Abstract

We study relative Seiberg-Witten moduli spaces and define relative invariants for a pair $(X, \Sigma)$ consisting of a smooth, closed, oriented 4-manifold $X$ and a smooth, closed, oriented 2-dimensional submanifold $\Sigma \subset X$ with positive genus. These relative Seiberg-Witten invariants are meant to be the counterparts of relative Gromov-Witten invariants. We also obtain a sum formula (aka a product formula) that relates the SW invariants of a sum $X$ of two closed oriented 4-manifolds $X_1$ and $X_2$ along a common oriented surface $\Sigma$ with dual self-intersections to the relative SW invariants of $(X_1, \Sigma)$ and $(X_2, \Sigma)$. Our formula generalizes Morgan-Szabó-Taubes' product formula.

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1 Introduction

Relative moduli spaces associated to a pair \((X,D)\) of a symplectic manifold/complex projective variety \(X\) and a (symplectic) divisor \(D\) have many applications in both symplectic and complex algebraic geometry. In particular, we use moduli spaces of stable relative maps to define relative Gromov-Witten (or GW) invariants. We also use them to derive a GW sum formula when a smooth one-parameter family of symplectic manifolds/varieties \(\{Y_\lambda\}_{\lambda \in \mathbb{C}}\) degenerates to a simple normal crossing variety \(Y_0 = \bigcup_{i \in \mathbb{Z}} X_i\).

In real dimension four, Seiberg-Witten (or SW) moduli spaces have been an excellent tool for exploring smooth closed orientable 4-manifolds. When \(X\) is symplectic, by a celebrated work of Taubes, we know that the SW invariants of \(X\) are equal to certain (possibly disconnected) GW invariants \([T1]\). This correspondence has been a great tool for classifying symplectic 4-manifolds; e.g. see \([L]\). In light of SW-GW correspondence, it is natural to seek a construction of relative SW invariants for every pair \((X,\Sigma)\) of a closed oriented 4-manifold \(X\) and a closed, oriented, possibly non-connected, 2-dimensional submanifold \(\Sigma \subset X\). It is also natural to expect a sum formula that expresses the SW invariants of a sum \(X\) of two oriented 4-manifolds \(X_1\) and \(X_2\) along a common oriented surface \(\Sigma\) with dual self-intersections in terms of the relative SW invariants of \((X_1,\Sigma)\) and \((X_2,\Sigma)\). It is expected that in the symplectic case and under Taubes’ GW-SW correspondence, such an SW sum formula should correspond to the well-known GW sum formula.

\[
\text{Seiberg-Witten invariants} \quad \xrightarrow{\text{Taubes' correspondence}} \quad \text{Gromov-Witten invariants}
\]

\[
? \quad \text{SW sum formula} \quad ? \quad \text{GW sum formula}
\]

? \quad \text{Relative SW invariants} \quad ? \quad \text{Relative GW invariants}

If \(\Sigma\) has genus \(g>0\) and trivial normal bundles in \(X_1\) and \(X_2\), then Morgan-Szabó-Taubes’ product formula \([MST, \text{Thm. 3.1}]\), when the restriction of the characteristic line bundle to \(\Sigma\) has degree \(2g - 2\), is a particular case of this picture. Applications of their formula include the proof of Thom conjecture \([MST]\) — also proved independently in \([KM1]\) using a vanishing argument — as well as construction of non-symplectic 4-manifolds with non-trivial SW invariants \([FS]\). Also, the vanishing result of \([OS, \text{Thm. 2.1}]\) is another particular case of the SW sum formula envisioned above.

In this paper, we introduce a setup, in two formats, for constructing relative SW moduli spaces and invariants without any major restriction on the topology of \((X,\Sigma)\). In one format, as formulated in Theorem B, we obtain relative SW invariants \(\overline{\text{SW}}^{X,\Sigma}\) and a sum formula that generalizes Morgan-Szabó-Taubes’ product formula, but its relation to the GW side is not clear to us. In the other format, formulated in Theorem A, we obtain relative SW invariants \(\text{SW}^{X,\Sigma}\) and a sum formula, subject to the regularity of the tunneling spaces, which should be equivalent to the GW side of the diagram above under Taubes’ GW-SW correspondence. We believe, moreover, that for certain choice of \(\Omega\) in (1.5), the relative invariants provided by these two approaches are equivalent.

Given a smooth, closed, oriented 4-manifold \(X\) and a transverse union \(\Sigma = \bigcup_{i \in \mathbb{Z}} \Sigma_i\) of positively intersecting, closed, oriented, 2-dimensional sub-manifolds \(\Sigma_i \subset X\), we can define a notion of logarithmic tangent bundle \(TX(-\log \Sigma)\) that coincides with the corresponding notions in algebraic geometry/symplectic topology whenever \((X,\Sigma)\) is complex/symplectic; see \([FMZ]\). There is a vector
bundle homomorphism

$$\iota : TX(-\log \Sigma) \longrightarrow TX,$$

covering $\text{id}_X$, which is an isomorphism away from $\Sigma$. In this paper, for simplicity, we assume that $\Sigma$ is smooth. We will then have

$$TX(-\log \Sigma)|_\Sigma \cong T\Sigma \oplus \mathbb{C}_\Sigma,$$

where $\mathbb{C}_\Sigma$ is the trivial complex line bundle $\Sigma \times \mathbb{C}$. In order to define relative SW invariants, our idea is to use the logarithmic tangent bundle $TX(-\log \Sigma)$ instead of the classical tangent bundle $TX$, as is traditionally used in SW theory. The advantage of this approach is two-fold. First of all, it is amenable to working with cylindrical-end manifolds, for which a good deal of literature is available at our disposal [MMR, T, MST, MOY, N, N1, S, KM]. We will review the essential material in Section 2 for the sake of coherence of the presentation and establishing the notation. In this context, we will work over the cylindrical-end manifold $X - \Sigma$ and simply use the logarithmic tangent bundle to distinguish different topological components of particular SW moduli spaces in dimensions 3 and 4.

From another perspective, using the logarithmic tangent bundle enables us to directly work over the closed manifold $X$ and generalize the results to arbitrary normal crossing case, where it is not feasible to study $X - \Sigma$ as a manifold with one cylindrical end. This approach is outlined in Section 4, where we use a so-called logarithmic connection on $TX(-\log \Sigma)$ and logarithmic connections on the related spinor bundles to derive a logarithmic version of the SW equations. We will treat the details of this direct approach and the general normal crossing case in a separate paper.

Here is an outline of the rest of the paper. In Section 3.1, we define the notion of a relatively canonical spin$^c$ structure for the logarithmic tangent bundle. Every other spin$^c$ structure will then be obtained by tensoring a relatively canonical spin$^c$ structure with a hermitian line bundle $E$ on $X$. Let $s$ be a spin$^c$ structure on $TX(-\log \Sigma)$ with spinor bundle $S = S^+ \oplus S^-$ and characteristic line bundle

$$L = L_s = \text{det}(S^+) = \text{det}(S^-).$$

Define the degree of $s$ along $\Sigma$ to be\(^1\) the integer

$$d(s) := (g - 1) - |m(s)|, \quad m(s) = \frac{1}{2} \deg(L_\Sigma), \quad (1.1)$$

where $g$ is the genus of $\Sigma$ and $L_\Sigma := L_s|_\Sigma$. The motivation for this definition comes from complex geometry: if $X$ is a symplectic 4-manifold, then $TX(-\log \Sigma)$ can be equipped with a complex structure. In this case, if $s_{\text{can}}$ is the canonical spin$^c$ structure on $TX(-\log \Sigma)$ and $s_E = s_{\text{can}} \otimes E$ is the canonical spin$^c$ structure twisted by a complex line bundle $E$, then $d(s_E)$ in (1.1) is simply the degree of $E_{\Sigma} := E|_\Sigma$ or of the Serre dual $K_\Sigma \otimes E^*_\Sigma$, depending on the sign of $\deg(L_\Sigma)$ being non-positive or positive. If we set $\beta = \text{PD}(c_1(E)) \in H_2(X, \mathbb{Z})$, then the spin$^c$ structure $s_E$ is determined by $\beta$ and $d(s_E)$ is equal to the product of homology classes $\Sigma \cdot \beta$ (or $\Sigma \cdot (K_\Sigma - \beta)$).

We focus next on $X - \Sigma$. The restriction of a suitable metric on $TX(-\log \Sigma)$ to $X - \Sigma$ gives $X - \Sigma$ the structure of a manifold with cylindrical end $[0, \infty) \times Y$, where $Y$ is a circle bundle over $\Sigma$. We will then proceed in two steps, which we call gliding and descent, as described below. Let

---

\(^1\)For simplicity, we are assuming that $\Sigma$ is connected. If $\Sigma$ is not connected, some of the numbers and structures in the following description, such as the integer $d(s)$, should be defined component-wise.
\( \mathcal{M}(X - \Sigma, s_{X - \Sigma}) \) denote the moduli space of monopoles on \( X - \Sigma \) with finite energy on the end with respect to the induced spin\(^c\) structure \( s_{X - \Sigma} := s|_{X - \Sigma} \) on \( X - \Sigma \). Different spin\(^c\) structures \( s \) and \( s' \) on \( TX(-\log \Sigma) \) may have the same restriction on \( X - \Sigma \). Therefore, for such \( s \) and \( s' \), \( \mathcal{M}(X - \Sigma, s_{X - \Sigma}) \) will be the same as \( \mathcal{M}(X - \Sigma, s'_{X - \Sigma}) \). In the following, by looking at the limits of the monopoles over the cylindrical end, we will choose a component of \( \mathcal{M}(X - \Sigma, s_{X - \Sigma}) \) corresponding to \( s \).

In temporal gauge, a monopole on \( X - \Sigma \) with finite energy on the end is the gradient flow line of the Chern-Simons-Dirac function \( \text{CSD} \) on the cylindrical end (see Section 2.5), where a stationary solution on the end corresponds to a monopole on \( Y \). Therefore, we have a limiting map

\[
\partial : \mathcal{M}(X - \Sigma, s_{X - \Sigma}) \to \mathcal{M}(Y, s_Y),
\]

where \( s_Y \) is the induced spin\(^c\) structure on \( Y \). Meanwhile, as we will see in Section 2.4, an irreducible monopole on \( Y \) in this setup actually descends to a monopole on \( \Sigma \), i.e., it will be the pullback of a solution of \( \text{SW}^2 \) on \( \Sigma \) via the bundle projection \( Y \to \Sigma \). Because of this, the moduli space \( \mathcal{M}(Y, s_Y) \) decomposes into a disjoint union of components

\[
\mathcal{M}(Y, s_Y) = \mathcal{J} \cup \bigcup_m \mathcal{M}(Y, s_Y)_m,
\]

where \( \mathcal{J} \cong \text{Map}(\Sigma, S^1) \cong \mathbb{T}^{2g} \) is the subspace of reducible solutions\(^2\) and the second union is over the set of integers\(^3\) \( m = m(s) \) for all spin\(^c\) structures \( s \) on \( TX(-\log \Sigma) \) that restrict to \( s_{X - \Sigma} \) on \( X - \Sigma \). The integer \( m(s) \) and the restriction \( s_{X - \Sigma} \) uniquely determine \( s \); therefore, for each \( s \), we define

\[
\mathcal{M}(X - \Sigma, s) := \partial^{-1}(\mathcal{M}(Y, s_Y)_{m(s)}) \subset \mathcal{M}(X - \Sigma, s_{X - \Sigma}).
\]

By definition, all the monopoles in \( \mathcal{M}(X - \Sigma, s) \) are automatically irreducible. We also define

\[
\mathcal{M}(X - \Sigma, s_{X - \Sigma}, \mathcal{J}) = \partial^{-1}(\mathcal{J}).
\]

Thus, we have a decomposition

\[
\mathcal{M}(X - \Sigma, s_{X - \Sigma}) = \mathcal{M}(X - \Sigma, s_{X - \Sigma}, \mathcal{J}) \cup \bigcup_s \mathcal{M}(X - \Sigma, s),
\]

where the second union runs over all spin\(^c\) structures \( s \) on \( TX(-\log \Sigma) \) that restrict to the fixed spin\(^c\) structure \( s_{X - \Sigma} \) on \( X - \Sigma \).

A monopole in \( \mathcal{M}(Y, s_Y)_{m(s)} \) can be identified with an effective divisor of degree \( d(s) > 0 \) on \( \Sigma \), or a single point if \( d(s) = 0 \). Depending on whether \( m(s) < 0 \) or \( m(s) > 0 \), this divisor is the zero set of a non-zero holomorphic section of \( E_\Sigma \) or \( K_\Sigma \otimes E_\Sigma^* \), respectively. For \( d > 0 \), let

\[
\text{Div}_d(\Sigma) \cong \text{Sym}^d(\Sigma)
\]

denote the space of effective divisors of degree \( d \) on \( \Sigma \); if \( d = 0 \), this space is taken to be a single point. By the argument above, we obtain a landing map

\[
b : \mathcal{M}(Y, s_Y)_{m(s)} \to \text{Div}_d(\Sigma)
\]

\(^2\)This component is empty if \( \Sigma \cdot \Sigma = 0 \) and \( \deg(L_\Sigma) \neq 2 - 2g \), and is equal to \( \mathbb{T}^{2g+1} \) if \( \Sigma \cdot \Sigma = 0 \) and \( \deg(L_\Sigma) = 2 - 2g \). Since the case \( \Sigma \cdot \Sigma = 0 \) is treated extensively in [MST], throughout the paper, we often implicitly or explicitly assume that \( \Sigma \cdot \Sigma \neq 0 \).

\(^3\)For \( m \not\in [1 - g, g - 1] \), the component \( \mathcal{M}(Y, s_Y)_m \) is empty, and for \( m = 0 \), we get the reducible component \( \mathcal{J} \) instead. Different \( m \)'s among non-trivial components differ by a multiple of \( \Sigma \cdot \Sigma \).
which assigns to each monopole on $Y$ the corresponding effective divisor on $\Sigma$ of degree $d(s)$ on which the non-zero spinor lands; this will be the “empty” divisor if $d(s)=0$. The landing map is a diffeomorphism. Combining the two steps, and using the nomenclature of Gromov-Witten theory, we obtain an evaluation map
\[
ev := b \circ \partial : M(X - \Sigma, s) \longrightarrow \text{Div}_{d(s)}(\Sigma).
\]

In order to make sure that the moduli space is cut transversely and contains no reducible solutions, we need to perturb the SW equations on $X - \Sigma$ in a way that (1.3) is still defined. This can be achieved by using either a \textit{compact perturbation} or an \textit{adapted perturbation}; see Section 2.5. In the first case, the perturbation term is a compactly supported self-adjoint 2-form $\eta_o$. In the second case, restricted to the neck $[0, \infty) \times Y$, the perturbation term is additionally equal to the self-adjoint part of the pullback of a non-trivial 2-form $\eta$ on $Y$. Furthermore, following [MST], we consider a special type of adapted perturbations $\eta$, where $\eta$ is obtained from a holomorphic 1-form $\nu$ on $\Sigma$; see Lemma 2.4. We denote the resulting moduli spaces by $M_{\eta}(X - \Sigma, s)$ and $M_{\eta}(X - \Sigma, s)$, respectively. The discussion leading to (1.3) readily generalizes to the compactly-perturbed moduli spaces $M_{\eta}(X - \Sigma, s)$. In the second case, CSD is circle-valued, but (1.3) is still defined and we get a major restriction on its image. Given a non-trivial holomorphic 1-form $\nu$, vanishing on $X - \Sigma$ with $\text{div}(\nu) = \{p_1, \ldots, p_{2g-2}\} \subset \Sigma$

(counted with multiplicities), let $S_d(\nu)$ (\(\equiv S_{2g-2-d}(\nu)\)) denote the set of subsets of $\text{div}(\nu)$ of size $d$.

In the case of $M_{\eta}(X - \Sigma, s)$, $J$ will be empty and (1.3) takes values in $S_{d(s)}(\nu)$. Therefore,

$$M_{\eta}(X - \Sigma, s_X - \Sigma) = \bigcup_{s} M_{\eta}(X - \Sigma, s).$$

**Remark 1.1.** We have assumed $g > 0$; otherwise, there is no such $\nu$. Furthermore, if $g = 0$, we have $M_{\eta}(X - \Sigma, s_X - \Sigma) = M_{\eta}(X - \Sigma, s_X - \Sigma, J)$.

**Theorem A.** Let $X$ be a smooth, closed, oriented 4-manifold and $\Sigma \subset M$ a smooth, closed, oriented surface with positive genus. If $b_X^{\Sigma} > 0$, for a generic compact perturbation $\eta_o$, $M_{\eta}(X - \Sigma, s)$ is a smooth orientable (but not necessarily compact) manifold of real dimension

$$d = \frac{(c_1(L) + \Sigma)^2 - 2\chi(X) - 3\sigma(X)}{4}.$$  

(1.4)

Furthermore, the evaluation map (1.3) is a smooth submersion.

Unlike the classic case in SW theory, the moduli spaces $M_{\eta}(X - \Sigma, s)$ are not necessarily compact. Because of the tunneling phenomenon (see [KM, Sec. 16] or [N, Sec. 4.4.2]), a sequence of monopoles in $M_{\eta}(X - \Sigma, s)$ will, after passing to a sub-sequence, “converge” to a finite ordered set of monopoles (called a \textit{broken trajectory} in [KM]), where the first one is a monopole on $X - \Sigma$ and the rest are non-trivial (i.e., non-stationary) monopoles on the cylinder $\mathbb{R} \times Y$. Furthermore,

(1) the limit at $+\infty$ of the monopole on $X - \Sigma$ defined in (1.2) coincides with the limit at $-\infty$ of the monopole on the first copy of $\mathbb{R} \times Y$, and

(2) the limit at $+\infty$ of the monopole on the $i$-th copy of $\mathbb{R} \times Y$ coincides with the limit at $-\infty$ of the monopole on the $(i+1)$-th copy of $\mathbb{R} \times Y$. 

5
In order to obtain a compact moduli space without boundary, we will consider the monopoles on the cylinder \( \mathbb{R} \times Y \) up to the natural \( \mathbb{C}^* \)-action generated by translation in the \( \mathbb{R} \)-direction and rotation in the \( Y \)-direction. We denote the space containing such limits by \( \overline{\mathcal{M}}_{\eta_0}(X - \Sigma, \mathfrak{s}) \). With the exception of taking quotients by \( \mathbb{C}^* = \mathbb{R} \times S^1 \) instead of just by \( \mathbb{R} \), this is the same compactification considered in [KM, OS]. The limiting map (1.2) on

\[
\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, \mathfrak{s}) \subset \mathcal{M}_{\eta_0}(X - \Sigma, \mathfrak{s}_{X - \Sigma})
\]

is taken at \(+\infty\) of the last monopole. This compactification is the direct analogue of the relative compactification of the moduli space of pseudo-holomorphic curves relative to \( \Sigma \) (with the contact orders all equal to one).

While we show that the complement

\[
\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, \mathfrak{s}) - \mathcal{M}_{\eta_0}(X - \Sigma, \mathfrak{s})
\]

is a finite union of strata of expected codimension at least 2, it is not clear to us if the tunneling spaces are always manifolds of the expected dimension; see Section 3.5. Therefore, without further restrictions (e.g. as in [OS]), it may not always be the case that, for generic \( \eta_0 \), \( \overline{\mathcal{M}}_{\eta_0}(X - \Sigma, \mathfrak{s}) \) is a \( C^0 \)-manifold of the expected dimension (1.4). Whenever the latter happens, the relative Seiberg-Witten invariants of \( (X, \Sigma) \) in the class of a logarithmic spin\(^c\) structure \( \mathfrak{s} \) are defined by integration on \( \overline{\mathcal{M}}_{\eta_0}(X - \Sigma, \mathfrak{s}) \) in the following way.

Let

\[
c \in H^2(\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, \mathfrak{s}), \mathbb{Z})
\]

denote the first Chern class of the natural circle bundle on the moduli space as in the classical case (see [M, Sec. 6.7] or [Sal, p. 249, (7.24)]). For \( \Omega \in H^*(\text{Div}_{d(\mathfrak{s})}(\Sigma)) \) satisfying

\[
\text{deg}(\Omega) + 2r = d,
\]

define

\[
\text{SW}^X_{\eta_0}(\mathfrak{s}; \Omega) := \int_{\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, \mathfrak{s})} c^r \wedge \text{ev}^*\Omega = \int_{\mathcal{M}_{\eta_0}(X - \Sigma, \mathfrak{s})} c^r \wedge \text{ev}^*\Omega.
\]

Under the assumption that the tunneling spaces have their expected dimensions, the inclusion \( \mathcal{M}_{\eta_0}(X - \Sigma, \mathfrak{s}) \subset \overline{\mathcal{M}}_{\eta_0}(X - \Sigma, \mathfrak{s}) \) defines a pseudo-cycle in the sense of [Z]; this justifies the second equality in (1.5). In this situation, unlike in the definition of the classical SW invariants, the condition \( b^r_{X - \Sigma} > 0 \) is sufficient to conclude that \( \text{SW}^X_{\eta_0}(\mathfrak{s}; -) \) is independent of the choice of generic \( \eta_0 \) and the cylindrical metric, since all the monopoles in \( \mathcal{M}_{\eta_0}(X - \Sigma, \mathfrak{s}) \) are automatically irreducible by definition. We will elaborate more on this point in Section 3.6. Therefore, we drop \( \eta_0 \) from the notation and denote the invariants by \( \text{SW}^X_{\mathfrak{s}}(\mathfrak{s}; -) \). When \( X \) is symplectic, we believe that a special case of (1.5), described below, is equal to certain relative Gromov-Witten invariant in the sense of Taubes’ correspondence.

Given a spin\(^c\) structure \( \mathfrak{s} \) on \( TX(-\log \Sigma) \) such that \( d(\mathfrak{s}) \geq 0 \) (otherwise the moduli space is empty), let \( t = \{t_1, \ldots, t_k\} \) be a (possibly empty) partition of \( d(\mathfrak{s}) \) into \( k = k(t) \) positive integers, i.e.,

\[
d(\mathfrak{s}) = t_1 + \cdots + t_k.
\]

Let

\[
\text{Div}_t(\Sigma) = \left\{ \sum_{i=1}^k t_i p_i : p_i \in \Sigma \; \forall \; i = 1, \ldots, k \right\} \subset \text{Div}_{d(\mathfrak{s})}(\Sigma)
\]
The dimension formula (1.7) differs from (1.4) by a term \( d(s) \). If we sort the numbers in (1.6) so that

\[
t_1 = \cdots = t_{i_1} < t_{i_1+1} = \cdots = t_{i_1+i_2} < \cdots < t_{i_1+\cdots+i_{n-1}+1} = \cdots = t_{i_1+\cdots+i_n}, \quad k = i_1 + \cdots + i_n,
\]
then

\[
\text{Div}_t(\Sigma) \cong \text{Sym}^{i_1}(\Sigma) \times \cdots \times \text{Sym}^{i_n}(\Sigma).
\]

For \( \Omega = \text{PD}(\text{Div}_t(\Sigma)) \) in (1.5), we define a particular type of relative SW invariants by

\[
\text{SW}^{X, \Sigma}(s; t) := \text{SW}^{X, \Sigma}(s; \text{PD}(\text{Div}_t(\Sigma))).
\]

If \( X \) is symplectic, as mentioned earlier, we can identify the set of spin\(^c\) structures \( s \) on \( TX(-\log \Sigma) \) with the set of homology classes \( \beta \in H_2(X, \mathbb{Z}) \). Under this identification, (1.4) simplifies to

\[
d = \beta \cdot \beta - K_X \cdot \beta
\]
and \( t \) is a partition of \( d(s) = \Sigma \cdot \beta \) (or \( \Sigma \cdot (K_X - \beta) \)) into a sum \( t_1 + \cdots + t_k \) of positive integers. Then we believe that, similar to Taubes’ GW-SW correspondence theorem, \( \text{SW}^{X, \Sigma}(\beta; t) \) is equal to certain relative Gromov-Witten invariant \( GW^{X, \Sigma}(\beta; t) \); it is a count of (possibly disconnected) \( J \)-holomorphic curves of degree \( \beta \) that intersect \( \Sigma \) in \( k \) points with tangency orders \( t = \{t_1, \ldots, t_k\} \) and pass through \( r \) generic points in \( X \).

In order to resolve the issues above regarding regularity of tunneling spaces and non-compactness, we may consider adapted perturbations \( \eta \) corresponding to holomorphic 1-forms \( \nu \) on \( \Sigma \). In this case, we show that the tunneling spaces are empty.

**Theorem B.** Let \( X \) be a smooth, closed, oriented 4-manifold and \( \Sigma \subset M \) a smooth, closed, oriented surface with positive genus. If \( b^+_X - \Sigma > 0 \), for every \( \nu \neq 0 \) and generic adapted perturbation \( \eta \), \( \mathcal{M}_{\eta}(X - \Sigma, s) \) is a smooth closed manifold of real dimension

\[
\bar{d} = \frac{(c_1(L) + \Sigma)^2 - 2\chi(X) - 3\sigma(X)}{4} - d(s).
\]

For different such \( \eta_\nu \) and \( \eta_{\nu'} \), \( \mathcal{M}_{\eta}(X - \Sigma, s) \) and \( \mathcal{M}_{\eta_\nu}(X - \Sigma, s) \) are smoothly cobordant.

In this case, for each point \( q \in S_{d(s)}(\nu) \), if \( \bar{d} \) is even, we define

\[
\text{SW}^{X, \Sigma}_{\nu}(s; q) := \int_{ev^{-1}(q) \subset \mathcal{M}_{\nu}(X - \Sigma, s)} c^{\bar{d}/2}
\]

By the second statement of Theorem B, the right-hand side (1.8) is independent of the choice of the compactly-supported part of \( \eta_\nu \), which justifies the notation on the left-hand side. Recall that, by definition, the elements of \( \mathcal{M}_{\eta}(X - \Sigma, s) \) are automatically irreducible. Furthermore, the finite sum

\[
\text{SW}^{X, \Sigma}(s) = \sum_{q \in S_{d(s)}(\nu)} \text{SW}^{X, \Sigma}_{\nu}(s; q)
\]
is independent of the choice of \( \nu \).

