A SHORT NOTE ON VECTOR BUNDLES ON CURVES

MARTIN KREIDL

ABSTRACT. In [BL94] Beauville and Laszlo give an interpretation of the affine Grassmannian for $\text{Gl}_n$ over a field $k$ as a moduli space of, loosely speaking, vector bundles over a projective curve together with a trivialization over the complement of a fixed closed point. In order to establish this correspondence, they use an abstract descent lemma, which they prove in [BL95]. It turns out, however, that one can avoid this descent lemma by using a simple approximation-argument, which leads to a more direct prove of the above mentioned correspondence.

1. INTRODUCTION

There is a well-known correspondence between points of the affine Grassmannian for $\text{Gl}_n$ and vector bundles on a projective curve together with certain trivializations. Let us recall this correspondence, as Beauville and Laszlo describe it in [BL94].

Let $X$ be a smooth projective curve over $k$, $p \in X$ be a closed point, and choose a uniformizer $z \in \mathcal{O}_{X,p}$. We fix these data for the rest of these notes. For every $k$-algebra $R$ we set

$$
\begin{align*}
X_R & := X \otimes_{\text{Spec} k} \text{Spec} R, \\
X_R^* & := \text{Spec}(\mathcal{O}_X(X - \{p\}) \otimes_k R), \\
D_R & := \text{Spec} R[[z]], \\
D_R^* & := R((z)).
\end{align*}
$$

These data determine a cartesian diagram of schemes

$$
\begin{array}{ccc}
D_R \\
\downarrow i \\
D_R^* & \xrightarrow{j} & X_R.
\end{array}
$$

Beauville and Laszlo prove the following

**Proposition 1** ([BL94], Proposition 1.4). The functor

$$L \text{GL}_n : R \mapsto \text{Gl}_n(R((z)))$$

on the category of $k$-algebras is isomorphic to the functor which associates to $R$ the set of isomorphism classes of triples $(E, \rho, \sigma)$, where $E$ is a vector bundle of rank $n$ over $X_R$, and $\rho$ and $\sigma$ are trivializations of $E$ over $X_R^*$ and $D_R$, respectively.

As a consequence they obtain

**Proposition 2** ([BL94], Proposition 2.1 and Remark 2.2). The affine Grassmannian for $\text{Gl}_n$, which is by definition the fpqc-sheafification of the functor $R \mapsto \text{Gl}_n(R((z)))/\text{Gl}_n(R[[z]])$, is isomorphic to the functor which associates to $R$ the set
of isomorphism classes of pairs \((E, \rho)\), where \(E\) is a vector bundle of rank \(n\) over \(X_R\), and \(\rho\) is a trivialization of \(E\) over \(X_R^\ast\).

The interesting part in the proof of Proposition 1 is to see why the data of trivial vector bundles of rank \(n\) on \(D_R\) and \(X_R^\ast\), respectively, together with a transition function over \(X_R^\ast\), determine a vector bundle on \(X_R\). This is not a classical descent situation, since if \(R\) is not Noetherian, \(D_R\) is in general not flat over \(X_R\). In [BL95] Beauville and Laszlo prove that descent holds nonetheless.

In the present notes we present an alternative proof of Proposition 1 using the following strategy. We define the subring \(A_R \subset R[[z]]\) as a certain localization of \(\mathcal{O}_{X,p} \otimes_k R\), which depends functorially on \(R\) and determines a flat neighborhood of the locus \(z = 0\) in \(X_R\). Let us write \(\Delta_R = \text{Spec } A_R\) and \(\Delta^R_R = \text{Spec } A_R[1/z]\). Then \(\Delta_R \coprod X_R^\ast \to X_R\) is an fppf-covering, and if we could replace \(D_R\) by \(\Delta_R\) and \(D^R_R\) by \(\Delta^R_R\) in the formulation of Proposition 1, then this proposition would immediately follow by faithfully flat descent. Indeed, we will show below how to arrive at this situation using a simple approximation argument. Moreover, the concrete situation will turn out to be not only fppf-local, but even Zariski-local, so that descent of vector bundles holds trivially.

