Validity of Goldstone theorem at two loops in noncommutative $U(N)$ linear sigma model

Yi Liao
Institut für Theoretische Physik, Universität Leipzig,
Augustusplatz 10/11, D-04109 Leipzig, Germany

Abstract

The scalar theory is ultraviolet (UV) quadratically divergent on ordinary spacetime. On noncommutative (NC) spacetime, this divergence will generally induce pole-like infrared (IR) singularities in external momenta through the UV/IR mixing. In spontaneous symmetry breaking theory this would invalidate the Goldstone theorem which is the basis for mass generation when symmetry is gauged. We examine this issue at two loop level in the $U(N)$ linear $\sigma$ model which is known to be free of such IR singularities in the Goldstone self-energies at one loop. We analyze the structures in the NC parameter ($\theta_{\mu\nu}$) dependence in two loop integrands of Goldstone self-energies. We find that their coefficients are effectively once subtracted at the external momentum $p = 0$ due to symmetry relations between 1PI and tadpole contributions, leaving a final result proportional to a quadratic form in $p$. We then compute the leading IR terms induced by NC to be of order $p^2 \ln(\theta_{\mu\nu})^2$ and $p^2 \ln \tilde{p}^2$ ($\tilde{p}_\mu = \theta_{\mu\nu} p^\nu$) which are much milder than naively expected without considering the above cancellation. The Goldstone bosons thus keep massless and the theorem holds true at this level. However, the limit of $\theta \to 0$ cannot be smooth any longer as it is in the one loop Goldstone self-energies, and this nonsmooth behaviour is not necessarily associated with the IR limit of the external momentum as we see in the term of $p^2 \ln(\theta_{\mu\nu})^2$.

PACS: 11.30.Qc, 02.40.Gh, 11.10.Gh
Keywords: noncommutative field theory, spontaneous symmetry breaking, UV/IR mixing
1 Introduction

Quantum field theory on noncommutative (NC) spacetime may be formulated in terms of the Moyal-Weyl correspondence \[1\]. Namely, one still works on commutative spacetime but replaces the usual product of functions by the star product,

\[
(f_1 \star f_2)(x) = \left[ \exp \left( \frac{i}{2} \theta_{\mu\nu} \partial_\mu \partial_\nu \right) f_1(x) f_2(y) \right]_{y=x},
\]

where \(x, y\) are the usual commutative coordinates and \(\theta_{\mu\nu}\) is a real, antisymmetric, constant matrix characterizing the noncommutativity of spacetime, \([\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}\). At the classical action level, while the star product in the bilinear terms may be identified with the usual one for rapidly decaying functions at spacetime infinity, it does modify the interaction terms by introducing a phase which in momentum space depends on \(\theta_{\mu\nu}\) and the momenta of fields involved. At the quantum level the phase results in a new feature never seen in ordinary field theory, the ultraviolet-infrared (UV/IR) mixing \[2\]. The basic mechanism for this occurrence may be understood as follows. When an otherwise UV divergent loop integral is multiplied by a phase depending on both the loop momentum \(k\) and the external momentum \(p\), e.g., \(\exp(i/2\theta_{\mu\nu} k^\mu p^\nu)\), it may become UV convergent due to the rapid oscillation of the phase in the UV regime. However, this improvement of the UV convergence is effective only for a nonvanishing external momentum (or more precisely for a nonvanishing NC momentum \(\tilde{p}_\mu = \theta_{\mu\nu} p^\nu\)). The hidden singularity from the UV loop momentum will reappear as a new form when the external momentum goes to the IR limit. Depending on the degree of divergence of the loop integral, this NC IR singularity may be pole-like, logarithmic, etc.

The above NC IR singularity, especially the pole-like one, may cause serious problems in NC field theory. It leads to a drastic modification to dispersion relation at low energy in perturbation theory which may make the theory not well-defined in the IR. When going beyond one loop level it may destroy or at least make unclear the renormalizability of the theory. Indeed, most of explicit model analyses made so far are restricted to the one loop level and their renormalization is considered for nonexceptional NC external momenta \[3 - 9\]. For exceptional ones we would have to choose a different subtraction scheme. The real \(\phi^4\) theory has been examined at two loops \[10\], but again the main concern is with the UV regime of loop momenta for nonexceptional NC momenta. For complex \(\phi^4\) theory with spontaneous symmetry breaking the IR behaviour becomes important as it
is related to the issue of whether the Goldstone theorem still holds true on NC space-time, namely, whether the masslessness of Goldstone bosons is stable against radiative corrections. This is a starting point to all attempts of realistic model building including weak interactions [11]. The complex $U(N) \sigma$ model has been studied in this context at one loop in Refs. [6], and it was found that there are no NC IR singularities at all in the self-energy of Goldstone bosons so that their masslessness is guaranteed at this order: both pole-like and logarithmic ones are cancelled in the mass correction due to the delicate relations governed by the spontaneously broken symmetries as occurring in the commutative theory. This is a surprising result since the scalar theory is UV quadratically divergent. It would be highly desirable to investigate whether this is a special feature at one loop, or more importantly whether NC IR singularities at higher orders, if any, endanger the masslessness of Goldstone bosons making the theorem no longer valid on NC spacetime. Naively speaking, this should not be surprising if it occurs. Beyond one loop, the would-be NC IR singularities for external momenta at one loop now appear as an internal part of higher loops; it is not clear whether they persist to be cancelled. Even worsely, they may combine with remaining massless Goldstone bosons to enhance the IR behaviour in external momenta. It is the purpose of the present work to clarify these problems by an explicit two loop analysis. Our main results may be summarized as follows. At two loop level, there are no IR terms more singular than $p^2 \ln \tilde{p}^2$, and individual stronger singularities at intermediate steps are finally cancelled due to symmetry relations. The Goldstone bosons thus keep massless and the Goldstone theorem holds valid at this order in perturbation theory. We also point out the difference between the NC singularity in the limit of $\theta_{\mu\nu} \to 0$ and the NC IR singularity in the limit of $p \to 0$. For the self-energies of Goldstone bosons obtained we have the NC behaviour of $p^2 \ln \theta_{\mu\nu}^2$ and $p^2 \ln \tilde{p}^2$. While the latter is leading in the IR limit, both are singular in $\theta$: we have NC singularities at higher orders independently of external momentum configurations.

In the next section we describe the NC $U(N)$ linear $\sigma$ model, whose Feynman rules are reproduced in appendix A. Then we present a detailed two loop analysis in section 3. Some examples of two loop integrals involving $\theta_{\mu\nu}$ are shown in appendix B. We conclude in the last section.
2 The model

We follow the same conventions as in Ref. [5] in describing the NC $U(N)$ linear $\sigma$ model. The complex scalar $\Phi$ is in the fundamental representation of $U(N)$ with

$$\mathcal{L} = (\partial_{\mu} \Phi)^\dagger \ast \partial^\mu \Phi + \mu^2 \Phi^\dagger \ast \Phi - \lambda \Phi^\dagger \ast \Phi \ast \Phi^\dagger \ast \Phi.$$ (2)

The spontaneous symmetry breaking is triggered by the non-vanishing scalar VEV, assuming $\mu^2, \lambda > 0$,

$$\Phi = \phi + \phi_0,$$

$$\phi = \left(\pi_1, \ldots, \pi_{N-1}, (\sigma + i\pi_0)/\sqrt{2}\right)^T,$$

$$\phi_0 = \left(0, \ldots, 0, v/\sqrt{2}\right)^T,$$

with $v = \sqrt{\mu^2/\lambda}$. The $\sigma$ field is the Higgs boson with mass $m = \sqrt{2\lambda v^2}$ and the $\pi_0$ and $\pi_i (i = 1, \ldots, N - 1)$ fields are the real and complex Goldstone bosons. We have ignored other possible orderings of interaction like $\Phi_i^\dagger \ast \Phi_j^\dagger \ast \Phi_i \ast \Phi_j$ which are problematic already at one loop [5][6]. In terms of the shifted fields, we have

