POSETS, PARKING FUNCTIONS AND THE REGIONS OF THE SHI ARRANGEMENT REVISITED

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ABSTRACT. The number of regions of the type $A_{n-1}$ Shi arrangement in $\mathbb{R}^n$ is counted by the intrinsically beautiful formula $(n + 1)^{n-1}$. First proved by Shi, this result motivated Pak and Stanley as well as Athanasiadis and Linusson to provide bijective proofs. We give a description of the Athanasiadis-Linusson bijection and generalize it to a bijection between the regions of the type $C_n$ Shi arrangement in $\mathbb{R}^n$ and sequences $a_1a_2\ldots a_n$, where $a_i \in \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$, $i \in [n]$. Our bijections naturally restrict to bijections between regions of the arrangements with a certain number of ceilings (or floors) and sequences with a given number of distinct elements. A special family of posets, whose antichains encode the regions of the arrangements, play a central role in our approach.

1. Introduction

A hyperplane arrangement $A$ is a finite set of affine hyperplanes in $\mathbb{R}^n$. The regions of $A$ are the connected components of the space $\mathbb{R}^n \setminus \cup_{H \in A} H$. In this paper we study Shi arrangements of type $A_{n-1}$ and $C_n$, which are affine hyperplane arrangements whose hyperplanes are parallel to reflecting hyperplanes of Coxeter groups. Denote by Cox$^A(n)$ the Coxeter arrangement of type $A_{n-1}$:

$$\text{Cox}^A(n) = \{x_i - x_j = 0 \mid 1 \leq i < j \leq n\}.$$

Note that the regions of Cox$^A(n)$ are naturally indexed by type $A_{n-1}$ permutations $w \in S_n$. Namely, if $C^A \subset \mathbb{R}^n$ is the dominant cone of Cox$^A(n)$ defined by $x_1 > x_2 > \cdots > x_n$, then

$$wC^A = \{x \in \mathbb{R}^n \mid x_{w(1)} > x_{w(2)} > \cdots > x_{w(n)}\}.$$

Thus, the number of regions of Cox$^A(n)$ is $|S_n| = n!$. The type $A_{n-1}$ Shi arrangement $S_n^A$ was first defined by Shi [Shi1]:

$$S_n^A = \text{Cox}^A(n) \cup \{x_i - x_j = 1 \mid 1 \leq i < j \leq n\}.$$

Shi [Shi1] proved the beautiful result that the number of regions of $S_n^A$ is $(n + 1)^{n-1}$. This statement is clearly deserving of a combinatorial proof; two different bijections proving this result were provided by Stanley [Sta1, Sta2] and Athanasiadis and Linusson [Ath-Lin]. We give a description of the Athanasiadis-Linusson bijection and generalize it to type $C_n$, thereby answering a question of Athanasiadis and Linusson [Ath-Lin, Section 4, Question 3]. We also study statistics naturally arising from the bijections. In their forthcoming work on parking spaces [Arm-Rei-Rho], Armstrong, Reiner and Rhoades provide the ultimate generalization of the Athanasiadis-Linusson bijection by constructing a uniform bijection for all crystallographic root systems.

We now review the definitions necessary to state our results.

A sequence $a = (a_1, a_2, \ldots, a_n) \in [n]^n$ is a parking function if and only if the increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ of $a_1, a_2, \ldots, a_n$ satisfies $b_i \leq i$. Denote by $PF(n)$ the set of all parking functions of length $n$. Let

$$A(n) = \{a = (a_1, a_2, \ldots, a_n) \mid a_i \in [n+1], i \in [n]\}.$$
Denote by $d(a)$ the number of distinct numbers contained in the sequence $a$.

A hyperplane $H$ is a wall of a region $R$ if it is the affine span of a codimension-1 face of $R$. A wall $H$ is called a floor if $H$ does not contain the origin and $R$ and the origin lie in opposite half-spaces defined by $H$. Denote by $f(R)$ the number of floors of $R$. A wall $H$ is called a ceiling if $H$ does not contain the origin and $R$ and the origin lie in the same half-spaces defined by $H$. Denote by $c(R)$ the number of floors of $R$. Denote by $R(\mathcal{H})$ the set of regions of the hyperplane arrangement $\mathcal{H}$.

Our first description of the type $A_{n-1}$ bijection proving that $|R(S_n^A)| = (n+1)^{n-1}$ yields a natural correspondence between the multiset $\mathcal{M}$ in which each element of $R(S_n^A)$ appears $n+1$ times and the sequences $\mathcal{A}(n)$, while the second description, analogous to that of Athanasiadis and Linusson \cite{Ath-Lin} is a direct correspondence between $R(S_n^A)$ and parking functions. The properties of these bijections yield Theorem 2

**Theorem 1.** \cite{Ath-Lin} There is a bijection between the regions of $S_n^A$ and parking functions of length $n$.

**Theorem 2.**

$$\sum_{R \in R(S_n^A)} q'^{(R)} = \sum_{R \in R(S_n^A)} q'^{(R)} = \frac{1}{n+1} \sum_{a \in \mathcal{A}(n)} q^{n-d(a)} = \sum_{a \in PF(n)} q^{n-d(a)}.$$ 

We use the techniques developed for the type $A_{n-1}$ case to construct bijective proofs for type $C_n$.

The **type $B_n$ and $C_n$ Coxeter arrangement** $\text{Cox}^{BC}(n)$ in $\mathbb{R}^n$ is defined as follows.

$$\text{Cox}^{BC}(n) = \{x_i - x_j = 0, x_i + x_j = 0, x_k = 0 \mid 1 \leq i < j \leq n, k \in [n]\},$$

$$= \{x_i - x_j = 0, x_i + x_j = 0, 2x_k = 0 \mid 1 \leq i < j \leq n, k \in [n]\}.$$

Just as in the type $A_{n-1}$ case, the regions of the arrangements $\text{Cox}^{BC}(n)$ naturally correspond to type $B_n$ permutations $w \in S_n^B$. Recall that $S_n^B$ is the group of all bijections $w$ of the set $[\pm n] = \{-n, -n+1, \ldots, -1, 1, \ldots, n-1, n\}$ onto itself such that

$$w(-i) = -w(i),$$

for all $i \in [\pm n]$ and composition as group operation. The notation $w = [a_1, \ldots, a_n]$ means $w(i) = a_i$, for $i \in [n]$, and is called the **window** of $w$.