2. Vector bundles on a smooth curve

Note that the choice of a uniformizer \(z \in \mathcal{O}_{X,p}\) determines an inclusion \((R \otimes_k \mathcal{O}_{X,p}) \subset R[[z]], R[[z]]\) being the completion with respect to the \(z\)-adic valuation. For each \(f \in (R \otimes_k \mathcal{O}_{X,p}) \cap R[[z]]\) we define \(S_{R,f} := (R \otimes_k \mathcal{O}_{X,p})_f \subset R[[z]]\). The union of all these rings, for varying \(f\), will be denoted \(A_R\). Writing \(\Delta_R := \text{Spec } A_R\) and \(\Delta^R_R := \text{Spec } A_R[1/z]\) we have a cartesian diagram

\[
\begin{array}{ccc}
\Delta_R & \xrightarrow{\psi} & X_R^\ast \\
\downarrow{\iota} & & \downarrow{\jmath} \\
\Delta_R & \xrightarrow{\varphi} & X_R \\
\end{array}
\]

Moreover we set \(U_{R,f} := \text{Spec } S_{R,f}\).

**Lemma 3.** The morphism \(D_R \coprod X_R^\ast \to X_R\) is surjective. Thus \(\Delta_R \coprod X_R^\ast \to X_R\) is an fppf-, and \(U_{R,f} \coprod X_R^\ast \to X_R\) is a Zariski-covering for each \(f \in (R \otimes_k \mathcal{O}_{X,p}) \cap R[[z]]\).

**Proof.** Let \(P\) be a point of \(X_R\) and let \(A = (\mathcal{O}_X \otimes R)_P\) be the local ring at \(P\). Either \(z\) is invertible in \(A\) — then \(P \in X_R^\ast\) — or \(z\) is in the maximal ideal \(p \subset A\). In the latter case we consider \(\text{can} : A \to \hat{A} = \lim A/z^N\) and the ideal \(\hat{p} = \lim p/z^N\).

Passing to the inverse limit over the short exact sequences

\[
0 \to p/(z^N) \to A/(z^N) \to A/p \to 0
\]

we obtain \(\text{can}^{-1}(\hat{p}) = p\), and the commutative square

\[
\begin{array}{ccc}
\text{Spec } \hat{A} & \xrightarrow{\text{can}} & \text{Spec } R[[z]] = D_R \\
\downarrow & & \downarrow \\
\text{Spec } A & \xrightarrow{\text{can}^{-1}} & X_R \\
\end{array}
\]

shows that \(\hat{p} \cap R[[z]] \subset R[[z]]\) is a preimage of \(P\) in \(D_R\). \(\square\)
Let $T$ be the functor on the category of $k$-algebras, which associates to a $k$-algebra $R$ the set of isomorphisms classes of triples $(E, \rho, \sigma)$, where $E$ is a vector bundle of rank $n$ on $X_R$, and
\[
\rho : O^n_{X_R} \xrightarrow{\sim} E|_{X_R},
\sigma : O^n_{\Delta_R} \xrightarrow{\sim} E|_{\Delta_R}
\]
are trivializations. To each isomorphism class $[(E, \rho, \sigma)] \in T(R)$ we may assign the respective ‘transition matrix over $\Delta_R$’. This is independent of the actual representative of $[(E, \rho, \sigma)]$ and hence determines a morphism of functors
\[
\Phi(R) : T(R) \to \text{Gl}_n(A_R[1/z]); \quad (E, \rho, \sigma) \mapsto \Gamma(X_R,(\rho|\Delta_R) \circ (\sigma^{-1}|\Delta_R)).
\]

