INTERACTIONS OF TIME DELAY AND SPATIAL DIFFUSION
INDUCE THE PERIODIC OSCILLATION OF THE VEGETATION
SYSTEM

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ABSTRACT. Empirical data exhibit a common phenomenon that vegetation biomass fluctuates periodically over time in ecosystem, but the corresponding internal driving mechanism is still unclear. Simultaneously, considering that the conversion of soil water absorbed by roots of the vegetation into vegetation biomass needs a period time, we thus introduce the conversion time into Klausmeier model, then a spatiotemporal vegetation model with time delay is established. Through theoretical analysis, we not only give the occurrence conditions of stability switches for system without and with diffusion at the vegetation-existence equilibrium, but also derive the existence conditions of saddle-node-Hopf bifurcation of non-spatial system and Hopf bifurcation of spatial system at the coincidence equilibrium. Our results reveal that the conversion delay induces the interaction between the vegetation and soil water in the form of periodic oscillation when conversion delay increases to the critical value. By comparing the results of system without and with diffusion, we find that the critical value decreases with the increases of spatial diffusion factors, which is more conducive to emergence of periodic oscillation phenomenon, while spatial diffusion factors have no effects on the amplitude of periodic oscillation. These results provide a theoretical basis for understanding the spatiotemporal evolution behaviors of vegetation system.

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1. **Introduction.** Vegetation, which has an important role in modifying and maintaining the ecological environment, is called “ecosystem engineer”, and has significant effect on richness and abundances of species and habitat \[8, 18\]. Climate conditions (such as light, rainfall, temperature, etc.), geographical environment (mountains, plains, etc.) and human activities (grazing, wanton felling, etc.) affect the growth and distribution of vegetation, which leads to form different vegetation distribution structures, namely vegetation patterns. Besides, empirical results show a common phenomenon that vegetation biomass fluctuates periodically over time \[19, 12\]. In recent decade years, with the increasingly serious phenomenon of desertification, the protection of vegetation on the ground becomes more and more necessary, so the research on internal driving mechanisms of emergences of these phenomena has become a hot direction. The study of vegetation system in arid and semi-arid area is helpful for us to better understand the internal mechanism of evolution of vegetation system, to a certain extent, which can provide a certain indicative role for vegetation protection, so as to provide a certain reference value for the study of ecosystem robustness. Mathematical models, especially partial differential equation models \[3, 25, 28, 5, 9, 14, 11\], have been proven to be a very useful tool in explaining the evolution mechanisms of biological system including vegetation ecosystem.

In 1997, Lefever and Lejeune established a single variable model including only vegetation biomass, which reveals the relationships of short-distance mutual promotion and long-distance mutual competition between individual and vegetation communities \[14\]. In reality, the competition of vegetation for water resources is an important factor to determine the biomass of vegetation in arid and semi-arid areas. Experimental results show that density and height of vegetation biomass directly depends on the moisture content of soil in water-limited areas, especially the vegetation coverage. In addition, High density biomass patches can increase the survival and outward expansion of the vegetation, which in turn indicates that vegetation plays a role in redistribution of water resources. Therefore, in 1999, Klausmeier first proposed to use the reaction-diffusion mathematical model including two variables (vegetation biomass \(N\) and soil water \(W\)) to explain the generation of vegetation patterns on a large spatial scale caused by the soil water flow and the uptake of vegetation for water resources \[11\], which becomes building blocks for almost all vegetation biomass-water models. Then utilizations of mathematical models to understand evolution of vegetation ecosystem have aroused great interest and research of theoretical ecologists and mathematicians \[29, 21, 2, 23, 27\].

In order to survival, vegetation competes with each other for water in arid and semi-arid regions due to scarcity of water resource. von Hardenberg et al. \[29\] thus introduced \(\nabla^2(W - \beta_1 N)\) term to simulate the redistribution of water resources caused by the competition among vegetation communities for water, which describes the feedback mechanism of soil-water diffusion, namely the positive feedback between vegetation and water redistribution, where \(\beta_1\) represents the feedback strength of soil-water diffusion, \(D_2\) is the coefficient of free diffusion of water when vegetation is not considered. The new reaction-diffusion model for vegetation dynamics is given by introducing soil-water diffusion feedback into Klausmeier model \[26, 17\]:

\[
\begin{align*}
\frac{\partial N}{\partial t} &= RJWN^2 - MN + D_1 \nabla^2 N, \\
\frac{\partial W}{\partial t} &= A - LW - RWN^2 + V \frac{\partial W}{\partial x} + D_2 \nabla^2 (W - \beta_1 N),
\end{align*}
\]

(1)
where $A$ is the rate of water production under natural conditions, and the main source is precipitation, $L$ is the evaporation coefficient of water in unit time, $M$ is the natural mortality coefficient of vegetation, $V$ is the speed at which water flows down a slope, $RWN^2$ describes the rate at which roots of vegetation absorb soil water, and $R$ is the coefficient of water uptake by vegetation roots, $J$ is the coefficient of conversion of water uptake into biomass of vegetation. In order to depict the diffusion process, the usual Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2}$ or $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in one- or two-dimension space is given, $D_1$ describes the diffusion coefficient of vegetation caused by clonal reproduction or seed diffusion. Since then, enormously may PDE models have been established to investigate the interaction between vegetation biomass and water resource.

The mathematical analysis of these vegetation-water models mainly focuses on pattern-formation theory, which explains the formation mechanism of various vegetation patterns in nature. Sherratt found that banded vegetation is a common pattern structure on hillsides landscape, which moved upward along the slope with constant speed, and slope gradient affects the migration speed of vegetation pattern [23]. Getzin et al. demonstrated that fairy circles of Namibia in arid grasslands were characterized by small-scale hexagonal ordering of circular bare-soil gaps whose driving mechanism is a positive feedback between vegetation and soil water with respect to vegetation biomass-dependent infiltration and water runoff [6]. Sun et al. revealed the reduction of the soil-water diffusion intensity led to structure transitions of vegetation pattern through intermediate states of gap, stripe (labyrinth) and spot distribution structures, and the growth of the vegetation depended on the critical value of feedback intensity [26]. For more details, we refer the readers to [10, 24, 30, 1, 20, 27] and the references therein.

In the aforementioned reaction-diffusion models, the conversion of water uptake by roots into vegetation biomass is assumed to occur immediately at time $t$. However, such transformation process needs to take a period of time to complete rather than occur promptly. To model such conversion time, we incorporate time delay effect into the interaction term between vegetation and water in vegetation equation, then give following model with time delay and feedback of soil-water diffusion based on Klausmeier model in one-dimension space:

$$
\begin{align*}
\frac{\partial N(x, t)}{\partial t} &= RJW(x, t - \tau)N^2(x, t - \tau) - MN(x, t) + D_1\nabla^2 N(x, t), \\
\frac{\partial W(x, t)}{\partial t} &= A - LW(x, t) - RW(x, t)N^2(x, t) + V \frac{\partial W(x, t)}{\partial x} \\
&\quad + D_2\nabla^2 (W(x, t) - \beta_1 N(x, t)), \quad t \geq 0, \ x \in (0, \pi), \\
\left. \frac{\partial N(x, t)}{\partial x} \right|_{x=0, \pi} &= 0, \quad \left. \frac{\partial W(x, t)}{\partial x} \right|_{x=0, \pi} = 0, \ t \geq 0, \\
N(x, t) = \phi_1(x, t) \geq 0, \ W(x, t) = \phi_2(x, t) \geq 0, \ (x, t) \in [0, \pi] \times [-\tau, 0],
\end{align*}
$$

where $N(x, t)$ and $W(x, t)$ represents the vegetation biomass and the density of water at time $t$ and position $x$, respectively, $\tau$ is conversion time, non-zero initial condition and homogeneous Neumann boundary conditions is chosen, $\phi_1(x, t)$ and $\phi_2(x, t)$ denote the initial vegetation biomass and density of soil water, respectively. For reaction-diffusion (RD) models, if we consider the factors including time delay, spatial heterogeneity, and so on, the dynamical behaviors become much more complicated [4, 13, 15, 16].
In addition, according to monthly data of vegetation coverage at 1 kilometer (km) spatial resolution calculated from normalized difference vegetation index (NDVI) of the 1 km monthly composite data product based on moderate resolution imaging spectroradiometer (MODIS) vegetation indices (MOD13A3) version 6, the evolution of vegetation coverage in Gansu Province from January 2000 to December 2018 shows the phenomenon of periodic fluctuation over time, and the corresponding period is one year (see Fig. 1). However, such phenomenon can not be studied by pattern-formation theory, and then a new theoretical perspective needs to be introduced. In this paper, we therefore mainly focus on periodic solutions of the system so as to explain such phenomena existing in the vegetation system from another perspective, which is different from the previous studies on vegetation pattern dynamics.

Figure 1. Time series of vegetation coverage in Gansu Province from January 2000 to December 2018.

The structure of this paper is as follows. In Section 2, we study locally asymptotic stabilities of three equilibria, and give the occurrence conditions of stability switches and saddle-node-Hopf bifurcation of non-spatial system at the vegetation-existence equilibrium and the coincidence equilibrium, respectively. In section 3, we analyze the existence conditions of stability switches and Hopf bifurcation of spatial system at the vegetation-existence equilibrium and the coincidence equilibrium, respectively. In Section 4, the numerical simulations are illustrated to verify theoretical results, and some new observations are exhibited. Finally, we give some conclusions and discussions in section 5.

