Describing the syntax of programming languages using conjunctive and Boolean grammars

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Abstract

A classical result by Floyd ("On the non-existence of a phrase structure grammar for ALGOL 60", 1962) states that the complete syntax of any sensible programming language cannot be described by the ordinary kind of formal grammars (Chomsky’s “context-free”). This paper uses grammars extended with conjunction and negation operators, known as conjunctive grammars and Boolean grammars, to describe the set of well-formed programs in a simple typeless procedural programming language. A complete Boolean grammar, which defines such concepts as declaration of variables and functions before their use, is constructed and explained. Using the Generalized LR parsing algorithm for Boolean grammars, a program can then be parsed in time $O(n^3)$ in its length, while another known algorithm allows subcubic-time parsing. Next, it is shown how to transform this grammar to an unambiguous conjunctive grammar, with square-time parsing. This becomes apparently the first specification of the syntax of a programming language entirely by a computationally feasible formal grammar.

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1 Introduction

Formal grammars emerged in the early days of computer science, when the development of the first programming languages required mathematical specification of syntax. The first and the most basic model, independently introduced by Chomsky [9] (as “phrase structure grammars”, later “context-free grammars”), and in the Algol 60 report [23] (as “metalinguistic formulae”, later “Backus–Naur form”), is universally recognized as the standard model of syntax, the ordinary kind of formal grammars.

Already in the Algol 60 report, a grammar was used to describe the complete lexical composition of the language and the basic elements of its syntax. More complicated syntactical requirements, such as the rules requiring declaration of variables before use, were described in plain words together with the semantics of the language. The hope of inventing a more sophisticated grammar that would describe the entire syntax of Algol 60 was buried by Floyd [12], who proved that no (ordinary) grammar could ensure declaration before use. Floyd considered strings of the following form, here translated from Algol 60 to C.

```c
main()
{
  int x ...
  x = x ...
  x ...
}
```

Such a string is a valid program if and only if \( i = j = k \), for otherwise the assignment statement would refer to an undeclared variable. This very simple fragment of the programming language is an instance of an abstract formal language \( L_1 = \{ a^n b^n c^n \mid n \geq 0 \} \), and since it is known that no ordinary formal grammar (that is, Chomsky’s “context-free grammar”) could be constructed for the latter language, one can infer that no such grammar can describe the whole programming language.

Many other simple syntactic conditions common to all programming languages are also beyond the scope of ordinary formal grammars. For instance, strings of the following form are valid programs if and only if \( i = k \) and \( j = \ell \), because otherwise the number of arguments in one of the calls would not match that in the function prototype.

```c
void f(int, ..., int); void g(int, ..., int); main() { f(0, ..., 0); g(0, ..., 0); }
```

This construct is modelled by the formal language \( L_2 = \{ a^m b^n c^n \mid m, n \geq 0 \} \), which is another standard example of a language not described by any (ordinary) grammar. The next example corresponds to mathing a pair of identifiers over a two-symbol alphabet; these strings, defined for any two identifiers \( w, w' \in \{a, b\}^\ast \), are valid programs if and only if \( w = w' \).

```c
main() { int w; w' = 0; }
```

This is an instance of the formal language \( L_3 = \{ wcw \mid w \in \{a, b\}^\ast \} \), investigated, in particular, by Sokolowski [43] and by Wotschke [49]. Not only it is not described by any (ordinary) grammar, it is not representable as an intersection of finitely many languages described by grammars [49].
Floyd’s [12] result, besides pointing out important limitations of grammars, can also be regarded as a formulation of a problem: that of defining an extended formal grammar model powerful enough to describe those constructs of programming languages that are beyond the power of ordinary grammars. Viewed in this way, the result of Floyd has prompted formal language theorists to search for such a model. Many early attempts ended up with seemingly promising new grammar families, which, at a closer examination, turned out to be too powerful to the point of being useless: this can be said of Chomsky’s [9] “context-sensitive grammars” and of van Wijngaarden’s [48] “two-level grammars”, which can simulate Turing machines. Accordingly, these models define computations, rather than any kind of syntactic structures. Even at their best, any grammars defined in these models are actually parsers that are executed as programs, rather than parser-independent specifications of the syntax; as such, they are beyond any formal analysis.

Besides these unsuccessful attempts, a few generalized grammar models capable of giving meaningful descriptions of syntax were discovered. The first such models were Aho’s indexed grammars [1], Fischer’s macro grammars [11] and tree-adjoining grammars by Joshi et al. [16]. Later, the ideas behind these models led to the more practical multi-component grammars [42, 47], which became a standard model in computational linguistics and receive continued attention. However, even though these models are powerful enough to define the above three abstract languages $L_1, L_2$ and $L_3$, the mere existence of grammars for these languages does not at all imply that a grammar for any programming language can be constructed. On the contrary, a simple extension of Floyd’s example shows that, again, no multi-component grammar can describe the set of well-formed programs. Consider the following strings, in which the same variable is referenced an unbounded number of times.

```plaintext
main() { int x; x = 0; ... x = 0; i >= 1; x = 0; j_1 >= 1; j_k >= 1; k >= 0; }
```

Such a string is a well-formed program if and only if all numbers $j_1, \ldots, j_k$ are equal to $i$. This is an instance of an abstract language $L_4 = \{(a^n b)^k \mid n \geq 0, k \geq 1\}$, which is known to have no multi-component grammar, because its commutative image is not a semilinear set [47 Sect. 4.2].

It is still possible to construct an indexed grammar for $L_4$. However, it is not difficult to present yet another simple case of programming language syntax that is beyond their expressive power. This time, consider strings of the following form.

```plaintext
int f(int, ...); main() { f(f(0, ...), ... , f(0, ...)); }
```

Such a string is a well-formed program if and only if $i = k = j_1 = \ldots = j_k$. This is an instance of an abstract language $L_5 = \{(a^n b)^n \mid n \geq 1\}$, which cannot be described by an indexed grammar [13].

Even though abstract languages, such as these, have occasionally been brought forward to claim practical relevance of new grammar models, they are not representative of the syntactic constructs in programming languages. In order to show that some kind of formal grammars are powerful enough to define those syntactic constructs, the only convincing demonstration would be a complete grammar for some programming language. This paper provides such a demonstration for the families of conjunctive grammars and Boolean grammars, constructing a complete grammar for the set of well-formed programs in a simple model procedural language featuring a single data type, a standard set of flow control statements and nested compound statements with rules for variable scope.
Conjunctive grammars \[24, 34\] extend Chomsky’s “context-free” grammars by allowing a conjunction of any syntactic conditions to be expressed in any rule. Consider that a rule \(\text{A} \rightarrow \text{BC}\) in an ordinary grammar states that if a string \(w\) is representable as \(\text{BC}\)—that is, as \(w = uv\), where \(u\) has the property \(B\) and \(v\) has the property \(C\)—then \(w\) has the property \(A\). In a conjunctive grammar, one can define a rule of the form \(\text{A} \rightarrow \text{BC} \& \text{DE}\), which asserts that every string \(w\) representable both as \(\text{BC}\) (with \(w = uv\)) and at the same time as \(\text{DE}\) (with \(w = xy\)) therefore has the property \(A\). The more general family of Boolean grammars \[29, 34\] further allows negation: a rule \(\text{A} \rightarrow \text{BC} \& \neg\text{DE}\) states that if a string is representable as \(\text{BC}\) (with \(w = uv\)), but is not representable as \(\text{DE}\), then it has the property \(A\). In this way, the properties of a string are defined independently of the context in which it occurs, in the same way as in Chomsky’s “context-free” grammars. These models differ only in the set of allowed Boolean operations, and for that reason, the familiar kind of grammars featuring disjunction only shall be called \textit{ordinary grammars} throughout this paper.

Even though conjunctive grammars have a Chomsky-like definition by term rewriting, whereas Boolean grammars are defined by language equations generalizing those by Ginsburg and Rice \[14\], their true meaning lies in logic. The understanding of ordinary grammars in terms of logical inference can be found, for instance, in Kowalski’s \[19\;\text{Ch. 3}\] textbook. In one of the first papers exploring more powerful logics inspired by grammars, Schuler \[40\] argues that the set of well-formed programs is Algol 60 can be described in his formalism \[41\]. For the modern logical understanding of grammars, the reader is referred to an important paper by Rounds \[37\], who explained different kinds of formal grammars as fragments of the FO(LFP) logic \[15, 46\]. Conjunctive grammars are another such fragment.

Conjunctive and Boolean grammars are important for two reasons. First, they enrich standard inductive definitions of syntax with important logical operations, which extend the expressive power of such definitions in a practically useful way; this shall be further supported in the present paper. At the same time, these grammars have generally the same parsing algorithms as ordinary grammars \[2, 29, 31, 32, 33\], and share the same subcubic upper bound on the time complexity of parsing \[35\], which makes them suitable for implementation. Based on these properties, Stevenson and Cordy \[44\] recently applied Boolean grammars to \textit{agile parsing} in software engineering.

The most practical parsing algorithm for Boolean grammars is a variant of the Generalized LR (GLR) \[31\], which runs in worst-case time \(O(n^4)\) and operates very similarly to the GLR for ordinary grammars \[45\]. Two implementations of Boolean GLR are known \[21, 26\]. In the literature, GLR parsers have sometimes been applied to analyzing programming languages symbol by symbol, without an intermediate layer of lexical analysis \[8, 10, 17\]: this is known as \textit{scannerless parsing} \[38\]. The Boolean grammar for a programming language constructed in this paper follows the same principle, and a Boolean GLR parser for the new grammar is not much different from the GLR operating on an ordinary grammar.

The theoretical work on conjunctive and Boolean grammars is reviewed in a recent survey paper \[34\]. This paper includes all the necessary definitions, given in Section \[2\] and illustrates the use of conjunction and negation on two examples presented in Section \[3\].

The model programming language is defined in Section \[4\] and every point of the definition is immediately expressed in the formalism of Boolean grammars. The grammar is designed for scannerless parsing, and for this reason alone, it is bound to be somewhat involved, with definitions of nested syntactic structure occasionally interleaved with simulation of finite automata. In the literature, this was metaphorically described as skipping “water” in search for “islands” \[17\]. In a few places, defining separation into tokens within the grammar results in rather awkward rules. This, however, is a general trait of scannerless parsing and not a fault of Boolean grammars.

A certain improvement to this grammar is presented in Section \[5\] where it is shown how
to eliminate negation and ambiguity in it. In other words, a Boolean grammar is reformulated as an unambiguous conjunctive grammar \[33\], which is a conceptually easier model with a better worst-case parsing complexity—namely, square time in the length of the input. In the literature on scannerless parsing, grammars are typically ambiguous, with attached external disambiguation rules \[7, 8, 38\]. From this perspective, this section demonstrates a new, entirely grammatical approach to disambiguating scannerless parsers for programming languages.

The paper is concluded with two kinds of research directions, suggested in Section 6. First, what kind of parsers could handle this or similar grammars in less than square time? Second, what kind of new grammar models could describe the syntax of programming languages more conveniently?

Note: An earlier form of the Boolean grammar in Section 4 was presented at the AFL 2005 conference held in Dobogókő, Hungary, and published in a local proceedings volume \[30\]. This paper supercedes that preliminary report.

2 Conjunctive and Boolean grammars

2.1 Conjunctive grammars

In ordinary formal grammars, rules specify how substrings are concatenated to each other, and one can define disjunction of syntactic conditions by writing multiple rules for a nonterminal symbol. In conjunctive grammars, this logic is extended to allow conjunction within the same kind of definitions.

Definition 1 (\[24, 34\]). A conjunctive grammar is a quadruple \(G = (\Sigma, N, R, S)\), in which:

- \(\Sigma\) is the alphabet of the language being defined;
- \(N\) is a finite set of nonterminal symbols, each representing a property of strings defined within the grammar;
- \(R\) is a finite set of rules, each of the form
  \[A \rightarrow \alpha_1 \& \ldots \& \alpha_m,\]
  where \(A \in N\), \(m \geq 1\) and \(\alpha_1, \ldots, \alpha_m \in (\Sigma \cup N)^*\);
- \(S \in N\) is a symbol representing the property of being a syntactically well-formed sentence of the language ("the initial symbol").

