Exotic $\bar{D}_s^{(*)}D^{(*)}$ molecular states and $sc\bar{c}$ tetraquark states with $J^P = 0^+, 1^+, 2^+$

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We have calculated the mass spectra for the $\bar{D}_s^{(*)}D^{(*)}$ molecular states and $sc\bar{c}$ tetraquark states with $J^P = 0^+, 1^+, 2^+$. The masses of the axial-vector $D_sD^*$, $D_s^*D$ molecular states and $D_s^{(*)}\bar{D}_s^{(*)}$ tetraquark states are measured as $3.98$ GeV, which are in good agreement with the mass of $Z_{cs}(3985)$ from BESIII [1]. In both the molecular and diquark-antidiquark pictures, our results suggest that there may exist two almost degenerate states, as the strange partners of the $X(3872)$ and $Z_c(3900)$. We propose to carefully examine the $Z_{cs}(3985)$ in future experiments to verify this. One may also search for more hidden-charm four-quark states with strangeness not only in the open-charm $\bar{D}_s^{(*)}D^{(*)}$ channels, but also in the hidden-charm channels $\eta_b K/K^*$, $J/\psi K/K^*$.

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I. INTRODUCTION

Very recently, the BESIII Collaboration announced a new structure near the $D^-D^{*0}$ and $D_s^-D^0$ thresholds in the $K^+$ recoil-mass spectra in $e^+e^-\rightarrow K^+ (D^-D^{*0} + D_s^-D^0)$ [1]. The pole mass and width of this $Z_{cs}(3985)$-resonance are measured as $(3982.5^{+1.8}_{-2.6} \pm 2.1)$ MeV and $(12.8^{+5.3}_{-4.3} \pm 3.0)$ MeV, respectively. Decaying into $D^-D^{*0}$ and $D_s^-D^0$ in S-wave, the spin-parity of $Z_{cs}(3985)$ is assumed to favor $J^P = 1^+$ and the quark content as $c\bar{c}s\bar{u}$ [1]. It will be the first candidate of the hidden-charm four-quark state with strangeness.

Recall the theoretical investigations of the hidden-charm four-quark states with strangeness, the compact tetraquark configuration $sc\bar{c}$ has already been studied in the color-magnetic interaction method [2] and QCD sum rules [3–10]. In Ref.[11], the authors investigated the charged charmonium-like structures with hidden-charm and open-strange channels using the initial single chiral particle emission mechanism. Their results suggested the existence of enhancement structures near the thresholds of $\bar{D}_s^{(*)}D_s^{(*)}$. In Ref.[12], an axial-vector hidden-charm $D^*-D_s^- - D^-D_s^*$ molecular state was also predicted to exist. Possible $DD_{s0}^{*0}(2317)$ and $D^*\bar{D}_s^{*0}(2460)$ molecules were studied in Ref.[13], in which their results disfavor the existence of such states.

A hadronic molecule is composed of two color-singlet hadrons by exchanging light mesons. This is a very useful configuration to study the nature of some exotic XYZ states and pentaquark states [14–19]. Since the $Z_{cs}(3985)$ lies very close to the mass thresholds of $D^-D^{*0}$ and $D_s^-D^0$, it is naturally studied in a molecular picture [20–27], as a partner state of $Z_c(3900)$ discovered by BESIII [28]. It is also explained as a compound mixture of four different four-quark configurations [29], or a reflection structure of charmed-strange meson $D_{cs}^*(2573)$ [30]. Besides, the production mechanisms of the hidden-charm four-quark states with strangeness are studied in Refs. [31, 32]. In Ref.[4], the authors studied the decay width of the $D_sD^*/\bar{D}_s^*D$ by calculating the three-point correlation functions in QCD sum rules. Their result of the total width suffers from a large uncertainty, although its central value is consistent with the experimental result of $Z_{cs}(3985)^-$. Such large uncertainty of the total width originated from the square of form factors, which is inherent and hard to be reduced in the method of three-point QCD sum rules. We also refers to the works [33–39] for recent studies on $Z_{cs}(3985)$ in other methods. In this work, we shall study the exotic $\bar{D}_s^{(*)}D^{(*)}$ molecular states and $sc\bar{c}$ tetraquark states with $J^P = 0^+, 1^+, 2^+$ in the method of QCD sum rules [40–42].

The paper is organized as follows. In Sec. II, we construct the interpolating currents for the $\bar{D}_s^{(*)}D^{(*)}$ molecular systems and $sc\bar{c}$ tetraquark systems with $J^P = 0^+, 1^+$ and $2^+$. In Sec. III, we calculate the correlation functions and spectral densities for these interpolating currents. We extract the masses for the $\bar{D}_s^{(*)}D^{(*)}$ molecular states and $sc\bar{c}$ tetraquark states by performing the QCD sum rule analyses in Sec. IV. The last section is a summary and discussion.

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II. INTERPOLATING CURRENTS

The color structures of a molecular field $[qQ][Qq]$ and a tetraquark field $[qQ][\bar{Q}\bar{q}]$ can be written via the SU(3) symmetry

$$
(3 \otimes \bar{3})_{[qQ]} \otimes (3 \otimes \bar{3})_{[Qq]} = (1 \otimes 8)_{[qQ]} \otimes (1 \otimes 8)_{[Qq]}
$$

$$
= (1 \otimes 1) \oplus (1 \otimes 8) \oplus (8 \otimes 1) \oplus (8 \otimes 8) = 1 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 10 \oplus 10 \oplus 27,
$$

$$
(3 \otimes 3)_{[qQ]} \otimes (\bar{3} \otimes \bar{3})_{[Qq]} = (6 \otimes \bar{3})_{[qQ]} \otimes (3 \otimes \bar{3})_{[Qq]}
$$

$$
= (6 \otimes 6) \oplus (3 \otimes 3) \oplus (6 \otimes 3) \oplus (\bar{3} \otimes \bar{3}) = (1 \otimes 8 \oplus 27) \oplus (1 \otimes 8 \oplus 8 \oplus 10) \oplus (8 \oplus 10),
$$

in which the color singlet structures come from the $(1_{[qQ]} \otimes 1_{[Qq]})$ and $(8_{[qQ]} \otimes 8_{[Qq]})$ terms for the molecular field, while from the $(6_{[qQ]} \otimes \bar{3}_{[Qq]})$ and $(\bar{3}_{[qQ]} \otimes 3_{[Qq]})$ terms for the tetraquark field. In this work, we shall consider the molecular and tetraquark interpolating currents with color structures $(1_{[qQ]} \otimes 1_{[Qq]})$ and $(\bar{3}_{[qQ]} \otimes 3_{[Qq]})$, respectively. To study the lowest lying molecular and tetraquark states, we use only S-wave mesonic and diquark fields to construct the molecular and tetraquark currents with the angular momentum $L = 0$ between two mesonic fields and also two diquark fields. Finally, we obtain the $D_1^{(*)} D^{(*)}$ molecular interpolating currents as

$$
J_1 = (\bar{c}_a \gamma_5 s_a)(\bar{q}_b \gamma_5 c_b), \quad J_1^P = 0^+,
$$

$$
J_2 = (\bar{c}_a \gamma_\mu s_a)(\bar{q}_b \gamma^\mu c_b), \quad J_2^P = 0^+,
$$

$$
J_{1\mu} = (\bar{c}_a \gamma_\mu s_a)(\bar{q}_b \gamma_5 c_b), \quad J_{1\mu}^P = 1^+,
$$

$$
J_{2\mu} = (\bar{c}_a \gamma_\mu s_a)(\bar{q}_b \gamma_5 c_b), \quad J_{2\mu}^P = 1^+,
$$

$$
J_{3\mu} = (\bar{c}_a \gamma_\mu s_a)(\bar{q}_b \sigma_\mu \gamma_5 c_b), \quad J_{3\mu}^P = 1^+,
$$

$$
J_{4\mu} = (\bar{c}_a \gamma_\mu s_a)(\bar{q}_b \gamma_5 c_b), \quad J_{4\mu}^P = 1^+,
$$

$$
J_{\mu \nu} = (\bar{c}_a \gamma_\mu s_a)(\bar{q}_b \gamma_\nu c_b), \quad J_{\mu \nu}^P = 2^+,
$$

and the $sc\bar{q}c$ tetraquark interpolating currents as

$$
\eta_1 = s^T_a C \gamma_5 c_b \left( q_a \gamma_5 c^T_b - q_b \gamma_5 c^T_a \right), \quad J_{\eta_1}^P = 0^+,
$$

$$
\eta_2 = s^T_a C \gamma_\mu c_b \left( q_a \gamma^\mu c^T_b - q_b \gamma^\mu c^T_a \right), \quad J_{\eta_2}^P = 0^+,
$$

$$
\eta_{1\mu} = s^T_a C \gamma_\mu c_b \left( q_a \gamma_5 c^T_b - q_b \gamma_5 c^T_a \right), \quad J_{\eta_{1\mu}}^P = 1^+,
$$

