ORIENTABILITY AND REAL SEIBERG-WITTEN INVARIANTS

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Abstract. We investigate Seiberg-Witten theory in the presence of real structures. Certain conditions are obtained so that integer valued real Seiberg-Witten invariants can be defined. In general we study properties of the real Seiberg-Witten projection map from the point of view of Fredholm map degrees.

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1. Introduction

After the much success of Seiberg-Witten theory, it is a natural problem to study the real version and its potential application in real algebraic geometry. In outline, the real version starts with a Kähler 4-manifold with anti-holomorphic involution (a real structure). Then one would like to understand the lifted action on the Seiberg-Witten moduli space as well as the invariant extracted from the real moduli space. This real Seiberg-Witten invariant should link to the real Gromov-Witten invariant counting real holomorphic curves through a Taubes type correspondence as in [17]. An invariant counting nodal rational curves in real rational surfaces is found in [21]. More recently Solomon [15] has defined Gromov-Witten invariants counting arbitrary degree real holomorphic curves from fixed Riemann surfaces. A Kähler manifold with a real structure is what physicists refer to as an orientifold [14], and the real Gromov-Witten invariants have been one of their main interests for the last few years. Compare with the original Gromov-Witten theory of Ruan-Tian [10].

One should first point out that such a real theory is not to be treated as an equivariant theory; for one thing, the lifted real map does not preserve the spin\(^c\) bundle in the Seiberg-Witten theory. Nevertheless the lifted map is conjugate linear in a proper sense. Consequently, among the standard issues of transversality, compactness, orientability and reducible solutions, only orientability requires a substantially new strategy to tackle. As a matter of fact, the real Seiberg-Witten
moduli space is not necessarily orientable or naturally oriented even if orientable. This is rather typical in real algebraic geometry: the real part of a real structure is usually non-orientable or un-oriented. It is well-known that the usual Seiberg-Witten invariant can be viewed as the degree of the projection map $\pi : M \to \Omega^2_+$, where $M$ is the parameterized moduli space. This is the case if the moduli space is 0-dimensional. In the real Seiberg-Witten theory, we will encounter the real Seiberg-Witten projection map $\pi_R : M_R \to (\Omega^2_+)_R$ defined on the real parameterized moduli space. We will undertake two approaches: the first is to place real moduli spaces in the real configuration space $B_R$ and seek conditions in terms of $B_R$ that will guarantee the orientability of real moduli spaces. To this end we have the following results (see Theorems 4.3 and 4.5).

**Theorem 1.1.** Let $X$ be a Kähler surface with a real structure $\sigma$ and $S$ a spin$^c$ structure compatible with $\sigma$. Fix orientations on $H^1_R(X, \mathbb{R}), H^2_R(X, \mathbb{R})$. If $H^1(X, \mathbb{R})$ is trivial or if $c_1(L)$ is divisible by 4 for the determinant $L$ of $S$, then the real Seiberg-Witten invariant is well-defined and takes integer values.

As an application we prove a real version of the Thom conjecture (Corollary 4.4) for smoothly embedded surfaces in $\mathbb{CP}^2$ that are equivariant with respect to the real structures.

The second approach is less conventional, where we focus on the parameterized moduli space $M_R$ itself without ever involving $B_R$. Though as a trade-off, we need to work with all perturbations in $(\Omega^2_+)_R$. The main goal here is to understand the critical point set and the regular value set of the projection $\pi_R$. Since $\pi_R$ is proper, its regular values form an open and dense subset of $(\Omega^2_+)_R$. Thus the complement forms “walls”, cutting the regular value set into chambers. In the absence of the orientability and hence the integer Seiberg-Witten invariant, the pattern of chambers and distribution of the chamber-wide Seiberg-Witten invariants become new geometry to investigate for our real Seiberg-Witten theory. Among the main results here, we prove the following (Theorems 5.3 and 5.4)

**Theorem 1.2.** Let $C_R^l$ denote the critical point set of $\pi_R$ and $C_R^l(l) = \{ x \in C_R | \dim \text{coker} D\pi_R(x) = l \}$. For each integer $l \geq 0$, $C_R^l(l) \subset M_R$ is a smooth Banach submanifold of codimension $kl$, where $k = \text{ind} D\pi_R + l$.

Here is an outline of the paper. In Section 2, after reviewing the setup and notations of the standard Seiberg-Witten projection, we discuss thoroughly how to lift a real structure from an almost complex manifold.
to its associated spin\textsuperscript{c} bundle, and apply the lifted real structure to the Seiberg-Witten theory. In Section 3 we determine the orientation bundle of real moduli spaces and examine the natural extension to the real configuration space. This illustrates precisely the difference between the usual and real Seiberg-Witten theories. In Section 4 we find sufficient conditions so that the real moduli spaces are orientable and oriented, thus defining integer valued real invariants. In Section 5 we demonstrate that the critical point set of \( \pi_R \) stratifies into immersed submanifolds of \( M_R \), which are of the expected co-dimensions. Under the assumption that \( \pi_R \) is non-orientable, we introduce chamberwise invariants and their distribution.

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2. Lifted real structures and Seiberg-Witten equations

2.1. Parameterized Seiberg-Witten moduli spaces. We first recall briefly the standard Seiberg-Witten theory and set up notations to be used; compare for example [8, 12]. Special care is placed on the differential of the projection map into the perturbation space. The calculations are often left out in the literature, partly because of the similarity with the previous Donaldson theory.

Start with the perturbed Seiberg-Witten equations for a general spin\textsuperscript{c} structure \( S = S^+ \oplus S^- \) with determinant \( L \), defined on an arbitrary smooth 4-manifold \( X \). The equations are:

\[
\begin{align*}
\mathcal{D}_A \Phi &= 0 \\
F^+_A &= q(\Phi) - h
\end{align*}
\]

where \( A \) is a connection on \( L \), \( \Phi \in \Gamma(S^+) \), and \( h \in \Omega^2 \) is a perturbation.

From now on, we will suppress Sobolev spaces throughout the paper in order to focus on the main issue of orientability. Given any solution \((A, \Phi)\) of (1), we have the following fundamental elliptic complex

\[
\begin{array}{cccc}
0 & \rightarrow & \Omega^0 & \rightarrow & \Omega^1 \oplus \Gamma(S^+) \rightarrow & \Omega^2_+ \oplus \Gamma(S^-) & \rightarrow & 0 \\
D^1 & & D^2 & & D^3 & & D^4 & & \end{array}
\]

where \( D^1 = D^1_{A,\Phi} \) is defined by \( D^1(f) = (2df, -f \Phi) \) for \( f \in \Omega^0 \), and \( D^2 = D^2_{A,\Phi} \) by

\[
D^2(a, \phi) = (d^+a - Dq_{\phi}(\phi), \mathcal{D}_A \phi + \frac{1}{2}a \cdot \Phi)
\]
for \((a, \phi) \in i\Omega^1 \oplus \Gamma(S^+)\). Here \(q(\Phi)\) has the differential
\[
Dq_{\Phi}(\phi) = \Phi \otimes \phi^* + \phi \otimes \Phi^* - \frac{<\Phi, \phi> + <\Phi, \phi>}{2} \text{Id}.
\]
Remark that only the first equation \(\partial A \Phi = 0\) is needed to show that (2) is a complex. Moreover the perturbation \(h\) does not appear explicitly in the formulas but certainly affects \(D^1, D^2\) through \(A, \Phi\) due to equations (1).

The perturbed SW equations (1) define the smooth function
\[
F: A \times \Gamma(S^+) \times i\Omega^2_+ \longrightarrow i\Omega^2_+ \times \Gamma(S^-)
\]
\((A, \Phi, h) \leftrightarrow (F_A^+ - q(\Phi) - h, \partial A \Phi)\).

At a point \((A, \Phi, h)\), the differential \(D_F A, \Phi, h: i\Omega_1^1 \oplus \Gamma(S^+) \longrightarrow i\Omega_2^1 \oplus \Gamma(S^-)\) is
\[
D_F(a, \phi, k) = D^2(a, \phi) - (k, 0).
\]
Here \((k, 0)\) is viewed as a vector in the direct sum. The standard transversality theorem says that 0 is a regular value of \(F\), when restricted to irreducibles \((A, \Phi, h), \Phi \neq 0\). Hence \(F^{-1}(0) \cap \{\Phi \neq 0\}\) is a smooth Banach manifold. The tangent space at such a point \((A, \Phi, h)\) is of curse \(T F^{-1}(0) = \ker D_F A, \Phi, h\).

Take the projection to the parameter space \(\pi: F^{-1}(0) \rightarrow i\Omega_+^2\), namely \(\pi(A, \Phi, h) = h\). We want to express the kernel and cokernel of its differential \(D\pi\) in terms of \(D^2\) introduced above. The differential at a point \((A, \Phi, h)\) with \(\Phi \neq 0\) is
\[
D\pi: \ker D_F A, \Phi, h \longrightarrow i\Omega_+^2
\]
\((a, \phi, k) \mapsto k\).

Then one can readily verify there is a natural isomorphism \(\ker D\pi = \ker D^2\), by using (1) and the inclusion \(i\Omega_1^1 \oplus \Gamma(S^+) \hookrightarrow i\Omega_+^2 \oplus i\Omega_+^2\).
Next one can relate the image sets \(\text{im}D\pi \subset i\Omega_+^2\) and \(\text{im}D^2 \subset i\Omega_+^2 \oplus \Gamma(S^-)\) as follows:
\[
\text{im}D\pi = \{p_1(\xi) \mid \xi \in \text{im}D^2 \text{ such that } p_2(\xi) = 0\},
\]
where \(p_1, p_2\) are projections of \(i\Omega_+^2 \oplus \Gamma(S^-)\) onto its factors. Thus the inclusion map \(i\Omega_+^2 \hookrightarrow i\Omega_+^2 \oplus \Gamma(S^-), k \mapsto (k, 0)\) induces a well-defined injective map
\[
\frac{i\Omega_+^2}{\text{im}D\pi} \longrightarrow \frac{i\Omega_+^2}{\text{im}D^2} \oplus \Gamma(S^-)/\text{im}D^2.
\]
Furthermore this map is surjective, which follows from (1) and \(\text{coker} D\pi = 0\) on \(F^{-1}(0)\) by the transversality theorem. Hence we have a natural isomorphism
\[
\text{coker} D\pi_A, \Phi, h \longrightarrow \text{coker} D^2 A, \Phi,
\]
induced by the inclusion \( i\Omega^2_+ \hookrightarrow i\Omega^2_+ \oplus \Gamma(S^-) \). Again the perturbation \( h \) does affect \( \text{coker} D^2_{A,\Phi} \) through the SW equations.

Consider the gauge \( \mathcal{G} \) action, \( g(A, \Phi) = ((g^2)^* A, g \cdot \Phi) \) for each \( g \in \mathcal{G} \). After modulo out the action, we have the parameterized moduli space \( \mathcal{M} = \mathcal{F}^{-1}(0)/\mathcal{G} \), which is however only a topological space. But the subspace \( \mathcal{M}^* \) of irreducible solutions is a smooth Banach manifold and the tangent space at a point \([A, \Phi, h]\) is

\[
\text{T} \mathcal{M}^* = \text{ker} \mathcal{D} \cap \text{ker}(D^1)^*,
\]

by using the slice \( \text{ker}(D^1)^* \) of the \( \mathcal{G} \) action. (It will be useful to keep in mind that \((D^1)^*(a, \phi) = 2d^* a - <\phi, \Phi>\). We have the new projection map \( \pi : \mathcal{M} \rightarrow i\Omega^2_+ \). By applying the slice to other discussions above we can summarize the main results here.

**Proposition 2.1.** (i) The tangent space of the parameterized irreducible moduli space \( \mathcal{M}^* \) at a point \([A, \Phi, h]\) is the following subspace of \( i\Omega^1 \oplus \Gamma(S^+) \oplus i\Omega^2_+ \):

\[
\text{T} \mathcal{M}^* = \{(a, \phi, k) \mid D^2(a, \phi) = (k, 0) \text{ and } (D^1)^*(a, \phi) = 0\}.
\]

(ii) The differential \( D\pi : \text{T} \mathcal{M}^* \rightarrow i\Omega^2_+ \) has the kernel and cokernel canonically identified with:

\[
\text{ker}(D\pi) = \mathcal{H}^1, \quad \text{coker} D\pi = \mathcal{H}^2,
\]

where \( \mathcal{H}^1 = \mathcal{H}^1_{A,\Phi}, \mathcal{H}^2 = \mathcal{H}^2_{A,\Phi} \) are the cohomology of the complex (2).

Because of the slice \( \text{ker}(D^1)^* \), we cannot use \( \text{ker} D^2 \) alone to characterize \( \text{ker} D\pi \) on \( \text{T} \mathcal{M}^* \), although we do have \( \text{coker} D^2 = \text{coker} D\pi \).

Quite often, it is advantageous to form a single elliptic operator converted from the basic complex (2):

\[
\delta = \delta_{A,\Phi} = D^2 \oplus (D^1)^* : i\Omega^1 \oplus \Gamma(S^+) \longrightarrow [i\Omega^2_+ \oplus \Gamma(S^-)] \oplus i\Omega^0.
\]

Remark that for \([A, \Phi, h] \notin \mathcal{M}^* \), (2) is not a complex but \( \delta = \delta_{A,\Phi} \) is still elliptic.

