MCMULLEN’S AND GEOMETRIC PRESSURES AND APPROXIMATING HAUSDORFF DIMENSION OF JULIA SETS FROM BELOW

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Abstract. We introduce new variants of the notion of geometric pressure for rational functions on the Riemann sphere, including non-hyperbolic functions, in the hope some of them occur useful to achieve a fast approximation from below of the hyperbolic Hausdorff dimension of Julia set.

Contents
1. Introduction 1
2. Definitions, the statement of a main theorem 8
2.1. Topological pressures 8
2.2. Geometric pressures 4
2.3. Hausdorff dimension 6
2.4. Main Theorem 6
3. Fuzzy (infimum) and pullback infimum tree pressures 7
4. McMullen’s pressures 11
5. Restricted fuzzy tree pressure 16
6. Final remarks, more geometric pressures and examples 17
6.1. On convergence 17
6.2. Double sampling pressures 19
6.3. Examples 21
References 22

1. Introduction

Iterating the action of a mapping \( f \), (roughly) increasing distances, for example a rational function on a neighbourhood of its ”chaotic” invariant Julia set.
set, leads its small subsets to large ones, roughly preserving shapes (dynamical "escalator"). So a long time behaviour under the action of $f$, provides an insight into the local structure of $J(f)$, here, in complex dimension 1, into its Hausdorff dimension characteristic.

A tool is a geometric pressure with respect to the potential $\phi = \phi_t = -t \log |f'|$ on Julia set, here for $t > 0$. The pressure (free energy) can be defined for any $\phi$ in a variational way

\begin{equation}
P_{\text{var}}(f, \phi) = \sup_{\mu} (h_{\mu}(f) + \int \phi \, d\mu),
\end{equation}

supremum over all $f$-invariant probability measures $\mu$ on $J(f)$. $h_{\mu}(f)$ means the measure-theoretical (Kolmogorov’s) entropy and $\mu$ for which the supremum is attained (if it exists) an equilibrium state. Below we provide different definitions. There is an analogy with equilibria in statistical physics, e.g. Ising model of ferromagnetic, where equilibria are distributed on the space of all configurations of + or - over $\mathbb{Z}^2$ with a potential depending on a hamiltonian function expressed in terms of interactions between elements of the configuration.

The founders of applications in dynamics are in particular Y. Sinai, D. Ruelle and R. Bowen (SRB measures). Here we consider forward trajectories so the ”configurations” are over $\mathbb{N}$. In particular a geometric application with the use of $\phi_t$, hence $\exp S_n(\phi_t) = |(f^n)'|^{-t}$, is to relate (roughly) an equilibrium measure (a mass) of a disc of diameter $|(f^n)'|^{-1}$ with this diameter, for each $t$ up to a normalizing coefficient $\exp nP(f, \phi_t)$, equal to 1 if $t = t_0$ a zero of the pressure $P(f, \phi_t)$, denoted also $P(f, t)$. This $t_0$ is called hyperbolic Hausdorff dimension of $J(f)$, $\text{HD}_{\text{hyp}}(f, t)$, defined as the supremum of Hausdorff dimensions of invariant hyperbolic subsets of $J(f)$.

An introduction to this theory is provided in [P-Ubook]. Closer to the content of the paper are [P-TAMS2] and [PRS], where geometric pressure for general rational functions was first defined and studied. See also [P-ICM].

The aim of this note is to introduce more variants of this notion, in particular some, close to McMullen’s pressure defined for hyperbolic rational functions in [McM-HD3], useful to numerically calculate Hausdorff dimension of the underlying Julia sets, estimate it from below.

The pressure function $t \mapsto P(f, t)$ will occur to be a limit from below of a sequence of functions specific to each notion of the pressure we introduce. Therefore their first zeros will converge to $\text{HD}_{\text{hyp}}(f, t)$ from below. There are two concepts in these notions of pressure. 1. To replace a potential along each trajectory by a ”fuzzy” one, mainly by replacing the value at a point by the infimum of the value in its small neighbourhood, or the smaller one at one of two sampling points close to it. 2. To restrict the pressure to trajectories not passing too close to the set of critical points.
The key issue we do not address here is how efficient is the calculation of these functions and how fast are the convergences. This will be dealt with in [DGT].

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2. Definitions, the statement of a main theorem

2.1. Topological pressures. Start recalling the basic definition of the topological pressure for a continuous transformation of a metric compact space and real continuous potential, see e.g. [Walters] or [PUbook].

Definition 2.1 (topological pressure via separated sets). Let \( f : X \to X \) be a continuous map and \( \phi : X \to \mathbb{R} \) a continuous real-valued function (potential) on \( X \). Consider for \( S_n \) defined below, in (2.2),

\[
P_{sep}(f, \phi) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \left( \sup_{y \in Y} \exp S_n \phi(y) \right),
\]

where the supremum is taken over all \( (n, \varepsilon) \)-separated sets \( Y \subset X \), that is such \( Y \) that for every \( y_1, y_2 \in Y \) with \( y_1 \neq y_2 \), \( \rho_n(y_1, y_2) \geq \varepsilon \), where \( \rho_n \) is the metric defined by \( \rho_n(x, y) = \max \{ \rho(f^j(x), f^j(y)) : j = 0, \ldots, n \} \).

We can slightly modify this definition, defining a fuzzy pressure or inf-pressure \( P_{sep}^0(f, \phi) \) by first replacing in

\[
S_n \phi(y) := \sum_{j=0}^{n-1} \phi(f^j(y))
\]

in (2.1), \( \phi(x) \) for \( x = f^j(y) \) for each \( j \), by \( \inf \{ \phi(z) : z \in B(x, \delta) \} \) thus defining

\[
S_n^\delta \phi(y) := \sum_{j=0}^{n-1} \inf \{ \phi(z) : z \in B(f^j(y), \delta) \}
\]

writing \( P_{sep}^\delta(f, \phi) \), and at the end defining

\[
P_{sep}^0(f, \phi) := \lim_{\delta \to 0} P_{sep}^\delta(f, \phi).
\]

By the uniform continuity of \( \phi \), an easy calculation gives \( P_{sep}(f, \phi) = P_{sep}^0(f, \phi) \).
A related notion is tree pressure (or Gurevitch pressure), interesting for a non-invertible \( f \), defined for an arbitrary \( z \in X \) by

\[
P_{\text{tree}}(f, \phi, z) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{y \in f^{-n}(z)} \exp S_n \phi(y).
\]

which also can be defined in a "fuzzy" way as a fuzzy tree pressure, another name infimum tree pressure,

\[
P^0_{\text{tree}}(f, \phi, z) := \lim_{\delta \to 0} P_{\delta, \text{tree}}(f, \phi, z)
\]

where

\[
P_{\delta, \text{tree}}(f, \phi, z) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{y \in f^{-n}(z)} \exp S_{\delta n} \phi(y).
\]

Later on we shall use the following easy, [PUbook, Chapter 4], Remark 2.2.

Suppose \( f : X \to X \) is open distance expanding topologically exact, namely for every open \( U \subset X \) there exists \( n \in \mathbb{N} \) such that \( f^n(U) = X \), and \( \phi : X \to \mathbb{R} \) be continuous. Then all the above pressures coincide and are independent of \( z \). So we can denote them just by \( P(f, \phi) \).

2.2. Geometric pressures. From now on we shall consider a rational transformation of the Riemann sphere \( f : \mathbb{C} \to \mathbb{C} \) and its restriction to the Julia set \( J(f) \). We shall consider geometric potentials \( \phi = \phi_t = -t \log |f'| \) for \( t > 0 \). Note that in this case we can write \( \exp S_n \phi_t \) in Definition 2.1 in the form \( |(f^n)'|^{-t} \). The derivative \( f' \) will be considered only with respect to the spherical Riemann metric. We shall use only its absolute value \( |f'| \) so there will be no ambiguity caused by its argument. The points \( x \in \mathbb{C} \) where \( f'(x) = 0 \) are called critical and their set denoted by Crit\((f)\). For \( c \in \text{Crit}(f) \) where in the complex plane coordinates \( f(z) = a(z - c)^\nu + b(z - c)^{\nu+1} + \ldots \), with \( a \neq 0 \), \( \nu = \nu(c) \) is called the multiplicity of \( f \) at \( c \). We shall consider also the post-critical set, \( \text{PC}(f) := \bigcup_{n=1}^{\infty} f^n(\text{Crit}(f)) \).

