A trajectory of a harmonic oscillator obeying the Schrödinger wave equation is exactly derived and illustrated. The trajectory resembles well the classical orbit between the turning points, and also runs through the tunneling region. The dynamics of the ‘particle’ motion and the wave function associated with the motion are proposed. The period of a round trip on the trajectory is exactly equal to that obtained in classical mechanics.

1 Introduction

In general the motion of a particle whose beam shows an interference phenomenon is described by the wave equation. On the other hand the orbit of the particle described in classical mechanics plays an important role in the fields such as electron optics, particle accelerators, radiation from electron beams and so forth. The significance of the scattering wave function, wave-optical approach, was clarified by comparison with the classical wave function, ray-optical approach, by Gordon. [1] Geometrical optics describes the movement of the light corpuscle in the space as the ray of light. Formulas for the ray of light derive from the eikonal equation. This corresponds to the Hamilton characteristic function for the particle motion, [2] which can be derived from the Schrödinger wave function in the WKB approximation for the system with the stationary potential.

The concept of the orbit or the ray of light in the wave phenomena gives a comprehensive image and physical insight of the process, although it has been an approximate idea. [3, 4, 5] The field ion microscope and scanning tunneling microscope are now available to image an atomic movement on solid surfaces. [6, 7] An atom can be picked up and moved to an arbitrary place. [8] These suggest the possibility of the precise description of the motion of the particle even in the atomic scale.
A proposal of extension of the light ray to the shadow region had been made in order to clarify the physics of the diffraction. The relation between the wave and the light ray or orbit should be investigated more carefully. Trajectory in the optical wave, extension of a ray of light, was developed by generalizing the eikonal to the mode characteristic function with dynamical assumptions.

If the motion of the particle is restricted to that in the classical region where classical mechanics is the case, the orbital motion is derived from the Hamilton characteristic function by the Hamilton-Jacobi theory for the system with the stationary potential. If the characteristic function could be generalized to that valid for every space region, it would be able to pinpoint the motion of the particle even in the tunneling region.

Along the similar way of thinking, Bohm proposed a quantum theory with “hidden variables” to suggest objective description of individual systems at a quantum level of accuracy. It has made a general scheme of the causal interpretation but not given a concrete trajectory with the dynamical behavior of a ‘particle’ in the space-time region from the wave equation. The goal of every causal theory or interpretation as to the quantum mechanics is a complete description of an individual real situation as it exists independently of acts of observation. The motivation is summarized in the Einstein’s feeling that the statistical prediction of the quantum theory is correct but by supplying the missing elements, it could be in principle got beyond statistics to a determinate theory. To fulfill the aim it might be necessary to exploit the suitable mathematical tools to describe the particle motion in quantum mechanics.

In the present paper, a trajectory of the harmonic oscillator in one and two dimensions is exactly derived from the Schrödinger wave function. The trajectory and dynamical motion are compared with the classical orbit and dynamical behavior in the classical region. The relation between the traveling waves associated with the ‘particle’ motion and usual stationary wave function is discussed.

2 Dynamics and wave function

The way to derive the trajectory from the wave equation is described and the significance of the wave function is discussed. The Schrödinger wave equation for a particle in a stationary potential, \( V \), in one spatial dimension, is

\[
i \hbar \frac{\partial}{\partial t} \Psi = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \right) \Psi \equiv H \Psi.
\] (1)

The equation is separable in \( t \) and \( x \). A wave function for a ‘stationary’ state is written as

\[
\Psi(x, t) = \exp(-iEt/\hbar)\Phi(x, E),
\] (2)

where \( E \) is a constant of separation. Function \( \Phi \) is a general solution of an ordinary differential equation of second order. The boundary condition that the function be bounded and continuous everywhere in the domain defined is not imposed on the function yet. Function \( \Phi \) is the ‘eigenfunction’ of the Hamiltonian operator \( H \). Constant \( E \) is the ‘eigenvalue’ and the energy of the ‘stationary’ state. The wave function \( \Psi \) determines the ‘eigenstate’, or the ‘mode’, specified by \( E \). The constant \( E \) should be called the mode parameter.
If the parameter takes specific, discrete, values, the wave function satisfies the boundary condition. The state with a specific parameter is usually called a stationary state or the mode, and function \( \Phi \) is the eigenfunction. For the present, the parameter is assumed to be a real number and the words, such as stationary, eigen, mode and particle are used with quotation marks.

