Variational Representations related to Quantum Rényi Relative Entropies

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Abstract

In this note, we focus on the variational representations of some matrix norm functions and matrix trace functions that are related to the quantum Rényi relative entropies. Concretely, by using the Hölder inequality and Young inequality for symmetric norms we give the variational representations of the function \((A, B) \mapsto \|B^{1/2}K^*A^pKB^{1/2}\|\) for symmetric norms. These variational expressions enable us to give some new proofs of the convexity/concavity of the trace function \((A, B) \mapsto \text{Tr}(B^{1/2}K^*A^pKB^{1/2})\) and some extensions of the Lieb’s theorems in terms of symmetric norms or symmetric anti-norms.

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1 Introduction

Let \(\mathcal{M}_n\) be the set of \(n \times n\) matrices and \(\mathcal{P}_n\) be the set of \(n \times n\) positive semi-definite matrices. A matrix \(A \in \mathcal{P}_n\) with \(\text{Tr} A = 1\) is called a density matrix. Many of the statements in this note are of special interest for density matrices but we will not make such restriction. For \(A, B \in \mathcal{P}_n\), the traditional relative Rényi entropy is defined as

\[
D_\alpha(A\|B) = \frac{1}{\alpha - 1} \log \text{Tr}(A^\alpha B^{1-\alpha}), \quad \alpha \in (0, \infty) \setminus \{1\}. \tag{1.1}
\]

A variant of the traditional relative Rényi entropy is called the sandwiched Rényi relative entropy which was introduced by Müller-Lennert, Dupuis, Szehr, Fehr, Tomamichel [19] and Wilde, Winter, Yang [23]. And this entropy is defined as

\[
\tilde{D}_\alpha(A\|B) = \frac{1}{\alpha - 1} \log \text{Tr} F_\alpha(A, B), \quad \alpha \in (0, \infty) \setminus \{1\}, \tag{1.2}
\]

where

\[
F_\alpha(A, B) = \text{Tr} \left( B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}} \right)^\alpha, \quad \alpha \in (0, \infty). \tag{1.3}
\]
The trace function $F_\alpha(A, B)$ is a parameterized version of the fidelity

$$F(A, B) = \text{Tr} \left( B^{\frac{1}{2}} A B^{\frac{1}{2}} \right)^{\frac{1}{2}} ,$$

and is called the sandwiched quasi-relative entropy. We should notice that (1.4)

$$\lim_{\alpha \to \infty} \tilde{D}_\alpha(A||B) = \| \log B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \| ,$$

where $\| \cdot \|$ is the operator norm. The expression in (1.5) coincides with the Thompson metric

$$d_T(A, B) = \max \{ \log \lambda_1(AB^{-1}), \log \lambda_1(BA^{-1}) \}$$
on $P_n$ (see [2]), and is closely related to the max-relative entropy

$$D_{\text{max}}(A||B) = \log \lambda_1(AB^{-1})$$
in quantum information theory [11]. Moreover,

$$\lim_{\alpha \to 1} \tilde{D}_\alpha(A||B) = \frac{1}{\text{Tr}A} \text{Tr}(A \log A - \log B) .$$

The expression in (1.6) is the quantum relative entropy introduced by Umegaki [22]. Audenaert and Datta [3] recently unified the above relative Rényi entropies and introduced the $\alpha - z$ Rényi entropy

$$D_{\alpha,z}(A||B) = \frac{1}{\alpha - 1} \log \text{Tr} \left( B^{\frac{1}{2z}} A^{\frac{1}{z}} B^{\frac{1}{2z}} \right)^z , \quad \alpha \in (0, \infty) \setminus \{1\}, \quad z > 0 .$$

The Rényi entropies or more generally quantum divergences $\mathcal{D} (\cdot||\cdot)$ should satisfy the monotonicity under the quantum channel, i.e., the completely positive trace preserving map to make them have operational meaning. That is

$$\mathcal{D}(\Phi(A)||\Phi(B)) \leq \mathcal{D}(A||B) ,$$

for all CPTP maps $\Phi$ and density matrices $A, B$. This inequality is also known as Data Processing Inequality. Essentially, the data processing inequality is equivalent to the joint convexity or concavity of the trace functions in the definition of the quantum divergence $\mathcal{D}$.

The trace function in the traditional relative Rényi entropy $D_\alpha$ is $\text{Tr}(A^\alpha B^{1-\alpha})$ which can be viewed as the tracial geometric mean and its concavity/convexity is given by the famous Lieb’s concavity theorem [18] and Ando’s convexity theorem [1]. The convexity of $F_\alpha(A, B)$ in the definition of the sandwiched Rényi relative entropy was established by Frank and Lieb in [13]. The trace function in the $\alpha - z$ Rényi entropy $D_{\alpha,z}$ is abstracted into

$$\Psi_{p,q,s}(A, B) = \text{Tr} \left( B^{\frac{2}{q}} K^s A^p K B^{\frac{2}{q}} \right)^s .$$
For which values of \( p, q, s \) does \( \Psi_{p,q,s}(A, B) \) satisfy convexity/concavity draw extensively attention in recent papers. We refer the readers to to [9] and also [8, 10, 24] for the whole story of development of the convexity theorems related to \( \Psi_{p,q,s}(A, B) \).

