A UNIQUENESS RESULT FOR 2-SOLITON SOLUTIONS OF THE KORTEWEG-DE VRIES EQUATION

JOHN P. ALBERT

Department of Mathematics, University of Oklahoma
Norman OK 73019, USA

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Abstract. Multisoliton solutions of the KdV equation satisfy nonlinear ordinary differential equations which are known as stationary equations for the KdV hierarchy, or sometimes as Lax-Novikov equations. An interesting feature of these equations, known since the 1970’s, is that they can be explicitly integrated, by virtue of being finite-dimensional completely integrable Hamiltonian systems. Here we use the integration theory to investigate the question of whether the multisoliton solutions are the only nonsingular solutions of these ordinary differential equations which vanish at infinity. In particular we prove that this is indeed the case for 2-soliton solutions of the fourth-order stationary equation.

1. Introduction. The Korteweg-de Vries (or KdV) equation,

\[ u_t = \frac{1}{4}(u_{xxx} + 6uu_x), \]

was first derived in the 1800’s as a model for long, weakly nonlinear one-dimensional water waves ([24], see also equation (283 bis) on p. 360 of [4]). It was not until the 1960’s, however, that the striking discovery was made that the equation has particle-like solutions known as solitons, whose interactions with each other are described by explicit multisoliton solutions [16, 19].

It is well-known that the profiles of multisoliton solutions, which are smooth functions that vanish rapidly at infinity, are critical points for variational problems associated with conserved functionals of KdV (see, e.g., [27]). By virtue of this property, the profiles are solutions of Lagrange multiplier equations, which take the form of nonlinear ordinary differential equations, sometimes known as Lax-Novikov equations, or as the equations for stationary solutions of a family of time-dependent equations known as the KdV hierarchy (see below for details). In this paper we investigate the problem of establishing a converse to this statement: is it true that if a solution to a stationary equation for the KdV hierarchy is, together with enough of its derivatives, square integrable on the real line, then must it be a profile of a multisoliton solution?

For the case of the KdV equation itself (the first equation in the hierarchy), it is an elementary exercise to prove that the only stationary solutions in \( L^2 \) are the well-known solitary-wave profiles. Here we give a proof that the answer is also

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affirmative for the case of the fourth-order stationary equation for the second, fifth-order, equation in the KdV hierarchy (see Theorem 5.2 below). Much of our proof easily generalizes to the other stationary equations for the hierarchy, but some work remains to be done to complete the proof in the general case.

Our proof proceeds by integrating the stationary equations, using the method developed in the pioneering work of Dubrovin [11], Its and Matveev [21], Lax [25], and Novikov [32] on solutions of the periodic KdV equation. An early survey of the work of these authors is [12], and more recent treatments are [2] and [18]. For a lively historical account of the development of the subject, we refer the reader to [29], in which it is noted that elements of the theory, including in particular equation (2.13), can be traced back at least as far the work of Drach [10] in 1919. Here we follow the approach of Gel’fand and Dickey, which first appeared in [17], and has received a nice expository treatment in Chapter 12 of Dickey’s book [8]. In this approach, the stationary equations, which have the structure of completely integrable Hamiltonian systems, are rewritten in action-angle variables, which reduces them to an easily integrable set of equations (see (5.22) below) first obtained by Dubrovin in [11]. (We remark that each stationary equation is a finite-dimensional completely integrable Hamiltonian system in the classical sense; unlike the time-dependent KdV equations which are in some sense [14, 15] infinite-dimensional completely integrable Hamiltonian systems.) Integrating Dubrovin’s equations shows that every smooth solution of the stationary equations must be expressible in the form given below in (3.22), which is known as the Its-Matveev formula [21]. It turns out that this part of the proof is valid for all stationary equations for the KdV hierarchy. We then conclude by determining which solutions of the Its-Matveev formula are nonsingular. The latter step we have so far only completed for the second stationary equation in the hierarchy: that is, for the equation for 2-solitons.

We emphasize that our interest here is not in constructing solutions of the stationary equations; all the solutions appearing in this paper are already well-known (see, for example, [30]). Rather, our focus is on showing that a corollary of the method used to integrate these equations is that the \( N \)-solution profiles are the only solutions with finite energy, at least in the case \( N = 2 \). Also, we have made an effort to give a self-contained presentation, which in fact relies entirely on elementary calculations.

The result we prove here has consequences for the variational properties of KdV multisolitons. As noted above, the fact that KdV 2-soliton profiles are solutions of a certain stationary equation in the KdV hierarchy (namely, equation (5.1) below) can be interpreted as saying that they are critical points for a certain variational problem, in which one conserved functional of the KdV equation is minimized subject to the constraint that two other functionals be held equal to given constants. As part of their proof of the stability of \( N \)-soliton solutions of KdV for general \( N \), Maddocks and Sachs [27] showed that 2-soliton profiles are actually local minimizers for this variational problem. However, the question of whether 2-soliton profiles (or more generally, \( N \)-soliton profiles) are global minimizers for their respective variational problems remained open.

As we show in an upcoming paper, 2-soliton profiles are indeed global minimizers for their associated variational problems, and in fact more is true: every minimizing sequence for the variational problem converges strongly in \( H^2 \) to the minimizing set, which consists solely of 2-soliton profiles with the same pair of amplitudes. In particular, it follows that this minimizing set is stable, thus giving another proof
of the stability result in [27]. The uniqueness result in the present paper enters into the proof of convergence of minimizing sequences in an essential way: one first performs a profile decomposition (cf. [28]) on the minimizing sequence; next, one shows that the component profiles also converge to minimizers, with potentially different values of the constraint functional than those specified in the original problem; and then applies the uniqueness result to conclude that each component profile must converge to a 1-soliton or 2-soliton profile. This implies that the original minimizing sequence has a subsequence converging to a set of multisoliton profiles, and one concludes by showing that a multisoliton profile satisfying the original constraints cannot be a minimizer unless it is a 2-soliton profile. Note that in this argument we have no control over the component profiles, and cannot for example assume that they are close to being 2-soliton profiles. Instead, what is required is a uniqueness result which makes no assumption on the Lagrange multipliers which appear in the equation.

In studying the properties of KdV solitons, an alternative approach to the alge- braic methods used here is the formalism of the inverse scattering theory, discovered by Gardner, Greene, Kruskal, and Miura in [15] and subsequent papers (see [31] for an overview of the two approaches). One might fairly ask whether inverse scattering methods could be used to derive our uniqueness result. In particular, the “trace identities” of Zakharov and Faddeev [35], which express the conserved func- tionals of KdV in terms of scattering data, seem to be relevant. An immediate technical obstacle to applying inverse scattering theory, however, is that it applies only to functions whose decay at infinity is faster than that of a typical $H^2$ func- tion (Deift and Trubowitz [6], for example, require their potentials $q(x)$ to satisfy $\int_{-\infty}^{\infty} (1 + x^2) |q(x)| \, dx < \infty$), and as noted above, for the application we have in mind, a uniqueness theory in $H^2$ is needed. A perhaps more fundamental obsta- cle to the use of inverse scattering theory to investigate the variational properties of multisolitons is that, in terms of scattering data, the problem becomes one of translating certain relations between the scattering data of two functions into information about the distance between the functions in $H^2$, and it is far from obvious how to do this.

We also remark that an alternate method that suggests itself to prove the unique- ness result is to simply count the dimensions of the stable and unstable manifolds of \eqref{5.1} at the origin in phase space, and use the fact that the set of 2-soliton profiles for a given choice of amplitudes has the same dimensions. Indeed, if one assumes in advance that $d_3$ and $d_5$ are such that equation \eqref{5.5} has distinct positive roots, then this method does give, after some work, that the only homoclinic solutions of \eqref{5.1} are 2-soliton profiles. Such an argument, however, becomes much more complicated for other choices of $d_i$, partly because center manifolds of dimension up to 4 can appear. Since the application envisaged above requires a proof valid for all possible choices of $d_i$, we have found it better not to proceed by this method, but rather to directly integrate \eqref{5.5} instead.

The plan of the paper is as follows. In Sections 2 and 3, for the reader’s conve- nience and to set notation, we review some of the basic properties of multisoliton solutions. In Section 2 we introduce the equations of the KdV hierarchy, and their associated stationary equations. (Here “stationary” means “time-independent”: stationary equations are equations for time-independent solutions of the KdV hierarchy. Coincidentally, they are also equations for stationary points of variational problems.) In Section 3 we define the $N$-soliton solutions of the KdV hierarchy, and
give a proof of the well-known fact that their profiles are actually solutions of stationary equations. Section 4 prepares for the main result by treating the elementary case of stationary solutions of the KdV equation itself. In Section 5 we prove the main result, which is that for the stationary equation for the fifth-order equation in the KdV hierarchy, the only \(H^2\) solutions are 1-soliton and 2-soliton profiles. A concluding section discusses the question of how to generalize the result to higher equations in the hierarchy and \(N\)-solitons for \(N > 2\).

2. The KdV hierarchy. We review here the definition of the KdV hierarchy, following the treatment of chapter 1 of [8].

Let \(A\) denote the differential algebra over \(C\) of formal polynomials in \(u\) and the derivatives of \(u\). That is, elements of \(A\) are polynomials with complex coefficients in the symbols \(u, u', u'', \ldots\); and elements of \(A\) can be acted on by a derivation \(\xi\), a linear operator on \(A\) which obeys the Leibniz product rule, and takes \(u\) to \(u'\), \(u''\), etc. We adopt the convention that primes also denote the action of \(\xi\) on any element of \(A\). Thus the expressions \(a'\) and \(\xi a\) are synonymous for \(a \in A\). Later we will substitute actual functions of \(x\) for \(u\), and then \(\xi\) will correspond to the operation of differentiation with respect to \(x\), so that \(u', u'', \ldots\), will denote the derivatives of these functions with respect to \(x\) in the usual sense.

If \(M\) is an integer, we define a pseudo-differential operator of order \(M\) to be a formal sum

\[
X = \sum_{i=-\infty}^{M} a_i \partial^i,
\]

where \(a_i \in A\) for each \(i\). Clearly the set \(P\) of all pseudo-differential operators has a natural module structure over the ring \(A\). We can also make \(P\) into an algebra by first defining, for each integer \(k\) and each \(a \in A\), the product \(\partial^k a\) as

\[
\partial^k a = a \partial^k + \binom{k}{1} a' \partial^{k-1} + \binom{k}{2} a'' \partial^{k-2} + \ldots,
\]

where \(\binom{k}{i} = \frac{k(k-1) \cdots (k-i+1)}{i!}\); and then extending this multiplication operation to all of \(P\) in the natural way:

\[
\left( \sum_{i=-\infty}^{M} a_i \partial^i \right) \left( \sum_{j=-\infty}^{N} b_j \partial^j \right) = \sum_{i=-\infty}^{M} \sum_{j=-\infty}^{N} a_i (\partial^i b_j) \partial^j
= \sum_{i=-\infty}^{M} \sum_{j=-\infty}^{N} \sum_{l=0}^{\infty} a_i \binom{i}{l} (\xi^l b_j) \partial^{i+j-l}.
\]

The last sum in the preceding equation is well-defined in \(P\) because each value of \(i + j - l\) occurs for only finitely many values of the indices \(i, j, l\). It can be checked that, with this definition of multiplication, \(P\) is an associative algebra with derivation \(\partial\). Interestingly, this algebra \(P\) was studied by Schur in [34], many years before its utility for the theory of integrable systems was discovered.

In particular we will have \(\partial \partial^{-1} = 1\) in \(P\). More generally, suppose \(X\) is given by (2.1) with \(a_M = 1\). Then \(X\) has a multiplicative inverse \(X^{-1}\) in \(P\); this may be verified by first observing that the order of \(X^{-1}\) must be \(-M\) and then using the equation \(XX^{-1} = 1\) to solve recursively for the coefficients \(b_i\) of \(X^{-1} = \sum_{i=-\infty}^{-M} b_i \partial^i\). Carrying out this process, one finds that the \(b_i\) are polynomials in the \(a_i\) and their
Observe that, in general, the commutator of an operator of order \( m \) is easily obtained by computing the term of order 0 in the expansion of \( Y = \sum_{i=-\infty}^{1} c_i \partial^i \). These coefficients will be uniquely determined if we specify that \( c_1 = 1 \), and in that case the operator \( Y \) so obtained will be denoted by \( X^1/m \). (All other solutions of \( Y^m = 1 \) are of the form \( \alpha X^1/m \), where \( \alpha \) is an \( m \)th root of unity.) For \( k \in \mathbb{Z} \) we then define \( X^{k/m} \) to be the \( k \)th power of \( X^1/m \). Since \( X \) is an integer power of \( X^1/m \) it follows immediately that \( X \) and \( X^{1/m} \) commute, and hence so do \( X \) and \( X^{k/m} \).

If in (2.1) we have \( a_i = 0 \) for all \( i < 0 \), then we say that \( X \) is a differential operator; obviously the product and sum of any two differential operators is again a differential operator. For general \( X \in \mathcal{P} \), the differential part of \( X \), denoted by \( X_+ \), is defined to be the differential operator obtained by omitting all the terms from \( X \) which contain \( \partial^i \) with negative \( i \). We also define \( X_- \) to be \( X - X_+ \). As usual, we define the commutator \( [X_1, X_2] \) of two elements of \( \mathcal{P} \) by \( [X_1, X_2] = X_1 X_2 - X_2 X_1 \).

Also, if \( X \) is given by (2.1), it will be useful to define the residue of \( X \), \( \text{Res} X \), to equal \( a_{-1} \). That is, \( \text{Res} X \in \mathcal{A} \) is the coefficient of \( \partial^{-1} \) in the expansion of \( X \). Finally, for \( X \) as in (2.1) we define \( \sigma_2(X) \in \mathcal{P} \) by

\[
\sigma_2(X) = a_{-1} \partial^{-1} + a_{-2} \partial^{-2}.
\]

The Korteweg-de Vries hierarchy can be defined in terms of fractional powers of the differential operator \( L \) given by

\[
L = \partial^2 + u.
\]

From the above considerations, \( L^{(2k+1)/2} \) is well-defined as an element of \( \mathcal{P} \) for each nonnegative integer \( k \). When we take its differential part, we obtain the operator \( (L^{(2k+1)/2})_+ \), which has the following important property.

**Lemma 2.1.** The commutator \( [(L^{(2k+1)/2})_+, L] \) is a differential operator of order 0; that is, a polynomial in \( u \) and its derivatives. In fact, it is given by the equation

\[
\left[ \left( L^{(2k+1)/2} \right)_+, L \right] = 2 \left( \text{Res} L^{(2k+1)/2} \right)'.
\]

**Proof.** As the commutator of two differential operators, \( [(L^{(2k+1)/2})_+, L] \) is a differential operator. Now

\[
\left[ \left( L^{(2k+1)/2} \right)_+, L \right] = \left[ L^{(2k+1)/2} - \left( L^{(2k+1)/2} \right)_- \right],
\]

and, as noted above, \( L^{(2k+1)/2} \) commutes with \( L \), so

\[
\left[ \left( L^{(2k+1)/2} \right)_+, L \right] = -\left[ \left( L^{(2k+1)/2} \right)_-, L \right].
\]

Observe that, in general, the commutator of an operator of order \( M_1 \) and an operator of order \( M_2 \) has order \( M_1 + M_2 - 1 \). Since the right hand side of (2.5) is a commutator of an operator of order \( -1 \) and an operator of order 2, it therefore has order 0.

Once it is established that both sides of (2.5) are equal to a differential operator of order 0, the identity (2.4) is easily obtained by computing the term of order 0 in the expansion of \( -\left[ \left( L^{(2k+1)/2} \right)_-, L \right] \).

