Chains of Quasi-Classical Informations for Bipartite Correlations and the Role of Twin Observables

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 Having the quantum correlations in a general mixed or pure bipartite state in mind, the part of information accessible by simultaneous measurement on both subsystems is shown never to exceed the part accessible by measurement on one subsystem, which, in turn is proved not to exceed the von Neumann mutual information. A particular pair of (opposite-subsystem) observables are shown to be responsible both for the amount of quasi-classical correlations and for that of the purely quantum entanglement in the pure-state case: the former via simultaneous subsystem measurements, and the latter through the entropy of coherence or of incompatibility, which is defined for the general case. The observables at issue are so-called twin observables. A general definition of the latter is given in terms of their detailed properties.

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As it is well known, quantum information theory is closely connected with the correlations inherent in an arbitrary bipartite state (mathematically: statistical operator) \( \rho_{12} \) of a composite system \( 1 + 2 \). The correlations have surprisingly many facets, and the relations among them are the subject of intense current investigations. This article is intended to make a contribution to the issue.

Let us define some quantitative elements of correlations. The subsystem states (reduced statistical operators) are \( \rho_s \equiv \text{Tr}_r \rho_{12}, \ s, s' = 1, 2; \ s \neq s' \) (“Tr” is a partial trace), and we have the three von Neumann entropies: \( S(n) \equiv -\text{Tr}_n (\rho_n \log \rho_n), \ n = 1, 2, 12 \).

One of the basic correlation or entanglement entities is the von Neumann mutual information:

\[
I(1 : 2) \equiv S(1) + S(2) - S(12). \tag{1}
\]

It is conjectured that it is the amount of total correlations \( \equiv I \).

For the purpose of notation, let us write down an arbitrary first-subsystem and an arbitrary second subsystem complete observable (Hermitian operator) with purely discrete spectra: \( A_1 = \sum_i a_i \ket{1_i}, \ B_2 = \sum_j b_j \ket{2_j} \). The measurement of \( A_1 \otimes 1 \) gives rise to the distant (as opposed to “direct”) state decomposition \( \rho_2 = \sum_i p_i \rho_2^i \), where \( p_i \equiv \text{Tr}[\rho_{12}(\ket{1_i} \bra{1_i} \otimes 1)] \) is the probability of the result \( a_i \), and \( \rho_2^i \equiv p_i ^{-1} \text{Tr}(\rho_{12}(\ket{1_i} \bra{1_i} \otimes 1)) \) is the opposite-subsystem state corresponding to this result if \( p_i > 0 \).

Entropy is concave \( \equiv (\text{section II.B there}) \), i. e., \( \sum_i p_i S(\rho_2^i) \leq S(2), \) and

\[
I(m1 \rightarrow 2)_A \equiv S(2) - \sum_i p_i S(\rho_2^i). \tag{2a}
\]

is the information gain about subsystem 2 on account of the direct measurement of the observable \( A_1 \) on subsystem 1. Symmetrically, one defines the symmetric quantity \( I(1 \leftarrow m2)_B \).

One, further, defines \( \equiv (\text{section II.B there}) \), the supremum taken over all complete \( A_1 \), as the largest amount of information (contained in the correlations) accessible by measurement of an observable on the first subsystem. Symmetrically, one defines the symmetric quantity \( I(1 \leftarrow m2) \equiv \sup\{I(1 \leftarrow m2)_B\} \) over all second-subsystem complete measurements.

If one performs simultaneous measurement of \( A_1 \otimes 1 \) and of \( (1 \otimes B_2) \) on \( \rho_{12} \) (denoted by \( (A_1 \wedge B_2) \)), then one deals with a classical discrete joint probability distribution \( p_{ij} \equiv \text{Tr}[\rho_{12}(\ket{i_1} \bra{i_1} \otimes \ket{j_2} \bra{j_2})] \).

It implies, in its turn, the mutual information \( I(m1 : m2) \equiv \sup\{I(m1 \leftarrow m2)_A \wedge B \wedge B \wedge \} \) via the Gibbs- Boltzmann-Shannon entropies \( H(A, B) \equiv -\sum_i p_i \log p_i, \ H(A) \equiv -\sum_i p_i \log p_i, \ H(B) \equiv -\sum_j p_j \log p_j, \) where \( p_i \equiv \sum_j p_{ij} \) and \( p_j \equiv \sum_i p_{ij} \) are the marginal probability distributions. Then

\[
I(m1 : m2)_{A\wedge B} \equiv H(A) + H(B) - H(A, B). \tag{3a}
\]

Finally,

\[
I(m1 : m2) \equiv \sup\{I(m1 : m2)_{A\wedge B}\} \tag{3b}
\]

over all choices of complete observables \( A_1 \) and \( B_2 \). This is the largest amount of information on a subsystem observable (contained in the quantum correlations) accessible by measurement of an observable on the opposite subsystem.

