A BIJECTIVE PROOF OF THE HOOK-LENGTH FORMULA FOR SHIFTED STANDARD TABLEAUX

Ilse Fischer

Institut für Mathematik der Universität Klagenfurt,
Universitätsstrasse 65-67, A-9020 Klagenfurt, Austria.
E-mail: Ilse.Fischer@uni-klu.ac.at

Abstract. We present a bijective proof of the hook-length formula for shifted standard tableaux of a fixed shape based on a modified jeu de taquin and the ideas of the bijective proof of the hook-length formula for ordinary standard tableaux by Novelli, Pak and Stoyanovskii [6]. In their proof Novelli, Pak and Stoyanovskii define a bijection between arbitrary fillings of the Ferrers diagram with the integers 1, 2, . . . , n and pairs of standard tableaux and hook tabloids. In our shifted version of their algorithm the map from the set of arbitrary fillings of the shifted Ferrers diagram onto the set of shifted standard tableaux is analog to the construction of Novelli, Pak and Stoyanovskii, however, unlike to their algorithm, we are forced to use the 'rowwise' total order of the cells in the shifted Ferrers diagram rather than the 'columnwise' total order as the underlying order in the algorithm. Unfortunately the construction of the shifted hook tabloid is more complicated in the shifted case. As a side-result we obtain a simple random algorithm for generating shifted standard tableaux of a given shape, which produces every such tableau equally likely.

1. Introduction

A partition of a positive integer n is a sequence of positive integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) with \( \lambda_1 + \lambda_2 + \cdots + \lambda_r = n \) and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \). The (ordinary) Ferrers diagram of shape \( \lambda \) is an array of cells with \( r \) left-justified rows and \( \lambda_i \) cells in row \( i \). Figure 1.a shows the Ferrers diagram corresponding to \( (4, 3, 3, 1) \).

If \( \lambda \) is a partition with distinct components (strict partition) then the shifted Ferrers diagram of shape \( \lambda \) is an array of cells with \( r \) rows, each row indented by one cell to the right with respect to the previous row and \( \lambda_i \) cells in row \( i \). Figure 1.b shows the shifted Ferrers diagram corresponding to \( (5, 4, 2, 1) \).

![The Ferrers diagram corresponding to (4, 3, 3, 1)](image1.png)

![The shifted Ferrers diagram corresponding to (5, 4, 2, 1)](image2.png)

Figure 1.
Given a partition $\lambda$ of $n$, respectively strict partition of $n$, a standard tableau, respectively a shifted standard tableau of shape $\lambda$, is a filling of the cells of the ordinary Ferrers diagram, respectively shifted Ferrers diagram, of shape $\lambda$ with $1, 2, \ldots, n$, such that the entries along rows and columns are increasing. Figure 2.a displays an example of a standard tableau of shape $(4, 3, 3, 1)$ and Figure 2.b displays an example of a shifted standard tableau of shape $(5, 4, 2, 1)$.

Once we have accepted these definitions it is a natural question to ask for the number of standard tableaux, respectively shifted standard tableaux, of a given shape $\lambda$. Surprisingly there exists a simple product formula for these numbers. It involves objects called hooks, which are defined in the following paragraph.

We label the cell in the $i$-th row and $j$-th column of the ordinary, respectively shifted, Ferrers diagram of shape $\lambda$ by the pair $(i, j)$. The hook of a cell $(i, j)$ in an ordinary Ferrers diagram is the set of cells that are either in the same row as $(i, j)$ and to the right of $(i, j)$, or in the same column as $(i, j)$ and below $(i, j)$, $(i, j)$ included. The dots in Figure 3.a indicate the hook of the cell $(2, 1)$. The hook of a cell $(i, j)$ in a shifted Ferrers diagram again includes all cells that are either in the same row as $(i, j)$ and to the right of $(i, j)$, or in the same column as $(i, j)$ and below $(i, j)$, $(i, j)$ included, but if this set contains the cell $(j, j)$ on the main diagonal, then also the cells of the $(j + 1)$-st row belong to the hook of $(i, j)$. The dots in Figure 3.b indicate the hook of cell $(1, 2)$. The hook-length $h_{i,j}$ of the cell $(i, j)$ is the number of cells in the hook of $(i, j)$.

Now we are in the position to state the hook-length formula.

**Theorem 1** ([1], [2]). The number of standard tableaux, respectively shifted standard tableaux, of shape $\lambda$ is

$$\frac{n!}{\prod_{(i,j)} h_{i,j}}.$$
where the product in the denominator is taken over all cells in the Ferrers diagram, respectively shifted Ferrers diagram, of shape $\lambda$.

Thus the number of standard tableaux of shape $(4, 3, 3, 1)$ is $\frac{11!}{7 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1188$ and the number of shifted standard tableaux of shape $(5, 4, 2, 1)$ is $\frac{12!}{9 \cdot 7 \cdot 6 \cdot 5 \cdot 2 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 176$.

For the history of the various proofs of the hook-length formula for ordinary standard tableaux see [3, page 400]. A recipe for an inductive proof of the hook-length formula for shifted standard tableaux can be found in [4, page 266]. In [5] it is shown that the nice probabilistic proof of the hook-length formula for ordinary standard tableaux in [3] has an analog for shifted standard tableaux.

A majority among the combinatorialists considers a bijective proof as the most aesthetic type of proof for an enumeration result. In [6] a bijective proof of the hook-length formula for ordinary standard tableaux based on a modified jeu de taquin is given. There exists a bijective proof of the hook-length formula for shifted standard tableaux as well [7], however, it makes use of the involution principle by Garsia and Milne. The aim of this paper is to present an involution principle-free bijective proof of the hook-length formula for shifted standard tableaux, which is in the spirit of the beautiful bijective proof in [6].

If we discover two sets $S, T$ of combinatorial objects with the same cardinality, we often believe that this fact is a projection of a canonical bijection between the two sets. Such a bijection is called a bijective proof of the equality $|S| = |T|$. Now suppose we are in the following more general situation: There exists an integer $h$ such that $h \cdot |S| = |T|$. Then this fact could be a projection of a canonical $h$ to 1 surjection from $T$ onto $S$, i.e. a map from $T$ onto $S$ where every element in $S$ is assigned to exactly $h$ elements in $T$. In the following $S_\lambda$ denotes the set of shifted standard tableaux of shape $\lambda$. A shifted tabloid of shape $\lambda$ is an (arbitrary) filling of the cells of the shifted Ferrers diagram of shape $\lambda$ with $1, 2, \ldots, n$. We denote the set of shifted tabloids of shape $\lambda$ by $T_\lambda$ and observe that its cardinality is $n!$.

In our main theorem (Theorem 2) we present a $\prod_{(i,j)} h_{i,j}$ to 1 surjection from $T_\lambda$ onto $S_\lambda$, which is clearly a proof of the hook-length formula for shifted standard tableaux. We prove Theorem 2 by introducing the set of hook tabloids $H_\lambda$ with $|H_\lambda| = \prod_{(i,j)} h_{i,j}$ and extending the surjection to a bijection from $T_\lambda$ to $S_\lambda \times H_\lambda$. The corresponding surjection from the set of ordinary tabloids onto the set of standard tableaux is similar to the surjection in the shifted case, see [3]; the extension in the shifted case is however far more complicated compared to the ordinary case.

2. Modified jeu de taquin

In this section we describe the ordering procedure which assigns to every ‘scrambled’ shifted tabloid $T \in T_\lambda$ an ‘ordered’ shifted standard tableau $S \in S_\lambda$. This map has the property that the number of shifted tabloids which are mapped to a fixed shifted standard tableau is $\prod_{(i,j)} h_{i,j}$. The ordering procedure is based on a modified jeu de taquin, which we have to describe first.

Notation. Let $T$ be a shifted tabloid of shape $\lambda$. Then $T_{i,j} = T_{(i,j)}$ denotes the entry of cell $(i, j)$. If the cell $(i, j)$ does not exist in the shifted Ferrers diagram of shape $\lambda$, let $T_{i,j} = \infty$. For every $e$, $1 \leq e \leq n$, there exists a unique cell $(i, j)$ in the shifted Ferrers diagram of shape $\lambda$ with $T_{i,j} = e$. We define $c_T(e) = (i, j)$.

Jeu de taquin in $T$ with entry $e$ and with respect to a set $D$. Let $1 \leq e \leq n$ and $D$ be a set of cells in the shifted Ferrers diagram of shape $\lambda$. We define the routine jeu de taquin in $T$ with entry $e$ and with respect to $D$ inductively. The output is another
JT Ferrers diagram we write JT shape \( \rho \).

Let \( \rho \) be such that \( \min(T;\rho) = T_{i,j} \leq \min(T_{i+1,j}, T_{i,j+1}) \). (In the latter case \( e \) is said to be stable. See Figure 4.) Otherwise let \( e \in \{(i+1,j), (i,j+1)\} \) be such that \( T_e = \min(T_{i+1,j}, T_{i,j+1}) \) and let \( T' \) denote the tabloid we obtain by exchanging the entries \( e \) and \( T_e \) in \( T \). (See Figure 5.) Next perform jeu de taquin in \( T' \) with entry \( e \) and with respect to \( D \) in order to obtain the final tabloid \( U \), i.e. we repeat this exchanging procedure with \( e \) and either its current neighbour to the right or below until \( e \) is either stable or in a cell of \( D \). (For an example see Figure 6.)

If \( D \) is the empty set we omit ‘with respect to \( D \)’. The output tabloid \( U \) is denoted by \( JT_D(T,e) \) and the cell of \( e \) in \( U \) is denoted by \( CJT_D(T,e) \). If \( D \) is the \( k \)-th row of the shifted Ferrers diagram we write \( JT_k(T,e) \) and \( CJT_k(T,e) \), respectively.

The total order. We define a total order on the cells of a shifted Ferrers diagram of shape \( \lambda \). This will be the order in which we perform the jeu de taquin just defined with the entries of a shifted tabloid. A cell \( p_1 \) comes before cell \( p_2 \) if \( p_1 \) is in a lower row than \( p_2 \) or if both are in the same row but \( p_1 \) is to the right of \( p_2 \). Phrased differently, to obtain the total order one starts with the rightmost cell in the last row and reads each row from right to left, beginning with the bottom row and continuing up to the first row. Figure 7 displays this total order for the shifted Ferrers diagram of shape \((5,4,2,1)\).

The \( \prod_{(i,j)} h_{i,j} \) to 1 map from \( T_{\lambda} \) onto \( S_{\lambda} \). Let \( T \) denote an arbitrary shifted tabloid of shape \( \lambda \). In order to construct the corresponding shifted standard tableau \( S \) we perform step
by step jeu de taquin with the entries of $T$ subject to the total order we have just defined and starting with the entry in the smallest cell. To be more accurate:

- $\rho$ = smallest cell with respect to the total order
- $S = T$
- Repeat $S = JT(S, S_\rho)$
- $\rho$ = successor of $\rho$ in the total order
- until $\rho = (1, 1)$.

**Example 1.** Consider the shifted tabloid

\[
T = \begin{array}{cccc}
11 & 4 & 9 & 8 \\
12 & 6 & 2 & 3 \\
10 & 5 & & \\
7 & & & \\
\end{array}
\]

of shape $(5, 4, 2, 1)$. We construct the corresponding shifted standard tableau. According to the algorithm above we start with performing jeu de taquin with 7, but 7 is stable. The same is true for the entry 5. Performing jeu de taquin with 10 results in $S = \begin{array}{cccc}
12 & 6 & 2 & 3 \\
5 & 7 & & \\
10 & & & \\
\end{array}$. Since 3 and 2 are stable the next change happens to be when performing jeu de taquin with entry 6.

\[
S = \begin{array}{cccc}
11 & 4 & 9 & 8 \\
12 & 2 & 3 & 6 \\
5 & 7 & & \\
\end{array}
\]

We perform jeu de taquin with 12 — $S = \begin{array}{cccc}
11 & 4 & 9 & 8 \\
2 & 3 & 6 & 12 \\
5 & 7 & & \\
10 & & & \\
\end{array}$, —

next with 8 — $S = \begin{array}{cccc}
11 & 4 & 9 & 8 \\
2 & 3 & 6 & 12 \\
5 & 7 & & \\
10 & & & \\
\end{array}$, — then with 9 — $S = \begin{array}{cccc}
11 & 4 & 1 & 6 \\
2 & 3 & 7 & 12 \\
5 & 9 & & \\
10 & & & \\
\end{array}$, — and

with 4 — $S = \begin{array}{cccc}
11 & 1 & 3 & 6 \\
2 & 4 & 7 & 12 \\
5 & 9 & & \\
10 & & & \\
\end{array}$, — and finally with 11 — $S = \begin{array}{cccc}
11 & 1 & 3 & 6 \\
2 & 4 & 7 & 12 \\
5 & 9 & & \\
10 & & & \\
\end{array}$.

Observe that the output tabloid $S$ is a shifted standard tableau by construction. We denote it by $\text{STAND\_SPLIT}(T)$. We are in the position to state our main theorem. Note that the bijective proof in [6] shows that a similar theorem is true for ordinary standard tableaux.
**Theorem 2.** The map $T \rightarrow \text{STAND\_SPLIT}(T)$ is a $\prod_{(i,j)} h_{i,j}$ to 1 map from the set of shifted tabloids $T_\lambda$ onto the set of shifted standard tableaux $S_\lambda$.

As an interesting side-result we obtain a random algorithm which produces every shifted standard tableau of a given shape with the same probability.

**Corollary 1.** The following algorithm produces every shifted standard tableau of a given shape $\lambda$ with the same probability.

1. Generate a permutation $\pi$ of $\{1, 2, \ldots, n\}$ subject to uniform distribution.
2. Construct the corresponding shifted tabloid $T_\pi$ of shape $\lambda$ by filling the elements from $\pi$ into the shifted Ferrers diagram of shape $\lambda$ rowwise from top to bottom and in each row from left to right.
3. Apply $\text{STAND\_SPLIT}(T_\pi)$ in order to obtain the shifted standard tableau.

Note that Novelli, Pak and Stoyanovskii [6] use another total order of the cells in the Ferrers diagram, they perform jeu de taquin columnwise from right to left and within a column from bottom to top. By 'transposing' their algorithm it is clear that the order we defined in the shifted case would also induce a $\prod_{(i,j)} h_{i,j}$ to 1 map in the ordinary case. However, computer experiments with the strict partition $(4, 3, 2, 1)$ have shown that the order defined by Novelli, Pak and Stoyanovskii is not admissible in the shifted case. Moreover it seems that the total order we have defined is the only admissible in the shifted case, whereas in the ordinary case there exist many total orders with the property that they induce a $\prod_{(i,j)} h_{i,j}$ to 1 map. We plan to discuss this phenomenon in a forthcoming paper.

We prove Theorem 2 by giving a bijective proof of the hook-length formula. For that purpose we rewrite the hook-length formula as

$$n! = |S_\lambda| \cdot \prod_{(i,j)} h_{i,j}.$$ 

We define combinatorial objects that correspond to $\prod_{(i,j)} h_{i,j}$ in this formula in our bijective proof. A *shifted hook tabloid* of shape $\lambda$ is a filling of the cells of the shifted Ferrers diagram of shape $\lambda$ with pairs of integers, such that the entry in a cell $\rho$ are the coordinates of a cell in the hook of $\rho$. (See Figure 8. This definition of a shifted hook tabloid has a natural analog for ordinary Ferrers diagram, which is equivalent to the definition of a hook function in [6].) We denote the set of shifted hook tabloids by $H_\lambda$. Since $|H_\lambda| = \prod_{(i,j)} h_{i,j}$ it suffices to find a bijection between the set of shifted tabloids $T_\lambda$ and the cartesian product of the set of shifted standard tableaux $S_\lambda$ and the set of shifted hook tabloids $H_\lambda$ to prove the hook-length formula for shifted standard tableaux.

$$T_\lambda \xrightarrow{\text{bijection}} S_\lambda \times H_\lambda$$

In order to prove Theorem 2 we construct such a bijection, where the shifted standard tableau is obtained from the shifted tabloid by the modified jeu de taquin described above. Unfortunately the second component of the bijection, i.e. the construction of the shifted hook tableau, is complicated. Thus it would be nice to find an easier-to-describe bijection. Also a shorter proof of Theorem 2 would be of interest. Maybe the present paper serves as an inspiration in this task.

We give some further definitions we need for the rest of the paper.

**Reverse jeu de taquin in $U$ with entry $e$ and with respect to a set $D$.** We define an inverse to jeu de taquin, which we need to construct the inverse of the map which assigns
Furthermore observe that \( U_{i,j} \) symbols the fixed row). Perform reverse jeu de taquin in \( T \) in the following sense: Let \( D \) be a shifted tabloid of shape \( \lambda \). Let \( e \leftarrow (i', j') \). Set \( T = U \) and stop if either \( (i', j') = (i, i) \) or \( (i', j') \in D \). Otherwise let \( \rho \in \{(i'-1, j'), (i', j'-1)\} \) be such that \( U_\rho = \max(U_{i'-1,j'}, U_{i',j'-1}) \) if \( i' \notin \{i,j\} \), \( \rho = (i', j') \) if \( i' = i \) and \( \rho = (i'-1, j') \) if \( i' = j' \) and let \( U' \) denote the tabloid we obtain by exchanging the entries \( e \) and \( U_\rho \) in \( U \). Next perform jeu de taquin in \( U' \) with entry \( e \) and with respect to \( D \) in order to obtain the final tabloid \( T \). (For an example read Figure 8 from right to left. The left shifted tabloid can be obtained from the right by performing reverse jeu de taquin with \( 64 \) and with respect to \( (1, 4) \).)

If \( D \) is the empty set we omit ‘with respect to \( D \)’. In the following the output tabloid \( T \) is denoted by \( RJT_D(U, e) \) and the cell of \( e \) in \( T \) is denoted by \( CRJT_D(U, e) \). If \( D \) is the \( k \)-th row we write \( RJT_k(U, e) \) and \( CRJT_k(U, e) \), respectively.

Let \( T \) be a shifted tabloid of shape \( \lambda \) and \( 1 \leq e \leq n \). The forward path of \( e \) in \( T \) with respect to \( D \) is the set of cells \( e \) comes across when performing jeu de taquin in \( T \) with \( e \) and with respect to \( D \). For example the forward path of \( 64 \) with respect to the cells on the main diagonal in the left tabloid of Figure 8 is \( \{(1, 4), (2, 4), (3, 4), (3, 5), (4, 5), (5, 5)\} \). Similarly the backward path in \( T \) of \( e \) with respect to \( D \) is the set of cells \( e \) comes across when performing reverse jeu de taquin in \( T \) with \( e \) and with respect to \( D \). Clearly the backward path of \( 64 \) with respect to \( (1, 4) \) in the right shifted tabloid in Figure 8 coincides with the forward path in the left shifted tabloid.

We need one more definition before we are able to relate jeu de taquin and reverse jeu de taquin. Let \( T \) be a shifted tabloid of shape \( \lambda \) and \( \rho \) a cell in the shifted Ferrers diagram of shape \( \lambda \). The shifted tabloid \( T \) is said to be ordered up to cell \( \rho \) if rows and columns are increasing in the subtabloid consisting of the cells that are smaller or equal to \( \rho \) with respect to the total order. (Note that in Figure 8 the left tabloid is ordered up to \( (1, 5) \)).

