Multi-Keyword Multi-Click Advertisement Option Contracts for Sponsored Search

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In sponsored search, advertisement (shortly ad) slots are usually sold by a search engine to an advertiser through an auction mechanism in which advertisers bid on keywords. In theory, an auction mechanism encourages advertisers to truthfully bid for keywords. However, keyword auctions have a number of problems including: the uncertainty in payment prices for advertisers; the volatility in the search engine’s revenue; and the weak loyalty between advertiser and search engine. In this paper we propose a special ad option contract that alleviates these problems. In our proposal, an advertiser can purchase an option in advance from a search engine by paying an upfront fee, known as the option price. He/she then has the right, but no obligation, to then purchase among the pre-defined set of keywords at fixed cost-per-clicks (CPCs) for a specified number of clicks in a specified period of time. The proposed option is closely related to a special exotic option in finance that contains multiple underlying assets (multi-keyword) and is also multi-exercisable (multi-click). There are several benefits of this novel structure: advertisers can have increased certainty in sponsored search; the search engine can raise the advertisers’ loyalty as well as obtain a stable and increased revenue over time. The proposed ad option can be implemented in conjunction with traditional keyword auctions. As such, the option price and corresponding fixed CPCs must be set such that there is no arbitrage. Therefore, we discuss the option pricing methods tailored to sponsored search that deal with the spot CPCs of targeted keywords in the generalized second price (GSP) auction. Our experimental results with the real sponsored search data validate the development and demonstrate that, compared to only keyword auctions, a search engine can have an increased expected revenue by selling an ad option.

Categories and Subject Descriptors: J.4 [Computer Applications]: Social and Behavior Science – Economics

General Terms: Theory, Algorithms, Experimentation

Additional Key Words and Phrases: Sponsored Search, Option Contract, Pricing Model, Revenue Analysis

1. INTRODUCTION

Sponsored search has become an important online advertising format [PWC 2013]. A search engine sells advertisement (shortly ad) slots in the search engine results pages (SERPs) generated in response to a user query. The terms in the query are referred to as keywords and the price of an ad slot is usually determined by a keyword auction [Fain and Pedersen 2006; Borgers et al. 2008; Varian 2009] such as the widely used generalized second price (GSP) auction [Edelman et al. 2007; Varian 2007]. In the
GSP auction, advertisers bid on keywords present in the query, and the highest bidder pays the price associated with the bid next to him/her.

Despite the success of keyword auctions, there are two major drawbacks. First, the uncertain and volatile bids make it difficult for advertisers to predict their campaign costs and thus complicate their business planning [Wang and Chen 2012]. Second, the pay-as-you-go nature of the auctions does not encourage a stable relationship between advertiser and search engine [Jank and Yahav 2010]: an advertiser can switch from one search engine to another in the next bidding at near-zero cost.

To alleviate these problems, we propose a multi-keyword multi-click ad option in this paper. This is a contract between an advertiser and a search engine. It consists of a non-refundable upfront fee, known as the option price, paid by the advertiser, in return for the right, but not the obligation, to subsequently purchase a fixed number of clicks for particular keywords for specified fixed cost-per-clicks (CPCs) during a specified period of time. From the advertiser's perspective, fixing the CPCs significantly reduces the uncertainty in cost of advertising campaigns. Moreover, for a keyword, if the spot CPC set by keyword auction falls below the fixed CPC, the advertiser is not obligated to exercise the option, but can, instead, participate in keyword auction. Thus, the option can be considered as an “insurance” that establishes an upper limit on the cost of advertising campaigns. From the search engine's perspective, the option is not only an additional guaranteed service provided for advertisers. We show that the search engine can, in fact, increase the expected revenue in the process of selling an option. Also, the option covers a specific period of time should encourage a more stable relationship between the advertiser and the search engine. An important question for us is to determine the option price and the fixed CPCs associated with candidate keywords in the advertiser's request list. Clearly if the option is priced too low, then significant loss in revenue may ensure. Moreover, this may create an arbitrage opportunity where the buyer of the option sells the clicks their targeted keywords to gain extra profits. Conversely, if the option is priced too high, then the advertiser will not purchase it. In this paper we consider a risk-neutral environment and price the option under the no-arbitrage objective [Björk 2009]. We use the Monte Carlo method to price the option with many keywords and show the closed-form pricing formulas for the cases of single and two keywords. Further, the effects of the ad option on the search engine's revenue is analyzed.

This paper has three major contributions. First, we propose a new mechanism to pre-sell ad slots in sponsored search, which provides the guaranteed delivery to advertisers. It naturally complements the current keyword auction mechanism and provides both the advertisers and search engine with an effective risk mitigation tool due to the bid price fluctuation. Although the proposed ad option belongs to a family of exotic option contracts in finance, its payoff function differs from existing exotic option contracts that we know from finance and other industries (see Table I for a detailed comparison): it can be exercised not once but multiple times during the contract period; it is not for a single keyword but multiple keywords; it also has multiple fixed CPCs and the advertiser can choose which keyword to buy at the corresponding fixed CPC later during the contract period. Second, we propose a generalized numerical pricing method for the proposed ad option (see Algorithm 1) due to the high dimensionality. Third, we demonstrate that theoretically and experimentally, compared to keyword auctions, a search engine can have an increased expected revenue by selling an ad option.

The rest of the paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the multi-keyword multi-click ad option, discusses the option pricing methods and analyzes the effects on search engine’s revenue. Section 4 presents our experimental evaluation and Section 5 concludes the paper. Some important mathematical formulas and results are given in Appendices A-C.
2. RELATED WORK

The work presented in this paper touches upon several streams of literature. We first review the prior work on options in finance and other industries, and then discuss the related work in the field of online advertising.

2.1. Option Contracts and Their Pricing

A standard option is a contract in which the seller grants the buyer the right, but not the obligation, to enter into a transaction with the seller to either buy or sell an underlying asset at a fixed price on or before a fixed date. The fixed price is called the strike price and the fixed date is called the expiration date. The seller grants this right in exchange for a certain amount of money, called the option price. An option is called a call option or a put option depending on whether the buyer is purchasing the right to buy or sell the underlying asset. The simplest option is the European option [Wilmott 2006], which can be exercised only on the expiration date. This differs from the American option [Wilmott 2006], which can be exercised at any time during the contract lifetime. Both European and American options are standard options.

In the beginning of the 1980s, standard options became better understood and their trading volume exploded. Financial institutions began to search for alternative forms of options, known as exotic options [Zhang 1998], to meet their particular business needs. Among them, two types of exotic options, multi-asset options and multi-exercise options are particularly relevant to the study of this paper.

Multi-asset options, also referred to as correlation options, are the options whose payoff is affected by at least two underlying assets. The underlying assets can be stocks, bonds, currencies, and indices such as the S&P-100, FTSE-100 etc. To illustrate the uniqueness of the proposed ad option, we compare its contract specifications to other relevant multi-asset options as well as standard options in Table I. The proposed ad option is novel, whose multi-asset setting is a generalized framework of the European dual-strike call option [Zhang 1998] as it not only has multiple strike prices (i.e., the fixed CPCs) but also allows the early exercise opportunity for advertisers. In Table I, we provide two possible formulas of payoff function for the proposed ad option. The first formula can be used to describe either exact match or broad match setting in sponsored search, depending on the match type of underlying spot CPCs. If we only have the exactly matched spot CPCs, the second formula enables us to create a broad match structure for computing the advertiser’s option payoff because the weight \( \omega_{ji} \) is interpreted as the probability that the \( i \)th broad matched keyword (i.e., the sub-phrase occurs in search queries) for the \( j \)th targeted ad keyword. The detailed discussion on the two formulas is given in Section 3.2.