The claimed chains of information inequalities, valid for every bipartite state \( \rho_{12} \), go as follows:

\[
0 \leq I(m1 : m2) \leq I(m1 \rightarrow 2) \leq \min\{I(1 : 2), S(2)\}. \tag{4a}
\]
Both in (4a) and in (4b) one has equality in the first inequality if and only if the state $\rho$ is uncorrelated, i.e., $\rho_{12} = \rho_1 \otimes \rho_2$.

The role of $S(2)$ in the last inequality in (4a) is obvious from (2a), and symmetrically for (4b).

In the classical discrete case both chains (4a) and (4b) contain only equalities, and one has $I(1 : 2) \leq S(1), S(2)$. As to the corresponding inequality in the quantum case, one cannot do better than $I(1 : 2) \leq 2S(1), 2S(2)$.\[\text{[3]}\]

The inequality $I(m1 : m2) \leq I(1 : 2)$ implied by (4a), together with the stated necessary and sufficient condition for equality in the first inequality in (4a), was proved in 1973 by Lindblad\[\text{[5]}\] (Theorem 2, there). The third inequality in (4b) with $I(1 : 2)$ was claimed and a proof was presented in \[\text{[6]}\]. (It is perhaps useful to have an independent verification like the one in this article.)

The second inequality in (4a) is being proved for the first time in this article I believe. For the sake of completeness, let me prove the entire chain.

The inequalities in (4a) are, essentially, a consequence of a result of Lindblad of classical value \[\text{[7]}\] (see Corollary there), and (4b) follows symmetrically. To explain this claim, let me introduce the so-called relative entropy of a quantum state (statistical operator) $\sigma$ with relation to another state (statistical operator) $\rho$:

$$S(\sigma|\rho) \equiv \text{Tr} \log \sigma - \text{Tr} \log \rho.$$ 

One has $0 \leq S(\sigma|\rho)$ with equality if and only if $\sigma = \rho$.

Lindblad's result involves the ideal measurement of an arbitrary complete or incomplete observable (Hermitian operator) $A$ with a purely discrete spectrum. Let its unique spectral form, i.e., the one without repetition in the characteristic values, be $A = \sum_i \alpha_i P_i$. Denoting by $T_A\sigma$ the state into which $\sigma$ changes due to the nonselective ideal measurement of $A$ in it, one has

$$T_A\sigma = \sum_i P_i^\dagger \sigma P_i \quad \text{(5)}$$

and Lindblad's result states that

$$S(T_A\sigma|T_A\rho) \leq S(\sigma|\rho). \quad \text{(6)}$$

One should note that also the RHS of (5) is a statistical operator. Hence, for any other observable $B = \sum_j \beta_j Q_j$, (6) implies, what may be called, the \textit{Lindblad chain}

$$S(T_B T_A\sigma|T_B T_A\rho) \leq S(T_A\sigma|T_A\rho) \leq S(\sigma|\rho).$$

One may even extend the measurements to operations \[\text{[8]}\].

The \textit{von Neumann mutual information} in any bipartite state $\rho_{12}$ can be expressed in terms of relative entropy:

$$I(1 : 2) = S(\rho_{12}|\rho_1 \otimes \rho_2). \quad \text{(7)}$$

(This known claim is easily checked utilizing $\log(\rho_1 \otimes \rho_2) = (\log \rho_1) \otimes 1 + 1 \otimes (\log \rho_2)$, which, in turn, is easily seen in spectral forms.)

I am going to demonstrate that the claimed chain of inequalities (4a) is a consequence of the Lindblad chain:

$$0 \leq S(T_A T_B \rho_{12}|T_A T_B (\rho_1 \otimes \rho_2)) \leq \quad \text{(4b)}$$

$$\leq S(T_A \rho_{12}|T_A (\rho_1 \otimes \rho_2)) \leq S(\rho_{12}|\rho_1 \otimes \rho_2) \quad \text{(8)}$$

with subsystem observables $A_1$ and $B_2$ that are complete in some subspaces $S_1$ and $S_2$ containing the ranges of the operators $\rho_1$ and $\rho_2$ respectively.

