Modelling weathering induced retreat of \( c-\phi \) cliffs with limited tensile strength

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Abstract. Natural cliffs subject to weathering induced retreat are typically made of hard soils and / or weak rocks exhibiting limited tensile strength. In this paper, the morphologic evolution of uniform \( c, \phi \) slopes subject to weathering is investigated for a range of values of tensile strengths employing the limit analysis upper bound method. This paper extends the analytical framework set up in [1, 2] by accounting for the limited tensile strength of the ground which was previously disregarded. The solutions were obtained by employing the kinematic method of limit analysis providing rigorous upper bounds to the true collapse values. The inclusion of tension cracks leads to modified analytical expressions of the energy balance equation (the balance between external work and dissipated energy) and as a consequence, of the function whose minimum provides the solution in terms of failure mechanisms and associated values of soil strength. Pre-existing cracks are considered, as well as cracks that form as part of the failure mechanism. It turns out that the presence of tension cracks may significantly alter the size of each landslide contributing to the retrogression of the slope. Results in the form of dimensionless ready-to-use charts are produced for any value of engineering interest of friction angle and slope inclination for the case of dry cracks. Moreover, upper bounds for values not included in the charts can be achieved either by interpolation from the charts or by running the minimisation of the analytical functions provided in the paper.

1. Introduction
Morphological evolution of cliffs (natural and excavated slopes) is a traditional subject in engineering geology and geomorphology. Modelling of the progressive retreat of cliffs has recently received considerable attention by the engineering community due to increasing coastal erosive processes caused by climate change and amplified environmental awareness at national and European level [1, 3]. The insurance industry needs reliable models for the prediction of the amount of cliff retreat over time for residential buildings located in exposed areas, whereas local authorities and decision makers
need to know the level of risk faced by the public infrastructure (e.g. coastal roads, pedestrian footpaths, car parks, etc.).

Cracks or fissures are widely present in soil slopes and can cause a significant decrease in their stability [2, 4, 5], as they provide preferential flow channels which increase the soil permeability and decrease the soil strength, they can form a part of the critical slip surface with no shear strength and when it is water filled an additional driving force is applied on the slope. They can be the result of a variety of phenomena, for instance low tensile resistance, cycles of wetting-drying [6, 7], desiccation [8-10] and weathering [11].

Weathering tend to turn hard rocks into soft rocks characterised by higher void ratios and reduced bond strengths; then soft rocks are transformed into granular residual soils typically by destroying the bonds keeping the rock grains together [12]. Note that also other natural processes, e.g. methane hydrate dissociation in methane hydrate bearing sands [13], can be conceptualized as weathering. Therefore, a common feature of weathering processes is the degradation of the mechanical properties of the geomaterials (rocks and soils) of which the slope face is made; which in turn leads to the occurrence of successive landslides. This paper aims at investigating and developing an engineering model of the morphological evolution of natural cliffs with limited tensile strength subject to weathering induced landslides.

Most stability analyses of slopes are mainly based on the conventional limit equilibrium method. The weakness with limit equilibrium methods, when it comes to studying slopes with cracks, is that they are not rigorous methods and are limited in their capacity of analysis, since they usually require the user to assume a crack depth and location in the slope. Physical modelling has also been used through the years to simulate the behaviour of slopes subject to weathering, the results of which can be used for the validation of numerical models [14, 15]. The use of numerical methods such as Finite Element Method [16, 17] and Material Point Method [18] to provide approximate solutions to the slope stability problems is also increased in the latest decades. The Discrete Element Method has also been employed [19], more recently for 3D analysis of the stability of rock slopes [20]. This has been made possible by computational advances in the DEM contact detection algorithms to deal with polyhedral blocks [21, 22]. However, in cases of rather uniform slopes subject to a few cracks, such a numerical approach is not justified. In this paper, the limit analysis upper bound method is used to derive the analytical law describing the evolution of slopes subject to weathering and to provide rigorous solutions. Soil strength is characterised by the Mohr-Coulomb failure criterion therefore only three parameters are needed to describe the soil properties (unit weight, friction angle and cohesion).
Soil strength can be expressed by two parameters, cohesion $c$ and internal friction angle $\varphi$, according to the Mohr-Coulomb yield criterion (Figure 1). Most of the models developed in the studies mentioned above, assume that the normal and shear stresses along the slip surface complies this criterion, which is the most used criterion for slope stability problems in cohesive soils. Thus, weathering can be taken into consideration by assuming that cohesion (and friction angle) reduces by time. In Figure 1, failure envelopes obtained from tests on a granitic rock subject to various degrees of weathering are illustrated [23]. As shown in Figure 1, weathering causes mainly a decrease in the cohesion and a lower decrease in the friction angle. In this work the case of cohesion only decrease is tackled.

