The scalar sector of the $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ model

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Abstract

A complete study of the Higgs sector of the $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ model is carried out, obtaining all possible cases of vacuum expectation values that permit the spontaneous symmetry breaking pattern $SU(3)_L \otimes U(1)_X \rightarrow SU(2)_L \otimes U(1)_Y \rightarrow U(1)_Q$. We find the most general Higgs potentials that contain three triplets of Higgs and one sextet. A detailed study of the scalar sector for different models with three Higgs triplets is done. The models end up in an electroweak two Higgs doublet model after the first symmetry breakdown; we find that the low energy limit depends on a trilinear parameter of the Higgs potential, and that the decoupling limit from the electroweak two Higgs doublet model to the minimal standard model can be obtained quite naturally.

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1 Introduction

Though the Standard Model (SM) is a good phenomenological theory and coincides very well with all experimental results, it leaves several unanswered questions which suggest that SM should be an effective model at low energies, originated from a more fundamental theory. Some of the unexplained aspects in SM are: the existence of three families, the mass hierarchy problem, the quantization of the electric charge, the large number of free parameters to fit the model, the absence of an explanation for the matter anti-matter asymmetry, the fact that...
the SM says nothing about the stability of the proton, gravity cannot be incorporated as a gauge theory \([4], [6]\), it cannot account on the neutrino deficit problem \([7], [8]\), etc.

Some of these problems can be understood introducing a larger particle content or enlarging the group of symmetry where the SM is embedded. The \(SU(5)\) grand unification model of Georgi and Glashow \([9]\) unifies the interactions and predicts the electric charge quantization; the group \(E_6\) unifies the electroweak and strong interactions and might explain the masses of the neutrinos \([10]\). Nevertheless, such models are for only one fermion family. Models with larger symmetries that may allow to understand the origin of the families have been proposed\([11]\). In some models, it is also possible to understand the number of families from the cancelation of quiral anomalies, necessary to preserve the renormalizability of the theory\([12]\). This is the case of the \(SU(3)_c \otimes SU(3)_L \otimes U(1)_X\), or 331 models, which are an immediate extension of the SM \([14], [15], [16], [17]\). There is a great variety of such models, which have generated new expectatives and possibilities to solve several problems of the SM.

Refs. \([18]\) have studied the different 331 models based on the criterium of cancelation of anomalies. Cancelation of chiral anomalies demands some conditions, for instance, the number of fermionic triplets must be equal to the number of anti-triplets\([19]\); notwithstanding, an infinite number of models with exotic charges are found. However, only two 331 models with no exotic charges exist, and they are classified according to the fermion assignment. In the present work, we find those possible 331 models from a different point of view.

As well as the cancelation of anomalies, the scalar sector should be taken into account. This sector couples to the fermions by means of the Yukawa Lagrangian generating additional constraints for the quantum numbers. From the Yukawa Lagrangian, the scalar fields neccesary to endow fermions with masses are selected. The vacuum is aligned in such a way that it respects the symmetry breaking scheme

\[
SU(3)_L \otimes U(1)_X \rightarrow SU(2)_L \otimes U(1)_Y \rightarrow U(1)_Q \quad (31 \rightarrow 21 \rightarrow 1)
\]  

where the different solutions to align the vacuum fit quantum numbers which are important in the determination of the electric charge and to determine the different 331 models.

In section 2 we examine the structure of the representations for fermions and vector bosons under the quiral group \(SU(3)_c \otimes SU(3)_L \otimes U(1)_X\). In section 3 we analyze the representations in the Higgs sector, as well as the possible vacuum alignments that generates the scheme of spontaneous symmetry breaking (SSB) \(31 \rightarrow 21 \rightarrow 1\). Along with these possibilities, section 4 shows the most general Higgs potentials for any type of 331 models containing three Higgs triplets and one Higgs sextet, necessary for all the fermions to acquire their masses. In section 5.2. we apply our general results about the scalar sector for a model of type \(\beta = 1/\sqrt{3}\) with three Higgs triplets, obtaining the complete scalar spectrum. Section 7 shows the behavior of the \(\beta = 1/\sqrt{3}\) model in the low energy regime. Finally, section 8 is regarded
for our conclusions.

2 Fermion and vector representations

2.1 Fermion representations

The fermions present the following general structure of transformations under the quiral group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$

\[
\begin{align*}
\hat{\psi}_L &= \{ \hat{\psi}_L : (3, 3, X_q^L) = (3, 2, X_q^L) \oplus (3, 1, X_q^L), \\
\hat{\psi}_L^* &= \{ \hat{\psi}_L^* : (3, 3^*, -X_q^L) = (3, 2^*, -X_q^L) \oplus (3, 1, -X_q^L), \\
\hat{\psi}_R &= \{ \hat{\psi}_R : (3, 1, X_q^R),
\end{align*}
\]

where the quarks $\hat{q}$ are color triplets $(3)$ and the leptons $\hat{\ell}$ are color singlets $(1)$. The second equality corresponds to the branching rules $SU(2)_L \subset SU(3)_L$. Both possibilities $3$ and $3^*$ are included in the flavor sector $(SU(3)_L)$, since the same number of fermion triplets and antitriplets must be present in order to cancel anomalies $[19]$. The generator of $U(1)_X$ commute with the matrices of $SU(3)_L$, therefore it should be a matrix proportional to the identity, where the factor of proportionality $X_p$ takes a value according to the representations of $SU(3)_L$ and the anomalies cancelation.

The electric charge is defined in general as a linear combination of the diagonal generators of the group

\[
\hat{Q} = \alpha T_3 + \beta T_8 + X \hat{I},
\]

with $T_3 = \frac{1}{2} \text{diag}(1, -1, 0)$ and $T_8 = \frac{1}{2\sqrt{3}} \text{diag}(1, 1, -2)$, where the normalization chosen is $Tr(T_\alpha T_\beta) = \frac{1}{2} \delta_{\alpha\beta}$, and $\hat{I} = \text{diag}(1, 1, 1)$ is the identity matrix. In this work we shall show that these quantum numbers can also be fixed by choosing the scalar sector and vacuum alignment to break the symmetry of the model.

2.2 Representation of gauge bosons

The gauge bosons associated to the group $SU(3)_L$ transform according to the adjoint representation and are written in the form

\[
W_\mu = W_\mu^a G_a = \frac{1}{2} \begin{bmatrix} W_\mu^3 + \frac{1}{\sqrt{3}} W_\mu^8 & \sqrt{2} W_\mu^+ & \sqrt{2} K_\mu^Q_1 \\ \sqrt{2} W_\mu^- & -W_\mu^3 + \frac{1}{\sqrt{3}} W_\mu^8 & \sqrt{2} K_\mu^Q_2 \\ \sqrt{2} K_\mu^{Q_1} & \sqrt{2} K_\mu^{Q_2} & -\frac{2}{\sqrt{3}} W_\mu^8 \end{bmatrix}.
\]
Therefore, the electric charge takes the general form

\[
Q_W \rightarrow \begin{bmatrix}
0 & 1 & \frac{1}{2} + \frac{\sqrt{3}}{2} \\
-1 & 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2} \\
-\frac{1}{2} - \frac{\sqrt{3}}{2} & \frac{1}{2} - \frac{\sqrt{3}}{2} & 0
\end{bmatrix}.
\] (2.4)

From Eq. (2.4) we see that the cases \( \beta = \pm \sqrt{3} \) lead to exotic charges and only the cases \( \beta = \pm 1/\sqrt{3} \) does not. As for the gauge field associated to \( U(1)_X \), it is represented as

\[
B_\mu = \mathbf{I}_{3 \times 3} B_\mu,
\] (2.5)

which is a singlet under \( SU(3)_L \), and its electric charge is given by

\[
Q_{B_\mu} \rightarrow 0.
\] (2.6)

From the previous expressions we see that three gauge fields with charges equal to zero are obtained, and in the basis of mass eigenstates they correspond to the photon, \( Z \) and \( Z' \). Two fields with charges \( \pm 1 \) associated to \( W^\pm \) and four fields with charges that depend on the choice of \( \beta \). Demanding that the model contains no exotic charges is equivalent to settle \( \beta = -1/\sqrt{3} \) [16], and \( \beta = 1/\sqrt{3} \). The Higgs potential of the model with \( \beta = 1/\sqrt{3} \) has been discussed recently by the third of Refs. [18], however we consider the most general Higgs potential with three Higgs triplets and we also show that it reduces to the 2HDM in a natural way when the decoupling limit is taken.

In both of the cases \( \beta = \pm 1/\sqrt{3} \), the fields \( K^\pm Q_1 \) and \( K^\pm Q_2 \) possess charges \( \pm 1 \) and 0. It is important to take into account the scalar sector and the symmetry breakings to fix this quantum number, which in turn determine the would-be Goldstone bosons associated to the gauge fields, with the same electric charge of the gauge fields that are acquiring mass in the different scales of breakdown. When \( \beta \) takes a value different from the previous ones we get fields with exotic electric charges.

### 2.3 Fermionic anomalies

The coefficient of triangular anomalies is defined as

\[
\frac{1}{2} A_{\alpha \beta \gamma} = Tr \left[ \left\{ G_\alpha (\hat{T}_\psi)_L, G_\beta (\hat{T}_\psi)_L \right\} G_\gamma (\hat{T}_\psi)_L 
- \left\{ G_\alpha (\hat{T}_\psi)_R, G_\beta (\hat{T}_\psi)_R \right\} G_\gamma (\hat{T}_\psi)_R \right]
\] (2.7)

where \( G_\alpha (\hat{T}_\psi)_h \) are the matrix representations for each group generator acting on the basis \( \psi \) with helicity left or right. In the particular case of the 331 gauge models,
The scalar sector of the 331 model

such conditions become

\[(a) \quad [SU(3)_c]^3 \to A_{\alpha\beta\gamma} = 0,\]
\[(b) \quad [SU(3)_c]^2 \otimes SU(3)_L \to A_{\alpha\beta\gamma} = 0,\]
\[(c) \quad [SU(3)_c]^2 \otimes U(1)_X \to \frac{1}{2} A_{\alpha\alpha0} = \pm 3X_q^L - \sum_{\text{singlets}} X_q^R = 0,\]
\[(d) \quad SU(3)_c \otimes [SU(3)_L]^2 \to A_{\alpha\beta\gamma} = 0,\]
\[(e) \quad SU(3)_c \otimes [U(1)_X]^2 \to A_{\alpha\alpha0} = 0,\]
\[(f) \quad SU(3)_c \otimes SU(3)_L \otimes U(1)_X \to \frac{1}{2} A_{\alpha\alpha0} = 0,\]
\[(g) \quad [SU(3)_L]^3 \to \frac{1}{2} A_{\alpha\alpha\beta} = 0,\]
\[(h) \quad [SU(3)_L]^2 \otimes U(1)_X \to \frac{1}{2} A_{\alpha\alpha0} = \sum_m (\pm X_{\ell(m)}^L) \pm 3X_q^L = 0,\]
\[(i) \quad SU(3)_L \otimes [U(1)_X]^2 \to A_{\alpha\alpha0} = 0,\]
\[(j) \quad [U(1)_X]^3 \to \frac{1}{2} A_{000} = 3 \sum_m (\pm X_{\ell(m)}^L)^3 \pm 9 (X_q^L)^3 - \sum_{\text{singlets}} (X_{\ell}^R)^3 - 3 \sum_{\text{singlets}} (X_q^R)^3 = 0.\]

When we impose for the fermionic representations of \(SU(3)_c \otimes SU(3)_L\) to be vector-like, then the number of color triplets should be equal to the number of color antitriplets, and same for the left triplets. In this case, all the conditions of Eqs. (2.8) are satisfied, except the ones that involve the quantum numbers associated to \(U(1)_X\). Considering a particle content as the one in table I, the non-trivial conditions are reduced to

\[\frac{1}{2} A_{\alpha\alpha0} = \pm 3X_q^L - \sum_{\text{singlets}} X_q^R = 0\]
\[\frac{1}{2} A_{\alpha\alpha0} = \sum_m (\pm X_{\ell(m)}^L) \pm 3X_q^L = 0\]
\[\frac{1}{4} A_{000} = 3 \sum_m (\pm X_{\ell(m)}^L)^3 \pm 9 (X_q^L)^3 - \sum_{\text{singlets}} (X_{\ell}^R)^3 - 3 \sum_{\text{singlets}} (X_q^R)^3 = 0\]

\[\text{[Grav]}^2 \otimes U(1)_X \to A_{\text{grav}} = 3 \sum_m (\pm X_{\ell(m)}^L) \pm 9X_q^L - \sum_{\text{singlets}} (X_{\ell}^R) - 3 \sum_{\text{singlets}} (X_q^R) = 0\]

where we have included the gravitational anomaly condition as well. These conditions still permit an infinite number of solutions for the quantum numbers, that can be characterized by the values of \(a\) and \(\beta\) in the definition of the electromagnetic charge Eq. (2.2). However, \(a = 1\) is required to obtain isospin doublets to embed properly the \(SU(2)_L \otimes U(1)_Y\) model into \(SU(3)_L \otimes U(1)_X\). Now, in order to restrict \(\beta\), we could demand that the exotic component of the triplet has ordinary electromagnetic charge, in whose case \(\beta = \pm 1/\sqrt{3}\). On the other hand, we can obtain
relations between $\beta$ and the quantum numbers associated to $U(1)_X$ by imposing

gauge invariance in the Yukawa sector, and by taking into account the symmetry

breaking pattern.

$$
q_L = \begin{pmatrix}
q_1^L \\
q_2^L \\
q_3^L
\end{pmatrix}, \quad Q_\psi = \begin{pmatrix}
\pm \frac{2}{3} \pm X_q^L \\
\mp \frac{2}{3} \pm X_q^L \\
\mp \frac{1}{3} \pm X_q^L
\end{pmatrix}, \quad Y_\psi = \begin{pmatrix}
\pm \frac{1}{6} \pm X_q^L \\
\mp \frac{1}{6} \pm X_q^L \\
\mp \frac{1}{3} \pm X_q^L
\end{pmatrix}
$$

| $q_L$ | $Q_\psi$ | $Y_\psi$ |
|-------|----------|----------|
| $\ell^{(m)}_L = \begin{pmatrix}
\ell_1^{(m)}_L \\
\ell_2^{(m)}_L \\
\ell_3^{(m)}_L
\end{pmatrix}$ | $\begin{pmatrix}
\pm \frac{1}{6} \pm X_q^{(m)}_L \\
\mp \frac{1}{6} \pm X_q^{(m)}_L \\
\mp \frac{1}{3} \pm X_q^{(m)}_L
\end{pmatrix}$ | $\begin{pmatrix}
\pm \frac{1}{6} \pm X_q^{(m)}_L \\
\mp \frac{1}{6} \pm X_q^{(m)}_L \\
\mp \frac{1}{3} \pm X_q^{(m)}_L
\end{pmatrix}$ |

Table 1: Particle content for the fermionic sector of the 331 model for a one family

scenario. $q$ refers to triplets of color, $m = 1, 2, 3$ and $n$ is arbitrary.

