NUMERICAL RADIUS INEQUALITIES OF $2 \times 2$ OPERATOR MATRICES

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Abstract. Several upper and lower bounds for the numerical radius of $2 \times 2$ operator matrices are developed which refine and generalize the earlier related bounds. In particular, we show that if $B, C$ are bounded linear operators on a complex Hilbert space, then

$$\frac{1}{2} \max \{\|B\|, \|C\|\} + \frac{1}{4} \|B + C^*\| - \|B - C^*\| \leq w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2} \max \{\|B\|, \|C\|\} + \frac{1}{2} \max \left\{ r_{\frac{1}{2}}(|B||C^*|), r_{\frac{1}{2}}(|B^*||C|) \right\},$$

where $w(\cdot)$, $r(\cdot)$ and $\|\cdot\|$ are the numerical radius, spectral radius and operator norm of a bounded linear operator, respectively. We also obtain equality conditions for the numerical radius of the operator matrix $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. As an application of results obtained, we show that if $B, C$ are self-adjoint operators then

$$\max \left\{ \|B + C\|^2, \|B - C\|^2 \right\} \leq \|B^2 + C^2\| + 2w(|B||C|).$$

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B}(\mathcal{H})$ be the collection of all bounded linear operators on $\mathcal{H}$. As usual the norm induced by the inner product $\langle \cdot, \cdot \rangle$ is denoted by $\|\cdot\|$. For $A \in \mathcal{B}(\mathcal{H})$, let $\|A\|$ be the operator norm of $A$, i.e., $\|A\| = \sup_{\|x\|=1} \|Ax\|$. For $A \in \mathcal{B}(\mathcal{H})$, $A^*$ denotes the adjoint of $A$ and $|A|, |A^*|$ respectively denote the positive part of $A, A^*$, i.e., $|A| = (A^*A)^{\frac{1}{2}}, |A^*| = (AA^*)^{\frac{1}{2}}$. The real part and the imaginary part of $A$ are denoted by $\Re(A)$ and $\Im(A)$ respectively so that $\Re(A) = \frac{A + A^*}{2}$ and $\Im(A) = \frac{A - A^*}{2i}$. The numerical range of $A$, denoted by $W(A)$, is defined as $W(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}$. It is well known that $W(A)$ is a compact subset of $\mathbb{C}$. The famous Toeplitz-Hausdorff theorem states that the numerical range is a convex set. The numerical radius of $A$, denoted by $w(A)$, is defined as $w(A) = \sup_{\|x\|=1} |\langle Ax, x \rangle|$. The numerical radius is a norm on $\mathcal{B}(\mathcal{H})$ satisfying

$$\frac{1}{2} \|A\| \leq w(A) \leq \|A\|, \quad (1.1)$$

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and so the numerical radius norm is equivalent to the operator norm. The inequality (1.1) is sharp, \( w(A) = \|A\| \) if \( A \) is normal and \( w(A) = \frac{\|A\|}{2} \) if \( A^2 = 0 \). The spectral radius of \( A \), denoted as \( r(A) \), is defined as \( r(A) := \sup_{\lambda \in \sigma(A)} |\lambda| \), where \( \sigma(A) \) is the spectrum of \( A \). Since \( \sigma(A) \subseteq \overline{W(A)} \), \( r(A) \leq w(A) \). For further basic properties on the numerical range and the numerical radius of bounded linear operators, we refer to [15]. Various refinements of (1.1) have been obtained recently, a few of them are in [7, 8, 9, 10, 11].

The direct sum of two copies of \( \mathcal{H} \) is denoted by \( \mathcal{H} \oplus \mathcal{H} \). If \( A, B, C, D \in \mathcal{B}(\mathcal{H}) \), then the operator matrix
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
can be considered as an operator on \( \mathcal{H} \oplus \mathcal{H} \), and is defined by
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} x = \begin{bmatrix}
A x_1 + B x_2 \\
C x_1 + D x_2
\end{bmatrix}, \forall x = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}.
\]

In this paper, we obtain several upper and lower bounds for the numerical radius of \( 2 \times 2 \) operator matrices. The bounds obtained here improve and generalize the earlier related bounds. We also obtain equality conditions for the numerical radius of \( \begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix} \), where ‘0’ denotes the zero operator on \( \mathcal{H} \). An application of some of our obtained bounds, we give norm inequalities for sums and differences of self-adjoint operators.

2. Main results

We begin this section with the following well known lemmas. The first lemma can be found in [17, Lemma 2.1].

**Lemma 2.1.** Let \( A, B, C, D \in \mathcal{B}(\mathcal{H}) \). Then
\[
(1) \quad w\left( \begin{bmatrix}
A & 0 \\
0 & D
\end{bmatrix} \right) = \max\{w(A), w(D)\}.
\]
\[
(2) \quad w\left( \begin{bmatrix}
A & B \\
B & A
\end{bmatrix} \right) = \max\{w(A + B), w(A - B)\}.
\]

In particular, \( w\left( \begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix} \right) = w(B) \).

The second lemma can be proved easily.