At \([A, \Phi, h] \in \mathcal{M}^* \), \( \text{ker} \delta = \mathcal{H}^1, \text{coker} \delta = \mathcal{H}^2 \). Hence Proposition 2.1 translates into the following:

**Corollary 2.2.** There are natural isomorphisms

\[
\text{ker} D\pi = \text{ker} \delta, \quad \text{coker} D\pi = \text{coker} \delta.
\]

Unlike \( D\pi \), \( \delta \) is defined between two fixed vector bundles, i.e. \( \delta \) can be viewed as a family of elliptic operators, which is another advantage over \( D\pi \).
2.2. Real spin\textsuperscript{c} structures. Let \((X, J)\) be an almost complex manifold of dimension \(2n\) and \(\sigma : X \to X\) a real structure, i.e. an anti-holomorphic involution, so \(\sigma^*J = -J\sigma : TX \to TX\). Endow \(X\) with a Hermitian metric that is preserved by both \(J\) and \(\sigma\), namely, \((Ju, Jv) = (u, v), (\sigma(u), \sigma(v)) = (u, v)\). It is well-known that \(X\) has a canonical spin\textsuperscript{c} structure \(P_{sp}\) associated with \(J\) and the metric. In this subsection, we consider a natural lifting of \(\sigma\) on the spin\textsuperscript{c} structure.

Let \(P_U \to X\) be the \(U(n)\)-bundle of complex frames and \(P_{so} \to X\) the \(SO(2n)\)-bundle of real frames, both of which use the metric on \(X\). There is a canonical inclusion \(\rho : U(n) \to SO(2n)\), given as \(\rho[a_{jk}] = [a_{jk}]\), where

\[
a_{jk} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}
\]

if the entry \(a_{jk} = x + iy\). Then \(P_{so} = P_U \times_\rho SO(2n)\). Note \(\rho(\overline{v}) = T \rho(u) T\), where \(T = T^{-1}\) is the diagonal \(2n \times 2n\) matrix

\[
\text{diag}\{1, -1, 1, -1, \cdots, 1, -1\}.
\]

Hence under \(\rho\), the complex conjugation on \(U(n)\) is transferred onto \(SO(2n)\) as \(v \mapsto \overline{v} := TvT\).

**Lemma 2.3.** There is a canonical involution lifting \(\tau : P_U \to P_U\) of \(\sigma\) which is conjugate in the sense that \(\tau(pu) = \tau(p)\overline{u}\) for \(p \in P_U, u \in U(n)\). Moreover, \(\tau\) induces a lifting on \(P_{so}\) (still denoted by \(\tau\)) satisfying \(\tau(pv) = \tau(p)\overline{v}\) for \(p \in P_{so}, v \in SO(2n)\).

**Proof.** Enough to show the first statement. Let \(\overline{P}_U\) denote the complex frame bundle of \((X, -J)\). Since \(\sigma : (X, J) \to (X, -J)\) is holomorphic, it induces a unique bundle isomorphism \(\sigma_* : P_U \to \overline{P}_U\). Note that \(\overline{P}_U = P_U \times_c U(n)\), where \(c(u) = \overline{u}\) is the conjugation map on \(U(n) \to U(n)\). Thus one can take \(\tau\) to be \(\sigma_*\) composed with the conjugation \(c : P_{so} \to \overline{P}_U\).

In particular, the lifting \(\tau : P_{so} \to P_{so}\) is not the induced map \(\sigma_* : P_{so} \to P_{so}\), since the latter is a \(SO(2n)\)-bundle isomorphism.

Recall the embedding \(\gamma : U(n) \to Spin^c(2n)\) can be defined as follows (cf. Lawson-Michelson [6]): if \(u \in U(n)\) is diagonalized as

\[
\text{diag}\{e^{i\theta_1}, \cdots, e^{i\theta_n}\}
\]

under a complex basis \((e_1, \cdots, e_n)\) of \(\mathbb{C}^n\), then \(\gamma(u) \in Spin(2n) \times \pm U(1) = Spin^c(2n)\) is

\[
\prod_k \left(\cos\frac{\theta_k}{2} + \sin\frac{\theta_k}{2} \cdot e_k \cdot J e_k\right) \times e^{\frac{i}{2} \sum_k \theta_k}.
\]
Since $\pi$ is diagonalized as $\{e^{-i\theta_1}, \cdots, e^{-i\theta_n}\}$ under the complex basis $(\tau_1, \cdots, \tau_n)$, $\gamma(\pi)$ is equal to
\[
\prod_k \left( \cos \frac{\theta_k}{2} - \sin \frac{\theta_k}{2} \cdot \tau_k \cdot J\tau_k \right) \times e^{-\frac{1}{2} \sum_k \theta_k},
\]
which is the same as
\[
\prod_k \left( \cos \frac{\theta_k}{2} + \sin \frac{\theta_k}{2} \cdot \epsilon\tau_k \cdot J\epsilon\tau_k \right) \times e^{-\frac{1}{2} \sum_k \theta_k},
\]
by noting that $J\tau_k = \tau_k = -J\epsilon\tau_k$. Here the conjugation on the $Spin(2n)$-factor is the restriction of that to the Clifford algebra $Cl(\mathbb{R}^{2n}) = Cl(\mathbb{C}^n)$, namely the one generated by the standard conjugation on $\mathbb{C}$. This means that only by conjugating both factors $Spin(2n)$ and $U(1)$, we obtain the conjugation on $Spin^c(2n)$ which is compatible with the conjugation on $U(n)$ via the inclusion $\gamma$. (One should emphasize that the other conjugation on $Spin^c(2n)$ coming from the $U(1)$-factor alone, as usually considered, is not what is required here.) In general we call this kind of conjugation coming from both factors a “diagonal conjugation”.

Now $Cl(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic to the matrix algebra $\mathbb{C}[2]$, and the diagonal conjugation on $Cl(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$ – from both factors, is compatible with the usual entry wise conjugation on $\mathbb{C}[2]$ under this isomorphism. Using the periodicity $Cl(\mathbb{R}^{2n}) \otimes \mathbb{C} \cong [Cl(\mathbb{R}^2)] \otimes \mathbb{C} \cong \mathbb{C}[2^n]$, it is not hard to check that the diagonal conjugation on $Cl(\mathbb{R}^{2n}) \otimes \mathbb{C}$ is compatible with the standard entry wise complex conjugation on $\mathbb{C}[2^n]$.

Next consider the standard complex spin representation
\[
Spin^c(2n) \hookrightarrow Cl(\mathbb{R}^{2n}) \otimes \mathbb{C} = \mathbb{C}[2^n] \hookrightarrow GL_c(V),
\]
where $V$ is a complex vector space of dimension $2^n$. Then the diagonal conjugation on $Spin^c(2n)$ is compatible with the complex conjugation on $V$.

Recall that the canonical spin$^c$ bundle is $P_{sp} = P_U \times_{\gamma} Spin^c(2n)$ and the associated spinor bundle is $S = P_{sp} \times V$.

**Proposition 2.4.** There is a canonical lifting $\tau : P_{sp} \rightarrow P_{sp}$ of $\sigma$, which satisfies
\[
\tau(pg) = \tau(p)\overline{g},
\]
where $p \in P_{sp}$ and $\overline{g}$ signifies the diagonal conjugation on $Spin^c(2n)$.

The induced lifting on $S$ (still denoted by $\tau$) is fiberwise complex anti-linear and compatible with the complex Clifford multiplication $(T^*X \otimes
Proof. The lifting on $P_{sp}$ is induced from the one given in Lemma 2.5. The compatibility holds because the lifting on $S$ is constructed via the spin$^c$-principal bundle and the diagonal conjugation on Spin$^c(2n)$ is compatible with the conjugation on $V$ as discussed above. \hfill $\square$

Note that the determinant line bundle of $P_{sp}$ also carries a natural anti-linear lifting of $\sigma$, since the isomorphism $U(1) \cong U(1)/ \pm 1$ preserves the complex conjugation. In fact, $\det P_{sp} = K^{-1}$ (anti-canonical bundle of $J$), which certainly has an anti-linear lifting. More generally, for any line bundle $L' \to X$ such that $\sigma^* c_1(L') = -c_1(L')$, the corresponding Spin$^c$ bundle $S_{L'} = S \otimes L'$ (with determinant $K^{-1} \otimes (L')^2$) has a canonical anti-linear lifting, compatible with the Clifford multiplication.

Consider now an arbitrary real vector space $W$ with an almost complex structure $J$. Given any linear map $\sigma : W \to W$ such that $\sigma \circ J = -J \circ \sigma$, we extend it sani-linearly on the complexification, $\tilde{\sigma} : W \otimes C \to W \otimes C$, so that $\tilde{\sigma}(w \otimes c) = \sigma(w) \otimes \bar{c}$. This contrasts with the usual linear extension of $J$ on the complexification, and is required by the following lemma.

Lemma 2.5. The extension $\tilde{\sigma}$ preserves the decomposition $W \otimes C = W^{1,0} \oplus W^{0,1}$ of the $\pm i$-eigen spaces of $J$.

Proof. Take any $w \in W^{1,0}$. Then $J(w) = iw$. Since $J(\sigma(w)) = -\sigma(J(w)) = -\sigma(iw) = i\sigma(w)$, we have $\sigma(w) \in W^{1,0}$. This establishes $\sigma : W^{1,0} \to W^{1,0}$. The second summand is similar. \hfill $\square$

Remark. Because of this lemma, from now on we will always take the anti-linear extension of an anti-holomorphic involution $\sigma$ on the complexification. We will also use $\sigma$ for the extension without the tilde sign.

Corollary 2.6. Identify the spinor bundle $S$ canonically with the cotangent bundle $\Lambda^{0,*}X = \bigoplus_r \Lambda^{0,r}X$ of $(0,*)$-forms as usual. Then the lifting $\tau$ on $S$ is equivalent to the anti-linear lifting $\sigma^*$ on $\Lambda^{0,*}X$.

Proof. From the early discussion,

$$S = P_{sp} \times V = (P_U \times_{\gamma} \text{Spin}^c(2n)) \times V = P_U \times_{\eta} V,$$

where the composition $\eta : U(n) \to GL_c(V)$ of $\gamma$ with the spin representation is the standard unitary representation on $V = \Lambda^c C^n$. Thus $S = \Lambda^* T_{\gamma}X$, where $T_{\gamma}X$ is the tangent bundle with almost complex structure $J$ and the wedge product is taken over $C$ fiberwisely. Clearly
the lifting $\tau$ on $S$ is equivalent to the lifting $\sigma_* : T_cX \to T_cX$ which is fiberwise anti-linear, since $\sigma_* \circ J = -J \circ \sigma_*$. The natural identification of $S = \Lambda^* T_cX$ with $\Lambda^{0,*} X$ is through

$$T_cX \quad \longrightarrow \quad \Lambda^{0,1} X$$

where the metric is used. Since $\sigma$ is an isometric involution, $\sigma_*(v) \mapsto (\bullet, \sigma_*(v)) = (\sigma_*(\bullet), v) = \sigma^*(v^*)$. Hence $\sigma^*$ is equivalent to the lifting $\sigma^*$ on $\Lambda^{0,1} X$. That $\sigma^*$ is anti-linear follows from $\sigma^* \circ J = -J \circ \sigma^*$ and $J = -i$ on $\Lambda^{0,1} X$.

2.3. **Seiberg-Witten equations with real structures.** Now we specialize to the case of our interest, that $(X, J)$ is a Hermitian 4-dimensional almost complex manifold with an isometric anti-holomorphic involution $\sigma$.

**Convention.** When no confusion is possible, we will often use $w$ for $\sigma(w), \sigma_*(w), \sigma^*(w)$ or more generally for $\tau(w)$, where $\tau$ is any induced map by $\sigma$. This is convenient and makes sense since the maps are often anti-linear.

For example, $\overline{i\alpha} = -i\overline{\alpha}$ interprets conveniently the formula $\sigma^*(i\alpha) = -i\sigma^*(\alpha)$ with $i\alpha \in \Omega^* \otimes \mathbb{C}$. In particular, if $i\alpha = F$ is the curvature 2-form of a unitary connection on a complex line bundle over $X$, then $\overline{F} = -i\overline{\alpha}$. The appearance of the $-$ sign here will save a lot of $-$ signs elsewhere.

**Lemma 2.7.** Suppose $L \to X$ is a complex line bundle and $\tau : L \to L$ is an anti-linear lifting of $\sigma$ (so $\sigma^* c_1(L) = -c_1(L)$). For any unitary connection $A$ on $L$ and its pull-back $\overline{A} = \tau^* A$, their curvatures satisfy $F_{\overline{A}} = \overline{F_A}$.

**Proof.** One can prove the lemma by direct calculations on the local connection matrices under a gauge. More convenient is to use the corresponding principal bundle $P$ of $L$. Then the lifting $\tau : P \to P$ satisfies $\tau(pg) = \tau(p)\overline{g}$ for $p \in P, g \in U(1)$. The connection 1-form $\omega_A$ is globally defined on $P$ with values in the Lie algebra $i\mathbb{R}$ of $U(1)$. Since the conjugation $g \mapsto \overline{g}$ induces the map $\xi \mapsto -\xi$ on $i\mathbb{R}$, which is compatible with the anti-complex linear extension of $\sigma$, the connection form of $\overline{A}$ is $\omega_{\overline{A}} = \overline{\omega_A}$. It follows that $F_{\overline{A}} = \overline{F_A}$. □

Thus if $A$ is equivariant under $\tau$, then its curvature obeys $F_{\overline{A}} = \overline{F_A}$.

Consider the canonical spin$^c$ bundle $S$ of $(X, J)$, with determinant bundle $L = K^{-1}$. By Proposition 2.4, we have a canonical anti-linear lifting $\tau$ on $S$ and $L$.

Under the previous remark, for a section $\Phi \in \Gamma(S^+)$, $\overline{\Phi}$ is the pull-back section $\tau^* \Phi := \tau^{-1} \circ \Phi \circ \sigma$. Similarly, $\overline{h} = \sigma^*(h)$ if $h \in i\Omega^2_+$. The
induced action of \( \sigma \) on the gauge group \( \mathcal{G} \) is \( g \mapsto \overline{g(x)} \), i.e. \( g(\sigma(x)) \), the long over line being the conjugation on \( S^1 \). With these actions understood, we have the following:

**Proposition 2.8.** (i) The gauge transformation \( \mathcal{G} \times \mathcal{C} \to \mathcal{C} \) is \( \sigma \)-equivariant, where \( \mathcal{C} = \mathcal{A} \times \Gamma(S^+) \) is the configuration space. Hence the quotient space \( \mathcal{B} = \mathcal{C}/\mathcal{G} \) has an induced involution \( \sigma \).

