Fast Adaptation Nonlinear Observer for SLAM

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Abstract—The process of simultaneously mapping the environment in three dimensional (3D) space and localizing a moving vehicle’s pose (orientation and position) is termed Simultaneous Localization and Mapping (SLAM). SLAM is a core task in robotics applications. In the SLAM problem, each of the vehicle’s pose and the environment are assumed to be completely unknown. This paper takes the conventional SLAM design as a basis and proposes a novel approach that ensures fast adaptation of the nonlinear observer for SLAM. Due to the fact that the true SLAM problem is nonlinear and is modeled on the Lie group of SLAM\(\text{n}(3)\), the proposed observer for SLAM is nonlinear and modeled on SLAM\(\text{n}(3)\). The proposed observer compensates for unknown bias attached to velocity measurements. The results of the simulation illustrate the robustness of the proposed approach.

I. INTRODUCTION

Simultaneous Localization and Mapping (SLAM) is a well-established problem in robotics and has been an active area of research over the past three decades [1]–[7]. The SLAM problem concerns a vehicle whose 1) pose (orientation and position) is unknown, traveling within 2) an unknown environment. This task is particularly important in GPS-denied applications, for instance, indoor applications, surveillance, and others. The localization and mapping process are performed via a set of measurements, typically, angular and translational velocities of the vehicle, and landmark measurements. It is apparent that sensor measurements are characterized by irregular behavior and the presence of uncertainties. Therefore, robust observers for SLAM are indispensable.

SLAM observation is traditionally tackled using Gaussian filters or nonlinear observers. Gaussian filters allow to observe the vehicle’s pose along with the surrounding landmarks. Examples of Gaussian filters for SLAM include the MonoSLAM approach that utilizes a single camera and real-time data [5], FastSLAM based on a scalable approach [8], extended Kalman filter (EKF) [9], and particle filter [4], among others. The Gaussian filters take a probabilistic approach to uncertainties present in measurements. It is worth mentioning that SLAM is an open problem and common issues are consistency [10], solution complexity [11], and landmarks in motion. However, the SLAM problem is highly nonlinear and constitutes a dual observation process comprised of pose and environment observation. Pose of a vehicle is composed of: orientation (attitude) and position. While attitude is represented relative to the Special Orthogonal Group \(\mathbb{SO}(3)\) [12], pose is described relative to the Special Euclidean Group \(\mathbb{SE}(3)\) [14]–[16]. Gaussian filters fail to account for the high nonlinearity of the SLAM problem. As such, SLAM observation problem is best addressed using nonlinear observers.

Recent advances in the area of nonlinear observers evolved directly on \(\mathbb{SO}(3)\) [12], [13], [17], [18] and \(\mathbb{SE}(3)\) [14], [15], [19], [20], which opened the door to proposing nonlinear observers for SLAM. An early study that proposed using the Lie group of \(\mathbb{SE}(3)\) as the true representation of the SLAM problem was presented in [21]. It was followed by two-staged observers, with nonlinear observer for pose estimation and Kalman filter for landmark estimation [22]. The true SLAM problem is nonlinear and is modeled on the Lie group of SLAM\(\text{n}(3)\). Nonlinear observers for SLAM on SLAM\(\text{n}(3)\) have been proposed in [1], [6], [7]. The innovative component of the observers in [6], [7] consists in the use of constant gains which do not allow for fast adaptation. Accordingly, this paper proposes a nonlinear observer for SLAM on SLAM\(\text{n}(3)\) that follows the structure of the work in [1], [6] with the main contributions as listed below:

1) A nonlinear observer for SLAM with fast adaptation that uses the available measurements of angular velocity, translational velocity, and landmarks.
2) Exponential convergence of the error component is guaranteed.
3) The closed loop error signals are guaranteed to be uniformly ultimately bounded.

The remainder of the paper is organized as follows: Section II introduces the nomenclature, overview of \(\mathbb{SO}(3)\) and \(\mathbb{SE}(3)\), and math notation. Section III defines the SLAM problem, available sensor measurements, and the true motion kinematics. Section IV presents nonlinear observer for SLAM on SLAM\(\text{n}(3)\) with fast adaptation. Section V reveals the robustness of the proposed observer. Lastly, the conclusion is contained in Section VI.