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \sigma)^2 - \frac{1}{2} m^2 \sigma^2 + \frac{1}{2} (\partial_{\mu} \pi_0)^2 + \frac{1}{2} \partial_{\mu} \pi_i^\dagger \partial^\mu \pi_i$$

$$- \lambda v \sigma (\sigma^2 + \pi_0^2 + 2\pi_i^\dagger \pi_i)$$

$$- \lambda \left(\frac{1}{4} (\sigma^4 + \pi_0^4) + \pi_i^\dagger \pi_i \pi_j^\dagger \pi_j\right)$$

$$- \lambda \left(\sigma^2 \pi_0^2 - \frac{1}{2} \sigma \pi_0 \sigma \pi_0 + (\sigma^2 + \pi_0^2) \pi_i^\dagger \pi_i\right)$$

$$- \lambda i \sigma [\sigma, \pi_0] \pi_j^\dagger \pi_j,$$ (4)

where we have suppressed the star notation and dropped terms which vanish upon integration over spacetime. The perturbation theory is based on the above Lagrangian. The one loop calculation has been done in Refs. [5][6]. We now proceed to consider two loop contributions in the next section.

3 Two loop contributions

There are three sets of contributions at two loop level: bare two loop diagrams, one loop diagrams with one insertion of counterterms determined at one loop, and the counterterms determined at two loops. It is clear that the third causes no IR problem. We start with the second which is just a one loop calculation.
3.1 One loop diagrams with one insertion of one loop counter-terms

The contributing diagrams are shown in Figs. 1 and 2 where the solid and dashed lines are for $\sigma$ and $\pi_{0,i}$ fields respectively. As we are explicitly including the tadpole contributions, we shall not impose the requirement of tadpole cancellation, nor introduce a counterterm for the VEV. The counterterms for the self-energies are respectively,

$$
\pi_0, \pi_j : \quad i[p^2 \delta Z_\phi - m^2 \delta \pi]
$$

$$
\sigma : \quad i[(p^2 - m^2) \delta Z_\phi - m^2 \delta \sigma].
$$

(5)

The vertex counterterms are obtained simply by attaching a factor of $\delta Z_\lambda$ to their Feynman rules. The quantities $\delta Z_{\phi, \lambda}, \delta_{\pi, \sigma}$ are renormalization constants whose details may be found, e.g., in Refs. [8]. For our purpose here, it is sufficient to know that

$$
\delta_\sigma - \delta_\pi = \delta Z_\lambda - \delta Z_\phi,
$$

(6)

which arises due to their different mass and renormalization.

---

(a) ![Diagram](a)  
(b) ![Diagram](b)  
(c) ![Diagram](c)  
(d) ![Diagram](d)  

Fig. 1: 1PI contributions to $\pi_0$ or $\pi_j$ self-energy.

(e) ![Diagram](e)  
(f) ![Diagram](f)  
(g) ![Diagram](g)  
(h) ![Diagram](h)  

Fig. 2: Tadpole contributions to $\pi_0$ or $\pi_j$ self-energy.
Let us first consider the part proportional to $\delta Z_\phi$. We have,
\[
[(1a) + (1b) + (1c) + (2a) + (2b)]_{\delta Z_\phi} = (\text{ one loop result }) \times (-\delta Z_\phi),
\]
which is free of NC IR singularities according to Refs. [5][6]. The remaining $\delta Z_\phi$ dependence will be given below together with that of $\delta Z_\lambda$. Next, consider the part proportional to $\delta Z_\lambda$. We have similarly,
\[
[(1e) + (1f) + (1g) + (2c) + (2d)]_{\delta Z_\lambda} = (\text{ one loop result }) \times \delta Z_\lambda,
\]
which again is safe. The remaining $\delta Z_\phi$ and $\delta Z_\lambda$ dependence is,
\[
[(1d)_{\delta Z_\phi} + (1h)] = \lambda m^2(\delta Z_\lambda - \delta Z_\phi) \\
\times \left\{ \begin{array}{l}
2\delta_{ij} [J(0, m) + \cdots], \text{ for } \pi_i^\dagger \pi_j \\
[J(0, m) + J_{\theta,p}(0, m) + \cdots], \text{ for } \pi_0 \pi_0
\end{array} \right.
\]
where the dots stand for the terms which are both UV (loop momentum) and IR (external momentum) finite. Following Ref. [5] we have introduced similar notations for integrals,
\[
J(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2}, \quad J(0, m) = \int \frac{d^4k}{(2\pi)^4} \frac{\cos(2k \wedge p)}{(k^2 - m^2)^2}, \\
J_{\theta,p}(0) = \int \frac{d^4k}{(2\pi)^4} \frac{\cos(2k \wedge p)}{(k^2)^2}, \quad J_{\theta,p}(m) = \int \frac{d^4k}{(2\pi)^4} \frac{\cos(2k \wedge p)}{(k^2 - m^2)^2},
\]
Our manipulations will be independent of schemes used to regularize divergences in the above integrals. Now we compute the $\delta_{\sigma,\pi}$ terms and obtain,
\[
[(1a) + (2a)]_{\delta \sigma} = +\lambda m^2 \delta_{\pi} \left\{ \begin{array}{l}
\delta_{ij} 2J(0) \\
2 \delta_{ij} 2J(0) \\
\delta_{ij} 2J(m) \\
\delta_{ij} 2[J(0, m) - J(0)] + \cdots
\end{array} \right.
\]
Using eqn. (3), the IR singularities are cancelled in the sum of eqns. (9) and (11) leaving behind an IR safe result proportional to $p^2$.

### 3.2 Two loop diagrams

Now we calculate the genuine two loop contributions. The 1PI diagrams are depicted in Figs. 3 and 4 where we only show topologically different graphs with the solid line
representing all scalar fields. The number appearing as a subscript refers to the number of diagrams actually involved.

\[
\begin{align*}
(a)_9 & \quad (b)_5 & \quad (c)_3 \\
\end{align*}
\]

Fig. 3 Two loop 1PI contributions to σ tadpole.

\[
\begin{align*}
(a)_9 & \quad (b)_5 & \quad (c)_4 & \quad (d)_6 \\
\end{align*}
\]

Fig. 4: Two loop 1PI contributions to π₀ or πₗ self-energy.

The 1PI σ tadpole is found to be,

\[
i T^{1\text{PI}} = i\lambda^2 v \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} T(k_i),
\]

\[
T(k_i) = T_a + T_b + T_c,
\]

where, using the notations \(D_m(q) = (q^2 - m^2)^{-1}\), \(D(q) = (q^2)^{-1}\) and \(K_{12} = \cos(2k_1 \wedge k_2)\), we have,

\[
T_a = +[3D_m^2(k_1)D_m(k_2) + D^2(k_1)D(k_2)][2 + K_{12}]
+3D_m^2(k_1)D(k_2) + D^2(k_1)D_m(k_2)][2 - K_{12}]
+(N - 1)[6D_m^2(k_1)D(k_2) + 4D^2(k_1)D(k_2) + 2D^2(k_1)D_m(k_2)]
+4(N - 1)ND^2(k_1)D(k_2),
\]

\[
T_b/m^2 = +[27/2D_m^2(k_1)D_m(k_2)D_m(k_1 + k_2) + 3/2D_m^2(k_1)D(k_2)D(k_1 + k_2)
+D^2(k_1)D(k_2)D_m(k_1 + k_2)][1 + K_{12}]
+(N - 1)[6D_m^2(k_1)D(k_1 + k_2) + 4D^2(k_1)D_m(k_1 + k_2)]D(k_2),
\]
\[ T_c = +[3D_m(k_1)D_m(k_2) + D(k_1)D(k_2)]D_m(k_1 + k_2)(1 + K_{12}) \\
+4(N - 1)D(k_1)D(k_2)D_m(k_1 + k_2). \quad (15) \]

Note that we can have \( \theta \) dependence beyond one loop even if the external momentum vanishes since there are independent loop momenta which can combine with the antisymmetric \( \theta_{\mu\nu} \). The result will depend on it through \( \theta^2 = \theta_{\mu\nu}\theta^{\mu\nu} \), etc. As long as we do not use the Lorentz covariance to choose a specific frame for external momenta, we can always treat \( \theta \) in integrals as if it were a Lorentz tensor.

