Abstract. We consider a general weak perturbation of a non-interacting quantum lattice system with a non-degenerate gapped ground state. We prove that the presence of isolated eigenvalues in the spectrum of the decoupled model leads to the existence of quasi-particles in the perturbed model. Also, a scattering theory for asymptotically free many-particle states is developed.

1 Introduction and results

We consider a quantum system on a lattice, which is a weak perturbation of a non-interacting system with a non-degenerate gapped ground state. A rigorous perturbation theory for ground states in such models was developed in [1, 5, 9, 10, 11, 16, 17, 23, 24]. In particular, it is known that the weakly interacting model has a unique gapped ground state. In the present paper we establish a quasi-particle picture in the corresponding sector of the model. We prove that if there are isolated eigenvalues in the spectrum of the decoupled system, then the weakly interacting system has particle-like states which are obtained perturbatively from the eigenvectors of the decoupled system. We then show that the existence of a quasi-particle subspace leads to subspaces, describing scattering states of asymptotically free finite collections of quasi-particles.

An explicit quasi-particle picture for quantum Ising model in a strong magnetic field was developed by Malyshev in [14] (see also [13, 15]). The existence of one-particle subspaces in more complicated quantum spin systems was later shown in [3, 26]. All these results for quantum spin systems rely on a special ground state renormalization, which makes the Hamiltonian into...
a generator of a Markov stochastic process (see \[2, 12, 18, 19, 20, 8, 25\] for results on quasi-particles for Markov processes). It appears, however, that the interpretation of the renormalized system as a stochastic process limits the applicability of the method and is not actually necessary for the study of quasi-particles. Indeed, in the present paper we show that the quasi-particle picture can be established for a quite general class of quantum models within the \(C^*\)-algebraic framework.

We give now precise definitions.

We consider a quantum “spin” system on the lattice \(\mathbb{Z}^\nu\). Suppose that for each \(x \in \mathbb{Z}^\nu\) there is a Hilbert space \(\mathcal{H}_x\) (possibly infinite-dimensional) associated with this site. For the restriction to a finite volume \(\Lambda \subset \mathbb{Z}^\nu\) we will use the notation

\[\mathcal{H}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{H}_x.\]

The (formal) Hamiltonian of the model consists of a non-interacting Hamiltonian and a perturbation:

\[H = H_0 + \Phi.\]

Here \(H_0\) is the free Hamiltonian:

\[H_0 = \sum_{x \in \mathbb{Z}^\nu} h_x.\]

We assume that each \(h_x\) is a non-negative self-adjoint, possibly unbounded operator on \(\mathcal{H}_x\) with a non-degenerate ground state \(\Omega_x \in \mathcal{H}_x\):

\[h_x \Omega_x = 0\]

and a spectral gap \(\geq 1\):

\[h_x|_{\mathcal{H}_x \cap \Omega_x} \geq 1\]  \hspace{1cm} (1)

(this is necessary and sufficient in order that the non-interacting Hamiltonian have a non-degenerate ground state and a spectral gap \(\geq 1\); here and in the sequel we slightly abuse the notation by denoting the one-dimensional subspace spanned by \(\Omega_x\) with the same symbol). In order to define the perturbation \(\Phi\) we fix a finite subset \(\Lambda_0 \subset \mathbb{Z}^\nu\) (range of the perturbation) and set

\[\Phi = \sum_{x \in \mathbb{Z}^\nu} \phi_x,\]  \hspace{1cm} (2)

where \(\phi_x\) is a self-adjoint bounded operator on \(\mathcal{H}_{\Lambda_0 + x}\) (here \(\Lambda_0 + x\) is a shift of \(\Lambda_0\)). We will assume that the perturbation is small in the sense that \(\sup_{x \in \mathbb{Z}^\nu} \|\phi_x\|\) is finite and small enough.

Under these assumptions the model has a unique infinite volume ground state with a spectral gap and an exponential decay of correlations. It can be
obtained rigorously as the limit of finite volume ground states. Let $\Lambda \subset \mathbb{Z}^\nu$ be a finite volume and $H_\Lambda$ the restriction of the Hamiltonian $H$ to $\Lambda$ with empty boundary conditions:

$$H_\Lambda := H_{\Lambda,0} + \Phi_\Lambda,$$

where

$$H_{\Lambda,0} := \sum_{x \in \Lambda} h_x, \quad \Phi_\Lambda := \sum_{x \in \mathbb{Z}^\nu \cap \Lambda_0 + x \subset \Lambda} \phi_x$$

(other boundary conditions can be used as well). Since $\Phi_\Lambda$ is bounded, $H_\Lambda$ is self-adjoint with $\text{Dom}(H_\Lambda) = \text{Dom}(H_{\Lambda,0})$. The following theorem shows that the presence of a non-degenerate ground state and a perturbative spectral estimate hold uniformly for all finite volumes:

**Theorem 1.** There exists a constant $c_1$, depending only on the perturbation range $\Lambda_0$, such that if $\sup_x \|\phi_x\| < c_1$ then for any finite $\Lambda$ the Hamiltonian $H_\Lambda$ has a non-degenerate ground state $\Omega_\Lambda$:

$$H_\Lambda \Omega_\Lambda = E_\Lambda \Omega_\Lambda, \quad H_\Lambda |_{H_\Lambda \otimes \Omega_\Lambda} > E_\Lambda 1,$$

where $E_\Lambda$ is the ground state energy. Moreover, let

$$\tilde{H}_\Lambda := H_\Lambda - E_\Lambda 1$$

be the renormalized Hamiltonian. There exists a constant $c_2 = c_2(\Lambda_0)$ such that

$$\text{Spec}(\tilde{H}_\Lambda) \subset \bigcup_{a \in \text{Spec}(H_{\Lambda,0})} \{ z : |z - a| \leq c_2 \sup_x \|\phi_x\| a \}.$$  \hspace{1cm} (4)

