The Wess-Zumino Model and the AdS$_4$/CFT$_3$ Correspondence

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Abstract

We consider the non-interacting massive Wess-Zumino model in four-dimensional anti-de Sitter space and show that the conformal dimensions of the corresponding boundary fields satisfy the relations expected from superconformal invariance. In some cases the irregular mode must be used for one of the scalar fields.

1 Introduction

The AdS/CFT correspondence, originally conjectured by Maldacena [1] formulates a duality between a field theory on anti-de Sitter space (AdS) and a conformal field theory (CFT) on its boundary. The most noted example is the duality between AdS type IIB string theory and $\mathcal{N} = 4$ super Yang-Mills theory [2]. The precise form of the AdS/CFT correspondence [3, 4] in classical approximation reads

$$\exp (-I_{\text{AdS}}[\phi]) = \langle \exp \left( \int d^d x \phi_0(x) O(x) \right) \rangle .$$

On the AdS side, the function $\phi_0$ represents the boundary value of the field $\phi$, whereas on the CFT side it couples as a current to the conformal field operator $O$. There is a characteristic relation between the mass of the AdS field $\phi$ and the conformal dimension of the CFT field $O$ [5]. This has been investigated for scalar [4, 6, 7], spinor [8, 9], vector [7, 10], graviton [11, 12, 13] and Rarita-Schwinger fields [14, 15, 16]. Supersymmetry and supergravity in the AdS/CFT context have been considered in [17, 18, 19, 20, 21, 22, 23, 24, 25].

For AdS supersymmetric field theories, supersymmetry relates the masses of fields in the same multiplet with each other. Hence, the AdS/CFT correspondence predicts that the conformal dimensions of the corresponding boundary CFT operators satisfy specific relations. On the other hand, superconformal symmetry imposes a condition on the conformal

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dimensions of the primary fields of a superconformal multiplet. One would expect that the AdS prediction coincides with the CFT condition, which would mean that the AdS/CFT correspondence couples an AdS super multiplet to a superconformal multiplet on the AdS boundary. However, to the best of our knowledge, no direct comparison has yet been made.

In this paper, we shall tackle this problem by looking at the non-interacting massive Wess-Zumino model in AdS\(_4\), finding the relations between the conformal dimensions of the corresponding scalar and spinor boundary operators and comparing them with the relations expected from superconformal invariance. We shall find in agreement with the classic AdS papers [26, 27] that in some cases one must use the irregular mode for one of the AdS scalar fields in order for the AdS/CFT correspondence to hold true. This modifies the standard prescription \([4]\), which uses only the regular modes.

Let us start with some preliminaries and use them to explain our notation. For simplicity, we shall consider AdS\(_4\) with Euclidean signature. As is well known [28], it can be constructed as a hyperboloid embedded into a five-dimensional Minkowski space with a metric tensor \(\eta_{AB}\) (\(A, B = -1, 0, 1, 2, 3\)), where

\[
\eta_{-1-1} = -1, \quad \eta_{\mu \nu} = \delta_{\mu \nu}, \quad \text{and} \quad \eta_{-1 \mu} = 0 \tag{1}
\]

\((\mu, \nu = 0, 1, 2, 3)\). Then, AdS\(_4\) can be defined by the embedding

\[
y^A y^B \eta_{AB} = -1, \quad y^{-1} > 0, \tag{2}
\]

where the “radius” of the hyperboloid has been chosen equal to 1 for simplicity. The metric

\[
ds^2 = dy^A dy^B \eta_{AB} \tag{3}
\]

represents the AdS metric, if one takes the \(y^\mu\) as AdS\(_4\) coordinates and defines \(y^{-1}\) via equation (2).

While the representation (3) proves useful for finding the AdS symmetries, a change of variables will reveal the conformal symmetry of the AdS boundary. Introducing the variables \(x^\mu\) by

\[
x^0 = \frac{1}{y^0 + y^{-1}}, \quad x^i = x^0 y^i \quad (i = 1, 2, 3), \tag{4}
\]

yields a representation of AdS\(_4\) as the upper half space \(0 < x^0 < \infty, x^i \in \mathbb{R}\) with the metric

\[
ds^2 = \frac{\delta_{\mu \nu}}{(x^0)^2} dx^\mu dx^\nu. \tag{5}
\]

The use of the Minkowski five-space suggests the introduction of \(4 \times 4\) gamma matrices \(\hat{\gamma}_A\) satisfying \(\{\hat{\gamma}_A, \hat{\gamma}_B\} = 2 \eta_{AB}\). The spin matrices of the corresponding Lorentz algebra in five dimensions are \(S_{AB} = \frac{i}{4}[\hat{\gamma}_A, \hat{\gamma}_B]\). The gamma matrices of the four-dimensional Euclidean Lorentz frame of AdS\(_4\) are given by

\[
\gamma_a = \hat{\gamma}_a \hat{\gamma}_{-1} \tag{6}
\]