Upon choosing loop momenta properly in some integrals, the 1PI self-energy of the charged Goldstone bosons \( \pi_i^+\pi_j \) is found to be,

\[ i\Sigma_{ij}^{1PI}(p) = i\lambda^2\delta_{ij} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} U(k_i, p), \]

\[ U(k_i, p) = \sum_{x=a}^b U_x, \quad (16) \]

\[
U_a = +[D_m^2(k_1)D_m(k_2) + D^2(k_1)D(k_2)](2 + K_{12}) \\
+D^2(k_1)D_m(k_2)D(k_1 + k_2)(1 + K_{12}) \\
+2(N - 1)[D_m^2(k_1) + D^2(k_1)]D(k_2) \\
+2N[D^2(k_1)D_m(k_2) + D^2(k_1)D(k_2)] \\
+4N^2D^2(k_1)D(k_2), \quad (17) \]

\[
U_b/m^2 = +[9/2D_m^2(k_1)D_m(k_2)D_m(k_1 + k_2) + 1/2D_m^2(k_1)D(k_2)D(k_1 + k_2) \\
+D^2(k_1)D_m(k_2)D(k_1 + k_2)](1 + K_{12}) \\
+2(N - 1)D_m^2(k_1)D(k_2)D(k_1 + k_2) \\
+4ND^2(k_1)D_m(k_2)D(k_1 + k_2), \quad (18) \]

\[
U_c/m^4 = +D_m^2(k_1)D(k_1 + p)[9D_m(k_2)D_m(k_1 + k_2) \\
+D(k_2)D(k_1 + k_2)](1 + K_{12}) \\
+4D^2(k_1)D_m(k_1 + p)D_m(k_2)D(k_1 + k_2) \\
+4(N - 1)D_m^2(k_1)D(k_1 + p)D(k_2)D(k_1 + k_2), \quad (19) \]

\[
U_d/m^2 = +2D_m^2(k_1)D(k_1 + p)D_m(k_2)(2 + K_{12}) \\
+2D^2_m(k_1)D(k_1 + p)D(k_2)(2 - K_{12}) \\
+2D^2(k_1)D_m(k_1 + p)[D_m(k_2) + D(k_2)] \\
+4(N - 1)D_m^2(k_1)D(k_1 + p)D(k_2) \\
+4ND^2(k_1)D_m(k_1 + p)D(k_2), \quad (20) \]

\[
U_e/m^2 = +2D_m(k_1)D(k_1 + p)D_m(k_2)D(k_2 + p)(1 + K_{12}), \quad (21) \]

\[
U_f = +[D_m(k_1)D_m(k_2) + D(k_1)D(k_2)]D(k_1 + k_2 - p)(1 + K_{12}) \\
+2D_m(k_1)D(k_2)D(k_1 + k_2 - p)(1 - K_{12}) \\
+4[(N - 1)D(k_1)D(k_2) \\
+D(k_1 + p)D(k_2 + p)K_{12}]D(k_1 + k_2 + p), \quad (22) \]

\[
U_g/m^4 = +6D_m(k_1)D(k_1 + p)D_m(k_2)D(k_2 + p)D_m(k_1 - k_2)(1 + K_{12}) \\
+4D_m(k_1)D(k_1 + p)D(k_2 + p)D_m(k_2)D(k_1 + k_2 + p)K_{12}, \quad (23) \]

8
\[ U_{h/m^2} = 4D_m(k_1)D(k_1 + p)[6D_m(k_2)D_m(k_1 + k_2) + 2D(k_2)D(k_1 + k_2) + 4D_m(k_2)D(k_1 + k_2 + p)](1 + K_{12}) + B D_m(k_1)D(k_1 + p)[(N - 1)D(k_2)D(k_1 + k_2) + D(k_2 - p)D(k_1 + k_2 - p)K_{12}] \]

Although the 1PI self-energy of the neutral Goldstone boson \( \pi_0 \) has the same set of diagrams as the charged one, it becomes more complicated due to multiplications of trigonometric functions involving the loop and external momenta and \( \theta \). To simplify our analysis of the NC IR behaviour, it is useful to cast these products into standard forms. For the self-energy at two loops, we have three independent momenta in the integrand, \( k_1, k_2 \) and \( p \) so that we can form two independent combinations with \( \theta, 2k_1 \wedge k_2 \) and \( 2k_1 \wedge p \). \( (2k_2 \wedge p \) is not independent as it can be obtained from \( 2k_1 \wedge p \) by \( k_1 \leftrightarrow k_2 \). We find that it is always possible by shifting and interchanging loop momenta properly so that the only \( \theta \) dependence in integrands enters through either the above \( K_{12} \) or \( K_1 = \cos(2k_1 \wedge p) \). For example, the simple-looking Fig. 4(e) in this case involves the following product,

\[
\cos(p \wedge k_1) \cos(p \wedge k_2) \\
\times \{2 \cos(k_1 \wedge k_2) \cos((k_1 + p) \wedge (k_2 + p)) - \cos((k_1 + k_2) \wedge p)\} \\
= 1/4 \{\cos(2k_1 \wedge k_2) + \cos[2k_1 \wedge k_2 + 2(k_1 - k_2) \wedge p] \\
+ \cos[2(k_1 + p) \wedge k_2] + \cos[2(k_2 + p) \wedge k_1] \\
+ \cos[2(k_1 - k_2) \wedge p] - \cos[2(k_1 + k_2) \wedge p]\},
\]

where the five non-standard forms may be transformed into the standard ones, e.g.,

\[
\cos[2k_1 \wedge k_2 + 2(k_1 - k_2) \wedge p] \rightarrow K_{12}, \text{ with } k_i \rightarrow -k_i - p, \\
\cos[2(k_1 + k_2) \wedge p] \rightarrow K_1, \text{ with } k_1 \rightarrow -k_1 - k_2.
\]

After this manipulation, the expression for the 1PI \( \pi_0 \) self-energy becomes very lengthy though it has a simpler structure in \( \theta \),

\[
i\Sigma^{1\text{PI}}_{\pi_0}(p) = i\lambda^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} V(k_i, p) \\
V(k_i, p) = \sum_{x=a} V_x,
\]

\[
V_a = D_m^2(k_1)[D_m(k_2)2(2 - K_1 + K_{12}) - D_m(k_2 + p)K_{12}] \\
+ D_m^2(k_1)[D(k_2)2(2 - K_1 - K_{12}) + D(k_2 + p)K_{12}] \\
+ D^2(k_1)[D_m(k_2)2(2 + K_1 - K_{12}) - D_m(k_2 + p)K_{12}] \\
+ D^2(k_1)[D(k_2)2(2 + K_1 + K_{12}) + D(k_2 + p)K_{12}] \\
+ 2(N - 1)\{[D_m^2(k_1)(2 - K_1) + D^2(k_1)(2 + K_1)]D(k_2) \\
+ D^2(k_1)[D_m(k_2) + D(k_2)]\} \\
+ 4(N - 1)N D^2(k_1)D(k_2),
\]

