WAVE BREAKING PHENOMENA AND GLOBAL EXISTENCE FOR THE WEAKLY DISSIPATIVE GENERALIZED CAMASSA-HOLM EQUATION

YONGHUI ZHOU$^{1,2}$ AND SHUGUAN JI$^{1,*}$

$^1$School of Mathematics and Statistics and Center for Mathematics and Interdisciplinary Sciences Northeast Normal University, Changchun, 130024, China
$^2$School of Mathematics and Statistics, Hexi University Zhangye 734000, China

(Communicated by Zhen Lei)

ABSTRACT. In this paper, we mainly study several problems on the weakly dissipative generalized Camassa-Holm equation. We first establish the local well-posedness of solutions by Kato’s semigroup theory. We then derive the necessary and sufficient condition of the blow-up of solutions and a criteria to guarantee occurrence of wave breaking. Moreover, when the solution blows up, we obtain the precise blow-up rate. We finally show that the equation has a unique global solution provided the momentum density associated with their initial datum satisfies appropriate sign conditions.

1. Introduction. The Camassa-Holm equation

$u_t - u_{txx} + 3uu_x = uu_{xxx} + 2u_xu_{xx}$

(1.1)

is a mathematical model describing the unidirectional propagation of shallow water waves, which was derived physically by Camassa and Holm [2]. It is well-known that equation (1.1) has bi-Hamiltonian structure [9] and complete integrability [2]. Its solitary waves are peaked [2], and they are orbitally stable and interact like solitons [7]. Particularly, after the Camassa-Holm equation is derived, there are a large number of papers to investigate its mathematical properties, such as local well-posedness [4, 12], global existence of strong solutions [4], wave breaking [3, 4, 6, 12], blow-up rate [6] and the existence of global weak solutions [1, 5, 16, 19].

The modified Camassa-Holm equation and the generalized Camassa-Holm equation have been studied by many authors in recent years. For example, Wu and Guo [18] considered the local well-posedness and wave breaking phenomena for the following modified Camassa-Holm equation

$u_t - u_{txx} + 3u^2u_x = uu_{xxx} + 2u_xu_{xx}$. 

(1.2)
Yin [20, 21, 22] considered the local-well posedness and wave breaking phenomena for the following generalized Camassa-Holm equation
\[ u_t - u_{txx} + ku_x + \frac{1}{2} (g(u))_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.3) \]
where \( k \) is a nonnegative parameter related to the critical shallow water speed.

In general, energy dissipation is inevitable in the real world. Ott and Sudan [15] investigated how Korteweg-de Vries (KdV) equation was modified by taking into account of the presence of dissipation and the effect of such dissipation on the solitary solution. Ghidaglia [11] studied the long time behavior of solutions to the weakly dissipative KdV equation as a finite-dimensional dynamical system. In 2009, Wu and Yin [17] considered the global existence and wave breaking phenomena for the following weakly dissipative Camassa-Holm equation
\[ u_t - u_{txx} + 3uu_x + \lambda (u - u_{xx}) = uu_{xxx} + u_xu_{xx}, \quad (1.4) \]
where \( \lambda \) is a nonnegative dissipative parameter. In 2020, Freire et al. [10] considered the local well-posedness and wave breaking phenomena for the following weakly dissipative modified Camassa-Holm equation
\[ u_t - u_{txx} + 3u^2u_x + \lambda (u - u_{xx}) = uu_{xxx} + 2u_xu_{xx}. \quad (1.5) \]

Inspired by the previous work, in this paper, we study the local well-posedness, wave breaking, blow-up rate, and global existence and uniqueness of solutions for the following weakly dissipative generalized Camassa-Holm equation
\[ u_t - u_{txx} + ku_x + \frac{1}{2} (g(u))_x + \lambda (u - u_{xx}) = 2u_xu_{xx} + uu_{xxx} \quad (1.6) \]
with the initial datum
\[ u(0, x) = u_0(x), \quad (1.7) \]
where \( g(u) \in C^\infty(\mathbb{R}, \mathbb{R}), g(0) = 0. \)

The rest of this paper is organized as follows. In Section 2, the local well-posedness result is obtained by using Kato’s theorem. In Section 3, we obtain a time-dependent conserved quantity. In Section 4, we derive the blow-up mechanism of solutions and give the blow-up rate of blow-up solutions. In Section 5, we establish the global existence and uniqueness of solutions.

**Notation.** Throughout this paper, all spaces of functions are over \( \mathbb{R} \) and for simplicity, we drop \( \mathbb{R} \) in our notation of function spaces if there is no ambiguity. Additionally, we denote by \( \| \cdot \|_s \) the norm in the Sobolev space \( H^s(\mathbb{R}) \).

2. **Local well-posedness.** In this section, we will prove the local well-posedness result by using Kato’s theorem.