3 Scalar representations

3.1 SSB scheme

From the phenomenological point of view, the SM is an effective theory at the
electroweak scale, hence, the SSB follows the scheme

$$SU(3)_L \otimes U(1)_X \xrightarrow{\Phi_1} SU(2)_L \otimes U(1)_Y \xrightarrow{\Phi_2} U(1)_Q$$

In the first breakdown there are five gauge fields that acquire mass proportional to

$< \Phi_1 >$ and in the second breakdown (electroweak breakdown), three gauge fields

acquire mass of the order of $< \Phi_2 >$, leaving a massless field associated to the

unbroken generator $Q$, the photon.

3.1.1 $SU(3)_L \otimes U(1)_X \rightarrow SU(2)_L \otimes U(1)_Y$ transition

In the first transition the VEV’s of $\Phi_1$ break the symmetry $SU(3)_L \otimes U(1)_X / SU(2)_L \otimes U(1)_Y$ where the hypercharge is defined as $\hat{Y} = \beta \hat{T}_8 + X \hat{T}$. The conditions that
should be satisfied in this breaking step are

\[
\begin{align*}
\left[ \hat{T}_1^L, \langle \Phi_1 \rangle_0 \right] &= 0, & \left[ \hat{T}_4^L, \langle \Phi_1 \rangle_0 \right] &\neq 0 \\
\left[ \hat{T}_2^L, \langle \Phi_1 \rangle_0 \right] &= 0, & \left[ \hat{T}_5^L, \langle \Phi_1 \rangle_0 \right] &\neq 0 \\
\left[ \hat{T}_3^L, \langle \Phi_1 \rangle_0 \right] &= 0, & \left[ \hat{T}_6^L, \langle \Phi_1 \rangle_0 \right] &\neq 0 \\
\left[ \beta \hat{T}_8^L + X \hat{T}_I, \langle \Phi_1 \rangle_0 \right] &= 0, & \left[ \beta \hat{T}_8^L - X \hat{T}_I, \langle \Phi_1 \rangle_0 \right] &\neq 0
\end{align*}
\]

These conditions reflect the fact that five bosons acquire mass and in order to preserve the number of degrees of freedom, at least five components in \( \Phi_1 \) are needed to represent the would be Goldstone bosons.

### 3.1.2  \( SU(2)_L \otimes U(1)_Y \to U(1)_Q \) transition

In the second transition the VEV’s of \( \Phi_2 \) breaks the symmetry \( SU(2)_L \otimes U(1)_Y / U(1)_Q \). The conditions that should be satisfied in this breakdown read

\[
\begin{align*}
\left[ \hat{T}_1^L, \langle \Phi_2 \rangle_0 \right] &\neq 0 \\
\left[ Q, \langle \Phi_2 \rangle_0 \right] &= 0, & \left[ \hat{T}_2^L, \langle \Phi_2 \rangle_0 \right] &\neq 0 \\
\left[ \hat{T}_3^L - (\beta \hat{T}_8^L + X \hat{T}_I), \langle \Phi_2 \rangle_0 \right] &\neq 0
\end{align*}
\]

where three gauge bosons acquire mass, from which \( \Phi_2 \) needs three components associated to the would be Goldstone bosons.

### 3.2 Yukawa terms

Other conditions fixed by the scalar sector are related to the fermion-fermion-scalar couplings that generate the fermion masses. The scalar fields have to be coupled to fermions by Yukawa terms invariant under \( SU(3)_L \otimes U(1)_X \); these couplings can be written as

\[
\begin{align*}
\overline{\psi}_L \psi_R \Phi : & \quad 3^* \otimes 1 \otimes \Phi = 1 \Rightarrow \Phi = 3, \\
\overline{\psi}_L (\psi_R) \Phi : & \quad 3^* \otimes 3^* \otimes \Phi = 1 \Rightarrow \Phi = 3 \otimes 3 = 3^* \oplus 6, \\
\overline{\psi}_R (\psi_R) \Phi : & \quad 1 \otimes 1 \otimes \Phi = 1 \Rightarrow \Phi = 1, \\
(\psi_R) \Phi : & \quad 1 \otimes 3^* \otimes \Phi = 1 \Rightarrow \Phi = 3.
\end{align*}
\]

Hence, in order to generate the masses of the fermions, the Higgs bosons should lie either in the singlet, triplet, antitriplet or sextet representations of \( SU(3)_L \).
3.3 Scalar representations for the first transition ($\Phi_1$)

The choice of the scalar sector should fulfill the following basic conditions

1. The VEV’s should accomplish the conditions (3.2) and (3.3) in the first and second transition respectively.

2. When the VEV’s are replaced in Eq. (3.4), the fermions should acquire the appropriate masses.

3. The number of components of $\Phi$ should be at least the number of would be Goldstone bosons for each transition of SSB.

3.3.1 Triplet representation 3

One of the possible solutions of the Eq. (3.4) is the fundamental representation 3 where the VEV’s can be written in the form

$$\langle \chi \rangle_0 = \begin{pmatrix} \nu_{\chi_1} \\ \nu_{\chi_2} \\ \nu_{\chi_3} \end{pmatrix}.$$

(3.5)

By demanding the conditions of Eq. (3.2) for the first breaking of symmetry, it is found that the vacuum should be aligned in the following way

$$\langle \chi \rangle_0 = \begin{pmatrix} 0 \\ 0 \\ \nu_{\chi_3} \end{pmatrix}.$$  

(3.6)

with the condition

$$X_{\chi} - \frac{\beta}{\sqrt{3}} = 0, \quad \beta \neq 0 \quad \nu_{\chi_3} \neq 0,$$

(3.7)

The choice of vacuum above generates the following mass terms

$$\mathcal{L}_Y = \Gamma_1 \bar{\psi}_L^i \psi_R^j \langle \chi^i \rangle + \Gamma_2 (\bar{\psi}_R^j \psi_L^i)^c \langle \chi^i \rangle + h.c.$$  

(3.8)

$$= \Gamma_1 \nu_{\chi^3} \bar{\psi}_L^3 \psi_R^3 \langle \chi^3 \rangle^c + h.c.$$  

(3.9)

3.3.2 Antisymmetric representation 3\(^*\)

Other possible representation to generate masses from Eq. (3.4) is the antisymmetric one 3\(^*\). If we use this representation for the first breaking of the symmetry, then the conditions of Eq. (3.2) should be satisfied, implying for the vacuum to be aligned as

$$\langle \chi^* \rangle_0 = \begin{pmatrix} 0 \\ 0 \\ \nu_{\chi_{3*}} \end{pmatrix}.$$  

(3.10)
with the condition
\[ X_{\chi^*} - \frac{\beta}{\sqrt{3}} = 0, \quad \beta \neq 0; \quad \nu_{\chi^*} \neq 0. \] (3.11)

The anti-triplet antisymmetric representation only admits mass Yukawa terms of the form
\[ \mathcal{L}_Y = \Gamma^i \epsilon_{ijk} \overline{\psi}_L^j (\psi_L^k)^c \langle \chi^k \rangle + h.c. \]
\[ = \Gamma^x \nu_{\chi^*} \epsilon_{ij3} \overline{\psi}_L^i (\psi_L^j)^c + h.c. \] (3.12)

### 3.3.3 Symmetric representation

As for the possibility of \( \Phi_1 = S \) symmetric, we have:
\[ \langle S^{ij} \rangle_0 = \begin{bmatrix} \nu_1 & \nu_2 & \nu_3 \\ \nu_2 & \nu_4 & \nu_5 \\ \nu_3 & \nu_5 & \nu_6 \end{bmatrix}, \] (3.13)
and requiring the conditions of Eq. (3.2), it is obtained
\[ \langle S^{ij} \rangle_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu_6 \end{bmatrix} \] (3.14)
with
\[ X_{S^{ij}} - \frac{\beta}{\sqrt{3}} = 0, \quad \beta \neq 0 \quad \nu_6 \neq 0. \] (3.15)

The mass terms generated from the Yukawa Lagrangian are given by
\[ \mathcal{L}_Y = \Gamma^{01} \overline{\psi}_L^i (\psi_L^j)^c \langle S^{ij} \rangle + h.c = \Gamma^x \nu_6 \overline{\psi}_L^3 (\psi_L^3)^c + h.c. \] (3.16)

### 3.4 Scalar representations for the second transition (\( \Phi_2 \))

The first breaking align the vacuum according to the conditions \( X[\Phi] - \beta/\sqrt{3} = 0 \) and impose a requirement on the quantum numbers of the representations \( X[\Phi] - \beta/\sqrt{3} = 0 \). This condition gives freedom to choice the electric charge of the fields of the scalar representations in different directions. To align the vacuum in the second breakdown according to the conditions (3.3) we shall write explicitly the electric charge of the components of the scalar fields.
\[ [\hat{Q}, \Phi] = [\hat{T}_3, \Phi] + \beta [\hat{T}_8, \Phi] + [\hat{X}, \Phi]. \]
The charges for the representations 3, 3* and 6 are given respectively by

\[ Q_3 = \begin{pmatrix} \frac{1}{2} + \frac{\beta}{2\sqrt{3}} + X_\Phi \\ -\frac{1}{2} + \frac{\beta}{2\sqrt{3}} + X_\Phi \\ -\beta \frac{1}{\sqrt{3}} + X_\Phi \end{pmatrix}, \]  
\[ (3.17) \]

\[ Q_{3^*} = \begin{pmatrix} -\frac{1}{2} - \frac{\beta}{2\sqrt{3}} - X_{\Phi^*} \\ \frac{1}{2} - \frac{\beta}{2\sqrt{3}} - X_{\Phi^*} \\ \beta \frac{1}{\sqrt{3}} - X_{\Phi^*} \end{pmatrix}, \]  
\[ (3.18) \]

\[ Q_6 = \begin{pmatrix} 1 + \frac{\beta}{\sqrt{3}} + 2X_{\Phi^{ij}} \\ -1 + \frac{\beta}{\sqrt{3}} + 2X_{\Phi^{ij}} \\ \frac{1}{2} - \frac{\beta}{2\sqrt{3}} + 2X_{\Phi^{ij}} \\ \frac{1}{2} - \frac{\beta}{2\sqrt{3}} + 2X_{\Phi^{ij}} \\ \frac{1}{2} - \frac{\beta}{2\sqrt{3}} + 2X_{\Phi^{ij}} \\ \frac{1}{2} - \frac{\beta}{2\sqrt{3}} + 2X_{\Phi^{ij}} \end{pmatrix}. \]  
\[ (3.19) \]

We should notice that the choice of a particular vacuum alignment fixes the value of the quantum number \( X_\Phi \), and it in turn fixes the \( \beta \) parameter according to the solutions (3.7), (3.11) and (3.15) respectively. The quantum numbers \( X \) for the fermions can be determined by means of the Yukawa sector, according to the couplings of fermions to the scalar sector and same for the gauge fields which acquire mass through the covariant derivative of the scalar fields. Therefore, it should exist a relation amongst both the fermion and gauge sectors with the scalar one. Owing to it, the quantum numbers of the scalar sector were settled free for the different symmetry breakings to fix them and find the different 331 models.

For this transition the vacuum should be chosen in the direction in which the scalar fields have null charge. From Eqs. (3.17)–(3.19) we observe that the components depend on the value assigned to \( \beta \) and \( X_\Phi \). We will align the vacuum arbitrarily and choice the quantum numbers \( \beta \) and \( X_\Phi \) that define the electric charge operator according to the conditions (3.3).

### 3.4.1 Fundamental representation 3

Like in section 3.3.1, the possibility for a triplet is considered but for the second transition. Thus, the conditions (3.3) should be fulfilled. In table 2 the possible vacuum alignments for a representation 3 are shown.

### 3.4.2 Antisymmetric representation 3*

For the anti-triplet antisymmetric representation, we get the same type of solutions as in table 2 and they are shown in table 3.
Table 2: Solutions for the second SSB with Higgs triplets in the fundamental representation $3$.

Table 3: Solutions for the second SSB with Higgs triplets in the antisymmetric representation $3^*$.  

3.4.3 Symmetric representation 6

For the representation of dimension 6 we have the solutions to the conditions shown in table 4.
Table 4: Vacuum alignments for the second SSB with Higgs sextets, and for all values of $\beta$.

In this way, we have found the Higgs bosons necessary to break the symmetry according to the scheme $SU(3)_L \otimes U(1)_X \rightarrow SU(2)_L \otimes U(1)_Y \rightarrow U(1)_Q$, and generate the masses of the fermions and gauge fields. This choice fixes the values of the quantum numbers $X$ and the value of $\beta$ needed to define the electric charge. Among the possible solutions we have $\beta = -\sqrt{3}, \sqrt{3}, -1/\sqrt{3}$ and $1/\sqrt{3}$. They correspond to the models in Refs. [14], [15], [16], and [18] respectively; the Higgs sector of the latter will be developed in detailed in this paper. For $\beta = \pm 1/\sqrt{3}$ the models does not contain exotic electric charges. For values of $\beta$ different from $\pm 1/\sqrt{3}$, exotic charges arise i.e. electric charges different from $\pm 1$ and 0.