In the development of operator convexity theorems staring from the Lieb’s concavity theorem, there are several powerful methodologies. By using the theory of Herglotz (Pick) functions, Epstein [12] not only proved the Lieb’s concavity theorem but also derived the concavity of the trace function \( \Upsilon_{p,1}(A) = \text{Tr}(K^*A^pK)^{1/p} \) for \( 0 < p < 1 \). Epstein’s method can be viewed as an analyticity method. We refer the readers to Hiai’s papers [14, 15] for the development of this method. The variational method is introduced by Carlen and Lieb [10] by using the tracial Young inequality and its reverse version. In [10], Carlen and Lieb proved that the trace function \( \Upsilon_{p,q}(A) = \text{Tr}(B^*A^pB)^{q/p} \) is convex for \( 1 \leq p \leq 2 \), and is concave for \( 0 \leq p, q \leq 1 \) when \( p = 2 \). These results are extensions of the Leib’s theorem and the Epstein’s theorem. In [13], by using variational method, Frank and Lieb proved that \( \Psi_{p,q,s}(A, B) \) is jointly convex for \( 1 \leq p \leq 2, -1 \leq q < 0 \) and \( s \geq \min\{1/(p-1), 1/(1+q)\} \); when \( p = 2 \), \( \Psi_{p,q,s}(A, B) \) is jointly convex for \( -1 \leq q < 0 \) and \( s \geq \min\{1/(p-1), 1/(1+q)\} \); and when \( 0 \leq p, q \leq 1, 0 \leq s \leq 1/(p+q) \), \( \Psi_{p,q,s}(A, B) \) is jointly concave. In [24], Zhang tackled with the Audenaert-Datta conjecture and the Carlen-Frank-Lieb conjecture. By using variational method, he proved that when \( -1 \leq q < 0, 1 \leq p \leq 2, (p, q) \neq (1, -1), s \geq 1/(p+q) \), the trace function \( \Psi_{p,q,s}(A, B) \) is jointly convex. Hence together with other known results, the full range of \( (p, q, s) \) for \( \Psi_{p,q,s}(A, B) \) to be joint convex/concave are given.

There are also other uses of the variational method in quantum information theory. The well-known Gibbs variational principle and the variational expressions established by Hiai-Petz [16], Tropp [21] and Shi-Hansen [20] enable one to establish the relationship between quantum entropies and the trace functions related to exponential/logarithm functions.

In this note, we focus on the variational method. We will give some different variational representations of \( \Psi_{p,q,s}(A, B) \) by using the Hölder inequality, Young inequality and their reverse versions. Especially, we give the critical points of the variational representations. These representations will make the proof of the convexity/concavity of \( \Psi_{p,q,s}(A, B) \) more clear and give some new extensions.

## 2 Reverse Hölder Inequality and Young Inequality

In this section, we consider the reverse Hölder inequality and Young Inequality for matrices. Set \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). A function \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R}_+ \) is called a symmetric gauge function if it satisfies the following conditions:

(i) \( \Phi \) is a norm on \( \mathbb{R}^n \),

(ii) \( \Phi(Px) = \Phi(x) \) for all \( x \in \mathbb{R}^n \), \( P \in S_n \),

(iii) \( \Phi(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) = \Phi(x_1, \ldots, x_n) \) if \( \varepsilon_j = \pm 1 \),
(iv) \( \Phi(1, 0, \ldots, 0) = 1 \).

Symmetric gauge function is convex on \( \mathbb{R}^n \) and is monotone on \( \mathbb{R}^n_+ \). Hence if \( x \prec_w y \) in \( \mathbb{R}^n_+ \), we have \( \Phi(x) \leq \Phi(y) \).

Recall the notations: \( |x| = (|x_1|, \ldots, |x_n|) \), and \( |x| \leq |y| \) if \( |x_i| \leq |y_i| \) for \( 1 \leq i \leq n \).

From the scalar reverse Young inequality we have
\[
1 \frac{1}{r} |x|^{r} \geq 1 \frac{1}{p} |x|^{p} + 1 \frac{1}{q} |y|^{q}, \quad \text{for} \quad r, p > 0, q < 0 \quad \text{and} \quad 1 \frac{1}{r} = 1 \frac{1}{p} + 1 \frac{1}{q}.
\]

Then it follows that for \( r, p > 0, q < 0 \) and \( 1 \frac{1}{r} = 1 \frac{1}{p} + 1 \frac{1}{q} \),
\[
1 \frac{1}{r} |xy|^r \geq 1 \frac{1}{p} |x|^p + 1 \frac{1}{q} |y|^q. \quad \text{(2.1)}
\]

**Theorem 2.1.** For \( x, y \in \mathbb{R}^n \) and symmetric gauge function \( \Phi \),
\[
\Phi (|xy|^r)^\frac{1}{r} \geq \Phi (|x|^p)^\frac{1}{p} \Phi (|y|^q)^\frac{1}{q}
\]
holds for \( r, p > 0, q < 0 \) with \( 1 \frac{1}{r} = 1 \frac{1}{p} + 1 \frac{1}{q} \).

**Proof.** By inequality (2.1), it is easy to see
\[
\frac{p}{r} |xy|^r - \frac{p}{q} |y|^q \geq |x|^p.
\]
Since \( p \frac{1}{r} - q > 0 \) and \( p \frac{1}{r} + (-q) = 1 \), it follows from the monotonicity and convexity of \( \Phi \) on \( \mathbb{R}^n_+ \) that
\[
1 \frac{1}{r} \Phi (|xy|^r) \geq 1 \frac{1}{p} \Phi (|x|^p) + 1 \frac{1}{q} \Phi (|y|^q).
\]

For \( t > 0 \), by replacing \( x, y \) by \( tx \) and \( t^{-1} y \), we have
\[
\Phi (|xy|^r) \geq \max_{t > 0} \left\{ \frac{r}{p} t^p \Phi (|x|^p) + \frac{r}{q} t^{-q} \Phi (|y|^q) \right\}.
\]

