THE DEGREE OF THE JACOBIAN LOCUS AND THE SCHOTTKY PROBLEM

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Abstract. We show that the degree of the images of the (level covers of) moduli space of principally polarized abelian varieties $A_g$ and of the Jacobian locus $J_g$ under the embedding $Th$ into the projective space by theta constants are equal to the top self-intersection numbers of one half the first Hodge class on them. This allows us to obtain an explicit formula for $\text{deg} Th(A_g)$ in all genera, compute $\text{deg} Th(J_g)$ for small $g$, and obtain an explicit upper bound for $\text{deg} Th(J_g)$ for all $g$.

Knowing $\text{deg} Th(A_g)$ allows us to effectively determine this subvariety, i.e. effectively obtain all polynomial equations satisfied by theta constants. Furthermore, combining the bound on $\text{deg} Th(J_g)$ with effective Nullstellensatz allows us to rewrite the Kadomtsev-Petviashvili (KP) partial differential equation as a system of algebraic equations for theta constants, and thus effectively obtain an algebraic solution to the Schottky problem.

1. Introduction

The Schottky problem, the question of characterizing Jacobians of Riemann surfaces among principally polarized abelian varieties (ppavs), was solved by Shiota more than a hundred years after the problem was first posed. Shiota completed the proof of Novikov’s conjecture: that a ppav is a Jacobian if and only if a certain modification of its associated theta function satisfies the KP equation [Sh]. However, this solution of the Schottky problem is not effective in the following sense: first, it requires choosing the values of certain $3g + 1$ parameters used to modify the theta function, and, secondly, verifying that the KP equation, a partial differential equation, holds for these values of parameters. Checking the validity of such an equation is a hard task — it entails verifying that the sum of a certain Fourier series is equal to zero at a given point, and this cannot be done effectively.
Another approach to the Schottky problem, the Schottky–Jung theory (see [Do], [Far], and [vG] for a review), if successful, would yield a system of algebraic equations for theta constants of the second order defining the locus of Jacobians inside the moduli space of ppavs, and thus would give an effective algebraic solution to the Schottky problem. However, so far this approach is only known to characterize Jacobians up to additional components.

Numerous other approaches have been developed, but despite much progress (see, for example, [Deb] and Arbarello’s appendix to [Mu1] for reviews), an entirely algebraic solution to the Schottky problem, in the spirit of the Schottky’s original approach, has not yet been obtained for genus higher than four. Some effective criteria to check that an abelian variety is not a Jacobian have been obtained, though (see [La2]), but if one produces an explicit ppav, there is no known way to show that it actually is a Jacobian, apart from explicitly constructing the corresponding curve.

In this paper we study the algebraic degrees of the image of the Jacobian locus and of the moduli space of ppavs in the projective space under the theta constants map, and obtain an effective algebraic solution to the Schottky problem (“effective” here means that there is an actual explicit finite procedure that produces the complete set of algebraic equations in theta constants defining the Jacobian locus).

The structure of the work is as follows. We introduce notations in section 2; in sections 3 and 4 we express the degrees in question as some intersection numbers on moduli spaces. In section 5 we compute and discuss these numbers for low genera. In section 6 we use the methods of intersection theory on the moduli space, together with Schumacher and Trapani’s [ScTr] and our [Gr] results on Weil-Petersson volumes to obtain an explicit upper bound for the degree of the Jacobian locus. In sections 7 and 8 we then show how the KP equation can be effectively rewritten as a system of algebraic equations for theta constants characterizing Jacobians, and also how to effectively obtain all relations among theta constants of general abelian varieties.

2. Notations

Fix some genus \( g > 0 \), which we will also interchangeably call the dimension. Let \( \Lambda \) be a lattice in \( \mathbb{C}^g \) of maximal rank, and let \( A = \mathbb{C}^g/\Lambda \) be the associated abelian variety. We will think of \( A \) as being principally polarized and of \( \Lambda \) as being generated over \( \mathbb{Z} \) by the unit vectors in all directions and the columns of a complex \( g \times g \) matrix \( \tau \), called the period matrix for \( A \). The period matrix \( \tau \) has to be
symmetric with positive definite imaginary part; the set of such \( \tau \) is the *Siegel upper half-space* \( \mathcal{H}_g \). Different period matrices may define isomorphic ppavs — this corresponds to a change of basis in \( \Lambda \) under the action of the symplectic group \( \Gamma_g := \text{Sp}(2g, \mathbb{Z}) \) on the Siegel upper half-space. Representing \( \gamma \in \Gamma_g \) as \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where \( A, B, C, D \in \text{Mat}_{g \times g}(\mathbb{Z}) \), with the symplectic condition being

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

the action of \( \Gamma_g \) on \( \mathcal{H}_g \) is given by \( \gamma \tau := (A\tau + B)(C\tau + D)^{-1} \). We denote by \( \mathcal{A}_g \) the fundamental domain for this action, which is the moduli space of ppavs of dimension \( g \) up to biholomorphisms: \( \mathcal{A}_g := \mathcal{H}_g / \Gamma_g \).

We also define *level* \( n \) subgroups of the symplectic group, for \( n \) a positive integer:

\[
\Gamma_g(n) := \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid \gamma \in \Gamma_g, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}
\]

\[
\Gamma_g(n, 2n) := \left\{ \gamma \mid \gamma \in \Gamma_g(n), \text{diag}(AB^t) \equiv \text{diag}(CD^t) \equiv 0 \pmod{2n} \right\}.
\]

A function \( f : \mathcal{H}_g \to \mathbb{C} \) is called a *modular form of weight* \( k \) with respect to a subgroup \( \Gamma \subset \Gamma_g \) if

\[
f(\gamma\tau) = \det(C\tau + D)^k f(\tau) \quad \forall \gamma \in \Gamma, \forall \tau \in \mathcal{H}_g.
\]

We define *theta functions of the second order* to be, for \( \varepsilon \in \frac{1}{2}\mathbb{Z}^g / \mathbb{Z}^g \), \( \tau \in \mathcal{H}_g \), and \( z \in \mathbb{C}^g \)

\[
\Theta[\varepsilon](\tau, z) := \sum_{n \in \mathbb{Z}^g} \exp(2\pi i(n + \varepsilon)^t \tau(n + \varepsilon) + 2\pi i(n + \varepsilon)^t z).
\]

The values of theta functions for \( z = 0 \) are called *theta constants*. These theta constants are modular forms of weight one half with respect to \( \Gamma_g(2, 4) \) (see \([\text{Ig1}]\)). Geometrically they are sections of a certain line bundle \( L \) over a finite cover \( \mathcal{A}_g^{2,4} := \mathcal{H}_g / \Gamma_g(2, 4) \) of \( \mathcal{A}_g \). Under the action of \( \Gamma_g(2, 4) \) all theta constants multiply by the same factor \( \det(C\tau + D)^{1/2} \), and thus the mapping

\[
Th : \tau \mapsto \{ \Theta[\varepsilon](\tau) \}_{\varepsilon \in \frac{1}{2}\mathbb{Z}^g / \mathbb{Z}^g}
\]

gives a well-defined map \( Th : \mathcal{A}_g^{2,4} \to \mathbb{P}^{2^g - 1} \), which is known to be generically injective (see \([\text{Sas}], \text{[Sa2]}\)). In \([\text{Sa2}]\) Salvati Manni claims the injectivity of the map, but he has informed us that there is a small gap in the proof. For our purposes here generic injectivity combined with the statement that \( Th \) is at most finite-to-one everywhere, (which follows from the fact that the theta-null map is an embedding, see
suffices. Since \( \Gamma_g(2,4) \) is a normal subgroup of \( \Gamma_g \), the level moduli space \( A_{g,4}^2 \) is a Galois cover of \( A_g \), of degree (see [Ig2]) equal to

\[
[\Gamma_g : \Gamma_g(2,4)] = 2g^2 + 2g \prod_{k=1}^g (2^k - 1).
\]

We denote by \( \mathcal{M}_g \) the moduli of Riemann surfaces of genus \( g \), by \( J : \mathcal{M}_g \to A_g \) — the Torelli map sending a curve \( X \) to its Jacobian \( J(X) \), and by \( J_g := J(\mathcal{M}_g) \subset A_g \) the locus of (isomorphism classes of) Jacobians within the moduli of ppavs. Let \( A_{g,irr}^2 \) be the locus of irreducible ppavs; then in fact the image \( J(\mathcal{M}_g) \subset A_{g,irr}^2 \).