If the forward trajectory of no critical point accumulates at \( J(f) \), that is there are no \( f \)-critical points in \( J(f) \), nor attracted to parabolic periodic orbits, then \( f|_{J(f)} \) is open expanding. Another term for this is hyperbolic. It means there exist \( C > 0, \lambda > 1 \) such that \( |(f^n)'(z)| > C\lambda^n \) for all \( z \in J(f) \) and \( n \in \mathbb{N} \). This is an easy case, covered by Remark 2.2.

We shall consider from now on a general case with critical points whose forward trajectories can accumulate at \( J(f) \). The above definitions of \( P_{\text{tree}}(f, \phi, z) \) and \( P^0_{\text{tree}}(f, \phi, z) \) above make sense even though \( \phi_t \) is infinite at critical points.
(yielding $\infty$) and for $z$ outside $J(f)$. The pressure $P_{\text{tree}}(f, \phi_t, z)$ does not depend on $z \in \mathbb{C}$ for non-exceptional $z$ (not in or fast accumulated by $\text{PC}(f)$, see Definition 2.10). See [P-TAMS2] and [PRS]. So we can denote it just by $P_{\text{tree}}(f, t)$. The independence of $P_{\text{tree}}^0(f, \phi, z)$ of non-exceptional $z$ also holds, via $P_{\text{hyp}}(f, t)$ and $P_{\text{tree}}(f, t)$, see Main Theorem 2.4, item 1.

To fix notation for $\phi = \phi_t = -t \log |f'|$ let us rewrite:

**Notation 2.3** (geometric tree pressure).

\[(2.6) \quad P_{\text{tree}}(f, t, z) = P_{\text{tree}}(f, \phi_t, z) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{v \in f^{-n}(z)} \Pi_n(t, v)\]

where

\[(2.7) \quad \Pi_n(t, v) := \prod_{k=1}^{n} \left| f'(f^{n-k}(v)) \right|^{-t} = |(f^n)'(v)|^{-t}.\]

and

**Notation 2.4** (geometric fuzzy tree pressure).

\[(2.8) \quad P_{\text{tree}}^0(f, t, z) := \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sum_{v \in f^{-n}(z)} \prod_{k=0}^{n} \inf \left\{|f'(y)|^{-t} : y \in B(f^{n-k}(v), \delta)\right\}.\]

We shall introduce also a new notion

**Definition 2.5** (pullback infimum tree pressure).

\[(2.9) \quad P_{\text{pullinf}}(f, t, z) := \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sum_{v \in f^{-n}(z)} \Pi_{n}^{\text{pullinf}}(t, v)\]

where

\[(2.10) \quad \Pi_{n}^{\text{pullinf}}(t, v) := \prod_{k=1}^{n} \inf \left\{|f'(y)|^{-t} : y \in \text{Comp}_{f^{n-k}(v)} f^{-k}(B(f^n(v), r))\right\}\]

where $\text{Comp}_x$ means the component of a set containing $x$.

The limit for $r \to 0$ exists since the family under it is monotone increasing as $r \to 0$ because infima in (2.10) are taken on shrinking sets.

Let the following definition of $P(f, -t \log |f'|)$, called hyperbolic pressure be considered as a default one, to be denoted $P(f, t)$, see e.g. [P-ICM]:

**Definition 2.6** (Hyperbolic pressure).

\[P(f, t) = P_{\text{hyp}}(f, t) := \sup_{X \in \mathcal{X}(f, J(f))} P(f|_X, -t \log |f'|),\]
where \( \mathcal{H}(f, J(f)) \) is defined as the space of all compact forward \( f \)-invariant (that is \( f(X) \subset X \)) hyperbolic topologically exact subsets of \( J(f) \) and repelling for \( f \), that is if a forward trajectory of a point is in a sufficiently small neighbourhood of \( X \) then it is entirely in \( X \). The repelling assumption can be omitted without influencing the resulting \( P_{\text{hyp}}(f, t) \).

2.3. **Hausdorff dimension.** From this definition and Bowen’s formula for every hyperbolic set \( X \), saying that Hausdorff dimension \( \text{HD}(X) \) is the only zero of the function \( t \mapsto P(f|_X, t) \), see [Bowen], it immediately follows that:

**Proposition 2.7.** (Generalized Bowen’s formula) The first zero \( t_0 \) of \( t \mapsto P_{\text{hyp}}(f, t) \) is equal to the hyperbolic dimension \( \text{HD}_{\text{hyp}}(J(f)) \), defined by

\[
\text{HD}_{\text{hyp}}(J(f)) := \sup_{X \in \mathcal{H}(f, K)} \text{HD}(X).
\]

Note that this zero exists, since all \( P(f|_X, -t \log |f'|) \) are decreasing, hence their limit \( P_{\text{hyp}}(f, t) \) is decreasing and \( \text{HD}(X) \leq 2 \) since \( X \subset \overline{C} \) of dimension 2.

2.4. **Main Theorem.** We shall prove in this paper the following (some definitions to be provided later on)

**Theorem 2.8 (Main Theorem).** Let \( f : \overline{C} \to \overline{C} \) be a rational function backward uniformly asymptotically stable. Then

1. \( P(f, t) = P_{\text{tree}}^0(f, t, z) = P_{\text{tree}}^0(f, t, z) \) for all non-exceptional \( z \in \overline{C} \)
2. \( P(f, t) = \hat{P}_{\text{McM}}(f, t) = P_{\text{McM}}^0(f, t) \leq P_{\text{McM}}^0(f, t) \) (restricted and fuzzy McMullen’s pressures) provided for the puzzle structure \( \mathcal{P} \) in the definition of McMullen’s pressures the diameters of the puzzle pieces of the renormalizations \( \mathcal{R}^N(\mathcal{P}) \) tend uniformly to 0 as \( N \to \infty \).
3. \( P(f, t) = P_{\text{pullinf}}^r(f, t) \) for all \( r \) small enough, provided \( f \) is backward uniformly asymptotically stable. Moreover for all \( r \) small enough, all non-exceptional \( z \) and every backward trajectory \( (z_n) \) of \( z \) (that is \( f(z_n) = z_{n-1} \) for all \( n \in \mathbb{N} \), and \( z_0 = z \))

\[
(2.11) \quad \frac{1}{n} \log \left( \frac{\Pi_{\text{pullinf}}(t, z_n)}{|(f^n)'(z_n)|^{-t}} \right) \to 0
\]

as \( n \to \infty \), uniformly with respect to the backward trajectory \( (z_n) \), where \( z_0 = z \).

The equalities in the item 1. say in particular that the pressures there do not depend on \( z \) for all non-exceptional \( z \).
Definition 2.9. \( f \) is said to be \textit{backward uniformly asymptotically stable},
abbreviated: (buas), if there exists \( r_0 > 0 \) such that for each \( z \in J(f) \), Julia
set, diameters of all pullbacks of \( B(z, r_0) \) (namely: components of preimages) for \( f^n \) tend
to 0 uniformly fast, with respect to \( z \) and to the pullback, as \( n \to \infty \).

Notice that this property is hereditary, that is if it holds for \( r \), then it
holds for every \( r \leq r \).

Definition 2.10. We call a point \( z \in \overline{\mathbb{C}} \) \textit{non-exceptional} if for each \( \epsilon > 0 \) and
\( n \) large enough \( B(z, \exp(-n\epsilon)) \) is disjoint from \( f^j(\text{Crit}(f)) \) for all \( j = 1, ..., n \).
In particular \( z \) is not \textit{post-critical} that is \( z \notin PC(f) = \bigcup_{n=1}^{\infty} f^n(\text{Crit}(f)) \).
The other points are called \textit{exceptional} and the set of all exceptional points
is denoted by \( E \).

It is clear from the definition that the exceptional set \( E \subset \hat{\mathbb{C}} \) has Hausdorff
dimension 0. Sometimes it is comfortable to consider \( z \in \mathbb{C} \setminus J(f) \), so close
to \( J(f) \) that it cannot be postcritical itself neither be accumulated by the
postcritical set. Then of course it must be non-exceptional.

Remark 2.11. \( \limsup_{n \to \infty} \) in the definition (2.3) can be replaced by \( \lim \),
which always exists, see [PUbook, Remark 12.5.18]. So the limits exist also
in the definition of \( P_{\text{pullinf}}(f, t, z) \) for \( z \in J(f) \), provided the property (buas).
It will follow easily from the proof of Theorem 2.8, see (2.11) and (3.4). The
same holds for \( P_{\text{inf W}}(f, t, z) \), to be defined in (3.5).

