On van der Waals friction: I. Between two atoms

Gabriel Barton
Department of Physics and Astronomy, University of Sussex, Brighton BN1 9QH, UK
E-mail: g.barton@sussex.ac.uk

New Journal of Physics 12 (2010) 113044 (10pp)
Received 14 June 2010
Published 26 November 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/11/113044

Abstract. Neglecting retardation and all other relativistic effects, we find the van der Waals force between two identical atoms mimicked as harmonic oscillators, constrained to move with constant relative velocity on fixed parallel tracks, starting infinitely far apart at time $t = -\infty$, each in its ground state. The component parallel to the motion has an irreversible part, producing finite excitation probabilities as $t \to +\infty$. In this sense, the calculation demonstrates that the atoms are subject to nonzero frictional resistance, with the immediate implication that friction is experienced also by an atom moving outside and parallel to the surface of an electrically dilute and nonabsorptive half-space. The next paper in this series finds the friction experienced by an atom outside a half-space which may be absorptive and electrically dense.

Contents

1. Introduction  
2. Perturbation theory  
3. Comments  

Appendix A. Isotropic three-dimensional oscillators  
Appendix B. Nondissipative half-spaces as dilute assemblies of oscillators  
Appendix C. Nonidentical oscillators  
References
1. Introduction

The recent literature contains both explicit controversies and implicit uncertainties or contradictions regarding quite basic features of the effects of constrained uniform relative motion on Casimir-type forces. Our objective is to try and cast some light on such questions by studying scenarios where distances and relative velocities are small enough for retardation to be neglected, with Drude-modelled half-spaces, and/or oscillator-modelled atoms subject only to van der Waals (VdW) forces. This limit of Casimir effects is governed by familiar nonrelativistic quantum mechanics: being simpler and less prone to mishandling, it can serve to check the incomparably more sophisticated calculations based on relativistic quantum field theory or statistical mechanics. References will be given later, in context.

This paper (to be cited as I) deals with the simplest case of two atoms on parallel trajectories. The next paper finds the drag forces, at constant separation, between an atom and a half-space (Barton 2010a; to be cited as II). Loosely speaking, I is a natural preliminary to II, even though it lacks an analogue to the dissipative properties of the half-space, which eventually play a dominant role there. We consider only zero temperature and, in II, pay no attention to the attractive components of the forces. It will become apparent that our main conclusion is that in both cases the drag force is nonzero, and not all that hard to find.

Here in I we start with two simple-harmonic oscillators 1 and 2, constrained to move with speeds $\pm u/2$ parallel to the $z$-axis, on trajectories a distance $a$ apart in the $x$-direction, opposite each other at time $t = 0$ (we shall call this passage). Our nonretarded regime posits $u/c \ll 1$, and $c$ never enters the formalism. For simplicity, we study only identical oscillators and in full detail only the case where both are polarized in the $y$-direction, i.e. normally to the plane of the trajectories. The rest of this section spells out the Hamiltonian. Section 2 calculates the expectation value $\langle \hat{F} \rangle$ of the force between them, taking them each to start in its ground state, and infinitely far apart. The result is displayed in equations (2.7)–(2.10) and discussed in section 3. Brief appendices consider also isotropic three-dimensional (3D) oscillators, and dilute half-spaces constructed from them.

The positions of the mass centres are written as $\mathbf{s}_{1,2} = (\pm a/2, 0, \pm ut/2)$, and their separation as $s = \sqrt{a^2 + (ut)^2}$. Figure 1 shows the layout. The internal coordinates are $\mathbf{r}_{1,2}$ and the dipole-moment operators $e \mathbf{r}_{1,2}$. (We use unrationalized Gaussian units.) In the dipole approximation, the Hamiltonian (after dropping an irrelevant constant part $2 \times (\mu u^2/4)$ stemming from the constraints: cf Barton and Calogeracos 1996) is

$$H = \frac{(p_1^2 + p_2^2)}{2m} + \frac{1}{2}m\omega^2 (y_1^2 + y_2^2) + \frac{e^2}{s^3} y_1 y_2, \quad [y_i, p_j] = i\hbar \delta_{ij}. \quad (1.1)$$

