ON THE OBSERVABILITY OF CONFORMABLE LINEAR TIME-INVARIANT CONTROL SYSTEMS

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Abstract. In this paper, we analyze the concept of observability in the case of conformable time-invariant linear control systems. Also, we study the Gramian observability matrix of the conformable linear system, its rank criteria, null space, and some other conditions. We also discuss some properties of conformable Laplace transform.

1. Introduction. A new interest has been developed to fractional-order systems in the area of control theory. One can find the uses and importance of fractional derivatives in the field of control theory, dynamical systems, nanotechnology, viscoelasticity, financial modeling, anomalous transport, and anomalous diffusion, see, for example, [2,4,6]. In modern control theory, controllability and observability have become the backbone, which was introduced by Kalman in 1960, see [8], and [9]. Without determining the solution, a control system can be classified with the help of observability and controllability. In this paper, we focus on the idea of observability. It has many thought-provoking applications. For example, in recent years, observability is used in wave equation with Ventcel dynamic condition, see [3] and also regional observability is scrutinized for Hadamard-Caputo time-fractional distributed parameter systems in [16].

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Let us take the linear time-invariant system as follows
\[\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\
y(t) &= Cx(t).
\end{align*}\] (1)

In the above system \(x(t) \in \mathbb{R}^{n \times 1}, u(t) \in \mathbb{R}^{m \times 1}, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times m}, C \in \mathbb{R}^{k \times n}, y(t) \in \mathbb{R}^{k \times 1}.\)

The solution of (1) is already defined in [15]. Also, many results have been attained related to the observability of linear time-invariant systems where a linear system is said to be observable if there exists at least one output provided that the initial state can be determined. There are no results of observability in the case of the Conformable Linear Time-invariant Control System (CLTCS). Not a long ago, the concept of fractional is also described in dynamical systems by [19]. While Khalil et al. [7] came up with an interesting theory that extent the limit definition of a derivative of a function termed as conformable derivative. Also, the Laplace transform is an advantageous technique for solving linear differential equations in conformable calculus. Abdeljawad [1] defined conformable Laplace transform and its essential properties are also mentioned in [17]. Very recently, many developments in this concept appears in the study of mathematical models because of conformable derivatives [18,20]. Many more applications of conformable derivatives are available in the literature. The stability analysis of conformable linear differential system found in [12,14]. For more applications of fractional and conformable derivatives readers can be referred to [11,13].

In this paper, we develop new properties of conformable Laplace transform. Also, we examined observability in CLTCS and obtained useful consequence from which we can quickly check whether the CLTCS is either observable or not. We consider the following CLTCS, that is
\[\begin{align*}
T_{t_0}^\alpha x(t) &= Ax(t) + Bu(t), \quad 0 < \alpha \leq 1, \\
y(t) &= Cx(t), \quad x(t_0) = x_0.
\end{align*}\] (2)

where \(y(t) \in \mathbb{R}^k, x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \(C \in \mathbb{R}^{k \times n}.\) We discuss observability of (2). Moreover, we study the Gramian observability matrix of CLTCS, its rank criteria, its null space, and some other properties.

The organization of this manuscript is as follows: In section 2, we examine some elementary facts about conformable Laplace and conformable calculus. In section 3, the concept of observability is generalized in the case of conformable linear time-invariant control systems, and also results of the observability of CLTCS are presented.