(ii) The SW function \( \mathcal{F} : \mathcal{C} \times i\Omega^2_+ \to i\Omega^2_+ \times \Gamma(S^-) \) is equivariant also. Hence \( (A, \Phi) \) is a SW solution with respect to \( h \) iff \( (\overline{A}, \overline{\Phi}) \) is a SW solution with respect to \( \overline{h} \).

(iii) The projection \( \pi : \mathcal{C} \times i\Omega^2_+ \to i\Omega^2_+ \) is equivariant, so is \( \pi : M \to i\Omega^2_+ \) after dividing gauge transformations.

**Proof.** The statements follow from Proposition 2.4 coupled with Lemma 2.7. \( \square \)

Note that we may also prove the proposition using Corollary 2.6.

Proposition 2.8 can be obviously extended from the canonical spin\(^c\) structure \( P_{sp} \) to a general one:

**Proposition 2.9.** If a principal \( U(1) \)-bundle \( \xi \) has an anti-linear lifting of \( \sigma \), then all three parts of 2.8 remains to be true for the twisted spin\(^c\) structure \( P_{sp} \times \xi \) of \( P_{sp} \) by \( \xi \).

**Remark.** It is important to point out that the parameterized moduli space \( M \) is not a complex or almost complex manifold, partly because \( i\Omega^2_+ \) is not so. Furthermore, the fibers of \( \pi \) do not have any obvious complex structure, except the unperturbed moduli space \( \pi^{-1}(0) \) on a Kähler surface \( X \). Nonetheless, it is convenient to say \( \pi : M \to i\Omega^2_+ \) is real which simply is taken to mean that \( \pi \) is \( \sigma \)-equivariant. By the same token, even though \( i\Omega^2_+ \) is not a complex space, we still call \( (i\Omega^2_+)_{R} := \text{Fix}(\sigma : i\Omega^2_+ \to i\Omega^2_+) \) the real space. Note that under the convention above, \( (i\Omega^2_+)_{R} \) consists of \( i \cdot (\sigma \text{-anti-invariant smooth forms}) \), namely \( i\alpha \) where \( \overline{\alpha} = -\alpha \in \Omega^2_+ \). Similar remark applies to \( (\mathcal{F})_{R} \) of other degrees. This is consistent with \( c_1(L) = \sigma^*c_1(L) = -c_1(L) \).

On various occasions it will be useful to define real liftings in a topological way, irrespective of any almost complex structure on \( X \). Proposition 2.4 motivates the following:

**Definition 2.10.** Let \( X \) be a smooth manifold of dimension \( 2n \) and \( \sigma : X \to X \) a smooth involution that admits a conjugate lifting on the frame bundle \( P_o \), \( \sigma(pv) = \sigma(p)\overline{v} \), where \( \overline{v} = T\overline{T}^{-1} \) as in (9).

(i) A spin structure on \( X \) is called **real compatible with** \( \sigma \) if the \( Spin(2n) \)-bundle \( P_s \) admits a conjugate lifting \( \tau \) of \( \sigma \), namely \( \tau(pg) = \overline{pg} \).
\( \tau(p)\overline{g} \), where for \( g \in \text{Spin}(2n) \), \( \overline{g} \) is the restriction of the complex conjugation from \( \text{Cl}(\mathbb{C}^n) \).

(ii) Similarly a spin\(^c\) structure on \( X \) is real compatible with \( \sigma \) if its principal \( \text{Spin}^c(\text{an}) \)-bundle \( P \) admits a conjugate lifting \( \tau \), \( \tau(pg) = \tau(p)\overline{g} \), where \( \overline{g} \) is the diagonal conjugation of \( g \in \text{Spin}^c(2n) \).

In both cases we will also call \( \tau \) (topological) real liftings. In terms of the spinor bundle \( S \), \( \overline{\tau} \) leads to an anti-linear involution lifting on \( S \) which is compatible with the Clifford multiplication on \( T^*X \otimes \mathbb{C} \hookrightarrow \text{End}_c(S) \). As before, \( \sigma \) should be extended as an anti-linear map on \( T^*X \otimes \mathbb{C} \) in order to have this compatibility. In particular for \( \dim X = 4 \) and a compatible spin\(^c\) structure \( P \), with the same induced action on the gauge group \( \mathcal{G} \) and Lemma 2.7 as in the previous section, Proposition 2.8 carries over to the new set-up. In particular, \( \mathcal{B} \) inherits an involution, and \( (A, \Phi) \) is a SW solution with perturbation \( h \) iff \( (\overline{A}, \overline{\Phi}) \) is with perturbation \( \overline{h} \).

3. Configuration spaces and determinant bundles

In this and next sections, to be definitive, we focus on a Kähler surface \((X, \omega, J)\) that has an isometric real structure \( \sigma \), thus \( \sigma^*\omega = -\omega, \sigma^* \circ J = -J \circ \sigma^* \). We will indicate when appropriate that many results below either remain to be true (for example those in Subsection 3.1) or can be modified suitably for an almost complex or symplectic manifolds.

Suppose that \( S = S^+ \oplus S^- \rightarrow X \) is a spin\(^c\) structure admitting a real lifting of \( \sigma \) (cf. Proposition 2.9). Let \( L = \det S^+ \) be the determinant bundle of the spin\(^c\) structure.

3.1. The real configuration and moduli spaces. Set \( \mathcal{C}^* = \mathcal{A}(L) \times \big( \Gamma(S^+) \setminus \{0\} \big) \). By Proposition 2.9 the induced involutions on \( \mathcal{C}^*, \mathcal{G} \), namely \( (A, \Phi) \mapsto (\overline{A}, \overline{\Phi}), g \mapsto \overline{g} \), are compatible:

\[ g \cdot (A, \Phi) = \overline{g}(\overline{A}, \overline{\Phi}) . \]

Thus we have the further induced involution \( \sigma \) on the configuration space \( \mathcal{B}^* = \mathcal{C}^*/\mathcal{G} \) and hence the fixed configuration space

\[ \mathcal{B}^{*\sigma} = \text{Fix}(\sigma : \mathcal{B}^* \rightarrow \mathcal{B}^*) \subseteq \mathcal{B}^* . \]

Moreover, we have the real configuration space defined as

\[ \mathcal{B}^*_R = \mathcal{C}^*_R/\mathcal{G}_R \]

namely the set of real points of \( \mathcal{C}^* \) modulo the real gauge group. It follows essentially from the compatibility above and the freeness of the \( \mathcal{G} \) action on \( \mathcal{C}^* \) that the natural map \( [(A, \Phi)]_R \mapsto [(A, \Phi)] \) gives
rise to an inclusion $\mathcal{B}_R^* \hookrightarrow \mathcal{B}^{*\sigma}$. (In the appendix, we organize and state the results for the general set-up.) In this paper, we will be mainly interested in the space $\mathcal{B}_R^*$ and its subspace of real Seiberg-Witten solutions.

From the standard Seiberg-Witten theory, e.g. the book [8], the gauge group $\mathcal{G}$ is naturally homotopic to $S^1 \times H^1$, where $H^1 = H^1(X, \mathbb{Z})$. As the classifying space of the group $\mathcal{G}$, $\mathcal{B}^*$ is weakly homotopic to $\mathbb{C}P^\infty \times K(H^1, 1)$. Since $\sigma$ induces the standard conjugation on the $S^1$-factor through $\mathcal{G}$, the induced action on $\mathbb{C}P^\infty$ is also the conjugation. Hence taking fixed points on both sides, we have

$$\mathcal{B}^{*\sigma} \sim \mathbb{R}P^\infty \times K(H^1, 1)^\sigma.$$ 

An similar argument will give the following result for the weak homotopy type of $\mathcal{B}_R^*$.

**Proposition 3.1.** There is a natural weak homotopy equivalence:

$$\mathcal{B}_R^* \sim \mathbb{R}P^\infty \times K(H^1_R, 1),$$

where $H^1_R = H^1(X, \mathbb{Z})^\sigma$.

**Proof.** The real constant gauges form a subgroup: $\mathbb{Z}_2 \subset \mathcal{G}_R$. For the quotient group, there is a natural bijection $\pi_0(\mathcal{G}_R/\mathbb{Z}_2) \to H^1_R$ given by $\xi \mapsto \rho_\xi$, where $\rho_\xi \in H^1(X, \mathbb{Z})^\sigma \subset H^1(X, \mathbb{R})^\sigma$ is defined as $\rho_\xi = [g^{-1}dg]$ for a gauge $g \in \xi$ such that $g^{-1}dg$ is a real harmonic 1-form. It follows that $\mathcal{G}_R$ is homotopic to $\mathbb{Z}_2 \times H^1_R$ and the classifying space $BG_R$ of $\mathcal{G}_R$ is weakly homotopic to $\mathbb{R}P^\infty \times K(H^1_R, 1)$. Since the real part $C_R$ is clearly contractible just as $\mathcal{C}^*$ is, the real configuration space $\mathcal{B}_R^* = C_R/\mathcal{G}_R$ is weakly homotopic to $BG_R$ hence to $\mathbb{R}P^\infty \times K(H^1_R, 1)$. □

**Remarks.** (1) The generator in $H^2(\mathcal{B}^*, \mathbb{Z})$ that comes from the $\mathbb{C}P^\infty$ factor restricts to a 2-torsion in $H^2(\mathcal{B}_R^*, \mathbb{Z})$. In fact the resulting complex line bundle on $\mathcal{B}_R^*$ is the complexification of the real line bundle corresponding to the generator in $H^1(\mathcal{B}_R^*, \mathbb{Z}_2)$ that comes from the $\mathbb{R}P^\infty$ factor.

(2) By Proposition 3.1, $\mathcal{B}_R^*$ is connected; in contrast, the fixed configuration space $\mathcal{B}^{*\sigma}$ is disconnected and contains $\mathcal{B}_R^*$ as a connected component.

At a real point $(A, \Phi) \in C_R^*$, the tangent space is

$$T_{A, \Phi}C_R^* = (i\Omega^1)_R \oplus \Gamma(S^+)_R,$$

where the subscript $R$ indicates the invariant subspaces under the extended $\sigma$-action. Linearizing the $\mathcal{G}_R$ action on $C_R^*$, we have

$$D_R^1 : (i\Omega^0)_R \to (i\Omega^1)_R \oplus \Gamma(S^+)_R$$
as the restriction of $D^1$ from the complex (2). Thus the tangent space $T_{[A,\Phi]}B^*_R = \ker(D^1_R)^*$, from which one sees that $B^*_R$ is an open subspace of $B^\sigma$.

The relevant complex for the real parameterized moduli space is

$$
0 \longrightarrow (i\Omega^0)_R \longrightarrow (i\Omega^1)_R \oplus \Gamma(S^+)_R \longrightarrow (i\Omega^2_+)_R \oplus \Gamma(S^-)_R \longrightarrow 0,
$$

by restricting the complex (2) to the real spaces.

By Proposition 2.9, the Seiberg-Witten functional restricts to the real spaces:

$$
\mathcal{F}_R : \mathcal{A}_R \times \Gamma(S^+)_R \times (i\Omega^2_+)_R \longrightarrow (i\Omega^2_+)_R \times \Gamma(S^-)_R.
$$

At a real point $(A, \Phi, h)$, the differential

$$
D\mathcal{F}_R : (i\Omega^1)_R \oplus \Gamma(S^+)_R \oplus (i\Omega^2_+)_R \longrightarrow (i\Omega^2_+)_R \oplus \Gamma(S^-)_R
$$

is $D\mathcal{F}_R(a, \phi, k) = D^2_R(a, \phi) - (k, 0)$. The usual proof of the transversality theorem can be adapted easily to show that $0$ is a regular value of $\mathcal{F}_R$ when restricted to irreducibles, hence $\mathcal{F}_R^{-1}(0) \cap \{\Phi \neq 0\}$ is a smooth Banach manifold. The tangent space at the real point $(A, \Phi, h)$ is $T\mathcal{F}_R^{-1}(0)^* = \ker D\mathcal{F}_R$.

Dividing by real gauge transformations, we have the parameterized real moduli spaces: $M_R = \mathcal{F}_R^{-1}(0)/G_R$. The real version of Proposition 2.1 becomes:

**Proposition 3.2.** (i) The tangent space of the parameterized irreducible real moduli space $M^*_R$ at a real point $[(A, \Phi, h)]$ is the following subspace of $(i\Omega^1)_R \oplus \Gamma(S^+)_R \oplus (i\Omega^2_+)_R$:

$$
TM^*_R = \{(a, \phi, k) \mid D^2_R(a, \phi) = (k, 0) \text{ and } (D^1_R)^*(a, \phi) = 0\}.
$$

(ii) The differential $D\pi_R$ of the projection map $\pi_R : M^*_R \rightarrow (i\Omega^2_+)_R$ has the kernel and cokernel canonically identified with:

$$
\ker(D\pi_R) = H^1_R, \quad \text{coker}D\pi_R = H^2_R,
$$

where $H^1_R, H^2_R$ are the cohomology of the complex (11).

A single elliptic operator converted from the basic complex (10) is

$$
\delta_R = D^2_R \oplus (D^1_R)^* : (i\Omega^1)_R \oplus \Gamma(S^+)_R \longrightarrow [(i\Omega^2_+)_R \oplus \Gamma(S^-)_R] \oplus (i\Omega^0)_R.
$$

The real version of Corollary 2.2 is

**Corollary 3.3.** There are natural isomorphisms

$$
\ker D\pi_R \cong \ker \delta_R, \quad \text{coker}D\pi_R \cong \text{coker} \delta_R.
$$
Thus the orientation bundle \( \det \pi_R = \bigwedge^{\max} \ker D \pi_R \otimes (\bigwedge^{\max} \coker D \pi_R)^* \) of the map \( \pi_R \) is naturally identified with the determinant bundle of \( \delta_R \):

(14) \[
\det \pi_R = \bigwedge^{\max} \ker \delta_R \otimes (\bigwedge^{\max} \coker \delta_R)^*
\]
on \( M^*_R \). This is the reason why the latter bundle will play a prominent role in the paper.