2. Limit analysis model

In order to study the influence due to the presence of cracks to the stability of a slope with uniform cohesion $c$ and internal friction angle $\varphi$ subject to weathering, the kinematic approach of limit analysis upper bound method has been adopted. As described by [24], an upper-bound solution can be obtained by considering a rotational discontinuity (Figure 2). A homogeneous slope, with zero pore pressure and constant unit weight is presented, with $H$ and $\beta$ being the height and the inclination of the slope respectively. Basic elements of the slope under study are illustrated in Figure 2. In this part, detailed calculations for the upper-bound limit analysis, considering dry cracks of any possible depth and location, departing from the upper surface of the slope are illustrated.

The weathering of the slope can cause decrease in the cohesion and the friction angle of the slope’s material. In the following, only the case of decreasing cohesion is illustrated. The calculations presented below refer to the case of a horizontal slope crest, $\alpha=0$. Since the limit analysis method is not able to give any information about the final geometry of the debris propagation after each landslide, the material accumulated at the slope toe cannot be taken into account in the model. It is therefore, assumed that the accumulated debris is taken away by atmospheric agents or erosion.

2.1. First failure mechanism

Concerning the occurrence of the first landslide and according to the failure mechanism assumed above, the region of soil B-C-D-E (Figure 2) rotates rigidly around a centre of rotation $P_1$, as yet undefined, with the material below the logarithmic spiral C-D and right of the vertical crack B-C remaining at rest. The calculations for the first failure mechanisms are obtained from [2], the mechanism is defined by three variables $x_1$, $z_1$, $y_1$ (Figure 2). For the needs of this paper, only the analytical expressions for the occurrence of the second failure mechanism will be illustrated, as the analytical expressions for the first failure can be found in [2].

2.2. Second (successive) failure mechanism

The analytical expressions for the second failure also appeal to every successive failure that will take place. As stated by [24], the stability number $N_s=\gamma H/c$ can be derived from the work rate balance equation:

$$\dot{W}_{ext} = \dot{W}_d$$  \[1\]

where $\dot{W}_{ext}$ and $\dot{W}_d$ are the external work rate and the internally dissipated energy along the failure line I-D respectively. Note that this equation only appeals for dry cracks, where the rate of external work is due to the soil weight only, in the case of water-filled cracks there is also the contribution of the hydrostatic water pressure filling the crack. According to the limit equilibrium, the angle between the displacement rate (velocity) vector $u$ of the soil mass sliding away and the failure line I-D must be always equal to $\varphi$ [24]. This condition is met by the adoption of a logarithmic spiral line, where:

$$r = r_x \exp[\tan \varphi(\theta - \theta_x)]$$  \[2\]

Some of the geometrical relationships, as arising from Figure 2, will be used for the calculations of the various work rates are given below. Where $r_x$, $r_y$, and $r_z$ are the radii of the spiral at the angles $x$, $y$
and z respectively, L₁, l₁, L₂, l₂ are the horizontal lengths as indicated in Figure 2 and H is the height of the slope.

