ON QUANTUM JACOBI IDENTITY

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November, 1996

Abstract. In this paper we present our variant of quantum antisymmetry and quantum Jacobi identity.

1. Introduction. The notion of “quantum Lie algebra,” related to the notion of quantum group, has recently been investigated by many researchers (see Bibliography on quantum Lie algebras.) It seems, however, that many important problems are still open.

Our goal in the subject is to reconstruct the category $E$ (of finitely generated representaions of quantum group corresponding to complex simple Lie algebra $g$) from such data as the quantization of the adjoint representation $V^h$, the quantum Casimir tensor $t^h$, the quantum Lie bracket $T^h$, the quantum commutativity morphism $\tilde{R}^h$, etc i.e. we would like to present the category $E$ as the category of modules over quantum Lie algebra.

As a first step towards the goal we present in this paper our version of what may be called “quantum antisymmetry” and “quantum Jacobi identity.” It turns out that the infinitesimal deformation resembles the classical Jacobi identity.

2. Notation. In this paper 1 stands for the identity morphism of some object. Each time it is either clear which object we mean, or we specify the object.

Let $g$ be a complex semisimple Lie algebra, and $t \in g \otimes g$ be the invariant symmetric tensor corresponding to the Killing form on $g$. Following notations and terminology of Kazhdan and Lusztig ([6]), we take $D$ and $E$ to be Drinfeld’s categories associated to $g$. We notice that the category $D$ has two structures of braided tensor category: a trivial structure $(D, \otimes, \sigma, 1 \otimes 1 \otimes 1)$, where $\sigma : U \otimes W \rightarrow U \otimes W$ is just the simple transposition of factors in $U \otimes W$, and a non-trivial structure $(D, \otimes, \tilde{R} = \sigma \circ \tilde{R} = \sigma \circ e^{\sqrt{-1}\pi h t}, \Phi = \Phi_{KZ}(t_{12}, t_{23}))$. Let $X : D \rightarrow E$ be the equivalence functor of Kazhdan and Lusztig. For two objects $U$ and $W$ in $D$, let $M_{U,W} : X(U) \otimes X(W) \rightarrow X(U \otimes W)$ be the “natural transformation” morphism providing the tensor structure of the functor $X$ (see 25.7. of [6].) Transforming the trivial structure in $D$ via the
functor \( X \) we will get the second structure of the braided tensor category in \( \mathcal{E} \). We will denote the corresponding commutativity morphism by \( \sigma^h \) and the corresponding associativity morphism by \( \Psi \). We will study the category \(( \mathcal{E}, \otimes, \sigma^h, \Psi \)) in more detail.

In the category \( \mathcal{D} \) we have a distinguished object \( V = g[[h]] \) and a distinguished morphism \([ , ] = T : V \otimes V \to V\), corresponding to the Lie bracket on \( g \), and a distinguished morphism \( \tilde{R} : V \otimes V \to V \otimes V \), \( \tilde{R} = \sigma \circ R_{VV} \), where \( R_{VV} \) is the representation in \( V \otimes V \) of the universal \( R \)-matrix \( R = e^{-\frac{1}{2} \pi h} \). The morphisms \( t^h, T^h, \) and \( \tilde{R}^h \) are defined as

\[
t^h = M_{VV}^{-1} \circ X(t) \circ M_{C[[h]],C[[h]]}, \quad T^h = X(T) \circ M_{VV}, \quad \tilde{R}^h = M_{VV}^{-1} \circ X(\tilde{R}) \circ M_{VV}
\]

where \( t \) is considered as a morphism \( t : C[[h]] \otimes C[[h]] \to V \otimes V \). We denote by \( V^h = X(V) \) and we set

\[
\Omega = -(T \otimes T) \circ (1_V \otimes t \otimes 1_V), \quad \Omega^h = M_{VV}^{-1} \circ X(\Omega) \circ M_{VV}, \quad \sigma^h = M_{VV}^{-1} \circ X(\sigma) \circ M_{VV} = \tilde{R}^h \circ e^{-\frac{1}{2} \pi h} \Omega^h
\]

where \( \Omega \) is considered as a morphism \( V \otimes V \to V \otimes V \) in \( \mathcal{D} \), and \( \Omega^h : V^h \otimes V^h \to V^h \otimes V^h \) is considered as a morphism in \( \mathcal{E} \).