The Korteweg-de Vries hierarchy is a set of partial differential equations, indexed by the natural numbers \( k = 0, 1, 2, 3, \ldots \), for functions \( u(x, t_{2k+1}) \) of two real derivatives.
variables $x$ and $t_{2k+1}$. The $k$th equation in the hierarchy is defined as

$$u_{t_{2k+1}} = 2 \left( \text{Res} \left( L^{(2k+1)/2} \right) \right)' . \quad (2.6)$$

Here the subscripted $t_{2k+1}$ denotes the derivative with respect to $t_{2k+1}$. Starting with $k = 0$, the first three in the hierarchy are given by:

$$\begin{align*}
u_t &= u', \\
u_t &= \frac{1}{4} (u'' + 6u'), \\
u_t &= \frac{1}{16} (u'''' + 10uu'' + 20uu' + 30u^2 u').
\end{align*} \quad (2.7)$$

The second equation in (2.7) is the KdV equation (1.1).

This definition of the hierarchy is due to Gelfand and Dickey, and leads to simple formulations and proofs of many properties of these equations, including the fact that they define commuting flows, which were formerly proved by more unwieldy methods. Also, natural modifications of the definition lead readily to more general hierarchies of equations (today called Gelfand-Dickey hierarchies), of which the Korteweg-de Vries hierarchy is just one, and which share many of the interesting integrability properties of the Korteweg-de Vries hierarchy [8, 9].

An important feature of the KdV hierarchy (2.6) is that the differential polynomials which appear on the right-hand side satisfy a simple recurrence relation. Following the notation of Chapter 12 of [8], let us define, for $k = 0, 1, 2, \ldots$,

$$R_{2k+1} = \frac{(-1)^k}{2} \text{Res} L^{(2k-1)/2},$$

so that the KdV hierarchy takes the form

$$u_{t_{2k+1}} = 4(-1)^{k+1} R'_{2k+3}. \quad (2.8)$$

Lemma 2.2. The differential polynomials $R_{2k+1}$ satisfy the recurrence relation

$$R''_{2k+1} + 4u R'_{2k+1} + 2u'R_{2k+1} = -4R'_{2k+3}, \quad (2.9)$$

for $k = 0, 1, 2, \ldots$, with initial condition $R_1 = 1/2$.

Proof. The proof of this lemma is essentially an exercise on the material in Section 1.7 of [8], but for the reader’s convenience we indicate the details here.

Let $\mathcal{C}$ denote the set of all formal Laurent series in $z$ of the form $\sum_{r = -\infty}^{\infty} X_r z^r$, where $X_r \in \mathcal{P}$ for $r \in \mathbb{Z}$. Then $\mathcal{C}$ inherits an operation of addition from $\mathcal{P}$, and if $S$ and $T$ are in $\mathcal{C}$ and all but finitely many of the coefficients in $T$ are zero, then the products $ST$ and $TS$ are defined in $\mathcal{C}$ by the usual term-by-term multiplication of series. Also, for $S = \sum_{r = -\infty}^{\infty} X_r z^r$ in $\mathcal{C}$ we define $S_+ = \sum_{r = -\infty}^{\infty} (X_r)_+ z^r$, $S_- = \sum_{r = -\infty}^{\infty} (X_r)_- z^r$, Res($S$) = $\sum_{r = -\infty}^{\infty}$ (Res $X_r$)z^r, and $\sigma_2(S) = \sum_{r = -\infty}^{\infty} \sigma(X_r)z^r$, where $\sigma_2$ is the operator defined in (2.2).

Let $\hat{L} = L - z\partial$, and define the map $H : \mathcal{C} \to \mathcal{C}$ by

$$H(X) = (\hat{L}X)_+ \hat{L} - \hat{L}(X\hat{L})_+ .$$

(Dickey [8] calls $H$ the Adler map, as it was introduced in section 4 of [1].) Since $(\hat{L}X)\hat{L} = \hat{L}(X\hat{L})$, it follows that $H(X) = (\hat{L}X)_- \hat{L} - \hat{L}(X\hat{L})_-$ for all $X$ in $\mathcal{C}$. Moreover, from the definition of $H$ and the fact that $L$ is a differential operator of order 2, one sees easily that $H(X) = H(\sigma_2(X))$ for all $X$ in $\mathcal{C}$. 

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Define
\[ T = \sum_{r=-\infty}^{\infty} L^{r/2} z^{-r-4}. \]

Clearly \( \hat{L}T = T \hat{L} = 0 \), so \( H(T) = 0 \), and hence also \( H(\sigma_2(T)) = 0 \). On the other hand, by observing that \( \sigma_2(L^{2k}) = 0 \) for all nonnegative integers \( k \), \( \sigma_2(L^{r/2}) = 0 \) for all integers \( r \leq -3 \), and \( \sigma_2(L^{-1}) = \partial^{-2} \), we can write \( \sigma_2(T) \) as
\[ \sigma_2(T) = R\partial^{-1} + \tilde{R}\partial^{-2}, \tag{2.10} \]

where \( R \) and \( \tilde{R} \) are in \( \mathbb{C} \) and
\[ R = \text{Res} \sigma_2(T) = \sum_{k=0}^{\infty} 2(-1)^k R_{2k+1} z^{-2k-3}. \tag{2.11} \]

Substituting (2.10) into the equation \( H(\sigma_2(T)) = 0 \), we find after a computation that
\[ 0 = H(\sigma_2(T)) = (-\tilde{R}'' + 2uR' + u'R - 2\tilde{R}^2)\partial, \tag{2.12} \]

and therefore \( \tilde{R} = -\frac{1}{2} R' \) and
\[ \frac{1}{2} R''' + 2uR' + u'R = 2\tilde{R}^2 R'. \tag{2.13} \]

Then substituting (2.11) into (2.13) gives (2.9) for \( k = 0, 1, 2, \ldots \). Finally, we can verify that \( R_1 = 1/2 \) by directly computing \( R_1 = \frac{1}{2} \text{Res} L^{-1/2} \).

Using the recurrence relation in Lemma 2.2, we find, for example, that the first few terms in the sequence \( \{ R_{2k+1} \} \) are
\[ R_1 = 1/2, \]
\[ R_3 = (-1/4)u, \]
\[ R_5 = (1/16)(u'' + 3u^2), \]
\[ R_7 = (-1/64)(u''' + 5u'^2 + 10uu'' + 10u^3). \tag{2.14} \]

Of particular interest are time-independent or stationary solutions of (2.8). If \( u \) is a such a solution, then \( u \) satisfies (2.8) with \( u_t = 0 \), and hence integration gives that \( u \) satisfies the equation \( R_{2k+3} = d \), where \( d \) is a constant, independent of \( x \) and \( t \). Letting \( d_1 = 2d \), we can rewrite this equation in the form
\[ d_1 R_1 - R_{2k+3} = 0. \]

More generally, we can view any solution of the equation
\[ d_1 R_1 + d_3 R_3 + d_5 R_5 + \cdots + d_{2N+3} R_{2N+3} = 0 \tag{2.15} \]
as a stationary solution of the equation
\[ u_t = d_3 R_1' + d_5 R_3' + \cdots + d_{2N+3} R_{2N+3}', \]
which itself can be considered to be an equation in the KdV hierarchy. For this reason, following [8], we refer to equations (2.15) as the stationary equations of the KdV hierarchy. (They are also sometimes called Lax-Novikov equations.)

Equation (2.15) is an ordinary differential equation of order \( 2N \), and can therefore be rewritten as a first-order system in phase space \( \mathbb{R}^{2N} \). It turns out that this system is of Hamiltonian form, and in fact is completely integrable in the sense that it has \( N \) independent integrals in involution with each other. In general, Liouville’s method provides a technique for actually integrating completely integrable systems:
that is, for explicitly finding the transformation from coordinates of phase space to action-angle variables. However, this integration involves solving a system of first-order partial differential equations. For the system (2.15), Dubrovin [11] introduced a change of variables under which this system of PDE’s has a simple form and is trivially solvable. This is the change of variables we use below in Section 5.

3. N-soliton profiles. Another key aspect of the KdV hierarchy is that the flows which it defines all commute with each other, at least formally. More precisely, one can check that the equations in (2.8) have the formal structure of Hamiltonian equations with respect to a certain symplectic form, and are all in involution with each other with respect to this form (see, for example, chapters 1 through 4 of [8]). This suggests the following. Assume that a function class \( S \) has been defined such that for each \( k \in \mathbb{N} \), the initial-value problem for equation (2.8) is well-posed on \( S \), and let \( S(t_{2k+1}) \) be the solution map for this problem, which to each \( \psi \in S \) assigns the function \( S(t_{2k+1})[\psi] = u(\cdot, t_{2k+1}) \in S \), where \( u \) is the solution of (2.8) with initial data \( u(x, 0) = \psi(x) \). Then in light of the formal structure mentioned above, one would expect that the solution operators \( S(t_{2k+1}) \) and \( S(t_{2l+1}) \) commute with each other as mappings on \( S \). Hence, for each \( \psi \in S \) and each \( l \in \mathbb{N} \), one should be able to define a simultaneous solution \( u(x, t_1, t_3, t_5, \ldots, t_{2l+1}) \) to all of the first \( l \) equations in the hierarchy by setting

\[
u(x, t_1, t_3, \ldots, t_{2l+1}) = S(t_1)S(t_3) \cdots S(t_{2l+1})\psi.
\]

This formal analysis, however, does not lead easily to concrete results about general solutions of the KdV hierarchy. For this reason there has historically been great interest in constructing and elucidating the structure of explicit solutions. In this section we review the definition and basic properties of an important class of such solutions, the N-soliton solutions.

To begin the construction of N-soliton solutions, let \( N \in \mathbb{N} \), and for \( 1 \leq j \leq N \) define the functions

\[
y_j(x) = e^{\alpha_j x} + a_j e^{-\alpha_j x},
\]

where \( \alpha_j \) and \( a_j \) are complex numbers satisfying

(i) for all \( j \in \{1, \ldots, N\} \), \( \alpha_j \neq 0 \) and \( a_j \neq 0 \),
(ii) for all \( j, k \in \{1, \ldots, N\} \), if \( j < k \) then \( \alpha_j \neq \alpha_k \) and \( 0 \leq \Re \alpha_j \leq \Re \alpha_k \).

We will use \( D(y_1, \ldots, y_N) \) to denote the Wronskian of \( y_1, \ldots, y_N \):

\[
D(y_1, \ldots, y_N) = \begin{vmatrix}
y_1 & \cdots & y_N \\
y_1' & \cdots & y_N' \\
\vdots & \ddots & \vdots \\
y_1^{(N-1)} & \cdots & y_N^{(N-1)}
\end{vmatrix}.
\]

Next we will construct an operator of the form (2.3) from the \( y_j \), using a technique known as the “dressing method” [8]. First, on any interval \( I \) where \( D \neq 0 \), we define a differential operator \( \phi \) of order \( N \) by

\[
\phi = \frac{1}{D} \begin{vmatrix}
y_1 & \cdots & y_N & 1 \\
y_1' & \cdots & y_N' & \partial \\
\vdots & \ddots & \vdots & \vdots \\
y_1^{(N-1)} & \cdots & y_N^{(N-1)} & \partial^{N-1} \\
y_1^{(N)} & \cdots & y_N^{(N)} & \partial^{N}
\end{vmatrix}.
\]
Here it is understood that the determinant in (3.4) is to be expanded along the final column, multiplying each operator $\partial^i$ by its corresponding cofactor on the left. In other words,
\[
\phi = \partial^N + W_{N-1}\partial^{N-1} + W_{N-2}\partial^{N-2} + \cdots + W_1\partial + W_0,
\]
where for $i = 1, \ldots, N - 1$,
\[
W_i = \frac{(-1)^{N+2+i}}{D} \begin{vmatrix}
  y_1 & \cdots & y_N \\
  y'_1 & \cdots & y'_N \\
  \vdots & \cdots & \vdots \\
  y_{i-1}^{(i-1)} & \cdots & y_N^{(i-1)} \\
  y_{i+1}^{(i+1)} & \cdots & y_N^{(i+1)} \\
  \vdots & \cdots & \vdots \\
  y_1^{(N)} & \cdots & y_N^{(N)}
\end{vmatrix},
\]
and
\[
W_0 = \frac{(-1)^{N+2}}{D} \begin{vmatrix}
  y'_1 & \cdots & y'_N \\
  y''_1 & \cdots & y''_N \\
  \vdots & \cdots & \vdots \\
  y_{1}^{(N)} & \cdots & y_{N}^{(N)}
\end{vmatrix}.
\]

Next, we slightly generalize the notion of pseudo-differential operator defined in Section 2 to include formal sums of type (2.1) in which the $a_i$ are no longer differential polynomials in a single variable $u$, but now are rational functions of the $n$ symbols $y_1, y_2, \ldots, y_n$ and their formal derivatives $y'_1, y'_2, y''_2$, etc. (forgetting for the moment that $y_1, y_2, \ldots, y_n$ are actually functions of $x$). The definitions of the multiplication and inverse operations on pseudo-differential operators given in Section 2 remain unchanged for this larger algebra. Thus, $\phi$ as defined in (3.4) has a formal inverse $\phi^{-1}$, which is a pseudo-differential operator whose coefficients are rational functions of $y_i$ and their derivatives, expressible as polynomials in $W_j$ and their derivatives. We now define $L$ as the formal pseudo-differential operator given by
\[
L = \phi \partial^2 \phi^{-1}.
\]

**Lemma 3.1.** The differential part of $L$ is
\[
L_+ = \partial^2 - 2W'_{N-1}.
\]

**Proof.** First observe that
\[
L = (\phi \partial^{-N}) \partial^2 (\phi \partial^{-N})^{-1}.
\]
Now we can write
\[
\phi \partial^{-N} = 1 + W_{N-1} \partial^{-1} + W_{N-2} \partial^{-2} + O(\partial^{-3}),
\]
where “$O(\partial^{-3})$” denotes terms containing $\partial^j$ with $j \leq -3$. Also, a computation shows that
\[
(\phi \partial^{-N})^{-1} = 1 - W_{N-1} \partial^{-1} + (W_{N-1}^2 - W_{N-2}) \partial^{-2} + O(\partial^{-3}).
\]
Equation (3.9) then follows easily by inserting (3.11) and (3.12) into (3.10) and carrying out the multiplication to determine the terms of nonnegative order. \qed
Lemma 3.2. Define $W_{-1} = 0$. Then
\[ L_\phi = - \sum_{j=0}^{N-1} \left( W_j'' + 2W_j'_{N-1}W_j \right) \partial^j. \] (3.13)

Proof. Since $L_\phi = L_+ L_-$, we have from (3.8) that
\[ L_\phi = \phi \partial^2 - L_+. \] (3.14)
The desired result follows by substituting (3.5) and (3.9) into the right-hand side, and carrying out the multiplications. \qed

So far, in discussing $\phi$ and $L$, we have considered them only as formal pseudo-differential operators with coefficients that are rational functions in the symbols $y_i$, $y_i'$, $y_i''$, etc. Now, however, we wish to “remember” the fact that these coefficients are specific functions of $x$. To this end we first observe that by (3.2) the functions $y_1, \ldots, y_N$ are analytic and linearly independent. Therefore, by a theorem of Peano (see [3]), their Wronskian $D$ cannot vanish identically on any open interval in $\mathbb{R}$. In particular, by continuity there exists an open interval $I$ on $\mathbb{R}$ such that $D(x) \neq 0$ for all $x \in I$. Therefore, on $I$ the right-hand-side of (3.4) defines a linear differential operator with smooth coefficients $W_i$ for $i = 1, \ldots, N-1$. To emphasize the distinction between the formal operator $\phi$ and its concrete realization, we introduce the notation $r(\phi)$ for the differential operator with smooth coefficients on $I$ obtained by remembering that the $y_i$ are certain functions of $x$.

More generally, if $X$ is any pseudo-differential operator whose coefficients are formal polynomials in $W_i$ and their derivatives, we define $r(X)$ to be the operator obtained by remembering that the coefficients of $X$ are actually smooth functions of $x$ on $I$. Thus $r$ defines an algebra homomorphism from $P$ to the algebra of pseudo-differential operators with coefficients that are smooth functions on $I$.

Although it is clear from Lemma 3.2 that $L_-$ is not zero as a formal pseudo-differential operator, nevertheless the coefficients of $L_-$ evaluate to zero when viewed as functions on $I$. That is, we have the following result.

