The Theory of
ULTRALOGICS
Standard Edition

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1993
(Part I latest revision, 3 JUL 2012.)
First formal announcement of many of these results appeared in

(1) Mathematical philosophy, Vol. 2 (No. 6) (1981), #81T-03-529, p. 527.
(2) A useful *-real valued function, Vol. 4 (No. 4) (1983), #83T-26-280, p. 318.
(3) Nonstandard consequence operators I, Vol. 5 (No. 1) (1984), #84T-03-61, p. 129.
(4) Nonstandard consequence operators II, Vol. 5 (No. 2) (1984), #84T-03-93, p. 195.
(5) D-world alphabets I, Vol. 5 (No. 4) (1984), #84T-03-320, p. 269.
(6) D-world alphabets II, Vol. 5 (No. 5) (1984), #84T-03-374, p. 328.
(7) A solution to the grand unification problem, Vol. 7 (No. 2) (1986), #86T-85-41, p. 238.

Some of the refereed papers relative to MA-model concepts and its mathematical construction.

(1) A special isomorphism between superstructures, Kobe J. Math., 10(2)(1993), 125-129.
(2) Fractals and ultrasmooth microeffects, J. Math. Physics, 30(4), April 1989, 805-808.
(3) Physics is legislated by a cosmogony, Speculations in Science and Technology, 11(1) (1988), 17-24.
(4) Nonstandard consequence operators, Kobe J. Math., 4(1)(1987), 1-14.
(5) Mathematical philosophy and developmental processes, Nature and System, 5(1/2)(1983), 17-36.

And many more.
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NOTES

Some of the following typographical errors (or confusions) may (do) appear in this edition.

[1] There may be a very few times when it is not apparent as to the necessary structure for the results stated. Of course, one can always stay with the EGS. I may have written $M$ when I meant $M_1$ or conversely.

[2] The actual reason for the inverse order discussed on page 10 is that it is useful when considering adjective reasoning discussed later in this edition.

[3] Much of this edition was written in the late 1980s with the exception of Chapter 2, sections 2.1 — 2.2 which was written in 1991. In the formal model, I wanted to differentiate between the actual set used for the words and the sets used for other analytical purposes. This is done by letting members of $\mathcal{W}$ symbols that are different than those in sets such as $\mathbb{N}$ from Chapter 2 on. This is the usual approach. However, as of the date of this version, by a special technique the set of symbols $\mathcal{W}$ or extended symbols $\mathcal{W}'$ are now included.

[4] I give two different superstructure constructions and certain processes used to obtain nonstandard models. It would have been better to concentrated on the second construction which is the one actually used.

[5] Page 80. In order to have $B \cap M_i = \emptyset$, $i \neq 0$, we can reserve a special symbol in our alphabet. Then only consider a $B$ that does not contain this special symbol within any of its members. Now use this special symbol, with or without spacing, to construct the $M_i$, $i \neq 0$. Of course, once again we interpret $\wedge$ in the axiom system for $S$ as this special symbol with or without spacing.
[6] Relative to the alphabet $A$. It is trivially obvious that one can include within this alphabet the written symbols used by an intelligent life form that uses a written language and deductive rules similar to those used by humankind. If one goes to the extreme and requires infinitely many such intelligent life forms, then generalized languages as discussed on page 87 can be utilized.

[7] Beginning in Chapter IV, due to the complexity of the first-order statements, I adopt the process of replacing set-theoretically defined predicates (abbreviations) such as $x \in y$, $x = y$, $1 \leq x < z$ etc. by $(x \in y)$, $(x = y)$, $(1 \leq x < z)$. Of course, the $=$ is interpreted as set-theoretic equality.
1. INTUITIVE CONCEPTS

1.1 The Alphabet, Words, and Choice Sets.

There exists at the instant of time you read this sentence a finite set of all the symbols you have previously used throughout your life for your various forms of communication and human deduction. You may also include frames from sound motion picture film, TV tape, and the like, if you wish, in the event you require visual or audio stimuli for your deductive processes. Let $A_h$ be the set of such symbols for a given human being at this instant of time and $H_t$ the set of all human beings who exist at this instant of time. Now let $\mathcal{A} = \bigcup \{A_h \mid h \in H_t\}$ be the alphabet for humanity at this instant of time. The set $\mathcal{A}$ may be enlarged to include all of the symbols which humanity has ever used, if we wished. However, for our purposes the set $\mathcal{A}$ will suffice. Observe that the set $\mathcal{A}$ is finite and among the numerous references relative to alphabets, I direct your attention to [3] [7] [12] [13] [21].

Assume that there is included in the set $\mathcal{A}$ a distinct symbol which represents a blank symbol, say something like $||$. Now following the above references, any finite string (with repetition) of elements from $\mathcal{A}$ which is nonempty is called a word. Let $\mathcal{W}$ be the set of all words. In certain applications of these mathematical methods, a word is also called an intuitive or naive readable sentence. Only a fragment of the set $\mathcal{W}$ is used in this investigation and, in most cases, this fragment will be the set of meaningful sentences or a portion of a formal language. The metalanguage may be assume to be written in a different color than is $\mathcal{W}$. Also, the concept of the empty word will NOT be employed.

Next, apply the concept that Markov calls the abstraction of identity [12] to two words $W_1$ and $W_2$. The two words are equal if they are composed of the same symbols in the same order written left to right or right to left etc. The finitary character of each word allows for such an identification. Another intuitive string concept required for this discussion is the notion of the juxtaposition operation or join of two strings such as $W_1$, $W_2$ and this is denoted by $W_1W_2$ or $W_2W_1$. Apparently, since about 1914 [22], these two intuitive string concepts relative to word theory have been accepted as a reasonable consequence of the finitary character of such forms.

Following Robinson’s procedure in [15] applied to $\mathcal{W}$ rather than to a formal language, assume that there exists an injection $i$ from $\mathcal{W}$ onto $\mathbb{N}$, the set of natural numbers. Intuitively, such an injection exists since $\mathcal{W}$ is countably infinite. Now the term “intuitive” is utilized to denote the words, various grammatical rules, the informal logical procedures, and the like used in ordinary communication. This is to differentiate our common modes of descriptive communication from the formal theory into which these intuitive objects are mapped. Select a fixed injection for this
and all other investigations. Notice that Robinson used any set $U$ of cardinality greater than or equal to that of his set of well-formed formulas. The procedure of intuitively mapping objects such as the set $\mathcal{W}$ onto concrete mathematical objects is well established and has been a major procedure in geometry since Descartes. The Cantor Axiom used by Hilbert and Birkhoff [2] for modern geometry assumes that such as map exists from the set of points in a straight line onto the real numbers. Evidently, this injection $i: \mathcal{W} \to \mathbb{N}$ falls into a category of similar content as these well-known geometric assumptions. Indeed, $i$ can be an into Gödel coding.

Prior to continuing this introductory section a brief discussion of the Set Theory being used and related matters appears useful. The general set theory being used is $\text{ZFC} = \text{ZF} + \text{AC}$. The $\text{ZF}$ (i.e. Zermelo-Fraenkel axioms) and the Axiom of Choice $\text{AC}$ may be found listed on pages 2 —19 of [5] among hundreds of other references. Within this general set theory we are working within a model for the axiom system $\text{ZFH} = \text{ZFA} + \text{AC} + A$ is countably infinite. The axiom system $\text{ZFA}$ is the Zermelo-Fraenkel axiom system with atoms (i.e. urelements or individuals). The set of atoms is the $A$ in the above formulation. All of the axioms are expressible in a first-order language with the predicates $\in$ and $=$, where $\emptyset$ and $A$ are constants. The almost completely written axioms or modifications to $\text{ZF} + \text{AC}$ axioms that yield $\text{ZFA} + \text{AC}$ may be found on pages 122 of [5], page 44 of [6] and, due to the use of individuals, the actual system studied throughout [20]. I point out that the language used is a first-order language with the logical axioms for equality.

Assuming the consistency of $\text{ZF}$, Gödel showed that there is a model in which the $\text{ZFC}$ axioms hold. Thus the consistency of $\text{ZF}$ implies the consistency of $\text{ZFC}$. Using $\text{ZFC}$ our model construction for $\text{ZFH}$ is a slight modification of that which appears in problem 1 on page 51 of [6]. In the modification, let $C$ be a countably infinite set of infinite subsets of the $\omega$ in the $\text{ZFC}$ model. Moreover, we bijectively map the order relation for $\omega$ onto $C - \{a_0\}$. Thus, for a bijection $f: (C - \{a_0\}) \to \omega$ define the well-ordering $<$ on $C - \{a_0\}$ as follows: For each $x, y \in (C - \{a_0\})$, $x < y$ iff $f(x) \in f(y)$. By the construction of the model, this well-ordering is also a member of this model. We also have need of an interpretation of $=$ for the set $(C - \{a_0\})$. This is to be the logical equality and as such is interpreted to be the identity relation on $(C - \{a_0\})$ which also exists within this model. The set $(C - \{a_0\})$ with its order and equality relation is to be considered the set of natural numbers $\mathbb{N}$ within this model. On the other hand, we also have within this model its $\omega$ that will be used when certain constructions are considered. The interpretation of $=$ for objects not in $(C - \{a_0\})$ is set-theoretic equality and the $\in$ in this model is the same as the $\in$ in our model for $\text{ZFC}$. All of this yields a model for the $\text{ZFH}$ axioms that are, hence, consistent relative to $\text{ZF}$. Set $\mathcal{W}$ will be intuitively considered an object in this model. It does not contain the empty set since the empty word is not used. The injection $i$ is to be considered as a intuitive map from $\mathcal{W}$ onto $\mathbb{N}$ within this model for $\text{ZFH}$. [See page 19 detailed refinements.]

All of the new results are obtained by using informal mathematical reasoning relative to $\text{ZFH}$, proving results by means of the “observer language” and using
acceptable mathematical procedures. We subscribe to the remarks quoted in chapter III of the work by Rosiowa and Sikorski [14] as well as the observations made by Stoll [18, p. 228] relative to Rosser’s philosophy of mathematics. That is that the procedures used, even though informal in nature, are capable of formalization, and such things as the “formal” proofs of the formal consistency of \( \text{ZFH} \) relative to \( \text{ZF} \) by means of model theory methods imply that the same informal but acceptable procedures which we use, when convincingly presented, will not produce any contradictions.

Now, intuitively, consider that there is a fixed dictionary \( D \) that uses a subset of an individual’s personal \( Y \subset A \). Let \( W_Y \) be the set of all words generated by \( Y \). Then descriptions of natural processes, of their behaviors and developments, as well as psychological descriptions for human behavior, philosophical descriptions for belief-systems, life styles and any other descriptions that are of interest are elements of the set \( \mathcal{P}(W_Y) \), the power set of \( W_Y \). Such sets are also informally called describing sets. The meanings of such sets are understandable by the individual and have great content. Now there exists a set \( B_Y \subset \mathcal{P}(W_Y) \) composed of all and only these describing sets. Evidently, the set \( B_Y \) is finite. Consider the set \( \mathcal{P}(B_Y) \). In some applications of the forthcoming mathematical results, the set \( \mathcal{P}(B_Y) \) is called a free (will) choice set. For the universal free (will) choice set, simply consider the set of all \( \mathcal{P}(B_Y) \) as \( Y \) varies over all humanity which exists at a given instant of time. This is still a finite set. Notice that all that has been done to obtain these free choice sets can easily be formulated with respect to the set \( \mathbb{N} \). Simply consider \( i[W_Y] \), and then the formal describing sets are the corresponding elements of \( \mathcal{P}(i[W_Y]) \) under the injection \( i \). The intuitive describing sets can be recaptured by considering the inverse image of \( i \) on these elements of \( \mathcal{P}(i[W_Y]) \). The (general) Axiom of Choice is not necessary in order to obtain these free choice sets. However, in the sequel, The Axiom of Choice is used in the construction of the NSP-structure. Consequently, The Axiom of Choice is utilized when we interpret, in some applications, the free choice sets as elements of the NSP-structure.

1.2 Readable Sentences.

For the purposes of this research, not only is the finitary concepts of the abstraction of identity and join accepted but evidently a third fundamental procedure needs to be introduced and investigated. Consider the symbol string ‘mathematics’. Now this can be obtained or “read” by numerous applications of the join operation with symbol strings of lesser length. For example, let \( W_1 = \text{math}; W_2 = \text{e}; W_3 = \text{mat}; W_4 = \text{ics} \). Then \( \text{mathematics} = W_1W_2W_3W_4 \). Observe that \( W_1, W_2, W_3, W_4 \) are the syllables for this word. Clearly, for writing purposes, we could consider \( \text{mathematics} = W_1W_2\ldots W_{11} \), where \( W_j, j = 1, \ldots, 11 \), are the single letters in the word. The necessity to consider intuitively a symbol string as composed of words of various length joined together from left to right leads, when \( i[W] \) is considered, to the concept of the set of (special) partial sequences.

Let nonempty \( H \subset i[W] \) and \( n \in \mathbb{N} \), where \( 0 \in \mathbb{N} \). For simplicity, let \( H^n = H^{[0,n]} \) be the set of all maps from the segment \([0,n]\) into \( H \). An element of \( H^n \) is
called a *partial sequence* even though this definition is a slight restriction of the one that usually appears in the literature. Now let \( P_H = \bigcup \{ H^n \mid n \in \mathbb{N} \} \). In general, if \( H = (i[I]) \), then the \( H \) notation will be deleted from such symbols as \( P_H \). The set \( P_H \) is a set of partial sequences and is a subset of \( P_{(i[I])} = P \).

Let \( P_1, P_2 \) denote the set-theoretic first and second coordinate projection maps. Then the following first-order sentences, where the usual assortment of set-theoretic abbreviations for subset, functions, domains, ranges, etc. are used, hold and represent two basic properties for the object \( P_H \).

\[
\forall x(x \in P_H) \rightarrow \exists y((y \in \mathbb{N}) \land (P_1[x] = [0, y]) \land (P_2[x] \subset H));
\]
\[
\forall x(x \subset \mathbb{N} \times H) \land \exists y((y \in \mathbb{N}) \land (P_1[x] = [0, y]) \land (w \in \mathbb{N}) \land (z \in H) \land (z_1 \in H) \land ((w, z) \in x) \land ((w, z_1) \in x) \rightarrow z = z_1) \rightarrow x \in P_H).
\]

(1.2.1)

I point out that ever since G"odel used a natural number coding for certain metamathematical concepts, interpreting naive or intuitive processes involving symbol strings as concepts relative to \( \mathbb{N} \) has prevailed and become accepted by mathematicians. As Kleene writes: “*Metamathematics has become a branch of number theory.*” [7, p. 205] Consequently, it is clearly justified to use partial sequences to discuss the “ordering” of symbol strings. This ordering will be associated with the finitary ordering obtained by joining words by juxtapositioning them in a specific intuitive order.

Consider \( f \in H^n, n \in \mathbb{N} \). Then \( f(0), f(1), \ldots, f(n) \in H \). The *order induced by \( f \)* is defined to be the simple inverse order generated by \( f \) applied to the simple order of \([0, n]\). Formally, for each \( f(j), f(k) \in f([0, n]) \), define the order induced by \( f \) to be \( f(k) \leq_f f(j) \) iff \( j \leq k \), where as usual the order \( \leq \) is the simple order on \([0, n]\) induced by the simple order on \( \mathbb{N} \). In general, the notation \( \leq_f \) will NOT be specifically used to denote this \( f \) induced order but, rather, the order will be indicated by writing the symbols \( f(n), \ldots, f(0) \) from left to right in an ordered fashion. This corresponds to what the intuitive ordering would be under the inverse of \( i, i^{-1} \), when it is used to recapture the original symbol string. Remember, that everything done with the join operation, partial sequences and the induced order is finitary in character. Apparently, care is required in the selection of the \( H \) if all the ways a given word may be partitioned for readability are to be investigated. It is required that \( i^{-1}[H] \) contain enough symbols for this purpose. However, an approach is now developed that eliminates this apparent difficulty.

Relate the induced ordering for some \( f \in P \) to the join in the following manner. The word \( W_0 \) to which \( f \in (i[I])^n, n \in \mathbb{N} \), corresponds is

\[
W_0 = (i^{-1}((f(n)))(i^{-1}((f(n-1))))) \cdots (i^{-1}((f(0)))).
\]

(1.2.2)

Using the abstraction of identity and this ordered join concept, a basic equivalence relation is obvious. For \( f, g \in P, f \in H^n, g \in H^n, \) let

\[
f \sim g \text{ iff }...\]
(1.2.3)

\[(i^{-1}((f(n))) \cdots (i^{-1}((f(0)))) = (i^{-1}((g(m))) \cdots (i^{-1}(((0)))).\]

Observe that this process is still finitary in nature and consequently effectively recognizable. Clearly, “~” is an equivalence relation on \(P\) since the identity = is such a relation on \(W\). For each \(f \in P\) let \([f]\) denote the corresponding basic equivalence class. Now if the cardinality of an intuitive word \(|W_0| = m+1\) = the total number of symbols in \(W_0\) including repetitions and \(W_0 = (i^{-1}((f(m))) \cdots (i^{-1}((f(0))))\), then each equivalence class is a nonempty finite set and theoretically each element in each equivalence class can be effectively recognized.

Recall that many of the intuitive concepts associated with word theory and algorithms [7] [12] [13] [22] have much less rigorously defined concepts than this equivalence relation even though such word theory concepts have been extensively employed. Before proceeding, however, here are some of the simple facts about these equivalence classes. Consider \([f]\) and let the corresponding word for \(f\) be \(W_0\), where \(|W_0| = m + 1\). Then there exists two unique maps \(f_m, f_0 \in P\) such that \(f_m \sim f_0, f_m \in (i(|W|))^m, f_0 \in (i(|W|))^0\) and

\[
W_0 = (i^{-1}((f_m(m))) \cdots (i^{-1}((f_m(0)))) = (i^{-1}(f_0(0))).
\]

Furthermore, for each \(k \in \mathbb{N}\) such that \(0 \leq k \leq m\), there exists \(g \in [f]\) such that \(g \in (i(|W|))^k\). And, for each \(k \in \mathbb{N}\) such that \(k > m\) there does not exist any \(g \in (i(|W|))^k\) such that \(g \in [f]\). Finally, how is a particular class \([f]\) to be intuitively interpreted? In order to interpret a class \([f]\) in \(W\), simply select any element in \([f]\), say \(f_n\), and effectively construct the word \((i^{-1}((f_n(n))) \cdots (i^{-1}((f_n(0)))) = W_0\). The word \(W_0\) is called the intuitive or naive interpretation for the class \([f]\).

Let \(E\) be set of all equivalence classes generated by \(\sim\) on \(P\). The set \(E\) is called the set of (formal) readable sentences. The term “readable sentence” is used in two contexts, the intuitive readable sentence that is a member of \(W\) and the corresponding formal readable sentence in \(E\). The terms “intuitive” or “formal” will not be used where no confusing would occur.

1.3 Human Deduction.

As has become the custom, the concepts of human deduction (i.e. reasoning) will first be discussed intuitively with respect to the set \(W\). Certain metatheorems relative to such processes will be established prior to associating these processes with the more formal set \(E\). Recall Tarski’s [21] basic axioms for the undefined (finite) consequence operators \(C\) on a set of meaningful sentences \(A\). As usual, let \(|A| = \) the cardinality of \(A\), the symbol \(\mathcal{P}(A)\) denote the powerset of \(A\), \(F(A)\) denote the set of all finite subsets of \(A\) and \(C: \mathcal{P}(A) \rightarrow \mathcal{P}(A)\). Tarski first bounds the cardinality of \(A\)

\[
(1) 0 < |A| \leq \aleph_0
\]
Now the map $C$ is a (finite) consequence operator if $A \neq \emptyset$ and

| Axiom  | Condition |
|--------|-----------|
| (2)    | for each $B \subset A$, then $B \subset C(B) \subset A$, |
| (3)    | for each $B \subset A$, then $C(C(B)) = C(B)$, |
| (4)    | for each $B \subset A$, then $C(B) = \bigcup\{C(F) \mid F \in F(B)\}$ |

Axioms (2), (3), (4) appear to be considerably more significant than does axiom (1) in these applications and the cardinality of the set of sentences considered is not, usually, so bounded. Actually the consequence operator is generated by a slightly more general concept which is called the deductive process. Let $A$ be a nonempty set. Then any nonempty $k \subset F(A) \times A$ is a deductive process. The term deduc
tive operator is also used. A deductive process $k \subset F(A) \times A$ is total if for each nonempty $B \subset A$ there exists some $F \in F(B)$ and some $a \in A$ such that $(F,a) \in k$. For a total $k \subset F(A) \times A$, let

\[(i) \quad C_k = \{(x,y) \mid x \in P(A) \land y \in P(A) \land y \neq \emptyset \land\]
\[
\forall w(w \in A \land w \in y \iff \exists z(z \in F(x) \land (z,w) \in k))\}
\]

(1.3.1)

And, $(B,\emptyset) \in C_k \iff B = \emptyset \iff \emptyset \notin P_1[k]$.

Notice that the definition of $C_k$ implies that $C_k: P(A) \to P(A)$.

Assume that $A$ is a nonempty arbitrary set that corresponds to Tarski’s set of meaningful sentences with the exception that axiom (1) need not hold for $A$.

**Theorem 1.3.1.** Let total $k \subset F(A) \times A$. Then $C_k$ satisfies Tarski’s axiom (4).

**Proof.** Obviously, if $B \subset A$, then $C_k(B) \subset A$. Let $C_k(B) = \emptyset$. Then $B = \emptyset$ implies that $C_k(B) = \bigcup\{\emptyset\} = \bigcup\{C_k(F) \mid F \in F(\emptyset)\} = \bigcup\{C_k(\emptyset)\}$. Consequently, assume that $C_k(B) \neq \emptyset$ and $x \in C_k(B)$. Then there exists some $F_0 \in F(B)$ such that $F_0 \subset B \subset A$ and $(F_0,x) \in k$. From the definition of $C_k$, this implies that $x \in C_k(F_0)$. Thus $C_k(B) \subset \bigcup\{C_k(F) \mid F \in F(B)\}$. On the other hand, assume that $x \in \bigcup\{C_k(F) \mid F \in F(B)\}$. Then there exists some $F_1 \in F(B)$ such that $x \in C_k(F_1)$. Hence there exists some $F_2 \in F(F_1) \subset F(B)$ and $(F_2,x) \in k$. Consequently, $x \in C_k(B)$ and this completes the proof.

Many of the usual deduction processes, such as propositional deduction, satisfy the following additional properties. Let $k \subset F(A) \times A$ be ordinary whenever

(i) if $(A_0,b) \in k$ and $A_0 \subset D \in F(A)$, then $(D,b) \in k$ and

(ii) if $\{(A_0,b_1),\ldots,(A_0,b_n)\} \in k$ and $\{(b_1,\ldots,b_n),c\} \in k$, then $(A_0,c) \in k$.

**Theorem 1.3.2.** Let total and ordinary $k \subset F(A) \times A$. If $C_k$ satisfies axiom (2) of Tarski, then $C_k$ is a consequence operator.

**Proof.** Since axiom (4) holds for $C_k$, now proceed to establish axiom (3). Tarski [21] has shown that axiom (4) implies that if $B \subset D \subset A$, then $C_k(A) \subset C_k(D)$. Thus assume $B \subset A$. Then $B \subset C_k(B)$ implies that $C_k(B) \subset C_k(C_k(B))$. Now if
\[ C_k(C_k(B)) = \emptyset, \text{ then this implies that } C_k(B) = \emptyset \text{ and } C_k(C_k(B)) = C_k(B). \]

Let \( x \in C_k(C_k(B)) \). Then there exists some finite \( F_0 \subset C_k(B) \) such that \( (F_0, x) \in k \). If \( F_0 = \emptyset \), then \( \emptyset \subset B \) implies \( x \in C_k(F_0) \subset C_k(B) \). Assume that \( F_0 \neq \emptyset \), say \( F_0 = \{a_1, \ldots, a_n\} \). Then there exists finitely many \( F_i \subset B, i = 1, \ldots, n \) such that \( (F_i, a_i) \in k \). Let \( F' = F_1 \cup \cdots \cup F_n \in F(B) \subset F(A) \). Then ordinary (i) implies that \( (F', a_i) \in k, i = 1, \ldots, n \). But \( (F_0, x) \in k \) and ordinary part (ii) implies that \( (F', x) \in k \). Consequently, \( x \in C_k(B) \) and this completes the proof. 