The function
\[
\varphi(t) = \frac{r}{p} t^p a + \frac{r}{q} t^{-q} b, \quad \text{where} \quad t, a, b > 0.
\]

gets its maximum at the point \( t_0 = (\frac{b}{a})^{\frac{1}{p+q}} \), and
\[
\varphi(t_0) = a^{\frac{p}{p+q}} b^{\frac{q}{p+q}}.
\]

Hence we have
\[
\Phi (|xy|^r)^\frac{1}{r} \geq \Phi (|x|^p)^\frac{1}{p} \Phi (|y|^q)^\frac{1}{q}.
\]
Now we consider the reverse Hölder inequality for symmetric norms. Denote \( s(A) \) the \( n \)-vector whose coordinates are the singular values of the matrix \( A \in \mathcal{M}_n \) in the decreasing order, i.e. \( s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A) \). Given a symmetric gauge function \( \Phi \) on \( \mathbb{R}_n \), the function
\[
\| A \| := \Phi(s(A))
\]
defines a symmetric norm (unitarily invariant norm) on \( \mathcal{M}_n \).

**Theorem 2.2.** For symmetric norm \( \| \cdot \| \) and matrices \( A, B \) with \( B \) invertible, we have
\[
\| |AB|^r \|^{\frac{1}{r}} \geq \| |A|^p \|^{\frac{1}{p}} \cdot \| |B|^q \|^{\frac{1}{q}}
\] (2.2)
holds for \( r, p > 0, q < 0 \) and \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).

**Proof.** By Gelfand-Naimark Theorem (see [17, Theorem 6.13]) we have
\[
(s_i(A)s_{n-i+1}(B)) \preceq \log s(AB).
\] (2.3)
Since for invertible matrix \( B, s_i^{-1}(B^{-1}) = s_{n-i+1}(B) \), it follows that
\[
(s_i(A)s_i^{-1}(B^{-1})) \preceq \log s(AB).
\]
And since \( r > 0 \), we have
\[
(s_i^r(A)s_i^{-r}(B^{-1})) \preceq \log s^r(AB).
\]
Thus for matrices \( A, B \), with \( B \) invertible and \( r, p > 0, q < 0 \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \),
\[
\Phi(s^r(AB))^\frac{1}{r} \geq \Phi((s_i^r(A)s_i^{-r}(B^{-1})))^\frac{1}{r}
\]
\[
\geq \Phi(s^p(A))^\frac{1}{p} \Phi(s^{-q}(B^{-1}))^\frac{1}{q}
\]
\[
= \Phi(s^p(A))^\frac{1}{p} \Phi(s^q(B))^\frac{1}{q}.
\]
Hence it follows that
\[
\| |AB|^r \|^{\frac{1}{r}} \geq \| |A|^p \|^{\frac{1}{p}} \cdot \| |B|^q \|^{\frac{1}{q}}.
\]

The following corollaries are easy to get.

**Corollary 2.3.** For matrices \( A, B \), with \( B \) invertible, the reverse Young inequality for symmetric norm
\[
\frac{1}{r} \| |AB|^r \| \geq \frac{1}{p} \| |A|^p \| + \frac{1}{q} \| |B|^q \|
\] (2.4)
holds for \( r, p > 0, q < 0 \) and \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).
Corollary 2.4. For matrices $A, B$, with $B$ invertible, the reverse tracial Hölder inequality
\[
(\text{Tr}|AB|^r)^{\frac{1}{r}} \geq (\text{Tr}|A|^p)^{\frac{1}{p}} (\text{Tr}|B|^q)^{\frac{1}{q}}
\]
holds for $r, p > 0, q < 0$ and \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).

Corollary 2.5. For matrices $A, B$, with $B$ invertible, the reverse tracial Young inequality
\[
\frac{1}{r} \text{Tr}|AB|^r \geq \frac{1}{p} \text{Tr}|A|^p + \frac{1}{q} \text{Tr}|B|^q
\]
holds for $r, p > 0, q < 0$ and \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).

A symmetric anti-norm $\| \cdot \|$ on $\mathcal{P}_n$ is a non-negative continuous function that is positive homogeneous, unitary invariant and satisfying the super-additivity:
\[
\|A + B\| \geq \|A\| + \|B\| \quad \text{for all } A, B \in \mathcal{P}_n.
\]
It is easy to see that a symmetric anti-norm on $\mathcal{P}_n$ is concave. And if $\| \cdot \|$ is a symmetric norm and $p > 0$, then for invertible matrix $A$, $\|A\| := \|A^{-p}\|^{-1/p}$ is a symmetric anti-norm. For more information about the symmetric anti-norm and also reverse Hölder inequality for matrix we refer the readers to [7].

Now we define
\[
\|A\|_p = \left( \sum_{i=1}^{n} s_i^p(A) \right)^{1/p}
\]
for $p \in \mathbb{R} \setminus \{0\}$. When $p < 0$, we set $A$ invertible, and in this case $\| \cdot \|_p$ is the negative Schatten anti-norm, which is a typical symmetric anti-norm. When $0 < p < 1$, it is a quasi-norm and is also a symmetric anti-norm. When $p \geq 1$, it is the Schatten $p$-norm.