Let $C^{BC} \subseteq \mathbb{R}^n$ be the **dominant cone** of $\text{Cox}^{BC}(n)$ defined by

$$C^{BC} = \{x \in \mathbb{R}^n \mid -x_n > -x_{n-1} > \cdots > -x_2 > -x_1 > x_1 > x_2 > \cdots > x_{n-1} > x_n\}.$$

Let

$$wC^{BC} = \{x \in \mathbb{R}^n \mid x_w(-n) > x_w(-n+1) > \cdots > x_w(-1) > x_w(1) > x_w(2) > \cdots > x_w(n)\},$$

where $\{x_1, \ldots, x_n\}$ are the standard coordinate functions on $\mathbb{R}^n$ and $x_{-i} = -x_i$ for $i < 0$. It follows that the number of regions of $\text{Cox}^{BC}(n)$ is $|S_n^B| = 2^n n!$.

The **type $C_n$ Shi arrangement** $S_n^C$ is as expected:

$$S_n^C = \text{Cox}^{BC}(n) \cup \{x_i - x_j = 1, x_i + x_j = 1, 2x_k = 1 \mid 1 \leq i < j \leq n, k \in [n]\}.$$

We construct bijections between the regions of the type $C_n$ Shi arrangement $S_n^C$ in $\mathbb{R}^n$ and sequences in the set

$$\mathcal{A}^C(n) = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}, i \in [n]\}.$$

Athanasiadis and Linusson \cite[Section 4, Question 3]{Ath-Lin} were the first to ask for the construction of such bijection in their paper dealing with the type $A_{n-1}$ case. The properties of our bijections yield Theorem 3.
Theorem 3. There is a bijection between the regions of $\mathcal{S}_n^C$ and sequences in the set $\mathcal{A}_C(n)$.

Theorem 4.

$$\sum_{R \in \mathcal{R}(\mathcal{S}_n^C)} q^{\mathcal{C}(R)} = \sum_{R \in \mathcal{R}(\mathcal{S}_n^C)} q^{\mathcal{J}(R)} = \sum_{a \in \mathcal{A}_C(n)} q^{n-d_C(a)};$$

where $d_C(a)$ is the number of distinct absolute values of the nonzero numbers appearing in $a$.

The outline of the paper is as follows. In Section 2 we explain the connection between the regions of Shi arrangements and posets of nonnesting partitions. In Section 3 we build on this connection to give a description of the Athanasiadis-Linusson bijection between the regions of the type $A_{n-1}$ Shi arrangement and parking functions, as well as prove Theorem 2. In Section 4 we generalize the contents of Section 3 to the type $C_n$ case, proving Theorems 3 and 4. Section 5 is the story of Section 3 without arrangements, only in terms of posets and sequences of type $A_{n-1}$. Section 6 similarly reiterates the basic thoughts in type $C_n$ on the level of posets and sequences.

2. Posets and the regions of Shi arrangements

Our bijections are based on a correspondence developed by Stanley in [Sta1] between the antichains of a certain family of posets $Q_w$, $w \in \mathcal{G}_n$, and regions of $\mathcal{S}_n^A$. In this section we explain this correspondence and its type $C_n$ extension. For a related bijection between the positive chambers of the Shi arrangement and order ideals of the root poset of corresponding type see [Arm] and [Cel-Pap]. For basic definitions about posets see [Sta3, Chapter 3].

2.1. Poset and the regions of $\mathcal{S}_n^A$. Each region of $\mathcal{S}_n^A$ lies in one of the cones of $\text{Cox}^A(n)$. We restrict our attention to the regions of $\mathcal{S}_n^A$ in an arbitrary cone $wC^A$, $w \in \mathcal{G}_n$. Each such region is uniquely determined by the set of its ceilings (or the set of its floors). The set of hyperplanes of $\mathcal{S}_n^A$ intersecting $wC^A$ is

$$\mathcal{H}_w = \{x_{w(i)} - x_{w(j)} = 1 | 1 \leq i < j \leq n, w(i) < w(j)\}.$$

There are two natural orders on the hyperplanes in $\mathcal{H}_w$; namely, hyperplane $H_1$ is less than hyperplane $H_2$ if all the points in $wC^A$ which are on the same (opposite) side of $H_1$ as the origin are also on the same (opposite) side of $H_2$ as the origin. Thus, $\mathcal{H}_w$ can be considered as a poset. The set of ceilings of some region of $\mathcal{S}_n^A$ in $wC^A$ is an antichain of this poset. As Theorem 5 states below, the reverse is also true, and so the antichains of $\mathcal{H}_w$ are in bijection with the regions of $\mathcal{S}_n^A$ in $wC^A$. To avoid any confusion we now (re-)define the poset we consider.

Let

$$Q_w = \{(i, j) : 1 \leq i < j \leq n, w(i) < w(j)\}$$

partially ordered by

$$(i, j) \leq (r, s) \text{ if } r \leq i < j \leq s.$$ We think of $(i, j) \in Q_w$ as the hyperplane $x_{w(i)} - x_{w(j)} = 1$ in $\mathcal{H}_w$. Note that in $Q_w$ we have $(i, j) \leq (r, s)$ if and only if all points in $wC^A$ which are on the same side of $x_{w(i)} - x_{w(j)} = 1$ as the origin are also on the same side of $x_{w(r)} - x_{w(s)} = 1$ as the origin.

We represent antichains of $Q_w$ as partitions of $[n]$, where we draw an arc $(i, j)$ in the diagram if $(i, j) \in Q_w$ is in the antichain. For basic definitions about partitions see [Sta3, Chapter 1]. Bijecting the regions of $\mathcal{S}_n^A$ to their set of ceilings, and the set of ceilings to the corresponding antichains in $Q_w$, which we represent as partitions, we obtain a labeling of the regions of $\mathcal{S}_n^A$ by partitions, as shown on Figure 1.
Figure 1. Labeling the regions of $S_n^A$ by partitions.