Multi-exercise options are a generalization of American options that provide the buyer more than one exercise opportunity and sometimes control over one or more other variables, e.g. the amount of the underlying asset exercised at a certain time. Multi-exercise options have become more prevalent over the past decade especially in the energy industry, such as water options and electricity options [Villinski 2004; Marshall et al. 2011; Marshall 2012]. Our proposed ad option is also a special case of multi-exercise options because it permits the option buyer to repeatedly exercise it to obtain clicks on targeted keywords. However, compared to the energy industry, the multi-exercise opportunity is more flexible in sponsored search. Advertisers can exercise the option at any time in the option lifetime, i.e. the exercise time is not pre-specified. There is no minimum number of clicks required for each exercise so there is no penalty fee if the advertiser does not exercise the minimum clicks. Also, there is no transaction fee in sponsored search. Thus in this paper, the value of an \( m \)-click ad option is the sum of \( m \) independent and identical 1-click ad options.
Table I. A brief comparison between the $n$-keyword 1-click ad option and other relevant options: the price of the $i$th underlying asset at time $t$ is denoted by $C_i(t)$, where $t$ is a continuous time point in period $[0, T]$ and $T$ is the contract expatriation date (if there is only one underlying asset we denote its price at time $t$ by $C(t))$; the strike price, i.e., the fixed payment price, of the $i$th underlying asset is denoted by $F_i$ (if there is only one strike price we denote it by $F$); the weight of $i$th asset in a basket-type option is denoted by $\omega_i$ (note that in the $n$-keyword 1-click ad option, $\omega_{ji}$ represents the weight of the $i$th broad matched keyword for the $j$th targeted ad keyword). A detailed summary of notations and terminology is given in Table II.

| Option contract | Payoff function | Underlying variable | Exercise opportunity | Early exercise | Strike price | Application area |
|-----------------|-----------------|---------------------|----------------------|---------------|--------------|------------------|
| $n$-keyword 1-click ad option (keyword exact or broad match) | $\max\{C_1(t) - F_1, \ldots, C_n(t) - F_n, 0\}$ | Multiple | Single | Yes | Multiple | Keywords |
| $n$-keyword 1-click ad option (keyword broad match) | $\max\left\{\sum_{i=1}^{k_1} \omega_{1i} C_1(t) - F_1, \ldots, \sum_{i=1}^{k_n} \omega_{ni} C_n(t) - F_n, 0\right\}$ | Multiple | Single | Yes | Multiple | Keywords |
| European standard call option [Wilmott 2006] | $\max\{C(T) - F, 0\}$ | Single | Single | No | Single | Equity stock, or index |
| American standard call option [Wilmott 2006] | $\max\{C(t) - F, 0\}$ | Single | Single | Yes | Single | Equity stock, or index |
| European basket call option [Krekel et al. 2006] | $\max\{\sum_{i=1}^{n} \omega_i C_i(T) - F, 0\}$ | Multiple | Single | No | Single | Index of equity stocks, bonds or foreign currencies |
| European dual-strike call option [Zhang 1998] | $\max\{C_1(T) - F_1, C_2(T) - F_2, 0\}$ | Double | Single | No | Double | Equity stocks, or indexes of equity stocks, bonds, or foreign currencies |
| European rainbow call on max option [Ouwehand and West 2006] | $\max\{\max\{C_1(T), \ldots, C_n(T)\} - F, 0\}$ | Multiple | Single | No | Single | Equity stocks, or indexes of equity stocks, bonds, or foreign currencies |
| European paying the best and cash option [Johnson 1987] | $\max\{C_1(T), C_2(T), F\}$ | Double | Single | No | Single | |
| European quotient call option [Zhang 1998] | $\max\{C_1(T)/C_2(T) - F, 0\}$ | Double | Single | No | Single | |
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Evaluating an option contract is difficult but imperative. Option pricing models constitute one of the most important building blocks of asset pricing theory. In 1900, Bachelier first proposed to use a continuous-time random walk process to price option contracts [Bachelier 1900]. Sixty-five years later, Samuelson replaced Bachelier’s assumption on the asset price with a geometric form, called geometric Brownian motion (GBM) [Samuelson 1965], thereby solving the problem of a negative asset price in option pricing. The research of Samuelson highly affected Black and Scholes and Merton, who then studied the risk-neutral pricing for European options. The Black-Scholes-Merton (BSM) option pricing formula [Black and Scholes 1973; Merton 1973] spurred research in this area and we discuss the option pricing methods based on the above developments in finance.

2.2. Guaranteed Contracts in Online Advertising

In online advertising, guaranteed contracts have been variously investigated. Previous studies include the ads matching algorithm, the optimal allocation of ad slots, the contract pricing method and the innovative contract mechanism design.

Feldman et al. proposed an algorithm to address the problem of ads selection and matching [Feldman et al. 2009], where a publisher’s objective is not only to fill the guaranteed contracts but also to deliver well-targeted impressions to advertisers. Ghosh et al. discussed the allocation of impressions between guaranteed and non-guaranteed channels and considered a publisher to act as a bidder and to bid for guaranteed contracts [Ghosh et al. 2009]. A good property of this setting is that the publisher acts as a bidder would possibly allocate impressions to online auctions only when the winning bids are high enough. The same allocation problem was discussed in [Balseiro et al. 2011] by considering stochastic control models. For a given impression price, the publisher decides whether to send it to ad exchanges or to assign it to an advertiser with a fixed price. The total revenue from ad exchanges and reservations are then maximized. Roels and Fridleirsdottir studied a similar framework to Balseiro et al., where the publisher can dynamically select which guaranteed buy requests to accept and then delivers the guaranteed impressions [Roels and Fridleirsdottir 2009]. Compared to [Balseiro et al. 2011], the uncertainty in advertisers’ buy requests and the traffic of websites are explicitly modeled in revenue maximization. A lightweight allocation framework was discussed in [Bharadwaj et al. 2012]. It intends to simplify the computations in optimization and to let real servers allocate ads efficiently and with little overhead.

Two statistical pricing methods of guaranteed contracts were discussed by Bharadwaj et al. for display advertising [Bharadwaj et al. 2010]. Based on the statistics of users’ visits to the webpage, the first method calculates the weighted average price while the second method considers the price that minimizes the variance of revenue between the negotiation prices and the spot prices from auctions. This study shows the user visit level can be employed to price guaranteed contracts while the effects of online auctions are not well considered.

Based on the posted prices, Engel and Tennenholtz discussed an interesting market design for display advertising contracts [Engel and Tennenholtz 2013], where advertisers’ utility function exhibits constant elasticity of substitution and goods are gross substitutes. The equilibrium price of a guaranteed contract can be finally approximated by some polynomial functions. Salomatin et al. proposed a guaranteed delivery framework tailored for sponsored search [Salomatin et al. 2012]. An advertiser is able to send the click demand and the budget in his/her request to a search engine, who will then decide the guaranteed service delivery according to query and position.

Financial derivatives, including option contracts, can provide more flexibility to advertisers. They were recently introduced into online service markets to hedge economic
risks and enhance the internal process efficiency [Meinl and Blau 2009]. To the best of our knowledge, Moon and Kwon proposed the first ad option contract in online advertising [Moon and Kwon 2010]. The option buyer (i.e., advertiser) is guaranteed the right to choose the minimum payment between the cost-per-mille (CPM) and the CPC once the click-through rate (CTR) \(^1\) is realised. This option structure is similar to an option paying the worst and cash [Zhang 1998], where the option payoff depends on the minimum of the two underlying assets. In the study of Moon and Kwon, the option price is determined by a negotiation between the advertiser and the publisher under the framework of a Nash bargaining game. Specifically, there are two utility functions (one for the advertiser and one for the publisher) and the objective of option pricing is to maximise the product of the two utility functions under their corresponding negotiation powers. Thus, the final option price is the optimal solution that maximises the negotiated joint utility. Wang and Chen proposed another ad option contract [Wang and Chen 2012]. It allows its buyer to select his/her preferred payment scheme (either the CPM or the CPC) as the underlying measurement for the fixed payment (i.e., the strike price). Therefore, an advertiser can have a different fixed payment to the ad format. For example, paying the fixed CPC in display advertising or paying the fixed CPM in sponsored search. Wang and Chen priced the option based on the one-step Binomial tree [Cox et al. 1979] where a publisher or a search engine can hedge the revenue risk as much as possible. Our research in this paper is significantly different to the previous studies. First, the option structure doesn’t consider the multi-payment schemes, instead, contains the unique features of sponsored search, such as the keyword broad match, the switch between multiple keywords, and the multiple exercising opportunities. Second, the developed option pricing methods are under the continuous-time framework and to eliminate the arbitrage between the spot (auction) market and the option market.

3. MULTI-KEYWORD MULTI-CLICK ADVERTISEMENT OPTION CONTRACTS

We first explain how the multi-keyword multi-click ad option works, then introduce the option pricing methods and finally provide a revenue analysis for search engine.

3.1. Option Contract Mechanism for Sponsored Search

We employ the following example to illustrate our idea. Suppose that a computer science department creates a new master degree program on “Web Science and Big Data Analytics” and is interested in an advertising campaign based around relevant search terms such as ‘MSc Web Science’, ‘MSc Big Data Analytics’ and ‘Data Mining’ etc. The campaign is to start immediately and last for three months and the goal is to generate at least 1000 clicks on the ad which directs users to the program’s homepage. The department (i.e., advertiser) does not know how the clicks will be distributed among the keywords, nor how much the campaign will cost if based on keyword auctions. However, with the ad option, the advertiser can submit a request to the search engine to lock-in the advertising cost. The request consists of the candidate keywords, the overall number of clicks needed, and the duration of the contract. The search engine responds with a price table for the option, as shown in Figure 1. It contains the option price and the fixed CPC for each keyword. The CPCs are fixed yet different across the candidate keywords. The contract is entered into when the advertiser pays the option price.

