Transformations of polar Grassmannians preserving certain intersecting relations

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Abstract

Let $\Pi$ be a polar space of rank $n \geq 3$. Denote by $G_k(\Pi)$ the polar Grassmannian formed by singular subspaces of $\Pi$ whose projective dimension is equal to $k$. Suppose that $k$ is an integer not greater than $n-2$ and consider the relation $R_{i,j}$, $0 \leq i \leq j \leq k+1$ formed by all pairs $(X, Y) \in G_k(\Pi) \times G_k(\Pi)$ such that $\dim_p(X \perp Y) = k-i$ and $\dim_p(X \cap Y) = k-j$ ($X \perp$ consists of all points of $\Pi$ collinear to every point of $X$). We show that every bijective transformation of $G_k(\Pi)$ preserving $R_{1,1}$ is induced by an automorphism of $\Pi$ and the same holds for the relation $R_{0,t}$ if $n \geq 2t \geq 4$ and $k = n - t - 1$. In the case when $\Pi$ is a finite classical polar space, we establish that the valencies of $R_{i,j}$ and $R_{i',j'}$ are distinct if $(i, j) \neq (i', j')$.

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1 Introduction

Let $V$ be an $n$-dimensional vector space $V$ (over a division ring). Denote by $G_k(V)$ the Grassmannian formed by $k$-dimensional subspaces of $V$. Suppose that $1 < k \leq n-k$. For every integer $i$ satisfying $1 \leq i \leq k$ we define the following relation

$$R_i := \{(X, Y) \in G_k(V) \times G_k(V) : \dim_l(X \cap Y) = k-i\}.$$ 

We write $\dim_l$ for the linear dimension (the dimension of vector spaces and their subspaces), since we want to distinguish it from the projective dimension (the dimension of projective spaces and their subspaces) which will be denoted by $\dim_p$. Note that $(X, Y) \in R_i$ if and only if the distance between $X$ and $Y$ in the Grassmann graph

$$\Gamma_k(V) = (G_k(V), R_i)$$

is equal to $i$.

The same relations can be defined on dual polar spaces. Let $\Pi$ be a polar space of rank $n$. Denote by $G_k(\Pi)$ the polar Grassmannian formed by singular subspaces of $\Pi$ whose projective dimension is equal to $k$. The associated dual polar space is formed by maximal singular subspaces, i.e. singular subspaces of dimension $n-1$. For any integer $i$ satisfying $1 \leq i \leq n$ we define

$$R_i := \{(X, Y) \in G_{n-1}(\Pi) \times G_{n-1}(\Pi) : \dim_p(X \cap Y) = n-1-i\}.$$
As above, we have \((X, Y) \in \mathcal{R}_i\) if and only if the distance between \(X\) and \(Y\) in the dual polar graph
\[\Gamma_{n-1}(\Pi) = (\mathcal{G}_{n-1}(\Pi), \mathcal{R}_i)\]
is equal to \(i\).

By [4], every automorphism of the Grassmann graph \(\Gamma_k(V)\) is induced by a semilinear automorphism of \(V\) or a semilinear isomorphism of \(V\) to the dual vector space \(V^*\) (the second possibility can be realized only in the case when \(n = 2k\)). Similarly, every automorphism of the dual polar graph \(\Gamma_{n-1}(\Pi)\) is induced by an automorphism of the polar space \(\Pi\). The latter was proved by Chow [4] for classical polar spaces only, but Chow’s method works in the general case [20, Section 4.6]. Some results closely related to these statements were obtained [7, 8, 9, 10, 11, 12, 16, 21, 22] and we refer [20] for a survey.

Every bijective transformation of \(\mathcal{G}_k(\Pi)\) preserving \(\mathcal{R}_i, \mathcal{R}_j\) is an automorphism of \(\Gamma_k(V)\) [2, 9]. For the relation \(\mathcal{R}_i, \mathcal{R}_j\) the same is not proved. However, all bijective transformations of \(\mathcal{G}_k(\Pi)\) preserving \(\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_m\) are automorphisms of \(\Gamma_k(V)\) for every integer \(m < k\) [17]. This is a generalization of the previous result; indeed, if \(m = k - 1\) then the considered above transformations preserve \(\mathcal{R}_k\). The same statement is proved for some dual polar spaces [15]. Results of similar nature were established for other objects [1, 5, 6, 13, 14].

Now suppose that \(k\) is an integer not greater than \(n - 2\) and consider the relation \(\mathcal{R}_{i,j}\), \(0 \leq i \leq j \leq k + 1\) formed by all pairs
\[(X, Y) \in \mathcal{G}_k(\Pi) \times \mathcal{G}_k(\Pi)\]
satisfying the following conditions
\[\dim_p(X^\perp \cap Y) = k - i \quad \text{and} \quad \dim_p(X \cap Y) = k - j\]
\((X^\perp\) consists of all points of \(\Pi\) collinear to every point of \(X\)). All automorphisms of the polar Grassmann graph
\[\Gamma_k(\Pi) = (\mathcal{G}_k(\Pi), \mathcal{R}_{0,1})\]
are described in [20, Section 4.6]. They are induced by automorphisms of \(\Pi\), except the case when \(n = 4, k = 1\) and our polar space is of type \(D\) (every \((n - 2)\)-dimensional singular subspace is contained in precisely two maximal singular subspaces). Also, every automorphism of so-called weak Grassmann graph
\[\Gamma^w_k(\Pi) = (\mathcal{G}_k(\Pi), \mathcal{R}_{0,1} \cup \mathcal{R}_{1,1})\]
is induced by an automorphism of \(\Pi\) [19].

We show that every bijective transformation of \(\mathcal{G}_k(\Pi)\) preserving \(\mathcal{R}_{i+1,1}\) is induced by an automorphism of \(\Pi\) (Theorem 4). Our second result (Theorem 2) states that the same holds for the relation \(\mathcal{R}_{0,t}\) if \(n \geq 2t \geq 4\) and \(k = n - t - 1\). The latter relation is equivalent to the fact that two elements with the relation \(\mathcal{R}_{0,t}\) span a maximal singular subspace.

