Structural features of photogravitational celestial mechanics

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Abstract. The features of mathematical models of controlled motion of a spacecraft with a solar sail, which take into account the translational and rotational motion of all parts of the structure, are considered. Equations for motion in a photogravitational field are used on the basis of the two-body problem, taking into account perturbations, or a more general model of the bounded three-body problem. The gravitational field is supplemented by the field of light pressure forces, which makes it possible to simulate real dynamics processes. When approximating the system of equations of perturbed motion for different orbits and parameters of the main bodies, control capabilities and stability conditions of motion are obtained at the specified orbits, as well as in the vicinity of libration points. Stability of orientation or rotation relative to the center of mass is provided by the moment of light pressure forces on the elements of the sail system or additional actions of special engines.

1. Introduction

The possibility of motion in space under the influence of light pressure and the basis of the theory of motion of spacecraft with a solar sail was put forward by Friedrich Arturovich Zander in 1924 [1, 2]. He considered the design of solar sails for such devices. The acceleration that gives the spacecraft the flow of sunlight depends on the ratio of the area of the sail to the mass of the entire structure. Opening the sail in orbit will cause the light pressure to partially compensate for the attraction of the Sun. If the gravitational field of attraction is supplemented by the field of light pressure forces, then for problems of celestial mechanics we can talk about a mathematical model of the photogravitational field [3, 4, 5].

In this simulation, it is sufficient to limit ourselves to taking into account two forces: the gravitational interaction of several bodies \(F_g(r_1, r_2, \ldots, r_k)\) and the light pressure \(F_p(r_1, r_2, n, u)\) on the body of the solar radiation flux. The resulting force is determined by the position of the spacecraft in space, as well as the orientation of all elements of the solar sail relative to the attracting centers and radiation centers. The gravitational forces can be modeled as the central field in the two-body problem; the geopotential when the spacecraft moves in the vicinity of the Earth; the two attracting centers of the three-body problem that affect the motion of the spacecraft (for example: Earth and Moon, Sun and planet, a double star).

The use of a solar sail will provide the spacecraft with a low-thrust engine, which has an almost unlimited supply of fuel. However, the sail has a drawback: unlike jet engines, we can...
not use its thrust in any direction with the same efficiency. It is necessary to orient the sail specifically to get the desired change in the parameters of the orbit in outer space.

Problems of motion control with a solar sail lead to the study of mathematical models of dynamics in the photogravitational fields of orbital motion and problems of controlling the rotation of the entire spacecraft complex relative to the center of mass [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. There are many projects of possible flight with a solar sail, taking into account the restrictions or additional conditions. Some of them have been successfully implemented.

Spacecraft motion control is most often considered divided into two parts: the problem of finding an algorithm for optimal maneuvering during the transition between specified orbits, and the problem of rotation relative to the center of mass to ensure the desired orientation of the hull together with the equipment, solar panels and jet engines [14, 15, 16, 17, 18, 19, 20, 21]. The use of light pressure results in a combination of tasks.

The influence of the main forces determines not only the orbital and rotational motion, but can also be used to implement control for interplanetary flight and maneuvering within the scope of another planet or to stabilize the orientation of the Spacecraft, which will shape the direction and thrust of the engine by solar energy with an unlimited supply (solar sail with a good mirror surface to reflect light flux) which depends on the distance to the source area $S$ and the shape of the surface elements sail Spacecraft [3, 15].

The classical criteria for optimal control problems in terms of energy consumption (fuel consumption) or the time of implementation of the transition process (speed) will not be relevant in the case when the main control element is only the light pressure. The energy of the light pressure forces can flow almost unlimited, and the value depends on the correct position of the solar sail elements. The power consumption will be required to control the rotation or turns under the action of moments of forces. This can be compensated for by using additional devices.

The main factor is taking into account the current position in the orbit to control the spacecraft rotations relative to the selected axes. At the same time, there may be new requirements for the time of movement to reach a given position in space and a convenient final orbit, or the conditions of special circumstances. This leads to new models and the construction of control laws [12, 13, 14, 15, 16] for the orbital and rotational motion of the structure, taking into account the acting forces.