We change to new canonical variables

$$p_\pm = \frac{p_1 \pm p_2}{\sqrt{2}}, \quad y_\pm = \frac{y_1 \pm y_2}{\sqrt{2}}, \quad (1.2)$$

$$H = \left\{ \frac{p_+^2}{2m} + \frac{1}{2}m\omega_+^2 y_+^2 \right\} + \left\{ \frac{p_-^2}{2m} + \frac{1}{2}m\omega_-^2 y_-^2 \right\}, \quad \omega^2 = \omega^2 (1 \pm \beta), \quad (1.3)$$

$$\beta = \alpha/s^3, \quad \alpha = e^2/m\omega^2, \quad (1.4)$$

where $\alpha$ is the zero-frequency polarizability.
Figure 1. Layout of the collision: the heavy dots indicate the oscillators. They are polarized at right angles to the plane of the trajectories. Their separation \( s = \sqrt{a^2 + (ut)^2} \) minimizes at passage, when \( t = 0 \).

One recalls

\[
y_{1, 2} = \sqrt{\frac{\hbar}{2m\omega}} (A_{1, 2}^+ + A_{1, 2}),
\]

where \( A \) and \( A^+ \) are the familiar step-down and step-up operators; the range parameters \( b_1 = b_2 = b = \sqrt{\hbar/2m\omega} \); and the corresponding expressions for \( y_\pm \) and \( b_\pm \) (with \( \omega \to \omega_\pm \)).

The dipole approximation underlying \( H \) assumes that \( b/a \ll 1 \). To proceed perturbatively, as we shall, one requires also \( \beta_{\text{max}} = \epsilon^2/m\omega a^3 = \alpha/a^3 \ll 1 \), sharpening the evident stability condition \( \beta_{\text{max}} < 1 \). We assume that both conditions are met.

To second order in \( \beta \), we approximate

\[
H \to H_+ + H_- + H_{\text{int}+} + H_{\text{int}-}, \quad H_\pm = \frac{p_\pm^2}{2m} + \frac{1}{2}m\omega^2 y_\pm^2, \quad H_{\text{int} \pm} = \frac{\epsilon^2}{2s^3} y_\pm^2,
\]

with \( H_{\text{int}+} + H_{\text{int}-} \) as the perturbation. For stationary oscillators, the mean VdW potential between them and the mean force on atom 1 read

\[
V_0 = -\frac{\hbar e^4}{8m^2\omega^3 s^6} = -\frac{\hbar \omega^2}{8s^6}, \quad F_0 = -\frac{\partial V_0}{\partial s} = -s \frac{3\hbar \omega^2}{4s^8}.
\]

2. Perturbation theory

The \( \pm \) oscillators contribute independently, and in the approximation governed by \( (1.6) \) equally. We calculate the contributions to energies and forces from just one of them, say from +, and then multiply by 2 to obtain the total. We work in the Schrödinger picture. It proves convenient to introduce dimensionless variables

\[
x \equiv ut/a, \quad v \equiv u/\omega a, \quad \lambda \equiv 2/v = 2\omega a/u,
\]

and we shall need

\[
\int_0^\infty d\xi \begin{bmatrix} \cos(\eta \xi)/(1 + \xi^2)^{3/2} \\ \sin(\eta \xi)/(1 + \xi^2)^{3/2} \\ \xi \sin(\eta \xi)/(1 + \xi^2)^{5/2} \end{bmatrix} = \begin{bmatrix} \eta K_1(\eta) \\ \{1 - \pi I_1(\eta)/2 + \pi L_1(\eta)/2\} \\ \eta^2 K_1(\eta)/3 \end{bmatrix},
\]

where \( K_1, I_1 \) and \( L_1 \) are the modified Bessel and Struve functions.