9
\[ V_b/m^2 = +9/2D_m^2(k_1)[D_m(k_2)D_m(k_1 + k_2)(2 - K_1 + 2K_{12})
- D_m(k_2 + p)D_m(k_1 + k_2 + p)K_{12}]
+ 1/2D_m^2(k_1)[\text{same as above except } m \to 0]
+ D^2(k_1)[D_m(k_2)D(k_1 + k_2)(2 + K_1 + 2K_{12})
+ D_m(k_2 + p)D(k_1 + k_2 + p)K_{12}]
+ 2(N - 1)[D_m^2(k_1)D(k_2)(2 - K_1)
+ 2D^2(k_1)D_m(k_2)]D(k_1 + k_2), \] (29)

\[ V_c/m^4 = +9/4D_m^2(k_1)D(k_1 + p)[D_m(k_2)D_m(k_1 + k_2)2(1 + K_1 + K_{12})
+ [D_m(k_2 + p)D_m(k_1 + k_2 + p) + (p \to -p)]K_{12}]
+ 1/2D_m^2(k_1)D_m(k_1 + p)[\text{same as above except } m \to 0 \text{ in each } 2\text{nd } D}
+ 1/4D_m^2(k_1)D(k_1 + p)[\text{same as above except } m \to 0]}
+ 2(N - 1)D_m^2(k_1)D(k_1 + k_2)D(k_2)(1 + K_1), \] (30)

\[ V_d/m^2 = +D_m^2(k_1)D(k_1 + p)[[D_m(k_2)(2 + 2K_1 + K_{12}) + D_m(k_2 + p)K_{12}]
+ [D(k_2)(2 + 2K_1 - K_{12}) - D(k_2 + p)K_{12}]]
+ D^2(k_1)D_m(k_1 + p)[[D_m(k_2)(2 + 2K_1 - K_{12}) - D_m(k_2 + p)K_{12}]
+ [D(k_2)(2 + 2K_1 + K_{12}) + D(k_2 + p)K_{12}]]
+ 2(N - 1)[D_m^2(k_1)D(k_1 + p) + D^2(k_1)D_m(k_1 + p)]D(k_2)(1 + K_1), \] (31)

\[ V_e/m^2 = +[D(k_1 + p)D_m(k_1) + D(k_1)D_m(k_1 + p)]
\times [D(k_2 + p)D_m(k_2) + D(k_2)D_m(k_2 + p)]K_{12}
+ D_m(k_2)[D(k_2 + p) - D(k_2 - p)]D_m(k_1 + k_2)D(k_1 + k_2 + p)K_1, \] (32)

\[ V_f = +[D_m(k_1 + p)D_m(k_2)D(k_1 + k_2)(3 - 4K_{12})
- 2[D_m(k_1)D_m(k_2 + p) + D_m(k_1 + p)D_m(k_2)]D(k_1 + k_2)K_1
+ 2D(k_1)D_m(k_2)D_m(k_1 + k_2 - p)K_1]
+ [2D_m(k_1)D_m(k_2) + 2D_m(k_1 - p)D_m(k_2 + p)
+ D_m(k_1 + p)D_m(k_2 + p)]D(k_1 + k_2 + p)K_{12}]
+ 1/3[\text{same as above except } \to + \text{ in first two lines}
\text{and all } m \to 0}]
+ 2(N - 1)D(k_2)[D(k_1 + p)D_m(k_1 + k_2) + D(k_1)D(k_1 + k_2 - p)K_1]
+ 2(N - 1)D(k_2)[D(k_1 + p)D_m(k_1 + k_2) - D_m(k_1)D(k_1 + k_2 - p)K_1], \] (33)

\[ V_g/m^4 = +3/2[D_m(k_1)D(k_1 + p)D_m(k_2)[D(k_2 + p)D_m(k_1 + k_2 - 2K_{12})
+ D_m(k_1)D(k_1 + p)D_m(k_2)[D(k_2 + p)D_m(k_1 + k_2 - p)K_1]
+ D_m(k_1 - p)D(k_1)D_m(k_2 - p)D(k_2)D_m(k_1 + k_2 - 2K_{12})]
+ [D_m(k_1)D(k_1 + p)D_m(k_2)]D(k_1 + k_2 - 2K_{12})
+ D_m(k_1 + k_2)D(k_1 + k_2 + p)D_m(k_2)D_m(k_1)]D(k_2 + p)K_1}\}
+ 1/2[\text{same as above but interchanging masses in}
\text{3rd and 4th } D \text{ and } m \to 0 \text{ in } 5\text{th } D}, \] (34)
\[ V_{ih}/m^2 = +3\{D_m(k_1)D(k_1 + p)D_m(k_2)D_m(k_1 + k_2)2(1 + K_1 + K_{12}) + D_m(k_1)D(k_1 + p)[D_m(k_2 + p)D_m(k_1 + k_2 + p) + (p \to -p)]K_{12} - [D_m(k_2)D(k_2 + p)D_m(k_1)D_m(k_1 + k_2) + (k_2 \to -k_1 - k_2)]K_1 - D_m(k_1 - p)D(k_1)[D_m(k_2)D_m(k_1 + k_2 - p) + (k_2 \to k_2 + p)]K_{12}\} + \{\text{same as above except } m \to 0 \text{ in 3rd and 4th } D \text{ and } - \to + \text{ in last two lines}\} + 2\{D(k_1)D_m(k_1 + p)[D_m(k_2 - p)D(k_1 + k_2 - p) - (p \to -p)]K_{12} + D(k_1 - p)D_m(k_1)[D_m(k_2)D(k_1 + k_2 - p) + (k_2 \to k_2 + p)]K_{12} + [D_m(k_2)D(k_1)D_m(k_1 + k_2 + p) + D_m(k_1)D(k_2)D_m(k_2 + p)]D(k_1 + k_2)K_1\} + 4(N - 1)D_m(k_1)D(k_1 + p)D(k_2)D(k_1 + k_2)(1 + K_1). \]

Including the tadpole contributions, we have the self-energies for Goldstone bosons,

\[ i \Sigma_{ij}(p) = i\lambda^2 \delta_{ij} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} [U(k_i, p) - T(k_i)], \]

\[ i \Sigma_{00}(p) = i\lambda^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} [V(k_i, p) - T(k_i)], \]

which have the following structure in \( \theta \),

\[ i \Sigma_{ij}(p) = i\lambda^2 \delta_{ij} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} [f_0(k_i, p) + f_{12}(k_i, p)K_{12}], \]

\[ i \Sigma_{00}(p) = i\lambda^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} [g_0(k_i, p) + g_{12}(k_i, p)K_1 + g_{12}(k_i, p)K_{12}], \]

where all \( \theta \) dependence resides in \( K \) factors. A crucial observation from the above explicit expressions is that all form factors \( f \) and \( g \) vanish at \( p = 0 \) so that we are effectively subtracting at \( p = 0 \) for each form factor when doing integrations,

\[ i \Sigma_{ij}(p) = i\lambda^2 \delta_{ij} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} [\overline{f}_0(k_i, p) + \overline{f}_{12}(k_i, p)K_{12}], \]

\[ i \Sigma_{00}(p) = i\lambda^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} [\overline{g}_0(k_i, p) + \overline{g}_{12}(k_i, p)K_1 + \overline{g}_{12}(k_i, p)K_{12}], \]

where \( \overline{f}_0(k_i, p) = f_0(k_i, p) - f_0(k_i, 0), \overline{g}_0(k_i, p) = g_0(k_i, p) - g_0(k_i, 0) \), etc. This cancellation in the \( \theta \) independent and \( K_{12} \) structures between the 1PI and tadpole contributions originates from symmetry relations among vertices summarized in the Ward identities. The vanishing of \( g_1(k_i, 0) \) is a feature of the 1PI part alone which fits in the requirement of the Goldstone theorem to be verified here. All of this also serves as a good test of the correctness of the calculation.