In passing note that if axiom (2) holds for \( C_k \) and (i) of ordinary holds for \( k \), then for each \( x \in A, (\{x\}, x) \in k \). Obviously, if for each \( x \in A, (\{x\}, x) \in k \), then axiom (2) holds for \( C_k \). In this case, we say that the deduction process is \textit{singular}. Combining the above results, we have the next theorem.

**Theorem 1.3.3.** If for nonempty \( A \) the deductive process \( k \subset F(A) \times A \) is a total, ordinary and singular, then \( C_k \) is a consequence operator for \( A \).

Is there an obvious deduction process generated by a given consequence operator? Let \( C \) be a consequence operator on \( \mathcal{P}(A) \). Define \( k_c \subset F(A) \times A \) as follows:

\[ (F, a) \in k_c \iff F \in F(A) \text{ and } a \in C(F). \quad (1.3.2) \]

By axiom (2), it follows that \( k_c \) is total.

**Theorem 1.3.4.** If \( C \) is a consequence operator on \( A \), then \( C_{k_c} = C \).

Proof. First assume that \( C(B) = \emptyset \). Then axiom (2) implies that \( B = \emptyset \). Hence \( \emptyset \notin P_1(k_c) \). From this it follows that \( C_{k_c}(\emptyset) = \emptyset = C(B) \). Now let \( B \subset A \) and suppose that \( x \in C(B) \). Then there exists some \( F \in F(B) \) such that \( x \in C(F) \). Since \( F \in F(A) \), then \( (F, x) \in k_c \). From the definition of \( C_{k_c} \), it follows that \( x \in C_{k_c}(B) \); which yields that \( C(B) \subset C_{k_c} \). Now if \( C_{k_c}(B) = \emptyset \), then \( C_{k_c}(B) \subset C(B) \). Hence suppose that \( y \in C_{k_c}(B) \). Then there exists \( F \in F(B) \) such that \( (F, y) \in k_c \). Since \( F \in F(A) \), then \( y \in C(F) \) implies that \( C_{k_c}(B) \subset C(B) \). Therefore, for each \( B \in \mathcal{P}(A) \), \( C_{k_c}(B) = C(B) \) implies that \( C_{k_c} = C \) and the proof is complete. 

From the above results, a consequence operator can be thought of as being determined by a deductive process and, in certain cases, conversely. When deductive processes are described in a metalanguage, then such properties as total, ordinary or singular can often be easily established. Sometimes we say that a deductive process \( k \subset F(A) \times A \) satisfies the Tarski axioms (2), (3), or (4) if \( C_k \) satisfies (2), (3), or (4), respectively.

Our next task is to find an appropriate correspondence between any \( k \subset F(\mathcal{W}) \times \mathcal{W} \) and some \( K \subset F(\mathcal{E}) \times \mathcal{E} \) such that the axioms of Tarski and set-theoretic properties are preserved. It is clear that this relation should be defined relative to the quotient map determined by \( \sim \). Thus for each \( w \in \mathcal{W} \), let \( f_w \in P \) be such that \( f_w \in (i|\mathcal{W})^0 \) and \( f_w(0) = i(w) \). Consider the intuitive bijection \( \Theta: \mathcal{W} \rightarrow \mathcal{E} \), the \textit{basic bijection}, defined by \( \Theta(w) = [f_w] \) for each \( w \in \mathcal{W} \). In the usual manner, extend \( \Theta \) to its corresponding set functions and the like. Since it is not assumed that all readers of this book are aware of these definitions, we present them in the
context of consequence operator theory. However, for the next few results, the map will not be restricted to the specially defined map \( \Theta \) but rather we establish them for any arbitrary injection \( \beta: A \to X \). Recall that

(i) if \( B \subset A \), then \( \beta[B] = \{ \beta(x) \mid x \in B \} \),

(ii) for \( B \subset \mathcal{P}(A) \), \( \beta[B] = \{ \beta[x] \mid x \in B \} \),

(iii) for \( k \subset F(A) \times A \), let \( \beta k = \{ (\beta[x], \beta(y)) \mid (x, y) \in k \} = \{ (z, w) \mid z \in F(x) \land w \in X \land \exists x \exists y(x \in F(A) \land y \in A \land z = \beta[x] \land w = \beta(y) \land (x, y) \in k) \}, \)

(iv) for \( C: \mathcal{P}(A) \to \mathcal{P}(A) \), let \( \beta C = \{ (\beta[x], \beta[y]) \mid (x, y) \in C \} \).

By considering the inverse, \( \beta^{-1} \), it is evident that if \( k \) is total on \( A \), then \( \beta k \) is total on \( \beta[A] \); if \( k \) is ordinary on \( A \), then \( \beta k \) is ordinary on \( \beta[A] \); and if \( k \) is singular on \( A \), then \( \beta k \) is singular on \( \beta[A] \).

Numerous propositions are immediate consequences of the set-theoretic definitions associated with the injection \( \beta \). However, even though the following propositions hold true from elementary algebraic results, we prove them explicitly for they refer to intuitive structures and metatheoretic results. Furthermore, they are of considerable importance to the foundations of this subject.

As before, let \( \beta \) be an injection on \( A \) into \( X \), let \( k \subset F(A) \times A \) and \( C: \mathcal{P}(A) \to \mathcal{P}(A) \).

**Theorem 1.3.5.** Let \( B \subset \beta[A] \). Then

\[
\beta^{-1}[(\beta C)(B)] = C(\beta^{-1}[B]), \\
(1.3.3) \quad (\beta C)(\beta[\beta^{-1}[B]]) = (\beta C)(B) = \beta(C(\beta^{-1}[B])].
\]

Proof. Let \( B \subset \beta[A] \). Then \( \beta^{-1}[\beta[B]] \in \mathcal{P}(A) \). Hence, \( (\beta^{-1}[B], C(\beta^{-1}[B])) \in C \). Consequently,

\[
(\beta[\beta^{-1}[B]], \beta(C(\beta^{-1}[B])) \in \beta C
\]

implies since \( \beta C \) is a map that

\[
\beta[C(\beta^{-1}[B])] = (\beta C)(\beta[\beta^{-1}[B]]) = (\beta C)(B).
\]

Thus

\[
\beta^{-1}[\beta[C(\beta^{-1}[B])]] = \beta^{-1}[(\beta C)(B)] = C(\beta^{-1}[B])
\]

and this completes the proof. \( \blacksquare \)

The next two results are important consequences of Theorem 1.3.5.

**Theorem 1.3.6.** If \( C \) is a consequence operator on \( \mathcal{P}(A) \), (i.e. it satisfies axioms (2), (3), (4) of Tarski), then \( \beta C \) is a consequence operator on \( \mathcal{P}(\beta[A]) \).

Proof. Observe that \( \beta[\mathcal{P}(A)] = \mathcal{P}(\beta[A]) \) and that \( \beta C: \mathcal{P}(\beta[A]) \to \mathcal{P}(\beta[A]) \).

(i) Let \( B \subset \beta[A] \). Then \( \beta^{-1}[B] \subset A \) implies that \( \beta^{-1}[B] \subset C(\beta^{-1}[B]) \subset A \). Hence \( B \subset \beta[C(\beta^{-1}[B])] = (\beta C)(B) \subset \beta[A] \).
(ii) Let $B \subset \beta[A]$. Then $\beta^{-1}[B] \subset A$ and $C(C(\beta^{-1}[B])) = C(\beta^{-1}[B])$. Consequently

$$\beta[C(C(\beta^{-1}[B]))] = (\beta C)(\beta[C(\beta^{-1}[B])]) = (\beta C)(\beta C)(\beta^{-1}[B]) = (\beta C)(\beta^{-1}[B]) = (\beta C)(\beta^{-1}[B]) = (\beta C)(B). \quad (1.3.7)$$

(iii) First, we show that $\beta[F(\beta^{-1}[B])] = f[B]$. Let $F \in F(\beta^{-1}[B])$. Then $\beta[F] \subset B$ and $\beta[F] \in F(B)$. Therefore, $\beta[F(\beta^{-1}[B])] \subset F(B)$. Conversely, let $F \in f(B)$. Then $\beta^{-1}[F] \in F(\beta^{-1}[B])$, since $|F| = |\beta^{-1}[F]|$. Thus $\beta[F(\beta^{-1}[B])] = F(B).

For each $B \subset \beta[A]$, $C(\beta^{-1}[B]) = \bigcup \{C(F) \mid F \in F(\beta^{-1}[B])\}$. This implies that

$$\beta[C(\beta^{-1}[B])] = \bigcup \{\beta[C(F)] \mid F \in F(\beta^{-1}[B])\} = \bigcup \{(\beta C)(\beta[F]) \mid F \in F(\beta^{-1}[B])\} = (\beta C)(\beta^{-1}[B]) = (\beta C)(\beta^{-1}[B]) = (\beta C)(B). \quad (1.3.8)$$

Results (i), (ii), (iii) imply that $(\beta C)$ is a consequence operator and the proof is complete. \[ \square \]

Theorem 1.3.7. If $k \subset F(A) \times A$ is total, then $\beta(C_k) = C_{\beta k}$.

Proof. Let $(x, y) \in \beta(C_k)$. Then $(\beta^{-1}[x], \beta^{-1}[y]) \in C_k$. Notice that $\beta^{-1}[y] = \emptyset$ iff $y = \emptyset$. Assume that $z \in \beta^{-1}[y]$. Then there exists some $w_z \in F(\beta^{-1}[x])$ such that $(w_z, z) \in k$. Hence, it follows that for each $\beta(z) \in y$ there exists some $\beta[w_z] \in F(x)$ such that $(\beta[w_z], \beta(z)) \in \beta k$. This leads to $(x, y) \in C_{\beta k}$. Now if $y = \emptyset$, then $\beta^{-1}[y] = \emptyset$ implies that $\beta^{-1}[x] = \emptyset$ and $x = \emptyset$. Hence, $(\emptyset, \emptyset) \in \beta(C_k)$ implies that $(\emptyset, \emptyset) \in C_k$. The definition gives $\emptyset \notin P_1(k)$ and $\beta[\emptyset] = \emptyset \notin P_1(\beta k)$. Thus $(\emptyset, \emptyset) \in C_{\beta k}$.

On the other hand, for each $z \in y \neq \emptyset$ there exists some $F \in F(x)$ such that $(F, z) \in \beta k$. Hence $(\beta^{-1}[F], \beta^{-1}(z)) \in k$ and again $z \in y$ iff $\beta^{-1}(z) \in \beta^{-1}[y]$; $\beta^{-1}[F] \in F(\beta^{-1}[x])$ imply that $(\beta^{-1}[x], \beta^{-1}[y]) \in C_k$. Also, if $y = \emptyset$, then $x = \emptyset$ (i.e. $(\emptyset, \emptyset) \in C_{\beta k}$) and $\emptyset \notin P_1(\beta k)$. Hence $\emptyset \notin P_1(k)$ and $(\emptyset, \emptyset) \in \beta(C_k)$ and the proof is complete. \[ \square \]

I will not continue with this piecemeal approach but rather use a more general result established within the next chapter, where $E$ will be defined on a set $\mathcal{W}$ $\mathcal{W} \cap \mathbb{N} = \emptyset$. 

\[ \text{The Theory of Ultralogics} \]
2. THE G-STRUCTURE

2.1 A Basic Construction.

A primary construction will be a superstructure. For $X$, a superstructure is constructed as follows: Let ground set $X = X_0$. Then by induction, let $X_{n+1} = X_n \cup \mathcal{P}(X_n)$. Now let $\mathcal{X} = \bigcup \{X_n \mid n \in \omega\}$. The set $\mathcal{X}$ is a superstructure over $X_0$. Within our model for ZFA a set $B$ is $X_0$-transitive if for each $x \in B$ either $x \in X_0$ or $x \subseteq B$.

**Theorem 2.1.1** For each $n \in \omega$, the set $X_n$ is $X_0$-transitive.

Proof. The proof is by induction. Let $n = 0$. The $X_0$ is $X_0$-transitive immediately from the definition. Assume that for $n$, the set $X_n$ is $X_0$-transitive. Consider $X_{n+1}$ in the above construction. We need only check any $x \in X_{n+1} - X_0 = (X_n \cup \mathcal{P}(X_n)) - X_0$. Hence, either $x \in X_n - X_0$ or $x \in \mathcal{P}(X_n) - X_0$. In the first case, $x \subseteq X_n$ by the induction hypothesis. In the second case, $x \subseteq X_n$ by the definition of the power set operator. Since $X_n \subseteq X_{n+1}$, it follows that $x \subseteq X_{n+1}$.

Thus by induction the proof is complete. $lacksquare$

**Theorem 2.1.2** For each $n \in \omega$, if $y \in x \in X_{n+1} - X_0$, then $y \in X_n$.

Proof. For $n = 0$, $y \in x \in X_0 \Rightarrow y \in X_0$. Assume that it holds for $n - 1$, $n \geq 1$. Let $y \in x \in X_{n+1} - X_0$. Then $x \in (X_n \cup \mathcal{P}(X_n)) - X_0$. If $x \in X_n - X_0$, then by the induction hypothesis $y \in X_{n-1}$. But $X_{n-1} \subseteq X_n$ implies that $y \in X_n$. If $x \in \mathcal{P}(X_n)$, then $x \subseteq X_n$ implies that $y \in X_n$. By induction the proof is complete. $lacksquare$

Obviously, since we have only used facts about ZF to establish Theorems 2.1.1, 2.1.2, these theorems hold for superstructures within ZF. Recall that if $A$ is a set of atoms, then this means that if $x \in A$, then $x \neq \emptyset$ and $y \in x$ is not defined. A nonempty ground set $X_0$ is $n$-atomic if $x \in X_0$ implies that $x \neq \emptyset$ and if $y \in x \in X_0$, then $y \notin X_n$. Two important observations relative to $n$-atomic. If $X_0$ is a set of atoms, then $X_0$ is $n$-atomic for each $n \in \omega$. If $X_0$ is $n$-atomic, then $X_0$ is $k$-atomic for each $k$ such that $0 \leq k \leq n$. For each $X_n$, $n \geq 0$, let $M_{X_n} = \{(x, y) \mid (x \in X_n) \land (y \in X_n) \land (x = y)\}$ and $E_{X_n} = \{(x, y) \mid (x \in X_n) \land (y \in X_n) \land (x = y)\}$, where the $=$ is set-theoretic equality on sets and the identity on atoms. In like manner, for ground set $Z$, defined $M_{Z_n}$ and $E_{Z_n}$ for the respective $Z_n$. For $n \geq 1$, an isomorphism $\beta_n$ from $\langle X_n, M_{X_n}, E_{X_n}, \emptyset \rangle$ onto $\langle Z_n, M_{Z_n}, E_{Z_n}, \emptyset \rangle$ is special if $\beta_n(X_k) = Z_k$, $0 \leq k \leq n - 1$, where $\emptyset$ is a set of atoms. Observe that since $X_k \subseteq X_{k+1} \subset X_n$, it follows that $X_k \in X_n$.

**Theorem 2.1.3** Let $A$ be a set of atoms. Suppose that for nonempty sets $X$, $Z$, $X \cap A = Z \cap A$ and there exists a bijection $\beta : X \rightarrow Z = \beta[X]$, where $\beta$ is a set-theoretic bijection and the identity on any atoms in $X \cap A$. Consider the sets $X_0 = X \cup A$, $Z_0 = Z \cup A$, and $A$ and $\emptyset$ as the constants that denote a set of atoms and the empty set in our ZFA model.
(i) If $X_0$, $Z_0$ are 0-atomic, then the structures $(X_0, M_{X_0}, E_{X_0})$ and $(Z_0, M_{Z_0}, E_{Z_0})$ are isomorphic.

(ii) For each $n \geq 1$, if $X_0$, $Z_0$ are $n$-atomic, then the structures $(X_n, M_{X_n}, E_{X_n}, A, \emptyset)$ and $(Z_n, M_{Z_n}, E_{Z_n}, A, \emptyset)$ are isomorphic and the isomorphism is special.

Proof. By $\varepsilon$-recursion, define the map $\zeta$ on $X$ as follows:

- For $x \in X$, $\zeta(x) = \beta(x)$;
- For $x \in X_0 - X$, $\zeta(x) = x$;
- For $x \in X - X_0$, $\zeta(x) = \zeta[x]$.

Let $\beta_n = \zeta|X_n$, where $n \in \omega$. We need only show that for each $n \in \omega$, if $X_0$ and $Z_0$ are $n$-atomic, then

- (A) $\beta_n$ is an isomorphism from $(X_n, M_{X_n}, E_{X_n})$ onto $(Z_n, M_{Z_n}, E_{Z_n})$;
- (B) if $n \geq 1$, then $\beta_n(A) = A$, $\beta_n(\emptyset) = \emptyset$,

and $\beta_n(X_k) = \beta_n(Z_k)(0 \leq k < n)$. Clearly $\beta_0$ is a bijection from $X_0$ onto $Z_0$. Since $X_0$ and $Z_0$ are 0-atomic, $M_{X_0} = M_{Z_0} = \emptyset$. Therefore (A) and, obviously, (B) hold for $n = 0$.

Assume that (A) and (B) hold for $n$, where $X_0$ and $Z_0$ are $n$-atomic. We show that (A) and (B) hold for $n + 1$, where now $X_0$ and $Z_0$ are $(n+1)$-atomic.

Notice that

$$[\dagger] \quad \text{for any set } x, x \in X_0 \text{ implies } x \not\in X_n,$$

for if $x \in X_0$, then $x \neq \emptyset$ and $x \cap X_n = \emptyset$ by the $n$-atomicity of $X_0$. Hence, it cannot be that $x \subset X_0$. Similarly,

$$[\dagger\dagger] \quad \text{for any set } z, z \in Z_0 \text{ implies } z \not\in Z_n.$$

Clearly, $\beta_{n+1}$ is a map from $X_{n+1}$ into $Z_{n+1}$. Suppose that $x, y \in X_{n+1}$ and $\zeta(x) = \zeta(y)$. Then $\zeta(x) = \zeta(y) \not\in Z_0$ or $\zeta(x) = \zeta(y) \in Z_0$. For the first case, $x, y \not\in X_0$. Hence, by Theorem 2.1.2, $x \subset X_n$ and $\zeta[x] = \zeta(x) = \zeta(y) = \zeta[y]$. Since $\beta_n$ is an injection, it follows that $x = y$. In the second case, it follows from $[\dagger\dagger]$ that $\zeta(x) \not\in Z_n$. But as was shown in the course of the first case, $x \not\in X_0$ implies that $\zeta(x) = \zeta[x] \subset Z_n$. Hence $x \in X_0$. The same argument shows that $y \in X_0$. Again the injectivity of $\beta_n$ implies that $x = y$. Consequently, $\beta_{n+1}$ is an injection from $X_{n+1}$ into $Z_{n+1}$.

To show that $\beta_{n+1}$ is a surjection, let $z \in Z_{n+1}$. If $z \in Z_n$, then the surjectivity of $\beta_n$ yields an $x \in X_n \subset X_{n+1}$ such that $z = \zeta(x) \in Z_{n+1}$. If $z \not\in Z_n$, then $z \subset Z_n$. Hence again by the surjectivity of $\beta_n$, we have that $z = \zeta[x]$, where $x = \beta_n^{-1}[z] \subset X_n$.

By $[\dagger]$, $x \not\in X_0$. Hence $z = \zeta[x] = \zeta(x) \in Z_{n+1}$.
If \( x, y \in X_{n+1} \), and \( x \neq y \), then \( y \notin X_0 \) by the \((n+1)\)-atomicity of \( X_0 \). Hence, \( \zeta(x) \in \zeta[y] = \zeta(y) \). Conversely, since \( \zeta|X_{n+1} = \beta_{n+1} \) is a bijection onto \( Z_{n+1} \), it suffices to assume that \( x, y \in X_{n+1} \); \( \zeta(x), \zeta(y) \in Z_{n+1} \) and \( \zeta(x) \in \zeta(y) \). Then from the \((n+1)\)-atomicity of \( Z_0 \), \( \zeta(y) \notin Z_0 \) implies that \( y \notin X_0 \). Hence, \( \zeta(x) \in \zeta[y] \), and, thus, \( \zeta(x) = \zeta(x') \) for some \( x' \in y \). By Theorem 2.1.2, \( x' \in X_n \). Since \( \beta_{n+1} \) is an injection, \( x = x' \). Thus \( x \in y \). It follows immediately from the definition of \( \beta \) that \( \beta_{n+1}(x) = \beta_{n+1}(y) \) if and only if \( x = y \). Consequently, (A) is established for \( n + 1 \).

In general, since \( A \subset X_0 \subset X_n \), we have that \( A \notin X_0 \) by [\( \dagger \)] . Therefore, \( \beta_{n+1}(A) = \zeta[A] = A \). The remainder of (B) is easily verified for \( n + 1 \) and by induction the proof is complete.

A criterion as to when a set \( X_0 \) is \( n \)-atomic for all \( n \in \omega \) is very useful. Obviously, if \( X_0 \) is a set of atoms, then \( X_0 \) is \( n \)-atomic for all \( n \in \omega \). For the definition of \( TC \), see page 54.

**Theorem 2.1.4** Suppose that \( \emptyset \notin TC(X_0) \). If there exists \( n \in \omega \) such that \( X_0 \) is not \( n \)-atomic, then there exists some \( y \in X_0 \) such that \( TC(y) \cap X_0 \neq \emptyset \).

Proof. Observe that a straightforward inductive argument shows that for each \( n \in \omega \), if \( x \in X_n \) and \( \emptyset \notin TC(\{x\}) \), then \( TC(\{x\}) \cap X_0 \neq \emptyset \). Assume the hypotheses of the theorem. Since \( X_0 \) is not \( n \)-atomic, there exists \( x, y \) such that \( x \in y \in X_0 \) and \( x \in X_n \). Since \( TC(\{x\}) \subset TC(X_0) \) and \( \emptyset \notin TC(X_0) \), we have \( \emptyset \notin TC(\{x\}) \). Hence \( TC(\{x\}) \cap X_0 \neq \emptyset \). Since \( TC(\{x\}) \subset TC(y) \), it follows that \( TC(y) \cap X_0 \neq \emptyset \).

Application of Theorems 2.1.3 and 2.1.4 can eliminate a great deal of tedious work. Intuitively, words in a language behave, in many respects, as if they are themselves atoms. We discuss sets of them, subsets of sets of them, etc. Since the symbol strings carry a positioning, unless we extend the intuitive set-theoretic structure to a much more complex one, it would be difficult to discuss the internal construction of a word in the most simplistic of set-theoretic languages. After all, as a set of elements \( \{\text{BOOTS}\} = \{\text{BOTS}\} \) have considerably different meanings. This is why the actual intuitive ordering is indicated by the partial sequences. On the other hand, if words seem to behave like atoms within our basic logic, then certain statements about the number of steps in a formal deduction or the “number” of words used for some purpose needs to be represented by relations with respect to the natural numbers.