Each concatenation \(\alpha_i\) in a rule \((1)\) is called a conjunct. If a grammar has a unique conjunct in every rule \((m = 1)\), it is an ordinary grammar (Chomsky’s “context-free”). If every conjunct contains at most one nonterminal symbol \((\alpha_1, \ldots, \alpha_m \in \Sigma^* N \Sigma^* \cup \Sigma^*)\), a grammar is called linear conjunctive. Multiple rules for the same nonterminal symbol may be presented in the usual notation, such as \(A \rightarrow \alpha_1 \& \ldots \& \alpha_m | \beta_1 \& \ldots \& \beta_n\), etc. As in ordinary grammars, the vertical line is essentially disjunction.

Each rule \((1)\) means that any string representable as each concatenation \(\alpha_i\) therefore has the property \(A\). This understanding can be equivalently formalized by term rewriting \[24\] and by language equations \[25\]. Consider the former definition, which extends Chomsky’s definition of ordinary grammars by string rewriting, using terms instead of strings.

Definition 2 (\[24\]). Let \(G = (\Sigma, N, R, S)\) be a conjunctive grammar, and consider terms over concatenation and conjunction, with symbols from \(\Sigma \cup N\) and the empty string \(\varepsilon\) as atomic terms. The relation of one-step rewriting on such terms \((\Rightarrow)\) is defined as follows.
Figure 1: Parse trees in conjunctive grammars: a subtree with root $A \rightarrow \alpha_1 \& \ldots \& \alpha_m$, representing $m$ parses of a substring $a_i \ldots a_j$.

- Using a rule $A \rightarrow \alpha_1 \& \ldots \& \alpha_m \in R$, with $A \in N$, any atomic subterm $A$ of any term may be rewritten by the term on the right-hand side of the rule, enclosed in brackets.

  \[ \ldots A \ldots \Rightarrow \ldots (\alpha_1 \& \ldots \& \alpha_m) \ldots \]

- A conjunction of several identical strings may be rewritten to one such string.

  \[ \ldots (w \& \ldots \& w) \ldots \Rightarrow \ldots w \ldots \quad (w \in \Sigma^*) \]

The language defined by a term $\varphi$ is the set of all strings over $\Sigma$ obtained from it in a finite number of rewriting steps.

\[ L_G(\varphi) = \{ w \mid w \in \Sigma^*, \varphi \Rightarrow^* w \} \]

The language described by the grammar is the language defined by its initial symbol.

\[ L(G) = L_G(S) = \{ w \mid w \in \Sigma^*, S \Rightarrow^* w \} \]

An important property of conjunctive grammars is that every string in $L(G)$ has a corresponding parse tree, which is exactly a proof tree in this logic theory. This is, strictly speaking, an acyclic graph rather than a tree in a mathematical sense. Its leaves (sinks) correspond to the symbols in $w$. Every internal node in this tree is labelled with some rule [1], and has as many ordered children as there are symbols in all conjuncts $\alpha_1, \ldots, \alpha_m$. Subtrees corresponding to different conjuncts in a rule define multiple interpretations of the same substring, and accordingly lead to the same set of leaves, as illustrated in Figure [1]. For succinctness, the label of an internal node can be just a nonterminal symbol, as long as the rule can be deduced from the children’s labels (this is always the case in ordinary grammars, but need not be true in conjunctive grammars).

### 2.2 Boolean grammars

The second family of grammars used in this paper are Boolean grammars, which further extend conjunctive grammars with a negation operator. To be precise, negation can be put over any conjunct in any rule, making it a negative conjunct.

**Definition 3 ([29, 31]).** A Boolean grammar is a quadruple $G = (\Sigma, N, R, S)$, where
• \( \Sigma \) is the alphabet;

• \( N \) is the set of nonterminal symbols;

• \( R \) is a finite set of rules of the form

\[
A \rightarrow \alpha_1 \& \ldots \& \alpha_m \& \neg \beta_1 \& \ldots \& \neg \beta_n
\]

with \( A \in N, m, n \geq 0, m + n \geq 1 \) and \( \alpha_i, \beta_j \in (\Sigma \cup N)^* \);

• \( S \in N \) is the initial symbol.

A conjunctive grammar is then a Boolean grammar, in which every conjunct is positive.

A rule (2) is meant to state that every string representable as each of \( \alpha_1, \ldots, \alpha_m \), but not representable as any of \( \beta_1, \ldots, \beta_n \), therefore has the property \( A \). This intuitive definition is formalized by using language equations, that is, by representing a grammar as a system of equations with formal languages as unknowns, and using a solution of this system as the language defined by the grammar. The definition of Boolean grammars exists in two variants: the simple one, given by the author [29], and the improved definition by Kountouriotis et al. [18] based on the well-founded semantics of negation in logic programming. Even though the simple definition handles some extreme cases of grammars improperly [18], it ultimately defines the same family of languages, and is therefore sufficient in this paper.

For every integer \( \ell \geq 0 \), let \( \Sigma^{\leq \ell} \) denote the set of all strings over \( \Sigma \) of length at most \( \ell \).

With an alphabet \( \Sigma \) fixed, the complement of a language \( L \subseteq \Sigma^* \) is the language \( \overline{L} = \{ w \mid w \in \Sigma^*, w \notin L \} \).

**Definition 4** (Okhotin [29]). Let \( G = (\Sigma, N, R, S) \) be a Boolean grammar, and consider the following system of equations, in which every symbol \( A \in N \) is an unknown language over \( \Sigma \).

\[
A = \bigcup_{A \rightarrow \alpha_1 \& \ldots \& \alpha_m \& \neg \beta_1 \& \ldots \& \neg \beta_n \in R} \left[ \bigcap_{i=1}^m \alpha_i \cap \bigcap_{j=1}^n \beta_j \right]
\]

Each symbol \( B \in N \) used in the right-hand side of any equation is a reference to a variable, and each symbol \( a \in \Sigma \) represents a constant language \( \{ a \} \).

Assume that, for every \( \ell \geq 0 \), there exists a unique vector of languages \((\ldots, L_A,\ldots)_{A \in N}\) with \( L_A \subseteq \Sigma^{\leq \ell} \), for which a substitution of \( L_A \) for \( A \), for all \( A \in N \), turns every equation (3) into an equality modulo intersection with \( \Sigma^{\leq \ell} \). Then the system is said to have a strongly unique solution, and, for every \( A \in N \), the language \( L_G(A) \) is defined as \( L_A \) from the unique solution of this system. The language described by the grammar is \( L(G) = L_G(S) \).

If, for some \( \ell \geq 0 \), the solution modulo \( \Sigma^{\leq \ell} \) is not unique, then the grammar is considered invalid.

Boolean grammars also define parse trees, which, however, reflect only positive components of a parse [29]. An internal node labelled with a rule [2] has children corresponding to the symbols in the positive conjuncts \( \alpha_1, \ldots, \alpha_m \), which represent multiple parses of that substring, exactly like in a conjunctive grammar. Negative conjuncts have no representation in the tree.

### 2.3 Ambiguity

Informally, a grammar is unambiguous if every string can be parsed in a unique way. For ordinary grammars, this is formalized by uniqueness of a parse tree. For conjunctive grammars, the same kind of definition would no longer be useful, because a grammar may define multiple parses for some substrings, only to eliminate those substrings later using intersection: in that
case, the parse tree can still be unique, but in terms of complexity of parsing, such grammars are ambiguous. For Boolean grammars, a parse tree represents only partial information on the parse, and a definition of ambiguity by parse tree uniqueness becomes completely wrong.

These observations led to the following definition.

**Definition 5** (33). A Boolean grammar $G = (\Sigma, N, R, S)$ is unambiguous, if

1. the choice of a rule for every single nonterminal $A$ is unambiguous, in the sense that for every string $w$, there exists at most one rule

   $A \rightarrow \alpha_1 \& \ldots \& \alpha_m \& \neg \beta_1 \& \ldots \& \neg \beta_n,$

   with $w \in L_G(\alpha_1) \cap \ldots \cap L_G(\alpha_m) \cap \overline{L_G(\beta_1)} \cap \ldots \cap \overline{L_G(\beta_n)}$ (in other words, different rules generate disjoint languages), and

2. all concatenations are unambiguous, that is, for every conjunct $X_1 \ldots X_{\ell}$ or $\neg X_1 \ldots X_{\ell}$ that occurs in the grammar, and for every string $w$, there exists at most one partition $w = u_1 \ldots u_{\ell}$ with $u_i \in L_G(X_i)$ for all $i$.

The concatenation unambiguity requirement applies to positive and negative conjuncts alike. For a positive conjunct belonging to some rule, this means that a string that is potentially generated by this rule must be uniquely split according to this conjunct. For a negative conjunct $\neg DE$, this condition requests that a partition of $w \in L_G(DE)$ into $L_G(D) \cdot L_G(E)$ is unique, even though $w$ is not defined by any rule involving this conjunct.

### 3 Language specification with conjunctive and Boolean grammars

All five abstract languages mentioned in the introduction are defined by conjunctive grammars: grammars for the first three, $L_1 = \{ a^nb^n c^n \mid n \geq 0 \}$, $L_2 = \{ a^m b^m a^n c^n \mid m, n \geq 0 \}$ and $L_3 = \{ wcw \mid w \in \{a,b\}^* \}$, appeared in the literature [24, 34], and grammars for the languages $L_4 = \{ (a^n b)^k \mid n \geq 0, k \geq 1 \}$ and $L_5 = \{ (a^n b)^n \mid n \geq 1 \}$ are given in Examples 1–2 below. What is more important, is that these syntactical elements of programming languages can be described in such a way that generalizes further to a grammar for an entire programming language.

A conjunctive grammar for $L_4$ is given below. In this language, all strings of the form $(a^n b)^k$, with $n \geq 0$ and $k \geq 1$, model $k - 1$ references to the same declaration $a^n$. In an ordinary grammar, one can define such strings only for $k = 2$, and a typical grammar will use a rule $C \rightarrow aCa$ to match the number of symbols $a$ in the first block (declaration) to that in the second block (reference). The following grammar arranges the same matching to be done between the first block and every subsequent block.

**Example 1.** The following conjunctive grammar describes the language $L_4 = \{ (a^n b)^k \mid n \geq 0, k \geq 1 \}$.

\[
\begin{align*}
S & \rightarrow SA\&Cb \mid A \\
A & \rightarrow aA \mid b \\
C & \rightarrow aCa \mid B \\
B & \rightarrow BA \mid b
\end{align*}
\]

The rules for $A$ and for $B$ define regular languages $L(A) = a^* b$ and $L(B) = b(a^*)^*$. Then, $C$ defines the language $L(C) = \{ a^n xa^n \mid n \geq 0, x \in b(a^*)^* \}$ representing a single identifier check, carried out in the standard way. The rules for $S$ arrange for all references to be checked by $C$. 


All strings \((a^n b)^k\) are defined by \(S\) inductively on \(k\). If \(k = 1\), then a string \(a^n b\), representing a lone declaration without references, is given by a rule \(S \rightarrow A\). For \(k \geq 2\), the rule \(S \rightarrow SA \& Cb\) imposes two conditions on a string. First, the conjunct \(SA\) declares the string to be a concatenation of a string \((a^n b)^{k-1}\) with any string \(a^m b \in L(A)\); this verifies that all earlier references are correct. The other conjunct \(Cb\) compares the number of symbols \(a\) in the first block \(a^n b\) (the declaration) to that in the last block \(a^m b\) (the last reference). This ensures that the string is actually \((a^n b)^k\), as desired.

The parse tree of the string \(aabaabaab\) is given in Figure 2. The parts emphasized by thick black lines show \(C\) comparing the number of symbols \(a\) in the first block to that in the second block and in the third block. The nodes in the upper part of the tree, labelled with different rules for \(S\), arrange these comparisons to be made.