The dimension formula (1.7) differs from (1.4) by a term \( d(s) \). Particularly, if \( X \) is symplectic, identifying the set of spin\(^c\) structures \( s \) on \( TX(-\log \Sigma) \) with the set of homology classes \( \beta \in H_2(X, \mathbb{Z}) \) again, (1.7) simplifies to

\[
\bar{d} = \beta \cdot \beta - (K_X + \Sigma) \cdot \beta.
\]
Note that $K_X + \Sigma$ is the logarithmic canonical line bundle of $(X, \Sigma)$. The integer $d$ is even if $d(\mathfrak{s}) = \Sigma \cdot \beta$ is even. It is not clear to us if SW is equal to certain count of $J$-holomorphic curves.

Next, we prove that there is a SW sum formula that relates the SW invariant of a sum $X = X_1 \#_\Sigma X_2$ of two closed oriented 4-manifolds $X_1$ and $X_2$ along a common oriented surface $\Sigma$ with dual self-intersections to the relative SW invariants of $(X_1, \Sigma)$ and $(X_2, \Sigma)$. Here, we only need to consider spin$^c$ structures on $X$ that are of "pullback type" on the separating hypersurface $Y$ of this sum manifold. Recall that the SW invariants of the connected sum $X$ of two closed oriented 4-manifolds $X_1$ and $X_2$ with $b_2^+(X_i) > 1$ are zero.

Suppose $X = X_1 \#_\Sigma X_2$ and $\mathfrak{s}, \mathfrak{s}'$ are two spin$^c$ structures on $X$. We say $\mathfrak{s}$ is equivalent to $\mathfrak{s}'$ if both restrict to the same spin$^c$ structures on $X_1 - \Sigma$ and $X_2 - \Sigma$. We denote the equivalence class of $\mathfrak{s}$ by $[\mathfrak{s}]$.

**Theorem C.** Suppose $X = X_1 \#_\Sigma X_2$ and $\mathfrak{s}$ is a spin$^c$ structure on $X$ that is of pullback type on the separating hypersurface $Y$. For each holomorphic 1-form $\nu \neq 0$ we have the sum formula

$$
\sum_{\mathfrak{s}' \in [\mathfrak{s}]} \text{SW}^X(\mathfrak{s}') = \sum_{[\mathfrak{s}] = [\mathfrak{s}_1] \#_\Sigma [\mathfrak{s}_2]} \sum_{q \in S_{d(\nu)}(\nu)} \varepsilon_{\mathfrak{s}_1, \mathfrak{s}_2, q} \text{SW}^{X_1, \Sigma}(\mathfrak{s}_1; q) \cdot \text{SW}^{X_2, \Sigma}(\mathfrak{s}_2; q). \quad (1.9)
$$

In (1.9), $\varepsilon_{\mathfrak{s}_1, \mathfrak{s}_2, q} \in \{\pm 1\}$ depend on the choice of orientations. Whenever the relative invariants (1.5) are defined, we get another sum formula

$$
\sum_{\mathfrak{s}' \in [\mathfrak{s}]} \text{SW}^X(\mathfrak{s}') = \sum_{[\mathfrak{s}] = [\mathfrak{s}_1] \#_\Sigma [\mathfrak{s}_2]} \varepsilon_{\mathfrak{s}_1, \mathfrak{s}_2} \sum_{\Omega} \text{SW}^{X_1, \Sigma}(\mathfrak{s}_1; \Omega) \cdot \text{SW}^{X_2, \Sigma}(\mathfrak{s}_2; \Omega^*) \quad (1.10)
$$

where $\Omega$ runs over the terms in the Künneth decomposition $\sum \Omega \otimes \Omega^*$ of the diagonal in $\text{Div}(\Sigma) \times \text{Div}(\Sigma)$. Under the GW-SW correspondence, the SW sum formula (1.10) should correspond to the well-known GW sum formula, with $X$ being the symplectic sum of $X_1$ and $X_2$ along $\Sigma$.

**Remark 1.2.** If $\mathfrak{s}$ is a spin$^c$ structure on $X$ that is not of pullback type on the separating hypersurface $Y$, then it follows from Fact 1 in [N1, p. 94] and the same convergence argument used in the proof of Theorem C that all the terms in (1.10) are zero.

**Remark 1.3.** The issue that the sum $\mathfrak{s}_1 \#_\Sigma \mathfrak{s}_2$ of two spin$^c$ structures on $X_1$ and $X_2$ is only well-defined up to the equivalence relation defined before Theorem C also appears in the GW sum formula: two homology classes $A_1 \in H_2(X_1, \mathbb{Z})$ and $A_2 \in H_2(X_2, \mathbb{Z})$ that have the same intersection number with $\Sigma$ can be glued together to produce a homology class $A \in H_2(X, \mathbb{Z})$; however, $A$ is only well-defined up to addition with homology classes associated to certain elements of $H_1(\Sigma, \mathbb{Z})$, known as rim tori. A natural question is whether we can refine the sum formulas (1.9) and (1.10) to express each individual $\text{SW}^X(\mathfrak{s}')$ in terms of some "refined relative invariants" of $X_1$ and $X_2$. In the context of GW sum formula, this problem is extensively studied in [FZ1, FZ2].

## 2 A tour of Seiberg-Witten theory

### 2.1 Spin$^c$ structures

Given an oriented riemannian rank $2n$ (real) vector bundle $V \rightarrow X$ over a smooth manifold $X$, a spin$^c$ structure $\mathfrak{s} = (\mathfrak{S}, \rho)$ on $V$ consists of a rank $2n$ (complex) hermitian vector bundle $\mathfrak{S} \rightarrow X$, called the spinor bundle, and a Clifford multiplication, which is a linear map of vector bundles

$$
\rho : V \rightarrow \text{End}(\mathfrak{S}), \quad V_x \ni v \rightarrow \rho(v) \in \text{End}(\mathfrak{S}|_x),
$$

where $\mathfrak{S}$ is a real vector bundle over $X$. The Clifford multiplication is a linear map of vector bundles $\rho : V \rightarrow \text{End}(\mathfrak{S})$, where $\text{End}(\mathfrak{S})$ is the space of linear maps from $\mathfrak{S}$ to itself. The Clifford multiplication $\rho$ is a natural way to extend the action of $\text{Spin}(4)$ on $\mathfrak{S}$ to the bundle $V$. This action is a homomorphism of Lie groups, and it is a fundamental property of spin$^c$ structures that the action of $\text{Spin}(4)$ on $\mathfrak{S}$ is compatible with the bundle structure $\rho$. This compatibility is what makes the bundle $\mathfrak{S}$ into a spin$^c$ structure on $X$.
satisfying
\[ \rho(v)^* + \rho(v) = 0, \quad \rho(v)^* \rho(v) = |v|^2 \text{id}, \quad \forall v \in V. \quad (2.1) \]

We sometimes denote the action of \( \rho(v) \) on \( \psi \in S \) by \( v \cdot \psi \) when there is no chance of confusion. Every such \( S \) admits a canonical splitting into rank \( 2^{n-1} \) hermitian vector bundles, \( S = S^+ \oplus S^- \), satisfying
\[ \rho(v)S^\pm|_x = S^\mp|_x, \quad \forall x \in X, \ v \in V_x. \]

In other words, we have
\[ \rho(v) = \begin{bmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{bmatrix}, \quad \gamma : V \rightarrow \text{Hom}(S^-, S^+), \]
where
\[ \gamma(v)^* \gamma(v) = |v|^2 \text{id}. \]

A spin\(^c\) isomorphism from \((S_0, \rho_0)\) to \((S_1, \rho_1)\) is a unitary bundle isomorphism \( f : S_0 \rightarrow S_1 \) which induces a bundle isomorphism \( \text{End}(f) : \text{End}(S_0) \rightarrow \text{End}(S_1) \) such that \( \text{End}(f) \circ \rho_0 = \rho_1 \). Denote by \( S^c(V) \) the set of isomorphism classes of spin\(^c\) structures on \( V \). Given a riemannian \( 2n\)-manifold \( X \), let \( S^c(X) \) be the set of isomorphism classes of spin\(^c\) structures on \( TX \). This is in fact the set of principal spin\(^c\) bundles lifting the principal tangent bundle of \( X \) up to bundle isomorphism.

If \((S, \rho)\) is a spin\(^c\) structure on \( V \) and \( n > 1 \), then
\[ \det(S^+) = \det(S^-) = L^0_2 \]
for a unique complex line bundle \( L_2 \), called the characteristic line bundle of the spin\(^c\) structure, for which \( c_1(L_2) \) is an integral lift of \( w_2(V) \). In fact, \( L_2 \) is the determinant line bundle of the principal Spin\(^c\)(2n)-bundle which lifts the principal SO(2n)-bundle associated to \( V \rightarrow X \). We define \( c_1(S) := c_1(L_2) \). Similarly, we say that the spin\(^c\) structure \( s \) is torsion if \( c_1(S) \) is torsion. When there is no chance of confusion, we will drop the subscript \( s \) and simply write \( L \).

Given a spin\(^c\) structure \( s_0 = (S_0, \rho_0) \), every other spin\(^c\) structure \( s = (S, \rho) \in S^c(V) \) has the form
\[ S \cong S_0 \otimes E, \quad \rho = \rho_0 \otimes \text{id}, \]
where \( E \) is a hermitian line bundle. We express the above relation between \( s \) and \( s_0 \) by writing \( s = s_0 \otimes E \). Note that \( L_2 = L_2 \otimes E \otimes 2 \) and the two spin\(^c\) structures \( s \) and \( s_0 \) are isomorphic if and only if \( c_1(E) = 0 \).

Every complex vector bundle \((V, J)\) equipped with a hermitian metric \((\cdot, \cdot)\) admits a canonical spin\(^c\) structure \( S_{\text{can}} = (S_{\text{can}}, \rho_{\text{can}}) \) with
\[ S_{\text{can}}^+ = \Lambda^{0,\text{even}} V^*, \quad S_{\text{can}}^- = \Lambda^{0,\text{odd}} V^*, \]
\[ \rho_{\text{can}}(v)\alpha = -\sqrt{2} t_{v} \alpha + \frac{1}{\sqrt{2}} (\langle \cdot, v \rangle \wedge \alpha). \quad (2.2) \]

As a result, the correspondence \( S_{\text{can}} \otimes E \leftrightarrow E \) determines a canonical bijection between isomorphism classes of spin\(^c\) structures and isomorphism classes of complex line bundles over \( X \).

Specifically, every almost-complex manifold \((X^{2n}, J)\) admits a canonical spin\(^c\) structure \( S_{\text{can}} \) with
\[ S_{\text{can}}^+ = \Lambda^{0,\text{even}} T^* X, \quad S_{\text{can}}^- = \Lambda^{0,\text{odd}} T^* X, \quad L_{S_{\text{can}}} = K_X^*, \]
\[ \rho_{\text{can}}(v)\alpha = -\sqrt{2} t_v \alpha + \frac{1}{\sqrt{2}} \langle \cdot, v \rangle \wedge \alpha. \]
In particular, if \((X, \omega_X)\) is a symplectic manifold, we can choose an \(\omega_X\)-compatible (or \(\omega_X\)-tame) almost-complex structure \(J\) and define \(s_{can}\) as above. In this case, \(K_X = \Lambda^{n,0} T^* X\) is the canonical bundle of \(X\) with respect to \(\omega_X\) and \(\langle \cdot, v \rangle = \omega_X(\cdot, iv + Jv)\) is the \((0,1)\)-form dual to \(iv + Jv \in T^{1,0} X\).

If \(J\) is \(\omega_X\)-tame but not \(\omega_X\)-compatible, then one has to replace \(\omega_X\) in the definition of \(\langle \cdot, v \rangle\) with

\[
\bar{\omega}_X(u, v) = \frac{1}{2} \left( \omega_X(u, v) + \omega_X(Ju, Jv) \right).
\]

Let \(g\) be a riemannian metric on \(X^{2n}\), \(\nabla\) the corresponding Levi-Civita connection, and \(s \in S^c(X)\). A hermitian connection \(\bar{\nabla}\) on \(S\) is called a spin\(^c\) connection if it is compatible with \(\nabla\) and the Clifford multiplication

\[
\bar{\nabla}_v(w \cdot \Phi) = w \cdot \bar{\nabla}_v \Phi + (\nabla_v w) \cdot \Phi, \quad \forall \Phi \in \Gamma(S), \ v, w \in \Gamma(TX).
\]

Every spin\(^c\) connection preserves \(S^\pm\). Every two such connections differ by an imaginary-valued 1-form \(\alpha \in \Omega^1(X, i\mathbb{R})\). Moreover, the gauge group \(\mathcal{G} = \text{Maps}(X, S^1)\) acts on the space of connections by

\[
(u^* \bar{\nabla})(\Phi) = u^{-1} \bar{\nabla}(u \Phi) = \frac{du}{u} \otimes \Phi + \nabla \Phi
\]

for \(\Phi \in \Gamma(S)\) and \(u \in \text{Maps}(X, S^1)\).

Every spin\(^c\) connection \(\bar{\nabla}\) on the spinor bundle \(S\) is uniquely determined by \(\nabla\) and a connection \(A\) on the characteristic line bundle \(L\) of the spin\(^c\) structure. This is essentially due to the fact that the underlying principal bundle of \(S\) (with fiber \(\text{Spin}^c(2n)\)) lifts the principal tangent bundle of \(X\) (with fiber \(\text{SO}(2n)\)) as a circle-bundle extension (corresponding to the line bundle \(L\)). This corresponds to the lifting of \(w_2(V)\) to the integral class \(c_1(L)\); see [M, Chap. 3]. The space of connections on the characteristic line bundle \(L\), denoted by \(\mathcal{A}(L)\), is an affine space with tangent space \(\Omega^1(X, i\mathbb{R})\). If \(A \in \mathcal{A}(L)\) and \(a \in \Omega^1(X, i\mathbb{R})\), then the curvature \(F_A \in \Omega^2(X, i\mathbb{R})\) and \(F_{A+a} = F_A + da\). The connection \(A\) on \(L\) induces a spin\(^c\) connection \(\bar{\nabla} = \nabla_A\) on \(S\) and their curvatures are related by [Sal, Sec. 6.1]

\[
F_A(v, w) = \frac{1}{2n} \text{tr}(F_{\nabla_A}(v, w)).
\]

The gauge group \(\mathcal{G} = \text{Maps}(X, S^1)\) acts on \(\mathcal{A}(L)\) by

\[
w^*A = \frac{du}{u} + A
\]

and leaves \(F_A\) invariant. The operator

\[
\partial_A : \Gamma(S^\pm) \rightarrow \Gamma(S^\mp), \quad \partial_A \Phi = \sum_{i=1}^{2n} e_i \cdot \bar{\nabla}_{e_i} \Phi,
\]

where \(e_1, \ldots, e_{2n}\) form an orthonormal frame for \(TX\), is the Dirac operator associated to \(A \in \mathcal{A}(L)\). It is a self-adjoint operator independent of the particular choice of \(e_1, \ldots, e_{2n}\).

If \((X, \omega_X, J)\) is Kähler, then

\[
\Gamma(S^+_{\text{can}} \otimes E) = \Omega^{0,\text{even}}(X, E), \quad \Gamma(S^-_{\text{can}} \otimes E) = \Omega^{0,\text{odd}}(X, E),
\]

and

\[
\frac{1}{\sqrt{2}} \partial_A = \partial_{A_E} + \partial_{A_E}^* : \Omega^{0,\text{even}}(X, E) \rightarrow \Omega^{0,\text{odd}}(X, E), \quad (2.3)
\]
where \( A_E \) is a hermitian connection on \( E \), \( \bar{\partial}_{A_E} = \nabla_{A_E}^{0,1} \) is the \( \bar{\partial} \)-operator associated to \( A_0 \), and \( A \) is the induced connection on the characteristic line bundle \( L \) of the spin\(^c\) structure. For example, in dimension 4, \( \mathcal{S}_{\text{can}} = \Lambda^{0,1} T^* X \), \( \det(\mathcal{S}_{\text{can}}) = K_X^* \) and \( L = \det(S^-) = \det(\mathcal{S}_{\text{can}} \otimes E) = K_X^* \otimes E^2. \)

Therefore, any hermitian connection \( A \) on \( L \) is equivalent to a hermitian connection \( A_E \) on \( E \) and we have \( A = A_{\text{can}} \otimes A_E^2 \), where \( A_{\text{can}} \) is the holomorphic hermitian connection on \( K_X^* \).

The identity (2.3) continues to hold in the symplectic case [Sal, Thm. 6.17]. If \((X, \omega_X)\) is a symplectic manifold, \( J \) is compatible with \( \omega_X \) (resp. tames \( \omega_X \)), and \( \nabla \) is the Levi-Civita connection of the metric \( g = \omega_X(\cdot, J\cdot) \) (resp. \( g = \bar{\omega}_X(\cdot, J\cdot) \)), then \( \nabla \) does not preserve \( \Omega^{0,1}(X) \) unless \((X, \omega_X, J)\) is Kähler, i.e., \( \nabla J = 0 \). However, there is a canonical hermitian connection on \( TX \), defined by

\[
\bar{\nabla}_v w := \nabla_v w - \frac{1}{2} J(\nabla_v J)w,
\]

which gives rise to a hermitian connection on \( \mathcal{S}_{\text{can}} = \Lambda^{0,1} T^* X \), compatible with the Clifford multiplication but not with \( \nabla \). However, it is possible to modify \( \bar{\nabla} \) further to produce a spin\(^c\) connection \( \bar{\nabla} \), compatible with both the Clifford multiplication and the Levi-Civita connection \( \nabla \)

\[
\bar{\nabla}_v \Phi := \bar{\nabla}_v \Phi + \frac{1}{2} \mu(J \nabla_v J) \Phi,
\]

where \( \Phi \in \Gamma(\mathcal{S}_{\text{can}}) = \Omega^{0,1}(X) \) and \( \mu : \mathfrak{so}(TX) \rightarrow \text{End}(\mathcal{S}_{\text{can}}) \) is characterized by

\[
[\mu(A), \rho_{\text{can}}(v)] = \rho_{\text{can}}(Av).
\]

When \( X \) is Kähler, the connections \( \bar{\nabla} \) and \( \bar{\nabla} \) coincide with the Levi-Civita connection \( \nabla \) on forms [Sal, pp. 198–199].

### 2.2 SW equations in dimension 4

Let \( X \) be a smooth closed connected oriented riemannian 4-manifold (with metric \( g \)) and \( s = (S, \rho) \) a spin\(^c\) structure on \( X \). The (unperturbed) Seiberg-Witten monopole equations are a system of first order differential equations for a pair \((A, \Phi)\) in the configuration space \( \mathcal{C}(X, s) = \mathcal{A}(L) \times \Gamma(S^+) \), where \( A \) is a connection on the characteristic line bundle of \( s \) and \( \Phi \) is a plus-spinor. The spaces \( \mathcal{A}(L) \) and \( \Gamma(S^+) \) are completed with respect to appropriate Sobolev norms, so that we will be working in the context of Banach spaces; see [M]. The Seiberg-Witten equations read

\[
\begin{align*}
E_A^+ &= (\Phi \Phi^*)_0, \\
\partial_A \Phi &= 0,
\end{align*}
\]

where \((\Phi \Phi^*)_0 \in \Gamma(\text{End}_0(\mathcal{S}^+)), \) defined by

\[
(\Phi \Phi^*)_0(w) = \langle \Phi, w \rangle \Phi - \frac{1}{2} |\Phi|^2 w \quad \forall \, w \in \Gamma(\mathcal{S}^+)
\]

is the trace-less part of \( \Phi \Phi^* \in \Gamma(\text{End}(\mathcal{S}^+)) \) and the two sides of the curvature equation in SW\(^4\) are identified via the bundle isomorphism

\[
\Lambda^{2+} T^* X \otimes \mathbb{C} \rightarrow \text{End}_0(\mathcal{S}^+), \quad \sum_{i<j} c_{ij} e_i^* \wedge e_j^* \rightarrow \sum_{i<j} c_{ij} \rho(e_i) \rho(e_j). \tag{2.4}
\]

We will occasionally denote the self-dual 2-form representing the quadratic term \((\Phi \Phi^*)_0\) by \( q(\Phi) \).
For an imaginary-valued self-dual 2-form $\eta \in \Omega^{2,+}(X, i\mathbb{R})$, the perturbed Seiberg-Witten monopole equations are

$$F_A^+ - \eta = (\Phi\Phi^*)_0, \quad \varphi_A \Phi = 0. \quad (\text{SW}^1_{\eta})$$

We call a solution to $\text{SW}^1_{\eta}$ an $\eta$-monopole. The set of $\eta$-monopoles is invariant under the action of $\mathcal{G} = \text{Maps}(X, S^1)$, which sends $(A, \Phi)$ to $(u^*A, u^{-1}\Phi)$. Here, $\mathcal{G}$ is of course completed with an appropriate Sobolev norm consistent with the completions of $\mathcal{A}(L)$ and $\Gamma(S^+)$. Let

$$\Omega^{2,+}(X, i\mathbb{R})_0 = \{\eta \in \Omega^{2,+}(X, i\mathbb{R}) : \exists A \in \mathcal{A}(L) \text{ s.t. } F_A^+ - \eta = 0\}.$$ 

This is an affine subspace of codimension $b^+$ in $\Omega^{2,+}(X, i\mathbb{R})$. An $\eta$-monopole for which the spinor $\Phi$ is identically zero is called reducible, in which case $\eta \in \Omega^{2,+}(X, i\mathbb{R})_0$, and irreducible otherwise. The Seiberg-Witten moduli space is the quotient

$$\mathcal{M}(X, \mathfrak{s}) = \mathcal{M}_{\eta, \mathfrak{s}}(X, \mathfrak{s}) := \{(A, \Phi) \in \mathcal{A}(L) \times \Gamma(S^+) \text{ satisfying SW}^1_{\eta}\}/\mathcal{G}.$$ 

If $b^+ > 0$, we can avoid reducible solutions of $\text{SW}^1_{\eta}$ with a generic perturbation $\eta$ and obtain a smooth closed orientable moduli space of real dimension

$$d = \frac{c_1(L)^2 - 2\chi(X) - 3\sigma(X)}{4}. \quad (2.5)$$

The deformation-obstruction complex of $\mathcal{M}(X, \mathfrak{s})$ at any $\eta$-monopole $(A, \Phi)$ is given by

$$0 \longrightarrow \Omega^0(X; i\mathbb{R}) \overset{D^0}{\rightarrow} \Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^+) \overset{D^1}{\rightarrow} \Omega^{2,+}(X; i\mathbb{R}) \oplus \Gamma(S^-) \longrightarrow 0. \quad (\mathcal{E}_X(A, \Phi))$$

The Banach spaces in $\mathcal{E}_X(A, \Phi)$ are tangent spaces to $\mathcal{G}, \mathcal{A}(L) \times \Gamma(S^+)$ and $\Omega^{2,+}(\Sigma; i\mathbb{R}) \times \Gamma(S^-)$ with appropriate completions, respectively, and the maps

$$D^0(f) = (2df, -f\Phi), \quad D^1(a, \phi) = \left(d^+a - D\phi q(\phi), \frac{1}{2}a \cdot \Phi + \varphi_A \phi\right)$$

are, respectively, the linearizations of the gauge group action and the map

$$\text{SW}^1_{\eta}(A, \Phi) = \left(F_A^+ - \eta - \Phi\Phi^*, \varphi_A \Phi\right).$$

We have used small letters $a$ and $\phi$ to distinguish tangent space variables at $(A, \Phi)$ from the point itself; we may occasionally adopt this convention later on.

The complex $\mathcal{E}_X(A, \Phi)$ is an elliptic complex and its index is $-d$. In fact, $H^0(\mathcal{E}_X(A, \Phi)) = \ker(D^0)$ is zero exactly when $\Phi \neq 0$ at least at some point, which is the definition of $(A, \Phi)$ being irreducible. The first cohomology $H^1(\mathcal{E}_X(A, \Phi)) = \ker(D^1)/\text{Im}(D^0)$ is the Zariski tangent space of the moduli space $\mathcal{M}(X, \mathfrak{s})$ and $H^2(\mathcal{E}_X(A, \Phi)) = \text{coker}(D^1)$ is its obstruction space, which is zero when $(A, \Phi)$ is a regular point of the moduli space.

The Seiberg-Witten invariants $\text{SW}(X, \mathfrak{s})$ are defined as the integral of a canonical cohomology class (a suitable power of the 2-form $c$, as in (1.5)) over $\mathcal{M}(X, \mathfrak{s})$, if this moduli space is even-dimensional and non-empty, and zero otherwise. The invariants are well-defined (independent of the choice of the metric $g$ and the generic perturbation term $\eta$) when $b^+ > 1$. If $b^+ = 1$, the formal dimension $d$ of the SW moduli space in (2.5) is even exactly when $b_1$ is even, in which case the SW invariants $\text{SW}^\pm(X, \mathfrak{s})$ depend on the chamber where the metric $g$ resides and satisfy a wall-crossing formula

$$\text{SW}^+(X, \mathfrak{s}) - \text{SW}^-(X, \mathfrak{s}) = -(-1)^{d/2}.$$
If \((X, \omega_X, J)\) is Kähler, using (2.3), the SW\(_4\) equations for the pair \((A, \Phi)\) take the form

\[
2(2F_{AE} - \eta)^{0,2} = \nabla^0 \Phi_2, \quad 4i(F_{A_{can}} + 2F_{AE}^+ - \eta)^{1,1} = (|\Phi_2|^2 - |\Phi_0|^2)\omega_X, \quad \bar{\partial}_{AE} \Phi_0 + \bar{\partial}^*_{AE} \Phi_2 = 0,
\]

where

\[
A = A_{can} \otimes A_E^2, \quad \Phi = (\Phi_0, \Phi_2) \in \Omega^{0,0}(X, E) \times \Omega^{0,2}(X, E).
\]

If \(\eta \in \Omega^{1,1} \cap \Omega^{2,+}\), then either \(\Phi_0 = 0\) or \(\Phi_2 = 0\). The latter will happen if

\[
c_1(E) \cdot \omega_X < \frac{1}{2} c_1(K_X) \cdot \omega_X,
\]

or \(\eta\) has a large (positive) multiple of \(-i\omega_X\); see [Sal, Sec. 12.2]. Recall that \(c_1(K_X) \cdot \omega_X \geq 0\) for every Kähler surface with \(b^+ > 1\). By adapting the arguments in [M, Sec. 7.2], we obtain a holomorphic description of the moduli space of \(\eta\)-monopoles in terms of holomorphic structures on \(E\) (or, equivalently, on the characteristic line bundle \(L\) of the spin\(^c\) structure) and non-zero holomorphic sections of \(E\) or \(K_X \otimes E^*\) (up to constant scalar multiples), depending on the case.

