Matching QCD and HQET at three loops

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QCD/HQET matching for the heavy-quark field [1] and heavy–light quark currents [2] with three-loop accuracy is discussed.

1. Heavy-quark field

QCD problems with a single heavy quark $Q$ can be treated in a simpler effective theory — HQET, if there exists a 4-velocity $v$ such that the heavy-quark momentum is $p = mv + k$ ($m$ is the on-shell mass) and the characteristic residual momentum is small: $k \ll m$. QCD operators can be written as series in $1/m$ via HQET operators; the coefficients in these series are determined by matching on-shell matrix elements in both theories.

At the tree level, the heavy-quark field is discussed. If there are no massive flavours except $Q$, then $Z_0 = 1$ because all loop corrections are scale-free. The QCD on-shell renormalization constant $Z_Q^{os}$ contains the single scale $m$ in this case; it has been calculated [6] up to three loops. The three-loop MS anomalous dimensions of $Q$ [8] and $Q$ [8] are also known. We have to express all three quantities $Z_Q^{os}(g_0^{(n_f+1)}, a_0^{(n_f+1)}), Z_Q(\alpha_s^{(n_f+1)}(\mu), \lambda^{(n_f+1)}(\mu)), Z_Q(\alpha_s^{(n_f)}(\mu), \lambda^{(n_f)}(\mu))$ via the same variables, say,

Therefore, the bare fields are related by

$$Q_0(x) = e^{-imv \cdot x} \left[ z_0^{1/2} \left( 1 + \frac{iD_\perp}{2m} \right) Q_{v0}(x) + O(\frac{1}{m^2}) \right], \quad (3)$$

where the bare matching coefficient is

$$z_0 = \frac{Z_Q^{os}(g_0^{(n_f+1)}, a_0^{(n_f+1)})}{Z_Q^{os}(g_0^{(n_f)}, a_0^{(n_f)})} \quad (4)$$

(we use the covariant gauge: the gauge-fixing term in the Lagrangian is $-\partial_\mu A_{\mu}^a(2a_0)$, and the free gluon propagator is $(-i/p^2)(g_{\mu\nu} - (1 - a_0)p_\mu p_\nu/p^2)$; the number of flavours in QCD is $n_f = n_t + 1$). The $O(1/m)$ matching coefficient in (3) is equal to the leading one, $z_0$; this reflects the reparametrization invariance [5]. The MS renormalized fields are related by the formula similar to (3), with the renormalized decoupling coefficient

$$z(\mu) = \frac{Z_Q(\alpha_s^{(n_f)}(\mu), \lambda^{(n_f)}(\mu))}{Z_Q(\alpha_s^{(n_f+1)}(\mu), \lambda^{(n_f+1)}(\mu))} z_0. \quad (5)$$

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\( \alpha_s^{(n)}(\mu), a^{(n)}(\mu) \), see \[10\]. The explicit result for the renormalized matching coefficient \( z(\mu) \) can be found in \[4\]. Gauge dependence first appears at three loops, as in \( Z_Q^{\infty} \) \[6\]. The requirement of finiteness of the renormalized matching coefficient \( 5 \) at \( \varepsilon \to 0 \) has allowed the authors of \[6\] to extract \( Z_Q \) from their result for \( Z_Q^{\infty} \).

In the large-\( \beta_0 \) limit (see Chapter 8 in \[11\] for a pedagogical introduction):

\[
\begin{align*}
z(\mu) &= 1 + \int_0^\beta \frac{d\beta}{\beta} \left( \frac{\gamma(\beta)}{2\beta} - \frac{\gamma_0}{2\beta_0} \right) \\
&\quad + \frac{1}{\beta_0} \int_0^{\infty} du \, e^{-u/\beta} S(u) + O \left( \frac{1}{\beta_0^2} \right),
\end{align*}
\]

where \( \beta = 3\alpha_s/(4\pi), \gamma = 2\alpha_s/(4\pi) + \cdots \) (differences of \( n_f \)-flavour and \( (n_f + 1) \)-flavour quantities can be neglected at the 1/\( \beta_0 \) order). The difference of the QCD and HQET anomalous dimensions \( \gamma = \gamma_Q - \gamma_Q \) (it is gauge invariant at this order) and the Borel image \( S(u) \) are \[12,13,11\]

