Do we need the $W^{(n>3)}$ constraints to solve the $(1,q)$ models coupled to 2D gravity?

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ABSTRACT

We prove that all the correlation functions in the $(1,q)$ models are calculable using only the Virasoro and the $W^{(3)}$ constraints. This result is based on the invariance of correlators with respect to an interchange of the order of the operators they contain. In terms of the topological recursion relations, it means that only two and three contacts and the corresponding degenerations of the underlying surfaces are relevant. An algorithm to compute correlators for any $q$ and at any genus is presented and demonstrated through some examples. On route to these results, some interesting polynomial identities, which are generalizations of Abel’s identity, were discovered.
1. **Introduction**

Two dimensional (2D) gravitational models in general and topological ones in particular attracted much attention in recent years.\[1\] This trend was motivated mainly by the search for a framework to investigate the non-perturbative behavior of string theory and to address fundamental questions of quantum gravity. Mathematical questions like the topological properties of certain moduli spaces were additional motivations for this research effort.\[2\]

The 2D gravitational models, which are often referred to as non-critical string models, have been investigated in a variety of methods. Among them one finds: (i) The continuum formulations of Liouville theory,\[3\] and the world-sheet light-cone gauge;\[4\] (ii) The discrete approach which started with the matrix models\[5\] and evolved into the application of KdV flows;\[6\] (iii) The $W^{(n)}$ constraints on the partition function;\[7,8\] (iv) The $G^r$ topological coset approach;\[9\] (v) The topological recursion relations approach;\[10,11\] (vi) The Kontsevitch integral formulation\[12\] for the (1, 2) model and its generalizations. Since only $W^{(n)}$ constraints and the topological recursion relations approach are directly related to our approach, we summarize certain developments only in these directions.

The KdV hierarchy\[13\] provides in principle a tool to completely solve the non-critical string models. In practice, however, it is not an easy task to solve the corresponding non-linear differential equations. It was found that the KdV $\tau$ function, which determines the partition function of topological gravity, the topological (1, $q = 2$) model, had to satisfy a set of constraints that obey half of a Virasoro algebra.\[7\] Whereas this set of constraints is enough to completely solve the $q = 2$ case, it is insufficient for $q > 2$. A generalization of the Virasoro constraints to a set of constraints associated with the $W^{(3)}$ algebra produced a recursion relations for the $q = 3$ case\[14,15\] which were shown to be in accordance with the KdV solution. Recursion relations based on higher $W^{(q)}$ constraints were not written down. Presumably the cumbersome expressions found for the case of two primaries discouraged people to proceed in this direction.
The topological recursion relation approach was initiated by Witten\cite{witten10} for the case of the topological gravity model which was conjectured to be related to the one matrix model. The main idea in this approach is that correlators of topological models should be determined from contacts between the operators and between them and possible degenerations of the underlying manifolds. By its nature it relies very little on the explicit details of the field theory description. The introduction of the contact algebra concept combined with the requirement for invariance of correlators under the interchange of the order of the operators, led E. and H. Verlinde\cite{verlinde11} to a full solution of the topological gravity theory. This was further developed in ref. \cite{montano16} where the contact algebra for the (1, 2) model was proven to be unique and the role of multicontacts for higher $q$ was emphasized. Further progress in this program of analyzing the general (1, $q$) models was made by D. Montano and G. Rivlis\cite{montano17}(MR). They provided recursion relations for any correlator of the (1, $q$) model on the sphere and for the case of $q = 3$ also for all higher genus. Their approach included the insertion of a “complete” set of states at each degeneration of the surface and summing over all the possible degenerations. However, this prescription does not enable one to deduce a full solution for $q > 3$ mainly due to the lack of a consistent normal ordering technique to handle divergences that occur frequently at higher genus.

In an attempt to generalize the topological recursion relations of ref. \cite{montano17} and to explicitly construct $W^{(n>3)}$ constraints we discovered that in fact one can compute correlators of $q > 3$ and on higher genus Riemann surfaces only with the aid of the Virasoro and $W^3$ constraints ($V$ and $W^3$). Equivalently one can use the MR method with $q = 3$, namely, only the two and three contacts. Note that for this particular value of $q$, it is free from any normal ordering problems. These observations were later translated into a proof based on induction for every correlator of the $q > 3$ models. The proof includes an algorithm for the explicit computation of any correlator at any genus. In the usual topological recursion relations a given correlator is expressed in terms of a sum of correlators of less operators or correlators at lower genus or products of the latter. The idea behind our result is to go
backward and express a correlator of operators which cannot be manipulated only by the $V$ and $W^3$ constraints as a part of the recursion sum of a correlator which include in addition to the original operators certain operators which are descendants of the first two primaries. For the latter correlator two different $W^3$ recursions are applied in such a way that the coefficients in front of the original correlator are different. Thus by taking the difference of the two sums one finds a recursion expression for the original correlator. Obviously we use here the requirement that a correlator is independent of the order of the operators it contains. It is interesting to note that this “physical” assumption translates into the commutativity of derivatives with respect to the coupling in the KdV language. This turns into a non-trivial condition when one construct the corresponding $W^{(n)}$ constraints.

The paper is organized as follows. In section 2 we review the application of $W_q$ constraints and the topological procedure of ref. [17] to the computation of correlators in the $(1,q)$ models. The relation to the KdV hierarchies is also briefly discussed. Section 3 is devoted to a proof that every correlation function for the $q = 4$ case can be computed using only the $V$ and $W^3$ constraints. The proof is based on an induction algorithm both in the number of operators in the correlator and the genus of the underlying Riemann surface. In section 4 the proof is generalized to the general $q > 4$ models by invoking a further induction in the minimal primary degree of the operators. A summary and several open questions are presented in section 5. Appendix A presents certain explicit calculations of correlators for $q = 4$. These computations demonstrate the usefulness of the methods introduced in the proofs presented in sections 3. In Appendix B we write down a proof of the polynomial identities which are used to derive the result for the general $(1,q)$ model.
2. **Recursion relations from \(W_q\) constraints and from the MR topological procedure**

The \((1, q)\) models are a class of topological models with \(q - 1\) primaries. They are believed to be a description of the \((1, q)\) minimal models coupled to 2D gravity. The correlation functions of these models can be determined by the \((1, q)\) KdV hierarchy. The basic differential operators of the KdV flows take the form \(P = D = \frac{\partial}{\partial x}\) and \(Q = D^q + x\) where \(x\) is the cosmological constant. The correlators of the theory are expectation values of products of primary operators and their descendants. These operators are denoted by \(P_{k, \alpha}\) where \(\alpha = 1, \ldots, q - 1\). The \(k = 0\) operators are the primaries and their descendants carry a positive integer \(k\). There is a \(U(1)\) “ghost number” charge associated with each operator

\[
gh(P_{k, \alpha}) = [(k - 1)q + (\alpha - 1)]
\]

such that non-trivial correlators \(< \prod_i^n P_{k_i, \alpha_i} >_g\) at \(x = 0\) have to obey the following “ghost number” conservation law

\[
\sum_i^n gh(P_{k_i, \alpha_i}) = 2(g - 1)(1 + q)
\]

