2 + 1 Abelian ‘Gauge Theory’ Inspired by Ideal Hydrodynamics

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Abstract

We study a possibly integrable model of abelian gauge fields on a two-dimensional surface $M$, with volume form $\mu$. It has the same phase space as ideal hydrodynamics, a coadjoint orbit of the volume-preserving diffeomorphism group of $M$. Gauge field Poisson brackets differ from the Heisenberg algebra, but are reminiscent of Yang-Mills theory on a null surface. Enstrophy invariants are Casimirs of the Poisson algebra of gauge invariant observables. Some symplectic leaves of the Poisson manifold are identified. The Hamiltonian is a magnetic energy, similar to that of electrodynamics, and depends on a metric whose volume element is not a multiple of $\mu$. The magnetic field evolves by a quadratically non-linear ‘Euler’ equation, which may also be regarded as describing geodesic flow on $\text{SDiff}(M,\mu)$. Static solutions are obtained. For uniform $\mu$, an infinite sequence of local conserved charges beginning with the hamiltonian are found. The charges are shown to be in involution, suggesting integrability. Besides being a theory of a novel kind of ideal flow, this is a toy-model for Yang-Mills theory and matrix field theories, whose gauge-invariant phase space is conjectured to be a coadjoint orbit of the diffeomorphism group of a non-commutative space.

Keywords: Gauge theory; coadjoint orbits; volume preserving diffeomorphisms; Euler equation; integrability.

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1 Introduction and Summary

The classical theory of gauge fields\(^1\) \(A_1(x^1, x^2, t)\) and \(A_2(x^1, x^2, t)\) we study in this paper, may be summarized in four equations. The hamiltonian is a gauge-invariant magnetic energy,

\[
H = \int \left( \frac{B}{\rho} \right)^2 \sigma \, \rho \, d^2x
\]

where \(B = \partial_1 A_2 - \partial_2 A_1\) is the magnetic field. \(\rho\) is a given volume element and \(g_{ij}\) a fixed metric on a two dimensional surface \(M\) such that \(\sigma = \rho^2/g\) is not a constant \((g = \det g_{ij})\). The Poisson bracket between gauge fields is a Lie algebra (independent of the metric)

\[
\{A_i(x), A_j(y)\} = \delta^2(x-y) \left[ A_j(y) \frac{\partial}{\partial y} \rho^{-1}(y) - A_i(x) \frac{\partial}{\partial x} \rho^{-1}(x) \right].
\]

Hamilton’s equation for time evolution of gauge-invariant observables, \(\dot{f} = \{H, f\}\) implies that the magnetic field evolves according to a non-linear ‘Euler’ equation

\[
\dot{B} = \nabla (B/\rho) \times \nabla (B\sigma/\rho).
\]

This theory has the same phase space and Poisson brackets as 2 + 1 ideal (inviscid and volume preserving) hydrodynamics, but a different hamiltonian. It may be integrable, since we find an infinite number of conserved quantities \(H_n = \int (B/\rho)^n \sigma \, \rho \, d^2x\) as well as an infinite number of Casimirs \(I_n = \int (B/\rho)^n \rho \, d^2x\) for uniform \(\rho\). It is remarkable that one can make this modification to 2 + 1 ideal flow, which is sometimes studied as a toy-model for turbulence, to get a potentially integrable system. However, our original motivation for studying this model was different. We argue below that it is the simplest ‘gauge theory’ that shares some quite deep, though unfamiliar, mathematical features of Yang-Mills theory.

The formulation of Yang-Mills theory in terms of gauge-invariant observables, and the development of methods for its solution are important and challenging problems of theoretical physics, since all the experimentally observed asymptotic states of the strong interactions are color-singlets. This problem has a long history stretching at least as far back as the work of Mandelstam\(^1\). Wilson loops are a natural choice for gauge-invariant variables, but they have trivial Poisson brackets on a spatial initial value surface, since the gauge field is canonically conjugate to the electric field on such a surface. More recently, it has been shown by Rajeev and Turgut\(^2\)\(^3\), that Wilson Loops of 3 + 1 dimensional Yang-Mills theory on a null initial value hypersurface satisfy a quadratic Poisson algebra with no need for electric field insertions. This is because the transverse components of the gauge field satisfy a non-trivial Poisson algebra among themselves, as opposed to the situation on a spatial surface. The Poisson algebra of Wilson loops is degenerate due to Mandelstam-like constraints. The gauge invariant phase space of Yang-Mills theory is conjectured to be a coadjoint orbit of this Poisson algebra. It is

\(^1\)After gauge-fixing, there will be a single propagating field degree of freedom.
still a challenge to write the hamiltonian in terms of these variables. However, this has been possible in dimensionally reduced versions\(^2\) such as adjoint scalar field theories coupled to quarks in \(1 + 1\) dimensions, as shown by Lee and Rajeev.\(^3\) In conjunction with 't Hooft’s large \(N\) approximation,\(^7\) viewed as an alternative classical limit, this is an approach to better understand the non-perturbative dynamics, especially of non-supersymmetric gauge theories. In such an approach to \(1 + 1\) QCD, the phase space of gauge-invariant meson variables is an infinite Grassmannian, a coadjoint orbit of an infinite dimensional unitary group.\(^8\) This allows one to understand baryons as well as mesons in the large \(N\) limit, going beyond the early work of 't Hooft.\(^9\) \(^8\) \(^10\) \(^11\).

However, the groups and Lie algebras whose coadjoint orbits are relevant to matrix field theories and Yang-Mills theory are poorly understood non-commutative versions of diffeomorphism groups.\(^3\) In the case of a multi-matrix model, the group is, roughly speaking, an automorphism group of a tensor algebra. The Lie algebra is a Cuntz-type algebra which can be thought of as an algebra of vector fields on a non-commutative space.\(^12\) \(^5\) \(^6\) \(^13\) \(^14\). However, it is still very challenging to find the proper mathematical framework for these theories and develop approximation methods to solve them even in the large \(N\) limit. To develop the necessary tools, it becomes worth while to practice on simpler theories whose gauge invariant phase space is the coadjoint orbit of a less formidable group. Here, we take a step in this direction by studying an abelian gauge theory whose phase space is a coadjoint orbit of the volume preserving diffeomorphism group of a two dimensional surface.

To put these remarks in perspective, recall the common classical formulation of Eulerian rigid body dynamics, ideal hydrodynamics, the KdV equation\(^15\) \(^16\) \(^17\) and the large \(N\) limit of two dimensional QCD.\(^8\) The phase space of each of these theories is a symplectic leaf of a degenerate Poisson manifold, which is the dual \(G^*\) of a Lie algebra. \(G^*\) always carries a natural Poisson structure. Symplectic leaves are coadjoint orbits of a group \(G\) acting on the dual of its Lie algebra \(G^*\). On any leaf, the symplectic structure is given by the Kirillov form. The appropriate groups in these examples are \(SO(3)\), the volume preserving diffeomorphism group of the manifold upon which the fluid flows, and the central extensions of \(\text{Diff}(S^1)\) and of an infinite dimensional unitary group, respectively. The coadjoint orbits for the rigid body and 2d QCD are well-known symplectic manifolds: concentric spheres and the infinite dimensional Grassmannian manifold. The observables in each case are real-valued functions on \(G^*\). The Poisson algebra of observables is degenerate, i.e. has a center consisting of Casimirs. The symplectic leaves can also be characterized as the level sets of a complete set of Casimirs. In each case, the hamiltonian is a quadratic function on the phase space and classical time evolution is given by Hamilton’s equations. Hamilton’s equations are non-linear despite a quadratic hamiltonian, since the Poisson brackets of observables are more complicated than the Heisenberg algebra. In exceptional cases such as the rigid body and the KdV equation, these non-linear equations are exactly integrable. In other cases, it is useful to develop approximation methods to solve them, that are adapted to the geometry of the phase space.

Our earlier remarks indicate that it may be fruitful to regard Yang-Mills theory and matrix field theories as Hamiltonian dynamical systems along the lines of the more well known ones listed in the last paragraph. As a toy-model in this direction, we seek a gauge theory where

\(^2\)See also the work of Karabali, Nair and Kim who have made significant progress with a gauge invariant Hamiltonian approach to \(2 + 1\) Yang-Mills theory.\(^9\)

\(^3\)This is not the structure group (sometimes called the gauge group) of the theory, which is still \(SU(N)\) or \(U(N)\). The gauge group plays little role in a gauge-invariant formulation of the theory.
the gauge fields satisfy a closed Poisson algebra, without any need for electric fields. We want a theory whose phase space is a coadjoint orbit of an ordinary diffeomorphism group, which is simpler than its non-commutative cousins. We would also like to understand in more detail the structure of the Poisson algebra of gauge-invariant observables, work out the equations of motion and try to solve them.

In this paper, we identify a classical theory of abelian gauge fields in two spatial dimensions, different from Maxwell theory. In particular, it is not Lorentz covariant, indeed, time plays the same role as in Newtonian relativity. The theory is defined by a two dimensional manifold $M$, a volume form $\mu$ and a metric $g_{ij}$ whose volume element $\Omega_g$ is not a multiple of $\mu$. The hamiltonian is a gauge-invariant magnetic energy, much like that of Maxwell theory. Unlike in electrodynamics, the gauge field is a 1-form on space, rather than on space-time. Thus, even before any gauge fixing, the gauge field has no time component. There is a magnetic field $B$, but no electric field, so to speak. After gauge fixing, there remains only one dynamical component of the gauge field. In this sense, the theory has the same number of degrees of freedom as $2 + 1$ electrodynamics. However, though the hamiltonian is quadratic in the gauge fields, the classical theory is nonlinear due to the ‘non-canonical’ Poisson algebra of gauge fields. Equations of motion are non-linear and comparable to those of a $2 + 1$ dimensional non-abelian gauge theory or ideal hydrodynamics.

The phase space of the theory is a coadjoint orbit of the volume preserving diffeomorphism group $\text{SDiff}(M,\mu)$ of the spatial two-dimensional manifold. Roughly speaking, this means that $\text{SDiff}(M,\mu)$ is a symmetry group of the Poisson algebra of observables. The gauge group (structure group) of the theory is $U(1)$. The inspiration for this lies in ideal hydrodynamics. Indeed, even before the diffeomorphism group of a manifold appeared in general relativity, it was relevant as the configuration space of a fluid. The theory we study is not the same as, but is motivated by $2 + 1$ dimensional ideal hydrodynamics, regarded as a hamiltonian system\cite{18, 19, 15, 20, 21, 22}. Though we arrived at it as a toy model for Yang-Mills theory, it turns out to have a nice geometric and possibly even integrable structure. We find two infinite sequences of conserved charges. The first set are Casimirs, analogues of the enstrophy invariants of ideal hydrodynamics. In addition, we find another infinite set of conserved charges which are not Casimirs but are in involution. The theory we study here can also be regarded as a theory of geodesics of a right-invariant metric on the volume preserving diffeomorphism group of a two dimensional manifold. However, the right invariant metric on $\text{SDiff}(M,\mu)$ implied by our hamiltonian is different from that arising in ideal hydrodynamics (the $L^2$ metric leading to ideal Euler flow) as well as the $H^1$ metric leading to averaged Euler flow\cite{23}.

Another way to view the current work is to recall that adding supersymmetry usually gives greater analytical control over gauge theories. But there may be other modifications of gauge theories that also lead to interesting toy-models or enhanced solvability. Our investigation concerns one such novel modification of gauge field Poisson brackets.

In Sec. 2 we introduce the space of abelian gauge fields $A_i dx^i$ on a two dimensional surface $M$ as the dual of the Lie algebra $\text{SVect}(M,\mu)$ of vector fields preserving a volume element $\mu = \rho d^2 x$. This ‘duality’ is known in hydrodynamics\cite{15}. The differentials $df^i = \rho^{-1} \frac{\delta f}{\delta A_i}$ of differentiable gauge-invariant observables $f(A)$ are shown to be volume preserving vector fields. In Sec. 3 we give the Poisson structure on gauge-invariant observables

$$\{f, g\} = \int d^2 x \rho A_i \left[ \rho^{-1} \frac{\delta f}{\delta A_j} \partial_j \left( \rho^{-1} \frac{\delta g}{\delta A_i} \right) - \rho^{-1} \frac{\delta g}{\delta A_j} \partial_j \left( \rho^{-1} \frac{\delta f}{\delta A_i} \right) \right]$$  \quad (5)
turning the space of gauge fields into a Poisson manifold. The Poisson brackets of gauge fields are obtained explicitly and compared with those of Yang-Mills theory on a spatial and null initial value hypersurface.

In Sec. 4 we give the coadjoint action of \( \text{SDiff}(M, \mu) \) and its Lie algebra \( \text{SVect}(M, \mu) \) on the Poisson manifold of gauge fields \( \text{SVect}(M, \mu)^* \), and show that the action is canonical i.e. preserves the Poisson structure. The moment maps generate the coadjoint action. The symplectic leaves of the Poisson manifold are coadjoint orbits. The enstrophy invariants of hydrodynamics \( I_n = \int_M \left( \frac{dA}{\mu} \right)^n \sigma \mu \) are an infinite sequence of Casimirs of the Poisson algebra. The coadjoint orbits of closed gauge field 1-forms are shown to be finite dimensional. Single-point orbits for simply connected \( M \) are found. We argue that all other orbits are infinite dimensional and try to characterize their isotropy subalgebras as well as tangent spaces.

In Sec. 5 we first review the choice of Hamiltonian leading to ideal Eulerian hydrodynamics in 2+1 dimensions. Then we propose a different gauge-invariant Hamiltonian depending on both \( \mu \) and a metric \( g_{ij} \), by analogy with the magnetic energy of Maxwell theory.

\[
H = \frac{1}{2} \int_M \left( \frac{F \wedge \ast F}{\Omega_g} \right) \mu = \int \left( \frac{B}{\rho} \right)^2 \sigma \rho \, d^2x
\]

where \( F = dA \) is the field strength, \( \ast F \) is its Hodge dual, and \( \Omega_g \) is the volume element of the metric \( g_{ij} \). Here \( \sigma = (\mu/\Omega_g)^2 = \rho^2/g \), \( g = \det g_{ij} \) and \( \rho \) is the density associated to \( \mu \). \( H \) is shown to determine a non-negative inner product on the dual of the Lie algebra \( \text{SVect}(M, \mu)^* \) and an inverse ‘inertia tensor’ by analogy with the rigid body. If \( M \) is simply connected, the inverse inertia operator is non-degenerate and could be inverted to get an inner product on the Lie algebra \( \text{SVect}(M, \mu) \). This could be extended to the diffeomorphism group \( \text{SDiff}(M, \mu) \) by right translations. Thus, the magnetic energy should define geodesic flow on \( \text{SDiff}(M, \mu) \) with respect to a right-invariant metric different from that coming from Eulerian hydrodynamics.

