Cosmology of Brane-Bulk Models in Five Dimensions

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(March, 2000)

Abstract

We study the cosmology of models with four space and one time dimension where our universe is a 3-brane and report a few results which extend existing work in several directions. Assuming a stable fifth dimension, we obtain a solution for the metric, which does not depend on any arbitrary parameters. We discuss some implications of this result.

I. INTRODUCTION

Over the last two years there has been a lot of interest in models where our universe is a 3-brane (a hyper-surface) embedded in a higher dimensional bulk. The standard model particles are confined to the brane whereas gravity propagates in the bulk. Such models were conjectured early on \cite{1} as interesting possibilities and have recently been argued as plausible solutions of type I string theories \cite{2}. One of the attractive features of these models is the intriguing possibility that the fundamental scale, $M$, identified as the string scale, could be lower than the Planck scale, $M_{Pl} = (8\pi G_N)^{-1/2}$, by several orders of magnitude \cite{3}, perhaps even of TeV range \cite{4}. This last possibility may provide a new way to solve the hierarchy problem between the electroweak scale and the scale of gravity i.e. both scales being of same order, there is no hierarchy to worry about. The large value of the Planck scale in this picture owes its origin to the existence of very large hidden extra dimensions in nature. The price one has to pay is of course that now one has to understand why the extra dimension(s) is(are) so large. The relation between the fundamental scale, the $M_{Pl}$ and the volume of the external space, $V_n$, is given by \cite{4}

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This picture leads to a modification of the inverse square law of gravity at small distances, \( r \sim V_n^{1/n} \) and can therefore be probed experimentally. If \( M \) is in the TeV range, string theories become accessible to collider tests. All these make the idea phenomenologically quite attractive. It is therefore interesting to study the cosmology of these models.

One of the first things that one needs to know in order to study the cosmology of these models is the time dependence of the metric. In addition, the metric will also have a dependence on the bulk coordinate \( y \), even when the bulk is totally empty, simply because the presence of a brane induces a nonzero curvature. It turns out that this fact leads to a variety of interesting consequences for the cosmology of such models [5–14]. In particular, the new bulk-brane picture seems to drastically modify the standard time evolution law in the brane-confined universe. The five dimensional Einstein equations first studied in [8] implies that the Hubble parameter \( H \) is proportional to the density on the brane, \( \rho \), instead of the usual \( H \sim \sqrt{\rho} \) of the standard big bang cosmology. Since the successes of the standard cosmology such as nucleosynthesis and the common understanding of subsequent evolution rely crucially on the assumption of \( H \sim \sqrt{\rho} \), a great deal of work has been devoted to understand how this standard behaviour of \( H \) can be recovered. Some ideas proposed to solve this problem include cancellation of bulk and brane cosmological constants [10], consideration of a non zero thickness of the brane [11] etc. It has also been noted that Friedmann equation could be recovered on the basis of fine tuning of cosmological constants even when the extra dimension is not stable [12]. However, in the process it has been found that a free parameter \( C \), a constant of integration, appears in this equation. It contributes to the evolution of the Hubble parameter in the form of an effective radiation term, jeopardizing the cosmic scale parameters. The same arbitrary parameter has been used to study cosmological evolution even when the bulk radius is stable [12].

We reexamine this in this paper. Writing the equations governing the evolution in time, \( t \) and the fifth coordinate, \( y \) of the various parameters defining the metric (the analogs of the Friedman-Robertson-Walker (FRW) scale factor), we show that it is possible to get a solution for the metric as a function of the bulk coordinate without any arbitrary constant, if the bulk radius is stable. This is the main result of this brief note. We then point out some of the implications of this result.