Note that for finite symplectic and hermitian polar spaces, the first result under some conditions was proved in [18, 27].

For a finite classical polar space, the valency of every \(\mathcal{R}_{i,j}\) was given by Stanton [23]. We establish that the valencies of \(\mathcal{R}_{i,j}\) and \(\mathcal{R}_{i',j'}\) are distinct if \((i, j) \neq (i', j')\) (Proposition 3).

## 2 Polar spaces

We recall some basic properties of polar spaces and refer [3, 20, 25] for their proofs.

Let \(P\) be a non-empty set whose elements are called points and \(E\) be a family formed by proper subsets of \(P\) called lines. Two distinct points joined by a line are said to be collinear.
Let \( \Pi = (P, \mathcal{L}) \) be a \textit{partial linear space}, i.e. each line contains at least two points and for any distinct collinear points \( p, q \in P \) there is precisely one line containing them, this line is denoted by \( pq \).

We say that \( S \subset P \) is a \textit{subspace} of \( \Pi \) if for any distinct collinear points \( p, q \in S \) the line \( pq \) is contained in \( S \). A \textit{singular} subspace is a subspace where any two distinct points are collinear. Note that the empty set and a single point are singular subspaces. Using Zorn lemma, we show that every singular subspace is contained in a certain maximal singular subspace.

From this moment we suppose that \( \Pi \) is a \textit{polar space}. This means that the following axioms hold:

\begin{itemize}
  \item [(P1)] each line contains at least three points,
  \item [(P2)] there is no point collinear to all points,
  \item [(P3)] if \( p \in P \) and \( L \in \mathcal{L} \) then \( p \) is collinear to precisely one point or all points of the line \( L \),
  \item [(P4)] every flag formed by singular subspaces is finite.
\end{itemize}

If there is a maximal singular subspace of \( \Pi \) containing more than one line then all maximal singular subspaces of \( \Pi \) are projective spaces of the same finite dimension \( \geq 2 \). We say that the \textit{rank} of \( \Pi \) is \( n \) if this dimension is equal to \( n - 1 \).

In the case when the rank of \( \Pi \) is not less than 4, every maximal singular subspace \( M \) can be identified with the projective space associated to a certain \( n \)-dimensional vector space \( V \) (over a division ring). Then every non-empty singular subspace \( S \subset M \) will be identified with the corresponding subspace of the vector space \( V \).

The collinearity relation on \( \Pi \) is denoted by \( \perp \). We write \( p \perp q \) if \( p, q \in P \) are collinear points and \( p \not\perp q \) otherwise. If \( X, Y \subset P \) then \( X \perp Y \) means that every point of \( X \) is collinear to all points of \( Y \). For every subset \( X \subset P \) satisfying \( X \perp X \) the minimal singular subspace containing \( X \) is called \textit{spanned} by \( X \) and denoted by \( (X) \). For every subset \( X \subset P \) we denote by \( X^\perp \) the subspace of \( \Pi \) formed by all points collinear to all points of \( X \).

\textbf{Fact 1} Let \( X \) be a subset of \( P \) satisfying \( X \perp X \) and spanning a maximal singular subspace \( M \). Then \( p \perp X \) implies that \( p \in M \).

\textbf{Fact 2} If \( M \) is a maximal singular subspace of \( \Pi \) then for every point \( p \in P \) such that \( p \not\in M \) we have
\[ \dim_p(p^\perp \cap M) = n - 2. \]

\textbf{Fact 3} For every singular subspace \( S \) there are maximal singular subspaces \( M \) and \( N \) such that \( S = M \cap N \).

\section{Results}

Let \( \Pi = (P, \mathcal{L}) \) be a polar space of rank \( n \geq 3 \). Recall that the polar Grassmannian formed by all \( k \)-dimensional singular subspaces of \( \Pi \) is denoted by \( G_k(\Pi) \). Suppose that \( k \leq n - 2 \). We consider the relation \( R_{i,j} \), \( 0 \leq i \leq j \leq k + 1 \) is formed by all pairs
\[ (X, Y) \in G_k(\Pi) \times G_k(\Pi) \]
satisfying
\[ \dim_p(X^\perp \cap Y) = k - i \quad \text{and} \quad \dim_p(X \cap Y) = k - j. \]

Since the equality
\[ \dim_p(X^\perp \cap Y) = \dim_p(Y^\perp \cap X) \]
holds for any pair \( X, Y \in G_k(\Pi) \), this relation is symmetric. Every automorphism of \( \Pi \) (a bijective transformation of \( P \) preserving \( L \)) induces a transformation of \( G_k(\Pi) \) which preserves all \( \mathcal{R}_{i,j} \).

First, we determine all automorphisms of the graph

\[ \Gamma_k'(\Pi) = (G_k(\Pi), \mathcal{R}_{1,1}). \]

**Theorem 1** Every automorphism of \( \Gamma_k'(\Pi) \) is induced by an automorphism of \( \Pi \).

**Remark 1** For \( k = 0 \) this statement is trivial. The edges of \( \Gamma_0'(\Pi) \) are pairs of non-collinear points of \( \Pi \) and every automorphism of this graph is an automorphism of the collinearity graph \( \Gamma_0(\Pi) \). It is well-known that the class of automorphisms of \( \Gamma_0(\Pi) \) coincides with the class of automorphisms of \( \Pi \).

In the case when \( k \in \{1, \ldots, n-3\} \), the distance between \( S, U \in G_k(\Pi) \) in the Grassmann graph \( \Gamma_k(\Pi) \) is equal to 2 if and only if \( (S, U) \) belongs to \( \mathcal{R}_{1,1} \cup \mathcal{R}_{0,2} \). The distance between \( S, U \in G_{n-2}(\Pi) \) in \( \Gamma_{n-2}(\Pi) \) is equal to 2 if and only if \( (S, U) \in \mathcal{R}_{1,1} \) (if \( k = n - 2 \) then \( \mathcal{R}_{0,2} \) is empty). Theorem 1 gives the following.

