THE ORBITAL EQUIVALENCE OF BERNOULLI ACTIONS AND THEIR SINAI FACTORS

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Abstract. Given a countable amenable group $G$ and $\lambda \in (0, 1)$, we give an elementary construction of a type-III$_{\lambda}$ Bernoulli group action. In the case where $G$ is the integers, we show that our nonsingular Bernoulli shifts have independent and identically distributed factors.

1. Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space. We say that a measurable map $T : \Omega \to \Omega$ is nonsingular if the measure given by $\mu \circ T^{-1}$ is equivalent to $\mu$ so that they have same null sets. The measure space endowed with a nonsingular map is a nonsingular dynamical system, which is a model for studying dynamics of a system which is not at equilibrium; in the special more widely studied case where $\mu(\Omega) = 1$ and $\mu$ is invariant, $\mu \circ T^{-1} = \mu$, we obtain a measure-preserving system which models dynamics at equilibrium. If $T$ is invertible, then we also refer to it as an automorphism and also say the system is invertible. We say that $T$ is ergodic if for all $E \in \mathcal{F}$, we have that if $\mu(E \triangle T^{-1}(E)) = 0$, then $\mu(E) = 0$ or $\mu(E^c) = 0$. The theory and stock of examples developed for the study of nonsingular dynamical systems has received considerable attention in the last decade, see the survey article by Danilenko and Silva [13] and its updated version [14]. We hope to add to the stock of useful examples in the study of the isomorphism class of a nonsingular system.

Ergodic invertible nonsingular dynamical systems are usually classed by their Krieger ratio set [42, 43], which are isomorphism invariants. We will give more involved definitions in Section 4. We say that $T$ of type-II if it admits an invariant $\sigma$-finite measure; if the invariant measure can be chosen to be finite, then $T$ is type-II$_1$, otherwise $T$ is type-II$_\infty$. If $T$ is not type-II, then we say if is type-III; type-III systems

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are further classified by a parameter $\lambda \in [0,1]$. The existence of a type-III system was a long standing open problem of Halmos [20], which was resolved by Ornstein [47] who exhibited a nonsingular odometer system that is of type-III. Ulrich Krengel and Benjamin Weiss asked what are the possible Krieger types of shift systems arising from independent sequences.

Let $\mathbb{N} = \{0,1,2,\ldots\}$ be the set of natural numbers. Let $A$ be a set which will usually be finite, countable, or a subset of $\mathbb{R}$ and let $(\rho_i)_{i \in \mathbb{N}}$ be a sequence of probability measures on $A$. A one-sided Bernoulli shift on $A$ is the system given by the product probability space $(A^\mathbb{N}, \mathcal{B}, \otimes_{i \in \mathbb{N}} \rho_i)$, endowed with the left-shift given by $(Ta)_i = a_{i+1}$ for all $i \in \mathbb{N}$, where $\mathcal{B}$ is the usual Borel product sigma-algebra. In the case where all the measures $\rho_i$ are identical, then we say the Bernoulli shift is an independent and identically (i.i.d.) system. Two-sided Bernoulli shifts are similarly obtained by replacing the natural numbers with the integers, with the left-shift becoming an invertible transformation.

Although, Bernoulli shifts are one of the most fundamental objects in ergodic theory and probability theory, it is difficult to exhibit type-III Bernoulli shifts. Two decades after Ornstein’s type-III odometer, Hamachi [21] constructed the first nonsingular Bernoulli type-III systems and then three decades later, Kosloff [35] constructed one that he could verify was type-III. Both of their constructions are Bernoulli shifts on two symbols, where the corresponding probabilities are defined inductively. See also Vaes and Wahl [55, Section 6] for examples of type-III nonsingular Bernoulli shifts where the probabilities are specified by an explicit formula. Type-III shifts also play a crucial role in Kosloff’s construction of type-III Anosov diffeomorphisms [38, 39].

In this paper, we will focus on the case $\lambda \in (0,1)$ and we will discuss briefly how we could deal with the case $\lambda = 1$ in Section 8. We say that a measurable map $f : \mathbb{R} \to [0,\infty)$ is a density if $\int_{\mathbb{R}} f(u) \, du = 1$, where $du$ represents integration with the usual Lebesgue measure. We will identify the density $f$ with the probability measure given by $E \mapsto \int_E f(u) \, du$.

**Theorem 1.** For every $\lambda \in (0,1)$, there exists a choice of densities $(f_n)_{n \in \mathbb{N}}$ such that the Bernoulli shift $(\mathbb{R}^\mathbb{N}, \mathcal{B}, \otimes_{n \in \mathbb{N}} f_n)$ is of type-III, where $f_n = 1_{(0,1)}$ for all $n < 0$.

We will also show that the (half-stationary) Bernoulli shift in Theorem 1 is power weakly mixing with a Maharam extension that is a power weakly mixing $K$-automorphism. See Section 6 for details.
Our construction is readily adapted to yield examples in the case of a countable number of symbols, and also in the more general setting of a countable amenable group. We give more precise definitions in Section 7.

**Theorem 2.** Let $A$ be a countable set. For every $\lambda \in (0, 1)$, there exists a Bernoulli shift on $(A^\mathbb{Z}, \mathcal{B}, \otimes_{n \in \mathbb{Z}} \rho_n)$ that is of type-III$_\lambda$.

**Theorem 3.** Let $G$ be a countable amenable group and $\lambda \in (0, 1)$. There exists a product measure $\otimes_{g \in G} f_g$ on $[0, 1]^G$ such that the corresponding Bernoulli action is nonsingular, ergodic and of stable type-III$_\lambda$.

In a recent article of Björklund, Kosloff, and Vaes [5], they proved that in the very special case when $G$ is a locally finite group, then for every $\lambda \in (0, 1)$ there is a type-III$_\lambda$ Bernoulli action on $\{0, 1\}^G$; their construction makes use of the locally finite assumption and when applied to Bernoulli shifts of $\mathbb{Z}$, and most of the other amenable groups, the resulting Bernoulli action is dissipative, hence not ergodic.

Let $(\Omega, \mathcal{F}, \mu, T)$ and $(\Omega', \mathcal{F}', \mu', T')$ be two nonsingular systems. We say that a measurable map $\phi : \Omega \to \Omega'$ is a **nonsingular factor** if $\phi$ is **equivariant** so that $\phi \circ T = T' \circ \phi$, and the push-forward $\mu \circ \phi^{-1}$ is equivalent to $\mu'$; in the case where $\mu' = \mu \circ \phi^{-1}$, we say that $\phi$ is a **measure-preserving factor**. If $\phi : \Omega \to \Omega'$ is a factor, then we also refer to $(\Omega', \mu', T')$ as a **factor** of the original system $(\Omega, \mu, T)$. Note that in the case of two measure-preserving systems, we will always assume that factors are also measure-preserving. Sinai [53] proved under the most general possible conditions when i.i.d. systems are factors of measure-preserving systems.

**Theorem 4 (Sinai factor theorem).** A non-atomic ergodic measure-preserving system has all i.i.d. systems of no greater Kolmogorov-Sinai entropy as factors.

Notice that Sinai’s theorem holds even in the non-invertible one-sided setting. Recall that Kolmogorov-Sinai entropy [34, 52] is defined for all measure-preserving systems and for an i.i.d. system associated with the finite probability space $(A, \rho)$ the entropy of the dynamical system given by the usual static Shannon entropy: $-\sum_{c \in A} \rho(c) \log(\rho(c))$.

The Sinai factor theorem was one of the early triumphs of entropy theory which has spectacular results in identifying and classifying i.i.d. systems in the measure-preserving context [16] and entropy has been referred to as “dynamical systems most glorious number” [30]. However, the role of entropy in the general nonsingular setting remains
unclear, with many results that may appear to violate intuition built from our better understanding of the measure-preserving case. There is a striking contrast between Krieger’s finite generator theorem [41] for the measure-preserving case and Krengel’s generator theorem [40] in the nonsingular case, where the former is a landmark result in entropy theory and the latter makes no reference to entropy at all. See [14, Section 9] for more information about entropy and other invariants in nonsingular dynamics.

It will become apparent that our constructions involving factors are grounded in the measure-preserving realm. Given a Borel set $E \subset \mathbb{R}$ the conditional density of $f$ on $E$ is given by the renormalized density

$$
\left( \int_{E} f(u)du \right)^{-1} f1_E.
$$

**Theorem 5.** Fix a Borel set $E \subset [0, 1]$, and a density $g$ with support $E$. There exists a measurable map $\phi : \mathbb{R}^N \to [0, 1]^N$ such that if $(f_n)_{n \in \mathbb{N}}$ is a sequence of densities with the same conditional density $g$ on the set $E$, with

$$
\sum_{n \in \mathbb{N}} \int_{E} f_n(u)du = \infty,
$$

and an associated Bernoulli shift $(\mathbb{R}^N, \mathcal{B}, \otimes_{n \in \mathbb{N}} f_n)$ that is nonsingular, then $\phi$ is a measure-preserving factor from the nonsingular system to an i.i.d. system given by the product of Lebesgue measure restricted to the unit interval.

Recall that a real-valued random variable $Z$ is continuous with density $f$ if the law of $Z$ given by $\mathbb{P}(Z \in \cdot)$ is absolutely continuous with respect to Lebesgue measure with $f$ as its density. Theorem 5 states that there exists a deterministic equivariant function $\phi$, which only depends on the set $E$ and the common density $g$, such that if $X = (X_i)_{i \in \mathbb{N}}$ is a sequence of continuous random variables all with the same conditional law given $E$, then $\phi(X)$ is a sequence of independent random variables that are all uniformly distributed on the unit interval.

The densities $f_n$ in Theorem 5 can be chosen to satisfy the conditions of Theorem 5 with a uniform distribution serving as the common conditional distribution. Thus Theorem 5 together with Theorem 5 give various examples of nonsingular Bernoulli shifts which have i.i.d. systems as factors.

**Corollary 6.** For every $\lambda \in (0, 1)$, there exists a type-III$_\lambda$ nonsingular Bernoulli shift which has all i.i.d. factors.
A closely related result is given by Rudolph and Silva [49, Theorem 2.1], where the machinery of joinings is adapted in the nonsingular setting to construct type-III_\lambda systems for \( \lambda \in (0, 1) \) as a nonsingular joinings of two measure-preserving systems; thus it follows that i.i.d. factors can be obtained from these type-III_\lambda systems, which are not given by a product measure.

Note that it is not known whether a Bernoulli shift on a finite number of symbols can exhibit all the different types.

**Question 1.** Let \( \lambda \in [0, 1] \). Does there exists a Bernoulli shift on a finite number symbols that is of type-III_\lambda?

We already know that the answer to Question 1 is yes for \( \lambda = 1 \). It turns out that Hamachi’s original example is also of type-III_1 [37, Section 4]. See Section 2.2 for more information.

We prove the following variant of Theorem 5 in the case of a finite number of symbols. Whereas, entropy did not play a role in the statement of Theorem 5, it will be prominent in the next theorem. Let \( A \) be a finite set, and \( E \subseteq A \). Let \( \beta \) be a probability measure on the finite set \( A \). Suppose \( \beta(E) > 0 \), then the conditional measure of \( \beta \) on \( E \) is defined via

\[
B \mapsto \frac{\beta(B \cap E)}{\beta(E)}.
\]

**Theorem 7** (Low entropy Sinai factor). Let \( A \) be a finite set. Let \( E \subset A \) have at least two elements and \( \rho \) be a probability measure on \( E \) with \( H(\rho) > 0 \). Let \( \delta > 0 \). There exists a measurable map \( \phi : A^\mathbb{N} \rightarrow \{0, 1\}^\mathbb{N} \) such that if \((p_n)_{n \in \mathbb{N}}\) is a sequence of probability measures on \( A \) with the same conditional probability \( \rho \) on \( E \), with the properties that

\[
p_n(E) \geq \delta > 0 \text{ for all } n \in \mathbb{N},
\]

and that the associated Bernoulli shift \((A^\mathbb{N}, \mathcal{B}, \bigotimes_{n \in \mathbb{N}} p_n)\) is nonsingular, then \( \phi \) is a measure-preserving factor from the nonsingular system to an i.i.d. system taking two values \( \{0, 1\} \).

2. **Explicit Constructions**

It will be fairly straightforward to state the densities that we will use to prove Theorem 1 and we will defer the proof to Section 5.

2.1. **The densities for \( 0 < \lambda < 1 \).** We define the densities that will be used to prove Theorem 1. For \( n \geq 2 \), set

\[
a_n := \frac{1}{(n + 4) \log(n + 4)}.
\]
Thus \( a_n \) decreases to zero with
\[
\sum_{n=2}^{\infty} a_n = \infty \quad \text{and} \quad \sum_{n=3}^{\infty} (a_{n-1} - a_n) < \infty. \quad (4)
\]

Let \( \mathcal{L}(A) = |A| \) denote the Lebesgue measure or length of an interval \( A \). Let \( \lambda \in (0, 1) \). Then
\[
\lambda a_n + a_n < 1, \quad \text{for all} \quad n \geq 2. \quad (5)
\]

Let \( \{A_n\}_{n=2}^{\infty} \) and \( \{B_n\}_{n=2}^{\infty} \) be decreasing sequences of open intervals of \([0, 1]\) satisfying:

(a) For all \( n \in \mathbb{N} \), \( A_n \cap B_n = \emptyset \).
(b) For all \( n \in \mathbb{N} \), \( A_{n+1} \subset A_n \) and \( B_{n+1} \subset B_n \).
(c) For all \( n \in \mathbb{N} \), \( |A_n| = a_n = \lambda^{-1} |B_n| \).

