Tighter Monogamy and Polygamy Relations of Quantum Entanglement in Multi-qubit Systems

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Abstract
We investigate the monogamy relations related to the concurrence, the entanglement of formation, convex-roof extended negativity, Tsallis-q entanglement and Rényi-α entanglement, the polygamy relations related to the entanglement of formation, Tsallis-q entanglement and Rényi-α entanglement. Monogamy and polygamy inequalities are obtained for arbitrary multipartite qubit systems, which are proved to be tighter than the existing ones. Detailed examples are presented.

Keywords Monogamy relations · Polygamy relations · Concurrence · Entanglement of formation · Negativity · Tsallis-q entanglement · Rényi-α entanglement

1 Introduction
Quantum entanglement is an essential feature of quantum mechanics, which distinguishes the quantum theory from the classical theory [3–7]. The quantification of quantum entanglement is a central issue in quantum information theory [1, 2]. As one of the fundamental differences between quantum entanglement and classical correlation, a key property of entanglement is that a quantum system entangled with one of the other systems limits its entanglement with the remaining ones. The monogamy of entanglement (MoE) gives rise to the structures of entanglement in the multipartite setting. Monogamy is also an essential feature allowing for security in quantum key distribution [8].

For a tripartite quantum state $\rho_{ABC}$, MoE is characterized as $\varepsilon(\rho_{A|BC}) \geq \varepsilon(\rho_{AB}) + \varepsilon(\rho_{AC})$, where $\rho_{AB} = \text{Tr}_C(\rho_{ABC})$ and $\rho_{AC} = \text{Tr}_B(\rho_{ABC})$ are reduced density matrices, and $\varepsilon$ is an entanglement measure. The well-known concurrence introduced in [9, 10] has an explicit expression for arbitrary two-qubit states. Based on this expression, Coffman, Kundu and Wootters [11] derived the famous genuine three-qubit entanglement monotone, three tangle, and conjectured an inequality for concurrence which describes the
monogamy feature of entanglement distribution in a multipartite quantum system. However, such monogamy relations are not always satisfied by any entanglement measures. It has been shown that the squared concurrence $C^2$, and the squared entanglement of formation $E^2$ do satisfies the monogamy relations, while the squared convex-roof extended negativity (CREN) $\tilde{N}^2$ satisfies the monogamy relations for multiqubit states [12–16].

Another important concept is the assisted entanglement, which is the amount dual to the bipartite entanglement measure. It has a dually monogamous property in multipartite quantum systems and gives rise to polygamy relations. For a tripartite state $\rho_{ABC}$, the usual polygamy relation is of the form,

$$\varepsilon_a(\rho_{AB}) \leq \varepsilon_a(\rho_A) + \varepsilon_a(\rho_{AC}),$$

where $\varepsilon_a$ is the corresponding measure of assisted entanglement associated to $\varepsilon$. Such polygamy inequality has been deeply investigated in recent years, and was generalized to multiqubit systems and classes of higher dimensional quantum systems [17–23, 28].

Some monogamy and polygamy inequalities related to the $\alpha$th power of entanglement measures have been also proposed. In [24–27], it is proved that the $\alpha$th power of concurrence and CREN satisfy the monogamy inequalities in multiqubit systems for $\alpha \geq 2$. It has also been shown that the $\alpha$th power of EoF satisfies monogamy relations when $\alpha \geq \sqrt{2}$. Besides, the $\alpha$th power of Tsallis-q entanglement and Rényi-$\alpha$ entanglement satisfy monogamy relations when $\alpha \geq 1$ for some cases [17, 24–27, 29]. The corresponding polygamy relations have also been established [19–21, 23, 30, 31].

In this paper, we present the monogamy inequalities in terms of the concurrence $C$, entanglement of formation $E$, convex-roof extended negativity $\tilde{N}$, Tsallis-q entanglement $T_q$, and Rényi-$\alpha$ entanglement $E_{\alpha}$. These inequalities are proved to be tighter than the existing ones.

## 2 Tighter Monogamy Relations for Concurrence

Let $\mathcal{H}_X$ denote a finite-dimensional complex vector space associated to a quantum subsystem $X$. Given a bipartite pure state $|\phi\rangle_{AB}$ in Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the concurrence is given by

$$C(|\phi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]},$$

where $\rho_A = \text{Tr}(|\phi\rangle_{AB}\langle\phi|)$ is the reduced density matrix obtained by tracing over the subsystem $B$ [32–34]. The concurrence for a bipartite mixed state $\rho_{AB}$ is defined by the convex roof extension,

$$C(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle),$$

where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum_i p_i |\phi_i\rangle \langle\phi_i|$, with $p_i \geq 0$, $\sum_i p_i = 1$, and $|\phi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. For any $N$-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$ in an $N$-qubit system $\mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, the concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{AB_1\cdots B_{N-1}}$ viewed as a bipartite state under the partition $A$ and $B_1, B_2, \cdots, B_{N-1}$, satisfies

$$C^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq C^\alpha(\rho_{AB_1}) + C^\alpha(\rho_{AB_2}) + \cdots + C^\alpha(\rho_{AB_{N-1}}).$$
for $\alpha \geq 2$, where $\rho_{AB_i} = \text{Tr}_{B_1 \cdots B_{i-1} B_{i+1} \cdots B_{N-1}} (\rho_{A B_1 \cdots B_{N-1}})$ [24]. The relation (3) is improved for $\alpha \geq 2$ [25]. If $C(\rho_{AB_i}) \geq C(\rho_{A|B_{i+1} \cdots B_{N-1}})$ for $i = 1, 2, \cdots, m$, and $C(\rho_{AB_i}) \leq C(\rho_{A|B_{j+1} \cdots B_{N-1}})$ for $j = m+1, \cdots, N-2$, $1 \leq m \leq N-3$, $N \geq 4$, then

\[
C^\alpha(\rho_{A|B_1 \cdots B_{N-1}}) \geq C^\alpha(\rho_{AB_1}) + (2^{\frac{\alpha}{2}} - 1)C^\alpha(\rho_{AB_2}) + \cdots + C^\alpha(\rho_{AB_N})
\]

\[
+ (2^{\frac{\alpha}{2}} - 1)^{m+1}C^\alpha(\rho_{AB_{m+1}}) + \cdots + C^\alpha(\rho_{AB_{N-2}})
\]

\[
+ (2^{\frac{\alpha}{2}} - 1)^m C^\alpha(\rho_{AB_{N-1}}).
\]

(4)

The relation (4) is further improved for $\alpha \geq 2$ as

\[
C^\alpha(\rho_{A|B_1 \cdots B_{N-1}}) \geq C^\alpha(\rho_{AB_1}) + (((1 + k)^{\frac{\alpha}{2}} - 1)\langle k^{\frac{\alpha}{2}} \rangle C^\alpha(\rho_{AB_2}) + \cdots + (((1 + k)^{\frac{\alpha}{2}} - 1)\langle k^{\frac{\alpha}{2}} \rangle)^m C^\alpha(\rho_{AB_{m+1}}) + \cdots + C^\alpha(\rho_{AB_{N-2}})
\]

\[
+ (((1 + k)^{\frac{\alpha}{2}} - 1)\langle k^{\frac{\alpha}{2}} \rangle)^m C^\alpha(\rho_{AB_{N-1}}).
\]

(5)

with $kC^2(\rho_{AB_i}) \geq C^2(\rho_{A|B_{i+1} \cdots B_{N-1}})$ for $i = 1, 2, \cdots, m$, and $C^2(\rho_{AB_i}) \leq kC^2(\rho_{A|B_{i+1} \cdots B_{N-1}})$ for $j = m+1, \cdots, N-2$, $1 \leq m \leq N-3$, $N \geq 4$, and $0 < k \leq 1$ [27].

In the following, we show that these monogamy relations for concurrence can become even tighter under some conditions. For convenience, we denote by $C_{AB_j} = C(\rho_{AB_j})$ for $j = 1, 2, \cdots, N-1$, and $C_{A|B_1 B_2 \cdots B_{N-1}} = C(\rho_{A|B_1 B_2 \cdots B_{N-1}})$. We first introduce the following lemma.

**Lemma 2.1** For any non-negative real number $x$ and $y$ satisfying $0 \leq y \leq x$, and real numbers $t$ and $s$ satisfying $t \geq 1$, $0 \leq s \leq 1$, we have

\[
(1 + x)^t - x^t \geq (1 + y)^t - y^t,
\]

(6)

\[
(1 + x)^s - x^s \leq (1 + y)^t - y^s.
\]

(7)

**Proof** Let $g(x, t) = (1 + x)^t - x^t$. Since $\frac{dg(x, t)}{dx} = t[(1 + x)^{t-1} - x^{t-1}] \geq 0$, the function $g(x, t)$ is increasing with respect to $x$. As $y \leq x$, $g(y, t) \leq g(x, t)$, we get the inequality (6). Similar to the proof of inequality (6), we can obtain the inequality (7).