Lemma 3.3. When $L$ is defined as in (3.8), with $\phi$ given by (3.4), then
\[ r(L_-) = 0. \]

Proof. From (3.13) we have that, as a formal pseudo-differential operator,
\[ L_\phi = \sum_{j=0}^{N-1} F_j \partial^j, \]
where each $F_j$ is a differential polynomial in $W_0, \ldots, W_{N-1}$. Therefore
\[ r(L_\phi) = \sum_{j=0}^{N-1} F_j(x) \partial^j, \]
where the $F_j(x)$ are smooth functions on $I$.

For all $i = 1, \ldots, N$, we see from (3.4) that
\[ r(\phi)y_i = 0, \] (3.15)
and since $\partial^2 y_i = \alpha_i^2 y_i$, then $r(\phi \partial^2)y_i = 0$ also. Therefore (3.14) implies that
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\[
\begin{align*}
  r(L_- \phi)y_i = & \sum_{j=0}^{N-1} F_j(x) \partial_j y_i(x) = 0 \text{ for } i = 1, \ldots, N. \tag{3.16}
\end{align*}
\]

Since \( D(x) = \det(\partial_j y_i(x)) \neq 0 \) for all \( x \in I \), it follows from (3.16) that \( F_j(x) = 0 \) for all \( x \in I \) and all \( j = 0, 1, \ldots, N - 1 \).

Now as a formal pseudo-differential operator, \( \phi \) is invertible, with inverse \( \phi^{-1} \) of the form

\[
\phi^{-1} = \sum_{k=0}^{\infty} B_k \partial^{-N-k},
\]

where the \( B_k \) are differential polynomials in \( W_0, \ldots, W_{N-1} \). Hence

\[
L_- = (L_- \phi) \phi^{-1} = \sum_{j=0}^{N-1} F_j \partial^j \sum_{k=0}^{\infty} B_k \partial^{-N-k} = \sum_{r=1}^{\infty} \sum_{j=0}^{N-1} (F_j(x) G_{j,r}(x)) \partial^{N-r},
\]

where each \( G_{j,r} \) is a finite linear combination of the \( B_k \) and their formal derivatives \( B'_k, B''_k, \ldots \). Since \( F_j(x) = 0 \) for all \( x \in I \), it follows that the coefficients \( \sum_{j=0}^{N-1} (F_j(x) G_{j,r}(x)) \) of \( r(L_-) \) are also identically zero on \( I \).

The following consequence of Lemma 3.3 will be useful in Section 5.

**Corollary 3.4.** If \( I \) is any interval such that \( D(x) \neq 0 \) for all \( x \in I \), then the equation

\[
W'_{N-1} - W^2_{N-1} + 2W_{N-2} + \sum_{i=1}^{N} \alpha_i^2 = 0 \tag{3.17}
\]

holds at all points of \( I \).

**Proof.** From Lemma 3.3 we have \( r(L_-) = 0 \), and hence each coefficient in the sum in (3.13) is identically zero as a function of \( x \) on \( I \). In particular,

\[
W''_{N-1} + 2W'_{N-2} - 2W'_{N-1} W_{N-1} = 0
\]

on \( I \). Integrating gives

\[
W'_{N-1} + 2W_{N-2} - W^2_{N-1} + C = 0,
\]

where \( C \) is a constant.

To evaluate \( C \), first assume that \( \alpha_1, \ldots, \alpha_N \) are positive numbers, and observe that since \( y_i \) behaves as \( x \to \infty \) like \( e^{\alpha_i x} \), we have that \( \lim_{x \to \infty} W_{N-1} = -d_1/d, \lim_{x \to \infty} W'_{N-1} = 0 \), and \( \lim_{x \to \infty} W_{N-2} = d_2/d \), where

\[
d = \begin{vmatrix}
  1 & \ldots & 1 \\
  \alpha_1 & \ldots & \alpha_N \\
  \alpha_1^2 & \ldots & \alpha_N^2 \\
  \vdots & \ldots & \vdots \\
  \alpha_1^{N-1} & \ldots & \alpha_N^{N-1}
\end{vmatrix}
\]
is the Vandermonde matrix of the numbers $\alpha_1, \ldots, \alpha_N$, and

$$
\begin{vmatrix}
1 & \ldots & 1 \\
\alpha_1 & \ldots & \alpha_N \\
\alpha_1^2 & \ldots & \alpha_N^2 \\
\vdots & \ddots & \vdots \\
\alpha_1^{N-2} & \ldots & \alpha_N^{N-2} \\
\alpha_1^{N} & \ldots & \alpha_N^{N} \\
\end{vmatrix} = d_1,
\begin{vmatrix}
1 & \ldots & 1 \\
\alpha_1 & \ldots & \alpha_N \\
\alpha_1^2 & \ldots & \alpha_N^2 \\
\vdots & \ddots & \vdots \\
\alpha_1^{N-3} & \ldots & \alpha_N^{N-3} \\
\alpha_1^{N-1} & \ldots & \alpha_N^{N-1} \\
\alpha_1^{N} & \ldots & \alpha_N^{N} \\
\end{vmatrix} = d_2.
$$

Therefore

$$
C = \lim_{x \to \infty} \left( W_{N-1}^2 - 2W_{N-2} - W'_{N-1} \right) = \left( \frac{d_1}{d} \right)^2 - 2 \left( \frac{d_2}{d} \right). \quad (3.18)
$$

But it follows from a classic exercise on Vandermonde matrices (see problem 10 on p. 99 of [33], or [26]) that

$$
d_1/d = \sum_{i=1}^{N} \alpha_i \quad \text{and} \quad d_2/d = \sum_{1 \leq i < j \leq N} \alpha_i \alpha_j.
$$

Substituting in (3.18), we obtain $C = \sum_{i=1}^{N} \alpha_i^2$, as desired. The result for general complex values of $\alpha_1, \ldots, \alpha_N$ then follows by analytic continuation. \(\square\)

**Remark.** Since (3.17) is a Ricatti equation, the substitution $W_{N-1} = -D'/D$ converts it to the following linear equation for $D$:

$$
D'' = \left( 2W_{N-2} + \sum_{i=1}^{N} \alpha_i^2 \right) D. \quad (3.19)
$$

Corollary 3.4 is therefore equivalent to the assertion that (3.19) holds when $D$ is given by (3.3) and (3.1).

**Corollary 3.5.** Suppose $D$ is given by (3.3), $\phi$ by (3.4), and $L$ by (3.8). Then

$$
r(L) = \partial^2 + u,
$$

where

$$
u = 2 \left( \frac{D'}{D} \right)'. \quad (3.20)
$$

**Proof.** From the definition of the determinant and the product rule, one easily sees that the derivative of $D$ is given by

$$
D' = \begin{vmatrix} y_1 & \ldots & y_N \\ y'_1 & \ldots & y'_N \\ \vdots & \ddots & \vdots \\ y_1^{(N-2)} & \ldots & y_N^{(N-2)} \\ y_1^{(N)} & \ldots & y_N^{(N)} \end{vmatrix}, \quad (3.21)
$$

which, as we see from (3.6), implies that $W_{N-1} = -D'/D$. Since $r(L) = r(L_+) + r(L_-)$, the desired result therefore follows from Lemma 3.1 and Lemma 3.3. \(\square\)

**Definition 3.6.** Let $y_j$ be given by (3.1), and assume (3.2) holds. Then we define

$$
\psi^{(N)}(x) = \psi^{(N)}(x; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N) = 2 \left( \frac{D'(y_1, \ldots, y_N)}{D(y_1, \ldots, y_N)} \right)', \quad (3.22)
$$

for all $x$ such that $D(y_1, \ldots, y_N) \neq 0$.\(\square\)
Introducing simple time dependencies into \( \psi^{(N)} \) yields a function which satisfies all the equations in the KdV hierarchy simultaneously.

**Theorem 3.7.** Let \( N \in \mathbb{N} \), and let \( a_j \), and \( \alpha_j \), \( j = 1, \ldots, N \) be complex numbers satisfying (3.2). Fix \( l \in \mathbb{N} \), and for \( 1 \leq j \leq N \), define the function \( \tilde{a}_j \) by

\[
\tilde{a}_j(x, t_3, t_5, \ldots, t_{2l+1}) = a_j \exp \left( -2(\alpha_j^3 t_3 + \alpha_j^5 t_5 + \cdots + \alpha_j^{2l+1} t_{2l+1}) \right).
\]  

(3.23)

Then let \( u \) be defined as a function of \( x, t_3, t_5, \ldots, t_{2l+1} \) by

\[
u = \psi^{(N)}(x; \tilde{a}_1, \ldots, \tilde{a}_N; \alpha_1, \ldots, \alpha_N).
\]  

(3.24)

Then for all \( k \in \{1, \ldots, l\} \), and at all points in its domain of definition, \( u \) satisfies the partial differential equation

\[
\frac{\partial u}{\partial t_{2k+1}} = \left[ \left( L^{(2k+1)/2} \right) + L \right] = 4(-1)^{k+1} R_{2k+3}^{(2k+1)}.
\]  

(3.25)

**Remark.** Using the fact that multiplication of \( y_i \) by the exponential of a linear function of \( x \) does not change the value of \( \psi^{(N)} \), one sees easily that (3.24) can also be written in the form

\[
u = 2 \left( \frac{D'(\tilde{y}_1, \ldots, \tilde{y}_N)}{D(\tilde{y}_1, \ldots, \tilde{y}_N)} \right),
\]  

where \( \tilde{y}_i(x) = e^{w_i x} + a_i e^{-w_i x} \) and \( w_i = \alpha_i x + \alpha_i^3 t_3 + \alpha_i^5 t_5 + \cdots + \alpha_i^{2l+1} t_{2l+1} \).

We will not need to make use of Theorem 3.7 in the present paper, and so do not include a proof here. But the reader may be interested to know that, using the tools defined in Section 2 above, a one-paragraph proof can be given. It may be found in [8], where it appears as the proof of part (ii) of Proposition 1.6.5 in [8], or in [9] as the proof of Proposition 1.7.5.

We are concerned here, rather, with the fact the functions \( \psi^{(N)} \) satisfy stationary equations of the form (2.15):

**Theorem 3.8.** Let \( \psi^{(N)} \) be as in Definition 3.6, and define constants \( s_0, s_1, \ldots, s_N \) by

\[
s_0 + s_1 x + s_2 x^2 + \cdots + s_N x^N = (x - \alpha_1^2)(x - \alpha_2^2) \cdots (x - \alpha_N^2);
\]  

(3.26)

in other words, \( s_i \) is the \( i \)th elementary symmetric function of \( N \) variables, evaluated on \(-\alpha_1^2, \ldots, -\alpha_N^2\). Then, on each interval of its domain of definition, the function

\[
u(x) = \psi^{(N)}(x; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N)
\]  

satisfies the ordinary differential equation in \( x \) given by

\[
s_0 R_3 - s_1 R_5 + s_2 R_7 + \cdots + (-1)^N s_N R_{2N+3} = C,
\]  

(3.27)

where \( C \) is a constant.

**Proof.** Define \( \tilde{a}_i \) by (3.23) with \( l = N \), and for \( i = 1, \ldots, N \) extend \( y_i \) to be a function of \( x, t_3, t_5, \ldots, t_{2N+1} \) by replacing \( a_i \) with \( \tilde{a}_i \) in (3.1). That is, set

\[
y_i = \exp(\alpha_i x) + a_i \exp \left( -\left( \alpha_i x + 2\alpha_i^3 t_3 + 2\alpha_i^5 t_5 + \cdots + 2\alpha_i^{2N+1} t_{2N+1} \right) \right).
\]  

(3.28)

Also extend \( u \) to be a function of \( x, t_3, t_5, \ldots, t_{2N+1} \) by (3.24). With \( \phi \) and \( W_i \) defined in terms of \( y_i \) as before, we have as in (3.20) that

\[
u = -2W_{N-1}.
\]  

(3.29)

For \( 1 \leq k \leq N \), let \( \partial_{2k+1} \) denote differentiation with respect to \( t_{2k+1} \).
For each $i$ from 1 to $N$, we apply to both sides of (3.15) the operator
\[
\tilde{\partial} = s_0 \partial + s_1 \partial_3 + s_2 \partial_5 + \cdots + s_N \partial_{2N+1}.
\]
There results the identity
\[
0 = (\tilde{\partial} r(\phi)) y_i + r(\phi)(\tilde{\partial} y_i),
\]
where
\[
\tilde{\partial} r(\phi) = (\tilde{\partial} W_{N-1}) \partial^{N-1} + (\tilde{\partial} W_{N-2}) \partial^{N-2} + \cdots + (\tilde{\partial} W_1) \partial + \tilde{\partial} W_0.
\]
But for all $k = 1, \ldots, N$ we have from (3.28) that
\[
\tilde{\partial} y_i = \alpha_i^{2k+1} z_i,
\]
where
\[
z_i = -2\alpha_i \exp \left[ -(\alpha_i x + 2\alpha_i^3 t_3 + 2\alpha_i^5 t_5 + \cdots + 2\alpha_i^{2N+1} t_{2N+1}) \right].
\]
Therefore
\[
\tilde{\partial} y_i = (s_0 \alpha_i + s_1 \alpha_i^3 + s_2 \alpha_i^5 + \cdots + s_N \alpha_i^{2N+1}) z_i.
\]
It follows from (3.26) and (3.31) that $\tilde{\partial} y_i = 0$, and so, by (3.30),
\[
(\tilde{\partial} r(\phi)) y_i = 0.
\]
Now $\tilde{\partial} r(\phi)$ is a linear differential operator in $x$ of order $N - 1$ or less, so as in the proof of Lemma 3.3, the fact that it takes all the functions $y_1, \ldots, y_N$ to zero means that all its coefficients must be identically zero. In particular, $\tilde{\partial} W_{N-1} = 0$, or, in other words,
\[
s_0 \partial W_{N-1} + s_1 \partial_3 W_{N-1} + \cdots + s_N \partial_{2N+1} W_{N-1} = 0.
\]
On the other hand, (3.25) and (3.29) tell us that
\[
-2\partial_{2k+1} W_{N-1} = 4(-1)^{k+1} R'_{2k+3}
\]
for $1 \leq k \leq l$. Integrating (3.33) with respect to $x$, we obtain
\[
\partial_{2k+1} W_{N-1} = 2(-1)^k R_{2k+3} + C_k,
\]
for $1 \leq k \leq l$, where $C_k$ is a constant of integration. Moreover, from (2.9) we have $R_3 = -u/4$, and hence
\[
\partial W_{N-1} = W'_{N-1} = 2R_3.
\]
Substituting (3.34) and (3.35) into (3.32) then gives (3.37).

In general, $\psi^{(N)}$ will have singularities at points where the denominator $D$ in (3.4) is equal to zero, but away from these points, $\psi^{(N)}$ is a smooth, and in fact analytic, function of its arguments. Our next task is to determine conditions on the parameters $\alpha$ and $a$ under which $D$ has no zeros on $\mathbb{R}$, or equivalently under which $\psi^{(N)}$ is a smooth function on all of $\mathbb{R}$.

For this purpose it will be useful to represent $D$ as an explicit sum of exponential functions. For given $N \in \mathbb{N}$, let $\{-1,1\}^N$ denote the set of functions $\epsilon$ from $\{1, \ldots, N\}$ to $\{-1,1\}$; thus $\{-1,1\}^N$ has cardinality $2^N$. For $\epsilon \in \{-1,1\}^N$, we denote the image of $j$ under $\epsilon$ by $\epsilon_j$, and define $S(\epsilon)$ to be the set of all $j \in \{1, \ldots, N\}$ such that $\epsilon_j = -1$. Also, for any ordered $N$-tuple $(r_1, \ldots, r_N)$, let
\[
V(r_1, \ldots, r_N) = \det \{r_j^{i-1}\}_{i,j=1}^N, \quad \prod_{1 \leq i < j \leq N} (r_j - r_i)
\]
be the corresponding Vandermonde determinant. Then expansion of the determinant in (3.3) yields the formula

\[
D = \sum_{{\epsilon \in \{-1,1\}^N}} \left[ \exp \left( \sum_{j=1}^{N} \epsilon_j \alpha_j x \right) \left( \prod_{j \in S(\epsilon)} a_j \right) V(\epsilon_1 \alpha_1, \ldots, \epsilon_N \alpha_N) \right]. \tag{3.36}
\]

**Lemma 3.9.** Let \( y_j \) be given by (3.1), and suppose that (3.2) holds. Suppose in addition that for all \( j \in \{1, \ldots, N\} \), \( \Re \alpha_j > 0 \). Then \( \psi^{(N)} \) and all of its derivatives are defined for all sufficiently large \( |x| \), and approach zero exponentially fast as \( |x| \to \infty \).