Using these sequences we define a sequence of functions \( f_n : [0, 1] \to \{\lambda^{-1}, 1, \lambda\} \). For all integers \( n \leq 1 \), set \( f_n \equiv 1 \). For \( n \geq 2 \), set
\[
f_n(u) := \begin{cases} 
\lambda, & u \in A_n, \\
\lambda^{-1}, & u \in B_n, \\
1, & u \in [0, 1] \setminus (A_n \cup B_n).
\end{cases} \quad (6)
\]

An easy calculation verifies that the \( f_n \) are densities. For all \( n \geq 2 \), we have
\[
\int_0^1 f_n(u) du = (1 - |A_n| - |B_n|) + \lambda |A_n| + \lambda^{-1} |B_n| = 1.
\]

In addition, with regards to Theorem 5, the strict inequality in (5) assures us that the set \( E \) for which the densities have the same conditional (uniform) distribution exists.

We will refer to Lebesgue measure \([0, 1]\) as the underlying probability measure, when comparing the construction given here to the construction given later in Section 2.2.1.

2.1.1. Brief outline of the Proof of Theorem 1. We will verify that the densities defined above witness Theorem 1 so that the Bernoulli shift \((\Omega, \mathcal{B}, \mu, T)\) is a type-III \( \lambda \), where \( \Omega = [0, 1]^\mathbb{Z} \) and \( \mu = \bigotimes_{n \in \mathbb{Z}} f_n \).

A straightforward application of Kakutani’s theorem on equivalence of product measures implies nonsingularity. To show that it is of the appropriate Krieger type, we employ a synthesis of methods which were recently developed for the study of Bernoulli shifts on two symbols. We show that the shift is conservative by using ideas appearing in Vaes and Wahl [55, Proposition 4.1] and Danilenko, Kosloff, and Roy [11, Proposition 2.5]. This is enough to imply ergodicity in the setting of a product measure, since the shift is a \( K \)-automorphism.
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It is well-known that the discrete Maharam extension of \( T \) is ergodic if and only if it is of type-III\( _\lambda \); see Theorem 18. We will show the stronger property that the Maharam extension is a conservative \( K \)-automorphism; we argue that the tail sigma-algebra is trivial by showing that the larger exchangeable sigma-algebra is trivial. In the course of our proof, we will also obtain that the action of the group of all finite permutations of the integers on \( (\Omega, \mathcal{B}, \mu) \) is ergodic.

2.2. Discrete Random Variables. Prior to Theorem 1, all constructions of type-III Bernoulli shifts were of type-III\( _1 \); we already mentioned in Question 1 that it is not known whether Bernoulli shifts on a finite number of symbols can exhibit all the different Krieger type. In fact under weak conditions, the behaviour of such shifts is severely limited in the following sense. Let \( A \) be a finite set. Let \( \rho_n \) be probability measures on \( A \). We say that the product measure \( \mu = \bigotimes_{n \in \mathbb{Z}} \rho_n \) satisfies the Doeblin condition if there exists \( \delta > 0 \) such that for all \( n \in \mathbb{Z} \) and \( a \in A \), we have \( \rho_n(\{a\}) > \delta \). In the case where \( A = \{0, 1\} \) it was shown in [5, Theorem B] that when \( \mu \) satisfies the Doeblin condition, then either \( \left( \{0, 1\}^\mathbb{Z}, \mathcal{B}, \mu, T \right) \) is dissipative (see Section 5.2) or it is ergodic and of type II\( _1 \) or III\( _1 \). Recently, Avraham-Re’em [11] extended this dichotomy to the broader class of inhomogeneous Markov shifts supported on mixing subshifts of finite type; a particular case of this result is the following.

**Theorem 8** (Avraham-Re’em). Let \( A \) be a finite set and \( (A^\mathbb{Z}, \mathcal{B}, \mu, T) \) be a nonsingular Bernoulli shift. If the product measure \( \mu \) satisfies the Doeblin condition then the system is either is dissipative or it is ergodic and of type II\( _1 \) or III\( _1 \).

The question arises whether Theorem 8 holds when the product measure does not satisfy the Doeblin condition, as in the countable case. When \( A = \{0, 1\} \) the following question arising from [5] is still open.

**Question 2.** What are the possible Krieger types of the Bernoulli shift \( \left( \{0, 1\}^\mathbb{Z}, \mathcal{B}, \bigotimes_{n \in \mathbb{Z}} \rho_n, T \right) \) with \( \lim_{|n| \to \infty} \rho_n(0) = 0 \)?

2.2.1. The probability mass function for the countable case. We will adapt the our construction in Section 2.1 to the countable setting by replacing the decreasing sequences of subsets of the unit interval with decreasing sequences of subsets of \( \mathbb{N} \).

Let \( 0 < \lambda < 1 \). For each integer \( n \geq 1 \), let

\[
a_n = \frac{1}{(n + 4) \log(n + 4)}.
\]
Let $\rho$ be the probability mass function on $\mathbb{N}$ defined by
\[
\rho(n) = \begin{cases} 
1 - (1 + \lambda)a_1, & n = 0, \\
a_k - a_{k+1}, & n = 2k \\
\lambda(a_k - a_{k+1}), & n = 2k + 1.
\end{cases}
\]
We will refer to $\rho$ as the underlying probability measure. For each integer $n \geq 1$, set
\[
A_n = 2\mathbb{N} \cap [2n, \infty) \quad \text{and} \quad B_n = (\mathbb{N} \setminus 2\mathbb{N}) \cap [2n-1, \infty),
\]
then
\[
\rho(A_n) = \sum_{k=n}^{\infty} (a_k - a_{k+1}) = a_n = \lambda^{-1} \rho(B_n).
\]
For each integer $n \geq 1$, let $f_n : \mathbb{N} \to \{\lambda^{-1}, 1, \lambda\}$ be defined via
\[
f_n(k) = \begin{cases} 
\lambda, & k \in A_n, \\
\frac{1}{\lambda}, & k \in B_n, \\
1, & k \in \mathbb{N} \setminus (A_n \cup B_n).
\end{cases}
\]
We collect the following useful observation for future reference.

**Remark 9.** Note that for all $n \geq 1$, we have
\[
\sum_{k=1}^{\infty} f_n(k) \rho(k) = \rho(\mathbb{N} \setminus (A_n \cup B_n)) + \lambda \rho(A_n) + \frac{1}{\lambda} \rho(B_n) = 1.
\]
Thus the functions $f_n$ are probability density functions with respect the underlying probability measure $\rho$. 

Finally, define the product measure $p$ on $\mathbb{N}^\mathbb{Z}$ by
\[
p_n = \begin{cases} 
\rho, & n \leq 0 \\
f_n \rho, & n \geq 1.
\end{cases}
\]

Let $\Omega = \mathbb{N}^\mathbb{Z}$, and $\mathcal{B}$ be the usual product sigma-algebra, and $\mu = \bigotimes_{n \in \mathbb{Z}} p_n$. We will show that the Bernoulli shift $(\Omega, \mathcal{B}, \mu, T)$ witnesses Theorem \[2\]. After we prove Theorem \[4\], we will see that the proof of Theorem \[2\] given in Section \[8\] will be a straightforward adaptation of the proof for the continuous random variables.

3. **The Proofs of Theorems \[5\] and \[7\]**

In our proof of Theorem \[5\], we will harness independent uniform random variables for every integer $n$ for which $x_n \in E \subset [0, 1]$; these uniform random variables will then be distributed to the other integers. Kalikow and Weiss \[28\] elegantly use similar ideas to construct explicit isomorphism of some infinite entropy processes.
Proof of Theorem 5. Let the conditional density of the $f_n$ on $E$ be $g$. Thus if $G : \mathbb{R} \rightarrow [0, 1]$ is the cumulative distribution function given by

$$G(v) = \int_{-\infty}^{v} g(u)du$$

we easily verify that if $V$ is a random variable with density $g$, then $G(V)$ is uniformly distributed in $[0, 1]$. We also note that by taking binary expansions, it is easy to see that there exists a measurable function $r : [0, 1] \rightarrow [0, 1]^\mathbb{N}$ such that if $U$ is uniformly distributed in $[0, 1]$, then $r(U)$ is a an i.i.d. sequence of random variables that are uniformly distributed in $[0, 1]$; for details see [29, Lemma 3.21].

Let $X = (X_i)_{i \in \mathbb{N}}$ be a sequence of independent continuous random variables with corresponding densities $(f_n)_{n \in \mathbb{N}}$. Call $s \in \mathbb{N}$ special if $X_s \in E$; thus conditional on the event that $s$ is special, $G(X_s)$ is uniformly distributed in $[0, 1]$; furthermore, conditional on the sequence of special integers $s_n$, the sequences of random variables $(r \circ G)(X_{s_n})_{n \in \mathbb{N}}$ are independent.

Thus the assumption on the densities $f_n$ make it is easy to independently assign a uniform random variable to each special natural number in an equivariant way. Observe that by (1) and the second Borel-Cantelli lemma there are infinitely many special natural numbers. It remains to independently assign each non-special natural number a uniform random variable, in an equivariant way.

We say that each $k \in \mathbb{N}$ reports to the smallest integer greater than or equal to $k$ that is special; thus if $s$ is special, then it reports to itself. If $k \in \mathbb{N}$ reports to the special integer $s$, we set

$$[\phi(X)]_k = [r(G(X_s))]_{s-k}.$$ 

It is easy to verify that $\phi$ satisfies the required properties. \qed

Our proof of Theorem 5 is slightly more involved than our proof of Theorem 3 since in the finite entropy regime we cannot replicate uniform random variables. At the special integers we only get i.i.d. discrete random variables; these random variables will be transformed using Theorem 4 into \textbf{bits}, that is, zero and ones, that will then be distributed using the following equivariant matching scheme.

Consider $d \in \mathbb{Z}^+$ and a subset $\Omega' \subset \{a, b\}^\mathbb{N}$ with $T(\Omega') \subset \Omega'$, where $T$ is the left-shift. We want to define a loop-free graph $G(\omega)$ on $\mathbb{N}$ with the following properties.

- If $m$ and $n$ are adjacent and $m < n$, then $\omega_m = b$ and $\omega_n = a$.
- Each vertex $n$ with $\omega_m = b$ is of degree 1.
- Each vertex $m$ such that $\omega_m = a$ has degree at most $d$. 


• If \( n, m \geq 1 \), then the vertices \( n \) and \( m \) are adjacent in \( G(\omega) \) if and only if \( n - 1 \) and \( m - 1 \) are adjacent in \( G(T\omega) \).

We call \( G \) a **degree-\( d \) equivariant matching scheme**. Thus \( G \) is a matching of \( a \)'s and \( b \)'s, where every \( b \) is matched to a unique \( a \), and each \( a \) has at most \( d \) partners.

**Proposition 10.** Consider the Bernoulli shift \((\{a,b\}^\mathbb{N}, \mathcal{B}, \otimes_{n \in \mathbb{N}} p_n)\), where \( p_n(a) \geq \delta > 0 \) for all \( n \in \mathbb{N} \). Let \( d \in \mathbb{Z}^+ \) be such that \( d \geq (1 - \delta)/\delta \). There exists a degree-\( d \) equivariant matching scheme \( G \) on a set of full measure.

Mešalkin [45] gave the first example of a nontrivial isomorphism between two i.i.d. systems and almost a half a century afterwards his map was adapted by Holroyd and Peres [24] to define a perfect equivariant matching scheme for the case of i.i.d. fair coin-flips. Our proof of Proposition 10 uses a similar adaptation of Mešalkin map. For related matchings constructions in probability theory see also [15, 23, 54].

**Proof of Proposition 10.** Let \( \omega \in \{a,b\}^\mathbb{N} \). We define the graph \( G(\omega) \) inductively in the following way. For each \( n \in \mathbb{N} \), if \( \omega_n = b \) and \( \omega_{n+1} = a \), then add the edge \( \{n, n+1\} \) to the graph; that is, we match an \( a \) and \( b \) if \( b \) is immediately followed by an \( a \). Now we disregard all the \( b \)'s that have been matched, and all the \( a \)'s that are already of degree \( d \), and repeat inductively.

Note that by definition if \( n < m \), and \( \omega_n = b = \omega_m \), then if \( (n, k) \) is an edge with \( k > m \), then \( (m, \ell) \) is an edge for some \( m < \ell \leq k \).

To show that every \( b \) is matched, let \( X_n \) be independent \( \{-1, d\} \)-valued random variables with \( \mathbb{P}(X_n = d) = p_n(a) \). Identify \( a \)'s with \( d \)'s and \( b \)'s and with \( -1 \)'s. Fix \( m \in \mathbb{N} \). Suppose \( X_m = -1 \). Let

\[
S_n := X_m + \cdots + X_{m+n}
\]

and let

\[
R := \inf \{k \geq 1 : S_k \geq 0\}.
\]

Then

\[
\mathbb{P} \{ (m, m + \ell) \text{ is not an edge for all } \ell \leq k \} = \mathbb{P}(R > k). \tag{8}
\]

It remains to show that the right hand side of (8) decays to zero as \( k \to \infty \). Let \( X'_n \) be i.i.d. \( \{-1, d\} \)-valued random variables with \( \mathbb{P}(X'_0 = d) = \delta \). Also assume that \( X'_m = -1 \) and similarly define \( S'_n \) and \( R' \). A standard coupling argument gives that

\[
\mathbb{P}(R > k) \leq \mathbb{P}(R' > k).
\]

Since

\[
\mathbb{E}X'_0 = d\delta - (1 - \delta) \geq 0,
\]

(8)

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\[
\mathbb{P}(R > k) \leq \mathbb{P}(R' > k).
\]

Since

\[
\mathbb{E}X'_0 = d\delta - (1 - \delta) \geq 0,
\]
if $\mathbb{E}X'_0 > 0$, then the law of large numbers gives that $R'$ is finite almost surely and if $\mathbb{E}X'_0 = 0$, then classical results of Chung and Ornstein \cite{7, 8} regarding the recurrence random walks imply that $R'$ is finite almost surely. □

Proof of Theorem 7. Let $d \in \mathbb{Z}^+$ be so that $d \geq (1 - \delta)/\delta$. Let $\alpha$ be a probability measure on $\{0, 1\}$ such that

$$(d + 1)H(\alpha) \leq H(\rho).$$

By Theorem 4, it follows there exists a factor $\psi$ from the i.i.d. system with common distribution $\rho$ to the i.i.d. system with common probability measure $\alpha$ on $\{0, 1\}$. Let $X = (X_i)_{i \in \mathbb{N}}$ be a sequence of independent random variables with corresponding probability mass functions $(p_i)_{i \in \mathbb{N}}$. Call $s \in \mathbb{N}$ special if $X_s \in E$; thus conditional on the event that $s$ is special, $(X_s)$ has p.m.f. $\rho$; furthermore, conditional on the sequence of special integers $s_n$, the sequence of random variables $Y = (X_{s_n})_{n \in \mathbb{N}}$ are independent. We apply the factor map $\psi$ on $Y$ to obtain at each special integer, $d$-independent bits with distribution $\alpha$.