**Proposition 4.** The morphism $\Phi(R)$ defined above is an isomorphism of functors.

**Proof.** We have to construct an inverse for $\Phi(R)$. To this end, we choose a matrix $g \in \text{Gl}_n(A_R[1/z])$ and consider the following diagram of quasi-coherent sheaves on $X_R$,
\[
\begin{array}{ccc}
E & \xrightarrow{\text{can}} & O^n_{X_R} \\
\downarrow & & \downarrow \\
O^n_{\Delta_R} & \xleftarrow{\text{can}} & O^n_{\Delta_R} \xrightarrow{g} O^n_{\Delta_R} \\
\end{array}
\]
where $E$ is uniquely determined up to isomorphism by requiring that the diagram be cartesian. (By abuse of notation we do not indicate the obvious push-forwards to $X_R$ in this diagram.) It is easy to check (by pullback to $\Delta_R$ and $X_R^*$, respectively) that this diagram determines trivializations of $E$ over $\Delta_R$ and $X_R^*$. The transition function for these two trivializations is equal to $g$ by construction.

To see that this construction indeed gives an inverse for $\Phi(R)$ it remains to check that $E$ is a vector bundle. This is immediate by Lemma 3 together with faithfully flat descent, or by the following elementary argument: the matrix $g$ involves only finitely many elements of $A_R[1/z]$, whence in fact $g \in S_{R,f}[1/z]$ for some $f \in (R \otimes_k O_{X,p}) \cap R[[z]]^\times$. This shows that $E$ can as well be obtained by gluing trivial bundles over $U_{R,f}$ and over $X_R^*$, respectively. Now, since $U_{R,f} \subset X_R$ is Zariski-open, this shows that $E$ is a vector bundle. \qed

3. ‘Formal’ descent of vector bundles

Let us now consider the situation introduced at the beginning in diagram (1.2), where we consider the formal neighborhood $D_R = \text{Spec } R[[z]]$ of $\text{Spec } R \times \{p\} \subset X_R$.

By $\hat{T}$ we denote the functor, which associates to every $k$-algebra $R$ the set of isomorphism classes of triples $(E, \rho, \sigma)$, where $E$ is a vector bundle of rank $n$ over $X_R$ and
\[
\rho : O^n_{X_R} \xrightarrow{\sim} E|_{X_R},
\sigma : O^n_{D_R} \xrightarrow{\sim} E|_{D_R}
\]
are trivializations.

As in the previous section, we obtain a functorial morphism $\hat{\Phi}(R) : \hat{T}(R) \to \text{Gl}_n(R((z)))$ by assigning to each triple $(E, \rho, \sigma)$ the corresponding transition function over $D^*_R$. 
Theorem 5. ([BL94], Proposition 1.4). The morphism $\hat\Phi$ is an isomorphism of functors.

Proof. In order to construct an inverse for $\hat\Phi$, i.e. to construct a triple $(E, \rho, \sigma)$ from a given $\gamma \in \text{Gl}_n(R((z)))$, we proceed exactly as in the proof of Proposition 4. The only non-trivial thing to check is that the quasi-coherent sheaf $E$, defined so to make the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\gamma} & O_{X_R}^n \\
\downarrow & & \downarrow \\
O_{D_R}^n & \xrightarrow{\text{can}} & O_{D_R}^n
\end{array}
\]

cartesian, is a vector bundle over $X_R$. We do this by reducing to a situation where Proposition 4 applies. More precisely, Lemma 6 below shows that every $\gamma \in \text{Gl}_n(R((z)))$ can be written as a product $\gamma = g \cdot \delta$, where $g \in \text{Gl}_n(A_R[1/z])$ and $\delta \in \text{Gl}_n(R[[z]])$.