Now we proceed to consider the NC IR behaviour in the total self-energies. The \( \overline{f}_0 \) and \( \overline{g}_0 \) terms are independent of \( \theta \), proportional to \( p^2 \) as in the commutative theory and thus IR safe. The \( \theta \) dependent terms are proportional to some factors quadratic in \( p \) due to
the subtraction, but in principle not necessarily to $p^2$. With the constant antisymmetric tensor $\theta_{\mu\nu}$, we may construct symmetric and antisymmetric ones by contraction, e.g., $\theta^2_{\mu\nu} = \theta_{\mu\rho} \theta^\rho_{\nu}$, which may be used to build new NC momenta like $\tilde{p}_\mu = \theta^2_{\mu\rho} p^\rho$. Thus the proportionality factors can be $p^2$, $p \cdot \tilde{p} = \tilde{p}^2$, etc. The task here is to show that negative powers of these scalars never appear in the final results so that the IR safety is guaranteed in the self-energies. We shall show that the above quadratic $p$ factors will be multiplied by a leading factor of order $\ln \theta^2$ or $\ln \tilde{p}^2$. (The $p^2 \ln p^2$ behaviour is not new as it already appears in commutative theory.) Thus the self-energies are IR safe but cannot go to the commutative limit smoothly.

Generally speaking, an integral with $K_1$ or $K_{12}$ will be IR safe in the external momentum if it is already convergent both superficially and in subgraphs without these factors. For divergent integrals either superficially or in subgraphs, we must be careful. Let us first consider the $K_{12}$ term. We shall demonstrate our calculation by some typical terms in $f_{12}(k_1, p)$. The case of $g_{12}$ is similar but much more complicated due to the momentum shifts which introduce $p$ in many propagators. Using $A_1(p)A_2(p) = A_1(p)A_2(p) + A_1(p)A_2(0)$, etc, we can always subtract sequentially, so that the only complication in $g_{12}$ lies in the polynomial $p$ dependence of Feynman parameter integrals. Concerning $f_{12}$, the contributions from Figs. (4a) and (4b) are cancelled by those of the tadpole. Fig. (4g) is finite superficially and in subgraphs without $K_{12}$ and is thus safe. The most dangerous is Fig. (4d) which has a quadratically divergent subgraph, which in turn may transmute into a pole-like singularity in the external momentum. Fortunately, this quadratic divergence is cancelled between the $\sigma$ and $\pi_0$ contributions proportional to,

$$\int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} D^2_m(k_1)[D_m(k_2) - D(k_2)]D(k_1 + p)K_{12}. \quad (39)$$

When computing loop integrals, we always work on Euclidean spacetime to simplify their analytic property. (Now $D_m(k) = (k^2 + m^2)^{-1}$.) Finishing the $k_2$ integral using $I_1$ in appendix B, we have,

$$\text{integral} = -2m^2(4\pi)^{-2} \int_0^1 dx \int \frac{d^4 k_1}{(2\pi)^4} D^2_m(k_1)D(k_1 + p)K_0 \left(\sqrt{xm^2\tilde{k}_1^2}\right). \quad (40)$$

There are preferred directions defined by $p_\mu$ and $\theta_{\mu\nu}$. To avoid complicated angular integration, we have to make some simplifying assumptions which should not alter the IR singularity drastically. As $\theta^2_{\mu\nu}$ is symmetric and semi-positive definite on Euclidean space, it may be diagonalized by an orthogonal rotation. We assume that it has a four-fold
degenerate eigenvalue of \( \eta^2 \), i.e., \( \theta_{\mu\nu}^2 = \eta^2 \delta_{\mu\nu} \) with \( \eta > 0 \) a small area scale characterizing NC. Then, \( \tilde{k}_1^2 = \eta^2 k_1^2 \) and the angular integral is much simplified,

\[
\int d\Omega_4 \ (k_1 + p)^{-2} = 4\pi \int_0^{\pi} d\omega \sin^2 \omega [k_1^2 + p^2 + 2\sqrt{k_1^2 p^2 \cos \omega}]^{-1} \\
= 2\pi^2 \left\{ \begin{array}{ll} 1/k_1^2 & (k_1^2 \geq p^2) \\ 1/p^2 & (k_1^2 \leq p^2) \end{array} \right..
\]

(41)

Including the subtraction \( D(k_1) \) term, we are thus integrating over \( k_1^2 \in [0, p^2] \). Note that this seems to be a special feature of angular integrals in four dimensions. For \( p^2 \ll m^2 \), we have for the dominant \( p \) dependent part,

\[
\text{integral} \approx m^{-2} (4\pi)^{-4} \int_0^1 dx \int_0^{p^2} dk_1^2 k_1^2 \left[ \frac{1}{p^2} - \frac{1}{k_1^2} \right] \ln(xm^2 \eta^2 k_1^2) \\
\approx -2^{-1} (4\pi)^{-4} m^{-2} p^2 \ln(\eta^2 m^2 p^2),
\]

(42)

which indeed vanishes as \( p \to 0 \) but singular as \( \theta \to 0 \). Integrals in Fig. (4e) can be similarly computed to arrive at the same conclusion.

Let us now consider the case when the two loop momenta are overlapping. For example, Fig. (4f) contains the following integral,

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D(k_1 + k_2)D_m(k_2)\overline{D_m(k_1 + p)}K_{12}.
\]

(43)

First finish \( k_2 \) integral using \( I_2 \) in appendix B,

\[
\text{integral} = \int \frac{d^4k_1}{(2\pi)^4} \overline{D_m(k_1 + p)} \\
\times 2(4\pi)^{-2} \int_0^1 dx \ K_0 \left( \sqrt{(1 - x)(k_1^2 x + m^2)k_1^2} \right),
\]

(44)

where the cosine factor disappears due to \( k_1 \cdot \tilde{k}_1 = 0 \). We are interested in the small \( p \) limit,

\[
\overline{D_m(k_1 + p)} = D_m(k_1) \left[ -2k_1 \cdot p D_m(k_1) \\ -p^2 D_m(k_1) + 4(k_1 \cdot p)^2 D_m^2(k_1) + O(p^3) \right].
\]

(45)