For any $2 \otimes 2 \otimes 2^{n-2}$ mixed state $\rho_{ABC} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, we have from relation (3)

\[
C^2_{A|B C} \geq C^2_{A B} + C^2_{A C}.
\]

Therefore, there exists $\mu \geq 1$ such that

\[
C^2_{A|B C} \geq C^2_{A B} + \mu C^2_{A C}.
\]

(8)

**Lemma 2.2** Let $l \geq 1$ be a real number. For any $2 \otimes 2 \otimes 2^{n-2}$ mixed state $\rho_{ABC} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, if $C^2_{A B} \geq 1C^2_{A C}$, we have

\[
C^\alpha_{A|B C} \geq C^\alpha_{A B} + ((\mu + l)^{\frac{\alpha}{2}} - l^{\frac{\alpha}{2}})C^\alpha_{A C},
\]

(9)

for all $\alpha \geq 2$.  

\[\square\]
Proof By straightforward calculation, we have
\[
C_{A|BC}^\alpha = (C_{A|BC}^2)^{\frac{\alpha}{2}} \geq (C_{AB}^2 + \mu C_{AC}^2)^{\frac{\alpha}{2}}
\]
\[
= \mu^{\frac{\alpha}{2}} C_{AC}^\alpha \left( (\mu^{-1}(C_{AB}^2/C_{AC}^2) + 1)^{\frac{\alpha}{2}} - (\mu^{-1}(C_{AB}^2/C_{AC}^2) - 1)^{\frac{\alpha}{2}} \right) + C_{AB}^\alpha
\]
\[
\geq \mu^{\frac{\alpha}{2}} C_{AC}^\alpha \left( (\mu^{-1}l + 1)^{\frac{\alpha}{2}} - (\mu^{-1}l - 1)^{\frac{\alpha}{2}} \right) + C_{AB}^\alpha
\]
\[
= [(l + \mu)^{\frac{\alpha}{2}} - (l)^{\frac{\alpha}{2}}] C_{AC}^\alpha + C_{AB}^\alpha,
\] (10)
where the second inequality is due to Lemma 2.1. We can also see that if \(C_{AB} = 0\), then \(C_{AC} = 0\), and the lower bound becomes trivially zero. \(\square\)

For multiqubit systems, we have the following theorems.

**Theorem 2.3** Let \(\mu_r \geq 1\) and \(l_r \geq 1\) be real numbers, \(1 \leq r \leq N - 2\). For any \(N\)-qubit mixed state \(\rho_{AB_1 \ldots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}\), if \(C_{AB_i}^2 \geq l_i C_{A|B_{i+1} \ldots B_{N-1}}^2\), \(C_{A|B_i \ldots B_{N-1}}^2 \geq C_{AB_i}^2 + \mu_i C_{A|B_{i+1} \ldots B_{N-1}}^2\) for \(i = 1, 2, \ldots, m\), and \(C_{A|B_{j+1} \ldots B_{N-1}}^2 \geq l_j C_{AB_j}^2\), we have
\[
C_{A|B_1 \ldots B_{N-1}}^\alpha \geq C_{AB_1}^\alpha + K_1 C_{AB_2}^\alpha + \cdots + K_{m-1} C_{AB_m}^\alpha
\]
\[
+ K_1 \cdots K_m (K_{m+1} C_{AB_{m+1}}^\alpha + \cdots + K_{N-2} C_{AB_{N-2}}^\alpha)
\]
\[
+ K_1 \cdots K_m C_{A|B_{m+1} \ldots B_{N-1}}^\alpha
\] (11)
for all \(\alpha \geq 2\), where \(K_r = (\mu_r + l_r)^{\frac{\alpha}{2}} - l_r^{\frac{\alpha}{2}}\) with \(1 \leq r \leq N - 2\).

Proof From Lemma 2.2, we have
\[
C_{A|B_1 \ldots B_{N-1}}^\alpha \geq C_{AB_1}^\alpha + K_1 C_{A|B_2 \ldots B_{N-1}}^\alpha
\]
\[
\geq C_{AB_1}^\alpha + K_1 C_{AB_2}^\alpha + K_1 K_2 C_{A|B_3 \ldots B_{N-1}}^\alpha \geq \cdots
\]
\[
\geq C_{AB_1}^\alpha + K_1 C_{AB_2}^\alpha + \cdots + K_1 \cdots K_{m-1} C_{AB_m}^\alpha
\]
\[
+ K_1 \cdots K_m C_{A|B_{m+1} \ldots B_{N-1}}^\alpha
\] (12)
Since \(C_{A|B_{j+1} \ldots B_{N-1}}^2 \geq l_j C_{AB_j}^2\), \(C_{A|B_j \ldots B_{N-1}}^2 \geq \mu_j C_{AB_j}^2 + C_{A|B_{j+1} \ldots B_{N-1}}^2\) for \(j = m + 1, \ldots, N - 2\), we get
\[
C_{A|B_{m+1} \ldots B_{N-1}}^\alpha \geq K_{m+1} C_{AB_{m+1}}^\alpha + C_{A|B_{m+2} \ldots B_{N-1}}^\alpha
\]
\[
\geq K_{m+1} C_{AB_{m+1}}^\alpha + K_{m+2} C_{AB_{m+2}}^\alpha + C_{A|B_{m+3} \ldots B_{N-1}}^\alpha \geq \cdots
\]
\[
\geq K_{m+1} C_{AB_{m+1}}^\alpha + K_{m+2} C_{AB_{m+2}}^\alpha + \cdots + \cdots
\]
\[
+ K_{N-2} C_{AB_{N-2}}^\alpha + C_{A|B_{N-1}}^\alpha
\] (13)
Combining (12) and (13), we complete the proof. \(\square\)

An immediate corollary of Theorem 2.3, we have in particular,

**Theorem 2.4** Let \(\mu_r \geq 1\) and \(l_r \geq 1\) be real numbers, \(1 \leq r \leq N - 2\). For any \(N\)-qubit mixed state \(\rho_{AB_1 \ldots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}\), if \(C_{AB_i}^2 \geq l_i C_{A|B_{i+1} \ldots B_{N-1}}^2\), \(C_{A|B_i \ldots B_{N-1}}^2 \geq C_{AB_i}^2 + \mu_i C_{A|B_{i+1} \ldots B_{N-1}}^2\) for all \(i = 1, 2, \ldots, N - 2\), then we have
\[
C_{A|B_1 \ldots B_{N-1}}^\alpha \geq C_{AB_1}^\alpha + K_1 C_{AB_2}^\alpha + \cdots + K_1 \cdots K_{N-2} C_{A|B_{N-1}}^\alpha,
\] (14)
for all $\alpha \geq 2$, where $K_r = (\mu_r + l_r)^{\frac{\alpha}{2}} - l_r^{\frac{\alpha}{2}}$ with $1 \leq r \leq N - 2$.

Remark 2.5 Since

$$(\mu + l)^{\frac{\alpha}{2}} - l^{\frac{\alpha}{2}} \geq (1 + l)^{\frac{\alpha}{2}} - l^{\frac{\alpha}{2}} \geq (2)^{\frac{\alpha}{2}} - l$$

(15)

for $\alpha \geq 2$, $\mu \geq 1$ and $l \geq 1$, we have $(1 + l)^{\frac{\alpha}{2}} - l^{\frac{\alpha}{2}} = \frac{(1+k)^{\frac{\alpha}{2}} - 1}{k^{\frac{\alpha}{2}}}$ if $l = \frac{1}{k}$ with $0 < k \leq 1$. In (15) the first equality holds when $\mu = 1$ and the second equality holds when $l = 1$. For given $l$, the bigger the $\mu$ is, the tighter the inequality in Theorem 2.3 is. Therefore, our new monogamy relation for concurrence is better than the ones in [25, 27].

Example 2.6 Let us consider the three-qubit state $|\phi\rangle_{ABC}$ in the generalized Schmidt decomposition from [35, 36],

$$|\phi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$

(16)

where $\lambda_i \geq 0$, $i = 0, 1, \ldots, 4$, and $\sum_{i=0}^{4} \lambda_i^2 = 1$. One gets $C_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $C_{AB} = 2\lambda_0\lambda_2$ and $C_{AC} = 2\lambda_0\lambda_3$. Setting $\lambda_0 = \lambda_3 = \lambda_4 = 1/\sqrt{3}$, $\lambda_2 = \sqrt{2/5}$ and $\lambda_1 = 0$, we have $C_{A|BC} = 4/5$, $C_{AB} = 2\sqrt{2/5}$ and $C_{AC} = 2/5$. Therefore,

$$C_{AB}^\alpha + (2^{\frac{\alpha}{2}} - 1)C_{AC}^\alpha = (2\sqrt{2/5})^\alpha + (2^{\frac{\alpha}{2}} - 1)(2/5)^\alpha.$$  

(17)

$$C_{AB}^\alpha + (((1 + k)^{\frac{\alpha}{2}} - 1)/k^{\frac{\alpha}{2}})C_{AC}^\alpha = (2\sqrt{2/5})^\alpha + (((1 + k)^{\frac{\alpha}{2}} - 1)/k^{\frac{\alpha}{2}})(2/5)^\alpha,$$

(18)

$$C_{AB}^\alpha + ((\mu + l)^{\frac{\alpha}{2}} - l^{\frac{\alpha}{2}})C_{AC}^\alpha = (2\sqrt{2/5})^\alpha + (\mu + l)^{\frac{\alpha}{2}} - l^{\frac{\alpha}{2}})(2/5)^\alpha.$$  

(19)

When $k = 0.5$ the lower bound (18) gives the best result. When $l = \frac{1}{k} = 2$, $\mu = 1$ the lower bound (19) gives the same result as (18). But when $l = \frac{1}{k} = 2$ and $1 < \mu \leq 2$, the lower bound (19) is better than (18). It can be seen that our result is better than the result (18) in [27] for $\alpha \geq 2$, hence better than (17) given in [25], see Fig. 1.

Fig. 1 From top to bottom, the first curve represents the concurrence of $|\phi\rangle_{A|BC}$ in Example 2.6, the third and fourth curves represent the lower bounds from [27] and [25], respectively. The second curve represents the lower bound from our result.
3 Tighter Monogamy and Polygamy Relations for EoF

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be $m$ and $n$ dimensional ($m \leq n$) vector spaces, respectively. The entanglement of formation (EoF) of a pure state $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is defined by $E(|\phi\rangle) = S(\rho_A)$, where $\rho_A = \text{Tr}(|\phi\rangle \langle \phi|)$ and $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ [37, 38]. For a bipartite mixed state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, the EoF is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i E(|\phi_i\rangle),$$

with the minimum taking over all possible pure state decomposition of $\rho_{AB}$.