\footnote{The writer must apologize for using the scaled variable ‘\( x \)’ to measure time, which with our present choice of coordinates governs relative displacements of the oscillators along the third axis, customarily called the ‘\( z \)’-axis: given his accumulated backlog here and in II, he can no longer remedy the discrepancy.}

New Journal of Physics 12 (2010) 113044 (http://www.njp.org/)
The perturbation links the ground state $|0_s\rangle$ only to the doubly excited state $|2_s\rangle$, with $\langle 2_s| y_+^2 |0_s\rangle = \sqrt{2}h/2m\omega$. Following textbook time-dependent perturbation theory\(^2\), we write the state vector as

$$\psi_s(t) \simeq \exp(-i\omega t/2) |0_s\rangle + \exp(-5i\omega t/2) c(t) |2_s\rangle, \quad (2.3)$$

$$c(t) = -\frac{i}{\hbar} \int_{-\infty}^{t} dt' \langle 2_s\rangle H_{\text{int}}(t') |0_s\rangle \exp(2i\omega t')$$

$$= -\frac{i}{\hbar} \int_{-\infty}^{t} dt' \frac{e^2}{2s^3(t')} \langle 2_s\rangle y_+^2 |0_s\rangle \exp(2i\omega t'),$$

$$c(t) = -\frac{i\sqrt{2}\omega}{4d^3} (h_r + \imath h_i)$$

$$\begin{pmatrix} h_r \\ h_i \end{pmatrix} = \frac{1}{v} \int_{-\infty}^{x} \frac{dx'}{(1 + x'^2)^{3/2}} \left[ -\sin(2x'/v) \right] \cos(2x'/v). \quad (2.5)$$

We see at once that

$$c(\infty) = -\frac{i\sqrt{2}\omega}{4d^3} \cdot \frac{1}{v} \int_{-\infty}^{\infty} \frac{dx' \cos(2x'/v)}{(1 + x'^2)^{3/2}} = -\frac{i\sqrt{2}\omega}{d^3} \cdot \frac{1}{v^2} K_1 \left( \frac{2}{v} \right). \quad (2.6)$$

The operator for the consequent force\(^3\) on atom 1, and its expectation value at time $t$, are

$$\hat{F}^{(+)} = -\frac{\partial H_{\text{int}}}{\partial \mathbf{s}} = s \left( \frac{3e^2 \hbar}{2s^5} \right) y^2, \quad \langle \hat{F}^{(+)} \rangle = \langle \psi_s | \hat{F} | \psi_s(t) \rangle, \quad \langle \hat{F} \rangle = 2 \langle \hat{F}^{(+)} \rangle. \quad (2.7)$$

Straightforward calculation using the first-order-perturbed state vector (2.3)–(2.5) yields the mean force accurately to second order in $H_{\text{int}}$:

$$\langle \hat{F}^{(+)} \rangle = s \frac{3e^2 \hbar}{4m^2\omega^2s^5} \text{Re} \left\{ -i \exp(-2i\omega t) \int_{-\infty}^{t} dt' \frac{e^2}{s^5} \exp(2i\omega t') \right\}$$

$$= F_0 (1 + x^2)^{3/2} f(x) = -|F_0| (1, 0, x) (1 + x^2) f(x, v), \quad (2.8)$$

$$f = \cos(2x/v) h_r(x, v) + \sin(2x/v) h_i(x, v). \quad (2.9)$$

The factors $(1 + x^2)^{3/2}$ and $(1 + x^2)$ in (2.9) are chosen to make the nontrivial factor $f(x)$ finite at infinity and thereby easier to display.

Mathematically speaking, we could admit $v$ large or small. But if, in atomic units, $\omega \sim 1$, while, realistically, $u \lesssim 1$ and $a \gg 1$, then evidently $v \ll 1$. This makes it tolerable for a rough exploration like ours to continue ignoring retardation effects, even though they become crucial, Casimir-like, when $s$ exceeds the emission wavelength, i.e. when $s \sim u|t| \gtrsim c/\omega \Rightarrow |x| \gtrsim c/\omega a = v(c/u)$.

\(^2\) One natural alternative is the adiabatic method, using the exact eigenvalues and eigenvectors of $H$ with the instantaneous values of $\beta(t)$. The gain would be more accuracy at low enough $v$; the cost would be incomparably more computation, and the loss of all hope of results in closed or semi-closed form. The present writer is much indebted to Victor Dodonov for advice about adiabatic methods generally.

