Quantum mechanical path integrals and thermal radiation in static curved spacetimes

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The propagator of a spinless particle is calculated from the quantum mechanical path integral formalism in static curved spacetimes endowed with event-horizons. A toy model, the Gui spacetime, and the 2D and 4D Schwarzschild black holes are considered. The role of the topology of the coordinates configuration space is emphasised in this framework. To cover entirely the above spacetimes with a single set of coordinates, tortoise coordinates are extended to complex values. It is shown that the homotopic properties of the complex tortoise configuration space imply the thermal behaviour of the propagator in these spacetimes. The propagator is calculated when end points are located in identical or distinct spacetime regions separated by one or several event-horizons. Quantum evolution through the event-horizons is shown to be unitary in the fifth variable.

I. INTRODUCTION

Quantum mechanical path integrals [1–4], also called first quantised path integrals, have been applied to some problems in curved spacetimes [5], such as to cosmological and black-holes issues [6–11]. A remarkable theoretical prediction in semi-classical gravity is that of the thermal and quantum radiation of black holes [12,13]. This result is recovered again in the present paper within the formalism of quantum mechanical path integrals. We shall consider some static spacetimes endowed with event-horizons such as the two-dimensional (2D) and four-dimensional (4D) Schwarzschild black-hole spacetimes, and a toy spacetime model, the Gui spacetime [14,15].

An advantage of the path integral formalism over the canonical approach of quantum field theory is that path integrals capture at once both the local and global properties of the base space in which they are calculated. They are thus quite useful in spaces endowed with a non-trivial topology, since they allow one to exploit relatively easily the homotopic properties of the space when computing the propagator.

The circle is the simplest example of a space with a non-trivial topology in which path integrals can be calculated [2,3]. In this case, paths are catalogued into homotopic classes according to their number of turns around the circle. This number is called the winding number (it is positive when the path turns in the anticlockwise direction, and negative when it turns in the clockwise direction). A path contribution to the path integral depends essentially on the homotopic class to which it belongs, i.e. its contribution is a function of its winding number. One can decompose the propagator as the sum of the individual contributions of the homotopic classes over the winding number with respect to the circle. The contributions of the homotopic classes can be calculated in this case and shown to be equal to the free propagator. In more complex spaces, path integrals are evaluated in a similar way. One needs sometimes to define several winding numbers [16], but the contributions of the homotopic classes may be quite difficult to calculate, as in curved backgrounds.

The explicit dependence of path integrals on homotopic properties may also be useful when physical or topological constraints are introduced in a base space whose topology is trivial. In that case, one usually prefers to work in a copy of the base space from which the points made unaccessible to the particle motion have been removed. If the cutout is not too severe, one will expect that the path integral when computed in this new space leads to the same result as when computed in the base space. In this case, the new space is called the “approximate space” of the problem. Because of the surgery performed, its homotopic properties may be quite different from that of the base space, and in particular may be non-trivial. Its distinct topology is then referred to as the “approximate topology” of the problem [17].

When the above surgery results in the removal of a single point, which is sufficient to transform a space with a trivial topology into a multiply connected space, the value of the path integral is not affected when moving to the approximate space. This point is generally a singularity of some kind. It may be where an exterior physical constraint such as a localised field has been applied, or it may be the singular point of a set of coordinates chosen to cover the base space.

An example of a base space constraint by a localised field is encountered in the Aharanov-Bohm effect [4,17,18]. In this case a vector potential \( A \) is localised on
a straight line of $\mathbb{R}^3$. If this line is removed from the base space, the resulting approximate space will be multiply connected. Its homotopic properties enable one to write the propagator as an infinite sum of effective propagators, where the sum is taken over the winding number with respect to the above line.

The origin of the 2D plane covered by circular coordinates is the simplest example of a singular coordinate point. If the origin is removed from the plane, the topology of the corresponding approximate space becomes that of $\mathbb{R}_+ \times S^1$, because the polar angle has an intrinsic periodical structure. The propagator may then be decomposed according to the winding number with respect to the origin, and expressed in terms of free propagators in circular coordinates $\Omega^2$.

The concept of approximate topology is not only useful when topological or physical constraints are applied on a system, but also when a particular calculation technique or method is considered.

Firstly, it is useful when WKB approximations are used or when asymptotic solutions of a problem are considered. One may find that either of these solutions break down and become singular at some points (for example at the turning points of a potential), although the exact solution is regular there. By removing these points, the topology of the base space is modified and one may then resort to the concept of approximate topology $\Omega^2$.

Secondly, if one considers a curved spacetime, one may prefer to work in its Euclidean section for simplicity. The topology of the Euclidean section may be quite different from the one of the Lorentzian section, as for example in the Schwarzschild black-hole case [22]. Although it is widely believed that the value of a path integral is not modified when moving to the Euclidean section, this has never been proved in general to the best of my knowledge.

Thirdly, the geodesic structure may transform the topology of the base space. For example, a star curves spacetime in such a way that two points in its space projection $\mathbb{R}^3$ are connected by an infinite number of geodesics. This gives a multiply connected structure to the spacetime although the set $\mathbb{R}^3$ is simply connected, even when the star itself is removed from $\mathbb{R}^3$.

I have enumerated several problems for which the concepts of approximate space and approximate topology are useful to consider when calculating path integrals. However, I believe that for one of these problems, these concepts are not very well-suited or appropriate. This happens when the base space is covered by a set of coordinates admitting one or several singular points. In this case, I believe that one should avoid adopting the point of view that it is the base space which is transformed into an approximate space by removing the coordinate singularities, and rather use the concept of the coordinates configuration space, and work in this space instead. The reason for doing this is that a choice of coordinates is a rather passive one, in so far that it does not require a modification of the base space in itself.

The coordinates configuration space is the entire set of values taken by the chosen coordinates. In this space, there are no restrictions to the particle motion stemming from the chosen set of coordinates, when these coordinates cover the whole space. It contains also the coordinate values of the singular points and so, contrary to the approximate space, it is a closed space in general. Since a singularity may correspond in general to a family of points in the coordinates configuration space, the topology of this space may still be quite different from that of the base space. For the same reason, there may also be an ambiguity regarding the actual particle trajectory in the coordinates configuration space when it moves through a coordinate singularity in the base space.

As an example, we consider again the 2D plane covered by circular coordinates. Its origin is included in the configuration space, and may be parametrised by the set of values $\{(r, \theta) \mid r = 0 \text{ and } \theta \in [0, 2\pi]\}$, where $r$ is the radius and $\theta$ the polar angle. The configuration space in these coordinates is an half-infinite cylinder whose radius is equal to unity, and where its boundary, a circle, parametrises the origin. If the particle goes through the origin in the base space, its trajectory on this circle is not a priori determined.

From now on I shall reserve the terms “approximate space” and “approximate topology” for an ad hoc or active change of space resulting from the choice of working in another space than the base space, and use the terms “associated space” and “associated topology” for the coordinates configuration space. In this sense, as it will become clear in the present paper, a change of coordinates (i.e. a passive transformation) implies a change of associated topology when this transformation possesses singular points.

The concepts of approximate and associated spaces are illustrated in Fig. 1, where it is shown also how they can be combined to account for the various situations one may encounter. In a first stage, one may choose to transform the base space into an approximate space to take into account the physical constraints or in order to use a given calculation technique. In a second stage, after a choice of coordinates has been made to cover the approximate space, one considers the coordinates associated space. Path integrals are calculated in its universal covering space because it is simply connected. This space is considered in the third stage. In the fourth stage, a change of coordinates may then be performed, resulting in a distinct covering space. In the last and fifth stage, one rebuilds the associated space in the new coordinates and study its possibly new topology.

In the literature, different approaches have been considered in curved backgrounds.

Troost and Van Dam [7], in their study of Rindler spacetime, considered its Euclidean section to compute...
the path integral and thus worked in an Euclidean approximate space. They considered firstly Euclidean Cartesian coordinates for which the associated topology is trivial. The path winding number with respect to the bifurcation of the event-horizons (i.e. the origin of Euclidean spacetime) is then introduced, and the propagator is expressed as a sum over this winding number. One sums path integrals with a modified action as in Ref. [10], which amounts to working in circular coordinates (i.e. Euclidean Rindler coordinates in that case) for which the associated topology is multiply connected. These winding number dependent path integrals are then calculated by solving the corresponding Schr"{o}dinger equation. The thermal properties of the propagator are then highlighted.

Hartle and Hawking [1] considered a black-hole background and continued analytically the Kruskal time coordinates to obtain a positive definite metric. The associated space has a non-trivial topology because of the periodic structure of the imaginary Kruskal time coordinate. They used the path integral to obtain the boundary conditions satisfied by the propagator. From these boundary conditions and from the Klein-Gordon equation, they deduced the analytical properties obeyed by the propagator. In this way the thermal properties of the propagator are revealed.

In an eternal black-hole background, one is generally interested in computing the propagator in tortoise coordinates to capture the thermal radiation. However, this task is made difficult by the fact that these coordinates only cover the exterior region of the black hole. The paths crossing the horizon cannot then be parametrised, and as a consequence the path integral cannot be adequately expressed in these coordinates. For this reason, one may choose to consider the path integral in Euclidean Kruskal coordinates which cover the whole black-hole spacetime, as did the above authors. However, in non-static spacetimes such as in collapsing black-hole spacetimes [2], the Euclidean approach is simply useless because in these cases the periodical structure of the Kruskal type coordinates is missing in the Euclidean section. A strict Lorentzian approach is then required. Furthermore, the physical interpretation of the results in the Euclidean section is not always obvious.

The philosophy of the present paper is to remain in the Lorentzian section of the considered spacetimes throughout, and to attempt to calculate the path integral in a direct way. This has been done in non-static backgrounds in Ref. [11]. The goal of this work is to extend the previous analysis to static backgrounds. I shall first consider the static Gui spacetime [12], which I believe is the simplest spacetime model incorporating event-horizons which exhibits thermal properties. This vanishing curvature spacetime is made up of four Minkowski quadrants glued together along the “event-horizons”. The analysis shall then be extended to the more realistic case of the Schwarzschild black hole. The similarities and differences between the static Gui spacetime and the Schwarzschild black-hole spacetime are discussed.

The key point of the Lorentzian approach of the present paper is the concept of associated topology discussed above. As shown in Ref. [13], this is different in coordinates related by a non-analytical transformation. In this method, one allows the tortoise type coordinates to take complex values in order to cover the entire spacetime. The resulting complex associated space is multiply connected essentially because these coordinates are singular at the event-horizon, in contrast to the Kruskal associated space which is homotopically trivial. A path crossing the event-horizon has a well defined winding number with respect to the event-horizon in the complex tortoise associated space. The propagator can then be written as a sum of path integrals over this winding number which can be evaluated in some instances. The thermal structure of the propagator becomes then obvious. The thermal properties of the vacuum in a black-hole background, for example, can thus be seen to follow from topological considerations.