**Corollary 1** Let \( f \) be a bijective transformation of \( G_{n-2}(\Pi) \) satisfying the following condition: the distance between \( S, U \in G_{n-2}(\Pi) \) in \( \Gamma_{n-2}(\Pi) \) is equal to 2 if and only if the distance between \( f(S) \) and \( f(U) \) in \( \Gamma_{n-2}(\Pi) \) is equal to 2. Then \( f \) is an automorphism of \( \Gamma_{n-2}(\Pi) \).

Suppose that \( k = n-t-1 \), where \( t \) is an integer satisfying \( n \geq 2t \geq 4 \). Then \( (S, U) \in \mathcal{R}_{0,t} \) is equivalent to the fact that \( S \) and \( U \) span a maximal singular subspace of \( \Pi \). Our second result describes all automorphisms of the graph

\[ \Gamma_k''(\Pi) = (G_k(\Pi), \mathcal{R}_{0,t}). \]

**Theorem 2** Every automorphism of \( \Gamma_k''(\Pi) \) is induced by an automorphism of \( \Pi \).

4 Clique

From this moment we suppose that \( k \in \{1, \ldots, n-2\} \). For every singular subspace \( N \) such that \( \dim_p N < k \) we denote by \( [N]_k \) the set of all elements of \( G_k(\Pi) \) containing \( N \). This subset is said to be a big star if \( N \in G_{k-1}(\Pi) \).

If \( N \) and \( M \) are singular subspaces satisfying

\[ N \subset M \quad \text{and} \quad \dim_p N < k < \dim_p M \]

then we denote by \( [N, M]_k \) the set of all \( S \in G_k(\Pi) \) such that \( N \subset S \subset M \). This subset is called a star if

\[ N \in G_{k-1}(\Pi) \quad \text{and} \quad M \in G_{n-1}(\Pi). \]

In the case when \( N = \emptyset \), we write \( [M]_k \) instead of \([N, M]_k \). We say that \( [M]_k \) is a top if \( M \in G_{k+1}(\Pi) \).

All maximal cliques of the Grassmann graph

\[ \Gamma_k(\Pi) = (G_k(\Pi), \mathcal{R}_{0,1}) \]

and the weak Grassmann graph

\[ \Gamma_k''(\Pi) = (G_k(\Pi), \mathcal{R}_{0,1} \cup \mathcal{R}_{1,1}) \]

are known [20, Subsection 4.5.1 and Subsection 4.6.2]. Every maximal clique of \( \Gamma_k''(\Pi) \) is a big star or a top. Every maximal clique of \( \Gamma_k(\Pi) \) is a star or a top. In the case when \( k = n-2 \), every star is contained in a certain top and all maximal cliques of \( \Gamma_k(\Pi) \) are tops.
Proposition 1 Every clique of $\Gamma'_k(\Pi)$ is contained in a big star.

Proof. If $C$ is a clique of $\Gamma'_k(\Pi)$ then it is a clique of $\Gamma''_k(\Pi)$. Hence $C$ is contained in a big star or a top. Since any two distinct elements of a top are non-adjacent vertices of $\Gamma'_k(\Pi)$, $C$ is a subset in a big star.

Let $N \in G_{k-1}(\Pi)$. For every $M \in G_{k+1}(\Pi)$ containing $N$ the subset $[N, M)_k$ is said to be a line. The big star $[N)_k$ together with all lines defined above is a polar space of rank $n - k$ [20, Lemma 4.4]. This polar space will be denoted by $\Pi_N$.

Lemma 1 Let $N \in G_{k-1}(\Pi)$. Two distinct elements of the big star $[N)_k$ are adjacent vertices of the graph $\Gamma'_k(\Pi)$ if and only if they are non-collinear points of the polar space $\Pi_N$.

Proof. Easy verification.

Proposition 2 Suppose that $k = n - t - 1$, where $t$ is an integer satisfying $n \geq 2t \geq 4$. If $S$ and $U$ are adjacent vertices of $\Gamma''_k(\Pi)$, then $M := \langle S \cup U \rangle$ is a maximal singular subspace of $\Pi$ and every clique of $\Gamma''_k(\Pi)$ containing $S$ and $U$ is a subset in $\langle M \rangle_k$.

Proof. It is clear that $S \perp U$ and $M$ is a maximal singular subspace. Let $C$ be a clique of $\Gamma''_k(\Pi)$ containing $S$ and $U$. Then for every $A \in C$ we have $A \perp S$ and $A \perp U$. By Fact I this implies that $A \subset M$. Hence $C$ is contained in $\langle M \rangle_k$.

5 Proof of Theorem I

Lemma 2 Let $p$ and $q$ be non-collinear points of $\Pi$ and let $t$ be a point of $\Pi$ collinear to at least one of the points $p, q$. Then there exists a point of $\Pi$ non-collinear to $p, q, t$.

Proof. Suppose that the statement fails and every point of $\Pi$ is collinear to at least one of the points $p, q, t$, in other words,

$$\mathcal{P} = p^\perp \cup q^\perp \cup t^\perp.$$ (1)

First we show that every maximal singular subspace of $\Pi$ contains at least one of the points $p, q, t$.

Let $M$ be a maximal singular subspace of $\Pi$. If each of the points $p, q, t$ does not belong to $M$ then

$$p^\perp \cap M, q^\perp \cap M, t^\perp \cap M$$

are $(n - 2)$-dimensional subspaces of $M$ (Fact 2) and $$ implies that $M$ is the union of these subspaces. The latter is impossible, since a projective space cannot be presented as the union of three hyperplanes.

By our assumption, $t$ is collinear to at least one of the points $p$ and $q$. Suppose that $q \perp t$. Since $p \not\perp q$, the line $qt$ contains the unique point $s$ collinear to $p$. One of the following possibilities is realized:

1. $s = t$,
2. $s \neq t$.
In the case (1), we take any point \( v \) on the line \( qt \) different from \( q \) and \( t \). It is clear that \( p \not\perp v \). By Fact 3, there exist maximal singular subspaces \( M \) and \( N \) such that \( M \cap N = \{ v \} \). Since \( p \not\perp v \), they do not contain \( p \). Then one of these subspaces contains \( q \) and the other contains \( t \). So, each of these subspaces contains two distinct points of the line \( qt \). This means that this line is contained in \( M \cap N \) which is impossible.