3.2. The real determinant line bundle. At a point \((A, \Phi) \in C^*\), let us decompose the operator \( \delta = \delta_{A, \Phi} : i\Omega^1 \oplus \Gamma(S^+) \to [i\Omega^0 \oplus i\Omega^2_+] \oplus \Gamma(S^-) \) defined in (7) as \( \delta = (\delta^X \oplus \partial A) + \eta \) where

(15) \[
\delta^X = (2d^*, d^+) : i\Omega^1 \to i\Omega^0 \oplus i\Omega^2_+
\]
depends on \( X \) only and \( \eta = \eta_\Phi \) is a zero-th order operator depending on \( \Phi \) only:

\[
\eta_\Phi(a, \phi) = (- < \phi, \Phi > - Dq_\Phi(\phi)) + \frac{1}{2} a \cdot \Phi.
\]

Note that \( \eta_\Phi = t\eta_\Phi \); in particular \( \eta_0 \) is the zero operator. If we set further \( \delta^L = \delta^L_A = \delta^X \oplus \partial A \), then \( \delta = \delta^L + \eta \) so that \( A, \Phi \) are separated in the two operators.

**Proposition 3.4.** There are suitable complex structures in the infinitely dimensional spaces \( i\Omega^1, i\Omega^0 \oplus i\Omega^2_+ \) such that the extended \( \sigma \) actions are anti-holomorphic on these spaces and \( \delta^X \) is complex linear. Moreover, the numerical index \( \text{ind} \delta_R \) is half of \( \text{ind} \delta \) namely

\[
\text{ind} \delta_R = \frac{1}{8} (c_1(L)^2 - 2c_X - 3s_X).
\]

**Proof.** The space \( i\Omega^1 \cong i\Omega^{0,1} \) has the induced complex structure by \( J \) under which \( \sigma \) is anti-holomorphic. Whereas there is a natural real linear isomorphism \( \Omega^0 \oplus \Omega^2_+ \cong \Omega^0 \oplus \Omega^0 \cdot \omega \oplus \Omega^{0,2} \), the latter being isomorphic to \( \Omega^0_c \oplus \Omega^{0,2} \) by viewing \( \omega = i \) on the complexification \( \Omega^0_c \).

Hence \( i\Omega^0 \oplus i\Omega^2_+ \) inherits a complex structure, under which \( \sigma \) is anti-holomorphic in view of \( \sigma^* \omega = -\omega \). Furthermore \( \delta^X \) is complex linear, since it is equivalent to \( \partial^* \oplus \partial : \Omega^{0,1} \to \Omega^0_c \oplus \Omega^{0,2} \) under the previous transformations. (In the symplectic case, they are equivalent up to a zeroth order operator.)

At a real point \((A, \Phi) \in C_R \), \( \delta^L = \delta^X \oplus \partial A \) is complex linear and real with respect to \( \sigma \). Hence \( \ker \delta^L, \coker \delta^L \) are complex vector spaces with real structure, and the real parts have half the dimensions, giving \( \text{ind} \delta^L_R = \text{ind} \delta^L / 2 \). Since \( \eta, \eta_R \) are zeroth order operators, the indices
remain the same for $\delta = \delta^L + \eta$ and $\delta_R = \delta_R^L + \eta_R$. Thus $\text{ind}\delta_R = \text{ind}\delta/2$ holds.

With respect to the extended real structure, the previously defined fixed point set $(i\Omega^1)_R$ is now the true real part of $i\Omega^1$. Clearly the real part of $i\Omega^0 \oplus i\Omega^2_\bot$ is

$$[i\Omega^0 \oplus i\Omega^2_\bot]_R = (i\Omega^0)_R \oplus (i\Omega^2_\bot)_R,$$

where the summands are fixed point sets of $\sigma$ (which are not real parts).

Note that for when $\Phi \neq 0$, $\eta$ is not a complex linear operator, because of the quadratic term $Dq_\Phi(\phi)$. (But it is $\sigma$-equivariant and so $\eta_R$ is defined, as we have used above.) Thus unlike $\ker\delta^L$, $\text{coker}\delta^L$, the spaces $\ker\delta^L$, $\text{coker}\delta^L$ are not necessarily complex vector spaces.

By Proposition 3.4, $\delta_R = \delta_A, \Phi; R$ is certainly a Fredholm operator, which is parameterized by $(A, \Phi) \in C_R$. As usual such a Fredholm family gives rise to the (real) determinant line bundle

$$\text{det}\text{ind}\delta_R = \bigwedge \ker\delta_R \otimes (\bigwedge \text{coker}\delta_R)^*,$$

which descends to the real configuration space $B^*_R$, since the real gauge group $G_R$ action lifts to the bundle. We still denote the descended bundle by $\text{det}\text{ind}\delta_R \to B^*_R$. (In the almost complex surface case, $\delta_R$ is still Fredholm, because $\ker\delta_R \subset \ker\delta$ and $\text{coker}\delta_R \subset \text{coker}\delta$ both are finite dimensional. The second inclusion uses $\text{coker}\delta_R = \ker\delta^*_R$, $\sigma$ is isometric, etc.) The bundle is an extension of $\text{det}\pi_R$ in view of 14.

Since $\pi_R$ is clearly a Fredholm map, by Sard-Smale theorem, the regular values of $\pi_R$ form a dense subset of $(i\Omega^2_\bot)_R$. For each regular value $h$, the corresponding real moduli space $M_R(h) = \pi_R^{-1}(h)$ is a smooth manifold. As in the usual situation, its orientation bundle is the restriction of $\text{det}\text{ind}\delta_R$ to $M_R(h) \subset B^*_R$. However, the orientation of $\text{det}\text{ind}\delta_R \to B^*_R$ is much more complicated in the current real case. Indeed we will see that the bundle is in general non-orientable (i.e. non-trivial).

Let $H^i_R(X, R)$ denote the real De Rham cohomology group with respect to $\sigma$, namely the space of $\sigma$-invariant closed forms modulo $\sigma$-invariant exact forms. While we define the fixed cohomology $H^i(X, R)^\sigma = \text{Fix}(\sigma^*: H^i(X, R) \to H^i(X, R))$. Similarly introduce $H^+_R(X, R)$ and $H^+_+(X, R)^\sigma$. The following is a simple consequence of the classical Hodge theorem, using only that $\sigma$ is isometric.

**Lemma 3.5.** There are natural isomorphisms

$$H^i_R(X, R) \cong H^i(X, R)^\sigma, \quad H^+_R(X, R) \cong H^+_+(X, R)^\sigma.$$
Proof. To show $H^i_R(X, R) \cong H^i(X, R)_{\sigma}$, it is enough to show that the natural inclusion $H^i_R(X, R) \hookrightarrow H^i(X, R)_{\sigma}$ is surjective. Take any fixed class in $H^i(X, R)_{\sigma}$ and represent it by the harmonic $i$-form $\alpha$. Hence $[\alpha] = [\alpha] \in H^i(X, R)_{\sigma}$. Since $\sigma$ preserves the metric on $X$, $\alpha$ is also harmonic. As each class has a unique harmonic representative, one must have $\alpha = \alpha$; hence $[\alpha]$ comes from a class in $H^i_R(X, R)$ that is represented also by $\alpha$.

The second isomorphism can be proved similarly. □

To orient the determinant $\det \ind \delta_R = \det \ind (\delta^L_R \oplus \eta_R) \to \mathcal{C}_R$, as in the usual case, there are two slightly different but equivalent approaches available. One is to deform the fiber of $\det \ind \delta_R$ at a given point $(A, \Phi) \in \mathcal{C}_R$ by deforming the operator $\delta_R = \delta_{A, \Phi; R}$ in a family:

$$\delta_{A, \Phi; R}(t) = \delta^L_{A, R} + t\eta_{A, R}, \quad 0 \leq t \leq 1,$$

so obtaining the deformed fibers $\det \ind \delta_{A, \Phi; R}(t)$ over the same point $(A, \Phi)$. The other is to first deform the point $(A, \Phi)$ itself in a path $(A, t\Phi), 0 \leq t \leq 1$, and then restrict the bundle $\det \ind \delta_R$ to the path, which becomes the bundle $\det \ind \delta_{A, \Phi; R} \to [0, 1]$. The two approaches are interchangeable through the relation

$$\delta_{A, \Phi; R} = \delta^L_{A, R} + \eta_{A, R} = \delta^L_{A, R} + t\eta_{A, R} = \delta_{A, \Phi; R}(t).$$

In the end, both approaches relate $\xi$ with the bundle $\det \ind \delta^L_R \to \mathcal{C}_R$ by taking $t = 0$, where we suppress again the subscript $A$ in the operator family. We shall adapt the second approach in the argument below, which is conceptually more clear.

As before, $\mathcal{A}_R$ denotes the space of real connections on $L$. Let $\mathcal{B}_R^L = \mathcal{A}_R / \mathcal{G}_R$. Note the $\mathcal{G}_R$ action has a stabilizer $\pm 1$ at every point in $\mathcal{A}_R$, hence $\mathcal{G}_R / \pm 1$ acts freely and $\mathcal{B}_R^L = \mathcal{A}_R / (\mathcal{G}_R / \pm 1)$ is a smooth Banach manifold. Clearly the natural forgetting map

$$p : \mathcal{B}_R^* \to \mathcal{B}_R^L, \quad [A, \Phi] \mapsto [A]$$

is a smooth map. The determinant bundle of the real Dirac operators

$$\mathcal{\bar{J}}_{A, R} : \Gamma(S^+)_{R} \to \Gamma(S^-)_{R}$$

parameterized by $A \in \mathcal{A}_R$ obviously descends to $\mathcal{B}_R^L$, which we denote by $\det \ind \mathcal{\bar{J}}_{A, R}$ or simply $\det \ind \mathcal{\bar{J}}_R$.

Theorem 3.6. Fix orientations on $H^1(X, R)_{\sigma}, H^+(X, R)_{\sigma}$. The determinant $\det \ind \delta_R \to \mathcal{B}_R^*$ is isomorphic to the pull-back bundle, $p^* \det \ind \mathcal{\bar{J}}_{A, R}$, via an isomorphism that is unique up to a positive continuous function. In other words, the bundle

$$\det \ind \delta_R \otimes (p^* \det \ind \mathcal{\bar{J}}_{A, R})^{-1} \to \mathcal{B}_R^*$$
Proof. Consider the real full configuration space \( \mathcal{B}_R = \mathcal{C}_R / \mathcal{G}_R \), which is Hausdorff and contains \( \mathcal{B}_L^R \) as a singular submanifold. Clearly one can extend the map \( p \) in (16) to \( \mathcal{B}_R \) as a continuous map, which in turn establishes a homotopy equivalence \( \mathcal{B}_R \cong \mathcal{B}_L^R \) through the standard deformation retraction

\[
\Theta : \mathcal{B}_R \times [0, 1] \to \mathcal{B}_R, \quad ([A, \Phi], t) \mapsto [A, t\Phi].
\]

Modulo the lifted \( \mathcal{G}_R \) action, the determinant bundle \( \xi \to \mathcal{C}_R \) descends to a continuous line bundle \( \text{det ind} \delta_R^L \to \mathcal{B}_R \). Using the retraction \( \Theta \), one sees that \( \text{det ind} \delta_R^L \) is isomorphic to the pull-back of \( \text{det ind} \delta_R^L |_{\mathcal{B}_L^R} = \delta_R^L \), where \( \delta_R^L = \delta_X^R \oplus \partial_{A,R} \) is understood to be parameterized by \( [A] \in \mathcal{B}_L^R \). Indeed the isomorphism can be obtained by deforming the points in \( \mathcal{B}_R \) as follows: take the fibers \( f, f' \) of \( \text{det ind} \delta_R^L \) over an arbitrary point \( [A, \Phi] \) and its projection \( [A, 0] \). Of course the topological line bundle \( \text{det ind} \delta_R^L \) is trivial along the path \( [A, t\Phi], 0 \leq t \leq 1 \). Any trivialization gives rise to an isomorphism between \( f \) and \( f' \). Moreover the isomorphisms obtained through different trivializations differ by positive constants. (So the correspondence of the orientations of \( f, f' \) is independent of the trivializations. Alternatively this orientation correspondence can be obtained through the two connected components of the set \( \text{det ind} \delta_R^L \setminus \{0 - \text{section}\} \) over the path.) Since the path depends continuously on the point \( [A, \Phi] \), one can choose a global bundle isomorphism \( \text{det ind} \delta_R^L \cong p^* \text{det ind} \delta_R^L \) on \( \mathcal{B}_R \), which is unique up to a positive continuous function. In particular the bundle

\[
\text{det ind} \delta_R^L \otimes (p^* \text{det ind} \delta_R^L)^{-1} \to \mathcal{B}_R^*
\]

is orientable with a canonical orientation.

