STATISTICS OF FINITE DEGREE COVERS OF TORUS KNOT COMPLEMENTS

ELIZABETH BAKER AND BRAM PETRI

ABSTRACT. In the first part of this paper, we determine the asymptotic subgroup growth of the fundamental group of a torus knot complement. In the second part, we use this to study random finite degree covers of torus knot complements. We determine their Benjamini–Schramm limit and the linear growth rate of the Betti numbers of these covers. All these results generalise to a larger class of lattices in $\text{PSL}(2,\mathbb{R}) \times \mathbb{R}$. As a by-product of our proofs, we obtain analogous limit theorems for high degree random covers of non-uniform Fuchsian lattices with torsion.

1. INTRODUCTION

A classical theorem due to Hempel [Hem87] states that the fundamental group of a tame 3-manifold is residually finite. As such, it has many finite index subgroups, or equivalently, the manifold has lots of finite degree covers.

In this paper we study the fundamental groups of torus knot complements and groups closely related to these. We ask two questions: How fast does the number of index $n$ subgroups grow as a function of $n$? And what are the properties of a random index $n$ subgroup and the corresponding degree $n$ cover?

1.1. Subgroup growth. We will study groups of the form

$$\Gamma_{p_1, \ldots, p_m} = \langle x_1, \ldots, x_m | x_1^{p_1} = x_2^{p_2} = \cdots = x_m^{p_m} \rangle.$$ 

When $\gcd(p, q) = 1$ and $p, q \geq 2$ then $\Gamma_{p,q}$ is the fundamental group of a $(p, q)$-torus knot complement. More generally, when $\sum_{j=1}^{m} \frac{1}{p_j} < m - 1$, $\Gamma_{p_1, \ldots, p_m}$ appears as a non-uniform lattice in $\text{PSL}(2,\mathbb{R}) \times \mathbb{R}$ (see for instance [Eck04, Proposition 7.2]).

The first of our questions asks for the subgroup growth of these groups. Writing $a_n(\Gamma)$ for the number of index $n$ subgroups of a group $\Gamma$, we will prove:

**Theorem 1.1.** Let $p_1, \ldots, p_m \in \mathbb{N}_{\geq 1}$ such that $\sum_{j=1}^{m} \frac{1}{p_j} < m - 1$. Then it holds that\(^1\)

$$a_n(\Gamma_{p_1, \ldots, p_m}) \sim A_{p_1, \ldots, p_m} \cdot n^{-1/2} \cdot \exp \left( \sum_{i=1}^{m} \sum_{0 < j < p_i} \frac{n^{j/p_i}}{j} \right) \cdot \left( \frac{n}{e} \right)^{n \left( m - 1 - \sum_{j=1}^{m} \frac{1}{p_j} \right)}$$

as $n \to \infty$, where

$$A_{p_1, \ldots, p_m} = \sqrt{2\pi} \exp \left( - \sum_{i : p_i \text{ even}} \frac{1}{2p_i} \right) \prod_{i=1}^{m} p_i^{-1/2}.$$ 

\(^1\)Here and throughout the paper, the notation $f(n) \sim g(n)$ as $n \to \infty$ will indicate that $\lim_{n \to \infty} f(n)/g(n) \to 1$. 

*Date: May 26, 2020.*
Note that all torus knot groups satisfy the condition on \( p_1, \ldots, p_m \in \mathbb{N}_{>1} \). In general, the only groups excluded by this condition are \( \Gamma_{2,2} \), the fundamental group of the Klein bottle, and \( \mathbb{Z} \). The subgroup growth of both of these groups is well understood.

The theorem above also generalizes to free products of the form
\[
\Gamma_{p_1, \ldots, p_{1,m_1}} * \cdots * \Gamma_{p_{r,1}, \ldots, p_{r,m_r}}
\]
where \( \sum_j p_{i,m_j} < m_i - 1 \) for all \( i = 1, \ldots, r \). In the case of torus knot groups, this corresponds to taking connected sums.

As is common in subgroup growth, our proof goes through counting homomorphisms \( \Gamma_{p_1, \ldots, p_m} \to S_n \), where \( S_n \) denotes the symmetric group on \( n \) letters. Given a group \( \Gamma \) and \( n \in \mathbb{N} \), we shall write
\[
h_n(\Gamma) = |\text{Hom}(\Gamma, S_n)|.
\]

Writing \( C_p \) for the cyclic group of order \( p \), the group \( \Gamma_{p_1, \ldots, p_m} \) is a central extension
\[
1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_{p_1, \ldots, p_m} \xrightarrow{\Phi} C_{p_1} * \cdots * C_{p_m} \longrightarrow 1
\]
where \( \Phi_{p_1, \ldots, p_m} \) sends \( x_j \) to a generator of \( C_{p_j} \). If we think of \( \Gamma_{p_1, \ldots, p_m} \) as a lattice in \( \text{PSL}(2, \mathbb{R}) \times \mathbb{R} \), then \( \Phi_{p_1, \ldots, p_m}(\Gamma_{p_1, \ldots, p_m}) \) is the projection onto \( \text{PSL}(2, \mathbb{R}) \) of \( \Gamma_{p_1, \ldots, p_m} \).

The number \( m - 1 - \sum_{i=1}^m \frac{1}{p_i} \), that appears everywhere throughout this paper is the absolute value of the orbifold Euler characteristic of \( \Phi_{p_1, \ldots, p_m}(\Gamma_{p_1, \ldots, p_m}) \setminus \mathbb{H}^2 \).

The kernel of \( \Phi_{p_1, \ldots, p_m} \) is generated by \( x_1^{p_1} \) (or equivalently \( x_i^{p_i} \) for any \( i = 1, \ldots, m \)). As such,
\[
h_n(\Gamma_{p_1, \ldots, p_m}) \geq h_n(C_{p_1} * \cdots * C_{p_m}) = \prod_{j=1}^m h_n(C_{p_j}, S_n).
\]

It turns out that asymptotically this bound is tight. In other words, a typical homomorphism factors through \( \Phi_{p_1, \ldots, p_m} \). We prove:

**Theorem 1.2.** \( p_1, \ldots, p_m \in \mathbb{N}_{>1} \) such that \( \sum_{j=1}^m \frac{1}{p_j} < m - 1 \). Then
\[
\frac{|\{ \rho \in \text{Hom}(\Gamma_{p_1, \ldots, p_m}, S_n); \rho \text{ factors through } \Phi_{p_1, \ldots, p_m} \}|}{h_n(\Gamma_{p_1, \ldots, p_m})} \longrightarrow 1
\]
as \( n \to \infty \).

The analogous result is also known to hold for orientable circle bundles over surfaces [LM00]. However, even if these are also central extensions of Fuchsian groups, the methods of Liskovets and Mednykh are quite different.

### 1.2. Random subgroups and covers

In the second part of our paper, we use our results to study random finite index subgroups of \( \Gamma_{p_1, \ldots, p_m} \). That is, since the number of index \( n \) subgroups of \( \Gamma_{p_1, \ldots, p_m} \) is finite, we can pick one uniformly at random and ask for its properties. Let us denote our random index \( n \) subgroup by \( H_n \). This is an example of an Invariant Random Subgroup (IRS) – i.e. a conjugation invariant Borel measure on the Chabauty space of subgroups of \( \Gamma_{p_1, \ldots, p_m} \) (for more details see Section 2.3).
Let us also fix a classifying space $X_{p_1, \ldots, p_m}$ for $\Gamma_{p_1, \ldots, p_m}$. For instance, if $p, q \geq 2$ and $\gcd(p, q) = 1$ we can take the corresponding torus knot complement. More generally, since $\Gamma_{p_1, \ldots, p_m}$ appears as a torsion-free lattice in $\text{PSL}(2, \mathbb{R}) \times \mathbb{R}$, we may take the manifold $\Gamma \setminus (\mathbb{H}^2 \times \mathbb{R})$. $H_n$ gives rise to a random degree $n$ cover of $X_{p_1, \ldots, p_m}$.

We will study three (related) problems:

- First, we will ask, given a conjugacy class $K \subseteq \Gamma_{p_1, \ldots, p_m}$, how many conjugacy classes of $H_n$ the set $K \cap H_n$ contains. We will denote this number by $Z_K(H_n)$. In topological terms, $K$ corresponds to a free homotopy class of loops in $X_{p_1, \ldots, p_m}$, $Z_K(H_n)$ is the number of closed lifts of that loop to the cover of $X_{p_1, \ldots, p_m}$ corresponding to $H_n$. We note that we count these lifts as loops and not as sets. In particular, if the corresponding element in $\Gamma_{p_1, \ldots, p_m}$ is non-primitive, some of these different lifts overlap.
- After this we will ask what IRS the random subgroup $H_n$ converges to as $n \to \infty$. In topological terms, this asks for the Benjamini–Schramm limit of the corresponding random cover of $X_{p_1, \ldots, p_m}$ (see Section 2.4 for a definition of Benjamini–Schramm convergence).
- Finally, we will study the asymptotic behaviour of the real Betti numbers $b_k(H_n; \mathbb{R})$ of $H_n$, or equivalently of the corresponding random cover of $X_{p_1, \ldots, p_m}$.

We will write

$$L_{p_1, \ldots, p_m} := \ker(\Phi_{p_1, \ldots, p_m}) \cong \mathbb{Z}.$$ 

For a torus knot this is the subgroup generated by the longitude. Since $L_{p_1, \ldots, p_m}$ is normal in $\Gamma_{p_1, \ldots, p_m}$, it’s also an IRS.

We will prove

**Theorem 1.3.** Let $p_1, \ldots, p_m \in \mathbb{N}_{\geq 1}$ be such that $\sum_{j=1}^{m} \frac{1}{p_j} < m - 1$.

(a) Let $K_1, \ldots, K_r \subseteq \Gamma_{p_1, \ldots, p_m}$ be distinct non-trivial conjugacy classes. Then $Z_{K_1}(H_n), \ldots, Z_{K_r}(H_n)$ are asymptotically independent as $n \to \infty$. Moreover, if $K_i \subseteq L_{p_1, \ldots, p_m}$ then

$$\lim_{n \to \infty} \mathbb{P}[Z_{K_i}(H_n) = n] = 1$$

- and if $K_i \nsubseteq L_{p_1, \ldots, p_m}$ then $Z_{K_i}(H_n)$ converges in distribution to a Poisson$(1)$-distributed random variable.

(b) $H_n$ converges to $L_{p_1, \ldots, p_m}$ as an IRS.

(c) We have that

$$\lim_{n \to \infty} \frac{b_k(H_n; \mathbb{R})}{n} = \begin{cases} m - 1 - \sum_{i=1}^{m} \frac{1}{p_i} & \text{if } k = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

in probability.

Recall that a random variable $X : \Omega \to \mathbb{N}$ is Poisson-distributed with parameter $\lambda > 0$ if and only if

$$\mathbb{P}[X = k] = \frac{\lambda^k e^{-\lambda}}{k!} \quad \forall k \in \mathbb{N}.$$ 

So (a) above gives us an explicit limit for the probability that a fixed curve lifts to any given number of curves in the cover. For example, if we denote the random degree $n$
cover of our \((p,q)\)-torus knot complement by \(X_{p,q}(n)\) and \(\gamma\) is any free homotopy class of closed curves in \(X_{p,q}(1)\) that is not freely homotopic to a power of the longitude we obtain:

\[
\lim_{n \to \infty} \mathbb{P}[\gamma \text{ lifts to exactly 3 closed curves in } X_{p,q}(n)] = \frac{1}{6e} = 0.0613 \ldots.
\]

(b) in particular implies that a random degree \(n\) cover of a torus knot complement does not converge to the universal cover of the given torus knot complement as \(n \to \infty\). This is different from the behaviour of random finite covers of graphs [DJPP13], surfaces [MP20] and many large volume locally symmetric spaces of higher rank [ABB+17], that all do converge to their universal covers.

(c) also has implications for the number of boundary tori in a random cover of a torus knot complement. Indeed, together with “half lives, half dies” [Hat07, Lemma 3.5], it also implies that the number of boundary components of a degree \(n\) cover is typically at most \(\left(1 - \frac{1}{p} - \frac{1}{q}\right) \cdot n + o(n)\).

Because all the results in the theorem above are really about the group \(\Gamma_{p_1,\ldots,p_m}\), we can also apply them to random covers of more general spaces \(Y_{p_1,\ldots,p_m}\) that have \(\Gamma_{p_1,\ldots,p_m}\) as their fundamental group (i.e. without assuming that \(Y_{p_1,\ldots,p_m}\) is a classifying space for \(\Gamma_{p_1,\ldots,p_m}\)). In that case, the random cover Benjamini–Schramm converges to the cover of \(Y_{p_1,\ldots,p_m}\) corresponding to \(L_{p_1,\ldots,p_m}\) and the normalised Betti numbers converge to the \(\ell^2\)-Betti numbers of that cover.

Finally, we note that we prove analogous results to Theorem 1.3 for random index \(n\) subgroups of non-cocompact Fuchsian groups.

**Theorem 1.4.** Let \(\Lambda\) be a non-cocompact Fuchsian group of finite covolume. Moreover, let \(G_n < \Lambda\) denote an index \(n\) subgroup, chosen uniformly at random.

(a) Let \(K_1, \ldots, K_r \subset \Lambda\) be distinct non-trivial conjugacy classes. Then, as \(n \to \infty\), the vector of random variables

\[
\left( Z_{K_1}(G_n), \ldots, Z_{K_r}(G_n) \right)
\]

converges in distribution to a vector of independent Poisson(1)-distributed random variables.

(b) \(G_n\) converges to the trivial group as an IRS.

Note that the analogue to Theorem 1.3(c) also holds here. However, a much stronger statement follows directly from multiplicativity of orbifold Euler characteristic.

The case of free groups in the theorem above is very similar to results on cycle counts in random regular graphs in the permutation model (see for instance [DJPP13] and also [Bol80] for a slightly different model), so our real contribution is to the case with torsion. For surface groups similar results have very recently been proved by Magee–Puder [MP20]. The case of cocompact Fuchsian groups with torsion is currently open.

1.3. **The structure of the proofs.** Our proofs start with the count of the number of homomorphisms \(\Gamma_{p_1,\ldots,p_m} \to S_n\). Because the presentation for our groups is very explicit, we are able to write down a closed (albeit somewhat involved) formula for \(h_n(\Gamma_{p_1,\ldots,p_m})\) (Proposition 3.1).
The formula we find expresses $h_n(\Gamma_{p_1,\ldots,p_m})$ as a sum, so the next step is to single out the largest term in this sum. The key technical results, which most of the paper rests on, are Lemmas 4.4 and 4.5, which determine the dominant term in the sum.

