Hamiltonian study of Supersymmetric Yang-Mills Quantum Mechanics*+

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Abstract

New results obtained within the recently developed approach to supersymmetric quantum mechanical systems are presented. The method does not suffer from the sign problem in any dimensions and is capable to provide any quantum observable with controllable systematic error. Discussed results include: the spectrum and Witten index of the D=4 system, and the spectrum of zero volume glueballs in higher ($4 \leq D \leq 10$) dimensions.

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New results obtained within the recently developed approach to supersymmetric quantum mechanical systems are presented. The method does not suffer from the sign problem in any dimensions and is capable to provide any quantum observable with controllable systematic error. Discussed results include: the spectrum and Witten index of the D=4 system, and the spectrum of zero volume glueballs in higher (4 ≤ D ≤ 10) dimensions.

We report on the new development in studying supersymmetric quantum mechanical systems. The main goal is to solve the ten (space-time) dimensional (D=10) supersymmetric Yang-Mills quantum mechanics (SYMQM) with the SU(N) gauge group \( \mathbb{SU}(N) \). At infinite N it provides relatively simple model of M-theory \( \mathbb{M} \)-theory. In spite of recent results that question the exact equivalence between the two, D=10 SYMQM possesses lot of fascinating properties required in M-theory (e.g. threshold bound states). Independently, these systems provide a good laboratory to study various consequences of supersymmetry.

We use the hamiltonian formalism adapted to supersymmetric systems with local gauge invariance \( \mathbb{M} \)-theory. It consists of two steps: 1) creation of a finite basis of gauge invariant states, and 2) algebraic calculation of the matrix representation of the hamiltonian, followed by its numerical diagonalization. Algebraic calculations are automated by implementing standard rules of quantum mechanics in a symbolic language like Mathematica. Faster, compiler based, version is now also available. Given the spectrum and energy eigenstates, any other relevant observable can be easily obtained using our computer based rules of quantum mechanics.

Such an approach necessarily introduces a cut-off - the size of the basis. We choose for it the gauge and rotationally invariant total number of bosonic quanta \( B = a_i^a a_i^{a \dagger} \). Hence the cut off basis consists of all states with \( B \leq N_{\text{cut}} \) and all allowed fermions. Changing the size of the basis gives the quantitative measure of the finite \( N_{\text{cut}} \) effects similarly to lattice calculations.

The technique has been applied to Wess-Zumino quantum mechanics, D=2 and D=4 SYMQM and to D = 5 - 10 YMQM, all based on the SU(2) gauge group. In all cases studied until now the spectrum of lower states converges with \( N_{\text{cut}} \) well within the reach of present computers. Clear SUSY signatures are seen. Similar methods have been independently developed for lower dimensional supersymmetric field theories. Since a notable progress, in increasing the Fock space, has been achieved by M. Campostrini. Here we report the new results for four (space-time) dimensional quantum mechanics. D=4 SYMQM.

The system is described by nine bosonic coordinates \( x_i^a(t) \), \( i=1,2,3; a=1,2,3 \) and six independent fermionic ones contained in the Majorana spinor \( \psi^a_{\alpha}(t) \), \( \alpha=1,...,4 \). Hamiltonian reads

\[
H = \frac{1}{2} p_i^a p_i^{a \dagger} + \frac{g^2}{4} \epsilon_{abc} x_i^a x_j^b x_k^c + \frac{i g}{2} \epsilon_{abc} \psi^a T^k \psi^b \psi^c \tag{1}
\]

with the Dirac \( \alpha \) matrices. We work in the Majorana representation of Itzykson and Zuber.

The system has rotational symmetry, with the spin(3) angular momentum \( J^i = \)

\( ^4 \text{See Uwe Trittmann contribution.} \)
the gauge invariance with the generators

\[ G_a = \epsilon_{abc} \left( x_b^k p^j_c - \frac{1}{4} \psi^T_a \Sigma^{jk} \psi_a \right), \quad \Sigma^{jk} = -\frac{i}{4} [\Gamma^j, \Gamma^k], \]

and supersymmetry generated by the charges

\[ Q_\alpha = (\Gamma^k \psi_a) \alpha p^j_k + ig \epsilon_{abc} (\Sigma^{jk} \psi_a) \alpha x^j_b x^k_c. \]

Generation of basis, construction of Majorana fermions and details of the calculation of hamiltonian matrix and other observables are described in [3]. Here we present new results for higher cut-offs \( N_{\text{cut}} \leq 8 \) in all fermionic sectors. Hamiltonian \( \mathcal{H} \) conserves fermion number

\[ F = \sum_i f^i_a f^i_a, \]

hence it suffices to diagonalize \( \mathcal{H} \) separately in each fermionic sector. Moreover, as a consequence of a particle-hole symmetry there are only four independent sectors.