Let \( \mathbb{N} \) be a set of individuals in our model for \( \mathsf{ZF+H} \) that is isomorphic to \( \omega \). The set \( \mathcal{W} \) is assumed to have symbols that represent aspects of the theory of natural numbers (or rational, real, etc.) In the usual manner, these are assumed to be different than those symbols from \( \mathbb{N} \) (or other formal sets) used to analyze the set \( \mathcal{W} \). Since the specific type of entity being employed is always obvious, a symbolic distinction will not, generally, be made. Relative to the symbols in countably infinite \( \mathcal{W} \), \( \mathcal{W} \cap \mathbb{N} = \emptyset \), \( \mathcal{W} \) is a set of atoms and \( \mathbb{N} \) is a disjoint countably infinite set of atoms. The set \( \mathbb{N} \) is the natural numbers within the “intuitive” and the “formal”
portion of this model. [See note [1] at end of this section.] Let \( X_0 = \mathcal{W} \cup \mathbb{N} \). It is a simply matter to show that separating the original set of atoms in this fashion is consistent relative to \( \text{ZF} \). Since \( \mathcal{W} \cup \mathbb{N} \) are atoms, \( X_0 \) is \( n \)-atomic for all \( n \in \omega \).

We now show that the set \( \mathcal{E} \cup \mathbb{N} = Z_0 \) satisfies the hypotheses of the contrapositive of Theorem 2.1.4. First consider \( \mathcal{E} \). Note that each member of \( \mathcal{E} \) is a nonempty set and is a finite set of partial functions. That is a finite set of nonempty sets of ordered pairs. Consider any \( y \in \mathcal{E} \). Let \( x_0 \in y \). Then \( x_0 \) is a nonempty finite set of ordered pairs. Let \( x_1 \in x_0 \). Then \( x_1 \) is a nonempty finite set containing one singleton and one doubleton set. Then if \( x_2 \in x_1 \), then \( x_2 \) is a nonempty finite set of atoms. Hence, if \( x_3 \in x_2 \), then \( x_3 \in i[\mathcal{W}] \). Now none of these sets is the empty set and for each \( y \in \mathcal{E} \), \( TC(y) \cap \mathcal{E} = \emptyset \). For each \( y \in \mathbb{N} \), \( TC(y) = \emptyset \). Since \( \mathcal{W} \cup \mathbb{N} \) are atoms, then \( Z_0 \) is \( n \)-atomic for each \( n \in \omega \). Thus, for our superstructure construction, Theorem 2.1.3 now applies for each \( n \in \omega \).

The above finite argument is considered an effective procedure as are inductive definitions. What can be claimed to be the effective procedure? Even though some might accept the effective procedure as the inductive definition of members in \( \mathcal{E} \), in reality, it is the concept of finite recognizability and the fact that members of \( \mathcal{E} \) can be constructed from a concrete physical symbol model. Finite recognizability is the same concept that allows for the acceptance that Gödel numbering generates an effective injection into \( \mathbb{N} \). If we assign \( g(\text{"("}) = 3 \), \( g(\text{"",}) = 5 \), and \( g(\text{"())"}) = 7 \), then unless it is accepted (i.e. recognized) that the string \( \text{"(()")} \) is different from the string \( \text{"()"} \), the relation determined by assigning to the strings \( 2^{35577} \) and \( 2^{43755} \) would not be a map. Using a concrete symbol model, then from the construction of \( \mathcal{E} \), no object that is either an atom, a nonempty set composed of one or two atoms, an ordered pair composed from these previous sets, or a nonempty set of such ordered pairs, is equal to any nonempty finite set of sets of such ordered pairs. Thus, \( Z_0 = \mathcal{E} \cup \mathbb{N} \) is \( n \)-atomic for every \( n \in \omega \). Due to (1.2.4), there is a bijection \( \theta : i[\mathcal{W}] \rightarrow \mathcal{E} \) that associates each member of \( i[\mathcal{W}] \) with a unique member of \( \mathcal{E} \). This is coupled with the identity map on \( \mathcal{W} \). This composition yields that bijection needed for Theorem 2.1.3. Consequently, by Theorem 2.1.3, for each \( n \in \omega \) the structures \( \langle X_n, \in_i, =, \mathbb{N}, \emptyset \rangle \) and \( \langle Z_n, \in_i, =, \mathbb{N}, \emptyset \rangle \) are isomorphic.

For each \( n \in \omega \), let \( (\mathcal{E} \cup \mathbb{N})_n \) be the \( n \)th level in a superstructure based upon ground set \( \mathcal{E} \cup \mathbb{N} \). Note that relative to a superstructure based upon \( \mathcal{W} \cup \mathbb{N} \) and \( (\mathcal{E} \cup \mathbb{N}) = X_n \), for each \( n \in \omega \), there is a \( m \in \omega \) such that \( (\mathcal{E} \cup \mathbb{N})_n \subset X_m \subset X_{m+1} \). Thus, we also have that \( (\mathcal{E} \cup \mathbb{N})_n \subset X_{m+1} \).

The intuitive properties for the deductive processes with which we are concerned can be described within a first-order language and all hold within some particular \( (\mathcal{W} \cup \mathbb{N})_n \). Hence, the same properties hold in the corresponding \( (\mathcal{E} \cup \mathbb{N})_n \) through application of the isomorphism which exists between these two structures. It is, in reality, by means of \( i \) and \( \theta \), that the basic logical properties within our intuitive theory become properties within the formal mathematical theory based upon \( \mathbb{N} \). (The term “informal” means a restriction to superstructure entities determined by \( \mathcal{W} \). The term “formal” means the entire superstructure.) Assuming finite
recognizability, the injection \( i : \mathcal{W} \rightarrow \mathbb{N} \) is extended, in the usual manner, to subsets of \( \mathcal{W} \). Certain constant symbols used to name objects with specific properties in the intuitive part of the superstructure, except for \( \mathbb{N} \), its elements and \( \emptyset \) are mapped by extended \( i \) into a formal superstructure such as \( \mathcal{X} \), where the ground set is \( X_0 = \mathcal{W} \cup \mathbb{N} \). The map \( \theta \) is also extended in the same manner as \( i \). Where applicable, the composition of \( i \) followed by \( \theta \) is denoted by \textbf{bold} type face. Also, except for members of such sets as \( \mathbb{N} \) and variables, most of the informal notation for functions and the like are also represented in the standard model by \textbf{bold} font. For this example, let \( L = \mathcal{W} \) and the consequence operator \( C : \mathcal{P}(L) \rightarrow \mathcal{P}(L) \). Then \( C : \mathcal{P}(L) \rightarrow \mathcal{P}(L) \) is also a consequence operator. This notational convention is followed throughout the remainder of this book.

From these results, if \( A \subset i[\mathcal{W}] \), then any intuitive deductive process \( k \subset F(A) \times A \) or any consequence operator \( C : \mathcal{P}(A) \rightarrow \mathcal{P}(A) \) becomes under the isomorphism a deductive process \( k \subset F(A) \times A \) or a consequence operator \( C : \mathcal{P}(A) \rightarrow \mathcal{P}(A) \). Notice that we do not need to consider the isomorphism on the operators \( F \) or \( \mathcal{P} \) since \( B \in F(A) \) if and only if a sentence, with appropriate constants, of the following type holds. \( B = \emptyset \lor \forall x (x \in rB \leftrightarrow (x = a_1 \land a_1 \in A) \lor (x = a_2 \land a_2 \in A) \lor \cdots \lor (x = a_n \land a_n \in A)) \). Hence \( B \in F(A) \) if and only if \( B = \emptyset \lor \forall x (x \in B \leftrightarrow (x = a_1 \land a_1 \in A) \lor (x = a_2 \land a_2 \in A) \lor \cdots \lor (x = a_n \land a_n \in A)) \), where the isomorphism does map \( \emptyset \) onto \( \emptyset \) at level \( n = 1 \). In like manner, the power set operator. (In most cases since it reveals an order, only \( E \) is employed.) Let \( A \subset \mathcal{W} \) and let \( K_A \) denote the set of all deductive processes defined for \( A \). Now let \( C_A \) denote the set of all consequence operators defined on \( \mathcal{P}(A) \). The set \( R_A = K_A \cup C_A \) is a set of all \textbf{intuitive human reasoning processes} while \( R_A = K_A \cup C_A \) is a set of \textbf{formal human reasoning processes}.
2.2 A Remark About 2.1

The basic intuitive procedure in establishing a formal model is not relative to structures with a universe $(E \cup \mathbb{N})$. What must be done is to express in a structure such as $(W \cup \mathbb{N}, \in, =, \emptyset)$ informal statements about our language $W$, where $W$ is termed as informal ground set of atoms disjoint from $\mathbb{N}$. For named objects within such a superstructure, the same bold face convention is used for the corresponding objects within any particular $(E \cup \mathbb{N})_n$ that involves only the members of $E$.

One additional remark is in order. In 1978 when the following concepts within the discipline termed nonstandard analysis were developed, they were in the mainstream of complexity. Today, many who work in this area would consider them to be very simplistic in nature. To the neophyte, however, they may seem to be somewhat difficult.

2.3 The Nonstandard Structure

Now that the general and basic concepts for the deductive processes and consequence operators have been developed, it is necessary to consider $W \cup \mathbb{N}$ as embedded into an additional structure. The same concept that every member of the following type of superstructure corresponds to a constant within our language is to be used. With respect to the previous convention, many of these constants will be denoted in bold.

Recall for a moment how $\mathbb{N}$ is obtained. Let set $A$ be our countably infinite set of atoms, disjoint from $W$, and $f: A \rightarrow \omega$ the bijection which exists from $A$ onto the set $\omega$ of natural numbers in our model for ZFA. Consider $f^{-1}: \omega \rightarrow A$ and use $f^{-1}$ to pass the order relation (and other necessary operations) on $\omega$ from $\omega$ to $A$. For example, this yields for each $x, y \in A$, $x \leq y$ iff $f(x) \subset f(y)$ and $A$ inherits all the order properties for $\omega$. Notice that since $f$ is a bijection that $f$ preserves equality. Denote this set $A$ by $\mathbb{N}$.

We obtain a nonstandard model for a slightly different superstructure with ground set $W \cup \mathbb{N}$ than considered in section 2.1. This is one of the two basic constructions that appear in the literature. The superstructure levels are slightly different [10, p. 40], [17, p. 110], [19, p. 23]. Let $X_0 = W \cup \mathbb{N}$ and by induction, let $X_{n+1} = \mathcal{P}(\bigcup\{X_k \mid k = 0, \ldots, n\})$. Finally, let $\mathcal{N} = \bigcup\{X_n \mid n \in \omega\}$. Consider a $\kappa$-adequate ultrafilter $\mathcal{U}$, where $\kappa > |\mathcal{N}|$. By Theorem 7.5.2 in [19] or Theorem 1.5.1 in [9] such an ultrafilter exists in our ZFH and is determined by the indexing set $J = F(\mathcal{P}(\kappa))$.

Consider the structure $\mathcal{M} = (\mathcal{N}, \in, =)$. [Note: Since every member of $\mathcal{N}$ is named by a constant, including the customary ones for specific objects, these constants are suppressed in the notation.] By Theorem 7.5.3 in [19] or Theorem 1.5.2 in [19] the ultrapower construction yields by definition 3.8.1 in [9] a structure $\mathcal{M}_1 = (\mathcal{N}', \in_{\mathcal{U}}, =_u)$ which is a nonstandard model for all sentences, $K_0$, in a first-order language $L$ with equality and predicates for $\in$ and $=$ which hold in $(\mathcal{N}, \in, =)$. Assume that the cardinality of the set of constants of $L \geq |\mathcal{N}|$. Moreover, by
means of sequences from $J$ onto $N'$, the structure $(N', \in, =)$ may be considered as isomorphically embedded into $M_1$ so that $M_1$ is also an elementary extension of the embedded $(N', \in, =)$. The structure $M_1$ is also an enlargement of $(N', \in, =)$. A proof of The Fundamental Result may be found on page 39 of [19] (Theorem 3.8.3) among other places. Now in [10], Theorem 3.8 establishes this for BOUNDED sentences which hold in $(N', \in, =)$. Notice that the interpretation map from the language onto $(N', \in, =)$ has been suppressed and each member of $N'$ is simply to be considered as named by the constants in $L$.

The next step is to realize either by analysis of the ultrapower construction directly or by interpreting the appropriate sentences [17, p. 119], that $=_{U}$ is an equivalence relation with the substitution property for $\in_{U}$. Thus passing to the equivalence class $[x]$ for each $x \in N'$, define $[x] \in [y]$ iff $x \in_{U} y$ for each $x, y \in N'$. w.

Now let $(X_n)'$ be the objects in $\{[x] \mid x \in N'\}$ that correspond to $X_n$ in $N'$ under the interpretation map $I$ followed by the quotient map for the equivalence relation as determined by $=_{U}$ (i.e. the “prime” mapping.) It follows that $(X_n)'$ behaves like atoms (urelements) and each $(X_n)'$, $n > 0$ is well-founded with respect to $\in'$. This comes from interpreting the appropriate bounded sentences such as the results of Lemma 2.1 (iv) [10, p. 40] where $R = X_0$ or property (iii) on page 23 of [19] in order to obtain the $\epsilon'$ well-founded for each $(X_n)'$, $n \geq 1$. For example, for each $n \geq 1$, the following sentence

$$\forall x((x \in \mathbb{N}) \rightarrow \neg\exists y((y \in X_n) \land (y \in x)))$$

(2.3.1)

holds in the structure $(N', \in, =)$. Lastly, each $(X_n)'$, $n \geq 1$, is well-founded with respect to $\in'$ since “If $x \in y \in X_n$, then $x \in X_0 \cup X_{n-1}$” (n ≥ 1) holds in $(N', \in, =)$. Consequently, the Mostowski Collapsing Lemma [17, p. 247] or [17, p. 120] can be inductively applied to each $(X_n)'$ and obtain a corresponding set $^*X_n$. Specify the set $^*X_0$ to correspond to $(X_0)'$ and we have a unique collapse. As a result of this, the structure $\langle \bigcup \{^*X_n \mid n \in \omega\}, \in, = \rangle = \langle ^*M, \in, = \rangle$ is a set-theoretic model for all bounded sentences that hold in $(N', \in, =)$. Recall that a bounded sentence in a first-order language $L$ is one for which each quantified variable is restricted to an element of $N$. The composition of the interpretation map $I^d$, the quotient map $t$ and the collapse yield the $^*$ map from the structure $(N', \in, =)$ into $^*M$ and maps any element $a \in N$ to the element $^*a$ preserving all of the usual properties for a normal, enlarging and comprehensive monomorphism. For each $B \in N'$, let $^*B = \{^*x \mid x \in B\}$. (This definition does not correspond to that used by some other authors.) The $^*$ notation is also not placed on elements of $X_0$ when they are considered as mapped into $^*X_0$ by the $^*$ map. Observe that for each $B \in N'$, $^*B \subset ^*B$. Technically, where used, $B \subset ^*B$ also means $^*B \subset ^*B$.

Now to complete the construction, begin with the set $Y_0 = ^*X_0$ and construct a superstructure with $Y_0$ as the ground set as defined in this first example. Let $Y_{n+1} = \mathcal{P}(\bigcup \{Y_n \mid n \in \omega\})$; and let $Y = \bigcup \{Y_n \mid n \in \omega\}$.

For the above, some general principles such as the Mostowski Collapsing Lemma have been used in order to obtain $^*M$; however, an explicit construction appears
on pages 83 - 88 and Theorem 6.3 in Hurd, London. Also
construction also appears on pages 42-49 in Loeb and Wolff,
"An Introduction to Nonstandard Real Analysis," Academic Press, Orlando. [Note: This
construction also appears on pages 42-49 in Loeb and Wolff, (eds) (2000), "Non-
standard Analysis for the Working Mathematician," Kluwer Academic Publishers,
London. Also $X_n(2.1) = X_0 \cup X_n(2.3), \ n \geq 1$.] This construction simply needs
to be restricted to our language with $\in$ and $=$, where $=$ is interpreted as set-
theoretic equality on sets and the identity on atoms. For the first superstructure,
constructed using the procedure in section 2.1, let $\mathcal{N} = \mathcal{X}$. The second superstruc-
ture constructed using this procedure has as its ground set $Y_0 = \mathcal{X}_0$ and, as before,
$Y = \bigcup \{ Y_n \mid n \in \omega \}$. This leads to the G-structure $\mathcal{Y} = \langle Y, \in, = \rangle$
where since I apply this to logical operations this structure is call the Grundlegend Structure.

Now to summarize. The consistency of ZF implies the consistency of ZFH
and one can apparently use a model of ZFH to obtain the nonstandard structure
$\mathcal{M} = \langle \mathcal{N}, \in, = \rangle$. The set $\mathcal{N}$ is dependent upon the of atoms $\mathbb{N}$, the atoms
of ZFH with the order induced by $\omega$. Any sentence in a appropriate first-order
language in which each quantified variable is restricted to an element of $\mathcal{N}$ (i.e.
bounded variable) will, when each constant is replaced by the $*$ of the constant,
give a true statement about the structure $\mathcal{M}$. Moreover, $\mathcal{M}$, at the least, has
bounds for all standardly definable concurrent relations. For notation, we denote
for each $n \in X_0$, $*n = n$. In addition, all properties of the $*$ map as listed in [10],
[17], [19], among other places, hold true. Next some unusual names for G-structure
objects will be adopted in order to reflect our application to languages and logics.

Recall some of the basic terminology associated with $\mathcal{Y}$. For each $A \in \mathcal{N}$, $A$ is
called a standard entity. The set $*A$ is often called an (internal) standard entity or
better still an extended standard entity in $\mathcal{Y}$. If $b \in *A$, $A \in \mathcal{N}$, then $b$ is called an
internal entity. Indeed, $b \in \mathcal{Y}$ is internal iff there is some $X_n$ such that $b \in *X_n$. Any
entity of $\mathcal{Y}$ which is not internal is called external. These terms are generally used
throughout nonstandard analysis, but for our present purposes they are modified
as follows: Any entity of $\mathcal{Y}$ is a subtle object, some appropriate members of $\mathcal{N}$ are
human objects and any entity in $\mathcal{Y}$ which is not the $\sigma$ of a member of $\mathcal{N}$ or the $*$
of a member of $\mathcal{N}$ is a purely subtle object. Please refer to the basic references [11],
[16], [19] for other terminology and the properties of $*$. So as to avoid symbolic
confusion, from this moment on, the entire or the major part of any symbol used to
represent objects within a language and within our intuitive model will be denoted
by Roman type.
2.4 General Interpretations

Throughout this work on ultralogics, \( L_0 \) will denote the usual set of propositional formulas (wff) constructed from the connectives \( \neg, \land, \lor, \rightarrow, \leftrightarrow \), say as done by Kleene [7, p. 108] and \( L_1 \) is a set of predicate formulas with equality considered as an extension of \( L_0 \) as say constructed on page 143 of [7]. We also use the usual assortment of set-theoretic abbreviations when we consider the special predicates \( \in \) and =. Of course, \( L_1 \) is called a first-order language. Assume that \( L_1 \subset W \) and that the set of all predicate symbols is a subset of \( \{ P_i \mid i \in \mathbb{N} \} \). It is important to realize that any intuitive set-theoretic deduction process, and the like, that is discussed relative to \( W \) is to be embedded by the map \( \theta \) to a corresponding process relative to \( E \). This also applies to a member of \( W \) and the \( i \) injection. The results of any \( * \)-transfer of statements which hold relative to \( E \) or \( i[W] \) are modeled in \( *M \). Also, any results relative to \( E \) or \( i[W] \) (i.e. with respect to standard objects) can be referred back to corresponding intuitive objects relative to \( W \) by means of either the maps \( i^{-1} \) or \( \theta^{-1} \). Moreover, in order to simplify notation somewhat any formal first-order statement that explicitly involves \( i(w) \) individuals will be written with the \( i \) deleted from the notation if no confusing results from such an omission.

For example, the sentence
\[
\forall x (x \in \mathbb{N} \land x \geq 1 \rightarrow \exists y (y \in P \land \forall w (w \in \mathbb{N} \land 0 < w \leq x \rightarrow y(w) = \text{very,}|| | y(0) = \text{just})))
\]
(2.4.1)
is a slight simplification of the following sentence
\[
\forall x (x \in \mathbb{N} \land x \geq 1 \rightarrow \exists y (y \in P \land \forall w (w \in \mathbb{N} \land 0 < w \leq x \rightarrow y(w) = i(\text{very,}|| | y(0) = i(\text{just}))))
\]
(2.4.2)

Most of the following investigation is concerned with specific elements of \( P \) and specific human reasoning processes with respect to \( E \). Using the previous example, the \( * \)-transfer yields
\[
\forall x (x \in *\mathbb{N} \land x \geq 1 \rightarrow \exists y (y \in *P \land \forall w (w \in *\mathbb{N} \land 0 < w \leq x \rightarrow y(w) = *i(\text{very,}|| | y(0) = *i(\text{just}))))
\]
(2.4.3)
and which holds in \( *M \). Notice that we do not place \( * \) on the order relation \( < \) since we assume that it is but an extension of the simple order \( < \) on \( \mathbb{N} \) satisfying all of the same first-order properties. Also note that \( *i(\text{very,}|| |) = i(\text{very,}|| |) \) etc. Let \( \nu \in *\mathbb{N} - \mathbb{N} = \mathbb{N}_\infty \) (i.e. the infinite numbers). Then there exists in \( *P \) a \( * \)-partial sequence, say \( f \), such that for each \( w \in *\mathbb{N} \), \( 0 < w \leq \nu \), \( f(w) = \text{very,} \) and \( f(0) = \text{just} \). Hence even though members of \( [f] \) are not readable sentence in our sense, we can read the elements in the range of \( f \) as well as reading the intuitive ordering when \( f \) is restricted to \( [0, n], n \in \mathbb{N} \). This gives an intuitive interpretation for such an \( f \) when it is so restricted to such standard segments as well as knowledge of the properties of the ordering when not so restricted. Observe also, that if \( f \in *P - P \), then there exists some \( \nu \in *\mathbb{N} \) such that \( f: [0, \nu] \rightarrow *i [ *W ], \) where \( [0, \nu] = \{ x \mid (x \in *\mathbb{N}) \land (0 \leq x \leq \nu) \} \). Often, in our formal statements, parentheses are suppressed and the strength of connectives notion is used.
2.5 Sets of Behavior Patterns

In certain applications of subtle consequence operators the following construction is useful. This is all relative to what is called *adjective reasoning* and any equivalent form. Let $B'$ denote a list of names or simple phrases that are used to identify specific behavior patterns. These terms are taken from a specific discipline language and are, as usual, to be considered as elements of $W$. For example, the set $B'$ could be taken from the discipline called *psychology* and each term could identify a specific human behavior pattern, as general as such concepts as "kind" or "generous." You may also include any synonyms that might be equivalent to the members of $B'$. Now consider $B$ constructed as follows: an element $b \in B$ if and only if $b$ is a *qualifiable* form of a member of $B'$. That is each $b \in B$ is a $b' \in B'$ where $b'$ is written in a form so that it can be modified by the word *very*. (Or, such words as "great," "greater.") Let $B = C_0$, $C_1 = \{\text{very}, ||c\mid c \in C_0\}$. By induction let $C_{n+1} = \{\text{very}, ||c\mid c \in C_n\}$. Then an intuitive set of *modified behavior patterns* is the set $BP = \bigcup\{C_n \mid n \in \mathbb{N}\}$. The *formal modified behavior patterns* is the set $BP = \bigcup\{C_n \mid n \in \mathbb{N}\}$. Notice that each $C_m$ is a finite set.

In certain cases, the intuitive set $BP$ is associated with a set of formal propositional statements in $W$. Let $L_0$ be our propositional language constructed from a denumerable set of atoms $\{P_i \mid i \in \mathbb{N}\}$. Since $B$ is finite, then there exists an injection $j: B \rightarrow \{P_i \mid i \in \mathbb{N}\}$ and $\{P_i \mid i \in \mathbb{N}\} - j[B]$ is denumerable. Let $V \in \{P_i \mid i \in \mathbb{N}\} - j[B]$. Let the symbol string "very, ||" correspond to the partial formula "$V \land$". Then proceed to construct $BP_0$ as follows: $E_0 = j[B]$; $E_{n+1} = \{(V \land x) \mid x \in E_n\}$. Then, finally, $BP_0 = \bigcup\{E_n \mid n \in \mathbb{N}\}$.