In the case (2), we take any point \( w \) on the line \( ps \) different from \( p \) and \( s \). As in the previous case, we consider maximal singular subspaces \( M \) and \( N \) such that \( M \cap N = \{ w \} \). One of these subspaces contains \( q \) or \( t \). Then at least one of the points \( q, t \) is collinear to \( w \). Since \( q \) and \( t \) both are collinear to \( s \) and \( w \) is on the line \( ps \), one of the points \( q, t \) is collinear to all points of the line \( ps \). Thus \( p \) is collinear to \( q \) or \( t \) which is impossible. \( \square \)

**Lemma 3** Suppose that \( N \in \mathcal{G}_{k-1}(\Pi) \). Let \( P, Q \in [N]_k \) be adjacent vertices of \( \Gamma'_k(\Pi) \) and let \( T \in [N]_k \) be a vertex of \( \Gamma'_k(\Pi) \) non-adjacent to at least one of the vertices \( P, Q \). Then there exists \( S \in [N]_k \) adjacent to \( P, Q, T \) in \( \Gamma'_k(\Pi) \).

**Proof.** By Lemma 1, \( P \) and \( Q \) are non-collinear points of \( \Pi_N \) and \( T \) is a point of \( \Pi_N \) collinear to at least one of the points \( P, Q \). We apply Lemma 2 to the polar space \( \Pi_N \) and get the claim. \( \square \)

Let \( f \) be an automorphism of \( \Gamma'_k(\Pi) \).

Show that \( f \) transfers big stars to subsets of big stars. We take any \( N \in \mathcal{G}_{k-1}(\Pi) \). Let \( P \) and \( Q \) be adjacent vertices of \( \Gamma'_k(\Pi) \) contained in the big star \([N]_k \). Then \( f(P) \) and \( f(Q) \) are adjacent vertices of \( \Gamma'_k(\Pi) \) contained in the big star \([N']_k \), where

\[
N' = f(P) \cap f(Q).
\]

We assert that \( f(T) \in [N']_k \) for every \( T \in [N]_k \) and prove this statement in several steps.

(i). First, we consider the case when \( T \) is a vertex of \( \Gamma'_k(\Pi) \) adjacent to both \( P \) and \( Q \). Then \( f(P), f(Q), f(T) \) form a clique in \( \Gamma'_k(\Pi) \) which, by Proposition 1, is contained in a certain big star \([N'\prime]_k \). We have

\[
N' = f(P) \cap f(Q) = N'\prime
\]

which gives the claim.

(ii). Consider the case when \( T \) is a vertex of \( \Gamma'_k(\Pi) \) adjacent to precisely one of the vertices \( P, Q \). Suppose that \( T \) is adjacent to \( P \). Lemma 3 implies the existence of a vertex \( S \in [N]_k \) in the graph \( \Gamma'_k(\Pi) \) adjacent to \( P, Q, T \). By (i), \( f(S) \) belongs to \([N']_k \). Then

\[
f(P) \cap f(S) = N'\prime
\]

and \( f(P), f(S), f(T) \) form a clique of \( \Gamma'_k(\Pi) \). As in (i), we show that \( f(T) \in [N']_k \).

(iii). Suppose that \( T \) is a vertex of \( \Gamma'_k(\Pi) \) non-adjacent to both \( P \) and \( Q \). As above, we consider \( S \in [N]_k \) which is a vertex of \( \Gamma'_k(\Pi) \) adjacent to \( P, Q, T \) and obtain that \( f(S) \in [N']_k \). We apply the arguments from (ii) to \( P, S, T \) and establish that \( f(T) \in [N']_k \).

So, \( f \) transfers big stars to subsets of big stars. The same arguments show that \( f^{-1} \) sends big stars to subsets of big stars. This means that \( f \) and \( f^{-1} \) both map big stars to big stars, i.e. there exists a bijective transformation \( g \) of \( \mathcal{G}_{k-1}(\Pi) \) such that

\[
f([N]_k) = [g(N)]_k
\]

for every \( N \in \mathcal{G}_{k-1}(\Pi) \).

Let \( U \in \mathcal{G}_k(\Pi) \). Then \( g \) transfers the top \( \langle U \rangle_{k-1} \) to the top \( \langle f(U) \rangle_{k-1} \). Indeed, we have

\[
N \in \langle U \rangle_{k-1} \iff U \in [N]_k \iff f(U) \in [g(N)]_k \iff g(N) \in \langle f(U) \rangle_{k-1}.
\]
Similarly, \( g^{-1} \) sends \( \langle U \rangle_{k-1} \) to the top \( (f^{-1}(U))_{k-1} \).

Therefore, \( g \) and \( g^{-1} \) both transfer tops to tops. Since for any two adjacent vertices of the Grassmann graph \( \Gamma_{k-1}(\Pi) \) there is a top containing them, \( g \) is an automorphism of \( \Gamma_{k-1}(\Pi) \). In some special cases, there are automorphisms of \( \Gamma_{k-1}(\Pi) \) which are not induced by automorphisms of \( \Pi \). However, every automorphism of \( \Gamma_{k-1}(\Pi) \) transferring tops to tops is induced by an automorphism of \( \Pi \) [20, Section 4.6.1]. Thus \( g \) is induced by an automorphism of \( \Pi \). An easy verification shows that this automorphism also induces \( f \).

6 Proof of Theorem 2

In this section we suppose that \( k = n - t - 1 \), where \( t \) is an integer satisfying \( n \geq 2t \geq 4 \).