Observe that the routines jeu de taquin and reverse jeu de taquin are inverse to each other in the following sense: Let \( T \) be a shifted tabloid and \( e \) an entry in \( T \) with \( c_T(e) = (i, j) \). Suppose \( T \) is ordered up to the predecessor of \( (i, j) \) in the total order. Perform jeu de taquin in \( T \) with \( e \) and with respect to \( D \) and obtain the tabloid \( U \). If we perform reverse jeu de taquin in \( U \) with \( e \) and with respect to \( (i, j) \) (\( i \) being the fixed row), we reobtain \( T \). In symbols

\[
T = RJT_{(i,j)}(JT_D(T,e), e).
\]

Furthermore observe that \( U \) is ordered up to \( (i, j) \) if \( D = \emptyset \).

Conversely: Let \( e \) be an entry weakly below the \( i \)-th row of a shifted tabloid \( T \) that is ordered up to \( (i, j) \) and assume that the backward path of \( e \) in \( T \) contains \( (i, j) \) (\( i \) being the fixed row). Perform reverse jeu de taquin in \( T \) with \( e \) and respect to \( (i, j) \) and denote
Figure 9. The backward paths of the entries in the shifted tabloid in Figure 2.b. Thus $1 < T 3 < T 2 < T 6 < T 5 < T 4 < T 12 < T 11 < T 9 < T 10 < T 7 < T 8$.

the output tabloid by $U$. If we perform jeu de taquin in $U$ with $e$ and with respect to $(i', j') = c_T(e)$, we reobtain $T$. In symbols

$$T = JT_{(i', j')}(RJT_{(i, j)}(T, e), e).$$

Furthermore observe that $U$ is ordered up to the predecessor of $(i, j)$. Therefore: In order to be able to reconstruct $T$ from $\text{STAND}_\text{SPLIT}(T)$ we have to store the endcells of the forward paths we obtain in the application of modified jeu de taquin to $T$ in the shifted hook tabloid. Since we want to obtain a bijection this storage has to be organized most efficiently. The fact that the endcells are not independent from each other (see Lemma 3) makes this storage non-trivial.

**Backward paths order.** We define the backward paths order on the entries of a shifted tabloid $T$. Let $e_1, e_2$ be two entries in $T$ and $P_1, P_2$ their backward paths ($1$ being the fixed row) in $T$. Furthermore let $(i, j)$ denote the smallest cell in $P_1 \cap P_2$ with respect to the total order. If either $(i + 1, j) \in P_1$ or $(i, j + 1) \in P_2$ we define $e_1 < T e_2$ and say that $e_2$ is greater than $e_1$ with respect to the backward paths order in $T$. For an example see Figure 9.

We introduce a manner-of-speaking: Let $T$ be a shifted tabloid and $H$ an accompanying partial shifted hook tabloid, a partial shifted hook tabloid being a filling of some cells of the shifted Ferrer diagram with entries satisfying the requirement for the entries in a shifted hook tabloid. Let $\rho$ be an entry of a cell $\sigma$ in the fixed $i$-th row of $H$, i.e. $H_\sigma = \rho$. We say that $T_\rho$ is a horizontal candidate in $T$ with respect to $H$, if $\rho$ is in the same column as $\sigma$ in the shifted Ferrers diagram. For example consider the shifted hook tabloid in Figure 8 and an arbitrary shifted tabloid $T$ of shape $(5, 4, 2, 1)$. If $i = 1$ then $T_{1, 2}$, $T_{4, 4}$ and $T_{2, 5}$ are the horizontal candidates. If $\rho$ is neither in the same row nor in the same column as $\sigma$, $T_\rho$ is a vertical candidate with respect to $H$. In our example in Figure 8 $T_{2, 2}$ is the only vertical candidate if $i = 1$.

The rest of the paper is organized as follows. In Section 3 we describe the Algorithm SPLIT that converts a shifted tabloid of shape $\lambda$ into a pair of a shifted standard tableau of shape $\lambda$ and a shifted hook tabloid of shape $\lambda$. In Section 4 we give some examples of the application of SPLIT. In Section 5 we describe the Algorithm MERGE that ‘merges’ a pair of a shifted standard tableau and a shifted hook tabloid to a shifted tabloid. In Section 6 we prove that the Algorithm MERGE is the inverse of the Algorithm SPLIT.

### 3. The Algorithm SPLIT

In this section we describe the Algorithm SPLIT that transforms a shifted tabloid into a pair of a shifted standard tableau and a shifted hook tabloid.

The construction of the shifted hook tabloid is more involved compared to the construction of the shifted standard tabloid, which we have already described. It depends on the endcells
of the forward paths we obtain in the course of performing jeu de taquin in the shifted tabloid and on the intersection of two such paths near the main diagonal. We have to modify the order of some steps in the construction of the shifted standard tableau so that the shifted hook tabloid can be built up simultaneously. But since these steps in question commute the modified algorithm for building the shifted standard tableau is equivalent to the origin algorithm.

Before we are in the position to describe the algorithm, we introduce two routines on a partial shifted hook tableau $H$ of shape $\lambda$.

**A shift from cell $(i, j)$ to cell $(i, j')$ in a partial shifted hook tabloid.** Let $(i, j)$, $(i, j')$ be two cells in $H$, $j \leq j'$. We define the term shift from $(i, j)$ to $(i, j')$ in $H$. The output of this operation is another partial shifted hook tabloid $H'$ which coincides with $H$ except for the cells $(i, k)$, $j \leq k \leq j'$. Let $j \leq k < j'$. If $H_{i, k+1} = (i', k+1)$, $i < i'$, or $H_{i, k+1} = (k+2, j')$ then set $H_{i, k}' = (i' - 1, k)$ or $H_{i, k}' = (k + 1, j' - 1)$, respectively. If $H_{i, k+1} = (i, l)$ then set $H_{i, k}' = (i, l)$. The cell $(i, j')$ is empty. We denote $H'$ by $\text{SHIFT}(H, i, j, j')$. For an example see Figure 10.

Whenever we perform jeu de taquin with an entry $e$ and with respect to a set $D$ in the shifted tabloid $T$ in the algorithms below, we store the end of the forward path in the accompanying partial shifted hook tableau $H$. If we are in the course of performing jeu de taquin with the entries in the $i$-th row and $(i', j')$ is the end of the forward path, this is either done by setting $H_{i, j'} = (i', j')$ or by setting $H_{i, j'+1} = (i', j')$. The latter possibility is used if $e$ was previously a vertical candidate. By reverse jeu de taquin the knowledge of the end of the forward path is enough to undo the performance of jeu de taquin. However, when $H_{i, j'}$, respectively $H_{i, j'+1}$, is occupied we have to perform a shift from the cell in $H$ that has previously pointed to $e$ to $(i, j')$, respectively $(i, i' - 1)$, in order to empty either $(i, j')$ or $(i, i' - 1)$. More accurate: If $e$ is either a horizontal candidate or no candidate in $(T, H)$ we denote by $\text{JS}_D(T, H, e)$ the pair of a shifted tabloid $T'$ and a shifted hook tabloid $H'$, which is obtained in the following way: $T' = JT_D(T, e)$; $H' = \text{SHIFT}(H, i, q, j')$ and $H_{i, j'}' = (i', j')$, where $q$ is the column of $e$ in $T$ and $c_T(e) = (i', j')$. If $e$ is a vertical candidate we denote by $\text{JS}_D(T, H, e)$ the pair of a shifted tabloid $T'$ and a shifted hook tabloid $H'$, which is obtained in the following way: $T' = JT_D(T, e)$, $H' = \text{SHIFT}(H, i, p - 1, i' - 1)$ and $H_{i, j'}' = (i', j')$, where $p$ is the row of $e$ in $T$ and $c_T(e) = (i', j')$. If $D$ is empty we write $\text{JS}(T, H, e)$ and if $D$ is the $k$-th row we write $\text{JS}_k(T, H, e)$.

**A transfer from cell $(i, j)$ to cell $(i, k)$ in a partial shifted hook tabloid.** Let $j \leq k \leq r$ and $H_{i, j} = (i', j')$ with either $i' = i$ or $j' = j$. We define the term transfer from cell $(i, j)$ to cell $(i, k)$ in $H$. The output of this operation is again another partial shifted hook tabloid $H'$ which coincides with $H$ except for the cells $(i, l)$, $j \leq l \leq k$. If $i' = i$ and $k \leq j'$ let $H_{i, k}' = (i', j')$, otherwise let $H_{i, k}' = (i' + k - j', k)$. For $j \leq l < k$ let $H_{i, l}' = (l + 1, l + 1)$.

![Figure 10. A shift from (1,2) to (1,6): The ends of the arrows indicate the entries of the cells at the origins of the arrows. The x denotes an empty cell.](image-url)
Figure 11. A transfer from (1,2) to (1,6): The ends of the arrows indicate the entries of the cells at the origins of the arrows.

We denote $H'$ by $\text{TRANS}(H,i,j,k)$. For an example see Figure 11. Whenever we apply a transfer from $(i,j)$ to $(i,k)$ in $H$ we have $H_{i,l} = (l,l)$ for $j < l \leq k$ as in the example. This operation we need in the algorithms for converting horizontal candidates into vertical candidates.

The Algorithm SPLIT is divided into $r$ steps, where in the $i$-th step we perform jeu de taquin with the entries in the $(r-i+1)$-st row. Within a row $i$, SPLIT is divided into 3 steps, SPLIT 1, SPLIT 2 and SPLIT 3. Assume we just start performing jeu de taquin with the entries in the $i$-th row. Let $T$ denote the shifted tabloid we have constructed so far and $H$ the partial shifted hook tabloid. At this point $T$ is ordered up to $(i+1,i+1)$ (as it is in the origin algorithm for constructing the shifted standard tableau in Section 3), the first $i$ rows of $H$ are empty and the last $r-i$ rows of $H$ form a shifted hook tabloid. In SPLIT 1 we perform jeu de taquin with the entries in the $i$-th row from right to left and with respect to the set $MD = \{(1,1),(2,2),\ldots,(r,r)\}$ of cells on the main diagonal.

**SPLIT 1.** Repeat for $j = \lambda_i + i - 1$ down to $j = i$: Set $(T,H) = JS_{MD}(T,H,T_{i,j})$.

After SPLIT 1 for every entry $e$ whose forward path terminates in SPLIT 1 in a cell $(k,k)$ on the main diagonal, we have $T_{k,k} = e$ and $H_{i,k} = (k,k)$ and therefore all unstable entries in $T$ are horizontal candidates with respect to $H$ after the application of SPLIT 1. This follows from the fact that whenever a forward path of an entry $T_{i,j}$ ends in a cell $(k,k)$ on the main diagonal, then the forward paths of the following entries $T_{i,g}$, $g < j$, end strictly left of the $k$-th column. (See Lemma 3.)
Thus the backward path of the same horizontal candidate includes the cell \((i, i')\) which implies \(T_{k-1,k-1} \leq T_{k-1,k}\) for \(i < k \leq i'\). If \(i' = i\) jump to Case 2.

Set \(U = T\). Repeat for \(g = i'\) down to \(g = i\): Set \(U = JT_{g+1}(U, T, g,g)\).

We distinguish between two cases. Let \(h\) be minimal, \(i \leq h \leq i'\), such that \(T_{h,h}\) is not in the \((h+1)\)-st row of \(U\). We continue with Case 2 below if \(h\) does not exist. We also continue with Case 2 if \(h = i'\) and there exists a horizontal candidate with respect to \(H\) strictly below the \(i'\)-th row of \(U\) which is smaller than \(T_{i',i'-1}\) with respect to the backward paths order in \(U\). In all other cases we continue with Case 1. (See Figure 12.)

Reject the tabloid \(U\) we constructed so far in SPLIT 2.

**SPLIT 2.** Choose \(i'\), \(i \leq i' \leq r\), maximal such that \(H_{i,k} = (k, k)\) and \(T_{k-1,k-1}\) is unstable (i.e. \(T_{k-1,k-1} > T_{k-1,k}\)) for \(i < k \leq i'\). If \(i' = i\) jump to Case 2.

Set \(U = T\). Repeat for \(g = i'\) down to \(g = i\): [ Let \(k\) be such that either \((g, k)\) or \((g+1, k)\) is the endcell of the forward path of \(T_{g,g}\) in the procedure for constructing \(U\). Set \(T = JT_{(g,k)}(T,H,T,g,g)\) and \(H_{i,g-1} = (g, k)\). (Note that the forward path of \(T_{g,g}\) ends in \((g,k),\)) ]

Let \((h,k) = CJT(T,h,h)\) and set \(T = JT(T,h,h)\). If \(h - k \leq i - i'\) let \(H_{i,i'} = (i, i - h + k)\) otherwise let \(H_{i,i'} = (i' + h - k, i')\).

Repeat for \(g = h - 1\) down to \(g = i\): [ Let \(H_{i,g} = CJT_{g+1}(T,T,g,g)\) and \(T = JT_{g+1}(T,T,g,g)\). ]

**Case 2.** Set \((T,H) = JS_{i'+1}(T,H,T,i',i')\).

Repeat for \(g = i' - 1\) down to \(g = i\): [ Let \(H_{i,g} = CJT_{g+1}(T,T,g,g)\) and \(T = JT_{g+1}(T,T,g,g)\). ]

Observe the following (See also Figure 12): Consider the backward paths with respect to the \(i'\)-th row of the horizontal candidates strictly below the \(i'\)-th row in the output tabloid of SPLIT 2. Then either one of these paths ends weakly to the left of the vertical candidate in the \(i'\)-th row or there exists no vertical candidate if and only if we were in Case 2 of SPLIT 2. (If we were in Case 1 of SPLIT 2 and the backward path of a horizontal candidate strictly below the \(i'\)-th row would end weakly to the left of the vertical candidate in the \(i'\)-th row after the application of Case 1 SPLIT 2 then \(h \neq i'\) by the cases distinction in SPLIT 2. Thus the backward path of the same horizontal candidate includes the cell \((i' + 1, i' + 1)\) on the main diagonal before the application of Case 1 in SPLIT 2 (by the argument in the proof of Lemma 3) which implies \(H_{i,i'+1} = (i' + 1, i' + 1)\) (by property AFTER, SPLIT" 1 in the proof of Claim 1 in Section 6) and this is not possible by the choice of \(i'\) for our assumptions imply that \(T_{i',i'}\) is unstable before the application of Case 1 in SPLIT 2. Furthermore there

![Figure 12](image-url)
always exists a vertical candidate after the performance of Case 1 of SPLIT 2.) In other words: We were in Case 2 of SPLIT 2 if and only if either the smallest horizontal candidate strictly below the row of the smallest vertical candidate is smaller than this smallest vertical candidate or there exists no vertical candidate in the output pair.

Besides the pair \((T, H)\) we need two sets of entries \(C'\) and \(C\) as an input for SPLIT 3. The set \(C'\) contains the vertical candidates together with the smallest candidate with respect to the backward paths order if it is unstable. The set \(C\) contains the horizontal candidates on the main diagonal of \(T\) which are not in \(C'\). Observe that the entries in \(C'\) are strictly above of the entries in \(C\), i.e. the maximal row of an entry in \(C'\) is smaller than the minimal row of an entry in \(C\). This will also be true in the course of performing SPLIT 3. Similarily all vertical candidates are strictly above of the entries in \(C\) before and while performing SPLIT 3.

**SPLIT 3.** Repeat the following until \(C \cup C' = \emptyset\).

1. If \(C' = \emptyset\) choose \(\bar{e} \in C\) such that the row of \(\bar{e}\) is minimal, set \(C' = C' \cup \{\bar{e}\}\) and \(C = C \setminus \{\bar{e}\}\).
2. If \(C \neq \emptyset\), choose \(e' = T_{h,h} \in C\) such that \(h\) is minimal.
   - Consider the forward path \(P\) of \(e\) in \(T\). We distinguish between three cases.
   - **Case 1.** \(P\) ends weakly left of \((h - 1)\)-st column or \(e'\) does not exist.
     - Set \((T, H) = \text{JS}(T, H, e)\) and \(C' = C' \setminus \{e\}\).
   - **Case 2.** \((h - 1, h - 1) \in P\) and \((h - 1, h) \in P\).
     - Set \((T, H) = \text{JS}_{(h-1,h-1)}(T, H, e)\).
     - Let \(h, h' \leq h'\), be maximal such that \(T_{j-1,j-1}\) is unstable and \(T_{j,j} \in C\) for \(h < j \leq h'\).
     - If \(e\) is a horizontal candidate set \(H = \text{TRANS}(H, i, h - 2, h' - 1)\), otherwise \(H = \text{TRANS}(H, i, h - 1, h')\).
     - Furthermore let \(C' = C' \cup \{T_{h,h}, T_{h+1,h+1}, \ldots, T_{h',h'}\}\) and \(C = C \setminus \{T_{h,h}, T_{h+1,h+1}, \ldots, T_{h',h'}\}\).
   - **Case 3.** \(P\) ends weakly right of column \(h\), but does not contain \((h - 1, h - 1)\). Set \(C' = C' \cup \{e'\}\) and \(C = C \setminus \{e'\}\).

In Figure 13 an example of Case 2 in SPLIT 3 is displayed.

Let \(T\) be a shifted tabloid ordered up to \((i + 1, i + 1)\) and \(H\) a partial shifted hook tabloid. Let \((T', H')\) denote the pair we obtain after the application of SPLIT 1, SPLIT 2 and SPLIT 3 to the \(i\)-th row of \((T, H)\). We denote \(T' = \text{STAND}_{\text{SPLIT}}(T, H, i) = \text{STAND}_{\text{SPLIT}}(T, i)\) and \(H' = \text{HOOK}_{\text{SPLIT}}(T, H, i)\). We are finally in the position to formulate the Algorithm

![Figure 13](image-url)
SPLIT. The input is a shifted tabloid $T$ of shape $\lambda$ and an empty shifted tabloid $H'$ of shape $\lambda$.

**SPLIT.** $T' = T$. Repeat for $i = r$ down to $i = 1$: $[ H' = \text{HOOK}_\text{SPLIT}(T', H', i) \quad \text{and} \quad T' = \text{STAND}_\text{SPLIT}(T', H', i). ]$

The output $T'$ and $H'$ of the algorithm is denoted by $\text{STAND}_\text{SPLIT}(T, H) = \text{STAND}_\text{SPLIT}(T')$ and $\text{HOOK}_\text{SPLIT}(T, H)$.

4. Examples

In this section we give five examples for the application of SPLIT. All examples are of shape $(11, 10, 9, 8, 7, 6, 5, 4, 3, 2)$ and the input shifted tabloid is ordered up to $(2, 2)$. We perform jeu de taquin with the entries in the first row of the shifted tabloid and simultaneously build up the first row of the shifted hook tabloid. Our examples cover all possible cases in SPLIT 2: In Example 2 we are in Case 2 of SPLIT 2 for $h$ does not exist, in Example 3 we are in Case 2 of SPLIT 2 for $i = i'$, in Example 4 we are in Case 1 of SPLIT 2 for $h = 1$, in Example 5 we are in Case 1 of SPLIT 2 with $h = i'$ and in Example 6 we are in Case 2 of SPLIT 2 with $h = i'$.