2. Dynamic analysis of non-spatial system (4). Through nondimensionalizing the model variables, let \( s = \sqrt{\frac{\mathcal{R}J}{L}} W \), \( n = \sqrt{\frac{\mathcal{R}}{L}} N \), \( T = Lt \), \( \tilde{\tau} = L\tau \), and we still use \( t \) and \( \tau \) instead of \( T \) and \( \tilde{\tau} \) for the convenience of symbol representation, then model (2) is simplified as:

\[
\begin{align*}
\frac{\partial n(x,t)}{\partial t} &= s(x,t - \tau)n^2(x,t - \tau) - mn(x,t) + d_1 \nabla^2 n(x,t), \\
\frac{\partial s(x,t)}{\partial t} &= a - s(x,t) - s(x,t)n^2(x,t) + v_1 \frac{\partial s(x,t)}{\partial x} + d_2 \nabla^2 (s(x,t) - \beta n(x,t)),
\end{align*}
\]

(3)

where \( a = \frac{\sqrt{\mathcal{R}J}}{L\sqrt{L}} A \), \( m = \frac{M}{L} \), \( v_1 = \frac{V}{L} \), \( \beta = \beta_1 J \), \( d_1 = \frac{D_1}{L} \), \( d_2 = \frac{D_2}{L} \).
In this paper, we mainly study the vegetation ecosystem on plain landscape, that is, water flowing downhill isn’t considered. Then we discuss the following system:

\[
\begin{cases}
\frac{\partial n(x,t)}{\partial t} = s(x,t - \tau)n^2(x,t - \tau) - mn(x,t) + d_1\nabla^2 n(x,t), \\
\frac{\partial s(x,t)}{\partial t} = a - s(x,t) - s(x,t)n^2(x,t) + d_2\nabla^2 (s(x,t) - \beta n(x,t)).
\end{cases}
\] (4)

Without considering the diffusion effect, namely setting spatial derivatives equal to zero, system (4) is transformed into the following ordinary differential equation (ODE) system:

\[
\begin{cases}
\frac{dn(t)}{dt} = s(t)n^2(t) - mn(t), \\
\frac{ds(t)}{dt} = a - s(t) - s(t)n^2(t).
\end{cases}
\] (5)

By some calculations and employing the linear analysis, we obtain the following results based on discussions in literatures [11, 24] (see Fig. 2): (a) system (5) always has a vegetation-extinction equilibrium \((E_0 = (0, a))\) for all constant \(a\) values, which is locally asymptotically stable, and (b) two vegetation-existence equilibria \(E_* = (n_*, s_*)\) and \(E^* = (n^*, s^*)\) for \(a > 2m\), where

\[
\begin{align*}
s_* &= \frac{a + \sqrt{a^2 - 4m^2}}{2}, \\
n_* &= \frac{m}{s_*}, \\
s^* &= \frac{a - \sqrt{a^2 - 4m^2}}{2}, \\
n^* &= \frac{m}{s^*}.
\end{align*}
\]

One can further obtain \(E_*\) is a saddle point which is unstable, but \(E^*\) is locally asymptotically stable for \(a > 2m\). Note that \(E_*\) coincides with \(E^*\) at \(a = 2m\), which becomes \(E_* = (1, m)\). Besides, these three equilibria correspond to spatially homogeneous equilibria of system (4), and \(E_*\) and \(E^*\) are called as unstable and stable uniform vegetation state, respectively.

**Figure 2.** Stability of equilibria of system (5) with respect to parameter \(a\) for \(m=0.8\), the solid line and dotted line respectively indicate that the equilibria are stable and unstable, except for pink dotted line \(a = 2m\).

### 2.1. Bifurcation analysis of the vegetation-existence equilibrium \(E^*\).

From the perspective of biology, we investigate the most meaningful equilibria \(E^*\) and \(E_*=\). To simplify the subsequent analysis, these two equilibria are denoted as \(\hat{E} = (\hat{n}, \hat{s})\). Now we substitute \(\bar{n} = n - \hat{n}\) and \(\bar{s} = s - \hat{s}\) into system (4), then linearize system
at $\hat{E}$, which gives:
\[
\begin{cases}
\frac{\partial \hat{n}(t)}{\partial t} = a_{11} \hat{n}(t) + b_{11} \hat{n}(t - \tau) + a_{12} \hat{s}(t - \tau), \\
\frac{\partial \hat{s}(t)}{\partial t} = a_{21} \hat{n}(t) + a_{22} \hat{s}(t),
\end{cases}
\] (6)
where $a_{11} = -m$, $b_{11} = 2m$, $a_{12} = \frac{m^2}{s^2}$, $a_{21} = -2m$, $a_{22} = - \frac{m^2 + \bar{s}^2}{s^2}$.

Therefore, the characteristic equation of non-spatial system (4) takes the form:
\[
\Delta(\tau) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} + (-b_{11}\lambda - a_{12}a_{21} + a_{22}b_{11})e^{-\lambda \tau} = 0, \tag{7}
\]

Since we mainly focus on the effect of conversion time on dynamic behavior of non-spatial system (4), then we choose $\tau$ as controlled parameter in the next content. To find the existence conditions of bifurcation, we substitute $\lambda = i\omega(\omega > 0)$ into characteristic equation (7), which derives
\[
-\omega^2 - (a_{11} + a_{22})i\omega + a_{11}a_{22} - b_{11}\cos(\omega \tau)i\omega - b_{11}\sin(\omega \tau)\omega + (b_{11}a_{22} - a_{12}a_{21})\cos(\omega \tau) - (b_{11}a_{22} - a_{12}a_{21})\sin(\omega \tau)i = 0, \tag{8}
\]
by separating the real part and the imaginary part of above Eq. (8), one can get
\[
\begin{cases}
-\omega^2 + a_{11}a_{22} = b_{11}\cos(\omega \tau) - (b_{11}a_{22} - a_{12}a_{21})\cos(\omega \tau), \\
-(a_{11} + a_{22})\omega = b_{11}\omega\cos(\omega \tau) + (b_{11}a_{22} - a_{12}a_{21})\sin(\omega \tau).
\end{cases}
\] (9)

Squaring and adding the above two Eqs., one can obtain:
\[
\omega^4 + p\omega^2 + q = 0,
\]
where $p = a_{11}^2 + a_{22}^2 - b_{11}^2$, $q = a_{11}a_{22}^2 - (b_{11}a_{22} - a_{12}a_{21})^2$. Furthermore, let $z = \omega^2$,
\[
z^2 + pz + q = 0, \tag{10}
\]
assume that Eq. (10) has two real roots $z_1$ and $z_2$, where
\[
z_1 = -p + \sqrt{p^2 - 4q}, \quad z_2 = -p - \sqrt{p^2 - 4q}, \quad z_1 > z_2.
\]
According to the properties of quadratic function, we have
\[
z_1 + z_2 = -p = \frac{1}{s^2}((m\sqrt{3} - 1)s^2 - m^2)((m\sqrt{3} + 1)s^2 + m^2),
\]
\[
z_1z_2 = q = \frac{m(m - \bar{s})(m + \bar{s})}{s^2}(m^2 + 3\bar{s}^2)
\] (11)

For the vegetation-existence equilibrium $E^*$, it is obvious that $m - s^* > 0$ holds for $a > 2m$, which further derive $z_1z_2 > 0$. On the basis of these results and mathematical analysis, we consider the following different cases:

(C1) If $a > 2m$ and $(m\sqrt{3} - 1)s^2 - m^2 < 0$ are satisfied, then Eq. (10) has no positive roots. On the basis of the stability of $E^*$ of system (5), then we know that the real parts of all roots of the characteristic Eq. (7) are negative for $\tau \geq 0$.

(C2) If $a > 2m$ and $(m\sqrt{3} - 1)(s^*)^2 - m^2 > 0$ are satisfied, then the characteristic Eq. (7) has two pairs of pure imaginary roots $\pm i\omega_{01}$ and $\pm i\omega_{02}$, and $\omega_{01} > \omega_{02}$, where $\omega_{0i} = \sqrt{z_i}, \ i = 1, 2$. 

From Eqs. (9), we can calculate the following results:

\[
\begin{align*}
\cos(\omega_0\tau) &= \frac{1}{2} \frac{\omega_0^2 s^2 + m\omega_0^2 i + m^2 + s^2}{s^2(\omega_0^2 i + 1)} = C(\omega_0), \\
\sin(\omega_0\tau) &= \frac{1}{2} \frac{\omega_0(-\omega_0^2 s^2 + m^2 + s^2)}{m \omega_0^2 (\omega_0^2 i + 1)} = S(\omega_0),
\end{align*}
\]

(12)

On the basis of above Eqs., we can calculate:

\[
\tau_{0i}^j = \begin{cases} 
\arccos(C(\omega_0)) + 2j\pi, & \text{when } S(\omega_0) \geq 0, \\
2\pi - \arccos(C(\omega_0)) + 2j\pi, & \text{when } S(\omega_0) < 0.
\end{cases}
\]

Then set \(\lambda(\tau) = r(\tau) + i\omega(\tau)\) be the eigenvalue of characteristic Eq. (7) near \(\tau_{0i}^j\), which is satisfied \(r(\tau_{0i}^j) = 0\) and \(\omega(\tau_{0i}^j) = \omega_0\) for \(i = 1, 2, j = 0, 1, 2, \ldots\).