In particular, this gives a lower bound for the gap between $0$ and the rest of the spectrum:

$$\tilde{H}_\Lambda |_{H_\Lambda \otimes \Omega_\Lambda} \geq (1 - c_2 \sup_x \|\phi_x\|) 1.$$  \hspace{1cm} (5)

Now the thermodynamic limit of the ground states can be performed; the limiting, infinite volume ground state is to be understood as a state ($\equiv$ normalized positive linear functional) on the algebra of local observables (see, e.g., [4]). Let $B(H_\Lambda)$ be the algebra of bounded operators in $H_\Lambda$ for any finite $\Lambda$, and

$$\mathcal{A}_\infty := \bigcup_{\Lambda \subset \mathbb{Z}^\nu, |\Lambda| < \infty} B(H_\Lambda)$$

the full local algebra. Let $\Lambda \not\rightarrow \mathbb{Z}^\nu$ mean that $\Lambda$ converges to $\mathbb{Z}^\nu$ in the sense that it eventually contains any finite subset. Then
Theorem 2. There exists the thermodynamic weak*-limit $\omega_\infty$ of the finite volume ground states:

$$\langle A\Omega, \Omega_A \rangle \xrightarrow{\Lambda/\mathbb{Z}^\nu} \omega_\infty(A), \quad A \in A_\infty.$$  

Throughout the paper by $\langle \cdot, \cdot \rangle$ we denote various scalar products.

Using appropriate definition of the infinite volume ground state, one can show that in our model it is unique: see [24].

Having found the infinite volume ground state, we use the GNS construction to define in a conventional way the Hilbert space $H_\infty$, describing local excitations of this ground state. Precisely, we consider the GNS triple $(H_\infty, \pi_\infty, \Omega_\infty)$ (Hilbert space, cyclic representation of $A_\infty$ in $H_\infty$, and the cyclic vector), associated with the state $\omega_\infty$ so that

$$\langle \pi_\infty(A)\Omega_\infty, \Omega_\infty \rangle = \omega_\infty(A).$$

Next we define the infinite volume Hamiltonian $H_\infty$ on $H_\infty$ as an appropriate limit of $\tilde{H}_\Lambda$; it is convenient to use the (week) resolvent convergence, because resolvent expansions are very relevant for the perturbation theory.

Theorem 3. There exists a (unique) self-adjoint operator $H_\infty$ on $H_\infty$, which is the weak resolvent limit of $\tilde{H}_\Lambda$ in the following sense: for any $A, B \in A_\infty$ and $z \in \mathbb{C} \setminus \mathbb{R}$

$$\langle (\tilde{H}_\Lambda - z)^{-1}A\Omega, B\Omega \rangle \xrightarrow{\Lambda/\mathbb{Z}^\nu} \langle (H_\infty - z)^{-1}\pi_\infty(A)\Omega_\infty, \pi_\infty(B)\Omega_\infty \rangle.$$  

(6)

Moreover, $H_\infty\Omega_\infty = 0$ and estimates [4], [5] hold with $\tilde{H}_\Lambda, H_\Lambda, \Omega_\Lambda$ replaced by $H_\infty, H_\infty, \Omega_\infty$, and $\text{Spec } (H_{\Lambda,0})$ replaced by

$$\text{Spec } (H_{\infty,0}) := \left\{ \sum_{x \in \mathbb{Z}^\nu} a_x \big| a_x \in \text{Spec } (h_x), a_x \neq 0 \text{ only for finitely many } x \right\}.$$  

Theorems 1-3 have been proved in [1, 10, 23].

We are now in a position to discuss the main topic of this paper, the quasi-particle excitations. In addition to the assumptions made above we assume now that the model is translationally invariant, i.e., $\mathcal{H}_x, h_x, \phi_x, \Omega_x$ are translates of some $\mathcal{H}, h, \phi, \Omega$. This implies that the ground state is translationally invariant and there is a representation of lattice shifts by unitary operators $U_x : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty, x \in \mathbb{Z}^\nu$, commuting with the Hamiltonian $H_\infty$. Let $\mathbb{T}^\nu$ be the $\nu$-dimensional torus of quasi-momenta (we will identify $\mathbb{T}$ with the segment $[0, 1]$) and consider the Hilbert space $L_2(\mathbb{T}^\nu)$. We shall say that an invariant with respect to $H$ and $U_x$ subspace $G \subset \mathcal{H}_\infty$ is a one-particle
**subspace** if there is a unitary $V: G \rightarrow \mathbb{L}_2(T^\nu)$ such that $V U_x V^*$ is the multiplication by $e^{2\pi i \langle x, \cdot \rangle}$ and $V H_\infty V^*$ the multiplication by some function $m(\cdot)$ (which is the energy-momentum relation of the quasi-particle). Equivalently, an invariant $G$ is a one-particle subspace if there exists an orthonormal basis $\xi_x, x \in \mathbb{Z}^\nu$, in $G$ such that $U_x \xi_y = \xi_{x+y}$; the unitary $V$ is then given by $$V: \xi_x \mapsto e^{2\pi i \langle x, \cdot \rangle}.$$ The natural situation in which one expects the presence of a one-particle subspace and which we will only deal with is when the non-interacting Hamiltonian $H_{\infty,0}$ has a spectrally isolated one-particle subspace and the perturbation is sufficiently small. We assume therefore now that the single-site Hamiltonian $h$ has an isolated non-degenerate eigenvalue $\mu$ and $\mu \notin \cup_{k>1} \{\mu_1 + \ldots + \mu_k | \mu_i \in \text{Spec}(h), \mu_i \neq 0\}$, so that $\mu$ is an isolated eigenvalue of $H_{\infty,0}$, corresponding to the one-particle subspace spanned by $w_x \otimes (\otimes_{y \neq x} \Omega_y), x \in \mathbb{Z}^\nu$, where $w$ is the eigenvector of $h$: $$hw = \mu w;$$ the function $m$ identically equals $\mu$ in this case. By Th.3 and (4), if $\|\phi\|$ is sufficiently small then the part of the spectrum of $H_\infty$, lying in the segment $[(1-c_2\|\phi\|)\mu, (1-c_2\|\phi\|)\mu]$, is separated by gaps from the rest of the spectrum. Let $\mathcal{H}_1$ be the spectral subspace of $H_\infty$, corresponding to this segment. Then

**Theorem 4.** If $\|\phi\|$ is sufficiently small, then $\mathcal{H}_1$ is a one-particle subspace, and the corresponding energy-momentum relation $m$ is a real analytic function.