**Example 2.** We consider the shifted tabloid

$$
T = \begin{array}{cccccccccc}
63 & 64 & 61 & 41 & 34 & 20 & 62 & 65 & 56 & 15 & 54 \\
1 & 2 & 3 & 4 & 7 & 8 & 13 & 16 & 18 & 24 \\
5 & 6 & 9 & 10 & 17 & 19 & 30 & 31 & 45 \\
11 & 12 & 21 & 23 & 26 & 32 & 37 & 47 \\
14 & 22 & 25 & 27 & 36 & 39 & 51 \\
28 & 29 & 35 & 40 & 44 & 52 \\
33 & 38 & 43 & 48 & 53 \\
42 & 46 & 49 & 55 \\
50 & 57 & 58 \\
59 & 60 \\
\end{array}
$$

First we have to apply SPLIT 1 to the first row. If we perform jeu de taquin with 54 and with respect to the cells on the main diagonal, the entry ends in cell $(7, 11)$.

$$
T = \begin{array}{cccccccccc}
63 & 64 & 61 & 41 & 34 & 20 & 62 & 65 & 56 & 15 & 24 \\
1 & 2 & 3 & 4 & 7 & 8 & 13 & 16 & 18 & 45 \\
5 & 6 & 9 & 10 & 17 & 19 & 30 & 31 & 47 \\
11 & 12 & 21 & 23 & 26 & 32 & 37 & 51 \\
14 & 22 & 25 & 27 & 36 & 39 & 52 \\
28 & 29 & 35 & 40 & 44 & 53 \\
33 & 38 & 43 & 48 & 54 \\
42 & 46 & 49 & 55 \\
50 & 57 & 58 \\
59 & 60 \\
\end{array}
$$

Thus we set $H_{1,11} = (7, 11)$ and therefore the first row of the shifted hook tabloid is

$$
H_1 = ((-, -), (\_ , -), (\_ , -), (-, -), (-, -), (-, -), (-, -), (-, -), (-, -), (7, 11)).
$$

The entry 15 is stable, thus the shifted tabloid does not change in the next step and we set $H_{1,10} = (1, 10)$, the cell of 15.

$$
H_1 = ((-, -), (\_ , -), (\_ , -), (-, -), (-, -), (-, -), (-, -), (1, 10), (7, 11))
$$
Next we perform jeu de taquin with 56.

\[
\begin{array}{ccccccccccccc}
63 & 64 & 61 & 41 & 34 & 20 & 62 & 65 & 15 & 18 & 24 \\
1 & 2 & 3 & 4 & 7 & 8 & 13 & 16 & 31 & 45 \\
5 & 6 & 9 & 10 & 17 & 19 & 30 & 37 & 47 \\
11 & 12 & 21 & 23 & 26 & 32 & 39 & 51 \\
14 & 22 & 25 & 27 & 36 & 44 & 52 \\
\end{array}
\]

\[
T =
\begin{array}{ccccccccccccc}
28 & 29 & 35 & 40 & 48 & 53 \\
33 & 38 & 43 & 49 & 54 \\
42 & 46 & 55 & 56 \\
50 & 57 & 58 \\
59 & 60 \\
\end{array}
\]

The forward path ends in cell (8,11) and therefore we want to set \( H_{1,11} = (8,11) \). However, \( H_{1,11} \) is occupied, thus we perform a shift from (1,9) to (1,11).

\[
H_1 = ((-, -), (-, -), (-, -), (-, -), (-, -), (-, -), (-, -), (1,10), (6,10), (8,11))
\]

If we perform jeu de taquin with 65 and with respect to the cells on the main diagonal, the entry gets stuck in (9,9) and remains unstable there.

\[
\begin{array}{ccccccccccccc}
63 & 64 & 61 & 41 & 34 & 20 & 62 & 13 & 15 & 18 & 24 \\
1 & 2 & 3 & 4 & 7 & 8 & 16 & 30 & 31 & 45 \\
5 & 6 & 9 & 10 & 17 & 19 & 32 & 37 & 47 \\
11 & 12 & 21 & 23 & 26 & 36 & 39 & 51 \\
14 & 22 & 25 & 27 & 40 & 44 & 52 \\
\end{array}
\]

\[
T =
\begin{array}{ccccccccccccc}
28 & 29 & 35 & 40 & 48 & 53 \\
33 & 38 & 43 & 49 & 54 \\
42 & 50 & 55 & 56 \\
65 & 57 & 58 \\
59 & 60 \\
\end{array}
\]

Again we want to set \( H_{1,9} = (9,9) \), but \( H_{1,9} \) is occupied and therefore we have to perform a shift.

\[
H_1 = ((-, -), (-, -), (-, -), (-, -), (-, -), (-, -), (1,10), (9,9), (6,10), (8,11))
\]

Next we perform jeu de taquin with 62 and this entry again gets stuck in a cell on the main diagonal. Then we perform jeu de taquin with 20, 34, 41, 61, 64 and finally with 63. These entries either end in a stable position or on the main diagonal. Simultaneously we built up the first row of the shifted hook tabloid. There we put down the endcells of the paths in the appropriate columns and whenever we want to use a cell of the shifted hook tabloid which is already occupied we perform the appropriate shift. This procedure results in

\[
\begin{array}{ccccccccccccc}
63 & 1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 \\
64 & 5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 45 \\
61 & 11 & 12 & 17 & 20 & 26 & 32 & 37 & 47 \\
41 & 14 & 21 & 23 & 27 & 36 & 39 & 51 \\
34 & 22 & 25 & 35 & 40 & 44 & 52 \\
\end{array}
\]

\[
T =
\begin{array}{ccccccccccccc}
28 & 29 & 38 & 43 & 48 & 53 \\
33 & 42 & 46 & 49 & 54 \\
62 & 50 & 55 & 56 \\
65 & 57 & 58 \\
59 & 60 \\
\end{array}
\]
and
\[ H_1 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,10),(3,7),(8,8),(9,9),(6,10),(8,11)\}. \]

We apply SPLIT 2 to this pair. Observe that \( i' = 5 \) since \( H_{1,j} = (j,j) \) for \( 1 \leq j \leq 5 \) and 63, 64, 61 and 41 are unstable. We perform jeu de taquin with 34, 41, 61, 64 and 63 in that order and with respect to the relative next row.

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 63 \\
11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
14 & 21 & 23 & 27 & 35 & 61 & 39 & 51 \\
22 & 25 & 29 & 41 & 40 & 44 & 52 \\
28 & 34 & 38 & 43 & 48 & 53 \\
33 & 42 & 46 & 49 & 54 \\
62 & 50 & 55 & 56 \\
65 & 57 & 58 \\
59 & 60 \\
\end{array}
\]

\[ U = \]

We observe that every entry changes row and therefore \( h \) does not exist. Consequently we are in Case 2 of SPLIT 2. We set \( T = U \). Since 34 is in cell (6,7), we set \( H_{1,7} = (6,7) \) after performing a shift, 41 is in cell (5,8) and therefore \( H_{1,4} = (5,8) \), 61 is in cell (4,9) and therefore \( H_{1,3} = (4,9) \), 64 is in cell (3,10) and therefore \( H_{1,2} = (3,10) \) and 63 is in cell (2,11) and therefore \( H_{1,1} = (2,11) \). Thus

\[ H_1 = \{(2,11),(3,10),(4,9),(5,8),(1,10),(2,6),(6,7),(8,8),(9,9),(6,10),(8,11)\}. \]

Finally we apply SPLIT 3. Observe that \( C' = \{34,41,61,64,63\} \) and \( C = \{62,65\} \). In the first step of SPLIT 3 \( e = 34 \) and \( e' = 62 \). The forward path of \( e \) ends left of the column of \( e' \) and therefore we are in Case 1. We obtain

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 63 \\
11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
14 & 21 & 23 & 27 & 35 & 61 & 39 & 51 \\
22 & 25 & 29 & 41 & 40 & 44 & 52 \\
28 & 33 & 38 & 43 & 48 & 53 \\
34 & 42 & 46 & 49 & 54 \\
62 & 50 & 55 & 56 \\
65 & 57 & 58 \\
59 & 60 \\
\end{array}
\]

\[ T = \]

and

\[ H_1 = \{(2,11),(3,10),(4,9),(5,8),(1,10),(2,6),(7,7),(8,8),(9,9),(6,10),(8,11)\}. \]

We delete 34 from \( C' \): \( C' = \{41,61,64,63\} \) and \( C = \{62,65\} \).

In the next step of SPLIT 3 we have \( e = 41 \) and \( e' = 62 \). Since the forward path of 41 ends weakly right of the column of 62 but does not contain (7,7), we are in Case 3. Here we do not change \( T \) or \( H \), we only delete 62 from \( C \) and put it into \( C' \): \( C' = \{62,41,61,64,63\} \) and \( C = \{65\} \).

Now \( e = 62 \) and \( e' = 65 \) and we are in Case 2 for the forward path of 62 contains (8,8) and (8,9). Therefore we want to change 62 into a vertical candidate by setting \( H_{1,7} = (8,8) \), but since \( H_{1,7} \) is occupied we perform a transfer from (1,7) to (1,8).

\[ H_1 = \{(2,11),(3,10),(4,9),(5,8),(1,10),(2,6),(8,8),(8,8),(9,9),(6,10),(8,11)\}. \]
The shifted tabloid $T$ does not change. Furthermore $C' = \{65, 62, 41, 61, 64, 63\}$, $C = \emptyset$ and we are always in Case 1 for the rest of the application of SPLIT 3.

Next $e = 65$ and its forward path ends in $(10, 11)$. Therefore we want to set $H_{1, 11} = (10, 11)$ and since $H_{1, 11}$ is occupied we perform a shift from the previous pointer $(1, 9)$ of 65 to $(1, 11)$. We obtain

\[
T = \begin{array}{ccccccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 63 \\
11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
14 & 21 & 23 & 27 & 35 & 61 & 39 & 51 \\
22 & 25 & 29 & 40 & 44 & 52 \\
\end{array}
\]

\[
H_1 = ((2, 11), (3, 10), (4, 9), (5, 8), (1, 10), (2, 6), (8, 8), (8, 8), (5, 9), (7, 10), (10, 11))
\]

and $C' = \{62, 41, 61, 64, 63\}$.

The forward path of $e = 62$ ends in $(9, 11)$. Therefore we want to set $H_{1, 8} = (9, 11)$ and since $H_{1, 8}$ is occupied we perform a shift from $(1, 7)$ to $(1, 8)$. This yields

\[
T = \begin{array}{ccccccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 63 \\
11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
14 & 21 & 23 & 27 & 35 & 61 & 39 & 51 \\
22 & 25 & 29 & 40 & 44 & 52 \\
\end{array}
\]

\[
H_1 = ((1, 10), (3, 10), (4, 9), (5, 8), (1, 10), (2, 6), (8, 8), (8, 8), (5, 9), (7, 10), (10, 11))
\]

and $C' = \{41, 61, 64, 63\}$.

In the next step $e = 41$ etc. . . .

Finally we obtain the following shifted standard tableau

\[
T = \begin{array}{ccccccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 47 \\
11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
14 & 21 & 23 & 27 & 35 & 39 & 48 & 51 \\
22 & 25 & 29 & 40 & 44 & 52 \\
\end{array}
\]

\[
H_1 = ((1, 10), (3, 5), (1, 5), (4, 4), (6, 9), (7, 9), (8, 11), (9, 11), (5, 9), (7, 10), (10, 11))
\]

and the following first row of $H$

\[
H_1 = ((1, 10), (3, 5), (1, 5), (4, 4), (6, 9), (7, 9), (8, 11), (9, 11), (5, 9), (7, 10), (10, 11))
\]
Example 3. We consider

\[
T = \begin{pmatrix}
64 & 35 & 8 & 12 & 36 & 49 & 51 & 1 & 34 & 63 & 54 \\
2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
17 & 18 & 24 & 28 & 29 & 37 & 41 \\
22 & 23 & 26 & 33 & 42 & 43 & 45 \\
32 & 38 & 40 & 46 & 47 & 53 \\
39 & 44 & 50 & 52 & 58 \\
48 & 55 & 56 & 61 \\
57 & 59 & 62 \\
60 & 65
\end{pmatrix}
\]

After the application of SPLIT 1 we obtain the pair

\[
T = \begin{pmatrix}
64 & 1 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
2 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
35 & 8 & 12 & 19 & 24 & 28 & 29 & 37 & 41 \\
17 & 18 & 23 & 26 & 33 & 34 & 43 & 45 \\
22 & 32 & 38 & 40 & 42 & 47 & 53 \\
36 & 39 & 44 & 46 & 52 & 54 \\
49 & 48 & 50 & 56 & 58 \\
51 & 55 & 59 & 61 \\
57 & 60 & 62 \\
63 & 65
\end{pmatrix}
\]

\[H_1 = ((1,1), (1,8), (3,3), (3,4), (3,5), (6,6), (7,7), (8,8), (4,9), (10,10), (6,11)).\]

Since \(i' = i\) we are in Case 2 of SPLIT 2. We move 64 to the second row

\[
T = \begin{pmatrix}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
64 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
35 & 8 & 12 & 19 & 24 & 28 & 29 & 37 & 41 \\
17 & 18 & 23 & 26 & 33 & 34 & 43 & 45 \\
22 & 32 & 38 & 40 & 42 & 47 & 53 \\
36 & 39 & 44 & 46 & 52 & 54 \\
49 & 48 & 50 & 56 & 58 \\
51 & 55 & 59 & 61 \\
57 & 60 & 62 \\
63 & 65
\end{pmatrix}
\]

and set \(H_{1,2} = (2,2)\) after performing the appropriate shift

\[H_1 = ((1,8), (2,2), (3,3), (3,4), (3,5), (6,6), (7,7), (8,8), (4,9), (10,10), (6,11)).\]

We set \(C' = \{64\}\) and \(C = \{35, 36, 49, 51, 63\}\). In the first step of SPLIT 3 we have \(e = 64\) and \(e' = 35\). Since the forward path of 64 includes (2,2) and (2,3) we are in Case 2 of SPLIT 3. The shifted tabloid \(T\) does not change, but we want to set \(H_{1,1} = (2,2)\) and therefore perform a transfer from (1,1) to (1,2) in \(H\).

\[H_1 = ((2,2), (1,8), (3,3), (3,4), (3,5), (6,6), (7,7), (8,8), (4,9), (10,10), (6,11))\]

Furthermore \(C' = \{35, 64\}\) and \(C = \{36, 49, 51, 63\}\).
Next $e = 35$ and $e' = 36$. Again we are in Case 2.

\[
T = \begin{array}{cccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
64 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
8 & 12 & 18 & 19 & 24 & 28 & 29 & 37 & 41 \\
17 & 22 & 23 & 26 & 33 & 34 & 43 & 45 \\
35 & 32 & 38 & 40 & 42 & 47 & 53 \\
36 & 39 & 44 & 46 & 52 & 54 \\
49 & 48 & 50 & 56 & 58 \\
51 & 55 & 59 & 61 \\
57 & 60 & 62 \\
63 & 65 \\
\end{array}
\]

We set $H_{1,4} = (5,5)$ after performing an appropriate shift and a transfer from $(1,4)$ to $(1,5)$.

$$H_1 = ((2,2),(1,8),(2,3),(5,5),(3,5),(6,6),(7,7),(8,8),(4,9),(10,10),(6,11))$$

We update $C' = \{36,35,64\}$ and $C = \{49,51,63\}$.

Next $e = 36$ and $e' = 49$ and since 36 is stable we are in Case 1. Thus $T$ and $H$ do not change, but $C' = \{35,64\}$ and $C = \{49,51,63\}$.

Therefore $e = 35$ and $e' = 49$ and we are in Case 1.

\[
T = \begin{array}{cccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
64 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
8 & 12 & 18 & 19 & 24 & 28 & 29 & 37 & 41 \\
17 & 22 & 23 & 26 & 33 & 34 & 43 & 45 \\
32 & 35 & 38 & 40 & 42 & 47 & 53 \\
36 & 39 & 44 & 46 & 52 & 54 \\
49 & 48 & 50 & 56 & 58 \\
51 & 55 & 59 & 61 \\
57 & 60 & 62 \\
63 & 65 \\
\end{array}
\]

Moreover $C' = \{64\}$ and $C = \{49,51,63\}$.

In the next step $e = 64$ and $e' = 49$. Because of the run of the forward path of 64 we are in Case 3. Thus $T$ and $H$ do not change, only $C' = \{49,64\}$ and $C = \{51,63\}$.

Therefore $e = 49$ and $e' = 51$. We are in Case 2 and $T$ does not change. We want to set $H_{1,6} = (7,7)$. Since $H_{1,6}$ is occupied by a pointer which does not point to 49, we perform a transfer from $(1,6)$ to $(1,7)$.

$$H_1 = ((2,2),(1,8),(2,3),(5,6),(3,5),(6,6),(7,7),(8,8),(4,9),(10,10),(6,11))$$

Moreover $C' = \{51,49,64\}$ and $C = \{63\}$.

Next $e = 51$ and $e' = 63$. We are in Case 1 for 51 is stable and thus $C' = \{49,64\}$ and $C = \{63\}$. 

$$H_1 = ((2,2),(1,8),(2,3),(5,6),(3,5),(7,7),(7,7),(8,8),(4,9),(10,10),(6,11))$$

Moreover $C' = \{51,49,64\}$ and $C = \{63\}$. 

Next $e = 51$ and $e' = 63$. We are in Case 1 for 51 is stable and thus $C' = \{49,64\}$ and $C = \{63\}$.
Therefore $e = 49$ and $e' = 63$. Again we are in Case 1.

\[
T = \begin{array}{cccccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
64 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
8 & 12 & 18 & 19 & 24 & 28 & 29 & 37 & 41 \\
17 & 22 & 23 & 26 & 33 & 34 & 43 & 45 \\
32 & 35 & 38 & 40 & 42 & 47 & 53 \\
36 & 39 & 44 & 46 & 52 & 54 \\
48 & 49 & 50 & 56 & 58 \\
51 & 55 & 59 & 61 \\
57 & 60 & 62 \\
63 & 65
\end{array}
\]

$H_1 = ((2, 2), (1, 8), (2, 3), (5, 6), (3, 5), (7, 8), (7, 7), (8, 8), (4, 9), (10, 10), (6, 11))$

Moreover $C' = \{64\}$ and $C = \{63\}$.

Now we have $e = 64$ and $e = 63$. We are in Case 2.

\[
T = \begin{array}{cccccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
4 & 7 & 9 & 11 & 16 & 21 & 25 & 29 & 30 & 31 \\
8 & 12 & 18 & 19 & 24 & 28 & 34 & 37 & 41 \\
17 & 22 & 23 & 26 & 33 & 42 & 43 & 45 \\
32 & 35 & 38 & 40 & 46 & 47 & 53 \\
36 & 39 & 44 & 50 & 52 & 54 \\
48 & 49 & 55 & 56 & 58 \\
51 & 57 & 59 & 61 \\
64 & 60 & 62 \\
63 & 65
\end{array}
\]

Since 64 was already a vertical candidate we want to set $H_{1,9} = (10, 10)$. For that purpose we perform a transfer from $(1, 9)$ to $(1, 10)$.

$H_1 = ((1, 8), (1, 2), (4, 5), (2, 4), (6, 7), (6, 8), (7, 7), (9, 9), (10, 10), (5, 10), (6, 11))$

We update $C' = \{63, 64\}$ and $C = \emptyset$. Now 63 is stable and $C' = \{64\}$ after the next step. We terminate with the pair

\[
T = \begin{array}{cccccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
4 & 7 & 9 & 11 & 16 & 21 & 25 & 29 & 30 & 31 \\
8 & 12 & 18 & 19 & 24 & 28 & 34 & 37 & 41 \\
17 & 22 & 23 & 26 & 33 & 42 & 43 & 45 \\
32 & 35 & 38 & 40 & 46 & 47 & 53 \\
36 & 39 & 44 & 50 & 52 & 54 \\
48 & 49 & 55 & 56 & 58 \\
51 & 57 & 59 & 61 \\
60 & 62 & 64 \\
63 & 65
\end{array}
\]

$H_1 = ((1, 8), (1, 2), (4, 5), (2, 4), (6, 7), (6, 8), (7, 7), (9, 11), (10, 10), (5, 10), (6, 11))$. 

