Frobenius Numbers and Automatic Sequences

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Abstract

The Frobenius number $g(S)$ of a set $S$ of non-negative integers with gcd 1 is the largest integer not expressible as a linear combination of elements of $S$. Given a sequence $s = (s_i)_{i \geq 0}$, we can define the associated sequence $G_s(i) = g(\{s_i, s_{i+1}, \ldots\})$. In this paper we compute $G_s(i)$ for some classical automatic sequences: the evil numbers, the odious numbers, and the lower and upper Wythoff sequences. In contrast with the usual methods, our proofs are based largely on automata theory and logic.

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1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the natural numbers. Let $S$ be a nonempty set of natural numbers with gcd $1$, possibly infinite. A classical result then says that every sufficiently large integer can be written as a linear combination of elements of $S$ with natural number coefficients. Then $g(S)$, the Frobenius number of $S$, is defined to be the greatest integer $t$ such that $t$ does not have such a representation. For example, $g(\{6, 9, 20\}) = 43$.

The Frobenius number has received a lot of attention in recent years. For a detailed discussion of the function $g$, see the books of Ramírez Alfonsín [15] and Rosales and García-Sánchez [17].

Let $s = (s_i)_{i \geq 0}$ be an increasing sequence of natural numbers such that gcd$(s_i, s_{i+1}, \ldots) = 1$ for all $i \geq 0$. For $i \geq 0$ define $G_s(i) = g(\{s_i, s_{i+1}, \ldots\})$, the Frobenius number of a final segment of $s$, beginning with $s_i$. Given a sequence $s$, it can be an interesting and challenging

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problem to compute the Frobenius numbers $G_s(i)$ exactly, or estimate their growth rate. Computing the Frobenius number is notoriously difficult because the problem is NP-hard [14].

For example, for the case where $s$ is 1, 4, 9, 16, ..., the sequence of squares, Dutch and Rickett [7] proved that $G_s(n) = o(n^{2+\epsilon})$ for all $\epsilon > 0$, and this bound was improved to $O(n^{2})$ by Moscariello [12].

In this paper we are interested in calculating $G_s(i)$ for some famous integer sequences $s$ whose characteristic sequence is automatic. (The characteristic sequence is the binary sequence $b = (b_i)_{i \geq 0}$ where $b_i = 1$ if $i$ occurs in $s$ and $b_i = 0$ otherwise.) By “automatic” we mean generated by a finite automaton in a certain way, described in the next section. In particular, we completely characterize the Frobenius number for the sequence of evil numbers, the sequence of odious numbers, and the lower and upper Wythoff sequences. Although our results are number-theoretic in nature, our approach is largely via automata theory and logic.

2 Automatic and synchronized sequences

A numeration system is a method for writing a non-negative integer $N$ as a linear combination $N = \sum_{0 \leq i \leq t} a_i d_i$ of some increasing sequence $d_0 = 1 < d_1 < d_2 < \cdots$ of integers, where the $a_i$ are chosen from a finite set that includes 0. Suppose

(a) the representation for $N$ is always unique (up to leading zeros);

(b) the set of all valid representations forms a regular language;

(c) there is a finite automaton recognizing the triples $(x, y, z)$ for which $x + y = z$, where the inputs to the automaton are representations of integer triples, padded with leading zeros, if necessary, to make them all of the same length.

If all three conditions hold, then we call the numeration system regular. A sequence $a = (a_i)_{i \geq 0}$ over a finite alphabet is automatic if it is computed by an automaton taking the representation of $i$ as input and returning $a_i$ as the output associated with the last state reached. For more about automatic sequences, see [3].

If a sequence is automatic, then there is a decision procedure for proving or disproving assertions about it, provided these assertions are phrased in first-order logic and using only the logical operations, comparisons of natural numbers, addition, and indexing into the sequence [6].

