Fractional integration with singularity on Light-cone,

II: critical index

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Abstract

In this second part of our project, we study a family of fractional integral operators whose kernels carry a critical index and have singularity on the light cone. As a result, we partially answered a question proposed by Oberlin earlier in 1989.

1 Introduction

Let

\[
\Omega^\alpha(x) = \begin{cases} 
  c_\alpha \left( \frac{1}{1 - |x|^2} \right)^{\left( \frac{n+1}{2n} \right) (n-\alpha)}, & |x| < 1, \\
  0, & \frac{\alpha}{n} > \frac{n-1}{n+1}
\end{cases}
\] (1.1)

whose Fourier transform equals

\[
\widehat{\Omega}^\alpha(\xi) = \left( \frac{1}{|\xi|} \right)^{\left( \frac{n+1}{2n} \right) (\alpha-\frac{1}{2})} J_{\frac{n+1}{2n} (\alpha-\frac{1}{2})} (2\pi|\xi|)
\] (1.2)

where \( \Gamma \) is Gamma function and \( J \) is Bessel function. More discussion of this classical kernel can be found in the books by Watson [4], Stein and Weiss [6] and Stein [7].

We consider \( \Omega^\alpha, 0 \leq \alpha \leq n \) by analytic continuation from (1. 2). Denote

\[
\Omega^\alpha_r(x) = \frac{1}{r^n} \Omega^\alpha \left( \frac{x}{r} \right), \quad |x| \neq 1
\] (1.3)

for every \( r > 0 \).

A fractional integral operator \( I_\alpha \) is defined by

\[
(I_\alpha f)(x,t) = \int_\Lambda f(x-u,t-r) \Omega^\alpha_r(u)|r|^{\left( \frac{n+1}{n} \right) (\alpha-1)} du dr,
\] (1.4)

\[ \Lambda = \{ (u, r) \in \mathbb{R}^n \times \mathbb{R} : r^2 > |u|^2 \} \]

whose kernel has singularity appeared on the light cone in \( \mathbb{R}^{n+1} \).
We always write $\mathcal{C}$ as a generic constant.

**Conjecture A** Let $I_{\alpha}$ be defined in (1.4) for $0 < \alpha < n$. We have

$$\|I_{\alpha}f\|_{L^q(R^{n+1})} \leq \mathcal{C} p q \|f\|_{L^p(R^{n+1})} \tag{1.5}$$

for $1 < p < q < \infty$ if and only if

$$\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} \tag{1.6}$$

and

$$1 - \frac{n+1}{2n} \left(1 - \frac{\alpha}{n}\right) < \frac{1}{p} < \frac{\alpha}{n} + \frac{n+1}{2n} \left(1 - \frac{\alpha}{n}\right). \tag{1.7}$$

Conjecture A is proved for $\frac{n}{n+1} \leq \alpha < n$ by Oberlin [3] who proposed the question of the analogue result for $0 < \alpha < \frac{n}{n+1}$. If $n = 1$, it is answered by a simple iteration argument [11]. We assume $n \geq 2$ in this paper. Our main result is given below.

**Theorem A** Let $I_{\alpha}$ be defined in (1.4) for $0 < \alpha < \frac{n}{n+1}$. We have

$$\|I_{\alpha}f\|_{L^q(R^{n+1})} \leq \mathcal{C} p q \|f\|_{L^p(R^{n+1})} \tag{1.8}$$

for $1 < p < q < \infty$ if

$$\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} \tag{1.9}$$

and

$$\frac{1}{2} < \frac{1}{p} < \frac{1}{2} + \frac{\alpha}{n}. \tag{1.10}$$

Figure 1: The shaded area above the two lines meeting at $\frac{1}{2}$ is where $I_{\alpha}$ known to be bounded.
We follow the framework developed by Oberlin [3], but improve a lemma that plays a principal role in the analysis by using Stein-Weiss theorem for fractional integrals instead of Hardy-Littlewood-Sobolev theorem used in [3]. This is done in section 2. Next, section 3 is devoted to certain regularity estimates with the line of duality. The proof is completed in section 4 after carrying out a combination of one-dimensional fractional integration with convolution inequalities on \( \mathbb{R}^n \) of which the original idea is given by Strichartz [9].

## 2 A principal Lemma

Consider \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) in this section. Omit on notations. Let \( \Omega^\alpha \) denote a distribution on \( \mathbb{R}^n \) whose dilated variant satisfies (1.3).