\[ r₁ = r₁ \exp[\tan \varphi(y₁ - x₁)] \]  

\[ r₂ = r₂ \exp[\tan \varphi(y₂ - x₂)] \]  

\[ H = r₃ \exp[\tan \varphi(y - x)] \sin y₁ - \sin x₁ \]  

\[ L₄ = r₄ \cos x₁ - \cos y₁ \exp[\tan \varphi(y₁ - x₁)] \]  

\[ l₅ = r₅ \cos x₁ - \cos z₁ \exp[\tan \varphi(z₁ - x₁)] \]  

\[ L₂ = r₂ \cos x₂ - \cos y₂ \exp[\tan \varphi(y₂ - x₂)] \]  

\[ l₆ = r₆ \cos x₂ - \cos z₂ \exp[\tan \varphi(z₂ - x₂)] \]  

The analytical expression of the stability factor \( N_s = \gamma H/c \), where \( \gamma \) is the unit weight of soil, H is the slope height and c the cohesion of the soil, can be obtained by calculating the rate of the dissipated energy along the logarithmic spiral shaped failure line I-D. Energy is dissipated only along the failure line I-D, according to the assumed rigid rotational mechanism:

\[ W_d = \omega r₂ \int_{y₂}^{y₁} \rho^2 d \theta \]  

\[ = \exp[2 \tan \varphi(z₂ - x₂)] \frac{\exp[2 \tan \varphi(y₂ - z₂)] - 1}{2 \tan \varphi} \]  

\[ = \omega r₂ \int_{y₂}^{y₁} f_d(x₂, y₂, z₂, \varphi) \]  

where \( \omega \) is the angular velocity.

After the region B-C-D-E has slipped away, further weathering will at some point induce a second landslide. The double logarithm spiral shaped area G-I-D-C-B will rigidly rotate around the center of rotation P₂, as yet undefined, with the material below the logarithmic spiral I-D and right of the vertical crack G–I remaining at rest. The equation for the energy balance for a double logarithmic region rigidly rotating was first presented in [25]. Here, the mechanism is defined by six variables \( x₁, z₁, y₁, x₂, z₂, y₂ \). The rate of external work due to the soil weight of region G-I-D-C-B is calculated as the work done by the region J-D-E subtracting the work done by the region B-C-D-E and the work done by the region J-I-J. A direct integration of the rate of the external work due to the soil of the regions mentioned above is very complicated. Instead the method of superposition by first finding the rates of works \( W₁, W₂, W₄, W₅, W₆ \) respectively. The rate of the work of G-I-E-J is given by six contributors \( W₁, W₂, W₄, W₅, W₆ \) respectively. The rate of the work of J-D-E is given by three contributors \( W₅, W₆ \) respectively. And, finally the rate of the region J-I-G is also given by three contributors \( W₄, W₅, W₆ \) respectively. Therefore, it can be summed up that the rate of external work due to the weight of the soil is a total of twelve different contributors, each one referring to the work done by the soil weight of the regions that are shown in Figure 2, given by the following equation:

\[ \dot{W}_{ext} = \dot{W}_1^n - \dot{W}_2^n - \dot{W}_3^n - \dot{W}_4^n - \dot{W}_5^n + \dot{W}_6^n - \dot{W}_7^n + \dot{W}_8^n + \dot{W}_9^n - \dot{W}_5^n + \dot{W}_4^n - \dot{W}_6^n \]
with \( n \) and \( o \) referring to the new and the old landslide respectively.

To find a solution to the problem, it is necessary to express all the various work contributions in terms of a common variable, here chosen as \( r_o \), so that the equation obtained will be linear in \( H \). The calculation of the work rates \( W_1^n \), \( W_2^n \) and \( W_3^n \) can be found in Chen (1975). For the demands of this paper only the first and the final expressions of all calculations are given; detailed calculations for each of the following expressions are available.

**Figure 2.** Second Failure mechanism: Region of soil G-I-D-C-B by rotates around point P2. Region of soil B-C-D-E has already slipped away when the first failure occurred.