Alternatively, the structure of comparisons defined in this grammar is illustrated in the informal diagram in Figure 3 (left), where the upper part shows how the rule \(S \rightarrow SA \& Cb\) recursively refers to \(S\) for shorter substrings. The lower part of the diagram illustrates the length equality defined by \(C\). All subsequent grammars in this paper shall be illustrated by similar diagrams.

A grammar for \(L_5\) can be obtained by reusing the grammar for \(L_4\) in the following way.

**Example 2.** The language \(L_5 = \{(a^n b)^n \mid n \geq 1\}\) is an intersection of \(L_4\) with the language \(L' = \{a^n b (a^* b)^n \mid n \geq 1\}\). A grammar for the latter language is not difficult to construct. Then it remains to combine the grammars for \(L_4\) and for \(L'\) with a conjunction operator.
Figure 3: (left) How the grammar in Example 1 defines strings of the form \((a^n b)^k\); (right) How the grammar in Example 3 defines strings of the form \(uczu\).

Even though the language in Example 1 is just one simple abstract language, the grammar construction technique for conjunctive grammars demonstrated in this example is sufficient to arrange all identifier checks in a simple programming language. Another essential element is the ability to compare identifiers over a multiple-symbol alphabet, which is modelled in the next example.

Example 3 (Okhotin [24]). The following conjunctive grammar describes the language \(\{ wcw \mid w \in \{a,b\}^* \}\).

\[
\begin{align*}
S & \rightarrow C \& D \\
C & \rightarrow XCX \mid c \\
D & \rightarrow aA \& aD \mid bB \& bD \mid cE \\
A & \rightarrow XAX \mid cEa \\
B & \rightarrow XBX \mid cEb \\
E & \rightarrow XE \mid \varepsilon \\
X & \rightarrow a \mid b
\end{align*}
\]

First, \(C\) defines the language of all strings \(xxy\), with \(x,y \in \{a,b\}^*\) and \(|x| = |y|\), and thus the conjunction with \(C\) in the rule for \(S\) ensures that the string consists of two parts of equal length separated by a center marker. The other conjunct \(D\) checks that the symbols in corresponding positions are the same. The actual language defined by \(D\) is \(L(D) = \{ uczu \mid u,z \in \{a,b\}^* \}\), and the rules for \(D\) define these strings inductively as follows: a string is in \(L(D)\) if and only if

- either it is in \(c\{a,b\}^*\) (the base case: no symbols to compare),
- or its first symbol is the same as the corresponding symbol on the other side, and the string without its first symbol is in \(L(D)\) (that is, the rest of the symbols in the left part correctly correspond to the symbols in the right part).

The comparison of a single symbol to the corresponding symbol on the right is done by the nonterminals \(A\) and \(B\), which generate the languages \(\{ xcvay \mid x,v,y \in \{a,b\}^*, |x| = |y| \}\) and \(\{ xcvby \mid x,v,y \in \{a,b\}^*, |x| = |y| \}\), respectively, and the above inductive definition is directly expressed in the rules for \(D\), which recursively refer to \(D\) in order to ensure that the same condition holds for the rest of the string.

The grammar in Example 3 essentially relies on having a center marker \(c\) between the first and the second \(w\). The same language without a center marker, \(\{ wv \mid w \in \{a,b\}^* \}\), cannot be described by this method (and possibly cannot be described by any conjunctive grammar at all). This center marker stands for the middle part of the program between the declaration of \(w\) and the reference to \(w\), and as long as the beginning and the end of this middle part is distinguishable from \(w\), the same grammar will work.
The last thing to demonstrate is the use of negation in Boolean grammars. Consider the following variant of Example 1, in which one has to ensure that the first block is not equal to any subsequent block. This is achieved by putting negation over the identifier comparison.

**Example 4.** The following Boolean grammar describes the language \( \{ a^{n_1} b a^{n_2} b \ldots a^{n_k} b \mid k \geq 1, n_1, \ldots, n_k \geq 0, n_2 \neq n_1, \ldots, n_k \neq n_1 \} \).

\[
S \rightarrow SA & \neg C b \mid A \\
A \rightarrow aA \mid b \\
C \rightarrow aCa \mid B \\
B \rightarrow BA \mid b
\]

This grammar can be rewritten without using negation by replacing \( C \) with a new nonterminal symbol that defines identifier inequality.

## 4 A model programming language and its grammar

For a quick introduction into the model programming language used in this paper, consider the following sample program in this language.

```plaintext
average(x, y) { return (x+y)/2; } 

factorial(n) 
{ 
    var i, product;
    i=1;
    product=1;
    while(i<=n) {
        product=product*i;
        i=i+1;
    }
    return product;
}

factorial2(n) 
{ 
    if(n>=2)
        return n*factorial2(n-1);
    else
        return 1;
}

main(arg)
{ 
    return average(factorial(arg), factorial2(arg));
}
```

This is a well-formed program. All functions and variables are defined before their use. The number of arguments in each function call matches that in the definition of that function: for example, `average` is defined with two arguments and called with two arguments. Each variable declaration has its scope of visibility: if `product` were declared and initialized inside the `while` statement, then the statement `return product;` would refer to an undeclared variable. There are no duplicate declarations.

In the rest of this section, a semi-formal definition of the syntax of this language is presented, with every point immediately expressed in the formalism of Boolean grammars.

### 4.1 Alphabet

A program is a finite string over an alphabet \( \Sigma \) that consists of the following 54 characters: 26 letters (\( \text{a, \ldots, z} \)), 10 digits (\( \text{0, \ldots, 9} \)), the space (\( \text{ } \)), and 17 punctuators (\( \text{“(”, “)”, {, }, “,”, “,”} \)).
The Boolean grammar to be constructed defines strings over this alphabet $\Sigma$. The grammar has the following 7 nonterminals that define some subsets of the alphabet referenced in the grammar.

\[
\begin{align*}
\text{anyletter} & \rightarrow \text{"a"} \mid \ldots \mid \text{"z"} \\
\text{anydigit} & \rightarrow \text{"0"} \mid \ldots \mid \text{"9"} \\
\text{anyletterdigit} & \rightarrow \text{anyletter} \mid \text{anydigit} \\
\text{anypunctuator} & \rightarrow \text{"("} \mid \text{"{"} \mid \text{",} \mid \text{","} \mid \text{".} \mid \text{"+"} \mid \text{"-"} \mid \text{"*"} \mid \text{"/"} \mid \text{"&"} \mid \text{"|"} \mid \text{"!"} \mid \text{"="} \mid \text{"<"} \mid \text{">"} \mid \text{"%"} \\
\text{anysymbol} & \rightarrow \text{"["} \mid \text{anyletter} \mid \text{anydigit} \mid \text{anypunctuator} \\
\text{anypunctuatorexceptrightpar} & \rightarrow \text{"("} \mid \text{"{"} \mid \text{",} \mid \text{","} \mid \text{".} \mid \text{"+"} \mid \text{"-"} \mid \text{"*"} \mid \text{"/"} \mid \text{"&"} \mid \text{"|"} \mid \text{"!"} \mid \text{"="} \mid \text{"<"} \mid \text{">"} \mid \text{"%"} \\
\text{anysymbol} & \rightarrow \text{anyletterdigit} \\
\text{anystring} & \rightarrow \text{anysymbol} \mid \varepsilon \\
\text{anylettermatches} & \rightarrow \text{anysymbol} \mid \text{anysymbol} \mid \text{anystring} \mid \text{anysymbol} \\
\text{safeendingstring} & \rightarrow \text{anystring} \mid \text{anystring} \mid \varepsilon
\end{align*}
\]

There are also the following three nonterminal symbols for simple regular sets of strings over $\Sigma$.

\[
\begin{align*}
\text{anystring} & \rightarrow \text{anysymbol anysymbol} \mid \varepsilon \\
\text{anylettermatches} & \rightarrow \text{anysymbol anysymbol} \mid \varepsilon \\
\text{safeendingstring} & \rightarrow \text{anystring anysymbol anysymbol anysymbol} \mid \varepsilon
\end{align*}
\]

The last nonterminal, safeendingstring, denotes a string that can be directly followed by an identifier or a keyword. It shall be used to ensure that names are never erroneously split in two, so that, for instance, an expression \text{varnish}; is never mistaken for a declaration \text{var nish;}.

In a usual parsing framework, this kind of conditions are be ensured by the “longest match” principle, according to which, a longer sequence of symbols should always be preferred to a shorter one. However, there is no way to attach this principle to a grammar, and the correct splitting of a program into tokens has to be expressed entirely within grammar rules.

### 4.2 Lexical conventions

The character stream is separated into tokens from left to right, with possible whitespace characters between them. Each time the longest possible sequence of characters that forms a token is consumed, or a whitespace character is discarded. The handling of whitespace is facilitated by a nonterminal \text{ws} representing a possibly empty sequence of whitespace characters.

\[
\begin{align*}
\text{ws} & \rightarrow \text{ws} \mid \varepsilon
\end{align*}
\]

This programming language has 28 token types. In the grammar, each of them is represented by a nonterminal symbol enclosed in frame (\text{[ ]}, etc.), which defines the set of all valid tokens of this type, possibly followed by whitespace characters.

First, there are 5 keywords: \text{var}, \text{if}, \text{else}, \text{while}, \text{return}.

\[
\begin{align*}
\text{Keyword} & \rightarrow \text{var} \mid \text{if} \mid \text{else} \mid \text{while} \mid \text{return} \\
\text{var} & \rightarrow \text{v a r} \text{ws} \\
\text{if} & \rightarrow \text{i f} \text{ws} \\
\text{else} & \rightarrow \text{e l s e} \text{ws} \\
\text{while} & \rightarrow \text{w h i l e} \text{ws} \\
\text{return} & \rightarrow \text{r e t u r n} \text{ws}
\end{align*}
\]
An **identifier** is a finite nonempty sequence of letters and digits that begins with a letter and is not a keyword. In a usual parsing framework, this could be specified by defining a certain order of preference between the rules. Boolean grammars are powerful enough to express this order explicitly, leading to the following description of the set of identifiers exactly according its definition.

\[
\text{ID} \rightarrow \text{Id}_1 \text{ ws} \& \neg \text{Keyword} \\
\text{Id}_1 \rightarrow \text{anyletter } | \text{Id}_1 \text{ anyletter } | \text{Id}_1 \text{ anydigit}
\]

A **number** is a finite nonempty sequence of digits.

\[
\text{NUM} \rightarrow \text{Num}_1 \text{ ws} \\
\text{Num}_1 \rightarrow \text{Num}_1 \text{ anydigit } | \text{anydigit}
\]

There are 21 tokens built from punctuator characters, namely, 13 binary infix operators ("+", 
"−", "∗", "/", "\&", ">", ">", "≤", "≥", "==", ">=", ">!"), 2 unary prefix operators ("−", 
"−!") and figure brackets ("{", "}”). The form of the middle part in

\[
\text{x} \text{mid } \text{y}
\]

is ensured by

\[
\text{x} \text{mid } \text{y}
\]

An **example adapted to handle identifiers and token separation rules in the model programming language shall now be constructed.**

The nonterminal symbol \(C\) defines the set of all strings \(wxwy\), where \(w\) is an identifier, \(x\) is an arbitrarily long middle part of the program between these two identifiers, and \(y\) is a possibly empty sequence of whitespace characters.

\[
C \rightarrow C_{\text{len}} \& C_{\text{iterate}} | C\
C_{\text{len}} \rightarrow \text{anyletterdigit } C_{\text{len}} \text{ anyletterdigit } | \text{anyletterdigit } C_{\text{mid}} \text{ anyletterdigit}
\]

The form of the middle part \(x\) in \(wxwy\) is ensured by \(C_{\text{mid}}\), which verifies that \(w\) is the longest

prefix and the longest suffix of \(wxw\) that is formed of letters and digits.

\[
C_{\text{mid}} \rightarrow \omega | \text{anypunctuator anystring anypunctuator} | \omega \text{ anystring } \omega \\
C_{\text{mid}} \rightarrow \omega \text{ anystring anypunctuator} | \text{anypunctuator anystring } \omega | \text{anypunctuator}
\]

In the definition of \(C_{\text{iterate}}\), each nonterminal symbol \(C_\sigma\), with \(\sigma \in \{a, \ldots, z, 0, \ldots, 9\}\), specializes in comparing one particular character (cf. \(A\) and \(B\) in Example 3, which could be called \(C_a\) and \(C_b\)).

\[
C_{\text{iterate}} \rightarrow C_\sigma \sigma \& C_{\text{iterate}} \sigma \quad \text{(for all } \sigma \in \{a, \ldots, z, 0, \ldots, 9\}\} \\
C_{\text{iterate}} \rightarrow \text{anyletterdigits } C_{\text{mid}} \\
C_\sigma \rightarrow \text{anyletterdigit } C_\sigma \text{ anyletterdigit} \quad \text{(for all } \sigma \in \{a, \ldots, z, 0, \ldots, 9\}\} \\
C_\sigma \rightarrow \sigma \text{ anyletterdigits } C_{\text{mid}} \quad \text{(for all } \sigma \in \{a, \ldots, z, 0, \ldots, 9\}\)}
As the first application of identifier comparison, consider describing a list of pairwise unique identifiers. In the sample program given in the beginning of this section, there is a function header `average(x, y)` and a variable declaration statement `var i, prod;`. Each of them contains a list of identifiers being declared, and no identifier may appear on the list twice.

