6$j$-symbols for symmetric representations of SO$(n)$
as the double series

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Abstract. The corrected triple sum expression of Ališauskas (1987) for the
recoupling (Racah) coefficients ($6j$-symbols) of the symmetric (most degenerate)
representations of the orthogonal groups SO$(n)$ (previously derived from the fourfold
sum expression of Ališauskas also related to result of Hormeß and Junker 1999)
is rearranged into three new different double sum expressions (related to the
hypergeometric Kampé de Fériet type series) and a new triple sum expression with
preferable summation condition. The Regge type symmetry of special $6j$-symbols
of the orthogonal groups SO$(n)$ in terms of special Kampé de Fériet $F_{1:3}^{1:4}$ series is
revealed. The recoupling coefficients for antisymmetric representations of symplectic
group Sp$(2n)$ are derived using their relation with the recoupling coefficients of the
formal orthogonal group SO$(-2n)$.

1. Introduction

The importance of $6j$ (Racah) coefficients of SU$(2)$ for the quantum angular momentum
theory is well known, as well as their applications in many branches of mathematical
physics, representation theory of Lie and quantum groups, in theory of orthogonal
polynomials and other special functions. The Racah coefficients ($6j$-symbols) and other
recoupling coefficients of the unitary SU$(n)$, orthogonal SO$(n)$ and symplectic Sp$(2n)$
groups of different rank are useful when calculating energy levels and transition rates
in atomic, molecular and nuclear theory (for example, in connection with Jahn–Teller
effect and structural analysis of atomic shells, see many papers of Judd and co-workers
[1–7], for description of multi-fermionic systems and in the microscopic nuclear theory
[8–14]) and in conformal field theory [15].

Special classes of coupling coefficients and $6j$-symbols of the SO$(n)$ groups were
considered by Ališauskas [16, 17], Junker and Hormeß [18, 19], with the fourfold [16, 17]
and triple [10] sum expressions for the recoupling coefficients with all most degenerate
(symmetric or class-one) irreducible representations. (Such $6j$-symbols have application
in the statistical physics, in the high-temperature expansion of the \( SO(n) \)-symmetric classical lattice models \([18–21]\)). Other special expressions for \( 6j \)-symbols of the \( SO(n) \) were also considered in \([22–25]\) and extended to the Racah coefficients of the quantum algebras \( O_q(n) \) \( [26] \).

The fourfold sum expressions (5.1)–(5.3) of \([14]\) and (2)–(10) of \([13]\) (cf. the integral representation in section 6 of \([18]\)) for the \( 6j \)-symbols of \( SO(n) \) with all six irreducible representations (irreps) symmetric are equivalent, taking into account the different expressions (11)–(15) of \([19]\) and (3.10a)–(3.10b) of \([17]\) for the integrals involving triplets of the Gegenbauer polynomials in terms of the very well-poised \( \tau F_6(1) \) or balanced \( 4F_3(1) \) hypergeometric series, related to the \( 6j \) coefficients of \( SU(2) \). The Biedenharn–Elliott identity \([27, 28]\) (see \([29–31]\)), used in two stages and related expansions allowed us \([16]\) to derive the triple sum expression (5.7) for the corresponding \( 6j \)-symbols of \( SO(n) \). Note that the phase factor \((-1)^{(g-e)/2}\) (where \( g \geq e \)) should be omitted in the right-hand side of this expression, in contrast with (5.5) of the same paper.

Expressions (5.3) and (5.7) of \([16]\) for the \( 6j \)-symbols of \( SO(n) \) are given as expansions in terms of three and two multiplied \( 6j \) coefficients of \( SU(2) \) (with some multiple of 1/4 parameters for odd \( n \)), respectively. The corresponding sums over the angular momentum type parameters resemble the usual expansions \([29, 30]\) of \( 9j \) and \( 12j \) coefficients of \( SU(2) \) in terms of \( 6j \) coefficients, which recently were rearranged by Rosengren \([32]\) (for the \( SU(1,1) \) group) and Ališauskas \([33–35]\) using the appropriate (less symmetric) expressions (29.1b) and (29.1c) of Jucys and Bandzaitis \([29]\) (see also (5) and (6) in section 9.2 of \([30]\)) for the Racah coefficients (related to the balanced hypergeometric \( 4F_3(1) \) series) and Dougall’s summation formula \([36]\) of the very well-poised \( 4F_3(−1) \) series. In \([33, 37]\), Dougall’s summation formula \([36, 37]\) of the very well-poised hypergeometric \( 5F_4(1) \) series, together with the corresponding expressions for the Racah coefficients, was suitable for rearrangement of \( 12j \) coefficients of \( SU(2) \). This way the total number of sums in expressions was reduced.

In this paper, the triple sum expression (5.7) of \([16]\) for the \( 6j \)-symbols of \( SO(n) \) with all six irreps symmetric is rearranged in a similar manner into the different double sum expressions of the hypergeometric (Kampé de Fériet \([38, 39]\)) type, as well as into the triple sum expression, with all three separate sums of the balanced \( 4F_3(1) \) type restricted by a single parameter.

In section 2, the main results of \([16]\) concerning the \( 6j \)-symbols of \( SO(n) \) are summarized and reconsidered in view of our objectives and some approaches used in \([33–35]\) in the case of \( 9j \) and \( 12j \) coefficients of \( SU(2) \). Three new double sum expressions for the renormalized \( 6j \)-symbols of \( SO(n) \) (specified in terms of so-called \( \alpha \)-graphs \( I_n(a,b,c|d,c,f) \) or related rational \( c_{a,b,e,d,c,f}^{(\alpha,n)} \) functions of \([19]\)) are derived in section 3, where the Regge \([10]\) type symmetry is also revealed (for \( n \geq 5 \)), as well as the role of the Bargmann–Shelepin \([11, 12]\) parameters, extended from the \( 6j \) coefficients of \( SU(2) \).
or SO(3). Triple sum expression for the renormalized $6j$-symbols of SO($n$) presented in section 4 sometimes may be more preferable, similarly as special expressions of the stretched or almost stretched $6j$-symbols of SO($n$).

In section 5, the renormalized $6j$-symbols of SO($n$) are expanded in terms of (numerator) Pochhammer symbols, as well as in terms of special class of Kampé de Fériet functions $F_{1:3;3}^{1:4;4} [:; 1, 1]$, which specific features and diversity are considered.

The recoupling coefficients for antisymmetric representations $\langle 1' \rangle$ of symplectic group Sp$(2n)$ are presented in appendix as formal analytical continuation of the recoupling coefficients for symmetric representations of the orthogonal group with negative rank SO($-2n$), in accordance with [3, 7, 22] (cf. also [43, 44]).

2. Preliminaries

In accordance with (5.3) of [16], we may express the $6j$-symbol of SO($n$) ($n \geq 4$) with all representations symmetric as follows:

$$\begin{aligned}
&\left\{ \begin{array}{ccc}
a & b & e \\
d & c & f
\end{array} \right\}_{\text{SO}(n)} = \left[ \frac{(2c + n - 2)(2d + n - 2)(2e + n - 2)}{8 d_c^{(n)} d_d^{(n)} d_e^{(n)}} \right]^{1/2} \\
&\times \left( \begin{array}{ccc}
c & d & e \\
0 & 0 & 0
\end{array} \right)_n^{-1} \sum_{l'} (-1)^{(c+d-e)/2+f+n+l'} (2l' + n - 3) \\
&\times \left\{ \begin{array}{ccc}
\frac{1}{2} b & \frac{1}{2} f + \frac{1}{4} n - 1 & \frac{1}{2} d + \frac{1}{4} n - 1 \\
\frac{1}{2} f + \frac{1}{4} n - 1 & \frac{1}{2}(b + n) - 2 & l' + \frac{1}{2} n - 2
\end{array} \right\} \\
&\times \left\{ \begin{array}{ccc}
\frac{1}{2} a & \frac{1}{2} f + \frac{1}{4} n - 1 & \frac{1}{2} c + \frac{1}{4} n - 1 \\
\frac{1}{2} f + \frac{1}{4} n - 1 & \frac{1}{2}(a + n) - 2 & l' + \frac{1}{2} n - 2
\end{array} \right\} \\
&\times \left\{ \begin{array}{ccc}
\frac{1}{2} a & \frac{1}{2} b + \frac{1}{4} n - 1 & \frac{1}{2} e + \frac{1}{4} n - 1 \\
\frac{1}{2} b + \frac{1}{4} n - 1 & \frac{1}{2}(a + n) - 2 & l' + \frac{1}{2} n - 2
\end{array} \right\} \left[ \frac{l'!(n-3)!}{(l' + n - 4)!} \right]^{1/2},
\end{aligned}$$

(2.1)

where in the right-hand side the usual $6j$ coefficients of SU$(2)$ [29–31] appear for $n$ even. Otherwise, for $n$ odd some integer linear combination of parameters of the type $a - l'$, $l'$ or $(b + d - f)/2$ are also restricting the summation intervals in extensions of the asymmetric (Jucys–Bandzaitis) expressions for $6j$ coefficients (as presented by (2.1a, b) and (2.2a, b) in [23] or [34] for $q = 1$), with some ratios of factorials $x!/y!$ turning into ratios of the gamma functions $\Gamma(x + 1)/\Gamma(y + 1)$ with half-integer arguments.