2. Preliminaries. In this section, we provide some basic concepts of conformable Laplace transform:

**Definition 2.1.** [1, Definition 5.1]Let \(t_0 \in \mathbb{R}, 0 < \alpha \leq 1\) and \(g : [t_0, \infty) \to \mathbb{R}\) be real-valued function. Then the conformable Laplace transform of order \(\alpha\) of \(g\) is presented as
\[L_{t_0}^\alpha \{g(t)\} (s) \equiv G_{t_0}^\alpha (s) := \int_{t_0}^\infty e^{-\frac{s(t-t_0)\alpha}{\alpha}} g(t) \, d\alpha (t,t_0).\] (3)
Lemma 2.2. [1, Lemma 5.2] Let \( g : [t_0, \infty) \to \mathbb{R} \) be a function such that \( L^0_\alpha \{ g(t) \} (s) = G^0_\alpha (s) \) exists. Then
\[
G^0_\alpha (s) = \mathcal{L} \left\{ g \left( t_0 + (at)^{\frac{1}{\alpha}} \right) \right\} (s)
\]
where “\( \mathcal{L} \)” represents classical Laplace transform.

Definition 2.3. A function \( g : [0, \infty) \to \mathbb{R} \) is of exponential order \( \beta \), if there exists a constant \( M > 0 \) and \( \beta \) such that for some \( t_0 \geq 0 \), we have \( |g(t)| \leq Me^{\beta t} \) for all \( t \geq t_0 \).

Example 2.4. Let \( g(t) = t^2 \), \( t > N \) is of exponential order for \( M = 1 \) and \( \beta = 3 \), we have \( |g(t)| = |t^2| = t^2 < e^{3t} \).

Lemma 2.5. For every function \( g \) of exponential order, we have
\[
L^0_\alpha \{ g(at) \} (s) = \left( \frac{1}{a} \right)^\alpha L^2_\alpha \{ g(t) \} \left( s \cdot \frac{1}{a} \right) . \tag{4}
\]

Proof. By Definition 3, we can see that
\[
L^0_\alpha \{ g(at) \} (s) = \int_{t_0}^{\infty} e^{-\frac{(at-t_0)^\alpha}{\alpha}} g(at)(t-t_0)^{\alpha-1} dt,
\]
by substituting \( at = u \), and \( t = t_0 \), we get (4). \( \square \)

Lemma 2.6. Consider \( L^0_\alpha \{ g(t) \} = G^0_\alpha (s) \) and \( a \) is any real number, then
\[
L^0_\alpha \left[ e^{\frac{a(t-t_0)^\alpha}{\alpha}} g(t) \right] = G^0_\alpha (s-a). \tag{5}
\]

Proof. By using Definition 3, we obtain 5. \( \square \)

Let us recall basic definitions and properties of conformable derivative.

Definition 2.7. [1, Definition 2.1] The conformable derivative starting from \( a \) of a function \( g : [a, \infty) \to \mathbb{R} \) of order \( 0 \leq \alpha \leq 1 \) is defined by
\[
(T^t_\alpha g)(t) = \lim_{\epsilon \to 0} \frac{g(t + \epsilon(t-a)^{1-\alpha}) - g(t)}{\epsilon}.
\]

Lemma 2.8. [7, Theorem 2.2] Let \( \alpha \in (0, 1] \) and \( g \) be a differentiable function at a point \( t > 0 \). Then
\[
T_\alpha g(t) = t^{1-\alpha} \frac{dg}{dt} (t).
\]

Lemma 2.9. [10] Let \( g(t) \) be a differentiable function,
\[
T_\alpha g(t), T^2_\alpha g(t), T^3_\alpha g(t) \ldots T^{(n-1)}_\alpha g(t), T^n_\alpha g(t)
\]
are continuous on \([0, \infty)\) which are of exponential order. Moreover, \( T^n_\alpha g(t) \) is piece-wise continuous on interval \([0, \infty)\), then
\[
L^0_\alpha \{ T^n_\alpha g(t) \} = s^n G_\alpha (s) - s^{n-1} g(0) - s^{n-2} T_\alpha g(0) - \cdots - T^{n-1}_\alpha g(0)
\]
where \( G_\alpha (s) = L^0_\alpha \{ g(t) \} \).