Considering the region P2JD:

\[
W_1^n = W_{P2JD} = \left( \frac{2}{3} \omega r_o \cos \theta \right) \left( \frac{1}{2} \gamma r_o^2 d \theta \right)
\]

\[
= \omega \gamma r_o^3 \exp\left[3\tan \phi (y_2 - x_2)\right] \left(3\tan \phi \cos y_2 + \sin y_2\right) - 3\tan \phi \cos x_2 - \sin x_2 \frac{3(1 + 9 \tan^2 \phi)}{3(1 + 9 \tan^2 \phi)}
\]

\[
= \omega \gamma r_o^3 f_1^n (x_2, y_2, \phi)
\]

Considering the region P2JE:
\[ W^n_2 = \dot{W}_{P;JE} = \left( \omega - \frac{1}{3}(2r_x \cos x_2 - L_z) \right) \left( \frac{1}{2} \gamma L z r_x \sin x_2 \right) \]
\[ = \omega \gamma r_x^3 \left( \frac{1}{6} \frac{L_z}{r_x} (2 \cos x_2 - \frac{L_z}{r_x}) \sin x_2 \right) \]  \hspace{1cm} [15]
\[ = \omega \gamma r_x^3 f^n_2 (x_2, y_2, \phi) \]

Considering the region P3ED:
\[ \dot{W}_3 = \dot{W}_{P3;ED} = (\omega - \frac{2}{3} r_x \cos y_2) \left( \frac{1}{3} \gamma H r_x \cos y_2 \right) \]
\[ = \omega \gamma r_x^3 \left( \frac{1}{3} \left[ \exp(\tan(y_2 - x_2) \sin y_2 - \sin x_2) \right] \cos^2 y_2 \exp(2 \tan \phi(y_2 - x_2) \right) \]
\[ = \omega \gamma r_x^3 f^n_3 (x_2, y_2, \phi) \]  \hspace{1cm} [16]

Considering the region P3JI, the rate of the external work is calculated for an infinitesimal slice, as illustrated in Figure 3:
\[ d\dot{W}_4^n = u_x dF_4 = \omega \left| x_{G4} - x_P \right| \gamma dA = \omega - \frac{1}{3} r^3 \gamma \cos \theta d \theta \]

where \( u_x \) is the displacement rate of the infinitesimal region, \( dF_4 \) is the weight force, and \( x_{G4} \) and \( x_P \) are the x coordinates of the gravity center of the soil region and of the center of rotation P2 respectively.

Integrating over the whole region, the rate of the work of the region P3JI is calculated:
\[ \dot{W}_4^n = \dot{W}_{P;JI} = (\omega - \frac{2}{3} r_0 \cos \theta) \left( \frac{1}{2} \gamma r_0^2 d \theta \right) \]
\[ = \omega \gamma r_x^3 \exp \left[ 3 \tan \phi(z_2 - x_2) \right] \left( \frac{3 \tan \phi \cos z_2 + \sin x_2 - 3 \tan \phi \cos x_2 - \sin x_2}{3(1 + 9 \tan^2 \phi)} \right) \]
\[ = \omega \gamma r_x^3 f^n_3 (x_2, z_2, \phi) \]  \hspace{1cm} [17]

**Figure 3.** Infinitesimal slice of logarithmic spiral regions.
Considering the region P$_2$JG:
\[ W^o_{5} = W_{P:AB} = \left[ \omega \frac{1}{3} (2r_2 \cos x_2 - l_2) \left( \frac{1}{2} \gamma l_2 r_2 \sin x_2 \right) \right. \]
\[ = \omega \gamma l_2 r_2 \frac{1}{6} \left( 2 \cos x_2 - \frac{l_2}{r_2} \right) \sin x_2 \]
\[ = \omega \gamma l_2 r_2 \frac{1}{3} \gamma_{5}^{o} (x_2, z_2, \varphi) \quad [18] \]

Finally, considering the region P$_2$GI:
\[ W^o_{6} = W_{P:GI} = \left( \omega \frac{2}{3} r_2 \cos z_2 \right) \left( \frac{1}{2} \gamma Hr_2 \cos z_2 \right) \]
\[ = \omega \gamma l_2 r_2 \frac{1}{3} \left[ \exp(\tan(\varphi - x_2) \sin z_2 - \sin x_2) \cos^2 z_2 \exp(2 \tan \varphi (z_2 - x_2)) \right. \]
\[ = \omega \gamma l_2 r_2 \frac{1}{3} \gamma_{6}^{o} (x_2, z_2, \varphi) \quad [19] \]