**Theorem 5.** [Sta1, Section 5], [Sta2, Theorem 2.1] There is a bijection between the regions of $S_n^A$ contained in the cone $wC^A$ and the antichains of the poset $Q_w$. In particular,

$$|R(S_n^A)| = \sum_{w \in S_n} j(Q_w),$$

where $j(Q_w)$ denotes the number of antichains of the poset $Q_w$.

**Proof.** It is clear from the above that there is an injective map from the regions of $S_n^A$ to the multiset of the antichains of the posets $Q_w$, $w \in S_n$. Since it is known that $|R(S_n^A)| = (n+1)^{n-1}$ [Shi1] and $\sum_{w \in S_n} j(Q_w) = (n+1)^{n-1}$ can be proved without reference to $S_n^A$ (see Corollary 21 in Section 5) the map also has to be surjective and Theorem 5 follows. \[\square\]

Studying the relations of the poset $Q_w$ we see that the antichains of $Q_w$ correspond to nonnesting $A_{n-1}$-partitions if we think of $(k, l) \in Q_w$ as an arc in a partition of $[n]$.

2.2. **Posets and the regions of $S_n^C$.** Pick a region $R$ of $S_n^C$ in the cone $wC^{BC}$ of Cox$^{BC}(n)$, $w \in S_n^B$. The set of hyperplanes of $S_n^C$ that intersect $wC^{BC}$ is

$$H_w^C = H_w^+ \cup H_w^{-} \cup H_w^s,$$

where
Theorem 6. The regions of $S_w^C$ contained in $wC_{n}^{BC}$ are in bijection with the antichains of $Q_w^C$. In particular,

$$|R(S_w^C)| = \sum_{w \in \mathcal{S}_n^B} j(Q_w^C),$$

where $j(Q_w^C)$ denotes the number of antichains of the poset $Q_w^C$.

**Proof.** It is clear from the above that there is an injective map from the regions of $S_w^C$ to the multiset of the antichains of the posets $Q_w^C$, $w \in \mathcal{S}_n^B$. Since it is known that $|R(S_w^C)| = (2n + 1)^n$ [Shi2] and $\sum_{w \in \mathcal{S}_n^B} j(Q_w^C) = (2n + 1)^n$ can be proved without reference to $S_w^C$ (see Corollary 25 in Section 6) the map also has to be surjective and Theorem 6 follows.

Note that the antichains of $Q_w^C$ correspond to nonnesting $C_n$-partitions if we think of $(k, l) \in Q_w^C$ as an arc in a partition of $[\pm n]$. Recall that a nonnesting $C_n$-partition of $[\pm n]$ can be thought of as a nonnesting diagram of arcs, which are drawn over the ground set $-n, -n + 1, \ldots, -2, -1, 1, 2, \ldots, n - 1, n$ (in this order) such that if there is an arc between $i$ and $j$, for $i, j \in [\pm n]$, then there is also an arc between $-j$ and $-i$ (there are no multiple arcs). See Figure 2 for an example.

**Figure 2.** A $C_5$-partition.
3. Sequences and Shi arrangements in type $A_{n-1}$

In this section we construct a bijection between the regions of $S_n^A$ and $(n+1)$-tuples of sequences $a_1 \ldots a_n$, $a_i \in [n+1]$, for $i \in [n]$, such that every such sequence appears in exactly one of the $(n+1)$-tuples. Furthermore, exactly one among the $n+1$ sequences assigned to a region is a parking function, thereby also leading to a bijection between the regions of $S_n^A$ and parking functions. The same bijection previously appeared in the paper by Athanasiadis and Linusson [Ath-Lin]. Our exposition makes the enumeration of the ceiling and floor statistic on the regions on $S_n^A$ transparent, and that it readily generalizes to bijections in the type $C_n$ case. The ceiling and floor statistics on Shi arrangements was also used and studied by Armstrong and Rhoades in their beautiful paper on the Shi and Ish arrangements [Arm-Rho]. The ideas of this section appear explicitly or implicitly in [Ath-Lin] and [Arm-Rho].

For ease of exposition we consider $n+1$ copies of the arrangement $S_n^A$, denoted by

$$(S_n^A)^{(1)} \sqcup \cdots \sqcup (S_n^A)^{(n+1)},$$

and biject the regions of $(S_n^A)^{(n+1)}$ defined as the regions of $(S_n^A)^{(1)}, \ldots, (S_n^A)^{(n)}$ and $(S_n^A)^{(n+1)}$ with the sequences $a_1 \ldots a_n$, $a_i \in [n+1]$, for $i \in [n]$.

The type of an $A_{n-1}$-partition $\pi$ is the integer partition $\lambda$ whose parts are the sizes of the blocks of $\pi$.

**Theorem 7.** [Ath] There is a bijection $b$ between the set of type $\lambda = (\lambda_1, \ldots, \lambda_d)$ nonnesting $A_{n-1}$-partitions and pairs $(S, g)$, where $S$ is a $d$-subset of $[n]$ and the map $g: S \to \{\lambda_1, \ldots, \lambda_d\}$ is such that $|g^{-1}(i)| = r_i$, $0 \leq i$.

**Proof.** Given a type $\lambda = (\lambda_1, \ldots, \lambda_d)$ nonnesting $A_{n-1}$-partition, let $S$ be the leftmost elements of its blocks. Let $g(s) = k$ if $s \in S$ is in a block of size $k$. It can be shown by induction on $n$ that the set $S$ and function $g$ defined this way uniquely determine the nonnesting partition they came from. \hfill \Box

Label each region of $(S_n^A)^{(n+1)}$ by the nonnesting $A_{n-1}$-partition corresponding to an antichain of $Q_w$, $w \in S_n$, as described in Section 2.1 and shown on Figure 3. Each region of $(S_n^A)^{(n+1)}$ is completely specified by a number $k \in [n+1]$ (specifying which copy of $S_n^A$ we are in in $(S_n^A)^{(n+1)}$), a permutation $w \in S_n$ (specifying the cone of $S_n^A$), and a nonnesting $A_{n-1}$-partition $\pi$ (specifying the ceilings of $R$ in $wC^A$). While we generally think of $\pi$ as on the vertices $1, 2, \ldots, n-1, n$, in this order, the $A_{n-1}$-partition $\pi$ also has $w$-labels $w(1), \ldots, w(n-1), w(n)$. See Figure 3.