*Proof.* Because \( \Re \alpha_j > 0 \) for all \( j \), the dominant term in (3.36) is \( V e^{Sx} \), where \( V = V(\alpha_1, \ldots, \alpha_N) \) and \( S = \sum_{j=1}^{N} \alpha_j \); all other terms have exponents with smaller real parts. In particular, we have \( D(x) \neq 0 \) whenever \( |x| \) is sufficiently large. The dominant terms in \( D'(x) \) and \( D''(x) \) are \( VSe^{Sx} \) and \( V S^2 e^{Sx} \), respectively; and so in the expression \( \psi^{(N)}(x) = 2(DD'' - (D')^2)/D^2 \), the denominator has dominant term \( V^2 e^{2Sx} \), while the coefficient of \( e^{2Sx} \) in the numerator is zero. It follows easily that \( \psi^{(N)}(x) \), together with all its derivatives, tends to zero exponentially fast as \( x \to \infty \). A similar argument applies as \( x \to -\infty \) (where the dominant term in \( D(x) \) is \( |a_1 \cdots a_N V|^{|Sx|} \)). \( \square \)

**Lemma 3.10.** Let \( y_j \) be given by (3.1), and suppose that (3.2) holds. Suppose also that for all \( j \in \{1, \ldots, N\} \), \( \alpha_j \) and \( a_j \) are real, and

\[
(-1)^{j-1} a_j > 0. \tag{3.37}
\]

Then \( \psi^{(N)}(x) \in H^1(\mathbb{R}) \).

*Proof.* Suppose that the \( \alpha_j \) and \( a_j \) are real and (3.37) holds. Then for \( 1 \leq j < k \leq N \) the factor \( \epsilon_k \alpha_k - \epsilon_j \alpha_j \) in \( V(\epsilon_1 \alpha_1, \ldots, \epsilon_N \alpha_N) \) has the same sign as \( \epsilon_k \). For a given \( k \), there are \( k-1 \) such factors in \( V(\epsilon_1 \alpha_1, \ldots, \epsilon_N \alpha_N) \), corresponding to the values \( 1 \leq j < k-1 \), so the sign of \( V(\epsilon_1 \alpha_1, \ldots, \epsilon_N \alpha_N) \) is \( \prod_{k \in S(\epsilon)} (-1)^{j-1} \). It then follows from (3.37) that the coefficient of each exponential in (3.36) is positive. Hence \( D > 0 \) for all \( x \in \mathbb{R} \), and it follows that \( \psi^{(N)}(x) \) is well-defined and smooth on all of \( \mathbb{R} \). Then from Lemma 3.9 it follows that \( \psi^{(N)} \in H^1(\mathbb{R}) \). \( \square \)

**Lemma 3.11.** Let \( y_j \) be given by (3.1), and suppose that (3.2) holds. Suppose also that for each \( j \in \{1, \ldots, N\} \), either (i) \( \alpha_j \) and \( a_j \) are real, (ii) \( \alpha_j \) is purely imaginary and \( |a_j| = 1 \), or (iii) there exists \( k \in \{1, \ldots, N\} \) such that \( \alpha_k = \alpha^*_k \) and \( a_k = a^*_k \). (These conditions can be summarized by saying that the numbers \( \alpha^*_j \) and \( (\log a_j)^2 \) are either real and of the same sign, or occur in complex conjugate pairs.) Then \( \psi^{(N)}(x) \) is real-valued at all points where it is defined.

*Proof.* In case (i) we have \( y_j^* = y_j \), in case (ii) we have \( y_j^* = (1/a_j)y_j \), and in case (iii) we have \( y_j^* = y_k \). It follows easily that the conjugate \( D^* \) of \( D = D(y_1, \ldots, y_N) \) is equal to a constant times \( D \) itself. Therefore \( (\psi^{(N)})^* = \psi^{(N)} \). \( \square \)

**Definition 3.12.** We say that \( \psi^{(N)}(x; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N) \) is an \( N \)-soliton profile if \( \psi^{(N)}(x) \) is real-valued for all \( x \in \mathbb{R} \), and \( \psi^{(N)}(x) \in H^1(\mathbb{R}) \). The corresponding time-dependent functions given by (3.24) are called \( N \)-soliton solutions of the KdV hierarchy. The numbers \( \alpha_1, \ldots, \alpha_N \) are called the wavespeeds of the \( N \)-soliton solution.
Remarks. (i) At least in the case when \( N = 2 \), it can be shown that, given that the conditions in (3.2) hold for \( a_j \) and \( \alpha_j \), the hypotheses on \( a_j \) and \( \alpha_j \) in Lemmas 3.10 and 3.11 are not only sufficient for \( \psi^{(N)} \) to be an \( N \)-soliton profile according to the above definition, but also necessary. We conjecture that these conditions on \( a_j \) and \( \alpha_j \) are also necessary in the case of general \( N \), although we have not proved this yet.

(ii) If the conjecture in the preceding remark is true, it then follows from Lemma 3.9 that \( N \)-soliton profiles, together with all their derivatives, approach zero exponentially fast as \( |x| \to \infty \).

(iii) By transforming the index in the outermost sum of (3.36) from \( \epsilon \) to \( \mu \), where \( \mu_j = \frac{1}{2} (\epsilon_j + 1) \) for \( j \in \{1, \ldots, N\} \), one can rewrite \( D \) in the form

\[
D = e^{px + q} D_1,
\]

where \( p \) and \( q \) are constants and

\[
D_1 = \sum_{\mu \in \{0,1\}^N} \exp \left( \sum_{i=1}^N 2\mu_i \alpha_i (x + \zeta_i) + \sum_{1 \leq i < j \leq n} \mu_i \mu_j A_{ij} \right),
\]

where \( \zeta_i \) and \( A_{ij} \) are real constants. Explicitly, one has

\[
e^{px + q} = \exp \left( -\sum_{i=1}^N \alpha_i x \right) V(\alpha_1, \ldots, \alpha_N) \prod_{j=1}^N |a_j|,
\]

\[
\zeta_i = \frac{1}{2\alpha_i} \log \left| \frac{V(\alpha_1, \ldots, \alpha_{i-1}, -\alpha_i, \alpha_{i+1}, \ldots, \alpha_N)}{a_i V(\alpha_1, \ldots, \alpha_N)} \right|, \quad (i = 1, \ldots, N),
\]

\[
A_{ij} = 2 \log \left| \frac{\alpha_j - \alpha_i}{\alpha_j + \alpha_i} \right|, \quad (i, j = 1, \ldots, N).
\]

(Here use has been made of the assumption (3.37).) Writing

\[
\psi^{(N)} = 2(D_1'/D_1)',
\]

one obtains the formula for \( N \)-soliton profile found in [27] or on page 55 of [20].

(iv) If \( \psi^{(N)} \) is an \( N \)-soliton solution, then the constant \( C \) in equation (3.27) is equal to zero; that is,

\[
s_0 R_3 - s_1 R_5 + s_2 R_7 + \cdots + (-1)^N s_N R_{2N+3} = 0 \tag{3.38}
\]

on \( \mathbb{R} \). This is seen by taking the limit of (3.27) as \( x \to \infty \), and observing that, for each \( k \geq 1 \), \( R_{2k+1} \) is a differential polynomial in \( u = \psi^{(N)} \) and its derivatives, with no constant term.

4. The stationary equation for \( N = 1 \). To set the stage for the analysis of (2.15) in the case \( N = 2 \), we now discuss the case \( N = 1 \). The result we prove in this section, Theorem 4.2, is a standard exercise in elementary integration, but writing out the proof in detail will serve to introduce the notation we use for the more complicated computations of the next section.

From (2.14), we have that in the case when \( N = 1 \), (2.15) is given by

\[
d \frac{d}{2} - \frac{d_3}{4} (u) + \frac{d_5}{16} (u'' + 3u^2) = 0. \tag{4.1}
\]

Suppose \( d_1, d_3, \) and \( d_5 \) are given real numbers, and suppose \( u \in L^2 \) is a real-valued solution of the ordinary differential equation (4.1) (in the sense of distributions). Then from (4.1) we see that \( d_5 \) must be nonzero, and by dividing by \( d_5 \) if necessary, we can assume that \( d_5 = 1 \). Also, multiplying both sides of (4.1) by a
test function $\phi_\tau(x) = \phi(x - \tau)$, where $\int_\mathbb{R} \phi(x) \, dx = 1$, and letting $\tau \to \infty$, we arrive at the conclusion that
\[
\lim_{\tau \to \infty} \int_\mathbb{R} u''_\tau = \lim_{\tau \to \infty} \int_\mathbb{R} u'' \phi_\tau = \lim_{\tau \to \infty} \int_\mathbb{R} (-3u^2 + 4du - 8d_1) \phi_\tau = -8d_1,
\]
from which it follows that $d_1 = 0$. Letting $C = d_3$, we can then write (4.1) as
\[
C \left( \frac{-u}{4} \right) + \frac{1}{16} (u'' + 3u^2) = 0.
\]

**Lemma 4.1.** Suppose $u \in L^2$ is a solution of (4.3) in the sense of distributions. Then $u$ must be in $H^s$ for all $s \geq 0$, and $u$ is analytic on $\mathbb{R}$.

**Proof.** Equation (4.3) can be rewritten as
\[
u - u'' = au + bu^2,
\]
where $a$ and $b$ are constants. Let $\mathcal{F}$ denote the Fourier transform, defined for $f \in L^1$ by $\mathcal{F}(f(k)) = \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx$, and extended to $L^2$ in the usual way. Letting $f = au$ and $g = bu^2$, taking the Fourier transform of (4.4), and dividing both sides by $(1 + k^2)^{1/2}$, we obtain
\[
(1 + k^2)^{1/2} \mathcal{F}u = \frac{1}{(1 + k^2)^{1/2}} (\mathcal{F}f + \mathcal{F}g).
\]
Since $f$ is in $L^2$, then $\mathcal{F}f$ and $\mathcal{F}f/(1 + k^2)^{1/2}$ are in $L^2$; and since $g$ is in $L^1$, then $\mathcal{F}g$ is bounded and continuous, and $\mathcal{F}g/(1 + k^2)^{1/2}$ is in $L^2$. Therefore $(1 + k^2)^{1/2} \mathcal{F}u \in L^2$, so $u \in H^1$. But it then follows that $u^2 \in L^2$, whence both $f$ and $g$ are in $L^2$, so (4.4) gives $u'' \in L^2$ and $u \in H^2$. Taking derivatives of (4.4) successively now easily gives that all higher-order derivatives of $u$ are in $L^2$, so that $u \in H^s$ for all $s \geq 0$.

As a particular consequence, we have that $u$ is a classical solution of (4.4) on $\mathbb{R}$. Therefore the fact that $u$ is analytic on $\mathbb{R}$ follows from the fundamental theorems of ordinary differential equations, given that the right-hand side of (4.4) is an analytic function of $u$ (see, e.g., section 1.8 of [5]).

**Theorem 4.2.** Suppose $C \in \mathbb{R}$, and suppose $u \in L^2$ is a real-valued solution of (4.3), in the sense of distributions. Suppose also that $u$ is not identically zero. Then $C > 0$, and there exists $K \in \mathbb{R}$ such that
\[
u = \frac{2C}{\cosh^2(\sqrt{C}x + K)}.
\]

**Remark.** We can also write (4.5) in the form
\[
u = \psi^{(1)}(x; a; \sqrt{C}),
\]
where $a = e^{-2K}$.

**Proof.** Suppose $u$ is an $L^2$ solution of (4.3) on $\mathbb{R}$, and is not identically 0. By Lemma 4.1, $u$ is analytic on $\mathbb{R}$, and is in $H^s$ for every $s \geq 0$. Hence, in particular, $u$ and all its derivatives tend to zero as $|x| \to \infty$.

Multiplying (4.3) by $u'$ and integrating gives
\[
(u')^2 = 4Cu^2 - 2u^3,
\]
where we have used the fact that $u \to 0$ and $u' \to 0$ as $x \to \infty$ to evaluate the constant of integration as zero. Letting $\zeta = -C + u/2$, we can rewrite (4.6) as
\[
(\zeta')^2 = -4\zeta(\zeta + C)^2.
\]
Since $u$ is analytic on $\mathbb{R}$, then so is $\zeta$. We know that $\zeta$ cannot be identically equal to $-C$ on $\mathbb{R}$, because $u$ is not identically zero. Also, if $C \neq 0$, then $\zeta$ cannot be identically equal to 0 on $\mathbb{R}$, because this would contradict the fact that $u \to 0$ as $x \to \infty$. Therefore the set $\{x \in \mathbb{R} : \zeta(x) = 0 \text{ or } \zeta = -C\}$ must consist of isolated points (or be empty). Hence there exists an open interval $I$ in $\mathbb{R}$ such that for all $x \in I$, $\zeta(x) \neq 0$ and $\zeta(x) + C \neq 0$. Also, from (4.7) it follows that $\zeta(x) < 0$ for $x \in I$.

Define $\Omega$ to be the domain in the complex plane given by
\[ \Omega = \mathbb{C} - \{z : \Re z \geq 0 \text{ and } \Im z = 0\}. \] (4.8)
Henceforth, for $z \in \Omega$ we will denote by $\sqrt{z}$ the branch of the square root function given by $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ when $z = re^{i\theta}$ with $r > 0$ and $0 < \theta < 2\pi$. Thus $\sqrt{z}$ is an analytic function on $\Omega$, and since $\zeta(x)$ takes values in $\Omega$, then $\sqrt{\zeta(x)}$ is an analytic function of $x$ on $I$.

From (4.7) we have that there exists a function $\theta : I \to (-1, 1)$ such that
\[ \zeta' = 2i\theta(x)\sqrt{\zeta}(\zeta + C) \] (4.9)
for all $x \in I$. Since $\zeta(x) + C \neq 0$ for all $x \in I$, it then follows from (4.9) that $\theta$ is analytic on $I$, and, since $\theta$ takes values in $(-1, 1)$, $\theta$ must therefore be constant on $I$.

Now define $v = -i\theta\sqrt{\zeta}$ on $I$, noting for future reference that, since $\zeta < 0$ on $I$, then $v$ is real-valued. Let $\alpha = \sqrt{C}$. We then have from (4.9) that
\[ \frac{v'}{C - v^2} = \frac{v'}{\alpha^2 - v^2} = 1. \] (4.10)
To integrate (4.10), we first fix $x_0 \in I$, let $V = v(x_0)$, and define
\[ L_{\alpha, V}(z) = \int_{-V}^{z} \frac{dw}{\alpha^2 - w^2}. \]
Since $V \neq \pm\alpha$, this defines $L_{\alpha, V}$ as a single-valued, analytic function of $z$ in some neighborhood of $V$. By shrinking $I$ if necessary, we may assume that $L_{\alpha, V}(v(x))$ is defined for all $x \in I$, and so (4.10) may be integrated to give
\[ L_{\alpha, V}(v(x)) = x - x_0 \] (4.11)
for $x \in I$.

