PRIME GEODESIC THEOREM OF GALLAGHER TYPE

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Abstract. We reduce the exponent in the error term of the prime geodesic theorem for compact Riemann surfaces from $\frac{3}{4}$ to $\frac{7}{10}$ outside a set of finite logarithmic measure.

1. Introduction

Let $\Gamma \subset PSL(2, \mathbb{R})$ be a strictly hyperbolic Fuchsian group and $F = \Gamma \backslash \mathcal{H}$ be the corresponding compact Riemann surface of a genus $g \geq 2$, where $\mathcal{H} = \{ z = x + iy : y > 0 \}$ denotes the upper half-plane equipped with the hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. The Selberg zeta function on $F$ is defined by the Euler product

$$Z(s) = Z_\Gamma(s) = \prod_{\{P_0\}k=0}^\infty (1 - N(P_0)^{-s-k}), \text{ Re}(s) > 1,$$

where the product is taken over all primitive conjugacy classes $\{P_0\}$ and $N(P_0)$ is the norm of a conjugacy class $P_0$. (See any of the standard references [9], [13] for a necessary background.)

The Selberg zeta function can be continued to the whole complex plane as a meromorphic function of a finite order. Its zeros $\rho = \frac{1}{2} + i\gamma$ are denumerable and closely related to the eigenvalues $\lambda$ of the Laplace-Beltrami operator on $F$. This operator being essentially self-adjoint, its eigenvalues are non-negative and tend to infinity. Therefore, there are finitely many of them that are less than $\frac{1}{4}$. We have $\gamma = \pm \sqrt{\lambda - \frac{1}{4}}$ for $\lambda \geq \frac{1}{4}$ and $\gamma = \mp i\sqrt{\frac{1}{4} - \lambda}$ for $\lambda < \frac{1}{4}$. So, the zeros of $Z$ are split into two parts: those lying on the critical line $\text{Re}s = \frac{1}{2}$ and the real zeros in the interval $[0, 1]$.

Compared to the Riemann zeta case, there are "too many zeros" of $Z$ in some sense [8, (6.14) on p. 113]. Let $N(t)$ denote the number of zeros $\rho = \frac{1}{2} + i\gamma$ such that $0 < \gamma \leq t$. The function $R(t)$, given by $N(t) = \frac{1}{2\pi} t^2 + R(t)$, grows as $O(t(\log t)^{-1})$, where $|\mathcal{F}|$ is the volume of $F$.

The norm $N(P_0)$ is determined by the length of the geodesic joining two fixed points, necessarily the same ones for all representatives of $P_0$. The statement about the number $\pi_0(x)$ of classes $\{P_0\}$ such that $N(P_0) \leq x$, for $x > 0$, is known as the prime geodesic theorem. It has been proved by Selberg [21] and Huber [10] and subsequently generalized to various settings. The references [16, 5, 8, 18, 6, 19] form an interesting range of samples.

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One should consult [9, Discussion 6.20., pp. 113-115 and Discussion 15.16., pp. 253-255] on the difficulties in improving Huber’s $O \left( x^{\frac{3}{4}} (\log x)^{-\frac{1}{2}} \right)$. The best estimate for the error term in the prime geodesic theorem up to now is $O \left( x^{\frac{3}{4}} (\log x)^{-1} \right)$ obtained by Randol [20]. Its analogue has been also established for higher dimensional hyperbolic manifolds [2], improving Park’s theorem [17, Th. 1.2.]. An important ingredient, implicitly [20] or explicitly [17], is the growth rate of the log-derivative of the Ruelle zeta vs. the Selberg zeta log-derivative [3].

Though the expected exponent on Riemann surfaces is $\frac{1}{2} + \varepsilon$, the above mentioned $\frac{3}{4}$ was successfully reduced only in the case of modular surfaces $\Gamma \not\subset \mathcal{H}$, $\Gamma \subset PSL(2, \mathbb{Z})$. Iwaniec [12] obtained $\frac{35}{48} + \varepsilon$, Luo and Sarnak [15] $\frac{7}{10} + \varepsilon$, Cai [4] $\frac{71}{102} + \varepsilon$, Soundararajan and Young [22] $\frac{25}{36} + \varepsilon$.

Following Gallagher’s [7] approach to the Riemann zeta, we shall prove that $\frac{7}{10}$ can be achieved for $\Gamma \subset PSL(2, \mathbb{R})$ outside a set of finite logarithmic measure.

2. Main result

Let $\psi(x) = \sum N(P) \leq x \Lambda(P)$ and $\psi_1(x) = \int_1^x \psi(t) dt$ be the Chebyshev resp. integrated Chebyshev function, as usual. Recall that the error term $O \left( x^{\frac{3}{4}} (\log x)^{-1} \right)$ in the prime geodesic theorem corresponds to $O \left( x^{\frac{3}{4}} \right)$ in the explicit formula for $\psi$.

