Abstract In this paper, we propose computational algorithms for a pseudo-formal linearization method for nonlinear dynamic systems and a nonlinear observer for nonlinear scalar measurement systems using Chebyshev interpolation. This pseudo-formal linearization method transforms a nonlinear autonomous dynamic system into an augmented linear one with respect to a linearization function that consists of polynomials of state variables. When linearizing, Chebyshev interpolation is exploited so that a computational algorithm is implemented. As an application of this method, a computational algorithm for a nonlinear observer is proposed. Numerical experiments indicate that the performance of the proposed method is superior to that of the previous method.

Keywords: nonlinear system, pseudo-formal linearization, nonlinear observer, Chebyshev interpolation, linearization function, computer algorithm

1. Introduction

Linearization methods for nonlinear dynamic systems employed to apply linear system theories [1]-[5] have been well studied for several decades. A formal linearization method [6], [7] is considered as one of them. This method has been expanded into a pseudo-formal linearization method by using an automatic choosing function and Chebyshev expansion [8].

In this paper, we propose a numerical computational algorithms for a pseudo-formal linearization method exploiting Chebyshev interpolation [9]. Introducing a formal linearization function and dividing the given domain into some subdomains, a given nonlinear autonomous dynamic system is transformed into some linearized systems on each subdomain using a formal linearization method based on Chebyshev interpolation. Then we unite these linearized systems on subdomains into a single linear system on the whole domain smoothly by using an automatic choosing function. As an application of this method, a computational algorithm for a nonlinear observer is synthesized. When an augmented measurement vector that consists of polynomials of measurement data is introduced, a measurement system is transformed into an augmented linear one by the pseudo-formal linearization using Chebyshev interpolation.

The advantages of these algorithms are that the coefficients of linearized systems are simply obtained by summation because of the orthogonality for a finite sum, and the running time for calculating the coefficients of the linearized systems can be markedly improved compared with the case of evaluating definite integrals for linearization in a previous work [8]. Therefore, this linearization method is easy to execute using the proposed computational algorithm. Its inversion is also simple, because the original state variables are included within the linearization function itself.

Numerical experiments indicate that the performance of the proposed method is superior to that of the previous method [8].

2. Pseudo-Formal Linearization

A nonlinear dynamic system is assumed to have the form

$$\Sigma_1 : \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in D \quad (1)$$
where \( t \) denotes time, \( \cdot = d/dt \), \( \mathbf{x} \) is an \( n \)-dimensional state vector, and \( \mathbf{f} \in \mathbb{R}^n \) is a sufficiently smooth nonlinear vector-valued function. \( \mathbf{D} \) is a domain denoted by the Cartesian product

\[
\mathbf{D} = \prod_{i=1}^{n} [m_i - p_i, m_i + p_i] \quad (m_i \in \mathbb{R}, p_i > 0)
\]

To apply a pseudo-formal linearization method, a vector-valued separable function is introduced as

\[
C : \mathbf{D} \to \mathbb{R}^L
\]

which is continuously differentiable, and here we let \( C = [I : 0] \) (\( I : L \times L \) unit matrix) for simplicity. Considering the nonlinearity of the given nonlinear system, this \( C(\mathbf{x}) \) can be determined. We let \( D \) be a domain of \( C^{-1} \) and divide into \((M + 1)\) subdomains:

\[
D = \bigcup_{k=0}^{M} D_k
\]

where \( D_M = D - \bigcup_{k=0}^{M-1} D_k \) and \( C^{-1}(D_0) \supseteq 0 \) (see Fig. 1). \( D_k (0 \leq k \leq M - 1) \) endowed with a lexicographic order is the Cartesian product

\[
D_k = \prod_{j=1}^{L} [a_{kj}, b_{kj}] \quad (a_{kj} < b_{kj})
\]

Figure 1 Pseudo-formal linearization

A pseudo-formal linearization uses an automatic choosing function of the sigmoid type as follows:

\[
I_k(\zeta) = \frac{1}{L} \left\{ 1 - \frac{1}{1 + \exp (2\mu(\zeta - a_{kj}))} \right\} \quad (0 \leq k \leq M - 1)
\]

so that

\[
\sum_{k=0}^{M} I_k(\zeta) = 1
\]

where

\[
\zeta = [\zeta_1, \ldots, \zeta_L]^T = C(\mathbf{x})
\]

and \( \mu \) is a positive real value.

