \beta + 2N^{-1} \left( \eta(x_1 - y) \nabla_1 p_1 \Psi_N, \eta(x_1 - y) \nabla_1 p_1 \Psi_N \right)_y \\
+ 2 \left( \eta(x_1 - y) \nabla_1 p_1 q_2 \Psi N, \eta(x_1 - y) \nabla_1 p_1 q_2 \Psi N \right)_y.
\]
Interchanging the order of integration we have

\begin{align}
\langle 94 \rangle \leq 2N^{-1} \langle \nabla_1 p_1 \Psi_N, \left( \int d^3y |\eta(x_1 - y)|^2 \right) \nabla_1 p_1 \Psi_N \rangle + 2\beta^b \\
+ 2 \langle \nabla_1 p_1 q_2 \Psi_N, \left( \int d^3y |\eta(x_1 - y)|^2 \right) \nabla_1 p_1 q_2 \Psi_N \rangle \\
= 2 \|\eta\|^2_2 \langle N^{-1}(\Psi_N, p_1(-\Delta_1)p_1 \Psi_N) + \langle \Psi_N, q_2 p_1(-\Delta_1)p_1 q_2 \Psi_N \rangle \rangle + 2\beta^b.
\end{align}

By virtue of \( p_1(-\Delta) p_1 = p_1 \|\nabla \varphi\|^2_2 \), this becomes

\begin{align}
\langle 97 \rangle \leq 2 \|\eta\|^2_2 \|\nabla \varphi\|^2_2 \langle N^{-1}(\Psi_N, p_1 \Psi_N) + \langle \Psi_N, q_2 p_1 q_2 \Psi_N \rangle \rangle + 2\beta^b \\
\leq \|\eta\|^2_2 C(\|\varphi\|_{H^2}) (\beta^a + \beta^b + N^{-1}) \leq \Lambda C(\|\varphi\|_{H^2}) (\beta + N^{-1})
\end{align}

and we obtain

\begin{align}
\langle 99 \rangle \leq \Lambda C(\|\varphi\|_{H^2}) (\beta + N^{-1}).
\end{align}

**VIII.2.2 Bound on \langle 94 \rangle:**

\begin{align}
\langle 94 \rangle \leq 2 \langle \Psi_N, p_1 \left( N^{-1} \hat{A}_\kappa^2(x_1) - A_\kappa^2(x_1, t) \right) q_1 \Psi_N \rangle \\
= 2 \langle \Psi_N, p_1 \left( N^{-1/2} \hat{A}_\kappa(x_1) - A_\kappa(x_1, t) \right) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) q_1 \Psi_N \rangle \\
\leq 2 \langle \Psi_N, p_1 \mathcal{A}^-(x_1, t) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) q_1 \Psi_N \rangle \\
+ 2 \langle \Psi_N, p_1 \mathcal{A}^+(x_1, t) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) q_1 \Psi_N \rangle
\end{align}

First, we deal with line \langle 103 \rangle:

\begin{align}
\langle 103 \rangle &= 2 \langle \Psi_N, p_1 \left( \eta \ast \mathcal{E}^- \right) (x_1, t) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) q_1 \Psi_N \rangle \\
&= 2 \langle \mathcal{E}^+(y, t)p_1 \Psi_N, \eta(y - x_1) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) q_1 \Psi_N \rangle \|_y \\
&\leq \|\mathcal{E}^+(y, t)p_1 \Psi_N\|^2_2 + \|\eta(y - x_1) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) q_1 \Psi_N \|^2_2 \\
&\leq \|\mathcal{E}^+(y, t)p_1 \Psi_N\|^2_2 + \|\eta\|^2_2 \left( \|N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \|^2 q_1 \Psi_N \right)
\end{align}

Making use of Lemma \[101\] and \((a + b)^2 \leq 2(a^2 + b^2)\), we obtain

\begin{align}
\langle 104 \rangle \leq \beta^b + 2 \|\eta\|^2_2 \left( \|A_\kappa\|^2_\infty \|q_1 \Psi_N\|^2 + \|N^{-1/2} \hat{A}_\kappa(x_1) q_1 \Psi_N\|^2 \right).
\end{align}

By means of \[\ref{7} \] and Lemma \[\text{VIII.2.5} \] this becomes

\begin{align}
\langle 104 \rangle \leq \Lambda^2 C(\|A_\kappa\|_\infty, \|\varphi\|_{H^2}, \|\varphi\|_{\infty}, \mathcal{E}_M) (\beta + \Lambda/N).
\end{align}

The second line is bounded by

\begin{align}
\langle 105 \rangle &= 2 \langle \Psi_N, p_1 \mathcal{A}^+(x_1, t) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) q_1 \Psi_N \rangle \\
&= 2 \langle \Psi_N, p_1 \left[ \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) \mathcal{A}^+(x_1, t) + \Lambda^2/(4\pi^2 N) \right] q_1 \Psi_N \rangle \\
&\leq 2 \langle \Psi_N, p_1 \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1, t) \right) \mathcal{A}^+(x_1, t) q_1 \Psi_N \rangle \\
+ 2\Lambda^2/(4\pi^2 N) \left\langle \Psi_N, p_1 q_1 \Psi_N \right\rangle.
\end{align}
Here, we have used the commutation relation

\[
\left[ A^+(x_1,t), \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1,t) \right) \right] = N^{-1} \left[ \hat{A}^+_\kappa(x_1), \hat{A}^-_\kappa(x_1) \right] = \Lambda^2 / (4\pi^2 N). \quad (110)
\]

Lemma VIII.1 and Lemma XI.1 lead to

\[
(105) \leq 2|\langle \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1,t) \right) p_1 \Psi_N, \int d^3 y \eta(x_1 - y) E^+(y,t) q_1 \Psi_N \rangle |
= 2|\langle q_1 \eta(x_1 - y) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1,t) \right) p_1 \Psi_N, E^+(y,t) \Psi_N \rangle |
\leq N^{-2} \left| \sum_{i=1}^{N} q_i \eta(x_i - y) \left( N^{-1/2} \hat{A}_\kappa(x_i) + A_\kappa(x_i,t) \right) p_i \Psi_N \right|^2 + \left| E^+(y,t) \Psi_N \right|^2
\leq N^{-2} \left| \sum_{i=1}^{N} q_i \eta(x_i - y) \left( N^{-1/2} \hat{A}_\kappa(x_i) + A_\kappa(x_i,t) \right) p_i \Psi_N \right|^2 + \beta^b \quad (111)
\]

Similar to the estimate of (97) one obtains

\[
(104) \leq N^{-1} \left| \eta(x_1 - y) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1,t) \right) p_1 \Psi_N \right|^2
+ \left| \eta(x_1 - y) \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1,t) \right) p_1 q_2 \Psi_N \right|^2 + \beta^b
\leq N^{-1} \left| \eta \right|^2 \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1,t) \right) p_1 \Psi_N \right|^2
+ \left| \eta \right|^2 \left( N^{-1/2} \hat{A}_\kappa(x_1) + A_\kappa(x_1,t) \right) p_1 q_2 \Psi_N \right|^2 + \beta^b
\leq CA \left( \beta^b + \left| A_\kappa \right|^2 \right) \beta^a + \left| N^{-1/2} A_\kappa(x_1) p_1 q_2 \Psi_N \right|^2
+ CA/N \left( \left| A_\kappa \right|^2 \right). \quad (112)
\]

By means of Lemma VIII.3 this is bounded by

\[
(103) \leq \Lambda^2 C(\left| \left| A_\kappa \right| \right|, \left| \varphi \right|_{H^2}, \left| \varphi \right|_{\infty}, E_M) (\beta + \Lambda/N). \quad (133)
\]

In total, we obtain

\[
(12) \leq (104) + (105) \leq \Lambda^2 C(\left| \left| A_\kappa \right| \right|, \left| \varphi \right|_{H^2}, \left| \varphi \right|_{\infty}, E_M) (\beta + \Lambda/N). \quad (144)
\]

VIII.2.3 Bound on (55):

Subsequently, we consider the term that arises from the direct interaction. Inserting the identity \( 1 = p_2 + q_2 \) and using the shorthand notation

\[
Z(x_1,x_2) := (N - 1) N^{-1} v(x_1 - x_2) - (v * |\varphi|^2)(x_1) \quad (115)
\]
The third term vanishes due to symmetry of the wave function under the interchange of $x_1$ and $x_2$ and we are left with

$$|95| \leq 2|\langle \Psi_N, p_1 p_2 Z(x_1, x_2) q_1 q_2 \Psi_N \rangle|$$

(117)

$$+ 2|\langle \Psi_N, p_1 p_2 Z(x_1, x_2) q_1 q_2 \Psi_N \rangle|$$

(118)

$$+ 2|\langle \Psi_N, p_1 q_2 Z(x_1, x_2) q_1 q_2 \Psi_N \rangle|.$$  

(119)

The first line is the most important. It is small because the direct interaction of the many-body system is well approximated by the mean-field potential. By means of

$$p_2 Z(x_1, x_2) p_2 = p_2 \left[ (N-1)N^{-1} v(x_1 - x_2) - (v * |\varphi|^2) (x_1) \right] p_2$$

$$= \left[ (N-1)N^{-1} - 1 \right] (v * |\varphi|^2) (x_1) p_2 = -N^{-1} (v * |\varphi|^2) (x_1) p_2$$  