Automatic sequences are restricted in scope because they take their values in a finite alphabet. However, it’s possible for automata to compute sequences taking their values in $\mathbb{N}$, the natural numbers, using a different meaning of “compute”. We say a sequence $(a_i)_{i \geq 0}$ is synchronized if there is a finite automaton recognizing exactly the representations of pairs $(i, a_i)$, read in parallel. If a sequence is synchronized, then we can compute it in linear time. If the underlying numeration system is base $k$, then the quantities $\limsup_i a_i/i$ and $\liminf_i a_i/i$ are computable [18]. For more about synchronized sequences, see [5].
3 The odious and evil numbers

Let $s_2(n)$ denote the sum of bits of $n$ when represented in base 2. Define $t(n) = s_2(n) \mod 2$, the famous Thue-Morse sequence [2]. The evil numbers 0, 3, 5, 6, 9, ... are those $n$ with $t(n) = 0$, and the odious numbers 1, 2, 4, 7, ... are those $n$ with $t(n) = 1$. (The somewhat painful terminology is from [4, p. 431].) More precisely we have [1]

\[
\begin{align*}
e_n &= 2n + t(n) \\
o_n &= 2n + 1 - t(n)
\end{align*}
\]

for $n \geq 0$. Observe that both sequences obey the gcd condition, because $\gcd(e_i, e_{i+1}, e_{i+2}) = \gcd(o_i, o_{i+1}, o_{i+2}) = 1$ for all $i \geq 0$. Additive properties of these numbers were studied previously in [13, Thm. 2].

Our first goal is to prove the following result:

**Theorem 1.** The function $G_e(n)$ is 2-synchronized.

In order to prove this we need the following lemmas:

**Lemma 2.** If $n \geq 1$ can be written as the sum of four evil numbers $\geq m$, then $n$ can be written as the sum of one, two, or three evil numbers $\geq m$.

**Remark 3.** When we speak of such sums we never insist that the sum be of distinct integers.

**Proof.** This assertion can be phrased as a first-order formula, namely,

\[
\forall m, n \ \text{evil}_4(m, n) \implies \text{evil}_{1,2,3}(m, n),
\]

where

\[
\begin{align*}
\text{evil}_{1,2,3}(m, n) &:= \exists j, k, \ell \ t(j) = t(k) = t(\ell) = 0 \land j, k, \ell \geq m \land (n = j \lor n = j + k \lor n = j + k + \ell) \\
\text{evil}_4(m, n) &:= \exists i, j, k, \ell \ t(i) = t(j) = t(k) = t(\ell) = 0 \land i, j, k, \ell \geq m \land n = i + j + k + \ell.
\end{align*}
\]

To prove this, we use the theorem-proving software called **Walnut**, where we can simply translate the statement of the previous paragraph into **Walnut**’s syntax and evaluate it.

```walnut
def evil123rep "Ej,k,l \ (T[j]=@0) & (T[k]=@0) & (T[l]=@0) & j>=m & k>=m & l>=m & (n=j | n=j+k | n=j+k+l)";

def evil4rep "Ei,j,k,l \ (T[i]=@0) & (T[j]=@0) & (T[k]=@0) & (T[l]=@0) & i >= m & j>=m & k>=m & l>=m & (n=j+k+l+m)";

eval evilcheck "Am,n $evil4rep(m,n) \implies $evil123rep(m,n)";
```

This returns `true`, so the lemma is proved.

**Remark 4.** To help in understanding the syntax of **Walnut**, we note the following:
Lemma 5. Let \( n \geq 1 \) be a non-negative integer linear combination of evil numbers \( \geq m \). Then \( n \) can be written as the sum of either one, two, or three evil numbers \( \geq m \).

Proof. Let \( n \) be written as a non-negative integer linear combination of evil numbers \( \geq m \). Without loss of generality choose a representation for \( n \) that minimizes \( s \), the sum of the coefficients. If this sum is at least 4, we can write \( n = u + v \) where \( u \) is the sum of 4 evil numbers and \( v \) is the sum of \( n - 4 \) evil numbers, all \( \geq m \). But then by Lemma 2, we can write \( u \) as the sum of 3 evil numbers \( \geq m \), so \( n \) is the sum of \( n - 1 \) evil numbers \( \geq m \), a contradiction. So \( s \) is no more than 3, as desired. \( \square \)

We can now prove Theorem 1.

Proof. It suffices to give a first-order definition of \( G_e(n) \). We can do this as follows:

\[ \text{def evilg } " (Aj (j>n) => $evil123rep(2*m,j)) \& \neg$evil123rep(2*m,n)" : \]

This gives a 58-state synchronized automaton computing \( G_e(n) \). \( \square \)

Now that we have a synchronized automaton, we can determine the asymptotic behavior of \( G_e(n) \).

