THE DYNAMICS OF GONOSOMAL EVOLUTION OPERATORS

AKMAL T. ABSALAMOV AND UTKIR A. ROZIKOV

ABSTRACT. In this paper we investigate the dynamical systems generated by gonosomal evolution operator of sex linked inheritance depending on parameters. Mainly we study dynamical systems of a hemophilia which is biological group of disorders connected with genes that diminish the body’s ability to control blood clotting or coagulation that is used to stop bleeding when a blood vessel is broken. For the gonosomal operator we describe all forms and give explicitly the types of fixed points. Moreover we study limit points of the trajectories of the corresponding dynamical system.

1. Introduction

In biology sex is determined genetically: males and females have different genes that specify their sexual morphology. In animals this is often accompanied by chromosomal differences. There are some sex linked systems which depend on temperature and even some of systems have sex change phenomenon, see [7] for more details. For mathematical models of bisexual population, see [3], [4], [5] and [6]. In [10] an algebra associated to a sex change is constructed.

In this paper we consider evolution of a hemophilia which is a lethal recessive $X$-linked disorder: a female carrying two alleles for hemophilia die. Therefore, if we denote by $X^h$ the gonosome $X$ carrying the hemophilia, there are only two female genotypes: $XX$ and $XX^h$ ($X^hX^h$ is lethal) and two male genotypes: $XY$ and $X^hY$. We have four types of crosses defined as

\[
\begin{align*}
XX \times XY & \rightarrow a_1XX, \ a_2XY, \\
XX \times X^hY & \rightarrow c_1XX^h, \ c_2XY, \\
XX^h \times XY & \rightarrow b_1XX, \ b_2XX^h, \ b_3XY, \ b_4X^hY, \\
XX^h \times X^hY & \rightarrow d_1XX^h, \ d_2XY, \ d_3X^hY.
\end{align*}
\]

Let $F = \{XX, XX^h\}$ and $M = \{XY, X^hY\}$ be sets of genotypes. Assume that state of the set $F$ is given by a real vector $(x, y)$ and state of $M$ by a real vector $(u, v)$. Then a state of the set $F \cup M$ is given by the vector $t = (x, y, u, v) \in \mathbb{R}^4$. If $t' = (x', y', u', v')$ is a state of the system $F \cup M$ in the next generation, then by the above rule we get the
evolution operator \( W : \mathbb{R}^4 \to \mathbb{R}^4 \) defined by

\[
W : \begin{cases}
x' = a_1 x + b_1 y,
y' = c_1 x + b_2 y + d_1 y,
u' = c_2 x + c_3 y + d_2 y,
v' = b_3 y.
\end{cases}
\tag{1.1}
\]

This example can be generalized as follows. Suppose that the set of female types is \( F = \{1, 2, ..., \eta\} \) and the set of male types is \( M = \{1, 2, ..., \nu\} \). Let \( x = (x_1, x_2, ..., x_\eta) \in \mathbb{R}_\eta^\eta \) be a state of \( F \) and \( y = (y_1, y_2, ..., y_\nu) \in \mathbb{R}_\nu^\nu \) be a state of \( M \). Consider \( p_{ir,j}^{(f)} \) and \( p_{ir,i}^{(m)} \) as some inheritance non-negative real coefficients (not necessarily probabilities) with

\[
\sum_{j=1}^\eta p_{ir,j}^{(f)} + \sum_{l=1}^\nu p_{ir,l}^{(m)} = 1
\]

and the corresponding evolution operator

\[
W : \begin{cases}
x_j' = \sum_{i=1}^\eta \sum_{r=1}^\nu p_{ir,j}^{(f)} x_i y_r, & j = 1, ..., n
y_l' = \sum_{i=1}^\eta \sum_{r=1}^\nu p_{ir,l}^{(m)} x_i y_r, & l = 1, ..., \nu.
\end{cases}
\tag{1.2}
\]

This operator is called gonosomal evolution operator.

The main problem for a given discrete-time dynamical system is to describe the limit points of the trajectory \( \{t^{(n)}\}_{n=0}^\infty \) for arbitrarily given \( t^{(0)} = (x, y) \in \mathbb{R}^{\eta+\nu} \), where

\[
t^{(n)} = W^n(t) = W(W(...W(t^{(0)})...))_{n}
\]

denotes the \( n \) times iteration of \( W \) to \( t^{(0)} \).