**Principal Lemma**  
Let \( 0 < \alpha < \frac{n}{n+1} \) and

\[
\frac{\alpha}{n} = \frac{1}{2} - \frac{1}{q}.
\]  
(2.1)

Suppose

\[
\| f \ast \Omega^\alpha_r \ast \Omega^\alpha_s \|_{L^q(\mathbb{R}^n)} \leq C_\alpha \left( \frac{1}{rs} \right)^{\frac{\alpha}{n}} \left( \frac{1}{|r-s|} \right)^{\frac{1}{2n+1}} \| f \|_{L^2(\mathbb{R}^n)}
\]  
(2.2)

for every \( r, s > 0 \). We have

\[
\left\{ \int_0^\infty r^{\frac{\alpha}{q}} \left( \int_{\mathbb{R}^n} |(f \ast \Omega^\alpha_r)(x)|^q \, dx \right) \frac{dr}{r} \right\}^{\frac{1}{q}} \leq C_\alpha \| f \|_{L^2(\mathbb{R}^n)}.
\]  
(2.3)

**Proof:** Define

\[
(Sf)(x, r) = |r|^{\frac{n}{2} - \frac{\alpha}{q}} (f \ast \Omega^\alpha_r)(x).
\]  
(2.4)

Observe that (2.3) is equivalent to

\[
\| Sf \|_{L^\infty(\mathbb{R}^{n+1})} \leq C_\alpha \| f \|_{L^2(\mathbb{R}^n)}.
\]  
(2.5)

We aim to prove (2.5) by showing \( S^*: L^\frac{q}{q+1}(\mathbb{R}^{n+1}) \rightarrow L^q(\mathbb{R}^n) \). This in turn can be deduced from

\[
SS^*: L^\frac{q}{q+1}(\mathbb{R}^{n+1}) \rightarrow L^q(\mathbb{R}^{n+1}).
\]  
(2.6)

Let \( g: \mathbb{R}^{n+1} \rightarrow \mathbb{R} \). From direct computation, we have

\[
(SS^*g)(x, r) = \int_{\mathbb{R}} |rs|^{\frac{n}{2} - \frac{\alpha}{q}} (g(\cdot, s) \ast \Omega^\alpha_{|r|} \ast \Omega^\alpha_{|s|})(x) \, ds.
\]  
(2.7)

Set

\[
-\gamma = \alpha - \frac{1}{q} - \left( \frac{n+1}{2n} \right) \alpha = \left( \frac{n-1}{2n} \right) \alpha - \frac{1}{q}.
\]  
(2.8)

Note that \( \frac{\alpha}{n} = \frac{1}{2} - \frac{1}{q} \) and \( 0 < \alpha < \frac{n}{n+1} \) imply

\[
\left( \frac{n-1}{2n} \right) \alpha - \frac{1}{q} = \left( \frac{n-1}{2n} \right) \alpha - \frac{1}{2} + \frac{\alpha}{n}
\]  
(2.9)

\[
= \left( \frac{n+1}{2n} \right) \alpha - \frac{1}{2} < 0.
\]
From (2.8)-(2.9), we have

$$0 < \gamma = \frac{1}{2} - \left(\frac{n + 1}{2n}\right) \alpha = \frac{1}{q} - \left(\frac{n - 1}{2n}\right) \alpha < \frac{1}{q}. \quad (2.10)$$

Moreover,

$$1 - 2\left(\frac{\alpha}{n} + \gamma\right) = 1 - 2\left[\frac{\alpha}{n} + \frac{1}{2} - \left(\frac{n + 1}{2n}\right)\alpha\right] = \left(\frac{n - 1}{n}\right) \alpha. \quad (2.11)$$

Recall a classical result obtained by Stein and Weiss [5].

**Stein-Weiss Theorem (1958)** Let $0 < a < N$ and $\gamma, \delta \in \mathbb{R}$. We have

$$\left\{\int_{\mathbb{R}^N} f(u) \left(\frac{1}{|u|}\right)^b \left(\frac{1}{|x-u|}\right)^{N-a} \left(\frac{1}{|x|}\right)^\gamma \, du \right\}^{\frac{1}{q}} \int \left\{ \int_{\mathbb{R}^N} |f(x)|^p \, dx \right\}^{\frac{1}{p}} \leq C_{p,q,a} \left\{ \int_{\mathbb{R}^N} |f(x)|^p \, dx \right\}^{\frac{1}{p}} \quad (2.12)$$

for $1 < p \leq q < \infty$ if

$$\gamma < \frac{N}{q}, \quad \delta < N\left(\frac{p - 1}{p}\right), \quad \gamma + \delta \geq 0 \quad (2.13)$$

and

$$\frac{a}{N} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{N}. \quad (2.14)$$