Corresponding to the level cover \( p : A_{g,4}^2 \to A_g \) we can consider the level cover \( p : \mathcal{M}_{g,4}^2 \to \mathcal{M}_g \) with the level Torelli embedding \( J_{g,4}^2 : \mathcal{M}_{g,4}^2 \to A_{g,4}^2 \), whose image we denote by \( J_{g,4}^2 \) (see [vG], [Do]). Schottky problem is the question of describing \( Th(J_{g,4}^2) \subset Th(A_{g,4}^2) \subset \mathbb{P}^{2g-1} \).

Recall that the moduli spaces \( \mathcal{M}_g \) and \( A_g \) are not compact, and this is where most of our trouble will come from. We will need to use the Deligne-Mumford compactification (see [DeMu], [HaMo]) \( \overline{\mathcal{M}}_g \) of the moduli space of curves, and a certain second Voronoi toroidal compactification \( \overline{A}_g \) of \( A_g \), such that the embedding \( J : \mathcal{M}_g \to A_g \) extends to a map \( \overline{J} : \overline{\mathcal{M}}_g \to \overline{A}_g \), injective on the locus of irreducible stable curves, and which becomes an embedding if we blow up the locus of reducible abelian varieties in \( A_g \) (see [Nam], chapter 9.D). There also exists another compactification \( \tilde{A}_g \) of \( A_g \), called the Satake, or minimal, compactification, from which the toroidal compactification can be obtained by a series of blowups. In particular there is a natural blow-down map \( b : \overline{A}_g \to \tilde{A}_g \), and the theta constants extend naturally to define a map \( \overline{Th} : \tilde{A}_g \to \mathbb{P}^{2g-1} \). Abusively we denote the compose map \( b \circ \overline{Th} : \overline{A}_{g,4}^2 \to \mathbb{P}^{2g-1} \) also by \( \overline{Th} \). We let \( \overline{\mathcal{M}}_{g,4}^2 \) be the Deligne-Mumford compactification of the level moduli space (i.e. the normalization of \( \overline{\mathcal{M}}_g \) in the field of functions of \( \mathcal{M}_{g,4}^2 \)) — see [DeMu], [Ber] for more details. We also have the extended level Torelli map \( \overline{J}_{g,4}^2 : \overline{\mathcal{M}}_{g,4}^2 \to \tilde{A}_{g,4}^2 \).

3. Theta metric

**Definition 3.1.** We call the **theta metric** the restriction of the Fubini-Study metric from the cotangent bundle of \( \mathbb{P}^{2g-1} \) to the cotangent bundle of the image \( \overline{Th}(\overline{A}_{g,4}^2) \). When we pull it back to \( \overline{A}_{g,4}^2 \), it is a metric on the bundle \( \mathcal{L} \) of theta constants. By abuse of notation we will also call theta metric the pullback of this metric to \( \overline{\mathcal{M}}_{g,4}^2 \) by the map \( \overline{J}_{g,4}^2 \).
Denote by $\Omega$ the curvature form of the Fubini-Study metric on $\mathbb{P}^{2g-1}$ and, by abuse of notation, its pullbacks to $\mathcal{A}^{2,4}_g$ and $\mathcal{M}^{2,4}_g$. It is given by (see [GrHa], section 0.2)

\begin{equation}
\Omega = \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \log \sum_{\varepsilon \in \mathbb{Z}^2} |\Theta[\varepsilon](\tau)|^2.
\end{equation}

By definition the degree of $\text{Th}(\mathcal{J}^{2,4}_g) \subset \mathbb{P}^{2g-1}$ is equal to the integral of the top power of the form $\Omega$ over it:

\begin{equation}
\deg \text{Th}(\mathcal{J}^{2,4}_g) = \int_{\text{Th}(\mathcal{J}^{2,4}_g)} \Omega^{2g-3}.
\end{equation}

Note that when computing the integral, we can as well integrate over the open part, while technically the degree only makes sense for closed subvarieties. As we do not know the image $\text{Th}(\mathcal{J}^{2,4}_g)$ explicitly (knowing it would be solving the Schottky problem), explicitly integrating over it is unmanageable. Thus we pull $\Omega$ back to $\mathcal{M}^{2,4}_g$, and try to integrate there. However, integrating over $\mathcal{M}^{2,4}_g$ is still unmanageable, and instead we express the integral as an intersection number on the compactification $\overline{\mathcal{M}}^{2,4}_g$. But we do not know the intersection theory on $\overline{\mathcal{M}}^{2,4}_g$ well enough to be able to compute this intersection number. Using the branched cover $p : \overline{\mathcal{M}}^{2,4}_g \to \overline{\mathcal{M}}_g$, we reduce the intersection-theoretic computation on $\overline{\mathcal{M}}^{2,4}_g$ to an intersection-theoretic computation on $\overline{\mathcal{M}}_g$, which is then manageable. In doing all of this, the main difficulty is dealing with the boundary contributions appropriately.

**Remark 3.2.** It is known that, up to torsion, over $\mathcal{A}^{2,4}_g$ the bundle $L$ of modular forms of weight one half is the square root of the determinantal line bundle of the Hodge bundle $H$ — the $g$-dimensional complex vector bundle over some $A$ being $H^{1,0}(A, \mathbb{C})$, the space of holomorphic one-forms on $A$ (see [FaCh], section 1.5). We would then like to say that the integral of the top power of $\Omega$ is just equal to the top self-intersection number of the divisor corresponding to the determinantal line bundle of the Hodge vector bundle. This indeed happens to be the case, but actually proving it requires some work, as the theta metric has a singularity near the boundary of the and it is not clear a priori how the bundle and the metric extend to the boundary. One possible way to deal with the problem is to extend the bundle and then smoothen the metric using complex-analytic techniques of Monge-Ampère operator applied to currents (see [BeTa]) to compute its total mass (see [Ra]). What we will do for our proof, however, is
study the boundary degeneration directly. In [Wo2] Wolpert deals with the similar problem for the Weil-Petersson metric.

**Remark 3.3.** There is a natural metric on the Hodge bundle over \( \mathcal{M}_g \), called the Hodge metric, discussed by Nag in [Nag]: for two abelian differentials \( \omega \) and \( \omega' \) on \( X \) we define their scalar product to be \( \int_X \omega \wedge \omega' \). We will show that Hodge and theta metrics are distinct by showing that their behaviors on the boundary are different. We will obtain explicit upper bounds for theta volumes, while at this time we were not able to obtain a bound for Hodge volumes or to express them as intersection numbers.

**Example 3.4.** To illustrate the potential problems arising, we provide a simple example (due to Yum-Tong Siu) illustrating the difference between the integral of the top power of a current and the top self-intersection number of the cohomology class it represents.

Indeed, on \( \mathbb{P}^2 \) with homogeneous coordinates \((x : y : z)\) consider the expression \( \partial \partial \log(|x|^2 + |y|^2) \) — it is a smooth form outside the point \((1 : 0 : 0)\) and can be trivially extended to a current \( \phi \) on \( \mathbb{P}^2 \). The class \(|\phi| \in H^2(\mathbb{P}^2) = \mathbb{Q} \) is clearly non-zero (in fact it is equal to the class of the Fubini-Study metric), and thus the intersection number \( \langle |\phi|^2 \rangle_{\mathbb{P}^2} \neq 0 \). On the other hand the integral \( \int_{\mathbb{P}^2 \setminus \{1:0:0\}} \phi \wedge^2 \) is zero, since the form \( \phi \) is exact on the chart where \( x \neq 0 \).