First of all, together with results by Müller [Müll97], these lemmas give us the asymptotic behaviour for $h_n(\Gamma_{p_1,\ldots,p_m})$ (Theorem 4.1). Moreover, they also imply that most homomorphisms $\Gamma_{p_1,\ldots,p_m} \to S_n$ factor through $\Phi_{p_1,\ldots,p_m}: \Gamma_{p_1,\ldots,p_m} \to C_{p_1} \ast \cdots \ast C_{p_m}$ (Theorem 1.2) and, using this fact, the subgroup growth of $\Gamma_{p_1,\ldots,p_m}$ (Theorem 1.1).

The idea behind the proofs of our results on random subgroups is to first prove the analogous results for random index $n$ subgroups of $C_{p_1} \ast \cdots \ast C_{p_m}$ and then use the fact that most index $n$ subgroups of $\Gamma_{p_1,\ldots,p_m}$ come from index $n$ subgroups of $C_{p_1} \ast \cdots \ast C_{p_m}$ to upgrade these into results about $\Gamma_{p_1,\ldots,p_m}$.

First, we prove Poisson statistics for the number of fixed points of an element $g \in C_{p_1} \ast \cdots \ast C_{p_m}$ under a random homomorphism $C_{p_1} \ast \cdots \ast C_{p_m} \to S_n$ (This is most of Theorem 1.4(a)). This uses the method of moments together with results by Volynets [Vol86] and independently Wilf [Wil86] on $h_n(C_p)$. Then we turn these into Poisson statistics for the variables $Z_K$, where $K \subseteq C_{p_1} \ast \cdots \ast C_{p_m}$ is a conjugacy class. This, together with Theorem 1.2 implies the statistics in Theorem 1.3(a). In order to keep the proof a little lighter, we did not compute explicit error terms for our Poisson approximation result in (a) and used the method of moments to prove it. Error terms could be made explicit using the error terms in Müller’s results [Müll96]. Moreover, the Chen–Stein method (see for instance [AGG89, BHJ92, DJPP13]) would probably give sharper bounds than the method of moments.

The fact that a conjugacy class $K \subseteq \Gamma_{p_1,\ldots,p_m}$ typically has very few lifts to $H_n$ if it does not lie in $L_{p_1,\ldots,p_m}$ and typically has $n$ lifts if it does (this is essentially Theorem 1.3(a)), implies that the IRS $H_n$ converges to $L_{p_1,\ldots,p_m}$ (Theorem 1.3(b)). Using results by Elek [Ele10] and Lück [Lüc94], we then also obtain that the normalised Betti numbers of $H_n$ converge to the $\ell^2$-Betti numbers of the cover of $\tilde{X}_{p_1,\ldots,p_m}/L_{p_1,\ldots,p_m}$.

Finally, in Section 5.5, we sketch how to complete the proof of Theorem 1.4.

1.4. Notes and references. As opposed to the case of 2-manifolds [Dix69, MP02, LS04], there are very few 3-manifolds for which the subgroup growth is well understood. For instance, to the best of our knowledge, there isn’t a single hyperbolic 3-manifold group $\Gamma$ for which the asymptotic behaviour of $a_n(\Gamma)$ is known. It does follow from largeness of these groups [Ago13] that the number

$$s_n(G) := \sum_{m \leq n} a_m(G)$$

grows faster than $(n!)^\alpha$ for some $\alpha > 0$, but even at the factorial scale, the growth (i.e. the optimal $\alpha$) is not known. In the more general settings of lattices in PSL$(2,\mathbb{C})$ it’s known in one very particular case [BPR20, Section 2.5.2]. One of the difficulties in determining $\alpha$ in general is that for a general hyperbolic 3-manifold, no proof for a factorial lower bound is known that does not rely on Agol’s work.

For Seifert fibred manifolds a little more is known: the subgroup growth of orientable circle bundles over surfaces was determined by Liskovets and Mednykh [LM00] and the subgroup growth of Euclidean manifolds can be derived from general results on the subgroup growth of virtually abelian groups [dSMS99, Sul16].
One can also ask for the number of distinct isomorphism types of subgroups, in which case even less is known [FPP+20].

Finally, results similar to our Theorems 1.1 and 1.2 are known to hold for Baumslag–Solitar groups [Kel20].

The geometry of a random cover of a graph is a classical subject in the study of random regular graphs (see for instance [AL02, Fri08, DJPP13, Pud15]). Moreover, it is known that, as $n \to \infty$, a random $2d$-regular graph sampled uniformly from the set of such graphs on $n$ vertices as a model is contiguous to the model given by a random degree $n$ cover of a wedge of $d$ circles [GJKW02, Wor99]. In other words, random covers are also a tool that can be used to study other models of random graphs.

Random covers of manifolds are much less well understood. Of course, random graph covers also give rise to random covers of punctured surfaces, so some of the graph theory results can be transported to this context. Very recently, Magee–Puder [MP20] and Magee–Naud–Puder [MNP20] studied random covers of closed hyperbolic surfaces. They proved that these covers Benjamini–Schramm converge to the hyperbolic plane and that the spectral gap of their Laplacian is eventually larger than $\frac{3}{16} - \epsilon$ for all $\epsilon > 0$ (given that his holds for the base surface).

More general random surfaces (see for instance [BM04, GPY11, Mir13, Pet17, PT18, MP19, MRR18, BCP19, GLMST19, Shr20]), random 3-manifolds (see for instance [DT06, Mah10, BBG+18, HV19]) and random knots (see for instance [EZ17, BKL+20]) have recently also received considerable attention.

Invariant Random Subgroups were introduced by Abért–Glasner–Virág in [AGV14], by Bowen in [Bow14] and under a different name by Vershik in [Ver12], but had been studied in various guises before (see the references in [AGV14]). Benjamini–Schramm convergence was introduced for graphs in [BS01] and for lattices in Lie groups in [ABB+17]. The fact that Benjamini–Schramm convergence implies convergence of normalised Betti numbers was proved for sequences of simplicial complexes in [Ele10], for sequences of lattices in [ABB+17] and for sequences of negatively curved Riemannian manifolds in [ABBG18].

Acknowledgement. We thank Jean Raimbault for useful remarks.
Another result we will need is on the asymptotic number of homomorphisms \( C_m \to S_n \) (or equivalently the number of elements of order \( m \) in \( S_n \)). The result we will use is due to Volynets [Vol86] and independently Wilf [Wil86] and fits into a large body of work, starting with classical results by Chowla–Herstein–Moore [CHM51], Moser–Wyman [MW55], Hayman [Hay56] and Harris–Schoenfeld [HS68] and culminating in a paper by Müller [Mül97] in which the asymptotic behaviour of \( h_n(G) \) as \( n \to \infty \) is determined for any finite group \( G \). It states:

**Theorem 2.2** (Volynets [Vol86], Wilf [Wil86]). Let \( m_1, \ldots, m_k \in \mathbb{N} \). Then

\[
h_n(C_m) \sim A_m \cdot \exp \left( \sum_{d|m, d<m} \frac{1}{d} n^{d/m} \right) \cdot \left( \frac{n}{e} \right)^n \left( 1 - \frac{1}{m} \right)
\]

as \( n \to \infty \). Here

\[
A_m = \begin{cases} 
  m^{-1/2} & \text{if } m \text{ odd} \\
  m^{-1/2} \exp \left( -\frac{1}{2m} \right) & \text{if } m \text{ even}
\end{cases}
\]

Finally, we will need two results due to Müller. The first in fact also implies the previous theorem:

**Theorem 2.3** (Müller [Mül97]). Let \( P(x) = \sum_{\mu=1}^{m} c_\mu x^\mu \in \mathbb{R}[x] \) be a polynomial with degree \( m \geq 1 \) and let \( \exp(P(z)) = \sum_{n=0}^{\infty} \alpha_n x^n \). Suppose further that

1. \( \alpha_n > 0 \) for all sufficiently large \( n \), and
2. \( c_\mu = 0 \) for \( m/2 < \mu < m \).

Then the coefficients \( \alpha_n \) satisfy the asymptotic formula

\[
\alpha_n \sim \frac{B_m}{\sqrt{2\pi n}} n_0^{-n/m} \exp \left( P(n_0^{1/m}) \right) \quad \text{as } n \to \infty,
\]

where \( n_0 := n / (mc_m) \) and

\[
B_m := \begin{cases} 
  m^{-1/2} & \text{if } m \text{ odd} \\
  m^{-1/2} \exp \left( -\frac{c_{m/2}}{8c_m} \right) & \text{if } m \text{ even}
\end{cases}
\]

The second result we will need is:

**Theorem 2.4** (Müller [Mül96]). Let \( p_1, \ldots, p_m \in \mathbb{N}_{>1} \) such that \( \sum_{i=1}^{m} \frac{1}{p_i} < m - 1 \). Then

\[
t_n(C_{p_1} \ast \cdots \ast C_{p_m}) \sim h_n(C_{p_1} \ast \cdots \ast C_{p_m}) \quad \text{as } n \to \infty.
\]

In fact, Müller also provides error terms and proves the theorem for more general groups; we refer to his paper for details.

### 2.2. Probability theory

For our Poisson approximation results, we will use the method of moments. Given a random variable \( Z : \Omega \to \mathbb{N} \) and \( k \in \mathbb{N} \), we will write

\[
(Z)_k = Z(Z - 1) \cdots (Z - k + 1).
\]

Moreover, recall that a sequence of random variables \( Z_n : \Omega_n \to \mathbb{N}^d \) is said to converge jointly in distribution to a random variable \( Z : \Omega \to \mathbb{N}^d \) if and only if

\[
P[Z_n \in A] \xrightarrow{n \to \infty} P[Z \in A] \quad \forall A \subset \mathbb{N}^d.
\]
The following theorem is classical. For a proof see for instance [Bol85].

**Theorem 2.5** (The method of moments). Let \( Z_{n,1}, Z_{n,2}, \ldots, Z_{n,r} : \Omega_n \to \mathbb{N}, n \in \mathbb{N} \) be random variables. If there exist \( \lambda_1, \ldots, \lambda_r > 0 \) such that for all \( k_1, \ldots, k_r \in \mathbb{N} \)

\[
\lim_{n \to \infty} \mathbb{E} \left[ (Z_{n,1})^{k_1} (Z_{n,2})^{k_2} \cdots (Z_{n,r})^{k_r} \right] = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_r^{k_r},
\]

then \( (Z_{n,1}, \ldots, Z_{n,r}) : \Omega_n \to \mathbb{N}^r \) converges jointly in distribution to a vector of random variables \( (Z_1, \ldots, Z_r) : \Omega \to \mathbb{N}^r \) where

- \( Z_i \sim \text{Poisson}(\lambda_i), i = 1, \ldots, r \)
- The random variables \( Z_1, \ldots, Z_r \) form an independent family.

2.3. **Invariant Random Subgroups.** We will phrase our results on random subgroups in the language of Invariant Random Subgroups. For a finitely generated group \( \Gamma \), \( \text{Sub}(\Gamma) \) will denote the Chabauty space of subgroups of \( \Gamma \) (see for instance [Gel18] for an introduction).

We will be interested in random index \( n \) subgroups of such a group \( \Gamma \). This corresponds to studying the measure \( \mu_n \) on \( \text{Sub}(\Gamma) \), defined by

\[
\mu_n = \frac{1}{a_n(\Gamma)} \sum_{H < \Gamma : [\Gamma:H] = n} \delta_H
\]

where \( \delta_H \) denotes the Dirac mass on \( H \in \text{Sub}(\Gamma) \).

\( \mu_n \) is an example of what is called an Invariant Random Subgroup (IRS) of \( \Gamma \) – i.e. a Borel probability measure on \( \text{Sub}(\Gamma) \) that is invariant under conjugation by \( \Gamma \). We will write IRS(\( \Gamma \)) for the space of IRS’s of \( \Gamma \) endowed with the weak-* topology. This space has been first studied under this name in [AGV14] and [Bow14] and under a different name in [Ver12].

We will also use a characterisation for convergence in IRS(\( \Gamma \)) terms of fixed points. This characterisation is probably well known, but we couldn’t find the exact statement in the literature (for instance [AGV14, Lemma 16] is very similar). We will provide a proof for the sake of completeness.

Given a function \( f : \text{Sub}(\Gamma) \to \mathbb{C} \), we will write \( \mu_n(f) \) for the integral of \( f \) with respect to \( \mu_n \) (all measures considered in our paper are finite sums of Dirac masses, so this is always well defined).

**Lemma 2.6.** Let \( \Gamma \) be a countable discrete group. Set

\[
\mu_n = \frac{1}{a_n(\Gamma)} \sum_{H < \Gamma : [\Gamma:H] = n} \delta_H.
\]

and let \( N \lhd \Gamma \). Then

\[
\mu_n \xrightarrow{n \to \infty} \delta_N \text{ in IRS}(\Gamma) \quad \iff \quad \begin{cases} \mu_n(Z_K) \xrightarrow{n \to \infty} o(n) & \forall \text{ conjugacy class } K \not\subset N \\ \mu_n(Z_K) \xrightarrow{n \to \infty} n & \forall \text{ conjugacy class } K \subset N \end{cases}
\]

**Proof.** We start with the fact that for \( g \in K \), \( \mu_n(\{H; g \in H\}) = \frac{1}{n} \mu_n(Z_K) \). Indeed, for any \( p \in \{1, \ldots, n\} \), the map \( \varphi \mapsto \text{Stab}_\varphi\{p\} \) gives an \( (n-1)! \)-to-1 correspondence...
between transitive homomorphisms $\Gamma \to S_n$ and index $n$ subgroups of $\Gamma$. $Z_K(\varphi)$ equals the number of fixed points of $\varphi(g)$ on $\{1, \ldots, n\}$. As such

$$\mu_n(\{H; g \in H\}) = \frac{1}{n \cdot t_n(\Gamma)} \sum_{p=1}^{n} \sum_{\varphi \in T_n(\Gamma)} 1_{g \in \text{Stab}_p(\varphi)}(\varphi) = \frac{1}{n} \mu_n(Z_K),$$

where

$$T_n(\Gamma) = \{ \varphi \in \text{Hom}(\Gamma, S_n); \varphi(\Gamma) \curvearrowright \{1, \ldots, n\} \text{ transitively}\}.$$

Now, the topology on $\text{Sub}(\Gamma)$ is generated by sets of the form

$$O_1(U) := \{ H \in \text{Sub}(\Gamma); H \cap U \neq \emptyset \}, \quad U \subset \Gamma$$

and

$$O_2(V) := \{ H \in \text{Sub}(\Gamma); H \cap V = \emptyset \}, \quad V \subset \Gamma \text{ finite,}$$

(see for instance [Gel18]). By the Portmanteau theorem, convergence $\mu_n \xrightarrow{w^*} \delta_N$ is equivalent to

$$\liminf_{n \to \infty} \mu_n(O) \geq \delta_N(O)$$

for every open set $O \subset \text{Sub}(\Gamma)$. This is equivalent to proving that $\mu_n(O) \to 1$ for every open set $O \subset \text{Sub}(\Gamma)$ such that $N \in O$. Since every open set is a union of sets of the form $O_1(U)$ and $O_2(V)$, $\mu_n \xrightarrow{w^*} \delta_N$ if and only if

$$\mu_n(O_1(U)) \to 1 \text{ when } U \cap N \neq \emptyset \quad \text{and} \quad \mu_n(O_2(V)) \to 1 \text{ when } V \cap N = \emptyset$$

for all $U \subset \Gamma$ and all finite $V \subset \Gamma$.