3. Fuzzy (infimum) and pullback infimum tree pressures

Proof of Theorem 2.8, the item 1. It follows immediately from known theory
and easy observations.

Indeed, the inequality \( P_{\text{hyp}}(f, t) \leq P_{\text{tree}}(f, t, z) \) follows from the trivial
observation that for every \( X \in \mathcal{H}(f, J(f)) \) in Definition 2.6 and \( z \in X \),
\[
P_{\text{tree}}(f|_X, -t \log |f'| \leq P_{\text{tree}}(f, -t \log |f'|, z)
\]
since in the former pressure we count only backward trajectories in \( X \) of
a non-exceptional \( z \in X \), whereas in the latter one all in \( J(f) \). A non-
exceptional \( z \in X \) exists since \( \text{HD}_{\text{hyp}}(J(f)) > 0 \) hence in its definition it is
sufficient to consider only \( X \) satisfying \( \text{HD}(X) \), and \( \text{HD}(E) = 0 \).

The same reasoning holds for \( P_{\text{hyp}}(f, t) \leq P^0_{\text{tree}}(f, t) \), defined in (2.8).
Take in account that for every backward trajectory \((z_n)\) of \( z \) in \( X \) we have
\[
\log |f'(z_n)| \leq \log |f'(v)| + \epsilon \text{ for every } v \in \overline{\mathbb{C}} \text{ with } \rho(x_n, v) \leq \delta, \text{ for every } \epsilon \text{ and }
\delta \text{ small enough. This is so because } X \text{ is disjoint hence, as being compact,}
\text{ bounded away from Crit} f.
The inequality $P_{\text{tree}}^0(f, t, z) \leq P_{\text{tree}}(f, t, z)$ holds trivially for every $z$ since
$
\inf\{v \in B(z_k, \delta) : |f'(y)|^{-1} \leq |f'(z_k)|^{-1} \text{ for all } k.\}$

The latter yields also the monotone increasing of $P_{\text{tree}}^\delta(f, t, z)$ as $\delta \to 0$.

The proof that $P_{\text{tree}}(f, t, z) \leq P_{\text{hyp}}(f, t)$ for non-exceptional $z \in J(f)$ is harder, fortunately it is known. First one assumes that $z$ is non-exceptional and additionally hyperbolic, which means by definition that there exists $r > 0$ and $\lambda > 1$ such that for every disc $B(z, \tau)$ there exists $n \in \mathbb{N}$ such that $f^n(B(z, \tau))$ is univalent and $f^n(B(z, \tau)) \supset B(f^n(z), r)$ [PRS Proposition 2.1].

An idea of the proof is to capture a large hyperbolic set using a "shad owing".

Next one proves that for two non-exceptional $z^1$ and $z^2$ the equality $P_{\text{tree}}(f, t, z^1) = P_{\text{tree}}(f, t, z^2)$ holds. See [P-TAMS2 Theorem 3.3] and [PRS Geometric Lemma]. An idea is to find a curve (or a curve for each $n$) joining $z_1$ to $z_2$ in $\overline{C}$ not fast accumulated by $f^n(\text{Crit } f)$, therefore with a controlable distortion for all branches of $f^{-n}$ on its adequate neighbourhoods.

So one concludes the equality between the tree pressures for every non-exceptional $z$ and justifies the definition of tree pressure

\begin{equation}
P_{\text{tree}}(f, t) := P_{\text{tree}}(f, t, z)
\end{equation}

for every non-exceptional $z$.

\[\square\]

**Proof of Theorem 2.8, the item 3.** The inequality $P_{\text{tree}}^{\text{pullinf}}(f, t, z) \leq P_{\text{tree}}(f, t, z)$ holds trivially for every $z \in \overline{C}$ as before since obviously $\inf\{v \in W_{k,z_k,r} : |f'(y)|^{-1} \leq |f'(z_k)|^{-1} \text{ for all } k.$

So let us prove the opposite inequality for every non-exceptional $z \in J(f)$, namely

$P_{\text{tree}}^{\text{pullinf}}(f, t, z) \geq P_{\text{tree}}(f, t, z).$

We apply for any pullback $\text{Comp}_x f^{-k}(B(f^k(x), \tau))$ the notation $W_{k,x,\tau}$. Write $r_n := \sup_{z \in J(f), v \in f^n(z)} \{\text{diam } W_{n,v,r}\}$ and $r_{\text{max}} := \max\{r_n : n = 1, 2, \ldots\}$ for all $n = 1, 2, \ldots$ Notice that

$(*)$ for $r \to 0$ we have $r_{\text{max}} \to 0$.

Indeed, by the backward uniform asymptotic stability, for all $r \leq r_0$ as in Definition 2.9,

\[\forall \epsilon > 0 \exists n(\epsilon) \forall n \geq n(\epsilon) \forall W_{n,v,r}, \; \text{diam } W_{n,v,r} \leq \epsilon.
\]

Notice also, that for every $z \in \overline{C}$ and every component $\text{diam} (\text{Comp } f^{-1}(B(z, \delta'))) \leq \delta$. By iterating a number of times smaller than $n(\epsilon)$ we conclude that there exists $0 < r' \leq r$ such that for every $n < n(\epsilon)$, $\text{diam } W_{n,v,r'} \leq \epsilon$. Thus, $(r')_{\text{max}} \leq \epsilon$. This proves $(*)$.

Having chosen $z \in J(f)$ consider an arbitrary backward trajectory $(z_n)$. To simplify notation assume that $z$ is non-periodic, hence $z_n$ determines $n$. 


Consider since now on $r$ such that $2r_{\text{max}} < r_0$. Define inductively a sequence of integers $k_j$. Let $k_1$ be the least $k \geq 1$ such that
\[ \tilde{W}^{k,1} := W_{k,z_k,2r} \]
contains a critical point. Pay attention to the coefficient 2 at $r$; it will guarantee bounded distortion on discs of radius $r$ of the branches $f^{-k}$, $k = 1, \ldots, k_1 - 1$ mapping $z$ to $z_k$.

Having defined $k_j$ we consider the least $k = k_{j+1} > k_j$ such that
\[ \tilde{W}^{k_{j+1},j+1} := W_{k_{j+1},z_k,2r} \]
contains a critical point.

Let $C$ be an upper bound of the distortion of $f^{k_{j+1} - k_j} - 1$ on $f(W_{k_{j+1},z_k,2r})$; by Koebe distortion lemma it is universal for all $r \leq r_0$, see [PUbook, Lemma 6.2.3]. Notice also that
\[ \frac{\text{diam } W_{k_{j+1},z_k,2r}}{\text{diam } f(W_{k_{j+1},z_k,2r})} \leq C |f'(y)|^{-1} \]
for all $y \in W_{k_{j+1},z_k,2r}$ for each $j$, and a constant $C$ depending on the maximal degree of criticality at critical points.

Finally notice that since $\text{diam } W_{k_j,z_k,2r} \to 0$, then by (*)
\[ \text{diam } \tilde{W}^{k_{j+1}} \to 0 \]
as $j \to \infty$, uniformly with respect to the backward trajectories $(x_n)$ for $x \in J(f)$. So $k_{j+1} - k_j \to \infty$ uniformly if the same critical point above appears, since otherwise the trajectory $(x_n)$ would not be in $J(f)$, see e.g. [P-PAMS, Lemma 1]. Taking in account that $f$ has only a finite number of critical points, we conclude that $\#\{j : k_j \leq n\}/n \to 0$.

Thus, for every $v \in W_{n,z_n,2r}$
\[ (3.2) \quad \Pi_n(1,v) = |(f^n)'(v)|^{-1} \geq (\exp -n\epsilon) \frac{\text{diam } W_{n,z_n,2r}}{r} \]
for all $n \geq n(\epsilon)$ uniformly with respect to the backward trajectory $(z_n)$, for all $\epsilon > 0$.

Finally we shall use the assumption that $z$ is non-exceptional, set $v = z_n$ and prove the inequality roughly opposite to (3.2):
\[ (3.3) \quad \frac{\text{diam } W_{n,z_n,2r}}{r} \geq C \exp -n\epsilon \frac{\text{diam } \text{Comp}_{z_n} f^{-n}(B(z,C \exp -n\epsilon))}{C \exp(-n\epsilon)} \geq \text{Const}(\exp -n\epsilon)|(f^n)'(z_n)|^{-1}. \]
The latter inequality follows from the the bounded distortion of the conformal mapping $f^{-n} : B(z, C^{-1} \exp(-n\varepsilon)) \to \text{Comp}_{z_n} f^{-n}(B(z, C^{-1} \exp -n\varepsilon))$ for a constant $C > 0$ and every $n$ since $z$ is non-exceptional.

