BASKET, FLAT PLUMBING AND FLAT PLUMBING BASKET SURFACES DERIVED FROM INDUCED GRAPHS

DONGSEOK KIM

Abstract. The existence of basket, flat plumbing and flat plumbing basket surfaces of a link was first proven from a braid representative of the link. In the present article, we show the existence of such surfaces from an induced graph of the link. Consequently, we define the basket number, flat plumbing number and flat plumbing basket number of a link. Then we provide several upper bounds for these plumbing numbers and study the relation between these plumbing numbers and the genera of links.

1. Introduction

Let \( L \) be a link in \( S^3 \). A compact orientable surface \( F \) is a Seifert surface of \( L \) if the boundary of \( F \) is isotopic to \( L \). The existence of such a surface was first proven by Seifert using an algorithm on a diagram of \( L \), named after him as Seifert’s algorithm [18]. Some of Seifert surfaces feature extra structures. Seifert surfaces obtained by plumbings annuli are one of the main subjects of this article. Even though a higher dimensional plumbing can be defined but here we will only concentrate on annuli plumbings. It is often called a Murasugi sum and it has been studied extensively for the fibreness of links and surfaces [7, 19]. Rudolph has introduced several plumbing Seifert surfaces [15, 16]. To show the existence of these plumbing surfaces of a link, it is common to present the link as the closure of a braid in classical Artin group [6, 9]. Furthermore, a few different ways to find its braid presentations were found by Alexander [2], Morton [13], Vogel [21] and Yamada [22]. In particular, the work of Yamada is closely related with Seifert’s algorithm and has been generalized to find another beautiful presentation of the braids groups [3]. Several authors showed the existence of basket surfaces, flat plumbing surfaces and flat plumbing basket surfaces using a braid representative of a link [6, 9]. For more terms in knot theory, we refer to [1].

We define the basket number, the flat plumbing number and the flat plumbing basket number of a link by constructing these plumbed surfaces using Theorem 2.1. Consequently, we provide some upper bounds for these plumbing numbers from braid representatives of links and canonical Seifert surfaces of links. We also compare these upper bounds for those which were obtained from a braid representative.

The outline of this paper is as follows. In section 2, we first review some preliminary definitions in graph theory, then we provide the definition of the induced graph \( \Gamma \) obtained from a Seifert surface \( F \). We also introduce an algorithm which will produce

2000 Mathematics Subject Classification. Primary 57M27; Secondary 05C05, 05C10.
This work was supported by Kyonggi University Research Grant 2010.
a spanning tree $T$ and a coloring $\kappa$ on $T$ with the desired property and we prove
the existence of such a spanning tree and a coloring in Theorem 2.1. In section 3, we review definitions
of these plumbing numbers and we find some upper bounds for these plumbing numbers by constructing plumbing surfaces from Seifert surfaces. In section 4, we study the relations between these plumbing numbers and the genera of a link.

2. Induced graphs of Seifert surfaces

We first review some preliminary definitions in graph theory in subsection 2.1. For more terms in graph theory, we refer to [8]. Then we provide the definition of the induced graph $\Gamma$ obtained from a Seifert surface $F$ in subsection 2.2. At last, we introduce an algorithm which will produce a spanning tree $T$ and a coloring $\kappa$ on $T$ with the desired property in subsection 2.3. At last, we prove Theorem 2.1 in subsection 2.4.

2.1. Preliminaries in graph theory. A graph $\Gamma$ is an ordered pair $\Gamma = (V(\Gamma), E(\Gamma))$ comprising a set $V(\Gamma)$ of vertices together with a set $E(\Gamma)$ of edges which are 2-element subsets of $V(\Gamma)$. An edge is multiple if there is another edge with the same end vertices; otherwise it is simple. A graph is a simple graph if it has no multiple edges or loops, a multigraph if it has multiple edges, but no loops. A graph is signed if there is a function $\mu : E(\Gamma) \to \{+, -\}$. A graph $\Gamma$ is finite if the cardinality of the vertices set $V(\Gamma)$ and the edge set $E(\Gamma)$ are finite. In the present article, a graph means to be a signed finite multigraph and a coloring on $\Gamma$ means to be a 2-edge coloring which is a function $\kappa : E(\Gamma) \to \{+, -\}$ unless stated differently. A graph $\Gamma$ is bipartite if vertices can be divided into two disjoint sets $S$ and $T$ such that every edge connects a vertex in $S$ to one in $T$. A subgraph of a graph $\Gamma$ is a graph whose vertex set is a subset of that of $\Gamma$, and whose adjacency relation is a subset of that of $\Gamma$ restricted to this subset. A subgraph $H$ is a spanning subgraph of a graph $\Gamma$ if it has the same vertex set as $\Gamma$. A tree is a connected acyclic simple graph. A spanning tree is a spanning subgraph that is a tree. A walk is an alternating sequence of vertices and edges, beginning and ending with a vertex, where each vertex is incident to both the edge that precedes it and the edge that follows it in the sequence, and where the vertices that precede and follow an edge are the end vertices of that edge. A walk is closed if its first and last vertices are the same, and open if they are different. A path is an open walk which is simple, meaning that no vertices (and thus no edges) are repeated. We will often omit vertices in path if there is no ambiguity. If it is possible to establish a path from any vertex to any other vertex of a graph, the graph is said to be connected. Every connected graph $\Gamma$ admits a spanning tree $T$ and for every edge $e \in \Gamma - T$, there exists a unique path in $T$ joining both ends of $e$.