Theorem 6. We have \( 4m \leq G_e(m) \leq 6m + 7 \) for all \( m \geq 0 \). These bounds are optimal, because they are attained for infinitely many \( m \).

Proof. We run the following Walnut commands, which all evaluate to true.

\[ \text{eval upperb } "Am,n $evilg(m,n) => n <= 6*m+7" : } \]
\[ \text{eval upperopt } "Ai Em,n (m>i) \& $evilg(m,n) \& n=6*m+7" :} \]
\[ \text{eval lowerb } "Am,n $evilg(m,n) => n >= 4*m" : } \]
\[ \text{eval loweropt } "Ai Em,n (m>i) \& $evilg(m,n) \& n=4*m" : }\]
Corollary 7. We have

\[
\begin{align*}
\inf_{i \geq 1} G_e(i)/i &= 4 \\
\sup_{i \geq 1} G_e(i)/i &= 7 \\
\liminf_{i \geq 1} G_e(i)/i &= 4 \\
\limsup_{i \geq 1} G_e(i)/i &= 6.
\end{align*}
\]

The sequence \((G_e(i))_{i \geq 0}\) has a rather erratic behavior. In particular we can prove

Theorem 8.

(a) The difference \(G_e(i + 1) - G_e(i)\) can be arbitrarily large.

(b) There are arbitrarily long blocks of indices on which \(G_e(i)\) is constant.

Proof. We use the following Walnut code.

```
eval evilgdiff "Ai Ej,m,n1,n2 (j>=i) & $evilg(m,n1) & $evilg(m+1,n2) & n2=n1+j":
eval evalmonotone "Ai Ej,m,u (j>=i) & $evilg(m,u) &
  (At,v ((t>m) & (t<m+j) & $evilg(t,v)) => u=v)":
```

Both return true.

We can carry out exactly the same analysis for the odious numbers. The analogues of Theorem 1 and Lemma 2 and 5 all hold. Here are the results:

Theorem 9. We have \(4m \leq G_o(m) \leq 6m - 1\) for all \(m \geq 1\). These bounds are optimal, because they are attained for infinitely many \(m\).

Corollary 10. We have

\[
\begin{align*}
\inf_{i \geq 1} G_o(i)/i &= 4 \\
\sup_{i \geq 1} G_o(i)/i &= 6 \\
\liminf_{i \geq 1} G_o(i)/i &= 4 \\
\limsup_{i \geq 1} G_o(i)/i &= 6.
\end{align*}
\]

Furthermore, the analogue of Theorem 8 also holds.

The first few terms of the sequences we have discussed in this section, together with their numbers from the On-Line Encyclopedia of Integer Sequences (OEIS) [20], are given below.

| OEIS number | OEIS number | \(n\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------------|-------------|------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| A001969     | \(e_n\)     |      | 0 | 3 | 5 | 6 | 9 | 10|12 |15 |17 |18 |20 |23 |24 |27 |29 |30 |
| A342581     | \(G_e(n)\)  |      | 7 | 7 |13 |14 |16 |31 |31 |31 |32 |55 |55 |55 |55 |61 |62 |
| A000069     | \(o_n\)     |      | 1 | 2 | 4 | 7 | 8 |11 |13 |14 |16 |19 |21 |22 |25 |26 |28 |31 |
| A342579     | \(G_o(n)\)  |      |−1| 5 |10 |17 |23 |23 |24 |34 |39 |39 |45 |46 |71 |71 |71 |71 |
4 Results for the Wythoff sequences

We can carry out a similar analysis for the lower and upper Wythoff sequences, defined as follows. Here we find substantially different behavior than for the odious and evil numbers.

Let \( \varphi = (1 + \sqrt{5})/2 \), the golden ratio. Define

\[
L_n = \lfloor n\varphi \rfloor
\]
\[
U_n = \lfloor n\varphi^2 \rfloor
\]

for \( n \geq 0 \). Here, instead of base-2 representation, all numbers are represented in Fibonacci representation (also called Zeckendorf representation) \([11, 22]\). In this representation a number is represented as a linear combination \( \sum_{2 \leq i \leq t} a_i F_i \) with \( a_i \in \{0,1\} \) and subject to the condition that \( a_i a_{i+1} = 0 \). We then define \( (n)_F = a_t a_{t-1} \cdots a_2 \) and \( [x]_F = \sum_{1 \leq i \leq t} a_i F_{t+2-i} \) if \( x = a_1 \cdots a_t \).