For convenience, we state Kato’s theorem in the form suitable for our purpose. Consider the abstract quasilinear evolution equation of the form
\[ \frac{dz}{dt} + A(z)z = f(z), \quad t > 0, \quad (2.1) \]
with the initial datum \( z(0) = z_0. \)

Let \( X \) and \( Y \) be Hilbert spaces such that \( Y \) is continuously and densely embedded in \( X \) and let \( Q : Y \rightarrow X \) be a topological isomorphism. \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) denote the norm of Banach space \( X \) and \( Y \), respectively. Let \( L(Y, X) \) denote the space of all bounded linear operators from \( Y \) to \( X \) (if \( Y = X \), it is abbreviated as \( L(X) \)). Let \( G(z) \) be the set of all negative generators of \( C_0 \)-semigroups on \( X \). More precisely, we denote by \( G(X, M, \beta) \) the set of all linear operators \( A \) in \( X \) such that \( -A \) generates...
Assume that (i)–(iii) hold. Given

Problem (where a maximal $P$-Set $u$

Lemma 2.1 (Kato’s theorem \[ A \text{ is continuous.} \]

In what follows, we prove the local well-posedness result. Moreover, the solution $u$ is bounded on any bounded sets in $Y$ and also extends to a map from $X$ onto $X$ and satisfies

\[
\|f(y) - f(z)\|_Y \leq \rho_1 \|y - z\|_Y, \quad y, z \in Y,
\]

where $\rho_1(i = 1, 2, 3)$ depend only on $\max \{\|y\|_Y, \|z\|_Y\}$ and $\rho_4$ depends only on $\max \{\|y\|_X, \|z\|_X\}$.

Lemma 2.1 (Kato’s theorem \[13\]). Assume that (i)–(iii) hold. Given $z_0 \in Y$, there exists a maximal $T > 0$ depending only on $\|z_0\|_Y$ and unique solution $z$ of equation (2.1) such that

\[ z = z(\cdot, z_0) \in C([0, T); Y) \cap C^1([0, T); X). \]

Moreover, the map $z_0 \mapsto z(\cdot, z_0)$ is continuous from $Y$ to $C([0, T); Y) \cap C^1([0, T); X)$.

In order to apply Kato’s theorem, we reformulate equation (1.6). With $y = u - u_{xx}, (1.6)$ is reduced to

\[
y_t + u_y + 2u_{xx}y + \lambda y = -\left( \frac{1}{2} g(u) - \frac{3}{2} u^2 + ku \right). \tag{2.2}
\]

Set $P(x) := \frac{1}{2} e^{-|x|}$, then we have $(1 - \partial_x^2)^{-1} y = \Lambda^{-2} y = P \ast y, \quad \forall y \in L^2(\mathbb{R})$. Using this identity, we can rewrite (2.2) as

\[
u_t + uu_x = -\partial_x \Lambda^{-2} \left( \frac{1}{2} g(u) - \frac{1}{2} u^2 + ku + \frac{1}{2} u_x^2 \right) - \lambda u, \tag{2.3}
\]

or its equivalent form

\[
u_t + uu_x = -\partial_x P \ast \left( \frac{1}{2} g(u) - \frac{1}{2} u^2 + ku + \frac{1}{2} u_x^2 \right) - \lambda u. \tag{2.4}
\]

In what follows, we prove the local well-posedness result.

Theorem 2.1 (Local well-posedness). Given $u_0(x) \in H^s(s > \frac{3}{2})$, then there exists a maximal $T = T(k, g, \lambda, u_0) > 0$ and a unique solution $u(t, x)$ of the Cauchy problem (1.6)–(1.7) such that

\[ u = u(\cdot, u_0) \in C([0, T); H^s) \cap C^1([0, T); H^{s-1}). \]

Moreover, the solution $u(t, x)$ depends continuously on the initial datum, i.e., the map

\[ u_0 \mapsto u(\cdot, u_0) : H^s \to C([0, T); H^s) \cap C^1([0, T); H^{s-1}) \]

is continuous.
Proof. It is enough to verify that
\[ A(u) = u \partial_x \] and
\[ f(u) = -\partial_x P \ast \left( \frac{1}{2} g(u) - \frac{1}{2} u^2 + ku + \frac{1}{2} u_x^2 \right) - \lambda u \]
satisfy conditions (i)–(iii) in Kato’s theorem. From the results proved in [22], we see that \( A(u) \) satisfies conditions (i)–(ii) in Kato’s theorem.

Let \( f(u) = h(u) - \lambda u \) with \( h(u) = -\partial_x P \ast \left( \frac{1}{2} g(u) - \frac{1}{2} u^2 + ku + \frac{1}{2} u_x^2 \right) \). Then for any norm \( \| \cdot \| \), we have
\[ \| f(u) - f(v) \| \leq \| h(u) - h(v) \| + \lambda \| u - v \|. \]
This means that \( f(u) \) satisfies condition (iii) in Kato’s theorem if and only if \( h(u) \) does, which follows from the results proved in [22]. This completes the proof of Theorem 2.1.

Similar to [22], we can easily prove that \( T \) may be chosen independent of \( s \) in Theorem 2.1. So we give the following theorem but omit the detail proof.