Next we examine the bundle \( \text{det ind} \delta_R^L \to \mathcal{B}_L^R \). Recall

\[
\text{det ind} \delta_R^L = \text{det ind} \delta_X^R \otimes \text{det ind} \partial_{A,R},
\]

and \( \text{det ind} \delta_X^R = \text{det ind}(2d_R^+ \oplus d_R^+) \) is a constant 1-dimensional vector space independent of \( A \). An orientation of \( \text{det ind} \delta_X^R \) is determined by orientations of the cohomology groups of the complex

\[
0 \longrightarrow (i\Omega_0)^R \longrightarrow (i\Omega_1)^R \longrightarrow (i\Omega_2)^R \longrightarrow 0.
\]

Since \( \ker \delta^X, \text{coker} \delta^X \) are complex vector spaces with natural orientations by Proposition 3.4, the orientations of the cohomology groups
of (18) are determined by those of the “imaginary part” complex

\[(19) \quad 0 \longrightarrow (\Omega^0)_R \longrightarrow (\Omega^1)_R \overset{2d_R}{\longrightarrow} (\Omega^2)_R \overset{d_R^+}{\longrightarrow} (\Omega^3)_R \longrightarrow 0.\]

By Lemma 3.3, the cohomology of the last complex is isomorphic to \(H^0(X, \mathbb{R})^\sigma, H^1(X, \mathbb{R})^\sigma, H^+(X, \mathbb{R})^\sigma\). Hence any orientations on \(H^1(X, \mathbb{R})^\sigma, H^+(X, \mathbb{R})^\sigma\) determine the isomorphism \(\text{det} \delta_R^L \cong \text{det} \delta_R^X\). The theorem is proved by coupling with (17) above. \qed

Note that \(\mathcal{B}_R^*\) is not homotopic to \(\mathcal{B}_R^L\), as \(\mathcal{B}_R^*\) and \(\mathcal{B}_R\) are not homotopic. The latter is so in spite that the complement \(\mathcal{B}_R^L\) has infinite codimensions. For example, the generator of \(H^1(\mathcal{B}_R^*, \mathbb{Z}_2)\) that comes from the \(\mathbb{RP}^\infty\) factor according to Proposition 3.1 does not extend over \(\mathcal{B}_R^L\), since it restricts non-trivially on the link of \(\mathcal{B}_R^L\) in \(\mathcal{B}_R\).

Unlike the complex Dirac operator family \(\mathcal{D}_A\), the real family \(\mathcal{D}_{A,R}\) in general produces non-orientable determinant bundle \(\text{det} \text{ind}_{A,R}\).

### 3.3. The real universal bundle.

We consider here more carefully the various universal bundles that are related to our index bundles in the previous subsections. First recall a universal complex line bundle \(\mathcal{L} \to \mathcal{B}^* \times X\) can be defined as the quotient bundle of \(\pi^*L \to \mathcal{C}^* \times X\) under the lifted \(\mathcal{G}\) action, where \(\pi: \mathcal{C}^* \times X \to X\) is the projection on the second factor. One may also define a universal bundle on \(\mathcal{B}^L \times X\) but the construction needs to be modified: the \(\mathcal{G}\) action on the space \(\mathcal{A}\) of connections is not free, so the quotient bundle of \(\pi^*L \to \mathcal{A} \times X\) is undefined. To overcome the problem, one needs to use the based connection space and pull back the universal bundle constructed there. More precisely choose any base point \(x_0 \in X\) and set \(\mathcal{B}_0^L = \mathcal{A}/\mathcal{G}(x_0)\) where the based gauge group \(\mathcal{G}(x_0) = \{g \in \mathcal{G}|g(x_0) = e\}\) acts freely. Thus the above construction yields again a universal bundle \(\mathbb{L} \to \mathcal{B}_0^L \times X\). On the other hand, there is a natural identification \(\mathcal{B}^L = \mathcal{B}_0^L\), through which one has the universal bundle \(\mathbb{L} \to \mathcal{B}^L \times X\) as a carry-over.

It is interesting to observe that the pull-back bundle \(\tilde{\mathbb{L}}\) is not isomorphic to \(\mathcal{L}\), where \(\tilde{\mathcal{L}}: \mathcal{B}^* \times X \to \mathcal{B}^L \times X\) is the forgetting map: \(([A], \Phi), x) \mapsto ([A], x)\). Such a discrepancy originates from the above varied construction of \(\mathbb{L}\). In fact, by construction \(\mathbb{L}\) restricts to a trivial bundle \(\mathbb{L}_{x_0}\) on the slice \(\mathcal{B}_0^L \times \{x_0\}\), hence the pull-back \(\tilde{\mathcal{L}}_{x_0} \to \mathcal{B}^* \times \{x_0\}\) is trivial as well. However the restriction \(\mathcal{L}_{x_0} \to \mathcal{B}^* \times \{x_0\}\) of \(\mathcal{L}\) is non-trivial, since \(\mathcal{L}_{x_0}\) is the quotient of the trivial bundle \(\mathbb{L}_{x_0} \to \mathcal{B}_0^* \times \{x_0\}\) under a free \(S^1\) action. Here \(\mathcal{B}_0^* = \mathcal{C}^*/\mathcal{G}(x_0)\) is the based irreducible configuration space and \(\mathbb{L}' \to \mathcal{B}_0^* \times X\) is the universal bundle, constructed similarly as \(\mathbb{L}\). There is a natural \(S^1\)
action on $\mathcal{B}^*$ with quotient $\mathcal{B}^*$ and the principal circle bundle $\mathcal{B}^*_0 \to \mathcal{B}^*$ (the based point fibration) associates exactly the vector bundle $L$. In other words, $c_1(\mathcal{L}_{x_0})$ is the generator of $H^2(\mathcal{B}^*)$ from the $\mathbb{CP}^\infty$-factor.

The real structure on $\mathcal{B}^* \times X$ lifts to an anti-linear isomorphism on $\mathcal{L}$. It follows that one has a real line bundle on $\mathcal{B}^\sigma \times X_R$ by restricting to the fixed points. Since $\mathcal{B}^*_R \subset \mathcal{B}^\sigma$, a further restriction gives us the anticipated real universal line bundle $\mathcal{L}_R \to \mathcal{B}^*_R \times X_R$.

Using the Stiefel-Whitney class $w_1(\mathcal{L}_R) \in H^1(\mathcal{B}^*_R \times X_R, \mathbb{Z}_2)$ and the slant product one defines a map,

$$\nu = w_1(\mathcal{L}_R) / : H_0(X_R, \mathbb{Z}_2) \to H^1(\mathcal{B}^*_R, \mathbb{Z}_2).$$

This is in addition to the usual map $\mu : H_0(X, \mathbb{Z}) \to H^2(\mathcal{B}^*, \mathbb{Z})$ using the slant product with $c_1(\mathcal{L})$. To make things less mysterious, let $\mathcal{L}_{R,x_0}$ denote the restriction of $\mathcal{L}_R$ to $\mathcal{B}^*_R \times \{x_0\}$ where $x_0 \in X_R$. Then $\nu(x_0) = w_1(\mathcal{L}_{R,x_0})$, much like $\mu(x_0) = c_1(\mathcal{L}_{x_0})$. Clearly the restriction of $\mu(x_0)$ to $\mathcal{B}^*_R$ is the complexification of $\nu(x)$, if they both are viewed as bundles. In the end the classes $\mu(x_0), \nu(x_0)$ both are independent of the point $x_0 \in X_R$, which we will simply call $\mu, \nu$, since they respectively come from the $\mathbb{CP}^\infty, \mathbb{RP}^\infty$ factors of $\mathcal{B}^*, \mathcal{B}^*_R$.

By analogous constructions, one has the universal spin$^c$ bundle

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- \to \mathcal{B}^L \times X$$

with $\det \mathcal{S}^+ = L$. The last bundle carries a tautological connection in the $X$ direction. As a consequence, one obtains the virtual index bundle $\text{ind}\delta_A \in K(\mathcal{B}^L)$ and its real version that was used in [8.2]. The standard Atiyah-Singer family index theorem can be applied to calculate the Chern character $\text{ch}(\text{ind}\delta_A) \in H^*(\mathcal{B}^L)$.

Return to the map $p : \mathcal{B}^*_R \to \mathcal{B}^L_R$ in (16). This is a smooth fibration with fibers $(\Gamma(W^+)_R - \{0\})/\{\pm 1\}$ homotopic to $\mathbb{RP}^\infty$. It will be useful to settle the question whether the determinant bundle $\text{det ind}\delta_R \to \mathcal{B}^*_R$ can be isomorphic to the bundle $\mathcal{L}_{R,x_0} \to \mathcal{B}^*_R$.

**Proposition 3.7.** The bundles $\mathcal{L}_{R,x_0}$ and $\text{det ind}\delta_R$ are never isomorphic. In other words, $\nu \neq w_1(\text{det ind}\delta_R)$.

**Proof.** As we have seen, on each fiber of $p$, the class $\nu = w_1(\mathcal{L}_{R,x_0})$ restricts to the generator of $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$. On the other hand, by Theorem 3.6 $\text{det ind}\delta_R \cong p^* \text{det ind}\delta_{A,R}$ is a pull-back bundle. Hence $\text{det ind}\delta_R$ restricts trivially on fibers of $p$ and can not be isomorphic to $\mathcal{L}_{R,x_0}$ as a result. \qed
4. Real Seiberg-Witten invariants in orientable cases

In 3.1 and 3.2 we introduced the projection \( \pi_R : M_R \to (i\Omega_+^2)_R \) from the parameterized irreducible real moduli space. This is a Fredholm map. So by the Sard-Smale theorem, for a generic perturbation \( h \in (i\Omega_+^2)_R \), the real moduli space \( M_R^*(h) = \pi_R^{-1}(h) \) is a smooth manifold of dimension
\[
m = \frac{1}{8} (c_1(L)^2 - 2e_X - 3s_X).
\]
(See Proposition 3.4.) As in the standard case, the same kind of a priori estimates can be applied to real solution pairs \((A, \Phi) \in B_R\) (simply by restriction) to show that each real moduli \( M_R^*(h) \) is compact, provided that \( h \) stays away from the real reducible wall \( W_R = ic^+ + \text{Im}d^+_R \subset (i\Omega_+^2)_R \).

Here \( c^+ \) is the unique (\( \sigma \) anti-invariant) self dual harmonic 2-form representing \( c_1(L) \) and \( d^+_R : (i\Omega^1)_R \to (i\Omega_+^2)_R \) as before. Note that \( W_R \) is an affine subspace of codimension
\[
b_R^+ := \dim H^+_R(X, i\mathbb{R}) = \dim H^+(X, \mathbb{R})^{-},
\]
where the superscript \( - \) indicates the \( \sigma \) anti-invariant part is used. For our Kähler manifold \( X \) case, one can apply the Hodge decomposition to show that \( b_R^+ = 1 + p_g \), with \( p_g \) the geometric genus of \( X \). Hence \( b_R^+ > 1 \) iff \( b^+ = 1 + 2p_g > 1 \).

Thus we have at least a \( \mathbb{Z}_2 \)-fundamental class \([M_R^*(h)] \in H_m(M, \mathbb{Z}_2)\) for each generic perturbation \( h \notin W_R \). Hence we can make the following definition.

**Definition 4.1.** Suppose \( \sigma \) is a real structure on a Kähler manifold \( X \) and \( \xi = S^+ \oplus S^- \) is a spin\(^c \) structure on \( X \), admitting a real lifting of \( \sigma \). One defines the \( \mathbb{Z}_2 \)-valued real Seiberg-Witten invariant to be the paring
\[
SW_R(\xi) = \langle [M_R^*(h)], \nu \cup \cdots \cup \nu \rangle,
\]
where the cup product is taken \( m \)-times and \( \nu \in H^1(\mathcal{B}^*, \mathbb{Z}_2) \) as before. If \( b_R^+ > 1 \) i.e. \( b^+ > 1 \), then \( SW_R(\xi) \) is independent of \( h \). Otherwise it is well-defined in each of the two chambers of \((i\Omega_+^2)_R - W_R\).

However, in view of the following result, such real Seiberg-Witten invariants are of limited usage in most situations.

**Proposition 4.2.** (i) When \( X \) is of general type and \( b^+ > 1 \), the invariant \( SW_R(\xi) \) is trivial unless \( m = 0 \).

(ii) If \( m = 0 \) (but for any \( X \)), \( SW_R(\xi) \) is the mod 2 reduction of the ordinary Seiberg-Witten invariant \( SW(\xi) \).
Proof. (i) By the standard complex surface theory, when \( m > 0 \), here the corresponding moduli space \( M_R(h) \) is empty with \( h = 0 \). Hence \( SW_R(\xi) = 0 \).

(ii) The main issue is that a generic real perturbation \( h \in (i\Omega^2_+)_R \) may not be generic in \( i\Omega^2_2 \) (namely the equivariant transversality fails). However the virtual neighborhood method can be applied so no generic perturbation is really necessary to compute \( SW(\xi) \). Thus one first uses a generic real perturbation \( h \) to compute \( SW_R(\xi) \). Then one applies a suitable neighborhood of the whole moduli space \( M(h) \) to compute \( SW(\xi) \) (without changing \( h \)). Furthermore, when \( m = 0 \), one can compare the two resulted invariants and prove \( SW_R(\xi) = SW(\xi) \mod 2 \). The precise argument can be carried out essentially in the same way as Ruan-Wang \([11]\). □

Therefore it makes more sense to obtain integer valued real Seiberg-Witten invariants. The orientability and orientation of \( M_R(h) \) now come to play, thus the line bundle \( \det \text{ind}_R \to B^*_R \) must be invoked.

But first, we have seen that the class \( \mu \in H^2(B^*,\mathbb{Z}) \) restricts to a \( 2 \)-torsion in \( H^2(B^*_R,\mathbb{Z}) \), while \( \nu \in H^1(B^*_R,\mathbb{Z}_2) \) simply does not lift to the \( \mathbb{Z} \) coefficients. Thus neither class will be useful in defining integer valued invariants through their pairings with the possible fundamental class \( [M^*_R(h)] \in H^m(B^*_R,\mathbb{Z}) \) \((m > 0)\). In other words, the most likely integer real Seiberg-Witten invariants come from the virtual dimension \( m = 0 \) real moduli spaces, even for any general almost complex manifold \( X \) admitting real structures. (One might use \( H^1(X,\mathbb{Z})^\sigma \) to pair the \( [M^*_R(h)] \), but it is not clear how useful the invariant will be.)

For the rest of the paper, we will consider all spin\(^c\) structures with virtual dimension 0, unless specifically indicated otherwise.

Theorem 4.3. Fix orientations on \( H^1_+(X,\mathbb{R}), H^1_-(X,\mathbb{R}) \). If \( H^1(X,\mathbb{R}) \) is trivial or more generally if the \( \sigma \) anti-invariant part \( H^1(X,\mathbb{R})^- \) is trivial, then the associated real Seiberg-Witten invariant is a well-defined integer, possibly chamberwise when \( b^+ = 1 \).