Our next goal will be to solve (4.11) for $v(x)$. Once this has been done, we can recover $u$ from the formula
\[ u = 2(\zeta + C) = 2(\alpha^2 - v^2). \] (4.12)
Consider first the case when $\alpha \neq 0$ (and hence $C \neq 0$). In this case, by choosing an appropriate branch of the complex logarithm function, we could express $L_{\alpha, V}(z)$ as
\[ L_{\alpha, V}(z) = \frac{1}{2\alpha} \log \left( \frac{\alpha + z}{\alpha + V} \right) - \log \left( \frac{\alpha - z}{\alpha - V} \right). \] (4.13)
However, this will not be necessary, since we really only need to use the fact that
\[ \exp(2\alpha L_{\alpha, V}(z)) = \left( \frac{\alpha + z}{\alpha - z} \right) \left( \frac{\alpha - V}{\alpha + V} \right) \] (4.14)
for all $z$ in some neighborhood of $V$. To see that (4.14) is true, define $f_1(z)$ to be the function on the left side of (4.14), and $f_2(z)$ to be the function on the right side. Then both $f_1$ and $f_2$ satisfy the differential equation $df/dz = (2\alpha/(\alpha^2 - z^2))f(z)$ in
some neighborhood of $V$, and both take the value 1 at $z = V$. Since a solution $f$ of the differential equation with a prescribed value at $V$ is unique on any neighborhood of $V$ where it is defined, $f_1$ must equal $f_2$ on some neighborhood of $V$.

Now multiplying both sides of (4.11) by $2\alpha$, taking exponentials, and using (4.14), one obtains

$$\frac{\alpha + v}{\alpha - v} = \left(\frac{\alpha + V}{\alpha - V}\right)e^{2\alpha(x-x_0)} = e^{2A},$$

(4.15)

where

$$A = \alpha(x - x_0) + M$$

(4.16)

and $M$ is any number such that

$$e^{2M} = \frac{\alpha + V}{\alpha - V}.$$  

(4.17)

Solving (4.15) for $v$, we find that

$$v = \frac{y'}{y}$$

where

$$y = \sinh A.$$  

(4.18)

Substituting into (4.12), we find that

$$u = 2\left(\frac{\alpha^2 y^2 - (y')^2}{y^2}\right) = 2(y'/y)'.$$  

(4.19)

Since $u$ is analytic on $\mathbb{R}$, then the function on the right side of (4.19) is extendable to an analytic function on $\mathbb{R}$. This implies that $y$ cannot have any zeroes on $\mathbb{R}$. We now have to determine the values of $C$ for which this is possible. We consider separately the subcases in which $C > 0$ and $C < 0$.

If $C > 0$, then $\alpha = \sqrt{C}$ is real, and from (4.16) and (4.17) we see that we can take

$$A = \alpha x + K + i\sigma\pi/2,$$  

(4.20)

where $K$ is real and either $\sigma = 0$ or $\sigma = 1$, according to whether $(V + \alpha)/(V - \alpha)$ is positive or negative. If $\sigma = 0$, then

$$y = \sinh(\alpha x + K),$$

which equals zero for some $x \in \mathbb{R}$, so $u$ has a singularity at this $x$. On the other hand, if $\sigma = 1$, then (4.18) gives

$$y = i \cosh(\alpha x + K),$$

which does not vanish at any point of $\mathbb{R}$. In this case the function $u$ given by (4.19) is nonsingular, and in fact we recover the solution given by (4.5).

If, on the other hand, $C < 0$, then $\alpha = i\sqrt{|C|}$ is purely imaginary, so

$$\left|\frac{\alpha + V}{\alpha - V}\right| = 1.$$  

It then follows from (4.16) and (4.17) that $A$ is purely imaginary, and we can write

$$A = i(\sqrt{|C|}x + K),$$  

(4.21)

where $K$ is real. Then (4.18) gives

$$y = i \sin (\sqrt{|C|}x + K),$$

contradicting the fact that $y$ cannot have any zeroes on $\mathbb{R}$. We conclude that $C$ cannot be negative.
It remains to show that $\alpha$ and $C$ cannot equal zero. For if they were, then integrating (4.10) would give $v = (x + K)^{-1}$, where $K$ is a constant, and so by (4.12),

$$u = \frac{-2}{(x + K)^2}$$

for all $x \in I$. But this contradicts the fact that $u$ is analytic on all of $\mathbb{R}$.

We have now shown that if (4.3) has a solution in $L^2$ that is not identically zero, then $C$ must be positive, and in that case the only solutions are those given by (4.5). So the proof is complete.

5. The stationary equation for $N = 2$. According to Theorem 3.8 and Definition 3.12, every $N$-soliton solution of the KdV hierarchy has profiles which are solutions of the stationary equation (2.15), or more specifically of (3.38). In this section, for the case $N = 2$, we prove a converse to this result: every $H^2$ solution of (2.15) with $N = 2$ must be either a 1-soliton profile or a 2-soliton profile.

Taking $N = 2$ in (2.15), we obtain from (2.14) the equation

$$\frac{d_1}{2} - \frac{d_3}{4}(u) + \frac{d_5}{16}(u'' + 3u^2) - \frac{d_7}{64}(u'''' + 5u'^2 + 10uu'' + 10u^3) = 0.$$  \hspace{1cm} (5.1)

We may assume in what follows that $d_7 \neq 0$, for otherwise we are back in the case $N = 1$, which has already been handled in section 4. Dividing by $d_7$ if necessary, we can therefore take $d_7 = 1$ without losing generality. We may also henceforth assume that $d_1 = 0$, since a computation similar to that given in (4.2) shows that this must be the case if (5.1) has a solution $u$ in $H^2$.

Lemma 5.1. Suppose $u \in H^2$ is a solution of equation (5.1) in the sense of distributions. Then $u$ must be in $H^s$ for all $s \geq 0$, and $u$ is analytic on $\mathbb{R}$. In particular, we have

$$\lim_{x \to \infty} u(x) = \lim_{x \to \infty} u'(x) = \lim_{x \to \infty} u''(x) = \lim_{x \to \infty} u'''(x) = 0.$$ \hspace{1cm} (5.2)

Proof. Taking $d_1 = 0$ in (5.1) and solving for $u'''$, we obtain

$$u''' = au + bu^2 + cu^3 + d(u')^2 + eu'' + fuu'',$$

where $a, b, c, d, e, f$ are constants. Since $u \in H^2$, then all the terms on the right-hand side of (5.3) are in $L^2$, so $u''' \in L^2$ as well. Hence $u \in H^4$, and this already yields (5.2). It also implies that $u$ is a classical solution of (5.1), so by fundamental theorems of ordinary differential equations, $u$ is analytic. Finally, taking derivatives of (5.3) successively and applying an inductive argument yields that $u \in H^s$ for all $s \geq 0$.

Theorem 5.2. Suppose $d_1 = 0$, $d_7 = 1$, and $d_3$ and $d_5$ are arbitrary real numbers, and suppose $u \in H^2$ is a nontrivial (i.e., not identically zero) distribution solution of equation (5.1).

Then either

(i) $u$ is a 1-soliton profile given by

$$u = \psi(1)(x; a; \sqrt{C}),$$ \hspace{1cm} (5.4)

where $C$ is a positive root of the quadratic equation

$$z^2 - d_5z + d_3 = 0.$$ \hspace{1cm} (5.5)

and $a$ is a real number such that $a > 0$; or
(ii) $u$ is a 2-soliton profile given by
\[
u = \psi^{(2)}(x; a_1, a_2; \sqrt{C_1}, \sqrt{C_2}),
\]
where $C_1$ and $C_2$ are roots of equation (5.5) with $0 < C_1 < C_2$, and $a_1$ and $a_2$ are real numbers such that $a_2 < 0 < a_1$.

**Proof.** Suppose $u$ is a nontrivial distribution solution of (5.1) with $d_7 = 1$ and $d_1 = 0$. By Lemma 5.1 we may assume that $u$ is analytic on $\mathbb{R}$ and satisfies (5.2).

Following Chapter 12 of [8] we define, for $x \in \mathbb{R}$ and $\zeta \in \mathbb{C}$,
\[
\hat{R}(x, \zeta) = \hat{R}_0 + \hat{R}_1 \zeta + \hat{R}_2 \zeta^2,
\]
where
\[
\hat{R}_0 = d_3 R_1 + d_5 R_3 + d_7 R_5 = \frac{d_3}{2} - \frac{d_5}{4} u + \frac{1}{16} (u'' + 3u^2)
\]
\[
\hat{R}_1 = d_5 R_1 + d_7 R_3 = \frac{d_5}{2} - \frac{1}{4} u
\]
\[
\hat{R}_2 = d_7 R_1 = \frac{1}{2}.
\]
We claim that
\[
\hat{R}''' + 4u\hat{R}' + 2u'\hat{R} + 4\zeta \hat{R}' = 0.
\]
Indeed, substituting (5.7) into (5.9) and using Lemma 2.2, we find that the left side of (5.9) is equal to
\[
-4(d_3 R_1' + d_5 R_3' + d_7 R_5') + \zeta(d_5 Q_1 + d_7 Q_3 + 4d_3 R_1') + \zeta^2(d_7 Q_1 + 4d_5 R_1') + \zeta^3(4d_7 R_1'),
\]
where $Q_1 = R_1''' + 4uR_1' + 2u'R_1 + 4R_3'$ and $Q_3 = R_3''' + 4uR_3' + 2u'R_3 + 4R_5'$. But since $u$ is a solution of (5.1) and $d_1 = 0$, we have that $d_3 R_1 + d_5 R_3 + d_7 R_7 = 0$, so the first term in (5.10) vanishes, and $Q_1$ and $Q_3$ are zero by virtue of Lemma 2.2. Since $R_1' = 0$, this proves (5.9).

Multiplying (5.9) by $\hat{R}$ and integrating with respect to $x$ gives
\[
2\hat{R}'' \hat{R} - \hat{R}'^2 + 4(u + \zeta)\hat{R}^2 = P(\zeta),
\]
where $P(\zeta)$ is a polynomial in $\zeta$ with coefficients that are independent of $x$. From (2.14), (5.2), and (5.8) we see that
\[
\lim_{x \to \infty} \hat{R}_0 = \frac{d_3}{2}
\]
\[
\lim_{x \to \infty} \hat{R}_1 = \frac{d_5}{2}
\]
\[
\lim_{x \to \infty} \hat{R}_2 = \frac{d_7}{2} = 1/2.
\]
Also,
\[
\lim_{x \to \infty} \hat{R}_i(x) = \lim_{x \to \infty} \hat{R}_i'(x) = 0 \quad \text{for } i = 0, 1, 2.
\]
Therefore, taking the limit of (5.11) as $x \to \infty$, we get that
\[
P(\zeta) = \zeta(d_3 + d_5 \zeta + \zeta^2)^2.
\]
Combining (5.11) and (5.13) gives
\[
2\hat{R}'' \hat{R} - \hat{R}'^2 + 4(u + \zeta)\hat{R}^2 = \zeta(d_3 + d_5 \zeta + \zeta^2)^2.
\]
Let $C_1$ and $C_2$ denote the (possibly repeated) roots of equation (5.5). As roots of a polynomial with real coefficients, $C_1$ and $C_2$ are either both real numbers or are
complex conjugates of each other, and we may assume they are ordered so that 
\( \Re C_1 \leq \Re C_2 \). Then

\[
d_3 = C_1 C_2 \quad \text{and} \quad d_5 = C_1 + C_2, \tag{5.15}
\]

and so, by (5.14),

\[
2 \hat{R}'' \hat{R} - \hat{R}^2 + 4(u + \zeta) \hat{R}^2 = \zeta(\zeta + C_1)^2(\zeta + C_2)^2. \tag{5.16}
\]

Let us now view the function \( \hat{R}(x, \zeta) \) as a polynomial in the complex variable \( \zeta \) with coefficients which are analytic functions of \( x \). Our next goal is to study the roots of this polynomial.

First, observe that since \( \hat{R}_0 \) is, like \( u \), analytic on \( \mathbb{R} \), then \( \hat{R}_0 \) is either identically zero on \( \mathbb{R} \) or has only isolated zeros. But if \( \hat{R}_0 \) is identically zero on \( \mathbb{R} \), then by (5.12) we must have \( d_3 = 0 \). The equation \( \hat{R}_0 = 0 \) in (5.8) is then seen to take the form of (4.3), with \( C \) replaced by \( d_5 \), and so it follows from Theorem 4.2 that \( d_5 > 0 \) and \( u \) is given by (5.4). Notice also that since \( d_4 = 0, d_5 \) is a positive root of (5.5). Thus statement (i) of the Theorem holds in this case. Therefore, we can, without loss of generality, assume that \( \hat{R}_0 \) has only isolated zeros, and hence there exists an open interval \( I \subseteq \mathbb{R} \) such that \( \hat{R}_0(x) \neq 0 \) for all \( x \in I \). It then follows that for all \( x \in I \), \( \zeta = 0 \) is not a root of \( \hat{R}(x, \zeta) \).

We claim that there exists at least one \( x_0 \in I \) such that the polynomial \( \hat{R}(x_0, \zeta) \) has distinct roots \( \zeta_1 \) and \( \zeta_2 \). For if this is not the case, then there exists a function \( \zeta_1(x) \) such that for all \( x \in I \),

\[
\hat{R}(x, \zeta) = \frac{1}{2}(\zeta - \zeta_1(x))^2. \tag{5.17}
\]

From (5.17) and (5.7) we have that \( \zeta_1(x)^2 = 2\hat{R}_0(x) \), and since \( \hat{R}_0(x) \) is nonzero for all \( x \in I \) it follows that \( \zeta_1(x) \) is analytic, and hence differentiable, as a function of \( x \). Thus we can differentiate (5.17) with respect to \( x \) to obtain

\[
\hat{R}'(x, \zeta) = -(\zeta - \zeta_1(x))\zeta_1'(x)
\]

for \( x \in I \). But then substituting \( \zeta = \zeta_1(x) \) into (5.16) gives

\[
0 = \zeta_1(x)(\zeta_1(x) + C_1)^2(\zeta_1(x) + C_2)^2.
\]

Since \( \zeta_1(x) \neq 0 \) for \( x \in I \), it follows that for all \( x \in I \), either \( \zeta_1(x) = -C_1 \) or \( \zeta_1(x) = -C_2 \). Since \( \zeta_1(x) \) is analytic on \( I \), the set of points where \( \zeta_1(x) \) takes a given value must either be a discrete subset of \( I \), or consist of all of \( I \). Since the union of two discrete subsets of \( I \) cannot equal all of \( I \), it must be that either 
\( \zeta_1(x) \equiv -C_1 \) on \( I \) or \( \zeta_1(x) \equiv -C_2 \) on \( I \). Hence (5.17) gives, for either \( j = 1 \) or \( j = 2 \),

\[
\hat{R}(x, \zeta) = \frac{1}{2}(\zeta^2 + 2C_j \zeta + C_j^2),
\]

and so, by (5.7),

\[
\hat{R}_1 = \frac{d_5}{2} - \frac{1}{4}u = C_j
\]

holds for all \( x \in I \). But this implies that \( u \) is constant on \( I \), and since \( u \) is analytic on \( \mathbb{R} \), then \( u \) must be constant on \( \mathbb{R} \). Then (5.2) gives that \( u \) is identically zero, contrary to our assumption that \( u \) is nontrivial. Thus the claim has been proved.

It now follows from standard perturbation theory [22] that, by shrinking \( I \) if necessary to a smaller neighborhood, we can assume that there exist analytic, nonzero
functions $\zeta_1$ and $\zeta_2$ on $I$ such that $\zeta_1(x) \neq \zeta_2(x)$ and $\hat{R}(x, \zeta_1(x)) = \hat{R}(x, \zeta_2(x)) = 0$ for all $x \in I$. We therefore have

$$\hat{R}(x, \zeta) = \frac{1}{2}(\zeta - \zeta_1(x))(\zeta - \zeta_2(x)), \quad (5.18)$$

for all $x \in I$. Also, since $\zeta_1$ and $\zeta_2$ are roots of a real polynomial, we have that either $\zeta_1$ and $\zeta_2$ are both real on $I$ or $\zeta_1^* = \zeta_2$ on $I$.