In Sec. 6 we find the equation of motion for the magnetic field \( B = \epsilon^{ij} \partial_i A_j \), \( \dot{B} = \nabla(B/\rho) \times \nabla(B\rho/g) \). This simple quadratically non-linear evolution equation is strikingly similar to the Euler equation of a rigid body \( \dot{L} = L \times \Omega, \ L = I\Omega \). It can be regarded as the ‘Euler equation’ for the group \( \text{SDiff}(M, \mu) \) with Hamiltonian given above. Remarkably, for a uniform measure \( \mu \) we find an infinite sequence \( H_n \) of conserved charges in involution, which are not Casimirs. The Hamiltonian is \( \frac{H^2}{2} \).

\[
H_n = \int_M \left( \frac{dA}{\mu} \right)^n \sigma \mu, \quad n = 1, 2, 3, \ldots
\]

In Sec. 7 we find some static solutions of the equations of motion. We show that for circularly symmetric \( \rho \) and \( g \), every circularly symmetric magnetic field is a static solution. We generalize this to the non-symmetric case as well. We also find a one parameter family of static solutions that are local extrema of energy even with respect to variations that are not restricted to the symplectic leaf on which the extremum lies. Some ideas for further study are given in Sec. 8.

Volume preserving diffeomorphisms and gauge theories have appeared together previously in the literature (see for example Ref. [24, 25]). Our investigation seems quite different, since \( \text{SDiff}(M, \mu) \) is not the gauge group of our theory but rather a symmetry of the Poisson algebra.
2 Volume Preserving Vector Fields to Gauge Invariant Observables

2.1 Lie Algebra of Volume Preserving Vector Fields

Let $M$ be a surface with local coordinates $x^i$, to be thought of as the space on which a fluid flows. A vector field on $M$ is regarded as the velocity field of a fluid at a particular time. The space of all vector fields on $M$ forms a Lie algebra $\text{Vect}(M)$ with Lie bracket

$$[u,v]^i = u^j \partial_j v^i - v^j \partial_j u^i$$  \hspace{1cm} (8)

$\text{Vect}(M)$ is the Lie algebra of the diffeomorphism group $\text{Diff}(M)$. Conservation of the mass of the fluid during its flow implies the continuity equation for its density $\rho(x,t)$

$$\frac{\partial \rho(x,t)}{\partial t} + \nabla \cdot (\rho u) = 0$$ \hspace{1cm} (9)

We are interested in flows where the density at any point of space does not depend on time. Using the continuity equation, this becomes $\nabla \cdot (\rho u) = 0$. We call such a flow volume preserving.

Geometrically, we are considering a flow that generates diffeomorphisms of $M$ that preserve a given volume form $4$. The density is constant along integral curves of $u$. To see the equivalence of this with the continuity equation for a volume preserving flow, recall that

$$\mathcal{L}_u \mu = (di_u + i_u d)\mu = d(i_u \mu)$$ \hspace{1cm} (10)

where $i_u$ is the contraction with $u$. Here $d\mu = 0$ since $\mu$ is a volume form. In local coordinates $\mu = \frac{1}{2} \mu_{ij} dx^i \wedge dx^j$ where $\mu_{ij} = -\mu_{ji}$ and $\epsilon_{ij}$ is antisymmetric with $\epsilon_{12} = 1$. So $\mu = \rho(x) \, d^2 x$ where $d x^1 \wedge d x^2 \equiv d^2 x$. Then $i_u \mu = \frac{1}{2} \mu_{ij} (u^i dx^j - dx^i u^j) = \mu_{ij} u^i dx^j$, so that

$$\mathcal{L}_u \mu = (di_u + i_u d)\mu = \partial_k (\mu_{ij} u^i) dx^k \wedge dx^j = \partial_i (\rho u^i) dx^1 \wedge dx^2$$ \hspace{1cm} (11)

Thus $\mathcal{L}_u \mu = 0$ becomes $\nabla \cdot (\rho u) = 0$. We will use the terms volume preserving and area preserving interchangeably since $M$ is a two dimensional surface. Some of what we say has a generalization to higher (especially even) dimensional $M$.

The properties $\mathcal{L}_{\alpha u} + \beta \mathcal{L}_u = \alpha \mathcal{L}_u + \beta \mathcal{L}_v$ and $\mathcal{L}_{[u,v]} = \mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u$ ensure that the space of volume-preserving vector fields $\mathcal{G} = \text{SVect}(M, \mu)$ forms a Lie subalgebra of $\text{Vect}(M)$. It is the Lie algebra of the group of volume preserving diffeomorphisms $G = \text{SDiff}(M, \mu)$.

Volume preserving flow is a mathematical implementation of the physical concept of incompressible flow, which occurs when the fluid speed is small compared to the speed of sound\footnote{A volume form must be non-degenerate. In 2 dimensions it is the same as an area form or a symplectic form.}. In particular, shock waves cannot form in incompressible flow, since shock waves involve supersonic flow. Under ordinary conditions, air flow in the atmosphere is incompressible. Volume preserving flow is sometimes referred to as divergence-free flow.

\footnote{Some authors consider only the special case where density is a constant, $\nabla \cdot u = 0$. Note also that the same fluid may support both compressible and incompressible flow under different conditions, so our definitions refer to the flow and not just to the fluid.}
If $M$ is simply connected, the volume preserving condition $\partial_i(\rho u^i) = 0$ may be solved in terms of a stream function $\psi$ satisfying $i_u \mu = d\psi$. $\psi$ is a scalar function on $M$ that serves as a ‘potential’ for the velocity field. In local coordinates

$$i_u \mu = d\psi \Rightarrow \mu_{ij} u^i = \partial_j \psi \quad (12)$$

Since $\mu$ is non-degenerate ($\rho \neq 0$), it can be inverted $\rho^{-1} \epsilon_{ij} \mu_{jk} = -\delta_i^k$ where $\epsilon_{ij}$ is a constant antisymmetric tensor with $\epsilon_{12} = 1$, $\epsilon_{ij} \epsilon_{jk} = -\delta_i^k$. This does not require a metric on $M$. Then

$$u^i = \rho^{-1} \epsilon^{ij} \partial_j \psi. \quad (13)$$

$u$ determines $\psi$ up to an additive constant, which can be fixed by a boundary condition. If $M$ is simply connected, then $\text{SVect}(M, \mu)$ may be identified with the space of stream functions.

Suppose two volume preserving vector fields $u, v$ have stream functions $\psi_u$ and $\psi_v$,

$$u^i = \rho^{-1} \epsilon^{ij} \partial_j \psi_u, \quad v^i = \rho^{-1} \epsilon^{ij} \partial_j \psi_v \quad (14)$$

Then their Lie bracket $[u, v]$ has stream function $\rho^{-1} \nabla \psi_v \times \nabla \psi_u$:

$$[u, v]^i = \rho^{-1} \epsilon^{il} \partial_l \left\{ \rho^{-1} \epsilon^{jk} (\partial_j \psi_v)(\partial_k \psi_u) \right\}$$

$$\psi_{[u, v]} = \rho^{-1} \epsilon^{jk} (\partial_j \psi_v)(\partial_k \psi_u) \quad (15)$$

### 2.2 Abelian Gauge Fields as the Dual of $\text{SVect}(M, \mu)$

The Lie algebra $\text{SVect}(M, \mu)$ is akin to the Lie algebra of angular velocities of a rigid body. The angular momenta are in the dual space to angular velocities, and satisfy the angular momentum Poisson algebra. As explained in [A], the dual of any Lie algebra is a Poisson manifold. This is interesting because the observables of a classical dynamical system are real-valued functions on a Poisson manifold. The dual of the Lie algebra $\mathcal{G} = \text{SVect}(M, \mu)$ is the space of abelian gauge fields modulo gauge transformations,

$$\mathcal{G}^* = \text{SVect}(M, \mu)^* = \Omega^1(M)/d\Omega^0(M). \quad (16)$$

This fact is well-known in hydrodynamics (see Ref. [15]), though it is usually not thought of in terms of gauge fields. To see this duality, we define the pairing $(A, u)$ between gauge fields and volume preserving vector fields by integrating the scalar $A(u)$ with respect to $\mu$

$$(A, u) = \mu_u(A) = \int_M A(u) \mu = \int_A u^i \rho \ d^2x \quad (17)$$

The pairing $\mu_u(A)$ is also called the moment map. It is a gauge-invariant pairing. Under a gauge transformation $A \mapsto A' = A + d\Lambda$ for any scalar $\Lambda(x)$

$$\mu_u(A') - \mu_u(A) = \int_M (\partial_i \Lambda) u^i \mu = - \int A \partial_i (\rho u^i) d^2x = 0 \quad (18)$$

since $u$ is volume preserving. We assume that gauge fields and gauge transformations $\Lambda$ vanish on the boundary $\partial M$ or at infinity. We make no such assumption about the vector fields.

Gauge Fixing: It is occasionally convenient to ‘gauge-fix’, i.e. pick a coset representative for $\Omega^1(M)/d\Omega^0(M)$. Under a gauge transformation, $A'_i = A_i + \partial_i \Lambda$. We can pick $\Lambda$ such that
symplectic leaf of $\text{SVect}(M, \mu)$, so that we are left with only one component of the gauge field $A_2'$. We can still make an $x^1$-independent ‘residual’ gauge transformation, $A_2'' = A_2' + \partial_2 \Lambda(x^2)$ to eliminate any additive term in $A_2'$ depending on $x^2$ alone. Suppose we have gauge fixed on a particular spatial initial value surface at time $t = 0$. Unlike in Yang-Mills theory, the equations of motion of our theory are purely dynamical. They only evolve the gauge-fixed fields forward in time, and do not contain any further constraints. In effect, after gauge fixing, we will be left with one propagating field degree of freedom.

2.3 Differentials of Gauge Invariant Charges are Volume Preserving Vector Fields

We should regard $\text{SVect}(M, \mu)^*$, the space of gauge fields modulo gauge transformations, as the Poisson manifold of some dynamical system. Real-valued functions on this space (i.e. gauge-invariant functions $f(A)$) are the observables. Given such an $f(A)$, we can define its differential $df(A)$

$$(df(A))^i = \rho^{-1}(x) \frac{\delta f}{\delta A_i(x)} = \rho^{-1} \delta_i f$$

For each equivalence class of gauge fields $[A] = \{A|A \sim A + d\Lambda\}$, the differential $df$ defines a vector field $df^i \partial_i$ on $M$. If $f(A)$ is non-linear, the vector field $(df(A))^i(x)$ changes as $A \in G^*$ changes. Suppose $f(A)$ is gauge invariant and differentiable. Then we can show that its differential $df$ is a volume preserving vector field on $M$: $\partial_i(df^i) = 0$. To see this, note that gauge invariance implies that the change in $f$ under any gauge transformation $\delta A_i = \partial_i \Lambda$ must vanish

$$0 = \delta f = \int \frac{\delta f}{\delta A_i(x)} \delta A_i(x) d^2x = \int \frac{\delta f}{\delta A_i(x)} \partial_i \Lambda d^2x = - \int \partial_i \left( \frac{\delta f}{\delta A_i(x)} \right) \Lambda(x) d^2x$$

Since $\Lambda(x)$ is arbitrary, it must follow that $\partial_i(\delta f/\delta A_i) = 0$. So the differential of a gauge invariant function can be regarded as an element of the Lie algebra $G = \text{SVect}(M, \mu)$.

The simplest gauge-invariant observable is the field strength 2-form

$$F = dA = \frac{1}{2} F_{ij} dx^i \wedge dx^j; \quad F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ij} B$$

where $B = \epsilon^{ij} \partial_i A_j = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field. The differential of $F$

$$((dF)(A))^k = \rho^{-1} \frac{\delta F_{ij}(x)}{\delta A_k(y)} = \rho^{-1} \left( \delta_j^k \partial_i \delta(x - y) - \delta_i^k \partial_j \delta(x - y) \right)$$

is a volume preserving vector field on $M$ for each $A$:

$$\partial_k(\rho(dF)^k) = \partial_k \left( \delta_j^k \partial_i \delta(x - y) - \delta_i^k \partial_j \delta(x - y) \right) = (\partial_i\partial_j - \partial_j\partial_i)\delta(x - y) = 0.$$  

The differential $(df(A))^i$ is regarded as a vector field on $M$ for each $A$ and should not be confused with the closely related exterior derivative $df$, which is a 1-form on $G^*$. However, as we will see later (Sec. 4 [A]), on any symplectic leaf of $\text{SVect}(M, \mu)^*$ with symplectic form $\omega$, the 1-form $df$ determines the canonical vector field $V_f$ via $\omega(V_f, .) = df(.)$. $V_f$ is a vector field on the leaf, and its relation to the differential is $V_f(A) = ad^\nu_{\mu} A$.}

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Similarly, the differential of the magnetic field

$$((dB)(A))^k = \rho^{-1} \frac{\delta B(x)}{\delta A_k(y)} = \rho^{-1} \varepsilon^{ij}\partial_i \delta^k_j \delta(x - y) = \rho^{-1} \varepsilon^{ik} \partial_i \delta(x - y)$$  \hspace{1cm} (24)$$

is volume preserving \( \partial_k(\rho dB)^k = \varepsilon^{ik} \partial_i \delta(x - y) = 0 \). We can regard the moment maps \( \mu_u(A) \) as linear gauge-invariant observables. The differential of \( \mu_u(A) \) is the volume preserving vector field \( u \), for all gauge fields \( A \).

Other gauge invariant observables \( f(A) \) we will be interested in are ‘charges’: integrals over \( M \) with respect to \( \mu \), of a local gauge-invariant scalar function \( \mathcal{F} \). \( \mathcal{F} \) can depend on \( A \) only through the field strength two form \( F = dA \). The analogue of the Chern-Simons 3-form, vanishes identically since \( A \) is a 1-form on space, not space-time. The quotient of \( dA \) and the non-degenerate volume 2-form \( \frac{dA}{\mu} \) is a scalar function on \( M \). Then

$$f(A) = \int \mathcal{F}(\sigma, (dA/\mu), v_i \partial_i (dA/\mu), w^{ij} \partial_i \partial_j (dA/\mu), \ldots) \mu$$  \hspace{1cm} (25)$$

where \( \sigma \) is a scalar function and \( v^i, w^{ij} \) etc are arbitrary but fixed contravariant tensor fields. We can get an explicit formula for the differential of such a gauge invariant charge. Using \( dA/\mu = B/\rho \) and

$$\frac{\partial B}{\partial A_i} = -\varepsilon^{ij} \partial_j \delta^2(x - y); \quad \frac{\delta \partial_k(B/\rho)(x)}{\delta A_i(y)} = -\varepsilon^{ij} \partial_k \left( \frac{1}{\rho} \partial_j \delta^2(x - y) \right); \ldots$$  \hspace{1cm} (26)$$

we get upon integrating by parts,

$$df(A)^i = \rho^{-1} \frac{\delta f}{\delta A_i} = \rho^{-1} \varepsilon^{ij} \partial_j \left[ \left( \frac{\partial \mathcal{F}}{\partial (B/\rho)} \right) - \left( \frac{1}{\rho} \partial_k \left( \frac{\partial \mathcal{F}}{\partial \partial_k(B/\rho)} \right) \right) + \ldots \right]$$  \hspace{1cm} (27)$$

Due to the anti-symmetry of \( \varepsilon^{ij} \), it follows that \( df \) is volume preserving \( \partial_i(\rho df^i) = 0 \). Moreover, if \( f \) is gauge-invariant and of the form assumed above, then its differential is also gauge-invariant.