(120)

one has

$$|116| \leq 2N^{-1} |\langle \Psi_N, p_1 (v * |\varphi|^2) (x_1) q_2 \Psi_N \rangle|$$

$$\leq 2N^{-1} \||v * |\varphi|^2\|(x_1) p_1 \Psi_N \||p_2 q_1 \Psi_N\| \leq 2N^{-1} \||v * |\varphi|^2\|_\infty.$$  

(121)

We decompose the interaction potential $v = v_1 + v_2$ into $v_1 \in L^2(\mathbb{R}^3)$ and $v_2 \in L^\infty(\mathbb{R}^3)$. Then

$$\||v * |\varphi|^2\|_\infty \leq \||v_1 * |\varphi|^2\|_\infty + \||v_2 * |\varphi|^2\|_\infty \leq \||v_1\||_2 \|||\varphi|^2||_2 + \||v_2||_\infty \|||\varphi|^2||_1$$

$$\leq \||v_1\||_2 \|||\varphi||_\infty \||\varphi||_2 + \||v_2||_\infty \|||\varphi||_2 \leq C(||\varphi||_\infty).$$  

(122)

holds due to Young’s inequality and we obtain

$$|117| \leq N^{-1} C(||\varphi||_\infty).$$  

(123)

Moreover, we have

$$p_1 Z^2(x_1, x_2) p_1 = p_1 (|\varphi_t\| \left( (N-1)N^{-1} v(x_2 - \cdot) - (v * |\varphi_t|^2) \right)^2 |\varphi_t\rangle)$$

$$\leq 2p_1 (|\varphi_t\| \left( v^2(x_2 - \cdot) + (v * |\varphi_t|^2) \right)^2 |\varphi_t\rangle)$$

$$\leq 2p_1 \left( \||v^2 * |\varphi_t|^2||_\infty + \||v * |\varphi_t|^2||_\infty \right) \leq p_1 C(||\varphi||_\infty)$$  

(124)

because of (117) and (118). This shows

$$\||p_1 Z^2(x_1, x_2) p_1||_{op} \leq C(||\varphi||_\infty)$$  

(125)
and allows us to estimate

$$\|115\| = 2\left| \langle q_2 Z(x_1, x_2) p_1 p_2 \Psi_N, q_1 \Psi_N \rangle \right| = 2(N - 1)^{-1} \left( \sum_{i=2}^{N} q_i Z(x_1, x_i) p_1 p_i \Psi_N, q_1 \Psi_N \right)$$

$$\leq N^{-1} \left| \sum_{i=2}^{N} q_i Z(x_1, x_i) p_1 p_i \Psi_N \right|^2 + 4 |q_1 \Psi_N|^2$$

$$= N^{-2} \left( \sum_{i=2}^{N} q_i Z(x_1, x_i) p_1 p_i \Psi_N \right) \sum_{j=2}^{N} q_j Z(x_1, x_j) p_1 p_j \Psi_N + 4 \beta^a$$

$$\leq \langle q_2 Z(x_1, x_2) p_1 p_2 \Psi_N, q_3 Z(x_1, x_3) p_1 p_3 \Psi_N \rangle + N^{-1} \left| q_2 Z(x_1, x_2) p_1 p_2 \Psi_N \right|^2 + 4 \beta^a$$

$$\leq \left| Z(x_1, x_2) p_1 p_2 q_3 \Psi_N \right|^2 + N^{-1} \left| Z(x_1, x_2) p_1 p_2 \Psi_N \right|^2 + 4 \beta^a$$

$$\leq \left| p_1 Z(x_1, x_2) p_1 \right|_{\text{op}} \left( \beta^a + N^{-1} \right) + 4 \beta^a$$

$$\leq C(|\varphi|_\infty) \left( \beta + N^{-1} \right).$$

The last term of (114) is bounded by

$$\|119\| = 2 \left| \langle Z(x_1, x_2) p_1 q_2 \Psi_N, q_1 q_2 \Psi_N \rangle \right|$$

$$\leq \langle \Psi_N, q_2 p_1 Z^2(x_1, x_2) p_1 q_2 \Psi_N \rangle + |q_1 q_2 \Psi_N|^2$$

$$\leq \left| p_1 Z^2(x_1, x_2) p_1 \right|_{\text{op}} |q_2 \Psi_N|^2 + \beta^a \leq C(|\varphi|_\infty) \beta.$$ (127)

This leads to

$$\|153\| \leq C(|\varphi|_\infty) \left( \beta + N^{-1} \right).$$ (128)

**VIII.3 Bound on $d_t \beta^b$:**

**Lemma VIII.8.** Let $v$ satisfy (A1), $(\varphi_t, A(t), E(t)) \in \mathcal{G}$ and $\Psi_{N,t}$ be the unique solution of (14) with initial data $\Psi_{N,0} \in (L^2_s(\mathbb{R}^{3N})) \otimes \mathcal{F}_p \cap \mathcal{D}(H_N)$. Then, there exists a monotone increasing function $C(t)$ of $\mathcal{E}_{M}[\varphi_t, u_t], |\varphi_t|_{H^2(\mathbb{R}^3)}$ and $|\varphi_t|_{L^\infty(\mathbb{R}^3)}$ such that

$$|d_t \beta^b(\Psi_{N,t}, u_t)| \leq \Lambda^4 C(t) \left( \beta(\Psi_{N,t}, \varphi_t, u_t) + \Lambda/N \right).$$ (129)

**Proof.** We would like to note that the following calculation can be carried out in more detail. We could for example write $\beta^b$ as

$$\beta^b(\Psi_{N,t}, u_t) = N^{-1} \langle \Psi_{N,t}, H_f \Psi_{N,t} \rangle + |u_t|_{\mathcal{B}}^2$$

$$- 2N^{-1/2} \text{Re} \langle \Psi_{N,t}, \left( \sum_{j=1,2} d^b k u_t(k, \lambda)|k|^{1/2} a^*(k, \lambda) \right) \Psi_{N,t} \rangle$$ (130)

and determine its derivative analogously to [12] Appendix 2.11]. Since we disregard photons with small energies, $\beta^b$ is well defined for $(\varphi_t, A(t), E(t)) \in \mathcal{G}$ and $\Psi_{N,t} \in \mathcal{D}(H_N) = \mathcal{D}(\sum_{i=1}^{N} (-\Delta_i) + H_f) \subset \mathcal{D}(H_f)$. This allows us to determine the derivative for many-body wave functions in $\mathcal{D}(H^2_N)$ (which is invariant due to Stone’s theorem) and extend the result later to $\mathcal{D}(H_N)$ by a standard density argument.

We compute the commutator

$$i \left[ H_N, \frac{a(k, \lambda)}{\sqrt{N}} \right] = -i |k| \frac{a(k, \lambda)}{\sqrt{N}} - \frac{2i}{N} \sum_{j=1}^{N} \tilde{\kappa}(k) \epsilon_{\lambda}(k) e^{-ikx_j} \left( i \nabla_j + \frac{\mathcal{A}(x_j)}{\sqrt{N}} \right)$$ (131)
by use of the canonical commutation relations \((\ref{eq:canonical})\) and observe that the Maxwell-Schrödinger system leads to

\[
\partial_t |k|^{1/2} \alpha_t(k, \lambda) = -i |k|^{3/2} \alpha_t(k, \lambda) + \frac{ie_\lambda(k)}{\sqrt{2}} \mathcal{F}[j](k) \tag{132}
\]

Then, we continue with

\[
d_t \beta^\lambda(t) = \sum_{\lambda = 1, 2} \int d^3k |k| \langle \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N, \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N \rangle
\]

\[
\begin{align*}
&= \sum_{\lambda = 1, 2} \int d^3k |k| \langle \left(\frac{H}{N}, N^{-1/2}a(k, \lambda)\right) \Psi_N, \left(\frac{H}{N}, N^{-1/2}a(k, \lambda)\right) \Psi_N \rangle \\
&+ \sum_{\lambda = 1, 2} \int d^3k d_t |k| \langle \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N, i \left[\frac{H}{N}, N^{-1/2}a(k, \lambda)\right] \Psi_N \rangle \\
&- \sum_{\lambda = 1, 2} \int d^3k |k| \langle (\partial_\alpha \alpha_t)(k, \lambda) \Psi_N, \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N \rangle \\
&- \sum_{\lambda = 1, 2} \int d^3k |k| \langle \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N, (\partial_\alpha \alpha_t)(k, \lambda) \Psi_N \rangle \\
&= 2 \sum_{\lambda = 1, 2} \int d^3k |k| \text{Re}(i \left[\frac{H}{N}, N^{-1/2}a(k, \lambda)\right] \Psi_N, \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N \rangle \\
&- 2 \sum_{\lambda = 1, 2} \int d^3k |k| \text{Re}(\partial_\alpha \alpha_t)(k, \lambda) \Psi_N, \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N \rangle
\]

The first terms cancels because the scalar product is purely imaginary.