Thus diagram (3.1) 'decomposes' likewise, and yields the big diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\gamma} & O_{X_R}^n \\
\downarrow & & \downarrow \\
O_{D_R}^n & \xrightarrow{\text{can}} & O_{D_R}^n
\end{array}
\]

The two small squares in this diagram are trivially cartesian, while the big rectangle coincides with the square (3.1), and is thus cartesian by definition of $E$. Consequently, the upper rectangle is cartesian, which proves that $E$ is nothing but the vector bundle corresponding to the transition matrix $g \in \text{Gl}_n(A_R[1/z])$ under the correspondence of Proposition 4.

Lemma 6. We have $\text{Gl}_n(R((z))) = \text{Gl}_n(A_R[1/z]) \cdot \text{Gl}_n(R[[z]])$.

Proof. We set $B := \cup_{P \in R[z]} R[z, z^{-1}, P^{-1}] \subset R((z))$ (Note that the ring $B \cap R[[z]]$ is equal to the ring $A_R$ in the case $X = \mathbb{P}^1_R$). Since $B \subset A_R[1/z]$, it suffices to check that $\text{Gl}_n(R((z))) = \text{Gl}_n(B) \cdot \text{Gl}_n(R[[z]])$. First we note that $\text{Gl}_n(R[[z]]) \subset \text{Gl}_n(R((z)))$ is open: Namely, $\det : \text{Mat}_n(R[[z]]) \to R[[z]]$ is continuous and $R$ carries the discrete topology, and thus $R^\times \subset R$ is open. This shows that $\text{Gl}_n(R[[z]]) \subset \text{Mat}_n(R[[z]]) \subset \text{Mat}_n(R((z)))$ are two open inclusions, so $\text{Gl}_n(R[[z]]) \subset \text{Gl}_n(R((z)))$ is as well open. As a second step we deduce from Lemma 7 below that $\text{Gl}_n(B) = \text{Gl}_n(R((z))) \cap \text{Mat}_n(B)$. Since $\text{Mat}_n(B) \subset \text{Mat}_n(R((z)))$ is dense and $\text{Gl}_n(R((z))) \subset \text{Mat}_n(R((z)))$ is open, we conclude that $\text{Gl}_n(B) \subset \text{Gl}_n(R((z)))$ is dense.

These two statements together imply that $\text{Gl}_n(B) \cdot \text{Gl}_n(R[[z]])$ is dense and closed in $\text{Gl}_n(R((z)))$, whence the lemma.

Lemma 7. The subring $B \subset R((z))$ defined above satisfies $B^\times = R((z))^\times \cap B$. 

Proof. We consider \( f \in R((z))^\times \cap B \). By multiplying with a suitable \( P \in R[z] \cap R[[z]]^\times \), we may reduce to the case \( f \in R((z))^\times \cap R[z, z^{-1}] \). Such an \( f \) has the form \( f = -N + Q \), where \( N \in R[z, z^{-1}] \) is a nilpotent Laurent polynomial and the leading coefficient of \( Q \in R((z))^\times \) is a unit in \( R \). Using the formula \( (-N + Q)(N^i + N^{i-1}Q + \cdots + Q^i) = (-N^i + Q^i) \) we may assume that \( f = Q^i \), i.e. has a leading coefficient in \( R^\times \). Multiplying with \( z^m \) for a suitable \( m \in \mathbb{Z} \) we obtain \( z^m f \in R[z] \cap R[[z]]^\times \), which is invertible in \( B \) by construction. \( \square \)

The property of the ring \( B \) which is exhibited in the last lemma is crucial for our strategy of approximation to work. This is what forces us to consider the, at first glance, rather artificial rings \( A_R \) instead of for example just \( \mathcal{O}_{X,p} \otimes R \). The latter would not contain the ring \( B \), and in particular would not have the property of Lemma 7.

References

[BL94] Arnaud Beauville and Yves Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. 164 (1994), no. 2, 385–419.
[BL95] Arnaud Beauville and Yves Laszlo, *Un lemme de descente*, C.R.A.S. 320 (1995), no. 3, 335–340.