The final result we discuss here concerns existence of asymptotically free many-particle scattering states. Let $\mathcal{F} = \mathcal{F}(\mathcal{H}_1)$ be the symmetric Fock space obtained from the above one-particle subspace $\mathcal{H}_1$ and describing various finite collections of these quasi-particles (in fact, though the indistinguishability of the particles is essential, the particular form of statistics is not relevant for the result below). The free evolution of a finite collection of quasi-particles is governed by the Hamiltonian $H_f$, acting on $\mathcal{F}$ and defined as the second quantization of $H_{\infty}|_{\mathcal{H}_1}$. We will show that the initial Hilbert space $\mathcal{H}_\infty$ contains invariant subspaces, describing asymptotically, in distant past or future, free collections of quasi-particles. We have to assume now that the energy momentum relation $m$ is non-constant, otherwise there is no scattering and the particle picture breaks down. Denote by $U_{f,x}: \mathcal{F} \rightarrow \mathcal{F}$ the second quantization of the unitary lattice shifts $U_x|_{\mathcal{H}_1}$ in the one-particle subspace. Then
Theorem 5. There exist isometric wave operators $W_{\pm} : \mathcal{F} \to \mathcal{H}_\infty$ such that $\text{Ran}(W_{\pm})$ are invariant subspaces of $H_\infty$, $U_x$, and $H_\infty|_{\text{Ran}(W_{\pm})} = W_{\pm}H_fW_{\pm}^*$, $U_x|_{\text{Ran}(W_{\pm})} = W_{\pm}U_f,xW_{\pm}^*$.

Our proofs of Theorems 4 and 5 closely follow the paper [25] and we will omit here some technical details. The proof of Theorem 4 is a modification of Malyshev-Minlos technique (see [2, 3, 8, 12, 15, 18, 19, 20, 25, 26]). The proof of Theorem 5 follows general ideas of scattering theory for quantum many-body systems. In [14], the scattering theory was based on the well-known construction of many-particle states given by Haag and Ruelle in the axiomatic quantum field theory [6, 7, 22]. Our exposition is closer to the spin wave scattering for the Heisenberg ferromagnet as presented in [21], section XI.14.

2 Preliminary results. Theorems 1-3

All theorems stated above rely heavily on perturbative expansions for the ground state, one-particle states and the Hamiltonian. We begin by briefly reviewing the proofs of Ths.1-3 (see [23] for details). From now on we adopt for brevity the following convention. We will denote by $c$ and $\epsilon$ various (generally different in different formulas) positive constants, which do not depend on the volume $\Lambda$, though may depend on the interaction range $\Lambda_0$. We write $\epsilon$ if this constant can be chosen arbitrarily small by choosing $\sup_x \|\phi_x\|$ small enough; on the other hand the constant $c$ is typically greater than 1 and does not depend on $\sup_x \|\phi_x\|$.

The ground state vector $\Omega_\Lambda$ of the Hamiltonian $H_\Lambda$ in a finite volume can be found using a suitable ansatz, reflecting smallness of correlations between distant spins.

For any $I \subset \Lambda$ set

$$\mathcal{H}'_I := \otimes_{x \in I} \mathcal{H}'_x, \quad \Omega_{I,0} := \otimes_{x \in I} \Omega_x$$

(with $\mathcal{H}'_\emptyset \equiv \mathbb{C}$). It follows that

$$\mathcal{H}_\Lambda \otimes \Omega_{\Lambda,0} = \bigoplus_{I \not\subset \Lambda} \mathcal{H}'_I \otimes \Omega_{\Lambda \setminus I,0}. \quad (7)$$

We will typically denote vectors from $\mathcal{H}'_I$ by $u_I, v_I$, etc. For each $u_I \in \mathcal{H}'_I$ we introduce a “creation” (or “spin raising”) operator $\hat{u}_I$ in $\mathcal{H}_I$ by

$$\hat{u}_I v = \langle v, \Omega_{I,0} \rangle u_I, \quad v \in \mathcal{H}_I$$
(with \( \hat{u}_\emptyset \) a scalar operator). A useful property of these operators is that for any \( I, J \) and \( u_I, v_J \)

\[
\hat{u}_I \hat{v}_J = \begin{cases} 
0 & \text{if } I \cap J \neq \emptyset, \\
\frac{u_I \otimes v_J}{I \cap J = \emptyset}. & \text{if } I \cap J = \emptyset.
\end{cases}
\]