Let us first prove that our conditions on the behaviour of $\mu_n(Z_K)$ imply convergence in $\text{IRS}(\Gamma)$.

We start by checking (2) for sets of the form $O_1(U)$. Suppose $g \in U \cap N$. Using (1) and writing $K$ for the conjugacy class of $g$,

$$\mu_n(\{H; H \cap U \neq \emptyset\}) \geq \mu_n(\{H; g \in H\}) = \frac{1}{n} \mu_n(Z_K) \to 1,$$

by our assumption on $\mu_n(Z_K)$.

Now we deal with sets of the form $O_2(V)$. We will write $K(g)$ for the conjugacy class of an element $g \in \Gamma$. (1) gives us

$$\mu_n(\{H; H \cap V \neq \emptyset\}) \leq \frac{1}{n} \sum_{g \in V} \mu_n(Z_{K(g)}) \xrightarrow{n \to \infty} 0,$$

by our assumptions on $\mu_n(Z_{K(g)})$. This proves the first direction.

For the other direction, suppose $g \in N$ then $\delta_N(O_1(\{g\})) = 1$ and hence by (2) and (1), we obtain

$$\liminf_{n \to \infty} \frac{1}{n} \mu_n(Z_K) = 1,$$

which proves that $\mu_n(Z_K) \sim n$ as $n \to \infty$. Moreover, if $K$ is a conjugacy class such that $K \not\subset N$ and $g \in K$, then by (2) and (1),

$$\liminf_{n \to \infty} \mu_n(O_2(\{g\})) = \liminf_{n \to \infty} 1 - \mu_n(O_1(\{g\})) = \liminf_{n \to \infty} 1 - \frac{1}{n} \mu_n(Z_K) \geq 1,$$

which proves that $\mu_n(Z_K) = o(n)$ as $n \to \infty$. □
2.4. **Benjamini–Schramm convergence.** Now suppose that — as many of the groups that we study do — $\Gamma$ admits a finite simplicial complex $X$ as a classifying space. Picking a 0-cell $x_0 \in X$ gives an identification $\Gamma \approx \pi_1(X, x_0)$. Moreover, an index $n$ subgroup $H < \Gamma$ gives rise to a pointed simplicial covering space

$$(Y_H, y_H) \to (X, x_0).$$

This means that the measure $\mu_n$ above also gives rise to a probability measure $\nu_n$ on

$$K_D = \left\{ (Y, y_0); \begin{array}{l} Y \text{ a connected simplicial complex in} \\ \text{which the degree of 0-cells is at} \\ \text{most } D, \; y_0 \in Y \text{ a 0-cell} \end{array} \right\} / \sim$$

for some $D > 0$, where two pairs $(Y, y_0) \sim (Y', y'_0)$ if there is a simplicial isomorphism $Y \to Y'$ that maps $y_0$ to $y'_0$. This set $K$ can be metrised by setting

$$d_K([Y, y_0], [Y', y'_0]) = \frac{1}{1 + \sup \left\{ R \geq 0; \text{The } R\text{-balls around } y_0 \text{ and } y'_0 \text{ are isomorphic as pointed simplicial complexes} \right\}}.$$ 

This allows us to speak of weak-* convergence of measures on $K_D$. If there is a pointed simplicial complex $[Z, z_0] \in K_D$ such that

$$\nu_n \overset{w^*}{\rightarrow} \delta_{[Z, z_0]} \quad \text{as } n \to \infty,$$

where $\delta_{[Z, z_0]}$ denotes the Dirac mass on $[Z, z_0]$, then we say that the random complex determined by $\nu_n$ Benjamini–Schramm converges (or locally converges) to $[Z, z_0]$.

We will write $BS(K_D)$ for the space of probability measures on $K_D$ endowed with the weak-* topology. The procedure described above describes a continuous map

$$IRS(\Gamma) \to BS(K_D),$$

for some $D > 0$, that depends on the choice of classifying space.

2.5. **Betti numbers.** One reason for determining Benjamini–Schramm limits, is that they help determine limits of normalised Betti numbers. We will exclusively be dealing with homology with real coefficients in this paper. Given a simplicial complex $X$, we will write

$$b_k(X) = \dim(H_k(X; \mathbb{R})).$$

In order to state Elek’s result, let $[\beta_1(R), o_1], \ldots, [\beta_M(R), o_M]$ denote all the complexes in $K_D$ that can appear as an $R$-ball of a complex in $K_D$. Note that this is a finite list, the length of which depends on $R$ and $D$. Moreover, given a finite simplicial complex $X$ of which all 0-cells degree at most $D$, we will write

$$\rho_{\beta_i(R)}(X) = \frac{\left| \{ x \in V(X); \text{The } R\text{-ball around } x \text{ is isomorphic to } \beta_i(R) \} \right|}{|V(X)|}, \quad i = 1, \ldots, M$$

where $V(X)$ denotes the set of 0-cells of $X$. Elek’s theorem now states:
Theorem 2.7 (Elek [Ele10, Lemma 6.1]). Fix $D > 0$ and let $(X_n)_n$ be a sequence of finite simplicial complexes in which the degree of every 0-cell is bounded by $D$. If $|V(X_n)| \to \infty$ and for all $R > 0$, for all $i$, $\rho_{\beta_i(R)}(X_n)$ converges as $n \to \infty$, then
\[
\lim_{n \to \infty} \frac{b_k(X_n)}{|V(X_n)|}
\]
exists for all $k \in \mathbb{N}$.

Often, an explicit limit for these normalised Betti numbers can be determined in terms of $\ell^2$-Betti numbers. We will not go into this theory very deeply in this paper and refer the interested reader to for instance [Lüc02] or [Kam19] for more information.

If $\Gamma$ is a group and $X$ is a finite $\Gamma$-CW complex, then we will write $b_k^{(2)}(X; \Gamma)$ for the $k$th $\ell^2$-Betti number of the pair $(X, \Gamma)$.

We will rely on the Lück approximation theorem [Lüc94] (see also [Kam19, Theorem 5.26]). If $\Gamma$ is a group and $\Gamma_i \triangleleft \Gamma$, $i \in \mathbb{N}$ are such that $[\Gamma : \Gamma_i] < \infty$ and $\Gamma_{i+1} < \Gamma_i$, $i \in \mathbb{N}$,
then we call $(\Gamma_i)_i$ a chain of finite index normal subgroups of $\Gamma$.

Theorem 2.8 (Lück approximation theorem). Let $\Gamma$ be a group and $X$ be a finite free $\Gamma$-CW complex. Moreover, let $(\Gamma_i)_i$ be a chain of finite index normal subgroups of $\Gamma$ and set
\[
\Theta = \bigcap_{i \in \mathbb{N}} \Gamma_i.
\]
Then
\[
\lim_{i \to \infty} \frac{b_k(X/\Gamma_i)}{[\Gamma : \Gamma_i]} = b_k^{(2)}(\Theta\backslash X; \Gamma/\Theta).
\]

In order to prove convergence of Betti numbers we are after (Theorem 1.3(c)), we will use the approximation theorems of Elek and Lück to deduce the following lemma. Like Lemma 2.6, this lemma is probably well known but, as far as we know, not available in the literature in this form, so we will provide a proof.

Lemma 2.9. Let $\Gamma$ be a group that admits a finite simplicial complex $X$ as a classifying space. Set
\[
\mu_n = \sum_{H < \Gamma \atop [\Gamma : H] = n} \delta_H.
\]
If there exists a normal subgroup $N \triangleleft \Gamma$ such that $\Gamma/N$ is residually finite and
\[
\mu_n \xrightarrow{n \to \infty} \delta_N
\]
in IRS(\Gamma). Then for every $\varepsilon > 0$ and every $k \in \mathbb{N}$,
\[
\mu_n \left( \left| \frac{b_k(H)}{n} - b_k^{(2)}(N\backslash \tilde{X}; \Gamma/N) \right| < \varepsilon \right) \xrightarrow{n \to \infty} 1,
\]
where $\tilde{X}$ denotes the universal cover of $X$. 
Proof. Recall that $V(X)$ denotes the set of 0-cells of $X$ and write $D$ for the maximal degree among these 0-cells. Fix a choice of 0-cell $x_0 \in V(X)$, to obtain an identification $\Gamma \simeq \pi_1(X,x_0)$ and denote the measure on $\mathcal{K}_D$ induced by $\mu_n$ by $\nu_n \in BS(\mathcal{K}_D)$. Finally, we will let $(Z,z_0) \to (X,x_0)$ denote the pointed cover corresponding to $N$.

For $g \in K \subset \Gamma$, where $K$ is a conjugacy class, $Z_K(H)$ equals the number of lifts of $x_0$ at which the loop in $X$ corresponding to $g$ lifts to a closed loop.

Now consider the set $W_R$ of all $g \in \Gamma$ that have translation distance at most $R$ on the universal cover $\tilde{X}$. This set consists of a finite number of conjugacy classes.

If $H \triangleleft \Gamma$ is such that $[\Gamma : H] = n$ and

$$\left\{ \begin{array}{ll} Z_K(H) = o(n) & \text{if } K \not\subset N \cap W_R \\ n - Z_K(H) = o(n) & \text{if } K \subset N \cap W_R \end{array} \right.$$ \hspace{1cm} (3)

then the number of lifts $y$ in the cover of $X$ corresponding to $H$, around which the $R$-ball $B_R(y)$ is not isometric to the $R$-ball $B_R(z_0)$ around $z_0 \in Z$ is $o(n)$ (this uses that $W_R$ consists of finitely many conjugacy classes).

Lemma 2.6 tells us that for any finite set of conjugacy classes, (3) is satisfied with asymptotic $\mu_n$-probability 1. So we obtain that for every $R, \varepsilon > 0$

$$\nu_n\left( \left\{ [Y,y]; \left\{ \frac{v \in V(Y)}{n} \text{ a lift of } x_0; B_R(v) \simeq B_R(z_0) \right\} > 1 - \varepsilon \right\} \right) \rightarrow 1.$$ \hspace{1cm} \hspace{1cm} (4)

Now, since $V(X)$ is finite we can repeat the argument finitely many times and obtain that for each $R > 0$ there is a finite list $B_{1R}, \ldots, B_{LR}$ of finite simplicial complexes and a finite list of densities $\rho_1, \ldots, \rho_L > 0$ such that

$$\nu_n\left( \left\{ [Y,y]; \forall i : \left| \frac{v \in V(Y)}{n} ; B_{iR}(v) \simeq B_i \right| - \rho_i \right| < \varepsilon \right\} \right) \rightarrow 1.$$ \hspace{1cm} \hspace{1cm} (4)

So, by Theorem 2.7, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if we fix any finite pointed complex $[Q,q] \in \mathcal{K}_D$ that satisfies $\left| \frac{v \in V(Q)}{n} ; B_{iR}(v) \simeq B_i \right| - \rho_i \right| < \delta$ for $i = 1, \ldots, L$, then

Using the fact that $\Gamma / N$ is residually finite, we can find a chain of normal subgroups $H_i < \Gamma / N$ of finite index such that $\cap_i H_i = \{e\}$. We lift this sequence of subgroups to a sequence $\tilde{H}_i < \Gamma$ and obtain a sequence of pointed covers $(Q_i,q_i) \to (X,x_0)$. Now, if we set

$$\eta_i = \frac{1}{[\Gamma : H_i]} \sum_{u \in (\Gamma / \tilde{H}_i) \cdot q_i} \delta_{[Q_i,u]} \in BS(\mathcal{K}_D),$$

then $\eta_i \overset{i \to \infty}{\to} \delta_N$ by construction. So, for (4), we can take a $(Q_i,q_i)$ deep in the sequence we just constructed. Moreover, by Theorem 2.8 we have

$$\frac{b_k(Q_i)}{n} \approx b_k^{(2)}(N \backslash \bar{X}; \Gamma / N),$$

which finishes the proof. \hfill \square
3. A Closed Formula

Our first objective is now to derive a closed formula for \( h_n(\Gamma_{p_1,\ldots,p_m}) \). In this section we will prove:

**Proposition 3.1.** Let \( n, p_1, \ldots, p_m \in \mathbb{N} \). Then

\[
h_n(\Gamma_{p_1,\ldots,p_m}) = n! \sum_{r_1,\ldots,r_m \geq 0} \prod_{1 \leq i \leq n} (r_i! \cdot l_i^r)^{m-1} \prod_{i=1}^m \sum_{k \in K(p_i,l_i)} \prod_{j=1}^{l_i} \frac{1}{(l \cdot j)^k j!}
\]

where

\[
K(p,l,r) = \left\{ k \in \mathbb{N}^p \mid \sum_{i=1}^p k_i \cdot i = r \text{ and } k_i = 0 \text{ whenever } \gcd(i \cdot l, p) \neq i \right\}.
\]

3.1. Counting roots. The main ingredient for the formula above is the count of the number of \( m^{th} \) roots of a given permutation \( \pi \in S_n \) – i.e. the number

\[
N_m(\pi) = \mid \{ \sigma \in S_n; \sigma^m = \pi \} \mid.
\]

Note that this number only depends on the conjugacy class of \( \pi \). The computation of \( N_m(\pi) \) is a classical problem, that to the best of our knowledge has been first worked out by Pavlov [Pav82]. For the sake of completeness, we will give a proof here.