By taking the derivative with respect to \(\tau\) on the characteristic equation (7), one can get

\[
\left\{2\lambda - (a_{11} + a_{22}) - b_{11}e^{-\lambda \tau} + \tau[b_{11}\lambda + (a_{12}a_{21} - a_{22}b_{11})]e^{-\lambda \tau}\right\} \frac{d\lambda}{d\tau} = 0,
\]

then one can further obtain:

\[
\begin{align*}
\left.\Re\left(\frac{d\lambda}{d\tau}\right)\right|_{\tau = \tau_{0i}^j}^{-1} &= -\Re\left\{\left.\left(2\lambda - (a_{11} + a_{22}) - b_{11}e^{-\lambda \tau}\right)\right|_{\tau = \tau_{0i}^j}\right\} \\
&= \Re\left\{\left.\left(2\lambda^2 - (a_{11} + a_{22})\lambda - b_{11}\right)\right|_{\tau = \tau_{0i}^j}\right\} \\
&= \Re\left\{\left.\left(2\omega_0^2 + a_{11}^2 + a_{22}^2 - b_{11}\right)\right|_{\tau = \tau_{0i}^j}\right\} \\
&= \Re\left\{\left.\left(\frac{2\omega_0^2 + p}{b_{11}^2\omega_0^2 + (a_{12}a_{21} - a_{22}b_{11})^2}\right)\right|_{\tau = \tau_{0i}^j}\right\}.
\end{align*}
\]

on the basis of the above analysis, for case (C2), since the expression \(2\omega_0^2 + p = \sqrt{p^2 - 4q} > 0\) holds, then the transversality conditions \(\Re\left(\frac{d\lambda}{d\tau}\right)\left|_{\tau = \tau_{0i}^j}\right. > 0\). Similarly, we can derive \(\Re\left(\frac{d\lambda}{d\tau}\right)\left|_{\tau = \tau_{0i}^j}\right. < 0\) due to the expression \(2\omega_0^2 + p = -\sqrt{p^2 - 4q} < 0\).

By the above analysis, we have the following result.

**Theorem 2.1.** If condition (C1) holds, the vegetation-existence equilibrium \(E^*\) of non-spatial system (4) is absolutely stable for \(\tau \geq 0\).

We further discuss the occurrence conditions of stability switching of non-spatial system (4).

**Lemma 2.1.** If the conditions (C2) and \(m > 1\) hold, then \(\tau_{01}^0 < \tau_{02}^0\).

**Proof.** To analyze the relationship between \(\tau_{01}^0\) and \(\tau_{02}^0\), we assume \(S(\omega_0) < 0\) and define the following expressions:

\[
\begin{align*}
C(\omega) &= \frac{1}{2} \frac{\omega^2 (s^*)^2 + m\omega^2 + m^2 + (s^*)^2}{(s^*)^2(\omega^2 + 1)} = G_1(\omega), \\
G_2(\omega) &= \frac{1}{\omega} G_1(\omega),
\end{align*}
\]

one can deduce \(\frac{dC(\omega)}{d\omega} = -\frac{m\omega(m-1)}{(s^*)^2(\omega^2 + 1)^2}\). Obviously, if \(m > 1\) holds, then \(C(\omega)\) is a monotone decreasing function associated with \(\omega\), which further states that \(G_1(\omega)\) is a
decrease function through the properties of compound function, namely \( \frac{dG_1(\omega)}{d\omega} < 0 \). By solving first-order derivative directly, then

\[
\frac{dG_2(\omega)}{d\omega} = -\frac{G_1(\omega)}{\omega^2} + \frac{1}{\omega} \frac{dG_1(\omega)}{d\omega} < 0,
\]

thus one can get \( \tau_{01} < \tau_{02}^0 \) for \( \omega_{01} > \omega_{02} \). Using the same method, we don’t give the monotonicity of \( G_2(\omega) \) for \( m < 1 \), then this case isn’t discussed.

\[\text{Theorem 2.2.}\] If conditions (C2) and \( m > 1 \) hold, then non-spatial system (4) gives rise to stability switches with the increase of \( \tau \), and the number of stability switches is finite, that is, there exists a positive integer \( l \) such that the vegetation-existence equilibrium \( E^* \) is locally asymptotically stable when \( \tau \in \{ \tau_{01}^0 \} \cup (\tau_{01}^0, \tau_{01}^2) \cup (\tau_{02}^1, \tau_{02}^2) \cup \cdots \cup (\tau_{0j-1}^1, \tau_{0j}^1) \), while the contrary situation for \( \tau \in \{ \tau_{01}^0, \tau_{02}^0 \} \cup (\tau_{01}^1, \tau_{01}^2) \cup (\tau_{02}^1, \tau_{02}^2) \cup \cdots \cup (\tau_{0j-2}^1, \tau_{0j}^1) \cup (\tau_{0j}^1, +\infty) \). Moreover, non-spatial system (4) undergoes Hopf bifurcation at \( E^* \) when \( \tau = \tau_{0j}^1, i = 1, 2, j = 0, 1, 2, \cdots \).

\[\text{Proof.}\] According to condition (C2), we obtain the transversality conditions

\[
\text{Re}\left( \frac{d\lambda}{d\tau} \Big|_{\tau=\tau_{0j}^1} \right) > 0 \quad \text{and} \quad \text{Re}\left( \frac{d\lambda}{d\tau} \Big|_{\tau=\tau_{0j}^2} \right) < 0,
\]

which further explains a pair of negative real roots cross the imaginary axis at \( \omega_{01} \) into the right-half plane at each critical point \( \tau = \tau_{0j}^1 \), while a pair of positive real roots cross the imaginary axis at \( \omega_{02} \) back into the left-half plane at each critical point \( \tau = \tau_{0j}^2 \), then stability switches occurs with \( \tau \) increasing. Besides, through analyzing

\[\tau_{0j+1}^{j+1} - \tau_{0j}^1 = \frac{2\pi}{\omega_{01}}, \quad \tau_{0j+1}^{j+1} - \tau_{0j}^2 = \frac{2\pi}{\omega_{02}},\]

one can get \( \tau_{0j+1}^{j+1} - \tau_{0j}^1 < \tau_{0j+1}^{j+1} - \tau_{0j}^2 \) due to \( \omega_{01} > \omega_{02} \), which indicates that this switches can not persist for the entire sequences \( \{ \tau_{01}^j \} \) and \( \{ \tau_{02}^j \} \), \( j = 0, 1, 2, \cdots \). Thus the number of stability switches is finite, set the number is \( l \), which satisfies

\[\tau_{01}^{j-1} < \tau_{01}^j < \tau_{01}^{j+1} < \tau_{02}^1 < \tau_{01}^{j-1} < \tau_{02}^2.\]

Thus the multiplicity of roots in the right-half plane is at least two, then non-spatial system (4) is unstable for \( \tau > \tau_{01}^1 \). On the basis of condition (C2) and the corresponding transversality conditions, it is easy to obtain that non-spatial system (4) experiences a Hopf bifurcation at \( E^* \) when \( \tau = \tau_{0j}^1, i = 1, 2, j = 0, 1, 2, \cdots \). 

2.2. Bifurcation analysis of the coincidence equilibrium \( E_{\ast \ast} \). For the coincidence equilibrium \( E_{\ast \ast} = (1, m) \), by introducing \( E_{\ast \ast} \) into the expressions \( a_{11}, b_{11}, a_{12}, a_{21} \) and \( a_{22} \), the corresponding characteristic Eq. (7) can be transformed into:

\[
\Delta(0, \tau) = \lambda^3 + (m + 2)\lambda + 2m - 2m(\lambda + 1)e^{-\lambda \tau}, \quad (13)
\]

The above characteristic Eq. (13) with \( \tau = 0 \) becomes the following equation:

\[
\Delta(0, 0) = \lambda^2 - (m - 2)\lambda, \quad (14)
\]

obviously, the two roots of characteristic Eq. (14) are \( \lambda_1 = 0 \) and \( \lambda_2 = m - 2 \). If \( m \neq 2 \) holds, then system(5) undergoes a saddle-node bifurcation at \( E_{\ast \ast} \).

For characteristic equation (13), it is easy to verify that \( \lambda_1 = 0 \) is a root. We next judge the positive and negative signs of the real parts of other roots, which is similar to the process of analyzing the roots of characteristic Eq. (7). By inserting
\( \lambda = i\omega (\omega > 0) \) into Eq. (13), and separating the real and imaginary parts, one can get
\[
\begin{align*}
-\omega^2 + 2m &= 2m\omega \sin(\omega \tau) + 2m\omega \cos(\omega \tau), \\
(m + 2)\omega &= 2m\omega \cos(\omega \tau) - 2m\sin(\omega \tau).
\end{align*}
\]
(15)

By adding the squares of both sides of the above two equations, we can obtain
\[
\omega^4 + (-3m^2 + 4)\omega^2 = 0.
\]

Let \( z = \omega^2 \), then the above equation can be transformed as:
\[
z^2 + (-3m^2 + 4)z = 0,
\]
(16)

it is easy to give that two roots of Eq. (16) are
\[z_1 = 0 \text{ and } z_2 = 3m^2 - 4 = (\sqrt{3}m + 2)(\sqrt{3}m - 2).\]

we shall discuss the sign of \( z_2 \) in the following:

(C3) If \( m < \frac{2\sqrt{3}}{3} \) holds, then Eq. (16) has no positive real root. On the basis of the stability of \( E_{\ast} \) of system (5), all roots of characteristic Eq. (13) have negative real parts except zero root for \( \tau \geq 0 \).

(C4) If \( \frac{2\sqrt{3}}{3} < m < 2 \) holds, then Eq. (16) has a positive root, which shows that characteristic Eq. (13) has a pair of pure imaginary root \( \pm i\omega_{00}, \) where \( \omega_{00} = \sqrt{z_2} \).