Note that the operator (1.2) describes evolution of a hemophilia. The dynamical system generated by the gonosomal operator (1.2) is complicated. In this paper we study the dynamical system generated by the gonosomal operator (1.1) which is a particular case of (1.2) corresponding to the case \( \eta = \nu = 2 \) and the coefficients

\[
p_{11,1}^{(f)} = a_1, \quad p_{11,2}^{(f)} = 0, \quad p_{11,1}^{(m)} = a_2, \quad p_{11,2}^{(m)} = 0,
p_{12,1}^{(f)} = 0, \quad p_{12,2}^{(f)} = c_1, \quad p_{12,1}^{(m)} = c_2, \quad p_{12,2}^{(m)} = 0,
p_{21,1}^{(f)} = b_1, \quad p_{21,2}^{(f)} = b_2, \quad p_{21,1}^{(m)} = b_3, \quad p_{21,2}^{(m)} = b_4,
p_{22,1}^{(f)} = 0, \quad p_{22,2}^{(f)} = d_1, \quad p_{22,1}^{(m)} = d_2, \quad p_{22,2}^{(m)} = d_3,
\tag{1.3}
\]

where \( a_1, a_2, c_1, c_2, b_1, b_2, b_3, b_4, d_1, d_2, d_3 \) are non-negative real numbers such that

\[
a_1 + a_2 = c_1 + c_2 = b_1 + b_2 + b_3 + b_4 = d_1 + d_2 + d_3 = 1.
\tag{1.4}
\]

**Remark 1.** An analogy of this problem was discussed in [9] for the classical case when

\[
a_1 = a_2 = c_1 = c_2 = \frac{1}{2}, \quad b_1 = b_2 = b_3 = b_4 = \frac{1}{4}, \quad d_1 = d_2 = d_3 = \frac{1}{3}.
\]
2. The types of the fixed points.

A point \( s \) is called a fixed point of the operator \( W \) if \( s = W(s) \). Let us find all the forms of the fixed points of \( W \) given by (1.1), i.e. we solve the following system of equations for \((x, y, u, v)\)

\[
\begin{align*}
    x &= a_1 xu + b_1 yu, \\
    y &= c_1 xv + b_2 yu + d_1 yv, \\
    u &= a_2 xu + c_2 xv + b_3 yu + d_2 yv, \\
    v &= b_4 yu + d_3 yv.
\end{align*}
\] (2.1)

It is easy to see that \( s_1 = (0, 0, 0, 0) \) is a solution of the system (2.1). If \( u = 0 \), then from the first equation we get \( x = 0 \). If \( y = 0 \), then from the last equation we get \( v = 0 \). Moreover, if \( x = y = 0 \), then the third and the last equations yield \( u = v = 0 \). If \( u = v = 0 \), then the first and the second equations give \( x = y = 0 \). That is why the fixed points of the operator (1.1) might be of the following forms:

I) \( s_1 = (0, 0, 0, 0) \),

II) \( s_2 = (x, 0, u, 0) \),

III) \( s_3 = (0, y, 0, v) \),

IV) \( s_4 = (0, y, u, 0) \),

V) \( s_5 = (x, y, u, v) \), where \( xyuv \neq 0 \).

Remark 2. For the given operator (1.1) the forms II), III) of the fixed points are uniquely defined. Indeed the system of equations (2.1) gives us the following:

\[
\begin{align*}
    (x, 0, u, 0) &= \left( \frac{1}{a_2}, 0, \frac{1}{a_1}, 0 \right) & \text{when} & & a_1, a_2 \in (0, 1), \\
    (0, y, 0, v) &= \left( 0, \frac{1}{d_3}, 0, \frac{1}{d_1} \right) & \text{when} & & d_2 = 0 \text{ and } d_1, d_3 \in (0, 1), \\
    (0, y, u, 0) &= \left( 0, \frac{1}{b_3}, \frac{1}{b_2}, 0 \right) & \text{when} & & b_1 = b_4 = 0 \text{ and } b_2, b_3 \in (0, 1).
\end{align*}
\]

Remark 3. For the given operator (1.1) the form V) of the fixed points might not be defined uniquely. To see this, consider the evolution operator

\[
W_0 : \begin{cases}
    x' = \frac{1}{2} xu, \\
    y' = \frac{1}{2} yu, \\
    u' = \frac{1}{2} xu + xv + \frac{1}{2} yu + \frac{1}{2} yv, \\
    v' = \frac{1}{2} yv.
\end{cases}
\]

One can check that \( s = (1, 2, 2, -\frac{1}{2}) \) and \( s = (2, 2, 2, -\frac{2}{3}) \) are fixed points of the operator \( W_0 \) which are of the form V).

In order to find the type of the fixed points of the operator (1.1) we consider the Jacobi matrix.
\[ J(s) = J_W = \begin{pmatrix}
    a_1u & b_1u & a_1x + b_1y & 0 \\
    c_1v & b_2u + d_1v & b_2y & c_1x + d_1y \\
    a_2u + c_2v & b_3u + d_2v & a_2x + b_3y & c_2x + d_2y \\
    0 & b_4u + d_3v & b_4y & d_3y
\end{pmatrix} \]

and the corresponding characteristic equation \( \det(J(s) - \lambda I) = 0 \). The characteristic equation has the form

\[
\lambda^4 - \lambda^3 p_1 + \lambda^2 p_2 + \lambda p_3 = 0, \tag{2.2}
\]

where

\[
p_1 = a_2x + (b_3 + d_3)y + (a_1 + b_2)u + d_1v,
\]

\[
p_2 = (a_1b_3 + a_1d_3 + b_2d_3 - b_4d_1 - a_2b_1)yu + a_1b_2u^2 + (a_1d_1 - b_1c_1)uv + (b_3d_3 - b_4d_2)y^2
\]

