Bounding Membership Inference
Anvith Thudi, Ilia Shumailov, Franziska Boenisch, and Nicolas Papernot
University of Toronto and Vector Institute

Abstract
Differential Privacy (DP) is the de facto standard for reasoning about the privacy guarantees of a training algorithm. Despite the empirical observation that DP reduces the vulnerability of models to existing membership inference (MI) attacks, a theoretical underpinning as to why this is the case is largely missing in the literature. In practice, this means that models need to be trained with DP guarantees that greatly decrease their accuracy.

In this paper, we provide a tighter bound on the positive accuracy (i.e., attack precision) of any MI adversary when a training algorithm provides $(\varepsilon, \delta)$-DP. Our bound informs the design of a novel privacy amplification scheme: an effective training set is sub-sampled from a larger set prior to the beginning of training. We find this greatly reduces the bound on MI positive accuracy. As a result, our scheme allows the use of looser DP guarantees to limit the success of any MI adversary; this ensures that the model’s accuracy is less impacted by the privacy guarantee. While this clearly benefits entities working with far more data than they need to train on, it can also improve the accuracy-privacy trade-off on benchmarks studied in the academic literature. Consequently, we also find that subsampling decreases the effectiveness of a state-of-the-art MI attack (LiRA) much more effectively than training with stronger DP guarantees on MNIST and CIFAR10. We conclude by discussing implications of our MI bound on the field of machine unlearning.

1 Introduction
Differential Privacy (DP) (Dwork et al., 2006) is employed extensively to reason about privacy guarantees in a variety of settings (Dwork, 2008). Recently, DP started being used to give privacy guarantees for the training data of deep neural networks (DNNs) learned through stochastic gradient descent (SGD) (Abadi et al., 2016). However, even though DP provides privacy guarantees and bounds the worst case privacy leakage, it is not immediately clear how these guarantees bound the accuracy of known existing forms of privacy infringement attacks.
At the time of writing, the most practical attack on the privacy of DNNs is Membership Inference (MI) (Shokri et al., 2017), where an adversary predicts whether or not a model used a particular data point for training; note that this is quite similar to the hypothetical adversary at the core of the game instantiated in the definition of DP. MI attacks saw a strong interest by the community, and several improvements and renditions were proposed since its inception (Sablayrolles et al., 2019; Choquette-Choo et al., 2021; Maini et al., 2021; Hu et al., 2021). Having privacy guarantees would desirably defend against MI, and in fact, current literature highlights that DP does give an upper bound on the performance of MI adversaries (Yeom et al., 2018; Erlingsson et al., 2019; Sablayrolles et al., 2019; Jayaraman et al., 2020).

In this paper, we first propose a tight bound on MI performance for training algorithms that provide $\varepsilon$-DP; in particular, the quantity of interest for our bound is MI positive accuracy (i.e. the MI attack’s precision). We focus on positive accuracy since predicting incorrectly often (as measured by accuracy) is not an effective attack strategy in practice, whereas being always correct that a given datapoint appears in a training set is advantageous. Bounding positive accuracy, as is done in this paper, removes this worst-case scenario by bounding the certainty any adversary can have when predicting positively. The key insight for our bound is that an MI adversary only ever has a finite set of points they suspect were used during training. This allows us to apply a counting lemma to simplify terms in the equation for MI positive accuracy. Our bound is tight up to the looseness of the DP guarantee holding across all datasets. We also extend our result to bound MI positive accuracy for $(\varepsilon, \delta)$-DP, a common relaxation of $\varepsilon$-DP.

Perhaps most importantly, in obtaining our bound, we expose how MI positive accuracy is influenced by a form of privacy amplification where the training dataset itself is sub-sampled from a larger dataset. Amplification is a technique pervasively found in work improving the analysis of DP learners like DP-SGD (Abadi et al., 2016). It allows one to prove tighter bounds on privacy leakage. Here, we observe the effect of our amplification on lowering MI positive accuracy is significantly stronger than the effect of previous forms of amplifications used to obtain tighter $\varepsilon$-DP guarantees.

In particular, amplification from subsampling achieves a greater effect on MI positive accuracy than batch sampling – a common privacy amplification scheme for training deep neural networks. This finding has immediate consequences for entities with far more data than they need to train on: subsampling data before training will greatly contribute to reducing the risk of membership inference. We also observe benefits for benchmarks commonly studied in the academic literature like MNIST and CIFAR10. For instance, we find that the success of a state-of-the-art MI attack (LiRA Carlini et al. (2022)) experiences an outweighted effect from subsampling compared to reducing the $\varepsilon$ in the DP guarantee. This confirms that studying MI positive accuracy exposes factors to control MI performance that allow training with higher accuracy than simply reasoning with the DP guarantee would allow.

Our bound also has consequences for the problem of unlearning (or data
forgetting in ML) introduced by Cao and Yang (2015). In particular the MI accuracy on the point to be unlearned is a popular measure for how well a model has unlearned it (Baumhauer et al., 2020; Graves et al., 2020; Golatkar et al., 2020b,a). However empirical verification of the MI accuracy can be open-ended, as it is subjective to the attack employed. This can lead one to underestimate the need for unlearning by overestimating how well a model has already unlearned a point. Theoretical bounds on all MI attacks, such as the one proposed in this work, circumvent this issue; a bound on the precision of MI attacks, that is, the probability that a data point was used in the training dataset, indicates a limitation for any entity to discern if the model has been trained on that data point. In the case when this probability is sufficiently low (where sufficiently is defined apriori), one can then claim to have unlearned by achieving a model sufficiently likely to have not come from training with the data point. Our analysis shows that, if dataset sub-sampling is used, one can unlearn under this definition by training with a relatively large $\varepsilon$-DP (and thus have less cost to accuracy performance than previously observed).

To summarize, our contributions are:

• We present a bound on MI positive accuracy (i.e. attack precision) for $\varepsilon$-DP and $(\varepsilon,\delta)$-DP. This bound is tight up to the looseness of the DP guarantee holding across all datasets.

• We further demonstrate how to lower this bound using a novel privacy amplification scheme built on dataset subsampling; we explain the theoretical benefits of this for entities that do not need to train on all their data.

• We show this new factor our bound highlights also has empirical benefits on CIFAR10 and MNIST: dataset subsampling has an outweighed effect on the success of a state-of-the-art MI attack (LiRA).

• We discuss the benefits of such bounds to machine unlearning as a rigorous way to use MI as a metric for unlearning.

2 Background

2.1 Differential Privacy

Differential privacy (DP) (Dwork et al., 2006) bounds how different the outputs of a function on adjacent inputs can be in order to provide privacy guarantees for the inputs. More formally, a function $F$ is $\varepsilon$-DP if for all adjacent inputs $x$ and $x'$ (i.e. inputs with Hamming distance of 1) we have for all sets $S$ in the output space:

$$P(F(x) \in S) \leq e^{\varepsilon} P(F(x') \in S).$$

(1)

There also is the more relaxed notion of $(\varepsilon,\delta)$-DP which is defined for the same setup as above, but introduces a parameter $\delta \in (0,1]$ such that $P(F(x) \in S) \leq$
\( e^\varepsilon \mathbb{P}(F(x') \in S) + \delta \). Notably, \((\varepsilon, \delta)\)-DP is used for functions where it is more natural to work with \(\ell_2\) metrics on the input space, having to do with how DP guarantees are obtained.

To achieve DP guarantees, one usually introduces noise to the output of the function \(F\). The amount of noise is calibrated to the maximal \(\ell_2\) or \(\ell_1\) difference between all possible outputs of the function on adjacent datasets (also called sensitivity). Significant progress was achieved on minimizing the amount of noise needed for a given sensitivity (Balle and Wang 2018), and on how DP guarantees scale when composing multiple DP functions (Dwork et al. 2010; Kairouz et al. 2015).

Abadi et al. (2016); Song et al. (2013); Bassily et al. (2014) demonstrated a method to make the final model returned by mini-batch SGD \((\varepsilon, \delta)\)-DP with respect to its training dataset by bounding the sensitivity of gradient updates during mini-batch SGD and introducing Gaussian noise to each update. This approach became the de-facto standard for DP guarantees in DNNs. However, the adoption is still greatly limited because of an observed trade-off between privacy guarantees and model utility. At the time of writing there is still no feasible way to learn with low \(\varepsilon\) and high accuracy; what is more, past work (Jagielski et al., 2020) observed a gap between observed and claimed privacy, which suggests that DP-analysis, so far, may be too loose.

However, more recently Nasr et al. (2021) showed (using statistical tests and stronger MI adversaries) that the current state of the art methods of achieving \((\varepsilon, \delta)\)-DP guarantees for deep learning are tight, in contrast to the gap Jagielski et al. (2020) observed. This suggests that there is not much more improvement to be gained by studying how to improve the \((\varepsilon, \delta)\)-DP guarantee from a given amount of noise or improving composition rules. Facing this, future improvements in DP training would lie in understanding the guarantees that DP provides against the performance of relevant attacks. Once more informed about the guarantees required during training, this enables the use of looser guarantees if one is only interested in defending against a specific set of attacks.\(^1\)

### 2.2 Membership Inference

Shokri et al. (2017) introduced an MI attack against DNNs, which leveraged shadow models (models with the same architecture as the target model) trained on similar data in order to train a classifier which, given the outputs of a model on a data point, predicts if the model was trained on that point or not. Since the introduction of this initial attack, the community has proposed several improved and variations of the original MI attack (Yeom et al. 2018; Salem et al. 2018; Sablayrolles et al. 2019; Truex et al. 2019; Jayaraman et al. 2020; Maini et al. 2021; Choquette-Choo et al. 2021).