2. The main problems of photogravitational celestial mechanics

Within the framework of the general classification of photogravitational problems, the specifics of controlling a spacecraft and solar sails are considered, which takes into account the translational and rotational motion of the entire structure, as well as the influence of light pressure on objects from radiating bodies (the Sun or binary stars) as a perturbing factor in the motion of bodies that depend on the position and orientation of the sail system, which allows us to model real processes in problems of celestial mechanics [2, 3, 4, 6, 7, 8, 9, 10].

(i) Heliocentric movements in the framework of the two-body problem or the study of orbital transitions taking into account light pressure, including heliocentric transitions from the Earth’s orbit to the Sun, planets, asteroids or comets. The control of the spacecraft motion taking into account the light pressure (orbital dynamics or transitions with low thrust of the pressure forces on the solar sail) is investigated. Controlling the position of the solar sail allows you to determine the optimal trajectories based on the cost or time of maneuvering.

(ii) Geocentric motions within the limited problem of three bodies in the Sun-Earth-SC system. The use of a sail for geocentric acceleration and departure to the Moon’s orbit or exit from the Earth’s sphere of action, the study of the stability and stabilization of the spacecraft position in the vicinity of libration points, the orbital correction or formation
of geosynchronous latitudinal orbits using solar sails, the use of a Solar sail as a spatial reflector to illuminate the polar regions on the Earth’s surface.

(iii) Motion within a limited problem of three bodies in a system with two radiating stars (a double star and a spacecraft). Nonstationary modifications of photogravitational two-or three-dimensional problems with variable physical parameters. Investigation of stability and the possibility of stabilization of the position in the vicinity of libration points, including for a star pair.

(iv) Rotational movements of the spacecraft when taking into account the pressure of solar radiation. Control of orientation and control of position in space under the influence of the moment of light pressure forces, taking into account changes in the shape and size of the sail or when using the relative movements of the elements.

Preliminary examples of kinematic constraints on the trajectory of a spacecraft with a solar sail to the near-solar regions or in the vicinity of the Earth are proposed in the form of equations of well-known classical curves: asteroid, cardioid or Viviani curve. They describe flights to the near-polar regions of the Sun to observe physics while flying over its pole. The comparison of different ”non-Keplerian” orbits was made by flight time and energy. The last criterion refers to the choice of such control of the angle of setting the sail and the orientation of the spacecraft, in which the energy costs for turns are minimal both in terms of the number of controlling turns, and in terms of the magnitude of their angles [13, 14, 15, 16, 17, 18, 19].

When designing geocentric orbits near the Earth or for placing spacecraft at the libration points of the system, a more general model of the photogravitational plane bounded three-body problem is required. There are known conditions and examples of the existence of stable libration points in problems with one or two radiating centers [7, 8, 9, 10, 11], which can form special orbits.

In the spatial problem with two radiating principal bodies (a double star), there are triangular libration points that are formally stable for almost all parameter values from the stability domain in the linear approximation. Taking into account the influence of light pressure from radiating bodies (star, Sun) as one of the main factors allows us to adequately model the real picture of the dynamics and evolution of particles in the field of binary star systems [10]. For the first time, the libration points in the photogravitational problem of three bodies with one radiating mass were studied by Radzievsky V.V. [8], and a nonlinear analysis of the stability of triangular libration points of the bounded photogravitational problem of three bodies with two radiating masses is given in [10].