This paper is arranged as follows: in section 2, we review Einstein equations in a diagonal metric, commonly used in the literature; in section 3, we look for solutions of Einstein equation keeping the bulk radius stable and obtain an explicit form for the metric which involves only the known FRW scale factor and no extra parameters. We recover the known behaviour for the Hubble parameter, i.e. \( H \) depends linearly on the brane energy density. We also show that the equation involving the brane-space like components can be derived from the other equations; they reduce to the acceleration equation for the FRW scale factor. Finally, we reconsider some examples already studied in [12] and rederive the exact solutions for the metric in the presence of a cosmological constant in the bulk. In section 5, we make some remarks on the general nonfactorizable nature of this solutions.
II. BASIC FRAMEWORK

The basic framework for our discussion is a five dimensional space-time where a flat (zero thickness) brane is localized at the position identified as \( y = 0 \) along the fifth dimension. Although in this paper, we assume that the extra dimension is compact, this is not crucial for our conclusions and our results also apply when the extra dimension is noncompact.

Since we are interested in the brane cosmology, we start by adopting the cosmological principle of isotropy and homogeneity in the three space dimensions of the brane. The presence of the brane clearly breaks the isotropy along the fifth dimension and this is reflected in the explicit \( y \) dependence of the metric tensor which we choose to have the following form:

\[
\begin{align*}
    ds^2 &= -n^2(y,t) dt^2 + a^2(y,t) \gamma_{ij} dx^i dx^j + b^2(y,t) dy^2. \\
\end{align*}
\] (2.1)

Here \( \gamma_{ij} = f(r) \delta_{ij} \), with \( f^{-1}(r) = 1 - kr^2 \) being the usual Robertson-Walker curvature term, where \( k = -1, 0, 1; \) \( t \) and \( x^i, \ i = 1, 2, 3 \) are the time and space-like coordinates along the brane respectively.

The five dimensional Einstein equations take the form

\[
G_{AB} = R_{AB} - \frac{1}{2} g_{AB} R = \kappa_5^2 \left[ \delta_{AB} + T_{\mu \nu} \delta_A^\mu \delta_B^\nu \delta(by) \right],
\] (2.2)

where \( \kappa_5^2 = 8\pi G_5 = M^{-3} \) is the five dimensional coupling constant of gravity, \( R_{AB} \) is the five dimensional Ricci tensor and \( R \) the scalar curvature, \( A, B = 0, 1, 2, 3 \) and \( \mu, \nu = 0, 1, 2, 3 \). In conformity with the usual practice, we identify the mass parameter \( M \) with the string scale. In the last expression the various source terms have been explicitly separated. \( T_{\mu \nu} \) and \( \delta_{AB} \) represent the stress-energy-momentum tensors of the brane and bulk respectively. For the scenarios we are going to discuss from now on, it is sufficient to work in the perfect fluid approximation to these tensors

\[
\begin{align*}
    \delta_{AB} &= \text{diag}(\rho_B, P_B, P_B, P_B, P_T), \\
    T_{\mu \nu} &= \text{diag}(\rho_b, p_b, p_b, p_b).
\end{align*}
\] (2.3)

where \( \rho_B, P_B \) and \( P_T \) represent the densities and pressure on the bulk and, respectively, \( \rho_b \) and \( p_b \) are those on the brane. Notice that assuming \( \delta_{04} = 0 \) avoids the complication of a matter flux along the fifth dimension.

So far, most of the attention paid to this model in literature appears to have focussed on the case where vacuum energy (a cosmological constant) is the only component of the bulk stress tensor. In general however, all the components of \( T_{\mu \nu}^B \) could be functions of time, if they are in the brane and could depend on both \( t \) and \( y \) if they describe the bulk [1]. Therefore, in our analysis, we will keep the energy-momentum tensor dependent on \( t \) (and \( y \) for the bulk terms). It is of course straightforward to see that more branes could be considered by including their corresponding stress tensors in Eq. (2.2) and our discussion below easily generalises to this case.

In order to solve the Einstein equations on the presence of the delta-function type densities, we proceed as follows. First we observe that the brane divides the bulk into two different domains, where the only source is \( \delta_{AB} \). We then solve the equations in each domain separately, and impose the boundary conditions at the brane to get the global solution.
This also helps to define the metric on the brane itself. First, the metric tensor, $g$, clearly should be continuous, i.e. the solutions must satisfy

$$g_{AB}(y = 0^-) = g_{AB}(y = 0^+).$$  \hspace{1cm} (2.4)$$

Next, as has already been noted, since $G_{AB}$ involves up to second derivatives on the metric tensor with respect of $y$, we must use them to match the delta function distributions \cite{8}. Technically speaking this means that the extrinsic curvature $K_{AB}$ in the Gauss-Codacci formulation \cite{13} should be discontinuous at the position of the branes. Then, by integrating Eq. (2.2) at both sides of the brane, we will get matching conditions for the first derivatives. This leads to the constraint

$$\int_0^{0^+} dy \ b \ G_{\mu\nu} = \kappa_5^2 T_{\mu\nu}. \hspace{1cm} (2.5)$$

To evaluate this integral we should assume that all other terms not involving second derivatives on $y$ are finite.