\(^1\)It is worth noting that Nasr et al. (2021) showed that current analytic upper bounds on DP guarantees are tight, measuring them empirically with various strong privacy adversaries. Although results do suggest that bounds match, the paper did not investigate how DP guarantees limit performance of the adversary.
As mentioned previously, Nasr et al. (2021) demonstrated that current analysis of DP-SGD can not be significantly improved, and so further privacy improvements will be in understanding how DP bounds privacy attacks. MI attacks are currently the main practical threat to the privacy of the data used to train DNNs. Hence, tighter bounds on MI attacks when using DP would help entities reason about what DP parameters to train with; this is particularly relevant when stronger DP guarantees are known to also reduce performance and fairness (Bagdasaryan et al., 2019).

In our paper, we will work with the following (abstract) notion of MI in giving our bounds. In particular, we consider MI as a decision problem over the space of models (i.e. weights) such that if the model is in a particular set $S$, it predicts $x^*$ is in the training dataset, else not. That is, to every adversary we associate a set $S$. We then have the following definitions of its "positive accuracy", i.e. accuracy when predicting $x^*$ is in the training set (also called precision or positive predictive value):

**Definition 1 (MI accuracy).** The positive accuracy of a MI adversary, associated to the set $S$, is $\mathbb{P}(x^* \in D|S)$. Similarly, the negative accuracy is $\mathbb{P}(x^* \notin D|S) = 1 - \mathbb{P}(x^* \in D|S)$,

where $D$ is the training dataset and $\mathbb{P}(x^* \in D|S)$ is the probability a data point $x^*$ was in the training dataset used to obtain the models in the set $S$ (i.e. this is a probability over datasets). We explain more about where the randomness is introduced (in particular the probability involved in obtaining a training dataset) in Section 3.1.

### 2.3 Previous Bounds

Before describing past bounds on MI accuracy, we have to formally define the attack setting past work considered. Two of the main bounds (Yeom et al., 2018; Erlingsson et al., 2019) focused on an experimental setup first introduced by Yeom et al. (2018). In this setting, an adversary $f$ is given a datapoint $x^*$ that is 50% likely to have been used to train a model $S$ or not. The adversary then either predicts 1 if they think it was used, or 0 otherwise. Let $b = 1$ or 0 indicate if the datapoint was or was not used for training, respectively; we say the adversary was correct if their prediction matches $b$. We then define the adversary’s advantage as improvement in accuracy over the 50% baseline of random guessing, or more specifically $2(A(f) - 0.5)$ where $A(f)$ is the accuracy of $f$.

For such an adversary operating in a setting where data is equally likely to be included or not in the training dataset, Yeom et al. (2018) showed that they could bound the advantage of the adversary by $e^\varepsilon - 1$ when training with $\varepsilon$-DP. In other words, they showed that they could bound the accuracy of the MI adversary by $e^\varepsilon/2$. Their proof used the fact that the true positive rate (TPR) and false positive rate (FPR) of their adversary could be represented as expectations over the different data points in a dataset, and from that introduced the
Erlingsson et al. (2019) improved on the bound developed by Yeom et al. (2018) for an adversary operating under the same condition by utilizing a proposition given by Hall et al. (2013) on the relation between TPR and FPR for an \((\varepsilon, \delta)\)-DP function. Based on these insights, Erlingsson et al. (2019) bounded the membership advantage by \(1 - e^{-\varepsilon} + \delta e^{-\varepsilon}\), which is equivalent to bounding the accuracy of the adversary by \(1 - e^{-\varepsilon}/2\) when \(\delta = 0\) (i.e. in \(\varepsilon\)-DP). Finally, improving on Erlingsson et al. (2019) by using propositions given by Kairouz et al. (2015), Humphries et al. (2020) bounded the membership advantage by \(e^{-\varepsilon+\delta}\). This translates to an accuracy bound of \(\frac{2e^{-\varepsilon+\delta}}{2e^{-\varepsilon}+1}\) which is the tightest MI accuracy bound we are aware of. We will see that as a corollary of our results we can match Humphries et al. (2020) bound on accuracy, but moreover, our proof shows this bound is tight upto the looseness of the DP guarantee holding over all datasets.

Other works, similar to our setup in Section 3.1, considered a more general setting where the probability of sampling a datapoint in the dataset can vary. For \(\varepsilon\)-DP, Sablayrolles et al. (2019) bounded the probability of a datapoint \(x^*\) being used in the training set of a model (i.e., the accuracy of an attacker who predicted the datapoint was in the dataset of the model which we refer to as positive accuracy) by \(P_{x^*}(1) + \frac{\varepsilon}{4}\) where \(P_{x^*}(1)\) is the probability of the datapoint being in the dataset. Recent concurrent work by Mahloujifar et al. (2022) improves the bound on positive accuracy (also called precision or positive predictive value) to \(\frac{1}{1+e^{-\varepsilon}}\) in the case of \(P_{x^*}(1) = 0.5\) and all other data is fixed. Our bound on positive accuracy both improves over Sablayrolles et al. (2019) and generalizes to a more general setting than the one Mahloujifar et al. (2022) is limited to.

Finally, Jayaraman et al. (2020) bounded the positive predictive value of an attacker on a model trained with \((\varepsilon, \delta)\)-DP when the FPR is fixed. Similarly, Jayaraman et al. further bounded MI advantage under the experiment described by Yeom et al. (2018) for a fixed FPR. Note, that both our work and the previously mentioned bounds are independent of FPR. In particular, Erlingsson et al. (2019) followed a similar technique to Jayaraman et al. (2020), but were able to drop the FPR term using a proposition relating TPR to FPR (Hall et al., 2013).

3 Bounds on Positive and Negative Accuracy of Membership Inference

3.1 The Setting

Our setting is more general than the one introduced by Yeom et al. (2018) and, as we note later, a specific instance of it can be reduced to their setting. In particular, we formalize how an entity samples data into the training dataset, and proceed with our analysis from there.
The intuition for our formalization is the existence of some finite data superset containing all the data points that an entity could have in their training dataset. Yet, any one of these datapoints only has some probability of being sampled into the training dataset. For example, this larger dataset could consist of all the users that gave an entity access to their data, and the probability comes from the entity randomly sampling the data to use in their training dataset. This randomness can be a black-box such that not even the entity knows what data was used to train. In essence, this is the setting Jayarajan et al. (2020) considers, though in their case, the larger dataset is implicit and takes the form of an arbitrary distribution. We can then imagine that the adversary (or perhaps an arbitrator in an unlearning setup) knows the larger dataset and tries to infer whether a particular point was used in the training dataset. The particular general MI attack we analyze and bound is based on this setting.

Specifically, let the individual training datasets $D$ be constructed by sampling from a finite countable set where all datapoints are unique and sampled independently, i.e. from some larger set $\{x_1, \ldots, x_N\}$. That is if $D = \{x_1, x_2, \ldots, x_n\}$ then the probability of sampling $D$ is $P(D) = P_{x_1}(1)P_{x_2}(1) \cdots P_{x_n}(1)P_{x_{n+1}}(0) \cdots P_{x_N}(0)$, where $P_{x_i}(1)$ is probability of drawing $x_i$ into the dataset and $P_{x_i}(0)$ is the probability of not.

We define $D$ as the set of all datasets. Let now $D_{x^*}$ be the set of all datasets that contain a particular point $x^* \in \{x_1, \ldots, x_N\}$, that is $D_{x^*} = \{D : x^* \in D\}$. Similarly let $D_{x^*}'$ be the set of all datasets that do not contain $x^*$, i.e. $D_{x^*}' = \{D' : x^* \notin D'\}$. Note $D = D_{x^*} \cup D_{x^*}'$ by the simple logic that any dataset has or does not have $x^*$ in it. We then have the following lemma.

**Lemma 1.** $D_{x^*}$ and $D_{x^*}'$ are in bijective correspondence with $P(D)P_{x^*}(0) = P(D')$ for $D \in D_{x^*}$ and $D' \in D_{x^*}'$ that map to each other under the bijective correspondence.

*Proof.* Note that for a given $D = \{x_1, \ldots, x_n\} \in D_{x^*}$, $D' = D/x^* \in D_{x^*}'$ is unique (i.e. the map by removing $x^*$ is injective) and similarly for a given $D' \in D_{x^*}'$, $D = D' \cup x^*$ is unique (i.e. the map by adding $x^*$ is injective). Thus, we have injective maps running both ways which are the inverses of each other. As a consequence, we have $D_{x^*}$ and $D_{x^*}'$ are in bijective correspondence.