Denote by $f(x) = H\left(1+\frac{\sqrt{x}}{2}\right)$, where $H(x) = -x \log_2 (x) - (1-x) \log_2 (1-x)$. It is obvious that $f(x)$ is a monotonically increasing function for $0 \leq x \leq 1$ which satisfies

$$f(\sqrt{x}) (x^2 + y^2) \geq f(\sqrt{x}^2) + f(\sqrt{y}^2),$$

$$f(x^2 + y^2) \leq f(x^2) + f(y^2),$$

where $f(\sqrt{x}) (x^2 + y^2) = [f(x^2 + y^2)]^{\sqrt{x}}$. It is showed in [10] that $E(|\phi\rangle) = f(C^2(|\phi\rangle))$ for $2 \otimes m$ ($m \geq 2$) pure state $|\phi\rangle$, and $E(\rho) = f(C^2(\rho))$ for two-qubit mixed state $\rho$.

EoF does not satisfy the inequality $E_{AB} \geq E_{AB} + E_{AC}$ [11]. In [39] it is shown that EoF is a monotonic function: $E^2(2^{\frac{C}{A|B_1B_2\cdots B_{N-1}}}) \geq E^2(\sum_{i=1}^{N-1} C_{AB_i})$. It is further proved that for $N$-qubit systems, $E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \geq E^{\alpha}_{AB_1} + E^{\alpha}_{AB_2} + \cdots + E^{\alpha}_{AB_{N-1}}$ for $\alpha \geq \sqrt{2}$, where $E^{\alpha}_{A|B_1B_2\cdots B_{N-1}}$ is the EoF of $\rho$ in bipartite partition $A|B_1B_2\cdots B_{N-1}$, and $E^{\alpha}_{AB_i}$, $i = 1, 2, \cdots, N-1$, is the EoF of the bipartite states $\rho_{AB_i} = \text{Tr}_{B_2\cdots B_{N-1}}(\rho)$ [24].

For $N$-qubit systems, the following monogamy relation has been obtained,

$$E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \geq E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} + (2^{\frac{\alpha}{2\sqrt{2}}} - 1)E^{\alpha}_{AB_2} + \cdots + (2^{\frac{\alpha}{2\sqrt{2}}} - 1)^{m-1}E^{\alpha}_{AB_m} + (2^{\frac{\alpha}{2\sqrt{2}}} - 1)^mE^{\alpha}_{(\rho_{ABN-1})}$$

for $\alpha \geq \sqrt{2}$, with the conditions $C(\rho_{AB_i}) \geq C(\rho_{A|B_1B_2\cdots B_{N-1}})$ for $i = 1, 2, \cdots, m$, and $C(\rho_{AB_j}) \leq C(\rho_{A|B_1B_2\cdots B_{N-1}})$ for $j = m+1, \cdots, N-2, 1 \leq m \leq N-3, N \geq 4$ [25]. The inequality (24) is a further improvement [27] as

$$E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \geq E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} + (((1+k)\frac{\alpha}{\sqrt{2}} - 1)/k\frac{\alpha}{\sqrt{2}})E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} + \cdots + (((1+k)\frac{\alpha}{\sqrt{2}} - 1)/k\frac{\alpha}{\sqrt{2}})^{m-1}E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} + (((1+k)\frac{\alpha}{\sqrt{2}} - 1)/k\frac{\alpha}{\sqrt{2}})^mE^{\alpha}_{A|B_1B_2\cdots B_{N-1}},$$

for $\alpha \geq \sqrt{2}$, with $kE^{\sqrt{2}}(\rho_{AB_i}) \geq E^{\sqrt{2}}(\rho_{A|B_1B_2\cdots B_{N-1}})$ for $i = 1, 2, \cdots, m$, and $E^{\sqrt{2}}(\rho_{AB_i}) \leq kE^{\sqrt{2}}(\rho_{A|B_1B_2\cdots B_{N-1}})$ for $j = m+1, \cdots, N-2, 1 \leq m \leq N-3, N \geq 4$ and $0 < k \leq 1$.

The corresponding entanglement of assistance (EoA) is defined in terms of the entropy of entanglement for a tripartite pure state $|\phi\rangle_{ABC}$,

$$E^a(|\phi\rangle_{ABC}) \equiv E^a(\rho_{AB}) = \max_{\{p_i, |\phi_i\rangle\}} \sum_i p_i E(|\phi_i\rangle).$$

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where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} = \text{Tr}_C(\ketbra{\phi})_{ABC} \ketbra{\phi} = \sum_i p_i \ketbra{\phi_i}_{AB} \ketbra{\phi_i} \) with \( p_i \geq 0 \) and \( \sum p_i = 1 \) [40]. For an arbitrary dimensional multipartite quantum state \( \rho_{AB_1B_2\cdots B_{N-1}} \), a general polygamy inequality of multipartite quantum entanglement was established in [21],

\[
E_a(\rho_{A|B_1B_2\cdots B_{N-1}}) \leq \sum_{i} E_a(\rho_{AB_i}).
\]  

(26)

In the following, we show that these monogamy and polygamy relations for EoF can become even tighter under some conditions. For convenience, we denote by \( E_{AB_j} = E(\rho_{AB_j}) \) for \( j = 1, 2, \cdots, N - 1 \), and \( E_{A|B_1B_2\cdots B_{N-1}} = E(\rho_{A|B_1B_2\cdots B_{N-1}}) \).

**Theorem 3.1** Let \( \mu_r \geq 1 \) and \( l_r \geq 1 \) be real numbers, \( 1 \leq r \leq N - 2 \). For any \( N \)-qubit mixed state \( \rho_{A_1B_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}} \), if \( E_{A|B_1B_2\cdots B_{N-1}} \geq l_1 E_{A|B_1B_2\cdots B_{N-1}} \),

\[
E_{A|B_1B_2\cdots B_{N-1}} \geq \sum_{i} E_{A|B_1B_2\cdots B_{N-1}}^{\alpha_i} + K_1 E_{A|B_1B_2\cdots B_{N-1}}^\alpha + \cdots + K_{m-1} E_{A|B_1B_2\cdots B_{N-1}}^{\alpha m}
\]

\[
+ K_1 \cdots K_m (E_{A|B_1B_2\cdots B_{N-1}}^{\alpha m+1} + \cdots + K_{N-2} E_{A|B_1B_2\cdots B_{N-1}}^{\alpha N-2})
\]

\[
+ K_1 \cdots K_m E_{A|B_1B_2\cdots B_{N-1}}^{\alpha N-1}
\]

\[
\text{for all } \alpha \geq \sqrt{2}, \text{ where } K_r = (\mu_r + l_r) \sqrt{2} - l_r \sqrt{2} \text{ with } 1 \leq r \leq N - 2.
\]

**Proof** Consider \( \alpha \geq \sqrt{2} \) and \( f^\sqrt{2}(x^2) \geq l f^\sqrt{2}(y^2) \). Due to inequality (21), there exists \( \mu \geq 1 \) such that \( f^\sqrt{2}(x^2 + y^2) \geq f^\sqrt{2}(x^2) + f^\sqrt{2}(y^2) \). Hence we have

\[
f^\alpha(x^2 + y^2) = [f^\sqrt{2}(x^2 + y^2)]^\alpha \leq [f^\sqrt{2}(x^2) + \mu f^\sqrt{2}(y^2)]^\alpha \sqrt{2}
\]

\[
= \mu ^\frac{\alpha}{\sqrt{2}} f^\alpha(y^2) [(\mu^{-1} (f^\sqrt{2}(x^2)/f^\sqrt{2}(y^2)))^\alpha] + f^\alpha(x^2)
\]

\[
\geq \mu ^\frac{\alpha}{\sqrt{2}} f^\alpha(y^2) [(\mu^{-1} l + 1) \frac{\alpha}{\sqrt{2}} - (\mu^{-1} l) \frac{\alpha}{\sqrt{2}}] + f^\alpha(x^2)
\]

\[
= [(\mu + l) \frac{\alpha}{\sqrt{2}} - l \frac{\alpha}{\sqrt{2}}] f^\alpha(y^2) + f^\alpha(x^2),
\]

(28)

where the second inequality is obtained from inequality (6). Let \( \rho = \sum_i p_i \ketbra{\phi_i} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}} \) be the optimal decomposition of \( E_{A|B_1B_2\cdots B_{N-1}}(\rho) \) for the \( N \)-qubit mixed state \( \rho \). Then from [25]

\[
E_{A|B_1B_2\cdots B_{N-1}} \geq f \left( C_{A|B_1B_2\cdots B_{N-1}}^2 \right).
\]

(29)
Therefore,
\[
E^\alpha_{A|B_1\cdots B_{N-1}} \geq f^\alpha(C^2_{A|B_1\cdots B_{N-1}})
\]
\[
\geq f^\alpha(C^2_{AB_1}) + K_1 f^\alpha(C^2_{AB_2}) + \cdots + K_1 \cdots K_{m-1} f^\alpha(C^2_{AB_m})
+ K_1 \cdots K_m (K_{m+1} f^\alpha(C^2_{AB_{m+1}}) + \cdots + K_N f^\alpha(C^2_{AB_{N-2}}))
+ K_1 \cdots K_m f^\alpha(C^2_{AB_{N-1}})
\]
\[
= E^\alpha_{AB_1} + K_1 E^\alpha_{AB_2} + \cdots + K_1 \cdots K_{m-1} E^\alpha_{AB_m}
+ K_1 \cdots K_m (K_{m+1} E^\alpha_{AB_{m+1}} + \cdots + K_N E^\alpha_{AB_{N-2}})
+ K_1 \cdots K_m E^\alpha_{AB_{N-1}},
\]
where the first inequality is due to (29), the second inequality is obtained, similar to the proof of Theorem 2.3, by using inequality (28). The last equality holds since for any $2 \otimes 2$ quantum state $\rho_{AB_i}$, $E(\rho_{AB_i}) = f[C^2(\rho_{AB_i})]$. \hfill \Box

In particular, we have

**Theorem 3.2** Let $\mu_r \geq 1$ and $l_r \geq 1$ be real numbers, $1 \leq r \leq N - 2$. For any $N$-qubit mixed state $\rho_{A_1B_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $E^\sqrt{3}_{A|B_1\cdots B_{N-1}} \geq E^\sqrt{3}_{AB_1} + \mu_i E^\sqrt{3}_{A|B_1\cdots B_{N-1}}$ for $i = 1, 2, \ldots, N - 2$, then
\[
E^\alpha_{A|B_1\cdots B_{N-1}} \geq E^\alpha_{AB_1} + K_1 E^\alpha_{AB_2} + \cdots + K_1 \cdots K_{m-1} E^\alpha_{AB_m}
+ K_1 \cdots K_m E^\alpha_{AB_{N-1}},
\]
for all $\alpha \geq \sqrt{2}$, where $K_r = (\mu_r + l_r) \frac{\alpha}{\sqrt{2}} - l_r \frac{\alpha}{\sqrt{2}}$ with $1 \leq r \leq N - 2$.

