Are high-dimensional entangled states robust to noise?

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High-dimensional entangled states are of significant interest in quantum science as they increase available bandwidth and can remain entangled in the presence of significant noise. We show that a system’s tolerance to noise can be significantly reduced by modest increases to the size of the Hilbert space, e.g., doubling the size of a \( d = 300 \) space leads to a reduction of the necessary detector efficiencies for entanglement certification of two orders of magnitude. We also demonstrate that knowledge of a single parameter, the signal-to-noise ratio, precisely links information theoretic measures of entanglement to a range of experimental parameters quantifying noise in a quantum communication system, enabling accurate predictions of its performance.

Introduction—Entanglement is considered one of the most important features of quantum information science and plays a central role in many quantum communication protocols \[1\]–[3]. The strong, non-classical correlations inherent to entanglement allow one to share information between two parties that is secure against the most sophisticated eavesdropping attacks, with information security even provided independent of the devices used [4]. Entanglement is widely used in many different branches of quantum information science, e.g., communication, simulation, computing, metrology, imaging, and quantum fundamentals [5]–[9]. However, entanglement is an extremely fragile resource, and sources of noise such as loss, background counts, and imperfect devices can result in its rapid deterioration. In recent years, high-dimensional entanglement has emerged as a way to increase the robustness of entanglement, and as a result, increase the resistance of entanglement-based quantum communication to noise [10]–[14], see Refs. [15]–[16] for an overview. In addition, encoding information in high-dimensional quantum systems offers an increased bandwidth over qubit encoding, allowing one to build efficient quantum networks and quantum cryptography systems that use the full information-carrying capacity of a photon [17]–[20].

Several recent works on high-dimensional entanglement have demonstrated a qualitative advantage over qubit entanglement in terms of information-capacity and noise-robustness [21]–[22]. While these experiments showcase two specific scenarios where high-dimensional entanglement provides an increased resistance to noise, a careful analysis outlining the precise noise conditions and Hilbert space dimension where such an advantage can or cannot be found is still lacking. Here, we formulate these conditions in terms of easily measurable experimental parameters, and establish the exact noise bounds and Hilbert space dimension where high-dimensional entanglement provides an advantage for entanglement certification in the presence of noise, and where it does not.

An often-used model for describing noise in an entangled state combines a target state \( |\phi\rangle \) with a maximally mixed state \( I \) denoting white noise: \( \hat{\rho} = p|\phi\rangle\langle\phi| + ((1 - p)/d^2)\hat{I} \). Here, \( p \) refers to the probability that the state remains intact [23]. Using this simple model, it is easy to see that the resistance to noise will increase linearly with system dimension \( d \)—the threshold \( p \) required to certify any entanglement is equal to \( 1/(d + 1) \) [10]. Thus, as the system dimension is increased, one can expect to overcome any amount of noise. However, in any realistic scenario, \( p \) is a dimension-dependent parameter that involves a complex interplay between noise attributed to the state, the channel, and the detection system.

We follow a two-step approach to developing an operational noise model for high-dimensional entanglement: First, we show how the seemingly complex relationship between different sources of noise in the state, channel, and detection system can be distilled into one operational quantity—the signal-to-noise ratio \( Q \), which we refer to as the quantum contrast. The quantum contrast is equal to the ratio between two-photon coincidence and accidental counts and is easily measured in experiment. An analysis of the functional form of \( Q \) that we present will allow an experimenter to optimise their source, channel, and detector specifications in order to achieve the best noise performance (highest \( Q \)) possible.

Second, any realistic implementation of entanglement necessitates practical methods for certifying it. Once we have incorporated noise into our model that is attributable to the state only, we construct measurements on the state that include noise attributed to the detection system. In this manner, we test a series of contemporary entanglement measures and analyse their performance in the presence of noise. In all cases, we find that the performance of the system, as quantified by the ability to certify (high-dimensional) entanglement, can be accurately predicted by knowledge of the system dimension \( d \) and only one experimentally measurable parameter: the quantum contrast \( Q \).

For high-dimensional entanglement, our results show that separating the dimension in which one wishes to observe the entanglement and the dimension of the Hilbert space allows one to tolerate more noise in an entanglement distribution system. However, depending on the entanglement certification method used, there is an op-
imum dimension where one finds the best possible noise performance. Interestingly, the largest increase in noise tolerance is obtained for only a modest increase in Hilbert space dimension. For non-locality based on the CGLMP inequality, and EPR entanglement based on conditional entropy, our results show that higher dimensions require higher signal-to-noise ratios. Additionally, our results can be used to immediately estimate the performance of an entanglement distribution system based on a single, easily measurable parameter $Q$, thus serving as a very powerful tool to help design and model future quantum information systems.

