Chaos of soliton systems
and special Lax pairs for chaos systems

Sen-yue Lou\textsuperscript{1,2,3,4}, Xiao-yan Tang\textsuperscript{2,3} and Ying Zhang\textsuperscript{3}

CCAST (World Laboratory), PO Box 8730, Beijing 100080, P. R. China

Physics Department of Shanghai Jiao Tong University, Shanghai 200030, P. R. China\textsuperscript{4}

\textsuperscript{3}Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

Abstract

In this letter, taking the well known (2+1)-dimensional soliton systems, Davey-Stewartson (DS) model and the asymmetric Nizhnik-Novikov-Veselov (ANNV) model, as two special examples, we show that some types of lower dimensional chaotic behaviors may be found in higher dimensional soliton systems. Especially, we derive the famous Lorenz system and its general form from the DS equation and the ANNV equation. Some types of chaotic soliton solutions can be obtained from analytic expression of higher dimensional soliton systems and the numeric results of lower dimensional chaos systems. On the other hand, by means of the Lax pairs of some soliton systems, a lower dimensional chaos system may have some types of higher dimensional Lax pairs. An explicit (2+1)-dimensional Lax pair for a (1+1)-dimensional chaotic equation is given.

PACS numbers: 02.30.Ik, 05.45.-a, 05.45.Ac, 05.45.Jn

In the past three decades, both the solitons \textsuperscript{1} and the chaos\textsuperscript{2} have been widely studied and applied in many natural sciences and especially in almost all the physics branches such as the condense matter physics, field theory, fluid dynamics, plasma physics and optics etc. Usually, one considers that the solitons are the basic excitations of the integrable models, and the chaos is the basic behavior of the nonintegrable models. Actually, the above consideration may not be complete especially in higher dimensions. When one says a model is integrable, one should emphasize two important facts. The first one is that we should point out the model is integrable under what special meaning(s). For instance, we say a model is Painlevé integrable if the model

\textsuperscript{1}Email: sylou@mail.sjtu.edu.cn

\textsuperscript{1}Mailing address
possesses the Painlevé property, and a model is Lax or IST (inverse scattering transformation) integrable if the model has a Lax pair and then can be solved by the IST approach. An integrable model under some special meanings may not be integrable under other meanings. For instance, some Lax integrable models may not be Painlevé integrable\[3, 4]. The second fact is that for the general solution of a higher dimensional integrable model, say, a Painlevé integrable model, there exist some lower dimensional *arbitrary* functions, which means any lower dimensional chaotic solutions can be used to construct exact solutions of higher dimensional integrable models.

In this letter, we will show that an IST integrable model and/or a Painlevé integrable model may have some lower dimensional reductions with chaotic behaviors and then a lower dimensional chaos system may have some higher dimensional Lax pairs.

To show our conclusions, we use the (2+1)-dimensional Davey-Stewartson (DS) equation\[5\]

\[
iu_t + 2^{-1}(u_{xx} + u_{yy}) + \alpha |u|^2u - uv = 0, \tag{1}
\]

\[
v_{xx} - v_{yy} - 2\alpha (|u|^2)_{xx} = 0, \tag{2}
\]

as a concrete example at first. The DS equation is an isotropic Lax integrable extension of the well known (1+1)-dimensional nonlinear Schrödinger (NLS) equation. The DS system is the shallow water limit of the Benney-Roskes equation\[5\], where \(u\) is the amplitude of a surface wave-packet and \(v\) characterizes the mean motion generated by this surface wave. The DS system (1) and (2) can also be derived from the plasma physics\[6\] and the self-dual Yang-Mills field. The DS system has also been proposed as a 2+1 dimensional model for quantum field theory\[7\]. It is known that the DS equation is integrable under some *special* meanings, namely, it is IST integrable and Painlevé integrable\[8\]. Many other interesting properties of the model like a special bilinear form, the Darboux transformation, finite dimensional integrable reductions, infinitely many symmetries and the rich soliton structures\[8, 9\] have also been revealed.