Let us first introduce some notation. Recall that the conjugacy class of a permutation \( \pi \in S_n \) is determined by its cycle type – the unordered partition of \( n \) given by the lengths of the cycles in a disjoint cycle decomposition of \( \pi \). In what follows the notation \( 1^{r_1}2^{r_2} \cdots n^{r_n} \) will denote the partition of \( n \) that has \( r_1 \) parts of size 1, \( r_2 \) parts of size 2, etcetera. \( K(1^{r_1}2^{r_2} \cdots n^{r_n}) \subset S_n \) will denote the corresponding conjugacy class. In this notation, we will often omit the sizes of which there are 0 parts and write \( i \) for \( i^1 \).

**Proposition 3.2** (Pavlov [Pav82]). Let \( m, n \in \mathbb{N} \) and \( \pi \in K(1^{r_1}2^{r_2} \cdots n^{r_n}) \subset S_n \). Then

\[
N_m(\pi) = \prod_{1 \leq i \leq n} \frac{r_i! \cdot l_i^r}{s.t. \ l_i > 0} \sum_{k \in K(m,l,r)} \prod_{i=1}^{l_i} \frac{1}{(l \cdot i)^k j!}
\]

where \( K(m,l,r) \) is as in Proposition 3.1.

Note that there may be an \( l \) such that \( r_l > 0 \) and \( K(m,l,r_l) = \emptyset \). In this case, \( N_m(\pi) = 0 \).

**Proof.** First observe that when \( \sigma \in K(k) \subset S_k \), then

\[
\sigma^m \in K\left( \left( \frac{k}{\gcd(k,m)} \right)^{\gcd(k,m)} \right) \subset S_k,
\]

which also describes what happens to the cycles in a general permutation \( \sigma \in S_n \) upon taking its \( m^{th} \) power.

This puts restrictions on which conjugacy classes \( K \) of \( S_n \) can contain \( m^{th} \) roots of \( \pi \). In order to describe these restrictions, we will split the cycles of the \( m^{th} \) root \( \sigma \) according to which cycles of \( \pi \) they contribute.
So first assume $\pi \in K(l^r) \subset S_{lr}$ - i.e. $\pi$ consists solely of $l$-cycles. If $\sigma \in K(1^{s_1} \cdots (lr)^{s_r})$ satisfies $\sigma^m = \pi$, then the observation above tells us that all cycles of $\sigma$ must have lengths that are multiples of $l$. Moreover,

$$\gcd(i \cdot l, m) \neq i \implies s_{il} = 0$$

In particular, we obtain that $s_{il} = 0$ for all $i > m$. Moreover, we have that $\sum_i s_{il} \cdot il = rl$.

We will now first completely work out the proof for $\pi \in K(l^r)$. The expression for a more general permutation can then be obtained by multiplying the result from this special case over all cycle lengths that appear in the permutation.

So, given a conjugacy class $K(1^{s_1} \cdots (lr)^{s_r})$ that satisfies these conditions, we must count the number of $m^{th}$ roots it contains. That is, for every $i$ such that $\gcd(i \cdot l, m) = i$, we must count how many $i \cdot l$-cycles $C$ we can build out of $i$ cycles of length $l$ from $\pi$ such that $C^m$ consists exactly of these cycles of $\pi$. We claim that the number of such cycles $C$, given $i$ cycles from $\pi$ is

$$l^{i-1}(i-1)!$$

Indeed, write

$$(\alpha_1 \alpha_2 \cdots \alpha_l)(\alpha_{l+1} \cdots \alpha_{2l}) \cdots (\alpha_{(i-1)l+1} \cdots \alpha_{il})$$

for these cycles from $\pi$. $C$ will be of the form

$$C = (\beta_1 \cdots \beta_{il})$$

Then taking some $1 \leq j \leq i \cdot l$, there are $i \cdot l$ choices for the value of $\beta_j$. Given a choice, we also know the value of $\beta_{j+m}, \beta_{j+2m}, \cdots, \beta_{j+lm}$, since supposing $\beta_j = \alpha_k$, we obtain

$$\alpha_{k+1} = \pi(\alpha_k) = C^m(\alpha_k) = C^m(\beta_j) = \beta_{j+m}.$$  

Hence, by assigning a value to one $\beta$, we have assignments for all $\beta$'s. In this way, we have $i \cdot l$ ways to assign the first $l$ values of $C$, $i \cdot l - l$ ways to assign the second $l$ values, and so on, until we have $l$ way to assign the last $l$ values of $C$. This results in

$$(i \cdot l) \cdot l(i-1) \cdots 1 = l^i \cdot i!$$

ways to place the elements of $C$ such that $C^m = \pi$.

However, by rotating the first item in $C$ through the $i \cdot l$ places without changing the order of elements, gives us equivalent cycles within $S_{il}$. There are $i \cdot l$ of these, and so after dividing out by these, the number of possible cycles $C$ such that $C^m = \pi$ is given by

$$\frac{l^i \cdot i!}{i \cdot l} = l^{i-1}(i-1)!,$$

which proves (5).

This implies that $\pi \in K(l^r)$ has

$$\prod_{i=1}^{r} \frac{r!}{i^{s_{il}} s_{il}} s_{il}^{i(i-1)}(i-1)!^{s_{il}} = r! \prod_{i=1}^{r} \frac{1}{i^{s_{il}} s_{il}}$$

$m^{th}$ roots $\sigma \in K(1^{s_1} \cdots (lr)^{s_r})$, where the extra factors account for the number of partitions of the cycles in $\pi$ into $s_{il}$ sets containing $i$ cycles and we used the fact that $\sum_i s_{il} \cdot i = r$ to obtain the second expression.
In order to simplify notation a little we write $k_i = s_{it}$. Summing over all conjugacy classes that contain $m^{th}$ roots of $\pi$, we get that $\pi \in K(l')$ has

$$\sum_{K(m,l,r)} r!l'! \prod_{i=1}^{r} \frac{1}{k_ik_ik_i!}$$

$m^{th}$ roots.

For a general permutation $\pi \in K(1^n \cdots n^n) \subset S_n$, we take the product of this expression over all cycle lengths that appear in $\pi$. \hfill \square

### 3.2. The proof of Proposition 3.1.

**Proof.** Given a conjugacy class $K \subset S_n$, we write $N_m(K)$ for the number of roots of an element $\pi \in K$. We have

$$h_n(\Gamma_{p_1,\ldots,p_m}) = \sum_{K \subset S_n \text{ a conjugacy class}} |K| \cdot N_{p_1}(K) \cdots N_{p_m}(K).$$

Using Proposition 3.2 and the fact that $|K(1^n \cdots n^n)| = n! / \prod_{i=1}^{n} i^{r_{l,i}}!$ gives the formula. \hfill \square

### 4. Asymptotics

The goal of this section is to prove Theorem 1.1 – the asymptotic number of index $n$ subgroups of $\Gamma_{p_1,\ldots,p_m}$ as $n \to \infty$.

First we will determine the asymptotic behaviour of $h_n(\Gamma_{p_1,\ldots,p_m})$. This is done by singling out the dominant term in the expression we found for it in Proposition 3.1. After that, we show that most homomorphisms are transitive, from which the asymptotic number of index $n$ subgroups directly follows (using Proposition 2.1)

#### 4.1. Homomorphisms. We will prove

**Theorem 4.1.** Let $p_1, \ldots, p_m \in \mathbb{N}_{>0}$ such that $\sum_{j=1}^{m} \frac{1}{p_j} < m - 1$. Then

$$h_n(\Gamma_{p_1,\ldots,p_m}) \sim B_{p_1,\ldots,p_m} \cdot \exp \left( \sum_{i=1}^{m} \sum_{0 < j < p_i \text{ s.t. } j/p} \frac{n^{j/p}}{j} \right) \cdot \left( \frac{n}{e} \right)^n \left( m - \sum_{i=1}^{m} \frac{1}{p_i} \right)^{m - 1} \prod_{i=1}^{m} p_i^{-1/2}$$

as $n \to \infty$, where

$$B_{p_1,\ldots,p_m} = \exp \left( - \sum_{i: p_i \text{ even}} \frac{1}{2p_i} \right) \prod_{i=1}^{m} p_i^{-1/2}.$$

Let us write

$$\tau_{p,l,r} = \sum_{k \in K(p,l,r)} \prod_{j=1}^{p} \frac{1}{(1, j)^{k_ik_j!}}$$

so that

$$h_n(\Gamma_{p_1,\ldots,p_m}) = n! \sum_{r_1,\ldots,r_n \geq 0} \prod_{1 \leq i \leq n} (r_i!)^{r_i m - 1} \prod_{i=1}^{m} \tau_{p_i,l,r_i}$$

by Proposition 3.1.
The first thing we shall need is a bound on these numbers \( \tau_{p,l,r} \). To this end, we consider the ordinary generating function for \( \tau_{p,l,r} \) for fixed \( p \) and \( l \), defined by

\[
F_{p,l}(x) = \sum_{r=0}^{\infty} \tau_{p,l,r} x^r.
\]

**Lemma 4.2.** Let \( l, r \in \mathbb{N} \). Then

\[
F_{p,l}(x) = \prod_{i \in I_{p,l}} \exp \left( \frac{x^i}{i \cdot l} \right),
\]

where \( I_{p,l} = \{ i \leq p \mid \gcd(i \cdot l, p) = i \} \).

**Proof.** By definition it holds

\[
F_{p,l}(x) = \sum_{r=0}^{\infty} \left( \sum_{K_{(p,l,r)}} \prod_{i=1}^{m} \frac{1}{(i \cdot l)^{k_i} k_i!} \right) x^r.
\]

Let the set \( I_{p,l} = \{ i \leq p \mid \gcd(i \cdot l, p) = i \} = \{ i_1, \ldots, i_m \} \). Then the above sum becomes

\[
F_{p,l}(x) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \prod_{j=1}^{m} \frac{1}{(i_j \cdot l)^{k_j} k_j!} x^{\sum_{j=1}^{m} k_j i_j} = \prod_{i \in I_{p,l}} \exp \left( \frac{x^i}{i \cdot l} \right).
\]

This, together with Theorem 2.3 implies

**Corollary 4.3.**

(a) Let \( p \in \mathbb{N} \). Then

\[
\tau_{p,1,n} \sim \frac{C_p}{\sqrt{2 \pi n}} \exp \left( \sum_{i \geq 0 \ s.t. i \mid p} \frac{n^{i/p}}{i} \right) \left( \frac{1}{n} \right)^{n/p} \text{ as } n \to \infty,
\]

where

\[
C_p = \begin{cases} 
  p^{-1/2}, & \text{if } p \text{ odd} \\
  p^{-1/2} \exp \left( -\frac{1}{2p} \right), & \text{if } p \text{ even}.
\end{cases}
\]

(b) Let \( p, l, r \in \mathbb{N} \). Then

\[
\tau_{p,l,r} \leq \left( \frac{1}{r \cdot l} \right)^{\frac{r}{p}} \cdot \exp \left( \sum_{i \mid p} \frac{(r \cdot l)^{i/p}}{i \cdot l} \right).
\]

**Proof.** Item (a) is a direct consequence of Theorem 2.3, using that \( \mathcal{K}(p,1,n) \) is non empty when \( p \leq n \) – i.e. that the symmetric group contains elements of order \( p \) whenever \( p \leq n \) – and that \( I_{p,1} \) consists of the divisors of \( p \).
For (b), observe that all the coefficients in \( F_{p,l} \) are non-negative. As such, \( \tau_{p,l,r} \leq F(x_0)/x_0^r \) for all \( x_0 \in (0, \infty) \). Setting \( x_0 = (r \cdot l)^{1/p} \) and using Lemma 4.2, we get

\[
\tau_{p,l,r} \leq \left( \frac{1}{r \cdot l} \right)^{\frac{r}{p}} \exp \left( \sum_{i \in I_{p,l}} \frac{(r \cdot l)^{i/p}}{i \cdot l} \right).
\]

We note that any \( i \) satisfying \( \gcd(i \cdot l, p) = i \) must also satisfy \( i|p \) and hence taking the product over \( i|p \) results in a bound on taking the product over \( i \in I_{r,l} \), which proves item (b). □

Note that our proof for (a) does not work for \( \tau_{p,l,n} \) with \( l \neq 1 \) and \( p \geq 2 \), since it does not hold that \( \tau_{p,l,n} \neq 0 \) for all large \( n \). To see this, let \( n \) be prime. Then the only \( i \in \mathbb{N} \) satisfying \( \gcd(i \cdot l, p) = i \) is when \( i = p \). Hence, the only vectors \( k \in K(p,l,n) \) have to be of the form \( k = (0, \ldots, 0, \frac{n}{p}) \). However, if \( n > p \) is prime then \( \frac{n}{p} \) will never be an integer.

The remainder of the proof of Theorem 4.1 now consists of proving that the term corresponding to

\[
(r_1, r_2, \ldots, r_n) = (n, 0, \ldots, 0)
\]

in (6) dominates the sum when \( n \to \infty \).

We start with the terms in which \( r_1 \) is “small”, this is the longest part of the proof.

**Lemma 4.4.** Let \( p_1, \ldots, p_m \in \mathbb{N}_{>0} \) such that \( m - 1 > \sum_{j=1}^{m} \frac{1}{p_j} \). Then for any \( \delta > (m + 2)/(m - 1 - \sum_{j=1}^{m} \frac{1}{p_j}) \), it holds that

\[
\frac{\sum_{r_1, \ldots, r_n \geq 0 \text{ s.t. } \sum_{i=1}^{m} r_i = n \text{ s.t. } r_i > 0} \prod_{1 \leq i \leq n} (r_i! l_i^{r_i})^{m-1} \prod_{i=1}^{m} \tau_{p_i,l,r_i} (n!)^{m-1} \prod_{i=1}^{m} \tau_{p_i,1,n}}{\prod_{i=1}^{m} \tau_{p_i,1,n}} \to 0
\]

as \( n \to \infty \).