Theorem 2.6. Suppose $r, p > 0$ and $q < 0$ satisfying $1/r = 1/p + 1/q$. Then for matrices $A, B$ with $B$ invertible,
\[
\|AB\|_r \geq \|A\|_p \|B\|_q.
\]

Proof. By calculation,
\[
\|AB\|_r = \left( \sum_{i=1}^{n} s_i^r(AB) \right)^{1/r} \\
\geq \left( \sum_{i=1}^{n} s_i^r(A) s_{n-i+1}^r(B) \right)^{1/r} \\
\geq \left[ \left( \sum_{i=1}^{n} (s_i^r(A))^\frac{q}{r} \right)^\frac{r}{q} \left( \sum_{i=1}^{n} (s_i^r(B))^\frac{q}{r} \right)^\frac{r}{q} \right]^\frac{1}{r} \\
\geq \left( \sum_{i=1}^{n} s_i^q(A) \right)^\frac{1}{q} \left( \sum_{i=1}^{n} s_i^q(B) \right)^\frac{1}{q} \\
= \|A\|_p \|B^{-1}\|_{-q}^{-1}.
\]
Hence the conclusion follows.
3 Variational representations of some matrix functionals

For symmetric norm \( \| \cdot \| \), consider the function

\[
\hat{\Psi}_{p,q,s}(A, B) = \left\| \left( B^\frac{s}{2} K^s A^p K B^\frac{s}{2} \right)^s \right\|
\]

for \( A, B \in \mathcal{P}_n, K \in \mathcal{M}_n \) and \( p, q, s \in \mathbb{R} \). We have the following variational representations:

**Theorem 3.1.** (i) Let \( s, p, q > 0; \) or \( s > 0, p, q < 0 \). Then

\[
\hat{\Psi}_{p,q,s}(A, B) = \left\{ \begin{array}{ll}
\inf_{Z > 0} \left\{ \left( Z^{-\frac{s}{2}} K^s A^p K Z^{-\frac{s}{2}} \right) \frac{(p+q)}{p} \right\} \cdot \left( Z^{\frac{s}{2}} B^q Z^{\frac{s}{2}} \right) \frac{(s+q)}{q} \right\}
\end{array} \right. \quad (3.1)
\]

(ii) Let \( s > 0, p > 0, q < 0 \) with \( p + q > 0; \) or \( s > 0, p < 0, q > 0 \) with \( p + q < 0 \). Then

\[
\hat{\Psi}_{p,q,s}(A, B) = \left\{ \begin{array}{ll}
\sup_{Z > 0} \left\{ \left( Z^{-\frac{s}{2}} K^s A^p K Z^{-\frac{s}{2}} \right) \frac{(p+q)}{p} \right\} \cdot \left( Z^{\frac{s}{2}} B^q Z^{\frac{s}{2}} \right) \frac{(s+q)}{q} \right\}
\end{array} \right. \quad (3.2)
\]

**Proof.** From Hölder inequality and Young inequality and their reverse versions, we have for \( S, T \in \mathcal{M}_n \),

\[
\| ST^{r_0} \| \leq \| S \|^{r_0} \frac{r_0}{r_1} \| T^{r_2} \|^{r_0} \quad (3.3)
\]

\[
\leq \frac{r_0}{r_1} \| S \|^{r_0} + \frac{r_0}{r_2} \| T \|^{r_2} \quad (r_0, r_1, r_2 > 0, \frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}); \quad (3.4)
\]

and for \( S, T \in \mathcal{M}_n \) with \( T \) invertible,

\[
\| ST^{r_0} \| \geq \| S \|^{r_0} \frac{r_0}{r_1} \| T \|^{r_2} \frac{r_0}{r_2} \quad (3.5)
\]

\[
\geq \frac{r_0}{r_1} \| S \|^{r_0} + \frac{r_0}{r_2} \| T \|^{r_2} \quad (r_0, r_1 > 0, r_2 < 0, \frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}). \quad (3.6)
\]

Set \( S = A^\frac{s}{2} K Z^{-\frac{s}{2}} \) and \( T = Z^\frac{s}{2} B^\frac{s}{2} \) and \( r_0 = 2s, r_1 = 2s(p + q)/p, r_2 = 2s(p + q)/q \) in inequality (3.3) and (3.4), then the conditions \( r_0, r_1, r_2 > 0 \) and \( \frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2} \) hold. Hence

\[
\hat{\Psi}_{p,q,s}(A, B) = \left\| \left( B^\frac{s}{2} K^s A^p K B^\frac{s}{2} \right)^s \right\|
\]

\[
= \left\| \left( B^\frac{s}{2} Z^\frac{s}{2} K Z^{-\frac{s}{2}} A^p K Z^{-\frac{s}{2}} B^\frac{s}{2} \right)^s \right\|
\]

\[
= \| A^\frac{s}{2} K Z^{-\frac{s}{2}} \cdot B^\frac{s}{2} \|^{r_0}
\]

\[
\leq \| A^\frac{s}{2} K Z^{-\frac{s}{2}} \|^{r_0} \cdot \| Z^\frac{s}{2} B^\frac{s}{2} \|^{r_0}
\]

\[
= \left\{ \left( Z^{-\frac{s}{2}} K^s A^p K Z^{-\frac{s}{2}} \right) \frac{(p+q)}{p} \right\} \cdot \left( Z^\frac{s}{2} B^q Z^\frac{s}{2} \right) \frac{(s+q)}{q} \right\}
\]

\[
\leq \frac{p}{p+q} \left\{ \left( Z^{-\frac{s}{2}} K^s A^p K Z^{-\frac{s}{2}} \right) \frac{(p+q)}{p} \right\} + \frac{q}{p+q} \left\{ \left( Z^\frac{s}{2} B^q Z^\frac{s}{2} \right) \frac{(s+q)}{q} \right\}.
\]
When
\[ Z = B^{-q_0} \frac{2}{p+q} (K^* A^p K), \]
we have
\[ \| (Z^{-1} K^* A^p K Z^{-1}) \frac{s(p+q)}{p} \| = \| (K^* A^p K) \frac{s(p+q)}{p} \| \]
\[ = \| (K^* A^p K) \frac{s(p+q)}{p} \| \]
\[ = \| (K^* A^p K) \frac{s(p+q)}{p} \| \]
\[ = \| (K^* A^p K) \frac{s(p+q)}{p} \| \]
\[ = \| (B^{\frac{2}{p}} K^* A^p K B^{\frac{2}{p}})^s \| ; \]
and
\[ \| (Z^{\frac{1}{2}} B^{\frac{1}{2}} Z^{\frac{1}{2}})^{s(p+q)} q \| = \| (B^{\frac{2}{p}} Z B^{\frac{2}{p}})^{s(p+q)} q \| = \| (B^{\frac{2}{p}} K^* A^p K B^{\frac{2}{p}})^{s(p+q)} q \| = \| (B^{\frac{2}{p}} K^* A^p K B^{\frac{2}{p}})^{s(p+q)} q \| . \]