The dimension

$$d_l^{(n)} = \frac{(2l + n - 2)(l + n - 3)!}{l!(n-2)!}$$

(2.2)

of the SO($n$) symmetric irreducible representation $l$ and special $3j$-symbols

$$\begin{pmatrix}
l_1 & l_2 & l_3 \\
0 & 0 & 0
\end{pmatrix}_n = (-1)^{\psi_n} \frac{1}{\Gamma(n/2)} \left[ \frac{(J + n - 3)!}{(n-3)!\Gamma(J + n/2)} \right]$$
\begin{equation}
\times \prod_{i=1}^{3} \frac{(l_i + n/2 - 1) \Gamma (J - l_i + n/2 - 1)}{d_i^{(n)}(J - l_i)!} \right]^{1/2} \tag{2.3a}
\end{equation}

\begin{equation}
= (-1)^\psi_n \frac{\nabla^{-1}_{n[0,1,2,3]}(l_1,l_2,l_3)}{\Gamma(n/2)} \left[ \frac{1}{(n-3)!} \prod_{i=1}^{3} \frac{l_i + n/2 - 1}{d_i^{(n)}} \right]^{1/2} \tag{2.3b}
\end{equation}

(see [16–18] and related special Clebsch–Gordan coefficients [19, 20]), used in (2.1) and further, are rational numbers (in part under the square root sign). In (2.3a), $J = \frac{1}{2}(l_1 + l_2 + l_3)$ and $J - l_i$ ($i = 1, 2, 3$) are non-negative integers. The triangular coefficient $\nabla^{-1}_{n[0,1,2,3]}(l_1,l_2,l_3)$ in (2.3b) may be expressed as follows

\begin{equation}
\nabla^{-1}_{n[0,1,2,3]}(a,b,e) = \left( \frac{\left[ \frac{1}{2}(b + e - a) \right]! \left[ \frac{1}{2}(a - b + e) \right]!}{\Gamma \left( \frac{1}{2}(b + e - a - n) \right) \Gamma \left( \frac{1}{2}(a - b + e + n) \right)} \right) \times \frac{\left[ \frac{1}{2}(a + b - e) \right]! \Gamma \left( \frac{1}{2}(a + b + e + n) \right)}{\Gamma \left( \frac{1}{2}(a + b - e + n) \right) \Gamma \left( \frac{1}{2}(a + b + e) + n - 3 \right)}^{1/2}. \tag{2.4}
\end{equation}

It is reasonable to take $\psi_n = 0$ for $n \geq 4$ (cf. [16, 17, 19]).

Six (from 24) elementary symmetry properties of the $6j$-symbols of $\text{SO}(n)$

\begin{equation}
\left\{ \begin{array}{ccc}
j_1 & j_2 & j_3 \\
l_1 & l_2 & l_3 \end{array} \right\}_{\text{SO}(n)} = \left\{ \begin{array}{ccc}j_a & j_b & j_c \\
l_a & l_b & l_c \end{array} \right\}_{\text{SO}(n)} \tag{2.5}
\end{equation}

are visible from expression (2.1) (see also (16) of [19]).

We may replace the last two factors (in the last line) on the right-hand side of (2.1) by

\begin{equation}
\nabla^{-1}_{n[0,3,5,6]}(a,b,e,0) \sum_{g \geq e} \left( \frac{(g + n - 3) \Gamma \left( \frac{1}{2}(g - e + n) \right) - 2 \left[ \frac{1}{2}(g + e) + n - 4 \right]!}{\left[ \frac{1}{2}(g - e) \right]! \Gamma \left( \frac{1}{2}(g + e + n) \right) \Gamma \left( \frac{1}{2}n - 2 \right)} \right) \times \left( -1 \right)^{(g-e)/2} \left( \frac{(n-3)! \left[ \frac{1}{2}(a - b + g) \right]! \left[ \frac{1}{2}(b - a + g) \right]!}{\left[ \frac{1}{2}(a - b + g) + n - 4 \right]! \left[ \frac{1}{2}(b - a + g) + n - 4 \right]!} \right)^{1/2} \times \left\{ \begin{array}{ccc} \frac{1}{2}(a + n) - 2 & \frac{1}{2}a & l' + \frac{1}{2}n - 2 \\
\frac{1}{2}(b + n) - 2 & \frac{1}{2}b & \frac{1}{2}(g + n) - 2 \end{array} \right\}, \tag{2.6}
\end{equation}

valid also in accordance with $q = 1$ version of expression (2.1b) of [32, 33] and Dougall’s summation formula (2.3.4.5) of [31] for special very well-poised $5F_4(1)$ series as presented by an extension of (A1a) of [25], or (A4a) of [31] (after replacing $\Gamma(-x)/\Gamma(-y)$ if necessary by $(-1)^{x-y} \Gamma(y+1)/\Gamma(x+1)$ for $x - y$ integer) with parameters

\begin{align*}
&j \rightarrow \frac{1}{2}(g + n) - 2, \quad p_1 \rightarrow -\frac{1}{2}(e + n) + 1, \quad p_2 \rightarrow \frac{1}{2}e, \\
p_3 \rightarrow -\frac{1}{2}(a + b + n), \quad p_4 \rightarrow \frac{1}{2}(a + b + n) - 2 - l' + z
\end{align*}
and integer $p_1 + p_4 + 1$, restricting the summation interval of $5F_4(1)$ series. Another triangular coefficient

$$
\nabla_{n[0,3,5,6]}(a, b; e, 0) = \binom{\Gamma \left(\frac{1}{2}(b + e - a + n) - 1\right) \Gamma \left(\frac{1}{2}(a - b + e + n) - 1\right)}{\left[\frac{1}{2}(b + e - a)\right]! \left[\frac{1}{2}(a - b + e)\right]!} \times \frac{\left[\frac{1}{2}(a + b - e)\right]! \Gamma \left(\frac{1}{2}(a + b + e + n)\right)}{\Gamma \left(\frac{1}{2}(a + b - e + n) - 1\right) \left[\frac{1}{2}(a + b + e) + n - 3\right]!}^{1/2}.
\tag{2.7}
$$

in (2.6) is related to $\nabla_{n[0,3,5,6]}(a, b; e, 0)$ as defined by (2.3) of [16], but coincides with it only for even $n$.

Hence, the Biedenharn–Elliott identity, applied to triplet of $6j$-coefficients of SU(2) in (2.1) with substituted (2.6), allowed us to present special $6j$-symbol of SO($n$) (cf. (5.7) of [16]) for $n > 4$ as follows:

$$
\begin{align*}
\{a & b e \cr
\{d & c f \}
\end{align*}
_{\text{SO}(n)} = \left[\frac{(2c + n - 2)(2d + n - 2)(2e + n - 2)}{8 d_c^{(n)} d_d^{(n)} d_e^{(n)}}\right]^{1/2} \binom{c d e}{0 0 0}^{-1} \\
\times & \nabla_{n[0,3,5,6]}(a, b; e, 0) \sum_{g=e}^{a+b} \frac{(g + n - 3)!}{\left[\frac{1}{2}(g - e)\right]! \Gamma \left(\frac{1}{2}(g + e + n)\right) \Gamma \left(\frac{1}{2}n - 2\right)} \\
\times & \left(\frac{\left[\frac{1}{2}(a + b + g)\right]! \left[\frac{1}{2}(b - a + g)\right]!(n - 3)!}{\left[\frac{1}{2}(a - b + g) + n - 4\right]! \left[\frac{1}{2}(b - a + g) + n - 4\right]!}\right)^{1/2} \\
\times & \left[\frac{1}{2}(g + e) + n - 4\right]! \binom{\frac{1}{2}c + \frac{1}{2}n - 1 & \frac{1}{2}a & \frac{1}{2}f + \frac{1}{2}n - 1 \\
\frac{1}{2}c & \frac{1}{2}(a + n) - 2 & \frac{1}{2}(g + n) - 2 \\
\frac{1}{2}c & \frac{1}{2}(b + n) - 2 & \frac{1}{2}(b + n) - 2 \\
\frac{1}{2}d + \frac{1}{2}n - 1 & \frac{1}{2}f + \frac{1}{2}n - 1 \\
\frac{1}{2}d + \frac{1}{2}n - 1 & \frac{1}{2}f + \frac{1}{2}n - 1}
\right),
\tag{2.8}
\end{align*}
$$

with product of two $6j$-coefficients of SU(2) (some parameters of which accept values multiple of 1/4 for odd $n$) in the right-hand side. It was suggested in [16] to use for them the most symmetric (Racah) expression (see [29–31]) useless, however, for rearrangement of (2.8).