**Remark 3.3** Since $(\mu + l) \frac{\alpha}{\sqrt{2}} - l \frac{\alpha}{\sqrt{2}} \geq (1 + l) \frac{\alpha}{\sqrt{2}} - l \frac{\alpha}{\sqrt{2}} \geq (2) \frac{\alpha}{\sqrt{2}} - l$, where $\alpha \geq \sqrt{2}$, $\mu \geq 1$, $l \geq 1$, we have $(1 + l) \frac{\alpha}{\sqrt{2}} - l \frac{\alpha}{\sqrt{2}} = ((1 + k) \frac{\alpha}{\sqrt{2}} - 1)/k \frac{\alpha}{\sqrt{2}}$ when $l = \frac{1}{k}$ with $0 < k \leq 1$. The first equality holds when $\mu = 1$ and the second equality holds when $l = 1$. For given $l$, the bigger the $\mu$ is, the tighter the inequality in Theorem 3.1 is. Hence, our new monogamy relation for EoF is better than the ones in [25, 27].

**Example 3.4** Let us again consider the three-qubit state $|\phi\rangle_{ABC}$ defined in Example 2.6 with $\lambda_0 = \lambda_3 = \lambda_4 = 1/\sqrt{5}$, $\lambda_2 = \sqrt{2/5}$ and $\lambda_1 = 0$. We have
\[
E_{A|B_1C} = -(4/5) \log_2(4/5) - (1/5) \log_2(1/5) \approx 0.721928,
\]
\[
E_{AB} = -((5 + \sqrt{17})/10) \log_2((5 + \sqrt{17})/10)
- ((5 - \sqrt{17})/10) \log_2((5 - \sqrt{17})/10) \approx 0.428710,
\]
\[
E_{AC} = -((5 + \sqrt{21})/10) \log_2((5 + \sqrt{21})/10)
- ((5 - \sqrt{21})/10) \log_2((5 - \sqrt{21})/10) \approx 0.250225.
\]
Thus,
\[
E^\alpha_{AB} + (2 \frac{\alpha}{\sqrt{2}} - 1) E^\alpha_{AC} = (0.428710)^\alpha + (2 \frac{\alpha}{\sqrt{2}} - 1)(0.250225)^\alpha,
\]
\[
E^\alpha_{AB} + (1 + k) \frac{\alpha}{k \sqrt{2}} - 1) E^\alpha_{AC} = (0.428710)^\alpha + (1 + k) \frac{\alpha}{k \sqrt{2}} - 1)(0.250225)^\alpha,
\]
\[
E^\alpha_{AB} + (\mu + l) \frac{\alpha}{\sqrt{2}} - l \frac{\alpha}{\sqrt{2}) E^\alpha_{AC} = (0.428710)^\alpha + ((\mu + l) \frac{\alpha}{\sqrt{2}} - l \frac{\alpha}{\sqrt{2})}(0.250225)^\alpha.
\]

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We see that our result is better than the one in [25, 27], see Fig. 2.

We can also provide tighter polygamy relations for the entanglement of assistance.

**Theorem 3.5** Let \(0 < \mu_r \leq 1\) and \(l_r \geq 1\) be real numbers, \(1 \leq r \leq N - 2\). For any \(N\)-qubit mixed state \(\rho_{A_1 \cdots A_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}\), if \(E_{aA|B_1 \cdots B_{N-1}} \geq l_1 E_{aA|B_1 \cdots B_{N-1}}\), \(E_{aA|B_1 \cdots B_{N-1}} \leq E_{aA|B_1 \cdots B_{N-1}} + \mu_i E_{aA|B_1 \cdots B_{N-1}}\) for \(i = 1, 2, \ldots, m\), and \(E_{aA|B_1 \cdots B_{N-1}} \geq l_j E_{aA|B_1 \cdots B_{N-1}}\), \(E_{aA|B_1 \cdots B_{N-1}} \leq \mu_j E_{aA|B_1 \cdots B_{N-1}}\) for \(j = m + 1, \ldots, N - 2\), \(1 \leq m \leq N - 3\), \(N \geq 4\), we have

\[
E_{aA|B_1 \cdots B_{N-1}}^\alpha \leq E_{aA|B_1 \cdots B_{N-1}}^\alpha + K_1 E_{aA|B_1 \cdots B_{N-1}}^\alpha + \cdots + K_{m-1} E_{aA|B_1 \cdots B_{N-1}}^\alpha
\]

\[
+ K_1 \cdots K_m (K_{m+1} E_{aA|B_1 \cdots B_{N-1}}^\alpha + \cdots + K_{N-2} E_{aA|B_1 \cdots B_{N-1}}^\alpha)
\]

\[
+ K_1 \cdots K_m E_{aA|B_1 \cdots B_{N-1}}^\alpha
\]

for all \(0 \leq \alpha \leq 1\), where \(K_r = (\mu_r + l_r)^\alpha - l_r^\alpha\) with \(1 \leq r \leq N - 2\).

Particularly, we have

**Theorem 3.6** Let \(0 < \mu_r \leq 1\) and \(l_r \geq 1\) be real numbers, \(1 \leq r \leq N - 2\). For any \(N\)-qubit mixed state \(\rho_{A_1 \cdots A_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}\), if \(E_{aA|B_1 \cdots B_{N-1}} \geq l_1 E_{aA|B_1 \cdots B_{N-1}}\), \(E_{aA|B_1 \cdots B_{N-1}} \leq E_{aA|B_1 \cdots B_{N-1}} + \mu_i E_{aA|B_1 \cdots B_{N-1}}\) for \(i = 1, 2, \ldots, N - 2\), then

\[
E_{aA|B_1 \cdots B_{N-1}}^\alpha \leq E_{aA|B_1 \cdots B_{N-1}}^\alpha + K_1 E_{aA|B_1 \cdots B_{N-1}}^\alpha + \cdots + K_1 \cdots K_{N-2} E_{aA|B_1 \cdots B_{N-1}}^\alpha
\]

for all \(0 \leq \alpha \leq 1\), where \(K_r = (\mu_r + l_r)^\alpha - l_r^\alpha\) with \(1 \leq r \leq N - 2\).

---

**Fig. 2** From top to bottom, the first curve represents the EoF \(E(|\phi\rangle_{A|BC})\), the third curve and the fourth curves represent the lower bounds from [27] and [25], respectively, the second curve represents the lower bound from our result.
4 Tighter Monogamy Relations for Negativity

Another well-known quantifier of bipartite entanglement is the negativity, which is based on the positive partial transposition (PPT) criterion. For a bipartite state $\rho_{AB}$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ the negativity is given by $N(\rho_{AB}) = (\|\rho_{AB}^T_A\| - 1)/2$, where $\rho_{AB}^T_A$ is the partial transpose with respect to the subsystem $A$, and $\|X\|$ denotes the trace norm of $X$, i.e., $\|X\| = \sqrt{XX^\dagger}$ [41].

For the purposes of discussion, we use the definition of negativity as

$$\tilde{N}(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i N(|\phi_i\rangle),$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\phi_i\rangle\}$ of $\rho_{AB}$.