During the contract period \([0, T]\), where \(T\) represents the contract expiration date (in terms of year format and is three months in this example), the advertiser has the right, at any time, to exercise portions of the contract, for example, to buy a requested number of clicks for a specific keyword. This right expires after time \(T\) or when the

\(^1\)The CTR is the ratio of the number of advertisements clicked on to the number of advertisements displayed.
Fig. 1. A schematic view of buying, selling and exercising a multi-keyword multi-click ad option.

total number of clicks have been purchased, whichever is sooner. For example, at time $t_1 \leq T$ the advertiser may exercise the right for 100 clicks on the keyword ‘MSc Web Science’. After receiving the exercise request, the search engine immediately reserves an ad slot for the keyword for the advertiser until the ad is clicked by a 100 times. In our current design, the search engine decides which rank position the ad should be displayed as long as the required number of clicks is fulfilled - we assume there are adequate search impressions within the period. It is also possible to generalise the
study in this paper and define a rank specific option where all the parameters (CPCs, option prices etc.) become rank specific.

The advertiser can switch among the candidate keywords and also monitor the keyword auction market. If, for example, the CPC for the keyword “MSc Web Science” drops below the fixed CPC, then the advertiser may choose to participate in the auction rather than exercise the option for the keyword. If later in the campaign, the spot price for the keyword ‘MSc Web Science’ exceeds the fixed CPC, the advertiser can then exercise the option.

The above example illustrates the flexibility of the proposed ad option. Specifically, (i) the advertiser does not have to use the option and can participate in keyword auctions as well, (ii) the advertiser can exercise the option at any time during the contract period, (iii) the advertiser can exercise the option up to the maximum number of clicks, (iv) the advertiser can request any number of clicks in each exercise provided the accumulated number of exercised clicks does not exceed the maximum number, and (v) the advertiser can switch among keywords at each exercise with no additional cost. Of course, this flexibility complicates the pricing of the option, which is discussed next.

### 3.2. Option Pricing Methods

The proposed multi-keyword multi-click ad option enables an advertiser to fix his/her advertising campaign cost beforehand, yet leave the decision of selecting suitable ad keywords and the exact timing to place the ad to later. Since the advertiser enjoys great flexibility in sponsored search, there is an intrinsic value associated with an ad option and the advertiser needs to pay an upfront fee to buy the option.

In the following discussion, we focus on the option pricing based on the first payoff function in Table I (i.e., \( \max\{C_1(t) - F_1, \cdots, C_n(t) - F_n, 0\} \)) while in Section 3.2.4 we briefly discuss how the introduced option pricing settings can be applied to the second payoff function in Table I (i.e., \( \max\{\sum_{i=1}^{k_1} \omega_{1i} C_{1i}(t) - F_1, \cdots, \sum_{i=1}^{k_n} \omega_{ni} C_{ni}(t) - F_n, 0\} \)).

#### 3.2.1. Underlying Stochastic Model

We adopt a widely used stochastic process to model the movement of spot CPCs of underlying ad keywords, denoted by \( C(t) \). Specifically, the keyword \( K_i \)’s CPC movement can be described by a multivariate Geometric Brownian Motion (GBM) [Samuelson 1965] as follows

\[
dC_i(t) = \mu_i C_i(t) dt + \sigma_i C_i(t) dW_i(t), \quad i = 1, \ldots, n, \tag{1}
\]
where \( \mu_t \) and \( \sigma_t \) are constant drift and volatility of the CPC respectively, and \( W_i(t) \) is a standard Brownian motion satisfying the conditions:

\[
\mathbb{E}(dW_i(t)) = 0, \\
\text{var}(dW_i(t)) = \mathbb{E}(dW_i(t)dW_i(t)) = dt,
\]
\[
\text{cov}(dW_i(t), dW_j(t)) = \mathbb{E}(dW_i(t)dW_j(t)) = \rho_{ij}dt,
\]

where \( \rho_{ij} \) is the correlation coefficient between the \( i \)th and \( j \)th keywords, such that \( \rho_{ii} = 1 \) and \( \rho_{ij} = \rho_{ji} \). The correlation matrix is denoted by \( \Sigma \), so that the covariance matrix is simply \( M\Sigma M \), where \( M \) is the matrix with the \( \sigma_i \) along the diagonal and zeros everywhere else. A brief summary of notations is given in Table II. For a detailed discussion of such stochastic settings, see [Wilmott 2006]. Later in Section 4, we discuss the GBM parameter estimation and its fitness testing. We choose GBM for its mathematical convenience and the limitations of GBM model are also indicated in Section 5.

\[3.2.2. \text{Risk-Neutral Valuation.} \] As described in Section 3.1, the proposed option framework allows the advertiser to exercise the option at any time for any clicks in the contract period. Therefore, the value of an \( n \)-keyword \( m \)-click ad option can be treated as the sum of \( m \) independent \( n \)-keyword \( 1 \)-click ad options. If we consider the advertiser buys the option at time 0, the option price \( \pi \) can be expressed as follows

\[
\pi = V(0, C(0); T, F, m) = mV(0, C(0); T, F, 1). \tag{2}
\]

Eq. (2) simplifies the pricing problem but it may not be viable for some electricity and water options because they require the option buyers to exercise the options a certain amount in some specified periods [Villinski 2004; Marshall et al. 2011; Marshall 2012].

Our focus now centres on the \( n \)-keyword \( 1 \)-click ad option. Consider if the advertiser exercises the option at the contract expiration date \( T \), then the option payoff\(^2\) is

\[
\Phi(C(T)) = \max\{C_1(T) - F_1, \ldots, C_n(T) - F_n, 0\}. \tag{3}
\]

By having Eq. (3), we can see if the advertiser would early exercise the ad option by the backward deduction. At time \( t < T \), the option value can be expressed as follows

\[
V(t, C(t); T, F, 1) = \begin{cases} 
\Phi(C(t)), & \text{early exercise,} \\
\mathbb{E}^\mathbb{Q}_t[e^{-r(T-t)}\Phi(C(T))], & \text{no early exercise.}
\end{cases}
\]

where \( \mathbb{E}^\mathbb{Q}_t[\cdot] \) is the conditional expectation with respect to time \( t \) under the risk-neutral probability \( \mathbb{Q} \) [Björk 2009]. If the ad option is early exercised, the option value is equal to its payoff at time \( t \). However, if the ad option is not exercised at time \( t \), the option value is equal to the discounted value of the expected payoff at expiration date \( T \). The comparison between \( \Phi(C(t)) \) and \( \mathbb{E}^\mathbb{Q}_t[e^{-r(T-t)}\Phi(C(T))] \) over the period \([0, T]\) can let us know the optimal decision for the advertiser, and we deduce the following inequality (see Appendix A for detailed proof):

\[
\Phi(C(t)) \leq \mathbb{E}^\mathbb{Q}_t[e^{-r(T-t)}\Phi(C(T))], \quad 0 \leq t \leq T. \tag{4}
\]

Eq. (4) illustrates, to gain the maximum option value, the advertiser will not exercise the option until its expiration date. Therefore, the option price should be computed at

\(^2\)The option payoff in sponsored search does not mean direct reward; it calculates how much cost can be reduced if the advertiser exercises the ad option. Therefore, the option payoff actually measures the difference of advertising cost between the option market and the spot (auction) market.
Algorithm 1 Multi-keyword multi-click ad option pricing via Monte Carlo simulation (where the notations and terminology are given in Table II).

```
function OptionPricingMC(K, C(0), Σ, M, n, T, r)
    ˜n ← 1000; # the number of simulated paths;
    for k ← 1 to ˜n do
        [z1,k, ..., zn,k] ← GeneratingMultivariateNoise(MVN[0, MΣM])
        for i ← 1 to n do
            Ci,k ← C_i(0) exp \{(r - \frac{1}{2}σ_i^2)T + σ_iz_i,k√T\}.
        end for
        G_k ← Φ([C_1,k, ..., C_n,k]).
    end for
    π ← me^{-rT}E_Q[Φ(C(T))] ≈ me^{-rT}(\frac{1}{n}∑_{k=1}^{n} G_k).
    return π
end function
```

the discounted value of the expected payoff at expiration date T. Together with Eq. (2), we obtain the following option pricing formula for the n-keyword m-click ad option:

\[
π = me^{-rT}E_Q[Φ(C(T))].
\] (5)

It is worth noting that Eq. (5) uses a classical work of risk-neutral pricing for European multi-asset options [McDonald and Siegel 1986; Wilmott 2006; Björk 2009]. For the reader’s convenience, we also give a simple proof of Eq. (5) in Appendix B. The proof given in Appendix B mimics the approach discussed by Black and Scholes [Black and Scholes 1973], in which we construct an advertising strategy for an advertiser and explain why the calculated option price rules out arbitrage [Varian 1987].

3.2.3. Monte Carlo Method. Eq. (5) can be expanded in the integration form as follows

\[
π = me^{-rT}(2πT)^{-n/2} |Σ|^{-1/2}\left(\prod_{i=1}^{n} σ_i\right)^{-1} \int_0^∞ \cdots \int_0^∞ \frac{Φ(\bar{C})}{\prod_{i=1}^{n} C_i} \exp \left\{-\frac{1}{2}ζ^TΣ^{-1}ζ\right\} d\bar{C},
\] (6)

where ζ = [ζ1, ..., ζn] and ζ_i = \frac{1}{σ_i√T}(ln{\bar{C}_i/C_i(0)}) - (r - \frac{σ_i^2}{2})T). Other notations are described in Table II.