\(^3\) The reader would be entitled to but might prefer not to be encumbered by double labels on $F$, namely $+ \text{ for the coupled oscillator responsible and } 1 \text{ for the atom on which the force acts.}$
To help visualize \( f \), we start with some symmetry and asymptotic properties of the auxiliary functions \( h_{i,r} \).

As \( x \to -\infty \), both vanish.

As \( x \to +\infty \), it is obvious that \( h_i \) vanishes, because its integrand is odd in \( x' \); moreover, it is easily seen that \( h_i(x) \) is even in \( x \). Equations (2.5) and (2.2) yield

\[
f(0, v) = h_i(0, v) = \frac{1}{v^2} \left( 2 \pi L_1(2/v) - \pi I_1(2/v) \right),
\]

\[
h_r(0, v \ll 1) = \frac{1}{2} + \frac{3}{8} v^2 + \frac{45}{32} v^4 + \cdots, \quad h_r(0, v \gg 1) = \frac{2}{v^2} - \frac{\pi}{v^3} + \frac{(8/3)}{v^4} + \cdots.
\]

By contrast, it proves convenient to write

\[
h_r(x, v) = h_r(0, v) + \Delta h_r(x, v), \quad \Delta h_r(-x, v) = -\Delta h_r(x, v),
\]

\[
h_r(0, v) = \frac{2}{\pi v^2} K_1 \left( \frac{2}{v} \right), \quad h_r(\infty, v) = 2 h_r(0, v),
\]

\[
h_r(0, v \ll 1) = \exp(-2/v) \sqrt{\pi} \left\{ v^{-3/2} + \frac{3}{16} v^{-1/2} - \cdots \right\},
\]

\[
h_r(0, v \gg 1) = \frac{1}{v} \left[ -2 \log v + 2 \gamma - 1 \right] \frac{1}{v^3} + \cdots,
\]

where \( \gamma \approx 0.5772 \) is Euler’s constant. Of most importance for us is that \( h_r(\infty, v) \) is nonzero, unlike \( h_i(\infty, v) \), but that \( h_r \) remains exponentially small for all \( x \) when \( v \) is small. Thus, given \( v \ll 1 \), in a straight competition \( h_i \) swamps \( h_r \) when \( x \sim \mathcal{O}(1) \), i.e. anywhere near passage.

3. Comments

(i) From our information about \( h_{i,r} \), one finds that

\[
f(x \ll -1, v) = \frac{1}{2} \left\{ \frac{1}{|x|^3} - \frac{(3/4 + 3v^2/2)}{|x|^5} \right\} + \cdots,
\]

\[
f(x \gg 1, v) = \sin \left( \frac{2x}{v} \right) h_r(\infty) + \frac{1}{2x^3} - \frac{(3/4 + 3v^2/2)}{x^5} + \cdots,
\]

with \( f(0, v) \) given by (2.11) and (2.12). Thus, when \( v \) is small \( f \) starts by rising proportionately to \( 1/|x|^3 \), is of the order of 1 near passage, and eventually settles to oscillating around a nonzero but exponentially small value, with frequency \( 2\omega \). Figure 2 plots \( f(x, v) \), with \( v = 0.4 \) chosen unrealistically high in order to make the oscillations more visible (and also to ease the numerics). One sees that \( f \) maximizes slightly past passage.

(ii) How the mean force varies follows from (2.9), (2.10) and (2.5). At passage the \( z \) component (i.e. the drag) vanishes simply because \( \langle \dot{F}_0 \rangle \), does; only the \( x \) component survives, whence

\[
2 |\langle \dot{F}^{(+)}_0 \rangle|/|\dot{F}_0| = 2 f(0, v) = 2 h_i(0, v) \simeq 1 + 3v^2/4 + \cdots,
\]

showing that the attraction is greater than it would be between stationary oscillators in the same positions. This tallies with Ferrell and Ritchie’s (1980) discovery that the attraction between an exterior atom and a nondissipative half-space is enhanced when the atom moves parallel to the surface. Likewise (by equation (5.3) of Barton 1977) the image force on a charge \( Q \) a distance \( a \) outside a half-space of
nondissipative plasma (surface-plasmon frequency $\omega_s$) changes to $-Q^2(1/4a^2 + 3u^2/16\omega_s^2a^4)$ when $Q$ moves with velocity $u$ parallel to the surface.

