Towards a classical proof of exponential lower bound for 2-probe smooth codes

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Abstract

Let $C : \{0,1\}^n \mapsto \{0,1\}^m$ be a code encoding an $n$-bit string into an $m$-bit string. Such a code is called a $(q,c,\epsilon)$ smooth code if there exists a decoding algorithm which while decoding any bit of the input, makes at most $q$ probes on the code word and the probability that it looks at any location is at most $c/m$. The error made by the decoding algorithm is at most $\epsilon$. Smooth codes were introduced by Katz and Trevisan [LK00] in connection with Locally decodable codes.

For 2-probe smooth codes Kerenidis and de Wolf [dWK03] have shown exponential in $n$ lower bound on $m$ in case $c$ and $\epsilon$ are constants. Their lower bound proof went through quantum arguments and interestingly there is no completely classical argument as yet for the same (albeit completely classical !) statement.

We do not match the bounds shown by Kerenidis and de Wolf but however show the following. Let $C : \{0,1\}^n \mapsto \{0,1\}^m$ be a $(2,c,\epsilon)$ smooth code and if $\epsilon \leq \frac{c^2}{8n^2}$, then $m \geq 2^{\frac{n}{320}c^2 - 1}$. We hope that the arguments and techniques used in this paper extend (or are helpful in making similar other arguments), to match the bounds shown using quantum arguments. More so, hopefully they extend to show bounds for codes with greater number of probes where quantum arguments unfortunately do not yield good bounds (even for 3-probe codes).

1 Introduction

We define smooth codes in a very similar way as defined by Katz and Trevisan [LK00] as follows:

Definition 1 Let $c > 1, 1/2 > \epsilon \geq 0$ and $q$ be an integer. We call a code $C : \{0,1\}^n \mapsto \{0,1\}^m$ to be a $(q,c,\epsilon)$ smooth code if there exists a non-adaptive probabilistic decoding algorithm $A$ such that:

1. For every $x \in \{0,1\}^n$ and every $i \in [n]$, we have:
   \[ \mathrm{Pr}[A(C(x),i) = x_i] \geq 1 - \epsilon \]

2. For every $i \in [n]$ and every $j \in [m]$, we have,
   \[ \mathrm{Pr}[A(.,i)`reads index`j] \leq c/m \]

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3. In every invocation A reads at most $q$ indices of the code non-adaptively.

Remark: Here we are considering only non-adaptive codes, however for constant probe codes, bounds for non-adaptive codes also imply bounds (up to constants) for adaptive codes [LK00].

Katz and Trevisan defined smooth codes in the context of locally decodable codes (ldc) and showed that existence of ldcs imply existence of smooth codes. Therefore lower bounds for smooth codes imply lower bounds for ldcs. However in our case this is not the case since the error that we are considering is much smaller and we are letting the smoothness parameter to be constant. Lower bounds for smooth codes when the error is allowed to be constant also imply bounds for corresponding ldcs.

For smooth codes, Kerenidis and de Wolf [dWK03], using interesting quantum information theoretic arguments, showed exponential in $n$ lower bound on $m$ where they let $c, \epsilon$ to be constants. This is one of the few nice examples of quantum arguments leading to classical results. However till now no completely classical argument for the same result is known. Unfortunately the quantum arguments of [dWK03] had a drawback that they could not be extended to imply similar bounds for smooth codes for higher number of probes, for instance these arguments fail to lead interesting bounds even for 3-probe smooth codes.

We attempt here a completely classical argument for showing exponential lower bound for 2-probe smooth codes but we fall short in terms of showing it for constant error. The result we show for smooth codes is the following:

**Theorem 1.1** Let $C : \{0, 1\}^n \mapsto \{0, 1\}^m$ be a $(2, c, \epsilon)$ smooth code and $\epsilon \leq \frac{\epsilon^2}{8n^2}$. Then, $m \geq 2^{-\frac{n}{320c^2}}$.

We hope that although the result here falls short of the desirable, the arguments presented here could be extended or similar arguments be made to match, via purely classical arguments, the bounds shown by [dWK03] and also more importantly in deriving interesting bounds for codes with higher number of probes (in particular for 3-probe codes).