Two families of gauge invariant charges which play an important role in our theory are \( I_n \) and \( H_n \) defined below. Let

$$I_n(A) = \int_M (dA/\mu)^n \mu = \int (B/\rho)^n \rho \, d^2 x = \int_M \left( \frac{B}{\rho} \right)^{n-1} dA, \quad n = 1, 2, 3, \ldots$$

$$\left( dI_n \right)^i = \frac{1}{\rho} \frac{\delta I_n}{\delta A_i} = \frac{1}{\rho} n \varepsilon^{ij} \partial_j \left( \frac{B/\rho}{\rho} \right)^{n-1}$$  \hspace{1cm} (28)$$

Their differentials are volume preserving \( \partial_i(\rho (dI_n(A))^i) = 0 \) since \( \varepsilon^{ij} \) is antisymmetric. Note that \( I_1 = \int_M dA = 0 \). We assume \( B \) vanishes sufficiently fast at infinity and do not consider \( I_n \) for \( n < 0 \). Given a scalar function \( \sigma \) on \( M \) we can construct additional gauge-invariant charges. These are similar to the \( I_n \), except that we multiply by \( \sigma \) before integrating over \( M \)

$$H_n(A) = \int (dA/\mu)^n \sigma \mu = \int (B/\rho)^n \sigma \rho \, d^2 x$$  \hspace{1cm} (29)$$

More generally, one can replace \( (B/\rho)^n \) by an arbitrary function of \( B/\rho \). The differential of \( H_n \) is volume preserving:

$$\left( dH_n(A) \right)^i = \frac{n \varepsilon^{ij}}{\rho} \partial_j (\sigma (B/\rho)^{n-1}).$$  \hspace{1cm} (30)$$
Gauge-invariant observables depending on the volume form $\mu$ and a metric $g_{ij}$ on $M$ will play an important role in determining the dynamics of our theory. They are given in Sec. 5.2.

There are other interesting gauge invariant observables such as the circulation $C_i(A) = \int_0^1 A_i x^{i(s)} ds$ and its exponential, the Wilson loop. Since these observables are concentrated on one dimensional curves on $M$, they may fail to be differentiable and their differentials exist only as distributional vector fields. They require a more careful analysis and are not considered in this paper. Henceforth, when we say observable, we will mean gauge invariant observables that are differentiable, in which case, their differentials are guaranteed to be volume preserving vector fields on $M$.

## 3 Poisson Brackets

### 3.1 Definition of Poisson Bracket on $\mathcal{G}^* = \text{SVect}(M, \mu)^*$

The Lie algebra structure of volume preserving vector fields $\mathcal{G} = \text{SVect}(M, \mu)$ can be used to define a Poisson structure on its dual (see Ref. 26, 27, 31). The dual space $\mathcal{G}^* = \text{SVect}(M, \mu)^* = \Omega^1(M)/d\Omega^0(M)$ of gauge field 1-forms then becomes a Poisson manifold. Observables are real-valued functions on it. The Poisson bracket (p.b.) between gauge-invariant observables $f(A)$ and $g(A)$ with volume preserving differentials $df$ and $dg$, is defined using the pairing $(A, u)$ between $\mathcal{G}$ and $\mathcal{G}^*$ and the Lie algebra bracket $[df, dg]$:

$$\{f, g\}(A) = (A, [df, dg]) = \int_M A_i [df, dg]^i \mu = \int d^2 x \rho A_i [df^j \partial_j dg^i - dg^j \partial_j df^i]$$

$$= \int d^2 x \rho A_i \left[ \rho^{-1} \frac{\delta f}{\delta A_i} \partial_j \left( \rho^{-1} \frac{\delta g}{\delta A_i} \right) - \rho^{-1} \frac{\delta g}{\delta A_i} \partial_j \left( \rho^{-1} \frac{\delta f}{\delta A_i} \right) \right]$$

$$= \int A_i \left[ (\delta^j f) \partial_j (\rho^{-1} \delta^i g) - (\delta^j g) \partial_j (\rho^{-1} \delta^i f) \right] d^2 x$$

(31)

where $\delta^i = \frac{\delta}{\delta A_i}$. The antisymmetry, linearity and Jacobi identity follow from the corresponding properties of the Lie bracket. The Leibnitz rule follows from the Leibnitz rule for differentials.

The p.b. (31) preserves the class of gauge-invariant functions. Suppose $f, g$ are gauge-invariant. Then $df, dg \in \text{SVect}(M, \mu)$. Recall (Sec. 2.3) that if $f$ and $g$ are gauge-invariant, then so are their differentials $df^i$ and $dg^i$. It follows that $[df, dg]$ is also gauge-invariant. Now, under a gauge transformation $A' = A + d\Lambda$,

$$\{f, g\}(A') = (A', [df, dg](A')) = (A', [df, dg](A)) = \int A'_i ([df, dg](A))^i \mu$$

$$\Rightarrow \{f, g\}(A') - \{f, g\}(A) = - \int A \partial_i ([df, dg](A))^i \rho d^2 x = 0$$

(32)

Thus (31) is a gauge invariant Poisson bracket.

A gauge-invariant observable $f(A)$ defines canonical transformations on the Poisson manifold $\mathcal{G}^*$ via the p.b. Suppose $g(A)$ is any observable, then its Lie derivative under the flow generated by $f$ is $L_f g(A) = \{f, g\}(A) = (A, [df, dg])$ (see Sec. 4).

Example: The p.b. of two moment maps $\mu_u(A)$ and $\mu_v(A)$ is

$$\{\mu_u, \mu_v\}(A) = \mu_{[u,v]}(A) = \int_M A_i (u^j \partial_j v^i - v^j \partial_j u^i) \mu$$

(33)
If \( u \) and \( v \) are volume preserving, then so is \([u,v]\); therefore, if \( \mu_u \) and \( \mu_v \) are gauge-invariant functions of \( A \), then so is \( \mu_{[u,v]} \). Moreover, the canonical transformation generated by the moment map \( \mu_u \) is just the Lie algebra coadjoint action (see (A) and Sec. 4)

\[
\mathcal{L}_{V_{\mu_u}} f(A) = \{\mu_u, f\}(A); \quad \mathcal{L}_{V_{\mu_u}} A = ad_u^* A.
\] (34)

### 3.2 Poisson Brackets of Gauge Fields

Equivalence classes of gauge fields are coordinates on our Poisson manifold \( G^* = SVect(M, \mu)^* \). Thus, an explicit formula for the ‘fundamental’ p.b. between components of the gauge field \( A_i(x) \) is useful. This will also facilitate a comparison with electrodynamics. We will show that the p.b. between gauge field components is

\[
\{A_i(x), A_j(y)\} = \delta^2(x - y) \left[ A_j(y) \frac{\partial}{\partial y} p^{-1}(y) - A_i(x) \frac{\partial}{\partial x} p^{-1}(x) \right],
\] (35)

where the derivatives act on everything to their right. Though this formula for \( \{A_i(x), A_j(y)\} \) looks a bit complicated, the rhs is linear in gauge fields. Thus, our Poisson algebra is actually a Lie algebra like the Lie algebra of angular momenta \( \{L_i, L_j\} = \epsilon_{ijk} L_k \).

Recall that electrodynamics is based on the Heisenberg algebra between the spatial components of the gauge field and the spatial components of the electric field. In two spatial dimensions we have (before any gauge fixing)

\[
\{A_i(x, t), E^j(y, t)\} = \delta^2(x - y) \delta_i^j \quad \{A_i(x, t), A_j(y, t)\} = 0 \quad \{E^i(x, t), E^j(y, t)\} = 0
\] (36)

While in electrodynamics the electric field is canonically conjugate to the gauge field, this is not the case in our theory. The components of the gauge field in our theory obey p.b. relations with each other, without being canonically conjugate. This is not unusual. For instance, the components of angular momentum form a closed Poisson algebra though none of them is canonically conjugate to another. This is a generic feature of degenerate Poisson manifolds where canonically conjugate variables can only be chosen on individual symplectic leaves (the concentric spheres in the case of angular momenta).

Gauge fields obeying p.b. not involving the electric field are not alien to conventional Yang-Mills theory. For example, in a coordinate system where initial values of fields are specified on a null cone at past time-like infinity, the transverse components of gauge fields satisfy p.b. among themselves as shown by Rajeev and Turgut\(^2\) \(^3\)

\[
\{A^a_{ib}(z, R), A^c_{jd}(z', R')\} = \frac{1}{2} \delta_a^c \delta_i^j q_{ij}(z) \delta(z - z') \text{ sgn} (R - R')
\] (37)

where \( z^i \) are transverse angular coordinates, \( q_{ij} \) is the round metric on \( S^2 \), \( R \) is a radial coordinate and \( a, b \) are color indices. In fact, our original motivation for studying the dynamical system in this paper was to find a toy-model that shared this feature with Yang-Mills theory.

Now we will establish \(^5\) using \(^3\) and the relation between the p.b. of functions and those between the ‘coordinates’ \( A_i(x) \)

\[
\{f, g\}(A) = \int d^2x d^2y \{A_i(x), A_j(y)\} \frac{\delta f}{\delta A_i(x)} \frac{\delta g}{\delta A_j(y)}
\] (38)
We rewrite \( \{f, g\}(A) \) from (31) to make it look like this

\[
\int d^2x A_j \left[ \frac{\delta f}{\delta A_j} \partial \left( \frac{1}{\rho} \frac{\delta g}{\delta A_j} \right) - \frac{\delta g}{\delta A_j} \partial \left( \frac{1}{\rho} \frac{\delta f}{\delta A_j} \right) \right] = \int d^2x^2 y \delta^2(x - y) A_j(y) \left[ \frac{\delta f}{\delta A_i(x)} \delta g \left( 1 - \rho \frac{\delta A_j(y)}{\delta A_j(y)} \right) - f \leftrightarrow g \right] = \int d^2x^2 y \delta^2(x - y) A_j(y) \left[ 1 - \rho \frac{\delta A_i(x)}{\delta A_i(x)} \delta A_j(y) \right] \left( f - g \right) = \int d^2x^2 y \delta^2(x - y) \left[ A_j(y) \frac{\partial}{\partial y^i} \rho^{-1}(y) - A_i(x) \frac{\partial}{\partial x^j} \rho^{-1}(x) \right] \frac{\delta f}{\delta A_i(x)} \frac{\delta g}{\delta A_j(y)}. \tag{39}
\]

Finally we read off the desired expression (35).

4 Structure of Poisson Algebra of Observables

In this section we give the canonical action of \( \text{SDiff}(M, \mu) \) (generated via p.b.) on the Poisson manifold \( G^* = \text{SVect}(M, \mu)^* \) (see also Ref. [15]). Symplectic leaves are coadjoint orbits of \( \text{SDiff}(M, \mu) \). The Poisson algebra of gauge invariant observables is degenerate. The enstrophy invariants \( I_n \) of hydrodynamics are an infinite number of Casimirs. They are constant on the coadjoint orbits. Then we try to identify some of the simpler coadjoint orbits. We show that there is always at least one single-point orbit, that of the pure gauge configuration. If \( M \) has non-vanishing first cohomology, then we show that there are finite dimensional symplectic leaves lying inside \( H^1(M) \setminus \{0\} \). This also shows that \( I_n \) could not be a complete set of coadjoint orbit invariants. If \( M \) is simply connected, we show that the only single-point orbits consist of the configurations for which \( dA/\mu \) is constant. For simply connected \( M \), we also argue that the orbit of \( [A] \in G^* \) for which \( dA/\mu \) is not constant, is infinite dimensional. We identify the isotropy sub-algebra and tangent space of such an orbit and give an example where \( dA/\mu \) is circularly symmetric. However, this analysis is far from complete. It would be useful to find a nice coordinate system on these orbits and get the symplectic structure in explicit form so as to study hamiltonian reduction.

4.1 Coadjoint Action of \( \text{SDiff}(M, \mu) \) on Poisson Algebra

The Poisson manifold \( G^* = \Omega^1(M)/d\Omega^0(M) = \{ A \in \Omega^1(M) \mid A \sim A + d\Lambda \} \) carries the coadjoint action of \( \text{SDiff}(M, \mu) \). For example, on a simply connected region \( M \), \( G^* \) may be identified with the space of scalar functions \( f = (dA/\mu) = (B/\rho) \) on \( M \). The coadjoint action is the pull-back action of volume preserving diffeomorphisms \( \phi \in \text{SDiff}(M, \mu) \) on functions \( \phi^* f = f \circ \phi \). The action on equivalence classes of gauge fields is also the pull back \( \text{Ad}_u^\phi[A] = [\phi^* A] \). For infinitesimal \( \phi(t) = 1 + ut \) we get the Lie algebra coadjoint action of \( u \in \text{SVect}(M, \mu) \) on \( G^* \)

\[
ad_u^\phi A = -\mathcal{L}_u A \tag{40}
\]

This action of \( \text{SDiff}(M, \mu) \) on \( G^* \) is canonical i.e. there is a function on \( G^* \) (a gauge-invariant observable) that generates the coadjoint action via the p.b. The generating function is the moment map \( \mu_u(A) \). To see this, suppose \( u \) is a volume preserving vector field, then

\[
\{\mu_u(A), A_j\} = -({\mathcal{L}_u A})_j. \tag{41}
\]
To show this we begin with the rhs of (41)

\[
\mathcal{L}_u A = d(i_u A) + i_u (dA) = d(u^i A_i) + \frac{1}{2} i_u \left\{ (\partial_j A_i - \partial_i A_j) dx^i \wedge dx^j \right\}
\]

\[
= (\partial_j A_i u^i + A_i \partial_j u^i) dx^j + \frac{1}{2} (\partial_j A_i - \partial_i A_j)(u^j dx^i - dx^j u^i)
\]

\[
= \left\{ A_i \partial_j u^i + u^i \partial_i A_j \right\} dx^j \tag{42}
\]

On the other hand, using (35), \{\mu_n (A), A_k(z)\} is equal to

\[
\int d^2 x \ d^2 y \ \{A_i(x), A_j(y)\} \frac{\delta \mu_n}{\delta A_i(x)} \frac{\delta A_k(z)}{\delta A_j(y)}
\]

\[
= \int d^2 x \ d^2 y \ d^2 (x - y) \left[ A_j(y) \frac{\partial}{\partial y^i} \rho^{-1}(y) - A_i(x) \frac{\partial}{\partial x^j} \rho^{-1}(x) \right] \rho(x) u^i(x) \delta^2(z - y)
\]

\[
= \int d^2 x \ A_k(x) \rho(y) u^i(y) \frac{\partial}{\partial y^i} \left( \rho^{-1}(y) \delta^2(y - z) \right) - \int d^2 x \ A_i(x) \delta^2(z - x) \frac{\partial u^i(x)}{\partial x^k}
\]

\[
= -\rho^{-1} \partial_i (A_k \rho u^i) - A_i \partial_k u^i = -u^i \partial_i A_k - A_i \partial_k u^i = -(\mathcal{L}_u A)_k \tag{43}
\]

We integrated by parts \((A = 0 \text{ on } \partial M)\) and used \(\partial_i (\rho u^i) = 0\). Thus, we obtain (41).