Typically, a parity symmetry $P : y \rightarrow -y$ is assumed as in the Horava-Witten model \cite{2}. Physically, this symmetry could be seen as a residual effect of the broken isotropy along the fifth dimension, as in the $S^1/Z_2$ orbifold construction of ref. \cite{4}. We will assume this hereafter. In the presence of other branes that explicitly break this symmetry, our discussions have to be reconsidered depending on the number of branes.

Let us now proceed to the details. Using the above form of the bulk stress tensor, the non trivial components of the Einstein equations (away from the brane) are given as \cite{8}

$$G_{00} = 3 \left\{ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - \frac{n^2}{b^2} \left[ \frac{a'}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) + \frac{a''}{a} \right] + k \frac{n^2}{a^2} \right\} = \kappa_5^2 n^2 \rho_B, \hspace{1cm} (2.6)$$

$$G_{ij} = \frac{a^2}{b^2} \left\{ \frac{a'}{a} \left( \frac{2n'}{n} + \frac{a'}{a} \right) - \frac{b'}{b} \left( \frac{n'}{n} + 2 \frac{a'}{a} \right) + 2 \frac{a''}{a} + \frac{n''}{n} \right\} \gamma_{ij} + \frac{a^2}{n^2} \left\{ \frac{\dot{a}}{a} \left( \frac{\dot{n}}{n} - \frac{\dot{a}}{a} \right) + \frac{\dot{b}}{b} \left( \frac{\dot{n}}{n} - 2 \frac{\dot{a}}{a} \right) - 2 \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} \right\} \gamma_{ij} - k \gamma_{ij} = \kappa_5^2 a^2 P_B \gamma_{ij}, \hspace{1cm} (2.7)$$

$$G_{04} = 3 \left\{ \frac{n'}{a} \frac{\dot{a}}{n} + \frac{a' \dot{b}}{a b} - \frac{\dot{a}'}{a} \right\} = 0, \hspace{1cm} (2.8)$$

$$G_{44} = \frac{3}{b^2} \left\{ \frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{b}{n^2} \left[ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\ddot{a}}{a} \right] - k \frac{b^2}{a^2} \right\} = \kappa_5^2 b^2 P_T; \hspace{1cm} (2.9)$$

where primes (dots) are used to denote derivatives with respect to $y$ ($t$). This system of equations is supplemented by the Bianchi identity $\hat{T}^A_{\ B;A} = 0$, which translates into the conservation laws \cite{11}

$$\dot{\rho}_B + 3 \frac{\dot{a}}{a} \left( \rho_B + P_B \right) + \frac{\dot{b}}{b} \left( \rho_B + P_T \right) = 0 \hspace{1cm} (2.10)$$

$$P'_T + 3 \frac{a'}{a} \left( P_T - P_B \right) + \frac{n'}{n} \left( P_T + \rho_B \right) = 0 \hspace{1cm} (2.11)$$

Next, by using Eq. (2.5), the following boundary conditions are easily obtained \cite{8}.
\[
\frac{\Delta a'}{ab} \bigg|_0 = -\frac{\kappa_5^2}{3} \rho_b, \tag{2.12}
\]
\[
\frac{\Delta n'}{nb} \bigg|_0 = \frac{\kappa_5^2}{3} (3 \rho_b + 2 \rho_b); \tag{2.13}
\]

where the left hand side of the above equations has to be evaluated at the position of the brane, and the function \(\Delta a'(0) = a'(0^+) - a'(0^-)\) give the size of the jump of the derivative of \(a(y)\). The same applies for \(\Delta n'\). Since we are assuming \(P\) parity, the jump on the above equations could be expressed in terms of the limiting value on one side of the brane, for instance by \(\Delta a'(0) = 2a'(0^+)\), and a similar relation for \(\Delta n'(0)\).