By solving Eq. (15), one can obtain
\[
\cos(\omega_{00}\tau) = \frac{1}{2}\frac{m\omega_{00}^2 + \omega_{00}^2 + 2m}{m(\omega_{00}^2 + 1)} \triangleq C(\omega_{00}),
\]
\[
\sin(\omega_{00}\tau) = \frac{1}{2}\frac{\omega_{00}(-\omega_{00}^2 + m - 2)}{m(\omega_{00}^2 + 1)} \triangleq S(\omega_{00}).
\]
(17)

Here, \( C(\omega_{00}) > 0 \) and \( S(\omega_{00}) < 0 \) can be easily verified.

Moreover, we can calculated the the critical value of \( \tau \) based on Eq. (17):
\[
\tau_{00}^j = \frac{2\pi - \arccos(C(\omega_{00}))) + 2j\pi}{\omega_{00}}, \quad j = 0, 1, 2, \cdots.
\]

Similarly, let \( \lambda(\tau) = r(\tau) + i\omega(\tau) \) be the root of characteristic Eq. (13) near \( \tau_{00}^j \), which satisfies \( r(\tau_{00}^j) = 0 \) and \( \omega(\tau_{00}^j) = \omega_{00} \), where \( j = 0, 1, 2, \cdots \).

Differentiating two sides of characteristic Eq. (13) with respect to \( \tau \) yields
\[
(2\lambda + m + 2 + 2m(\lambda + 1)\tau e^{-\lambda\tau} - 2me^{-\lambda\tau}) \frac{d\lambda}{d\tau} + 2m\lambda(\lambda + 1)e^{-\lambda\tau} = 0,
\]
then one can further obtain:
\[
\text{Re}\left(\frac{d\lambda}{d\tau}\bigg|_{\tau = \tau_{00}^j}\right)^{-1} = -\text{Re}\left\{\lambda \left(\frac{2\lambda + m + 2 - 2me^{-\lambda\tau}}{2m\lambda(\lambda + 1)e^{-\lambda\tau} - 2m} - \frac{\tau}{\lambda}\right)\bigg|_{\tau = \tau_{00}^j}\right\} = \text{Re}\left\{-\left(\frac{2\lambda + m + 2 + 2m(\lambda + 1)\tau e^{-\lambda\tau} - 2me^{-\lambda\tau}}{2m\lambda(\lambda + 1)}\right)\bigg|_{\tau = \tau_{00}^j}\right\} = \text{Re}\left\{-\left(\frac{2\omega_{00}^2 - 3m^2 + 4}{2m(\omega_{00}^2 + 1)}\right)\bigg|_{\tau = \tau_{00}^j}\right\} = \text{Re}\left\{\frac{\omega_{00}^2}{2m(\omega_{00}^2 + 1)}\right\}.
\]

According condition (C4), we obtain \( \text{Re}\left(\frac{d\lambda}{d\tau}\bigg|_{\tau = \tau_{00}^j}\right) > 0 \) because of the expression \( \omega_{00}^2 - 3m^2 + 4 = 0 \).

On the basis of results, the following conclusions are given:
Theorem 2.3. The bifurcation behaviors of non-spatial system (4) at the coincidence equilibrium $E_{**}$ are as follows:

1. If condition (C3) is satisfied, non-spatial system (4) undergoes a saddle-node bifurcation at $E_{**}$ for $\tau \geq 0$;

2. If condition (C4) is satisfied, non-spatial system (4) undergoes a saddle-node-Hopf bifurcation at $E_{**}$ when $\tau = \tau_{j0}^*$, $j = 0, 1, 2, \cdots$.

3. Dynamic analysis of system (4). In this section, we discuss bifurcation behaviors of spatial system (4) at $E^*$ and $E_{**}$, and denote these two equilibria as $\hat{E} = (\hat{n}, \hat{s})$. By inserting $\hat{n} = n - \bar{n}$ and $\hat{s} = s - \bar{s}$ into system (4) and linearizing system at $\hat{E}$, then we can give

$$\begin{aligned}
\frac{\partial \hat{n}(t)}{\partial t} &= a_{11}\hat{n} + b_{11}\hat{n}(t - \tau) + a_{12}\hat{s}(t - \tau) + d_2 \nabla^2 \hat{n}, \\
\frac{\partial \hat{s}(t)}{\partial t} &= a_{21}\hat{n} + a_{22}\hat{s} + d_2 \nabla^2 (\hat{s} - \beta \hat{n}).
\end{aligned} \tag{18}$$

Based on the method of variable separation and Neumann boundary conditions, spatiotemporal perturbations $\hat{n}$ and $\hat{s}$ can be expressed as:

$$\hat{n} = \hat{n} e^{\lambda t} \cos kx, \quad \hat{s} = \hat{s} e^{\lambda t} \cos kx,$$

then insert into linearized model (18), one can derive the following characteristic equation for system (4):

$$\Delta(k, \tau) = \lambda^2 + ((d_1 + d_2)k^2 - (a_{11} + a_{22}))\lambda + (d_1 d_2 k^4 - (a_{11} d_2 + a_{22} d_1)k^2 + a_{11} a_{22}) + (-b_1 \lambda - (a_{12} \beta + b_{11}) d_2 k^2 - a_{12} a_{21} + a_{22} b_{11}) e^{-\lambda \tau} = 0. \tag{19}$$

3.1. Bifurcation analysis of the vegetation-existence equilibrium $E^*$. Eq. (19) with $\tau = 0$ is equivalent to the following characteristic equation:

$$\Delta(k, 0) = \lambda^2 + ((d_1 + d_2)k^2 - (a_{11} + b_{11} + a_{22}))\lambda + (d_1 d_2 k^4 - (a_{22} d_1 + (a_{11} + b_{11} + a_{12} \beta) d_2)k^2 + a_{11} a_{22} - a_{12} a_{21} + a_{22} b_{11} = 0, \tag{20}$$

assume that $\lambda_1$ and $\lambda_2$ are two roots of Eq. (20), that is, for $\forall k$,

$$\begin{aligned}
\lambda_1 + \lambda_2 &= -((d_1 + d_2)k^2 - (a_{11} + b_{11} + a_{22})) , \\
\lambda_1 \lambda_2 &= d_1 d_2 k^4 - (a_{22} d_1 + (a_{11} + b_{11} + a_{12} \beta) d_2)k^2 + a_{11} a_{22} - a_{12} a_{21} + a_{22} b_{11} ,
\end{aligned}$$

where $a_{11} a_{22} - a_{12} a_{21} + a_{22} b_{11} = \frac{1}{\sqrt{2\pi} \tau} (m - s^*)(m + s^*) > 0$ because of $m - s^* > 0$ for $a > 2m$.

If conditions (B0) $a_{11} + b_{11} + a_{22} < 0$ and (B1) $a_{22} d_1 + (a_{11} + b_{11} + a_{12} \beta)d_2 < 0$ hold, it is easy to obtain $\lambda_1 + \lambda_2 < 0$ and $\lambda_1 \lambda_2 > 0$, which indicates that $E^*$ of system (4) with $\tau = 0$ is locally asymptotically stable for any $k$.

Let $\lambda = \omega (\omega > 0)$ be a root of characteristic Eq. (19), then $\omega$ satisfies the following equation:

$$\begin{aligned}
& - \omega^2 + ((d_1 + d_2)k^2 - (a_{11} + a_{22}))\omega + d_1 d_2 k^4 - (a_{11} d_2 + a_{22} d_1) + a_{11} a_{22} \\
& - b_1 \cos(\omega \tau) \omega - b_{11} \sin(\omega \tau) \omega - (a_{12} \beta + b_{11}) d_2 k^2 \cos(\omega \tau) + (a_{12} \beta + b_{11}) \\
& - d_2 k^2 \sin(\omega \tau) + (b_{11} a_{22} - a_{12} a_{21}) \cos(\omega \tau) - (b_{11} a_{22} - a_{12} a_{21}) \sin(\omega \tau) = 0, \tag{21}
\end{aligned}$$
By separating real and imaginary parts of the above equation, one can get
\[
\begin{cases}
(a_1 + d_2)k^2 \omega - (a_1 + a_22) \omega = b_{11} \omega \cos(\omega \tau) - ((a_{12} \beta + b_{11})d_2 k^2 + a_{12}a_{21} - b_{11} a_{22}) \sin(\omega \tau), \\
-d_1 d_2 k^4 - (a_{11} d_2 + a_{22} d_1) k^2 + a_{11} a_{22} - \omega^2 = b_{11} \omega \sin(\omega \tau) + ((a_{12} \beta + b_{11})d_2 k^2 + a_{12} a_{21} - b_{11} a_{22}) \cos(\omega \tau).
\end{cases}
\] (22)

By adding and squaring of Eqs. (22), the following equation is given:
\[
\omega^4 + P(k) \omega^2 + Q(k) = 0,
\] (23)
where
\[
P(k) = (d_1^2 + d_2^2)k^4 - 2(a_{11} d_1 + a_{22} d_2)k^2 + a_{11}^2 + a_{22}^2 - b_{11}^2, \\
Q(k) = d_1 d_2 k^4 - (a_{11} d_2 + a_{22} d_1) k^2 + a_{11} a_{22} + a_{12} a_{21} - b_{11} a_{22}.
\]

Here, \(a_{11} a_{22} + a_{12} a_{21} - b_{11} a_{22} = \frac{m(m^2 + 3x^2)}{(x^2)^2} > 0\). So it is easy to verify \(Q_1(k) > 0\) for any \(k\).