**Remark.** We identify objects \( V, V \otimes C[[h]] \) and \( C[[h]] \otimes V \) in \( \mathcal{D} \) as well as \( V^h, V^h \otimes X(C[[h]]) \) and \( X(C[[h]]) \otimes V^h \) in \( \mathcal{E} \).

### 3. Classical Jacobi identity.

By the definition of Lie algebra, our distinguished morphism \( T \) satisfies the antisymmetry property

\[
(3.1) \quad T \circ \sigma = -T
\]

and the classical Jacobi identity

\[
(3.2) \quad T \circ (T \otimes 1_V) = T \circ (1_V \otimes T) \circ (1_V \otimes V \otimes V - \sigma \otimes 1_V)
\]

The morphism \( \sigma \) obviously satisfies the quantum Yang-Baxter equation.

\[
(3.3) \quad \sigma_{12} \circ \sigma_{23} \circ \sigma_{12} = \sigma_{23} \circ \sigma_{12} \circ \sigma_{23}
\]

### 4. Quantum Jacobi identity.

Now let us apply the Kazhdan-Lusztig functor \( X \) to our distinguished morphisms \( T \) and \( \sigma \) and see what happens to the identities (3.1), (3.2) and (3.3). Let us denote

\[
M = (M_{VV} \otimes V) \circ (1_{V^h} \otimes M_{VV}), \quad N = (M_{V^h} \otimes V) \circ (M_{VV} \otimes 1_{V^h})
\]

so \( M, N : X(V) \otimes X(V) \otimes X(V) \to X(V \otimes V \otimes V) \) are isomorphisms in \( \mathcal{E} \). Note that by Proposition 25.8 of [6], we have

\[
(4.1) \quad X(\Phi) = M \circ N^{-1} = (M + N) \circ N^{-1} + 1
\]

where \( \Phi \) here is considered as a morphism \( V \otimes V \otimes V \to V \otimes V \otimes V \) in \( \mathcal{D} \). We put

\[
(4.2) \quad \Psi = M^{-1} \circ N = M^{-1} \circ (N - M) + 1
\]
Theorem 4.1. The following identities hold for morphisms in $\mathcal{E}$

\begin{equation}
(4.3) \quad T^h \circ \sigma^h = -T^h, \quad (\sigma^h)^2 = 1
\end{equation}

\begin{equation}
(4.4) \quad T^h \circ (T^h \otimes 1) = T^h \circ (1 \otimes T^h) \circ \Psi \circ (1_{V^h} \otimes \sigma^h \otimes V^h - \sigma^h \otimes 1)
\end{equation}

\begin{equation}
(4.5) \quad \Psi \circ \sigma_{12}^h \circ \Psi^{-1} \circ \sigma_{23}^h \circ \Psi \circ \sigma_{12}^h = \sigma_{23}^h \circ \Psi \circ \sigma_{12}^h \circ \Psi^{-1} \circ \sigma_{23}^h \circ \Psi
\end{equation}

Proof. Follows from definitions, identities (3.1), (3.2) and (3.3), and an elementary diagram search. □

The identity (4.3) is our variant of quantum antisymmetry and the identity (4.4) is our variant of quantum Jacobi identity.

5. The associator $\Psi$. It is well known (due to Le and Murakami [8]; we use the exposition in [5]) that the morphism $\Phi : V \otimes V \otimes V \to V \otimes V \otimes V$ can be presented as

\begin{equation}
(5.1) \quad \Phi = 1 + \sum_{k=2}^{\infty} E_k(\Omega_{12}, \Omega_{23})h^k
\end{equation}

where $E_k(\Omega_{12}, \Omega_{23})$ is a homogeneous polynomial (of degree $k$) in $\Omega_{12}$ and $\Omega_{23}$. The explicit form of the polynomials $E_k$ can be found in the references. For example,

\begin{equation}
(5.2) \quad E_2(\Omega_{12}, \Omega_{23}) = -\frac{\zeta(2)}{(2\pi \sqrt{-1})^2} [\Omega_{12}, \Omega_{23}] = \frac{1}{24}[\Omega_{12}, \Omega_{23}]
\end{equation}