Let us notice that if we use the above boundary conditions, we may evaluate equation (2.8) on the brane to get the conservation equation
\[
\dot{\rho}_b + 3 \frac{\dot{a}}{a} (p_b + \rho_b) = 0. \tag{2.14}
\]

This is a general result that is independent of the bulk content. In the subsequent discussion, we will assume that the bulk is stable and use the freedom of coordinate redefinition to set \(b = 1\). This condition will simplify the Einstein equations making it easier to extract its physical meaning, as we will see later.

### III. Master Equations of Brane Cosmology

Once we assume that the fifth dimension is stable, we can follow standard procedure as in four dimensional cosmology to reduce the equations given in the last section to a minimal set. First, notice that the conservation laws (2.10), (2.11) and (2.14), will have \(\dot{b} = 0\). Taking \(b = 1\), we notice that the \(G_{00}\) component (Eq. 2.6) of the Einstein equations is already the equivalent of the Friedmann equation, with the Hubble parameter defined as a function of \(y\). However, the presence of the brane will require that the term with second derivative be regularized by extracting the divergent part. The Friedmann equation then has the form
\[
H^2(y) := \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa_5^2}{3} n^2 \rho_B + n^2 \left[ \left( \frac{a'}{a} \right)^2 + \frac{a''_R}{a} \right] - k \frac{n^2}{a^2}; \tag{3.1}
\]

where \(a''_R\) stands for the regular part of the function. Clearly, we recognize an expression similar to that given early [12]. However, let us stress that there is no unknown constant of integration as in [12]. Also, we may identify the regular term as the contribution of the Weyl tensor of the bulk [13]. Since this expression is continuous, thanks to parity symmetry, we may evaluate it on the brane to get the effective Friedmann equation of our universe, where the time component of the metric tensor is chosen to be \(n_0 = 1\):
\[
H_0^2 = \frac{\kappa_5^2}{3} \rho_{B0} + \left( \frac{\kappa_0^2}{6} \rho_b \right)^2 + \left( \frac{a''_R}{a} \right)_0 - k \frac{n^2}{a_0^2}; \tag{3.2}
\]

where the subindex 0 stand for the evaluation at \(y = 0\). As already known [8], this expression has the squared dependence on the brane density. On the other hand, it also has a
dependence on the metric outside (i.e. in the bulk), through \( a'y' \), which however could be evaluated, once we solve for \( a \) as a function of \( y \). Particularly, for simple cases as an empty bulk, the \( y \) dependence can be explicitly extracted and this term evaluated.

Next, let us consider the equation involving \( G_{ij} \), Eq. (2.7). Again, we separate the singular and the regular parts of the second derivative term and by introducing (3.1) we reduce this equation to a form equivalent to the acceleration equation given by

\[
\frac{\ddot{a}}{a} = -\frac{\kappa_5^2}{6} (3P_B + \rho_B) n^2 + n^2 \left( \frac{a' n'}{a n} \right) + H \frac{\dot{n}}{n} + n^2 \left( \frac{a''}{a} + \frac{n''}{n} \right).
\]  

(3.3)

Note that, as in the standard FRW cosmology, this is not an independent equation. Indeed, it can be derived by taking the time derivative of (3.1) and combining that expression with the energy conservation law (2.10) and \( G_{04} \) equation (2.8). This derivation holds regardless of whether the bulk radius is constant or changing with time. As a result, this equation does not provide any extra information, but it will be useful in what follows.