3. Double sum expressions for $6j$-symbols of SO($n$)

Nevertheless, we may rearrange (2.8) expressing the second $6j$-coefficient of SU(2) by means of (2.1a) of [13, 14] and the first one by means of (2.1b) of [13, 14]. In this case the factors, depending on the summation parameter $j = (g + n)/2 - 2$ and distributed in the numerators and denominators of different $6j$-coefficients under the square root, cancel, together with the asymmetric triangle coefficients

$$
\nabla(xyj) = \binom{(x + y - j)!(x - y + j)!(x + y + j + 1)!}{(y + j - x)!}^{1/2}
\tag{3.1a}
$$
\[
\Gamma(x + y - j + 1) \Gamma(x - y + j + 1) \Gamma(x + y + j + 2)
\]
\[
\Gamma(y + j - x + 1)
\]
\[
1/2.
\]
(3.1b)

Then we again may use the Dougall’s summation formula for very well-poised \( {}_5F_4(1) \) series as presented by (A1b) of [35] (see (A4b) of [33]) with parameters

\[
j \rightarrow \frac{1}{2}(g + n) - 2, \quad p_1 \rightarrow -\frac{1}{2}(e + n) + 1, \quad p_2 \rightarrow \frac{1}{2} e,
\]
\[
p_3 \rightarrow \frac{1}{2}(f - b - c) - 1 + z_2, \quad p_4 \rightarrow \frac{1}{2}(b + c - f + n) - 2 + z_1.
\]

Two more different rearrangements of (2.8) are also possible in the following ways:
The second version may be obtained when we express the last 6\(j\)-coefficients of SU(2) in the right-hand side

\[
\left\{ \begin{array}{l}
\frac{1}{2}(a + n) - 2 \\
\frac{1}{2}f + \frac{1}{4}n - 1 \\
\frac{1}{2}d + \frac{1}{4}n - 1
\end{array} \right\}
\]
\[
\left\{ \begin{array}{l}
\frac{1}{2}(b + n) - 2 \\
\frac{1}{2}c + \frac{1}{4}n - 1 \\
\frac{1}{2}b
\end{array} \right\}
\]

(3.2)

(with transposed parameters) by means of (2.2a) of [33, 34] and use extended version of (A1a) of [33] (or (A4a) of [33]) with parameters

\[
j \rightarrow \frac{1}{2}(g + n) - 2, \quad p_1 \rightarrow -\frac{1}{2}(e + n) + 1, \quad p_2 \rightarrow \frac{1}{2} e,
\]
\[
p_3 \rightarrow z_2 - \frac{1}{2}(b + c + f + n), \quad p_4 \rightarrow \frac{1}{2}(b + c - f + n) - 2 + z_1.
\]

The third version may be obtained when we express the 6\(j\)-coefficients of SU(2) in the right-hand side of (2.8)

\[
\left\{ \begin{array}{l}
\frac{1}{2}(g + n) - 2 \\
\frac{1}{2}a \\
\frac{1}{2}f + \frac{1}{4}n - 1 \\
\frac{1}{2}d + \frac{1}{4}n - 1
\end{array} \right\}
\]
\[
\left\{ \begin{array}{l}
\frac{1}{2}(b + n) - 2 \\
\frac{1}{2}c + \frac{1}{4}n - 1 \\
\frac{1}{2}b
\end{array} \right\}
\]

\[
\times \left\{ \begin{array}{l}
\frac{1}{2}(a + n) - 2 \\
\frac{1}{2}f + \frac{1}{4}n - 1 \\
\frac{1}{2}d + \frac{1}{4}n - 1
\end{array} \right\}
\]

(3.3)

by means of (2.2a) and (2.1a) of [34], respectively, and use an extended version of (A1b) of [35] with parameters

\[
j \rightarrow \frac{1}{2}(g + n) - 2, \quad p_1 \rightarrow -\frac{1}{2}(e + n) + 1, \quad p_2 \rightarrow \frac{1}{2} e,
\]
\[
p_3 \rightarrow \frac{1}{2}(c - d) - 1 - z_2, \quad p_4 \rightarrow \frac{1}{2}(a + b + n) - 2 - z_1.
\]

In all three cases the summation intervals over \(j\) (or \(g\)) are restricted by non-negative integers \(p_1 + p_4 + 1\). In contrast with the case of 9\(j\) and 12\(j\) coefficients of SU(2) (see [33–35]), the formal summation intervals over \(g\) cannot exceed \(\frac{1}{2} \min(a + b - e, c + d - e)\) (determined by triangular conditions) in main and replaced by (3.2) versions of (2.8). Nevertheless, the possible superfluous terms arising for \(g = c + d + 2, c + d + 4, \ldots\) in the third version of (2.8) (with 6\(j\) coefficients replaced by (3.3)) are unimportant, since in this case the sum over \(z_2\) turns into 0, in accordance with Karlssons summation formula [37] (cf. section 2 of [33, 34]).
Now it is convenient to write the $6j$-symbol of SO($n$) in terms of so-called $\alpha$-graph
$I_n(a, b, e|d, c, f)$ or related quantity $c_{a,b,e; d,c,f}^{(a,n)}$ (see [19])
\[
\begin{align*}
\{ a & \ b \ c \ e \ \} \\
\{ d & \ c \ f \ \}_{SO(n)} = c_{a,b,e; d,c,f}^{(a,n)} \\
&\times \begin{pmatrix} a & c & f \\ 0 & 0 & 0 \end{pmatrix}_n^{-1}
\end{align*}
\]
(3.4a)
\[
= \n_{[0,1,2,3]}(a, b, e) \n_{[0,1,2,3]}(a, c, f) \n_{[0,1,2,3]}(b, d, f)
\times \n_{[0,1,2,3]}(c, d, e) \frac{[(n-3)!]^2 \Gamma(n/2)}{(a, b, c, d, e, f)_{[n]}} c_{a,b,e; d,c,f}^{(a,n)}
\]
(3.4b)
where
\[(a, b, c, d, e, f)_{[n]} = \frac{1}{n!} (2a + n - 2)(2b + n - 2)(2c + n - 2)
\times (2d + n - 2)(2e + n - 2)(2f + n - 2),\]
and the quantities
\[
c_{a,b,e; d,c,f}^{(a,n)} = d^{n}_{a} d^{n}_{b} d^{n}_{c} d^{n}_{d} d^{n}_{e} d^{n}_{f} I_n(a, b, e|d, c, f)
\]
\[
= d^{n}_{a} d^{n}_{b} d^{n}_{c} d^{n}_{d} d^{n}_{e} d^{n}_{f} \int_{SO(n)} dg_1 \int_{SO(n)} dg_2 \int_{SO(n)} dg_3 D^{a}_{00}(g_1)
\times D^{b}_{00}(g_2)D^{c}_{00}(g_3) D^{d}_{00}(g_2^{-1}g_3) D^{e}_{00}(g_3^{-1}g_1) D^{f}_{00}(g_1^{-1}g_2)
\]
(3.5)
are rational numbers and the triangular coefficients $\n_{[0,1,2,3]}(l_1, l_2, l_3)$ are defined by
(2.4). Here $D_{00}^l(g)$ are the zonal spherical functions [17] of irrep $l$ of SO($n$). In our
phase system with $\psi_n = 0$ the phase factor $(-1)^{d+e+f}$ of (2) of [19] vanishes.

From (2.8) after summation over $g$ we obtain three following different expressions for coefficients (3.5):
\[
c_{a,b,e; d,c,f}^{(a,n)} = (a, b, c, d, e, f)_{[n]} \frac{1}{n!} \left[ \frac{1}{2}(a + c + f) + n - 3 \right] \Gamma \left( \frac{1}{2}(a + c + f + n) - 1 \right)
\times \frac{\Gamma \left( \frac{1}{2}(b + e - a + n) - 1 \right) \Gamma \left( \frac{1}{2}(a - b + e + n) - 1 \right)}{\Gamma^{3} \left( \frac{1}{2}n \right) \left[ \frac{1}{2}(b + e - a) \right]! \left[ \frac{1}{2}(a - b + e) \right]! (-1)^{(b+e-f)/2}}
\times \sum_{z_1, z_2} (-1)^{z_1+z_2} \Gamma \left( \frac{1}{2}(b + d + f + n) - 1 + z_1 \right) \left[ \frac{1}{2}(a + c - f) + z_1 \right]!
\times \frac{\Gamma \left( f + \frac{1}{2}n - 1 - z_1 \right) \Gamma \left( \frac{1}{2}(b + c + e + f + n) - 1 - z_2 \right)}{\left[ \frac{1}{2}(b + c - e - f) + z_1 \right] \Gamma \left( \frac{1}{2}(b + c + e - f + n) + z_1 \right) \left[ \frac{1}{2}(d + f - b) - z_1 \right]!}
\times \left[ \frac{1}{2}(b + d - f) - z_2 \right] \Gamma \left( \frac{1}{2}(a + c + f + n) - 1 - z_2 \right)
\times \left[ \frac{1}{2}(a + c + f - n) - 1 \right] \Gamma \left( \frac{1}{2}(b + d - f) - z_2 \right) \Gamma \left( \frac{1}{2}(a + c + f + n) - 1 + z_2 \right)
\times \left[ \frac{1}{2}(b + d - f) - z_2 \right] \Gamma \left( \frac{1}{2}(a + c + f + n) - 1 - z_2 \right)
\]
\[
\times \left[ \frac{1}{2}(a + c + f - n) - 1 \right] \Gamma \left( \frac{1}{2}(b + d - f) - z_2 \right) \Gamma \left( \frac{1}{2}(a + c + f + n) - 1 + z_2 \right)
\]
\[
\times \left[ \frac{1}{2}(b + d - f) - z_2 \right] \Gamma \left( \frac{1}{2}(a + c + f + n) - 1 - z_2 \right)
\]
\[
\times \left[ \frac{1}{2}(a + c + f - n) - 1 \right] \Gamma \left( \frac{1}{2}(b + d - f) - z_2 \right) \Gamma \left( \frac{1}{2}(a + c + f + n) - 1 + z_2 \right)
\]
\[
\times \left[ \frac{1}{2}(b + d - f) - z_2 \right] \Gamma \left( \frac{1}{2}(a + c + f + n) - 1 - z_2 \right)
\begin{align}
\times & \frac{(z_1 + z_2)!}{\frac{1}{2}(e + f - b - c) + z_2} \Gamma \left( f + \frac{1}{2}n + z_2 \right) \Gamma \left( \frac{1}{2}n - 1 + z_1 + z_2 \right) \tag{3.6a} \\
& = (a, b, c, d, e, f) \left[ \frac{1}{2}(a + c + f) + n - 3 \right] ! \Gamma \left( \frac{1}{2}(a + c - f + n) - 1 \right) \\
& \times \frac{\Gamma \left( \frac{1}{2}(b + e - a + n) - 1 \right) \Gamma \left( \frac{1}{2}(a - b + e + n) - 1 \right)}{\Gamma^3 \left( \frac{1}{2}n \right) \left[ \frac{1}{2}(b + e - a) \right] ! \left[ \frac{1}{2}(a - b + e) \right] !} \left( -1 \right)^{(a-b-c+d)/2} \\
& \times \sum_{z_1, z_2} (-1)^{z_1 + z_2} \Gamma \left( \frac{1}{2}(b + d - f + n) - 1 + z_1 \right) \left\{ \frac{1}{2}(a + c - f) + z_1 \right\} ! \\
& \times \frac{\Gamma \left( f + \frac{1}{2}n - 1 - z_1 \right) (f - z_1 - z_2)!}{\Gamma \left( \frac{1}{2}(b + c + e - f + n) + z_1 \right)} \\
& \times \frac{\Gamma \left( f + \frac{1}{2}n - 1 - z_1 - z_2 \right) \left[ \frac{1}{2}(b + d - f + n) - 1 + z_1 \right] \left[ \frac{1}{2}(a + c - f + n) - 1 \right] \left[ \frac{1}{2}(c + f - a - z_2) \right] !}{\Gamma \left( \frac{1}{2}(b + c + e - f + n) + n - 3 - z_2 \right) !} \\
& \times \frac{\Gamma \left( \frac{1}{2}(b + d - f + n) - z_2 \right) \left[ \frac{1}{2}(a + c + f) + n - 3 - z_2 \right] !}{\Gamma \left( \frac{1}{2}(b + c + e - f + n) - n + 3 - z_2 \right) !} \tag{3.6b} \\
& = (a, b, c, d, e, f) \left[ \frac{1}{2}(a + c + f - a + n) - 1 \right] ! \Gamma \left( \frac{1}{2}(a - c + f + n) - 1 \right) \\
& \times \frac{\Gamma \left( \frac{1}{2}(b + e - a + n) - 1 \right) \Gamma \left( \frac{1}{2}(a - b + e + n) - 1 \right)}{\Gamma^3 \left( \frac{1}{2}n \right) \left[ \frac{1}{2}(b + e - a) \right] ! \left[ \frac{1}{2}(a - b + e) \right] !} \left( -1 \right)^{(a+d-e-f)/2} \\
& \times \sum_{z_1, z_2} (-1)^{z_1 + z_2} \Gamma \left( \frac{1}{2}(a + b + c - d + n) - 1 - z_1 \right) \left( a - z_1 \right) ! \\
& \times \frac{1}{2}(a + b + c + d) + n - 3 - z_1 \right] \Gamma \left( \frac{1}{2}(a + b + e + n) - z_1 \right) \\
& \times \frac{1}{2}(a + c - f) - z_1 \right] \Gamma \left( \frac{1}{2}(a + c + f + n) - z_1 \right) \left[ \frac{1}{2}(b + d - f) - z_2 \right] ! \\
& \times \Gamma \left( \frac{1}{2}(d + e - c + n) - 1 + z_2 \right) \left[ \frac{1}{2}(c + d + e) - z_2 \right] ! \Gamma \left( \frac{1}{2}(a - b + c + d + n) - 1 + z_2 \right) \\
& \times \Gamma \left( \frac{1}{2}(a + b + c + d + n) - 1 - z_2 \right) \left[ \frac{1}{2}(a + b + c - d) - z_1 - z_2 \right] ! \\
& \times \Gamma \left( d + \frac{1}{2}n + z_2 \right) \Gamma \left( \frac{1}{2}(a + b + c - d + n) - 1 - z_1 - z_2 \right) ! \tag{3.6c}
\end{align}

