More Sublattices of the Lattice of Local Clones

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Abstract We investigate the complexity of the lattice of local clones over a countably infinite base set. In particular, we prove that this lattice contains all algebraic lattices with at most countably many compact elements as complete sublattices, but that the class of lattices embeddable into the local clone lattice is strictly larger than that: For example, the lattice $M_{2^\omega}$ is a sublattice of the local clone lattice.

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1 Local Clones

1.1 Defining Local Clones

Fix a countably infinite base set $X$, and denote for all $n \geq 1$ the set $X^n = \{ f : X^n \to X \}$ of $n$-ary operations on $X$ by $\mathcal{O}^{(n)}$. Then the union $\mathcal{O} := \bigcup_{n \geq 1} \mathcal{O}^{(n)}$ is the set of all...
finitary operations on $X$. A clone $C$ is a subset of $\mathcal{O}$ satisfying the following two properties:

- $C$ contains all projections, i.e., for all $1 \leq k \leq n$ the operation $\pi_k^n \in \mathcal{O}(n)$ defined by $\pi_k^n(x_1, \ldots, x_n) = x_k$, and
- $C$ is closed under composition, i.e., whenever $f \in C$ is $n$-ary and $g_1, \ldots, g_n \in C$ are $m$-ary, then the operation $f(g_1, \ldots, g_n) \in \mathcal{O}(m)$ defined by
  $$(x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$
  is also an element of $C$.

Since arbitrary intersections of clones are again clones, the set of all clones on $X$, equipped with the order of inclusion, forms a complete lattice $\text{Cl}(X)$. In this paper, we are not interested in all clones of $\text{Cl}(X)$, but only in clones which satisfy an additional topological closure property: Equip $X$ with the discrete topology, and $\mathcal{O}(n) = X^{X^n}$ with the corresponding product topology (Tychonoff topology), for every $n \geq 1$. A clone $C$ is called locally closed or just local iff each of its $n$-ary fragments $C \cap \mathcal{O}(n)$ is a closed subset of $\mathcal{O}(n)$. Equivalently, a clone $C$ is local iff it satisfies the following interpolation property:

For all $n \geq 1$ and all $g \in \mathcal{O}(n)$, if for all finite $A \subseteq X^n$ there exists an $n$-ary $f \in C$ which agrees with $g$ on $A$, then $g \in C$.

Again, taking the set of all local clones on $X$, and ordering them according to set-theoretical inclusion, one obtains a complete lattice, which we denote by $\text{Cl}_{\text{loc}}(X)$: This is because intersections of clones are clones, and because arbitrary intersections of closed sets are closed. We are interested in the structure of $\text{Cl}_{\text{loc}}(X)$, in particular in how complicated it is as a lattice.

Before we start our investigations, we give an alternative description of local clones which will be useful. Let $f \in \mathcal{O}(n)$ and let $\rho \subseteq X^m$ be a relation. We say that $f$ preserves $\rho$ iff $f(r_1, \ldots, r_n) \in \rho$ whenever $r_1, \ldots, r_n \in \rho$; here, the $m$-tuple $f(r_1, \ldots, r_n)$ is calculated componentwise, i.e., it is the $m$-tuple whose $i$-th component is obtained by applying $f$ to the $n$-tuple consisting of the $i$-th components of the tuples $r_1, \ldots, r_n$. For a set of relations $\mathcal{R}$, we write $\text{Pol}(\mathcal{R})$ for the set of those operations in $\mathcal{O}$ which preserve all $\rho \in \mathcal{R}$. The operations in $\text{Pol}(\mathcal{R})$ are called polymorphisms of $\mathcal{R}$. The following is due to [14], see also the textbook [17].

**Theorem 1** $\text{Pol}(\mathcal{R})$ is a local clone for all sets of relations $\mathcal{R}$. Moreover, every local clone is of this form.

Similarly, for an operation $f \in \mathcal{O}(n)$ and a relation $\rho \subseteq X^m$, we say that $\rho$ is invariant under $f$ iff $f$ preserves $\rho$. Given a set of operations $\mathcal{F} \subseteq \mathcal{O}$, we write $\text{Inv}(\mathcal{F})$ for the set of all relations which are invariant under all $f \in \mathcal{F}$. Since arbitrary intersections of local clones are local clones again, the mapping on the power set of $\mathcal{O}$ which assigns to every set of operations $\mathcal{F} \subseteq \mathcal{O}$ the smallest local clone $(\mathcal{F})_{\text{loc}}$ containing $\mathcal{F}$ is a closure operator, the closed elements of which are exactly the local clones. Using the operators $\text{Pol}$ and $\text{Inv}$ which connect operations and relations, one obtains the following well-known alternative for describing this operator (confer [14] or [17]).