\[
\begin{align*}
\gamma(\beta) &= -2\frac{\beta}{\beta_0} F(-\beta, 0) = \\
2C_F \frac{\beta}{\beta_0} \frac{1 + \beta(1 + \frac{2}{3}\beta)}{B(2 + \beta, 2 + \beta)\Gamma(3 + \beta)\Gamma(1 - \beta)} S(u) &= F(0, u) - F(0, 0) = \\
-6C_F \left[ e^{(u+5/3)/\beta} \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} (1-u^2) - \frac{1}{2u} \right].
\end{align*}
\]

This Borel image has infrared renormalon poles at each positive half-integer \( u \) and at \( u = 2 \). Therefore, the integral in \( \beta_0 \) is not well defined. Comparing its residue at the leading pole \( u = 1/2 \) with the residue of the static-quark self-energy at its ultraviolet pole \( u = 1/2 \) \[14\], we can express the renormalon ambiguity of \( z(\mu) \) as

\[
\Delta z(\mu) = \frac{3}{2} \frac{\Delta \bar{\Lambda}}{m}
\]

(\( \bar{\Lambda} \) is the ground-state meson residual energy). The matching coefficient is gauge invariant at the order 1/\( \beta_0 \). Expanding \( \gamma(\beta) \) and \( S(u) \) and integrating, we obtain confirm the contributions with the highest power of \( n_f \) in each term in our three-loop result, and predict such a contribution at \( \alpha_s^4 \).

Numerically, in the Landau gauge at \( n_f = 4 \)

\[
\begin{align*}
z(m) &= 1 - \frac{4}{3} \frac{\alpha_s^4(\mu)}{\pi} \\
&\quad - (16.6629 - 4.5421) \left( \frac{\alpha_s^4(\mu)}{\pi} \right)^2 \\
&\quad - (153.4076 + 42.6271 - 61.5397) \left( \frac{\alpha_s^4(\mu)}{\pi} \right)^3 \\
&\quad - (195.4013 + \cdots) \left( \frac{\alpha_s^4(\mu)}{\pi} \right)^4 + \cdots
\end{align*}
\]

(\( \beta_0 \) is for \( n_f = 4 \) flavours). Naive nonabelianization \[12\] works rather well at two and three loops. Numerical convergence of the series is very poor; this is related to the infrared renormalon at \( u = 1/2 \).

Now let us consider the relation between the MS renormalized electron field in QED and the Bloch–Nordsieck electron field. The bare matching coefficient \( z_0 = Z_Q^{\infty} \) is gauge invariant to all orders, see \[6\]. In the Bloch-Nordsieck model, due to exponentiation, \( \log Z_\psi = (3 - \alpha(0)\alpha(0)/(4\pi\varepsilon) \) (where the 0-flavour \( \alpha(0) \) is equal to the on-shell \( \alpha \approx 1/137 \)). In the full QED, \( \log Z_\psi = a^{(1)}(1)/(4\pi\varepsilon) + (\text{gauge-invariant higher terms}) \) (see \[11\] for the proof; this has been demonstrated up to four loops by the direct calculation \[15\]). The gauge dependence cancels in \( \log(Z_\psi/Z_\psi) \) because of the QED decoupling relation \( a^{(1)}(1) = a^{(0)}(0) \). Therefore, the renormalized matching coefficient \( z(\mu) \) in QED is gauge invariant to all orders. The three-loop result is presented in \[11\].
2. Heavy–light currents

Now we shall consider renormalized heavy–light QCD quark currents

\[ j(\mu) = Z_j^{-1}(\mu) j_0 , \quad j_0 = \bar{q}_0 \Gamma Q_0 , \]  

where \( \Gamma \) is a Dirac matrix. They can be expressed via operators in HQET

\[ j(\mu) = C_\Gamma(\mu) j(\mu) + \frac{1}{2m} \sum_i B_i(\mu) O_i(\mu) + O \left( \frac{1}{m^2} \right) , \]  

where

\[ j(\mu) = \hat{Z}_j^{-1}(\mu) j_0 , \quad j_0 = \bar{q}_0 \Gamma Q_{v0} , \]

and \( O_i \) are dimension-4 HQET operators with appropriate quantum numbers. The leading-order matching coefficients \( C_\Gamma \) have been calculated up to two loops [12,10].