**Remark 2.1** The constraints in (2.13)-(2.14) are also necessary conditions. See section 4 of [12].

We have

$$\left\| SS^* g \right\|_{L^q((\mathbb{R}^{n+1})^+)} = \left\{ \int \int_{\mathbb{R}^{n+1}} \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} |r|^{a - \frac{1}{q}} \left( g(r,s) \ast \Omega^a_r \ast \Omega^a_s \right) \, ds \right] \, dx \, dr \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int \left\{ \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |r|^{a - \frac{1}{q}} \left( g(r,s) \ast \Omega^a_r \ast \Omega^a_s \right) \, ds \right] \, dx \right\}^{\frac{1}{q}} \, dr \right\}^{\frac{1}{q}} \quad \text{by Minkowski integral inequality}$$

$$\leq C_a \left\{ \int \left[ \int_{\mathbb{R}} \left( \frac{1}{|r|} \right)^{\frac{n+1}{2n}} \left( \frac{1}{|r-s|} \right)^{\frac{n+1}{2n}} \left( \frac{1}{|s|} \right) \left\| g(r,s) \right\|_{L^q_{\mathbb{R}^n}} \, ds \right] \, dr \right\}^{\frac{1}{q}} \quad \text{by (2.2)}$$

$$= C_a \left\{ \int \left[ \int_{\mathbb{R}} \left( \frac{1}{|r|} \right)^{\gamma} \left( \frac{1}{|r-s|} \right)^{1-2(\frac{\gamma}{n}+\gamma)} \left( \frac{1}{|s|} \right) \left\| g(r,s) \right\|_{L^q_{\mathbb{R}^n}} \, ds \right] \, dr \right\}^{\frac{1}{q}} \quad \text{by (2.8)-(2.11)}$$

$$\leq C_a \left\{ \int \left\| g(r) \right\|_{L^q_{\mathbb{R}^n}}^{\frac{1}{q}} \, dr \right\}^{\frac{q-1}{q}} \quad \text{by Stein-Weiss theorem} \quad (2.15)$$

where the last inequality is obtained by taking into account that $N = 1$, $p = \frac{q}{q-1}$, $\delta = \gamma$ and $a = 2\left(\frac{n}{n} - \gamma\right)$ in (2.13)-(2.14). \qed
3 Regularity estimate with respect to the line of duality

First, recall some well known estimates for Bessel functions: For $\lambda > -\frac{1}{2}$ and $\rho > 0$, we have

$$J_{\lambda+it}(\rho) = \left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}} \cos\left(\rho - \frac{\pi}{2}(\lambda + it) - \frac{\pi}{4}\right) + e_t(\rho),$$

(3.1)

$$|e_t(\rho)| \leq C a^{\frac{1}{2}} \begin{cases} \rho^{-1}, & 0 < \rho \leq 1, \\ \rho^{-\frac{3}{2}}, & \rho > 1 \end{cases}$$

(3.2)

where $a$ is some positive constant. Moreover,

$$|J_{\lambda+it}(\rho)| \leq C a^{\frac{1}{2}} \left(\frac{1}{\rho}\right)^{\frac{1}{2}}, \quad \lambda \geq -\frac{1}{2}.$$  

(3.3)

A discussion of (3.1)-(3.3) can be found in Lemma 7, section 2 of the paper by Oberlin [3] and several references cited there.

Let $\Omega^\alpha$ be defined by analytic continuation from (1.2). We rewrite $\Omega^\alpha = \mathcal{E}^\alpha + \mathcal{E}_{\xi}^\alpha$ by splitting

$$\hat{\Omega}^\alpha(\xi) = \hat{\mathcal{E}}^\alpha(\xi) + \hat{\mathcal{E}}_{\xi}^\alpha(\xi),$$

$$\hat{\mathcal{E}}^\alpha(\xi) = \frac{1}{\pi} \left(\frac{1}{|\xi|}\right)^{(\frac{n+1}{2n})\alpha} \cos\left(2\pi|\xi| - \frac{\pi}{2} \left(\frac{n+1}{2n}\right)\alpha - \frac{\pi}{2}\right),$$

(3.4)

$$\hat{\mathcal{E}}_{\xi}^\alpha(\xi) = \left(\frac{1}{|\xi|}\right)^{(\frac{n+1}{2n})\alpha - \frac{1}{2}} e_0(2\pi|\xi|).$$