Our goal in what follows is to obtain a second-order system of differential equations for $\zeta_1$ and $\zeta_2$, which can then be integrated explicitly to find $\zeta_1$ and $\zeta_2$. Once this is accomplished, it is easy to recover $u$, since (5.7) and (5.18) imply that $\zeta_1 + \zeta_2 = -2\hat{R}_1$, and hence

$$u = 2(\zeta_1 + \zeta_2 + d_5). \quad (5.19)$$

For $i = 1, 2$, we have $\hat{R}(x, \zeta_i(x)) = 0$, and hence it follows from (5.16) that

$$\hat{R}'(x, \zeta_i(x))^2 = -\zeta_i(\zeta_i + C_1)^2(\zeta_i + C_2)^2. \quad (5.20)$$

Differentiating (5.18) with respect to $x$, we obtain

$$\hat{R}'(x, \zeta) = -\frac{1}{2}[(\zeta - \zeta_1)\zeta_1'(x) + (\zeta - \zeta_2)\zeta_2'(x)].$$

Therefore

$$\hat{R}'(x, \zeta_1(x)) = -\frac{1}{2}(\zeta_1(x) - \zeta_2(x))\zeta_1'(x)$$

$$\hat{R}'(x, \zeta_2(x)) = -\frac{1}{2}(\zeta_2(x) - \zeta_1(x))\zeta_2'(x).$$

From (5.20) we then have that

$$(\zeta_1 - \zeta_2)^2(\zeta_1')^2 = -4\zeta_1(\zeta_1 + C_1)^2(\zeta_1 + C_2)^2$$

$$(\zeta_1 - \zeta_2)^2(\zeta_2')^2 = -4\zeta_2(\zeta_2 + C_1)^2(\zeta_2 + C_2)^2. \quad (5.21)$$

Note that, since $C_1$ and $C_2$ are either both real numbers, or are complex conjugates of one another, it follows from (5.21) that if $\zeta_1$ and $\zeta_2$ are real, they must necessarily take negative values at all points of $I$. Since $\zeta_1$ and $\zeta_2$ are nonzero functions on $I$, it follows that both $\zeta_1$ and $\zeta_2$ map $I$ into the domain $\Omega$ in the complex plane defined in (4.8). Also, since $\zeta_1 - \zeta_2$ has no zeros in $I$, we have from (5.21) that

$$\zeta_1' = \frac{2i\theta_1(x)\sqrt{\zeta_1(\zeta_1 + C_1)(\zeta_1 + C_2)}}{\zeta_1 - \zeta_2}$$

$$\zeta_2' = \frac{2i\theta_2(x)\sqrt{\zeta_2(\zeta_2 + C_1)(\zeta_2 + C_2)}}{\zeta_1 - \zeta_2}, \quad (5.22)$$

where $\theta_i(x) \in \{-1, 1\}$ for $i = 1, 2$. Here, as throughout the paper, we use $\sqrt{\zeta}$ to denote the analytic branch of the square root function on $\Omega$ defined after (4.8).

The change of variables from $u$ to $(\zeta_1, \zeta_2)$, which reduces the stationary equation (5.1) to the separable system (5.22), is due originally to Dubrovin in [11] (see also [13, 21, 31], and chapter 12 of [8]). These authors use the same change of variables (or, more precisely, its generalization to the case of general $N$) to, among other things, determine the time evolution of finite-gap solutions of the Korteweg-de Vries hierarchy.
Again using the analyticity of $\zeta_1$ and $\zeta_2$, and taking $I$ smaller if necessary, we can reduce consideration to the following two cases: either there exist $i, j \in \{1, 2\}$ such that
\[
\zeta_i(x) + C_j = 0 \quad \text{for all } x \in I, \tag{5.23}
\]
or, for all $i, j \in \{1, 2\}$,
\[
\zeta_i(x) + C_j \neq 0 \quad \text{for all } x \in I. \tag{5.24}
\]

Suppose (5.23) holds, with for example $i = 1$ and $j = 1$; the argument for other choices of $i$ and $j$ is exactly similar. Then from (5.22) we obtain
\[
\zeta'_i = -2i\theta_2(x)\sqrt{\zeta_2}(\zeta_2 + C_2). \tag{5.25}
\]
We know that $\zeta_2$ is not identically equal to $-C_2$ on $I$, for otherwise (5.15), (5.19), and $\zeta_1 \equiv -C_1$ would imply that $u$ is identically equal to 0 on $I$, and hence also on $\mathbb{R}$. Therefore, by taking $I$ smaller if necessary, we may assume that $\zeta_2$ is never equal to $-C_2$ on $I$.

It then follows from (5.25) that $\theta_2(x)$ is analytic on $I$. But since $\theta_2$ takes values in $\{-1, 1\}$, the only way this can happen is if $\theta_2$ is constant on $I$. Setting $v = i\theta_2\sqrt{\zeta_2}$ in (5.25), we obtain
\[
\frac{v'}{C_2 - v^2} = 1,
\]
which is equation (4.10) with $C$ replaced by $C_2$.

Moreover, we also know that $C_1$ and $C_2$ must be real, since otherwise we would have $C_2 = C_1^*$, and together with $\zeta_1 \equiv -C_1$ and $\zeta_1 = \zeta_2^*$ this would imply $\zeta_2 = -C_2$ on $I$. Since $C_1$ is real, then $\zeta_1 \equiv -C_1$ implies that $\zeta_1$ is real, so $\zeta_2$ must also be real on $I$. Since we also know that $\zeta_2$ is never equal to zero on $I$, it follows from (5.21) that $\zeta_2 < 0$ on $I$. Hence $v$ is real-valued on $I$. Therefore we can reprise the proof of Theorem 4.2 from (4.10) onwards, replacing $C$ by $C_2$ throughout (notice that by (5.19) we have $u = 2(\zeta_2 + C_2) = 2(C_2 - v^2)$ in this case as well). The conclusion is that $C_2 > 0$ and that
\[
u = \frac{2C_2}{\cosh^2(\sqrt{\zeta_2}x + K)}.
\]

In light of the remark following Theorem 4.2, we obtain the formula (5.4) for $u$, and this must hold on the entire line. Thus the statement of the theorem is proved in this case.

Therefore we may assume henceforth that (5.24) holds for all $i, j \in \{1, 2\}$. In this case, the right-hand sides of both equations in (5.22) are never zero on $I$, so as in the preceding cases it follows from (5.22) that $\theta_1$ and $\theta_2$ are both analytic and hence constant functions on $I$, with value either $-1$ or $1$. Set
\[
\begin{align*}
v_1 &= -i\theta_1\sqrt{\zeta_1} \\
v_2 &= i\theta_2\sqrt{\zeta_2}, \tag{5.26}
\end{align*}
\]
and define $\alpha_j$, for $j = 1, 2$, to be complex numbers such that
\[
\alpha_j^2 = C_j. \tag{5.27}
\]
For definiteness we will choose $\alpha_j$ to be the square root of $C_j$ given by $\alpha_j = |C_j|^{1/2}e^{i\theta/2}$, where $-\pi < \theta \leq \pi$ and $C_j = |C_j|e^{i\theta}$. In particular, this choice guarantees that if $C_1^* = C_2$, then $\alpha_1^* = \alpha_2$.

Since $\zeta_1$ and $\zeta_2$ are nonzero on $I$, so are $v_1$ and $v_2$. Also, as noted above after (5.21), either $\zeta_1$ and $\zeta_2$ are both negative at all points of $I$, or $\zeta_1^* = \zeta_2$ on $I$. In the former case, we have that $v_1$ and $v_2$ are real-valued on $I$. In the latter case, we
see by taking the conjugate of the first equation in (5.22), comparing the result to the second equation in (5.22), and using the fact that $\sqrt{\overline{z}} = -\overline{\sqrt{z}}^*$ on $\Omega$, that $\theta_1 = -\theta_2$ on $I$. Therefore from (5.26) we obtain that $v_1^* = v_2$ on $I$.

We can now rewrite (5.22) as the following system for $v_1$ and $v_2$:

$$
\begin{align*}
v_1' &= (\alpha_1^2 - v_1^2)(\alpha_2^2 - v_1^2)\frac{v_2 - v_1}{v_2^2 - v_1^2}, \\
v_2' &= (\alpha_1^2 - v_2^2)(\alpha_2^2 - v_2^2)\frac{v_2 - v_2}{v_2^2 - v_2^2},
\end{align*}
$$

(5.28)

where either $v_1$ and $v_2$ are both real-valued on $I$, or $v_1^* = v_2$ on $I$. Choose $x_0 \in I$, and define

$$
\begin{align*}
V_1 &= v_1(x_0) \\
V_2 &= v_2(x_0).
\end{align*}
$$

(5.29)

(For future reference we note that, as values of $v_1$ and $v_2$, $V_1$ and $V_2$ must both be nonzero, and either $V_1$ and $V_2$ both real or $V_1^* = V_2$.) Recalling that $v_1^2 \neq v_2^2$ on $I$, we have that the right-hand sides of the equations in (5.28) define analytic functions of $v_1$ and $v_2$ on $I$. Therefore the system (5.28), together with the initial data (5.29), uniquely determines $v_1$ and $v_2$ on some neighborhood of $x_0$ on $I$. Furthermore, from $v_1$ and $v_2$ one can then recover $u$ via (5.19) as

$$
u(x) = 2(-v_1^2(x) - v_2^2(x) + \alpha_1^2 + \alpha_2^2).
$$

(5.30)

We will complete the proof of Theorem 5.2 by explicitly solving the initial-value problem (5.28) and (5.29) for $v_1$ and $v_2$, and then showing that, of the functions $u$ which arise from these solutions via (5.30), the only ones which extend to $H^1$ functions on $\mathbb{R}$ are those given by (5.6).

The system (5.28) can be integrated by separating the variables $v_1$ and $v_2$. Since (5.24) holds on $I$, we have that $v_i(x) \neq \alpha_j$ on $I$ for $i, j \in \{1, 2\}$. This allows us to rewrite (5.28) in the form

$$
\begin{align*}
\frac{v_1'}{(\alpha_1^2 - v_1^2)(\alpha_2^2 - v_1^2)} + \frac{v_2'}{(\alpha_1^2 - v_2^2)(\alpha_2^2 - v_2^2)} &= 0 \\
\frac{-v_1^2v_1'}{(\alpha_1^2 - v_1^2)(\alpha_2^2 - v_1^2)} + \frac{-v_2^2v_2'}{(\alpha_1^2 - v_2^2)(\alpha_2^2 - v_2^2)} &= 1.
\end{align*}
$$

(5.31)

To compute the solutions of (5.31) in the cases when the quantities $\alpha_i^2$ coincide or are zero, it will be helpful to consider as well a system in which the values of the constants $\alpha_i$ are slightly perturbed. For any positive number $\delta$ and complex number $z_0$, let $B_\delta(z_0)$ denote the open ball of radius $\delta$ centered at $z_0$ in $\mathbb{C}$. Choose $\delta$ to be any positive number such that $\delta < \frac{1}{\min\{|\alpha_i - V_j| : i, j = 1, 2\}}$ and $\delta < |V_1 - V_2|$. For each $j \in \{1, 2\}$, we define functions $G_j = G_j(\beta_1, \beta_2, v)$ and $H_j = H_j(\beta_1, \beta_2, v)$ on $B_\delta(\alpha_1) \times B_\delta(\alpha_2) \times B_\delta(V_j)$ by

$$
\begin{align*}
G_j(\beta_1, \beta_2, v) &= \int_{V_j} dw \frac{dw}{(\beta_1^2 - w^2)(\beta_2^2 - w^2)}, \\
H_j(\beta_1, \beta_2, v) &= \int_{V_j} -w^2 dw \frac{dw}{(\beta_1^2 - w^2)(\beta_2^2 - w^2)}.
\end{align*}
$$

(5.32)

From the definition of $\delta$ we know that the integrands in (5.32) are nonsingular functions of $w$ on $B_\delta(V_j)$, so $G_j$ and $H_j$ are well-defined and are analytic functions of $\beta_1$, $\beta_2$, and $v$ on their domains, as long as the integrals in their definition are
take over paths from $V_j$ to $v$ which lie within $B_δ(V_j)$. We can then define $F : C^2 \times C^2 \times \mathbb{R} \to C^2$, with domain $U_F = B_δ(α_1) \times B_δ(α_2) \times B_δ(V_1) \times B_δ(V_2) \times \mathbb{R}$, by setting

$$F(β_1, β_2, v_1, v_2, x) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} G_1(β_1, β_2, v_1) + G_2(β_1, β_2, v_2) \\ H_1(β_1, β_2, v_1) + H_2(β_1, β_2, v_2) - x + x_0 \end{bmatrix}. \quad (5.33)$$

**Lemma 5.3.** There exist numbers $δ_1 \in (0, δ)$ and $δ_2 > 0$ such that for each $(β_1, β_2, x)$ in $B_δ(α_1) \times B_δ(α_2) \times \{ |x - x_0| < δ_1 \} \subseteq C^2 \times \mathbb{R}$, there is a unique pair $(v_1, v_2)$ in $B_δ(V_1) \times B_δ(V_2) \subseteq C^2$ satisfying

$$F(β_1, β_2, v_1, v_2, x) = 0. \quad (5.34)$$

The functions $v_1(β_1, β_2, x)$ and $v_2(β_1, β_2, x)$ so defined are analytic functions of their arguments, and for each $(β_1, β_2) ∈ B_δ(α_1) \times B_δ(α_2)$, are solutions of the system of ordinary differential equations

$$v_1' = \frac{v_1}{(β^2_1 - v_1^2)(β^2_2 - v_2^2)} + \frac{v_2}{(β^2_2 - v_2^2)(β^2_2 - v_2^2)} = 0 \quad v_2' = \frac{-v_1 v_2'}{(β^2_1 - v_1^2)(β^2_2 - v_2^2)} + \frac{-v_2 v_2'}{(β^2_2 - v_2^2)(β^2_2 - v_2^2)} = 1, \quad (5.35)$$

on $\{ |x - x_0| < δ_1 \}$, with initial conditions

$$v_1(β_1, β_2, x_0) = V_1 \quad v_2(β_1, β_2, x_0) = V_2. \quad (5.36)$$

**Proof.** A computation of the Jacobian of $F$ with respect to the variables $v_1$ and $v_2$, reveals that, at all points $P = (β_1, β_2, v_1, v_2, x)$ in the domain $U_F$ of $F$, we have

$$\nabla_{v_1, v_2} F(P) = \begin{bmatrix} \frac{∂F_1}{∂v_1} & \frac{∂F_1}{∂v_2} \\ \frac{∂F_2}{∂v_1} & \frac{∂F_2}{∂v_2} \end{bmatrix} \quad (5.37)$$

The determinant of the matrix in (5.37) is $(v_1^2 - v_2^2) / \prod_{i,j=1,2} (β_i^2 - v_j^2)$, and is therefore nonzero for all $P ∈ U_F$. In particular, when $P_0 = (α_1, α_2, V_1, V_2, x_0)$ we have that $\nabla_{v_1, v_2} F(P_0)$ is an invertible map from $C^2$ to $C^2$. Moreover, $F(P_0) = 0$. The assertions of the Lemma concerning the existence, uniqueness, and analyticity of the functions $v_1$ and $v_2$ which satisfy (5.34) therefore follow from the Implicit Function Theorem (cf. §15 of [7]). Equations (5.35) then follow by differentiating both sides of (5.34) with respect to $x$. The initial conditions (5.36) are a consequence of the uniqueness assertion for the $v_i$ and the fact that

$$F(β_1, β_2, V_1, V_2, x_0) = 0$$

for each $(β_1, β_2) ∈ B_δ(α_1) \times B_δ(α_2)$. 

Motivated by (5.30), we now define, for each $(β_1, β_2, x) ∈ B_δ(α_1) \times B_δ(α_2) \times \{ |x - x_0| < δ_1 \}$,

$$u(β_1, β_2, x) = 2(-v_1(β_1, β_2, x)^2 - v_2(β_1, β_2, x)^2 + β_1^2 + β_2^2). \quad (5.38)$$
Suppose have the following Lemma.

Proof. By Lemma 5.3, the functions $u$ for where $\beta x u$ functions $\beta$ analytic and hence continuous in $u$. Then the solution $u$ of the initial value problem for (5.28) that $\beta\alpha_1,\alpha_2,\beta$ for $x-x_0<\delta_1$. Finally, the assertion that $u(x)$ is given by the limit in (5.39) follows from the fact that $u(\beta_1,\beta_2,x)$ is analytic and hence continuous in $\beta_1$ and $\beta_2$.