So if we use \(\left[\nabla_1, e_\lambda(k) e^{ik_1x_1}\right] = 0\) (recall Definition \((\ref{eq:definition})\) and the symmetry of the wave function, we get

\[
d_t \beta^\lambda(t) = 4 \text{Re} \sum_{\lambda = 1, 2} \int d^3k \Psi_N, i \sqrt{\frac{|k|}{2}} \tilde{\epsilon}(k) e_\lambda(k) e^{ik_1x_1} \left(i \nabla_1 + N^{-1/2} \hat{A}_\lambda(x_1)\right) \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N \rangle \\
+ 2 \text{Re} \sum_{\lambda = 1, 2} \int d^3k \Psi_N, i \sqrt{\frac{|k|}{2}} \tilde{\epsilon}(k) e_\lambda(k) (2\pi)^{1/2} \mathcal{F}[j](k) \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) \Psi_N \rangle. \tag{134}
\]

Inserting the identity \(1 = p_1 + q_1\) and

\[
\sum_{\lambda = 1, 2} \int d^3k \sqrt{\frac{|k|}{2}} \tilde{\epsilon}(k) e_\lambda(k) e^{ik_1x_1} \left(N^{-1/2}a(k, \lambda) - \alpha_t(k, \lambda)\right) = (\kappa \ast \mathcal{E}^+) \tag{135}
\]

lead to

\[
d_t \beta^\lambda(t) = 4 \text{Re} \langle \Psi_N, p_1 N^{-1/2} \hat{A}_\lambda(x_1) \kappa(y - x_1) p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y} \\
+ 2 \text{Re} \langle \Psi_N, p_1 (\kappa(y - x_1) i \nabla_1 + i \nabla_1 \kappa(y - x_1)) p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y} \\
+ 2 \text{Re} \langle \Psi_N, \left(\int d^3z \kappa(y - z) j(z)\right) \mathcal{E}^+(y, t) \Psi_N \rangle_{y} \\
+ 4 \text{Re} \langle \Psi_N, q_1 (\kappa(y - x_1) i \nabla_1 p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]

23
+ 4\text{Re}\langle \Psi_N, q_1 N^{-1/2} \hat{A}_\kappa(x_1) \kappa(y - x_1) p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y} \\
+ 4\text{Re}\langle \Psi_N, \left(i \nabla_1 + N^{-1/2} \hat{A}_\kappa(x_1) \right) \kappa(y - x_1) q_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
(136)

With the relations
\[
p_1 N^{-1/2} \hat{A}_\kappa(x_1) \kappa(y - x_1) p_1 = p_1 \int d^3 z |\varphi|^2(z) N^{-1/2} \hat{A}_\kappa(z) \kappa(y - z),
\]
\[
p_1 (\kappa(y - x_1) i \nabla_1 + i \nabla_1 \kappa(y - x_1)) p_1 = -2p_1 \int d^3 z \kappa(y - z) \text{Im}[\varphi^* \nabla \varphi](z),
\]
\[
j = 2 (\text{Im}(\varphi^* \nabla \varphi) - |\varphi|^2 A_\kappa)
(137)

we obtain
\[
d_t \beta^h(t) = -4\text{Re} \int d^3 z |\varphi|^2(z) \langle \Psi_N, q_1 N^{-1/2} \hat{A}_\kappa(z) \kappa(y - z) \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
+ 4\text{Re} \int d^3 z \text{Im}[\varphi^* \nabla \varphi](z) \langle \Psi_N, q_1 \kappa(y - z) \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
+ 4\text{Re} \int d^3 z |\varphi|^2(z) \langle \Psi_N, \kappa(y - z) \left( N^{-1/2} \hat{A}_\kappa(z) - A_\kappa(z, t) \right) \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
+ 4\text{Re} \langle \Psi_N, q_1 \kappa(y - x_1) i \nabla_1 p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
+ 4\text{Re} \langle \Psi_N, q_1 N^{-1/2} \hat{A}_\kappa(x_1) \kappa(y - x_1) p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
+ 4\text{Re} \langle \Psi_N, \left(-i \nabla_1 - N^{-1/2} \hat{A}_\kappa(x_1) \right) \kappa(y - x_1) q_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
(138)

Subsequently, we estimate each line separately:
\[
|138| \leq 4 \int d^3 z |\varphi|^2(z) \langle \Psi_N, q_1 N^{-1/2} \hat{A}_\kappa(z) \kappa(y - z) \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
\leq 4 \int d^3 y \int d^3 z |\varphi|^2(z) \kappa(y - z) \langle \langle \mathcal{E}^+(y, t) \Psi_N \kappa(y - z) \rangle_{y^y} \rangle \langle \langle \hat{A}_\kappa(z) q_1 \Psi_N \rangle_{y^y} \rangle
\leq 4 \int d^3 y \int d^3 z |\varphi|^2(z) \| \mathcal{E}^+(y, t) \Psi_N \| \| \kappa(y - z) \| \langle \langle \hat{A}_\kappa(z) q_1 \Psi_N \rangle_{y^y} \rangle
\leq 2 \int d^3 y \int d^3 z |\varphi|^2(z) \| \mathcal{E}^+(y, t) \Psi_N \|^2
+ 2 \int d^3 z |\varphi|^2(z) \| \hat{A}_\kappa(z) q_1 \Psi_N \|^2 \left( \int d^3 y (\kappa(y - z))^2 \right)
= 2 \langle \Psi_N, \mathcal{E}^-(y, t) \mathcal{E}^+(y, t) \Psi_N \rangle_{y^y}
+ 2 \| \kappa \|^2 \int d^3 z |\varphi|^2(z) \| \hat{A}_\kappa(z) q_1 \Psi_N \|^2
(144)

With the help of Lemma VIII.5 and (57) this becomes
\[
|138| \leq \Lambda^4 C(\| \varphi \|_{H^2}, \| \varphi \|_{\infty}, \mathcal{E}_M) \left( \beta + \Lambda/N \right).
(145)
Similarly,}

\[
\left| \mathbf{139}\right| \leq 4\int d^{3}y\int d^{3}z |\kappa(y-z)||\varphi(z)||\nabla\varphi(z)||\langle q_{1}\Psi_{N}, \mathcal{E}^{+}(y,t)\Psi_{N}\rangle|
\]

\[
\leq 4\int d^{3}y\int d^{3}z |\kappa(y-z)||\varphi(z)||\nabla\varphi(z)||\mathcal{E}^{+}(y,t)\Psi_{N}|| |q_{1}\Psi_{N}|
\]

\[
\leq 2\int d^{3}y\int d^{3}z |\nabla\varphi(z)|^{2}||\mathcal{E}^{+}(y,t)\Psi_{N}||^{2}
\]

\[
+2\int d^{3}y |\varphi(z)|^{2}||q_{1}\Psi_{N}||^{2}\left(\int d^{3}y |\kappa(y-z)|^{2}\right)
\]

\[
\leq 2||\nabla\varphi||^{2}\beta^{b} + 2||\kappa||^{2}\beta^{a} \leq \Lambda^{3}C(||\varphi||)\beta
\quad (146)
\]

and

\[
\left| \mathbf{140}\right| \leq 4\int d^{3}y\int d^{3}z |\varphi|^{2}(z)||\kappa(y-z)||\langle \mathcal{A}(z,t)\Psi_{N}, \mathcal{E}^{+}(y,t)\Psi_{N}\rangle|
\]

\[
\leq 4\int d^{3}y\int d^{3}z |\varphi|^{2}(z)||\kappa(y-z)|||\mathcal{A}(z,t)\Psi_{N}|| ||\mathcal{E}^{+}(y,t)||
\]

\[
\leq 2\int d^{3}z |\varphi|^{2}(z)||\mathcal{A}(z,t)\Psi_{N}||^{2}\int d^{3}y |\kappa(y-z)|^{2}
\]

\[
+2\int d^{3}z |\varphi|^{2}(z)\int d^{3}y ||\mathcal{E}^{+}(y,t)||^{2}
\]

\[
\leq 2\beta^{b} + 2||\kappa||^{2}\int d^{3}z |\varphi|^{2}(z)||\mathcal{A}(z,t)\Psi_{N}||^{2}.
\quad (147)
\]

Linearity and \((a + b)^{2} \leq 2(a^{2} + b^{2})\) lead to

\[
\left| \mathbf{140}\right| \leq 2\beta^{b} + 4||\kappa||^{2}\int d^{3}z |\varphi|^{2}(z)\left(||\mathcal{A}^{+}(z,t)\Psi_{N}||^{2} + ||\mathcal{A}^{-}(z,t)\Psi_{N}||^{2}\right).
\quad (148)
\]

By means of the commutation relation

\[
[\mathcal{A}^{+}(z,t), \mathcal{A}^{-}(z,t)] = N^{-1}\left[\hat{A}_{\kappa}^{+}(z), \hat{A}_{\kappa}^{-}(z)\right] = \Lambda^{2}/(4\pi^{2}N)
\quad (149)
\]

we calculate

\[
\int d^{3}z |\varphi|^{2}(z)||\mathcal{A}^{-}(z,t)\Psi_{N}||^{2} = \int d^{3}z |\varphi|^{2}(z)\langle \Psi_{N}, \mathcal{A}^{+}(z,t)\mathcal{A}^{-}(z,t)\Psi_{N}\rangle
\]

\[
= \int d^{3}z |\varphi|^{2}(z)\langle \Psi_{N}, (\mathcal{A}^{-}(z,t)\mathcal{A}^{+}(z,t) + \Lambda^{2}/(4\pi^{2}N)) \Psi_{N}\rangle
\]

\[
= \int d^{3}z |\varphi|^{2}(z)||\mathcal{A}^{+}(z,t)\Psi_{N}||^{2} + \Lambda^{2}/(4\pi^{2}N)
\quad (150)
\]