(iii) The mean total internal excitation energy of the two oscillators produced by the fly-past is

$$E = 2 \times 2\hbar \omega |c(\infty)|^2 = \left[ \frac{\hbar \omega \alpha^2}{a^6} \right] G, \quad G = \frac{8}{v^4} K_1^2 \left( \frac{2}{v} \right) = \frac{\lambda^4}{2} K_1^2(\lambda),$$

(3.3)

$$G (v \ll 1) \simeq (2\pi/v^3) \exp(-4/v), \quad G (v \gg 1) \simeq 2/v^2. \quad (3.4)$$

$G$ reaches its one maximum 0.198 at $v = 1.50$. We see that perturbation theory is adequate for all speeds as long as $E/\hbar \omega \sim (\alpha/a^3)^2 \ll 1$. This purely classical condition is met if $\alpha$ is of the order of atomic units, while $a$ even if small is macroscopic.

It is reassuring to check explicitly that $E$ agrees with the total work $W$ done against the drag force:

$$W = \int_{-\infty}^{\infty} 2a \, dx \cdot \langle \hat{F}_z \rangle = \left[ \frac{3\hbar \omega \alpha^2}{2a^6} \right] \int_{-\infty}^{\infty} dx \cdot \frac{xf(x)}{(1+x^2)^{5/2}}.$$ 

Dropping from the integrand the parts odd in $x$, we find that

$$W = \left[ \frac{3\hbar \omega \alpha^2}{2a^6} \right] [2h_r(0,v)] \left[ \int_0^{\infty} dx x \sin(2x/v) (1+x^2)^{5/2} \right]$$

$$= \left[ \frac{3\hbar \omega \alpha^2}{2a^6} \right] \left[ \frac{4}{v^2} K_1 \left( \frac{2}{v} \right) \right] \left[ \frac{4}{3v^2} K_1 \left( \frac{2}{v} \right) \right], \quad (3.5)$$

which tallies with (3.3) as it should. We note that $E$ is quadratic in the admixture amplitudes $c(t)$, while $\langle \hat{F}_z \rangle$ and thereby $W$ are linear: the identity $E = W$ is in effect the optical theorem for the rudimentary collision process we are considering.
(iv) It is tempting to try and identify a purely dissipative part of the force, although no choice can be wholly compelling, because only the total force can be measured. One perhaps plausible candidate is the part $\Phi$ of $\langle \hat{F}_z \rangle$ that is irreversible in the sense that it is even in time, i.e. even in $x$, by contrast to the parallel component $(F_0)_z$ of the static force, which is odd. In view of (2.9), (2.10) and (2.13) we need merely replace $f$ by its $x$-odd part, readily visualized by mentally adapting figure 2. Explicitly,

$$\Phi_1 = -2 |F_0| x (1 + x^2) \phi(x, v), \quad \phi = \frac{\sin(2x/v)h_\omega(0) = \sin(2x/v)2K_1(2/v)/v^2}. \quad (3.6)$$

We stress that at any given time $\Phi$, like $\langle \hat{F}_z \rangle$ itself, depends implicitly on the (scaled) distance $x$ at all earlier times; and we leave open the question whether or to what accuracy it might be possible to identify dissipative effects that depend only on the instantaneous value or only on the very recent past of $x$.

(v) The constraints to uniform motion along straight lines might, at first sight, be expected to admit similarities mainly to collisions between fast atoms at large impact parameters $a$. Since we have been concerned mainly with small $v = u/a\omega$, the question is worth a closer look.