All calculations of the external works mentioned above, refer to the occurrence at the second landslide and have been achieved by calculating the moment of each soil region around the point P$_2$. The calculations for the remaining six regions that refer to the occurrence of the first landslide should also be achieved by calculating the moment of each soil region around the point P$_2$. Note that for the calculation of the rates of the external work $\dot{W}_1^o$, $\dot{W}_2^o$ and $\dot{W}_3^o$, the front of the slope is considered vertical for the simplicity of the calculations and that is the reason why the inclination of the slope $\beta$ does not appear in the calculations. After subtracting $\dot{W}_2^o$ and $\dot{W}_3^o$ from $\dot{W}_1^o$ the result is the same.

For the region P$_1$FD, the rate of the external work is calculated for an infinitesimal slice, similar to $\dot{W}_4^o$, by considering this time the point P$_2$ as the center of rotation:
\[ d\dot{W}_1^o = \left( \omega \frac{2}{3} r_2 \cos \theta - r_1 \cos y_1 + r_2 \cos y_2 \right) \left( \frac{1}{2} \gamma r_2 \theta \right) d\theta \]

And after integration by parts, manipulations and substitutions ([3][5]) of the obtained expression:
\[ \dot{W}_1^o = \dot{W}_{P:FD} \]
\[ = \omega \gamma l_2 r_2 \left[ \frac{\left( \frac{r_2}{x_2} \right)^2 \left[ \exp(\tan \varphi (y_2 - x_2) \cos y_2 \exp(2 \tan \varphi (y_1 - x_1) - 1) \right]}{4 \tan \varphi} \right. \]
\[ + \left. \left( \frac{r_2}{x_2} \right)^3 \exp(3 \tan \varphi (y_1 - x_1)) \left( \sin y_1 + 3 \tan \varphi \cos y_1 - \sin x_1 - 3 \tan \varphi \cos x_1 \right) \right] \]
\[ \left. + \left( \frac{r_2}{x_2} \right)^3 \left[ \exp(\tan \varphi (y_1 - x_1) \cos y_1 \exp(2 \tan \varphi (y_1 - x_1) - 1) \right] \right] \]
\[ = \omega \gamma l_2 r_2 \frac{1}{3} \gamma_{1}^{o} (x_1, y_1, x_2, y_2, \varphi) \quad [20] \]

Considering the region P$_1$FE:
\[ \dot{W}_2^o = \dot{W}_{P:FE} = \left[ \frac{1}{3} (2r_1 \cos x_1 - L_1 + r_2 \cos y_2 - r_2 \cos y_1) \right] \left( \frac{1}{2} \gamma L_1 r_1 \sin x_1 \right) \]

And after manipulations and substitutions ([3][5][8]) of the obtained expression, the following expression is obtained:
\[
\dot{W}_2^o = \omega \gamma r_{z_2}^3 \left( \frac{1}{2} \frac{r_{x_1}}{r_{z_2}} \right)^2 \sin x_1 \left[ \frac{1}{r_{x_1}} \left( \cos x_1 - 2 \exp(\tan \varphi(y_1 - x_1)) \cos y_1 + \exp(\tan \varphi(y_2 - x_2)) \cos y_2 \right) \right]
\]

\[
= \omega \gamma r_{z_2}^3 f_2^o(x_1, y_1, x_2, y_2, \varphi)
\]

Considering the region \(P_{1ED}\):
\[
\dot{W}_3^o = \dot{W}_{P_{1ED}} = (\omega \frac{2}{3} r_\gamma \cos y_1 - r_\gamma \cos y_1 + r_\gamma \cos y_2) \left( \frac{1}{2} \gamma Hr_\gamma \cos y_1 \right)
\]

And after manipulations and substitutions ([3][5][7]) of the obtained expression:
\[
\dot{W}_3^o = \omega \gamma r_{z_2}^3 \left( \frac{1}{2} \frac{r_{x_1}}{r_{z_2}} \right)^2 \left[ \exp(\tan \varphi(y_1 - x_1)) \sin y_1 - \sin x_1 \right] \exp(\tan \varphi(y_1 - x_1)) \cos y_1
\]

\[
= \omega \gamma r_{z_2}^3 f_3^o(x_1, y_1, x_2, y_2, \varphi)
\]