**Theorem 2.2.** The maximal \( T \) in Theorem 2.1 may be chosen independent of \( s \) in the following sense. If
\[ u \in C([0,T); H^s) \cap C^1([0,T); H^{s-1}) \]
is a solution of equation (1.6) and \( u_0(x) \in H^{s_1} \) for some \( s_1 \neq s, s_1 \geq \frac{3}{2} \), then
\[ u \in C([0,T); H^{s_1}) \cap C^1([0,T); H^{s_1-1}) \]
and with the same \( T \). In particular, if \( u_0 \in H^\infty = \cap_{s \geq 0} H^s \), then \( u \in C([0,T); H^\infty) \).

3. Time-dependent conserved quantity. In this section, we will find a time-dependent conserved quantity of solutions of the Cauchy problem (1.6)–(1.7), which is crucial in the investigation of the global existence and wave breaking phenomena of solutions.

**Theorem 3.1.** Assume that \( u(t,x) \) is a solution of the Cauchy problem (1.6)–(1.7) such that \( g(u), u \) and its derivatives up to second order go to 0 as \( x \to \pm \infty \). Let
\[ H(t) := \int_{\mathbb{R}} (u^2 + u_x^2) \, ds. \]
Then, for any \( t \in [0,T) \), we have \( H(t) = e^{-2\lambda t} H(0) \).

**Proof.** Multiplying (1.6) by \( 2u \), we get
\[ 2u_t - 2u_{txx} + 2ku_x + u (g(u))_x + 2\lambda (u^2 - uu_x) = 4uu_xu_{xx} + 2u^2 u_{xxx}. \quad (3.1) \]
Integrating (3.1) with respect to \( x \) over \( \mathbb{R} \), we get
\[ \frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) \, dx + 2\lambda \int_{\mathbb{R}} (u^2 + u_x^2) \, dx = 0, \quad (3.2) \]
here we used the relations that
\[ \begin{align*}
2 \int_{\mathbb{R}} u_{txx} \, dx &= -2 \int_{\mathbb{R}} u_x u_{xxx} \, dx, \\
2 \int_{\mathbb{R}} uu_{xx} \, dx &= -2 \int_{\mathbb{R}} u_x^2 \, dx, \\
\int_{\mathbb{R}} u_x^2 u_{xxx} \, dx &= -2 \int_{\mathbb{R}} u u_x u_{xxx} \, dx, \\
\int_{\mathbb{R}} u g(u)_x \, dx &= - \int_{\mathbb{R}} G'(u(x)) \, dx = 0,
\end{align*} \]
where \( G(u) = \int_0^u g(v) \, dv. \)
Integrating (3.2) with respect to $t$ over $[0, t]$, we get
\[ \int_\mathbb{R} (u^2 + u_x^2) dx = e^{-2\lambda t} \int_\mathbb{R} (u_0^2 + u_{0x}^2) dx, \]
i.e., $H(t) = e^{-2Mt}H(0)$. This completes the proof of Theorem 3.1.
\[ \square \]

4. Wave breaking and blow-up rate. In this section, we will derive the necessary and sufficient condition of wave breaking, give a sufficient condition on the initial data to guarantee the occurrence of wave breaking and obtain the blow-up rate of blow-up solutions.

In what follows, we derive the necessary and sufficient condition of wave breaking. To this goal, we give the following three lemmas.

**Lemma 4.1** ([14]). If $r > 0$, then $H^r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is an algebra. Moreover
\[ \|fg\|_r \leq c \left( \|f\|_{L^\infty(\mathbb{R})} \|g\|_r + \|f\|_r \|g\|_{L^\infty(\mathbb{R})} \right), \]
where $c$ is a constant depending only on $r$.

**Lemma 4.2** ([14]). If $r > 0$, then
\[ \|\Lambda^r f\|_{L^2(\mathbb{R})} \leq c \left( \|\partial_x f\|_{L^\infty(\mathbb{R})} \|\Lambda^{r-1} g\|_{L^2(\mathbb{R})} + \|\Lambda^r f\|_{L^2(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} \right), \]
where $c$ is a constant depending only on $r$.

**Lemma 4.3** ([8]). If $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$. Then for any $s > \frac{1}{2}$, we have
\[ \|F(u)\|_r \leq \tilde{F}(\|u\|_{L^\infty(\mathbb{R})}) \|u\|_r, \quad u \in H^r(\mathbb{R}), \]
where $\tilde{F}$ is a monotone increasing function depending only on $F$ and $r$.

**Theorem 4.1.** Let $u_0(x) \in H^s(\mathbb{R})$ be given. And assume that $T$ is the maximal existence time of the corresponding solution $u(t, x)$ of the Cauchy problem (1.6)–(1.7). If there exists a positive constant $M > 0$ such that
\[ \limsup_{t \to T} \sup_{x \in \mathbb{R}} \|u_x(t, x)\|_{L^\infty} \leq M, \]
then the $H^s$-norm of $u(t, \cdot)$ does not blow up on $[0, T)$.

**Proof.** Let $u(t, x)$ be a solution of the Cauchy problem (1.6)–(1.7), which is guaranteed by Theorem 2.1. Applying the operator $\Lambda^s$ to (2.3), multiplying by $2\Lambda^s u$ and integrating the corresponding equation by parts on $\mathbb{R}$, we have
\[ \frac{d}{dt} (u, u)_s = -2(uu_x, u)_s + 2(f(u), u)_s, \]
where $f(u) = -\partial_x \Lambda^{-2} \left( \frac{1}{2} g(u) - \frac{1}{2} u^2 + ku + \frac{1}{2} u_x^2 \right) - \lambda u$.