**Theoretical model**— We introduce a formalism that takes into account the common sources of noise in photonic systems: multi-photon pairs in the state [24, 28], imperfect measurements with dark counts, background noise, and non-university collection efficiency. The affect of multi-photon pairs can be modelled by taking into account $\mu$, the photon pair generation rate per mode per detection event. In the case that $\mu \ll 1$, the higher order multi-photon terms do not significantly contribute. Noise in a realistic detector can be attributed to dark counts inherent to its inner workings, as well as counts from background sources of light [29–31]. These two sources of detector noise can be combined into a single constant quantity $n$. An additional source of noise in the measurement stems from loss in the channel and the quantum efficiency of the detector, which can be combined into the single quantity $\eta$. In general, $n$ and $\eta$ can be dimension-dependent. However, for simplicity we consider them to be constants.

If we make the assumption that the efficiencies and noise levels in each channel are the same, then the number of two-photon coincidences and accidentals per detection window is given by

$$N_{\text{coi}}^{(j,k)} = \eta^2 \delta_{j,k} \mu (1 + \mu) + (n + \eta \mu)^2,$$

where $j$ and $k$ are indices for the different modes of the photons, and $1 \leq j,k \leq d$, with $d$ equal to the dimension of the Hilbert space. In order to quantify the quality of the photon pair source, we define the signal-to-noise ratio or the quantum contrast $Q$ as the ratio of coincidences $N_{\text{coi}}^{(j)}$ to accidentals $N_{\text{coi}}^{(j,k \neq j)}$

$$Q = 1 + \frac{\mu (1 + \mu)}{n \eta + \mu}.$$

Normally $n/\eta$ is very much less than unity, and there is maximum value given by $Q_{\text{max}} = 1 + \eta^2/[4n(\eta - n)]$. This value is achieved when $\mu = n/(\eta - 2n)$. We see here that the upper limit to $Q$ is determined only by detector noise and the detector/channel efficiency, which when fixed, tell us the optimal pair generation rate to operate at (see figure 1).

**Target state**— A standard approach to verification of an entangled state is the use of an entanglement witness $\hat{W}$, where the witness can be constructed from local measurements. If the expectation value of $\text{Tr}[\hat{W} \rho] < 0$, then the state $\rho$ is confirmed to be entangled. In high dimensions, $k$-dimensional entanglement in a $d$-dimensional space can be tested for by using the following witness [32]:

$$\hat{W}_k = ((k - 1)/d) \hat{I} - |\Phi\rangle \langle \Phi|,$$

where $|\Phi\rangle$ is the target state. To construct this witness, we use the Gell-Mann matrices appropriate for all possible two-dimensional subspaces in the $d$-dimensional space. In a $d$-dimensional space, there are $(d$ choose $2)$ subspaces to consider, so this method involves a large number of measurements.

We calculate that the expectation value of $\hat{W}_k$ depends only on $k,d$, and the quantum contrast $Q$:

$$\text{Tr}[\hat{W}_k \rho] = \frac{k - 2}{d} + \left(\frac{d - 1}{d}\right) \left(\frac{2 - Q}{d + 1 + Q}\right).$$

For entanglement to be verified, the lower limit to the quantum contrast for $k$-dimensional entanglement in a $d$-dimensional space is given by

$$Q > \frac{(d - 1)k}{d - k + 1}.$$

We see here that as $k$ increases, entanglement certification in $k = d$ dimensions necessitates an accompanying increase in the required quantum contrast, see Fig. 2. However, significant reductions in the required quantum contrast are achieved when we allow the dimension of the Hilbert space to increase with respect to the dimension of the entanglement, i.e. if $k < d$. In the limit that $d \to \infty$, we see that the required contrast $Q \to k$. This means that if we measure a system with a contrast of $Q$, it can have at most $|Q|$ dimensions of entanglement.

**Two mutually unbiased bases**— Recently, it was shown that measurements in two mutually unbiased bases
(MUBs) are sufficient to provide a lower bound to the fidelity with respect to a target state $F$ and certify high-dimensional entanglement [22]. This method is desirable in comparison to the one above as significantly fewer measurements are required.