2 Preliminaries

In this section we briefly review some of the information theory facts that will be useful for us in our proofs in the next section. For a good introduction to information theory, please refer to the fine book by Cover and Thomas [CT91]. We let our random variables to be finite valued. Let $X,Y$ be random variables. We will let $H(X), H(X|Y)$ represent the entropy of $X$ and the conditional entropy of $X$ given $Y$. We let $I(X : Y) \triangleq H(X) + H(Y) - H(XY) = H(X) - H(X|Y)$ represent the mutual information between $X$ and $Y$. We will use the fact $I(X : Y) \geq 0$, alternately $H(X) + H(Y) \geq H(XY)$, alternately $H(X) \geq H(X|Y)$, several times without explicitly mentioning it. We will also use the monotonicity of entropy i.e. $H(XY) \geq H(X)$, alternately $H(Y) \geq I(X : Y)$ several times without explicitly mentioning it.

Let $X$ be an $m$ valued random variable, then it follows easily that $H(X) \leq \log_2 m$ (below we always take logarithm to the base 2).

For random variables $X_1, \ldots, X_n$, we have the following *chain rule of entropy*:

$$H(X_1, \ldots X_n) = \sum_{i=1}^{n} H(X_i|X_1 \ldots X_{i-1})$$ (1)
Similarly for random variables $X_1, \ldots, X_n, Y$, we have the following \textit{chain rule of mutual information}:

$$I(X_1 \ldots X_n : Y) = \sum_{i=1}^{n} I(X_i : Y|X_1 \ldots X_{i-1})$$ (2)

Let $X, Y, Z$ be random variables. Then we have the following important \textit{monotonicity relation} of mutual information:

$$I(X Y : Z) \geq I(X : Z)$$ (3)

All the above mentioned relations also hold for conditional random variables for example, for random variables $X, Y, Z$, $I(X : Y|Z) \geq 0$, $H(X|Y|Z) \geq H(X|Z)$ and so on. Again we may be using the conditional versions of the above relations several times without explicitly mentioning it.

For correlated random variables $X, Y$, we have the following Fano’s inequality. Let $\epsilon \triangleq \Pr[X \neq Y]$ and let $|X|$ represent the size of the range of $X$. Then

$$H(\epsilon) + \epsilon \log(|X| - 1) \geq H(X|Y)$$ (4)

For $0 \leq p \leq 1/2$, we have the bound $H(p) \leq 2\sqrt{p}$.

### 3 Proof of Theorem 1.1

Let $X \triangleq X_1 \ldots X_n$ be a random variable uniformly distributed in $\{0, 1\}^n$ (corresponding to the input being encoded) and $X_i$ correspond to the $i$-th bit of $X$. This implies that $X_i$’s are distributed independently and uniformly in $\{0, 1\}$. Let $Y \triangleq Y_1 \ldots Y_m$ be a random variable (correlated with $X$) corresponding to the code, i.e $Y = C(X)$. Here $Y_j, j \in [m]$ corresponds to the $j$-th bit of the code.

Let $A$ be as in Definition 1. Let $0 \leq \epsilon < 1/2$, for $i \in [n]$ let $E_2^i$ be the graph on $[m]$ consisting of edges $(j, k)$ such that,

$$\Pr[A(C(X), i) = X_i|A \text{ reads } (Y_j, Y_k)] \geq 1 - \epsilon$$ (5)

Following interesting fact can be shown using arguments of Katz and Trevisan [LK00]:

**Lemma 3.1** Let $C : \{0, 1\}^n \mapsto \{0, 1\}^m$ be a $(2, c, \epsilon)$ smooth code. Let $E_{2i}^i$ be as described above. Then for each $i \in [n]$, $E_{2i}^i$ has a matching $M_i$ of size at least $\frac{m}{4}$.  