To select out some chaotic behaviors of the DS equation, we make the following transformation

\[
v = v_0 - f^{-1}(f_{x'x'} + f_{y'y'} + 2f_{x'y'}) + f^{-2}(f_{x'}^2 + 2f_{y'}f_{x'} + f_{y'}^2), \tag{3}
\]

\[
u = gf^{-1} + u_0 \tag{4}
\]

with real \(f\) and complex \(g\), where \(x' = (x+y)/\sqrt{2}, y' = (x-y)/\sqrt{2}\), and \(\{u_0, v_0\}\) is an arbitrary seed solution of the DS equation. Under the transformation (3) and (4), the DS system (1) and (2) is transformed to a general bilinear form:

\[
(D_{x'x'} + D_{y'y'} + 2iD_1)g \cdot f + u_0(D_{x'x'} + 2D_{x'y'} + D_{y'y'})f \cdot f
+2\alpha u_0gg^* + 2\alpha u_0^2g^*f - 2v_0gf + G_1f^0 g = 0 \tag{5}
\]
\[2(D_{x'y'} + \alpha|u_0|^2)f \cdot f + 2\alpha gh + 2\alpha g f u_0^* + 2\alpha u_0 g^* f - G_1 f f = 0, \tag{6}\]

where \( D \) is the usual bilinear operator \([10] \) defined as \( D^m x A \cdot B \equiv (\partial_x - \partial_{x_1})^m A(x)B(x_1)|_{x_1=x}, \) and \( G_1 \) is an arbitrary solution of \(-16\alpha(u_{0x'} + u_{0y'})(u_{0x'}^* + u_{0y'}^*) + G_{1x'x'} + G_{1y'y'} + 2G_{1x'y'} - 4\alpha(D_{x'x'} + D_{y'y'} + 2D_{x'y'})u_0 \cdot u_0^* = 0. \) For the notation simplicity, we will drop the “primes” of the space variables later.

To discuss further, we fix the seed solution \( \{u_0, \ v_0\} \) and \( G_1 \) as

\[u_0 = G_1 = 0, \quad v_0 = p_0(x, t) + q_0(y, t), \tag{7}\]

where \( p_0 \equiv p_0(x, t) \) and \( q_0 \equiv q_0(y, t) \) are some functions of the indicated variables.

To solve the bilinear equations (5) and (6) with (7) we make the ansatz

\[f = C + p + q, \quad g = p_1 q_1 \exp(ir + is), \tag{8}\]

where \( p \equiv p(x, t), \ q \equiv q(y, t), \ p_1 \equiv p_1(x, t), \ q_1 \equiv q_1(y, t), \ r \equiv r(x, t), \ s \equiv s(y, t) \) are all real functions of the indicated variables and \( C \) is an arbitrary constant. Substituting (8) into (5) and (6), and separating the real and imaginary parts of the resulting equation, we have

\[2p_x q_y - \alpha p_1^2 q_1^2 = 0, \tag{9}\]

\[q_1 p_{1xx} + p_1 q_{1yy} - p_1 q_1 (2r_t + 2s_t + 2(p_0 + q_0) + s_y^2 + r_x^2))(C + p + q) + p_1 q_{1yy} - 2q_1 q_y = 0 \tag{10}\]

\[(-q_1 (2r_x p_1 + 2p_1 t + p_1 r_{xx}) - p_1 (2s_y q_{1y} + 2q_{1t} + q_{1s_{yy}}))(C + p + q) + 2q_1 p_1(q_t + s_y q_y) + 2q_1 p_1(r_{xx} + p_t) = 0 \quad \tag{11}\]