In order to recognize the meaning of the first relative entropy in (8), we make use of the auxiliary claim that for complete or incomplete subsystem observables $A_1 = \sum_i a_i P_i$ and $B_2 = \sum_j b_j Q_j$ (unique spectral forms), and for any bipartite state $\rho_{12}$, one has

$$T_A \rho_1 = \text{Tr}_2(T_A T_B \rho_{12}), \quad T_B \rho_2 = \text{Tr}_1(T_A T_B \rho_{12}), \quad \text{(9)}$$

where $\rho_s, s = 1, 2$, are the subsystem states of $\rho_{12}$. Relations (9) are proved in Appendix 1.

Further, one can argue with Lindblad\[\text{[9]}\] (Theorem 2, there) as follows. Making use of (1), (7) and (9), one obtains

$$S(T_A T_B (\rho_{12}|\rho_1 \otimes \rho_2)) = S(T_A \rho_1) + S(T_B \rho_2) - S(T_A T_B \rho_{12})$$

Taking $T_A$ and $T_B$ in explicit form (cf (5) \textit{mutatis mutandis}), we see that we have a mixture of \textit{orthogonal} pure states. The so-called \textit{mixing property} of entropy allows us to write it as the sum of the so-called \textit{mixing entropy} (that of the statistical weights) and the average entropy \[\text{[10]}\] (see Section II.F. and II.B. there). Since pure states have zero entropy, one obtains:

$$\text{LHS} = H(A) + H(B) - H(A, B) = I(m1 : m2)_{A \land B}.$$ 

Next, we turn to the \textit{second relative entropy} in (8). Utilizing again relations (9) (this time with $B \equiv 1$), (7) and (1), one obtains

$$S(T_A \rho_{12}|T_A (\rho_1 \otimes \rho_2)) = S(T_A \rho_1) + S(\rho_2) - S(T_A \rho_{12}). \quad \text{(10)}$$

Since $A_1 = \sum_i a_i |i\rangle \langle i|$, for $p_i \equiv \text{Tr}(|i\rangle \langle i| \rho_{12}) > 0$, one has

$$|i\rangle \langle i| \rho_{12} |i\rangle \langle i| \rho_{12} |i\rangle \langle i| = p_i |i\rangle \langle i| \rho_1 \otimes \rho_2^i, \quad \text{(11a)}$$

and

$$\rho_1^i \equiv p_i^{-1} \text{Tr}_1 (|i\rangle \langle i| \rho_{12}) |i\rangle \langle i|. \quad \text{(11b)}$$

(The tensor factor ”$\otimes 1$” is repeatedly omitted because no confusion can arise.) The validity of (11a) is straightforward to check in any pair of orthonormal and complete subsystem bases.

On account of (11a) and the fact that both $T_A \rho_1$ and $T_A \rho_{12}$ are orthogonal mixtures of states (cf (5)) with the same statistical weights, we can apply the mixing property of entropy both to $S(T_A \rho_1)$ and to $S(T_A \rho_{12})$. Then, the LHS of (10) becomes equal to

$$H(A) + S(\rho_2) - (H(A) + \sum_i p_i S(\rho_2^i)) = I(m1 \rightarrow 2)_A$$
The chain (8) can now be rewritten as

\[ 0 \leq I(m1 : m2)_{A \wedge B} \leq \leq I(m1 \rightarrow 2)_A \leq I(1 : 2). \]  

(12)

(The symmetric chain is derived symmetrically.)

The inequality

\[ I(m1 : m2)_{A \wedge B} \leq I(m1 \rightarrow 2)_A \]

has the obvious physical interpretation that, in general, only part of the quantum information gain about subsystem 2 due to the measurement of \( A_1 \) can be realized as information about a concrete complete observable \( B_2 \).

The same inequality implies that the quantum information gain \( I(m1 \rightarrow 2)_A \) is an upper bound to any concrete information \( I(m1 : m2)_{A \wedge B} \) about some \( B_2 \).

Taking the suprema in (12), and having (2a) in mind, one obtains (4a).

In [1] \( I(m1 \rightarrow 2) \) is interpreted as the quasi-classical part of the amount of quantum correlations in any bipartite state \( \rho_{12} \). The authors define the so-called relative entropy of entanglement \( E_{RE}(\rho_{12}) \equiv \inf \{ S(\rho_{12} | \sigma_{12}) \} \), where the infimum is taken over all separable states \( \sigma_{12} \), as a measure of (purely quantum) entanglement (cf also [3]). Since \( I(1 : 2) = S(\rho_{12} | \rho_1 \otimes \rho_2) \), obviously, \( E_{RE}(\rho_{12}) \leq I(1 : 2) \).