The wave function of the form of

\[
\Phi(x, E) = \sqrt{\rho(x, E)} \exp[i W(x, E)]
\]  

(3)
is sought, where functions \( \rho \) and \( W \) are real number. Function \( W \) satisfies approximately

\[
E = \frac{1}{2m} \left( \hbar \frac{\partial W}{\partial x} \right)^2 + V
\]

(4)
in the WKB approximation. The region where this equation holds should be called the classical region. This is the Hamilton-Jacobi equation with energy \( E \). If the solution for \( \hbar W \) of this equation is written as \( W_{cl} \), the Hamilton characteristic function is given by

\[
W_{cl}(x, E) = \int_x^x \sqrt{2m(E - V)} \, dx.
\]

(5)

Function \( W \) satisfies the nonlinear equation (4) in the reference of Bohm, expressed in the form \( S = -Et + \hbar W \), the Hamilton-Jacobi-like equation with the “quantum-mechanical” potential. Function \( W \) is assumed to satisfy the condition that in the classical region of \( x \)

\[
\hbar W(x, E) \simeq W(x, E)_{cl},
\]

(6)

except an additional constant independent of \( x \) and the mode parameter. If function \( W \) can be determined uniquely, function \( W \) should be named the mode characteristic function (mcf) for the system. The mcf \( W(x) \) is derived from the wave function \( \Phi(x) \) as follows.

Since the wave equation is the ordinary differential equation of 2nd order, the function \( \Phi \) is composed of two linearly independent solutions, say \( u_1 \) and \( u_2 \),

\[
\Phi(x, E) = a u_1(x, E) + b u_2(x, E),
\]

(7)

where \( a \) and \( b \) are complex constants. Constants \( a \) and \( b \) should be determined by the following assumptions from the theoretical point of view.

Assumption 1: Function \( \hbar W \) in the classical region must be as approximate to the corresponding characteristic function \( W_{cl} \) as possible in the sense of relation (6).

Assumption 2: The results derived from the equations of motion for \( W \) defined by Eq. (8) should be as approximate to the ones in classical mechanics as possible.

Function \( W(x) \) should be determined as the phase or argument of function \( \Phi \) like Eq. (3). The function \( \Psi \) consisting of this \( \Phi \) may be said to represent the traveling wave associated with the motion of a ‘particle’ in the ‘mode’ \( E \).

The equation of motion for the ‘particle’ is assumed

\[
\frac{\partial W}{\partial E} = \frac{1}{\hbar} (t - t_0),
\]

(8)
where \( t_0 \) is a constant (independent of \( t \)) that should be determined by the initial condition for the system. Variable \( t \) should be the dynamical time for the system.

Dynamical time \( t \) should be determined so as to increase monotonically as the ‘particle’ moves. From equation (8), a phase velocity

\[
\frac{\partial x}{\partial t} = \left( \hbar \frac{\partial^2 W(x, E)}{\partial x \partial E} \right)^{-1},
\]  

(9)
is obtained. If the ‘particle’ starts from a position, say \( x_0 \), at an initial time and \( \partial t/\partial x \geq 0 \) for \( x \leq x_b \), it could be considered that the ‘particle’ runs from \( x_0 \) to \( x_b \). Here \( x_b \) stands for a turning point or an endpoint of the potential. After reaching \( x_b \), the ‘particle’ returns to \( x_0 \) with the mcf of \(-W(x, E) + 2W(x_b, E)\), which guarantees the monotone increase of time \( t \) given by equation (8).

The wave function associated with the ‘particle’ motion in a bound state should be described as follows. If the function \( \Phi (3) \) represents the motion of the ‘particle’ traveling to the right, function \( \Phi^* = \sqrt{\rho} \exp[-i W] \), which is also the solution of the wave equation, stands for the motion to the reverse direction. Let the endpoints of the potential in the \( x \) coordinate be \( a \) and \( b \). Then the ‘particle’ moves in the region \( a \leq x \leq b \). Let it start from a point \( x_0 \) to the positive \( x \) direction and the mcf be \( W(x) \). The traveling wave associated with the returning motion from \( b \) to \( x_0 \) or \( a \) should be given by \( \sqrt{\rho(x, E)} \exp(i [-W(x, E) + 2W(b, E)]) \).