The next step in the proof of Theorem 5.2 is to explicitly determine the functions $u(\beta_1,\beta_2,x)$ defined in (5.38). Generically, we will have that $\beta_1^2$ and $\beta_2^2$ are distinct and nonzero, even if this is not true when $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$. In this case, we have the following Lemma.

Lemma 5.5. Suppose $(\beta_1,\beta_2,x) \in B_{\delta_1}(\alpha_1) \times B_{\delta_1}(\alpha_2) \times \{|x-x_0|<\delta_1\}$, and suppose that $\beta_1^2 \neq \beta_2^2$, and $\beta_j \neq 0$ for $j=1,2$. Then

$$u(\beta_1,\beta_2,x) = 2(D'/D)',$$

where

$$D = D(\beta_1,\beta_2,x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

with

$$y_j = \sinh(\beta_j(x-x_0)+M_j)$$

for $j=1,2$. Here $M_1$ and $M_2$ can be taken to be any complex numbers satisfying

$$e^{2M_j} = \frac{(\beta_1+V_1)(\beta_j+V_2)}{(\beta_j-V_1)(\beta_j-V_2)}$$

for $j=1,2$.

Proof. Under the stated assumptions on $(\beta_1,\beta_2,x)$, we have, for $v \in B_\delta(V_j)$,

$$G_j(\beta_1,\beta_2,v) = \frac{1}{\beta_2^2 - \beta_1^2} \left[ L_{\beta_1,v_j}(v) - L_{\beta_2,v_j}(v) \right]$$

$$H_j(\beta_1,\beta_2,z) = \frac{1}{\beta_2^2 - \beta_1^2} \left[ -\beta_1^2 L_{\beta_1,v_j}(v) + \beta_2^2 L_{\beta_2,v_j}(v) \right],$$

where $L_{\beta,v}$ is defined in (4.13). Let $v_j$ denote $v_j(\beta_1,\beta_2,x)$ for $j=1,2$, and define

$$w_j = 2\beta_j L_{\beta_1,v_j}(v_1) + 2\beta_j L_{\beta_2,v_j}(v_2).$$

Substituting (5.44) into (5.33), we see that (5.34) can be rewritten as a linear system for $w_1$ and $w_2$:

$$\beta_2 w_1 - \beta_1 w_2 = 0$$

$$-\beta_1 w_1 + \beta_2 w_2 = 2(\beta_2^2 - \beta_1^2)(x-x_0).$$
Since $\beta_1 \neq \beta_2$, the system has a unique solution, given by
\[ w_j = 2\beta_j(x - x_0) \quad \text{for } j = 1, 2. \quad (5.46) \]

Now from (4.14) and (5.45) we have that
\[ e^{w_j} = \left( \frac{\beta_j + v_1}{\beta_j - v_1} \right) \left( \frac{\beta_j + v_2}{\beta_j - v_2} \right) \left( \frac{\beta_j - V_1}{\beta_j + V_1} \right) \left( \frac{\beta_j - V_2}{\beta_j + V_2} \right). \]
Therefore after exponentiating both sides of (5.46), we obtain, for $j = 1, 2$,
\[ \frac{(\beta_j + v_1)(\beta_j + v_2)}{(\beta_j - v_1)(\beta_j - v_2)} = \frac{(\beta_j + V_1)(\beta_j + V_2)}{(\beta_j - V_1)(\beta_j - V_2)} e^{2\beta_j(x - x_0)} = e^{2(\beta_j(x - x_0) + M_j)}, \quad (5.47) \]
where $M_1$ and $M_2$ are any complex numbers such that (5.43) holds.

If we now define
\[ W_1 = -(v_1 + v_2), \]
\[ W_0 = v_1 v_2, \]
then (5.47) can be rewritten as
\[ \begin{bmatrix} y_1' & y_2' \\ y_2' & y_2' \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \end{bmatrix} = -\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix}, \quad (5.48) \]
where $y_j$ is given by (5.42) for $j = 1, 2$. Solving (5.48) by Cramer’s rule, we find that $W_1$ and $W_0$ are given by (3.6) and (3.7), with $N = 2$. Therefore we can conclude from Corollary 3.4 that
\[ W_1' = W_1^2 - 2W_0 - \beta_1^2 - \beta_2^2. \quad (5.49) \]

On the other hand, from (5.38) we have
\[ u(\beta_1, \beta_2, x) = 2(2W_0 - W_1^2 + \beta_1^2 + \beta_2^2), \quad (5.50) \]
and from (5.49) and (5.50) it then follows that $u(\beta_1, \beta_2, x) = -2W_1'$. By (3.6), we have $W_1' = -D'/D$, with $D$ given by (5.41). Then (5.40) follows.

We now determine the conditions on $\beta_1$ and $\beta_2$ under which (5.40) defines a nonsingular solution on $\mathbf{R}$, or in other words, under which $D(\beta_1, \beta_2, x)$ has no zeroes on $\mathbf{R}$.

**Lemma 5.6.** Suppose $(\beta_1, \beta_2) \in B_{\delta_1}(\alpha_1) \times B_{\delta_2}(\alpha_2)$, and suppose that $\beta_1^2 \neq \beta_2^2$, and $\beta_j \neq 0$ for $j = 1, 2$. Suppose also that $0 \leq \Re \beta_1 \leq \Re \beta_2$, and either $\beta_1^2$ and $\beta_2^2$ are both real numbers, or $\beta_1 = \beta_2$.

If, as a function of $x$, $u(\beta_1, \beta_2, x)$ can be analytically continued to an analytic function on the entire real line, then $0 < \beta_1 < \beta_2$, and there exist numbers $a_1$ and $a_2$ with $a_2 < 0 < a_1$ such that
\[ u(\beta_1, \beta_2, x) = \psi(x; a_1, a_2; \beta_1, \beta_2) \]
on $\mathbf{R}$.

**Proof.** Observe that, according to Lemma 5.5, if $u(\beta_1, \beta_2, x)$ can be analytically continued to an analytic function on the real line, then the function $D(\beta_1, \beta_2, x)$ defined in (5.41)–(5.43) must be nonzero for all $x \in \mathbf{R}$. In fact, from (5.40) it is easy to see that, at any point $x$ where $D(\beta_1, \beta_2, x)$ has a zero, $u(\beta_1, \beta_2, x)$ will have a pole of order two.

If $\beta_1^* = \beta_2$, then since either $V_1$ and $V_2$ are both real or $V_1^* = V_2$, we see from (5.43) that we can choose $M_1$ and $M_2$ so that $M_1^* = M_2$. Define $K_j = -\beta_j x_0 + M_j$ and
\[ A_j = \beta_j(x - x_0) + M_j = \beta_j x + K_j, \]
so that \( A_1^* = A_2 \) and, by (5.42), \( y_j = \sinh A_j \) for \( j = 1, 2 \). Let \( a = \Re \alpha_1 \), \( b = \Im \alpha_1 \), \( P = \Re \alpha_1 \), and \( Q = \Im \alpha_1 \). Then from (5.41), we have

\[
D = \beta_2 \sinh(A_1) \cosh(A_2) - \beta_1 \cosh(A_1) \sinh(A_2)
\]

\[
= \beta_1^* \sinh(A_1^*) \cosh(A_2^*) - \beta_1 \cosh(A_1) \sinh(A_2^*)
\]

\[
= 2 \Im (\beta_1^* \sinh(A_1^*) \cosh(A_1^*))
\]

\[
= a \sin(2(bx + Q)) - b \sin(2(ax + P)).
\]

Therefore \( D \) changes sign as \( x \) goes from large negative values to large positive values, and hence \( D \) must equal zero for some \( x \in \mathbb{R} \). Therefore, as remarked above, \( u(\beta_1, \beta_2, x) \) cannot be continued to an analytic function on \( \mathbb{R} \).

Next, suppose \( \beta_1^2 \) and \( \beta_2^2 \) are real and of opposite sign, say \( \beta_1^2 < 0 < \beta_2^2 \). In this case \( \beta_1 \) is purely imaginary, say \( \beta_1 = ib \), and \( \beta_2 \) is real. Then when \( M_1 \) and \( M_2 \) are chosen so that (5.43) holds, it follows that \( (e^{2M_1})^* = e^{-2M_1} \) and \( (e^{2M_2})^* = e^{2M_2} \). We can thus take \( M_1 \) to be purely imaginary, and \( M_2 \) so that either \( M_2 \) is real or \( \Im M_2 = \pi/2 \). Now the same arguments as in equations (4.20) to (4.21) in Section 4 show that

\[
y_1 = i \sin(bx + K_1) \quad (5.51)
\]

for some real \( K_1 \), and either

\[
y_2 = \sinh(\beta_2 x + K_2) \quad (5.52)
\]

or

\[
y_2 = i \cosh(\beta_2 x + K_2) \quad (5.53)
\]

for some real \( K_2 \). If (5.51) and (5.52) hold, the equation \( D = 0 \) becomes

\[
\frac{\tan(bx + K_1)}{b} = \frac{\tanh(\beta_2 x + K_2)}{\beta_2},
\]

which has infinitely many solutions on \( \mathbb{R} \). If, on the other hand, (5.51) and (5.53) hold, then the equation \( D = 0 \) becomes

\[
\frac{\tan(bx + K_1)}{b} = \frac{\coth(\beta_2 x + K_2)}{\beta_2},
\]

which again has infinitely many solutions.

Now suppose that \( \beta_1^2 \) and \( \beta_2^2 \) are both negative, say \( \beta_1^2 < \beta_2^2 < 0 \). Then \( \beta_1 \) and \( \beta_2 \) are both purely imaginary, say \( \beta_j = ib_j \) for \( j = 1, 2 \), and in (5.43) we can choose \( M_1 \) and \( M_2 \) to be purely imaginary. Thus

\[
y_j = i \sin(b_j x + K_j)
\]

on \( I \) for \( j = 1, 2 \), where \( K_1 \) and \( K_2 \) are real, so \( D \) has a zero at any point \( x \in \mathbb{R} \) which satisfies the equation

\[
\frac{\tan(b_1 x + K_1)}{b_1} = \frac{\tan(b_2 x + K_2)}{b_2}. \quad (5.54)
\]

Since \( \beta_1^2 \neq \beta_2^2 \), then \( b_1 \neq b_2 \), in which case it is easy to see that equation (5.54) always has solutions.

We have now shown that, under the stated assumptions on \( \beta_1 \) and \( \beta_2 \), \( u(\beta_1, \beta_2, x) \) can be continued to an analytic function on \( \mathbb{R} \) only if \( \beta_1^2 \) and \( \beta_2^2 \) are both positive, with therefore \( 0 < \beta_1 < \beta_2 \). In this case, when \( M_1 \) and \( M_2 \) are chosen to satisfy (5.43), we will have that \( (e^{2M_1})^* = e^{-2M_1} \) for \( j = 1, 2 \). Then, as in (5.52) and (5.53), for each \( j \) we have that either

\[
y_j = \sinh(\beta_j x + K_j)
\]
or
\[ y_j = i \cosh(\beta_j x + K_j), \]
where \( K_1 \) and \( K_2 \) are real. There are therefore four cases to consider, which, according to Definition 3.6, can be summarized as follows:
\[ u = \psi^{(2)}(x; a_1, a_2; \beta_1, \beta_2), \]
where \( a_j \) are nonzero numbers given by \( a_j = \pm e^{-2K_j} \) for \( j = 1, 2 \).

If \( a_1 \) and \( a_2 \) are both positive, then \( D = 0 \) holds at any point \( x \) where
\[ \beta_2 \tanh(\beta_2 x + K_2) - \beta_1 \tanh(\beta_1 x + K_1) = 0, \]
and since the left-hand side of (5.55) changes sign from negative to positive as \( x \) increases from large negative values to large positive values, there must exist solutions to (5.55) in \( \mathbb{R} \). Similarly, if \( a_1 \) and \( a_2 \) are both negative, then \( D = 0 \) when
\[ \beta_1 \tanh(\beta_2 x + K_2) - \beta_2 \tanh(\beta_1 x + K_1) = 0, \]
which again must have at least one solution in \( \mathbb{R} \). If \( a_1 < 0 < a_2 \), then \( D = 0 \) when
\[ \tanh(\beta_1 x + K_1) \tanh(\beta_2 x + K_2) = \frac{\beta_1}{\beta_2}, \]
which must have a solution, since the fraction on the right-hand side is between 0 and 1, and the function on the left-hand side attains the value zero at the points \( x = -K_j/\beta_j, j = 1, 2 \), and approaches 1 as \( x \to \pm \infty \).

Finally, if \( a_2 < 0 < a_1 \), then \( D = 0 \) only if
\[ \tanh(\beta_1 x + K_1) \tanh(\beta_2 x + K_2) = \frac{\beta_2}{\beta_1}. \]
But (5.56) has no solutions, since the fraction on the right-hand side is greater than 1, and the function on the left-hand side always takes values less than 1. (Alternatively, in this final case we could deduce from Lemma 3.10 that \( D \) is never 0 on \( \mathbb{R} \).)

5.1. The nondegenerate case. In this subsection we consider the case in which the roots \( C_1 \) and \( C_2 \) of (5.5) are nondegenerate: that is, when \( C_1 \neq C_2 \), \( C_1 \neq 0 \), and \( C_2 \neq 0 \). Recall that, as mentioned following equation (5.14), we can assume that either \( C_1 \) and \( C_2 \) are both real, with \( C_1 < C_2 \), or \( C_1^* = C_2 \). From our definition of \( \alpha_j \) in (5.27) and the remarks following, we have then that \( \alpha_1 \) and \( \alpha_2 \) are distinct and both nonzero, with \( 0 \leq \Re \alpha_1 \leq \Re \alpha_2 \); and either \( \alpha_1^2 \) and \( \alpha_2^2 \) are both real, with \( \alpha_1^2 < \alpha_2^2 \), or \( \alpha_1^* = \alpha_2 \).

According to Corollary 5.4, \( u(x) = u(\alpha_1, \alpha_2, x) \) on some neighborhood of \( x_0 \), and therefore \( u(\alpha_1, \alpha_2, x) \) can be analytically continued to a function which is analytic on all of \( \mathbb{R} \). It then follows from Lemma 5.6 that \( 0 < \alpha_1 < \alpha_2 \), and
\[ u(x) = \psi^{(2)}(x; a_1, a_2; \alpha_1, \alpha_2) \]
for some numbers \( a_1 \) and \( a_2 \) with \( 0 < a_1 < a_2 \).

We have therefore proved Theorem 5.2 in the nondegenerate case, by showing that the only possible solutions in this case are given by (5.6).

To complete the proof of Theorem 5.2, it remains to show that the degenerate cases, when \( C_1 \) and \( C_2 \) can coincide or vanish, cannot arise under the assumption that (5.1) has a nontrivial solution \( u(x) \) in \( H^2(\mathbb{R}) \). Since any \( H^2 \) solution must be analytic on \( \mathbb{R} \), to accomplish this it is enough to show that in the degenerate cases, any locally analytic solution of (5.1) extends analytically to a function with a singularity on \( \mathbb{R} \).
5.2. The degenerate case $C_1 = C_2 \neq 0$. In this subsection we consider the case when equation (5.5) has a nonzero double root, so that $C_1 = C_2 = C \neq 0$ in (5.31). In this case, $C$ must be real. Define $\alpha$ to be a square root of $C$, following the convention for choice of square roots set after (5.27). From Corollary 5.4 we have that

$$u(x) = u(\alpha, \alpha, x) = \lim_{\epsilon \to 0} u(\alpha, \alpha + \epsilon, x) \tag{5.57}$$

for $|x - x_0| < \delta_1$.