Then we substitute \(Z = \omega^2\) into Eq. (23), and get
\[
Z^2 + P(k)Z + Q(k) = 0, \quad \forall \ k \in N.
\] (24)

From condition (B1), it is obvious that \(Q_2(k) > 0\) holds for \(\forall \ k\). But the sign of \(P(k)\) is not uncertain. Then we give the following cases:

**Lemma 2.2.** If there exists a \(k_0 \in N = 1, 2, \ldots\) such that

(i) \((H1)\) (d_1^2 + d_2^2)k_0^4 - 2(a_{11} d_1 + a_{22} d_2)k_0^2 + a_{11}^2 + a_{22}^2 - b_{11}^2 > 0\) holds, then the real parts of all roots of characteristic Eq. (19) are negative.

(ii) \((H2)\) (d_1^2 + d_2^2)k_0^4 - 2(a_{11} d_1 + a_{22} d_2)k_0^2 + a_{11}^2 + a_{22}^2 - b_{11}^2 < 0\) holds, then characteristic equation (19) has two pairs of pure imaginary \(\pm i \omega_{k_0,i}\), where \(\omega_{k_0,i} = \sqrt{Z_{k_0,i}}, \ i = 1, 2\).

**Proof.** Assume that \(Z_{k,1}\) and \(Z_{k,2}\) are two roots of Eq. (24), where
\[
Z_{k,1,2} = \frac{1}{2} \left(-P(k) \pm \sqrt{(P(k))^2 - 4Q(k)}\right),
\]
and \(Z_{k,2} < Z_{k,1}\). On the basis of above discusses, one can obtain that \(Q(k) > 0\) holds for \(\forall \ k\).

(i) When condition \((H1)\) is satisfied, namely \(P(k) > 0\) at \(k = k_0\), then we can derive \(Z_{k_0,2} < Z_{k_0,1} < 0\), which further shows that characteristic Eq. (19) has no pure imaginary roots. Moreover, combining with the fact that the real parts of all roots of Eq. (19) with \(\tau = 0\) are negative, so all roots of Eq. (19) have negative parts [22].

(ii) While when condition \((H2)\) is satisfied, then we derive \(P(k) < 0\) for \(k = k_0\), which further indicates Eq. (24) has two positive roots \(0 < Z_{k_0,2} < Z_{k_0,1}\). Therefore, one can calculate
\[
\omega_{k_0,1} = \sqrt{Z_{k_0,1}}, \quad \omega_{k_0,2} = \sqrt{Z_{k_0,2}},
\]
then Eq. (19) has two pairs of pure imaginary roots \(\pm i \omega_{k_0,i}, \ i = 1, 2\). This completes the proof. \(\Box\)

Under the condition of Lemma 2.2., by solving Eqs. (22), then we obtain:
\[
\cos(\omega_{k_0,i} \tau) = \frac{A_1 b_{11} \omega_{k_0,i} + B_1 B_2}{b_{11}^2 \omega_{k_0,i} + B_2^2} \triangleq C(\omega_{k_0,i}),
\]
\[
\sin(\omega_{k_0,i} \tau) = \frac{-B_1 b_{11} \omega_{k_0,i} + A_1 B_2}{b_{11}^2 \omega_{k_0,i} + B_2^2} \triangleq S(\omega_{k_0,i}),
\]
where $A_1 = (d_1 + d_2)k_0^2\omega_{k_0,i} - (a_{11} + a_{22})\omega_{k_0,i}$, $B_1 = d_1d_2k_0^4 - (a_{11}d_2 + a_{22}d_1)k_0^2 + a_{11}d_2 - \omega_{k_0,i}^2$, $B_2 = (a_{12} + b_{11})d_2k_0^2 + a_{12}b_{21} - b_{11}a_{22}$, Moreover, we further compute the values of $\tau$ based on the above two formulas,

$$
\tau_{k_0,i}^j = \begin{cases}
\arccos(C(\omega_{k_0,i}) + 2j\pi), & \text{when } S(\omega_{k_0,i}) \geq 0, \\
\arccos(C(\omega_{k_0,i}) + 2j\pi), & \text{when } S(\omega_{k_0,i}) < 0, \\
\end{cases} \quad (25)
$$

We set $\lambda(\tau) = r(\tau) + i\omega(\tau)$ be the root of characteristic Eq. (19) near $\tau_{k_0,i}^j$, which satisfies $r(\tau_{k_0,i}^j) = 0$ and $\omega(\tau_{k_0,i}^j) = \omega_{k_0,i}$ for $j = 0, 1, 2, \cdots$.

**Lemma 2.3.** Assume that condition (H2) is satisfied, then the transversality condition

$$
\text{Re}(\frac{d\lambda}{d\tau} | \tau = \tau_{k_0,i}^j) > 0 \text{ and } \text{Re}(\frac{d\lambda}{d\tau} | \tau = \tau_{k_0,2}^j) < 0
$$

are obtained.

**Proof.** Differentiating the both sides of characteristic Eq. (19) associated with $\tau$ yields

$$
\text{Re}(\frac{d\lambda}{d\tau} | \tau = \tau_{k_0,i}^j) = \text{Re}(\frac{2\lambda + (d_1 + d_2)k_0^2 - (a_{11} + a_{22}) - b_{11}e^{-\lambda\tau}}{b_{11}\lambda + (a_{12} + b_{11})d_2k_0^2 + a_{12}a_{21} - b_{11}a_{22}\lambda e^{-\lambda\tau} - \lambda}|_{\tau = \tau_{k_0,i}^j})
$$

$$
= \text{Re}(\frac{2\lambda e^{\lambda\tau} + (d_1 + d_2)k_0^2 e^{\lambda\tau} - (a_{11} + a_{22}) e^{\lambda\tau} - b_{11}}{b_{11}\lambda + (a_{12} + b_{11})d_2k_0^2 + a_{12}a_{21} - b_{11}a_{22}\lambda}|_{\tau = \tau_{k_0,i}^j})
$$

$$
= \text{Re}(\frac{2\lambda e^{\lambda\tau} + (d_1 + d_2)k_0^2 e^{\lambda\tau} - (a_{11} + a_{22}) e^{\lambda\tau} - b_{11}}{b_{11}\lambda + (a_{12} + b_{11})d_2k_0^2 + a_{12}a_{21} - b_{11}a_{22}\lambda}|_{\tau = \tau_{k_0,i}^j})
$$

$$
= \frac{2\omega_{k_0,i}^2 + (d_1^2 + d_2^2)k_0^4 - 2(d_1a_{11} + d_2a_{22})k_0^2 + a_{11}^2 + a_{22}^2 - b_{11}^2}{b_{11}^2\omega_{k_0,i}^2 + [(a_{12} + b_{11})d_2k_0^2 + a_{12}a_{21} - b_{11}a_{22}]^2}
$$

$$
= \frac{2\omega_{k_0,i}^2 + P_{k_0}}{b_{11}^2\omega_{k_0,i}^2 + [(a_{12} + b_{11})d_2k_0^2 + a_{12}a_{21} - b_{11}a_{22}]^2},
$$

obviously, we can give $\text{Re}(\frac{d\lambda}{d\tau} | \tau = \tau_{k_0,i}^j) = \frac{\sqrt{P_{k_0}^2 - 4Q_{k_0,1}}}{\sqrt{P_{k_0,2}^2 - 4Q_{k_0,2}}}$, while $\text{Re}(\frac{d\lambda}{d\tau} | \tau = \tau_{k_0,2}^j) = \frac{\sqrt{P_{k_0,2}^2 - 4Q_{k_0,2}}}{\sqrt{P_{k_0,2}^2 - 4Q_{k_0,2}}}$, Lemma 2.3. is proved.

The above discussions are summarized as:

**Theorem 2.4.** For system (4), the following conclusions are given:

(i) If conditions (B0), (B1) and (H1) are satisfied, the vegetation-existence steady state $E^*$ of system (4) is locally asymptotically stability when $\tau \geq 0$.

(ii) If conditions (B0), (B1) and (H2) are satisfied, system (4) shall potentially experience stability switches at the vegetation-existence steady state $E^*$.

By comparing the bifurcation behaviors of non-spatial and spatial system (4) at $E^*$, we find that the introduction of spatial diffusion is able to maintain stability switches.
3.2. Bifurcation analysis of the coincidence equilibrium $E_{\ast\ast}$. According to the method of solving characteristic equation in section 2.2, we can give the characteristic equation of system (4) at $E_{\ast\ast}$:

$$\Delta(k, \tau) = \lambda^2 + ((d_1 + d_2)k^2 + m + 2)\lambda + (d_1d_2k^4 + (md_2 + 2d_1)k^2 + 2m)(2m\lambda + (\beta + m)d_2k^2 + 2m)e^{-\lambda \tau} = 0.$$  (26)

Meanwhile, it can be verified that $\lambda = 0$ is not a root of Eq. (26).

By the same method in section 3.1, we can derive

$$\omega^4 + P(k)\omega^2 + Q(k) = 0,$$  (27)

where $P(k) = (d_1^2 + d_2^2)k^4 - 2(md_1 + 2d_2)k^2 - (3m^2 - 4)$, $Q_1(k) = d_1d_2k^4 + (2d_1)k^2 + (\beta + 3m)d_2k^2 + 4m > 0$, $Q_2(k) = d_1d_2k^4 + (2d_1 - (\beta + m)d_2)k^2$, and $Q(k) = Q_1(k)Q_2(k)$. Obviously, the signs of $P(k)$ and $Q_2(k)$ are not determined.