and obtain

\[
\left| \mathbf{140}\right| \leq 2\beta^{b} + ||\kappa||^{2}\Lambda^{2}/(4\pi^{2}N) + 8 ||\kappa||^{2}\int d^{3}z |\varphi|^{2}(z)||\mathcal{A}^{+}(z,t)\Psi_{N}||^{2}.
\quad (151)
\]
Then, we use (140) and estimate
\[
\int d^3 z |\varphi|^2(z) \left| \mathbf{A}^+(z, t) \Psi_N \right|^2 = \int d^3 z |\varphi|^2(z) \langle \Psi_N, \mathbf{A}^-(z, t) \mathbf{A}^+(z, t) \Psi_N \rangle
\]
\[
= \int d^3 z |\varphi|^2(z) \langle \Psi_N, \int d^3 \eta \eta \mathcal{E}^-(y, t) \int d^3 l \eta(z - l) \mathcal{E}^+(l, t) \Psi_N \rangle
\]
\[
\leq \int d^3 y \int d^3 z \int d^3 l |\varphi|^2(z) |\eta(z - y)||\eta(z - l)||\mathcal{E}^+(y, t) \mathcal{E}^+(l, t) \Psi_N \rangle^2 \int d^3 y |\eta(z - l)|^2
\]
\[
+ 1/2 \int d^3 z |\varphi|^2(z) \int d^3 y \left| \mathcal{E}^+(y, t) \Psi_N \right|^2 \int d^3 l |\eta(z - l)|^2
\]
\[
\leq ||\eta||_2^2 \int d^3 y \left| \mathcal{E}^+(y, t) \Psi_N \right|^2 \leq ||\eta||_2 \beta^b \leq C \Lambda \beta^b. \quad (152)
\]
This yields
\[
(140) \leq 2 \beta^b + ||\kappa||_2^2 \Lambda^2 / (\pi^2 N) + C \Lambda ||\kappa||_2 \beta^b \leq C \Lambda^4 (\beta + \Lambda / N). \quad (153)
\]
The next terms of \(d \omega \beta(t)\) are bounded by
\[
(141) \leq 4 \langle \kappa(y - x_1) q_1 \Psi_N, i \nabla_1 p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y}\]
\[
\leq 2 \langle \kappa(y - x_1) q_1 \Psi_N, i \nabla_1 p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y} + 2 \langle \mathcal{E}^+(y, t) \Psi_N, p_1 (\Delta(t) \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
= 2 \langle q_1 \Psi_N, \left( \int d^3 y |\kappa(y - x_1)|^2 \right) q_1 \Psi_N \rangle + 2 \langle \mathcal{E}^+(y, t) \Psi_N, p_1 (\Delta(t) \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
= 2 \langle \Psi_N, q_1 \Psi_N \rangle + 2 \langle \mathcal{E}^+(y, t) \Psi_N, p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
\leq 2 \langle \Psi_N, \kappa(y - x_1) \rangle + 2 \langle \mathcal{E}^+(y, t) \Psi_N, p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
\leq \Lambda^3 \mathcal{C} \langle \varphi \rangle_{H^2} \beta \quad (154)
\]
and
\[
(142) \leq 4 \langle \kappa(y - z_1) N^{-1/2} \hat{A}_{\kappa}(x_1) q_1 \Psi_N, p_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
\leq 2 \langle \kappa(y - x_1) N^{-1/2} \hat{A}_{\kappa}(x_1) q_1 \Psi_N \rangle_{y} + 2 \langle \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
= 2 \langle N^{-1/2} \hat{A}_{\kappa}(x_1) q_1 \Psi_N, \left( \int d^3 y |\kappa(y - x_1)|^2 \right) N^{-1/2} \hat{A}_{\kappa}(x_1) q_1 \Psi_N \rangle
\]
\[
+ 2 \langle \Psi_N, \mathcal{E}^-(y, t) \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
\leq 2 \beta^b + \Lambda^4 \mathcal{C} \langle \varphi \rangle_{H^2} \beta \quad (155)
\]
Here, we made use of Lemma VIII.5
\[
(143) \leq 4 \langle \Psi_N, \left( -i \nabla_1 - N^{-1/2} \hat{A}_{\kappa}(x_1) \right) \kappa(y - x_1) q_1 \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
= 4 \langle N^{-1} \sum_{j=1}^N q_j \kappa(y - x_j) \left( -i \nabla_j - N^{-1/2} \hat{A}_{\kappa}(x_j) \right) \Psi_N, \mathcal{E}^+(y, t) \Psi_N \rangle_{y}
\]
\[
\leq 2 \langle N^{-1} \sum_{j=1}^N q_j \kappa(y - x_j) \left( -i \nabla_j - N^{-1/2} \hat{A}_{\kappa}(x_j) \right) \rangle_{y}^2 + 2 \langle \mathcal{E}^+(y, t) \Psi_N \rangle_{y}^2. \quad (156)
\]
According to Lemma X.1 and Lemma VIII.3 this is bounded by

$$|\text{143}| \leq C(\|\varphi\|_{H^2}, \|\varphi\|_\infty, \mathcal{E}_M) (\beta + \Lambda/N).$$ \hfill (157)

### VIII.4 Bound on $d_t\beta$

The Maxwell-Schrödinger equations are a conserved system and its energy does not change during the time evolution

$$\mathcal{E}_M[\varphi_t, u_t] = \mathcal{E}_M[\varphi_0, u_0].$$ \hfill (158)

Moreover, $\beta^c$ is a constant of motion because the self-adjointness of the Pauli-Fierz Hamiltonian gives rise to a strongly continuous unitary group $\{e^{-iH_N}\}_{t \in \mathbb{R}}$ such that $\Psi_{N,t} = e^{-iH_N}\Psi_{N,0}$ and

$$\beta^c(\Psi_{N,t}, \varphi_t, u_t) = \left\| (N^{-1}H_N - \mathcal{E}_M[\varphi_t, u_t]) \Psi_{N,t} \right\|^2$$

$$= \left\| (N^{-1}H_N - \mathcal{E}_M[\varphi_0, u_0]) e^{-iH_N}\Psi_{N,0} \right\|^2$$

$$= \left\| e^{-iH_N} (N^{-1}H_N - \mathcal{E}_M[\varphi_0, u_0]) \Psi_{N,0} \right\|^2 = \beta^c(\Psi_{N,0}, \varphi_0, u_0). \hfill (159)$$

This allows us to bound the time derivative of $\beta(t)$ by

**Lemma VIII.9.** Let $v$ satisfy (A1), $(\varphi_t, A(t), E(t)) \in \mathcal{G}$ and $\Psi_{N,t}$ be the unique solution of (10) with initial data $\Psi_{N,0} \in (L^2(\mathbb{R}^3) \otimes F_p) \cap \mathcal{D}(H_N)$. Then, there exists a monotone increasing function $C(t)$ of $\|A_s\|_\infty$, $\mathcal{E}_M[\varphi_t, u_t]$, $\|\varphi_t\|_{H^2(\mathbb{R}^3)}$ and $\|\varphi_t\|_{L^\infty(\mathbb{R}^3)}$ such that

$$\left| d_t\beta(\Psi_{N,t}, \varphi_t, u_t) \right| \leq \Lambda^4 C(t) (\beta(\Psi_{N,t}, \varphi_t, u_t) + \Lambda/N),$$

$$\beta(\Psi_{N,t}, \varphi_t, u_t) \leq e^{\Lambda^4 \int_0^t dsC(s)} (\beta(\Psi_{N,0}, \varphi_0, u_0) + \Lambda/N) \hfill (160)$$

holds for any $t \geq 0$.

**Proof.** The first inequality is a direct consequence of Lemma VIII.7 Lemma VIII.8 and (159). Then, we apply Gronwall’s inequality and obtain

$$\beta(\Psi_{N,t}, \varphi_t, u_t) \leq e^{\Lambda^4 \int_0^t dsC(s)} (\beta(\Psi_{N,0}, \varphi_0, u_0) + \Lambda/N). \hfill (161)$$

\hfill \Box

### IX Initial conditions

In this section, we show that $\beta(\Psi_{N,0}, \varphi_0, u_0)$ is small for the initial states of Theorem II.2.

**Lemma IX.1.** Let $\Psi_{N,0} \in \mathcal{D}(H_N) \cap (L^2(\mathbb{R}^3) \otimes F_p)$, $\varphi_0 \in H^2(\mathbb{R}^3)$ with $\|\varphi_0\| = 1$ and $\alpha_0 \in \mathfrak{h}$ such that $A(x, 0) \in H^2(\mathbb{R}^3, \mathbb{C}^3)$ and $E(x, 0) \in H^1(\mathbb{R}^3, \mathbb{C}^3)$. Then

$$\beta^a(\Psi_{N,0}, \varphi_0) \leq T_{L^2(\mathbb{R}^3)}|\gamma_{N,0}^{(1,0)}| - |\varphi_0| \langle \varphi_0 \rangle = a_N, \hfill (162)$$

$$\beta^b(\Psi_{N,0}, u_0) = N^{-1} \langle W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}, H_f W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0} \rangle = b_N, \hfill (163)$$

$$\beta^c(\Psi_{N,0}, \varphi_0, u_0) = c_N. \hfill (164)$$

In particular for $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ (and $\Lambda/N \leq C$) we have

$$\beta(\Psi_{N,0}, \varphi_0, u_0) \leq C \Lambda^4 N^{-1}. \hfill (165)$$

27
Before we prove Lemma IX.1 we recall some well-known properties of Weyl operators (13).

**Lemma IX.2.** Let \( f, g \in \mathfrak{h} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \).

(i) \( W(f) \) is a unitary operator and

\[
W^*(f) = W^{-1}(f) = W(-f).
\] (166)

(ii) We have

\[
W^*(f) a(k, \lambda) W(f) = a(k, \lambda) + f(k, \lambda),
\]

\[
W^*(f) a^*(k, \lambda) W(f) = a^*(k, \lambda) + f^*(k, \lambda).
\] (167)

(iii) From (ii) we see that coherent states are eigenvectors of annihilation operators

\[
a(k, \lambda) W(f) \Omega = f(k, \lambda) W(f) \Omega.
\] (168)

Subsequently, we compute the expectation values of the vector potential, the field energy and higher moments.