**Proof.** This will follow from Corollary 4.3. Let us write

\[
S(n, \delta) = \sum_{r_1, \ldots, r_n \geq 0 \text{ s.t. } \sum_{i=1}^{m} r_i = n \text{ s.t. } r_i > 0} \prod_{1 \leq i \leq n} (r_i! l_i^{r_i})^{m-1} \prod_{i=1}^{m} \tau_{p_i,l,r_i}.
\]
Corollary 4.3(b) implies that

$$S(n, \delta) \leq \sum_{r_1, \ldots, r_n \geq 0 \atop \text{s.t. } \sum_j r_j \cdot l = n \text{ and } r_1 \leq n - \delta} \prod_{1 \leq i \leq m} (r_i | p_i |)^{m-1} \prod_{i=1}^m (r_i \cdot l - r_i / p_i) \exp \left( \sum_{j \mid p_i} \frac{(r_i \cdot l)^{j/p_i}}{j \cdot l} \right)$$

\leq \sum_{r_1, \ldots, r_n \geq 0 \atop \text{s.t. } \sum_j r_j \cdot l = n \text{ and } r_1 \leq n - \delta} \left\{ (r_1!)^{m-1} \exp \left( \sum_{i=1}^m \sum_{j \mid p_i} \frac{r_i^{j/p_i}}{j} \right) \cdot \prod_{2 \leq i \leq n \atop \text{s.t. } r_i > 0} (r_i \cdot l)^{r_i \left( m - 1 - \sum_i \frac{1}{p_i} \right) \exp \left( \sum_{j \mid p_i} \frac{(r_i \cdot l)^{j/p_i}}{j \cdot l} \right)} \right\}.$$

In the product above, we have $r_i l \leq n$. Using this and the fact that $(r_i l)^{j/p_i}/l \leq r_i$ to bound the exponential factors, we obtain

$$S(n, \delta) \leq \sum_{r_1, \ldots, r_n \geq 0 \atop \text{s.t. } \sum_j r_j \cdot l = n \text{ and } r_1 \leq n - \delta} \left\{ (r_1!)^{m-1} \exp \left( \sum_{i=1}^m \sum_{j \mid p_i} \frac{r_i^{j/p_i}}{j} \right) \right.$$ 

$$\cdot \prod_{2 \leq i \leq n \atop \text{s.t. } r_i > 0} n^{r_i \left( m - 1 - \sum_i \frac{1}{p_i} \right) \exp \left( \sum_{j \mid p_i} \frac{(r_i \cdot l)^{j/p_i}}{j \cdot l} \right)} \right\}.$$

Now we use that $\sum_{i \geq 2} r_i \leq \frac{n-r_1}{2}$ and get

$$S(n, \delta) \leq \sum_{r_1, \ldots, r_n \geq 0 \atop \text{s.t. } \sum_j r_j \cdot l = n \text{ and } r_1 \leq n - \delta} \left\{ (r_1!)^{m-1} \exp \left( \sum_{i=1}^m \sum_{j \mid p_i} \frac{r_i^{j/p_i}}{j} \right) \right.$$ 

$$\cdot n^{\frac{n-r_1}{2} \left( m - 1 - \sum_i \frac{1}{p_i} \right)} \left\{ (r_1!)^{m-1} \exp \left( \sum_{i=1}^m \sum_{j \mid p_i} \frac{r_i^{j/p_i}}{j} \right) \right.$$ 

$$\leq \exp \left( \pi \sqrt{\frac{n-r_1}{3}} \right) \sum_{r_1=0}^{n-\delta} \left\{ (r_1!)^{m-1} \exp \left( \sum_{i=1}^m \sum_{j \mid p_i} \frac{r_i^{j/p_i}}{j} \right) \right.$$ 

$$\cdot n^{\frac{n-r_1}{2} \left( m - 1 - \sum_i \frac{1}{p_i} \right)} \right\}.$$

using the fact that the number of partitions of $n$ is bounded by $\exp(\pi \sqrt{2n/3})$ (see for instance [Apo76, Theorem 14.5]). Using Robbins’s [Rob55] version of Stirling’s approximation, one can write $r_1! \leq C \cdot \sqrt{r_1} (r_1/e)^{r_1}$ for some universal constant $C > 0$, whenever $r_1 > 0$. Moreover, the term corresponding to $r_1 = 0$ in the sum above is smaller than that corresponding to $r_1 = n - \delta$, if we increase the constant $C$ a little.
(depending on \(p_1, \ldots, p_m\)), we may write
\[
S(n, \delta) \leq C \cdot n^{-\frac{m}{2}} \sum_{r_1=1}^{n-\delta} n^{\frac{n-r_1}{2}} \left( (m-1) - \frac{1}{p_1} \right) \exp \left( -(m-1)r_1 + \sum_{i=1}^{m} \sum_{j \geq 0 \text{ s.t. } j/p_i} \frac{r_{i,j}^{j/p_i}}{j} + C \cdot (n - r_1) \right)
\]

On the other hand, Corollary 4.3(a), together with Stirling’s approximation, implies that
\[
(n!)^{m-1} \prod_{i=1}^{m} \tau_{p_i, 1,n} \sim \frac{C_{p_1} \cdots C_{p_m}}{\sqrt{2\pi n}} \exp \left( -(m-1)n + \sum_{i=1}^{m} \sum_{j \geq 0 \text{ s.t. } j/p_i} \frac{n^{j/p_i}}{j} \right) n^{\left( (m-1) - \frac{1}{p_1} \right)}
\]
So, there is a constant \(C > 0\), depending on \(p_1, \ldots, p_m\) only, such that
\[
\frac{S(n, \delta)}{(n!)^{m-1} \prod_{i=1}^{m} \tau_{p_i, 1,n}} \leq C \cdot n^{-\frac{m}{2}} \sum_{r_1=1}^{n-\delta} n^{\frac{n-r_1}{2}} \left( (m-1) - \frac{1}{p_1} \right) \exp \left( \sum_{i=1}^{m} \sum_{j \geq 0 \text{ s.t. } j/p_i} \frac{r_{i,j}^{j/p_i}}{j} \right) + C \cdot (n - r_1)
\]
\[
\leq D \cdot n^{-\frac{m+2}{2}} \sum_{r_1=1}^{n-\delta} \left( \frac{n}{D} \right)^{\frac{n-r_1}{2}} \left( (m-1) - \frac{1}{p_1} \right)
\]
for some \(D > 0\), depending on \(p_1, \ldots, p_m\) only. This tends to 0 as \(n \to \infty\), using our assumption on \(\delta\).

For the remaining terms in the sum, we have:

**Lemma 4.5.** Let \(p_1, \ldots, p_m \in \mathbb{N}_{>0}\) such that \(m - 1 > \sum_{i=1}^{m} \frac{1}{p_i}\). Then for any \(\delta > 0\), it holds that
\[
\sum_{r_1, \ldots, r_m \geq 0 \text{ s.t. } \sum_{i=1}^{m} r_i = n} \prod_{i=1}^{m} \frac{r_i!}{r_i^{r_i}} \prod_{i=1}^{m} \tau_{p_i, 1, r_i} \left( \frac{n!}{(n!)^{m-1}} \prod_{i=1}^{m} \tau_{p_i, 1, n} \right)^{m-1} \rightarrow 0
\]
as \(n \to \infty\).

**Proof.** The crux is that \(2^{r_2} \cdots n^{r_n}\) is a partition of \(n - r_1\), which is a uniformly bounded number in the sum we consider. As such, there exists some constant \(C > 0\), depending on \(p_1, \ldots, p_m\) only such that
\[
\sum_{r_1, \ldots, r_m \geq 0 \text{ s.t. } \sum_{i=1}^{m} r_i = n \text{ s.t. } r_i > 0} \prod_{i=1}^{m} \frac{r_i!}{r_i^{r_i}} \prod_{i=1}^{m} \tau_{p_i, 1, r_i} \leq C \sum_{r_1 = n-\delta}^{n} \left( \frac{r_1!}{n!} \right) \prod_{i=1}^{m} \tau_{p_i, 1, r_i}.
\]
Because this is a finite sum, we may apply Corollary 4.3(a), which implies that
\[
\frac{\tau_{p_i, 1, r_i}}{\tau_{p_i, 1, n}} \leq D \exp \left( \sum_{j \geq 0 \text{ s.t. } j/p_i} \frac{r_{i,j}^{j/p_i} - n^{j/p_i}}{j} \right) \frac{n^{n/p_i}}{r_1^{r_1/p_i}} \leq D' n^{n-r_1/p_i}
\]
for two constants $D, D' > 0$. Filling this in, we see that there exists a constant $C' > 0$ such that
\[
\frac{\sum_{r_1, \ldots, r_n \geq 0 \text{ s.t. } \sum_r |r| = n} \prod_{1 \leq l \leq n} (r_l! l! r_l)^{m-1}}{\prod_{l=1}^m \tau_{p_l, 1, r_l} \text{ s.t. } r_l > 0}
\leq C' \sum_{r_1 = n-\delta}^{n-1} (n-\delta)^{(n-r_1)(m-1)} n^{n-r_1} \frac{1}{r_1!}.
\]

The latter tends 0 as $n \to \infty$, using our assumption that $m - 1 > \sum \frac{1}{p_i}$. \hfill \Box

We are now ready to prove the asymptotic equivalent for $h_n(\Gamma_{p_1, \ldots, p_m})$.

**Proof of Theorem 4.1.** Lemmas 4.4 and 4.5 imply that $h_n(\Gamma_{p_1, \ldots, p_m})$ is asymptotic to the term in (6) corresponding to $(r_1, \ldots, r_n) = (n, 0, \ldots, 0)$. Corollary 4.3(a) together with Stirling’s formula thus imply the theorem. \hfill \Box

Recall that
\[
\Phi_{p_1, \ldots, p_m} : \Gamma_{p_1, \ldots, p_m} \to C_{p_1} \ast \cdots \ast C_{p_m}
\]
is the surjection that sends the generator $x_i \in \Gamma_{p_1, \ldots, p_m}$ to a generator of the $i$th factor on the right.

The lemmas above also prove:

**Theorem 1.2.** $p_1, \ldots, p_m \in \mathbb{N}_0$ such that $\sum_{j=1}^m \frac{1}{p_j} < m - 1$. Then
\[
\left| \frac{1}{h_n(\Gamma_{p_1, \ldots, p_m})} \left\{ \rho \in \text{Hom}(\Gamma_{p_1, \ldots, p_m}, S_n); \exists \rho_0 \in \text{Hom}(C_{p_1} \ast \cdots \ast C_{p_m}, S_n) \text{ s.t. } \rho = \rho_0 \circ \Phi_{p_1, \ldots, p_m} \right\} \right| \to 1
\]
as $n \to \infty$.

**Proof.** This can be done indirectly by comparing Theorem 4.1 to the asymptotic equivalent for $h_n(C_{p_1} \ast \cdots \ast C_{p_m})$ due to Volynets [Vol86] and independently Wilf [Wil86]. The fact that these two sequences are asymptotic to each other implies the result.

It can also be seen directly from Lemmas 4.4 and 4.5. Indeed, they imply that $h_n(\Gamma_{p_1,\ldots, p_m})$ is asymptotic to the term corresponding to $(r_1, r_2, \ldots, r_n) = (n, 0, \ldots, 0)$ in (6). In the proof of this formula, these vectors $(r_1, r_2, \ldots, r_n)$ that are summed over correspond to the conjugacy classes that roots are counted of. The term that determines the asymptotic are the roots of unity in $S_n$, i.e. maps
\[
\varphi : \Gamma_{p_1, \ldots, p_m} = \langle x_1, \ldots, x_m | x_1^{p_1} = \ldots = x_m^{p_m} \rangle \longrightarrow S_n
\]
such that $\varphi(x_i^{p_i})$ is the identity element in $S_n$. These are exactly the maps that factor through $\Phi_{p_1, \ldots, p_m}$. \hfill \Box

4.2. **Subgroups.** We are now ready to prove our main theorem – the asymptotic behaviour of the number of index $n$ subgroups of $\Gamma_{p_1, \ldots, p_m}$. We shall do this by showing that $h_n(\Gamma_{p_1, \ldots, p_m}) \sim t_n(\Gamma_{p_1, \ldots, p_m})$ as $n \to \infty$, that is for large $n$ most of the homomorphisms from $\Gamma_{p_1, \ldots, p_m}$ to $S_n$ are transitive. After that, Proposition 2.1, together with Theorem 4.1 gives the asymptote.
Theorem 1.1. Let \( p_1, \ldots, p_m \in \mathbb{N}_{>0} \) such that \( \sum_{j=1}^{m} \frac{1}{p_j} < m - 1 \). Then it holds that

\[
a_n(\Gamma_{p_1, \ldots, p_m}) \sim A_{p_1, \ldots, p_m} \cdot n^{-1/2} \cdot \exp \left( \sum_{i=1}^{m} \sum_{10 < j < p_i \text{ s.t. } j \mid p_i} \frac{n^j/p_i}{j} \right) \cdot \left( \frac{n}{e} \right)^n \cdot (m-1-\sum_{i=1}^{m} \frac{1}{p_i})
\]
as \( n \to \infty \), where

\[
A_{p_1, \ldots, p_m} = \sqrt{2\pi} \exp \left( - \sum_{i: p_i \text{ even}} \frac{1}{2p_i} \right) \prod_{i=1}^{m} p_i^{-1/2}.
\]

Proof. The quickest way to prove that most homomorphisms are transitive, is to use the fact that asymptotically almost all homomorphisms \( \Gamma_{p_1, \ldots, p_m} \) factor through the homomorphism

\[
\Phi : \Gamma_{p_1, \ldots, p_m} \to C_{p_1} \ast \cdots \ast C_{p_m}.
\]

Müller (Theorem 2.4) proved that asymptotically almost all homomorphisms \( C_{p_1} \ast \cdots \ast C_{p_m} \to S_n \) are transitive, which, together with Proposition 2.1 and Stirling’s approximation, gives the result.

For a more direct proof (that essentially goes along the same lines as that of Müller), we can use that the number of transitive homomorphisms \( G \to S_n \) can be recursively computed from the sequence \( (h_n(G))_n \). That is, we have (for a proof see [LS03, Lemma 1.1.3]):

\[
\frac{h_n(\Gamma_{p_1, \ldots, p_m}) - t_n(\Gamma_{p_1, \ldots, p_m})}{n-1} = \sum_{k=1}^{n-1} \binom{n-1}{k-1} t_k(\Gamma_{p_1, \ldots, p_m}) \cdot h_{n-k}(\Gamma_{p_1, \ldots, p_m}).
\]

Combining this with the bounds from Theorem 4.1, a further computation and Proposition 2.1 also gives the result. \( \square \)

5. Random subgroups and covers

In this section we will study the properties of random index \( n \) subgroups of \( \Gamma_{p_1, \ldots, p_m} \) and random degree \( n \) covers of torus knot complements.