**Lemma 4** Let \( S, U \) be adjacent vertices of \( \Gamma^n_k(\Pi) \). Let also \( M \) be the maximal singular subspace spanned by \( S \) and \( U \). If \( T \in \langle M \rangle_k \) is a vertex of \( \Gamma^n_k(\Pi) \) non-adjacent to at least one of the vertices \( S, U, T \), then there exists a vertex \( Q \) of \( \Gamma^n_k(\Pi) \) adjacent to \( S, U, T \).

**Proof.** We suppose that \( T \) is non-adjacent to \( S \) (the case when \( T \) is non-adjacent to \( U \) is similar). First of all, we show that the general case can be reduced to the case when \( N := S \cap U \) and \( T \) are disjoint. Indeed, if the projective dimension of \( W := N \cap T \) is equal to \( w \geq 0 \) then \( S, U, T \) can be naturally identified with vertices of the graph \( \Gamma^n_{k-w-1}(\Pi_W) \) which is isomorphic to the subgraph of \( \Gamma^n_k(\Pi) \) induced on \( |W|_k \).

So, we suppose that \( N \) and \( T \) are disjoint. Since \( n \geq 4 \), we identify \( M \) with the projective space associated to an \( n \)-dimensional vector space \( V \). Every non-empty singular subspace of \( M \) will be identified with the corresponding subspace of the vector space \( V \). We set

\[
m := \dim_{\mathbb{F}} N = n - 2t
\]

and suppose that

\[
\dim_{\mathbb{F}} (S \cap T) = i, \quad \dim_{\mathbb{F}} (U \cap T) = j.
\]

Note that \( i > m \), since \( S \) and \( T \) are non-adjacent vertices of \( \Gamma^n_k(\Pi) \).

First we consider the case when \( i + j = n - t \), i.e. \( T \) is spanned by \( S \cap T \) and \( U \cap T \).

Since \( \dim_{\mathbb{F}} (S/N) = \dim_{\mathbb{F}} (U/N) = t \), we can choose vectors

\[
x_1, \ldots, x_t \in S \setminus N \quad \text{and} \quad y_1, \ldots, y_t \in U \setminus N
\]

such that

\[
S = N + \langle x_1, \ldots, x_t \rangle, \quad U = N + \langle y_1, \ldots, y_t \rangle, \quad T = \langle x_1, \ldots, x_t \rangle + \langle y_{t-j+1}, \ldots, y_t \rangle.
\]

The vectors \( x_1 + y_1, \ldots, x_t + y_t \) are linearly independent. We define

\[
Q := N + \langle x_1 + y_1, \ldots, x_t + y_t \rangle.
\]

An easy verification shows that

\[
S + Q = U + Q = T + Q = M
\]

which implies that \( Q \) is a vertex of \( \Gamma^n_k(\Pi) \) adjacent to \( S, U, T \).

Now suppose that

\[
l := n - t - (i + j) > 0.
\]

Then \( t = i + j - m + l \). Since \( i > m \), we have \( i + j - m > 0 \). We choose linearly independent vectors

\[
x_1, \ldots, x_{i+j-m}, x'_1, \ldots, x'_l \in S \setminus N
\]

and suppose that

\[
\dim_{\mathbb{F}} (S \cap T) = i, \quad \dim_{\mathbb{F}} (U \cap T) = j.
\]

Note that \( i > m \), since \( S \) and \( T \) are non-adjacent vertices of \( \Gamma^n_k(\Pi) \).

First we consider the case when \( i + j = n - t \), i.e. \( T \) is spanned by \( S \cap T \) and \( U \cap T \).

Since \( \dim_{\mathbb{F}} (S/N) = \dim_{\mathbb{F}} (U/N) = t \), we can choose vectors

\[
x_1, \ldots, x_t \in S \setminus N \quad \text{and} \quad y_1, \ldots, y_t \in U \setminus N
\]

such that

\[
S = N + \langle x_1, \ldots, x_t \rangle, \quad U = N + \langle y_1, \ldots, y_t \rangle, \quad T = \langle x_1, \ldots, x_t \rangle + \langle y_{t-j+1}, \ldots, y_t \rangle.
\]

The vectors \( x_1 + y_1, \ldots, x_t + y_t \) are linearly independent. We define

\[
Q := N + \langle x_1 + y_1, \ldots, x_t + y_t \rangle.
\]

An easy verification shows that

\[
S + Q = U + Q = T + Q = M
\]

which implies that \( Q \) is a vertex of \( \Gamma^n_k(\Pi) \) adjacent to \( S, U, T \).

Now suppose that

\[
l := n - t - (i + j) > 0.
\]

Then \( t = i + j - m + l \). Since \( i > m \), we have \( i + j - m > 0 \). We choose linearly independent vectors

\[
x_1, \ldots, x_{i+j-m}, x'_1, \ldots, x'_l \in S \setminus N
\]
and linearly independent vectors
\[ y_1, \ldots, y_{i+j-m}, y'_1, \ldots, y'_l \in U \setminus N \]
such that
\[ T = \langle x_1, \ldots, x_i \rangle + \langle y_{i-m+1}, \ldots, y_{i+j-m} \rangle + \langle x'_1 + y'_1, \ldots, x'_l + y'_l \rangle. \]

We denote by \( Q \) the subspace spanned by \( N \) and the vectors
\[ x_1 + y'_1, x'_1 + y'_2, \ldots, x'_{i-1} + y'_i, x'_i + y_1, x_2 + y_2, \ldots, x_{i+j-m} + y_{i+j-m}. \]
Using the equalities
\[
S = N + \langle x_1, \ldots, x_{i+j-m}, x'_1, \ldots, x'_l \rangle,
\]
\[
U = N + \langle y_1, \ldots, y_{i+j-m}, y'_1, \ldots, y'_l \rangle,
\]
we establish that \( S + Q = U + Q = M \). To complete the proof we need to check that \( T + Q = M \).