Because the functions \( p_0, \ p, \ p_1 \) and \( r \) are only functions of \( \{x, \ t\} \) and the functions \( q_0, \ q, \ q_1 \) and \( s \) are only functions of \( \{y, \ t\} \), the equation system (9), (10) and (11) can be solved by the following variable separated equations:

\[p_1 = \delta_1 \sqrt{2\alpha^{-1} c_1^{-1} p_x}, \quad q_1 = \delta_2 \sqrt{c_1 q_y}, \quad (\delta_1^2 = \delta_2^2 = 1), \tag{12}\]

\[p_t = -r_x p_x + c_2, \quad q_t = -s_y q_y - c_2. \tag{13}\]

\[4(2r_t + r_x^2 + 2p_0)p_x^2 + p_{xx}^2 - 2p_{xxx} p_x + c_0 p_x^2 = 0, \tag{14}\]

\[4(2s_t + s_y^2 + 2q_0)q_y^2 + q_{yy}^2 - 2q_y q_{yyy} - c_0 q_y^2 = 0. \tag{15}\]
In Eqs. (12)–(15), $c_1$, $c_2$ and $c_0$ are all arbitrary functions of $t$.

Generally, for a given $p_0$ and $q_0$ the equation systems \{(12), (14)\} and \{(13), (15)\} may not be integrable. However, because of the arbitrariness of $p_0$ and $q_0$, we may treat the functions $p$ and $q$ as arbitrary while $p_0$ and $q_0$ are determined by (14) and (15). Because $p$ and $q$ are arbitrary functions, in addition to the stable soliton selections, there may be various chaotic selections. For instance, if we select $p$ and $q$ are solutions of (16) and (17),

$$p_{\tau_1\tau_1} = \left(p_{\tau_1\tau_1} + (c + 1)p_{\tau_1}^2\right)p^{-1} - (p^2 + bc + b)p_{\tau_1} - (b + c + 1)p_{\tau_1\tau_1} + pc(ba - b - p^2),$$

$$q_{\tau_2\tau_2} = \left(q_{\tau_2\tau_2} + (\gamma + 1)q_{\tau_2}^2\right)q^{-1} - (q^2 + \beta\gamma + \beta)q_{\tau_2} - (\beta + \gamma + 1)q_{\tau_2\tau_2} + qc(\beta\alpha - \beta - q^2),$$

where $\omega_1$, $\omega_2$, $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ are all arbitrary constants, then

$$c_0 = c_2 = 0, \quad r = -\omega_1(x + \omega_1 t/2), \quad s = -\omega_2(y + \omega_2 t/2),$$

$$p_0 = -4^{-1}\left(cp_0^3 p_{\tau_1} - 1 + p^2 - bcp_{\tau_1}^2 (a - 1)p + b(c + 1) + (b + c + 1)p_{\tau_1\tau_1} + p_{\tau_1}^2 \right),$$

$$q_0 = -4^{-1}\left(\gamma q_0^3 q_{\tau_2} - 1 + q^2 - \beta\gamma q_{\tau_2} (a - 1)q + \beta(\gamma + 1) + (\beta + \gamma + 1)q_{\tau_2\tau_2} q_{\tau_2}^{-1}\right).$$

Substituting (8) with (12)–(20) into (3) and (4), we get a general solution of the DS equation

$$u = \delta_1 \delta_2 \sqrt{2\alpha^{-1}p_{\tau_1} q_{\tau_2}} \exp(-i(\omega_1 x + \omega_2 y + \frac{1}{2}(\omega_1^2 + \omega_2^2)t))(C + p + q)^{-1},$$

$$v = p_0 + q_0 - (q_{\tau_2\tau_2} + p_{\tau_1\tau_1})(C + p + q)^{-1} + (q_{\tau_2}^2 + 2q_{\tau_2} p_{\tau_1} + p_{\tau_1}^2)(C + p + q)^{-2}$$

where $p_0$ and $q_0$ are determined by (19) and (20), while $p$ and $q$ are given by (16) and (17).

It is straightforward to prove that (16) and (17) is equivalent to the well known chaos system, the Lorenz system [11]:

$$p_{\tau_1} = -c(p - g), \quad g_{\tau_1} = p(a - h) - g, \quad h_{\tau_1} = pg - bh.$$  \hspace{1cm} (23)

Actually, after canceling the functions $g$ and $h$ in (23), one can find (16) immediately.