**Lemma 2.2.** Let \( A, D \in \mathcal{B}(\mathcal{H}) \). Then
\[
\left\| \begin{bmatrix}
A & 0 \\
0 & D
\end{bmatrix} \right\| = \left\| \begin{bmatrix}
0 & A \\
D & 0
\end{bmatrix} \right\| = \max\{\|A\|, \|D\|\}.
\]

The third lemma can be found in [16, pp. 75-76] which is a mixed Schwarz inequality.

**Lemma 2.3.** Let \( A \in \mathcal{B}(\mathcal{H}) \). Then
\[
|\langle Ax, x \rangle| \leq \langle |A|x, x \rangle \rangle^{1/2} \langle |A^*|x, x \rangle \rangle^{1/2}, \forall x \in \mathcal{H}.
\]

The fourth lemma involving positive operators can be found in [20, Cor. 2].

**Lemma 2.4.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be positive. Then
\[
\|A + B\| \leq \max\{\|A\|, \|B\|\} + \|A^{1/2} B^{1/2}\|.
\]
Our first result can be stated as the following theorem.

**Theorem 2.5.** Let $B, C \in \mathcal{B}(\mathcal{H})$. Then

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2} \max\{\|B\|, \|C\|\} + \frac{1}{2} \max\left\{ r^{\frac{1}{2}}(\|B\|\|C^*\|), r^{\frac{3}{2}}(\|B^*\|\|C\|) \right\}. $$

This inequality is sharp.

**Proof.** Let $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$. Then from Lemma 2.3 we have that

$$\left\langle\left[ \begin{array}{cc} 0 & B \\ C & 0 \end{array} \right] x, x \right\rangle \leq \left\langle\left[ \begin{array}{cc} 0 & B \\ C & 0 \end{array} \right] x, x \right\rangle^{\frac{1}{2}} \left\langle\left[ \begin{array}{cc} 0 & C^* \\ B^* & 0 \end{array} \right] x, x \right\rangle^{\frac{1}{2}} \leq \frac{1}{2} \left( \left\langle\left[ \begin{array}{cc} 0 & B \\ C & 0 \end{array} \right] x, x \right\rangle + \left\langle\left[ \begin{array}{cc} 0 & C^* \\ B^* & 0 \end{array} \right] x, x \right\rangle \right) = \frac{1}{2} \left\langle\left[ \begin{array}{cc} 0 & B \\ C & 0 \end{array} \right] + \left[ \begin{array}{cc} 0 & C^* \\ B^* & 0 \end{array} \right] \right\rangle x, x \right\rangle \leq \frac{1}{2} w\left(\begin{bmatrix} |C| + |B^*| & 0 \\ 0 & |B| + |C^*| \end{bmatrix}\right) \leq \frac{1}{2} \max\{\|\|C\| + |B^*||, \|\|B\| + |C^*||\}. $$

By considering the supremum over all $\|x\| = 1$, we get

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2} \max\{\|\|C\| + |B^*||, \|\|B\| + |C^*||\}. \quad (2.1)$$

Now it follows from Lemma 2.4 that

$$\|\|C\| + |B^*|| \leq \max\{\|B\|, \|C\|\} + \|\|C\|^{\frac{1}{2}}|B^*|^{\frac{1}{2}}\|$$

and

$$\|\|B\| + |C^*|| \leq \max\{\|B\|, \|C\|\} + \|\|B\|^{\frac{1}{2}}|C^*|^{\frac{1}{2}}\|. $$

Hence, from (2.1) we get,

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2} \max\{\|B\|, \|C\|\} + \frac{1}{2} \max\{\|\|B\|^{\frac{1}{2}}|C^*|^{\frac{1}{2}}\|, \|\|C\|^{\frac{1}{2}}|B^*|^{\frac{1}{2}}\|\}. $$

If $A, B \in \mathcal{B}(\mathcal{H})$ are positive, then $r^{\frac{1}{2}}(AB) = \|A^{1/2}B^{1/2}\|$, (see [6, Lemma 2.5]). Therefore,

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2} \max\{\|B\|, \|C\|\} + \frac{1}{2} \max\left\{ r^{\frac{1}{2}}(|B||C^*|), r^{\frac{1}{2}}(|C||B^*|) \right\}. $$

This is the required inequality. To show that the inequality is sharp, we consider $C = 0$ so that $w\left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}\right) \leq \frac{\|B\|}{2}$, which is actually equal. \qed
Remark 2.6. In particular, considering $B = C$ in Theorem 2.5 and using Lemma 2.1, we get the inequality (see [6, Th. 2.1])

$$w(B) \leq \frac{1}{2} \|B\| + \frac{1}{2} r^\frac{1}{2}(\|B\|).$$

Thus Theorem 2.5 generalizes [6, Th. 2.1].

We next obtain a lower bound for the numerical radius of the operator matrix $
\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}$.

