The Sum of Squares Law

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Abstract

We show that when projecting an edge-transitive $N$-dimensional polytope onto an $M$-dimensional subspace of $\mathbb{R}^N$, the sums of the squares of the original and projected edges are in the ratio $N/M$.

Statement

Let $X \subset \mathbb{R}^N$ a set of points that determines an $N$-dimensional polytope. Let $E$ denote the number of its edges, and $\sigma$ the sum of the squares of the edge lengths. Let $S$ be an $M$-dimensional subspace of $\mathbb{R}^N$, and $\sigma'$ the sum of the squares of the lengths of the projections, onto $S$, of the edges of $X$.

Let $G$ be the group of proper symmetries of the polytope $X$ (that is, no reflections). If $G$ acts transitively on the set of edges of $X$, then:

$$\sigma' = \sigma \cdot \frac{M}{N}.$$

The orthogonality relations

The basic result used in our proof is the so-called orthogonality relations in the context of representations of groups. The form of these relations that we need is the following:

**Theorem 1.** Let $\Gamma : G \to V \times V$ be an irreducible unitary representation of a finite group $G$. Denoting by $\Gamma(R)_{nm}$ the matrix elements of the linear map $\Gamma(R)$ with respect to an orthonormal basis of $V$, we have:

$$\sum_{R \in G} \Gamma(R)^*_{nm} \Gamma(R)_{n'm'} = \delta_{nn'}\delta_{mm'} \frac{|G|}{\dim V},$$

where the $^*$ denotes complex conjugation.

A proof of these relation can be found in standard books on representation theory, for instance [1, p. 79] or [2, p. 14]. See also the Wikipedia article \textcolor{blue}{http://en.wikipedia.org/wiki/Schur_orthogonality_relations}.

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Proof of the sum of squares law

The idea is apply the orthogonality relations (1) to the group $G$ of proper symmetries of the polytope $X$, considering its standard representation on the space $\mathbb{R}^N$ (i.e., $R \cdot x = R(x)$). This representation is clearly unitary, since the elements of the group are rotations and hence orthogonal transformations. Also, the representation is irreducible, since $G$ takes a given edge to all the other edges, which do not lie on any proper subspace due to the assumption of $X$ being an $N$-dimensional polytope.

We can assume that the edge lengths of $X$ are all equal to 1. Let $\{v_1, \ldots, v_N\}$ be an orthonormal basis for $\mathbb{R}^N$ such that $v_1$ coincides with the direction of one of the edges $e$ of $X$. Then, for $R \in G$, let $\Gamma(R)$ be the matrix of $R$ in that basis, that is:

$$R(v_j) = \sum_{i=1}^N \Gamma(R)_{ij} v_i.$$ 

Since this is an orthonormal basis, we have:

$$\Gamma(R)_{ij} = \langle R(v_j), v_i \rangle,$$

where $\langle , \rangle$ denotes the standard inner product in $\mathbb{R}^N$. In particular, for $j = 1$:

$$\Gamma(R)_{i1} = \langle R(e), v_i \rangle \quad (i = 1, \ldots, N). \tag{2}$$

Note that this is exactly the length of the projection of each edge onto the line spanned by $v_i$. Now, from equation (1), by putting $n' = n$ and $m' = m$, we get:

$$\sum_{R \in G} |\Gamma(R)_{nm}|^2 = \frac{|G|}{N}. \tag{3}$$

Using the $\Gamma$s given by the previous equation:

$$\sum_{R \in G} \langle R(e), v_i \rangle^2 = \frac{|G|}{N} \quad (i = 1, \ldots, N). \tag{4}$$