**Proposition 3.5.** The form \( \Omega \) extends smoothly to the compactifications \( \overline{\mathcal{M}}^{2,4}_g \) and \( \overline{A}^{2,4}_g \). We denote these extensions by \( \overline{\Omega} \).

**Proof.** Theta constants extend holomorphically to the Satake compactification \( \overline{A}^{2,4}_g \) (see [FaCh], chapter V). Thus the curvature form \( \Omega \) extends smoothly to \( \overline{\Omega} \) on the Satake compactification. Then \( \overline{\Omega} := \delta^*(\overline{\Omega}) \) is a smooth extension of \( \Omega \) to \( \overline{A}^{2,4}_g \). \( \square \)

4. **Theta constants at the boundary**

**Definition 4.1.** We denote by \( D_i \) the boundary divisor of \( \overline{\mathcal{M}}_g \) consisting of the closure in \( \overline{\mathcal{M}}_g \) of the locus of stable curves that are bouquets of two curves of genera \( i \) and \( g - i \) respectively. We also denote by \( D_0 \) the (closure of) the locus of irreducible nodal curves in \( \overline{\mathcal{M}}_g \). By Poincaré duality on \( \overline{\mathcal{M}}_g \) each of these divisors determines a class in \( H^2(\overline{\mathcal{M}}_g) \). We denote the corresponding cohomology classes by \( \delta_i \) and \( \delta_{irr} \), respectively. Let us also denote by \( \lambda \) the first Hodge class, i.e. the first Chern class of the Hodge bundle on \( \overline{\mathcal{M}}_g \).
We would now want to say that the divisor of the bundle $L$ of modular forms of weight one half is equal to the divisor of some specific theta constant, which is its section, and thus that the divisor of one theta constant is equal to the divisor of curves which have some theta constant vanishing, divided by $2^g$. However, as pointed out to us by Carel Faber, this argument does not necessarily have to work, and this is where we have to worry about level structures. The problem is that the divisor of one theta constant is not defined on $\mathcal{M}_g$, and that the covering map $p$ is branched exactly over the boundary and over the locus of reducible abelian varieties: see [Do], [vG] and [Sa2]. However, this argument can be salvaged and leads to the solution.

**Proposition 4.2.** The form $\Omega$ is invariant under the change of level structure, i.e. under the deck transformations of the cover $p : \mathcal{A}^2,4_g \to \mathcal{A}_g$. Since the cover $p$ is unbranched over $\mathcal{A}_g^{\text{irr}}$, see [Sa2], on $\mathcal{A}^2,4_{g,\text{irr}}$ the form $\Omega$ is a pullback of a smooth form on $\mathcal{A}_g^{\text{irr}}$, $\Omega = p^*\omega$ for some $\omega \in H^{1,1}(\mathcal{A}_g^{\text{irr}})$. This also holds for $\mathcal{M}_g$ and we abusively use the same notations there.

**Proof.** The proof easily follows from the transformation laws for theta functions (see, for example [LaB], 8.6.1). A very detailed study of the transformations of theta constants is done in [Sa1], [Sa2], and the following proof is basically a quote from there.

The action of the deck transformations group $\Gamma_g/\Gamma_g(2,4)$ on the set of theta constants of the second order is as follows:

— if $\gamma \in \Gamma_g(2)/\Gamma_g(2,4)$, then (apart from the modular factor that is the same for all theta constants, and thus would not matter for the Fubini-Study metric) $\gamma$ acts on each $\Theta[\varepsilon]$ by multiplying it by a sign, which may be different for different $\varepsilon$, but still does not matter for computing $\Omega$.

— if $\gamma \notin \Gamma_g(2)/\Gamma_g(2,4)$, then in addition to all of the above, the action of $\gamma$ permutes theta constants $\Theta[\varepsilon]$ in a certain way, but this again does not change the form $\Omega$.

Thus the form $\Omega$ on $\mathcal{A}^2,4_g$ is invariant under the deck transformations. Thus over the locus where the cover is unbranched it is the pullback of some smooth form, i.e. on $\mathcal{A}^2,4_{g,\text{irr}}$ it must be the (invariant) pullback of some form $\omega$ on $\mathcal{A}_g^{\text{irr}}$. \qed

**Proposition 4.3.** Extend the form $\omega$ trivially to a current on $\mathcal{A}_g$, and denote by $[\omega] \in H^{1,1}(\mathcal{A}_g)$ its cohomology class. The class $[\Omega] \in H^{1,1}(\mathcal{A}^2,4_g)$ is invariant under the deck transformations, and $[\Omega] = p^*[\omega]$. 
Proof. Since the form $\Omega$ is invariant under deck transformations, the divisor class $[\Omega]$ it represents is also invariant, and is the pullback of the divisor class $[\omega]$ of the form of which $\Omega$ is the pullback. This works perfectly when the cover is not branched. Over the branching locus we still have the invariance of $\Omega$ under the deck transformations, but now $\omega$ may actually have singularities, while its pullback $\bar{\Omega}$ stays smooth. However, the above statement about divisor classes still makes sense: the closed currents also represent cohomology classes, and we just need to compute these. Finally this means that we still have $[\Omega] = p^{*}[\bar{\omega}]$. □

Remark 4.4. The local picture near the branching locus is as follows: suppose locally the branching order is $B$, so that in local coordinates the covering map $p : z \rightarrow w$ is $w = p(z) = z^B$. Then for any $\epsilon \geq 0$ the pullback of the current $|w|^{-2+2/B+\epsilon}dw \wedge d\overline{w}$ is the smooth form

$$B^{2}|z|^{-2B+2+Be}z^{-B-1}\overline{z}^{B-1}dz \wedge d\overline{z} = B^{2}|z|^{Be}dz \wedge d\overline{z}.$$

Combining all the equalities between integrals and intersection numbers that we have obtained and using the fact that $\Omega$ is smooth on $A^{2,4}_{g}$, we get

**Theorem 4.5.** The degree of the Jacobian locus is

$$(4.1) \quad \deg Th(J^{2,4}_{g}) = \int_{\mathcal{M}^{2,4}_{g}} \Omega^{3g-3} = \langle [\Omega]^{3g-3} \rangle_{\mathcal{M}^{2,4}_{g}}^{\mathcal{M}^{2,4}_{g}} = \deg p \langle [\bar{\omega}]^{3g-3} \rangle_{\mathcal{M}^{2,4}_{g}}$$

and similarly for the degree of $Th(A^{2,4}_{g})$.

Thus finally we are left with the question of computing the class $[\bar{\omega}] \in H^{1,1}(\overline{\mathcal{M}}_{g})$ and also of $[\bar{\omega}] \in H^{1,1}(\overline{A}_{g})$.

Before proceeding to do this in the next section, let us give an analytic description of the picture for the moduli of curves. The complex coordinates near the boundary of the moduli space are given by the plumbing construction (introduced by Bers in [Be], see also [Ma], [Wo2]): we take a stable curve $X$, cut out a small neighborhood of every node $p_{i}$, which locally looks like $\{zw = 0\}$ in $\mathbb{C}^{2}$, and replace it by the neighborhood $\{zw = s_{i}\}$ for some $s_{i} \in \mathbb{C}$. Then $\{s_{i}\}$ and the rest of the complex coordinates $\{t_{i}\}$ on the moduli given by the Bers embedding and deformation theory give complex coordinates in a neighborhood of this nodded curve.

In these coordinates, the asymptotic behavior of the period matrix was obtained by Fay and by Yamada (see [Ya], [Ta]). In particular, if $X$ is an irreducible stable curve with one node, i.e. $X \in D_{0}$, then in
a neighborhood of \( X \) the period matrix stays bounded except for \( \tau_{11} \), which grows as \( \frac{\log s_1}{2\pi i} \) as \( s_1 \to 0 \).