A Seifert surface $F$ gives a rise to a natural signed graph, which is called the induced graph $G(F)$ by collapsing a disc to a vertex and a twisted band to a signed edge. By separating discs by the local orientation, it can be considered as a bipartite graph. If the Seifert surface is connected, then its induced graph is also connected. The spanning tree of the induced graphs plays a key role in the research of plumbing surfaces. If we use a braid representative of a link, its induced graph of the canonical
2.2. Induced graphs from Seifert surfaces. A Seifert surface $F_L$ of an oriented link $L$ by applying Seifert’s algorithm to a link diagram $D(L)$ as shown in Figure 1 (i) is called a canonical Seifert surface. From such a canonical Seifert surface, we construct an induced graph $\Gamma(F_L)$ by collapsing discs to vertices and half twist bands to signed edges as illustrated in Figure 1 (i). From arbitrary Seifert surfaces, these process can be done too. Since a link $L$ is tame and its Seifert surface $F_L$ is compact, the induced graph $\Gamma(F_L)$ is finite. By considering the local orientation as indicated on each vertices in Figure 1 (ii), $\Gamma(F_L)$ is a bipartite graph. For bipartite graphs, it is easy to see that the length of a closed path is always even. It is fairly easy to see that the number of Seifert circles (half twisted bands), denoted by $s(F_L)(c(F_L))$, is the cardinality of the vertex set, $V(\Gamma(F_L))$ (edge set $E(\Gamma(F_L))$, respectively). A spanning tree $T$ of $\Gamma(F_L)$ is depicted in Figure 1 (iii). The number of edges of a spanning tree of a connected graph with $n$ vertices is $n - 1$. One can see that the length of the path joining both end vertices of $e \in \Gamma(F_L)$ is odd.

However, in the case of $7_5$, these exist a spanning tree which is a path, thus, any alternating signing $\kappa$ on the spanning path will satisfy Theorem 2.1. To obtain a flat plumbing basket surface from Seifert surfaces, we need to find a spanning tree $T$ with a coloring $\kappa : E(T) \rightarrow \{+, -\}$ such that for any $e \in E(G) - T$, there exists a path $P$ joining both end vertices of $e$ in $T$ whose coloring is alternating. The exitance of such a spanning tree $T$ with a coloring $\kappa$ can be stated in the language of graph theory as in Theorem 2.1. In the following subsection, we will provide a proof of Theorem 2.1.

2.3. An algorithm to produce a desired spanning tree and a coloring. Let $\Gamma$ be a connected finite bipartite graph. First, we pick a spanning tree $T$, a vertex $v \in V(\Gamma)$ and a depth coloring $\kappa : E(T) \rightarrow \{+, -\}$. of $T$ which is defined as follows. For a tree $T$ with a root $v$, we define the depth $d(u)$ of a vertex $u$ to be the distance

![Figure 1](image-url)
between $u$ and $v$ and the depth $d(e)$ of an edge $e = (v_i, v_j)$ to be the maximum of the depth of $v_i$ and $v_j$. For $e \in T$, the depth coloring $\kappa_d(e)$ is the sign of $(-1)^{d(e)}$ as illustrated in Figure 2 (iii). For a tree drawn with respect to the depth, one can see that the depth of vertices in the path joining $u$ and $v$ has the minimum at $v_t$ and it is unique. If so, we say two vertices $u$ and $v$ have the least common ancestor $v_t$. The edges of the same depth of its end vertex $v$ is called the children of $v$.

First, we explain an algorithmic process to change $(T, \kappa_d)$ to a desired $(\overline{T}, \overline{\kappa}_d)$. Let $e$ be an edge in $\Gamma - T$ and $P_e$ be the unique simple path in $T$ joining both end vertices of $e$. Because $\Gamma$ is bipartite, the length of $P_e$ must be odd and there are two possibilities. If the least common ancestor $v_e$ of the end vertices of $e$ is one of the the end vertices of $e$, then the path $P$ has alternating signs with respect to the depth coloring $\kappa_d$. Otherwise, the path $P_e$ is the union of two paths joining $v_e$ and both end vertices of $e$ and each of these two paths has alternating signs with respect to the depth coloring $\kappa_d$ but two children of $v_e$ have the same sign. Since the lengths of these two paths are odd and even, we can choose the shorter one. We remove one of children edges.
Figure 3. (i) A part of spanning tree $T$ around $f, f_c, P_1, P_2$, (ii) resulting $\overline{T}$ by the process described in the proof and (iii) a new diagram of $\overline{T}$ with respect to the depth.

of $v_e$ which is belong to the shorter path and add $e$ to get a new spanning tree. This process does depend on the order of $e$’s, thus we do start an edge for which the least common ancestor has the minimal depth and if two have the same depth, we use an order given by the labeling of vertices as illustrated in Figure 2 (iv – vi).