The additive properties of the upper and lower Wythoff sequences were studied previously in \([10, 19]\).

**Theorem 11.** The functions \( L_n \) and \( U_n \) are Fibonacci synchronized.

**Proof.** We start by showing that the function \( n + 1 \) is Fibonacci synchronized. We can construct an automaton inc\((x,y)\) that computes the relation \( y = x + 1 \) for \( x \) and \( y \) in Fibonacci representation. Using the following easily-proven identities,

(a) \([x00(10)^i]_F + 1 = [x0102^i]_F\); and

(b) \([x0(01)^i]_F + 1 = [x0102^{i-1}]_F\),

we can obtain the incrementer depicted in Figure 4 below.

![Figure 1: Incrementer for Fibonacci representation](image)
Next, we use the identities

\[
[(n)_0F] = [(n + 1)\alpha] - 1
\]

\[
[(n)_1F] = [(n + 1)\alpha^2] - 1
\]

for \(n \geq 0\), whose proof can be found, for example, in [16]. Substituting \(n - 1\) for \(n\), this gives us formulas for \(\lfloor n\alpha \rfloor\) and \(\lfloor n\alpha^2 \rfloor\) in terms of shifting and incrementation in the Fibonacci representation. Shifting can be carried out using the Fibonacci synchronized DFA below:

![Diagram of Fibonacci synchronized DFA]

Figure 2: Shifter for Fibonacci representation

So we get synchronized automata for (a) and (b) as follows:

```python
def fibna "?msd_fib ((s=0)&(n=0)) | Et,u $fibinc(u,n) & $shift(u,t) & $fibinc(t,s)"
def fibna2 "?msd_fib ((s=0)&(n=0)) | Et,u,v,w $fibinc(u,n) & $shift(u,t) & $shift(t,v) & $fibinc(v,w) & $fibinc(w,s)"
```

with the automata depicted below in Figure 3:

Our next goal is to prove that \(G_L(n)\) is Fibonacci-synchronized (that is, there is an automaton recognizing the pairs \((n, G_L(n))\) represented in the Fibonacci numeration system). We start with some lemmas.

**Lemma 12.** Let \(n \geq 1\) be a non-negative integer linear combination of lower Wythoff numbers \(\geq m\). Then \(n\) can be written as the sum of either one or two lower Wythoff numbers \(\geq m\).

**Proof.** It can be carried out with Walnut in analogy with the proof of Lemma 5.

```python
def lower12rep "?msd_fib Ej,k (F[j-1]=@0) & (F[k-1]=@0) & j>=m & k>=m & (n=j|n=j+k)"
```
We can now prove

**Theorem 13.** The function $G_L(n)$ is Fibonacci synchronized.

**Proof.** We use the Walnut commands:

def lowerunrep "?msd_fib (Aj (j>n) => $lower12rep(m,j)) & ~$lower12rep(m,n)":
def lowerg "?msd_fib Et $fibna(m,t) & $lowerunrep(t,n)":

This gives us a synchronized automaton with 24 states computing $G_L(n)$.

We can now use this automaton to determine the behavior of $G_L(n)$.

---

Figure 3: Synchronized automata for $[n\alpha]$ (top) and $[n\alpha^2]$ (bottom).

```
**Theorem 14.** We have \(-3 \leq G_L(n) - 2L_n \leq 1\), and the upper and lower bounds are achieved infinitely often.

**Proof.** We use the following Walnut commands:

```walg
eval lowerb1 "?msd_fib Am En,r $lowerg(m,n) & $fibna(m,r) & n+3=2*r":
eval lowerb2 "?msd_fib Am En,r $lowerg(m,n) & $fibna(m,r) & n=2*r+1":
eval lowerbinf1 "?msd_fib As Em,n,r (m>=s) & $lowerg(m,n) & $fibna(m,r) & n+3=2*r":
eval lowerbinf2 "?msd_fib As Em,n,r (m>=s) & $lowerg(m,n) & $fibna(m,r) & n<=2*r+1":
```

**Corollary 15.** We have \(\lim_{n \to \infty} G_L(n)/n = 1 + \sqrt{5}\).

**Theorem 16.** We have \(G_L(n+1) - G_L(n) \in \{0, 2, 3, 5, 6, 8\}\), and furthermore each difference occurs infinitely often.

**Proof.** We use the following Walnut commands:

```walg
eval lowerdiff "?msd_fib Am Eu,v $lowerg(m,u) & $lowerg(m+1,v) & (v=u+v=2|v=u+3|v=u+5|v=u+6|v*u+8)":
def ldi "?msd_fib Am Et,u,v (t>=m) & $lowerg(t,u) & $lowerg(t+1,v) & v=u+d":
eval lowerdiffinfcheck "?msd_fib $ldi(0) & $ldi(2) & $ldi(3) & $ldi(5) & $ldi(6) & $ldi(8)":
```

**Theorem 17.** There exists a Fibonacci automaton of 11 states computing the first difference \(G_L(n+1) - G_L(n)\).

The automaton is depicted in Figure 4 below.

![Fibonacci automaton](image.png)

**Figure 4:** Fibonacci automaton computing \(G_L(n+1) - G_L(n)\)

Now we turn to the upper Wythoff sequence. The results are completely analogous to the results for the lower Wythoff sequence, and the proofs are also analogous. We omit the details.
Lemma 18. Let \( n \geq 1 \) be a non-negative integer linear combination of upper Wythoff numbers \( \geq m \), for \( m \geq 3 \). Then \( n \) can be written as the sum of either one, two, or three Wythoff numbers \( \geq m \).

Remark 19. Lemma 18 fails for \( m = 2 \) because 8 is not the sum of one, two, or three Wythoff numbers \( \geq 2 \), while it is the sum of four (since \( 8 = 2 + 2 + 2 + 2 \)).

Theorem 20. The function \( G_U(n) \) is Fibonacci synchronized.

Theorem 21. We have \(-5 \leq G_U(n) - 3U_n \leq 20\), and these upper and lower bounds are achieved infinitely often.

Corollary 22. We have \( \lim_{n \to \infty} G_U(n)/n = (9 + 3\sqrt{5})/2 \).

Theorem 23. We have \( G_U(n + 1) - G_U(n) \in \{0, 3, 5, 8, 11, 13, 18, 21, 23, 26, 31\} \) for \( n \geq 1 \), and furthermore each difference occurs infinitely often.

Theorem 24. There exists a Fibonacci automaton computing the first difference \( G_U(n + 1) - G_U(n) \).

The first few terms of the sequences we have discussed in this section, together with their numbers from the OEIS, are given below.

| OEIS number | \( n \)       | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|-------------|---------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| A000201     | \( L_n \)     | 0  | 1  | 3  | 4  | 6  | 8  | 9  | 11 | 12 | 14 | 16 | 17 | 19 | 21 | 22 |
| A342715     | \( G_L(n) \)  | -1 | -1 | 5  | 7  | 13 | 15 | 15 | 20 | 23 | 26 | 31 | 31 | 39 | 41 | 41 |
| A001950     | \( U_n \)     | 0  | 2  | 5  | 7  | 10 | 13 | 15 | 18 | 20 | 23 | 26 | 28 | 31 | 34 | 36 |
| A342716     | \( G_U(n) \)  | -1 | 3  | 16 | 19 | 42 | 42 | 42 | 55 | 58 | 76 | 79 | 79 | 110| 110| 110|

5 A counterexample

In all of the examples we have seen so far, if a sequence had automatic characteristic sequence, then the characteristic sequence of the associated Frobenius sequence was also automatic. It is natural to conjecture this might always be the case. However, we now prove

Theorem 25. Let \( s_i = 2^i + 1 \) for \( i \geq 0 \) and \( s = (s_i)_{i \geq 0} \). Then \( G_s(i) = 2^{2i} + 2^i + 1 \) for \( i \geq 1 \).

Proof. It suffices to prove that \( 2^{2i} + 2^i + 1 \) cannot be written as a non-negative integer linear combination of \( 2^i + 1, 2^{i+1} + 1, \ldots 2^{2i} + 1 \), while every larger integer can be so expressed.

Suppose \( 2^{2i} + 2^i + 1 = a_0(2^i + 1) + a_1(2^{i+1} + 1) + \cdots + a_i(2^{2i} + 1) \) with \( a_0, \ldots, a_i \) non-negative integers. Considering both sides modulo \( 2^i \), we see the left-hand side is 1, while the right-hand side is \( a_0 + a_1 + \cdots + a_i \). So either \( a_0 + a_1 + \cdots + a_i = 1 \) or \( a_0 + a_1 + \cdots + a_i \geq 2^i + 1 \).