satisfying \(\{\gamma_a, \gamma_b\} = 2 \delta_{ab}, (a, b = 0, 1, 2, 3)\). The corresponding spin matrices are \(S_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]\). Covariant gamma matrices are defined by \(\Gamma_\mu = e^a_\mu \gamma_a\) and covariant spin matrices by \(\Sigma_{\mu \nu} = e^a_\mu e^b_\nu S_{ab}\).
Finally, let us give a short outline of the rest of the paper. The AdS$_4$ symmetry algebra and its $\mathcal{N} = 1$ grading will be derived in section 2. In section 3 we recast these algebras in the form of conformal and superconformal algebras and recall the relations between the conformal weights of the primary fields in a superconformal multiplet. The AdS$_4$ superspace is constructed in section 4. In section 5 we consider the Wess-Zumino model in AdS$_4$ and calculate the conformal dimensions of the corresponding boundary fields. We refer our readers to the appendices A and B for information on Grassmann variables and the calculation of Killing spinors, respectively.

2 Symmetry Algebra and its $\mathcal{N} = 1$ Grading

The AdS Symmetries are easiest found considering the embedding (2). In fact, equation (2) is invariant under Lorentz transformations of the Minkowski five-space, which are of the form $(y')^A = R^A_B y^B$, where the matrix $R$ satisfies $R^T \eta R = \eta$ and $R_{-1} > 0$. For the purposes of this paper we consider only the connected subgroup of such matrices, namely the Lie group $SO(4,1)$. An infinitesimal transformation, $R = 1 + M$, has the form

$$
\delta y^A = M_B^A y^B = \frac{1}{2} \omega^{CD} (M_{CD})_B^A y^B, \quad (7)
$$

where the generators $(M_{CD})_B^A = \delta^A_C \eta_{BD} - \delta^A_D \eta_{BC}$ form the standard basis of the $so(4,1)$ algebra and satisfy the commutation relations

$$
[M_{AB}, M_{CD}] = \eta_{AD} M_{BC} + \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC}. \quad (8)
$$

One can show from the equations (4) and (7) that the infinitesimal change of the coordinates $x^\mu$ is given by

$$
\delta x^0 = -x^0 (\lambda + 2a_i x^i), \quad \delta x^i = -x^i (\lambda + 2a_j x^j) + a^i x^2 - b^i + \omega^{ij} x^j, \quad (9)
$$

where $x^2 = x^\mu x^\nu \delta_{\mu\nu}$ and where the parameters $a^i$, $b^i$ and $\lambda$ are defined by

$$
a^i = \frac{1}{2} (\omega^{0i} + \omega^{-1i}), \quad b^i = \frac{1}{2} (\omega^{0i} - \omega^{-1i}) \quad \text{and} \quad \lambda = \omega^{-10}, \quad (10)
$$

respectively. Obviously, the transformations (9) reduce to infinitesimal conformal transformations on the boundary, $x^0 = 0$.

It is straightforward to find the (complex) $\mathcal{N} = 1$ grading of the $so(4,1)$ algebra (8). First, introduce the fermionic generators $Q^\alpha$ ($\alpha = 1, 2, 3, 4$), which transform as $so(4,1)$ spinors, i.e.

$$
[M_{AB}, Q^\alpha] = -(S_{AB})_\beta^\alpha Q^\beta. \quad (11)
$$

Then, the superalgebra closes with the anti-commutator (see appendix A for notation)

$$
\{Q^\alpha, Q^\beta\} = -2(\hat{S}^{AB} \hat{C}^{-1})^{\alpha\beta} M_{AB}. \quad (12)
$$
We would like to add two remarks at this point. First, the validity of equation (12) is conditional upon the fact that we grade the five-dimensional Minkowski algebra. For higher dimensions (e.g. AdS$_5$) one would have to introduce additional bosonic operators to obtain closure of all Jacobi identities [29]. Second, the equations (8), (11) and (12) define the complex superalgebra $B(0/2)$, whose real form is $osp(1,4)$ [30]. Unfortunately, $osp(1,4)$ does not contain $so(4,1)$ in its even part, which means in other words that no Majorana spinors exist for our Minkowski five-space. However, $osp(1,4)$ contains $so(3,2)$, which is the symmetry group of AdS$_4$ with Minkowski signature. Resorting to a Wick rotation at the end to make our results valid, we shall ignore this fact and formally carry out the analysis.

### 3 Conformal and Superconformal Algebra

As mentioned in section 2, the AdS symmetry group acts as the conformal group on the AdS boundary. In this section, we shall for completeness explicitly show the isomorphisms between the $d = 3$ conformal algebra and $so(4,1)$ as well as between their $\mathcal{N} = 1$ superalgebras. Let us introduce the conformal basis of $so(4,1)$ by defining

\[
D = M_{-10}, \quad K_i = M_{0i} + M_{-1i}, \\
L_{ij} = M_{ij}, \quad P_i = M_{0i} - M_{-1i}.
\]