(a) Straight lines mean small angles of deviation $\vartheta$. To estimate $\vartheta$ for slow motion, we call the atomic mass $M$, and calculate the total transverse impulse $I_x$ from the static VdW force (1.7):

$$\vartheta \sim \frac{I_x}{Mu} = \frac{1}{Mu} \int_{-\infty}^{\infty} dt F_{0t} = \frac{15\pi}{128} \left[ \frac{\hbar \omega}{(Mu^2/2)} \left( \frac{\alpha}{a^3} \right)^2 \right]. \quad (3.7)$$

As was explained just below (1.5), perturbation theory and the dipole approximation have already committed us to $\alpha/a^3 \ll 1$. Thus, although ultimately $\vartheta$ must rise with falling $u$, it remains small unless the incident kinetic energy falls far below the atomic excitation $\hbar \omega$.

(b) Roughly constant speed means that the frictional energy loss given by (3.3) and (3.4) is well below the incident kinetic energy. For small $v$ this condition reads

$$v \ll 1 : \frac{\Delta E}{(Mu^2/2)} \simeq 2\pi \left[ \frac{\hbar \omega}{(Mu^2/2)} \left( \frac{\alpha}{a^3} \right)^2 \right] v^3 \exp(-4/v) \ll 1. \quad (3.8)$$

By (3.7) the contents of the first pair of square brackets must be taken as small in any case; and for small $v$ the contents of the second pair are also small. Thus condition (3.7) is both necessary and sufficient.

(vi) Hoye and Brevik (1992, 2010a, 2010b) have considered two oscillators that at time $t = 0$ are uncoupled and uncorrelated and, as regards their internal energies, appear to be governed by the appropriate canonical density matrix for temperature $T$. The writer understands their conclusions to imply that at nonzero $T$ the subsequent evolution (towards thermodynamic equilibrium?) is influenced by friction between the oscillators, but that at $T = 0$ it is not4. In their perturbative method the response function held to govern such friction is approximated by keeping only the (nonzero) term independent of plus another term linear in $t$; this contrasts sharply with the drag force (2.9) in our scenario, which vanishes at passage.

Since the initial state, the time spans they admit and the purpose of their calculation are so different from ours, it seems possible that the discrepancy between the results is merely

---

4 A sentence a few lines below their equation (2.12) states that ‘According to the present oscillator model there is thus no friction force at zero temperature’.
semantic. For instance, applied to study unconstrained oscillators each initially in its ground state and heading towards a collision with large enough impact parameter, one prime role of the friction identified in the present paper would be to transfer energy between translational and internal motions; and that role may well be partly or wholly preempted by Hoye and Brevik’s choice of initial density matrix.

However, in view of the manifold current controversies about quantum-governed frictional forces generally, it seems well worth exploring whether such differences reflect substantive disagreement or only a confusion of terms.

**Appendix A. Isotropic three-dimensional oscillators**

If the oscillators 1 and 2 are 3D and isotropic, the noninteracting part of the Hamiltonian is modified in an obvious way, and one must use the familiar 3D dipole–dipole interaction

\[
\tilde{H}_{\text{int}} = e^2 \left\{ \frac{(r_1 \cdot r_2)}{s^3} - \frac{3}{s^5} (r_1 \cdot s)(r_2 \cdot s) \right\} = \tilde{H}_{\text{int}+} + \tilde{H}_{\text{int}-},
\]

(A.1)

and replace the force operator \( \hat{F} \) in (2.7) by \(-\partial \tilde{H}_{\text{int}}/\partial s\). The analysis proceeds as in the text, but proves far more laborious, because it must account separately for paired excitations in the \(xx\)-, \(yy\)-, \(zz\)- and \(xz\)-directions.