Let $f : \mathbb{R}^n \to \mathbb{R}$. We aim to prove (2.3) for $f * \Omega^\alpha$ by showing that $f * \mathcal{E}^\alpha + \mathcal{E}_{\xi}^\alpha$ and $f * \mathcal{E}^\alpha + \mathcal{E}_{\xi}^\alpha$ satisfy the crucial estimate in (2.2). Namely,

$$\left\|f * \mathcal{E}^\alpha + \mathcal{E}_{\xi}^\alpha\right\|_{L^q(\mathbb{R}^n)} \leq C_{\alpha} \left(\frac{1}{r s}\right)^{\left(\frac{n+1}{2n}\right)\alpha} \left(\frac{1}{|r - s|}\right)^{\left(\frac{n+1}{n}\right)\alpha} \left\|f\right\|_{L^\infty(\mathbb{R}^n)}$$

(3.5)

and

$$\left\|f * \mathcal{E}^\alpha + \mathcal{E}_{\xi}^\alpha\right\|_{L^q(\mathbb{R}^n)} \leq C_{\alpha} \left(\frac{1}{r s}\right)^{\left(\frac{n+1}{2n}\right)\alpha} \left(\frac{1}{|r - s|}\right)^{\left(\frac{n+1}{n}\right)\alpha} \left\|f\right\|_{L^\infty(\mathbb{R}^n)}$$

(3.6)

for every $r, s > 0$.

From direct computation, we have

$$\left(f * \mathcal{E}^\alpha + \mathcal{E}_{\xi}^\alpha\right)(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi) \left(\frac{1}{|\xi|}\right)^{(\frac{n+1}{2n})\alpha - \frac{1}{2}} e_0(2\pi r|\xi|) \left(\frac{1}{|s|}\right)^{(\frac{n+1}{n})\alpha - \frac{1}{2}} \overline{e_0}(2\pi s|\xi|) d\xi$$

(3.7)

$$= \left(\frac{1}{r s}\right)^{\left(\frac{n+1}{2n}\right)\alpha} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi) \left(\frac{1}{|\xi|}\right)^{(\frac{n+1}{2n})\alpha} \left(|\xi|\right)^2 \left(s|\xi|\right)^2 e_0(2\pi r|\xi|) \overline{e_0}(2\pi s|\xi|) d\xi.$$

Without lose of generality, we assume $r \geq s$ whereas $r^{-1} \leq (r - s)^{-1}$. Recall the asymptotic estimate of $e_t$ given in (3.2).
Case 1 Suppose $|\xi| \leq (2\pi \tau)^{-1}$. We have
\[
\left( \frac{1}{|\xi|} \right)^{\frac{n+1}{2}} (r|\xi|)^{\frac{1}{2}} (s|\xi|)^{\frac{1}{2}} e_{\nu}(2\pi r|\xi|) \mathcal{E}_{\nu}(2\pi s|\xi|)
\leq C \left( \frac{1}{|\xi|} \right)^{\frac{n+1}{2}} \leq C \left( \frac{1}{r} \right)^{\frac{n+1}{2}} \left( \frac{1}{|\xi|} \right)^{2\alpha} (2\pi \leq \frac{1}{r|\xi|}) \tag{3. 8}
\leq C \left( \frac{1}{|r-s|} \right)^{\frac{n+1}{2}} \left( \frac{1}{|\xi|} \right)^{2\alpha}.
\]

Case 2 Suppose $|\xi| > (2\pi \tau)^{-1}$ and $|\xi| \leq (2\pi s)^{-1}$. We have
\[
\left( \frac{1}{|\xi|} \right)^{\frac{n+1}{2}} (r|\xi|)^{\frac{1}{2}} (s|\xi|)^{\frac{1}{2}} e_{\nu}(2\pi r|\xi|) \mathcal{E}_{\nu}(2\pi s|\xi|)
\leq C \left( \frac{1}{r} \right)^{\frac{n+1}{2} + 1} \leq C \left( \frac{1}{r} \right)^{\frac{n+1}{2}} \left( \frac{1}{|\xi|} \right)^{2\alpha} (r|\xi| > \frac{1}{2\pi}) \tag{3. 9}
\leq C \left( \frac{1}{|r-s|} \right)^{\frac{n+1}{2}} \left( \frac{1}{|\xi|} \right)^{2\alpha}.
\]