**Figure 3.** Labeling of the regions of $(S_n^A)^{(n+1)}$ by nonnesting partitions.
Lemma 8. The number of regions of $S_n^A$ containing the nonnesting $A_{n-1}$-partition $\pi$ of type $\lambda$ is equal to

\[ (n)_{\lambda_1, \ldots, \lambda_d}. \]

Proof. In this proof we effectively count the number of permutations $w \in S_n$ such that $\pi$ is an antichain in the poset $Q_w$, since the latter is equal to the of number regions of $S_n^A$ containing the nonnesting $A_{n-1}$-partition $\pi$. Given a nonnesting $A_{n-1}$-partition $\pi$ of type $\lambda$ there are $(n)_{\lambda_1, \ldots, \lambda_d}$ ways to choose the values of the $w$-labels which go into the blocks of $\pi$. Since in each block the $w$-labels increase, equation (1) follows. \qed

Given a type $\lambda = (\lambda_1, \ldots, \lambda_d)$ nonnesting $A_{n-1}$-partition $\pi$, denote by $S_\pi$ the set and $g_\pi$ the function from Theorem 7. Let $\hat{S}(\pi)$ be the multiset consisting of $\lambda_i$ copies of each element of $g_\pi^{-1}(\lambda_i)$, for each part in the set (not multiset!) $\{\lambda_1, \ldots, \lambda_d\}$. An $n$-shifted permutation of $\hat{S}(\pi)$ is a permutation of the elements of the multiset $\hat{S}(\pi)$ such that each entry is increased by $k \in \{0, 1, \ldots, n\}$ and taken modulo $n + 1$. For example the 2-shifted permutations of $\{\{1, 2\}\}$ are $12, 21, 23, 32, 31, 13$. 

Figure 4. The $w$-labels of partitions labeling of the regions of $S_3^A$ are shown for the permutations 132 and 321.
Theorem 9. There is a bijection $\phi$ between the regions of $(S_n^A)^{(n+1)}$ labeled by the nonnesting $A_{n-1}$-partition $\pi$ of type $\lambda$ and $n$-shifted permutations of the multiset $\bar{S}(\pi)$.

Proof. There are multiple ways to set up this bijection. We present one way here and note how to define a family of bijections satisfying Theorem 9.

Given the nonnesting $A_{n-1}$-partition $\pi$, a permutation $w$ for which $\pi$ is an antichain in $Q_w$, and an integer $k \in [n+1]$ specifying which copy of $S_n^A$ we are in in $(S_n^A)^{(n+1)}$, order the blocks of $\pi$ by increasing size. The blocks of the same size are ordered lexicographically according to the $w$-labels on them. Order the numbers in the multiset $\bar{S}(\pi)$ so that the numbers with less multiplicities are smaller. Among the numbers with the same multiplicity order them according to the natural order on integers. The previous two orders yield a correspondence $b_3$ between the blocks $B$ of $\pi$ and the numbers from $\bar{S}(\pi)$. (This correspondence could of course be defined in several ways leading to different bijections.) Let the values of the $w$-labels of $B$ specify the positions that the number $b_3(B) + k - 1 \mod n + 1$ is taking.

Correspondence $b_3$ could also be naturally defined by the bijection given in Theorem 7. As it turns out both descriptions of $b_3$ in the type $A_{n-1}$ case are the same.

Figure 5 shows the construction of the bijection on $(S_3^A)^{(1)}$. For example, consider the region $R$ in $(132)C^A$ with ceiling $x_1 - x_3 = x_{w(1)} - x_{w(2)} = 1$. The $w$-labels are written above the partition and the numbers corresponding to the elements of the blocks are below the partition and are circled individually. Then, to get the sequence corresponding to the region, read the circled numbers in the order specified by the $w$-labels. The resulting 3-digit sequence is circled on Figure 5.

For the restriction of the bijection to $(S_3^A)^{(1)}$, see Figure 6.

The above defined map is a bijection between the regions of $(S_n^A)^{(n+1)}$ labeled by the nonnesting $A_{n-1}$-partition $\pi$ of type $\lambda$ and $n$-shifted permutations of the multiset $\bar{S}(\pi)$, which can be shown by writing down an explicit inverse, or by noting that it is injective and the domain and codomain are equinumerous.

Extend the map $\phi$ defined in the proof of Theorem 9 to a map between all regions of $(S_n^A)^{(n+1)}$ and the set of sequences $A(n) = \{a_1 \ldots a_n | a_i \in [n+1], i \in [n]\}$.

Theorem 10. (cf. [Ath-Lin]) The map $\phi : R((S_n^A)^{(n+1)}) \to A(n)$ is a bijection.

Theorem 11. (cf. [Ath-Lin]) The restriction of the bijection $\phi$ to the first copy of the Shi arrangement $S_n^A(1)$ is a bijection between the regions of the Shi arrangement and parking functions.

We leave the details of the proofs of Theorems 10 and 11 to the reader. Hint: see [Sta4], Exercise 5.49.

Corollary 12.

$$\sum_{\lambda \vdash n} \binom{n+1}{d} \frac{n!}{m_\lambda(n - d + 1)!} \prod_{i=1}^{n} r_i! = (n + 1)^n,$$

where $m_\lambda = \prod_{i=1}^{n} r_i!$, if $r_i$ denotes the number of parts of $\lambda$ equal to $i$.