The external work for the region \(P_{1FC}\) is calculated similar to that of \(\dot{W}_1^o\) and:
\[
d\dot{W}_4^o = (\omega \frac{2}{3} r_\theta \cos \theta - r_\gamma \cos y_1 + r_\gamma \cos y_2) \left( \frac{1}{2} \gamma r_\theta^2 d \theta \right)
\]

And after integration by parts, manipulations and substitutions ([4][6]) of the obtained expression:
\[
\dot{W}_4^o = \dot{W}_{P_{1FC}} = \left[ \left( \frac{r_{x_1}}{r_{z_2}} \right)^2 \frac{\exp(\tan \varphi(y_2 - x_2)) \cos y_2 (\exp(2 \tan \varphi(z_1 - x_1)) - 1)}{4 \tan \varphi} \right]
\]

\[
= \omega \gamma r_{z_2}^3 \left( \frac{r_{x_1}}{r_{z_2}} \right)^3 \exp(3 \tan \varphi(z_1 - x_1)) (\sin z_1 + 3 \tan \varphi \cos z_1) - \sin x_1 - 3 \tan \varphi \cos x_1
\]

\[
- \left( \frac{r_{x_1}}{r_{z_2}} \right)^3 \frac{\exp(\tan \varphi(y_1 - x_1)) \cos y_1 (\exp(2 \tan \varphi(z_1 - x_1)) - 1)}{4 \tan \varphi}
\]

\[
= \omega \gamma r_{z_2}^3 f_4^o(x_1, z_1, x_2, z_2, \varphi)
\]

Considering the region \(P_{1FB}\):
\[
\dot{W}_5^o = \dot{W}_{P_{1FB}} = \left[ \frac{1}{3} \left( 2r_\gamma \cos x_1 - l_1 - r_\gamma \cos y_1 + r_\gamma \cos y_2 \right) \right] \left( \frac{1}{2} \gamma \mu l_1 r_\gamma \sin x_1 \right)
\]

And after manipulations and substitutions ([4][6][9]) of the obtained expression and dividing by \(r_{z_2}^3\), the following expression is obtained:
\[
\dot{W}_5^o = \omega \gamma r_{z_2}^3 \left( \frac{1}{2} \frac{r_{x_1}}{r_{z_2}} \right) \sin x_1 \left( \frac{r_{x_1}}{r_{z_2}} \left( 2 \cos x_1 - \frac{l_1}{r_{z_2}} \right) - \exp(\tan \varphi(y_1 - x_1)) \cos y_1 + \exp(\tan \varphi(y_2 - x_2)) \cos y_2 \right)
\]

\[
= \omega \gamma r_{z_2}^3 f_5^o(x_1, y_1, z_1, x_2, y_2, z_2, \varphi)
\]

Finally, considering the region \(P_{1BC}\):
\[ W_6^{\alpha} = W_{P: BC} = (\omega^2 \frac{2}{3} r_i \cos z_i - r_i \cos y_i + r_i \cos y_i)(\frac{1}{2} \gamma H r_i \cos z_i) \]

And after manipulations and substitutions [4][6] of the obtained expression and dividing by \( r_3^3 \), the following expression is obtained:

\[
\tilde{W}_6^{\alpha} = \alpha e r_3^3 \left[ \exp(\tan \varphi(z_1 - x_1)) \sin z_1 - \sin x_1 \right] \exp(\tan \varphi(z_1 - x_1) \cos z_1 \cos \phi)
\]

\[
\quad \left[ \exp(\tan \varphi(y_2 - x_2)) \cos y_2 + \frac{r_1}{r_2} (\frac{2}{3} \exp(\tan \varphi(z_1 - x_1) \cos z_1) - \exp(\tan \varphi(y_1 - x_1) \cos y_1)) \right] [25]
\]