It was shown that the \(\tau\) function of the KdV models is constraint by a set of operators which furnish an anomaly free half Virasoro algebra.\(^{[7,8]}\). These constraints completely determine all the correlators of the \((1, 2)\) model, but are not enough to do a similar job for models with \(q > 2\). Since the \(W_q\) algebra forms a closed algebra which includes the Virasoro generators it was proposed that invoking \(W_q\) constraints on the \(\tau\) function would lead to a complete derivation of recursion relations for the \((1, q)\) models. Indeed, for the case of two primaries, \(q = 3\), recursion relations were derived using this approach.\(^{[14]}\) However, once one uses the explicit operators, it becomes clear that the use of \(W_q\) constraints for \(q > 3\) is not practical since the corresponding constraints are very complicated. In fact, to the best of
our knowledge they were not explicitly derived for \( q > 4 \). We now briefly summarize the \( W_q \) constraints approach.\[7\]

To write down the constraints one uses a set of \( \alpha = 1, \ldots, q - 1 \) scalar fields \( \phi_{\alpha} \) which are defined by their mode expansion

\[
\partial \phi_{\alpha}(z) = \sum_{n \in \mathbb{Z}} a_{n+\frac{\alpha}{q}} z^{-(n+\frac{\alpha}{q}+1)}.
\]

By identifying the modes with the following differential operators

\[
a_{-n-\frac{\alpha}{q}} = \frac{\sqrt{q}}{\lambda}(n + \frac{\alpha}{q})t_{n,\alpha}, \quad a_{n+\frac{\alpha}{q}} = \frac{\lambda}{\sqrt{q}} \frac{\partial}{\partial t_{n,\alpha}}, \quad n \geq 0
\]

one derives the following algebra

\[
[a_{n+\frac{\alpha}{q}}, a_{m+\frac{\beta}{q}}] = (n + \frac{\alpha}{q})\delta_{n+m+\alpha+\beta}. \tag{4}
\]

One then writes the operators \( W^l \) as a polynomials of order \( l \) in \( \partial \phi_{\alpha} \) and imposes the condition that they obey the relevant part of the \( W_q \) algebra. For instance it is straightforward to check that \( L_n \) which is defined by

\[
W^2(z) = T(z) = \sum_{n} L_n z^{-n-2} = \sum_{\alpha} \frac{1}{2} : \partial \phi_{\alpha}(z) \partial \phi_{q-\alpha}(z) : + \frac{q^2 - 1}{24q} \frac{1}{z^2}
\]

takes the form

\[
L_n = \frac{1}{2} \sum_{\alpha=1}^{q-1} \sum_{k=-\infty}^{\infty} a_{k+\frac{\alpha}{q}} a_{n-1-k+\frac{\alpha}{q}} \tag{5}
\]

and obey for \( n \geq -1 \) the anomaly free Virasoro algebra. A similar result for the \( W_n^3 \) with \( n \geq -2 \) was derived in ref. [15]. The constraints on \( \tau \), the square root of the partition function of the perturbed model is then \( W^l_n \tau = 0 \). Any given correlator can be expressed as a differential operator acting on the log of the partition function as follows

\[
\langle P_{k_1,\alpha_1} \ldots P_{k_n,\alpha_n} \rangle = \frac{\partial}{\partial t_{k_1,\alpha_1}} \ldots \frac{\partial}{\partial t_{k_n,\alpha_n}} \log Z = \sum_{g} \lambda^{2g-2} \langle P_{k_1,\alpha_1} \ldots P_{k_n,\alpha_n} \rangle_g
\]

We then express one of the derivatives using the constraint operators to derive a recursion relation for the correlator after setting all the \( t_{k,\alpha} \) to their critical
values, namely, $t_{k,\alpha} = 0$ apart from $t_{1,1}$. For instance let us calculate the correlation function $\langle P_{0,1}P_{0,1}P_{0,3}\rangle_0$ in the $(1,4)$ model. For this we use the constraint $L_{-1} = \frac{1}{2} \sum_{\alpha=1}^{3} \sum_{k=-\infty}^{\infty} a_k + \frac{\alpha}{4} a_{-2-k} + \frac{1}{4 \alpha}$. At the critical point $t_{1,1} = -\frac{4}{5}$ and the only negative $a$'s that survives are $t_{0,1} \sim a_{-1+\frac{3}{4}}$, $t_{0,3} \sim a_{-1+\frac{1}{4}}$ and $t_{1,1} \sim a_{-2+\frac{3}{4}}$.

Thus, we can write $\{\lambda^{-2} \times \frac{3}{4} \times t_{0,1} \times t_{0,3} + (1 + \frac{1}{4}) \times t_{1,1} \times \frac{\partial}{\partial t_{0,1}} + \text{other terms}\} \cdot Z = 0$. Therefore we can express the derivative of $Z$ with respect to $t_{0,1}$ as

$$\frac{\partial}{\partial t_{0,1}} Z = \frac{3}{4} \lambda^{-2} t_{0,1} t_{0,3} Z + (\text{other terms}) Z$$

and so $\langle P_{0,1}P_{0,1}P_{0,3}\rangle = \frac{\partial}{\partial t_{0,3}} \frac{\partial}{\partial t_{0,1}} \frac{\partial}{\partial t_{0,1}} \log Z = \frac{3}{4} \lambda^{-2}$. Expanding both sides in powers of $\lambda$ we finally get $\langle P_{0,1}P_{0,1}P_{0,3}\rangle_0 = \frac{3}{4}$.