We now turn to the equation involving \( G_{44} \), Eq. (2.9). In conjunction with the equation involving \( G_{04} \), this represents the new ingredient of the brane cosmology and can be a window to understand the \( y \) dependence of the cosmological parameters. Notice that our procedure is in contrast to that used in previous works where the Friedmann equation has been obtained from \( G_{44} \) component. If we substitute Eqs. (3.1) and (3.3) in the equation for the \( G_{44} \) component (Eq. (2.9)), we find a simple equation that governs the behaviour of \( a(y, t) \) and \( n(y, t) \) on the bulk (into each one of the domains). It is given by

\[
3 \frac{a''}{a} + \frac{n''}{n} = \frac{\kappa_5^2}{3} (3P_B - 2P_T - \rho_B).
\]

(3.4)

This expression is supplemented by \( G_{04} \) which trivially reduces to

\[
n(y, t) = \lambda(t) \dot{a}(y, t),
\]

(3.5)

where \( \lambda(t) \) is an arbitrary function of time. Using the freedom of fixing the gauge on the coordinate system, we can set \( n_0 = 1 \) in which case we get \( \lambda = \dot{a}_0^{-1} \). However in most of the results this choice is not necessary at all.

Let us emphasize that Eqs. (3.1), (3.4) and (3.5) form the set of master equations, in the sense that they determine the \( t \) and \( y \) dependence of the metric. We study the solutions of these equations in the next section.

**IV. EXACT SOLUTIONS ON THE BULK**

As a simple application of our master equations let us reconsider the cases already studied in the literature. Let us assume that the stress tensor of the bulk gets contribution only from a cosmological constant, \( \hat{T}^{AB} = -\Lambda_B \delta^A_B \). Then, Eq. (3.1) reduces into

\[
3 \frac{a''}{a} + \frac{n''}{n} = -\frac{2}{3} \kappa_5^2 \Lambda_B.
\]

(4.1)

This equation, together with the scaling equation (3.5), can be solved in a straightforward manner. For \( \Lambda_B = 0 \) we get a linear solution just as in [8]
\[ a(y, t) = A|y| + B; \quad n = \lambda \left( \dot{A}|y| + \dot{B} \right). \]  

Here we have already imposed \( P \) (parity) symmetry on the solution. Next, by using the boundary conditions (2.12) and (2.13) we get the final result

\[ a(y, t) = a_0 \left( 1 - \frac{\kappa_5^2}{6} \rho_b |y| \right); \quad \text{and} \quad n(y, t) = n_0 \left( 1 + \frac{\kappa_5^2}{6} (3\rho_b + 2\rho_b)|y| \right). \]  

(4.3)

Notice that in the expression for \( n \) requiring consistency with the scaling (3.5) leads to the conservation law

\[ \dot{\rho}_b + 3H_0(\rho_b + \rho_b) = 0. \]  

(4.4)

Note that \( a''_R = 0 \) in this case, and there is no obvious way to get the correct Friedmann equation (3.2) (i.e. linear rather than squared dependence of \( H^2 \) on \( \rho_b \)).

Next, we assume that \( \Lambda_B \) is non zero. Again (4.1) is easy to solve, and after using the boundary conditions, we get for \( \Lambda_B < 0 \)

\[ a(y, t) = a_0 \left( \cosh(\mu|y|) - \frac{\kappa_5^2}{6\mu} \rho_b \sinh(\mu|y|) \right), \]

\[ n(y, t) = n_0 \left( \cosh(\mu|y|) + \frac{\kappa_5^2}{6\mu} (3\rho_b + 2\rho_b) \sinh(\mu|y|) \right); \]  

(4.5)

where \( \mu^2 = -\frac{\kappa_5^2}{6} \Lambda_B / 6 \). We recognize those solution presented by Binetruy et al. in [12] when they take their integration constant as zero. Our finding is that this is the only allowed solution with a stable extra dimension. In this case, the condition (3.3) reduces to (4.4). For \( \Lambda_B > 0 \) the solutions have a similar form, with the hyperbolic functions replaced by cos and sin respectively.