2.4. A main theorem for the algorithm. To construct flat plumbing surfaces from Seifert surfaces, we need to find a spanning tree of the induced graph with a special property. In present article, we provide a condition to have flat plumbing surfaces from (canonical) Seifert surfaces as follows. A much more general version of Theorem 2.1 was proven in the language of graph theory [10] but, here we will provide an elementary and constructive proof.

Theorem 2.1. For a connected bipartite graph $\Gamma$, there exist a spanning tree $T$ and a vertex $v$ such that for any $e \in \Gamma - T$, the unique path $P_e$ in $T$ joining both end vertices of $e$ has an alternating signs with respect to the depth coloring $\kappa : E(T) \rightarrow \{+, -\}$.

Proof. Suppose the algorithmic process described previously does not work. Let $\Gamma$ be a counterexample. For a vertex $v$ and a spanning tree $T$ of $\Gamma$, we define $\eta(\Gamma, T, v)$ to be

$$\eta(\Gamma, T) = \sum_{u \in V(\Gamma)} d_v(u)$$

where $d_v(u)$ is the distance between $u$ and $v$, the number of edges in the path joining $u$ and $v$. Since $\Gamma$ is a finite graph, there exist a vertex $v$ and a spanning tree $T$ such that $\eta(\Gamma, T, v)$ is maximal. Let $A = \{e \in \Gamma - T\}$ the least common ancestor $v_e$ of
end vertices of $e$ is not one of end vertices of $e$. If $|A| = 0$, then $(\Gamma, T, \kappa_d)$ satisfies that for any $e \in \Gamma - T$, the unique path in $T$ joining both end vertices of $e$ has an alternating signs. Thus, it contradicts the hypothesis of $(\Gamma, T)$ is a counterexample. Suppose if $|A| \neq 0$, then there exists an edge $f \in \Gamma - T$ such that the least common ancestor $v_f$ of end vertices of $f$ is not one of end vertices of $f$. Among all such $f \in A$, we pick $f$ for which it has the minimum $d(v_f)$. Let $P_e$ be the path in $T$ joining both end vertices of $e$. Since $e \in \mathcal{C}$, the path $P_e$ is union of two paths $P_1$ $(P_2)$ joining end vertices $u_1$ $(u_2$, respectively) of $f$, we further assume the path $P_1$ has the shorter length as depicted in Figure 3 (i). If we replace $f \in \Gamma - T$ by a child edge $f_c$ of the $v_f$ which belongs to the shorter path $P_1$, the we get a new spanning tree $\overline{T}$. Using the same vertex $v$ as a root, we have a new depth $\overline{d}$ on $\overline{T}$. We name vertices that the edge $f_c$ is $(v_f, v_{c_1})$ and $f = (f_1, f_2)$ where the vertex $f_i$ belongs to $P_i$, $i = 1, 2$. We will claim that $\eta(\Gamma, T, v) < \eta(\Gamma, \overline{T}, v)$ as follows. We denote that $\Gamma - T = \{f, e_1, e_2, \ldots, e_n\}$ and $\Gamma - \overline{T} = \{f_c, e_1, e_2, \ldots, e_n\}$. For a vertex $u \in \Gamma$, we divide cases 1) an ancestor of $u$ is not in $P_1$, 3) an ancestor of $u$ is in $P_1$. Let us deal with each case separately.

Case 1: if an ancestor of $u$ is not in $P_1$, it is easy to see that the depth of the path joining $u$ and $v$ has not been change. Thus, we have $d_v(u) = \overline{d}_v(u)$.

Case 2: if an ancestor of $u$ is in $P_1$, we choose $g$ to have the maximal depth among all such ancestors of $u$. Let $Q$ be the path joining $u$ and $g$ and $R$ be the path joining $f_1$ and $g$. Then,

$$d_v(u) = d_v(g) + l(Q) \leq d_v(f_1) + l(Q) < \overline{d}_v(f_1) + l(Q) + l(R) = \overline{d}_v(u).$$

Therefore, by summing all $u \in \Gamma$, we have $\eta(\Gamma, T, v) < \eta(\Gamma, \overline{T}, v)$. But it contradicts that $(\Gamma, T, v)$ has the maximum $\eta(\Gamma, T, v)$. It completes the proof of Theorem 2.1. \qed

\section{Plumbing numbers}

There are several interesting plumbing surfaces \cite{16} but we would like to discuss three plumbing surfaces and their plumbing numbers. We will review the definitions of these plumbing surfaces and define corresponding plumbing numbers and prove theorems for the upper bounds of each plumbing number.