Proof. Consider the usual reducible wall \( W = ic^+ + \text{Im}d^+ \subset i\Omega^2_+ \), consisting of all perturbations whose Seiberg-Witten equations contain reducible solutions. It is well-known that the map

\[
B^L = A/G \to W, A \mapsto F^+_A
\]

is a trivial fibration with fiber the torus \( T = H^1(X,\mathbb{R})/H^1(X,2\pi i\mathbb{Z}) \), see \([12]\) for example. In particular, \( B^L \) is homotopic to \( T \), since \( W \) is contractible. Similarly the real version says that \( B^L_R \) is homotopic to the fixed torus \( T_R = H^1(X,\mathbb{R})^\sigma/H^1(X,2\pi i\mathbb{Z})^\sigma \). By Lemma 3.5.
dim \mathcal{I}_R = \dim H^0_R(X, i\mathbb{R}) = \dim H^1(X, \mathbb{R})^-. Hence \mathcal{B}_R^L is contractible under the assumption in the theorem. Thus the bundle \det \text{ind}_{\mathcal{I}_R} \rightarrow \mathcal{B}_R^L is trivial and oriented. By Theorem 3.6, \det \text{ind}_{\mathcal{I}_R} is trivial and oriented, based on the orientations of \( H^1_R(X, \mathbb{R}), H^1_R(X, \mathbb{R}) \).

Let \( M^*_R \subset \mathcal{B}_R^* \) be a regular real Seiberg-Witten moduli space associated with a generic real perturbation \( h \in (i\Omega^2_{+})_R \setminus W_R \). By assumption, \( \dim M_R = 0 \). Hence at any point \([A, \Phi] \in \dim M_R, \ker \delta_R = \text{coker} \delta_R = \{0\} \) and the fiber \( \det \text{ind}_{\mathcal{I}_R} \) over \([A, \Phi] \) has a canonical orientation. As in the proof of Theorem 3.6, there is a topological line bundle \( \det \text{ind}_{\mathcal{I}_R} \) over the continuous path \([A, t\Phi], 0 \leq t \leq 1 \) in \( \mathcal{B}_R \). Any trivialization of the bundle yields a unique correspondence between orientations of the fibers over \([A, 0], [A, \Phi] \). Then we define \( \text{sign}(A, \Phi) = 1 \) if the canonical orientation over \([A, \Phi] \) matches the orientation of the fiber \( \det \text{ind}_{\mathcal{I}_R} = \det \text{ind}_{\partial A, \mathbb{R}} \) over \([A, 0] \) in \( \mathcal{B}_R \); otherwise define \( \text{sign}(A, \Phi) = -1 \). Then the real Seiberg-Witten invariant is defined to be the algebraic sum \( \sum \text{sign}(A, \Phi) \) over all points in \( M_R \). That the sum is independent of \( h \) (on each chamber if \( b^+ = 1 \)) follows from the standard cobordism argument, since the sign function \( \text{sign}(A, \Phi) \) is continuous and \( \mathcal{B}_R^L \) is certainly connected.

Note the theorem still holds for any almost complex manifold \( X \) (with \( b^+ = 1 \) replaced by \( b^+_R = 1 \)).

To seek an immediate application of the theorem, we consider a real version of the Thom conjecture. Let \( \Sigma \) be a closed oriented surface with a smooth orientation-reversing involution \( \tau \). One can show that the fixed point set \( \Sigma^\tau \) consists of disjoint circles and \( \Sigma \setminus \Sigma^\tau \) has at most two components, see for example [22]. Set \( k_{\Sigma} \) to be the number of such circles. Call \( \tau \) or \( \Sigma^\tau \) dividing if \( \Sigma \setminus \Sigma^\tau \) has exactly two components. In this case, let \( g^+_\Sigma \) denote the genus of either component.

**Corollary 4.4.** Suppose \( \Sigma \hookrightarrow \mathbb{CP}^2 \) is embedded smoothly and equivariantly with respect to \( \tau \) and the complex conjugation on \( \mathbb{CP}^2 \). Assume \( \tau \) is dividing and \( |\Sigma| \in H_2(\mathbb{CP}^2) \) is also represented by an algebraic curve \( C \) of degree \( d > 2 \). Then \( 2g^+_\Sigma + k_{\Sigma} \geq \frac{(d-1)(d-2)}{2} + 1 \). In addition, if \( C \) is a dividing real curve in \( \mathbb{CP}^2 \) and \( k_{\Sigma} = k_C \) (the number of ovals in \( C_R = C \cap \mathbb{RP}^2 \)), then \( g^+_\Sigma \geq g^+_C \).

**Proof.** This is essentially an adaptation of the Kronheimer-Mrowka argument [5] to our real Seiberg-Witten solutions.

Let \( X = \mathbb{CP}^2 # d^2 \mathbb{CP}^2 \) be a blown-up at \( d^2 \) points in \( \Sigma_R = \Sigma \cap \mathbb{RP}^2 \) and \( \tilde{\Sigma} \) be the internal connected sum with the \( d^2 \) real exceptional spheres \( E_i \). Clearly \( X \) carries a real structure under which \( \tilde{\Sigma} \) is invariant. This real structure has a canonical anti-holomorphic lifting.
on the line bundle $L = 3H - E$, where $H$ is the hyperplane divisor of $\mathbb{CP}^2$ and $E = \sum E_i$. Thus the canonical spin$^c$ structure $S$ on $X$ with determinant $L$ admits a real lifting. By Theorem 4.3 above, the real Seiberg-Witten invariant of $S$ is well-defined on the two chambers. The standard argument from Taubes [16] shows that the real Seiberg-Witten invariant is 1 on the main chamber, since the solution from [16] for a large real perturbation is also real.

Choose an invariant metric on $\tilde{\Sigma}$ with a constant scalar curvature $s_0$. Since the real Seiberg-Witten invariant is non-trivial, by the argument of [5] there is a real Seiberg-Witten solution $(A, \Phi)$ on $X$, satisfying $|F_A| \leq -2\pi s_0$ in a neighborhood of $\tilde{\Sigma} \subset X$. Let $\Sigma^+$ be one component of $\Sigma \setminus \Sigma^*$; similarly define $\tilde{\Sigma}^+$. Because $A$ hence $F_A$ is (anti) invariant under the real structure, we have the following calculation:

$$3d - d^2 = c_1(L)[\tilde{\Sigma}] = 2 \int_{\tilde{\Sigma}^+} \frac{i}{2\pi} F_A,$$

from which we have

$$-(3d - d^2) \leq 2 \int_{\tilde{\Sigma}^+} \frac{1}{2\pi} |F_A| \leq 2 \int_{\tilde{\Sigma}^+} (-s_0) = 2(g_\Sigma^+ + k_\Sigma - 2)$$

where the last equation is the Gauss-Bonnet formula on the surface $\tilde{\Sigma}^+$ with boundary. Note that $\tilde{\Sigma}^+$ is just $\Sigma^+$ connected sum with $d^2$ half disks and $\partial \tilde{\Sigma}^+$ is $\partial \Sigma^+$ connected sum with $d^2$ semi-circles. Hence $g_\Sigma^+ = g_\Sigma$, $k_\Sigma^+ = k_\Sigma$. From the computations above, one arrives at

$$g_\Sigma^+ + k_\Sigma \geq \frac{(d - 1)(d - 2)}{2} + 1. \quad (21)$$

For a real dividing algebraic curve $C \subset \mathbb{CP}^2$, the Euler characteristic satisfies $\chi_C = 2\chi_{C^+}$. In terms of genus, this translates into $2 - 2g_C = 2(2 - 2g_C^+ - k_C)$, which leads to

$$g_C^+ + k_C = g_C + 1 = \frac{(d - 1)(d - 2)}{2} + 1. \quad (22)$$

If $\Sigma$ is confined by $k_\Sigma = k_C$, then the last equation implies $g_\Sigma^+ \geq g_C^+$ in view of (21).

**Remark.** (1) In the case that $\tau$ is non-dividing, the corollary remains to be true if one replaces $g_\Sigma^+$ with the handle number of the quotient surface $\Sigma/\tau$, which is a non-orientable surface with boundary consisting of $k_\tau$ circles.

(2) From (22), one has the Harnack inequality

$$k_C \leq \frac{(d - 1)(d - 2)}{2} + 1$$
which gives the upper bound for the number of ovals in any real algebraic curves $C_R$ of degree $d$. It seems reasonable to conjecture the inequality holds true for any smooth equivariantly embedded surface $\Sigma \subset \mathbb{CP}^2$:

$$k_\Sigma \leq \frac{(d-1)(d-2)}{2} + 1,$$

as long as $[\Sigma] = [C] = dH$.

Without the assumption $H^1(X, \mathbb{R}) = 0$, Theorem 4.3 can be generalized as follows.

**Theorem 4.5.** Fix orientations on $H^1_R(X, \mathbb{R}), H^+_R(X, \mathbb{R})$. If the determinant $L = \det S^+$ of the spin$^c$ structure has its Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ divisible by 4, then the real Seiberg-Witten invariant is a well-defined integer (chamberwise when $b^+ = 1$).

**Proof.** It is enough to show that the line bundle $\det \text{ind} \mathcal{J}_{A,R} \to \mathcal{B}_R^c$ is orientable with a unique orientation. Then by Theorem 3.6, $\det \text{ind} \delta_R \to \mathcal{B}_R^c$ is oriented. Furthermore, the real Seiberg-Witten invariant can be constructed exactly the same way as the proof of Theorem 4.3.

Fix a base connection $A_0 \in \mathcal{B}_R^c$, the fiber of the map (20) over $0 \in W$ is naturally diffeomorphic to $T = H^1(X, i\mathbb{R})/H^1(X, 2\pi i\mathbb{Z})$. Thus we have a complex line bundle $\eta \to T$, using the diffeomorphism to pull back $\det \text{ind} \mathcal{J}_A$. To prove the theorem we need to show that the associated real line bundle $\eta_R \to T_R$ is oriented uniquely.

Since $X$ is Kähler, $H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$. It follows that $H^1(X, i\mathbb{R})$ is naturally isomorphic to $H^{0,1}$ as real vector spaces. This endows a natural complex structure on $H^1(X, i\mathbb{R})$ and hence on $T$. Then $T$ becomes the Picard variety of degree zero holomorphic bundles on $X$. Since $\sigma$ is a real structure on $X$, its induced map on the complex torus $T$ is now a real structure as well. (Indeed $\sigma$ induces a real structure on $H^{0,1}$ as seen before.)

Fix a $\sigma$-compatible complex basis on $H^1(X, i\mathbb{R})$ from that on $H^{1,0}$. The tangent bundle of $T$ is naturally isomorphic to the trivial bundle $T \times H^1(X, i\mathbb{R})$. Hence $T$ carries a natural spin structure that is real compatible with $\sigma$ in the sense of Definition 2.10. In turn this spin structure will determine a canonical square root of $\eta$ if $\eta$ has one. It is a classical fact that square roots of $\eta$ are in one-to-one correspondence with spin structures on $\eta$, for example from [1]. Hence, assuming $\eta$ has a square root, there is a well-defined spin structure on $\eta$, which is real compatible with $\sigma$, because the spin structure on $T$ is so. Applying
the main result in Wang [20], we see that the real line bundle η_\mathbb{R} is orientable with a well-defined orientation.

It remains to show that η has a square root, namely \( c_1(\eta) \in H^2(T, \mathbb{Z}) \) is divisible by 2. Apply the Atiyah-Singer family index theorem to the universal spin^c bundle S on \( T \times X \subset B^L \times X \) from Subsection 3.3. Thus \( ch(\text{ind}A) = \int_X \hat{A}(X)\exp(L/2) \). As in [7, 9], one computes the integral routinely, obtaining

\[
\text{(23)} \quad c_1(\text{ind}A) = \frac{1}{2} \sum_{i<j} < c_1(L)\alpha_i\alpha_j, [X] > \beta_i\beta_j
\]

where \( \{\alpha_i\} \) is any basis of \( H^1(X, \mathbb{Z}) \) and \( \{\beta_i\} \) is the induced dual basis in \( H^1(T, \mathbb{Z}) \). From our assumption 4, \( c_1(\text{ind}A) \) is then divisible by 2, so is \( c_1(\eta) \) and the proof is finished. In fact let us take any complex line bundle \( K \) on \( X \) with \( K^2 = L \). From \( w_2(X) \equiv c_1(L) \equiv 0 \mod 2 \), \( X \) is spin. Since \( 2|c_1(K) \in H^2(X, \mathbb{Z}) \), \( c_1(K) \) is a characteristic element. Thus there is a spin^c structure on \( X \) with determinant \( K \). Repeating the above argument for this new spin^c structure, one sees the analogy of formula (23) implies that

\[
\frac{1}{2} \sum_{i<j} < c_1(K)\alpha_i\alpha_j, [X] > \beta_i\beta_j
\]

are all integers for any \( i < j \). It follows that \( c_1(\text{ind}A) \) is an even class. □

Note that for the theorem, it is not enough to assume only \( 2|c_1(L) \), because then the bundle \( K \) in the last part of the proof will not be characteristic and \( c_1(\text{ind}A) \) may not be divisible by 2.

Theorem 4.5 can be extended to symplectic manifolds such that \( b_1(X) \) is even.

To give some examples with the Chern class \( c_1(X) \) divisible by 4, we can take a product of Riemann surfaces, \( \Sigma_g \times \Sigma_h \), with odd genera \( g, h \). Here both \( \Sigma_g \) and \( \Sigma_h \) carry real structures. If \( c_1(X) \) is divisible by 4, we can get additional examples by taking any branched cover of \( X \) along a branched locus \( C \subset X \) that is preserved by the real structure and such that \( 4|[C] \).