$$
\eta_{2\mu} = s^T_a C \gamma_\mu c_b \left( q_a \gamma^\mu c^T_b - q_b \gamma^\mu c^T_a \right), \quad J_{\eta_{2\mu}}^P = 1^+,
$$

$$
\eta_{3\mu} = s^T_a C \gamma_\alpha c_b \left( q_a \sigma_\alpha \gamma_5 c^T_b - q_b \sigma_\alpha \gamma_5 c^T_a \right), \quad J_{\eta_{3\mu}}^P = 1^+,
$$

$$
\eta_{4\mu} = s^T_a C \sigma_\alpha c_b \left( q_a \gamma_\alpha c^T_b - q_b \gamma_\alpha c^T_a \right), \quad J_{\eta_{4\mu}}^P = 1^+,
$$

$$
\eta_{\mu \nu} = s^T_a C \gamma_\mu c_b \left( q_a \gamma_\nu c^T_b - q_b \gamma_\nu c^T_a \right), \quad J_{\eta_{\mu \nu}}^P = 2^+,
$$

in which $a, b$ denote color indices and $q$ is an up or down quark. The mesonic field $q_a \sigma_\alpha \gamma_5 q_a$ in $J_{3\mu}$ and $J_{4\mu}$ can couple to both the vector channel $J^P = 1^-$ ($q_a \sigma_\alpha \gamma_5 q_a$) and axial-vector channel $J^P = 1^+$ ($q_a \sigma_\alpha \gamma_5 q_a$). We pick out its S-wave vector component by multiplying a vector mesonic field $q \gamma_\alpha q$, so that the molecular operators carry the positive parity. Similar situation happens for the tetraquark currents $\eta_{1\mu}$ and $\eta_{4\mu}$. The molecular currents in Eq. (2) are not independent of the diquark-antidiquark currents in Eq. (3). Actually, a molecular current can be rewritten in terms of a sum over diquark-antidiquark currents via Fierz transformation with some suppression factors. In this work, we shall establish both for the mass spectra for these two different configurations. Using the interpolating currents in Eqs. (2)-(3), we shall study the masses for the $D_1^{(*)} D^{(*)}$ molecular states and $sc\bar{q}c$ tetraquark states in the following.
III. QCD SUM RULES

In this section, we study the two-point correlation functions of the scalar, axial-vector and tensor interpolating currents above. For the scalar currents, the correlation function is

$$\Pi(p^2) = i \int d^4x e^{ip\cdot x} \langle 0 \left[ T[J(x)J^\dagger(0)] \right] 0 \rangle,$$

and for the axial-vector current

$$\Pi_{\mu\nu}(p^2) = i \int d^4x e^{ip\cdot x} \langle 0 \left[ T[J_{\mu\nu}(x)J_{\rho\sigma}^\dagger(0)] \right] 0 \rangle.$$

The correlation function $$\Pi_{\mu\nu}(p^2)$$ in Eq. (5) can be rewritten as

$$\Pi_{\mu\nu}(p^2) = \left( \frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \Pi_1(p^2) + \frac{p_\mu p_\nu}{p^2} \Pi_0(p^2),$$

where $$\Pi_0(p^2)$$ and $$\Pi_1(p^2)$$ are the scalar and vector current polarization functions corresponding to the spin-0 and spin-1 intermediate states, respectively. The correlation function for the tensor current $$J_{\mu\nu}(x)$$ is

$$\Pi_{\mu\nu,\rho\sigma}(p^2) = i \int d^4x e^{ip\cdot x} \langle 0 \left[ T[J_{\mu\nu}(x)\delta_{\rho\sigma}(0)] \right] 0 \rangle,$$

which can be expressed as

$$\Pi_{\mu\nu,\rho\sigma}(p^2) = \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{2}{3} \eta_{\mu\nu} \eta_{\rho\sigma} \right) \Pi_2(p^2) + \cdots,$$

where

$$\eta_{\mu\nu} = \frac{p_\mu p_\nu}{p^2} - g_{\mu\nu},$$

and $$\Pi_2(p^2)$$ is the tensor current polarization functions related to the spin-2 intermediate states, and the “…” represents other spin-0 or spin-1 states.

At the hadronic level, the correlation function can be described via the dispersion relation

$$\Pi(p^2) = \frac{(p^2)^N}{\pi} \int_{4m^2}^{\infty} \text{Im} \Pi(s) \frac{ds}{s^N (s - p^2 - i\epsilon)} + \sum_{n=0}^{N-1} b_n p^{2n},$$

where $$b_n$$ is the subtraction constant. In QCD sum rules, the imaginary part of the correlation function is defined as the spectral function

$$\rho(s) = \frac{1}{\pi} \text{Im} \Pi(s) = f_H^2 \delta(s - m_H^2) + \text{QCD continuum and higher states},$$

in which the “pole plus continuum parametrization” is used. The parameters $$f_H$$ and $$m_H$$ are the coupling constant and mass of the lowest-lying hadronic resonance $$H$$ respectively

$$\langle 0 | J_H | H \rangle = f_H,$$

$$\langle 0 | J_{\mu} | H \rangle = f_H \epsilon_{\mu},$$

$$\langle 0 | J_{\mu\nu} | H \rangle = f_H \epsilon_{\mu\nu},$$

with the polarization vector $$\epsilon_{\mu}$$ and polarization tensor $$\epsilon_{\mu\nu}$$.

On the other hand, we can calculate the correlation function $$\Pi(p^2)$$ and spectral density $$\rho(s)$$ by means of operator product expansion (OPE) at the quark-gluon level. To evaluate the Wilson coefficients, we adopt the propagator of light quark in coordinate space and the propagator of heavy quark in momentum space

$$iS_Q(x) = \frac{i\delta^{ab}}{2\pi^2x^2} \hat{\epsilon} + \frac{i\delta^{ab}}{32\pi^2} \frac{\lambda_{ab}^n}{2} g_{s} G_{\mu\nu}^{n} \frac{1}{x^2} (\sigma^{\mu\nu} \hat{x} + \hat{x} \sigma^{\mu\nu}) - \frac{\delta^{ab} x^2}{12} \langle \bar{q} g_s \sigma \cdot G q \rangle - \frac{m_q \delta^{ab}}{4\pi^2 x^2},$$

$$iS_Q(p) = \frac{i\delta^{ab}}{p - m_Q} + \frac{i\delta^{ab}}{4} g_{s} \frac{\lambda_{ab}^n}{2} G_{\mu\nu}^{n} \frac{\sigma^{\mu\nu}}{12} (\hat{p} + m_Q) + \frac{\sigma^{\mu\nu}}{12} \langle g_s^2 G^2 \rangle m_Q \frac{p^2 + m_Q \hat{p}}{(p^2 - m_Q^2)^2},$$

$$+ \frac{1}{12} \delta^{ab},$$

$$\cdot \langle \bar{q} g_s \sigma \cdot G q \rangle \frac{1}{12} \langle \bar{q} g_s \sigma \cdot G q \rangle - \frac{m_q \delta^{ab}}{4\pi^2 x^2}.$$
where $q$ is $u$, $d$ or $s$ quark and $Q$ represents the $c$ or $b$ quark. The superscripts $a,b$ denote the color indices and $\tilde{x} = x^\mu \gamma_\mu$, $\tilde{p} = p^\mu \gamma_\mu$. In this work, we calculate the Wilson coefficients up to dimension eight condensates at the leading order in $\alpha_s$. In Ref. [43], the NLO perturbative corrections to the correlation functions for the $sc\bar{q}c$ tetraquark systems have been studied and their results show that such contributions are numerically small. The spectral densities for the interpolating currents in Eqs. (2)-(3) are evaluated and listed in the appendix A. The tetraquark currents $\eta_1(x)$, $\eta_2(x)$, $\eta_1(x)$, $\eta_2(x)$ are the same with $\eta_2(x)$, $\eta_4(x)$, $\eta_2(x)$, $\eta_4(x)$ for the $sc\bar{q}b$ systems in Ref. [44], by replacing the bottom quark to charm quark $b \rightarrow c$. Thus we don’t list the spectral densities for these four tetraquark currents in the appendix A. To improve the convergence of the OPE series and suppress the contributions from continuum and higher states region, the Borel transformation is applied to the correlation function at both the hadron and the quark-gluon levels. The QCD sum rules are then established as

\[ \mathcal{L}_k (s_0, M_B^2) = f_H^2 m_H^2 e^{-m_H^2/M_B^2} = \int_{4m^2}^{s_0} ds e^{-s/M_B^2} \rho(s) s^k, \]  

in which $M_B$ represents the Borel mass introduced by the Borel transformation and $s_0$ is the continuum threshold. The mass of the lowest-lying hadron can be thus extracted as

\[ m_H (s_0, M_B^2) = \sqrt{\frac{\mathcal{L}_1 (s_0, M_B^2)}{\mathcal{L}_0 (s_0, M_B^2)}}, \]

which is the function of two parameters $M_B^2$ and $s_0$. We shall discuss the detail to obtain suitable parameter working regions in QCD sum rule analyses in next section.