Now if the larger set of datapoints is $\{x_1 \cdots x_{n-1}, x^*, x_n \cdots x_N\}$ letting $D = \{x_1 \cdots x_{n-1}\} \cup x^*$ and $D' = \{x_1 \cdots x_{n-1}\}$ be any pair of datasets that map to each other by the above bijective map, then note $P(D) = P_{x_1}(1)P_{x_2}(1) \cdots P_{x_{n-1}}(1)P_{x^*}(1) \cdots P_{x_{n+1}}(0) \cdots P_{x_N}(0)$ and $P(D') = P_{x_1}(1)P_{x_2}(1) \cdots P_{x_{n-1}}(1)P_{x^*}(0) \cdots P_{x_{n+1}}(0) \cdots P_{x_N}(0)$. In particular we have $P(D)P_{x^*}(0) = P(D')$.

Once some dataset $D$ is obtained, we call $H$ the training function which takes in $D$ and outputs a model $M$ as a set of weights in the form of a real vector. Recall that $H$ is $\varepsilon$-DP if for all adjacent datasets $D$ and $D'$ and any set of model(s) $S$ in the output space of $H$ (i.e. some weights) we have: $P(H(D) \in S) \leq e^\varepsilon P(H(D') \in S)$. It should be noted that from now on we assume that
the set $S$ has a non-zero probability to be produced by $H$. This is sensible as we are not interested in MI attacks on sets of models that have 0 probability to come from training; note also if $\Pr(H(D) \in S) = 0$, then $\Pr(H(D') \in S) = 0$ for all adjacent $D'$ as $0 \leq \Pr(H(D') \in S) \leq e^{-\varepsilon} \Pr(H(D) \in S) = 0$, and thus the probability is 0 for all countable datasets as we can construct any dataset by removing and adding a data point (which does not change the probability if it is initially 0) countably many times.

### 3.2 Our Bounds

We now proceed to use Lemma 1 to bound the positive and negative accuracy of MI, as stated in Definition 1, for a training function $H$ that is $\varepsilon$-DP under the data-sampling setting defined earlier. Our approach differs from those we discussed in Section 2.3 in that we now focus on the definition of the conditional probability $\Pr(x^* \in D | S)$ as a quotient; finding a bound then reduces to finding a way to simplify the quotient with the $\varepsilon$-DP definition, which we achieve using Lemma 1.

What follows are the bounds, with Section 5, 6, and 7 expanding on the consequences of the bounds.

**Theorem 1** (DP bounds MI positive accuracy). For any MI attack on datapoint $x^*$ (i.e. with any associated set $S$), and the training process $H$ is DP with $\varepsilon$, its positive accuracy is upper-bounded by

$$1 + \frac{e^{-\varepsilon} \Pr(x^* \in 1)}{\Pr(x^* \in 1)}$$

and lower bounded by

$$1 + \frac{e^{\varepsilon} \Pr(x^* \in 1)}{\Pr(x^* \in 1)},$$

where $\Pr(x^* \in 1)$ is the probability of drawing $x^*$ into the dataset.

The high level idea for the proof is to rewrite positive accuracy as a fraction, and use Lemma 1 (applicable since we only deal with countable sets) to simplify terms alongside the $\varepsilon$-DP condition. The $\varepsilon$-DP gives inequalities in both directions, providing us with the lower- and upper-bounds.

**Proof.** The positive accuracy of an adversary $f$ associated to any set $S$ is:

$$\Pr(x^* | S) = \frac{\sum_{D \in \mathcal{D}_{x^*}} \Pr(H(D) \in S) \Pr(D)}{\sum_{D \in \mathcal{D}} \Pr(H(D) \in S) \Pr(D)}.$$  \hspace{1cm} (2)

By the observation $\mathcal{D} = \mathcal{D}_{x^*} \cup \mathcal{D}_{x^*}'$ we have that the denominator can be split into $\sum_{D \in \mathcal{D}_{x^*}} \Pr(H(D) \in S) \Pr(D) + \sum_{D' \in \mathcal{D}_{x^*}'} \Pr(H(D') \in S) \Pr(D')$.

By Lemma 1 we can replace the $D' \in \mathcal{D}_{x^*}'$ in the second sum by $D \in \mathcal{D}_{x^*}$, and replace $\Pr(D')$ by $\Pr(D) \frac{\Pr(x^* \in 0)}{\Pr(x^* \in 1)}$. For $\Pr(H(D') \in S)$ note by $H$ being $\varepsilon$-DP we have $\Pr(H(D') \in S) \geq e^{-\varepsilon} \Pr(H(D) \in S)$ and so with the previous replacements

---

2We can obtain a bound against all points by using the largest $\Pr(x^* \in 1)$.

3Note that the previous bounds apply to any adversary. However if one wants to give a higher "existence lower-bound", i.e an instantiated adversary with a given accuracy, there is the trivial adversary which always predicts the datapoint was used to train and obtains positive accuracy $\Pr(x^* \in 1)$.  

8
we have that the denominator is greater than \((1 + \frac{e^{-\epsilon P_{x^*}(0)}}{P_{x^*}(1)}) \cdot \sum_{D \in \mathcal{D}_{x^*}} \mathbb{P}(H(D) \in S) \mathbb{P}(D)\).

Thus, the positive accuracy of \(f\) is \(\leq \frac{1}{1 + \frac{e^{-\epsilon P_{x^*}(0)}}{P_{x^*}(1)}}\) (i.e. the upper bound). If instead we used the fact that \(\mathbb{P}(H(D') \in S) \leq e^\epsilon \mathbb{P}(H(D) \in S)\), we would find that the positive accuracy of \(f\) is \(\geq \frac{1}{1 + \frac{e^{-\epsilon P_{x^*}(0)}}{P_{x^*}(1)}}\) (i.e. the lower bound).

By the definition of negative accuracy, we have the following corollary:

**Corollary 1** (DP bounds MI negative accuracy). For any MI attack on datapoint \(x^*\) (i.e. with any associated set \(S\)), and the training process \(H\) is DP with \(\epsilon\), then the negative accuracy is upper-bounded by \(\left(1 + \frac{e^{-\epsilon P_{x^*}(1)}}{P_{x^*}(0)}\right)^{-1}\) and lower-bounded by \(\left(1 + \frac{e^{-\epsilon P_{x^*}(1)}}{P_{x^*}(0)}\right)^{-1}\), where \(P_{x^*}(1)\) is the probability of drawing \(x^*\) into the dataset.

**Proof.** Immediately follows from Theorem 1 and Definition 1, as if \(\mathbb{P}(x^*|S) \geq \frac{1}{1 + \frac{e^{-\epsilon P_{x^*}(0)}}{P_{x^*}(1)}}\) then

\[
1 - \mathbb{P}(x^*|S) \leq 1 - \frac{1}{1 + \frac{e^{-\epsilon P_{x^*}(0)}}{P_{x^*}(1)}} = \frac{1}{1 + \frac{e^{-\epsilon P_{x^*}(1)}}{P_{x^*}(0)}}.
\]

Similarly, we get \(1 - \mathbb{P}(x^*|S) \geq 1 - \frac{1}{1 + \frac{e^{-\epsilon P_{x^*}(1)}}{P_{x^*}(0)}} = \frac{1}{1 + \frac{e^{-\epsilon P_{x^*}(0)}}{P_{x^*}(1)}}\).

Note that in the case \(P_{x^*}(1) = P_{x^*}(0) = 0.5\), the bounds given by Theorem 1 and Corollary 1 are identical. Therefore, as \(f(x^*, S)\) must output either 0 or 1, we have a more general claim that the attack accuracy (maximum of positive or negative accuracy) is always bounded by the same values given by Theorem 1.

**Remark on tightness** Note that the bounds in Theorem 2 are tight if all the DP inequalities we use (specifically \(\mathbb{P}(H(D') \in S) \geq e^{-\epsilon} \mathbb{P}(H(D) \in S) \forall D \in \mathcal{D}_{x^*}\)) are tight. This is due to those DP inequalities being the only inequalities appearing in the proof. Hence, any future improvement would have to go beyond the DP-condition and incorporate the looseness of the DP guarantee on different datasets. This means that our bound is optimal when no such knowledge is provided. Recent concurrent work by Mahloujifar et al. (2022) does just that by studying MI accuracy bounds given by a privacy mechanism, beyond just the \((\epsilon, \delta)\)-DP guarantees it yields. We remark now that MI accuracy, which is what Mahloujifar et al. (2022) studies, and MI positive accuracy (what we study) have divergent analyses in the case of \((\epsilon, \delta)\)-DP, which will be seen in the following section (§4).
4 Limitations of \((\varepsilon, \delta)\)-DP

We now illustrate the shortcomings of \((\varepsilon, \delta)\)-DP in giving similar bounds as Theorem 1. We give a practical counter-example that shows for a specific MI attack on an \((\varepsilon, \delta)\)-DP logistic regression there is no bound on its positive accuracy, unlike what follows from our bound for \(\varepsilon\)-DP; note, however, that we do have bounds on MI accuracy (as opposed to positive accuracy) when using \((\varepsilon, \delta)\)-DP as seen by Yeom et al. (2018); Erlingsson et al. (2019).