### 4.2 Center of Poisson Algebra

From Sec. 11 the symplectic leaves of SVect\((M, \mu)^*\) are homogeneous symplectic manifolds, identified with coadjoint orbits of SDiff\((M, \mu)\). Is the Poisson algebra of functions on SVect\((M, \mu)^*\) degenerate? What are the Casimirs that constitute its center? Casimirs are unchanged under canonical transformations. So they are constant on symplectic leaves. A complete set of such Casimirs would allow us to distinguish between distinct leaves. Since the symplectic leaves are coadjoint orbits of SDiff\((M, \mu)\), it suffices to find observables that commute with the moment maps \(\mu_n(A)\), which generate the coadjoint action.

We first observe that the center of the Poisson subalgebra of linear observables \(\mu_n(A)\) is trivial. \(\{\mu_n, \mu_n\}(A) \) vanishes for all \(A\) iff \([u, v] = 0\). But the center of SVect\((M, \mu)\) is trivial. This is seen by writing \(u\) and \(v\) in terms of their stream functions \(\psi_u, \psi_v\). Taking some simple choices for \(\psi_v\) in the condition \([u, v] = 0\) will imply that \(\psi_u = 0\). Thus none of the \(\mu_n(A)\) lie in the center of the Poisson algebra of gauge-invariant functions, since they do not even commute with each other. Thus we need to look elsewhere to find the Casimirs of our Poisson algebra. It turns out that the charges \(I_n = \int (dA/\mu)^n \) are central observables. The monomials \((dA/\mu)^n\) can be replaced by any scalar function of \((dA/\mu)\). We found that \(I_n\) are Casimirs by showing that they Poisson commute with the moment maps. However, this involves lengthy calculations. For eg., in 12 we show that \(I_n\) commute with each other and in 13 we show that \(I_2\) commutes with \(\mu_n\) (A) for uniform \(\mu\).

However, we found that \(I_n\) are closely related to the enstrophy invariants of hydrodynamics, which are known to be conserved quantities in Eulerian hydrodynamics. There is a simple argument in Ref. 15 based on earlier work 19, 20, 22 that leads to the conclusion that \(I_n\) are constant on coadjoint orbits. The argument is that the action of SDiff\((M, \mu)\) is merely to change coordinates in the integral defining \(I_n\). Since this integral is independent of the choice of coordinates, \(I_n\) must be invariant under the coadjoint action. Though \(I_n\) for \(n = 1, 2, 3, \ldots\) are all Casimirs, they are not a complete set of orbit invariants. Their level sets do not necessarily
distinguish the coadjoint orbits (see Sec. 4.3). For some remarks on additional invariants, see Sec. 9 of Ref. [15]. Our explicit calculations of \( \{ I_n, \mu_u \} \) given in B and C suggested to us how \( I_n \) could be modified in order to get an independent infinite sequence of conserved quantities (not Casimir invariants) for our choice of hamiltonian; see Sec. 6.3.

### 4.3 Finite Dimensional Symplectic Leaves in \( H^1(M) \)

The simplest symplectic leaf in the dual of the Lie algebra of the rotation group is the origin of angular momentum space \( \{ L_i = 0 \} \). It consists of a single point. Of course, one can also characterize the orbit \( \{ L_i = 0 \} \) as the zero set of the Casimir \( L^2 = 0 \). Can we get a similar explicit characterization of the simplest symplectic leaves of \( \text{SVect}(M, \mu)^* \)?

**Orbits of Closed One Forms and Zero Set of Casimirs \( I_n \)**

The simplest symplectic leaf in \( \text{SVect}(M, \mu)^* \) should be the orbit of exact 1-forms. Suppose \( A = d\Lambda \) is an exact 1-form. Then under the coadjoint action of \( u \in \text{SVect}(M, \mu) \) it goes to

\[
A' = A + ad^*_u A
\]

\[
A'_i = A_i - (\mathcal{L}_u A)_i = \partial_i \Lambda - (\partial_j \Lambda)(\partial_i \omega^j) + u^i \partial_j \partial_i \Lambda = \partial_i (\Lambda - u^j \partial_j \Lambda)
\]

We see that a pure gauge is mapped to a pure gauge under the Lie algebra coadjoint action. Thus the tangent space to the orbit of exact 1-forms is trivial. So pure gauges form a single-point orbit.

After the pure gauges, the next simplest configurations are closed but inexact 1-forms \( dA = 0, A \neq d\Lambda \). Suppose \( M \) is a manifold with non-vanishing first cohomology. What is the orbit of an element of \( H^1(M) \)? In particular, is the orbit a finite dimensional manifold? Does the orbit lie within \( H^1(M) \)? The answers to both these questions is affirmative. Suppose \( A \) is an exact 1-form, \( F = dA = 0 \). Under the Lie algebra coadjoint action, \( A' = A + ad^*_u A = A - \mathcal{L}_u A \).

\[
dA' = d(A - \mathcal{L}_u A) = -d(d_i u + i_u d) A = -d_i u dA = 0
\]

The tangent space to the orbit of a closed 1-form contains only closed 1-forms. Since the space of closed 1-forms on a two-dimensional manifold is finite dimensional, the tangent space to the orbit must be finite dimensional. Moreover, we have shown that the exact 1-forms form a single point orbit by themselves. Thus, at the infinitesimal level, the orbit of a non-trivial element of \( H^1(M) \) must lie within \( H^1(M) \setminus \{0\} \). This view will be reinforced by considering the zero set of Casimirs

\[
I_n = \int (dA/\mu)^n \mu = 0
\]

\( I_n \) are constant on coadjoint orbits. Thus, the orbits must be contained in their level sets. \( I_{2n} \) is the integral of a positive quantity and therefore vanishes iff \( dA = 0 \). If \( dA = 0 \) then \( I_{2n+1} \) is also zero. Thus the level set \( I_n = 0 \) is the space of closed 1-forms on \( M \). Thus, the coadjoint orbit of a closed one form must lie in \( H^1(M) \). Since we already established that the pure gauges form a single-point orbit, this means \( I_n \) cannot be a complete set of Casimirs if \( H^1(M) \) is non-trivial.
For example, if $M$ is the plane, then $H^1(\mathbb{R}^2)$ consists only of the equivalence class of pure gauge configurations and the zero set of $I_n$ contains only one single-point orbit $[A] = 0$. The same is true of the 2-sphere $S^2$ which has trivial first cohomology. For the 2-torus $H^1(T^2) \cong \mathbb{R}^2$. For $T^2$, there must be symplectic leaves which are submanifolds of $\mathbb{R}^2 \setminus \{0\}$. It is interesting to find the orbits of cohomologically non-trivial gauge fields more explicitly as well as the induced symplectic structure. However, we do not investigate this further since the Hamiltonian we pick \[66\] vanishes on closed 1-forms. There is no interesting dynamics on the finite dimensional symplectic leaves we have described above. Therefore, we turn to the case where $M$ is simply connected and try to characterize the orbits of gauge fields that are not closed.

4.4 Symplectic Leaves when $M$ is Simply Connected

In the case of the rigid body, symplectic leaves other than $L_i = 0$ are concentric spheres of non-zero radius, all two dimensional symplectic manifolds. These leaves may also be characterized as non-zero level sets of $L^2$. By analogy, we would like to find the orbits in $S\text{Vect}(M, \mu)^*$ of 1-forms that are not closed. They must lie within non-zero level sets of $I_n$. Can we say something more about them, such as whether they are finite dimensional? We address these questions below, assuming $M$ is simply connected.

By Poincare’s lemma, all the information in a gauge field on a simply connected $M$ can be stored in the 2-form field strength, $F = dA$. Then, the dual of the Lie algebra $S\text{Vect}(M, \mu)^*$ may be identified with the space of scalar functions on $M$

$$S\text{Vect}(M, \mu)^* = \{ f = \frac{dA}{\mu} \}$$

(47)

The coadjoint action of $\phi \in S\text{Diff}(M, \mu)$ is just the pull back

$$Ad^* \phi f = \phi^* f = f \circ \phi$$

(48)

and the Lie algebra coadjoint action of $u \in S\text{Vect}(M, \mu)$ is

$$f \mapsto f + ad_u^* f = f - L_u f = f - u^i \partial_i f$$

(49)

Thus the coadjoint orbit of $f$ and the tangent space to the orbit at $f$ are

$$O_f = \{ f \circ \phi \mid \phi \in S\text{Diff}(M, \mu) \}$$

$$T_f O = \{ f - u^i \partial_i f \mid u \in S\text{Vect}(M, \mu) \}$$

(50)

The picture that emerges from our analysis below, is that there are two types of orbits when $M$ is simply connected. Orbits of the first type contain only a single point, namely a constant function $f = dA/\mu = c$. In the case of the plane with the uniform measure, the only one-point orbit is $\{ f = (dA/\mu) = 0 \}$, consisting of the pure gauge configuration. The orbits of the second type are all infinite dimensional. They are the orbits of non-constant functions $f$. In the case of the plane with uniform measure, we expect an infinite number of such orbits of the second type, each contained within a level set of $\{ I_n, n = 1, 2, 3, \ldots \}$, though we do not rule out the existence of more than one such orbit in any one level set of the $I_n$. 

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Single Point Orbits

We show that constant functions \( f = (dA/\mu) = c \) are the only single point coadjoint orbits in the dual of the Lie algebra \( \text{SVect}(M, \mu)^* \) when \( M \) is simply connected. They lie within the level sets \( I_n = \text{Vol}(M, \mu) c^n \) where \( \text{Vol}(M, \mu) = \int_M \mu \).

Suppose \( f \) is a constant. Then \( \mathcal{L}_u f = 0 \). So the constant functions form single point orbits. The case \( f \) corresponds to the closed and exact 1-forms \( A \) which we already identified as a single point leaf if \( M \) is simply connected. Non-zero constant functions are not admissible elements of \( \text{SVect}(M, \mu)^* \) if \( M = \mathbb{R}^2 \) and \( \mu \) is the uniform measure. But they are allowed if \( M \) is compact or if \( \mu \to 0 \) at infinity on non-compact \( M \). Note that \( I_n = \int f^n \mu = \text{Vol}(M, \mu) c^n \) if \( f = c \). So the constant functions lie in the level sets with \( I_n = \text{Vol}(M, \mu) c^n \).

We can go one step further and show that constant functions are the only single point orbits if \( M \) is simply connected. Suppose \( f \) is a single point orbit. Then \( \mathcal{L}_u f \) must vanish for all volume preserving vector fields \( u \). Since \( M \) is simply connected, any such vector field can be written in terms of a stream function \( u^i = \rho^{-1} \varepsilon^{ij} \partial_j \psi \). Then denoting derivatives by subscripts,

\[
\mathcal{L}_u f = \rho^{-1} \varepsilon^{ij} (\partial_j \psi)(\partial_i f) = \rho^{-1} (f_x \psi_y - f_y \psi_x)
\]

This must be true for all stream functions \( \psi \). Taking \( \psi = x \) and \( \psi = y \) successively tells us that \( f \) must be independent of both \( x \) and \( y \) and hence a constant. We conclude that the only single point orbits in \( \mathcal{G}^* = \text{SVect}(M, \mu)^* \) are the constant functions \( f = (dA/\mu) = c \) when \( M \) is simply connected.

Orbit and Stabilizer of Non-constant Element of \( \text{SVect}(M, \mu)^* \).

The stabilizer algebra \( \text{Stab}(f) \) or isotropy subalgebra of a function \( f \) is the set of all volume preserving vector fields \( u^i \) which leave \( f \) fixed under the coadjoint action. The coset space \( \text{SVect}(M, \mu)/\text{Stab}(f) \) then has the same dimension as the tangent space to the orbit containing \( f \). For \( u \) to be in \( \text{Stab}(f) \) we need \( \mathcal{L}_u f = 0 \). Since \( M \) is simply connected we can express \( u \) in terms of its stream function \( u^i = \rho^{-1} \varepsilon^{ij} \partial_j \chi \). The condition \( \mathcal{L}_u f = 0 \) becomes

\[
\varepsilon^{ij}(\partial_i f)(\partial_j \chi) = 0 \Rightarrow \chi_x f_y - \chi_y f_x = 0
\]

which in vector notation\(^7\) says that \( \nabla f \times \nabla \chi = 0 \). Colloquially, the gradient of \( \chi \) must be everywhere parallel to the gradient of \( f \). \( \chi = cf \) is clearly a solution for any real number \( c \), so the isotropy subalgebra is at least one dimensional, provided \( f \) is not a constant. The product of two solutions as well as real linear combinations of solutions are again solutions to this linear PDE. To better understand the general solution of this PDE, let us first consider the specific example of a function \( f(r, \theta) \) that is circularly symmetric.

Example: Stabilizer and Orbit of Circularly Symmetric Function

For example, let \( f = (dA/\mu) \) be a non-constant function depending only on the radial coordinate \( r = \sqrt{x^2 + y^2} \). Let us call its orbit by the name \( \mathcal{O}_f \). The gradient \( \nabla f \) points radially, so to

\(^7\)We emphasize that this equation is independent of any metric on \( M \).
speak. The isotropy subalgebra of any such function \( f \) is the space of stream functions \( \chi \) with \( \nabla f \times \nabla \chi = 0 \). The solutions are stream functions \( \chi(r) \) that are independent of \( \theta \)

\[
\text{Stab}(f) = \{ \chi(r, \theta) \mid \partial_\theta \chi = 0 \}
\] (53)

In this case, the isotropy subalgebra is infinite dimensional. The coset space

\[
\text{SVect}(M, \mu)/\text{Stab}(f) = \{ \psi(r, \theta) | \psi \sim \psi + \chi(r) \}
\] (54)

has the same dimension as the tangent space to the orbit \( T_f \mathcal{O}_f \) at \( f \). Though the stabilizer is infinite dimensional, the orbit of \( f(r) \) is infinite dimensional as well. For e.g. the coset space can be parameterized using an infinite number Fourier coefficients

\[
\psi(r, \theta) = \sum_{n=1}^{\infty} \psi_n^{(c)}(r) \cos n\theta + \sum_{n=1}^{\infty} \psi_n^{(s)}(r) \sin n\theta
\] (55)

We omitted the \( \theta \) independent additive term in order to quotient out by \( \text{Stab}(f) \).

To summarize, the isotropy subalgebra of a non-constant circularly symmetric function consists of all stream functions that are circularly symmetric. Moreover, the tangent space to the orbit may be identified with the coset space in which two stream functions are identified if they differ by one depending on \( r \) alone.

**Non-constant Functions have Infinite Dimensional Orbits**

Using the circularly symmetric example as a guide, we can characterize the isotropy subalgebra of a general non-constant function. The condition for \( \chi \) to be in the isotropy subalgebra is \( f_x \chi_y - f_y \chi_x = 0 \). We will describe the set of all solutions \( \chi \) to this linear PDE in a region of \( M \) where \( df \) is never the zero 2-form. Such a region is guaranteed to exist by an analogue of the inverse function theorem, since we assumed that \( f \) was not a constant. Our answer is that the isotropy subalgebra of \( f \) consists of all stream functions \( \chi \) which are constant along level sets of \( f \). The level sets of \( f \) are necessarily one dimensional curves in such a region.