Example 4. Let

\[
T = \begin{array}{cccccccccccccccc}
22 & 15 & 37 & 21 & 44 & 11 & 14 & 13 & 34 \\
1 & 2 & 4 & 5 & 7 & 10 & 17 & 19 & 28 & 32 \\
3 & 6 & 8 & 9 & 23 & 27 & 31 & 38 & 41 \\
12 & 16 & 18 & 24 & 29 & 36 & 40 & 53 \\
20 & 25 & 26 & 30 & 46 & 47 & 54 \\
33 & 35 & 42 & 48 & 50 & 57 \\
39 & 43 & 49 & 51 & 60 \\
45 & 52 & 58 & 63 \\
56 & 59 & 64 \\
61 & 65 \\
\end{array}
\]

After the application of SPLIT 1 we obtain

\[
T = \begin{array}{cccccccccccccccc}
22 & 1 & 2 & 4 & 5 & 7 & 10 & 13 & 14 & 28 & 32 \\
15 & 3 & 6 & 8 & 9 & 17 & 19 & 31 & 34 & 41 \\
37 & 11 & 16 & 18 & 23 & 27 & 36 & 38 & 53 \\
12 & 20 & 24 & 26 & 29 & 40 & 47 & 54 \\
21 & 25 & 30 & 42 & 46 & 50 & 55 \\
33 & 35 & 43 & 48 & 51 & 57 \\
39 & 44 & 49 & 58 & 60 \\
45 & 52 & 59 & 63 \\
56 & 61 & 64 \\
62 & 65 \\
\end{array}
\]

and the first row of the shifted hook tabloid is

\[
H_1 = ((1, 1), (2, 2), (3, 3), (1, 4), (5, 5), (1, 10), (1, 9), (7, 8), (1, 10), (10, 10), (6, 11)) .
\]

We apply SPLIT 2 to the pair. Observe that \(i' = 3\) since \(H_{1,j} = (j, j)\) for \(1 \leq j \leq 3\) and the entries 22, 15 are unstable.

If we perform jeu de taquin with 37, 15 and 22 and with respect to the relative next row we observe that 22 does not change row and thus \(h = 1\). Therefore we are in Case 1 of SPLIT 2. We perform jeu de taquin with 37, 15 and 22 in that order and with respect to the last cell in the current row of the forward path in the course of constructing \(U\). This results in

\[
T = \begin{array}{cccccccccccccccc}
1 & 2 & 4 & 5 & 7 & 10 & 13 & 14 & 22 & 28 & 32 \\
3 & 6 & 8 & 9 & 15 & 17 & 19 & 31 & 34 & 41 \\
11 & 37 & 16 & 18 & 23 & 27 & 36 & 38 & 53 \\
12 & 20 & 24 & 26 & 29 & 40 & 47 & 54 \\
21 & 25 & 30 & 42 & 46 & 50 & 55 \\
33 & 35 & 43 & 48 & 51 & 57 \\
39 & 44 & 49 & 58 & 60 \\
45 & 52 & 59 & 63 \\
56 & 61 & 64 \\
62 & 65 \\
\end{array}
\]

Since 37 moved to cell (3, 4) we set \(H_{1,2} = (3, 4)\), since 15 moved to (2, 6) we set \(H_{1,1} = (2, 6)\) and since 22 moved to (1, 9) we set \(H_{1,3} = (1, 9)\).

\[
H_1 = ((2, 6), (3, 4), (1, 9), (1, 4), (5, 5), (1, 10), (1, 9), (7, 8), (1, 10), (10, 10), (6, 11))
\]
After the application of SPLIT 3 we obtain

\[
T = \begin{array}{cccccccccccc}
1 & 2 & 4 & 5 & 7 & 10 & 13 & 14 & 22 & 28 & 32 \\
3 & 6 & 8 & 9 & 15 & 17 & 19 & 31 & 34 & 41 \\
11 & 12 & 16 & 18 & 23 & 27 & 36 & 38 & 53 \\
20 & 21 & 24 & 26 & 29 & 40 & 47 & 54 \\
25 & 30 & 35 & 42 & 46 & 50 & 55 \\
33 & 37 & 43 & 48 & 51 & 57 \\
39 & 44 & 49 & 58 & 60 \\
45 & 52 & 59 & 63 \\
56 & 61 & 64 \\
62 & 65 \\
\end{array}
\]

and

\[
H_1 = ((2,6), (1,9), (4,4), (1,4), (6,7), (1,10), (1,9), (7,8), (1,10), (10,10), (6,11)) .
\]

Example 5. Let

\[
T = \begin{array}{cccccccccccc}
20 & 9 & 64 & 7 & 33 & 54 & 65 & 13 & 61 & 50 & 46 \\
1 & 2 & 4 & 8 & 11 & 14 & 16 & 22 & 24 & 37 \\
3 & 5 & 10 & 12 & 18 & 23 & 26 & 29 & 41 \\
6 & 15 & 17 & 21 & 25 & 30 & 36 & 43 \\
19 & 27 & 28 & 34 & 35 & 39 & 53 \\
31 & 32 & 38 & 42 & 44 & 55 \\
40 & 45 & 48 & 51 & 58 \\
47 & 49 & 52 & 59 \\
56 & 57 & 62 \\
60 & 63 \\
\end{array}
\]

After the application of SPLIT 1 we obtain the pair

\[
T = \begin{array}{cccccccccccc}
20 & 1 & 2 & 4 & 8 & 11 & 13 & 16 & 22 & 24 & 37 \\
9 & 3 & 5 & 10 & 12 & 14 & 23 & 26 & 29 & 41 \\
64 & 6 & 15 & 17 & 18 & 25 & 30 & 36 & 43 \\
7 & 19 & 21 & 28 & 34 & 35 & 39 & 46 \\
33 & 27 & 32 & 38 & 42 & 44 & 53 \\
31 & 40 & 45 & 48 & 50 & 55 \\
54 & 47 & 49 & 51 & 58 \\
65 & 52 & 57 & 59 \\
56 & 60 & 62 \\
61 & 63 \\
\end{array}
\]

and

\[
H_1 = ((1,1), (2,2), (3,3), (4,4), (5,5), (1,8), (7,7), (8,8), (5,9), (10,10), (4,11)) .
\]
Next we apply SPLIT 2. Observe that $i' = 4$ (since 7 is stable) and that

\[
\begin{array}{cccccccccccc}
1 & 2 & 4 & 7 & 8 & 11 & 13 & 16 & 22 & 24 & 37 \\
3 & 5 & 20 & 10 & 12 & 14 & 23 & 26 & 29 & 41 \\
6 & 9 & 15 & 17 & 18 & 25 & 30 & 36 & 43 \\
64 & 19 & 21 & 28 & 34 & 35 & 39 & 46 \\
33 & 27 & 32 & 38 & 42 & 44 & 53 \\
31 & 40 & 45 & 48 & 50 & 55 . \\
54 & 47 & 49 & 51 & 58 \\
65 & 52 & 57 & 59 \\
56 & 60 & 62 \\
61 & 63
\end{array}
\]

Thus $h = 4$ and no backward paths with respect to the 4-th row of a horizontal candidate strictly below the 4-th row ends weakly left of 64. Therefore we are in Case 1 of SPLIT 2. We set $U = T$. Since 7 original ended in (4, 4) we set $H_{1,4} = (4, 4)$, since 64 is in (4, 4) we set $H_{1,3} = (4, 4)$, since 9 is in (3, 4) we set $H_{1,2} = (3, 4)$ and since 20 is in (2, 4) we set $H_{1,1} = (2, 4)$.

\[
H_1 = ((2, 4), (3, 4), (4, 4), (5, 5), (1, 8), (7, 7), (8, 8), (5, 9), (10, 10), (4, 11))
\]

After the performance of SPLIT 3 we obtain the pair

\[
\begin{array}{cccccccccccc}
1 & 2 & 4 & 7 & 8 & 11 & 13 & 16 & 22 & 24 & 37 \\
3 & 5 & 9 & 10 & 12 & 14 & 23 & 26 & 29 & 41 \\
6 & 15 & 17 & 18 & 20 & 25 & 30 & 36 & 43 \\
19 & 21 & 28 & 32 & 34 & 35 & 39 & 46 \\
27 & 31 & 38 & 42 & 44 & 50 & 53 \\
33 & 40 & 45 & 48 & 54 & 55 \\
47 & 49 & 51 & 57 & 58 \\
52 & 56 & 59 & 62 \\
60 & 61 & 64 \\
63 & 65
\end{array}
\]

and

\[
H_1 = ((2, 3), (3, 7), (3, 3), (5, 5), (6, 9), (1, 8), (3, 7), (9, 11), (10, 11), (10, 10), (4, 11)) .
\]

Example 6. We consider

\[
\begin{array}{cccccccccccc}
26 & 4 & 13 & 31 & 47 & 24 & 58 & 65 & 53 & 25 & 60 \\
1 & 2 & 3 & 6 & 8 & 9 & 11 & 14 & 22 & 29 \\
5 & 7 & 10 & 16 & 17 & 20 & 23 & 30 & 36 \\
12 & 15 & 18 & 19 & 21 & 33 & 41 & 43 \\
27 & 28 & 32 & 37 & 38 & 42 & 49 \\
34 & 35 & 39 & 40 & 46 & 51 . \\
44 & 45 & 48 & 52 & 54 \\
50 & 55 & 57 & 61 \\
56 & 59 & 62 \\
63 & 64
\end{array}
\]
After the application of SPLIT 1 we obtain

\[
\begin{array}{cccccccccccccccc}
26 & 1 & 2 & 3 & 6 & 8 & 9 & 11 & 14 & 22 & 29 \\
4 & 5 & 7 & 10 & 16 & 17 & 20 & 23 & 25 & 36 \\
13 & 12 & 15 & 18 & 19 & 21 & 30 & 41 & 43 \\
31 & 24 & 28 & 32 & 33 & 38 & 42 & 49 \\
& 25 & 34 & 35 & 37 & 40 & 46 & 51 \\
& 47 & 39 & 45 & 48 & 52 & 54 \\
& 44 & 50 & 53 & 57 & 60 \\
& 58 & 55 & 59 & 61 \\
& 56 & 62 & 64 \\
& 63 & 65 \\
\end{array}
\]

\[T = \]

and

\[H_1 = ((1, 1), (2, 2), (3, 3), (4, 4), (3, 5), (6, 6), (1, 9), (8, 8), (6, 9), (6, 10), (10, 11)).\]

Next we apply SPLIT 2. Observe that \(i' = 2\) and that

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 6 & 8 & 9 & 11 & 14 & 22 & 25 & 29 \\
4 & 5 & 7 & 10 & 16 & 17 & 20 & 23 & 26 & 36 \\
13 & 12 & 15 & 18 & 19 & 21 & 30 & 41 & 43 \\
31 & 24 & 28 & 32 & 33 & 38 & 42 & 49 \\
& 27 & 34 & 35 & 37 & 40 & 46 & 51 \\
& 47 & 39 & 45 & 48 & 52 & 54 \\
& 44 & 50 & 53 & 57 & 60 \\
& 58 & 55 & 59 & 61 \\
& 56 & 62 & 64 \\
& 63 & 65 \\
\end{array}
\]

\[U = \]

Thus \(h = 2\) and the backward path with respect to the 2-nd row of the horizontal candidate 13 ends weakly left of 26. We are in Case 2 of SPLIT 2 and set \(T = U\) and

\[H_1 = ((2, 10), (2, 2), (3, 3), (4, 4), (3, 5), (6, 6), (1, 9), (8, 8), (6, 9), (6, 10), (10, 11)).\]

After the application of SPLIT 3 we finally obtain

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 6 & 8 & 9 & 11 & 14 & 22 & 25 & 29 \\
4 & 5 & 7 & 10 & 16 & 17 & 20 & 23 & 26 & 36 \\
12 & 13 & 15 & 18 & 19 & 21 & 30 & 41 & 43 \\
24 & 27 & 28 & 32 & 33 & 38 & 42 & 49 \\
31 & 34 & 35 & 37 & 40 & 46 & 51 \\
& 39 & 44 & 45 & 48 & 52 & 54 \\
& 47 & 50 & 53 & 57 & 60 \\
& 55 & 56 & 59 & 61 \\
& 58 & 62 & 64 \\
& 63 & 65 \\
\end{array}
\]

\[T = \]

and

\[H_1 = ((2, 10), (3, 4), (3, 3), (2, 4), (5, 5), (1, 9), (7, 7), (5, 8), (9, 9), (6, 10), (10, 11)).\]
an example for a reshift from (1, λ) shifted Ferrers diagram of shape λ shifted hook tabloid for (1, λ) if (1, λ) encounters. However, in the following two situations the reshift is indeed a partial shifted hook tabloid H if H ≤ i. If H_{i,k-1} = (i', k-1) or H_{i,k-1} = (k,j') then set H'_{i,k} = (i'+1,k) or H'_{i,k} = (k+1,j'+1), respectively. If H_{i,k-1} = (i,l), l ≥ k, then set H'_{i,k} = (i,l). The cell (i,j) in H' remains empty. We denote H' by RSHIFT(H, i, j', j). For an example for a reshift from (1, 6) to (1, 2) read Figure 10 from right to left.

Observe that SHIFT and RSHIFT are inverse to each other in the following sense: Let (i, j, j') be a shifted tabloid and (i, j, j') be a partial shifted hook tabloid in which only cell (i, j) is empty. Then

\[ H = \text{RSHIFT(SHIFT(H, i, j', j)), i, j', j}. \]

If H' is a partial shifted hook tabloid where only cell (i, j') is empty then

\[ H' = \text{SHIFT(RSHIFT(H, i, j', j), i, j, j')}. \]

In general the output of a reshift in a partial shifted hook tabloid need not to be a partial shifted hook tabloid for (i' + 1, k), respectively (k + 1, j' + 1), need not to be a cell in the shifted Ferrers diagram of shape λ if (i', k - 1), respectively (k, j'), is a cell in this diagram. However, in the following two situations the reshift is indeed a partial shifted hook tabloid and we apply the reshift only if one of these situations encounters.

1. Let T be a shifted tableau and H an accompanying partial shifted hook tabloid. Let e be a vertical candidate in T with respect to H and the i-th row (meaning that the pointer to e is in the i-th row of H) and perform reverse jeu de taquin (i being the fixed row) with e in T and with respect to a set D. Let (g, j) denote the cell of e in the output tabloid and (g', j') denote the cell of e in T. Suppose that e is the maximal vertical candidate in T under the vertical candidates in the rows h, g ≤ h ≤ g', with respect to the backward paths order of T. Then the reshift from cell (i, g' - 1) to cell (i, g - 1) in H produces another partial shifted hook tabloid.

2. Again let T be a shifted tableau and H an accompanying partial shifted hook tabloid. Furthermore we assume in this situation that there exists no vertical candidate in T with respect to H. Let e be a horizontal candidate in T with respect to H and the i-th row and perform reverse jeu de taquin (i being the fixed row) with e in T and with respect to a set D. Let (g, j) denote the cell of e in the output tabloid and (g', j') denote the cell of e in T. Suppose that e is the smallest horizontal candidate in T under the horizontal candidates in the columns k, j ≤ k ≤ j', with respect to the backward paths order of T. Then the reshift from cell (i, j') to cell (i, j) in H produces another partial shifted hook tabloid.

(In order to see that use Lemma 1.)

As jeu de taquin in SPLIT is mostly followed by a shift in the accompanying shifted hook tabloid, reverse jeu de taquin in MERGE is mostly followed by a reshift. Let T be a shifted tableau, H a shifted hook tabloid, e a candidate and D a set of cells. If e is a horizontal candidate then (T', H') = RJS_D(T, H, e) is obtained as follows: T' = RJT_D(T, e);
$H' = \text{RSHIFT}(H, i, q, j')$ and $H'_{i,j'} = (i', j')$, where $q$ is the column of $e$ in $T$ and $c_T(e) = (i', j')$. If $e$ is a vertical candidate then $(T', H') = \text{RJS}_D(T, H, e)$ is obtained as follows: $T' = \text{RJT}_D(T, e)$, $H' = \text{RSHIFT}(H, i, p - 1, i' - 1)$ and $H'_{i,i'-1} = (i', j')$, where $p$ is the row of $e$ in $T$ and $c_T'(e) = (i', j')$. Observe that JS and RJS are each other’s respective inverses.

A retransfer from cell $(i, k)$ to cell $(i, j)$ in a partial shifted hook tabloid. Let $j \leq k \leq r$ and $H_{i,k} = (i', k')$ with either $i' = i$ or $k' = k$. We define the term retransfer from cell $(i, k)$ to cell $(i, j)$ in $H$. The output of this operation is again another partial shifted hook tabloid $H'$ which coincides with $H$ except for the cells $(i, l)$, $j \leq l \leq k$. If $k' = k$ and $i' - i \geq k - j$ let $H'_{i,j} = (i' + j - k, j)$, otherwise $H'_{i,j} = (i, k' + i - i')$. For $j < l \leq k$ set $H'_{i,l} = (l, l)$. We denote $H'$ by $\text{RTRANS}(H, i, k, j)$. For an example of a retransfer from $(1, 6)$ to $(1, 2)$ read Figure 11 from right to left. Again, whenever we will apply a retransfer from $(i, k)$ to $(i, j)$ in $H$, we have $H_{i,l} = (l + 1, l + 1)$ for $j \leq l < k$ as in the example.

Observe that TRANS and RTRANS are inverse to each other in the following sense. Let $i \leq j \leq k \leq \lambda_i + i - 1$. If $H$ is a shifted hook tabloid such that $H_{i,j}$ is in the same row or column as $(i, j)$ and with $H_{i,l} = (l, l)$ for $j < l \leq k$ then

$$H = \text{RTRANS}(\text{TRANS}(H, i, j, k), i, k, j).$$

If $H'$ is a shifted hook tabloid such that $H'_{i,k}$ is in the same row or column as $(i, k)$ and with $H'_{i,l} = (l + 1, l + 1)$ for $j \leq l < k$ then

$$H' = \text{TRANS}(\text{RTRANS}(H', i, k, j), i, j, k).$$

Now we are in the position to state the Algorithm MERGE. It is divided into $r$ steps, where in the $i$-th step we perform reverse jeu de taquin with the entries in the $i$-th row. Immediately before we perform the $i$-th step of MERGE to a pair $(T, H)$, $T$ is ordered up to $(i, i)$, the first $(i - 1)$ rows of $H$ are empty and the last $r - i + 1$ rows form a shifted hook tabloid. Within a row $i$, MERGE is divided into 3 steps. The 3 parts of MERGE in a fixed row $i$ are numbered in reverse order to emphasize the connection between SPLIT $k$ and MERGE $k$, $k = 1, 2, 3$. All the horizontal and vertical candidates below are meant to be with respect to the $i$-th row, i.e. their pointers are in the $i$-th row of the current shifted hook tabloid. The marker '(*)' in MERGE 3 is needed in the proof that SPLIT 3 and MERGE 3 are inverse to each other in Subsection 6.6.
MERGE 3. Set $z = i$. Repeat the following until there exists no vertical candidate strictly below the $z$-th row.

