Relativistic diffusion with friction on a pseudoriemannian manifold

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April 1, 2010

Abstract

We study a relativistic diffusion equation on the Riemannian phase space defined by Franchi and Le Jan. We discuss stochastic Ito (Langevin) differential equations as a perturbation by noise of the geodesic equation. We show that the expectation value of the angular momentum and the energy grow exponentially fast. We discuss drifts leading to an equilibrium. As an example we consider a particle in de Sitter universe. It is shown that the relativistic diffusion of momentum in de Sitter space is the same as the relativistic diffusion on the Minkowski mass-shell with the temperature proportional to the de Sitter radius. We study a diffusion process with a drift corresponding to the Jüttner or quantum equilibrium distributions. We show that such a diffusion has a bounded expectation value of angular momentum and energy. The energy and the angular momentum tend exponentially fast to their equilibrium values.

1 Introduction

In this paper we discuss relativistic dynamics of a particle in general relativity which moves in a medium (gas) of other particles. We assume that the interaction with a gas consists of frequent elastic collisions which in a Markov limit (no memory) can be approximated by a diffusion process. In the framework of the special theory of relativity the diffusion problem has been formulated and solved a long time ago by Schay [1] and Dudley [2] under the assumption that the diffusion is evolving in the proper time on the phase space and preserves the mass $p^2$. The diffusion problem on a general pseudoriemannian manifold has been formulated recently in a geometric framework of diffusions on fiber bundles by Franchi and Le Jan [3]. The theory has been further developed and applied in [4] [5][6][7] (for a general coordinate dependent discussion of relativistic diffusions on the pseudoriemannian manifolds see [8]).
We follow the method of our earlier papers [9][10][11] to define the relativistic Brownian motion on the submanifold of fixed $p^2$ of the Riemannian phase space. With our choice of coordinates the generator of the process and the stochastic equations coincide with those of Franchi and Le Jan [3]. The diffusion process describes a random perturbation of the geodesic motion. We show that in a spherically symmetric static background the expectation value of the angular momentum and the energy grow exponentially fast. The model is rather unphysical without a friction. Following our earlier paper [9] we discuss the friction terms leading to the Jüttner and quantum distributions (diffusions with Jüttner equilibrium distribution are also discussed in [12][7]). We show that such a friction ensures finite expectation values of energy and angular momentum.

The diffusion process describes an evolution of the particle in a gas around a star (or a black hole) before achieving an equilibrium with the surrounding medium (this may be the gas of Hawking radiation [13] from the black hole). In ref.[10] we studied photon diffusion in an electron gas. Here, we discuss a soluble example of a particle diffusion in de Sitter universe. We obtain a surprising result that the diffusion of momentum is described by the same equation as the relativistic diffusion on the Minkowski mass-shell with a friction leading to the Jüttner distribution. The corresponding temperature is proportional to the radius of de Sitter space. The relativistic diffusion (without friction) in the Schwarzschild metric has been studied earlier in [3] and in the Gödel universe in [4]. For reviews on the relativistic diffusion see [14][15].

The paper is organized as follows. In sec.2 we discuss the geometry of the mass-shell. In sec.3 we construct the generator of the diffusion on the mass-shell as the Laplace-Beltrami operator on the level surface in the cotangent bundle. In sec.4 we restrict ourselves to isotropic metrics. We discuss the stochastic Ito equations in these coordinates. In sec.5 we obtain the transport equations in the laboratory time corresponding to the diffusion equations in the proper time. We discuss friction forces which lead to equilibrium measures determined by basic principles of the statistical physics. In sec.6 a time evolution of the energy and the angular momentum is discussed. In sec.7 we take the limit of zero mass. The diffusion in de Sitter space is discussed in sec.8. In two appendices we give a derivation of some results applying methods of the stochastic calculus.

2 Riemannian geometry of the mass-shell

We are interested in random perturbations of the dynamics of relativistic particles of mass $m$ moving in a gravitational field. The dynamics on a pseudoriemannian manifold $\mathcal{M}$ (the geodesic motion) is described by the equations [16]

\[ \frac{dx^\mu}{d\tau} = \frac{1}{m} g^{\mu\nu} p_\nu, \]

\[ \frac{dp_\rho}{d\tau} = -\frac{1}{2m} g^{\rho\sigma} p_\sigma p_\nu. \]
where $\mu = 0, 1, 2, 3$ and
\[ ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \] (3)
is the pseudoriemannian metric on $\mathcal{M}$. The four-momentum $p(\tau)$ of a relativistic particle defines the mass by the relation
\[ p^2 = g^{\mu\nu}p_\mu p_\nu = m^2c^2. \] (4)
If eq.(4) is satisfied then from eq.(1) it follows that $\tau$ has the meaning of the proper time (which is invariant under coordinate transformations). Eq.(4) defines a submanifold in the cotangent bundle. We take the pseudoriemmanian metric on the cotangent bundle $\mathcal{M} \times T\mathcal{M}^*$ as the product metric on $\mathcal{M}$ and $T\mathcal{M}^*$
\[ ds^2 = ds_x^2 + ds_p^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g^{\mu\nu}(x)dp_\mu dp_\nu. \] (5)
The metric on the submanifold (4) is inherited from the product metric on the cotangent bundle. It can be obtained by expressing $p_0$ by the spatial components of the momenta (here $g^{\mu\nu}$ is the inverse matrix to the one defining the metric on $T\mathcal{M}$; we choose the convention that the spatial terms in $ds^2$ have an opposite sign)
\[ ds_p^2 = g^{\mu\nu}(x)dp_\mu dp_\nu = g^{00}dp_0 dp_0 + 2g^{0k}dp_0 dp_k - g^{jk}dp_j dp_k = -\gamma^{jk}dp_j dp_k \] (6)
\((j, k = 1, 2, 3). We assume that $g^{00} > 0$ and $g^{j0} = 0$ (static metrics can be chosen in this form). Then, we obtain
\[ \gamma^{jk} = g^{jk} - \omega^{-2}g^{jr}g^{kn}p_r p_n \] (7)
We have
\[ D = \det(\gamma^{jk}) = \det(g^{jk})\omega^{-2} \] (8)
and
\[ \gamma_{jk} = g_{jk} + m^{-2}c^{-2}p_j p_k. \] (9)
The measure invariant under all coordinate transformations of the submanifold (4) of $T\mathcal{M}^*$ reads [17]
\[ p_0^{-1}dxdp. \] (10)