If \(\Phi_2 = 0\), the unperturbed SW equations reduce to the vortex equations

\[
F_{AE}^{0,2} = 0, \quad 4i(F_{A_{can}} + 2F_{AE})^{1,1} = -|\Phi_0|^2\omega_X, \quad \bar{\partial}_{AE} \Phi_0 = 0.
\]

Similar results hold in the symplectic case. In the almost-complex case, the SW\(_4\) equations for the pair \((A_{can} \otimes A_E^2, \Phi)\) take the form

\[
(F_{A_{can}} + 2F_{AE})^+ - \eta = q(\Phi), \quad \bar{\partial}_{AE} \Phi_0 + \bar{\partial}^*_{AE} \Phi_2 = 0.
\]

### 2.3 SW equations in dimension 2

In this and the following sections, suppose \((\Sigma, j, \omega)\) is a closed Riemann surface of genus \(g\), equipped with a complex structure \(j\) and a Kähler form \(\omega\). As a Kähler manifold of complex dimension one (real dimension two), \(\Sigma\) carries a canonical spin\(^c\) structure \((U_{can}, \rho_{can})\), where \(U_{can}\) is a hermitian vector bundle of rank 2, and any other spin\(^c\) structure \((U, \rho)\) on \(\Sigma\) is obtained by twisting this canonical spin\(^c\) structure by a hermitian line bundle \(E\). The characteristic line bundle is \(L = \det(U)\) and the spinor bundle \(U = U^+ \oplus U^-\) splits as a direct sum of two complex hermitian line bundles. We have

\[
U^+_{can} = C_\Sigma, \quad U^-_{can} = K^*_\Sigma, \quad \text{and} \quad L_{can} = K^*_\Sigma,
\]

where \(K = \Lambda^{1,0} T^* \Sigma\) is the canonical line bundle of \(\Sigma\) and \(C_\Sigma\) is the trivial line bundle, and

\[
U^+ = E, \quad U^- = K^*_\Sigma \otimes E, \quad \text{and} \quad L = K^*_\Sigma \otimes E^2.
\]

When there is a chance of confusion, we may add a subscript and, for example, denote the spinor bundles \(U^\pm\) or the line bundle \(L\) by \(U^\pm_E\) or \(L_E\), respectively.

**Remark 2.1.** It is also possible to define a canonical spin structure on \(\Sigma\) by simply lifting its principal \(SO(2)\)-tangent-bundle to a \(Spin(2)\)-bundle as a non-trivial double cover. One can then tensor with \(\mathbb{C}\) to obtain a spin\(^c\) structure on \(\Sigma\). This is the approach taken in [N1, p. 93]. The spinor bundles obtained in this case are \(U^- = K^{-1/2}_\Sigma\) and \(U^+ = K^{1/2}_\Sigma\) and the characteristic line bundle is trivial. This complexification of the canonical spin structure is obviously different from our canonical spin\(^c\) structure above, which is induced by the (almost-) complex structure on \(\Sigma\), but it can easily be offset by an additional twist by the line bundle \(K^{-1/2}_\Sigma = \sqrt{K^*_\Sigma}\). We have chosen here to work with the canonical spin\(^c\) structure, rather than the canonical spin structure, because of its compatibility with the overall framework of complex structures. □
Now, consider the set of pairs \((A, \Psi) \in \mathcal{A}(L) \times \Gamma(U)\) satisfying the equations

\[ F_A = \frac{|\Psi_+|^2 - |\Psi_-|^2}{2} \imath \omega, \quad \partial_A \Psi = 0, \quad (\text{SW}^2) \]

where \(\Psi = (\Psi_+, \Psi_-)\) is a section of \(U = U^+ \oplus U^-\). The set of solutions to these equations is invariant under the action of the gauge group \(G = \text{Maps}(\Sigma, S^1)\) on \(\mathcal{A}(L) \times \Gamma(U)\). As before, we call a solution reducible if the spinor \(\Psi\) is identically zero.

If \(A_{\text{can}}\) is the holomorphic hermitian connection on \(L_{\text{can}} = K^*_\Sigma\), then any connection on \(L = K^*_\Sigma \otimes E^2\) is of the form \(A = A_{\text{can}} \otimes A_E\), where \(A_E \in \mathcal{A}(E)\), and the corresponding Dirac operator \(\bar{\partial}_A\) on sections of \(U = U^+ \oplus U^-\) takes the form

\[ \frac{1}{\sqrt{2}} \bar{\partial}_A = \begin{bmatrix} 0 & \bar{\partial}_{A_E}^* \\ \bar{\partial}_{A_E} & 0 \end{bmatrix}. \]

The Dirac equation then splits into two equations \(\bar{\partial}_{A_E} \Psi_+ = 0\) and \(\bar{\partial}_{A_E}^* \Psi_- = 0\), and the curvature equation turns into

\[ F_{A_{\text{can}}} + 2F_{A_E} = \frac{|\Psi_+|^2 - |\Psi_-|^2}{2} \imath \omega. \]

Since \(\bar{\partial}_{A_E} \bar{\partial}_{A_E} = 0\) and \(\bar{\partial}_{A_E} \Psi_+ = 0\), we conclude that \(A_E\) induces a holomorphic structure on \(E\) for which \(\Psi_+\) is a holomorphic section. Similarly, using the involution on the spin\(^c\) structures induced by complex conjugation, which sends the characteristic line bundle to its inverse, we can see that the equation \(\bar{\partial}_{A_E}^* \Psi_- = 0\) implies that \(\bar{\Psi}_-\) is a holomorphic section of \(K_\Sigma \otimes E^*\), the Serre dual of \(E\). Since line bundles of negative degree can not have non-zero holomorphic sections, we conclude that one of \(\Psi_+\) or \(\Psi_-\) is identically zero, unless \(0 \leq d = \text{deg}(E) \leq 2g - 2\).

We will also be interested in another variant of the Seiberg-Witten equations

\[ F_A = \frac{|\Psi_+|^2 - |\Psi_-|^2}{2} \imath \omega, \quad \partial_A \Psi = 0, \quad \text{and} \quad \Psi_+ \bar{\Psi}_- = \imath \nu, \quad (\text{SW}^2_{\nu}) \]

where \(\nu\) is a fixed holomorphic 1-form. Notice that \(\Psi_+ \bar{\Psi}_-\), as a section of \(U^+ \otimes (U^-)^* = K_\Sigma\), identifies with a holomorphic 1-form on \(\Sigma\). This variant of the SW equations is closely related to the perturbed SW equations in higher dimensions; see Lemma 2.4.

### 2.4 SW equations on a circle bundle over a Riemann surface

Similarly to 2-manifolds, for a closed 3-manifold \(Y\), a spin\(^c\) structure \(s = (\mathcal{S}, \rho)\) consists of a rank 2 hermitian bundle \(\mathcal{S}\) and a Clifford multiplication map \(\rho\): \(TY \rightarrow \text{End}(\mathcal{S})\) satisfying (2.1), which result from a principal Spin\(^c(3)\)-lifting of the principal frame bundle of \(TY\). Note that, unlike dimensions 2 and 4, the spinor bundle \(\mathcal{S}\) does not decompose into plus- and minus-spinor bundles. We are interested in the case where \(Y\) is a circle bundle over a Riemann surface \(\Sigma\). In this subsection, to simplify notations, we will temporarily use the shorthand \(\pi\) to denote the bundle projection

\[ \pi = \pi_{Y, \Sigma}: Y \rightarrow \Sigma. \]

Viewing \(Y\) as a principal \(U(1)\)-bundle, a choice of a principal \(U(1)\)-connection gives rise to a decomposition

\[ TY \cong T^{\text{ver}}Y \oplus T^{\text{hor}}Y, \quad (2.6) \]
where the vertical summand is tangent to the $S^1$ fibers $\ker(d\pi)$ and the horizontal summand $T^\text{hor}Y \cong \pi^* T\Sigma$ is a lifting of the tangent space of $\Sigma$ which is equivariant under the $U(1)$ action. We will denote this connection by $i\alpha$, where $\alpha$ is a $U(1)$-invariant real 1-form on $Y$ whose restriction to each fiber is $d\theta$.

Under such a decomposition, $Y$ admits a canonical spin$^c$ structure $(S_{\text{can}}, \rho_{\text{can}})$, where
\[
S_{\text{can}} = \mathbb{C}_Y \oplus \pi^*(K^*_\Sigma)
\]
is the pullback of the canonical spinor bundle $U_{\text{can}} = \mathbb{C}\oplus K^*_\Sigma$ on $\Sigma$, the Clifford map $\rho_{\text{can}}$ on $T^\text{ver}Y$ sends the generator $\partial\theta$ to
\[
\begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix},
\]
and $\rho_{\text{can}}$ on $T^\text{hor}Y$ is the pullback of the canonical $\rho_{\Sigma,\text{can}}$ as in (2.2). The isomorphism class of this spin$^c$ structure is independent of the choice of the principal $U(1)$-connection and lifts the canonical Spin$^c(2)$ structure on $\Sigma$ to the canonical Spin$^c(3)$ structure on $Y$. Every other spin$^c$ structure on $Y$ is obtained by tensoring with a complex line bundle on $Y$.

**Remark 2.2.** By the Gysin long exact sequence
\[
H^2(\Sigma, \mathbb{Z}) \longrightarrow H^2(Y, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathbb{Z}) \longrightarrow 0,
\]
we see that, unless $\Sigma = S^2$, there are complex line bundles on $Y$ that are not the pullback of a complex line bundle on $\Sigma$. In fact, the Gysin exact sequence implies that if the Euler number $\ell$ of the circle-bundle $Y \longrightarrow \Sigma$ is non-zero, then
\[
H^2(Y, \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus (\mathbb{Z}/\ell\mathbb{Z}),
\]
where the torsion part is generated by the pullback $\pi^*\omega$ of the volume form on $\Sigma$. Therefore, the SW theory on $Y$ involves a larger class of spin$^c$ structures than those on $\Sigma$. In the setup that we will be developing for relative SW theory in Section 3.1, we will be working with complex line bundles that are defined over the entire $X$. Therefore we will be dealing only with complex line bundles on $Y$ that are pullbacks of those on $\Sigma$. Unless the circle bundle $Y \longrightarrow \Sigma$ is trivial, these line bundles on $Y$ will always be torsion. Moreover, it follows from [N1, p. 94, Fact 1] that spin$^c$ structures on $Y$ that are not of pullback type result in trivial relative invariants, as we will see in Section 3.1.

**Remark 2.3.** If the degree $\ell = \deg(Y)$ of the $U(1)$-bundle $Y \longrightarrow \Sigma$ is not zero and $E$ and $E'$ are two complex line bundles on $\Sigma$, then
\[
\pi^*(E) \cong \pi^*(E') \quad \text{if and only if} \quad \deg(E) \equiv \deg(E') \mod \ell.
\]
Unless $\deg(E) = \deg(E')$, such an isomorphism does not extend to the disk bundle $D$ over $\Sigma$ which has $Y$ as its boundary, where $D$ is defined using the same $U(1)$-cocycles of $Y \longrightarrow \Sigma$, only with circle fibers replaced by unit disks.

The general setup of Seiberg-Witten equations in dimension 3 is in most respects analogous to those in dimensions 2 and 4. For a closed, oriented, riemannian 3-manifold $Y$, equipped with a spin$^c$ structure $s = (S, \rho)$, consider the following (perturbed) Seiberg-Witten equations for a pair
(B, Ψ), consisting of a connection on the characteristic line bundle \( L = \det(S) \) and a section of the spinor bundle \( S \)

\[
F_B - \eta = (\Psi \Psi^*)_0, \quad \mathcal{D}_B \Psi = 0, \quad (\text{SW}_3^B)
\]

where \( \eta \in \Omega^2(Y, i\mathbb{R}) \) is a closed 2-form and the two sides of the curvature equation are identified via the isomorphism

\[
\Lambda^2 T^*Y \otimes \mathbb{C} \xrightarrow{\cong} \text{End}_0(S), \quad \sum_{i \prec j} c_{ij} e_i^* \wedge e_j^* \rightarrow \sum_{i \prec j} c_{ij} \rho(e_i) \rho(e_j),
\]

which is the adjoint of the Clifford multiplication.

As in Sections 2.2 and 2.3, we consider the following deformation-obstruction complex, as described in [MST], to understand the infinitesimal structure of the \( \text{SW}_3^Y \) moduli space at each solution \((B, \Psi)\),

\[
0 \rightarrow \Omega^0_\nu(Y; i\mathbb{R}) \xrightarrow{D^0} \Omega^1_\nu(Y; i\mathbb{R}) \oplus \Gamma_1(S) \xrightarrow{D^1} \Omega^1_\nu(Y; i\mathbb{R}) \oplus \Gamma_0(S)/D^0(\Omega^0_\nu(Y; i\mathbb{R})) \rightarrow 0, \quad (\mathcal{E}_Y(B, \Psi))
\]

where a subscript \( k \) indicates that the space has been completed using the \( L^2_k \) norm, \( D^0 \) is the linearization of the gauge group action, and \( D^1 \) is induced by the linearization of

\[
\text{SW}_3^Y(B, \Psi) = \left( \ast (F_B - \eta - q(\Psi)), \mathcal{D}_B \Psi \right).
\]

The cohomologies of this complex, as before, determine irreducibility, the Zariski tangent space and the obstruction space at a solution \((B, \Psi)\), respectively. Moreover, as is the case for any elliptic complex on an odd-dimensional manifold, the index of \( \mathcal{E}_Y(B, \Psi) \) is zero.

In what follows, we are going to describe how solutions of the \( \text{SW}_3^Y \) equations on a circle bundle \( \pi: Y \rightarrow \Sigma \) are related to those of the \( \text{SW}_3^Y \) equations on \( \Sigma \), if the spin\(^c\) structure \( s_\nu \) on \( Y \) is of pullback type and \( \eta \) is obtained from \( \nu \) in a particular way. Such a spin\(^c\) structure is specified by tensoring the canonical spin\(^c\) structure on \( Y \) with a line bundle \( \pi^*E \), where \( E \) is a hermitian line bundle of degree \( d \) on \( \Sigma \). However, as we have seen in remark 2.3, any other line bundle on \( \Sigma \) whose degree is \( d \) modulo \( \ell = \text{deg}(Y) \) will also pull back to the same \( \pi^*E \) on \( Y \). Therefore, given a fixed spin\(^c\) structure \( s_\nu \) on \( Y \), the choice of \( E \) on \( \Sigma \) is not unique. We will indicate this ambiguity by a subscript \([E]\) in our notation; for example, \( s_\nu = s_{\nu,[E]} \) is meant to emphasize that the spin\(^c\) structure depends on the equivalence class \([E]\) under the relation given in remark 2.3.

With this notation in place, fix the spin\(^c\) structures \( s_{\Sigma,E} \) on \( \Sigma \) and \( s_{Y,[E]} \) on \( Y \) as above. These are obtained by twisting the canonical spin\(^c\) structures on \( \Sigma \) and \( Y \) by \( E \) and \( \pi^*E \), respectively. Then the spinor bundle \( U = U_E \) on \( \Sigma \) pulls back to the spinor bundle \( S = S_{[E]} \) on \( Y \) and we have

\[
U = U^+ \oplus U^- = E \oplus (K^*E \otimes E) \quad \text{and} \quad L = L_{\Sigma,E} = K^*E \otimes E^{\otimes 2}
\]

\[
S = \pi^*U^+ \oplus \pi^*U^- = \pi^*E \oplus \pi^*(K^*E \otimes E) \quad \text{and} \quad L_{Y,[E]} = \pi^*L,
\]

where \( L \) and \( \pi^*L \) are the characteristic line bundles of the spin\(^c\) structures on \( \Sigma \) and \( Y \), respectively.

We will follow the construction laid out in [N1] to explicitly write out the Dirac operator and the Seiberg-Witten equations on \( Y \). Start from a constant-curvature conformal metric on \( \Sigma \) and use the splitting (2.6) and the connection 1-form \( \alpha \) on \( Y \) as the first element of an orthonormal frame to equip \( Y \) with a riemannian metric \( g \). Now fix a \( g \)-compatible connection on the tangent bundle of \( Y \) and couple that with any unitary connection \( B = B_{\text{can}} \otimes B_{\Sigma}^2 \) on the characteristic line bundle \( \pi^*L \) on \( Y \) to define the Dirac operator \( \mathcal{D}_B \) on \( Y \). According to the decomposition of \( S \) in (2.8) above,
any spinor on $Y$ splits as a section $\Psi = (\Psi_+, \Psi_-)$ of $\pi^*U^+ \oplus \pi^*U^-$ --- $Y$. With respect to this splitting, the Dirac operator $\bar{\partial}_B : \Gamma(S) \to \Gamma(S)$ on $Y$ takes the form

$$\bar{\partial}_B = \left[ \begin{array}{cc} \bar{\nabla}_{\partial_\theta} & \sqrt{2}\bar{\partial}_{B_E} \\ \sqrt{2}\bar{\partial}_{B_E} & \nabla_{\partial_\theta} \end{array} \right] + \lambda 1,$$  \hspace{1cm} (2.9)

where $\lambda$ is some constant depending on the choice of the $g$-compatible connection on $Y$, $\nabla_{\partial_\theta}$ is the covariant derivative in the fiber direction, and $\bar{\partial}_{B_E}$ is defined with respect to the lift of the complex structure on $\Sigma$ to the horizontal tangent space, so that it is zero in the fiber direction [N1, (2.7)]. Moreover, if we choose our $g$-compatible connection on $Y$ to be the adiabatic connection, that is, one which acts trivially in the fiber direction $T_{ver}Y$, then $\lambda = 0$. Using this (non-Levi-Civita) connection on $Y$, there is an explicit description of the solutions of the (perturbed) SW$^3_{\eta}$ equations, which we state below. This is essentially based on [MST, Sec. 5, pp. 719–726], [N1, Sec. 3.2, pp. 92–97], [N2, Sec. 3, pp. 356–372], and [MOY, Sec. 5, pp. 701–722] and continues to hold for perturbation terms of the form

$$\eta = \imath (\ast n),$$  \hspace{1cm} (2.10)

where $n = \pi^*(\nu + \tau)$ is the pullback of the real 1-form associated to a holomorphic 1-form $\nu$.

**Lemma 2.4.** With notation as above, and using the adiabatic connection ($\lambda=0$), for every solution $(B, \Psi)$ of SW$^3_{\eta}$ we have

$$\Psi_+ \bar{\Psi}_- = \imath \pi^* \nu,$$

both $\Psi_+$ or $\Psi_-$ are covariantly constant in the vertical direction, $\bar{\partial}_{B_E}^* (\Psi_-) = 0$ and $\bar{\partial}_{B_E} (\Psi_+) = 0$. The curvature equation of SW$^3_{\eta}$ simplifies to

$$F_B = \frac{|\Psi_+|^2 - |\Psi_-|^2}{2} \imath \pi^* \omega,$$  \hspace{1cm} (2.11)

where $\omega$ is a Kähler form on $\Sigma$ with constant curvature.

**Proof.** In the following, to keep the notation simple, we write $\bar{\partial}_B$ instead of $\bar{\partial}_{B_E}$. By (2.9), for $\lambda=0$, the Dirac equation in SW$^3_{\eta}$ becomes

$$-\imath \nabla_{\partial_\theta} \Psi_+ + \sqrt{2}\bar{\partial}_B \Psi_- = 0$$

$$\sqrt{2}\bar{\partial}_B \Psi_+ + \imath \nabla_{\partial_\theta} \Psi_- = 0$$  \hspace{1cm} (2.12)

Following the local calculations in [MST, pp. 722–723] and [N1, Appendix D], fix a local orthonormal coframe $\alpha_1$ and $\alpha_2$ on $\Sigma$ such that $d\alpha_1 = \kappa \alpha_1 \wedge \alpha_2$ and $d\alpha_2 = 0$, where $\kappa$ is a non-negative constant. Note that $\omega = \alpha_1 \wedge \alpha_2$. By pulling back to $Y$, we can think of $\alpha_1$ and $\alpha_2$ as 1-forms on $Y$. Let

$$\alpha_{1,0} = \alpha_1 + \imath \alpha_2 \quad \text{and} \quad \alpha_{0,1} = \alpha_1 - \imath \alpha_2$$  \hspace{1cm} (2.13)

denote the corresponding $(1, 0)$ and $(0, 1)$ forms, respectively. The triple $(\alpha_1, \alpha_2, \alpha)$ is an orthonormal coframe for $Y$ with dual frame $(\zeta_1, \zeta_2, \partial_\theta)$. In this notation we have

$$\bar{\partial}_B = \alpha_{0,1} \otimes \frac{1}{2}(\nabla_{\zeta_1} + \imath \nabla_{\zeta_2}).$$  \hspace{1cm} (2.14)
The curvature equation of $SW_3^3$ reads:

\[
-i(F_{a_1,a_2} - \eta \alpha_{a_1,a_2}) = \frac{1}{2}(|\Psi_+|^2 - |\Psi_-|^2)
\]

where

\[
F_B = F_{a_1,a_2} \alpha_1 \wedge \alpha_2 + F_{a_1,a} \alpha_1 \wedge \alpha + F_{a_2,a} \alpha_2 \wedge \alpha,
\]

\[
\eta = \eta_{a_1,a_2} \alpha_1 \wedge \alpha_2 + \eta_{a_1,a} \alpha_1 \wedge \alpha + \eta_{a_2,a} \alpha_2 \wedge \alpha.
\]

Applying $\bar{\partial}_B$ to the first equation in (2.12) we get

\[
-i\bar{\partial}_B \nabla_{\partial_b} \Psi_+ + \sqrt{2}\bar{\partial}_B \bar{\partial}_B \Psi_- = 0.
\]

For $\eta$ as in (2.10), by the curvature equation

\[
\bar{\partial}_B \nabla_{\partial_b} - \nabla_{\partial_b} \bar{\partial}_B = (F_{a_1,a} + iF_{a_2,a})\alpha_{0,1},
\]

and using the second equation in (2.15), we can rewrite (2.16) as

\[
-i\nabla_{\partial_b} \bar{\partial}_B \Psi_+ + 2(\bar{\Psi}_+ \Psi_- + i\pi^* \overline{\Psi})\Psi_+ + \sqrt{2}\bar{\partial}_B \bar{\partial}_B \Psi_- = 0.
\]

Applying the second equation in (2.12) to the first term we get

\[
-\frac{1}{\sqrt{2}} \nabla_{\partial_b} \nabla_{\partial_b} \Psi_- + 2(\bar{\Psi}_+ \Psi_- + i\pi^* \overline{\Psi})\Psi_+ + \sqrt{2}\bar{\partial}_B \bar{\partial}_B \Psi_- = 0.
\]

Since rotation in the circle direction acts by isometries on $TY$, we have $\nabla_{\partial_b}^* = -\nabla_{\partial_b}$. Taking the inner product with $\Psi_-$ then gives

\[
\frac{1}{\sqrt{2}} \|\nabla_{\partial_b} \Psi_-\|^2 + 2\|\bar{\Psi}_+ \Psi_- + i\pi^* \overline{\Psi}\|^2 + \sqrt{2}\|\bar{\partial}_B \Psi_-\|^2 = 0.
\]

Here, to get the middle term, we have used the fact that

\[
\langle (i\pi^* \overline{\Psi}, \bar{\Psi}_+ \Psi_- + i\pi^* \overline{\Psi}) \rangle = \int_Y i\pi^* \overline{\Psi} \wedge F_B = 0,
\]

because $\overline{\Psi}$ is a closed 1-form on $\Sigma$ and $L$ is pulled back from $\Sigma$; see [MST, p. 726]. Lemma 2.4 now follows from (2.17) and the first equation in (2.15).

Lemma 2.4 implies that an irreducible solution $(B, \Psi)$ of $SW_3^3$ is indeed pulled back from a pair of counterparts $(A, \Psi)$ on $\Sigma$ satisfying $SW_2^3$, with some abuse of notation on the spinor component. Such a solution $(A, \Psi)$ on $\Sigma$ defines a holomorphic structure on $\mathbb{U}^+ = E$ via $\bar{\partial}_A^E$, for which $\Psi_+$ is a holomorphic section. Analogous statements hold for the Serre dual bundle $K_\Sigma \otimes E^*$ and $\overline{\Psi}_-$. Therefore, to understand irreducible $\eta$-monopoles on $Y$, we need to understand irreducible solutions of $SW_2^3$ on $\Sigma$.

A word of caution here is in order though. In the discussion below, for the sake of specificity, we are going to fix the spin^c structure $\alpha_{3,3}$ on $\Sigma$. As a result, the line bundle $E = E_\Sigma$ and its degree $d = \deg(E)$, as well as the characteristic line bundle $L = L_\Sigma = K_\Sigma^* \otimes E^2$ of $\deg(L) = 2 - 2g + 2d$, are
also fixed. However, if \( \text{deg}(Y) \neq 0 \), recall that the spin\(^c\) structure \( s_{Y, [E]} \) depends on the equivalence class of \( E \); therefore, to find all the solutions of \( \text{SW}^3 \) on \( Y \) corresponding to the spin\(^c\) structure \( s_{Y, [E]} \), we need to account for all the solutions to the \( \text{SW}^2 \) equations on \( \Sigma \) corresponding to each line bundle in the equivalence class \([E]\). We denote the irreducible component of \( \mathcal{M}(Y, s_{Y, [E]}) \) corresponding to the line bundle \( E \) by \( \mathcal{M}(Y, s_{Y, E}) \).

Another point to note in the following discussion is that the re is a natural involution in Seiberg-Witten theory which sends the characteristic line bundle \( L \) to its inverse. Therefore, it will be enough to focus on the case where \( \text{deg}(L) \leq 0 \), that is, when \( d \leq g - 1 \).