Using the Fourier-Jacobi expansion (see [FaCh], chapter V) for theta functions

\[
\Theta[\varepsilon_1 \varepsilon'] \left( \frac{\tau_{11}}{\xi}, \frac{\xi'}{\tau'} \right) = \sum_{m \in \mathbb{Z}} \exp \left( 4\pi i \tau_{11}(2m + \varepsilon_1)^2 \right) \Theta[\varepsilon'](\tau', (m + \frac{\varepsilon_1}{2})\xi),
\]

we can then compute the asymptotics of the growth of the form \( \omega \) near the boundary \( D_0 \) of \( \mathcal{M}_g \) — this computation is very similar to the one in [Do], and we omit the easy details. The result is given by

**Proposition 4.6.** The form \( \omega \) has a singularity of type \( |s_1|^{-3/2}ds_1 \wedge d\overline{s_1} \) near the boundary component \( D_0 \subset \mathcal{M}_g \). The volume form \( \omega^{(3g-3)} \) has a singularity of the same order.

**Remark 4.7.** For comparison, for the Weil-Petersson volume form the singularity is (see [Ma]) \( |s_1|^{-2}(\log |s_1|)^{-3}ds_1d\overline{s_1} \), and for the curvature of the Hodge metric the singularity is \( |s_1|^{-2}(\log |s_1|)^{-2}ds_1d\overline{s_1} \). The last fact is easy and well-known; to see it we just notice that the Hodge metric is \( \det \text{Im}\tau \sim \log |s_1| \), and thus its curvature form blows up as

\[
\partial \overline{\partial} \log \det \text{Im}\tau \sim \partial \overline{\partial} \log (|s_1|).
\]

Thus the Hodge curvature has the worst singularity, while \( \omega \) has the mildest singularity of the three.

Notice also that for the Weil-Petersson and Hodge metrics \( |s_1|^{-1} \) is taken in the power 2, while for \( \omega \) it is taken to the power 3/2. This indicates that while potentially there can be delta-function contributions of the boundary to the volume computation for Hodge and Weil-Petersson metrics, for the theta metric the boundary can be excluded, as the singularity there is only branching, as we already know. As we will explain below, this basically translates to the fact that the divisor class of \( [\omega] \) lives in \( H^2(\mathcal{M}_g) \), uncompactified, i.e. does not have any extra terms coming from the boundary divisors.

### 5. The Degree Computations

**Theorem 5.1.** In \( H^2(\mathcal{M}_g) \) we have \( [\omega] = \lambda/2 \). Combined with theorem 4.3 this means that the degree of the Jacobian locus is

\[
\deg \text{Th}(\mathcal{J}_g^{2,4}) = \deg p \left( (\lambda/2)^{3g-3} \right)_{\mathcal{M}_g}.
\]

**Proof.** Analytically this follows from the fact that the singularity of the current \( \omega \) near the boundary of \( \mathcal{M}_g \) is milder than \( |z|^{-2}dz \wedge d\overline{z} \), so that there is no delta function on the boundary (i.e. in some sense the integral of the volume form \( \omega^{3g-3} \) over the boundary is zero) and
thus there are no $\delta_{\text{irr}}$ summands in $[\omega]$. The fact that there are no $\delta_i$ summands in $[\omega]$, either, follows similarly from the smoothness of the form $\Omega$ on $\mathcal{A}_{g}^{2,4}$ — note that similarly to the behavior near $\partial \mathcal{A}_{g}$, the form $\omega$ may blow up near $D_i$, as the cover $\Phi$ is branched there.

To make this proof rigorous, however, one needs to also study the vanishing near higher-order degenerations and, though doable, this becomes more cumbersome, unless we try to quote directly the fact that any branching-type singularities do not matter for computing the total mass of a current, for which we were not able to find a reference.

Thus let us present a rigorous proof specific to the problem at hand. Basically it stems from the fact that the form $\Omega$ extends smoothly to the Satake compactification of $\mathcal{M}_g$ (or of $\mathcal{A}_g$), and there the boundary is not a collection of divisors, but higher codimension, and thus there are no “boundary classes” in $H^2$. Indeed, let us use the basis $\{E, L_i\}$ for $H^2(\mathcal{M}_g)$ constructed by Wolpert in [Wo1]: here $E$ is the family of elliptic tails, i.e. a varying 1-pointed elliptic curve attached to a fixed genus $g - 1$ curve with 1 marked point, and $L_i$ are obtained by attaching a varying 4-pointed sphere to one or two fixed Riemann surfaces of lower genera.

Then by definition we have

$$\omega|[\mathcal{L}_i]]_{\mathcal{M}_g} = \omega|[\mathcal{J}(\mathcal{L}_i)]|_{\mathcal{A}_g} = (\text{deg } p)^{-1}[\Omega](p^* \circ \mathcal{J}(\mathcal{L}_i))_{\mathcal{A}_g^{2,4}}$$

$$= (\text{deg } p)^{-1}[\Omega](b \circ p^* \circ \mathcal{J}(\mathcal{L}_i))_{\mathcal{A}_g^{2,4}} = (\text{deg } p)^{-1}[\Omega](0) = 0$$

because the Jacobian of a bouquet of smooth curves does not keep track of the point of attachment, and, more generally, the image of the level cover of the family $\mathcal{L}_i$ (in which the curves differ only by the choice of points where the fixed components are attached to the sphere), is locally constant in the level cover of Satake compactification, i.e. is simply a collection of points.

Checking the intersection matrix of $H^2(\overline{\mathcal{M}_g})$ with $H_2(\overline{\mathcal{M}_g})$ in [Wo1], we see that it follows that $[\omega]$ is proportional to $\lambda$. Then either computing the interaction with $\mathcal{E}$ or recalling that over $\mathcal{M}_g$ the theta constant bundle is the square root of the determinant of Hodge bundle we see that in fact $[\omega] = \lambda/2 \in H^2(\overline{\mathcal{M}_g})$. □

The top self-intersection numbers of the class $\lambda$ on $\overline{\mathcal{M}_g}$ were computed by Carel Faber using his program described in [Fab]. For the degrees of the Jacobian locus in genera $1 \ldots 7$ one then gets

$$1, 1, 16, 208896, 282654670848,$$

$$23303354757572198400, 87534047502300588892024209408.$$
These numbers were known before in genera one through three: in genera one and two there are no equations by dimension considerations, and the answer of 16 (obtained by rewriting the Schottky’s original equation in terms of $\Theta[\varepsilon]$) for genus three is explained in \[vGvdG\]. The number in genus four is already large and thus rather worrying. However, this is the “total degree” of all equations needed to define $Th(J^{2,4}_4) \subset \mathbb{P}^{15}$, and not only the degree of the additional equations defining $Th(J^{2,4}_4) \subset Th(A^{2,4}_g)$.

**Theorem 5.2.** The later degree, $\deg Th(A^{2,4}_g) = \deg Th(A^{2,4}_g)$, is equal to $\deg p$ times the top self-intersection number of $\lambda/2$ on $\overline{A}_g$.

**Proof.** Analytically this works the same way as for the moduli of curves in the previous theorem, so this is again not very rigorous. A rigorous proof of the fact that the top self-intersection of $\lambda$ does not include anything on $\partial A_g$ is essentially contained in \[vdG\].

A self-contained rigorous proof is as follows: it is known from \[Mu2\] (see also \[Hu\] for a review and an interesting discussion of related issues) that the Picard group of $H^2(\overline{A}_g)$ is generated by $\lambda$ and the class of the boundary $\delta$. To show that the class $[\varpi]$ is proportional to $\lambda$, i.e. does not contain a boundary summand, we can use the basis $J(E), J(L_0)$ for $H^2(\overline{A}_g)$ — the fact that this is a basis follows from the non-degeneracy of the $2 \times 2$ intersection matrix of these two homology classes with $\lambda, \delta$, a basis for $H^2(\overline{A}_g)$ — and note that the intersection of $[\varpi]$ with $J(L_0)$ is zero, as above.