**Theorem 2.7.** Let $B, C \in B(\mathcal{H})$. Then

$$w\left(\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}\right) \geq \frac{1}{2} \max\{\|B\|, \|C\|\} + \frac{1}{4} \|B + C^*\| - \|B - C^*\|.$$  

**Proof.** We note that for any bounded linear operator $T$, $w(T) \geq \|\Re(T)\|$ and $w(T) \geq \|\Im(T)\|$. So we have, $w\left(\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}\right) \geq \|\frac{B+C^*}{2}\|$ and $w\left(\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}\right) \geq \|\frac{B-C^*}{2}\|$. Then

$$w\left(\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}\right) \geq \frac{1}{2} \max\{\|B + C^*\|, \|B - C^*\|\}$$

$$= \frac{1}{4}(\|B + C^*\| + \|B - C^*\|) + \frac{1}{4} \|B + C^*\| - \|B - C^*\|$$

$$\geq \frac{1}{4} \|(B + C^*) \pm (B - C^*)\| + \frac{1}{4} \|B + C^*\| - \|B - C^*\|.$$  

This implies that

$$w\left(\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}\right) \geq \frac{1}{2} \max\{\|B\|, \|C\|\} + \frac{1}{4} \|B + C^*\| - \|B - C^*\|.$$  

This completes the proof. \qed

Remark 2.8. In particular, considering $B = C$ in Theorem 2.7, we get

$$w(B) \geq \frac{\|B\|}{2} + \frac{1}{4} \|B + B^*\| - \|B - B^*\|.$$  

Clearly, this is an improvement of the first inequality in (1.1), i.e., $w(B) \geq \frac{\|B\|}{2}$.

Next, we need the following lemma, known as Buzano’s extension of Schwarz inequality (see [12]).

**Lemma 2.9.** If $x, y, e \in \mathcal{H}$ with $\|e\| = 1$, then

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} (\|x\| \|y\| + |\langle x, y \rangle|).$$  

Using the above lemma we prove the following theorem.
Theorem 2.10. If $B, C \in B(\mathcal{H})$, then

$$w^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{4} \max \left\{ \|B\|^2 + |C^*|^2, \|B^*\|^2 + |C|^2 \right\} + \frac{1}{2} \max \left\{ w(|B||C^*|), w(|C||B^*|) \right\}.$$ 

This inequality is sharp.

Proof. Let $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$. Then,

$$\left| \left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x \right\rangle \right|^2 \leq \left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x \right\rangle \left\langle \begin{bmatrix} 0 & C^* \\ B^* & 0 \end{bmatrix} x, x \right\rangle, \text{ by Lemma 2.3}$$

$$= \left\langle \begin{bmatrix} |B| & 0 \\ 0 & |C| \end{bmatrix} x, x \right\rangle \left\langle \begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} x, x \right\rangle$$

$$\leq \frac{1}{2} \left\langle \begin{bmatrix} |C|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} x, x \right\rangle \left\langle \begin{bmatrix} |B^*|^2 & 0 \\ 0 & |C^*|^2 \end{bmatrix} x, x \right\rangle$$

$$+ \frac{1}{2} \left\langle \begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} x, x \right\rangle \left\langle \begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} x, x \right\rangle$$

$$\leq \frac{1}{4} \left( \left\langle \begin{bmatrix} |C|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} |B^*|^2 & 0 \\ 0 & |C^*|^2 \end{bmatrix} x, x \right\rangle \right)$$

$$+ \frac{1}{2} \left\langle \begin{bmatrix} |B^*||C| & 0 \\ 0 & |C^*||B| \end{bmatrix} x, x \right\rangle$$

$$= \frac{1}{4} \left( \left\langle \begin{bmatrix} |C|^2 + |B^*|^2 & 0 \\ 0 & |B|^2 + |C^*|^2 \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} |B^*|^2 & 0 \\ 0 & |C^*|^2 \end{bmatrix} x, x \right\rangle \right)$$

$$+ \frac{1}{2} \left\langle \begin{bmatrix} |B^*||C| & 0 \\ 0 & |C^*||B| \end{bmatrix} x, x \right\rangle$$

$$\leq \frac{1}{4} \max \{ \| |C|^2 + |B^*|^2 \|, \| |B|^2 + |C^*|^2 \| \} + \frac{1}{2} \max \{ w(|B^*||C|), w(|C^*||B|) \}.$$
Taking supremum over all \( x \in \mathcal{H}, \|x\| = 1 \), we get

\[
w^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{4} \max \{ \| |B|^2 + |C^*|^2\|, \| |B^*|^2 + |C|^2\| \} \\
+ \frac{1}{2} \max \{ w(|B||C^*|), w(|C||B^*|) \}.
\]

To show that the inequality is sharp, we consider \( C = 0 \). Then we get,

\[
w^2 \left( \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) \leq \frac{1}{4} \| B \|^2, \text{ which is actually equal. This completes the proof.} \]

\[\square\]

**Remark 2.11.** In particular, considering \( B = C \) in Theorem 2.10 and using Lemma 2.1, we get the inequality [3, Th. 2.5]

\[
w^2(B) \leq \frac{1}{4} \| |B|^2 + |B^*|^2\| + \frac{1}{2} w(|B||B^*|).
\]

Thus Theorem 2.10 generalizes [3, Th. 2.5].

Our next result reads as follows.