\[
+ (a_2d_3 - b_1c_2)xy + (a_2b_2 - b_3c_1)xy + (a_2d_1 - c_1d_1 - a_1c_2)xv + (b_3d_1 - b_2d_2 - b_1c_2)yv,
\]

\[
p_3 = a_2d_3xy + b_4c_1xu + (2a_2d_1 + c_1d_3 - b_3c_1 + a_1c_2 - b_2c_2)xv + (a_1b_4d_2 - a_1b_3d_3)y^2u
\]

\[
+ (a_1b_4d_1 - a_1b_2d_3 - b_1b_4c_1)yu^2 + (a_1b_2d_2 - a_1b_3d_1 + b_1b_3c_1)yuv
\]

\[
+ (a_2b_1c_1 + a_2b_2c_2 - a_1a_2d_1)xuv + (b_2b_4c_2 - a_2b_3d_3 - a_1a_2d_3)yuv
\]

\[
+ (b_4c_2d_1 - b_4c_1d_2 + b_3c_3d_3 - a_2d_1d_3)xuv + a_2c_1x^2(b_4u + d_3v)
\]

\[
+ (c_2d_1 - c_1d_2)v^2(a_1x + b_1y) + b_1c_2d_3y^2v.
\]

Clearly, \( \lambda = 0 \) is the all eigenvalues of the fixed point \( s_1 \). Thus \( s_1 \) is attracting fixed point.

**Lemma 1.** \( \lambda = 0 \) and \( \lambda = 2 \) are eigenvalues for the fixed points \( s_2, s_3, s_4, s_5 \).

**Proof.** From the equation (2.2) it is clear that \( \lambda = 0 \) is an eigenvalue for all forms of the fixed points. If \( s_2 = (x, 0, u, 0) \) is a fixed point with \( xu \neq 0 \) then the coefficients of the equation (2.2) simplify as

\[
p_1 = a_2x + (a_1 + b_2)u,
\]

\[
p_2 = a_1b_2u^2 + (a_2b_2 - b_1c_1)xu,
\]

\[
p_3 = a_1b_4c_1xu^2 + a_2b_1c_1x^2u.
\]

Hence we get

\[
8 - 4p_1 + 2p_2 + p_3 = (4 - 2b_2u - b_4c_1xu)(2 - a_2x - a_1u) = 0
\]

which implies that \( \lambda = 2 \) is an eigenvalue.

If \( s_3 = (0, y, 0, v) \) is a fixed point with \( yv \neq 0 \), then the coefficients of the equation (2.2) simplify as:

\[
p_1 = (b_3 + d_3)y + d_1v,
\]

\[
p_2 = b_3d_3y^2 + (b_3d_1 - b_1c_2)yv,
\]

\[
p_3 = b_1c_2d_1yv^2 + b_1c_2d_3y^2v.
\]

Consequently,

\[
8 - 4p_1 + 2p_2 + p_3 = (4 - 2b_3y - b_1c_2yv)(2 - d_3y - d_1v) = 0
\]
Therefore the obtained equations \((2.3)\) and \((2.4)\) are well defined. Now from the second and these values into the first, the third and the last equations in the system \((2.1)\), we obtain
\[
p_1 = (b_3 + d_3)y + (a_1 + b_2)u,
\]
\[
p_2 = a_1b_2u^2 + b_3d_3y^2 + (a_1b_3 + a_1d_3 + b_2d_3)yu,
\]
\[
p_3 = -a_1b_3d_3y^2u - a_1b_2d_3yu^2.
\]
Therefore,
\[
8 - 4p_1 + 2p_2 + p_3 = (4 - 2a_1u - 2d_3y - a_1d_3yu)(2 - b_3y - b_2u) = 0
\]
which shows \(\lambda = 2\) is an eigenvalue.