4 Higgs Potential

4.1 Three Higgs triplets

The triplet field $\chi$ and the antitriplet $\chi^*$ only introduce VEV on the third component for the first transition, as it is indicated by the solutions (3.7) and (3.11) respectively, it induces the masses of the exotic fermionic components. In the second transition, it is observed in the tables 2 and 3 that pairs of solutions are obtained according to the
value of $\beta$. A careful analysis of such solutions shows that the pair of multiplets is necessary to be able to give masses to the quarks of type up and down respectively*. Therefore, in the second transition is necessary to introduce two triplets $\rho$ and $\eta$ associated to each pair of solutions in the tables 2 and 3†. In table 5 it is shown the minimal contents of Higgs bosons necessary to break the symmetry for the different values of $\beta$ and $X$.

$$V_{\text{higgs}} = \mu_1^2 \chi^i \chi_i + \mu_2^2 \rho^i \rho_i + \mu_3^2 \eta^i \eta_i + \mu_4^2 (\chi^i \rho_i + h.c) + f (\chi_i \rho_j \eta_k \varepsilon^{ijk} + h.c) + \lambda_1 (\chi^i \chi_i)^2 + \lambda_2 (\rho^i \rho_i)^2 + \lambda_3 (\eta^i \eta_i)^2 + \lambda_4 \chi^i \chi_j \rho^j \rho_j + \lambda_5 \chi^i \chi_i \eta^j \eta_j + \lambda_6 \rho^i \rho_j \eta^j \eta_j + \lambda_7 \chi^i \eta^j \eta^j \chi_j + \lambda_8 \chi^i \rho^j \rho_j \chi_j + \lambda_9 \chi^i \rho_j \eta^j \eta_j + \lambda_{10} \chi^i \chi_j \chi_j + \lambda_{11} \rho^i \rho_j + \lambda_{12} \eta^i \eta_j (\chi^i \rho_j + h.c) + \lambda_{13} (\chi_i \rho_j \chi_i \rho_j + h.c) + \lambda_{14} (\eta^i \chi_i \rho_j \eta_j + h.c).$$

(4.1)

*After the first breaking, the model 331 is reduced to the SM with two Higgs doublets [20, 21].
†The choice of three triplets does not ensure necessarily the masses of all leptons, an example is the model of Pisano and Pleitez, where an additional sextet is needed [14].
2. For $\beta = -\frac{1}{\sqrt{3}}$

\[
V_{\text{higgs}} = \mu_1^2 \chi^i \chi_i + \mu_2^2 \rho^i \rho_i + \mu_3^2 \eta^i \eta_i + \mu_4^2 \left( \chi^i \eta_i + h.c. \right) + f \left( \chi_i \rho_j \eta_k \varepsilon^{ijk} + h.c. \right)
\]
\[
+ \lambda_1 (\chi^i \chi_i)^2 + \lambda_2 (\rho^i \rho_i)^2 + \lambda_3 (\eta^i \eta_i)^2 + \lambda_4 \chi^i \chi_i \rho^j \rho_j + \lambda_5 \chi^i \chi_i \eta^j \eta_j
\]
\[
+ \lambda_6 \rho^i \rho^j \eta^k \eta_k + \lambda_7 \chi^i \eta^j \chi_j + \lambda_8 \chi^j \chi_j \eta^i \eta_i + \lambda_9 \rho^i \rho^j \eta^k \rho^k + \eta^k \eta^k
\]
\[
+ \lambda_{10} \chi^i \chi_i (\chi^j \eta_j + h.c.) + \lambda_{11} \eta^i \eta_i (\eta^j \chi_j + h.c.) + \lambda_{12} \rho^i \rho_i (\chi^j \eta_j + h.c.)
\]
\[
+ \lambda_{13} (\chi^i \eta^j \chi_j + h.c.) + \lambda_{14} (\rho^i \chi^j \rho_j + h.c.). \tag{4.2}
\]

3. For $\beta = \sqrt{3}$

\[
V_{\text{higgs}} = \mu_1^2 \chi^i \chi_i + \mu_2^2 \rho^i \rho_i + \mu_3^2 \eta^i \eta_i + f \left( \chi_i \rho_j \eta_k \varepsilon^{ijk} + h.c. \right) + \lambda_1 (\chi^i \chi_i)^2 + \lambda_2 (\rho^i \rho_i)^2
\]
\[
+ \lambda_3 (\eta^i \eta_i)^2 + \lambda_4 \chi^i \rho^j \rho_j + \lambda_5 \chi^i \eta^j \eta_j + \lambda_6 \rho^i \rho^j \eta^k \eta_k + \lambda_7 \chi^i \eta^j \chi_j + \lambda_8 \rho^i \rho^j \eta^k \eta_k
\]
\[
+ \lambda_{10} \chi^i \rho^j \chi_j + \lambda_{11} \eta^i \rho^j \eta_j + \lambda_{10} (\rho^i \chi^j \rho_j + h.c.). \tag{4.3}
\]

4. For $\beta = -\sqrt{3}$

\[
V_{\text{higgs}} = \mu_1^2 \chi^i \chi_i + \mu_2^2 \rho^i \rho_i + \mu_3^2 \eta^i \eta_i + f \left( \chi_i \rho_j \eta_k \varepsilon^{ijk} + h.c. \right) + \lambda_1 (\chi^i \chi_i)^2 + \lambda_2 (\rho^i \rho_i)^2
\]
\[
+ \lambda_3 (\eta^i \eta_i)^2 + \lambda_4 \chi^i \rho^j \rho_j + \lambda_5 \chi^i \eta^j \eta_j + \lambda_6 \rho^i \rho^j \eta^k \eta_k + \lambda_7 \chi^i \eta^j \chi_j + \lambda_8 \rho^i \rho^j \eta^k \eta_k
\]
\[
+ \lambda_{10} \chi^i \rho^j \chi_j + \lambda_{11} \eta^i \rho^j \eta_j + \lambda_{10} (\rho^i \chi^j \rho_j + h.c.). \tag{4.4}
\]

5. For $\beta$ arbitrary (different from $\pm \sqrt{3}, \pm 1/\sqrt{3}$)

\[
V_{\text{higgs}} = \mu_1^2 \chi^i \chi_i + \mu_2^2 \rho^i \rho_i + \mu_3^2 \eta^i \eta_i + f \left( \chi_i \rho_j \eta_k \varepsilon^{ijk} + h.c. \right) + \lambda_1 (\chi^i \chi_i)^2 + \lambda_2 (\rho^i \rho_i)^2
\]
\[
+ \lambda_3 (\eta^i \eta_i)^2 + \lambda_4 \chi^i \rho^j \rho_j + \lambda_5 \chi^i \eta^j \eta_j + \lambda_6 \rho^i \rho^j \eta^k \eta_k + \lambda_7 \chi^i \eta^j \chi_j + \lambda_8 \rho^i \rho^j \eta^k \eta_k
\]
\[
+ \lambda_{10} \chi^i \rho^j \chi_j + \lambda_{11} \eta^i \rho^j \eta_j. \tag{4.5}
\]

### 4.2 Higgs sextet

In some circumstances, the choice of three triplets may be not enough to provide all lepton masses. Hence, an additional sextet is considered, which should be compatible with any of the solutions of the table. The choice of one of these solutions depend on the fermionic sector to which we want to give masses. Once again, when we evaluate all possible linear combinations among the four fields $\chi, \rho, \eta$ and the sextet $S$, additional terms are obtained which should be added to the Higgs potentials found for the case of three triplets, and according to the chosen value of $\beta$. They are shown in tables.
Table 7: Additional terms in the Higgs potential of Eq. (4.1) for $\beta = \frac{1}{\sqrt{3}}$, when a Higgs sextet is included.
\[ V(S) = \mu_S^2 S_{ij} S_{ij} + S_{ij} S_{ij} (\lambda_{15} \chi^k \chi_k + \lambda_{16} \rho^k \rho_k + \lambda_{17} \eta^k \eta_k) + \lambda_{18} \chi_i S_{ij} S^{jk} \chi_k + \lambda_{19} \rho_i S_{ij} S^{jk} \rho_k + \lambda_{20} \eta_i S_{ij} S^{jk} \eta_k + \lambda_{21} (S^{ij} S_{ij})^2 + \lambda_{22} S^{ij} S_{jk} S^{kl} S_{li} \]

Table 8: Additional terms in the Higgs potential of Eq. (4.4) for \( \beta = \sqrt{3} \), when a Higgs sextet is included.

\[ V(S) = \mu_S^2 S_{ij} S_{ij} + \frac{3}{2} (\eta^i S_{ij} \chi^j + h.c.) + \lambda_{15} (\chi^i \eta^j \chi_j + h.c.) + \lambda_{18} \chi_i S_{ij} S^{jk} \chi_k + \lambda_{19} \rho_i S_{ij} S^{jk} \rho_k + \lambda_{20} \eta_i S_{ij} S^{jk} \eta_k + \lambda_{21} (S^{ij} S_{ij})^2 + \lambda_{22} S^{ij} S_{jk} S^{kl} S_{li} \]

Table 9: Additional terms in the Higgs potential of Eq. (4.4) for \( \beta = -\sqrt{3} \), when a Higgs sextet is included.

\[ V(S) = \mu_S^2 S_{ij} S_{ij} + \frac{3}{2} (\rho^i S_{ij} \chi^j + h.c.) + \lambda_{15} (\chi^i \rho^j \chi_j + h.c.) + \lambda_{18} \chi_i S_{ij} S^{jk} \chi_k + \lambda_{19} \rho_i S_{ij} S^{jk} \rho_k + \lambda_{20} \eta_i S_{ij} S^{jk} \eta_k + \lambda_{21} (S^{ij} S_{ij})^2 + \lambda_{22} S^{ij} S_{jk} S^{kl} S_{li} \]

Table 10: Additional terms in the Higgs potential of Eq. (4.3) for \( \beta \neq \pm \frac{1}{\sqrt{3}} \) and \( \beta \neq \pm \sqrt{3} \), when a Higgs sextet is included.
5 331 model with $\beta = 1/\sqrt{3}$

5.1 Fermionic content

Before describing in detail the 331 model with $\beta = 1/\sqrt{3}$, let us check that it could provide a realistic scenario. As we saw in section 3, we found the Higgs structures and vacuum alignments that satisfy the gauge invariance, the symmetry breaking pattern and the adequate assignment of masses. The quantum numbers associated to $U(1)_X$ for three Higgs triplets in the specific case of $\beta = 1/\sqrt{3}$, can be determined from Eq. (3.7) for the triplet associated to the first transition and from table 2 for the two triplets of the second transition. On the other hand, gauge invariance in the Yukawa sector provides some relations among the $X_{\Phi}$ and $X_f$ numbers associated to $U(1)_X$, corresponding to scalars and fermions respectively. Further, cancelation of fermionic anomalies and the requirement that the first two components of the fermion triplets of $SU(3)_L$ transform as doublets of the SM, provide the complete requirements to write down the quantum number assignments for the fermionic sector of the model. From the fermionic content described in table 1 with $\beta = 1/\sqrt{3}$ we get the quantum numbers of table 11 for a one family framework. This table shows us that the standard model fermions acquire the appropriate quantum numbers and charges, and that the exotic fermions do not have exotic charges either.

5.2 Scalar mass spectrum for $\beta = 1/\sqrt{3}$

The masses of the scalar sector arise from the implementation of the SSB over the Higgs potentials. A detailed calculation of the scalar spectrum is done in Ref. [13] in models with $\beta = -\sqrt{3}$ and $\beta = -\frac{1}{\sqrt{3}}$, that coincide for these values of $\beta$, with the solutions of table 5 for the triplets, and of table 4 for the sextet. In this article, we examine in detail a model with $\beta = \frac{1}{\sqrt{3}}$, which has no exotic charges. With three Higgs triplets, the most general potential is given by Eq. (4.1), where the solution in table 5 is presented more explicitly in table 12.

The conditions for the minimum of the potential are given by

$$\frac{\partial \langle V_{higgs} \rangle}{\partial \nu^2} = 0, \quad \frac{\partial \langle V_{higgs} \rangle}{\partial \nu^3} = 0, \quad \frac{\partial \langle V_{higgs} \rangle}{\partial \nu^\eta} = 0, \quad \frac{\partial \langle V_{higgs} \rangle}{\partial \nu^\chi} = 0,$$

(5.1)

and evaluated when all fields are equal to zero, we get

$$\mu_1^2 = -2\lambda_1 \nu_\chi^2 - \lambda_4 \nu_\eta^2 - \lambda_5 (\nu_{\rho^2}^2 + \nu_{\rho^3}^2) - 2\lambda_{10} \nu_\chi \nu_{\rho^3} - 2\lambda_{10} \nu_\chi \nu_{\rho^3} - f \left( \frac{\nu_\eta \nu_{\rho^3}}{\nu^2 \nu_\chi} + \frac{\nu_\eta \nu_{\rho^3}}{\nu^2 \nu_\chi} \right),$$

$$\mu_2^2 = -2\lambda_2 (\nu_{\rho^2}^2 + \nu_{\rho^3}^2) - \lambda_5 \nu_\chi^2 - \lambda_6 \nu_\eta^2 - 2\lambda_{11} \nu_\chi \nu_{\rho^3} - f \frac{\nu_\eta \nu_{\rho^3}}{\nu_\chi},$$

$$\mu_3^2 = -2\lambda_3 \nu_\eta^2 - \lambda_4 \nu_\chi^2 - \lambda_6 (\nu_{\rho^2}^2 + \nu_{\rho^3}^2) - 2\lambda_{12} \nu_\chi \nu_{\rho^3} - f \frac{\nu_\eta \nu_{\rho^3}}{\nu_\chi},$$

$$\mu_4^2 = -\lambda_8 \nu_\chi \nu_{\rho^3} - \lambda_{10} \nu_\chi^2 - \lambda_{11} (\nu_{\rho^2}^2 + \nu_{\rho^3}^2) - 2\lambda_{12} \nu_\chi \nu_{\rho^3} - 2\lambda_{13} \nu_\chi \nu_{\rho^3} + f \frac{\nu_\eta \nu_{\rho^3}}{\nu_\rho^2}. \quad (5.2)$$
| representation | $Q_{\psi}$ | $Y_{\psi}$ | $X_{\psi}$ |
|----------------|-----------|-----------|-----------|
| $q_L = \begin{pmatrix} u \\ d \\ J \end{pmatrix}_L : 3$ | \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} | \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \end{pmatrix} | $X^L_q = 0$ |
| $q_R^{(1)} = u_R : 1$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $X^L_u = \frac{2}{3}$ |
| $q_R^{(2)} = d_R : 1$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $X^L_d = -\frac{1}{3}$ |
| $q_R^{(3)} = J_R : 1$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $X^L_J = -\frac{1}{3}$ |

| $\ell_L^{(1)} = \begin{pmatrix} e^- \\ -\nu_e \\ E^0_1 \end{pmatrix}_L : 3^*$ | \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} | \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix} | $X^L_{\ell^{(1)}} = \frac{1}{3}$ |
| $\ell_L^{(2)} = \begin{pmatrix} E^0_2 \\ -E^+_1 \\ e^+ \end{pmatrix}_L : 3^*$ | \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} | \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} | $X^L_{\ell^{(2)}} = -\frac{2}{3}$ |
| $\ell_L^{(3)} = \begin{pmatrix} E^-_1 \\ -E^0_3 \\ E^0_4 \end{pmatrix}_L : 3^*$ | \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} | \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix} | $X^L_{\ell^{(3)}} = \frac{1}{3}$ |