We can deduce a closed form solution to Eq. (6) for the cases when n ≤ 2. If n = 1, Eq. (6) is equivalent to the Black-Scholes-Merton (BSM) pricing formula for an European call option [Black and Scholes 1973; Merton 1973]. If n = 2, Eq. (6) contains a bivariate normal distribution and the option price can be obtained by employing the pricing formula for a dual-strike European call option [Zhang 1998]. The discussed mathematical formulas are explained in details in Appendix C.

If an ad option contains more than two candidate keywords, i.e., n ≥ 3, taking integrals in Eq. (6) is computationally difficult. In such a case, we resort to numerical techniques to approximate the option price. Algorithm 1 illustrates our Monte Carlo method. Consider ˜n number of simulations, for each simulation, we generate a vector of multinormal noise and then calculate the CPC values at time T. As described in Eq. (4), there is no need to gerenate the whole paths in each simulation, we can only consider the CPCs on the expiration date and calculate the corresponding option payoff. Therefore, by having ˜n payoffs at time T, the option price π can then be approximated numerically. Algorithm 1 is lightweight and computationally fast.
3.2.4. Discussion. In the above discussion, the option payoff function in Eq. (3) can be used for both keyword exact match and keyword broad match settings, which depends on the type of the spot prices \( C(T) \) used. Also as described in Table I, if we only have the exactly matched \( C(T) \), we can still construct a broad match structure for the option, similar to Eq. (3), the option payoff function on time \( T \) is

\[
\Phi(C(T)) = \max \left\{ \sum_{i=1}^{k_1} \omega_{1i} C_{1i}(T) - F_1, \ldots, \sum_{i=1}^{k_n} \omega_{ni} C_{ni}(T) - F_n, 0 \right\}.
\]

(7)

where \( \omega_{ji} \) is the probability that the \( i \)th broad matched keyword (i.e., the sub-phrase occurs in search queries) for the \( j \)th targeted ad keyword. We can still use Eq. (1) to model the underlying CPCs’ movement but the selected keywords need to be uniquely distinctive from each other. For simplicity, we denote them by \( C(T) \). The correlation matrix is the correlation between these distinctive underlying keywords, denoted by \( \Sigma \). By having the distinctive underlying keywords in Eq. (1), the option price \( \pi_0 \) can be calculated by Algorithm 1. Here that some simulated CPCs on time \( T \) may be used more than once to calculate the difference between \( \sum_{i=1}^{k_j} \omega_{ji} C_{ji}(T) \) and \( F_j, j = 1, \ldots, n \).

3.3. Revenue Analysis for Search Engine

As described earlier, we can also loosely consider the proposed ad option as a kind of insurance for advertisers. This insurance does not come without a cost because an advertiser needs to pay an upfront option price; therefore, the ad option is also beneficial to the search engine’s revenue. In following discussion, we analyze the impact of an ad option on the search engine’s revenue. To simplify our discussion, we provide a functional analysis on the 1-keyword 1-click ad option in this section and leave the empirical investigation of the \( n \)-keyword cases in Section 4.

Let \( D(F) \) be the difference between the expected revenue from ad option and the expected revenue from keyword auctions, we then have

\[
D(F) = \left( C(0)\mathcal{N}[\zeta_1] - e^{-rT} F \mathcal{N}[\zeta_2] + e^{-rT} \mathbb{E}_0^C[C(T)] \right) \mathbb{P}(\mathbb{E}_0^C[C(T)] \geq F) \\
= \text{Discounted value of expected revenue from option if } \mathbb{E}_0^C[C(T)] \geq F \\
+ \left( C(0)\mathcal{N}[\zeta_1] - e^{-rT} F \mathcal{N}[\zeta_2] + e^{-rT} \mathbb{E}_0^C[C(T)] \right) \mathbb{P}(\mathbb{E}_0^C[C(T)] < F) \\
= \text{Discounted value of expected revenue from option if } \mathbb{E}_0^C[C(T)] < F \\
- \frac{e^{-rT} \mathbb{E}_0^C[C(T)]}{\mathbb{E}_0^C[C(T)]}.
\]

(8)

where \( \mathcal{N}[\cdot] \) represents the cumulative probability of a standard normal distribution.

Let us take a look at the boundary values first. If \( F = 0 \), the option price \( \pi \) achieves its maximum value \( e^{-rT} \mathbb{E}_0^C[C(T)] \); therefore, \( D(F) \to 0 \). If \( \pi = 0 \), the fixed CPC \( F \) is as large as possible, and \( \mathbb{P}(\mathbb{E}_0^C[C(T)] \geq F) \to 0 \) and \( D(F) \to 0 \). Since

\[
\ln\{C(T)/C(0)\} \sim N((r - \sigma^2/2)T, \sigma^2T),
\]

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we can have
\[ P(\mathbb{E}_0^Q[C(T)] \geq F) = P\left( C(0) \exp\{ (r - \frac{1}{2} \sigma^2)T \} \geq F \right) \]
\[ = P\left( \ln \left\{ \frac{F}{C(0)} \right\} - (r - \frac{1}{2} \sigma^2)T \leq 0 \right) \]
\[ = P\left( \ln \left\{ \frac{C(T)}{C(0)} \right\} - (r - \frac{1}{2} \sigma^2)T \leq \ln \left\{ \frac{F}{C(0)} \right\} + (r - \frac{1}{2} \sigma^2)T + \sigma W(T) \right) \]
\[ \approx \mathcal{N}\left[ \frac{1}{\sqrt{2 \pi}} \left( \ln \left\{ \frac{C(0)}{F} \right\} + (r - \frac{1}{2} \sigma^2)T \right) \right] = \mathcal{N}[\zeta_2]. \tag{9} \]

Substituting Eq. (9) into Eq. (8) gives
\[ D(F) = C(0) \mathcal{N}[\zeta_1] - e^{-rT} \mathbb{E}_0^Q(C(T)) \mathcal{N}[\zeta_2] \]
\[ \geq C(0) \mathcal{N}[\zeta_1] - e^{-rT} \mathbb{E}_0^Q(C(T)) \mathcal{N}[\zeta_1] \] (because \( \mathcal{N}[\zeta_1] \geq \mathcal{N}[\zeta_2] \))
\[ \geq C(0) \mathcal{N}[\zeta_1] - e^{-rT} C(0) e^{(r-\frac{1}{2} \sigma^2)T} \mathcal{N}[\zeta_1] \]
\[ = C(0) \mathcal{N}[\zeta_1] (1 - e^{-\frac{1}{2} \sigma^2 T}) > 0, \tag{10} \]
suggesting that the search engine can have an increase expected revenue if he/she sells the click via an option rather than through an auction. We then take the derivative of \( D(F) \) with respect to \( F \) and assign its value to zero:
\[ \frac{\partial D(F)}{\partial F} = C(0) \frac{\partial \mathcal{N}[\zeta_1]}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial F} - e^{-rT} \mathcal{N}[\zeta_2] \frac{\partial \mathcal{N}[\zeta_1]}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial F} + e^{-rT} P(\mathbb{E}_0^Q[C(T)] \geq F) \frac{\partial P(\mathbb{E}_0^Q[C(T)] \geq F)}{\partial F} - e^{-rT} P(\mathbb{E}_0^Q[C(T)] \geq F) = 0. \tag{11} \]

Since \( \frac{\partial \mathcal{N}(x)}{\partial x} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \), the following equation holds
\[ \frac{\partial \mathcal{N}[\zeta_2]}{\partial \zeta_2} / \frac{\partial \mathcal{N}[\zeta_1]}{\partial \zeta_1} = \exp\left\{ \frac{1}{2} (\zeta_1^2 - \zeta_2^2) \right\} = \frac{C(0)e^{-rT}}{F}. \tag{12} \]

Taking the derivative of \( \zeta_1 \) and \( \zeta_2 \) with respect to \( F \) gives
\[ \frac{\partial \zeta_1}{\partial F} = \frac{\partial}{\partial F} \left( \ln \left\{ \frac{C(0)}{F} \right\} + (r + \frac{1}{2} \sigma^2)T \right) = -\frac{1}{F \sigma \sqrt{T}}. \tag{13} \]
\[ \frac{\partial \zeta_2}{\partial F} = \frac{\partial \zeta_1}{\partial F} \frac{\partial \sigma}{\partial F} = -\frac{1}{F \sigma \sqrt{T}}. \tag{14} \]

By substituting Eqs. (12)-(14) into Eq. (11), we find that \( D(F) \) achieves its maximum or minimum value at \( F = \mathbb{E}_0^Q[C(T)] \). Further taking the second derivative of \( D(F) \) with respect to \( F = \mathbb{E}_0^Q[C(T)] \) gives
\[ \frac{\partial^2 D(F)}{\partial F^2} = \frac{\partial P(\mathbb{E}_0^Q[C(T)] \geq F)}{\partial F} = \frac{\partial \mathcal{N}[\zeta_2]}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial F} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \zeta_2^2} \frac{1}{F \sigma \sqrt{T}} < 0. \]

Therefore, if the fixed CPC is set as same as the estimated spot CPC on the contract expiration date (i.e., \( F = \mathbb{E}_0^Q[C(T)] \)), the search engine can have a maximized profit.
4. EXPERIMENTS

This section presents our evaluation results. We first describe the dataset, then conduct assumption and fairness tests, and finally investigate the option effect on the search engine's revenue.