**IV. NUMERICAL ANALYSIS**

In this section, we perform the QCD sum rule analyses for the $\bar{D}^* \bar{D}^*$ molecular and $sc\bar{q}c$ tetraquark systems by using the interpolating currents in Eqs. (2)-(3). We use the values of quark masses and various QCD condensates as follows [45-53]

\[ m_u(2\text{GeV}) = (2.2^{+0.5}_{-0.4}) \text{MeV}, \]
\[ m_d(2\text{GeV}) = (4.7^{+0.5}_{-0.3}) \text{MeV}, \]
\[ m_s(2\text{GeV}) = (3.5^{+0.5}_{-0.2}) \text{MeV}, \]
\[ m_s(m_c) = (1.275^{+0.025}_{-0.035}) \text{GeV}, \]
\[ m_b(m_b) = (4.18^{+0.04}_{-0.03}) \text{GeV}, \]
\[ \langle \bar{q}q \rangle = -(0.24 \pm 0.03)^3 \text{GeV}^3, \]
\[ \langle \bar{q}g^a \cdot Gq \rangle = -M_0^2 \langle \bar{q}q \rangle, \]
\[ M_0^2 = (0.8 \pm 0.2) \text{GeV}^2, \]
\[ \langle \bar{s}s \rangle/\langle \bar{q}q \rangle = 0.8 \pm 0.1, \]
\[ \langle g_s^2 G \rangle = (0.48 \pm 0.14) \text{GeV}^4, \]

where the $u, d, s$ quark masses of are the current quark masses obtained in the $\overline{MS}$ scheme at the scale $\mu = 2$ GeV. We use the running mass in the $\overline{MS}$ scheme for the charm quark, which is different from the value of pole quark mass. Various literatures prove that the use of $\overline{MS}$ mass of the charm quark can lead to very good predictions for the masses of XYZ states in the framework of QCD sum rules [15, 54].

To establish a stable mass sum rule, one should find appropriate parameter working regions first, i.e, the continuum threshold $s_0$ and the Borel mass $M_B^2$. The threshold $s_0$ can be determined via the minimized variation of the hadronic mass $m_H$ with the Borel mass $M_B^2$. The lower bound on Borel mass $M_B^2$ can be fixed by requiring a reasonable OPE convergence while its upper bound is determined through a sufficient pole contribution. The pole contribution is defined as

\[ \text{PC} (s_0, M_B^2) = \frac{\mathcal{L}_0 (s_0, M_B^2)}{\mathcal{L}_0 (\infty, M_B^2)}. \]
where \( L_0 \) has been defined in Eq. (14).

We use the \( \bar{D}^s D^* \) molecular current \( J_2(x) \) with \( J^P = 0^+ \) as an example to show the detail of the numerical analysis. For this current, the dominant non-perturbative contribution to the correlation function comes from the quark condensate \( \langle \bar{q}q \rangle \) and \( \langle \bar{s}s \rangle \). In Fig. 1, we show the contributions of the perturbative term and various condensate terms to the correlation function. It is clear that the Borel mass \( M_B^2 \) should be large enough to ensure the convergence of OPE series. Here, we require that the highest dimension condensate contribution to be less than 10%,

\[
\Pi(\bar{q}q, \bar{s}s)(M_B^2, \infty) < 10\% ,
\]

which results in \( M_B^2 \geq 2.6 \text{GeV}^2 \).

As mentioned above, the variation of the output hadron mass \( m_H \) with \( s_0 \) and \( M_B^2 \) corresponding to the current \( J_2(x) \) in the \( \bar{D}^s D^* \) system with \( J^P = 0^+ \). In Fig. 2, the mass sum rules are established to be very stable in these parameter regions and the hadron mass for the \( \bar{D}^s D^* \) molecule with \( J^P = 0^+ \) can be obtained as

\[
m_{\bar{D}^s D^* 0^+} = 4.11 \pm 0.14 \text{GeV} ,
\]

in which the error comes from the uncertainties of the continuum threshold \( s_0 \), Borel mass \( M_B \), the various condensates
and quark masses. After performing similar analyses, we obtain the numerical results for all the other interpolating currents in Eqs. (2)-(3) and collect them in Table I.

Table I: The numerical results for the $\bar{D}_s^{(*)}D^{(*)}$ molecular and diquark-antiquark $sc\bar{q}c$ tetraquark systems.

| System | Current | $J^P$ | $s_0$(GeV²) | $M_B^2$(GeV²) | $m_H$(GeV) | PC(%) |
|--------|---------|-------|-------------|--------------|-----------|-------|
| $\bar{D}_s D$ | $J_1$ | $0^+$ | 18.0 ± 2.0 | 1.6 ~ 3.6 | 3.74 ± 0.13 | 52.5 |
| $\bar{D}_s^* D^*$ | $J_2$ | $0^+$ | 20.5 ± 2.0 | 2.6 ~ 3.4 | 4.11 ± 0.14 | 42.4 |
| $\bar{D}_s^* D$ | $J_{1\mu}$ | $1^+$ | 20.7 ± 2.0 | 2.1 ~ 2.5 | 3.99 ± 0.12 | 68.2 |
| $D_s D^*$ | $J_{2\mu}$ | $1^+$ | 20.5 ± 2.0 | 2.1 ~ 2.5 | 3.97 ± 0.11 | 67.7 |
| $\bar{D}_s^* D^*$ | $J_{3\mu}$ | $1^+$ | 21.5 ± 2.0 | 2.8 ~ 3.6 | 4.22 ± 0.14 | 40.1 |
| $\bar{D}_s D^*$ | $J_{4\mu}$ | $1^+$ | 21.5 ± 2.0 | 2.8 ~ 3.6 | 4.22 ± 0.14 | 40.0 |
| $\bar{D}_s^* D^*$ | $J_{\mu\nu}$ | $2^+$ | 23.0 ± 2.0 | 2.8 ~ 4.3 | 4.34 ± 0.13 | 48.7 |

In Table I, the mass of scalar $\bar{D}_s D$ molecular state is predicted to be slightly below the open-charm threshold $T_{D_s D} = 3.84$ GeV, implying that it can only decay into the hidden-charm channel $\eta_c K$. The scalar $\bar{D}_s^* D^*$ state is predicted to be very close to $T_{D_s^* D^*} = 4.12$ GeV, however, it can decay into $\bar{D}_s D$ and $\eta_c K$ final states kinematically in S-wave. The masses for the $\bar{D}_s^* D^*$ molecular states with $J^P = 1^+, 2^+$ are significantly above the corresponding open-charm thresholds.

The masses obtained from the axial-vector molecular currents $J_{1\mu}$ and $J_{2\mu}$ are $m_{D_s^* D^*} = (3.99 ± 0.12)$ GeV, $m_{D_s D^*} = (3.97 ± 0.11)$ GeV, which are almost degenerate with each other. One may wonder whether these two currents $J_{1\mu}$ and $J_{2\mu}$ could couple to the same physical molecular state or not. In QCD sum rules, this can be specified by studying the following off-diagonal correlation function

$$\Pi_{12\mu\nu}^M (p^2) = i \int d^4xe^{ipx} \left\langle 0 \left| T \left[ J_{1\mu}(x) J_{J_{2\nu}}^{\dagger}(0) \right] \right| 0 \right\rangle .$$

Our calculation shows that this off-diagonal correlation function $\Pi_{12\mu\nu}^M (p^2) = 0$ at the leading order of $\alpha_s$ for the axial-vector molecular currents $J_{1\mu}$ and $J_{2\mu}$, including the perturbative term and all contributions from various non-perturbative condensates. According to Ref. [43], the NLO perturbative correction is numerically small and thus $\Pi_{12\mu\nu}^M (p^2)$ is still negligible comparing to the diagonal correlators $\Pi_{11\mu\nu}^M (p^2)$ and $\Pi_{22\mu\nu}^M (p^2)$ at the next leading order of $\alpha_s$. Such a result implies that $J_{1\mu}$ and $J_{2\mu}$ may couple to different physical states.

We also study the $sc\bar{q}c$ tetraquark systems with $J^P = 0^+, 1^+, 2^+$. In Fig. 3, we show the variations of the tetraquark mass with $s_0$ and $M_B^2$ for the current $\eta_{1\mu}(x)$ with $J^P = 1^+$, and the mass sum rules are very stable and reliable at the chosen parameter regions. For the interpolating currents in Eq. (3), we collect the numerical results for these $sc\bar{q}c$ tetraquark systems in Table I. It is shown that the mass spectra for the $sc\bar{q}c$ tetraquarks are very similar with the $\bar{D}_s^{(*)}D^{(*)}$ molecular states. For the axial-vector $sc\bar{q}c$ tetraquark systems, the extracted masses from $\eta_{1\mu}(x)$ and $\eta_{2\mu}(x)$ are almost the same with the $\bar{D}_s^* D$ and $\bar{D}_s D^*$ molecular states, which are consistent with the mass of $Z_{cs}(3985)^-$ from BESIII [1]. It is interesting to examine the off-diagonal correlation function for $\eta_{1\mu}(x)$ and $\eta_{2\mu}(x)$

$$\Pi_{12\mu\nu} (p^2) = i \int d^4xe^{ipx} \left\langle 0 \left| T \left[ J_{1\mu}(x) J_{J_{2\nu}}^{\dagger}(0) \right] \right| 0 \right\rangle .$$

(21)
The calculation indicates that the perturbative term and the quark condensate terms in $\Pi_{EM}^T (p^2)$ are equal to zero. This off-diagonal correlation function $\Pi_{EM}^T (p^2)$ is very small, suggesting that the currents $\eta_{1\mu}(x)$ and $\eta_{2\mu}(x)$ cannot strongly couple to the same physical state.