Essentially the same view of \( I(m2 \rightarrow 1) \) as in [1], is, independently, taken in [3]. The latter authors call the difference

\[ \delta(m2 \rightarrow 1) \equiv I(1 : 2) - I(m2 \rightarrow 1) \]

"quantum discord", and they interpret it as the truly quantum part of the total amount of correlations \( I(1 : 2) \). (It is inaccessible to subsystem measurement.)

Next we apply the derived chain of quasi-classical informations to pure states. They represent a simple enough case to gain detailed insight.

**Quasi-classical informations in bipartite pure states.**

We turn now to a general pure state \( \rho_{12} \equiv | \Phi \rangle_{12} \langle \Phi |_{12} \). Let us write \( | \Phi \rangle_{12} \) as a Schmidt decomposition [10, 11] into biorthogonal state vectors:

\[ | \Phi \rangle_{12} = \sum_i \sqrt{\frac{1}{\delta_i}} | i \rangle_1 | i \rangle_2. \]  

(14)

Taking

\[ A_1 = \sum_i a_i | i \rangle_1 \langle i |_1, \quad 0 \neq a_i \neq a_{i'} \neq 0 \text{ for } i \neq i', \quad \text{(15a)} \]

\[ B_2 = \sum_i b_i | i \rangle_2 \langle i |_2, \quad 0 \neq b_i \neq b_{i'} \neq 0 \text{ for } i \neq i', \quad \text{(15b)} \]

one obtains for the induced classical discrete probability distribution (cf (3a)): \( p_{ij} = \delta_{ij} r_i \). Then

\[ I(m1 : m2)_{A \wedge B} = H(A) = H(B) = H(A, B) = S(1) \]

\[ = S(2) = I(m1 \rightarrow 2) = I(1 \leftarrow m2) \]  

(16)

(cf (4a) and (4b) without \( I(1 : 2) \)). It is seen from (3b) that \( I(m1 : m2)_{A \wedge B} \) is a lower bound to all quantities in the chains (4a) and (4b), and it reaches its highest possible value \( S(1) = S(2) \) in \( | \Phi \rangle_{12} \) (cf (16)). Hence, it equals not only \( I(m1 : m2) \), but also \( I(m1 \rightarrow 2) \) and \( I(1 \leftarrow m2) \).

Besides, also

\[ \delta(m1 \rightarrow 2) = \delta(1 \leftarrow m2) = S(1) = S(2) \]  

(17)

(because \( I(1 : 2) = 2S(1) = 2S(2) \)). The same quantity, called entropy of entanglement and denoted by \( E(\rho_{12}) \) was obtained in [12].

Returning to the above quasi-classical informations in \( | \Phi \rangle_{12} \), one can say that the pair \( (A_1, B_2) \) of opposite subsystem observables (15a) and (15b) actually realize, in simultaneous measurement, the entire part of the total correlations that is available for subsystem measurement. This pair of observables has noteworthy properties. Next, we resort to a sketchy presentation of them in the general case.

**Twin observables with respect to a general bipartite state.**

Let us now turn to a concise but sufficiently detailed definition of twin observables, which is wider than the one given in previous work [11, 12]. All necessary proofs are provided in Appendix 2.

Let \( \rho_{12} \) be an arbitrary given bipartite state, and let \( A_1 \) and \( B_2 \) be opposite-subsystem observables (Hermian operators) having the following three properties with respect to \( \rho_{12} \):

(i) The operators \( A_1 \) commute with the corresponding reduced statistical operators: \( [A_1, \rho_1] = 0, \quad [B_2, \rho_2] = 0 \).

(ii) On account of the commutations, the (topological closures \( \mathcal{R}(\rho_i) \) of the) ranges \( \mathcal{R}(\rho_i), i = 1, 2 \), are invariant subspaces for \( A_1 \) and \( B_2 \) respectively, and the operators have purely discrete spectra in them. These are precisely the detectable parts of the respective spectra of \( A_1 \) and \( B_2 \), i. e., they consist of those characteristic values that have positive probability in \( \rho_{12} \).

(iii) The detectable parts of the spectra of \( A_1 \) and \( B_2 \) consist of an equal number of characteristic values, i. e., they are of the same power.