To see this, we argue as follows. In a region where \( f(x, y) \) has no local extrema, the level sets of \( f \) are one dimensional. These level curves foliate the region. Pick a coordinate \( \Theta(x, y) \) along the level curves. Then \( \partial_\Theta f = f_\Theta = 0 \). Also pick a coordinate \( R \) ‘transversal’ to the level curves of \( f \). What this means is that \( \partial_R \) and \( \partial_\Theta \) should be linearly independent at each point \( (R, \Theta) \). There is a lot of arbitrariness in the choice of these coordinates, and one certainly does not require any metric to define them. The volume form \( \mu \) is non-degenerate, so the volume enclosed by the mini-parallelogram determined by the tangent vectors \( \partial_R \) and \( \partial_\Theta \) is non-zero. Denoting partial derivatives by subscripts, we get

\[
\mu(\partial_R, \partial_\Theta) = \rho dx \wedge dy(\partial_R, \partial_\Theta)
\]

\[
= \frac{1}{2} \rho(dx \otimes dy - dy \otimes dx) \left( \frac{1}{R_x} \frac{\partial}{\partial x} + \frac{1}{R_y} \frac{\partial}{\partial y} \right)
\]

\[
= \frac{1}{2} \rho \left( \frac{1}{R_x} - \frac{1}{R_y} \right) = \frac{1}{2} \rho \left( \frac{R_y \Theta_x - R_x \Theta_y}{R_x \Theta_y R_y \Theta_x} \right) \neq 0
\] (56)

Since \( \rho \) is non-vanishing, this means

\[
R_y \Theta_x - R_x \Theta_y \neq 0
\] (57)
The condition for $\chi$ to lie in the isotropy subalgebra becomes (using $f_\Theta = 0$)

$$f_x \chi_y - f_y \chi_x = (f_R R_x + f_\Theta \Theta_x) (\chi_R R_y + \chi_\Theta \Theta_y) - x \leftrightarrow y$$

$$= f_R R_x (\chi_R R_y + \chi_\Theta \Theta_y) - f_R R_y (\chi_R R_x + \chi_\Theta \Theta_x)$$

$$= f_R \chi_\Theta (R_x \Theta_y - R_y \Theta_x)$$

(58)

We have already shown that $R_x \Theta_y - R_y \Theta_x \neq 0$. Moreover, $f$ is non-constant and $f_\Theta = 0$ so we must have $f_R \neq 0$. We conclude that $\chi_\Theta = 0$. In other words, the stream functions $\chi$ in the isotropy subalgebra are constant on level sets of $f$.

$$\text{Stab}(f) = \{ \text{Stream functions } \chi \text{ constant on level sets of } f \}$$

(59)

It is clear that the space of such stream functions is closed under linear combinations and also under multiplication, as we observed earlier. The tangent space to the orbit at $f$ is then identified with the coset space

$$T_f \mathcal{O}_f = \text{SVect}(M, \mu)/\text{Stab}(f) = \{ \psi | \psi \sim \psi + \chi, \chi \text{ constant on level sets of } f \}$$

(60)

Both the stabilizer and the orbit are infinite dimensional, provided $f$ is not constant, for the same reason as given in the example where $f$ was circularly symmetric. Thus, non-constant $f = dA/\mu$ have infinite dimensional orbits.

5 Hamiltonian

Having specified observables (gauge-invariant functions), their p.b. and phase space (coadjoint orbit of SDiff$(M, \mu)$), we need to pick a gauge invariant hamiltonian.

5.1 Hamiltonian Leading to Ideal Hydrodynamics

There is more than one interesting way to pick a hamiltonian. The classic choice leading to ideal hydrodynamics, requires a positive metric $g_{ij}$ on $M$ which defines a positive, symmetric, inner product on $\mathcal{G} = \text{SVect}(M, \mu)$

$$\langle u, v \rangle_\mathcal{G} = \int_M g(u, v) \mu = \int_M g_{ij} u^i v^j \mu.$$ 

(61)

The hamiltonian of Eulerian hydrodynamics is then

$$H(u) = \frac{1}{2} \langle u, u \rangle_\mathcal{G} = \frac{1}{2} \int g_{ij} u^i v^j \rho \, d^2 x.$$ 

(62)

There is no a priori relation between the metric $g$ and the volume form $\mu$. $\Omega_g = \sqrt{\det g} dx^1 \wedge dx^2$ need not equal $\mu$. If $\mu = \Omega_g$, the theory is particularly natural as well as non-trivial. The hamiltonian defines an inertia operator $I$ (generalization of the inertia tensor of a rigid body, see Ref. [15]) from the Lie algebra $\mathcal{G} = \text{SVect}(M, \mu)$ to its dual $\mathcal{G}^* = \Omega^1(M)/d\Omega^0(M)$. Suppose $u, v$ are volume preserving vector fields. Then the inertia operator is defined by the equation $\langle Iu, v \rangle_\mathcal{G} = (u, v)_\mathcal{G}$. In other words,

$$\int (Iu)_j v^j \mu = \int g_{ij} u^i v^j \mu \Rightarrow \int ((Iu)_j - g_{ij} u^i) v^j \mu = 0$$

(63)

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Since this is true for an arbitrary volume preserving \( v \), \( (Iu_j - g_{ij}u^i) \) must be a total derivative: 
\[
(Iu_j - g_{ij}u^i) = \partial_j \Lambda
\]
for some scalar function \( \Lambda \) which vanishes on the boundary of \( M \) or at infinity. We see that the metric does not determine the 1-form \( Lu \) uniquely, but rather up to an exact 1-form. Thus, the image \( Lu \) of a volume preserving vector field \( u \in G \) is an equivalence class in \( G^* = \text{gauge fields modulo gauge transformations} \). A coset representative is given by the 1-form \( A_i = g_{ij}u^j \). The equation of motion is the well-known Euler equation
\[
\frac{\partial u}{\partial t} = -\nabla u - \nabla p \quad \mathcal{L}_u \mu = 0
\]
(64)
Here \( \nabla u \) is the covariant derivative (with respect to \( g_{ij} \)) of \( u \) along itself. The two equations fix the pressure \( p(x,t) \) up to an additive constant. The Euler equations of hydrodynamics have a geometric interpretation\cite{15}. The configuration space of a volume preserving fluid flowing on the manifold \( M \) is the group of volume preserving diffeomorphisms \( \text{SDiff}(M,\mu) \). The inner product (61) on the Lie algebra of this group can be extended to a right-invariant metric on the whole group by right translations by group elements. Then, by the least action principle, the time evolution of the fluid is given by geodesics on \( \text{SDiff}(M,\mu) \) with respect to this metric.

### 5.2 Hamiltonian as Magnetic Energy

Now we propose a hamiltonian different from that of Eulerian hydrodynamics. It is a magnetic energy, natural from the point of view of Yang-Mills theory. The result will still be a theory of geodesics on \( \text{SDiff}(M,\mu) \), but with respect to a different right-invariant metric on this group than that implied by ideal hydrodynamics. Suppose \( M \) is a 2-dimensional surface with volume form \( \mu \) and metric \( g_{ij} \). The metric did not play any role so far since the phase space and poisson structure are independent of it. But to specify the dynamics, we need the metric. Any two dimensional manifold is conformally flat, so the information in \( g_{ij} \) is encoded in its volume form \( \Omega_g = \sqrt{g} dx^1 \wedge dx^2 \), where \( g = \det g_{ij} \). We do not assume that \( \mu \) is equal to \( \Omega_g \). Indeed, if that is the case, the dynamics is trivial. By analogy with Yang-Mills theory, we pick the manifestly gauge-invariant magnetic energy
\[
H = \frac{1}{2} \int \left( \frac{F \wedge *F}{\Omega_g} \right) \mu
\]
(65)
as our hamiltonian. It can be written in a variety of equivalent ways
\[
H = \frac{1}{2} \int (F/\Omega_g)^2 \mu = \frac{1}{2} \int (*F)^2 \mu = \frac{1}{2} \int (F/\mu)^2 \sigma \mu = \frac{1}{4} \int F_{ij} F^{ij} \mu = \frac{1}{2} \int \left( \frac{B^2 \sigma}{\rho} \right) d^2 x.
\]
(66)
We find the last formula most useful in calculations. Here \( \sigma \) is the positive scalar function on \( M \) given by the square of the quotient of the two volume forms.
\[
\sigma = (\mu/\Omega_g)^2 = \rho^2 / g
\]
(67)
We will see that for the dynamics to be non-trivial, \( \sigma \) must not be constant. The hamiltonian in (66) is the simplest choice that is gauge-invariant, quadratic in gauge fields and involves two derivatives. It is easy to see the equivalence of the various formulae for the magnetic energy in...
Let \( \epsilon^{ij} = g^{ik} g^{jk} \epsilon_{kl}, \) \( \epsilon^{ij} = -\epsilon^{ji}, \) \( \epsilon^{12} = 1/g. \) Moreover, \( g \epsilon^{ij} = \epsilon^{ij}. \) The Hodge dual of \( F \) is the scalar
\[
\ast F = \ast(dx^i \wedge dx^j) \frac{1}{2} F_{ij} = \frac{1}{2} \sqrt{g} \epsilon^{ij} F_{ij} = \frac{1}{2} \sqrt{g} \sum_{ij} F_{ij} \epsilon_{ij} = \frac{B}{\sqrt{g}}
\]
(68)
\( F \wedge \ast F \) is the 2-form \( (B^2/\sqrt{g})dx^1 \wedge dx^2. \) \( ((F \wedge \ast F)/\Omega_g) = B^2/g \) is a scalar. Hence
\[
(F \wedge \ast F) / \Omega_g = (\ast F)^2 = (F/\Omega_g)^2 = \frac{B^2}{g} \sigma^2 = (F/\mu)^2 \sigma
\]
(69)
Moreover, \( F_{ij} F^{ij} = 2(F_{12} F^{12}). \) But \( F^{12} = g^{1i} g^{2j} F_{ij} = (g^{11} g^{22} - g^{12} g^{21}) F_{12} = g^{-1} F_{12} = B/g. \) Thus \( F_{ij} F^{ij} = 2B^2/g. \) This shows the equivalence of all the forms of the Hamiltonian given in (66).

The Hamiltonian (66) involves only the spatial components of the field strength since there is no time component for the gauge field in our theory. So there is no analogue of electric energy. Moreover, our theory is not relativistically covariant unlike electrodynamics. The dynamics determined by (66) is not equivalent to Eulerian hydrodynamics, though both theories share the same phase space. To see this, note that if \( \Omega_g = \mu, \) then (66) reduces to the Casimir \( I_2 \) and has trivial dynamics while the Hamiltonian (62) continues to have non-trivial dynamics.

The differential of the Hamiltonian (66) is (\( \frac{\partial H(A)}{\partial A_i(x)} = -\epsilon^{ij} \partial_j \delta^2(x - y) \))
\[
(dH(A))^i = \frac{\delta H}{\rho \delta A_i(x)} = \frac{1}{\rho} \epsilon^{ij} \partial_j (B \sigma / \rho)
\]
(70)
For each \( A, \) \( dH(A) \) is volume preserving \( \partial_i(\rho dH^i) = \epsilon^{ij} \partial_i \partial_j (B \sigma / \rho) = 0. \) \( H \) generates a 1-parameter (time) family of diffeomorphisms of a coadjoint orbit of \( \text{SDiff}(M, \mu), \) which serves as the phase space. The Hamiltonian vector field \( V_H \) is a vector field on a coadjoint orbit \( \mathcal{O}. \) At each tangent space \( T_{[A(x)]} \mathcal{O} \) to an orbit, \( V_H([A]) \) is given by the coadjoint action of the Lie algebra element \( dH(A), \) \( V_H = ad^*_H(A). \) The change in an observable \( f \) under such an infinitesimal canonical transformation is the Lie derivative with respect to \( V_H \)
\[
\frac{df(A)}{dt} = -\mathcal{L}_{V_H} f = ad^*_H f = \{H, f\} = (A, [dH, df]) = \int A_i [dH, df]^i \mu
\]
(71)
Explicit equations of motion in local coordinates are given in Sec. 6.

### 5.3 Inertia Operator and Inner Product on \( G^* \) from Hamiltonian

Recall (Sec. 5.1) that the Hamiltonian of ideal hydrodynamics (62) defines an inner product on the Lie algebra of volume preserving vector fields via the inertia operator. Here, the magnetic energy defines a positive inner product \( \langle ., . \rangle_{G^*} \) on the dual of the Lie algebra, \( G^* = \Omega^1(M)/d\Omega^0(M), \) via an inverse inertia operator \( h, \) obtained below. This inner product is degenerate if \( M \) has non-vanishing first cohomology. Away from degeneracies, one can in principle invert \( h \) to obtain an inertia operator and an inner product \( \langle ., . \rangle_G \) on the Lie algebra \( G = \text{SVect}(M, \mu). \) It should be possible to extended \( \langle ., . \rangle_G \) to a metric on the group \( G = \text{SDiff}(M, \mu) \) by means of right translations. The equations of motion in \( G^* = \text{SVect}(M, \mu)^* \)
that energy describes geodesics on \( \text{SDiff}(\Sigma) \) with respect to this right-invariant metric. In this sense, the dynamics determined by the magnetic energy describes geodesics on \( \text{SDiff}(\Sigma) \). Eulerian hydrodynamics also describes geodesics on \( \text{SDiff}(\Sigma) \), but with respect to a different right-invariant metric\(^{[15]} \). Here, we obtain the inverse of the inertia operator and the inner product on \( \text{SVec}(\Sigma) \) explicitly. Let us begin by writing the Hamiltonian as a quadratic form on gauge fields. The result is

\[
H = \frac{1}{2} \int A_i h^{ij} A_i \rho \, d^2x \quad \text{with} \quad h^{ij} = \frac{\varepsilon^{ij} \varepsilon_{kl}}{\rho} \left[ (\partial_j \sigma) \partial_k + \frac{\sigma}{\rho} \partial_j \partial_k \right] \quad (72)
\]

\( h^{ij} \) plays the role of inverse inertia operator, mapping equivalence classes of gauge fields to volume preserving vector fields by raising the index \( A_j \rightarrow h^{ij} A_j = (dH(A))^i \). To get \((72)\), note that \( H \) may be written in terms of its differential,

\[
H = \frac{1}{2} (A, dH(A)) = \frac{1}{2} \mu_{dH}(A) = \frac{1}{2} \int A_i (dH(A))^i \rho \, d^2x. \quad (73)
\]

This is easily checked by integrating by parts assuming \( B \rightarrow 0 \) on \( \partial M \):

\[
H = \frac{1}{2} \int (B^2 \sigma/\rho) d^2x = \frac{\varepsilon^{ij}}{2} \int (B \sigma/\rho) \partial_i A_j d^2x = \frac{\varepsilon^{ij}}{2} \int \partial_j (B \sigma/\rho) A_i d^2x
\]

\[
= \frac{1}{2} \int A_i dH^i \rho d^2x. \quad (74)
\]

As a consequence,

\[
H = \frac{1}{2} \int A_i (hA)^i \mu \quad \text{where} \quad (hA)^i = h^{ij} A_j = dH^i = \frac{\varepsilon^{ij}}{\rho} \partial_j (B \sigma/\rho) \quad (75)
\]