There are 8 Dirac structures giving non-vanishing quark currents in 4 dimensions:

\[ \Gamma = 1 , \quad \gamma_5 , \quad \gamma_\mu , \quad \gamma_\mu \gamma_5 , \quad \gamma_\mu \gamma_5 , \quad \gamma_\mu \gamma_\nu \gamma_5 , \quad \gamma_\mu \gamma_\nu \gamma_\rho \gamma_5 , \]  

where \( \gamma_\mu = \gamma_\mu - \gamma_t \gamma_5 \gamma_\mu \). The last four of them can be obtained from the first four by multiplying by the 't Hooft–Veltman \( \gamma_5^{HV} \). We are concerned with flavour non-singlet currents only, therefore, we may also use the anticommuting \( \gamma_5^{AC} \) (there is no anomaly). The currents renormalized at a scale \( \mu \) with different prescriptions for \( \gamma_5 \) are related by [17]

\[ \left( q \gamma_5^{AC} Q \right)_\mu = Z_P(\mu) \left( q \gamma_5^{HV} Q \right)_\mu , \]  

\[ \left( q \gamma_5^{AC} \gamma_\mu Q \right)_\mu = Z_A(\mu) \left( q \gamma_5^{HV} \gamma_\mu Q \right)_\mu , \]  

\[ \left( q \gamma_5^{AC} \gamma_\mu \gamma_5 Q \right)_\mu = Z_T(\mu) \left( q \gamma_5^{HV} \gamma_\mu \gamma_5 Q \right)_\mu , \]

where the finite renormalization constants \( Z_{P,A,T} \) can be reconstructed from the differences of the anomalous dimensions of the currents. Multiplying \( \Gamma \) by \( \gamma_5^{AC} \) does not change the anomalous dimension, therefore,

\[ Z_T(\mu) = 1 ; \]  

\[ Z_{P,A}(\mu) \] are known up to three loops [17].

The anomalous dimension of the HQET current [12] does not depend on the Dirac structure \( \Gamma \). Therefore, there are no factors similar to \( Z_{P,A} \) in HQET. Multiplying \( \Gamma \) by \( \gamma_5^{AC} \) does not change the matching coefficient. Therefore, the matching coefficients for the currents in the second row of \( \Gamma \) can be obtained from those for the first row. In the \( v \) rest frame

\[ Z_P(\mu) = \frac{C_\gamma^{AC}(\mu)}{C_\gamma^{HV}(\mu)} = \frac{C_\gamma(\mu)}{C_\gamma^{\alpha\gamma\gamma\gamma}(\mu)} , \]  

\[ Z_A(\mu) = \frac{C_\gamma^{AC\alpha\gamma}(\mu)}{C_\gamma^{HV\gamma\gamma}(\mu)} = \frac{C_\gamma^{\alpha\gamma}(\mu)}{C_\gamma^{\alpha\gamma\gamma\gamma}(\mu)} , \]  

\[ Z_T(\mu) = \frac{C_\gamma^{AC\alpha\gamma\gamma}(\mu)}{C_\gamma^{HV\gamma\gamma\gamma}(\mu)} = \frac{C_\gamma^{\alpha\gamma\gamma}(\mu)}{C_\gamma^{\alpha\gamma\gamma\gamma}(\mu)} = 1 . \]

In particular, \( C_{\gamma_\mu^\mu}(\mu) = C_{\gamma_\mu^\mu}(\mu) \). In the following we shall consider only the matching coefficients for the first 4 Dirac structures in [13].