The conditions
\[
x_1, \ldots, x_i, y_{i-m+1}, \ldots, y_{i+j-m} \in T \quad \text{and} \quad x_2 + y_2, \ldots, x_{i+j-m} + y_{i+j-m} \in Q
\]
imply that \( x_p \in T + Q \) for every \( p \) and \( y_p \in T + Q \) if \( p \geq 2 \). Since \( x_1 \in T \) and \( x_1 + y'_1 \in Q \), we have \( y'_1 \in T + Q \). Then \( x'_1 + y'_1 \in T \) implies that \( x'_1 \in T + Q \). Step by step, we establish that all \( x'_q \) and \( y'_q \) belong to \( T + Q \). The conditions \( x'_1 \in T + Q \) and \( x'_1 + y_1 \in Q \) guarantee that \( y_1 \in T + Q \). Therefore, \( Q + T \) coincides with \( S + U = M \).

Let \( f \) be an automorphism of \( \Gamma''_k(\Pi) \). Show that for every \( M \in \mathcal{G}_{n-1}(\Pi) \) there exists \( M' \in \mathcal{G}_{n-1}(\Pi) \) such that
\[
f(\langle M \rangle_k) \subset \langle M' \rangle_k.
\]
We choose \( S, U \in \langle M \rangle_k \) such that \( M \) is spanned by \( S \cup U \). Then \( S \) and \( U \) are adjacent vertices of \( \Gamma''_k(\Pi) \) and the same holds for \( f(S) \) and \( f(U) \). Hence
\[
M' := \langle f(S) \cup f(U) \rangle
\]
is a maximal singular subspace. We assert that \( f(T) \in \langle M' \rangle_k \) for every \( T \in \langle M \rangle_k \) and prove this statement in several steps.

(i). If \( T \in \langle M \rangle_k \) is a vertex of \( \Gamma''_k(\Pi) \) adjacent to \( S \) and \( U \) then \( f(S), f(U), f(T) \) form a clique in \( \Gamma''_k(\Pi) \) and, by Proposition 2, we have \( f(T) \in \langle M' \rangle_k \).

(ii). Now, let \( T \in \langle M \rangle_k \) be a vertex of \( \Gamma''_k(\Pi) \) adjacent to precisely one of the vertices \( U \) and \( S \). Suppose that \( T \) is adjacent to \( U \) and non-adjacent to \( S \). Let \( Q \in \langle M \rangle_k \) be a vertex of \( \Gamma''_k(\Pi) \) adjacent to \( S, U, T \) (Lemma 4). By (i), \( f(Q) \) belongs to \( \langle M' \rangle_k \). Then
\[
\langle f(U) \cup f(Q) \rangle = M'.
\]
We apply (i) to \( U, Q, T \) and establish that \( f(T) \) belongs to \( \langle M' \rangle_k \).

(iii). Consider the case when \( T \) is a vertex of \( \Gamma''_k(\Pi) \) non-adjacent to both \( S \) and \( U \). As above, we consider \( Q \in \langle M \rangle_k \) which is a vertex of \( \Gamma''_k(\Pi) \) adjacent to \( S, U, T \). Then \( f(Q) \in \langle M' \rangle_k \) and (2) holds. We apply (ii) to \( U, Q, T \) and obtain that \( f(T) \in \langle M' \rangle_k \).

We apply the above arguments to \( f^{-1} \) and establish the existence of a bijective transformation \( g \) of \( \mathcal{G}_{n-1}(\Pi) \) such that
\[
f(\langle M \rangle_k) = \langle g(M) \rangle_k
\]
for every \( M \in \mathcal{G}_{n-1}(\Pi) \).
Let \( U \) be a singular subspace of \( \Pi \) such that \( k < \dim_p U < n - 1 \). Then there exist \( M_1, M_2 \in \mathcal{G}_{n-1}(\Pi) \) satisfying \( U = M_1 \cap M_2 \). We have
\[
\langle U \rangle_k = \langle M_1 \rangle_k \cap \langle M_2 \rangle_k,
\]
\[
f(\langle U \rangle_k) = \langle g(M_1) \rangle_k \cap \langle g(M_2) \rangle_k = \langle g(M_1) \cap g(M_2) \rangle_k.
\]
We set \( g(U) := g(M_1) \cap g(M_2) \)
and get an extension of \( g \) to a transformation of
\[
\mathcal{G}_{k+1}(\Pi) \cup \cdots \cup \mathcal{G}_{n-1}(\Pi) \quad (3)
\]
such that
\[
f(\langle U \rangle_k) = \langle g(U) \rangle_k
\]
for every \( U \) belonging to (3). We apply the same arguments to the pair \( g^{-1}, f^{-1} \) and show that \( g \) is bijective. It is clear that \( f \) is inclusion preserving, i.e.
\[
S \subset U \iff f(S) \subset f(U)
\]
for any \( S, U \) belonging to (3). The latter guarantees that \( f \) sends \( \mathcal{G}_i(\Pi) \) to itself for every integer \( i \) satisfying \( k < i \leq n - 1 \). Therefore, \( f \) and \( f^{-1} \) both send tops to tops which implies that \( f \) is induced by an automorphism of \( \Pi \).

### 7 Valency of the relation for finite classical polar spaces

Let \( V \) be an \( n \)-dimensional vector space over the finite field \( \mathbb{F}_q \) together with a non-degenerate symplectic, hermitian or symmetric form of Witt index \( d \geq 2 \). For every \( m \in \{1, \ldots, d\} \) we denote by \( \mathcal{N}_m \) the set of all \( m \)-dimensional totally isotropic subspaces. The associated polar space \( \Pi \) is defined as follows: the point set is \( \mathcal{N}_1 \) and the lines are defined by elements of \( \mathcal{N}_2 \) (the line corresponding to \( S \in \mathcal{N}_2 \) consists of all elements of \( \mathcal{N}_1 \) contained in \( S \)). This is a polar space of rank \( d \) and every \( \mathcal{G}_i(\Pi) \) can be naturally identified with \( \mathcal{N}_{i+1} \). For more information, see [24, 26].