3 Diffusion on the cotangent bundle

We define the Laplace-Beltrami operator on the cotangent bundle as usual by means of the metric. The metric (5) is a product of the metrics whereas the Laplace-Beltrami operator $\Delta$ is a sum $\Delta_x + \Delta_p$. The part of the Laplace-Beltrami containing derivatives over the momenta reads
\[ \Delta_p = g_{\mu\nu} \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu}. \] (11)
In order to define the Laplace-Beltrami operator on the submanifold (4) of the cotangent bundle we have to restrict the operator (11) to this submanifold. A way to do it is to treat the condition (4) as a definition of the level surface in the cotangent bundle. In such a case the Laplace-Beltrami operator on the level surface is the same as the one defined by the Riemannian geometry of the Riemannian mass-shell discussed in sec.2.

\[ \Delta_H = D^{-\frac{1}{2}} \frac{\partial}{\partial p_j} \gamma_{jk} D^{\frac{1}{2}} \frac{\partial}{\partial p_k}, \]  

where \( D \) is defined in eq.(8) and \( \gamma_{jk} \) in eq.(7).

Explicitly,

\[ \Delta_H = (g_{jk} + m^{-2}c^{-2} p_j p_k) \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} + 3m^{-2}c^{-2} p_k \frac{\partial}{\partial p_k}. \]  

The diffusion equation is

\[ \partial_\tau \phi_\tau = \mathcal{G} \phi_\tau \]  

with the generator

\[ \mathcal{G} = g^{\mu\nu} p_\mu \frac{\partial}{\partial x^\nu} - \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu \frac{\partial}{\partial p_j} + \frac{\kappa^2 m^2 c^2}{2} \Delta_H. \]  

The operator (15) generates a diffusive dynamics which is a perturbation of the geodesic motion (1)-(2) by a diffusion in the momentum space. A coordinate free frame bundle definition of the diffusion (14) has been obtained earlier in [3](Lemma 3.1).

We still transform eq.(15) in order to express the diffusion on the Riemannian mass-shell as a perturbation of a diffusion on the Minkowski mass-shell (we apply this transformation in sec.8). For this purpose let us introduce the tetrads \( f_j^a \) by the equation

\[ f_{j1} = f_j^a f_i^a \]  

\((a = 1, 2, 3)\). We define the inverse \( f_j^a \)

\[ f_j^a f_j^b = \delta_a^b. \]  

We change coordinates on the cotangent bundle

\[ p_j = f_j^a p_a \]  

while \( x^\nu = x'^\nu \). The coordinate vector fields take the form

\[ \frac{\partial}{\partial p_j} = f_j^a \frac{\partial}{\partial p_a}, \]  

\[ \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x'^\nu} + \omega^b_{\nu a p_b} \frac{\partial}{\partial p_a}, \]
\( \omega^b_{\nu a} = \frac{\partial f_a^l}{\partial x'^{\nu}} f^b_l. \) (21)

The vector field of eq. (20) can be considered as a horizontal lift of the coordinate vector field \( \partial_{\nu} \) from \( \mathcal{M} \) to \( \mathcal{M} \times T\mathcal{M}^* \).

4 Relativistic diffusion in isotropic coordinates

We shall consider highly symmetric manifolds. Such manifolds admit special coordinate systems. We restrict ourselves to the isotropic spherically symmetric metric[16] from now on. Then,

\[
\text{d}s^2 = A^{-2}(|x|) c^2 \text{d}t^2 - B^{-2}(|x|) \text{d}x^2. \quad (22)
\]

In these coordinates

\[
\triangle_H = (B^{-2} \delta_{jk} + m^{-2} c^{-2} p_j p_k) \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} + 3 m^{-2} c^{-2} p_k \frac{\partial^2}{\partial p_k}. \quad (23)
\]

The diffusion equation in the isotropic coordinates reads

\[
\partial_{\tau} \phi = \frac{1}{m} A \omega \frac{\partial}{\partial x^a} \phi - \frac{1}{m} B^2 p_j \frac{\partial}{\partial p_j} \phi + \frac{1}{2 m} \frac{\partial B^2}{\partial x^p} p^2 \frac{\partial \phi}{\partial p_j} - \frac{1}{2 m} \frac{\partial \ln A}{\partial x^j} \omega^2 \frac{\partial \phi}{\partial p_j} + \frac{m^2 c^2}{2} \triangle_H \phi. \quad (24)
\]

where in the generator (15) we expressed \( p_0 \) as

\[
p_0 = A^{-1} (B^2 p^2 + m^2 c^2)^{\frac{1}{2}} \equiv A^{-1} \omega. \quad (25)
\]

We can express the solution of the diffusion equation (24) as an expectation value \( E[..] \) over the sample paths of a diffusion process \( (x_{\tau}, p_{\tau}) \) starting from \( (x, p) \) [18]

\[
\phi_{\tau}(x, p) = E[\phi(x_{\tau}, p_{\tau})]. \quad (26)
\]

The stochastic process can be defined as a solution of a stochastic equation. In order to write down the stochastic equations we must calculate the square root of the matrix \( \gamma_{jl} \), i.e., to find a tetrad \( e^a_j \)

\[
e^a_j e^a_l = \gamma_{jl}. \quad (27)
\]