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II. Preliminaries and Math Notation

A. Nomenclature

\{I\} \quad \text{Inertial-frame}

\{B\} \quad \text{Body-frame}

\mathbb{R} \quad \text{Set of real numbers}

\mathbb{R}_+ \quad \text{Set of nonnegative real numbers}

\mathbb{R}^{n \times m} \quad \text{Set of real numbers with dimension } n \text{-by-} m

\|y\| \quad \text{Euclidean norm } \|y\| = \sqrt{y^\top y}, \forall y \in \mathbb{R}^n

SO(3) \quad \text{Special Orthogonal Group of order 3}

SE(3) \quad \text{Special Euclidean Group of order 3}

B. Preliminaries

The Special Orthogonal Group SO(3) is described by

\[ SO(3) = \{ R \in \mathbb{R}^{3 \times 3} | R R^\top = I_3, \det(R) = +1 \} \]

Note that \( R \in SO(3) \) is expressed relative to \{B\}. The Special Euclidean Group SE(3) is defined as

\[ SE(3) = \left\{ T = \begin{bmatrix} R & P \\ 0_{1 \times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4} \mid R \in SO(3), P \in \mathbb{R}^3 \right\} \]

where \( P \in \mathbb{R}^3 \) refers to rigid-body’s position. \( P \) is defined relative to \{I\}. \( T \in SE(3) \) is commonly known as a homogeneous transformation matrix that describes rigid-body’s pose and is given by

\[ T = \begin{bmatrix} R & P \\ 0_{1 \times 3} & 1 \end{bmatrix} \in SE(3) \] (1)

so(3) is the Lie-algebra of SO(3) with

\[ so(3) = \left\{ [y]_x \in \mathbb{R}^{3 \times 3} \mid [y]_x^\top = -[y]_x, y \in \mathbb{R}^3 \right\} \] (2)

where \([y]_x\) refers to a skew symmetric matrix such that

\[ [y]_x = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \in so(3), \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \]

\( \mathfrak{se}(3) \) is the Lie-algebra of SE(3) where

\[ \mathfrak{se}(3) = \left\{ U \right\}_\wedge \in \mathbb{R}^{4 \times 4} \mid \exists u_1, u_2 \in \mathbb{R}^3 : [U]_\wedge = \begin{bmatrix} [u_1]_x & u_2 \\ 0_3 & 0 \end{bmatrix} \right\} \]

\([\cdot]_\wedge \) is a wedge operator that follows \([\cdot]_\wedge : \mathbb{R}^6 \to \mathfrak{se}(3) \) such that

\[ [U]_\wedge = \begin{bmatrix} [u_1]_x & u_2 \\ 0_3 & 0 \end{bmatrix} \in \mathfrak{se}(3), \quad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^6 \] (3)

Let \( \|R\|_1 \) be a normalized Euclidean distance of \( R \in SO(3) \) where

\[ \|R\|_1 = \frac{1}{4} \text{Tr} \{ I_3 - R \} \in [0, 1] \] (4)

III. Problem Formulation

SLAM estimation problem concerns simultaneous observation of the vehicle’s pose and landmarks within the environment. Fig. 1 illustrates the SLAM observation problem.

Define \( R \in SO(3) \) as the rigid-body’s attitude and \( P \in \mathbb{R}^3 \) as the rigid-body’s translation for all \( R \in \{B\} \) and \( P \in \{I\} \). Assume that the map contains a family of \( n \) landmarks, and let \( p_i \) represent the \( i \)th landmark location where \( p_i \in \{I\} \) for all \( i = 1, 2, \ldots, n \). The observation problem can be solved given a set of measurements in the body-frame. The measurement of \( p_i \) is given by

\[ y_i = R^\top (p_i - P) + b_i^n + n_i^n \in \mathbb{R}^3 \] (5)

where \( R \), \( P \) refers to the rigid-body’s orientation, \( P \) describes its translation, and \( p_i \) refers to the landmark’s location. Additionally, \( b_i^n \) defines unknown constant bias and \( n_i^n \) defines unknown random noise attached to the measurement with \( y_i, b_i^n, n_i^n \in \{B\} \).

Assumption 1. Assume three or more landmarks are available for measurement.