In general, recursion relations of products of descendant and primary operators for $q = 3$ are composed of single and double contact terms, single and double surface degeneration terms, degeneration combined with contact terms and certain Kronecker delta function terms.[14]

The MR approach[17] is based on inserting a “complete” set of states at each degeneration of the surface and summing over all the possible degenerations. Auxiliary operators of negative ghost number were introduced in order to define an adjoint operator on the Hilbert space of states $P_i^\dagger = P_{-i}$ where $i = qk_i + \alpha_i$, the metric $\langle P_i P_j \rangle_0 = \eta_{ij} = |i| \delta_{(i+j)}$ and the identity operator $I = \sum_{i,j \neq 0, \text{mod}(q)} |P_i \rangle \langle P_j|$. It was shown[17] that on the sphere correlators with “anti-states” ($i < 0$) could consistently be set to zero apart from the metric and the one point function $\langle P_{-q-1} \rangle_0$. A recursion relation for any given correlator on the sphere was then derived using the following procedure. One operator is chosen to be the marked operator. This operator comes in contact with all possible degenerations of the surface, the number of which is determined by the primary field from which the marked operator descends. At each degeneration a complete set of states is introduced. The recursion relation is then written in terms of a “degeneration equation” which states that the sum over all the contacts with degenerations determined by the marked operator vanishes. The negative ghost number operator disappears from the correlators by performing “effective (multiple) contacts” with the marked
operator. The effective contact terms were found to be

\[
\{\{P_i \prod_{j=1}^{n} P_{i_j}\}\} = (-1)^{n-1} n! \binom{\alpha}{n} \frac{\prod_{j=1}^{n} i_j}{q^n} P_{i'}
\]

\[i' = i + \sum_{j=1}^{n} i_j - n(q + 1)\]  \hspace{1cm} (6)

where \(i = kq + \alpha\). On higher genus Riemann surfaces, one encounter infinities which follows from counting ambiguity. This situation occurs for \(q \geq 4\). It happens on the torus and even more frequently on surfaces with higher genus. Therefore, an application of the topological procedure for the general case at any higher genus Riemann surface is still missing.

3. Correlation Functions in the (1,4) Model

In the previous section correlation functions were shown to follow from the \(W^3\) constraints as well as from the MR topological procedure. We recall that these methods are not adequate already for \(q = 4\) the former due to regularization problems. We now show that in fact using the Viraosro and \(W^3\) constraints one can solve for every correlator in the (1,4) model. We state this result as the following theorem.

**Theorem** - In the (1,4) model every correlation function at any genus can be computed using the Virasoro and \(W^3\) (\(V\) and \(W^3\)) constraints only.

**Proof** - By induction on the genus. We first prove that the theorem holds for \(g = 0\), then assuming that it holds for certain \(g \geq 0\) we show that it holds also for \(g + 1\). For \(g = 0\) we make use of the following lemma:

**Lemma 1** - In the (1,4) model any correlation function at \(g = 0\) can be computed using the Virasoro and \(W^3\) constraints only.

**Proof of Lemma 1** - By induction on the number \(n\) of operators in the correlation function. The lemma is proven for \(n = 3\) and then assuming it is true for some \(n \geq 3\) we prove that it holds also for \(n + 1\). For \(g = 0\), due to ghost number
conservation, any correlation function includes at least 3 operators. Correlators of only \( \alpha = 1, 2 \) primaries and of their descendants are obviously determined by the Virasoro and \( W^3 \) constraint. The only non-trivial three point function that includes a primary with \( \alpha = 3 \) or its descendants, is \( \langle P_{0,1}P_{0,1}P_{0,3} \rangle \). For this case the Virasoro constraint is sufficient (see the example in section 2), so the claim is proven for \( n = 3 \).

The induction hypothesis—Suppose that the claim in lemma 1 is correct for correlators that contain \( n \) operators, \( n \geq 3 \). Consider a correlation function that contains \( n + 1 \) operators. If one of these operators have \( \alpha = 1, 2 \), the Virasoro or \( W^3 \) recursion relation can be used to calculate the correlator. The result will necessarily contain correlators with \( n \) operators or less. This is a consequence of a contact that reduces the number of operators, or of splitting the correlator into two or three correlators among which the remaining \( n \) operators are distributed. Therefore, all the correlation functions that appear after one step of the recursion fulfill the conditions of the induction assumption. Thus, they are completely determined by the Virasoro and \( W^3 \) recursions. Consider a correlation function that contains \( n + 1 \) operators, none of which is with \( \alpha = 1, 2 \). In the \((1, 4)\) model the remaining \( \alpha \) is 3 and thus the correlation function is of the form \( \langle \prod_{i=1}^{n+1} P_{k_i,3} \rangle_0 \). Consider the following two correlation functions:

\[
\langle P_{0,2}P_{k_1+1,2} \prod_{i=2}^{n+1} P_{k_i,3} \rangle_0 , \quad \langle P_{k_1+1,2}P_{0,2} \prod_{i=2}^{n+1} P_{k_i,3} \rangle_0
\]

Both of them can be computed using the \( W^{(3)} \) recursion. For the first one we get

\[
\langle P_{0,2}P_{k_1+1,2} \prod_{i=2}^{n+1} P_{k_i,3} \rangle_0 = (2k_1 + 3)\langle \prod_{i=1}^{n+1} P_{k_i,3} \rangle_0 + A
\]

(7)

where \( A \) denotes a set of correlators containing the operator \( P_{k_1+1,2} \) and \( n \) other operators (as a consequence of a contact), or \( n - 1 \) other operators (as a consequence of two contact), or \( n \) operators without \( P_{k_1+1,2} \) (again as a consequence of two
contact), or a product of 2 or 3 correlators (as a consequence of a split) each of them contains \( n \) operators at the most. All of these cases can be computed according to the induction hypothesis.

For the second correlation function we get

\[
\langle P_{k_1+1,2}P_{0,2} \prod_{i=1}^{n+1} P_{k_i,3}\rangle_0 = \langle \prod_{i=1}^{n+1} P_{k_i,3}\rangle_0 + B
\]

(8)

Where the terms in B are calculable in the same way as in A. We can use now the fact that the correlation function does not depend on the order of the operators and to subtract (7) from (8). We get:

\[
\langle \prod_{i=1}^{n+1} P_{k_i,3}\rangle_0 = \frac{1}{(2k_1+2)}(B - A)
\]

(9)

Where all the terms on the r.h.s can be computed according to the induction hypothesis.

Thus we have proven that a correlation function containing \( n + 1 \) operators can be computed using the \( V \) and \( W^3 \) recursions only, and this completes the proof of the induction step, and thus of lemma 1.

Returning to the main theorem, we now use again a proof by induction. We take as our induction hypothesis that the claim of the theorem is correct for genus \( g \geq 0 \) and we would like to show that it holds also for \( g + 1 \). To prove the induction step we use the following lemma.

**Lemma 2**-If any correlation function up to genus \( g \) can be computed by the Virasoro and \( W^{(3)} \) recursion relations only, then any correlation function at genus \( g + 1 \) can also be computed using these recursion relations only.