Case 3 Suppose $|\xi| > (2\pi s)^{-1}$. We have
\[
\left( \frac{1}{|\xi|} \right)^{\frac{n+1}{2}} (r|\xi|)^{\frac{1}{2}} (s|\xi|)^{\frac{1}{2}} e_{\nu}(2\pi r|\xi|) \mathcal{E}_{\nu}(2\pi s|\xi|)
\leq C \left( \frac{1}{r} \right)^{\frac{n+1}{2} + 2} \leq C \left( \frac{1}{r} \right)^{\frac{n+1}{2} + 1} \left( \frac{1}{|\xi|} \right)^{2\alpha} \tag{3. 10}
\leq C \left( \frac{1}{r} \right)^{\frac{n+1}{2}} \left( \frac{1}{|\xi|} \right)^{2\alpha} \leq C \left( \frac{1}{|r-s|} \right)^{\frac{n+1}{2}} \left( \frac{1}{|\xi|} \right)^{2\alpha}.
\]

Because $\frac{2\alpha}{n} = \frac{q-1}{q} - \frac{1}{2}$, we have $|\xi|^{-2\alpha}$ to be a $L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ Fourier multiplier. Indeed, its inverse Fourier transform equals $C_{\alpha}|x|^{2\alpha-n}$. Furthermore, $\frac{q-1}{q} - \frac{1}{2} = \frac{n}{2} - \frac{1}{q}$. Hence that $|\xi|^{-\alpha}$ is simultaneously a $L^q(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ Fourier multiplier.

By using (3. 8)-(3. 10) and applying Hardy-Littlewood-Sobolev inequality [1]-[2] together with Plancherel theorem, we have
\[
\left\{ \int_{\mathbb{R}^n} \left| \frac{f * E_{\alpha}^n * \overline{E_{\alpha}^n}}{r} \right|^q dx \right\}^{\frac{1}{q}}
\leq C_{\alpha} \left( \frac{1}{r} \right)^{\frac{n+1}{2}} \left( \frac{1}{|r-s|} \right)^{\frac{n+1}{2}} \left\{ \int_{\mathbb{R}^n} \left| f(x) \right|^q dx \right\}^{\frac{q-1}{q}}. \tag{3. 11}
\]

\[6\]
Now, we return to estimate
\[
(f \ast \mathcal{E}_r \ast \overline{\mathcal{E}}_s)(x) = \frac{1}{\pi^2} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \mathcal{F}(\xi) \left( \frac{(2\pi)^\alpha}{|\xi|} \right)^\alpha \cos \left( 2\pi r|\xi| - \frac{\pi}{2} \left( \frac{n+1}{2n} \right) \alpha - \frac{\pi}{2} \right) d\xi
\]
\[
= (\frac{1}{s|\xi|})^{\alpha} \cos \left( 2\pi s|\xi| - \frac{\pi}{2} \left( \frac{n+1}{2n} \right) \alpha - \frac{\pi}{2} \right) d\xi
\]
\[
= \mathcal{C}_\alpha \left( \frac{1}{r s} \right)^{\alpha} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \mathcal{F}(\xi) \left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \sin \left( 2\pi(r+s)|\xi| \right) d\xi
\]
\[
+ \mathcal{C}_\alpha \left( \frac{1}{r s} \right)^{\alpha} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \mathcal{F}(\xi) \left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \cos \left( 2\pi(r-s)|\xi| \right). \tag{3. 12}
\]
From (3. 12), it is suffice to show that
\[
\left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \sin \left( 2\pi|\xi| \right), \quad \left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \cos \left( 2\pi|\xi| \right) \tag{3. 13}
\]
are $L^{\frac{q}{n}}(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ Fourier multipliers. Indeed, we have
\[
\left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \mathcal{F}(\xi) \left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \cos \left( 2\pi(r-s)|\xi| \right) d\xi \right|^q dx \right\}^{\frac{1}{q}}
\]
\[
= |r-s|\left( \frac{n+1}{n} \right)^\alpha \left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \mathcal{F}(\xi) \left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \cos \left( 2\pi(r-s)|\xi| \right) d\xi \right|^q dx \right\}^{\frac{1}{q}}
\]
\[
= |r-s|\left( \frac{n+1}{n} \right)^\alpha \left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \mathcal{F}(\xi) \left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \cos \left( 2\pi(r-s)|\xi| \right) d\xi \right|^q \right\}^{\frac{1}{q}}
\]
\[
= |r-s|\left( \frac{n+1}{n} \right)^\alpha + \frac{\alpha}{2} \left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f((r-s)u) \left( \int_{\mathbb{R}^n} e^{2\pi i (x-u) \cdot \xi} \left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \cos \left( 2\pi|\xi| \right) d\xi \right) du \right|^q dx \right\}^{\frac{1}{q}}
\]
\[
\leq \mathcal{C}_\alpha |r-s|\left( \frac{n+1}{n} \right)^\alpha + \frac{\alpha}{2} \left\{ \int_{\mathbb{R}^n} \left| f((r-s)x) \right|^q dx \right\}^{\frac{1}{q} - \frac{\alpha}{q}} \quad \text{by assumption}
\]
\[
= \mathcal{C}_\alpha \left( \frac{1}{|r-s|} \right)^{(\frac{n+1}{n})\alpha - \frac{\alpha}{q} - \frac{(n+1)}{2} \alpha} \left\{ \int_{\mathbb{R}^n} \left| f(x) \right|^q dx \right\}^{\frac{1}{q} - \frac{\alpha}{q}} \quad x \rightarrow (r-s)^{-1} x
\]
\[
= \mathcal{C}_\alpha \left( \frac{1}{|r-s|} \right)^{(\frac{n+1}{n})\alpha} \left\{ \int_{\mathbb{R}^n} \left| f(x) \right|^q dx \right\}^{\frac{1}{q} - \frac{\alpha}{q}}. \tag{3. 14}
\]
A vice versa estimate works for $\left( \frac{1}{|\xi|} \right)^{(\frac{n+1}{n})\alpha} \sin \left( 2\pi|\xi| \right)$ with $r-s$ replaced by $r+s$ for which $|r-s| \leq |r+s|$. 
Define an analytic operator \( f \ast \Omega \) for \( 0 \leq \text{Re}(z) \leq 1 \). We hereby repeat an argument given by Strichartz [10].