We obtain

\[
e^a_j = B^{-1}(\delta^a_j + \frac{1}{mc} (\omega - mc) p^{-2} p_j p_a). \quad (28)
\]

Then, Ito stochastic differential equations [18] corresponding to the representation (26) of the solution of the diffusion equation (24) read (see Appendix...
A stochastic equations for the relativistic diffusion have been obtained also in the frame bundle formalism of ref.[3], Theorem 3.2)

\[ \frac{dx^0}{d\tau} = \frac{1}{m} A \omega \]

(29)

\[ \frac{dx}{d\tau} = - \frac{1}{m} B^2 \mathbf{p} \]

(30)

\[ dp_j = \frac{1}{m} B \partial_j B \mathbf{p}^2 d\tau - \frac{1}{m} \omega^2 \partial_j \ln \left( Ad\tau + \frac{3e^2}{2m} \mathbf{p}^2 \right) + mcqB^{-1} db_j + \kappa B^{-1}(\omega - mc) \mathbf{p}^{-2} p_j \mathbf{p} dB. \]

(31)

Here, \( \mathbf{b}(s) \) denotes the Brownian motion defined as the Gaussian process with values in \( \mathbb{R}^3 \) and the covariance

\[ E[b_a(s)b_c(\tau)] = \delta_{ac} \min(s, \tau) \]

5 Invariant measure and kinetic equations

We define an evolution of the measure \( d\sigma = dx d\mathbf{p} \Phi \) by the equality (see [9])

\[ \langle \phi_\tau \rangle_\sigma = \int dx d\mathbf{p} \Phi \phi_\tau = \int dx d\mathbf{p} \Phi \tau \phi. \]

(32)

Then,

\[ \partial_\tau \Phi_\tau = \mathcal{G}^* \Phi_\tau \]

(33)

where \( \mathcal{G}^* \) is the adjoint of \( \mathcal{G} \) in \( L^2(dx d\mathbf{p}) \). We say that the measure \( \sigma \) is invariant if

\[ \langle \phi_\tau \rangle_\sigma = \langle \phi \rangle_\sigma \]

(34)

is independent of \( \tau \). In such a case

\[ \mathcal{G}^* \Phi = 0. \]

(35)

Eq.(35) is a transport equation in the laboratory time \( x^0 \). A normalizable invariant measure needs a dissipation. We add such a dissipation term \( K_j d\tau \) to the stochastic equation (31). This is equivalent to adding the friction

\[ K = K_j \frac{\partial}{\partial p_j} \]

(36)

to the diffusion equation (24).

In isotropic coordinates eq.(35) reads

\[ \begin{align*}
- \frac{1}{m} A^2 p_0 \frac{\partial}{\partial x_0} \Phi &+ \frac{1}{m} \frac{\partial}{\partial x_i} B^2 p_j \Phi - \frac{1}{2m} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\partial}{\partial p_j} \Phi + \frac{1}{2m} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\partial}{\partial p_j} \Phi \\
+ \frac{e^2}{2} &\frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} \left( B^{-2} \delta_{jk} + m^{-2} c^{-2} p_j p_k \right) \Phi - \frac{3e^2}{2} \frac{\partial}{\partial p_k} p_k \Phi - \frac{\partial}{\partial p_k} K_k \Phi = 0.
\end{align*} \]

(37)
In the search of the equilibrium distribution $\Phi_E$ we demand that the dynamic part and the diffusion part of the equation (37) cancel separately. So, for the dynamic part

$$-\frac{1}{m} A^2 p_0 \frac{\partial}{\partial x} \Phi_E + \frac{1}{m} \frac{\partial}{\partial x} B^2 p_j \Phi_E - \frac{1}{2m} \frac{\partial B^2}{\partial p_j} \Phi_E + \frac{1}{2m} \frac{\partial A^2}{\partial p_j} p_0^2 \Phi_E = 0.$$  

(38)

Then, from the diffusion part

$$K_k = \kappa^2 m^2 c^2 \Phi_E^{-1} \left( \frac{1}{2} \frac{\partial}{\partial p_j} (B^{-2} \delta_{jk} + m^{-2} c^{-2} p_j p_k) - \frac{3}{2} (mc)^{-2} p_k \right) \Phi_E. \quad (39)$$

Any function of deterministic constants of motion is a solution of the dynamic equation (38) (static metric with a spherical symmetry has the angular momentum and the energy as constants of motion). Let us consider the solution as a function of $p_0$ (25) in the form (Jüttner [19] and quantum statistical distributions are of this form)

$$\Phi_E = A \omega^{-1} \exp( f(\beta c A^{-1} \omega)). \quad (40)$$

From eq.(39) we obtain

$$K_k = \kappa^2 \frac{2}{mc} p_k \beta c A^{-1} \omega f'(\beta c A^{-1} \omega). \quad (41)$$

(diffusions with Jüttner equilibrium distribution have been discussed also in [12][7]). Explicitly, with the friction (39) the diffusion equation is

$$\partial_\tau \phi = \frac{1}{m} A \omega \frac{\partial}{\partial x} \phi - \frac{1}{m} B^2 p_j \frac{\partial}{\partial x} \phi + \frac{1}{2m} \frac{\partial B^2}{\partial p_j} \frac{\partial \phi}{\partial p_j} - \frac{1}{2m} A^{-2} \frac{\partial A^2}{\partial x} \omega^2 \frac{\partial \phi}{\partial p_j} + \frac{\omega^2}{2} (B^{-2} \delta_{jk} + m^{-2} c^{-2} p_j p_k) \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} \phi + \kappa^2 \left( \frac{3}{2} + \frac{1}{2} \omega c A^{-1} f'(\beta c A^{-1} \omega) \right) p_k \frac{\partial}{\partial p_k} \phi. \quad (42)$$

Then, the stochastic equation for the process $p_\tau$ (26) solving the diffusion equation (42) reads (see Appendix A)

$$dp_j = \frac{1}{m} B \partial_\tau^x B p_j d\tau - \frac{1}{m} \omega^2 \partial_\tau^x \ln A d\tau + \frac{\kappa^2}{2} \omega c A^{-1} f' d\tau + \frac{2}{2} p_j d\tau + m \kappa \omega B^{-1} db_j + \kappa B^{-1} (\omega - mc) p_j p_k db. \quad (43)$$

The equilibrium distributions (40) (Jüttner or quantum) are not invariant under Lorentz transformations even in the flat (Minkowski) case. The notion of the equilibrium depends on the Lorentz frame where the particle equilibrates. We make this frame dependence explicit in [20].