Similarly to the Proof of Theorem 5, it remains to distribute the bits from the special integers, in an equivariant way, so that each integer has a bit; this can be accomplished via Proposition 10. Each special integer retains one bit and allows the remaining $d$ bits to be distributed to the non-special integers according to the given equivariant matching; any remaining bits are discarded. □

Remark 11. At the outset, we invoked Sinai’s factor theorem in our proof of Theorem 7. If one required a more constructive proof, and an explicit factor map, we could instead appeal to del Junco’s finitary version of Sinai’s theorem \cite{25, 26}. However, then we must assume that the set $E$ contains at least three symbols and require also that resulting Bernoulli shift be on three symbols. Keane and Smorodinsky’s celebrated finitary isomorphisms \cite{31, 32} do not have a symbol restriction, but are not one-sided. ◊

4. Krieger’s ratio set

In what follows, it will be convenient to think of an invertible non-singular dynamical systems $T = (T^n)_{n \in \mathbb{Z}}$ as the integer group action. Let $G$ be a group and write $\mu \sim \nu$ for two equivalent measures. A nonsingular group action is a measure space $(\Omega, \mathcal{F}, \mu)$ endowed with a group action $T = (T_g)_{g \in G}$ such that $T_g \circ T_h = T_{gh}$ and $\mu \circ T_g \sim \mu$ for all $g, h \in G$; we say it is ergodic for all $E \in \mathcal{F}$ and all $g \in G$, we have that if $\mu(\Delta T_g(E)) = 0$, then either $\mu(E) = 0$ or $\mu(E^c) = 0$. We
say that $T$ is \textbf{conservative} if for every $A \in \mathcal{F}$ with positive measure, there exist a $g \in G$ that is not the identity, with $\mu(A \cap T^{-g}A) > 0$, and otherwise we say that $T$ is \textbf{dissipative}. We will often identify $T_g$ with the $g$.

We say that two nonsingular group actions $(\Omega, \mathcal{F}, \mu, (T_g)_{g \in G})$ and $(\Omega', \mathcal{G}, \nu, (S_h)_{h \in H})$ are \textbf{orbit equivalent} if there exists a measurable bijection $\phi : \Omega \rightarrow \Omega'$ such that $\nu \sim \mu \circ \phi^{-1}$ and $\phi(\text{orb}_G(x)) := \text{orb}_H(\phi(x))$ for $\mu$-almost all $x \in \Omega$, where $\text{orb}_G(x) = \{T_g(x) : g \in G\}$. Motivated by problems in von Neumann algebras, Dye [17, 18] proved in the setting of a probability preserving system that any two Abelian discrete group actions are orbit equivalent.

Following the work of Araki and Woods [3], who were again motivated by the Murray-von Neumann classification problem, Krieger [42, 43] extended Dye's celebrated result to the nonsingular setting. For an ergodic nonsingular action a number $r \in \mathbb{R}$ is an \textbf{essential value} for $T$ if for all $A \in \mathcal{F}$ with $\mu(A) > 0$ and $\epsilon > 0$ there exists $g \in G$ such that

$$
\mu \left( A \cap T_g^{-1}A \cap \left[ \left| \log \frac{d\mu \circ T_g}{d\mu} - r \right| < \epsilon \right] \right) > 0.
$$

The \textbf{Krieger ratio set} $e(T)$ is the collection of all essential values of $T$. The ratio set is a closed subgroup of $\mathbb{R}$ hence it is of

- type-II or type-III$_0$: $e(T) = \{0\}$;
- type III$_\lambda$: $e(T) = \{n \log \lambda : n \in \mathbb{Z}\}$ for some $0 < \lambda < 1$; or
- type III$_1$: $e(T) = \mathbb{R}$.

The Krieger types are invariants for orbit equivalence and are a complete invariant when $e(T)$ is nonempty and $e(T) \neq \{0\}$; this classification holds for any discrete amenable group action. See [9] for background and more details.

We will use the following lemma to verify whether a given number is an essential value for $T$. The \textbf{orbital equivalence relation} of the action $T$ is the Borel subset $\mathcal{O}_T \subset X \times X$ defined by

$$(x, y) \in \mathcal{O}_T \text{ if and only if there exists } g \in G \text{ such that } T_gx = y.$$  

The \textbf{full group} $[T]$ consists of all nonsingular automorphisms $V$ of $(\Omega, \mathcal{F}, \mu)$ such that for almost all $x \in \Omega$, we have $(x, V(x)) \in \mathcal{O}_T$. Let $A \in \mathcal{F}$. We say that an injective nonsingular map $V : A \rightarrow V(A) \in \mathcal{F}$ such that $(x, V(x)) \in \mathcal{O}_T$ for all $x \in A$ is a \textbf{partial transformation} with domain $A$ and range $V(A)$. The collection of partial transformations will be denoted by $[[T]]$. 
Recall that a collection of subsets of $\mathcal{F}$ is $\mu$-dense if for every $\varepsilon > 0$ and every $F \in \mathcal{F}$ there exists a $F'$ from the collection such that $\mu(F' \triangle F) < \varepsilon$.

**Lemma 12** (Approximation). Let $(\Omega, \mathcal{F}, \mu, T)$ be a nonsingular ergodic group action. Let $\mathcal{G} \subset \mathcal{F}$ a countable semi-ring such that the ring generated by $\mathcal{G}$ is $\mu$-dense in $\mathcal{F}$. If there exists $0 < \delta < 1$ such that for each $A \in \mathcal{G}$ and $\varepsilon > 0$ there is:

- a subset $B \subset A$ with $\mu(V(B)) > \delta \mu(A)$ and
- a partial transformation $V : B \to A$ such that $(x, Vx) \in O_T$

and for all $x \in B$, $\left| \frac{d\mu \circ V}{d\mu}(x) - r \right| < \varepsilon$,

then $r \in e(T)$.

**Proof.** The proof is a routine extension of the second half of [6, Lemma 2.1], where we allow $V \in [[T]]$, rather than requiring that $V \in [T]$. See also [12, Lemma 2.1]. \qed

### 5. Proof of Theorem 1

#### 5.1. Nonsingularity and Kakutani’s theorem.

Let $(\Omega, F, \mu, T)$ be an invertible nonsingular system. We write

$$T' := \frac{d\mu \circ T}{d\mu}.$$ 

Let $\Omega = [0, 1]^\mathbb{Z}$. Given a sequence of densities $f_n : [0, 1] \to (0, \infty)$, let $\mu = \bigotimes_{n \in \mathbb{Z}} f_n$ be the associated product measure. Since $\mu \circ T = \bigotimes_{n \in \mathbb{Z}} f_{n-1}$ is also a product measure, it follows from Kakutani’s theorem [27] on equivalence of product measures that $\mu$ is $T$ nonsingular if and only if

$$\sum_{n \in \mathbb{Z}} \int_0^1 \left( \sqrt{f_n(u)} - \sqrt{f_{n-1}(u)} \right)^2 du < \infty. \quad (9)$$

With the densities $f_n$ defined in Section 2 by Kakutani’s theorem, for $\mu$-almost every $x \in \Omega$ and for all $n \in \mathbb{Z}$, we have

$$(T^n)'(x) := \frac{d\mu \circ T^n}{d\mu}(x) = \prod_{k \in \mathbb{Z}} \frac{f_{k-n}(x_k)}{f_k(x_k)} \in \{\lambda^n : n \in \mathbb{Z}\}. \quad (10)$$

In particular, we have $e(T) \subset \{\lambda^n : n \in \mathbb{Z}\}$ and thus to show that $T$ is type III$\lambda$ it is enough to show that $T$ is ergodic and that $\lambda \in e(T)$.

**Lemma 13.** With the densities $f_n$ defined in Section 2, the associated Bernoulli shift is nonsingular.
Proof. Setting $A_1 = B_1 = \emptyset$, we see that

$$\sum_{n \in \mathbb{Z}} \int_0^1 \left( \sqrt{f_n(u)} - \sqrt{f_{n-1}(u)} \right)^2 du = \sum_{n=2}^{\infty} \int_0^1 \left( \sqrt{f_n(u)} - \sqrt{f_{n-1}(u)} \right)^2 du$$

$$= \sum_{n=2}^{\infty} \left( \lambda |A_{n-1} \setminus A_n| + \frac{1}{\lambda} |B_{n-1} \setminus B_n| \right)$$

$$\leq \frac{\lambda}{6 \log 6} + \sum_{n=3}^{\infty} 2\lambda(a_{n-1} - a_n).$$

The finiteness of the right-hand side follows from (4) and thus Kakutani theorem implies the desired nonsingularity. \qed

5.2. **Conservativity and ergodicity.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $T : \Omega \to \Omega$ be a nonsingular transformation. It is well-known that if $T$ is ergodic and $\mu$ is non-atomic, then $T$ is conservative, whereas the converse fails. However, there is a partial converse, in the case that $T$ is an $K$-automorphism in the sense of the Kolmogorov zero-one law. Suppose that $T$ is invertible. Following Silva and Thieullen [51, Definition 4.5], we say that $T$ is a $K$-automorphism if there exists $\mathcal{G} \subset \mathcal{F}$ such that:

- $\mu|_{\mathcal{G}}$ is $\sigma$-finite and; $T^{-1}\mathcal{G} \subset \mathcal{G}$;
- $\bigcap_{n \in \mathbb{N}} T^{-n}\mathcal{G} = \{\emptyset, X\} \mod \mu$;
- $\bigvee_{n \in \mathbb{N}} T^n\mathcal{G} = \mathcal{F} \mod \mu$;
- $T'$ is $\mathcal{F}$ measurable.

The first three conditions are that $T$ is a natural extension of an endomorphism with a trivial tail field while the fourth comes to ensure that the natural extension is unique up to measure theoretic isomorphism of nonsingular systems.

We will make use of the following proposition from Silva and Thieullen [51, Proposition 4.8]; see also Parry [48].

**Lemma 14** (Silva and Thieullen). A $K$-automorphism is ergodic if and only if it is conservative.

**Lemma 15.** Let $\lambda \in (0, 1)$. Let $f_n$ be the densities defined in Section 2, let $\mu = \otimes_{n \in \mathbb{Z}} f_n$ and consider the associated Bernoulli shift. Then with

$$c(\lambda) = 2(\lambda^3 - 1 + \lambda^{-2} - \lambda),$$

for all $n \in \mathbb{N}$, we have

$$\int_{\Omega} \left( \frac{1}{(T^n)} \right)^2 d\mu \leq \exp \left( -c(\lambda) \sum_{k=2}^{n+1} a_k \right)$$
\textit{Proof.} Let \( n \in \mathbb{N} \). Since \( \mu \) is a product measure and \( f_k = f_{k-n} \) for all \( k \geq 1 \),
\[
\int_{\Omega} \left( \frac{1}{(T^n)'(x)} \right)^2 d\mu(x) = \prod_{n=1}^{\infty} \int_{0}^{1} \left( \frac{f_k(u)}{f_{k-n}(u)} \right)^2 f_k(u) du
\]

First note that for \( 2 \leq k \leq n + 1 \), \( f_{k-n} \equiv 1 \) and thus,
\[
\int_{0}^{1} \left( \frac{f_k(u)}{f_{k-n}(u)} \right)^2 f_k(u) du = \int_{0}^{1} f_k(u)^3 du
\]
\[
= \int_{0}^{1} \left( \lambda^3 1_{A_k} + \lambda^{-3} 1_{B_k} + 1_{(A_k \cup B_k)^c} \right) du
\]
\[
= 1 + (\lambda^3 - 1) |A_k| + (\lambda^{-3} - 1) |B_k|
\]
\[
= 1 + (\lambda^3 - 1 + \lambda^{-2} - \lambda) |A_k|
\]
\[
\leq \exp \left( \frac{c(\lambda)}{2} a_k \right).
\]

For every \( k \geq n + 2 \), since \( B_k \subset B_{k-n} \) and \( A_k \subset A_{n-k} \) we see that,
\[
\left( \frac{f_k}{f_{k-n}} \right)^2 f_k = \lambda 1_{A_k} + \lambda^{-2} 1_{A_{k-n} \setminus A_k} + \lambda^{-1} 1_{B_k} + \lambda^2 1_{B_{k-n} \setminus B_k} + 1_{(A_{k-n} \cup B_{k-n})^c}.
\]