5. Seiberg-Witten projection maps

In the initial part of the section we work with the most general real set-up, assuming only that \( (X, \sigma) \) is any smooth 4-manifold with involution and \( P_{sp} \) is a spin^c structure that is endowed with a real compatible lifting of \( \sigma \) in the sense of Definition 2.10. Then the Seiberg-Witten equations inherit a real structure as in Proposition 2.8. So far we have studied the real Seiberg-Witten moduli spaces by studying the ambient configuration space \( B^*_\mathbb{R} \), with the approach parallel to the standard theory. In this section, we will shift our focus and investigate the moduli spaces directly without going over \( B^*_\mathbb{R} \). More
precisely let $Q = i\Omega^2_+ \setminus W, Q_R = (i\Omega^2_+)_R \setminus W_R$ denote the complements of the (real) reducible walls. Then we will analyze systematically the Seiberg-Witten projection and its real version:

$$\pi : M \to Q, \pi_R : M_R \to Q_R,$$

where $M, M_R$ are the parameterized (irreducible) moduli spaces. Note that both projections are proper smooth Fredholm maps by the usual compactness theorem. (In comparison, the full projection $M \to i\Omega^2_+$ is only a continuous proper map, while the restriction to the irreducible ones $M^* \to i\Omega^2_+$ is smooth but not proper.)

5.1. The structures of critical points and critical values. In this subsection, we can actually consider an arbitrary Fredholm index of $\pi$, i.e. the virtual dimension $\text{ind}\delta$ of the moduli space is any integer. In fact, a point of our approach is to extract possibly additional information from $\pi$ or $\pi_R$ in the case of a negative virtual index where the usual Seiberg-Witten invariant fails to yield any information. Compare with Shevchishin [13] where the moduli space of pseudo-holomorphic curves was studied.

First we consider the general situation. Let $E, F$ be Banach bundles over $M$, and $\ell : E \to F$ be a Fredholm bundle homomorphism of constant index $m = \text{ind}\ell, x \in M$. Then using connections on $E, F$, one can define a pointwise linear map

$$\nabla \ell : T_x M \to \operatorname{Hom}(\ker \ell_x, \operatorname{coker} \ell_x)$$

for each $x \in M$, which is actually independent of the connections chosen, see Lemma 1.3.1 of Shevchishin [13]. The following basic result is used on page 50 of [13] without proof:

**Lemma 5.1.** Let $C(l) = \{ x \in M \mid \dim \operatorname{coker} \ell_x = l \}$ for a fixed integer $l \geq 0$. If $\nabla \ell_x$ is surjective for all $x \in C(l)$, then $C(l) \subset M$ is a submanifold of codimension $(m + l)l$.

**Proof.** We sketch for the case where $E = M \times U, F = M \times V$ are trivial product bundles, which is what we require in our applications. The general case can be dealt with using suitable modifications.

Consider the Banach space $\text{Fred}(U, V)_m$ of all Fredholm operators of index $m$. The subset

$$W = \{ g \in \text{Fred}(U, V)_m \mid \dim \operatorname{coker} g = l \}$$

is a submanifold of codimension $(m + l)l$. The map $\ell$ becomes $M \to \text{Fred}(U, V)_m$ and $\nabla \ell = p \circ d\ell \circ i$, where $i : \ker \ell_x \hookrightarrow U, p : V \to \operatorname{coker} \ell_x$. 
One may check that $\ell$ is transversal to $W$ iff $\nabla \ell$ is surjective on $N(\ell)$, by noting that the tangent space of $W$ is

$$T_g W = \{ h \in Hom(U, V) \mid h \text{ maps } \ker g \text{ to } \im g \}.$$  

It follows then from the usual transversality theorem that $C(l) = (\ell^{-1}(W))$ is a submanifold of codimension $(m + l)l$.  

**Corollary 5.2.** If $\nabla \ell$ is always surjective at any point $x \in M$, then $M$ is stratified by submanifolds $C(l), l = 0, 1, \ldots$.

Return to our parameterized Seiberg-Witten moduli space $M$ and the projection into the perturbation space $\pi : M \to Q$ as in Subsection 2.1. Let $C$ denote the critical point set of $\pi$ and $C(l) = \{ x \in M \mid \dim \coker D\pi_x = l \}$.

**Theorem 5.3.** For each $l = 0, 1, 2, \ldots$, $C(l) \subset M$ is a Banach submanifold of codimension $kl$, where $k = \ind D\pi + l$.

**Proof.** From Corollary 2.2, it is the same to show that $C(l) = \{ x \in M \mid \dim \coker \delta_x = l \}$ is a codimension $kl$ submanifold of $M$. We can of course view $\delta : M \to \text{Fred}(U, V)$, where $U = i\Omega^1 \oplus \Gamma(S^+), V = i\Omega^0 \oplus i\Omega_+^2 \oplus \Gamma(S^-)$ (The suitable Sobolev spaces are suppressed without harm). To apply Lemma 5.1, we need to show $\nabla \delta_x$ is surjective.

Let us compute the differential $d\delta_x : T_x M \to Hom(U, V)$. Take a point $x = (A, \Phi, h) \in C(l)$, a tangent vector $\xi = (a, \phi, k) \in T_x M$ and $(a', \phi') \in U$. Then we have in $V$ that:

$$d\delta_x(\xi)(a', \phi') = (i < \phi, \phi' >, Dq_\phi(\phi'), 2^{-1}a' \cdot \phi + 2^{-1}a \cdot \phi').$$

Consider $\nabla \delta_x : T_x M \to Hom(\ker \delta_x, \coker \delta_x)$, with $\nabla \delta_x(\xi)$ equal to the composition

$$d\delta_x(\xi)(a', \phi') \to \ker \delta_x \hookrightarrow U \to V \to \coker \delta_x.$$  

We need to show that by choosing $\xi$ suitably, $\nabla \delta_x(\xi)$ can realize all linear functions $f(a', \phi')$ from $\ker \delta_x$ to $\coker \delta_x$. Note that each of the three components of $d\delta_x$ from (24) is non-degenerate bilinear in the two sets of variables $\{a, \phi\}$ and $\{a', \phi'\}$. Hence each component can realize all linear functions of one set of variables $\{a', \phi'\}$ when the other set $\{a, \phi\}$ is suitably chosen. Of course this does not mean that all three components can simultaneously realize arbitrarily given three functions. However, after composing with the projection map $p$, only two components are actually independent. Moreover, when we restrict
to \( \ker \delta \), the two variables \( a', \phi' \) are not independent either. Therefore, essentially just one independent variable from the set \( \{ a, \phi \} \) is needed in order for the composition (25) to realize all linear functions \( f \) as indicated above. It would seem that we have a redundant variable from \( \{ a, \phi \} \), but remember \( \xi = (a, \phi, k) \in T_x M \) must satisfy two equations

\[
D_x^2(a, \phi) = (k, 0), (D_x^1(a, \phi))^* = 0
\]

according to Proposition (2.1). So actually we only have one essentially independent variable available from \( \xi \), and this is good enough here. □

Remark. Even when \( X \) is a Kähler manifold, \( \ker D\pi, \coker D\pi \) may be of odd dimensions at a non-trivial perturbation \( h \).

Next we take up the set up with a real structure, so we have the real Fredholm map \( \pi_R : M_R \to Q_R \). Let \( C_R \) be the critical point set of \( \pi_R \) and \( C_R(l) \) be the subset of points at which \( \dim \coker D\pi_R = l \). Thus \( C_R(0) \) is the set of regular points of \( \pi_R \). The real version of Theorem 5.3 holds under the same proof:

**Theorem 5.4.** For each \( l \), \( C_R(l) \subset M_R \) is a Banach submanifold of co-dimension \( l(\text{ind} D\pi_R + l) \).

In particular, when the virtual dimension \( \text{ind} D\pi_R = 0 \), the subset \( C_R(1) \) is a co-dimension 1 submanifold in \( M_R \).

5.2. Degree of Seiberg-Witten projection map. In this subsection, we study the projection \( \pi_R \) from a functional analytic point of view. Suppose in general that \( f : M \to N \) is a proper smooth Fredholm map of index 0 between two Banach manifolds. In order to define an integer degree of \( f \), the most natural approach is to impose certain oriented manifold structures on \( M, N \) and require \( f \) to preserve these structures. The only subtlety here is that the general linear group \( GL(E) \) of an \( \infty \)-dimensional Hilbert space \( E \) is contractible, thus connected, by a classical result of Kuiper. Hence, one needs to reduce the structure group of \( TM, TN \) to the smaller subgroup \( GL_c(E) \) of compact linear isomorphisms which has two connected components, so that the orientability may be imposed. This was the approach initiated by K.D. Elworthy and A.J. Tromba in the 1970s.

More recently, Fitzpatrick, Pejsachowicz, and Rabier [4] realized that the orientability of \( M, N \) is often un-natural to impose and not necessary either for the sole purpose of defining a degree for \( f \). Instead, all needed is the orientability of the map \( f \) itself. In [4], they introduced the parity of \( f \) along a path with two ends at regular points of \( f \). This is a functional analytic concept which involves parametrices and the
Leray-Schauder mod-2 degree. Then \( f \) is called \textit{orientable} if the parity is always 1 along any loop.

On the other hand, the geometric point of view is to characterize the orientability of \( f \) as that of the determinant line bundle
\[
\text{det } f = \wedge^{\max} \ker Df \otimes (\wedge^{\max} \text{coker } Df)^* 
\]
over \( M \) using the Fréchet derivative \( Df : TM \to f^*TN \). It is proved in \cite{19} that the two kinds of orientability mentioned above are actually equivalent. Namely, \( \text{det } f \) is a trivial line bundle iff \( f \) is orientable in the sense of \cite{4}. Let \( C_f \subset M \) denote the set of critical points where \( \text{coker } Df \) is 1-dimensional and \( R_f \) the set of regular points of \( f \). Then the equivalence in turn leads to the following (see \cite{19}):

\textbf{Proposition 5.5.} Suppose that \( C_f \) is a co-dimension 1 submanifold of \( M \) and \( R_f \neq \emptyset \). Then the line bundle \( \text{det } f \) is trivial iff there is a continuous sign function \( \epsilon : R_f \to \{ \pm 1 \} \), such that for any path \( \gamma \subset M \) with both ends in \( R_f \) and transversal to \( C_f \), the sign \( \epsilon \) will change whenever \( \gamma \) crosses \( C_f \).

Naturally the parity of \( f \) along a path between two regular points can now be determined by \( \epsilon \). Each \( \epsilon \) is called an \textit{orientation} of \( f \) in \cite{4}. By \cite{19}, this corresponds canonically to an orientation of \( \text{det } f \). Proposition 5.5 gives a convenient criterion for the orientability and orientation of \( \text{det } f \) in terms of signs at regular points only.

From here on we understand that \( f \) is \textit{oriented} if \( f \) carries a sign function \( \epsilon \) as in Proposition 5.5. Then the integer degree is defined to be
\[
\deg f = \sum_{x \in f^{-1}(y)} \epsilon(x),
\]
where \( y \in N \) is a regular value.

Recall from \cite{4} that an oriented homotopy is a smooth Fredholm map \( H : M \times [0,1] \to N \) that carries an orientation. Using determinant bundles, it is easy to see that a homotopy \( H \) is orientable (oriented) iff some section \( H_t : M \times \{ t \} \to N \) is orientable (oriented respectively). Note that \( \text{det } f \) is not exactly homotopy invariant in the usual sense; instead we should utilize the following (see \cite{4}):

\textbf{Proposition 5.6.} Suppose \( f \) is an oriented Fredholm map of index zero.

\textbf{(Homotopy Invariance)} The degree \( \deg f \) is invariant under any proper and oriented homotopy \( H \). Hence the absolute value \( | \deg f | \) is homotopic invariant regardless of orientation.

\textbf{(Reduction)} If \( P \subset N \) is a submanifold transversal to \( f \), then the restriction \( f|_P : f^{-1}(P) \to P \) is a Fredholm map with an induced
orientation. Moreover, \( R_f \cap f^{-1}(P) \) gives all regular points of \( f|_P \) and consequently \( \deg f = \deg f|_P \).

We now return to our Seiberg-Witten projection \( \pi_R : M_R \to Q_R \), assuming the virtual dimension is zero. The main point is that \( \pi_R : M_R \to Q_R \) can be orientable, although \( \det \text{ind} \delta_R \to B_R \) may well be non-trivial, making Section 4 inapplicable. (This is in analogy with [21] where only rational curves are given suitable signs). In other words we can expand the definition from Section 4:

**Definition 5.7.** When \( \pi_R \) is oriented, the real Seiberg-Witten invariant \( \text{SW}_R(P_{sp}) \) is defined to be the degree of \( \pi_R \).

By Proposition 5.6 with fixed orientations on \( H^1_R(X, \mathbb{R}) \) and \( H^+_R(X, \mathbb{R}) \), \( \text{SW}_R(P_{sp}) \) is independent of metrics on \( X \). Without fixing the orientations, the absolute value \( |\text{SW}_R(P_{sp})| \) is still well-defined.

To detect the orientability, from Theorem 5.4 and Proposition 5.5, it is enough to give a continuous sign assignment \( \epsilon \) at the regular points of \( \pi_R \) such that \( \epsilon \) changes whenever crossing the submanifold \( C_R(1) \).

In general it is still rather difficult to find a suitable \( \epsilon \). Nonetheless one immediate result within reach is a real blow up formula, which we describe next. Let \( \hat{X} = X \# \mathbb{CP}^2 \) be the blow-up of \( X \) at a real point. Then \( \sigma \) extends smoothly over \( \hat{X} \) as an involution, which further lifts to the spin\(^c\) bundle \( \hat{P}_{sp} \) on \( \hat{X} \). Let \( \hat{\pi}_R : \hat{M}_R \to \hat{Q}_R \) be the real Seiberg-Witten projection in the spin\(^c\) structure on \( \hat{X} \). Here is the real version of the usual blow up formula, the counter part of which is much harder to prove for real rational curves in [21].