The wave function observed at \( x(\geq x_0) \) is assumed to be the superposition of the traveling waves of either motion \([\Phi] \)

\[
\Phi(x, E) - \Phi^*(x, E) \exp[i 2W(b, E)].
\]

(10)

This is finite, zero, at \( b \). If the ‘particle’ turns at \( a \) and runs to \( x_0 \) or \( b \), the mcf should be given by \( W(x, E) - 2W(a, E) + 2W(b, E) \). The wave function for \( x(\leq x_0) \) is thus to be written as

\[
- \Phi^*(x, E) \exp[i 2W(b, E)] + \Phi(x, E) \exp(i [-2W(a, E) + 2W(b, E)]).
\]

(11)

This is finite, zero, at \( a \). The wave functions \([\Phi^*] \) and \([\Phi] \) are bounded for \( a \leq x \leq b \) for any ‘mode’, although they might not be continuous at \( x_0 \).

For \( x(\geq x_0) \) the traveling wave associated with a round trip of the ‘particle’ gets the shift in phase by \(-2W(a, E) + 2W(b, E)\). The wave function associated with \( N \) round trips of the ‘particle’ motion would result in, being averaged for one cycle,

\[
\frac{1}{N} \frac{1 - \exp(i N[-2W(a, E) + 2W(b, E)])}{1 - \exp(i [-2W(a, E) + 2W(b, E)])} \times (\Phi(x, E) - \Phi^*(x, E) \exp[i 2W(b, E)]).
\]

(12)

If the number \( N \) is large, this shows a sharp resonance if \( E \) satisfies

\[
-2W(a, E) + 2W(b, E) = 2\pi \times \text{integer}.
\]

(13)

This resonance condition gives rise to the stationary state and the eigenfunction. This is the exact version of the Bohr-Sommerfeld quantum condition. An approximate but a little general expression for it had been presented previously.[17] It might as well be interpreted that the observed wave should be proportional to the wave function mentioned above.
3 Simple harmonic oscillator

The trajectory of the simple harmonic oscillator is discussed. The wave equation for the oscillator with mass \( m \) for energy \( E \) is written as

\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m}{2} \omega^2 x^2 \right) \Phi(x) = E \Phi(x).
\] (14)

A solution of this equation of the form (7) that leads to function (3) is

\[
\Phi^{(+)}(x) = \left[ M(a, \frac{1}{2}, z) - i \frac{\Gamma(1-a)}{\sqrt{\pi \Gamma\left(\frac{1}{2} - a\right)}} V(a, \frac{1}{2}, x) \right] \exp(-z/2),
\] (15)

where \( \Gamma(x) \) is the gamma function and

\[
a = \frac{1}{4} (1 - 2E/\hbar \omega), \quad z = m\omega x^2/\hbar.
\] (16)

Functions \( M \) and \( V \) are linearly independent confluent hypergeometric functions. The former is the Kummer function, which is expressed in the series form

\[
M(a, \frac{1}{2}, z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(\frac{1}{2})_k k!}
\] (17)

and the latter is defined as

\[
V(a, \frac{1}{2}, x) = \Gamma(-\frac{1}{2}) \sqrt{m\omega/\hbar} x M(\frac{1}{2} + a, \frac{3}{2}, z).
\] (18)

Expression (15) is the ‘eigenfunction’ with ‘eigenvalue’ \( E \), not always bounded at \( x = \infty \), of Eq. (14).

The mcf \( W(x, E) \) satisfying two assumptions mentioned in section 2 is given by

\[
W(x, E) = \arctan \left[ \frac{-\Gamma(1-a) V(a, \frac{1}{2}, x)}{\sqrt{\pi \Gamma\left(\frac{1}{2} - a\right)} M(a, \frac{1}{2}, z)} \right] = \arctan[F(a, \frac{1}{2}, x)]
\] (19)

This mcf multiplied by \( \hbar \) is well approximate to \( W_{cl}(x, E) \) with \( W_{cl}(0, E) = 0 \) in the classical region or in the neighborhood of \( x = 0 \). The validity of the mcf will be recognized in the following discussion.