Then, an element $M \in so(4,1)$ takes the form

\[
M = \frac{1}{2} \omega^{AB} M_{AB} = \lambda D + a^i K_i + b^i P_i + \frac{1}{2} \omega^{ij} L_{ij},
\]

with the parameters $a_i$, $b_i$ and $\lambda$ given by equation (10). One easily finds from equation (13) the commutation relations of $D$, $P_i$, $K_i$ and $L_{ij}$, which are given by

\[
\begin{align*}
[D, P_i] & = -P_i, \\
[D, K_i] & = K_i, \\
[L_{ij}, P_k] & = \delta_{jk} P_i - \delta_{ik} P_j, \\
[L_{ij}, K_k] & = \delta_{jk} K_i - \delta_{ik} K_j, \\
[P_i, K_j] & = 2(\delta_{ij} D - L_{ij}), \\
[L_{ij}, L_{kl}] & = \delta_{il} L_{jk} + \delta_{jk} L_{il} - \delta_{ik} L_{jl} - \delta_{jl} L_{ik}, \\
[P_i, P_j] & = [K_i, K_j] = [D, L_{ij}] = 0.
\end{align*}
\]

Equations (15) are the standard representation of the conformal algebra [31].

The $\mathcal{N} = 1$ grading of the conformal algebra (15) is well known in the literature [29], but again, a direct comparison with the superalgebra given in section 2 seems useful. This is done by choosing a particular representation of the five-dimensional Clifford algebra of matrices $\hat{\gamma}_A$. Choosing

\[
\hat{\gamma}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, \quad \hat{\gamma}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\gamma}_{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

(16)
where $\sigma_i$ are the Pauli spin matrices and $\mathbf{1}$ is the $2 \times 2$ unit matrix, one easily finds from the definition (13) the spinor representations of the conformal basis elements, which are

$$
\hat{S} (L_{ij}) = \frac{1}{2} \begin{pmatrix} \sigma_{ij} & 0 \\ 0 & \sigma_{ij} \end{pmatrix}, \quad \hat{S} (D) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

$$
\hat{S} (P_i) = \begin{pmatrix} 0 & 0 \\ \sigma_i & 0 \end{pmatrix}, \quad \hat{S} (K_i) = \begin{pmatrix} 0 & -\sigma_i \\ 0 & 0 \end{pmatrix}.
$$

Splitting the spinor operator $Q^\alpha$ into two 2-component spinors,

$$Q^\alpha = \begin{pmatrix} q \\ s \end{pmatrix},$$

we find from equation (11) the commutators

$$[L_{ij}, q^\alpha] = -\frac{1}{2} (\sigma_i q)^\alpha, \quad [L_{ij}, s^\alpha] = -\frac{1}{2} (\sigma_i s)^\alpha,$$

$$[D, q^\alpha] = -\frac{1}{2} q^\alpha, \quad [D, s^\alpha] = \frac{1}{2} s^\alpha,$$

$$[P_i, q^\alpha] = 0, \quad [P_i, s^\alpha] = -\sigma_i q^\alpha,$$

$$[K_i, q^\alpha] = (\sigma_i s)^\alpha, \quad [K_i, s^\alpha] = 0.$$

Furthermore, the charge conjugation matrix $\hat{C}$ has the form

$$\hat{C} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix},$$

where $c$ is the charge conjugation matrix in three dimensions. Hence, using the identity

$$\hat{S}^{AB} M_{AB} = -2 \hat{S} (D) D + \hat{S} (K_i) P_i + \hat{S} (P_i) K_i + \hat{S} (L_{ij}) L_{ij},$$

equations (12) and (17) yield the anticommutators

$$\{ q^\alpha, q^\beta \} = 2 (\sigma_i c^{-1})^{\alpha\beta} P_i,$$

$$\{ s^\alpha, s^\beta \} = -2 (\sigma_i c^{-1})^{\alpha\beta} K_i,$$

$$\{ q^\alpha, s^\beta \} = (2c^{-1} D - \sigma_i L_{ij})^{\alpha\beta}.$$

Equations (15), (18) and (20) form the $N = 1$ superconformal algebra in three dimensions [29].

Obviously, the operators $L_{ij}, P_i$ and $q^\alpha$ form the three-dimensional $N = 1$ Poincaré superalgebra. Let us therefore consider a scalar super-Poincaré multiplet consisting of the scalar fields $\mathcal{O}$ and $\mathcal{F}$ and the spinor field $\chi$, which satisfy the supersymmetry relation

$$q^\alpha \mathcal{O} = \chi^\alpha,$$

$$q^\alpha \chi^\beta = (\sigma_i c^{-1})^{\alpha\beta} \partial_i \mathcal{O} + (c^{-1})^{\alpha\beta} \mathcal{F},$$

$$q^\alpha \mathcal{F} = - (\sigma_i \partial_i \chi)^\alpha.$$

5
Imposing conformal symmetry on the multiplet means that the scaling dimensions of the fields must satisfy

\[ \Delta_O = \Delta_\chi + \frac{1}{2} = \Delta_F + 1. \]  

This relation is obtained by acting with the commutator \([D, q]\) on the fields \(O\) and \(\chi\). Notice that the spinor operator \(s\) is expressed in terms of \(K_i\) and \(q\) and thus does not introduce new fields into the multiplet.