In what follows, the modeling of human reasoning processes is often approached from two different points of view. First, from the viewpoint of such sets is $BP$, as well as many others, we have the constructed set of *meaningful sentences* in the sense of Tarski. Thus, such strings or symbols become our formal language and a simple observer language (i.e. metalanguage) is used to investigate deductive processes on $BP$. These are mapped to the formal deductive processes on $BP$. However, many of these deductive processes on a given $BP$ can be associated with other formal processes in $L_0$, especially with respect to $BP_0$. Hence, whenever possible it is acknowledged that there are at least two "models" for various $BP$ type statements, among others, that are being investigated. The basic model (and probably the simplest) is that based on $BP$. Then a somewhat more complex model is based on $L_0$. The purest is probably more comfortable with the formal languages $L_0$ and $L_1$. I feel, however, that $BP$ is as meaningful a set of sentences formed by constructive methods as is the set $L_0$ and the various forms in $BP$ are easily recognized.

‡This is an important fact. Let $X_{n+1}(X) = X_n(X) \cup P(X_n(X))$, $n \geq 0$, $X_0(X) = X$ (Def. 2.1) and $X_{n+1} = P(X_0 \cup \cdots X_n)$, $n \geq 0$, $X_0 = X$ (Def. 2.3), where $X$ is the set of individuals. For Def. 2.3, we also have that $X_p \subseteq X_n$, $1 \leq p \leq n$ and $X_{n+1} = P(X \cup X_n)$, $n \geq 0$. We show by induction that $X_n(X) = X \cup X_n$, $n \geq 0$. First, let $n = 0$ on the right. Then $X_0(X) = X$, $X_0 = X \Rightarrow X_0(X) = X \cup X_0$. Now for the specific inductive form, let
$n = 0$. Then $X_1(X) = X \cup \mathcal{P}(X_0(X)) = X \cup \mathcal{P}(X) = X \cup \mathcal{P}(X_0) = X \cup X_1$. Assume result holds for $n$. Then $X_{n+1}(X) = X_n(X) \cup \mathcal{P}(X_n(X)) = X \cup X_n \cup \mathcal{P}(X \cup X_n) = X \cup X_n \cup X_{n+1} = X \cup X_{n+1}$ and the result follows by induction.

[1] (14 DEC 2012). The set $W$ (and later $W'$) was added to the ground set on this date. This has been done to provide an additional formal structure to enhance analysis. Using the members of a language itself as constituents of a ground set for a model is well established [13, p. 70]. However, it is the set $E$ that is generally more significant for our purposes than members of the language itself since they represent the significant aspects of the formation of “words” whether they be formed by symbols, diagrams, images or coded sensory information. Hence, in this theory, members of $E$ and $E'$ still remain the basic form for a “word” or “hyper-word.”

After developing the basic aspects of this approach, it was discovered that Robinson [15, section 3] also developed a nonstandard approach to sets of symbols. I have noted this in more recent versions. (Also see Geiser, J. T. (1968). "Nonstandard logic," J. Symbolic Logic 33(2):236-250.) The idea of incorporating $E$ as a way to include how languages are constructed is not part of the Robinson foundations.

The set $W$ can contain the language for various mathematics theories such as an appropriate portion $T(\omega)$ of the theory of natural numbers. Each member of $T(\omega)$ corresponds to objects in $N$. As mentioned, one can consider members of $W$ as written in a different color than any other symbols used for any other purposes. In some cases, the “prime” notion for the symbols expressing statements about members of $N$ other than members of $W$ is employed. For example, the expressions $2' < 3'$ and $T'(\omega')$ are in extended $W'$. External to $M$, one can state that the expression $2' < 3'$ is a member of $T'(\omega')$. Or we state that $2' < 3'$ holds. (One can actually include an additional model for this purpose.) This corresponds directly to a statement $2 < 3$ that “holds” in $M$ where $2, <, 3$ are names for the corresponding “formal” objects. In general, if used, “primed” statements of this type are expressed directly in terms of the corresponding “not primed” expressions. Robinson keeps the statements used to discuss behavior of the members of his set of symbols distinct from those in $W$ by simply defining such a set and leaving the rest to one’s intuition.

The use of the embedding $i$ now seems of little significance. The embedding was used so that $E$ could simply be considered as entities from the theory of natural numbers with its long history of empirical consistency. In the beginning of nonstandard analysis where simplified type theory was employed and formal set-theory was not considered, such a consistency notion might be useful. But since formal set-theory is now being considered, any consistency considerations depends upon the assume consistency of the set-theory axioms being employed. Hence, as demonstrated, the removing of the function $i$ from both the foundations and expressions should not effect any of the interpreted results.

If $i$ is so removed, then members of $E$ are still equivalence classes but they are now partial sequences of members of the language $W$ rather than the codes
produced by application of $i$. From the viewpoint of the nonstandard model, this would mean that rather than an ultraword being considered as a partial hypersquence of members of $^*\mathbb{N}$ with some “symbols” being represented by members of $^*\mathbb{N} - \mathbb{N}$, some $^*$symbols are represented by members of $^*\mathcal{W} - \mathcal{W}$.

There is, of course, a bijection $w: \mathcal{W} \rightarrow \mathcal{E}$, there $w(a) = [g]$ and there is a $f \in [g]$ such that $f \in T^0$ and $f(0) = i(a)$ (or simply $f(0) = a$. This bijection may be useful for further developments of this Theory of Ultra-logics.
3. DEDUCTIVE PROCESSES

3.1 Introduction.

We approach the investigation of various special deductive processes by defining them intuitively for some \( A \subset \mathcal{W} \) as being \( k \subset F(A) \times A \) or \( C:\mathcal{P}(A) \to \mathcal{P}(A) \). These sets are all considered mapped to objects relative to \( \mathcal{P}(\mathcal{E}) \times \mathcal{P}(\mathcal{E}) \) for formal investigation. In at least one case, a map \( C:\mathcal{P}(A) \to \mathcal{P}(A) \) is defined for each nonempty \( A \in *\mathcal{N} \) and it is shown (trivially) that such a map is a consequence operator. Any \( C \in *\mathcal{N} \) which satisfies (in *\( \mathcal{N} \)) axioms (2), (3), (4) or their *-transform is a subtle consequence operator or subtle reasoning process, where for convenience \( C \) is restricted to \( A \subset *\mathcal{E} \).

3.2 The Identity Process.

Let \( A \subset \mathcal{E} \) be any nonempty set. For each \( B \subset A \), define \( I(B) = B \). Obviously, this is the identity operator from \( \mathcal{P}(A) \) onto \( \mathcal{P}(A) \).

**Theorem 3.2.1.** Let \( A \subset \mathcal{E} \) be nonempty. Then the identity operator on \( \mathcal{P}(A) \) is a consequence operator.

**Proof.** Let \( B \subset A \). Then \( I(B) = B \) implies that \( B \subset I(B) \subset A \). Moreover, \( I(I(B)) = I(B) = B \) for each \( B \subset A \). Finally, \( B = I(B) = \bigcup \{ F \mid F \in F(B) \} = \bigcup \{ I(F) \mid F \in F(B) \} \) and the result follows.

Let \( H \) be a nonempty set of Tarski type deductive processes. That is if \( h \in H \), then \( h \subset F(\mathcal{E}_1) \times \mathcal{E}_1 \) for some \( \mathcal{E}_1 \subset \mathcal{E} \). Also let \( H_0 \) be a nonempty set of consequence operators on some \( \mathcal{P}(\mathcal{E}_2) \) for \( \mathcal{E}_2 \subset \mathcal{E} \). Then \( *H \cup *H_0 = D_0 \) is considered a set of subtle reasoning processes. Notice that if \( C \in H_0 \), then \( *C \in D_0 \) may not be a consequence operator under our definition. The first reason for this is that axiom (4) *-transforms to read that for every internal subset of \( B \subset *\mathcal{E}_2 \), \( *C(B) = \bigcup \{ *C(F) \mid F \in *F(B) \} \). For the sentence

\[
\forall x(x \in \mathcal{P}(\mathcal{E}_2) \to \forall w(w \in \mathcal{E}_2 \to (w \in C(x) \leftrightarrow \\
\exists y(y \in F(x) \land w \in C(y))))).
\]  

(3.2.1)

holds in \( \mathcal{M} \); hence in *\( \mathcal{M} \). As is well known a *-finite set need not be finite. However, there is at least one map from \( \mathcal{P}(A) \) into \( \mathcal{P}(A) \) for any \( A \subset *\mathcal{E}_2 \) which is a true consequence operator as shown by Theorem 3.2.1. Consider any infinite \( A \subset \mathcal{E}_2 \). Then no map \( C:\mathcal{P}( *A \to \mathcal{P}( *A \) can be written as an extended standard map (i.e. the star of a standard map) from \( \mathcal{P}(A) \) into \( \mathcal{P}(A) \). This follows from the next result.

**Theorem 3.2.2.** Let infinite \( A \subset \mathcal{E}_2 \) and \( G:\mathcal{P}(A) \to \mathcal{P}(A) \). Then there exists a subset of *A upon which *G is not defined.

**Proof.** Let infinite \( A \subset \mathcal{E}_2 \), \( G:\mathcal{P}(A) \to \mathcal{P}(A) \) and \( D(G) \) be the domain of \( G = P_1[G] \). For an appropriate \( X_m \), the sentence
when the elementary properties of the $*$-map are applied. The $*$-transfer reads

\[(3.2.3) \quad \forall x(x \in X_m \rightarrow (x \in D( *G) \leftrightarrow x \in *P(A))),\]

holds in $\mathcal{M}$; hence in $*\mathcal{M}$. The *-transfer reads

\[(3.2.2) \quad \forall x(x \in X_m \rightarrow (x \in D(G) \leftrightarrow x \in P(A))).\]

when the elementary properties of the *-map are applied. The $^*G$ is only defined on the internal subsets of $^*A$. Since $^*A - A$ is external then this result follows.]

**Corollary 3.2.2.1** There exist purely subtle reasoning processes.

As to the cardinality of $\mathcal{E}$, it follows immediately that since each $x \in \mathcal{E}$ is finite, then $|\mathcal{E}| = |\mathbb{N}|$. Note the following that will be used throughout this investigation. Recall that the identification $^*X_0 = X_0$ is being used. Then if $f \in P_H$ it follows, since $f$ is a finite sequence of members of $X_0$ that $^*f = f$ under this identification. Also, since for each $f \in P_H$ the set $[f]$ is finite, then $^*[f] = [f]$. Thus $^*\mathcal{E} = \mathcal{E}$. This reduction of finite sets of finite sets of partial sequences continues to other cases such as $^*(F(P_H)) = \{^*A \mid A \in F(P_H)\} = \{|A \mid A \in F(P_H)\} = F(P_H)$. With the above results in mind, it follows that each $x \in ^*\mathcal{E} - \mathcal{E}$ is a nonfinite *-finite subset of $^*P$ for the sentence $\forall x(x \in \mathcal{E} \rightarrow x \in F(P))$ holds in $^*\mathcal{M}$. Consequently, $^*\mathcal{E} \subseteq ^*(F(P))$ and $\mathcal{E} \subseteq F(P)$ imply that $^*\mathcal{E} - \mathcal{E} \subseteq ^*(F(P)) - F(P)$. Let infinite $A \subseteq \mathcal{E}$. Then it is an important fact that there exists a *-finite $B \in ^*(F(A))$ such that $^*A \subseteq B \subseteq ^*A \subseteq ^*\mathcal{E}$. For let $Q = \{(x, y) \mid y \in F(A) \land x \in A \land x \in y\}$. Assume that $(a_1, b_1), \ldots, (a_n, b_n) \in Q$. Then letting $b = b_1 \cup \cdots \cup b_n$ it follows that $(a_1, b), \ldots, (a_n, b) \in Q$. Therefore, $Q$ is a standard concurrent relation. Thus there exists some $B \in ^*(F(A))$ such that $^*x \in B$ for each $x \in A$. Now internal $B$ is not finite since $^*A$ is not finite. Indeed, as is well known $|B| = |(0, \nu)| \geq 2^\omega$, where $\nu \in ^*\mathbb{N} - \mathbb{N} = \mathbb{N}_\infty$. Thus $|^*A| \geq 2^\omega$. From the above remarks, it also follows that for $E \in ^*\mathcal{E} - \mathcal{E}$, $|E| \geq 2^\omega$. A few other useful results are easily obtained. For example,

(i) if $P : P(A) \rightarrow Y$ and $B \in P(A)$, then $^*(P(B)) = ^*P( ^*B)$.

(ii) Consider the finite power set operator $F$. If $F : P(A) \rightarrow Y$ and $B \in P(A)$, then $^*(F(B)) = ^*F( ^*B)$.

(iii) If $C$ is a map from $P(A)$ into $P(B)$, then for $D \in P(A)$ it follows that $^*(C(D)) = ^*C( ^*D)$.

The proofs of (i), (ii) and (iii) are easily obtained. Indeed, all three follow from the formal definition of a map. First, assuming that $A, Y \subseteq \mathcal{N}$. All the objects with which we shall be concerned will also be members of $\mathcal{N}$. Indeed, if necessary to obtain bounded sentences, we know that there is some $n \in \mathbb{N}$ such that everything needed to characterize (i), (ii), and (iii) are members of $X_0 \cup X_n$. For example, consider (i). Then the two sentences $\forall x(x \in P(A) \rightarrow \exists y(y \in Y \land (x, y) \in P))$, and $\forall x \forall y \forall z(x \in P(A) \land y \in P)$.
$Y \land z \in Y \land (x, y) \in P \land (x, z) \in P \rightarrow y = z$ imply, by \(*\)-transfer, that \(*P\) is a map from \(*P(A)\) into \(*Y\). Further, \((B, P(B)) \in P\) implies that \((B, *(P(B))) \in P\). Hence, since \(*P\) is a map, mapping notation yields that \(*P(B)) = \P(A)\). Now \((ii)\) follows in like manner. Indeed, the following set of sentences shows that \(*F\) generates the hyperfinite subsets of any internal subset of \(*A*\).

\[
\forall x \forall w(x \in P(A)) \land w \in P(A) \rightarrow ((A, x) \in F \leftrightarrow \\
x = \emptyset \lor \exists y \exists z(z \in X_n \land y \in \mathbb{N} \land \forall v(v \in A \land v \in x \rightarrow \\
\exists i(i \in \mathbb{N} \land 0 \leq i \leq y \land z(i) = v)))
\] (3.2.4)

It is well known that, in general, if \(A = \bigcup\{B_i \mid i \in \mathbb{N}\}\), then \(*A\) is singular and \(*C_\alpha\) satisfies axiom (2) of the Tarski axioms. Let \(A \in \mathcal{N}, A \subset X_n\). If \(B\) is a partition of \(A\), then \(*B\) is a partition of \(*A*\).

Proof. The sentences

\[
\forall x(x \in X_n \rightarrow (x \in A \leftrightarrow \exists y(y \in B \land x \in y))) \\
\forall x \forall y(x \in B \land y \in B \rightarrow x = y \lor x \cap y = \emptyset)
\]

hold in \(\mathcal{M}\); hence in \(\mathcal{M}\). Thus by \(*\)-transfer, \(*B\) is a partition of \(*A*\) and \(*A = \bigcup\{ *B \} = *(\bigcup\{B\})\). This completes the proof. \(\blacksquare\)

### 3.3 Adjective Reasoning

(Also see page 35.)

Following the ideas of Tarski (20) it appears that the set \(BP\) is a meaningful set of sentences. Define for \(BP\) an intuitive deductive process as follows: Let \(A \in F(BP)\). Then \(A \vdash_a b\), \(b \in BP\) if \(b \in A\) or \(b\) is obtained from some \(x \in A\) be removing a finite number (\(\neq 0\)) of “very,|||” strings from \(x\). Due to its form, this process \(\vdash_a\) is termed adjective reasoning. Denote the relation in \(F(BP) \times BP\) obtained by \(\vdash_a\) by the symbol “\(a\).” Let nonempty \(B \subset BP\). Then for each \(b \in B\) it follows that \((\{b\}, b) \in a\). Hence “\(a\)” is singular and \(C_\alpha\) satisfies axiom (2) of the Tarski axioms. Let \((A, b) \in a\) and \(A \subset B \in F(BP)\). Then \((B, b) \in a\) since \(b\) is obtained entirely from an element in \(A\). Assume that \((A, b_1), \ldots, (A, b_n) \in a\) and that \((\{b_1, \ldots, b_n\}, c) \in a\). Then \(c\) is either some \(b_i\), \(i = 1, \ldots, n\); or \(c\) is obtained from some \(b_i\) by removing finitely many “very,|||” symbol strings. But either this \(b_i\) \(\in A\) or this \(b_i\) is obtained from some \(x \in A\) also by removing finitely many “very,|||” symbol strings. Thus \(c\) is either an element of \(A\) or is obtained from \(A\) by removing finitely many “very,|||” symbol strings from a member of \(A\). Thus \((A, c) \in a\) and \(C_\alpha\) is a consequence operator on \(P(BP)\) by Theorem 1.3.3. For the next results, recall that when no confusion might occur the set \(*D*\) is denoted by \(D\).

A remark concerning notation is necessary. Two special abbreviations are used in certain explicit formal sentences. The first is the symbol \([x]\) for \(x \in P\). This denotes the unique object \(z\) that satisfies the sentence

\[
\exists z(z \in E \land x \in z \land x \in P),
\]

(3.3.1)
where $\exists! z A(z)$ means $\exists z (z \in \mathcal{E} \land \forall y(y \in \mathcal{E} \rightarrow (A(y) \leftrightarrow z = y)))$. Now in the first formula in the proof of Theorem 3.3.2 the formula $\exists! z_1 \exists! z_2 ((z_1 \in \mathcal{E}) \land (z_2 \in \mathcal{E}) \land (y \in z_1) \land (y \in z_2) \land (z_2 \in C_n(\{z_1\}))$ could be inserted. Also, the notation $\{x\}$ denotes the unique singleton set that satisfies the following for any $A \in \mathcal{N}$,

$$\forall z (z \in A \rightarrow \exists! x (x \in P(A) \land \forall w (w \in A \land w \in x \rightarrow w = z))).$$

(3.3.2)

We could insert for $z_2 \in C_n(\{z_1\})$ the formula

$$\exists! z_3 (z_3 \in P(\mathcal{E}) \land \forall w_3 (w_3 \in \mathcal{E} \land w_3 \in z_3 \rightarrow w_3 = z_1) \land z_2 \in C_n(z_3)).$$

(3.3.3)

Of course, these formulas are not inserted, but the appropriate abbreviations are used when needed. Recall that only constants which represent elements of $\mathcal{N}$ are "starred" in either the $*$-transform or any explicit partial formula obtained from the more general statement. All the internal objects which are not standardly internal take on constant names from an extended language. Thus if $A \in \mathcal{E} - \mathcal{E}$, then $A = [a]$, where $a \in P$. The same holds for any singleton set. If $a \in \mathcal{A} - A$, then we write $\{a\}$. For each $A \in \mathcal{N}$ and any nonempty finite $\{a_1, \ldots, a_n\} \subset A$ it follows that $\star \{a_1, \ldots, a_n\} = \{\star a_1, \ldots, \star a_n\} \subset \{\star x | x \in A\} = \sigma A$.

In what follows, other simplifying processes are employed when writing formal sentences. In many cases, these sentences do not appear to be written in the special bounded form. In all cases, the additional formal expressions can be easily added. In general, this is done by the addition of another $A$ type expression and an equivalent formula obtained, or $A \rightarrow$ when $\leftrightarrow$ appears. Many of the missing expressions are of these types. Here is one example of this process. Let $T = i[W]$.

Consider the set of all "natural number" intervals (i.e. segments) $H_1 = \{0, n | n \in \mathbb{N}\}$. From the construction of the superstructure it follows that there exists some $m \in \omega$ such that $\mathbb{N} \times T \subset X_0 \cup X_m$. Hence $P(\mathbb{N} \times T) \subset P(X_0 \cup X_m) = X_{m+1}$, where $m \geq 1$. Since no atoms are in $\mathbb{N} \times T$, the set $\mathbb{N} \times T \subset X_m$. For each $x \in H_1$, $T_x \in P(\mathbb{N} \times T) \subset X_{m+1}$. Obviously, $H_1 \subset X_2$. Hence, we also have that $H_1 \in X_{m+1}$. Observe that for each $x \in H_1$, $w \in T_x \subset x \times T \subset \mathbb{N} \times T \subset X_0 \cup X_m$ implies since no atoms are involved that $w \in X_m \subset X_{m+1}$. Thus, all of the objects being considered in an expression of the type $w \in T_x$ are all members of the set $X_{m+1}$. Notice that in the formal language $w \in T_x$ is a 2-place predicate replaceable by $\exists y \exists z (y \in \mathbb{N} \land 0 \leq y \leq x \land z \in T \land (y, z) \in w) \land \forall y_1 \forall z_1 (y_1 \in \mathbb{N} \land z_1 \in T \land x_1 \in T \land (y_1, z_1) \in w \land (y_1, x_1) \in w \rightarrow x_1 = z_1)$. Suppose that you have a formula with the expression "$\land (w \in T_x)$" as a subformula. Then replace it by " $\land (w \in X_m)$ \land (w \in T_x)". Now the explicit specially constructed formula usually used in the literature for such bounded formula is obtained by expanding the finite sequences of "$\land$" into the equivalent forms "$(\rightarrow (\ldots ))$" since recall that for the propositional calculus an expressing such as $P \land Q \rightarrow S$ is equivalent to $P \rightarrow (Q \rightarrow S)$. With these processes, all of the formula that seem to have quantified variables with missing bounding objects can be modified into an equivalent bounded form. Further, there are equivalent formula such as $\forall x(x \in C \rightarrow \exists y(y \in x\ldots))$.
where \( C \) is standard that express the requirement that the quantified variables vary
over members of our superstructure. Also, \( N \) and \( *N \) are closed under the basic
set-theoretic operations.

For each \( f \in P \), let \([f]\) denote the equivalence class in \( E \) containing \( f \). For each
\( g \in \ *P \), let \([g]\) denote the equivalence class containing \( g \) and determined by the
partition \( E \) of \( \ *P \).

**Theorem 3.3.1.** For each \( f \in P \), it follows that \( *[f] = [\ *f] = [\ *\ f] = [f] \).

Proof. Let \( f \in P \). Then \( f \in [f] \in E \) implies that \( *f \in \ *P \) and \( *f \in \ *[f] \in E \)
by properties of the \(*\)-map. Now the sentence

\[
f \in P \rightarrow \exists z \in E \land f \in z
\]

(3.3.4)

holds in \( M \); hence in \( *M \). Thus there is a unique set \( A \in \ *E \) such that \( *f \in \ A \in \ *E \).
This set is denoted by \( [f] \) since it contains \( f \), and \( \ *E \) is a partition. The uniqueness
implies that \( *[f] = [\ *f] \) and the finite nature of \( f \) yields that \([\ *f] = [f] \) under our
conventions. \( \qed \)

**Theorem 3.3.2.** There exists a purely subtle \( d \in \ *BP − BP \) such that
\( *Ca(\{d\}) \cap BP = \) an infinite set and \( *Ca(\{d\}) \cap (\ *BP − BP \) = an infinite set.