If \(s_5 = (x, y, u, v)\) is a fixed point with \(xyuv \neq 0\), then from the first and the last equations of the system \((2.1)\) we find \(x = \frac{b_1yu}{1-a_1u} \) and \(v = \frac{b_4yu}{1-d_3y}\). Substituting these values to other equations in the system \((2.1)\), we obtain
\[
(a_1b_4d_2 - a_1b_3d_3)y^2u + b_1c_2d_3y^2v - a_1a_2d_3xyu + a_2d_3xy = (-a_1b_1 - a_1d_3)yu - (b_3d_3 - b_4d_2)y^2 + b_1c_2yv + a_2x + a_1u + (b_3 + d_3)y - 1 - a_1a_2xu,
\]
\[
(a_1b_4d_1 - a_1b_2d_3 - b_1b_1c_1)yu^2 = (a_1 + b_2)u + d_3y - 1 - a_1b_2u^2 - (a_1d_3 + b_2d_3 - b_4d_1)yu.
\]
Note that we have \(u \neq \frac{1}{a_1}\) and \(y \neq \frac{1}{d_3}\), for otherwise the first and the last equations in the system \((2.1)\) would give us \(y = 0\) and \(u = 0\), contradicting to the condition \(xyuv \neq 0\). Therefore the obtained equations \((2.3)\) and \((2.4)\) are well defined. Now from the second and the last equations in the system \((2.1)\) we find \(y = \frac{c_1x}{1-b_2u - d_1v} \) and \(v = \frac{b_4yu}{1-d_3y}\). Substituting these values into the first, the third and the last equations in the system \((2.1)\), we obtain
\[
a_1b_2u^2 + (a_1d_1 - b_1c_1)uw = (a_1 + b_2)u + d_1v - 1,
\]
\[
(b_4c_2d_1 - b_4c_1d_2 + b_5c_1d_3 - a_2d_3d_4)xuv + (b_2b_3c_2 - a_2b_3d_3)xyu + a_2d_3xy + (a_2d_1 - b_1c_3)xv = a_2x + d_3y + b_2u + d_1v - 1 - b_2d_3yu - a_2b_2xu + b_4c_2xy - d_3d_3yv,
\]
\[
b_4c_1xu + c_1d_3xv = 1 - b_2u - d_1v.
\]
Moreover, from the first and the third equations in the system \((2.1)\) we find \(x = \frac{b_1yu}{1-a_1u}\) and \(u = \frac{c_2x + d_3v}{1-a_2x - b_3y}\). Substituting these values into the second and the last equations in the system \((2.1)\) we obtain
\[
(a_1b_2d_2 - a_1b_3d_1 + b_1b_3c_1)yu + (a_2b_1c_1 + a_1b_2c_2 - a_1a_2d_1)xuv + (a_2d_1 - b_2c_2)xv = a_2x + b_3y + a_1u + d_1v - 1 - a_1b_3yu - (a_1d_1 - b_1c_1)uw - (b_3d_1 - b_2d_2)yv - a_1a_2xu,
\]
\[
(b_3d_3 - b_4d_2)y^2 + (a_2b_3 - b_4c_2)xy = a_2x + (b_3 + d_3)y - 1.
\]
Taking into account all the obtained equations and the system (2.1) we get

\[
8 - 4p_1 + 2p_2 + p_3 = \frac{1 - d_1v}{u}(u - a_2xu - c_2xv - b_3yu - d_2yv) \\
+ \left(\frac{d_2v}{u} - \frac{a_2x}{y}\right)(y - c_1xv - b_2yu - d_1yv) \\
+ \frac{c_2v}{u}(x - a_1xu - b_1yu) = 0.
\] (2.10)

This shows that \(\lambda = 2\) is an eigenvalue for nonzero fixed point of operator (1.1). Lemma 1 is proved. \(\square\)

**Remark 4.** We have proved the Lemma 1 for the case when \(b_1, b_4, c_1, c_2x + d_2y\) are nonzero. In case when some of the numbers \(b_1, b_4, c_1, c_2x + d_2y\) are zero, then Lemma 1 can be proven similarly. For instance if we have only \(b_4 = 0\) then the last equation of the system (2.1) gives us \(y = \frac{1}{d_1}\) as \(xuyv \neq 0\). In this case we can rewrite (2.10) as

\[
8 - 4p_1 + 2p_2 + p_3 = \frac{1 - d_1v}{u}(u - a_2xu - c_2xv - b_3yu - d_2yv) \\
+ \left(\frac{d_2v}{u} - \frac{a_2x}{y} + \frac{1}{y}\right)(y - c_1xv - b_2yu - d_1yv) \\
+ \frac{c_2v}{u}(x - a_1xu - b_1yu) = 0.
\]

**Conjecture 1.** \(\lambda = 0\) and \(\lambda = 2\) are eigenvalues of the gonosomal evolution operator (1.2) corresponding to nonzero fixed points.

**Lemma 2.** If either \(p_1 - p_2 = 3\) or \(3p_1 - p_2 = 7\) holds, then \(s = (x, y, u, v)\) is a nonhyperbolic fixed point. Otherwise it is a saddle point in the case \(x^2 + y^2 + u^2 + v^2 \neq 0\).

**Proof.** For other roots of the equation (2.2) we have

\[
\lambda_{3,4} = -1 + \frac{p_1 \pm \sqrt{p_1^2 + 4p_1 - 4p_2 - 12}}{2} \\
= -1 + \frac{p_1 \pm \sqrt{p_1^2 + 4(p_1 - p_2 - 3)}}{2} \\
= -1 + \frac{p_1 \pm \sqrt{(p_1 - 4)^2 + 4(3p_1 - p_2 - 7)}}{2},
\]

which completes the proof. \(\square\)
Corollary 1. It holds that

\[
s_2 \text{ is } \begin{cases} 
\text{saddle} & \text{if } b_4c_1 \neq a_2(a_1 \pm b_2), \\
\text{nonhyperbolic} & \text{if } b_4c_1 = a_2(a_1 \pm b_2), 
\end{cases}
\]

\[
s_3 \text{ is } \begin{cases} 
\text{saddle} & \text{if } b_1c_2 \neq d_1(d_3 \pm b_3), \\
\text{nonhyperbolic} & \text{if } b_1c_2 = d_1(d_3 \pm b_3), 
\end{cases}
\]

\[
s_4 \text{ is } \begin{cases} 
\text{saddle} & \text{if } a_1d_3 \neq -1 \pm (a_1b_3 + b_2d_3), \\
\text{nonhyperbolic} & \text{if } a_1d_3 = -1 \pm (a_1b_3 + b_2d_3).
\end{cases}
\]

Corollary 2. Let \( s = (x, y, u, v) \) be a fixed point for the operator (1.2) such that \( x^2 + y^2 + u^2 + v^2 \neq 0 \). Then it is either nonhyperbolic or saddle point. Furthermore the gonosomal evolution operator (1.1) does not have repelling fixed points.