[ Let $e$ be the vertical candidate strictly below the $z$-th row, which is maximal with respect to the backward paths order.

If $c_T(e) = (h, h)$: [ Let $h'$ be minimal such that $T_{j,j}$ is a vertical unstable candidate for $h' \leq j < h$. If $e = T_{h,h}$ is unstable and the backward path of smallest horizontal candidate $e'$ strictly below the $h$-th row contains $(h + 1, h + 1)$ set $(T, H) = \text{RJS}_{h+1}(T, H, e')$ (**) and $H = \text{RTRANS}(H, i, h, h') - 1$, otherwise set $H = \text{RTRANS}(H, i, h, h')$. Furthermore $e = T_{h', h'}$. ]

If $e$ is a vertical candidate repeat the following:

[ If $c_T(e) = (k, k)$, $k \neq z + 1$, set $(T, H) = \text{RJS}_{k+1}(T, H, e)$. Set $(T, H) = \text{RJS}_{MD,z+1}(T, H, e)$ until either

1. $e$ is in row $z + 1$,
2. there exists a vertical candidate strictly below the row of $e$,
3. $c_T(e) = (l, l)$ for an $l$ and the backward path of a horizontal candidate includes $(l + 1, l + 1)$,
4. $c_T(e) = (l, l)$ for an $l$ and $T_{l-1,l-1}$ is an unstable vertical candidate.

If $e$ is in row $z + 1$ set $z = z + 1$. ]

Observe that in case the if-condition (If $c_T(e) = (h, h): [...])$ at the beginning of a step of MERGE 3 is true, there exists no vertical candidate strictly below the $h$-th row by the maximality of $e$.

MERGE 2. Let $i''$ be such that for $i + 1 \leq h \leq i''$ there exists a vertical candidate entry in the $h$-th row and all other candidates are horizontal.

We distinguish between two cases. In Case 2 we consider the case when the smallest horizontal candidate strictly below the $i''$-th row is smaller than the smallest vertical candidate or $i'' = i$.

Case 1. [ Set $a = \text{True}$. Let $H_{i,i''} = (h, k)$. We define $T_{i,k-h+i}$ to be a vertical candidate in row $i$.

Repeat for $g = i$ to $g = i''$: [ If $a = \text{True}$ let $e$ be the vertical candidate in row $g$ or $\min(g + 1, i'')$ that is maximal with respect to the backward paths order. If $a = \text{False}$ let $e$ be the vertical candidate in row $g$. Let $T = \text{RJT}(T, e)$. If the backward path includes $(g + 1, g + 1)$ set $a = \text{False}$. If $e$ was the candidate in the $(g + 1)$-st row and $T_{g,k}$ is the vertical candidate in row $g$, then let $H_{i,g} = (g + 1, k + 1)$. ]

For $i \leq g \leq i''$, let $H_{i,g} = (g, g)$. ]

Case 2. [ Repeat for $g = i$ to $g = i'' - 1$: [ Let $e$ be the entry in cell $H_{i,g}$ of $T$ and set $T = \text{RJT}(T, e)$. ] Let $i \leq i' \leq i''$, be such that in the previous step no entry moved to $(i', i')$. Set $H = \text{RTRANS}(H, i, i'', i')$. For every $g$, $i \leq g < i'$, let $H_{i,g} = (g, g)$. ]

MERGE 1. Repeat for $j = i$ to $j = \lambda_i + i - 1$: [ Let $e$ be the minimal horizontal candidate with respect to the backward paths order in $T$. Set $(T, H) = \text{RJS}_{(i,j)}(T, H, e)$ and erase the entry in cell $(i, j)$ of $H$. ]

Let $T$ be a shifted tabloid which is ordered up to $(i, i)$ and $H$ a partial shifted hook tabloid, where no entry in the $i$-th row is empty. Let $(T', H')$ denote the pair we obtain after the
application of MERGE 3, MERGE 2 and MERGE 1 to the \(i\)-th row of \((T, H)\). We denote \(T' = \text{MERGE}(T, H, i)\) and note that \(H'\) is equal to \(H\) with the entries in the \(i\)-th row deleted.

We invite the reader to apply MERGE 3, MERGE 2 and MERGE 1 to the first rows of the output pairs in the examples in Section 4 in order to find out that these applications result in the input pairs.

We are finally in the position to formulate the Algorithm MERGE. The input is a shifted standard tableaux \(T\) of shape \(\lambda\) and a shifted hook tabloid \(H\) of shape \(\lambda\).

\[
\text{MERGE. Set } T' = T \text{ and } H' = H. \text{ Repeat for } i = 1 \text{ to } i = r: \quad [T' = \text{MERGE}(T', H', i) \text{ and delete the } i\text{-th row of } H'.]
\]

The output tabloid \(T'\) of the algorithm is denoted by \(\text{MERGE}(T, H)\). Note that the output tabloid \(H'\) is empty.

In the remaining section we aim to show the following theorem.

\textbf{Theorem 3.} Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)\) be a partition with distinct components and \(1 \leq i \leq r\).

1. Let \(T\) be a shifted tabloid of shape \(\lambda\) which is ordered up to \((i + 1, i + 1)\) and \(H\) a partial shifted hook tabloid of shape \(\lambda\), where the cells of the \(i\)-th row are empty. Let \(T' = \text{STAND\_SPLIT}(T, H, i)\) and \(H' = \text{HOOK\_SPLIT}(T, H, i)\). Then \(T = \text{MERGE}(T', H', i)\).

2. Let \(T\) be a shifted tabloid of shape \(\lambda\) which is ordered up to \((i, i)\) and \(H\) a partial shifted hook tabloid of shape \(\lambda\), where no cell in the \(i\)-th row is empty. Let \(T' = \text{MERGE}(T, H, i)\) and \(H'\) denote the partial shifted hook tabloid \(H\) with the entries in the \(i\)-th row deleted. Then \(T = \text{STAND\_SPLIT}(T', H', i)\) and \(H = \text{HOOK\_SPLIT}(T', H', i)\).

This theorem, once it is proved, shows that SPLIT is the desired algorithm, i.e. a bijection from \(T_\lambda\) onto \(S_\lambda \times H_\lambda\), for MERGE is its inverse.

In the following we fix a strict partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)\) and a row \(i, 1 \leq i \leq r\).

6. The Proof of Theorem 3

6.1. Variantes of SPLIT 1 and SPLIT 2

We modify SPLIT 1 and SPLIT 2 such that after this modification MERGE \(k\) is the inverse of SPLIT \(k, k = 1, 2, 3\). The original version of SPLIT and the modification of SPLIT, where we replace SPLIT \(k, k = 1, 2,\) by their modifications described below, are equivalent, since we only change the order of some commuting steps and swap the beginning of SPLIT 3 to the end of SPLIT 2.

We start with a modification of SPLIT 2. Consider the following Algorithm POST-SPLIT 2, which we apply to a pair to which we have applied SPLIT 2.
POST-SPLIT 2. We continue with Case \( k, k = 1, 2 \), if we were in Case \( k \) in SPLIT 2.

**Case 1.** If \( T_{i', j'} \) is a vertical candidate, let \( i'' < i', i'' \leq i' \), be maximal such that for \( i'' < j < i' \), \( T_{j,j} \) is unstable and \( T_{j+1,j+1} \) is a horizontal candidate and set \( H = \text{TRANS}(H, i, i', i'') \). Otherwise omit this step and set \( i'' = i' \).

**Case 2.** If \( T_{i'+1, i'' + 1} \) is a horizontal candidate, let \( i'' < i' + 1 \leq i'' \), be maximal such that for \( i' + 1 \leq j < i'' \), \( T_{j,j} \) is unstable, \( T_{j+1,j+1} \) is a horizontal candidate and set \( H = \text{TRANS}(H, i', i'', i'') \). Furthermore set \( (T, H) = J\text{S}_i(T, H, T_{i''+1,i''+1}) \), where \( (i'' + 1, k) \) was the last cell in the forward path of \( T_{i''+1,i''+1} \) in the \((i'' + 1)\)-st row. Otherwise omit this step and set \( i'' = i' \).

In both cases: \( U = T \). If we were in Case 2 and \( i'' \neq i' \) set \( U = J\text{T}_{g+2}(U, T_{i''+1,k}) \). Repeat for \( g = i'' \) down to \( g = i' + 1 \): [ Let \( e \) be the vertical candidate in row \( g \) of \( T \) and set \( U = J\text{T}_{g+1}(U, e) \). ] Reject \( U \).

Repeat for \( g = i'' \) down to \( g = i' + 1 \): [ Let \( e \) be the vertical candidate in row \( g \) of \( T \) and set \( (T, H) = J\text{S}_{(g,l)}(T, H, e) \), where \((g,l)\) was the last cell in the forward path of \( e \) in the \( g \)-th row in the shifted tabloid \( U \) in the previous step. (Note that the forward path of \( e \) ends in \((g,l)\).) ]

Again the following holds for \( (T, H) \) after the application of POST-SPLIT 2: We were in Case 2 of POST-SPLIT 2 if and only if either the smallest horizontal candidate strictly below the \( i'' \)-th row is smaller than every vertical candidate or no vertical candidate exists.

Furthermore observe that the application of SPLIT 2 and SPLIT 3 is equivalent to the application of SPLIT 2, POST-SPLIT 2 and SPLIT 3, for POST-SPLIT 2 is the beginning of SPLIT 3 roughly speaking.

**Touching from above.** Let \( (T, H) \) denote a pair to which we apply SPLIT 2, POST-SPLIT 2 and SPLIT 3. Assume that we are in Case 2 in a certain step of SPLIT 3. Then we say \( e \) touches \( e' \) from above and for \( h < j \leq h' \), \( T_{j-1,j-1} \) touches \( T_{j,j} \) from above. If \( e \) is a horizontal candidate before the application of Case 2 we say that \((h - 1, h - 1)\) is the place of change for \( e \) and \((j, j)\) is the place of change for \( T_{j,j} \), \( h \leq j < h' \). If \( e \) is already a vertical candidate before the application of Case 2 we say that \((j, j)\) is the place of change for \( T_{j,j} \), \( h \leq j \leq h' \). Furthermore: If we are in Case 1 of SPLIT 2 let \( T_{g,g}, i \leq g \leq i'' \), be the exceptional entries, where \( T \) denotes the shifted tabloid at the beginning of SPLIT 2. If we are in Case 2 of SPLIT 2 let \( T_{g,g}, i \leq g \leq i'' \), and \( g \neq i', i' + 1 \), be the exceptional entries, \( T \) again being the shifted tabloid at the beginning of SPLIT 2. If \( i' \neq i'' \) and we are in Case 2 of SPLIT 2 then the vertical candidate in row \( i' + 1 \) after the application of SPLIT 2 is defined to be an entry which touches from above with place of change \((i' + 1, i' + 1)\).

We replace SPLIT 1 by the following variante. Again the input is a shifted tabloid \( T \), which is ordered up to \((i + 1, i + 1)\) and a partial shifted hook tabloid \( H \), where the \( i \)-th row is empty.

**Variante of SPLIT 1.** Repeat for \( j = \lambda_i + i - 1 \) down to \( j = i \): We set \( (T, H) = J\text{S}_D(T, H, T_{i,j}) \), where the set \( D \) is defined as follows. If \( T_{i,j} \) is exceptional or touched from above let \( D = \text{MD} \). If \( T_{i,j} \) is neither exceptional nor touched from above but touches from above, let \( D = \{ \rho \} \), where \( \rho \) is its place of change. In all other cases \( D = \emptyset \).]

The change of SPLIT 1 forces us to replace SPLIT 2 by the following slight modification. This is because for a pair \((T, H)\) which falls into Case 2 in the original version of SPLIT 2,
the entry $T_{e',e}$ either moves to a stable position or to its place of change if we apply the 

**Variante of SPLIT 1.** Choose $i', i \leq i' \leq r$, maximal such that $H_{i,k} = (k,k)$ and $T_{k-1,k-1}$ is unstable (i.e. $T_{k-1,k-1} > T_{k-1,1}$) for $i < k \leq i'$. If $i' = i$ stop.

Set $U = T$. Repeat for $g = i'$ down to $g = i$: Set $U = JT_{g+1}(U, T_{g,g})$.

We distinguish between two cases. Let $h$ be minimal, $i \leq h \leq i'$, such that $T_{h,h}$ is not in the $(h+1)$-st row of $U$. We continue with Case 2 below if $h$ does not exist or $T_{i',i'}$ is stable and the backward path with respect to the $i'$-th row of a horizontal candidate in row $i' + 1$ or below ends weakly to the left of the cell of $T_{i'-1,i'-1}$ in the $i'$-st row in $U$. In all other cases we continue with Case 1.

Reject the tabloid $U$ we constructed so far in this step.

**Case 1.** Repeat for $g = i'$ down to $g = h + 1$: [ Let $k$ be such that either $(g, k)$ or $(g+1, k)$ is the endcell of the forward path of $T_{g,g}$ in the procedure for constructing $U$. Set $T = JT_{i,g+1}(T, T_{g,g})$ and $H_{i,g-1} = (g,k)$. ]

Let $(h,k) = CJT(T,T_{h,h})$ and set $T = JT(T,T_{h,h})$. If $h - k \leq i - i'$ let $H_{i,i'} = (i, i + h + k) otherwise let $H_{i,i'} = (i' + h - k, i')$.

Repeat for $g = h + 1$ down to $g = i$: [ Let $H_{i,g} = CJT_{g+1}(T, T_{g,g})$ and $T = JT_{g+1}(T, T_{g,g})$. ]

**Case 2.** If $T_{i',i'}$ is unstable, set $i' = i' + 1$. Repeat for $g = i' - 1$ down to $g = i$: [ Let $H_{i,g} = CJT_{g+1}(T, T_{g,g})$ and $T = JT_{g+1}(T, T_{g,g})$. ]

Now we apply POST-SPLIT 2.

Check that if we replace SPLIT 1 and SPLIT 2 in SPLIT by the variante above, this variante of SPLIT is equivalent to the original version. In the following SPLIT denotes this variante and SPLIT $k$, $k = 1, 2, 3$, its parts. Furthermore SPLIT' and SPLIT'' $k$, $k = 1, 2, 3$, denote the versions that were valid before this paragraph. Note that after the application of SPLIT 2 the following holds: If we were in Case 1 and $T_{v',v' + 1}$ is a horizontal candidate then $T_{v',v'}$ is a stable vertical candidate, if we were in Case 2 and $T_{v',v' + 1}$ and $T_{v' + 1,v' + 2}$ are horizontal candidates then $T_{v',v' + 1}$ is stable.

In Section 3, before the description of SPLIT 3, we define a set $C$, which we need as an input for SPLIT 3. Note that we can omit the candidates in $C$ that neither touch nor are touched from above in the application of SPLIT 3 and consequently these entries are omitted in the following.

### 6.2. The main lemmas I

The lemmas in this subsection are ordered in such a way that the proofs of SPLIT I and MERGE I are inverse to each other. They have analogs that are used for showing that SPLIT III and MERGE II are each other's respective inverses as we see in Subsection 6.4.

We introduce some notation concerning the relative position of an entry and a path. Let $e$ and $e'$ be two entries in a shifted tabloid $T$ in the $i$-th row and $c_T(e) = (i,j)$, $c_T(e') = (i,k)$. If $j < k$, resp. $j \leq k$, we say that $e'$ is right, resp. weakly right, of $e$. If $P$ is a path in the shifted tabloid $T$ then $e'$ is said to be right, respectively weakly right, of $P$, if $P$ includes an entry $e$ such that $e'$ is right, respectively weakly right, of $e$. Similar definitions apply to left, weakly left, above, weakly above, below and weakly below. If the formulations of our lemmas and corollaries include phrases in brackets $[\ldots]$, the assertions are true with and without these phrases.
Figure 14. The full line indicates the backward path of entry $z$, the circles indicate the centers of the traversed cells. The dashed line indicates the border: The entries greater than $z$ with respect to the backward path order are located north-east of this border and the entries smaller than $z$ are located south-west of this border. Moreover observe that $g$ is greater than $z$ for its backward path (dotted line) enters the column of $z$ weakly above of $z$ and $s$ is smaller than $z$ for its backward path enters the row of $z$ weakly left of $z$.

In the following lemma we characterize the largest and the smallest entry with respect to the backward paths order of a given set of entries. See Figure 14.

Lemma 1. Let $Z$ be a set of entries in a shifted tabloid $T$.

1. Let $s \in Z$ and $P_s$ denote the backward path of $s$. Then $s$ is the smallest entry in $Z$ with respect to the backward paths order, if and only if every $z \in Z$ is either \{weakly right and\} above of $P_s$ or the backward path of $z$ enters the column of $s$ weakly above of $s$.

2. Let $g \in Z$ and $P_g$ denote the backward path of $g$. Then $g$ is the greatest entry in $Z$ with respect to the backward paths order, if and only if every $z \in Z$ is either left \{and weakly below\} of $P_g$ or the backward path of $z$ enters the row of $g$ weakly left of $g$.

We extend the definition of ‘entry $e$ touches entry $e'$ from above’: Let $T$ be a shifted tabloid, $e$, $e'$ two entries in $T$ and $D$ a set of cells in the associated shifted Ferrers diagram. Let $P$ be the forward path of $e$ with respect to $D$ and $Q$ the backward path of $e'$ in $T$. If there exists a $k$ such that $(k-1, k-1), (k-1, k) \in P$ and $(k, k) \in Q$, we say that $e$ touches $e'$ from above with respect to $D$. If we want to refer to the restricted definition for touching from above which was valid before this paragraph we add the phrase ‘in the application of SPLIT’. Note that there exist pairs $(T, H)$ in which an entry $e$ touches another entry $e'$ from above in the sense we just defined, but there is no step in the application of SPLIT 3 to $(T, H)$ where $e$ touches $e'$ from above in the original sense.

The following lemma is fundamental. It is illustrated in Figure 15.
Lemma 2. Let \((i,j-1)\) and \((i,j)\) be two cells in a shifted tabloid \(T\), which is ordered up to \((i,j)\). Let \(P_{i,j-1}\) denote the forward path of \(T_{i,j-1}\) in \(T\) with respect to the set \(D\) and \(e\) the entry of the endcell of \(P_{i,j-1}\) in \(T\). Let \(e\) be an entry in \(T\) whose backward path \((i being the fixed row)\) contains \((i,j)\). If \(T_{i,j-1}\) does not touch \(e\) from above with respect to \(D\), then \(e\) is right \([\mathrm{and}\ \mathrm{weakly}\ \mathrm{above}]\) of \(P_{i,j-1}\) or the backward path of \(e\) enters the column of \(e'\) weakly above of \(e'\). (See Figure 14.)

Proof. If \(P_{i,j-1}\) consists solely of south steps there is nothing to prove. Thus we assume that there is at least one east step in \(P_{i,j-1}\).

We show the following: Let \(e''\) be below of \(P_{i,j-1}\). If \(T_{i,j-1}\) does not touch \(e''\) from above with respect to \(D\) then every cell in the backward path \(P\) of \(e''\) \((i being the fixed row)\) in a column greater than \(j-1\) is below of \(P_{i,j-1}\).