In other words, for all $r > 0$ small enough,

$$\lim_{n \to \infty} \frac{1}{n} \log \prod_{W} \inf_{y \in B(f^{-k}(W), \delta)} |f'(y)|^{-t} = 1,$$

the convergence uniform over all backward trajectories of $z$.

Taking power $t$ and summing both inequalities (3.2) (taking $\inf_{v}$ and (3.3) over $z_n$ for $n \to \infty$ and $\varepsilon \to 0$ yields the inequality $\geq$ in Theorem 2.8, hence the equality.

Notice that in $P_{\text{Pull}}^\text{Pullinf}$ some components $W_{n,z_n,r}$ can contain many elements of $f^{-n}(z)$ thus being counted many times, but the number of these times is upper bounded by $\exp \tau n$ for $\tau$ arbitrarily small and $n$ large, again due to scarcity of "critical" times: $k_{j+1} - k_j \to \infty$ for each critical point. This justifies

**Definition 3.1.** [pullback infimum W-tree pressure]

$$P_{\text{Tree}}^\text{PullW}(f, t, z) := \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sum_{W_n \in \mathcal{W}} \prod_{k=1}^{n} \inf_{y \in W_n} |f'(y)|^{-t}.$$

where $\mathcal{W}_n$ is the family of all pullbacks of $B(z, r)$ for $f^n$, The limit in (3.5) for $r \to 0$ exists due to obvious monotonicity, compare Definition 2.5.

Thus we obtain

**Corollary 3.2.** $P_{\text{Tree}}^\text{PullW}(f, t, z) = P_{\text{Tree}}(f, t)$ for every non-exceptional $z$.

**Remark 3.3.** The inequality (3.2) was proved in $P_{\text{Tree}}^\text{Pullinf}$ and named a "telescope lemma". However there the exponential convergence of $(f^n)'(v)|^{-1}$ to 0 was assumed and nothing assumed on diam $W_{n,z_n,r}$. Here the uniform convergence diam $W_{n,z_n,r} \to 0$ is a priori assumed, but nothing about the derivatives.

**Remark 3.4.** In $P_{\text{Tree}}^0(f, t)$ a fraction $\prod_{k=0}^{n} \inf_{y \in B(f^{-k}(v), \delta)} |f'(y)|^{-1}$ can be very close to 0, unlike in (2.11). So an analogon of (2.11) need not hold. See Remark 4.6 and Section 5.
Define now McMullen’s pressure. Assume there is a puzzle structure for
\( f \) (a Markov partition with singularities). Namely there exists a covering of
a neighbourhood of \( J(f) \) by a family \( \mathcal{P} \) of closed topological Jordan discs
\( P_i \), small enough that none of them contains more than one critical point,
whose interiors \( \text{Int} P_i \) intersect \( J(f) \) and are mutually disjoint, and such
that if \( f(\text{Int} P_i) \) intersects \( \text{Int} P_j \) then \( f(\text{Int} P_i) \supset \text{Int} P_j \). We assume also
that all the maps \( f|_{\text{Int} P_i} \) are proper. We allow critical points to belong to
the boundaries of \( P_i \).

Note that this definition has some differences from McMullen’s one. Firstly,
we do not use any measure in it. Secondly, McMullen assumed an expanding
property, we do not. On the other hand he assumed \( f \) to be continuous
(conformal) only piecewise, that is on each \( P_i \).

Following McMullen [McM-HD3] define a refinement \( \mathcal{R}(\mathcal{P}) \) by
\[
\text{cl} \left( \text{Int} f^{-1}(\mathcal{P}) \cup \mathcal{P} \right)
\]
that is the family of the closures of all components of the sets \( f^{-1}(\text{Int} P_i) \cup \text{Int} P_i \). This is also a covering of a neighbourhood
of \( J(f) \) (maybe smaller than the one for \( \mathcal{P} \)) by the complete \( f \)-invariance
of \( J(f) \), and has a puzzle structure. We consider consecutive refinements
\( \mathcal{R}^N(\mathcal{P}) \) and assume that the diameters of their elements shrink to 0 uni-
formly.

For \( P_{N,i}, P_{N,j} \in \mathcal{R}^N(\mathcal{P}) \) denote and number the closures of the components
of \( \text{Int} P_{N,i} \cap \text{Int} f^{-1}(P_{N,j}) \) by \( P_{N+1,i,j,s} \) where \( s = 1, \ldots, s(i,j) = s_N(i,j) \).
Sometimes we shall omit \( s \), to simplify notation. Of course the number of
the components \( s(i,j) \) is larger than 1 if and only if there is an \( f \)-critical
point \( c \in \text{Int} P_{N,i} \setminus \text{Int} f^{-1}(P_{N,j}) \) (remember that we assume the puzzle pieces
are small enough each to contain at most one \( f \)-critical point in \( \text{Int} P_{N,i} \)).
In the later case \( s(i,j) = \nu(c) \) (the multiplicity of \( f \) at \( c \)). If the intersec-
tion is empty, we set \( s(i,j) = 0 \) In this notation the number of components
\( P_{N+1,i} \in \mathcal{R}^N(\mathcal{P}) \) is \( \sum_{i,j} s(i,j) \).

In McMullen’s hyperbolic setting all \( s(i,j) \) are 0 or 1 since there are no
critical points on stage. The same holds in Example 6.11.

With each \( \mathcal{P} \) and \( N \) as above distinguish a point \( y_{N,i} \) in each \( \text{Int} P_{N,i} \).
With \( \mathcal{R}^N(\mathcal{P}) \) we associate the matrix \( \mathcal{R}^N(T) \) with the entries

\[
(4.1) \quad a_{ij} = \begin{cases} |f'(y_{N,i})|^{-1} & \text{if } s(i,j) > 0, \\ 0 & \text{if } s(i,j) = 0. \end{cases}
\]
If \( T \) is a rank \( M \) matrix, then \( \mathcal{R}(T) \) is a rank \( \sum_{i,j=1}^{M} s(i,j) \) matrix. Similarly \( \mathcal{R}^{N+1}(T) \) for \( \mathcal{R}^{N}(T) \) in place of \( T \). If we consider a simplified integer-valued rank \( M \) matrix \( \hat{T} \), with each \( ij \) entry equal to \( s(i,j) \), then in the directed graph interpretation the vertices for \( \mathcal{R}(\hat{T}) \) are the edges of the graph of \( \hat{T} \). In the notation above \( t \) corresponds to an edge \((i,j,s)\). This is a derived graph concept, see [Ore], except the multiplicities \( s(i,j) \) of the edges for \( \hat{T} \), each giving rise to \( s(i,j) \) vertices for the graph of \( \mathcal{R}(\hat{T}) \).

Let \( \lambda(\mathcal{R}^{N}(T)) \) denote the spectral radius of \( \mathcal{R}^{N}(T) \). For each \( t > 0 \) we use the notation replacing above \( T \) by \( T^{t} \), the matrix with each entry being the adequate entry for \( T \) raised to the power \( t \). Similarly we define \( \mathcal{R}^{N}(T)^{t} \). In particular we denote its spectral radius by \( \lambda(\mathcal{R}^{N}(T)^{t}) \). Due to the topological exactness of \( f \) on \( J(f) \)

\[
\lambda(\mathcal{R}^{N}(T)^{t}) := \lim_{n \to \infty} \sqrt[n]{||((\mathcal{R}^{N}(T)^{t})^{n})_{ij}||} = \lim_{n \to \infty} \sqrt[n]{||((\mathcal{R}^{N}(T)^{t})^{n})_{ij}||}
\]

independently of an arbitrary position \( ij \) for \( 1 \leq i, j \leq \text{rank}((\mathcal{R}^{N}(T)^{t})) \). Indeed, in our situation the topological exactness says that all the matrices \( \mathcal{R}^{N}(T) \) are primitive, that is \((\mathcal{R}^{N}(T)^{n})_{ij} > 0 \) for \( n \) large enough for all \( i, j \). Useful is the term \((i_{0}, ..., i_{n})\) being admissible, which means that \( \mathcal{R}^{N}(T)_{i_{k}i_{k-1}} > 0 \) for all \( k = 1, ..., n \). Using this we can say that \( \mathcal{R}^{N}(T) \) is primitive if for all \( n \) large enough and all \( i, j \) there exists an admissible sequence \((i_{0}, ..., i_{n})\) such that \( i_{0} = j \) and \( i_{n} = i \).