**Proof of lemma 2**- We now apply an induction on the number \( n \) of operators in the correlation function. For \( n = 1 \), if this operator has \( \alpha = 1, 2 \) then the correlator can be computed using the \( V \) and \( W^3 \) recursion relations, having on the
r.h.s correlators at genus \( g \) or less. If this operator has \( \alpha = 3 \), then the correlation function we want to compute is \( \langle P_{k,3} \rangle_{g+1} \), with \( k = \frac{1}{2}(5g + 1) \) due to ghost-number conservation. This obviously implies that these one point functions are non-trivial only on even genus surfaces. Consider the two functions

\[
\langle P_{k+1,2}P_{0,2} \rangle_{g+1} \quad , \quad \langle P_{0,2}P_{k+1,2} \rangle_{g+1}
\]

As in the case of \( g = 0 \), we can compute them using the \( W^{(3)} \) recursion:

\[
\langle P_{k+1,2}P_{0,2} \rangle_{g+1} = \langle P_{k,3} \rangle_{g+1} + A \tag{10}
\]

\[
\langle P_{0,2}P_{k+1,2} \rangle_{g+1} = (2k + 3)\langle P_{k,3} \rangle_{g+1} + B \tag{11}
\]

where all the terms in A and B are correlation functions at genus \( g \) or less. Subtracting (10) from (11), we get

\[
\langle P_{k,3} \rangle_{g+1} = \frac{1}{(2k + 2)}(B - A) \tag{12}
\]

where according to the lemma assumption all the terms on the r.h.s can be computed using the \( V \) and \( W^3 \) recursion relations. This completes the proof for a one point function. On odd genera similar arguments hold for the two point function. We now proceed by invoking the induction hypothesis which states that any correlation function at genus \( g + 1 \) with \( n \) operators can be computed using the \( V \) and \( W^3 \) recursion relations only.

Consider a correlation function at genus \( g + 1 \) with \( n + 1 \) operators. If one of them has \( \alpha = 1, 2 \), then the correlation function can be computed by the \( V \) and \( W^3 \) recursion relations. The latter relates the correlator to a sum of correlators which are either at genus \( g + 1 \) with \( n \) operators or less, or \( n + 1 \) operators or less at genus \( g \) or less. These correlators are thus completely determined according to the induction assumption. If, on the other hand, all the operators are of the \( \alpha = 3 \) type ,namely, \( \langle \prod_{i=1}^{n+1} P_{k_i,3} \rangle_{g+1} \), one repeats the treatment introduced for the \( g = 0 \).
case and derive the same conclusions. We have thus proven lemma 2 and hence the theorem that every correlator of the (1, 4) model can be computed with the use of the \( V \) and \( W^3 \) constraints only. Note that rather than inserting \( P_{k_1+1,2} P_{0,2} \) in eqn. (8), one can use the operators \( P_{k_1,2} P_{1,2} \). This leads to the same conclusions as above, but cannot be generalized to the higher (1, \( q \)) models. In appendix A we demonstrate this result by computing several correlation functions using only the \( V \) and \( W^3 \) constraints. The results are in complete agreement with the \( KdV \) results.\(^{[18]}\)

4. Correlation Function For General (1, \( q \)) Model

We now generalize the result of the previous section to any arbitrary (1, \( q \)) model. The proof now involves three stages of induction steps, one for the genus, another one for the number of operators and a third one, which was not needed in the (1, 4) case, for \( \alpha = \min_{i=1..N} \{ \alpha_i \} \).

Consider the correlator \( \langle \prod_{i=1}^{N} P_{k_i,\alpha_i} \rangle_g \) such that \( \alpha \geq 3 \). Note that unlike the previous section here we denote the number of operators in a correlator by \( N \) rather than \( n \). Without a loss of generality we assume \( \alpha_1 = \alpha \). To apply the method used for \( q = 4 \) we have to introduce several additional operators into a correlator that could be related to the original one. Obviously those operators should be \( P_{0,2} \) or their descendants. In fact, only primaries can do the job since the contacts which contain only them vanish and thus their contacts necessarily involve some of the original operators. Thus, a situation where we end up with more than \( N \) operators is avoided. We would like to get the operator \( P_{k,\alpha} \) from consecutive contacts of the \( P_{0,2} \) primaries with an operator which is also a descendant of \( \alpha = 2 \). This requirement determines uniquely the minimal set of additional operators. The starting point is, therefore, the following two correlators

\[
\langle P_{0,2}^{\alpha-2} P_{k+\alpha-2,2} \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle_g , \quad \langle P_{k+\alpha-2,2} P_{0,2}^{\alpha-2} \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle_g
\]

As before, we want to subtract the two correlators, and to extract a recursion
relation for the original correlator. After repeatedly using the $W^3$ constraint for each $P_{0,2}$ we get

$$C\langle \prod_{i=1}^{N} P_{k_i,\alpha_i} \rangle_g = (C_1 - C_2)\langle \prod_{i=1}^{N} P_{k_i,\alpha_i} \rangle_g = B - A \quad (14)$$

where $C$ is a numerical coefficient and $B - A$ is a sum of correlators. Next we want to show that (i) $C \neq 0$ for any allowed values of $N$, $k_i$ and $\alpha_i$ and that (ii), after exhausting all the $P_{0,2}$'s, the expression for $B - A$ includes terms with $g' < g$ or with $N' < N$ or with $g' = g$, $N' = N$ but with $\alpha' < \alpha$ where $\alpha' = \min_{i=1..N'} \{\alpha_i\}$. Glancing at the recursion relation derived from the $W^3$ constraint one can convince oneself that any contact (or any two contact) reduces the number of operators, while a split (or a two split, or contact accompanied by a split) reduces either the genus or the number of operators or both. Since each of the other correlators involves a contact (or a two contact) between $P_{0,2}$ and the $\prod_{i=2}^{N} P_{k_i,\alpha_i}$ or a split where the $\prod_{i=2}^{N} P_{k_i,\alpha_i}$ are distributed among the different correlators, it is clear that after exhausting all the $P_{0,2}$'s we must have $g' \leq g$ or $N' \leq N$. So let us look at terms with $N' = N$ and $g' = g$. Clearly at least one of the operators in the correlator should be different from the original one, otherwise this term would contribute to $C$. This occurs in one of the successive applications of the $W^3$ constraint due to a contact or a split, when one of the $P_{k_i,\alpha_i}$ disappear from the product (possibly with a bunch of $P_{0,2}$) and another operator appears instead. At least one more new operator must be formed in order to keep $N' = N$ and it can be formed only from some of the remaining $P_{0,2}$ and $P_{k+\alpha-j,j}$ where the last operator is what became of the original $P_{k+\alpha-2,2}$ at this stage. From ghost number counting it is clear that this new formed operator has $\alpha' < \alpha$, and that completes the proof of claim (ii). In order to prove claim (i), we will compute $C$ explicitly. First we will obtain a general expression for correlators of the type $\langle P_{n,2}^n P_{k,\alpha} \rangle_0$ where $n(1-q) + (k-1)q + \alpha - 1 = -2(q+1)$ and therefore $k = n-2$, $\alpha = q-n-1$, $2 \leq n < q-1$. The general $W^3$ recursion for correlators
of this type is:

\[ \langle P^n_{0,2} P_{k,\alpha}\rangle_0 = \frac{1}{q^2} (2q(kq + \alpha)) \langle P^{n-1}_{0,2} P_{k-1,\alpha+1}\rangle_0 - 4(n - 1)(kq + \alpha) \langle P^{n-2}_{0,2} P_{k-2,\alpha+2}\rangle_0 \]

\[ - (kq + \alpha) \sum_{m=2}^{n-3} \binom{n-1}{m} \langle P^m_{0,2} P_{k,\alpha_1}\rangle_0 \langle P^{n-m-1}_{0,2} P_{k,\alpha_2}\rangle_0 \]