**Proof:** Using the definition of smooth code we have,

$$1 - \epsilon \leq \Pr[A(C(X), i) = X_i|A(C(X), i) \text{ reads } E_{2i}^i] \Pr[A(C(X), i) \text{ reads } E_{2i}^i]$$

$$+ \Pr[A(C(X), i) = X_i|A(C(X), i) \text{ reads complement of } E_{2i}^i] \Pr[A(C(X), i) \text{ reads complement of } E_{2i}^i]$$

$$\leq \Pr[A(C(X), i) \text{ reads } E_{2i}^i] + (1 - 2\epsilon)(1 - \Pr[A(C(X), i) \text{ reads } E_{2i}^i])$$

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This implies $\Pr[A(C(X), i) \text{ reads } E^2_{i}] \geq 1/2$. For an edge $e \in E^2_{i}$, let $P_e \triangleq \Pr[A(C(X), i) \text{ reads } e]$. This implies $\sum_{e \in E^2_{i}} P_e \geq 1/2$. Furthermore since $C$ is a $(2, c, \epsilon)$ smooth code, for every $j \in [m]$, it implies $\sum_{e \in E^2_{i} | j \in e} P_e \leq c/m$. Let $V$ be a vertex cover of $E^2_{i}$. Therefore,

$$1/2 \leq \sum_{e \in E^2_{i} | e \cap V \neq \emptyset} \sum_{j \in V} \sum_{e \in E^2_{i} | j \in e} P_e \leq |V| c/m$$

This implies that minimum vertex cover of $E^2_{i}$ has size at least $m/2c$. This now implies that $E^2_{i}$ has a matching of size at least $m/4c$.

We start with the following claim.

**Claim 3.2** Let $(j, k) \in M_i$ and $\epsilon' \triangleq \sqrt{\epsilon}$. Then, $I(X_i : Y_j Y_k) \geq 1 - \epsilon'$.

**Proof:**

$$I(X_i : Y_j Y_k) = H(X_i) - H(X_i | Y_j Y_k) \geq 1 - H(2\epsilon) \quad \text{(from (1) and (5))}$$

$$\geq 1 - \sqrt{\epsilon} \quad \text{(from the bound } H(p) \leq 2\sqrt{p})$$

**\[\Box\]**

We make the following claim which roughly states that the information about various $X_i$s do not quite go into the individual bits of $Y$. For $i \in [n]$ let, $X_i \triangleq X_1 \ldots X_{i-1}$.

**Claim 3.3**

$$\sum_{i \in [n]} \sum_{(j, k) \in M_i} I(X_i : Y_j | \tilde{X}_i) + I(X_i : Y_k | \tilde{X}_i) \leq m$$

**Proof:**

$$\sum_{i \in [n]} \sum_{(j, k) \in M_i} I(X_i : Y_j | \tilde{X}_i) + I(X_i : Y_k | \tilde{X}_i) \leq \sum_{i \in [n]} \sum_{j \in [m]} I(X_i : Y_j | \tilde{X}_i) \quad \text{(since } M_i \text{s are matchings)}$$

$$= \sum_{j \in [m]} \sum_{i \in [n]} I(X_i : Y_j | \tilde{X}_i)$$

$$= \sum_{j \in [m]} I(X : Y_j) \quad \text{(from (2))}$$

$$\leq m \quad \text{(since } \forall j \in [m], Y_j \text{ is a binary random variable)}$$

**\[\Box\]**

We now have the following claim which roughly states that for a typical edge $(j, k) \in M_i$ there is a substantial increase in correlation between $Y_j$ and $Y_k$ after conditioning on $X_i$.

**Claim 3.4** Let $\epsilon' \leq \frac{c}{n}$. Then,

$$E_{i \in [n], (j, k) \in U M_i} [I(Y_j : Y_k | X_i \tilde{X}_i) - I(Y_j : Y_k | \tilde{X}_i)] \geq 1 - 5c/n$$
Proof: Let \((j, k) \in M_i\). Since \(X_i\) and \(\tilde{X}_i\) are independent random variables, this implies \(I(X_i : \tilde{X}_i) = 0\) and we get:

\[
I(X_i : Y_j Y_k) \leq I(X_i : \tilde{X}_i Y_j Y_k) \quad \text{(from 3)} \\
= I(X_i : \tilde{X}_i) + I(X_i : Y_j Y_k \mid \tilde{X}_i) \quad \text{(from 2)} \\
= I(X_i : Y_j \mid \tilde{X}_i) + I(X_i : Y_k \mid \tilde{X}_i) + I(Y_j : Y_k \mid X_i \tilde{X}_i) - I(Y_j : Y_k \mid \tilde{X}_i) \quad \text{(from 2)}
\]