At last define McMullen’s pressure, see [McM-HD3] in the hyperbolic case.

\[
P_{\text{McM}}(f, t) := \lim_{N \to \infty} \sup \log \lambda(\mathcal{R}^{N}(T)^{t}).
\]

**Warning.** Unfortunately in presence of critical points in \( J(f) \) this notion has deficiencies if the distinguished points are critical or close to critical ones, making \( P_{\text{McM}}(f, t) \) too big, bigger than \( P(f, t) \). A remedy is to consider

**Definition 4.1** (restricted McMullen’s pressure). Define the restricted McMullen’s pressure as

\[
\widehat{P}_{\text{McM}}(f, t) := \lim_{N \to \infty} \log \lambda(\widehat{\mathcal{R}}^{N}(T)^{t}),
\]

where in each \( \widehat{\mathcal{R}}^{N}(T) \) we consider all the entries at the positions \( ij \) such that

\[
\frac{\text{dist}(P_{N,i}, \text{Crit}(f))}{\text{diam } P_{N,i}} \geq A(N)
\]

the same as in \( \mathcal{R}^{N}(T) \), and all other to be 0. Here \( A(N) \) is an arbitrary sequence of numbers tending to \( \infty \) as \( N \to \infty \) such that \( A(N)\text{diam}(\mathcal{R}^{N}(\mathcal{P})) \to 0 \), where \( \text{diam} \) here denotes supremum of diameters of the sets of a partition.
The limit in (4.4) exists since the ranks of the matrices are growing, acquiring growing number of entries (puzzle pieces), and the puzzle pieces already considered become split with growing $N$, the distinguished points may move but the movements distances decrease to 0. A detailed proof relies on (4.8) and (4.9). In fact we do not need it a priori to prove any limit (for a convergent subsequence) is equal to $P(f,t)$ as in Proof of Theorem 4.4. Still a deficiency of this notion is that with $y_{N,i}$ arbitrary, even far from Crit($f$), there is no reason that the sequence is monotone increasing, nor that its elements do not exceed $P(f,t)$ slightly.

So, consider also suggested in [DGT], see here Example 6.11.

**Definition 4.2** (fuzzy McMullen’s pressures). To define $P^0_{McM}(f,t)$, fuzzy McMullen’s pressure (another name: infimum McMullen’s pressure), just replace $|f'(y_{N,i})|^{-t}$ by $\inf_{y \in P_{N,i}} |f'(y)|^{-t}$ for each $N$ and $P_{N,i}$ in the definition of McMullen’s pressure in $a_{ij}$, see (4.1). Namely consider adequate matrices $(R_N(T)^{\inf})$ and their eigenvalues $\lambda_{N,t}^{\inf}$ and

$$(4.6) \quad P^0_{McM}(f,t) := \lim_{N \to \infty} \log \lambda_{N,t}^{\inf}.$$ 

The limit exists since the sequence is monotone increasing, because when the puzzle pieces split for $N$ growing, infima are taken on smaller sets.

**Definition 4.3** (fuzzy restricted McMullen’s pressure). To define it, denoted $\hat{P}^0_{McM}(f,t)$, keep unchanged the entries $a_{ij}$ in the matrices accompanying $P^0_{McM}(f,t)$ for $i$ satisfying (4.5), putting 0 elsewhere. The monotone increasing holds as before.

Now we can complete the proof of Theorem 2.8.

**Theorem 4.4.** In the setting above for each $t > 0$

$$P(f,t) = \underline{P}_{McM}(f,t) = P^0_{McM}(f,t) = P^0_{McM}(f,t) \leq P_{McM}(f,t).$$

**Proof.** We prove the following. For any non-exceptional $z \in J(F)$

$$(4.7) \quad P_{hyp}(f,t) \leq \underline{P}_{McM}(f,t) = P^0_{McM}(f,t) \leq P_{tree}(f,t,z) \leq P_{hyp}(f,t).$$

Consider an arbitrary set $X \in \mathcal{H}(f,J(f))$. Let $N$ be large so that every $P_{N,i} \in \mathcal{R}^N(\mathcal{P})$ has diameter less than $\delta \ll \text{dist}(X,\text{Crit}(f)$, as in (4.5), so that $f'(x)$ for $v \in B(x,\delta)$ for each $x \in X$ is close to 1 (compare Proof of Theorem 5.11). Consider now only the puzzle pieces intersecting $X$. Thus, passing to limits we obtain $P_{hyp}(f,t) \leq \underline{P}_{McM}(f,t)$. 

Similarly, considering puzzle pieces $P_{N,i}$ satisfying \eqref{1.5} and distinguished $y_{N,i}$ in them, we prove the equality $\widehat{P}_{\text{McM}}(f,t) = P_{\text{McM}}^0(f,t)$. Clearly only the $\leq$ part is non-trivial. Here is the proof:

For any $v \in P = P_{N,i}$ we have, close to a critical point $c$ of multiplicity $\nu$, assuming for simplification that $f(x) = (x - c)^{\nu}$, writing $y$ for $y_{N,i}$

\begin{equation}
|f'(v)/f'(y)|^{1/(\nu - 1)} - 1 \leq \left| \frac{\text{dist}(v,c)}{\text{dist}(y,c)} - 1 \right| \leq \frac{|\text{dist}(v,c) - \text{dist}(y,c)|}{\text{dist}(y,c)} \leq \frac{\text{diam}(P)}{\text{dist}(P,c)} \leq A(N)^{-1},
\end{equation}

see \eqref{1.5}. So $|f'(v)/f'(y)| \to 1$ as $N \to \infty$. Without the simplification we can write $f(z) = a(z - c)^{\nu}(h(z))$ for an analytic map $h(z) = 1 + a_1(z - c) + \ldots$ in a neighbourhood of $c$, so

\begin{equation}
\begin{aligned}
f'(z) &= a\nu(z - c)^{\nu - 1}h(z) + a(z - c)^{\nu}h'(z) \\
&= a(h(z)^{\nu} + (z - c)h'(z))(z - c)^{\nu - 1} \\
&= a(1 + a_1(z - c) + \ldots)\nu + (z - c)(a_1 + \ldots))(z - c)^{\nu - 1} \\
&= a\nu(z - c)^{\nu - 1}(1 + O(z - c)).
\end{aligned}
\end{equation}

So for $\text{dist}(P,c)$ small enough, for all $N$, $|f'(v)/f'(y)| \leq 2A(N)^{-1}$. Further from $c$, i.e. in a domain bounded away from $\text{Crit}(f)$, $\log |f'|$ is Lipschitz continuous, so for large $N$, $\text{dist}(v,y)$ implies $|\log |f'(v)| - \log |f'(y)||$ small, so $|f'(v)/f'(y)|$ is close to 1, namely it tends to 1 as $N \to \infty$.

The inequality $\widehat{P}_{\text{McM}}^0(f,t) \leq P_{\text{McM}}^0(f,t)$ is obvious because the matrices associated to the former one have just zeros replacing non-zero terms in the latter ones.

To prove the inequality $P_{\text{McM}}^0(f,t) \leq P_{\text{tree}}(f,t)$, first for each $P_{N,i} \in \mathcal{R}^N(\mathcal{P})$ select an arbitrary non-exceptional point $z^{N,i}$ in it. Consider an arbitrary sequence of integers $i_0, \ldots, i_n$, admissible, that is such that $\text{Int} P_{N,i_k} \cap f^{-1}(P_{N,i_{k+1}}) \neq \emptyset$ for all $k = 1, \ldots, n$. It contributes in the matrix $(\mathcal{R}^N(T)^{f})^n$ as corresponding to the path in the related directed graph with edges joining consecutively the vertices $i_n, \ldots, i_0$.

Consider $z_n \in \bigcap_{k=0}^n f^{-(n-k)}(\text{Int} P_{N,i_k})$ (maybe disconnected !) and $z_k := f^{n-k}(v_n)$ for all $k = n - 1, \ldots, 0$, with common $z = z_0 = z^{N,i_0}$. In fact the number of possible points $z_n$ is equal to $\prod_{k=0}^{n-1} \text{deg}(f |_{P_{N,i_k+1} \cap f^{-1}(P_{N,i_k})})$.