without the visible symmetry properties of 6j-symbols of the orthogonal SO(n) group.
These expressions are valid for $n \geq 4$ (and probably for $c_{a,b,c,d,e,f}$). When $n = 4$, the numerator and denominator factorials (gamma functions) depending on $z_1 + z_2$ cancel and 6j-symbols of SO(4) split into product (in this case equal to the square) of two 6j
coefficients of SU(2).

All separate sums over \( z_1 \) or \( z_2 \) in \((3.6a)\)–\((3.6c)\) correspond to the terminating balanced (Saalschützian) \( {}_5F_4(1) \) series (cf. \([30, 37]\)), with summation intervals restricted by

\[
\frac{1}{2} \min(a - c + f, d + f - b) \quad \text{and} \quad \frac{1}{2} (d + e - c) \quad \text{for} \quad \frac{1}{2} (b + c - e - f) \geq 0
\]

\((3.7a)\)
or by

\[
\frac{1}{2} \min(a + b - e, c + d - e) \quad \text{and} \quad \frac{1}{2} (b + d - f) \quad \text{for} \quad \frac{1}{2} (b + c - e - f) \leq 0
\]

\((3.7b)\)
in \((3.6a)\), by

\[
\frac{1}{2} \min(a - c + f, d + f - b, a + b - e, c + d - e) \quad \text{and} \quad \frac{1}{2} \min(b - d + f, c + f - a)
\]

\((3.7c)\)
in \((3.6b)\) and by

\[
\frac{1}{2} \min(a + b - e, a + c - f) \quad \text{and} \quad \frac{1}{2} \min(b - d + f, c - d + e)
\]

\((3.7d)\)
in \((3.6c)\).

Using \((3.4b)\), together with expression \((3.6a)\) or \((3.6b)\) for coefficients \(c_{a,b,c}^{(a,n)} \) (after cancelling the dimensions of irreps and the \((a, b, c, d, e, f)_{[n]} \) type factors), the Regge type symmetry (cf. \([40]\))

\[
\begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix}_{SO(n)} = \begin{pmatrix} s_3 - a & s_3 - b & e \\ s_3 - d & s_3 - c & f \end{pmatrix}_{SO(n)}
\]

\((3.8)\)
of special \(6j\)-symbols of \(SO(n)\) (where \( s_3 = \frac{1}{2} (a + b + c + d) \)) may be checked. Otherwise, using \((3.4b)\), together with expression \((3.6c)\), the usual and Regge type symmetries

\[
\begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix}_{SO(n)} = \begin{pmatrix} a & s_1 - e & s_1 - b \\ d & s_1 - f & s_1 - c \end{pmatrix}_{SO(n)}
\]

\((3.9a)\)

\[
= \begin{pmatrix} a & s_1 - f & s_1 - c \\ d & s_1 - e & s_1 - b \end{pmatrix}_{SO(n)} = \begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix}_{SO(n)}
\]

\((3.9b)\)

where \( s_1 = \frac{1}{2} (b + c + e + f) \), are visible. The symmetries \((3.8)\) and \((3.9a)\) correspond to some column transpositions of Shelepin’s \([12] \) \( 4 \times 3 \) \( R \)-array

\[
\begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix} = \begin{vmatrix} a + b - e & a + c - f & b + d - f & c + d - e \\ a - c + f & a - b + e & d + e - c & d + f - b \\ b - d + f & c - d + e & b + e - a & c + f - a \end{vmatrix}
\]

\((3.10a)\)

\[
\times \begin{vmatrix} 2r_{11} & 2r_{12} & 2r_{13} & 2r_{14} \\ 2r_{21} & 2r_{22} & 2r_{23} & 2r_{24} \\ 2r_{31} & 2r_{32} & 2r_{33} & 2r_{34} \end{vmatrix}
\]

\((3.10b)\)
of \(6j\) coefficients (cf. \((29.32)\) of \([29]\) or \((12)\) in section 9.1 of \([30]\)). Array \((3.10a)\) (cf. also \([41]\)) is also convenient for description of 144 symmetries of \(6j\)-symbols of \(SO(n)\) under
arbitrary transpositions of its columns or rows. Since all the entrees of (3.10a) are even integers for $n \geq 4$, integer parameters $r_{ik} = \beta_i - \alpha_k$ ($i = 1, 2, 3; j = 1, 2, 3, 4$) may be more convenient, as well as the most symmetric parameterization

$$\begin{align*}
\alpha_1 &= \frac{1}{2}(c + d + e), \quad \alpha_2 = \frac{1}{2}(b + d + f), \quad \alpha_3 = \frac{1}{2}(a + c + f), \quad \alpha_4 = \frac{1}{2}(a + b + e), \\
\beta_1 &= \frac{1}{2}(a + b + c + d), \quad \beta_2 = \frac{1}{2}(a + d + c + f), \quad \beta_3 = \frac{1}{2}(b + c + e + f),
\end{align*}$$

(3.11)

with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \beta_1 + \beta_2 + \beta_3$ (cf. [29]).

Expression (3.6a) includes the minimum of terms, when minimal values are accepted by the parameters in the same (the first or the second) row of array (3.10a) ($r_{i3}$ and $r_{i1}$ or $r_{i4}$, for $i = 1$ or 2). Otherwise, (3.6b) and (3.6c) include the minimum of terms, when minimal values are accepted by the parameters in the same column (respectively, $r_{i1}$ and $r_{31}$ or $r_{i4}$ and $r_{34}$, $i = 1$ or 2 for (3.6b), or $r_{i1}$ and $r_{31}$ or $r_{14}$ and $r_{34}$ in the last case). As a rule (with single exception in each case), the definite triplets of numerator and denominator factorials or gamma functions, depending on summation parameters $z_1$, $z_2$ and $z_1 + z_2$, form in (3.6a), (3.6b) and (3.6c) the binomial coefficients (e.g., $z_1!$, $z_2!$ and $(z_1 + z_2)!$), their analytical continuation or beta functions which respond to relations (5.9a) or (5.9b) between the parameters of special Kampé de Fériet functions $F_{1:3:3;1,1,1}$, considered in section 5.