Proof. Kreweras [Kre, Theorem 4] proved that the number of noncrossing partitions of $[n]$ of type $\lambda$ is equal

$$\frac{n!}{m_\lambda(n - d + 1)!},$$

where $d$ denotes the number of parts of $\lambda$. Athanasiadis [Ath, Theorem 3.1] gave a bijection between noncrossing and nonnesting $A_{n-1}$-partitions which preserves type. Thus, the total number of nonnesting $A_{n-1}$-partitions of type $\lambda$ labeling the regions of $(S_n^A)^{(n+1)}$ is

$$\binom{n+1}{d} \frac{n!}{m_\lambda(n - d + 1)!} = \binom{n+1}{d} \frac{n!}{m_\lambda(n - d + 1)!}.$$
Equation (3) together with Lemma 8 and Theorem 10 imply equation (2). □

Theorem 2 from the introduction is a corollary of the proofs of Theorems 9, 10 and 11. For further details see Section 5, and in particular Theorem 18.

**Theorem 2.**

\[
\sum_{R \in R(S_n^A)} q^{c(R)} = \sum_{R \in R(S_n^A)} q^{f(R)} = \frac{1}{n+1} \sum_{a \in A(n)} q^{n-d(a)} = \sum_{a \in PF(n)} q^{n-d(a)}.
\]

4. **Sequences and Shi arrangements in type C**

In this section we construct a bijection between the regions of \(S_n^C\) and the set of sequences \(A_n^{\pm} = \{a_1 \ldots a_n | a_i \in \{\pm i \} \cup \{0\}, i \in [n]\}\). Our proof yields enumeration of regions by the ceiling and floor statistic, which we express in a generating function form.

The type of a \(C_n\)-partition \(\pi\) is the integer partition \(\lambda\) whose parts are the sizes of the nonzero blocks of \(\pi\), including one part for each pair of blocks \(\{B, -B\}\). The zero block is a block \(B\) such that \(B = -B\). Figure 7 shows a nonnesting \(C_5\)-partition with blocks \(\{2\}, \{-2\}, \{-1, -4\}, \{1, 4\}, \{-3, 3, 5\}\). The last block is a zero block, and so the type of this partition is (2, 1).

The following theorem is based on a bijection of Fink and Iriarte [Fin-Iri] between noncrossing and nonnesting \(C_n\)-partitions which preserves type and a bijection of Athanasiadis [Ath] between
Theorem 13. There is a bijection between the set of type $\lambda = (\lambda_1, \ldots, \lambda_d)$ nonnesting $C_n$-partitions and pairs $(S, g)$, where $S$ is a $d$-subset of $[n]$ and the map $g : S \to \{ \lambda_1, \ldots, \lambda_d \}$ is such that $|g^{-1}(i)| = r_i$, $0 \leq i$.

Proof. \cite{Fin-Iri} Theorem 2.4] establishes a type-preserving bijection $b_1$ between nonnesting and noncrossing $C_n$-partitions, and \cite{Ath} Theorem 2.3] provides a bijection $b_2$ between the set of type $\lambda = (\lambda_1, \ldots, \lambda_d)$ noncrossing $C_n$-partitions and pairs $(S, g)$, where $S$ is a $d$-subset of $[n]$ and the map $g : S \to \{ \lambda_1, \ldots, \lambda_d \}$ is such that $|g^{-1}(i)| = r_i$, $0 \leq i$. \hfill $\square$
Label each region of $\mathcal{S}_n^C$ by the nonnesting $C_n$-partition corresponding to an antichain of $Q^C_w$, $w \in \mathfrak{S}_n^D$, as described in Section 2.2. Each region of $\mathcal{S}_n^C$ is completely specified by a nonnesting $C_n$-partition $\pi$ and $w \in \mathfrak{S}_n^B$. While we generally think of $\pi$ as on the vertices $-n,-n+1,\ldots,-1,1,2,\ldots,n-1,n$, in this order, the $C_n$-partition $\pi$ also has $w$-labels $w(-n),w(-n+1),\ldots,w(-1),w(1),\ldots,w(n-1),w(n)$.

**Lemma 14.** The number of regions of $\mathcal{S}_n^C$ containing the nonnesting $C_n$-partition $\pi$ of type $\lambda$ is equal to

$$\left(\begin{array}{c} n \\ \lambda_1, \ldots, \lambda_d, n-|\lambda| \end{array}\right) \prod_{i=1}^d 2^{\lambda_i}.$$

**Proof.** In this proof we effectively count the number of signed permutations $w \in \mathfrak{S}_n^B$ such that $\pi$ is an antichain in the poset $Q^C_w$, since the latter is equal to the number of regions of $\mathcal{S}_n^C$ containing the nonnesting $C_n$-partition $\pi$. Given a nonnesting $C_n$-partition $\pi$ of type $\lambda$ there are $(\lambda_1, \ldots, \lambda_d, n-|\lambda|)$ ways to choose the absolute values of the $w$-labels which go into the blocks of $\pi$. Let $2(n-\lambda)$ be the size of the zero block of $\pi$. The signs and order of the $w$-labels in the zero block of $\pi$ are determined: there have to be $(n-\lambda)$ positive numbers in increasing order followed by their $(n-\lambda)$ negatives in increasing order. Each nonzero block is comprised of a possibly empty sequence of positive $w$-labels in increasing order followed by a possibly empty sequence of negative $w$-labels in increasing order. There are exactly $2^{\lambda_i}$ ways to decide the signs among $\lambda_i$ numbers, and once the signs are decided so is the order. Thus, equation (4) follows. \hfill \Box

Given a type $\lambda = (\lambda_1, \ldots, \lambda_d)$ nonnesting $C_n$-partition $\pi$, denote by $S_\pi$ the set and $g_\pi$ the function from Theorem 13. Let $S(\pi)$ be the multiset consisting of $n-\lambda$ 0’s, and $\lambda_i$ copies of each element of $g_\pi^{-1}(\lambda_i)$, for each part in the set (not multiset!) $\{\lambda_1, \ldots, \lambda_d\}$. A marked permutation of $S(\pi)$ is a permutation of the elements of the multiset $S(\pi)$ such that each nonzero entry has a $\pm$ sign in addition. For example the marked permutations of $\{(0,1,1)\}$ are $011,101,110,0-1-1,-1-10,01-1,10-1,0-11,-101,-110$ (we omitted the $+$ signs).