For \( 0 \leq i \leq j \leq m \), denote by \( n_{i,j} \) the valency of the relation
\[
\{(S,U) \in \mathcal{N}_m \times \mathcal{N}_m : \dim_i(S^\perp \cap U) = m - i, \ \dim_i(S \cap U) = m - j\},
\]
where \( S^\perp \) is the orthogonal complement to \( S \). If \( m < d \) then (4) coincides with the relation \( \mathcal{R}_{i,j} \). In the case when \( m = d \), we have \( S^\perp \cap U = S \cap U \) for any \( S, U \in \mathcal{N}_m \) and (4) is the relation \( \mathcal{R}_i \) defined on the dual polar space.

The valency \( n_{i,j} \) is computed in [23] as follows:
\[
n_{i,j} = q^{2i+(n-2m-2j+\frac{1}{2}i+\frac{1}{2}j-\mu-\nu)} \left[ \begin{array}{c} m \\ i \end{array} \right] \left[ \begin{array}{c} j \\ i \end{array} \right] \prod_{s=0}^{j-i-1} (q^{m_s-m_s-s} - 1)(q^{m_s-m_s-s+1} + 1)(q^{i+1} - 1)^{-1},
\]
where \( \mu = \frac{1}{4}m - d \) and \( \nu \) is a number such that \( \mu + \nu \) equals \( 0, \frac{1}{2}, 1 \) in the symplectic, hermitian and symmetric cases, respectively.

**Proposition 3** All valencies \( n_{i,j} \) are pairwise distinct.
Lemma 5 Let $f(x)$ be a polynomial of degree at least 1 over the rational number field $\mathbb{Q}$.

(i) If $f(x)$ has the following two factorizations:

$$f(x) = (x^{r_1} - 1)^{i_1}(x^{r_2} - 1)^{i_2} \cdots (x^{r_t} - 1)^{i_t} = (x^{t_1} - 1)^{j_1}(x^{t_2} - 1)^{j_2} \cdots (x^{t_h} - 1)^{j_h},$$

where $r_1 > \cdots > r_t$ and $t_1 > \cdots > t_h$, then $h = l$, $r_s = t_s$ and $i_s = j_s$, $s = 1, \ldots, l$.

(ii) If $f(x)$ has the following two factorizations:

$$f(x) = (x^{r_1} + 1)(x^{r_2} - 1)^{i_2} \cdots (x^{r_t} - 1)^{i_t} = (x^{t_1} - 1)^{j_1} \cdots (x^{t_h} - 1)^{j_h},$$

then there exists an $s \in \{1, \ldots, h\}$ such that $t_s = 2r$.

(iii) If $f(x)$ has the following two factorizations:

$$f(x) = (x^{k_1} + 1) \cdots (x^{k_f} + 1)(x^{r_1} - 1)^{i_1} \cdots (x^{r_t} - 1)^{i_t} = (x^{t_1} - 1)^{j_1} \cdots (x^{t_h} - 1)^{j_h},$$

where $k_1, \ldots, k_f$ are distinct odd integers, then there exist $s_1, \ldots, s_f \in \{1, \ldots, h\}$ such that $t_{s_f} = 2k_f$.

Proof. (i) Consider the term of second minimum degree of $f(x)$, we have

$$(-1)^{i_1+\cdots+i_t} x^{r_1} = (-1)^{j_1+\cdots+j_h} x^{t_h},$$

which implies that $j_h = i_t$ and $t_h = r_t$. So

$$(x^{r_1} - 1)^{i_1} \cdots (x^{r_t-1} - 1)^{i_t-1} = (x^{t_1} - 1)^{j_1} \cdots (x^{t_h-1} - 1)^{j_h-1},$$

By induction, (i) holds.

(ii) The polynomial $(x^r - 1)f(x)$ has the following two factorizations:

$$(x^r - 1)f(x) = (x^{2r} - 1)(x^{r_1} - 1)^{i_1} \cdots (x^{r_t} - 1)^{i_t} = (x^r - 1)(x^{t_1} - 1)^{j_1} \cdots (x^{t_h} - 1)^{j_h},$$

So (ii) holds by (i).

The proof of (iii) is similar to that of (ii), and will be omitted. \qed

Proof of Proposition 3 Suppose $n_{a,b} = n_{a',b'}$. Since $q$ is a prime power, we obtain

$$b^2 + a(n - 2m - 2b + \frac{3}{2}a + \frac{1}{2} - \mu - \nu) = b'^2 + a'(n - 2m - 2b' + \frac{3}{2}a' + \frac{1}{2} - \mu - \nu),$$

and

$$\left[ \frac{m}{b} \right] \prod_{s=0}^{b-a-1} \left( q^{\frac{3}{2}m - m - s} - 1 \right) \left( q^{\frac{3}{2}m - m - s} + 1 \right)$$

$$\left[ \frac{m}{b'} \right] \prod_{s=0}^{b'-a'-1} \left( q^{\frac{3}{2}m - m - s} - 1 \right) \left( q^{\frac{3}{2}m - m - s} + 1 \right) - 1.$$ (5)

Simplifying (5), we have

$$(b - b')(2b + 2b' - 4a') = (a' - a)(2l + 3a' + 3a - 4b),$$

where $l = n - 2m + \frac{1}{2} - \mu - \nu$. Write

$$f_{i,j}(x) = \prod_{s=0}^{m-j} \left( x^{n-2m-2\mu-2s} - 1 \right) \left( x^{n-2m-2\nu-2s} + 1 \right),$$

$$g_{i,j}(x) = \prod_{s=1}^{m-j} \left( x^{2s} - 1 \right) \prod_{s=1}^{i} \left( x^{2s} - 1 \right) \prod_{s=1}^{i-j} \left( x^{2s} - 1 \right)^2.$$
The equality (9) implies that 
\[ f_{a,b}(q)g_{a',\nu}(q) = f_{a',\nu}(q)g_{a,b}(q) \]
for all prime powers \( q \), so
\[ f_{a,b}(x)g_{a',\nu}(x) = f_{a',\nu}(x)g_{a,b}(x). \] (8)

Computing the degree of this polynomial, we have
\[ (b - b')(2l - 5b - 5b' + m + 8a') = (a' - a)(-2l - 5a - 5a' + 8b). \] (9)