Writing this out we get,

\[
(hA)^i = \frac{\varepsilon^{ij} \varepsilon_{kl}}{\rho} \partial_j \left( \frac{\sigma}{\rho} \partial_k A_l \right) = \frac{\varepsilon^{ij} \varepsilon_{kl}}{\rho} \left[ (\partial_j (\sigma/\rho)) \partial_k + (\sigma/\rho) \partial_j \partial_k \right] A_l \quad (76)
\]

From this we read off the inverse of the inertia operator \((72)\). Through \( h \), the magnetic energy naturally defines a symmetric positive inner product on the dual of the Lie algebra, \( G^* = \Omega^1(M)/d\Omega^0(M) \). This inner product may be written in the following equivalent ways

\[
\langle A, \tilde{A} \rangle_{G^*} = (A, h \tilde{A}) = \mu_{h \tilde{A}}(A) = \int A_i h^{ij} \tilde{A}_j \rho \, d^2x = \int B \tilde{B} \left( \frac{\sigma}{\rho} \right) d^2x \quad (77)
\]

Here \( B, \tilde{B} \) are the magnetic fields corresponding to \( A, \tilde{A} \). Positivity of the inner product is ensured since \( \sigma(x) = (\mu/\Omega_g)^2 = (\rho^2/g) \geq 0 \) and the integrand is positive

\[
\langle A, A \rangle_{G^*} = 2H(A) = \int (B^2 \sigma/\rho) d^2x \geq 0 \quad (78)
\]

Symmetry of the inner product is shown by establishing the last equality in \((77)\):

\[
\int A_i (hA)^i \rho d^2x = \epsilon^{ij} \int A_i \partial_j (B \sigma/\rho) d^2x = -\epsilon^{ij} \int \partial_j A_i (B \sigma/\rho) d^2x = \int (B \tilde{B} \sigma/\rho) d^2x. \quad (79)
\]

Inner product \((77)\) is degenerate precisely on gauge fields in the first cohomology of \( M \), \( H^1(M) = \{ B = 0, A \neq dA \} \). If \( M \) is simply connected, \((77)\) is a non-degenerate inner product on \( G^* = \text{SVec}(M, \mu)^* \). Inverting it gives an inner product on \( G \) which may be extended by right translation to a right invariant metric on \( \text{SDiff}(M, \mu) \).
6 Equations of Motion

Before working out the equations of motion in detail, it would be prudent to convince ourselves that unlike the Casimir $I_2$, the magnetic energy $H$ does not lie in the center of the Poisson algebra for non-constant $\sigma$. We show this in $\text{(14)}$ by exhibiting a moment map that has non-vanishing p.b. with $H$.

6.1 Time Evolution of Magnetic Field

The time evolution of the magnetic field is given by the equation

$$\dot{B} = \{H, B\} = \varepsilon^{ij} \partial_i (B/\rho) \partial_j (B\sigma/\rho) = \nabla (B/\rho) \times \nabla (B\sigma/\rho)$$

(80)

where $\sigma = \rho^2 / g$. It is manifestly gauge-invariant. Though the hamiltonian is quadratic, (80) is non-linear since the p.b. of gauge fields (35) is linear in gauge fields. The ‘interactions’ are partly encoded in the Poisson structure and partly in the hamiltonian, so to speak. (80) is an analogue of the Euler equation $\dot{L} = L \times \Omega$ for the rigid body, (see $\text{(A)}$). Indeed, the equation of motion for the magnetic field can be regarded as the Euler equation for the group $\text{SDiff}(M, \mu)$ with respect to the metric $\text{(77)}$ defined by the hamiltonian (66). The time evolution of the magnetic field can also be written as an equation of motion for the field strength $F = dA$ defined in $\text{(21)}$

$$\dot{F} = \frac{1}{2} \varepsilon^{ij} \partial_i (F/\mu) \partial_j (F\sigma/\mu) \varepsilon_{kl} dx^k \wedge dx^l$$

(81)

The formula (80) for $\dot{B}$ is obtained using the p.b. formula (31) using the relation $\delta B(z) = -\varepsilon^{jk} \frac{\partial B(z-y)}{\partial y_j}$, assuming $A = 0$ on $\partial M$. After some integration by parts,

$$\frac{dB}{dt} = \{H, B\} = -\varepsilon^{jk} \varepsilon^{il} \partial_k \left[ \rho^{-1} (\partial_i A_j) + A_i (\partial_j \rho^{-1}) + \frac{1}{\rho} A_i \partial_j \right] \partial_l (B\sigma/\rho)$$

(82)

Expanding out derivatives of products, eliminating $A$ in favor of $B = \varepsilon^{ij} \partial_i A_j$ and after a lot of cancelations, one arrives at the advertised evolution equation.

6.2 Time Evolution of Moment Maps

We have already argued that the Casimirs $I_n = \int (dA/\mu)^n \mu$ are constant on symplectic leaves. Thus, they are conserved under time evolution $\frac{dI_n}{dt} = \{H, I_n\} = 0$, independent of the choice of hamiltonian. As for the moment maps $\mu_u$, we can show using $\text{(31)}$ that

$$\frac{d\mu_u(A)}{dt} = \{H, \mu_u\} = \int d^2 x A_i \left[ \varepsilon^{jk} (\partial_j u^i) + \varepsilon^{ik} w^j \rho^{-1} (\partial_j \rho) - \varepsilon^{ik} w^j \partial_j \right] \partial_k (B\sigma/\rho)$$

(83)

For $\rho = 1$ this reduces to

$$\frac{d\mu_u(A)}{dt} = \int d^2 x A_i \left[ \varepsilon^{jk} (\partial_j u^i) - \varepsilon^{ik} w^j \partial_j \right] \partial_k (B\sigma)$$

(84)

The rhs is gauge-invariant even though the gauge field appears explicitly.
6.3 Infinitely Many Conserved Charges in Involution

We find an infinite set of conserved charges

$$\frac{dH_n}{dt} = \{H, H_n\} = 0; \quad n = 1, 2, 3, \ldots$$

(85)

for a uniform $\mu (\rho = 1)$ and $g_{ij}$ an arbitrary metric\(^8\). We suspect that a similar result holds for non-uniform $\mu$. The conserved quantities are

$$H_n = \int (dA/\mu)^n \sigma \mu = \int B^n \sigma d^2 x,$$

(86)

where $\sigma = (\mu/\Omega g)^2 = (1/g)$ and $g = \det g_{ij}$. Furthermore, we find that $H_n$ are in involution $\{H_m, H_n\} = 0$. The hamiltonian is $H = \frac{1}{2} H_2$. The presence of infinitely many conserved charges in involution suggests an integrable structure underlying the dynamics determined by the magnetic energy. The hamiltonian of ideal hydrodynamics\(^{[02]}\) is not known nor expected to have any conserved quantities besides the $I_n$ and their close relatives, which are constant on coadjoint orbits. What is remarkable about $H_n$ is that they are not constant on coadjoint orbits but still conserved quantities for the magnetic energy hamiltonian\(^{[03]}\). To establish this we investigate the time evolution of $H_n$.

$$\frac{dH_n}{dt} = \{H, H_n\} = \int A_i \left[ \frac{\delta H}{\delta A_j} \partial_j \left( \rho^{-1} \frac{\delta H_n}{\delta A_i} \right) - \frac{\delta H_n}{\delta A_j} \partial_j \left( \rho^{-1} \frac{\delta H}{\delta A_i} \right) \right] d^2 x$$

(87)

Using

$$\frac{\delta H_n}{\delta A_i} = n \varepsilon^{ij} \partial_j \left( \sigma(B/\rho)^{n-1} \right); \quad \frac{\delta H}{\delta A_i} = \varepsilon^{ij} \partial_j \left( \sigma B/\rho \right)$$

(88)

We get

$$\frac{dH_n}{dt} = n \varepsilon^{jk} \varepsilon^{il} \int A_i \left[ \partial_k (\sigma B/\rho) \partial_j (\rho^{-1} \partial_l (\sigma(B/\rho)^{n-1})) - \partial_k (\sigma(B/\rho)^{n-1}) \partial_j (\rho^{-1} \partial_l (\sigma B/\rho)) \right] d^2 x$$

(89)

For $\rho = 1$ this becomes

$$\frac{dH_n}{dt} = n \varepsilon^{jk} \varepsilon^{il} \int A_i \left[ \partial_k (\sigma B) \partial_j (\sigma B^{n-1}) - \partial_k (\sigma B^{n-1}) \partial_j (\sigma B) \right] d^2 x$$

(90)

In the gauge $A_1 = 0$ , $A_2 \equiv A$, $B = \partial_x A$ this may be written as

$$\frac{dH_n}{dt} = n \int dxdy A \left[ \partial_x (B\sigma) \partial_{x^2} (B^{n-1} \sigma) - \partial_x (B^{n-1} \sigma) \partial_{x^2} (B\sigma) + \partial_y (B\sigma) \partial_x^2 (B^{n-1} \sigma) + \partial_y (B^{n-1} \sigma) \partial_x^2 (B\sigma) \right]$$

(91)

We prove in\(^{[04]}\) that $H_n$, $n = 1, 2, 3, \ldots$ are conserved by showing that the rhs of (91) vanishes. Having found an infinite number of conserved quantities we wanted to know whether their p.b.

\(^8\) $\sigma = \rho^2/g$ must not grow too fast at $\infty$ and the magnetic field must vanish at $\infty$. 

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generates new conserved quantities. In our case, we discovered (again for \( \rho = 1 \)), that the conserved quantities \( H_n \) are in involution, i.e. they mutually Poisson commute \( \{H_m, H_n\} = 0 \). The proof of this is somewhat lengthy and is relegated to E.

Intuitively, \( H_n \) are independent of each other since they are like average values of different powers of \( B \). Furthermore, \( H_n \) are independent of the Casimirs \( I_m \). \( I_m \) contain no information about the metric \( g_{ij} \) while \( H_n \) depend on the metric via \( \sigma \). For example, we show in D that \( H_2 \) is not a Casimir. More generally, it would be nice to prove that \( I_n \) and \( H_m \) are functionally independent by showing that on every tangent space to an orbit, the cross product of their gradients is non-vanishing.

### 7 Static Solutions

#### 7.1 Zero Energy Configurations

The magnetic energy is \( H = \int_M (dA/\mu)^2 \sigma \mu = f(B^2/\rho)\sigma d^2x \) where \( \sigma = (\mu/\Omega_g)^2 = (\rho^2/g) \geq 0 \). Thus, the energy is non-negative, \( H \geq 0 \). Moreover, if \( A \) is closed, the energy automatically is a global minimum \( dA = 0 \Rightarrow H = 0 \). Moreover, since \( H \) is the integral of the square of \( dA \), weighted by a positive function, closed gauge fields are the only configurations with zero energy. Any such closed gauge field is a static solution to the equations of motion (irrespective of \( \mu \) and \( g_{ij} \))

\[
\frac{df}{dt}_{dA=0} = \{H, f\}_{dA=0} = 0
\]

As discussed in Sec. 4.3, the closed gauge fields constitute one or more (according as \( M \) is simply connected or not) symplectic leaves of the Poisson manifold. They correspond to the zero set of Casimirs \( I_n = \int (dA/\mu)^n \mu = 0 \). Thus, the hamiltonian vanishes on all the symplectic leaves with \( I_n = 0 \) and there is no interesting dynamics to speak of. If \( M \) is the plane, then the pure gauges are the only ones with zero energy. The leaf/leaves with \( I_n = 0 \) are the analogue of the \( L^2 = 0 \) point at the origin \( L_i = 0 \) of the angular momentum Poisson manifold. At that point, the energy \( E = \sum_i L_i^2/2I_i \) of the rigid body vanishes as well.

It is interesting to find minima of energy on more interesting symplectic leaves. Initial conditions determine which symplectic leaf is the phase space of the theory. The general problem of finding the minimum of energy on a given symplectic leaf (perhaps specified through values of invariants such as \( I_n \)) is potentially quite interesting and difficult. One would first have to find which gauge fields or magnetic field configurations satisfy the constraints and lie on the specified leaf. This is similar to the problem we solved in the large \( N \) limit of 2d QCD where we found the minimum of energy on the symplectic leaf with baryon number equal to one \([11, 10]\). Here we do the opposite, find a few static solutions and then determine which orbit they lie on.

#### 7.2 Circularly Symmetric Static Solutions

Suppose \( M = \mathbb{R}^2 \). If both the volume forms \( \mu \) and \( \Omega_g \) are circularly symmetric, then any circularly symmetric initial magnetic field will remain unchanged with time. To see this, note that \( \sigma = \rho^2/g \) depends only on the radial coordinate \( r \). Suppose that at \( t = 0 \), \( B(r,t=0) \)
depends only on \( r \). The initial value problem for \( B \) given in (80) is
\[
\dot{B} = \nabla (B/\rho) \times \nabla (B\sigma/\rho)
\] (93)

Due to circular symmetry, both the gradients point radially and their cross product vanishes. Thus \( \dot{B} = 0 \) and we have a static solution \( B(r) \)!

We already met the \( B(r) = 0 \) static solution before, it lies on a one point symplectic leaf of pure gauge configurations. However, the static solutions corresponding to non-constant \( B(r) \) lie on infinite dimensional symplectic leaves which we found previously (Sec. 4.4). The values of Casimirs on the orbit containing a circularly symmetric static solution \( B(r) \) are
\[
I_n = 2\pi \int_0^{\infty} (B(r)/\rho(r))^n \rho(r) r \, dr
\] (94)

By a judicious choice\(^9\) of \( B(r) \) one should be able to find a static solution \( B(r) \) for given \( \rho(r) \) that lies on an orbit with practically any desired value for the invariants \( I_n \). More generally, by an argument similar to the one given in Sec. 4.4 we see that magnetic fields for which \( (B/\rho) \) and \( (B\sigma/\rho) \) have common 1-dimensional level sets, are static solutions of (80).

### 7.3 Some Other Local Extrema of Energy

For a gauge field to be an extremum of energy on a given symplectic leaf, the variation of energy in directions tangential to the leaf must vanish. There is no need for variations in other, let alone all, directions to vanish. However, though it is not necessary, if \([A]\) is such that all variations of the hamiltonian \( H(A) \) vanish, then, \( A \) must be a local extremum of energy. Such an extremum \( \frac{\delta H}{\delta A_k} = 0 \) is automatically a static solution to the equations of motion for any gauge-invariant observable \( f \)
\[
\frac{df}{dt} = \{H, f\} = \int d^2 x d^2 y \{A_i(x), A_j(y)\} \frac{\delta H}{\delta A_i(x)} \frac{\delta f}{\delta A_j(y)} = 0
\] (95)

These extrema are given by solutions of
\[
\frac{\delta H}{\delta A_k} = 2\varepsilon^{i3} \partial_j (B\sigma/\rho) = 0 \quad \Rightarrow \quad \partial_1 (B\sigma/\rho) = 0 \text{ and } \partial_2 (B\sigma/\rho) = 0
\] (96)

The only solutions are \( B = (c\rho/\sigma) = (cg/\rho) \) where \( c \) is a constant and \( g = \det g_{ij} \). These extrema lie on leaves where the Casimirs take the values
\[
I_n = c^n \int \left( \frac{\Omega g}{\mu} \right)^{2n} \mu = c^n \int \sigma^{-n} \rho \, d^2 x
\] (97)

We do not yet understand the physical meaning of these extrema of energy. They have a finite energy if \( B = (c\rho/\sigma) \) vanishes at infinity sufficiently fast.