3. The \( \omega \)-limit set and the main results.

The problem of describing the \( \omega \)-limit set of a trajectory is of great importance in the theory of dynamical systems.

Proposition 1. The point \( s = (0, 0, \ldots, 0) \in \mathbb{R}^{n+\nu} \) is a fixed point for the operator (1.2). If \( \delta \in [0, 4) \) and the coefficients of the operator (1.2) are nonnegative real numbers, then for any initial point \( t \in Q_\delta \), we have

\[
\lim_{n \to \infty} W^n(t) = (0, 0, \ldots, 0),
\]

where

\[
Q_\delta = \{(x_1, \ldots, x_\eta, y_1, \ldots, y_\nu) \in \mathbb{R}^{n+\nu} : \sum_{j=1}^{\eta} x_j + \sum_{l=1}^{\nu} y_l \leq \delta, x_j \geq 0, y_l \geq 0, j = 1, \eta, l = 1, \nu\}
\]

Proof. It is not difficult to see that \( s = (0, 0, \ldots, 0) \in \mathbb{R}^{n+\nu} \) is an attracting fixed point for the operator (1.2). If \( t \in Q_\delta \), then from (1.2) we get \( x_j' \geq 0, y_l' \geq 0 \), for \( j = 1, \eta, l = 1, \nu \), and

\[
\sum_{j=1}^{\eta} x_j' + \sum_{l=1}^{\nu} y_l' = \sum_{j=1}^{\eta} x_j \cdot \sum_{l=1}^{\nu} y_l \leq \frac{1}{4} \left( \sum_{j=1}^{\eta} x_j + \sum_{l=1}^{\nu} y_l \right)^2 \leq \frac{\delta^2}{4} < \delta.
\]

Therefore

\[
t' = (x_1', \ldots, x_\eta', y_1', \ldots, y_\nu') \in Q_\frac{\delta^2}{4} \subset Q_\delta.
\]

Denoting \( f(\delta) = \frac{\delta^2}{4} \), we can write

\[
W^n(Q_\delta) \subset W^{n-1}(Q_{f(\delta)}) \subset W^{n-2}(Q_{f^2(\delta)}) \subset \ldots \subset Q_{f^n(\delta)}.
\]

Since \( \lim_{n \to \infty} f^n(\delta) = \lim_{n \to \infty} 4(\frac{\delta}{4})^{2^n} = 0 \), we get

\[
\lim_{n \to \infty} W^n(Q_\delta) \subset Q_0 = \{(0, 0, \ldots, 0)\},
\]

which completes the proof. \( \square \)
If $a_1 \in (0,1)$, then for any initial point $t_0 = (x_0, 0, u_0, 0)$ for the operator (1.1) we have
\[
\lim_{n \to \infty} W^n(t_0) = \left( \frac{1}{a_2} (a_1 a_2 x_0 u_0)^{2n-1}, 0, \frac{1}{a_1} (a_1 a_2 x_0 u_0)^{2n-1}, 0 \right)
\]
\[
\begin{cases}
(0, 0, 0, 0), & \text{if } |x_0 u_0| < \frac{1}{a_1 y_2}, \\
(\frac{1}{a_2}, 0, \frac{1}{a_1}, 0), & \text{if } |x_0 u_0| = \frac{1}{a_1 y_2}, \\
+\infty, & \text{if } |x_0 u_0| > \frac{1}{a_1 y_2}.
\end{cases}
\]

If $a_1 = 0$, then $W(t_0) = (0, 0, x_0 u_0, 0)$ and $W^n(t_0) = (0, 0, 0, 0)$ for all $n \geq 2$. If $a_1 = 1$, then $W(t_0) = (x_0 u_0, 0, 0, 0)$ and $W^n(t_0) = (0, 0, 0, 0)$ for all $n \geq 2$. Thus for the cases $a_1 = 0$ and $a_1 = 1$, we have
\[
\lim_{n \to \infty} W^n(t_0) = (0, 0, 0, 0).
\]

If $d_1, d_3 \in (0,1)$ and $d_2 = 0$, then for any initial point $t_0 = (0, y_0, 0, v_0)$ we have
\[
\lim_{n \to \infty} W^n(t_0) = \left( 0, \frac{1}{d_3} (d_1 d_3 y_0 v_0)^{2n-1}, 0, \frac{1}{d_1} (d_1 d_3 y_0 v_0)^{2n-1} \right)
\]
\[
\begin{cases}
(0, 0, 0, 0), & \text{if } |y_0 v_0| < \frac{1}{d_1 y_3}, \\
(0, \frac{1}{d_3}, 0, \frac{1}{d_1}), & \text{if } |y_0 v_0| = \frac{1}{d_1 y_3}, \\
+\infty, & \text{if } |y_0 v_0| > \frac{1}{d_1 y_3}.
\end{cases}
\]