Proof. Let “just” be a member of BP and consider the sentence

\[
\forall x (x \in \mathbb{N} \land x > 0 \rightarrow \exists y (y \in T^x \land \forall w (w \in \mathbb{N} \land
0 < w \leq x \rightarrow y(w) = \text{very}, ||| \land g(0) = \text{just}) \land
\forall z (z \in \mathbb{N} \land z \leq x \rightarrow \exists y_1 (y_1 \in T^z \land \forall w_1 (w_1 \in \mathbb{N} \land
0 < w_1 \leq z \rightarrow y_1(w_1) = \text{very}, ||| \land y_1(0) = \text{just})
\)

\]

(3.3.5)

which holds in \( M \); hence in \( \ *M \). So, let \( \nu \in \ *\mathbb{N} − \mathbb{N} \). Then there exists some
\(*\)-partial sequence \( f \in (\ *T^\nu \) − \( P \) such that \( f(w) = i(\text{very}, |||) \) for each \( 0 < w \leq \nu \)
and \( f(0) = i(\text{just}) \). Also for each \( n \in \mathbb{N}, n > 0 \), there exists a partial sequence
\( f_n \in T^n \) such that for each \( n \in \mathbb{N}, \) where \( 0 < w \leq n, \) \( f_n(w) = i(\text{very}, |||) \) and
\( f_n(0) = i(\text{just}) \). Notice that if \( n, m \in \mathbb{N} \) \( \land \) \( n \neq m, \) then \( *[f_n] \neq *[f_m] \).
Now for each \( n \in \mathbb{N}, \) \( n > 0, \) \( *[f_n] \in \ *BP \). Application of Theorem 3.3.1 implies
that \( *[f_n] = [\ *f_n] = [f_n] \) and the above sentence yields that for each such \( n \in \mathbb{N}, \)
\( *[f_n] \in \ *Ca(\{[f]\}) \). Consequently, \( *Ca(\{[f]\}) \cap BP = \) an infinite set.

Consider the infinite set \( R = \{\nu_n \mid \nu_n = \nu − n \land n \in \mathbb{N} \land n \geq 1\} \subset \ *\mathbb{N} − \mathbb{N} \). By
\(*\)-transform of the above, for each \( n \in \mathbb{N}, n \geq 1 \), there exists some \( g_n \in (T)^{\nu_n} − P \)
such that \( g_n(w) = i(\text{very}, |||) \) for each \( w \in \ *\mathbb{N}; \) \( 0 < w \leq \nu_n < \nu \) and \( g_n(0) = i(\text{just}) \).
Observe that if \( n, m \in \mathbb{N}, \) \( n \neq m, \) then \( \nu_n \neq \nu_m \). Moreover, the following sentence

\[
\forall x \forall y (x \in \mathbb{N} \land y \in \mathbb{N} \land x > 0 \land y > 0 \land x \neq y →
\forall w \forall w_1 \forall z \forall z_1 (w \in T^x \land w_1 \in T^y \land z \in \mathbb{N} \land z_1 \in \mathbb{N} \land
0 < z \leq x \land 0 < z_1 \leq y \land w(z) = \text{very}, ||| \land w_1(z_1) = \text{very}, |||
\)
The concept of adjective deduction, which is obviously isomorphic to a subsystem of ordinary propositional deduction, was originally introduced to give a measure of the strength of various behavioral properties. These intuitive strengths may not be codifiable by a numerical measure. Thus, intuitively, “very, ||| very, ||| bold” is a stronger concept than “very, ||| bold”. The exact same process can be applied to physical concepts as well. Even though it may not be possible to measure the combined strengths of all of the intuitive forces that may be altering the appearance of a physical entity such as a thunderhead, the term “very, |||” could be replaced by other terms such as “greater, |||” or “weaker, |||” coupled with terms such as “force” and the like. The same type of \( \vdash \) analysis would follow.

With respect to the above remarks, later in this book, we consider the reasoning process called simply S, which is an axiomatically presented subsystem of propositional deduction. The process S is closely associated with adjective reasoning, if the set BP is constructed in a different manner and from different objects. One of the minor problems with these constructions is their relation to formal languages and the use of parentheses within such formal languages. Another illustration of the use of these nonstandard methods that does parallel Robinson’s original work along this line requires BP to be formally embedded into a propositional language with the insertion and removal of such parentheses.
the number of “(” so placed. (2) Place the same number of “)” after the “)” as your count in step (1). Denote this new symbol by $A_i$. Note that $A_i \in L_0$.

**Example.** Suppose that you are given $A = V \land V \land V \land V \land b$. Then $A_i = (V \land (V \land (V \land (V \land b))))$.

This process of considering a method of inserting parentheses and doing it in an ordered effective manner is no more complex and no less effective than Kleene’s concept of “closure with respect to (just) $x_1, \ldots, x_q$” on page 105 of [8]. Now to define in the obvious manner $\vdash^i \subset F(L') \times L'$. First, consider $BP_{0(\ell)} = \{x_\ell \mid x \in BP_0 - j[B]\} \cup j[B]$. Then $BP_{0(\ell)} \subset L_0$. For $F \in F(L')$, consider $F_i = \{x_\ell \mid x \in F \cap (BP_{0(\ell)}) \cup ((F - (BP_0 - j[B])) \subset L_0$. Then (i) if $F_i \vdash B \in L_0$, define $F \vdash^i B$. (ii) If $F_i \vdash B_1 \in L_0$ and $D$ is $B_1$ with all of the parentheses removed and $D \in BP_0$, then let $F \vdash^i D$. (iii) Finally, remove superfluous parenthesis if you wish [7, p. 74]. Only the procedure in this paragraph is to be used to obtain a $B \in L'$ such that $F \vdash^i B$.

Obviously, $\vdash^i$ is closely related to $\vdash$, since it is well known that for $A$, $B$, $C \in L_0, \vdash (A \land (B \land C) \leftrightarrow ((A \land B) \land C)$. By an abuse of notation we often write $\vdash$ for $\vdash^i$, $S_0$ for $S_0'$ and $L_0$ for $L'$ as well as suppressing parentheses insertion and removal for the elements of $BP_0$. The next result follows from the fact that for $A$, $B \in L_0$, $A \land B \vdash^i B$.

**Theorem 3.4.1** There exists a purely subtle $d \in *BP_0 - BP_0$ such that $*S_0\{d\} \cap BP_0 = \text{an infinite set}$ and also $*S_0\{d\} \cap (BP_0 - BP_0) = \text{an infinite set}$.

Proof. Make the following changes in the formal first-order sentences explicitly given in the proof of Theorem 3.3.2. First, let $b \in j[B]$ Now for the every “just” substitute the symbol “b”. Then for every “very,” string substitute the symbols “V \land “. With these substitutions made, the proof is exactly as for Theorem 3.3.2.

**Corollary 3.4.1.1** There exists a purely subtle $d \in *L_0 - L_0$ such that $*S_0\{d\} \cap L_0 = \text{an infinite set}$ and also $*S_0\{d\} \cap (L_0 - L_0) = \text{an infinite set}$.

[Remark: The above theorems for the propositional consequence operator also hold for the consequence operator $S$ and other such variations discussed later in this book.]

It is easily shown that for a propositional formula, say $A$, that $A \vdash^i A \land \cdots \land A$, with any $n \in N$ number of connectives $\land$.

**Theorem 3.4.2** For any $q \in L_0$ it follows that $*S_0\{q\} \cap L_0 = \text{an infinite set}$ and $*S_0\{q\} \cap (L_0 - L_0) = \text{an infinite set}$.

Proof. Let $q = [f], f \in T^0$, $f(0) = i(A)$, $A \in L_0$. The sentence
\[
\forall z(z \in \mathbb{N} \land z > 0 \rightarrow \exists y(y \in T \land \forall w(w \in \mathbb{N} \land 0 < w \leq z \rightarrow y(w) = i(A) \land y(0) = i(A) \land [y] \in S_0\{q\}))
\]
holds in $M$; hence in $*M$. Now proceed in the same manner as in the proof of Theorem 3.3.2, making the obvious changes, starting with the statement, “Also for each $n \in \mathbb{N}, n > 0, \ldots$” This completes the proof.
3.5 Modus Ponens Reasoning

The reasoning termed *Modus Ponens* (MP) is, of course, the major step in propositional deduction. One can, however, get more basic than $S_0$ and define MP reasoning to produce a subsystem of $S_0$ in the following (intuitive) manner. Simply let MP be the same deduction process as determines $S_0$ but with no axiom schemata. Use the symbol MP to represent the consequence operator obtained from this process. Then, for each $A \subseteq L_0$, it follows that $A \subseteq \text{MP}(A) \subseteq S_0(A)$. Thus for each internal $B \subseteq *L_0$, $B \subseteq *\text{MP}(B) \subseteq *S_0(B) \subseteq *L_0$.

Besides applying MP to $L_0$, it is straightforward to apply it to certain meaningfully constructed collections of intuitive readable sentences. For example, consider the set of symbols $B_1 = \{ \text{if}\,||\text{perfect},||\text{then}\,||\text{x. } \mid x \in \text{BP}\}$. Now apply MP to any finite subset of $\text{BP} \cup B_1$. Clearly, we can associate MP deduction formally to $\text{BP}_0$ in a meaningful way. Simply let $c \in \{ P_i \mid i \in \mathbb{N}\} - (\{ V \} \cup \{ B \})$ and $B'' = \{ c \to x \mid x \in \text{BP}_0\}$, etc. We leave to the reader the simple consequences of MP deduction in this case.

3.6 Predicate Deduction

In this section, predicate deduction in $L_1$, say as defined by Kleene on page 82 page [7], is briefly discussed relative to lengths of formal proofs. Robinson mentions [15, p. 25], what is well known from Gödel’s work, that using formal predicate deduction there is for each $n \in \mathbb{N}$ a readable sentence in $L_1$ that is provable as a theorem from the empty set of hypotheses, but requires $n$ or more steps.

Let $S_1$ denote the operator determined by predicate deduction with respect to $L_1$. Then $S_1(\emptyset)$ is the set of all provable formula (i.e. theorems). Of course, all the properties of $S_1$ are now referred to $E$. Hence there exists a relation $R_{L_1} \subseteq \mathbb{N} \times S_1(\emptyset)$ with the property that $(x, y) \in R_{L_1}$ if $y \in S_1(\emptyset), y \in L_1, x \in \mathbb{N}$ and $x$ is the length of a formal proof that yields $y \in S_1(\emptyset)$.

**Theorem 3.6.1** For each $\nu \in *\mathbb{N} - \mathbb{N}$ there is a subsequence $d \in *L_1, d \in S_1(\emptyset)$ and for each $\lambda \in *\mathbb{N}$, $\lambda < \nu, (\lambda, d) \notin *R_{L_1}$. Moreover, there exists some $\nu_0 \geq \nu$ and $(\nu_0, d) \in *R_{L_1}$.

Proof. As stated above the sentence

$$\forall x (x \in \mathbb{N} \to \exists y (y \in S_1(\emptyset) \land \forall w (w \in \mathbb{N} \land 0 \leq w \leq x \to
(w, y) \notin R_{L_1}) \land \exists z (z \in \mathbb{N} \land z \geq x \land (z, y) \in R_{L_1})))$$

(3.6.1)

holds in $\mathcal{M}$; hence in $*\mathcal{M}$. The result follows by *-transfer.

We now investigate a little more deeply what is meant by the “length of a formal proof.” There exists a partial sequence $f'$ of elements of $L_1$ such that the domain $D(f') = [0, n], n \in \mathbb{N}$, rather than $n \in \mathbb{N}$, the range of $f' = \text{Run}(f') \subseteq L_1$ and $f'(n) = A \subseteq S_1(\emptyset)$ and the length of the formal proof that yields $A \subseteq S_1(\emptyset)$ is $n + 1$. Of course, $f'$ actually gives the elements of $L_1$ that appear in such a specific formal proof. Now relate this intuitive partial sequence $f'$ to a corresponding partial sequence in $P$ in the following manner. Let $(x, y) \in f_{L_1} \text{ iff } y = i(w)$ and $(x, w) \in f'$. 


Denote by \( P_{L_1} \subset P \) the set of all such length of proof sequences. Then \( (x, y) \in ^*R_{L_1} \) iff \( x \in ^*\mathbb{N} \), \( y \in ^*S_1(\emptyset) \) and there is some \( f \in ^*P_{L_1} \) such that \( y = [f(x)] \). By *-transfer the hyperlength of the proof would be \( x + 1 \). The set \( P_{L_1} \) may be used to characterize the concept intuitively associated with the proof length for objects in \( S_1(\emptyset) \). These sequences have other properties as well but these will not be considered in this investigation. With this in mind, then \(^*P_{L_1}\) represents the subtle concept of proof length for elements in \(^*S_1(\emptyset)\). It’s the proof length concept we employ in one application of the results from this chapter. Theorem 3.6.1 can now be stated in an alternate form.

**Theorem 3.6.2** For each \( \nu \in ^*\mathbb{N} - \emptyset \) there is a subtle \( d \in ^*S_1(\emptyset) \) such that for each \( \lambda \in ^*\mathbb{N} - \emptyset \), \( \lambda < \nu \), there does not exist some \( g \in ^*P_{L_1} \) such that \( d = [g(\lambda)] \). Moreover, there exists some \( \nu_0 \in ^*\mathbb{N} - \emptyset \) such that \( \nu_0 \geq \nu \) and some \( f \in ^*P_{L_1} \) such that \( d = [f(\nu_0)] \).

**Corollary 3.6.2.1** There exists \( d \in ^*S_1(\emptyset) \), \( f \in ^*P_{L_1} \) and \( \nu \in ^*\mathbb{N} - \emptyset \) such that \( d = [f(\nu)] \) and for each \( g \in ^*P_{L_1} \) and each \( \nu' \in ^*\mathbb{N} \) such that \( d = [g(\nu')] \), \( |D(g)| \geq 2^{\omega} \).

[Remarks: It should be apparent to the reader that statements that hold in \( \mathcal{M} \) relative to consequence operators or deductive processes are obtained from the corresponding intuitive reasoning processes by application of \( \theta \). The proofs that these statements hold in the intuitive case are straightforward or obvious, and are omitted in all cases. Also, you might wonder about the term “ultralogics” since it has not been specifically defined as yet. The term is reserved for various special subtle consequence operators to be used in various cosmogony investigations that will be discussed later in this book.]
4. SPECIAL DEDUCTIVE PROCESSES

4.1 Introduction.

There are certain words that intuitively denote an upper [resp. lower] bound to such concepts as “stronger” [resp. “weaker”]. With respect to certain philosophic studies, one such concept is the notion of “perfect” when associated with a language like BP. In what follows, this “perfect” associated with BP is used as a prototype for these other cases. Two types of deductive processes associated with this prototype will be introduced, a very trivial one followed by a much more interesting and significant procedure.

4.2 Reasoning From the Perfect Type W

First, an intuitive extension of BP is defined. Let $BPC | = BP \cup \{ \text{perfect} \}$ and for convenience denote the readable string “perfect” by the single c. Now we define type W reasoning from the perfect by considering an intuitively defined operator, $\Pi_W$, from $\mathcal{P}(BPC |)$ into $\mathcal{P}(BPC |)$.

For any finite $F \subset BPC |$:

(i) if $c \in F$, then $\Pi_W(F) = BPC |$;

(ii) if $c \not\in F$, then $\Pi_W(F) = F$;

(iii) and for arbitrary $A \subset BPC |$, let

$$\Pi_W(A) = \bigcup \{ \Pi_W(F) \mid F \in F(A) \}$$

Theorem 4.2.1 The map $\Pi_W : \mathcal{P}(BPC |) \to \mathcal{P}(BPC |)$ is a consequence operator.

Proof. Let $A \subset BPC |$. Clearly, axiom (4) holds by the definition. Let $a \in A$. Then $\{a\} \in F(A)$. Now if $a \neq c$, then $\Pi_W(\{a\}) = \{a\}$. If $a = c$, then $\Pi_W(\{a\}) = BPC |$. In these two cases, (iii) of the definition yields that $a \in \Pi_W(A)$. Thus, even when $A = \emptyset$, it follows that $A \subset \Pi_W(A) \subset BPC |$ and axiom (2) holds.

Since axiom (4) holds and $A \subset \Pi_W(A)$, it follows that $\Pi_W(A) \subset \Pi_W(\Pi_W(A))$. Now either $\Pi_W(A) = A$; in which case $\Pi_W(\Pi_W(A)) = \Pi_W(A) = A$ or $\Pi_W(A) = BPC |$; in which case $\Pi_W(\Pi_W(A)) = \Pi_W(BPC |) = BPC | = \Pi_W(A)$. Thus axiom (3) holds and this completes the proof.

Recall that $T = i[W]$ and if $w \in W$, then $f_w \in P$ denotes the partial sequence which is an element of $T^0$ and $f_w(0) = i(w)$, $w = [f_w]$. Also, due to their finitary character, each $x \in E$ is often identified with $*x \in \sigma E$.

Theorem 4.2.2 For each internal $B \subset BPC |$ if $c = [ *f_c] = [f_c]$, then $*\Pi_W(B) = *BPC | = *BP \cup \{c\} = *BP \cup \{[f_c]\}$, where $[f_c] \in \sigma E$ and $\sigma E = E$ under the basic identification of $*\mathbb{N}$ with $\mathbb{N}$.

Proof. Simply consider the sentence
(4.2.1)  \( \forall x (x \in \mathcal{P}(\text{BPC}) \land c \in x \rightarrow \Pi_W([f_c]) = \text{BPC}) \)

that holds in \( \mathcal{M} \); hence in \( \mathcal{M} \). The result follows by *-transfer. \[\]

**Corollary 4.2.2.1** The set \( \star \Pi_W([f_\nu]) = \Pi_W([\star f_\nu]) = \star \text{BPC} \).

**Corollary 4.2.2.2** For each \( b \in i[B] \) and each \( \nu \in \star \mathbb{N} - \mathbb{N} \) there exists a subtle \( f^b \in (\star T)^\nu - P \) such that for each \( x \in \star \mathbb{N} \), where \( 0 < x \leq \nu \), \( f^b(x) = i(\text{very,}|||) \) and \( f^b(0) = b \). Moreover, \( [f^b] \in \star \Pi_W([f_\nu]) \).

For each \( b \in i[B] \) and a fixed \( \nu \in \star \mathbb{N} - \mathbb{N} \), apply the axiom of choice and let \( f^b \) denote one of the subtle objects that exists by Corollary 4.2.2.2 and satisfies the stated properties. Since \( B \) is finite, the set \( F_\nu = \{[f^b] \mid b \in i[B]\} \) is internal. The next result is obvious.

**Theorem 4.2.3** For each \( \nu \in \star \mathbb{N} - \mathbb{N} \), internal \( F_\nu \subset \star \Pi_W([f_\nu]) \).

Observe that there exist, at least, \( 2^\omega \) distinct \( F_\nu \) sets.

### 4.3 Strong Reasoning From the Prefect

For the second type of reasoning from the perfect, our attention will be restricted to \( L' = L_0 \cup \text{BP}_0 \) and the set \( \text{BP}_0 \) that bijectively corresponds to \( \text{BP} \). Let a specific \( c \in \{P_i \mid i \in \mathbb{N}\} - (\{V\} \cup j[B]) \). Correspond \( c \) to the readable sentence “prefect.” Let

\[
C_1 = \{(c \to x_i) \mid x \in \text{BP}_0 - j[B]\} \cup \{(c \to x) \mid x \in j[B]\} \cup \{c\}.
\]

\[
C = \{c \to x \mid x \in \text{BP}_0 - j[B]\} \cup \{c \to x \mid x \in j[B]\} \cup \{c\}.
\]

Why do we go through the following exercise of inserting and removing parentheses so as to conform more closely to the formula of a formal language? The basic reason is related to some of the results later in this book that refer to counting of symbols by means of the partial sequences. Clearly, parenthesis insertion does correspond to the increase strength idea of adjective reasoning, as does the ordering of the very,||| symbols by the partial sequences. However, in certain deductive processes, all of the axioms for the propositional logic are not used. Hence even though it is certainly of no importance, due to equivalence, when all of the usual axioms are used to write a formal \( (V \land (V \rightarrow b)) \) as \( V \land V \rightarrow b \), it may not be possible to establish this equivalence for these restrictive deductive processes. The process we now outline simply removes this formal difficulty at the cost of a more involved finitary process.

Let \( \text{BPC}_0 = \text{BP}_0 \cup C \) and \( \text{BPC}_{0\ell} = \text{BP}_{0\ell} \cup C_1 \). The axioms are elements of the set \( \text{Ax} = \{(V \land x) \rightarrow x \mid x \in \text{BP}_{0\ell}\} \) with the suppression of the outer most parentheses for simplicity in application of MP. Let \( \vdash_\pi \) denote ordinary propositional deduction but only using the axiom set \( \text{Ax} \) and only formula from the set \( \text{Ax} \cup \text{BPC}_{0\ell} \) in the steps of any proof. For a specific \( A_{\ell} \in \text{BP}_{0\ell} \), let \( A \) be the element of \( \text{BP}_0 \) formed by removing all parentheses from \( A_{\ell} \). Define \( \Pi : \mathcal{P}(\text{BPC}_0) \rightarrow \mathcal{P}(\text{BPC}_0) \) as follows: Let finite \( F \subset \text{BPC}_0 \subset L' \) and \( F_{\ell} = (\text{BP}_0 \cap F_{\ell} \cup (C \cap F)_{\ell} \). Then for any \( D \subset \text{BPC}_0 \), \( x \in \Pi(D) \) iff there exists a finite \( F \subset D \) and \( A \in \text{BPC}_{0\ell} \) such that
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$F_\ell \vdash A_\ell$ and if $A_\ell \in C_\ell$, then $X = A$ or if $A_\ell \in \text{BP}_0\ell$, then $A = X$. It is not difficult to show that $\Pi$ is a consequence operator since it is the restriction of formal $\vdash_\pi$ to $\text{BPC}_\ell$ and $\Pi$ is called strong reasoning from the perfect.

By way of a reminder, *-transfer and the fact that $E_0 = j[B]$ and $i[E_0]$ are finite imply that $f \in *\text{BP}_0 - *E_0$ iff there exists some $n \in *\mathbb{N}, n > 0$, and $b \in i[E_0]$ such that $f(w) = i(V \land)$ and $f(0) = b$ for each $w \in *\mathbb{N}$ such that $0 < w \leq n$. Further, for each $b \in i[E_0]$ and each $a \in i[E_0]$ and each $n \in *\mathbb{N}, n > 0$, there exists a $f \in *\text{BP}_0 - *E_0$ such that for each $w \in *\mathbb{N}$, where $0 < w \leq n$, $f(w) = i(V \land)$ and $f(0) = b$. Notice also that $[g] \in *E_0$ iff $g \in (\ast T)^0$, $g(0) = b \in i[E_0]$ and $|g| = 1$. The set $*E_0$ being nonempty and finite implies that $*E_0 = \{ *[g_1], \ldots, *[g_n] \} = *E_0 = E_0$. Finally, due to the identification of each specific $i(e)$, it follows that $g = g$, where, as usual, this follows from the finitary character of each equivalence class.

We know that for fixed $n \in *\mathbb{N}$ and any $b \in i[E_0]$ there exists a unique $g^b \in (\ast T)^n$ such that $g^b(0) = b$ and if $w \in *\mathbb{N}$ and $0 < w \leq n$, then $g^b(w) = i(V \land)$. Let $G_n = \{ [g^b] | b \in i[E_0] \}$ for each $n \in *\mathbb{N} \setminus \{ 0 \}$. Now $G_n$ has the same general properties as the previously defined set $F_n$. In particular, each $G_n$ is internal and if $\nu \in \mathbb{N}_{\infty}$, then $G_\nu$ is purely subtle.

**Theorem 4.3.1.** If $n \in *\mathbb{N} \setminus \{ 0 \}$, $m \in *\mathbb{N} \setminus \{ 0 \}$ are such that $0 \leq m \leq n$ and $f \in (\ast T)^m$ has the property that for each $w \in *\mathbb{N}$, where $0 < w \leq m$, $f(w) = i(V \land)$ and $f(0) = b \in i[E_0]$, then $[f] \in *\Pi(G_n)$. Moreover, if $[g] \in i[E_0]$, then $[g] \in *\Pi(G_n)$.