Table 11: Particle content for the fermionic sector of the 331 model with $\beta = 1/\sqrt{3}$ for a one family scenario.

| $\chi$ | $\xi_{\chi^0} \pm i\xi_{\chi^0}$ | $\pm 1$ | $\pm \frac{1}{2}$ | $\frac{1}{3}$ | 0 |
|--------|---------------------------------|---------|-----------------|-------------|---|
|        | $\xi_{\chi^0} \pm i\xi_{\chi^0}$ | 0       | $\pm \frac{1}{2}$ | 0 | $\nu_{\chi^0}$ |
| $\rho$ | $\xi_{\rho^2} \pm i\xi_{\rho^2}$ | $\pm 1$ | $\pm \frac{1}{2}$ | 1 | 0 |
|        | $\xi_{\rho^3} \pm i\xi_{\rho^3}$ | 0       | $\pm \frac{1}{2}$ | 0 | $\nu_{\rho^2}$ |
| $\eta$ | $\xi_{\eta^2} \pm i\xi_{\eta^2}$ | 0       | $\mp \frac{1}{2}$ | $\mp \frac{2}{3}$ | $\nu_{\eta^2}$ |
|        | $\eta^2 \mp i\eta^2$ | $\mp 1$ | $\mp \frac{1}{2}$ | 0 | 0 |

Table 12: Quantum numbers of three scalar triplets for $\beta = \frac{1}{\sqrt{3}}$.
The scalar sector of the 331 model

These parameters are again replaced in the Higgs potential. If the potential is derived twice and then evaluated for all scalar fields such that \( \hat{\Phi}_i = 0 \), we find the square mass terms. For the neutral scalar fields we have that

\[
M^2_{\hat{\Phi}_i \hat{\Phi}_j} = \frac{\partial^2 V_{\text{higgs}}}{\partial \hat{\Phi}_i \partial \hat{\Phi}_j} \bigg|_{\hat{\Phi}_i = 0}
\]

(5.3)

with \( \hat{\Phi}_i = \zeta, \zeta_x, \zeta_{\rho^2}, \zeta_\eta \) (pseudoscalars) and \( \hat{\Phi}_i = \xi\chi, \xi_x, \xi_{\rho^2}, \xi_\eta \) (real). For the charged components we have

\[
M^2_{\hat{\Phi}_i^* \hat{\Phi}_j} = \frac{\partial^2 V_{\text{higgs}}}{\partial \hat{\Phi}_i^* \partial \hat{\Phi}_j} \bigg|_{\hat{\Phi}_i = 0}
\]

(5.4)

with \( \hat{\Phi}_i = \chi^+, \rho^+, \eta^{2+}, \eta^{3+} \). From the equations above, we obtain the mass matrices \( M^2_{\zeta \zeta} \) for the imaginary sector, \( M^2_{\xi \xi} \) for the scalar real sector, and \( M^2_{\phi^\pm} \) for the charged scalar sector. They are written explicitly in appendix A Eqs. (A.1), (A.2) and (A.3), respectively.

5.2.1 Diagonalization of the charged sector

To obtain the physical spectrum of scalar particles, we should diagonalize the mass matrices. For the matrix \( M^2_{\phi^\pm} \) in (A.3) it is obtained that \( \det(M^2_{\phi^\pm}) = 0 \), which guarantees null eigenvalues associated to the would be Goldstone bosons. The eigenvalue equation is

\[
\det(M^2_{\phi^\pm} - P I) = 0.
\]

(5.5)

Resolving this equation, a degeneration with \( P_1 = P_2 = 0 \), is observed; generating two would be Goldstone bosons associated to two charged gauge fields that acquire mass \( W^+_\mu, K^+_\eta \). The eigenvectors \( V_i \) associated to each eigenvalue are obtained from

\[
(M^2_{\phi^\pm} - P I) \cdot V_i = 0,
\]

(5.6)

When solving this equation for the null eigenvalue modes (see appendix A.1), and using the basis \( \chi^\pm, \rho^\pm, \eta^{2\pm}, \eta^{3\pm} \), it leads to the following combinations, except for a normalization factor

\[
\phi_1^\pm \approx ( -\nu_{\rho^2} \rho^\pm + \nu_\eta \eta^{2\pm} ),
\]

\[
\phi_2^\pm \approx ( \nu_\chi \chi^\pm + \nu_{\rho^2} \rho^\pm - \nu_\eta \eta^{3\pm} ),
\]

(5.7)

In the \( R_\xi \) gauge, \( -\partial_\mu X_\mu - \alpha M_X G_X |^2 /\alpha \), the would be Goldstone boson \( G_X \) of the gauge field \( X_\mu \) can be defined if we demand from the bilinear term of the gauge fixing to be canceled with the bilinear term coming from the kinetic scalar Lagrangian. In this case we observe that the equation above coincides directly with the first two relations of the Eq. (B.13) in appendix B where the would be Goldstone
bosons of the theory have been obtained from the bilinear term of scalar-gauge fields that appear in the kinetic term of the scalar fields. Therefore, it is deduced that $\phi_1^\pm$ endows the electroweak charged gauge bosons $W_\mu^\pm$ with mass; while $\phi_2^\pm$ gives masses to the exotic gauge bosons $K_\mu^\pm$.

The rest of the eigenvalues are complicated expressions no easy to visualize directly. To find a solution with a clear significance, we should bear in mind that the exotic particles (non observed phenomenologically at low energies) acquire their masses in the first transition of SSB, which happens at very high energies respect to the second transition. It means that [13]:

$$\langle \chi \rangle_0 \gg \langle \rho \rangle_0, \langle \eta \rangle_0.$$  \hspace{1cm} (5.8)

Under this approximation, it is valid to keep only terms involving $\nu_\chi$ in the matrix $A.3$. If the approximation of only quadratic terms $\nu_\chi^2$ is held, it is obtained the matrix shown in appendix A.1, eq. (A.11). As it is observed, this approximation cancels all the components, except the term in the position $M_{\rho \rho}^2(4, 4)$, corresponding to the field $\eta^{3\pm}$, from which information about one of the non-zero eigenvalues is lost. To obtain a better approximation we should allow additional terms but without demanding linear orders in $\nu_\chi$. One way to have this, is by considering the coefficient $f$ of the cubic term in the scalar potential eq. (4.1) under the following approximation [13]:

$$|f| \approx \nu_\chi.$$  \hspace{1cm} (5.9)

It means that terms of the form $f \nu_\chi$ in the mass matrix eq. $A.3$ are of the order $O(\nu_\chi^2)$, and should be preserved in the approximation, from which we get the matrix of eq. $A.12$. As it was mentioned in Ref. [13], the approximation (5.9) avoids the introduction of another mass scale apart from the ones defined by the two transitions in eq. (3.11). Effectively, when we study the first transition only, we obtain Higgs square masses of the order of $\nu_\chi^2$ and $f \nu_\chi$, which can be considered of the same order of magnitude if $f \sim \nu_\chi$.

In the matrix $A.12$, it is observed that the component $\eta^{3\pm}$ decouples, and the mass matrix is reduced to $M_{\rho \rho}^2$ eq. $A.13$ of appendix A.1. Getting the eigenvalue $P_3 = \lambda_7 \nu_\chi^2 - f \nu_\chi \nu_{\rho}^2$, i.e.,

$$h_1^\pm = \eta^{3\pm}, \quad M_{h_1^\pm}^2 \approx \lambda_7 \nu_\chi^2 - f \nu_\chi \nu_{\rho}^2,$$  \hspace{1cm} (5.10)

and the last eigenvalue is

$$P_4 \approx -f \nu_\chi \left( \frac{\nu_{\eta}}{\nu_{\rho}^2} + \frac{\nu_{\rho}^2}{\nu_{\eta}} \right),$$  \hspace{1cm} (5.11)

whose eigenvector in the basis $(\rho^\pm, \eta^{2\pm})$ is:

$$h_2^\pm \approx C_\theta \rho^\pm + S_\theta \eta^{2\pm},$$  \hspace{1cm} (5.12)
The scalar sector of the 331 model

with

\[
S_\theta = \sin \theta = \frac{\nu_\rho^2}{\sqrt{\nu_\rho^2 + \nu_\eta^2}}, \quad C_\theta = \cos \theta = \frac{\nu_\eta}{\sqrt{\nu_\rho^2 + \nu_\eta^2}}.
\]

(5.13)

Using the hierarchy of the VEV’s in Eq. (5.8) we finally find the particles spectrum of the scalar charged sector and their masses which we summarize in Table 13. The fields \(\phi_1^\pm\) and \(\phi_2^\pm\) correspond to four charged and massless would-be Goldstone bosons, associated to four charged gauge bosons \(W_\mu^\pm\) and \(K_\mu^\pm\) respectively; while \(h_1^\pm\) and \(h_2^\pm\) appears in the spectrum as four charged Higgs bosons with the masses indicated in Table 13, of the order of the scale of the first SSB. The Higgs bosons \(h_2^\pm\) have real and positive masses if \(f < 0\).

| Charged scalars | Squared masses | Features |
|-----------------|----------------|----------|
| \(\phi_1^\pm\) = \(-S_\theta \rho^\pm + C_\theta \eta^\pm\) | \(M_{\phi_1^\pm}^2 = 0\) | Goldstone associated to \(W_\mu^\pm\) |
| \(\phi_2^\pm \simeq \chi^\pm\) | \(M_{\phi_2^\pm}^2 = 0\) | Goldstone associated to \(K_\mu^\pm\) |
| \(h_1^\pm \simeq \eta^{3\pm}\) | \(M_{h_1^\pm}^2 \simeq \lambda_7 \nu_\chi^2 - f \nu_\chi \nu_\eta \nu_\eta\) | Higgs |
| \(h_2^\pm \simeq S_\theta \rho^\pm + C_\theta \eta^{2\pm}\) | \(M_{h_2^\pm}^2 \simeq -f \nu_\chi \left(\frac{\nu_\eta}{\nu_\rho^2} + \frac{\nu_\rho^2}{\nu_\eta}\right)\) | Higgs |

Table 13: Physical spectrum of charged scalar particles, for three Higgs triplets and \(\beta = 1/\sqrt{3}\).

5.2.2 Diagonalization of the imaginary sector

Now we proceed to diagonalize the matrix \(M_{\zeta \zeta}^2\) of Eq. (A.12) appendix A which has null determinant associated to the would be Goldstone bosons of the neutral massive gauge fields. Solving the eigenvalues equation for \(M_{\zeta \zeta}^2\) we get the eigenvalues indicated in appendix A.2 Eq. (A.14); three of them are degenerate with eigenvalue zero.

Since the triplets of the scalar fields \(\chi\) and \(\rho\) have the same quantum numbers they can be rotated to eliminate one VEV [21]. With this basis it is simpler to find the mass eigenstates of the scalar sector

\[
\begin{pmatrix}
0 \\
\nu_\chi^2 \\
\nu_\chi^3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 \\
\nu_\chi^2 \\
\nu_\chi^3
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 \\
\nu_\rho^2 \\
\nu_\rho^3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 \\
\nu_\rho^2 \\
0
\end{pmatrix}.
\]

(5.14)
The solution of the characteristic equation generates the eigenvectors written in their complete form in Eq. (A.15). For the first three degenerate eigenvalues \( P_1 = P_2 = P_3 = 0 \), we have the corresponding eigenvectors associated to the three would be Goldstone bosons. Using the basis of the matrix \( M^2_\xi \) Eq. (A.1), we have

\[
\begin{align*}
\phi^0_1 &= N^0_1 (\nu_{\rho^2} \xi_{\rho^3} + \nu_{\chi} \xi_{\nu^0}) , \\
\phi^0_2 &= N^0_2 (\nu_{\chi} \xi_{\nu^0} - \nu_{\eta} \xi_{\eta}) , \\
\phi^0_3 &= N^0_3 (-\nu_{\rho^2} \xi_{\rho^2} + \nu_{\eta} \xi_{\eta}) \quad (5.15)
\end{align*}
\]

When we compare the first and the third of these combinations with the ones obtained in the appendix B Eq.(B.14), they could be identified with \( \phi^0_1 \) and \( \phi^0_3 \) which are associated to the longitudinal modes of the gauge fields \( K^0_\mu \) and \( Z^\prime_\mu \) respectively. As can be seen in Eq. (5.15), the vector \( \phi^0_2 \) is not orthogonal to \( \phi^0_3 \). Orthogonalizing, we get

\[
\phi^0_2 = N^0_2 \left( \nu_{\chi} \xi_{\nu^0} - \frac{\nu^2_{\rho^2}}{\nu_{\nu^2} + \nu^2_{\eta}} \xi_{\rho^2} - \frac{\nu^2_{\rho^2}}{\nu_{\rho^2} + \nu^2_{\eta}} \xi_{\eta} \right) . \quad (5.16)
\]

which correspond to the longitudinal component of \( Z^\prime_\mu \). The other two eigenvalues different from zero in Eq. (A.14), correspond to two massive neutral Higgs bosons, whose combination with respect to the basis of the matrix (A.1) are the vectors \( \phi^0_4 \) and \( \phi^0_5 \) which we rename as \( h^0_1 \) and \( h^0_2 \) respectively.

\[
\begin{align*}
h^0_1 &= N^0_4 (\nu_{\eta} \nu_{\rho^2} \xi_{\rho^3} + \nu_{\chi} \nu_{\eta} \xi_{\rho^2} + \nu_{\chi} \nu_{\rho^2} \xi_{\eta}) , \\
h^0_2 &= N^0_5 (\nu_{\chi} \xi_{\rho^3} - \nu_{\rho^2} \xi_{\nu^0}) . \quad (5.17)
\end{align*}
\]

If additionally, we implement the approximations of Eqs. (5.8) and (5.9) in the results obtained, the particle spectrum can be written as indicated in table 14. The fields \( \phi^0_1, \phi^0_2 \) and \( \phi^0_3 \) are the three neutral and massless would be Goldstone bosons, that dissapear from the particle spectrum to give masses to the three neutral gauge bosons \( K^0_\mu, Z^\prime_\mu \) and \( Z_\mu \), respectively; while \( h^0_1 \) and \( h^0_2 \) appear in the spectrum as two neutral Higgs bosons with the masses indicated in table 14. It is taken \( f < 0 \) to define the mass of \( h^0_1 \) real and positive.