In order to find the coefficients \( C_\Gamma(\mu) \), we equate matrix elements of the left- and right-hand side of \( \mu \) from the heavy quark with momentum \( p = m v + k \) to the light quark with momentum \( k_q \):

\[ <q(k_q)|j(\mu)|Q(m v + k)> = C_\Gamma(\mu) <q(k_q)|j(\mu)|Q_v(k)> + O \left( \frac{k_q}{m} \right) . \]

The on-shell matrix elements are

\[ <q(k_q)|j(\mu)|Q(p)> = \bar{u}_q(k_q) (p, k_q) u(p) \times Z_j^{-1}(\mu) Z_Q^{1/2} Z^{1/2}_q , \]  

\[ <q(k_q)|j(\mu)|Q_v(k)> = \bar{u}_q(k_q) (k, k_q) u_v(k) \times Z_j^{-1}(\mu) Z_Q^{1/2} Z^{1/2}_q , \]
where $\Gamma(p, k_q)$ and $\hat{\Gamma}(k, k_q)$ are the bare vertex functions, and $\hat{Z}_q$ differs from $Z_q$ because there are no $Q$ loops in HQET. The difference between $u(mv + k)$ and $u_c(k)$ is of order $k/m$, and can be neglected. It is most convenient to use $k = k_q = 0$, then the $O(1/m)$ term is absent. The QCD vertex has two Dirac structures:

$$\Gamma(mv, 0) = \Gamma \cdot (A + B\gamma) .$$

This leads to

$$\tilde{u}(0)\Gamma(mv, 0)u(mv) = \tilde{u}(0)\Gamma u(mv) ,$$

$$\Gamma(mv, 0) = A + B .$$

The HQET vertex has just one Dirac structure. Therefore,

$$C\Gamma(\mu) = \frac{\tilde{\Gamma}(mv, 0)Z_j^{-1}(\mu)Z_{\bar{Q}}^{1/2}Z_q^{1/2}}{\tilde{\Gamma}(0, 0)Z_j^{-1}(\mu)Z_{\bar{Q}}^{1/2}Z_q^{1/2}} . \tag{19}$$

If all flavours except $Q$ are massless, all loop corrections to $\tilde{\Gamma}(0, 0)$, $\tilde{Z}_Q$, and $\tilde{Z}_q$ contain no scale and hence vanish: $\tilde{\Gamma}(0, 0) = 1$, $\tilde{Z}_Q = 1$, $\tilde{Z}_q = 1$. The quantities $\Gamma(mv, 0)$, $Z_q$, and $Z_q$ contain a single scale $m$: $Z_Q$ has been calculated up to 3 loops in [6], $Z_q$ in [10], and $\Gamma(mv, 0)$ in the present work [2]. The MS renormalization constants $Z_j$ [7] and $Z_j$ [18] (for all $\Gamma$) are also known to 3 loops.

If there is another massive flavour ($c$ in $b$-quark HQET), then $\tilde{\Gamma}(0, 0)$, $\tilde{Z}_Q$, and $\tilde{Z}_q$ contain a single scale $m_c$. The first two quantities have been calculated up to 3 loops in [19]; the last one is known from [10]. The quantities $\Gamma(mv, 0)$, $Z_q$, and $Z_q$ now contain 2 scales, and are non-trivial functions of $x = m_c/m$. The renormalization constant $Z_Q$ has been calculated in this case, up to 3 loops, in [21] (the master integrals appearing in this case are discussed in Ref. [21]). The other two quantities are found in this work [2].

The bare on-shell QCD quantities $\tilde{\Gamma}(mv, 0)$, $Z_Q$, and $Z_q$ are expressed via $g_0^{(n_f)}$ (and $m_c^{(n_f)}$) if it is non-zero; we re-express it via the on-shell mass $m_c)$. They don’t contain $\mu$. The MS QCD renormalization constant $Z_j$ is expressed via $\alpha_s^{(n_f)}(\mu)$. The bare on-shell HQET quantities $\hat{\Gamma}(0, 0)$, $\hat{Z}_Q$, and $\hat{Z}_q$ are expressed via $g_0^{(n_f-1)}$ and $m_c^{(n_f-1)}$ (they are trivial at $m_c = 0$); we re-express $m_c^{(n_f-1)}$ via the on-shell mass $m_c$ (which is the same in both theories). These bare quantities also don’t contain $\mu$. Finally, the MS HQET renormalization constant $\hat{Z}_j$ is expressed via $\alpha_s^{(n_f-1)}(\mu)$. We re-express all the quantities in [19] via $\alpha_s^{(n_f-1)}(\mu)$, see [10].