In particular, they commute. For any \( v \in \mathcal{H}_\Lambda \) such that \( \langle v, \Omega_{\Lambda,0} \rangle = 1 \) there exists a unique collection \( \{ v_I \in \mathcal{H}'_I \}_{\emptyset \neq I \subset \Lambda} \) such that \( \exp(\sum_{\emptyset \neq I \subset \Lambda} \hat{v}_I) \Omega_{\Lambda,0} = v \) (these \( v_I \) can be obtained by truncation from components of \( v \) appearing in the decomposition (7)). Let \( \tilde{\Omega}_\Lambda \) be the ground state vector of \( H_\Lambda \) normalized so that \( \langle \tilde{\Omega}_\Lambda, \Omega_{\Lambda,0} \rangle = 1 \) (i.e., \( \tilde{\Omega}_\Lambda := \frac{\Omega_{\Lambda}}{\langle \Omega_{\Lambda}, \Omega_{\Lambda,0} \rangle} \)). Initially the non-degeneracy of the ground state and its non-orthogonality to \( \Omega_{\Lambda,0} \) is clear from the usual finite-volume perturbation theory for sufficiently weak perturbations in each particular volume, and it can be shown that the estimate for the perturbation which ensures this property can actually be chosen uniform in the volume. Let \( \{ v_I^{(\Lambda,gs)} \in \mathcal{H}'_I \}_{\emptyset \neq I \subset \Lambda} \) be the corresponding collection such that

\[
\tilde{\Omega}_\Lambda = \exp\left( \sum_{\emptyset \neq I \subset \Lambda} \hat{v}_I^{(\Lambda,gs)} \right) \Omega_{\Lambda,0}.
\]

**Lemma 1.** For any \( \epsilon > 0 \), if \( \sup_x \| \phi_x \| \) is sufficiently small then

\[
\max_{x \in \Lambda} \sum_{I \subset \Lambda : x \in I} \| H_{I,0} v_I^{(\Lambda,gs)} \| \epsilon^{-(d_I+1)} \leq 1,
\]

where \( d_I \) is the minimal length of a connected graph containing \( I \).

The idea of the proof is to rewrite the Schrödinger equation \( H_\Lambda \tilde{\Omega}_\Lambda = E_\Lambda \tilde{\Omega}_\Lambda \) as a fixed point equation for a suitable (non-linear) mapping on the space of collections, which is then shown to be a contraction in the set specified by Eq.(8).

Having found the ground state \( \tilde{\Omega}_\Lambda \), it is convenient for the study of the operator \( H_\Lambda \) to use for vectors \( v \in \mathcal{H}_\Lambda \) the expansion

\[
v = \sum_{I \subset \Lambda} \hat{v}_I \tilde{\Omega}_\Lambda.
\]

For any \( v \), there exists a unique collection such that (9) holds: indeed, if we introduce the operators \( P_{\Lambda,I} : \mathcal{H}_\Lambda \rightarrow \mathcal{H}'_I \) by

\[
u = \sum_{I \subset \Lambda} (P_{\Lambda,I} u) \otimes \Omega_{\Lambda \setminus I,0}, \quad \forall u \in \mathcal{H}_\Lambda,
\]
then

\[ v_K = P_{\Lambda,K} \sum_{J \subset \Lambda} \hat{v}_J \Omega_{\Lambda,0} \]

\[ = P_{\Lambda,K} \exp\left( - \sum_{\emptyset \neq I \subset \Lambda} \hat{v}_I^{(\Lambda,g,s)} \right) \sum_{J \subset \Lambda} \hat{v}_J \exp\left( \sum_{\emptyset \neq I \subset \Lambda} \hat{v}_I^{(\Lambda,g,s)} \right) \Omega_{\Lambda,0} \]

\[ = P_{\Lambda,K} \exp\left( - \sum_{\emptyset \neq I \subset \Lambda} \hat{v}_I^{(\Lambda,g,s)} \right) v. \quad (10) \]

Using expansion (9), we rewrite the renormalized Hamiltonian \( \tilde{H}_\Lambda \) as

\[ \tilde{H}_\Lambda = \tilde{H}_{\Lambda,0} + \tilde{\Phi}_\Lambda, \quad (11) \]

where \( \tilde{H}_{\Lambda,0} \) is the “diagonal” part defined by

\[ \tilde{H}_{\Lambda,0} \hat{v}_I \tilde{\Omega}_\Lambda = \hat{H}_{I,0} \hat{v}_I \tilde{\Omega}_\Lambda. \quad (12) \]

For the renormalized perturbation \( \tilde{\Phi}_\Lambda \) we then have

\[ \tilde{\Phi}_\Lambda \hat{v}_I \tilde{\Omega}_\Lambda = (\tilde{H}_\Lambda - \tilde{H}_{\Lambda,0}) \hat{v}_I \tilde{\Omega}_\Lambda \]

\[ = \tilde{H}_\Lambda \hat{v}_I \tilde{\Omega}_\Lambda - \tilde{H}_{I,0} \hat{v}_I \tilde{\Omega}_\Lambda \]

\[ = [\tilde{H}_\Lambda, \hat{v}_I] \tilde{\Omega}_\Lambda + [\tilde{\Phi}_\Lambda, \hat{v}_I] \tilde{\Omega}_\Lambda - \tilde{H}_{I,0} \hat{v}_I \tilde{\Omega}_\Lambda \]

\[ = [\tilde{\Phi}_\Lambda, \hat{v}_I] \tilde{\Omega}_\Lambda. \]

It is convenient to write the operator \( \tilde{\Phi}_\Lambda \) in the form

\[ \tilde{\Phi}_\Lambda \hat{v}_I \tilde{\Omega}_\Lambda = \sum_{J \subset \Lambda} (F_{\Lambda v_I})_J \tilde{\Omega}_\Lambda \quad (13) \]

with some \( (F_{\Lambda v_I})_J \in \mathcal{H}'_J \), thereby expanding the image vector as in (9). Using (10), we find that

\[ (F_{\Lambda v_I})_J = P_{\Lambda,J} \exp\left( - \sum_{\emptyset \neq K \subset \Lambda} \hat{v}_K^{(\Lambda,g,s)} \right) [\Phi_\Lambda, \hat{v}_I] \tilde{\Omega}_\Lambda \]

\[ = P_{\Lambda,J} \exp\left( - \sum_{\emptyset \neq K \subset \Lambda} \hat{v}_K^{(\Lambda,g,s)} \right) [\Phi_\Lambda, \hat{v}_I] \exp\left( \sum_{\emptyset \neq K \subset \Lambda} \hat{v}_K^{(\Lambda,g,s)} \right) \Omega_{\Lambda,0}. \]