Let \( \Omega^\alpha, 0 \leq \alpha \leq n \) be defined by analytic continuation from (1.2). From (1.1), we have

\[
\| f \ast \Omega^{\alpha+\text{Im}(z)} \|_{L^\infty(\mathbb{R}^n)} \leq C \| e^{\text{Im}(z) \log |\text{Im}(z)|} \| f \|_{L^1(\mathbb{R}^n)},
\]

On the other hand, the estimate of Bessel functions in (3.3) and Plancherel theorem imply

\[
\| f \ast \Omega^{\alpha+\text{Im}(z)} \|_{L^2(\mathbb{R}^n)} \leq C_0 \| e^{\text{Im}(z)} \| f \|_{L^2(\mathbb{R}^n)}', \quad a > 0.
\]

By applying Stein interpolation theorem [8], we obtain

\[
\| f \ast \Omega^{\beta} \|_{L^q(\mathbb{R}^n)} \leq C_\beta \| f \|_{L^{q'}(\mathbb{R}^n)}
\]

whenever

\[
1/p^* = 1/2 \left( 1 + \beta/n \right), \quad 1/q^* = 1/2 \left( 1 - \beta/n \right)
\]

for \( 0 < \beta < n \). In particular, if \( \beta = 2\alpha \) we have \( q^* = q \) and \( p^* = q/(q-1) \).

Recall from (3.4). We write

\[
\hat{f}(\xi) \Omega^{2\alpha}(\xi) = \hat{f}(\xi) \Omega^{2\alpha}(\xi) + \hat{f}(\xi) \Omega^{2\alpha}(\xi)
\]

where

\[
\Omega^{2\alpha}(\xi) = C_\alpha \left( \frac{2\mu}{|\xi|} \right)^{2\alpha} \cos (2\pi |\xi|) = C_\alpha \left( \frac{2\mu}{|\xi|} \right)^{2\alpha} \sin (2\pi |\xi|),
\]

\[
\Omega^{2\alpha}(\xi) = \left( \frac{2\mu}{|\xi|} \right)^{2\alpha} e_o (2\pi |\xi|).
\]

Observe that

\[
\left| \left( \frac{2\mu}{|\xi|} \right)^{2\alpha} e_o (2\pi |\xi|) \right| \leq C \left( \frac{2\mu}{|\xi|} \right)^{2\alpha}
\]

can be deduced from (3.8)-(3.10). Therefore, \( \Omega^{2\alpha}(\xi) \) is an \( L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \) Fourier multiplier. Together with the estimate in (3.17), we must have \( \Omega^{2\alpha}(\xi) \) as another desired Fourier multiplier.