**Theorem 2.12.** Let \( B, C \in \mathcal{B}(\mathcal{H}) \). Then

\[
w^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \geq \frac{1}{4} \max \{ \| |B|^2 + |C^*|^2\|, \| |B^*|^2 + |C|^2\| \} \\
+ \frac{1}{8} \| B + C^* \|^2 - \| B - C^* \|^2 \).
\]

**Proof.** We note that for any bounded linear operator \( T \), \( w(T) \geq \|\Re(T)\| \) and \( w(T) \geq \|\Im(T)\| \). So we have, \( w \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \geq \|\frac{B + C^*}{2}\| \) and \( w \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \geq \|\frac{B - C^*}{2}\| \).
\[ \| \frac{B - C^*}{2i} \|. \] Then,
\[
\begin{aligned}
w^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) & \geq \frac{1}{4} \max \{ \| B + C^* \|^2, \| B - C^* \|^2 \} \\
& = \frac{1}{8} (\| B + C^* \|^2 + \| B - C^* \|^2) + \frac{1}{8} (\| B + C^* \|^2 - \| B - C^* \|^2) \\
& = \frac{1}{2} \left( \left\| \frac{B + C^*}{2} \right\|^2 + \left\| \frac{B - C^*}{2i} \right\|^2 \right) \\
& + \frac{1}{8} \| B + C^* \|^2 - \| B - C^* \|^2 \\
& = \frac{1}{2} \left( \left\| 2 \mathbb{R} \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right\|^2 + \left\| 2 \mathbb{I} \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right\|^2 \right) \\
& + \frac{1}{8} \| B + C^* \|^2 - \| B - C^* \|^2 \\
& \geq \frac{1}{2} \left( \left\| 2 \mathbb{R} \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right\|^2 + \left\| 2 \mathbb{I} \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \right\|^2 \right) \\
& + \frac{1}{8} \| B + C^* \|^2 - \| B - C^* \|^2 \\
& = \frac{1}{4} \left\| \begin{bmatrix} |C|^2 + |B|^2 & 0 \\ 0 & |B|^2 + |C|^2 \end{bmatrix} \right\|^2 \\
& + \frac{1}{8} \| B + C^* \|^2 - \| B - C^* \|^2 \\
& = \frac{1}{4} \max \{ \| |B|^2 + |C|^2 \|, \| |B^*|^2 + |C|^2 \| \} \\
& + \frac{1}{8} \| B + C^* \|^2 - \| B - C^* \|^2.
\end{aligned}
\]

This completes the proof. \( \square \)

The following necessary condition for the equality of \( w \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \) follows from Theorem 2.12.

**Proposition 2.13.** If \( B, C \in \mathcal{B}(\mathcal{H}) \), then
\[
\begin{aligned}
w^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) &= \frac{1}{4} \max \{ \| |B|^2 + |C|^2 \|, \| |B^*|^2 + |C|^2 \| \} \\
& \quad + \frac{1}{8} \| B + C^* \|^2 - \| B - C^* \|^2.
\end{aligned}
\]

implies that \( \| B + C^* \| = \| B - C^* \|. \)
Remark 2.14. In [4, Th. 2.2], the authors obtained that
\[
\begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix}
\geq \frac{1}{4} \max \left\{ \|B\|^2 + |C^*|^2, \|B\|^2 + |C|^2 \right\}.
\]
Clearly, Theorem 2.12 refines [4, Th. 2.2].

Our next improvement of [4, Th. 2.2] is as follows.

Theorem 2.15. If \(B, C \in B(\mathcal{H})\), then
\[
\begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix}
\geq \frac{1}{8} \max \left\{ \|B + C^*\|^2, \|B - C^*\|^2 \right\} + \frac{1}{2} \|B - C\| \|B - C^*\|
\geq \frac{1}{4} \max \left\{ \|B\|^2 + |C^*|^2, \|B\|^2 + |C|^2 \right\}.
\]
The inequalities are sharp.

Proof. Let \(S = \begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix}\). First inequality follows from \(w(S) \geq \|\Re(S)\|\) and \(w(S) \geq \|\Im(S)\|\). We next prove the second inequality. Clearly,
\[
\frac{1}{8} \|S\|^2 + |S^*|^2 = \frac{1}{2} \|\Re^2(S) + \Im^2(S)\|.
\]
Now, from Lemma 2.4, we get
\[
\|\Re^2(S) + \Im^2(S)\| \leq \max\{\|\Re^2(S)\|, \|\Im^2(S)\|\} + \|\Re(S)\| \|\Im(S)\|
= \max\{\|\Re(S)\|^2, \|\Im(S)\|^2\} + \|\Re(S)\| \|\Im(S)\|.
\]
Hence, we have
\[
\frac{1}{8} \|S\|^2 + |S^*|^2 \leq \frac{1}{2} \max\{\|\Re(S)\|^2, \|\Im(S)\|^2\} + \frac{1}{2} \|\Re(S)\| \|\Im(S)\|
\leq \frac{1}{2} \max\{\|\Re(S)\|^2, \|\Im(S)\|^2\} + \frac{1}{2} \|\Re(S)\| \|\Im(S)\|
= \frac{1}{2} \max\{\|\Re(S)\|^2, \|\Im(S)\|^2\} + \frac{1}{2} \|\Re(S)\| \|\Im(S)\|.
\]
This implies that
\[
\frac{1}{4} \max\left\{ \|B\|^2 + |B^*|^2, \|B\|^2 + |C^*|^2 \right\} \leq \frac{1}{2} \max\left\{ \left| \frac{B + C^*}{2} \right|^2, \left| \frac{B - C^*}{2i} \right|^2 \right\}
\]
that is,
\[
\frac{1}{4} \max\left\{ \|B\|^2 + |C^*|^2, \|B\|^2 + |C|^2 \right\} \leq \frac{1}{8} \max\left\{ \|B + C^*\|^2, \|B - C^*\|^2 \right\}
+ \frac{1}{8} \|B + C^*\| \|B - C^*\|.
\]
This is the second inequality of the theorem. To show that the inequalities are sharp, we consider \( C = 0 \). Then we get \( w^2 \left( \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) \geq \frac{1}{4} \| B \|^2 \), which is actually equal. This completes the proof. □

The following sufficient condition for the equality of \( w \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \) follows from Theorem 2.10 and Theorem 2.15.