### 4 Construction of AdS Superspace

In order to obtain the \((N = 1)\) supersymmetric extension of AdS\(_4\), one first introduces Grassmann coordinates \(\hat{\theta}^\alpha\) in addition to the AdS coordinates \(x^\mu\). Then one postulates that the symmetry algebra of the superspace is given by the graded Lie algebra constructed in section 2. Because the symmetry algebra determines infinitesimal coordinate transformations, one can determine the latter from the knowledge of the algebra. The method to be used has been described by Zumino \([32]\) and applies to any group \(G\) with a subgroup \(H\).

In the case at hand, an element \(g \in G\) is uniquely represented by

\[ g = e^{\hat{\xi} Q} h(x), \]  

where \(\hat{\xi}\) is some Grassmann coordinate spinor, whose relation with \(\hat{\theta}\) will be defined later and \(h(x) \in H = SO(4, 1)\) is a Lie-algebra valued function of the coordinates \(x^\mu\). Then, by virtue of the group axioms, one can write

\[ g_0 g = g' = e^{\hat{\xi}' Q} h(x'), \]  

and consider the transformations \(\hat{\xi} \rightarrow \hat{\xi}'\) and \(x \rightarrow x'\) as induced by the group element \(g_0\). We shall in the following use the abbreviations \(\Theta = \hat{\xi} Q\) and \(M = \frac{1}{2} \omega^{AB} M_{AB}\). In the case of \(g_0 \in H\), i.e. an even transformation, equation (24) takes the form \(e^M e^{\Theta} h(x) = e^{\Theta'} e^{M'} h(x)\), where \(M'\) and \(\Theta'\) are determined by the Baker-Campbell-Hausdorff formula. For infinitesimal \(M\) one finds \(M' = M\) and \(\Theta' = \Theta + [M, \Theta]\). By definition, the even part \(e^M h(x) = h(x')\) yields equation (9) and thus does not contain new information, whereas the odd part yields a linear transformation of the Grassmann coordinates, namely

\[ \delta \hat{\xi}^\alpha = \frac{1}{2} \omega^{AB} (\hat{S}_{AB})^\alpha_{\beta} \hat{\xi}^\beta. \]  

On the other hand, for \(g_0 = e^R\) with the abbreviation \(R = \hat{\xi} Q\), one writes

\[ e^{R} e^{\Theta} h(x) = e^{\Theta'} e^{M} h(x). \]  

Then, for infinitesimal \(R\) one finds \([32]\)

\[ \delta \Theta = \left(1 + \frac{1}{3} \Theta^2 - \frac{1}{45} \Theta^4\right) \land R. \]
\[ M = \left( -\frac{\Theta}{2} + \frac{\Theta^3}{24} \right) \wedge R, \]  
(28)

where the notation
\[ 1 \wedge Y = Y, \quad X \wedge Y = [X,Y], \quad X^2 \wedge Y = [X,[X,Y]], \quad \text{etc.} \]

has been used. Equations (27) and (28) are evaluated explicitly using the anti-commutator (12) and various Fierz identities listed in appendix A, leading to

\[ \delta \hat{\xi}^\alpha = \hat{\varepsilon}^\alpha \left[ 1 - \frac{5}{6} \hat{\xi} \hat{\xi} - \frac{1}{9}(\hat{\xi} \hat{\xi})^2 \right] - \frac{1}{6}(\hat{\gamma}^A \hat{\varepsilon})^\alpha(\hat{\xi} \hat{\gamma}_A \hat{\xi}) \]  
(29)

and

\[ M = - \left( 1 + \frac{1}{6} \hat{\xi} \hat{\xi} \right) (\hat{\xi} \hat{S}^{AB} \hat{\varepsilon}) M_{AB}, \]  
(30)

respectively. The transformation formula (29) can be simplified by defining

\[ \hat{\theta} = \hat{\xi} \left( 1 - \frac{1}{3} \hat{\xi} \hat{\xi} \right). \]  
(31)

While \( \hat{\theta} \) still is an \( SO(4,1) \) spinor, i.e. it transforms under even transformations as

\[ \delta \hat{\theta}^\alpha = \frac{1}{2} \omega^{AB}(\hat{S}_{AB})^\alpha_\beta \hat{\theta}^\beta, \]  
(32)

the odd transformation laws, equations (29) and (30), become

\[ \delta \hat{\theta}^\alpha = \varepsilon^\alpha \left[ 1 - \hat{\theta} \hat{\theta} - \frac{1}{2}(\hat{\theta} \hat{\theta})^2 \right], \]  
(33)

and

\[ M = - \left( 1 + \frac{1}{2} \hat{\theta} \hat{\theta} \right) (\hat{\theta} \hat{S}^{AB} \hat{\varepsilon}) M_{AB}, \]  
(34)