V. CONCLUSION

To study the hidden-charm four-quark systems with strangeness, we have calculated the mass spectra for the $\bar{D}s D^*$ molecular states and $sc \bar{c}q$ tetraquark states with $J^P = 0^+, 1^+, 2^+$ in the framework of QCD sum rules. We construct the corresponding molecular and tetraquark interpolating currents, calculate their two-point correlation functions and spectral densities up to dimension eight condensates at the leading order of $\alpha_s$. The quark condensates are found to be the most important non-perturbative contribution to the correlation functions for both molecular and tetraquark systems.

One may wonder if the two-meson scattering states can contribute to the correlation functions in our calculations. In general, the interpolating currents can couple to all structures with the same quantum numbers, including resonances, two-meson scattering states and continuum. And thus these structures will give contributions to the correlation functions. However, it has been demonstrated that the two-meson scattering states cannot saturate the QCD sum rules, while only exotic four-quark states can saturate the QCD sum rules. Moreover, the contributions from the two-meson scattering states to the correlation functions are numerically negligible [43, 55].

Our results show that the masses of the axial-vector $\bar{D}s D^*$, $\bar{D}^* s D$ molecular states and the $sc \bar{c}q$ tetraquark states from $\eta_{1\mu}, \eta_{2\mu}$ are calculated in good agreement with the mass of $Z_{cs}(3985)^-$. The present calculations are difficult for distinguishing the nature of $Z_{cs}(3985)^-$ from the molecular and diquark-antidiquark configurations. In both the molecular and diquark-antidiquark pictures, our results suggest that there may exist two almost degenerate states, as the strange partners of the $X(3872)$ and $Z_{cs}(3900)$. We propose to carefully examine the $Z_{cs}(3985)$ in future experiments to verify this. One can also search for more hidden-charm four-quark states with strangeness not only in the open-charm $\bar{D}^s_1 D^s_1$ channels, but also in the hidden-charm channels $\eta_c K/K^*, J/\psi K/K^*$. 

**Note added:** After we finished this work, the LHCb Collaboration has reported two new charged resonances $Z_{cs}(4000)^+$ and $Z_{cs}(4220)^+$ in the $J/\psi K^+$ final states [56]. Their masses and decay widths are measured as $M_{Z_{cs}(4000)^+} = 4003 \pm 6_{-14}^{+14}$ MeV, $\Gamma_{Z_{cs}(4000)^+} = 131 \pm 15 \pm 26$ MeV and $M_{Z_{cs}(4220)^+} = 4216 \pm 24_{-30}^{+43}$ MeV, $\Gamma_{Z_{cs}(4220)^+} = 233 \pm 52_{-73}^{+97}$ MeV, while their spin-parity quantum numbers are identified to prefer $J^P = 1^+$. These masses and spin-parity are consistent with the axial-vector $D_s D^*$ ($D^*_s D$), $D^*_c D^*$ molecular states and $1_{[sc]} + 0_{[\bar{q}\bar{q}]} (0_{[sc]} + 1_{[\bar{q}\bar{q}]})$, $1_{[sc]} + 1_{[\bar{q}\bar{q}]}$) tetraquark states that we have predicted in Table I.

According to LHCb’s observation, the decay width of $Z_{cs}(4000)$ is much larger than that of $Z_{cs}(3985)$ observed by BESIII [1]. LHCb found no evidence that the $Z_{cs}(4000)$ and $Z_{cs}(3985)$ are the same state, although their masses are very close to each other. It this is true, they may be identified as the strange partners of the $X(3872)$ and $Z_{cs}(3900)$ with $J^{PC} = 1^{++}$ and $J^{PC} = 1^{+-}$ respectively. We propose to carefully examine the $Z_{cs}(4000)$ and $Z_{cs}(3985)$ in future experiments to verify this.
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Appendix A: The spectral densities

In this appendix, we list the spectral densities for the $D_s^{(*)}D^{(*)}$ and $sc\bar{c}$ systems with $J^{P} = 0^+, 1^+$ and $2^+$. The spectral density includes the perturbative term, quark condensate, gluon condensate, quark-gluon mixed condensate, four-quark condensate and dimension eight condensate

$$\rho(s) = \rho_0(s) + \rho_1(s) + \rho_2(s) + \rho_3(s) + \rho_4(s) + \rho_5(s),$$

in which the superscripts stand for the dimension of various condensates.

1. Spectral densities for $J_1$

$$\rho_0^{J_1}(s) = \frac{3}{2048\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 (m_\pi^2 (\alpha + \beta) - \alpha \beta s)^3 (m_\rho^2 (\alpha + \beta) - 3 \alpha \beta s),$$

$$\rho_1^{J_1}(s) = -\frac{3m_c}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 (m_\pi^2 (\alpha + \beta) - \alpha \beta s) 2 (m_\rho^2 (\alpha + \beta) - 5 \alpha \beta s) \left( \frac{m_s}{\alpha^2 \beta^2} + \frac{m_q}{\alpha^3 \beta^2} \right),$$

$$\rho_2^{J_1}(s) = -\frac{3(sq)}{128\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_\pi^2 (\alpha + \beta) - \alpha \beta s)^2 \frac{2(1 - \alpha - \beta)(m_\pi^2 (\alpha + \beta) - 2 \alpha \beta s)m_c}{\alpha \beta^2}$$

$$+ \frac{2m_c(m_\pi^2 (\alpha + \beta) - 2 \alpha \beta s)m_s}{\alpha \beta^2} + \frac{2m_c(m_\pi^2 (\alpha + \beta) - 2 \alpha \beta s)m_s}{\alpha \beta^2}.$$

$$\rho_3^{J_1}(s) = \frac{(g_s^2 G^2) m_c^2}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 (m_\pi^2 (\alpha + \beta) - 3 \alpha \beta s) \left( \frac{1}{\alpha^3} + \frac{1}{\beta^3} \right),$$

$$\rho_4^{J_1}(s) = \frac{3(g_s^2 G^2) m_c^2}{2048\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)(m_\pi^2 (\alpha + \beta) - \alpha \beta s)(m_\rho^2 (\alpha + \beta) - 2 \alpha \beta s) \left( \frac{1}{\alpha^2 \beta} + \frac{1}{\alpha \beta^2} \right),$$

$$\rho_5^{J_1}(s) = \frac{3(sg_s \sigma \cdot G^s m_c)}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( m_\pi^2 (\alpha + \beta) - 3 \alpha \beta s \right) \left( \frac{1}{\beta - \frac{2(1 - \alpha - \beta)}{\beta^2}} + \frac{2m_q m_c}{\beta} \right),$$

$$\rho_6^{J_1}(s) = \frac{3(sg_s \sigma \cdot G q m_c)}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( m_\pi^2 (\alpha + \beta) - 3 \alpha \beta s \right) \left( \frac{1}{\alpha} - \frac{2(1 - \alpha - \beta)}{\alpha^2} \right) + \frac{2m_q m_s}{\alpha},$$

$$\rho_7^{J_1}(s) = \frac{(s \bar{q} g_s \sigma \cdot G q)}{512\pi^4} \left( (s - 2m_c^2) m_q - 6m_c^2 m_s \right) \sqrt{1 - \frac{4m_c^2}{s}}$$

$$+ \frac{(s \bar{q} g_s \sigma \cdot G s)}{512\pi^4} \left( (s - 2m_c^2) m_s - 6m_c^2 m_q \right) \sqrt{1 - \frac{4m_c^2}{s}},$$

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\[
\rho_{J_1}(s) = \frac{\langle \bar{s}s \rangle \langle \bar{q}q \rangle}{32\pi^2} (2m_c^2 + m_cm_q + m_cm_s) \sqrt{1 - \frac{4m_c^2}{s}},
\]

\[
\Pi_{J_1}(M_B^2) = \frac{\langle \bar{s}s \rangle \langle \bar{q}q \rangle m_c^3}{32\pi^2} \int_0^1 d\alpha \left( \frac{m_q}{1 - \alpha} + \frac{m_q}{\alpha} \right) e^{-\frac{m^2}{\alpha(1-\alpha)M_B^2}},
\]

\[
\Pi_{J_4}(M_B^2) = \frac{m_c^4}{64\pi^2} \int_0^1 d\alpha \left( \frac{\langle \bar{s}s \rangle \langle \bar{q}q \sigma \cdot Gq \rangle + \langle \bar{q}q \sigma \cdot Gs \rangle}{(1-\alpha)^2 M_B^2} - \frac{2\langle \bar{s}s \rangle \langle \bar{q}q \sigma \cdot Gq \rangle}{(1-\alpha)m_c^2} - \frac{2\langle \bar{q}q \rangle \langle \bar{q}g_s \sigma \cdot Gs \rangle}{\alpha m_c^2} \right) e^{-\frac{m^2}{\alpha(1-\alpha)M_B^2}},
\]

where
\[
\alpha_{\text{min}} = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4m_c^2}{s}}, \quad \alpha_{\text{max}} = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4m_c^2}{s}}, \quad \beta_{\text{min}} = \frac{\alpha m_c^2}{\alpha s - m_c^2}, \quad \beta_{\text{max}} = 1 - \alpha,
\]