To make use of Chebyshev interpolation, the state vector \( \mathbf{x} \) is changed into \( \mathbf{y} \), so that \( \mathbf{y} \) has the basic domain of the Chebyshev polynomials \( \mathcal{D}_0 = \prod_{i=1}^{n} [-1, 1] \)

and \( \mathbf{y} \) is rewritten as

\[
\mathbf{y} = \mathcal{P}^{-1}(\mathbf{x} - \mathcal{M}(\mathbf{k})) \in \mathcal{D}_0
\]

where

\[
\mathcal{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_L \\ y_{L+1} \end{bmatrix}, \quad \mathcal{M}(\mathbf{k}) = \begin{bmatrix} m_1^{(k)} \\ \vdots \\ m_L^{(k)} \\ m_{L+1}^{(k)} \end{bmatrix}
\]

The given dynamic system (Eq. (1)) becomes

\[
\dot{\mathbf{y}}(t) = \mathcal{P}^{-1} \mathbf{f}(\mathcal{P}(\mathbf{k}) \mathbf{y}(t) + \mathcal{M}(\mathbf{k}))
\]

Here we define an \( N \)-th order formal linearization function that consists of polynomials defined by

\[
\phi(\mathbf{x}) = [x_1, x_2, \ldots, x_n, \frac{x_1^2}{2}, \frac{x_2^2}{2}, \ldots, \frac{x_n^2}{2}, \ldots, x_1 x_2 \ldots x_n, \frac{x_1 x_2 \ldots x_n}{r_1 r_2 \ldots r_n}, \ldots, \frac{x_1 x_2 \ldots x_n}{r_1 r_2 \ldots r_n}, \ldots, \frac{x_1 x_2 \ldots x_n}{r_1 r_2 \ldots r_n}, \ldots, \frac{x_1 x_2 \ldots x_n}{r_1 r_2 \ldots r_n}]^T
\]

From Eq. (1), the derivative of each element of \( \phi \) becomes

\[
\dot{\phi}_{(r_1 \ldots r_n)}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{x}} \phi_{(r_1 \ldots r_n)}(\mathbf{x}) \cdot \dot{\mathbf{x}}
\]

\[
= \frac{\partial}{\partial \mathbf{y}} \mathcal{P}^{-1}(\mathbf{x} - \mathcal{M}(\mathbf{k})) \cdot \mathbf{f}(\mathbf{x})
\]

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\[ \times \phi_{(r_1 \cdots r_n)}(P^{(k)}y + M^{(k)}) \cdot f(P^{(k)}y + M^{(k)}) \]
\[ \triangleq G_{(r_1 \cdots r_n)}^{(k)}(y) \quad (9) \]

To this \( G_{(r_1 \cdots r_n)}^{(k)}(y) \), we apply Chebyshev interpolation [9] as follows. The Chebyshev polynomials \( T_q() \) are defined as
\[ T_q(y_i) = \cos(q \cdot \cos^{-1} y_i) \quad (q = 0, 1, 2, \cdots) \quad (10) \]
or
\[ T_0(y_i) = 1, \ T_1(y_i) = y_i, \ T_2(y_i) = 2y_i^2 - 1, \]
and their polynomial representation [10] in terms of powers of \( y_i \) is given by
\[ T_q(y_i) = \sum_{s=0}^{\lfloor q/2 \rfloor} d^{(q)}_s y_i^{q-2s} \quad (11) \]
where
\[ d^{(q)}_s = \begin{cases} \frac{(-1)^{s+1}}{s+1} (q-s)^\frac{q-2s-1}{2} & (2s < q) \\ \frac{(-1)^s}{s!} & (q = 2s) \end{cases} \]
and \( \lfloor \cdot \rfloor \) is the floor function. Applying Chebyshev interpolation up to the \( N \)th order on each subdomain \( D_k \), \( G_{(r_1 \cdots r_n)}^{(k)}(y) \) is approximated by
\[ G_{(r_1 \cdots r_n)}^{(k)}(y) \approx \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C^{(k)}(q_1 \cdots q_n) T_{q_1}(y_1) \cdots T_{q_n}(y_N) \]
\[ \triangleq G_{(r_1 \cdots r_n)}^{(k)}(y) \quad (12) \]
where
\[ T_{q_1 \cdots q_n}(y) = T_{q_1}(y_1) T_{q_2}(y_2) \cdots T_{q_n}(y_n), \]
\[ C^{(k)}(q_1 \cdots q_n) = \frac{2^{N-\gamma}}{\pi} \prod_{i=1}^{N}(N+1-\gamma) \sum_{j_1=0}^{N} \cdots \sum_{j_n=0}^{N} \sum_{j_i=0}^{\lfloor q_i/2 \rfloor} \cdots \sum_{j_n=0}^{\lfloor q_n/2 \rfloor} \]
\[ \times G_{(r_1 \cdots r_n)}^{(k)}(y_{j_11} \cdots y_{j_n1}), \]
\[ \gamma = \{ \text{the number of } q_i = 0 : 1 \leq i \leq n \} \]
The interpolating points \( \{ y_{ij} \} \) are set to be
\[ y_{ij} = \cos \frac{2ji + 1}{2N+2} \pi \quad (i = 1, \cdots, N, \ j_i = 0, \cdots, N) \quad (14) \]