4. Expressions for $6j$-symbols of $SO(n)$ with summation restricted by single parameter

The double sum expressions of $6j$-symbols of $SO(n)$ may be inconvenient, when $r_{i1} \ll r_{ik}$ ($k = 2, 3, 4$) and $r_{i1} \ll r_{i1}$ ($i = 2, 3$). In the stretched case of $6j$-symbols of $SO(n)$ (with $r_{i1} = 0$) we obtain

$$c_{a,b,a+b,d,c,f}^{(\alpha,n)} = \frac{(a, b, c, d, e, f)[n] \Gamma \left( a + \frac{1}{2}n - 1 \right) \Gamma \left( b + \frac{1}{2}n - 1 \right)}{(n - 3)! \Gamma^3 \left( \frac{3}{2}n \right) \nabla^2 \left( \frac{1}{2}a + \frac{1}{2}c + \frac{1}{4}n - 1, \frac{1}{2}f + \frac{1}{4}n - 1 \right)}$$

$$\times \frac{\nabla^2 \left( \frac{1}{2}(e + n) - 2, \frac{1}{2}c + \frac{1}{4}n - 1, \frac{1}{2}d + \frac{1}{4}n - 1 \right)}{\nabla^2 \left( \frac{1}{2}b, \frac{1}{2}d + \frac{1}{4}n - 1, \frac{1}{2}f + \frac{1}{4}n - 1 \right) \Gamma \left( e + \frac{1}{2}n \right)}$$

(4.1)

(cf. (5.4) of [10]), since some parameter from sets (3.7a)–(3.7c) turns into 0 (possibly after using some symmetry property of $6j$-symbols), together with fixed corresponding summation parameter, when other sum turns into summable balanced $3F_2(1)$ series (see [30, 37] and Appendix A of [33]). When vanishing linear combination of parameters of the stretched $6j$-symbol does not belong to sets (3.7a)–(3.7c) the summation of special cases of (3.6a)–(3.6c) is more difficult.

In a near to the stretched case with $e = a + b - 2$, the sum over $z_1$ in (3.6a) includes two terms and we have the $4F_3(1)$ type sums over $z_2$ corresponding to the $6j$ coefficients
of SU(2). Using for them the most symmetric (Racah) expression (see [29–31]) we derive the following expression for special coefficients (3.5):

\[
c_{a,b,a+b-2,d,e,f}^{(a,n)} = \frac{(a,b,c,d,e,f)_{[n]} \Gamma \left(a + \frac{1}{2}n - 2\right) \Gamma \left(b + \frac{1}{2}n - 2\right)}{(n-3)! \Gamma^3 \left(\frac{1}{2}n\right) \nabla^2 \left(\frac{1}{2}a, \frac{1}{2}c + \frac{1}{4}n - 1, \frac{1}{2}f + \frac{1}{4}n - 1\right)} \\
\times \frac{\nabla^2 \left(\frac{1}{2}(e + n) - 2, \frac{1}{2}c + \frac{1}{4}n - 1, \frac{1}{2}d + \frac{1}{4}n - 1\right)}{64 \Gamma \left(e + \frac{1}{2}n + 1\right) \nabla^2 \left(\frac{1}{2}b, \frac{1}{2}d + \frac{1}{4}n - 1, \frac{1}{2}f + \frac{1}{4}n - 1\right)} \times \{2a(c + d - e)(e - c + d + n - 2) [(c + d - e + n - 4)(b + d - f) \\
\times (a + c - f + n - 4) - (c + d + e + 2n - 4)(a - c + f)(b - d + f)] \\
+(2c + n)(a - c + f)(c + f - a + n - 2) [(c + d + e + 2n - 4) \\
\times (b - d + f)(a - c + f + n - 4) - (b + d - f)(c + d - e)] \times (a + c - f + n - 4)\}\right). \tag{4.2}
\]

Expressions (4.1) and (4.2) cover all but the last entrees of table 1 of [19].

Otherwise, all three summation intervals in expression (2.8) are restricted by \(r_{11} = \frac{1}{2}(a + b - e)\). The sum over \(g\) in (2.8) turn into the very well-poised hypergeometric \(\tau F_6(1)\) series when we express the 6\(j\)-coefficients of SU(2) in the right-hand side of (2.8)

\[
\left\{ \begin{array}{l}
\frac{1}{2}(b + n) - 2 \\
\frac{1}{2}d + \frac{1}{4}n - 1 \\
\frac{1}{2}f + \frac{1}{4}n - 1 \\
\frac{1}{2}a \\
\frac{1}{2}(g + n) - 2 \\
\frac{1}{2}f + \frac{1}{4}n - 1 \\
\frac{1}{2}c + \frac{1}{4}n - 1 \\
\frac{1}{2}d + \frac{1}{4}n - 1
\end{array} \right\} \times \left\{ \begin{array}{l}
\frac{1}{2}(g + n) - 2 \\
\frac{1}{2}b \\
\frac{1}{2}(a + n) - 2 \\
\frac{1}{2}f + \frac{1}{4}n - 1 \\
\frac{1}{2}c + \frac{1}{4}n - 1 \\
\frac{1}{2}d + \frac{1}{4}n - 1
\end{array} \right\} \right). \tag{4.3}
\]

by means of (2.1a) (with inverted summation parameter) and (2.2a) of [33, 34], respectively. Using Watson’s transformation formula (2.5.1) of [37] or (6.10) of [48], we rearrange the sum over \(g\) into balanced \(4 F_3(1)\) hypergeometric series (see also the related transition between expression (C3) for the 6\(j\) coefficients [49] in terms of (3.1a) and expression (2.1a) of [33, 34]), with inverted sum. Then instead of (2.8) we obtain the following triple sum expression for coefficients (3.5):

\[
c_{a,b,c,d,e,f}^{(a,n)} = (a,b,c,d,e,f)_{[n]} \frac{\Gamma \left(\frac{1}{2}(a + b + e) + n - 3\right) \Gamma \left(\frac{1}{2}(a + b - e + n) - 1\right)}{(n-3)! \Gamma \left(\frac{1}{2}(a + c + f + n)\right) \frac{1}{2}(a + c - f)!} \\
\times \frac{\Gamma \left(\frac{1}{2}(b + e - a + n) - 1\right) \Gamma \left(\frac{1}{2}(a - b + e + n) - 1\right)}{\Gamma^3 \left(\frac{1}{2}n\right) \left[\frac{1}{2}(b + e - a)!\right] \left[\frac{1}{2}(a + b + e)!\right]} \\
\times \frac{\Gamma \left(\frac{1}{2}(d - b + f + n) - 1\right)}{\left[\frac{1}{2}(b - d + f)!\right]} \sum_{z_1,z_2,z_3} \frac{(-1)^{(a+b-e)/2+z_1+z_2+z_3}(a-z_1)!}{\Gamma \left(\frac{1}{2}(c + f - a + n) - 1 + z_1\right) (b - z_2)!} \\
\times \frac{\Gamma \left(\frac{1}{2}(c + f - a + n) - 1 + z_1\right)}{\left[\frac{1}{2}(a - c + f) - z_1\right]! \left[\frac{1}{2}(b + d - f) - z_2\right]!} \frac{1}{(a + b + n - 3 - z_1 - z_2)!}.
\]
\[
\times \left[ \frac{1}{2} (a + b - e) - z_3 \right] \Gamma \left( a + b + \frac{1}{2} (d - c - e + n) - 1 - z_1 - z_2 - z_3 \right) \\
\times \frac{\frac{1}{2} (a + b - e) - z_1 - z_3 \left[ \frac{1}{2} (a + b - e) - z_2 - z_3 \right] \Gamma \left( e + \frac{1}{2} n + z_3 \right) \\
\times \frac{\frac{1}{2} (a + b + c + d) + n - 3 - z_2 \right] \Gamma \left( \frac{1}{2} (c - d + e + n) - 1 + z_3 \right) \\
\Gamma \left( \frac{1}{2} (b + d + f + n) - z_2 \right) \Gamma \left( \frac{1}{2} (a + b - e + n) - 1 - z_3 \right),
\]

(4.4)

where the separate sums are the balanced \( _4F_3 \) series. The total number of terms in (4.4) does not exceed \( \frac{1}{6} (r_{11} + 1)(r_{11} + 2)(2r_{11} + 3) \) (but may be surpassed by \( (r_{11} + 1)(r_{13} + 1) \) of (3.6a) or \( (r_{11} + 1)(r_{31} + 1) \) of (3.6b) or (3.6c)). This expression does exhibit no usual or Regge type symmetry of \( 6j \)-symbols of \( \text{SO}(n) \).

It should be noted that all three summation intervals for the triple sum expressions, derived directly from (2.1) after used diverse expressions for the Racah coefficients (together with Dougall’s summation formula \([36, 37]\) of \( _5F_4 \) series) are never restricted by a single parameter.