**Unperturbed case.** This is the case of \( \text{SW}^3 \) equations \((\eta = 0)\). Let us first consider the case of \( \text{deg}(L) < 0 \). If the equations have an irreducible solution, then it will be the pullback of a solution \((A, \Psi)\) of \( \text{SW}^2 \) on \( \Sigma \). If \( \Psi_+ = 0 \), then integrating the curvature equation of \( \text{SW}^2 \)

\[
F_A = \frac{|\Psi_+|^2 - |\Psi_-|^2}{2} \omega
\]

over \( \Sigma \) yields \( \text{deg}(L) = 2(1 - g + d) \geq 0 \). Thus, when \( \text{deg}(L) < 0 \), we conclude that \( \Psi_+ \) must be non-zero. If the spinor \( \Psi_+ \) is non-zero, it will be unique up to a constant multiple if its zero set \( \text{Div}(\Psi_+) \), which is an effective divisor on \( \Sigma \), is prescribed. Moreover, the norm of this constant multiple can be determined from the curvature equation above by integrating over \( \Sigma \). Therefore, by fixing \( \text{Div}(\Psi_+) \), the plus spinor \( \Psi_+ \), and hence the pair \((A, \Psi)\) on \( \Sigma \), will be unique up to gauge equivalence. We conclude that the solution \([(B, \Psi)]\) on \( Y \) will be determined by the zero set of \( \Psi_+ \) on \( \Sigma \), which is a point in \( \text{Div}_d(\Sigma) \), the space of effective divisors of degree \( d \) on \( \Sigma \). In other words, each monopole in \( \mathcal{M}(Y, s_{Y, E}) \) can be identified with a point in the symmetric \( d \)-fold product \( \text{Sym}^d(\Sigma) \), where its plus spinor “lands”. The process above defines a smooth landing map

\[
b : \mathcal{M}(Y, s_{Y, E}) \longrightarrow \text{Div}_d(\Sigma) \cong \text{Sym}^d(\Sigma),
\]

which turns out to be a diffeomorphism. The inverse \( b^{-1} \) is constructed in the following way. Let \( D \) be an arbitrary effective divisor in \( \text{Div}_d(\Sigma) \). Corresponding to this divisor, we will have a unique holomorphic line bundle \([D]\) on \( \Sigma \) of degree \( d \), which is topologically the same as \( E \), together with a holomorphic section vanishing at \( D \), which is unique up to scalar multiplication. Using the induced holomorphic connection on \( E \) and the holomorphic section, we obtain a solution \((A, \Psi)\) to \( \text{SW}^2 \), unique up to gauge equivalence. We can then pull \((A, \Psi)\) back to a solution for \( \text{SW}^3 \) on \( Y \).

Similarly, in the case of \( \text{deg}(L) > 0 \), we can conclude in the same vein that \( \Psi_- \) must be non-zero. In this case, instead of \( \Psi_+ \), we will look at the holomorphic section \( \overline{\Psi}_- \) of the Serre dual line bundle \( K_\Sigma \otimes E^* \) and

\[
d = \text{deg}(\mathbb{U}^+) = \text{deg}(E) = (g - 1) + \frac{1}{2} \text{deg}(L)
\]

will be replaced by

\[
\text{deg}(\mathbb{U}^-) = \text{deg}(K_\Sigma \otimes E^*) = 2g - 2 - d = (g - 1) - \frac{1}{2} \text{deg}(L)
\]

as the degree of the effective divisor \( \text{Div}(\overline{\Psi}_-) \). This justifies our definition of \( d(\mathfrak{s}) \) as in (1.1) in the introduction. The landing map \( b : \mathcal{M}(Y, s_{Y, E}) \rightarrow \text{Sym}^{d(\mathfrak{s})}(\Sigma) \) will similarly be defined as in (2.20). Meanwhile, note that the existence of a non-zero holomorphic section for \( \mathbb{U}^+ \) (respectively \((\mathbb{U}^-)^*\)) implies that \( d \geq 0 \) (respectively \( d \leq 2g - 2 \)). Therefore, we have just shown that there are no
irreducible solutions to the $\text{SW}^3$ equations unless $0 \leq d = \deg(E) \leq 2g - 2$. Moreover, when $d = 0$ (respectively $d = 2g - 2$), the line bundle $\mathbb{U}^+$ (respectively $\mathbb{U}^-$) is trivial, so such a non-zero section will be constant. Therefore, if $d = 0, 2g - 2$, the irreducible solution to $\text{SW}^3$ will be unique up to gauge equivalence, unless $g = 1$, in which case $\deg(L) = 0$ and there will be no irreducible solution at all, as will be discussed below. For consistency of notation throughout this paper, we take $\text{Div}_d(\Sigma) \cong \text{Sym}^d(\Sigma)$ to consist of a single point when $d = 0$.

Finally, we consider the case of $\deg(L) = 0$, that is, when $L = C_\Sigma$, $E = \sqrt{K_\Sigma}$ and $d = g - 1$. If $\text{SW}^3$ has an irreducible solution, by integrating the curvature equation (2.19), we conclude that the spinors $\Psi_+$ and $\Psi_-$ have the same $L^2$-norms on $\Sigma$. Since the product of these spinors is zero, they both have to be identically zero. This means that all the solutions to $\text{SW}^3$, corresponding to $E = \sqrt{K_\Sigma}$, are reducible and correspond to flat connections on the torsion line bundle $\pi^*L$ on $Y$. The space of such flat connections, up to gauge equivalence, is homeomorphic to the Jacobian torus $\mathcal{H}^1(\Sigma, S^1) = \mathcal{H}^1(\Sigma, \mathbb{R})/\mathcal{H}^1(\Sigma, \mathbb{Z}) = \mathbb{T}^{2g}$ when $\deg(Y) \neq 0$. Moreover, if $\deg(L)$ is not a multiple of $\deg(Y)$, the reducible component will be non-degenerate in the sense defined in [MOY, Def. 5.15, p. 717]. If $\deg(L)$ is a multiple of $\deg(Y)$, this space can be identified with the theta divisor $W_{g-1}$ inside the Jacobian $J_{g-1}(\Sigma)$ (see Fact 3 in [N1, p. 95], [GH]). If $\deg(Y) = 0$, that is, when $Y = S^1 \times \Sigma$ is a trivial circle bundle over $\Sigma$, $\pi^*L$ will be topologically trivial and its space of flat connections can be identified with the torus $\mathbb{T}^{2g+1}$ as the space of $S^1$-representations of the fundamental group of $Y$ (see [N2, p. 369]).

Now, let $\mathcal{J}$ denote the component of reducible solutions to $\text{SW}^3$, that is, $\mathcal{J}$ is the torus $\mathbb{T}^{2g}$ when $\deg(Y) \neq 0$, the torus $\mathbb{T}^{2g+1}$ when $\deg(Y) = 0$ and $E = \sqrt{K_\Sigma}$, and empty otherwise. The preceding discussion shows that we can write the moduli space of unperturbed $\text{SW}^3$ equations as

$$\mathcal{M}(Y, s_{Y,[E]}) = \mathcal{J} \cup \bigcup_{E \in [E]} \mathcal{M}(Y, s_{Y,E}),$$

where $\mathcal{M}(Y, s_{Y,E}) \cong \text{Sym}^{d(s_{Y,E})}(\Sigma)$ is the irreducible component$^4$ for which $(\Psi_+, \overline{\Psi}_-)$ are holomorphic sections of $(E, K_\Sigma \otimes E^*)$ and $\mathcal{J}$ is the reducible component as above.

When $\deg(Y) = 0$, the pullback spin$^c$ structure on $Y$ is uniquely identified by $E$ (i.e., $[E] = \{E\}$) and the relation above can be simplified. If $E = \sqrt{K_\Sigma}$, we simply get an irreducible moduli space

$$\mathcal{M}(Y, s_{Y,E}) \cong \text{Sym}^{d(s_{Y,E})}(\Sigma)$$

and if $E = \sqrt{K_\Sigma}$, the moduli space is just the reducible component $\mathbb{T}^{2g+1}$. Of course, a perturbation of the $\text{SW}^3$ equations will help us get rid of these reducible solutions, as explained below.

**Perturbed case.** This is the case of $\text{SW}^3_\eta$ equations with $\eta = i(n) \neq 0$. As before, we conclude from Lemma 2.4 that both spinors $\Psi_+$ and $\Psi_-$ are covariantly constant in the direction of the circle, $(B, \Psi)$ is a pullback of $(A, \Psi)$ from $\Sigma$, and $\partial_A^B (\Psi_-) = \partial_A^B (\Psi_+) = 0$. This time, however, none of $\Psi_\pm$ can be identically zero, so our solution is not reducible. Moreover, since $\Psi_+ \overline{\Psi}_- = iv$, knowing one of the spinors $\Psi_+$ and $\Psi_-$ will uniquely identify the other. Again, consider the zero set $\text{Div}(\Psi_+)$ of the holomorphic spinor $\Psi_+$ on $\Sigma$. This divisor will determine $\Psi_+$ up to a constant multiple. The norm of this constant multiple can be determined from (2.19) by integrating over $\Sigma$, as in [MST, Cor. 5.5, p. 726]. Therefore, by fixing $\text{Div}(\Psi_+)$, the plus spinor $\Psi_+$, and hence the pair $(A, \Psi)$ on $\Sigma$, will be uniquely determined up to gauge equivalence. As in the unperturbed case

$^4$We are using a different notation here compared to the notation in the introduction. There, we denoted $\mathcal{M}(Y, s_{Y,E})$ by $\mathcal{M}(Y, s_{Y,m})$, where $m = \deg(L)/2$. 

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above, for degree reasons, the equations have no solution unless $0 \leq \deg(E) \leq 2g - 2$. We also have

$$\text{Div}(\Psi_+) + \text{Div}(\Psi_-) = \text{Div}(\nu),$$

so, unlike the unperturbed case above, $\text{Div}(\Psi_+)$ is constrained to a sum of $d$ points in the zero set $\text{Div}(\nu)$. Then $\text{Div}(\Psi_-)$ is the sum of the remaining $2g - 2 - d$ points in the complement of $\text{Div}(\Psi_+)$ in $\text{Div}(\nu)$. Therefore, $M_\eta(Y,s_{Y,E})$ is a finite set of regular points and the reducible component $J$ is empty.

2.5 SW equations on a four-manifold with a cylindrical end

Let $M$ be a smooth connected oriented riemannian 4-manifold with a cylindrical end\(^5\), that is, one which is orientation-preserving isometric to the cylinder $\mathbb{R}_+ \times Y$ outside a compact submanifold $M_0$ with non-empty interior, where $Y$ is a closed oriented riemannian 3-manifold. Moreover, the boundary $\partial M_0$ has a collar neighborhood diffeomorphic to $(-1,0] \times Y$, where $\partial M_0$ corresponds to $\{0\} \times Y$. Use the above isometry and diffeomorphism to identify the cylindrical end (also called the neck) and the collar with $\mathbb{R}_+ \times Y$ and $(-1,0] \times Y$, respectively, and let $t$ denote the longitudinal “time coordinate” on the neck $\mathbb{R}_+ \times Y$. We can view $t$ as the projection map onto the first factor $t: \mathbb{R}_+ \times Y \rightarrow \mathbb{R}_+$, so that $t(t,y) = t$ for $(t,y) \in \mathbb{R}_+ \times Y$, and smoothly extend it to the entire manifold $t: M \rightarrow [-1,\infty)$ using the collar diffeomorphism such that $t^{-1}(\mathbb{R}_+)$ is the neck, $t^{-1}((-1,0])$ is the collar, and $t^{-1}(-1)$ is the rest of the manifold. We will have $M_0 = t^{-1}([-1,0])$ and $\partial M_0 = t^{-1}(0)$.

We are going to consider the Seiberg-Witten equations on this cylindrical-end manifold $M$, where we perturb the equations using an adapted perturbation term

$$\tilde{\eta} = \eta_0 + \beta(t)\eta^+, \tag{2.21}$$

in which

- $\eta_0$ is a purely imaginary self-dual 2-form on $M$ with compact support,
- $\eta^+$ denotes the self-dual part of the pullback of a closed 2-form $\eta \in \Omega^2(Y,i\mathbb{R})$ under the natural projection $\mathbb{R}_+ \times Y \rightarrow Y$,
- $\beta(t)$ is a smooth cutoff function on $M$ which is identically 1 on the neck, non-negative on the collar, and zero otherwise. It is defined as the composition of a smooth increasing function $\beta: [-1,\infty) \rightarrow [0,1]$ satisfying $\beta(-1) = 0$ and $\beta(0) = 1$ with the map $t: M \rightarrow [-1,\infty)$.

An adapted perturbation will enable us to control how the Seiberg-Witten equations are perturbed on the compact piece and the neck, and to study the equations on the cylinder $\mathbb{R}_+ \times Y$ (with the product metric) as a stand-alone problem. Fixing a spin$^c$ structure on the cylinder will induce a spin$^c$ structure $s_Y$ on $Y$ by restriction, where the spinor bundle $S_Y$ is identified with both $S^+$ and $S^-$, which are themselves identified via Clifford multiplication by $dt$. Conversely, a spin$^c$ structure on $Y$ induces one on the cylinder in an obvious way via pullback; we will thus use the notation $s_Y$ interchangeably for both. A connection on the cylinder is called temporal if it acts as $\partial_t$ in the time direction; this can always be arranged by a suitable gauge transformation.

Therefore, any solution $(A,\Phi)$ of $\text{SW}_4^{\eta^+}$ on the cylinder $I \times Y$, where $I \subset \mathbb{R}$ is an interval and $A$ is in temporal gauge, can be identified with a path $(B(t),\Psi(t))$ in the configuration space on $Y$.

\(^5\)One may allow more than one cylindrical end. The results will readily generalize.
Moreover, these paths are the (upward) gradient flow lines for the Chern-Simons-Dirac function on this configuration space, as defined below. Fix a background connection $B_0$ on $L_Y$ and, for
\[(B, \Psi) \in C(Y, s_Y) = A(L_Y) \times \Gamma(S_Y),\]
define
\[\text{CSD}_\eta(B, \Psi) = \int_Y F_{B_0} \wedge b - \int_Y \eta \wedge b + \frac{1}{2} \int_Y b \wedge db + \int_Y \langle \Psi, \tilde{\phi}_B \Psi \rangle d\text{vol},\]
where $b = B - B_0$ is a 1-form on $Y$. A different choice of $B_0$ would change $\text{CSD}_\eta$ only by a constant and therefore the gradient on $C(Y, s_Y)$ is independent of this choice. Now, the gradient flow lines of \(\text{CSD}_\eta\) identify with solutions to $\text{SW}^3_\eta$, with temporal gauge and the stationary points correspond to solutions to $\text{SW}^3_\eta$. Note that changing the orientation on $Y$ reverses the direction of the flow lines and that $\text{CSD}_\eta$ is locally constant on $\mathcal{M}_\eta(Y, s_Y)$ as the zero locus of the gradient, hence constant on each connected component.

The function $\text{CSD}_\eta$ is not necessarily invariant under the gauge transformation $u: Y \to S^1$ though and will change by
\[\text{CSD}_\eta(u \cdot (B, \Psi)) - \text{CSD}_\eta(B, \Psi) = 4\pi^2 \left( [u] \cup \left( c_1(s_Y) - \frac{1}{2\pi} [\eta] \right) \right) [Y],\]
where $[u]$ denotes the homotopy class of $u: Y \to S^1$ as a cohomology class in $H^1(Y, \mathbb{Z})$. Thus, $\text{CSD}_\eta$ descends to a real-valued function on the quotient if $c_1(s_Y) = \frac{1}{2\pi} [\eta]$, and to a circle-valued function otherwise. In the former case, the value of $\text{CSD}_\eta$ is increasing along the flow lines in the quotient.

Now let $(A, \Phi)$ be a solution to $\text{SW}^4_\eta$ on $I \times Y$, where $A$ is in temporal gauge, and identify that with $(B(t), \Psi(t))$ in the configuration space on $Y$, as above. The topological energy of $(A, \Phi)$ is defined by any of the following equivalent formulas:
\[E(A, \Phi) = \int_I ||\dot{B}||^2 + ||\dot{\Psi}||^2 = \int_I ||\nabla \text{CSD}_\eta(B(t), \Psi(t))||^2 = \text{CSD}_\eta(B(t_1), \Psi(t_1)) - \text{CSD}_\eta(B(t_0), \Psi(t_0)),\]
where in the last equation the interval $I$ is assumed to be $I = [t_0, t_1]$. When working with four-manifolds $M$ with cylindrical ends, we will only consider solutions with finite energy on the ends and $\mathcal{M}_\eta(M, s_M)$ denotes the moduli space of finite-energy monopoles on $M$ modulo gauge transformations. We will complete the relevant underlying spaces using appropriate weighted Sobolev norms, as in [S] or [N].

If $[A, \Phi] = [B(t), \Psi(t)]$ is a finite-energy $\eta^+$-monopole on $\mathbb{R}_+ \times Y$, one can show that the limit $[B, \Psi] = \lim_{t \to \infty} [B(t), \Psi(t)]$ exists and is an $\eta$-monopole on $Y$, using the same methods established in [MMR] and [T]. Moreover, if the limit $[B, \Psi]$ is irreducible, the convergence to the asymptotic limit is exponential. This defines a limiting map
\[\partial: \mathcal{M}_{\hat{\eta}}(M, s_M) \to \mathcal{M}_\eta(Y, s_Y).\]  \hfill (2.22)

**Remark 2.5.** If $b^+(M) > 0$, a generic choice of the compact perturbation $\eta_0$ will ensure that all $\hat{\eta}$-monopoles on $M$ are irreducible and strongly regular, in the sense defined in [N, Def. 4.3.20, p. 383]. This implies that, for a generic compact perturbation $\eta_0$, the moduli space $\mathcal{M}_{\hat{\eta}}(M, s_M)$ is a smooth manifold [N, Prop. 4.4.1, p. 405] and the limiting map above is a submersion [N, Rmk. 4.4.2].
A finite-energy monopole on $\mathbb{R} \times Y$ is called a tunneling. To simplify notation in the following discussion, let us drop the reference to spin$^c$ structures and assume $\eta = 0$. The real line $\mathbb{R}$ acts on $\mathcal{M}(\mathbb{R} \times Y)$ by translations and we denote the quotient, that is, the moduli space of unparametrized tunnelings, by $\tilde{\mathcal{M}}(\mathbb{R} \times Y)$. Corresponding to the two ends of the cylinder $\mathbb{R} \times Y$, there are two induced limiting maps $\partial_{\pm}: \tilde{\mathcal{M}}(\mathbb{R} \times Y) \to \mathcal{M}(Y)$ as $t \to \pm \infty$, since the limiting maps are $\mathbb{R}$-invariant. If $C_{\pm}$ are two components of $\mathcal{M}(Y)$, we will denote the moduli space of such tunnelings from $C_{-}$ to $C_{+}$ by $\tilde{\mathcal{M}}(C_{-}, C_{+}) \subset \tilde{\mathcal{M}}(\mathbb{R} \times Y)$. If the spin$^c$ structure $s_Y$ is torsion, so that CSD is a real-valued function on the moduli space, then $\tilde{\mathcal{M}}(C_{-}, C_{+})$ will be empty if CSD($C_{-}$) $> \text{CSD}(C_{+})$, but there may be non-trivial tunnelings when CSD($C_{-}$) $< \text{CSD}(C_{+})$. We will use these tunnelings to construct a suitable compactification for $\mathcal{M}_{\partial_0}(M, s_M)$ in Section 3.6.

The moduli space $\mathcal{M}_{\partial_0}(M, s_M)$ in general is not compact and has components of different dimensions. At any point of the moduli space, the expected real dimension can be calculated using the Atiyah-Patodi-Singer index theorem [APS, Thm. 3.10] and is given by

$$d = \frac{c_1(A)^2 - 2\chi(M) - 3\sigma(M)}{4} + \xi,$$  \hspace{1cm} (2.23)

where $\xi$ stands for $\frac{1}{2}(\eta(0) + h)$, i.e., half of the sum of the APS $\eta$-invariant and the dimension of the kernel of $\text{SW}_{\eta}^3$ operator on $Y$ at the given point.\(^6\) The term $c_1(A)^2$ in (2.23) denotes the integral of the Pontrjagin form as in [APS, Thm. 4.2] and is given by

$$c_1(A)^2 := -\frac{1}{4\pi^2} \int_M F_A \wedge F_A.$$  \hspace{1cm} (2.24)

The signature $\sigma(M)$ is defined as in [AS, Prop. 7.1] and equals $b^+_M - b^-_M$, the difference of dimensions of maximal positive- and negative-definite subspaces in $H^2_c(M, \mathbb{R})$ with respect to the Poincaré intersection pairing. The dimension (2.23) is not a topological quantity and changes as we move from one component of $\mathcal{M}_{\partial_0}(M, s_M)$ to another. We will fix this problem on $M = X - \Sigma$ by choosing a spin$^c$ structure for $TX(-\log \Sigma)$ over the entire $X$.

### 2.6 Gluing monopoles

In this section, which is based on [S1], [S2], [S], we state a gluing theorem for SW monopoles on a closed 4-manifold $X$ if it can be “decomposed” as a union of two cylindrical-end manifolds $X_{\pm}$, in the sense described below. Proof of the sum formula in Theorem C follows from this gluing theorem and a convergence result. Such a gluing statement is also needed to prove that the compactified moduli space $\overline{\mathcal{M}}_{\partial_0}(X - \Sigma, s)$ is a $C^0$-manifold.

Let $Y$ be a smooth closed connected oriented riemannian 3-manifold. Suppose $X_+$ and $X_-$ are two smooth connected oriented riemannian 4-manifolds with cylindrical ends $\mathbb{R}_+ \times Y$ and $\mathbb{R}_- \times Y$, respectively. Note that we can also think of $X_-$ as a cylindrical-end manifold with cylindrical end $\mathbb{R}_+ \times Y$ at the expense of changing the orientation on $Y$. The collars of $X_{\pm}$, as in Section 2.5, are identified with $(-1, 0] \times Y$ and $[0, 1) \times Y$, respectively. For each $T \gg 0$, let $X_{T, \pm}$ denote the closed riemannian 4-manifold obtained by identifying

$$X_{T, +} = X_+ - (T, \infty) \times Y \quad \text{and} \quad X_{T, -} = X_- - (-\infty, -T) \times Y$$

---

\(^6\)We have opted to work with the notation convention $\xi$ of [N1] instead to avoid confusion regarding the terms $h$ and $\eta$ we have used elsewhere in the paper.
As a special case, when the perturbation form

\[ \eta \]

further that the perturbation terms are chosen generically so that

\[ \mathcal{M} \]

which is an embedding onto an open subset of

\[ \mathcal{M} \]

smooth manifold. Then, for sufficiently large

\[ T \]

every point and the limiting maps are submersions, therefore transversal, so the fiber product is a

\[ \mathcal{M} \]

Suppose

\[ X \]

Theorem 2.6.

The following is an immediate consequence of the gluing theorem in [S1, S].

X

on the cylindrical-end pieces

\[ X \]

components of

\[ X \]

restricted to

\[ X \]

We can meanwhile use these forms to define adapted perturbations

\[ \hat{\eta} \]

in which

\[ \hat{\eta}_r = \eta_{o,-} + \beta_r \cdot \eta^+ + \eta_{o,+}, \]  \hspace{1cm} (2.26)

in which

• \( \eta_{o,+} \) is a purely imaginary self-dual 2-form on \( X_+ - (-1, \infty) \times Y \) with compact support,

• \( \eta_{o,-} \) is a purely imaginary self-dual 2-form on \( X_- - (-\infty, 1) \times Y \) with compact support,

• \( \eta^+ \) denotes the self-dual part of the pullback of a closed 2-form \( \eta \in \Omega^2(Y, i\mathbb{R}) \) under the natural projection \( \mathbb{R} \times Y \rightarrow Y \). We can make sense of \( \beta_r \cdot \eta^+ \) as a 2-form on \( X_r \) in an obvious way.

We can meanwhile use these forms to define adapted perturbations \( \hat{\eta}_\pm \) on \( X_\pm \) as in (2.21). When restricted to \( X_{r,+} \) and \( X_{r,-} \) as subsets of \( X_r \), the perturbations \( \hat{\eta}_\pm \) coincide with \( \hat{\eta}_r \), respectively. The following is an immediate consequence of the gluing theorem in [S1, S].

**Theorem 2.6.** Suppose \( Y \) is a circle bundle over a Riemann surface, as in Section 2.4. Let

\[ \mathcal{M}(Y, s_Y) \]

denote an irreducible component of \( \mathcal{M}(Y, s_Y) \) and \( \mathcal{M}(Y, s_\pm) \) be the union of those components of \( \mathcal{M}(Y, s_\pm) \) that map into \( \mathcal{M}(Y, s_Y) \) under the limiting map (2.22). Assume moreover that the perturbation terms are chosen generically so that \( \mathcal{M}(Y, s_\pm) \) are regular at every point and the limiting maps are submersions, therefore transversal, so the fiber product is a smooth manifold. Then, for sufficiently large \( T \), there is a natural gluing map

\[ \mathcal{M}(Y, s_+) \times \mathcal{M}(Y, s_-) \rightarrow \mathcal{M}(Y, s), \]

which is an embedding onto an open subset of \( \mathcal{M}(Y, s) \).

As a special case, when the perturbation form \( \eta \) on \( Y \) is zero, we will have compact perturbations on the cylindrical-end pieces \( X_\pm \) and the gluing map above is written as a map

\[ \mathcal{M}(Y, s_+) \times \mathcal{M}(Y, s_-) \rightarrow \mathcal{M}(Y, s), \]

which is an embedding onto an open subset of \( \mathcal{M}(Y, s) \), with \( \eta_{o,r} = \eta_{o,-} + \eta_{o,+} \).
3 Relative Seiberg-Witten moduli spaces

In Section 3.1, we define the notion of logarithmic tangent bundles and spin$^c$ structures. Then in Section 3.2, using the moduli spaces defined in Section 2, we elaborate on the definition and properties of the relative moduli spaces $\mathcal{M}_0(X - \Sigma, s)$ and $\mathcal{M}_q(X - \Sigma, s)$. In Section 3.3, we use [N1] to find topological formulae for the dimensions of the moduli spaces $\mathcal{M}_0(X - \Sigma, s)$ and $\mathcal{M}_q(X - \Sigma, s)$, as well as $\mathcal{M}_0(X - \Sigma, s_{X - \Sigma}, J)$. This involves a highly non-trivial calculation of the $\xi$-invariant in formula (2.23), which is carried out in Section 3.4. In Section 3.5, we give a holomorphic description of the tunneling moduli spaces. This plays a key role in proving that certain relative moduli spaces are indeed compact. In Section 3.6, we describe the compactification of $\mathcal{M}_0(X - \Sigma, s)$. In Section 3.7, we show how the results from the earlier sections come together to prove Theorems A–C.