This \( G_{(r_1 \cdots r_n)}^{(k)}(y) \) in Eq. (12) is expressed by the use of the polynomial representation of the Chebyshev polynomials (Eq. (11)) and the binomial theorem in terms of \( x_i (i = 1, \cdots, n) \) as
\[ G_{(r_1 \cdots r_n)}^{(k)}(y) = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C^{(k)}(q_1 \cdots q_n) \]
\[ \times \sum_{s_1=0}^{\lfloor q_1/2 \rfloor} \cdots \sum_{s_n=0}^{\lfloor q_n/2 \rfloor} \sum_{s_n=0}^{\lfloor q_n/2 \rfloor} \frac{(-1)^{s_1}}{s_1!} \cdots \frac{(-1)^{s_n}}{s_n!} \]
\[ \times q_1 - 2s_1 \frac{q_1}{p_1^{(k)}}(q_1 - 2s_1 - j_1) \cdots \frac{q_n}{p_n^{(k)}}(q_n - 2s_n - j_n) \]
\[ \times x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \quad (15) \]
where \( \binom{i}{j} = \frac{i!}{j!(i-j)!} \) is the binomial coefficient. Substituting this \( G_{(r_1 \cdots r_n)}^{(k)}(y) \) into Eq. (9) yields
\[ \dot{\phi}_{(r_1 \cdots r_n)}(x) \approx \sum_{j_1=0}^{N} \cdots \sum_{j_n=0}^{N} A^{(k)}(j_1 \cdots j_n) \dot{\phi}_{(j_1 \cdots j_n)}(x) \]
where
\[ A^{(k)}(j_1 \cdots j_n) = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} \sum_{s_1=0}^{\lfloor q_1/2 \rfloor} \cdots \sum_{s_n=0}^{\lfloor q_n/2 \rfloor} C^{(k)}(q_1 \cdots q_n) \]
\[ \times d_{q_1}^{(i_1)} \cdots d_{q_n}^{(i_n)} \frac{q_1}{p_1^{(k)}}(q_1 - 2s_1 - j_1) \cdots \frac{q_n}{p_n^{(k)}}(q_n - 2s_n - j_n) \]
Thus, \( \dot{\phi}_{(r_1 \cdots r_n)}(x) \) on a subdomain \( D_k \) is approximated by the formal linearization function as
\[ \dot{\phi}_{(r_1 \cdots r_n)}(x) \approx \left[ A^{(k)}(10 \cdots 0), A^{(k)}(01 \cdots 0), \cdots, A^{(k)}(j_1 \cdots j_n) \right], \]
\[ \cdots, A^{(k)}_{(r_1, \cdots, r_n)} \phi(x) + A^{(k)}_{(r_1, \cdots, r_n)} y = 0 \quad (k = 1, \cdots, M) \quad (L-3) \]

and a linear dynamic equation with respect to \( \phi \) is obtained as

\[ \dot{\phi}(x) \approx A^{(k)} \phi(x) + b^{(k)} \quad (i = 1, \cdots, n, j_i = 0, \cdots, N) \quad (L-3.3) \]

where

\[ A^{(k)} = \left[ A^{(k)}_{(r_1, \cdots, r_n)} \right], \quad b^{(k)} = \left[ A^{(k)}_{(r_1, \cdots, r_n)} \right] \]

We unite \((M + 1)\) linearized systems (Eq. (17)) on subdomains into a single linear system on the whole domain by using Eq. (5) as

\[ \dot{\phi}(x) = \sum_{k=0}^{M} \phi(x)I_k(\zeta) \approx \sum_{k=0}^{M} \left( A^{(k)} \phi(x) + b^{(k)} \right)I_k(\zeta) \]

where

\[ \hat{A}(\zeta) = \sum_{k=0}^{M} A^{(k)}I_k(\zeta), \quad \hat{b}(\zeta) = \sum_{k=0}^{M} b^{(k)}I_k(\zeta) \]

Finally, a pseudo-formal linearization system is defined as

\[ \Sigma_2: \dot{z}(t) = \hat{A}(\zeta)z(t) + \hat{b}(\zeta), \quad z(0) = \phi(x(0)) \quad (L-3.5) \]

From Eq. (8), its inversion is carried out using

\[ \dot{x}(t) = [I, 0, \cdots, 0]z(t) \quad (L-3.6) \]

as the approximated value of \( x(t) \), where \( I \) is the \( n \times n \) unit matrix.