Similarly, set $Z = \omega^2$, then Eq. (27) can be transformed as:

$$Z^2 + P(k)Z + Q(k) = 0, \text{ for } \forall k \in N.$$  (28)

Lemma 2.4. If there exists a $k_0 \in N = 1, 2, \cdots$ such that

$$\text{(H3)} \ d_1d_2k_0^4 + (2d_1 - (\beta + m)d_2)k_0^2 < 0$$

holds, then characteristic equation (26) has a pair of pure imaginary roots $\pm i\omega_{k_0}$, where $\omega_{k_0} = \sqrt{Z_{k_0, 1}}$.

Proof. Assume that above Eq. (28) has two roots $Z_{k, 1}$ and $Z_{k, 2}$, that is,

$$Z_{k, 1} = \frac{1}{2} \left( -P(k) + \sqrt{(P(k))^2 - 4Q(k)} \right), \ Z_{k, 2} = \frac{1}{2} \left( -P(k) - \sqrt{(P(k))^2 - 4Q(k)} \right).$$

If condition (H3) is satisfied, one can derive that $Q(k_0) < 0$ based on $Q_1(k_0) > 0$ for $\forall k$, and then we can obtain $Z_{k, 1} > 0 > Z_{k, 2}$ when $k = k_0$. So we can get $\omega_{k_0} = \sqrt{Z_{k_0, 1}} > 0$, which indicates that characteristic Eq. (26) has a pair of pure imaginary roots $\pm i\omega_{k_0}$. This completes the proof.

Through the same calculation process, the following expressions are given:

$$\cos(\omega_{k_0} \tau) = \frac{A_1b_1 + B_1B_2}{b_1^2 + C_{k_0}}, \ \sin(\omega_{k_0} \tau) = \frac{-B_1b_1 + A_1B_2}{b_1^2 + C_{k_0}},$$

where $A_1 = (d_1 + d_2)k_0^2 \omega_{k_0} + (m + 2)\omega_{k_0}$, $B_1 = d_1d_2k_0^4 + (md_2 + 2d_1)k_0^2 + 2m - \omega_{k_0}^2$, $B_2 = (\beta + 2m)d_2k_0^2 + 2m$. Moreover, we further compute the values of $\tau$ based on the above two formulas.

$$\tau_{k_0}^j = \begin{cases} \frac{\arccos(C(\omega_{k_0})) + 2j\pi}{\omega_{k_0}}, & \text{when } S(\omega_{k_0}) \geq 0, \\ \frac{2\pi - \arccos(C(\omega_{k_0})) + 2j\pi}{\omega_{k_0}}, & \text{when } S(\omega_{k_0}) < 0, \ j = 0, 1, 2, \cdots . \end{cases}$$  (29)

Through the same analysis process, we can draw the following conclusion:

Lemma 2.5. Assume that condition (H3) is satisfied, then the transversality condition

$$\text{Re} \left( \frac{d\lambda}{d\tau} \bigg|_{\tau = \tau_{k_0}^j} \right) > 0$$

is obtained.

Theorem 2.5. If condition (H3) holds, the following statements are given:
when \( 0 < \tau < \tau_{ko}^0 \), while \( E_{ss} \) becomes unstable when \( \tau > \tau_{ko}^0 \);

(iii) System (4) undergoes a spatially homogeneous Hopf bifurcation at \( E_{ss} \) when \( \tau = \tau_{ko}^j, j = 0, 1, 2, \ldots \).

With the introduction of spatial diffusion (namely, \( d_1 \neq 0 \) and \( d_2 \neq 0 \)), different from bifurcation results of non-spatial system (4) at \( E_{ss} \), it is interesting to find that, instead of the occurrences of the saddle-node and the saddle-node-Hopf bifurcations, system (4) undergoes Hopf bifurcation, which shows that the spatial diffusion has a profound influence on the bifurcation behavior near \( E_{ss} \).

4. Main results. In this section, on the basis of theoretical results of the above section, a series of numerical simulations is performed as follow based on the method that system (4) is discretized in time and space. In numerical simulation process, we assume that the spatial range is \([0, \pi]\) and diffusions of the vegetation and soil water at each boundary are zero. Since we mainly focus on the influence of time delay on dynamic behaviors, the parameter \( \tau \) is selected as the control parameter. In the following, the dynamic behaviors of system (4) without and with diffusion are shown.

4.1. Dynamic simulations of non-spatial vegetation system. In this subsection, we provide simulation results for Theorem 2.3 and investigate the effects of rainfall capacity \( a \) and the natural mortality of the vegetation \( m \) on the critical value \( \tau_{k0}^0 \) of the bifurcation parameter \( \tau \), the amplitude and the period of bifurcating period solutions under the case \( a = 2m \). By choosing parameters as \( m = 1.4 \), 1.6 and 1.8, we plot the time series and phase plane figures of system (4) without diffusion when \( \tau > \tau_{k0}^0 \) in Fig. 3, which shows that a family of periodic solutions bifurcates from the coincidence equilibrium \( E_{ss} \). It is easy to find that the critical bifurcation value, the amplitude and the period decrease with the increase of \( a \) and \( m \), which indicates that the increases of rainfall capacity and the natural mortality accelerate the periodic oscillation behaviors of the vegetation and soil water in unit time, and reduce the vegetation biomass. Furthermore, the change of the vegetation biomass over time is opposite to density of soil water and slightly lags behind the change of soil water from Fig. 3 because of uptake feedback mechanism between the vegetation and soil water, that is, the vegetation need to absorb soil water in order to their own growth, and then reduce the density of soil water. With the decrement of water soil to a certain extent, it shall lead to the decline in the vegetation biomass, which bring a gradual increase in the density of soil water, such process rolls on in cycles over time.

In order to understand the occurrences of stability switches of system (4) without diffusion, we set \( a = 3.61, m = 1.8 \), then we can calculate the existence-vegetation equilibrium \( E^* = (1.0774, 1.6707) \), \( \tau_{01}^j = 2.6943, 5.6309, 8.5674, 11.5040 \) and \( \tau_{02}^j = 9.0307, 18.1641, 27.2976, 36.4310 \), where \( j = 0, 1, 2, 3 \). Besides, we further get \( \Re(d_{\tau} \big|_{\tau = \tau_{01}^j}) = 17.6114 > 0 \) and \( \Re(d_{\tau} \big|_{\tau = \tau_{02}^j}) = -4.6514 < 0 \). According to Theorem 2.2., the existence-vegetation equilibrium \( E^* \) is locally asymptotically stable when \( \tau \in [0, \tau_{01}^0) \), while it becomes unstable when \( \tau \in (\tau_{01}^0, +\infty) \), and Hopf bifurcation emerges at \( \tau = \tau_{01}^0 \), and the corresponding direction is forward. These phenomena are shown in Fig. 4, we can see that one switch occur from stable state to unstable state with the increase of \( \tau \), and finally to unstable state.
4.2. Dynamic simulations of vegetation system. Now we focus on that the combined effect of time delay and spatial diffusion on spatiotemporal dynamics of system (4), especially periodic solutions. By setting parameters as $a = 2.8$, $m = 1.4$, $\beta = 0.0$, $d_1 = 0.02$, $d_2 = 0.2$, we get the coincidence equilibrium $E_{ss} = (1.1.4)$ and $\tau_2^0 = 4.0707$ based on the expressions (29) of the critical bifurcation parameter $\tau$ for $j = 0$. Then when $\tau < \tau_2^0$, the vegetation biomass and the density of soil water eventually tend to the coincidence equilibrium $E_{ss}$ at different spatial locations in Fig. 5(a) and (d), which means that $E_{ss}$ is asymptotically stable. With the
increase of $\tau$ to exceed the critical value $\tau^0$, $E_{**}$ becomes unstable, then system (4) undergoes a family spatially homogeneous periodic solutions at $E_{**}$ for Fig. 5(b), (c), (e) and (f). Meanwhile (c) and (f) capture part of figures (b) and (e) from $t = 2800$ to $t = 2850$ so as to show the periodic oscillation behavior over time and space more clearly. Besides, we take $a = 3.2$, $m = 1.6$ and $a = 3.6$, $m = 1.8$ when other parameters remain invariant except $\tau$, then similar results are obtained. These results shows that the conversion time induces the interaction between the vegetation and soil water in the form of periodic fluctuations, which depicts the internal driving factors that affect the development of vegetation ecosystem.

![Figure 5.](image)

**Figure 5.** The evolutions of vegetation biomass and the density of soil water with time and space: (a) $\tau = 2 < \tau^0$; (b) $\tau = 4.2 > \tau^0$. Other parameters are $a = 2.8$, $m = 1.4$, $\beta = 0.0$, $d_1 = 0.02$, $d_2 = 0.2$. The initial conditions are $n(x,t) = 1.2$ and $s(x,t) = 1.6$, $(x,t) \in [0, \pi] \times [-\tau, 0]$.