The second section is devoted to a review of the path integral formalism and to the definition of the notations used in the present paper. In the third section, this formalism is applied to the static Gui spacetime. The path integral is computed there according to the Lorentzian approach. The propagator between points belonging to different or similar quadrants of this spacetime are obtained and the thermal and hermitian properties of these propagators are analysed. In the fourth section, the Schwarzschild black hole is investigated and the results obtained are generalised. The propagator is computed exactly far away from the black hole in the 2D case and some results are given in the 4D case.

II. REVIEW OF THE PATH INTEGRAL FORMALISM

A relativistic quantum particle is allowed to go virtually both forwards and backwards in time [24], contrary to the non-relativistic case. In addition to the spacetime parameters, it is then necessary to introduce a supplementary parameter \( s \), the so-called fifth parameter, to parametrise the path of a particle. This may be interpreted in the massive case as the proper time of the particle, and in general it plays the same role than that of the time parameter in the non-relativistic framework. This fifth parameter allows to describe more complex processes than in the non-relativistic case, for example a spontaneous creation of particles.

A Schrödinger equation for which \( s \) is the evolution parameter can be written for a spinless relativistic particle moving on a curved spacetime. This one-particle dynamical equation does not depend explicitly on the mass \( m \)
of the particle, and this enables one to treat the massless and massive cases on an equal footing. It is related to the Klein-Gordon equation by an imaginary Laplace transform with respect to $s$, whose conjugate variable is $m^2$. The propagator for this Schrödinger equation can be written as a quantum mechanical path integral as in the non-relativistic case. The time-ordered two-point correlation function of the underlying quantum field theory may then be obtained from this propagator.

As explained in Section 3, the path integral formalism is quite useful in spaces endowed with a non-trivial topology. In this context, the set over paths on which the sum is taken is of crucial importance in the definition of the path integral. This set can naturally be fixed by the space itself. Its choice is equivalent to the one of the propagator boundary conditions when solving directly the Schrödinger equation, and to the choice of the vacuum state in quantum field theory.

To fix the ideas, one considers a connected curved spacetime $\mathcal{M}$, endowed with the metric $g$, which may be covered entirely with a set of coordinates denoted by $q$. The associated space $Q$ in these coordinates is defined to be the set of values taken by $q$, and the induced metric on $Q$ is denoted by $g_q$. The space $Q$ should always be connected and may be multiplied connect in general.

The total probability amplitude for a quantum particle of mass $m$ to move within $\mathcal{M}$ from an initial point $q_i \in Q$ to a final point $q_f \in Q$ is given by the propagator $G_q(q_i; q_f; m^2)$ in $q$ coordinates. In curved spacetimes, the propagator for a spinless relativistic particle satisfies to the Klein-Gordon equation

\[ \left( \hbar^2 \Box + \hbar^2 \xi R + m^2 \right) G_q(q'; q''; m^2) = \|g\|^{-1/2} \delta(q' - q''), \]  

where $\Box = \nabla \mu \nabla^\mu$ if $\nabla_\mu$ denotes the covariant derivative associated to the metric $g$, where $\delta$ is the Dirac function, and with $\xi = 0$ and $\xi = (D-2)/[4(D-1)]$ correspond to the minimal and conformal coupling respectively if $D$ is the spacetime dimension. One introduces a $s$-dependent propagator $K_q(q'; q''; s)$ by the imaginary Laplace transform

\[ G_q(q'; q''; m^2) = \frac{i}{\hbar} \int_0^\infty ds \exp(-im^2s/\hbar) K_q(q'; q''; s). \]

By postulating that the propagator $G_q(q_i; q_f; m^2)$ vanishes if $m^2 < 0$, one obtains by inverting this last equation,

\[ \theta(s) K_q(q'; q''; s) \]

\[ = \frac{1}{2\pi i} \int_0^\infty dm^2 \exp(im^2s/\hbar) G_q(q'; q''; m^2), \]

where $\theta$ is the step or Heaviside function. One then checks from Eq. (3) that $K_q$ satisfies to the Schrödinger equation

\[ i\hbar \partial_s K_q(q'; q''; s) = \hbar^2 (\Box + \xi R) K_q(q'; q''; s), \]

if and only if the $s$-parameter boundary condition for the propagator is given by

\[ \lim_{s \to 0} K_q(q'; q''; s) = \|g\|^{-1/2} \delta(q' - q''). \]

The propagator $K_q(q_i; q_f; s)$ is interpreted as the probability amplitude for the particle to move from the initial end point $q_i$ to the final end point $q_f$ within a parameter time $s$. It is clear that $K_q(q'; q''; s) = K_q(q''; q'; -s)$.

One now introduces the one-particle Hilbert space $\mathcal{H}$ and the localised states $|q\rangle \in \mathcal{H}$, where $q \in Q$, which satisfy to the orthogonality relation

\[ < q' | q'' > = \|g\|^{-1/2} \delta(q' - q''). \]

The operator of evolution $U(s)$ acting on $\mathcal{H}$ is defined from the propagator $K_q$ by

\[ K_q(q'; q''; s) = < q'' | U(s) | q' >. \]

It satisfies by definition to $U(s)^{-1} = U(-s)$. Since $U(0) = I$, Eqs. (3) and (7) imply consistently the $s$-boundary condition, Eq. (8). The evolution operator is unitary, i.e. $U(s)^{-1} = U(s)^\dagger$, if and only if

\[ K_q(q'; q''; s)^* = K_q(q''; q'; -s). \]

Furthermore, from Eq. (2), we see that this last equation is satisfied if and only if

\[ G_q(q'; q''; m^2)^* = G_q(q''; q'; m^2). \]

In other words, $K_q$ describes a unitary evolution in the fifth parameter if and only if $G_q$ is a hermitian operator.

The general form of $K_q$ is obtained by solving directly the Schrödinger equation, Eq. (3), and is given in Ref. 3 by

\[ K_q^{\text{vac}}(q'; q''; s) = \frac{1}{(4\pi \hbar s)^{D/2}} \sqrt{\Delta(q'; q'')} F(q'; q''; s) \]

\[ \times \sum_{\gamma \in Q} \exp \left[ \frac{i}{\hbar} \frac{\sigma_0(q'; q'', \gamma)^2}{4s} \right], \]

*The Hamiltonian $H$ is defined by $U(s) = \exp(-i\hbar H/\hbar)$. From Eq. (3), one obtains $G = (H + m^2)^{-1}$. The operator $H$ is hermitian if and only if $U(s)$ is unitary, and $G$ is hermitian if and only $H$ is hermitian (the issues regarding the domain of the operators are not considered here).
where \( \gamma \) denotes an arbitrary geodesic joining the end points \( q' \) and \( q'' \), \( \sigma_\gamma (q'; q''; \gamma) \) is the proper arc length between \( q' \) and \( q'' \) along the particular geodesic \( \gamma \), \( \Delta(q'; q'') \) is the Van Vleck-Morette determinant and \( F(q'; q''; s) \) is a function whose general expression is unknown. The definition of the biscalars \( \Delta(q'; q'') \) and \( F(q'; q''; s) \) can be found in Ref. [3]. For our purpose, it is sufficient to realise that these two functions are equal to unity in a flat spacetime covered by any set of curvilinear coordinates. One notices that the explicit dependence of Eq. (10) on the spacetime dimension \( D \) is rather trivial.

A distinguished geodesic \( \gamma_0 \) in \( Q \) is the one whose length \( \sigma(q'; q''; \gamma_0) \) tends to zero when the end point \( q' \) approaches \( q'' \). Its contribution has necessarily to be included in the sum in Eq. (10) in order that the \( s \)-boundary condition, Eq. (5), is satisfied. It is the only contribution which is singular when the parameter \( s \) vanishes. The contributions of all the others geodesics individually satisfy to the Schrödinger equation, Eq. (3), and do not modify the \( s \)-boundary condition. In consequence, one can a priori include in the sum in Eq. (10) only those geodesics one wishes besides the geodesic \( \gamma_0 \). The spacetime boundary conditions will be determined crucially by this choice. However, instead of having to estimate whether the resulting spacetime boundary conditions are physically acceptable or not, one rather chooses as a rule to include in Eq. (10) the contributions of all the geodesics contained within \( Q \). This simple rule is legitimised by the fact that a particle has, by definition, virtual access to all the points of the connected associated space \( Q \).

The particular solution in Eq. (10) defines a vacuum state \( |\text{vac} > \) of the underlying quantum field theory. This vacuum defines a Fock space \( \mathcal{F} \) which may be related to the associated space \( Q \). Notice that Fock spaces corresponding to distinct associated spaces may not be necessarily unitarily equivalent [25].

The simplest example of a propagator is the one in a flat spacetime covered by Minkowski coordinates. In this case, the geodesic joining the two end points is unique. The relevant vacuum is the Minkowski one denoted by \( | M > \) and one has from Eq. (10)

\[
K^M_q (q'; q''; s) = K_0 (q'; q''; s) = \frac{1}{(4\pi \hbar s)^{D/2}} \exp \left[ \frac{i}{\hbar} \frac{(q'' - q')^2}{4s} \right],
\]

where \( K_0 \) denotes the free propagator.

In the general case, the propagator \( K_q (q_i; q_f; s) \) is written in a rather symbolic way as a sum over the paths \( |q> \) contained within \( Q \) and joining the end points \( q_i \) and \( q_f \) within a parameter time \( s = s_f - s_i \),

\[
K^{\text{vac}}_q (q_i; q_f; s) = \sum_{q_i \rightarrow q_f} \exp \left( \frac{i}{\hbar} S_{q_i} [q] \right),
\]

where the covariant action \( S_{q_i} [q] \) is given by

\[
S_{q_i} [q] = \frac{1}{4} \int_{s_i}^{s_f} d\omega g_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu.
\]

It is conjectured that the vacuum state \( |\text{vac} > \) defined in this sum is the same as the one defined by Eq. (10).