As a result, a pseudo-formal linearization algorithm is obtained as follows.

**Pseudo-Formal Linearization Algorithm**

**L-1** Given

\[ \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in D \]

**L-2** Set

\[ L, C(x), M, \mu, N, M^{(k)}, P^{(k)} \quad (k = 1, \cdots, M) \]

**L-3**

\[ (L-3.1) \]

\[ I_k(\zeta) = \prod_{j=1}^{L} \left( 1 - \frac{1}{1 + \exp(2\mu(\zeta - a_{kj}))} \right) - \frac{1}{1 + \exp(-2\mu(\zeta - b_{kj}))} \quad (0 \leq k \leq M - 1) \]

**L-3.2**

\[ T_q(y_{ij}) = \cos(q \cdot \cos^{-1} y_{ij}), \quad y_{ij} = \cos \frac{2j_i + 1}{2N + 2} \pi \]

\[ d^q_s = \left\{ \begin{array}{ll}
(-1)^{s}2^{q-2s-1} & \text{if } 2s < q \\
(-1)^s & \text{if } q = 2s
\end{array} \right. \]

**L-3.4**

\[ C^{(k)}(r_1, \cdots, r_n)(y) = \frac{\partial}{\partial y_i} P^{(k)-1} \hat{A}(r_1, \cdots, r_n)(P^{(k)}y + M^{(k)}) \]

\[ \times (P^{(k)}y + M^{(k)}) \]

**L-3.5**

\[ = \frac{2^{n-\gamma}}{n!} \sum_{i=1}^{n} \prod_{j=0}^{N} \sum_{j_z=0}^{N} \cdots \sum_{j_1=0}^{N} G^{(k)}_{(r_1, \cdots, r_n)}(y_{j_1} \cdot y_{2j_1} \cdots \cdot y_{nj_1}) \]

\[ \times T_{q_1}(y_{j_1})T_{q_2}(y_{2j_1}) \cdots T_{q_n}(y_{nj_1}) \]

\[ \gamma = \{ \text{the number of } q_i = 0 : 1 \leq i \leq n \} \]

**L-4**

\[ A^{(k)}_{(r_1, \cdots, r_n)} = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} \sum_{s_1=0}^{[q_1/2]} \cdots \sum_{s_n=0}^{[q_n/2]} C^{(k)}_{(r_1, \cdots, r_n)} \]

\[ \frac{d^{q_1}_{s_1} \cdots d^{q_n}_{s_n}}{P^{(k)}_{1}^{-2s_1} \cdots P^{(k)}_{n}^{-2s_n}} \left( q_1 - 2s_1 \right) \left( -M^{(k)}_{1} \right)^{q_1 - 2s_1 - j_1} \]

\[ \cdots \left( q_n - 2s_n \right) \left( -M^{(k)}_{n} \right)^{q_n - 2s_n - j_n} \]

**L-5**

\[ \hat{A}(\zeta) = \sum_{k=0}^{M} A^{(k)}I_k(\zeta), \quad \hat{b}(\zeta) = \sum_{k=0}^{M} b^{(k)}I_k(\zeta) \]

**L-6**

\[ \dot{x}(t) = [I, 0, \cdots, 0]z(t) \quad (L-6) \]

In the next section, we synthesize an algorithm for a nonlinear observer as an application of the linearization.
3. Nonlinear Observer