4.1. Data and Experimental Design

The data used in the experiments are collected from Google AdWords by using its Traffic Estimation service [Google 2013]. When an advertiser submits his/her ad keywords, budget, and other settings such as keyword match type and targeted ad location, the Traffic Estimation will return a list of data values, including estimated CPC, clicks, global impressions, local impressions and position etc. Such values are recorded for the period from 26/11/2011 to 14/01/2013, for a total of 557 keywords across US and UK markets. Note that in the data 21 keywords have missing values and 115 keywords’s CPCs are all 0.

For each market, we split the data into 4 experimental groups and each group has one training, one development, and one test set, as illustrated in Table III. The training set is used to: (i) select the keywords with non-zero CPCs; (ii) test the statistical properties of the underlying dynamic and estimate the model parameters. We then price ad options and simulate the corresponding buying and selling transactions in the development set. Finally, the test set is used as the baseline to examine the priced ad options.

4.2. Parameter Estimation and Option Pricing

In the experiments, we use the method suggested by Paul Wilmott to estimate the GBM parameters [Wilmott 2006, Sec. 11.3]. For the ad keyword $K_i$, the volatility $\sigma_i$ is the sample standard deviation of change rates of log CPCs and the correlation is

$$\rho_{ij} = \frac{\sum_{k=1}^{m} (y_i(k) - \bar{y}_i)(y_j(k) - \bar{y}_j)}{\sqrt{\sum_{k=1}^{m} (y_i(k) - \bar{y}_i)^2} \sqrt{\sum_{k=1}^{m} (y_j(k) - \bar{y}_j)^2}},$$

where $m$ is the size of training data and $y_i(t_k)$ is the $k$th change rate of log CPCs. Figure 2 illustrates an empirical example, where the ad keywords are

$$\left\{ K_1 = \text{‘canon cameras’}, K_2 = \text{‘nikon camera’}, K_3 = \text{‘yahoo web hosting’} \right\}.$$

The model parameters are estimated as follows

$$\sigma = \begin{pmatrix} 0.2263 \\ 0.4521 \\ 0.2136 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1.0000 & 0.2341 & 0.0242 \\ 0.2341 & 1.0000 & -0.0540 \\ 0.0242 & -0.0540 & 1.0000 \end{pmatrix}.$$

Table III. Overview of experimental settings of data.

| Market | Group | Training set (31 days) | Dev&test set (31 days) |
|--------|-------|------------------------|------------------------|
| US     | 1     | 25/01/2012-24/02/2012   | 24/02/2012-25/03/2012   |
|        | 2     | 30/03/2012-29/04/2012   | 29/04/2012-31/05/2012   |
|        | 3     | 10/06/2012-12/07/2012   | 12/07/2012-17/08/2012   |
|        | 4     | 10/11/2012-11/12/2012   | 11/12/2012-10/01/2013   |
| UK     | 1     | 25/01/2012-24/02/2012   | 24/02/2012-25/03/2012   |
|        | 2     | 30/03/2012-29/04/2012   | 29/04/2012-31/05/2012   |
|        | 3     | 12/06/2012-13/07/2012   | 13/07/2012-19/08/2012   |
|        | 4     | 18/10/2012-22/11/2012   | 22/11/2012-24/12/2012   |

3The data is available at: http://www.computational-advertising.org [Yuan and Wang 2012].
A high contextual relevance of keywords normally means that they have a high substitu-ational degree to each other, such as ‘canon cameras’ and ‘nikon camera’, whose CPCs move in the same direction with correlation 0.2341. The other keyword ‘yahoo web hosting’ is contextually less relevant to the formers and also has very low price correlations to them. This example shows that the contextual relevance of keywords has an impact on their CPCs movement. Based on the estimated parameters, we draw a sample of simulated paths of 3-dimensional GBM in Figure 2(a) for 31 days (where the x-axis is expressed in terms of year value). Recall that the option payoff at any time \( t \) in the contract lifetime is \( \max\{C_1(t) - F_1, \ldots, C_n(t) - F_n, 0\} \). In Figure 2(b), we plot the price difference between the spot CPC and the fixed CPC of each targeted keyword (i.e., \( C_i(t) - F_i, i = 1, \ldots, n \)) and also indicate the corresponding option daily payoffs (shown by the cyan curve). It suggests that switching between keywords would help the advertiser to maximise the benefits of an ad option. Repeating the above simulations 50 times generates 50 simulated values of each keyword for each day, as shown in Figure 2(c). We also calculate 50 option payoffs and calculate their daily mean values to obtain the final option price accordingly in Figure 2(d).

To examine the fairness (i.e., no-arbitrage) of the calculated option price, we can construct a risk-less value difference process by delta hedging \( \partial V / \partial C \) (see Appendix B) and check if any arbitrage exists [Wilmott 2006]. The hedging delta of the 1-keyword 1-click ad option is

\[
\frac{\partial V}{\partial C} = \mathcal{N}\left( \frac{1}{\sigma \sqrt{T}} \left( \ln \left\{ \frac{C(0)}{F} \right\} + (r + \frac{\sigma^2}{2})T \right) \right). \tag{16}
\]

For the \( n \)-keyword 1-click option, the hedging delta of each keyword can be computed by the Monte Carlo method, i.e., \( \partial V / \partial C_i = \mathbb{E}[\partial V(T, C(T))/\partial C_i(T)] \). According to Appendix B, we can define the 31-day growth rate of the value difference process as \( \tilde{\gamma} = (\Pi(t_{31}) - \Pi(t_0)) / \Pi(t_0) \), and compare \( \tilde{\gamma} \) to the risk-less bank interest rate \( r = 5\% \) (equivalent to \( \tilde{r} = 4.12\% \) per 31 days return\(^4\)). The arbitrage detection criteria is

\[
\tilde{\gamma} \leq |\tilde{r} + \varepsilon| \Rightarrow \text{arb doesn’t exist} : \text{arb exists}, \tag{17}
\]

where the notation \( \varepsilon \) is the model variation threshold (e.g., we set \( \varepsilon = 5\% \) in our experiments). We define the identified arbitrage \( \alpha \) as the excess return between \( \tilde{\gamma} \) and \( |\tilde{r} + \varepsilon| \). Therefore, if arbitrage exists, we have

\[
\alpha = \begin{cases} 
\tilde{\gamma} - (\tilde{r} + \varepsilon), & \text{if } \tilde{\gamma} \geq \tilde{r} + \varepsilon, \\
\tilde{\gamma} - (\tilde{r} - \varepsilon), & \text{if } \tilde{\gamma} \leq \tilde{r} - \varepsilon.
\end{cases} \tag{18}
\]

Therefore, a positive \( \alpha \) means that the advertiser buys an option can obtain arbitrage while a negative \( \alpha \) indicates the case of making arbitrage by selling an option.

Table IV presents the overall results of our arbitrage test under the GBM, in which we generate 100 simulated paths for each keyword and examine the options using delta hedging. There are 99.76\% (1-keyword), 93.06\% (2-keyword) and 92.71\% (3-keyword) options fairly priced. Only a small number of options exhibits arbitrage and most of the mean arbitrage values lie within 5\%, such as shown in Figure 3. The existence of small arbitrage in our tests under the GBM dynamics may be due to two reasons. First, the stability of process simulations in both option pricing and arbitrage test. Second, the ad keywords are randomly selected for the 2 or 3-keyword options, so a significant difference of keywords CPCs generates a large variation of calculated option payoffs that triggers a certain arbitrage.

\(^4\)The relationship between the continuous compounding \( r \) and the return per 31 days \( \tilde{r} \) is: \( 1 + \tilde{r} = e^{r \times 30/365} \) [Hull 2009].
4.3. Model Validation and Robustness Test

We now examine the GBM assumptions and investigate if arbitrage exists when the keywords in an option do not follow the GBM.