Following this, we provide a bound on MI positive accuracy when using \((\varepsilon, \delta)\)-DP, but the bound introduces a necessary dependence. We say necessary as the previous example showed MI positive accuracy becomes unbounded as \(P(S|x^*)\) becomes small.

4.1 Positive Accuracy is not always bounded

Consider a set of two (scalar) points \(\{x_1, x_2\}\) which are drawn into the training set \(D\) with \(P_{x_1}(1) = 1\) and \(P_{x_2}(1) = 0.5\); that is \(x_1\) is always in the training set, and \(x_2\) has a 50% chance of being in the training set. Let model \(M\) be a single dimensional logistic regression without bias defined as \(M(x) = wx\) initialized such that for cross-entropy loss \(L\), \(\nabla L|_{x_1} \approx 0\) (i.e. set \(x_1 = \{(10^6, 1)\}\) and the weights \(w = 10^6\) so that \(M(x) = 10^6x\) and thus the softmax output of the model is approximately 1 on \(x_1\) and thus gradient is approximately 0). Conversely set \(x_2\) such that the gradient on it is less than \(-1\) (i.e. for the above setting set \(x_2 = \{(10^6, 0)\}\)).

Now, train the model to \((1, 10^{-5})\)-DP following Abadi et al. (2016) with \(\eta = 1\), sampling rate of 100%, a maximum gradient norm of 1, for one step. Note these are just parameters which we are allowed to change under the DP analysis, and we found the noise we would need for \((1, 10^{-5})\)-DP is 4.0412. Then consider running a MI attack on the above setup where if for some threshold \(\alpha\) the final weights \(W\) are s.t if \(W \leq \alpha\) one classifies those weights as having come from the dataset with \(x_2\), otherwise not. Do note that here we use \(W\) for the final weights as opposed to \(w\) to emphasize that we are now talking about a random variable. The intuition for this attack is that if the dataset only contain \(x_1\) then the weights do not change, but if the dataset contains \(x_2\) we know the resulting gradient is negative (by construction) and thus decreases the weights (before noise).

By the earlier setting on how training datasets are constructed note that \(D = \{x_1\}\) or \(D = \{x_1, x_2\}\), and we will denote these \(D_1\) and \(D_2\) respectively. Note that if \(M\) trained on \(D_1\) following the suggested data points and initial weights, we have the distribution of final weight \(W_{D_1} = N(10^6, \sigma) = N(10^6, 4.0412)\) where \(\sigma\) denotes the noise needed for \((1, 10^{-5})\)-DP as stated earlier. Similarly \(W_{D_2} = N(10^6 - 1, 4.0412)\), since the maximum gradient norm is set to 1.

For the above MI attack we can then bound the positive accuracy as a function of \(\alpha\) by:

\[
P(D_2|W \leq \alpha) = \frac{P(W_{D_2} \leq \alpha) \cdot P(D_2)}{P(W_{D_2} \leq \alpha) \cdot P(D_2) + P(W_{D_1} \leq \alpha) \cdot P(D_1)}
\]
Figure 1: Impact of threshold on positive accuracy and accuracy.

\[ \text{Positive Accuracy vs Threshold} \]

(a) Positive accuracy as a function of the threshold

(b) Accuracy as a function of the threshold

\[ \text{Accuracy vs Threshold} \]

\( \phi(W_{D_2}, \alpha) \cdot 0.5 \)

Note \( \phi(W, \alpha) \) is the (Gaussian) cumulative function of random variable \( W \) up to \( \alpha \). We plot this in Figure 1a, and unlike Theorem 1, note how it is not bounded by anything less than 1 and goes to 1 as the threshold \( \alpha \) decreases (i.e. \( \forall m \in [0, 1) \exists \alpha \text{ s.t } S = (-\infty, \alpha] \) yields positive accuracy greater than \( m \)).

4.2 An \( (\varepsilon, \delta) \)-DP Bound

We have seen so far that MI positive accuracy is not always bounded for \( (\varepsilon, \delta) \)-DP. Nevertheless, Theorem 1 can be generalized to \( (\varepsilon, \delta) \)-DP if we introduce a \( \mathbb{P}(S|x^\ast) \) dependence.

**Theorem 2.** For any MI attack on datapoint \( x^\ast \) (i.e. with any associated set \( S \)), and the training process \( H \) is DP with \( (\varepsilon, \delta) \), its positive accuracy is upper-bounded by

\[
\mathbb{P}(x^\ast | S) \leq (1 + \frac{e^{-\varepsilon \mathbb{P}(x^\ast | 1)}}{\mathbb{P}(x^\ast | 1)}) - \delta e^{-\varepsilon \mathbb{P}(x^\ast | 0)}
\]

so long as the right-hand side is positive.

The proof follows the same strategy as Theorem 1, but also multiplies by 1 in a particular way to deal with the \( \delta \) terms. Do note however that some form of a \( \mathbb{P}(S|x^\ast) \) dependence in the bound is necessary since the MI positive accuracy, as shown in a previous subsection, becomes unbounded as \( \mathbb{P}(S|x^\ast) \) gets arbitrarily small. That is, for there to exist a bound on MI positive accuracy, we need a lower bound on \( \mathbb{P}(S|x^\ast) \); the bound in Theorem 2 shows this is in fact sufficient.

**Proof.** Following the proof of Theorem 1, we have the positive accuracy of any adversary \( f \) (i.e. associated to any set \( S \)) is:

\[
\mathbb{P}(x^\ast | S) = \frac{\sum_{D \in \mathcal{D}_x} \mathbb{P}(H(D) \in S) \mathbb{P}(D)}{\sum_{D \in \mathcal{D}_x} \mathbb{P}(H(D) \in S) \mathbb{P}(D) + \sum_{D' \in \mathcal{D}_x} \mathbb{P}(H(D') \in S) \mathbb{P}(D')}
\] (3)
Now using that $P(H(D) \in S) \geq e^{-\varepsilon}P(H(D) \in S) - e^{-\varepsilon}\delta$ and $P(D') = P(D|x^*(0))$ we have the denominator is $\geq \sum_{D \in D_x^*}(1 + \frac{e^{-\varepsilon}P(x^*(0))}{P_x^*(1)})P(D) - \delta e^{-\varepsilon}\sum_{D' \in D_x^*}P(D')$.

Note that

$$\delta e^{-\varepsilon}\sum_{D' \in D_x^*}P(D') = \delta e^{-\varepsilon}\sum_{D' \in D_x^*}P(D') \sum_{D \in D_x^*}P(H(D) \in S)P(D) \sum_{D \in D_x^*}P(H(D) \in S)P(D).$$

So now cancelling the $\sum_{D \in D_x^*}P(H(D) \in S)P(D)$ in the numerator and denominator, and noting $\sum_{D \in D_x^*}P(H(D) \in S)P(D) = P(S|x^*)$ and $\sum_{D' \in D_x^*}P(D') = P_x^*(0)$, we have $P(x^*|S) \leq (1 + \frac{e^{-\varepsilon}P(x^*(0))}{P_x^*(1)} - \frac{\delta e^{-\varepsilon}P(x^*(0))}{P(S|x^*)})^{-1}$. Note this is only true if the right-hand side of the bound is positive, as the fact $a/b \leq a/c$ if $c \leq b$ (which we used) is only true if $c$ is positive when $a$ and $b$ is positive.

Remarks on $(\varepsilon, \delta)$-DP As shown in Theorem $2$ we can obtain bounds on MI positive accuracy, but we must accept a probability of failure. This is not the case in bounds on MI accuracy with $(\varepsilon, \delta)$-DP, illustrating a difference between the formal study of the two. We will proceed to focus on $\varepsilon$-DP in the other sections of the paper, however everything follows analogously for $(\varepsilon, \delta)$-DP if one lower-bounds $P(S|x^*)$, i.e. accepts $X$ probability of failure, and so only studies $P(S|x^*) > X$.

4.3 Comparison with $(1 - \delta)$ Approach to $(\varepsilon, \delta)$ Positive Accuracy Bounds

As a last point of discussion on $(\varepsilon, \delta)$-DP, we provide a comparison to Mahloujifar et al. (2022) and outline how our analysis yields a more relevant (and stronger) approach to bounding failure rates when using $(\varepsilon, \delta)$-DP. Following Eq. 15 in Mahloujifar et al. (2022) which states the $\varepsilon$-DP conditions holds for any fixed $(D, D')$ pair with probability $(1 - \delta)$, the bounds in Theorem $\Pi$ trivially apply to $(\varepsilon, \delta)$-DP with probability $(1 - (\delta))^N$ where $N$ is the number of $(D, D')$ pairs appearing from the random sampling. Note this $N$ could be quite large, i.e. if $n$ is the number of datapoints in the larger set, $N = \sum_{i=1}^{n-1} \binom{n-1}{i}$ if every point is being randomly sampled. The scenario studied by Mahloujifar et al. (2022) bypasses the $N$ dependence by only considering one $(D, D')$ pair, i.e. all the other points in the dataset are fixed and only the datapoint in question is being randomly sampled. This assumption made by Mahloujifar et al. (2022) is not applicable in practice as we do not know which datapoints the adversary is going to test: hence, we would have to resort to randomly sampling all of them. Instead, our failure analysis in Section $\Pi$ particularly Theorem $2$ conveniently bypasses this problem by telling us low probability states are what lead to failures (removing the $N$ dependence in the failure rate).