It is known that a state $\rho$ existing in a $d$-dimensional Hilbert space is entangled in at least $k$ dimensions if the fidelity $F(\rho, \Phi)$ of the state $\rho$ with respect to a target state $\Phi$ is greater than $(k-1)/d$. This condition is equivalent to the condition that $\text{Tr}[W_{k}\rho] < 0$. For the state that we consider in this work, the two bases that one considers are mutually unbiased with respect to each other, and as we consider a state with a uniform modal distribution, the quantum contrasts calculated in each basis are the same. We calculate that the lower bound to the fidelity $F$ is given by

$$F(\rho, \Phi) \geq \frac{Q - d + 1}{Q + d - 1}. \quad (5)$$

If this result is combined with the requirement for the fidelity, we find the relationship between the quantum contrast and $k$-dimensional entanglement in a $d$-dimensional space, i.e. the signal-to-noise must satisfy

$$Q > \frac{(d - 1)(d + k - 1)}{d - k + 1}. \quad (6)$$

Again, we see that as $k$ is increased, the minimum contrast that is required increases. Importantly, as shown in Fig. 3, we see that in order to certify $k$-dimensional entanglement, a minimum quantum contrast of $Q_{\text{opt}} = 3k + 2\sqrt{2}(k - 2)(k - 1) - 4$ is necessary, which is obtained when the dimension of the Hilbert space we are working in is set to $d_{\text{opt}} = \left[\sqrt{2k^2 - 3k + 2} + k - 1\right]$.

We see that $d_{\text{opt}}$ can be found by increasing $k$ by around $(1 + \sqrt{2}) \approx 2.41$ times, at which we have significantly increased our system’s ability to tolerate noise. The gain can be quantified as the ratio of $Q$ when $d = k$ and when $d = d_{\text{opt}}$. This ratio is approximately $(2/(3 + 2\sqrt{2}))k \approx 0.343k$, so if $k = 1000$, increasing $d$ by 2.41 times reduces $Q$ by 343, which is greater than 2.5 orders of magnitude. From an experimental perspective, this decrease in $Q$ permits the use of a detector with a significantly lower detection efficiency/channel loss, or a higher dark/background count rate.

As an example of the gains this provides, consider a multi-outcome detector, such as a single-photon detector array or EMCCD camera [33] [34]. These detectors are commonly used for measurements on spatial entanglement as they allow multi-outcome measurements in two MUBs, i.e. the position and momentum bases. Certifying $k = 1000$ entanglement in a $d = 1000$ space using two MUBs requires a signal-to-noise of approximately $2 \times 10^6$. Achieving this signal-to-noise requires, for example, a noise per detection event of $n = 1 \times 10^{-7}$ and an efficiency of $\eta = 80\%$. Increasing the dimension of the space by a factor 2.41, (e.g. moving from a 32 by 32 array of pixels to a 49 by 49 grid), reduces the necessary $Q$ by $\approx$343 times, and, for example, the allowable detector noise can increase two orders of magnitude to $n = 1 \times 10^{-5}$ while the efficiency also drops to 23%.

In order to verify that knowledge of the signal-to-noise ratio provides an accurate estimate of system performance, we reanalyzed the data presented in Ref. [22] to calculate $Q_{\text{exp}}$ for the measurements in each of the MUBs and used this to predict the system performance. In each dimension $(d = 3, 5, 7, \text{ and } 11)$, we see that knowledge of $Q_{\text{exp}}$ gives an estimate of $(k_{\text{pred}} = 3, 5, 6, \text{ and } 9)$ respectively, which are the exact values of entanglement dimensionality obtained. The data for $Q_{\text{exp}}$ and $k_{\text{pred}}$ are shown in the supplementary information, including the values of $Q_{\text{opt}}$ and $d_{\text{opt}}$. This confirms that knowledge of a single, easily obtainable, experimentally measured parameter ($Q_{\text{exp}}$) provides a very accurate prediction of system performance.

We can now compare the required signal-to-noise ratio for the certification of $k$-dimensional entanglement using the target witness and the two-MUB fidelity witness. This is given by the ratio of Eq. 4 and Eq. 6 and is equal to $1 + (d - 1)/k$, thus the target witness always requires a lower $Q$. However, the two-MUB fidelity witness has the advantage that it requires significantly fewer measurements and access to only two MUBS. We see there is
a trade-off between the required signal-to-noise and the necessary number of measurements – it is not possible to have both a low required $Q$ with only a few measurements.

Mutually unbiased bases – The two-MUB witness can be extended to incorporate measurements in more mutually unbiased bases, and performing more measurements in more MUBs strengthens the resistance to noise. In $d$ dimensions, there exist at most $d + 1$ mutually unbiased bases [35]. If all of these are used, the fidelity is given by

$$F(\rho, \Phi) \geq \frac{Q + \frac{1}{2} - 1}{Q + d - 1} \tag{7}$$

and the corresponding condition for $k$ dimensions of entanglement in a $d$-dimensional space is then identical to that for the target state witness, see Eq. [4]. Consequently, whether the entanglement witness based on the target state or the witness based on $d + 1$ MUBs is used, the minimum contrast required for verification of $k$ dimensions of entanglement is equal to $k$.