4.3.1. Checking the Underlying GBM Assumptions. There are two GBM assumptions that need to be verified [Marathe and Ryan 2005]: (i) normality of change rates of log CPCs; and (ii) independence from previous data. Normality can be either graphically checked by histogram/Q-Q plot or statistically verified by the Shapiro-Wilk test.
Table IV. A test of arbitrage for ad options under the GBM: \( n \) is the number of keywords, \( N \) is the number of options priced in a group, \( P(\alpha) \) is percentage of options in a group found arbitrage, and the \( E[\alpha] \) is the average arbitrage value of the options found arbitrage, where the arbitrage \( \alpha \) is defined by Eq. (18) and the risk-less bank interest rate \( r = 5\% \).

| \( n \) | Group | US market | UK market |
|-----|-------|-----------|-----------|
|     |       | \( N \)  | \( P(\alpha) \) | \( E[\alpha] \) | \( N \)  | \( P(\alpha) \) | \( E[\alpha] \) |
| 1   | 1     | 94       | 0.00%     | 0.00%     | 76     | 0.00%     | 0.00%     |
|     | 2     | 64       | 0.00%     | 0.00%     | 45     | 0.00%     | 0.00%     |
|     | 3     | 94       | 1.06%     | 0.75%     | 87     | 0.00%     | 0.00%     |
|     | 4     | 112      | 0.89%     | -0.37%    | 53     | 0.00%     | 0.00%     |
| 2   | 1     | 47       | 4.26%     | 1.63%     | 38     | 0.00%     | 0.00%     |
|     | 2     | 32       | 9.38%     | 0.42%     | 22     | 4.55%     | 13.41%    |
|     | 3     | 47       | 4.26%     | 0.84%     | 43     | 4.65%     | 0.82%     |
|     | 4     | 56       | 5.36%     | 3.44%     | 26     | 23.08%    | -6.22%    |
| 3   | 1     | 31       | 0.00%     | 0.00%     | 25     | 4.00%     | 0.00%     |
|     | 2     | 21       | 4.76%     | -1.38%    | 15     | 0.00%     | 0.00%     |
|     | 3     | 31       | 0.00%     | 0.00%     | 29     | 3.45%     | -1.12%    |
|     | 4     | 37       | 10.81%    | 3.87%     | 17     | 35.29%    | -2.54%    |

Fig. 3. An empirical example of the arbitrage analysis under GBM dynamic for the US market.

test [Shapiro and Wilk 1965]. To examine independence, we employ the autocorrelation function (ACF) [Tsay 2005] and the Ljung-Box statistic [Ljung and Box 1978]. To illustrate the procedure, Figure 4 gives an example of the keyword ‘insurance’. Figure 4 (a)-(b) exhibit the movement of CPCs and log change rates while Figure 4 (c)-(d) show that the stated two assumptions are satisfied in this case.

The GBM assumptions were checked for the training data of all ad keywords. As shown in Figure 5, there are 14.25% and 17.20% of keywords in US and UK markets that satisfy the GBM, respectively. Thus about 15.73% of keywords that can be effectively priced into an option contract under the assumption of GBM dynamics. It is
worth mentioning that not all keywords follow the GBM. We use the GBM dynamics in our model mainly for the mathematical convenience; but nonetheless, we give our investigation on the arbitrage opportunities under non-GBM dynamics in order to test how robust of our pricing model is for non-GMB keywords.

4.3.2. Examining Arbitrage for Non-GBM Dynamics. Let us test several popular stochastic processes (together with the real data) to check the arbitrage from the options of non-GBM keywords. Table V shows the candidate dynamics (the CEV, the MRD, the CIR and HWV models). Each dynamic model represents a certain feature of time series data, such as mean-reverting, constant volatility and square root volatility [Hull 2009]. The arbitrage tests here are slightly different from that of GBM. We estimate the dynamic parameters from the data in the test sets instead of the learning sets and treat the real data as one single path of the dynamics. Therefore, the simulated

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data has the same drift, volatility and correlations as the real test data. Since the simul-
ation parameters are significantly different to the pricing parameters, we are able to examine the arbitrage multiple times when the real-world environment does not follow the GBM. Also for the candidate dynamics, several hypothesis tests have been employed to check if the simulated path and real data come from a same distribution. These tests include the Wilcoxon test [Wilcoxon 1945], Ansari-Bradley test [Mood et al. 1974] and Two-sample Kolmogorov-Smirnov test [Justel et al. 1997]. Figure 6 summarizes results of the dynamics’s fitness testing, where the y-axis represents the mean percentage of simulated paths not rejected by the hypothesis tests. Even though the three tests give different absolute percentages, the dynamics’ performance is similar, i.e. the CEV model has the best simulations of the real data, followed by the MRD model. The CIR and HWV models are very close.

Table VI presents the arbitrage testing results for non-GBM dynamics, where most of experimental groups exhibit arbitrage. The CEV model gives the best no-arbitrage performance, showing that 78.65% of CEV-based keywords can be fairly priced by using the GBM-based option pricing model. About 53.05% of CIR and about 43% of MRD or HWV based options have no arbitrage. For single-keyword options, the fairness percentage is more than 85% across all groups. However, this figure drops to around 38% for multi-keyword options (36.27% for 2-keyword options and 42% for 3-keyword options). For the identified arbitrage, many groups (especially single-keyword options) show small arbitrage values around 10% while arbitrage explodes in some groups.
Table V. Tested non-GBM dynamics: $k_i = 0.5$ while other parameters are learned from the training data.

| Dynamic                                      | Stochastic differential equation (SDE) |
|----------------------------------------------|----------------------------------------|
| Constant elasticity of variance (CEV) model  | $dC_i(t) = \mu_i C_i(t)dt + \sigma_i C_i(t)^{1/2} dW_i(t)$ |
| Mean-reverting drift (MRD) model             | $dC_i(t) = \mu_i - \kappa C_i(t) dt + \sigma_i C_i(t)^{1/2} dW_i(t)$ |
| Cox-Ingersoll-Ross (CIR) model               | $dC_i(t) = \mu_i - \kappa C_i(t) dt + (\sigma_i)^2 C_i(t)^{1/2} dW_i(t)$ |
| Hull-White/Vasicek (HWV) model               | $dC_i(t) = \kappa (C_i(t) - \theta) dt + \sigma_i dW_i(t)$ |

Table VI. Overview of delta hedging arbitrage testing for non-GBM dynamics: same notations as in Table IV.

| Market | n | Group | $N$ | real data + CEV simu | real data + MRD simu | real data + CIR simu | real data + HWV simu |
|--------|---|-------|-----|-----------------------|----------------------|----------------------|----------------------|
|        |   |       |     | $\pi$ (arb) | Mean arb | $\pi$ (arb) | Mean arb | $\pi$ (arb) | Mean arb | $\pi$ (arb) | Mean arb |
| US     | 1 | 1     | 74  | 2.70% | -1.97% | 8.11% | -0.38% | 75.68% | 1.94% | 56.76% | -0.01% |
|        | 2 | 2     | 55  | 0.00% | 0.00% | 1.82% | 1.80% | 16.36% | 1.86% | 9.09% | -0.93% |
|        | 3 | 3     | 77  | 9.09% | -10.26% | 5.19% | -7.93% | 42.86% | 0.85% | 23.38% | -1.51% |
|        | 4 | 4     | 108 | 3.70% | 1.38% | 2.78% | 4.27% | 7.41% | 3.99% | 8.33% | 2.84% |
|        |   |       |     | 21.32% | 18.06% | 13.26% | 10.54% | 7.88% | 5.96% | 4.24% | 2.84% |
|        | 2 | 1     | 37  | 24.32% | 3.83% | 81.08% | -4.09% | 97.30% | -16.24% | 97.30% | -13.24% |
|        | 2 | 2     | 27  | 37.04% | 6.01% | 70.37% | 5.36% | 85.19% | 11.01% | 85.19% | 10.51% |
|        | 3 | 3     | 38  | 31.58% | 5.97% | 31.58% | -0.41% | 73.68% | -6.91% | 57.89% | -5.96% |
|        | 4 | 4     | 54  | 29.63% | 5.95% | 81.48% | 6.61% | 94.44% | 16.78% | 94.44% | 16.25% |
|        |   |       |     | 24.62% | 20.34% | 18.00% | 15.83% | 10.89% | 8.84% | 6.85% | 5.51% |
|        | 3 | 1     | 24  | 45.83% | 1.36% | 79.17% | -4.04% | 100.00% | -19.99% | 100.00% | -17.44% |
|        | 2 | 2     | 18  | 11.11% | -2.00% | 22.22% | -5.11% | 55.56% | -4.39% | 72.22% | -2.22% |
|        | 3 | 3     | 25  | 24.00% | 7.01% | 32.00% | -3.71% | 84.00% | -11.14% | 76.00% | -10.26% |
|        | 4 | 4     | 36  | 16.67% | 3.91% | 30.56% | 2.66% | 83.33% | 2.73% | 88.89% | 3.38% |
|        |   |       |     | 16.59% | 13.38% | 12.00% | 9.85% | 8.89% | 7.85% | 6.85% | 5.51% |
| UK     | 1 | 1     | 58  | 0.00% | 0.00% | 0.00% | 0.00% | 74.14% | 1.95% | 55.17% | -1.60% |
|        | 2 | 2     | 35  | 0.00% | 0.00% | 2.86% | 1.51% | 22.86% | 1.65% | 14.29% | 2.29% |
|        | 3 | 3     | 72  | 5.56% | -5.78% | 1.39% | -10.38% | 29.17% | 1.03% | 18.06% | -0.32% |
|        | 4 | 4     | 50  | 4.00% | 5.55% | 6.00% | 4.47% | 10.00% | 4.56% | 8.00% | 3.45% |
|        |   |       |     | 4.19% | 3.38% | 3.00% | 2.66% | 1.95% | 1.60% | 1.60% | 1.60% |
|        | 2 | 1     | 29  | 37.93% | -0.76% | 62.07% | -5.52% | 89.66% | -14.55% | 72.41% | -11.39% |
|        | 2 | 2     | 17  | 47.06% | 2.18% | 92.35% | 5.00% | 100.00% | 9.87% | 100.00% | 8.62% |
|        | 3 | 3     | 36  | 19.44% | 2.71% | 33.33% | -1.78% | 75.00% | -5.24% | 61.11% | -3.58% |
|        | 4 | 4     | 25  | 64.00% | 6.71% | 96.00% | 9.01% | 92.00% | 21.17% | 92.00% | 20.04% |
|        |   |       |     | 36.04% | 33.04% | 30.04% | 25.04% | 19.04% | 16.04% | 14.04% | 12.04% |
|        | 3 | 1     | 19  | 26.32% | -1.56% | 84.21% | -5.21% | 100.00% | -16.33% | 78.90% | -16.34% |
|        | 2 | 2     | 11  | 18.18% | 0.40% | 18.18% | -1.09% | 63.64% | -1.28% | 63.64% | -1.05% |
|        | 3 | 3     | 24  | 16.67% | 3.45% | 25.00% | -1.61% | 79.17% | -9.14% | 66.67% | -9.23% |
|        | 4 | 4     | 16  | 37.50% | 7.83% | 43.75% | 7.37% | 81.25% | 0.68% | 81.25% | 8.64% |
In summary, Tables IV and VI have illustrated that our option pricing model is effective and robust for the real sponsored search data. As in Figure 7, when the keywords satisfy the GBM (15.73%), the pricing model ensures that 95.17% options are fairly priced under 5% arbitrage precision. For the non-GBM keywords, the best CEV model gives 78.65% fairness while the worst CIR model is with 31.97%. Overall, the best expected fairness of option pricing for all keywords is 81.25% while the worst is 41.91%. Also, the increase of the number of ad keywords in an ad option increases the likelihood of arbitrage. This is confirmed by the fact that expected fairness drops from 86.83% (99.76% GBM and 83.60% non-GBM for single-keyword options) to 43.69% (2-keyword options) and 53.39% (3-keyword options), respectively.