If the space \( Q \) is multiply connected, one introduces its covering space \( \tilde{Q} \) which is always simply connected. To define it, one considers the homotopic classes of paths of \( Q \). By definition, the paths of a given homotopic class can be deformed continuously into one another, but it is not possible to do so for paths belonging to distinct homotopic classes. The holonomy group \( \Gamma \) is the set of all the homotopic classes. By definition, the covering space \( \tilde{Q} \) contains the points denoted by \( \tilde{q}'' \), where \( q \) is an arbitrary point of \( Q \) and where the index \( \nu \) ranges over all the elements of the holonomy group \( \Gamma \). These points are called the images in \( \tilde{Q} \) of the point \( q \in Q \). One has then \( \tilde{Q} = Q / \Gamma \), and the elements of \( \Gamma \) can thus also be thought of as applications relating the different image points \( \tilde{q}'' \). For example, one defines the element \( \gamma''_\nu \in \Gamma \) and the base point \( \tilde{q} \equiv \tilde{q}''_0 = 0 \) in such a way that \( \tilde{q}'' = \gamma''_\nu (\tilde{q}) \). Paths in \( \tilde{Q} \) with identical initial and final end points but with distinct end points in \( \tilde{Q} \) belong to different homotopic classes. The covering space \( \tilde{Q} \) is endowed by the metric \( \tilde{g}_\tilde{q} \) defined naturally by \( \tilde{g}(\tilde{q}''_\nu) = g(q) \).

The sum over paths in Eq. (12) is rewritten in \( \tilde{Q} \) by taking into account its possible multiply connected topology, i.e. by summing over the classes of paths, or equivalently over the images \( \tilde{q}''_\nu \),

\[
\sum_{q_i \rightarrow \tilde{q}_i} \exp \left( \frac{i}{\hbar} S_{\tilde{q}_i} [\tilde{q}] \right) = \sum_{\nu} \sum_{\tilde{q}_i} \tilde{g}_\tilde{q}_i \exp \left( \frac{i}{\hbar} S_{\tilde{q}_i} [\tilde{q}] \right).
\]

The propagator \( \tilde{K}_\tilde{q} \) in the covering space \( \tilde{Q} \) is defined by the sum over paths appearing in the right hand side (RHS) of this last equation,

\[
\tilde{K}_\tilde{q} (\tilde{q}''; \tilde{q}'''; s) = \sum_{\tilde{q}_i \rightarrow \tilde{q}_j} \exp \left( \frac{i}{\hbar} S_{\tilde{q}_i} [\tilde{q}] \right).
\]

A new vacuum \( |\text{vac} > \) and a new propagator \( K^{\text{vac}}_q \) are defined by the identification \( \tilde{K}_\tilde{q} \equiv K^{\text{vac}}_q \). One thus obtains the general and important result [26]

\[
K^{\text{vac}}_q (q_i; q_f; s) = \sum_{\nu} K^{\text{vac}}_\nu (\tilde{q}_i; \tilde{q}_f; s).
\]

In general, the covering space \( \tilde{Q} \) is not necessarily real but may be complex. From Eq. (14), we deduce that if \( \tilde{Q} \)
is the complex conjugate of itself, i.e. $\tilde{Q}^* = \tilde{Q}$, evolution will be unitary, i.e. Eq. (8) will be satisfied. There will be end points for which the evolution is not unitary if and only if $\tilde{Q}^* \neq \tilde{Q}$.

The sum over paths is defined as a path integral, i.e. as an infinite dimensional integral. Following Ref. [4] one writes

$$\sum_{\tilde{q} \to s} \exp \left( \frac{i}{\hbar} S_{\tilde{g}_q} [\tilde{q}] \right) = \int_{\tilde{q}}^{q''} D\tilde{g}_q [\tilde{q}] \exp \left( \frac{i}{\hbar} S_{\tilde{g}_q} [\tilde{q}] \right),$$

(17)

where $S_{\tilde{g}_q} [\tilde{q}] = \int_{s_i}^{s_f} d\omega \tilde{L}_{\tilde{g}_q} [\tilde{q}, \tilde{q}]$, and where the RHS of this last equation is defined by

$$\int_{\tilde{q}}^{q''} D\tilde{g}_q [\tilde{q}] \exp \left( \frac{i}{\hbar} S_{\tilde{g}_q} [\tilde{q}] \right) = \lim_{N \to \infty} \left( \frac{N}{4\pi \hbar s} \right)^{\frac{2N}{N}} \times \prod_{j=1}^{N-1} \int_{\tilde{q}} L_{\tilde{g}_q} [\tilde{q}, \tilde{q}] [\tilde{q}],
$$

(18)

where $s_j = s_i + js/N$, $\tilde{q}_j = \tilde{q}(s_j)$ $(j = 0, 1, ..., N)$, $N \in \mathbb{N}$. It is assumed that each integral in the exponential is evaluated along the image-geodesic connecting $\tilde{q}_{j-1}$ and $\tilde{q}_j$ which ensures that the path integral is defined in a covariant way. This rule implies that the Lagrangian $\tilde{L}_{\tilde{g}_q} [\tilde{q}, \tilde{q}]$ is given by

$$\tilde{L}_{\tilde{g}_q} [\tilde{q}, \tilde{q}] = \frac{1}{4} \tilde{g}_{\mu\nu}(\tilde{q}) \tilde{q}^\mu \tilde{q}^\nu - \hbar^2 (\xi - 1/3) R(\tilde{q}).$$

(19)

With the definition of Eq. (18), the term $\hbar^2 R/3$ must be added to the usual Lagrangian to take into account the effect of the curvature in the path integral in order to get to right value for the propagator.

III. GUI’S SPACETIME

A. Spacetime model

The $y$-$\xi$ spacetime of Gui, which I shall call Gui’s spacetime, was introduced in Ref. [4]. Although this spacetime is rather pathological in its nature, its study is instructive for many reasons. Firstly, the path integral can be calculated in this background in a direct way as shown in the present paper. Secondly, I believe that it is the generic example of a spacetime exhibiting thermal properties in the sense that fields and states on this spacetime behave as if they were immersed in a thermal bath contained in a Minkowski background [4][3]. And finally, this spacetime shares some similarities with the Schwarzschild black-hole spacetime, although their global causal properties are different. These points are clarified in this section and in the next one.

The Gui spacetime is a vanishing scalar curvature spacetime with a non-trivial structure. In $D$ dimensions, it is defined by the line element

$$ds_h^2 = \frac{1}{\kappa^2} \frac{dx^+ dx^-}{x^+ x^-} - (dx)^2,$$

(20)

where $\kappa > 0$, $x^\pm = x^0 \pm x^1$, $x \in \mathbb{R}^n (D = n + 2)$, and where $h$ denotes the Gui metric. The $x$ coordinates are Kruskal type coordinates. This metric is rather pathological since it is singular on the two hyperplanes given by $x^+ = 0$ and by $x^- = 0$. These shall be called “event-horizons” for this reason. They divide Gui’s spacetime into four quadrants, which are individually isomorphic to the Minkowski spacetime (see below). These are denoted by $R, F, L$ and $P$ (the right, future, left and past quadrants); see Fig. [2]. Each of these quadrants is causally disconnected from the others in the classical sense, because the proper distance of an event in one of these quadrants to the bordering “event-horizons” is infinite. It is thus not possible for an observer located in a given quadrant to infer the existence of the others quadrants by performing a classical experiment. However, a quantum experiment in a given quadrant may a priori be influenced by the presence of the others, because in a quantum framework a particle may virtually tunnel through the “event-horizons”. One thus expects that the quadrants are causally connected in this quantum sense only.

B. Real tortoise coordinates

In Gui’s spacetime, the Kruskal associated space shall be denoted by $X$. It is given by

$$X = \{ (x^+, x^-, x) \in \mathbb{R}^D \},$$

(21)

and it is simply connected. Its covering space $\tilde{X}$ is thus isomorphic to it. In the covering space $X$, the line element in Eq. (20) is written in the form

$$ds_h^2 = \frac{1}{\kappa^2} \frac{\tilde{d}x^+ \tilde{d}x^-}{\tilde{x}^+ \tilde{x}^-} - (d\tilde{x})^2.$$ 

(22)

We now perform a change of coordinates in the covering space $\tilde{X}$. In each quadrant, one introduces the tortoise type coordinates $\tilde{y}_a$ by

\begin{align*}
\text{in quad. } R: & \quad \begin{cases} 
\tilde{x}^0 = + (1/\kappa) \exp (\kappa \tilde{y}_0^1) \sinh (\kappa \tilde{y}_0^0), \\
\tilde{x}^1 = + (1/\kappa) \exp (\kappa \tilde{y}_0^1) \cosh (\kappa \tilde{y}_0^0) 
\end{cases} \\
\text{in quad. } F: & \quad \begin{cases} 
\tilde{x}^0 = + (1/\kappa) \exp (\kappa \tilde{y}_0^1) \cosh (\kappa \tilde{y}_0^0), \\
\tilde{x}^1 = + (1/\kappa) \exp (\kappa \tilde{y}_0^1) \sinh (\kappa \tilde{y}_0^0), 
\end{cases}
\end{align*}

(23)

where $\tilde{y}_a$ are defined by

$$\tilde{y}_a = \int_{x^a = 0}^{x^a} dx^a,$$

(24)
in quad. $L$: \[ \begin{align*}
\tilde{x}^0 &= -(1/\kappa) \exp (\kappa \tilde{y}_R^0) \sinh (\kappa \tilde{y}_L^0), \\
\tilde{x}^1 &= -(1/\kappa) \exp (\kappa \tilde{y}_R^1) \cosh (\kappa \tilde{y}_L^1),
\end{align*} \tag{25} \]

in quad. $P$: \[ \begin{align*}
\tilde{x}^0 &= -(1/\kappa) \exp (\kappa \tilde{y}_R^0) \cosh (\kappa \tilde{y}_L^0), \\
\tilde{x}^1 &= -(1/\kappa) \exp (\kappa \tilde{y}_R^1) \sinh (\kappa \tilde{y}_L^1),
\end{align*} \tag{26} \]

and by $\tilde{x} = \tilde{y}_a \ (a = R, F, L, P)$. These transformations shall be called “Rindler transformations”. In these equations, one has chosen the signs of $\tilde{y}_R^0$ and $\tilde{y}_L^1$ in such a way that the coordinates $\tilde{y}_a$ behave in a continuous way when the corresponding spacetime event is moved from one quadrant to another and when subjected to a slight displacement. For example, when the event $(\tilde{x}^0, \tilde{x}^1)$ is moved towards the origin $(\tilde{x}^0, \tilde{x}^1) = (0, 0)$, one has $\tilde{y}_a^1 \to -\infty$ in any quadrant $a$. In a similar way, the coordinates $\tilde{y}_R^0$ and $\tilde{y}_P^1$, for example, are both positive in any small region containing a segment of the “event-horizon” RF. The Rindler coordinates of a point on the “event-horizon” RF or LF are $(\tilde{y}_R^0, \tilde{y}_L^1) = (+\infty, -\infty)$ when $a = R, F$ or $a = L, P$ respectively, and the ones of a point on the “event-horizon” PR or and FL are $(\tilde{y}_R^0, \tilde{y}_L^1) = (-\infty, -\infty)$ when $a = R, P$ or $a = L, F$ respectively. The couple $(\tilde{y}_R^0, \tilde{y}_L^1) = (c, -\infty) \ (c \in \mathbb{R}, \ a = R, F, L, P)$ parametrises necessarily the bifurcation or origin $(\tilde{x}^0, \tilde{x}^1) = (0, 0)$.