3. The model of the equations of motion and controls

The heliocentric motion of the Spacecraft with solar sail in flight to the Sun, the planets, asteroids or comets; and to create a special orbit motion in the vicinity of the Sun given the force of light pressure leads to equations of motion in the form of

$$\frac{d^2x_i}{dt^2} + \frac{\mu}{r^3}x_i = \frac{\partial U}{\partial x_i} + f_i, \quad i = 1, 2, 3,$$

where we have used the notation: \(x_i\) — Cartesian coordinates SC, an \(r\) — module is a radius-vector, \(\mu\) is the gravitational parameter of the Central body, the function \(U\) is determined by the perturbation of the potential strength, and function \(f_i(x, u(t, x))\) — non-potential acceleration and the contribution of control \(u(t, x)\), including the action of light pressure or jet engines on the active sites, the decomposition of the axes of the orbital coordinate system [21]. In this case, the forces of light pressure on the elements of the sail system and the moments relative to the
center of mass of the system determine the vector values:

\[ \mathbf{F} = \sum_i F_i = \sum_i k_i \mathbf{S}_i \frac{b(\theta_i)}{r^2} \mathbf{n}_i(\theta_i), \quad \mathbf{M} = \sum_i \rho_i \times \mathbf{F}_i(r, \theta_i). \]

Equations (1) can be rewritten as a system of first-order differential equations in normal form

\[ \frac{dx_i}{dt} = x_{i+3}, \quad \frac{dx_{i+3}}{dt} = -\frac{\mu}{r^3} x_i + \frac{\partial U}{\partial x_i} + f_i, \quad i = 1, 2, 3, \]

where \( f_i \) are the projections of the acceleration vector caused by the action of all perturbing forces. The equations can also be written as a system of Lagrange equations of the second kind

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = -\frac{\partial H}{\partial \dot{q}_i} + Q_i, \quad i = 1, 2, 3, \quad (2) \]

for known functions of kinetic and potential energy, as well as non-potential forces, where the generalized coordinates contain Cartesian coordinates or a set of parameters that determine the position of the selected system of bodies. It is possible to transform equations (2) to the form of a system of canonical Hamilton equations

\[ \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad i = 1, 2, 3, \quad (3) \]

The kinetic energy is determined by the speed of movement of the center of mass of the body, in which all its mass is considered to be concentrated, and the energy of relative rotation. For rotational motion relative to the center of mass, the kinetic energy of the body and the kinetic moment \( \mathbf{K} = \mathbf{I} \omega \) are determined by the moments of inertia and the angular velocity

\[ T = \frac{1}{2} \left( I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 \right), \quad K^2 = \left( I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2 \right). \]

The Euler equations of rotation of a body can generally be written as

\[ \mathbf{I} \dot{\omega} + \omega \times \mathbf{I} \omega = \mathbf{M}(u). \]

In the case of possible oscillations [14, 22], while maintaining the orientation of one of the main axes of the orthogonal plane of motion with an angular orbital velocity \( \omega_0 \), the change in the kinetic moment, taking into account the moment of forces, can be investigated using the equation:

\[ I_z \dot{\omega}_z = -k \omega_0^2 \left( I_x - I_y \right) \sin \varphi + M_z \left( t, u(t, \varphi) \right). \]

If we introduce the control moments \( u_i \) \( (i = 4, 5, 6) \), relative to the main axes of inertia, and use the projections of the kinetic moment \( x_7 = I_x \omega_x, x_8 = I_y \omega_y, x_9 = I_z \omega_z \), as unknowns, then the equations of motion for the considered set of generalized coordinates can be represented in normal form:

\[ \dot{x}_i = x_{i+3}, \quad \dot{x}_{i+3} = -\frac{\mu x_i}{r^3} + f_i(t, x_i, u_i), \quad i = 1, 2, 3, \]

\[ \dot{x}_7 = \beta_1 x_8 x_9 + u_4, \quad \dot{x}_8 = \beta_2 x_9 x_7 + u_5, \quad \dot{x}_9 = \beta_3 x_8 x_7 + u_6, \quad \beta_1 + \beta_2 + \beta_3 = 0. \]

The contribution of the light pressure is determined by the angle of deviation of the normal vector \( \mathbf{n} \) from the direction of the flow of sunlight \( \mathbf{e} \). By turning the sail, we get the opportunity
to change the direction of the thrust vector and control the spacecraft, but this also changes the magnitude.

Motion in the central gravitational field has a solution that, in the absence of disturbing forces, is determined by the initial values of the radius vector, the velocity vector, and the gravitational parameter of the central body. This defines the constant Kepler elements \( K = (a, e, i, \Omega, \omega, M_0) \), which allow us to calculate Cartesian coordinates and velocities for unperturbed motion at an arbitrary moment in time.