4.4. Effects on Search Engine’s Revenue

Let’s start with the case of 1-keyword options. The example of keyword ‘equity loans’ in Figure 8(a) illustrates (other keywords exhibit the similar pattern) the conclusions from our theoretical analysis in Section 3.3 that (i) the revenue difference between option and auction is always positive and (ii) that when the fixed CPC $F = \mathbb{E}_i^0(C(T)) = 4.5022$, the revenue difference $D(F)$ achieves its maximum (the corresponding option price $\pi = 0.1088$) and the two boundary values are approximately zero.

We further examine non-GBM cases. Figure 8(b)-(e) shows that when the fixed CPC is close to zero, the revenue difference $D(F) \to 0$. This is because when the fixed CPC approximates zero, it is almost certain that the option will be used in the contract period. As such, the only income for the keyword is from the option price, which in this case is close to the CPC in the auction market (discounted back to $t=0$). On the other hand, if the fixed CPC is very high, it is almost certain that the option won’t be used. In this case, the option price $\pi \to 0$ and the probability of exercising the option $\mathbb{P}(\mathbb{E}_i^0(C(T)) \geq F) \to 0$. Hence, $D(F)$ would be zero. However, under the non-GBM
Fig. 9. An empirical example of analysing the search engine’s revenue for the keywords ‘iphone4’ and ‘dr martens’, where $\rho = 0.0259$.

Fig. 10. An empirical example of analysing the search engine’s revenue for the keywords ‘non profit debt consolidation’ and ‘canon 5d’, where $\rho = 0.2247$.

dynamics, the point $F = \mathbb{E}_t^Q [C(T)]$ is not the optimal value that gives the maximum $D(F)$, which indicates that arbitrage may occur.

Next, Figure 9 illustrates the case where we have 2 candidate keywords: ‘iphone4’ and ‘dr martens’, where $r = 5\%$ and $\rho = 0.0259$. In Figure 9(a), we see that the higher the fixed CPCs the lower is the option price. This property is the same as for the single-keyword options. Also, the calculated option price achieves maximum when all the fixed CPCs are zeros. Figure 9(b) then shows the revenue difference curve of the search engine, where the red star represents the value when $F_1 = \mathbb{E}_t^Q [C_1(T)]$ and $F_2 = \mathbb{E}_t^Q [C_2(T)]$. The expected revenue differences are all above zero, showing that this 2-keyword ad option is beneficial to the search engine’s revenue. However, an interesting point to discuss is that the red star point is not the maximum difference revenue, which is different from single-keyword options. This may be due to the fact that the underlying CPCs move in a correlated manner and the advertiser switches his/her exercising from one to another. The revenues’ difference curve in Figure 9(b) is very smooth while Figure 10(b) shows a bit volatile pattern because the underlying correlation increases. Above all, the properties of the revenue difference are similar to those of single-keyword options and they are all positive.
It would be impossible to graphically examine the revenue difference for higher dimensional ad options (i.e., \( n \geq 3 \)). However, based on the earlier discussions, we can summarize two properties. First, there are boundary values of the revenue differences. If every \( F_i \to 0 \), \( D(F) \to 0 \); if every \( F_i \to \infty \), \( D(F) \to 0 \). Second, there exists a maximum revenue difference value even though this may not at the point \( F_i = E_Q[C_i(T)] \).

Overall, we are able to say that a proper setting of fixed CPCs by a search engine can increase the ad revenue compared to keyword auctions.

5. CONCLUDING REMARKS

In this paper, we propose a novel monetization mechanism for sponsored search that benefits both advertiser and search engine. On the one hand, an advertiser is able to secure advertising service delivery in the future and can be released from auction campaigns, thereby reducing the uncertainty in advertising cost. On the other hand, a search engine benefits by selling future ad slots in advance and receiving a stable and increased expected revenue over time. In addition, the search engine may also increase the advertisers' loyalty through contractual relationships, which has the potential to boost the revenue in the long-term period. Hence, we believe the proposed option mechanism is a good complement to keyword auctions in sponsored search.

There are two major limitations of the development in this paper. First, like other methods based on the (multivariate) GBM, the candidate keywords' prices may not follow it exactly. Some price features, such as jumps and volatility clustering etc., cannot be captured effectively [Samuelson 1965; Marathe and Ryan 2005]. However, GBM can also be employed for pricing ad options in sponsored search as our experimental results show that it is reasonably robust in terms of arbitrage tracking. Second, some other model assumptions may not be valid in the real-world sponsored search environment, such as infinitely divisible clicks, continuous-time perfect delta hedging, constant bank interest rate etc.

Our work leaves several directions for future research. First, to address the discussed limitations, the stochastic processes tailored to some specific keywords are worth studying, such as the jump-diffusion model [Kou 2002] and the multivariate variance Gamma model [Luciano and Schoutens 2007]. Second, the optimal dynamic pricing and allocation of ad options may further consolidate our work as we can discuss how to manipulate the limited inventories in front of the uncertain demand [Gallego and van Ryzin 1994]. In addition, to consider the competitions among advertisers who have similar needs, pricing ad options from a game-theoretic perspective may be of interest [Hucki and Kolokoltsov 2007].