Proof. Let $T_\alpha g(t)$ is continuous for $t \geq 0$, then

$$L^0_\alpha [T_\alpha g(t)] = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-s\tau} \frac{\tau}{\alpha} T_\alpha g(t) \, d\tau = -g(0) + sG_\alpha(s),$$

and

$$L^0_\alpha [T_\alpha^2 g(t)] = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-s\tau} \frac{\tau^2}{\alpha} T_\alpha g(t) \, d\tau = s^2G_\alpha(s) - T_\alpha g(0) - sg(0).$$

Moreover,

$$L^0_\alpha [T_\alpha^n g(t)] = s^nG_\alpha(s) - s^{n-1}g(0) - s^{n-2}T_\alpha g(0) - \cdots - T_\alpha^{n-1}g(0).$$

proved. \hfill \square

There are some more properties of conformable Laplace transform:

**Lemma 2.10.** For every function $g$ of exponential order, we have

$$T^s_\alpha \left\{ L^t_\alpha g(t)(s) \right\} = -s^{1-\alpha} L^t_\alpha \left\{ \frac{(t-t_0)^{\alpha}}{\alpha} g(t) \right\}(s).$$

Proof. It is easy to verify by taking conformable derivative with respect to $s$ of conformable Laplace transform. \hfill \square

**Lemma 2.11.** Let $A \in \mathbb{C}^{n \times n}$. Then conformable derivative of conformable exponential matrix is as follows:

$$T^t_\alpha \left( e^{A(t-a)^{\alpha}} \right) = Ae^{\frac{(t-a)^{\alpha}}{\alpha}},$$

where

$$e^{\frac{(t-a)^{\alpha}}{\alpha}} = \sum_{k=0}^{\infty} \frac{A^k(t-a)^{k\alpha}}{\alpha}.$$

**Definition 2.12.** [10] If $g$ and $h$ are piecewise continuous on $[0, \infty)$, then a product called convolution of $g$ and $h$ denoted by $g \ast h$ is defined as:

$$(g \ast h)_\alpha(t) = \frac{1}{\alpha} \int_0^t g(\tau)h(t-\tau)d\tau.$$

**Theorem 2.13.** [10] If $f$ and $g$ are piecewise continuous on the interval $[0, \infty)$ and are of exponential order, then we have

$$L^t_\alpha \{ f \ast g \} = L^t_\alpha \{ f(t) \} \, L^t_\alpha \{ g(t) \} = F^t_\alpha(s)G^t_\alpha(s).$$

Next, the basic concepts related to the observability of linear time-invariant system are given:

Consider the following linear time invariant system of the following form

$$\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) =Cx(t)
\end{cases}$$

(8)
where \( x(t) \) is \( n \times 1 \) vector called state vector, \( u(t) \) is \( m \times 1 \) input signal and \( y(t) \) is output signal. The solution of equation (8) with initial condition \( x(t_0) = x_0 \) has the following form

\[
x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t}e^{A(t-s)}B(s)u(s)\,ds
\]

where \( e^{At} \) is an exponential matrix.

**Definition 2.14.** [15, Definition 9.7] If from corresponding response \( y(t) \) on \( t \in [t_0, t_f] \), any initial state \( x(t_0) = x_0 \) is determined uniquely, then the linear state equation (8) is known as observable on \( [t_0, t_f] \).

**Definition 2.15.** The observability Gramian matrix \( W(t_0, t_f) \) for a linear time-invariant system is given as

\[
W(t_0, t_f) = \int_{t_0}^{t_f}e^{A^Tt}C^TCe^{At}\,dt.
\]

**Lemma 2.16.** [15, Theorem 9.8] The necessary and sufficient condition for the linear state equation (8) to be observable on \( [t_0, t_f] \) is that the observability Gramian matrix is invertible.

For the next result, let us recall the unobservable subspace as follows:

**Definition 2.17.** [15, Definition 18.10] For scalar matrix \( A \) and \( C \) with appropriate dimension the subspace \( \mathcal{N} \subset X \) given by

\[
\mathcal{N} = \bigcap_{k=0}^{n-1} \ker [CA^k]
\]

is unobservable subspace for system (8).