If there exists a positive constant $M > 0$ such that
\[ \limsup_{t \to T} \sup_{x \in \mathbb{R}} \|u_x(t, x)\|_{L^\infty} \leq M, \]
Then we have
\[ \|(uu_x, u)_s\| = \|(\Lambda^s uu_x, \Lambda^s u)_0\| \]
\[ = \|[(\Lambda^s, u_{ux}, \Lambda^s u_0) + (u\Lambda^s u_x, \Lambda^s u)_0] \]
\[ \leq c \|(\Lambda^s, u)_{ux}\|_{L^2} \|\Lambda^s u\|_{L^2} + \|u_x\|_{L^\infty} \|\Lambda^s u\|_{L^2}^2 \]
\[ \leq c \|(u_x)_{L^\infty} \|\Lambda^{s-1} u_x\|_{L^2} + \|\Lambda^s u\|_{L^2}^2 \|u_x\|_{L^\infty} \|u\|_s + cM \|u\|_s^2 \]
\[ \leq c \|(u_x)_{L^\infty} \|u\|_s + M \|u\|_s \|u\|_s + cM \|u\|_s^2 \]
\[ (f(u), u)_s = \left( -\partial_x \Lambda^{-2} \left( \frac{1}{2} g(u) - \frac{1}{2} u^2 + k u + \frac{1}{2} u_x^2 \right) - \lambda u, u \right)_s \]
\[ \leq c \|u\|_s \left( \left\| \partial_x \Lambda^{-2} \left( \frac{1}{2} g(u) - \frac{1}{2} u^2 + k u + \frac{1}{2} u_x^2 \right) \right\|_s + \lambda \|u\|_s \right) \]
\[ \leq c \|u\|_s \left( \|g(u)\|_{s-1} + \|u_x^2\|_{s-1} + \|u^2\|_{s-1} + k \|u\|_{s-1} + \lambda \|u\|_s \right) \]
\[ \leq c \|u\|_s \left( \|g(u)\|_{L^\infty} \|u\|_{s-1} + \|u_x\|_{s-1} \|u\|_{L^\infty} + \|u\|_{s-1} \|u\|_{L^\infty} \right) \]
\[ + k \|u\|_{s-1} + \lambda \|u\|_s \]
\[ \leq (c + \lambda) \|u\|_s^2, \] (4.3)

where we used Lemmas 4.1 and 4.3 with \( s = r \). Combining (4.1)–(4.3), we get
\[ \frac{d}{dt} \|u\|_s^2 \leq (c + \lambda) \|u\|_s^2. \] (4.4)

Applying Gronwall’s inequality, we have
\[ \|u\|_s^2 \leq e^{(c+\lambda)t} \|u_0\|_s^2. \]

This completes the proof of Theorem 4.1.

**Theorem 4.2.** Let \( u_0(x) \in H^s(s > \frac{3}{2}) \) be given. And assume that \( T \) is the maximal existence time of the corresponding solution \( u(t, x) \) of equation (1.6) with initial datum \( u_0(x) \). Then \( T \) is finite if and only if
\[ \liminf_{t \to T} \inf_{x \in \mathbb{R}} \{u_x(t, x)\} = -\infty. \]

**Proof.** In the following proof, we only prove the case of \( s = 2 \), since we can obtain the same conclusion for \( s > \frac{3}{2} \) by using denseness.

Note that \( y = u - u_{xx} \), we get
\[ \|u\|_{H^2}^2 \leq \|y\|_{L^2}^2 = \int_{\mathbb{R}} (u - u_{xx})^2 dx = \int_{\mathbb{R}} (u^2 + u_{xx}^2 + 2u_x y)^2 dx. \]

Multiplying 2\( y \) to both sides in (2.2), we get
\[ 2yy_t + 2u_yy_x + 2u_yy^2 + 2\lambda y^2 = -2 \left( \frac{1}{2} g(u) - \frac{3}{2} u^2 + k u \right)_x y. \] (4.5)

Integrating (4.5) with respect to \( x \) over \( \mathbb{R} \), we obtain
\[ \frac{d}{dt} \int_{\mathbb{R}} y^2 dx = -2 \int_{\mathbb{R}} uyy_t dx - 2 \int_{\mathbb{R}} u_x y^2 dx - 2\lambda \int_{\mathbb{R}} y^2 dx \]
\[ - 2 \int_{\mathbb{R}} \left( \frac{1}{2} g(u) - \frac{3}{2} u^2 + k u \right)_x y dx \]
\[ = - \int_{\mathbb{R}} u_x y^2 dx - 2\lambda \int_{\mathbb{R}} y^2 dx - 2 \int_{\mathbb{R}} \left( \frac{1}{2} g(u) - \frac{3}{2} u^2 + k u \right)_x y dx. \] (4.6)
Note that
\[
2 \int_{\mathbb{R}} \left( \frac{1}{2} g(u) - \frac{3}{2} u^2 + ku \right)_x \ y dx \leq \int_{\mathbb{R}} \left( \frac{1}{2} g(u) - \frac{3}{2} u^2 + ku \right)_x^2 + y^2 \ dx
\]
\[
\leq \left\| \frac{1}{2} g(u) - \frac{3}{2} u^2 + ku \right\|_{L^2}^2 + \| y \|_{L^2}^2
\]
\[
\leq c_1 \| u \|_1^2 + \| y \|_{L^2}^2
\]
\[
\leq c_2 \| y \|_{L^2}^2,
\]
where \(c_1, c_2 > 0\) represent specific constants.