6 Evolution of the energy and the angular momentum

In the isotropic coordinates we define the angular momentum

$$L = x \times p. \quad (44)$$
If \( A(|x|) \) and \( B(|x|) \) depend only on \(|x|\) then the metric is spherically symmetric. In such a case when \( \kappa = 0 \) then the angular momentum is a constant of motion. First, we consider the proper time evolution of the angular momentum in a stochastic model without a friction. From eq.(31) we obtain a stochastic equation for the time evolution of the angular momentum when \( \kappa \neq 0 \) (see Appendix B)

\[
dL_\tau = \frac{3\kappa^2}{2}L_\tau d\tau + \kappa B^{-1}(\omega - mc)p^{-2}L_\tau (pd) + \kappa mcB^{-1}x \times db. \tag{45}
\]

In the spherical coordinates \((r, \theta, \phi)\) eq.(45) can be expressed in the form

\[
dp_\phi = \frac{3\kappa^2}{2}p_\phi d\tau + \kappa e^a_\phi db_a. \tag{46}
\]

We write eqs.(45)-(46) in an integral form and apply the basic property of the Ito integral [18]

\[
E[\int F db] = 0. \tag{47}
\]

Then, it follows from eq.(45) that

\[
\exp(-\frac{3\kappa^2}{2}\tau)E[L_\tau] = \text{const}. \tag{48}
\]

Eq.(48) means that the angular momentum grows exponentially in time.

In a static metric when \( \kappa = 0 \) the energy \( p_0 = A^{-1}\omega \) is also a constant of motion. We calculate the change of energy during the proper time diffusion (31) in a model without friction. Differentiating eq.(25) we obtain (see Appendix B)

\[
dp_0 = \frac{3}{2}\kappa^2p_0 d\tau + \kappa A^{-1}Bpd. \tag{49}
\]

It follows from eqs.(49) and (47) that

\[
\mathcal{E}(\tau) = \frac{3}{2}\kappa^2 \int_0^\tau \mathcal{E}(s)ds \tag{50}
\]

where

\[
\mathcal{E}(s) = E[p_0(s)].
\]

Eq.(50) has the solution

\[
\mathcal{E}(\tau) = \mathcal{E}(0) \exp(\frac{3}{2}\kappa^2 \tau) \tag{51}
\]

We calculate a change of the angular momentum (in a spherically symmetric metric) resulting from the diffusion (43) with friction (Appendix B)

\[
dL_\tau = \frac{\kappa^2}{2}\omega c \beta A^{-1}f'L\tau + \frac{3\kappa^2}{2}L_\tau d\tau + \kappa B^{-1}(\omega - mc)p^{-2}L_\tau (pd) + \kappa mcB^{-1}x \times db. \tag{52}
\]
\( f' \) is negative for Jüttner as well as for quantum equilibrium distributions so that the angular momentum does not increase as it did in eq.(48). In the spherical coordinates we have

\[
dp = \frac{p_\phi \kappa^2}{2} \omega c \beta A^{-1} f' d\tau + \frac{3\kappa^2}{2} p_\phi d\tau + e_\phi d\alpha. \tag{53}
\]

\( \omega \) is growing linearly with \( p_\phi \) for large \( p_\phi \). For Jüttner distribution \( f' = -1 \) and for large \( p_\phi \) eq.(53) is of the form

\[
dp = \alpha p_\phi d\tau - \delta p_\phi^2 d\tau + e_\phi d\alpha. \tag{54}
\]

The stochastic equation for the energy is also of the form (54). We have from eq.(43) (see Appendix B)

\[
dp_0 = \frac{3}{2} \kappa^2 p_0 d\tau + \frac{\kappa^2}{2} c \beta p_0^2 f' d\tau - \frac{c^2}{2} A^{-2} f' d\tau + \kappa A^{-1} B \phi d\beta. \tag{55}
\]

We are going to prove that the expectation value of the energy is bounded in time. For this purpose the Lyapunov methods [21][22] are needed. We must find a non-negative increasing to infinity Lyapunov function \( V \) such that for constants \( a \geq 0 \) and \( K > 0 \)

\[
G V \leq -KV + a. \tag{56}
\]

The lhs of eq.(56) is a time derivative of \( E[V(p)] \). The integral form of the inequality (56) reads [22]

\[
E[V(p_\tau)] \leq \exp(-K\tau)V(p) + \frac{a}{K}(1 - \exp(-K\tau)). \tag{57}
\]

We choose \( V(p) = p_0 = A^{-1} \omega \). Then, by direct calculations

\[
G p_0 = \frac{3}{2} \kappa^2 p_0 + \frac{\kappa^2}{2} c \beta p_0^2 f' - \frac{c^2}{2} A^{-2} f'. \tag{58}
\]

Eq.(58) is also a consequence of eqs.(26),(47) and (55) as

\[
dE[p_0] = E[dp_0] = G p_0 d\tau
\]

If \( f' < 0 \) (true for Jüttner as well as quantum equilibrium distributions) then from eq.(58) the inequality (56) results. As a consequence of eq.(57) the energy is bounded in time. The proof assumes that the stochastic equation (43) for \( p \) has solutions for arbitrary time. If \( A \) and \( B^{-1} \) are bounded functions then an extension of the solution to arbitrarily large time can easily be shown by means of the Lyapunov function method (choose \( V(p) = p^2 \) as the Lyapunov function). We could also apply the Lyapunov argument to the stochastic equation for the angular momentum (53) assuming that \( A \) and \( B^{-1} \) are bounded (with \( p_\phi^2 \) as the Lyapunov function). The problem with singular \( A \) and \( B^{-1} \) (the case of the Schwarzschild solution) is more complicated. The diffusion can terminate in the
singularity at finite time. The properly behaving diffusion can be constructed on the Kruskal extension of the Schwarzschild solution [3]. Summarizing, without the friction the expectation value of the energy and the angular momentum grow exponentially fast whereas with the friction, leading to the Jüttner or Bose-Einstein equilibrium distribution, the energy and the angular momentum are bounded in time.