By \cite{16}, we define the top plumbing as follows. Let $\alpha$ be a proper arc on a Seifert surface $S$. Let $D_{\alpha}$ be a 3-cell on top of $S$ along a tubular neighborhood $C_{\alpha}$ of $\alpha$ on $S$. Let $A_n \subset D_{\alpha}$ be an annulus such that $A_n \cap \partial D_{\alpha} = C_{\alpha}$. A top plumbing on $S$ along a path $\alpha$ is the new surface $S' = S \cup C_{\alpha}$ where $A_n, C_{\alpha}, D_{\alpha}$ satisfies the previous conditions. Thus, two consecutive plumbings are non-commutative in general. Rudolph found a few interesting results on top and bottom plumbings in \cite{16}.

Throughout the section, we will assume all plumbings are top plumbing and all links are not splittable, \textit{i.e.}, Seifert surfaces are connected. Otherwise, we can handle each connected component separately.

\subsection{Basket number}

Let $A_n \subset S^3$ denote an $n$-twisted unknotted annulus. A Seifert surface $\mathcal{F}$ is a basket surface if $\mathcal{F} = D_2$ or if $\mathcal{F} = \mathcal{F}_0 \ast_{\alpha} A_n$ which can be constructed by plumbing $A_n$ to a basket $\mathcal{F}_0$ along a proper arc $\alpha \subset D_2 \subset \mathcal{F}_0$ \cite{16}. First, we define
The following example demonstrates that the inequality in Theorem 3.1 is sharp.

**Example 3.2.** Let $K$ be the figure-8 knot which is presented by $\sigma_1 \sigma_2 \sigma_1 \sigma_2 \in B_3$. Then by applying Theorem 3.1, its basket number is less than or equal to 2. However, the only links of the basket number 1 are closures of two string braids and it is well known that the braid index of the figure-8 knot is 3. Therefore, the basket number of the figure-8 knot is 2.

**3.2. Flat plumbing number.** A Seifert surface $F$ is a flat plumbing surface if $F = D_2$ or if $F = F_0 \star_\alpha A_0$ which can be constructed by plumbing $A_0$ to a flat plumbing surface $F_0$ along a proper arc $\alpha \subset F_0$. Remark that the gluing regions

![Diagram](image-url)
Figure 5. (i) adding extra edges to $T$ such that for any $e \in E(\Gamma) - T$, there exists a path in $\overline{T}$ with an alternating signs joining both ends of $e$, (ii) plumbing all edges in $E(\Gamma) - T$, (iii) a new induced graph $\overline{\Gamma(7_5)}$ by adding another extra edge of the opposite sign for each edge in step (i) for Reidemeister move II and (iv) a new knot diagram of $7_5$ and its Seifert surface $F_{7_5}$ corresponding to $\overline{\Gamma(F_{7_5})}$.

in the construction are not necessarily contained in $D_2$. Hayashi and Wada showed every oriented links in $S^3$ bound flat plumbing surfaces by finitely many flat plumbings $[9]$. Thus, we can define the flat plumbing number of $L$, denoted by $fp(L)$, to be the minimal number of flat annuli to obtain a flat plumbing surface. By using the original work of $[9]$, one can find an upper bound from a braid word representative of the link $L$ in the following theorem.

Theorem 3.3. ($[9]$) Let $L$ be an oriented link which is a closed $n$-braid with a braid word $W$ where the length of $W$ is $m$, then there exists a flat plumbing surface $F$ obtained by at most $m + n - 1$ flat plumbings such that $\partial F$ is isotopic to $L$, i.e., $fp(L) \leq m + n - 1$.

Now we want to find an upper bound for flat plumbing number of $L$ by using Seifert surface $F_L$. For a Seifert surface of closed braid is a path, thus, Theorem 2.1 can be directly obtained with changing anything. For a general Seifert surface, it could be drastically changed and that is a reason that we have proven Theorem 2.1. Obviously, we can directly apply Theorem 2.1 from the beginning but for the existence of a flat plumbing surface from Seifert surface, we have much more flexibility because the arc $\alpha$ we are plumbing along can pass the band we have made by previous plumbing. For example, every spanning tree of $\Gamma(F_{7_5})$ in Figure 1 is a tree, but Theorem 3.3 can not be directly used to construct a flat plumbing surface because there exist two consecutive edges of the same sign which will produce a full twist which obstructs the existence of 3 ball in flat plumbing. Thus, from a given spanning tree, we want to add extra edges to $T$ as depicted in Figure 5 (i) which can be obtained by a flat plumbing along think lines such that for any $e \in E(\Gamma) - T$, there exists a path in $\overline{T}$ with an alternating signs joining both ends of $e$ to prevent to have a full twist in flat plumbing. If we add extra edges of opposite sign for every edge in $T$, resulting
The graph $T$ must have this property. By plumbing all edges in $E(\Gamma) - T$ as shown in Figure 5 (ii), we have a flat plumbing surface but its boundary is not isotopic to the original link $L$ because we added extra bands to make flat plumbing. However, adding two adjacent edges of different signs is, in fact, a Reidemeister type II move which guarantees the resulting link is isotopic to the original one, so at last, we can add an edge of an opposite sign as illustrated in Figure 5 (iii) if we added an extra edge in step (a). The resulting diagram for a knot $7_5$ is given in Figure 5 (iv) which was obtained from the disc which is the union of disc $a, b, c$ and $d$ and twisted bands represented by the spanning tree $T$ by 8 flat plumbing as described above.