The equation of motion is given by Eq. (8), or

\[
t = \frac{1}{\omega} \left[ \Psi(1-a) - \Psi(\frac{1}{2} - a) - \frac{\partial \log M(\frac{1}{2} + a, \frac{3}{2}, z)}{\partial a} + \frac{\partial \log M(a, \frac{1}{2}, z)}{\partial a} \right] \times \frac{\Gamma(1-a) M(\frac{1}{2} + a, \frac{3}{2}, z)}{\Gamma(\frac{1}{2} - a) M(a, \frac{1}{2}, z)[1 + F(a, \frac{1}{2}, x)^2]},
\] (20)

where \( \Psi(x) = d \log \Gamma(x)/dx \) is the psi function. The oscillator has been assumed to be at \( x = 0 \) at \( t = 0 \).
By using the asymptotic form of the confluent hypergeometric functions, \([18]\) it is obtained for \(x\) large
\[
F(a, 1/2, x) \approx \tan \left[ \frac{\pi}{4} \left( 1 + \frac{2E}{\hbar \omega} \right) \right].
\] (21)

It is thus obtained that \(t(x = \infty) = \pi/2\omega\). The \(x\) dependence of function \(t(x)\) is monotone everywhere, as proved by a computer calculation. Therefore it can be considered that the oscillator moves between the end points, \(x = -\infty\) and \(\infty\), without interruption.

If the oscillator starts from \(x = 0\) at \(t = 0\) and goes to \(x = \infty\), it returns there and comes back to \(x = -\infty\). Then it goes back to \(x = 0\). The mcf \(W(x, E)\) for one cycle is written as follows:
\[
W(x, E) = \begin{cases} 
\arctan[F(a, 1/2, x)] & (x = 0 \to \infty), \\
2W(\infty, E) - \arctan[F(a, 1/2, x)] & (x = \infty \to -\infty), \\
4W(\infty, E) + \arctan[F(a, 1/2, x)] & (x = -\infty \to x = 0).
\end{cases}
\] (22)

These are determined so that time \(t\) increases as the oscillator runs along the trajectory. From the asymptotic form of \(F(a, 1/2, x)\) for \(x\) large, \((21)\), and \(F(a, 1/2, 0) = 0\), it is seen that the period of the motion is equal to \(2\pi/\omega\) that is just equal to that in classical mechanics. The oscillator runs throughout the space between \(x = -\infty\) and \(\infty\) and the phase velocity \(dx/dt\) in the tunneling region is very large.

The variation of the mcf after one cycle stands for the change of the phase of the wave function. It is at any point
\[
-2W(-\infty, E) + 2W(\infty, E) = 4W(\infty, E) = \pi \left( 1 + \frac{2E}{\hbar \omega} \right).
\] (23)

The resonance condition of the wave function \((13)\) leads to the eigenvalues of energy. Thus it holds for the stationary state
\[
E = \hbar \omega \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \cdots \quad (24)
\]

If equation \((20)\) is solved inversely for \(x\) as a function of \(t\), the function \(x(t)\) is well approximated by the classical oscillation
\[
x = \sqrt{2E/m\omega^2} \sin \omega t.
\] (25)

Expressions \((20)\) and \((25)\) are illustrated in Fig. 1. The discrepancy between the two occurs at about times, \(t = \pi/2\omega \times \) half integer, when the oscillator is running through the tunneling region.

The similarity suggests that the mcf \((19)\), satisfying assumptions in section 2, is the correct one for the harmonic oscillator.

4 Wave function

The wave function \((15)\) multiplied by \(\exp(-iEt/\hbar)\) describes a wave traveling to the right. According to the mcf \((22)\) it may be considered that it represents the wave associated with the oscillator motion for \(x = 0\) to \(\infty\).
Figure 1: Position versus time of a harmonic oscillator with $E = \frac{3}{2}\hbar\omega$. Solid line stands for the trajectory and broken line for the corresponding classical orbit.