respectively. Using equations (14), (19) and (6) one finds

\[ \lambda = - \left( 1 + \frac{1}{2} \hat{\theta} \hat{\theta} \right) \left( \hat{\theta} \gamma_0 \hat{\varepsilon} \right), \]
\[ a^i = -\frac{1}{2} \left( 1 + \frac{1}{2} \hat{\theta} \hat{\theta} \right) \left[ \hat{\theta} \gamma^i (1 - \gamma_0) \hat{\varepsilon} \right], \]
\[ b^i = \frac{1}{2} \left( 1 + \frac{1}{2} \hat{\theta} \hat{\theta} \right) \left[ \hat{\theta} \gamma^i (1 + \gamma_0) \hat{\varepsilon} \right], \]
\[ \omega^{ij} = -2 \left( 1 + \frac{1}{2} \hat{\theta} \hat{\theta} \right) (\hat{\theta} \hat{S}^{ij} \hat{\varepsilon}). \]  
(35)
Thus, the supersymmetry transformation $\delta x^\mu$ is given by equation (33) using the parameters of equation (35). Although this solves the problem of finding the superspace transformations, a space-time covariant formulation would be much more desirable. Such a formulation involves the Killing spinors, which are calculated in appendix B. In fact, it is easy to show from equation (B.6) that the quantity $\hat{\eta} \Gamma^\mu \Lambda^{-1} \hat{\varepsilon}$ is a Killing vector. On the other hand, because also $\delta x^\mu$ is a Killing vector and is linear in $\hat{\varepsilon}$, it must have exactly this form with $\hat{\eta}$ being a function of $\hat{\theta}$ only. A direct comparison using equations (9), (35), (B.3), (B.4) and (B.6) shows that

$$\delta x^\mu = \left( 1 + \frac{1}{2} \hat{\theta} \hat{\theta} \right) \hat{\theta} \Gamma^\mu \Lambda^{-1} \hat{\varepsilon}. \quad (36)$$

Equations (33) and (36) represent the supersymmetry transformation of the AdS superspace in a space-time covariant form. It is with this form that one can hope to effectively carry out the calculations involving superfields. Moreover, it will allow our formal results to be carried over to the Lorentzian signature case, where Majorana spinors exist. Notice that $\delta x^\mu$ in equation (36) is generically complex for Euclidean signature, because no Majorana spinors exist in this case.

Let us conclude this section by finding the invariant integral measure for integration over the AdS$_4$ superspace. First, we observe that the bosonic part $d^4x \sqrt{g(x)} = d^4x (x^0)^{-4}$ is in itself invariant under any variable transformation, i.e. also under the supersymmetry transformation (33). For the fermionic part of the integral measure, let us make the ansatz $d^4\hat{\theta} \rho(\hat{\theta})$ and demand that it be invariant under the transformation $\hat{\theta} \rightarrow \hat{\theta}' = \hat{\theta} + \delta \hat{\theta}$, where $\delta \hat{\theta}$ is given by equation (33). From equation (33) follows that

$$d^4\hat{\theta}' = d^4\hat{\theta} \left[ 1 - \frac{1}{2} \left( 1 + \hat{\theta} \hat{\theta} \right) \hat{\theta} \right]. \quad (37)$$

Multiplying equation (37) with $\rho(\hat{\theta}')$ and expanding to terms linear in $\hat{\varepsilon}$ we find the equation

$$\left[ 1 - \hat{\theta} \hat{\theta} - \frac{1}{2} (\hat{\theta} \hat{\theta})^2 \right] \frac{\partial}{\partial \hat{\theta}_a} \rho = 2 \hat{\theta}^a (1 + \hat{\theta} \hat{\theta}) \rho,$$

whose solution up to a multiplicative constant is

$$\rho(\hat{\theta}) = 1 + \hat{\theta} \hat{\theta} + \frac{3}{2} (\hat{\theta} \hat{\theta})^2. \quad (38)$$

It is straightforward to show that $d^4\hat{\theta} \rho(\hat{\theta})$ is also invariant under the bosonic transformation (32). Hence, the expression

$$d^4x \sqrt{g(x)} d^4\hat{\theta} \left[ 1 + \hat{\theta} \hat{\theta} + \frac{3}{2} (\hat{\theta} \hat{\theta})^2 \right] \quad (39)$$

is the invariant superspace integration measure.
5 The Wess-Zumino Model

Let us start this section with the expansion of a chiral superfield in powers of the Grassmann variables $\hat{\theta}$ in order to identify its scalar and spinor field contents. Because of the existence of previous work \cite{33, 34, 26, 27} only the results will be given. However, it should be noted that our derivation differs in some points from \cite{33, 34}. Keck \cite{33} coupled a spinor field directly to the $SO(4, 1)$ spinor variable $\hat{\xi}$ of section 4, thereby demanding that the spinor field too be an $SO(4, 1)$ instead of a Lorentz spinor. On the other hand, Ivanov and Sorin \cite{34} considered the Killing spinor $\theta$ (see appendix B) as the independent Grassmann variable, which can directly be coupled to a Lorentz spinor field. However, the complicated transformation rule for $\theta$ under supersymmetry transformations is a minor drawback of their very complete formulation, which led us to consider the $SO(4, 1)$ spinor $\hat{\theta}$ as the independent superspace variable and realize the coupling to Lorentz spinor fields via a matrix $\Lambda(x)$, which is calculated in appendix B. We feel that this treatment combines the nice features of both references, \cite{33} and \cite{34}. In addition, it yields the Killing spinor $\theta$ as a side product.