Let $\delta$ be the total number of the edges in $T$ for which we have to add extra edges of opposite sign such that for any $e \in E(G) - T$, there exists a path with an alternating signs joining both end vertices of $e$. By considering the minimum of $\delta$ over all spanning trees of $\Gamma(L)$, we choose $T$ for which we can obtain the minimum $fp(L)$. Then by summarizing the these facts, we obtain the following theorem.

**Theorem 3.4.** Let $F$ be a canonical Seifert surface of a link $L$ with $s(F)$ Seifert circles and $c(F)$ half twisted bands. Let $\Gamma$ be the induced graph of a Seifert surface $F$. Let $T$ be a spanning tree with the minimum $\delta$ as described above. Then the flat plumbing number of $L$ is bounded by $c(F) - s(F) + 1 + 2\delta$, i.e.,

$$fp(L) \leq c(F) - s(F) + 1 + 2\delta.$$ 

If we plumb two annuli for each edge in a spanning tree $T$ to have the property that for any twisted band $t(e)$ of sign $\epsilon$ which is not a part of the disc $D$, the maximum of $\delta$ is $s(F) - 1$ and in such case, we have the largest upperbound for the flat plumbing number of $L$ as $fp(L) \leq c(F) + s(F) - 1$.

### 3.3. Flat plumbing basket number.

A Seifert surface $F$ is a flat plumbing basket surface if $F = D_2$ or if $F = F_0 \ast_\alpha A_0$ which can be constructed by plumbing $A_0$ to a basket $F_0$ along a proper arc $\alpha \subset D_2 \subset F_0$. We say that a link $L$ admits a flat plumbing basket presentation if there exists a flat plumbing basket $F$ such that $\partial F$ is equivalent to $L$. In [6], it is shown that every link admits a flat plumbing basket presentation. So we can define the flat plumbing basket number of $L$, denoted by $fpbk(L)$, to be the minimal number of flat annuli to obtain a flat plumbing basket surface of $L$. By presenting a link by a closed braid, we have the following theorem.

**Theorem 3.5.** ([6]) Let $L$ be an oriented link which is a closed $n$-braid with a braid word $\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1W$ where the length of $W$ is $m$ and $W$ has $s$ positive letters, then there exists a flat plumbing basket surface $F$ with $m+2s$ bands such that $\partial F$ is isotopic to $L$, i.e., $fpbk(L) \leq m + 2s$.

The key ingredient of theorem is that each crossing corresponding to a negative letter can be deplumed by removing $A_0$ annulus as shown in Figure 4. But crossing corresponding to a positive letter can be deplumed by removing three $A_0$ annuli as shown in Figure 6.

By choosing an efficient braid representative of $L$ and a disc, we can improve the upper bound in Theorem 3.5 as follows. Let $L$ be an oriented link which is a closed
n-braid with a braid word $W$. Since we can add $\sigma_i\sigma_i^{-1}$, without loss of the generality, we assume $\sigma_i$ and $\sigma_i^{-1}$ appear in $W$, $i = 1, 2, \ldots, n - 1$. Let $a_i(1)$ and $a_i(-1)$ be the number of $\sigma_i$ and $\sigma_i^{-1}$ respectively in $W$. If $a_i(1) - a_i(-1) = 0$, we set $\epsilon_i = 1$. Otherwise, we set $\epsilon_i = -\frac{a_i(1) - a_i(-1)}{|a_i(1) - a_i(-1)|}$ for $i = 1, 2, \ldots, n - 1$. Then, we choose a disc which is presented by $\sigma_{\epsilon_1} \sigma_{\epsilon_2} \ldots \sigma_{\epsilon_{n-1}}$. The main idea of Theorem 3.5 is that each $\sigma_i$ in $W$ can be replaced by $\sigma_i^{-1}$ by plumbing two flat annuli. Since we replace each $\sigma_{\epsilon_i}$ in $W$ by $\sigma_i^{-\epsilon_i}$ by plumbing two flat annuli, we have the following theorem.

**Theorem 3.6.** Let $L$ be an oriented link which is a closed $n$-braid with a braid word $W$ which contains $\sigma_i$ and $\sigma_i^{-1}$, $i = 1, 2, \ldots, n-1$. Let $a_i, \epsilon_i$ be the numbers described above. Then the flat plumbing basket number of $L$ is bounded by $\sum_{i=1}^{n-1} a_i(-\epsilon_i) + 2(a_i(\epsilon_i) - 1)$, i.e.,

$$fpbk(L) \leq \sum_{i=1}^{n-1} a_i(-\epsilon_i) + 2(a_i(\epsilon_i) - 1).$$

One can easily find that as long as the braid word $\beta$ satisfies the hypothesis of both theorems, $\sum_{i=1}^{n-1} a_i(-\epsilon_i) + 2(a_i(\epsilon_i) - 1) \leq m + 2s$. Furthermore, the following example demonstrates that the upper bound in Theorem 3.6 is better than one in Theorem 3.5.