A nonlinear dynamic system is the same as Eq. (1) and a measurement equation is assumed to be
\[ \eta(t) = h(x(t)) \] (21)
where \( \eta \) is a scalar measurement and \( h \in \mathbb{R} \) is a nonlinear function and continuously differentiable. To improve the accuracy of estimation, an \( n \)th order measurement vector \( Y \) [7] is introduced as
\[ Y = [\eta, \eta^2, \cdots, \eta^r]^T \] (22)
We apply Chebyshev interpolation up to the \( N \)th order to each element of \( Y \) on the same subdomain \( D_k \) as in the state space. Then we have
\[ Y_r = \frac{h'(x)}{r!} \]
\[ \approx \frac{N}{r!} \sum_{q_i=0}^{N} \cdots \sum_{q_n=0}^{N} C^<(q_{1\cdots q_n}) \cdot T_{q_{1\cdots q_n}}(y) \triangleq G^<(r)(y) \] (23)
where
\[ C^<(q_{1\cdots q_n}) = \frac{2^{n-\gamma}}{n!} \prod_{i=1}^{n} (N+1) \sum_{j_1=0}^{q_1} \cdots \sum_{j_n=0}^{q_n} \]
\[ \times \sum_{j_n=0}^{N} C^<(r)(y_{1j_1}, y_{2j_2}, \cdots, y_{nj_n}) \]
\[ \times T_{q_{1j_1}}(y_{1j_1}) T_{q_{2j_2}}(y_{2j_2}) \cdots T_{q_{nj_n}}(y_{nj_n}) \] (24)
\[ \gamma = \{ \text{the number of } q_i = 0 : 1 \leq i \leq n \} \]
This \( G^<(r)(y) \) is also expressed by the use of the polynomial representation of the Chebyshev polynomials and the binomial theorem in terms of \( x_i \) \( (i=1, \cdots, n) \) as
\[ G^<(r)(y) = \left\{ \begin{array}{l}
\sum_{q_i=0}^{N} \cdots \sum_{q_n=0}^{N} C^<(q_{1\cdots q_n}) \left\{ \sum_{s_1=0}^{q_1/2} d_{s_1} q_1^{q_1 - 2s_1} \right. \\
\left. \cdots \sum_{s_n=0}^{q_n/2} d_{s_n} q_n^{q_n - 2s_n} \right\} \\
\end{array} \right. \]
\[ = \left[ \begin{array}{l}
\sum_{q_i=0}^{N} \cdots \sum_{q_n=0}^{N} C^<(q_{1\cdots q_n}) \left\{ \sum_{s_1=0}^{q_1/2} d_{s_1} q_1^{q_1 - 2s_1} \right. \\
\left. \cdots \sum_{s_n=0}^{q_n/2} d_{s_n} q_n^{q_n - 2s_n} \right\} \\
\end{array} \right] \]
\[ = \left[ \begin{array}{l}
\sum_{q_i=0}^{N} \cdots \sum_{q_n=0}^{N} \left\{ \sum_{s_1=0}^{q_1/2} d_{s_1} q_1^{q_1 - 2s_1} \right. \\
\left. \cdots \sum_{s_n=0}^{q_n/2} d_{s_n} q_n^{q_n - 2s_n} \right\} \\
\end{array} \right] \]
\[ = \sum_{q_i=0}^{N} \cdots \sum_{q_n=0}^{N} \left\{ \sum_{s_1=0}^{q_1/2} d_{s_1} q_1^{q_1 - 2s_1} \right. \\
\left. \cdots \sum_{s_n=0}^{q_n/2} d_{s_n} q_n^{q_n - 2s_n} \right\} \]

Thus, \( Y_r \) on a subdomain \( D_k \) is approximated by the formal linearization function as
\[ Y_r = \frac{h'(x)}{r!} \approx \left[ H^0(r), H^1(r), \cdots, H^N(r) \right] \phi(x) + H^0(r) \] (26)
and a linear measurement equation with respect to \( \phi \) is obtained as
\[ Y \approx \left[ H^0(r), H^1(r), \cdots, H^N(r) \right] \phi(x) + \left[ H^0(r) \right] \] (27)
We unite \( M+1 \) linearized systems (Eq. (27)) on subdomains into a single linear system on the whole domain by using Eq. (5) as
\[ Y \approx \left[ \sum_{k=0}^{M} H^k(r) \phi(x) + \sum_{k=0}^{M} d^k(r) \right] \sum_{k=0}^{M} \sum_{k=0}^{M} H^k(I_k(\zeta)) \] (28)
Therefore, a pseudo-formal linearization system for the measurement equation is approximately derived as
\[
Y(t) = \hat{H}(\zeta)z(t) + \tilde{d}(\zeta) \tag{29}
\]
where
\[
\hat{H}(\zeta) = \sum_{k=0}^{M} H^{(k)} I_k(\zeta), \quad \tilde{d}(\zeta) = \sum_{k=0}^{M} d^{(k)} I_k(\zeta)
\]
To the linearized systems in Eqs. (19) and (29), the linear observer theory is applied. Thus, an identity observer is synthesized as
\[
\dot{\hat{z}}(t) = \hat{A}(\zeta)\hat{z}(t) + \hat{b}(\zeta) + K(Y(t) - \hat{H}(\zeta)\hat{z}(t) - \tilde{d}(\zeta)) \tag{30}
\]
where \(\zeta = C(\hat{z})\) and \(K^{(k)}\) is the observer gain on a subdomain \(D_k\) given by
\[
K^{(k)} = S^{(k)-1} P^{(k)} H^{(k)T}
\]
\(P^{(k)}\) satisfies the matrix Riccati equation
\[
A^{(k)} P^{(k)} + P^{(k)} A^{(k)T} + Q^{(k)} - P^{(k)} H^{(k)T} S^{(k)} H^{(k)} P^{(k)} = 0 \tag{31}
\]
where \(Q^{(k)}\) and \(S^{(k)}\) are arbitrary real symmetric positive-definite matrices.