Our main result is given by the following theorem that substantially improves [14, Th. 1.] and [11 Th. 2.]

**Theorem.** Let $\Gamma \subset PSL(2, \mathbb{R})$ be a strictly hyperbolic Fuchsian group. There exists a set $G$ of finite logarithmic measure such that

$$
\psi(x) = x + \sum_{\frac{7}{10} < \rho < 1} \frac{x^\rho}{\rho} + O \left( x^{\frac{3}{4}} (\log x)^{\frac{3}{5} + \varepsilon} \right) \quad (x \to \infty, x \notin G),
$$

where $\varepsilon > 0$ is arbitrarily small.

**Proof.** As the starting point, we shall take Hejhal’s explicit formula for $\psi_1$ with an error term [9 Th. 6.16. on p. 110]

$$
\psi_1(x) = \alpha_0 x + \beta_0 x \log x + \alpha_1 + \beta_1 \log x + F \left( \frac{1}{x} \right) + \frac{x^2}{2} + \sum_{\rho < \rho + 1} \frac{x^{\rho+1}}{\rho (\rho + 1)} + O \left( \frac{x^2 \log x}{T} \right) \quad (x \to \infty),
$$

where $F(x) = (2g - 2) \sum_{k=2}^{\infty} \frac{2k+1}{k(k-1)^2} x^{1-k}$.

The asymptotics of $\psi$ is conveniently derived from the asymptotics of $\psi_1$ via the relation

$$
\int_{x-h}^x f(t) dt \leq f(x) h \leq \int_x^{x+h} f(t) dt
$$

valid for any non-decreasing function $f$, where $h > 0$. 

We have

\[ \psi(x) \leq \frac{1}{h} \int_x^{x+h} \psi(t) \, dt = x + \sum_{\frac{1}{2} < \rho < 1} \frac{x^\rho}{\rho} + O(\log x) + O(h) \]

(1)

\[ + \frac{1}{h} \left| \sum_{\text{Re}(\rho) = \frac{1}{2}, |\gamma| \leq T} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} \right| + O \left( \frac{x^2 \log x}{hT} \right) . \]

Now, for \( Y < T \),

\[ \sum_{\text{Re}(\rho) = \frac{1}{2}, |\gamma| \leq T} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} = \sum_{\text{Re}(\rho) = \frac{1}{2}, |\gamma| \leq T} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} + \sum_{\text{Re}(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} . \]

The trivial bound for the first sum on the right hand side is given by

\[ \frac{1}{h} \left| \sum_{\text{Re}(\rho) = \frac{1}{2}, |\gamma| \leq T} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} \right| = O \left( x^{\frac{3}{2}} \right) . \]

(2)

The second sum is split into

\[ \sum_{\text{Re}(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} - \sum_{\text{Re}(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{x^{\rho+1}}{\rho(\rho+1)} . \]

Let

\[ D_v^T = \left\{ x \in [T, eT) : \sum_{\text{Re}(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{x^{\rho+1}}{\rho(\rho+1)} > x^\alpha (\log x)^\beta (\log \log x)^\beta \right\} , \quad 1 < \alpha < \frac{3}{2} , \beta > 0 . \]

Then,

\[ \mu^x D_v^T = \int_{D_v^T} \frac{dt}{t} = \int_{D_v^T} t^{2\alpha} (\log t)^{2\beta} (\log \log t)^{2\beta} \frac{dt}{t^{1+2\alpha}(\log t)^{2\beta}(\log \log t)^{2\beta}} \]

\[ \leq \frac{1}{(\log T)^{2\beta}(\log \log T)^{2\beta}} \int_{D_v^T} \sum_{\text{Re}(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{t^{\rho+1}}{\rho(\rho+1)} \left( \frac{t^{3-2\alpha}}{t^{1+2\alpha}(\log t)^{2\beta}(\log \log t)^{2\beta}} \right)^2 t^{3-2\alpha} dt \]

\[ = O \left( \frac{T^{3-2\alpha}}{(\log T)^{2\beta}(\log \log T)^{2\beta}} \right) \int_{T}^{eT} \sum_{\text{Re}(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{t^{\rho+1}}{\rho(\rho+1)} \left( \frac{t^{3-2\alpha}}{t^{1+2\alpha}(\log t)^{2\beta}(\log \log t)^{2\beta}} \right)^2 dt / t^4 . \]
According to Koyama [14, p. 79],
\[
\int_{T}^{eT} \left| \sum_{\Re(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{t^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{dt}{t^4} = O \left( \frac{1}{T} \right).
\]

Taking \( Y = T^{3-2\alpha}(\log T)^{1-2\beta}(\log \log T)^{1-2\beta+\varepsilon} \), we obtain
\[
\mu^X D_T^Y \ll \frac{1}{\log T (\log \log T)^{1+\varepsilon}}.
\]

For \( n = \lfloor \log x \rfloor \), \( T = e^n \), denote \( E_n = D_T^Y \). Then, \( \mu^X E_n \ll \frac{1}{n(\log n)^{1+\varepsilon}} \) and \( \mu^X \cup E_n \ll \sum \frac{n(\log n)^{1+\varepsilon}}{n(\log n)^{1+\varepsilon}} < \infty \).