Let us now describe one formal polynomial map. Let $\mathbb{C}[A, B]$ be the ring of formal polynomials in two formal non-commuting variables over $\mathbb{C}$. We will construct a map $\alpha : \mathbb{C}[A, B] \to \mathbb{C}[C, D, Z, Z^{-1}]$, $\alpha(1) = 1$ where $C$, $D$, and $Z$ are also formal pairwise non-commuting variables, $ZZ^{-1} = Z^{-1}Z = 1$, as follows

\[ \alpha : A^{n_1} B^{m_1} A^{n_2} B^{m_2} \ldots A^{n_j} B^{m_j} \mapsto C^{m_1} Z^{-1} D^{m_1} Z \ldots Z C^{m_j} Z^{-1} D^{m_j} Z \]

where $n_i, m_i \in \mathbb{Z}_{\geq 0}$, for $1 \leq i \leq j$. We will now construct polynomials $G_k$ in variables $\Omega_{12}^h, \Omega_{23}^h, \Psi$ and $\Psi^{-1}$ as follows: consider $E_k(\Omega_{12}, \Omega_{23})$ as a polynomial in $A = \Omega_{12}$ and $B = \Omega_{23}$, take $\alpha(E_k)$, and substitute $\Psi$ and $\Psi^{-1}$ for $Z$ and $Z^{-1}$, and $\Omega_{12}^h$ and $\Omega_{23}^h$ for $C$ and $D$ (multiplication is composition of functors). Finally, $F_k = -\Psi \circ G_k$.

Theorem 5.1.

a. $\Psi = 1 + \sum_{k=2}^{\infty} F_k(\Omega_{12}^h, \Omega_{23}^h, \Psi, \Psi^{-1})h^k$

b. $\Psi^{-1} = 1 + \sum_{k=2}^{\infty} G_k(\Omega_{12}^h, \Omega_{23}^h, \Psi, \Psi^{-1})h^k$.

Proof. a. Notice that $\Psi^{-1} - 1 = N^{-1} \circ (M - N)$. A diagram search shows that $X(E_k(\Omega_{12}, \Omega_{23})) = -M \circ F_k(\Omega_{12}^h, \Omega_{23}^h, \Psi, \Psi^{-1}) \circ N^{-1}$. This observation combined with (4.1), (4.2) and (5.1) implies the claim by elementary calculations. Part b. is proved in an analogous way. □
Theorem 5.2. $\Psi, \Psi^{-1} \in C[[\hbar]][[\Omega^h_{12}, \Omega^h_{23}]]$

Proof. We will prove the claim by showing that $\Psi \mod h^n \in C[\hbar]/(h^n)[[\Omega^h_{12}, \Omega^h_{23}]]$ and $\Psi^{-1} \mod h^n \in C[\hbar]/(h^n)[[\Omega^h_{12}, \Omega^h_{23}]]$ for any $n \in \mathbb{Z}_{>0}$. The latter is done by induction on $n$. Indeed, $\Psi = 1 \mod h^2$, $\Psi^{-1} = 1 \mod h^2$, and if $\Psi \mod h^{n-2}$ and $\Psi^{-1} \mod h^{n-2}$ are polynomials in $\Omega^h_{12}, \Omega^h_{23}$, then

$$
\Psi = 1 + \sum_{k=2}^{n-1} F_k(\Omega^h_{12}, \Omega^h_{23}, \Psi \mod h^{n-2}, \Psi^{-1} \mod h^{n-2})h^k \mod h^n
$$

$$
\Psi^{-1} = 1 + \sum_{k=2}^{n-1} G_k(\Omega^h_{12}, \Omega^h_{23}, \Psi \mod h^{n-2}, \Psi^{-1} \mod h^{n-2})h^k \mod h^n
$$

are also polynomials in $\Omega^h_{12}, \Omega^h_{23}$. Following this algorithm we can get an explicit expression for $\Psi \mod h^n$ and $\Psi^{-1} \mod h^n$ for any $n \in \mathbb{Z}_{>0}$. □

6. Acknowledgement. I am very grateful to my advisor Igor Frenkel for the formulation of the problem and very useful discussions and to S. Majid, S. L. Woronowicz, P. Etingof and V. Protsak for numerous useful discussions. I would like to thank the organizers of the Program on Representation Theory and Its Applications to Mathematical Physics at ESI for inviting me and giving me the opportunity to speak on the topic. I am also grateful to ESI for their hospitality and financial support.

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