**Theorem 15.** There is a bijection $\phi$ between the regions of $\mathcal{S}_n^C$ labeled by the nonnesting $C_n$-partition $\pi$ of type $\lambda$ and marked permutations of the multiset $S(\pi)$.

**Proof.** There are multiple ways to set up this bijection. We present two natural ways here and note how to define a family of bijections satisfying Theorem 15.

Given the nonnesting $C_n$-partition $\pi$ and a signed permutation $w$ for which $\pi$ is an antichain in $Q^C_w$, order the (pair of) blocks $\{B,-B\}$ of $\pi$ as follows. If there is a zero block, then it comes first. The other blocks are ordered by increasing size ($|B|$), and the blocks of the same size are ordered lexicographically according to the $w$-labels on them (on $B$s). Order the numbers in the multiset $S(\pi)$ so that the 0’s come first, and among the other numbers the numbers with less multiplicities are smaller. Among the numbers with the same multiplicity order them according to the natural order on integers. The previous two orders yield a correspondence $b_3$ between the blocks $B$ of $\pi$ and the numbers from $S(\pi)$. (This correspondence could of course be defined in several ways leading to different bijections.) Let the absolute values of the $w$-labels of $B$ specify the positions that the number $b_3(B)$ is taking. For the $w$-labels of nonzero blocks which are negative add 1 to the number in the corresponding spot.

Correspondence $b_3$ could also be naturally defined by the bijection given in Theorem 13.

The above defined maps are bijections between the regions of $\mathcal{S}_n^C$ labeled by the nonnesting $C_n$-partition $\pi$ of type $\lambda$ and marked permutations of the multiset $S(\pi)$, which can be shown by writing down explicit inverses, or by noting that they are injective and the domains and codomains are equinumerous. \hfill \Box
Extend the map $\phi$ defined in the proof of Theorem 15 to a map between all regions of $S_n^C$ and the set of sequences $A^C(n) = \{a_1 \ldots a_n | a_i \in [\pm n] \cup \{0\}, i \in [n] \}$, to obtain the following corollaries as in the type $A_{n-1}$ case.

**Theorem 16.** The map $\phi : R(S_n^C) \rightarrow A^C(n)$ is a bijection.

**Corollary 17.**

$$\sum_{\lambda \vdash n} \frac{n!}{m_\lambda(n-d)!} \left( \lambda_1, \ldots, \lambda_d, n-|\lambda| \right) \prod_{i=1}^{d} 2^{\lambda_i} = (2n+1)^n.$$

**Proof.** Athanasiadis [Ath] proved that the number of nonnesting $C_n$-partitions of type $\lambda$ is

$$\frac{n!}{m_\lambda(n-d)!}$$

which together with Lemma 14 and Theorem 16 imply the above equality. \quad \Box

Theorem 4 is a corollary of the proofs of Theorems 13, 15 and 16 (use the second definition of $b_3$ in the proof of Theorem 15). For further details see Section 6, and in particular Theorem 22.

**Theorem 4.**

$$\sum_{R \in R(S_n^C)} q^{c(R)} = \sum_{R \in R(S_n^C)} q^{f(R)} = \sum_{a \in A^C(n)} q^{n-d^C(a)}.$$

### 5. Posets and sequences in type $A_{n-1}$

In this section we revisit the type $A_{n-1}$ world of posets $Q_w$, $w \in S_n$, and parking functions of length $n$ and state their relation explicitly without the mention of arrangements. Much of the considerations of this section appear in the work of Athanasiadis and Linusson [Ath-Lin] and Armstrong and Rhoades [Arm-Rho] either explicitly or implicitly. We highlight our perspective on the relation of the posets and sequences and study their properties in detail. We carry out a similar agenda for the posets $Q_w^C$, $w \in S_n^B$, and sequences in $A^C(n)$ in the next section.

Recall that $Q_w = \{(i, j) : 1 \leq i < j \leq n, w(i) < w(j)\}$ is partially ordered by

$$(i, j) \leq (r, s) \text{ if } r \leq i < j \leq s.$$

We explore the refinements of the equation

$$\sum \limits_{w \in S_n} j(Q_w) = (n+1)^{n-1},$$

which follows from Theorems 5 and 10. In the process we reiterate the proof of equation (5) without reference to arrangements.

Partition the set of parking functions of length $n$, $PF(n)$, according to the cardinality of the set $\{a_1, a_2, \ldots, a_n\}$. Let $S_k(n) = \{(a_1, a_2, \ldots, a_n) \in PF(n) : |\{a_1, a_2, \ldots, a_n\}| = k\}$. Then

$$PF(n) = \bigcup_{k=1}^{n} S_k(n).$$

Partition the multiset of antichains $M(n)$ of $Q_w$, $w \in S_n$, according to the cardinality of the antichains. Let $M_k(n) = \{\{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} : M(n)\}$. Then

$$M(n) = \bigcup_{k=0}^{n-1} M_k(n).$$
The following theorem can be deduced from the work of Athanasiadis and Linusson \cite{Ath-Lin} and Armstrong and Rhoades \cite{Arm-Rho}.

**Theorem 18.**

\[ |S^c_k(n)| = |M^c_{n-k}(n)|, \quad k \in [n]. \]

We prove Theorem \ref{thm:18} by providing a bijection between the sets \( S^c_k(n) \) and \( M^c_{n-k}(n) \), \( k \in [n] \). Before proceeding to the proof of Theorem \ref{thm:18} we partition the sets \( S^c_k(n) \) and \( M^c_{n-k}(n) \), \( k \in [n] \), further.