3.1 Logarithmic tangent bundles and relative spin$^c$ structures

Suppose $X$ is a closed oriented 4-manifold and $\Sigma$ is a closed oriented surface of genus $g$ in $X$. Equip $\Sigma$ with a Kähler structure $(j, \omega)$ compatible with the orientation. The normal bundle $\pi : N = \frac{TX|_\Sigma}{T\Sigma} \to \Sigma$ is an oriented vector bundle of real rank 2. Choose a complex structure $i$ on $N$ compatible with the orientations and let $(\bar{\nabla}, \bar{\partial})$ be a hermitian metric and a compatible complex linear connection. By abuse of notation, we also let $\bar{\nabla} v$ denote $|v|^2$ for every $v \in N$. The connection $\nabla$ gives rise to a decomposition of $TN$ into horizontal and vertical directions $TN = T^{\text{hor}}N \oplus T^{\text{ver}}N \cong \pi^*T\Sigma \oplus \pi^*N$. (3.1)

The tuple $(i, \varrho, \nabla)$ also gives rise to a 1-form $\alpha$ on $N - \Sigma$ such that $\alpha(\partial_\theta) = 1$, $\alpha$ is zero on the horizontal sub-space, and $d\alpha = \pi^*F$ with $[F] = -2\pi_1(N)$. The 2-form $\omega_N = \pi^*\omega_\Sigma + \frac{1}{2}d(\varrho \alpha) = \pi^*\omega_\Sigma + \frac{1}{2}\varrho \pi^*F + \frac{1}{2}d\varrho \wedge \alpha$ (3.2)
on $N$ is closed and non-degenerate in a sufficiently small neighborhood $D$ of $\Sigma$ in $N$. Via decomposition (3.1), define $J_N$ to be the almost complex structure on $N$ given by $\pi^*j$ on $\pi^*T\Sigma$ and $\pi^*i$ on $\pi^*N$. The tuple $(D, \omega_D := \omega_N|_D, J_D := J_N|_D)$ is Kähler. In other words, the $\bar{\partial}$ operator associated to $\nabla$,

$$\bar{\partial}_\nabla \zeta = \nabla^{0,1} \zeta = \frac{1}{2}(\nabla \zeta + i\nabla \zeta \circ j), \quad \forall \, \zeta \in \Gamma(\Sigma, N),$$

is integrable and defines a holomorphic structure on $N$. Locally around every point $p \in \Sigma$, if $z$ is a local holomorphic coordinate with $z(p) = 0$ and $\zeta$ is a holomorphic section with $\zeta(p) \neq 0$, then $N \ni (z, w\zeta(z)) \rightarrow (z, w) \in \mathbb{C}^2$ (3.4)
defines the holomorphic structure associated to the almost complex structure $J_N$.

Let $\Upsilon : D \to X$, $\Upsilon|_\Sigma = \text{id}$, $d\Upsilon|_\Sigma = \text{id}$,
be an identification of a neighborhood $D$ of $\Sigma$ in $X$ with a neighborhood of that in $N$. Via this identification, the previous paragraph shows that a neighborhood of $\Sigma$ in $X$ can be equipped with a Kähler structure of a standard form. If $(X, \omega_X)$ is a symplectic manifold and $\Sigma$ is a symplectic submanifold with $\omega = \omega_X|\Sigma$, by Symplectic Neighborhood Theorem [MS, Thm. 3.4.10], we can choose $\Upsilon$ to also satisfy

$$\Upsilon^* \omega_X = \omega_{D}.$$  

Then, via the identification $\Upsilon$, we can extend the complex structure $J_D$ to a compatible almost complex structure on the entire $X$ compatible with $\omega_X$.

If $X$ is a holomorphic surface and $\Sigma$ is a smooth holomorphic curve in $X$, the log tangent sheaf $TX(−\log \Sigma)$ is a sub-sheaf of the holomorphic tangent sheaf $TX$ that is equal to $TX$ away from $\Sigma$ and is generated by

$$\partial_z^{\log} := z\partial_z \quad \text{and} \quad \partial_w$$  

in any local holomorphic chart $(z, w): V \to \mathbb{C}^2$ around a point of $\Sigma$ with $\Sigma \cap V \equiv (z = 0)$. Since $TX(−\log \Sigma)$ is locally free, it is the sheaf of holomorphic sections of a holomorphic vector bundle $TX(−\log \Sigma)$. The inclusion $TX(−\log \Sigma) \subset TX$ gives rise to a holomorphic homomorphism

$$\iota: TX(−\log \Sigma) \to TX$$  

which is an isomorphism away from $\Sigma$.

Since $TX$ and $TX(−\log \Sigma)$ only differ in a neighborhood of $\Sigma$, the notion of logarithmic tangent bundle can be defined for every pair $(X, \Sigma)$ of an oriented 4-manifold and an oriented 2-dimensional submanifold. For such $(X, \Sigma)$ and any choice of $(\Upsilon, j, i, \vartheta, \nabla)$ as above, define

$$TX(−\log \Sigma) = (\Upsilon^{-1 \ast}(\pi^*TS^\ast \oplus \mathbb{C}_D) \sqcup TX|_{X−\Sigma})/\sim,$$

$$\Upsilon^{-1 \ast}(\pi^*TS^\ast \oplus \mathbb{C}_D) \ni (\Upsilon(v), u \oplus c) \sim (\Upsilon(v), d_{\vartheta} \Upsilon(u + cv)) \in TX|_{X−\Sigma},$$  

where $\mathbb{C}_D$ is the trivial complex line bundle on $D$ and in the last equation we think of $u + cv$ as a tangent vector in $T_h\mathcal{N}$ via the right-hand side of identification (3.1). Instead of the global definition in (3.7), one may as well define $TX(−\log \Sigma)$ via local holomorphic charts (3.4) in a neighborhood of $\Sigma$ as in (3.5) and identify that with $TX|_{X−\Sigma}$ on the overlap as in the second line of (3.7).

The riemannian metric $\omega(\cdot, j \cdot)$ on $T\Sigma$ and the standard metric on $\mathbb{C}_N$ give us a riemannian metric on $\pi^*TS^\ast \oplus \mathbb{C}_D$. Via (3.7), that can be extended to a metric $h_X$ on the entire $TX(−\log \Sigma)$. The restriction of this metric to $X−\Sigma$ via the identification

$$TX(−\log \Sigma)|_{X−\Sigma} \cong T(X−\Sigma)$$  

defines a complete riemannian metric $h_{X−\Sigma}$ on $X−\Sigma$, giving it the structure of a riemannian 4-manifold with a cylindrical end $[0, \infty) \times Y$, where $Y$ is the unit circle bundle in $\mathcal{N}$ and $[0, \infty)$ is parametrized by $t = −\frac{1}{2} \log(\rho)$. Note that in this paper, $Y$ gets the orientation induced by $X−\Sigma$, that is, the one induced by its cylindrical end $[0, \infty) \times Y$ using the “outward-normal-first” convention. This is opposite to the orientation $Y$ gets as $\partial D$; therefore, in calculations related to the degree of the circle bundle $Y \to \Sigma$, we have

$$\deg(Y) = −\Sigma \cdot \Sigma.$$  

The dual space

$$T^X(\log \Sigma) := TX(−\log \Sigma)^*$$
is the logarithmic cotangent space. Along $\Sigma$, the space $\Omega^1_{\log}(X)$ of smooth sections of $T^*X(\log \Sigma)$ is locally generated by $dz/z$ and $du$ and their complex conjugates.

If $(X, \omega_X)$ is a symplectic manifold, $\Upsilon$ is a symplectic identification, and $J$ is an almost complex structure on $X$ such that $\Upsilon^*J = J_D$, the complex structures $\pi^*j \oplus i_{\mathbb{C}}$ on $\pi^*T\Sigma \oplus \mathbb{C}_D$ and $J$ on $TX|_{X-\Sigma}$ match on the overlap region and define a complex structure on $TX(-\log \Sigma)$. Then, we have

$$c(TX(-\log \Sigma)) = c(TX)/(1 + PD(\Sigma)).$$

This identity follows from an isomorphism of vector bundles

$$TX(-\log \Sigma) \oplus \mathcal{O}(\Sigma) \cong TX \oplus \mathbb{C}_X,$$

where $\mathcal{O}(\Sigma)$ is the complex line bundle on $X$ with $c_1(\mathcal{O}(\Sigma)) = PD(\Sigma)$ and $\mathbb{C}_X$ is the trivial complex line bundle.

In the symplectic case, the hermitian metric on $T\Sigma$ and the standard hermitian metric on $\mathbb{C}_D$ give us a hermitian metric on $\pi^*T\Sigma \oplus \mathbb{C}_D$. Via (3.7), the latter can be extended to a hermitian metric $h_X$ on the entire $TX(-\log \Sigma)$. The restriction of this metric to $X-\Sigma$ defines a complete hermitian metric $h_{X-\Sigma}$ on $X-\Sigma$.

In order to define relative SW invariants, we use the moduli spaces $\mathcal{M}_{\theta}(M, s_M)$, or indeed their compactification, where $M = X-\Sigma$ is equipped with the cylindrical-end metric $h_{X-\Sigma}$. As mentioned in Section 2.5, the moduli space $\mathcal{M}_{\theta}(M, s_M)$ often has components of different dimensions. In order to choose a component of fixed dimension, we use a spin$^c$ structure $s_X$ on $X$ that restricts to $s_M$ on $M$. By [N1, p. 94, Fact 1], if $s_M$ does not admit such an extension over $D$, then $\mathcal{M}_{\theta}(M, s_M)$ is empty.

Let $s_X = (S_X, \rho_X)$ be a spin$^c$ structure on $TX(-\log \Sigma)$. We say $s_X$ is relatively canonical if $\Upsilon^*s_X$ is the canonical spin$^c$ structure of the complex vector bundle $\Upsilon^*TX(-\log \Sigma) \cong \pi^*T\Sigma \oplus \mathbb{C}_D$.

Therefore, if $s_X$ is relatively canonical, by (2.2),

$$\Upsilon^*S_X^+ = \Lambda^{0,0}(\pi^*T\Sigma \oplus \mathbb{C}_D) \oplus \Lambda^{0,2}(\pi^*T\Sigma \oplus \mathbb{C}_D) \cong \mathbb{C}_D \oplus \pi^*\Lambda^{0,1}\Sigma,$$

$$\Upsilon^*S_X^- = \Lambda^{0,1}(\pi^*T\Sigma \oplus \mathbb{C}_D) \cong \mathbb{C}_D \oplus \pi^*\Lambda^{0,1}\Sigma.$$ (3.10)

Observe that $\Upsilon^*S_X^\pm$ coincide with the pullback of the canonical spinor bundle $S_{\Sigma,\text{can}}$ on $\Sigma$ in Section 2.3 and that $d(s_X) = 0$ by (1.1). If $X$ is a symplectic manifold equipped with an $(\omega_X, \Upsilon)$-compatible almost complex structure $J$, then $TX(-\log \Sigma)$ has the structure of a complex vector bundle and the canonical spin$^c$ structure is globally well-defined.

### 3.2 The main component

Fix a relatively canonical spin$^c$ structure $s_{X,\text{can}} = (S_{X,\text{can}}, \rho_{X,\text{can}})$ on $TX(-\log \Sigma)$; this will be the globally canonical spin$^c$ structure if $X$ is a symplectic manifold. Every other spin$^c$ structure $s = s_{X,E} = (S_{X,E}, \rho_{X,E})$ can be obtained by tensoring with a hermitian line bundle $E$ on $X$. In the neighborhood $D$ of $\Sigma$, we may assume

$$E_D = \pi^*_D s E_{\Sigma}.$$
By (3.10), we have
\[ T^* S_{X,E}^+ \cong T^* S_{X,E}^- \cong \pi^*(E + (K^* \otimes E)) = \pi^*(S_{\Sigma, E}), \] (3.11)
that is, \( T^* S_{X,E}^* |_{D- \Sigma} \), where \( D - \Sigma \cong [0, \infty) \times Y \), coincide with the pullback of the spinor bundle on \( \Sigma \) corresponding to \( E_{\Sigma} \) in Sections 2.4 and 2.5. In this case, (1.1) gives us
\[
d(s_{X,E}) = \begin{cases} \deg(E_{\Sigma}) & \text{if } \deg(L_{\Sigma}) \leq 0, \\ \deg(K_{\Sigma} - E_{\Sigma}) & \text{if } \deg(L_{\Sigma}) \geq 0. \end{cases}
\]

Associated to \( s \), we are interested in two types of moduli spaces.

(1) Let \( M_{\eta_0}(X - \Sigma, s_{X - \Sigma}) \) denote the moduli space of monopoles on \( X - \Sigma \) with finite energy on the end with respect to the induced spin\(^c\) structure \( s|_{X-\Sigma} \) on \( X - \Sigma \), where \( \eta_0 \) is a perturbation with compact support, as in Section 2.5. Different spin\(^c\) structures \( s \) and \( s' \) on \( TX(- \log \Sigma) \) may have the same restriction on \( X - \Sigma \). Therefore, for such \( s \) and \( s' \), \( M_{\eta_0}(X - \Sigma, s_{X - \Sigma}) \) will be the same as \( M_{\eta_0}(X - \Sigma, s'_{X - \Sigma}) \). As we described in the introduction, by looking at the limits of the monopoles over the cylindrical end, we choose a component \( M_{\eta_0}(X - \Sigma, s) \) of \( M_{\eta_0}(X - \Sigma, s_{X - \Sigma}) \) corresponding to \( s \); here we assume that \( (\Sigma, \Sigma, \deg(L)) \neq (0, 0) \), otherwise there is no such component and we will have to use an adapted perturbation term instead. This alternative perturbation is described below.

(2) Consider an adapted perturbation
\[ \eta_\nu := \hat{\eta}(\eta_0, \eta) = \eta_0 + \beta(t)\eta^+ \]
as in (2.21), where
\[ \eta = i(\ast_y n), \quad n = \pi^*(\nu + \nu') \]
comes from a holomorphic 1-form \( \nu \), as in (2.10). In fact, on the cylindrical end we have
\[ \eta^+ = i(dt \wedge n + \ast_y n). \] (3.12)
Let \( M_{\eta_0}(X - \Sigma, s_{X - \Sigma}) \) denote the moduli space of monopoles on \( X - \Sigma \) with finite energy on the end with respect to the induced spin\(^c\) structure \( s|_{X-\Sigma} \) on \( X - \Sigma \). As before, by looking at the limits of the monopoles over the cylindrical end, we choose a component \( M_{\eta_0}(X - \Sigma, s) \) of \( M_{\eta_0}(X - \Sigma, s_{X - \Sigma}) \) corresponding to \( s \). In this case, all the monopoles are irreducible.

In both cases, there are limiting maps
\[ M_{\eta_0}(X - \Sigma, s) \to M(Y, s_{Y,E}), \quad M_{\eta_0}(X - \Sigma, s) \to M_{\eta}(Y, s_{Y,E}), \]
i.e., if \( (A, \Phi) \) is a finite-energy solution of \( SW_{\eta_0}^4 \) (resp. \( SW_{\eta_0}^2 \)), then, restricted to the cylindrical part \( D - \Sigma \cong [0, \infty) \times Y \), this solution identifies with the gradient flow line \( (B(t), \Psi(t)) \) of CSD (resp. \( CSD_{\eta} \)) such that the exponentially-converging limit
\[ \pi^*[A_\infty, \Psi_\infty] = [B, \Psi] = \lim_{t \to \infty} [B(t), \Psi(t)] \] (3.13)
is induced by the pullback of an irreducible solution \( (A_\infty, \Psi_\infty) \) of \( SW^2 \) (resp. \( SW^2_{\nu} \)) for \( s_{Y,E} \). Recall that an irreducible monopole on \( Y \) can be identified, up to gauge equivalence, with a divisor on \( \Sigma \). In the compactly supported case (1), every divisor in \( \text{Div}_{d(s)}(\Sigma) \) can arise as a limit, but in the adapted case (2), the limiting map takes values in \( S_{d(s)}(\Sigma) \).
Remark 3.1. Since the spin\(^c\) structure \(s\), and thus the characteristic line bundle \(L\), is defined over the entire \(X\), it is natural to ask whether

(Q1) there exists a pair \((A_X, \Phi_X)\), consisting of a connection on \(L\) and a section of \(S^+\), such that

\[
(A_X, \Phi_X)|_{X-\Sigma} = (A, \Phi) \quad \text{and} \quad (A_X|_{T\Sigma}, \Phi_X|_{\Sigma}) = (A_{\infty}, \Psi_{\infty}).
\]

(Q2) there exists a system of equations, defined over the entire \(X\), whose solutions are gauge equivalent to such pairs \((A_X, \Phi_X)\).

Note that we already know from (3.13) that \(\Psi_{\infty}\) and \(\Phi\) define a continuous section \(\Phi_X\) of \(S^+\). We will come back to these questions in Section 4.

In the following subsections, we summarize the relevant conclusions from the introduction and Sections 2.4 and 2.5 and extend them to prove Theorems A–C. For simplicity, we assume that \(\Sigma\) is connected. If \(\Sigma\) is not connected, some of the related numbers and structures should be treated component-wise.

3.3 Dimension

As seen in Section 2.5, and according to (2.23), if \(b_{X-\Sigma}^+ > 0\), for generic \(\eta_0\), \(M_{\eta_0}(X - \Sigma, s)\) and \(M_{\eta_0}(X - \Sigma, s)\) are smooth (but probably not closed) manifolds of real dimension

\[
\frac{c_1(A)^2 - 2\chi(X - \Sigma) - 3\sigma(X - \Sigma)}{4} + \xi. \tag{3.14}
\]

While \(\chi(X - \Sigma)\) and \(\sigma(X - \Sigma)\) are topological quantities, \(c_1(A)^2\) and \(\xi\) depend (at least) on the asymptotic behavior of the solution. In order to (topologically) fix the asymptotic behaviour, we have considered a spin\(^c\) structure on \(TX(-\log \Sigma)\), instead of \(T(X - \Sigma)\), resulting in a characteristic bundle \(L\) that is defined over the entire \(X\). Below, we show that \(c_1(A)^2\) is equal to the topological quantity \(c_1(L)^2\) and calculate \(\xi\) for each of \(M_{\eta_0}(X - \Sigma, s)\) and \(M_{\eta_0}(X - \Sigma, s)\).

Lemma 3.2. We have

\[
d = \dim_{\mathbb{R}} M_{\eta_0}(X - \Sigma, s) = \frac{c_1(L)^2 - 2\chi(X - \Sigma) - 3\sigma(X - \Sigma)}{4} + \frac{\Sigma \cdot \Sigma - 3\text{sign}(\Sigma \cdot \Sigma)}{4} + d(s)
\]

\[
= \frac{(c_1(L) + \Sigma)^2 - 2\chi(X) - 3\sigma(X)}{4}
\]

Proof. We use the formula (3.14), identify \(c_1(A)^2\) with the topological quantity \(c_1(L)^2\), and use the explicit computation of \(\xi\) in [N1, (3.27)].

Claim. The integral \(c_1(A)^2\) in (2.24) is equal to \(c_1(L)^2\).

Proof. Fix a \(U(1)\)-connection \(A_X\) on \(L\) whose restriction to \(T\Sigma\) is the limiting connection \(A_{\infty}\) defined in (3.13). Since \(L\) is the characteristic line bundle of the spin\(^c\) structure \(s\) defined on the entire \(X\), \(c_1(L)^2\) is a well-defined topological quantity that coincides with the integral

\[
\frac{-1}{4\pi^2} \int_X F_{A_X} \wedge F_{A_X}.
\]
We conclude that, assumption in the first line of the proof, imaginary-valued 1-forms on \( \Sigma \in \text{deg}(Y) \) in our notation, which is equal to \( \frac{1}{2} \text{deg}(L_{\Sigma}) \).

In conclusion, by \([N1, (3.27)]\), if \( \text{deg}(L_{\Sigma}) \leq 0 \), we have

\[
\xi = \left(-\frac{\varepsilon_{\Sigma,X}}{2} + \text{deg}(E_{\Sigma}) + (1 - g)\right) + \left(-\frac{1}{2}\right) + \frac{1}{4}(\Sigma \cdot \Sigma - \varepsilon_{\Sigma,X}) \\
= \text{deg}(E_{\Sigma}) + \frac{1}{4}(\Sigma \cdot \Sigma - 3\varepsilon_{\Sigma,X}).
\]

(3.16)

We conclude that,

\[
d = \frac{c_1(L)^2 - 2\chi(X) - 3\sigma(X)}{4} + (1 - g) + \frac{3}{4}\varepsilon_{\Sigma,X} - \frac{3}{4}\varepsilon_{\Sigma,X} + \frac{1}{4}\Sigma \cdot \Sigma + \text{deg}(E_{\Sigma})
\]

\[
= \frac{c_1(L) + \Sigma)^2 - 2\chi(X) - 3\sigma(X)}{4}.
\]

Similar calculations give us the dimension formula for the case \( \text{deg}(L_{\Sigma}) \geq 0 \).

For the component \( \mathcal{M}_{\nu}(X - \Sigma, J) \) ending at reducible solutions, \([N1, (3.29)]\) yields the following.

**Lemma 3.3.** If \((g - 1) \neq 0 \text{ modulo } \ell = -\Sigma \cdot \Sigma \) and \( \frac{1}{2}c_1(L_Y) \) is a torsion class \([k] \in H^2_{\text{tor}}(Y, \mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z}, \) with \( 0 < k < |\ell| \), we have

\[
d_{\mathcal{J}} = \dim_{\mathbb{R}} \mathcal{M}_{\nu}(X - \Sigma, J) = \frac{c_1(A)^2 - 2\chi(X - \Sigma) - 3\sigma(X - \Sigma)}{4} + (g - \frac{1}{2}) + \frac{1}{4}(\Sigma \cdot \Sigma - \varepsilon_{\Sigma,X}) + \frac{k^2}{\Sigma \cdot \Sigma} - k \cdot \varepsilon_{\Sigma,X}.
\]

(3.17)
If $\Sigma$ is not connected, each component contributes by the fact or in the second line. If $(g - 1) \equiv 0 \pmod{\ell}$, then $k = 0$ and the dimension formula will be slightly different. Note that, in the formula above, $A$ does not extend to a connection on $L$ over the entire $X$. If $(g - 1) \equiv 0 \pmod{\ell}$, then $k = 0$ and the dimension formula will be slightly different. Note that, in the formula above, $A$ does not extend to a connection on $L$ over the entire $X$. If

$$\frac{1}{2} \deg(L_{s_k}|_\Sigma) = k,$$

then [N1, (2.13)] and an integral formula similar\footnote{with $\lim_{t \to -\infty} a(t) = \frac{2i k}{\alpha}$ instead of 0.} to (3.15) implies

$$c_1(A)^2 - c_1(L_{s_k})^2 = -\frac{4k^2}{\Sigma \cdot \Sigma}.$$ 

We can therefore rewrite (3.17) as the topological formula

$$d(J) = \frac{c_1(L_{s_k})^2 - 2\chi(X - \Sigma) - 3\sigma(X - \Sigma)}{4} +$$

$$(g - \frac{1}{2}) + \frac{1}{4}(\Sigma \cdot \Sigma - \varepsilon_{\Sigma,X}) - k \cdot \varepsilon_{\Sigma,X}. \tag{3.18}$$

We will use this in Section 3.6 to explain why $d(J)$ does not contribute to the sum formula (1.10).

Finally, we prove the following result for the dimension of $M_{\eta}\nu(X - \Sigma, s)$.

**Lemma 3.4.** We have

$$\tilde{d} = \dim R M_{\eta}\nu(X - \Sigma, s) = \frac{c_1(L)^2 - 2\chi(X - \Sigma) - 3\sigma(X - \Sigma)}{4} + \frac{1}{4}(\Sigma \cdot \Sigma - 3\varepsilon_{\Sigma,X}) =$$

$$= \frac{(c_1(L) + \Sigma)^2 - 2\chi(X) - 3\sigma(X)}{4} - d(s).$$

The explicit computation of $\xi$ in [N1, (3.27)] does not apply to monopoles in $M_{\eta}\nu(X - \Sigma, s)$, because starting at [N1, Sec. 3.3], the author assumes $\psi_-$ is zero. In the following, we adapt the calculations in [N1] to monopoles in $M_{\eta}\nu(X - \Sigma, s)$ and prove the following proposition. With (3.19) in place of (3.16), the proof of Lemma 3.4 is similar to the proof of Lemma 3.2.

**Proposition 3.5.** The $\xi$-invariant of the monopoles in $M_{\eta}\nu(X - \Sigma, s)$ is equal to

$$\xi = \frac{1}{4}(\Sigma \cdot \Sigma - 3\varepsilon_{\Sigma,X}). \tag{3.19}$$

**3.4 Proof of Proposition 3.5**

In this subsection, in order to prove (3.19), we follow and modify the relevant details of [N1]. To keep the proof short, and to enable the reader to directly compare with [N1], we use the same setup and notation in [N1] with slight simplifications. In particular, our 3-manifold $Y$ is denoted by $N$ in this subsection.