The Wess-Zumino multiplet is given by the scalar fields $A$, $B$, $F$, $G$ and the Dirac spinor field $\psi$. Their supersymmetry transformations are easiest found by considering chiral superfields. Therefore, let us define

\begin{equation}
A = A_L + A_R, \quad B = A_L - A_R, \quad F = F_L + F_R, \quad G = F_L - F_R,
\end{equation}

and let us introduce the chiral projection operators

\begin{equation}
L = \frac{1}{2}(1 - i\gamma_{-1}) \quad \text{and} \quad R = \frac{1}{2}(1 + i\gamma_{-1}).
\end{equation}

Then, the left and right handed chiral superfields are given by

\begin{equation}
\Phi_L(x, \hat{\theta}) = A_L + \hat{\theta} \Lambda L \psi + (\hat{\theta} \Lambda L^{-1} \hat{\theta}) F_L - \frac{i}{2}(\hat{\theta} \Lambda \hat{\Gamma} \mu L^{-1} \hat{\theta}) D_\mu A_L + \frac{1}{2}(\hat{\theta} \hat{\theta} \hat{\Lambda} \hat{\Gamma} \mu L^{-1} \hat{\theta}) D_\mu A_L + \frac{1}{8} (\hat{\theta} \hat{\theta})^2 D_\mu D_\mu A_L,
\end{equation}

\begin{equation}
\Phi_R(x, \hat{\theta}) = A_R + \hat{\theta} \Lambda R \psi + (\hat{\theta} \Lambda R^{-1} \hat{\theta}) F_R + \frac{i}{2}(\hat{\theta} \hat{\Lambda} \hat{\Gamma} \mu R^{-1} \hat{\theta}) D_\mu A_R + \frac{1}{2}(\hat{\theta} \hat{\theta} \hat{\Lambda} \hat{\Gamma} \mu R^{-1} \hat{\theta}) D_\mu A_R + \frac{1}{8} (\hat{\theta} \hat{\theta})^2 D_\mu D_\mu A_R,
\end{equation}

respectively. It is straightforward to show using the transformation rules \cite{33} and \cite{34} that the supersymmetry transformations of the left handed superfield components are given by

\begin{equation}
\delta A_L = -\varepsilon L \psi, \quad \delta (L \psi) = -2L(\hat{\theta} A_L + F_L)\varepsilon, \quad \delta F_L = \varepsilon L \psi - \varepsilon \hat{\theta} L \psi,
\end{equation}

where we introduced the Killing spinor $\varepsilon_\alpha = (\hat{\varepsilon} \Lambda)_\alpha$. It takes somewhat more effort to show that all terms in the expansion \cite{12} transform correctly. To find the supersymmetry transformations of $A_R$, $F_R$ and $R \psi$, simply replace $L$ with $R$ in equation \cite{44}.
For the Wess-Zumino model we also introduce “conjugate” superfields by defining
\[
\bar{\Phi}_R = \bar{A}_L + \theta R \bar{\psi} + (\theta R \theta) \bar{F}_L + \cdots ,
\]
\[
\bar{\Phi}_L = \bar{A}_R + \theta L \bar{\psi} + (\theta L \theta) \bar{F}_R + \cdots ,
\]
(45)
where we used the Killing spinor $\theta_\alpha = (\hat{\theta} \Lambda)_\alpha$ and where the dots indicate terms similar to those in equations (42) and (43).

A manifestly supersymmetric action is then given by the expression
\[
S = \int d^4x \sqrt{g(x)} d^4 \bar{\theta} \bar{\phi} \left( \bar{\Phi}_L \Phi_R + \bar{\Phi}_R \Phi_L - m (\bar{\Phi}_L \Phi_L + \bar{\Phi}_R \Phi_R) \right) ,
\]
(46)
which describes the non-interacting Wess-Zumino model with a mass term. After inserting the integration measure (38) we can perform the Berezin integration and re-express the result in terms of the fields $A, B, F$ and $G$. Hence, we obtain (up to a multiplicative constant and surface terms, which have been dropped)
\[
S_{\text{bulk}} = \int d^4x \sqrt{g} \left[ \frac{1}{2} \bar{\psi} \left( \mathcal{D} - \mathcal{D} \right) \psi + D_\mu A \bar{A}^\mu A + D_\mu B \bar{B}^\mu B 
\right.
\]
\[ -3\bar{A}A - 3\bar{B}B - AF - \bar{A}F - BG - \bar{B}G
\]
\[ -m (\bar{\psi} \psi - 3\bar{A}A + 3\bar{B}B - AF - \bar{A}F + BG + \bar{B}G) \] .
(47)

Solving the equations of motion for the auxiliary fields $F$ and $G$ gives
\[
F = (m - 1)A \quad \text{and} \quad G = -(m + 1)B.
\]
(48)
Similar relations hold for $\bar{F}$ and $\bar{G}$. Substituting equation (48) back into the action (47) yields the on-shell supersymmetric action
\[
S_{\text{bulk}} = \int d^4x \sqrt{g} \left[ \frac{1}{2} \bar{\psi} \left( \mathcal{D} - \mathcal{D} \right) \psi - m\bar{\psi} \psi
\right.
\]
\[ + D_\mu A \bar{A}^\mu A + (m^2 + m - 2)\bar{A}A
\]
\[ + D_\mu B \bar{B}^\mu B + (m^2 - m - 2)\bar{B}B \] .
(49)

The mass parameter $m$ describes the mass of the fermion $\psi$. Moreover, for $m = 0$ the scalar fields $A$ and $B$ are conformally coupled. The bulk action $S_{\text{bulk}}$ has to be accompanied for the AdS/CFT correspondence by a surface term derivable from the variational principle.