To describe the motion taking into account perturbations, we can use \( K(t) = (a, e, i, \Omega, \omega, M_0) \) — the osculating elements, when the elements of the spacecraft orbit are functions of time. You can use differential equations, where the right-hand sides are determined by the current values of the elements and the projections of the perturbing accelerations on the axis of the orbital coordinate system.

If we consider the motion in the vicinity of the Earth, then the directions of the two main forces do not coincide, but we can assume in the first approximation that the luminous flux determines an almost constant pressure force collinear to the line that passes through the two main bodies of the system. The position of the sail plane allows you to form the direction of the control force \( P(t, u) \) to change the trajectory or stabilize in the vicinity of special libration points. Then we can use the equations of motion in the framework of the limited circular problem of three bodies

\[
\ddot{x} - 2\eta \dot{y} = \frac{\partial U}{\partial x} + P_1, \quad \ddot{y} + 2\eta \dot{x} = \frac{\partial U}{\partial y} + P_2, \quad \ddot{z} = \frac{\partial U}{\partial z} + P_3.
\]

The position of the center SC an infinitesimal mass relative to the main bodies (Earth and Sun) of mass \( \mu < 1 \) and \( (1 - \mu) \) in a rotating barycentric Cartesian coordinate system is determined by the radius vectors \( r = (x, y, z), r_1 = (x + \mu, y, z), r_2 = (x - 1 + \mu, y, z) \).

The force function of the gravitational interaction has the form

\[
U = \frac{1}{2} \eta^2 (x^2 y^2) + \kappa^2 \left( \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \right).
\]

Here is the constant angular velocity of rotation of the coordinate system relative to the center of mass of the system together with the main bodies. Thus, there is an additional simplification when \( P_1 = \text{const}, P_2 = P_3 = 0 \). When moving in the vicinity of the Earth, the intermediate orbits of the Hill problem can be used for an approximate solution [11]. In particular, when studying the stability of motion in the vicinity of libration points.

Similar equations and properties can be used to study spacecraft flights when it is necessary to take into account the light pressure from two sources in a binary star system. Let us consider the differential equations of the three-body problem after conversion to the canonical form (3) and obtain a decomposition of the Hamilton function \( H(q, p) \) with respect to generalized coordinates \( q_i \) and impulses \( p_i \), which in our case is represented as a series of degrees of perturbation in the vicinity of the point under consideration [10], then we obtain

\[
H = H_2 + H_3 + H_4 + \cdots.
\]

Here \( H_m \) are homogeneous polynomials of degree \( m \) \((m = 2, 3, 4, \ldots)\) with respect to \( q_i \) and \( p_i \). We assume that there are no resonances of the 3-rd and 4-th orders in the system. After applying the Birkhoff’s transform and limiting the expansion to the 4-th order inclusive, the Hamilton function can be written as

\[
H^* = \omega_1 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2, \quad 2r_i = q_i^2 + p_i^2, \quad i = 1, 2.
\]
According to the Arnold-Moser theorem [7, 10], when the conditions are simultaneously met

\[ k_1 \omega_1 + k_2 \omega_2 \neq 0, \quad C(\omega_1, \omega_2) = c_{20} \omega_2^2 + c_{11} \omega_1 \omega_2 + c_{02} \omega_1^2 \neq 0, \]

where \( k_1, k_2 \) are integers, determine the order of the resonance, and \( c_{ij} \) are the coefficients of the normal form, the Lyapunov stability of the original system is preserved everywhere. The exception is the sets of points corresponding to resonances of the 3-rd \((\omega_1 = 2 \omega_2)\) and 4-th \((\omega_1 = 3 \omega_2)\) orders. At resonance \( \omega_1 = 2 \omega_2 \), the normalized Hamiltonian will take the form

\[ H = 2\omega_1 r_1 - \omega_2 r_2 + A(\omega_1, \omega_2) r_2 \sqrt{r_1} \sin (\varphi_1 + \varphi_2) + O((r_1 + r_2)^2), \]

where, in the flat three-body problem, the expression \( A(\omega_1, \omega_2) \) for positive \( \lambda_1, \lambda_2 \) does not vanish anywhere. It follows that the triangular libration points are Lyapunov-stable everywhere, with the exception of the set of points for which the resonance is realized.