In tortoise coordinates, the line element is the Minkowski one in any quadrant,
\[ \text{d}s_h^2 = d\tilde{y}_a^+ d\tilde{y}_a^- - (d\tilde{y}_a)^2, \quad a = R, F, L, P. \tag{27} \]

This shows that each of the quadrants is isomorphic to Minkowski spacetime as stated above. The tortoise associated spaces $\mathcal{Y}_a$ are given by
\[ \mathcal{Y}_a = \{ (y^+, y^-, y) \in \mathbb{R}^D \}, \quad a = R, F, L, P. \tag{28} \]

They are simply connected and thus isomorphic to their covering spaces $\tilde{\mathcal{Y}}_a (a = R, F, L, P)$. The full tortoise covering space is given by $\tilde{\mathcal{Y}}_R \cup \tilde{\mathcal{Y}}_F \cup \tilde{\mathcal{Y}}_L \cup \tilde{\mathcal{Y}}_P$, but it is not connected. This implies that a path crossing an “event-horizon” cannot be parametrised with only one set of tortoise coordinates, and that a path integral cannot be expressed in these coordinates. In this sense, the parametrisation of Gui’s spacetime in terms of real tortoise coordinates is not satisfactory.

C. Complex tortoise coordinates

As shown in this section, it is possible to parametrise the entire Gui spacetime with only one set of tortoise coordinates, denoted by $\tilde{y}$, if one allows them to take complex values. These coordinates take their values in a connected complex covering space, denoted by $\tilde{\mathcal{Y}}$. One requires that in the entire Gui spacetime they satisfy exclusively to the transformation given in Eq. (24).
\[ \begin{align*}
\tilde{x}^0 &= (1/\kappa) \exp (\kappa \tilde{y}) \sinh (\kappa \tilde{y}^0), \\
\tilde{x}^1 &= (1/\kappa) \exp (\kappa \tilde{y}) \cosh (\kappa \tilde{y}^0),
\end{align*} \tag{29} \]

$\forall (\tilde{x}^0, \tilde{x}^1) \in \mathbb{R}^2, \forall \tilde{y} \in \tilde{\mathcal{Y}}$.

To construct the space $\tilde{\mathcal{Y}}$, one first considers the reciprocal of this last transformation given in quadrant $R$ by
\[ \begin{align*}
\tilde{y}^0 &= \frac{1}{2\kappa} \ln \left( \frac{-\tilde{x}^+}{\tilde{x}^-} \right), \\
\tilde{y}^1 &= \frac{1}{2\kappa} \ln \left( -\kappa^2 \tilde{x}^+ \tilde{x}^- \right).
\end{align*} \tag{30} \]

The functions $\tilde{y}^0 = \tilde{y}^0(\tilde{x}^0, \tilde{x}^1)$ and $\tilde{y}^1 = \tilde{y}^1(\tilde{x}^0, \tilde{x}^1)$ are then continued analytically from quadrant $R$ to the other quadrants by performing complex rotations of $180^\circ$ in the $\tilde{x}^\pm$ complex planes to connect the positive and negative values of $\tilde{x}^\pm$. Since these functions depend on two complex variables, their analytical continuations will not necessarily be unique. In the covering space, it is natural to treat the logarithm as a multivalued function. A cut is thus not fixed in the $\tilde{x}^\pm$ complex planes, i.e. the argument of the complex variables is not bounded and takes any value ranging from $-\infty$ to $+\infty$. The logarithm function is consequently defined here by
\[ \ln (\tilde{x}^\pm) = \ln |\tilde{x}^\pm| + i \arg (\tilde{x}^\pm), \tag{31} \]

where $\arg (\tilde{x}^\pm) \in \mathbb{R}$. As a point of departure, we now assume that in quadrant $R$ the values of $\tilde{y}^0$ and $\tilde{y}^1$ are given by Eq. (10) where $\text{Im} \tilde{y}^0 = \text{Im} \tilde{y}^1 = 0$. 

\[ |\tilde{x}^\pm| \in \mathbb{R}. \]

\[ \text{arg} (\tilde{x}^\pm) \in \mathbb{R}. \]
One decides first whether the analytic continuation is performed to the other quadrants according to the sequence $R \rightarrow F \rightarrow L \rightarrow P$ or to the opposite one $R \rightarrow P \rightarrow L \rightarrow F$ (in short $(R, F, L, P)$ and $(R, P, L, F)$ respectively). The results do not actually depend on the chosen sequence. According to the first sequence, one continues in the first step in the $\tilde{x}^+$ complex variable from quadrant $R$ to quadrant $F$. According to the second sequence, the analytic continuation is performed in the first step with respect to the $\tilde{x}^-$ complex variable from quadrant $R$ to quadrant $P$.

To fix the ideas, we choose the first sequence. When continuing analytically from quadrant $R$ to quadrant $F$, the sign of $\tilde{x}^-$ is reversed and becomes positive. They are two ways of implementing this change of sign in the $\tilde{x}^-$ complex plane: either we add or subtract $\pi$ to the argument of $\tilde{x}^-$. The values of the logarithm are different in these two cases,

$$\ln (-\tilde{x}^-) = \ln (\tilde{x}^-) \pm i\pi. \quad (32)$$

The analytic continuation can thus be performed either in the anticlockwise or clockwise directions of the $\tilde{x}^-$ complex plane. There are two alternatives. The choice of one alternative determines how the extension is performed from quadrant $R$ to quadrant $F$, and from quadrant $L$ to quadrant $P$ as well.

Next we analytically continue from quadrant $F$ to quadrant $L$. The sign of $\tilde{x}^+$ is reversed in this process and becomes negative. The analytic continuation is done this time in the $\tilde{x}^+$ complex plane. Again there are two ways of implementing this change of sign: either we continue in the anticlockwise or clockwise directions of the $\tilde{x}^+$ complex plane. One has

$$\ln (\tilde{x}^+) = \ln (-\tilde{x}^+) \pm i\pi. \quad (33)$$

This choice determines how the continuation is performed from quadrant $F$ to quadrant $L$, and from quadrant $P$ back to quadrant $R$ as well. Again, there are two alternatives.

Thus there are in total four ways of analytically continuing the functions $\tilde{y}^0(\tilde{x}^0, \tilde{x}^1)$ and $\tilde{y}^1(\tilde{x}^0, \tilde{x}^1)$ from quadrant $R$ to the other quadrants. The different analytic continuations are given in Table I. They are distinguished by the directions in the $\tilde{x}^\pm$ complex planes with respect to which the analytical continuations have been performed (the symbols $+$ and $-$ mean “anticlockwise” and “clockwise” respectively). The couple $(+,-)$ for instance designates the analytical continuation which has been performed in the anticlockwise direction of the $\tilde{x}^+$ complex plane and in the clockwise direction of the $\tilde{x}^-$ complex plane. It is not difficult to see that when the sequence $(R, F, L, P)$ is changed to the sequence $(R, P, L, F)$, the directions of the analytical continuation in both $\tilde{x}^\pm$ complex planes are reversed. For example, the analytical continuation $(+, -)$ for the sequence $(R, F, L, P)$ is identical to the analytical continuation $(-, +)$ for the sequence $(R, P, L, F)$.

When the analytic continuation has been performed from quadrant $R$ to the other quadrants according either to the sequence $(R, F, L, P)$ or $(R, P, L, F)$, we arrive back in quadrant $R$. The values taken there by either the function $\tilde{y}^0(\tilde{x}^0, \tilde{x}^1)$ or $\tilde{y}^1(\tilde{x}^0, \tilde{x}^1)$ obtained from the analytic continuation procedure may or may not be different from their departure values. If these values are different, the function will be a multivalued function; if they are not, it will be a single valued function. In the former case, the analytic continuation procedure is repeated an infinite number of times to obtain all the relevant values. In Table I, we see that either the function $\tilde{y}^0(\tilde{x}^0, \tilde{x}^1)$ or $\tilde{y}^1(\tilde{x}^0, \tilde{x}^1)$ is a multivalued function, not both of them. Furthermore, the values of the multivalued function differ by $i\beta\nu$, where $\beta = 2\pi/\kappa$ and $\nu \in \mathbb{Z}$. From now on, the analytical continuations will always be denoted with respect to the sequence $(R, F, L, P)$.

We now analyse in detail the analytical continuation denoted by $(+, -)$. In that case, we see from Table I that the complex coordinates $\tilde{y}^0 \in \tilde{\gamma}$ can be identified with the real coordinates $\tilde{y}_a \in \tilde{\gamma}_a$ in such a way that

$$\begin{cases}
(\tilde{y}^0)^\nu = \tilde{y}^0 + i\beta\nu, \\
(\tilde{y}^1)^\nu = \tilde{y}^1,
\end{cases} \quad (34)$$

where $\nu \in \mathbb{Z}$ and

\begin{align*}
\text{in quad. } R: & \begin{cases}
\tilde{y}^0 = \tilde{y}^0
\\
\tilde{y}^1 = \tilde{y}^1
\end{cases}, \\
\text{in quad. } F: & \begin{cases}
\tilde{y}^0 = \tilde{y}^0 + i\beta/4, \\
\tilde{y}^1 = \tilde{y}^1 - i\beta/4,
\end{cases} \\
\text{in quad. } L: & \begin{cases}
\tilde{y}^0 = \tilde{y}^0 + i\beta/2, \\
\tilde{y}^1 = \tilde{y}^1
\end{cases}, \\
\text{in quad. } P: & \begin{cases}
\tilde{y}^0 = \tilde{y}^0 + i3\beta/4, \\
\tilde{y}^1 = \tilde{y}^1 - i\beta/4
\end{cases}
\end{align*} \quad (35) to (38)

where $\beta = 2\pi/\kappa$ and $\tilde{y}^\nu = \tilde{y} = \tilde{y}_a$ $(a = R, F, L, P)$. Indeed, Eq. (39) and Eqs. (33) to (38) imply the Rindler transformations given in Eqs. (23) to (29).