The calculation of the $\xi$-invariant and thus the dimension formula [N1, (3.26)] involves contributions from several spectral flow calculations in [N1, Sec. 2.2-3.4]. Up to [N1, Sec. 3.3], the results apply to an arbitrary spinor $\phi = (\phi_-, \phi_+)$. Starting in [N1, Sec. 3.3], the author assumes $\phi_- = 0$; that will have major impacts on some of the calculations. Compared to [N1, (3.26)], we get a different value for $\text{SF}_+$ and the dimension of the asymptotic limit set is zero. In order to find $\text{SF}_+$ in our case, we similarly want to calculate the spectral flow of

$$\tilde{H}_t = \tilde{H}_0 + t P_\phi \quad t \in [0, 1], \tag{3.20}$$
but this time, $\phi = \phi_- \oplus \phi_+$ has two non-zero components in $\mathcal{K}^{-1/2} \otimes L$ and $\mathcal{K}^{1/2} \otimes L$ with $\bar{\phi}_- \phi_+ = \nu \in H^0(\mathcal{K})$. Here $\mathcal{K}$ and $L$ are pullback to $N$ of the canonical line bundle and an arbitrary complex line bundle on $\Sigma$, respectively. The operator $\tilde{H}_0$ is the same but $\mathcal{P}_\phi = \mathcal{P}_{\phi_+} + \mathcal{P}_{\phi_-}$ is different. As a result, the resonance matrix $\mathcal{R}_\phi$ used at the top of [N1, p. 97] is different, but it is defined on the same space $\text{Ker} \tilde{H}_0$. Also, in this case $\text{Ker} \tilde{H}_1 = 0$ since $(A, \phi)$ is a regular point. However, similarly to [N1, p. 97, STEP 1], there is no spectral flow along the path $(t \rightarrow \tilde{H}_t)_{t \in (0, 1]}$. The only contribution is from $t = 0$.

Before moving forward to find the generalization of [N1, Lemma 3.2] and calculate the spectral flow contribution $\text{SF}_+$ at $t = 0$, let us set up the notation, recall the definition of $\tilde{H}_0$, and find the generalization of $\mathcal{P}_\phi$. These operators are linear maps from

$$\Gamma(\mathcal{S}_L \oplus \mathcal{I}^1T^*N \oplus \Lambda^0T^*N)$$

(3.21)

to itself, where

$$\mathcal{S}_L = \mathcal{S}_{L} \oplus \mathcal{S}_L^+ = (\mathcal{K}^{-1/2} \otimes L) \oplus (\mathcal{K}^{1/2} \otimes L).$$

We have $\phi = \phi_- \oplus \phi_+ \in \mathcal{S}_L$ and an arbitrary element in (3.21) is written\(^8\) as

$$\Xi = (\psi_- \oplus \psi_+) \oplus \imath a \oplus \imath f.$$

Furthermore, we write

$$\imath a = \imath h\varphi + \frac{\omega - \bar{\omega}}{2}$$

where $\{\varphi, \varphi_1, \varphi_2\}$ is the local orthonormal coframe of $T^*N$ in [N1, Sec 2.1], and $\omega$ is a complex multiple of the $(1, 0)$ form $\frac{1}{\sqrt{2}}(\varphi_1 + \imath \varphi_2)$ as in [N1, Appendix D]. At the monopole $(A, \phi)$, the operators $\tilde{H}_0$ and $\mathcal{P}_\phi$ are given by

$$\tilde{H}_0 = \begin{bmatrix} D_A & 0 & 0 \\ 0 & -\ast d & d \\ 0 & d^\ast & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_\phi(\Xi) = \begin{bmatrix} c(\imath a)\phi - \imath f\phi \\ \hat{\phi}(\phi, \psi) \\ \imath \text{Im} \langle \phi, \psi \rangle \end{bmatrix}.$$  

Using the same calculations as in [N1, (D.1)-(D.3)], we have

$$\mathcal{P}_\phi(\Xi) = (\Psi_- \oplus \Psi_+) \oplus (\imath H\varphi + \frac{\Omega - \bar{\Omega}}{2}) \oplus \imath F,$$

where

$$\begin{bmatrix} \Psi_- \oplus \Psi_+ \\ \imath H\varphi + \frac{\Omega - \bar{\Omega}}{2} \\ \imath F \end{bmatrix} = \begin{bmatrix} (\bar{\omega}\phi_+ - (h + \imath f)\phi_-) \oplus (-\omega\phi_- + (h - \imath f)\phi_+) \\ \imath \text{Re}(\langle \phi_+, \psi_+ \rangle - \langle \phi_-, \psi_- \rangle)\varphi - \frac{1}{\sqrt{2}}(\bar{\psi}_-\phi_+ + \bar{\phi}_-\psi_+) \\ \imath \text{Im}(\langle \phi_+, \psi_+ \rangle + \langle \phi_-, \psi_- \rangle) \end{bmatrix}.$$

The calculation of the spectral flow contribution at $t = 0$ of the path $\{\tilde{H}_t\}_{t \in [0, 1]}$ is done in the following way in [N1, Step 2]. The spectral data of $\tilde{H}_t$ can be organized in families depending analytically on $t$. Denote by $Z$ the set of all pairs $(\Xi(t), \lambda(t))$ where $\lambda(t)$ is a real eigenvalue of

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\(^8\)Compared to [N], we have simplified $\dot{\psi}$ and $\dot{a}$ to $\psi$ and $a$.  

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\( \tilde{\mathcal{H}}_t, \Xi(t) \) is an eigenvector of unit length corresponding to \( \lambda(t) \), \( \lambda(0) = 0 \), and \( t \mapsto (\Xi(t), \lambda(t)) \) is analytic. We have \( \#Z = \dim \ker \tilde{\mathcal{H}}_0 \) and

\[
(\Xi(t); \lambda(t)) = (\Xi_0 + t\Xi_1 + \ldots; \lambda_0 + t\lambda_1 + \ldots + t^{m+1} + \ldots),
\]

(3.22)

where the integer \( m \) is the order of the pair \( (\Xi(t), \lambda(t)) \). A pair is called degenerate if \( m > 1 \) and non-degenerate otherwise. Since \( \ker \tilde{\mathcal{H}}_t = 0 \) for \( t > 0 \), all pairs have finite order (so in our case \( Z = Z^* \)). The spectral flow \( SF_+ \) is then determined by

\[-SF_+ = \#\{(\Xi(t), \lambda(t)): \lambda_m < 0\}.\]

For \( (\Xi(t), \lambda(t)) \) in (3.22), the equation \( \tilde{\mathcal{H}}_t\Xi(t) = \lambda(t) \) expands to

\[
\begin{align*}
\tilde{\mathcal{H}}_0(\Xi_0) &= 0, \\
\tilde{\mathcal{H}}_0(\Xi_1) + P_0(\Xi_0) &= 0, \\
& \vdots \\
\tilde{\mathcal{H}}_0(\Xi_{m-1}) + P_0(\Xi_{m-2}) &= 0, \\
\tilde{\mathcal{H}}_0(\Xi_m) + P_0(\Xi_{m-1}) &= \lambda_m \Xi_0, \\
& \vdots
\end{align*}
\]

(3.23)

If \( |\Xi_0| = 1 \), note that

\[
\lambda_m = \langle \lambda_m \Xi_0, \Xi_0 \rangle = \langle P_0(\Xi_{m-1}), \Xi_0 \rangle.
\]

(3.24)

We have

\[
\ker \tilde{\mathcal{H}}_0 \cong H^0(K^{1/2} \otimes L^0) \oplus H^0(K^{1/2} \otimes L) \oplus H^0(\Sigma, \mathbb{R}) \oplus H^1(\Sigma, \mathbb{R}),
\]

where we have identified \( H^0(K^{-1/2} \otimes L) \) with \( H^0(K^{1/2} \otimes L^*) \). We will also identify \( H^0(\Sigma, \mathbb{R}) \) with \( \mathbb{R} \) and \( H^1(\Sigma, \mathbb{R}) \) with the space of holomorphic 1-forms \( \omega \) via

\[
H^0(\Sigma, \mathbb{R}) \ni a \mapsto \omega \quad \text{s.t.} \quad ia = \frac{\omega - \overline{\omega}}{2}.
\]

For each \( \Xi_0 \in \ker \tilde{\mathcal{H}}_0 \), let \( R_{\phi}(\Xi_0) \) denote the projection to \( \ker \tilde{\mathcal{H}}_0 \) of \( P_0(\Xi_0) \). The operator \( R_{\phi} \) is called the resonance form/matrix. For

\[
\Xi_0 = (\psi_- \oplus \psi_+) + \frac{\omega - \overline{\omega}}{2} \oplus i \phi \in \ker \tilde{\mathcal{H}}_0,
\]

the quadratic form

\[
Q(\Xi_0) := \langle R_{\phi}\Xi_0, \Xi_0 \rangle = \langle P_0\Xi_0, \Xi_0 \rangle = \sqrt{2} f \Im \left( \langle \phi_+, \psi_+ \rangle + \langle \phi_-, \psi_- \rangle \right) - \Re \left( \langle \overline{\psi}_+ \phi_+ + \psi_+ \phi_- \rangle \omega \right)
\]

generalizes [N1, Lemma 3.2]. Non-degenerate pairs correspond to the case where \( m = 1 \) in (3.23), whereas degenerate pairs correspond to the kernel of \( R_{\phi} \) for which \( m > 1 \). The contribution to the spectral flow of the non-degenerate pairs is equal to the number of negative eigenvalues of the quadratic form \( Q \). In the following, we find the number of negative eigenvalues of \( Q \) and the kernel of \( R_{\phi} \).

Let

\[
V_- = H^0(K^{1/2} \otimes L^0), \quad V_+ = H^0(K^{1/2} \otimes L), \quad K = H^1(\Sigma, \mathbb{R}).
\]
Consider the multiplication maps
\[ m_+: V_+ \to K, \quad \psi_+ \mapsto \overline{\psi_+\phi_+}, \]
\[ m_-: V_- \to K, \quad \psi_- \mapsto \overline{\psi_-\phi_-}. \]

The second term in \( Q \) takes the form
\[ (f, v^+ , v^- , \omega) \mapsto -\text{Re}( (m_+(\psi_-) + m_- (\psi_+)) \overline{\omega}). \]

Let \( R_\pm \) denote the range of \( m_\pm \). Since \( \phi_+\phi_- = \nu \) (\( \nu \) is the perturbation term), we get
\[ R_+ \cap R_- = \mathbb{C} \cdot \nu. \]

Decompose \( K \) into \( \mathbb{C} \cdot \nu \oplus W_\pm \oplus W_0 \), where \( W_\pm \) is the intersection of orthogonal complement of \( \mathbb{C} \cdot \nu \) with \( R_+ + R_- \) and \( W_0 \) is the orthogonal complement of \( R_+ + R_- \). As in [N1, p. 99-100], we can choose complex bases \( v_1^\pm , \ldots , v_d^\pm \) for \( V_\pm \) such that

- \( v_1^\pm = \phi_\pm \),
- for \( v^\pm = \sum_{i=1}^{d_\pm} z_i^\pm v_i^\pm \), the first term in \( Q \) takes the form
  \[ (f, v^+ , v^- , \omega) \mapsto f \text{Im}(z_1^+ + z_1^-), \]
- under \( m_\pm \), \( \sum_{i=2}^{d_\pm} z_i v_i^\pm \) has image in \( W_\pm \).

Then, for \( \omega = \lambda \nu \oplus \omega_\pm \oplus \omega_0 \in \mathbb{C} \cdot \nu \oplus W_\pm \oplus W_0 \), we have
\[ Q(f, v^+, v^-, \omega) = f \text{Im}(z_1^+ + z_1^-) - \text{Re}( (\overline{z_1^+ z_1^-}) \lambda) |\nu|^2 - \text{Re} \langle \cdot , \cdot \rangle |W_\pm \times W_\pm|, \]
where the last term denotes the quadratic form \( \text{Re} \langle \cdot , \cdot \rangle \) on \( W_\pm \). We conclude that
\[ \dim_{\mathbb{R}} \ker Q = 1 + 2 \dim_{\mathbb{C}} W_0 = 3 + 2(g - (d_+ - 1) - (d_- - 1) - 1) = 2(g - (d_+ + d_- - 1)) + 1 \]
\[ \dim_{\mathbb{R}} \ker Q = 3 + 2 \dim_{\mathbb{C}} W_\pm = 2(d_+ + d_- - 1) + 1. \]

Therefore, (A) the spectral flow contribution of the non-degenerate pairs is
\[ -2(d_- + d_+ - 1) - 1, \]
and, (B) the number of degenerate pairs is equal to
\[ 2(g - (d_+ + d_- - 1)) + 1. \]

Since \( \dim_{\mathbb{R}} \ker \tilde{H}_t = 0 \) for \( t > 0 \), there can be at most
\[ x < \dim_{\mathbb{R}} \ker Q = 2(g - (d_+ + d_- - 1)) + 1 \]
degenerate pairs contributing to the spectral flow. Thus, we get
\[ -\text{SF}_+ = x + 2(d_- + d_+ - 1) + 1. \]
\[ (3.25) \]
Lemma 3.6. We have

\[ x = \begin{cases} 
(g - (d_+ + d_- - 1)) + 1 & \text{if } \ell > 0; \\
(g - (d_+ + d_- - 1)) + 0 & \text{if } \ell < 0.
\end{cases} \tag{3.26} \]

Therefore,

\[ -\text{SF}_+ = \begin{cases} 
g + d_+ + d_- + 1 & \text{if } \ell > 0; \\
g + d_+ + d_- & \text{if } \ell < 0.
\end{cases} \tag{3.27} \]

First, we use Lemma 3.6 to finish the proof of Proposition 3.5. Moving to [N1, Sec 3.4], in (3.21), the first term involves \( \tilde{H}_0 \) so it is as considered there, the second term is (3.25), and the third term (i.e., SF(\( \tilde{H}_1 \to \mathcal{O}_u \))) vanishes for the same reason\(^9\) as in [N1, Cor. D.3]. In [N1, (3.25)], replacing \(-\text{SF}_+ \) with (3.27), since the dimension of the asymptotic moduli space is zero, we get

\[ \xi = (d_- + d_+ + \text{SF}_+) + -g - \frac{1}{2} + \frac{1}{4}(\Sigma \cdot \Sigma - e_{\Sigma,X}) \]

\[ = g + \frac{1 - e_{\Sigma,X}}{2} + -g - \frac{1}{2} + \frac{1}{4}(\Sigma \cdot \Sigma - e_{\Sigma,X}) \]

\[ = \frac{1}{4}(\Sigma \cdot \Sigma - 3e_{\Sigma,X}). \]

That finishes the proof of Proposition 3.5. \( \square \)

It is just left to prove Lemma 3.6, which comes next.

Proof. Compared to the degenerate case in [N1, p. 101], in our case, \( \text{Ker } \tilde{H}_1 = 0 \), and \( \text{dim}_\mathbb{R} \text{Ker } \mathcal{R} = 2(g - (d_+ + d_- - 1)) + 1 \); \( \text{Ker } \mathcal{R}_\phi \) is the direct sum of

(1) the complex linear subspace \( W_0 \), and

(2) the 1-dimensional linear subspace generated by \( e_0 = \phi_+ \oplus -\phi_- \).

For \( \Xi_0, \Xi'_0 \in \text{Ker}(\mathcal{R}_\phi) \), the bilinear form

\[ B(\Xi_0, \Xi'_0) = \langle \Xi_0, \mathcal{P}_\phi \Xi'_1 \rangle = -\langle \tilde{\mathcal{H}}_0 \Xi, \Xi'_1 \rangle \]

in [N1, p. 102] is well-defined and symmetric. These sub-spaces (1) and (2) above are orthogonal with respect to \( B \). Restricted to \( W_0 \), for \( \Xi_0 = \Xi_0(\omega) = 0 \oplus \frac{\omega \Xi}{2} \oplus 0 \), we have

\[ \mathcal{P}_\phi(\Xi_0) = \begin{bmatrix} -\overline{\omega} \phi_+ & -\omega \phi_- \\ 0 & 0 \end{bmatrix}. \]

Therefore, the \( \psi \)-component \( \psi^\omega = \psi^\omega_+ \oplus \psi^\omega_- \) of the next term \( \Xi_1 \) in the expansion (3.22) satisfies

\[ \partial \psi^\omega_+ = \overline{\omega} \phi_+ \quad \text{and} \quad \partial \psi^\omega_- = \omega \phi_- , \]

where \( \partial \) and \( \overline{\partial} \) are the operators at the bottom of [N1, p. 79]. We conclude that

\[ -B(\Xi_0(\omega_1), \Xi_0(\omega_2)) = \text{Re} \langle \psi^\omega_1, D_A \psi^\omega_2 \rangle = \text{Re} \left( \langle \psi^\omega_2, \overline{\partial} \psi^\omega_1 \rangle + \langle \psi^\omega_2, \partial \psi^\omega_1 \rangle \right) = \text{Re} \left( \langle \psi^\omega_2, \overline{\omega}_1 \phi_+ \rangle + \langle \psi^\omega_1, \overline{\omega}_2 \phi_+ \rangle \right). \]

\(^9\)We get the same identity \( \int_N |df|^2 + |f|^2 |\phi| \text{ dvol}_N = 0 \) as in [N1, p. 121].
Fix a complex basis \( \omega_1, \ldots, \omega_k \) for \( W_0 \). Consider the complex \( k \times k \) symmetric matrix

\[
X + iY = \left( \langle \omega_j^\omega, \overline{\omega_i} \phi_+ \rangle + \langle \omega_j^\omega, \overline{\omega_i} \phi_- \rangle \right)_{i,j \in \{1, \ldots, k\}}.
\]

Since

\[
\psi_+^\omega + i \psi_-^\omega = \psi_+^\omega + i \psi_-^\omega
\]

with respect to the real basis \( \{e_0, \omega_1, i \omega_1, \ldots, \omega_k, i \omega_k\} \) for \( \text{Ker} \ R_\phi \), the matrix \( B \) is given by

\[
\begin{bmatrix}
\alpha & 0 & 0 \\
0 & X & -Y \\
0 & -Y & -X
\end{bmatrix}.
\]

(3.28)

The real number \( \alpha \) depends on \( \ell \) in the following way. Similarly to [N1, (3.16)-(3.18)], corresponding to \( \Xi_0 = e_0 \) the solution \( \Xi_1 = \psi \oplus ia \oplus if \) of the following set of equations

\[
\begin{align*}
D_A \psi &= 0 \\
- * da + df + (|\phi_+|^2 + |\phi_-|^2) \varphi &= 0 \\
d^* a &= 0
\end{align*}
\]

satisfies \( \tilde{H}_0 \Xi_1 + P_\phi \Xi_0 = 0 \). The above equation has a unique solution for which

\[
a = \frac{1}{2\ell} (|\phi_-|^2 + |\phi_+|^2) \varphi.
\]

Furthermore,

\[
B(\Xi_0) = -\frac{1}{2\ell} \int_N |\phi_-|^2 + |\phi_+|^2.
\]

We conclude that \( \alpha \) is negative if and only if \( \ell > 0 \). Also, if \( \begin{bmatrix} 0 \\ u \end{bmatrix} \) is an eigenvector of (3.28) with eigenvalue \( \lambda \), then \( \begin{bmatrix} 0 \\ -v \end{bmatrix} \) is another eigenvector with eigenvalue \( -\lambda \). Similarly to the non-degenerate (order 1) case, the negative eigenvalues of \( B \) correspond to the spectral flow contributions of order-2 eigenvectors in (3.23). If \( B \) has no zero eigenvalue, this finishes the proof of Lemma 3.6. Otherwise, we need to continue inductively in the following way.

Inductively, starting with \( \mathcal{R}_k^1 = \mathcal{R}_k \), for \( \Xi_0 \in \text{Ker} \mathcal{R}_k^k \), let \( \mathcal{R}_k^{k+1}(\Xi_0) \in \text{Ker} \mathcal{R}_k^k \) denote the orthogonal projection of \( P_\phi(\Xi_k) \) to \( \text{Ker} \mathcal{R}_k^k \). Eigenvectors of order \( k + 1 \) in (3.23) correspond to negative eigenvalues of the symmetric bilinear form

\[
B^k(\Xi_0, \Xi'_k) = \langle \Xi_0, P_\phi \Xi'_k \rangle
\]

(3.29)

and eigenvectors of order \( > k + 1 \) in (3.23) correspond to \( \text{Ker} \mathcal{R}_k^{k+1} \). For \( k \geq 2 \), \( \text{Ker} \mathcal{R}_k^k \) is a complex linear subspace of \( W_0 \). For the same reason as above, the number of positive and negative eigenvalues of (3.29) are the same. This inductive process ends in finite steps. \( \square \)
3.5 Tunneling spaces

Suppose \( \pi : \mathcal{N} \to \Sigma \) is a hermitian line bundle over a genus \( g \) surface and \( P \) is the projectivization of \( \mathcal{N} \). Let \( \Sigma_+ \) denote the section at infinity. We have

\[
\ell := \deg(\mathcal{N}) = \Sigma_+^2 = -\Sigma_+^2 \quad \text{and} \quad c_1(K_P) = -\text{PD}(\Sigma_- + \Sigma_+) + (2g-2)\text{PD}(F),
\]

where \( F \) is the fiber class, and the self-intersections numbers on the left are taken in \( P \). The second homology of \( P \) is generated by \( \Sigma_\pm \) and \( F \) satisfying the relation

\[
\Sigma_+ = \Sigma_- - \ell F.
\]

Let \( \Sigma = \Sigma_- \cup \Sigma_+ \). We have \( P - \Sigma = \mathbb{R} \times Y \), where \( Y \) is the unit circle bundle in \( \mathcal{N} \). We may think of \( P - \Sigma \) as a 4-manifold with two cylindrical ends modeled over the dual \( S^1 \)-bundles \( Y \) at \( +\infty \) and \( Y^* \) at \( -\infty \). Since \( P \) is a Kähler surface, every spin\(^c \) structure \( s \) on \( TP(- \log \Sigma) \cong \pi^*T\Sigma_\pm \oplus \mathbb{C}_P \) corresponds to a homology class

\[
\beta = [a\Sigma_- + b_+ F] = [a\Sigma_+ + b_- F], \quad b_- - b_+ = a\ell.
\]

The logarithmic canonical bundle of \( (P, \Sigma) \) is

\[
K_P(\log \Sigma) = K_P \otimes \mathcal{O}(\Sigma) = \pi^*K_\Sigma.
\]

The moduli space \( \mathcal{M}(P - \Sigma, s) \) has expected dimension

\[
d_{\beta} = \beta \cdot \beta - \beta \cdot K_P = (a + 1)(b_- + b_+) + 2a(1 - g)
\]

and admits a natural \( \mathbb{R} \times S^1 \)-action. The quotient with respect to \( \mathbb{R} \) is the unparametrized space \( \mathcal{M}(C_-, C_+) \) of tunnelings between \( C_- \) and \( C_+ \) described in Section 2.5, where \( C_\pm \) depends on the number \( b_\pm \). The moduli space is empty unless

\[
0 \leq \beta \cdot \Sigma_\pm = b_\pm \leq 2g - 2.
\]

If \( a = 0 \), then \( b_+ = b_- \) and \( \mathcal{M}(X - \Sigma, s) \cong \text{Sym}^b_+(\Sigma_-) \cong \text{Sym}^b_-(\Sigma_+) \) consists of only fiber-wise constant solutions. The action of \( \mathbb{R} \times S^1 \) is trivial in this case.

Similarly, we are interested in perturbed moduli spaces \( \mathcal{M}_{\eta_0}(P - \Sigma, s) \). In this case, the expected dimension is

\[
\tilde{d}_{\beta} = \beta \cdot \beta - \beta \cdot K_P(\log \Sigma) = a(b_- + b_+ + 2(1 - g)).
\]

The quotient with respect to \( \mathbb{R} \) is the unparametrized space \( \mathcal{M}(C_-, C_+) \) of tunnelings between two points \( C_- \in S_{b_-}(\nu) \) and \( C_+ \in S_{b_+}(\nu) \).

**Remark 3.7.** By Thom isomorphism, we have \( H^3_\mathbb{Z}(P - \Sigma) \cong H^1(Y) \). It follows that \( b^3_{P - \Sigma} = 0 \). Therefore, the example above does not fit in the framework of Theorem A.

**Proposition 3.8.** Suppose \((A, \Phi) \in \mathcal{M}(P - \Sigma, s) \) or \( \mathcal{M}_{\eta_0}(P - \Sigma, s) \). Then the connection \( A \) defines a holomorphic structure on the twisting complex line bundle \( \mathcal{O}(\beta) \), \( \Phi_+ \) is a holomorphic section of \( \mathcal{O}(\beta) \), \( \overline{\Phi}_- \) is a holomorphic section of \( K_P(\log \Sigma) \otimes \mathcal{O}(-\beta) \), and \( \Phi_+ \overline{\Phi}_- = i\pi^*\nu \in K_P(\log \Sigma) \cong \pi^*K_\Sigma \).
Proof. Comparing the notations in Section 2.4 and 3.1, the connection 1-form \( \alpha \) on \( Y \) is the same as \( \alpha \) after (3.1). By (3.3), it defines an integrable almost complex structure \( J \) on \( N \) and thus on \( \mathbb{R} \times Y \). On \( \mathbb{R} \times Y \), \( J \) acts by the complex structure of \( \Sigma \) on the horizontal subspace of every slice \( \{t\} \times Y \), and \( J \partial_t = \partial_y \). If \( \ell \neq 0 \), the cylindrical metric is not Kähler: the associated \((1,1)\)-form is \( dt \wedge \alpha + \pi^* \omega \) and \( d(dt \wedge \alpha + \pi^* \omega) = -dt \wedge d\alpha \neq 0 \) because \( \mathrm{id} \alpha \) is the curvature of the principal \( U(1) \)-bundle \( Y \).