\(^9\)Finding \( B(r) \) for given \( I_n, \rho(r) \) is similar to the Classical Moment Problem.
8 Discussion

A summary of the paper was given in the introduction Sec. 1. Here, we mention a few directions for further study. We would like to know whether there is a deeper integrable structure that would explain the presence of an infinite number of conserved charges in involution for the uniform volume measure $\mu$. The extension to an arbitrary volume form seems likely. Are there any time-dependent exact solutions of the non-linear evolution equation? What is the Poisson algebra of loop observables and can the Hamiltonian be written in terms of them? Can this model be quantized and is there a non-abelian extension? An extension to $3 + 1$ dimensions is possible, though the Poisson algebra has only one analytically known Casimir, the Hopf or link invariant. Is there a Lorentz covariant theory along these lines? Can the idea that gauge fields be thought of as dual to volume preserving vector fields be exploited in any other context?

How is our gauge theory related to hydrodynamics and turbulence? Recently, Jackiw, et. al. [28] have studied perfect fluids and certain non-abelian extensions. Our gauge theory shares the same phase space as ideal hydrodynamics, but the two theories have different hamiltonians. However, similar methods may be useful in the study of both theories. For example, Iyer and Rajeev [29] (see also Sec. 11.D of Ref. [15]) have proposed a statistical approach to 2 dimensional turbulence, based on a matrix regularization of the phase space. It may be possible to use a similar regularization for our gauge theory. For an $N \times N$ matrix regularization to be integrable, it would appear that we need $O(N^2)$ conserved quantities, while $I_n$ and $H_m$ furnish only $2N$ conserved quantities. It is unclear what this implies for the integrability of the continuum theory we have proposed in this paper. On the other hand, Polyakov has proposed a theory of turbulence in $2 + 1$ dimensions based on conformal invariance [30]. Since the group of conformal transformations and area preserving transformations are disjoint except for isometries, it appears unlikely that there is any direct relation of our work to Polyakov’s.

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A Poisson Manifolds and Coadjoint Orbits

We collect a few facts about classical mechanics [31, 15] that we use, to make the paper self-contained and fix notation.

A.1 Poisson Manifolds and Symplectic Leaves

The basic playground of classical mechanics is a Poisson manifold. It is a manifold $M$ with a product $\{\cdot,\cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ (the Poisson bracket p.b.) on the algebra of observables. $\{\cdot,\cdot\}$ is bilinear, skew symmetric and satisfies the Jacobi identity and Leibnitz rule. $\mathcal{F}(M)$ is a class of real-valued functions (say, $C^\infty(M)$). Any such function $f : M \to \mathbb{R}$ generates canonical transformations on $M$. A canonical transformation is a flow on $M$, associated to the
canonical vector field \( V_f \). The Lie derivative of any function \( g(A) \) along the flow is given by the p.b. \( \mathcal{L}_f g(A) = \{f, g\}(A), \ A \in M \). Flow lines of the canonical transformation generated by \( f \) are integral curves of \( V_f \).

Often, Poisson algebras of observables are degenerate. They have a center (Casimirs) which have zero p.b. with all observables. In such a situation, the Poisson manifold as a whole cannot serve as the phase space of a physical system, since it would not be a symplectic manifold. Rather, it is the symplectic leaves of a Poisson manifold that can serve as phase spaces. The symplectic leaf of a point \( A \in M \) is the set of all points of \( M \) reachable from \( A \) along integral curves of canonical vector fields. On a symplectic leaf, the Poisson structure is non-degenerate and can be inverted to define a symplectic structure, a non-degenerate closed 2-form \( \omega \). Indeed, if \( \xi, \eta \) are tangent vectors at \( A \) to a symplectic leaf, then the symplectic form at \( A \) is \( \omega(\xi, \eta) = \{f, g\}(A) \) where \( f \) and \( g \) are any two functions whose canonical vector fields at \( A \) coincide with \( \xi \) and \( \eta \); i.e. \( \xi = V_f|_A, \eta = V_g|_A \). Moreover, on any symplectic leaf, \( \omega(V_f,. ) = df(.) \) where \( df \) is the exterior derivative of \( f \).

The Hamiltonian vector field \( V_H \) is the canonical vector field of the Hamiltonian \( H : M \to \mathbb{R} \). \( V_H \) generates time evolution \( \frac{df}{dt} = \{H, f\} = \mathcal{L}_{V_H} f \). Irrespective of the Hamiltonian, time evolution always stays on the same symplectic leaf.

A.2 Coadjoint Orbits in the Dual of a Lie Algebra

A natural example of a Poisson manifold, that occurs in many areas of physics, is the dual of a Lie algebra. The symplectic leaves of the dual of a Lie algebra are the coadjoint orbits of the group. To understand this, suppose \( G \) is a group, \( \mathcal{G} \) its Lie algebra and \( \mathcal{G}^* \) the dual of the Lie algebra. Then we have a bilinear pairing between dual spaces \( (A, u) \in \mathbb{R} \) for \( A \in \mathcal{G}^* \) and \( u \in \mathcal{G} \). Suppose \( f, g \) are two real-valued functions on \( \mathcal{G}^* \). Then their p.b. is defined using the differential \( df(A) \in \mathcal{G}^* \)

\[
\{f, g\}(A) = \langle A, [df, dg]\rangle
\]  

(98)

\([df, dg]\) is the commutator in \( \mathcal{G} \). This turns \( \mathcal{G}^* \) into a Poisson manifold, which is often degenerate. The symplectic leaves are coadjoint orbits of the action of \( G \) on \( \mathcal{G}^* \). To see this, first define the inner automorphism \( A_g : G \to G, \ A_g h = ghg^{-1} \) which takes the group identity \( e \) to itself. The group adjoint representation \( Ad_g : G \to G \) is the linearization of the inner automorphism at \( e : Ad_{g e} = A_{g e} |_e \) and is \( Ad_g u = gug^{-1} \) for matrix groups. The group coadjoint representation \( Ad_g^* : \mathcal{G}^* \to \mathcal{G}^* \) is defined as \( (Ad_g^* A, u) = (A, Ad_g u) \). The group coadjoint orbit of \( A \in \mathcal{G}^* \) is

\[
\mathcal{O}_A = \{Ad_g^* A \mid g \in G\}
\]  

(99)

The Lie algebra adjoint representation is \( ad_u : \mathcal{G} \to \mathcal{G} \) where \( ad_u = \frac{df}{dt}|_{t=0} Ad_{g(t)} \) for a curve \( g(t) \) on the group with \( g(0) = e \) and \( \dot{g}(0) = u \) and takes the form \( ad_u v = [u, v] \) for matrix groups. The Lie algebra coadjoint representation \( ad_u^* : \mathcal{G}^* \to \mathcal{G}^* \) is defined by \( (ad_u^* A, v) = (A, ad_u v) \). The Lie algebra coadjoint orbit of \( A \in \mathcal{G}^* \) is the tangent space to \( A \) to the group coadjoint orbit of \( A \).

\[
\{ad_u A \mid u \in \mathcal{G}\} = T_A \mathcal{O}_A
\]  

(100)

Thus, a tangent vector \( \xi \) to a coadjoint orbit at \( A \) may be written as \( \xi = ad_u A \) for some (not necessarily unique) \( u \in \mathcal{G} \).
There is a natural symplectic structure on coadjoint orbits, which turns them into homogeneous symplectic leaves of $G^*$. The symplectic form (Kirillov form) $\omega$ acting on a pair of tangent vectors to the orbit $O_A$ at $A$ is given by $\omega(\xi, \eta) = (A, [u, v])$ where $u, v \in G$ are any two Lie algebra elements such that $\xi = \ad^*_u A$ and $\eta = \ad^*_v A$. Thus, the coadjoint orbits are symplectic manifolds which foliate $G^*$ in such a way as to recover the Poisson structure on the whole of $G^*$. The different coadjoint orbits in $G^*$ are not necessarily of the same dimension, but are always even dimensional if their dimension is finite.

The canonical vector field $V_f$ of an observable $f : G^* \to \mathbb{R}$ at a point $A \in G^*$ is given by the Lie algebra coadjoint action of the differential $df(A)$, $V_f(A) = \ad^*_A df(A)$.

\[
\mathcal{L}_{V_f} g(A) = \{f, g\}(A) = (A, [df, dg]) = (A, \ad^*_A dg) = (\ad^*_A A, dg)
\]

In particular, infinitesimal time evolution is just the coadjoint action of the Lie algebra element $dH$, the differential of the Hamiltonian. For example, if $\mu_u(A) = (A, u)$ is the moment map for $u \in G$, then the canonical vector field $V_{\mu_u}(A) = \ad^*_A A$. In other words, infinitesimal canonical transformations generated by moment maps are the same as Lie algebra coadjoint actions.

Observables in the center of the Poisson algebra (Casimirs) are constant on coadjoint orbits. They are invariant under the group and Lie algebra coadjoint actions. To show that an observable is a Casimir, it suffices to check that it commutes with the moment maps which generate the coadjoint action.

**Example: Eulerian Rigid Body** Let $\Omega^i$ be the components of angular velocity of a rigid body in the co-rotating frame. The components of angular velocity lie in the Lie algebra $\mathfrak{g}$ of the rotation group $G = SO(3)$. The dual space to angular velocities consists of angular momenta $L_i$ with the pairing (or moment map) $(L, \Omega) = L_i \Omega^i$. The space of angular momenta is the dual $SO(3)^* \cong \mathbb{R}^3$. The latter carries a Poisson structure $\{L_i, L_j\} = \delta_{ij}$. Observables are real-valued functions of angular momentum $f(L)$ and satisfy the p.b.

\[
\{f, g\}(L) = \sum_{i,j} \{L_i, L_j\} \frac{\partial f}{\partial L_i} \frac{\partial g}{\partial L_j}
\]

The space of angular momenta $\mathbb{R}^3$ must be a degenerate Poisson manifold, since it is not even dimensional. Indeed, the symplectic leaves are concentric spheres centered at the point $L_i = 0$ as well as the point $L_i = 0$. These symplectic leaves are the coadjoint orbits of $SO(3)$ acting on the space $\mathbb{R}^3$ of angular momenta. The symplectic form on a sphere of radius $r$ is given by $r \sin \theta d\theta \wedge d\phi$. The Casimirs are functions of $L^2 = \sum_i L_i^2$ and are constant on the symplectic leaves, they are invariant under the coadjoint action of the rotation group on $\mathbb{R}^3$. The Hamiltonian is $H = \sum_i L_i^2/2I_i$ if the axes are chosen along the principle axes of inertia. $I_i$ are the principle moments of inertia, the eigenvalues of the inertia operator $I_{ij} : \mathbb{R} \to \mathbb{R}^3$ which maps angular velocities to angular momenta $L_i = I_{ij} \Omega^j$. The equations of motion $\dot{L} = \ad^*_L L = \ad^*_I L = \ad^*_L L = L \times \Omega$ are $\dot{L}_i = \{H, L_i\}$, $\dot{L}_1 = a_{23} L_2 L_3$ and cyclic permutations thereof, where $a_{ij} = I^{-1}_j - I^{-1}_i$.

**B The Charges $I_n$ are in Involution**

We show that the charges $I_n$ are in involution $\{I_m, I_n\} = 0$. To calculate

\[
\{I_m, I_n\} = \int_M A_i \left[ \frac{\delta I_m}{\delta A_j} \partial_j \left( \frac{1}{\rho} \frac{\delta I_n}{\delta A_i} \right) - \frac{\delta I_n}{\delta A_j} \partial_j \left( \frac{1}{\rho} \frac{\delta I_m}{\delta A_i} \right) \right] d^2 x
\]

(103)
Therefore we get

$$I_m = \int (B/\rho)^m \rho d^2 x \Rightarrow \frac{\delta I_m}{\delta A_i} = m \varepsilon^{ij} \partial_j ((B/\rho)^{m-1})$$

(104)

So the p.b. becomes

$$\{I_m, I_n\} = mn \varepsilon^{ij} \varepsilon^{jk} \int_M d^2 x A_i \left[ \partial_k (B/\rho)^{m-1} \partial_j \left( \frac{1}{\rho} \partial_l (B/\rho)^{n-1} \right) - m \leftrightarrow n \right]$$

(105)

The first term in square brackets $\partial_k (B/\rho)^{m-1} \partial_j \left( \frac{1}{\rho} \partial_l (B/\rho)^{n-1} \right)$ can be written as

$$= (m-1)(n-1)(B/\rho)^{m-2} \partial_k (B/\rho) \partial_j \left\{ \rho^{-1}(B/\rho)^{n-2} \partial_l (B/\rho) \right\}$$

$$= (m-1)(n-1)(B/\rho)^{m+n-4} \partial_j (\rho^{-1}) \partial_k (B/\rho) \partial_l (B/\rho)$$

$$+ (m-1)(n-1)(n-2)(B/\rho)^{m+n-5} \partial_k (B/\rho) \partial_j (B/\rho) \partial_l (B/\rho)$$

$$+ (m-1)(n-1)(B/\rho)^{m+n-4} \rho^{-1} \partial_k (B/\rho) \partial_j \partial_l (B/\rho)$$

(106)

The 1st and 3rd terms are symmetric under $m \leftrightarrow n$ and therefore do not contribute to $\{I_m, I_n\}$. Therefore we get ($I_1 = 0$, so we can ignore $m, n = 1$)

$$\frac{\{I_m, I_n\}}{mn(m-1)(n-1)(n-m)} = \int d^2 x \frac{(B/\rho)^{m+n-5}}{\rho} \left[ \varepsilon^{ij} \varepsilon^{jk} A_i \partial_l (B/\rho) \partial_j (B/\rho) \partial_k (B/\rho) \right]$$

Now the term in square brackets vanishes identically due to antisymmetry of $\varepsilon^{jk}$. We conclude that $\{I_m, I_n\} = 0$. Thus, $I_n, n = 1, 2, 3 \ldots$ are an infinite number of charges in involution.