12
5 The Effect of $\mathbb{P}_{x^*}(1)$

We now focus on the effect of $\mathbb{P}_{x^*}(1)$. Section 5.1 explains how the privacy amplification, i.e., lowering of our MI positive accuracy bound, we observe from decreasing $\mathbb{P}_{x^*}(1)$ is fundamentally different than the privacy amplification on MI from batch sampling. Section 5.2 outlines the practical consequences of this for a defender.

5.1 A New Privacy Amplification for MI

Our bound given by Theorem 1 can be reduced by decreasing $\mathbb{P}_{x^*}(1)$ or $\varepsilon$. Furthermore, note batch sampling, the probability for a data point to be used in the batch for a given training step, reduces MI positive accuracy as it reduces $\varepsilon$. So dataset sub-sampling ($\mathbb{P}_{x^*}(1)$) and batch sampling both decrease our bound, and we term their effect "privacy amplification" (for MI) as they decrease privacy infringement (analogous to how "privacy amplification" for DP refers to methods that reduce privacy loss). However, is the effect of dataset sub-sampling and batch sampling different?

Before proceeding, it is useful to get a sense of the impact $\mathbb{P}_{x^*}(1)$ has on our bound. We plot $\mathbb{P}_{x^*}(1)$ against our positive MI accuracy bound given by Theorem 1 in Figure 2 for different $\varepsilon$. Notably, for a specific case when $\mathbb{P}_{x^*}(1)$ is small, we get that the positive accuracy is bounded by 6.9% for $\varepsilon = 2$ and $\mathbb{P}_{x^*}(1) = 0.01$ (i.e. by sampling with low enough probability, we certify that they will be correct at most 6.9% of the time).

We now turn to comparing the effect of batch sampling to the effect of $\mathbb{P}_{x^*}(1)$ (note $\mathbb{P}_{x^*}(0)/\mathbb{P}_{x^*}(1) = (1-\mathbb{P}_{x^*}(1))/\mathbb{P}_{x^*}(1)$). First it is worth noting that the two amplification methods are mostly independent, i.e. decreasing $\mathbb{P}_{x^*}(0)/\mathbb{P}_{x^*}(1)$ mostly places no restriction on the sampling rate for batch sizes (with some exception). Nevertheless we can ignore this restriction for the time being as we are interested in their independent mathematical behaviours. Including $q$ for DP batch privacy amplification, we can compare its impact to $\mathbb{P}_{x^*}(1)$ by looking at the term $e^{-q\varepsilon}\mathbb{P}_{x^*}(0)/\mathbb{P}_{x^*}(1)$ in the bound given by Theorem 1; the goal is to maximize this to make the upper bound as small as possible. In particular, we see that decreasing $q$ increases this term by $O(e^{-t})$ where as decreasing $\mathbb{P}_{x^*}(1)$ increases this term by $O((1 - t)/t)$, which is slower than $O(e^{-t})$ up to a point, then faster (do note that we are looking at the order as the variable $t$ decreases). Figure 8 plots this relation, however note that the specific values are subject to change with differing constant. Nevertheless what does not change with the constants are the asymptotic behaviours, and in particular we see $\lim_{t \to 0} O((1 - t)/t) = \infty$ where as $\lim_{t \to 0} O(e^{-t}) = \text{constant}$.

Thus, we can conclude the effects of data sampling and batch sampling are different to our bound. Therefore, data sampling presents a complementary privacy amplification scheme for MI positive accuracy. As a last remark, note

---

4We say "mostly" as this is true up to a point. In particular the expectation of the training dataset size decreases with smaller dataset sampling probabilities, and thus the lowest possible batch sampling rate $1/n$ increases in expectation.
that the same comparison holds more generally when comparing the effect of $\varepsilon$ and $\mathbb{P}_{x^*}(1)$.

![Figure 2](image)

Figure 2: Our upper bound on MI positive accuracy as a function of $\mathbb{P}_{x^*}(1)$

![Figure 3](image)

Figure 3: Comparing the DP amplification observed by decreasing batch probability (given by $e^{-t}$) to the amplification we observe from decreasing $\mathbb{P}_{x^*}(1)$ (given by $(1-t)/t$).

### 5.2 Usefulness for a Defender

We now explain one course of action a defender can take in light of this new privacy amplification for MI. In particular note that an upper bound on $\mathbb{P}_{x^*}(1)$ translates to an upper bound on the relation found in Theorem 1 (as the bound is monotonically increasing with $\mathbb{P}_{x^*}(1)$); hence one can, in practice, focus on giving smaller upper-bounds on $\mathbb{P}_{x^*}(1)$ to decrease MI positive accuracy.

A possible approach to this can be described as follows: assume a user (the defender) is given some sample $D \subset \{x_1, \cdots, x_N\}$ drawn with some unknown distribution. In particular, the user does not know the individual probabilities for the points being in $D$ and whether these points were drawn independently.
However, assume that the user obtains from $D$ the training dataset $D_{\text{train}}$ by sampling any point from $D$ independently with probability $T$. Our bound then bounds the conditional probability $\mathbb{P}(x^*|S, D)$ with $\mathbb{P}_{x^*}(1) = T$. Note that $\mathbb{P}(x^*|S) \leq \mathbb{P}(x^*|S, D)$ as $x^* \in D$, so our upper-bound also bounds $\mathbb{P}(x^*|S)$.

That is, one can use our bound to give upper-bounds for points sampled from unknown (not necessarily i.i.d.) distributions if independent sub-sampling (i.e., as needed for Lemma 1) is applied after drawing from this unknown distribution.

This does come with some drawbacks. In general one wants to train with more data, but by further sampling with probability $T$ we reduce our expected training dataset size. As a consequence, a user will have to make the decision between how low they can make $T$ (in conjunction with the $\varepsilon$ parameter they choose) compared to how small a dataset they are willing to train on. We leave this type of decision making for future work.

### 5.3 Empirical Attack Case Study

To explore the practical effect of $\mathbb{P}_{x^*}(1)$ on MI positive accuracy when training with different $(\varepsilon, \delta)$-DP guarantees, we implement the current state-of-the-art membership inference attack: LiRA [Carlini et al. 2022]. In the original paper, the attack was not evaluated against DP trained models, so we reimplement the attack ourselves on MNIST and CIFAR10. In particular, we follow Algorithm 1 in Carlini et al. [2022] but now the training function is DP-SGD, varying $\varepsilon$ with fixed $\delta = 10^{-5}$, and using the hyperparameters of Tramer and Boneh [2020] to train the models. We train 20 shadow models for each $\varepsilon$ setting (i.e. reuse the same shadow models for all the points we test), sampling their training sets uniformly with 50% probability from the original training set. Our target models use the same training function as the shadow models, but we now vary the $\mathbb{P}_{x^*}(1)$ to capture the effectiveness of the attack on target models with different sampling rates.

We evaluate the attack by computing the scores (outputted by Algorithm 1 [Carlini et al. 2022]) for the first 1000 training points of MNIST and CIFAR10 respectively. In each setting we consider different thresholds for acceptance according to a quantile based approach: we consider accepting the top $i * 2.5\%$ quantile of scores for every $i \in [40]$, and report the max precision (positive accuracy) of all these different thresholds (i.e. select the highest precision threshold).

Figure 4 plots the average positive accuracy (from 5 repeated trials) of the LiRA attack for different $\varepsilon$ as a function of the $\mathbb{P}_{x^*}(1)$ used for the target model. We further plot the positive accuracy of the baseline attack of always predicting a datapoint is in the training dataset. As we see the LiRA attack consistently outperforms the baseline, but importantly, we see that the sampling is the dominant factor dictating how much we outperform the baseline.

To illustrate this outweighed effect of sampling over $\varepsilon$, Figure 5 plots the average improvement (over $\varepsilon$ and trials) against the baseline attack. What we see is that both low and high sampling rates see less of an improvement over the baseline than medium sampling rates. In particular, this shows low sampling disproportionately mitigates membership inference. This behaviour is largely
Figure 4: Plots of the positive accuracy of the LiRA attack on MNIST and CIFAR10.

Figure 5: Plots of the positive accuracy improvement of the LiRA attack over the baseline attack on MNIST and CIFAR10. We include the 95% confidence interval (over the epsilon and trials).

consistent with what we expect from our bounds (see Figure 2).