The basic idea is to prove that a random index \( n \) subgroup of \( C_{p_1} \ast \cdots \ast C_{p_m} \) (as an element of \( \text{IRS}(C_{p_1} \ast \cdots \ast C_{p_m}) \)) converges to the trivial subgroup. This, together with Theorem 1.2 will then imply that a random index \( n \) subgroup of \( \Gamma_{p_1, \ldots, p_m} \) converges to \( L_{p_1, \ldots, p_m} \). Both of these results will be quantitative in the sense that we have control over the number of conjugacy classes a given conjugacy class of either \( C_{p_1} \ast \cdots \ast C_{p_m} \) or \( \Gamma_{p_1, \ldots, p_m} \) lifts to in a random index \( n \) subgroup of the corresponding subgroup (Theorem 1.3(a)). This then immediately implies the fact that a random degree \( n \) cover of \( X_{p_1, \ldots, p_m} \) Benjamini–Schramm converges to \( X_{\Phi_{p_1, \ldots, p_m}} \). Combined with Lemma 2.9, this convergence implies our result on Betti numbers.

5.1. Set-up. Given a group \( \Gamma \) and \( n \in \mathbb{N} \), we will write

\[
A_n(\Gamma) = \{ H < \Gamma; [\Gamma : H] = n \},
\]

so that \( a_n(\Gamma) = |A_n(\Gamma)| \). Moreover, if \( K \subset \Gamma \) is a conjugacy class then we will write

\[
Z_K : A_n(\Gamma) \to \mathbb{N}
\]
for the random variable that measures the number of conjugacy classes that $K$ splits into, i.e.

$$Z_K(H) = |(K \cap H)/H|$$

where $H$ acts on $K \cap H$ by conjugation. Note that if we fix any $g \in K$ and $\varphi : \Gamma \to S_n$ is a transitive homomorphism corresponding to $H$ (cf. Proposition 2.1), then

$$Z_K(H) = |\{j \in \{1, \ldots, n\}; \varphi(g) \cdot j = j\}|.$$

Our goal now is to show that these random variables are asymptotically Poisson-distributed.

5.2. Poisson statistics for random elements of $\text{Hom}(C_{p_1} \ast \cdots \ast C_{p_m}, S_n)$. Our first step is to enlarge our probability space and prove our results there. Concretely, the expression for $Z_K$ in terms of fixed points is well-defined for any homomorphism, not just for transitive ones. As such, we can interpret $Z_K$ as a random variable $Z_K : \text{Hom}(C_{p_1} \ast \cdots \ast C_{p_m}, S_n) \to \mathbb{N}$ as well, where we equip $\text{Hom}(C_{p_1} \ast \cdots \ast C_{p_m}, S_n)$ with the uniform measure $\mathbb{P}^{\text{Hom}}_n$. We will denote the expected value with respect to this measure by $E^{\text{Hom}}_n$.

We have:

**Theorem 5.1.** Let $p_1, \ldots, p_m \in \mathbb{N}$ and let $K_1, \ldots, K_r \subset C_{p_1} \ast \cdots \ast C_{p_m}$ be distinct conjugacy classes. Then for any $k_1, \ldots, k_r \in \mathbb{N}$ we have

$$\lim_{n \to \infty} E^{\text{Hom}}_n \left[ (Z_{K_1})^{k_1} \cdots (Z_{K_r})^{k_r} \right] = 1.$$

Before we prove this theorem, we observe that this immediately implies that on $\text{Hom}(C_{p_1} \ast \cdots \ast C_{p_m}, S_n)$, the random variables $Z_K$ are asymptotically Poisson-distributed and independent.

**Corollary 5.2.** Let $p_1, \ldots, p_m \in \mathbb{N}$ and let $K_1, \ldots, K_r \subset C_{p_1} \ast \cdots \ast C_{p_m}$ be distinct conjugacy classes. Then, as $n \to \infty$, the vector of random variables $(Z_{K_1}, \ldots, Z_{K_r}) : \text{Hom}(C_{p_1} \ast \cdots \ast C_{p_m}, S_n) \to \mathbb{N}^r$ converges jointly in distribution to a vector

$$(Z_{K_1}^\infty, \ldots, Z_{K_r}^\infty) : \Omega \to \mathbb{N}^r$$

of independent Poisson$(1)$-distributed variables.

**Proof.** This is direct from Theorem 5.1 together with Theorem 2.5. \qed

**Proof of Theorem 5.1.** We will write $\Lambda = C_{p_1} \ast \cdots \ast C_{p_m}$ and $H_n(\Lambda) = \text{Hom}(\Lambda, S_n)$. Let us once and for all fix $g_i \in K_i$ for $i = 1, \ldots, r$ and write these elements as words in the generators $x_1, \ldots, x_m$, i.e. we write

$$g_i = x_{i_1}^{s_{i_1}} \cdots x_{i_{l_i}}^{s_{i_{l_i}}}, \quad i = 1, \ldots, r,$$

where $x_{i_{l_i}}$ and $x_{i_{l_i+1}}$ are distinct for all $t = 1, \ldots, l_i - 1$. By potentially changing the conjugate, we may also assume that $x_{i_{l_i}} \neq x_{i_{l_i+1}}$. Moreover, we will choose the unique
and satisfies $s_i t < \frac{p_i}{j_i}$ for all $t = 1, \ldots, l_i$. We will write $|g_i|$ for the word length of $g_i$. So

$$|g_i| = \sum_{j=1}^{l_i} s_{ij}.\]

Now, if we want $v \in \{1, \ldots, n\}$ to be a fixed point of $\varphi(g_i)$ for some $\varphi \in \mathcal{H}_n(\Lambda)$, then there need to be sequences $(w_{t,0} w_{t,1} \ldots w_{t,s_{ij}})$, for $t = 1, \ldots, l_i$, such that

$$\begin{cases} \varphi(x_{ij})(w_{t,q}) = w_{t,q-1}, & q = 1, \ldots, s_{ij}, \\ w_{1,1} = w_{l_i,s_{ij}} = v. \end{cases}$$

(7)

In other words, if we want $v$ to be a fixed point of $g_i$, then certain sequences (for which there are many choices) need to appear in the disjoint cycle decompositions of the images of the generators $x_1, \ldots, x_m$. Figure 1 gives an example of the situation. We will call such a sequence of sequences corresponding to $v$ being a fixed point for $g_i$ a $g_i$-cycle based at $v$. The sequences $(w_{t,0} w_{t,1} \ldots w_{t,s_{ij}})$ appearing in the cycle will be called the words in the cycle. The elements from $\{1, \ldots, n\}$ appearing in the words will be called the labels in them. If $\varphi$ satisfies (7) for a given $g_i$-cycle $\omega$, we will say that $\varphi$ satisfies $\omega$.

Observe that the random variable $(Z_{K_1})_{k_1} \cdots (Z_{K_r})_{k_r} : \mathcal{H}_n(\Gamma) \to \mathbb{N}$ counts $r$-tuples $(F_1, F_2, \ldots, F_r)$ where $F_i$ is a sequence of $k_i$ distinct fixed points of $\varphi(g_i)$. As such, we may write

$$\mathbb{E}_n^{\text{Hom}} [(Z_{K_1})_{k_1} \cdots (Z_{K_r})_{k_r}] = \sum_{\alpha \in \mathcal{A}(n)} \mathbb{E}_n [\mathbb{I}_\alpha],$$

where

$$A(n) = \left\{ \alpha = (\alpha_1, \ldots, \alpha_r); \quad \alpha_i \text{ a } k_i \text{-tuple of } g_i \text{-cycles based at different elements of } \{1, \ldots, n\} \right\}$$

and

$$\mathbb{I}_\alpha : \mathcal{H}_n(\Lambda) \to \{0,1\}$$

satisfies $\mathbb{I}_\alpha(\varphi) = 1$ if and only if $\varphi$ satisfies all the $g_i$-cycles contained in $\alpha$ for all $i = 1, \ldots, r$. Note that many of these indicators are constant 0 functions, because the combination of labels involved leads to a contradiction about the properties of $\varphi(x_j)$ for some $j \in \{1, \ldots, m\}$.

We will write

$$A(n) = A_1(n) \sqcup A_2(n)$$

FIGURE 1. $v$ is a fixed point of $\varphi(x_2 x_1 x_2^2)$.
where

\[ A_1(n) = \{ \alpha \in A; \text{ every label appears at most once in } \alpha \} \]

and

\[ A_2(n) = A(n) \setminus A_1(n). \]

The remainder of the proof now consists of proving two facts, namely

\[
\lim_{n \to \infty} \sum_{a \in A_1(n)} \mathbb{E}_n^{\text{Hom}}[1_a] = 1 \quad \text{and} \quad \lim_{n \to \infty} \sum_{a \in A_2(n)} \mathbb{E}_n^{\text{Hom}}[1_a] = 0.
\]

We start with estimating \( \mathbb{E}_n^{\text{Hom}}[1_a] \) for \( \alpha \in A_1 \). Observe that

\[
\mathbb{E}_n^{\text{Hom}}[1_a] = \left\{ \varphi \in \mathcal{H}_n(\Lambda); \varphi \text{ satisfies all the } g_i\text{-cycles contained in } \alpha \text{ for all } i = 1, \ldots, r \right\} / h_n(\Lambda).
\]

In order to count the numerator on the right hand side, we need to count the number of ways to complete the information given in \( \alpha \) to a homomorphism \( \Lambda \to S_n \). We do this as follows.

The words from the \( g_i \)-cycle must appear as parts of cycles in a disjoint cycle decomposition of the \( x_j \)'s. So, a choice needs to be made for the lengths of these cycles, which words appear together in a cycle, and which other labels appear in these cycles. Once these cycles have been completed, this determines \( m \) homomorphisms \( C_{p_j} \to S_{D_i} \), where \( D_j \) depends on the chosen cycle lengths. To complete this into a homomorphism \( C_{p_j} \to S_n \), we have the choice out of \( h_{n-D_j}(C_{p_j}, S_n) \) homomorphisms. This, as \( n \to \infty \), gives a total of

\[
\sim \prod_{j=1}^m \sum_{\{S_1, \ldots, S_t\} \models W_j(\alpha)} \sum_{d_1, \ldots, d_t | p_j} \sum_{d_1 \geq \sum \in S_q} C(S, d) \cdot n^{\sum d_i - \sum \in W_j(\alpha) \ell(w)} h_{n - \sum d_q}(C_{p_j})
\]

ways to complete the information in \( \alpha \) to a homomorphism, where

- \( W_j(\alpha) \) is the set of words that appear in \( \alpha \) and pose a condition on \( \varphi(x_j) \),
- the notation \( \{S_1, \ldots, S_t\} \models W_j(\alpha) \) means that \( \{S_1, \ldots, S_t\} \) forms a set partition of \( W_j(\alpha) \) (these are the groups of words that are going to appear together in cycles in \( \varphi(x_j) \)),
- the numbers \( d_1, \ldots, d_t \) are going to be the lengths of the cycles containing the words in the sets \( \{S_1, \ldots, S_t\} \),
- \( \ell(w) \) is the number of labels in a word \( w \),
- \( C(S, d) \) is a combinatorial constant that counts the number of ways to distribute the words over cycles in according to \( \{S_1, \ldots, S_t\} \) and \( d_1, \ldots, d_t \). Moreover, if the set partition \( \{S_1, \ldots, S_t\} \) consists of singletons and \( d_1 = d_2 = \ldots = d_t = p_j \) then \( C(S, d) = 1 \)
- and we have already made one simplification: the powers of \( n \) should in reality take the form of a falling factorial. However, since we are only interested in asymptotics and all the products involved are of fixed bounded length, we replaced them by powers of \( n \), whence the “\( \sim \)”.
Now we notice that all the sums and products involved are finite, we may apply Theorem 2.2 to single out the largest term. This implies that, as \( n \to \infty \),

\[
\mathbb{E}_n^{\text{Hom}}[I_{a}] \sim \prod_{j=1}^{m} n \sum_{w \in W_j(a)} p_j - \ell(w) \frac{h_n - |W_j(a)| \cdot p_j(C_p)}{h_n(C_p)}
\]

\[
\sim \prod_{j=1}^{m} n \sum_{w \in W_j(a)} p_j - \ell(w) \ \cdot \frac{n}{n} - |W_j(a)| \cdot p_j \cdot \left(1 - \frac{1}{n}\right)
\]

\[
= \prod_{j=1}^{m} n \sum_{w \in W_j(a)} 1 - \ell(w)
\]

\[
= \prod_{i=r}^{n} n^{-|g_i| \cdot k_i}
\]

Another important thing to observe is that \( \mathbb{E}_n^{\text{Hom}}[I_{a}] \) is constant on \( A_1(n) \): it does not depend on the labels involved. This implies that

\[
\sum_{a \in A_1(n)} \mathbb{E}_n^{\text{Hom}}[I_{a}] \sim |A_1(n)| \cdot \prod_{i=r}^{n} n^{-|g_i| \cdot k_i}
\]

as \( n \to \infty \). Moreover,

\[
|A_1(n)| = n \cdot \left(n - 1\right) \cdots \left(n - \left(\sum_{i=1}^{r} |g_i| \cdot k_i\right) + 1\right),
\]

it is the number of ways to label the \( g_i \)-cycles with distinct elements from \( \{1, \ldots, n\} \). Together with (8), this proves our claim that

\[
\lim_{n \to \infty} \sum_{a \in A_1(n)} \mathbb{E}_n^{\text{Hom}}[I_{a}] = 1.
\]

In order to prove that the other term tends to zero, we argue in a similar fashion. Indeed, we will think of the \( g_i \)-cycles as labelled graphs: the vertices are the labels and the edges are determined by the conditions in (7). In this language the graphs in \( A_1(n) \) are exactly those that consist of disjoint circuits. The graphs in \( A_2(n) \) come in finitely many isomorphism types and all have more edges than vertices.

We write

\[
\sum_{a \in A_2(n)} \mathbb{E}_n^{\text{Hom}}[I_{a}] = \sum_{G} \sum_{a \in A_G(n)} \mathbb{E}_n^{\text{Hom}}[I_{a}]
\]

where the sum is over isomorphism types, types \( G \) of graphs appearing in \( A_2(n) \) and \( A_G(n) \) consists of all \( a \in A_2(n) \) whose graph has isomorphism type \( G \).

Suppose \( G \) is such an isomorphism type with \( v(G) \) vertices and \( e(G) \) edges. Again \( \mathbb{E}_n^{\text{Hom}}[I_{a}] \) is the same for all \( a \in A_G(n) \). Moreover, with exactly the same arguments as above we have

\[
\mathbb{E}_n^{\text{Hom}}[I_{a}] \sim n^{-e(G)} \text{ as } n \to \infty \forall a \in A_G(n) \quad \text{and} \quad |A_G(n)| \leq n^{v(G)}.
\]

Because \( v(G) < e(G) \) for all \( G \) appearing in the sum, the sum indeed tends to zero, which finishes the proof.

\[\square\]
5.3. Poisson statistics for random subgroups of $C_{p_1} \cdots C_{p_m}$ and Benjamini–Schramm convergence. From the above we also obtain that $Z_K$ are asymptotically independent Poisson-distributed variables when seen as random variables on the set of index $n$ subgroups of $C_{p_1} \cdots C_{p_m}$.