Conditional entropy – We also analyse conditional entropies, commonly used in the confirmation of EPR entanglement and in steering inequalities, in the context of the signal-to-noise [3 36 37]. EPR entanglement can be confirmed if the measurements in two mutually unbiased bases violate the following inequality [35]:

$$H_1(X|Y) + H_2(X|Y) \geq \log_2 d,$$

where $H_1(X|Y)$ and $H_2(X|Y)$ are the conditional entropies in each of the bases. We use the computational basis and a mutually unbiased basis, and for the state that we consider, the conditional entropies in each basis are the same. We find that the sum of the conditional entropies is given by

$$H_1(X|Y) + H_2(X|Y) = 2\log_2 (Q + d - 1) - \frac{2Q}{Q + d - 1},$$

and therefore

$$\log_2 (Q + d - 1) - \frac{Q}{Q + d - 1} \log_2 Q - \frac{1}{2}\log_2 d < 0. \tag{9}$$

There is no analytical solution to this, but numerical solutions show that if EPR entanglement is to be confirmed as the dimension increases, so too does the required quantum contrast. The EPR criteria does not separate $k$-dimensional entanglement from the size of the Hilbert space $d$.

High-dimensional non-locality based on the CGLMP inequality – Finally, we consider the CGLMP inequality [5 10] that can be used for establishing non-locality, under the fair-sampling assumption. Here, local measurements are performed on each photon from an entangled pair to establish the high-dimensional Bell parameter $S_d$. Local hidden variable theories are consistent with $S_d \leq 2$, whereas quantum mechanics permits a violation of this inequality. In their work, CGLMP showed that maximally entangled quantum systems in high dimensions can achieve a theoretical value of $S_d(QM)$, which would lead to a violation of the inequality [10]. However, in any experimental verification, the achievable $S_d$ is modified by imperfections in the system. The maximum value of $S_d$ that is achievable in a system with a signal-to-noise ratio of $Q$ is

$$S_d = \frac{Q - 1}{Q - 1 + d} S_d(QM), \tag{10}$$

Therefore, in order to violate the local hidden variable inequality, the quantum contrast must satisfy

$$Q > 1 + d\frac{2}{S_d(QM) - 2}. \tag{11}$$

We see here that as $d$ increases, so too does the required contrast.

It is known that under certain assumptions, the CGLMP inequality can also be used as a dimension witness for entanglement [5], and therefore, as in the fidelity witness case, the separation of $k$ and $d$ may provide a decrease to the required signal-to-noise and an advantage for high-dimensional states for high-dimensional non-locality. It will also be interesting to consider the implications of this work for other Bell inequalities, such as the one developed for maximally entangled states [39].

Discussion and conclusion— This work serves to answer a simple question: “Are high-dimensional states robust
to noise? In reality, the answer to this question is more nuanced than a simple “yes” or “no”, and our results provide a clear demonstration of the advantages and disadvantages. When we consider the isotropic state, we find that the total noise that the state can tolerate decreases as the dimension of the state increases, and therefore, it is obvious to claim that such states are robust to noise. However, a counter argument can be put forward when we recognize that the necessary experimental signal-to-noise tends to increase as we increase the dimension of the state in which we wish to observe entanglement or non-locality.

Significant benefits for entanglement certification emerge when we see that the dimension of the state in which we wish to establish entanglement (k) and the dimension of the space of the measurements (d) are not required to be the same. The signal-to-noise requirements for k-dimensional entanglement decrease as both d and the total number of mutually unbiased bases used in the measurement increases. Both the target state witness and the d + 1 MUB witness show that the required signal-to-noise ratio for k-dimensional entanglement reduces from $k^2 - k$ (for d = k) down to k (for d = ∞). If only two MUBs are used, the witness still shows a reduction in the required Q, and an optimal dimension for noise resistance can be found.

By distilling the information about the performance a high-dimensional entanglement system into a single operational parameter, we provide a powerful tool that links theory and experiment, allowing us to accurately predict system performance and choose an optimum entanglement measurement strategy. The parameter Q encapsulates all the key experimental information, such as detector/channel efficiencies, background noise, and photon pair generation rate. We analytically show that a minimum quantum contrast must be achieved in order to verify non-locality under the fair-sampling loophole and certify entanglement in k dimensions. In general, the certification of entanglement in higher dimensions requires an increase to the required signal-to-noise. However, we show that a small increase in the operational Hilbert space dimension allows us to tolerate significantly larger increases in noise, allowing the use of inefficient detectors or operation in extremely noisy environments. This work demonstrates that high-dimensional entanglement has the potential to push the boundaries of entanglement-based quantum communication systems, bringing them from the confines of laboratory proof-of-principle demonstrations to the realm of practical, real-world implementations under extreme conditions of noise.

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