Then it follows that
\[ \tilde{\Psi}_{p,q,s}(A, B) = \| (B^{\frac{2}{p}} K^* A^p K B^{\frac{2}{p}})^s \| \]
\[ = \inf_{Z > 0} \left\{ \left\| (Z^{-1} K^* A^p K Z^{-1}) \frac{s(p+q)}{p} \right\| + \frac{q}{p+q} \right\} \]
\[ = \inf_{Z > 0} \left\{ \left\| (Z^{-1} K^* A^p K Z^{-1}) \frac{s(p+q)}{p} \right\| + \frac{q}{p+q} \right\} . \]

Under the conditions of (ii), set
\[ r_0 = 2s, r_1 = \frac{2s(p+q)}{p}, r_2 = \frac{2s(p+q)}{q} . \]

Then we have \( r_0 > 0, r_1 > 0, r_2 < 0 \) and \( \frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2} \). Now set \( S = A^{\frac{2}{p}} K Z^{\frac{1}{2}} \) and \( T = Z^{\frac{1}{2}} B^{\frac{1}{2}} \) in inequalities (3.5) and (3.6). Following a similar argument as above, we can obtain
\[ \tilde{\Psi}_{p,q,s}(A, B) = \| (B^{\frac{2}{p}} K^* A^p K B^{\frac{2}{p}})^s \| \]
\[ = \sup_{Z > 0} \left\{ \left\| (Z^{-1} K^* A^p K Z^{-1}) \frac{s(p+q)}{p} \right\| + \frac{q}{p+q} \right\} . \]
Theorem 3.2. (i) Let $s, q > 0$ with $s \leq 1/q$. Then

$$
\Psi_{p,q,s}(A, B) = \begin{cases}
\inf_{Z > 0} \left\{ \left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\|^{1-sq} \cdot \left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\|^{sq} \right\} \\
\inf_{Z > 0} \left\{ (1 - sq) \left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\| + sq \left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\| \right\} 
\end{cases}
$$

(3.7)

(ii) Let $s > 0, q < 0$. Then

$$
\Psi_{p,q,s}(A, B) = \begin{cases}
\sup_{Z > 0} \left\{ \left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\|^{1-sq} \cdot \left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\|^{sq} \right\} \\
\sup_{Z > 0} \left\{ (1 - sq) \left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\| + sq \left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\| \right\} 
\end{cases}
$$

(3.8)

Proof. Under the conditions of (i), set $r_0 = 2s, r_1 = \frac{2s}{1-sq}, r_2 = \frac{2}{q}$, then $r_0, r_1, r_2 > 0$ and $\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}$ hold. Now set $S = A^\frac{1}{2}KZ^{-\frac{1}{2}}$ and $T = Z^{\frac{1}{2}}B^2$ in inequality (3.3) and (3.4), then we have

$$
\Psi_{p,q,s}(A, B) = \left\| (B^{\frac{1}{2}}K^*A^pKB^{\frac{1}{2}})^s \right\| \\
= \left\| (B^{\frac{1}{2}}Z^{\frac{1}{2}}K^{-\frac{1}{2}}K^*A^pA^{\frac{1}{2}}KZ^{-\frac{1}{2}}Z^{\frac{1}{2}}B^{\frac{1}{2}})^s \right\| \\
\leq \left\| A^{\frac{1}{2}}KZ^{-\frac{1}{2}} \right\|^{r_1} \cdot \left\| Z^{\frac{1}{2}}B^{\frac{1}{2}} \right\|^{r_2} \\
= \left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\|^{1-sq} \cdot \left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\|^{sq} \\
\leq (1 - sq)\left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\| + sq \left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\|.
$$

When

$$
Z = B^{-q/2-sq}(K^*A^pK),
$$
we have

$$
\left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\| = \left\| (B^{\frac{1}{2}}K^*A^pKB^{\frac{1}{2}})^s \right\|;
$$
and

$$
\left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\| = \left\| B^{\frac{1}{2}}K^*A^pKB^{\frac{1}{2}} \right\|^s.
$$
Hence it follows that

$$
\Psi_{p,q,s}(A, B) = \left\| (B^{\frac{1}{2}}K^*A^pKB^{\frac{1}{2}})^s \right\| \\
= \begin{cases}
\inf_{Z > 0} \left\{ \left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\|^{1-sq} \cdot \left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\|^{sq} \right\} \\
\inf_{Z > 0} \left\{ (1 - sq) \left\| (Z^{-\frac{1}{2}}K^*A^pKZ^{-\frac{1}{2}}) \right\| + sq \left\| (Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}}) \right\| \right\} 
\end{cases}
$$