If $d_1 = d_2 = 0$, then $W(t_0) = (0, 0, y_0 v_0)$ and $W^n(t_0) = (0, 0, 0, 0)$ for all $n \geq 2$. If $d_2 = d_3 = 0$, then $W(t_0) = (0, y_0 v_0, 0, 0)$ and $W^n(t_0) = (0, 0, 0, 0)$ for all $n \geq 2$. Thus for the cases $d_1 = d_2 = 0$ and $d_2 = d_3 = 0$, we have
\[
\lim_{n \to \infty} W^n(t_0) = (0, 0, 0, 0).
\]

If $b_2, b_3 \in (0,1)$ and $b_1 = b_4 = 0$, then for any initial point $t_0 = (0, y_0, u_0, 0)$ we have
\[
\lim_{n \to \infty} W^n(t_0) = \left( 0, \frac{1}{b_3} (b_2 b_3 y_0 u_0)^{2n-1}, 0, \frac{1}{b_2} (b_2 b_3 y_0 u_0)^{2n-1} \right)
\]
\[
\begin{cases}
(0, 0, 0, 0), & \text{if } |y_0 u_0| < \frac{1}{b_2 b_3}, \\
(0, \frac{1}{b_3}, \frac{1}{b_2}, 0), & \text{if } |y_0 u_0| = \frac{1}{b_2 b_3}, \\
+\infty, & \text{if } |y_0 u_0| > \frac{1}{b_2 b_3}.
\end{cases}
\]

If $b_1 = b_2 = b_4 = 0$, then $W(t_0) = (0, 0, y_0 u_0, 0)$ and $W^n(t_0) = (0, 0, 0, 0)$ for all $n \geq 2$. If $b_1 = b_3 = b_4 = 0$, then $W(t_0) = (0, y_0 u_0, 0, 0)$ and $W^n(t_0) = (0, 0, 0, 0)$ for all $n \geq 2$. Hence for cases $b_1 = b_2 = b_4 = 0$ and $b_1 = b_3 = b_4 = 0$ we have
\[
\lim_{n \to \infty} W^n(t_0) = (0, 0, 0, 0)
\]

Lemma 3. Let
\[Q_4 = \{(x, y, u, v) \in \mathbb{R}^4 : x \geq 0, y \geq 0, u \geq 0, v \geq 0, x + y + u + v \leq 4\}.
\]
For any initial point $t \in Q_4$ if there exists $k \geq 0$ such that
\[
(a_1 - \frac{1}{2}) x^{(k)} u^{(k)} + (c_1 - \frac{1}{2}) x^{(k)} v^{(k)} + (b_1 + b_2 - \frac{1}{2}) y^{(k)} u^{(k)} + (d_1 - \frac{1}{2}) y^{(k)} v^{(k)} \neq 0
\]
then
\[
\lim_{n \to \infty} W^n(t) = s_1 = (0, 0, 0, 0).
\] (3.2)

Proof. Since \( t \in Q_4 \) and
\[
x' + y' + u' + v' = (x + y)(u + v) \leq \left( \frac{x + y + u + v}{2} \right)^2 = 4,
\]
we have \( x + y = 2, u + v = 2 \). Otherwise, \( x' + y' + u' + v' < 4 \) and (3.2) follows by Proposition \[\]Hence
\[
x' + y' + u' + v' = 4,
\]
where
\[
x' + y' = 2 + (a_1 - \frac{1}{2}) xu + (c_1 - \frac{1}{2}) x v + (b_1 + b_2 - \frac{1}{2}) y u + (d_1 - \frac{1}{2}) y v
\]
and
\[
u' + v' = 2 - \left( (a_1 - \frac{1}{2}) xu + (c_1 - \frac{1}{2}) x v + (b_1 + b_2 - \frac{1}{2}) y u + (d_1 - \frac{1}{2}) y v \right).
\]
If \( (a_1 - \frac{1}{2}) xu + (c_1 - \frac{1}{2}) x v + (b_1 + b_2 - \frac{1}{2}) y u + (d_1 - \frac{1}{2}) y v \neq 0 \), then
\[
W^2(t) \in Q_4 - \left[ (a_1 - \frac{1}{2}) xu + (c_1 - \frac{1}{2}) x v + (b_1 + b_2 - \frac{1}{2}) y u + (d_1 - \frac{1}{2}) y v \right]^2
\]
and (3.2) again follows by Proposition \[\]Repeating this argument we get that, if
\[
(a_1 - \frac{1}{2}) x'u' + (c_1 - \frac{1}{2}) x'v' + (b_1 + b_2 - \frac{1}{2}) y'u' + (d_1 - \frac{1}{2}) y'v' \neq 0,
\]
then
\[
W^3(t) \in Q_4 - \left[ (a_1 - \frac{1}{2}) x'u' + (c_1 - \frac{1}{2}) x'v' + (b_1 + b_2 - \frac{1}{2}) y'u' + (d_1 - \frac{1}{2}) y'v' \right]^2.
\]
Otherwise, we iterate the argument again and conclude that if there exists \( k \geq 0 \) such that
\[
(a_1 - \frac{1}{2}) x^{(k)}u^{(k)} + (c_1 - \frac{1}{2}) x^{(k)}v^{(k)} + (b_1 + b_2 - \frac{1}{2}) y^{(k)}u^{(k)} + (d_1 - \frac{1}{2}) y^{(k)}v^{(k)} \neq 0,
\]
then (3.2) is satisfied. \[\]