5.4 Effect on Accuracy

To understand the benefits of low sampling (which allows higher $\varepsilon$) versus its effect of smaller training sets for common datasets, we plotted accuracy vs. $P_{x^*}(1)$ for CIFAR10 and MNIST with different fixed target bounds derived from Theorem 2 (setting $\delta = 10^{-5}$ always, and considering at most 10% failure, i.e. $P(S \mid x^*) = 0.1$). Note the fixed target bounds were chosen to be those achieved when setting $P_{x^*}(1) = 0.5$ and taking $\varepsilon = 3, 2, 1$. The models and training hyperparameters (learning rate, mini-batch size) are those used by Tramèr and Boneh [2020], with the hyperparameters being those reported to perform the best for $\varepsilon = 3$. The results (plotting the average over 5 trials) are shown in Figures 6 to 9, where note final accuracy is the accuracy after training for 100 epochs and best accuracy is the highest achieved accuracy during the
training run. We observe that MNIST often finds an optimal accuracy with $P_{x^*}(1) = 0.5$. CIFAR10 suffers somewhat more disproportionately from having smaller epsilons than having smaller datasets, tending to have an optimal accuracy with $P_{x^*}(1) = 0.3$. This is even more apparent when looking at the final accuracy for CIFAR10, where $P_{x^*}(1) = 0.1$ often performs best. We conclude that CIFAR10 performance benefits from low sampling (with some fixed target MI bound), though further work is needed to understand the benefits for other harder/larger datasets.

Nevertheless, we believe our results illustrate a general point of consideration when implementing DP for common DNNs: to balance the trade-off between dataset size and $\varepsilon$.

### 6 Importance to Data Deletion

The ability to decrease MI accuracy, i.e. the ability for an arbitrator to attribute a data point to a model, has consequences for machine unlearning and data deletion. Cao and Yang [2015] were the first to describe a setting where it is important for the model to be able to "forget" certain training points. The authors focused on the cases where there exist efficient analytic solutions to
Figure 8: Plots of the best accuracy vs. $P_x^*(1)$ at differing fixed bounds for MNIST. We include the 95% confidence interval (over the trials).

Figure 9: Plots of the final accuracy vs. $P_x^*(1)$ at differing fixed bounds for MNIST. We include the 95% confidence interval (over the trials).
this problem. The topic of machine unlearning was then extended to DNNs by Bourtoule et al. (2019) with the definition that a model has unlearned a data point if after the unlearning, the distribution of models returned is identical to the one that would result from not training with the data point at all. This definition was also stated earlier by Ginart et al. (2019) for other classes of machine learning models.

Given that unlearning is interested in removing the impact a data point had on the model, further work employed MI accuracy on the data point to be unlearned as a metric for how well the model had unlearned it after using some proposed unlearning method [Baumhauer et al., 2020, Graves et al., 2020, Golatkar et al., 2020]. Yet, empirical estimates on the membership status of a datapoint are subjective to the concrete MI attacks employed. Indeed it may be possible that there exists a stronger attack. The issue of empirical MI for measuring unlearning was recently illustrated by Thudi et al. (2021) where they highlighted undesirable disconnects between it and other measures of unlearning.

Applying our result to unlearning. Analytic bounds to MI attacks, on the other hand, resolve the subjectivity issue of MI as a metric for unlearning as they bound the success of any adversary. In particular one could give the following definition of an unlearning guarantee from a formal MI positive accuracy bound:

Definition 2 (B-MI Unlearning Guarantee). An algorithm is B-MI unlearnt for \( x^* \) if \( P(\hat{x}^* \notin D|S) \geq B \), i.e. the probability of \( x^* \) not being in the training dataset is greater than \( B \).

Therefore, our result bounding positive MI accuracy has direct consequences on the field of machine unlearning. In particular if \( P(x^* \in D|S) \) is sufficiently low, that is the likelihood of \( S \) coming from \( x^* \) is low, then an entity could claim that they do not need to delete the users data since their model is most likely independent of that data point as it most likely came from a model without it: i.e. leading to plausible deniability. Note we defined this type of unlearning as a B-MI unlearning guarantee. This is similar to the logic presented by Sekhari et al. (2021) where unlearning is presented probabilistically in terms of \((\varepsilon, \delta)\)-unlearning.

We also observe an analogous result to Sekhari et al. (2021) where we can only handle a maximum number of deletion requests before no longer having sufficiently low probability. To be exact, let us say we do not need to undergo any unlearning process given a set of data deletion request \( \hat{x}^* \) if \( P(\hat{x}^* \notin D|S) \geq B \) for some \( B \) (i.e. we are working with probability of not having that set of data in our training set which we want to be high). Note that we sampled data independently, thus if \( \hat{x}^* = \{x^*_1, x^*_2, \ldots, x^*_m\} \), then \( P(\hat{x}^* \notin D|S) = P(x^*_1 \notin D|S) \cdots P(x^*_m \notin D|S) \).

Now, for simplicity, assume the probability of drawing all points into the datasets are the same, so that for all \( x^*_i \) we have the same bound given by Corollary 1 that is \( P(x^*_i \notin D|S) > L \) for some \( L \leq 1 \). Then we have \( P(\hat{x}^* \notin D|S) > L \) for some \( L \leq 1 \).
$D(S) \geq L^m$ and so an entity does not need to unlearn if $L^m \geq B$, i.e. if $m \leq \frac{\ln 1/B}{\ln 1/L} = \frac{\ln B}{\ln L}$. This gives a bound on how many deletion requests the entity can avoid in terms of the lower bound given in Corollary 1. In particular, note that if $\{x_1 \cdots x_N\}$ is the larger set of data points an entity is sampling from, and $P_x(1) = c/N \forall x \in \{x_1 \cdots x_N\}$, then the lower bound given by Corollary 1 is $\left(1 + \frac{e^{-\epsilon}}{1-\frac{\epsilon}{N}}\right)^{-1}$.

**Corollary 2 (B-MI Unlearning Capacity).** If $\{x_1 \cdots x_N\}$ is the larger set of data points an entity is sampling from, and $P_x(1) = c/N \forall x \in \{x_1 \cdots x_N\}$, then one can delete $m \leq \frac{\ln B}{\ln L(c)}$ with $L(c) = \left(1 + \frac{e^{-\epsilon}}{1-\frac{\epsilon}{N}}\right)^{-1}$ and satisfy B-MI unlearning.

Sekhari et al. (2021) showed that with typical DP the deletion requests grow linearly with the size of the training (in the above case $c$ represents the expected training set size). We thus compare a linear line w.r.t to $c$ to $\frac{\ln B}{\ln L(c)}$ (given by Corollary 2) in Figure 10 to observe their respective magnitude: we fix $B = 0.8$, $N = 10000$ and $\epsilon = 1$ as we are interested in asymptotics. We observe that our deletion capacity is significantly higher for low expected training dataset sizes and is marginally lower than a linear trend for larger training set sizes.

![Deletion Capacity vs. Expected Training Set Size](image)

Figure 10: Comparing our deletion capacity trend to the trend Sekhari et al. (2021) describes. In particular our number of deletions degrades with training size while theirs increasing.

### 7 Discussion

We break our discussion into two themes. The first concerns discussion on how bounds on MI precision can be directly applied to bound other attacks/properties of interest; the objective is to understand the utility of only studying MI attacks. In §7.1 we discuss how our bounds can be applied to bound ”group” membership inference. In §7.2 we discuss how membership inference precision
bounds limit the confidence any reconstruction attack can have, and point out an alternative question of if MI precision bounds can lead to lower-bounds on the error of "interesting" reconstruction attacks. Lastly, in §7.3 we remark that MI accuracy bounds lead to bounds on the generalization gap of machine learning models, and point out this approach to bounding properties of interest could be generally applicable.

The second half of the discussion focuses on our work in relation to past work.

7.1 Generalizing to Group Membership Inference

One could consider a more general membership inference problem, where instead of asking if we trained on a particular datapoint $x^*$, we ask if we trained on any $x^* \in G$ for some set $G \subset \mathbb{D} := \{x_1, \ldots, x_N\}$. For example, this could be relevant when the set $G$ defines a subgroup: e.g. all people born in a particular city. Hence, inference reveals how much a final model can leak about these specific group features. Bounding the precision of this group membership is the same as bounding $P(G \cap D \neq \emptyset | S)$. We can obtain a bound on this as a corollary of Theorem 1; the crux of the proof is noting our bounds are invariant to the other datapoints in the larger set of possible datapoints, and this gives us a type of independence at the level of applying our bounds.

**Corollary 3 (Group Membership Inference).** Consider a subset of the set of possible data $G \subset \mathbb{D}$, and let $D$ be the training set obtained following $\varepsilon$-DP setting we have considered so far in the paper, described in Section 3.1. We then have $P(G \cap D \neq \emptyset | S) \leq 1 - \prod_{x^* \in G} \left(1 + \frac{e^{\varepsilon P(x^*)}(1)}{e^{\varepsilon P(x^*)}(0)}\right)$.

**Proof.** Let $G = \{x_{i_1}, \ldots, x_{i_g}\}$ for $g = |G|$. Note $P(G \cap D \neq \emptyset | S) = 1 - P(x_{i_j} \notin D, \forall j \in [g]|S)$. Now note $P(x_{i_j} \notin D, \forall j \in [g]|S) = \prod_{j \in [g]} P(x_{i_j} \notin D|S, x_{i_{j'}} \notin D \text{ for } j' < g)$. Note the condition $x_{i_{j'}} \notin D \text{ for } j' < g$ restricts us to considering $\mathbb{D} := \mathbb{D} \setminus \{x_{i_{j'}} \text{ for } j' < g\}$, but our bounds in Corollary 1 did not depend on what $\mathbb{D}$ exactly were. So we can apply the lower-bound from Corollary 1 for each term in the product, and hence get the upper-bound $P(G \cap D \neq \emptyset | S) \leq 1 - \prod_{x^* \in G} \left(1 + \frac{e^{\varepsilon P(x^*)}(1)}{e^{\varepsilon P(x^*)}(0)}\right)^{-1}$ as desired.