Equalities (7) and (9) imply that
\[ (b - b')(2l + 2m - b - b') = (a' - a)(2l + a + a'). \] (10)

Suppose \( b \neq b' \). Without loss of generality, we assume that \( b > b' \). Since \( 2l + 2m - b - b' > 0 \) and \( 2l + a + a' > 0 \), by (10) one gets \( a' > a \) and
\[ b - a - b' + a' \geq 2. \] (11)

Write
\[
\begin{align*}
k(x) &= \prod_{s = b' - a' + 1}^{b-a} (x^{2s} - 1), \\
h(x) &= \prod_{s = b' - a'}^{b-1} (x^{n-2m-2\mu-2s} - 1)(x^{n-2m-2\nu-2s} + 1) \prod_{s = m-b+1}^{m-b'} (x^{2s} - 1) \prod_{s = a+1}^{a'} (x^{2s} - 1).
\end{align*}
\]

The equation (8) implies that \( k(x) = h(x) \).

Set
\[ h_1(x) = x^{2n-4m-4\nu-4b'+4a'} - 1, \quad h_2(x) = x^{2n-4m-4\nu-4b+4a+4} - 1. \]

By Lemma (5) there exist \( s_1, s_2 \in \{b' - a' + 1, \ldots, b - a\} \) such that \( h_1(x) = x^{2s_1} - 1 \) and \( h_2(x) = x^{2s_2} - 1 \). It follows that
\[ s_1 = n - 2m - 2\nu - 2b' + 2a', \quad s_2 = n - 2m - 2\nu - 2b + 2a + 2. \]

By (11) we have \( s_1 - s_2 = 2b - 2a - 2b' + 2a' - 2 > b - a - (b' - a' + 1) \), a contradiction. So \( b = b' \) and \( a = a' \), as desired. \( \square \)

**Remark 2** It is well known that \( \mathcal{N}_m \) has a structure of a symmetric association scheme, each relation of which has form (3). By Theorem (4) every automorphism of this scheme is induced by an automorphism of the associated polar space \( \Pi \).

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**References**

[1] P. Abramenko, H. Van Maldeghem, *On opposition in spherical buildings and twin buildings*, Ann. Comb. 4 (2000), 125–137.

[2] A. Blunck, H. Havlicek, *On bijections that preserve complementarity of subspaces*, Discrete Math. 301 (2005), 46–56.

[3] F. Buekenhout, A. M. Cohen, *Diagram geometry*, Springer 2013.
[4] W.L. Chow, *On the geometry of algebraic homogeneous spaces*, Ann. of Math. 50 (1949), 32–67.

[5] E. Govaert, H. Van Maldeghem, *Distance-preserving maps in generalized polygons. Part I: Maps on flags*, Beiträge Algebra Geom. 43 (2002), 89–110.

[6] E. Govaert, H. Van Maldeghem, *Distance-preserving maps in generalized polygons. Part II: Maps on points and/or lines*, Beiträge Algebra Geom. 43 (2002), 303–324.

[7] H. Havlicek, *On isomorphisms of Grassmann spaces*, Mitt. Math. Ges. Hamburg. 14 (1995), 117–120.

[8] H. Havlicek, *Chow’s theorem for linear spaces*, Discrete Math. 208/209 (1999), 319–324.

[9] H. Havlicek, M. Pankov, *Transformations on the product of Grassmann spaces*, Demonstratio Math. 38 (2005), 675–688.

[10] W.-l. Huang, *Adjacency preserving transformations of Grassmann spaces*, Abh. Math. Sem. Univ. Hamburg 68 (1998), 65–77.

[11] W.-l. Huang, *Adjacency preserving mappings of invariant subspaces of a null system*, Proc. Amer. Math. Soc. 128 (2000), 2451–2455.

[12] W.-l. Huang, *Characterization of the transformation group of the space of a null system*, Results Math. 40 (2001), 226–232.

[13] W.-l. Huang, H. Havlicek, *Diameter preserving surjections in the geometry of matrices*, Linear Algebra Appl. 429 (2008), 376–386.

[14] W.-l. Huang, *Bounded distance preserving surjections in the geometry of matrices*, Linear Algebra Appl. 433 (2010), 1973–1987.

[15] W.-l. Huang, *Bounded distance preserving surjections in the projective geometry of matrices*, Linear Algebra Appl. 435 (2011), 175–185.

[16] A. Kreuzer, *On isomorphisms of Grassmann spaces*, Aequationes Math. 56 (1998), 243–250.

[17] M. H. Lim, *Surjections on Grassmannians preserving pairs of elements with bounded distance*, Linear Algebra Appl. 432 (2010), 1703–1707.

[18] W. Liu, C. Ma, K. Wang, *Full automorphism group of generalized unitary graphs*, Linear Algebra Appl. 437 (2012) 684–691.

[19] M. Pankov, K. Prażmowski, M. Żynel, *Geometry of polar Grassmann spaces*, Demonstratio Math., 39 (2006), 625–637.

[20] M. Pankov, *Grassmannians of classical buildings*, Algebra and Discrete Math. 2, World Scientific, Singapore, 2010.

[21] M. Pankov, *Metric characterization of apartments in dual polar spaces*, J. Combin. Theory Ser. A 118 (2011), 1313–1321.

[22] M. Pankov, *Embeddings of Grassmann graphs*, Linear Algebra Appl. 436 (2012), 3413–3424.

[23] D. Stanton, *Some q-Krawtchouk polynomials on Chevalley groups*, Amer. J. Math. 102 (1980), 625–662.
[24] D.E. Taylor, *The geometry of the classical groups*, Heldermann Verlag, Berlin, 1992.

[25] J. Ueberberg, *Foundations of incidence geometry. Projective and polar spaces*, Springer 2011.

[26] Z. Wan, *Geometry of classical groups over finite fields*, Science Press, Beijing, New York, 2002.

[27] L. Zeng, Z. Chai, R. Feng, C. Ma, *Full automorphism group of the generalized symplectic graph*, Sci. China Math., doi: 10.1007/s11425-013-4651-8.