Such lists are described by a nonterminal symbol $Z_{\text{distinct-id}}$ in a way that reminds of Example 1, although this time, instead of comparing all elements to the first element, the grammar compares every pair of identifiers on the list, ensuring that they are all distinct. This is done by a two-level iteration: first, $Z_{\text{distinct-id}}$ sets up a comparison of every element to all previous elements, to be carried out by the nonterminal `no-multiple-declarations`.

$$Z_{\text{distinct-id}} \rightarrow Z_{\text{distinct-id}} \ ID \ & \ no\text{-}multiple\text{-}declarations \ | \ ID$$

On the second level of iteration, `no-multiple-declarations`, applied to a prefix of the list, ensures that no previous element coincides with the last element of the prefix.

$$no\text{-}multiple\text{-}declarations \rightarrow \ ID \ ID \ no\text{-}multiple\text{-}declarations \ & \ \neg \ C \ | \ ID$$

The actual test for inequality is done by appropriately negating $C$.

### 4.4 Expressions

Arithmetical expressions are formed of identifiers and constant numbers using binary operators, unary operators, brackets, function calls and assignment. The definition of an expression ($E$) is formalized in the grammar in the standard way. No efforts are yet made to make the grammar unambiguous, and the grammar given here allows the operators to be evaluated in any order.

**Basic expressions:** any identifier is an expression, and any number is an expression.

$$E \rightarrow ID \ | \ NUM$$

**Expression enclosed in brackets:** ($e$) is an expression for every expression $e$.

$$E \rightarrow ( E )$$

**Binary operation:** $e_1 \ op \ e_2$ is an expression for every binary operator $op$ and for all expressions $e_1, e_2$.

$$E \rightarrow E \ op \ E \quad (op \in \{+,-,*,/,%,#,\&\&,\|,\land,\land,\rangle,\langle,\rangle,\langle,\rangle,\neq\})$$

**Unary operation:** $op \ e$ is an expression for every unary operator and for all expressions $e$.

$$E \rightarrow \neg \ E \ | \ \neg \ E$$

**Assignment:** $x = e$ is an expression for every identifier $x$ and expression $e$.

$$E \rightarrow ID \ ::= \ E$$

**Function call:** $f (e_1, \ldots, e_k)$ is an expression for all identifiers $f$ and expressions $e_1, \ldots, e_k$, with $k \geq 0$. This is a call to the function $f$ with the arguments $e_1, \ldots, e_k$. For later reference, it is also denoted by a separate nonterminal symbol $E^{\text{call}}$.

$$E \rightarrow E^{\text{call}}$$

$$E^{\text{call}} \rightarrow ID \ (Z_{\text{expr}})$$

The nonterminal $Z_{\text{expr}}$ defines (possibly empty) lists of expressions separated by commas.

$$\begin{align*}
Z_{\text{expr}} & \rightarrow Z_{\text{expr}}^{1+} \ | \ \varepsilon \\
Z_{\text{expr}}^{1+} & \rightarrow Z_{\text{expr}}^{1+} \ | \ E \ | \ E
\end{align*}$$
4.5 Statements

The model programming language has the following six types of statements ($S$). The rules of the grammar defining their form are standard. The rules describing the conditional statement have the dangling else ambiguity; this will be corrected later in Section 5.2.

**Expression-statement:** $e$ ; is a statement for every expression $e$.

$$S \rightarrow E ;$$

**Compound statement:** $\{ s_1 s_2 \ldots s_k \}$ is a statement for all $k \geq 0$ and for all statements $s_1, \ldots, s_k$.

$$S \rightarrow \{ S^* \}$$

$$S^* \rightarrow S^* S \mid \varepsilon$$

**Conditional statement:** $\text{if} (e) s$ and $\text{if} (e) s \text{ else } s'$ are statements for every expression $e$ and for all statements $s, s'$.

$$S \rightarrow \text{if} (E) S$$

$$S \rightarrow \text{if} (E) S \text{ else } S$$

**Iteration statement:** $\text{while} (e) s$ is a statement for every expression $e$ and statement $s$.

$$S \rightarrow \text{while} (E) S$$

**Declaration statement:** $\text{var} \ x_1, \ldots, x_k ;$ is a statement for every $k \geq 1$ and for all identifiers $x_1, \ldots, x_k$. Declaration statements are also denoted by a separate nonterminal symbol $S^{\text{var}}$.

$$S \rightarrow S^{\text{var}}$$

$$S^{\text{var}} \rightarrow \text{var} \to Z_{\text{distinct-id}} ;$$

**Return statement:** $\text{return} \ e$ ; is a statement for every expression $e$.

$$S \rightarrow S^{\text{return}}$$

$$S^{\text{return}} \rightarrow \text{return} E ; \& \text{returnstatementfix}$$

$$\text{returnstatementfix} \rightarrow \text{return} \text{ anypunctuator anystring} \mid \text{return} \text{ anystring}$$

A conjunction with returnstatementfix is necessary to ensure that the keyword return is always followed by some punctuator or whitespace character. Only in this case, a string may be parsed as a return statement. Without this condition, the first conjunct $\text{return} E ;$ would define, for instance, a string returnable; as if it were a return statement returnable;. This detail could be managed without using conjunction, but is a more complicated way.

The grammar also defines a subclass of returning statements ($S_r$), that is, those that may terminate their execution only by a return statement. A return statement itself is returning.

$$S_r \rightarrow S^{\text{return}}$$

A conditional statement is returning, if both branches are returning.

$$S_r \rightarrow \text{if} (E) S_r \text{ else } S_r$$
A compound statement is returning, if so is its last constituent. The set of returning compound statements is also denoted by a separate nonterminal symbol, $S_{r}^{\text{compound}}$.

$$
S_{r} \rightarrow S_{r}^{\text{compound}}
$$

$$
S_{r}^{\text{compound}} \rightarrow \{ \ S^{*} S_{r} \ \}
$$

### 4.6 Function declarations

A function declaration begins with a header of the form $f(\ x_{1}, \ldots, x_{k})$, where $k \geq 0$ and $f, x_{1}, \ldots, x_{k}$ are identifiers. The identifier $f$ is the name of the function, and the identifiers $x_{1}, \ldots, x_{k}$ are its formal arguments. The arguments must be distinct, hence the grammar refers to a list of pairwise distinct identifiers ($Z_{\text{distinct-id}}$).

$$
F_{\text{header}} \rightarrow \text{id} ( \ Z_{\text{distinct-id}} \ )
$$

$$
F_{\text{header}} \rightarrow \text{id} ( \ )
$$

A function declaration ($F$) is a header ($F_{\text{header}}$) followed by a returning compound statement ($S_{r}^{\text{compound}}$), called the body of the function.

$$
F \rightarrow F_{\text{header}} S_{r}^{\text{compound}} \ & \text{id} \ ( \ all-variables-declared \ )
$$

The second conjunct in the latter rule refers to the nonterminal symbol $all-variables-declared$ representing the conditions on variable declaration. Intuitively, one can interpret this rule in the sense that $F_{\text{header}} S_{r}^{\text{compound}}$ first defines a certain structure, and then $all-variables-declared$ processes that structure to verify declaration of variables before use. Then, for every string being defined by $all-variables-declared$, one can assume that it is already of the form $F_{\text{header}} S_{r}^{\text{compound}}$. This makes the rules for $all-variables-declared$, presented below, easier to construct and understand.

### 4.7 Declaration of variables before use

For each function, the goal is to check that every reference to a variable in the function body is preceded by a declaration of a variable with the same name. A reference is an identifier occurring in an expression. Declarations take place in the list of function arguments and in var statements; in the latter case, the reference should be in the scope of this declaration, that is, within the same compound statement as the var statement or in any nested statements, and occurring later than the var statement. Another related thing to check is that no declaration is in the scope of another declaration of the same variable. The purpose of the nonterminal symbol $all-variables-declared$ is to check all these conditions for a particular function.

The rules for the nonterminal $all-variables-declared$ iterate over all prefixes (of the function body) that end with an identifier, as illustrated in Figure 4. This is done generally in the same way as in Example 1, with the following details to note. First, the function body is split into tokens again, and all irrelevant tokens are skipped. Special care has to be exercised when skipping a number or a keyword, because the characters forming them might actually be a suffix of an identifier to be checked; this possibility is ruled out in the rule for $all-variables-declared-safe$.

$$
all-variables-declared \rightarrow \text{all-variables-declared-safe} \ [\text{NUM}]
$$

$$
all-variables-declared \rightarrow \text{all-variables-declared-safe} \ [\text{Keyword}]
$$

$$
all-variables-declared \rightarrow \text{all-variables-declared-safe} \ [\text{id}] \ [\text{safeendingstring}]
$$

$$
all-variables-declared-safe \rightarrow \text{all-variables-declared-safe} \ & \text{id} \ [\text{safeendingstring}]
$$
Figure 4: How all-variables-declared processes all prefixes of the function body, applying these-variables-not-declared to declarations and this-variable-declared to references.

Any punctuator character is skipped, unless it is a semicolon concluding a var statement.

\[ \text{all-variables-declared} \rightarrow \text{all-variables-declared} \text{ anypunctuator ws} \& \neg \text{safeendingstring} \text{ S}^{\text{var}} \]

It is important to distinguish between identifiers representing declarations and identifiers representing references. Once a var statement is found, these-variables-not-declared shall verify that none of the variables defined here have previously been defined; this is one of the two cases illustrated in Figure 4.

\[ \text{all-variables-declared} \rightarrow \text{these-variables-not-declared} ; \& \text{all-variables-declared-safe} \text{ S}^{\text{var}} \]

The other case shown in Figure 4 is that for each reference found, a nonterminal this-variable-declared is invoked to check that this variable has an earlier declaration.

\[ \text{all-variables-declared} \rightarrow \text{this-variable-declared} \& \text{all-variables-declared-safe} \text{ ID} \]

Finally, once the whole function body is processed, only the list of arguments in its header remains. This list may be empty, hence there are two terminating rules.

\[ \text{all-variables-declared} \rightarrow \text{Z}_{\text{distinct-id}} ( ) | ) \]

Thus, for every prefix ending with a reference to a variable, the nonterminal this-variable-declared is used to match it to a declaration of a variable that occurs inside this prefix. This process is illustrated in Figure 5. This time, the rules iterate over all suffixes of the current prefix, beginning at different tokens and ending with the identifier being checked. First, the rules for this-variable-declared search for a declaration among the function’s arguments, and if it is found, it is left to match the identifiers using C.

\[ \text{this-variable-declared} \rightarrow \text{ID} \text{ C} \text{ this-variable-declared} | C \]

If failed, the nonterminal declared-inside-function is invoked to look for a declaration in a suitable var statement.

\[ \text{this-variable-declared} \rightarrow \text{ID} ) \{ \text{ declared-inside-function} \]

If the function has no arguments, the search for a var statement in the body begins by the following rule.

\[ \text{this-variable-declared} \rightarrow ) \{ \text{ declared-inside-function} \]
While searching for a \texttt{var} statement, variable scopes have to be observed, and for that purpose, the rules for \texttt{declared-inside-function} parse the function body according to the nested structure of statements. First, any complete statement may be ignored: this means that the desired variable is not declared there.