From Claim 3.2, we get,

\[
(1 - \epsilon') \sum_i |M_i| \leq \sum_i \sum_{(j, k) \in M_i} I(X_i : Y_j Y_k) \\
\leq \sum_i \sum_{(j, k) \in M_i} I(X_i : Y_j \mid \tilde{X}_i) + I(X_i : Y_k \mid \tilde{X}_i) + I(Y_j : Y_k \mid X_i \tilde{X}_i) - I(Y_j : Y_k \mid \tilde{X}_i)
\]

Claim 3.3 now implies:

\[
\sum_i \sum_{(j, k) \in M_i} I(Y_j : Y_k \mid X_i \tilde{X}_i) - I(Y_j : Y_k \mid \tilde{X}_i) \geq (1 - \epsilon') \sum_i |M_i| - m \\
\geq (\sum_i |M_i|)(1 - \epsilon' - \frac{m}{\sum_i |M_i|}) \geq (\sum_i |M_i|)(1 - c/n - 4c/n) \quad \text{(from Lemma 3.1)}
\]

Applying Markov’s inequality on the above claim we get:

Claim 3.5 Let \(0 < \delta_1, \delta_2 \leq 1\). There exists a set \(GOOD \subseteq [n]\) and sets \(GOOD_i \subseteq M_i\) such that:

1. \(|GOOD| \geq (1 - \delta_1)n\) and \(i \in GOOD_i, \mathbb{E}_{(j, k) \in M_i}[I(Y_j : Y_k \mid X_i \tilde{X}_i) - I(Y_j : Y_k \mid \tilde{X}_i)] \geq 1 - \frac{5c}{\delta_1 n}\)

2. \(\forall i \in GOOD_i, |GOOD_i| \geq (1 - \delta_2)|M_i|\) and for \((j, k) \in GOOD_i\), \(I(Y_j : Y_k \mid X_i \tilde{X}_i) - I(Y_j : Y_k \mid \tilde{X}_i) \geq 1 - \frac{5c}{\delta_1 \delta_2 n}\)

Let \(\delta_1 = \delta_2 = 1/2\). Let \(\tilde{\epsilon} \triangleq \frac{20c}{n}\). Therefore for \(i \in GOOD\) and \((j, k) \in GOOD_i\) we have from above,

\[
I(Y_j : Y_k \mid X_i \tilde{X}_i) - I(Y_j : Y_k \mid \tilde{X}_i) \geq 1 - \tilde{\epsilon}\tag{6}
\]

We fix \(GOOD\) to have exactly \(\frac{1}{2} - 2\) elements. For \(i \in GOOD\), let \(a_i\) be the index of \(i\) in \(GOOD\). For \(i \notin GOOD\), let \(a_i\) be the index of largest \(i' < i\) in \(GOOD\). For \(j \in [m], i \in [n]\), let \(S_j^i \triangleq \{l \in [m] : H(Y_j \mid X_l \tilde{X}_i) \leq a_i \tilde{\epsilon}\}\). Let \(S_j^0 \triangleq \{j\}\).

We show the following main lemma.

Lemma 3.6 Let \(i \in GOOD, (j, k) \in GOOD_i\). Then,

1. \(S_j^{i-1} \cap S_k^{i-1} = \emptyset\)
2. $S_j^{i-1} \cup S_k^{i-1} \subseteq S_j^i \cap S_k^i$.