From equation of motion we have

$$i\partial_\alpha \alpha = i\partial_\alpha j_0^\alpha = m_{0}j_0 = m(\mu)j(\mu) , \tag{20}$$

$$mC_f(\mu) = m(\mu)C_1(\mu) . \tag{21}$$

The ratio $m(\mu)/m$ has been calculated at three loops numerically [24] and then analytically [23] (the analytical results [23,6] were later independently confirmed in [24], and then in several other papers); for $m_c \neq 0$, $m(\mu)/m$ has been found in [20].

The matching coefficients have been calculated up to 2 loops in [12], and to 3 loops in the present work [2]. Analytical expressions are long; numerically, at $m_c = 0$ and $\mu = m$ we have

$$C_1^{(2)} = 7.55 + 1.09 = 8.64 ,$$

$$C_2^{(2)} = -5.47 + 3.06 = -2.41 ,$$

$$C_3^{(2)} = -9.87 + 1.53 = -8.34 ,$$

$$C_4^{(2)} = -14.13 + 2.42 = -11.70 ,$$

$$C_1^{(3)} = 64.74 + 75.34 - 38.16 = 101.92 ,$$

$$C_2^{(3)} = -37.25 - 10.72 + 29.74 = -18.23 ,$$

$$C_3^{(3)} = -88.92 - 46.34 + 45.34 = -89.92 ,$$

$$C_4^{(3)} = -123.61 - 63.57 + 63.22 = -123.96$$

(in the middle part of each formula, terms with descending powers of $\theta_0^{(n_f-1)}$ are shown separately). Naive nonabelianization [12] works reasonably well.
Table 1
Master integrals with 5 lines

|       | 5.1, 5.1a | 5.2, 5.2a | 5.3, 5.3a | 5.4, 5.4a |
|-------|-----------|-----------|-----------|-----------|
| $\varepsilon^{-3}$ | DE | DE | DE | DE |
| $\varepsilon^{-2}$ | DE | DE | DE | DE |
| $\varepsilon^{-1}$ | DE | DE | DE | DE |
| $\varepsilon$ | DE | NEW | MB | DE |
| $\varepsilon^2$ | DE | x | x | DE |

At $m_c \neq 0$, results are expressed via the master integrals depending on $x = m_c/m$ [21]. Their status is summarized in the Tables 1–4 in this paper. In the present work [2], we were able to obtain exact analytical expressions (via harmonic polylogarithms of $x$) for $O(1)$ terms in the master integrals 5.2, 5.2a, from the requirement of finiteness of the matching coefficients. Therefore, the Table 3 in [21] should be now replaced with the following Table. (DE means the method of paper. In the present work [2], we were able to obtain exact analytical expressions (via harmonic polylogarithms of $x$) for $O(1)$ terms in the master integrals 5.2, 5.2a, from the requirement of finiteness of the matching coefficients. Therefore, the Table 3 in [21] should be now replaced with the following Table.)

The corresponding HQET matrix elements in the $v$ rest frame are

$$
<0| (\bar{q}_5\gamma_5^ACQ)_\mu |B(\vec{k})>_{nr} = -iF(\mu),
$$

$$
<0| (\bar{q}\gamma^\alpha Q)_\mu |B(\vec{k})>_{nr} = iF(\mu)\epsilon^\alpha,
$$

$$
<0| (\bar{q}\sigma^{\alpha\beta}Q)_\mu |B(\vec{k})>_{nr} = f_B^T(\mu)(p^\alpha e^\beta - p^\beta e^\alpha).
$$

These two matrix elements are characterized by a single hadronic parameter $F(\mu)$ due to the heavy-quark spin symmetry. From [20] we have [12]

$$
\frac{f_B^P(\mu)}{f_B} = \frac{m_B}{m(\mu)},
$$

where we may replace $m_B$ by the on-shell $b$-quark mass $m$, neglecting power corrections.