Expanding this expression into a commutator series and using Lemma 1, one finds that

\[ \sum_{J \subset \Lambda} \|(F_{\Lambda v_I})_J\| e^{-(d_J+1)} \leq c \sup_x \|\phi_x\| \|I\| \|v_I\|, \quad (14) \]
where by $d_{J,I}$ we denote the minimal length of a graph connecting all sites in $J$ to some sites in $I$. In particular, this estimate shows that $\tilde{\Phi}_\Lambda$ is a relatively bounded, in some special sense, perturbation of $\tilde{H}_{\Lambda,0}$. Indeed, we introduce a new norm $\| \cdot \|$ in $\mathcal{H}_\Lambda$ by

$$\| \sum_{I \subset \Lambda} \hat{u}_I \tilde{\Omega}_\Lambda \| := \sum_{I \subset \Lambda} \| u_I \|.$$ 

Then by (14) for any $v \in \mathcal{H}_\Lambda$

$$\| \tilde{\Phi}_\Lambda v \| \leq c \sup_x \| \phi_x \| \| \tilde{H}_{\Lambda,0} v \|.$$

We assume that $\sup_x \| \phi_x \|$ is small so that $c \sup_x \| \phi_x \| < 1$. If $z \in \mathbb{C}$ lies outside of the union of circles standing on the r.h.s. of (4), then $\| \tilde{\Phi}_\Lambda (\tilde{H}_{\Lambda,0} - z)^{-1} \| < 1$ and hence the resolvent $(\tilde{H}_{\Lambda} - z)^{-1}$ exists and is given by the exponentially convergent series

$$(\tilde{H}_{\Lambda,0} - z)^{-1} \sum_{k=0}^{\infty} (-1)^k (\tilde{\Phi}_\Lambda (\tilde{H}_{\Lambda,0} - z)^{-1})^k. \quad (15)$$

That proves the spectral estimate (4).

We consider now the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^\nu$. It easily follows from the proof of Lemma 1 that the quantities $v_I^{(\Lambda,gs)}$, defining the ground state in finite volumes, have limits

$$v_i^{(\infty,gs)} := \lim_{\Lambda \nearrow \mathbb{Z}^\nu} v_i^{(\Lambda,gs)}.$$ 

Using this, Theorem 2 follows immediately by cluster expansions. As a by-product of cluster expansions, one obtains an exponential decay of correlations in the ground state:

$$|\omega_\infty(A_1 A_2) - \omega_\infty(A_1) \omega_\infty(A_2)| \leq c |A_1| + |A_2| \epsilon \text{dist}(A_1, A_2) \| A_1 \| \| A_2 \|, \quad A_i \in \mathcal{B}(\mathcal{H}_\Lambda). \quad (16)$$

Iterating this inequality, one finds that

$$|\omega_\infty(A_1 \cdots A_n) - \omega_\infty(A_1) \cdots \omega_\infty(A_n)| \leq (n - 1) c |A_1| + \cdots + |A_n| \epsilon \text{min}_{i \neq j} \text{dist}(A_i, A_j), \quad A_i \in \mathcal{B}(\mathcal{H}_\Lambda). \quad (17)$$

Now we discuss the thermodynamic limit of the Hamiltonian. Recall that $(\mathcal{H}_\infty, \pi_\infty, \Omega_\infty)$ is the GNS triple associated with the ground state $\omega_\infty$. The limiting Hamiltonian $H_\infty$ is conventionally defined on the vectors of the form $\pi_\infty(A) \Omega_\infty$, with $A \in \mathcal{A}_\infty$ such that $[H, A] \in \mathcal{A}_\infty$, by

$$H_\infty \pi_\infty(A) \Omega_\infty := \pi_\infty([H, A]) \Omega_\infty.$$
It is convenient to take \( A = \hat{u}_I \) here. Consider the space \( \mathcal{U} \) of finite linear combinations of operators \( \hat{u}_I \). It is easy to see that the set \( \pi_\infty(\mathcal{U})\Omega_\infty \) is dense in \( \mathcal{H}_\infty \). Extending the formulas (11), (12), (13), we have

\[
H_\infty\pi_\infty(\hat{u}_I)\Omega_\infty = \pi_\infty(H_{I,0}u_I)\Omega_\infty + \pi_\infty\left( \sum_{J \subset 2^\mathbb{Z}, |J|<\infty} (\hat{F}_\infty u_I)_J \right)\Omega_\infty,
\]

where

\[
(\hat{F}_\infty u_I)_J := \lim_{\Lambda \to \mathbb{Z}^2} (\hat{F}_\Lambda u_I)_J.
\]

By this formula the operator \( H_\infty \) is densely defined on the subspace spanned by vectors \( \pi_\infty(\hat{u}_I)\Omega_\infty \) with \( u_I \in \mathcal{H}_{I,0} \cap \text{Dom} (H_{I,0}) \). The estimate (14) remains valid for the limiting quantities \( (\hat{F}_\infty u_I)_J \), so the resolvent expansion of the form (15) for \( (H_\infty - z)^{-1} \), with the same values of \( z \) as before, is exponentially convergent on vectors from \( \pi_\infty(\mathcal{U})\Omega_\infty \). The essential self-adjointness of \( H_\infty \) and the resolvent convergence (6) follow by standard arguments.

3 Proof of Theorem 4

The orthogonal projector onto the spectral subspace \( \mathcal{H}_1 \), corresponding to the neighborhood of the point \( \mu \) as defined in the Introduction, is given by

\[
P_{\mathcal{H}_1} = -(2\pi i)^{-1} \int_{\gamma} (H_\infty - z)^{-1} dz,
\]

where \( \gamma \) is a contour in the complex plane surrounding this neighborhood. Since \( \gamma \) lies outside of the union of circles standing on the r.h.s. of (14), we can use the resolvent expansion (15) to estimate projections of vectors \( \hat{u}_I\Omega_\infty \).