Partition the set of parking functions of length \( n \) with \( k \) distinct numbers \( S^c_k(n) \), \( k \in [n] \), according to the \( k \) distinct numbers appearing in the sequence \( (a_1, a_2, \ldots, a_n) \), and the number of times they appear. If \( \{a_1, a_2, \ldots, a_n\} = \{c_1 < c_2 < \cdots < c_k\} \) and \( c_i \) appears \( o_i \) times in \( (a_1, a_2, \ldots, a_n) \), \( i \in [k] \), let

\[ S^c_{k(o)}(n) = \{(a_1, a_2, \ldots, a_n) \in S_k(n) | \{a_1, a_2, \ldots, a_n\} = \bigsqcup_{i=1}^k \bigsqcup_{j=1}^{o_i} \{\{c_i\}\} \}, \]

where \( c = (c_1 < \ldots < c_k), \quad o = (o_1, \ldots, o_k), \quad o_i > 0, \quad for \quad i \in [k], \quad and \quad \sum_{i=1}^k o_i = n. \)

Given an antichain \( a = \{(i_1, j_1), \ldots, (i_{n-k}, j_{n-k})\} \in M_{n-k}(n) \), \( k \in [n] \), it naturally corresponds to a nonnesting partition \( \pi_a \) of \([n]\) with \( k \) blocks, where the arc diagram of \( \pi_a \) consists of the arcs \((i_1, j_1), \ldots, (i_{n-k}, j_{n-k})\). Order the \( k \) blocks of \( \pi_a \) according to their smallest elements \( c_i, \quad i \in [k], \quad c_1 < \ldots < c_k \). Let \( o_i > 0, \quad i \in [k] \), be the number of elements in the \( i^{th} \) block of \( \pi_a \). Denote by \( c(a) = (c_1 < \ldots < c_k) \) and \( o(a) = (o_1, \ldots, o_k) \). Partition the multiset of antichains of length \( n-k \) of \( Q_w, \quad w \in \mathfrak{S}_n \), \( M_{n-k}(n) \), \( k \in [n] \), according to \( c = (c_1 < \ldots < c_k), \quad o = (o_1, \ldots, o_k), \quad o_i > 0, \quad for \quad i \in [k], \quad and \quad \sum_{i=1}^k o_i = n, \) as described above. Let

\[ M^c_{n-k}(n) = \\{(a = \{(i_1, j_1), \ldots, (i_{n-k}, j_{n-k})\} \in M_{n-k}(n) | c(a) = c, o(a) = o\}, \]

where \( c = (c_1 < \ldots < c_k), \quad o = (o_1, \ldots, o_k), \quad o_i > 0, \quad for \quad i \in [k], \quad and \quad \sum_{i=1}^k o_i = n. \)

**Lemma 19.** \textbf{AHH} The vectors \( c(a) = c \) and \( o(a) = o \), where \( c = (c_1 < \ldots < c_k), \quad o = (o_1, \ldots, o_k), \quad k \in [n], \quad o_i > 0, \quad for \quad i \in [k], \quad \sum_{i=1}^k o_i = n, \quad c_1 = 1, \quad and \quad c_i \in \{c_{i-1} + 1, c_{i-1} + 2, \ldots, o_1 + \cdots + o_{i-1} + 1\}, \quad for \quad i \in \{2, \ldots, k\}, \) uniquely determine the antichain \( a. \)

The following theorem can be deduced from the work of Athanasiadis and Linusson \cite{Ath-Lin} and Armstrong and Rhoades \cite{Arm-Rho}.

**Theorem 20.**

\[ |\mathcal{S}^c_{k(o)}(n)| = |\mathcal{M}^c_{n-k}(n)| = \binom{n}{o_1, \ldots, o_k}, \]

where \( k \in [n], \quad c = (c_1 < \ldots < c_k), \quad o = (o_1, \ldots, o_k), \quad o_i > 0, \quad for \quad i \in [k], \quad and \quad \sum_{i=1}^k o_i = n. \)

**Proof.** A bijective proof can be given using Theorem \ref{thm:17} and the ideas of Theorem \ref{thm:9}. The enumeration is in Lemma \ref{lem:8}. Note that arrangements do not enter the proof. \( \square \)

**Proof of Theorem \ref{thm:18}** Straightforward corollary of Theorem \ref{thm:20} since

\[ S_k(n) = \sum_{c,o} S^c_{k(o)}(n) = \sum_{c,o} M^c_{n-k}(n) = M_{n-k}(n), \]

where \( c = (c_1 < \ldots < c_k), \quad o = (o_1, \ldots, o_k), \quad k \in [n], \quad o_i > 0, \quad for \quad i \in [k], \quad \sum_{i=1}^k o_i = n. \) \( \square \)

**Corollary 21.**

\[ \sum_{w \in \mathfrak{S}_n} j(Q_w) = (n + 1)^{n-1}. \]

**Proof.** Theorems \ref{thm:20} and \ref{thm:18} extend to a bijection between

\[ \mathcal{M}A(n) = \bigsqcup_{k=1}^n \bigsqcup_{c,o} M^c_{n-k}(n) \quad and \quad \mathcal{PF}(n) = \bigsqcup_{k=1}^n \bigsqcup_{c,o} S^c_{k(o)}(n), \]

the cardinalities of which are \( \sum_{w \in \mathfrak{S}_n} j(Q_w) \) and \( (n + 1)^{n-1} \), respectively. \( \square \)
6. Posets and sequences in type $C_\pi$

In this section we revisit the type $C_\pi$ world of posets $Q_w^C$, $w \in \mathfrak{S}_n^B$, and sequences in $\mathcal{A}_n^C$ and state their relation explicitly without the mention of arrangements.

Recall that

$$Q_w^C = \{(i,j), (-j,-i) \mid i < j, 0 < w(i) \leq |w(j)|\}$$

is partially ordered by $(i,j) \leq (r,s)$ if $r \leq i < j \leq s$.

We explore the refinements of the equation

$$\sum_{w \in \mathfrak{S}_n^B} j(Q_w^C) = (2n + 1)^n,$$

which follows from Theorems 6 and 16. In the process we reiterate the proof of equation (6) without reference to arrangements.