**Remark 5.3.** Corrado De Concini explained to us that the last theorem can also be proven algebraically in a different way. Indeed, the degree of a variety is the number of points in its intersection with a generic linear subspace of complementary dimension. Since the boundary of the Satake compactification has high codimension, such a linear subspace can be chosen not to intersect the boundary, so the boundary would not matter, and we can compute the intersection number on the Satake compactification (where there is only one divisor), which it follows from the work \[vdG\] is the same. Such an argument does not work for the moduli space of curves, as there we do not know how to compute intersection numbers on the Satake compactification.

Carel Faber has brought the work of van der Geer \[vdG\] to our attention. One of the results of that paper is a closed formula for the top self-intersection of $\lambda$ on $A_g$:

**Proposition 5.4** (\[vdG\]). Denoting $N := g(g + 1)/2$, in the orbifold sense (which, as explained to us by Carel Faber, means we will later
need to multiply these by two, due to the presence of $x \to -x$ involution on every abelian variety) the intersection numbers we get are

\begin{equation}
\langle \lambda^N \rangle_{A_g} = \langle \lambda^N \rangle_{A_g} = (-1)^N N! \prod_{k=1}^{g} \frac{\zeta(1-2k)}{2((2k-1)!!)}.
\end{equation}

where $\zeta(1-2k)$ are the values of Riemann’s zeta function at negative odd integer points (equal to some products of Bernoulli numbers and factorials), and double factorial denotes the product $(2k-1)!! := (2k-1)(2k-3)(2k-5) \cdots 3 \cdot 1$.

Multiplying these intersection numbers by $2(\deg p)/2^N$ we can compute $\deg Th(A^{2,4}_g)$ for any $g$. In particular for genera $1 \ldots 7$ we get

\begin{equation}
1, 1, 16, 13056, 123471424, 197972857997555419746140160.
\end{equation}

In genera one and two this agrees with the fact that there are no equations, and the only relation in genus 3 has degree 16 (see [vGvdG], p. 623).

We also compute the ratio $\frac{\deg Th(J^{2,4}_g)}{\deg Th(A^{2,4}_g)}$ for genera $1 \ldots 7$:

\begin{equation}
1, 1, 1, 16, \frac{2976}{13}, \frac{202742400}{223193}, \frac{8678490624}{19627855}.
\end{equation}

The 1’s we get for genera one through three correspond to the fact that $M_g = A^{irr}_g$ for $g \leq 3$. For genus four $Th(J_4)$ is defined within $Th(A_4)$ by one extra equation, Schottky’s original relation, which is of degree 16 (see [vGvdG]), so our computation produces the correct result. The fact that the ratios we get in higher genera are not integer yields the following

**Corollary 5.5.** The variety $Th(J^{2,4}_g)$ is not a complete intersection within $Th(A^{2,4}_g)$ for genera $5, 6, 7$.

**Remark 5.6.** There is a nuance here: in genus four the defining equation is of degree 16 in theta-nulls, not in theta constants of the second order. To rewrite it algebraically in terms of $\Theta$’s, it has to be multiplied by three conjugates and becomes of degree 64 (see [vGvdG]), but only one of the four isomorphic components of the locus it then defines is the Schottky locus. It is also possible to perform computations analogous to ours for the embedding of $A^{1,8}_g$ by theta-nulls — we will then get a larger factor corresponding to the degree of the level (4,8) cover instead of the level (2,4) cover, but the ratio of the degrees of the Jacobian locus and of the moduli of abelian varieties will stay
the same, 16. We will study $\text{Th}(A^{2,4}_4)$ and $\text{Th}(J^{2,4}_4)$ in more detail in another work.

Our degree computations also allow us to prove one curious result:

**Proposition 5.7.** Consider the (level cover of) the locus of 1-reducible abelian varieties: $A^{2,4}_1 \times A^{2,4}_{g-1} \subset A^{2,4}_g$. Its image under the theta map is not a complete intersection in $\text{Th}(A^{2,4}_g)$ for $g = 4, 5, 6, 7$. Similarly for the moduli space of curves $\text{Th}(J^{2,4}(D_1))$ is not a complete intersection in $\text{Th}(J^{2,4}_g)$ for $g = 4, 5, 6, 7$.

**Proof.** To prove this we note that from the considerations above it follows that

$$\deg \text{Th}(A^{2,4}_1 \times A^{2,4}_{g-1}) = \deg p \left\langle \frac{\lambda}{2} \right\rangle_{A_1} \left\langle \frac{\left(\frac{\lambda}{2}\right)^{g(g-1)/2}}{A_{g-1}} \right\rangle_{A_g},$$

and, evaluating the intersection number on $A_1$, that

$$\frac{\deg \text{Th}(A^{2,4}_1 \times A^{2,4}_{g-1})}{\deg \text{Th}(A^{2,4}_g)} = \frac{2^{g-1} \langle \lambda^{g(g-1)/2} \rangle_{A_{g-1}}}{24 \langle \lambda^{g(g+1)/2} \rangle_{A_g}}$$

and similarly for the moduli spaces of curves. These ratios are not integer for the values of $g$ in question, and thus one variety cannot be a complete intersection inside the other. □

This proposition is of interest for the following reason: if the locus of reducible abelian varieties were a complete intersection, i.e. were given inside $\text{Th}(A^{2,4}_g)$ by $g-1$ equations, then restricting this to the Jacobian locus we could potentially get a cover of $M^{2,4}_g$ by $g-1$ affines, each corresponding to the non-vanishing of one of the equations, and would prove Looijenga’s conjecture that the cohomology of the uncompactified $M_g$ has properties similar to that of a $(g-1)$-dimensional variety. Thus the proposition shows that this approach to proving Looijenga’s conjecture is basically hopeless.

6. Intersection number estimates

We have reduced the computation of the degree of the locus of Jacobians to the computation of the theta volumes, which we then expressed as an intersection number on $M^{2,4}_g$. Now we will obtain an explicit upper bound for these intersection numbers.

**Theorem 6.1.** The degree of the Jacobian locus can be bounded explicitly:

$$\deg \text{Th}(J^{2,4}_g) < C(g),$$
where $C(g)$ is an explicit function of $g$ with the leading growth order $2^{2g^2}$.

To obtain the bound we will use the results from previous work on Weil-Petersson volumes. Let $\kappa$ be the first Mumford’s tautological cohomology class on $\mathcal{M}_{g,n}$. For us it is just the class of the Kähler form of the Weil-Petersson metric, divided by $2\pi^2$ (see [Wo1]): $\kappa = [\omega_{WP}] / 2\pi^2$. Then algebraically instead of looking at $\text{Vol}_{g,n}$, the Weil-Petersson volume of $\mathcal{M}_{g,n}$ (see [Gr]), we can look at the top self-intersection number

$$\langle \kappa^{3g-3+n} \rangle_{\mathcal{M}_{g,n}} = \frac{(3g - 3 + n)! \text{Vol}_{g,n}}{(2\pi^2)^{3g-3+n}}.$$

**Remark 6.2.** This equality is not at all trivial: the Weil-Petersson form $\omega_{WP}$ is only smooth on $\mathcal{M}_{g}$, and extends to $\overline{M}_g$ as a closed positive current, and we get the same kind of problem as we had when extending $\Omega$. The Weil-Petersson volume is the total Monge-Ampère mass (see [BeTa], [Ra]) of this current, which in general might not be equal to the top self-intersection number of its cohomology class. The equality, proven in [Wo2] and further discussed in [Wo3], follows from the fact that the form $\omega$ extends to the boundary smoothly in the smooth structure near the boundary defined by certain Fenchel-Nielsen coordinates, and the extension has the same Čech cohomology class as the trivial extension of $\omega$ as a current in complex coordinates.