\textit{declared-inside-function} → \text{S} \textit{declared-inside-function}

If the reference being checked is inside a compound statement, then the following rule moves the search one level deeper into a nested compound statement.

\textit{declared-inside-function} → \{ \textit{declared-inside-function} \}

For \textit{if} and \textit{while} statements, a nested scope is entered through an extra nonterminal \texttt{declared-inside-function-nested}, which indicates that the current statement is not directly within a compound statement.

\textit{declared-inside-function} → \texttt{if} (\texttt{E}) \texttt{declared-inside-function-nested}
\textit{declared-inside-function} → \texttt{if} (\texttt{E}) \texttt{S} \texttt{else} \texttt{declared-inside-function-nested}
\textit{declared-inside-function} → \texttt{while} (\texttt{E}) \texttt{declared-inside-function-nested}

The rules for \texttt{declared-inside-function-nested} process potentially nested \texttt{if} and \texttt{while} statements and get back to \textit{declared-inside-function} as soon as a compound statement begins.

\textit{declared-inside-function-nested} → \{ \textit{declared-inside-function} \}
\textit{declared-inside-function-nested} → \texttt{if} (\texttt{E}) \texttt{declared-inside-function-nested}
\textit{declared-inside-function-nested} → \texttt{if} (\texttt{E}) \texttt{S} \texttt{else} \texttt{declared-inside-function-nested}
\textit{declared-inside-function-nested} → \texttt{while} (\texttt{E}) \texttt{declared-inside-function-nested}

If a \texttt{var} statement is encountered, the desired declaration may be there. In this case, the nonterminal \texttt{declared-inside-function} is used to find the correct declaration among the variables listed in this \texttt{var} statement, and \texttt{C} is invoked to match identifiers.

\textit{declared-inside-function} → \texttt{var} \texttt{\_} \texttt{declared-in-this-statement}
\textit{declared-in-this-statement} → \texttt{ID} \texttt{\_} \texttt{declared-in-this-statement}
\textit{declared-in-this-statement} → \texttt{C \& ignore-remaining-variables skip-part-of-this-scope}
After skipping the remaining variables declared in this `var` statement (`ignore-remaining-variables`), the middle part between the declaration and the reference is described by a nonterminal `skip-part-of-this-scope`, which ensures that the reference stays in the scope of the declaration.

```
ignore-remaining-variables → ID ID ignore-remaining-variables | ID ;
skip-part-of-this-scope → skip-part-of-this-scope { S* }
skip-part-of-this-scope → skip-part-of-this-scope {
skip-part-of-this-scope → skip-part-of-this-scope anycharexceptbracesandspace WS
```

Now consider the other nonterminal `these-variables-not-declared`, which is used in the rules for `all-variables-declared` for any prefix ending with a declaration, in order to ensure that none of these variables have been declared before. Here negation comes in particularly useful, because the condition that an identifier is in the scope of a variable with the same name has already been expressed as `this-variable-declared`, and now it is sufficient to negate it. The rules for `these-variables-not-declared` iterate over all variables declared at this point.

```
these-variables-not-declared → these-variables-not-declared , ID & ¬ this-variable-declared
the iteration terminates after checking the first variable declared in this var statement.
these-variables-not-declared → safeendingstring var → ID & ¬ this-variable-declared
```

### 4.8 Declaration of functions before use

Another kind of references to be matched to their declarations are calls to functions. Whenever a function is called, somewhere earlier in the program there should be a function header that opens a declaration of a function with the same name and with the same number of arguments. Furthermore, a program may not contain multiple declarations of functions sharing the same name and the same number of arguments.

Checking these conditions requires matching each function call to a suitable earlier function declaration. Similarly to the rules for `all-variables-declared`, this is done by considering all prefixes of the program that end with a function call or a function header. For that purpose, all tokens except right parentheses are being skipped.

```
function-declarations → function-declarations anypunctuatorexceptrightpar WS
function-declarations → function-declarations-safe Keyword
function-declarations → function-declarations-safe ID
function-declarations → function-declarations-safe NUM
function-declarations → ε
function-declarations-safe → function-declarations & safeendingstring
```

Whenever a right parenthesis is found, there are three possibilities, each handled in a separate rule for `function-declarations`.

First, this right parenthesis could be the last character of a function call expression, for which one should find a matching function declaration. This case is identified by the following two conditions: the current substring should end with a function call expression (`Ecall`), and at the same time the entire substring should not be of the form `F* Fheader`. The latter condition is essential, because otherwise (in this model programming language) a function call is
indistinguishable from a list of arguments in a function header.

\[ \text{function-declarations} \rightarrow \text{function-declarations} \] & safeendingstring \( \text{E}^{\text{call}} \) &
\& \neg F^* \text{F}_{\text{header}} \& F^* \text{this-function-declared-here} \]

This case is illustrated in Figure 6.

The concatenation in the last conjunct of the rule splits the current prefix of the program into zero or more irrelevant function declarations \( (F^*) \) followed by a substring that begins with a header of the desired function and ends with the function call expression, with these two sharing the same name and having same number of arguments \( (\text{this-function-declared-here}) \). The comparison of identifiers \( (\text{same-function-name}) \) and of the number of arguments \( (\text{same-number-of-arguments}) \) is carried out in the following rules.

\[ \text{this-function-declared-here} \rightarrow \text{same-function-name} \& \text{same-number-of-arguments} \]
\[ \text{same-function-name} \rightarrow C [ Z_{\text{expr}} ] \]
\[ \text{same-number-of-arguments} \rightarrow \text{id} [ n_{\text{of-arg-equal}} ] \]
\[ \text{same-number-of-arguments} \rightarrow \text{id} [ n_{\text{of-arg-equal-0}} ] \]

Here, the nonterminal \( n_{\text{of-arg-equal}} \) handles the case of one or more arguments, whereas \( n_{\text{of-arg-equal-0}} \) corresponds to the case of a call to a function with zero arguments.

\[ n_{\text{of-arg-equal-0}} \rightarrow ) \text{anystring} ( \]
\[ n_{\text{of-arg-equal}} \rightarrow \text{id} ( n_{\text{of-arg-equal}} E ) \]
\[ n_{\text{of-arg-equal}} \rightarrow \text{id} ( ) \text{anystring} ( E ) \]

The second possibility with a right parenthesis encountered in \( \text{function-declarations} \) is when it is the last character of a function header. In this case, the grammar should ensure that no other functions with the same name are declared. The second conjunct of the following rule verifies that the current substring ends with a function header rather than with a function call, whereas the third conjunct negates the condition of having an earlier declaration of the same function.

\[ \text{function-declarations} \rightarrow \text{function-declarations} \] & \( F^* \text{F}_{\text{header}} \& \neg F^* \text{this-function-declared-here} \]

Figure 7 demonstrates how this rule detects multiple declarations of the same function, so that the substring is not defined by \( \text{function-declarations} \). For it to be defined, it should have no partition into \( F^* \text{this-function-declared-here} \).
The last case in function-declarations is when a substring ends with a right parenthesis, but it neither marks an end of a function call expression, nor is a part of a function header. This can happen in several ways: it could be a subexpression enclosed in parentheses, or a part of a for or a while statement. In each case, there is nothing to check, and the right parenthesis is skipped like any other token. The case is identified by not ending with $E^{\text{call}}$.

$$\text{function-declarations} \rightarrow \text{function-declarations} \; \& \; \neg \text{safe ending string} \; E^{\text{call}}$$

The check for function declaration before use, as implemented in the nonterminal symbol function-declarations, ensures that both the name and the number of arguments match. When the same condition is used to define a duplicate declaration, two declaration are considered duplicate if they agree both the name in the number of arguments. This means that functions can be overloaded.

### 4.9 Programs

It remains to give the rules describing the set of well-formed programs in the model programming language. A program is a finite sequence of function declarations, which contains one function with the name main, with one argument.

A sequence of function declarations and a declaration of the main function are defined by the following rules.

$$F^* \rightarrow F^* \; F \; | \; \varepsilon$$

$$F_{\text{main}} \rightarrow \text{main ws } \begin{array}{c} \text{ID} \end{array} \; \text{ws}^{\text{compound}} \; \& \; \text{ID} \; \begin{array}{c} \text{all-variables-declared} \end{array}$$

Finally, a single rule for the initial symbol Program defines what a well-formed program is. This rule also defines possible whitespace characters occurring before the first token.

$$\text{Program} \rightarrow \text{ws} \; F^* \; F_{\text{main}} \; F^* \; \& \; \text{ws} \; \text{function-declarations}$$

This completes the grammar.

**Proposition 1.** The set of well-formed programs in the model programming language is described by a Boolean grammar with 117 nonterminal symbols and 361 rules.

The grammar constructed above can be used with any of the several known parsing algorithms for Boolean grammars. First, there is a simple extension of the Cocke–Kasami–Younger algorithm, with the running time $O(n^3)$ in the length of the input [29]. Like in the case of ordinary grammars, this algorithm can be accelerated to run in time $O(n^{\omega})$ [35], where $\omega < 3$ is the exponent in the complexity of matrix multiplication. The most practical algorithm is the
GLR [31], which has worst-case running time $O(n^4)$, but may run faster for some grammars and inputs, if a particular parse goes on partially deterministically.

The grammar has been tested on a large set of positive and negative examples using one of the existing implementations of GLR parsing for Boolean grammars [26]. The parser contains 754 states and operates in generally the same way as GLR parsers for ordinary grammars.

5 Eliminating negation and ambiguity

The grammar for the model programming language given in Section 4 uses negation several times and contains quite a lot of syntactical ambiguity. Disregarding these shortcomings made grammar construction easier.

In general, unintended ambiguity is always undesirable, and the use of negation can also be viewed as an unnecessary complication. The purpose of this section is to explain how the grammar can be rewritten as an unambiguous conjunctive grammar describing exactly the same language. Besides being a conceptually clearer model, unambiguous conjunctive grammars also have a better upper bound on the parsing complexity.

5.1 Negation in auxiliary definitions

At two occasions, the grammar uses the negation in the definitions of basic constructs. The same definitions now have to be reformulated without the negation.

First, in Section 4.2, the rule defining identifiers (ID) uses negation to describe a regular language \{a, ..., z\}{a, ..., z, 0, ..., 9}^* \{var, if, else, while, return\}. The same language can be recognized by a 21-state finite automaton, which can in turn be simulated in the grammar.