Unobservable subspace for equation (8) contains a slight extension of our inverse-image notation:

\[
\mathcal{N} = \ker [C] \cap A^{-1} \ker [C] \cap \cdots \cap A^{-(n-1)} \ker [C],
\]

where \( A^{-1}(w) \) is an inverse image of \( w \) under \( A \).

**Proposition 2.18.** [15] For any \( t_\alpha > 0 \)

\[
\mathcal{N} = \ker [W(0, t_\alpha)].
\]

**Corollary 2.19.** [15, Corollary 18.12] The necessary and sufficient condition for the linear state equation (8) to be observable on the interval \( [t_0, t_f] \) is that \( \mathcal{N} = 0 \).

Let us consider the conformable linear time invariant system of the below form

\[
T^\alpha_n x(t) = Ax(t) + f(t), \quad 0 < \alpha \leq 1
\]

where \( x, f : (a, b) \rightarrow \mathbb{R}^n \) are vector functions and \( A \) is an \( n \times n \) matrix. The general solution of the conformable non-homogeneous system (10) is presented as:

\[
x(t) = e^{A(t-t_0)^{\alpha}}c + \int_{t_0}^{t}e^{A(t-s)^{\alpha}}e^{-A(t-t_0)^{\alpha}}f(s)(s-t_0)^{-1}\,ds
\]
where \( e^{A(t-t_0)\alpha} = \sum_{k=0}^{\infty} \frac{A^k(t-t_0)^{\alpha k}}{\alpha^k k!} \) and \( c \) is a constant vector.

**Example 2.20.** The solution \( x(t) \) of the following conformable linear time invariant system

\[
T_\alpha^n x(t) = \begin{bmatrix} -1 & 0 \\ -2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t, \quad 0 < \alpha \leq 1, \tag{12}
\]

is expressed as

\[
x(t) = \begin{bmatrix} e^{-2\sqrt{t}} \left( \frac{te^{10t}}{5} - \frac{e^{10t}}{50} + \frac{51}{50} \right) \\ \frac{1}{2} e^{-2\sqrt{t}} \left( \frac{te^{10t}}{5} - \frac{e^{10t}}{50} \right) + \frac{1}{2} e^{6\sqrt{t}} \left( te^{2t} - \frac{1}{2} e^{2t} - \frac{1}{2} \right) \end{bmatrix}. \tag{13}
\]

**Remark 2.21.** It is possible that the solution obtained in (11) is \( \alpha \)-differentiable but not differentiable. For example, consider \( A = 1 \), \( f(t) = \sqrt{t} \) for all \( t \in [0,T] \), \( \alpha = \frac{1}{2} \), \( x_0 = 1 \), \( t_0 = 0 \) in (10) then the solution (11) becomes

\[
x(t) = e^{2\sqrt{t}} - \sqrt{t}. \tag{14}
\]

It is noted that the solution \( x \) is not differentiable at \( t = 0 \). However,

\[
T_{\frac{1}{2}} x(t) = e^{2\sqrt{t}}
\]

It is clear that the solution obtained in (14) is \( \frac{1}{2} \)-differentiable while not differentiable at \( t = 0 \).

### 3. Observability in conformable case.

**Theorem 3.1.** The necessary and sufficient condition for the linear state equation (2) to be observable on \([t_0, t_f]\) is that the conformable observability Gramian matrix

\[
W(t_0, t_f) = \int_{t_0}^{t_f} \left( C e^{A \left( \frac{(t-t_0)\alpha}{\alpha} \right)} \right)^T C e^{A \left( \frac{(t-t_0)\alpha}{\alpha} \right)} d_{\alpha} t \tag{15}
\]

is invertible.