**Theorem 5.3.** Let $p_1, \ldots, p_m \in \mathbb{N}$ such that $\sum_{i=1}^m \frac{1}{p_i} < m - 1$ and let $K_1, \ldots, K_r \subset C_{p_1} \cdots C_{p_m}$ be distinct conjugacy classes. Then, as $n \to \infty$, the vector of random variables

$$(Z_{K_1}, \ldots, Z_{K_r}) : \mathcal{A}_n(C_{p_1} \cdots C_{p_m}) \to \mathbb{N}^r$$

converges jointly in distribution to a vector of

$$(Z_{K_1}^\infty, \ldots, Z_{K_r}^\infty) : \Omega \to \mathbb{N}^r$$

of independent Poisson(1)-distributed variables.

**Proof.** We will again write $\Lambda = C_{p_1} \cdots C_{p_m}$. Using the $(n - 1)!$-to-1 correspondence between transitive permutation representations $\Gamma \to S_n$ and index $n$ subgroups of $\Gamma$ (i.e. Proposition 2.1), what we need to prove is that for all $A \subset \mathbb{N}^r$,

$$\frac{|\{ \varphi \in \text{Hom}(\Lambda, S_n); (Z_{K_1}, \ldots, Z_{K_r})(\varphi) \in A \}|}{t_n(\Lambda)} \xrightarrow{n \to \infty} \mathbb{P}[(Z_{K_1}^\infty, \ldots, Z_{K_r}^\infty) \in A] .$$

We have

$$\frac{|\{ \varphi \in \text{Hom}(\Lambda, S_n); (Z_{K_1}, \ldots, Z_{K_r})(\varphi) \in A \}|}{t_n(\Lambda)} \leq \frac{|\{ \varphi \in \text{Hom}(\Lambda, S_n); (Z_{K_1}, \ldots, Z_{K_r})(\varphi) \in A \}|}{h_n(\Lambda)} \cdot \frac{h_n(\Lambda)}{t_n(\Lambda)} \xrightarrow{n \to \infty} \mathbb{P}[(Z_{K_1}^\infty, \ldots, Z_{K_r}^\infty) \in A] ,$$

by Corollary 5.2 and Theorem 2.4 (note that this uses that $\sum_{i=1}^m \frac{1}{p_i} < m - 1$).

Likewise,

$$\frac{|\{ \varphi \in \text{Hom}(\Lambda, S_n); (Z_{K_1}, \ldots, Z_{K_r})(\varphi) \in A \}|}{t_n(\Lambda)} \geq \frac{|\{ \varphi \in \text{Hom}(\Lambda, S_n); (Z_{K_1}, \ldots, Z_{K_r})(\varphi) \in A \}|}{h_n(\Lambda)} \cdot \frac{h_n(\Lambda) - t_n(\Lambda)}{t_n(\Lambda)} \xrightarrow{n \to \infty} \mathbb{P}[(Z_{K_1}^\infty, \ldots, Z_{K_r}^\infty) \in A] ,$$

again by Corollary 5.2 and Theorem 2.4, which proves the result. \qed

Our next goal is to use this to prove convergence of a random index $n$ subgroup of $C_{p_1} \cdots C_{p_m}$:

**Corollary 5.4.** Let $p_1, \ldots, p_m$ be such that $\sum_{i=1}^m \frac{1}{p_i} < m - 1$. Then the IRS

$$\mu_n = \frac{1}{a_n(C_{p_1} \cdots C_{p_m})} \sum_{H < C_{p_1} \cdots C_{p_m}} \delta_H$$

for $|C_{p_1} \cdots C_{p_m}:H| = n$. 

[Note: The text appears to be cut off at the end of the paragraph, but it continues as follows:]
 Lemma 2.6. □

The distribution of the random index

Proof. Let us write

\[ \mu_n(Z_K) \leq \frac{h_n(C_{p_1} \ast \cdots \ast C_{p_m})}{\ell_n(C_{p_1} \ast \cdots \ast C_{p_m})} \cdot \mathbb{E}^{\text{Hom}[Z_K]} = o(n) \]

as \( n \to \infty \), by Theorem 2.4 combined with Theorem 5.1. So the corollary follows from Lemma 2.6. □

5.4. Statistics for \( \Gamma_{p_1, \ldots, p_m} \). Now we are ready to prove our results on the properties of random index \( n \) subgroups of \( \Gamma_{p_1, \ldots, p_m} \). Let us start with the statistics of the variables \( Z_K : A_n(\Gamma_{p_1, \ldots, p_m}) \to \mathbb{N} \). Given a sequence of random variables \( X_n, Y_n : \Omega \to \mathbb{N} \), we will say \( X_n \) and \( Y_n \) are asymptotically independent as \( n \to \infty \) if

\[ \lim_{n \to \infty} \mathbb{P}(X_n \in A \text{ and } Y_n \in B) - \mathbb{P}(X_n \in A) \cdot \mathbb{P}(Y_n \in B) = 0 \quad \forall A, B \subset \mathbb{N}. \]

**Theorem 1.3(a).** Let \( p_1, \ldots, p_m \in \mathbb{N}_{>1} \) such that \( \sum_{i=1}^{m} \frac{1}{p_i} < m - 1 \). Moreover, let \( K_1, \ldots, K_r \subset \Gamma_{p_1, \ldots, p_m} \) be distinct non-trivial conjugacy classes. Then, as \( n \to \infty \), the random variables \( Z_{K_i} : A_n(\Gamma_{p_1, \ldots, p_m}) \to \mathbb{N}, i = 1, \ldots, r \) are asymptotically independent. Moreover,

- if \( K_i \subset L_{p_1, \ldots, p_m} \) then
  \[ \lim_{n \to \infty} \mathbb{P}[Z_{K_i}(H_n) = n] = 1 \]
- and if \( K_i \not\subset L_{p_1, \ldots, p_m} \) then \( Z_{K_i}(H_n) \) converges in distribution to a \( \text{Poisson}(1) \)-distributed random variable.

**Proof.** Let us write \( \Gamma = \Gamma_{p_1, \ldots, p_m} \) and

\[ \mathcal{T}_n(\Gamma) = \{ \varphi \in \text{Hom}(\Gamma, S_n); \varphi(\Gamma) \sim \{1, \ldots, n\} \text{ transitively} \}. \]

The distribution of \( Z_{K_i} \) is the same on \( \mathcal{T}_n(\Gamma) \) as it is on \( A_n(\Gamma) \). By Theorem 1.2, as \( n \to \infty \) a typical element of \( \mathcal{T}_n(\Gamma) \) factors through \( \Phi_{p_1, \ldots, p_m} \). So the limiting distribution of the \( Z_{K_i} \) is the same as that on

\[ \mathcal{T}_n(\Gamma)^\Phi := \{ \varphi \in \mathcal{T}_n(\Gamma); \varphi \text{ factors through } \Phi_{p_1, \ldots, p_m} \}. \]

Now if \( K_i \subset L_{p_1, \ldots, p_m} = \ker(\Phi_{p_1, \ldots, p_m}) \) then \( Z_{K_i} \) is constant and equal to \( n \) on \( \mathcal{T}_n(\Gamma)^\Phi \). If \( K_i \not\subset L_{p_1, \ldots, p_m} \) then the limiting distribution of \( Z_{K_i} \) on \( \mathcal{T}_n(\Gamma)^\Phi \) is given by Theorem 5.3. Finally, Theorem 5.3 gives us the asymptotic independence among the \( Z_{K_i} \) for \( K_i \not\subset L_{p_1, \ldots, p_m} \) and the independence of the whole set follows from the fact that constant random variables are independent of any other random variable. □

Next, we determine the limit of a random index \( n \) subgroup of \( \Gamma_{p_1, \ldots, p_m} \) as an IRS:

**Theorem 1.3(b).** Let \( p_1, \ldots, p_m \in \mathbb{N}_{>1} \) be such that \( \sum_{i=1}^{m} \frac{1}{p_i} < m - 1 \). Then the IRS

\[ \mu_n = \frac{1}{a_n(\Gamma_{p_1, \ldots, p_m})} \sum_{H < \Gamma_{p_1, \ldots, p_m}} \delta_H \xrightarrow{w^*} \delta_{\Gamma_{p_1, \ldots, p_m}} \]

as \( n \to \infty \).
Proof. Write
\[ A_n(\Gamma_{p_1, \ldots, p_m}) = A_{n,1} \cup A_{n,2}, \]
where
\[ A_{n,1} = \Phi_{p_1, \ldots, p_m}^{-1}(A_n(C_{p_1} \ast \cdots \ast C_{p_m})) \quad \text{and} \quad A_n(\Gamma_{p_1, \ldots, p_m}) \setminus A_{n,1} \]
If \( f : \text{Sub}(\Gamma_{p_1, \ldots, p_m}) \to \mathbb{R} \) is a continuous function then
\[ \mu_n(f) = \frac{1}{a_n(\Gamma_{p_1, \ldots, p_m})} \sum_{H \in A_n(C_{p_1} \ast \cdots \ast C_{p_m})} f(\Phi_{p_1, \ldots, p_m}^{-1}(H)) + \frac{1}{a_n(\Gamma_{p_1, \ldots, p_m})} \sum_{H \in A_{n,2}} f(H). \]
Since \( f \) is bounded and \( \mu_n(A_{n,2}) \to 0 \) (by Theorem 1.2), the second term tends to 0 as \( n \to \infty \). The first term tends to \( f(\ker(\Phi_{p_1, \ldots, p_m})) \) by Corollary 5.4, which proves the theorem. \( \square \)

Finally, we will determine the limits of the normalised Betti numbers. We have:

**Theorem 1.3(c).** Let \( p_1, \ldots, p_m \in \mathbb{N}_{>1} \) be such that \( \sum_{i=1}^{m} \frac{1}{p_i} < m - 1 \). For every \( \varepsilon > 0 \) it holds that
\[ \lim_{n \to \infty} \mu_n \left( \frac{b_k(H; \mathbb{R})}{n} - b_k^{(2)}(X_{p_1 \ldots, p_m}^\Phi; C_{p_1} \ast \cdots \ast C_{p_m}) \right) < \varepsilon \]
for all \( k \in \mathbb{N} \).

Moreover,
\[ b_k^{(2)}(X_{p_1 \ldots, p_m}^\Phi; C_{p_1} \ast \cdots \ast C_{p_m}) = \begin{cases} m - 1 - \sum_{i=1}^{m} \frac{1}{p_i} & \text{if } k = 1, 2 \\ 0 & \text{otherwise} \end{cases} \]

It follows from the fact that \( \mu_n \) converges to \( \delta_{L_{p_1, \ldots, p_m}} \) together with Lemma 2.9 that the normalised Betti numbers of a random index \( n \) subgroup converge to those of the cover corresponding to \( L_{p_1, \ldots, p_m} \). So the only thing that we still have to prove is that the latter vanish, which is the content of the following lemma.

Recall that \( \Gamma / L_{p_1, \ldots, p_m} \simeq C_{p_1} \ast \cdots \ast C_{p_m} \). We have:

**Lemma 5.5.** Let \( X_{p_1 \ldots, p_m} \) be a classifying space for \( \Gamma_{p_1, \ldots, p_m} \) and let \( X_{p_1 \ldots, p_m}^\Phi \to X_{p_1 \ldots, p_m} \) denote the cover corresponding to \( L_{p_1, \ldots, p_m} \triangleleft \Gamma_{p_1, \ldots, p_m} \). Then
\[ b_k^{(2)}(X_{p_1 \ldots, p_m}^\Phi; C_{p_1} \ast \cdots \ast C_{p_m}) = \begin{cases} m - 1 - \sum_{i=1}^{m} \frac{1}{p_i} & \text{if } k = 1, 2 \\ 0 & \text{otherwise} \end{cases} \]

Proof. Since the \( \ell^2 \)-Betti numbers do not depend on the choice of classifying space, we identify \( \Gamma_{p_1, \ldots, p_m} \) with a lattice in \( \text{PSL}(2, \mathbb{R}) \times \mathbb{R} \) and set \( X_{p_1, \ldots, p_m} = \Gamma_{p_1, \ldots, p_m} \setminus \left( \mathbb{H}^2 \times \mathbb{R} \right) \).

This gives us an identification
\[ X_{p_1, \ldots, p_m} = \mathbb{H}^2 \times \mathbb{R}/\mathbb{Z}. \]

The action of \( C_{p_1} \ast \cdots \ast C_{p_m} \) on \( X_{p_1, \ldots, p_m} \) preserves the factors. The action on \( \mathbb{R}/\mathbb{Z} \) is through the quotient \( C_{p_1} \ast \cdots \ast C_{p_m} \to C_{p_1} \times \cdots \times C_{p_m} \). The kernel \( \Lambda \) of this quotient is a free group that acts trivially on \( \mathbb{R}/\mathbb{Z} \).

Because \( X_{p_1, \ldots, p_m} \) is three-dimensional and \( \Lambda \) is infinite,
\[ b_k^{(2)}(X_{p_1, \ldots, p_m}^\Phi; \Lambda) = 0 \quad \text{for } k \in \{0, 4, 5, \ldots\} \]
(see for instance [Lüc02, Theorem 1.35(8)] or [Kam19, Theorem 3.18(ii)]). Moreover, since
\[ b_k^2(H^2; \Lambda) = \begin{cases} -\chi(\Lambda \setminus H^2) & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \]
(see for instance [Lüc02, Example 3.16] or [Kam19, Exercise 3.3.1]) where \( \chi \) denotes Euler characteristic. So, the Künneth formula (see for instance [Lüc02, Theorem 1.35(4)] or [Kam19, Theorem 3.18(iii)]) gives us that
\[ b_1^2(X_{p_1 \cdots p_m}; \Lambda) = b_2^2(X_{p_1 \cdots p_m}; \Lambda) = -\chi(\Lambda \setminus H^2) \quad \text{and} \quad b_3^2(X_{p_1 \cdots p_m}; \Lambda) = 0. \]
Since both orbifold Euler characteristic and \( \ell^2 \)-Betti numbers are multiplicative with respect to finite index subgroups (see for instance [Lüc02, Theorem 1.35(9)] or [Kam19, Theorem 3.18(iv)] for the latter), the lemma follows.

\[ \square \]

Proof of Theorem 1.3(c). This is now direct from Theorem 1.3(b) and Lemmas 5.5 and 2.9.

\[ \square \]

5.5. Random index \( n \) subgroups of Fuchsian groups. In this last section we discuss applications of our results to random subgroups of Fuchsian groups. We have:

Theorem 1.4. Let \( \Lambda \) be a non-cocompact Fuchsian group of finite covolume. Moreover, set
\[ \mu_n = \sum_{H < \Lambda, [\Lambda:H]=n} \delta_H. \]

(a) Let \( K_1, \ldots, K_r \subset \Lambda \) be distinct non-trivial conjugacy classes. Then, as \( n \to \infty \), the vector of random variables
\[ \left( Z_{K_1}, \ldots, Z_{K_r} \right) : A_n(\Lambda) \to \mathbb{N}^r \]
converges in distribution to a vector of independent Poisson(1)-distributed random variables.