Note an analogous bound can be obtained for $(\varepsilon, \delta)$-DP by using Theorem 2 to get a lower-bound on $P(X^* \notin D|S)$, which is what we needed for the proof.

7.2 Applying to Data Reconstruction

An interesting question is how training with DP impacts the effectiveness of data reconstruction attacks. The goal of a data reconstruction attack is, given a model (and/or the sequence of models used to obtain it), to have an algorithm that outputs a training point: more realistically, a point close to a training...
point. In this sense, the attack "reconstructed" a training datapoint. An immediate consequence of our MI positive accuracy bounds is a statement that data reconstruction attacks are not trustworthy; even when they reconstruct an image close to a possible datapoint, the probability of the datapoint actually being a training datapoint is bounded.

**Corollary 4** (Data Reconstruction not Trustworthy). *Consider a deterministic "data reconstruction" algorithm $R$ that takes a model (or the training sequence of models) $M$ and outputs a datapoint. We say it "reconstructs" a training datapoint out of the set of possible datapoints $D := \{x_1, \cdots, x_N\}$ if for some metric $d$, $d(R(M), x_i) \leq \varepsilon$ for some $\varepsilon \geq 0$. For any $x_i$, consider the set $S := \{M : d(R(M), x_i) \leq \varepsilon\}$ of models it claims to have trained on $x_i$. By Theorem\[2\] when training with $(\varepsilon, \delta)$-DP, the probability $x_i$ was actually used to train models in $S$ is $\mathbb{P}(x^* \mid S) \leq \left(1 + e^{-\varepsilon} \mathbb{P}_x^*(1) - \frac{\delta e^{-\varepsilon} \mathbb{P}_x^*(0)}{\mathbb{P}(S \mid x^*)}\right)^{-1}$.*

*Proof.* Just an application of Theorem\[2\] \qed

However, we believe a more interesting question is if, considering a specific class of reconstruction algorithms $R$, we can lower bound the reconstruction error (with high probability). We believe this should be possible, but currently are not sure what an appropriate class of $R$ is. A naive idea is that $d(R(M), x) \leq \varepsilon$ iff $x \in D$, that is, the algorithm can only reconstruct datapoints used in the training set. However, the above corollary states no such attacks can exist when training with $\varepsilon$-DP. So it stands this characterization might be too powerful. We leave finding fruitful classes of reconstruction attacks to study for future work.

### 7.3 We can Bound Generalization Error

Yeom et al. (2018) studied MI adversaries in the $\mathbb{P}_x^*(1) = 0.5$ setting that took advantage of a generalization gap $R_g$ between training datapoints and non-training datapoints, and showed an adversary whose accuracy is $(R_g/B + 1)/2$ (Theorem 2 in Yeom et al. (2018)). Note this accuracy is upper and lower-bounded by Theorem\[1\] as the theorem bounds the accuracy of all adversaries when $\mathbb{P}_x^*(1) = 0.5$. Thus we have the following corollary.

**Corollary 5** (MI bounds Generalization Error). *For loss function upper-bounded by $B$ and $\varepsilon$-DP training algorithm, we have $(1 + e^\varepsilon)^{-1} \leq (R_g/B + 1)/2 \leq (1 + e^{-\varepsilon})^{-1}$ where $R_g$ is the generalization gap (Definition 3 in Yeom et al. (2018), where their $D$ is our larger set, and their $S$ is the specific training set sampled from the larger set).*

This tells us how well the $\mathbb{P}_x^*(1) = 0.5$ sampled dataset generalizes to the entire larger set. We believe this general procedure of using specific adversaries with accuracies dependent on values of interest (e.g. the generalization gap here) and applying more general bounds to get a range for those values of interest is something future work can expand on.

---

\footnote{One could consider random algorithms, but fixing the random seed the bound will apply. As the bound applies for every random seed, it also applies in expectation.}
Figure 11: Comparing the upper bound to MI performance we achieved matching Humphries et al. (2020) to that given by Erlingsson et al. (2019) and Sablayrolles et al. (2019) (note $\mathbb{P}_{x^*}(1) = 0.5$ here). In particular note we are tighter for all $\varepsilon$.

7.4 Our Bounds Match and Improve Earlier Results

Note that the setting described in Section 3.1 is equivalent to the MI experiment defined by Yeom et al. (2018) when $\mathbb{P}_{x_i}(1) = 0.5 \forall x_i \in \{x_1, \ldots, x_N\}$. Moreover as a consequence of Corollary 1, we have the positive accuracy bounds in Theorem 1 also bound MI accuracy for $\mathbb{P}_{x}(1) = 0.5$. Note the bound we obtain is $(1 + e^{-\varepsilon})$ which matches the bound given by Humphries et al. (2020), the previous tightest bound (see Figure 11). One of the contributions of our proof is it explicitly demonstrates where the looseness in this bound comes from, i.e., looseness of the DP inequality over all $(D, D')$ pairs.

Do note our positive accuracy bounds extend to the setting with a biased sampler. We compare our bound with the positive accuracy bound derived by Sablayrolles et al. (2019) in Figure 12 which demonstrates that our bound is indeed tighter. A reference for past bounds is given in Table 1.

| Paper                  | Analytic Form       | Type               |
|------------------------|---------------------|--------------------|
| Yeom et al. (2018)     | $e^\varepsilon / 2$ | Accuracy           |
| Erlingsson et al. (2019)| $1 - e^{-\varepsilon} / 2$ | Accuracy           |
| Humphries et al. (2020)| $(1 + e^{-\varepsilon})$ | Accuracy           |
| Sablayrolles et al. (2019)| $\mathbb{P}_{x^*}(1) + \varepsilon/4$ | Positive Accuracy  |
| Our Work               | $\left(1 + \frac{e^{-\varepsilon} \mathbb{P}_{x^*}(0)}{\mathbb{P}_{x^*}(1)}\right)^{-1}$ | Positive Accuracy  |

Table 1: Bounds found in prior work.
Figure 12: Sablayrolles et al. (2019) upper bound on MI positive accuracy as a function of $P_{x^*}(1)$ compared to our bound. Note that we are still tighter for all probabilities.

7.5 MI Accuracy vs. MI Advantage

Previous work, particularly Yeom et al. (2018) and Erlingsson et al. (2019), focused on membership advantage, which is essentially an improvement in accuracy of an adversary $f$ (with associated set $S$) over the random guess of 50% of drawing a point in the training dataset. More specifically, if we let $A(f)$ denote the accuracy of $f$, then membership advantage is computed as $2(A(f) - 0.5)$.

We can generalize this to ask what the membership advantage of the positive accuracy of $f$ compared to the baseline $P_{x^*}(1)$ is for values other than only 0.5.

Theorem 1 gives us an upper bound on the positive accuracy and thus an upper bound on the positive advantage of $f$ defined as

$$A_d(f) = 2 \left( P_{x^*}(0) - P_{x^*}(1) \right).$$

We plotted this advantage as a function of $P_{x^*}(1)$ for different fixed $\varepsilon$ in Figure 13. We observe that the advantage clearly depends on $P_{x^*}(1)$, and in fact for different $\varepsilon$, the $P_{x^*}(1)$ resulting in the maximum advantage changes. In particular, $P_{x^*}(1) = 0.5$ is not close to the maximum advantage for large $\varepsilon$, which shows how the fixed experiment proposed by Yeom et al. (2018) does not necessarily give the maximum advantage an adversary could have, whereas our result allows us to.

However, it should be noted that higher advantage here does not mean a higher upper bound on MI accuracy; as we already saw in Figure 2, the upper bound on accuracy increases monotonically with $P_{x^*}(1)$, in contrast to the bump observed with membership advantage. This serves to differentiate the study of MI advantage and the study of MI accuracy for future work.

8 Conclusion

In this work, we provide a tight (upto the looseness in the DP guarantee across different datasets) bound on MI positive accuracy against ML models trained
with $\varepsilon$-DP, and bound MI positive accuracy when training with $(\varepsilon, \delta)$-DP accepting some necessary probability of failure. Our bound highlights that intricacies of dataset construction are of importance for model MI vulnerability. Indeed, based on our findings, we develop a privacy amplification scheme that just requires one to sub-sample their training dataset from a larger pool of possible data points. While the benefits of such a mechanism for performance are clear for entities with far more data than they need to train on, we empirically illustrated performative benefits for CIFAR10 (having also evaluated on MNIST).

Based on our results, entities training their ML models with DP can employ looser privacy guarantees (and thereby preserve their models’ accuracy better) while still limiting the success of MI attacks. Finally, our bound, and more generally bounds on positive MI accuracy, can also be applied to characterize a model’s ability to process unlearning requests when defining machine unlearning as achieving a model with low probability of having been derived from a particular data point to be unlearned.