If \(u_x(t, x)\) is bounded from below on \([0, T) \times \mathbb{R}\), i.e., there exists \(N > 0\) such that \(u_x(t, x) > -N\) on \([0, T) \times \mathbb{R}\). Then we get
\[
\frac{d}{dt} \int_{\mathbb{R}} y^2 dx \leq (N + 2\lambda + c_2) \| y \|_{L^2}^2.
\]

Applying Gronwall’s inequality, we get
\[
\| y \|_{L^2}^2 \leq e^{(N + 2\lambda + c_2)t} \| y_0 \|_{L^2}^2.
\]

By using the Sobolev embedding theorem \(H^2 \hookrightarrow L^\infty\), we get
\[
\| u_x \|_{L^\infty} \leq \| u \|_2 \leq \| y \|_{L^2}^2 \leq e^{(N + 2\lambda + c_2)t} \| y_0 \|_{L^2}^2.
\]

By Theorem 4.1, we deduce that the solution exists globally in time.

On the other hand, if \(u_x(t, x)\) is unbounded from below, by applying Theorem 4.1 and using the Sobolev embedding theorem \(H^s \hookrightarrow L^\infty\) \((s > 1/2)\), we infer that the solution will blow up in finite time. This completes the proof of Theorem 4.2.

In what follows, we will find a sufficient condition for the wave breaking by investigating the initial datum. To this end, we need the following lemma.

**Lemma 4.4 ([3]).** Let \(T > 0\) and \(v(t, x) \in C^1([0, T); H^2(\mathbb{R}))\) be a given function. Then, for any \(t \in [0, T)\), there exists at least one point \(\xi(t) \in \mathbb{R}\) such that
\[
m(t) = \inf_{x \in \mathbb{R}} v_x(t, x) = v_x(t, \xi(t))
\]
and the function \(m(t)\) is almost everywhere differentiable in \([0, T)\), with
\[
m'(t) = v_{xx}(t, \xi(t)), \quad \text{a.e. on} \quad [0, T).
\]

**Theorem 4.3.** Let \(u_0 \in H^s(s > \frac{1}{2})\) be given. Assume that \(T\) is the maximal existence time of the corresponding solution \(u(t, x)\) of equation (1.6) with initial datum \(u_0(x)\). If there exists some \(x_0 \in \mathbb{R}\) such that
\[
u'_0(x_0) < -\lambda - \sqrt{\lambda^2 + 2K},
\]
where
\[
K = \sup_{|v| \leq C_2, \| v \|_1} \left| g(v) \right| + \| u_0 \|_1^2 + \sqrt{2k} \| u_0 \|_1^2,
\]
then the wave breaking will occur for the Cauchy problem (1.6)–(1.7). Moreover, the maximal existence time of the corresponding solution \(u(t, x)\) for the Cauchy problem (1.6)–(1.7) is estimated above by \(T'\), where
\[
T' = \frac{1}{\sqrt{\lambda^2 + 2K}} \ln \frac{\nu_0^0(0) + \lambda - \sqrt{\lambda^2 + 2K}}{\nu_0^0(0) + \lambda + \sqrt{\lambda^2 + 2K}}.
\]
Proof. Differentiating (2.3) with respect to \( t \) and using the relation \( \partial^2_t \Lambda^{-2} = \Lambda^{-2} - 1 \), we obtain
\[
 u_{tx} = -\frac{1}{2} u_x^2 - u_{xx} \Lambda^{-2} \left( \frac{1}{2} g(u) + \frac{1}{2} u_x^2 - \frac{1}{2} u^2 + ku \right) + \frac{1}{2} g(u) - \frac{1}{2} u^2 + ku - \lambda u_x.
\] (4.9)

According to Lemma 4.4, there is at least one point \( \xi(t) \in \mathbb{R} \) satisfying
\[
 u_x(t, \xi(t)) = \inf_{x \in \mathbb{R}} u_x(t, x).
\]

Let
\[
 m(t) = \inf_{x \in \mathbb{R}} u_x(t, x) = u_x(t, \xi(t)).
\]
Then for any given \( t \in (0, T) \), we obtain \( u_{xx}(t, \xi(t)) = 0 \). Therefore, we have
\[
m_t = -\frac{1}{2} m^2 \Lambda^{-2} \left( \frac{1}{2} g(u) + \frac{1}{2} u_x^2 - \frac{1}{2} u^2 + ku \right)
+ \frac{1}{2} g(u) - \frac{1}{2} u^2 + ku - \lambda m, \text{ a.e. on } [0, T].
\] (4.10)