We perform some explicit (although approximate) calculations of the expectation value of the energy for the Jüttner equilibrium distribution ($f' = -1$). From eq.(55) we obtain an equation

$$E(\tau) = \frac{3}{2} \kappa^2 \int_0^\tau ds E(s) - \frac{\kappa^2 c^2}{2} \int_0^\tau ds E[p_0(s)]$$

$$+ \frac{1}{2} c^2 \kappa^2 m^2 c^2 \int_0^\tau ds [A^{-2}].$$

(59)

Eq.(59) is not a closed equation for the expectation value of the energy because $p_0$ is inside the expectation values on the rhs of eq.(59). We need a perturbative method. A linearization of eqs.(43) or (55) is not a proper tool because the $p_0^2$ term is crucial for the equilibration. We can apply an expansion which assumes small $\kappa$ and finite $\kappa^2 c^2 \beta$ and $\kappa^2 m^2 c^2$. In such an expansion the stochastic force in eqs.(43) and (55) can be neglected in the lowest order (another method often applied in stochastic equations is based on time averaging which eliminates stochastic forces at the lowest order). Then,

$$E[p_0^2] \simeq E[p_0]^2$$

(60)

( in general $E[p_0^2] \geq E[p_0]^2$ ). Eq.(59) with the approximation (60) is equivalent to the differential equation

$$\frac{d}{d\tau} E(\tau) = \frac{3}{2} \kappa^2 E(\tau) - \frac{\kappa^2 c^2}{2} E(\tau)^2 + \frac{1}{2} c^2 \kappa^2 m^2 c^2 A^{-2}.$$ 

(61)

There is still $p_0$ in the argument of $A^{-2}$. Applying the same reasoning as in eq.(60)(expectation value of a function is approximately a function of the expectation value) we would obtain a closed integro-differential equation (61) for $E(\tau)$. Such an equation could be approached by iterative methods. We make a simplifying assumption that $A^{-2}$ is a slowly varying function of its argument

$$E[A^{-2}(x_{\tau})] \simeq A^{-2}(x)$$

(62)

where the rhs does not depend on time. The solution of eq.(61) with the approximation (62) is

$$E(\tau) = \left( (E(0) - \epsilon_-) \epsilon_+ \exp(\tau \kappa^2 (\epsilon_+ - \epsilon_-)) - \epsilon_-(E(0) - \epsilon_+) \right)$$

$$\left( (E(0) - \epsilon_-) \exp(\tau \kappa^2 (\epsilon_+ - \epsilon_-)) - E(0) + \epsilon_+ \right)^{-1}$$

(63)

where

$$\epsilon_\pm = \frac{3}{2} \frac{c^2}{\beta} \pm \left( m^2 c^2 A^{-2} + \frac{9}{4 c^2 \beta^2} \right)^{\frac{1}{2}}.$$ 

(64)
In the limit $\tau \to \infty$ we obtain
\[
\mathcal{E}(\infty) = \epsilon_+ = \frac{3}{2c\beta} + \left( m^2c^2A^{-2} + \frac{9}{4c^2\beta^2} \right)^{\frac{1}{2}}
\]  \hspace{1cm} (65)

In the limit of large $\beta$ and for the Schwarzschild solution [16] ($G$ is the Newton constant, $M$ is the mass of the source of gravity)
\[
\mathcal{E}(\infty) \simeq \frac{3}{2c\beta} + mcA^{-1} \simeq \frac{3}{2c\beta} + mc - \frac{GMm}{rc}.
\]  \hspace{1cm} (66)

This is the classical non-relativistic equipartition of energy in a gravitational potential $\frac{M}{r}$. In the limit of zero mass
\[
\mathcal{E}(\infty) = \frac{3}{c\beta} = \left( \int d\mathbf{p} \exp(-c\beta|\mathbf{p}|) \right)^{-1} \int d\mathbf{p} |\mathbf{p}| \exp(-c\beta|\mathbf{p}|)
\]
we obtain the relativistic equipartition of energy $\epsilon = c\mathcal{E}(\infty)$ of massless particles (the result is correct and could be derived rigorously on the basis of an exact solution of eq.(80) and a calculation of its expectation values in the next section).

The approach to the equilibrium is exponential with the speed $\kappa^2(\epsilon_+ - \epsilon_-)$. An exponential decay to the equilibrium of relativistic diffusions could be proved by means of general methods of stochastic differential equations as discussed in [7], sec.3.2. Another method could be based on our results in [9]. We have shown in [9] (sec.9) that the time evolution of the relativistic diffusion can be described as the time evolution in imaginary time quantum mechanics. The approach to the equilibrium is equivalent to the approach to the ground state (this is shown explicitly in the model of [10]) with the speed equal to the eigenvalue of the first excited state of the Hamiltonian. There are well-developed methods in quantum mechanics for a study of this eigenvalue.