(b) \( \mu_n \longrightarrow \delta_{\{e\}} \) as an IRS.

Proof sketch. First of all note that non-cocompact Fuchsian group of finite covolume are exactly groups of the form \( F_r \ast C_{p_1} \ast \cdots \ast C_{p_m} \), with \(-r + m - 1 - \sum_{i=1}^{m} 1/p_i < 0\), where \( F_r \) denotes the free groups on \( r \) generators.

If \( r = 0 \), (a) and (b) are the content of Theorem 5.3 and Corollary 5.4 respectively. If \( r > 0 \), the proof of Theorem 5.3 needs to be adapted slightly: \( r \) of the generators are now allowed to have any permutation of their image and not just permutations of a fixed order. With exactly the same strategy (and slightly easier computations, which we leave to the reader) the analogue of Theorem 5.3 can now be proved (if \( m = 0 \), much better bounds are in fact available [DJPP13]). In order to prove the analogue of Corollary 5.4, the only new ingredient that is needed is that \( t_n(\Gamma)/h_n(\Gamma) \to 1. \)

When \( m = 0 \), this is a direct consequence of Dixon’s theorem [Dix69]. For the remaining cases, the proof has not been written down, but a similar strategy does the
trick. Indeed, the results by by Volynets–Wilf (Theorem 2.2) together with Stirling’s approximation that for \( p > 1, \)

\[
h_n(F_r \ast C_{p_1} \ast \cdots \ast C_{p_m}) \sim B \cdot n^{r/2} \cdot \exp \left( \sum_{i=1}^{m} \left( \sum_{d|p_i, d < p_i} \frac{1}{d} \right) \right) \cdot \left( \frac{n}{e} \right)^{n + \sum_{i=1}^{m} 1 - \frac{1}{p_i}} ,
\]
as \( n \to \infty, \) where \( B \) is a constant depending on \(( r, p_1, \ldots, p_m ).\) We have

\[
1 - \frac{t_n(\Lambda)}{h_n(\Lambda)} = \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{t_k(\Lambda)h_{n-k}(\Lambda)}{h_n(\Lambda)}
\]
(see for instance [LS03, Lemma 1.1.3]). Combining the two, we get that there exists a constant \( A > 0 \) such that

\[
1 - \frac{t_n(\Lambda)}{h_n(\Lambda)} \leq A \sum_{k=1}^{n-1} \binom{n}{k}^{1-r-m} \sum_{i=1}^{m} \frac{1}{p_i} \exp \left( \sum_{i=1}^{m} \left( \sum_{d|p_i, d < p_i} \left( n - k \right)^{d/p_i} + k^{d/p_i} - n^{d/p_i} \right) \right) \to 0,
\]
as \( n \to \infty, \) which settles the remaining cases. □

REFERENCES

[ABB+17] Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. On the growth of \( L^2 \)-invariants for sequences of lattices in Lie groups. Ann. of Math. (2), 185(3):711–790, 2017.

[ABBG18] Miklós Abért, Nicolas Bergeron, Ian Biringer, and Tsachik Gelander. Convergence of normalized betti numbers in nonpositive curvature. Preprint, arXiv: 1811.02520, 2018.

[AGG89] R. Arratia, L. Goldstein, and L. Gordon. Two moments suffice for Poisson approximations: the Chen-Stein method. Ann. Probab., 17(1):9–25, 1989.

[Ago13] I. Agol. The virtual Haken conjecture. Doc. Math., 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.

[AGV14] Miklós Abért, Yair Glasner, and Bálint Virág. Kesten’s theorem for invariant random subgroups. Duke Math. J., 163(3):465–488, 2014.

[AL02] Alon Amit and Nathan Linial. Random graph coverings. I. General theory and graph connectivity. Combinatorica, 22(1):1–18, 2002.

[Apo76] Tom M. Apostol. Introduction to analytic number theory. Springer-Verlag, New York-Heidelberg, 1976. Undergraduate Texts in Mathematics.

[BBG+18] Hyungryul Baik, David Bauer, Ilya Gekhtman, Ursula Hamenstädt, Sebastian Hensel, Thorben Kastenholz, Bram Petri, and Daniel Valenzuela. Exponential torsion growth for random 3-manifolds. Int. Math. Res. Not. IMRN, (21):6497–6534, 2018.

[BCP19] Thomas Budzinski, Nicolas Curien, and Bram Petri. Universality for random surfaces in unconstrained genus. Electron. J. Combin., 26(4):Paper No. 4.2, 34, 2019.

[BHJ92] A. D. Barbour, Lars Holst, and Svante Janson. Poisson approximation, volume 2 of Oxford Studies in Probability. The Clarendon Press, Oxford University Press, New York, 1992. Oxford Science Publications.

[BKL+20] Sebastian Baader, Alexandra Kjuchukova, Lukas Lewark, Filip Misev, and Arunima Ray. Average four-genus of two-bridge knots. Proc. Amer. Math. Soc., to appear, 2020+.

[BM04] Robert Brooks and Eran Makover. Random construction of Riemann surfaces. J. Differential Geom., 68(1):121–157, 2004.

[Bol80] Béla Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. European J. Combin., 1(4):311–316, 1980.

[Bol85] Béla Bollobás. Random graphs. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1985.
[Bow14] Lewis Bowen. Random walks on random coset spaces with applications to Furstenberg entropy. *Invent. Math.*, 196(2):485–510, 2014.

[BPR20] H. Baik, B. Petri, and J. Raimbault. Subgroup growth of right-angled Artin and Coxeter groups. *J. London Math. Soc.* (2), to appear, 2020+.

[BS01] Itai Benjamini and Oded Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13, 2001.

[CHM51] S. Chowla, I. N. Herstein, and W. K. Moore. On recursions connected with symmetric groups. *I. Canad. J. Math.*, 3:328–334, 1951.

[Dix69] J. D. Dixon. The probability of generating the symmetric group. *Math. Z.*, 110:199–205, 1969.

[DJPP13] Ioana Dumitriu, Tobias Johnson, Soumik Pal, and Elliot Paquette. Functional limit theorems for random regular graphs. *Probab. Theory Related Fields*, 156(3-4):921–975, 2013.

[dSMS99] M. P. F. du Sautoy, J. J. McDermott, and G. C. Smith. Zeta functions of crystallographic groups and analytic continuation. *Proc. London Math. Soc.* (3), 79(3):511–534, 1999.

[DT06] Nathan M. Dunfield and William P. Thurston. Finite covers of random 3-manifolds. *Invent. Math.*, 166(3):457–521, 2006.

[Eck04] Beno Eckmann. Lattices, $l_2$-Betti numbers, deficiency, and knot groups. *Enseign. Math. (2)*, 50(1-2):123–137, 2004.

[Ele10] Gábor Elek. Betti numbers are testable. In *Fete of combinatorics and computer science*, volume 20 of *Bolyai Soc. Math. Stud.* pages 139–149. János Bolyai Math. Soc., Budapest, 2010.

[EZ17] Chaim Even-Zohar. Models of random knots. *J. Appl. Comput. Topol.*, 1(2):263–296, 2017.

[FPP+20] S. Friedl, J. Park, B. Petri, J. Raimbault, and A. Ray. On distinct finite covers of 3-manifolds. *Indiana Univ. Math. J.*, to appear, 2020+.

[Fri08] Joel Friedman. A proof of Alon’s second eigenvalue conjecture and related problems. *Mem. Amer. Math. Soc.*, 195(910):viii+100, 2008.

[Gel18] T Gelander. A lecture on invariant random subgroups. In *New Directions in Locally Compact Groups*, pages 186–204. Cambridge Univ. Press, 2018.

[GJKW02] Catherine Greenhill, Svante Janson, Jeong Han Kim, and Nicholas C. Wormald. Permutation pseudographs and contiguity. *Combin. Probab. Comput.*, 11(3):273–298, 2002.

[GLMST19] Clifford Gilmore, Etienne Le Masson, Tuomas Sahlsten, and Joe Thomas. Short geodesic loops and $L^p$ norms of eigenfunctions on large genus random surfaces. Preprint, arXiv: 1912.09961, 2019.

[GPY11] Larry Guth, Hugo Parlier, and Robert Young. Pants decompositions of random surfaces. *Geom. Funct. Anal.*, 21(5):1069–1090, 2011.

[Hat07] Allen Hatcher. Notes on basic 3-manifold topology. Lecture notes, available at: http://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html, 2007.

[Hay56] W. K. Hayman. A generalisation of Stirling’s formula. *J. Reine Angew. Math.*, 196:67–95, 1956.

[Hem87] J. Hempel. Residual finiteness for 3-manifolds. In *Combinatorial group theory and topology (Alta, Utah, 1984)*, volume 111 of *Ann. of Math. Stud.*, pages 379–396. Princeton Univ. Press, Princeton, NJ, 1987.

[HS68] Bernard Harris and Lowell Schoenfeld. Asymptotic expansions for the coefficients of analytic functions. *Illinois J. Math.*, 12:264–277, 1968.

[HV19] Ursula Hamenstädt and Gabriele Viaggi. Small eigenvalues of random 3-manifolds. Preprint, arXiv: 1903.08031, 2019.

[Kam19] Holger Kammeyer. *Introduction to $\ell^2$-invariants*, volume 2247 of *Lecture Notes in Mathematics*. Springer, Cham, 2019.

[Kel20] Andrew James Kelley. Subgroup growth of all Baumslag-Solitar groups. *New York J. Math.*, 26:218–229, 2020.

[LM00] Valery Liskovets and Alexander Mednykh. Enumeration of subgroups in the fundamental groups of orientable circle bundles over surfaces. *Comm. Algebra*, 28(4):1717–1738, 2000.

[LS03] A. Lubotzky and D. Segal. *Subgroup growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003.
[LS04] M. W. Liebeck and A. Shalev. Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks. *J. Algebra*, 276(2):552–601, 2004.

[Lüc94] W. Lück. Approximating $L^2$-invariants by their finite-dimensional analogues. *Geom. Funct. Anal.*, 4(4):455–481, 1994.

[Lüc02] Wolfgang Lück. *L₂-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

[Mah10] Joseph Maher. Random Heegaard splittings. *J. Topol.*, 3(4):997–1025, 2010.

[Mir13] Maryam Mirzakhani. Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus. *J. Differential Geom.*, 94(2):267–300, 2013.

[MNP20] Michael Magee, Frédéric Naud, and Doron Puder. A random cover of a compact hyperbolic surface has relative spectral gap $\frac{3}{16} - \epsilon$. Preprint, arXiv:2003.10911, 2020.

[MP02] T. W. Müller and J. C. Puchta. Character theory of symmetric groups and subgroup growth of surface groups. *J. London Math. Soc. (2)*, 66(3):623–640, 2002.

[MP19] Maryam Mirzakhani and Bram Petri. Lengths of closed geodesics on random surfaces of large genus. *Comment. Math. Helv.*, 94(4):869–889, 2019.

[MP20] Michael Magee and Doron Puder. The asymptotic statistics of random covering surfaces. Preprint, arXiv: 2003.05892, 2020.

[MRR18] Howard Masur, Kasra Rafi, and Anja Randecker. Expected covering radius of a translation surface. Preprint, arXiv: 1809.10769, 2018.

[Mül96] Thomas Müller. Subgroup growth of free products. *Invent. Math.*, 126(1):111–131, 1996.

[Mül97] Thomas Müller. Finite group actions and asymptotic expansion of $e^{P(z)}$. *Combinatorica*, 17(4):523–554, 1997.

[MW55] Leo Moser and Max Wyman. On solutions of $x^d = 1$ in symmetric groups. *Canadian J. Math.*, 7:159–168, 1955.

[Pav82] A. I. Pavlov. On the limit distribution of the number of solutions of the equation $x^k = a$ in the symmetric group $S_n$. *Mat. Sb. (N.S.)*, 117(159)(2):239–250, 288, 1982.

[Pet17] Bram Petri. Random regular graphs and the systole of a random surface. *J. Topol.*, 10(1):211–267, 2017.

[PT18] Bram Petri and Christoph Thäle. Poisson approximation of the length spectrum of random surfaces. *Indiana Univ. Math. J.*, 67(3):1115–1141, 2018.

[Pud15] Doron Puder. Expansion of random graphs: new proofs, new results. *Invent. Math.*, 201(3):845–908, 2015.

[Rob55] Herbert Robbins. A remark on Stirling’s formula. *Amer. Math. Monthly*, 62:26–29, 1955.

[Shr20] Sunrose Shrestha. The topology and geometry of random square-tiled surfaces. Preprint, arXiv: 2005.00099, 2020.

[Sul16] Diego Sulca. Zeta functions of virtually nilpotent groups. *Israel J. Math.*, 213(1):371–398, 2016.

[Ver12] A. M. Vershik. Totally nonfree actions and the infinite symmetric group. *Mosc. Math. J.*, 12(1):193–212, 216, 2012.

[Vol86] L. M. Volynets. The number of solutions of the equation $x^k = e$ in a symmetric group. *Mat. Zametki*, 40(2):155–160, 286, 1986.

[Wil86] Herbert S. Wilf. The asymptotics of $e^{P(z)}$ and the number of elements of each order in $S_n$. *Bull. Amer. Math. Soc. (N.S.)*, 15(2):228–232, 1986.

[Wor99] N. C. Wormald. Models of random regular graphs. In *Surveys in combinatorics, 1999 (Canterbury)*, volume 267 of *London Math. Soc. Lecture Note Ser.*, pages 239–298. Cambridge Univ. Press, Cambridge, 1999.

Tübingen, Germany, elizabeth.baker@gmx.net

Institut de Mathématiques de Jussieu-Paris Rive Gauche, Sorbonne Université, Paris, France, bram.petri@imj-prg.fr