**Proposition 2.16.** Let \( B, C \in \mathcal{B}(\mathcal{H}) \). If \( |B||C^*| = |B^*||C| = 0 \), then
\[
w^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) = \frac{1}{4} \max \left\{ \|B\|^2 + |C^*|^2, \|B^*\|^2 + |C|^2 \right\}.
\]

**Remark 2.17.** In particular, considering \( B = C \) in Theorem 2.15 and using \( w \left( \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) = w(B) \), we get
\[
w^2(B) \geq \frac{1}{8} \left[ \max \left\{ \|B + B^*\|^2, \|B - B^*\|^2 \right\} + \|B + B^*\||B - B^*\| \right]
\geq \frac{1}{4} \|B\|^2 + |B^*|^2.
\]
Thus Theorem 2.15 generalizes [5, Th. 2.10].

For next result we need the following lemma (see [4, Th. 2.4]).

**Lemma 2.18.** If \( A, B \in \mathcal{B}(\mathcal{H}) \), then
\[
\|A + B\|^2 \leq 2 \max \left\{ \|A\|^2 + \|B\|^2, \|A^*\|^2 + \|B^*\|^2 \right\}.
\]

**Theorem 2.19.** If \( B, C \in \mathcal{B}(\mathcal{H}) \), then
\[
w^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \geq \frac{1}{4\sqrt{2}} \left[ \|B + C^*\|^4 + \|B - C^*\|^4 \right]^\frac{1}{2}
\geq \frac{1}{4} \max \left\{ \|B\|^2 + |C^*|^2, \|B^*\|^2 + |C|^2 \right\}.
\]

The inequalities are sharp.

**Proof.** Let \( \mathcal{S} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \). First inequality follows from \( w(\mathcal{S}) \geq \|\Re(\mathcal{S})\| \) and \( w(\mathcal{S}) \geq \|\Im(\mathcal{S})\| \). We next prove the second inequality. Clearly,
\[
\frac{1}{4} \|\mathcal{S}\|^2 + |\mathcal{S}^*|^2 = \frac{1}{2} \|\Re^2(\mathcal{S}) + \Im^2(\mathcal{S})\|.
\]
Now, from Lemma 2.18, we have
\[
\|\Re^2(\mathcal{S}) + \Im^2(\mathcal{S})\| \leq \sqrt{2} \|\Re^4(\mathcal{S}) + \Im^4(\mathcal{S})\|^\frac{1}{2}
\leq \sqrt{2} \left[ \|\Re(\mathcal{S})\|^4 + \|\Im(\mathcal{S})\|^4 \right]^\frac{1}{2}.
\]
Hence, we have
\[
\frac{1}{4} \|\mathcal{S}\|^2 + |\mathcal{S}^*|^2 \leq \frac{1}{\sqrt{2}} \left[ \|\Re(\mathcal{S})\|^4 + \|\Im(\mathcal{S})\|^4 \right]^\frac{1}{2}.
\]
This implies that
\[
\frac{1}{4} \left\| \begin{bmatrix} |C|^2 + |B^*|^2 & 0 \\ 0 & |B|^2 + |C^*|^2 \end{bmatrix} \right\| \leq \frac{1}{\sqrt{2}} \left( \left\| \frac{B + C^*}{2} \right\|^4 + \left\| \frac{B - C^*}{2i} \right\|^4 \right)^{\frac{1}{2}},
\]
that is,
\[
\frac{1}{4} \max \left\{ \left\| |B|^2 + |C^*|^2 \right\|, \left\| |B^*|^2 + |C|^2 \right\| \right\} \leq \frac{1}{4\sqrt{2}} \left( \left\| B + C^* \right\|^4 + \left\| B - C^* \right\|^4 \right)^{\frac{1}{2}}.
\]
This is the second inequality of the theorem. To show that the inequalities are sharp, we consider \( C = 0 \). Then we get
\[
\omega^2 \left( \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) \geq \frac{1}{4}\left\| B \right\|^2,
\]
which is actually equal. This completes the proof. \( \square \)

Remark 2.20. In particular, considering \( B = C \) in Theorem 2.19, we get the inequality (see [5, Th. 2.13])
\[
\omega^2(B) \geq \frac{1}{4\sqrt{2}} \left( \left\| B + B^* \right\|^4 + \left\| B - B^* \right\|^4 \right)^{\frac{1}{2}} \geq \frac{1}{4}\left\| B \right\|^2 + \left\| B^* \right\|^2
\]
and so Theorem 2.19 is a generalization of [5, Th. 2.13].

For our next result we need the following lemmas.