Lemma 4. Let
\[
\Delta = \{ (x, y, u, v) \in \mathbb{R}^4 : x \geq 0, y \geq 0, u \geq 0, v \geq 0, x + y + u + v > 4 \}.
\]

For any initial point \( t \in \Delta \),
(i) if there exists \( k \geq 0 \) such that \( (x^{(k)} + y^{(k)})(u^{(k)} + v^{(k)}) < 4 \), then (3.2) holds.
(ii) if \( \max\{a_1 a_2 xu, b_2 b_3 yu, d_1 d_3 yv\} > 1 \), then \( \lim_{n \to \infty} W^n(t) = \infty \), i.e. at least one coordinate of \( W^n(t) \) tends to \( \infty \) as \( n \to \infty \).

Proof. Part (i) of this lemma simply follows from the identity
\[
x^{(k+1)} + y^{(k+1)} + u^{(k+1)} + v^{(k+1)} = (x^{(k)} + y^{(k)})(u^{(k)} + v^{(k)}),
\]
and by the Proposition \[\]We prove the claim in (ii) for the case
\[
\max\{a_1 a_2 xu, b_2 b_3 yu, d_1 d_3 yv\} = a_1 a_2 xu > 1.
\]
Other cases can be proven similarly. To this end, observe that for any \( t = (x, y, u, v) \in \mathbb{R}^4 \) with \( x \geq 0, y \geq 0, u \geq 0, v \geq 0 \) we get from (1.1) that

\[
x^{(k+1)} \geq a_1 x^{(k)} u^{(k)}, \quad u^{(k+1)} \geq a_2 x^{(k)} u^{(k)}, \quad k = 0, 1, \ldots.
\]

By iterating these inequalities, we obtain

\[
x^{(k+1)} \geq \frac{1}{a_2} [a_1 a_2 x u]^{2^k}, \quad u^{(k+1)} \geq \frac{1}{a_1} [a_1 a_2 x u]^{2^k}, \quad k = 0, 1, \ldots.
\]

This completes the proof. \( \Box \)

Part (ii) of Lemma 4 can be generalized as follows.

**Proposition 2.** Let the coefficients of the operator (1.2) and the coordinates of an initial point \( t \) be nonnegative real numbers. If

\[
\max_{1 \leq i, r \leq \eta, 1 \leq j, l \leq \nu} \{ p_{ij,r}^{(f)} p_{ij,l}^{(m)} x_r y_l \} > 1,
\]

then \( \lim_{n \to \infty} W^n(t) = \infty \), i.e. at least one coordinate of \( W^n(t) \) tends to \( \infty \) as \( n \to \infty \).

**Proof.** We prove the claim only for the case

\[
\max_{1 \leq i, r \leq \eta, 1 \leq j, l \leq \nu} \{ p_{ij,r}^{(f)} p_{ij,l}^{(m)} x_r y_l \} = p_{11,1}^{(f)} p_{11,1}^{(m)} x_1 y_1 > 1.
\]

Other cases can be proven similarly. To this end, observe that for any initial point \( t \in P \), we get from (1.2) that

\[
x_1^{(k+1)} \geq p_{11,1}^{(f)} x_1^{(k)} y_1^{(k)}, \quad y_1^{(k+1)} \geq p_{11,1}^{(m)} x_1^{(k)} y_1^{(k)}, \quad k = 0, 1, \ldots.
\]

By iterating these inequalities, we obtain

\[
x_1^{(k+1)} \geq \frac{1}{p_{11,1}^{(m)}} [p_{11,1}^{(f)} p_{11,1}^{(m)} x_1 y_1]^{2^k}, \quad y_1^{(k+1)} \geq \frac{1}{p_{11,1}^{(f)}} [p_{11,1}^{(f)} p_{11,1}^{(m)} x_1 y_1]^{2^k}, \quad k = 0, 1, \ldots.
\]