For any bipartite pure state $|\phi\rangle_{AB}$, the negativity is given by $N(|\phi\rangle_{AB}) = 2 \sum_{i<j} \sqrt{\lambda_i \lambda_j} = (\text{Tr}(\sqrt{\rho_A}))^2 - 1$, where $\lambda_i$ are the eigenvalues of the reduced density matrix of $|\phi\rangle_{AB}$. For any bipartite pure state $|\phi\rangle_{AB}$ in $d \otimes d$ with Schmidt rank two, $|\phi\rangle_{AB} = \sqrt{\lambda_0} |00\rangle + \sqrt{\lambda_1} |11\rangle$, one has

$$N(|\phi\rangle_{AB}) = \|\phi\langle \phi|^T_B\| - 1 = 2\sqrt{\lambda_0 \lambda_1} = \sqrt{2[1 - \text{Tr}(\rho_A^2)]} = C(|\phi\rangle_{AB}).$$

In other words, the negativity is equivalent to the concurrence for any pure state with Schmidt rank two. Consequently it follows that for any two-qubit mixed state $\rho_{AB} = \sum_i p_i |\phi_i\rangle \langle \phi_i|$, $\tilde{N}(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i N(|\phi_i\rangle)$

$$= \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle)$$

$$= C(\rho_{AB}).$$

(39)

Recently, the monogamy relations satisfied by the $\alpha$th ($\alpha \geq 2$) power of negativity for $N$-qubit systems have been studied [25]. If $\tilde{N}(\rho_{AB_i}) \geq \tilde{N}(\rho_{A|B_i+1\cdots B_{N-1}})$ for $i = 1, 2, \cdots, m$, and $\tilde{N}(\rho_{AB_j}) \leq \tilde{N}(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1, \cdots, N-2, 1 \leq m \leq N-3, N \geq 4$, one has

$$\tilde{N}^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq \tilde{N}^\alpha(\rho_{A|B_1}) + (2^\frac{\alpha}{2} - 1)\tilde{N}^\alpha(\rho_{AB_2}) + \cdots + (2^\frac{\alpha}{2} - 1)^{m-1}\tilde{N}^\alpha(\rho_{AB_m})$$

$$+ (2^\frac{\alpha}{2} - 1)^{m+1}\tilde{N}^\alpha(\rho_{AB_{m+1}}) + \cdots + \tilde{N}^\alpha(\rho_{AB_{N-2}})$$

$$+ (2^\frac{\alpha}{2} - 1)^{m}\tilde{N}^\alpha(\rho_{AB_{N-1}}).$$

(40)

This relation is further improved to be

$$\tilde{N}^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq \tilde{N}^\alpha(\rho_{A|B_1}) + (((1+k)^{\frac{\alpha}{2}} - 1)/k^{\frac{\alpha}{2}})\tilde{N}^\alpha(\rho_{AB_2}) + \cdots$$

$$+ (((1+k)^{\frac{\alpha}{2}} - 1)/k^{\frac{\alpha}{2}})^{m-1}\tilde{N}^\alpha(\rho_{AB_m})$$

$$+ (((1+k)^{\frac{\alpha}{2}} - 1)/k^{\frac{\alpha}{2}})^{m+1}\tilde{N}^\alpha(\rho_{AB_{m+1}}) + \cdots + \tilde{N}^\alpha(\rho_{AB_{N-2}})$$

$$+ (((1+k)^{\frac{\alpha}{2}} - 1)/k^{\frac{\alpha}{2}})^{m}\tilde{N}^\alpha(\rho_{AB_{N-1}}).$$

(41)

with $k\tilde{N}^2(\rho_{AB_i}) \geq \tilde{N}^2(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \cdots, m$, and $\tilde{N}^2(\rho_{AB_j}) \leq k\tilde{N}^2(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1, \cdots, N-2, 1 \leq m \leq N-3, N \geq 4$ and $0 < k \leq 1$ [27].

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Similar to the consideration of concurrence, we have the following result. For convenience, we denote \( \tilde{N}_{AB} = \tilde{N}(|\rho\rangle_{AB}) \) for \( j = 1, 2, \ldots, N - 1 \), and \( \tilde{N}_{A|B_1B_2\cdots B_{N-1}} = \tilde{N}(\rho_{A|B_1B_2\cdots B_{N-1}}) \).

**Theorem 4.1** Let \( \mu_r \geq 1 \) and \( l_r \geq 1 \) \( (1 \leq r \leq N - 2) \) be real numbers. For any \( N \)-qubit mixed state \( \rho_{A_1\cdots A_N} \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_N} \), if \( \tilde{N}_{A_{i+1}\cdots A_N}^2 \geq l_i \tilde{N}_{A_{i+1}\cdots A_N}^2 \), then

\[
\tilde{N}_{A_{i+1}\cdots A_N}^2 \geq \tilde{N}_{A_{i+1}\cdots A_N}^2 + \mu_i \tilde{N}_{A_{i+1}\cdots A_N}^2 \quad \text{for} \quad i = 1, 2, \ldots, m, \quad \text{and} \quad \tilde{N}_{A_{N-1}A_N}^2 \geq l_1 \tilde{N}_{A_{N-1}A_N}^2 \quad \text{for} \quad j = m+1, \ldots, N-1, \quad 1 \leq m \leq N-3, \quad N \geq 4,
\]

\[
\text{then } \quad \tilde{N}_{A_{1}\cdots A_{N}}^\alpha \geq \tilde{N}_{A_{1}\cdots A_{N}}^\alpha + K_1 \tilde{N}_{A_{2}\cdots A_{N}}^\alpha + \cdots + K_{m-1} \tilde{N}_{A_{m}\cdots A_{N}}^\alpha + K_{m} \cdots K_{m-1} \tilde{N}_{A_{m+1}\cdots A_{N}}^\alpha
\]

\[
+ K_{m} \cdots K_{m-1} \tilde{N}_{A_{m+1}\cdots A_{N}}^\alpha (42)
\]

for all \( \alpha \geq 2 \), where \( K_r = (\mu_r + l_r) - l_r^2 \) with \( 1 \leq r \leq N - 2 \).

In particular, we have

**Theorem 4.2** Let \( \mu_r \geq 1 \) and \( l_r \geq 1 \) \( (1 \leq r \leq N - 2) \) be real numbers. For any \( N \)-qubit mixed state \( \rho_{A_1\cdots A_N} \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_N} \), if \( \tilde{N}_{A_{i+1}\cdots A_N}^2 \geq l_i \tilde{N}_{A_{i+1}\cdots A_N}^2 \), then \( \tilde{N}_{A_{i+1}\cdots A_N}^2 \geq \tilde{N}_{A_{i+1}\cdots A_N}^2 + \mu_i \tilde{N}_{A_{i+1}\cdots A_N}^2 \) \( \text{for} \quad i = 1, 2, \ldots, N - 2, \quad \text{we have} \quad \tilde{N}_{A_{N-1}A_N}^2 \geq l_1 \tilde{N}_{A_{N-1}A_N}^2 \quad \text{for} \quad j = m+1, \ldots, N-1, \quad 1 \leq m \leq N-3, \quad N \geq 4,
\]

\[
\text{then } \quad \tilde{N}_{A_{1}\cdots A_{N}}^\alpha \geq \tilde{N}_{A_{1}\cdots A_{N}}^\alpha + K_1 \tilde{N}_{A_{2}\cdots A_{N}}^\alpha + \cdots + K_{m-1} \tilde{N}_{A_{m}\cdots A_{N}}^\alpha + K_{m} \cdots K_{m-1} \tilde{N}_{A_{m+1}\cdots A_{N}}^\alpha (43)
\]

for all \( \alpha \geq 2 \), where \( K_r = (\mu_r + l_r) - l_r^2 \) with \( 1 \leq r \leq N - 2 \).

**Example 4.3** Let us consider the state in Example 2.6 with \( \lambda_0 = \lambda_3 = \lambda_4 = 1/\sqrt{5} \) \( \lambda_2 = \sqrt{2/5} \) and \( \lambda_1 = 0 \). We have \( \tilde{N}_{A_1BC} = 4/5 \), \( \tilde{N}_{AB} = 2\sqrt{2/5} \) and \( \tilde{N}_{AC} = 2/5 \). Therefore,

\[
\tilde{N}_{A_1B_1}^\alpha + ((1+k)^2 - 1)/(k^2) \tilde{N}_{A_2C}^\alpha = (2\sqrt{2/5})^\alpha + ((1+k)^2 - 1)/(k^2)(2/5)^\alpha, (44)
\]

\[
\tilde{N}_{A_1B_1}^\alpha + ((1+k)^2 - 1)/k^2 \tilde{N}_{A_2C}^\alpha = (2\sqrt{2/5})^\alpha + ((1+k)^2 - 1)/k^2)(2/5)^\alpha, (45)
\]

\[
\tilde{N}_{A_1B_1}^\alpha + ((\mu + l)^2 - l^2) \tilde{N}_{A_2C}^\alpha = (2\sqrt{2/5})^\alpha + ((\mu + l)^2 - l^2)(2/5)^\alpha. (46)
\]

Our result is better than the one given in [25, 27] for \( \alpha \geq 2 \), see Fig. 3.

## 5 Tighter Monogamy and Polygamy Relations for Tsallis-\( q \) Entanglement

The Tsallis-\( q \) entanglement of a bipartite pure state \( |\phi\rangle_{AB} \) is given by

\[
T_q(|\phi\rangle_{AB}) = S_q(\rho_A) = \frac{1}{q-1}(1 - \text{Tr}(\rho_A^q)), (47)
\]

where \( q > 0 \) and \( q \neq 1 \) [17]. \( T_q(\rho) \) converges to the von Neumann entropy when \( q \) tends to 1. \( \lim_{q \to 1} T_q(\rho) = -\text{Tr}\rho \log_2 \rho = S(\rho) \). For a bipartite mixed state \( \rho_{AB} \), the Tsallis-\( q \) entanglement is defined as \( T_q(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i T_q(|\phi_i\rangle) \), with the minimum taken over
Fig. 3 From top to bottom, the first curve represents the negativity $\tilde{N}(|\phi\rangle_{A|BC})$, the third and fourth curves represent the lower bounds from [27] and [25], respectively, the second curve represents the lower bound from our result on all possible pure state decompositions of $\rho_{AB}$. Yuan et al. presented an analytic relationship between the Tsallis-$q$ entanglement and concurrence for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$,

$$ T_q(|\phi\rangle_{AB}) = g_q(C^2(|\phi\rangle_{AB})), $$

where $g_q(x)$ is defined as

$$ g_q(x) = \frac{1}{q-1} \left[ 1 - \left( \frac{1+\sqrt{1-x}}{2} \right)^q - \left( \frac{1-\sqrt{1-x}}{2} \right)^q \right], $$

with $0 \leq x \leq 1$ [42]. For a $2 \otimes m$ pure state $|\phi\rangle$, it has been showed that $T_q(|\phi\rangle) = g_q(C^2(|\phi\rangle))$, and if $\rho$ is a two-qubit mixed state, then $T_q(\rho) = g_q(C^2(\rho))$ [17]. Therefore, (48) holds for any $q$ such that $g_q(x)$ in (49) is monotonically increasing and convex. Moreover, we have $g_q(x^2 + y^2) \geq g_q(x^2) + g_q(y^2)$ with $2 \leq q \leq 3$.