Now $P_{\text{McM}}^0(f,t) \leq P_{\text{tree}}(f,t, z)$ follows from the obvious $\inf_{y \in P_{N,i_k}} |f'(y)|^{-1} \leq |f'(z_k)|^{-1}$ for each $k = 1, \ldots, n$. Indeed the only issue is that we took care only for the sequences $(i_0, \ldots, i_k)$ with common $i_0$. Considering all the sequences
the constant factor \( \#(\mathcal{R}^N(P)) \) appears. However it disappears when \( n \to \infty \) in the definition of the spectral radius.

The last inequality in (4.7) is known if \( z \in J(f) \) is hyperbolic and non-exceptional, with a proof via capturing hyperbolic subsets via shadowing; it was mentioned in Proof of Theorem 2.8, referring to [PRS].

Finally, notice that the inequality, say, \( P_{\text{McM}}^0(f, t) \leq P_{\text{McM}}(f, t) \) is obvious.

\[ \square \]

**Remark 4.5** (Fuzzy multiple McMullen’s pressure). Notice that replacing the sequence \( y_{N,i_k} \) for admissible \((i_0, \ldots, i_n)\) we do not exploit all appropriate \( z_n \). So in the definition of the matrix \( \mathcal{R}^N(T) \) we could consider each entry \( a_{ij} \) multiplied by \( \deg(f|_{P_{N,i_k}}) = s_N(i,j) \). For a related notion of pressure, to be called fuzzy multiple McMullen’s pressure, denoted \( P_{\text{multMcM}}^0(f, t) \) we also get the upper bound by \( P_{\text{tree}}(f, t) \).

**Remark 4.6.** We cannot prove for each sequence \( i_1, \ldots, i_n \) admissible for the matrix \( \mathcal{R}^N(P) \) neither the inequality (4.10) \[ \lim_{N \to \infty} \inf_{n \to \infty} \frac{1}{n} \log \left( \prod_{k=1}^{n} \left| f'(y_{N,i_k}) \right| \right) \leq 0 \] nor the opposite one with \( \liminf \) replaced by \( \limsup \), unlike in e.g. (3.4). A problem is, that the expressions \( \left| (f^n)'(z_n) \right| \) must be replaced by \( \prod_{k=1}^{n} \left| f'(y_{N,i_k}) \right| \), where domains of \( f' \) to which \( y_{N,i_k} \) respectively belong, are larger than the components of \( f^{-k}(B(f^n(z_n), r)) \) for large \( k \), so the latter products can be too large or too small. Namely the choices of \( y_{N,i_k} \) for \( P_{N,i_k} \) close to \( \text{Crit}(f) \), are much further from \( \text{Crit}(f) \) than \( z_k \), or much closer.

A way out of this trouble would be to replace one telescope in the proof of (3.4) by a sequence of telescopes. But for this, to prove e.g. \( \leq 0 \) in (4.10) we need to know that each \( z_{n_j} \) starting a new telescope is non-exceptional (with the same constants) to obtain (3.3), which can be impossible.

So, a way we have chosen to avoid \( \prod_{k=1}^{n} \left| f'(y_{N,i_k}) \right| \) too large, has been just to get rid of the trouble-making backward trajectories, by considering the restricted McMullen’s pressure.

A way to avoid \( \prod_{k=1}^{n} \left| f'(y_{N,i_k}) \right| \) too small, is to consider fuzzy McMullen’s pressure replacing \( \left| f'(y_{N,i}) \right|^{-1} \) by \( \inf \{ |f'(y)|^{-1} : y \in P_{N,i} \} \), see Definition 4.2. Without this it can just happen that \( P_{\text{McM}}(f, t) > P(f, t) \), see the Warning preceding Definition 4.1 (or the restricted one as in that Definition). For another remedy, replacing the infimum or one point \( y_{N,i} \) by pairs of distinguished points, see Section 6.

**Remark 4.7.** Notice that each set

\[ X_N = \bigcap_{n=1}^{\infty} \bigcup_{i_0, \ldots, i_n} \bigcap_{k=0}^{n-1} f^{-(n-k)}(P_{N,i_k}) \]
the summation over all \((i_0, ..., i_k)\) admissible for the restricted matrix, i.e. \(\mathcal{R}^N(T)\), see (4.5).

Notice that each \(X_N\) is hyperbolic for \(f|_{X_N}\). Indeed, inverses of 
\((f|_{\text{Int}P_{i_0, ..., i_n}})^n\) where \(P_{i_0, ..., i_n} := \bigcap_{k=0}^{n-1} f^{-(n-k)}(P_{i_k})\), for each \(i_n\) is a Montel normal family of holomorphic maps on \(P_{i_0, ..., i_n}\). This is so, because the ranges omit \(f^k(\text{Crit}(f)), k = 0, ..., n\), i.e. eventually more than 2 points in \(\mathbb{C}\). So their limits must be points since otherwise a limit domain \(U\) would not intersect \(J(f)\) as all \(f^n(U')\) for some open \(U' \subset U\) intersecting \(J(f)\) and \(n\) large enough, are bounded in \(P_{i_0, ..., i_n}\), contradicting the definition of Julia set. On the other hand \(U\) must intersect \(J(f)\) by its backward invariance and compactness. This implies uniform convergence of \(|(f^n)'|^{-n}\) on \(X_N\) to 0, hence hyperbolicity.

Notice however that \(X_N\) need not be repelling, like in Definition 2.6 equivalently: the maps \(f|_{X_N}\) need not be open. See [PUbook, Example 4.5.5] where a question of a small extension of \(X_N\) to an invariant set on which \(f\) is open was discussed.

Notice finally that the sequence of the sets \(X_N\) is monotone increasing with respect to inclusion and that the pressures satisfy

\[ P(f|_{X_N}, -t \log |f'|) = \log \lambda(\mathcal{R}^N(T))^t. \]

5. Restricted fuzzy tree pressure

One more definition of pressure might be useful for computations, close to the restricted McMullen’s pressure and to the fuzzy tree pressure as in (2.4) and (2.5), for the potentials \(-t \log |f'|\), making sense for all rational maps. Namely define

\[ \hat{P}^0_{\text{tree}}(f, t, z) := \lim_{\Delta \to 0} \hat{P}_{\text{tree}}^\Delta(f, t, z), \]

where

\[ \hat{P}_{\text{tree}}^\Delta(f, t, z) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_v \hat{\Pi}_\delta(v), \]

where the sum is over all \(v \in f^{-n}(z)\) such that \(\text{dist}(f^{n-k}(v), \text{Crit}(f)) > \Delta\) for all \(1 \leq k \leq n\).

where

\[ \hat{\Pi}_\delta(v) := \prod_{k=1}^{n} |f'(v_k)|^{-t} \]
MCMULLEN'S AND GEOMETRIC Pressures

where $\hat{v}_k$ is a point in $\text{cl} \ B(f^{n-k}(v), \delta)$ where $|f'|^{-1}$ takes infimum, and

\[ \delta = o(\Delta). \]

Existence of the limit in (5.1) follows from the monotone increasing, which is obvious since the infima are taken on shrinking sets. Notice that if we do not mind about the monotonicity, we can choose $\hat{v}_k$ arbitrarily (randomly) in the ball.

**Theorem 5.1.** $\hat{P}^0_{\text{tree}}(f, t, z)$ does not depend on non-exceptional $z$. So (omitting writing $z$) we have

\[ \hat{P}^0_{\text{tree}}(f, t) = P(f, t) \]

**Proof.** The proof is similar to the proof of Theorem 4.4. The inequality $\hat{P}^0_{\text{tree}}(f, t, z) \leq P^0_{\text{tree}}(f, t, z)$ is obvious, just more backward branches of $z$ in the latter pressure are considered. Also $P^0_{\text{tree}}(f, t, z) \leq P_{\text{tree}}(f, t, z)$ is obvious, it was mentioned already in Theorem 2.8 the item 1.

It is left to prove $P_{\text{hyp}}(f, t) \leq \hat{P}^0_{\text{tree}}(f, t, z)$. For this, consider an arbitrary hyperbolic $X \subset J(f)$. It is enough to consider a non-exceptional point $z \in X$ (or just a preimage under an iterate of $f$, arbitrarily close to $X$, of an priori given point $z$). We prove $P_{\text{tree}}(f|X, t, z) \leq \hat{P}^0_{\text{tree}}(f, t, z)$. It is a repetition of the proof of $P_{\text{hyp}}(f, t) \leq \hat{P}^0_{\text{tree}}(f, t, z)$ in Theorem 4.4. For $\Delta = \text{dist}(X, \text{Crit})$ we use $\delta = o(\Delta)$ to assure that for $x \in X$ and $y \in B(x, \delta)$ for $x, y$ close to a critical point $c \in \mathbb{C}$ with a multiplicity $\nu$ for $f$, the ratio

\[ |f'(x)|/|f'(y)| \leq \text{Const} \left( |x - c|/|y - c| \right)^{\nu - 1} \leq 1 + \text{Const} \left( \frac{\delta}{\Delta} \right)^{\nu - 1} \]

is close to 1. For more details see [4.9]. In other words the difference of the potentials $-t \log |f'(x)| - (-t \log |f'(y)|)$ is small.