Adding and subtracting \( \lambda 1_{A_{k-n} \setminus A_k} + \lambda^{-1} 1_{B_{k-n} \setminus B_k} \) to the right hand side shows that
\[
\left( \frac{f_k}{f_{k-n}} \right)^2 f_k = \left[ (\lambda 1_{A_{k-n}} + \lambda^{-1} 1_{B_{k-n}} + 1_{(A_{k-n} \cup B_{k-n})^c}) + (\lambda^{-2} - \lambda) 1_{A_{k-n} \setminus A_k} \right.
\]
\[
+ (\lambda^2 - \lambda^{-1}) 1_{B_{k-n} \setminus B_k}
\]
\[
= f_{k-n} + (\lambda^{-2} - \lambda) 1_{A_{k-n} \setminus A_k} + (\lambda^2 - \lambda^{-1}) 1_{B_{k-n} \setminus B_k}
\]

Integrating we see that for all \( k \geq n + 2 \),
\[
\int_{0}^{1} \left( \frac{f_k(u)}{f_{k-n}(u)} \right)^2 f_k(u) du = \left[ \int_{0}^{1} f_{k-n}(u) du + (\lambda^{-2} - \lambda) |A_{k-n} \setminus A_k| \right.
\]
\[
+ (\lambda^2 - \lambda^{-1}) |B_{k-n} \setminus B_k|
\]
\[
= 1 + (\lambda^{-2} - \lambda - \lambda (\lambda^2 - \lambda^{-1})) (|A_{k-n}| - |A_k|)
\]
\[
\leq \exp \left( \frac{c(\lambda)}{2} (a_{k-n} - a_k) \right).
\]
Combining the inequalities we have obtained, we see that
\[
\int_{\Omega} \left( \frac{1}{(T^n)^{\prime}} \right)^2 \, d\mu \leq \exp \left( \frac{c(\lambda)}{2} \left( \sum_{k=2}^{n+1} a_k + \sum_{k=n+2}^{\infty} (a_{k-n} - a_k) \right) \right) = \exp \left( c(\lambda) \sum_{k=2}^{n+1} a_k \right). \tag{11}
\]

**Lemma 16.** The Bernoulli shift associated with the densities defined in Section 2 is conservative.

**Proof.** By (3)
\[
\sum_{k=2}^{n} a_k = \log \log(n) + O(1).
\]

By Lemma 15 there exists \( k > 1 \) such that for all \( n \in \mathbb{N} \),
\[
\int_{\Omega} \left( \frac{1}{(T^n)^{\prime}} \right)^2 \, d\mu \leq k \left( \log(n) \right)^{c(\lambda)}.
\]

This implies via Markov inequality that
\[
\mu\{ x \in \Omega : (T^n)^{\prime}(x) < n^{-1} \} = \mu\{ x \in \Omega : (T^n)^{\prime}(x)^2 > n^{-2} \} \\
\leq \frac{k \left( \log(n) \right)^{c(\lambda)}}{n^2}.
\]

By the first Borel-Cantelli lemma, for almost every \( x \in \Omega \) there exists \( n(x) \in \mathbb{N} \) such that for all \( n \geq n(x) \), we have \( (T^n)^{\prime}(x) \geq \frac{1}{n} \). Since the harmonic series diverges, the comparison test gives that for almost every \( x \in \Omega \), we have
\[
\sum_{n=1}^{\infty} (T^n)^{\prime}(x) = \infty.
\]

It follows from the Hopf criteria [1, Proposition 1.3.1] that \( T \) is conservative. \( \square \)

**Corollary 17.** The Bernoulli shift associated with the densities defined in Section 2 is nonsingular, conservative, and ergodic.

**Proof.** We already know the associated Bernoulli shift is nonsingular from Lemma 13. Product measures satisfy Kolomogorov’s zero-one law [33, Appendix] and thus all nonsingular Bernoulli shifts are \( K \)-automorphisms. The result follows from Lemma 16 and Lemma 14. \( \square \)
5.3. Maharam extensions. A useful duality in studying the ratio set is given by the following skew product. The Maharam extension of a nonsingular group action \((\Omega, \mathcal{F}, \mu, (T_g)_{g \in G})\) is a \(G\) action on \(\Omega \times \mathbb{R}\), given by
\[
\tilde{T}_g(x, u) = \left(T_gx, u - \log \frac{d\mu \circ T_g}{d\mu}(x)\right),
\]
which preserves the measure given by
\[
\nu(A \times I) := \mu(A) \int_I e^u du,
\]
for all \(A \in \mathcal{F}\) and intervals \(I \subset \mathbb{R}\). By the celebrated result of Maharam [44], in the case where \(G = \mathbb{Z}\), the \(\mathbb{Z}\)-action \((\Omega, \mathcal{F}, \mu, T)\) is conservative if and only if its Maharam extension is conservative with respect to \(\nu\).

In a related result that is not required for the proof of Theorem 1, we will show in Section 6 that the product of nonsingular transformations are conservative if and only if their the product of their Maharam extensions are conservative.

Let \(\lambda \in (0, 1)\). Consider the nonsingular system \((\Omega, \mathcal{F}, \mu, T)\). Suppose that for \(\mu\)-almost every \(x \in \Omega\), we have
\[
\varphi_\mu(x) := \log_\lambda \left(\frac{d\mu \circ T}{d\mu}(x)\right) \in \mathbb{Z}. \tag{12}
\]
Then we define the **discrete Maharam extension** on \(\Omega \times \mathbb{Z}\) by
\[
\tilde{T}(x, n) := (Tx, n - \varphi_\mu(x)). \tag{13}
\]
The discrete Maharam extension preserves the measure \(\tilde{\mu}\) such that for all \(A \in \mathcal{F}\) and \(n \in \mathbb{Z}\), we have \(\tilde{\mu}(A \times \{n\}) = \lambda^n \mu(A)\). More generally, in the context of a group action \((T_g)_{g \in G}\) if
\[
\varphi_\mu(x, g) := \log_\lambda \left(\frac{d\mu \circ T_g}{d\mu}(x)\right) \in \mathbb{Z} \tag{14}
\]
for all \(g \in G\), then we define the **discrete Maharam of the \(G\)-action** by
\[
\tilde{T}_g(x, n) := (T_gx, n - \varphi_\mu(x, g)).
\]

**Theorem 18.** Let \(\lambda \in (0, 1)\). Let \((\Omega, \mathcal{F}, \mu, (T_g)_{g \in G})\) be a nonsingular group action such that (14) holds. Then the nonsingular system is conservative if and only if the discrete Maharam extension of the \(G\)-action is conservative and furthermore the nonsingular system is of type-\(\text{III}_\lambda\) if and only if the discrete Maharam extension is ergodic.

**Proof.** A proof of this theorem, in a more general setting, is given in the monograph of Klaus Schmidt [50, Corollary 5.4, Theorem 5.5]. \(\square\)
Note that by (10) the Bernoulli shift defined in Section 2 satisfies (12).

**Theorem 19.** The discrete Maharam extension associated with the nonsingular Bernoulli shift defined in Section 2 is a $K$-automorphism.

**Proof of Theorem 19.** Let $\lambda \in (0,1)$. Consider the Bernoulli shift defined in Section 2. We already know it is nonsingular, conservative, and ergodic by Corollary 17. By Theorem 18 its Maharam extension is also conservative. By Theorem 19, the associated discrete Maharam extension is a $K$-automorphism and thus by Lemma 14 it is ergodic. Thus by Theorem 18 the nonsingular Bernoulli shift is of type-III$_\lambda$. □

It remains to prove Theorem 19. We will use both directions of Theorem 18 in our proof of Theorem 19. Let $\lambda \in (0,1)$. Consider the Bernoulli shift defined in Section 2 and its one-sided version defined as follows. Let $\mathcal{B}_+ = \mathcal{B}([0,1]^\mathbb{N})$ be the product sigma-algebra for $\Omega_+ = [0,1]^\mathbb{N}$, $\mu_+ := \otimes_{n \in \mathbb{N}} f_n$, and $S$ be $T$ restricted to $[0,1]^\mathbb{N}$, then $([0,1]^\mathbb{Z}, \mathcal{B}_+, \mu, T)$ is the natural extension of $([0,1]^\mathbb{N}, \mathcal{B}_+, \mu, S)$ and the latter has a trivial tail field, by Kolmogorov’s zero-one law.

Recall that for the two-sided system, we defined

$$\varphi_\mu(x) := \log_\lambda \left( \frac{d\mu \circ T}{d\mu} (x) \right) = \log_\lambda \left( \prod_{k \in \mathbb{Z}} \frac{f_{k-1}(x_k)}{f_k(x_k)} \right) \in \mathbb{Z}.$$ 

Note that the restriction of $(T^n)'$ to $\Omega_+$ is given by

$$\prod_{k \in \mathbb{N}} \frac{f_{k-n}(x_k)}{f_k(x_k)}$$

is $\mathcal{B}_+$ measurable; in a slight abuse of notation we will also continue to denote the restriction of $\varphi_\mu$ to $\Omega_+$ by $\varphi_\mu$. The discrete Maharam extension of $T$ is the natural extension of the skew product extension of $S$ by the restriction of $\varphi_\mu$, defined by on $\Omega_+ \times \mathbb{Z}$ given by

$$S_{\varphi_\mu}(x,n) := (Sx, n - \varphi_\mu(x)).$$

Note that $S_{\varphi_\mu}$ preserves the measure $\tilde{\mu}_+$ which is the restriction of $\tilde{\mu}$ to $\Omega_+ \times \mathbb{Z}$. Therefore in order to prove that $\tilde{T}$ is a $K$-automorphism, it suffices to show that $S_{\varphi_\mu}$ has a trivial tail field.

It is well-known that the Hewitt-Savage zero-one law [22] implies Kolmogorov’s zero-one law. Similarly, we will prove that the exchangeable sigma field of $S_{\varphi_\mu}$ is trivial. As the tail field is a subset of the exchangeable $\sigma$-field, this will establish that the tail field is trivial and consequently that $\tilde{T}$ is a $K$-automorphism.
5.4. **The ergodic action of the permutation group.** We say that a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ of the integers fixes an element of $n \in \mathbb{N}$ if $\sigma(n) = n$. Let $\Sigma$ be the subgroup of all permutations of $\mathbb{N}$ that fix all but a finite number of elements of $\mathbb{N}$. Let $(f_n)_{n \in \mathbb{N}}$ be a collection of densities and consider the product space $([0, 1]^\mathbb{N}, \mathcal{B}, \otimes_{n \in \mathbb{N}} f_n)$. The group $\Sigma$ acts on this space via $\sigma(x)_n = x_{\sigma(n)}$ for all $x \in [0, 1]^\mathbb{N}$ and all $n \in \mathbb{N}$.

**Lemma 20.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of densities all with domain $[0, 1]$. Set $g_n(x) := \text{ess inf}_{x \in [0, 1]} f_n(x)$ and $G_n(x) := \text{ess sup}_{x \in [0, 1]} f_n(x)$, where these essential bounds are taken with respect to Lebesgue measure. If for all $n \in \mathbb{N}$, we have the following strict bounds

$$0 < \inf_{n \in \mathbb{N}} g_n(x) \leq \sup_{n \in \mathbb{N}} G_n(x) < \infty,$$

then $(\Omega, \mathcal{B}, \otimes_{n \in \mathbb{N}} f_n, \Sigma)$ is ergodic.

In our proof of Lemma 20, we will verify a condition that Aldous and Pitman [2, Condition (c)] refer to as tameness. Let $u \wedge v$ denote the minimum of two real numbers $u, v \in \mathbb{R}$.

**Theorem 21** (Aldous and Pitman). Let $(\Omega, \mathcal{F})$ be a probability space endowed with a sequence of probability measures $(\mu_n)_{n \in \mathbb{N}}$. Set

$$\mathcal{F}_0 := \bigcap_{n \in \mathbb{N}} \{ C \in \mathcal{F} : \mu_n(C) \in \{0, 1\} \}.$$

Suppose there exists a probability measure $\nu$ such that for every $B \notin \mathcal{F}_0$ there exists $\delta > 0$ such that

$$\sum_{n \in \mathbb{N}} r_n(B, \delta) = \infty,$$

where

$$r_n(B, \delta) := \inf \{ \mu_n(B \setminus C) \wedge \mu_n(B^c \setminus C) : \nu(C) \leq \delta \}. \quad (15)$$

Then the exchangeable sigma-field associated with the sequence $(\mu_n)_{n \in \mathbb{N}}$ is trivial.

**Remark 22.** Theorem 21 follows from [2, Theorem 1.1.2]. Note that in the language of probability theory, the triviality of the exchangeable sigma-field for the process $([0, 1]^\mathbb{N}, \mathcal{B}, \otimes_{n \in \mathbb{N}} f_n)$ is the ergodicity of the $\Sigma$ action. A sequence of probability measures satisfying the conditions of Theorem 21 is said to be $\nu$-tame.

Recall that $\mathcal{L}$ denotes the usual Lebesgue measure on $\mathbb{R}$. 
Proof of Lemma 20. Let $c \in (0, 1)$ be such that for Lebesgue-almost all $u \in [0, 1]$ for all $n \in \mathbb{Z}$, we have $c < f_n(u) < c^{-1}$. For all Borel sets $B \subset [0, 1]$, set

$$\mu_n(B) = \int_B f_n(u) du = \int_B f_n(u) d\mathcal{L}(u),$$

so that

$$c \mathcal{L}(B) \leq \mu_n(B) \leq c^{-1} \mathcal{L}(B).$$

It is easy to check that for all $B \in \mathcal{B}$ with $0 < \mathcal{L}(B) < 1$ if

$$\delta(B) = \delta := (\mathcal{L}(B) \wedge \mathcal{L}(B^c))/2,$$

then for all $n \in \mathbb{N}$, we have $r_n(B, \delta) \geq c \delta$, where $r_n$ is defined as in (15). Thus tameness is verified with Lebesgue measure and by Theorem 21 and Remark 22 we obtain the desired ergodicity. □

Theorem 23. Let $\lambda \in (0, 1)$ and $(f_n)_{n \in \mathbb{N}}$ be the sequence of densities defined in Section 2. Then the system $([0, 1]^\mathbb{N}, \mathcal{B}_+; \otimes_{n \in \mathbb{N}} f_n, \Sigma)$ is of type-$\text{III}_\lambda$.