7 The limit $m \to 0$

There is a substantial simplification of stochastic equations in the limit $m \to 0$. We have studied this diffusion on the Minkowski space-time in [10]. We discuss a representation of the Poincare group for diffusing massless particles in [11]. It is shown that the helicity does not mix with the diffusion. Hence, our equations without a spin can describe diffusion of photons as well as the ultrarelativistic behaviour of massive particles with a spin (so it can be useful in a description of heavy ion collisions [23]). The limit $m \to 0$ seems singular when applied to the generator (13). However, when we consider the time evolution $\exp(\tau m^2 \Delta_H)$ then the limit $m \to 0$ exists (it corresponds to a rescaling of an affine time parameter which in the massless case does not have the meaning of the proper time anyhow). The diffusion generator in the limit $m \to 0$ reads
\[
\Delta_H = p_j p_k \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} + 3p_k \frac{\partial}{\partial p_k}.
\]  \hspace{1cm} (67)
The differential operator (67) defined on the manifold $M$ does not depend on the metric. It is degenerate as an elliptic operator because the quadratic form $(p_j a_j)^2$ is degenerate. As a consequence of this degeneration the diffusion in the configuration space is trivial. The time evolution of a massless particle on the Minkowski space describes a deterministic motion with a velocity of light on a straight line (as discussed in more detail in [20]). For this reason it has been rejected by Dudley [2] (sec.11). However, in the momentum space it describes a diffusion of the energy. We have shown in [10] that the diffusion of massless particles is just a linearization of the Kompaneetz diffusion well-known in astrophysics [24]. On a curved manifold the motion of a massless particle in configuration space is non-trivial but still deterministic. The stochastic Ito differential equations (29)-(31) take the simple form

\[
\frac{dx^0}{d\tau} = AB|p|, \quad (68)
\]

\[
\frac{dx}{d\tau} = -B^2p, \quad (69)
\]

\[
dp_j = B\partial^j B|p|^2d\tau - \omega^2\partial^j \ln A d\tau + \frac{3}{2}p_j d\tau + \kappa p_j db. \quad (70)
\]

There is only one Brownian motion $b$ for all $j$ components of $p_j$ (a consequence of the degenerate generator (67)).

The transport equation reads

\[
-A^2 p_0 \frac{\partial \Phi}{\partial x^0} + B^2 p_j \frac{\partial \Phi}{\partial x^j} - \frac{\partial}{\partial x^0} \left( \frac{\partial}{\partial p_j} |p|^2 \Phi \right) + \frac{\partial^2}{\partial x^j \partial p_k} p_j p_k \Phi - \frac{\partial^2}{\partial p_j \partial p_k} K_k \Phi = 0. \quad (71)
\]

where the friction $K_k$ is determined by the equilibrium distribution $\Phi_E$

\[
K_k = \kappa^2 \Phi_E^{-1} \left( \frac{1}{2} \frac{\partial}{\partial p_j} p_j p_k - \frac{3}{2} p_k \right) \Phi_E. \quad (72)
\]

We discuss two examples of the equilibrium measures

\[
\Phi_L = p_0^{-1} \exp(-\beta \phi p_0) \quad (73)
\]

and

\[
\Phi_E = p_0^{-1} \exp(-\beta p_0). \quad (74)
\]

In the first case

\[
K_k = -\frac{\kappa^2}{2} \beta \phi p_k p_0. \quad (75)
\]

Hence, the stochastic equation for the angular momentum reads

\[
d\phi = \frac{3\kappa^2}{2} p_0 d\tau - \frac{\kappa^2}{2} \beta \phi p^2 d\tau + \kappa p_0 db. \quad (76)
\]
Let us write

\[ p_\phi = \exp(u). \]  

(77)

The solution of eq.(76) is

\[ u_\tau = u + \kappa^2 \tau + \kappa b \tau - \ln \left(1 + \frac{\kappa^2}{2} \beta_\phi \exp(u) \int_0^\tau ds \exp(\kappa^2 s + \kappa b s)\right). \]  

(78)

In the case (74) we obtain

\[ K_k = -\frac{\kappa^2}{2} \beta p_k p_0. \]  

(79)

Hence, the equation for \( p_0 \) reads

\[ dp_0 = \frac{3\kappa^2}{2} p_0 d\tau - \frac{\kappa^2}{2} p_0^2 d\tau + \kappa p_0 db. \]  

(80)

It has the same solution as eq.(76) (just replace \( \beta_\phi \) by \( \beta \) in eq.(78)). The correlation functions of polynomials of \( p_0(s)^{-1} \) and \( p_\phi(s)^{-1} \) can be calculated explicitly. The expectation values have a limit \( \tau \to \infty \) expressed as moments of the invariant measures (73)-(74). The stochastic equations for \( p_\phi \) and \( p_0 \) do not depend on the metric. Their solutions and expectation values are discussed in [20].

8 A particle diffusing on de Sitter space

In this section we discuss a relativistic particle diffusing on a background metric of the cosmological type

\[ ds^2 = (dx^0)^2 - B^{-2}(dx)^2. \]  

(81)

It is useful to apply the transformations (19)-(20) in order to reduce the stochastic equations to a perturbation of the diffusion on the Minkowski mass-shell. For simplicity of the formulae we restrict ourselves to \( B \) which depends only on time. Then, the diffusion equation (13)-(14) after the transformation (19)-(20) has the generator

\[ G = \frac{1}{2} \kappa^2 m^2 c^2 \Delta^{Min} + \frac{1}{m} p_0 \frac{\partial}{\partial x^0} - \frac{1}{m} B p_k \frac{\partial}{\partial p_k} + \frac{1}{m} p_0 \frac{\partial B}{\partial x^0} B^{-1} p_k \frac{\partial}{\partial p_k} \]  

(82)

where

\[ \Delta^{Min} = (\delta_{jk} + m^{-2} c^{-2} p_j p_k) \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} + 3m^{-2} c^{-2} p_k \frac{\partial}{\partial p_k} \]  

(83)

(expressed in eqs.(19)-(20) by primed momenta) is the generator of the diffusion on the Minkowski mass-shell. In eq.(82)

\[ p_0 = \sqrt{p^2 + m^2 c^2} \equiv \omega_1. \]  

(84)
where \( \omega_1 \) denotes the energy on the Minkowski space-time (with the spatial metric tensor equal to 1).