### 5.2.3 Diagonalization of the real sector

Finally, we take the matrix \( M^2_\xi \) in (A.2) with null determinant too. The eigenvalues obtained are not simple. So we use the approximations of the Eqs. (5.8) and (5.9), obtaining the matrix given in appendix A.3 Eq. (A.16), which decouples in a null eigenvalue (would-be Goldstone boson) and the two submatrices \( M^2_{\xi_{\rho^2} \xi_{\eta}} \) and \( M^2_{\xi_{\chi} \xi_{\rho^3}} \) indicated in (A.17). In this manner, it is defined

\[
\phi^0_4 \approx \xi_{\nu^0} , \quad (5.18)
\]
The scalar sector of the 331 model

| Imaginary scalars          | Square masses  | Features                                |
|----------------------------|----------------|-----------------------------------------|
| \(\phi_1^0 \simeq \zeta_\alpha\) | \(M_{\phi_1^0}^2 = 0\) | Goldstone asociado a \(\overline{K}_\mu^0\) |
| \(\phi_2^0 \simeq \zeta_\chi\) | \(M_{\phi_2^0}^2 = 0\) | Goldstone asociado a \(Z'_\mu\)         |
| \(\phi_3^0 = -S_\theta \zeta_\rho^2 + C_\theta \zeta_\eta\) | \(M_{\phi_3^0}^2 = 0\) | Goldstone asociado a \(Z_\mu\)         |
| \(h_1^0 \simeq C_\theta \zeta_\rho^2 + S_\theta \zeta_\eta\) | \(M_{h_1^0}^2 \simeq -\frac{2f_\eta}{\nu_\eta^2 + \nu_\rho^2} (\nu_\eta^2 + \nu_\rho^2)\) | Higgs                                    |
| \(h_2^0 \simeq \zeta_\rho^3\) | \(M_{h_2^0}^2 \simeq -\frac{2f_\eta \nu_\rho}{\nu_\rho^2} + 2(\lambda_8 - 2\lambda_{13}) \nu_\chi^2\) | Higgs                                    |

Table 14: Physical spectrum of the pseudoscalar fields, for three Higgs triplets and \(\beta = 1/\sqrt{3}\).

when comparing with \(\phi_1^0\) of Appendix B Eq. (B.14) it is deduced a coupling to the gauge boson \(\overline{K}_\mu^0\). The submatrix \(M_{\xi_\rho^2 \xi_\eta}^2\) contains an eigenvalue zero. However, it does not correspond to a Goldstone boson. Such eigenvalue arose from the effect of the quadratic approximation in (A.16). To extract information for this eigenvalue, we take all the terms of the sector \(M_{\xi_\rho^2 \xi_\eta}^2\) from \(M_{\xi_\xi}^2\), obtaining an exact submatrix written in (A.18). Based on this we get the eigenvalues \((P_2, P_3)\) associated to the matrix \(M_{\xi_\rho^2 \xi_\eta}^2\) of (A.18) and \((P_4, P_5)\) for the matrix \(M_{\xi_\chi \xi_\rho}^2\) of (A.17):

\[
\begin{align*}
P_2 &= \frac{1}{2} \left[ M_{22} + M_{11} + \sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2} \right] \equiv \frac{1}{2} \left[ M_D + \sqrt{M_C} \right], \\
P_3 &= \frac{1}{2} \left[ M_{22} + M_{11} - \sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2} \right] \equiv \frac{1}{2} \left[ M_D - \sqrt{M_C} \right], \\
P_4 &= \Lambda_+ + \sqrt{\Lambda_-^2 + 16\lambda_{10}^2} \nu_\chi^2, \\
P_5 &= \Lambda_+ - \sqrt{\Lambda_-^2 + 16\lambda_{10}^2} \nu_\chi^2, \\
\end{align*}
\]

(5.19)

with

\[
\Lambda_\pm \equiv 4\lambda_1 \pm \left( \lambda_8 + 2\lambda_{13} - \frac{f_\eta}{\nu_\chi^2} \right).
\]

(5.20)

and \(M_{ij}\) denote the elements of the matrix \(M_{\xi_\rho^2 \xi_\eta}^2\). The non-normalized eigenvectors are written respectively as
\[ h_3^0 = S_\alpha \xi_\rho^2 + C_\alpha \xi_\eta, \]
\[ h_4^0 = C_\alpha \xi_\rho^2 - S_\alpha \xi_\eta, \]
\[ h_5^0 \approx (\Lambda - \sqrt{\Lambda^2 + 16\lambda_1^2}) \xi_\chi + 4\lambda_1 \xi_\rho^3, \]
\[ h_6^0 \approx (\Lambda - \sqrt{\Lambda^2 + 16\lambda_1^2}) \xi_\chi + 4\lambda_1 \xi_\rho^3, \quad (5.21) \]

where \( S_\alpha \equiv \sin \alpha, \) \( C_\alpha \equiv \cos \alpha. \) Being \( \alpha \) a new mixing angle defined as:

\[
\sin 2\alpha = \frac{2M_{12}}{\sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2}} \\
\cos 2\alpha = \frac{M_{22} - M_{11}}{\sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2}}. \quad (5.22)
\]

the equations above define finally the physical spectrum of the scalar real fields, as indicated in table 15.

| Scalars | Square masses | Features |
|---------|--------------|----------|
| \( \phi_4^0 \approx \xi_\chi^0 \) | \( M_{\phi_4^0}^2 = 0 \) | Goldstone of \( \widetilde{K}_0^\mu \) |
| \( h_3^0 \approx S_\alpha \xi_\rho^2 + C_\alpha \xi_\eta \) | \( M_{h_3^0}^2 = \frac{1}{2} [M_D + \sqrt{M_C}] \) | Higgs |
| \( h_4^0 \approx C_\alpha \xi_\rho^2 - S_\alpha \xi_\eta \) | \( M_{h_4^0}^2 = \frac{1}{2} [M_D - \sqrt{M_C}] \) | Higgs |
| \( h_5^0 \approx \Lambda_1 \xi_\chi + 4\lambda_1 \xi_\rho^3 \) | \( M_{h_5^0}^2 \approx \Lambda_1 \nu_\chi^2 \) | Higgs |
| \( h_6^0 \approx \Lambda_2 \xi_\chi + 4\lambda_1 \xi_\rho^3 \) | \( M_{h_6^0}^2 \approx \Lambda_2 \nu_\chi^2 \) | Higgs |

Table 15: Physical spectrum of masses for the real scalar fields, with three Higgs triplets and with \( \beta = 1/\sqrt{3}. \) We use the definitions of Eqs. (5.23).

\( \phi_4^0 \) is a neutral would-be Goldstone boson associated to the gauge field \( \widetilde{K}_0^\mu \) and \( h_3^0, h_4^0, h_5^0, h_6^0 \) are four neutral Higgs bosons with masses given in table 15 with the condition \( f < 0 \) in order for \( h_4^0 \) to acquire a positive defined mass.

Finally, we are able to summarize the particle spectrum of the scalar sector and the would be Goldstone bosons in table 16. Taking into account the following definitions.
The scalar sector of the 331 model

\[ \Lambda_{1,2} \equiv \left( \Lambda_\pm \pm \sqrt{\Lambda^2 + 16\lambda_1^2} \right) \]

\[ \Lambda_\pm = 4\lambda_1 \pm \left( \lambda_8 + 2\lambda_{13} - \frac{f\nu_\eta}{\nu_{\chi}^2 + \nu_\eta^2} \right) \]

\[ \nu_\chi \gg \nu_{\rho^2}, \nu_{\rho^3}, \nu_{\eta}; \quad |f| \approx \nu_\chi; \quad f < 0, \]

\[ S_\theta = \frac{\nu_{\rho^2}}{\sqrt{\nu_{\rho^2}^2 + \nu_\eta^2}}, \quad C_\theta = \frac{\nu_\eta}{\sqrt{\nu_{\rho^2}^2 + \nu_\eta^2}}, \]

\[ \sin 2\alpha = \frac{2M_{12}}{\sqrt{M_C}}, \quad \cos 2\alpha = \frac{M_{22} - M_{11}}{\sqrt{M_C}}, \]

\[ M_D \equiv M_{11} + M_{22}; \quad M_C = (M_{11} - M_{22})^2 + 4M_{12} \]

\[ M_{ij} = \left( M_{\xi_{\rho^2}} \right)_{ij} \quad (5.23) \]

We see then that the scalar spectrum consists of eight would be Goldstone bosons (four charged and four neutral ones) plus ten physical Higgs bosons (four charged and six neutral ones).

6 Trilinear Higgs-Gauge Bosons terms for \( \beta = 1/\sqrt{3} \)

The term \( 2 \) in Eq. (B.1) contains cubic couplings between scalar and gauge bosons. We consider these couplings for the SM gauge bosons: \( W_\mu^\pm, Z_\mu \), and \( A_\mu \), with masses given by [18]:

\[ M_{W^\pm} = g \sqrt{\frac{\nu_{\rho^2}^2 + \nu_\eta^2}{2}}, \quad M_Z = g \sqrt{\frac{\nu_{\rho^2}^2 + \nu_\eta^2}{2C_W}}. \quad (6.1) \]

In Eq. (B.1) there are not cubic couplings of electroweak bosons with the scalar boson \( \chi \) (that acquire VEV in the 3\textsuperscript{rd} component), as can be seen in Eq. (B.5), where the SM gauge bosons are in the 2\times2 superior components, while in the third component (3\textsuperscript{rd} row and column) are the exotic gauge bosons which get their masses
through $\nu_\chi$. Using Eqs. (B.5) and (B.7) for $\Phi = \rho$ and $\eta$ in Eq. (B.1), it is found

$$L_{\text{trilinear}}^{SM} = g^2 (\nu_{\rho^2} \xi_{\rho^2} + \nu_{\eta} \xi_{\eta}) W^\mu W^\nu + 2a_4^2 (\nu_{\rho^2} \xi_{\rho^2} + \nu_{\eta} \xi_{\eta}) Z^\mu Z_\mu$$

$$+ \frac{g}{\sqrt{2}} a_1 \left(-\nu_{\rho^2} \rho^- + \nu_{\eta} \eta_2\right) A^\mu W^\nu + h.c$$

$$+ \frac{g}{\sqrt{2}} (a_2 + a_4) \left(-\nu_{\rho^2} \rho^- + \nu_{\eta} \eta_2\right) Z^\mu W^\nu$$

$$- \frac{ig}{\sqrt{2}} \partial^\mu \rho^- W^\nu \xi_{\rho^2} - \frac{ig}{\sqrt{2}} \partial^\mu \xi_{\rho^2} W^\nu \rho^-$$

$$- \frac{ig}{\sqrt{2}} \partial^\mu \eta_2^+ W^\nu \xi_{\eta} - \frac{ig}{\sqrt{2}} \partial^\mu \xi_{\eta} W^\nu \eta_2 + h.c. \quad (6.2)$$

where $a_1, a_2$ and $a_4$ are defined in Eq. (B.7). The terms in Eq. (6.2) are in the weak basis, which are related to the physical basis through Table 16. In the physical basis, and after some algebraic manipulation using Eq. (B.7) in Eq. (6.2), it is finally

| Scalars | Square masses | Feature |
|---------|---------------|---------|
| $\phi_1^\pm = -S_\theta \rho^\pm + C_\theta \eta^2 \pm$ | $M_{\phi_1^\pm}^2 = 0$ | Goldstone |
| $\phi_2^\pm = \chi^\pm$ | $M_{\phi_2^\pm}^2 = 0$ | Goldstone |
| $\phi_1^0 \sim \zeta_{\chi}$ | $M_{\phi_1^0}^2 = 0$ | Goldstone |
| $\phi_2^0= -S_\theta \zeta_{\phi} + C_\theta \zeta_{\eta}$ | $M_{\phi_2^0}^2 = 0$ | Goldstone |
| $\phi_3^0 = -S_\theta \zeta_{\phi^2} + C_\theta \zeta_{\eta^2}$ | $M_{\phi_3^0}^2 = 0$ | Goldstone |
| $h_1^\pm \sim \eta^3 \pm$ | $M_{h_1^\pm}^2 \sim \lambda_7 \nu_\chi^2 - f \nu_\chi \nu_\eta^2$ | Higgs |
| $h_2^\pm \sim C_\theta \rho^\pm + S_\theta \eta^2 \pm$ | $M_{h_2^\pm}^2 \sim -f \nu_\chi (\nu_\rho^2 + \nu_\eta^2)$ | Higgs |
| $h_1^0 \sim C_\theta \zeta_{\phi^2} + S_\theta \zeta_{\eta^2}$ | $M_{h_1^0}^2 \sim -2f \nu_\chi \nu_{\rho^2}^2 (\nu_\eta^2 + \nu_{\rho^2}^2)$ | Higgs |
| $h_2^0 \sim \zeta_{\rho^3}$ | $M_{h_2^0}^2 \sim -2f \nu_{\rho^2} \nu_{\eta^2}^2 + 2(\lambda_7 - 2\lambda_1) \nu_\chi^2$ | Higgs |
| $h_3^0 \sim S_\alpha \zeta_{\phi^2} + C_\alpha \zeta_{\eta^2}$ | $M_{h_3^0}^2 \sim \nu_\eta \nu_{\rho^2}$ | Higgs |
| $h_4^0 \sim C_\alpha \zeta_{\phi^2} - S_\alpha \zeta_{\eta^2}$ | $M_{h_4^0}^2 \sim f \nu_\chi$ | Higgs |
| $h_5^0 \sim \Lambda_1 \chi_4 + 4\lambda_{10} \zeta_{\phi^3}$ | $M_{h_5^0}^2 \sim \Lambda_1 \nu_\chi^2$ | Higgs |
| $h_6^0 \sim \Lambda_2 \chi_4 + 4\lambda_{10} \zeta_{\phi^3}$ | $M_{h_6^0}^2 \sim \Lambda_2 \nu_\chi^2$ | Higgs |