Partition $\mathcal{A}_n^C$ according to the number of nonzero absolute values in the set $\{a_1, a_2, \ldots, a_n\}$, denoted by $d^C(a)$ for $a = a_1a_2 \ldots a_n$. Let $S_k^C(n) = \{(a_1, a_2, \ldots, a_n) \in \mathcal{A}_n^C : d^C(a) = k\}$. Then

$$\mathcal{A}_n^C = \bigcup_{k=0}^{n} S_k^C(n).$$

Partition the multiset of antichains $\mathcal{M}_n^C$ of $Q_w^C$, $w \in \mathfrak{S}_n^B$, according to the number of pairs $(i,j), (-j,-i)$, $i \leq j$, in the antichains. Denote by $p(\pi)$ the number of pairs $(i,j), (-j,-i), i \leq j$, in the antichain $\pi \in \mathcal{M}_n^C$. Let $M_k^C(n) = \{\{\pi \in \mathcal{M}_n^C \mid p(\pi) = k\}\}$. Then

$$\mathcal{M}_n^C = \bigcup_{k=0}^{n} M_k^C(n).$$

**Theorem 22.**

$$|S_k^C(n)| = |M_{n-k}^C(n)|, \ k \in \{0\} \cup [n].$$

We prove Theorem 22 by providing a bijection between the sets $S_k^C(n)$ and $M_{n-k}^C(n)$, $k \in \{0\} \cup [n]$. Before proceeding to the proof of Theorem 22 we partition the sets $S_k^C(n)$ and $M_{n-k}^C(n)$, $k \in \{0\} \cup [n]$, further.

Partition $S_k^C(n)$, $k \in \{0\} \cup [n]$, according to the $k$ distinct nonzero absolute values of the numbers appearing in the sequence and the number of times they appear. If

$$\{|a_1|, |a_2|, \ldots, |a_n|\} \{0\} = \{c_1 < c_2 < \cdots < c_k\}$$

and $c_i$ appears $o_i$ times in $(|a_1|, |a_2|, \ldots, |a_n|)$, $i \in [k]$, let

$$S_k^{c,o}(n) = \{(a_1, a_2, \ldots, a_n) \in S_k^C(n) \mid \{|a_1|, |a_2|, \ldots, |a_n|\} = \cup_{i=1}^{k} \cup_{j=1}^{o_i} \{c_i\} \cup_{i=1}^{n-k-o} \{0\}\},$$

where $c = (c_1 < \ldots < c_k)$, $o = (o_1, \ldots, o_k)$, $o_i > 0$, for $i \in [k]$, and $\sum_{i=1}^{k} o_i \leq n$.

Given an antichain $a \in M_{n-k}^C(n)$, $k \in \{0\} \cup [n]$, it naturally corresponds to a nonnesting $C_\pi$-partition $\pi_a$ of $[\pm n]$ with $k$ pairs of nonzero blocks. Let $(S_{\pi_a}, g_{\pi_a})$ be the pair of $k$-set and function corresponding to $\pi_a$ under the bijection described in Theorem 13. Let

$$S_{\pi_a} = \{c_1 < \ldots < c_k\} \text{ and } o_i = g_{\pi_a}(c_i), i \in [k].$$

Denote $c(a) = (c_1 < \ldots < c_k)$ and $o(a) = (o_1, \ldots, o_k)$.

Partition the multiset $M_{n-k}^C(n)$, $k \in \{0\} \cup [n]$, according to $c = (c_1 < \ldots < c_k)$, $o = (o_1, \ldots, o_k)$, $o_i > 0$, for $i \in [k]$, and $\sum_{i=1}^{k} o_i \leq n$, as described above. Let
Lemma 23. The vectors $c(a) = c$ and $o(a) = o$, where $c = (c_1 < \ldots < c_k)$, $o = (o_1, \ldots, o_k)$, $c_1 > 0$, for $i \in [k]$, and $\sum_{i=1}^{k} o_i \leq n$.

$M_{n-k}^{c,o}(n) = \{a \in M_{n-k}(n) | c(a) = c, o(a) = o\}$,
where $c = (c_1 < \ldots < c_k)$, $o = (o_1, \ldots, o_k)$, $c_i > 0$, for $i \in [k]$, and $\sum_{i=1}^{k} o_i \leq n$.

Lemma 23. The vectors $c(a) = c$ and $o(a) = o$, where $c = (c_1 < \ldots < c_k)$, $o = (o_1, \ldots, o_k)$, $k \in \{0\} \cup [n]$, $o_1 > 0$, for $i \in [k]$, $\sum_{i=1}^{k} o_i \leq n$, uniquely determine the antichain $a$.

Proof. Lemma 23 follows readily since Theorem 13 establishes a bijection.

Theorem 24. $|S_k^{c,o}(n)| = |M_{n-k}^{c,o}(n)| = \binom{n}{o_1, \ldots, o_k, n - \sum_{j=1}^{k} o_j} 2^{\sum_{j=1}^{k} o_j}$, where $k \in \{0\} \cup [n]$, $c = (c_1 < \ldots < c_k)$, $o = (o_1, \ldots, o_k)$, $o_i > 0$, for $i \in [k]$, and $\sum_{i=1}^{k} o_i \leq n$.

Proof. A bijective proof can be given using Theorem 13 and the ideas of Theorem 15. The enumeration is in Lemma 14. Note that arrangements do not enter the proof.

Proof of Theorem 22. Straightforward corollary of Theorem 24 since

$S_k^{c,o}(n) = \sum_{c,o} S_k^{c,o}(n) = \sum_{c,o} M_{n-k}^{c,o}(n) = M_{n-k}^{c,o}(n)$,
where $c = (c_1 < \ldots < c_k)$, $o = (o_1, \ldots, o_k)$, $k \in [n]$, $o_i > 0$, for $i \in [k]$, $\sum_{i=1}^{k} o_i \leq n$.

Corollary 25. $\sum_{w \in \mathcal{B}_n^c} j(Q_w^c) = (2n + 1)^n$.

Proof. Theorems 24 and 22 extend to a bijection between

$\mathcal{M}_c(n) = \bigcup_{k=0}^{n} \cup_{c,o} M_{n-k}^{c,o}(n)$ and $\mathcal{A}_c(n) = \bigcup_{k=0}^{n} \cup_{c,o} S_{c}^{c,o}(n)$,
the cardinalities of which are $\sum_{w \in \mathcal{B}_n^c} j(Q_w^c)$ and $(2n + 1)^n$, respectively.

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