In the generator of the diffusion (82) the diffusion part remains the same as on the Minkowski mass-shell. The motion on the de Sitter space is described by the drift. So, the classical geodesic equations, when expressed in the coordinates (19)-(20), are perturbed by the relativistic Brownian motion defined on the mass-shell \( p_0^2 - \mathbf{p}^2 = m^2 c^2 \).

The stochastic equations for the diffusion with the generator (82) read

\[
\frac{dx^0}{d\tau} = \frac{1}{m} \omega_1, \quad \frac{dx}{d\tau} = -\frac{B^2}{m} \mathbf{p}
\]  

and

\[
dp_j = \frac{1}{m} \omega_1 \frac{\partial \ln B}{\partial x^0} p_j d\tau + \frac{3m^2}{2} p_j d\tau + mc\kappa db_j + \kappa (\omega_1 - mc) \mathbf{p}^{-2} p_j \mathbf{p} db \quad (87)
\]

In general, it is difficult to derive an explicit solution of such diffusion equations. There is one remarkable exception if (this is the metric for a causally connected part of de Sitter space sometimes called the Bondi-Hoyle universe [16])

\[
ds^2 = (dx^0)^2 - \exp(\frac{2c}{c R} x^0) dx^2
\]  

(\( R \) is the radius of the pseudosphere in the embedding of de Sitter space in a five-dimensional Minkowski space).

Then, the drift in eq.(87) corresponding to the metric (88) is

\[
K_j = -\frac{1}{mcR} \omega_1 p_j
\]  

exactly the same as the one for the relativistic model on the Minkowski mass-shell (eq.(42) \( B = A = 1 \)) with the friction leading to the Jüttner equilibrium distribution \( f' = -1 \) and the inverse temperature

\[
\kappa^2 \beta = \frac{2}{mc^2 R}.
\]  

As \( \frac{1}{\beta} = \sqrt{-\frac{4}{3} \Lambda} \), we obtain a relation between the temperature and the cosmological constant \( \Lambda \). Eq.(37) for the equilibrium distribution has now the solution

\[
\Phi_E(p) = \omega_1^{-1} \exp \left( -\omega_1 \frac{2}{m \kappa^2 c^2} \sqrt{\frac{1}{3 \Lambda}} \right)
\]

The model of a massless particle diffusing in de Sitter space is explicitly soluble. Eqs.(85)-(87) read

\[
\frac{dx^0}{d\tau} = |\mathbf{p}|
\]  

14
\frac{dx}{d\tau} = -\exp\left(\frac{2x_0}{cR}\right)p 
\tag{93}

and

\frac{dp_j}{d\tau} = -\frac{1}{cR}|p|p_j d\tau + \frac{3\nu^2}{2}p_j d\tau + \kappa p_j db. \n\tag{94}

From eq.(94) it follows that $p_\tau = |p_\tau|n$, where $n$ is a time independent unit vector. The equation for $|p|$ is the same as eq.(76) with the solution

$$\ln |p_\tau| = \ln |p| + \kappa^2 \tau + \kappa b - \ln \left(1 + \frac{1}{Rc^2}|p| \int_0^\tau ds \exp(\kappa^2 s + \kappa b_s)\right). \n\tag{95}
$$

We can calculate expectation values of functions of $p_\tau$ using the formulae of Yor [25] for the probability distribution of exponentials of the Brownian motion and the integrals of exponentials (we have calculated some expectation values in [20]). The correlation functions of $|p_\tau|^{-1}$ can be expressed by elementary functions. The equilibrium measure reads

$$\Phi_E(p) = |p|^{-1}\exp\left(-\frac{2}{R\kappa^2 c}|p|\right).$$

It is rather surprising that the momentum evolution in de Sitter space tends to an equilibrium as if a certain friction was involved in a geodesic diffusion on de Sitter space.

9 Summary

We have derived basic equations concerning the relativistic diffusion with friction in a gravitational background (adding friction to the diffusion equations of Franchi and Le Jan [3]). Such equations can describe the dynamics of particles around stars and black holes in the presence of a gas of some other particles. The gravitational effects must be sufficiently strong to be detectable by experiments. The most impressive data apply to CMBR spectrum. Then, the diffusion of photons scattered on charged particles in the intergalactic space can distort the CMBR spectrum [27]. An application of the relativistic diffusion to the CMBR spectrum distortion (Sunyaev-Zeldovich effect) is discussed in [10]. The effect of the gravitational field studied in sec.7 is rather weak to be detected in a near future. Nevertheless, from the derivation of the (Kompaneetz) diffusion equation [24] it can be seen that if the quantum electrodynamics in a background metric is considered then the resulting diffusion process should be the one perturbing the geodesic motion. The relativistic diffusion can be a useful tool for a selection of a relativistic approximation to multiparticle interactions determined by the form of the equilibrium resulting from the relativistic quantum field theory at finite temperature. We discussed as an example a diffusion in de Sitter space. We obtained a surprising relation between the diffusion on the de Sitter space and the diffusion on the flat space in a heat bath of non-zero
temperature. There is a well-known relation between the particle temperature in the de Sitter space and the radius of this space [26]. Such a relation results from a quantum theory and it is different from the one derived in this paper.

The main aim of this paper is a discussion of the friction terms which lead to an equilibrium. It is not clear what is the physical meaning of the diffusion in the proper time because in contradistinction to the deterministic dynamics in random dynamics the proper time associated with an observer moving with a particle is a random variable. The transport equation (sec.5) expressed in the laboratory time has a clear physical meaning. We discuss the transport equation in more detail in [20]. In particular, we show that the diffusion in the laboratory time can be obtained from the diffusion in the proper time by an integration over the proper time in a similar way as this is done in Feynman’s relativistic dynamics [28] (for a discussion of the relation between the proper time dynamics and laboratory time dynamics see also [29]).