Proof. First, since $G_n$ is internal, $n \in *\mathbb{N} \setminus \{ 0 \}$, and $*\text{BP}_0 \subset *\text{BP}_0 \cup C = *\text{BPC}_0$ and $G_n \subset *\text{BP}_0$, it follows that $G_n$ is in the domain of $*\Pi$.

Let $A \in \text{BP}_0$, $A = V \land \cdots \land V \land b, b \in E_0$ and that are $n \geq 1$, $(n \in \mathbb{N})$ connectives $\land$. We prove by induction that for any $n \in \mathbb{N}$ that $0 < w \leq n$, the symbol string $B = V \land \cdots \land V \land b$ with $w \geq 1$ connectives $\land$ or $B = b$ has the property that $A_\ell \vdash_\pi B_\ell$, where if $B = b$, then $B_\ell = b$.

**Case 1.** Let $n = 1$. Then $A = V \land b$. Consider $A_\ell = (A \land b)$. The following is a proof that $A_\ell \vdash_\pi b$. (i) $(V \land b)$, (ii) $(V \land b) \rightarrow b$, (iii) $b$. A proof composed of step (i) only yields the trivial result that $A_\ell \vdash_\pi A_\ell$.

**Case (n + 1).** Suppose that the result holds for $n$, and $A = V \land \cdots \land V \land b$ has $n + 1$ connectives $\land$. The formula $A_\ell = (V \land (A_\ell \land (V \land \cdots)))$. Let $B_\ell = (V \land (A_\ell \land (V \land \cdots)) \cdot \cdots) \cdot n$ connectives $\land$. The following is a proof that $A_\ell \vdash_\pi B_\ell$. (i) $A_\ell = (V \land b_\ell)$, (ii) $(V \land b_\ell) \rightarrow b_\ell$, (iii) $b_\ell$. Thus $A_\ell \vdash_\pi B_\ell$. From the induction hypothesis, $B_\ell \vdash_\pi E_\ell$, where $E_\ell$ has $w$ connectives $\land$ such that $0 < w \leq n$, or $E_\ell = b$. Since $\vdash_\pi$ is transitive, it follows that $A_\ell \vdash_\pi E_\ell$. The trivial proof using step (i) yields that $A_\ell \vdash_\pi A_\ell$, and the basic result above follows by induction.

We have shown that for each $b \in i[E_0]$, the following sentences

$\forall y \forall x \forall w \forall z (x \in \mathbb{N}) \land (x > 0) \land (y \in \mathbb{N}) \land (0 < y \leq x) \land (w \in T^e) \land \forall w_1 ((w_1 \in \mathbb{N}) \land (0 < w_1 \leq x) \rightarrow (w(w_1) = i(V \land))) \land$
\[(w(0) = b) \land (z \in T^y) \land \forall z_1((z_1 \in \mathbb{N}) \land (0 < z_1 \leq y) \rightarrow \)

\[(z(z_1) = i(V \land) \land (z(0) = b) \rightarrow ([z] \in \Pi((\{x\}))). \]

\[\forall x \forall y \forall \forall z((x \in \mathbb{N}) \land (x > 0) \land (w \in T^x) \land \forall w_1((w_1 \in \mathbb{N}) \land (0 < w_1 \leq x) \rightarrow w(w_1) = i(V \land) \land (w(0) = b) \land \]

\[(z \in T^0) \land (z(0) = b) \rightarrow ([z] \in \Pi((\{x\}))). \]

hold in \(M\), hence in \({}^*M\). Since the singleton subsets of \(G_n\) are \(\text{*-finite}\), it follows that \(\bigcup\{ \ast\Pi([g^0]) \mid b \in \{E_0\} \} \subset \ast\Pi(G_n)\) by \(\ast\text{-transfer}\) of Axiom (4). Let \(n \in \mathbb{N} \setminus \{0\}, f \in (T)^m, 0 < m \leq n, f(w) = i(V \land)\) for each \(w \in \mathbb{N}, 0 < w \leq m\) and \(f(0) = b\). Then by \(\ast\text{-transfer}\) of sentence (4.3.1), we have that

\[(f) \in \ast\Pi([g^0]) \subset \ast\Pi(G_n). \]

For \(g \in T_0\) such that \(g(0) = b\), \(\ast\text{-transfer}\) of sentence (4.3.2) yields \([g] = \ast\Pi([g^0]) \subset \ast\Pi(G_n)\) and this completes the proof. \(\square\)

**Corollary 4.3.1.1** If \(\nu \in \mathbb{N}_\infty\), then \(BP_0 \cap G_\nu = \emptyset, BP_0 \subset \ast\Pi(G_\nu)\) and \(BP_0 \cup G_\nu \subset \ast\Pi(G_\nu)\).

For \(A \in L_0\), let \(\#(A)\) denote the length of the formula \(A\).

**Theorem 4.3.2** Let \(B \subset BP_0, A = V \land \cdots \land V \land b, b \in E_0, where there are n \geq 1 connections \land and for each Z \in B, \#(A) > \#(Z). Then A \notin \Pi(B).\)

Proof. (Note: In this proof certain of the indicated parentheses may be superfluous.) Assume the hypothesis of the theorem. We show that there does not exist a finite \(C \subset B\) such that \(F_\lambda \vdash \pi A_\lambda\). Assume that there exists a finite \(C \subset B\) such that \(F_\lambda \vdash \pi A_\lambda\). A relation \(R_m \subset BP_{0i} \times BP_{0i}\) is called an \(m\text{-chained sequence of MP processes}\) if there exists an \(m \in \mathbb{N}, m \geq 1\) such that \((X, Y) \in R_m\) has the form \((A_i(A_i-1))\) for \(i = 1, \ldots, m, \) where \(A_0 = A_\lambda\). Also each \(A_i = V \land (A_{i-1})\) and \(A_i\) is a step in the proof of \(F_\lambda \vdash \pi A_\lambda\) where \(i = 1, \ldots, m\). We now show by induction that, for each \(m \in \mathbb{N}, m \geq 1\), there exists an \(m\text{-chained sequence of MP processes in the proof that } F_\lambda \vdash \pi A_\lambda.\)

**Case m = 1.** We know that \(A_1 \in BP_{0i}\) implies that \(A_1\) is not an instance of the use of an axiom since no axioms appear in \(BP_{0i}\). Since \(A \notin B\) then \(A_1 \notin F_\lambda\). Thus \(A_1\) being the last step in the proof implies that \(A_1\) is the conclusion of an MP process with premises \((D) \rightarrow (A_\lambda)\) and \(D\). Assume that \(D = c\). Then the single step which contains the primitive \(c\) could not be an axiom nor an assumption since \(c \notin B\). Hence \(c\) would be the conclusion of a prior MP process. Therefore a prior step would be of the form \((E) \rightarrow c\). This is impossible; all steps must be formula in \(Ax \cup BP_{0i}\). Thus \(D \neq c\) implies that \((D) \rightarrow (A_\lambda)\) is an instance of an axiom or the conclusion of a prior MP process. However, since no step can be of the form \((E) \rightarrow ((D) \rightarrow (A_\lambda))\), it follows that \((D) \rightarrow (A_\lambda)\) must be an instance of an axiom. Consequently, \(D = V \land (A_\lambda) = A_1 \in BP_{0i}\) and \(D\) is a step in the proof. Therefore, \(R_1 = (A_1, (A_1) \rightarrow (A_\lambda))\), \(A_0 = A_\lambda\) is a 1-\text{chained sequence of MP processes.}\n
Assume the result holds for \(m\).
Case \( m + 1 \). Let \( R_m = \{(A_m, (A_m) \rightarrow (A_{m-1})), \ldots, (A_1, ((A_1) \rightarrow (A_0)))\} \) be an \( m \)-chained sequence of MP processes. Now \( A_m = V \land (A_{m-1}) \) implies by a simple induction proof that \( \#(A_m) > \#(A_0) = \#(A_1) \). Hence, \( A_m \notin B \). Thus \( A_m \notin F \). Moreover, \( A_m \notin C \), \( A_m \notin Ax \) for the primary connective is \( \land \). This implies that \( A_m \in B P_0 \) and \( A_m \) must be the conclusion of some MP process with premises \((D) \rightarrow (A_m) \) and \( D \). As in case \( m = 1 \), it follows that \( D \neq c \) and that \((D) \rightarrow (A_m) \) must be an instance of an axiom. Consequently, \( D = V \land (A_m) \). Let \( D = A_{m+1} \). Since \( D \) is a step in the proof, \( D \in B P C_0 \). Thus, \( R_{m+1} = \{(A_{m+1}, (A_{m+1}) \rightarrow (A_m))\} \cup R_m \) is an \( m + 1 \)-chained sequence of MP processes.

The length of the proof that \( F_i \vdash_\pi A_i \) is some finite number, say \( n \in \mathbb{N} \), \( n \geq 1 \). The above shows that there exists an \( n+1 \)-chained sequence of MP processes for this proof. Since \( |P_1(R_{m+1})| = n + 1 \) (note that for each \( i \), \( j \in \mathbb{N} \) such that \( 0 \leq i < j \leq n + 1 \), \( \#(A_i) < \#(A_j) \)) and each element of \( P_1(R_{m+1}) \) is a distinct step in the proof, this contradicts the fact that the proof length is \( n \). Consequently, there does not exist a finite \( F \subset B \) such that \( F_i \vdash_\pi A_i \). Therefore \( A \notin \Pi(B) \) and this completes the proof of this theorem. □

Under our embedding, Theorem 4.3.2 is interpreted by \( \theta \) as embedded into \( M \). When this is done the length of a formula \( A \), \( \#(A) \), is the length of the preimage \( A \) of the map \( i \) associated with the special partial sequence \( f_A(0) = i(A) \). Let \( G = \{G_n \mid (n > 0) \land (n \in \mathbb{N}) \} \cup \{G_0\} \), \( G_0 = E_0 \), and \( i[E_0] = m + 1 \). Indeed, \( E_0 = j[B] = \{b_0, \ldots, b_m\} \). Let \( G \subset X_n \). The following sentences hold in \( M \); hence in \( \star M \).

\[
\forall x(x \in X_n \rightarrow (x \in G) \leftrightarrow \exists y_0 \exists y_1 \cdots \exists y_m((y \in \mathbb{N}) \land (y \geq 0) \land (y_0 \in T^y) \\
\land \cdots \land (y_m \in T^y) \land \forall w(w \in \mathbb{N}) \land (0 < w \leq y) \rightarrow (y_0(w) = i(V \land)) \land (y_0(0) = b_0) \\
\land \cdots \land (y_m(0) = b_m \land ((y_0 \in x) \land \cdots \land (y_m \in x)))).
\] (4.3.4)

Sentence (4.3.4) can also be written as

\[
\forall x(x \in X_n \rightarrow (x \in G \leftrightarrow \exists y((y \in \mathbb{N}) \land (y \geq 0) \land A(y))));
\forall y((y \in \mathbb{N}) \land (y \geq 0) \rightarrow A(y)),
\] (4.3.5)

where \( A(y) \) is the obvious expression taken from (4.3.4). The objects that exist for each “\( y \)” in the \( A(y) \) expression (i.e. the \( y_j \in T^y \), \( j = 0, \ldots, m \)) are unique with respect to the property expressed in \( A(y) \). Obviously, for each \( n \in \mathbb{N} \), \( G_n \in \star G \). Moreover, there exists a bijection \( F: \mathbb{N} \rightarrow G \) such that \( F(n) = G_n \). Now let \( n, m \in \mathbb{N} \), \( n, m \) and \( A = V \land \cdots \land b \), \( b \in E_0 \) has \( m \) connectives \( \land \). Then \( \#(A) > \#(Z) \) for each \( |f| \in G_n \). It follows from Theorem 4.3.2 that \( [f_A] \notin \Pi(G_n) \).

**Theorem 4.3.3** If \( n, m \in \star \mathbb{N} \), \( 0 \leq n < m \), \( b \in i[E_0] \) and \( f \in (\star T)^m \) such that \( f(w) = i(V \land) \) for each \( w \in \star \mathbb{N} \), \( 0 < w \leq m \), and \( f(0) = b \), then it follows that \( [f] \notin \star \Pi(G_n) \).

**Proof.** Let \( b \in i[E_0] \). From the above discussion, the following sentence

\[
\forall x \forall y \forall z((x \in \mathbb{N}) \land (y \in \mathbb{N}) \land (0 \leq y < x) \land (z \in T^x) \land
\]
holds in $\mathcal{M}$; hence in $\star\mathcal{M}$. The result follows by $\star$-transfer.]

**Corollary 4.3.3.1** For each $n \in *\mathbb{N}$, $\star\Pi(G_n) \neq \star\Pi(BP_0)$ and $\star\Pi(G_n) = \bigcup\{G_x \mid (x \in *\mathbb{N}) \land (0 \leq x \leq n)\}$. Proof. Since $G_n \subset \star BP_0$, $\star\Pi(G_n) \subset \star\Pi(BP_0)$. Theorems 4.3.1 and 4.3.3 along with the above discussion completely characterizes the elements of $\star\Pi(G_n)$. This completes the proof.

For $\nu \in \mathbb{N}_\infty$, let $G_\nu = \bigcup\{G_x \mid (x \in \mathbb{N}_\infty) \land (x < \nu)\}$. Then $G_\nu$ is a purely subtle external object. This follows from the fact that $\star\Pi(G_\nu)$ is internal, $BP_0$ is external and $\star\Pi(G_\nu) = G_\nu \cup BP_0 \cup G_\nu$. Moreover, observe that if $\lambda, \nu \in \mathbb{N}_\infty$, $\nu > \lambda$, then $\star\Pi(G_\lambda) \cap G_\nu = \emptyset$ and that $G_\nu$ and $BP_0$ are not in the domain of $\star\Pi$ unless we extend $\star\Pi$, say by the identity operator.

**Theorem 4.3.4** The set $\star\Pi(BP_0) = \star BP_0$.

Proof. (Note once again that some superfluous parentheses may have been added to some formula in this proof.) It is known that $\star BP_0 \subset \star\Pi(BP_0)$. Let finite $F \subset BP_0$, $A \in C$ and assume that $F \vdash A$. Then $A = c$ or $A = c \rightarrow (x)$, $x \in BP_0$.

**Case 1.** Assume that $A = c$. Since $c \notin F \subset BP_0$ and $A = c$ is not an instance of an axiom, $A = c$ must be the conclusion of an MP process. Thus a prior step is of the form $(D) \rightarrow c$. This is impossible for $(D) \rightarrow c \notin Ax \cup BP_0$.

**Case 2.** Assume that $A = c \rightarrow (x)$, $x \in BP_0$. Again $c \rightarrow (x)$ is the conclusion of an MP process. This is impossible since no formula of the type $(D) \rightarrow (c \rightarrow (x))$ is an element of $Ax \cup BP_0$. Hence by $\star$-transfer of the appropriate first-order sentence, after the $\theta$ embedding, it follows that $\star\Pi(BP_0) \subset \star BP_0$.

**Corollary 4.3.4.1** The set $\star\Pi(BP_0) \neq \star\Pi(BPC_0)$.

It is easy to see that $\Pi(C) = BPC_0$. For let $A \in BP_0$ and consider the proof (1) $c$, (2) $c \rightarrow (A_\lambda)$, (3) $A_\lambda$. Thus $\{c, c \rightarrow (A_\lambda)\} + \pi A_\lambda$ yields that $A \in \Pi(C)$. Hence $\Pi(C) = BPC_0$. Also, $\star\Pi(C) = \star BP_0$.

**Theorem 4.3.5** Let internal $A \subset *BP_0$ and internal $B \subset *BPC_0$. Then $\star\Pi(A \cup B) = *BPC_0$ if $*C \subset B$.

Proof. For the sufficiency, let internal $A \subset *BP_0$, internal $*C \subset B$. Then $A \cup B$ is internal and $\star\Pi(C) = BPC_0$.

So $\star\Pi(A \cup B) \subset \star\Pi(B) \subset \star\Pi(A \cup B) \subset BPC_0$. Thus $*BPC_0 = \star\Pi(A \cup B)$.

For the necessity, assume that internal $A \subset *BP_0$, internal $B \subset *BPC_0$ and that $\star\Pi(A \cup B) = *BPC_0$. Let $A_1 \subset BP_0$, $B_1 \subset BPC_0$ and $\Pi(A_1 \cup B_1) = BPC_0$. It follows from Theorem 4.3.4 that $B_1 \not\subset BP_0$. Indeed, given any finite $F \subset A_1 \cup (B_1 \cap BP_0)$. If $D \in \Pi(F)$, then $D \in BP_0$. Thus only for a finite $F_1 \subset B_1 \cap C$ can there be an $E \in C$ such that $E \in \Pi(F_1)$. Hence all that needs to be shown is
that \( C \subseteq \Pi(B_1 \cap C) \) implies that \( B_1 \cap C = C \). So, assume that \( B_1 \cap C \neq C \). Hence either \( c \notin B_1 \cap C \) or there exists some \( A_1 \in BP_0 \) such that \( c \rightarrow (A_1) \notin (B_1 \cap C) \).

**Case 1.** Assume that \( c \notin B_1 \cap C \) and \( F \) is any finite subset of \( B_1 \cap C \) such that \( F \vdash c \). Of course, \( c \) is the last step in a formal proof. \( c \) is the conclusion of some MP process since \( c \) is not an assumption nor an axiom. Thus some formula of the form \( (D) \rightarrow c \) must be in a prior step in the formal proof. This is impossible since no formula of this form is an element of \( BPC_0 \).

**Case 2.** Assume that there exists some \( A_1 \in BP_0 \) such that \( c \rightarrow (A_1) \notin B_1 \cap C \) and there exists finite \( F \subseteq B_1 \cap C \) such that \( F \vdash c \rightarrow (A_1) \). Again \( c \rightarrow (A_1) \) is not an assumption nor an axiom. Consequently, \( c \rightarrow (A_1) \) is the conclusion of an MP process. Thus there exists some formula of the form \( (D) \rightarrow (c \rightarrow (A_1)) \) in a prior step. Again this is impossible.

These two cases imply that \( B_1 \cap C = C \). Therefore, \( C \subseteq B_1 \) implies the sentence

\[
\forall x \forall y ((x \in P(BP_0)) \land (y \in P(BPC_0)) \land
\quad (\Pi(x \cup y) = BPC_0) \rightarrow (C \subseteq y))
\]

(4.3.7)

holds in \( \mathcal{M} \); hence in \( \star \mathcal{M} \). The result follows from \( \star \)-transfer. \( \square \)

Note that all of the results in this section hold for BP and BPC, where \( C \) is constructed without parentheses.

### 4.4 Order

We briefly look at two special types of order relations, the “number of symbols” order and the “better than” order. Previously the concept of the length of a formula or word \( A \) (i.e. \( #(A) \)) was introduced. This type of order has few properties unless it is restricted to certain interesting types of subsets.

Let nonempty \( B, D \subseteq BPC \) (or \( BPC_0 \)), then define \( B \leq_\# D \) if for each \( b \in B \) and for each \( d \in D \), it follows that \( #(b) \leq #(d) \). This order is obviously a pre-order in the sense that it is reflexive and transitive. However, in general, it should probably not be considered a partial order since antisymmetry does not imply set equality although it does imply that all the symbol strings have equal length in both \( B \) and \( D \). Also other pre-orders of this type appear not to be partial orders for the same reason. If \( \leq_\# \) is restricted to certain collections of sets, then it does become a useful partial order under set equality.

Consider the collection \( \{G_n \mid n \in \star \mathbb{N}\} \). Then the pre-order \( \leq_\# \) restricted to this set is isomorphic to the simple order of \( \star \mathbb{N} \). Indeed, \( G_n \leq_\# G_m \text{iff } n \leq m \), where \( n, m \in \star \mathbb{N} \) and \( \leq \) is the usual extension of the simple order induced on \( \mathbb{N} \) by \( \omega \). Moreover, notice that \( G_n = G_m \text{iff } n = m \), and \( G_n \not\leq_\# G_m \text{iff } G_n \cap G_m = \emptyset \).

For the collection \( \{G_\nu \mid \nu \in \mathbb{N}_\infty\} \), it follows that \( G_\zeta \subseteq G_\lambda \text{ iff } \zeta \leq \lambda \). Thus \( \{G_\nu \mid \nu \in \mathbb{N}_\infty\} \) is ordered by inclusion when the simple order of the subscripts is considered. Notice also that \( \bigcap \{G_\nu \mid \nu \in \mathbb{N}_\infty\} = \emptyset \).

Let fixed \( \nu \in \mathbb{N}_\infty \). Then there exist infinitely many \( G_\lambda \) which differ only be a finite set of subtle objects. Simply consider the set \( \{G_{\nu+n} \mid n \in \mathbb{N}\} \). If \( n, m \in \mathbb{N} \) and
$m > n$, then $|G_{v+n} - G_{v-n}| = (m-n)|G_0| \in \mathbb{N}$. Also there exist infinitely many sets “longer than” any $G_{v+n}$, where $n \in \mathbb{N}$ or strictly containing any $G_{v+n}$. To see this consider $\nu^2 < \nu^3 < \cdots < \nu^n < \cdots$, $n \in \mathbb{N}$, and observe that $\nu^2 - \nu = \nu(\nu-1) \in \mathbb{N}_\infty$. Thus the length of an interval $[\nu^n, \nu]$, longer than any $n$, for any of $\ast$ as well as all of the very, which is simply a restriction of $\ast$-propositional deduction, one has that attribute $b$ of infinite natural numbers, such that $\nu \leq \lambda$, $\ast \Pi(G_\nu) \subset \ast \Pi(G_\lambda)$ and conversely.

The “better than” order is only defined for comparable readable sentences. For this research, the domain of definition is restricted to the set $\text{BP}_0$ [resp. $\text{BP}$]. Two elements $[f]$, $[g] \in \text{BP}_0$ [resp. $\text{BP}$] are comparable if there exists $b \in i[E_0]$ [resp. $i[B]$] such that $f^b \in [f]$ and $g^b \in [g]$. Recall that $f^b$ and $g^b$ are unique element of $T^n$ and $T^m$, respectively, where $n$ and $m$ count the number “$\forall$” [resp. “very,” $|||$”] symbol strings. The $f^b$, $g^b$ are restricted to $T^n$, $T^m$, where $\nu = n, m > 0$. For example, $0 < x \leq n$, $f^b(x) = i(\text{very}, |||$) and $f^b(0) = b \in i[B]$. For two comparable objects $[f]$, $[g]$ define $[f] \leq_B [g]$ if $n \leq m$. Two nonempty sets $A, D \subset \text{BP}_0$ [resp. $\text{BP}$] have the property that $A \leq_B D$ if for each $[f] \in A$ there exists some $[g] \in D$ such that $[f] \leq_B [g]$. This is the better than pre-order and usually $[f] \leq_B [g]$ is stated as follows: “$[g]$ is better than $[f]$” or some similar expression. Actually, for the $\text{BP}_0$ [resp. $\text{BP}$], the “better than order” is a partial order and, in some cases, it is equivalent to the $\leq_\#$ order. Of course, $\leq_B$ and $\leq_\#$ are $\ast$-transferred to $\ast \mathcal{M}$.

For each $b \in B$, let $C_b = \{ x \mid (x \in \text{BP}) \land (b \leq_B x) \}$.

**Theorem 4.4.1** There exists a purely subtle $c \in \ast C_b$ such that $C_b \ast \leq_B \{c\}$.

Proof. The sentence

$$\forall x (x \in \mathbb{N} \rightarrow \exists y (y \in T^x \land [y] \in C_b \land \forall z \forall w (z \in \mathbb{N} \land z \leq x \land w \in T^z \land [w] \in C_b \rightarrow [w] \leq_B [y]))$$

holds in $\mathcal{M}$; hence, in $\ast \mathcal{M}$.