For this particular parametrisation, the covering space $\tilde{\gamma}$ should therefore contain the points

$$(\tilde{y}^0, \tilde{y}^1, \tilde{y}) \in A_{\nu/4} \times B_{\nu} \times \mathbb{R}^n, \quad (39)$$
where $\nu \in \mathbb{Z}$, and where the sets $A_\nu$ and $B_\nu$ are defined by

$$A_\nu = \mathbb{R} + i\beta \nu, \quad (40)$$

$$B_\nu = \begin{cases} \mathbb{R}, & \text{when } \nu \text{ is even,} \\ \mathbb{R} - i\beta/4, & \text{when } \nu \text{ is odd}. \end{cases} \quad (41)$$

However, the covering space $\tilde{\mathcal{Y}}$ cannot only be composed of the joining of the sets in Eq. (39) over $\nu \in \mathbb{Z}$, because it has to be connected. The regions which have not yet been parametrised are the “event-horizons”, which connect the four quadrants together. They have to be included in the space $\tilde{\mathcal{Y}}$ to make it connected. One can convince oneself that the points of $\tilde{\mathcal{Y}}$ parametrising the “event-horizons” are

$$(\tilde{y}^0, \tilde{y}^1, \tilde{y}) \in H_0 \times H_1 \times \mathbb{R}^n, \quad (42)$$

where

$$H_0 = \mathbb{C} \cup \{-\infty + i\mathbb{R}\} \cup \{+\infty + i\mathbb{R}\}, \quad (43)$$

$$H_1 = -\infty + i[-\beta/4,0]. \quad (44)$$

The set $\{+\infty + i\mathbb{R}\} \times H_1 \times \mathbb{R}^n$ parametrises the “event-horizons” RF and LP excluding the bifurcation. So do the set $\{-\infty + i\mathbb{R}\} \times H_1 \times \mathbb{R}^n$ the “event-horizons” FL and PR. The bifurcation is exclusively parametrised by the set $\mathbb{C} \times H_1 \times \mathbb{R}^n$. The full connected covering space $\mathcal{Y}$ in complex tortoise coordinates is then given by

$$\mathcal{Y} = \left\{ (\tilde{y}^0, \tilde{y}^1, \tilde{y}) \in \left[ \bigcup_{\nu \in \mathbb{Z}} A_{\nu/4} \times B_{\nu} \times \mathbb{R}^n \right] \bigcup [H_0 \times H_1 \times \mathbb{R}^n] \right\}. \quad (45)$$

In summary, the sets $\bigcup_{\nu \in \mathbb{Z}} A_{\nu} \times \mathbb{R} \times \mathbb{R}^n$ and $\bigcup_{\nu \in \mathbb{Z}} A_{\nu+1/2} \times \mathbb{R} \times \mathbb{R}^n$ cover the right and left quadrants respectively, and the sets $\bigcup_{\nu \in \mathbb{Z}} A_{\nu+1/4} \times B_- \times \mathbb{R}^n$ and $\bigcup_{\nu \in \mathbb{Z}} A_{\nu+3/4} \times B_- \times \mathbb{R}^n$ the future and past quadrants. The set $H_0 \times H_1 \times \mathbb{R}^n$ parametrises the “event-horizons” including the bifurcation. A projection of the complex covering space $\mathcal{Y}$ is shown on the LHS of Fig. 3.

For the sake of completeness, we now give the parametrisations corresponding to the other analytic continuations of Table 1. The $(+, +)$ parametrisation is given by Eq. (34) where

in quad. $R$: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_R, \\ \tilde{y}^1 = \tilde{y}^1_R. \end{cases} \quad (46)$$

in quad. F: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_F + i3\beta/4, \\ \tilde{y}^1 = \tilde{y}^1_F + i\beta/4, \end{cases} \quad (47)$$

in quad. L: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_L + i\beta/2, \\ \tilde{y}^1 = \tilde{y}^1_L, \end{cases} \quad (48)$$

in quad. P: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_P + i\beta/4, \\ \tilde{y}^1 = \tilde{y}^1_P + i\beta/4, \end{cases} \quad (49)$$

where $\beta = 2\pi/\kappa$ and $\tilde{y}^\nu = \tilde{y} = \tilde{y}_a$ ($a = R, F, L, P$).

The $(+, +)$ parametrisation is given by

$$\begin{cases} (\tilde{y}^0)^\nu = \tilde{y}^0, \\ (\tilde{y}^1)^\nu = \tilde{y}^1 + i\beta\nu, \end{cases} \quad (50)$$

where $\nu \in \mathbb{Z}$ and

in quad. $R$: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_R, \\ \tilde{y}^1 = \tilde{y}^1_R. \end{cases} \quad (51)$$

in quad. F: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_F - i\beta/4, \\ \tilde{y}^1 = \tilde{y}^1_F + i\beta/4, \end{cases} \quad (52)$$

in quad. L: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_L, \\ \tilde{y}^1 = \tilde{y}^1_L + i\beta/2, \end{cases} \quad (53)$$

in quad. P: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_P - i\beta/4, \\ \tilde{y}^1 = \tilde{y}^1_P + i3\beta/4; \end{cases} \quad (54)$$

while the $(-, -)$ parametrisation is given by Eq. (50) where

in quad. $R$: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_R, \\ \tilde{y}^1 = \tilde{y}^1_R. \end{cases} \quad (55)$$

in quad. F: $$\begin{cases} \tilde{y}^0 = \tilde{y}^0_F + i\beta/4, \\ \tilde{y}^1 = \tilde{y}^1_F + i3\beta/4, \end{cases} \quad (56)$$
by the differences between these and to familiarise ourselves with them, I defined and represented four paths, denoted by \(\gamma_1\) to \(\gamma_4\), in the Kruskal associated space \(\mathcal{X}\) (see Fig. 3) and in the tortoise associated and covering spaces \(\mathcal{Y}\) and \(\widetilde{\mathcal{Y}}\) (see Figs. 2 and 5). An advantage of the winding number defined with respect to the “event-horizons” over the one defined with respect to the origin is that the former is always well defined. In the Kruskal associated space \(\mathcal{X}\), we see that the path \(\gamma_1\) crosses the “event-horizon” \(RF\) twice and that its winding number with respect to the origin vanishes. However, in the covering or associated spaces, \(\mathcal{Y}\) or \(\widetilde{\mathcal{Y}}\), we see that its winding number with respect to the “event-horizons” is +1. The path \(\gamma_2\) crosses the “event-horizons” \(RF\) and \(FL\) once and then passes through the origin before returning to the quadrant \(R\). In this case, the winding number with respect to the “event-horizons” is +1, and the one with respect to the origin is not defined. The path \(\gamma_3\) crosses the four “event-horizons” \(RF\), \(FL\), \(LP\) and \(PR\) once. Its winding numbers with respect to the “event-horizons” and to the origin are both equal to +1. Finally, we see that the path \(\gamma_4\) goes through the origin twice. In this case, the winding number with respect to the “event-horizons” vanishes, and the one with respect to the origin is again not defined.

From Eq. (13), we see that the complex tortoise covering space for the parametrisation \((+,-)\) is periodic in the imaginary time direction. One draws the same conclusion for the parametrisation \((-,+).\) The corresponding covering space can also be constructed in the latter case and found to be similar but not identical to the former case. For the parametrisations \((+,-)\) and \((-,+),\) as one can easily convince oneself, the complex tortoise covering space is periodic in the imaginary space direction. The associated spaces are similar but not identical in these two last cases as well.

E. Applying the path integral formalism

From Eq. (12), the propagator in Gui’s spacetime and in Kruskal coordinates is expressed as a sum over paths in the associated space \(\mathcal{X}\) given in Eq. (21),

$$K_x^{Kr}(x_i, x_f; s) = \sum_{z \in \mathcal{X}} \exp \left( i \nu h_x [z] \right),$$

where the metric \(h_x\) in Kruskal coordinates is defined in Eq. (24). This sum over paths defines a Kruskal-like vacuum \(K_r\). It needs to be properly defined by using a principal value because the metric and its determinant are singular at the “event-horizons” (see Appendix).

In tortoise coordinates, we now consider the 16 propagators \(K_{y,ab}^{Kr}(y_i, a; y_f, b; s)\), where \(a, b \in \{R, F, L, P\}\) and where the end points \(y_i, a\) and \(y_f, b\) belong to the quadrants \(a\) and \(b\) respectively. They describe the propagation from quadrant \(a\) to quadrant \(b\) in tortoise coordinates. For simplicity, we shall actually drop the subscript \(a\) in \(y_i, a\) when writing the propagator since the quadrants are already specified as substricts of the propagator. One writes thus \(K_{y,ab}^{Kr}(y_i, a; y_f, b; s) \equiv K_{ab}^{Kr}(y_i; y_f; s)\). Since the
propagator is a biscal, one has when \( x_i \) and \( x_f \) belong respectively to quadrants \( a \) and \( b \),

\[
K_{ab}^{Kr}(y(x_i); y(x_f); s) = K_{ab}^{Kr}(x_i; x_f; s)
\]  
(61)

where \( y_i = y(x_i) \) and \( y_f = y(x_f) \), if \( y = y(x) \) is the inverse of the Rindler transformations given in Eqs. (23) to (24). It is clear that \( K_{ab}^{Kr}(y_i; y_f; s) \) is redefined (see Appendix). The parametrisation in Eq. (61), one performs a change of coordinates to the complex tortoise coordinates covering the entire spacetime. The relevant connected associated space to consider is then \( \eta \) in any chosen parametrisation. From Eq. (61), one has then

\[
\sum_{\eta \in \gamma} \exp \left( \frac{i}{\hbar} S_{\eta}(x) \right) = \sum_{\eta \in \gamma} \exp \left( \frac{i}{\hbar} S_{\eta}(y) \right),
\]

(62)

where \( \eta \) is the Minkowski metric. When this sum over paths is written in the complex tortoise covering space \( \gamma \), one sums over all the final end point,

\[
\sum_{\eta \in \gamma} \exp \left( \frac{i}{\hbar} S_{\eta}(y) \right) = \sum_{\eta \in \gamma} \int_{\gamma_N} D_\eta[y] \exp \left( \frac{i}{\hbar} S_{\eta}(y) \right),
\]

(63)

where \( N = \eta \). Here \( \nu \) is the winding number with respect to the “event-horizons”. This path integral is also badly defined and needs to be redefined (see Appendix). The contribution of the homotopic class of winding number \( \nu \) is the free propagator \( K_0 \) with end points \( y_i \) and \( y_f \) (see Appendix),

\[
\int_{\gamma_N} D_\eta[y] \exp \left( \frac{i}{\hbar} S_{\eta}(y) \right) = K_0(y_i, y_f; s).
\]

(64)

One has \( K_0 = K_y^M = K_0 \), where \( M > 1 \) is the Minkowski vacuum and where \( K_0 \) is the free propagator; see Eq. (24). By adding the propagators of Eq. (64), the total propagator is obtained

\[
K_{ab}^{Kr}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0(y_i, y_f; s).
\]

(65)

We now restrict ourselves to the parametrisation \((+,-)\), which is periodic in the imaginary time direction; see Eq. (24). In this case, Eq. (64) becomes

\[
K_{ab}^{Kr}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0(y_i, \bar{y}_f + i\beta \nu, \bar{y}_f; s).
\]

(66)