A wave traveling to the left associated with the oscillator motion from $x = \infty$ to $-\infty$ should be given by

$$\Phi(-)(x, E) = \left[ M(a, 1/2, z) + i \frac{\Gamma(1 - a)}{\sqrt{\pi}\Gamma(1/2 - a)} V(a, 1/2, x) \right] \times \exp \left[ -z/2 + i 2W(\infty, E) \right].$$

(26)

A linear combination with $\Phi^+(x, E)$ like expression (10),

$$\Phi_1(x, E) = \Phi^+(x, E) - \Phi^-(x, E)$$

$$= 2 \exp(-i\pi a - 1/2z) \left[ M(a, 1/2, z) \cos \pi a + \frac{\Gamma(1 - a)}{\sqrt{\pi}\Gamma(1/2 - a)} V(a, 1/2, x) \sin \pi a \right],$$

(27)

gives rise to a wave function finite for $0 \leq x \leq \infty$. The boundedness at $x = \infty$ is seen as follows. Expression (27) can be rewritten as

$$\Phi_1(x, E) = \frac{2\sqrt{\pi}}{\Gamma(1/2 - a)} \exp(-i\pi a - z/2) x U(a + 1/2, 3/2, z).$$

(28)

Function $U$ is the Kummer function. [18] From the asymptotic form of function $U$ for $x$ large [18] it is found that

$$\Phi_1(x, E) \simeq \frac{2\sqrt{\pi}}{\Gamma(1/2 - a)} z^{-a} \exp(-i\pi a - 1/2z),$$

(29)

which tends to zero as $x$ tends to infinity.

For $x$ negative, an associated wave for the oscillator motion reflected at $x = -\infty$ and going back to $x = 0$ should be written as

$$\Phi^+(x, E)'$$

$$= \left[ M(a, 1/2, z) - i \frac{\Gamma(1 - a)}{\sqrt{\pi}\Gamma(1/2 - a)} V(a, 1/2, x) \right] \exp[-z/2 + i 4W(\infty, E)].$$

(30)
The wave function for x negative is

\[ \Phi_2(x, E) = \Phi^+(x, E)' - \Phi^-(x, E) \]

\[ = 2 \exp(-i3\pi a - \frac{1}{2}z) \left[ M(a, \frac{1}{2}, z) \cos \pi a + \frac{\Gamma(1 - a)}{\sqrt{\pi \Gamma(\frac{1}{2} - a)}} V(a, \frac{1}{2}, x) \sin \pi a \right]. \] (31)

Since \( M(a, \frac{1}{2}, z) \) is an even function of \( x \) and \( V(a, \frac{1}{2}, x) \) is an odd, there is a symmetry between functions \( \Phi_1 \) and \( \Phi_2 \)

\[ \Phi_2(x, E) = \Phi_1(-x, E) \exp(-i2\pi a). \] (32)

A wave function \( \Phi_1 \) except a constant but \( E \)-dependent factor is shown in Fig. 2 for several \( E \)'s.

![Figure 2: Wave function \( \Phi_1(x, E)e^{i\pi a} \) vs \( x \) for \( E = \frac{1}{2}\hbar\omega \times \) (a) 1, (b) 1.5, (c) 2, (d) 2.5, (e) 3.](image)

It can be seen from the figure that function \( \Phi_1 \) is smoothly continuous to function \( \Phi_2 \) at \( x = 0 \) if \( E = \frac{1}{2}\hbar\omega \times [1 \text{ or } 3] \), which is the eigenstate.

If \( 4W(\infty, E) \) is a multiple of \( 2\pi \), in general, expression (27) becomes smoothly continuous to equation (31) at \( x = 0 \). The two expressions constitute the eigenfunction for the stationary state finite and smoothly continuous for any space point with the eigenvalue \( E \) discrete.

5 Two dimensional motion

The simple harmonic oscillator runs only between the end points, \( x = -\infty \) and \( x = \infty \). Except that it runs through tunneling regions, it oscillates like a classical particle. Here, the harmonic oscillator moving in the space of two dimension is studied. The extension is straightforward if the partial differential equation is separable in variables.
The wave equation for the harmonic oscillator in two dimensional space with energy $E$ is given by

$$E \Phi(x, y) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{m}{2} \left( \omega_1^2 x^2 + \omega_2^2 y^2 \right) \right] \Phi(x, y),$$ \hfill (33)

where $m$ is the mass, and $\omega_1$ and $\omega_2$ are proper frequencies. By introducing a constant of separation of variables, $E_2$, the above equation can be decomposed into two equations

$$E_1 \Phi_1(x, E_1) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m}{2} \omega_1^2 x^2 \right) \Phi_1(x, E_1),$$ \hfill (34)

$$E_2 \Phi_2(y, E_2) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{m}{2} \omega_2^2 y^2 \right) \Phi_2(y, E_2),$$ \hfill (35)

where $E_1 = E - E_2$.