Now let $v$ be any unit vector. We’ll show that the above equality holds for $v$ as it does for $v_i$. To see this, write $v$ as a linear combination of the basis vectors $v_i$: $v = \sum a_i v_i$. Since $\|v\| = 1$, we have $\sum a_i^2 = 1$. Then:

$$\sum_R \langle R(e), v \rangle^2 = \sum_R \langle R(e), \sum_i a_i v_i \rangle^2 = \sum_R \left( \sum_i a_i \langle R(e), v_i \rangle \right)^2$$

$$= \sum_R \left( \sum_i a_i^2 \langle R(e), v_i \rangle^2 + 2 \sum_{i<j} a_i a_j \langle R(e), v_i \rangle \langle R(e), v_j \rangle \right)$$

$$= \sum_i a_i^2 \sum_R \langle R(e), v_i \rangle^2 + 2 \sum_{i<j} a_i a_j \sum_R \langle R(e), v_i \rangle \langle R(e), v_j \rangle$$

$$= \frac{|G|}{N} + 2 \sum_{i<j} a_i a_j \sum_R \Gamma(R)_{i1} \Gamma(R)_{j1},$$

due to eqs. (4) and (2). Now it turns out that the second term is 0. This is an immediate consequence of eq. (1) with $n = i$, $n' = j$, $m = m' = 1$. Therefore, the equality:

$$\sum_R \langle R(e), v \rangle^2 = \frac{|G|}{N} \tag{5}$$
holds for any unit vector $v$.

Now let $S$ be the projection subspace of dimension $M > 1$, and let’s denote by $P_S : \mathbb{R}^N \to S$ the projection operator. Choose an orthonormal basis $\{u_1, \ldots, u_M\}$ of $S$. Then:

$$P_S(R(e)) = \sum_{i=1}^{M} b_i u_i,$$

with

$$b_i = \langle P_S(R(e)), u_i \rangle = \langle R(e), u_i \rangle.$$

Therefore,

$$\sum_{R \in G} \|P_S(R(e))\|^2 = \sum_{R \in \cup_{l=1}^{E} C_l} \sum_{i} b_i^2 = \sum_{R \in \cup_{l=1}^{E} C_l} \sum_{i} \langle R(e), u_i \rangle^2 = \sum_{R \in \cup_{l=1}^{E} C_l} \left( \sum_{i} \langle R(e), u_i \rangle^2 \right) = |G| \cdot \frac{M}{N},$$

where the last equality is because of eq. [5].

To obtain the required result, we observe that $G$ can be partitioned in $E$ “cosets” of the same cardinality $k$, where $E$ is the number of edges of $X$. To see this, let $H = \{g \in G \mid g \cdot e = e\}$ be the subgroup of $G$ that leaves edge $e$ invariant. Then the coset $RH = \{g \in G \mid g \cdot e = R(e)\}$ is the subset of elements of $G$ that send edge $e$ to edge $R(e)$. Denote the cardinality of $H$ by $k$. Since there are $E$ edges and the action is edge-transitive, there are $E$ cosets, each of cardinality $k$. Therefore, $|G| = kE$. Denoting the edges by $e_1, \ldots, e_E$, and the corresponding cosets by $C_1, \ldots, C_E$ (so that $R(e) = e_l$ for $R \in C_l$), we have:

$$\sum_{R \in G} \|P_S(R(e))\|^2 = \sum_{R \in \cup_{l=1}^{E} C_l} \|P_S(R(e))\|^2 = \sum_{l=1}^{E} \sum_{R \in C_l} \|P_S(R(e))\|^2 = \sum_{l=1}^{E} k \|P_S(e_l)\|^2 = k \sum_{l=1}^{E} \|P_S(e_l)\|^2.$$

On the other hand, we saw that the left-hand side of this equation equals $|G| \cdot M/N$, which is $kE \cdot M/N$. Equating this to the above and canceling the factor $k$, we obtain:

$$\sigma' = \sum_{l=1}^{E} \|P_S(e_l)\|^2 = E \cdot \frac{M}{N} = \sigma \cdot \frac{M}{N},$$

which completes the proof.

References

[1] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*. Springer-Verlag, New York, 1985.

[2] J.-P. Serre. *Linear Representations of Finite Groups*. Springer-Verlag, New York, 1977.