**C I_2 is a Casimir of the Poisson Algebra**

To find the infinitesimal change of $I_n$ under the coadjoint action, we calculate $\{I_n, \mu_u\}$. If this vanishes for all volume preserving $u$, then $I_n$ would be constant on symplectic leaves and hence a Casimir. Recall (28) that the differential of $I_n$ is

$$\left( dI_n(A) \right)^i = n \rho^{-1} \varepsilon^{ij} \partial_j (B/\rho)^{n-1}$$

(107)

and the differential of the moment map is $(d\mu_u)^i = u^i$. Their Lie bracket is

$$[u, dI_n]^i = u^j \partial_j (n \rho^{-1} \varepsilon^{ik} \partial_k (B/\rho)^{n-1}) - n \rho^{-1} \varepsilon^{ik} \partial_k (B/\rho)^{n-1} \partial_j u^i$$

$$= n \left[ \varepsilon^{ik} u^j \partial_j (\rho^{-1} \partial_k (B/\rho)^{n-1}) - \rho^{-1} \varepsilon^{ik} \partial_k (B/\rho)^{n-1} \partial_j u^i \right]$$

(108)

Thus $\{\mu_u, I_n\} = \int A_i [u, dI_n] \rho d^2 x$ gives

$$\{\mu_u, I_n\} = n \int d^2 x A_i \left[ \varepsilon^{ik} u^j \partial_j (\rho^{-1} \partial_k (B/\rho)^{n-1}) - \varepsilon^{ik} \partial_k (B/\rho)^{n-1} \partial_j u^i \right].$$

(109)

For $\rho = 1$ this becomes

$$\{\mu_u, I_n\} = n \int d^2 x A_i \left[ \varepsilon^{ik} u^j (\partial_k \partial_j B^{n-1}) - \varepsilon^{jk} \partial_j (B/\rho)^{n-1} \right]$$

(110)
Specializing to \( n = 2 \), and writing \( u \) in terms of its stream function, \( u^i = \varepsilon^{ij} \partial_j \psi \),

\[
\{I_2, \mu_u\} = 2 \int d^2x \left[ \varepsilon^{il} \varepsilon^{jk} A_i(\partial_l \partial_j \psi) (\partial_k B) - \varepsilon^{ik} \varepsilon^{jl} A_i(\partial_k \partial_j B)(\partial_l \psi) \right] \tag{111}
\]

\( \{I_2, \mu_u\} \) is gauge-invariant, so we calculate it in the gauge \( A_1 = 0 \), \( B = \partial_1 A_2 \) and denote \( A_2 = A \), \( x^1 = x \), \( x^2 = y \) and derivatives by subscripts.

\[
\frac{1}{2} \{I_2, \mu_u\} = \int dx dy A \left[ \psi_y A_{xxx} - \psi_x A_{xyy} + \psi_y A_{xx} - \psi_{xx} A_{xy} \right] \tag{112}
\]

The idea is to integrate by parts and show that this expression vanishes. Let us temporarily call the second factor of \( A \) by the name \( \tilde{A} \). Write \( \frac{1}{2} \{I_2, \mu_u\} \) as a sum of four terms \( T_1 + T_2 + T_3 + T_4 \)

\[
T_1 = \int dx dy A \psi_y \tilde{A}_{xxx} ; \quad T_2 = - \int dx dy A \psi_x \tilde{A}_{xyy} \\
T_3 = \int dx dy A \psi_{xy} \tilde{A}_{xx} ; \quad T_4 = - \int dx dy A \psi_{xx} \tilde{A}_{xy} . \tag{113}
\]

We will show that \( T_1 + T_2 = - T_3 - T_4 \). Integrating by parts till there are no derivatives on \( \tilde{A} \),

\[
T_1 = - \int dx dy \tilde{A} \left( \psi_y A_{xxx} + 3A_{xx} \psi_{xy} + 3A_x \psi_{xyy} + A \psi_{xxx} \right) \\
T_2 = \int dx dy \tilde{A} \left( \psi_x A_{xyy} + 2A_{xy} \psi_{xy} + 2A_x \psi_{xyy} + A_{xx} \psi_{xy} + A_y \psi_{xxx} + A \psi_{xxx} \right) \\
T_3 = \int dx dy \tilde{A} \left( \psi_{xy} A_{xx} + 2A_x \psi_{xyy} + A \psi_{xyy} \right) \\
T_4 = - \int dx dy \tilde{A} \left( A_{xy} \psi_{xx} + A_y \psi_{xxx} + A_x \psi_{xy} + A \psi_{xxx} \right) . \tag{114}
\]

Using the fact that \( \tilde{A} = A \) we get for \( T_1 \) and \( T_2 \)

\[
2 \times T_1 = - \int dx dy \tilde{A} \left( 3A_{xx} \psi_{xy} + 3A_x \psi_{xyy} + A \psi_{xxx} \right) \\
2 \times T_2 = \int dx dy \tilde{A} \left( 2A_{xy} \psi_{xy} + 2A_x \psi_{xyy} + A_{xx} \psi_{xy} + A_y \psi_{xxx} + A \psi_{xxx} \right) \tag{115}
\]

\( T_3 \) and \( T_4 \) give us the identities

\[
\int dx dy \tilde{A} A_x \psi_{xyy} = - \int dx dy \tilde{A} A \psi_{xxx} \\
\int dx dy \tilde{A} A_y \psi_{xxx} = \int dx dy \tilde{A} A_x \psi_{xyy} \tag{116}
\]

Use these to simplify \( T_1 \) and \( T_2 \) by eliminating \( \psi_{xxx} \) and \( \psi_{xxx} \) in favor of \( \psi_{xy} \):

\[
2 \times T_1 = - \int dx dy \tilde{A} \left( 3A_{xx} \psi_{xy} + A_x \psi_{xyy} \right) \\
2 \times T_2 = \int dx dy \tilde{A} \left( 2A_{xy} \psi_{xx} + A_{xx} \psi_{xy} + A_x \psi_{xyy} \right) \tag{117}
\]

Adding these and setting \( A = \tilde{A} \)

\[
T_1 + T_2 = \int dx dy A \left( A_{xy} \psi_{xx} - A_{xx} \psi_{xy} \right) \tag{118}
\]
Meanwhile by definition,

\[ T_3 + T_4 = \int dxdy A \left( A_{xx} \psi_{xy} - A_{xy} \psi_{xx} \right) \]

Thus \( \{ I_2, \mu_u \} = 2(T_1 + T_2 + T_3 + T_4) = 0 \) (119)

We conclude that \( I_2 \) lies in the center of the Poisson algebra for constant \( \rho \).

\section*{D \quad \boldsymbol{H} = \frac{1}{2} \int (dA/\mu)^2 \sigma \mu \text{ is not a Casimir}}

\( I_2 = \int (dA/\mu)^2 \mu \) turned out to be in the center of the Poisson algebra of gauge-invariant functions. Here we show that the magnetic energy \( \boldsymbol{H} = \frac{1}{2} \int (dA/\mu)^2 \sigma \mu \) with a non-constant \( \sigma = \rho^2/g \), does \textit{not} lie in the center, and therefore leads to non-trivial time evolution. We do this by giving an explicit example of a gauge-invariant function with which it has a non-vanishing p.b. Consider \( \rho = 1 \), then from (83)

\[ \{ H, \mu_u \} = \int d^2x A \left[ \varepsilon^{ijk} (\partial_j u^i) \partial_k (B \sigma) - \varepsilon^{ijk} u^i (\partial_j \partial_k B \sigma) \right] \] (120)

In gauge \( A_1 = 0, \ A_2 \equiv A \) and with \( u^i = \varepsilon^{ij} \partial_j \psi \) we get \( (x^1 = x, x^2 = y) \)

\[ \{ H, \mu_u \} = \int dxdy A \left[ (\partial_x \partial_y \psi) \partial_x (B \sigma) \right. \] 
\[ \left. + (\partial_y \psi) \partial_y (B \sigma) \right) \] (121)

Now for the simple choices \( \psi = xy, \ A = \sigma = e^{-(x^2+y^2)/2} \) we have \( B = \partial_x A = -xe^{-(x^2+y^2)/2} \).

The p.b. can be calculated exactly to yield

\[ \{ H, \mu_u \} = \int dxdye^{-3(x^2+y^2)/2} \left( 1 - 4x^4 - 2y^2 + 4x^2(2 + y^2) \right) = \frac{2\pi}{27} \] (122)

Thus \( H \) \textit{does not} lie in the center of the Poisson algebra. We also checked that \( H \) transforms non-trivially under many other generators \( \mu_u \) of the coadjoint action.

\section*{E \quad \boldsymbol{H}_n \text{ are in Involution for Uniform Measure}}

Suppose \( \mu \) is the uniform measure \( (\rho = 1) \). We prove that the charges

\[ H_n = \int (dA/\mu)^n \sigma \mu = \int (B/\rho)^n \sigma \rho d^2x = \int B^n \sigma d^2x \] (123)

are in involution

\[ \{ H_m, H_n \} = 0 \quad \text{for} \quad m, n = 0, 1, 2, 3, \ldots \] (124)

An immediate corollary is that \( H_n \) are conserved quantities since the hamiltonian is \( \frac{1}{2} H_2 \). Here \( B \to 0 \) at \( \infty \) and \( \sigma = (\mu/\Omega_g)^2 = 1/g \) must be such that these integrals converge. The proof involves explicitly computing the p.b. and integrating by parts several times. Using

\[ \frac{\delta H_n}{\delta A_i} = \delta^i H_n = n \varepsilon^{ij} \partial_j (\sigma (B/\rho)^{n-1}) \] (125)
After collecting terms, this becomes

\[ \{H_m, H_n\} = \int A_i \left[ \delta^i H_m \partial_j (\rho^{-1} \delta^i H_n) - m \leftrightarrow n \right] d^2 x = mn \varepsilon^{il} \varepsilon^{jk} \int A_i \left[ \partial_l (\sigma B^m) \partial_j (\sigma B^n) - m \leftrightarrow n \right] d^2 x \]

(126)

For \( \rho = 1 \) this becomes

\[ \frac{\{H_{m+1}, H_{n+1}\}}{(m+1)(n+1)} = \varepsilon^{il} \varepsilon^{jk} \int A_i B^{m+n-1} \left[ B (\partial_k \sigma) (\partial_l \sigma) (\partial_j B) + \sigma B (\partial_k \sigma) (\partial_l \partial_j B) \right. \\
\left. + (n + m - 3) \sigma (\partial_k \sigma) (\partial_j \partial_l \sigma) \right] d^2 x \]

(127)

Now we expand out the derivatives of products of \( B \) and \( \sigma \) and eliminate terms that vanish due to antisymmetry of \( \varepsilon^{jk} \) or antisymmetry in \( m \) and \( n \). We get

\[ \frac{\{H_{m+1}, H_{n+1}\}}{mn(m-n)} = \varepsilon^{il} \varepsilon^{jk} \int A_i B^{m+n-1} \left[ B (\partial_k \sigma) (\partial_l \sigma) (\partial_j B) + \sigma B (\partial_k \sigma) (\partial_l \partial_j B) \right. \\
\left. + (n + m - 3) \sigma (\partial_k \sigma) (\partial_j \partial_l \sigma) \right] d^2 x \]

(128)

Since this p.b. is gauge-invariant, calculate in the gauge \( A_1 = 0 \), call \( A_2 = A \), \( B = \partial_1 A \) and denote derivatives by subscripts \((x^1 = x, x^2 = y)\)

\[ \{H_{m+1}, H_{n+1}\} = \varepsilon^{jk} (m + 1)(n + 1)(m-n) \int dx dy \ AB^{m+n-2} \left[ B \sigma_x B_j \sigma_k \right. \\
\left. + \sigma B B_j x \sigma_k + (n + m - 1) \sigma B x B_j \sigma_k + \sigma B \right] d^2 x \]

(129)

After collecting terms, this becomes

\[ \{H_m, H_n\} = mn(m-n) \int AB^{m+n-4} \left[ B \sigma_y B \sigma_x + (n + m - 3) B \sigma_x \right] \\
\left. + \sigma B \left( B \sigma_y - B \sigma_x + B \sigma_y - B \sigma_x \right) \right] d^2 x \]

(130)

To simplify it we define \( k = m + n - 4 \) and use identities such as \( B^k B_x = (B^{k+1})_x / (k+1) \) and \( \sigma \sigma_x = \frac{1}{2} \sigma^2_x \) to combine the factors of \( B \) and \( \sigma \) to write

\[ \frac{1}{mn(m-n)} \{H_m, H_n\} = \sum_{i=1}^{8} T_i \quad \text{where} \]

\[ T_1 = \frac{1}{k+2} \int A (B^{k+2})_x \sigma_x \sigma_y \quad T_2 = - \frac{1}{k+2} \int A (B^{k+2})_y \sigma_x \sigma_x \]

\[ T_3 = \frac{1}{2} \int A (B^{k+1})_x B (\sigma^2)_y \quad T_4 = - \frac{1}{2} \int A (B^{k+1})_y B (\sigma^2)_x \]

\[ T_5 = \frac{1}{2} \int AB^{k+1} B (\sigma^2)_y \quad T_6 = - \frac{1}{2} \int AB^{k+1} B (\sigma^2)_x \]

\[ T_7 = \frac{1}{k+2} \int A (B^{k+2})_x \sigma_x \sigma_y \quad T_8 = - \frac{1}{k+2} \int A (B^{k+2})_y \sigma_x \sigma_x \]

(131)

Numerically we find \( T_1 + T_7 + T_2 + T_8 = -(T_3 + T_5 + T_4 + T_6) \). To see this analytically, integrate by parts with the aim of eliminating \( A \) in favor of \( B = \partial_x A \). Using similar identities as before,

\[ T_1 = -T_7 - \frac{1}{2} \int \frac{(B^{k+3})_x (\sigma^2)_y}{k+3} - \frac{1}{2} \int \frac{A (B^{k+2})_x (\sigma^2)_y}{k+2} \]
\[ T_2 = -T_8 + \frac{1}{2} \int \frac{(B^{k+3})_y(\sigma^2)_x}{k+3} + \frac{1}{2} \int \frac{A(B^{k+2})_{xy}(\sigma^2)_x}{k+2} \]

\[ T_3 = -T_5 - \frac{1}{2} \int \frac{(B^{k+3})_x(\sigma^2)_y}{k+3} - \frac{1}{2} \int \frac{A(B^{k+2})_{xy}(\sigma^2)_x}{k+2} \]

\[ T_4 = -T_6 + \frac{1}{2} \int \frac{A_y(B^{k+2})_x(\sigma^2)_x}{k+2} + \frac{1}{2} \int \frac{A(B^{k+2})_{xy}(\sigma^2)_x}{k+2} \]  

(132)

Then

\[ T_3 + T_5 + T_4 + T_6 = -\frac{1}{2} \int \frac{(B^{k+3})_y(\sigma^2)_x}{k+3} + \frac{1}{2} \int \frac{A_y(B^{k+2})_x(\sigma^2)_x}{k+2} \]

\[ T_1 + T_7 + T_2 + T_8 = \frac{1}{2(k+2)} \int A \left\{ (B^{k+2})_{xy}(\sigma^2)_x - (B^{k+2})_{xx}(\sigma^2)_y \right\} \]

\[ \{H_m, H_n\} = mn(m-n)(U_1 + U_2 + U_3 + U_4) \]  

(133)

where \( T_3 + T_5 + T_4 + T_6 \equiv U_1 + U_2 \) and \( T_1 + T_7 + T_2 + T_8 \equiv U_3 + U_4 \). Finally, integration by parts shows that

\[ U_2 = -U_3 - \frac{1}{2(k+2)} \int A(B^{k+2})_{xy}(\sigma^2)_y \]  

(134)

and that \( U_2 + U_3 = -U_4 - U_1 \). We conclude that \( \sum_{i=1}^{4} U_i = 0 \) and therefore \( \{H_m, H_n\} = 0 \). Since the hamiltonian is half the second charge, \( H = \frac{1}{2} H_2 \), we have shown that \( H_n \) for \( n = 1, 2, 3, \ldots \) are an infinite number of conserved quantities, which moreover, are in involution!

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