5. **Expansions in terms of Pochhammer symbols and Kampé de Fériet series**

Using the parameters \( r_{ik} = \beta_i - \alpha_k \) of modified Shelepin’s \([12]\) \( R \)-array (3.10b) (together with invariant parameters \([3, 11]\)) and Pochhammer symbols, we may rewrite expressions for Regge symmetrical quantities in the following form:

\[
\frac{c_{a,b,c,d,e,f}^{(a,n)}}{(a, b, c, d, e, f)_{[n]}} = \frac{(-1)^{\alpha_1 - \alpha_3 (a_3 + n - 3)}!}{\Gamma^3(n/2)(n - 3)!r_{11}!r_{12}!r_{13}!r_{14}!r_{21}!r_{33}!} \\
\times \Gamma \left[ \frac{r_{22} + \tau, r_{23} + \tau, r_{24} + \tau, r_{32} + \tau, r_{33} + \tau, r_{34} + \tau, r_{34} + \tau}{\alpha_2 + \tau + 1, \alpha_3 + \tau + 1, \alpha_4 + \tau + 1} \right] \\
\times \sum_{x_1, x_2} \left( \frac{r_{11}}{x_1} \right) \left( \frac{r_{13}}{x_2} \right) (-1)^{x_1 + x_2} \Gamma (-r_{14}, r_{22} + 1, r_{23} + \tau, x_1) \\
\times (-r_{21}, -\alpha_4 - \tau, r_{34} + \tau)_{r_{11} - x_1} \\
\times (r_{24} + \tau, -r_{12} - r_{13} = \tau + 1)_{x_1} (-\alpha_2 - \tau, r_{32} + \tau)_{r_{13} - x_2} \\
\times (\beta_2 - \beta_1 + x_2 + 1)_{x_1} (-r_{21} - \tau - x_2 + 1, r_{11} - x_1) \\
\times (1)^{\beta_1 - \beta_3 (\alpha_1 + n - 3)}! \\
\times \Gamma^3(n/2)(n - 3)!r_{11}!r_{12}!r_{14}!r_{21}!r_{31}!r_{33}! \\
\times \Gamma \left[ \frac{r_{12} + \tau, r_{22} + \tau, r_{23} + \tau, r_{24} + \tau, r_{33} + \tau, r_{34} + \tau, r_{34} + \tau}{\alpha_2 + \tau + 1, \alpha_3 + \tau + 1, \alpha_4 + \tau + 1} \right] \\
\times \sum_{x_1, x_2} \left( \frac{r_{11}}{x_1} \right) \left( \frac{r_{31}}{x_2} \right) (-1)^{x_1 + x_2} \Gamma (-r_{14}, r_{22} + 1, r_{25} + \tau, x_1) \\
\times (-r_{21}, r_{34} + \tau, -\alpha_4 - \tau)_{r_{11} - x_1} (-\alpha_2 - \tau, -\alpha_3 - n + 3)_{x_2} \\
\times (r_{24} + \tau, \alpha_1 + n - 2)_{r_{31} - x_1} (r_{34} - x_2 + 1, r_{11} - x_1) \tag{5.1a}
\]
where $\tau = n/2 - 1$. For products of several gamma functions in numerator and denominator and Pochhammer symbols (appearing here only in the numerator) we use the notations

$$
\Gamma \left[ \left( \begin{array}{c} a_1, a_2, \ldots, a_A \\ b_1, b_2, \ldots, b_B \end{array} \right) \right] = \frac{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_A)}{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_B)},
$$

(5.2)

$$(a_1, a_2, \ldots, a_A)_k = (a_1)_k(a_2)_k\ldots(a_A)_k = \frac{\Gamma(a_1+k)\Gamma(a_2+k)\ldots\Gamma(a_A+k)}{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_A)}.
$$

(5.3)

Now we may express the terminating double hypergeometric series in (5.1a)–(5.1c) in terms of special Kampé de Fériet function $F_{1:3;3}^{1:4;4}[\ldots;1,1]$, which is defined as follows:

$$
F_{1:3;3}^{1:4;4} \left[ \begin{array}{c} a_1 \\ c_1 \end{array} ; (b), (b'); (d), (d') ; x, y \right] = \sum_{s,t}^{\infty} \frac{(a_1)_{s+t}}{s!t!(c_1)_{s+t}} \prod_{j=1}^{d} (b_j)_s (b'_j)_t \prod_{j=1}^{d'} (d_j)_t x^s y^t.
$$

(5.4)

and is terminating, because some separate numerator parameters are equal to negative integers: e.g., $b_i = -m$ and $b'_i = -n$ in (5.4), with $m$ and $n$ positive integers. In both cases some of the denominator parameters may be negative integers, but they should be smaller than the parameters responsible for the termination of series. Both separate series are balanced $\text{5}_F(1)$ series with parameters satisfying conditions

$$
c_1 - a_1 = 1 + \sum_{i=1}^{4} b_i - \sum_{j=1}^{3} d_j = 1 + \sum_{i=1}^{4} b'_i - \sum_{j=1}^{3} d'_j = \tau - 1 = n/2 - 2.
$$

(5.5)

Further we denote $r_{jk} + \tau$ by $\hat{r}_{jk}$, $a_k + \tau$ by $\hat{a}_k$ and $\beta_j + \tau$ by $\hat{\beta}_j$ and write expression for (5.1a) in terms of special Kampé de Fériet series as follows:

$$
\frac{c_{(a,b,c,d,e,f)}^{(a,n)}}{(a,b,c,d,e,f)_{[n]}} = \frac{(a_3 + n - 3)!}{\Gamma^3(n/2)(n - 3)!r_{11}!r_{12}!r_{13}!r_{14}!r_{33}!(\beta_2 - \beta_1)!}
$$

(5.5)
\[\begin{align*}
\times & \Gamma \left[ \hat{r}_{21}, \hat{r}_{22}, \hat{r}_{23}, \hat{r}_{24}, \hat{r}_{34}, r_{11}, \hat{r}_{32} + r_{13} \right] \\
& \times F_{1:3;3}^{1:4:4} \left[ \beta_2 - \beta_1 + 1, -r_{11}, -r_{14}, -\hat{r}_{23}, r_{22} + 1 \\
& \beta_2 - \beta_1, -r_{11} - r_{14}, -\hat{r}_{23}, r_{22} + 1; \\
& \hat{r}_{21}, \hat{r}_{24}, -\hat{r}_{12} + 1 \right] \\
& \times F_{1:3;3}^{1:4:4} \left[ -r_{11} - r_{23} + 1, -r_{11} - r_{23} + 1, \hat{r}_{34}, r_{11} + r_{13} \right] \\
& \times F_{1:3;3}^{1:4:4} \left[ -r_{11} - r_{23} + 1, -r_{11} - r_{23} + 1, \hat{r}_{34}, r_{11} + r_{13} \right] \\
& = \frac{(-1)^{\alpha_1 - \alpha_3}((\alpha_3 + n - 3)!(r_{11} + r_{22})!)(r_{11} + r_{23})!}{\Gamma^3(n/2)(n - 3)!r_{12}!r_{13}!r_{12}!r_{21}!r_{22}!r_{23}!r_{33}!r_{34}!(\alpha_1 - \alpha_4)!} \\
& \times F_{1:3;3}^{1:4:4} \left[ -r_{34} - r_{11} + 1, -r_{34} - r_{11} + 1, \hat{r}_{34}, r_{34} + r_{11} + 1 \right] \\
& = \frac{(-1)^{\beta_1 - \beta_3}((\beta_3 + n - 3)!(r_{34} + r_{11})!)(\beta_3 + n - 3)!}{\Gamma^3(n/2)(n - 3)!r_{12}!r_{13}!r_{12}!r_{21}!r_{22}!r_{23}!r_{33}!r_{34}!(\beta_2 - \beta_1)!} \\
& \times F_{1:3;3}^{1:4:4} \left[ -r_{34} - r_{11} + 1, -r_{34} - r_{11} + 1, \hat{r}_{34}, r_{34} + r_{11} + 1 \right]
\end{align*}\]

where parameters \(r_{11}\) and \(r_{13}\) of array (3.10a) are responsible for the termination of series in both cases (and may be replaced by \(r_{21}\) and \(r_{23}\) in the second (5.6b) case).

Similarly, we write expression for (5.1b) in terms of special Kampé de Féret series as follows:

\[\begin{align*}
\frac{c_{a,b,c,d,e,f}^{(a,n)}}{(a, b, c, d, e, f)_{[n]}} = & \frac{(-1)^{\beta_1 - \beta_3}((\beta_3 + n - 3)!(\beta_3 + n - 3)!)}{\Gamma^3(n/2)(n - 3)!r_{12}!r_{13}!r_{12}!r_{21}!r_{22}!r_{23}!r_{33}!r_{34}!(\beta_2 - \beta_1)!} \\
& \times F_{1:3;3}^{1:4:4} \left[ -r_{34} - r_{11} + 1, -r_{34} - r_{11} + 1, \hat{r}_{34}, r_{34} + r_{11} + 1 \right] \\
& = \frac{1}{(\alpha_1 - \alpha_4)!} \frac{1}{(\alpha_1 - \alpha_4)!} \\
& \times F_{1:3;3}^{1:4:4} \left[ -r_{34} - r_{11} + 1, -r_{34} - r_{11} + 1, \hat{r}_{34}, r_{34} + r_{11} + 1 \right]
\end{align*}\]

where parameters \(r_{11}\) and \(r_{31}\) are responsible for the termination of series in both cases (and may be replaced by \(r_{14}\) and \(r_{34}\) in the first (5.7a) case).
Finally, we write expression for (5.1) in terms of special Kampé de Fériet series as follows:

\[
\frac{c_{a,b,c,d,e,f}^{(α,ν)}}{(a, b, c, d, e, f)_{[n]}} = \frac{(-1)^{β_1-β_3}(β_1 + n - 3)!(r_{11} + r_{22})!(r_{11} + r_{32})!}{Γ^3(n/2)(n-3)!r_{11}!r_{12}!r_{21}!r_{22}!r_{31}!r_{32}!r_{33}!r_{34}!} \\
\times Γ \left[ \hat{r}_{21}, \hat{r}_{22}, \hat{r}_{23}, \hat{r}_{24}, \hat{r}_{33}, \hat{r}_{34}, \hat{r}_{32} + r_{11} \right] \left[ \hat{α}_3 + 1, \hat{α}_4 + 1, \hat{α}_2 - r_{31} + 1, \hat{β}_2 - \hat{β}_3 \right] \times F_{1:3:3}^{1:4:4} \left[ \begin{array}{c}
-\hat{r}_{32} - r_{11} + 1 \\
-\hat{r}_{32} - r_{11} + 1, -r_{11} - r_{12} - \hat{α}_3 - \hat{α}_4 \\
-\hat{r}_{32} - r_{11} + 1, -\hat{β}_1 - n + 3, -r_{11} - r_{22} \\
-\hat{r}_{32} - r_{31}, \hat{r}_{24}, \hat{r}_{23} \end{array} \right], (5.8a)
\]

where parameters \(r_{11}\) and \(r_{31}\) are responsible for the termination of series in both cases (and may be replaced by \(r_{12}\) and \(r_{32}\) in the first (5.8a) case). However, the possible indefiniteness (appearing, e.g., with negative integer arguments of \((β_2 - β_1)!\) or \(Γ(\hat{α}_1 - \hat{α}_2)\) and never troubling in expressions (5.1a)–(5.1c)) should be kept in attention in expressions (5.6a)–(5.8b) in terms of special Kampé de Fériet series.