The Tsallis-$q$ entanglement satisfies the following relation,

$$ T_{qA|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} T_{qAB_i}, $$

where $i = 1, 2, \cdots, N-1, 2 \leq q \leq 3$ [17]. It is further proved in [42] that

$$ T_{qA|B_1B_2\cdots B_{N-1}}^2 \geq \sum_{i=1}^{N-1} T_{qAB_i}^2, $$

with $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$.

Recently, it has been proven that, for $N$-qubit mixed systems,

$$ T_q^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq T_q^\alpha(\rho_{AB_1}) + (2^\alpha - 1) T_q^\alpha(\rho_{AB_2}) + \cdots + (2^\alpha - 1)^{m-1} T_q^\alpha(\rho_{AB_m}) $$

$$ + (2^\alpha - 1)^m (T_q^\alpha(\rho_{ABm+1}) + \cdots + T_q^\alpha(\rho_{ABN-2})) $$

$$ + (2^\alpha - 1)^m T_q^\alpha(\rho_{ABN-1}). $$

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where $\alpha \geq 1$, $2 \leq q \leq 3$, under the conditions that $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1}B_{N-1}})$ for $i = 1, 2, \ldots, m$, and $C(\rho_{A|B_j}) \leq C(\rho_{A|B_{j+1}B_{N-1}})$ for $j = m + 1, \ldots, N - 2$, $1 \leq m \leq N - 3$ and $N \geq 4$ [25]. Later, the inequality (52) is further improved as

$$T^\alpha_q(\rho_{A|B_1\cdots B_{N-1}}) \geq T^\alpha_q(\rho_{AB_1}) + \left(1 + (1 + k)\alpha - \frac{1}{k}\right)T^\alpha_q(\rho_{AB_2}) + \cdots + \left(1 + (1 + k)\alpha - \frac{1}{k}\right)T^\alpha_q(\rho_{AB_N})$$

$$+ \frac{T^\alpha_q(\rho_{AB_{m+1}})}{k} + \cdots + \frac{T^\alpha_q(\rho_{AB_{N-2}})}{k} + \frac{T^\alpha_q(\rho_{AB_{N-1}})}{k},$$

(53)

where $\alpha \geq 1$, $2 \leq q \leq 3$, under the conditions that $kT^\alpha_q(\rho_{AB_i}) \geq T^\alpha_q(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \ldots, m$, and $T^\alpha_q(\rho_{AB_j}) \leq kT^\alpha_q(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m + 1, \ldots, N - 2$, $1 \leq m \leq N - 3$, $N \geq 4$ and $0 < k \leq 1$ [27].

As a dual quantity to the Tsallis-q entanglement, the Tsallis-q entanglement of assistance (TEoA) is defined by $T_q(\rho_{AB}) = \max_{\{\psi_i, |\phi_i\rangle\}} \sum_i p_i T_q(|\phi_i\rangle)$, where the maximum is taken over all possible pure state decompositions of $\rho_{AB}$ [14]. If $1 \leq q \leq 2$ or $3 \leq q \leq 4$, the function $g_q$ defined in (49) satisfies

$$g_q(x^2 + y^2) \leq g_q(x^2) + g_q(y^2),$$

(54)

which leads to the Tsallis polygamy inequality for any multi-qubit state $\rho_{AB_1B_2\cdots B_{N-1}}$ [30],

$$T^\alpha_q(\rho_{AB_1B_2\cdots B_{N-1}}) \leq \sum_{i=1}^{N-1} T^\alpha_q(\rho_{A|B_iB_{i+1}\cdots B_{N-1}}).$$

(55)

Taking a similar consideration to concurrence, we have the tighter monogamy and polygamy relations related to the Tsallis-q entanglement as following. For convenience, we denote by $T^\alpha_q(\rho_{AB_j}) = T^\alpha_q(\rho_{AB_1})$ for $j = 1, 2, \ldots, N - 1$, and $T^\alpha_q(\rho_{A|B_1B_2\cdots B_{N-1}}) = T^\alpha_q(\rho_{A|B_1B_2\cdots B_{N-1}})$.

**Theorem 5.1** Let $\mu_r \geq 1$ and $l_r \geq 1$ ($1 \leq r \leq N - 2$) be real numbers. For any $N$-qubit mixed state $\rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $T^\alpha_q(\rho_{AB_i}) \geq l^\alpha_{i-1} T^\alpha_q(\rho_{AB_1B_{i+1}\cdots B_{N-1}})$, $T^\alpha_q(\rho_{A|B_{i+1}\cdots B_{N-1}}) \geq T^\alpha_q(\rho_{A|B_1B_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \ldots, m$, and $T^\alpha_q(\rho_{A|B_1B_2\cdots B_{N-1}}) \geq l^\alpha_{N-2} T^\alpha_q(\rho_{A|B_1B_2\cdots B_{N-1}})$ for $j = m + 1, \ldots, N - 2$, $1 \leq m \leq N - 3$, $N \geq 4$, then

$$T^\alpha_{qA|B_1B_2\cdots B_{N-1}} \geq T^\alpha_{qAB_1} + K_1 T^\alpha_{qAB_2} + \cdots + K_{m-1} T^\alpha_{qAB_{m+1}} + K_{m} T^\alpha_{qAB_{m+1}} + \cdots + K_{N-2} T^\alpha_{qAB_{N-2}} + K_{N-1} T^\alpha_{qAB_{N-1}},$$

(56)

for all $\alpha \geq 1$ and $2 \leq q \leq 3$, where $K_r = (\mu_r + l^\alpha_{r-1})^\alpha - l^\alpha_{r-1}$ with $1 \leq r \leq N - 2$.

The above theorem gives rise to, in particular,

**Theorem 5.2** Let $\mu_r \geq 1$ and $l_r \geq 1$ ($1 \leq r \leq N - 2$) be real numbers. For any $N$-qubit mixed state $\rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $T^\alpha_q(\rho_{A|B_1B_{i+1}\cdots B_{N-1}}) \geq l^\alpha_{i-1} T^\alpha_q(\rho_{A|B_1B_{i+1}\cdots B_{N-1}})$, $T^\alpha_q(\rho_{A|B_{i+1}\cdots B_{N-1}}) \geq T^\alpha_q(\rho_{A|B_1B_{i+1}\cdots B_{N-1}})$ for all $i = 1, 2, \ldots, N - 2$, then

$$T^\alpha_{qA|B_1B_2\cdots B_{N-1}} \geq T^\alpha_{qAB_1} + K_1 T^\alpha_{qAB_2} + \cdots + K_{N-2} T^\alpha_{qAB_{N-1}}$$

(57)

for all $\alpha \geq 1$ and $2 \leq q \leq 3$, where $K_r = (\mu_r + l^\alpha_{r-1})^\alpha - l^\alpha_{r-1}$ with $1 \leq r \leq N - 2$.\n
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Example 5.3 Let us consider the state in Example 2.6 with \( \lambda_0 = \lambda_3 = \lambda_4 = 1/\sqrt{5} \), \( \lambda_2 = \sqrt{2}/5 \) and \( \lambda_1 = 0 \). For \( q = 2 \), we have \( T_{2A|BC} = 8/25 \), \( T_{2AB} = 4/25 \), and \( T_{2AC} = 2/25 \). Then

\[
T_{2AB}^\alpha + (2^\alpha - 1)T_{2AC}^\alpha = \left(\frac{4}{25}\right)^\alpha + (2^\alpha - 1)\left(\frac{2}{25}\right)^\alpha, \tag{58}
\]

\[
T_{2AB}^\alpha + \frac{((1 + k)^\alpha - 1)}{k^\alpha}T_{2AC}^\alpha = \left(\frac{4}{25}\right)^\alpha + \frac{((1 + k)^\alpha - 1)}{k^\alpha}\left(\frac{2}{25}\right)^\alpha, \tag{59}
\]

\[
T_{2AB}^\alpha + ((\mu + l)^\alpha - l^\alpha)T_{2AC}^\alpha = \left(\frac{4}{25}\right)^\alpha + ((\mu + l)^\alpha - l^\alpha)\left(\frac{2}{25}\right)^\alpha. \tag{60}
\]

We see that our result is better than the one given in [25, 27] for \( \alpha \geq 1 \), see Fig. 4.

For the Tsallis-\( q \) entanglement of assistance (TEoA), we have

**Theorem 5.4** Let \( 0 < \mu_r \leq 1 \) and \( l_r \geq 1 \) (\( 1 \leq r \leq N - 2 \)) be real numbers. For any \( N \)-qubit mixed state \( \rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}} \), if \( T_{aqAB_i} \geq l_i T_{aqA|B_{i+1}\cdots B_{N-1}} \), \( T_{aqA|B_{1}\cdots B_{N-1}} \leq T_{aqAB_i} + \mu_i T_{aqA|B_{i+1}\cdots B_{N-1}} \) for \( i = 1, 2, \cdots, m \), and \( T_{aqA|B_{j+1}\cdots B_{N-1}} \geq l_j T_{aqAB_j}, T_{aqA|B_{j}\cdots B_{N-1}} \leq \mu_j T_{aqAB_j} + T_{aqA|B_{i+1}\cdots B_{N-1}} \) for \( j = m+1, \cdots, N-2 \), \( 1 \leq m \leq N-3 \), \( N \geq 4 \), we have

\[
T_{aqA|B_{1}\cdots B_{N-1}}^\alpha \leq T_{aqAB_1}^\alpha + K_1 T_{aqAB_2}^\alpha + \cdots + K_{m-1} T_{aqAB_{m-1}}^\alpha + K_m T_{aqAB_{m+1}}^\alpha + \cdots + K_{N-2} T_{aqAB_{N-2}}^\alpha + K_1 \cdots K_m T_{aqAB_{N-1}}^\alpha \tag{61}
\]

for all \( 0 \leq \alpha \leq 1 \) with \( 1 \leq q \leq 2 \) and \( 3 \leq q \leq 4 \), where \( K_r = (\mu_r + l_r)^\alpha - l_r^\alpha \), \( 1 \leq r \leq N-2 \).