For the normalized eigenvector \( w_x \) of \( h \) at the site \( x \), using estimate (14) we obtain

\[
P_{\mathcal{H}_1} \hat{w}_x \Omega_\infty = \hat{w}_x \Omega_\infty + \sum_I v^{(w_x)}_I \Omega_\infty,
\]

with some vectors \( v^{(w_x)}_I \) such that

\[
\sum_I \| H_{I,0} v^{(w_x)}_I \| e^{-d_{I,0}(x)+1} \leq 1.
\]

The first term in (19) comes from the leading term in the resolvent expansion; the second term, coming from the rest of the expansion, is small due to (14). It is easy to see that the vectors \( P_{\mathcal{H}_1} \hat{w}_x \Omega_\infty \) are total in \( \mathcal{H}_1 \). Indeed, since \( \mathcal{U}\Omega_\infty \) is dense in \( \mathcal{H}_\infty \), it suffices to show that for any \( u_I \) we can expand
\[ P_{H_1} \hat{u}_I \Omega_\infty = \sum_j \hat{v}_j^{(u_I)} \Omega_\infty, \quad (21) \]

where

\[ \sum_j \| H_{j,0} v_j^{(u_I)} \| e^{-(d_{j,I}+1)} \leq \| u_I \|. \]

In particular, \( \sum_j \| v_j^{(u_I)} \| \leq \| u_I \|/2 \) for such \( u_I \). Now, on the r.h.s. of (21), we single out the contribution spanned by \( \hat{w}_x \Omega_\infty, x \in \mathbb{Z}^\nu \), and apply \( P_{H_1} \) to both sides of the equality. Since \( P_{H_1}^2 = P_{H_1} \), we see that the projection of an irrelevant vector is represented as a sum of relevant projections plus projections of irrelevant vectors with a smaller total norm; the iteration of this procedure yields the desired expansion of \( P_{H_1} \hat{u}_I \Omega_\infty \) in terms of \( P_{H_1} \hat{w}_x \Omega_\infty \).

Now we orthogonalize the vectors \( P_{H_1} \hat{w}_x \Omega_\infty, x \in \mathbb{Z}^\nu \). Let \( G \) be the Gram matrix of this family:

\[ G_{xy} = \langle P_{H_1} \hat{w}_x \Omega_\infty, P_{H_1} \hat{w}_y \Omega_\infty \rangle. \]

Using the decay of correlation estimate (16) and the expansion (19) with the estimate (20), one finds that

\[ |G_{xy} - \delta_{xy}| \leq \epsilon^{|x-y|+1}. \quad (22) \]

We obtain an orthogonal family \( \xi_x \) by applying \( G^{-1/2} \) to the vectors \( P_{H_1} \hat{w}_x \Omega_\infty \):

\[ \xi_x = \sum_y G_{xy}^{-1/2} P_{H_1} \hat{w}_y \Omega_\infty. \]

It is easy to see that the exponential estimate (22) for \( G \) implies similar estimate for \( G^{-1/2} \); as a result, we have for \( \xi_x \) an expansion analogous to (19) with an exponential estimate analogous to (20):

\[ \xi_x = \hat{w}_x \Omega_\infty + \sum_I \hat{v}_I^{(\xi_x)} \Omega_\infty, \quad (23) \]

where

\[ \sum_I \| H_{I,0} v_I^{(\xi_x)} \| e^{-(d_{I,|x|-1})+1} \leq 1. \quad (24) \]

We have thus shown that \( H_1 \) is a one-particle subspace, and it only remains to check that the energy-momentum relation is analytic. It follows from the exponential decay of \( \langle H_\infty \xi_x, \xi_y \rangle \) at \( |x-y| \to \infty \), which in turn follows from the above expansion (23), (24) using decay of correlations (16).
4 Proof of Theorem 5

An important point in the proof of Theorem 5 is the observation that distant excitations of the ground state evolve independently. Recall that the Hamiltonian $H_\infty$ is defined by $H_\infty A \Omega_\infty = [H, A] \Omega_\infty$, where $A$ is any local operator such that $[H, A]$ is bounded. If operators $A_i$ act on $H_{I_i}$, $i = 1, \ldots, n$, and the distance between different $I_i$'s is greater than the range of interaction, then

$$[H, A_1 A_2 \cdots A_n] = A_2 \cdots A_n [H, A_1] + \ldots + A_1 \cdots A_{n-1} [H, A_n].$$

In particular, if $A_i = \hat{v}_{I_i}$ with some $v_{I_i} \in H'_{I_i} \cap \text{Dom}(H_{I_i,0})$, where $I_i$'s are separated by distances not less than the interaction range, then

$$H_\infty \hat{v}_{I_1} \hat{v}_{I_2} \cdots \hat{v}_{I_n} \Omega_\infty = \hat{v}_{I_2} \cdots \hat{v}_{I_n} H_\infty \hat{v}_{I_1} \Omega_\infty + \ldots + \hat{v}_{I_1} \cdots \hat{v}_{I_{n-1}} H_\infty \hat{v}_{I_n} \Omega_\infty. \quad (25)$$

In the previous section we found a basis $\xi_x$ in the one-particle subspace $H_1$, whose vectors have the well-localized expansion (23), (24). The time evolution of a quasi-particle state is straightforwardly described in the momentum picture, where such a state is identified with a function on the torus $T^\nu$ of quasi-momenta. In this picture we can form one-particle states, propagating as wave packets in certain directions, and, using (24), these states can be shown to be spatially well-localized. If we consider a few such wave packets, moving in different direction away from each other, then in the distant past or future they will form far separated excitations of the ground state. By (25), the evolution of the corresponding “product” state in $H_\infty$ is then asymptotically close to the product of respective one-particle evolutions. Precisely this relation can be stated as the existence of isometric wave operators intertwining the second quantization of the one particle evolution with the evolution generated by $H_\infty$ in a subspace of $H_\infty$.