**Remark 5.2.** Introducing $\hat{P}^0_{\text{tree}}$ between $P_{\text{hyp}}$ and $P^0_{\text{tree}}$ shows directly how to omit the problem for individual backward trajectories, see Proof of Theorem 2.8 the item 1 in Section 3, Remark 3.4, and compare Remark 4.6.

6. **Final remarks, more geometric pressures and examples**

6.1. **On convergence.**

**Remark 6.1.** It is obvious that the sequence of functions $t \mapsto \hat{P}^\Delta_{\text{tree}}(f, t, z)$ in (5.1) converges uniformly (locally) as $\Delta \to 0$ and clear that the limit $P(f, t)$ is non-increasing, by e.g. the $P_{\text{hyp}}(f, t)$ definition. So calculating these functions and their first zeros we obtain as the limit the first zero of $P(f, t)$, which is $\text{HD}_{\text{hyp}}(J(f))$, see Proposition 2.7. Unfortunately we do not
know the speed of the convergence for general $f$. For some classes of maps $f$ the situation is better, i.e. topological Collet-Eckmann maps, Remark 6.3.

**Remark 6.2.** It might be worthy to use instead, the functions

$$t \mapsto \log \lambda(\hat{R}^N(T)),$$

as $N \to \infty$, and their zeros.

Note that their zeros can be calculated as solutions of the equation $\lambda(\Lambda^t) = 1$ if all the entries of a primitive matrix $\Lambda$ are nonnegative, here for $\Lambda = \hat{R}^N(T)$, see [McM-HD3, Practical considerations]. See also Remark 4.7.

**Remark 6.3.** It may happen that $\text{HD}_{\text{hyp}}(J(f)) < \text{HD}(J(f))$, see [Lyu, Subsection 2.13.2], so the methods here are not adequate to estimate $\text{HD}(J(f))$, unless e.g. $f$ is topological Collet-Eckmann, see e.g. [P-ICM], where $P(f, t)$ has only one zero, denote it $t_0$, and $\text{HD}_{\text{hyp}}(J(f)) = \text{HD}(J(f))$.

Notice that in this case at $t_0$ the derivative $dP/dt(t_0) < 0$ so the convergence of approximations, say $\text{HD}(X_N) \to t_0$ is faster than if the left derivative of $dP(f, t)$ at $t_0$ were 0.

**Conclusions and considerations.** As noted in Section 1, our aim is to approximate the geometric pressure $P(f, t)$ from below by quantities depending on a parameter $\delta$ for tree or $N$ for McMullen’s pressures. If approximating quantities exceed $P(f, t)$, if we do not know how far are the quantities from the limit, we do not know how big might an error be in our estimates of $P(f, t)$ from below. In these estimates we use $P(f, t) = P_{\text{tree}}(f, t, z)$. To be safe we want also the numbers under the $\lim sup_n$, see (2.6), be as small as possible. To this end we can choose any $z$ close to $J(f)$ but outside it, compare [DGT], hence not only non-exceptional, but not accumulated by forward trajectories of critical points at all. Notice that if $z_1, z_2$ are like $z$ and belong to the same component $B$ of Fatou set, then all the ratios $\Pi_n(t, v_1)/\Pi_n(t, v_2)$ for corresponding $v_i \in f^{-n}(z_i), i = 1, 2$ are uniformly bounded. Corresponding in the sense that for a curve $\gamma$ joining $z_1$ to $z_2$ in $B \setminus \text{PC}(f)$ each $v_1$ and $v_2$ are the end points of a lift of $\gamma$ for $f^n$. This happens for polynomials, where both $z_i$ belong to the basin of $\infty$.

As we noted in Section 1 among the notions of appropriate geometric pressures we introduced to approximate (calculate) $\text{HD}(J(f))$ from below, the infimum (fuzzy) and/or restricted pressures are appropriate, in particular $\tilde{P}_{\text{MCM}}^0(f, t), P_{\text{MCM}}^0(f, t), P_{\text{multMCM}}^0(f, t), \tilde{P}_{\text{tree}}^0(f, t, z)$ and $P_{\text{tree}}^0(f, t, z)$ might occur useful, since elements of the sequences defining them, depending on $\delta$ or $N$, do not exceed $P(f, t)$.

Another pressure: $P_{\text{MCM}}(f, t)$ is ”almost” monotone increasing, because of bounded distortion in the puzzle pieces satisfying (4.3). This distortion,
responsible for possible decreasing shrinks to 0, as \( N \to \infty \) with the speed depending on \( A(N) \).

Some may increase the speed of approximation, but may lead to results exceeding \( P(f,t) \), as it may happen with \( P_{\text{McM}}(f,t) \), see [3]. This is so because of the use of distinguished points where \( |f'| \) can be too small (its inverse too large). See Remark [4.6]. On the other hand, considering \( P_{\text{pullinf}}(f,t) \) requires finding infima in sets shrinking with the time of iteration, which might be computationally awkward.

### 6.2. Double sampling pressures.

A remedy to avoid the quantities exceeding \( P(f,t) \) and an excessive complexity of calculations would be something between, e.g. double (or multiple) - sampling variants.

**Definition 6.4.** Define double sampling tree pressure \( P^*_{\text{tree}}(f,t,z) \) similarly to \( P_{\text{tree}}^0(f,t,z) \) but replacing infima by minima over two points, namely

\[
P^*_{\text{tree}}(f,t,z) := \limsup_{\delta \to 0} P_{\text{tree}}^{*,\delta}(f,t,z), \quad \text{where}
\]

\[
P_{\text{tree}}^{*,\delta}(f,t,z) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{v \in f^{-n}(z)} \Pi_{n,\delta}^{*,\delta}(v),
\]

where

\[
\Pi_{n,\delta}^{*,\delta}(v) := \prod_{k=1}^{n} \min(|f'(v_{k,1})|^{-t}, |f'(v_{k,2})|^{-t}),
\]

where \( v_{k,1}, v_{k,2} \in B(f^{n-k}(v), \delta) \), are symmetric to each other with respect to \( v_k = f^{n-k}(v) \) and \( \text{dist}(v_{k,1}, v_{k,2}) \geq \delta \). Compare [3,4]. Thus in place of one distinguished point in \( B(v_k, \delta) \), we choose two.

If there is a critical point \( c \) close to \( v_k \), then at least one \( v_{k,i} \) is further from \( c \) than \( v_k \). Even \( |f'(v_{k,i})| \geq |f'(v_k)| \), hence \( |f'(v_{k,i})|^{-t} \leq |f'(v_k)|^{-t} \). At other points \( v_i \), for which \( \delta = o(\text{dist}(v_i, \text{Crit}(f))) \), this inequality holds up to a factor \( 1 + \epsilon \), where \( \epsilon = O(\delta) \). Then \( |f'(v_{k,i})|^{-t}|f'(v_k)|^{-t} \approx 1 \) for \( i = 1, 2 \). So

\[
\frac{\Pi_{n,\delta}^{*,\delta}(v)}{|(f^n)'(v)|^{-t}} \leq 1 + \epsilon,
\]

So, taking \( n \to \infty \) next summing over \( v \) and taking \( \delta \to 0 \) we obtain

**Proposition 6.5.** For every non-exceptional \( z \), \( P_{\text{hyp}}(f,t) \leq P^*_{\text{tree}}(f,t,z) \leq P_{\text{tree}}(f,t,z) \).