If \( x \in [e^n, e^{n+1}) \setminus E_n \), one gets
\[
\left( \sum_{\Re(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{x^{\rho+1}}{\rho(\rho+1)} \right) \leq x^\alpha (\log x)^\beta (\log \log x)\beta.
\]

We are interested in achieving \( h < x^{\frac{1}{\beta}} \). If it happens that \( x + h \in [e^n, e^{n+1}) \setminus E_n \), one shall have
\[
\sum_{\Re(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{(x + h)^{\rho+1}}{\rho(\rho+1)} = O \left( x^\alpha (\log x)^\beta (\log \log x)\beta \right)
\]
as well, since \( x + h < 2x \).

The other possibility is that \( x + h \in [e^{n+1}, e^{n+2}) \). In that case, we proceed as follows
\[
\sum_{\Re(\rho) = \frac{1}{2}, Y < |\gamma| \leq T} \frac{(x + h)^{\rho+1}}{\rho(\rho+1)} = \sum_{\Re(\rho) = \frac{1}{2}, Y < |\gamma| \leq eT} \frac{(x + h)^{\rho+1}}{\rho(\rho+1)} - \sum_{\Re(\rho) = \frac{1}{2}, T < |\gamma| \leq eT} \frac{(x + h)^{\rho+1}}{\rho(\rho+1)}.
\]

For \( x + h \in [e^{n+1}, e^{n+2}) \setminus E_{n+1} \), we get
\[
\left( \sum_{\Re(\rho) = \frac{1}{2}, Y < |\gamma| \leq eT} \frac{(x + h)^{\rho+1}}{\rho(\rho+1)} \right) = O \left( x^\alpha (\log x)^\beta (\log \log x)\beta \right).
\]

To estimate \( \sum_{\Re(\rho) = \frac{1}{2}, T < |\gamma| \leq eT} \frac{(x + h)^{\rho+1}}{\rho(\rho+1)} \), we shall consider
\[
D_T^{e^2T} = \left\{ x \in [eT, e^{2T}) : \left| \sum_{\Re(\rho) = \frac{1}{2}, T < |\gamma| \leq eT} \frac{x^{\rho+1}}{\rho(\rho+1)} \right| > x^\alpha (\log x)^\beta (\log \log x)\beta \right\}.
\]
By the argumentation above leading to the estimate of $\mu^x D_T^e$, we get

$$\mu^x D_T^e \ll \frac{T^{3-2\alpha}}{(\log T)^{2\beta} (\log \log T)^{2\beta}} \frac{1}{T} \ll \frac{1}{T^{2\alpha-2}}.$$  

Recall that $n = \lfloor \log x \rfloor$, $T = e^n$ and denote $F_{n+1} = D_T^e$. Notice that $\mu^x F_{n+1} < \frac{1}{e^{(2\alpha-2)n}}$ and the series $\sum \frac{1}{e^{(2\alpha-2)n}}$ converges since $\alpha > 1$. Thus, if we additionally assume $x + h \in [e^{n+1}, e^{n+2}) \setminus F_{n+1}$, we get

$$\sum_{\Re(\rho) = \frac{1}{2}, \sigma < \rho < 1} (x + h)^{\rho+1} = O \left( x^{\alpha}(\log x)^{\beta} (\log \log x)^{2\beta} \right).$$

Looking back at (1), and taking into account the relations (2), (3), (4) and (5), we are left to optimize

$$h, \frac{x \log x}{h}, x Y \text{ and } \frac{x^n (\log x)^{\beta} (\log \log x)^{\delta}}{h}, \text{ i.e.,}$$

$$h, x^{\frac{1}{2}} \cdot x^{3-2\alpha} (\log x)^{1-2\beta} (\log \log x)^{1-2\beta} \text{ and } \frac{x^n (\log x)^{\beta} (\log \log x)^{\delta}}{h}$$

since $\alpha > 1$, $T \approx x$ and $Y = O \left( x^{3-2\alpha} (\log x)^{1-2\beta} (\log \log x)^{1-2\beta+\epsilon} \right)$. Choosing $h \approx x^{\frac{1}{2}} (\log x)^{\frac{1}{2}} (\log \log x)^{\frac{1}{2}}$, we get $\frac{1}{2} + 3 - 2\alpha = \frac{7}{2}$ and $1 - 2\beta = \frac{5}{2}$. Hence, $\alpha = \frac{7}{2}$ and $\beta = \frac{5}{2}$. This completes the proof since the sets $E = \cup E_n$ and $F = \cup F_n$ have finite logarithmic measure.

\[\square\]

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