Table 16: Physical scalar spectrum after both transitions, for three Higgs triplets with $\beta = 1/\sqrt{3}$. Taking into account the definitions in Eqs. (B.25a).
found

\[ L_{\text{trilinear}}^{SM} = g M_{W^\pm} \cos (\theta - \alpha) W^{\mu} W^{\mu+} h_3^0 + g M_{W^\pm} \sin (\theta - \alpha) W^{\mu} W^{\mu+} h_4^0 \]

\[ + \frac{g}{2 C_W} M_Z \cos (\theta - \alpha) Z^{\mu} Z^{\mu} h_3^0 \]

\[ - (e M_{W^\pm} A^{\mu} W^{\mu+} \phi_1^- + g M_Z S^2 W^{\mu+} Z^{\mu} \phi_1^- + h.c) \]

\[ - \frac{g}{2} (p - k)^{\mu} \cos (\theta - \alpha) W^{\mu} h_2^0 + \frac{g}{2} (p - q)^{\mu} \sin (\theta - \alpha) W^{\mu} h_2^0 h_3^0 \]

\[ - \frac{i g}{2} (p - r)^{\mu} W^{\mu} h_2^0 h_4^0 + h.c, \]

where the electric charge has been defined as \( e = g S_W \), and \( \theta \) and \( \alpha \) are given by Eqs. (5.13) and (5.22). It worths to note that these vertices are the same as the ones obtained in the Standard Model with two Higgs doublets (2HDM). In the notation of Ref. [21], the Higgs bosons of the 2HDM have the following correspondence with the Higgs notation shown here

\[ h_3^0 \rightarrow h^0, \quad h_4^0 \rightarrow H^0, \quad h_1^0 \rightarrow A^0, \quad h_2^0 \rightarrow H^\pm, \quad \phi_1^- \rightarrow G^\pm_W, \quad \phi_3^0 \rightarrow G^0_Z \]

The couplings of the charged Higgs bosons to the vertex \( A^{\mu} W^{\mu} \) vanish because of the conservation of electromagnetic charge and only the couplings to the would-be Goldstone bosons can exist since they are not physical fields.

In fact, we can realize that the approximation \( f \nu_x > v^2_{\rho}, \nu^2_{\eta}; \) implies that the boson \( h_3^0 \) (very massive Higgs) decouple from the vertices, and that \( h_3^0 \) couple in the same way as the SM Higgs boson, without any other constraint. In this limit, it is obtained that

\[ \tan 2\alpha \approx \frac{2 v^2_{\rho} \nu_\eta}{v^2_\eta - v^4_{\rho}} \]

i.e. \( \theta \approx \alpha \), and the \( L_{\text{trilinear}}^{SM} \) is given by

\[ L_{\text{trilinear}}^{SM} = g M_{W^\pm} W^{\mu} W^{\mu+} h_3^0 + \frac{g}{2 C_W} M_Z Z^{\mu} Z^{\mu} h_3^0 \]

\[ - (e M_{W^\pm} A^{\mu} W^{\mu+} \phi_1^- - g M_Z S^2 W^{\mu+} Z^{\mu} \phi_1^- + h.c) \]

\[ - \left( \frac{g}{2} (p - k)^{\mu} W^{\mu} h_2^0 h_3^0 + \frac{i g}{2} (p - r)^{\mu} W^{\mu} h_2^0 h_4^0 + h.c. \right). \] (6.5)

Where Eq. (6.5) shows explicitly the fact that the couplings of \( h_3^0 \) are SM-like when the limit \( f \nu_x > v^2_{\rho}, \nu^2_{\eta}; \) is taken.

### 7 Low energy limit for \( \beta = 1/\sqrt{3} \)

We shall discuss briefly some important properties of the low energy limit of this model. As we see in table 16, the Higgs scalar masses depend on some parameters of the Higgs potential, the vacuum expectation value \( \nu_\chi \) at the higher scale, and
the vacuum expectation values at the electroweak scale ($\nu_1$, $\nu_2$) as it must be. Notwithstanding, there is a crucial parameter that determines the scale in which the Higgs bosons lie, i.e. the trilinear coupling constant $f$ defined in Eq. (4.1). For arbitrary values of it (or more precisely for arbitrary values of $f\nu_\chi$) some of the Higgs bosons would lie on a new scale different from the breakdown ones. If we argue a naturalness criterium, we could assume that $f\nu_\chi$ should lie either in the electroweak or $SU(3)_L \otimes U(1)_X$ breaking scale. Since these 331 models break to a $SU(2)_L \otimes U(1)_Y$ two Higgs doublet model; the value assumed for $f\nu_\chi$ determines the scales for the $SU(2)_L \otimes U(1)_Y$. Higgs bosons which can be identified as the $h_{3/2}^\pm$, $h_1^0$, $h_3^0$ and $h_4^0$ fields of table 16, where $\phi_1^\pm$ and $\phi_3^0$ are the corresponding would-be Goldstone bosons. We also see in table 16 that $h_3^0$ is the only Higgs field that depends exclusively on the vacuum expectation values at the electroweak scale, so it corresponds to the SM Higgs. The scale of the other $SU(2)_L \otimes U(1)_Y$ Higgs bosons depend on the value assumed by $f\nu_\chi$. If $f\nu_\chi \sim O(\nu_2^2)$ the remaining $SU(2)_L \otimes U(1)_Y$ Higgs bosons belong to the higher breaking scale and for sufficiently large values of $\nu_2^2$, the $h_{3/2}^\pm$, $h_1^0$ and $h_4^0$ fields become heavy modes, in addition by taking $f\nu_\chi$ large enough we obtain automatically that the two mixing angles of the Higgs sector in the remnant two Higgs doublet model become equal, from which the decoupling limit at the electroweak scale to the minimal SM follows naturally [25]. Otherwise, if we assume $f\nu_\chi \sim O(\nu_{EW}^2)$ the diagonalization process for the scalar sector would be much more complicated, but a naive analysis shows that we would get a spectrum of $SU(2)_L \otimes U(1)_Y$ Higgs bosons lying roughly on the EW scale, obtaining a non-decoupling two Higgs doublet model.

8 Conclusions

For the 331 models, the set of equations arising from anomalies predicts that the number of triplets and antitriples must be equal in order to cancel anomalies. These equations lead to an infinite set of possible solutions. If we impose for the model not to have exotic electromagnetic charges, the number of solutions is reduced to only two. On the other hand, the quantum numbers may be restricted by demanding gauge invariance for the Yukawa couplings of the scalar fields to fermions (classical constraints) as well as from the cancelation of fermionic anomalies (quantum constraints), obtaining models with ordinary and exotic charges. Letting free the parameters $X$ and $\beta$ that define the electromagnetic charge, we get all possible vacuum alignments in both symmetry breakings. When we choice the VEV’s according to the breakdown scheme $331 \rightarrow 321 \rightarrow 1$ the values for $X$ and $\beta$ are found, obtaining the models already studied in the literature plus other ones with exotic charges. Moreover, it is found that one scalar triplet is necessary for the first breaking and two scalar triplets for the second one, the two triplets for the second breaking contain two doublets of $SU(2)_L$ for the second breaking that give masses to the up and down quarks respectively. In some cases it is necessary to introduce an additional
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scalar sextet in order to endow the neutrinos with masses.

Furthermore, we present the most general Higgs potentials for the models with \( \beta = \pm \sqrt{3} \), \( \beta = \pm 1/\sqrt{3} \) and \( \beta \) arbitrary with three scalar triplets and one scalar sextet. In particular, the case of \( \beta = 1/\sqrt{3} \) is analyzed, getting the scalar spectrum in the \( R_\zeta \)-gauge; such spectrum consists of ten massive Higgs bosons as well as eight would be Goldstone bosons necessary to break the eight generators of the SSB scheme demanded.

Besides, in the framework of the \( \beta = 1/\sqrt{3} \) model, we calculate the trilinear couplings of the Higgs bosons to the SM gauge bosons, finding that such couplings are identical to the ones of the standard two Higgs doublet model (2HDM). It worths to say that \( A_\mu W^\pm_\mu \) are not coupled to the charged Higgs bosons in consistence with electromagnetic charge conservation.

As for the low energy limit \( (\nu^2_x >> \nu^2_\rho, \nu^2_\eta) \), the Higgs potential of the 331 model with \( \beta = 1/\sqrt{3} \) is also reduced to the Higgs potential of the 2HDM in this limit. Thus, this model ends up in a \( SU(2)_L \otimes U(1)_Y \) two Higgs doublet model after the first symmetry breaking. We see that the behavior of the scalar sector of these models at low energies is strongly determined by the trilinear coupling \( f \) of the Higgs potential. For instance, by setting \( f \nu_\chi \sim O(\nu^2_{EW}) \) all the \( SU(2)_L \otimes U(1)_Y \) Higgs bosons belong to the electroweak scale, obtaining a non-decoupling two Higgs doublet model. By contrast, the assumption \( f \nu_\chi \sim O(\nu^2_\chi) \) leaves one of these Higgs bosons at the electroweak scale (the SM Higgs), and the other four ones would lie on the scale \( O(\nu_\chi) \) of the first transition; additionally, if the scale \( O(\nu_\chi) \) is sufficiently large, we obtain that the mixing angles of the Higgs potential are equal, from which the couplings of the light Higgs boson to fermions and gauge fields become identical to the SM couplings. Therefore, the natural assumption \( f \nu_\chi \sim O(\nu^2_\chi) \gg \nu^2_\rho, \nu^2_\eta \) leads automatically to the minimal SM at low energies.

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A Scalar masses

i) Imaginary sector

The mass matrix $M^2_{\zeta\zeta}$ in the basis $\zeta_\chi, \zeta_\rho^2, \zeta_\rho^3, \zeta_\eta, \zeta_\chi^0$ is given by

\[
M^2_{\zeta\zeta}(1, 1) = (2\lambda_8 - 4\lambda_{13})\nu_{\rho^3}^2 - f\left(\frac{2\nu_\eta\nu^2_{\rho^2}}{\nu_\chi} + \frac{2\nu_\eta\nu^2_{\rho^3}}{\nu_\rho^3}\right)
\]

\[
M^2_{\zeta\zeta}(1, 2) = M^2_{\zeta\zeta}(2, 1) = -2f\nu_\eta
\]

\[
M^2_{\zeta\zeta}(1, 3) = M^2_{\zeta\zeta}(3, 1) = (2\lambda_8 + 4\lambda_{13})\nu_{\chi_3}\nu_{\rho^3} + 2f\nu_{\rho^3}\nu_{\eta}\nu_{\rho^2}/\nu_{\rho^3}
\]

\[
M^2_{\zeta\zeta}(1, 4) = M^2_{\zeta\zeta}(4, 1) = -2f\nu_{\rho^2}
\]

\[
M^2_{\zeta\zeta}(1, 5) = M^2_{\zeta\zeta}(5, 1) = (2\lambda_8 - 4\lambda_{13})\nu_{\rho^3}\nu_{\rho^3}
\]

\[
M^2_{\zeta\zeta}(2, 2) = -2f\frac{\nu_{\eta}\nu_\chi}{\nu_{\rho^2}}
\]

\[
M^2_{\zeta\zeta}(2, 3) = M^2_{\zeta\zeta}(3, 2) = 0
\]

\[
M^2_{\zeta\zeta}(2, 4) = M^2_{\zeta\zeta}(4, 2) = -2f\nu_{\chi}
\]

\[
M^2_{\zeta\zeta}(2, 5) = M^2_{\zeta\zeta}(5, 2) = 2f\frac{\nu_{\eta}\nu_{\rho^3}}{\nu_{\rho^2}}
\]

\[
M^2_{\zeta\zeta}(3, 3) = (2\lambda_8 - 4\lambda_{13})\nu_\chi^2 - 2f\frac{\nu_{\chi}\nu_\eta}{\nu_{\rho^2}}
\]

\[
M^2_{\zeta\zeta}(3, 4) = M^2_{\zeta\zeta}(4, 3) = 0
\]

\[
M^2_{\zeta\zeta}(3, 5) = M^2_{\zeta\zeta}(5, 3) = (2\lambda_8 - 4\lambda_{13})\nu_\chi^2 - 2f\frac{\nu_{\chi}\nu_\eta}{\nu_{\rho^2}}
\]

\[
M^2_{\zeta\zeta}(4, 4) = -2f\frac{\nu_{\chi}\nu_{\rho^2}}{\nu_\eta}
\]

\[
M^2_{\zeta\zeta}(4, 5) = M^2_{\zeta\zeta}(5, 4) = 2f\nu_{\rho^3}
\]

\[
M^2_{\zeta\zeta}(5, 5) = (2\lambda_8 - 4\lambda_{13})\nu_{\rho^3}^2 - f\left(\frac{2\nu_\eta\nu_{\rho^2}}{\nu_\chi} + \frac{2\nu_\eta\nu_{\rho^3}^2}{\nu_\chi\nu_{\rho^2}}\right)
\]  
(A.1)
ii.) Real sector