Let $\nu \in \ast \mathbb{N} - \mathbb{N}$, then there is a $f \in (\ast T)^\nu$ and a purely subtle $c = [f] \in \ast C_b$, where $[f]$ satisfies the remainder of the $\ast$-transformed (4.4.1) statement. Let $a \in C_b \subset \ast C_b$. Then there is some $m \in \mathbb{N}$ and some $g \in T^m$ such that $a = [g]$. Thus, since $m \leq \nu$, then $[g] \ast \leq_B [f]$. Consequently, $C_b \ast \leq_B \{c\}$.

The following is somewhat trivial and is not formalized as a theorem. Consider the usual representation for $c = [f]$, $f \in (\ast T)^\nu$, $\nu \in \ast \mathbb{N} - \mathbb{N}$. Intuitively, members of $\ast C_b(\{c\})$, are obtained by removing $\ast$-finitely many (including 0) $i(\text{very}, |||$) from $c$. Let $n \in \mathbb{N}$. Then $\{ x \mid (x \in \ast \mathbb{N}) \land (n \leq x \leq \nu) \}$ is $\ast$-finite. By $\ast$-transfer of the appropriate sentence, you have the following for each $m \in \mathbb{N}$. If $m = 0$, then $[g] \in \ast C_b(\{c\})$, where $g(0) = b$. If $m \geq 1$, then $[g] \in \ast C_a(\{c\})$, where $g(0) = b$ and, for each $j \in \mathbb{N}$, such that $1 \leq j \leq m$, $g(j) = i(\text{very}, |||$). Thus, for $\ast C_a(\{c\})$, which is simply a restriction of $\ast$-propositional deduction, one has that attribute $b$ as well as all of the very, $|||$ · · · very, $|||b$ attributes are rationally related to $c$. When Theorem 4.4.1 is interpreted, then $c$ is stronger than, better than, greater than, $b$ or any of these standard strengthens of the basic $b$. 

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5. CONSEQUENCE OPERATORS

5.1 Basic Definitions

Recall once again the Tarski [21] cardinality independent axioms for a finite consequence operator \( C : \mathcal{P}(A) \to \mathcal{P}(A) \) on a nonempty set of meaningful sentences \( A \).

\[
\begin{align*}
&\text{(2) If } B \subset A, \text{ then } B \subset C(B) \subset A, \\
&\text{(3) If } B \subset A, \text{ then } C(C(B)) = C(B), \\
&\text{(4) If } B \subset A, \text{ then } C(B) = \bigcup \{ C(F) \mid F \in F(B) \}
\end{align*}
\]

The modern theory of consequence operators (the term finite dropped) alters axiom (4) and replaces it by axiom

\[
\text{(5) If } B, D \subset A, \text{ if } B \subset D \text{ then } C(B) \subset C(D)
\]

It is very important to state that axioms (2), (3) and (4) imply axiom (5). Thus a finite consequence operator is a consequence operator, but not conversely. All things that can be established for consequence operators without any further axioms hold for finite consequence operators. For this reason, some of the following results will be established for consequence operators in general. Of course, consequence operators need not be restricted only to objects that are considered to be language. In the theory here being developed, \( A \) can be of two types. Either \( A \subset \mathcal{W} \) or \( A \subset \mathcal{E} \). I shall, however, continue to use Roman notation for all of the objects related to \( \mathcal{W} \) so as to differentiate them from the other mathematical entities. In all that follows, the symbol \( \mathcal{C}' \) will denote the set of all consequence operators defined on some specified \( \mathcal{P}(A) \) and the symbol \( \mathcal{C}'_f \) the set of a finite consequence operators. Obviously, \( \mathcal{C}'_f \subset \mathcal{C}' \), where, if no mention is made of any other possible, it will also be assumed that each member of \( \mathcal{C}' \) is defined on the same \( \mathcal{P}(A) \).

On the sets \( \mathcal{C}' \) and \( \mathcal{C}'_f \), we can define a significant partial order. For \( C_1, C_2 \in \mathcal{C}' \) let \( C_1 \leq C_2 \) if \( C_1(X) \subset C_2(X) \) for each \( X \in \mathcal{P}(A) \). This partial order, I term the stronger than order. The partial order defined on \( \mathcal{C}'_f \) is the restricted stronger than order.

A great deal has been discovered about algebras \( \langle \mathcal{C}', \leq \rangle \) and \( \langle \mathcal{C}'_f, \leq \rangle \). For example, one can define a compatible meet operation as follows: For each \( C_1, C_2 \in \mathcal{C}' \) [resp. \( \mathcal{C}'_f \)], let the map \( C_1 \land C_2 : \mathcal{P}(A) \to \mathcal{P}(A) \) be defined by \( (C_1 \land C_2)(X) = C_1(X) \cap C_2(X) \), where \( X \subset A \). Each of these algebras has the same upper unit and the same lower unit. The lower unit is but the identity map on \( \mathcal{P}(A) \). The upper unit \( U \) is the map defined by \( U(X) = A \) for each \( X \subset A \). These algebras are both meet semi-lattices.

Our interest in the above two algebras is not in any deep investigation into
there different properties but, rather, will be restricted two chains. In general, \( \langle C', \leq \rangle \) is not closed under composition. [24] However, for chains there is a very simple relation between the stronger than order and composition.

**Theorem 5.1.1** Let \( D \subset C' \). Then \( D \) is a chain in \( \langle C', \leq \rangle \) iff for each \( C_1, C_2 \in D \) either the composition \( C_1C_2 = C_1 \) or \( C_2C_1 = C_2 \).

Proof. For the necessity, assume that hypothesis. Suppose that \( C_1 \leq C_2 \).
Then for each \( X \subset A \), \( X \subset C_1(X) \subset C_2(X) \) implies that \( C_2(X) \subset C_2(C_1(X)) \subset C_2 \). Hence \( C_2C_1 = C_2 \). In like manner, if \( C_2 \leq C_1 \).

For the sufficiency, let \( C_2C_1 = C_2 \). Then for each \( X \subset A \), \( C_1(X) \subset C_2(C_1(X)) = (C_2C_1)(X) = C_2(X) \) implies that \( C_1 \leq C_2 \). In like manner for \( C_1C_2 = C_1 \) and this completes the proof. \( \square \)

5.2 Basic \( \sigma \) Properties

Since consequence operators are relations between sets, it becomes more essential to incorporate, to a certain degree, the \( \sigma \) operator into much of our discussion. Since we wish to maintain symbolic consistency and avoid trivialities, assume that nonfinite \( A \subset W \). One important result that will be used many times without further elaboration, uses the finitary construction of the equivalence class \( [f] \in \mathcal{E} \). From our previous discussion, a readable sentence \( [f] \) behaves as follows: \( \ast [f] = \ast F([f]) = [f] \) under the identification of the natural numbers. Now if \( \emptyset \neq A \subset \mathcal{E} \), then \( \ast A = \{ \ast [f] \mid [f] \in A \} = \{ [f] \mid [f] \in A \} = A \). The following result brings together various facts relative to the \( \sigma \) operator all of which follow easily from the definitions and characterizing properties. The proofs will be omitted.

**Theorem 5.2.1**

(i) Let \( A \in \mathcal{N} \). Then \( \sigma (F(A)) = F(\sigma A) \). If also \( A \subset (W \cup \mathcal{E}) \), then \( \sigma (F(A)) = F(A) \).

(ii) Let \( C \in C', B \subset X \subset W \).

(a) \( \sigma (C(B)) = C(B) \).

(b) \( \ast C \mid \{ \ast A \mid A \in \mathcal{P}(X) \} = \{ \ast A, \ast B \mid (A, B) \in C \} = \sigma C \).

(c) If \( F \in F(B) \), then \( \sigma (C(F)) \subset \langle \sigma C \rangle \langle \sigma F \rangle = \langle \sigma C \rangle (F) \). Also \( \sigma (C(B)) \subset \sigma (C \langle B \rangle) \) and, in general, \( \sigma (C(B)) \neq \sigma (C \langle B \rangle) \), \( \sigma (C(F)) \neq \sigma (C \langle F \rangle) \).

(d) If \( C \in C'_f \), then \( \sigma (C(B)) = \bigcup \{ \sigma (C(F)) \mid F \in (C(B)) \} = \bigcup \{ \sigma (C(F)) \mid F \in F(C(B)) \} \).

(A duplicate theorem holds, where \( A \in \mathcal{N} \) and \( C \in C \), where \( C \) set of consequence operators defined on subsets of \( W \) if \( W \) is included as a subset of the ground set. The difference is that the “bold” notion does not appear.)

Throughout the reminder of this section, in order to escape trivialities, we remove the upper unit \( U \) from the collections of consequence operators. Let \( C = C' \setminus \{ U \} \) and \( C_f = C'_f \setminus \{ U \} \). One of the consequences of this last requirement shows that if \( D \in \mathcal{N} \) satisfies the axioms for a consequence operator and \( G \subset C \), then there
does not exist a consequence operator \( C \in ^*G \) such that \(^*D = C\). To see this simply note that from Theorem 5.2.1 \(^*D\) is defined on extended standard sets while each member of \(^*G\) is defined on the internal subsets of \( A \). Since \( A \) is not finite, there exists internal subsets of \( A \) that are not equal to any extended standard set. There is also one useful general fact. Consider any sets \( A, B \in \mathcal{N} \) such that \( A \subseteq B \). Then \(^*A \subseteq ^*B\). For suppose that there exists some \( X \in ( ^*A \subseteq ^*B\) such that \( X \subseteq ^*B\). Then \( X \subseteq ^*B\) for some \( D \in B \). Thus \(^*D \subseteq ^*A\) implies that \( D \subseteq A \). From this, we have the contradiction that \( X = ^*D \subseteq ^*A\). Also note that there does not exist \( D \in \mathcal{N} \) such that \(^*D \subseteq ( ^*B \subseteq ^*B\) (i.e. each member of \( ^*B \subseteq ^*B\) is an internal nonstandard object or a pure subtle object.)

5.3 Major Results

For the algebras \( \langle C, \leq \rangle \) and \( \langle f, \leq \rangle \) two types of chains will be studied. Denote by \( K \) any nonempty chain contained in either of these algebras and by \( K_\infty \) a chain with the following property. For each \( C \in K_\infty \) there exists \( C' \in K_\infty \) such that \( C < C'\).

**Theorem 5.3.1** There exists \( C_0 \in ^*K \) such that for each \( C \in K \), \(^*C \leq C_0\). There exists some \( C_\infty \in ^*K_\infty \) such that \( C_\infty \) is a purely subtle consequence operator and for each \( C \in K_\infty \), \(^*C < C_\infty\). Each member of \(^*K\) and \(^*K_\infty\) are subtle consequence operators.

Proof. Let \( R = \{(x, y) \mid \langle x \in K \rangle \wedge \langle y \in K \rangle \wedge (x \leq y)\} \) and \( R_\infty = \{(x, y) \mid \langle x \in K_\infty \rangle \wedge \langle y \in K_\infty \rangle \wedge (x \leq y)\} \). In the usual manner, it follows that \( R \) is concurrent on \( K \) and \( R_\infty \) is concurrent on \( K_\infty \). Consequently, there is some \( C_0 \in ^*K \) and some \( C_\infty \in ^*K_\infty \) such that for each \( C \in K \) and each \( C' \in K \), \(^*C \leq C_0\), \(^*C' < C_\infty\) since \( ^*M \) is an enlargement. Further, it follows that \( C_\infty \in ^*K_\infty \) implies that \( C_\infty \) is a purely subtle consequence operator. Note that each member of \(^*K \cup ^*K_\infty\) is defined on the set of all internal subsets of \(^*A\). This completes our proof. \( \Box \)

Notice that \( C_\infty \) is stronger than or “more powerful than” any \( C \in K_\infty \) in the following sense. If \( B \subseteq A \), then for each \( C \in K_\infty \), it follows that \( C(B) \subseteq ^*\langle C(B) \rangle = ^*C\langle ^*B \rangle \subseteq C_\infty\langle ^*B \rangle \). Also for each \( C \in K_\infty \) there exists some internal \( E_C \subseteq ^*A \) and \(^*C\langle E_C \rangle \subseteq C_\infty\langle E_C \rangle \). Recall that for \( C \in C \), a set \( B \subseteq A \) is a \( C\)-deductive system if \( C(B) = B \). Also, when we write the \(^*-\)operator on any map \( f \) in the form \(^*f(x)\) this always means \( ^*f(x) \) rather than \(^*(f(x))\).

**Theorem 5.3.2** Let \( C \in C_\infty \) and \( B \subseteq A \subseteq W \). Then there exists a \(^*-\)finite \( F \in ^*\langle F(B) \rangle \) such that \( C(B) \subseteq C(F) \subseteq C\langle ^*B \rangle = C\langle ^*B \rangle \) and \(^*C(F) \cap A = C(B) = C(F) \cap C(B)\).

Proof. Consider the binary relation \( Q = \{(x, y) \mid \langle x \in C(B) \rangle \wedge \langle y \in F(B) \rangle \wedge (x \in C(y))\} \). By axiom (4), the domain of \( Q \) is \( C(B) \). Let \( (x_1, y_1), \ldots, (x_n, y_n) \in Q \). By Theorem 1 in [5, p. 64] (i.e. axiom (5)) we have that \( C(y_1) \cup \cdots \cup C(y_n) \subseteq C(y_1 \cup \cdots \cup y_n) \). Since \( F = y_1 \cup \cdots \cup y_n \in F(B) \), then \( (x_1, F), \ldots, (x_n, F) \in Q \). Thus \( Q \) is concurrent on \( C(B) \). Hence there is some \( F \in ^*\langle F(B) \rangle \) such that
σ(C(B)) = C(B) ⊂ *C(F) ⊂ *C(∗B) = *C(B)). Since σA = A, *C(F) ∩ A = C(B) = *C(F) ∩ C(B).

**Corollary 5.3.2.1** If C ∈ Cf and B ⊂ A ⊂ W is a C-deductive system, then there exists a *-finite F ⊂ *B such that *C(F) ∩ A = B.

**Corollary 5.3.2.2** Let C ∈ Cf. Then there exists a *-finite F ⊂ *A such that for each B ⊂ A, *C(F) ∩ B = B.

Proof. In Theorem 5.3.2, let the “B” be equal to A. Then there exists some *-finite F ⊂ *A such that *C(F) ∩ A = C(A) = A. Thus *C(F) ∩ A ∩ B = *C(F) ∩ B = A ∩ B = B.

**Theorem 5.3.3** Let B ⊂ A ⊂ W.

(i) There exists a *-finite FB ∈ *(F(B)) and a subtle consequence operator CB ∈ *K such that for all C ∈ K, σ(C(B)) = C(B) ⊂ CB(FB).

(ii) There exists a purely subtle consequence operator C∞B ∈ *K∞ such that for all C ∈ K∞, σ(C(B)) = C(B) ⊂ C∞B(FB).

Proof. (i) Consider the binary relation Q = {((x, z), (y, w)) | (x ∈ K) ∧ (y ∈ K) ∧ (w ∈ F(B)) ∧ (z ∈ x(w)) ∧ (x(w) ⊂ y(w))}. Let nonempty \{((x1, z1), (y1, w1)), . . . , ((xn, zn), (yn, wn))\} ⊂ Q. Notice that F = w1 ∪ . . . ∪ wn ∈ F(B) and the set R = \{x1, . . . , xn\} has a largest member D with respect to the E embedded ≤ ordering for the consequence operators. It follows that zj ⊂ xi(wi) ⊂ xi(F) ⊂ D(F) for each i = 1, . . . , n. Hence \{((x1, z1), (D, F)), . . . , (xn, zn), (D, F))\} ⊂ Q implies that Q is concurrent on its domain. Consequently, there exists some (CB, FB) ∈ *K × *(F(B)) such that for each (x, z) ∈ domain of Q, (*C, *b) ∈ *Q. Therefore, each (u, v) ∈ σ(domain of Q), (u, v), (CB, FB) ∈ *Q. Let arbitrary C ∈ K and b ∈ C(B). Then there exists some F′ ∈ F(B) such that b ∈ C(F′). Thus (*C, *b) ∈ σ(domain of Q). Consequently, for each C ∈ K and b ∈ C(B), b = *b ∈ *C(FB) ⊂ CB(FB).

This all implies that for each C ∈ K, *C(B)) = C(B) ⊂ CB(FB).

(ii) Change the relation Q to Q′ be adding the additional requirement to Q that x ≠ y. Replace the D in (i) with any D′ that is greater than and not equal to the largest member of R. Such a D′ exists in K∞ from the definition of K∞. Continue the proof in the same manner as in (i) to obtain C∞B and FB. The fact that C∞B is a purely subtle consequence operator follows as in the proof of Theorem 5.3.

**Corollary 5.3.3.1** There exists a [resp. purely] subtle consequence operator CA ∈ *K [resp. *K∞] and a *-finite FA ∈ *(F(A)) such that for all C ∈ K [resp. K∞] and each B ⊂ A, B ⊂ C(B) ⊂ CA(FA).

Proof. Simply let the “B” in Theorem 5.3.3 be equal to A. Then there exists a [resp. purely] subtle CA ∈ *K [resp. *K∞] such that for all C ∈ K [resp. K∞], C(A) ⊂ CA(FA). If B ⊂ A and C ∈ K [resp. K∞], then B ⊂ C(B) ⊂ C(A). Thus for each B ⊂ A and C ∈ K [resp. K∞], B ⊂ C(B) ⊂ CA(FA) and this completes the proof.
Relative to the above results, it is well known that for $B \subseteq A$ that there exists a *-finite $F_1 \subseteq ^*B$ such that $B \subseteq F_1 \subseteq ^*B$. Thus for any $C \in \mathcal{C}$ it follows that $B \subseteq C(F_1) \subseteq ^*C( ^*B)$. One significance of the above results is that the $C^\infty_B$ is purely subtle and, thus, not the same as any extended standard consequence operator.

5.4 Applications

In what follows, let denumerable $L$ be a language constructed from a denumerable set of primitive symbols $\{P_i \mid i \in \mathbb{N}\}$. As to the construction of $L$ it is, at least, constructed from the binary operation $\to$. Deduction over $L$ is defined in the usual sense. Only finitely many steps are allowed, and if any axiom schema are used, then they do not yield statements of the form $P_i$ or $P_i \to P_j$, $i \neq j$. Further, deduction from premises is also allowed. There are many examples of such languages. Propositional languages with denumerably many atoms. Indeed, in a predicate language with, at least, one predicate the list of all predicates can be considered the set of primitives from which $L$ is constructed. Of course, simple natural languages are isomorphic to $L$ in the usual sense. There will be one modification, however. The modification is in a rule of inference. Define the $MP_n$, $n \in \mathbb{N}$ rule of inference on $L$ as follows:

If two previous steps of a demonstration (or proof) are of the form $A, A \to B$ where for each $P_i$ in the primitive expansions of $A$, $A \to B$, $i \leq n$, then the formula $B$ may be written down as the next step.

No other type of MP rule is used.

Given a set of hypotheses $\mathcal{H} \subseteq L$ and $X \in L$, denote by the symbol $\mathcal{H}^{\rightarrow}_n X$ this deductive process. It is immediate that $\vdash_n$ determines a finitary consequence operator $C_n$ on $\mathcal{P}(L)$. Suppose that $\vdash$ deduction on $\mathcal{P}(L)$ has all of the above properties with the exception that the MP rule of inference is the ordinary modus ponens in unrestricted form. Let $S_0$ denote the consequence operator determined by $\vdash$.

It is a simple matter to show that for any $B \subseteq L$, $S_0(B) = \bigcup\{C_n(B) \mid n \in \mathbb{N}\}$. Let $B \subseteq L$. Suppose that $X \in S_0(B)$. Then $B \vdash X$. Now in the formal proof of this fact when all of the formula are written in primitive form there is a maximum $P_i$ subscript, say $m \in \mathbb{N}$. It follows immediately that the same steps yield a formal proof that $B \vdash_m X$ using the $MP_m$ in place of any MP step that appears in the formal proof. Thus $X \in \bigcup\{C_n(B) \mid n \in \mathbb{N}\}$. Clearly, if $B \vdash_n X$, then $B \vdash X$. This implies $S_0(B) = \bigcup\{C_n(B) \mid n \in \mathbb{N}\}$. Another interesting result is that if $B$ is an $S_0$-deductive system, then $S_0(B) = \bigcup\{C_n(B) \mid n \in \mathbb{N}\} = B$ implies that for each $n \in \mathbb{N}$, $C_n(B) = B$. Thus $B$ is also a $C_n$-deductive system.

Let $n, m \in \mathbb{N}$, $n \leq m$, $B \subseteq L$. Suppose that $X \in C_n(B)$. If there are any $MP_n$ steps in the formal proof, then these steps can also be obtained by application of $MP_m$. On the other hand, if no steps were obtained by the $MP_n$ rule, then the exact same steps yield a formal proof that $B \vdash_m X$. From this we have that for each
B ⊂ L, C_n(B) ⊂ C_m(B). Hence, C_n ≤ C_m. Therefore, \{C_n \mid n ∈ \mathbb{N}\} is a chain of consequence operators.

Now to show that this chain is of type K_∞. Let n < m and let B = \{P_n, P_n → P_m\}. First, no member of B can be obtained as an instance of an axiom. Further, it cannot be the case that B ⊨ n P_m for MP_n does not apply to P_n → P_m or, indeed, any formula containing P_m. Therefore, P_m \notin C_n(B). Obviously, P_m ∈ C_m(B). Hence, C_n(B) \neq C_m(B) implies that C_n < C_m. Thus this chain is of type K_∞. Further, note that P_m ∈ S_0(B). Thus for each C ∈ \{C_n \mid n ∈ \mathbb{N}\} there exists some B ⊂ L such that C(B) \neq S_0(B) and, clearly C ≤ S_0. Hence, in general, C < S_0 for all C ∈ \{C_n \mid n ∈ \mathbb{N}\}.

**Theorem 5.4.1** Let L and K_∞ = \{C_n \mid n ∈ \mathbb{N}\} be defined as above. Then there exists a purely subtle consequence operator C_∞ ∈ *K_∞ and a *-finite F ∈ *(F(L)) such that for each B ⊂ L and each C ∈ K_∞

(i) C(B) ⊂ C_∞(F),

(ii) S_0(F) ⊂ C_∞(F) ⊂ *S_0(F) ⊂ *S_0(*L),

(iii) *S_0(F) ∩ L = C_∞(F) ∩ L.

Proof. (i) is but Corollary 5.3.3.1. From (i), it follows that \( \bigcup \{ \mathcal{C}(\mathcal{C}(B)) \mid C ∈ K_∞ \} = \bigcup \{ (\mathcal{C}(B)) \mid C ∈ K_∞ \} = S_0(B) = \mathcal{S}_0(B) \subset C_∞(F) \) and the first part of (ii) holds. By *-transfer C_∞ < *S_0 and C_∞ and *S_0 are defined on all internal subsets of *L. Hence, C_∞(F) ⊂ *S_0(F) ⊂ *S_0(*L) and this completes (ii). (iii) follows immediately from (ii) and this completes the proof.

For this application, let L be a predicate type language and M any set-theoretic structure in which the predicates and constants are interpreted in the usual manner. A finite consequence operator defined on \( \mathcal{P}(L) \) is sound for M if whenever B ∈ \( \mathcal{P}(L) \) has the property that M ⊨ B, then M ⊨ C(B). As usual, T(M) = \{x \mid (x ∈ L) \land (M ⊨ x)\}. Obviously, if C is sound for M, then T(M) is a C-deductive system.