One can establish a one-to-one map between the two detectable parts of the spectra such that the corresponding characteristic values, denoted by the same index \( i \), satisfy for all value of \( i \) one of the following four conditions:

(a) The information-theoretic condition:

\[ p_{ii'} \equiv \text{Tr} \rho_{12} P_i P_{i'} = \delta_{i,i'} p_i, \]
where $P_i^j$ is the characteristic projector of $A_1$ corresponding to the detectable characteristic value $a_i$ and symmetrically for $P_i^j$ and $b_j$ of $B_2$; and $p_i \equiv \text{Tr}\rho_1 P_i^1$ is the probability of $P_i^1$ in $\rho_{12}$.

(b) The measurement-theoretic condition:

$$P_i^1 \rho_{12} P_i^1 = P_i^2 \rho_{12} P_i^2.$$  

(c) The condition in terms of quantum logic:

$$\text{Tr}[\rho_2(P_i^1 P_i^2)] = 1,$$

where $\rho_2(P_i^1) \equiv p_i^{-1} \text{Tr}_1 \rho_{12} P_i^1$ is the conditional state of subsystem 2 when the event $P_i^1$ occurs.

(d) The algebraic condition:

$$P_i^1 \rho_{12} = P_i^2 \rho_{12}.$$  

The four conditions in property (iii) are equivalent.

If $A_1$ and $B_2$ do have the mentioned three properties, then we call them twin observables for $\rho_{12}$. If all characteristic values of $A_1$ and $B_2$ in $\mathcal{R}(\rho_1)$ and $\mathcal{R}(\rho_2)$ respectively are nondegenerate, i. e., if $\forall i_1, i_2 \in \mathcal{R}(\rho_1)$, we say that $A_1$ and $B_2$ are complete twin observables with respect to $\rho_{12}$.

Comments on the four conditions in property (iii).

(a) The probability distribution $p_i = p_i \rho_{12}$ is the best possible classical information channel: a so-called lossless and noiseless one. It is obvious that the correspondence between the detectable parts of the spectra is unique.

(b) The detectable characteristic values $a_i$ of $A_1$ and $b_i$ of $B_2$ are equally probable in $\rho_{12}$. Besides, the ideal measurement of $A_1$ and that of $B_2$ (actually of $(A_1 \otimes B_2)$) convert $\rho_{12}$ into the same state (cf the general formula of L"uders for ideal measurement [13]). This makes possible so-called distant measurement [13]: One can measure $B_2$ in $\rho_{12}$ without any dynamical influence on the second subsystem by just measuring $A_1$ on the first subsystem (or vice versa) in the state $\rho_{12}$ of the bipartite system.

(c) For an arbitrary event (projector) $E_2$ for subsystem 2 one can write

$$\text{Tr}[\rho_{12} P_i^1 E_2] = p_i \text{Tr}[\rho_2(P_i^1 E_2)],$$

i. e., one can factorize coincidence probability into probability of the condition $P_i^1$ and conditional probability of the event $E_2$ (in analogy with classical physics). The conditional state $\rho_2(P_i^1)$, when giving probability one, extends the absolute implication in quantum logic (which is $E \leq F \iff EF = E$, $E$ and $F$ projectors) by state-dependent implication [14]. This makes $P_i^1$ and $P_i^2$ to imply each other $\rho_{12}$-dependently.

(d) Since the detectable characteristic values of twin observables $A_1$ and $B_2$ are arbitrary, one can choose them equal: $\forall i : a_i = b_i$. Then the algebraic condition strengthens into

$$A_1 \rho_{12} = B_2 \rho_{12}.$$  

This case was studied in detail in previous work [11], [13]. It was shown that the stronger algebraic condition implies all three above properties, i. e., that it by itself makes $A_1$ and $B_2$ twin observables (as defined in this article) with the additional property (iv): $\forall i : a_i = b_i$. It was also shown that in the pure state case the multiplicities of $a_i$ and $b_i$ necessarily coincide, but they need not be equal in the mixed-state case.

Without property (iv) twin observables have a wider scope of potential application.

Let us return to the above discussion of quasi-classical informations inherent in a given pure state vector $|\Phi\rangle_{12}$. In view of the information-theoretic condition in property (iii) of twin observables, it clearly follows from the above discussion of (15a) and (15b) that one is dealing with twin observables.

One can say that it is the pair $(A_1, B_2)$ of twin observables given by (15a) and (15b) that realizes, in simultaneous measurement, the entire quasi-classical information.

The ideal nonselective measurements of $A_1$, that of $B_2$, and that of $A_1 \cup B_2$ each convert $|\Phi\rangle_{12}$ into one and the same mixed state

$$\rho_{12} = \sum_i r_i |i\rangle_1 \langle i| \otimes |i\rangle_2 \langle i|$$  

(cf (14)).