If the statement were false there would exist integers \(g,l\) such that \((g,l), (g,l+1)\) \(P_{i,j-1}\) and \((g,l+1), (g+1,l+1)\) \(P\). Since \(T_{i,j-1}\) does not touch \(e''\) from above with respect to \(D\) the cell \((g,l)\) is not a cell in the main diagonal and thus \((g+1,l)\) is a cell in the shifted Ferrers diagram. From \((g,l), (g,l+1)\) \(P_{i,j-1}\) it follows that \(T_{g,l+1} < T_{g+1,l}\), which is a contradiction to \(T_{g,l+1} > T_{g+1,l}\), which follows from \((g,l+1), (g+1,l+1)\) \(P\). (Later we will often refer to this important argument as the argument in the proof of Lemma 3.)

If the statement in the lemma were false, \(e\) is below of \(P_{i,j-1}\) or the backward path of \(e\) enters the column of \(e'\) below of \(e'\). By the assertion we have just proved every entry in the backward path \(P\) of \(e\) which is in a column greater than \(j-1\) and weakly left of the column of \(e'\) must be below of \(P_{i,j-1}\). But, since \((i,j)\) is contained in the backward path of \(e\) and \((i,j)\) is not below of \(P_{i,j-1}\), this is a contradiction if \(P_{i,j-1}\) does not consist solely of south steps.

Note that if we are in the situation of Lemma 3 and \(e\) enters the column of \(e'\) weakly above of \(e'\) then \(e\) enters the column of \(e'\) weakly above uppermost cell in \(P_{i,j-1}\) and the column of \(e'\) by the proof of the lemma.
Remark 1. Mostly we apply the lemma in the following situation. Let \((i, j - 1)\) and \((i, j)\) be two cells in a shifted tabloid \(T\) which is ordered up to the predecessor of \((i, j)\) in the total order. Furthermore let \(P_{i,j}\) denote the forward path of \(T_{i,j}\) in \(T\) with respect to a set \(D\) and let \(P_{i,j-1}\) denote the forward path of \(T_{i,j-1}\) in \(T'\) with respect to a set \(D'\), where \(T'\) denotes the shifted tabloid we obtain after performing jeu de taquin in \(T\) with \(T_{i,j}\) and with respect to \(D\). Let \(e'\) denote the entry in the last cell of \(P_{i,j-1}\) in \(T'\). Observe that \(P_{i,j}\) coincides with the backward path of \(T_{i,j}\) in \(T'\) (\(i\) being the fixed row) except for the part of the \(i\)-th row left of \((i, j)\). Thus, if \(T_{i,j-1}\) does not touch \(T_{i,j}\) from above with respect to \(D'\) in \(T'\), then \(T_{i,j}\) is right \([\text{and weakly above}]\) of \(P_{i,j-1}\) in \(T'\) or the backward path of \(T_{i,j}\) in \(T'\) enters the column of \(e'\) weakly above of \(e'\) by Lemma 2.

Observe that Figure 14 and Figure 16 coincide after a shift of the (dashed) domain border in Figure 14 by the vector \((1, -1)\). This leads us to the following.

Corollary 2. Let \((i, j - 1), (i, j), T\) and \(T'\) be as in Remark 1. Furthermore let \(Z'\) be a set of cells in the last \(r - i + 1\) rows of \(T'\), such that \(T_{i,j}\) is the smallest entry under the entries in the cells of \(Z'\) with respect to the backward paths order in \(T'\). Let \(T''\) denote the shifted tabloid we obtain after performing jeu de taquin with \(T_{i,j-1}\) in \(T'\) and with respect to \(D'\) and let \(j'\) denote the column of \(T_{i,j-1}\) in \(T''\). Let \(Z''\) denote the set of cells we obtain from \(Z'\) by first replacing every cell \((h, k) \in Z'\) with \(k \leq j'\) by \((h - 1, k - 1)\) and then deleting the replacing cells in the \((i - 1)\)-st row. If \(T_{i,j-1}\) touches no entry in a cell in \(Z'\) from above with respect to \(D'\) in \(T'\) then \(T_{i,j-1}\) is smaller than every entry in a cell in \(Z''\) with respect to the backward paths order in \(T''\).

Proof. By Lemma 1 (1), Lemma 2 and Remark 1 it is clear that either the entry in the cell of \(Z''\) that came from \(T_{i,j}\) is greater than \(T_{i,j-1}\) with respect to the backward paths order in \(T''\) or the cell that came from \(T_{i,j}\) was deleted in the course of constructing \(Z''\). (This is due to the following observation: Let \(\rho\) be a cell which is right and weakly above of a path \(P\) then \(\rho - (1, 1)\) is weakly right and above of \(P\) or \(P\) has no cell in the row of \(\rho - (1, 1)\).)

Now suppose that \(e\) is an entry in \(T'\) weakly below the \(i\)-th row that it is greater than \(T_{i,j}\) with respect to the backward paths order in \(T'\). Furthermore suppose that \(T_{i,j-1}\) does not touch \(e\) from above with respect to \(D'\) in \(T'\). Thus, by the relative position of \(e\) and
$T_{i,j}$ by Lemma 4, and since the backward path (i being the fixed row) of $T_{i,j}$ in $T'$ includes $(i,j)$, the backward path of $e$ in $T'$ includes $(i,j)$ as well. (Observe that the backward path of $e$ coincides with the backward path of $T_{i,j}$ after they intersect.) By Lemma 2 this implies that $e$ is either right or weakly above of $P_{i,j-1}$ in $T'$ or the backward path of $e$ enters the column of the end $e'$ of $P_{i,j-1}$ in $T'$ weakly above of $e'$. From this the assertion follows by Lemma 4 (1).

In Section 3 after SPLIT' 1 we claim that for an entry $e$ whose forward path terminates in a cell $(k,k)$ in the course of SPLIT' 1, we have $T_{k,k} = e$ and $H_{i,k} = (k,k)$ after the application of SPLIT' 1. This is also true for SPLIT 1 and proved in the subsequent lemma.

**Lemma 3.** Let $T$ be a shifted tabloid which is ordered up to $(i+1, i+1)$ and $H$ a partial shifted hook tableau where the $i$-th row is empty. Let $(T^j, H^j)$ denote the pair we obtain in the course of applying SPLIT 1 to $(T, H)$ in the $i$-th row after performing jeu de taquin with $T_{i,j}$. Assume that the forward path of $T_{i,j}$ in $T^{j+1}$ contains cells on the main diagonal and let $(k,k)$ be the first of these in the run of the forward path. Let $h < j$ be such that the forward path $P$ of $T_{i,h}$ in $T^{h+1}$ ends weakly right of the $k$-th column. Furthermore suppose that the forward path of $T_{i,l}$ in $T^{l+1}$ does not contain a cell on the main diagonal for $h < l < j$. Then $T_{i,h}$ touches $T_{i,j}$ from above in $T^{h+1}$, to be more accurate $(k−1, k−1), (k−1, k) ∈ P$ and $(k,k)$ is the cell of $T_{i,j}$ in $T^{h+1}$.

**Proof.** We show that the backward path of $T_{i,j}$ in $T^{h+1}$ does not contain a cell $(g,g)$ on the main diagonal with $i < g < k$. Furthermore this backward path contains $(i,h+1)$ and $(k,k)$. The assertion is then a consequence of Lemma 2. Let $T$ in Lemma 2 equal $T^{h+1}$, $j$ equal $h+1$ and let $D = \{(g,k)\}$, where $(g,k)$ is the first cell in the $k$-th column in the forward path $P$ of $T_{i,h}$ in $T^{h+1}$. By Lemma 2 $T_{i,j}$ is either right of $P$ or the backward path of $T_{i,j}$ enters column $k$ weakly above of $(g,k)$ or $T_{i,h}$ touches $T_{i,j}$ from above in $T^{h+1}$. Since $g < k$ and $T_{i,j}$ is either strictly right of the $k$-th column in $T^{h+1}$ or in $(k,k)$ the first two options are impossible and thus $T_{i,h}$ touches $T_{i,j}$ from above. The only possible place for touching is $(k−1, k−1), (k−1, k), (k,k)$ for the backward path of $T_{i,j}$ in $T^{h+1}$ does not contain a cell on the main diagonal strictly between the $i$-th and the $k$-th column.

Observe that the backward path (i being the fixed row) of $T_{i,j}$ in $T^j$ contains no cell $(g,g)$ with $i < g < k$ and it contains $(i,j)$ and therefore $(i,h+1)$, for it coincides with the forward path of $T_{i,j}$ in $T^{j+1}$ except for the part strictly left of $(i,j)$. If $j−1 = h$ there is nothing to prove. Otherwise the forward path of $T_{i,j−1}$ in $T^j$ contains no cell on the main diagonal and thus the backward path of $T_{i,j}$ in $T^{j−1}$ does not contain a cell on the main diagonal strictly between column $i$ and $k$ and it contains $(i,j−1)$, for this backward path coincides with the backward path of $T_{i,j}$ in $T^j$ until it meets the backward path of $T_{i,j−1}$ in $T^{j−1}$ and after that it coincides with the backward path of $T_{i,j−1}$ in $T^{j−1}$. The claim at the beginning of the proof now follows by induction with respect to $j–h$.

Now we are ready to show the assertion before the formulation of Lemma 3 (We use the notation of Lemma 3): Assume that the forward path of $T_{i,k}$ in $T^{k+1}$ contains a cell $\rho$ on the main diagonal and let $h < k$ be such that the forward path of $T_{i,h}$ in $T^{h+1}$ ends weakly right of the column of $\rho$. Let $D$ be the appropriate set of cells in SPLIT 1 such that $T^h = JT_D(T^{h+1}, T_{i,h})$, then it suffices to show that the forward path of $T_{i,h}$ in $T^{h+1}$ with respect to $D$ ends strictly left of the column of $\rho$. We show the assertion by induction with respect to $k−h$. To this end let $j, j > h$, be minimal such that the forward path of $T_{i,j}$ in $T^{j+1}$ with respect to the appropriate set in SPLIT 1 contains a cell $\rho'$ on the main diagonal. By Lemma 3 the forward path of $T_{i,h}$ in $T^{h+1}$ with respect to $D$ ends in a column strictly

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left of the column of \( \rho' \). If \( k = j \) the assertion follows immediately. Otherwise the column of \( \rho' \) is smaller than the column of \( \rho \) by induction since \( k - j < k - h \) and the assertion follows too.

Actually the corollary also implies the following inversion of the statement above: If \( H_{i,k} = (k, k) \) after the application of SPLIT 1, then \( T_{k,k} \) is an entry with which we have performed jeu de taquin in the course of applying SPLIT 1 and \( (k, k) \) is the end of its forward path.

In fact this is a consequence of the following observation which can be shown by Lemma 2 and Lemma 3. Assume that the backward path of a horizontal candidate in \((T^j, H^j)\) contains \((k, k)\). Then there exists an \( l, j \leq l \leq \lambda_i + i - 1 \), such that the forward path of \( T_{i,l} \) in \( T^{l+1} \) with respect to the appropriate \( D \) in SPLIT 1 contains \((k, k)\) and \( T_{i,l} \) is the smallest horizontal candidate in \((T^j, H^j)\), whose backward path contains \((k, k)\).

6.3. SPLIT 1 and MERGE 1 are each other’s respective inverse

We define property AFTER_SPLIT 1 for a pair \((T, H)\) and will observe that the application of SPLIT 1 yields a pair with that property. A pair \((T, H)\) of a shifted tabloid \( T \) and a partial shifted hook tabloid \( H \) has property AFTER_SPLIT 1 if

1. the last \( r - i + 1 \) rows of \( H \) form a shifted hook tabloid,
2. there exists no vertical candidate (with respect to the \( i \)-th row) in \((T, H)\),
3. the subtabloid of \( T \) consisting of the last \( r - i + 1 \) rows is standard, except for \( T_{k,k} \in C \) there might hold \( T_{k,k} \geq T_{k,k+1} \), where \( C \) is the set of entries \( T_{k,k} \) with \( H_{i,k} = (k, k) \).
4. for \( T_{k,k} \in C \) such that \( T_{k,k} \) either proves to be exceptional or touched from above in the application of SPLIT and every diagonal cell \((j, j) \neq (k, k)\) in the backward path of \( T_{k,k} \), there exists a horizontal candidate \( e \) in \( T \) whose backward path includes \((j, j)\) and \( e < T T_{k,k} \), and
5. \( T_{k,k} \in C \) is unstable and neither exceptional nor touched from above in the application of SPLIT, then \( T_{k,k} \) touches another entry from above in the application of SPLIT and is in its place of change.

Claim 1.

1. Let \( T \) be a shifted tabloid which is ordered up to \((i + 1, i + 1)\) and \( H \) a partial shifted hook tabloid such that the \( i \)-th row is empty. Let \((T', H')\) denote the pair we obtain after the application of SPLIT 1. Then \((T', H')\) has property AFTER_SPLIT 1 and the application of MERGE 1 to \((T', H')\) yields \((T, H)\).

2. Let \((T', H')\) be a pair with property AFTER_SPLIT 1 and \((T, H)\) denote the pair we obtain after the application of MERGE 1. Then \( T \) is ordered up to \((i + 1, i + 1)\) and \( H \) is a partial shifted hook tabloid such that the \( i \)-th row is empty and the application of SPLIT 1 to \((T, H)\) yields \((T', H')\).

Proof.

**Part 1.** Let \((T^j, H^j)\) denote the pair we obtain after the performance of jeu de taquin with \( T_{i,j} \) and the corresponding shift in the partial shifted hook tabloid in SPLIT 1 and set \((T^{\lambda_i+1}, H^{\lambda_i+1}) = (T, H)\). By construction the current cell of \( T_{i,j+1} \) in \( T^{j+1} \) is in \( \{H_{i,j+1}^{j+1}, H_{i,j+2}^{j+1}, \ldots, H_{i,\lambda_i+1}^{j+1}\} \). Furthermore by induction we assume that \( T_{i,j+1} \) is the smallest horizontal candidate in a cell in \( \{H_{i,j+1}^{j+1}, H_{i,j+2}^{j+1}, \ldots, H_{i,\lambda_i+1}^{j+1}\} \) with respect to the backward paths order in \( T^{j+1} \).

Let \( D \) be the set such that \( T^j \) is obtained from \( T^{j+1} \) in SPLIT 1 by performing jeu de taquin with \( T_{i,j} \) and with respect to \( D \). By construction no horizontal candidate in a cell in \( \{H_{i,j+1}^{j+1}, H_{i,j+2}^{j+1}, \ldots, H_{i,\lambda_i+1}^{j+1}\} \) is touched from above by \( T_{i,j} \) with respect to \( D \) in \( T^{j+1} \). Let \( k \) be minimal such that the backward path in \( T^{j+1} \) of a horizontal candidate with respect
to $H^{j+1}$ contains $(k, k)$. Then the forward path of $T_{i, j}$ with respect to $D$ ends strictly left of the column of $k$ by Lemma 3 (See also the last paragraph in the previous subsection.) Thus, by Corollary 2, $T_{i, j}$ is smaller or equal than every horizontal candidate in a cell in \{$H^j_{i,j}, H^j_{i,j+1}, \ldots, H^j_{i,\lambda_i-1}$\} with respect to the backward paths order in $T^j$. Therefore if we apply the $j - i + 1$-st step of MERGE 1 to the pair $(T^j, H^j)$ we reobtain $(T^{j+1}, H^{j+1})$.

We define the following algorithm PRE-MERGE 1. The input is a pair $(T', H')$ with the property AFTER SPLIT 1.

**PRE-MERGE 1.** Repeat the following: [Let $(j, j)$ be such that $H'_{i,j} \neq (j, j)$ but the backward path of a horizontal candidate in $T'$ contains $(j, j)$. If such a $j$ does not exist stop. Otherwise let $e$ be the smallest horizontal candidate with that property and set $(T', H') = RJS_{(j,j)}(T', H', e)$].

Observe that the application of PRE-MERGE 1 and MERGE 1 to $(T', H')$ yields the same result as the application of MERGE 1 to $(T', H')$ for the smallest horizontal candidate whose backward path contains $(j, j)$ is the smallest horizontal candidate with respect to the backward paths order weakly right of the $j$-th column. Let $(T'', H'')$ denote the pair we obtain after the application of PRE-MERGE 1 to $(T', H')$. Then $(T'', H'')$ has the properties 1 – 3 from AFTER SPLIT and for every cell $(k, k)$ on the main diagonal such that the backward path of a horizontal candidate contains $(k, k)$, $T''_{k,k}$ is itself a horizontal candidate.

We denote the property of $(T'', H'')$ by AFTER SPLIT 1. (If we apply SPLIT 1 to a pair $(T, H)$, $T$ ordered up to $(i + 1, i + 1)$ and the $i$-th row of $H$ is empty, then the output pair has property AFTER SPLIT 1.)

We show the following: Let $(T, H)$ denote the pair we obtain after the application of MERGE 1 to $(T'', H'')$. If we apply SPLIT 1 to $(T, H)$ we reobtain $(T'', H'')$.

Let $(T''_{i,j}, H''_{i,j})$ denote the pair we obtain in the course of applying MERGE 1 to $(T'', H'')$ after performing reverse jeu de taquin with respect to $\{i, j - 1\}$ and the corresponding reshift in the partial shifted hook tabloid, and set $(T''_{i,j}, H''_{i,j}) = (T'', H'')$. By induction we assume that the backward path of every horizontal candidate in a cell in $\{H''_{i,j}, H''_{i,j+1}, \ldots, H''_{i,\lambda_j-1}\}$ in $T''_{i,j}$ contains $(i, j)$ and it contains no cell on the main diagonal except for possibly $c_{T''_{i,j}}$ and $(i, j)$. Let $e$ be the smallest horizontal candidate in a cell in $\{H''_{i,j}, H''_{i,j+1}, \ldots, H''_{i,\lambda_j-1}\}$ with respect to the backward paths order in $T''_{i,j}$. Furthermore let $P$ be its backward paths in $T''_{i,j}$ with respect to $(i, j)$. If $j'$ is the column of $e$ in $T''_{i,j}$, the subtabloid of $T''_{i,j}$ consisting of the first $j'$ columns is ordered up to $(i, j)$ by property AFTER SPLIT 1 and the minimality of $e$. Thus $e = T''^{j+1}_{i,j}$ and performing jeu de taquin with $e$ in $T''_{i,j+1}$ and with respect to the cells on the main diagonal together with the corresponding shift in $H''^{j+1}$ results in the pair $(T''_{i,j}, H''^{j})$. By the minimality of $e$, after the application of PRE-MERGE 1 to $(T''_{i,j}, H''^{j})$ with the construction of the reshift in $H''^{j}$ the entries in the cells in $\{H''^{j+1}_{i,j+1}, H''^{j+1}_{i,j+2}, \ldots, H''^{j+1}_{i,\lambda_i-1}\}$ are right [and weakly above] of $P$ or their backward paths in $T''^{j+1}$ enter column $j'$ weakly above of $c_{T''_{i,j}}(e)$. Thus, by the argument from Lemma 2, the backward paths of the entries in these cells in $T''^{j+1}$ contain $(i, j + 1)$.) Likewise it is easy to see that no backward path of these entries contains a cell on the main diagonal except for possibly its origin and terminus.

We define another Algorithm POST-SPLIT 1 which we apply to a pair $(T'', H'')$ with property AFTER SPLIT 1.
POST-SPLIT 1. Repeat the following: Let \( T''_{j,j} \) be an unstable horizontal candidate which is neither exceptional nor touched from above and not already in its place of change (if \( T''_{j,j} \) has a place of change). If \( T''_{j,j} \) does not exist stop. Otherwise: If \( T''_{j,j} \) has a place of change \( \rho \) let \( D = \{ \rho \} \) otherwise let \( D = \emptyset \). Set \( (T'', H'') = JS_D(T'', H'', T''_{j,j}). \)

Now the assertion follows since PRE-MERGE 1 and POST-SPLIT 1 are inverse to each other (in order to show that use Lemma 3) and since the application of SPLIT' 1 and POST-SPLIT 1 is equivalent to the application of SPLIT 1.