If we also assume a brane tension, \( \Lambda_b \), the Friedmann equation on the brane becomes

\[ H_0^2 = \frac{\kappa_5^4}{18} \Lambda_b \rho_b + \frac{\kappa_5^4}{36} \rho_b^2 - \frac{k}{a_0^2} + \frac{\kappa_5^4}{36} \Lambda_b^2 + \frac{\kappa_5^2}{6} \Lambda_B. \]  

(4.6)

We emphasize that this result shows that \( C \), the unknown constant found in [12] is actually zero. A fine tuning among the cosmological constants [10]

\[ \Lambda_B = -\frac{\kappa_5^2}{6} \Lambda_b^2 \]  

(4.7)

may linearize the Friedmann equation in the limit where \( \rho_b \ll \Lambda_b \). As it is clear, this will work only if the vacuum energy of the bulk is negative. Moreover, to get the right expression of the Friedmann equation as in standard cosmology, an extra fine tuning [10]

\[ \Lambda_b = 6 \frac{\kappa_4^2}{\kappa_5^2} \]  

(4.8)

is required, where \( \kappa_4^2 = M_P^{-2} \). With those new assumptions the solution (4.5) reduces into
\begin{align*}
a &= a_0 \left( e^{-\mu |y|} - \frac{\rho_b}{\Lambda_b} \sinh(\mu |y|) \right) \\
n &= n_0 \left( e^{-\mu |y|} + \frac{3 \rho_b + 2 \rho_b}{\Lambda_b} \sinh(\mu |y|) \right)
\end{align*}

Clearly, if we neglect the contribution of the brane densities, we identify the Randall-Sundrum static solutions \[6\]
\begin{align*}
a &= a_0 e^{-\mu |y|} \quad \text{and} \quad n = n_0 e^{-\mu |y|}.
\end{align*}

So far we have only been concerned with finding solutions of the five dimensional cosmological equations with an empty bulk. If we wanted to include a bulk field, e.g. to provide a new picture of inflation \[15\], one would have a modified form of the Equation 3.4:
\begin{align*}
3 \frac{a''}{a} + \frac{n''}{n} &= -2 \frac{\kappa_5^2}{3} \left( V(\phi) + \Lambda_B \right).
\end{align*}

Clearly, if the scalar field evolves very slowly, e.g in the inflation epoch, the solution will be approximately of the form \[15\], but now with
\begin{align*}
\mu^2 &= -\frac{\kappa_5^2}{6} \left( V(\phi) + \Lambda_B \right).
\end{align*}

By setting the fine tuning \[4.7\] on \[3.2\] we get
\begin{align*}
H_0^2 &= \frac{\kappa_5^2}{6} V(\phi(0)) - \frac{k}{a_0^2} + \frac{\kappa_5^4}{18} \Lambda_b \rho_b + \frac{\kappa_5^4}{36} \rho_b^2.
\end{align*}

As expected, the inflaton potential contributes linearly to last equation, and the effective potential is just the value of the bulk potential on the brane.

**V. NON FACTORIZATION OF THE METRIC PARAMETERS**

Before closing the present discussion, let us comment briefly on the nature of the exact solutions. First we point out that all the solutions presented on the previous section are not factorizable. This is in fact a general property of this class of models as we will show now. For this purpose, let us start with case when the bulk radius \(b\) is time dependent and we will see that factorization of the metric parameters \(n(y, t)\) and \(a(y, t)\) will not be consistent with a stable \(b\). For this purpose, we start with the identity
\begin{align*}
\frac{d}{dt} \left( \frac{a'}{a} \right) &= \frac{\dot{a}}{a} - \frac{\dot{a}}{a} \frac{a'}{a};
\end{align*}

which, together with the \(G_{04}\) equation leads to
\begin{align*}
\frac{d}{dt} \left( \frac{a'}{a} \right) &= H \left( \frac{n'}{n} - \frac{a'}{a} \right) + \frac{\dot{b}}{b} \frac{a'}{a}.
\end{align*}

Note that this expression reduces to Eq. \[2.14\] on the brane.
Now, let us assume that $a$ and $n$ are factorizable i.e.

$$a = a_0(t)\beta_1(y) \quad n = n_0(t)\beta_2(y). \quad (5.3)$$

One can then easily see that

$$\frac{d}{dt} \left( \frac{a'}{a} \right) = 0. \quad (5.4)$$

This can be used to rewrite Eq. (5.2) as

$$\frac{\dot{b}}{b} + H_0 \left( \frac{\beta_1 \beta_2'}{\beta_1' \beta_2} - 1 \right) = 0. \quad (5.5)$$