### 4 Proof of Theorem A

Let \( \Omega^\alpha \) be defined by analytic continuation from (1.2) for \( 0 < \alpha < \frac{n}{n+1} \) whose dilated variant \( \Omega^\alpha_r, r > 0 \) satisfies (1.3). Consider \( f: \mathbb{R}^{n+1} \rightarrow \mathbb{R} \).

From (3.17)-(3.18), after changing dilations, we find

\[
\| f(\cdot,t) \ast \Omega^\alpha \|_{L^p(\mathbb{R}^n)} \leq C_\alpha \| f(\cdot,t) \|_{L^p(\mathbb{R}^n)}, \quad r \neq 0, \quad t \in \mathbb{R}
\]
for
\[
\frac{1}{p_0} = \frac{1}{2} \left( 1 + \frac{\alpha}{n} \right), \quad \frac{1}{q_0} = \frac{1}{2} \left( 1 - \frac{\alpha}{n} \right).
\]
(4.2)

Note that (4.1)-(4.2) hold for 0 < \alpha < n.

By taking the supremum over \( r \neq 0 \) in (4.1), we have the following mixed-norm estimate
\[
\left\| |r|^\alpha \left\| f(\cdot, t) * \Omega^a_{\beta} \right\|_{L^p(B^n)} \right\|_{L^q(B, \lvert x \rvert^{-1} dr)} \leq C_\alpha \left\| f(\cdot, t) \right\|_{L^p(B^n)}, \quad t \in \mathbb{R}.
\]
(4.3)

On the other hand, set
\[
\frac{\alpha}{n} = \frac{1}{2} - \frac{1}{q_1}.
\]
(4.4)

From the previous section, we have \( f(\cdot, t) * \Omega^a \) satisfying (2.3) by applying Principal Lemma. This is equivalent to
\[
\left\| |r|^\alpha \left\| f(\cdot, t) * \Omega^a_{\beta} \right\|_{L^p(B^n)} \right\|_{L^q(B, \lvert x \rvert^{-1} dr)} \leq C_\alpha \left\| f(\cdot, t) \right\|_{L^p(B^n)}, \quad t \in \mathbb{R}.
\]
(4.5)

Let 0 < \theta < 1 and denote
\[
\frac{1}{s} = \frac{\theta}{q_1}, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.
\]
(4.6)

Note that \( s > q \) since \( \theta q_1^{-1} < q^{-1} \) for 0 < \theta < 1.

From (4.3) and (4.5), by applying Riesz interpolation theorem (in mixed norm), we have
\[
\left\{ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} \left| (f(\cdot, t) * \Omega^a_{\beta})(x) \right|^q dx \right)^{\frac{1}{q}} \frac{dr}{|r|^s} \right\} \leq C_\alpha \left\| f(\cdot, t) \right\|_{L^q(B^n)}.
\]
(4.7)

We aim to prove
\[
\left\| I_\alpha f \right\|_{L^q(B^{n+1})} \leq C_{p,q} \left\| f \right\|_{L^q(B^{n+1})} \quad \iff \quad \left\| I_\alpha f \right\|_{L^p(B^{n+1})} \leq C_{p,q} \left\| f \right\|_{L^p(B^{n+1})}
\]
(4.8)

for
\[
\begin{align*}
\frac{1}{p} - \frac{1}{q} &= \frac{\alpha}{n} = \frac{q - 1}{q} - \frac{p - 1}{p}, \quad \frac{1}{p_1} < \frac{1}{p} < \frac{1}{p_0}, \quad \frac{1}{q_1} < \frac{1}{q} < \frac{1}{q_0}
\end{align*}
\]
(4.9)

where
\[
\begin{align*}
\frac{1}{2} - \frac{\alpha}{n} < \frac{1}{q} < \frac{1}{2} - \frac{\alpha}{2n} &\iff \frac{1}{2} + \frac{\alpha}{2n} < \frac{q - 1}{q} < \frac{1}{2} + \frac{\alpha}{n},
\end{align*}
\]
(4.10)

from (4.2), (4.4) and (4.6).

Furthermore, it is easy to verify that all previous estimates hold for the adjoint operator \( I_\alpha \). Hence that
\[
\left\| I_\alpha f \right\|_{L^q(B^{n+1})} \leq C_{p,q} \left\| f \right\|_{L^q(B^{n+1})} \quad \iff \quad \left\| I_\alpha f \right\|_{L^p(B^{n+1})} \leq C_{p,q} \left\| f \right\|_{L^p(B^{n+1})}.
\]
(4.11)
From (4. 8)-(4. 11), the boundedness of $I_\alpha$ for $\frac{1}{2} < \frac{1}{p} < \frac{1}{2} + \frac{\alpha}{n}$ follows by duality and interpolation.