We now compute the limit in (5.57). Since $\alpha \neq 0$, by taking $\epsilon$ positive and sufficiently small, we may assume that $\beta_1 = \alpha$ and $\beta_2 = \alpha + \epsilon$ satisfy the hypotheses of Lemma 5.5. We thus obtain that, for $|x - x_0| < \delta_1$,

$$u(\alpha, \alpha + \epsilon, x) = 2(D(\alpha, \alpha + \epsilon, x)'/D(\alpha, \alpha + \epsilon, x))' \tag{5.58}$$

where for all complex numbers $s$ and $t$ we define $D(s, t, x)$ by

$$D(s, t, x) = t \sinh A(s, x) \cosh A(t, x) - s \sinh A(t, x) \cosh A(s, x), \tag{5.59}$$

with

$$A(s, x) = s(x - x_0) + M(s)$$

and

$$e^{M(s)} = \frac{(s + V_1)(s + V_2)}{(s - V_1)(s - V_2)}. \tag{5.60}$$

Since the right side of (5.60) is nonzero for $s = \alpha$, we may assume that $M(s)$ is defined and analytic for $s$ in some neighborhood of $\alpha$.

Observe that for $\epsilon > 0$ we can rewrite equation (5.58) in the form

$$u(\alpha, \alpha + \epsilon, x) = 2(D_1(\epsilon, x)'/D_1(\epsilon, x))' \tag{5.61},$$

where

$$D_1(\epsilon, x) = \frac{D(\alpha, \alpha + \epsilon, x)}{\epsilon} = \frac{D(\alpha, \alpha + \epsilon, x) - D(\alpha, \alpha, x)}{\epsilon}.$$

Let us define

$$D_1(0, x) = \lim_{\epsilon \to 0} D_1(\epsilon, x) = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon = 0} D(\alpha, \alpha + \epsilon, x).$$

Computing the derivative gives

$$D_1(0, x) = \frac{1}{2} \sinh(2\alpha(x - x_0) + 2M(\alpha)) - \alpha(x - x_0) - \alpha M'(\alpha) \tag{5.62}$$

for all $x \in \mathbb{R}$. Choose $I_1$ to be any nonempty subinterval of $(x_0 - \delta_1, x_0 + \delta_1)$ such that $D_1(0, x) \neq 0$ on the closure of $I_1$. Then, as $\epsilon$ goes to 0, $D_1(\epsilon, x)$ will converge to $D_1(0, x)$ uniformly on $I_1$, and the derivatives with respect to $x$ of $D_1(\epsilon, x)$ will converge to the corresponding derivatives of $D_1(0, x)$ uniformly on $I_1$ as well. It then follows from (5.57) and (5.61) that

$$u(x) = 2(D_1(0, x)'/D_1(0, x))' \tag{5.63}$$

for $x \in I_1$.

We now show that (5.63) extends analytically to a singular function on $\mathbb{R}$. For this it is enough to show that $D_1(0, x)$ has at least one zero on $\mathbb{R}$.

If $C$ is positive, then $\alpha$ is a positive real number. Recalling that either $V_1$ and $V_2$ are both real or $V_1^* = V_2$, we see that $e^{M(\alpha)}$ is real. Depending on whether the
right-hand side of (5.60) is positive or negative at \( s = \alpha \), we can choose \( M(\alpha) \) to either be real, or to have imaginary part \( \pi/2 \). Also since (5.60) implies
\[
M'(s) = \frac{(s + V_1) + (s + V_2)}{(s + V_1)(s + V_2)} - \frac{(s - V_1) + (s - V_2)}{(s - V_1)(s - V_2)},
\]
\( M'(\alpha) \) is real in any case.

If \( M(\alpha) \) is real, from (5.62) we see that \( D_1(0, x) > 0 \) for \( x \) large and positive, and \( D_1(0, x) < 0 \) for \( x \) large and negative. Therefore there must exist at least one \( x \in \mathbb{R} \) for which \( D_1(0, x) = 0 \). On the other hand, if \( M(\alpha) = K + i\pi/2 \) for \( K \) real, then (5.62) gives
\[
D_1(0, x) = -\frac{1}{2} \sinh(2\alpha(x - x_0) + 2K) - \alpha(x - x_0) - \alpha M'(\alpha),
\]
and again we see that \( D_1(0, x) = 0 \) for some \( x \in \mathbb{R} \).

If \( C \) is negative, then \( \alpha = i\gamma \) for some positive number \( \gamma \). We see from (5.60) that \( e^{M(i\gamma)} \) has modulus 1, and so \( M(i\gamma) \) can be taken to be purely imaginary: say \( M(i\gamma) = i\eta_1 \) for \( \eta_1 \in \mathbb{R} \). From (5.64) it is readily checked that \( M'(i\gamma) = M(i\gamma) \), and hence \( M'(i\gamma) \) is real: say \( M'(i\gamma) = \eta_2 \) for \( \eta_2 \in \mathbb{R} \). Then from (5.62) we have that
\[
D_1(0, x) = i \left( \frac{1}{2} \sin(2\gamma(x - x_0) + 2\eta_1) - \gamma(x - x_0) - \gamma\eta_2 \right).
\]
The quantity in parentheses in (5.65) is a real-valued function of \( x \) which is positive for large negative values of \( x \), and is negative for large positive values of \( x \). Therefore \( D_1(0, x) \) must equal zero for some \( x \in \mathbb{R} \).

We have shown, then, that whenever \( C_1 = C_2 \neq 0 \), \( u(x) \) is given on some open interval by the function on the right hand side of (5.63), which however cannot be extended analytically to a function on all of \( \mathbb{R} \). This contradicts the fact that \( u(x) \) is analytic on \( \mathbb{R} \). Thus no nontrivial \( H^2 \) solutions to (5.1) can exist in this case.

5.3. The degenerate case when \( C_1 = 0 \) or \( C_2 = 0 \) (but not both). Here we consider the cases when either \( 0 = C_1 < C_2 \) or \( C_1 < C_2 = 0 \).

Suppose first that \( C_1 = 0 \) and \( C_2 > 0 \), and let \( \alpha = \sqrt{C_2} \). From Corollary 5.39 we have that
\[
u(x) = u(0, \alpha, x) = \lim_{\epsilon \to 0} u(\epsilon, \alpha, x) \tag{5.66}
\]
for \( |x - x_0| < \delta_1 \). For \( \epsilon \) sufficiently small, we can take \( \beta_1 = \epsilon \) and \( \beta_2 = \alpha \) in Lemma 5.5, obtaining
\[
u(\epsilon, \alpha, x) = 2(D(\epsilon, \alpha, x)' / D(\epsilon, \alpha, x))', \tag{5.67}
\]
where \( D \) is as defined in (5.59). Again, we may assume that \( M(s) \) in (5.60) is defined and analytic for \( s \) in some neighborhood of \( \alpha \) and for \( s \) in some neighborhood of 0. Note that we can take \( M(0) = 0 \), so \( A(0, x) = 0 \).

As in (5.61), we can write, for all \( \epsilon > 0 \),
\[
u(\epsilon, \alpha, x) = 2(D_2(\epsilon, x)' / D_2(\epsilon, x))', \tag{5.68}
\]
where
\[
D_2(\epsilon, x) = D(\epsilon, \alpha, x) \left( \frac{D(\epsilon, \alpha, x)}{\epsilon} - \frac{D(0, \alpha, x)}{\epsilon} \right)
\]
and \( D(0, \alpha, x) = 0 \). Define
\[
D_2(0, x) = \lim_{\epsilon \to 0} D_2(\epsilon, x) = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon = 0} D(\epsilon, \alpha, x).
\]
We find by differentiating that
\[ D_2(0, x) = \alpha \cosh(\alpha(x - x_0) + M(\alpha)) [x - x_0 + M'(0)] - \sinh(\alpha(x - x_0) + M(\alpha)). \] 
(5.69)

As in Subsection 5.2, on any subinterval of \(|x-x_0| < \delta_1\} where \(D_2(0, x)\) is bounded away from zero, it follows from (5.66) and (5.68) that
\[ u(x) = u(0, \alpha, x) = 2(D_2(0, x)' / D_2(0, x))^\prime. \] 
(5.70)

Now \(D_2(0, x) = 0\) at any point \(x\) for which
\[ \alpha(x - x_0 + M'(0)) = \tanh(\alpha(x - x_0) + M(\alpha)). \] 
(5.71)

Again as in Subsection 5.2, since \(\alpha\) is real then \(M'(\alpha)\) is real and \(M(\alpha)\) can be taken to either be real or to have imaginary part \(\pi/2\). Since \(\tanh(x + i\pi/2) = \tanh(x)\) for all \(x \in \mathbb{R}\), in either case (5.71) takes the form
\[ \alpha x - \beta_1 = \tanh(\alpha x - \beta_2), \]
where \(\beta_1\) and \(\beta_2\) are real numbers, and hence must have a solution at some point in \(\mathbb{R}\). It follows then from (5.70) that \(u\) cannot be analytically continued to all of \(\mathbb{R}\).

There remains to consider the case when \(C_1 < 0\) and \(C_2 = 0\). Let \(\alpha = i\beta\), where \(\beta > 0\) and \(\beta^2 = |C_1|\). From Corollary 5.39 we have
\[ u(x) = u(0, \alpha, x) = \lim_{\epsilon \to 0} u(\alpha, \epsilon, x), \]
and Lemma (5.5) gives, for \(\epsilon\) sufficiently small,
\[ u(\alpha, \epsilon, x) = 2(D(\alpha, \epsilon, x)' / D(\alpha, \epsilon, x))^\prime = 2(D(\epsilon, \alpha, x)' / D(\epsilon, \alpha, x))^\prime, \]
provided \(|x-x_0| < \delta_1\), where \(D\) is still given by (5.59). Hence \(u(\alpha, \epsilon, x)\) is still given by the right-hand side of (5.67), and the calculations in (5.68) to (5.71) apply to \(u(\alpha, \epsilon, x)\), the only difference being that now \(\alpha = i\beta\) is purely imaginary. In this case, as seen in Subsection 5.2, \(M(\alpha)\) can be taken to be purely imaginary, and so (5.71) can be rewritten in the form
\[ \beta x - \beta_1 = \tan(\beta x - \beta_2), \]
where \(\beta_1\) and \(\beta_2\) are real numbers. This equation has (infinitely many) real solutions, and so again \(u\) cannot be analytically continued to all of \(\mathbb{R}\).

We have thus proved that when either \(C_1\) or \(C_2\) (but not both) is zero, then \(u\) cannot be analytically continued to \(\mathbb{R}\). So under the assumption that \(u\) is an \(H^2\) solution of (5.1), this case cannot arise.

5.4. The degenerate case \(C_1 = C_2 = 0\). Finally we consider the case when \(C_1\) and \(C_2\) are both zero. Then, by Corollary (5.4),
\[ u(x) = u(0, 0, x) = \lim_{\alpha \to 0} u(0, \alpha, x), \] 
(5.72)
where \(u(0, \alpha, x)\) is given for \(\alpha > 0\) by (5.70) with (5.69). To emphasize the dependence of \(D_2(0, x)\) on \(\alpha\), let us denote \(D_2(0, x)\) by \(D_3(\alpha, x)\) in what follows. That is,
\[ D_3(\alpha, x) = \alpha \cosh(\alpha(x - x_0) + M(\alpha)) [x - x_0 + M'(0)] - \sinh(\alpha(x - x_0) + M(\alpha)). \]

The limit in (5.72) is more singular than those in preceding sections, because \(D_3(\alpha, x)\) has a zero of order three at \(\alpha = 0\). That is, we have
\[ D_3(0, x) = \frac{\partial D_3}{\partial \alpha}(0, x) = \frac{\partial^2 D_3}{\partial \alpha^2}(0, x) = 0. \]
Therefore, to obtain a formula for \( u(0,0,x) \), we should define

\[
D_4(\alpha, x) = \frac{D_3(\alpha, x)}{\alpha^3},
\]

and we will have

\[
u(0,0,x) = 2\left( D_4(0,x)' / D_4(0,x) \right)',
\]

where

\[
D_4(0,x) = \lim_{\alpha \to 0} D_4(\alpha, x) = \frac{1}{6} \frac{\partial^3}{\partial \alpha^3} \bigg|_{\alpha=0} D_3(\alpha, x).
\] (5.73)

An elementary but fairly tedious computation of the derivative in (5.73) shows that

\[
D_4(0,x) = \frac{1}{3} (x - x_0 + M'(0))^3 - \frac{1}{6} M'''(0),
\]

and as \( M'(s) \) is real for all real \( s \) by (5.64), we have that \( M'(0) \) and \( M'''(0) \) are real. Clearly then \( D_4(0,x) \) has a zero at some \( x \in \mathbb{R} \), and so \( u(x) = u(0,0,x) \) cannot be extended to an analytic function on \( \mathbb{R} \).

To summarize, we have now shown that in all the degenerate cases, when \( C_1 \) or \( C_2 \) are zero or when \( C_1 = C_2 \), (5.1) cannot have an \( H^2 \) solution on \( \mathbb{R} \); and in the nondegenerate case the only possible solutions are given by (5.6). This then completes the proof of Theorem 5.2. \( \square \)

6. The stationary equation for general \( N \). We conclude with a few comments as to how the results above may be generalized to arbitrary stationary equations of the KdV hierarchy. In view of the first remark following Definition 3.12, it is natural to conjecture the following generalization of Theorems 4.2 and 5.2: if \( u \in H^{2N-2} \) is a nontrivial distribution solution of the stationary equation

\[
d_3 R_3 + d_5 R_5 + d_7 R_7 + \cdots + d_{2n+3} R_{2n+3} = 0,
\]

then \( u \) must be a \( k \)-soliton profile for the KdV hierarchy, for some \( k \in \{1, 2, \ldots, N\} \). More precisely, there must exist real numbers \( \alpha_j \) and \( a_j \), with \((-1)^{j-1} a_j > 0 \) for \( j = 1, \ldots, k \), such that

\[
u(x) = \psi^{(k)}(x; a_1, \ldots, a_k; \alpha_1, \ldots, \alpha_k).
\]

Much of the proof given above for Theorem 5.2 generalizes immediately to arbitrary \( N \). The extension of Lemma 5.1 to arbitrary \( N \), with \( H^2 \) replaced by \( H^{2N-2} \), is obvious. Let \( C_1, \ldots, C_N \) be the roots of the equation

\[
d_{2N+3} z^N - d_{2N+1} z^{N-1} + d_{2N-1} z^{N-2} - \cdots \pm d_3,
\] (6.1)

and let \( \alpha_j \) be the square roots of the \( C_j \), suitably defined. The generalization to arbitrary \( N \) of the definition of \( \bar{R}(x, \zeta) \) is already given in [8], along with the proof that in the case when \( \bar{R}(x, \zeta) \) has distinct roots \( \zeta_1, \ldots, \zeta_N \) at some \( x_0 \), they satisfy an analogue of the system (5.22).

Using induction and an argument like that given above in Section 5, we can assume that the \( \zeta_j \) are in fact distinct, since otherwise (5.22) reduces to a system with a smaller value of \( N \). From (5.22) one then obtains a generalization of the system (5.31) for functions \( v_j \) which are suitably defined square roots of the functions \( -\zeta_j \). Lemma 5.3, Corollary 5.4, and Lemma 5.5 all generalize straightforwardly to arbitrary \( N \). We thus obtain that any solution \( u(x) \) of (2.15) is given by

\[
u(x) = u(\alpha_1, \ldots, \alpha_n, x) = \lim_{\beta_1 \to \alpha_1, \ldots, \beta_N \to \alpha_N} \psi^{(N)}(x; a_1, \ldots, a_N; \beta_1, \ldots, \beta_N),
\] (6.2)
for some complex numbers $a_1, \ldots, a_N$, where the limit is taken through values of $(\beta_1, \ldots, \beta_N)$ such that the $\beta_j$ are distinct and all non-zero.

To complete the proof of the conjectured general result, then, it would remain to do two things. First, establish an analogue of Lemma 5.6, or in other words establish the conjecture mentioned in the first remark after Definition 3.12; and second, show that in the degenerate cases when some of the $C_j$ coincide or are equal to zero, the functions $\psi^{(N)}(x; a_1, \ldots, a_N; \beta_1, \ldots, \beta_N)$ in (6.2) converge to a limit which cannot be analytically continued to all of $\mathbb{R}$.

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E-mail address: jalbert@ou.edu