On \( \mathbb{R} \times Y \), using the same notation as in the proof of Lemma 2.4, for

\[
(A, \Phi) = (B(t), \Psi(t))_{t \in \mathbb{R}}, \quad \text{where} \quad \Psi(t) = (\Psi_+(t), \Psi_-(t)),
\]

the Dirac operator has the form\(^\text{10}\)

\[
\hat{\phi}_A = \begin{bmatrix}
\nabla_{-\partial_t - i\partial_y} & \sqrt{2}\bar{\partial}_B t(t)
\n\sqrt{2}\bar{\partial}_B t(t) & \nabla_{-\partial_t + i\partial_y}
\end{bmatrix}.
\]

Therefore,

\[
\hat{\phi}_A \Phi = 0 \Leftrightarrow \begin{cases}
\sqrt{2}\bar{\partial}_B t(t) \Psi_+(t) - \nabla_{\partial_t - i\partial_y} \Psi_-(t) = 0 \\
\sqrt{2}\bar{\partial}_B t(t) \Psi_-(t) - \nabla_{\partial_t + i\partial_y} \Psi_+(t) = 0
\end{cases}
\]

By [KM, (4.9)], the unperturbed curvature equation has the form

\[
*_{Y} \hat{B}(t) + \hat{F}_B(t) = (\Psi(t)\Psi(t)^*)_0.
\]

By (3.12), the curvature equation, when perturbed using \( \nu \), has the form

\[
*_{Y} \left( \hat{B}(t) - i\nu \right) + (\hat{F}_B(t) - *_{Y} i\nu) = (\Psi(t)\Psi(t)^*)_0.
\]

Note that \( \hat{B}(t) \) is a family of imaginary valued 1-forms \( \beta(t) \) on \( Y \). The curvature equation (3.36) now reads:

\[
-i(F_{\alpha_1,\alpha_2,t} + \beta_{\alpha,t}) = \frac{1}{2}(|\Psi_+(t)|^2 - |\Psi_-(t)|^2)
\]

\[
\left(\frac{1}{2}(F_{\alpha_2,\alpha,t} - iF_{\alpha_1,\alpha,t} + \beta_{(0,1),t})\alpha_{0,1} - i\nu \right) = \bar{\Psi}_+(t)\Psi_-(t),
\]

where \( \alpha_{0,1} \) is defined as in (2.13) and

\[
F_{B(t)} = F_{\alpha_1,\alpha_2,t} + \alpha_1 \wedge \alpha_2 + F_{\alpha_1,\alpha,t} + F_{\alpha_2,\alpha,t} + \alpha_1 \wedge \alpha, \\
\beta(t) = \beta_{\alpha,t} + \beta_{(0,1),t} \alpha_{0,1} - \overline{\beta_{(0,1),t} \alpha_{1,0}}.
\]

Applying \( \bar{\partial}_B(t) \) to the second equation in (3.34) we get

\[
-\bar{\partial}_B(t) \nabla_{\partial_t + i\partial_y} \Psi_+(t) + \sqrt{2}\bar{\partial}_B t(t) \bar{\partial}_B(t) \Psi_-(t) = 0.
\]

Commuting the differential operators \( \nabla_{\partial_t + i\partial_y} \) and \( \bar{\partial}_B(t) \) produces a curvature term

\[
\nabla_{\partial_t + i\partial_y} \bar{\partial}_B(t) - \bar{\partial}_B(t) \nabla_{\partial_t + i\partial_y} = (2\beta_{(0,1),t} + (F_{\alpha_2,\alpha,t} - iF_{\alpha_1,\alpha,t}))\alpha_{0,1}
\]

\(^\text{10}\)Note that \( \lambda = 0 \) because we are using the adiabatic connection on \( Y \).
and using the second equation in (3.37), we can rewrite (3.38) as
\[-\nabla_{\partial_t+i\partial_b} \bar{\partial}_{B(t)} \Psi_+ + 2(\bar{\Omega}_+(t)\Psi_-(t) + i\pi^*\nu)\Psi_+(t) + \sqrt{2}\bar{\partial}_{B(t)} \bar{\partial}_{B(t)}^* \Psi_-(t) = 0.\]
Applying the first equation in (3.34) to the first term and dividing by \sqrt{2} we get
\[-\frac{1}{2} \nabla_{\partial_t+i\partial_b} \nabla_{\partial_t-i\partial_b} \Psi_-(t) + \sqrt{2}(\bar{\Omega}_+(t)\Psi_-(t) + i\pi^*\nu)\Psi_+(t) + \bar{\partial}_{B(t)} \bar{\partial}_{B(t)}^* \Psi_-(t) = 0. \tag{3.39}\]
Similarly, we get
\[-\frac{1}{2} \nabla_{\partial_t-i\partial_b} \nabla_{\partial_t+i\partial_b} \Psi_+(t) + \sqrt{2}(\Psi_+(t)\bar{\Omega}_-(t) - i\pi^*\nu)\Psi_-(t) + \bar{\partial}_{B(t)} \bar{\partial}_{B(t)}^* \Psi_+(t) = 0. \tag{3.40}\]
Define
\[\bar{\partial}^{\text{ver}} = \nabla_{\partial_t+i\partial_b}\]
Note that the first terms in (3.39) and (3.40) are
\[\frac{1}{2} \bar{\partial}^{\text{ver}} \bar{\partial}^{\text{ver}} + \text{ and } \frac{1}{2} \bar{\partial}^{\text{ver}} \bar{\partial}^{\text{ver}},\]
respectively. Also note that the \(\bar{\partial}\)-operator associated to the connection \(A\) over the entire complex manifold \(\mathbb{R} \times Y\) is
\[\bar{\partial}_A = (\partial t - i\alpha) \otimes \frac{1}{2} \nabla_{\partial_t+i\partial_b} + \bar{\partial}_B \]
\[= ((\partial t - i\alpha) \otimes \frac{1}{2} \nabla_{\partial_t+i\partial_b} + (\alpha_0,1 \otimes \frac{1}{2} \nabla_{\xi_1+i\xi_2}) : \Omega^0(\mathcal{O}(\mathcal{B})) \rightarrow \Omega^{0,1}(\mathcal{O}(\mathcal{B}))).\]
Therefore, if \(\Psi_+\) satisfies \(\bar{\partial}^{\text{ver}} \Psi_+ = 0\) and \(\bar{\partial}_B \Psi_+ = 0\), then it satisfies \(\bar{\partial}_A \Psi_+ = 0\). We have a similar statement for \(\Psi_-\). As in the proof of Lemma 2.4, we want to take the inner product of, say (3.40), with \(\Phi_+\) over the entire \(\mathbb{R} \times Y\) and conclude that
\[\bar{\partial}^{\text{ver}} \Phi_+ = 0, \quad \bar{\partial}_A \Phi_+ = 0, \quad \text{and} \quad \Phi_+ \Phi_+ - \pi^* \omega. \tag{3.41}\]
Towards this goal, we will work over \([-T_1, T_2] \times Y\) and then let \(T_i \rightarrow \infty\). To find the boundary terms, consider the complex (2,1)-form
\[\Omega = -i \left< \Phi_+, \bar{\partial}^{\text{ver}} \Phi_+ \right> (dt + i\alpha) \wedge \pi^* \omega.\]
Since \(\mathbb{R} \times Y\) is a holomorphic manifold, we have
\[d\Omega = \bar{\partial} \Omega = 2 \left( \left< \bar{\partial}^{\text{ver}} \Phi_+, \bar{\partial}^{\text{ver}} \Phi_+ \right> - \left< \Phi_+, \bar{\partial}^{\text{ver}} \bar{\partial}^{\text{ver}} \Phi_+ \right> \right) dt \wedge dvol_Y.\]
By the same reasoning as in (2.18), and integration by parts corresponding to the 3-form above, we get
\[0 = \int_{[-T_1, T_2] \times Y} \left< \Phi_+, \frac{1}{2} \bar{\partial}^{\text{ver}} \bar{\partial}^{\text{ver}} \Phi_+ + \bar{\partial}_{B(t)} \bar{\partial}_{B(t)}^* \Phi_+ + \sqrt{2}(\Phi_+ \Phi_+ - i\pi^*\nu)\Phi_+ \right> dt \wedge dvol_Y \]
\[= \int_{[-T_1, T_2] \times Y} \left( \frac{1}{2} \bar{\partial}^{\text{ver}} \Phi_+^2 + |\bar{\partial}_{B(t)} \Phi_+|^2 + \sqrt{2}\Phi_+ \Phi_+ - i\pi^*\nu \right) dt \wedge dvol_Y \tag{3.42}\]
\[-\frac{1}{4} \int_{T_2} \left< \Phi_+, \bar{\partial}^{\text{ver}} \Phi_+ \right>_{t=T_2} dvol_Y + \frac{1}{4} \int_{(-T_1) \times Y} \left< \Phi_+, \bar{\partial}^{\text{ver}} \Phi_+ \right>_{t=-T_1} dvol_Y.\]
The integral in the second line is non-negative and is an increasing function of each $T_i$. The integrals in the third line converge to 0 as $T_i \rightarrow \infty$. Thus the integral in the second line must be zero. We conclude that $\bar{\partial} \nu_{\nu} \Phi_+ = 0$, $\bar{\partial}_{B(Y)} \Phi_+ = 0$, and $\Phi_+ \bar{\Phi}_- = i \pi^* \nu$, from which (3.41) follows. Moreover, the curvature of $A$ will be a $(1,1)$-form by (3.37), so $A$ defines a holomorphic structure. Note meanwhile that the boundary terms in the third line turn out to be zero, because $\bar{\partial} \nu_{\nu} \Phi_+ = 0$. □

**Corollary 3.9.** Unless $b_- = b_+$, the moduli space $\mathcal{M}_{\eta}(P - \Sigma, s)$ is empty. If $b_+ = b_- = b$, which happens when $a = 0$, then $\mathcal{M}_{\eta}(P - \Sigma, s) \cong S_b(\nu)$ consists of fiber-wise constant solutions.

**Proof.** The zero sets of $\Phi_+$ and $\bar{\Phi}_-$ represent the homology classes

\[ [a \Sigma_- + b_+ F] \quad \text{and} \quad [-a \Sigma_- + (2g - 2 - b_+) F], \]

respectively. Therefore, we must have $a=0$. If $a = 0$, the restrictions of $\Phi_+$ and $\bar{\Phi}_-$ to each fiber of $P$ are constant sections. Therefore, both are pullbacks from $\Sigma$. □

**Remark 3.10.** In the case of $\mathcal{M}(P - \Sigma, s)$, if $a > 0$, then $\Phi_- = 0$ and if $a < 0$, then $\Phi_+ = 0$. Furthermore, if $a > 0$, then $0 \leq b_\pm \leq g - 1$, and if $a < 0$, then $g - 1 \leq b_\pm \leq 2(g - 1)$. The two cases are related by Serre duality.

### 3.6 Compactification

As we mentioned in the introduction and Sec. 2.5, the moduli spaces $\mathcal{M}_s(X - \Sigma, s)$ (where $s$ means either a compact perturbation or an adapted perturbation) are not necessarily compact. Because of the tunneling phenomenon (see [KM, Sec. 16] or [N, Sec. 4.4.2]), a sequence of monopoles in $\mathcal{M}(X - \Sigma, s)$ will, after passing to a sub-sequence, “converge” to a finite ordered set of monopoles where the first one is a monopole on $X - \Sigma$ and the rest are non-trivial monopoles on the cylinder $\mathbb{R} \times Y$. For an adapted perturbation $\eta_\nu$, with $\nu \neq 0$, $\mathcal{M}_{\eta_\nu}(X - \Sigma, s)$ is compact by Corollary 3.9. For a compact perturbation $\eta_\nu$, however, the relative moduli spaces $\mathcal{M}_{\eta_\nu}(X - \Sigma, s)$ have the same associated tunneling spaces as in the unperturbed case and these tunneling spaces can be non-trivial.

In the following, we will assume that $\Sigma \cdot \Sigma \neq 0$ to ensure that the spin$^c$ structure on $Y$ is torsion and thus CSD is real-valued and increasing along the flow lines on $\mathcal{M}(Y, s_Y)$, as discussed in Sec. 2.5. The case of $\Sigma \cdot \Sigma = 0$ is slightly different and simpler and is already discussed in [MST].

Define $\mathcal{M}_{q_\nu}(X - \Sigma, s)$ to be the space of all level-$k$ broken trajectories for $s$ in the following sense. A level-$k$ element of $\mathcal{M}_{q_\nu}(X - \Sigma, s)$ is a tuple

\[ \left( [A_0, \Phi_0]; [A_1, \Phi_1]; \ldots; [A_k, \Phi_k] \right) \]  

in the fiber product

\[ \mathcal{M}_{q_\nu}(X - \Sigma, s_{X - \Sigma}) \times_{ev} \mathcal{M}(\mathbb{R} \times Y, s_{R \times Y})/\mathbb{C}^* \times_{ev} \mathcal{M}(\mathbb{R} \times Y, s_{R \times Y})/\mathbb{C}^* \]

such that

(1) the $\mathbb{C}^*$-action on $\mathcal{M}(\mathbb{R} \times Y, s_{R \times Y})$ is given by the translation symmetry on $\mathbb{R}$ and $S^1$-rotation symmetry on $Y$;

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(2) the evaluation map $ev$ at the ends takes value in a divisor space $\text{Div}_m(\Sigma)$ as in (1.3), if the limit is irreducible, or in $\mathcal{J} \cong \mathbb{T}^{2g}$ (or $\mathbb{T}^{2g+1}$), if the limit is reducible;

(3) the limit at $+\infty$ of the last monopole belongs to $\text{Div}_{d(\sigma)}(\Sigma)$.

It is known that $\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, s)$ has a natural sequential convergence topology that is Hausdorff and compact (e.g. see [KM, Sec. 16]). Compared to the compactification in [KM], the only difference here is that we take the quotient by the action of $\mathbb{C}^*$ instead of $\mathbb{R}$, therefore our compactification has no boundary.

For a broken trajectory as in (3.43), let $C_i$ denote the component of the limit at $+\infty$ of $[A_i, \Phi_i]$ for $i \in \{0, \ldots, k\}$. Therefore, the limit at $-\infty$ of $[A_i, \Phi_i]$ belongs to $C_{i-1}$ for $i \in \{1, \ldots, k\}$ and $C_k = \text{Div}_{d(\sigma)}(\Sigma)$. The $C_i$'s are in fact components of the decomposition

$$\mathcal{M}(Y, s_Y) = \mathcal{J} \cup \bigcup_m \mathcal{M}(Y, s_Y)_m.$$  \hspace{1cm} (3.45)

Using the notation of Sec. 2.5, for $i \in \{1, \ldots, k\}$, each $[A_i, \Phi_i]$ is an element of

$$\mathcal{M}(C_{i-1}, C_i)/\mathbb{C}^* = \tilde{\mathcal{M}}(C_{i-1}, C_i)/S^1.$$}

Recall from Sec. 2.5 that the components $C_0, \ldots, C_k$ are ordered by the value of CSD in the sense that CSD($C_0$) $<$ CSD($C_1$) $< \cdots <$ CSD($C_k$).

If none of these limits belongs to $\mathcal{J}$, it follows from Lemma 3.2 that the expected dimension of such level-$k$ configurations is $2k$ lower than the expected dimension of $\mathcal{M}_{\eta_0}(X - \Sigma, s)$.

On the other hand, depending on sign($\Sigma \cdot \Sigma$), the reducible component $\mathcal{J}$ appears in two ways:

- If $\Sigma \cdot \Sigma < 0$, then CSD takes its minimum on $\mathcal{J}$;
- If $\Sigma \cdot \Sigma > 0$, then CSD takes its maximum on $\mathcal{J}$.

To see this, first note that we can assume CSD($\mathcal{J}$) = 0 by taking our background connection to be a flat connection on $L_Y$. To evaluate CSD at an irreducible solution $(B, \Psi)$, recall that it is the pullback of a monopole on $\Sigma$, so the holonomy of $B$ is $\exp(2\pi ic/\ell)$, where $c = \text{deg}(L_Y)$ and $\ell = \text{deg}(Y) = -\Sigma \cdot \Sigma$. Therefore, for some flat connection $B_0$ on $L_Y$, we have $B - B_0 = (c/\ell)i\alpha$, where $i\alpha$ is the connection 1-form associated to decomposition (2.6). Since $\frac{i}{\pi}d(i\alpha)$ and $\ell \pi^* \omega$ both represent the pullback of the (integral) Euler class of $Y$, a straightforward calculation shows that CSD($B, \Psi$) is a positive multiple of $c^2/\ell$ and the two items above follow. As a result,

- If $\Sigma \cdot \Sigma < 0$, then only $C_0$ can be equal to $\mathcal{J}$;
- If $\Sigma \cdot \Sigma > 0$, then none of $C_0, \ldots, C_k$ is equal to $\mathcal{J}$.

In the first case, where $C_0 = \mathcal{J}$, the expected dimension of level-$k$ configurations is $2k + 1$ lower than the expected dimension of $\mathcal{M}_{\eta_0}(X - \Sigma, s)$ by lemmas 3.2 and 3.3. To summarize, we have the following theorem.

**Theorem 3.11.** The moduli space $\mathcal{M}_{\eta_0}(X - \Sigma, s)$ naturally has a Hausdorff compactification $\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, s)$ with a “boundary” of expected real codimension at least two. The moduli space $\mathcal{M}_{\eta_0}(X - \Sigma, s)$ is compact.
Remark 3.12. It may happen that (3.45) has only one irreducible component for topological reasons. If further $\Sigma \cdot \Sigma > 0$, this implies that $\mathcal{M}_{\eta_0}(X - \Sigma, s)$ is compact. For example, if $\Sigma \cdot \Sigma > 2g - 2$, since different $m$’s in (3.45) differ by a multiple of $\Sigma \cdot \Sigma$ and $\mathcal{M}(Y, s_v)_m$ is empty unless $|m| \leq g - 1$, we conclude that $\mathcal{M}_{\eta_0}(X - \Sigma, s)$ is compact. This point has been used in [OS] to show that certain decompositions of 4-manifolds cannot happen.

Remark 3.13. Subject to the regularity of the tunneling spaces, we get a gluing map

$$\mathcal{M}_{\eta_0}(X - \Sigma, s_{X-\Sigma}) \xrightarrow{ev \times ev} (\mathcal{M}(\mathbb{R} \times Y, s_{\mathbb{R} \times Y})/\mathbb{C}^*) \times (S^1)^k \Delta^k$$

as in (2.27), where $\Delta \subset \mathbb{C}$ is a sufficiently small disk and the fiber product with the $i$-th copy of $\Delta$ is with respect to the $S^1$-actions on $\Delta$ and the $i$-th tunneling space. This gluing map gives the compactified moduli space $\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, s)$ the structure of a closed $C^0$-manifold in which level-$k$ configurations in (3.44) are embedded as a codimension $2k$ (or $2k + 1$) submanifold. □

We finish this section by some comments on the contribution of the component $\mathcal{M}_{\eta_0}(X - \Sigma, s_{X-\Sigma}; J)$, consisting of monopoles ending at reducibles, to the relative invariants (1.5) and the resulting sum formula (1.10).

By the discussion above, if $\Sigma \cdot \Sigma > 0$, all the monopoles appearing in $\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, s)$ are automatically irreducible. Therefore, whenever the invariants are defined, the condition $b^+_X > 0$ is sufficient for the relative invariants $SW_{X,\Sigma}^{X,\Sigma}(s; -)$ to be independent of the choice of generic $\eta_0$ and the cylindrical metric. However, the component $\mathcal{M}_{\eta_0}(X - \Sigma, s_{X-\Sigma}; J)$ may appear in two ways in this paper.

Firstly, if $\Sigma \cdot \Sigma < 0$, it contributes to the level-$k$ strata of the compactified moduli space $\overline{\mathcal{M}}_{\eta_0}(X - \Sigma, s)$, where $[A_0, \Phi_0] \in \mathcal{M}_{\eta_0}(X - \Sigma, s_{X-\Sigma}, J)$, $[A_1, \Phi_1] \in \mathcal{M}(J, C_1)/\mathbb{C}^*$, and the rest of the tunnelings in (3.43) begin and end at irreducibles. The expected codimension of such a configuration is $2k+1 \geq 3$. Therefore, in this case, reducibles happen in high codimension and do not contribute to (1.5).

Secondly, the component $\mathcal{M}_{\eta_0}(X - \Sigma, s_{X-\Sigma}, J)$ would appear in the proof of the alternative sum formula (1.10) in the following way. Proving (1.10) revolves around the following observation. Let $X$ be the connected sum of $X_1$ and $X_2$ along $\Sigma$ and $s$ be a spin$^c$ structure on $X$. Consider the fiber product

$$\mathcal{M}_{\eta_1}(X_1 - \Sigma, s|_{X_1-\Sigma}) \times_{\mathcal{M}(Y_1, s_{Y_1})} \mathcal{M}_{\eta_2}(X_2 - \Sigma, s|_{X_2-\Sigma})$$

(3.46)

over $\mathcal{M}(Y, s_Y)$ via the limiting maps (1.2), where $\eta_1$ and $\eta_2$ are generic compact perturbations on $X_1 - \Sigma$ and $X_2 - \Sigma$, respectively, and $\eta = \eta_1 + \eta_2$ is the resulting perturbation on $X$.

The fiber product (3.46) decomposes into a union of main components, indexed by various pairs of spin$^c$ structures $(s_1, s_2)$ on $TX_1(-\log \Sigma)$ and $TX_2(-\log \Sigma)$, respectively, which restricted to $(s|_{X_1-\Sigma}, s|_{X_2-\Sigma})$ and have the same degree on $\Sigma$,

$$\bigcup_{s_1 \neq s_2 = s} \mathcal{M}_{\eta_1}(X_1 - \Sigma, s_1) \times_{\Div(s_1) = \Div(s_2)} \mathcal{M}_{\eta_2}(X_2 - \Sigma, s_2),$$

(3.47)

and the fiber product

$$\mathcal{M}_{\eta_1}(X_1 - \Sigma, s|_{X_1-\Sigma}, J) \times_J \mathcal{M}_{\eta_2}(X_2 - \Sigma, s|_{X_2-\Sigma}, J).$$

(3.48)

---

11 Here, $X$ can be thought of as any of the glued manifolds $X_T$ in Theorem 2.6 for some sufficiently large $T$, with $\eta_T = \eta$. 

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The union in (3.47) already embeds into \( \mathcal{M}_\eta(X, s) \) by the Gluing Theorem 2.6. Assuming that (3.48) also embeds into \( \mathcal{M}_\eta(X, s) \), as in [OS], we can conclude that \( \mathcal{M}_\eta(X, s) \) is the same as (3.46) except for a subset of real codimension 2 (consisting of broken trajectories involving non-trivial tunnelings). This follows from a convergence result similar to the one used in the construction of the compactification \( \overline{\mathcal{M}}_{\eta_\circ}(X - \Sigma, s) \) above; see further below.

By Lemma 3.2, the expected dimension of each component in (3.47) matches that of \( \mathcal{M}_\eta(X, s) \). However, by Lemma 3.3, the expected dimension of (3.48) is 1 less than the expected dimension of \( \mathcal{M}_\eta(X, s) \). This explains why (3.48) does not contribute to the sum formula (1.10). This observation is used in [OS] to prove their main theorem.

With notation as above, in general, a sequence of monopoles \( [A_{T_i}, \Phi_{T_i}] \) in \( \mathcal{M}(X_{T_i}, s) \), where \( \lim_{i \to \infty} T_i = \infty \), will, after passing to a sub-sequence, “converge” to a finite chain of monopoles where the first one is a monopole on \( X_1 - \Sigma \), the last one is a monopole on \( X_2 - \Sigma \), and the rest are non-trivial monopoles on the cylinder \( \mathbb{R} \times Y \). The proof is more or less identical to the proof of [KM, Thm. 16.1.3] in the following sense. Every compact sub-domain \( K \) of \( X_1 - \Sigma \) can be identified with a sub-domain of \( X_T \) for \( T > T_K \). Restricted to \( K \), after passing to a sub-sequence, \( \{[A_{T_i}, \Phi_{T_i}]\}_{T_i > T_K} \) converges to a monopole \( [A_{1,K}, \Phi_{1,K}] \). By considering a sequence of exhausting compact sets \( K \), using unique continuity and a diagonal argument, we get a monopole \( [A_1, \Phi_1] \) over \( X_1 - \Sigma \) whose restriction to any \( K \) is \( [A_{1,K}, \Phi_{1,K}] \). Similarly, we get a monopole over \( X_2 - \Sigma \).

However, as \( T_i \to \infty \), the energy of \( [A_{T_i}, \Phi_{T_i}] \) may concentrate at several locations along the expanding cylinder \( [0,T_i] \times Y \subset X_{T_i} \). The same proof as in [KM, Thm. 16.1.3] gives us the connecting tunneling monopoles between the monopoles on \( X_1 - \Sigma \) and \( X_2 - \Sigma \).

More precisely, define \( \mathcal{M}_{\eta_\infty}(X_\infty, s) \) to be the space of all level-\( k \) broken trajectories \( (k \geq 0) \) for \( s \) in the following sense. A level-\( k \) element of \( \mathcal{M}_{\eta_\infty}(X_\infty, s) \) is a tuple

\[
([A_0, \Phi_0]; [A_1, \Phi_1]; \ldots; [A_k, \Phi_k]; [A_{k+1}, \Phi_{k+1}])
\]

in the fiber product

\[
\mathcal{M}_\eta(X_1 - \Sigma, s_{X_1 - \Sigma}) \times_{ev} \mathcal{M}(\mathbb{R} \times Y, s_{Y}) \times_{ev} \mathcal{M}_\eta(X_2 - \Sigma, s_{X_2 - \Sigma}),
\]

such that

1. the \( \mathbb{C}^* \)-action on \( \mathcal{M}(\mathbb{R} \times Y, s_{Y}) \) is given by the translation symmetry on \( \mathbb{R} \) and \( S^1 \)-rotation symmetry on \( Y \);

2. the evaluation map \( ev \) at the ends takes values in a divisor space \( \text{Div}_m(\Sigma) \) as in (1.3), if the limit is irreducible, or in \( J \cong \mathbb{T}^{2g} \) (or \( \mathbb{T}^{2g+1} \)), if the limit is reducible.

An analogous construction exists for adapted perturbations \( \eta_\nu \), in which the intermediate tunneling spaces are empty and we simply have the fiber product of \( \mathcal{M}_{\eta_\nu}(X_1 - \Sigma, s_{X_1 - \Sigma}) \).

With notation as above, we have the following compactness theorem.

**Theorem 3.14.** The union

\[
\bigcup_{T \in [0, \infty)} \mathcal{M}_{\eta_T}(X_T, s)
\]

has a natural sequential convergence topology that is Hausdorff and compact.

Replacing \( \mathbb{C}^* \) with \( \mathbb{R} \), the same theorem holds for a decomposition of \( X \) along an arbitrary 3-dimensional manifold \( Y \).
3.7 Conclusion; proofs of Theorems A–C

Except for the orientability of the relative moduli spaces \( \mathcal{M}_+ (X - \Sigma, s) \), so far we have discussed all the steps that go into proving Theorems A–C. In this section, we wrap up the proofs with some comments on the orientation problem. The SW moduli spaces of closed 4-manifolds are orientable; a choice of orientation on \( H^1(X) \) and \( H^2_\sharp (X) \) determines an orientation on the moduli space. Unfortunately, a simple statement like this for the SW moduli spaces of cylindrical-end manifolds does not exist in the literature; we refer to [N, Sec. 4.4.3] and [KM, Sec. 20] for a rather lengthy discussion of the problem. The problem is on the cylindrical part, where it is not clear if all the operators \( \mathcal{H}_i \) in (3.20) are Fredholm. In Section 4, we introduce a different setup for constructing relative moduli spaces which, among other things, could provide a short answer to the orientation problem as well; see the discussion after Definition 4.4.