This equation can be integrated in a straightforward manner and leads to

$$b = a_0^{-\beta_3(y)}; \quad (5.6)$$

where $\beta_3$ is given in terms of $\beta_{1,2}$ by the expression between parenthesis in (5.5). From this, we conclude that the factorization ansatz works when the bulk radius, $b$ is time dependent since clearly $a_0$ grows with time. A stable $b$ would then mean that $\beta_3$ must equal zero, which is possible only if $\beta_1 = \beta_2$. Furthermore, since Eq. (5.4) holds everywhere, we may evaluate it on the brane and then conclude using Eq. (2.12) that $\dot{\rho}_b = 0$. From this, it follows that factorizability of $n(y, t)$ and $a(y, t)$ proposed above implies that only a cosmological constant may be present in the brane. Since we require a true time dependent density in the brane to describe realistic cosmology e.g. the transition from a radiation to matter dominated universe, the exact solution to the metric can not be factorizable.

Parenthetically, let us note that if we had chosen a more general form for the exact solutions

$$a = a_0(t)\beta_1(y, t) \quad n = n_0(t)\beta_2(y, t); \quad (5.7)$$

where the functions $\beta_{1,2}$ satisfy the boundary condition $\beta_1(0, t) = \beta_2(0, t) = 1$, no such restriction on brane energy density would emerge. This is indeed the form of the solutions discussed above. For this case, the five dimensional scalar curvature can be written as

$$R = \beta_2^{-2}R_{(4)} + \ldots; \quad (5.8)$$

where $R_{(4)}$ is the four dimensional scalar curvature formed by $a_0$ and $n_0$, the dots represent extra terms.

At this point, we note a puzzling feature with regard to the true definition of the Newton’s constant. One can define the Newton’s constant in two ways: one by identifying the coefficient in front of the energy density in Friedmann equation and another way is by integrating over the fifth dimension in the action integral. We call the first definition a “local” one whereas the second one we call “global”. Clearly in the presence of matter, the first one gives a constant $G_N$ whereas the second gives a time-dependent $G_N$. We can check this easily as follows. Consider the gravity action in five dimensions
\[ S = \int d^4 x dy \sqrt{-g} \frac{1}{2\kappa_5^2} R, \quad (5.9) \]

Using the relation
\[ \sqrt{-g} = \sqrt{-g_4} \beta_1^3 \beta_2 \quad (5.10) \]
(with \( g_4 \) the determinant of the corresponding four dimensional metric) and Eq. (5.8), we obtain
\[ \frac{1}{\kappa_4^2} = \frac{1}{\kappa_5^2} \int dy \, \beta_1^3 \beta_2^{-1} \quad (5.11) \]

adopting the “global” definition of Newton’s constant. The integral in the last equation is taken over the whole fifth dimension. It is clear from the above expression, that this leads, in general to a time dependent Newton’s constant whereas if we used Friedmann equation, we would get it to be time independent. We wish to note that when one uses the static solution, a lá Randall and Sundrum, there is no time dependence due to the absence of matter and hence no puzzle.

VI. CONCLUSIONS

In summary, we have analyzed Einstein equations for cosmology in five dimensions within a brane-bulk picture. Restricting to the case of a stable bulk radius, we extract the generalized Friedmann equation which does not contain any integration constant, but a term that involves a regular part of the second derivative over the spatial component of the metric along the brane coordinates. We then evaluate this second derivative using the complete set of the master equations for five dimensional cosmology and find that for the case of a stable bulk radius, there is no arbitrary constant in the Friedmann equation in the brane. This makes it easier to interpret the brane cosmology as the standard big bang picture. An advantage of our analysis is that we do not need to make any explicit assumptions regarding the bulk content. In this sense it generalizes the results presented before in the literature where only a cosmological constant was assumed to be present in the bulk to obtain the Friedmann equation. Finally we make some remarks on the general nonfactorizable nature of the exact solutions.

Acknowledgements. The work of RNM is supported by a grant from the National Science Foundation under grant number PHY-9802551. The work of APL is supported in part by CONACyT (México). The work of CP is supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP). We wish to thank C. Van de Bruck and S. Pastor for discussions.
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