**Example 3.7.** Let $\beta = \sigma_1\sigma_2\sigma_3\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_2\sigma_3\sigma_3$ in $B_3$. This braid word is reduced. Since $\beta$ is the exact shape as stated in Theorem 3.5, we find that an upper bound for the flat plumbing basket number of $L = \bar{\beta}$ is $m + 2s = 12 + 2 \cdot 9 = 30$. Since $\beta$ satisfies the hypothesis of Theorem 3.6, we can directly calculated that for all $i = 1, 2, 3$, $\epsilon_i = -1$ and $a_1(1) = 3, a_2(1) = 5, a_3(1) = 4, a_1(-1) = a_2(-1) = a_3(-1) = 1$. Thus, the upper bound in Theorem 3.6 is $12 + 2 \cdot 0 = 12$.

Now we want to find an upper bound for the flat plumbing basket number of $L$ by using Seifert surface $F_L$. For a flat plumbing basket surface, $\alpha$ has to stay in the disk $D$ which was fixed from the beginning, thus we have to choose a disk $D$ carefully.
The induced graph $\Gamma(\mathcal{F})$ of the Seifert surface $\mathcal{F}$ of a closed braid is a path. Thus, there is no ambiguity about the choice of a spanning tree for $\Gamma(\mathcal{F})$, a spanning path is only spanning tree of $\Gamma(\mathcal{F})$. For a general Seifert surface, it could be drastically changed. For example, every spanning tree of $\Gamma(7_5)$ in Figure 1 is a tree, but none of these can be used to construct a flat plumbing basket surface neither Theorem 3.5 nor Theorem 3.6 because there exist two consecutive edges of the same sign which will produce a full twist that is prohibited in flat plumbing.

**Example 3.8.** A reduced braid representative $\beta$ of $7_5$ is $\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1^3 \in B_2$, thus, we find that an upper bound of the flat plumbing basket number of $7_5$ by applying Theorem 3.5 is 19 and an upper bound of the flat plumbing basket number of $7_5$ by applying the process described above is 8.

**Proof.** Thus, we want to have a spanning tree with an alternating signs as depicted in Figure 7 (i) to prevent to have a full twist in flat plumbing. However, adding two adjacent edges of different signs is, in fact, a Reidemeister type II move which guarantees the resulting link is isotopic to the original one, so we can add two adjacent edges of different signs if the original sign and the sign in alternating signs are different to get a new induced graph $\Gamma(\mathcal{F}_{7_5})$ as illustrated in Figure 7 (ii) and a new link diagram of $7_5$ in Figure 7 (iii). We start with the disc $D$ which is a union of four discs $a, b, c$ and $d$ joined by half twisted band represented by $T$ with alternating signs as indicated in a think line in Figure 7 (ii). The order of flat plumbing matters, thus, let us show the construction of a flat plumbing basket surface of $7_5$ in step by step. We first plumb $A_0$ to $D$ along a path represented by edges between disc $b$ and disc $c$ from the bottom. After then, the order of plumbing does not matter anymore because there is no ambiguity of a path presented by edges and the annuli never leave two adjacent disc. The Seifert surface $\mathcal{F}_{7_5}$ of a knot $7_5$ in Figure 7 (iii) can be obtained by plumbing eight $A_0$. Thus, the flat plumbing basket number of $7_5$ is less than or equal to 8. □
Let us deal with general cases for the flat plumbing basket number of $L$ by using canonical Seifert surface $F_L$. Let $\Gamma$ be the induced graph of a canonical Seifert surface $F_L$ of $L$. Using Theorem 2.1, there is a spanning tree $T$ and a coloring $\kappa$ on $T$ such that for any $e \in E(\Gamma) - T$, the unique path in $T$ joining both ends of $e$ has an alternating signs. Let $\mu : E(\Gamma) \rightarrow \{+,-\}$ be coloring representing the sign of edges in $\Gamma$. Let

$$B = \{e \in T \mid \mu(e) \neq \kappa(e)\}.$$ 

First if an edge $e$ in $T$ belongs to $B$, then we have to isotop the link by a type II Reidemeister move as shown in Figure 7 (iii). Since we can completely reverse the sign of all edges in the spanning tree $T$, we may assume the total number of type II Reidemeister moves in the process is less than or equal to $\left\lceil \frac{s(F_L) - 1}{2} \right\rceil$. Let $\gamma$ be the minimum of the cardinality of the set $B$ and $|T| - |B|$. Now we set $D$ the disc corresponding to the spanning tree $T$ as depicted in Figure 7 (iii). Let

$$C = \{e \in E(\Gamma) - T \mid \mu(e) \neq \sum_{f \in P_e} \kappa(f)\}.$$ 

If an edge $e$ in $E(\Gamma) - T$ belongs to $C$, then we can plumb a flat annulus along a curve $\alpha$ corresponding to the path $P_e$ in the spanning tree $T$. Otherwise, we need to add three flat annuli to make the half twisted band presented by the edge $e$ as shown in Figure 4 [6]. By plumbing all edges in $E(\Gamma) - T$ as described, we have a flat plumbing basket surface of $L$. Then by summarizing above description of flat plumbing surface of $L$, we obtain the following theorem.