(15)

Let us now prove by induction that the outcome of this recursion is given by

\[ \langle P^n_{0,2} P_{k,\alpha}\rangle_0 = 2^n q^{n-1} \prod_{i=1}^{n} (i q - n - 1) \delta_{k,n-2} \delta_{\alpha,q-n-1} = \frac{2^n \Gamma(n - \frac{n+1}{q})}{\Gamma(1 - \frac{n+1}{q})} \delta_{k,n-2} \delta_{\alpha,q-n-1} \]

(16)

It is straightforward to check that the values of the correlators for the first few cases agree with that expression. The results are the following

\[ \langle P^2_{0,2} P_{0,q-3}\rangle_0 = \frac{4(q - 3)}{q} \quad n = 2 \]

\[ \langle P^3_{0,2} P_{1,q-4}\rangle_0 = \frac{8(2q - 4)(q - 4)}{q^2} \quad n = 3 \]

\[ \langle P^4_{0,2} P_{2,q-5}\rangle_0 = \frac{16(3q - 5)(2q - 5)(q - 5)}{q^3} \quad n = 4 \]

Assuming now that (16) holds for \( n - 1 \) and then using the recursion relation (15) one finds that (16) is correct also for \( n \), provided that the following is an identity

\[ \prod_{i=1}^{n-2} (i q - n - 1) = \prod_{i=1}^{n-2} (i q - n) - \frac{1}{2} \sum_{m=1}^{n-2} \binom{n-1}{m} \prod_{i=1}^{m-1} (i q - m - 1) \prod_{j=1}^{n-m-2} (j q - n + m) \]

\[ = - \frac{1}{2} \sum_{m=0}^{n-1} \binom{n-1}{m} \prod_{i=1}^{m-1} (i q - m - 1) \prod_{j=1}^{n-m-2} (j q - n + m) \]

(17)

where in the second line we use the extrapolation of \( \prod_{i=n_f}^{n_i} = \prod_{i=n_f}^{n_i} \prod_{i=n_j}^{n_f} \) to the case that \( n_i > n_f \). It is interesting to note that for the unphysical case of \( q = 0 \) this
relation reduces to the identity of Abel

\[ 2(n + 1)^{n-2} = \sum_{m=0}^{n-1} \binom{n-1}{m} (m+1)^{m-1} (n-m)^{(n-m-2)} \]  

(18)

which counts the number of trees formed from \( n + 1 \) points where two given points are connected by a link. The identity (17) will be shown to be a special case of a more general identity which is proven in appendix B.

Equipped with the expression for the \( \langle P_{n,0}^n P_{k,\alpha} \rangle_0 \) we now proceed to write down a recursion relation for a correlator of the type \( \langle P_{0,2}^n P_{k+n,\alpha-n} \prod_{i=2}^{N} P_{k_i, \alpha_i} \rangle_g \) and consider only terms that contribute to \( C_1 \)

\[
\langle P_{0,2}^n P_{k+n,\alpha-n} \prod_{i=2}^{N} P_{k_i, \alpha_i} \rangle_g = \frac{1}{q^2} \{ 2q[(k+n)q + \alpha - n] \langle P_{0,2}^{n-1} P_{(k,\alpha)(n-1)} \prod_{i=2}^{N} P_{k_i, \alpha_i} \rangle_g \\
- 4(n-1)[(k+n)q + \alpha - n] \langle P_{0,2}^{n-2} P_{(k,\alpha)(n-1)} \prod_{i=2}^{N} P_{k_i, \alpha_i} \rangle_g \\
- 2[(k+n)q + \alpha - n] \sum_{m=2}^{n-1} \binom{n-1}{m} \langle P_{0,2}^m P_{(k,\alpha)} \rangle_0 \langle P_{0,2}^{n-1-m} P_{(k,\alpha)(n-m-1)} \prod_{i=2}^{N} P_{k_i, \alpha_i} \rangle_g + \Delta_1 \}
\]  

(19)

where \( P_{(k,\alpha)(l)} \) means \( P_{k+l,\alpha-l} \) and \( \Delta_1 \) stands for terms that do not contribute to \( C_1 \).

Once again we make use of an induction procedure and a polynomial identity to prove a result. This time it is the general expression for \( C_1 \), the contribution of this correlator to \( C \), which takes the form

\[ C_1 = (\frac{2}{q})^n \prod_{i=1}^{n} ((k+i)q + \alpha - n) \]

The cases of \( n = 1 \) and \( n = 2 \) can be easily shown to obey this rule. When one assumes it holds for \( n-1 \) and uses the recursion (19) one proves the result for \( n \)
if the following expression is an identity.

\[
\prod_{i=1}^{n-1}[(k+i)q + \alpha - n + 1] - (n-1) \prod_{i=1}^{n-2}[(k+i)q + \alpha - n + 2] \\
- \sum_{m=2}^{n-1} \binom{n-1}{m} \prod_{i=1}^{m-1}(i q - m - 1) \prod_{i=1}^{n-1-m}[(k+i)q + \alpha - n + 1 - m] \\
= - \sum_{m=0}^{n-1} \binom{n-1}{m} \prod_{i=1}^{m-1}(i q - m - 1) \prod_{i=1}^{n-1-m}[(k+i)q + \alpha - n + 1 - m] \\
= \prod_{i=1}^{n-1}[(k+i)q + \alpha - n]
\]

The proof of this identity is given in appendix B. Note that for the special value \( \alpha + 2 + kq = 0 \) this identity turns into the previous one given in (17).