By taking the same parameters as those in Fig. 5, we further show evolutions of the vegetation and soil water in the two dimensions of time and space with respect to $\tau$, namely spatiotemporal evolution diagrams, where the first row and the second row represent the spatiotemporal evolution laws of the vegetation and soil water respectively, and the corresponding color bars instruct the biomass of vegetation and the density of soil water. We observe that a series of vertical lines parallel to the $y$-axis are formed at the initial period of time for $\tau < \tau^0$, their ranges gradually are widened with the evolution of time, and the corresponding color is deepened. The vertical line disappears from $t = 200$ and the color turns into dark blue (i.e., the density reached 1.0) in Fig. 5(a). When $\tau$ is greater than the critical value $\tau^0$, the color changes from red to yellow and then to light green in the direction parallel to the $y$-axis, and light green lasts for a period of time. Next, dark green and dark gray vertical lines emerge for a short time, and then formed regular brown-gray vertical lines. In order to clearly observe the evolution law from time $t = 2200$, we intercept the spatiotemporal evolution Fig. 6(b) from $t = 2800$
Comparisons of dynamic simulations of the vegetation system without or with diffusion. Considering that spatial diffusion can more truly reflect the evolution processes of the vegetation and soil water, then we consider the potential impact of spatial diffusion on delayed vegetation system by comparative
Figure 7. The variation of the value of $\tau_c$ for three groups $a$ and $m$: (a) with $\beta$ when $d_1 = 0.02$, $d_2 = 0.2$; (b) with $d_2$ when $\beta = 0.0$, $d_1 = 0.02$. (c) The variation of the critical bifurcation parameter $\tau_c$, the period and the amplitude of spatially homogeneous periodic solutions with respect to $m$. Other parameters as $d_1 = 0.02$, $d_2 = 0.2$, $\beta = 0.0$, $a = 2m$ and $\tau = 4.2$.

We find that the value of the critical bifurcation parameter $\tau^0_c$ decreases with the increase of $a$ and $m$, which is consistent with results of system (4) without diffusion (see Table 1). Moreover, we can obtain that the value of the critical bifurcation parameter of system (4) is greater than that of system (4) without diffusion for $m \in [1.46, 2]$ by Table 1 when other parameters are $\beta = 0.0$, $d_1 = 0.02$, $d_2 = 0.2$ and $k = 2$, but the opposite case occur for $m = 1.40, 1.43$, which indicates that the spatial diffusion increases the value of critical bifurcation parameter and makes the vegetation system difficult to generate Hopf bifurcation for $m \in [1.46, 2]$. Meanwhile, the change of imaginary part of a pair of pure imaginary root with $m$ is just contrary to the change of the critical bifurcation parameter. Through analyzing system (4) with and without diffusion, we can find that the generation of the bifurcating period solutions depends on the change of time delay in two systems.

On the basis of Theorem 2.1.-2.3., we study that the stability of $E^*$ and $E_{**}$ of system (4) without diffusion in different parameter ranges. The curves $m(\sqrt{3} - 1)(s^*)^2 - m^2 = 0$ (black dotted line) and $a = 2m$ (red line) divide the $a - m$ parameter plane into three parts in Fig. 8(a): $R_1$ is an absolutely stable region, that is, $E^*$ of system (4) without diffusion is asymptotically stable for all $\tau \geq 0$; $R_2$ is conditionally stable region, and stability switches occurs; there are no two vegetation-existence equilibria in region $R_3$ due to $a \geq 2m$ not satisfied. Besides, two equilibria $E^*$ and $E_{**}$ merge into one equilibrium $E_{**} = (1, m)$ on the line of $a = 2m$, at the same time, system (4) without diffusion undergoes saddle-node-Hopf bifurcation at $E_{**}$. In addition, we further consider the influence of spatial diffusion on the partition of $a - m$ parameter plane in Fig. 8(b) based on Theorem 2.4., which is divided into three parts by curves $a = 2m$ and $P(k) = 0$ with different $k$. $E^*$ of system (4) is asymptotically stable for all $\tau \geq 0$ in region $D_1$, which consists of $a = 2m$, $P(k) = 0$ and $xy$ axis. Corresponding to $R_2$, region $D_2$ is conditionally stable, then stability switches emerges for system (4). There don’t exist two vegetation-existence equilibria again in region $D_3$. Moreover, it is easy to observe that region $D_1$ becomes wider, region $D_2$ is lessened with the increase of $k$, while $D_3$ remains unchanged. Moreover, system (4) may undergo Hopf bifurcation on the line of $a = 2m$. 

Table 1. Comparison of $\omega_{00}, \tau_{00}, \omega_2$ and $\tau_2$ of system (4) without and with diffusion.

| Order | $m$ | $\omega_{00}$ | $\tau_{00}$ | $\omega_2$ | $\tau_2$ |
|-------|-----|---------------|-------------|------------|---------|
| 1.    | 1.40 | 1.3711        | 4.2650      | 1.4055     | 4.0708  |
| 2.    | 1.43 | 1.4611        | 3.9880      | 1.4570     | 3.9233  |
| 3.    | 1.46 | 1.5475        | 3.7526      | 1.5095     | 3.7835  |
| 4.    | 1.49 | 1.6310        | 3.5494      | 1.5630     | 3.6507  |
| 5.    | 1.52 | 1.7121        | 3.3717      | 1.6173     | 3.5247  |
| 6.    | 1.55 | 1.7909        | 3.2145      | 1.6725     | 3.4051  |
| 7.    | 1.58 | 1.8679        | 3.0742      | 1.7285     | 3.2916  |
| 8.    | 1.61 | 1.9433        | 2.9480      | 1.7852     | 3.1839  |
| 9.    | 1.64 | 2.0171        | 2.8337      | 1.8426     | 3.0817  |
| 10.   | 1.67 | 2.0897        | 2.7295      | 1.9006     | 2.9847  |
| 11.   | 1.70 | 2.1610        | 2.6341      | 1.9591     | 2.8927  |
| 12.   | 1.73 | 2.2313        | 2.5462      | 2.0181     | 2.8053  |
| 13.   | 1.76 | 2.3006        | 2.4650      | 2.0775     | 2.7223  |
| 14.   | 1.79 | 2.3690        | 2.3897      | 2.1373     | 2.6435  |
| 15.   | 1.82 | 2.4366        | 2.3195      | 2.1974     | 2.5687  |
| 16.   | 1.85 | 2.5035        | 2.2540      | 2.2578     | 2.4975  |
| 17.   | 1.88 | 2.5697        | 2.1926      | 2.3184     | 2.4299  |
| 18.   | 1.91 | 2.6352        | 2.1350      | 2.3791     | 2.3655  |
| 19.   | 1.94 | 2.7001        | 2.0807      | 2.4400     | 2.3042  |
| 20.   | 1.97 | 2.7645        | 2.0295      | 2.5010     | 2.2460  |
| 21.   | 2.00 | 2.8284        | 1.9811      | 2.5620     | 2.1903  |

Figure 8. Region divisions of parameter space $a - m$: (a) for system (4) without diffusion; (b) for system (4) with parameters as $\beta = 0.1, d_1 = 0.02, d_2 = 0.2$.

According to Theorem 2.4., the sign of expression $P(k)$ determines the stability of the vegetation-existence equilibrium $E^*$ of system (4). Fig. 9 shows that the value of expression $P(k)$ increases with the increasing of wave number $k$ for different parameters $a, d_2$ and $m$. From Fig. 9(a), we observe that expression $P(k)$ becomes smaller with the decrease of $a$, while the interval of $P(k) < 0$ with respect to $k$ increases, which indicates that stability switches is easy to emerge in system (4).
because of these parameter ranges belonging to region $D_2$ of Fig. 8(b), and then the stability interval of $E^*$ reduces. Fig. 9(b) presents a similar result to Fig. 9(a). Fig. 9(c) shows expression $P(k)$ decreases with the increase of $m$ and less than zero for some $k$ when $m = 1.8$, which illustrates that the increase of $m$ doesn’t easily makes the vegetation-existence $E^*$ of system (4) lose stability.

![Figure 9](image)

**Figure 9.** The variation of the sign of expression $P(k)$ with $k$: (a) under different $a = 4.0$, 3.8 and 3.61. Other parameter as $m = 1.8$, $d_1 = 0.02$, $d_2 = 0.2$; (b) under different $d_2 = 1$, 0.6 and 0.2. Other parameter as $a = 3.61$, $m = 1.8$, $d_1 = 0.02$; (c) under different $m = 1.8$, 1.6 and 1.4. Other parameter as $a = 3.61$, $d_1 = 0.02$, $d_2 = 0.2$.

The occurrence of the spatially homogeneous Hopf bifurcation at the coincidence equilibrium $E_{eq}$ of system (4) depends on the sign of expression $Q_2(k)$ based on Theorem 2.5., Fig. 10 thus shows the variation of expression $Q_2(k)$ with wave number $k$ for different parameters $\beta$, $d_2$ and $m$. We find that the value of $Q_2(k)$ is always less than 0 for all $k > 0$, which indicates that the spatially homogeneous Hopf bifurcation is easy to emerge in system (4). Furthermore, the value of $Q_2(k)$ becomes smaller with the increase of $\beta$, $d_2$, $m$, or $k$, which indicates that the increase of these four parameters easily makes $E_{eq}$ of system (4) lose stability.