A more interesting use of negation is in the rule for no-multiple-declarations, where it is applied to C in order to express identifier inequality. To eliminate the negation here, one should define a new nonterminal symbol $\tilde{C}$ that would describe all strings $uxvy$, where $u$ and $v$ are distinct identifiers, $x$ is the middle part of the program between these two identifiers, and $y$ is a possibly empty sequence of whitespace characters. The first possibility for $u$ and $v$ not to be equal is if they are of different length; this case is handled in the rules for $\tilde{C}_{\text{len}>}$ ($|u| > |v|$) and for $\tilde{C}_{\text{len}<}$ ($|u| < |v|$).

$$ \tilde{C} \rightarrow \tilde{C}_{\text{len}<} \mid \tilde{C}_{\text{len}>} \mid \tilde{C} $$

$$ \tilde{C}_{\text{len}>} \rightarrow \text{anyletterdigit} \, \tilde{C}_{\text{len}>} \mid \text{anyletterdigit} \, C_{\text{len}} $$

$$ \tilde{C}_{\text{len}<} \rightarrow \tilde{C}_{\text{len}<} \, \text{anyletterdigit} \mid C_{\text{len}} \, \text{anyletterdigit} $$

Otherwise, if $|u| = |v|$, then $\tilde{C}_{\text{iterate}}$ begins comparing the characters of $u$ and $v$ in the same way as done in $C_{\text{iterate}}$, using $C_{\sigma}$ to check each character.

$$ \tilde{C} \rightarrow C_{\text{len}} \, \& \, \tilde{C}_{\text{iterate}} $$

$$ \tilde{C}_{\text{iterate}} \rightarrow C_{\sigma} \, \& \, \tilde{C}_{\text{iterate}} \, \sigma \quad \text{(for all } \sigma \in \{a, ..., z, 0, ..., 9\}) $$

The iteration in $\tilde{C}_{\text{iterate}}$ is stopped when a pair of mismatched characters is encountered, that is, if $u = u'\sigma w$ and $v = v'\tau w$, for some $u'$ and $v'$.

$$ \tilde{C}_{\text{iterate}} \rightarrow C_{\sigma} \, \tau \quad \text{(for all } \sigma, \tau \in \{a, ..., z, 0, ..., 9\}, \text{ with } \sigma \neq \tau) $$
5.2 Two standard ambiguous constructs

The rules in Section 4.4 include two standard cases of ambiguous definitions in programming languages. The first of them concerns the expressions. The given definition is ambiguous, because the precedence and the associativity of operators are not defined. One could make the rules for $E$ unambiguous by rewriting them in the standard way, introducing a new nonterminal symbol for each level of precedence.

The rules defining the conditional statement feature another classical kind of ambiguity, known as the “dangling else” ambiguity. Indeed, a string such as $\text{if}(x) \text{ if}(y) \ s; \ \text{else} \ t;$ can be parsed in two different ways, depending on whether the last $\text{else}$ clause binds to the first or to the second $\text{if}$ statement. This ambiguity can also be resolved in the standard way, by introducing a variant of $S$ called $S_{\text{not-if-then}}$, which should define all statements except those of the form if-then, without an $\text{else}$ clause. Then, all rules for $S$ describing statements other than conditional statements are preserved, whereas the rules describing conditional statements take the following form.

\[
S \rightarrow \text{if} \ (E) S \\
S \rightarrow \text{if} \ (E) S_{\text{not-if-then}} \text{ else } S
\]

The new nonterminal symbol $S_{\text{not-if-then}}$ does not have a rule of the former type (if-then), but otherwise, it has all the same rules as $S$.

5.3 Variable declarations

The rules requiring declaration of variables before reference, as given in Section 4.7, essentially use negation to ensure that a variable has not been declared before ($\neg \text{this-variable-declared}$). Furthermore, there is some subtle ambiguity in the rules for $\text{this-variable-declared}$, declared-inside-function and declared-in-this-statement. Both shortcomings shall now be corrected by reimplementing parts of the grammar.

First, consider the ambiguity, which manifests itself on any program containing a variable with multiple declarations. Even though any such program is ultimately considered ill-formed, according to Definition 5, this is still ambiguity, in the sense that some substrings still have multiple parses. This affects the complexity of parsing. Consider the following ill-formed function declaration.

\[
f(\text{arg, arg}) \{ \ \text{var arg; var arg, arg; arg=0; } \}
\]

When checking the reference to $\text{arg}$, the nonterminal $\text{this-variable-declared}$ has to handle the underlined substring that ends with that reference. First, there is an ambiguity between the rules $\text{this-variable-declared} \rightarrow C$ and $\text{this-variable-declared} \rightarrow \text{ID} \ \text{this-variable-declared}$. The former matches the first argument of the function to the reference, whereas the latter ignores the first argument and looks for a declaration of $\text{arg}$ later. For this particular string, both conditions hold at the same time, hence the ambiguity. Next, there is a similar ambiguity between matching the last argument of the function ($\text{this-variable-declared} \rightarrow C$) and looking for a declaration of $\text{arg}$ inside the function body ($\text{this-variable-declared} \rightarrow \text{ID} \ \{ \}$ declared-inside-function). Later on in the grammar, there is an ambiguity between the first and the second $\text{var}$ statements: in other words, the ambiguity in the choice between declared-inside-function $\rightarrow S$ declared-inside-function and declared-inside-function $\rightarrow \text{var} \ \text{declared-in-this-statement}$. Finally, when the second $\text{var}$ statement is analyzed by declared-in-this-statement, one can use the first or the second identifier in it: this is the ambiguity between the two rules for declared-in-this-statement.
In each case, the ambiguity could be resolved by using negation to set the precedence of the two conditions explicitly. In general, given two rules \( A \rightarrow \alpha \) and \( A \rightarrow \beta \), and assuming that the former has higher precedence, the latter rule would be replaced with \( A \rightarrow \beta \& \neg \alpha \) [33 Prop. 2]. The goal is to do the same without using negation.

The solution proposed here is to introduce a negative counterpart to each of the three nonterminals that need to be negated (\( \text{this-variable-declared}, \text{declared-inside-function}, \text{declared-in-this-statement} \)). The rules for this negative counterpart implement a dual version of the rules for its original positive version, and define the opposite condition. A general transformation that achieves this effect is known only for the special class of linear conjunctive grammars [27 Thm. 5]. Although the grammar considered here is not exactly linear, the idea of that general method shall be used to dualize the particular rules of this grammar.

The new nonterminal symbols representing negations of the required conditions shall be called \( \text{this-variable-not-declared}, \text{not-declared-inside-function} \) and \( \text{not-declared-in-this-statement} \). First, consider how these nonterminals shall be used in the existing grammar. The rule for \( \text{these-variables-not-declared} \) with an explicit negation is rewritten using the negative nonterminal.

\[
\text{these-variables-not-declared} \rightarrow \text{these-variables-not-declared} & \text{id} \& \text{this-variable-not-declared}
\]

The ambiguity between the first three rules for \( \text{this-variable-declared} \) is resolved by allowing earlier declarations \( (C) \) only if there are no later declarations \( \text{id} \), \( \text{this-variable-declared} \) or \( \text{id} \) \( \text{not-declared-inside-function} \). Thus, the rule \( \text{this-variable-declared} \rightarrow C \) is replaced with the following two rules.

\[
\text{this-variable-declared} \rightarrow C \& \text{id} \& \text{this-variable-not-declared}
\]

\[
\text{this-variable-declared} \rightarrow C \& \text{id} \& \text{not-declared-inside-function}
\]

This prioritizes later declarations over earlier declarations. The same principle is followed in disambiguating the definitions of \( \text{declared-inside-function} \) and \( \text{declared-in-this-statement} \). For the former, every \text{var} statement is processed for declarations only if this variable is not declared later.

\[
\text{declared-inside-function} \rightarrow \text{var} \_ \text{declared-in-this-statement} \& S \text{ not-declared-inside-function}
\]

For \( \text{declared-in-this-statement} \), a declaration is matched to a reference \( (C) \) only if no later declaration in this \text{var} statement is suitable.

\[
\text{declared-in-this-statement} \rightarrow C \& \text{ignore-remaining-variables skip-part-of-this-scope} \& \\
\& \text{id} \& \text{not-declared-in-this-statement}
\]

\[
\text{declared-in-this-statement} \rightarrow C \& \text{id} \& \text{skip-part-of-this-scope}
\]

It remains to define the rules for the negative versions of the three nonterminals in question. The rules for \( \text{this-variable-not-declared} \) ensure that the identifier being checked is different from each of the function’s arguments.

\[
\text{this-variable-not-declared} \rightarrow \tilde{C} \& \text{id} \& \text{this-variable-not-declared}
\]

Then the search proceeds into the body of the function.

\[
\text{this-variable-not-declared} \rightarrow \tilde{C} \& \text{id} \& \text{not-declared-inside-function}
\]

\[
\text{this-variable-not-declared} \rightarrow \text{id} \& \text{not-declared-inside-function}
\]
Turning to not-declared-inside-function, let $S_{\text{not-var}}$ be a new nonterminal that denotes all well-formed statements except var statements. These are the statements that not-declared-inside-function is allowed to skip without consideration: whatever declarations are made inside such a statement, the reference being checked is not in their scope.

$$\text{not-declared-inside-function} \rightarrow S_{\text{not-var}} \text{ not-declared-inside-function}$$

The rules for $S_{\text{not-var}}$ are the same as the rules for $S$, given in Section 4.5 except for the rule for a var statement, which is not included. Next, not-declared-inside-function navigates through the nested structure of statements in the same way as declared-inside-function.

$$\text{not-declared-inside-function} \rightarrow \{ \text{ not-declared-inside-function} \}$$
$$\text{not-declared-inside-function} \rightarrow \text{ if } (E) \text{ not-declared-inside-function-nested}$$
$$\text{not-declared-inside-function} \rightarrow \text{ if } (E) S \text{ else } \text{ not-declared-inside-function-nested}$$
$$\text{not-declared-inside-function} \rightarrow \text{ while } (E) \text{ not-declared-inside-function-nested}$$
$$\text{not-declared-inside-function-nested} \rightarrow \{ \text{ not-declared-inside-function} \}$$
$$\text{not-declared-inside-function-nested} \rightarrow \text{ if } (E) \text{ not-declared-inside-function-nested}$$
$$\text{not-declared-inside-function-nested} \rightarrow \text{ if } (E) S \text{ else } \text{ not-declared-inside-function-nested}$$
$$\text{not-declared-inside-function-nested} \rightarrow \text{ while } (E) \text{ not-declared-inside-function-nested}$$

When a var statement is encountered, not-declared-in-this-statement is invoked to verify that this statement does not declare the variable under consideration. At the same time, not-declared-inside-function recursively refers to itself to make sure that this variable is also not declared in any subsequent statements.

$$\text{not-declared-inside-function} \rightarrow \text{ var } \sim \text{ not-declared-in-this-statement} \& \& S_{\text{var}} \text{ not-declared-inside-function}$$

Unlike the nonterminal declared-inside-function, which ends the iteration by finding a suitable declaration, here the iteration ends in a short substring without variable declarations.

$$\text{not-declared-inside-function} \rightarrow \text{ anystringwithoutbracesandsemicolons}$$
$$\text{not-declared-inside-function-nested} \rightarrow \text{ anystringwithoutbracesandsemicolons}$$

Finally, not-declared-in-this-statement applies $\tilde{C}$ to each identifier in the current var statement, in order to ensure that all of them are different from the identifier in the end of the substring.

$$\text{not-declared-in-this-statement} \rightarrow \tilde{C} \& \text{ID} \& \text{not-declared-in-this-statement}$$
$$\text{not-declared-in-this-statement} \rightarrow \tilde{C} \& \text{ID} ; \text{anystringwithoutbraces skip-part-of-this-scope}$$

5.4 Function declarations

The rules describing declaration of functions, given in Section 4.8, suffer from the same kind of problems as the rules for variable declarations. This part of the grammar shall be reconstructed similarly to what was done in the above Section 5.3.

First, consider the conjunct $F^* \text{ this-function-declared-here}$ in one of the rules for function-declarations, which concatenates a prefix with zero or more irrelevant function declarations to
a substring that begins with a declaration of the desired function, and ends with a call to that
function. This concatenation is ambiguous, because the function being called may have multiple
declarations, as demonstrated in the following example.

```c
function() { return 0; } function() { return 1; } main(arg) { return function(); }
```

Here the underlined substring has two partitions as $F^* \text{this-function-declared-here}$, corresponding
to the first and the second declaration of `function()`.