There are the definite linear dependencies

\[
a_1 = d_1 = d_1' = d_j + d_j' - 1 \quad (j = 2, 3), \]
\[
c_1 = b_i + b_i' - n + 4 = b_i + b_i' \quad (i = 1, 2, 3) \quad (5.9a)
\]

between parameters of each special Kampé de Fériet series in (5.6a), (5.7a) and (5.8b), as well as the relations

\[
a_1 = d_1 = d_1' = d_2 + d_2' - 1 = d_3 + d_3' - n + 3, \]
\[
c_1 = b_i + b_i' \quad (i = 1, \ldots, 4) \quad (5.9b)
\]

between parameters of each special Kampé de Fériet series in (5.6b), (5.7b) and (5.8a). Of course, parameters \(a_1 = d_1 = d_1'\) (which are integers depending on the distance between the rows or columns of Shelepin's array in (5.6a) or in (5.7b) and (5.8b), respectively) are never responsible for the termination of series. There is absent any correlation between the type of dependencies (5.9a)–(5.9b) and the types of parameters \(a_1\) and \(c_1\), the last being non-positive integers in (5.6b), (5.7a) and (5.8a). Nevertheless,
expressions (5.6a) and (5.7a) (as well as (5.6b) and (5.7b)) are mutually related with respect to the substitution (hook reflection)

\[ d \to -d - n + 2, \quad (5.10a) \]

leaving invariant dimension \( d_d^{(n)} \) and character of irrep \( d \) of SO\((n)\), when (5.8b) is invariant under this substitution, which corresponds to the transposition \( \hat{r}_{32} \leftrightarrow \alpha_1 + n - 2 \) (together with \( -r_{31} \leftrightarrow -\hat{\alpha}_2 \)) of parameters. Otherwise, expressions (5.6a) and (5.8b) (as well as (5.6b) and (5.8a)) are mutually related with respect to the hook reflections

\[ c \to -c - n + 2, \quad d \to -d - n + 2, \quad f \to -f - n + 2, \quad (5.10b) \]

leaving invariant dimensions and characters of irreps \( c, d \) and \( f \).

The transposition \( r_{11} \leftrightarrow r_{14} \) (together with \( \hat{r}_{24} \leftrightarrow \hat{r}_{21} \)) in (5.6a) gives Regge symmetry (3.8), as well as \( r_{11} \leftrightarrow r_{14} \) (together with \( r_{31} \leftrightarrow r_{34} \)) in (5.7a). The transposition \( r_{11} \leftrightarrow r_{21} \) (together with \( r_{13} \leftrightarrow r_{23} \)) in (5.6b) gives Regge symmetry

\[
\begin{pmatrix}
  a & b & e \\
  d & c & f
\end{pmatrix}_{SO(n)} \leftrightarrow \begin{pmatrix}
  a & s_1 - c & s_1 - f \\
  d & s_1 - b & s_1 - e
\end{pmatrix}_{SO(n)}, \quad (5.11)
\]

which corresponds to the transposition of rows in array (3.10b), as well as the transposition \( r_{11} \leftrightarrow r_{21} \) (together with \( \hat{r}_{14} \leftrightarrow \hat{r}_{24} \)) in (5.7b). Otherwise, transpositions \( r_{11} \leftrightarrow r_{12} \) (together with \( r_{31} \leftrightarrow r_{32} \)) and \( \hat{\alpha}_3 \leftrightarrow \hat{\alpha}_4 \) (together with \( \hat{r}_{23} \leftrightarrow \hat{r}_{24} \)) are the generators of symmetries (3.9a) and (3.9b) for (5.8a).

Further, the transposition \( -r_{14} \leftrightarrow \hat{r}_{23} \) (together with \( -r_{13} \leftrightarrow \hat{r}_{24} \)) in (5.6a) or with \( r_{31} \leftrightarrow \hat{\alpha}_2 \) in (5.7a), respectively), the transpositions \( r_{11} \leftrightarrow \hat{\alpha}_4 \) (together with \( -r_{32} \leftrightarrow \hat{r}_{32} \) or with \( \hat{r}_{14} \leftrightarrow \alpha_1 + n - 2 \)) and \( -r_{21} \leftrightarrow \hat{r}_{34} \) (together with \( r_{13} \leftrightarrow \hat{\alpha}_2 \) or with \( \hat{r}_{24} \leftrightarrow -r_{31} \)) in (5.6b) or (5.7b), respectively, and the interchange of \( r_{11} \) and \( \hat{\alpha}_4 \) (together with \( -r_{32} \leftrightarrow \hat{r}_{23} \)) or the interchange of \( r_{12} \) and \( \hat{\alpha}_3 \) (together with \( -r_{31} \leftrightarrow \hat{r}_{24} \)) in (5.8a) correspond to some hook reflections. The remaining from 3! or 4! transpositions, leaving invariant dependencies (5.9a) or (5.9b), correspond to compositions of some Regge symmetries and hook reflections, as well as the interchanges of the sets \( b_1, b_2, b_3, b_4; d_2, d_3 \) and \( b'_1, b'_2, b'_3, b'_4; d'_2, d'_3 \) in (5.4).

6. Concluding remarks

In this paper, we considered the recoupling coefficients and 6\( j \)-symbols of the orthogonal SO\((n)\) groups for all six representations corresponding to the spherical or hyperspherical harmonics of these groups. Corrected triple sum expression of Alīšauskas [10] (which was previously derived from the fourfold sum expression of [11], related to later expression of [13]) as an expansion in terms of the 6\( j \) coefficients of SU\((2)\), with possible multiple of 1/4 parameters for odd \( n \), here has been rearranged into three different double
hypergeometric series of the Kampé de Fériet $F_{1:3;3}^{1:4:4}$ type with the moderate \((2\times2)\) symmetry which (together with hidden usual symmetry) nevertheless is resolving for all 144 Regge type symmetries. The different double sum expressions are mutually related with respect to the substitutions, generalizing the “mirror reflection” symmetry \(j \to -j - 1\) of the angular momentum theory \([29]\). Note that more general double $F_{1:3;3}^{1:4:4}$ series (with balanced by condition \((5.5)\) parameters and four (from 8) separate relations of the type \((5.9a)\) or \((5.9b)\)) appeared as the doubly stretched 12\(j\) coefficients of the second kind \([35]\) for SU(2) as presented by expressions \((2.5a)-(2.5c)\) (with \(z_1 = z_2 = 0\)) for \(j_1 = k_1 - l_1 = l_2 - k_3\), or by expression \((2.9)\) for \(k_1 = j_1 + l_1 = j_2 + l_3\), although our 6\(j\)-symbols of SO\((n)\) cannot be reduced to special 12\(j\) coefficients of SU(2).

However, two non-positive integer parameters are necessary for termination of these double series, in contrast with special double hypergeometric series $F_{1:1;1}^{1:2:2}$ of the Kampé de Fériet type \([34, 4, 6]\) which correspond to the stretched 9\(j\) coefficients of SU(2), and may terminate for fixed single integer non-positive parameter, restricting all summation parameters. Single integer non-positive parameter is sufficient for restricting of all summation parameters only in the triple sum expression, derived in section 4. (This parameters appears in the double sums over \(z_1, z_3\) and over \(z_2, z_3\), which again may be treated as the double hypergeometric series $F_{1:3;3}^{1:4:4}$ of the Kampé de Fériet type).