![Figure 10](image)

**Figure 10.** The variation of the sign of expression $Q_2(k)$ with $k$: (a) under different $\beta = 0.2$, 0.1 and 0.0. Other parameter as $a = 3.2$, $m = 1.6$, $d_1 = 0.02$, $d_2 = 0.2$; (b) under different $d_2 = 1$, 0.6 and 0.2. Other parameter as $\beta = 0.1$, $a = 3.2$, $m = 1.6$, $d_1 = 0.02$; (c) under different $m = 1.8$, 1.6 and 1.4. Other parameter as $\beta = 0.1$, $a = 2m$, $d_1 = 0.02$, $d_2 = 0.2$. 
4.4. Other interesting results of vegetation system with diffusion. In this subsection, we find that system (4) experiences a family of spatially heterogeneous periodic solutions for initial value depending on space in Fig. 11(a) and (b) when parameters are still set as $a = 2.8$, $m = 1.4$, $\beta = 0.0$, $d_1 = 0.02$, $d_2 = 0.2$. The emergence of spatially heterogeneous periodic solutions is independent of time delay, and the corresponding period and amplitude have the same results. Furthermore, we can observe that the spatial and temporal distributions of vegetation biomass are inhomogeneous at the beginning stage, and the distribution is not regular. Over time the distribution presents a regular structure in whole space $[0, \pi]$ at about $t = 160$, that is, periodic oscillations are formed in space (see Fig. 11(c)), but at the same spatial location the distribution in the direction of time finally tends to steady state (see Fig. 11(f)), which indicates that the spatial heterogeneity distributions of vegetation biomass and soil water at initial period has a significant effect on the spatial distribution in the later stage, namely a periodic oscillation pattern. Moreover, it is also obvious that the spatial distribution of vegetation biomass and soil water presents an anti-phase spatial distribution in Fig. 11(c), that is, vegetation biomass increases rapidly while the density of soil water decreases quickly in the intervals $(0, 0.8)$ and $(1.6, 2.4)$, and the vegetation biomass reduce while the density of soil water increases in the intervals $(0.8, 1.6)$ and $(2.4, 3.2)$. The reason is that the growth of vegetation biomass leads to more soil water being uptake by roots, and then the soil water is decremented, the contrary case is that the density of soil water increases when vegetation biomass decreases, then result in the formation of periodic oscillation pattern in space. Such phenomenon is consistent with that in the fairy circles of Namibia, which is caused by the uptake-diffusion feedback mechanism\[6\].

In the initial period of time, although the distribution of the vegetation is uneven, the biomass gap is relatively small. Then the biomass gap becomes obvious in the whole region over time from $t = 120$, and five red and blue strips is formed in the direction parallel to time, where the middle color of the red strip is relatively dark, and the red color becomes lighter to the left and right sides of the spatial position (i.e., the biomass decreases from the middle stripe to the left and right sides). Similarly, the biomass increases from the middle stripe to the left and right sides. We further find that the biomass of the red strip is higher than that of the blue one, which clearly describes that the spatial evolution of vegetation over time presents obvious spatial heterogeneity and forms regular high-low density alternation. Simultaneously, the density of soil water follows the same spatiotemporal evolution rule, which is composed of three orange and two blue strips alternately, and the colors of these two stripes are lighter than those in the Fig.(e) by comparison, which shows that the maximum density of soil water is less than that of vegetation, while the minimum value is greater than that of vegetation (see Fig.11(f)).

Now, we mainly focus on the influences of diffusion feedback strength $\beta$, diffusion coefficient of soil water $d_2$ and natural mortality $m$ on the amplitude and the period of spatially heterogeneous periodic solutions. Keep the time delay $\tau = 2$ and the initial conditions invariant in simulation process. Fig. 12(a) shows that the amplitude corresponding to soil water decreases slowly with the increase of $\beta$ when $m = 1.4$, 1.6 and 1.8, while the amplitude corresponding to the vegetation increases obviously. Actually, the size of $\beta$ describes the capacity of water uptake by vegetation roots, so the stronger the uptake capacity is, the less the density of soil water becomes, and then such situation promotes the growth of the vegetation. As
Figure 11. The evolutions of vegetation biomass and the density of soil water with time and space: (a)(d) vegetation biomass, (b)(e) the density of soil water. The evolutions of vegetation biomass and the density of soil water: (c) with space at $t = 700$, (f) with time at $x = 20$. The initial conditions are $n(x, t) = u(i, j) = 1.2 + 0.1\cos(2i)$ and $s(x, t) = v(i, j) = 1.6 - 0.2\cos(2i)$, $(x, t) \in [0, \pi] \times [-\tau, 0]$. Parameters: $a = 2.8$, $m = 1.4$, $\beta = 0.0$, $d_1 = 0.02$, $d_2 = 0.2$, $\tau = 2$.

As a result, the amplitude corresponding to the vegetation increases and the amplitude corresponding to soil water declines. Moreover, the amplitude corresponding to soil water increases with $m$ value becoming large for the same $\beta$ ($\beta \in [0, 0.25]$), and the amplitude of $m = 1.4$ is less than that of $m = 1.6$ and $m = 1.8$ for the vegetation, but the amplitude becomes as close as possible with the increase of $\beta$, especially almost the same for $\beta \geq 0.15$. Here, the parameter $m$ represents the ratio between the natural mortality coefficient $M$ of the vegetation and the evaporation coefficient of water $L$. One of the possible reasons for the increase of $m$ is that the evaporation of water is reduced, which makes the soil retain more water, then indirectly promotes the vegetation to absorb more water, that is to say, it accelerates the growth of the vegetation. Besides, the amplitude corresponding to the vegetation is much greater than that of soil water, which explains that limited water resources can support the growth of the vegetation in arid and semi-arid areas.

Fig. 12(b) presents a similar phenomenon to Fig. 12(a), this may be because the increase of diffusion rate of soil water accelerates more water flowing to low density areas, such water is quickly absorbed by the vegetation, resulting in a decrease in water density and an increase in vegetation. However, the specifics are quite different: with the increase of $d_2$, the increase rate of the amplitude corresponding to the vegetation becomes slower, and the reduction rate of the amplitude corresponding to soil water is faster in Fig. 12(b). Moreover, we observe that three curves about the vegetation was very close for $m = 1.4$, 1.6 and 1.8 when $d_2 \leq 0.02$, then the distance among the three curves is gradually enlarged with the increase of $d_2$. On the contrary, the three curves for soil water gets closer and closer, which likely
caused by the variety in the coupling effect between diffusion and evaporation of soil water. Furthermore, it is obvious from the illustration that the amplitude value corresponding to soil water at $m = 1.8$ is greater than that of the vegetation when $d_2 \in (0.05, 0.08)$.

Next, we investigate the variation of the amplitude with the parameter $m$ under different $\beta$. Fig. 12(b) shows that the amplitude corresponding to soil water gradually increases with the increase of $m$, the amplitude corresponding to vegetation first increases and then decreases slowly for $\beta = 0.1$ and $0.2$, but it always increases for $\beta = 0.0$. The possible reasons is that the reduction of evaporation and the increase of rainfall make soil retain more water, which increases the density of soil water. At the same time, such situation also promotes the vegetation to absorb more water, then accelerates the growth of the vegetation. However, when the diffusion feedback of soil water is considered, the ability of vegetation to uptake water becomes stronger, which forms opposite effect with the reduction of evaporation for the vegetation and soil water. Then the amplitude corresponding to soil water decreases with the increase of $\beta$ for the same evaporation, but the difference between the three curves is not obvious. Besides, when $m$ increased to a certain extent, instead the growth of the vegetation is inhibited.

5. Conclusion and discussion. In view of the fact that the evolution of vegetation with time presents periodic oscillation pattern, but the relevant research on driving mechanism is relatively limited. Furthermore, the conversion of soil water uptake by roots of the vegetation into their own biomass takes a period of time to complete, then conversion time delay is introduced to the basic Klausmeier model, and we don’t consider the flow of soil water along the slope. To study the generated mechanism of such phenomenon, we need to find a new theoretical method which is different from pattern-formation theory. Through theoretical analyses and numerical simulations, we find that the conversion delay induces non-spatial system (4) to generate stability switches at the vegetation-existence equilibrium $E^*$ and saddle-
node-Hopf bifurcation at the coincidence equilibrium $E_{**}$, and also makes spatial system (4) produce a family of spatially homogeneous Hopf bifurcation, which indicates that the conversion delay induces the interaction of the vegetation and soil water in the pattern of periodic oscillation. Besides, an interesting phenomenon, a family of spatially heterogeneous periodic solutions, is shown by numerical results, that is, the corresponding solution is periodic fluctuation in spatial region and steady in time.

On the basis of previous analysis, the emergence of Hopf bifurcation in two systems depends on the variation of the conversion delay $\tau$. By comparing theoretical and numerical results of system (4) without and with diffusion, the obtained results demonstrate that the increase of spatial diffusion factors ($\beta$, $d_2$) can reduce the critical value of conversion delay, but it has no effect on the period and amplitude of periodic oscillation. Furthermore, the increases of other parameters ($a$, $m$) result in decrease of the critical value but the contrary situation for the amplitudes of two systems. Additionally, absolutely stable, conditionally stable and non-existent regions for $E^*$ and $E_{**}$ are shown in $a - m$ parameters plane for two systems, and stability switches emerge in two systems, we further find that spatial diffusion reduces the emergence range of stability switches. The numerical results show that the sign of the expression $Q_2(k)$ is negative with the change of wave number $k$ under different parameters, which explains that the increases of these parameters are beneficial to the occurrence of the spatially homogeneous Hopf bifurcation. Besides, spatial diffusion factors and other parameters show a complex effects on the amplitudes of the spatially heterogeneous periodic solutions.

Currently, the researches on vegetation system focus on the analysis of various vegetation pattern structures produced in ecosystem by using pattern-formation theory, which is a nonuniform stationary instability. Different from this aspect, we study the instability characteristics of vegetation system from a new perspective, that is, the behavior of periodic solution of the vegetation system. The spatiotemporal evolution of the vegetation shows spatial homogeneity and time periodic oscillation behaviors, which largely depends on the conversion delay. Meanwhile, other parameters affect the amplitude of periodic oscillation and the critical value of conversion delay. In addition, the numerical simulations show novel results that system (4) gives rise to a family spatially heterogeneous periodic solution, and changes of vegetation biomass and the density of soil water in spatial region present anti-phase state, which is consistent with the phenomenon observed in Nambia [7]. In the following work, we expect to give the generated mechanism of such phenomenon from the perspective of dynamics.

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