From Eq. (8), the estimate \(\hat{x}(t)\) of the nonlinear observer becomes
\[
\dot{\hat{x}}(t) = [I, 0, \cdots, 0] \hat{z}(t) \tag{32}
\]
As a result, a nonlinear observer algorithm is obtained as follows.

**Observer Algorithm**

(O-1) Given
\[
\begin{align*}
\dot{x}(t) &= f(x(t)), \quad x(0) = x_0 \in D \\
\eta(t) &= h(x(t))
\end{align*}
\]

(O-2) Set
\[
L, C(x), M, \mu, N, M^{(k)}, P^{(k)}, \ell, Q^{(k)}, S^{(k)}
\]
\((k = 1, \cdots, M)\)

(O-3)

(O-3.1)
\[
I_k(\zeta) = \prod_{j=1}^{L} \left(1 - \frac{1}{1 + \exp(2\mu(\zeta_j - a_{kj}))}\right)
\]

\[
T_{\eta}(y_{ij}) = \cos(q \cdot \cos^{-1} y_{ij}), \quad y_{ij} = \cos \frac{2\zeta_j + 1}{2N + 2\pi}
\]
\((i = 1, \cdots, n, \quad j = 0, \cdots, N)\)

\[
\begin{align*}
\sum_{j=0}^{N} C_{r_1 \cdots r_n}^{(k)}(y) &= \frac{\partial}{\partial y} P^{(k)-1} \phi_{r_1 \cdots r_n}(P^{(k)} y + M^{(k)}) \\
&\times f(P^{(k)} y + M^{(k)}) \\
&+ h'(P^{(k)} y + M^{(k)}) \frac{r!}{r!} \tag{33}
\end{align*}
\]

\[
C_{r_1 \cdots r_n}^{(k)}(y) = \frac{2^{n-\gamma}}{\prod_{i=1}^{n} (N + 1)} \sum_{j_1=0}^{N} \sum_{j_2=0}^{N} \cdots \sum_{j_n=0}^{N}
\]
\[
\sum_{j_n=0}^{N} C_{r_1 \cdots r_n}^{(k)}(y_{j_1 j_2 \cdots j_n}) \times T_{q_1}(y_{j_1 j_2}) \times T_{q_2}(y_{j_2 j_3}) \cdots T_{q_n}(y_{j_n j_1})
\]
\[
\gamma = \{\text{the number of } q_i = 0 : 1 \leq i \leq n\}
\]

\[
A_{r_1 \cdots r_n}^{(k)j_1 \cdots j_n} = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} \sum_{s_1=0}^{[q_1/2]} \cdots \sum_{s_n=0}^{[q_n/2]}
\]
\[
\frac{d_{q_1}^{(q_1 \cdots q_n)}}{P^{(k)q_1 \cdots q_n-2s_n}} \frac{d_{q_2}^{(q_2)}}{P^{(k)q_2-2s_2-j_1} \cdots P^{(k)q_n-2s_n-j_n}} (-M_{s_1}^{(k)})^{q_1-2s_1-j_1} \cdots (-M_{s_n}^{(k)})^{q_n-2s_n-j_n}
\]

\[
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\]
\[ H^{(k)}(\eta) = \sum_{q=0}^{N} \left. \sum_{s=0}^{[q/2]} \left. \sum_{j=0}^{[q_n/2]} C^{(k)}(q_1, q_2) \right\} \right\} \]

\[
\frac{d^{(q_1)} x_1 \ldots d^{(q_n)} x_n}{P^{(k)}_{n=2} \ldots P^{(k)}_{n=2-n}} \left( q_{n-2s_1} j_1 \right) \left( -M_n^{(k)} \right)_{n-2s_1-j_1} \]