The first term vanishes since we cannot make up a vector out of \( \theta_{\mu\nu} \) and \( \delta_{\mu\nu} \). It also vanishes by the \( k_1 \to -k_1 \) symmetry. The third term is an integral involving \( k_{1\mu}k_{1\nu} \) which may be simplified as follows. The result of the integral is composed of the \( \delta_{\mu\nu} \), \( \theta_{\mu\nu} \), \( \theta^2_{\mu\nu} \), \( \theta^3_{\mu\nu} \), ... terms, where odd products of \( \theta \) actually cannot appear due to symmetry. As we are also interested in the small \( \theta \) limit, the \( \theta^2 \) and higher terms may be dropped; namely, we may replace \( k_{1\mu}k_{1\nu} \) by \( k_1^2 \delta_{\mu\nu}/4 \). For \( \theta^2_{\mu\nu} = \eta^2 \delta_{\mu\nu} \), the above argument becomes exact since only one structure \( \delta_{\mu\nu} \) is possible. We continue to work in this assumption so that
the angular integration may be finished explicitly. Together with the second term, we have \( \overline{D_m(k_1 + p)} \rightarrow -m^2 p^2 D_m^3(k_1) \). To help identify the relevant integration region, we first rescale \( \eta k_1^2 = y \) so that a small \( \eta \) will not interfere with a large \( k_1^2 \) in \( K_0 \). We have,

\[
\text{integral} \approx -2(4\pi)^{-4} p^2 \epsilon \int_0^\infty dy \ y(y + \epsilon)^{-3} \int_0^1 dx \ K_0 \left( \sqrt{(1-x)(xy + \epsilon)y} \right),
\]

with \( \epsilon = m^2 \eta \). As \( K_0 \) decays exponentially at a large argument and explodes at a small one, only the latter region is important. After some calculation, we obtain the following leading term,

\[
\text{integral} \approx 2^{-1}(4\pi)^{-4} p^2 \ln(m^4 \eta^2),
\]

which is IR finite but singular as \( \theta \rightarrow 0 \). Fig. (4b) in the neutral Goldstone case is convergent without the \( K_{12} \) factor after it is once subtracted at \( p = 0 \), and thus safe in the IR limit. Figs. (4c, h) are similarly done. More examples of integrals, especially those appearing in Figs. (4a) with many massless propagators that may cause an IR problem in the virtual loop momentum, are given in appendix B.

The neutral Goldstone boson self-energy has an additional contribution containing \( K_1 \). Since \( K_1 \) involves only one of the loop momenta \( k_1 \), there is a big difference between overlapping and non-overlapping integrals. For the latter, it is easy to identify the leading singular terms since they are essentially a product of two one-loop integrals. Fig. (4d) belongs to this category. For example, consider the integral,

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D_m(k_2)D_m^2(k_1)\overline{D(k_1 + p)K_1}.
\]

It is quadratically divergent in \( k_2 \) as in commutative theory, which is not harmful at all to the Goldstone theorem. It is also regular and vanishes in the NC IR limit. For the overlapping case, let us begin with an integral appearing in Fig. (4c),

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D_m(k_1)D_m(k_2)D_m(k_1 + k_2)\overline{D(k_1 + p)K_1}.
\]

Integration over \( k_2 \) gives a usual logarithmic UV divergence plus a \( \ln k_1^2 \) term. However, the resulting \( k_1 \) integral is again regular and vanishes in the limit of \( p \rightarrow 0 \). Fig. (4e) is proportional to the following integral,

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D_m(k_1 + k_2)D_m(k_2)
\times[D(k_1 + k_2 + p) - D(k_1 + k_2 - p)]D(k_2 + p)K_1
\]

\[
- \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D_m(k_1 + k_2)D_m(k_2)
\times[D(k_2 - p) - D(k_2 + p)]D(k_1 + k_2 - p)K_1.
\]

(50)
It becomes, using $I_2$ in appendix B,

$$\text{integral} = 2(4\pi)^{-2} \int_0^1 dx \ K_0 \left( \sqrt{(1-x)(m^2+p^2x)p^2} \right) \times \int \frac{d^4k_2}{(2\pi)^4} D_m(k_2) [D(k_2-p) - D(k_2+p)] \cos(k_2 \cdot \hat{p}),$$

(51)

which vanishes by $k_2 \to -k_2$.

Now consider overlapping integrals arising in Figs. (4f, h) which are nontrivial in the NC IR limit. For example, Fig. (4f) contains the following one,

$$\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D(k_1) D_m(k_1 + k_2) \overline{D_m(k_2 + p)} K_1$$

$$= 2(4\pi)^{-2} \int \frac{d^4k_2}{(2\pi)^4} \overline{D_m(k_2 + p)}$$

$$\times \int_0^1 dx \ K_0 \left( \sqrt{x(m^2+(1-x)k_2^2)p^2} \right) \cos(xk_2 \cdot \hat{p}).$$

(52)

Note that only the small $k_2$ region is important for $K_0$ where the cosine factor may be ignored for the leading term. A similar calculation to eqn. (44) leads to,

$$\text{integral} \approx 2^{-1}(4\pi)^{-4} p^2 \ln(m^2p^2).$$

(53)

And for the purely massless case, we obtain,

$$\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D(k_1) D(k_1 + k_2) \overline{D(k_2 + p)} K_1$$

$$\approx 2^{-1}(4\pi)^{-4} p^2 \ln(p^2\hat{p}^2).$$

(54)

Integrals from Fig. (4h) can be similarly computed whose results are presented in appendix B. Figs. (4a, b) have no contributions containing $K_1$ after subtraction while Fig. (4g) is regular in the limit of $p \to 0$.

4 Conclusion

The scalar theory is UV quadratically divergent whether or not its symmetry is spontaneously broken. On NC spacetime this virtual UV quadratic divergence may transmute into a pole-like IR singularity in external momenta. If this occurrence persists in scalar theory with spontaneous symmetry breaking, it may spoil the validity of the Goldstone theorem which is utilized to generate mass through the Higgs mechanism when the symmetry is gauged. On naive grounds there is no reason why this should not happen. This is especially the case when one goes beyond one loop level where the richer structure in
the NC parameter $\theta_{\mu\nu}$ may produce NC IR singularities not appearing at one loop and the singularities may even be enhanced by virtual massless Goldstone bosons in the extra loop.

We have made a complete analysis of the above problem at two loop level by studying the self-energies of the Goldstone bosons in the NC $U(N)$ linear $\sigma$ model. We found that the integrands in loop integrals have three types of $\theta$ dependence, i.e., $\theta$ independent, involving the two loop momenta ($K_{12} = \cos(2k_1 \wedge k_2)$), and involving one loop and one external momentum ($K_1 = \cos(2k_1 \wedge p)$). Our crucial observation is that the form factors of the above structures vanish in the limit of $p \to 0$. This implies that they are effectively once subtracted at $p = 0$. The subtraction arising from symmetry relations in 1PI and tadpole contributions cancels the most singular terms that are harmful to the theorem leaving behind a result proportional to a quadratic form in $p$. We have computed in detail the leading terms in the coefficients of the above form. We observed that delicate cancellation also occurs between the Higgs and Goldstone bosons that prevents harmful terms in the coefficients. The masslessness of virtual Goldstone bosons is not a problem; its IR behaviour can always be separated from the one induced by NC. The final leading IR terms in the Goldstone self-energies induced by NC are of order $p^2 \ln \theta^2$ and $p^2 \ln \tilde{p}^2$ so that the Goldstone theorem still holds true at two loop level. Since the basic mechanism for this mild IR behaviour originates from symmetry relations amongst vertices of the Higgs and Goldstone bosons, it seems rather natural to expect that the theorem should also be valid beyond two loop level. On the other hand, the limit of $\theta \to 0$ cannot be smooth at two loops and beyond, and this nonsmooth behaviour in $\theta$ is not necessarily associated with the IR limit of the external momentum as we saw in the leading term $p^2 \ln \theta^2$ from the $K_{12}$ part.

Acknowledgements

I would like to thank K. Sibold for many helpful discussions and for reading the manuscript carefully.