**Proof.** Consider the solution of the system (2) as

\[
x(t) = e^{A \left( \frac{(t-t_0)\alpha}{\alpha} \right)} x_0 + \int_{t_0}^{t} e^{A \left( \frac{(t-s)\alpha}{\alpha} \right)} e^{A \left( \frac{(s-t_0)\alpha}{\alpha} \right)} B u(s) (s-t_0)^{1-\alpha} ds
\]

then \( y(t) \) becomes

\[
y(t) = C e^{A \left( \frac{(t-t_0)\alpha}{\alpha} \right)} x_0 + \int_{t_0}^{t} C e^{A \left( \frac{(t-s)\alpha}{\alpha} \right)} e^{A \left( \frac{(s-t_0)\alpha}{\alpha} \right)} B u(s) (s-t_0)^{1-\alpha} ds.
\]

Letting \( u(t) = 0 \) implies that

\[
y(t) = C e^{A \left( \frac{(t-t_0)\alpha}{\alpha} \right)} x_0.
\]
Multiplying the solution expression by \( e^A \left( \frac{(t-t_0)^\alpha}{\alpha} \right) C \) on both sides and conformable integration yields

\[
\int_{t_0}^{t_f} \left( e^A \left( \frac{(t-t_0)^\alpha}{\alpha} \right) C \right)^T y(t) \, dt = W(t_0, t_f)x_0.
\]

The left hand side is determined from \( y(t) \) on \( t \in [t_0, t_f] \) and therefore (16) represents a linear algebraic equation for \( x_0 \). Then \( x_0 \) is determined uniquely if \( W(t_0, t_f) \) is invertible.

Conversely: Suppose \( W(t_0, t_f) \) is not invertible, then there is a non-zero \( n \times 1 \) vector \( x_b \) provided that

\[
W(t_0, t_f)x_b = 0.
\]

This implies

\[
x_b^T W(t_0, t_f) x_b = 0.
\]

By the definition of norm, it follows that

\[
\left\| e^A \left( \frac{(t-t_0)^\alpha}{\alpha} \right) x_b \right\|^2 = 0.
\]

So, \( x(t_0) = x_0 + x_b \) gives the zero response of input for equation (2) on \( [t_0, t_f] \) as \( x(t_0) = x_0 \), and the state equation fails to be observable on \( [t_0, t_f] \).

Let us consider the conformable exponential matrix of following form

\[
e^A \left( \frac{(t-t_0)^\alpha}{\alpha} \right) = \sum_{k=0}^{\infty} \frac{A^k (t-t_0)^{k\alpha}}{\alpha^k k!}.
\]

If \( A \) is diagonalizable matrix with distinct eigenvalues then \( A^k = PD^k P^{-1} \), equation (18) implies

\[
e^A \left( \frac{(t-t_0)^\alpha}{\alpha} \right) = P \begin{bmatrix}
\sum_{k=0}^{\infty} \frac{\lambda_1^k (t-t_0)^{k\alpha}}{\alpha^k k!} & 0 & \cdots & 0 \\
0 & \sum_{k=0}^{\infty} \frac{\lambda_2^k (t-t_0)^{k\alpha}}{\alpha^k k!} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{\lambda_n^k (t-t_0)^{k\alpha}}{\alpha^k k!}
\end{bmatrix} P^{-1}
\]
Therefore conformable exponential matrix can be decompose as
\[
e^{A \alpha (t-t_0)^\alpha} = P \left[ \begin{array}{ccc}
e^{\frac{\lambda_1(t-t_0)^\alpha}{\alpha}} & 0 & \ldots & 0 \\
0 & e^{\frac{\lambda_2(t-t_0)^\alpha}{\alpha}} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & e^{\frac{\lambda_n(t-t_0)^\alpha}{\alpha}}
\end{array} \right] P^{-1}.
\]

**Example 3.2.** Consider a linear state equation
\[
T_\alpha x(t) = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} x(t) + Bu(t)
y(t) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x(t).
\]
by equation (15) observability Gremian matrix becomes
\[
W(0,1) = \begin{bmatrix} 0.88526 & 0.9377 \\ 0.9377 & 4.89225 \end{bmatrix}.
\]
We easily compute that \(\text{det}[W(0,1)] = 3.4516 \neq 0\) and so it is invertible.