Remark 24. It is straightforward to verify that for every $\sigma \in \Sigma$, if $\mu = \otimes_{n \in \mathbb{N}} f_n$ then for $\mu$-almost every $x \in [0, 1]^\mathbb{N}$, we have

$$\frac{d\mu \circ \sigma}{d\mu}(x) = \prod_{n \in \mathbb{N}} \frac{f_{\sigma(n)}(x_n)}{f_n(x_n)},$$

where this product is a finite product, with only a finite number of non-unit terms, since $\sigma$ fixes all but a finite number of integers. For example, if $\sigma_{a,b}$ is the transposition of $a, b \in \mathbb{Z}$, then

$$\frac{d\mu \circ \sigma_{a,b}}{d\mu}(x) = \frac{f_a(x_b) f_b(x_a)}{f_a(x_a) f_b(x_b)}. \quad (16)$$

We will use Lemma 12 to prove Theorem 23 by defining partial transformations that will be given by a finite number of disjoint transpositions so that an explicit computation can be performed via (16). Let $N > 0$ be an integer. For a finite number of intervals $I_0, \ldots, I_N \subseteq [0, 1]$, we say that the subset of $\Omega_+ = [0, 1]^\mathbb{N}$ given by

$$\{x \in \Omega_+ : x_k \in I_k \text{ for every integer } k \in [0, N]\}$$

is a cylinder set specified up to $N$; if all the intervals $I_0, \ldots, I_N$ have rational endpoints, we say that it is rational. The collection $\mathcal{G}$ of all rational cylinders is a countable semi-ring, and the ring generated by $\mathcal{G}$ is dense in $\mathcal{B}_+$ the product sigma-algebra for $\Omega_+$.
Proof of Theorem 23. We already have ergodicity from Theorem 20 and nonsingularity follows from Remark 24.

Since for all $\sigma \in \Sigma$, we have that
\[
\log \left( \frac{d\mu \circ \sigma}{d\mu} \right) \in \log \lambda \mathbb{Z},
\]
it is sufficient to show that $\log \lambda$ is an essential value; we will accomplish this by verifying the conditions of Lemma 12 with $\mathcal{G}$ the rational cylinders and $\delta = 1/2$.

Let $I \in \mathcal{G}$ be a cylinder set specified up to $N$. Recall the sets $A_n, B_n \subset [0, 1]$ that were used to define $f_n$ in Section 2. Set
\[
C_n := [0, 1] \setminus (A_n \cup B_n).
\]

In what follows it will be convenient to use the language of random variables. We endow measurable space $(\Omega_+, \mathcal{B}_+)$ with the probability $\mathbb{P} = \mu$ and the expectation
\[
\mathbb{E}Y = \int Y(x) d\mathbb{P}(x)
\]
for a random variable $Y : \Omega \to \mathbb{R}$. Thus with $X : \Omega \to \Omega$ as the identity $X(x) = x$ the sequence $X = (X_n)_{n \in \mathbb{Z}}$ is a sequence of continuous independent random variables with density functions given by $(f_n)_{n \in \mathbb{N}}$.

For $n \geq N + 1$ and $n \not\in \{2^k : k \in \mathbb{N}\}$, let $Y_n : \Omega \to \{-1, 0, 1\}$ be defined by
\[
Y_n := \mathbf{1}_{C_n}(X_n) \mathbf{1}_{A_n}(X_{2n}) - \mathbf{1}_{A_n}(X_n) \mathbf{1}_{C_n}(X_{2n})
= \mathbf{1}[(X_n, X_{2n}) \in C_n \times A_n] - \mathbf{1}[(X_n, X_{2n}) \in A_n \times C_n].
\]

We also set $Y_{2k} \equiv 0$ for all $k \in \mathbb{N}$. We claim that
\[
\lim_{M \to \infty} \sum_{k=N+1}^{M} Y_k = \infty \quad \text{in probability.} \tag{17}
\]

Since $C_n \subset C_{2n}$ and $A_{2n} \subset A_n$, we have
\[
\int_{C_n} f_{2^n}(u) du = |C_n|
\]
and
\[
\int_{A_n} f_{2^n}(u) du = \int_{A_n \setminus A_{2n}} 1 du + \lambda |A_{2n}|.
\]

Since $X$ is an independent sequence, we have
\[ \mathbb{E} Y_n = \int_{C_n} f_n(u) \, du \cdot \int_{A_n} f_{2^n}(u) \, du - \int_{A_n} f_n(u) \, du \cdot \int_{C_n} f_{2^n}(u) \, du \]
\[ = (1 - \lambda) |C_n| (|A_n| - |A_{2^n}|) \]
\[ = \frac{1 - \lambda}{(n + 4) \log(n + 4)} \left( 1 - \frac{\lambda + 1}{(n + 4) \log(n + 4)} \right) + O \left( \frac{1}{2^n n} \right) \]

for \( n \not\in \{2^k : k \in \mathbb{N}\} \). Recall that \( Y_{2^k} \equiv 0 \) for all \( k \in \mathbb{N} \). Thus \( \sum_{k=N+1}^{M} \mathbb{E} Y_k \sim (1 - \lambda) \log \log M \) as \( M \to \infty \), and since \( 0 < \lambda < 1 \), we have \( \lim_{M \to \infty} \sum_{k=N}^{M} \mathbb{E} Y_k = \infty \).

Since for each \( n \not\in \{2^k : k \in \mathbb{N}\} \) the random variable \( Y_n \) is a function of \((X_n, X_{2^n})\) the random variables \((Y_n)_{n \in \mathbb{N}}\) are independent, a similar calculation gives that

\[ \text{var} \left( \sum_{k=N+1}^{M} Y_k \right) = \sum_{k=N+1}^{M} \text{var}(Y_k) \]
\[ = \sum_{k=N+1}^{M} \left( \mathbb{E} \left( Y_k^2 \right) - \mathbb{E} (Y_k)^2 \right) \]
\[ = (\lambda + 1 + o(1)) \log \log M. \]

In particular,

\[ \text{var} \left( \sum_{k=N+1}^{M} Y_k \right) = o \left( \left( \mathbb{E} \left( \sum_{k=N+1}^{M} Y_k \right) \right)^2 \right) \text{ as } M \to \infty. \]

Hence Chebyshev’s inequality gives,

\[ \mathbb{P} \left( \sum_{k=N+1}^{M} Y_k < \sqrt{\log \log M} \right) = O \left( \frac{\text{var} \left( \sum_{k=N+1}^{M} Y_k \right)}{\left( \mathbb{E} \left( \sum_{k=N+1}^{M} Y_k \right) \right)^2} \right) \text{ as } M \to \infty. \]

Hence (17) holds.

The divergence to infinity in (17) and the fact that the random variables in the sum are \(-1, 0, 1\) valued implies there exist \( M > N + 1 \) such that the set

\[ \mathcal{E} := \left\{ x \in \Omega_+ : \text{there is an integer } K \in [N + 1, M] \text{ with } \sum_{j=N+1}^{K} Y_j = 1 \right\}, \]
satisfies \( P(E) > \frac{1}{2} \). Note that \( E \) depends on the random variables \((X_{N+1}, X_{2N+1}), \ldots, (X_M, X_{2M})\). For \( x \in E \) set
\[
\tau(x) := \min \left\{ n \geq N + 1 : \sum_{j=N+1}^{K} Y_j(x) = 1 \right\}.
\]

Now we specify a partial transformation \( V \) with domain \( D := E \cap I \) which satisfies the conditions of Lemma \[12\]. Let \( V : D \to I \) be defined by
\[
(Vx)_j := \begin{cases} 
x_{2j}, & j \in [N + 1, \tau(x)] \text{ and } Y_j(x) \neq 0 \\
x_{\log_2 j}, & \log_2 j \in [N + 1, \tau(x)] \text{ and } Y_{\log_2 j}(x) \neq 0 \\
x_j, & \text{otherwise.}
\end{cases}
\]
Thus \( V \) is simply the application of the transposition of some dyadic pair.

Since \( I \) is specified up to \( N \), the events \( E \) and \( I \) depend on a disjoint collections of the coordinates \( X = (X_n)_{n \in \mathbb{N}} \) and are thus independent. Therefore we have
\[
P(D) = P(E)P(I) > \frac{1}{2}P(I),
\]
verifying the first condition of Lemma \[12\]. Secondly, for \( x \in D \), we have
\[
\frac{d\mu \circ V}{d\mu}(x) = \prod_{k \in [N+1, \tau(x)], \ Y_k(x) \neq 0} \frac{d\mu \circ \sigma_{k,2k}}{d\mu}(x) = \prod_{k \in [N+1, \tau(x)], \ Y_k(x) \neq 0} f_k(x_{2k}) f_k(x_k) = \lambda Y_k(x),
\]
where the last inequality follows from the fact that if \( Y_k(x) = 1 \), then \( x_{2k} \in A_k \) and \( x_k \notin (A_k \cup B_k) \) so that \( f_k(x_{2k}) = \lambda \) and \( f_k(x_k) = 1 \) and similarly if \( Y_k(x) = -1 \), then \( f_k(x_{2k}) = 1 \) and \( f_k(x_k) = \lambda \). Hence
\[
\frac{d\mu \circ V}{d\mu}(x) = \lambda \sum_{k=N+1}^\tau Y_k(x) = \lambda,
\]
and we have verified (an epsilon free version of) the second condition of the lemma.

Clearly \( V \) is nonsingular and by construction \((x, Vx) \in O \Sigma \) for all \( x \in D \). It remains to verify that \( V \) is injective. Let \( x, x' \in D \) be such
that \( Vx = Vx' \) and assume without loss of generality that \( \tau(x) \leq \tau(x') \).
Since for all \( N + 1 \leq k \leq \tau(x) \), we have
\[
Y_k(Vx) = -Y_k(x)
\]
it follows that for all \( N + 1 \leq k \leq \tau(x) \), we have \( Y_k(x) = Y_k(x') \). Hence
\[
\sum_{k=N}^{\tau(x)} Y_k(x') = \sum_{k=N}^{\tau(x)} Y_k(x) = 1
\]
so that the minimality of \( \tau \) gives that \( \tau(x) = \tau(x') \). In addition, the subsets where \( V \) fixes the coordinates of \( x \) and \( x' \) are the same, since
\[
\{ k \in [N + 1, \tau(x)] : Y_k(x) \neq 0 \} = \{ k \in [N + 1, \tau(x)] : Y_k(x') \neq 0 \}.
\]
For the other indices, by definition, \( Vx \) and \( Vx' \) are permutations of the coordinates of \( x \) and \( x' \), respectively. Since \( Vx = Vx' \) and permutation are injective, it follows that \( x = x' \) as desired. \( \Box \)

Let \( \lambda \in (0, 1) \). Let \((\Omega_+, \mathcal{B}_+, \mu_+, S)\) be the one-sided Bernoulli shift from Section 2. The tail equivalence relation on \( \Omega_+ \) is given by
\[
T := \{(x, x') \in \Omega_+ \times \Omega_+ : \text{there exists } n \in \mathbb{N} \text{ with } S^n x = S^n x' \}.
\]
The exchangeable equivalence relation is given by
\[
E := \{(x, x') \in \Omega_+ \times \Omega_+ : \text{there exists } \sigma \in \Sigma \text{ with } \sigma(x) = x' \}.
\]

**Remark 25.** Note that \( E \) is countable and coarser than \( T \). \( \Diamond \)

Let \( S_{\varphi_{\mu}} \) be its discrete Maharam extension. Let \( \psi : \Omega_+ \times \Omega_+ \to \mathbb{Z} \) be the tail cocycle associated to \( \varphi_{\mu} \), defined by
\[
\psi(x, x') := \sum_{n=0}^{\infty} (\varphi_{\mu} \circ S^n(x) - \varphi_{\mu} \circ S^n(x')).
\]

**Remark 26.** Note that for all \((x, z), (x', z') \in \Omega_+ \times \mathbb{Z}\), there exists \( n \in \mathbb{N} \) such that
\[
S^n_{\varphi_{\mu}}(x, z) = S^n_{\varphi_{\mu}}(x', z') \text{ if and only if } (x, x') \in T \text{ and } z + \psi(x, x') = z'.
\]

For \( \sigma \in \Sigma \), let \( \psi_{\sigma}(x) := \psi(x, \sigma(x)) \). The discrete Maharam extension of the \( \Sigma \)-action on \( \Omega_+ \times \mathbb{Z} \) is given by
\[
\sigma(x, n) := (\sigma(x), n - \log_{\lambda} (\sigma'(x)))
\]

**Lemma 27.** Let \( \lambda \in (0, 1) \). Let \((\Omega_+, \mathcal{B}_+, \mu_+, S)\) be the one-sided Bernoulli shift from Section 2. For every \( \sigma \in \Sigma \), we have
\[
\psi_{\sigma} = \log \frac{d\mu \circ \sigma}{d\mu}.
\]
where $\psi$ is the associated tail cocycle. Furthermore, if the Maharam extension of the $\Sigma$-action is ergodic with respect to $\hat{\mu}_+$, then

$$\bigcap_{n=1}^{\infty} S^{-n} (\mathcal{B}([0,1]^\mathbb{N}) \times \mathcal{B}(\mathbb{Z})) = \{\emptyset, \Omega_+ \times \mathbb{Z}\} \mod \hat{\mu}_+,$$

so that discrete Maharam extension of the left-shift is a $K$-automorphism.