2. Spectral densities for \( J_2 \):

\[
\rho_{J_2}^{0a}(s) = \frac{3}{512\pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \left( \frac{1 - \alpha - \beta}{\alpha^2 \beta^3} \right) (m_c^2(\alpha + \beta) - \alpha \beta s)^3 (m_c^2(\alpha + \beta) - 3\alpha \beta s),
\]

\[
\rho_{J_2}^{0b}(s) = -\frac{3m_c}{512\pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta (1 - \alpha - \beta)^2 (m_c^2(\alpha + \beta) - \alpha \beta s)^2 (m_c^2(\alpha + \beta) - 5\alpha \beta s) \left( \frac{m_q}{\alpha^2 \beta^3} + \frac{m_c}{\alpha^3 \beta^2} \right),
\]

\[
\rho_{J_2}^{3a}(s) = \frac{3\langle \bar{s}s \rangle}{64\pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta (m_c^2(\alpha + \beta) - \alpha \beta s) \left( 2(1 - \alpha - \beta)(m_c^2(\alpha + \beta) - 2\alpha \beta s) m_c \right) \frac{\alpha^2}{\alpha^3 \beta},
\]

\[
\rho_{J_2}^{3b}(s) = -\frac{3\langle \bar{q}q \rangle}{64\pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta (m_c^2(\alpha + \beta) - \alpha \beta s) \left( 2(1 - \alpha - \beta)(m_c^2(\alpha + \beta) - 2\alpha \beta s) m_c \right) \frac{\alpha^2}{\alpha^2 \beta},
\]

\[
\rho_{J_2}^{4}(s) = \frac{\langle \bar{q}qGG \rangle m_c^2}{1024\pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta (1 - \alpha - \beta)^2 (2m_c^2(\alpha + \beta) - 3\alpha \beta s) \left( \frac{1}{\alpha^3} + \frac{1}{\beta^3} \right),
\]

\[
\rho_{J_2}^{5a}(s) = \frac{3 \langle \bar{s}g_s \sigma \cdot Gs \rangle m_c}{128\pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \left( \frac{2m_c^2(\alpha + \beta) - 3s \alpha \beta}{\beta} \right) m_c + \frac{3 \langle \bar{q}g_s \sigma \cdot Gq \rangle m_c}{128\pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \left( \frac{2m_c^2(\alpha + \beta) - 3s \alpha \beta}{\alpha} \right) m_c,
\]

\[
\rho_{J_2}^{5b}(s) = \frac{\langle \bar{q}g_s \sigma \cdot Gq \rangle}{128\pi^4} \left( (s - 2m_c^2) m_q - 6m_c^2 m_s \right) \sqrt{1 - \frac{4m_c^2}{s}} + \frac{\langle \bar{s}g_s \sigma \cdot Gs \rangle}{128\pi^4} \left( (s - 2m_c^2) m_s - 6m_c^2 m_q \right) \sqrt{1 - \frac{4m_c^2}{s}},
\]
\[ \rho_{J_{1\nu}}^{6a}(s) = \frac{3}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)^2}{\alpha^3\beta^3} (m_c^2 + m_\nu + m_s) \sqrt{1 - \frac{4m_\nu^2}{s}}, \]

\[ \Pi_{J_{2}}^{6b}(M_B^2) = \frac{\langle \bar{s}s \rangle \langle \bar{q}q \rangle m_c^3}{16\pi^2} \int_0^1 d\alpha \left( \frac{m_\nu + m_s}{\alpha} \right) e^{-\frac{m_\nu^2}{\alpha(1 - \alpha)M_B^2}}, \]

\[ \Pi_{J_{2}}^{8}(M_B^2) = \frac{m_c^4}{16\pi^2} \int_0^1 d\alpha \frac{\langle \bar{s}s \rangle \langle \bar{q}q \rangle \cdot Gq}{(1 - \alpha)^2 M_B^2} e^{-\frac{m_\nu^2}{\alpha(1 - \alpha)M_B^2}}, \]

3. Spectral densities for \( J_{1\nu} \):

\[ \rho_{J_{1\nu}}^{6a}(s) = \frac{3}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)^2}{\alpha^3\beta^3} (m_c^2 + m_\nu + m_s) \cdot (m_c^2 + m_\nu + m_s) \sqrt{1 - \frac{4m_\nu^2}{s}}, \]

\[ \rho_{J_{1\nu}}^{6b}(s) = -\frac{3m_c}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( 1 - \alpha - \beta \right)^2 (m_c^2 + m_\nu + m_s) \cdot \frac{(m_c^2 + m_\nu + m_s)}{\alpha^3\beta^3}, \]

\[ \rho_{J_{1\nu}}^{3a}(s) = -\frac{3\langle \bar{s}s \rangle}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2 + m_\nu + m_s) \cdot \frac{(m_c^2 + m_\nu + m_s)}{\alpha^3\beta^3}, \]

\[ \rho_{J_{1\nu}}^{3b}(s) = \frac{3\langle \bar{q}q \rangle}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2 + m_\nu + m_s) \cdot \frac{(m_c^2 + m_\nu + m_s)}{\alpha^3\beta^3}, \]

\[ \rho_{J_{1\nu}}^{4a}(s) = \frac{(g_2^2GG)m_c^2}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( 1 - \alpha - \beta \right)^2 (m_c^2 + m_\nu + m_s) \cdot \frac{1}{\alpha^3} \left( 1 + \frac{1}{\beta^3} \right), \]

\[ \rho_{J_{1\nu}}^{4b}(s) = \frac{(g_2^2GG)m_c^2}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( 1 - \alpha - \beta \right)^2 (m_c^2 + m_\nu + m_s) \cdot \frac{3(m_c^2 + m_\nu + m_s)}{\alpha^3\beta^3}, \]

\[ \rho_{J_{1\nu}}^{5a}(s) = \frac{3\langle \bar{s}g_\sigma \cdot Gs \rangle m_c}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{2m_c^2 + 3m_\nu + 3m_s}{\alpha}, \]

\[ \rho_{J_{1\nu}}^{5b}(s) = \frac{3\langle \bar{q}u \sigma \cdot Gq \rangle m_c}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left[ (m_c^2 + m_\nu + m_s) \cdot \frac{1}{\beta} - \frac{2(1 - \alpha - \beta)}{\beta^2} + \frac{2m_c m_s}{\beta} \right], \]
\[ \rho_{J_{2\mu}}^{s}(s) = \frac{\langle \bar{s}g_{s} \cdot Gs \rangle}{768\pi^4} (s-m_{c}^{2})m_{s}-9m_{c}^{2}m_{q} \sqrt{1-\frac{4m_{c}^{2}}{s}} + \frac{\langle \bar{q}g_{s} \cdot Gq \rangle}{768\pi^4} (s-m_{c}^{2})m_{q}-9m_{c}^{2}m_{s} \sqrt{1-\frac{4m_{c}^{2}}{s}} , \]

\[ \rho_{J_{2\mu}}^{6a}(s) = \frac{3}{128\pi^2} (4m_{c}^{2}+2m_{c}m_{q}+m_{c}m_{s}) \sqrt{1-\frac{4m_{c}^{2}}{s}} , \]

\[ \Pi_{J_{2\mu}}^{6b}(M_{B}^{2}) = -\frac{3}{32\pi^2} \left( \frac{\langle \bar{s}g_{s} \cdot Gs \rangle + \langle \bar{q}g_{s} \cdot Gq \rangle (1-\alpha)M_{B}^{2}}{1-\alpha} \right) e^{-\frac{m_{c}^{2}}{(1-\alpha)M_{B}^{2}}} , \]

\[ \Pi_{J_{2\mu}}^{3}(M_{B}^{2}) = \frac{m_{c}^{4}}{64\pi^2} \int_{0}^{1} \alpha \left( \frac{\langle \bar{q}g_{s} \cdot Gs \rangle + \langle \bar{s}g_{s} \cdot Gq \rangle (1-\alpha)M_{B}^{2}}{1-\alpha} \right) e^{-\frac{m_{c}^{2}}{(1-\alpha)M_{B}^{2}}} , \]