(3.7)
Similarly, under the conditions of (ii) and using inequalities (3.5) and (3.6), we can get

$$
\Psi_{p,q,s}(A, B) = \left\| \left(B^\frac{1}{2} K^* A^p K B^\frac{1}{2} \right)^s \right\|
$$

\[
\begin{align*}
\sup_{Z > 0} & \left\{ \left( \left( Z - \frac{1}{2} K^* A^p K Z - \frac{1}{2} \right)^{s(p+q)\frac{1}{p}} \right)^{\frac{p}{p+q}} \cdot \left( \left( Z^\frac{1}{2} B^q Z^\frac{1}{2} \right)^{s(p+q)\frac{1}{q}} \right)^{\frac{q}{p+q}} \right\} \\
& \sup_{Z > 0} \left\{ (1 - sq) \left( Z - \frac{1}{2} K^* A^p K Z - \frac{1}{2} \right)^{s(p+q)\frac{1}{p}} + sq \left( Z^\frac{1}{2} B^q Z^\frac{1}{2} \right)^{s(p+q)\frac{1}{q}} \right\}. 
\end{align*}
\]

Consider the trace function

$$
\Psi_{p,q,s}(A, B) = \text{Tr} \left( B^\frac{1}{2} K^* A^p K B^\frac{1}{2} \right)^s,
$$

for $A, B \in P_n, K \in M_n$ and $p, q, s \in \mathbb{R}$. We have the following variational representations:

**Corollary 3.3.**

(i) Let $s, p, q > 0$; or $s > 0, p, q < 0$. Then

$$
\Psi_{p,q,s}(A, B) = \left\{ \left( \text{Tr} \left( Z - \frac{1}{2} K^* A^p K Z - \frac{1}{2} \right)^{s(p+q)\frac{1}{p}} \right)^{\frac{p}{p+q}} \cdot \left( \text{Tr} \left( Z^\frac{1}{2} B^q Z^\frac{1}{2} \right)^{s(p+q)\frac{1}{q}} \right)^{\frac{q}{p+q}} \right\}.
$$

(ii) Let $s > 0, p > 0, q < 0$ with $p + q > 0$; or $s > 0, p < 0, q > 0$ with $p + q < 0$. Then

$$
\Psi_{p,q,s}(A, B) = \left\{ \left( \text{Tr} \left( Z - \frac{1}{2} K^* A^p K Z - \frac{1}{2} \right)^{s(p+q)\frac{1}{p}} \right)^{\frac{p}{p+q}} \cdot \left( \text{Tr} \left( Z^\frac{1}{2} B^q Z^\frac{1}{2} \right)^{s(p+q)\frac{1}{q}} \right)^{\frac{q}{p+q}} \right\}.
$$

**Corollary 3.4.**

(i) Let $s, q > 0$ with $s \leq 1/q$. Then

$$
\Psi_{p,q,s}(A, B) = \left\{ \left( \text{Tr} \left( Z - \frac{1}{2} K^* A^p K Z - \frac{1}{2} \right)^{s\frac{1}{q}} \right)^{1 - sq} \cdot \left( \text{Tr} \left( Z^\frac{1}{2} B^q Z^\frac{1}{2} \right)^{s\frac{1}{q}} \right)^{sq} \right\}.
$$

(ii) Let $s > 0, q < 0$. Then

$$
\Psi_{p,q,s}(A, B) = \left\{ \left( \text{Tr} \left( Z - \frac{1}{2} K^* A^p K Z - \frac{1}{2} \right)^{s\frac{1}{q}} \right)^{1 - sq} \cdot \left( \text{Tr} \left( Z^\frac{1}{2} B^q Z^\frac{1}{2} \right)^{s\frac{1}{q}} \right)^{sq} \right\}.
$$
Set \( s = t, p = 1, q = \frac{1-t}{t} \) and \( K = I \) in Corollary 3.3 or Corollary 3.4, we can obtain:

**Corollary 3.5.** Let \( A, B \in \mathcal{P}_n \). If \( 0 \leq t \leq 1 \), then

\[
F_t(A, B) = \text{Tr} \left( B^{\frac{1-t}{2}} AB \frac{1-t}{2} \right)^t
\]

\[
= \begin{cases}
\inf_{Z \in \mathcal{P}_n} \left\{ t \left( \text{Tr} Z^{-\frac{1}{2}} AZ^{-\frac{1}{2}} \right)^t \left( \text{Tr} \left( Z^{\frac{1}{2}} B \frac{1-t}{2} Z^{\frac{1}{2}} \right)^{1-t} \right) \right. \\
\left. \inf_{Z \in \mathcal{P}_n} \left\{ t \text{Tr} Z^{-\frac{1}{2}} AZ^{-\frac{1}{2}} + (1-t) \text{Tr} \left( Z^{\frac{1}{2}} B \frac{1-t}{2} Z^{\frac{1}{2}} \right)^{1-t} \right\} \right. \\
\sup_{Z \in \mathcal{P}_n} \left\{ t \left( \text{Tr} Z^{-\frac{1}{2}} AZ^{-\frac{1}{2}} \right)^t \left( \text{Tr} \left( Z^{\frac{1}{2}} B \frac{1-t}{2} Z^{\frac{1}{2}} \right)^{1-t} \right) \right. \\
\left. \sup_{Z \in \mathcal{P}_n} \left\{ t \text{Tr} Z^{-\frac{1}{2}} AZ^{-\frac{1}{2}} + (1-t) \text{Tr} \left( Z^{\frac{1}{2}} B \frac{1-t}{2} Z^{\frac{1}{2}} \right)^{1-t} \right\} \right. \\
\end{cases}
\]

If \( t \geq 1 \), then

\[
F_t(A, B) := \text{Tr} \left( B^{\frac{1-t}{2}} AB \frac{1-t}{2} \right)^t
\]

\[
= \begin{cases}
\sup_{Z \in \mathcal{P}_n} \left\{ t \left( \text{Tr} Z^{-\frac{1}{2}} AZ^{-\frac{1}{2}} \right)^t \left( \text{Tr} \left( Z^{\frac{1}{2}} B \frac{1-t}{2} Z^{\frac{1}{2}} \right)^{1-t} \right) \right. \\
\left. \sup_{Z \in \mathcal{P}_n} \left\{ t \text{Tr} Z^{-\frac{1}{2}} AZ^{-\frac{1}{2}} + (1-t) \text{Tr} \left( Z^{\frac{1}{2}} B \frac{1-t}{2} Z^{\frac{1}{2}} \right)^{1-t} \right\} \right. \\
\end{cases}
\]

The variational expressions of \( F_t(A, B) \) for \( t \in (0, 1) \) was obtained in [13]. See also [4] and [6].