**Proof: Part 1:** Let $l \in S_j^{i-1} \cap S_k^{i-1}$. Using standard information theoretic relations it follows:

$$H(Y_k Y_j | Y_i \hat{X}_i) \leq H(Y_k | Y_i \hat{X}_i) + H(Y_j | Y_i \hat{X}_i) \leq 2(a_i - 1) \hat{\epsilon}$$

Since $(j, k) \in \text{GOOD}_i$ and from (11),

$$H(Y_k | \hat{X}_i) \geq H(Y_k | X_i \hat{X}_i) \geq I(Y_k : Y_j | X_i \hat{X}_i) \geq 1 - \hat{\epsilon}$$

Similarly $H(Y_j | \hat{X}_i) \geq 1 - \hat{\epsilon}$. Therefore again from (11),

$$H(Y_j Y_k | \hat{X}_i) = H(Y_j \hat{X}_i) + H(Y_k | \hat{X}_i) - I(Y_j : Y_k | \hat{X}_i) \geq 2 - 2\hat{\epsilon} - \hat{\epsilon} = 2 - 3\hat{\epsilon}$$

Now,

$$I(Y_i : Y_j Y_k | \hat{X}_i) = H(Y_j Y_k | \hat{X}_i) - H(Y_j Y_k | Y_i \hat{X}_i) \geq 2 - 3\hat{\epsilon} - 2(a_i - 1) \hat{\epsilon} \geq 2 - 2(a_i + 1) \hat{\epsilon} > 1 \text{ (since } a_i \leq \frac{1}{2e} - 2)$$

This is a contradiction since $Y_i$ is a binary random variable.

**Part 2:** We show $S_j^{i-1} \cup S_k^{i-1} \subseteq S_j^i$ and $S_j^{i-1} \cup S_k^{i-1} \subseteq S_k^i$ follows similarly. It is easily seen that $S_j^{i-1} \subseteq S_j^i$. Let $l \in S_k^{i-1}$. Since $(j, k) \in \text{GOOD}_i$, from (11),

$$H(Y_j | Y_k X_i \hat{X}_i) = H(Y_j | X_i \hat{X}_i) - I(Y_j : Y_k | X_i \hat{X}_i) \leq 1 - (1 - \hat{\epsilon}) = \hat{\epsilon}$$

Now,

$$H(Y_j | Y_i X_i \hat{X}_i) \leq H(Y_j Y_i | Y_i X_i \hat{X}_i)$$

$$= H(Y_k | Y_i X_i \hat{X}_i) + H(Y_j | Y_k Y_i X_i \hat{X}_i) \text{ (from (11))}$$

$$\leq H(Y_k | Y_i \hat{X}_i) + H(Y_j | Y_k X_i \hat{X}_i)$$

$$\leq (a_i - 1) \hat{\epsilon} + \hat{\epsilon} = a_i \hat{\epsilon}$$

Hence $l \in S_j^i$ and therefore $S_k^{i-1} \subseteq S_j^i$.

Our theorem now finally follows.

**Proof:** [Theorem 1.1] Let $i \in \text{GOOD}$. Since $\epsilon \leq \frac{\epsilon^2}{8n^2}$, Claim 3.4 holds. Lemma 3.3 implies that for $(j, k) \in \text{GOOD}_i$, either $|S_j^i| = 2|S_j^{i-1}|$ or $|S_k^i| = 2|S_k^{i-1}|$. Then,

$$\sum_j \log |S_j^{i-1}| + |\text{GOOD}_i| \leq \sum_j \log |S_j^i| \quad (7)$$
Let \( \tilde{i} \) be the largest \( i \in \text{GOOD} \). Now,

\[
\left( \frac{n}{40c} - 2 \right) \frac{m}{8c} \leq \sum_{i \in \text{GOOD}} \frac{|M_i|}{2} \quad \text{(from Lemma 3.1)}
\]

\[
\leq \sum_{i \in \text{GOOD}} |\text{GOOD}_i| \quad \text{(from Claim 3.5) and } \delta_2 = 1/2
\]

\[
\leq \sum_j \log |S_j| \quad \text{(from (7))}
\]

\[
\leq m \log m
\]

\[
\Rightarrow m \geq 2\left( \frac{n}{40c} - 2 \right) \frac{1}{8c} \geq 2\frac{n}{320c} - 1
\]

4 Conclusion

We have attempted here a classical proof of an already known theorem \([dWK03]\) which however has been shown using quantum arguments. We hope that the arguments used here are helpful in matching the result derived using quantum arguments. The need for a classical proof is also due to the fact that the quantum arguments do not help us to derive interesting bounds for codes with higher number of probes, in particular even for 3-probe codes.

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