A Feynman rules

For completeness, we list below the Feynman rules for the vertices in the noncommutative $U(N)$ linear $\sigma$ model with the scalar field in the fundamental representation, which were first given in Ref. [5]. All momenta are incoming and shown in the parentheses of the
corresponding particles. There are no changes in propagators.

\[
\sigma \sigma(p_1) \sigma(p_2) = -i6 \lambda v \ c_{12} \\
\sigma \pi_0(p_1) \pi_0(p_2) = -i2 \lambda v \ c_{12} \\
\sigma \pi^\perp_i(p_1) \pi_j(p_2) = -i2 \lambda v \delta_{ij} \ e_{12} \\
\sigma(p_1) \sigma(p_2) \sigma(p_3) \sigma(p_4) = -i2 \lambda (c_{12} c_{34} + c_{31} c_{24} + c_{23} c_{14}) \\
\pi_0(p_1) \pi_0(p_2) \pi_0(p_3) \pi_0(p_4) = -i2 \lambda (c_{12} c_{34} + c_{31} c_{24} + c_{23} c_{14}) \\
\sigma(p_1) \sigma(p_2) \pi_0(p_3) \pi_0(p_4) = -i2 \lambda (2c_{12} c_{34} - c_{13,24}) \\
\sigma(p_1) \sigma(p_2) \pi^\perp_i(p_3) \pi_j(p_4) = -i2 \lambda \delta_{ij} \ c_{12} e_{34} \\
\pi_0(p_1) \pi_0(p_2) \pi^\perp_i(p_3) \pi_j(p_4) = -i2 \lambda \delta_{ij} \ c_{12} e_{34} \\
\pi^\perp_i(p_1) \pi^\perp_j(p_2) \pi_k(p_3) \pi_l(p_4) = -i2 \lambda [\delta_{ik} \delta_{jl} e_{13} e_{24} + \delta_{il} \delta_{jk} e_{14} e_{23}]
\]

(55)

where the following notations are used: \(p \wedge q = \theta_{\mu \nu} p^{\mu} q^{\nu}/2\), \(c_{ij} = \cos(p_i \wedge p_j)\), \(s_{ij} = \sin(p_i \wedge p_j)\), \(c_{ij,kl} = \cos(p_i \wedge p_j + p_k \wedge p_l)\), \(s_{ij,kl} = \sin(p_i \wedge p_j + p_k \wedge p_l)\) and \(e_{ij} = \exp(-ip_i \wedge p_j)\).

### B Some examples of two loop integrals involving \(\theta\)

We start on Euclidean spacetime where the integrals have a simpler analytic property. We start with the one loop integrals that have been computed by many authors in the literature.

\[
I_1(\rho^2, m) = \mu^{4-n} \int \frac{d^nk}{(2\pi)^n} (k^2 + m^2)^{-2} \cos(k \cdot \rho),
\]  

(56)

where \(\rho_\mu\) will be identified later with \(\theta_{\mu \nu} q^{\nu}\) with \(q\) a loop or external momentum. Using the Schwinger parameter integral to exponentiate the denominator, completing the square and shifting \(k\), we have

\[
I_1 = \int_0^\infty d\alpha \ \alpha \ \mu^{4-n} \left[ \frac{d^nk}{(2\pi)^n} \right] \exp[-\alpha(k^2 + m^2) + ik \cdot \rho]
= \int_0^\infty d\alpha \ \alpha \ \exp\left[-\alpha m^2 - \frac{\rho^2}{4\alpha}\right] \ \mu^{4-n} \left[ \frac{d^nk}{(2\pi)^n} \right] \ \exp[-\alpha k^2]
= \int_0^\infty d\alpha \ \alpha \ \exp\left[-\alpha m^2 - \frac{\rho^2}{4\alpha}\right] \ \mu^{4-n} \left[ \frac{1}{(4\pi\alpha)^{n/2}} \right] K_{n/2-2}(\sqrt{m^2 \rho^2})
= 2(4\pi)^{-2} \left[ \frac{4\pi \mu^2}{m^2} \frac{1}{2} \sqrt{m^2 \rho^2} \right]^{2-n/2} K_{n/2-2}(\sqrt{m^2 \rho^2}),
\]

(57)

where the remaining parameter integral has been expressed in terms of the modified Bessel function \([12],\)

\[
K_{\nu}(t) = \frac{1}{2} \left( \frac{t}{2} \right)^\nu \int_0^\infty d\alpha \ \alpha^{-1-\nu} \ \exp\left[-\alpha - \frac{t^2}{4\alpha}\right], \ |\arg t| < \frac{\pi}{2}, \ \text{Re} \ t^2 > 0.
\]

(58)
For $n = 4$, we have $I_1 = 2(4\pi)^{-2}K_0(\sqrt{m^2\rho^2})$ which is finite except for $\rho^2 \to 0$ since $K_0(x) \to -\ln x$ as $x \to 0$. This is the UV/IR mixing; the virtual UV singularity is regularized at the cost of introducing an IR singularity in the external momentum. Consider the case of $m = 0$. The virtual IR singularity in $I_1$ may be regularized either by a small mass or working in $n$ dimensions. In the latter case, a similar calculation leads to

$$I_1(\rho^2,0) = (4\pi)^{-2} (\pi \mu^2 \rho^2)^{2-n/2} \Gamma(n/2-2)$$
$$= (4\pi)^{-2} \left[ \Gamma(n/2-2) - \ln(\pi \mu^2 \rho^2) + O(n/2-2) \right],$$

(59)

where the first term is the virtual IR divergence and the second is the would-be virtual UV divergence regularized by the non-vanishing external momentum $\rho$. More interesting is the case when $\rho$ carries the momentum of a second loop involving massless particles so that the virtual IR singularity may be enhanced. Using the above result we also obtain,

$$I_2(\rho^2,m) = \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} [(k+p)^2 + m^2]^{-2} \cos(k \cdot \rho)$$
$$= I_1(\rho^2,m) \cos(p \cdot \rho).$$

(60)

In the following we give the integrals appearing in Fig. (4a). We shall assume $\theta^2_{\mu\nu} = \eta^2 \delta_{\mu\nu}$ throughout for simplicity.

$$I_3(m,m) = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} D^2_m(k_1) \overline{D_m(k_2+p)} K_{12}$$
$$\approx 2^{-1}(4\pi)^{-4} \rho^2 \ln(m^4 \eta^2),$$

(61)

which is computed using $I_1$ and the argument employed to simplify the $k_1$ integral in eqn. (43). Using $I_1$ and eqn. (41), we have

$$I_3(m,0) = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} D^2_m(k_1) \overline{D_m(k_2+p)} K_{12}$$
$$\approx -2^{-1}(4\pi)^{-4} \rho^2 \ln(m^2 \rho^2 \eta^2).$$

(62)

Now consider the integral,

$$I_3(0,m) = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} D^2(k_1) \overline{D_m(k_2+p)} K_{12}.$$

(63)

This integral looks dangerous since there is a virtual IR singularity in $k_1$ due to masslessness which may be mixed up with that coming from the $k_2$ loop to enhance the final IR singularity in $p$. The masslessness may be regularized either by a small mass or in dimensional regularization. In the first case, the result is obtained from $I_3(m,m)$ by setting the first $m$ to be the small mass. It is clear that the two IR singularities are separated from
each other. In the second case, we proceed as follows. Using \( I_1(\tilde{k}_2^2, 0) \) we have

\[
I_3(0, m) = \int \frac{d^4k_2}{(2\pi)^4} \frac{D_m(k_2 + p)}{D_m(k_2 + p)} \times (4\pi)^{-2} \left[ \Gamma(n/2 - 2) - \ln(\pi \mu^2 \tilde{k}_2^2) + O(n/2 - 2) \right],
\]

where the first and second terms are respectively from the IR and UV regions of \( k_1 \). Only the second one is of interest here since the first is proportional to \( p^2 \) and thus does not affect our main arguments on the Goldstone theorem. Finishing the \( k_2 \) integral as before, we have the contribution from the small \( k_2 \),

\[
I_3(0, m) \approx 2^{-1}(4\pi)^{-4} p^2 \ln(\mu^2 m^2 \eta^2),
\]