If \( p_{11,1}^{(f)} p_{11,1}^{(m)} = 0 \) then we go to the other cases. Proposition 2 is proved. \( \Box \)
Let us make the notations

\[ O = \{(0,0,u,v) \in \mathbb{R}^4 : u, v \in \mathbb{R}\} \cup \{(x,y,0,0) \in \mathbb{R}^4 : x, y \in \mathbb{R}\} \]

\[ I = \{(x,y,u,v) \in \mathbb{R}^4 : y = v = 0\} \]

\[ J = \{(x,y,u,v) \in I : x = u\} \]

\[ P = \{(x,y,u,v) \in \mathbb{R}^4 : x \geq 0, y \geq 0, u \geq 0, v \geq 0\} \]

\[ P_0 = \{(x,y,u,v) \in P : (x+y)(u+v) < 4\} \]

\[ Q_a = \{(x,y,u,v) \in P : x + y + u + v \leq a\}, \quad a \in [0,4] \]

\[ N = \{(x,y,u,v) \in \mathbb{R}^4 : x \leq 0, y \leq 0, u \leq 0, v \leq 0\} \]

\[ N_0 = \{(x,y,u,v) \in \mathbb{R}^4 : x \leq 0, y \leq 0, u \geq 0, v \geq 0\} \]

\[ N_1 = \{(x,y,u,v) \in \mathbb{R}^4 : x \geq 0, y \geq 0, u \leq 0, v \leq 0\} \]

\[ \Delta_0 = \{(x,y,u,v) \in P : x + y + u + v > 4, \quad \max\{a_1a_2xu, b_2b_3yu, d_1d_3yv\} > 1\}. \]

The sets \( I, J, P \) and \( Q_a \), where \( a \in [0,4] \), are invariant with respect to the operator \( \mathbf{1.1} \). Moreover, we have

\[ W(O) = \{(0,0,0,0)\}, \quad W(Q_a) \subset Q_{\frac{a}{4}}, \quad W(N) \subset P, \quad W(N_0) \subset N, \quad W(N_1) \subset N. \]

Summarizing above observations, we get the following result.

**Theorem 1.** If \( t = (x,y,u,v) \in \mathbb{R}^4 \) is such that

(i) one of the following conditions is satisfied

1) \( t \in P_0 \),
2) \( t \in Q_4 \) and Lemma 3 holds,
3) \( t \in N, \quad W(t) \in P_0 \),
4) \( t \in N_0, \quad W^2(t) \in P_0 \),
5) \( t \in N_1, \quad W^2(t) \in P_0 \),

then

\[ \lim_{n \to \infty} W^n(t) = s_1 = (0,0,0,0). \]

(ii) one of the following conditions is satisfied

1) \( t \in \Delta_0 \),
2) \( t \in N, \quad W(t) \in \Delta_0 \),
4) \( t \in N_0, \quad W^2(t) \in \Delta_0 \),
5) \( t \in N_1, \quad W^2(t) \in \Delta_0 \),

then

\[ \lim_{n \to \infty} W^n(t) = \infty, \]

i.e. at least one coordinate of \( W^n(t) \) tends to \( \infty \).
4. Conclusion

We have considered the dynamical systems of a hemophilia generated by gonosomal evolution operator of sex linked inheritance in $\mathbb{R}^4$ depending on parameters and studied their trajectory behavior. In Section 2 it is proven that operator (1.1) has a unique attracting fixed point and the other fixed points might be either nonhyperbolic or saddle. We note that the union of sets for initial points considered in Theorem 1 does not cover $\mathbb{R}^4$ and the question of description of the entire $\omega$-limit sets for the fixed points $s_2, s_3, s_4, s_5$ is remained as an open problem. However, due to the eigenvalues which we have found in section 2 we can give exact measure of stable and unstable manifolds of those fixed points, see [1] for more details. The dynamical systems considered in this paper are interesting as they are examples for nonlinear higher dimensional discrete-time dynamical systems that have not been fully understood yet.

References

[1] R. L. Devaney, An introduction to chaotic dynamical system, Westview Press, 2003.
[2] R. N. Ganikhodzhaev, F.M. Mukhamedov, U.A. Rozikov, Quadratic stochastic operators and processes: results and open problems. Inf. Dim. Anal. Quant. Prob. Rel. Fields., 14(2) (2011), 279-335.
[3] S. Karlin, Mathematical models, problems and controversics of evolutionary theory. Bull. Amer. Math. Soc. (N.S) 10(2) (1984), 221-274.
[4] M. Ladra, U.A. Rozikov, Evolution algebra of a bisexual population, Jour. Algebra. 378(2013), 153-172.
[5] Y. I. Lyubich, Mathematical structures in population genetics, Springer-Vergar, Berlin 1992.
[6] M. L. Reed, Algebraic structure of genetic inheritance, Bull. Amer. Math. Soc. (N.S.) 34 (2) (1997) 107-130.
[7] U. A. Rozikov, Evolution operators and algebras of sex linked inheritance. Asia Pacific Math. Newsletter. 3 (1) (2013), 6-11.
[8] U. A. Rozikov, U.U. Zhamilov, Volterra quadratic stochastic operators of bisexual population. Ukraine Math. Jour., 63(7) (2011), 985-998.
[9] U. A. Rozikov, R. Varro, Dynamical systems generated by a gonosomal evolution operator, Discontinuity, Nonlinearity and Complexity 2016, V.5, p. 173-185.
[10] R. Varro, Gonosomal algebra, 2015, Jour. Algebra, 447 (2016), p. 1-30.

Samarkand State University, Boulevard str., 140104, Samarkand, Uzbekistan.
E-mail address: absalamov@gmail.com

Institute of Mathematics, 81, Mirzo Ulug’bek str., 100170, Tashkent, Uzbekistan.
E-mail address: rozikovu@yandex.ru