The mass matrix $M_{ξξ}^2$ in the basis $ξ_χ, ξ_ρ^2, ξ_η, ξ_ν$ is given by

$$M_{ξξ}^2(1, 1) = 8λ_1ν_χ^2 + 8λ_{10}ν_χν_ρ^2 + (2λ_8 + 4λ_{13})ν_ρ^4$$

$$- f \left( \frac{2ν_ην_ρ^2}{ν_χ} + \frac{2ν_ην_ρ^3}{ν_χν_ρ^2} \right)$$

$$M_{ξξ}^2(1, 2) = M_{ξξ}^2(2, 1) = 4λ_5ν_χν_ρ^2 + 4λ_{11}ν_ρ^3ν_ρ^4 + 2fν_η$$

$$M_{ξξ}^2(1, 3) = M_{ξξ}^2(3, 1) = (4λ_5 + 2λ_8 + 4λ_{13})ν_χν_ρ^4 + 4λ_{10}ν_ρ^2 + 4λ_{11}ν_ρ^3 + 2f \frac{ν_ην_ρ^3}{ν_ρ^2}$$

$$M_{ξξ}^2(1, 4) = M_{ξξ}^2(4, 1) = 4λ_4ν_χν_η + 4λ_{12}ν_η$$

$$M_{ξξ}^2(1, 5) = M_{ξξ}^2(5, 1) = (2λ_8 + 4λ_{13})ν_ρ^5ν_ρ^3 + 4λ_{10}ν_χν_ρ^2$$

$$M_{ξξ}^2(2, 2) = 8λ_2ν_ρ^2 - 2f \frac{ν_ην_χ}{ν_ρ^2}$$

$$M_{ξξ}^2(2, 3) = M_{ξξ}^2(3, 2) = 8λ_2ν_ρ^2ν_ρ^3 + 4λ_{11}ν_χν_ρ^2$$

$$M_{ξξ}^2(2, 4) = M_{ξξ}^2(4, 2) = 4λ_6ν_ην_ρ^2 + 2fν_χ$$

$$M_{ξξ}^2(2, 5) = M_{ξξ}^2(5, 2) = 4λ_{11}ν_ρ^2 + 2f \frac{ν_ην_ρ^3}{ν_ρ^2}$$

$$M_{ξξ}^2(3, 3) = (2λ_8 + 4λ_{13})ν_χ^2 + 8λ_{11}ν_χν_ρ^3 + 8λ_2ν_ρ^2 - 2f \frac{ν_χν_η}{ν_ρ^2}$$

$$M_{ξξ}^2(3, 4) = M_{ξξ}^2(4, 3) = 4λ_6ν_ην_ρ^3 + 4λ_{12}ν_ην_χ$$

$$M_{ξξ}^2(3, 5) = M_{ξξ}^2(5, 3) = (2λ_8 + 4λ_{13})ν_χν_ρ^2 + 4λ_{11}ν_ρ^2ν_ρ^3 - 2fν_η$$

$$M_{ξξ}^2(4, 4) = 8λ_3ν_η^2 - 2f \frac{ν_χν_ρ^2}{ν_η}$$

$$M_{ξξ}^2(4, 5) = M_{ξξ}^2(5, 4) = 4λ_{12}ν_ην_ρ^2 - 2fν_ρ^3$$

$$M_{ξξ}^2(5, 5) = (2λ_8 + 4λ_{13})ν_ρ^4 - f\left( \frac{2ν_ην_ρ^4}{ν_χ} + \frac{2ν_ην_ρ^2}{ν_χν_ρ^2} \right)$$

(A.2)

iii.) Charged sector

The mass matrix $M_{φ±}^2$ in the basis $χ^±, ρ^±, η^{±2}, η^{±3}$ reads
\[ M_{\phi \pm}^2 (1, 1) = \lambda_7 \nu_\eta^2 - f \left( \frac{\nu_\eta \nu_\rho^2}{\nu_\chi} + \frac{\nu_\eta \nu_\rho^3}{\nu_\chi} \right) \]

\[ M_{\phi \pm}^2 (1, 2) = M_{\phi \pm}^2 (2, 1) = \lambda_{14} \nu_\eta^2 + f \frac{\nu_\eta \nu_\rho^3}{\nu_\rho^2} \]

\[ M_{\phi \pm}^2 (1, 3) = M_{\phi \pm}^2 (3, 1) = \lambda_{14} \nu_\eta \nu_\rho^2 + f \nu_\rho^3 \]

\[ M_{\phi \pm}^2 (1, 4) = M_{\phi \pm}^2 (4, 1) = \lambda_7 \nu_\chi \nu_\eta + \lambda_{14} \nu_\eta \nu_\rho^3 - f \nu_\rho^2 \]

\[ M_{\phi \pm}^2 (2, 2) = \lambda_9 \nu_\eta^2 - f \frac{\nu_\eta \nu_\chi}{\nu_\rho^2} \]

\[ M_{\phi \pm}^2 (2, 3) = M_{\phi \pm}^2 (3, 2) = \lambda_9 \nu_\eta \nu_\rho^2 - f \nu_\chi \]

\[ M_{\phi \pm}^2 (2, 4) = M_{\phi \pm}^2 (4, 2) = \lambda_9 \nu_\eta \nu_\rho^3 + \lambda_{14} \nu_\chi \nu_\eta \]

\[ M_{\phi \pm}^2 (3, 3) = \lambda_9 \nu_\rho^2 - f \frac{\nu_\rho \nu_\chi}{\nu_\eta} \]

\[ M_{\phi \pm}^2 (3, 4) = M_{\phi \pm}^2 (4, 3) = \lambda_9 \nu_\rho \nu_\rho^3 + \lambda_{14} \nu_\chi \nu_\rho^2 \]

\[ M_{\phi \pm}^2 (4, 4) = \lambda_7 \nu_\chi^2 + 2 \lambda_{14} \nu_\chi \nu_\rho^3 + \lambda_9 \nu_\rho^2 - f \frac{\nu_\chi \nu_\rho^2}{\nu_\eta} \] \hspace{1cm} (A.3)

The previous matrices are singular, it is associated to the would be Goldstone bosons of the theory (at least one null eigenvalue).

\[ \det(M_{\xi \xi}^2) = \det(M_{\zeta \zeta}^2) = \det(M_{\phi \pm}^2) = 0. \] \hspace{1cm} (A.4)

**A.1 Diagonalization of the charged sector**

The matrix \( M_{\phi \pm}^2 \) in Eq. \( \text{(A.3)} \) presents null eigenvalues. For the eigenvalue \( P_1 = 0 \), the equation \( \text{(5.6)} \) is block reduced to:

\[
\begin{pmatrix}
1 & 0 & 0 & \frac{\nu_\chi}{\nu_\eta} \\
0 & 1 & \frac{\nu_\rho^2}{\nu_\eta} & \frac{\nu_\rho^3}{\nu_\eta} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = 0,
\] \hspace{1cm} (A.5)

it produces two rows of zeros because of the fact that the zero eigenvalue is two-fold degenerate, such that only two of the four equations defined by \( \text{(5.6)} \) are linearly independent, as can be seen in Eq. \( \text{(A.3)} \). Since there are four variables, we take a particular solution among the infinite ones, for instance

\[ x_3 = 1; \quad x_4 = 0, \] \hspace{1cm} (A.6)

and when we replace in \( \text{(A.5)} \), it is found

\[ x_1 = 0; \quad x_2 = -\frac{\nu_\rho^2}{\nu_\eta}, \] \hspace{1cm} (A.7)
obtaining the solution $\phi^\pm_1$ written in Eq. (5.7).

To find out the solution of the other degenerate eigenvalue $P_2 = 0$, we choose

$$x_3 = 0; \quad x_4 = -1,$$

and from Eq. (A.8), we find

$$x_1 = \frac{\nu_\chi}{\nu_\eta}; \quad x_2 = \frac{\nu_\omega}{\nu_\eta},$$

obtaining $\phi^\pm_2$ in Eq. (5.7).

For the other two eigenvalues we utilize the approximation given by Eqs. (5.8). Keeping only quadratic order in $\nu_\chi$, it is obtained from the matrix in Eq. (A.3):

$$M^2_{\phi^\pm} \approx \begin{bmatrix}
\chi^\pm & \rho^\pm & \eta^{2\pm} & \eta^{3\pm} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_7 \nu_\chi^2
\end{bmatrix},$$

which is block reduced in the following way

$$M^2_{\chi^\pm} = 0$$

$$M^2_{\rho^\pm\eta^{2\pm}} \approx \begin{bmatrix}
-f \frac{\nu_\omega \nu_\chi}{\nu_\rho^2} & -f \nu_\chi \\
-f \nu_\chi & -f \frac{\nu_\rho^2 \nu_\chi}{\nu_\eta}
\end{bmatrix},$$

$$M^2_{\eta^{3\pm}} \approx \lambda_7 \nu_\chi^2 - f \frac{\nu_\chi \nu_\rho^2}{\nu_\eta},$$

The eigenvalue $P_3 = \lambda_7 \nu_\chi^2 - f \nu_\chi \frac{\nu_\rho^2}{\nu_\eta}$, is associated to the decoupled component $\eta^{3\pm}$ as it is written in Eq. (5.10). The square matrix in Eq. (A.13) contains one of the null degenerate eigenvalues. The other one is the $P_4$ eigenvalue written in Eq. (5.11), whose eigenvector is given by Eq. (5.12).
A.2 Diagonalization of the imaginary sector

As for the matrix in Eq. (A.1), its eigenvalues read

\[ P_1 = 0, \]
\[ P_2 = 0, \]
\[ P_3 = 0, \]
\[ P_4 = -\frac{2f}{\nu_\eta\nu_\rho^2\nu_\chi} (\nu_\chi^2\nu_\eta^2 + \nu_\chi^2\nu_\rho^2 + \nu_\rho^2\nu_\eta^2 + \nu_\rho^2\nu_\eta^2), \]
\[ P_5 = -\frac{2}{\nu_\rho^2\nu_\chi} \left[ (2\lambda_{13} - \lambda_8) \nu_\rho^3\nu_\chi^3 + f\nu_\eta\nu_\chi^2 + f\nu_\eta (\nu_\rho^2 + \nu_\rho^2) + (2\lambda_{13}\nu_\rho^2\nu_\rho^2 - \lambda_8\nu_\rho^3\nu_\rho^2 + 2\lambda_{13}\nu_\rho^3 - \lambda_8\nu_\rho^3) \nu_\chi \right], \quad (A.14) \]

and their corresponding eigenvectors are

\[
V_1 \approx \begin{pmatrix}
0 \\
0 \\
\nu_\rho^2 \\
\nu_\eta\nu_\rho^3 \\
\nu_\chi\nu_\rho^2
\end{pmatrix}, \quad V_2 \approx \begin{pmatrix}
\nu_\chi\nu_\rho^2 \\
0 \\
-\nu_\eta (\nu_\rho^2 + \nu_\rho^2) \\
-\nu_\chi\nu_\rho^2\nu_\rho^3
\end{pmatrix},
\]
\[
V_3 \approx \begin{pmatrix}
0 \\
\nu_\rho^2 \\
-\nu_\eta \\
0
\end{pmatrix}, \quad V_4 \approx \begin{pmatrix}
\nu_\eta\nu_\rho^3 \\
\nu_\chi\nu_\eta \\
0 \\
\nu_\chi\nu_\rho^2 \\
-\nu_\rho^3\nu_\eta
\end{pmatrix},
\]
\[
V_5 \approx \begin{pmatrix}
-\nu_\rho^3 \\
0 \\
\nu_\chi \\
0 \\
-\nu_\rho^2
\end{pmatrix}. \quad (A.15)
\]

A.3 Diagonalization of the real sector

The equation for the eigenvalues associated to the matrix (A.2), has no simple solutions. Then, we take the approximations (5.8) and (5.9), such that we keep only
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quadratic order in \( \nu_\chi \)

\[
M_{\xi \xi}^2 \approx \begin{bmatrix}
\xi_\chi & \xi_\rho^2 & \xi_\rho^3 & \xi_\eta & \xi_\chi^0 \\
8\lambda_1 \nu_\chi^2 & 0 & 4\lambda_{10} \nu_\chi^2 & 0 & 0 \\
0 & -2f \frac{\nu_\chi \nu_\eta}{\nu_\rho^2} & 0 & 2f \nu_\chi & 0 \\
4\lambda_{10} \nu_\chi^2 & 0 & (2\lambda_8 + 4\lambda_{13}) \nu_\chi^2 - 2f \frac{\nu_\chi \nu_\eta}{\nu_\rho^2} & 0 & 0 \\
0 & 2f \nu_\chi & 0 & -2f \frac{\nu_\chi \nu_\rho^2}{\nu_\eta} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(A.16)

and the following decoupled submatrices arise

\[
M_{\xi_\rho^2 \xi_\eta}^2 \approx \begin{bmatrix}
\xi_\rho^2 & \xi_\eta \\
-2f \frac{\nu_\chi \nu_\eta}{\nu_\rho^2} & 2f \nu_\chi \\
2f \nu_\chi & -2f \frac{\nu_\chi \nu_\rho^2}{\nu_\eta} \\
\end{bmatrix},
\]

\[
M_{\xi_\chi \xi_\rho^3}^2 \approx \begin{bmatrix}
\xi_\chi & \xi_\rho^3 \\
8\lambda_1 \nu_\chi^2 & 4\lambda_{10} \nu_\chi^2 \\
4\lambda_{10} \nu_\chi^2 & (2\lambda_8 + 4\lambda_{13}) \nu_\chi^2 - 2f \frac{\nu_\chi \nu_\eta}{\nu_\rho^2} \\
\end{bmatrix},
\]

\[
M_{\xi_\chi^0}^2 = 0
\]

(A.17)

The first square matrix in Eq. (A.17) has one null eigenvalue. However, this first null eigenvalue arises due to the approximations given by Eqs. (5.8, 5.9) applied to (A.2). Therefore, we take all terms in the submatrix \( M_{\xi_\rho^2 \xi_\eta}^2 \) in Eq. (A.17)

\[
M_{\xi_\rho^2 \xi_\eta}^2 = \begin{bmatrix}
\xi_\rho^2 & \xi_\eta \\
8\lambda_1 \nu_\chi^2 \nu_\rho^2 & 4\lambda_{10} \nu_\chi^2 \nu_\rho^2 + 2f \nu_\chi \\
4\lambda_{10} \nu_\chi^2 \nu_\rho^2 + 2f \nu_\chi & 8\lambda_3 \nu_\eta^2 - 2f \frac{\nu_\chi \nu_\rho^2}{\nu_\eta} \\
\end{bmatrix},
\]

(A.18)

whose eigenvalues are \( P_2 \) and \( P_3 \) in Eq. (5.19). We should observe precisely that \( P_2 \) is a small value that does not depend on \( \nu_\chi \). Owing to it, \( P_2 \) vanishes for the first approximation of \( M_{\xi_\rho^2 \xi_\eta}^2 \) in (A.17).
Bilinear Gauge-Goldstone bosons terms for $\beta = 1/\sqrt{3}$