**Proposition 3.3.** There exist analytic scalar functions \(a_0(t), a_1(t), \ldots, a_{n-1}(t)\) such that
\[
e^{A \alpha (t-t_0)^\alpha} = \sum_{k=0}^{n-1} a_k \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^k.
\]

**Proof.** The \(n \times n\) matrix differential equation
\[
T_\alpha X(t) = AX(t), \ X(t_0) = I
\]
has the unique solution
\[
X(t) = e^{A \alpha (t-t_0)^\alpha}.
\]
We can establish equation (20) by showing that the existence of scalar analytic functions \(a_0(t), a_1(t), \ldots, a_{n-1}(t)\) such that
\[
\sum_{k=0}^{n-1} a_k \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^k = \sum_{k=0}^{n-1} a_k \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^{k+1}, \ \sum_{k=0}^{n-1} a_k (0) A^k = I.
\]
The Cayley-Hamilton theorem implies
\[
A^n = a_0 I - a_1 A - \cdots - a_{n-1} A^{n-1}
\]
where \(a_0, a_1, \ldots, a_{n-1}\) are the coefficients in the characteristic polynomial of \(A\). Then, equation (21) can be presented as:
\[
\sum_{k=0}^{n-1} a_k \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^k = \left[ \sum_{k=0}^{n-2} a_k \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^{k+1} + a_{n-1} \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^n \right]
\]
\[
= \sum_{k=0}^{n-2} a_k \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^{k+1} + \sum_{k=0}^{n-1} a_k a_{n-1} \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^k
\]
\[
= \sum_{k=1}^{n-1} a_{k-1} \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^k + \sum_{k=1}^{n-1} a_k a_{n-1} \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^k
\]
\[
- a_0 a_{n-1} \left(\frac{(t-t_0)^\alpha}{\alpha}\right) I
\]
subsequent condition. Comparing coefficients of same powers of \( A \) separately. Comparing coefficients of same powers of \( A \) gives the time-invariant linear state equation:

\[
\sum_{k=0}^{n-1} \frac{a_k}{\alpha} (t-t_0)^\alpha A^k = \left[-a_0 a_{n-1} \left(\frac{(t-t_0)^\alpha}{\alpha}\right) I \right] + \sum_{k=1}^{n-1} a_k \left(\frac{(t-t_0)^\alpha}{\alpha}\right) + a_k a_{n-1} \left(\frac{(t-t_0)^\alpha}{\alpha}\right) A^k \] , \sum_{k=1}^{n-1} a_k (0) = I. \tag{22}
\]

Equation (22) can be solved by taking the coefficient equation for each power of \( A \) separately. Comparing coefficients of same powers of \( A \) gives the time-invariant linear state equation:

\[
\begin{bmatrix}
\dot{a}_0 (t) \\
\dot{a}_1 (t) \\
\vdots \\
\dot{a}_n (t)
\end{bmatrix} = 
\begin{bmatrix}
0 & \cdots & 0 & -a_0 \\
1 & \cdots & 0 & -a_1 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & -a_{n-1}
\end{bmatrix} 
\begin{bmatrix}
a_0 (t) \\
a_1 (t) \\
\vdots \\
a_{n-1} (t)
\end{bmatrix} + 
\begin{bmatrix}
a_0 (0) \\
a_1 (0) \\
\vdots \\
a_{n-1} (0)
\end{bmatrix} = 
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

Thus the existence of an analytic solution to this linear state equation exhibits the existence of analytic functions \( a_0 (t), a_1 (t), \ldots, a_{n-1} (t) \) that satisfy equation (22).