In [Gr] we used Penner’s decorated Teichmüller theory and his earlier work [Pe] on the subject to show that for $n$ fixed and $g$ large $\text{Vol}_{g,n} < c^g g^{2g}$ for some explicit constant $c$. In [ScTr] Schumacher and Trapani use ampleness of $\kappa$ (which follows from the Weil-Petersson metric being Kähler and extendable smoothly to $\overline{M}_g$), and the knowledge of some explicit effective divisors on $\overline{M}_{g,n}$ to obtain lower bounds on Weil-Petersson volumes. Indeed, since $\kappa$ is ample on $\overline{M}_g$, we have $\langle \kappa^{3g-4+D} \rangle_{\overline{M}_g} \geq 0$ for any effective divisor $D$. Since the Weil-Petersson metric on the boundary of the moduli restricts to the Weil-Petersson metric on the lower-dimensional moduli spaces, choosing some particular $D$’s then yields estimates on $\text{Vol}_{g,n+1}$ in terms of $\text{Vol}_{g,n}$, and for $\text{Vol}_{g,0}$ in terms of $\text{Vol}_{m,k}$ for $m < g$, which, using our upper bounds on $\text{Vol}_{g,n}$ for $n > 0$, then implies that $\text{Vol}_{g,0} \leq c^g g^{2g}$, and using Penner’s lower bounds for $\text{Vol}_{g,1}$ also yields $\text{Vol}_{g,n} \geq A^g g^{2g}$ for all $n$, including the no-puncture case. In particular Schumacher and Trapani prove
(theorem 2 in [ScTr]) that

\[ 14 \kappa^{3g-3} \geq \kappa^{3g-4} \cdot \sum_{i=0}^{g/2} \delta_i. \]

We will now use the Hodge index inequality in the following form:

**Lemma 6.3** ([Dem], lemma 5.3, see also [La1]). Let \( X \) be a complex variety of dimension \( N \), and let \( E_1 \ldots E_N \) be any nef divisors (i.e. intersecting any effective curve non-negatively) on \( X \). Then for the intersection numbers we have

\[ \langle E_1 \cdots E_N \rangle_X^N \geq \langle E_1 \rangle_X \cdots \langle E_N \rangle_X. \]

If we set \( E_1 = \ldots = E_p = \alpha \) and \( E_{p+1} = \ldots = E_N = \beta \), this general inequality becomes

\[ \langle \alpha^p \cdot \beta^{N-p} \rangle_X^N \geq \langle \alpha^p \rangle_X \langle \beta^{N-p} \rangle_X. \]

**Proof of theorem 6.1.** Applying the last version of Hodge index theorem for \( p = 1 \), \( \alpha = \lambda \) and \( \beta = \kappa \) yields

\[ \langle \lambda^{3g-3} \rangle_{\mathcal{M}_g} \langle \kappa^{3g-4} \rangle_{\mathcal{M}_g} \leq \langle \lambda \kappa^{3g-4} \rangle_{\mathcal{M}_g} \]

From [Wo1] we recall that in \( H^2(\mathcal{M}_g) \)

\[ 12 \lambda = \kappa + \delta_{irr} + \delta_1/2 + \sum_{i>1} \delta_i. \]

Substituting this in the previous formula, and then using Schumacher and Trapani’s estimate (6.2), we get

\[ \langle \lambda \kappa^{3g-4} \rangle_{\mathcal{M}_g} = \frac{1}{12} \left( \kappa + \delta_{irr} + \delta_1/2 + \sum_{i>1} \delta_i \right) \kappa^{3g-4} \leq \frac{15}{12} \langle \lambda \kappa^{3g-3} \rangle_{\mathcal{M}_g}. \]

Substituting this in (6.5) and using the upper bound of \( c^g g^{2g} \) for the Weil-Petersson volume \( \text{Vol}_{g,0} = \langle \kappa^{3g-3} \rangle/(3g-3)! \), we then end up with

\[ \langle \lambda^{3g-3} \rangle \leq \frac{\langle \lambda \kappa^{3g-3} \rangle_{\mathcal{M}_g}}{(\kappa^{3g-3})_{\mathcal{M}_g}} \leq \left( \frac{5}{4} \right)^{3g-3} \langle \kappa^{3g-3} \rangle < (3g-3)! C^g g^{2g}, \]

where \( C := 5^3 c/4^3 \) is a new explicit constant.

Combining this with theorem 6.1, we then obtain the following bound for the degree

\[ \text{deg} \, Th(J_g^{2,4}) = \int_{\mathcal{M}_g^{2,4}} \Omega^{3g-3} < (3g-3)! C^g g^{2g} 2^{g^2-g+3} \prod_{k=1}^{g} (2k - 1) \]
which is absolutely huge, but explicit and finite nonetheless. □

7. ELIMINATING THE UNCERTAINTIES IN THE KP EQUATION

Shiota has obtained a solution to the Schottky problem via the KP equation for the theta functions:

Theorem 7.1 ([SH]). A principally polarized abelian variety is the Jacobian of a Riemann surface if and only if there exist three vectors \( u \neq 0, v, w \in \mathbb{C}^g \), and a constant \( c \in \mathbb{C} \) such that the theta function \( \theta(\tau, z) := \theta\left[ \begin{matrix} 0 \\ \end{matrix} \right](\tau, z) \) of this ppav satisfies the following differential equation (the KP equation) for all values of \( z \):

\[
\theta_{uuuu} \theta - 4\theta_{uuu} \theta_u + 3\theta_{uu} \theta_{uu} + \theta_u \theta_w - 4\theta_{uu} \theta + 3\theta_{vv} \theta - 3\theta_v \theta_v + 8c\theta^2 = 0,
\]

where the subscript denotes differentiation with respect to the \( z \) variables in the direction of the vector indicated.

Using the addition theorem and other properties of theta functions, this equation was reformulated in terms of theta constants of the second order by Dubrovin. We will use the following convention:

**Definition 7.2.** For \( u, v \in \mathbb{C}^g \) we denote by

\[
(7.2) \quad uv\partial \Theta[\varepsilon](\tau) := \sum_{i,j=1}^{g} u_i v_j \frac{\partial \Theta[\varepsilon]}{\partial \tau_{ij}}
\]

the value of the bilinear form determined by the matrix \( \frac{\partial}{\partial \tau_{ij}} \Theta[\varepsilon] \) at the pair of vectors \( (u, v) \). We also define \( uvwx\partial^2 \Theta[\varepsilon](\tau) \) similarly, as the convolution of rank four tensors \( u \otimes v \otimes w \otimes x \) and \( \frac{\partial^2}{\partial \tau_{ij} \partial \tau_{kl}} \Theta[\varepsilon](\tau) \).

In these notations the KP equation can be reformulated as follows:

**Proposition 7.3 ([Du]).** The theta function of a principally polarized abelian variety with period matrix \( \tau \) satisfies the KP equation (7.1) if and only if for the same \( u, v, w \) and \( c \) all theta constants of the second order satisfy the following differential equations:

\[
(7.3) \quad u^4 \partial^2 \Theta[\varepsilon](\tau) + \left( \frac{3}{4}v^2 - uw \right) \partial \Theta[\varepsilon](\tau) + c\Theta[\varepsilon](\tau) = 0 \quad \forall \varepsilon.
\]

**Remark 7.4.** Dubrovin in [Du] indicates and Sasaki in [Sas] proves that the \( g(g+1)/2 \times 2^g \) matrix \( \{ \frac{\partial}{\partial \tau_{ij}} \Theta[\varepsilon], \Theta[\varepsilon] \} \) has maximal rank, \( g(g+1)/2 \), for irreducible abelian varieties. It then follows that if the KP equation in Dubrovin’s formulation has a solution with \( u = 0 \), i.e. if we have \( v^2 \partial \Theta[\varepsilon] + 4c\Theta[\varepsilon]/3 = 0 \) valid for all \( \varepsilon \), then we would need to have \( v = 0 \) and \( c = 0 \). Thus in the sequel we do not have to worry about the \( u \neq 0 \) condition in the KP equation.
The KP equation does not directly allow one to verify whether an explicitly given principally polarized abelian variety is a Jacobian. First, one needs to be able to eliminate the unknowns \( u, v, w \) and \( c \) from the system of equations (7.3). Second, even after we eliminate the unknowns, we end up with a system of non-linear differential equations for theta constants, the validity of which we need to check at the point \( \tau \). We will now deal with these problems.