As it is easily seen, the same pair of observables (15a) and (15b) are complete twin observables not only with respect to $|\Phi\rangle_{12}$, but also regarding $\rho_{12}$. Also (16) holds true for the latter. Again, the same pair of twin observables "carry" the entire subsystem-measurement-accessible part of information. But instead of (17), we have zero quantum discord. There is no subsystem-measurement-inaccessible part of information. (No wonder, we are dealing with a biorthogonal separable mixed state in (18).)

In view of the fact that twin observables have a variety of particular properties, one may wonder if the pair given by (15a) and (15b) is, perhaps, of some relevance also for the quantum discord in $|\Phi\rangle_{12}$ (cf (14)). To reach an answer in the affirmative, we must first introduce entropy of coherence.

Entropy of coherence or of incompatibility. To begin with, we should notice that the difference between (14) and (18) lies in coherence, which is present in the former and absent in the latter. One may wonder if coherence can be given a precise and general definition.

I suggest to consider the following quantity as the amount of coherence or of incompatibility between a given observable $A = \sum_i a_i P_i$ (in the unique spectral form) and a given quantum state $\rho$, and call it the entropy of coherence or of incompatibility:

$$E_C(A, \rho) \equiv S(T_A \rho) - S(\rho)$$  

(cf (5)), i. e., the increase of entropy in ideal nonselective measurement of $A$ in $\rho$. 


That the RHS of (19) is always nonnegative and zero if and only if $A$ and $\rho$ commute (compatibility) was proved in [13] (pp. 380-387) for complete $A$. That for any state $\rho$ and for any incomplete observable $A$ there always exists a complete one $B$ such that the former is a function of the latter and such that $T_A \rho = T_B \rho$ was proved in [16] (Theorem 2. there). Hence, the RHS of (19) is always nonnegative also for incomplete observables, and it is zero if and only if $[A, \rho] = 0$. (Namely, the commutation is sufficient for $T_A \rho = \rho$, and hence for zero LHS of (19). On the other hand, the mentioned zero implies, as stated, commutation with $B$, and hence also with $A$.)

Utilizing the mixing property of entropy, we can rewrite (19) as

$$E_C(A, \rho) = H(A) - \left( S(\rho) - \sum_i w_i S(\rho_i) \right),$$

(20)

where $\forall i : w_i = \text{Tr} P_i^* \rho, P_i = \rho P_i / w_i$ (for $w_i > 0$) and $H(A) = H(w_i)$ is the mixing entropy, which is simultaneously, also the entropy of the observable $A$ in $\rho$.

It was proved in [14] (Theorem 2, there) that, whenever $S(\rho) < \infty$, the second term on the RHS of (20) is, in its turn, always nonnegative, and zero if and only if $\forall i : S(\rho_i) = S(\rho)$. (This condition is satisfied, e. g., when $\rho$ and all $\rho_i$ are pure states, like in the case of measurement in a pure state.) On the other hand, the above discussion shows that the mentioned second term never exceeds the first; and they are equal if and only if $[A, \rho] = 0$.

If $A$ is complete and $\rho$ mixed or pure, then the states $\rho_i$ are pure and

$$E_C(A, \rho) = H(A) - S(\rho).$$

(21)

If $\rho$ is pure and $A$ is incomplete or complete, the states $\rho_i$ are again pure, and

$$E_C(A, \rho) = H(A).$$

(22)

If both $A$ is complete, i. e., $\forall i : P_i = |i\rangle\langle i|$, and $\rho$ is pure, i. e., $\rho = |\phi\rangle\langle \phi|$, then

$$E_C(A, \rho) = H(|f_i|^2),$$

(23a)

where

$$|\phi\rangle = \sum_i f_i |i\rangle$$

(23b)

is the relevant expansion.

Now we may face the question if the twin observables given by (15a) and (15b) have anything to do with quantum discord in $|\Phi\rangle_{12}$.

**Purely quantum information and coherence in bipartite pure states.** The entropy of coherence of $(A_1 \otimes 1)$ given by (15a) or of its twin observable $(1 \otimes B_2)$ (cf (15b)) in $|\Phi\rangle_{12}$ (cf (14)) is $H(A) = H(B) = S(1) = S(2)$, which equals the relative entropy of entanglement $E_{RE}(|\Phi\rangle_{12})$ or the quantum discord $\delta(m2 \rightarrow 1)$ in this state. In $\rho_{12}'$ given by (18) the analogous coherence entropies are zero (because $[(A_1 \otimes 1), \rho_{12}'] = [(1 \otimes B_2), \rho_{12}'] = 0$).