**Proof of Theorem A.** The fact that, for generic \( \eta_0 \), \( \mathcal{M}_{\eta_0} (X - \Sigma, s) \) is a smooth manifold and the evaluation map (1.3) is a submersion is proved in [N]; see Remark 2.5. The dimension formula (1.4) is derived in Lemma 3.2 by simplifying Nicolaescu’s formula [N1, (3.27)]. Since all of the non-standard analysis happens on the cylindrical part of \( X - \Sigma \), the orientability of \( \mathcal{M}_{\eta_0} (X - \Sigma, s) \) can be proved in the same way as in [KM, Cor. 20.4.1]. Here, for the sake of completeness, we present an ad hoc way of proving orientability. We show that \( \mathcal{M}_{\eta_0} (X - \Sigma, s) \) embeds in the moduli space \( \mathcal{M}_{\eta_0} (X, s') \) of the closed-up manifold \( X \) with a related classical spin\(^c\) structure \( s' \) on \( TX \). Therefore, since \( X \) is closed, \( \mathcal{M}_{\eta_0} (X, s') \) can be oriented by a choice of homology orientation on \( H^1(X, \mathbb{Z}) \oplus H^2_\sharp (X, \mathbb{Z}) \), which in turn induces an orientation on its submanifold \( \mathcal{M}_{\eta_0} (X - \Sigma, s) \) of the same dimension.

In the context of the gluing map (2.27), consider the natural decomposition of \( X = X_r \) to the cylindrical-end manifolds \( M_+ = X - \Sigma \) and \( M_- = \mathcal{N} \), where \( \mathcal{N} \) is the normal bundle of \( \Sigma \subset X \), as defined in Sec. 3.1. With notation as in Sec. 3.5, the cylindrical-end manifold \( \mathcal{N} \) can also be seen as \( P - \Sigma_+ \). Recall that \( \mathcal{N} \) has a canonical spin\(^c\) structure determined by its almost-complex structure, and any spin\(^c\) structure \( s_{\mathcal{N}} \) on \( \mathcal{N} \) can be obtained by twisting this canonical spin\(^c\) structure with \( E \), where \( E \) is the pullback of a line bundle of degree \( d \) on \( \Sigma \). We will be interested in the moduli space \( \mathcal{M}(\mathcal{N}, s_{\mathcal{N}}) \) of the cylindrical-end manifold \( \mathcal{N} \), where \( s_{\mathcal{N}} \) is determined by the line bundle \( E \) with \( d = d(s) \). By gluing the pair \( s_+ = s \) on \( M_+ \) and \( s_- = s_{\mathcal{N}} \) on \( M_- = \mathcal{N} \), we get a spin\(^c\) structure \( s' \) on \( X \). If \( \mathcal{M}(\mathcal{N}, s_{\mathcal{N}}) \) is regular, the gluing map (2.27) then gives an embedding

\[
\mathcal{M}_{\eta_0} (X - \Sigma, s) \times_{\mathcal{M}(Y, s_{Y,E}) \cong \text{Div}_d(\Sigma)} \mathcal{M}(\mathcal{N}, s_{\mathcal{N}}) \to \mathcal{M}_{\eta_0} (X, s').
\]

If we show that \( \mathcal{M}(\mathcal{N}, s_{\mathcal{N}}) \) has a component isomorphic to \( \text{Div}_d(\Sigma) \), for that component, the left-hand side of the map above is simply \( \mathcal{M}_{\eta_0} (X - \Sigma, s) \). Therefore, we get an embedding of equi-dimensional manifolds

\[
\mathcal{M}_{\eta_0} (X - \Sigma, s) \to \mathcal{M}_{\eta_0} (X, s').
\]

The following lemma finishes the proof.

**Lemma 3.15.** For the spin\(^c\) structure \( s_{\mathcal{N}} \) defined by \( E \) as above, where \( 0 < |d - g + 1| \leq g - 1 \), the moduli space \( \mathcal{M}(\mathcal{N}, s_{\mathcal{N}}) \) is non-empty and has a regular component diffeomorphic to the irreducible component \( \mathcal{M}(Y, s_{Y,E}) \cong \text{Div}_d(\Sigma) \) of \( \mathcal{M}(Y, s_Y) \).

**Proof.** As we have seen, if \( 0 < |d - g + 1| \leq g - 1 \), then \( \mathcal{M}(Y, s_{Y,E}) \cong \text{Div}_d(\Sigma) \) is an irreducible component of \( \mathcal{M}(Y, s_Y) \), and if \( d = g - 1 \), we get the reducible component. Let us start by constructing a family of metrics on \( Y \) as follows. Recall that \( TY \) decomposes as in (2.6), where the
vertical sub-bundle is given as the kernel of a real 1-form $\alpha$ on $Y$ and the horizontal sub-bundle identifies with $\pi^* T\Sigma$. Let $g_\Sigma$ denote the Kähler metric on $\Sigma$ corresponding to the Kähler form $\omega$. We can now use $\alpha$ and $g_\Sigma$ to define a family of metrics $g_\lambda = (\lambda \alpha) \otimes (\lambda \alpha) + \pi^* g_\Sigma$ on $Y$ for any $\lambda > 0$.

Observe meanwhile that, according to the descent process discussed in Section 2.4, if $(A, \Psi)$ is an irreducible monopole on $\Sigma$, then its pullback $(B, \Psi)$ via $Y \longrightarrow \Sigma$ is also an irreducible monopole on $Y$, regardless of which of the metrics $g_\lambda$ is used on $Y$.

Now, we can view $\mathcal{N}$ topologically as a quotient of $[0, \infty) \times Y$, where $\{0\} \times Y$ is identified with $\Sigma$ via the bundle projection $Y \longrightarrow \Sigma$. Using the family of metrics defined above, we can equip $\mathcal{N}$ with a cylindrical-end riemannian metric as follows. Start by considering the following family of metrics $g_r$ on the slices $Y_r = \{r\} \times Y$ of $\mathcal{N}$, $r > 0$,

$$g_r = \alpha_r \otimes \alpha_r + \pi^* g_\Sigma, \quad \alpha_r = \beta(r) \alpha,$$

where $\beta: (0, \infty) \longrightarrow [0, 1]$ is a smooth increasing function such that $\beta(r) = 0$ for $r \leq 1/2$ and $\beta(r) = 1$ for $r \geq 1$. In other words, $g_r = g_\lambda$ for $\lambda = \beta(r)$. As a result, this family of metrics has shrinking fibers on $Y$ as $r \to 0$ in one direction, and stabilizes at $g_r = g_1$ for $r \geq 1$ in the other direction. Note by the way that the adiabatic connection on each slice $Y_r$ is compatible with the metric $g_r$ by construction. Now define a “radial” metric $g$ on $\mathcal{N} - \Sigma$ by

$$g = dr^2 + g_r = dr \otimes dr + \alpha_r \otimes \alpha_r + \pi^* g_\Sigma.$$ \hfill (3.51)

An elementary local calculation in polar coordinates [P, Ch. 1, Sec. 3.4] shows that $g$ has a limit as $r \to 0$ and smoothly extends to a (non-degenerate) riemannian metric on the entire $\mathcal{N}$. In a neighborhood of $r \approx 0$, we can formally write $g = dr^2 + r^2 d\theta^2 + g_\Sigma$, where the metric $g$ collapses to $dr^2 + g_\Sigma$ at $r = 0$ as the slices $Y_r$ collapse to $\Sigma$. The metric $g$ in a neighborhood of $\Sigma \subset \mathcal{N}$ is the Kähler metric associated to the Kähler form (3.2). Moreover, $g$ is clearly cylindrical for $r \geq 1$, where we can formally write $g = dr^2 + d\theta^2 + g_\Sigma$.

We will now construct a monopole on $\mathcal{N}$ from an arbitrary smooth irreducible monopole $(A, \Psi)$ on $\Sigma$. To begin with, let us pull $(A, \Psi)$ back via $\pi: Y \longrightarrow \Sigma$ to obtain a smooth irreducible monopole $(B, \Psi)$ on $Y$, which is invariant under the circle action on $Y$. We can extend this to a configuration on $[0, \infty) \times Y$ as a constant family $(B, \Psi)$ on each slice $Y_r$, which in turn will descend to a smooth configuration $(\tilde{B}, \tilde{\Psi})$ on the quotient $\mathcal{N}$. We will show that $(\tilde{B}, \tilde{\Psi})$ satisfies the SW$^4$ equations on $\mathcal{N}$. On $(0, \infty) \times Y$, the spinor $\tilde{\Psi}$ is certainly harmonic, because it is harmonic on $Y$ and constant in the radial direction (0, $\infty$). Moreover, the spinor bundles on the slices $Y_r$ are identified with the plus- and minus-spinor bundles on $(0, \infty) \times Y$ via Clifford multiplication by the unit cotangent vector $d\theta$ and, since we are using the adiabatic connection on each slice, the curvature $F_B$ has no component in the radial direction and the curvature equation in SW$^4$ reduces to the curvature equation on $Y$. We conclude that $(\tilde{B}, \tilde{\Psi})$ satisfies the SW$^4$ equations on $\mathcal{N} - \Sigma$. Therefore, by smoothness, it will satisfy the equations over the entire $\mathcal{N}$.

We have just showed that $\mathcal{M}(\mathcal{N}, s_{\mathcal{N}})$ is non-empty, so we can consider the limiting map

$$\partial: \mathcal{M}(\mathcal{N}, s_{\mathcal{N}}) \longrightarrow \mathcal{M}(Y, s_{Y,E}) \cong \text{Sym}^d(\Sigma)$$

for the cylindrical-end manifold $\mathcal{N}$. We have in fact constructed a right inverse to the limiting map in the previous paragraph, so $\text{Sym}^d(\Sigma)$ embeds into $\mathcal{M}(\mathcal{N}, s_{\mathcal{N}})$. A calculation using Lemma 3.2
shows that the expected dimension of \( \mathcal{M}(\mathcal{N}, s, \nu) \) is equal to \( 2d \), which matches the dimension of \( \text{Sym}^d(\Sigma) \). We conclude that at least a component of \( \mathcal{M}(\mathcal{N}, s, \nu) \) is diffeomorphic to \( \text{Sym}^d(\Sigma) \) via the limiting map and the lemma follows.

**Proof of Theorem B.** As before, the fact that \( \mathcal{M}_{\eta, \nu}(X - \Sigma, s) \) is a smooth manifold for a generic choice of the compactly-supported part \( \eta_o \) is proved in [N]; see Remark 2.5. The dimension formula (1.7) is proved in Lemma 3.4. Compactness, as stated in Theorem 3.11, is derived from Corollary 3.9. Orientability can be proved as above. The fact that, for different such \( \eta, \nu \) and \( \eta, \nu' \), \( \mathcal{M}_{\eta, \nu}(X - \Sigma, s) \) and \( \mathcal{M}_{\eta, \nu'}(X - \Sigma, s) \) are smoothly cobordant follows from considering a 1-parameter family of perturbations connecting \( \eta, \nu \) and \( \eta, \nu' \), as in the classic case.

**Proof of Theorem C.** The sum formula is a direct consequence of the gluing theorem and convergence/compactness argument at the end of Section 3.6. In other words, for sufficiently large \( T \), the gluing map

\[
\bigcup_{[s] = s_1 \# s_2} \mathcal{M}_{\eta_1, \nu}(X_1 - \Sigma, s_1) \times S_{d(s_1)} = S_{d(s_2)} \mathcal{M}_{\eta_2, \nu}(X_2 - \Sigma, s_2) \rightarrow \mathcal{M}_{\eta_T, \nu}(X_T, s)
\]

which is an embedding by Theorem 2.6, is also onto by Theorem 3.14, thus is an identification of closed manifolds. Here \( \eta_1, \nu \) and \( \eta_2, \nu \) are adapted perturbations on \( X_1 - \Sigma \) and \( X_2 - \Sigma \) corresponding to a holomorphic 1-form \( \nu \) on \( \Sigma \) and \( \eta_T, \nu \) is the resulting perturbation term on \( X_T \).

To see the surjectivity of the gluing map, note that \( X_T \)'s can be identified with each other in a natural way by re-scaling the neck in the time direction. Therefore, a given monopole in \( \mathcal{M}_{\eta, \nu}(X, s) \) can be identified with a monopole in \( \mathcal{M}_{\eta_T, \nu}(X_T, s) \) for any \( T \); these monopoles are compatible with each other and have the same energy on the neck. As \( T \rightarrow \infty \), these monopoles converge to an element of the fiber product on the left, which is the desired inverse image of the original monopole on \( X \approx X_T \) under the gluing map above.

As mentioned before, the \( \pm \) signs \( \varepsilon(s_1, s_2, q) \in \{\pm 1\} \) in (1.9) depend on the choices of orientations. For a meticulous discussion of how to orient the fiber products to be consistent with the gluing, see [KM, 20.5].

### 4 Logarithmic SW equations

In this section, we explain our idea for a direct construction of relative SW moduli spaces to bypass the issues related to working with non-closed manifolds. We will define logarithmic SW equations by replacing \( TX \) with \( TX(-\log \Sigma) \) and considering “logarithmic connections”. This construction can easily be generalized to the normal crossings case, where it is hard to work with \( X - \Sigma \) as a cylindrical-end manifold. Through this construction, it should be possible to address the orientation problem more systematically.

Let \((X, \Sigma)\) be as in the previous sections and \( s = (S, \rho) \) be a spin\(^c\) structure on \( TX(-\log \Sigma) \). If \( W \rightarrow X \) is a vector bundle over \( X \), a connection \( \nabla \) on \( W \) is a bi-linear map

\[
\nabla : \Gamma(X, TX) \times \Gamma(X, W) \rightarrow \Gamma(X, W), \quad (\xi, \zeta) \rightarrow \nabla_\xi \zeta,
\]

that is tensorial in the first input and satisfies the Leibniz rule in the second input. In the classical theory, the construction of SW moduli space involves a riemannian metric on \( TX \), a hermitian metric on \( S \), and compatible connections on \( S \) and \( TX \). The latter is usually fixed to be the Levi-Civita connection.
Definition 4.1. Let \((X, \Sigma)\) be a pair of a closed oriented 4-manifold \(X\) and a closed oriented 2-dimensional submanifold \(\Sigma \subset X\). For any vector bundle \(W \rightarrow X\), a \textit{logarithmic} connection \(\nabla\) on \(W\) is a bi-linear map
\[ \nabla : \Gamma(X, TX(-\log \Sigma)) \times \Gamma(X, W) \rightarrow \Gamma(X, W), \quad (\xi, \zeta) \rightarrow \nabla_\xi \zeta, \] (4.1)
that is tensorial in the first input and satisfies the Leibniz rule in the second input.

As in the classical case, in any local trivialization, we have \(\nabla = d\iota + \Theta\), where \(\iota\) is the homomorphism in (3.6) and \(\Theta\) is a matrix of logarithmic 1-forms. Therefore, globally, every two logarithmic connections \(\nabla\) and \(\nabla'\) on \(W\) differ by an \(\text{End}(W)\)-valued logarithmic 1-form \(\Theta \in \Gamma(X, T^*X(\log \Sigma) \otimes \text{End}(W))\).

Definition 4.2. In the presence of a metric \(\langle \cdot, \cdot \rangle\) on \(W\), we say \(\nabla\) in (4.1) is compatible with the metric if
\[ d\iota(\xi) \langle \zeta_1, \zeta_2 \rangle = \langle \nabla_\xi \zeta_1, \zeta_2 \rangle + \langle \zeta_1, \nabla_\xi \zeta_2 \rangle. \]

With the definition above, for \((X, \Sigma)\) and a spin\(^c\) structure \(s = (S, \rho)\) on \(TX(-\log \Sigma)\), we need logarithmic connections \(\nabla\) and \(\nabla^S\) on \(TX(-\log \Sigma)\) and \(S\) which are compatible with the metrics on \(TX(-\log \Sigma)\) and \(W\), respectively, as well as with the Clifford multiplication:
\[ \nabla^S(\xi \cdot \Phi) = \xi \cdot \nabla^S \Phi + (\nabla\xi) \cdot \Phi. \]

As in the classical case, a compatible connection \(\nabla^S\) on \(S\) is uniquely determined by \(\nabla\) and a logarithmic connection \(A\) on the characteristic line bundle \(L_s\).

Associated to a logarithmic connection \(\nabla^S\) as above we define the \textit{logarithmic Dirac operator} to be
\[ \partial^{\log} : \Gamma(S^+) \rightarrow \Gamma(S^-), \quad \partial^{\log} \Phi = \sum_{i=1}^{4} e_i \cdot \nabla^S_{e_i} \Phi, \] (4.2)
where \(e_1, \ldots, e_4\) is an orthonormal basis for \(T_xX(-\log \Sigma)\). The metric considered on \(TX(-\log \Sigma)\) is the one described before (3.8): on a neighborhood \(D\) in \(N\), identified with a neighborhood of \(\Sigma\) in \(X\) using the map \(\Upsilon\), the metric is the direct sum of the pullback of the Kähler metric on \(T\Sigma\) and the standard riemannian metric on \(C_D\) via the identification
\[ \Upsilon^*TX(-\log \Sigma) = \pi^*T\Sigma \oplus C_D. \]

We take \(\nabla\) to be the direct sum connection on \(D\) and the Levi-Civita connection outside a larger neighborhood \((1 + \varepsilon)D\) and splice them in the middle using a convex combination with suitable smooth coefficients. For each fixed-radius circle bundle \(Y \subset D\), the restriction of \(\nabla\) to \(TY\) is the adiabatic connection mentioned in Section 2.4. Therefore, restricted to \(X - \Sigma\), via the identification (3.8), \(\nabla|_{T(X - \Sigma)}\) is the connection considered in the definition of relative moduli spaces \(\mathcal{M}(X - \Sigma, s)\).

As expected, the logarithmic Dirac operator (4.2) is not elliptic on the entire \(X\) in the classical sense of the word. The principal symbol of (4.2), which is a function on the cotangent bundle \(T^*X\), is zero on the dual space \(T\Sigma^+ \subset T^*X|_\Sigma\) of \(T\Sigma\), while it is non-zero everywhere else. This is essentially due to the fact that the homomorphism \(\iota\) maps \(\partial^z\) to \(z\partial_z\) in \(TX\), which is zero along \(\Sigma\). We expect though that a \textit{logarithmic} elliptic theory could be developed for \(\partial^{\log}\) that paves the...
way for working directly over $TX(-\log \Sigma)$ instead. In such a theory, the principal symbol of $\hat{\phi}^{\log}$ would rather be a function on the logarithmic cotangent bundle $T^*X(\log \Sigma)$, which, by analogy, is Clifford multiplication by that cotangent vector.

Next, we define the curvature of a logarithmic connection (4.1).

**Lemma 4.3.** For $\xi_1, \xi_2 \in \Gamma(X, TX(-\log \Sigma))$, there exists a unique

$$\xi := [\xi_1, \xi_2] \in \Gamma(X, TX(-\log \Sigma))$$

such that

$$\iota(\xi) = [\iota(\xi_1), \iota(\xi_2)].$$

(4.3)

**Proof.** Fix local holomorphic coordinates $(z, w) : V \rightarrow \mathbb{C}^2$ around a point of $\Sigma$ with $\Sigma \cap V = \{z \equiv 0\}$. For $a = 1, 2$, if

$$\xi_a = f_{a,1} \frac{\partial^{\log}}{\partial z} + f_{a,2} \frac{\partial^{\log}}{\partial \bar{z}} + f_{a,3} \partial_w + f_{a,4} \partial_{\bar{w}}$$

then

$$\iota(\xi_a) = f_{a,1} \partial_z + f_{a,2} \bar{\partial}_z + f_{a,3} \partial_w + f_{a,4} \partial_{\bar{w}}.$$

We have

$$[f_{1,1} \partial_z, f_{2,1} \partial_z] = \left(f_{1,1} \frac{\partial(z f_{2,1})}{\partial z} - f_{2,1} \frac{\partial(z f_{1,1})}{\partial z}\right) \partial_z,$$

$$[f_{1,2} \bar{\partial}_z, f_{2,1} \partial_z] = \left(f_{1,2} \frac{\partial(f_{2,1})}{\partial z} z \right) \partial_z - \left(f_{2,1} \frac{\partial f_{1,2}}{\partial z} \partial_z \partial \bar{z},$$

$$[f_{1,3} \partial_w, f_{2,1} \partial_z] = \left(f_{1,3} \frac{\partial(f_{2,1})}{\partial w} \right) \partial_z - \left(f_{2,1} \frac{\partial f_{1,3}}{\partial z} \partial_z \partial \bar{w},$$

$$[f_{1,4} \partial_{\bar{w}}, f_{2,1} \partial_z] = \left(f_{1,4} \frac{\partial(f_{2,1})}{\partial \bar{w}} \right) \partial_z - \left(f_{2,1} \frac{\partial f_{1,4}}{\partial z} \partial_z \partial \bar{w}.$$

Similarly, we see that (4.3) holds for the rest of the terms. Uniqueness follows from continuity and the fact that (4.3) is the same as ordinary bracket away from $\Sigma$.

It follows from Lemma 4.3 that given a logarithmic connection $\nabla$ as in (4.1), the curvature equation

$$F_\nabla(\xi_1, \xi_2)\zeta = \nabla_{\xi_1} \nabla_{\xi_2} \zeta - \nabla_{\xi_2} \nabla_{\xi_1} \zeta - \nabla_{[\xi_1, \xi_2]} \zeta$$

is well-defined. Furthermore, $F_\nabla$ is tensorial; i.e.,

$$F_\nabla \in \Gamma(X, \Lambda^2 T^*X(\log \Sigma) \otimes \text{End}(\mathcal{W})).$$

Note that $T^*X(\log \Sigma)$ and thus $\Lambda^2 T^*X(\log \Sigma)$ inherit metrics from $TX(-\log \Sigma)$. Therefore, the star operator $*$ on $\Lambda^2 T^*X(\log \Sigma)$ is defined. As in the classical case, let $\Lambda^{2,+} T^*X(\log \Sigma)$ denote the subspace of self-dual elements. As in (2.4), the homomorphism

$$\Lambda^{2,+} T^*X(\log \Sigma) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \text{End}_0(S^+), \quad \sum_{i<j} c_{ij} e_i^* \wedge e_j^* \mapsto \sum_{i<j} c_{ij} \rho(e_i) \rho(e_j)$$

(4.4)

is an isomorphism.
**Definition 4.4.** Let \((X, \Sigma)\) be a pair of a closed oriented 4-manifold \(X\) and a closed oriented 2-dimensional submanifold \(\Sigma \subset X\). Let \(s = (S, \rho)\) be a spin\(^c\) structure on \(TX(−\log \Sigma)\). We define the (unperturbed) Logarithmic Seiberg-Witten (or LSW) monopole equations to be the system of equations

\[
F_A^+ = (\Phi \Phi^*)_0, \quad \partial_A^\log \Phi = 0, \quad (\text{SW}_{\log}^4)
\]

for a pair \((A, \Phi)\) in the configuration space \(\mathcal{C}(X, s) = \mathcal{A}_{\log}(L) \times \Gamma(\mathbb{S}^+)\), where \(A\) is a logarithmic connection on the characteristic line bundle of \(s\), \(\Phi\) is a plus-spinor, and the curvature equation is with respect to the identification (4.4).

The Logarithmic Seiberg-Witten moduli space is the quotient

\[
\mathcal{M}(X, \Sigma, s) := \{(A, \Phi) \in \mathcal{A}_{\log}(L) \times \Gamma(\mathbb{S}^+) \text{ satisfying } \text{SW}_{\log}^4\} / \mathcal{G}.
\]

The restriction of \(\text{SW}_{\log}^4\) to \(X - \Sigma\) is the classical cylindrical-end \(\text{SW}^4\) equations considered in the previous sections. We expect that

\[
\mathcal{M}(X, \Sigma, s) = \mathcal{M}(X - \Sigma, s); \quad (4.5)
\]

i.e., the answer to questions (1) and (2) in Remark 3.1 is positive. Furthermore, local calculations suggest that

- the operator
  \[
  \partial_{\log} : \Gamma^2(\mathbb{S}^+) \rightarrow \Gamma^2(\mathbb{S}^-)
  \]
  is Fredholm (even though it is not elliptic), where the Banach space completions are with respect to a classical metric on \(X\) and the given metric on \(W\);

- the sequence
  \[
  0 \rightarrow \Omega^0(X; \mathbb{R}) \xrightarrow{d} \Omega^1_{\log}(X; \mathbb{R}) \xrightarrow{d^+} \Omega^2_{\log}(X; \mathbb{R}) \rightarrow 0 \quad (4.6)
  \]
  has finite cohomology.

If this is the case, then the index of \(\mathcal{E}_{X, \Sigma}(A, \Phi)\) is the sum of the indices of \(\partial_{\log}\) and (4.6), and orienting the cohomology classes of (4.6) orients the moduli space \(\mathcal{M}(X, \Sigma, s)\). We hope to shed more light on these questions in the near future.

The equality (4.5) extends to the perturbed moduli spaces considered in this paper. If \(\eta_\nu\) is a self-dual 2-form supported in \(X - \Sigma\), the definition of \(\mathcal{M}_{\eta_\nu}(X, \Sigma, s)\) is a straightforward generalization of \(\mathcal{M}(X, \Sigma, s)\). Likewise, an adapted perturbation \(\eta_\nu\) on \(X - \Sigma\) extends to a logarithmic self-dual 2-form on \(X\) with a non-trivial residue (1-form) on \(\Sigma\). Then, the definition of \(\mathcal{M}_{\eta_\nu}(X, \Sigma, s)\) is a straightforward generalization of \(\mathcal{M}(X, \Sigma, s)\) obtained by replacing the first equation in \(\text{SW}_{\log}^4\) with

\[
F_A^+ = (\Phi \Phi^*)_0 - \eta_\nu \in \Omega^2_{\log}(X; \mathbb{R}).
\]

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