From the above discussions, some interesting things are worth emphasizing:

Firstly, because of the arbitrariness of the functions $p$ and $q$, any types of other lower dimensional systems may be used to construct the exact solutions of the DS system.

Secondly, the lower dimensional chaotic behaviors may be found in many other higher dimensional soliton systems. For instance, by means of the direct substitution or the similar discussions as for the DS equation, one can find that $(\tau_1 \equiv x + \omega_1 t)$

$$u = 2p_x w_y (p + w)^{-2},$$  \hspace{1cm} (24)
with \( w \) being an arbitrary function of \( y \), and \( p \) being determined by the (1+1)-dimensional extension of the Lorenz system

\[
p_t = -p_{xxx} + p^{-1}[p_{xx}p_x + (c + 1)p_x^2] - p^2p_x - (b + c + 1)p_{xx} - pc(b - ba + p^2),
\]

solves the following IST and Painlevé integrable KdV equation which is known as the ANNV model \([12]\)

\[
u_t + u_{xxx} - 3(uv)_x = 0, \quad u_x = v_y.
\]

It is clear that the Lorenz system (16) is just a special reduction of (27) with \( p = p(x + b(c + 1)t) \equiv p(\tau_1) \). Actually, \( p \) of equations (24) may also be arbitrary function of \( \{x, t\} \) after changing (25) appropriately \([12]\). In other words, any lower dimensional chaotic behavior can also be used to construct exact solutions of the ANNV system.

The third thing is more interesting. The Lax pair plays a very important and useful role in integrable models. Nevertheless, there is little progress in the study of the possible Lax pairs for chaos systems like the Lorenz system. In Ref. \([3]\), Chandre and Eilbeck had given out the Lax pairs for two discrete non-Painlevé integrable models. In Ref. \([4]\), we have found some Lax pairs for some non-integrable Schwarzian equations. Now from the above discussions, we know that both the Lax pairs of the DS equation and those of the ANNV system may be used as the special higher dimensional Lax pairs of arbitrary chaos systems like the Lorenz system (16) and/or the generalized Lorenz system (26) by selecting the fields appropriately like (21)-(22) and/or (24)-(25). For instance, the (1+1)-dimensional generalized Lorenz system (26) has the following Lax pair

\[
\psi_{xy} = u\psi, \quad \psi_t = -\psi_{xxx} + 3v\psi_x
\]

with \( \{u, v\} \) being given by (24) and (25). From (24)-(27), we know that a lower dimensional chaos system can be considered as a consistent condition of a higher dimensional linear system. For example, \( \psi_{xyt} = \psi_{txy} \) of (28) just gives out the generalized Lorenz system (26).

Now a very important question is what the effects of the lower dimensional chaos to the higher dimensional soliton systems are. To answer this question, we use the numerical solutions of the Lorenz system to see the behaviors of the corresponding solution (24) of the ANNV equation by taking \( w = 200 + \tanh(y - y_0) \equiv w_s \) and \( p = p(\tau_1) \equiv p(X) \) as the numerical solution of the Lorenz system (16). Under the selection \( w = w_s \), (24) is a line soliton solution located at \( y = y_0 \). Due to the entrance of the function \( p \), the structures of the line soliton become
Figure 1: Plot of the period two line soliton solution (24) of the ANNV equation with (29), $p = p(\tau)$, and $a = 350$, $b = 8/3$, $c = 10$ at $t = 0$.

very complicated. For some types of the parameters, the solutions of the Lorenz system have some kinds of periodic behavior, then the line soliton solution (24) with $w = w_x$ becomes an $x$ periodic line soliton solution that means the solution is localized in $y$ direction (not identically equal to zero only near $y = y_0$) and periodic in $x$ direction. Fig. 1 shows the behavior of the periodic two line soliton solution and $p$ being a periodic two solution of the Lorenz system (16) with $a = 350$, $b = 8/3$ and $c = 10$. In some other types of the parameter ranges, the solution of the Lorenz system becomes chaotic and then the line soliton of the ANNV system becomes chaotic line soliton which is localized in $y$ direction and chaotic in $x$ direction. Fig. 2 displays the chaotic behavior of the amplitude of the line soliton located at $y = y_0$ with $a = 60, b = 8/3$ and $c = 10$. The parameters we used here are the same as those used in literature[11].