Since the operator \( \Lambda^{-2} \) preserves positivity, we get \( \Lambda^{-2} u_x^2 \geq 0 \). Note that
\[
 \|g(u)\|_{L^\infty} \leq \sup_{|v| \leq \frac{1}{2} \|u_0\|_1} |g(v)|,
\]
\[
 \|\Lambda^{-2}(g(u))\|_{L^\infty} \leq \|g(u)\|_{L^\infty} \leq \sup_{|v| \leq \frac{1}{2} \|u_0\|_1} |g(v)|,
\]
\[
 \|\Lambda^{-2} u^2\|_{L^\infty} \leq \frac{1}{2} \|u^2\|_{L^1} \leq \frac{1}{2} \|u_0\|^2_1,
\]
and
\[
 2u^2(t, \xi(t)) = 2 \left| \int_{-\infty}^{\xi(t)} u(t, y) u_x(t, y) dy - \int_{\xi(t)}^{+\infty} u(t, y) u_x(t, y) dy \right|
\leq \int_{-\infty}^{\xi(t)} \left( u^2(t, y) + u_x^2(t, y) \right) dy + \int_{\xi(t)}^{+\infty} \left( u^2(t, y) + u_x^2(t, y) \right) dy
= \|u\|^2_{H^1}.
\]
Therefore, for a.e. \( t \in [0, T) \), (4.10) is reduced to
\[
m'(t) \leq -\frac{1}{2} m^2(t) - \lambda m(t) + K
= -\frac{1}{2} \left( m(t) + \lambda + \sqrt{\lambda^2 + 2K} \right) \left( m(t) + \lambda - \sqrt{\lambda^2 + 2K} \right).
\] (4.11)
By (4.8), we have \( m(0) < -\lambda - \sqrt{\lambda^2 + 2K} \), thereby \( m'(0) < 0 \). From the continuity of \( m(t) \) with respect to \( t \), it can be obtained that for any \( t \in [0, T) \), there is \( m'(t) < 0 \). Therefore
\[
m(t) < -\lambda - \sqrt{\lambda^2 + 2K}.
\]
Then, by solving the above inequality, we obtain
\[
 \frac{m(t) + \lambda + \sqrt{\lambda^2 + 2K}}{m(t) + \lambda - \sqrt{\lambda^2 + 2K}} e^{\frac{\lambda + \sqrt{\lambda^2 + 2K} t}{\lambda - \lambda \sqrt{\lambda^2 + 2K}}} - 1 \leq \frac{2\sqrt{\lambda^2 + 2K}}{m(t) + \lambda - \sqrt{\lambda^2 + 2K}} \leq 0.
\]
Due to
\[
 0 < \frac{m(0) + \lambda + \sqrt{\lambda^2 + 2K}}{m(0) + \lambda - \sqrt{\lambda^2 + 2K}} < 1,
\]

there exists

\[ T \leq \frac{1}{\sqrt{\lambda^2 + 2K}} \ln \frac{m(0) + \lambda - \sqrt{\lambda^2 + 2K}}{m(0) + \lambda + \sqrt{\lambda^2 + 2K}} =: T', \]

such that

\[ \liminf_{t \to T} m(t) = -\infty. \]

Applying Theorem 4.2, the solution \( u(t, x) \) of the Cauchy problem (1.6)–(1.7) blows up in finite time. This completes the proof of Theorem 4.3.

**Theorem 4.4.** Under the condition of Theorem 4.3. If \( T \) is finite, then we have

\[ \liminf_{t \to T} \inf_{x \in \mathbb{R}} u_x(t, x)(T - t) = -2. \]  

**Proof.** From (4.11), we get

\[ -K \leq m'(t) + \frac{1}{2} m^2(t) + \lambda m(t) \leq K, \text{ a.e. on } [0, T). \]  

Therefore

\[ -K - \frac{1}{2} \lambda^2 \leq m'(t) + \frac{1}{2} (m(t) + \lambda)^2 \leq K + \frac{1}{2} \lambda^2, \text{ a.e. on } [0, T). \]  

Let \( \sigma \in (0, \frac{1}{2}) \). According to Theorem 4.3, we get \( \liminf_{t \to T}(m(t) + \lambda) = -\infty \), there is some \( t_0 \in (0, T) \) with

\[ m(t_0) + \lambda < 0 \text{ and } (m(t_0) + \lambda)^2 > \frac{1}{\sigma} \left( K + \frac{1}{2} \lambda^2 \right). \]

From Lemma 4.4, we can obtain \( m(t) \) is locally Lipschitz and \( m(t) < 0, t \in [0, T) \), then there exists some \( \alpha > 0 \) such that

\[ (m(t) + \lambda)^2 > \frac{1}{\sigma} \left( K + \frac{1}{2} \lambda^2 \right), \text{ } t \in (t_0, t_0 + \alpha). \]

Thus, we get

\[ m'(t) \leq \left( \sigma - \frac{1}{2} \right) (m(t) + \lambda)^2 < 0, \text{ a.e. on } (t_0, t_0 + \alpha). \]  