10 Appendix A: Stochastic equations

In this Appendix we explain the relation between the diffusion equation (24) and the stochastic equations (29)-(31) (see [18] for a complete theory). The main observation is that the solution at time $\tau$ of the stochastic differential equations (assuming it is unique) is causal, i.e., depends on the initial conditions and values of the Brownian motion $b(s)$ at $s \leq \tau$. As a consequence (of the Markov property) eq.(26) determines a semigroup $T_{\tau}\phi = \phi_{\tau}$. In such a case in order to prove that $\phi_{\tau}$ of eq.(26) is the solution of eq.(24) it is sufficient to check that the initial conditions are the same and the generators coincide. The initial condition for eq.(24) follows from the choice of the initial conditions for the stochastic equations. Then, the calculation of the generator is reduced to the solution of the stochastic equations at arbitrarily small time and the calculation of $d\phi$. Let us consider eqs.(29)-(30) for $x^\mu$ and assume eq.(31) in a general form

$$dp_j = B_j d\tau + m c e_j^a db_a,$$

(96)

where $e_j^a$ are defined in eqs.(27)-(28) (together with the $\gamma_{jk}$ of eq.(9) and $g_{jk} = B^{-2}\delta_{jk}$ for isotropic coordinates). As a next step we need to calculate (for a small $\tau$)

$$\phi(x_\tau, p_\tau) - \phi(x, p) \simeq \phi(x + \Delta x, p + \Delta p) - \phi(x, p),$$

(97)

where from eqs.(29)-(30) $\Delta x^0 = \frac{1}{m} A \omega \Delta \tau$, $\Delta x = -\frac{1}{m} B^2 p \Delta \tau$ and from eq.(96)

$$\Delta p_j = B_j \Delta \tau + m c e_j^a (b_a(\Delta \tau) - b_a(0)),$$

(98)

here $b_a(0) = 0$. We expand eq.(97) in a Taylor series in $\Delta x^\mu$ and $\Delta p_j$ (or what is the same in $\Delta \tau$ and $b_a(\Delta \tau)$). After the expansion we calculate the expectation values involving the Brownian motion (see the expectation value below eq.(31);
we use $E[b_a] = 0$ and $E[b_a(\triangle \tau)b_c(\triangle \tau)] = \delta_{ac}\triangle \tau$). We must take into account terms till the second order in $\triangle p$ which give

$$E[\triangle p_j \triangle p_k] = m^2 c^2 \kappa^2 \epsilon_j^b \epsilon_k^c E[b_a(\triangle \tau)b_c(\triangle \tau)] = m^2 c^2 \kappa^2 \epsilon_j^b \epsilon_k^c \triangle \tau = m^2 c^2 \kappa^2 \gamma_{jk} \triangle \tau.$$  

(99)

Now, collecting the terms of order $\triangle \tau$ we obtain for $\tau = \triangle \tau$

$$\phi_\tau - \phi = \frac{1}{\mu} A \omega \frac{\partial}{\partial x^j} \phi \triangle \tau - \frac{1}{\mu} B^2 p_j \frac{\partial}{\partial x^j} \phi \triangle \tau + B_j \frac{\partial \phi}{\partial p_j} \triangle \tau + m^2 c^2 \kappa^2 \frac{1}{2} \kappa_{jk} \gamma \frac{\partial}{\partial p_j} \phi \triangle \tau. \quad (100)$$

Dividing by $\triangle \tau$ and taking the limit $\triangle \tau \to 0$ we obtain eq.(24) with

$$B_j = \frac{1}{2 m} p_j \frac{\partial B^2}{\partial x^j} - \frac{1}{2 m} \omega^2 \frac{\partial \ln A}{\partial x^j} + \frac{3}{2} \kappa^2 p_j.$$  

11 Appendix B: Time evolution of the energy

We calculate the change of the energy

$$dp_0 = d(A^{-1} \omega) = -\omega A^{-2} \partial_j A d x^j - A^{-1} \omega^{-1} B \partial_j B p^2 d x^j + A^{-1} B^2 \omega^{-1} p_j \circ dp_j. \quad (101)$$

Here, the circle denotes the Stratonovitch differential [18]

$$h \circ dg = h dg + \frac{1}{2} dh dg \quad (102)$$

The correction term to the Stratonovitch differential is

$$\frac{1}{2} d(\omega^{-1} p_j) dp_j = \frac{3}{2} m^2 c^2 \kappa^2 B^{-2} \omega^{-1} d \tau. \quad (103)$$

After an insertion of the differentials (29)-(30) and (43) on the rhs of eq.(101) the deterministic (\kappa independent terms) cancel identically (as they should because $p_0$ is a constant of motion when the stochastic perturbation is absent). There remains

$$dp_0 = \frac{3}{2} \kappa^2 A^{-1} \omega^{-1} B^2 p^2 d \tau + \frac{3}{2} m^2 c^2 \kappa^2 A^{-1} \omega^{-1} d \tau + \frac{\kappa^2}{2} A^{-2} (\omega^{-1} B^2 c \beta f p^2 d \tau + \kappa A^{-1} B p d \beta). \quad (104)$$

This is eq.(55) (after elementary transformations).

A proof of the formula (52) for the angular momentum is much simpler. We have

$$dL = d x \times p + x \times dp = x \times dp \quad (105)$$

We assume that the functions $A$ and $B$ in the isotropic metric are spherically symmetric. Then, in $dp$ (defined in eq.(43)) the terms independent of $\kappa$ are parallel to $x$ because gradients of $A$ and $B$ are parallel to $x$. The $\kappa$-dependent terms in (105) give eq.(45).
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