**Proof of Theorem 19.** Theorem 23 together with Theorem 18 implies that the Maharam extension of the $\Sigma$-action is ergodic, thus the hypothesis of Lemma 27 is satisfied and the discrete Maharam extension of the left-shift is a $K$-automorphism and by Lemma 14 it is ergodic; finally, by Theorem 18 this implies that the Bernoulli shift is of type-III$_\lambda$. $\square$

**Proof of Lemma 27.** Let $\sigma \in \Sigma$ and suppose that for all $m \geq n$, the integer $m$ is fixed by $\sigma$. Then for all $m \geq n$ and $x \in \Omega_+$, we have $S^m \sigma(x) = S^m(x)$. Thus for $\mu$-almost all $x \in \Omega_+$, we have

$$\psi(x, \sigma(x)) = \sum_{k=0}^{n-1} (\varphi_\mu \circ S^k(x) - \varphi_\mu \circ S^k(\sigma(x)))$$

$$= \log \left( \prod_{k=0}^{n-1} \frac{d\mu \circ T}{d\mu} (T^k x) \right) - \log \left( \prod_{k=0}^{n-1} \frac{d\mu \circ T}{d\mu} (T^k \sigma(x)) \right)$$

$$= \log \left( (T^n)'(x) \right) - \log \left( (T^n)'(\sigma(x)) \right).$$

Again, since $\sigma$ fixes all integers $m \geq n$ and $f_{k-n} \equiv 1$ for all $k \leq n$, for $\mu$-almost all $x \in \Omega_+$, we have

$$\frac{(T^n)'(x)}{(T^n)'(\sigma(x))} = \prod_{k=1}^{n} \frac{f_{k-n}(x_k)}{f_{k-n}(x_{\sigma(k)})} \frac{f_k(x_{\sigma(k)})}{f_k(x_k)}$$

$$= \prod_{k=1}^{n} \frac{f_k(x_{\sigma(k)})}{f_k(x_k)} = \frac{d\mu \circ \sigma}{d\mu}(x).$$

The first claim is verified.

The first claim gives that the equivalence relation $\mathcal{R}$ on $\Omega_+ \times \mathbb{Z}$ given by

$$\{((x,z),(x',z')) : \exists \sigma \in \Sigma \text{ with } \sigma(x) = x' \text{ and } z' = z - \psi(x,\sigma(x))\}$$
is in fact the tail equivalence relation for the Maharam extension of the 
$\Sigma$-action given by
\[
\{(x, z), (x', z') : \exists \sigma \in \Sigma \text{ such that } \sigma(x, z) = (x', z')\}.
\]
By Remarks 25 and 26, $R$ is countable and is coarser than tail relation of $S_{\varphi \mu}$, given by
\[
\{(x, z), (x', z') : \exists n \in \mathbb{N} \text{ such that } S^n_{\varphi \mu}(x, z) = (x', z')\}.
\]
Therefore, $I$, the sigma-field of invariant sets for the equivalence relation $R$ contains the tail sigma-field for $S_{\varphi \mu}$. Hence the assumption that the Maharam extension of the $\Sigma$-action is ergodic yields the desired triviality. □

Remark 28. It was proved in [37, Theorem 3.2] and [12] that a conservative nonsingular half stationary Bernoulli shift on two symbols without an absolutely continuous invariant probability measure has a Maharam extension that is a $K$-transformation. The idea of Kosloff’s proof and extensions given in [12] do not extend in an obvious way in the setting of Theorem 19, since the tail equivalence relation for $S$ is not countable and not nonsingular; see [37, Section 2.3] for more details.

6. THE CONSERVATIVE INDEX OF MAHARAM’S EXTENSION

The following theorem can be viewed as an addition to Maharam’s celebrated result on recurrence of the Maharam extension.

Theorem 29. Let $(\Omega, \mathcal{F}, \mu, T)$ and $(\Omega', \mathcal{C}, \kappa, R)$ be two nonsingular transformations on probability spaces. The product $T \times R$ is conservative if and only if the direct product of their Maharam extensions is conservative.

We will use the following characterization of conservativity due to Hopf [1, Proposition 1.1.6].

Proposition 30. Let $T$ be a measure-preserving transformation of a sigma-finite measure space $(\Omega, \mathcal{F}, \mu)$. Fix $F : \Omega \to (0, \infty)$ be integrable. Then $T$ is conservative if and only if for $\mu$-almost all $x \in \Omega$, we have
\[
\sum_{k=1}^{\infty} F \circ T^k(x) = \infty.
\]

Proof Theorem 29. We write
\[
(T^k)' = \frac{d\mu \circ T^k}{d\mu} \text{ and } (R^k)' = \frac{d\kappa \circ R^k}{d\kappa}.
\]
and let $\nu, \nu_\kappa$ and $\nu_\mu \times \kappa$ be the invariant measures for the Maharam extension of $T$, $R$ and $T \times R$ respectively. Let $V = \tilde{T} \times \tilde{R}$ be the Maharam extension of $T \times R$.

Let $F : \Omega \times \Omega' \times \mathbb{R} \to (0, \infty)$ be given by
\[
F(x, z, t) := e^{-2|t|}
\]
and let $G : (\Omega \times \mathbb{R}) \times (\Omega' \times \mathbb{R}) \to (0, \infty)$ be given by
\[
G((x, t_1), (z, t_2)) := e^{-2|t_1|}e^{-2|t_2|}.
\]

Since $F$ and $G$ are integrable with respect to $\nu_\mu \times \kappa$ and $\nu_\mu \times \nu_\kappa$, respectively, Proposition 30 applies.

Notice that for all $k \in \mathbb{N}$ and all $x,y,t_1,t_2$, we have
\[
\sum_{k=1}^{\infty} F \circ V^k(x, z, t) = \sum_{k=1}^{\infty} e^{-2|t - \log((T^k)'(x)(R^k)'(z))|}
\leq e^{2|t|} \sum_{k=1}^{\infty} \left((T^k)'(x)(R^k)'(z)\right)^{-2}
\]
and
\[
G\left(\tilde{T}^k(x, t_1), \tilde{R}^k(z, t_2)\right) \geq e^{-2(|t_1|+|t_2|)} \left((T^k)'(x)(R^k)'(z)\right)^{-2}.
\]

If $T \times R$ is conservative, then by Maharam’s theorem $\tilde{T} \times \tilde{R}$ is conservative, and thus by Proposition 30, it is immediate that $\tilde{T} \times \tilde{R}$ is conservative.

The other direction can be proved in a similar way, but we give a more direct argument. By permuting the coordinates, we can regard $\tilde{T} \times \tilde{R}$ as a transformation of $\Omega \times \Omega' \times \mathbb{R}^2$. If $T \times R$ is dissipative, then there exists a wandering set $W \subset \Omega \times \Omega'$ measurable with $(\mu \times \kappa)(W) > 0$ and the collection $\{(T \times R)^n(W)\}_{n \in \mathbb{N}}$ pairwise disjoint. The set $\tilde{W} = W \times \mathbb{R}^2$ is then a wandering set for $\tilde{T} \times \tilde{R}$ of positive $\nu_\mu \times \kappa$ measure and thus $\tilde{T} \times \tilde{R}$ is dissipative.

Given a non-singular transformation $(\Omega, \mathcal{F}, \mu, T)$, denote by $T^{(k)}$ its iterated transformation $T \times T \times \cdots \times T$ ($k$ times). We say that $T$ has **conservative index** $k$ if $T^{(k)}$ is a conservative transformation and $T^{(k+1)}$ is dissipative. Let $\tilde{T}$ denote the Maharam extension. Thus Maharam’s theorem gives that $c(T) = 1$ if and only if $c(\tilde{T}) = 1$. Denote the conservative index of $T$ by $c(T)$. The following is an answer to a question of Kosloff who asked whether $c(T) > 2$ implies that $c(\tilde{T}) > 2$; see also [12, Question 7].
Corollary 31. For every nonsingular transformation \((\Omega, \mathcal{F}, \mu, T)\), we have \(c(T) = c(\tilde{T})\).

Recall that \(T\) is **power weakly mixing** if for all \(n_1, n_2, \ldots, n_k \in \mathbb{Z}\), \(T^{n_1} \times T^{n_2} \times \cdots \times T^{n_k}\) is ergodic. Power weakly mixing type-III Bernoulli shifts were constructed by Kosloff [36, Theorem 7]. The Bernoulli shifts we constructed are also weakly power mixing.

Corollary 32. Let \(T\) be the Bernoulli shift given in Theorem 1. Then \(T\) is weakly power mixing.

Proof. Let \(\lambda \in (0, 1)\) and \(\mu = \bigotimes_{n \in \mathbb{Z}} f_n\) be the corresponding product measure and \(n_1, \ldots, n_k \in \mathbb{Z} \setminus \{0\}\). Then, writing \(\mu^{\otimes k}\) for the \(k\) fold product measure of \(\mu\) and \(S = T^{n_1} \times \cdots \times T^{n_k}\). As \(S\) is a direct product of \(K\)-automorphisms it is a \(K\)-automorphism. It follows from Lemma 15 that there exists \(c = c(\lambda) > 0\) such that for all \(r \in \mathbb{N}\), we have

\[
\int_{\Omega^k} \left( \frac{1}{(S^r)^\gamma} \right)^2 \, d\mu^{\otimes k} = \prod_{j=1}^{k} \int_{\Omega} \left( \frac{1}{(T^{n_j})^\gamma} \right)^2 \, d\mu \\
\leq \prod_{j=1}^{k} (\log((r+4)|n_j|))^c \leq (M + \log(r+4))^{kc},
\]

where \(M := \max_{j \in \{1,\ldots,k\}} \log(|n_j|)\). Since

\[
\sum_{r=1}^{\infty} \frac{(M + \log(r+4))^{kc}}{r^2} < \infty,
\]

a similar argument as in the proof of Lemma 16 shows that \(S\) is conservative. Since \(S\) is a conservative \(K\)-automorphism, by Proposition 14 it is hence ergodic. Hence \(T\) is power weakly mixing. \(\square\)

We give the following application of Theorem 29.

Corollary 33. Let \(T\) be the Bernoulli shift given in Theorem 1. Then the Maharam extension of \(T\) is a power weakly mixing \(K\)-automorphism.

Proof. Let \(n_1, n_2, \ldots, n_k \in \mathbb{Z}\). The product \(\tilde{T}^{n_1} \times \tilde{T}^{n_2} \times \cdots \times \tilde{T}^{n_k}\) is a product of \(K\)-automorphisms by [12] [37] and is thus a \(K\)-automorphism. By Corollary 32, \(T^{n_1} \times T^{n_2} \times \cdots \times T^{n_k}\) is conservative hence by an inductive application of Theorem 29, \(\tilde{T}^{n_1} \times \tilde{T}^{n_2} \times \cdots \times \tilde{T}^{n_k}\) is conservative. By Proposition 14 a conservative \(K\)-automorphism is ergodic, hence \(\tilde{T}^{n_1} \times \tilde{T}^{n_2} \times \cdots \times \tilde{T}^{n_k}\) is ergodic. \(\square\)
7. Countable amenable groups

7.1. Introduction. Benjamin Weiss and Andrey Alpeev asked whether for every countable amenable group and $0 \leq \lambda \leq 1$ there exists an explicit action of $G$ which is type-III$_\lambda$. We give a positive (partial) answer to this question in Theorem 3 by producing type-III$_\lambda$ Bernoulli action of $G$ for $\lambda \in (0, 1)$. The case where $\lambda = 1$ is known from [5, 55].

Let $G$ be a countable group. We will sometimes write $G \curvearrowright (\Omega, \mathcal{F}, \mu)$ to denote a group action on the measured space $(\Omega, \mathcal{F}, \mu)$. The group $G$ acts on $\Omega = [0, 1]_G$ via $(T_g x) h := x_{g^{-1}h}$ for all $x \in \Omega$ and all $g, h \in G$. We endow $\Omega$ with the usual Borel product sigma-algebra, $\mathcal{B}$. Let $(f_i)_{i \in G}$ be a collection of probability densities with domain $[0, 1]$ and let $\mu = \bigotimes_{i \in G} f_i$. Then we say that $G$ is a Bernoulli action on $(\Omega, \mathcal{B}, \mu)$.

Recall that an action $G \curvearrowright (\Omega, \mathcal{F}, \mu)$ is of stable type-III$_\lambda$ if for every ergodic probability preserving $G$ action on $(\Omega', \mathcal{C}, \nu)$, the diagonal action $G \curvearrowright (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{C}, \mu \times \nu)$ is ergodic and of type-III$_\lambda$.

For a finite set $F$, let $|F|$ denote its cardinality. Recall that a Følner sequence for a countable group $G$ is a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of $G$ such that for all $g \in G$, we have

$$\lim_{n \to \infty} \frac{|g \triangle g F_n|}{|F_n|} = 0.$$

Følner [19] proved that a $G$ is amenable if and only it admits it Følner sequence. Recall the statement of Theorem 3.

**Theorem 3.** Let $G$ be a countable amenable group and $\lambda \in (0, 1)$. There exists a product measure $\bigotimes_{g \in G} f_g$ on $[0, 1]^G$ such that the corresponding Bernoulli action is nonsingular, ergodic and of stable type-III$_\lambda$.

We will prove Theorem 3 by adapting the densities defined in Section 2 to the amenable group setting. The main difference in this more general setting is that when we no longer have a notion of the $K$-property.