**Lemma IX.3.** Let \( \alpha_0 \in \mathfrak{h} \) such that \( u_0 \in \mathfrak{h} \) and \( E^\pm_n(x, t) \) be defined by (152). Moreover, let

\[
\gamma^\pm_k^\Lambda(x) := \int d^3k |\tilde{k}(k)|^2 |k|^{-1} e^{ikx} (\delta_{il} - k_i k_l |k|^{-2}) \quad \text{with} \quad ||\gamma^\pm_k^\Lambda||_2^2 \leq \Lambda.
\] (169)

Then

\[
\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1/2} \hat{A}_\kappa(x) W(\sqrt{N}\alpha_0)\Omega \rangle_{\mathcal{F}_p} = A_\kappa(x, 0),
\]

\[
\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1} \hat{A}_\kappa^2(x) W(\sqrt{N}\alpha_0)\Omega \rangle_{\mathcal{F}_p} = A_\kappa^2(x, 0) + \Lambda^2/(4\pi^2 N),
\]

\[
\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1} \hat{A}_\kappa(x) \hat{A}_\lambda(y) W(\sqrt{N}\alpha_0)\Omega \rangle_{\mathcal{F}_p} = A_\kappa(x, 0) A_\lambda(y, 0) + (2N)^{-1} \gamma^\pm_k^\Lambda(x - y),
\]

\[
\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1} H_f W(\sqrt{N}\alpha_0)\Omega \rangle_{\mathcal{F}_p} = ||u_0||^2_{\mathfrak{h}},
\]

\[
\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1/2} H^2_f W(\sqrt{N}\alpha_0)\Omega \rangle_{\mathcal{F}_p} = ||u_0||^4_{\mathfrak{h}} + N^{-1} |||| \cdot 1/2 u_0||^2_{\mathfrak{h}},
\]

\[
\langle W(\sqrt{N}\alpha_0)\Omega, N^{-3/2} \hat{A}_\kappa(x) H_f W(\sqrt{N}\alpha_0)\Omega \rangle_{\mathcal{F}_p} = A_\kappa(x, 0) ||u_0||^2_{\mathfrak{h}} - i N^{-1} E^+_n(x, 0),
\]

\[
\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1/2} \hat{A}_\kappa^2(x) H_f W(\sqrt{N}\alpha_0)\Omega \rangle_{\mathcal{F}_p} = (A^2_\kappa(x, 0) + \Lambda^2/(4\pi^2 N)) ||u_0||^2_{\mathfrak{h}} - 2i N^{-1} A_\kappa(x, 0) E^+_n(x, 0).
\] (170)

**Proof.** The proof is a simple application of the canonical commutation relations (9) and part (ii) from Lemma IX.2.

**Proof of Lemma IX.1** Relation (102) directly follows from Lemma VII.1. In view of Lemma IX.2 we calculate

\[
\beta^b(\Psi_N, 0, u_0) = \sum_{\lambda=1,2} \int d^3k \bigg| \bigg| (N^{-1/2} a(k, \lambda) - \alpha_0(k, \lambda)) \Psi_N,0 \bigg| \bigg|^2
\]

\[
= \sum_{\lambda=1,2} \int d^3k \bigg| \bigg| W^{-1}(\sqrt{N}\alpha_0) \left( (N^{-1/2} a(k, \lambda) - \alpha_0(k, \lambda)) W(\sqrt{N}\alpha_0) \right) W^{-1}(\sqrt{N}\alpha_0) \Psi_{N,0} \bigg| \bigg|^2
\]

\[
= \sum_{\lambda=1,2} \int d^3k \bigg| \bigg| N^{-1/2} a(k, \lambda) W^{-1}(\sqrt{N}\alpha_0) \Psi_{N,0} \bigg| \bigg|^2
\]

\[
= N^{-1} \langle W^{-1}(\sqrt{N}\alpha_0) \Psi_{N,0}, H_f W^{-1}(\sqrt{N}\alpha_0) \Psi_{N,0} \rangle = b_N.
\] (171)
Equation (163) is solely the definition of $\beta$. In the following, we are interested in initial data $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N} \alpha_0)\Omega$ of product type. First, we notice that

$$
\beta_a(\Psi_{N,0}, \varphi_0) = \langle \Psi_{N,0}, q_1^a\Psi_{N,0} \rangle = \langle \varphi_0, \varphi_0 \rangle_{L^2(\mathbb{R}^3)} - \langle \varphi_0, \varphi_0 \rangle_{L^2(\mathbb{R}^3)} = 0
$$

(172)

because the scalar product factorizes for product states and $q_1$ only acts on the Hilbert space of the first charged particle. Then, we follow

$$
\beta^a(\Psi_{N,0}, \varphi_0) = \langle \Psi_{N,0}, q_1^a\Psi_{N,0} \rangle = \langle \varphi_0, \varphi_0 \rangle_{L^2(\mathbb{R}^3)} - \langle \varphi_0, \varphi_0 \rangle_{L^2(\mathbb{R}^3)} = 0
$$

for $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N} \alpha_0)\Omega$ from (171). To show that the product structure suppresses the fluctuations of the energy per particle around its mean value is more elaborate. Nevertheless, the idea of the proof simple and in the spirit of the law of large numbers from probability theory. We bound $\beta$ by

$$
\beta(0) = \langle (N^{-1} H - \mathcal{E}_M) \Psi_{N,0}, (N^{-1} H - \mathcal{E}_M) \Psi_{N,0} \rangle - \langle \Psi_{N,0}, N^{-1} H \Psi_{N,0} \rangle - \langle \Psi_{N,0}, N^{-1} H \Psi_{N,0} \rangle \leq CA^2 N^{-1}
$$

(174)

and show that

(i) $\langle \Psi_{N,0}, N^{-1} H \Psi_{N,0} \rangle - \mathcal{E}_M[\varphi_0, u_0] \leq CA^2 N^{-1}$

(ii) $\langle N^{-1} H \Psi_{N,0}, N^{-1} H \Psi_{N,0} \rangle - \mathcal{E}_M[\varphi_0, u_0] \leq CA^2 N^{-1}$

holds for states of product type.

(i) The mean value of the energy per particle

For ease of notation we denote $A_\kappa(\cdot, 0)$, $E_\kappa^+(\cdot, 0)$, $\|u_0\|^2$, $\|u_0\|^2$ by $A_\kappa(\cdot)$, $E_\kappa(\cdot)$, $\|u_0\|^2$, $\|u_0\|^2$ in the following. The mean value of the energy per particle is given by

$$
\langle \Psi_{N,0}, N^{-1} H \Psi_{N,0} \rangle = \langle \Psi_{N,0}, N^{-1} \sum_{j=1}^N \left(-i \nabla_j - N^{-1/2} A_\kappa(x_j)\right) \Psi_{N,0} \rangle
$$

$$
+ \langle \Psi_{N,0}, 1/(2N^2) \sum_{j \neq k} v(x_j - x_k) \Psi_{N,0} \rangle
$$

$$
+ \langle \Psi_{N,0}, N^{-1} H f \Psi_{N,0} \rangle.
$$

(175)

Due to symmetry and the product structure of $\Psi_{N,0}$ this becomes

$$
\langle \Psi_{N,0}, N^{-1} H \Psi_{N,0} \rangle = \langle \varphi_0, (\Delta) \varphi_0 \rangle
$$

$$
+ 2i\langle \varphi_0, \langle W(\sqrt{N} \alpha_0)\Omega, N^{-1/2} A_\kappa W(\sqrt{N} \alpha_0)\Omega \rangle F_p \cdot \nabla \varphi_0 \rangle
$$

$$
+ \langle \varphi_0, \langle W(\sqrt{N} \alpha_0)\Omega, N^{-1} A_\kappa^2 W(\sqrt{N} \alpha_0)\Omega \rangle F_p \varphi_0 \rangle
$$

$$
+ (N - 1)/(2N) \langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle
$$

$$
+ \langle W(\sqrt{N} \alpha_0)\Omega, N^{-1} H f W(\sqrt{N} \alpha_0)\Omega \rangle F_p.
$$

(176)

Lemma [X,3] gives

$$
\langle \Psi_{N,0}, N^{-1} H \Psi_{N,0} \rangle = \|(-i \nabla - A_\kappa) \varphi_0\|^2 + 1/2\langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle + \|u_0\|^2
$$

$$
+ \Lambda^2/(4\pi^2 N) - 1/(2N) \langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle.
$$

(177)

Now we can pull the sums together to get

$$
\langle \Psi_{N,0}, N^{-1} H \Psi_{N,0} \rangle = \mathcal{E}_M[\varphi_0, u_0] + \Lambda^2/(4\pi^2 N) - 1/(2N) \langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle,
$$

(178)

where $\langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle \leq C(||\varphi_0||_\infty)$ holds according to [122].
(ii) The second moment of the energy per particle