We illustrate the difference by quoting the total excitation energy. It is relatively accessible because the various \(c(\infty)\) are much easier to determine than the \(c(t)\). Eventually one finds that (3.3) is replaced by

\[
\tilde{E} = \left[ \frac{\hbar \omega a^2}{a^6} \right] \tilde{G},
\]

(A.2)

\[
\tilde{G} = \lambda^6 K_6^2(\lambda) + \lambda^5 K_0(\lambda) K_1(\lambda) + (\lambda^4 + \lambda^6) K_1^2(\lambda),
\]

(A.3)

\[
\tilde{G}(v \ll 1) \simeq \frac{32\pi}{v^5} \exp(-4/v), \quad \tilde{G}(v \gg 1) \simeq \frac{4}{v^2}.
\]

(A.4)

\( \tilde{G} \) reaches its one maximum 3.00 at \( v \simeq 0.878 \). It may be compared to its 1D analogue \( G \) from (3.3) and (3.4): one sees that at low speeds it exceeds \( G \) by a factor of \( 16/v^2 \), i.e. by at least two orders of magnitude.

Appendix B will need \( \tilde{E} \) at low speeds spelled out in terms of \( a \):

\[
v \ll 1 : \tilde{E} \simeq 32\pi \frac{\hbar \omega a^2}{u^5 a} \exp(-4\omega a/u).
\]

(A.5)

**Appendix B. Nondissipative half-spaces as dilute assemblies of oscillators**

Consider two half-spaces \( L \) and \( R \), separated by a gap of width \( \zeta \), envisaging each as an optically (or rather electrically) dilute assembly of randomly distributed simple-harmonic, isotropic, 3D oscillators, \( \bar{n} \) per unit volume, all with frequency \( \omega \) and electrostatic polarizability \( \alpha \); and work only to leading order in \( \bar{n} \). Then for the surface-plasmon frequency one has \( \omega_S \simeq \omega \), and for the relative dielectric function at zero frequency \( \varepsilon \simeq 1 + 4\pi \bar{n} \alpha \).
We derive the frictional power loss $P$ per unit area at low relative velocity, $v \ll 1$, by summing the energy dissipated by pairs of otherwise noninteracting atoms, one in $L$ and the other in $R$. To suit the choice of axes more common in the literature, the motion is now taken to be parallel to the $x$-axis, while the separation $a$ is measured in the $z$-direction.

Accordingly, the oscillators move at constant velocity $\pm u/2$ in straight lines parallel to the $x$-axis. Start by considering just one pair, 1 and 2. The perpendicular distance between their trajectories is $a = \sqrt{y^2 + Z^2}$, where $y = y_2 - y_1$ and $Z = z_1 + z_2$, with $z_{1,2}$ measuring perpendicular distance from the median plane into the half-space. Call the $x$-component of the force between them $\tilde{F}_x$. Their joint excitation energy $\tilde{E}$ generated between times $\mp \infty$ is given by (A.5). To find the total force per unit area say on the half-space $R$, call it $F_{||} = P/u$, we (i) integrate $\tilde{F}_x$ over all pairs of trajectories; (ii) integrate again so as to allow for all the oscillators along each trajectory, which amounts to implementing $\int_{-\infty}^{\infty} dx \ldots$ as already done in (A.2); and multiply by $\hat{n}^2$. Thus

$$F_{||} = \hat{n}^2 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \tilde{F}_x = \hat{n}^2 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \int_{-\infty}^{\infty} dy \tilde{E}. \quad (B.1)$$

We set $\int_{-\infty}^{\infty} dy \ldots = 2 \int_{-\infty}^{\infty} da (a/\sqrt{a^2 - Z^2}) \ldots$, $a = \mu Z$, $z_{1,2} = (\xi/2)t_{1,2}$, $Z = \zeta T/2$; introduce a new scaled variable

$$\tilde{v} \equiv u/\omega \zeta; \quad (B.2)$$

and use (A.4a). Then

$$N(\tilde{v}) \equiv 16 \frac{\hbar \omega \hat{n}^2 \alpha^2}{\zeta^3} N(\tilde{v}), \quad (B.3)$$

$$N(\tilde{v}) = \int_1^\infty \frac{d\mu}{\sqrt{\mu^2 - 1}} \int_1^\infty \int_1^\infty \frac{dt_1 dt_2}{(t_1 + t_2)^3} \tilde{G}(\Lambda), \quad \Lambda \equiv \frac{\mu (t_1 + t_2)}{\tilde{v}}. \quad (B.4)$$

The triple integral seems intractable except asymptotically.