## 4 Extensions of Lieb’s Concavity Theorems

Now we consider the convexity or concavity of \( \tilde{\Psi}_{p,q,s}(A, B) \) and \( \Psi_{p,q,s}(A, B) \) by using the variational method. Before doing so, we recall some convexity/concavity theorems about the matrix function

\[
\Upsilon_{p,s}(A) = \text{Tr} \left( K^s A^p K^{-s} \right).
\]

Here we only consider the case of \( s > 0 \).

**Theorem 4.1.** Fix \( K \in \mathcal{M}_n \). Then for \( A \in \mathcal{P}_n \) we have

(i) If \( 0 \leq p < 1 \) and \( 0 < s \leq 1/p \), then \( \Upsilon_{p,s} \) is concave.

(ii) If \( -1 \leq p \leq 0 \) and \( s > 0 \), then \( \Upsilon_{p,s} \) is convex.

(iii) If \( 1 \leq p \leq 2 \) and \( s \geq 1/p \), then \( \Upsilon_{p,s} \) is convex.

We now firstly recover the following well-known conclusions (Theorem 4.2) by using Corollary 3.4. For more information of Theorem 4.2 we refer the readers to [9, 24].

**Theorem 4.2.** Let \( K \in \mathcal{M}_n \) be arbitrary, and \( A, B \in \mathcal{P}_n \).

(i) If \( 0 \leq p, q \leq 1 \) and \( 0 < s \leq 1/(p + q) \), then \( \Psi_{p,q,s}(A, B) \) is jointly concave.

(ii) If \( -1 \leq p, q \leq 0 \) and \( s > 0 \), then \( \Psi_{p,q,s}(A, B) \) is jointly concave.
(iii) If \(1 \leq p \leq 2, -1 \leq q \leq 0, (p, q) \neq (1, -1), \) and \(s \geq 1/(p + q), \) then \(\Psi_{p,q,s}(A, B)\) is jointly convex.

**Proof.** Under the conditions of (i), we have
\[
0 \leq p \leq 1, \quad \frac{s}{1 - sq} = \frac{1}{s^{-1} - q} \leq \frac{1}{p}, \quad \text{and} \quad 0 \leq q \leq 1.
\]
Hence it follows from Theorem 4.2 (i) that
\[
\text{Tr}(Z^{-\frac{1}{2}}K^*APZ^{-\frac{1}{2}})^{\frac{1}{1-sq}}
\]
is concave in \(A.\) And
\[
\text{Tr}(Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}})^{\frac{1}{q}}
\]
is concave in \(B.\) Since \((1 - sq)\) and \(sq\) are both positive, we have
\[
(1 - sq)\text{Tr}(Z^{-\frac{1}{2}}K^*APKZ^{-\frac{1}{2}})^{\frac{1}{1-sq}} + sq\text{Tr}(Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}})^{\frac{1}{q}}
\]
is concave in \((A, B).\) Then by the variational representation (3.11) and Lemma 2.3 of [10] we have \(\Psi_{p,q,s}(A, B)\) is jointly concave.

Under the conditions of (ii), we have
\[
-1 \leq p \leq 0, \quad \frac{s}{1 - sq} = \frac{1}{s^{-1} - q} > 0, \quad \text{and} \quad -1 \leq q \leq 0.
\]
Hence it follows from Theorem 4.2 (ii) that
\[
\text{Tr}(Z^{-\frac{1}{2}}K^*APKZ^{-\frac{1}{2}})^{\frac{1}{1-sq}}
\]
is convex in \(A.\) And it follows from Theorem 4.2 (i) that
\[
\text{Tr}(Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}})^{\frac{1}{q}}
\]
is concave in \(B.\) Since \((1 - sq)\) is positive and \(sq\) is negative, we have
\[
(1 - sq)\text{Tr}(Z^{-\frac{1}{2}}K^*APKZ^{-\frac{1}{2}})^{\frac{1}{1-sq}} + sq\text{Tr}(Z^{\frac{1}{2}}B^qZ^{\frac{1}{2}})^{\frac{1}{q}}
\]
is convex in \((A, B).\) Hence by the variational representation (3.12) and Lemma 2.3 of [10] we have \(\Psi_{p,q,s}(A, B)\) is jointly convex.