In summary, though some (2+1)-dimensional soliton systems, like the DS equation and the ANNV equation, are Lax and IST integrable, and some special types of soliton solutions can be found by IST and other interesting approaches[8], any types of chaotic behaviors may still be allowed in some special ways. Especially, the famous chaotic Lorenz system and its (1+1)-dimensional generalization are derived from the DS equation and the ANNV equation. Using the numerical results of the lower dimensional chaotic systems, we may obtain some types of nonlinear excitations like the periodic and chaotic line solitons for higher dimensional soliton systems. On the other hand, the lower dimensional chaos systems like the generalized Lorenz system may have some particular Lax pairs in higher dimensions.

It is also known that both the ANNV system and the DS systems are related to the Kadomtsev-Petviashvili (KP) equation while the DS and the KP equation are the reductions of the self-dual Yang-Mills (SDYM) equation. So both the KP and the SDYM equations may
Figure 2: Plot of the chaotic line soliton solution (24) of the ANNV equation with the same conditions as Fig. 1 except for $a = 60$.

possess arbitrary lower dimensional chaotic behaviors induced by arbitrary functions. From the results of this paper, various important and interesting problems should be studied further. For instance, what on earth is the complete integrability and what kinds of information about chaos can be obtained from some types of special higher dimensional Lax pairs?

The author is in debt to thanks the helpful discussions with the professors Q. P. Liu and G-x Huang. The work was supported by the National Outstanding Youth Foundation of China (No.19925522), the Research Fund for the Doctoral Program of Higher Education of China (Grant. No. 2000024832) and the National Natural Science Foundation of Zhejiang Province of China.

References

[1] Y. S. Kivshar and B. A. Malomend, Rev. Mod. Phys. 61, 765 (1989).

[2] J. P. Gollub and M. C. Cross, Nature, 404, 710 (2000); R. A. Jalabert and H. M. Pastawski, Phys. Rev. Lett. 86, 2490 (2001).

[3] C. Chandre and J. C. Eilbeck, Does the existence of a Lax pair imply integrability, Preprint (1997).

[4] S-y Lou, X-y Tang, Q-P Liu and T. Fukuyama, nlin.SI/0108045 (2001).

[5] A. Davey and K. Stewartson, Proc. R. Soc. A, 338, 101 (1974).
[6] K. Nishinari and K. Abe and J. Satsuma, J. Phys. Soc. Japan, 62, 2021 (1993).

[7] C. L. Schultz, M. J. Ablowitz and D. BarYaacov, Phys. Rev. Lett. 59, 2825 (1987).

[8] M. Boiti, J. J. P. Leon, L. Martina and F. Penpinelli, Phys. Lett. 132A, 432 (1988); A. S. Fokas and P. M. Santini, Phys. Rev. Lett. 63, 1329 (1989); R. A. Leo, G. Mancarella, G. Soliani and L. Solombrino, J. Math. Phys. 29 (1988) 2666.

[9] J. Hietarinta, Phys. Lett. 149A, 133 (1990); V. B. Matveev and M. A. Salle, *Darboux Transformation and Solitons*, (Springer-Verlag 1991); Z-x Zhou, Inverse Problems, 14, 1371 (1998); S-y Lou and X-b Hu, J. Phys. A: Math. Gen. 27, L207 (1994); S-y Lou, and Lu J-z, J. Phys. A: Math. Gen. 29, 4209 (1996).

[10] R. Hirota, Phys. Rev. Lett. 27, 1192 (1971).

[11] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Appl. Math. Sci. 41, (1982 Springer-Verlag New York).

[12] M. Boiti, J. J. P. Leon, M. Manna and F. Penpinelli, Inverse Problems, 2, 271(1986); S-y Lou and H-y Ruan, ibid. 34, 305 (2001).