The next step is obviously the computation of the contribution to \( C \) from \( \langle P_{k+n,\alpha-n} P_{0,2} \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle g \). The recursion relation in this case reads

\[
\langle P_{k+n,\alpha-n} P_{0,2} \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle g = \frac{1}{q^2} \{ 4 n \langle P_{0,2} P_{k,\alpha} \rangle \langle P_{0,2}^{n-1} P_{k,\alpha} \rangle \langle P_{0,2}^{n-2} P_{k,\alpha} \rangle \langle P_{0,2}^{n-3} P_{k,\alpha} \rangle \langle P_{0,2}^{n-4} P_{k,\alpha} \rangle \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle g \\
+ 2 q \sum_{m=2}^{n} \binom{n}{m} \langle P_{0,2}^{m} P_{k,\alpha} \rangle \langle P_{0,2}^{n-m} P_{k,\alpha} \rangle \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle g - 8 \langle P_{0,2}^{n-2} P_{k,\alpha} \rangle \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle g \\
- 4 n \sum_{m=2}^{n-1} \binom{n-1}{m} \langle P_{0,2}^{m} P_{k,\alpha} \rangle \langle P_{0,2}^{n-m-1} P_{k,\alpha} \rangle \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle g \\
- 3 \sum_{a=2}^{n-2} \sum_{b=2}^{n-a} \binom{n}{a} \binom{n-a}{b} \langle P_{0,2}^{a} P_{k_1,\alpha_1} \rangle \langle P_{0,2}^{b} P_{k_2,\alpha_2} \rangle \prod_{i=2}^{N} P_{k_i,\alpha_i} \rangle g + \Delta_2 \}
\]

with \( \Delta_2 \) denotes terms that do not contribute to \( C_2 \). And so the contribution to \( C_2 \) is given by

\[
\left( \frac{2}{q} \right)^n \{ 2 n \prod_{i=1}^{n-1}[(k+i)q + \alpha - n + 1] + 2 \sum_{m=2}^{n} \binom{n}{m} \prod_{i=1}^{m-1}(i q - m - 1) \prod_{i=1}^{n-m}[(k+i)q + \alpha - n + m] 
\].
\[-2 \binom{n}{2} \prod_{i=1}^{n-2} [(k+i)q + \alpha - n + 2] - 2n \sum_{m=2}^{n-1} \binom{n-1}{m} \prod_{i=1}^{m-1} (iq - m - 1) \prod_{i=1}^{n-m-1} [(k+i)q + \alpha - n + m + 1] \]

\[-\sum_{a=2}^{n-a} \binom{n}{a} \binom{n-a}{b} \prod_{i=1}^{a-1} (iq - a - 1) \prod_{i=1}^{b-1} (iq - b - 1) \prod_{i=1}^{n-a-b} [(k+i)q + \alpha - n + a + b] \}

The sums of products and triple products in this expression can be simplified using eqn. (20). Substituting \( n = \alpha - 2 \) into \( C_1 - C_2 \) we finally get

\[ C = C_1 - C_2 = 2^{\alpha-2} \prod_{i=1}^{\alpha-2} (k + i). \quad (22) \]

This completes the proof of claim (i) since obviously \( C \) does not vanish. Having the explicit expression for \( C \) is obviously also useful for the full determination of the correlators. Note that \( C \) is independent of \( q \). In particular eqn. (9) is a special case of (22) for \( \alpha = 2 \). The procedure just described to write down a recursion relation for \( \langle \prod_{i=1}^{N} P_{k_i, \alpha_i} \rangle_g \) constitutes a proof to the following lemma.

**Lemma 3:** In any \((1, q)\) model a recursion relation can be written for \( \langle \prod_{i=1}^{n} P_{k_i, \alpha_i} \rangle_g \) with \( \alpha \geq 3 \), \( \alpha = \min\{\alpha_i\} \) using only the \( W^{(3)} \) recursion relations so that all the terms on the right hand side will be of the form \( \langle \prod_{j=1}^{n'} P_{k'_j, \alpha'_j} \rangle_{g'} \) with \( g' < g \) or \( n' < n \) or \( g' = g \), \( n' = n \) and \( \alpha' = \alpha - 1 \) where \( \alpha' = \min\{\alpha'_j\} \).

Using Lemma 3 and the analogs of lemmas 1, 2, proven in the previous section for the \((1, 4)\) model, we can now generalize the theorem that any correlator can be computed using \( V \) and \( W^{3} \) constraints proven above for the \((1, 4)\) model to any \( q \). As for Lemma 1, the \( g = 0 \) case, the smallest number of operators in any non-trivial correlator is still three. But it is not necessary that all \( \alpha = 1, 2 \) if all the operators are primaries and \( q \geq 8 \). In the latter case we have to use the commutator construction given above in lemma 3. Once the three point function is computed using the \( V \) and \( W^{3} \) constraints, lemma 1 follows just as for the \( q = 4 \) models.
5. SUMMARY AND DISCUSSION

Topological properties of certain moduli spaces can be easily written down in terms of correlators of topological quantum field theories. However, the field theory realization does not shed much light on the explicit evaluation of these topological characteristics since only in a limited number of cases the relevant correlators could be computed. One such example is the moduli space of punctured Riemann surfaces which corresponds to the theory of pure topological gravity, the topological $(1,2)$ minimal model. Expectation values of “physical operators” in this model were determined via recursion relations which originally\cite{10} led to a solution on the sphere and later turned into a full solution with the introduction of the contact algebra concept\cite{11}. A different, even though not unrelated approach, to this model was introduced via the Virasoro constraints. The equivalence of the two approaches was proven in ref. \cite{17} together with a generalization to the $(1,q)$ models. The main advantage of the latter approach was the introduction of a prescription for recursion relations which was much simpler to use than the method of $W_q$ constraints. In fact these constraints were never written down for $q > 4$ and already for $q \leq 4$ the resulting expressions are very cumbersome. The MR approach failed short in computing correlators for $q > 3$ on Riemann surfaces at genus greater than zero. The reason for that was the lack of a regularization scheme to handle divergences that appear at $q > 3$.

The question that was investigated in the present work was whether one could solve for the correlators of higher $q$ models at any genus using only the part of the MR technique which is otherwise needed for $q = 2, 3$, or equivalently using only the $V$ and $W^3$ constraints. Recall that normal ordering in the Virasoro case ($q = 2$) is well known and the $W^3$ case is free from infinities. Indeed, we proved by induction that the $V$ and $W^3$ are sufficient for a complete solution of the models. The fact that it is unnecessary to use higher $W_q$ information was proven here only within the framework of the observables of the topological $(1,q)$ models. The implications to other domains where $W_q$ algebras are involved, has to be further investigated.
One feature which played an important role in the present derivation is obviously the interchange of the order of operators inside correlation functions.

There are certain open questions and related topics that are still awaiting further investigation. Among them one can find: (i) The search for a general structure for classes of correlators like for instance eqn. (16) and their interpretation as intersection numbers or similar topological properties of the corresponding moduli spaces. (ii) A related question is whether there are additional independent polynomial identities. Certain generalization of those presented in this work have been already discovered and they will be presented in a future publication. (iii) Another challenge is to transcribe our results directly in terms of the corresponding KdV flows.