In view of Claim 1 it remains to show the following:

1. Let \((T, H)\) be a pair with the property AFTER_SPLIT 1 and let \((T', H')\) denote the pair we obtain after the application of SPLIT 2 and SPLIT 3 to \((T, H)\). We have to show that if we apply MERGE 3 and MERGE 2 to \((T', H')\) we reobtain \((T, H)\).
2. Let \((T, H)\) be such that \( T \) is ordered up to \((i, i)\) and the subtabloid of \( H \) consisting of the last \( r - i + 1 \) rows is a shifted hook tabloid. Let \((T', H')\) denote the pair we obtain after the application of MERGE 3 and MERGE 2. We have to show that \((T', H')\) has property AFTER_SPLIT 1 and that the application of SPLIT 2 and SPLIT 3 to \((T', H')\) yields \((T, H)\).

6.4. THE MAIN LEMMAS II

The following lemma is similar to Lemma 3.

Lemma 4. Let \((i, m)\) and \((i - 1, n)\), \(m \leq n\), be two cells in a shifted tabloid \( T \) such that the subtabloid of \( T \) consisting of the last \( r - i + 2 \) rows and without the entries in \((i, m)\) and \((i - 1, n)\) has increasing rows and columns. Let \( P_{i-1},n \) denote the forward path of \( T_{i-1},n \) in \( T' \) with respect to a set, where \( T' \) denotes the shifted tabloid we obtain after performing jeu de taquin in \( T \) with \( T_{i,m} \) and with respect to \( D \). Furthermore let \( e \) denote the entry in the last cell of \( P_{i-1},n \) in \( T' \). Then \( T_{i,m} \) is [weakly left and] below of \( P_{i-1},n \) \( = P_{i-1},n \cup \{(i - 1, m),(i - 1, m + 1),\ldots,(i - 1, n - 1)\} \) \( \in T' \) or the backward path of \( T_{i,m} \) with respect to \((i, m)\) in \( T' \) enters the row of \( e \) weakly left of \( e \).

Proof. Similar to the proof of Lemma 3. Observe that if the backward path of \( T_{i,m} \) in \( T' \) enters the row of \( e \) weakly left of \( e \) then it enters the row of \( e \) weakly left of the leftmost cell of \( P_{i-1},n \) in the row of \( e \). 

Corollary 3. Let \((i - 1, n), (i, m), T, T' \) be as in Lemma 3. Furthermore let \( Z' \) be a set of cells weakly below the \( i-th \) row, which includes the cell of \( T_{i,m} \) in \( T' \), such that \( T_{i,m} \) is the greatest entry in a cell of \( Z' \) in \( T' \) with respect to the backward paths order. Let \( T'' \) denote the shifted tabloid we obtain after performing jeu de taquin with \( T_{i-1},n \) in \( T' \) and with respect to a set and let \( i' \) denote the row of \( T_{i-1},n \) in \( T'' \). Let \( Z'' \) denote the set of cells we obtain from \( Z' \) by replacing every cell \((h,k)\) in \( Z' \) with \( h \leq i' \) by \((h - 1, k - 1)\). Then \( T_{i-1},n \) is greater than every entry in a cell of \( Z'' \) in \( T'' \) with respect to the backward paths order.

Proof. Similar to the proof of Corollary 3.

6.5. SPLIT 2 AND MERGE 2 ARE EACH OTHER’S RESPECTIVE INVERSE

We define property AFTER_SPLIT 2 for a pair \((T, H)\) and will observe that a pair to which we have applied SPLIT 1 and SPLIT 2 has that property. A pair \((T, H)\) of a shifted tabloid \( T \) and a partial shifted hook tabloid \( H \) has property AFTER_SPLIT 2 if

1. the last \( r - i + 1 \) rows of \( H \) form a shifted hook tabloid,
2. there exists an $i''$ such that for every $i + 1 \leq g \leq i''$ there is a vertical candidate in row $g$ and no other vertical candidates exist. For $i + 1 \leq g < g + 1 \leq i''$ the vertical candidate in row $g$ is greater than the vertical candidate in row $g + 1$ with respect to the backward paths order ($i$ being the fixed row). If we perform jeu de taquin with the vertical candidates in row $i'', i'' - 1, \ldots, i + 1$ in that order, then every candidate is either stable or its forward path starts with a step into the next row. Moreover: $T_{r', r''}$ is a stable vertical candidate or $T_{r'' + 1, r'' + 1}$ is a stable horizontal candidate or $T_{r'' + 2, r'' + 2}$ is not a horizontal candidate.

3. the subtabolid of $T$ consisting of the last $r - i + 1$ rows is standard, except for $T_{p, q} \in C' \cup C$ there might hold $T_{p, q} > \min(T_{p + 1, q}, T_{p, q + 1})$, where $C'$ and $C$ are the two sets we define before the application of SPLIT 3 to $(T, H)$ (see Section 3 before the description of SPLIT 3).

4. $e$ is a horizontal candidate which is touched from above in the application of SPLIT, then for every cell $(j, j) \neq c_T(e)$ with $i'' < j$ in the backward path of $e$, there exists a horizontal candidate $e'$ with $e' \prec_T e$ and the backward path of $e'$ contains $(j, j)$, and

5. $e$ is an unstable horizontal candidate which is not touched from above in the application of SPLIT then $e$ touches from above in the application of SPLIT and is in its place of change.

A remark on Property (2) of AFTER.SPLIT 2: Note that the vertical candidate in row $g$ is greater than the vertical candidate in row $g + 1$, $i + 1 \leq g < g + 1 \leq i''$ in a pair to which we have applied SPLIT 2 by Lemma 4.

**Claim 2.**

1. Let $(T, H)$ be a pair with property AFTER.SPLIT 1 and let $(T', H')$ denote the pair we obtain after the application of SPLIT 2. Then $(T', H')$ has property AFTER.SPLIT 2 and the application of MERGE 2 to $(T', H')$ results in $(T, H)$.

2. Let $(T', H')$ be a pair with property AFTER.SPLIT 2 and $(T, H)$ denote the pair we obtain after the application of MERGE 2. Then $(T, H)$ has property AFTER.SPLIT 1 and the application of SPLIT 2 to $(T, H)$ results in $(T', H')$.

**Proof.** Left to the reader. Observe that we are in Case 2 of SPLIT 2, respectively Case 2 of MERGE 2, if and only if either the smallest horizontal candidate with respect to the backward paths order in $(T', H')$ in a row strictly below the lowest vertical candidate is smaller than the smallest vertical candidate or there exists no vertical candidate. The rest follows from the fact that jeu de taquin and reverse jeu de taquin are inverse to each other and from the fact that (with one exception) the endcells of the forward paths are not distorted by a shift in the accompanying shifted hook tabloid. The exception is the exceptional vertical candidate in the $h$-th row in Case 1 of SPLIT 2, whose endcell is stored in the $i$-th row. There we need Corollary 3.

### 6.6. Irreducible pairs

**Touching from below.** We define an analog to ‘$e$ touches $e'$ from above’ for the algorithm MERGE. Suppose we are at the beginning of a step of MERGE 3 and the if-condition (If $c_T(e) = (h, h) : [\ldots]$) is true. In the following commands in the description of MERGE 3 we distinguish between two cases. If we are in the first case we say that $e'$ touches $e$ from below and for $h' < j \leq h$, $T_{j, j}$ touches $T_{j - 1, j - 1}$ from below. Furthermore for $h' \leq j \leq h$, let $(j, j)$ be the place of change for $T_{j, j}$. In the second case we say that for $h' < j \leq h$, $T_{j, j}$ touches $T_{j - 1, j - 1}$ from below and $(j, j)$ is the place of change for $T_{j, j}$.
We define the exceptional entries with respect to MERGE 2. Suppose we have applied MERGE 2, Case 1. Define \( T_{g,g} \) to be exceptional for \( i \leq g \leq i'' \). Now suppose we have applied MERGE 2, Case 2. For \( i \leq g \leq i'' \) and \( g \neq i', i' + 1 \) we define \( T_{g,g} \) to be exceptional and \( T_{i'+1,i'+1} \) is said to be touched from below by the smallest candidate strictly below the \((i'+1)\)-st row if \( i' \neq i'' \).

A step of SPLIT 3 includes the choice of a protagonist \( e \in C' \) and the performance of either Case 1, Case 2 or Case 3. A step of MERGE 3 is defined analogously. The maximal vertical candidate \( e \) at the beginning of a step of MERGE 3 is said to be the first protagonist of the step and the candidate \( e \) after the performance of the if-condition (If \( c_T(e) = (h,h) : [\ldots] \) is said to be the second protagonist. Clearly the second protagonist can be equal to the first protagonist in a step of MERGE 3.

We first show that SPLIT 3 and MERGE 3 are inverse to each other for a certain class of pairs \((T,H)\).

**Irreducible pair with respect to SPLIT.** Let \((T,H)\) be a pair with property AFTER_SPLIT 2. Suppose that in the course of applying SPLIT 3 to \((T,H)\) every protagonist which touches an entry from above was either in \( C' \) at the beginning of SPLIT 3 or is itself touched by an entry from above in the application of SPLIT. Observe that this is equivalent to the fact that we are never in Case 3 in SPLIT 3 and that \( C' \neq \emptyset \) at the beginning of every step of SPLIT 3. Then \((T,H)\) is said to be irreducible with respect to SPLIT. In Figure 7 the application of SPLIT 3 to an irreducible pair is illustrated.

**Irreducible pair with respect to MERGE.** Let \( T \) be a shifted tableau which is ordered up to \((i,i)\) and \( H \) a partial shifted hook tableau, such that the subtableau consisting of the last \( r - i + 1 \) rows is a shifted hook tableau. If every protagonist \( e \) (except for possibly the last) in MERGE 3 either touches an entry from below or terminates in row \( z + 1 \) in a certain step of MERGE 3, then \((T,H)\) is said to be irreducible with respect to MERGE. Note that this is equivalent to the following: Suppose the second protagonist in a step of MERGE 3 is a horizontal candidate. Then this is either the last step in the application of MERGE 3 to \((T,H)\) or this second protagonist is equal to \( e' \) in the if-condition at the beginning of the next step of MERGE 3.

**Claim 3.** Let \((T,H)\) be irreducible with respect to SPLIT and let \((T',H')\) denote the pair we obtain after the application of SPLIT 3. Then \( T' \) is ordered up to \((i,i)\), the last \( r - i + 1 \) rows of \( H' \) form a shifted hook tableau and the application of MERGE 3 to \((T',H')\) results in \((T,H)\).

**Proof.** Let \( C_{\text{start}}' \) denote the set \( C' \) at the beginning of the application of SPLIT 3 to \((T,H)\). Suppose we are in the course of applying SPLIT 3: We choose in a certain step entry \( e_1 \) to be the protagonist and obtain the pair \((U,J)\), in the next step we choose \( e_2 \) to be the protagonist and obtain \((U',J')\). Let \( z' \) be such that there exists a vertical candidate in row \( z' \) of \((U,J)\) and in \( C_{\text{start}}'' \), which is not the protagonist in and before the step of \( e_1 \) and if there exists a vertical candidate in the \((z'+1)\)-st row of \((U,J)\) and in \( C_{\text{start}}'' \), this entry was the protagonist in or before the step of \( e_1 \). If \( z' \) does not exist, set \( z' = i \). Let \( z'' \) be the corresponding quantity for the step of \( e_2 \) and observe that \( z'' = z' - 1 \) iff \( e_2 \) is the vertical candidate of \((U,J)\) in the \( z'' \)-th row. Otherwise \( z'' = z' \). If we are in Case 2 in the step of \( e_k \) let \( h_k \) and \( h'_k \) denote \( h \) and \( h' \) in the description of Case 2 of SPLIT 3, \( k = 1,2 \).

By induction we assume the following:

1. Suppose we are in Case 1 in the step of \( e_1 \). Then \( e_1 \) is greater or equal than every vertical candidate strictly below the \( z' \)-th row of \((U,J)\) with respect to the backward
Figure 17. The application of SPLIT 3 to a irreducible pair. A full circle indicates a candidate in $C'$, an empty circle indicates a candidate in $C$ at the beginning of the application. The lines are the forward paths of jeu de taquin, the numbers near them indicate their order in the application of SPLIT 3. If the tip of an arrow touches the border of a cell, this indicates that the entry in the cell is noticed to be unstable at that point in the application of the algorithm.
paths order and if $e_1$ is horizontal candidate then $e_2$ is the greatest vertical candidate strictly below the $z''$-th row of $(U, J)$.

2. Suppose we are in Case 2 in the step of $e_1$. Then $e_2 = U_{h'_1,h'_1}$ is greater or equal than every vertical candidate strictly below the $z'$-th row of $(U, J)$ and if $e_2$ is a horizontal candidate in $(U, J)$ then $U_{h'_1-1,h'_1-1}$ is the greatest vertical candidate strictly below the $z'$-th row of $(U, J)$.

We aim to show that the application of one step of MERGE 3 to $(U', J')$ with $z = z''$ either yields

1. $(U, J)$ or
2. the pair we obtain before the performance of TRANS in the step of $e_1$ (in which we are in Case 2 of SPLIT 3 and $e_1$ is a horizontal candidate at the beginning of the step of $e_1$) or
3. a pair which results in $(U, J)$ after the application of a second step of MERGE 3 until the marker $(*)$ (see the description of MERGE 3).

Furthermore we show the following (induction step):

1. Suppose we are in Case 1 in the step of $e_2$. Then $e_2$ is greater or equal than every vertical candidate in $(U', J')$ strictly below the $z''$-th row and if $e_2$ is a horizontal candidate then $e_2 = U_{k,k}$ for a $k$ and $U'_{k-1,k-1} = U_{k-1,k-1}$ is the greatest vertical candidate of $(U', J')$ strictly below the $z''$-th row.

2. Suppose we are in Case 2 in the step of $e_2$. In this case $U'_{h'_2,h'_2}$ is greater or equal than every vertical candidate of $(U', J')$ strictly below the $z''$-th row of $(U', J')$ and if $U'_{h'_2,h'_2}$ is a horizontal candidate then $U'_{h'_2-1,h'_2-1}$ is the greatest vertical candidate of $(U', J')$ strictly below the $z''$-th row.

We distinguish between the two cases according to whether or not we are in Case 1 in the step of $e_1$.

Case 1. There are two cases. Either

1. $e_1, e_2 \in C'_{start}$, $e_2$ is in the $z'$-th row of $U$ and $e_1$ was either the vertical candidate of $C'_{start}$ in the $(z'+1)$-st row or the unique horizontal candidate of $C'_{start}$ in $(T, H)$ or
2. $e_2$ is unstable in $U$ and the backward path of $e_1$ in $U$ contains $(k + 1, k + 1)$.

Case 1: Clearly $e_2$ is the greatest vertical candidate in $(U, J)$ weakly below the $z'$-th row for $e_1$ is greater or equal than every vertical candidate in $(U, J)$ strictly below the $z'$-th row by induction and $e_2$ is greater than $e_1$ by property AFTER_SPLIT 2. If $e_2$ changes row in the step of $e_2$, the first cell in row $z'+1$ in its forward path is weakly right of the first cell in row $z'+1$ in $(U, J)$ in the backward path of $e_1$ by property AFTER_SPLIT 2. Thus $e_2$ is also the greatest vertical candidate in $(U', J')$ strictly below the $z''$-th row by Corollary 3, if we are not in Case 2 in the step of $e_2$ and in this case it is obvious that $e_2$ is the first and the second protagonist in the application of a step of MERGE 3 to $(U', J')$ with $z = z''$ (check that the if-condition at the beginning of the step of MERGE 3 is either not fulfilled or its application leaves $(U', J')$ unchanged in this step). Otherwise $U'_{h_2',h_2'}$ is the greatest vertical candidate in $(U', J')$ strictly below the $z''$-th row for firstly $e_2 = U'_{h_2-1,h_2-1}$ is the greatest vertical candidate strictly below the $z'$-th row and strictly above the $h_2$-th row of $(U', J')$ by Corollary 3, secondly there exists no vertical candidate strictly below the $h_2$-th row and thirdly $U'_{h_2',h_2'}$ is greater than $U'_{h,h}$ for $h_2 - 1 \leq h < h_2'$. Thus $U'_{h_2',h_2'}$ is the first protagonist in the application of a step of MERGE 3 to $(U', J')$. Consequently $e_2$ is the second protagonist.
in the application of a step of MERGE 3 to \((U', J')\) if we are in Case 2 in the step of \(e_2\). It remains to show that the loop in the application of a step of MERGE 3 with \(z = z''\) to \((U', J')\) terminates with ‘1’. \(e\) is in row \(z + 1\): Observe that the forward path of \(e_2\) in \(U\) does not contain a cell \((k, k)\) such that there exists a vertical candidate strictly below the \(k\)-th row for such a vertical candidate would be greater than \(e_2\) in \(U\) by the arguments from Lemma 3, which is a contradiction to the maximality of \(e_2\) weakly below the \(z'\)-th row in \((U, J)\). Thus the loop does not stop with ‘2.’. Furthermore the forward path of \(e_2\) in \((U, J)\) with respect to the appropriate set does not contain a cell \((k, k)\) (except for possibly the endcell of the forward path of \(e_2\) with respect of the emptyset) with \(h_2 - 1 \neq k\) such that the backward path in \(U\) of a horizontal candidate contains \((k + 1, k + 1)\), for otherwise \(e_2\) touches the minimal horizontal candidate whose backward paths contains \((k + 1, k + 1)\) from above in the application of SPLIT by Lemma 3, which implies \(J_i,k+1 = (k + 1, k + 1)\) and this is a contradiction to the choice of \(h_2\). Therefore the loop does not stop with ‘3.’. Finally, if \(k\) is the row of \(e_2\) in \(U'\), then the subtabloid of \(U\) consisting of the cells strictly below of the \(z'\)-th row and weakly above of the \(k\)-th row is standard and consequently the loop does not stop with ‘4.’. Therefore the application of one step of MERGE 3 to \((U', J')\) with \(z = z''\) yields \((U, J)\) in this case.

\(2\). There are two cases: either \(e_1\) is a vertical candidate in \((U, J)\) or \(e_1\) is a horizontal candidate in \((U, J)\).

Let \(e_1\) be a vertical candidate, then \(e_1\) is the greatest vertical candidate strictly below the \(z'\)-th row of \((U, J)\) by the induction hypothesis. If we are not in Case 2 in the step of \(e_2\) then \(e_2\) is the greatest vertical candidate strictly below the \(z''\)-th row of \((U', J')\) by Corollary 3 (since \(e_2\) is unstable in \((U, J)\)) and thus the first and the second protagonist in the application of a step of MERGE 3 to \((U', J')\) with \(z = z''\). If we are in Case 2 in the step of \(e_2\) then again \(U'_{h_2', h_2'}\) is the greatest vertical candidate strictly below the \(z''\)-th row of \((U', J')\) and with this the first protagonist in the application of a step of MERGE 3 to \((U, J)\). Thus \(e_2\) is the second protagonist in the application of a step of MERGE 3 to \((U', J')\) with \(z = z''\) in this case. It remains to show that the loop at the end of the application of a step of MERGE 3 to \((U', J')\) stops with ‘2.’ and at the cell of \(e_2\) in \(U\). This is left to the reader, for it is similar to the previous case.