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ACM Journal Name, Vol. x, No. x, Article x, Publication date: January 2014.
APPENDICES

A. PROOF OF THE NO-EARLY EXERCISE PROPERTY FOR THE AD OPTION

Eq. (3) can be rewritten as \( \Phi(x) = \max\{x - f, 0\} \), where \( x' = [x_1, \ldots, x_n] \) and \( f' = [f_1, \ldots, f_n] \). It is not difficult to find that \( \Phi(x) \) is multivariate convex. Let \( 0 \leq \lambda < 1 \) and let \( y' = [y_1, \ldots, y_n] \), if the elements of vector \( a = y - x \) are all non-negative, we have

\[
\Phi(\lambda x + (1 - \lambda)y) \leq \lambda \Phi(x) + (1 - \lambda)\Phi(y).
\]

If taking \( y' = (0, \ldots, 0) \), and using the fact that \( \Phi(0) = 0 \), we obtain

\[
\Phi(\lambda x) \leq \lambda \Phi(x), \quad \text{for all } x_i \geq 0, \ 0 \leq \lambda \leq 1.
\]

For \( 0 \leq s \leq t \leq T \), we have \( 0 \leq e^{-r(t-s)} \leq 1 \), and then

\[
\begin{align*}
\mathbb{E}_s^Q[e^{-r(t-s)}\Phi(X(t))] &\geq \mathbb{E}_s^Q[\Phi(e^{-r(t-s)}X(t))] \\
&\geq \Phi(\mathbb{E}_s^Q[e^{-r(t-s)}X(t)]) \quad \text{(by the Jensen's Inequality)} \\
&= \Phi(e^{-rs}\mathbb{E}_s^Q[e^{-r}X(t)]),
\end{align*}
\]

where \( \mathbb{E}_s^Q[\cdot] \) is the conditional expectation with respect to time \( s \) under the risk-neutral probability \( Q \). As \( e^{-r}X(t) \) is a martingale under \( Q \) [Björk 2009], we have

\[
\Phi(e^{-rs}\mathbb{E}_s^Q[e^{-r}X(t)]) = \Phi(e^{-rs}e^{-rs}X(s)) = \Phi(X(s)).
\]

Therefore, we obtain

\[
\mathbb{E}_s^Q[e^{-r(t-s)}\Phi(X(t))] \geq \Phi(X(s)),
\]

showing that \( e^{-r(t-s)}\Phi(X(t)) \) is a sub-martingale under \( \mathbb{Q} \). This tells that we can price the proposed ad option as same as its European structure, focusing on the payoff on the contract expiration date. For the detailed definitions of martingale and sub-martingale, see [Björk 2009].

B. DERIVATION OF THE AD OPTION PRICING FORMULA

As the proposed ad option complements the existing keyword auctions, there may exist a situation that some advertisers make guaranteed profits from the difference of costs between the option and auctions without taking any risk. This situation is called the arbitrage opportunity [Varian 1987]. Therefore, we must fairly evaluate the option so that arbitrage is eliminated.

We consider the advertiser buys a \( n \)-keyword \( m \)-click ad option. At time \( t \), the difference between the option value and the market value of candidate keywords can be expressed as

\[
\Pi(t) = V(t, C(t); F, T, m) - \sum_{i=1}^{n} \psi_i(t)C_i(t), \quad (19)
\]

where \( \psi_i(t) \) represents the number of clicks needed for the keyword \( K_i \) such that \( \sum_i \psi_i(t) = m \). Here we call \( \Pi(t) \) as the value difference process. As in Eq. (3) we consider the value of a \( n \)-keyword \( m \)-click option as the sum of \( m \) independent \( n \)-keyword 1-click options, for the mathematical convenience, we can rewrite Eq. (19) as follows

\[
\Pi(t) = m\left(V(t, C(t); F, T, 1) - \sum_{i=1}^{n} \Delta_i C_i(t)\right), \quad (20)
\]

where \( \Delta_i \) represents the probability that the click goes for the keyword \( K_i \) and \( \sum_i \Delta_i = 1 \). The changes of \( \Pi \) over a very short period of time \( dt \) is

\[
d\Pi(t) = m\left(\frac{\partial V}{\partial t} dt + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij} C_i C_j \frac{\partial^2 V}{\partial C_i \partial C_j} dt + \sum_{i=1}^{n} \frac{\partial V}{\partial C_i} dC_i - \sum_{i=1}^{n} \Delta_i dC_i\right). \quad (21)
\]
We can remove the uncertain components in $d\Pi(t)$ if choosing $\Delta_i = \partial V / \partial C_i$. This is called delta hedging in financial option pricing [Wilmott 2006]. Therefore, $\Pi(t)$ is now a risk-less process over time

$$d\Pi(t) = m \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij} C_i C_j \frac{\partial^2 V}{\partial C_i \partial C_j} \right) dt. \tag{22}$$

Consider if the advertiser has no initial fund. He/she borrows the money from a bank at the risk-less bank interest rate $r$. Then the interest of this debt is

$$d\Pi(t) = r\Pi(t)dt = rm \left( V - \sum_{i=1}^{n} \frac{\partial V}{\partial C_i} C_i \right) dt. \tag{23}$$

Eqs. (22) and (23) should be equal otherwise arbitrage exists. If the risk-less growth rate of the value difference process is larger than the risk-less bank interest rate, the advertiser can obtain arbitrage by: (i) borrowing the money from bank at interest rate $r$ to buy an ad option first; (ii) selling the ad option later to repay the bank interest. In the case when the risk-less growth rate of the value difference process is smaller than the risk-less bank interest rate, the advertiser can obtain the risk-less surplus by: (i) selling short an ad option first and deposit the revenue into bank; (ii) using the deposit money to buy the clicks of underlying keywords later. In either case, the advertiser can finally receive a risk-less surplus; therefore, arbitrage exists.

Solving Eqs. (22)-(23) gives a parabolic partial differential equation (PDE) for the no-arbitrage equilibrium as follows

$$\frac{\partial V}{\partial t} + r \sum_{i=1}^{n} \frac{\partial V}{\partial C_i} C_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial C_i \partial C_j} \sigma_i \sigma_j \rho_{ij} C_i C_j - rV = 0.$$

The PDE satisfies the boundary condition in Eq. (3). We can employing the multidimensional Feynman-Kac stochastic representation [Björk 2009] to obtain the solution

$$V(t, C(t); F, T, m) = e^{-r(T-t)} \mathbb{E}^Q_t[\Phi(C(T))],$$

where $\mathbb{E}^Q_t[\cdot]$ is the conditional expectation with respect to time $t$ under the risk-neutral probability $Q$. Under this, the process $C_i(t)$ is rewritten as

$$dC_i(t) = rC_i(t)dt + \sigma_i C_i(t)dW^Q_i(t),$$

where $W^Q_i(t)$ is the standard Brownian motion under $Q$. Therefore, the option price $\pi$ can be calculated by

$$\pi = V(0, C(0); F, T, m) = mV(0, C(0); F, T, 1) = me^{-rT} \mathbb{E}^Q_0[\Phi(C(T))].$$

C. OPTION PRICING FORMULAS FOR SPECIAL CASES

If there is only one candidate keyword (i.e. $n = 1$), Eq. (6) is equivalent to the Black-Scholes-Merton (BSM) pricing formula for an European call option [Black and Scholes 1973; Merton 1973], so we have

$$\pi = mC(0) \mathcal{N}[\zeta_1] - mFe^{-rT} \mathcal{N}[\zeta_2], \tag{24}$$

where $\zeta_1 = \frac{1}{\sigma \sqrt{T}} \left( \ln \{C(0)/F\} + (r + \frac{\sigma^2}{2})T \right)$ and $\zeta_2 = \zeta_1 - \sigma \sqrt{T}$.

If there are two candidate keywords (i.e. $n = 2$), Eq. (6) contains a bivariate normal distribution. We can use the formula from a dual-strike European call option [Zhang...
1998] to calculate the option price, given by

\[ \pi = mC_1(0) \int_{-\infty}^{\zeta_1} f(u) \mathcal{N} \left[ \frac{q_1(u) + \rho u}{\sqrt{1 - \rho^2}} \right] du 
+ mC_2(0) \int_{-\infty}^{\zeta_2} f(v) \mathcal{N} \left[ \frac{q_2(v) + \rho v}{\sqrt{1 - \rho^2}} \right] dv 
- me^{-rT} \left( F_1 \int_{-\infty}^{\zeta_1} f(u) \mathcal{N} \left[ \frac{q_1(u) + \rho u}{\sqrt{1 - \rho^2}} \right] du 
+ F_2 \int_{-\infty}^{\zeta_2} f(v) \mathcal{N} \left[ \frac{q_2(v) + \rho v}{\sqrt{1 - \rho^2}} \right] dv \right), \tag{25} \]

where

\[ q_1(u) = \frac{1}{\sigma_2 \sqrt{T}} \left( \ln \left( \frac{F_2 - F_1 + C_1(0)e^{(r - \frac{1}{2}\sigma_1^2)T - u\sigma_1\sqrt{T}}}{C_2(0)} \right) - (r - \frac{1}{2}\sigma_2^2)T \right), \]

\[ q_2(u) = \frac{1}{\sigma_1 \sqrt{T}} \left( \ln \left( \frac{F_1 - F_2 + C_2(0)e^{(r - \frac{1}{2}\sigma_2^2)T - v\sigma_2\sqrt{T}}}{C_1(0)} \right) - (r - \frac{1}{2}\sigma_1^2)T \right), \]

\[ \zeta_1 = \frac{1}{\sigma_1 \sqrt{T}} \left( \ln \{C_1(0)/F_1\} + (r - \frac{1}{2}\sigma_1^2)T \right), \]

\[ \zeta_2 = \frac{1}{\sigma_2 \sqrt{T}} \left( \ln \{C_2(0)/F_2\} + (r - \frac{1}{2}\sigma_2^2)T \right). \]

Eq. (25) appears somewhat complicated and we can further approximate the option price by using some polynomial functions, see [Zhang 1998] for the detailed discussion.