7.2. The explicit construction. We will now proceed with the construction of the density functions on $[0, 1]$. Let $\lambda \in (0, 1)$ and $G = \{g_n\}_{n \in \mathbb{N}}$ be a countable amenable group which we enumerate. A straightforward application of Følner’s characterization of amenability implies that there exists a pairwise disjoint Følner sequence $(F_n)_{n \in \mathbb{N}}$ satisfying
the property that for all integers $0 \leq k < n$, we have

$$\max\left(\frac{|F_n \triangle g_k^{-1}F_n|}{|F_n|}, \frac{|F_n \triangle g_k F_n|}{|F_n|}\right) < \frac{1}{n},$$

(18)

and that the union of Følner sets leaves an infinite subset of $G$, so that $G \setminus \bigcup_{n \in \mathbb{N}} F_n$ is infinite. We will also assume that $|F_n| \geq 4$ for all $n \in \mathbb{N}$.

Recall that $\mathcal{L}$ denotes Lebesgue measure. Let $(A_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$ be a decreasing sequences of open intervals of $[0, 1]$ such that $A_1$ and $B_1$ are disjoint, so that any two of $A_m$ and $B_n$ are disjoint; furthermore, we specify that

$$\mathcal{L}(A_n) = \lambda^{-1} \mathcal{L}(B_n) = \frac{1}{n \log(n + 1)} \frac{1}{\#|F_n|}.$$

Since $\#|F_n| \geq 4$ it follows that

$$\mathcal{L}(B_n) + \mathcal{L}(A_n) < 1.$$

Let $(\hat{f}_n)_{n \in \mathbb{N}}$ be a sequence of $\{\lambda^{-1}, 1, \lambda\}$-valued densities on $[0, 1]$ given by

$$\hat{f}_n(u) := \begin{cases} 
\lambda, & u \in A_n \\
\lambda^{-1}, & u \in B_n \\
1, & u \in [0, 1] \setminus (A_n \cup B_n)
\end{cases}$$

and $(f_g)_{g \in G}$ be the sequence of functions indexed by $G$ given by

$$f_g := \begin{cases} 
\hat{f}_n & \text{if there exists } n \in \mathbb{N} \text{ with } g \in F_n \\
1_{[0,1]} & \text{otherwise.}
\end{cases}$$

(19)

We will show that the Bernoulli action $G \curvearrowleft (\Omega, \mathcal{B}, \mu)$ witnesses Theorem 3 where $\Omega = [0, 1]^G$ and $\mu = \bigotimes_{g \in G} f_g$.

7.3. The Proof of Theorem 3. Our approach to the proof of Theorem 3 is similar to the proof of Theorem 1 and in what follows we display the main components, and highlight the differences with regards to our treatment of Theorem 1 and postpone the proofs of the components parts until Section 7.4.

Lemma 34. Let $\lambda \in (0, 1)$. The Bernoulli action defined in Section 7.2 is nonsingular and conservative.

Lemma 34 follows from similar calculations given earlier for the proofs of Lemma 13 and 16.

Again, in order to prove that $\lambda$ is an essential value, we will analyze the action the permutation group rather than the shift action of the group itself. Denote by $\Sigma_G$ the group of finite permutations on $G$; that
is, those that fix all but a finite number of elements of $G$. This group acts on $[0, 1]^G$ by setting $(\sigma x)_g := x_{\sigma(g)}$ for all $g \in G$ and all $x \in [0, 1]^G$. Note that action is nonsingular with Radon-Nykodym derivative

$$\sigma'(x) = \frac{d\mu \circ \sigma}{d\mu}(x) = \prod_{h \in G} f_h(x_{\sigma(h)})/f_h(x_h).$$

Furthermore, we have the following version of Theorem 23.

**Proposition 35.** Let $\lambda \in (0, 1)$ and $(f_g)_{g \in G}$ be the sequence of functions defined by (19) in Section 7.2. Then $(\Omega, \mathcal{B}, \otimes_{g \in G} f_g, \Sigma_G)$ is of Krieger type-III$_\lambda$.

In order exchange Proposition 35 for a statement about the actual group action we will apply the Hopf method argument as in Avraham-Re'em [4, Section 4]. Let $G$ be a countable group. We write $g_n \to \infty$ if for every finite $H \subset G$ and for all $n$ sufficiently large we have $g_n \notin H$. Let $\Phi : G \to \mathbb{R}$. If for all sequences $g_n \to \infty$, the $\lim_{n \to \infty} \Phi(g_n)$ exists and has the same value $L$, then we write $\lim_{g \to \infty} \Phi(g) = L$.

Let $(\Omega, \mathcal{B}, \mu)$ be a standard measure space, where $d$ is a complete separable metric on $\Omega$ that generates $\mathcal{B}$. Suppose $G \curvearrowright (\Omega, \mathcal{B}, \mu)$. A pair of points $(x, x') \in \Omega \times \Omega$ is an asymptotic pair if

$$\lim_{g \to \infty} d(gx, gx') = 0.$$  

We say that an action of another countable group $\Gamma$ on $(\Omega, \mathcal{B}, \mu)$ is an action by $G \curvearrowright (\Omega, \mathcal{B}, \mu)$ asymptotic pairs if for every $\gamma \in \Gamma$ for $\mu$-almost every $x \in \Omega$, we have

$$\lim_{g \to \infty} d(gx, g\gamma x) = 0.$$  

**Remark 36.** Consider the Bernoulli action $G \curvearrowright (\Omega, \mathcal{B}, \mu)$ given in Section 7.2. Note that $\Sigma_G \curvearrowright (\Omega, \mathcal{B}, \mu)$ is an action by $G \curvearrowright (\Omega, \mathcal{B}, \mu)$ asymptotic pairs. Let $\sigma \in \Sigma_G$ and write

$$H := \{ h \in G : \sigma(h) \neq h \}.$$  

For every $x \in \Omega$, for all $g \in G$, the set $\{ h \in G : (gx)_h \neq (g\sigma x)_h \}$ is contained in $gH$, which is finite, since $H$ is finite. Hence for all $x \in \Omega$ and any $g_n \to \infty$, we have that for all $n$ sufficiently large

$$d(g_n x, g_n \sigma x) = 0.$$  

Let $\lambda \in (0, 1)$. Consider the action $G \curvearrowright (\Omega, \mathcal{F}, \mu)$ as in Section 7.2. Recall that we defined the discrete Maharam extension of a group
action in Section 5.3. Thus the \textit{discrete Maharam extension of the G-action} on $\Omega \times Z$ is given by
\[ g(x, n) := (g(x), n - \log_\lambda (g'(x))) \]
and the \textit{discrete Maharam extension of the $\Sigma_G$-action} on $\Omega \times Z$ is given by
\[ \sigma(x, n) := (\sigma(x), n - \log_\lambda (\sigma'(x))). \]

\textbf{Lemma 37.} Let $\lambda \in (0, 1)$. Consider the action $G \acts (\Omega, \mathcal{B}, \mu)$ as in Section 7.2. The discrete Maharam extension of the $\Sigma_G$-action on $(\Omega \times Z)$ is an action of asymptotic pairs of the discrete Maharam extension of the $G$-action on $(\Omega \times Z)$.

\textit{Proof of Theorem 3.} Let $\lambda \in (0, 1)$. Consider the action $G \acts (\Omega, \mathcal{B}, \mu)$ as in Section 7.2. Let $G \acts (\Omega', \mathcal{C}, \nu)$ be an additional ergodic probability preserving transformation. Since
\[ \frac{d(\mu \times \nu) \circ g}{d(\mu \times \nu)} (x, y) = \frac{d\mu \circ g}{d\mu} (x) = g'(x), \]
the discrete Maharam extension of the diagonal action on the product space $G \acts (\Omega \times \Omega', \mathcal{B} \times \mathcal{C}, \mu \times \nu)$ is given by
\[ g(x, y, n) = (gx, gy, n - \log_\lambda (g'(x))), \]
for all $(x, y, n) \in \Omega \times \Omega' \times Z$. We specify that for $\sigma \in \Sigma_G$ the action on the product space $\Omega \times \Omega'$ is given by ignoring the second coordinate so that
\[ \sigma(x, y) = (\sigma x, y). \]
Thus the discrete Maharam extension of $\Sigma_G$ is given by
\[ \sigma(x, y, n) = (\sigma x, y, n - \log_\lambda (\sigma'(x))). \]
As in Remark 36, the $\Sigma_G$-action is an action by asymptotic pairs of the discrete Maharam extension of the diagonal $G$-action.

Let $F : \Omega \times \Omega' \times Z \to [0, 1]$ be a $G$-invariant function with respect to $\mu \times \nu$, the measure associated with discrete Maharam extension. We will show that $F$ is a constant. By \cite[Lemma 4.4]{4}, $F$ is also $\Sigma_G$-invariant. Proposition 35 together with Theorem 18 give that discrete Maharam extension of $\Sigma_G \acts (\Omega, \mathcal{B}, \mu)$ is ergodic. Hence it follows from the definition of $\Sigma_G$-action on $\Omega \times \Omega' \times Z$, which is the identity on $\Omega'$, and ergodicity that there exists $\psi : \Omega' \to [0, 1]$ such that $F(x, y, n) = \psi(y)$. Since $F$ is $G$-invariant it follows that for all $g \in G$ and almost every $y \in \Omega'$, that $\psi(y) = \psi(gy)$, from which the assumed ergodicity of $G \acts (\Omega', \mathcal{C}, \nu)$ yields that $F$ is a constant almost surely.
Hence
\[ G \bowtie \left( \Omega \times \Omega' \times \mathbb{Z}, \mathcal{B} \times \mathcal{C} \times \mathbb{Z}, \mu \times \nu \right) \]
is ergodic. From another application of Theorem 18 we obtain that
\[ G \bowtie \left( \Omega \times \Omega', \mathcal{B} \times \mathcal{C}, \mu \times \nu \right) \]is of type-III \( \lambda \). Since the system \((\Omega', \mathcal{C}, \nu)\)
was arbitrary, we have the desired stability. \( \square \)

7.4. The remaining proofs for Theorem 3

Proof of Lemma 34. Let \( g \in G = \{g_k\}_{k \in \mathbb{N}} \) and \( n := n(g) \in \mathbb{N} \) such that \( g = g_n \). We will show that the action is nonsingular by applying Kaku-
tani’s theorem on the equivalence of product measures; furthermore we
will obtain that there exists \( K(\lambda) \geq 6 \lambda \) such that
\[
\sum_{h \in G} \int_0^1 \left( \sqrt{f_{g^{-1}h}(u)} - \sqrt{f_h(u)} \right)^2 \, du < K(\lambda) \log \log(n(g) + 1). \tag{20}
\]
Let
\[ H_k(g) := F_k \triangle (gF_k \cup g^{-1}F_k). \]
Note that if \( h \not\in \bigcup_{k \in \mathbb{N}} H_k(g) \), then \( f_{g^{-1}h} = f_h \). Thus for \( h \in H_k(g) \), we have
\[
\int_0^1 \left( \sqrt{f_{g^{-1}h} - \sqrt{f_h}} \right)^2 \, du \leq \int_0^1 \left( \sqrt{f_k} - 1 \right)^2 \, du \\
\leq \lambda^{-1} (\mathcal{L}(A_k) + \mathcal{L}(B_k)) \\
\leq \frac{2}{\lambda k \log(k+1)} \frac{1}{\#|F_k|},
\]
the second inequality follows from the simply inequalities
\[ \sqrt{\lambda^{-1} - 1}, 1 - \sqrt{\lambda} < \lambda^{-1/2}. \]
Consequently,
\[
\sum_{h \in G} \int_0^1 \left( \sqrt{f_{g^{-1}h} - \sqrt{f_h}} \right)^2 \, du = \sum_{k=1}^\infty \sum_{h \in H_k(g)} \int_0^1 \left( \sqrt{f_{g^{-1}h}} - \sqrt{f_h} \right)^2 \, du \\
\leq 2\lambda^{-1} \sum_{k=1}^\infty \sum_{h \in H_k(g)} \frac{\#|H_k(g)|}{\#|F_k|} \frac{1}{k \log(k)}.
\]
By (18),
\[
\frac{\#|H_k(g)|}{\#|F_k|} \leq \begin{cases} 
\frac{2}{k}, & k \geq n(g), \\
3, & 1 \leq k \leq n(g).
\end{cases}
\]
Hence

\[
\sum_{h \in G} \int_0^1 \left( \sqrt{f_{g^{-1}h}} - \sqrt{f_h} \right)^2 \, du \leq \frac{6}{\lambda} \sum_{k=1}^{n(g)} \frac{1}{k \log(k+1)} + 4 \sum_{k=n(g)}^{\infty} \frac{1}{k^2 \log(k+1)} \\
\leq \frac{6}{\lambda} \left( \log \log(n(g) + 1) + n(g)^{-1} \right),
\]

which concludes the proof of (20) from which nonsingularity follows.

Note that if \( h \notin H_k(g) \), then

\[
\int_0^1 \left( \frac{f_h(u)}{f_{g^{-1}h}} \right)^2 f_h(u) \, du = \int_0^1 f_h(u) \, du = 1.
\]

If \( h \in H_k(g) \), then as in the proof of Lemma 15, there exists \( c(\lambda) > 0 \) such that

\[
\int_0^1 \left( \frac{f_h(u)}{f_{g^{-1}h}} \right)^2 f_h(u) \, du \leq \exp \left( c(\lambda) \log(A_k) \right).
\]

Thus

\[
\int_{\Omega} \left( \frac{1}{(T_g)'(x)} \right)^2 \, d\mu(x) = \prod_{k=1}^{\infty} \prod_{h \in H_k(g)} \int_0^1 \left( \frac{f_h(u)}{f_{g^{-1}h}} \right)^2 f_h(u) \, du \\
\leq \exp \left( c(\lambda) \sum_{k=1}^{\infty} \log(A_n) \# |H_k(g)| \right) \\
\leq \exp \left( K(\lambda)c(\lambda) \log(n(g) + 1) \right),
\]

where the last bound is the same as in the proof of (20).