To eliminate \( u, v, w, \) and \( c \) from (7.3), treat them as unknowns, and all expressions in theta constants and derivatives — as given coefficients. Then we have a system of \( M := 2g \) polynomial equations in \( N := 3g + 1 \) variables:

\[
\{ f_i(x_1, \ldots, x_N) = 0 \} \text{ for } i = 1 \ldots M, \text{ with } \deg f_i = 4, \text{ and we ask whether it admits a solution.}
\]

The problem can be looked at in two different ways. On one hand, we can think of this as a problem of eliminating variables \( x_1, \ldots, x_N \) from the system of equations. Once we eliminate all of them, we end up with a system of relations among the coefficients of \( f_i \). On the other hand, instead of doing elimination we can ask a global question whether this system has a solution — this amounts to dealing with Nullstellensatz. The currently available techniques in Nullstellensatz and elimination (see [CLO]) actually yield similar results, but using Nullstellensatz is much easier. We use the following version of the effective Nullstellensatz:

**Theorem 7.5 (Ko).** For \( f_i \in \mathbb{C}[x_1, \ldots, x_N], \deg f_i \leq d \) the system \( \{ f_i(x_1, \ldots, x_N) = 0 \}, i = 1, \ldots, M \) does not have a solution in \( \mathbb{C}^N \) if and only if there exist \( c_1, \ldots, c_M \in \mathbb{C}[x_1, \ldots, x_N] \) such that \( \sum c_i f_i = 1 \) and \( \deg(c_if_i) \leq d^N \forall i \).

Applying this to our situation allows us to effectively eliminate the variables from the KP equation:

**Lemma 7.6.** Using the effective Nullstellensatz to eliminate the unknowns in (7.3), we obtain a system of finitely many parameterless non-linear differential equations and inequalities for the theta constants, which is equivalent to the KP equation for the theta function.

**Proof.** The total degree of each \( f_i \) is equal to 4, and there are \( 3g + 1 \) variables, so by Kollár’s result we can choose all \( c_i \) to have total degree at most \( 4^{3g} \). The number of monomials in \( 3g + 1 \) variables of total degree at most \( 4^{3g} \) is \( (4^{3g}+3g+1)^{3g+1} \). Thus the total number of variables we have, which is the total number of undetermined coefficients of all \( c_i \) together, is \( K := 2^g(4^{3g}+3g+1)^{3g+1} \). The condition \( 1 = \sum c_i f_i \) gives one equation for the coefficient of each monomial of \( \sum c_i f_i \) — there are
that there can be, as this sum is a polynomial of total degree at most $4 + 4^g$. So we get a system of $L$ linear equations for $K$ unknowns, all but one of which have no constant term.

Such a system does not have a solution if and only if the one equation with the non-zero constant term is a linear combination of all the others. Verifying that this one equation (call it $R$ — we think of it as a row of a matrix) is a linear combination of all the others (call the matrix of those $A$) amounts to verifying that the rank of the matrix $A$ together with $R$ is the same as the rank of $A$ by itself.

Computing the rank by checking the vanishing and non-vanishing of the determinants of minors of increasing size, this verification can be done effectively. Thus the condition that the rank of $A$ together with $R$ is the same as the rank of $A$ is just a certain system of polynomial equations and inequalities for the entries of $A$ and $R$. □

8. Solving the Schottky Problem effectively

For the purposes of this section let us fix the genus/dimension $g$, denote $N := 2^g - 1$, and drop $g$ and 2, 4 in all notations. All polynomials considered are homogeneous.

We are dealing with the following loci: $J \subset A$, and $Th(J) \subset Th(A) \subset P^N$. In the previous sections of this work we obtained a formula for $L := \deg Th(A)$, and an explicit upper bound $K > \deg Th(J)$.

We want to determine the algebraic equations defining these loci in $P^N$, i.e. the ideals $I(Th(A)), I(Th(J)) \subset C[x_0, \ldots, x_N]^{\text{hom}}$, in the algebra of homogeneous polynomials in $N + 1$ variables. Notice that for any algebraic subvariety $Z \subset P^N$ the ideal $I(Z)$ is generated by its elements of degree at most $\deg Z$. This crude observation will allow us to effectively obtain the ideals we are interested in.

We will use the following generalization of Bezout’s theorem:

Lemma 8.1 ([Ha], Theorem 1.7.7). Let $H$ be a hypersurface in $P^N$ and let $Z$ be an irreducible algebraic subvariety of $P^N$. If $Z$ is not contained in $H$, let us denote by $\{Z_i\}$ the irreducible components of $H \cap Z$. Then the following equality holds:

$$\deg H \cdot \deg Z = \sum_i \text{mult}_{Z_i}(H \cap Z) \cdot \deg Z_i,$$

where $\text{mult}_{Z_i}(H \cap Z)$ denotes the multiplicity with which the intersection $H \cap Z$ contains $Z_i$.

What follows from this lemma is that if a polynomial (defining $H$) vanishes at a point of the subvariety $Z$ to a high enough order along $Z$
(this is the high multiplicity condition), then it vanishes identically on $Z$.

**Theorem 8.2.** A system of generators of $I(Th(A))$ can be obtained effectively.

**Proof.** Let $P \in \mathbb{C}[\Theta[\varepsilon]]_{\leq L}$ be a homogeneous polynomial in theta constants of degree at most $L$ with undetermined coefficients. We would like to know whether $P \in I(Th(A))$. Fix some specific $p \in H_g$ — for example, the point of the moduli space corresponding to the Jacobian of the hyperelliptic curve for which the period matrix is computed in [Sc]. The $(L^2 + 1)$-jet of $Th(A)$ at $Th(p)$ is the linear span of the set of all partial derivatives of all theta constants $\Theta[\varepsilon]$ with respect to all $\tau_{ij}$ to order $L^2 + 1$, evaluated at $p$ — thus it can be computed effectively. The conditions for $P$ to vanish along $Th(A)$ at the point $Th(p)$ to order $L^2 + 1$ are thus effectively a finite system of algebraic equations for the coefficients of $P$. Using the effective Nullstellensatz (possible as the number of equations and their degree are explicit functions of an explicit $L$) then allows us to effectively choose the generators for the ideal of all $P$ that satisfy these equations (see [EiLa], [Ko]), which will thus be a basis for $I(Th(A))$. □

Now for the Schottky problem:

**Theorem 8.3.** A system of generators of $I(Th(J))$ can be obtained effectively.

**Proof.** In the previous section we effectively reformulated the KP equation as a finite system of explicit parameterless polynomial equations and inequalities for theta constants and their first and second derivatives. Let $S$ denote this resulting system of polynomial equations in theta constants and their derivatives. We will not need to look at the inequalities in detail.

Since the KP gives a solution to the Schottky problem, the locus $J \subset A$ locally near $p$ is the locus of solutions of $\{S = 0\}$ in $A$ — the inequalities may only serve to cut away extra components of the solution set of $\{S = 0\}$ at a certain distance from $p$. Recall that the locus of Jacobians is irreducible.