Following the standard setting of scattering theory for a pair of Hilbert spaces (see [21]), we begin by introducing the operator $T : \mathcal{F} \to H_\infty$, defining an approximate “product” of one-particle states. Let $\mathcal{F}_n$ be the $n$-particle subspace of the bosonic Fock space $\mathcal{F}$. The normalized symmetrized products $(\xi_{x_1} \otimes \cdots \otimes \xi_{x_n})^{sym}$ form an orthonormal basis in $\mathcal{F}_n$. Using expansion (23), we set

$$T : (\xi_{x_1} \otimes \cdots \otimes \xi_{x_n})^{sym} \mapsto \prod_{k=1}^n (\hat{w}_{x_k} + \sum_i \hat{v}^{(\xi_{x_k})}_{f}) \Omega_\infty.$$  

This operator does not generally extend to a bounded operator on $\mathcal{F}$, but it is well-defined on vectors of the form $\sum_{n=1}^N k_{x_1,\ldots,x_n} (\xi_{x_1} \otimes \cdots \otimes \xi_{x_n})^{sym}$, where $N$ is finite and $\sum_{n=1}^N |k_{x_1,\ldots,x_n}| < \infty$; this will be sufficient for us. In the Fock space $\mathcal{F}$ we find a dense subset $S$ such that the quantities

$$W_t f := \exp\{iH_\infty\} T \exp\{-iH_f\} f$$
are well-defined for \( f \in S \) and \( t \in \mathbb{R} \), and there exist isometric limits

\[
W_\pm f := \lim_{t \to \pm \infty} W_t f.
\]  

(26)

Then \( W_\pm : \mathcal{F} \to H_\infty \) are the desired isometric wave operators, implementing the intertwining relation \( W_\pm H_f = H_\infty W_\pm \). The intertwining relation for lattice translations, \( W_\pm U_{f,x} = U_x W_\pm \), is obvious because \( TU_{f,x} = U_x T \).

The dense set \( S \) is spanned by collections of quasi-particles forming smooth wave packets drifting away from each other. Precisely, recall that by definition of the one-particle subspace we have a unitary \( V : H_1 \to L^2(\mathbb{T}^\nu) \) defining the momentum picture. If \( f \in H_1 \) is such that \( V f \in C^\infty(\mathbb{T}^\nu) \), then a standard stationary phase estimate shows that, roughly speaking, this wave packet propagates with velocities from the set \( \{2\pi \nabla m(p) | p \in \text{supp} (V f)\} \):

**Lemma 2** (21). Let \( V f \in C^\infty(\mathbb{T}^\nu) \) and \( \mathcal{O} \subset \mathbb{R}^\nu \) be an open set containing \( \{2\pi \nabla m(p) | p \in \text{supp} (V f)\} \). Then for arbitrarily large \( a \) there exists a constant \( c = c(f, \mathcal{O}, a) \) such that

\[
|\langle \exp \{itH_\infty\} f, \xi \rangle| = \left| \int_{\mathbb{T}^\nu} e^{itm(p)-2\pi i(x,p)} V f(p) dp \right| \leq c(1 + |x| + |t|)^{-a}
\]

for \( x/t \notin \mathcal{O} \).

This Lemma shows that if \( \exp \{itH_\infty\} f \) is expanded as \( \sum_{x \in \mathbb{Z}^\nu} k_x \xi_x \), then the coefficients \( k_x \) are small for \( x \notin \{2\pi t \nabla m(p) | p \in \text{supp} (V f)\} \).

Consider now the subset \( S_n \subset \mathcal{F}_n \) formed by vectors \( (f_1 \otimes \cdots \otimes f_n)^{\text{sym}} \) such that \( V f_k \) are \( C^\infty \) and

\[
\nabla m(p_k) \neq \nabla m(p_l) \quad \text{for} \quad p_k \in \text{supp} (V f_k), \quad p_l \in \text{supp} (V f_l), \quad k \neq l.
\]

We define \( S \) as the linear span of \( \cup_n S_n \). Since the energy-momentum relation \( m(p) \) is a non-constant analytic function, it is easy to see that \( S_n \) is a total subset of \( \mathcal{F}_n \) and hence \( S \) is dense in \( \mathcal{F} \).

We will show now that the wave operators (26) exist for \( f \in S_n \) (and hence for \( f \in S \)). By the Cook method, it suffices to prove that

\[
\int_{\mathbb{R}} \| (H_\infty T - T H_f) \exp \{-itH_f\} f \| dt < \infty.
\]

(27)

The question is whether the integrand falls off fast enough as \( t \to \infty \). We will see that it is actually \( o(|t|^a) \) for any \( a \).

Note first that for any \( f \in \mathcal{H}_1 \) we can expand \( f_t := \exp \{-itH_\infty\} f \) as

\[
f_t = \sum_i \hat{v}_i^{(f_t)} \Omega_\infty
\]

(28)
by first expanding $f_t$ in $\xi_x$ and then expanding the latter as in (28). If $V f \in C^\infty(T^*)$ and $O$ is an open vicinity of $\{2\pi \nabla m(p) | p \in \text{supp}(V f)\}$, then it follows from Lemma 3 and estimate (24) that in (28) the contribution from the terms with $I \notin tO$ is small: precisely, for any $a$

$$\sum_{I \notin tO} \|H_{I,t}v^{(f_t)}_I\| = o(|t|^a), \text{ as } t \to \infty.$$  

(29)