Related recoupling coefficients of the symplectic groups Sp\((2n)\) with all six irreps antisymmetric are also given in Appendix as the double series which correspond to the orthogonal groups SO\((-2n)\) of negative dimension, with more rich restriction structure. These Sp\((2n)\) invariants never form the complete recoupling matrices, as well as the SO\((n)\) invariants considered in this paper as the main objects. Note the essential difference between our 6\(j\)-symbols and the 6\(j\)-symbols of the orthogonal SO\((n)\) groups for all six irreps antisymmetric and 6\(j\)-symbols of the symplectic groups Sp\((2n)\) with all six irreps symmetric as expressed \([3, 5, 22]\) in terms of the hypergeometric (single) \(4F_3(1)\) series with all 144 Regge type symmetries visible immediately. Alternating single sums appear in the Sp\((2n)\) type, but these \(4F_3(1)\) series are balanced (Saalschützian) only in the Sp\((2)\) or SU\((2)\) case, when these 6\(j\)-symbols also are forming the complete recoupling matrices.
Appendix A. Recoupling coefficients for antisymmetric representations of Sp(2n)

As it was demonstrated by Judd et al [7] the recoupling function†

\[
U \left( \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right)_{Sp(2n)} (AM_A, FM_F|CM_C) = \sum_{M_B, M_D, M_E} (DM_D, BM_B|FM_F)
\times (AM_A, BM_B|EM_E)(EM_E, DM_D|CM_C)
\]

(A.1)
of the symplectic group Sp(2n) for all six antisymmetric representations \(\langle 1^\nu \rangle\) \((\nu \leq n)\), with dimension (cf. [32])

\[
d^{(2n)}_{\langle 1^\nu \rangle} = \frac{2(2n + 1)! (n - \nu + 1)}{\nu! (2n - \nu + 2)!}
\]

(A.2)
may be expressed as an analytical continuation of the recoupling coefficients for symmetric representations of the orthogonal group of the negative rank SO(−2n), in accordance with [3, 4, 22] (cf. also [13, 14]) as follows:

\[
U \left( \begin{array}{ccc} \langle 1^a \rangle & \langle 1^b \rangle & \langle 1^e \rangle \\ \langle 1^d \rangle & \langle 1^c \rangle & \langle 1^f \rangle \end{array} \right)_{Sp(2n)} = (-1)^{\chi_{12}} \left( d^{(2n)}_{\langle 1^e \rangle} d^{(2n)}_{\langle 1^f \rangle} \right)^{1/2} \left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\}_{SO(−2n)}
\]

(A.3)
(ignoring possible phase factor \((-1)^{\chi}\)).

It is most convenient to perform the analytical continuation of expressions (5.1a)–(5.1c), together with (3.4b) and (2.4), using the same integer parameters \(r_{ik} = \beta_i - \alpha_k\) \((i = 1, 2, 3; j = 1, \cdots, 4)\) of array (3.10b), or the most symmetric parametrization (3.11). Hence replacing \(n\) by \(-2n\) and \(\tau\) by \(-n - 1\) we write the following three expressions for the recoupling coefficients of the symplectic group:

\[
U \left( \begin{array}{ccc} \langle 1^a \rangle & \langle 1^b \rangle & \langle 1^e \rangle \\ \langle 1^d \rangle & \langle 1^c \rangle & \langle 1^f \rangle \end{array} \right)_{Sp(2n)} = (-1)^{\chi_{12}} \left( d^{(2n)}_{\langle 1^e \rangle} d^{(2n)}_{\langle 1^f \rangle} \right)^{1/2} \frac{N n!^3 (2n + 2)!}{(2n + 2 - \alpha_3)! r_{11}! r_{12}! r_{13}! r_{14}! r_{21}! r_{31}! (n - r_{22} + 1)!}
\times \frac{(n - n - \alpha_2)! (n - \alpha_3)!(n - \alpha_4)!}{(n - r_{23} + 1)!(n - r_{24} + 1)!(n - r_{32} + 1)!(n - r_{33} + 1)!(n - r_{34} + 1)!
\times \sum_{x_1, x_2} \left( \begin{array}{c} r_{11} \\ x_1 \end{array} \right) \left( \begin{array}{c} r_{13} \\ x_2 \end{array} \right) (-1)^{x_1 + x_2} (-r_{14} + r_{22} + 1, r_{23} - n - 1)_{x_1}
\times (-r_{21} - \alpha_4 + n + 1, r_{34} - n - 1)_{r_{11} - x_1}
\times (r_{24} - n - 1, -r_{12} + n + 2)_{x_2} (-\alpha_2 + n + 1, r_{32} - n - 1)_{r_{13} - x_2}
\times (\beta_2 - \beta_1 + x_2 + 1)_{x_1} (-r_{21} + n + 2 - x_2)_{r_{11} - x_1}
\times (2n - \alpha_1 + 2)! r_{11}! r_{12}! r_{14}! r_{21}! r_{31}! r_{33}! (n - r_{12} + 1)!(n - r_{22} + 1)!
\times \frac{N n!^3 (2n + 2)!}{(2n - \alpha_1 + 2)! r_{11}! r_{12}! r_{14}! r_{21}! r_{31}! r_{33}! (n - r_{12} + 1)!(n - r_{22} + 1)!}
\]

(A.4a)

† Which here is defined in a formal way with arbitrary labels of the basis states, without needed internal and external multiplicity labels.
\[
\times \frac{(n - \alpha_2)! (n - \alpha_3)! (n - \alpha_4)!}{(n - r_{23} + 1)!(n - r_{24} + 1)! (n - r_{33} + 1)! (n - r_{34} + 1)!} \\
\times \sum_{x_1, x_2} \left( \frac{r_{11}}{x_1} \right) \left( \frac{r_{31}}{x_2} \right) (-1)^{x_1 + x_2} (-r_{14}, r_{22} + 1, r_{23} - n - 1)_{x_1} \\
\times (-r_{21}, r_{34} - n - 1, -\alpha_4 + n + 1)_{r_{11} - x_1} (-\alpha_2 + n + 1, -\alpha_3 + 2n + 3)_{x_2} \\
\times (r_{24} - n - 1, \alpha_1 - 2n - 2)_{r_{31} - x_2} (r_{34} - x_2 + 1)_{r_{11} - x_1} \\
\times (-r_{34} - r_{11} + n + x_2 + 2)_{x_1} (A.4b)
\]

\[
= \frac{(-1)^{\chi_{12} + \beta_{1} - \beta_{3} + 1} \left( d_{(1\epsilon)}^{(2n)} d_{(1\epsilon)}^{(2n)} \right)^{1/2} N n! (2n + 2)!}{(2n - \alpha_1 + 2)! r_{11}! r_{12}! r_{21}! r_{31}! r_{33}! r_{34}! (n - r_{22} + 1)!} \\
\times \frac{(n - r_{23} + 1)! (n - r_{24} + 1)! (n - r_{33} + 1)! (n - r_{34} + 1)!}{(n - \alpha_2)! (n - \alpha_3)! (n - \alpha_4)!} \\
\times \sum_{x_1, x_2} \left( \frac{r_{11}}{x_1} \right) \left( \frac{r_{31}}{x_2} \right) (-1)^{x_1 + x_2} (-r_{12}, -\alpha_3 + n + 1, -\alpha_4 + n + 1)_{x_1} \\
\times (r_{32} - n - 1, r_{22} + 1, \alpha_1 - 2n - 2)_{r_{11} - x_1} \\
\times (r_{23} - n - 1, r_{24} - n - 1)_{x_2} (-\alpha_2 + n + 1, -r_{21} + n + 2)_{r_{31} - x_2} \\
\times (-r_{32} - r_{11} + n + x_2 + 2)_{x_1} (r_{32} - x_2 + 1)_{r_{11} - x_1}, (A.4c)
\]

where actually some ratios of gamma functions \( \Gamma(-y)/\Gamma(-x) \) turned into the ratios of factorials \((-1)^{x-y}x!/y!\) only in the factor under the square root of \((3.4b)\) (together with \((2.4)\)) which absolute value appeared as

\[
N = \left[ \frac{\prod_{i=1}^{3} \prod_{k=1}^{4} r_{ik}! (n + 1 - r_{ik})!}{(2n + 2)!! n!} \right]^{1/2} \left[ \prod_{k=1}^{4} (2n + 2 - \alpha_k)! \right]^{1/2},
\]

in the dimension factors and in the factors before the summation sign of \((5.1a)-(5.1c)\). Intervals of summation in \((A.4a)-(A.4c)\) are restricted not only by conditions \((3.7a)-(3.7d)\) (e.g., \(r_{11} \geq 0\) and \(r_{13} \geq 0\), or \(r_{31} \geq 0\)) but also by conditions of the type \(n + 1 - r_{ik} \geq 0\). Therefore it may be convenient to write \((A.4a)-(A.4c)\) as factorial series, which summation intervals are determined by non-negative arguments of the denominator factorials. Some factorials before the sum signs cancel. All the separate sums in such new version of \((A.4a)-(A.4c)\) are alternating, with exception of the sum over \(x_1\) in \((A.4c)\). Of course, the recoupling coefficients of the symplectic group \(\text{Sp}(2n)\) with all six antisymmetric irreps vanish unless \(n - \alpha_k \geq 0\). The number of different recoupling coefficients of \(\text{Sp}(2n)\) for fixed \(n\) is finite, in contrast with the number of \(6j\)-symbols of \(\text{SO}(n)\), considered in this paper.

The symmetry properties of the recoupling functions of the symplectic group \(\text{Sp}(2n)\) may be more complicated, e.g.

\[
\left( d_{(1\epsilon)}^{(2n)} d_{(1\epsilon)}^{(2n)} \right)^{-1/2} U \left( \begin{array}{ccc} 1^n & \{1^a\} & \{1^b\} \\ 1^d & \{1^i\} & \{1^c\} \end{array} \right)_{\text{Sp}(2n)} = (-1)^{\chi_{13} - \chi_{12} + (b+c-e-f)/2}
\]
\[ \times \left( a_{12n}^{2n} d_{12n}^{2n} \right)^{-1/2} U \left( \begin{array}{ccc} \langle 1^a \rangle & \langle 1^b \rangle & \langle 1^c \rangle \\ \langle 1^d \rangle & \langle 1^e \rangle & \langle 1^f \rangle \end{array} \right)_{\text{Sp}(2n)}. \]  

(A.5)

The $6j$-symbols of $\text{Sp}(2n)$ in the case of all antisymmetric irreps are real and invariant under all permutations of the type (2.5), if we choose $\chi_{13} = \beta_2$, $\chi_{12} = \beta_1$ as defined by (3.11).

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