In the former case we would have \(2^{2i} + 2^{i} + 1 = 2^{j} + 1\) for some \(j, i \leq j \leq 2i\), which is clearly impossible. In the latter case we would have
\[
2^{2i} + 2^{i} + 1 = a_0(2^{i} + 1) + a_1(2^{i+1} + 1) + \cdots + a_i(2^{2i} + 1) \geq (a_0 + a_1 + \cdots + a_i)(2^{i} + 1) \geq (2^{i} + 1)^2,
\]
which is also impossible. This shows \(2^{2i} + 2^{i} + 1\) is not representable.

We now argue that if \(2^{2i} + 2^{i} + 1 < x \leq 2^{2i} + 2^{i+1} + 2\), then \(x\) has a representation. This suffices to show that all \(x > 2^{2i} + 2^{i+1} + 2\) can then be represented by adding the appropriate multiple of \(2^{i} + 1\).

Given a particular linear combination
\[
x = a_0(2^{i} + 1) + a_1(2^{i+1} + 1) + \cdots + a_i(2^{2i} + 1),
\]
call its weight \(a_0 + \cdots + a_i\). We now repeat the following transformation: given a linear combination \(x = a_0(2^{i} + 1) + \cdots + a_i(2^{2i} + 1)\), find the largest nonzero \(a_j\) in the combination. Then form the linear combination of \(x + 1\) by adding \(2 \cdot (2^{j-1} + 1) - (2^{j} + 1) = 1\) to the representation for \(x\). Doing so increases the weight of the linear combination by 1, because we add 2 to one coefficient and subtract 1 from another.

Now let us start the process with the number \((2^{2i} + 1) + (2^{i} + 1)\), which has a representation of weight 2. When we carry out the transformation of the previous paragraph once, the 1 coefficient of \(2^{2i} + 1\) in the linear combination disappears and a 2 appears as the coefficient of \(2^{2i-1} + 1\). Doing it twice more causes this 2 to disappear, and a 4 appears as the coefficient of \(2^{2i-2} + 1\). This process continues for a total of \(1 + 2 + 4 + \cdots + 2^{i-1} = 2^i - 1\) times, eventually resulting in the representation \(2^{2i} + 2^{i} + 2^{i} - 1 = 2^{2i} + 2^{i+1} + 1\) as \((2^{i} + 1)(2^{i} + 1)\) of weight \(2^{i} + 1\). Finally, \(2^{2i} + 2^{i+1} + 2\) has the representation \((2^{2i} + 1) + (2^{i+1} + 1)\). This gives us \(2^{i} + 1\) consecutive representable numbers, as desired, and completes the proof.

We have now shown that \(G_s(i) = 2^{2i} + 2^{i} + 1\) for \(i \geq 1\). Hence we get our desired counterexample: the characteristic sequence of \((2^{i} + 1)_{i \geq 0}\) is automatic, as the set of its base-2 representations is specified by the regular expression \(10^*1\). But the characteristic sequence of \((G_s(i))_{i \geq 1} = (2^{2i} + 2^{i} + 1)_{i \geq 1}\) is not automatic, as the set of its base-2 representations is of the form \(\{10^*10^i1 : i \geq 0\}\), which can easily be seen to be non-regular using a standard tool from formal language theory called the pumping lemma [9, Lemma 3.1].

Remark 26. Theorem 25, in more generality, also appears in a recent paper of Song [21]. However, the proof was omitted there, so we give it here.

6 Concluding remarks

We conjecture that the analogue of Corollary 15 holds for all Beatty sequences.

For other results of additive number theory based on automata theory, see [13].

All the Walnut code we used is available from the author’s website, https://cs.uwaterloo.ca/~shallit/papers.html.
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