It seems straightforward to read off from equation (49) the conformal dimensions of the boundary operators coupling to the boundary values of $A, B$ and $\psi$. However, care must be taken when specifying boundary conditions. According to [26, 27], because the fields $A, B$ and $\psi$ are to form an irreducible representation of $osp(1, 4)$ (for Minkowski signature), one must use the irregular mode for one of the scalar fields, if $|m| < \frac{1}{2}$. Therefore, let us not exclude the irregular modes for the scalar fields. Then the conformal dimensions of the
boundary operators are given by \[ \frac{1}{2}, \frac{3}{2}, \frac{9}{2} \]

\[
\Delta_A = \frac{3}{2} \pm \left| \frac{1}{2} + m \right|,
\]

\[
\Delta_B = \frac{3}{2} \pm \left| \frac{1}{2} - m \right|,
\]

\[
\Delta_\psi = \frac{3}{2} + |m|,
\]

(50)

where the plus and minus signs correspond to using the regular and irregular modes, respectively. Let us consider the case \( m \geq 0 \). For \( m < 0 \) only the roles of \( A \) and \( B \) interchange. Comparing the values (50) with equation (22) we can identify the boundary fields corresponding to the AdS fields \( A, B, \) and \( \psi \) with the primary conformal fields \( O, F \) and \( \chi \), respectively. Moreover, if \( m < \frac{1}{2} \), the irregular mode must be used for \( B \) in order to make this identification.

In conclusion, we found for a simple example that the AdS/CFT correspondence relates fields of AdS supersymmetry multiplets to the primary fields of superconformal multiplets. This fact was derived by explicitly constructing the AdS supersymmetric model and comparing its predictions with the relations expected from super CFT. In some cases, irregular modes must be considered for AdS fields, changing the standard prescription of the AdS/CFT correspondence. This conclusion stems from both, a pure AdS point of view and the AdS/CFT correspondence. We feel that a more general treatment should be attempted in future work.

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A Spinor Grassmann Variables

This appendix summarizes our notations and useful formulae for Grassmannian spinor variables in five dimensions. We shall concentrate on facts which do not depend on the signature of the five-dimensional metric, thus avoiding explicit matrix representations and the introduction of complex conjugate spinors. Most of the following is derived from information on Clifford algebras and their representations, which can be found in [30].

A spinor \( \hat{\theta} \) has components \( \hat{\theta}^\alpha (\alpha = 1, 2, 3, 4) \), which are Grassmannian variables, i.e. the components of any two spinors \( \hat{\theta} \) and \( \hat{\eta} \) satisfy

\[
\left\{ \hat{\theta}^\alpha, \hat{\eta}^\beta \right\} = 0.
\]

(A.1)

Spinor matrices usually carry an upper and a lower index, such as \( \delta^\alpha_\beta, (\hat{\gamma}_A)^\alpha_\beta \) etc. However, indices can be lowered and raised with the charge conjugation matrix and its inverse,
respectively:

\[
\hat{\theta}_\alpha = \hat{C}_{\alpha \beta} \hat{\theta}^\beta, \quad \hat{\theta}^\alpha = (\hat{C}^{-1})^{\alpha \beta} \hat{\theta}_\beta.
\]  (A.2)

For \( D = 5 \) the charge conjugation matrix is anti-symmetric. One can now define the scalar product of two spinors by

\[
\hat{\eta} \hat{\theta} = \hat{\eta}_\alpha \hat{\theta}^\alpha = -\hat{\eta}^\alpha \hat{\theta}_\alpha = \hat{\theta} \hat{\eta}.
\]  (A.3)

The vector space of \( 4 \times 4 \) matrices with only lower indices is spanned by 16 matrices, which can conveniently be chosen to be i) the anti-symmetric charge conjugation matrix \( \hat{C}\ ), ii) the 5 anti-symmetric matrices \((\hat{C}\hat{\gamma}_A)\) and iii) the 10 symmetric matrices \((\hat{C}\hat{S}_{AB})\) [36]. The symmetry properties of the latter two follow directly from

\[
\hat{C}\hat{\gamma}_A = \hat{\gamma}_A^T \hat{C},
\]  (A.4)

\[
\hat{C}\hat{S}_{AB} = -\hat{S}_{AB}^T \hat{C}.
\]  (A.5)