\[ \ldots \left( q_{n-2s_n} / j_n \right) \left( -M_n^{(k)} \right)_{n-2s_n-j_n} \]

\[ \text{(O-4)} \]

\[ A^{(k)} = \left[ A^{(k)}(q_1 \ldots q_n) \right], \quad b^{(k)} = \left[ b^{(k)}(q_1 \ldots q_n) \right], \]

\[ H^{(k)} = \left[ H^{(k)}(\eta) \right], \quad d^{(k)} = \left[ H^{(k)}(\eta) \right] \]

\[ (k = 1, \ldots, M) \]

\[ \tilde{A}(\zeta) = \sum_{k=0}^{M} A^{(k)} I_k(\zeta), \quad \tilde{b}(\zeta) = \sum_{k=0}^{M} b^{(k)} I_k(\zeta), \]

\[ \tilde{H}(\zeta) = \sum_{k=0}^{M} H^{(k)} I_k(\zeta), \quad \tilde{d}(\zeta) = \sum_{k=0}^{M} d^{(k)} I_k(\zeta) \]

\[ \text{(O-5)} \]

\[ P^{(k)} \text{ by solving the Riccati equation:} \]

\[ A^{(k)} P^{(k)} + P^{(k)} A^{(k)^T} + Q^{(k)} \]

\[ -P^{(k)} H^{(k)^T} S^{(k)} H^{(k)} P^{(k)} = 0 \quad (k = 1, \ldots, M) \]

\[ \text{(O-6)} \]

\[ \dot{z}(t) = \tilde{A}(\zeta) \dot{z}(t) + \tilde{b}(\zeta) + K(Y(t) - \tilde{H}(\zeta) \dot{z}(t) - \tilde{d}(\zeta)) \]

where

\[ K^{(k)} = S^{(k)-1} P^{(k)} H^{(k)^T} \]

\[ \text{(O-7)} \]

\[ \dot{x}(t) = [I, 0, \ldots, 0] \dot{z}(t) \]

4. Numerical Experiments

To show the effectiveness of this method, numerical experiments of a pendulum system are illustrated.

4.1 Pseudo-formal linearization

Consider a pendulum system, that consists of a point mass \( m \text{[kg]} \) suspended from a support by a massless rod of length \( l \text{[m]} \). Let \( \theta \text{[rad]} \) denote the angle subtended by the rod and the vertical axis through the pivot point and the coefficient of friction be \( k \text{[kg/s]} \). The equation of motion in the tangential direction can be written as

\[ ml \ddot{\theta}(t) = -mg \sin(\theta(t)) - kl \dot{\theta}(t) \]

where \( g \) is the acceleration due to gravity. Let us take the parameters as \( l = 9.81 \text{, } m = 1 \text{, } k = 1 \) and the state variable as \( x_1 = \theta \) and \( x_2 = \dot{\theta} \), so the state equation is

\[ \dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\sin(x_1(t)) - x_2(t) \end{bmatrix} = f(x(t)) \]

We set \( C(x) = x_1 \) in Eq. (2) because of its highest nonlinearity \( \sin(x_1) \) and \( L = 1 \). Let the domain \( D \) be

\[ D = [-34\pi, 34\pi] \times [-2.6, 0.4] \]

We divide the whole domain of \( x_1 \) into three subdomains \((M = 2)\) as

\[ D_0 = [-34\pi, -14\pi], \quad D_1 = [-14\pi, 14\pi], \quad D_2 = [14\pi, 34\pi] \]

The system parameters are set as

\[ M^{(0)} = \left[ \begin{array}{c} -\pi \\ -0.34 \end{array} \right], \quad M^{(1)} = \left[ \begin{array}{c} 0 \\ -0.34 \end{array} \right], \quad M^{(2)} = \left[ \begin{array}{c} \pi \\ -0.34 \end{array} \right] \]

\[ P^{(k)} = \left[ \begin{array}{cc} \pi/4 & 0 \\ 0 & 0.47 \end{array} \right] \quad (k = 0, 1, 2) \]

Figure 2 shows the true value \( x(t) \), which is a solution of the given system (Eq. (35)), and the approximated values \( \hat{x}(t) \) given by Eq. (20) when the order of the linearization function \( N \) is varied from 1 to 3. \( \hat{x}(N = 3\text{(old))} \) denotes the results obtained in our previous work [8] for comparison.