The spectrum of the theory (Fig.1) is rather rich and behaves differently in different sectors. In \( F = 0 \) sector our spectrum converges to the well known classic results of Lüscher and Münster for zero volume glueballs [5] providing a satisfactory test of the whole approach [6]. For \( F = 1 \) we again observe fast convergence with increasing \( N_{\text{cut}} \) – our bases are adequate to extract the ”infinite volume” limit of lower glueballs and gluinoballs in both sectors. One also observes emergence of the supersymmetric multiplets spanned across different fermionic sectors [5]. In ”fermion rich” sectors \( F = 2, 3 \), situation is even more interesting. In addition to quickly convergent with \( N_{\text{cut}} \) states we observe levels with strong \( N_{\text{cut}} \) dependence whose energies tend to zero. In fact the cutoff dependence can be used to identify discrete and continuous spectra. It was found that localized, discrete bound states have fast (e.g. exponential) convergence with \( N_{\text{cut}} \), while continuous, non localized spectrum is signaled by slow \( O(1/N_{\text{cut}}) \) dependence. Therefore our results confirm (and quantify) famous SUSY predictions of the continuous spectrum in this model [7]. We also establish that the SUSY vacuum is the lowest, \( J = 0 \), state in the \( F = 2 \) sector, that it belongs to the continuum hence is non normalizable, and in fact is doubly degenerate \( (F = 2 \text{ and } 4 \text{ sectors}) \).

The Witten index

\[ I_W(T) = \sum_i (-1)^{F_i} \exp (-T E_i), \]

provides a global measure of the restoration of supersymmetric charges and angular momentum operator.

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tion/violation of SUSY. It is discontinuous at $T = 0$ with $I_W(0)$ depending on the regularization. For discrete spectrum it is time independent and equals to the number of ground states. For continuous spectrum it may have a mild $T$ dependence, still $I_W(\infty)$ gives the number of localized ground (unbalanced by SUSY) states. The index shown in Fig.2 was obtained directly from our spectrum. Compared to [3] we observe further slow approach to the large $N_{\text{cut}}$ limit, consistent with all above properties. The physical (bulk) value $\lim_{T \to 0} I_W(T)$ seems to point towards 1/4 calculated from nonabelian integrals [8] and marked by the flat line. There is no indication of the threshold bound state in agreement with Fig.1 and with theoretical expectations.

From four to ten space-time dimensions. To assess the feasibility of our approach to the full D=10 theory we have calculated the spectra of the purely bosonic hamiltonians in all intermediate dimensions $4 \leq D \leq 10$. Thereby results shown in Fig.3 extend for the first time the spectrum of the zero volume glueballs to higher dimensions. This generalization is straightforward in our approach, however it demands larger, but realistic, computer effort. In all dimensions we see fast convergence with $N_{\text{cut}}$ indicating the localized nature of glueball bound states. Since our cutoff respects rotational symmetry for all $d=D-1$, we observe exact SO($d$) degeneracies in the spectra. The ordering of the first three multiplets, shown in Fig.3, remains the same as in $d = 3$: scalar, two dimensional symmetric tensor with dimension $g = d(d+1)/2 - 1$, and scalar again. The largest uncertainty (i.e. that of the third $d=9$ level) is around 25% at present and diminishes gradually to below 1% at $d = 3$. Turning on $d$ as another parameter allows for further confrontation with analytical calculations. For example the simple eye-ball estimate from Fig.3 suggest the power dependence $E \sim d^{-1.5}$ to be compared with the mean field result at low temperature $d^{4/3}$ [3].

We conclude that calculations in higher dimensions including $d = 9$ are perfectly realistic even if more time consuming. Consequently a path towards the quantitative study of the full D=10 supersymmetric theory is now available.

Figure 3. Zero volume glueballs in higher dimensions.

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