Lemma 2.21. ([21, p. 20]). Let \( A \in \mathcal{B}(\mathcal{H}) \) be positive, i.e., \( A \geq 0 \). Then
\[
\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle,
\]
for all \( r \geq 1 \) and for all \( x \in \mathcal{H} \) with \( \|x\| = 1 \).

Lemma 2.22. Let \( x, y, e \in \mathcal{H} \) with \( \|e\| = 1 \). Then we have, for \( 0 \leq \alpha \leq 1 \)
\[
\left| \langle x, e \rangle \langle e, y \rangle \right|^2 \leq \frac{1 + \alpha}{4} \|x\|^2 \|y\|^2 + \frac{1 - \alpha}{4} \|\langle x, y \rangle\|^2 + \frac{1}{2} \|x\| \|y\| \|\langle x, y \rangle\|.
\]

Proof. From Lemma 2.9, we have
\[
\left| \langle x, e \rangle \langle e, y \rangle \right|^2 \leq \frac{1}{4} \left( \|x\| \|y\| + \|\langle x, y \rangle\| \right)^2
\]
\[
\quad = \frac{1}{4} \left( \|x\|^2 \|y\|^2 + 2 \|x\| \|y\| \|\langle x, y \rangle\| + \|\langle x, y \rangle\|^2 \right)
\]
\[
\quad = \frac{1}{4} \left( \|x\|^2 \|y\|^2 + 2 \|x\| \|y\| \|\langle x, y \rangle\| + \alpha \|\langle x, y \rangle\|^2 + (1 - \alpha) \|\langle x, y \rangle\|^2 \right)
\]
\[
\quad \leq \frac{1}{4} \left( \|x\|^2 \|y\|^2 + 2 \|x\| \|y\| \|\langle x, y \rangle\| + \alpha \|x\|^2 \|y\|^2 + (1 - \alpha) \|\langle x, y \rangle\|^2 \right)
\]
\[
\quad \leq \frac{1 + \alpha}{4} \|x\|^2 \|y\|^2 + \frac{1 - \alpha}{4} \|\langle x, y \rangle\|^2 + \frac{1}{2} \|x\| \|y\| \|\langle x, y \rangle\|,
\]
as desired. \( \square \)

Now, we are in a position to prove our next result.
Theorem 2.23. If $B, C \in \mathcal{B}(\mathcal{H})$, then for $0 \leq \alpha \leq 1$, we have

$$w^4 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1 + \alpha}{8} \max \left\{ \|B\|^4 + \|C^*\|^4, \|B^*\|^4 + \|C\|^4 \right\} + \frac{1 - \alpha}{4} w^2(BC), w^2(CB) \right\} + \frac{1}{4} \max \left\{ \|B\|^2 + \|C^*\|^2, \|B^*\|^2 + \|C\|^2 \right\} \times \max \{w(BC), w(CB)\}.$$

Proof. Let $S = \left[ \begin{array}{cc} 0 & B \\ C & 0 \end{array} \right]$. Let $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$. Then it follows from Lemma 2.22 that

$$|\langle Sx, x \rangle|^4 = |\langle Sx, x \rangle\langle x, S^*x \rangle|^2 \leq \frac{1 + \alpha}{8} \|Sx\|^2 \|S^*x\|^2 + \frac{1 - \alpha}{4} |\langle S^2x, x \rangle|^2 + \frac{1}{2} \|Sx\| \|S^*x\| |\langle S^2x, x \rangle| \leq \frac{1 + \alpha}{8} (\|S\|^4 + \|S^*\|^4) + \frac{1 - \alpha}{4} |\langle S^2x, x \rangle|^2 + \frac{1}{4} (\|S\|^2 + \|S^*\|^2) |\langle S^2x, x \rangle|,$$

using Lemma 2.21

$$= \frac{1 + \alpha}{8} \left( \begin{array}{cc} |C|^4 + |B^*|^4 & 0 \\ 0 & |B|^4 + |C^*|^4 \end{array} \right) x, x \right\} + \frac{1 - \alpha}{4} \left( \begin{array}{cc} BC & 0 \\ 0 & CB \end{array} \right) x, x \right\}^2 + \frac{1}{4} \left( \begin{array}{cc} |C|^2 + |B^*|^2 & 0 \\ 0 & |B|^2 + |C^*|^2 \end{array} \right) x, x \right\} |\left( \begin{array}{cc} BC & 0 \\ 0 & CB \end{array} \right) x, x \right\} \leq \frac{1 + \alpha}{8} \sum \left( \begin{array}{cc} |C|^4 + |B^*|^4 & 0 \\ 0 & |B|^4 + |C^*|^4 \end{array} \right) x, x \right\} + \frac{1 - \alpha}{4} \sum \left( \begin{array}{cc} BC & 0 \\ 0 & CB \end{array} \right) x, x \right\} \leq \frac{1 + \alpha}{8} \max \{ \|B\|^4 + \|C^*\|^4, \|B^*\|^4 + \|C\|^4 \} + \frac{1 - \alpha}{4} \max \{w^2(BC), w^2(CB)\} + \frac{1}{4} \max \{\|B\|^2 + \|C^*\|^2, \|B^*\|^2 + \|C\|^2 \} \times \max \{w(BC), w(CB)\}.$$
Taking supremum over all \( x \in \mathcal{H}, \|x\| = 1 \), we get
\[
\begin{aligned}
\quad w^4 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \\
\quad \leq \frac{1 + \alpha}{8} \max \left\{ \| |B|^4 + |C^*|^4\|, \| |B^*|^4 + |C|^4\| \right\} \\
\quad + \frac{1 - \alpha}{4} \max \left\{ w^2(BC), w^2(CB) \right\} \\
\quad + \frac{1}{4} \max \left\{ \| |B|^2 + |C^*|^2\|, \| |B^*|^2 + |C|^2\| \right\} \times \max \left\{ w(BC), w(CB) \right\}.
\end{aligned}
\]
\[
\square
\]

In particular, considering \( B = C \) in Theorem 2.23, we get the following corollary.