Particularly, one has

![Fig. 4](https://example.com/fig4.png)

**Fig. 4** From top to bottom, the first curve represents the Tsallis-\( q \) entanglement \( T_q(\rho_{A|BC}) \), the third and fourth curves represent the lower bounds from [27] and [25], respectively, the second curve represents the lower bound from our result.
Theorem 5.5 Let $0 < \mu_r \leq 1$ and $l_r \geq 1$ ($1 \leq r \leq N - 2$) be real numbers. For any $N$-qubit mixed state $\rho_{AB_1 \cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $T_{aqAB} \geq l_1 T_{aqA} |B_{i+1} \cdots B_{N-1}$, $T_{aqA} |B_{i} \cdots B_{N-1} \leq T_{aqAB} + \mu_j T_{aqA} |B_{j+1} \cdots B_{N-1}$ for all $i = 1, 2, \ldots, N - 2$, then

$$T_{aqA} |B_1 \cdots B_{N-1} \leq T_{aqAB} + \kappa_1 T_{aqAB_2} + \cdots + \kappa_1 \cdots \kappa_{N-2} T_{aqAB_{N-1}}$$

for all $0 \leq \alpha \leq 1$ with $1 \leq q \leq 2$ and $3 \leq q \leq 4$, where $\kappa_r = (\mu_r + l_r)^{\alpha} - l_r^{\alpha}$, $1 \leq r \leq N - 2$.

6 Tighter Monogamy and Polygamy Relations for Rényi-$\alpha$ Entanglement

For a bipartite pure state $|\phi\rangle_{AB}$, the Rényi-$\alpha$ entanglement is defined as $E_{\alpha}(|\phi\rangle_{AB}) = S_{\alpha}(\rho_A)$, where $S_{\alpha}(\rho) = \frac{1}{1 - \alpha} \log_2(\text{Tr} \rho^{\alpha})$ for any $\alpha > 0$ and $\alpha \neq 1$, and $\lim_{\alpha \to 1} S_{\alpha}(\rho) = S(\rho) = -\text{Tr} \rho \log_2 \rho$ [43]. For a bipartite mixed state $\rho_{AB}$, the Rényi-$\alpha$ entanglement is given by $E_{\alpha}(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i E_{\alpha}(|\phi_i\rangle)$, where the minimum is taken over all possible pure state decompositions of $\rho_{AB}$. For each $\alpha > 0$, one has $E_{\alpha}(\rho_{AB}) = f_{\alpha}(C(\rho_{AB}))$, where $f_{\alpha}(x) = \frac{1}{1 - \alpha} \log \left[ \frac{1 - \sqrt{1 - x^2}}{2} \right]^2 + \frac{1 + \sqrt{1 - x^2}}{2} \right]^2$ is a monotonically increasing and convex function [29]. For $\alpha \geq 2$ and any $N$-qubit state $\rho_{AB_1B_2 \cdots B_{N-1}}$, one has $E_{\alpha}(\rho_{A|B_1B_2 \cdots B_{N-1}}) \geq E_{\alpha}(\rho_{AB}) + E_{\alpha}(\rho_{A|B_2}) + \cdots + E_{\alpha}(\rho_{A|B_{N-1}})$ [17].

The Rényi-$\alpha$ entanglement of assistance (REoA), a dual quantity to the Rényi-$\alpha$ entanglement, is defined as $E_{\alpha \alpha}(\rho_{AB}) = \max_{\{p_i, |\phi_i\rangle\}} \sum_i p_i E_{\alpha}(|\phi_i\rangle)$, where the maximum is taken over all possible pure state decompositions of $\rho_{AB}$. For $\alpha \in \left[ \frac{-1}{2}, \frac{-1}{3} \right]$ and any $N$-qubit state $\rho_{AB_1B_2 \cdots B_{N-1}}$, a polygamy relation of multi-partite quantum entanglement in terms of REoA has been presented [23], $E_{\alpha \alpha}(\rho_{A|B_1B_2 \cdots B_{N-1}}) \leq E_{\alpha \alpha}(\rho_{AB}) + E_{\alpha \alpha}(\rho_{A|B_2}) + \cdots + E_{\alpha \alpha}(\rho_{A|B_{N-1}})$.

We propose the following monogamy and polygamy relations for the Rényi-$\alpha$ entanglement, which are tighter than the previous results. For convenience, we denote by $E_{\alpha \alpha} \rho_{AB} = E_{\alpha \alpha} \rho_{A|B_1B_2 \cdots B_{N-1}}$ for $j = 1, 2, \ldots, N - 1$, and $E_{\alpha \alpha} \rho_{A|B_1B_2 \cdots B_{N-1}} = E_{\alpha \alpha} \rho_{A|B_1B_2 \cdots B_{N-1}}$.

Theorem 6.1 Let $\mu_r \geq 1$ and $l_r \geq 1$ ($1 \leq r \leq N - 2$) be real numbers. For any $N$-qubit mixed state $\rho_{AB_1 \cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $E_{\alpha \alpha} \rho_{A|B_1B_2 \cdots B_{N-1}} \geq E_{\alpha \alpha} \rho_{AB} + \mu_j E_{\alpha \alpha} \rho_{A|B_{j+1} \cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $E_{\alpha \alpha} \rho_{A|B_{i+1} \cdots B_{N-1}} \geq E_{\alpha \alpha} \rho_{AB} + \mu_j E_{\alpha \alpha} \rho_{A|B_{j+1} \cdots B_{N-1}}$ for $j = m + 1, \ldots, N - 2$, $1 \leq m \leq N - 3, N \geq 4$, we have

$$E_{\alpha \alpha} \rho_{A|B_1 \cdots B_{N-1}} \geq E_{\alpha \alpha} \rho_{AB} + K_1 E_{\alpha \alpha} \rho_{A|B_2} + \cdots + K_{m-1} E_{\alpha \alpha} \rho_{A|B_m} + K_{m} \cdots K_{m-1} \cdots K_{m-2} E_{\alpha \alpha} \rho_{A|B_{N-1}}$$

(63)

for all $\alpha \geq 1$ and $\alpha \geq 2$, where $K_r = (\mu_r + l_r)^{\alpha} - l_r^{\alpha}$, $1 \leq r \leq N - 2$. 

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Theorem 6.2 Let $\mu_r \geq 1$ and $l_r \geq 1$ ($1 \leq r \leq N - 2$) be real numbers. For any $N$-qubit mixed state $\rho_{AB_1\ldots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $E_{\tilde{\alpha}AB_i} \geq l_i E_{\tilde{\alpha}AB_i} + \mu_i E_{\tilde{\alpha}AB_i + \cdots + B_{N-1}}$ for all $i = 1, 2, \ldots, N - 2$, then
\[
E_{\tilde{\alpha}AB_i} \geq E_{\tilde{\alpha}AB_1} + K_1 E_{\tilde{\alpha}AB_2} + \cdots + K_{N-1} E_{\tilde{\alpha}AB_{N-1}}
\]
for all $\alpha \geq 1$ and $\tilde{\alpha} \geq 2$, where $K_r = (\mu_r + l_r)^{\alpha} - l_r^{\alpha}$, $1 \leq r \leq N - 2$.

Example 6.3 Let us consider the state in Example 2.6 with $\lambda_0 = \lambda_3 = \lambda_4 = 1/\sqrt{5}$, $\lambda_2 = \sqrt{2/5}$ and $\lambda_1 = 0$. For $\tilde{\alpha} = 2$, we have $E_{\tilde{\alpha}AB} = \log_2(25/17) \approx 0.556393$, $E_{\tilde{\alpha}AC} = \log_2(25/17) \approx 0.556393$ and $E_{\tilde{\alpha}BC} = \log_2(25/17) \approx 0.556393$. Then
\[
E_{\tilde{\alpha}AB} + E_{\tilde{\alpha}AC} = (0.251539)^\alpha + (0.120294)^\alpha,
\]
\[
E_{\tilde{\alpha}AB} + ((1+k)^{\alpha} - 1)/k^{\alpha})E_{\tilde{\alpha}AC} = (0.251539)^\alpha + ((1+k)^{\alpha} - 1)/k^{\alpha})(0.120294)^\alpha,
\]
\[
E_{\tilde{\alpha}AB} + ((\mu + l)^{\alpha} - l^{\alpha})E_{\tilde{\alpha}AC} = (0.251539)^\alpha + ((\mu + l)^{\alpha} - l^{\alpha})(0.120294)^\alpha,
\]
which show that our result is better than the one given in [25, 27] for $\alpha \geq 1$, see Fig. 5.