In order to look up functions unambiguously, instead of using concatenation to get to the
desired declaration at once, one should process all declarations iteratively, one by one, in the
same way as for variable declarations. This shall be done in a new nonterminal \texttt{this-function-declared},
which reimplements the concatenation $F^* \text{this-function-declared-here}$. The rules for
\texttt{this-function-declared} shall iteratively consider all substrings that begin with various function
declarations and end with the reference being checked, and apply \texttt{this-function-declared-here}
to every such substring. Furthermore, doing this unambiguously by the same method as in
Section 5.3 requires negative counterparts of these two nonterminals, called \texttt{this-function-not-declared}
and \texttt{this-function-not-declared-here}.

According to this plan, the three rules for \texttt{function-declarations} dealing with the right paren-
thesis are rewritten as follows. First, if this is a function call, then the new nonterminal \texttt{this-
function-declared} verifies that there is a declaration of the function being called. In order to make
sure that the rule indeed deals with a function call rather than with a declaration, an extra con-
junct states that this prefix of the program is a sequence of function declarations followed by
a header and an incomplete compound statement, using the nonterminal \texttt{skip-part-of-this-scope}
defined in Section 4.7.

\[
\text{function-declarations} \rightarrow \text{function-declarations} \& \text{safeendingstring } E^{\text{call}} \& F^*F_{\text{header}} \{ \text{skip-part-of-this-scope} \& \text{this-function-declared} \}
\]

Second, if this is a function header, then the negative version of the new nonterminal \texttt{(this-
function-not-declared)} shall ensure that there are no earlier declarations of any functions with
the same name and the same number of arguments.

\[
\text{function-declarations} \rightarrow \text{function-declarations} \& F^*F_{\text{header}} \& \text{this-function-not-declared}
\]

The third case is when the string ending with a right parenthesis is not of the form
“safeendingstring $E^{\text{call}}$”. Since negation is no longer allowed, one has to list all possibilities
of how a prefix of a well-formed program could be of such a form; the following list is inferred
from the syntax of this model programming language. First of all, this prefix may end with an
expression enclosed in brackets; in this case, the expression must be preceded by some punctu-
ator character (which is either an operator or a bracket within a larger expression, or the last
character of some syntactical unit other than an expression) or by a \texttt{return} keyword.

\[
\text{function-declarations} \rightarrow \text{function-declarations} \& \text{safeendingstring } \text{anystring anypunctuator} \text{ws} ( \text{} E \text{})
\]

\[
\text{function-declarations} \rightarrow \text{function-declarations} \& \text{safeendingstring } \text{return} E \text{)}
\]

The remaining possibility is that the right parenthesis under consideration is a part of an \texttt{if} or
a \texttt{while} statement.

\[
\text{function-declarations} \rightarrow \text{function-declarations} \& \text{safeendingstring } \text{if} (\text{} E \text{})
\]

\[
\text{function-declarations} \rightarrow \text{function-declarations} \& \text{safeendingstring } \text{while} (\text{} E \text{})
\]
The next goal is to define the rules for the new nonterminals `this-function-declared` and `this-function-not-declared`. The first rule for `this-function-declared` skips any function declarations, as long as there is still a declaration of this function later on.

\[
\text{this-function-declared} \rightarrow F \text{ this-function-declared}
\]

The second rule describes a substring that begins with a desired function declaration and ends with a reference to the same function: this condition is checked by the nonterminal `this-function-declared-here`, using the rules defined in Section 4.8.

\[
\text{this-function-declared} \rightarrow \text{this-function-declared-here} \& F \text{ this-function-not-declared}
\]

In order to keep the grammar unambiguous, the second conjunct of this rule ensures that no later function declaration matches this reference. The last rule handles a special case, where the substring contains a function header and a part of the same function’s body ending with its recursive call to itself.

\[
\text{this-function-declared} \rightarrow \text{this-function-declared-here} \& F \text{ header } \{ \text{skip-part-of-this-scope} \}
\]

Here the nonterminal `skip-part-of-this-scope`, reused from Section 4.7, ensures that the substring contains no other function declarations.

Now consider checking a function for having no declaration (`this-function-not-declared`). This is done for strings of the same form as for `this-function-not-declared`: that is, for a substring beginning with zero or more function declarations, and continued either with a function header or with an incomplete function declaration ending with a function call. One has to verify that the function in the end of the substring has no earlier declarations. For substrings that begin with a function declaration, the first rule states that the function is declared neither here, nor later.

\[
\text{this-function-not-declared} \rightarrow \text{this-function-not-declared-here} \& F \text{ this-function-not-declared}
\]

Once all earlier function declarations are processed in this way, eventually there is one of two base cases to handle. First, `this-function-not-declared` may have to deal with a substring that consists of just one function header. It has already been checked that this function has no earlier declarations, and the iteration ends here.

\[
\text{this-function-not-declared} \rightarrow F_{\text{header}}
\]

The second case is when a substring begins with a function header and continues with an incomplete function body ending with a function call expression. Then, the following rule ensures that this is not a valid recursive call to itself.

\[
\text{this-function-not-declared} \rightarrow \text{this-function-not-declared-here} \& F_{\text{header}} \{ \text{skip-part-of-this-scope} \}
\]

Finally, it remains to write down the rules for `this-function-not-declared-here`. Dually to the rule for `this-function-declared-here` from Section 4.8, here one can say that either the two functions have different names, or they have the same name but a different number of arguments.

\[
\text{this-function-not-declared-here} \rightarrow \text{different-function-name}
\]
\[
\text{this-function-not-declared-here} \rightarrow \text{same-function-name} \& \text{different-number-of-arguments}
\]
Name mismatch is tested using $\bar{C}$.

$$\text{different-function-name} \rightarrow \bar{C} \left( \{ \text{Zexpr} \} \right)$$

Mismatch in the number of arguments is described by the same kind of rules as for match ($n$-of-arg-equal, $n$-of-arg-equal-0).

$$\text{different-number-of-arguments} \rightarrow \text{id} \left( \text{n-of-arg-less-0} \right) | \text{id} \left( \text{n-of-arg-greater-0} \right)\text{id} \left( \text{n-of-arg-less} \right) | \text{id} \left( \text{n-of-arg-greater} \right)$$

5.5 The main function

The rules for the main function defined in Section 4.9 are ambiguous, because the program may contain multiple declarations for the main function. To be precise, the concatenation $F^n F_{\text{main}} F^n$ in the rule for Program is ambiguous, because there are as many partitions as there are main functions declared in a program (which is obviously ill-formed).

The rule for Program is therefore rewritten by using a new nonterminal symbol $F^{*m*}$ that represents any sequence of function declarations that contains a main function.

$$\text{Program} \rightarrow \text{ws} F^{*m*} \& \text{ws} \text{function-declarations}$$

The rules for $F^{*m*}$ add function declarations, one by one, as long as they are not for a main function. Once the last declaration of a main function in the program is located, it does not matter which functions are defined before it.

$$F^{*m*} \rightarrow F^{*m*} F_{\text{main}} | F^n F_{\text{main}}$$

(note that if the first rule were replaced with $F^{*m*} \rightarrow F^{*m*} F$, then the grammar would become ambiguous again) Finally, it remains to define a declaration of a function that is not a main function, and to do this without using the negation. The first possibility is that the function’s name is not main. Assume that the set of all identifiers other than main is defined in a new nonterminal symbol $\text{id}_{-\text{main}}$ with its rules simulating a finite automaton.

$$F_{\text{-main}} \rightarrow \text{id}_{-\text{main}} \text{ws} \left( \{ \text{Zdistinct-id} \} \right) S_r^{\text{compound}} \& \text{id} \left( \text{all-variables-declared} \right)$$

The other case is when the function is called main, but has none or at least two arguments.

$$F_{\text{-main}} \rightarrow \text{main} \text{ws} \left( \{ \text{Z}_{\text{2+}}{\text{distinct-id}} \} \right) S_r^{\text{compound}} \& \text{id} \left( \text{all-variables-declared} \right)$$

$$F_{\text{-main}} \rightarrow \text{main} \text{ws} \left( \{ \text{Z}_{\text{2+}}{\text{distinct-id}} \} \right) S_r^{\text{compound}} \& \text{id} \left( \text{all-variables-declared} \right)$$

The rules defining $Z^{2+}_{\text{distinct-id}}$ are a variant of those for $Z_{\text{distinct-id}}$.

This completes the last correction to the grammar.

**Proposition 2.** The set of well-formed programs in the model programming language is described by an unambiguous conjunctive grammar with 187 nonterminal symbols and 3828 rules.
There are so many rules because of the two finite automaton simulations for $ID$ and for $ID_{-\text{main}}$, and also because of the many rules of the form $\tilde{C}_{\text{iterate}} \rightarrow C_\sigma \tau$ needed to compare identifiers for inequality.

Even though the size of the grammar has increased ten-fold in comparison with the original ambiguous Boolean grammar, the overall impact on the worst-case parsing complexity is beneficial. Indeed, for unambiguous conjunctive grammars, as well as for unambiguous Boolean grammars, the theoretical upper bound on their parsing complexity is $O(n^2)$, where $n$ is the length of the program \[33\]. This is a significant improvement over cubic-time or slightly better performance of the existing algorithms on highly ambiguous grammars. In particular, the GLR parsing algorithm for Boolean grammars \[31\] works in square time on any unambiguous conjunctive grammar (even though no formal proof of that fact has ever been given). On some inputs, such as on the following program, the GLR parser for the grammar from Section \[4\] is forced into cubic-time behaviour, whereas the unambiguous GLR parser works significantly faster due to its guaranteed square-time performance.

```
main(x) { return x+x+ \ldots+x; }
```

The grammar-dependent constant factor in the parsing complexity is small in comparison, as it is only linear in the size of a GLR parser’s table.

No proof that the given grammar is unambiguous is given in this paper, just like there is no proof that it describes exactly the desired programming language. Although it is possible to prove this kind of results by hand, for such a large grammar, that would hardly be practical. For ordinary grammars (“context-free” according to Chomsky), there exists several automated methods for ambiguity detection \[6, 39\]. Investigating any methods of this kind for conjunctive and Boolean grammars is one of the many open problems suggested by this work.

### 5.6 A possible linear conjunctive grammar

It is possible that the grammar presented in this section could be further reconstructed into a linear conjunctive grammar. This is a simpler model notable for its equivalence to a family of cellular automata \[28\]. The best known complexity upper bound for linear conjunctive grammars is still $O(n^2)$, so this transformation is of a purely theoretical interest.

Some parts of the grammar, such as identifier comparison in Section \[4.3\] are already almost linear conjunctive, and require only very obvious transformations. Another kind of inessential non-linearity is caused by the nonterminal symbols representing tokens, which are being concatenated throughout the grammar. Since each of them defines a regular language, those languages can be directly expressed within linear rules.

The rules defining the nested structure of expressions, statements and functions, given in Sections \[4.4\]-\[4.6\] are essentially non-linear. However, that structure is simple enough to be recognized by input-driven pushdown automata \[22\], also known under the name of visibly pushdown automata \[3\]—and those automata can in turn be simulated by linear conjunctive grammars \[36\].

The rules concerned with declaration before use, in their final form given in Sections \[5.3\]-\[5.4\] have many essentially non-linear concatenations. As a representative example, consider the rule $\text{not-declared-inside-function} \rightarrow \text{if} \left( E \right) \left\{ \right. \text{not-declared-inside-function-nested}$. Strictly speaking, it concatenates 6 nonterminal symbols. However, four of them represent tokens and accordingly define only regular sets, whereas $E$ is a bracketed construction recognized by an input-driven pushdown automaton. It is conjectured that this and all other such concatenations in the grammar can be simulated by linear conjunctive rules by further elaborating the known simulation of input-driven pushdown automata \[36\].
If the suggested transformation works out, the resulting linear conjunctive grammar will be large and incomprehensible. It should then be regarded as a kind of “machine code” that implements the grammar given in Sections 4.1–5.5.