The kinetic term of the Higgs Lagrangian is:

$$L_H = (D^\mu \Phi)^\dagger D_\mu \Phi$$

$$= \frac{1}{\sqrt{3}} (\partial^\mu \Phi)^\dagger (D_\mu \Phi) + (D^\mu \Phi)^\dagger (\partial_\mu \Phi) \right)$$

$$+ \Phi^\dagger (g W^\mu + g' B^\mu X_\Phi)^\dagger (g W_\mu + g' B_\mu X_\Phi) \Phi, \quad (B.1)$$

with

$$D_\mu = \partial_\mu + ig W^\alpha_\mu G_\alpha + ig' X_\Phi B_\mu, \quad (B.2)$$

The term 1 in Eq. (B.1) contains couplings among gauge fields and the derivatives of the scalar fields $\partial_\mu \Phi$, from which we obtain information about each would-be Goldstone boson coupled to the corresponding massive gauge boson. The term 2 generates the masses of the gauge bosons and couplings with the physical Higgs fields. The term 1 is evaluated to identify the would-be Goldstone bosons obtained in the diagonalization of the mass matrix in the scalar sector, and its corresponding gauge boson that acquires mass. The gauge bosons of the model with $\beta = 1/\sqrt{3}$ read

$$W^\alpha_\mu G_\alpha = \frac{1}{2} \left[ \begin{array}{cccc} W^3_\mu + \frac{1}{\sqrt{3}} W^8_\mu & -\sqrt{2} W^8_\mu & \sqrt{2} K^0_\mu \\ -\sqrt{2} W^8_\mu & -W^3_\mu + \frac{1}{\sqrt{3}} W^8_\mu & \sqrt{2} K^0_\mu \\ \sqrt{2} K^0_\mu & \sqrt{2} K^0_\mu & -\frac{2}{\sqrt{3}} W^8_\mu \end{array} \right],$$

$$B_\mu = IB_\mu = \left[ \begin{array}{ccc} B_\mu & 0 & 0 \\ 0 & B_\mu & 0 \\ 0 & 0 & B_\mu \end{array} \right], \quad (B.3)$$

where the neutral electroweak basis is related to the physical bosons by

$$\left[ \begin{array}{c} A_\mu \\ Z_\mu \\ Z'_\mu \end{array} \right] = \left[ \begin{array}{ccc} S_W & \frac{1}{\sqrt{3}} S_W & C_W \sqrt{1 - \frac{1}{3} T^2_W} \\ -C_W & \frac{1}{\sqrt{3}} S_W T_W & S_W \sqrt{1 - \frac{1}{3} T^2_W} \\ 0 & -\sqrt{1 - \frac{1}{3} T^2_W} & \frac{1}{\sqrt{3}} T_W \end{array} \right] \left[ \begin{array}{c} W^3_\mu \\ W^8_\mu \\ B_\mu \end{array} \right], \quad (B.4)$$

where the Weinberg angle $\theta_W$ is defined by $\tan \theta_W = T_W = \frac{\sqrt{2} g'}{\sqrt{3} g^2 + g'^2}$. For the fields $\Phi = \chi$ and $\rho$ of table 12 ($X_\Phi = \frac{1}{3}$), the covariant derivative Eq.
The scalar sector of the 331 model

(B.2) in the basis of the physical gauge fields is

\[
D_\mu = i \partial_\mu + \begin{bmatrix}
  a_1 A_\mu + a_2 Z_\mu + a_3 Z'_\mu & -\frac{g}{\sqrt{2}} W^+_{\mu} & \frac{g}{\sqrt{2}} K^+_{\mu} \\
  -\frac{g}{\sqrt{2}} W^-_{\mu} & a_4 Z_\mu + a_3 Z'_\mu & \frac{g}{\sqrt{2}} K^0_{\mu} \\
  \frac{g}{\sqrt{2}} K^-_{\mu} & \frac{g}{\sqrt{2}} K^0_{\mu} & \frac{\sqrt{3 g^2 + g'^2}}{3} Z'_\mu
\end{bmatrix},
\]

(B.5)

For \( \Phi = \eta \ (X_\Phi = -\frac{2}{3}) \), we have

\[
D_\mu = i \partial_\mu + \begin{bmatrix}
  -a_4 Z_\mu - a_5 Z'_\mu & \frac{g}{\sqrt{2}} W^+_{\mu} & \frac{g}{\sqrt{2}} K^+_{\mu} \\
  \frac{g}{\sqrt{2}} W^-_{\mu} & -a_1 A_\mu - a_2 Z_\mu - a_5 Z'_\mu & \frac{g}{\sqrt{2}} K^0_{\mu} \\
  \frac{g}{\sqrt{2}} K^-_{\mu} & \frac{g}{\sqrt{2}} K^0_{\mu} & -a_4 A_\mu - a_6 Z_\mu - 2 a_5 Z'_\mu
\end{bmatrix},
\]

(B.6)

and

\[
ak_1 = g Sw, \quad a_2 = \frac{g}{2} C_W \left( T_W^2 - 1 \right),
\]

\[
ak_3 = \frac{g'}{2 \sqrt{3}} \left( \frac{T_W^2 - 1}{T_W} \right), \quad a_4 = \frac{g}{2} C_W \left( T_W^2 + 1 \right)
\]

\[
ak_5 = \frac{g'}{2 \sqrt{3}} \left( \frac{T_W}{T_W^2} \right), \quad a_6 = \frac{g'}{\sqrt{3}} \left( \frac{1}{T_W} \right).
\]

(B.7)

With the covariant derivative for the fields \( \chi, \rho \) and \( \eta \), we have that the terms of the bilinear scalar-gauge mixing coming from the term 1 of Eq. (B.1) are given respectively by

\[
(\partial^\mu \chi)^\dagger (D_\mu \langle \chi \rangle_0) + h.c. = \]

\[
\frac{ig}{\sqrt{2}} \partial_\mu (\nu_\chi \chi^-) K^+_{\mu} + \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\chi \chi^+) K^-_{\mu} + \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\chi \xi^-_0 - i \nu_\chi \xi^0) K^0_{\mu}
\]

\[- \frac{ig}{\sqrt{2}} (\nu_\chi \partial_\mu \xi^0_0 + i \nu_\chi \partial_\mu \xi^0) K^0_{\mu} + \frac{\sqrt{3 g^2 + g'^2}}{3} \partial_\mu (2 \nu_\chi \xi^-) Z'_\mu,
\]

(B.8)

\[
(\partial^\mu \rho)^\dagger (D_\mu \langle \rho \rangle_0) + h.c. = \]

\[
\frac{ig}{\sqrt{2}} \partial_\mu (\nu_\rho \rho^-) K^+_{\mu} + \frac{ig}{\sqrt{2}} (\partial_\mu \nu_\rho \rho^+) K^-_{\mu} + \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\rho \rho^-) W^+_{\mu}
\]

\[+ \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\rho \rho^+) W^-_{\mu} + \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\rho \xi^-_0 - \nu_\rho \xi^0 - i \nu_\rho \xi^0_0 - i \nu_\rho \xi^0) K^0_{\mu}
\]

\[- \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\rho \xi^0_0 - \nu_\rho \xi^0 - i \nu_\rho \xi^- + i \nu_\rho \xi^0_0) K^0_{\mu} + a_4 \partial_\mu (2 \nu_\rho \xi^0) Z'_\mu
\]

\[+ \frac{\sqrt{3 g^2 + g'^2}}{3} \partial_\mu \left( 2 \nu_\rho \xi^0_0 + \frac{6 a_3}{\sqrt{3 g^2 + g'^2}} \nu_\rho \xi^0 \right) Z'_\mu,
\]

(B.9)
\[(\partial^\mu \eta)(D_\mu \langle \eta \rangle_0) + h.c = \]
\[
-\frac{ig}{\sqrt{2}} \partial_\mu (\nu_\eta \eta^3^-) K^\mu_+ + \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\eta \eta^3^+) K^-_\mu + \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\eta \eta^2^-) W^\mu_+
\]
\[
-\frac{ig}{\sqrt{2}} \partial_\mu (\nu_\eta \eta^2^+) W^-_\mu - a_4 \partial_\mu (2\nu_\eta \zeta_\eta) Z_\mu - \frac{a_4 \sqrt{3g^2 + 4g'^2}}{3g} \partial_\mu (2\nu_\eta \zeta_\eta) Z'_\mu.
\]

Adding the Eqs. \[(B.8-B.10)\] we can find the would-be Goldstone bosons in the $R_\zeta$ gauge associated to the gauge fields that acquire masses

\[
(\partial^\mu \Phi)(D_\mu \langle \Phi \rangle_0) + h.c = 
\]
\[
\frac{ig}{\sqrt{2}} \partial_\mu (-\nu_\rho^2 \rho^- + \nu_\eta \eta^2^-) W^\mu_+ - \frac{ig}{\sqrt{2}} \partial_\mu (-\nu_\rho^2 \rho^+ + \nu_\eta \eta^2^+) W^-_\mu
\]
\[
+ \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\chi \chi^0 + \nu_\rho^3 \xi_\rho^2 - \nu_\rho^2 \xi_\rho^3 - i\nu_\chi \zeta_\chi^0 - i\nu_\rho^3 \zeta_\rho^2 - i\nu_\rho^2 \zeta_\rho^3) K^0_\mu
\]
\[
- \frac{ig}{\sqrt{2}} \partial_\mu (\nu_\chi \chi^0 + \nu_\rho^3 \xi_\rho^2 - \nu_\rho^2 \xi_\rho^3 + i\nu_\chi \zeta_\chi^0 + i\nu_\rho^3 \zeta_\rho^2 + i\nu_\rho^2 \zeta_\rho^3) \bar{K}^0_\mu
\]
\[
+ \frac{g\sqrt{3g^2 + 4g'^2}}{\sqrt{3g^2 + g'^2}} \partial_\mu (\nu_\rho^3 \zeta_\rho^2 - \nu_\eta \zeta_\eta) Z_\mu
\]
\[
+ \frac{2\sqrt{3g^2 + g'^2}}{3} \partial_\mu \left( \nu_\chi \chi^0 + \nu_\rho^3 \zeta_\rho^3 + \frac{1}{2} \frac{2g'^2 - 3g^2}{3g^2 + g'^2} \nu_\rho^2 \zeta_\rho^2 - \frac{1}{2} \frac{3g^2 + 4g'^2}{3g^2 + g'^2} \nu_\eta \zeta_\eta \right) Z'_\mu.
\]

In the couplings with $K^0_\mu$ and $\bar{K}^0_\mu$, it is observed a mixing amongst the real and imaginary scalars, which is not produced in the mass matrices of the scalar sector, as it is confirmed by Eqs. \[(A.1)\] and \[(A.2)\]. To have it, we make the rotation

\[
\frac{ig}{\sqrt{2}} \partial_\mu \left( \nu_\chi \chi^0 + \nu_\rho^3 \xi_\rho^3 - \nu_\rho^2 \xi_\rho^2 - i\nu_\rho^3 \zeta_\rho^3 - i\nu_\rho^2 \zeta_\rho^3 \right) K^0_\mu
\]
\[
- \frac{ig}{\sqrt{2}} \partial_\mu \left( \nu_\chi \chi^0 + i\nu_\rho^3 \xi_\rho^3 - \nu_\rho^2 \xi_\rho^2 + i\nu_\rho^3 \zeta_\rho^3 \right) \bar{K}^0_\mu
\]
\[
= \frac{ig}{\sqrt{2}} \partial_\mu \left( \nu_\chi \chi^0 + \nu_\rho^3 \xi_\rho^3 - \nu_\rho^2 \xi_\rho^2 \right) \tilde{K}^0_\mu
\]
\[
- \frac{ig}{\sqrt{2}} \partial_\mu \left( i\nu_\chi \chi^0 + \nu_\rho^3 \zeta_\rho^3 + i\nu_\rho^2 \zeta_\rho^3 \right) \bar{\tilde{K}}^0_\mu.
\]

with $\tilde{K}^0_\mu = K^0_\mu - \bar{K}^0_\mu$ and $\bar{\tilde{K}}^0_\mu = K^0_\mu + \bar{K}^0_\mu$. The couplings in Eqs. \[(B.11)\] and \[(B.12)\], indicates the following combinations of the would-be Goldstone bosons.
The scalar sector of the 331 model

\[ \phi_1^\pm \sim \nu_\rho \rho^\pm - \nu_\eta \eta^{2\pm} \]

\[ \phi_2^\pm \sim \nu_\chi \chi^{\pm} + \nu_\rho \rho^\pm - \nu_\eta \eta^{3\pm} \]

\[ \phi_1^0 \sim (\nu_\chi \zeta^0 + \nu_\rho \zeta^\rho) \]

\[ \phi_2^0 \sim \left( \nu_\chi \zeta^0 + \nu_\rho \zeta^\rho + \frac{1}{2} \frac{g'^2 - 3g^2}{3g^2 + g'^2} \nu_\rho \zeta^\rho - \frac{1}{2} \frac{3g^2 + 4g'^2}{3g^2 + g'^2} \nu_\eta \zeta^\eta \right) \]

\[ \phi_3^0 \sim (\nu_\rho \zeta^\rho - \nu_\eta \zeta^\eta) \]

\[ \phi_4^0 \sim \nu_\chi \zeta^0 + \nu_\rho \zeta^\rho - \nu_\rho \zeta^\rho \]

(B.13)

that couples to the corresponding physical gauge bosons \( W_\mu^\pm, K_\mu^\pm, K_\mu^0, Z_\mu, Z_\mu, K_\mu^0 \) respectively.

Rotating the VEV to the basis where \( \nu_\rho^3 \to 0 \), as in Eq. (B.14), it is found

\[ \phi_1^\pm \sim \nu_\rho \rho^\pm - \nu_\eta \eta^{2\pm} \]

\[ \phi_2^\pm \sim \nu_\chi \chi^{\pm} - \nu_\eta \eta^{3\pm} \]

\[ \phi_1^0 \sim i \left( \nu_\chi \zeta^0 + \nu_\rho \zeta^\rho \right) \]

\[ \phi_2^0 \sim i \left( \nu_\chi \zeta^0 + \frac{1}{2} \frac{3g^2 + 4g'^2}{3g^2 + g'^2} \nu_\rho \zeta^\rho - \frac{1}{2} \frac{3g^2 + 4g'^2}{3g^2 + g'^2} \nu_\eta \zeta^\eta \right) \]

\[ \phi_3^0 \sim i (\nu_\rho \zeta^\rho - \nu_\eta \zeta^\eta) \]

\[ \phi_4^0 \sim \nu_\chi \zeta^0 - \nu_\rho \zeta^\rho \]

(B.14)

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