One can easily derive the matrix identity

\[
\delta^\alpha_\gamma \delta^\beta_\delta = -\frac{1}{4} \hat{C}_\gamma^\delta (\hat{C}^{-1})^{\alpha \beta} - \frac{1}{4} (\hat{C}\hat{\gamma}_A)_{\gamma \delta} (\hat{\gamma}^A \hat{C}^{-1})^{\alpha \beta} - \frac{1}{2} (\hat{C}\hat{S}_{AB})_{\gamma \delta} (\hat{S}^{AB} \hat{C}^{-1})^{\alpha \beta},
\]  (A.6)

which leads to various Fierz identities. Moreover, one can simplify products involving two or more identical spinor factors. In particular, one finds

\[
(\hat{\theta} \hat{\eta})(\hat{\theta} \hat{\varepsilon}) = -\frac{1}{4} (\hat{\theta} \hat{\theta})(\hat{\eta} \hat{\varepsilon}) - \frac{1}{4} (\hat{\theta} \hat{\gamma}_A \hat{\theta})(\hat{\eta} \hat{\gamma}_A^T \hat{\theta}),
\]  (A.7)

\[
(\hat{\theta} \hat{\eta})(\hat{\theta} \hat{\gamma}_A \hat{\theta}) = -(\hat{\theta} \hat{\theta})(\hat{\eta} \hat{\gamma}_A \hat{\theta}).
\]  (A.8)

**B Calculation of the Killing Spinor**

In this appendix we shall calculate the matrix \( \Lambda(x) \), which relates Lorentz and \( SO(4,1) \) spinors with each other. It will turn out that the Lorentz spinor derived from the \( SO(d+1,1) \) spinor \( \theta \) automatically is a Killing spinor.

According to equation (32) \( \theta \) is an \( SO(4,1) \) spinor. However, a spinor field \( \psi(x) \) conventionally is a Lorentz spinor, i.e. it transforms as a spinor under rotations of the local Lorentz frame. Hence, we introduce a matrix \( \Lambda(x) \) and demand that the product \( \theta \Lambda(x) \psi(x) \) be a scalar with respect to symmetry transformations. The matrix \( \Lambda(x) \) can be calculated using the knowledge of the transformation laws under the \( SO(4,1) \) symmetries. Thus,

\[
\delta \left( \theta \Lambda(x) \psi(x) \right) = -\delta \theta_\alpha \Lambda(x)^\alpha_\beta \psi(x)^\beta - \delta x^\mu \partial_\mu \Lambda(x)^\alpha_\beta \psi(x)^\beta
\]

\[
\equiv \hat{\theta}_\alpha \Lambda(x)^\alpha_\beta \delta \psi(x)^\beta,
\]  (B.1)

where \( \delta \theta \) and \( \delta x^\mu \) are given by equations (32) and (4), respectively, and

\[
\delta \psi = -\delta x^\mu D_\mu \psi + \frac{1}{2} D^\nu \delta x^\mu \Sigma_{\mu \nu} \psi
\]
As the parameters $a_i, b_i, \lambda$ and $\omega_{ij}$ in $\delta x$ are independent, equation (B.1) yields the following system of equations for $\Lambda$:

\[
\begin{align*}
\hat{S}(P_i)\Lambda + \partial_i \Lambda &= 0, \\
\hat{S}(D)\Lambda + x^\mu \partial_\mu \Lambda &= 0, \\
\hat{S}(K_i)\Lambda - 2x^\mu \Lambda S_{i\mu} + (2x_ix^\mu \partial_\mu - x^2 \partial_i) \Lambda &= 0, \\
\hat{S}(L_{ij})\Lambda - \Lambda S_{ij} + (x_i \partial_j - x_j \partial_i) \Lambda &= 0.
\end{align*}
\] (B.2)

The solution of equations (B.2) is not unique, but any solution will suffice. A solution of equations (B.2) is

\[
\Lambda(x) = \frac{\sqrt{x_0}}{2} (1 + \gamma_0) - \frac{1}{2\sqrt{x_0}} (1 - \gamma_0) + \frac{x^i}{2\sqrt{x_0}} \gamma_i (1 - \gamma_0).
\] (B.3)

It is also useful to know the inverse $\Lambda^{-1}$, which is easily found to be

\[
\Lambda^{-1}(x) = \frac{1}{2\sqrt{x_0}} (1 + \gamma_0) - \frac{\sqrt{x_0}}{2} (1 - \gamma_0) + \frac{x^i}{2\sqrt{x_0}} \gamma_i (1 - \gamma_0).
\] (B.4)

Consider the spinor $(\hat{\theta} \Lambda)_\alpha$, which by construction is a Lorentz spinor. Since $\hat{\theta} \hat{\eta} = \hat{\theta} \Lambda \Lambda^{-1} \hat{\eta}$, one finds

\[
(\Lambda^{-1}\hat{\theta})^\alpha = (C^{-1})^{\alpha \beta} (\hat{\theta} \Lambda)_\beta,
\] (B.5)

where $C^{-1}$ is the inverse of the charge conjugation matrix $C$ for Lorentz spinors. Equation (B.5) yields $C = \Lambda^T \hat{C} \Lambda$, which, together with equation (B.3), leads to $C = -\hat{C}$.

Finally, one can check explicitly from the expression (B.3) that

\[
D_\mu (\hat{\theta} \Lambda)_\alpha = \frac{1}{2} (\hat{\theta} \Lambda \Gamma_\mu)_\alpha,
\] (B.6)

which shows that $\theta(x) = \Lambda^{-1}(x)\hat{\theta}$ is a Killing spinor.

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