Consequently, the propagators \( K_{ab}^{Kr} \) are periodic in the imaginary time coordinate,

\[
K_{ab}^{Kr}(y_i; y_f, y_i, y_f; s) = K_{ab}^{Kr}(y_i; y_f + i\beta \nu, y_f, y_f; s).
\]

(67)

The propagators \( K_{ab}^{Kr}(y_i; y_f; s) \) are obtained by replacing in Eq. (66) the points \( \bar{y}_i \) and \( \bar{y}_f \) by their base-point values given in Eqs. (33) to (35). One then deduces that

\[
K_{ab}^{Kr}(y_i; y_f; s) = K_{ab}^{Kr}(y_i; y_f; s)
\]

\[
= K_{ab}^{Kr}(y_i; y_f; s) = K_{ab}^{Kr}(y_i; y_f; s)
\]

(68)

where \( \nu \) is the winding number with respect to the “event-horizons”. This path integral is also badly defined and needs to be redefined (see Appendix). The parametrisation in Eq. (68) is periodic in the imaginary time direction, so that Eq. (66) is also true in this case. By applying this equation to the base-point values given in Eqs. (40) to (49), one obtains further the decompositions,

\[
K_{ab}^{Kr}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0(y_i, y_f + i\beta \nu, y_f, y_f; s),
\]

(69)

\[
K_{ab}^{Kr}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0(y_i, y_f + i\beta \nu, y_f, y_f; s),
\]

(70)

where the property \( K_0(y_i, \bar{y}_f + \bar{z}; y_f; s) = K_0(y_i, \bar{y}_f + \bar{z}; s) \) \((\bar{z} \in \mathbb{R}^0)\) has been used. The propagators in Eq. (68) describe a steady flux of thermal radiation of temperature \( T = \hbar k/(2\pi k) \), where \( k \) is the Boltzmann constant.

The parametrisation \((-,+)\) is also periodic in the imaginary time direction, so that Eq. (66) is also true in this case. By applying this equation to the base-point values given in Eqs. (40) to (49), one obtains further the decompositions,

\[
K_{ab}^{Kr}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0(y_i, y_f + i\beta \nu, y_f, y_f; s),
\]

(71)

\[
K_{ab}^{Kr}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0(y_i, y_f + i\beta \nu, y_f, y_f; s),
\]

(72)

\[
K_{ab}^{Kr}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0(y_i, y_f + i\beta \nu, y_f, y_f; s),
\]

(73)

Since \( K_0(y_i; y_f; s) = K_0(y_i; y_f; s) \) and \( K_0(y_i, \bar{y}_f + \bar{z}; s) = K_0(y_i, \bar{y}_f + \bar{z}; s) \) we deduce that the decompositions in Eqs. (71) and (72) are equivalent to the ones in Eqs. (43) and (44) respectively. From the results obtained in this section up to now, one can check that the propagator is hermitian, i.e. that

\[
K_{ab}^{Kr}(y_i; y_f; s)^* = K_{ba}^{Kr}(y_f; y_i; s),
\]

(74)

\( \forall a, b \in \{ R, F, L, P \} \), as it should be since \( \bar{Y}^* = \bar{Y} \) (see discussion after Eq. (44)).
The parametrisations \((+, +)\) and \((-,-)\) are periodic in the imaginary space direction; see Eq. (55). Consequently, we have from Eq. (55),

\[
K^R_y(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0 \left( y_i, y_f^0, y_f^1 + i\beta\nu, y_f; s \right). \quad (75)
\]

In a similar way than above, and from the base-point values given in Eqs. (64) to (67) and (68) to (71) respectively, we deduce from this last equation the decompositions

\[
K^R_R(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0 \left( y_i, y_f^0, y_f^1 + i\beta\nu, y_f; s \right), \quad (76)
\]

\[
K^R_F(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0 \left( y_i, y_f^0 - i\beta/4, y_f^1 + i\beta(\nu + 1/4), y_f; s \right), \quad (77)
\]

\[
K^R_L(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0 \left( y_i, y_f^0 + i\beta/4, y_f^1 + i\beta(\nu + 3/4), y_f; s \right), \quad (78)
\]

\[
K^R_K(y_i; y_f; s)
= \sum_{\nu \in \mathbb{Z}} K_0 \left( y_i, y_f^0 - i\beta/4, y_f^1 + i\beta(\nu + 3/4), y_f; s \right), \quad (80)
\]

\[
= \sum_{\nu \in \mathbb{Z}} K_0 \left( y_i, y_f^0 + i\beta/4, y_f^1 + i\beta(\nu + 1/4), y_f; s \right). \quad (81)
\]

The decompositions in Eqs. (77) and (78) are equivalent, so are those in Eqs. (80) and (81).

**F. Non-static Gui’s spacetime**

We now review the non-static Gui spacetime and give some new results. This spacetime has been introduced and studied in Ref. [11]. In \(D\) dimension, it is defined by the metric

\[
ds^2_\gamma = -\frac{dx^+ dx^-}{\kappa x^-} - (dx)^2, \quad (82)
\]

where \(\kappa > 0\), \(x^+ = x^0 \pm x^t\), and \(x \in \mathbb{R}^n, (D = n+2)\). The regions \(I\) and \(II\) are defined by the half-planes \(x^- < 0\) and \(x^- > 0\) respectively. The “event-horizon” is located at \(x^- = 0\). Two sets of tortoise type coordinates \(y_i\) and \(y_f\) are defined by

- in region \(I\): \(x^-(y_i^-) = -\exp(-\kappa y_i^-)\), \(\quad (83)\)
- in region \(II\): \(x^-(y_f^-) = +\exp(-\kappa y_f^-)\), \(\quad (84)\)

and by \(x^+ = y_i^{1/2}, x = y_i, u\). The complex tortoise coordinates \(\tilde{y} \in \tilde{Y}\) are defined in the covering space by

\[
\begin{cases}
(\tilde{y}^+) = \tilde{y}^+ + i\beta\nu, \\
(\tilde{y}^-) = \tilde{y}^- + i\beta\nu,
\end{cases} \quad (85)
\]

where

- in region \(I\): \(\tilde{y}^+ = y_i^+, \quad (86)\)
- in region \(II\): \(\tilde{y}^- = y_i^-\), \(\quad (87)\)

and by \(\tilde{y}^\nu = \tilde{y} = y_i, u\). One deduces from Eqs. (63) to (67) that

\[
\tilde{x}^- (\tilde{y}^-) = -\exp(-\kappa \tilde{y}^-), \quad (88)
\]

in the joining of regions \(I\) and \(II\). The complex covering space \(\tilde{Y}\) is then given by

\[
\tilde{Y} = \{ (\tilde{y}^+, \tilde{y}^-) \in \mathbb{R} \times \bigcup_{\nu \in \mathbb{Z}} A_{\nu/2} \}, \quad (89)
\]

where

\[
A_\nu = \mathbb{R} + i\beta\nu, \quad (90)
\]

\[
H = +\infty + i\mathbb{R}. \quad (91)
\]

The sets \(\bigcup_{\nu \in \mathbb{Z}} A_{\nu}\) and \(\bigcup_{\nu \in \mathbb{Z}} A_{\nu+1/2}\) parametrise regions \(I\) and \(II\) respectively. The axis \(H\) at infinity parametrises the “event-horizon”.

Since the covering space \(\tilde{Y}\) is the complex conjugate of itself, quantum evolution is unitary in the fifth parameter, even through the event-horizon. And because the covering space is periodic in the imaginary \(\tilde{y}\) direction, one has from Eq. (68)

\[
K^R_y(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0 \left( \tilde{y}_i, \tilde{y}_f^0, \tilde{y}_f^1 + i\beta\nu, \tilde{y}_f; s \right), \quad (92)
\]

where \(\beta = 2\pi/\kappa\); see Ref. [11]. One then defines the 4 propagators \(K^R_{ab}(y_i; y_f; s)\), where \(a, b \in \{I, II\}\) and \(y_i\) and \(y_f\) belong to regions \(a\) and \(b\) respectively. They describe the propagation from region \(a\) to region \(b\).

In a similar way as in Sec. III E, one considers the base-point values given in Eqs. (80) and (81). From Eq. (92), the propagators are then given by

\[
K^R_{I,I}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0 \left( \tilde{y}_i, \tilde{y}_f^0, \tilde{y}_f^1 + i\beta\nu, \tilde{y}_f; s \right), \quad (93)
\]

\[
K^R_{I,II}(y_i; y_f; s) = \sum_{\nu \in \mathbb{Z}} K_0 \left( y_i, y_f^0 + i\beta(\nu + 1/2), y_f^1 + i\beta\nu, y_f; s \right), \quad (94)
\]
IV. ETERNAL BLACK HOLE

A. Two-dimensional case

The 2D Schwarzschild black-hole line element is given by [13]

$$ds_g^2 = \frac{1}{\kappa^2} \left[ 1 - \frac{2M}{r} \right] du^+ du^-,$$

(95)

where $\kappa = (4M)^{-1}$ and

$$[u^+/u^-] = \exp(t/2M),$$

(96)

$$u^+ u^- = -(r - 2M) \exp(r/2M),$$

(97)

if $t$ and $r$ are the time and radius coordinates; see Fig. 3. The $u$ coordinates are Kruskal coordinates. The tortoise coordinates $v_a$ ($a = R, F, L, P$) are defined by the Rindler transformations, i.e., by Eqs. (23) to (26) when $x$ is replaced by $u$ and $y_a$ by $v_a$. The scalar curvature vanishes asymptotically far away from the black hole. The Kruskal and tortoise associated spaces, denoted by $\mathcal{U}$ and $\mathcal{V}_a$ respectively ($a = R, F, L, P$), are given by

$$\mathcal{U} = \{ u \in \mathbb{R}^2 \mid r(u) > 0 \},$$

(98)

$$\mathcal{V}_a = \{ v_a \in \mathbb{R}^2 \},$$

(99)

$$\mathcal{V}_b = \{ v_b \in \mathbb{R}^2 \mid r(v) > 0 \},$$

(100)

These spaces have been restricted to the spacetime events for which the radius is strictly positive. The spaces $\mathcal{V}_F$ and $\mathcal{V}_P$ parametrise the interior regions of the black hole. The spaces $\mathcal{U}$ and $\mathcal{V}_a$ are clearly isomorphic to their covering spaces $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}_a$ ($a = R, F, L, P$).

The vacuum defined by the sum over paths,

$$K_v^{Kr}(v_i; v_f; s) = \sum_{v_i, v_f} \exp \left( \frac{i}{\hbar} S_{v_i} [\tilde{u}] \right),$$

(101)

is the Kruskal vacuum $|Kr>$.