In the spatial problem with two radiating masses, the triangular libration points are formally stable for almost all parameter values from the stability domain in the linear approximation. The exceptions are, in addition to the values of the parameters corresponding to the studied resonances, perhaps those values from the linear stability region at which resonances above the fourth order are realized.

4. Control and stability modeling
The rotational movements of the hull SC well as the solar sail systems under the action of the solar radiation pressure forces determine the control capabilities and orbital dynamics for a given trajectory, orientation and stabilization in space.

Taking into account the light pressure on the sail results in stability conditions that can be used to control the movement. The effect of perturbations can be compensated by changing the size or reflective properties of the elements of SC sail, as well as their relative position. This creates additional moments of force that can be used as controls.

Of particular interest is the case of placing the spacecraft in the vicinity of the Euler-Lagrange libration points, where the small forces of the active perturbations will determine the nature of the motion and stability. There are analytical and numerical methods for studying and analyzing the basic properties of equations that allow us to obtain exact or approximate solutions to a set of conditions.

5. The inverse problem of dynamics photogravitational
Considering the inverse problem for finding the forces for a known or given motion in the case of two interacting bodies, we find the control accelerations that can realize the trajectory in the central field

\[ f_i = \frac{d^2 x_i}{dt^2} + \frac{\mu}{r^3} x_i = u_i(t), \quad i = 1, 2, 3, \]

when the parametric functions \( x_i \) are pre-defined for \( t \in [0, T] \). Assuming that the required control accelerations are created by the solar sail [5], we write them as

\[ u_1 = -\frac{q}{r^2} \cos \gamma_1 \cos \gamma_2, \quad u_2 = \frac{q}{r^2} \sin \gamma_1 \cos \gamma_2, \quad u_3 = \frac{q}{r^2} \sin \gamma_2, \]

where \( \gamma_1 \) and \( \gamma_2 \) are angles of the three-dimensional orientation of the normal vector to the shadow side mirror on both sides of the sail, and the dimensionless factor \( k \) depends on the pressure of solar radiation on the surface of the sails SC. From here we get the formulas for finding the angles and building the control.
6. Plotting trajectories with the selected control

As examples, some results of solving the inverse dynamics problem are obtained. A simulation of the trajectories of SC with a controlled solar sail to reach the helio-polar regions or fly over the North and South poles of the Sun and return to near-Earth orbit is presented [12, 14]. Flights over the plane of the ecliptic (over the poles of the Sun) along given curves were considered. For example Astroid:

\[ x(t) = r_0 \cos^3 t, \quad y(t) = 0, \quad z(t) = h_2 \sin^3 t. \]

Figures 1, 2, and 3 show the Astroid and control parameters. Here \( r_0 \) — the radius of the Earth’s orbit, the spacecraft starts from a point \((r_0, 0, 0)\), \( h_2 \leq r_0 \) expresses the scale along the axis of the application.

Solving the inverse problem of dynamics when moving along a given trajectory allows you to get an initial approximation for control and evaluate the possibility of implementing a system of elements of the spacecraft sail for the selected model. With this solution, you can also refine the processes of rotation or control of the turns of the entire structure relative to the center of mass.

7. Conclusion

The functional features of the dynamics and the possibility of controlling the angles of inclination of the solar sail elements, which are necessary for flight, when the trajectory is set in an analytical parametric form as a certain law of motion, are considered. Changing the angle of inclination allows you to change the direction and magnitude of the thrust vector. Additionally, you can take into account various characteristics in shape, size, and physical properties, which complicate the process of mathematical modeling using classical analytical dynamics tools. This can be provided by a fairly complex controlled process or by a given program for optimal system control.

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