To apply the same argument as in theorem 8.2 we need to compute the $(K^2 + 1)$-jet of the set $Th(\{S = 0\})$ at $Th(p)$ effectively. To do it we need to differentiate the equations of the system $S$ with respect to some $\Theta[\varepsilon]$ a number of times (no more than $K^2 + 1$ times), and then
evaluate at $p$. Noticing that
\[
\left. \frac{\partial}{\partial \Theta[\epsilon]} \left( \frac{\partial \Theta[\delta]}{\partial \tau_{ij}} \right) \right|_p = \sum_{kl} \left. \left( \frac{\partial \Theta[\epsilon]}{\partial \tau_{kl}} \right) \right|_p^{-1} \cdot \left. \frac{\partial^2 \Theta[\delta]}{\partial \tau_{kl} \partial \tau_{ij}} \right|_p,
\]
we see that the $(K^2 + 1)$-jet of $Th(\{S = 0\})$ near $Th(p)$ only depends on the values of theta constants and their derivatives with respect to $\tau$ at $p$; thus this jet can be computed since we know all the equations of the system $S$ explicitly, and the values of theta constants and their derivatives at $p$ are also explicit numbers. Thus the same argument as in the previous theorem applies, and we have an effective way of solving the Schottky problem algebraically. □

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References

[Ber]  Bernstein, M.: Moduli Of Curves With Level Structure, Ph.D. dissertation, Harvard University, 1999
[Be]  Bers, L.: Spaces of degenerating Riemann surfaces, Discontinuous groups and Riemann surfaces, Ann. Math. Stud. 79 (1974), 43–55
[CLO] Cox, D., Little, J., O’Shea, D.: Using algebraic geometry, Graduate Texts in Mathematics 185, Springer-Verlag, New York 1998

[BeTa] Bedford, E. and Taylor, B.A.: The Dirichlet problem for a complex Monge-Ampère equation, Invent. Math. 37 (1976), 1–44
[Deb] Debarre, O.: The Schottky problem: an update, Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), 57–64, Math. Sci. Res. Inst. Publ., 28, Cambridge Univ. Press, Cambridge, 1995

[DeMu] Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math. 36 (1969) 75–109

[Dem] Demailly, J.-P.: A numerical criterion for very ample line bundles, J. Differential Geom. 37 (1993), no. 2, 323–374

[Do] Donagi, R.: Big Schottky, Invent. Math. 89 (1987), 569–599

[Du] Dubrovin, B.A.: Theta functions and non-linear equations, Russ. Math. Surveys 36 (1981) 2, 11–92

[EiLa] Ein, L., Lazarsfeld, R.: A geometric effective Nullstellensatz, Invent. Math. 137 (1999), 427–448

[Fab] Faber, C.: Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians. New trends in algebraic geometry (Warwick, 1996), 93–109, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999

[FaCh] Faltings, G. and Chai, C.-L.: Degeneration of abelian varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete 22. Springer-Verlag, Berlin, 1990

[Far] Farkas, H.: Schottky-Jung theory, Theta functions — Bowdoin 1987, 459–483

[vG] van Geemen, B.: The Schottky problem and second order theta functions, Workshop on Abelian Varieties and Theta Functions (Spanish) (Morelia, 1996), 41–84

[vGvdG] van Geemen, B., van der Geer, G.: Kummer varieties and the moduli spaces of abelian varieties, Amer. J. of Math. 108 (1986), 615–642

[vdG] van der Geer, G.: Cycles on the moduli space of abelian varieties, Moduli of curves and abelian varieties, 65–89, Aspects Math., E33, Vieweg, Braunschweig, 1999

[GrHa] Griffiths, P., Harris, J.: Principles of algebraic geometry, Pure and Applied Mathematics. Wiley-Interscience, New York, 1978

[Gr] Grushevsky, S.: An explicit upper bound for Weil-Petersson volumes of the moduli spaces of punctured Riemann surfaces, Math. Ann. 321 (2001), 1–13

[Ha] Hartshorne, R.: Algebraic geometry. Graduate Texts in Mathematics, 52. Springer-Verlag, New York-Heidelberg, 1977

[HaMo] Harris, J. and Morrison, D.: Moduli of curves. Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998

[Hu] Hulek, K.: Nef Divisors on Moduli Spaces of Abelian Varieties Complex analysis and algebraic geometry, 255–274, de Gruyter, Berlin, 2000

[Ig1] Igusa, J.-I.: Theta functions. Die Grundlehren der mathematischen Wissenschaften, Band 194. Springer-Verlag, New York-Heidelberg, 1972

[Ig2] Igusa, J.-I.: On the graded ring of theta-constants, Amer. J. of Math 86 (1964), 219–246

[Ko] Kollár, J.: Sharp effective Nullstellensatz, J. Amer. Math. Soc. 1 (1988), 963–975

[Kou] Kouvidakis, A.: Theta line bundles and the determinant of the Hodge bundle, Trans. Amer. Math. Soc. 352 (2000) 6, 2553–2568
[LaBi] Lange, H., Birkenhake, Ch.: Complex Abelian varieties, Grundlehren der mathematischen Wissenschaften 302, Springer-Verlag, New York 1992

[La1] Lazarsfeld, R.: Lectures on linear series. With the assistance of Guillermo Fernández del Busto. IAS/Park City Math. Ser., 3, Complex algebraic geometry (Park City, UT, 1993), 161–219, AMS, Providence, RI, 1997

[La2] Lazarsfeld, R.: Lengths of periods and Seshadri constants of abelian varieties, Math. Res. Lett. 3 (1996) 4, 439–447

[Ma] Masur, H.: The extension of the Weil-Petersson metric to the boundary of Teichmüller space, Duke Math. J. 43 (1976) 3, 623–635

[Mu1] Mumford, D.: The red book of varieties and schemes. Second, expanded edition. Includes the Michigan lectures (1974) on curves and their Jacobians. With contributions by Enrico Arbarello. Lecture Notes in Mathematics, 1358. Springer-Verlag, Berlin, 1999

[Mu2] Mumford, D.: On the Kodaira dimension of the Siegel modular variety. In: Algebraic geometry — open problems. Proceedings, Ravello 1982. Springer Lecture Notes 997, 348–375.

[Nag] Nag, S.: Canonical measures on the moduli spaces of compact Riemann surfaces, Proc. Indian Acad. Sci (Math. Sci.), 99 (1989) 2, 103–111

[Nam] Namikawa, Yu.: Toroidal compactification of Siegel spaces, Lecture Notes in Mathematics, 812. Springer-Verlag, New York 1980

[Pe] Penner, R.: Weil-Petersson volumes, J. Differential Geometry 35 (1992) 559–608

[Ra] Rashkovskii, A.: Total masses of mixed Monge-Ampère currents, Michigan Math. J. 51 (2003) 1, 169–185

[Sa1] Salvati Manni, R.: On the projective varieties associated with some subrings of the ring of thetanullwerte, Nagoya Math. J. 133 (1994), 71–83

[Sa2] Salvati Manni, R.: Modular varieties with level 2 theta structure, Amer. J. of Math. 116 (1994), 1489–1511

[Sas] Sasaki, R.: Modular forms vanishing at the reducible points of the Siegel upper-half space, J. Reine und Angew. Math. 345 (1983), 111–121

[Sc] Schindler, B.: Period matrices of hyperelliptic curves, Manuscripta Math. 78 (1993), 4, 369–380

[S] Schottky, F.: Zur Theorie der Abelschen Functionen vor vier Variablen, J. Reine und Angew. Math. 102 (1888), 304–352

[ScJu] Schottky, F., and Jung, H.: Neue Satze über Symmetralfunctionen und die Abel’schen Funktionen der Riemann’schen Theorie, Akad. Wiss. Berlin, Phys. Math. Kl. (1909), 282–297

[ScTr] Schumacher, G. and Trapani, S.: Estimates of Weil-Petersson volumes via effective divisors, Commun. Math. Phys. 222 (2001) 1, 1–7

[Sh] Shiota, T.: Characterization of Jacobian varieties in terms of soliton equations, Invent. Math. 83 (1986) 2, 333–382

[Ta] Taniguchi, M.: On the singularity of the periods of abelian differentials with normal behavior under pinching deformation, J. Math. Kyoto Univ. 31 (1991) 4, 1063–1069

[Wo1] Wolpert, S.: On the homology of the moduli space of stable curves, Ann. of Math. 118 (1983), 491–523

[Wo2] Wolpert, S.: On the Weil-Petersson geometry of the moduli space of curves, Amer. J. of Math. 107 (1985) 4, 969–997
THE DEGREE OF THE JACOBIAN LOCUS AND THE SCHOTTKY PROBLEM

[Wo3] Wolpert, S.: On obtaining a positive line bundle from the Weil-Petersson class, Amer. J. of Math. 107 (1985) 6, 1485–1507

[Ya] Yamada, A.: Precise variational formulas for abelian differentials, Kodai Math. J. 3 (1980), 1, 114–143

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