By integrating (4.15) with respect to \( t \) on \( [t_0, t_0 + \alpha] \), we obtain

\[ m(t_0 + \alpha) \leq m(t_0) < 0. \]  

Then, it is obvious that

\[ m(t_0 + \alpha) + \lambda \leq m(t_0) + \lambda < 0. \]  

According to (4.17), we get

\[ (m(t_0 + \alpha) + \lambda)^2 \geq (m(t_0) + \lambda)^2 > \frac{1}{\sigma} \left( K + \frac{1}{2} \lambda^2 \right) \]  

for \( t \in (t_0, t_0 + \alpha) \). By the continuous extension method, we have

\[ (m(t) + \lambda)^2 > \frac{1}{\sigma} \left( K + \frac{1}{2} \lambda^2 \right), \text{ } t \in [t_0, T). \]

From (4.14) and (4.19), we get

\[ -\frac{1}{2} - \sigma < \frac{m'(t)}{(m(t) + \lambda)^2} < -\frac{1}{2} + \sigma, \text{ a.e. on } [t_0, T). \]
For $t \in (t_0, T)$, integrating (4.20) with respect to $t$ over $(t, T)$, we get
\[
\left( -\frac{1}{2} - \sigma \right) (T - t) < \frac{1}{m(t) + \lambda} < \left( -\frac{1}{2} + \sigma \right) (T - t), \quad t \in [t_0, T).
\]
Let $\sigma \to 0$, then we get
\[
\liminf_{t\to T} (m(t) + \lambda) (T - t) = -2,
\]
namely
\[
\liminf_{t\to T} m(t)(T - t) = -2.
\]
This completes the proof of Theorem 4.4. \hfill \Box

5. **Global existence.** In this section, we will prove the global existence and uniqueness of solutions of the Cauchy problem (1.6)–(1.7). To do this, we need to consider a related problem to (1.6)–(1.7) as follows
\[
\begin{cases}
\frac{dq(t,x)}{dt} = u(t, q(t,x)), \quad (t, x) \in [0, T) \times \mathbb{R}, \\
q(0, x) = x, \quad x \in \mathbb{R}.
\end{cases}
\tag{5.1}
\]
Applying the classical results in the theory of ODEs, we can obtain the following results on $q(t,x)$ which is very important to prove the global existence of solutions.

**Lemma 5.1.** Let $u_0(x) \in H^s(s > \frac{3}{2})$, and $T > 0$ be the maximal existence time of the corresponding solution $u(t,x) \in H^s(s > \frac{3}{2})$ of equation (1.6). Then problem (5.1) has a unique solution $q(t,x) \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$, and $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with
\[
q_x(t, x) = e^{\int_0^t u_x(\tau,q(\tau,x))d\tau} > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.
\]
Furthermore, we have
\[
y(t, q(t,x))q_x^2(t, x) = y_0 e^{-\lambda t} - \int_0^t e^{\lambda(s-t)} \left( \frac{1}{2}g(u) - \frac{3}{2}u^2 + ku \right)_x q_x^2(s,x)ds.
\]

**Proof.** For fixed $x \in \mathbb{R}$, we deal with an ordinary differential equation. By using the Sobolev embedding theorem, we have that $u \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$. Therefore the classical results in the theory of ODEs yield the first assertion. Differentiating (5.1) with respect to $x$, we obtain
\[
\begin{cases}
\frac{d}{dt}q_x(t, x) = u_x(t, q(t,x))q_x(t, x), \quad (t, x) \in [0, T) \times \mathbb{R}, \\
q_x(0, x) = 1, \quad x \in \mathbb{R}.
\end{cases}
\tag{5.2}
\]
From (5.2), we get
\[
q_x(t, x) = e^{\int_0^t u_x(\tau,q(\tau,x))d\tau} > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.
\]
Differentiating $y(t, q(t,x))q_x^2(t, x)$ with respect to $t$, using (2.2) and (5.2), we get
\[
\frac{d}{dt}(y(t, q(t,x))q_x^2(t, x)) = -\lambda y(t, q(t,x))q_x^2(t, x) - \left( \frac{1}{2}g(u) - \frac{3}{2}u^2 + ku \right)_x q_x^2(t, x).
\]
Integrating it with respect to $t$ over $[0, t]$, we obtain
\[
y(t, q(t,x))q_x^2(t, x) = y_0 e^{-\lambda t} - \int_0^t e^{\lambda(s-t)} \left( \frac{1}{2}g(u) - \frac{3}{2}u^2 + ku \right)_x q_x^2(s,x)ds.
\]
This completes the proof of Lemma 5.1. \hfill \Box
Theorem 5.1. Let \( u_0 \in H^s \left( s > \frac{3}{2} \right) \), and \( y_0 = u_0 - u_0_{xx} \). Assume that:
(a) there exists a point \( x_0 \in \mathbb{R} \) such that \( y_0(x) \leq 0 \) on \((\infty, x_0]\) and \( y_0(x) \geq 0 \) on \([x_0, +\infty)\);
(b) \( \text{sgn}(y) = \text{sgn}(y_0) \).