Here, we consider the case of zero input that is \( u (t) = 0 \), then equation (2) becomes:

\[
\begin{cases}
T_0^t x (t) = Ax (t) \\
y (t) = C x (t)
\end{cases} \tag{23}
\]

**Theorem 3.4.** The conformable time-invariant linear state equation (CTLSE) (23) is observable on \([t_0, t_f] \) if and only if the \( np \times n \) observability matrix holds the subsequent condition

\[
\text{rank} \begin{bmatrix}
C \\
CA \\
\vdots \\
C A^{n-1}
\end{bmatrix} = n. \tag{24}
\]

If the following conformable linear time-invariant system

\[
T_0^t x (t) = Ax (t) + f (t), \ 0 < \alpha \leq 1. \tag{25}
\]

has a solution which is differentiable, then Lemma 2.8 yields that

\[
(t-t_0)^{1-\alpha} x' (t) = Ax (t) + f (t). \tag{26}
\]

That is classical linear time-varying system

\[
x' (t) = (t-t_0)^{\alpha-1} Ax (t) + (t-t_0)^{\alpha-1} f (t). \tag{27}
\]

We know that the solution of equation (25) has the following form

\[
x (t) = e^{A \frac{(t-t_0)^\alpha}{\alpha}} c + \int_{t_0}^{t} e^{A \frac{(t-t_0)^\alpha}{\alpha}} e^{-A \frac{(t-t_0)^\alpha}{\alpha}} f (s) (s-t_0)^{\alpha-1} ds. \tag{28}
\]

On the other hand, if we take classical derivative of equation (28), we get

\[
x' (t) = A e^{A \frac{(t-t_0)^\alpha}{\alpha}} (t-t_0)^{\alpha-1} c + e^{A \frac{(t-t_0)^\alpha}{\alpha}} e^{-A \frac{(t-t_0)^\alpha}{\alpha}} f (t) (t-t_0)^{\alpha-1} + 
\int_{t_0}^{t} (t-t_0)^{\alpha-1} A e^{A \frac{(t-t_0)^\alpha}{\alpha}} e^{-A \frac{(t-t_0)^\alpha}{\alpha}} f (s) (s-t_0)^{\alpha-1} ds
\]
\begin{align*}
= (t - t_o)^{\alpha-1} A \left[ e^{A \frac{(t-t_o)^\alpha}{\alpha}} c + \int_{t_0}^{t} e^{A \frac{(s-t_o)^\alpha}{\alpha}} e^{-A \frac{(s-t_0)^\alpha}{\alpha}} f(s)(s-t_0)^{\alpha-1} \, ds \right] \\
+ f(t)(t - t_o)^{\alpha-1}.
\end{align*}

Finally, we have
\begin{equation}
x' = (t - t_o)^{\alpha-1} A x + (t - t_o)^{\alpha-1} f(t).
\end{equation}

Hence, the solution \( x(t) \) satisfies equation (26). This shows that solution \( x(t) \) of conformable initial value problem is also the solution of classical differential equation (26).

**Corollary 3.5.** Let \( x(t) \) is differentiable then CTLCS in (2) is equivalent to time-varying control system in (29), which implies that system (29) is observable if and only if
\begin{equation*}
\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.
\end{equation*}

**Proposition 3.6.** For any \( t_f > 0 \)
\begin{equation*}
\mathcal{N} = \ker [W(t_0, t_f)],
\end{equation*}
where
\begin{equation*}
W(t_0, t_f) = \int_{t_0}^{t_f} e^{A T \frac{(t-t_0)^\alpha}{\alpha}} C^T C e^{A \frac{(t-t_0)^\alpha}{\alpha}} dt.
\end{equation*}