4. Spectral densities for \( J_{2\mu} \):

\[ \rho_{J_{2\mu}}^{3a}(s) = \frac{3}{4096\pi^6} \int_{\alpha_{min}}^{\alpha_{max}} \int_{\beta_{min}}^{\beta_{max}} \cos \left( \frac{2\alpha m_{c} \cos(\alpha) + \beta m_{c} \sin(\alpha)}{\alpha \beta} \right) m_{s} \]

\[ \rho_{J_{2\mu}}^{3b}(s) = -\frac{3}{256\pi^4} \int_{\alpha_{min}}^{\alpha_{max}} \int_{\beta_{min}}^{\beta_{max}} \cos \left( \frac{2\alpha m_{c} \cos(\alpha) + \beta m_{c} \sin(\alpha)}{\alpha \beta} \right) m_{s} \]

\[ \rho_{J_{2\mu}}^{4a}(s) = \frac{g_{s}^{2}G_{c}^{2}m_{c}^{2}}{4096\pi^6} \int_{\alpha_{min}}^{\alpha_{max}} \int_{\beta_{min}}^{\beta_{max}} \cos \left( \frac{2\alpha m_{c} \cos(\alpha) + \beta m_{c} \sin(\alpha)}{\alpha \beta} \right) m_{s} \]

\[ \rho_{J_{2\mu}}^{4b}(s) = \frac{g_{s}^{2}G_{c}^{2}m_{c}^{2}}{4096\pi^6} \int_{\alpha_{min}}^{\alpha_{max}} \int_{\beta_{min}}^{\beta_{max}} \cos \left( \frac{2\alpha m_{c} \cos(\alpha) + \beta m_{c} \sin(\alpha)}{\alpha \beta} \right) m_{s} \]
5. Spectral densities for $J_{3\mu}$:

\[ \rho_{J_{3\mu}}^{0\mu}(s) = \frac{9}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left\{ (1 - \alpha - \beta)^2 \left( m_c^2(\alpha + \beta) - \alpha\beta s \right)^2 \left( m_c^2(\alpha + \beta) - 5\alpha\beta s \right) \right\}^{\frac{1}{2}} / \alpha^3 \beta^3, \]

\[ \rho_{J_{3\mu}}^0(s) = -\frac{9m_c}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 (m_c^2(\alpha + \beta) - \alpha\beta s)^2 \left( \frac{m_c^2(\alpha + \beta) - 3\alpha\beta s m_c}{\alpha^2\beta^2} - \frac{\alpha\beta s m_c}{\alpha^2\beta^2} \right), \]

\[ \rho_{J_{3\mu}}^{3\mu}(s) = \frac{3(\bar{s}s)}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha\beta s) \left\{ 4(1 - \alpha - \beta) s m_c + 12m_c^2 m_q + 3(m_c^2(\alpha + \beta) - 3\alpha\beta s) s m_c \right\} / \beta, \]

\[ \rho_{J_{3\mu}}^3(s) = -\frac{3(\bar{q}q)}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( m_c^2(\alpha + \beta) - \alpha\beta s \right)^2 \left[ 2(1 - \alpha - \beta)(3m_c^2(\alpha + \beta) - 5\alpha\beta s) m_c \right] / \alpha^2 \beta, \]

\[ \rho_{J_{3\mu}}^{4\mu}(s) = \frac{3(g^2GG)m_c^2}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 (m_c^2(\alpha + \beta) - 2\alpha\beta s) / \alpha^3, \]

\[ \rho_{J_{3\mu}}^4(s) = \frac{(g^2GG)^2}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta) (m_c^2(\alpha + \beta) - \alpha\beta s) \left( \frac{3m_c^2(\alpha + \beta) - 5\alpha\beta s}{\alpha^2\beta^2} \right) / \alpha^2 \beta^2, \]

\[ \rho_{J_{3\mu}}^{5\mu}(s) = -\frac{3(\bar{s}g\sigma\cdot Gs)m_c}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta s \alpha, \]
6. Spectral densities for $J_{4\mu}$:

$$\rho_{J_{4\mu}}^{5b}(s) = \frac{(qg_s \cdot Gq) m_c}{256 \pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left[ (3m^2_c(\alpha + \beta) - 4\alpha\beta) \left( \frac{3}{\alpha} + \frac{2(1 - \alpha - \beta)}{\alpha^2} \right) - \frac{6m_cm_s}{\alpha} \right]$$

$$\rho_{J_{4\mu}}^{5c}(s) = \frac{(qg_s \cdot Gq)}{256 \pi^4} \left[ ((s - m^2_c) m_q - 9m^2_cm_s) \sqrt{1 - \frac{4m^2_c}{s}} \right. + \frac{(qg_s \cdot Gs)}{256 \pi^4} \left. \left[ ((s - m^2_c) m_s - 9m^2_cm_q) \sqrt{1 - \frac{4m^2_c}{s}} \right] \right.$$

$$\rho_{J_{4\mu}}^{6a}(s) = \frac{3(\bar{s}s)(\bar{q}q)}{64\pi^2} (4m^2_c + m_cm_s) \sqrt{1 - \frac{4m^2_c}{s}}$$

$$\Pi_{J_{4\mu}}^{5b}(M^2_B) = \frac{(\bar{s}s)(\bar{q}q)m^3_c}{32\pi^2} \int_0^1 d\alpha \left( \frac{m_q}{1 - \alpha} + \frac{m_s}{\alpha} \right) e^{\alpha(1-\alpha)M^2_B}$$

$$\Pi_{J_{4\mu}}^{5c}(M^2_B) = \frac{m^4_c}{64\pi^2} \int_0^1 d\alpha \left( \frac{3(\bar{s}s)(\bar{q}q) + (\bar{q}q)(\bar{s}s)}{(1 - \alpha)^2 M^2_B} + \frac{2(\bar{s}s)(\bar{q}g_s \cdot Gq)}{(1 - \alpha)m^2_c} \right) e^{\alpha(1-\alpha)M^2_B}$$

$$\rho_{J_{4\mu}}^{3a}(s) = \frac{3(\bar{q}q)}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m^2_c(\alpha + \beta) - \alpha\beta s) \left[ \frac{4(1 - \alpha - \beta)sm_c}{\alpha} + 12m^2_cm_s + 3(m^2_c(\alpha + \beta) - 3\alpha\beta s)m_q \right]$$

$$\rho_{J_{4\mu}}^{3b}(s) = \frac{3(\bar{s}s)g^2GGm^2_c}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m^2_c(\alpha + \beta) - \alpha\beta s) \left[ \frac{2(1 - \alpha - \beta)(3m^2_c(\alpha + \beta) - 5\alpha\beta s)}{\alpha^2 \beta^2} - \frac{12m^2_cm_q + 3(m^2_c(\alpha + \beta) - 3\alpha\beta s)m_s}{\alpha \beta} \right]$$

$$\rho_{J_{4\mu}}^{3a}(s) = \frac{3(g^2GGm^2_c)}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m^2_c(\alpha + \beta) - \alpha\beta s) \left[ \frac{3m^2_c(\alpha + \beta) - 5\alpha\beta s}{\alpha^2 \beta} \right]$$

$$\rho_{J_{4\mu}}^{3b}(s) = \frac{g^2GGm^2_c}{4096\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)(m^2_c(\alpha + \beta) - \alpha\beta s) \left[ \frac{3m^2_c(\alpha + \beta) - 5\alpha\beta s}{\alpha^2 \beta} \right]$$

$$\rho_{J_{4\mu}}^{5a}(s) = -\frac{3(\bar{q}g_s \cdot Gq)m_c}{256\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \beta s \beta$$
\[ \rho_{J_{\mu\nu}}(s) = \frac{(q_g \cdot G_s) m_c}{256 \pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} da \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \left[ (3m_c^2(\alpha + \beta) - 4s\alpha\beta) \left( \frac{3}{\beta} + \frac{2(1 - \alpha - \beta)}{\beta^2} \right) - \frac{6m_c m_q}{\beta} \right], \]

\[ \rho_{J_{\tau\nu}}(s) = \frac{(q_g \cdot G_s) m_s}{256 \pi^4} \left( (s - m_c^2) m_s - 9m_c^2 m_q \right) \sqrt{1 - \frac{4m_s^2}{s}}, \]

\[ \rho_{J_{\mu\nu}}(s) = \frac{3(s \bar{s})(q\bar{q})}{64 \pi^2} (4m_c^2 + m_c m_q) \sqrt{1 - \frac{4m_c^2}{s}}, \]

\[ \Pi_{J_{\mu\nu}}^6(M_B^2) = \frac{(s \bar{s})(q\bar{q}) m_c^3}{32 \pi^2} \int_0^1 da \left( \frac{m_s}{1 - \alpha} + \frac{m_q}{\alpha} \right) e^{-m_c^2 \alpha \bar{\alpha} M_B^2}, \]

\[ \Pi_{J_{\mu\nu}}^8(M_B^2) = \frac{m_c^4}{64 \pi^2} \int_0^1 da \left( \frac{3(s \bar{s})(q\bar{q})_g G_s + (q\bar{q})_s G_s - 2(q\bar{q})_g G_s - G_s \bar{G}}{(1 - \alpha)^2 M_B^2} \right) e^{-m_c^2 \alpha \bar{\alpha} M_B^2}, \]