Now let \(e_1\) be a horizontal candidate, then \(e_1\) is greater than every vertical candidate strictly below the \(z'\)-th row of \((U, J)\) and \(e_2\) is the greatest vertical candidate strictly below the \(z'\)-th row of \((U, J)\) by the induction hypothesis. Thus either \(e_2\) (if we are in Case 1 in the step of \(e_2\)) or \(U'_{h_2', h_2'}\) (if we are in Case 2 in the step of \(e_2\)) is the greatest vertical candidate strictly below the \(z''\)-th row of \((U', J')\) and therefore the application of a step of MERGE 3 to \((U', J')\) with \(z = z''\) yields \((U, J)\). Note that the loop in MERGE 3 terminates with ‘3.’ and at the cell of \(e_2\) in \(U\), \(e_1\) being the horizontal candidate in ‘3.’.

**Case 2.** Clearly \(e_2 = U_{h_2', h_2'}\) and \(e_2\) is greater or equal than every vertical candidate strictly below the \(z' = z''\)-th row of \((U, J)\) with respect to the backward paths order by the induction hypothesis. Thus either \(e_2\) (if we are in Case 1 in the step of \(e_2\)) or \(U'_{h_2', h_2'}\) (if we are in Case 2 in the step of \(e_2\)) is greater or equal than every vertical candidate strictly below the \(z''\)-th row in \((U', J')\).

Suppose \(e_2\) is a vertical candidate in \((U, J)\). Then either \(e_2\) or \(U'_{h_2', h_2'}\) is the first protagonist in the application of a step of MERGE 3 with \(z = z''\) to \((U', J')\) and therefore \(e_2\) is the second protagonist. Thus the application of a step of MERGE 3 yields \((U, J)\). Observe that the loop in MERGE 3 terminates with ‘4.’ and at the cell of \(e_2\) in \(U\).
Suppose \( e_2 \) is a horizontal candidate in \((U, J)\). First we suppose that we are in Case 1 in the step of \( e_2 \). Then \( U'_{h_1' - 1, h_1' - 1} \) is the greatest vertical candidate strictly below the \( z'' \)-th row of \((U', J')\) and consequently the first protagonist in the application of a step of MERGE 3 to \((U, J)\). Thus \( e' = e_2 \) in the if-condition at the beginning of this step of MERGE 3 and \( e_1 \) is the second (horizontal) protagonist. We obtain the pair which we had immediately before the performance of TRANS in the step of \( e_1 \). Now suppose we are in Case 2 in the step of \( e_2 \). Clearly \( U'_{h_2' - 1, h_2' - 1} \) is the greatest vertical candidate strictly below the \( z'' \)-th row of \((U', J')\) and consequently the first protagonist in the application of a step of MERGE 3 to \((U', J')\) with \( z = z'' \). The second protagonist is the horizontal candidate \( e_2 \) and since \( U'_{h_1' - 1, h_1' - 1} \) is the first protagonist in the next step of MERGE 3 we have \( e' = e_2 \) in the if-condition at the beginning of this next step. Thus the application of another step of MERGE 3 until the step of MERGE 3 is true and we have just applied the appropriate commands, we set the first protagonist in the next step of MERGE 3 we have Consequently the first protagonist in the application of a step of MERGE 3 to \((U, J)\) was fulfilled. After the if-condition and its commands we let \((U, J)\) be irreducible with respect to MERGE and \((T', H')\) denote the pair we obtain after the application of MERGE 3 to \((T, H)\). Then \((T', H')\) has property AFTER-SPLIT 2 and the application of SPLT 3 to \((T', H')\) yields \((T, H)\).

**Proof.** In the course of applying MERGE 3 to \((T, H)\) we construct two sets \( C' \) and \( C \). We start with \( C = C' = \emptyset \). If the if-condition (If \( c_T(e) = (h, h) \colon \ldots \)) at the beginning of a step of MERGE 3 is true and we have just applied the appropriate commands, we set \( C = C' \cup \{ T_{h'+1, h'+1}, T_{h'+2, h'+2}, \ldots, T_{h+1, h+1} \} \) and \( C' = C' \setminus \{ T_{h'+1, h'+1}, T_{h'+2, h'+2}, \ldots, T_{h+1, h+1} \} \) if we are in the first case in these commands, and we set \( C = C \cup \{ T_{h'+1, h'+1}, T_{h'+2, h'+2}, \ldots, T_{h,h} \} \) and \( C' = C' \setminus \{ T_{h'+1, h'+1}, T_{h'+2, h'+2}, \ldots, T_{h,h} \} \) if we are in the second case in these commands. After the if-condition and its commands we let \( C' = C' \cup \{ e \} \), whether or not the if-condition was fulfilled.

Suppose that after a certain step in the application of MERGE 3 to \((T, H)\) we obtain the pair \((U', J')\), the sets \( C' \) and \( C \) and a row \( z \). First we show that the second protagonist \( e \) of this step is also the protagonist in the application of one step of SPLT 3 to \((U', J')\) with the sets \( C' \) and \( C \). This is equivalent to the fact that \( e \) is the candidate in \( C' \) whose row is maximal.

If the second protagonist is a horizontal candidate in a certain step of MERGE 3, then it is the horizontal candidate \( e' \) in the if-condition at the beginning of the next step (if we are not in the final step) of MERGE 3 and thus leaves \( C' \) at that point, for \((T, H)\) is irreducible with respect to MERGE 3. Consequently \( C' \) includes at most one horizontal candidate and if this is the case then all other candidates in \( C' \) are strictly above the row of the unique horizontal candidate. (In order to see the second assertion in the previous sentence note that whenever \( c_T(e) = (h, h) \) at the beginning of a step of MERGE 3 then there exists no vertical candidate strictly below the \( h \)-th row by the maximality of \( e \).) Therefore it suffices to show the assertion for second protagonists \( e \) that are vertical candidates.

In a step of MERGE 3 the second protagonist either terminates in row \( z + 1 \) or in a cell \((k, k)\) on the main diagonal, if the protagonist is a vertical candidate. In the latter case by the maximality of the second protagonist (observe that after the if-condition in a step of MERGE 3 the second protagonist is greater or equal than every vertical candidate strictly below the \( z \)-th row; this is because the backward path of the first protagonist always contains the cell of the second protagonist at the beginning of a step of MERGE 3) every backward path of a vertical candidate strictly below the \( k \)-th row in the tableau we obtain after this step contains \((k + 1, k + 1)\) by the argument in the proof of Lemma 2 and therefore these vertical candidates are greater than \( e \) with respect to the backward paths order. However, the vertical second protagonist is still greater than every vertical candidate strictly below
the $z$-th row and strictly above the $k$-th row, since this part of the shifted hook tabloid remains unchanged in this step of MERGE 3. Consequently in the course of MERGE 3 the second vertical protagonists (and with this the elements of $C'$) are arranged in increasing rows (as long as the first protagonist is equal to the second protagonist) until from time to time former second vertical protagonists leave $C'$ after the performance of the commands in the if-condition. We conclude that the second protagonist in a step of MERGE 3 either starts (by its maximality and the arguments in this paragraph) and ends (by the stopping criteria ‘4. $c_T(e) = (l, l)$ and $T_{l-1,l-1}$ is an unstable vertical candidate’ in MERGE 3 and the arguments in this paragraph) below the lowest vertical candidate in $C'$ or is equal to the lowest vertical candidate in $C'$, whose backward path in this step ends below the second lowest vertical candidate in $C''$ (if it exists) by the fourth stopping criteria. Consequently the row of the vertical second protagonist in a step of MERGE 3 is always below the other vertical candidates in $C'$ and by the observation about the horizontal candidates in $C''$ in the previous paragraph this proves our claim in the second paragraph of the proof.

Let $(U, J)$ denote the pair before the step of MERGE 3, in which $e$ is the second protagonist and let $C'_{before}$ and $C_{before}$ denote the accompanying sets. We have to show that if we apply one or two steps of SPLIT 3 to $(U', J')$ with the sets $C'$ and $C'$, we reobtain $(U, J)$ together with $C'_{before}$ and $C_{before}$ or a pair of a shifted tabloid and a shifted hook tabloid together with the two sets which we have obtained in the application of MERGE 3 to $(T, H)$ before the pair $(U, J)$. Proving this claim is now a matter of considering every circumstance in the step of $e$ in MERGE 3. (Case distinction: First suppose that we are in the first case in the if-condition at the beginning of the step of MERGE 3 (this leaves no choice open for the rest of the step) and second combine the second case in the if-condition and the case that the if-condition is false with each of the four stopping criterions of the loop in MERGE 3.)

Thus we have shown that for irreducible pairs SPLIT 3 and MERGE 3 are inverse to each other. Now we show the assertion for reducible pairs by induction with respect to the number of protagonists not contained in $C'$ at the beginning of the application of SPLIT 3 which touch from above but are not touched from above in the application of SPLIT, respectively with respect to the number of second protagonists unequal to the last second protagonist which neither touch an entry from below nor terminate in row $z + 1$ in the application of MERGE 3.

### 6.7. Reducible pairs

Observe that the pair in Example 3 in Section 4 is reducible. In our proof of Claim 5 we sometimes refer to that example. Moreover in Figure 18 the application of SPLIT 3 to a reducible pair is illustrated.

For the proof of Claim 5 we need the following observation: Suppose we are in Case 3 of SPLIT 3 and let $(U, J)$ denote the current pair before the application of Case 3. Then the protagonist $e$ in the description of Case 3 of SPLIT 3 is a vertical candidate. In order to show that suppose that $e$ was the first horizontal candidate in the course of SPLIT 3 which contradicts this assertion. Then we are either in the first step of SPLIT 3 or $e$ joined $C'$ in the previous step of SPLIT 3 in which we were in Case 2 and $e$ equals $T_{h', h'}$ in the description of Case 2. We only consider the latter case here, for the first case is similar. By the relative position of $e$ and $e'$ in the current step of SPLIT 3 and the argument from Lemma 3 the backward path of $e'$ in $(U, J)$ contains $(h' + 1, h' + 1)$. By Lemma 3 $e$ touches the smallest horizontal candidate $e''$ in $(U, J)$ whose backward path in $U$ contains $(h' + 1, h' + 1)$ from above in the application of SPLIT 3 and this is a contradiction, since $e''$ is weakly above of the row of $e'$ by its minimality and by the fact that $e'$ is on the main diagonal.
Figure 18. The application of SPLIT 3 to a reducible pair. (See caption of Figure 17 for an explanation.) Observe that for distinct irreducible 'parts' different line styles are used.
Claim 5. Let \((T, H)\) be a pair with property \text{AFTER}\_\text{SPLIT} 2 and \((T', H')\) the pair we obtain after the application of \text{SPLIT} 3. Then \(T'\) is ordered up to \((i, i)\), the subtabloid of \(H'\) consisting of the last \(r - i + 1\) rows is a shifted hook tabloid and the application of \text{MERGE} 3 to \((T', H')\) yields \((T, H)\).

Proof. Let \(C'_\text{start}\) and \(C_\text{start}\) denote \(C'\) and \(C\) at the beginning of the application of \text{SPLIT} 3 to \((T, H)\). We assume that every entry in \(C_\text{start}\) is either touched from above or touches an entry from above in the application of \text{SPLIT} 3. By Claim 3 we furthermore assume that \((T, H)\) is reducible with respect to \text{SPLIT}.

Let \(e\) be the candidate in \(C_\text{start}\) with maximal row \(k\), which is not touched from above in the application of \text{SPLIT} \((e = 49\) in Example 3). Furthermore, if it exists, let \(e' \in C_\text{start}\) be the entry with minimal row strictly below the \(k\)-th row such that \(e'\) is touched from above in a certain step in the application of \text{SPLIT} 3 by a protagonist which is at the beginning of the step in a row strictly above of the \(k\)-th row \((e' = 63\) in the example). Let \(C_1\) denote the entries in \(C_\text{start}\) which are strictly above the \(k\)-th row and weakly below the row of \(e'\) and \(C_2\) denote the entries in \(C_\text{start}\) which are strictly below the \(k\)-th row and strictly above the row of \(e'\) in \(T\).

Let \text{SPLIT} 3.1 denote the application of \text{SPLIT} 3 with \(C' = C'_\text{start}\) and \(C = C_1\) to \((T, H)\) and let \text{SPLIT} 3.2 denote the application of \text{SPLIT} 3 with \(C' = \{e\}\) and \(C = C_2\) to \((T, H)\). Observe that the application of \text{SPLIT} 3 with \(C' = C'_\text{start}\) and \(C = C_\text{start}\) to \((T, H)\) is equivalent to the simultaneous application of \text{SPLIT} 3.1 and \text{SPLIT} 3.2, where after every step of \text{SPLIT} 3.2, \(i = 1, 2\), we can decide to make either the next step of \text{SPLIT} 3.1 or the next step of \text{SPLIT} 3.2, but whenever a protagonist in \text{SPLIT} 3.1 moves weakly below the \(k\)-th row the application of \text{SPLIT} 3.2 has already terminated. This is because \(k\) is maximal and there exists a step of \text{SPLIT} 3 in which we are either in Case 3 with \(e\) being the \(e'\) in the description of Case 3 of \text{SPLIT} 3 or \(C' = \emptyset\) at the beginning of the step and \(e\) is the candidate with minimal row in \(C\) at that point.

Observe that after the termination of \text{SPLIT} 3.2 and before a protagonist of \text{SPLIT} 3.1 moves weakly below the \(k\)-th row, \(e\) is the greatest vertical candidate under the vertical candidates that arose in the application of \text{SPLIT} 3.2 by the proof of Claim 3, for \(e\) is the final protagonist in \text{SPLIT} 3.2 for \(k\) is maximal. The first protagonist of \text{SPLIT} 3.1 that moves weakly below of the \(k\)-th row is a vertical candidate (since whenever we are in Case 3 in \text{SPLIT} 3 the protagonist is a vertical candidate; this is proved at the beginning of this subsection) and greater than \(e\) with respect to the backward paths order by Corollary 3. Moreover every subsequent protagonist of \text{SPLIT} 3.1, which moves weakly below the \(k\)-th row, is greater than the previous. Thus the vertical candidates from \text{SPLIT} 3.1 dominate the vertical candidates from \text{SPLIT} 3.2 once they come weakly below \(k\)-th row.

Let \(V_1\) denote the vertical pointers that came from \text{SPLIT} 3.1 and let \(V_2\) denote the vertical pointers that came from \text{SPLIT} 3.2 after the application of \text{SPLIT}. By the induction hypothesis we reobtain \((T, H)\) if we apply \text{MERGE} 3 to the pointers in \(V_1\) (this procedure is denoted by \text{MERGE} 3.1) and, seperately, to the pointers in \(V_2\) (this procedure is denoted by \text{MERGE} 3.2) such that when we start applying it to the pointers originated in \(V_2\) the lowest vertical pointer that came from \(V_1\) is strictly above the \(k\)-th row. If we apply \text{MERGE} 3 not seperately to the vertical pointers in \(V_1\) and \(V_2\), but to all vertical pointers in \((T', H')\) at once, then the first steps of this application are equal to the first steps of \text{MERGE} 3.1 until the vertical pointers of \text{MERGE} 3.1 have moved strictly above the \(k\)-th row, for the candidates originated in \(V_1\) dominate the candidates originated in \(V_2\) with respect to the backward paths order until they are strictly above the \(k\)-th row as we saw in the previous paragraph. □
Claim 6. Let \((T, H)\) be such that \(T\) is ordered up to \((i, i)\), the subtabloid of \(H\) consisting of the last \(r - i + 1\) rows is a shifted hook tabloid and let \((T', H')\) denote the pair we obtain after the application of MERGE 3. Then \((T', H')\) has property AFTER_SPLIT 2 and the application of SPLIT 3 to \((T', H')\) yields \((T, H)\).

Proof. By Claim 4 we assume that \((T, H)\) is reducible with respect to MERGE 3. Let \(e\) be the protagonist in the application of MERGE 3 to \((T, H)\) which neither touches an entry from below nor terminates in row \(z + 1\), whose row \(k\) of the place of change is maximal. We bipartite the vertical pointers of \((T, H)\) into two sets \(V_1\) and \(V_2\). The set \(V_2\) includes the vertical pointer in \((T, H)\) that results in \(e\) and if \(p_1 \in V_2\) and the vertical pointer \(p_2\) results in a protagonist that touches the protagonist that corresponds to \(p_1\) from below then \(p_2 \in V_2\). All other vertical pointers are in \(V_1\). Observe that by the algorithm a candidate associated with \(V_2\) can be a protagonist only after the candidates associated with \(V_1\) have moved strictly above the \(k\)-th row or changed to horizontal candidates: By the proof of Claim 3 and 4 a candidate \(e'\) associated with \(V_2\) and unequal to \(e\) can only be a protagonist if \(e\) was already a protagonist and has moved to a row above of \(e'\). Therefore we only have to show that \(e\) is a protagonist only after the pointers associated with \(V_1\) have moved strictly above the \(k\)-th row or changes to horizontal candidates. But this is obvious since if \(e\) is a protagonist in a step of MERGE 3 then every vertical candidate weakly below the \(k\)-th row belongs to \(V_2\). (After \(e\) is protagonist for the first time and until it changes into a horizontal candidate, \(e\) is greater than every vertical candidate strictly above of the current row of \(e\) and strictly below the \(z\)-th row. Moreover if there exist vertical candidates in this phase, which are strictly below the current row of \(e\) then they are greater than \(e\) with respect to the backward paths order and their places of change are strictly below the current row of \(e\). See the proof of Claim 3 and 4.)

We denote the application of MERGE 3 to \((T, H)\) restricted to the vertical pointers in \(V_1\) by MERGE 3.1 and the application of MERGE 3 to \((T, H)\) restricted to the vertical pointers in \(V_2\) by MERGE 3.2. Then the application of MERGE 3 is equivalent to the simultaneous application of MERGE 3.1 and MERGE 3.2, where MERGE 3.2 starts after the vertical pointers that came from \(V_1\) are strictly above of the \(k\)-th row or have changed to horizontal candidates.

By induction hypothesis we reobtain \((T, H)\) from \((T', H')\) if we apply SPLIT 3 simultaneously to the protagonists of MERGE 3.1 together with the minimal horizontal candidates \(e'\) from the if-condition (we denote this restricted application of SPLIT 3 by SPLIT 3.1) and to the protagonists of MERGE 3.2 together with these horizontal candidates (we denote this restricted application of SPLIT 3 by SPLIT 3.2) such that when a candidate in SPLIT 3.1 moves weakly below the \(k\)-th row the application of SPLIT 3.2 has already finished. Note that \(e\) is not touched from above in the application of SPLIT 3 to \((T, H)\) (if we apply it to all candidates at once), for \(e\) does not touch from below in the application of MERGE 3 to \((T, H)\). Consequently if we apply SPLIT 3 unrestricted to \((T, H)\), then this application starts with the application of SPLIT 3.1, until we are either in Case 3 of SPLIT 3 with the \(e'\) in the description of Case 3 in SPLIT 3 being equal to \(e\) or SPLIT 3.1 has terminated. In the first case we perform SPLIT 3.2 and after the termination of SPLIT 3.2 we continue with the application of SPLIT 3.1. In the second case the set \(C'\) is empty immediately after the termination of SPLIT 3.1 and \(e\) is the candidate in \(C\) with minimal row. Thus we terminate with the application of SPLIT 3.2 in this case.

\[\Box\]

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