**Corollary 2.24.** If \( B \in \mathcal{B}(\mathcal{H}) \), then for \( 0 \leq \alpha \leq 1 \),
\[
w^4(B) \leq \frac{1 + \alpha}{8} \| |B|^4 + |B^*|^4\| + \frac{1 - \alpha}{4} w^2(B^2) + \frac{1}{4} \| |B|^2 + |B^*|^2\| w(B^2).
\]

**Remark 2.25.** For every \( 0 \leq \alpha \leq 1 \), we have
\[
w^4(B) \leq \frac{1 + \alpha}{8} \| |B|^4 + |B^*|^4\| + \frac{1 - \alpha}{4} w^2(B^2) + \frac{1}{4} \| |B|^2 + |B^*|^2\| w(B^2)
\leq \frac{1 + \alpha}{8} \| |B|^4 + |B^*|^4\| + \frac{1 - \alpha}{4} \| B^2 \| + \frac{1}{4} \| |B|^2 + |B^*|^2\| B^2
\leq \frac{1 + \alpha}{8} \| |B|^4 + |B^*|^4\| + \frac{1 - \alpha}{4} \left\| \frac{|B|^2 + |B^*|^2}{2} \right\|
\quad + \frac{1}{4} \| |B|^2 + |B^*|^2\| \left\| \frac{|B|^2 + |B^*|^2}{2} \right\| B^2 \| \leq \frac{1}{2} \| |B|^2 + |B^*|^2\|
\leq \frac{1 + \alpha}{8} \| |B|^4 + |B^*|^4\| + \frac{1}{8} \| |B|^4 + |B^*|^4\|
\quad + \frac{1}{4} \| |B|^4 + |B^*|^4\|, \quad \left\| \frac{|B|^2 + |B^*|^2}{2} \right\| \leq \frac{|B|^4 + |B^*|^4}{2}
\leq \frac{1}{2} \| |B|^4 + |B^*|^4\|.
\]

Hence, Corollary 2.24 refines the earlier related inequality \( w^4(B) \leq \frac{1}{2} \| |B|^4 + |B^*|^4\| \), (see [14], for \( r = 2 \)).

We next obtain the following estimation for an upper bound of the numerical radius of general \( 2 \times 2 \) operator matrices, i.e.,
\[
w \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right).
\]


Theorem 2.26. If $A, B, C, D \in \mathcal{B}(\mathcal{H})$, then for $0 \leq \alpha \leq 1$

$$w^4 \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 8 \max \left\{ w^4(A), w^4(D) \right\} + (1 + \alpha) \max \left\{ \|B\|^4 + \|C^*\|^4, \|B^*\|^4 + \|C\|^4 \right\} + 2(1 - \alpha) \max \left\{ w^2(BC), w^2(CB) \right\} + 2 \max \left\{ \|B\|^2 + \|C^*\|^2, \|B^*\|^2 + \|C\|^2 \right\} \times \max \left\{ w(BC), w(CB) \right\}.$$ 

Proof. Let $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$. Now have by convexity of $f(t) = t^4$,

$$\left| \left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} x, x \right\rangle \right|^4 \leq \left( \left| \left\langle \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} x, x \right\rangle \right|^4 + \left| \left\langle \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix} x, x \right\rangle \right|^4 \right)^4 \leq 8 \left( \left| \left\langle \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} x, x \right\rangle \right|^4 + \left| \left\langle \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix} x, x \right\rangle \right|^4 \right)^4 \leq 8w^4 \left( \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \right) + 8w^4 \left( \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix} \right) = 8 \max \left\{ w^4(A), w^4(D) \right\} + 8w^4 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right).$$

Taking supremum over all $x \in \mathcal{H}, \|x\| = 1$ we have,

$$w^4 \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 8 \max \left\{ w^4(A), w^4(D) \right\} + 8w^4 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right).$$

Therefore, by using Theorem 2.23, we get

$$w^4 \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 8 \max \left\{ w^4(A), w^4(D) \right\} + (1 + \alpha) \max \left\{ \|B\|^4 + \|C^*\|^4, \|B^*\|^4 + \|C\|^4 \right\} + 2(1 - \alpha) \max \left\{ w^2(BC), w^2(CB) \right\} + 2 \max \left\{ \|B\|^2 + \|C^*\|^2, \|B^*\|^2 + \|C\|^2 \right\} \times \max \left\{ w(BC), w(CB) \right\}.$$ 

\[ \square \]