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APPENDIX A

Computation of Correlators of the $(1,4)$ model using the $V$ and $W^3$ recursion relations

The algorithm of computing correlators, $\langle \prod_{i}^{N} P_{k_{i},3} \rangle_g$ that otherwise would require the use of $W_4$ constraint, using only the $V$ and $W^3$ ones is demonstrated here in three examples at genus $g = 0,1,2$

$(i)$. As the first example we compute is $\langle P_{0,3}^5 \rangle_0$. So we look at $\langle P_{0,2} P_{1,2} P_{0,3}^4 \rangle_0$ and $\langle P_{1,2} P_{0,2} P_{0,3}^4 \rangle_0$. For the first one we use the $W^{(3)}_{-2}$ constraint:

$$
\sum_{k} \sum_{l} a_{k+\frac{1}{4}} a_{l+\frac{1}{4}} + \sum_{k} \sum_{l} a_{k+\frac{1}{4}} a_{l+\frac{3}{4}} a_{-3-k-l+\frac{1}{4}} + \sum_{k} \sum_{l} a_{k+\frac{1}{4}} a_{l+\frac{1}{4}} a_{-4-k-l+\frac{1}{4}}
$$

The negative $a$’s that survives at the critical point are $a_{-2+\frac{1}{4}} \sim t_{1,2}$, $a_{-1+\frac{1}{4}} \sim t_{0,3}$
, \(a_{-1} + 1 \sim t_{0,2}\), and \(a_{-2,4} \sim t_{1,1}\). So the relevant terms in the constraint are

\[
2 \times \frac{2}{\lambda} \times 3 \times t_{1,2} \times 3 \times t_{0,3} \times \frac{\partial}{\partial t_{0,1}} + 2 \times \frac{2}{\lambda} \times (1 + \frac{1}{4}) \times t_{1,1}^{2} \times \frac{\partial}{\partial t_{0,2}} + 2 \times \frac{2}{\lambda} \times 3 \times t_{1,2} \times (1 + \frac{1}{4}) \times t_{1,1} \times \frac{\partial}{\partial t_{0,3}}
\]

Translating this to correlators we get

\[
\langle P_{0,2} P_{1,2} P_{0,3}^{4} \rangle_{0} = 3 \langle P_{0,3}^{5} \rangle_{0} - 9 \langle P_{0,1} P_{0,3}^{3} \rangle_{0}
\]

Similarly we get for the second correlator

\[
\langle P_{1,2} P_{0,2} P_{0,3}^{4} \rangle_{0} = -\frac{3^{3}}{4} \langle P_{0,2} P_{0,3}^{2} \rangle_{0} - 3 \langle P_{0,1} P_{0,3}^{3} \rangle_{0} + \langle P_{0,3}^{5} \rangle_{0}
\]

combining these results we get

\[
\langle P_{0,3}^{5} \rangle_{0} = \frac{1}{2} \left( 6 \langle P_{0,1} P_{0,3}^{3} \rangle_{0} - \frac{3^{3}}{4} \langle P_{0,2} P_{0,3}^{2} \rangle_{0} \right)
\]

Inserting the following values, which can be computed directly from the constraints

\[
\langle P_{0,1} P_{0,3}^{3} \rangle_{0} = 0 \quad , \quad \langle P_{0,2} P_{0,3}^{2} \rangle_{0} = -\frac{9}{4}
\]

We finally get:

\[
\langle P_{0,3}^{5} \rangle_{0} = \frac{243}{32} = \left( \frac{3}{2} \right)^{5}
\]

(ii) \(\langle P_{1,3} P_{0,3} \rangle_{1}\).

We have

\[
\langle P_{0,2} P_{2,2} P_{0,3} \rangle_{1} = 2 \times \frac{5}{2} \times \langle P_{1,3} P_{0,3} \rangle_{1} - 2 \times \frac{5}{2} \times \frac{3}{4} \times \langle P_{1,1} \rangle_{1} - \frac{5}{2} \times \frac{1}{4} \times \langle P_{0,1} P_{0,1} P_{0,3} \rangle_{0}
\]

And also

\[
\langle P_{2,2} P_{0,2} P_{0,3} \rangle_{1} = 2 \times \frac{1}{2} \times \langle P_{1,3} P_{0,3} \rangle_{1} - 2 \times \frac{1}{2} \times \frac{3}{4} \times \langle P_{1,1} \rangle_{1} - \frac{1}{2} \times \frac{1}{4} \times \langle P_{0,1} P_{0,1} P_{0,3} \rangle_{0}
\]
\[ +2 \times \frac{1}{4} \times \langle P_{0,2} P_{0,2} P_{0,3} P_{0,3} \rangle_0 - 2 \times \frac{1}{4} \times \frac{3}{4} \times \langle P_{0,2} P_{0,2} P_{0,1} \rangle_0 \]

And thus
\[ \langle P_{1,3} P_{0,3} \rangle_1 = \frac{1}{4} \left( 3 \langle P_{1,1} \rangle_1 + \frac{1}{2} \langle P_{0,1} P_{0,1} P_{0,3} \rangle_0 \right) + \frac{1}{2} \langle P_{0,2} P_{0,2} P_{0,3} P_{0,3} \rangle_0 - \frac{3}{8} \langle P_{0,2} P_{0,2} P_{0,1} \rangle_0 \]

And if we substitute the values (computed directly from the \( W_3 \) constraints):
\[ \langle P_{1,1} \rangle_1 = \frac{5}{32} \quad , \quad \langle P_{0,1} P_{0,1} P_{0,3} \rangle_0 = \frac{3}{4} \]
\[ \langle P_{0,2} P_{0,2} P_{0,3} P_{0,3} \rangle_0 = -\frac{9}{4} \quad , \quad \langle P_{0,2} P_{0,2} P_{0,1} \rangle_0 = 1 \]

We get:
\[ \langle P_{1,3} P_{0,3} \rangle_1 = -\frac{21}{128} \]

\( (iii) \) Our third example is \( \langle P_{3,3} \rangle_2 \). From the constraints we get:
\[ \langle P_{1,2} P_{3,2} \rangle_2 = -\frac{175}{1024} - \frac{7}{4} \langle P_{0,1} P_{2,1} \rangle_1 - \frac{7}{4} \langle P_{0,3} P_{1,3} \rangle_1 + 7 \langle P_{3,3} \rangle_2 \]
\[ \langle P_{3,2} P_{1,2} \rangle_2 = -\frac{1}{16} \langle P_{0,2} P_{0,2} P_{1,2} \rangle_0 - \frac{75}{1024} - \frac{3}{4} \langle P_{0,1} P_{2,1} \rangle_1 \]
\[ + 3 \langle P_{3,3} \rangle_2 + \frac{1}{2} \langle P_{0,3} P_{0,2} P_{0,2} P_{1,2} \rangle_1 + \frac{1}{2} \langle P_{1,2} P_{0,2} P_{0,2} P_{1,3} \rangle_1 - \frac{3}{4} \langle P_{0,3} P_{1,3} \rangle_1 \]

And so we get:
\[ \langle P_{3,3} \rangle_2 = \frac{1}{4} \left( \frac{25}{256} + \langle P_{0,1} P_{2,1} \rangle_1 + \langle P_{0,3} P_{1,3} \rangle_1 - \frac{1}{16} \langle P_{0,2} P_{0,2} P_{1,2} \rangle_0 \right) \]
\[ + \frac{1}{2} \langle P_{0,3} P_{1,2} \rangle_1 + \frac{1}{2} \langle P_{1,3} P_{0,2} P_{1,2} \rangle_1 \right) = -\frac{263}{1024} \]
APPENDIX B