**Theorem 3.9.** Let $\Gamma$ be an induced graph of canonical Seifert surface $F$ of a link $L$ with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Let $T$ be a spanning tree of $\Gamma$ and $\kappa$ a coloring on $T$ chosen in Theorem 2.1. Let $\gamma$ be the minimum of $|B|$ and $|T| - |B|$ and $\delta$ be $|C|$. Then the flat plumbing basket number of $L$ is bounded by $3(E(\Gamma) - V(\Gamma)) + 2(\gamma - \delta) + 3$, i.e.,

$$fpbk(L) \leq 3(E(\Gamma) - V(\Gamma)) + 2(\gamma - \delta) + 3.$$ 

**Proof.** We claim that there exists a coedge $e \in \Gamma - T$ such that the half twisted band represented by $e$ can be depumbed. We induct on the number of coedges in $\Gamma - T$ and the number of Seifert circles in lexicographic order. If there is no coedge in $\Gamma - T$ or there is only one Seifert circle, then the Seifert surface $F$ is a disc $D_2$ so it is a flat plumbing basket surface. Suppose there exist at least two Seifert circles and at least one coedge in $\Gamma - T$. We divide cases depend on the existence of two concentric Seifert circles. First we remark that the induced graph $\Gamma$ may not be planar since there might exist some vertices which are corresponding to Seifert circles which are concentric as in the canonical Seifert surface of a closed braid. If there exist at least two Seifert circles which are concentric, there exists at least one edge in $T$ and one edge $e$ in $\Gamma - T$ between two vertices which are corresponding two adjacent Seifert circles. Then the half twisted band represented by $e$ can be depumbed as we have proven for Theorem 3.6. Suppose that there exist no Seifert circles which are concentric, then
the induced graph $\Gamma$ is planar. Since $\Gamma$ is finite, there exists a coedge $e \in \Gamma - T$ such that the disc bounded by the $P_e$, the path joining the both end vertices of $e$ in $T$ and $e$ does not contain any other coedges of $\Gamma - T$. Our next claim is that the interior of the disc bounded by the $P_e$ and $e$ does not contain any other edges of $\Gamma$. Since the interior of the disc bounded by the $P_e$ and $e$ does not contain any coedges of $\Gamma - T$, if there exist an edge in the interior of the disc bounded by the $P_e$ and $e$, it must be the edge in $T$. But there does not exist any coedge and it is a part of tree $T$, there exists a vertex of valency 1. But it can be removed by untwisting, which reduces the number of Seifert circles. By the induction hypothesis, we can see that the second claim is true. Finally, we can see that coedge $e \in \Gamma - T$ such that the half twisted band represented by $e$ can be depumbed where the 3-ball $D_\alpha$ can be chosen along $P_e$ as depicted in Figure 8.

Now, we simply count the number of flat plumbings required for each edges in $\Gamma - T$. By using Reidemeister moves, we added two edges and this will contribute $2\gamma$ flat plumbings. For edges in $C$, their cardinality is $\delta$ and each contributes a flat plumbing. For the rest coedges, it counts $E(\Gamma) - (V(\Gamma) - 1) - \delta$ and they require three flat plumbings. By adding them all, we get $3(E(\Gamma) - V(\Gamma)) + 2(\gamma - \delta) + 3$. It completes the proof of the theorem.

\[ \square \]

Since $E(\Gamma), V(\Gamma)$ are prefixed, the minimum can be attained for the minimality of $\gamma - \delta$ by choosing a suitable spanning tree and a vertex. We may attain a better upper bound for Theorem 3.9 if we use the method in the proof of Theorem 3.6 by considering all possible pairs of (spanning tree $T$, coloring $\kappa$ on $T$) obtained from Theorem 2.1 as illustrated in the following example.
Figure 9. (i) A link diagram whose induced diagram is given in Figure 2 (i), (ii) a signed bipartite graph $\Gamma$ with a fixed vertex $v$, (iii)–(viii) all possible spanning tree $T_i$ for which Theorem 2.1 holds for a fixed root $v$, $i = 1, 2, \ldots, 6$ (ix) a spanning tree $\overline{T}$ for which Theorem 2.1 holds for a fixed root $v_2$.

Example 3.10. For the link as depicted in Figure 9 (i) for which the induced graph is given in Figure 2 (i), if we fix the vertex $v$ and the depth coloring with a root $v$, the upperbound attained in Theorem 3.4 is 11. If we consider all vertices of $\Gamma$ and spanning trees and coloring, the upperbound attained in Theorem 3.9 is 9 as depicted in Figure 9 (ix).