7. Spectral densities for \( J_{\mu\nu} \):

\[ \rho_{J_{\mu\nu}}^6(s) = \frac{3(s \bar{s})(q\bar{q})}{64 \pi^2} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} da \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \left[ (1 - \alpha - \beta)^2 \frac{(m_c^2(\alpha + \beta) - \alpha \beta s)^3}{\alpha \beta^3} \right] \]

\[ \rho_{J_{\mu\nu}}^8(s) = \frac{5(q_g^2 GG_s m_c^2)}{1024 \pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} da \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \left[ (1 - \alpha - \beta)^2 \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) \right], \]

\[ \rho_{J_{\mu\nu}}^4(s) = \frac{5(q_g^2 GG_s m_c^2)}{2048 \pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} da \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \left[ (1 - \alpha - \beta)(m_c^2(\alpha + \beta) - \alpha \beta s) \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) \right] \]

\[ \times \left( (1 - \alpha - \beta)(m_c^2(\alpha + \beta) - \alpha \beta s) - 4(m_c^2(\alpha + \beta) - 2\alpha \beta s) \right). \]
\[\rho_{3\mu}^5(s) = \frac{3(\bar{g}\sigma \cdot Gs)m_c}{128\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{3(m_c^2(\alpha + \beta) - 2\alpha\beta s)m_c + 2(2m_c^2(\alpha + \beta) - 3\alpha\beta s)m_c}{\beta},\]
\[+ \frac{5(\bar{g}\sigma \cdot Gq)m_c}{128\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{3(m_q^2(\alpha + \beta) - 2\alpha\beta s)m_q + 2(2m_q^2(\alpha + \beta) - 3\alpha\beta s)m_q}{\alpha},\]

\[\rho_{3\mu}^5(s) = \frac{5(\bar{q} g_s \cdot Gq) m_q}{256\pi^4} (s - 2m_c^2) m_q - 30m_c^2 m_s \sqrt{1 - \frac{4m_c^2}{s}},\]
\[+ \frac{5(\bar{g} g_s \cdot GSs)}{256\pi^4} (s - 2m_s^2) m_s - 30m_c^2 m_q \sqrt{1 - \frac{4m_s^2}{s}}.,\]

\[\rho_{3\mu}^6(s) = \frac{5(\bar{g} g_s \cdot Gq)}{32\pi^2} (4m_c^2 + m_c m_q + m_c m_s) \sqrt{1 - \frac{4m_c^2}{s}},\]

\[\Pi_{3\mu}^6(M_B^2) = \frac{5m_c^4}{32\pi^2} \int_0^1 da \left( \frac{m_q}{1 - \alpha} + \frac{m_s}{\alpha} \right) e^{\frac{-m_s^2}{\alpha(1 - \alpha)M_B^2}},\]

\[\Pi_{3\mu}^8(M_B^2) = \frac{5m_c^4}{32\pi^2} \int_0^1 da \frac{\langle \bar{g} g_s \cdot Gq \rangle}{(1 - \alpha)^2 M_B^2 + \frac{\alpha^3 M_B^2}{\alpha^2}},\]

8. Spectral densities for \(\eta_{3\mu}^5\):

\[\rho_{\eta_{3\mu}^5}(s) = \frac{3}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)^2}{\alpha^3 \beta^3} \left( m_c^2(\alpha + \beta) - \alpha\beta s \right)^3 \left( m_c^2(\alpha + \beta) - 5\alpha\beta s \right),\]

\[\rho_{\eta_{3\mu}^b}(s) = \frac{3m_c}{256\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 \left( m_c^2(\alpha + \beta) - \alpha\beta s \right)^2 \left( \frac{m_c^2(\alpha + \beta) - 2\alpha\beta s}{\alpha^3 \beta^2} - \frac{\alpha\beta s m_q}{\alpha^2 \beta^3} \right),\]

\[\rho_{\eta_{3\mu}^a}(s) = \frac{(\bar{q} q)}{64\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha\beta s) \left[ \frac{4(1 - \alpha - \beta) sm_c}{\beta} + \frac{12m_c^2 m_q + 3(m_c^2(\alpha + \beta) - 3\alpha\beta s)m_q}{\alpha^3 \beta^2} \right],\]

\[\rho_{\eta_{3\mu}^b}(s) = \frac{(\bar{q} q) GG}{64\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha\beta s) \left[ \frac{2(1 - \alpha - \beta)(3m_c^2(\alpha + \beta) - 5\alpha\beta s)m_c}{\alpha^2 \beta} - \frac{12m_c^2 m_q + 3(m_c^2(\alpha + \beta) - 3\alpha\beta s)m_q}{\alpha^3 \beta^2} \right],\]

\[\rho_{\eta_{3\mu}^a}(s) = \frac{(g_s Gq)}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 \left( m_q^2(\alpha + \beta) - 2\alpha\beta s \right) \left( \frac{1}{\alpha^3} + \frac{1}{\beta^3} \right),\]

\[\rho_{\eta_{3\mu}^b}(s) = \frac{(g_s Gq) GG}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_q^2(\alpha + \beta) - \alpha\beta s) \left[ 3 + \frac{4(1 - \alpha - \beta)}{\beta} - \frac{3(1 - \alpha - \beta)^2}{4\beta^2} \right],\]

\[\rho_{\eta_{3\mu}^a}(s) = \frac{5(\bar{g} g_s \cdot Gq)m_c}{192\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (3m_c^2(\alpha + \beta) - 4s\alpha\beta) \left( \frac{1 - \alpha + 2\beta}{\alpha \beta} \right),\]
\[ \rho_{\eta_{\mu}}^{5b}(s) = -\frac{(\bar{g}g_\sigma Gs)m_c}{384\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 + 5\alpha - \beta), \]

\[ \rho_{\eta_{\mu}}^{5c}(s) = \frac{(\bar{g}g_\sigma Gs)}{256\pi^4} \left( (s - m_c^2) m_s - 9m_c^2 m_q \right) \sqrt{1 - \frac{4m_c^2}{s}} + \frac{(\bar{q}g_\sigma Gq)}{256\pi^4} \left( (s - m_c^2) m_q - 9m_c^2 m_s \right) \sqrt{1 - \frac{4m_c^2}{s}}, \]

\[ \rho_{\eta_{\mu}}^{6a}(s) = \frac{3(\bar{q}q)(\bar{s}s)}{16\pi^2} (4m_c^2 + m_c m_q) \sqrt{1 - \frac{4m_c^2}{s}}, \]

\[ \Pi_{\eta_{\mu}}^{6b}(M_B^2) = \frac{(\bar{q}q)(\bar{s}s)m_c^3}{24\pi^2} \int_0^1 d\alpha \left( \frac{m_s}{1 - \alpha} + \frac{m_q}{\alpha} \right) e^{-\frac{m_c^2}{(1 - \alpha) M_B^2}}, \]

\[ \Pi_{\eta_{\mu}}^{8}(M_B^2) = \frac{m_s^4}{96\pi^2} \int_0^1 d\alpha \left( \frac{6((\bar{q}q)(\bar{s}s) Gs) + (\bar{s}s)qg_\sigma Gq)}{(1 - \alpha)^2 M_B^2} + \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle Gs + 2 \langle \bar{s}s \rangle \langle \bar{q}q \rangle Gq}{(1 - \alpha) m_c^2} \right) e^{-\frac{m_c^2}{(1 - \alpha) M_B^2}}, \]

9. Spectral densities for \( \eta_{\mu} \):

\[ \rho_{J_{\mu}}^{0a}(s) = \frac{3}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta) (m_c^2(\alpha + \beta) - \alpha \beta s)^3(m_c^2(\alpha + \beta) - 5\alpha \beta s), \]

\[ \rho_{J_{\mu}}^{0b}(s) = -\frac{3m_c}{256\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)(m_c^2(\alpha + \beta) - \alpha \beta s)^2 \left( \frac{m_s}{\alpha^3 \beta^3} - \frac{\alpha \beta m_s}{\alpha^2 \beta^3} \right), \]

\[ \rho_{\eta_{\mu}}^{3a}(s) = \frac{(\bar{s}s)}{64\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha \beta s) \left[ \frac{4(1 - \alpha - \beta) m_s}{\beta} + \frac{12 m_c^2 m_q + 3(m_c^2(\alpha + \beta) - 3\alpha \beta s m_s)}{\alpha \beta} \right], \]

\[ \rho_{\eta_{\mu}}^{3b}(s) = -\frac{(\bar{q}q)}{64\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha \beta s) \left[ \frac{2(1 - \alpha - \beta)(3m_c^2(\alpha + \beta) - 5\alpha \beta s m_s)}{\alpha^2 \beta} \right. \]

\[ \left. - \frac{12 m_c^2 m_s + 3(m_c^2(\alpha + \beta) - 3\alpha \beta s m_q)}{\alpha \beta} \right], \]