Given $f \in L^p(\mathbb{R}^{n+1})$ and $g \in L^{q'}(\mathbb{R}^{n+1})$, we have

$$
\int \int_{\mathbb{R}^{n+1}} (I_\alpha f)(x,t)g(x,t)dxdt
$$

$$
= \int \int_{\mathbb{R}^{n+1}} \left\{ \int_{\mathbb{R}} (f(\cdot,t-r) \star \Omega_{\rho(r)}^\alpha)(x) |r|^{\frac{\alpha n}{n-1}} dr \right\} g(x,t)dxdt
$$

$$
= \int \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}^n} (f(\cdot,t) \star \Omega_{\rho(t)}^\alpha)(x)g(x,t+r)dx \right\} |r|^{\frac{\alpha n}{n-1}} drdt \quad t \rightarrow t + r
$$

$$
\leq \int \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^n} \left( f(\cdot,t) \star \Omega_{\rho(t)}^\alpha \right)(x) \|g(\cdot,t+r)\|_{L^p(\mathbb{R}^n)} \right\} \|f(\cdot,t) \star \Omega_{\rho(t)}^\alpha\|_{L^{q'}(\mathbb{R}^n)} |r|^{\frac{\alpha n}{n-1}} drdt
$$

$$
\leq \int \left\{ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} \left( f(\cdot,t) \star \Omega_{\rho(t)}^\alpha \right)(x) \|g(\cdot,t+r)\|_{L^p(\mathbb{R}^n)} \right)^\frac{\alpha}{p} dr \right\} \frac{1}{\|f(\cdot,t) \star \Omega_{\rho(t)}^\alpha\|_{L^{q'}(\mathbb{R}^n)}} |r|^{\frac{\alpha n}{n-1}} drdt
$$

$$
\leq \left\{ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} \left( g(\cdot,t+r) \frac{|r|^{\frac{\alpha n}{n-1}}}{\|f(\cdot,t) \star \Omega_{\rho(t)}^\alpha\|_{L^{q'}(\mathbb{R}^n)}} \right) \right)^\frac{\alpha}{p} dr \right\} \frac{1}{\|f(\cdot,t) \star \Omega_{\rho(t)}^\alpha\|_{L^{q'}(\mathbb{R}^n)}} |r|^{\frac{\alpha n}{n-1}} drdt
$$

by using Hölder inequality 3 times.

Observe that

$$
(n+1)\left(\frac{\alpha}{n}\right) - 1 + \frac{1}{s} - \alpha \left(\frac{s}{s-1}\right) = \left(\frac{s}{s-1}\right)\left(\frac{\alpha}{n}\right) - 1
$$

(4. 13)

where $\frac{s}{s-1} < \frac{q}{q-1}$.

By applying Hardy-Littlewood-Sobolev inequality [1]-[2] on $\mathbb{R}$, we have

$$
\left\{ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} \left( g(\cdot,t+r) \frac{|r|^{\frac{\alpha n}{n-1}}}{\|f(\cdot,t) \star \Omega_{\rho(t)}^\alpha\|_{L^{q'}(\mathbb{R}^n)}} \right) \right)^\frac{\alpha}{p} dr \right\} \frac{1}{\|f(\cdot,t) \star \Omega_{\rho(t)}^\alpha\|_{L^{q'}(\mathbb{R}^n)}} |r|^{\frac{\alpha n}{n-1}} drdt
$$

(4. 14)

$$
\leq C_{p,q} \left\{ \int_{\mathbb{R}} \|g(\cdot,t)\|^\frac{\alpha}{q} dr \right\} \frac{1}{\|f(\cdot,t) \star \Omega_{\rho(t)}^\alpha\|_{L^{q'}(\mathbb{R}^n)}} \leq C_{p,q} \|g\|_{L^{q'}(\mathbb{R}^n)}.
$$

On the other hand, (4. 7) implies

$$
\left\{ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} \left( f(\cdot,t) \star \Omega_{\rho(t)}^\alpha \right)(x) \|g(\cdot,t+r)\|_{L^p(\mathbb{R}^n)} \right)^\frac{\alpha}{p} dr \right\} \frac{1}{\|f(\cdot,t) \star \Omega_{\rho(t)}^\alpha\|_{L^{q'}(\mathbb{R}^n)}} |r|^{\frac{\alpha n}{n-1}} drdt \leq C_\alpha \|f\|_{L^p(\mathbb{R}^{n+1})}.
$$

(4. 15)
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