Thus, in every pure bipartite state $|\Phi\rangle_{12}$ it is not only true that a pair of twin observables $A_1$ and $B_2$ "carries" the quasi-classical part of correlations, i. e., the one accessible by subsystem measurement, but it is also true that the same twin observables "carry" also the subsystem-measurement-inaccessible part of correlations, i. e., the quantum entanglement, via the amount of coherence of any of the twin observables in the bipartite state.

**APPENDIX: 1**

Proof of relations (9) is based on $\sum_j (Q_j^1)^2 = 1$, and on $\text{Tr}_2[(\rho_{12} Q_j^2)Q_j^2] = \text{Tr}_2[Q_j^2(\rho_{12} Q_j^2)]$:

$$T_A \rho_1 \equiv \sum_i P_i^1 (\text{Tr}_2(\rho_{12})) P_i^1 = \sum_i P_i^1 [\text{Tr}_2(\sum_j Q_j^2 \rho_{12} Q_j^2)] P_i^1 = \text{Tr}_2 \sum_i P_i^1 \sum_j Q_j^1 \rho_{12} Q_j^2 P_i^1 \equiv \text{Tr}_2 T_A T_B \rho_{12}. $$

The second relation in (9) is proved symmetrically.

**APPENDIX: 2**

Proofs for the initial claims in the definition of twin observables.

As well known, statistical operators, in particular, the reduced ones, have purely discrete spectra and their spectral forms (with distinct characteristic values) read: $\rho_s = \sum_k r_k^s \phi_k$, $s = 1, 2$. As a consequence of the commutations in property (i), one has $\forall k : [A_1, Q_k^1] = 0$, $[B_2, Q_k^2] = 0$. Since the range projectors $Q_s$ of $\rho_s$ are $Q_s = \sum_k Q_k^s$, $s = 1, 2$ (all $r_k^s$ are positive), one has also $[A_1, Q_1] = 0$, $[B_2, Q_2] = 0$. Hence, the (topological closures of the) ranges $R(\rho_s) \left( \tilde{R}(\rho_s) = R(Q_s) \right), s = 1, 2$ are invariant subspaces for $A_1$ and $B_2$ respectively. Further, since also the characteristic subspaces $R(Q_k^1)$ of $\rho_1$ are invariant for $A_1$, and they are necessarily finite dimensional (because $\sum_k d_k^1 r_k^1 = \text{Tr} \rho_1 = 1$, where $d_k^1$ is the multiplicity of $r_k^1$), only discrete characteristic values of $A_1$ appear in $\tilde{R}(\rho_1)$, and symmetrically for $B_2$.

Let $\sum_l a_l P_l^1$ be the discrete part of the spectral form (with distinct characteristic values) of $A_1$. This operator and $A_1 Q_1$ act equally in $\tilde{R}(\rho_1)$. Further, as already proved, all spectral projectors of $A_1$ belonging to its (possible) continuous spectrum are subprojectors of the null-space projector $Q_1^1$. Hence, $A_1 Q_1 = \sum_l a_l (P_l^1 Q_1)$. Omitting all terms in which $P_l^1 Q_1 = 0$, and changing the index from $l$ to $i$ in the remaining sum, one obtains the
spectral form $A_1Q_1 = \sum a_i (P_i^1 Q_1)$. Obviously, $A_1$ has those and only those characteristic values $a_i$ in $\mathcal{R}(\rho_1)$ for which $P_i^1 Q_1 \neq 0$.

On the other hand, the detectable discrete characteristic values $a_n$ of $A_1$ in $\rho_{12}$ are those for which $0 < p_n = \text{Tr}(\rho_1 P_i^1)$. One can always write $\rho_1 = \rho_1 Q_1$. Therefore, $p_n = \text{Tr}[\rho_1 (P_i^1 Q_1)]$. If $P_i^1 Q_1 = 0$, then $p_n = 0$. If $P_i^1 Q_1 \neq 0$, and we substitute the spectral form $p_n = \sum_k r_k^1 Q_1^k$, then $p_n = \sum_k r_k^1 \text{Tr}(P_i^1 Q_1^k)$. (We omit $Q_1$ because $Q_1 Q_1^k = Q_1^k$.) Since $\sum_k P_i^1 Q_1^k = P_i^1 Q_1$, which is nonzero by assumption, not all $P_i^1 Q_1^k$ can be zero. The nonzero terms $r_k^1 \text{Tr}(P_i^1 Q_1^k)$ are obviously positive. Thus, $p_n > 0$, and $a_n$ is detectable. This bears out the claim that precisely the detectable values of $A_1$ in $\rho_{12}$ appear as its characteristic values in $\mathcal{R}(\rho_1)$. (Thus, we can write $i$ instead of $n$ like in the preceding passage.)