the same as we get from \( I_3(m, m) \) using the small mass regularization. Without giving further details, we have,

\[
I_3(0, 0) = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D^2(k_1) D(k_2 + p) K_{12}
\approx -2^{-1}(4\pi)^{-4} p^2 \ln(\mu^2 p^2 \eta^2) + \cdots,
\]

where the dots are the usual terms in commutative theory that vanish in dimensional regularization. Following are the examples of integrals appearing in Figs. (4c, e, h),

\[
I_4 = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D^2_m(k_1) D_m(k_2) D_m(k_1 + k_2) \frac{D(k_1 + p)}{D(k_2 + p)} K_{12}
\approx -2^{-1}(4\pi)^{-4} m^{-2} p^2 \ln(m^2 p^2 \eta^2),
\]

\[
I_5 = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D_m(k_1) D_m(k_2) D(k_1 + p) D(k_2 + p) K_{12}
\approx -2^{-1}(4\pi)^{-4} m^{-2} p^2 \ln(m^2 p^2 \eta^2),
\]

\[
I_6 = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D_m(k_1) D_m(k_2) D_m(k_1 + k_2) \frac{D(k_1 + p)}{D(k_2 + p)} K_{12}
\approx -2^{-1}(4\pi)^{-4} m^{-2} p^2 \ln(p^2 \eta^2).
\]

In the following we list integrals involving \( K_1 \) that arise in Fig. (4h) and are nontrivial in the NC IR limit.

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D_m(k_1) D_m(k_2) D_m(k_1 + k_2) \frac{D(k_2 + p)}{D_m(k_2 + p)} K_1
\approx 2^{-1}(4\pi)^{-4} m^{-2} p^2 \ln(p^2 \eta^2),
\]

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D(k_1) D(k_2) D_m(k_1 + k_2) \frac{D_m(k_2 + p)}{D(k_2 + p)} K_1
\approx 2^{-1}(4\pi)^{-4} m^{-2} p^2 \ln(m^2 p^2 \eta^2),
\]

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D_m(k_1) D(k_2) D(k_1 + k_2) \frac{D(k_2 + p)}{D_m(k_2 + p)} K_1
\approx 2^{-1}(4\pi)^{-4} m^{-2} p^2 \ln(p^2 \eta^2),
\]

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} D(k_1) D_m(k_2) D(k_1 + k_2) \frac{D(k_2 + p)}{D(k_2 + p)} K_1
\approx 2^{-1}(4\pi)^{-4} m^{-2} p^2 \ln(p^2 \eta^2).
\]

19
References

[1] For reviews, see: M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. 73 (2001) 977 [hep-th/0106048]; R. J. Szabo, *Quantum field theory on noncommutative spaces*, [hep-th/0109162].

[2] S. Minwalla, M. V. Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, J. High Energy Phys. 02 (2000) 020 [hep-th/9912072]; I. Ya. Aref’eva, D. M. Belov and A. S. Koshelev, *Two-loop diagrams in noncommutative φ⁴ theory*, Phys. Lett. B476 (2000) 431 [hep-th/9912075]; M. V. Raamsdonk and N. Seiberg, *Comments on noncommutative perturbative dynamics*, J. High Energy Phys. 03 (2000) 035 [hep-th/0002186]; A. Matusis, L. Susskind and N. Toumbas, *The UV/IR connection in the noncommutative gauge theories*, J. High Energy Phys. 12 (2000) 002 [hep-th/0002073].

[3] C. P. Martin and D. Sanchez-Ruiz, *The one-loop UV divergent structure of U(1) Yang-Mills theory on noncommutative R⁴*, Phys. Rev. Lett. 83 (1999) 476 [hep-th/9903077]; T. Krajewski and R. Wulkenhaar, *Perturbative quantum gauge fields on the noncommutative torus*, Int. J. Mod. Phys. A15 (2000) 1011 [hep-th/9903187]; M. M. Sheikh-Jabbari, *Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus*, J. High Energy Phys. 06 (1999) 015 [hep-th/9903107]; M. Hayakawa, *Perturbative analysis on infrared aspects of noncommutative QED on R⁴*, Phys. Lett. B478 (2000) 394 [hep-th/9912094] and *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R⁴*, [hep-th/9912167]; A. Matusis, L. Susskind and N. Toumbas, in Ref. [2]; I. Ya. Aref’eva, D. M. Belov, A. S. Koshelev and O. A. Rychkov, *Renormalizability and UV/IR mixing in noncommutative theories with scalar fields*, Phys. Lett. B487 (2000) 357.

[4] A. Armoni, *Comments on perturbative dynamics of non-commutative Yang-Mills theory*, Nucl. Phys. B593 (2001) 229 [hep-th/0005208]; L. Bonora and M. Salizzoni, *Renormalization of noncommutative U(N) gauge theories*, Phys. Lett. B504 (2001) 80 [hep-th/0011088]; C. P. Martin and D. Sanchez-Ruiz, *The BRS invariance of noncommutative U(N) Yang-Mills theory at the one-loop level*, Nucl. Phys. B598 (2001) 348 [hep-th/0012024].

[5] B. A. Campbell and K. Kaminsky, *Noncommutative field theory and spontaneous symmetry breaking*, Nucl. Phys. B581 (2000) 240 [hep-th/0003137] and *Noncommutative linear sigma models*, *ibid*. B 606 (2001) 613 [hep-th/0102022].
[6] F. Ruiz Ruiz, *UV/IR mixing and the Goldstone theorem in noncommutative field theory*, hep-th/0202011.

[7] F. J. Petriello, *The Higgs mechanism in noncommutative gauge theories*, Nucl. Phys. B 601 (2001) 169 [hep-th/0101109].

[8] Y. Liao, *One loop renormalization of spontaneously broken U(2) gauge theory on noncommutative spacetime*, J. High Energy Phys. 11 (2001) 067 [hep-th/0110112]; *One loop renormalizability of spontaneously broken gauge theory with a product of gauge groups on noncommutative spacetime: the U(1) × U(1) case*, J. High Energy Phys. 04 (2002) 042 [hep-th/0201135].

[9] For efforts towards renormalization to all orders, see, for example: I. Chepelev and R. Roiban, *Renormalization of quantum field theories on noncommutative Rd*. 1. Scalars, J. High Energy Phys. 05 (2000) 037 [hep-th/9911098]; *Convergence theorem for noncommutative Feynman graphs and renormalization*, J. High Energy Phys. 03 (2001) 001 [hep-th/008090]; L. Griguolo and M. Pietroni, *Hard noncommutative loops resummation*, Phys. Rev. Lett. 88 (2002) 071601 [hep-th/0102070]; *Wilsonian renormalization group and the noncommutative IR/UV connection*, J. High Energy Phys. 05 (2001) 032 [hep-th/0104217]; S. Sarkar, *On the UV renormalizability of noncommutative field theories*, hep-th/0202171.

[10] I. Ya. Arefeva, D. M. Belov and A. S. Koshelev, in Ref. [2]; A. Micu and M. M. Sheikh-Jabbari, *Noncommutative Φ 4 theory at two loops*, J. High Energy Phys. 01 (2001) 025 [hep-th/0008057].

[11] See for example: M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, *Noncommutative gauge field theories: a no-go theorem*, Phys. Lett. B526 (2002) 132 [hep-th/0107037] and *Noncommutative standard model: model building*, hep-th/0107055. For discussions on the unitarity problem in the latter work, see: J. L. Hewett, F. J. Petriello and T. G. Rizzo, *Noncommutativity and unitarity violation in gauge boson scattering*, hep-ph/0112003.

[12] I. S. Grashteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th edition, Academic Press.