Polynomial identities

The steps toward the proof of Lemma 3 included the use of the identities

$$-\frac{1}{2} \sum_{m=0}^{n-1} \binom{n-1}{m} \prod_{i=1}^{m-1} (iq - m - 1) \prod_{j=1}^{n-m-2} (jq - n + m) = \prod_{i=1}^{n-2} (iq - n - 1) \quad (B.1)$$

and

$$- \sum_{m=0}^{n-1} \binom{n-1}{m} \prod_{i=1}^{m-1} (iq - m - 1) \prod_{i=1}^{n-1-m} [(k + i)q + \alpha - n + 1 + m]$$

$$= \prod_{i=1}^{n-1} [(k + i)q + \alpha - n].$$

As stated in section 4, identity (B.1) is a special case of (B.2) at $\alpha + 2 + kq = 0$, and Abel’s identity (18) is the $q = 0$ case of (B.1). We thus present here the proof of eqn.(B.2).

Proof of identity (B.2)*

A product of the form $\prod_{i=1}^{n}[iq - \gamma]$ can be represented in the following form

$$\prod_{i=1}^{n}[iq + \gamma]x^{\frac{\gamma}{q} - (n+1)} = (-q)^n \left( \frac{d}{dx} \right)^n [x^{\frac{\gamma}{q} - 1}] \equiv f(x, n, q, \gamma) \quad (B.3)$$

In terms of this representation (B.2) takes the form

$$\left( \frac{d}{dx} \right)^{n-1} [x^{\frac{\alpha - \gamma}{q} - 1}] = - \sum_{m=0}^{n-1} \binom{n-1}{m} \left( \frac{d}{dx} \right)^{m-1} [x^{\frac{\alpha + 1}{q} - 1}] \left( \frac{d}{dx} \right)^{n-m-1} [x^{\frac{\alpha - m - 1}{q} - 1}] \quad (B.4)$$

where $\beta = \alpha + kq$

* The identity was proven by O.Kenet.
It is easy to check that the identity holds for \( n = 1, 2 \). We now assume that it holds for \( n \) and we examine now the case of \( n + 1 \). In the RHS that implies an additional differentiation with respect to \( x \) and thus we now take the derivative of the RHS

\[
\text{RHS} = - \sum_{m=0}^{n-1} \binom{n-1}{m} \left( \frac{d}{dx} \right)^m \left[ x^{\frac{m+1}{q} - 1} \left( \frac{d}{dx} \right)^{n-m-1} \left[ x^{\frac{n-\beta-m-1}{q} - 1} \right] \right]
\]

(B.5)

We now replace the summation index \( m \) in the first term as follows \( m \rightarrow n - m \) and we choose a particular value for \( \beta \), \( \beta = -2 \). The two terms are now identical apart from the combinatorial factor and the summation range. The latter is easily fixed and using \( \binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m} \) one gets

\[
\text{RHS} = - \sum_{m=0}^{n} \binom{n}{m} \left( \frac{d}{dx} \right)^m \left[ x^{\frac{m+1}{q} - 1} \left( \frac{d}{dx} \right)^{n-m-1} \left[ x^{\frac{n-\beta-m-1}{q} - 1} \right] \right] \quad \text{(B.6)}
\]

which is exactly the RHS of the identity (B.2) for \( n + 1 \) at the particular value of \( \beta = \beta_0 = -1 \). Differentiating the LHS at \( \beta = -2 \) also takes the form of the LHS at \( n + 1 \) and \( \beta = \beta_0 = -1 \). The identity was thus proven for the particular value \( \beta_0 \).

To prove the identity for any value of \( \beta \) we now show that if it holds for some particular value \( \beta_0 \) it holds also for \( \beta_0 - q \). By repeating this process the identity which is a polynomial in \( \beta \) is shown to hold in infinitely many points and thus it should be an exact identity. It is easy to realize that

\[
f(x, n, q, \gamma - q) = xf(x, n, q, \gamma) + nf(x, n - 1, q, \gamma)
\]

and thus inserting it into eqn.(B.2) with the use of the assumption that holds for \( n - 1 \) for any \( \gamma \) we get that the identity holds for the shifted \( \beta \). This completes the proof of the identity.
REFERENCES

1. For a review of the subject see for instance P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, LA-UR-93-1722, SPhT/93-061, hep-th/9306153.

2. R. Dijkgraaf “Intersection theory, integrable hierarchies and topological field theory” IASSNS-HEP-91/91, and references therein.

3. F. David *Mod. Phys. Lett.* **A3** (1988) 1651;
   J. Distler and H. Kawai *Nucl. Phys.* **B321** (1989) 509.

4. A. M. Polyakov, *Mod. Phys. Lett.* **A2** (1987) 893.

5. D. Gross and A. A. Migdal *Phys. Rev. Lett.* **64** (1990) 127;
   M. Douglas and S. Shenker *Nucl. Phys.* **B335** (1990) 635;
   E. Brezin and V. A. Kazakov, *Phys. Lett.* **236B** (1990) 144.

6. M. Duglas, *Phys. Lett.* **238B** (1990) 176.

7. R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Nucl. Phys.* **B348** (1991) 435; *Nucl. Phys.* **B352** (1991) 59.

8. M. Fukuma, H. Kawai and R. Nakayama, *Int. Jou. of Mod. Phys.* **A6** (1991) 1385.

9. M. Spigelglas and S. Yankielowicz, *Nucl. Phys.* **B393** (1993) 301.
   E. Witten, *Comm. Math. Phys.* **144** (1992) 189.

10. E. Witten, *Nucl. Phys.* **B340** (1990) 281.

11. E. Verlinde, and H. Verlinde, *Nucl. Phys.* **B348** (1991) 457.

12. M. Kontsevich *Comm. Math. Phys.* **147** (1992) 1.

13. For a review see for instance A. Morozov “Integrability and Matrix Models” ITEP-M2/93, hep-th9303139.

14. J. Goeree, *Nucl. Phys.* **B358** (1991) 737.

15. K. Li, *Nucl. Phys.* **B 354** (1991) 725-739.
16. K. Aoki, D. Montano, J. Sonnenschein, *Int. Jour. of Mod. Phys.* A7 (1992) 1755.

17. D. Montano and G. Rivlis, “Solving topological 2-D quantum gravity using Ward identities” UCB-PTH-92-30 [hep-th/9210106].

18. D. Montano and G. Rivlis, *Nucl. Phys.* B360 (1991) 524.

19. Abel, “Oeuvres Comtletes”, Cristiana, C. Grondhal 1893.