Corollary 5.3.2.2 implies that there exists *-finite F ∈ *(T(M)) such that *C(F) ∩ L = T(M). Notice that F being *-finite implies that F is *-recursive. Moreover, F is a *-axiom system for *C(F) and we do not lack knowledge about the behavior of F since any formal property about C or recursive sets, among others, must hold true for *C or F when properly interpreted. If L is a first-order language with at least one predicate, then its associated consequence operator S_1 is sound for first-order structures. Theorem 5.4.1 not only yields a *-finite F_1 but a purely subtle consequence operator C_1 such that F_1 is a *-axiom system for C_1(F_1) and *S_1(F_1). In this case, we have that *S_1(F_1) ∩ L = T(M) = C_1(F_1) ∩ L. As strange as it may appear, by use of internal and external objects, the nonstandard logics \{ *C, *L \}, \{ C_1, *L \}, \{ *S_1, *L \} technically by-pass a portion of Gödel’s first incompleteness theorem. Of course, this incompleteness theorem still holds under an internal interpretation.

By definition b ∈ S_0(B), B ⊂ L iff there is a finite length proof of b from the premises B. Thus for each b ∈ *(T(M)) there exists a *-finite length proof of b from the *-finite F_1. If we let *M be an enlargement with the \( \aleph_1 \)-isomorphism
property, among others, then each *-finite length proof is either externally finite or externally infinite. Further, all externally infinite proof lengths would be of the same cardinality.

{Remark: Using the customary notation in this chapter, the relation \( \leq \) has not been starred in \( ^*N \). If this omission is confusing, the * can be easily inserted. When these two different order relations are compared, the * notation becomes necessary. For example, the relation \( ^*\leq \) in \( ^*N \) is NOT an extension, in the usual sense, of the relation \( \leq \) as defined in \( N \), although it is an extension of \( ^\sigma \leq \). Also notice that if we had restricted our attention to \( C_f \), then the partial order \( \leq \) is characterized totally by the finite subsets of \( A \). This is useful since that \( ^*\leq \) is characterized by the *-finite subsets of \( ^*A \). It’s clear that our concept of a consequence-type operator must be generalized slightly. Let \( B \) and \( B_0 \) be two families of sets. Then if \( f: B \rightarrow B_0 \) satisfies axioms (2)(3)(4) or (2)(3)(5) or the *-transform of these axiom systems, then \( f \) is a subtle consequence operator. I also point out that, unfortunately, there are many typographical errors in reference [24].}
6. ASSOCIATED MATERIAL

6.1 Perception

In this section, the theory of ultralogics is applied to one aspect of subliminal perception. What is needed is an interpretation scheme. When subsets of $\mathcal{E}$ are concerned the conscious objects are subsets [resp. elements of] $\sigma \mathcal{E} = \mathcal{E}$. The subconscious objects are nonstandard internal subsets [resp. elements of] $\mathcal{E}$. Moreover, subconscious objects can contain conscious objects and the union of a subconscious set and a finite conscious set is a subconscious set. The unconscious objects are external nonstandard subsets of $\mathcal{E}$. Like definitions apply to members of $\mathcal{E} \times \mathcal{E}$ and so forth. In what follows, only strong reasoning from the perfect is considered. You may assume that it is defined on a natural language “very,|||” or a formal language “$V \land$” and the like.

As to some sort of interpretation procedure the following seems adequate. Let $\llbracket \rrbracket$ denote an interpretation symbol. First, we have subperception and the better than ordering. Let $A \subset \mathcal{E}$ be one of the above defined objects in the domain of $\star \Pi$. Let internal $D \subset \star \Pi(A)$ and standard $\sigma \mathcal{E} \subset \star \Pi(A)$. Assume that $\sigma \mathcal{E} \leq_B D$ and that each member of $i^{-1}[E]$ is a sentence which is distinctly comparable by the “very,|||” symbol string. [Note I am not differentiating between the object and a constant representing that object.] One might interpret the following: $\llbracket \forall x((x \in \sigma \mathcal{E}) \rightarrow \exists y((y \in D) \land (x \leq_B y))) \rrbracket$ : “You are (I am, we are, etc) subperceptibly aware that for each conscious (known) object (element, member) of (in) $\sigma \mathcal{E}$ there exists an object (element, member) of (in) $D$ which is better than that conscious object of (in) $\sigma \mathcal{E}$.”

Let $a \in \sigma \mathcal{E}$. Then: $\llbracket \exists y((y \in D) \land (a \leq_B y)) \rrbracket$ : “You are (I am, we are) subperceptibly aware that there exists a conscious (known) object (element, member) of (in) $D$ which is better than $a$. “ Note that $a = [f_0] \in \mathcal{B} \mathcal{P}$ and $[a] = i^{-1}(f_0(0))$. Another example is $\llbracket \sigma \mathcal{E} \leq_B D \rrbracket$ : “You are (I am, we are) subperceptibly aware that $D$ is better than $\sigma \mathcal{E}$. “

For another example, let $\sigma F \subset \star \Pi(A)$. Then $\llbracket ([\sigma \mathcal{E} < |\sigma F|) \land (\sigma \mathcal{E} \leq_B D) \land (\sigma F \leq_B D) \rrbracket$ : “The result that $D$ is better than $\sigma F$ is a stronger subperceptible property than $D$ is better than $\sigma \mathcal{E}$. “

We also have the idea of general subperception. In this case, we use some of the meaningful set-theoretic terminology. Let internal nonstandard $A, B \subset \star \mathcal{B} \mathcal{P}$. Now any elementary set-theoretic relation existing between $A$ and $B$ can be subperceptibly interpreted as $\llbracket A \subset B \rrbracket$ : “You are (I am, we are) subperceptible aware of the following: $A$ is contained in $B$. “ Also you might interpret relations between standard objects as a complete awareness.

6.2 Existence

Some philosophers of science differentiate between theoretical entities and those
that are assumed to exist in objective reality. In the original work in ultralogics, these two concepts were disjointly modeled. This was done as follows: Consider \( \sigma[f_0] = [f_0] \in \mathcal{E} \) to be the unique partial sequence with the property that \( \sigma^{-1}(f_0(0)) = \text{externally} ||\text{exists}||\text{reality} = [[f_0]] \). For \( A \subset ^* \mathbf{BP} \), let \( (A)_R = \{(x,[f_0]) \mid x \in A\} \). Then for any \( E \subset ^* \mathbf{BP} \), define \( RR(E,A) = \{(x,y) \mid (x \in E) \land (y \in (A)_R)\} \) to be the "realism relation." These definitions are then extended to \(^* \mathbf{BP} \times ^* \mathbf{BP} \) in such a manner that \( (A)_R \times (B)_R \) is considered to be isomorphic to \((A \times B)_R\). I now believe that this is a waste of effort. The difference lies in the interpretation and not in the mathematical structure. Thus, under the interpretation, if one wishes to differentiate between these two concepts, one simply includes "existence in objective reality" as a part of the interpretation for some entities and the statement "theoretical entities" for other distinct entities.

6.3 An Alternate Approach

What is presented in this section is mainly of historical interest although this author's first research into nonstandard analysis used this alternate approach. This approach utilizes a pseudo-set theory and has essentially been replaced by the superstructure approach. Some years ago, certain applications employ this alternate approach due to its use of a basic language that is somewhat more expressive than the \( \in \), language. However, what might be gained in an additional freedom of expression will lead to a more complex array of extensions, definitions and the requirement that extreme care be exercised.

All of our constructions are within \( \mathbf{ZFC} \). We utilize the transitive closure operator, denoted by \( TC \). Let \( V \) be a set. (Note: This definition also applies to atoms and sets containing atoms.) The transitive closure of \( V \) is obtained by an inductive construction using the union operation. Let \( V_0 = V \) and for each \( i \in \mathbb{N} \), let \( V_{i+1} = \cup V_i \). Then the transitive set \( TC(V) = \bigcup \{V_i \mid i \in \mathbb{N}\} \). The set \( TC(V) \) for a set \( V \) has the property that if \( W \) is another transitive set such that \( V \subset W \), then \( V \subset TC(V) \subset W \). Define the superstructure operator, denoted by \( SS \), on \( V \) as \( SS(V) = \bigcup \{U_i \mid i \in \mathbb{N}\} \), where \( U_0 = V \), \( U_{i+1} = U_i \cup P(U_i) \), \( i \in \mathbb{N} \). Recall that this is the first type of superstructure defined in Chapter 2. To correspond to our previous investigation, let \( V = W \cup \mathbb{N} \) and \( \mathcal{N}_1 = SS(TC(V)) \). (If \( V \) is a set of atoms, then \( TC(V) = V \).) Let the structure \( \mathcal{M}_1 = (\mathcal{N}_1, \in, =, \cup, \cap) \) where \( \in, = \) are the usual set-theoretic membership and set equality relations restricted to \( \mathcal{N}_1 \) and \( \cup, \cap \) are two ternary relations, the "applying a function to its argument" and "ordered pair creation", respectively. (Of course, \( =, \cup, \cap \) can all be defined in terms of \( \in \).) Notice that \( \mathcal{M}_1 \) is a fragment of our \( \mathbf{ZFC} \) model.

Consider a \( \kappa \)-adequate ultrafilter, where \( \kappa > |\mathcal{N}_1| \). By Theorem 7.5.2 in [19] or 1.5.1 in [9], such an ultrafilter \( \mathcal{U} \) exists in our \( \mathbf{ZFC} \) model and is determined by the indexing set \( J = F(\mathcal{P}(\kappa)) \). By the ultrapower or ultralimit construction, a first-order structure \( \mathcal{M}_2 = (\mathcal{N}^J, \in, =, \cup, \cap, \cup \mathcal{U}, \cap \mathcal{U}) \) is obtained of the same type as is \( \mathcal{M}_1 \), but \( \mathcal{M}_2 \) is a nonstandard model for the set of all sentences, \( K_0 \), in our first-order language \( L \), with predicates \( \in, =, \cup, \cap \) which hold in \( \mathcal{M}_1 \). (Theorem 3.8.3 in [19])
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Note that the cardinality of the set of constants in $L \geq |\mathcal{M}_1|$. Further, the members in $\mathcal{N}$ are interpreted by constants in an extended language $L'$. By the axioms of our ZFC set-theory, the relation “=” is an equivalence relation with substitution for $\in$, $ap$, $pr$ and, hence, $=_{U}$ has these properties for $\in_{U}$, $ap_{U}$, $pr_{U}$. Consequently, we shift the structure $\mathcal{M}_2$ (i.e. contract it) [13, p. 83] and obtain a structure $'\mathcal{M} = \langle J_1, \epsilon, =, 'ap, 'pr \rangle$, where $= \epsilon$ is the original equality in our ZFC model. Note that members of $'\mathcal{N}$ are still interpreted by constants in $L'$ as before. [In [4] and [11], a structure isomorphic to $'\mathcal{M}$ is obtained by application of the compactness theorem for a first-order language.]

We next isomorphically embed $\mathcal{M}_1$ into $'\mathcal{M}$ in the same manner as outlined in [4, p. 22]. However, please notice that the following notation differs from that used in this reference. First, let $I$ be the the original interpretation map from $L$ onto $\mathcal{M}_1$ and let $'I$ by the composition of the extended ultrapower interpretation map and the contraction interpretation map restricted to the constants in $L$.

Now for each “$a$” that is a constant in $L$, define $^\sigma(I(a)) = I(a) \in '\mathcal{N}$. Assume that $a$, $b$ are constants in $L$ that represent the same element and consider the well-formed formula $(a = b)$. Then $\mathcal{M}_1 \models (a = b)$ iff $'\mathcal{M} \models (a = b)$ implies that $^\sigma(I(a)) = ^\sigma(I(b))$. Thus the map $\sigma$ is well-defined. Again let $a$, $b$ be constants in $L$. Then $\mathcal{M}_1 \models \neg(a = b)$ iff $'\mathcal{M} \models \neg(a = b)$ implies that $^\sigma(I(a)) \neq ^\sigma(I(b))$ iff $I(a) \neq I(b)$. Thus $\sigma$ is injective. It is immediately clear that $\mathcal{M}_1 \models (a \in b)$ iff $'\mathcal{M} \models (a \in b)$ implies that $I(a) \in I(b)$ iff $^\sigma(I(a)) \epsilon ^\sigma(I(b))$ and, in like manner, for the relations “$ap$” and “$pr$”. Consequently, $\sigma$ is an isomorphic embedding of $\mathcal{M}_1$ into $'\mathcal{M}$. For convenience in all that follows, we suppress the interpretation map notation and simply use the constants of the language $L$ (and the extended language $L'$) to represent members in $'\mathcal{N} = \{ x \mid \exists a \in L \land (^\sigma(I(a)) = I(a)) \land (x = I(a)) \}$. Of course, $'\mathcal{N} \subset '\mathcal{N}$ and $\mathcal{M}_1$ is isomorphic to $\mathcal{M} = \langle N, (\epsilon)/\sigma, =, (ap)/\sigma, (pr)/\sigma \rangle$. [Note: $(\epsilon)/\sigma$ is the relation restricted to members of $N$, etc.]

Now to continue this construction. First, by Theorem 1.5.2 in [9], $'\mathcal{M}$ is an enlargement. For each $p \in '\mathcal{N}$, let $^*p = \{ x \mid (x \in '\mathcal{N}) \land (x \epsilon p) \}$ and $^*\mathcal{N} = \{ ^*p \mid p \in '\mathcal{N} \}$. Notationally, let $^*^*\mathcal{N} = ^*\mathcal{N} \cup N_1 \cup '\mathcal{N}$ and define $^0^*\mathcal{N} = SS(TC( ^*^*\mathcal{N} ))$. Obviously, $^*\mathcal{N} \cup N_1 \cup '\mathcal{N} \cup ^*^*\mathcal{N} \subset TC( ^*^*\mathcal{N} )$. Since $TC( ^*^*\mathcal{N} )$ is a member of $SS(TC( ^*^*\mathcal{N} ))$, then this implies, by Theorem 2.6 (iii) in [4], that $^*\mathcal{N}, ^*^*\mathcal{N}, '\mathcal{N}, '^*\mathcal{N}, ^*^*^*\mathcal{N} \in ^0^*\mathcal{N}$. Finally, applying Theorem 2.6 (ii) and 2.10 in [4] to the transitive set $^0^*\mathcal{N}$, one obtains that $^0^*\mathcal{N}$ is closed under finite power set iteration and finite Cartesian products. The structure $^0^*\mathcal{M} = \langle ^0^*\mathcal{N}, \epsilon, =, ap, pr \rangle$ that is a fragment of our ZFC model is the G-structure in this alternative approach.

The model $^0^*\mathcal{M}$ contains all of the set-theoretic objects needed for this investigation. The fact that we are only interested in semantic consistency allows us to consider all of the structure $\mathcal{M}$ as the standard model in which sentences from $L$ are interpreted and $'\mathcal{M}$ the nonstandard model for sentences from $L$. Of course, we can always return to $\mathcal{M}_1$ by application of $\sigma^{-1}$.

There is one important notational convention that is continually employed. The $\sigma$ map is suppressed when considering members of $\mathcal{N}$. That is to say that for
each constant \( a \in L \), \( \sigma(I(a)) = I(a) = a \). The use of \( \mathcal{M} \) can be made more efficient since there should be no great difficulty if you consider \( (\epsilon) / \sigma, (\prime \text{pr}) / \sigma, (\prime \text{ap}) / \sigma \) to be the same as the ZFC model relations \( \in, \prime \text{ap}, \prime \text{pr} \) for there is no first-order differences between these structures.

The fact that we are actually working with the restricted \( \epsilon, \prime \text{ap}, \prime \text{pr} \) can be determined by the additional result that the only objects to which the restriction of these relations apply are members of \( \mathcal{N} \). The only other objects to which nonrestricted \( \epsilon, \prime \text{ap}, \prime \text{pr} \) apply are elements of \( \mathcal{N}' \) that are not members of \( \mathcal{N} \). The actual \( \in, \prime \text{ap}, \prime \text{pr} \) are used in all other contexts such as the following important definition as previously stated. For each \( A \in \mathcal{N} \), \( \star A = \{(x \in \mathcal{N}) \land (x \in A)\} \). This is the beginning of certain technical features for this model. What is significant as we define some of these technical terms is that all of the objects within this and other nonstandard investigations are set-theoretic members of \( 0\mathcal{M} \) and, of course, \( 0\mathcal{M} \) is in our ZFC model.

Rather than force the reader to seek out references [4] or [11], I reproduce here the more significant definitions required to relate many of our results to the \( \in, \prime \text{ap}, \prime \text{pr} \) operations within the structure \( 0\mathcal{M} \). Let \( p \in \mathcal{N}' \). If \( p \notin \mathcal{N} \), then \( p \) is called a nonstandard object or entity. If \( p \in \mathcal{N} \), then \( p \) is called a standard object. If \( S \subset \star P \) and there exists some \( Q \in \mathcal{N}' \) such that \( S = \star Q \), then \( S \) is called an internal object or set. Observe that \( P \in \mathcal{N}' \) that is not a \( \prime \)atom is an internal subset of itself. Also it is often the case that each element of \( \star P \) is called internal for if \( P \in \mathcal{N} \), then there exists some \( X_n \) such that \( P \in X_n \) and \( X_n \) is \prime-transitive. Thus if \( p \in P \), then \( p \in X_n \) implies that \( p \in \mathcal{N}' \). Intuitively internal means that there exists a symbolic name in \( L' \) for the object that generates, under the given definitions, the second corresponding object.

This generation of the second corresponding object is of a special nature. Let \( f \in \mathcal{N}' \) be an \( \prime \)n-ary relation where \( n > 1 \). Thus \( f \) satisfies in \( L' \) the appropriate sentence that defines such an object. Extend \( f \) in the following manner. Let \( f^* = \{(a_1, \ldots, a_n) \mid (a_1 \in \mathcal{N}') \land \cdots \land (a_n \in \mathcal{N}') \land ((a_1 \prime \text{pr} \cdots \prime \text{ap} a_n) \in f)\} \). In general, \( \prime f \neq f^* \). For the many properties associated with this definition, refer to references [4] [11]. I note that in [4] one of the important properties for such an extension of \( f \) relative to the \( i' \)th projection [Theorem 4.5 (vii)] is stated on one side of the equation incorrectly. However, the proof goes through correctly and one should correct the statement of that small portion of the theorem to show that the \( i' \)th projection of the \( n \)-ary relation \( f^* = \star (of \ the \ i' \)th projection of \( f \). [Note: there are two theorems in [4] that are proved incorrectly, even though the theorem statement is correct. The proofs were corrected when these results were published.] Any \( n \)-ary relation that is produced by an extension that has the \( \star \) on the right is called an internal \( n \)-ary relation. Notice that what this actually means is that there is a name for the \( \prime \)n-ary relation in the extended language \( L' \).

Our major interest and application for this model will be confined to objects in \( \mathcal{E} \) as well as in \( \mathcal{E} - \mathcal{E} \), and a fixed power set iteration or Cartesian products of these objects. The use of the \( \prime \)-ing process is different in this model than it is in
the model utilized in the previous sections of this chapter and previous chapters of this book. For example, it is important to realize that the set \( f \in \mathcal{E} \) is a finite set of functions. Thus \( \forall x(x \in [f] \leftrightarrow (x = a_1) \lor \ldots \lor (x = a_n)) \) holds in \( M \). Therefore, \( \forall x(x \in * [f] \leftrightarrow (x = a_1) \lor \ldots \lor (x = a_n)) \) implies that \( * [f] = [f] \) and \([f] \) is internal. Observe that the symbols \( a_1, \ldots, a_n \) do not carry the * notation as would be necessary, prior to our identification process, in the previous sections of this chapter and previous chapters. Further, each \( g \in [f] \) is a finite set of ordered pairs, as previously. Thus \( *g = g \) and \( g \) is internal.

Even though the above property seems to be a nice property, the nonstarring of standard objects, it turns out that the partition concept must be handled differently. Indeed, Theorem 3.2.3 is not true in this model. If \( A, B \in \mathcal{N} \) and \( B \) is a partition of \( A \), then \( *B \) is not a partition of \( *A \). What is needed is to consider the set \( D = \{ *x \mid x \in *B \} \). The \( D \) is a partition for \( *A \), but \( D \) is not in general an internal set. On the other hand, each element of \( *B \) is an internal set as is each finite subset. It is interesting to note that we require a different extension definition for the consequence operators when \( ^0M \).

Let \( C: \mathcal{P}(A) \rightarrow \mathcal{P}(B) \) be a standard set-valued map. One must be more careful with the extensions of such set-valued maps than the other maps since the types of objects contained in the ordered pairs are of significance. Observe that \( C^* \) is composed of ordinary ordered pairs of \( 's \) sets and as such these sets are in \( ^0M \) and contain \( 's \) elements. However, these sets may also contain ordinary elements as well. Assume that \( (P, Q) \in C^* \), and that \( P, Q \) are not finite standard sets, then \( (P, Q) \neq ( 'P, 'Q) \). Consequently, in general, a map such as \( C \) can be extended to a map \( \mathcal{C} = \{( 'P, 'Q) \mid (P, Q) \in C^* \} \). In this case, \( 'P, 'Q \) contain only \( 's \) elements from \( P \) and \( Q \). This different interpretation occurs because the “starring” process in the first model used in this analysis is distinct from the “starring” process as employed with respect to \( ^0M \). In fact, the * process in the first model is a renaming of the standard language objects as they are interpreted within that model and is a member of the extended language \( L' \). The other members of \( L' \) are restricted to internal members of our model.

More importantly, with the first model all of the relations \( \epsilon, 'pr, 'ap \) have been replaced by the ordinary \( \epsilon, pr \) and \( ap \) within the ZFH model and the standard model has been embedded into the structure that what would have been the \( \epsilon, 'pr, 'ap \) defined objects denoted by members of \( L \) are so altered that they become the original relations restricted to entities that are isomorphically related to the original standard objects. One can say that the first model alters the objects with a “minimal” language change. This alternate approach requires a much larger language change but is more expressive in character.

The above extension processes lead to three distinct objects \( \mathcal{C}, C^* \) and \( \mathcal{C} \) within \( ^0M \). If \( C: \mathcal{P}(A) \rightarrow \mathcal{P}(B) \) is a consequence operator, then \( C^* \) and \( \mathcal{C} \) are interesting but distinct member of \( ^0M \). They both satisfy (extended) Tarski type axioms. If we were to continue this development, then the map \( \mathcal{C} \) appears to be the most appropriate for such an investigation. However, \( \mathcal{C} \) is, in general, an
external object. It does not satisfy the internal defining method for 'n-ary relations that requires the variables to vary over '\mathcal{N}'. Observe that '*' is defined for all of the internal subsets (within $^0\mathcal{M}$) of '*A. With respect to the first model, '*' is restricted to its internal subsets of '*A as well. Now '*' is an internal map in $^0\mathcal{M}$ that is defined on "internal entities" that are 'subsets of A and it yields 'subsets of B which, when viewed from the structure $^0\mathcal{M}$, could contain many non- 'elements. Notice that we cannot obtain any information about these other objects by simply transferring, by the *-transfer method, information from the standard model. These objects could be investigated by a more careful analysis of the exact construction of $^0\mathcal{M}$.

Finally, this alternate approach also requires a more specific definition for the "standard restriction process." The definition of the "standard restriction" for $^0\mathcal{M}$ would depend upon which type of extension '*' or '*' is used. It is clear that all of this as well as the appropriate definitions for human, subtle and purely subtle entities can be successfully accomplished.

[NOTE: In Chapters 7 - 11, in most cases, the symbol $A_1 = i[\mathcal{W}]$. Also in Chapter 9, the structure $\mathcal{M}_1$ being used is incorrectly denoted. The superstructure $\mathcal{N}$ is to have either $\mathcal{W} \cup \mathbb{R}$ or $\mathcal{W} \cup \mathbb{Q}$ as a ground set. It is useful to extend the language to $\mathcal{W}'$ that includes symbols for $\mathbb{R}$ or $\mathbb{Q}$. Further in Section 9.1, $r \in ^*A_1 \simeq ^*\mathbb{R}$ should read $r \in ^*A_1$. Notice that if one chooses to use $\mathcal{W}'$, then $r$ corresponds to an $r' \in ^*\mathcal{W}'$.}
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