As in Gui’s case, one introduces complex tortoise coordinates. For the parametrisation $(+, -)$, these are defined by Eqs. (12) to (13), where $\tilde{y}$ is replaced by $\tilde{v}$, and $\tilde{y}_a$ by $\tilde{v}_a$. The Rindler transformation in the covering space then becomes

$$\tilde{u}^0 = (1/\kappa) \exp(\kappa \tilde{y}^0) \sinh(\kappa \tilde{u}^0),$$

$$\tilde{u}^1 = (1/\kappa) \exp(\kappa \tilde{y}^1) \cosh(\kappa \tilde{u}^0);$$

see Eq. (29). The connected covering space $\widetilde{\mathcal{V}}$ in complex tortoise coordinates is consequently given by

$$\tilde{\mathcal{V}} = \{ (\tilde{v}^0, \tilde{v}^1) \in$$

$$\left[ \bigcup_{\nu \in \mathbb{Z}} A_{\nu/4} \times B_{\nu} \right] \bigcup \{ H_0 \times H_1 \mid r(v) > 0 \},$$

(103)

where $A_\nu$, $B_\nu$, $H_0$ and $H_1$ are defined in Eqs. (10), (11), (13) and (14); see also Eq. (15). The space $\mathcal{V}$ is shown along with the corresponding associated space $\mathcal{V}$ in Fig. 4.

Since the covering space is invariant under the action of the elements of the holonomy group $\Gamma$, given by

$$\Gamma = \{ \gamma^\nu : (\hat{v}^0, \hat{v}^1) \rightarrow (\hat{v}^0 + i \beta \nu, \hat{v}^1), \nu \in \mathbb{Z} \},$$

(104)

the propagator satisfies

$$\tilde{K}_v(\hat{v}^0; \hat{v}_f^0 + i \beta \nu, \hat{v}_f^1; s) = \tilde{K}_v(\hat{v}^0; \hat{v}_0^0 - i \beta \nu, \hat{v}_0^1; \hat{v}_f; s),$$

(105)

where $\nu \in \mathbb{Z}$. One obtains furthermore from Eq. (16)

$$K_v^{Kr}(v_i; v_f; s) = \sum_{\nu \in \mathbb{Z}} K_v^B(\tilde{v}^0_i; \tilde{v}_f^0 + i \beta \nu, \tilde{v}^1_f; s),$$

(106)

where $|B>$ is the Boulware vacuum [13]. Contrary to Gui’s case, the propagator $K_v^B$ is not the free one, since $g_v$ is not the Minkowski metric. From this last equation, one obtains similar results than that of Gui’s spacetime for $D = 2$, i.e., Eqs. (18) to (21) remain true if $K_0$ is replaced by $K_v^B$. One has then

$$K_{rr}^{Kr}(v_i; v_f; s) = \sum_{\nu \in \mathbb{Z}} K_v^B(v_i^0 + i \beta \nu, v_i^1; s),$$

(107)

$$K_v^{Kr}(v_i; v_f; s) = \sum_{\nu \in \mathbb{Z}} K_v^B(v_i^0 + i \beta(\nu + 1/4), v_i^1 - i \beta/4; s),$$

(108)

$$K_{rl}^{Kr}(v_i; v_f; s) = \sum_{\nu \in \mathbb{Z}} K_v^B(v_i^0 + i \beta(\nu + 1/2), v_i^1; s),$$

(109)

$$K_{rp}^{Kr}(v_i; v_f; s) = \sum_{\nu \in \mathbb{Z}} K_v^B(v_i^0 + i \beta(\nu + 3/4), v_i^1 - i \beta/4; s),$$

(110)

and other results similar to those of Sec. III E.

We now concentrate our attention on a connected region $\mathcal{R}$ located far away from the black hole (i.e. where $r \gg 2M$, $\forall t \in \mathbb{R}$). In $\mathcal{R}$, the line element, Eq. (83), becomes

$$ds_g^2 \approx \frac{1}{\kappa^2} \frac{du^+ du^-}{u^+ u^-}.$$

(111)

Under the identifications $u \equiv x$, the asymptotic form of the 2D black-hole line element in $\mathcal{R}$, Eq. (114), is identical to the Gui metric, Eq. (24), when $D = 2$.

In the 2D Gui spacetime, there is only one geodesic joining any two points in the Kruskal associated space. If one assumes that these points are located in $\mathcal{R}$ and that they are on the same side of the black hole, this unique
geodesic will be entirely contained within $\mathcal{R}$. Since spacetime is asymptotically flat in $\mathcal{R}$, one may write from Eq. (10),
\[
K_{u}^{Kr}(u_{i}; u_{f}; s) \cong \frac{1}{4\pi \hbar s} \exp \left[ \frac{i}{\hbar} \sigma_{g_{u}}(u_{i}, u_{f}) \right].
\]
(112)
The exact equivalent expression can be written in Gui’s spacetime,
\[
K_{x}^{Kr}(x_{i}; x_{f}; s) = \frac{1}{4\pi \hbar s} \exp \left[ \frac{i}{\hbar} \sigma_{h_{x}}(x_{i}, x_{f}) \right].
\]
(113)

From now on, I shall use the notation $\cong$ to compare two functions which tend to each other in a given limit when their arguments are identified, and also the term “asymptotically equivalent” to refer to this property. For example, from Eqs. (21) and (111), the metrics $h_{x}$ and $g_{u}$ are asymptotically equivalent in $\mathcal{R}$, i.e. $h_{x} \cong g_{u}$, when the coordinates $x$ and $u$ are identified, i.e. when $x = u$.

Returning to Eqs. (12) and (13), and since under our assumption the unique geodesic is entirely contained within $\mathcal{R}$, $h_{x} \cong g_{u}$ implies that $\sigma_{h_{x}} \cong \sigma_{g_{u}}$ in $\mathcal{R}$. Consequently, the propagators in Eqs. (12) and (13) are asymptotically equivalent in this region as well, i.e. $K_{x}^{Kr}(x_{i}; x_{f}; s) \cong K_{u}^{Kr}(u_{i}; u_{f}; s)$ in $\mathcal{R}$.

In terms of sum over paths, we have
\[
\sum_{u_{i} \rightarrow u_{f}} \exp \left( \frac{i}{\hbar} S_{g_{u}}[u] \right) \cong \sum_{x_{i} \rightarrow x_{f}} \exp \left( \frac{i}{\hbar} S_{h_{x}}[x] \right).
\]
(114)
Thus although one sums over paths that leave the region $\mathcal{R}$ and cross the event-horizon, the propagators of the two problems are nevertheless asymptotically equivalent. As it will become clear below, this is not true in higher dimensional spacetimes for topological reasons. Equations (33) and (114) implies then that the propagator $K_{v}^{Kr}$ in tortoise coordinates is asymptotically equal in region $\mathcal{R}$ to the propagator of a steady flux of thermal radiation. In terms of Schwarzschild coordinates, one has
\[
K_{v}^{Kr}(t_{i}, r_{i}; t_{f}, r_{f}; s) \cong \sum_{\nu \in \mathbb{Z}} K_{0}(t_{i}, r_{i}; t_{f} + i\beta \nu, r_{f}; s).
\]
(115)
Furthermore, from Eqs. (60) and (114) one also has the asymptotic relation
\[
K_{v}^{Kr}(t_{i}, r_{i}; t_{f}, r_{f}; s) \approx \sum_{\nu \in \mathbb{Z}} K_{0}(t_{i}, r_{i}; t_{f} + i\beta \nu; r_{f}; s).
\]
(116)

In conclusion, the 2D Gui space is the approximate space for the 2D Schwarzschild black hole asymptotically far away from it, in the sense of section B.

### B. Four dimensional case

In the 4D case, the Schwarzschild black-hole line element is given by
\[
\frac{ds^{2}}{\kappa^{2}} = \frac{1}{1 - 2M/r} \left( \frac{du^{+} du^{-}}{u^{+} u^{-}} - r^{2} d\Omega^{2} \right),
\]
(117)
where $\Omega$ is the solid angle, $\kappa = (4M)^{-1}$ and where $r$ and $t$ are also given by Eqs. (68) and (77). In a similar way as that of the 2D case, one defines real and complex tortoise coordinates. One then finds that the covering space in complex tortoise coordinates is given by
\[
\tilde{\mathcal{V}} = \left\{ (v^{0}, v^{1}, \tilde{\Omega}) \in \bigcup_{\nu \in \mathbb{Z}} A_{\nu/4} \times B_{\nu} \times S^{2} \cup \{ H_{0} \times H_{1} \times S^{2} \} : r(v) > 0 \right\};
\]
(118)
see Eq. (103). Equations (107) to (110) obtained in the 2D case can be easily generalised to the 4D case. One then has
\[
K_{rR}^{Kr}(v_{i}; v_{f}; s) = \sum_{\nu \in \mathbb{Z}} K_{v}^{B}(v_{i}; v_{f}^{0} + i\beta \nu, v_{f}^{1}, \Omega_{f}; s),
\]
(119)
\[
K_{rF}^{Kr}(v_{i}; v_{f}; s) = \sum_{\nu \in \mathbb{Z}} K_{v}^{B}(v_{i}; v_{f}^{0} + i\beta(\nu + 1/4), v_{f}^{1} - i\beta/4, \Omega_{f}; s),
\]
(120)
\[
K_{rL}^{Kr}(v_{i}; v_{f}; s) = \sum_{\nu \in \mathbb{Z}} K_{v}^{B}(v_{i}; v_{f}^{0} + i\beta(\nu + 1/2), v_{f}^{1}, \Omega_{f}; s),
\]
(121)
\[
K_{rP}^{Kr}(v_{i}; v_{f}; s) = \sum_{\nu \in \mathbb{Z}} K_{v}^{B}(v_{i}; v_{f}^{0} + i\beta(\nu + 3/4), v_{f}^{1} - i\beta/4, \Omega_{f}; s),
\]
(122)
and other results similar to those of Sec. IIII.

In the 4D black-hole spacetime, there are a infinite number of geodesics joining two end points, contrary to the 2D case. This gives a multiply connected structure to this spacetime. In its space projection, the geodesics have a well defined winding number $\mu$ around the origin $r = 0$; see Fig. 4. The geodesics will not cross the horizon if the end points are located outside the black hole. In this case, one writes from Eq. (11)
\[
K_{u}^{Kr}(u_{i}; u_{f}; s) = \frac{1}{4\pi \hbar s} \sqrt{\Delta(u_{i}; u_{f})} \left( \begin{array}{c}
F(u_{i}; u_{f}; s) \\
\times \sum_{\mu \in \mathbb{Z}} \exp \left[ \frac{i}{\hbar} \sigma_{g_{u}}(u_{i}, u_{f}; \mu) \right]
\end{array} \right),
\]
(123)
where \( \sigma_{g_\ast}(v_i, v_f; \mu) \) is the proper arc length between the end points along the geodesic of winding number \( \mu \). Some of these geodesics probe spacetime close to the black hole, where Gui’s metric for \( D = 4 \), Eq. (20), and the 4D black-hole metric, Eq. (117), differ significantly. Since these geodesics are not contained entirely within the region \( \mathcal{R} \), Eq. (114), obtained in the 2D case, is not true in the 4D case. Physically, this means that the potential barrier close to the black hole modifies the properties of the radiation. If its influence is neglected, as is often done in the 2D black-hole case.