Under the conditions of (iii), we have
\[
1 \leq p \leq 2, \quad \frac{s}{1 - sq} = \frac{1}{s^{-1} - q} \geq \frac{1}{p}, \quad \text{and} \quad -1 \leq q \leq 0.
\]
Hence it follows from Theorem 4.2 (iii) that
\[
\text{Tr}(Z^{-\frac{1}{2}}K^*APKZ^{-\frac{1}{2}})^{\frac{1}{1-sq}}
\]
is convex in $A$. And it follows from Theorem 4.2 (i) that
\[
\text{Tr}(Z^{\frac{1}{2}} B^q Z^{\frac{1}{2}})^{\frac{1}{q}}
\]
is concave in $B$. Since $(1 - sq)$ is positive and $sq$ is negative, we have
\[
(1 - sq)\text{Tr}(Z^{-\frac{1}{2}} K^* A^p K Z^{-\frac{1}{2}})^{\frac{1}{1 - sq}} + sq\text{Tr}(Z^{\frac{1}{2}} B^q Z^{\frac{1}{2}})^{\frac{1}{q}}
\]
is convex in $(A, B)$. Hence by the variational representation (3.12) and Lemma 2.3 of [10] we have $\Psi_{p,q,s}(A, B)$ is jointly convex.

More generally, Hiai [14] proved the following results, which can be viewed as extensions of the Epstein’s theorem for symmetric (anti-)norms.

**Theorem 4.3.** Set $K \in M_n$. And set $\| \cdot \|_1$ be symmetric anti-norm and $\| \cdot \|$ be symmetric norm for matrix.

(i) If $0 \leq p < 1$ and $0 < s \leq 1/p$, then $\|(K^* A^p K)^*\|_1$ is concave for $A \in \mathcal{P}_n$.

(ii) If $-1 \leq p \leq 0$ and $s > 0$, then $\|(K^* A^p K)^*\|$ is convex for $A \in \mathcal{P}_n$.

(iii) If $1 \leq p \leq 2$ and $s \geq 1/p$, then $\|(K^* A^p K)^*\|$ is convex for $A \in \mathcal{P}_n$.

We now consider some extensions of the Lieb’s concavity theorem for matrix norm or anti-norm.

**Theorem 4.4.** Let $K \in M_n$ be arbitrary, and $A, B \in \mathcal{P}_n$. If $-1 \leq p, q \leq 0$ and $s > 0$, then for symmetric norm $\| \cdot \|$, 
\[
\tilde{\Psi}_{p,q,s}(A, B) = \| (B^q K^* A^p K B^q)^* \|
\]
is jointly convex.

**Proof.** Under the above conditions, we have
\[
-1 \leq p < 0, \quad \frac{s(p + q)}{p} > 0, \quad \text{and} \quad -1 \leq q < 0, \quad \frac{s(p + q)}{q} > 0.
\]
Hence it follows from Hiai’s results Theorem 4.3 (ii) that
\[
\|(Z^{-\frac{1}{2}} K^* A^p K Z^{-\frac{1}{2}})^{\frac{s(p + q)}{p}}\|
\]
is convex in $A$, and
\[
\|(Z^{\frac{1}{2}} B^q Z^{\frac{1}{2}})^{\frac{s(p + q)}{q}}\|
\]
is convex in $B$. Hence,
\[
\frac{p}{p + q} \|(Z^{-\frac{1}{2}} K^* A^p K Z^{-\frac{1}{2}})^{\frac{s(p + q)}{p}}\| + \frac{q}{p + q} \|(Z^{\frac{1}{2}} B^q Z^{\frac{1}{2}})^{\frac{s(p + q)}{q}}\|
\]
is convex. Then by the variational representation (3.2) we have $\tilde{\Psi}_{p,q,s}(A, B)$ is jointly convex. \qed
Theorem 4.5. Let $K \in \mathcal{M}_n$ and $A, B \in \mathcal{P}_n$. Let $\| \cdot \|$ be a symmetric anti-norm which ensures the Hölder inequality:

\[
\| |AB|^r \|_1^{\frac{1}{r}} \leq \| |A|^r_1 \cdot |B|^{r_2}_2 \|_1^{\frac{1}{r_2}} \quad \text{for} \quad r, r_1, r_2 > 0, \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}. \tag{4.1}
\]

Then for $0 \leq p, q \leq 1$ and $0 < s \leq 1/(p+q)$, we have

\[
(A, B) \mapsto \|(B^\frac{p}{2}K^*A^pKB^\frac{q}{2})^s\|
\]

is jointly concave.

Proof. Since

\[
0 \leq p \leq 1, \quad \frac{s(p+q)}{p} \leq \frac{1}{p}, \quad \text{and} \quad 0 \leq q \leq 1, \quad \frac{s(p+q)}{q} \leq \frac{1}{q},
\]

it follows from Theorem 4.3 (i) that

\[
\|(Z^{-\frac{1}{p}}K^*A^pKZ^{-\frac{1}{p}})^{\frac{s(p+q)}{p}}\|
\]

is concave in $A$, and

\[
\|(Z^{-\frac{1}{q}}B^qZ^{-\frac{1}{q}})^{\frac{s(p+q)}{q}}\|
\]

is concave in $B$. The Hölder inequality (4.1) ensures the corresponding Young inequality and hence a similar version of variational representation for symmetric anti-norm:

\[
\|(B^\frac{p}{2}K^*A^pKB^\frac{q}{2})^s\| = \inf_{Z \in \mathcal{P}_n} \left\{ \frac{p}{p+q} \|(Z^{-\frac{1}{p}}K^*A^pKZ^{-\frac{1}{p}})^{\frac{s(p+q)}{p}}\|_1 + \frac{q}{p+q} \|(Z^{-\frac{1}{q}}B^qZ^{-\frac{1}{q}})^{\frac{s(p+q)}{q}}\|_1 \right\}
\]

Hence, it follows that $\|(B^\frac{p}{2}K^*A^pKB^\frac{q}{2})^s\|$ is jointly concave in $(A, B)$.

Remark 4.6. The Schatten quasi-norm $\| \cdot \|_p$ for $p \in (0, 1)$ is a symmetric anti-norm, which is concave and satisfying the Hölder inequality and also the reverse Hölder inequality.

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