Subsequently, we show that the second moment of the energy per particle approximately equals the energy of the effective system squared. We split the double sum, arising from the second moment of the many-body Hamiltonian into its diagonal and off-diagonal part. The diagonal only consists of \( N \) constituents and has a subleading contribution for large \( N \). On the contrary, there are \( N^2 \) elements from the off-diagonal which give rise to \( \mathcal{E}_M^2 \). In order to organize the estimate, we first decompose the second moment of the energy per particle as well as the effective energy squared into pieces:

\[
\langle N^{-1} H_N \Psi_{N,0}, N^{-1} H_N \Psi_{N,0} \rangle = 
N^{-2} \sum_{j,k} \left( -i \nabla_j - N^{-1/2} \hat{A}_n(x_j) \right)^2 \Psi_{N,0}, \left( -i \nabla_k - N^{-1/2} \hat{A}_n(x_k) \right)^2 \Psi_{N,0} \rangle 
+ (4N^4)^{-1} \sum_{i \neq j, k \neq l} \langle v(x_i - x_j) \Psi_{N,0}, v(x_k - x_l) \Psi_{N,0} \rangle 
+ N^{-2} \langle \Psi_{N,0}, H_f^2 \Psi_{N,0} \rangle 
+ N^{-3} \sum_{j, k \neq l} \text{Re} \langle \left( -i \nabla_j - N^{-1/2} \hat{A}_n(x_j) \right)^2 \Psi_{N,0}, v(x_k - x_l) \Psi_{N,0} \rangle 
+ 2N^{-2} \sum_j \text{Re} \langle \left( -i \nabla_j - N^{-1/2} \hat{A}_n(x_j) \right)^2 \Psi_{N,0}, H_f \Psi_{N,0} \rangle 
+ N^{-3} \sum_{j \neq k} \text{Re} \langle v(x_j - x_k) \Psi_{N,0}, H_f \Psi_{N,0} \rangle 
\]  

(179)

and

\[
\mathcal{E}_M^2[\varphi_0, u_0] = \langle \varphi_0, (-i \nabla - \mathcal{A}_n) \varphi_0 \rangle^2 
+ 1/4 \langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle^2 
+ ||u_0||^4 
+ \langle \varphi_0, \left( (-i \nabla - \mathcal{A}_n)^2 \right) \varphi_0 \rangle \langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle 
+ 2\langle \varphi_0, (-i \nabla - \mathcal{A}_n)^2 \varphi_0 \rangle ||u_0||^2 
+ \langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle ||u_0||^2. 
\]  

(185)

(186)

(187)

(188)

(189)

(190)

Then, we estimate the difference between the corresponding expressions and obtain

\[
|\langle N^{-1} H_N \Psi_{N,0}, N^{-1} H_N \Psi_{N,0} \rangle - \mathcal{E}_M^2[\varphi_0, u_0] | \leq C \Lambda^4 N^{-1}. 
\]  

(191)

\[ |\text{(179)} - \text{(185)}| \leq C \Lambda^4 / N: \]

The off-diagonal part of (179) is given by

\[
\langle \left( -i \nabla_1 - N^{-1/2} \hat{A}_n(x_1) \right)^2 \Psi_{N,0}, \left( -i \nabla_2 - N^{-1/2} \hat{A}_n(x_2) \right)^2 \Psi_{N,0} \rangle 
= \langle (-\Delta_1) \Psi_{N,0}, (-\Delta_2) \Psi_{N,0} \rangle 
+ 2i \langle (-\Delta_1) \Psi_{N,0}, N^{-1/2} \hat{A}_n(x_2) \nabla_2 \Psi_{N,0} \rangle 
- 2i \langle N^{-1/2} \hat{A}_n(x_1) \nabla_1 \Psi_{N,0}, (-\Delta_2) \Psi_{N,0} \rangle 
+ \langle (-\Delta_1) \Psi_{N,0}, N^{-1} \hat{A}_n^2(x_2) \Psi_{N,0} \rangle 
\]  

(192)

(193)

(194)

(195)

(196)
In order to evaluate the last three lines, we use that

\begin{align}
&\langle \varphi_0, (-\Delta) \varphi_0 \rangle^2 + 4i\langle \mathbf{A}_\kappa \varphi_0, \nabla \varphi_0 \rangle \langle \varphi_0, (-\Delta) \varphi_0 \rangle,
&\langle \varphi_0, \mathbf{A}_\kappa^2 \varphi_0 \rangle \langle \varphi_0, (-\Delta) \varphi_0 \rangle + \Lambda^2/(2\pi^2 N) ||\nabla \varphi_0||^2,
&-2/N \int d^3 x \int d^3 y \varphi_0^*(x) \varphi_0(y) \gamma_{kl}^A(x-y) \nabla^k \varphi_0(x) \nabla^l \varphi_0(y).
\end{align}

In order to evaluate the last three lines, we use that

\begin{align}
N^{-3/2} &\langle W(\sqrt{N} \alpha_0) \Omega, \hat{\mathbf{A}}^i_\kappa(x) \hat{\mathbf{A}}^i_\kappa(y) W(\sqrt{N} \alpha_0) \Omega \rangle_{F_p} = \mathbf{A}_\kappa^2(x) \mathbf{A}_\kappa^2(y) + \Lambda^2/(4\pi^2 N) \mathbf{A}_\kappa^4(y)
+ N^{-4} \sum_{j=1}^3 \gamma_{ij}^A(x-y) \hat{\mathbf{A}}^i_\kappa(x),
\end{align}

and

\begin{align}
N^{-2} &\langle W(\sqrt{N} \alpha_0) \Omega, \hat{\mathbf{A}}^2_\kappa(x) \hat{\mathbf{A}}^2_\kappa(y) W(\sqrt{N} \alpha_0) \Omega \rangle_{F_p} = \mathbf{A}_\kappa^2(x) \mathbf{A}_\kappa^2(y) +
\Lambda^2/(4\pi^2 N) \left( \mathbf{A}_\kappa^2(x) + \mathbf{A}_\kappa^2(y) \right) + 2/N \sum_{k,l=1}^3 \gamma_{kl}^A(x-y) \mathbf{A}_\kappa^k(x) \mathbf{A}_\kappa^l(y)
+ N^{-2} \left( \sum_{k,j=1}^3 |\gamma_{kj}^A(x-y)|^2 + \Lambda^4/(2\pi^4) \right),
\end{align}

can also be obtained by the canonical commutation relations (1) and Lemma [X.2]. Consequently, we have

\begin{align}
&\langle \varphi_0, \mathbf{A}_\kappa^2 \varphi_0 \rangle \langle \varphi_0, \mathbf{A}_\kappa \nabla \varphi_0 \rangle + i\Lambda^2/(2\pi^2 N) \langle \varphi_0, \mathbf{A}_\kappa \nabla \varphi_0 \rangle,
+ N^{-2} \int d^3 x \int d^3 y \varphi_0^*(x) \varphi_0(y) \gamma_{kl}^A(x-y) A^k_\kappa(x) \varphi(x) (\nabla^l \varphi)(y),
\end{align}

and

\begin{align}
&\langle \varphi_0, \mathbf{A}_\kappa^2 \varphi_0 \rangle^2 + \Lambda^4/(2\pi^2 N) \langle \varphi_0, \mathbf{A}_\kappa^2 \varphi_0 \rangle + \Lambda^4/(16\pi^4 N^2)
+ 2/N \int d^3 x \int d^3 y \varphi_0^*(x) \varphi_0(y) \gamma_{kl}^A(x-y) \phi(x) \phi(y).
\end{align}

Pulling all the pieces together and using that all error terms are bounded by $C\Lambda^4/N$ under the assumptions of Lemma [X.1] gives

\begin{align}
||1924 - \langle \varphi_0, (-i \nabla - \mathbf{A}_\kappa)^2 \varphi_0 \rangle^2 || \leq C\Lambda^4/N.
\end{align}

Since the diagonal part of (179) is of order $N^{-1}$, this implies

\begin{align}
|179 - 185| \leq C\Lambda^4/N.
\end{align}
|180| - |186| \leq C/N:

By virtue of the symmetry of the wave function and \(v(-x) = v(x)\) we can write line (180) as

\[
(4N^4)^{-1} \sum_{i \neq j, k \neq l} \langle v(x_i - x_j) \Psi_{N,0}, v(x_k - x_l) \Psi_{N,0} \rangle = \\
= 1/4 \langle v(x_1 - x_2) \Psi_{N,0}, v(x_3 - x_4) \Psi_{N,0} \rangle \\
- (6N^2 - 11N + 6)N^{-3} \langle v(x_1 - x_2) \Psi_{N,0}, v(x_3 - x_4) \Psi_{N,0} \rangle \\
+ (N - 1)N^{-3}/2 \langle v(x_1 - x_2) \Psi_{N,0}, v(x_1 - x_2) \Psi_{N,0} \rangle \\
+ (N - 1)(N - 2)N^{-3} \langle v(x_1 - x_2) \Psi_{N,0}, v(x_1 - x_3) \Psi_{N,0} \rangle.
\]

(208)