In this paper it was shown that it is possible to evaluate quantum mechanical path integrals in spacetimes endowed with event-horizons. In order to do so, we worked in tortoise coordinates, for which the metric looks like the Minkowski metric, at least in the region of interest, for example far away from the black hole. We then introduced complex tortoise coordinates to cover the entire spacetime at once. The global properties of the path integral, which are related to the boundary conditions of the propagator, have then been exploited to obtain its thermal features, via the spacetime non-trivial associated topology in complex tortoise coordinates.

An advantage of using complex tortoise coordinates relies on the fact that some global issues can be addressed in these coordinates, such as the calculation of the propagator whose end points are located in spacetime regions separated by one or several event-horizons. It is not clear to me that it is possible to do this within the framework of a strict Euclidian approach. The hermiticity property of the propagator can then be analysed globally. In particular, it was shown that quantum evolution in the fifth variable is unitary through the event-horizons in both the static and non-static versions of the Gui and Schwarzschild spacetimes.

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**APPENDIX**

In this Appendix, one redefines and compute the sums over paths in Gui’s spacetime.

In the Kruskal covering space \( \tilde{\mathcal{X}} \), the sum over paths in Eq. (64) is defined as a path integral whose integration over the variable \( \tilde{x}^\pm \) is performed by using the principal value, i.e. one defines

\[
\sum_{x_i \rightarrow x_f^\pm} \exp \left( \frac{i}{\hbar} S_{h_x} [x] \right) = \int_{\tilde{x}_i}^{\tilde{x}_f} D_{h_{\tilde{x}}} [\tilde{x}] \exp \left( \frac{i}{\hbar} \tilde{S}_{h_{\tilde{x}}} [\tilde{x}] \right),
\]

if the space \( \tilde{\mathcal{X}}' \) is given by

\[
\tilde{\mathcal{X}}' = \{ (\tilde{x}^+, \tilde{x}^-, \tilde{x}) \in \mathbb{R} \times \mathbb{R}' \times \mathbb{R}^n \},
\]

where \( \mathbb{R}' = \lim_{\epsilon \rightarrow 0} (-\infty, -\epsilon] \cup [+\epsilon, +\infty) \).
In the covering space \( \tilde{Y} \), since the sum over paths on
the RHS of Eq. (\ref{eq:propagator}) is also badly defined when written
as a path integral, one introduces the space \( \tilde{Y}' \) by
\[
\tilde{Y}' = \{ (\eta^0, \eta^1, \tilde{y}) \in \bigcup_{\mu \in \mathbb{Z}} A_{\mu/4} \times B_{\mu} \times \mathbb{R}^n \},
\]
i.e. one removes from the covering space \( \tilde{Y} \) the set
parametrising the “event-horizon” as in Kruskal coordi-
mates. One then rewrites the sum over paths in the form
\[
\sum_{\eta} \exp \left( i \frac{\eta}{\hbar} S_\eta[\tilde{y}] \right) = \sum_{\nu' \in \mathbb{Z}} \int_{\tilde{y}_{i \nu}}^{\tilde{y}_{f \nu}} D\tilde{y}[\tilde{y}] \exp \left( i \frac{\tilde{y}}{\hbar} S_{\tilde{y}}[\tilde{y}] \right),
\]
where \( \tilde{y} = \eta \). In this last equation, we have taken into
account the multiply connected nature of \( Y \) by summing
over the homotopic classes of winding number \( \nu \).

In Eq. (\ref{eq:propagator}) each path integral in the sum contains the term
\[
\int_{\tilde{y}'} d^D \tilde{y}_j \exp \left( i \frac{\eta}{\hbar} \sum_{k=1}^{N} \int_{\tilde{y}_{k-1}}^{\tilde{y}_k} d\omega \tilde{L}_g(\tilde{y}, \dot{\tilde{y}}) \right)
\]
\[
= \sum_{\mu \in \mathbb{Z}} \int_{A_{\mu/4}} d\tilde{y}_j \int_{B_{\mu}} d\tilde{y}_j \int_{\mathbb{R}^n} d\tilde{y}_j \times \exp \left( i \frac{\tilde{y}}{\hbar} \sum_{k=1}^{N} \int_{\tilde{y}_{k-1}}^{\tilde{y}_k} d\omega \tilde{\eta}_{\mu} \tilde{\eta}^{\mu} (\omega) \tilde{y}' (\omega) \right).
\]
Since the integrand in the RHS of this last equation does not
actually depend on the imaginary value of \( \tilde{y}_j \), one has
\[
\int_{\tilde{y}'} d^D \tilde{y}_j \exp \left( i \frac{\eta}{\hbar} \sum_{k=1}^{N} \int_{\tilde{y}_{k-1}}^{\tilde{y}_k} d\omega \tilde{\eta}_{\mu} \tilde{\eta}^{\mu} (\omega) \tilde{y}' (\omega) \right)
\]
\[
= C \int_{\mathbb{R}^D} d^D \tilde{y}_j \exp \left( i \frac{\eta}{\hbar} \sum_{k=1}^{N} \int_{\tilde{y}_{k-1}}^{\tilde{y}_k} d\omega \tilde{\eta}_{\mu} \tilde{\eta}^{\mu} (\omega) \tilde{y}' (\omega) \right),
\]
where \( C \) is an infinite constant. This constant is removed
by renormalising the path integral to take into account
the fact that the integration is performed over an infi-
te number of copies of \( \mathbb{R}^D \). The contribution of
the homotopic class of winding number \( \nu \) is then the free
propagator \( \tilde{K}_0 \) with arguments \( \tilde{y}_i \) and \( \tilde{y}'_i \),
\[
\int_{\tilde{y}_i}^{\tilde{y}_f} D\tilde{y}[\tilde{y}] \exp \left( i \frac{\tilde{y}}{\hbar} S_{\tilde{y}}[\tilde{y}] \right) = \tilde{K}_0 (\tilde{y}_i; \tilde{y}'_f; s).
\]
Equation (\ref{eq:propagator}) is finally obtained from Eqs. (\ref{eq:propagator}) and
(\ref{eq:propagator}).
FIG. 1. The relationship between the base, approximate, associated and covering spaces ($\Gamma$ is the holonomy group). A change of coordinates admitting a singularity modifies the spacetime associated topology.

FIG. 2. The Kruskal associated space $\mathcal{X}$ of Gui’s spacetime. The Kruskal coordinates $x$ and the tortoise coordinates $y_\alpha$ are shown ($\alpha = L, F, R, P$). The arrows indicate the direction of increasing values.
FIG. 3. On the left hand side is shown a section of the complex tortoise covering space $\tilde{Y}$ of Gui’s spacetime. The similar section of the complex tortoise associated space $\tilde{\gamma}$ is shown on the right hand side.

FIG. 4. The paths $\gamma_1$ to $\gamma_4$ in the Kruskal associated space $\tilde{X}$. 
FIG. 5. On the left hand side are shown the paths $\gamma_1$ and $\gamma_2$ in the complex tortoise covering space $\tilde{\mathcal{Y}}$. These paths are shown in the complex tortoise associated space $\mathcal{Y}$ on the right hand side.
FIG. 6. On the left hand side are shown the paths $\gamma_3$ and $\gamma_4$ in the complex tortoise covering space $\widetilde{\mathcal{Y}}$. These paths are shown in the complex tortoise associated space $\mathcal{Y}$ on the right hand side.

FIG. 7. The Kruskal associated space $\mathcal{U}$ of the Schwarzschild black-hole spacetime. The regions $F$ and $P$ are the black-hole interior regions, and the regions $R$ and $L$ are the black-hole exterior regions.
FIG. 8. On the left hand side is shown a section of the complex tortoise covering space $\tilde{\mathcal{V}}$ of the Schwarzschild black-hole spacetime. The similar section of the complex tortoise associated space $\mathcal{V}$ is shown on the right hand side.

FIG. 9. A path is shown in the space projection of a Schwarzschild black-hole spacetime. The path winding number $\mu$ with respect to the black-hole singularity is $+2$. 
| Quadrant | Analytic continuations |
|----------|------------------------|
|          | (+,+)       | (+,-)       | (-,+)       | (-,-)       |
| Quadrant R | \( \text{Im } \tilde{y}^0 \) | 0           | \( \beta \nu \) | \( \beta \nu \) | 0           |
|          | \( \text{Im } \tilde{y}^1 \) | \( \beta \nu \) | 0           | 0           | \( \beta \nu \) |
| Quadrant F | \( \text{Im } \tilde{y}^{0} \) | \( -\beta/4 \) | \( +\beta/4+\beta \nu \) | \( +3\beta/4+\beta \nu \) | \( +\beta/4 \) |
|          | \( \text{Im } \tilde{y}^{1} \) | \( +\beta/4+\beta \nu \) | \( -\beta/4 \) | \( +\beta/4 \) | \( +3\beta/4+\beta \nu \) |
| Quadrant L | \( \text{Im } \tilde{y}^{0} \) | 0           | \( +\beta/2+\beta \nu \) | \( +\beta/2+\beta \nu \) | 0           |
|          | \( \text{Im } \tilde{y}^{1} \) | \( +\beta/2+\beta \nu \) | 0           | 0           | \( +\beta/2+\beta \nu \) |
| Quadrant P | \( \text{Im } \tilde{y}^{0} \) | \( -\beta/4 \) | \( +3\beta/4+\beta \nu \) | \( +\beta/4+\beta \nu \) | \( +\beta/4 \) |
|          | \( \text{Im } \tilde{y}^{1} \) | \( +3\beta/4+\beta \nu \) | \( -\beta/4 \) | \( +\beta/4 \) | \( +\beta/4+\beta \nu \) |

**TABLE I.** The four possible ways to continue analytically the functions \( \tilde{y}^0 = \tilde{y}^0(x^0, x^1) \) and \( \tilde{y}^1 = \tilde{y}^1(x^0, x^1) \) in the complex tortoise covering space \((\beta = 2\pi/\kappa, \nu \in \mathbb{Z})\); see Eq. (30). The couple of signs on the first line indicates the directions (anticlockwise (+) or clockwise (−)) with respect to which the corresponding analytical extensions has been performed in the \( \tilde{x}^+ \) and \( \tilde{x}^- \) complex planes respectively assuming the sequence \((R, F, P, L)\).