Unfortunately we cannot prove the monotone increasing of \( P_{\text{tree}}^{*,\delta}(f,t,z) \) as \( \delta \to 0 \), nor \( P_{\text{tree}}^{*,\delta}(f,t,z) \leq P^*_{\text{tree}}(f,t,z) \). For \( v_k \) close to a critical point \( c \), if
dist\((v_k, c) = C\delta\), we have dist\((v_{k,i}, c) \geq \delta\sqrt{C^2 + 1/2} = \text{dist}(v_k, c)\sqrt{C^2 + 1/4}

for \(i = 1\) or \(2\). Hence

\[
\frac{|f'(v_{k,i})|^{-t}}{f'(v_k)} \leq \left( \frac{C}{\sqrt{C^2 + 1/4}} \right)^{t\nu(c)} (1 + O(\delta)).
\]

However we cannot achieve this gain far from \(\text{Crit}(f)\). A remedy would be to consider triple sampling McMullen’s pressures, with \(v_{k,i}, i = 1, 2, 3\), at the vertices of an equilateral triangle centered at \(v_k\). Then \(f'\) in a small neighbourhood of \(v_k\) is almost affine, so the inequality \(|f'(v_{k,i})| \geq f'(v_k)\) holds for some \(i\) provided there is no point \(x_0\) where \(f''(x) = 0\). If the latter case takes place, assume \(f'(x) = a(x - x_0)m(x) + \ldots\) for \(a \neq 0\) and an integer \(m(x) > 1\). To cope with this case if it happens, consider \(m\)-sampling tree pressure, with \(v_{k,i}, i = 1, \ldots, m\) at the vertices of a regular \(m\)-gon centered at \(v_i\). Then, for \(m = 3\max\{m(x) : f''(x) = 0\}\), there exists \(i\) such that \(|f'(v_{k,i})| \geq |f'(v_k)|\). So we can skip \(\epsilon\) in (6.2) and then conclude with \(P^{*}\text{tree}(f, t, z) \leq P(f, t)\) for each \(\delta\) for \(m\)-sampling tree pressure.

Definition 6.6 (double sampling McMullen’s pressure). Similarly we define double sampling McMullen’s pressure \(P^{*}_{\text{McM}}(f, t)\). For \(P = P_{N,i}\) for which \(\text{Int} P_{N,i} \cap f^{-1}(\text{Int} P_{N,j}) \neq \emptyset\) we consider two points \(v_{P,1}, v_{P,2} \in B(P, r_{N,P})\) where \(r_{N,P} := A(N)(\text{diam} P)\) for an arbitrary sequence \(A(N) \rightarrow \infty\) as \(N \rightarrow \infty\), but \(A(N) \text{diam} (\mathcal{R}^N(\mathcal{P})) \rightarrow 0\), compare (4.5). The points \(v_{P,1}, v_{P,2}\) are chosen symmetric with respect to an arbitrary point \(z_p\) in \(P\) and \(\text{dist}(v_{P,1}, v_{P,2}) \approx r_{N,P}\) (i.e. far from \(P\) compared to its diameter). We distinguish such a pair only for \(P\) such that \(\text{dist}(P, \text{Crit}(f)) \leq r_{N,P}\). In this case we need to do so because of the arbitrariness of the choices of \(z_p\).

Now define the matrices \(\mathcal{R}^N(T)^*\) by changing in \(\mathcal{R}^N(T)\) defined at the beginning of Section 4 the entries \(|f'(y_{N,i})|^{-1}\) with distinguished points \(y_{N,i}\) to \((d_{ij}) = \min\{|f'(v_{P,1})|^{-1}, |f'(v_{P,2})|^{-1}\}\) for \(P\) close to \(\text{Crit}(f)\) as above. Finally define

\[
P^{*}_{\text{McM}}(f, t) = \lim_{N \rightarrow \infty} \log \lambda((\mathcal{R}^N(T)^*)^t).
\]

Consider now an arbitrary non-exceptional \(z \in J(f)\) not belonging to the boundary of any puzzle piece of any generation. Then for each \(z_k \in f^{-k}(z) \in P_{N,i}\), we have \(a_{ij} = |f'(z_k)|^{-t}(1 + \epsilon)\), where \(\epsilon \rightarrow 0\) as \(N \rightarrow \infty\), compare (6.2). So indeed it holds:

Proposition 6.7. \(P_{\text{hyp}}(f, t) \leq P^{*}_{\text{McM}}(f, t) \leq P_{\text{tree}}(f, t, z) = P_{\text{tree}}(f, t)\).

Definition 6.8. In fact we could distinguish \(v_{P,1}, v_{P,2} \in \text{Int} P\). Indeed, assume \(d := \text{dist}(v_{P,1}, v_{P,2}) \geq \frac{1}{2} \text{diam} P\). So \(\text{dist}(z_k, v_{P,i}) \leq 2d\) for \(z_k \in P\) for both \(i = 1\) and \(i = 2\). But \(\text{dist}(c, v_{P,i}) \geq d/2\) for \(i = 1\) or \(i = 2\) for each
point \( c \) close to \( P \), in particular a critical one. So for such \( \iota \) we get due to the triangle inequality, skipping indices, \[ \frac{\text{dist}(c, z)}{\text{dist}(c, v)} \leq 1 + \frac{\text{dist}(v, z)}{\text{dist}(c, v)} \leq 1 + \frac{2d}{d/2} \leq 5. \]

So, for \( \nu \) denoting the multiplicity of \( f \) at \( c \),
\[
\left| f'(z_k) \right|^{-1} \geq \text{Const} 5^{-(\nu-1)}, \quad \text{hence} \quad \left| f'(v_{P,\iota}) \right|^{-1} \leq \text{Const} 5^{\nu-1} \left| f'(z_k) \right|^{-1}.
\]

Since this occurrence happens rarely with \( N \) large the constant \( \text{Const}^{-1} 5^{\nu-1} \) does not matter.

Note that the shape of \( P \) can be very distorted, making finding above \( v_{P,1}, v_{P,2} \in \text{Int} P \) difficult. So instead we can consider in the definition taking \( v_{P,1}, v_{P,2} \in B(P, \text{diam} P) \).

Unfortunately these constructions do not allow to prove neither monotonicity nor that the elements of the sequence in (6.3) do not exceed \( P(f, t) \) (though discrepancies seem low), unless we modify the definition to an m-sampling McM-pressure, as in the tree pressure case in Definition 6.4.

6.3. Examples.

**Example 6.9.** For each non-renormalizable polynomial \( f \), say with connected Julia set and all periodic orbits repelling, one considers Yoccoz puzzle construction, namely a covering \( \mathcal{P} \) of a neighbourhood of \( J(f) \) whose pieces have boundaries consisting of equipotential lines for the Green’s function in the basin of \( \infty \) and external radii to fixed points dissecting \( J(f) \). Assume these points are not postcritical. Diameters of consecutive pullbacks of these pieces shrink uniformly to 0, so the assumptions of Theorem 2.8, item 2, are satisfied. See e.g. [KvS] for this and more general cases.

**Remark 6.10.** There is no need in the definition of McMullen’s pressures that the consecutive puzzle structures are of the form \( \mathcal{R}^N(\mathcal{P}) \). One can just take any sequence \( \mathcal{P}_N \) of puzzle structure coverings such that diameters of their elements tend uniformly to 0. Then in Example 1 one may allow infinitely renormalizable case, with so-called a priori complex bounds condition, allowing to find such a sequence, see [Lyu] and references therein.

**Example 6.11.** For the Feigenbaum map \( f_{\text{Feig}}(z) = z^2 + c_{\text{Feig}} \) where \( c \approx -1.40155 \), infinitely renormalizable, where \( c_{\text{Feig}} \) is the limit of the decreasing sequence of the period doubling real parameters, a different puzzle structure is used, see [DS] and references therein. The critical point 0 is in the boundary of four first generation puzzle pieces adjacent at it, so all restrictions of \( f_{\text{Feig}} \) to all generations puzzle pieces are injective, the integer parameter \( s \) is
not needed, see beginning of Section 4. In fact in [DGT] it is announced that for 
\( t = \text{HD}_{\text{hyp}}(J(f)) \) it holds \( \log \lambda_{N,t}^0 \to \delta_{\text{cr}}^0 \) as \( N \to \infty \), with \( \lambda_{N,t}^0 \) defined in our Definition 4.2 and \( \delta_{\text{cr}} \) being the critical exponent of Poincaré series. The latter is equal to \( \text{HD}(J(f)) \) and Minkowski dimension (box dimension) of \( J(f) \) provided the area of \( J(f) \) is zero (which is the case by [DS] for \( f_{\text{Feig}}(z) \)), for the latter see [Bishop]. (There, Witney critical exponent appears, but it is straightforward that it is Minkowski dimension of \( J(f) \), by Koebe’s 1/4 lemma. In [Bishop], actions by Kleinian groups were considered.)

See [DS], Figures 1,2 and a precise description there.

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