**Proof.** We have to show that
\begin{equation*}
\mathcal{N} = \ker [W(t_0, t_f)].
\end{equation*}
We prove it by showing that \( \mathcal{N} \subset \ker [W(t_0, t_f)] \) and \( \ker [W(t_0, t_f)] \subset \mathcal{N} \). Let \( x_0 \in \mathcal{N} \). By Definition 2.17 \( x_0 \in \bigcap_{k=0}^{n-1} \ker [CA^k] \). Then
\begin{equation*}
CA^k x_0 = 0, \text{ with } k = 0, 1, \ldots, n - 1.
\end{equation*}
Also
\begin{equation*}
\sum_{k=0}^{n-1} CA^k a_k \left( \frac{(t-t_0)^\alpha}{\alpha} \right) x_0 = 0.
\end{equation*}
By applying Proposition 3.3, we get
\begin{equation*}
C e^{A \frac{(t-t_0)^\alpha}{\alpha}} x_0 = 0.
\end{equation*}
Also
\begin{equation*}
\int_{t_0}^{t_f} \left( C e^{A \frac{(t-t_0)^\alpha}{\alpha}} \right)^T C e^{A \frac{(t-t_0)^\alpha}{\alpha}} x_0 = 0.
\end{equation*}
This implies
\begin{equation*}
x_0 \in \ker [W(t_0, t_f)].
\end{equation*}
It shows that
\begin{equation*}
\mathcal{N} \subset \ker [W(t_0, t_f)].
\end{equation*}
Now consider, \( x_0 \in \ker [W (t_0, t_f)] \)
\[
W (t_0, t_f) x_0 = 0
\]

Moreover,
\[
x_0^T W (t_0, t_f) x_0 = 0
\]
\[
\int_{t_0}^{t_f} x_0^T e^{A^T \left( \frac{(t-t_0)^\alpha}{\alpha} \right)} C^T C e^{A \left( \frac{(t-t_0)^\alpha}{\alpha} \right)} x_0 dt = 0
\]
\[
\int_{t_0}^{t_f} \left\| C e^{A \left( \frac{(t-t_0)^\alpha}{\alpha} \right)} x_0 \right\|^2 dt = 0.
\]

By applying definition of norm, we get
\[
C e^{A \left( \frac{(t-t_0)^\alpha}{\alpha} \right)} x_0 = 0.
\]

Differentiating \( k \)-times and by substituting \( t = t_0 \), we get
\[
CA^k x_0 = 0, \; k = 0, 1, \ldots, n - 1.
\]

This implies
\[
x_0 \in \ker [C] \cap \ker [CA] \cap \ldots \cap \ker [CA^{n-1}]
\]
\[
x_0 \in \mathcal{N}.
\]

Hence
\[
ker [W (t_0, t_f)] \subset \mathcal{N} \tag{31}
\]

From equations (30) and (31)
\[
\mathcal{N} = ker [W (t_0, t_f)].
\]

\( \square \)

**Corollary 3.7.** The necessary and sufficient condition for the linear state equation (2) to be observable for \( [t_0, t_f] \) is that \( \mathcal{N} = \{0\} \).

**Proof.** Suppose linear state equation (2) is observable on \( [t_0, t_f] \). We have to prove that \( \mathcal{N} = 0 \). Since system is observable from Theorem 3.1 it follows that conformable observable Gramian matrix is invertible. Therefore
\[
ker [W (0, t_0)] = \{0\}.
\]

By using Proposition 3.6, we have
\[
\mathcal{N} = \{0\}.
\]

Conversely: Suppose \( \mathcal{N} = \{0\} \). By using Proposition 3.6, we have
\[
ker [W (0, t_0)] = \{0\}.
\]

It follows that conformable observable Gramian matrix \( W (0, t_0) \) is invertible matrix. Then by Theorem 3.1 linear state equation (2) is observable. \( \square \)
4. Conclusion. The control problem for integer-order systems has been subjected to several investigations. However, much less interest is paid to the novel extended mathematical description, named as conformable systems. In the present article, some existing results in the literature, connected to observability for linear integer-order systems, are generalized into conformable derivative. Gramian and Kalman type rank conditions have been obtained for conformable linear control systems. The findings of the current work point to a promising research direction at the intersection of control theory and conformable operators.

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