Proof. If we fix the vertex $v$ and the depth coloring with a root $v$, there are six possible spanning tree for which Theorem 2.1 holds as illustrated in Figure 9 (iii–viii). From spanning tree $T_1, T_2$, we have $\gamma - \delta = 1$. For the others, we have
γ − δ = 2. Therefore, the upperbound for the flat plumbing basket number attained in Theorem 3.9 for the fixed vertex v is 11. By considering all vertices in Γ, we find that the upperbound for the flat plumbing basket number attained in Theorem 3.9 is 9 where we find one of such a spanning tree T for which Theorem 2.1 holds and attains the minimum γ − δ = 0 as shown in Figure 9 (ix). Let us remark that we ignore the edge between v and v₁ since it can simply be removed by Reidemeister move I. □

4. Relations between plumbing numbers

First we will look at the relations between three plumbing numbers. Fundamental inequalities regarding these three basket numbers are

\[ bk(L) \leq fp(L) \leq f pbk(L). \]

For the second inequality, there exists a link that the inequality is proper [6]. For the first inequality, we consider \( L_{2n} \), the closure of \( (σ₁)^{2n} \in B_2 \). For \( n \neq 0 \), it is a nontrivial link and it can be obtained by plumbing one annulus \( A_{2n} \), so its basket number is 1. On the other hand, it is fairly easy to see that any link whose flat plumbing number is less than 3 is trivial. Thus, the first equality is proper for \( L_{2n} \).

Then naturally we can ask the following question.

**Question 4.1.** Are there links for which the difference of plumbing numbers is arbitrary large?

Next, we compare these plumbing numbers with genus and canonical genus of a link. Let us recall the definitions first. The genus of a link \( L \) is the minimal genus among all Seifert surfaces of \( L \), denoted by \( g(L) \). A Seifert surface \( F \) of \( L \) with the minimal genus \( g(L) \) is called a minimal genus Seifert surface of \( L \). A Seifert surface of \( L \) is said to be canonical if it is obtained from a diagram of \( L \) by applying Seifert’s algorithm. Then the minimal genus among all canonical Seifert surfaces of \( L \) is called the canonical genus of \( L \), denoted by \( g_c(L) \). A Seifert surface \( F \) of \( L \) is said to be free if the fundamental group of the complement of \( F, \pi_1(\mathbb{S}^3 - F) \) is a free group. Then the minimal genus among all free Seifert surfaces of \( L \) is called the free genus for \( L \), denoted by \( g_f(L) \). Since any canonical Seifert surface is free, we have the following inequalities.

\[ g(L) \leq g_f(L) \leq g_c(L). \]

There are many interesting results about the above inequalities [3, 5, 11, 12, 14, 17].

By applying Theorem 3.4, we obtain the following corollary.

**Corollary 4.2.** Let \( F \) be a canonical Seifert surface of a link \( L \) with \( s(F) \) Seifert circles and \( c(F) \) half twisted bands and let \( l \) be the number of components of \( L \). Let \( Γ \) be the induced graph of a Seifert surface \( F \). Let \( T \) be a spanning tree with the minimum \( β \) as described in Theorem 3.4. Then,

\[ 2g(L) + l - 1 \leq fp(L) \leq 2g_c(L) + 2β + l - 1. \]
Proof. Let \( V, E \) and \( F \) be the numbers of vertices, edges and faces, respectively in a minimal canonical embedding of \( \Gamma \). From Theorem 3.4, we find \( fp(F) \leq \gamma(F) - s(F) + 1 + 2\beta = E - V + 1 + 2\beta = (E - V - F) + F + 1 + 2\beta = 2g_c(L) - 2 + F + 1 + 2\beta = 2g_c(L) + 2\beta + l - 1 \). Since a flat plumbing surface is also a Seifert surface of \( L \), the first inequality follows from the definition of the genus of \( L \). \( \square \)

By applying Theorem 3.9 we obtain the following corollary using a similar argument used in Corollary 4.2.

**Corollary 4.3.** Let \( \Gamma \) be an induced graph of canonical Seifert surface \( \mathcal{F} \) of a link \( L \) with the vertex set \( V(\Gamma) \) and the edge set \( E(\Gamma) \) and let \( l \) be the number of components of \( L \). Let \( T \) be a spanning tree and \( \kappa \) a coloring on \( T \) chosen in Theorem 2.1. Let \( \gamma \) be the minimum of \( |\mathcal{B}| \) and \( |T| - |\mathcal{B}| \) and \( \delta \) be \( |\mathcal{C}| \). Then,

\[
2g(L) + l - 1 \leq fpbk(L) \leq 6g_c(L) + 2(\gamma - \delta) + 3l - 3.
\]

**Acknowledgments**

The author would like to thank Tsuyoshi Kobayashi, Jaeun Lee, Young Soo Kwon and Myoungsoo Seo for helpful discussion and Lee Rudolph for his comments. A critical error in the first version was pointed by Tsuyoshi Kobayashi. Author has learnt critical facts about the induced graphs from Jaeun Lee and Young Soo Kwon. The \TeX{} macro package \texttt{PSTricks} [20] was essential for typesetting the equations and figures.

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\textsc{Department of Mathematics, Kyonggi University, Suwon, 443-760 Korea}

\textit{E-mail address: dongseok@kgu.ac.kr}