6 Afterthoughts

The main purpose of this paper was to demonstrate that the expressive power of conjunctive and Boolean grammars is sufficient to describe some of the harder syntactic elements of programming languages. This becomes the first successful experience of constructing a complete formal grammar from a practically parsable class for a programming language.

As a first experience, it has many shortcomings. The model programming language is quite restricted, and enriching its syntax even a little bit, while staying within Boolean grammars, would be challenging, if at all possible: for instance, introducing type checking would be problematic, and it is quite possible that doing this would already be beyond the power of Boolean grammars. Some parts of the resulting description are quite natural, some are admittedly awkward. Grammar maintenance would be difficult, as a small change in the syntax of the language—such as, for instance, changing the punctuators used in a single language structure—may require rewriting other parts of the grammar in a non-trivial way. Square-time parsing may be fast enough for some applications, but it is still too slow in comparison with standard parsers for programming languages. These observations suggest the following research directions.

First, could there exist a substantially faster parsing algorithm that would still be applicable to (some variant of) the grammar given in this paper? The existing linear-time algorithms for subclasses of conjunctive grammars are too restrictive. If the model programming language defined here were parsed by a standard human-written program, that parser would maintain a symbol table of some sort, filling in new entries upon reading declarations, and looking it up for every reference. This suggests that if a prospective parsing algorithm is to parse this language substantially faster than in square time, then it will most likely need some advanced data structures. Would it be possible to adapt the GLR algorithm to index its graph-structured stack using a symbol table? The grammar in Section 5 demonstrates the kind of rules such an algorithm is expected to handle.

The second research direction is motivated by the awkward parts of the grammar and by the limitations of the model programming language. These can be regarded as signs of imperfection of conjunctive and Boolean grammars, and from this perspective, the goal is to find a new grammar formalism, in which all that could be done better. Any such formalisms must maintain efficient parsing algorithms, and would likely be found by further extending conjunctive or Boolean grammars. Over the years this paper has been under preparation, Barash found out that the grammar(761,477),(991,691) in this paper can be improved and the model language extended, if the grammar model is augmented with operators for referring to the contexts of a substring. The resulting new grammar model has a purely logical definition and still has a cubic-time parsing algorithm. It would be interesting to see any other new attempts to define a suitable grammar model.

The last suggested topic concerns conjunctive and Boolean grammars as they are, and their applications. The grammar constructed in this paper is an extreme case of using these models, meant to demonstrate the possibility of describing a number of constructs. It remains to find more practical ways of using these grammar construction methods.

For instance, it could be possible to define a slightly higher-level notation for syntax based on conjunctive and Boolean grammars. In the grammar presented in this paper, handling some simple issues of lexical analysis turned out to be quite inconvenient, because every rule of the grammar had to define the splitting of a string into tokens and implement the longest match principle. If a traditional lexical analyzer could prepare a program for parsing—for
example, by ensuring that there is exactly one space character between every two tokens—then
a Boolean grammar may describe only the well-tokenized inputs, and this would simplify the
definition. Since the family of formal languages defined by Boolean grammars is known to be
closed under inverse finite transductions [20], there are no theoretical obstacles to combining a
finite transducer and a Boolean grammar in this way. The question is to design a convenient
syntactic formalism along these lines.

In general, conjunctive and Boolean grammars should not be difficult to implement in exist-
ing projects, since the widely used GLR parsing algorithm can handle conjunctive and Boolean
rules [21, 31]. Besides parsing algorithms, no other essential grammar technologies and tools
have yet been developed for Boolean grammars. Grammar maintenance, ambiguity detection,
attribute evaluation, etc.—none of these subjects have yet been addressed. Investigating these
problems, perhaps finding out some ways to extend the ideas from ordinary grammars to gram-
mars with Boolean operations, is another research direction.

References

[1] A. V. Aho, “Indexed grammars” Journal of the ACM, 15:4 (1968), 647–671.

[2] T. Aizikowitz, M. Kaminski, “LR(0) conjunctive grammars and deterministic synchronized
alternating pushdown automata” Journal of Computer and System Sciences, 82:8 (2016),
1329–1359.

[3] R. Alur, P. Madhusudan, “Visibly pushdown languages” ACM Symposium on Theory of
Computing (STOC 2004, Chicago, USA, June 13–16, 2004), 202–211.

[4] M. Barash, “Programming language specification by a grammar with contexts” Non-
classical Models of Automata and Applications (NCMA 2013, Umeå, Sweden, 13–14 August
2013), Österreichische Computer Gesellschaft, 294 (2013), 51–67.

[5] M. Barash, A. Okhotin, “An extension of context-free grammars with one-sided context
specifications” Information and Computation, 237 (2014), 268–293.

[6] H. J. S. Basten, “Tracking down the origins of ambiguity in context-free grammars” Theo-
retical Aspects of Computing (ICTAC 2010, Natal, Rio Grande do Norte, Brazil, September
1–3, 2010), LNCS 6255, 76–90.

[7] A. Begel, S. L. Graham, “XGLR—an algorithm for ambiguity in programming languages”.
Science of Computer Programming, 61:3 (2006), 211–227.

[8] M. van den Brand, J. Scheerder, J. J. Vinju, E. Visser, “Disambiguation filters for scanner-
less generalized LR parsers” Compiler Construction (CC 2002, Grenoble, France, April
8–12, 2002), LNCS 2304, 143–158.

[9] N. Chomsky, “On certain formal properties of grammars” Information and Control, 2:2
(1959), 137–167.

[10] G. Economopoulos, P. Klint, J. Vinju, “Faster scannerless GLR parsing” Compiler Con-
struction (CC 2009, York, United Kingdom, March 22–29, 2009), LNCS 5501, 126–141.

[11] M. J. Fischer, “Grammars with macro-like productions” 9th Annual Symposium on Switch-
ing and Automata Theory (SWAT 1968, Schenectady, New York, USA, 15–18 October
1968), 131–142.
[12] R. W. Floyd, “On the non-existence of a phrase structure grammar for ALGOL 60”, Communications of the ACM, 5 (1962), 483–484.

[13] R. H. Gilman, “A shrinking lemma for indexed languages”, Theoretical Computer Science, 163:1–2 (1996), 277–281.

[14] S. Ginsburg, H. G. Rice, “Two families of languages related to ALGOL”, Journal of the ACM, 9 (1962), 350–371.

[15] N. Immerman, “Relational queries computable in polynomial time”, Information and Control, 68:1–3 (1986), 86–104.

[16] A. K. Joshi, L. S. Levy, M. Takahashi, “Tree adjunct grammars”, Journal of Computer and System Sciences, 10:1 (1975), 136–163.

[17] L. C. L. Kats, M. de Jonge, E. Nilsson-Nyman, E. Visser, “Providing rapid feedback in generated modular language environments”, OOPSLA 2009 (Orlando, Florida, USA, 25–29 October 2009), 445–464.

[18] V. Kountouriotis, Ch. Nomikos, P. Rondogiannis, “Well-founded semantics for Boolean grammars”, Information and Computation, 207:9 (2009), 945–967.

[19] R. Kowalski, Logic for Problem Solving, North-Holland, Amsterdam, 1979.

[20] T. Lehtinen, A. Okhotin, “Boolean grammars and GSM mappings”, International Journal of Foundations of Computer Science, 21:5 (2010), 799–815.

[21] A. Megacz, “Scannerless Boolean parsing”, 6th Workshop on Language Descriptions, Tools, and Applications (LDTA 2006), Electronic Notes in Theoretical Computer Science, 164:2 (2006), 97–102.

[22] K. Mehlhorn, “Pebbling mountain ranges and its application to DCFL-recognition”, Automata, Languages and Programming (ICALP 1980, Noordwijkerhout, The Netherlands, 14–18 July 1980), LNCS 85, 422-435.

[23] P. Naur (Ed.), J. W. Backus, J. H. Wegstein, A. van Wijngaarden, M. Woodger, F. L. Bauer, J. Green, C. Katz, J. McCarthy, A. J. Perlis, H. Rutishauser, K. Samelson, B. Vauquois, “Revised report on the algorithmic language ALGOL 60”, Communications of the ACM, 6:1 (1963), 1–17.

[24] A. Okhotin, “Conjunctive grammars”, Journal of Automata, Languages and Combinatorics, 6:4 (2001), 519–535.

[25] A. Okhotin, “Conjunctive grammars and systems of language equations”, Programming and Computer Science, 28:5 (2002), 243–249.

[26] A. Okhotin, “Whale Calf, a parser generator for conjunctive grammars”, Implementation and Application of Automata (Proceedings of CIAA 2002, Tours, France, July 3–5, 2002), LNCS 2608, 213–220.

[27] A. Okhotin, “On the closure properties of linear conjunctive languages”, Theoretical Computer Science, 299:1–3 (2003), 663–685.

[28] A. Okhotin, “On the equivalence of linear conjunctive grammars to trellis automata”, RAIRO Informatique Théorique et Applications, 38:1 (2004), 69–88.
A. Okhotin, “Boolean grammars”, Information and Computation, 194:1 (2004), 19–48.

A. Okhotin, “On the existence of a Boolean grammar for a simple programming language”, Automata and Formal Languages (Proceedings of AFL 2005, 17–20 May 2005, Dobogókő, Hungary).

A. Okhotin, “Generalized LR parsing algorithm for Boolean grammars”, International Journal of Foundations of Computer Science, 17:3 (2006), 629–664.

A. Okhotin, “Recursive descent parsing for Boolean grammars”, Acta Informatica, 44:3-4 (2007), 167–189.

A. Okhotin, “Unambiguous Boolean grammars”, Information and Computation, 206 (2008), 1234–1247.

A. Okhotin, “Conjunctive and Boolean grammars: the true general case of the context-free grammars”, Computer Science Review, 9 (2013), 27–59.

A. Okhotin, “Parsing by matrix multiplication generalized to Boolean grammars”, Theoretical Computer Science, 516 (2014), 101–120.

A. Okhotin, “Input-driven languages are linear conjunctive”, Theoretical Computer Science, 618 (2016), 52–71.

W. C. Rounds, “LFP: A logic for linguistic descriptions and an analysis of its complexity”, Computational Linguistics, 14:4 (1988), 1–9.

D. J. Salomon, G. V. Cormack, “Scannerless NSLR(1) parsing of programming languages”, Proceedings of the ACM SIGPLAN 1989 Conference on Programming Language Design and Implementation (PLDI 1989), SIGPLAN Notices, 24:7 (1989), 170–178.

S. Schmitz, “An experimental ambiguity detection tool”, Science of Computer Programming, 75:1–2 (2010), 71–84.

P. F. Schuler, “Weakly context-sensitive languages as model for programming languages”, Acta Informatica, 3 (1974), 155–170.

P. F. Schuler, “Inductive definability in formal language theory”, Journal of Computer and System Sciences, 16 (1978), 400–412.

H. Seki, T. Matsumura, M. Fujii, T. Kasami, “On multiple context-free grammars”, Theoretical Computer Science, 88:2 (1991), 191–229.

S. Sokolowski, “A method for proving programming languages non context-free”, Information Processing Letters, 7:2 (1978), 151–153.

A. Stevenson, J. R. Cordy, “Parse views with Boolean grammars”, Science of Computer Programming, 97:1 (2015), 59–63.

M. Tomita, “An efficient augmented context-free parsing algorithm”, Computational Linguistics, 13:1 (1987), 31–46.

M. Y. Vardi, “The complexity of relational query languages”, STOC 1982, 137–146.

K. Vijay-Shanker, D. J. Weir, A. K. Joshi, “Characterizing structural descriptions produced by various grammatical formalisms”, 25th Annual Meeting of the Association for Computational Linguistics (ACL 1987), 104–111.
[48] A. van Wijngaarden, “Orthogonal design and description of a formal language”, Technical Report MR 76, Mathematisch Centrum, Amsterdam, 1965.

[49] D. Wotschke, “The Boolean closures of deterministic and nondeterministic context-free languages” in W. Brauer (Ed.), Gesellschaft für Informatik e. V., 3. Jahrestagung 1973, LNCS 1, 113–121.