Then the Cauchy problem (1.6)–(1.7) has a unique global solution \( u \).

Proof. By the relation \( u(t, x) = P * y \) with \( P(x) = \frac{1}{2} e^{-|x|} \), we get
\[
\begin{align*}
    u(t, x) &= \frac{1}{2} \int_{\mathbb{R}} e^{-|x-\xi|} y(t, \xi) d\xi \\
    &= \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{\xi} y(t, \xi) d\xi + \frac{1}{2} e^{x} \int_{x}^{+\infty} e^{-\xi} y(t, \xi) d\xi. 
\end{align*}
\]
Thus, we get
\[
    u_x(t, x) + u(t, x) = e^x \int_{x}^{+\infty} e^{-\xi} y(t, \xi) d\xi \geq 0
\]
and
\[
    u_x(t, x) - u(t, x) = -e^{-x} \int_{-\infty}^{x} e^{\xi} y(t, \xi) d\xi \geq 0.
\]
By using the Sobolev embedding theorem, we get
\[
    u_x \geq -\|u\|_{L^\infty} \geq -\frac{\sqrt{2}}{2} ||u_0||_{1}.
\]
Combining the above inequality, and using Theorems 2.1 and 4.2, we obtain that the Cauchy problem (1.6)–(1.7) has a unique global solution \( u \). This completes the proof of Theorem 5.1.

Remark 1. In Lemma 5.1, we observe how the presence of \( g(u), u^2 \) and \( ku \) affect the investigation of global existence and uniqueness of solutions of equation (1.6). If \( g(u) = 3u^2 - 2ku \), then we can immediately conclude that \( \text{sgn}(y) = \text{sgn}(y_0) \) and the second condition in the Theorem 5.1 would hold automatically as a consequence of the first condition.

Acknowledgement. The authors sincerely thank the editors and the anonymous referees for very careful reading and for providing many inspiring and valuable comments and suggestions which led to much improvement in the earlier version of this paper.

REFERENCES

[1] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, Arch. Ration. Mech. Anal., 183 (2007), 215–239.
[2] R. Camassa and D. Holm, An integrable shallow water equation with peaked soliton, Phys. Rev. Lett., 71 (1993), 1661–1664.
[3] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math., 18 (1998), 229–243.
[4] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, Commun. Pure Appl. Math., 51 (1998), 475–504.
[5] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, Commun. Math. Phys., 211 (2000), 45–61.
[6] A. Constantin and J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, Math. Z., 233 (2000), 75–91.
[7] A. Constantin and W. A. Strauss, Stability of peakons, Commun. Pure Appl. Math., 53 (2000), 603–610.
[8] A. Constantin and L. Molinet, The initial value problem for a generalized Boussinesq equation, *Differ. Integral Equ.*, **15** (2002), 1061–1072.

[9] A. Fokas and B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Phys. D.*, **4** (1981), 47–66.

[10] I. L. Freire, N. S. Filho, L. C. Souza and C. E. Toffoli, Invariants and wave breaking analysis of a Camassa-Holm type equation with quadratic and cubic non-linearities, *J. Differ. Equ.*, **269** (2020), 56–77.

[11] J. M. Ghidaglia, Weakly damped forced Korteweg-de Vries equations behave as a finite dimensional dynamical system in the long time, *J. Differ. Equ.*, **74** (1988), 369–390.

[12] Y. Li and P. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Differ. Equ.*, **162** (2000), 27–63.

[13] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in *Spectral Theory and Differential Equations*, Lecture Notes in Math, Springer, Berlin, 1975.

[14] T. Kato and G. Ponce, Commutator estimate and the Euler and Navier-Stokes equations, *Commun. Pure. Appl. Math.*, **41** (1988), 891–907.

[15] E. Ott and R. N. Sudan, Damping of solitary waves, *Phys. Fluids*, **13** (1970), 1432–1434.

[16] E. Wahlén, Global existence of weak solutions to the Camassa-Holm equation, *Int. Math. Res. Not.*, **2006** (2006), 1–12.

[17] S. Wu and Z. Yin, Global existence and blow up phenomena for the weakly dissipative Camassa-Holm equation, *J. Differ. Equ.*, **246** (2009), 4309–4321.

[18] X. Wu and B. Guo, The Cauchy problem of the modified CH and DP equations, *IMA J. Appl. Math.*, **80** (2015), 906–930.

[19] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, *Commun. Pure. Appl. Math.*, **53** (2000), 1411–1433.

[20] Z. Yin, On the Blow-up scenario for the generalized Camassa-Holm equation, *Commun. Partial Differ. Equ.*, **29** (2004), 867–877.

[21] Z. Yin, Well-posedness and blow-up phenomena for the periodic generalized Camassa-Holm equation, *Commun. Pure Appl. Anal.*, **3** (2004), 501–508.

[22] Z. Yin, On the Cauchy problem for the generalized Camassa-Holm equation, *Nonlinear Anal.*, **66** (2007), 460–471.

Received July 2021; revised September 2021; early access November 2021.

E-mail address: zhouyh318@nenu.edu.cn
E-mail address: jishuguan@hotmail.com