\[ \rho_{\eta_{\mu}}^{4a}(s) = \frac{m_c^2}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)(m_c^2(\alpha + \beta) - 2\alpha \beta s) \left( \frac{1}{\alpha^3} + \frac{1}{\beta^3} \right), \]

\[ \rho_{\eta_{\mu}}^{4b}(s) = \frac{m_c^2}{1024\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha \beta s) \left( 3 + \frac{4(1 - \alpha - \beta)}{\beta} - 3(1 - \alpha - \beta)^2 \right), \]

\[ \rho_{\eta_{\mu}}^{5a}(s) = \frac{(\bar{g}g_\sigma Gs)m_c}{192\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (3m_c^2(\alpha + \beta) - 4s\alpha \beta) \left( \frac{1 - \alpha + 2\beta}{\alpha \beta} \right), \]

\[ \rho_{\eta_{\mu}}^{5b}(s) = -\frac{(\bar{q}g_\sigma Gq)m_c}{384\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 + 5\alpha - \beta), \]
\[
\rho_{\eta_{\mu\nu}}^{5c}(s) = \frac{\langle \bar{s}g_s \cdot Gs \rangle}{256\pi^4} (s - m_c^2) m_s - 9m_c^2 m_q) \sqrt{1 - \frac{4m_c^2}{s}},
\]
\[
+ \frac{\langle \bar{q}g_s \cdot Gq \rangle}{256\pi^4} (s - m_c^2) m_q - 9m_c^2 m_s) \sqrt{1 - \frac{4m_c^2}{s}},
\]
\[
\rho_{\eta_{\mu\nu}}^{6a}(s) = \frac{3\langle \bar{q}q \rangle (\bar{s}s)}{16\pi^2} (4m_c^2 + m_c m_s) \sqrt{1 - \frac{4m_c^2}{s}},
\]
\[
\Pi_{\eta_{\mu\nu}}^{6b}(M_B^2) = \frac{\langle \bar{q}q \rangle (\bar{s}s) m_c^3}{24\pi^2} \int_0^1 da \left( \frac{m_s}{1 - \alpha} + \frac{m_q}{\alpha} \right) e^{\frac{-m_c^2}{\alpha(1 - \alpha)M_B^2}},
\]
\[
\Pi_{\eta_{\mu\nu}}^{8}(M_B^2) = -\frac{m_c^4}{96\pi^2} \int_0^1 da \left[ \frac{6\langle \bar{s}g_s \cdot Gs \rangle + \langle \bar{s}s \rangle \langle qg_s \cdot Gq \rangle}{(1 - \alpha)^2 M_B^2} + \frac{2\langle \bar{q}q \rangle (\bar{s}g_s \cdot Gs) + \langle \bar{s}s \rangle (\bar{q}g_s \cdot Gq)}{(1 - \alpha) m_c^2} \right] e^{\frac{-m_c^2}{\alpha(1 - \alpha)M_B^2}},
\]

10. Spectral densities for \(\eta_{\mu\nu}^a\):
\[
\rho_{\eta_{\mu\nu}}^{5a}(s) = -\frac{5}{768\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( 1 - \alpha - \beta \right)^2 (m_c^2(\alpha + \beta) - \alpha \beta s)^3 \left( (\alpha + 2)(m_c^2(\alpha + \beta) - \alpha \beta s) - 3(m_c^2(\alpha + \beta) - 3\alpha \beta s) \right)
\]
\[
\rho_{\eta_{\mu\nu}}^{5b}(s) = -\frac{15m_c^2}{384\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 (m_c^2(\alpha + \beta) - \alpha \beta s)^2 (m_c^2(\alpha + \beta) - 4\alpha \beta s) \left( \frac{m_s}{\alpha^2 \beta^3} + \frac{m_q}{\alpha^3 \beta^2} \right)
\]
\[
\rho_{\eta_{\mu\nu}}^{3a}(s) = -\frac{5\langle \bar{s}s \rangle}{16\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha \beta s) \left[ \frac{(1 - \alpha - \beta)(m_c^2(\alpha + \beta) - 3\alpha \beta s)m_c}{\alpha^2 \beta} - \frac{2m_c^2 m_q - \alpha \beta s m_s + (1 - \alpha - \beta)(m_c^2(\alpha + \beta) - s \alpha \beta ) m_s}{\alpha \beta} \right]
\]
\[
\rho_{\eta_{\mu\nu}}^{3b}(s) = -\frac{5\langle \bar{q}q \rangle}{16\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha \beta s) \left[ \frac{(1 - \alpha - \beta)(m_c^2(\alpha + \beta) - 3\alpha \beta s)m_c}{\alpha \beta^2} - \frac{2m_c^2 m_s - \alpha \beta s m_q + (1 - \alpha - \beta)(m_c^2(\alpha + \beta) - s \alpha \beta ) m_q}{\alpha \beta} \right]
\]
\[
\rho_{\eta_{\mu\nu}}^{4a}(s) = \frac{5\langle \bar{q}G \rangle G m_c^2}{768\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 \left[ (1 - \alpha - \beta)(m_c^2(\alpha + \beta) - \alpha \beta s) \left( \frac{1}{3\alpha^3} + \frac{1}{3\beta^3} \right) - \left( \frac{\beta s}{2\alpha^2} + \frac{\alpha s}{2\beta^2} \right) \right]
\]
\[
\rho_{\eta_{\mu\nu}}^{4b}(s) = \frac{5\langle \bar{q}G \rangle G m_c^2}{12288\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} da \int_{\beta_{\min}}^{\beta_{\max}} d\beta (m_c^2(\alpha + \beta) - \alpha \beta s) \left[ (m_c^2(\alpha + \beta) - 3\alpha \beta s) \left( \frac{1}{\alpha \beta^2} + \frac{2(1 - \alpha - \beta)^2}{\alpha \beta} \right) + \frac{4(m_c^2(\alpha + \beta) - \alpha \beta s)(1 - \alpha - \beta)(\alpha + \beta)}{\alpha \beta^2} \right] + \frac{4(m_c^2(\alpha + \beta) - \alpha \beta s)(1 - \alpha - \beta)(\alpha + \beta)}{\alpha \beta^2},
\]
\[ p_{5a}^{\eta_{\mu \nu}}(s) = \frac{5(\bar{s}g_s \cdot G_s)m_c}{96\pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{3(m_c^2(\alpha + \beta) - 2\alpha\beta s)m_c + 2(2m_c^2(\alpha + \beta) - 3\alpha\beta s)m_s}{\beta}, \]
\[ + \frac{5(\bar{q}g_s \cdot G_q)m_c}{96\pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{3(m_c^2(\alpha + \beta) - 2\alpha\beta s)m_c + 2(2m_c^2(\alpha + \beta) - 3\alpha\beta s)m_q}{\alpha}, \]
\[ p_{5b}^{\eta_{\mu \nu}}(s) = \frac{5(\bar{q}g_s \cdot G_q)}{192\pi^4} \left( (s - 2m_c^2) m_q - 30m_c^2m_s \right) \sqrt{1 - \frac{4m_c^2}{s}}, \]
\[ + \frac{5(\bar{s}g_s \cdot G_s)}{192\pi^4} \left( (s - 2m_c^2) m_s - 30m_c^2m_q \right) \sqrt{1 - \frac{4m_c^2}{s}}, \]
\[ p_{5c}^{\eta_{\mu \nu}}(s) = \frac{5((\bar{s}g_s \cdot G_s) + (\bar{q}g_s \cdot G_q))m_c}{384\pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{7(m_c^2(\alpha + \beta) - 6\alpha\beta s)(\alpha + 5(1 - \alpha + \beta))}{\alpha \beta}, \]
\[ \Pi_{6a}^{\eta_{\mu \nu}}(s) = \frac{5(\bar{s}s)(\bar{q}q)m_c^3}{24\pi^2} \left( 4m_c^2 + m_c m_q + m_c m_s \right) \sqrt{1 - \frac{4m_c^2}{s}}, \]
\[ \Pi_{6b}^{\eta_{\mu \nu}}(M_B^2) = \frac{5(\bar{s}s)(\bar{q}q)m_c^3}{12\pi^2} \int_0^1 d\alpha \left( \frac{m_q}{1 - \alpha} + \frac{m_s}{\alpha} \right) e^{-\frac{m_c^2}{\alpha(1 - \alpha)M_B^2}}, \]
\[ \Pi_{8}^{\eta_{\mu \nu}}(M_B^2) = \frac{5m_c^4}{24\pi^2} \int_0^1 d\alpha \left[ \frac{(\bar{s}s)(\bar{q}q \cdot G_q) + (\bar{q}q)(\bar{s}g_s \cdot G_s)}{(1 - \alpha)^2 M_B^2} - \frac{(\bar{s}s)(\bar{q}g_s \cdot G_q) + (\bar{q}g_s \cdot G_q)}{12\alpha} \right] e^{-\frac{m_c^2}{\alpha(1 - \alpha)M_B^2}}, \]