**Acknowledgements**

We would like to acknowledge our sponsors, who support our research with financial and in-kind contributions: Alfred P. Sloan Foundation, Amazon, Apple, Canada Foundation for Innovation, CIFAR through the Canada CIFAR AI Chair program, DARPA through the GARD program, Intel, Meta, NFRF through an Exploration grant, and NSERC through the Discovery Grant and COHESA Strategic Alliance, and the Ontario Early Researcher Award. Resources used in preparing this research were provided, in part, by the Province of Ontario, the Government of Canada through CIFAR, and companies sponsoring the Vector Institute. We would like to thank members of the CleverHans Lab for their feedback.
References

Martin Abadi, Andy Chu, Ian Goodfellow, H Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. 2016. Deep learning with differential privacy. In Proceedings of the 2016 ACM SIGSAC conference on computer and communications security. 308–318.

Eugene Bagdasaryan, Omid Poursaeed, and Vitaly Shmatikov. 2019. Differential privacy has disparate impact on model accuracy. Advances in Neural Information Processing Systems 32 (2019), 15479–15488.

Borja Balle and Yu-Xiang Wang. 2018. Improving the Gaussian mechanism for differential privacy: Analytical calibration and optimal denoising. In International Conference on Machine Learning. PMLR, 394–403.

Raef Bassily, Adam Smith, and Abhradeep Thakurta. 2014. Private empirical risk minimization: Efficient algorithms and tight error bounds. In 2014 IEEE 55th Annual Symposium on Foundations of Computer Science. IEEE, 464–473.

Thomas Baumhauer, Pascal Schöttle, and Matthias Zeppelzauer. 2020. Machine unlearning: Linear filtration for logit-based classifiers. arXiv preprint arXiv:2002.02730 (2020).

Lucas Bourtoule, Varun Chandrasekaran, Christopher A Choquette-Choo, Hengrui Jia, Adelin Travers, Baiwu Zhang, David Lie, and Nicolas Papernot. 2019. Machine unlearning. arXiv preprint arXiv:1912.03817 (2019).

Yinzhi Cao and Junfeng Yang. 2015. Towards making systems forget with machine unlearning. In 2015 IEEE Symposium on Security and Privacy. IEEE, 463–480.

Nicholas Carlini, Steve Chien, Milad Nasr, Shuang Song, Andreas Terzis, and Florian Tramer. 2022. Membership inference attacks from first principles. In 2022 IEEE Symposium on Security and Privacy (SP). IEEE, 1897–1914.

Christopher A Choquette-Choo, Florian Tramer, Nicholas Carlini, and Nicolas Papernot. 2021. Label-only membership inference attacks. In International Conference on Machine Learning. PMLR, 1964–1974.

Cynthia Dwork. 2008. Differential privacy: A survey of results. In International conference on theory and applications of models of computation. Springer, 1–19.

Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. 2006. Calibrating noise to sensitivity in private data analysis. In Theory of cryptography conference. Springer, 265–284.

Cynthia Dwork, Guy N Rothblum, and Salil Vadhan. 2010. Boosting and differential privacy. In 2010 IEEE 51st Annual Symposium on Foundations of Computer Science. IEEE, 51–60.
Úlfar Erlingsson, Ilya Mironov, Ananth Raghunathan, and Shuang Song. 2019. That which we call private. *arXiv preprint arXiv:1908.03566* (2019).

Antonio Ginart, Melody Y Guan, Gregory Valiant, and James Zou. 2019. Making ai forget you: Data deletion in machine learning. *arXiv preprint arXiv:1907.05012* (2019).

Aditya Golatkar, Alessandro Achille, and Stefano Soatto. 2020a. Eternal sunshine of the spotless net: Selective forgetting in deep networks. In *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*. 9304–9312.

Aditya Golatkar, Alessandro Achille, and Stefano Soatto. 2020b. Forgetting outside the box: Scrubbing deep networks of information accessible from input-output observations. In *European Conference on Computer Vision*. Springer, 383–398.

Laura Graves, Vineel Nagisetty, and Vijay Ganesh. 2020. Amnesiac Machine Learning. *arXiv preprint arXiv:2010.10981* (2020).

Rob Hall, Alessandro Rinaldo, and Larry Wasserman. 2013. Differential privacy for functions and functional data. *The Journal of Machine Learning Research* 14, 1 (2013), 703–727.

Hongsheng Hu, Zoran Salcic, Gillian Dobbie, and Xuyun Zhang. 2021. Membership Inference Attacks on Machine Learning: A Survey. *arXiv preprint arXiv:2103.07853* (2021).

Thomas Humphries, Matthew Rafuse, Lindsey Tulloch, Simon Oya, Ian Goldberg, Urs Hengartner, and Florian Kerschbaum. 2020. Differentially Private Learning Does Not Bound Membership Inference. *arXiv preprint arXiv:2010.12112* (2020).

Matthew Jagielski, Jonathan Ullman, and Alina Oprea. 2020. Auditing differentially private machine learning: How private is private sgd? *arXiv preprint arXiv:2006.07709* (2020).

Bargav Jayaraman, Lingxiao Wang, Katherine Knipmeyer, Quanquan Gu, and David Evans. 2020. Revisiting membership inference under realistic assumptions. *arXiv preprint arXiv:2005.10881* (2020).

Peter Kairouz, Sewoong Oh, and Pramod Viswanath. 2015. The composition theorem for differential privacy. In *International conference on machine learning*. PMLR, 1376–1385.

Saeed Mahloujifar, Alexandre Sablayrolles, Graham Cormode, and Somesh Jha. 2022. Optimal Membership Inference Bounds for Adaptive Composition of Sampled Gaussian Mechanisms. *arXiv preprint arXiv:2204.06106* (2022).
Pratyush Maini, Mohammad Yaghini, and Nicolas Papernot. 2021. Dataset inference: Ownership resolution in machine learning. *arXiv preprint arXiv:2104.10706* (2021).

Milad Nasr, Shuang Song, Abhradeep Thakurta, Nicolas Papernot, and Nicholas Carlini. 2021. Adversary instantiation: Lower bounds for differentially private machine learning. *arXiv preprint arXiv:2101.04535* (2021).

Alexandre Sablayrolles, Matthijs Douze, Cordelia Schmid, Yann Ollivier, and Hervé Jégou. 2019. White-box vs black-box: Bayes optimal strategies for membership inference. In *International Conference on Machine Learning*. PMLR, 5558–5567.

Ahmed Salem, Yang Zhang, Mathias Humbert, Pascal Berrang, Mario Fritz, and Michael Backes. 2018. MI-leaks: Model and data independent membership inference attacks and defenses on machine learning models. *arXiv preprint arXiv:1806.01246* (2018).

Ayush Sekhari, Jayadev Acharya, Gautam Kamath, and Ananda Theertha Suresh. 2021. Remember what you want to forget: Algorithms for machine unlearning. *arXiv preprint arXiv:2103.03279* (2021).

Reza Shokri, Marco Stronati, Congzheng Song, and Vitaly Shmatikov. 2017. Membership inference attacks against machine learning models. In *2017 IEEE Symposium on Security and Privacy (SP)*. IEEE, 3–18.

Shuang Song, Kamalika Chaudhuri, and Anand D Sarwate. 2013. Stochastic gradient descent with differentially private updates. In *2013 IEEE Global Conference on Signal and Information Processing*. IEEE, 245–248.

Anvith Thudi, Gabriel Deza, Varun Chandrasekaran, and Nicolas Papernot. 2021. Unrolling sgd: Understanding factors influencing machine unlearning. *arXiv preprint arXiv:2109.13398* (2021).

Florian Tramèr and Dan Boneh. 2020. Differentially private learning needs better features (or much more data). *arXiv preprint arXiv:2011.11660* (2020).

Stacey Truex, Ling Liu, Mehmet Emre Gursoy, Lei Yu, and Wenqi Wei. 2019. Demystifying membership inference attacks in machine learning as a service. *IEEE Transactions on Services Computing* (2019).

Samuel Yeom, Irene Giacomelli, Matt Fredrikson, and Somesh Jha. 2018. Privacy risk in machine learning: Analyzing the connection to overfitting. In *2018 IEEE 31st Computer Security Foundations Symposium (CSF)*. IEEE, 268–282.
A Broader Impact

Our observation of an independent benefit beyond DP from sampling datapoints with low probability could potentially allow entities with access to significant amounts of data, significantly more than one can reasonably train on, to claim privacy against membership inference by default for high $\epsilon$ when training with $\epsilon$-DP (if they sample i.i.d in some part of the data selection pipeline). However, practically most entities use $(\epsilon,\delta)$-DP for privacy, and as noted this already comes with a probability of failure for the bound. This probability of failure should be clearly explained to the entity contributing training data. Nevertheless, our work could, in the case where privacy is primarily about membership inference, allow for looser DP parameters avoiding the degradation to performance typically seen with DP. However we emphasize this is only for membership inference, and there are other aspects to privacy beyond membership inference.