Moreover, if $g_t := H_\infty \sum_{I \notin tO} \tilde{v}^{(f_t)}_I \Omega_\infty$, then by (18) and estimate (14) we have that $g_t = \sum_I \tilde{v}^{(g_t)}_I \Omega_\infty$ with

$$\sum_I \|v^{(g_t)}_I\| = o(|t|^a).$$  

(30)

Now suppose that $f \in S_n$, $f = (f_1 \otimes \cdots \otimes f_n)^{sym}$. Then $\exp\{-itH_f\}f = (f_1, t \otimes \cdots \otimes f_n)^{sym}$, where $f_{k,t} := \exp\{-itH_\infty\}f_k$. Choose the corresponding open sets $O_k$, $k = 1, \ldots, n$, so that they don’t overlap. Then using (29) one finds that

$$H_\infty T \exp\{-itH_f\}f = H_\infty \sum_{I_1 \subset tO_1} \tilde{v}^{(f_{1,t})}_{I_1} \cdots \sum_{I_n \subset tO_n} \tilde{v}^{(f_{n,t})}_{I_n} \Omega_\infty + o(|t|^a)$$  

(31)

for any $a$. On the other hand, using (30) one finds that

$$TH_f \exp\{-itH_f\}f$$  

(32)

$$= \left( \sum_{I_2 \subset tO_2} \tilde{v}^{(f_{2,t})}_{I_2} \cdots \sum_{I_n \subset tO_n} \tilde{v}^{(f_{n,t})}_{I_n} \right) H_\infty \sum_{I_1 \subset tO_1} \tilde{v}^{(f_{1,t})}_{I_1} \Omega_\infty + \ldots$$  

$$+ \sum_{I_1 \subset tO_1} \tilde{v}^{(f_{1,t})}_{I_1} \cdots \sum_{I_{n-1} \subset tO_{n-1}} \tilde{v}^{(f_{n-1,t})}_{I_{n-1}} \right) H_\infty \sum_{I_n \subset tO_n} \tilde{v}^{(f_{n,t})}_{I_n} \Omega_\infty \right)$$  

$$+ o(|t|^a).$$

By (25), the first term in the r.h.s. of (31) equals the expression in brackets in (32), therefore the integrand in (27) is indeed $o(|t|^a)$ for any $a$, and the wave operators $W^\pm_t$ exist on $S$.

It remains to show that the wave operators are isometric. It suffices to prove that for any $f^{(l)} \in S_n$, $l = 1, 2$, we have

$$\lim_{t \to \infty} \langle T \exp\{-itH_f\}f^{(1)}, T \exp\{-itH_f\}f^{(2)} \rangle = \langle f^{(1)}, f^{(2)} \rangle.$$  

If $n_1 = n_2 = n$, then, if $f^{(l)} = (f_1^{(l)} \otimes \cdots \otimes f_n^{(l)})^{sym}$, the r.h.s. equals $\sum_{\sigma} \Pi_{k=1}^n (f_1^{(\sigma(k))}, f_2^{(\sigma(k))})$, where the sum is over permutations of $n$ elements; otherwise the r.h.s. vanishes.
Expanding in $\xi_x$, we see that the l.h.s. equals

$$\sum_{x_1^{(1)} \in \mathcal{O}_x^{(1)}} \sum_{k=1}^{n_1} \prod_{k=1}^{n_1} \langle e^{-itH f_k^{(1)}}, \xi_{x_1^{(1)}}(1) \rangle \prod_{k=1}^{n_2} \langle \xi_{x_1^{(2)}}(2), e^{-itH f_k^{(2)}} \rangle \times \langle T(\xi_{x_1^{(1)}}(1) \otimes \cdots \otimes \xi_{x_1^{(1)}}(1))^{sym}, T(\xi_{x_1^{(2)}}(2) \otimes \cdots \otimes \xi_{x_1^{(2)}}(2))^{sym} \rangle = \delta_{\{x_1^{(1)}\},\{x_2^{(2)}\}} + o(|t|)^a).$$

Here $\mathcal{O}_x^{(l)}$ are open sets for functions $f_k^{(l)}$, disjoint for equal values of $l$. We can restrict the summation to $x_k^{(l)} \in t\mathcal{O}_x^{(l)}$, with the remainder being $o(|t|)^a$ for any $a$, by Lemma 3. For this range of $x_k^{(l)}$, using expansions for $\xi_x$, the decay of correlations (17), and the fact that the minimal distance between such points with common $l$ grows linearly in $t$, one can establish an asymptotic orthogonality of the vectors in the last scalar product of (33):

$$\langle T(\xi_{x_1^{(1)}}(1) \otimes \cdots \otimes \xi_{x_1^{(1)}}(1))^{sym}, T(\xi_{x_2^{(2)}}(2) \otimes \cdots \otimes \xi_{x_2^{(2)}}(2))^{sym} \rangle = \delta_{\{x_1^{(1)}\},\{x_2^{(2)}\}} + o(|t|)^a),$$

with some $\gamma < 1$, where $\delta_{\{x_1^{(1)}\},\{x_2^{(2)}\}}$ equals 1 if $\{x_1^{(1)}, k = 1, \ldots, n_1\} = \{x_2^{(2)}, k = 1, \ldots, n_2\}$ as sets and 0 otherwise. It follows that for $n_1 \neq n_2$ the expression (33) converges to 0, and for $n_1 = n_2 = n$ it converges to

$$\sum_{\sigma} \sum_{x_k \in \mathcal{O}_x^{(l)}} \prod_{k=1}^{n} \langle e^{-itH f_k^{(1)}}, \xi_{x_k} \rangle \prod_{k=1}^{n} \langle \xi_{x_{\sigma(k)}}, e^{-itH f_k^{(2)}} \rangle = \sum_{\sigma} \prod_{k=1}^{n} \langle f_k^{(1)}, f_{\sigma(k)}^{(2)} \rangle,$$

which completes the proof.

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