Our main result is the ratio $f_{B^*}/f_B$. At $m_c = 0$

$$
\frac{f_{B^*}}{f_B} = 1 - \frac{1}{2}C_F \frac{\alpha_s^4(m)}{\pi} + \frac{1}{2}C_F r_F + C_A r_A + T_F n_l r_l + T_F r_h C_F \left( \frac{\alpha_s^4(m)}{\pi} \right)^2 + \left( C_F^2 r_{FF} + C_F C_A r_{FA} + C_A^2 r_{AA} + C_F T_F n_l r_{FL} + C_F T_F r_{Fh} + C_A T_F n_l r_{Al} + C_A T_F r_{Ah} + T_F^2 n_l^2 r_{ll} + T_F^2 n_l r_{lh} + T_F^2 r_{hh} \right) C_F \left( \frac{\alpha_s^4(m)}{\pi} \right)^3 + \mathcal{O} \left( \frac{\alpha_s^4(m)}{m} \right),
$$

where

$$
r_F = \frac{1}{3} \pi^2 \log 2 - \frac{1}{2} \zeta_3 - \frac{4}{9} \pi^2 + \frac{31}{48},
$$

$$
r_A = \frac{1}{6} \pi^2 \log 2 + \frac{1}{4} \zeta_3 + \frac{1}{9} \pi^2 - \frac{263}{144},
$$

$$
r_l = \frac{19}{36}, \quad r_h = \frac{1}{9} \pi^2 - \frac{41}{36}.$$
\[ r_{FF} = -\frac{8}{3} a_4 - \frac{1}{9} \log^2 2 - \frac{2}{9} \pi^2 \log^2 2 + \frac{19}{6} \pi^2 \log 2 + \frac{25}{12} \zeta_5 - \frac{1}{9} \pi^2 \zeta_3 + \frac{11}{8} \zeta_3 - \frac{43}{1080} \pi^4 - \frac{43}{24} \pi^2 - 289 \]
\[ r_{FA} = -\frac{20}{9} a_4 - \frac{5}{24} \log^2 2 - \frac{5}{27} \pi^2 \log^2 2 + \frac{305}{108} \pi^2 \log 2 - \frac{115}{48} \zeta_5 + \frac{1}{12} \pi^2 \zeta_3 - \frac{899}{144} \zeta_3 + \frac{817}{12960} \pi^4 - \frac{2233}{648} \pi^2 + \frac{4681}{864} , \]
\[ r_{AA} = 864 (\text{some terms}) \]
\[ r_{FI} = 864 (\text{some terms}) \]
\[ r_{FH} = 864 (\text{some terms}) \]
\[ r_{AI} = 864 (\text{some terms}) \]
\[ r_{Ah} = 864 (\text{some terms}) \]
\[ r_{Il} = 864 (\text{some terms}) \]
\[ r_{hh} = 864 (\text{some terms}) \]

Examples of matching coefficients at a large number of loops are given.

Naive nonabelianization \[12] works reasonably well.

Asymptotics of the perturbative coefficients for the matching coefficients at a large number of loops \( l \gg 1 \) have been investigated in Ref. \[25]\ in a model-independent way. The results contain three unknown normalization constants \( N_{0,1,2} \sim 1 \). The asymptotics of the perturbative coefficients for \( f_{B^+}/f_B \) contain \( N_0 \) and \( N_2 \); in the case of \( m/\bar{m} \) it contains only \( N_0 \):

\[ \left( \frac{f_{B^+}}{f_B} \right)^{(n+1)} L=\frac{-5}{3} = -\frac{14}{27} \left( 1 + O \left( \frac{1}{n} \right) \right) \]
\[ + \frac{2}{7} \left( \frac{50}{3} \right)^{-9/25} \left[ 1 + O \left( \frac{1}{n} \right) \right] \frac{N_2}{N_0} \]
\[ \times \left( \frac{m}{\bar{m}} \right)^{(n+1)} L=\frac{-5}{3} . \]

The coefficient of \( N_2/N_0 \) is about 0.08 at \( n = 2 \), and it seems reasonable to neglect this contribution. Neglecting also \( 1/n \) corrections, we obtain \[26]\n
\[ \left( \frac{f_{B^+}}{f_B} \right)^{(3)} L=\frac{-5}{3} = -\frac{14}{27} \cdot 56.37 = -29.23 . \]

Our exact result \(-37.787\) agrees with this prediction reasonably well. However, \( 1/n \) corrections are large and tend to break this agreement. It is natural to expect that \( 1/n^2 \) (and higher) corrections are also substantial at \( n = 2 \).

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