Hence by Markov’s inequality, the sequence of sets

\[
C_n := \{ x \in \Omega : (T_{g_n})'(x) < n^{-1} \}
\]

satisfies

\[
\mu(C_n) = \mu \{ x \in \Omega : ((T_{g_n})'(x))^{-2} > n^{-2} \} \leq \frac{\log(n + 1)K(\lambda)c(\lambda)}{n^2}.
\]

The first Borel-Cantelli lemma implies that for \( \mu \)-almost all \( x \in \Omega \) there exists \( N(x) \in \mathbb{N} \) such that for all \( k > N(x) \), we have \((T_{g_k})'(x) \geq k^{-1}\).

Again, the divergence of the harmonic series gives

\[
\sum_{g \in G} (T_g)'(x) = \sum_{n \in \mathbb{N}} (T_{g_n})'(x) = \infty,
\]

for \( \mu \)-almost every \( x \in \Omega \). By a routine extension of Hopf’s criteria [1] Proposition 1.3.1 to the case of a general countable group action, the Bernoulli action is conservative. \( \Box \)
Our proof of Proposition 35 will be an adaptation of the proof of Theorem 23 and in our proof we will point out the necessary modifications. Consider the action \( G \acts (\Omega, \mathcal{B}, \mu) \) given in Section 7.2. Given a finite subset \( H \subset G \) and open subintervals \( \{I_h\}_{h \in H} \) with rational endpoints, we say that the set

\[ I = \{ x \in \Omega : \text{for all} \ h \in H \text{ we have} \ x_h \in I_h \}. \]

is a rational cylinder set determined on \( H \). As in Section 5.4, the collection \( G \) of rational cylinders is a countable semiring and the ring generated by \( G \) is dense in \( \mathcal{B} \).

**Proof of Proposition 35.** We first note that the ergodicity of the non-singular system \( (\Omega, \mathcal{B}, \bigotimes_{g \in G} f_g, \Sigma) \) follows from Lemma 20, since in its proof we can replace \( N \) be \( G \).

We will show that \( \lambda \) is an essential value for the \( \Sigma \)-action and this is done by verifying the conditions of Lemma 15 with \( G \) the rational cylinders and \( \delta = 1/2 \). Let \( H \subset G \) finite and \( I \) be a rational cylinder determined on \( H \).

In order to mimic the proof of Theorem 23, we will need to fix certain useful enumerations of subsets of \( G \). Enumerate \( \bigcup_{n \in \mathbb{N}} F_n = (\ell_n)_{n \in \mathbb{N}} \) so that each element of \( F_m \) has a greater index than each element of \( F_n \), if \( m > n \). Specifically, write \( n-1 = 0 \) and for \( k \in \mathbb{N} \), let \( n_k = n_{k-1} + (\# |F_k|) \). Then for all \( k \in \mathbb{N} \) we choose an enumeration

\[ \{\ell_j\}_{j=n(k-1)}^{n_k-1} = F_k. \]

We will write for \( A_j = A_n \), \( B_j = B_n \), and \( C_j = C_n \) if \( \ell_j \in F_n \), or equivalently \( j \in [n_{k-1}, n_k) \). Let \( (h_j)_{j=1}^{\infty} \) be an enumeration of the countably infinite set \( G \setminus \bigcup_{n \in \mathbb{N}} F_n \) and \( N \in \mathbb{N} \) such that

\[ H \subset \left( \{\ell_j\}_{j=0}^{nN-1} \cup \{h_j\}_{j=0}^{nN-1} \right) := H'. \]

With these enumerations we define the random variables as in the Proof of Theorem 23. Let \( X : \Omega \rightarrow \Omega \) be the identity function, \( X(x) = x \) so that \( (X_g)_{g \in G} \) are a collection of independent continuous random variables with densities \( (f_g)_{g \in G} \). For \( j \in \mathbb{Z} \), let \( Y_j : \Omega \rightarrow \{-1, 0, 1\} \) be given by

\[ Y_j = 1_{A_j} (X_{h_j}) 1_{C_j} (X_{\ell_j}) + 1_{A_j} (X_{\ell_j}) 1_{C_j} (X_{h_j}). \]

For all \( n \geq n_N \), let \( Z_n := \sum_{j=n_N}^{n} Y_j \).

As in the Proof of Theorem 23 by elementary expectation-variance calculations it is easy to verify that

\[ \mu(Z_j \geq 1) \rightarrow 1 \text{ as } j \rightarrow \infty. \]
By \((21)\) let \(J \geq n_N\) be such that the set
\[
E := \{ x \in \Omega : \exists k \in [n_N, J], \, Z_k = 1 \}
\]
satisfies \(\mu(E) \geq \frac{1}{2}\). Let \(D := I \cap E\). Note that \(I\) is determined on \(H \subset H'\) and depends on \((X_g)_{g \in H}\) and \(E\) depends on \((X_g)_{g \in G \setminus H'}\). Since \(X\) is an independent sequence of random variables, the events \(E\) and \(I\) are independent so that \(\mu(D) \geq \frac{1}{2} \mu(I)\). Let \(\tau : D \to [n_N, J]\) be given by \(\tau(x) := \min\{l \in [n_N, J] : Z_l = 1\}\) and \(V : D \to I\) be given by
\[
(Vx)_g := \begin{cases} 
x_{\ell_j}, & \exists j \in [n_N, \tau(x)], \, Y_j \neq 0 \text{ and } g = h_j, 
x_{h_j}, & \exists j \in [n_N, \tau(x)], \, Y_j \neq 0 \text{ and } g = \ell_j, 
x_g, & \text{otherwise.}
\end{cases}
\]
As in the proof of Theorem 23 an easy calculation shows that for all \(x \in D\), we have
\[
\frac{d\mu \circ V}{d\mu} = \lambda^{Z_{\tau(x)}} = \lambda.
\]
Again, from the proof of Theorem 23 it is routine to verify that \(V\) is injective.

By Lemma 12 \(\lambda\) is an essential value for the \(\Sigma_G\)-action and since the Radon-Nykodym derivatives of the \(\Sigma_G\)-action are in \(\lambda^2\) we conclude that \((\Omega, B, \otimes_{g \in G} f_g, \Sigma_G)\) is of Krieger type-III\(\lambda\).

**Proof of Lemma 37**: For \(\sigma \in \Sigma_G, \, g \in G\), and for almost every \(x \in \Omega = [0, 1]^G\), we have
\[
g' \circ \sigma(x) = \prod_{h \in G} \frac{f_{g^{-1}h}(x_{\sigma(h)})}{f_h(x_{\sigma(h)})}
= \prod_{h \in G, \, \sigma(h) \neq h} \frac{f_{g^{-1}h}(x_{\sigma(h)})}{f_{g^{-1}h}(x_h)} \frac{f_h(x_h)}{f_h(x_{\sigma(h)})} \prod_{h \in G} \frac{f_{g^{-1}h}(x_h)}{f_h(x_h)}
= \frac{g'(x)}{\sigma'(x)} \prod_{h \in G, \, \sigma(h) \neq h} \frac{f_{g^{-1}h}(x_{\sigma(h)})}{f_{g^{-1}h}(x_h)}
= \frac{g'(x)}{\sigma'(x)} \prod_{h \in G, \, \sigma(h) \neq h} \frac{f_{g^{-1}h}(x_{\sigma(h)})}{f_{g^{-1}\sigma(h)}(x_{\sigma(h)})}.
\]
where the last equality comes from rearranging the terms in the denominators in the finite product. Since for all \(u \in [0, 1]\), the set
\[
\{g \in G : f_g(u) \neq 1\}
\]
is finite we see that
\[
\lim_{g \to \infty} \prod_{h \in G} \frac{f_{g^{-1}h}(x_{\sigma(h)})}{f_{g^{-1}\sigma(h)}(x_{\sigma(h)})} = 1.
\]
Hence for all \(\sigma \in \Sigma_G\) and for almost all \(x \in \Omega\), we have
\[
\lim_{g \to \infty} \left[ \log_\lambda(g' \circ \sigma(x)) - \log(\sigma'(x)) - \log_\lambda(g'(x)) \right] = 0. \tag{22}
\]
This shows that for \(\tilde{\mu}\)-almost every \((x, n) \in \Omega \times \mathbb{Z}\), we have
\[
d_{\Omega \times \mathbb{Z}}[g(x, n), g(\sigma(x, n))] = d_\Omega(gx, g\sigma x) + \left| \log_\lambda(g'(x)) - [\log_\lambda(g'(\sigma(x))) - \log_\lambda(\sigma'(x))] \right|.
\]
By Remark \(36\), the first term on the right tends to 0 as \(g \to \infty\). The second one tends to 0 by (22).

8. Concluding remarks

8.1. The Proof of Theorem 2. The proof of Theorem 2 follows from a routine modification of the proof of Theorem 1, which we outline below.

Proof of Theorem 2 Let \(\lambda \in (0, 1)\) and consider the Bernoulli shift \((\Omega, \mathcal{B}, \mu, T)\) given in Section 2.2.1. By Remark 9, the functions \((f_n)_{n \in \mathbb{Z}^+}\) embedded in the definition of the product measure \(\mu\) are densities with respect the underlying probability measure \(\rho\).

The Proof of Theorem 2 follows from replacing the underlying probability measure, Lebesgue measure on \([0, 1]\), in Section 5 with the new underlying measure \(\rho\). For example, integrals with respect to Lebesgue measure in Lemma 13 become integrals (or weighted sums) with respect to \(\rho\) and in Lemma 20 we verify tameness with respect to \(\rho\).

With this substitution, the proof is the same.

8.2. Type-III\(_1\) examples on \([0, 1]^\mathbb{Z}\). By considering a certain mixtures of our previous densities from Section 2.1 it is not difficult to write down type-III\(_1\) Bernoulli shifts of a similar form.

Let \(0 < c < 1 < M\) and a consider a sequence of functions \(f_n : [0, 1] \to (c, M)\) such that
\[
\sum_{n=1}^\infty \int_0^1 \left( \sqrt{f_n} - \sqrt{f_{n-1}} \right)^2 du < \infty
\]
and
\[
\sum_{n=1}^\infty \int_0^1 \left( \sqrt{f_n} - 1 \right)^2 du = \infty.
\]
These conditions imply that if we set $f_n = 1$ for all $n < 0$, then shift is a nonsingular $K$-automorphism with respect to $\mu = \otimes_{n \in \mathbb{Z}} f_n$ and the $\Sigma$-action is ergodic since the tameness condition of Aldous and Pitman holds as the functions are uniformly bounded from above and below and Theorem 21 applies.

In addition, if there exists $a < 1$ such that

$$\int_{[0,1]} \left( \frac{1}{(T^n)^r} \right)^2 d\mu = O(n^a),$$

then the shift is conservative and ergodic. Using the argument with asymptotic pairs, in order that the shift will be type-III$_1$, it is sufficient that the $\Sigma$-action is of type-III$_1$. The following two constructions satisfy all these conditions, and we state them without proof. The first construction will be in the spirit of the constructions given for the Proof of Theorem 21 whereas the latter construction resembles constructions given in for a Bernoulli shift on two symbols.

**Example 38.** Let $0 < \delta < \lambda < 1$ be two numbers such that $\log(\delta)$ and $\log(\lambda)$ are linearly independent over $\mathbb{Q}$. Let $A_n, B_n, D_n, E_n$ be a decreasing sequence of disjoint intervals such that

$$\mathcal{L}(A_n) = \mathcal{L}(D_n) = \lambda^{-1} \mathcal{L}(B_n) = \mu^{-1} \mathcal{L}(E_n) = \frac{1}{(n + 4) \log(n + 4)}.$$

Set for $n \leq 1$, $f_n \equiv 1$. For $n \geq 2$, let $f_n : [0, 1] \to \{\delta, \lambda, 1, \lambda^{-1}, \delta^{-1}\}$ be given by

$$f_n(u) = \begin{cases} 
\lambda, & u \in A_n, \\
\lambda^{-1}, & u \in B_n \\
\delta, & u \in D_n, \\
\delta^{-1}, & u \in E_n \\
1, & \text{otherwise}
\end{cases}$$

The proof of conservativity of the shift is similar to the prove of Proposition 16. With an argument similar to the proof of Theorem 23 it follows that $\log(\delta)$ and $\log(\lambda)$ are essential values for the $\Sigma$-action. Since the essential values are a closed subgroup of $\mathbb{R}$ the assumed rational independence gives that then the $\Sigma$-action is of type-III$_1$. 

\[ \Diamond \]

**Example 39.** Set

$$\lambda_n := 1 - \frac{1}{\sqrt{n \log n}}.$$
Let $f_n : [0, 1] \to \{\lambda_n, 2 - \lambda_n\}$ be defined by

$$f_n(u) = \begin{cases} 
\lambda_n, & 0 \leq u \leq \frac{1}{2}, \\
2 - \lambda_n, & \frac{1}{2} \leq u \leq 1.
\end{cases}$$

It is not difficult to verify that the resulting Bernoulli shift will be of type-$\text{III}_1$. 

8.3. **Type-$\text{III}_0$ Bernoulli shifts.** Type-$\text{III}_0$ systems are much less understood than the other type-$\text{III}$ systems and have not been treated in this paper. It is known that one can construct products of odometers [10] that are of type-$\text{III}_0$, but little is known about the possibility of a Bernoulli shift of this type.

**Question 3.** Does there exist a Bernoulli shift that is of type-$\text{III}_0$?

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