**Proof of equivalence of the four conditions** will be given via the following closed chain of implications: (a) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (a).

**LINK (a) $\Rightarrow$ (d).**

Let

$$\rho_{12} = \sum w_k |\Phi_k^{12}\rangle \langle \Phi_k^{12}|$$

be a (convex linear) decomposition of $\rho_{12}$ into ray projectors. (For instance, the $|\Phi_k^{12}\rangle$ can be the characteristic state vectors of $\rho_{12}$.) If a projector $E$ is probability-one in $\rho_{12}$, then so is it in each $|\Phi_k^{12}\rangle$ (as seen from $1 = \text{Tr}(\rho_{12} E) = \sum w_k \text{Tr}(|\Phi_k^{12}\rangle \langle \Phi_k^{12}| E)$ and $\sum w_k = 1$). Further,

$$1 = \langle \Phi_k^{12}| E |\Phi_k^{12}\rangle \Rightarrow 0 = \langle \Phi_k^{12}| E^{\perp} |\Phi_k^{12}\rangle \Rightarrow$$

$$||E^{\perp} |\Phi_k^{12}\rangle||^2 = 0 \Rightarrow E^{\perp} |\Phi_k^{12}\rangle = 0 \Rightarrow E |\Phi_k^{12}\rangle = |\Phi_k^{12}\rangle.$$

The sum $\sum_i P_i^1 (\sum_j P_j^2)$ of all detectable values of $A_1$ ($B_2$) is a probability-one projector in $\rho_{12}$. Therefore,

$$\forall k : |\Phi_k^{12}\rangle = \sum_i P_i^1 |\Phi_k^{12}\rangle = \sum_i P_j^2 |\Phi_k^{12}\rangle,$$

and

$$|\Phi_k^{12}\rangle = \sum_i P_i^1 |\Phi_k^{12}\rangle = \sum_i P_j^2 |\Phi_k^{12}\rangle.$$  \hfill (A.2)

Assuming the validity of condition (a), and utilizing (A.1), we have

$$i \neq i' \Rightarrow 0 = p_{i'i'} \equiv \text{Tr}\rho_{12} P_{i'i'}^1 P_{i'i'}^2$$

$$= \sum k w_k (|\Phi_k^{12}\rangle \langle \Phi_k^{12}|) P_{i'i'}^1 P_{i'i'}^2 |\Phi_k^{12}\rangle.$$  \hfill (A.3)

If one utilizes condition (b), this expression becomes $p_{i'i'}^{-1} p_i$, i.e., condition (c) follows.

**LINK (c) $\Rightarrow$ (a).**

Let us return to the argument given in the proof of the link \((a) \Rightarrow (d)\), and to (A.1). It was shown that a probability-one projector $E$ in $\rho_{12}$ is such an event also in each $|\Phi_k^{12}\rangle$, and $\forall k : E |\Phi_k^{12}\rangle = |\Phi_k^{12}\rangle$. Then, (A.1) implies

$$E \rho_{12} = \rho_{12}. \hfill (A.5)$$

Assuming the validity of (c), $P_i^1$ is a probability-one projector in $\rho_{2}(P_i^1)$, hence, on account of the adjoint of (A.5), one has

$$\rho_2 P_i^1 = \rho_2 P_i^1.$$  \hfill (A.6)

The LHS of condition (a), due to (A.6), implies

$$p_{i'i'} \equiv \text{Tr}(\rho_{12} P_{i'i'}^1 P_{i'i'}^2) = p_{i'i'} \text{Tr}(\rho_2 P_i^1)$$

$$= p_{i'i'} \left[ \rho_2 P_i^1 \right] = \delta_{i,i'} p_i.$$  \hfill \(\forall\)

Thus, (a) is derived.

**Proof of the stronger algebraic relation.**

Since $\rho_{12} = (\sum_i P_i^1) \rho_{12}$, one has $A_1 \rho_{12} = (\sum_i a_i P_i^1) \rho_{12}$. Assuming then property (iv), i.e., $\forall i : a_i = b_i$, and utilizing condition (d), one further obtains

$$A_1 \rho_{12} = (\sum_i b_i P_i^2) \rho_{12} = B_2 \rho_{12}.$$  \hfill \(\square$$

The last equality is due to the fact that for the second subsystem one has the symmetric argument. Thus, the stronger algebraic relation is derived.
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