The product structure of the initial state, (76) and (182) give

\[
\langle v(x_1 - x_2) \Psi_{N,0}, v(x_3 - x_4) \Psi_{N,0} \rangle = \langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle \leq C(\|\varphi_0\|_\infty), \\
\|v(x_1 - x_2) \Psi_{N,0}\|^2 = \langle \varphi_0, (v^2 * |\varphi_0|^2) \varphi_0 \rangle \leq C(\|\varphi_0\|_\infty)
\]

and we conclude

\[
|\langle 4N^4 \rangle^{-1} \sum_{i \neq j, k \neq l} \langle v(x_i - x_j) \Psi_{N,0}, v(x_k - x_l) \Psi_{N,0} \rangle | \\
= 1/4 \langle v(x_1 - x_2) \Psi_{N,0}, v(x_3 - x_4) \Psi_{N,0} \rangle \\
- (6N^2 - 11N + 6)N^{-3} \langle v(x_1 - x_2) \Psi_{N,0}, v(x_3 - x_4) \Psi_{N,0} \rangle \\
+ (N - 1)N^{-3}/2 \langle v(x_1 - x_2) \Psi_{N,0}, v(x_1 - x_2) \Psi_{N,0} \rangle \\
+ (N - 1)(N - 2)N^{-3} \langle v(x_1 - x_2) \Psi_{N,0}, v(x_1 - x_3) \Psi_{N,0} \rangle.
\]

(210)

|181| - |187| \leq C/N:

This bound results from Lemma [IX.3] because

\[
N^{-2} \langle \Psi_{N,0}, H_f^2 \Psi_{N,0} \rangle = N^{-2} \langle W(\sqrt{N}\alpha_0)\Omega, H_f^2 W(\sqrt{N}\alpha_0)\Omega \rangle = ||u_0||^4 + N^{-1}|||v||^2 / u_0|^2.
\]

(211)

|182| - |188| \leq CA^2/N:

Line (182) simplifies to

\[
N^{-3} \sum_{j,k \neq l} \text{Re} \langle -i\nabla_j - N^{-1/2} \hat{A}_\kappa(x_j) \rangle^2 \Psi_{N,0}, v(x_k - x_l) \Psi_{N,0} \rangle \\
= (N - 1)(N - 2)N^{-2} \text{Re} \langle -i\nabla_1 - N^{-1/2} \hat{A}_\kappa(x_1) \rangle^2 \Psi_{N,0}, v(x_2 - x_3) \Psi_{N,0} \rangle \\
+ 2(N - 1)N^{-2} \text{Re} \langle -i\nabla_1 - N^{-1/2} \hat{A}_\kappa(x_1) \rangle^2 \Psi_{N,0}, v(x_2 - x_1) \Psi_{N,0} \rangle \\
= (1 - 3(N - 2)N^{-2}) \langle \varphi_0, (-i\nabla - A_\kappa)^2 \varphi_0 \rangle (v * |\varphi_0|^2) \varphi_0 \\
+ (N - 1)(N - 2)N^{-3} \Lambda^2/\langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle \\
+ 2(N - 1)N^{-2} \text{Re} \langle -i\nabla_1 - N^{-1/2} \hat{A}_\kappa(x_1) \rangle^2 \Psi_{N,0}, v(x_2 - x_1) \Psi_{N,0} \rangle.
\]

(212)

Consequently the estimate follows because \(\left|\langle -i\nabla_1 - N^{-1/2} \hat{A}_\kappa(x_1) \rangle^2 \Psi_{N,0} \right|\) is finite under the assumptions of Lemma [IX.1]

32
\[ |(183) - (189)| \leq CA^2/N. \]

Similar to the previous calculations we obtain
\[
2N^{-2} \sum_{j=1}^{N} \text{Re} \langle -i\nabla_j - N^{-1/2} A_\kappa(x_j) \rangle^2 \Psi_{N,0}, H_f \Psi_{N,0} \rangle
= 2 \text{Re} \langle -\Delta_1 + 2iN^{-1/2} A_\kappa(x_1) + N^{-1} A^2_\kappa(x_1) \rangle \Psi_{N,0}, N^{-1} H_f \Psi_{N,0} \rangle
= 2 \langle \varphi_0, (-i\nabla - A_\kappa)^2 \varphi_0 \rangle \|u_0\|^2
+ 2N^{-1} \text{Re} \left( \Lambda^2 / (4\pi^2) \|u_0\|^2 - 4 \langle \nabla \varphi_0, \nabla^+ E_\kappa(x) \varphi_0 \rangle - 2i \langle A_\kappa \varphi_0, \nabla^+ \varphi_0 \rangle \right). \tag{213}
\]

By means of
\[
\langle A_\kappa \varphi_0, \nabla^+ \varphi_0 \rangle \leq \|\varphi_0\|_{\infty} \|E^+_\kappa\| \|A_\kappa\|_{\infty},
\|
\langle \nabla \varphi_0, \nabla^+ E_\kappa \varphi_0 \rangle \leq \|\varphi_0\|_{\infty} \|E^+_\kappa\| \|
\langle \nabla \varphi_0 \rangle , \tag{214}
\]
and
\[
\|E^+_\kappa\|_2^2 = \frac{1}{2} \sum_{\lambda=1,2} \int_{|k| \leq \Lambda} d^3 k |k| \alpha_0(k,\lambda)^2 \leq \|u_0\|^2 \tag{215}
\]
the inequality follows.

\[ |(184) - (190)| \leq C/N; \]

Making use of symmetry and Lemma [X.3] one has
\[
N^{-3} \sum_{j \neq k} \text{Re} \langle v(x_k - x_k) \Psi_{N,0}, H_f \Psi_{N,0} \rangle = (N - 1)N^{-2} \langle v(x_1 - x_2) \Psi_{N,0}, H_f \Psi_{N,0} \rangle
= (1 - N^{-1}) \langle \varphi_0, (v * |\varphi_0|^2) \varphi_0 \rangle \|u_0\|^2. \tag{216}
\]

This shows the last inequality and altogether we obtain
\[
\langle -N^1 H_N \Psi_{N,0}, N^{-1} H_N \Psi_{N,0} \rangle - \mathcal{E}^2_M[\varphi_0, u_0] \leq CA^4 N^{-1}, \tag{217}
\]
which proves Lemma [X.4]. \qed

**X  Proof of Theorem II.2**

Let \( v \) satisfy (A1), \((\varphi_t, A(t), E(t)) \in \mathcal{G} \) and \( \Psi_{N,t} \) be the unique solution of (I) with initial data \( \Psi_{N,0} \in (L^2_2(\mathbb{R}^3) \otimes \mathcal{F}_p) \cap \mathcal{D}(H_N) \). According to Lemma [VIII.9] and Lemma [IX.4] there is a monotone increasing function of \( \|\varphi_s\|_{H^2(\mathbb{R}^3)} \), \( \|A_\kappa(s)\|_{\infty} \) and \( \mathcal{E}_M[\varphi_s, u_s] \) such that
\[
\beta(\Psi_{N,t}, \varphi_t, u_t) \leq e^{A^4 \int_0^t ds C(s)} (a_N + b_N + c_N + \Lambda/N). \tag{218}
\]

The energy \( \mathcal{E}_M[\varphi_s, u_s] = \mathcal{E}_M[\varphi_0, u_0] \) is a finite constant of motion. Moreover, we have \( \|A_\kappa\|_{\infty} \leq \|A\|_{H^1(\mathbb{R}^3)} \). This displays that \( C(s) \) only depends on \( \|\varphi_s\|_{H^2(\mathbb{R}^3)} \) and \( \|A\|_{H^2(\mathbb{R}^3)} \). We choose for a given time \( t \geq 0 \) the number \( N \) of charges large enough so that \( \beta(\Psi_{N,t}, \varphi_t, u_t) \leq 1 \) and obtain
\[
\text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t|\rangle \langle \varphi_t| \leq \sqrt{a_N + b_N + c_N + \Lambda/N} e^{A^4 \int_0^t ds C(s)}
\text{Tr}_{0}|\gamma_{N,t}^{(0,1)} - |u_t|\rangle \langle u_t| \leq \sqrt{a_N + b_N + c_N + \Lambda/N} 6(1 + \|u_t\|_0) e^{A^4 \int_0^t ds C(s)}. \tag{219}
\]
by Lemma VII.1. Then, we recall (37) and derive

\[ \text{Tr}_{L^2(\mathbb{R}^3)} |\gamma^{(1,0)}_{N,t} - |\varphi_t\rangle\langle \varphi_t| \leq \sqrt{a_N + b_N + c_N + N^{-1} \Lambda e^{A^4} \int_0^t ds C(s)} \]
\[ \text{Tr}_b |\gamma^{(0,1)}_{N,t} - |u_t\rangle\langle u_t| \leq \sqrt{a_N + b_N + c_N + N^{-1} \Lambda C(s) e^{A^4} \int_0^t ds C(s)} \]

(220)

where \( C(s) \) depends on \( ||\varphi||_{L^2(\mathbb{R}^2)} \), \( ||A||_{L^2(\mathbb{R}^3)} \) and \( ||E||_{L^2(\mathbb{R}^3)} \). For initial states of product type \( \Psi_{N,0} = \varphi_0^\otimes N \otimes W(\sqrt{N} \alpha_0) \Omega \) this becomes

\[ \text{Tr}_{L^2(\mathbb{R}^3)} |\gamma^{(1,0)}_{N,t} - |\varphi_t\rangle\langle \varphi_t| \leq N^{-1/2} \Lambda^2 e^{A^4} \int_0^t ds C(s) \]
\[ \text{Tr}_b |\gamma^{(0,1)}_{N,t} - |u_t\rangle\langle u_t| \leq N^{-1/2} \Lambda^2 C(s) e^{A^4} \int_0^t ds C(s). \]

(221)