By following the discussion of section 3, the mcf in the ‘mode’ $(E, E_2)$, is obtained as

$$W(x, y, E, E_2) = W_1(x, E, E_2) + W_2(y, E_2),$$ \hfill (36)

where

$$W_1(x, E, E_2) = \arctan F \left( \frac{1}{4} (1 - 2E_1/\hbar \omega_1), 1/2, x \right),$$ \hfill (37)

$$W_2(y, E_2) = \arctan F \left( \frac{1}{4} (1 - 2E_2/\hbar \omega_2), 1/2, y \right).$$ \hfill (38)

The equations of motion are given by equation (8), or

$$t = t_0 + \hbar \frac{\partial W_1(x, E, E_2)}{\partial E} = t_2 + \hbar \frac{\partial W_2(y, E_2)}{\partial E_2},$$ \hfill (39)

where $t_0$ and $t_2$ are constants to be determined by the initial condition. The whole trajectory for $-\infty \leq x \leq \infty$ and $-\infty \leq y \leq \infty$ can be obtained by using mcf’s (22) for $x$ and $y$ coordinates. The projected motion of the ‘particle’ onto the $x$ or $y$ coordinate is the periodic one with period of $2\pi/\omega_1$ or $2\pi/\omega_2$, respectively. If the ratio $\omega_1/\omega_2$ is irrational, the trajectory in the two dimensional space is not closed as is the case in classical mechanics.

An example of the trajectory for the system with parameters $E_1 = 3/2\hbar \omega_1$ and $E_2 = 2\hbar \omega_2$ and $\omega_1/\omega_2 = 0.8$ is shown in Fig. 3. The oscillator is set to start from the origin at $t = 0$, or $t_0 = t_2 = 0$. In the figure the corresponding classical orbit with the same parameters is also drawn. It could be seen that the bigger the values of the parameters, the closer the trajectory and the classical orbit with each other, which shows the correspondence principle.

\section{Conclusion}

By generalizing the argument on optical wave, the dynamics that should figure out a ray of ‘particle’, a trajectory, in a ‘mode’ of the Schrödinger wave equation of the completely separable form, especially in one spatial dimension, has been proposed. The dynamics should be determined by the mcf and dynamical assumptions on it. The trajectory thus


Figure 3: The trajectory of a two-dimensional harmonic oscillator with $E_1 = \frac{3}{2} \hbar \omega_1$, $E_2 = \frac{5}{2} \hbar \omega_2$ and $\omega_1/\omega_2 = 0.8$ (solid line), and the corresponding classical orbit (broken line).

determined resembles well the orbit of the corresponding state in classical mechanics in the classical region. It runs also through a tunneling region. This is verified for the harmonic oscillator system. The period of a ‘particle’ in the oscillator is exactly equal to that in classical mechanics.

The wave function bounded everywhere but not always continuous for any bound state could be made by superposing the traveling waves associated with the ‘particle’ motion. The eigenfunction of the stationary state should be interpreted to be the wave function of the resonating state in the potential of the system. It is to be noted that all solutions of the wave equation are necessary in order to derive the mcf to get the trajectory. This suggests the role of all the solutions of the wave equation. Since the trajectory is determined by the phase or argument of the traveling wave function, it does not always show all the path of the energy transfer, which might be necessary for the real particle motion in the wave equation.

The accordance of these characteristics between classical and quantum mechanics suggests the validity of the dynamics defined here. The main difference from the Bohm’s theory[4] stems from the dynamical assumption[8]. The significance and the consistency with the quantum theory should be verified for a more important system such as the one with the Coulomb potential. By treating the scattering problem, the statistical but an in principle determinate nature in quantum theory will be shown on getting the cross section.

It suggests the consistent existence of the trajectory in wave mechanics and the significance of the relation between the particle motion and the wave function, although there may remain a lot to be considered about the observation process of particle and wave phenomena.

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