Substituting equations [7][12][13][14][15][16][17][18][19][20][21][22][23][24][25] into equation [1] and dividing by \( \omega \) and \( r_3^3 \), and rearranging, the stability factor, \( N_s = γH/c \), is obtained as

\[
\gamma H = g(x_1, y_1, z_1, x_2, y_2, z_2, \varphi) = \frac{f d}{(f_1 - f_2 - f_3 - f_4 + f_5 + f_6 - f_4 + f_2 + f_3 + f_4 - f_5 - f_6)} [\exp(\tan \varphi(y_2 - x_2) \sin y_2 - \sin x_2)] [26]
\]

**Figure 4.** Failure lines relative to the different mechanisms considered.

This second (and every latter) mechanism could pass through any point since the current slope profile is no longer straight, as presented in Figure 4. To take this possibility into consideration, the slope that was created after the occurrence of the first landslide, has been divided into a discrete number of points (n) and each point has been assumed as the toe of a slope with height, \( h_i \). In order to find the most critical mechanism among all the possible failure mechanisms the minimum stability number must be obtained. The minimum of \( g(x_1, y_1, z_1, x_2, y_2, z_2, \varphi) \) [26], provides the upper bound for the stability factor, which corresponds to the maximum value of the dimensionless cohesion \( c/(γH) \). The critical cohesion values, \( c_i \), and angles, \( x_i, y_i \), and \( z_i \), associated with the critical log spiral, should be determined for all \( n \) slopes of different height, \( h_i \), where the parameter \( y_i \) assumes a different value associated with each of the \( n \) slopes assumed. The most critical failure mechanism among the \( n \) potential mechanisms is the one with the highest cohesion value.
2.3. **Failure mechanism including crack formation**

Apart from the case of pre-existing cracks, the case of slopes with a crack forming in the soil during failure, for a tension cut-off (where \( f_t = 0 \)) and slopes with soil tensile strength limited to half of that described by the classical Mohr–Coulomb yield condition are tackled.

For any failure mechanism including the crack-opening the expenditure of energy needed to open the crack should be added to the energy dissipation along the logarithmic spiral. The final expression of this energy is given in [5]:

\[
W_{ef} = r_{xz}^2 \frac{\omega_f}{\omega_t} \int \sin(x_z) \left( f_c - \frac{1 - \sin \theta}{2} + f_t \frac{\sin \theta - \sin \varphi}{1 - \sin \varphi} \right) \cos \theta \ d \theta \\
= \omega \rho r_{xz}^2 f_d (x_2, z_2, \theta_2, \varphi)
\]

where: \( f_c = \frac{2c \cos \varphi}{1 - \sin \varphi} \) and \( f_t = \frac{2c \cos \varphi}{1 + \sin \varphi} \), the unconfined compressive strength and the tensile strength of the soil respectively, as described by the classical Mohr-Coulomb yield condition.

The final expression to calculate the stability number \( N_s = \gamma H / c \), including the energy required for the formation of the crack is:

\[
\frac{\gamma H}{c} = g(x_1, y_1, z_1, x_2, y_2, z_2, \theta_2, \varphi) \\
= \frac{f_t + f_d}{f_1^a - f_2^a - f_3^a - f_4^a + f_1^u + f_2^u + f_3^u + f_4^u - f_5^a - f_6^a} \left[ \exp(\tan \varphi(y_2 - x_i)) \sin y_2 - \sin x_i \right]
\]

3. **Cohesion decrease**

As stated by [23] it is reasonable, as a first approximation, to assume that weathering causes mainly a decrease of cohesion, and to a much lesser extent of the friction angle. For the needs of this paper only the cohesion decrease is tackled and the slopes are assumed to be homogeneous.

Computations were carried out for a wide range of parameters (friction angle \( \varphi \) and initial slope inclination \( \beta \)) for slopes with pre-existing cracks, slopes with no tensile strength (tension cut-off) where crack forming requires work to open and slopes with soil tensile strength limited to half of that described by the classical Mohr–Coulomb yield condition. Results for the evolution of the stability number from the analysis for ten successful failures for initial slope inclination \( \beta = 60^\circ \) and friction angles \( \varphi = 20^\circ \) and \( 40^\circ \) are illustrated in Figure 5.