At low speeds ($\Lambda \gg 1$), by (A.4a)

$$N(\tilde{v} \ll 1) \approx \frac{\pi}{\tilde{v}^3} \int_1^\infty \frac{d\mu}{\sqrt{\mu^2 - 1}} \left[ \int_1^\infty dt_1 \exp(-2\mu t_1/\tilde{v}) \right]^2 \approx \frac{\pi^{3/2}}{8\sqrt{2}} \tilde{v}^{-5/2} \exp(-4/\tilde{v}). \quad (B.5)$$

The last integral could have been expressed exactly in terms of modified Bessel functions: we have cited the leading term, which at this point is all that is warranted.

A study of the friction between two half-spaces (Barton 2010b) reveals that (B.3) and (B.5) tally with the appropriate limit of the result given by Pendry (2010). It shows also that the perturbative approximation embodied in (B.3) and (B.4) is unreliable at high speeds. If one wants it nevertheless, it is quickest to revert to (B.1), use (A.4b), and scale $(z_{1,2}, y) = (\xi/2)(t_{1,2}, r)$:

$$F_{||} = \frac{16 \hbar \hat{n}^2 e^4}{m^2 \omega_S \zeta u^2} \int_1^\infty dt_1 \int_1^\infty dt_2 \int_0^\infty dr \frac{1}{[r^2 + (t_1 + t_2)^2]^2} = \frac{\pi \hbar \hat{n}^2 e^4}{m^2 \omega_S \zeta u^2}. \quad (B.6)$$

5 With apologies for $t_{1,2}$ unrelated to time.
Appendix C. Nonidentical oscillators

If the values of $m$, $\omega$, $e$ for the primary oscillators 1 and 2 differ, a linear and canonical change of variables similar to (1.1) still turns the exact Hamiltonian into the Hamiltonian for two mutually uncoupled oscillators, call them i and ii. The analyses for i and for ii proceed independently, exactly as in sections 1 and 2, but for each in terms of its own frequency and dipole-moment operator: instead of simply multiplying the results for either oscillator by 2, one must now combine two expressions having the same structure but different parameters.

On the other hand, although we have just seen that technically speaking the specialization to identical oscillators is not all that restrictive, it becomes very restrictive indeed as soon as one thinks of using the expressions for just two such atoms in order to find the interaction of one atom with a dilute half-space made of others. One would then be confined to the highly exceptional case where an atomic transition frequency is in exact resonance with the surface-plasmon frequency of the half-space, known to require special treatment (Barton 1978). By contrast, as regards the interaction between two such half-spaces, the scenario where they are identical turns out to be not at all unrepresentative, and in any case is the one normally considered first. The theory (generalized to isotropic 3D oscillators) was outlined in appendix B.

However, the most serious shortcoming of all such applications is that they can say nothing about atoms coupled to dissipative (e.g. ohmically conducting) materials. The reason is simply that isolated atoms are not subject to dissipation, whence no amount of information about interactions between just two atoms can afford a grip on the interaction of one atom with such a half-space.

References

Barton G 1977 J. Phys. A: Math. Gen. 10 601
Barton G 1978 Solid State Commun. 27 95
Barton G 2010a New J. Phys. 12 113045 (next paper, cited as II)
Barton G 2010b On van der Waals friction between half-spaces (in preparation)
Barton G and Calogeracos A 1996 Proc. R. Soc. A 452 1167
Ferrell T L and Ritchie R H 1980 Phys. Rev. A 21 1305
Hoye J S and Brevik I 1992 Physica A 181 413
Hoye J S and Brevik I 2010a Europhys. Lett. 91 60003
Hoye J S and Brevik I 2010b arXiv:1009.3135v1 [quant-ph]
Pendry J B 2010 New J. Phys. 12 033028 (equation (11))

That is why II cannot exploit the results of the present paper.

Since our model discards all relativistic effects from the outset, there is no Maxwell field: atomic excited states cannot radiate, and have zero width.

New Journal of Physics 12 (2010) 113044 (http://www.njp.org/)