**Theorem 5.8.** If \( \pi_R : M_R \to Q_R \) is an orientable Fredholm map, then so is \( \hat{\pi}_R : \hat{M}_R \to \hat{Q}_R \). Moreover an orientation of \( \pi_R \) induces one for \( \hat{\pi}_R \) and the real Seiberg-Witten invariant remains the same: \( \text{SW}_R(P_{sp}) = \text{SW}_R(\hat{P}_{sp}) \).

**Proof.** One just needs to make sure that the usual proof can be carried out equivariantly with respect to our real structures. Let \( S^2 = \mathbb{CP}^1 \) be given the standard complex conjugation. The degree \(-1\) line bundle on \( S^2 \) has a natural real lifting, which preserves the standard Hermitian fiber metric. Thus the disk bundle \( N \) inherits the real structure, which of course is the restriction of the complex conjugation to the neighborhood of \( \mathbb{CP}^1 \subset \mathbb{CP}^2 \). Attach a long cylinder \([1, r] \times S^3 \) to the boundary \( \partial N = S^3 \) and let \( N_r \) denote the resulted manifold with the extended real structure.

Fix a small 4-disk \( D \subset X \) at the real blow up point in \( X \) and let \( D_r \) denote manifold with a long cylinder attached. Attach this cylinder as
well as the infinite cylinder $[1, \infty) \times S^3$ to the boundary $S^3 = \partial (X \setminus D)$ so we get two more manifolds $X_r, X_\infty$. Note that $D_r, X_r, X_\infty$ all inherit real structures from $X$

Fix a large enough $r$ and diffeomorphisms $X \approx X_r \cup D_r, \hat{X} \approx X_r \cup N_r$ (glue the long necks together). Without loss of generality we assume all perturbations on $X$ and $\hat{X}$ have compact supports on $X_r$, namely they are trivial near the blow up point and the exceptional curve $S^2$ respectively. Thus we have identified the (real) perturbation spaces $Q_R \approx \hat{Q}_R$. Via the standard gluing process, every irreducible Seiberg-Witten solution on $X$ and on $\hat{X}$ both correspond to a unique finite energy solution on $X_\infty$. Thus we have the usual diffeomorphism $M \approx \hat{M}$ between the parameterized moduli spaces. The gluing between $X_r$ and $D_r, N_r$ requires the use of a canonically defined reducible solution $(A_0, 0)$ in the spin$^c$ structure over $N$. One checks easily that this solution is equivariant with respect to the real structure. Thus one has a diffeomorphism $M_R \approx \hat{M}_R$ by restriction. Since $\hat{\pi}_R, \tilde{\pi}_R$ commute under the previous diffeomorphisms $M_R \approx \hat{M}_R, Q_R \approx \hat{Q}_R$, the orientability of one certainly implies that of the other.

In particular if the spin$^c$ structure $P_{sp}$ satisfies the condition in Theorem 4.5 then the blow up spin$^c$ structure $\hat{P}_{sp}$ on $X \# CP^2$ has an integer real Seiberg-Witten invariant. Note the determinant of $\hat{P}_{sp}$ is no longer divisible by 4; thus Theorem 4.5 is not applicable. Some of the standard applications of the blow up formula can be readily extended to our real case.

Another observation to make is about reductions. Assume here that $(X, \omega)$ is a symplectic real 4-manifold. Recall $Q_R \subset (i\Omega^0)_{R, -\omega} \oplus (i\Omega^{0, 2})_{R}$. Let $\pi^1_R, \pi^2_R$ be respectively the compositions of $\pi_R$ with the projections onto the two factors. It is straightforward to show the following

**Proposition 5.9.** The map $\pi_R$ is orientable (oriented) iff $\pi^1_R, \pi^2_R$ are orientable (oriented respectively).

The Fredholm map $\pi^1_R : M_R \rightarrow (i\Omega^0)_{R, -\omega}$ involves the generalized Taubes perturbation [16], whereas $\pi^2_R : M_R \rightarrow (i\Omega^{0, 2})_{R}$ involves a Witten type perturbation [23]. With either perturbation, the Seiberg-Witten equations can be decomposed nicely, and the orientability can be sorted out in special situations.

5.3. **The non-orientable case: chamberwise invariants.** The setup here is an almost complex 4-manifold $(X, J)$ together with a real structure. It is a basic fact that $J$ maps isomorphically the tangent space $TX_R$ to the normal space of $X_R$. Applying the same to the domain and range of $\delta = \delta_{A, \Phi}$ and in view of Corollary 3.3, we see
easily that regular points of $\pi_R$ are real regular points of $\pi$. Thus we can restrict the usual orientation of $\pi$ to $\pi_R$. Namely the orientation of the standard Seiberg-Witten theory gives the continuous sign map $\epsilon_c$ for $\pi$, and by restriction we obtain a sign map $\epsilon$ defined at the regular points of $\pi_R$. (Note that $D\pi$ is not complex linear, hence not all signs of $\epsilon_c$ are positive.) Then one can apply the criterion in Proposition 5.5 together with Theorem 5.4 and seek to determine when $\epsilon$ is in fact an orientation for $\pi_R$. We believe this should work for a class of real almost complex 4-manifolds that include cases in Theorems 4.3 and 4.5, although we have not checked the details. (The last claim is essentially in view of the deformation of $\delta_R$ to the linear operator $\delta^L_R$.)

What we are interested more is about the opposite case that the sign map $\epsilon$ is not an orientation for $\pi_R$, as it will bring up new geometry to study. Specifically let $Z, T$ be respectively the sets of regular values and critical values of $\pi_R$. Since $\pi_R$ is proper, $Z$ is open and dense in $Q_R$ by the Sard-Smale theorem. Call connected components of $Z$ the chambers, which are divided by the wall $T$. Take any regular value $h \in Z$, we can count the signed points in $\pi_R^{-1}(h)$ using our map $\epsilon$. Note that $\pi_R^{-1}(Z)$ is generally a proper subset of the regular point set $C_R(0)$ of $\pi_R$, so we could require the map $\epsilon$ be defined in a smaller set than $C_R(0)$. Obviously the resulted number is independent of regular values in the same chamber. Hence it makes sense to define the chamberwise real Seiberg-Witten invariant for a real almost complex 4-manifold. For example, in the Taubes chamber that contains $i\omega, r$ a large constant, the real Seiberg-Witten invariant takes value $\pm 1$, since the only (regular) Seiberg-Witten solution from Taubes’ argument [16] is also real.

In the non-orientable $\epsilon$ case, the real Seiberg-Witten invariant will vary from chamber to chamber. The pattern and distribution of the invariant then become the new geometry to investigate. The essential issue is to give a “wall crossing formula” that describes the change between two neighboring chambers in $Z$. More precisely take any path $\Gamma = \{\gamma(t) \in Q_R, -1 \leq t \leq 1\}$ that is transversal to $\pi_R$, so that $\Gamma' = \pi_R^{-1}(\Gamma)$ is a submanifold consisting of finitely many arcs. Suppose all points, except $\gamma(0)$, in $\Gamma$ are regular values and $\gamma(-1), \gamma(1)$ belong to different chambers. We need to examine the restriction $\tilde{\pi} : \Gamma' \rightarrow \Gamma$ of $\pi_R$. Since $\pi_R$ is transversal to the 1-dimensional $\Gamma$, $\dim \ker D\pi_R$ is at most 1 at any point in $\Gamma'$. Hence at any critical point of $\tilde{\pi}$, $\dim \ker D\pi_R$ is exactly 1. The converse is also true; therefore the critical point set of $\tilde{\pi}$ equals $C_R(1) \cap \Gamma'$. 

Set \( h_{\pm} = \gamma(\pm 1), q = \gamma(0) \). The pre-image points \( \tilde{\pi}^{-1}(h_{\pm}) \) all carry signs according to \( \epsilon \). To describe the invariant change between the two chambers means to compare the two sets of signs here. Take an arc component \( \eta \) of \( \Gamma' \). Along \( \eta \), the only possible critical point of \( \tilde{\pi} \) is \( p \in \eta \cap \tilde{\pi}^{-1}(q) \). If \( p \) is not a critical point, of course the two ends of \( \eta \) should have the same sign by continuity of \( \epsilon \). Otherwise we can determine its type:

**Proposition 5.10.** If \( p \) is a critical point of \( \tilde{\pi} \) along \( \eta \), then \( p \) is a non-saddle point. Namely \( \tilde{\pi} \) has either a local maximum or a local minimum at \( p \), under suitable re-parameterizations of \( \eta \) and \( \Gamma \).

**Proof.** Here we adapt a Kuranishi type argument of a finite dimensional reduction (which also reflects how the Leray-Schauder mod-2 degree is defined). In essence, this is due to the fact that the only non-linear part of the Seiberg-Witten equations is the quadratic term \( q(\Phi) \).

Let \( L = D\pi_R(p) : T_pM_R \rightarrow T_qQ_R \) be the differential at \( p \). Up to diffeomorphisms and locally around \( p \in M_R \), we can decompose

\[
\pi_R(u, v) = (Lu, \psi(u, v)) \in \text{im}L \times \text{coker}L,
\]

where \( (u, v) \in \ker L^\perp \times \ker L \) and \( \psi \) is a function with \( D\psi(0, 0) = 0 \). Recall the critical point \( p \in C_R(1) \cap \Gamma' \), meaning that \( \text{coker}L \) and hence \( \ker L \) are both 1-dimensional spaces. Certainly \( \psi \) depends on the various choices made. But as in the original Donaldson theory, the quadratic part of the restriction \( f(v) = \psi(0, v) : \ker L \rightarrow \text{coker}L \) is intrinsic, namely after re-parameterizations \( \ker L \approx \mathbb{R}, \text{coker}L \approx \mathbb{R} \), we always have

\[
f(v) = \pm v^2 + O(v^2).
\]

On the other hand, our spaces \( \eta, \Gamma \) are also 1-dimensional. Applying the Implicit Function Theorem if necessary and in view of (26), we can assume that locally \( \eta = \ker L, \Gamma = \text{coker}L \) and \( \tilde{\pi} = f \). By (27), \( \tilde{\pi} \) has a local maximum or minimum at \( p = 0 \).

It follows from Proposition 5.10 that \( \epsilon \) is an orientation iff every such \( \eta \) must have opposite signs at its two ends. Other than Proposition 5.10, we have not yet determined any precise wall crossing formula but conjecture that the invariant change should be independent of neighboring chambers.

**Appendix: Real classes and classes of real points**

We lay down the following useful algebraic set up once for all. It has scattered widely in the literature that deals with real structures.
Assume that $C$ is a set and $\sigma : C \to C$ an involution. Write $\overline{x} = \sigma(x)$ for convenience, where $x \in C$. Analogously for a group $G$, let $\sigma' : G \to G, g \mapsto \overline{g}$ be an involution such that

$$\overline{gh} = \overline{g}\overline{h}, \overline{1} = 1, \text{ for } g, h \in G.$$ (Namely $\sigma'$ is a group homomorphism.) Suppose $G$ acts freely on $C$ and the involution actions are compatible in the sense that $\sigma g x = g \sigma x$ for $g, h \in G, x \in C$.

From this, $\sigma$ and $\sigma'$ induce an involution $\sigma_* : B \to B$, the quotient set $B = C/G$.

We need to introduce additional sets. If $\sigma, \sigma', \sigma_*$ are viewed as real structures, then the set of real classes should be $B_{\sigma_*} := \text{Fix}(\sigma_* : B \to B) \subset B$, while the set of classes of real points should be the quotient $B_R := C_R/G_R = \text{Fix} \sigma/\text{Fix} \sigma'$.

There is a natural inclusion $B_R \hookrightarrow B_{\sigma_*}$. The main purpose of the Appendix is to generalize the set $B_R$ as well as the inclusion.

Define a subgroup $U = \{g \in G : \overline{gg} = 1\}$ of $G$ and its quotient $\tilde{U} = U/\sim$, where $g \sim \overline{h}g^{-1}h$ for some $h \in G$. Any $g \in U$ yields involutions $\sigma_g : C \to C, x \mapsto \overline{gx}$ and $\sigma'_g : G \to G, h \mapsto \overline{g^{-1}hg}$ which are compatible in the above sense. (Note that all elements in $\tilde{U}$ have order 2.) One can view $\sigma_g, \sigma'_g$ as shifted real structures by $g$. With these new real structures, we introduce the set

$$B_g = \text{Fix} \sigma_g/\text{Fix} \sigma'_g,$$

generalizing that $B_1 = B_R$. Given a class $\xi \in \tilde{U}$, we introduce a subset of $B$:

$$B_\xi = \{[x] \in B : \overline{x} = gx \text{ for some } g \in \xi\}.$$ With the right set up at hands, one can verify easily the following statements.

**Proposition.** (i) The subsets $B_\xi$, with $\xi \in \tilde{U}$, are mutually disjoint.

(ii) Clearly each $B_\xi \subset B_{\sigma_*}$; moreover there is a natural decomposition:

$$B_{\sigma_*} = \coprod_{\xi \in \tilde{U}} B_\xi = \coprod_{[\xi], \xi \neq [1]} B_\xi.$$

(iii) There is a natural bijection $B_R \to B_{[1]}, [x] \mapsto [x]$. In particular, we have an inclusion $B_R \hookrightarrow B_{\sigma_*}$. More generally, we have a natural
bijection $B_g \to B^[[g]], [x]_g \mapsto [x]$, where $g \in U, [g] \in \tilde{U}$. Thus we can rephrase the previous decomposition as

$$B^\sigma = \bigcoprod_{[g] \in \tilde{U}} B_g = B_R \bigcoprod ( \bigcoprod_{[g] \neq [1]} B_g ).$$

In topological applications, one usually expects that $\bigcoprod_{[g] \neq [1]} B_g$ constitutes a small subset of $B^\sigma$ relative to $B_R$. The proportion has been applied to the real and fixed configuration spaces $B_R, B^\sigma$ in Section 3, in which $G$ acts freely on $C^\ast$.

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