Correspondingly, for $E_{\tilde{\alpha}a\tilde{a}}$ we have

Theorem 6.4 Let $0 < \mu_r \leq 1$ and $l_r \geq 1$ ($1 \leq r \leq N - 2$) be real numbers. For any $N$-qubit mixed state $\rho_{AB_1\ldots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $E_{a\tilde{\alpha}AB_i} \geq l_i E_{a\tilde{\alpha}AB_i} + \mu_i E_{a\tilde{\alpha}AB_i + \cdots + B_{N-1}}$ for all $i = 1, 2, \ldots, m$, and $E_{a\tilde{\alpha}A|_{B_j+1\ldots B_{N-1}}} \geq l_j E_{a\tilde{\alpha}AB_j} + E_{a\tilde{\alpha}AB_j + \cdots + B_{N-1}}$ for $j = m + 1, \ldots, N - 2$, $1 \leq m \leq N - 3$, $N \geq 4$, we have
\[
E_{a\tilde{\alpha}A|B_1\ldots B_{N-1}} \leq E_{a\tilde{\alpha}AB_1} + K_{1} E_{a\tilde{\alpha}AB_2} + \cdots + K_{m-1} E_{a\tilde{\alpha}AB_m} + K_{m} E_{a\tilde{\alpha}AB_{m+1}} + \cdots + K_{N-2} E_{a\tilde{\alpha}AB_{N-1}} + K_{N-1} E_{a\tilde{\alpha}AB_N}
\]

Fig. 5 From top to bottom, the first curve represents the Rényi-$\alpha$ entanglement $E_{\tilde{\alpha}(|\psi\rangle_{A|BC})}$, the third and fourth curves represent the lower bounds from [27] and [25], respectively, the second curve represents the lower bound from our result.
for all $0 \leq \alpha \leq 1$ and $\frac{\sqrt{7} - 1}{2} \leq \hat{\alpha} \leq \frac{\sqrt{13} - 1}{2}$, where $K_r = (\mu_r + l_r)^{\alpha} - l_r^{\alpha}$, $1 \leq r \leq N - 2$.

**Theorem 6.5**  Let $0 < \mu_r \leq 1$ and $l_r \geq 1$ ($1 \leq r \leq N - 2$) be real numbers. For any $N$-qubit mixed state $\rho_{AB_1 \cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $E_{a\hat{a}AB_1 \cdots B_{N-1}} \geq l_i E_a a_{AB_1 \cdots B_{N-1}}$, then

$$E^a_{a\hat{a}A|B_1 \cdots B_{N-1}} \leq E^a_{a\hat{a}AB_1} + K_1 E^a_{a\hat{a}A|B_1} + \cdots + K_{N-2} E^a_{a\hat{a}AB_{N-1}},$$

for all $0 \leq \alpha \leq 1$ and $\frac{\sqrt{7} - 1}{2} \leq \hat{\alpha} \leq \frac{\sqrt{13} - 1}{2}$, where $K_r = (\mu_r + l_r)^{\alpha} - l_r^{\alpha}$, $1 \leq r \leq N - 2$.

7 Conclusion

We have provided tighter monogamy inequalities with respect to the concurrence, entanglement of formation, convex-roof extended negativity, Tsallis-q entanglement and Rényi-$\alpha$ entanglement, we have also provided tighter polygamy inequalities with respect to the entanglement of formation, Tsallis-q entanglement and Rényi-$\alpha$ entanglement. Monogamy and polygamy inequalities play significant roles in characterizing the entanglement distributions and shareability in multipartite quantum systems. Tighter monogamy relations imply finer characterizations of the entanglement distribution. Our approach may also be used to study the monogamy properties related to other quantum correlations, and provides a useful way to understand the property of multipartite entanglement.

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**References**

1. Plenio, M.B., Virmani, S.: An introduction to entanglement measures. e-print arXiv:quant-ph/0504163 (2005)
2. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81, 865 (2009)
3. Mintert, F., Kuś, M., Buchleitner, A.: Concurrence of mixed bipartite quantum states in arbitrary dimensions. Phys. Rev. Lett. 92, 167902 (2004)
4. Chen, K., Albeverio, S., Fei, S.M.: Concurrence of arbitrary dimensional bipartite quantum states. Phys. Rev. Lett. 95, 040504 (2005)
5. Breuer, H.P.: Separability criteria and bounds for entanglement measures. J. Phys. A.: Math. Gen. 39, 11847 (2006)
6. Breuer, H.P.: Optimal entanglement criterion for mixed quantum states. Phys. Rev. Lett. 97, 080501 (2006)
7. de Vicente, J.I.: Lower bounds on concurrence and separability conditions. Phys. Rev. A 75, 052320 (2007)
8. Pawłowski, M.: Security proof for cryptographic protocols based only on the monogamy of Bells inequality violations. Phys. Rev. A 82, 032313 (2010)
9. Hill, S., Wootters, W.K.: Entanglement of a pair of quantum bits. Phys. Rev. Lett. 78, 5022 (1997)
10. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80, 2245 (1998)
11. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A 61, 052306 (2000)
12. Osborne, T.J., Verstraete, F.: General monogamy inequality for bipartite qubit entanglement. Phys. Rev. Lett. 96, 220503 (2006)
13. Bai, Y.K., Ye, M.Y., Wang, Z.D.: Entanglement monogamy and entanglement evolution in multipartite systems. Phys. Rev. A 80, 044301 (2009)
14. de Oliveira, T.R., Cornelio, M.F., Fanchini, F.F.: Monogamy of entanglement of formation. Phys. Rev. A 89, 034303 (2014)
15. Kim, J.S., Das, A., Sanders, B.C.: Entanglement monogamy of multipartite higher-dimensional quantum systems using convex-roof extended negativity. Phys. Rev. A 79, 012329 (2009)
16. Liu, F.: Monogamy relations for squared entanglement negativity. Commun. Theor. Phys. 66, 407 (2016)
17. Kim, J.S.: Tsallis entropy and entanglement constraints in multiqubit systems. Phys. Rev. A 81, 062328 (2010)
18. Buscemi, F., Gour, G., Kim, J.S.: Polygamy of distributed entanglement. Phys. Rev. A 80, 012324 (2009)
19. Gour, G., Meyer, D.A., Sanders, B.C.: Deterministic entanglement of assistance and monogamy constraints. Phys. Rev. A 72, 042329 (2005)
20. Gour, G., Bandyopadhay, S., Sanders, B.C.: Dual monogamy inequality for entanglement. J. Math. Phys. 48, 012108 (2007)
21. Kim, J.S.: General polygamy inequality of multiparty quantum entanglement. Phys. Rev. A 85, 062302 (2012)
22. Kim, J.S.: Tsallis entropy and general polygamy of multiparty quantum entanglement in arbitrary dimensions. Phys. Rev. A 94, 062338 (2016)
23. Song, W., Zhou, J., Yang, M., Zhao, J.L., Li, D.C., Zhang L.H., Cao, Z.L.: Polygamy relation for the Rényi- entanglement of assistance in multi-qubit systems. arxiv:quant-ph/1703.02858
24. Zhu, X.N., Fei, S.M.: Entanglement monogamy relations of qubit systems. Phys. Rev. A 90, 024304 (2014)
25. Jin, Z.X., Li, J., Li, T., Fei, S.M.: Tighter monogamy relations in multiqubit systems. Phys. Rev A 97, 032336 (2018)
26. Jin, Z.X., Fei, S.M.: Tighter entanglement monogamy relations of qubit systems. Quantum Inf. Process. 16, 77 (2017)
27. Yang, L.M., Chen, B., Fei, S.M., Wang, Z.X.: Tighter constraints of multiqubit entanglement. Commun. Theor. Phys., p 71 (2019)
28. Jin, Z.X., Fei, S.M.: Superactivation of monogamy relations for nonadditive quantum correlation measures. Phys. Rev A 99, 032343 (2019)
29. Kim, J.S., Sanders, B.C.: Monogamy of multi-qubit entanglement using Rényi entropy. J. Phys. A.: Math. Theor. 43, 445305 (2010)
30. Kim, J.S.: Generalized entanglement constraints in multi-qubit systems in terms of Tsallis entropy. Ann. Phys. 373, 197 (2016)
31. Luo, Yu., Li, Y.M.: Hierarchical polygamy inequality for entanglement of Tsallis q-entropy. Commun. Theor. Phys., p 52 (2018)
32. Uhlmann, A.: Fidelity and concurrence of conjugated states. Phys. Rev. A 62, 032307 (2000)
33. Rungta, P., Bužek, V., Caves, C.M., Hillery, M., Milburn, G.J.: Universal state inversion and concurrence in arbitrary dimensions. Phys. Rev. A 64, 042315 (2001)
34. Albeverio, S., Fei, S.M.: A note on invariants and entanglements. J. Opt. B.: Quantum Semiclass Opt. 3, 223 (2001)
35. Acín, A., Andrianov, A., Costa, L., Jané, E., Latorre, J.I., Tarrach, R.: Generalized Schmidt decomposition and classification of three-quantum-bit states. Phys. Rev. Lett. 85, 1560 (2000)
36. Gao, X.H., Fei, S.M.: Estimation of concurrence for multipartite mixed states. Eur. Phys. J. Spec. Topics 159, 71 (2008)
37. Bennett, C.H., Bernstein, H.J., Popescu, S., Schumacher, B.: Concentrating partial entanglement by local operations. Phys. Rev. A 53, 2046 (1996)
38. Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K.: Mixed-state entanglement and quantum error correction. Phys. Rev. A 54, 3824 (1996)
39. Bai, Y.K., Zhang, N., Ye, M.Y., Wang, Z.D.: Exploring multipartite quantum correlations with the square of quantum discord. Phys. Rev. A 88, 012123 (2013)
40. Cohen, O.: Unlocking hidden entanglement with classical information. Phys. Rev. Lett. 80, 2493 (1998)
41. Vidal, G., Werner, R.F.: Computable measure of entanglement. Phys. Rev. A 65, 032314 (2002)
42. Yuan, G.M., Song, W., Yang, M., Li, D.C., Zhao, J.L., Cao, Z.L.: Monogamy relation of multi-qubit systems for squared Tsallis-q entanglement. Sci. Rep. 6, 28719 (2016)
43. Vidal, G.: Entanglement monotones. J. Mod. Opt. 47, 355 (2000)

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