Remark 2.27. It follows from [13] that $w(CB) \leq \frac{1}{2} \|B\|^2 + \|C^*\|^2$ and $w(BC) \leq \frac{1}{2} \|B^*\|^2 + \|C\|^2$. Therefore, clearly it follows that the inequality obtained in Theorem 2.26 is stronger than the recently obtained inequality [2, Th. 3.1], that is,

$$w^4 \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq 8 \max \left\{ w^4(A), w^4(D) \right\} + (1 + \alpha) \max \left\{ \|B\|^4 + \|C^*\|^4, \|B^*\|^4 + \|C\|^4 \right\} + (3 - \alpha) \max \left\{ \|B\|^2 + \|C^*\|^2, \|B^*\|^2 + \|C\|^2 \right\} \times \max \left\{ w(BC), w(CB) \right\}.$$
3. Application

As application of results obtained bounds in Section 2, we develop some norm inequalities for sums and differences of self-adjoint operators. Note that if $B, C \in \mathcal{B}(\mathcal{H})$ are positive then $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \frac{\|B+C\|}{2}$, (see [1, Cor. 3]). Now we prove the following proposition, though it is known the proof given here is simple and different.

Proposition 3.1. If $B, C \in \mathcal{B}(\mathcal{H})$ are positive, then

(i) $\|B - C\| \leq \|B + C\|,$

(ii) $\max\{|\|B\||, |\|C\||\} \leq \frac{\|B + C\|}{2} + \frac{\|B - C\|}{2}.$

Proof. From the first inequality in 2.19, we have

$$\frac{\|B + C\|^2}{4} \geq \frac{1}{4\sqrt{2}} \left[\|B + C\|^4 + \|B - C\|^4\right]^{\frac{1}{2}}.$$ 

This implies that $\|B - C\| \leq \|B + C\|,$ i.e, (i). Now from Theorem 2.7 we have,

$$\frac{\|B + C\|}{2} \geq \frac{1}{2} \max\{|\|B\||, |\|C\||\} + \frac{1}{4}(\|B + C\| - \|B - C\|).$$

Therefore, using (i) we have,

$$\frac{\|B + C\|}{2} \geq \frac{1}{2} \max\{|\|B\||, |\|C\||\} + \frac{1}{4}(\|B + C\| - \|B - C\|).$$

This completes the proof of (ii). $\square$

Next we prove the following.

Theorem 3.2. Let $B, C \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then,

$$\max\left\{\|B + C\|^2, \|B - C\|^2\right\} \leq \|B^2 + C^2\| + 2w(|\|B||\|C||).$$

Proof. We have

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \geq \Re\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$$

and

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \geq \Im\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$$

so that

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \geq \left\|\frac{B + C}{2}\right\|$$

and

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \geq \left\|\frac{B - C}{2}\right\|$$

respectively. Therefore,

$$\frac{1}{4} \max\left\{\|B + C\|^2, \|B - C\|^2\right\} \leq w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right).$$
Hence, using Theorem 2.10 we get
\[
\max\left\{ \|B + C\|^2, \|B - C\|^2 \right\} \leq \|B^2 + C^2\| + 2w(\|B\|||C|).
\]
This completes the proof. 

\[\square\]

**Remark 3.3.** (i) It follows from the triangle inequality of the numerical radius that if \(B, C \in \mathcal{B}(\mathcal{H})\) are self-adjoint, then
\[
\max\left\{ \|B + C\|^2, \|B - C\|^2 \right\} \leq \|B^2 + C^2\| + 2w(BC).
\]

(ii) Clearly, if \(B, C\) are positive then the inequalities in Theorem 3.2 and Remark 3.3(i) are same. In [18], Kittaneh proved that if \(B, C \in \mathcal{B}(\mathcal{H})\) are positive, then
\[
\|B + C\| \leq \frac{1}{2} \left[ \|B\| + \|C\| + \sqrt{\left(\|B\| - \|C\|\right)^2 + 4\|B^{1/2}C^{1/2}\|^2} \right].
\]

In the example given below, we note that the bound obtained in Theorem 3.2 (for positive operators) is better than that in [18]. Consider \(B = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}\) and \(C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\). Then, Theorem 3.2 gives \(\|B + C\| \leq 5\), whereas [18] gives \(\|B + C\| \leq 3 + \sqrt{5}\).

**Remark 3.4.** Let \(B, C \in \mathcal{B}(\mathcal{H})\) be self-adjoint. It follows from Theorem 3.2 and Remark 3.3(i) that if \(\|B + C\| = \|B\| + \|C\|\), then
\[
(i) \quad \|B^2 + C^2\| = \|B\|^2 + \|C\|^2,
\]
\[
(ii) \quad w(|B||C|) = \|BC\| = \|B\|||C|\| = w(BC).
\]

The converse of the above result does not hold, in general. As for example consider \(B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) and \(C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\). Then we see that \(\|B^2 + C^2\| = \|B\|^2 + \|C\|^2 = 2\) and \(w(|B||C|) = \|BC\| = \|B\|||C|\| = w(BC) = 1\), but \(0 = \|B + C\| \neq \|B\| + \|C\| = 2\). We note that (see [18]) when \(B, C\) are positive, then \(\|B + C\| = \|B\| + \|C\|\) if and only if \(\|BC\| = \|B\|||C|\|\).

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