XI Appendix

Lemma XI.1. Let \( \Psi_{N,t} \in (L^2_+(\mathbb{R}^3)) \cap D(\mathcal{H}_N) \) and \( (\varphi_t, A(t), E(t)) \in \mathcal{G} \). Then

\[ \int d^3y \left( \left| \left( N^{-1/2} \hat{E}_N^+(y) - E_N^+(y,t) \right) \Psi_{N,t} \right|^2 = \langle \Psi_{N,t}, E^-(y,t) E^+(y,t) \Psi_{N,t} \rangle_{y} \right) \leq \beta^b(t). \]

(222)

For \( \hat{G} \in \{ A^+, A^-, E^+, E^- \} \) one obtains

\[ \left| \hat{G}(y,t) \nabla_1 p_1 \Psi_{N,t} \right|_{L^2(\mathbb{R}^3)}^2 \leq C \left( \left| \nabla_1 \varphi_t \right|_{L^2(\mathbb{R}^3)}^2 \left| \hat{G}(y,t) \Psi_{N,t} \right|_{L^2(\mathbb{R}^3)}^2 \right), \]
\[ \left| \hat{G}(x_1,t) p_1 \Psi_{N,t} \right|_{L^\infty(\mathbb{R}^3)}^2 \leq C \left( \left| \varphi_t \right|_{L^\infty(\mathbb{R}^3)}^2 \left| \hat{G}(y,t) \Psi_{N,t} \right|_{L^2(\mathbb{R}^3)}^2 \right). \]

(223)

Proof. The first inequality is proven by

\[ \left| E^+(y,t) \Psi_N \right|_{L^2(\mathbb{R}^3)}^2 = 1/2 \sum_{\lambda=1,2} \int d^3k \hat{k}(k) |k|^{1/2} \epsilon_\lambda(k) \sum_{\mu=1,2} \int d^3l \hat{l}(l) |l|^{1/2} \epsilon_\mu(l) \int d^3y e^{i(l-k)y} \]
\[ \times \langle \left( N^{-1/2} a(k,\lambda) - \alpha_t(k,\lambda) \right) \Psi_N, \left( N^{-1/2} a(l,\mu) - \alpha_t(l,\mu) \right) \Psi_N \rangle \]
\[ = (2\pi)^3/2 \int d^3l \hat{k}(k) |k|^3 \sum_{\lambda,\mu} \epsilon_\lambda(k) \epsilon_\mu(k) \times \]
\[ \times \langle \left( N^{-1/2} a(k,\lambda) - \alpha_t(k,\lambda) \right) \Psi_N, \left( N^{-1/2} a(k,\mu) - \alpha_t(k,\mu) \right) \Psi_N \rangle \]
\[ = 1/2 \sum_{\lambda=1,2} \int d^3k |k|^3 \langle \left( N^{-1/2} a(k,\lambda) - \alpha_t(k,\lambda) \right) \Psi_N \rangle^2 \leq \beta^b. \]

(224)

We continue with

\[ \left| \hat{G}(y,t) \nabla_1 p_1 \Psi_N \right|_{L^2(\mathbb{R}^3)}^2 = \int d^3y \left| \hat{G}(y,t) \nabla_1 p_1 \Psi_N \right|_{L^2(\mathbb{R}^3)}^2 \]
\[ = \int d^3y \left( \hat{G}(y,t) \Psi_N, p_1 (\Delta_1) p_1 \hat{G}(y,t) \Psi_N \right). \]

(225)

So if we use \( p_1 (\Delta_1) p_1 = p_1 ||\nabla \varphi||^2 \) we get

\[ \left| \hat{G}(y,t) \nabla_1 p_1 \Psi_N \right|_{L^2(\mathbb{R}^3)}^2 = ||\nabla \varphi||^2 \int d^3y \left| p_1 \hat{G}(y,t) \Psi_N \right|_{L^2(\mathbb{R}^3)}^2 \leq ||\nabla \varphi||^2 \int d^3y \left| \hat{G}(y,t) \Psi_N \right|_{L^2(\mathbb{R}^3)}^2. \]

(226)
In the same way, we apply

\[ p_1(\hat{G}(x_1,t))^\ast \hat{G}(x_1,t) p_1 = p_1 \int d^3y |\varphi(y)|^2 (\hat{G}(y,t))^\ast \hat{G}(y,t) \]

(227)

to show the third inequality

\[ \left\| \hat{G}(x_1,t)p_1 \Psi_N \right\|^2 = \int d^3y |\varphi(y)|^2 \left\langle \Psi_N, p_1(\hat{G}(y,t))^\ast \hat{G}(y,t) \Psi_N \right\rangle \]
\[ = \int d^3y |\varphi(y)|^2 \left\| p_1 \hat{G}(y,t) \Psi_N \right\|^2 \leq \left\| \varphi \right\|_\infty^2 \left\| \hat{G}(y,t) \Psi_N \right\|_{y}^2 . \]

(228)

\[ \square \]

Acknowledgments

We thank Dirk André Deckert, Jan Dereziński, Detlef Dürr, Marco Falcioni, Maximilian Jeblick, Vytautas Matulevičius, and Alessandro Michelangeli for many helpful remarks. We are deeply grateful to Vytautas Matulevičius for valuable discussions at the early stage of this project and to Alessandro Michelangeli for helpful remarks concerning the Maxwell-Schrödinger system. N.L. gratefully acknowledges financial support by the Cusanuswerk and the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 694227). The article appeared in slightly different form in one of the author’s (N.L.) PhD thesis [12].

References

[1] Ammari, Z., Falcioni, M.: Bohr’s correspondence principle in quantum field theory and classical renormalization scheme: the Nelson model, arXiv:1602.03212 (2016).

[2] Anapolitanos, I., Mott, M.: A simple proof of convergence to the Hartree dynamics in Sobolev trace norms, arXiv:1608.01192 (2016).

[3] Einstein, A.: Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt, Annalen der Physik, 17, 132-148 (1905).

[4] Falcioni, M.: Classical limit of the nelson model with cutoff, arXiv:1205.4367, J. Math. Phys, 54 no. 1, 012303 (2013).

[5] Frank, R. L., Gang, Z.: Derivation of an effective evolution equation for a strongly coupled polaron, arXiv:1505.03059 (2015).

[6] Frank, R. L., Schlein, B.: Dynamics of a strongly coupled polaron, arXiv:1311.5814, Lett. Math. Phys. 104, 911 - 929 (2014).

[7] Ginibre, J., Nironi, F., Velo, G.: Partially classical limit of the Nelson model, arXiv:math-ph/0411046, Ann. H. Poincaré, 7, 21 - 43 (2006).

[8] Hiroshima, F.: Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, Ann. H. Poincaré, 3, 171-201 (2002).

[9] Jeblick, M., Leopold, N., Pickl, P.: Derivation of the Time Dependent Gross-Pitaevskii Equation in Two Dimensions, arXiv:1608.05326 (2016).
[10] Knowles, A.: *Limiting dynamics in large quantum systems*, http://www.unige.ch/~knowles/thesis.pdf Ph.D thesis (2009).

[11] Knowles, A., Pickl, P.: *Mean-Field Dynamics: Singular Potentials and Rate of Convergence*, arXiv:0907.4313, Comm. Math. Phys. **298**, 101-139 (2010).

[12] Leopold, N.: *Effective Evolution Equations from Quantum Mechanics*, https://edoc.ub.uni-muenchen.de/21926/, Ph.D thesis (2018).

[13] Leopold, N., Pickl, P.: *Mean-field limits of particles in interaction with quantized radiation fields*, arXiv:1806.10843 (2018).

[14] Matulevičius, V.: *Maxwell’s Equations as Mean Field Equations*, http://www.mathematik.uni-muenchen.de/~bohmmech/theses/Matulevicius_Vytautas_MA.pdf, master thesis (2011).

[15] Michelangeli, A.: *Equivalent definitions of asymptotic 100% BEC*, Nuovo Cimento Sec. B. **123**, 181-192 (2008).

[16] Michelangeli, A., Olgiati, A.: *Mean-field quantum dynamics for a mixture of Bose-Einstein condensates*, arXiv:1603.02435 (2016).

[17] Nakamura, M., Wada, T.: *Global Existence and Uniqueness of Solutions to the Maxwell-Schrödinger Equations*, arXiv:math/0607039, Comm. Math. Phys. **276**, 315-339 (2007).

[18] Pickl, P.: *A simple derivation of mean field limits for quantum systems*, arXiv:0907.4464, Lett. Math. Phys. **97**, 151–164 (2011).

[19] Pickl, P.: *Derivation of the time dependent Gross-Pitaevskii equation without positivity condition on the interaction*, arXiv:0907.4466, J. Stat. Phys. **140**, 76–89 (2010).

[20] Pickl, P.: *Derivation of the time dependent Gross-Pitaevskii equation with external fields*, arXiv:1001.4894, Rev. Math. Phys., 27, 1550003 (2015).

[21] Rodnianski, I., Schlein, B.: *Quantum fluctuations and rate of convergence towards mean field dynamics*, arXiv:0711.3087, Comm. Math. Phys. **291**, 31–61 (2009).

[22] Šindelka, M.: *Derivation of coupled Maxwell-Schrödinger equations describing matter-laser interaction from first principles of quantum electrodynamics*, Phys. Rev. A **81**, no. 3, 033833 (2010).

[23] Spohn, H.: *Dynamics of charged particles and their radiation field*, Cambridge University Press, Cambridge (2004), ISBN 0-521-83697-2.

[24] Teufel, S.: *Effective N-body Dynamics for the Massless Nelson Model and Adiabatic Decoupling without Spectral Gap*, arXiv:math-ph/0203046, Ann. Henri Poincaré, **3**, 939-965, (2002).