To clarify the difference in the approximation errors in Fig. 2, Fig. 3 shows the integral square errors of the estimation

\[ J(t) = \int_{0}^{t} (x(\tau) - \hat{x}(\tau))^T (x(\tau) - \hat{x}(\tau)) d\tau \]

for the various orders in these cases. Table 1 shows running times for calculating the coefficients of the
Figure 3 Integral square errors of estimation compared with those of previous method

pseudo-formal linearization by using the proposed algorithm and those in the previous work [8] and the difference between them. We used a DELL Optiplex 3040 (Intel Core i7 CPU, 3.4GHz, 8 GB RAM) and the software was PTC Mathcad.

Table 1 Running times for the linearization

| N   | Proposed(s) | Previous(s) | Diff.(s) |
|-----|-------------|-------------|----------|
| 1   | 0.0160      | 0.0780      | -0.0620  |
| 2   | 0.0310      | 1.8440      | -1.8130  |
| 3   | 0.0780      | 13.139      | -13.061  |

From Fig. 3 and Table 1, we see that the proposed method has an accuracy of linearization similar to the previous method [8], but the running time for the linearization is much shorten. This is because the previous method requires numerical integration techniques, whereas the proposed method simply employs the finite sum on the interpolation points (Eq. (14)). The new method reduces the running time to about 6/1000 of that for the previous method when N = 3 for example. Therefore, when the performance is considered in terms of both the approximation error J(t) and the running time, this newly proposed method is much better than the previous ones.

4.2 Nonlinear observer

A nonlinear system is the same as Eq. (35) and a measurement equation is assumed to be

\[ \eta = \sin(x_1(t)) \equiv h(x(t)) \]  

(37)

For the nonlinear observer, we set

\[ \hat{x}(0) = [1.6, 0.3]^T, \quad Q^{(k)} = 0.5I \]

\[ S^{(k)} = 0.1 \quad (k = 0, 1, 2) \]

Figure 4 shows the true value x(t) and the estimates \( \hat{x}(t) \) in Eq. (32) obtained by the proposed algorithm

when the order \( \ell \) in Eq. (22) is varied from 1 to 2. \( \hat{x}(N = 2, \ell = 1 (\text{old})) \) is the result obtained in the previous work [8]. Figure 5 shows the integral square errors of the estimation

\[ J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^T(x(\tau) - \hat{x}(\tau))d\tau \]

in these cases.

Figures. 4 and 5 show that the performance of the nonlinear observer is almost the same as that of the previous method [8] and the estimation error of the nonlinear observer is improved as the order \( \ell \) increases.

5. Conclusions

We proposed a computational algorithm for the pseudo-formal linearization method by using Chebyshev interpolation and synthesized an algorithm for a nonlinear observer as its application. Numerical experiments showed that the performance of this new method is improved compared with that of the previous method [8]. It is left for future studies to extend this method to multidimensional measurement systems.
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measurement equation is assumed to be

A nonlinear system is the same as Eq. (35) and a

4.2 Nonlinear observer

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The new method reduces the running time to about

The software was PTC Mathcad.

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From Fig. 3 and Table 1, we see that the pro-

previous method [8]. It is left for future studies to ex-

method is improved compared with that of the pre-

nonlinear observer as its application. Numerical ex-

Figures 4 and 5 show that the performance of the

previous work [8]. Figure 5 shows the integral square

Figure 3 shows the true value

Figure 4 shows the true value

Table 1 shows running times for the linearization

Table 1 shows running times for the linearization

| N | Proposed(s) | Previous(s) | Diff.(s) |
|---|-------------|-------------|---------|
| 0 | 0.0780 | 13.139 | -13.061 |
| 2 | 0.0310 | 1.8440 | -1.8130 |
| 4 | 0.0160 | 0.0780 | -0.0620 |

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Kazuo Komatsu received his B.S. degree in computer science and Dr. Eng. degree in electrical engineering from Kyushu Institute of Technology in 1985 and 1995, respectively. He is currently a Professor at the Department of Human-Oriented Information Systems Engineering of the National Institute of Technology, Kumamoto College. His research interests include formal linearization for nonlinear systems and its applications. He is a member of the RISP.

Hitoshi Takata received his B.S. degree in electrical engineering from Kyushu Institute of Technology in 1968 and his M.S. and Dr. Eng. degrees in electrical engineering from Kyushu University in 1970 and 1974, respectively. He is currently a Professor Emeritus at Kagoshima University. His research interests include the control, linearization, and identification of nonlinear systems.