![Figure 5. Preliminary results for the evolution on the stability number for 10 successive failures for initial slope angles \( \beta = 60^\circ \) and friction angle \( \varphi = 20^\circ \) and \( 40^\circ \).](image)
To make meaningful comparisons, the outcome of calculations in terms of the critical height for selected geometrical parameters is also presented in Table 1. The values of the crest retreat normalised by the initial slope height, the sliding area normalised by the square of the initial height and the associated crack depth normalised by the height are listed for initial slope inclination $\beta=60^\circ$ and friction angles $\phi=20^\circ$ and $40^\circ$ for the three different cases of the appearance of the crack.

Table 1. Preliminary Results for Associated Crest Retreat, Sliding Area and Crack Depth for Slopes With Initial Inclination $\beta$ and Friction Angle $\phi$, for slopes with pre-existing cracks, slopes with no tensile strength where crack forming requires work to open and slopes with soil tensile strength limited to half of that described by the classical Mohr–Coulomb yield condition

| $\beta = 60^\circ$ | $\phi = 20^\circ$ | $\phi = 40^\circ$ |
|-------------------|-------------------|-------------------|
| CR/H | A/H$^2$ | d/H | CR/H | A/H$^2$ | d/H |
| --- | --- | --- | --- | --- | --- |
| Failure | Pre-existing crack | Tension cut-off | Limited tensile strength |
| 1 | 0.316 | 0.335 | 0.339 | 0.133 | 0.163 | 0.167 |
| 2 | 0.630 | 0.218 | 0.238 | 0.254 | 0.058 | 0.119 |
| 3 | 0.864 | 0.115 | 0.189 | 0.330 | 0.021 | 0.084 |
| 4 | 1.038 | 0.061 | 0.145 | 0.377 | 0.007 | 0.055 |
| 5 | 1.167 | 0.034 | 0.110 | 0.406 | 0.003 | 0.036 |
| 6 | 1.262 | 0.018 | 0.083 | 0.424 | 0.001 | 0.022 |
| 7 | 1.333 | 0.010 | 0.060 | 0.435 | 0.000 | 0.013 |
| 8 | 1.385 | 0.005 | 0.045 | 0.442 | 0.000 | 0.008 |
| 9 | 1.423 | 0.003 | 0.033 | 0.446 | 0.000 | 0.005 |
| 10 | 1.451 | 0.002 | 0.025 | 0.448 | 0.000 | 0.003 |

Note that for the case of slopes with pre-existing crack the most critical mechanism for the second failure is the one that passes through point C (Figure 6, mechanism G-C) with a crack almost as deep.
as the one from the previous mechanism. The evolution of the slope then will follow the line C-I (Figure 6 - dashed gray lines illustrate the successive log-spirals), until a point where the mechanism J-K will become more critical. Since the analytical formulation for a slope profile made up of more than one log-spiral lines becomes prohibitive, two profiles for the initiation of the second failure were compared. The profiles were F-C-D and I-C-D and the failure mechanisms starting from these two profiles were calculated, giving very close results. Slightly more critical solutions in terms of stability numbers are obtained by assuming that the initial profile for the second failure is F-C-D and was adopted for all the results presented in this paper.

![Figure 6. Failure lines relative to the different mechanisms considered for \( \beta = 60^\circ \).](image)

4. Conclusions
The presence of cracks may heavily affect the evolution of slope subject to weathering. Consideration of crack formation as part of the failure mechanism in the slope is also important. In this paper, a model based on the kinematic approach of limit analysis to predict evolution of slopes with limited tensile strength has been proposed. The location and the depth of the cracks as well as the most critical failure mechanism for every failure were calculated via an optimisation procedure. Solutions were provided for three different cases: determination of the evolution of slopes exhibiting pre-existing cracks, slopes with no tensile strength (tension cut-off), and slopes of limited soil tensile strength. With the presented model it is possible to account for the presence of cracks into the prediction of the evolution of natural slopes subject to a series of successive failures caused by the progressive degradation of soil strength over time.

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