The true motion kinematics of the rigid-body’s attitude and position and a group of \( n \)-landmarks are given by [1], [7]

\[ \dot{T} = T[U]_\wedge \]

\[ \dot{p}_i = R v_i, \quad \forall i = 1, 2, \ldots, n \]

and in detailed form

\[ \begin{cases} \dot{R} = R[\Omega]_x \\ \dot{P} = RV \\ \dot{p}_i = R v_i, \quad \forall i = 1, 2, \ldots, n \end{cases} \] (6)

where \( U = [\Omega^\top, V^\top]^\top, \Omega \in \mathbb{R}^3 \) defines the rigid-body’s true angular velocity, \( V \in \mathbb{R}^3 \) defines its true translational velocity, and \( v_i \in \mathbb{R}^3 \) defines the true linear velocity of the \( i \)th
landmark. Note that each of \( \Omega, V, v_i \in \{ B \} \). The measurements of angular and translational velocity can be described as

\[
\begin{align*}
\Omega_m &= \Omega + b_\Omega + n_\Omega \in \mathbb{R}^3 \\
V_m &= V + b_V + n_V \in \mathbb{R}^3
\end{align*}
\]

(7)

where \( b_\Omega \) defines unknown constant bias and \( n_\Omega \) denotes unknown random noise attached to the angular velocity, while \( b_V \) defines unknown constant bias and \( n_V \) denotes unknown random noise attached to the translational velocity. Note that the measurements of angular and translational velocities are expressed with respect to \( \{ B \} \). All landmarks are assumed to be fixed, thus \( v_i = 0_{3 \times 1} \), \( \forall i = 1, 2, \ldots, n \).

**Assumption 2.** (Uniform boundedness of \( b_\Omega \) and \( b_V \)) Assume that \( b_\Omega \) and \( b_V \) are subset of \( A_b \) with \( b_\Omega, b_V \in A_b \subset \mathbb{R}^3 \), where \( b_\Omega \) and \( b_V \) are ultimately bounded by \( \Gamma \).

**A. Error in Attitude, Position, and Landmark**

Consider \( \hat{R} \) to be an estimate of the true orientation \( (R) \), \( \hat{P} \) an estimate of the true rigid-body’s position \( (P) \), and \( \hat{p}_i \) an estimate of the true location of the \( i \)th landmark \( (p_i) \). Consider defining the error in pose observation as

\[
T = \hat{T}^{-1} = \begin{bmatrix}
\hat{R} & \hat{P} \\
0_3^T & 1
\end{bmatrix} \begin{bmatrix}
R^T & -R^T P \\
0_3^T & 1
\end{bmatrix}
\]

which is equivalent to

\[
\begin{align*}
\hat{R} &= \hat{R} R^T \\
\hat{P} &= \hat{P} - \hat{R} P
\end{align*}
\]

(9)

Consider defining the error in the \( i \)th landmark observation as

\[
\begin{bmatrix}
\epsilon_i \\
0
\end{bmatrix} = \begin{bmatrix}
\hat{p}_i \\
1
\end{bmatrix} - \begin{bmatrix}
\hat{R} & \hat{P} \\
0_3^T & 1
\end{bmatrix} \begin{bmatrix}
\hat{p}_i \\
1
\end{bmatrix}, \quad \forall i = 1, 2, \ldots, n
\]

(10)

\[
\begin{align*}
\hat{p}_i &= \hat{p}_i - \hat{R} p_i, \quad \hat{P} = \hat{P} - \hat{R} P \\
\hat{b}_\Omega &= \hat{b}_\Omega - \hat{b}_\Omega, \quad \hat{b}_V = \hat{b}_V - \hat{b}_V
\end{align*}
\]

(11)

**Definition 1.** Define \( x \in \mathbb{R}^3 \) as a unit-axis rotating at an angle of \( \theta \in \mathbb{R} \) in a 2-sphere \( \mathbb{S}^2 \). Angle-axis representation is one of the methods of attitude representation which has the map of \( \mathcal{R}_\theta : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{SO}(3) \) \([23], [24]\)

\[
\mathcal{R}_\theta (\theta, x) = I_3 + \sin(\theta) \begin{bmatrix} x \end{bmatrix}_\times + (1 - \cos(\theta)) \begin{bmatrix} x \end{bmatrix}_\times^2 \in \mathbb{SO}(3)
\]

From (10), and consistently with Lemma 4 \([24]\), consider defining

\[
\theta_i = 2 \tan^{-1}(||\epsilon_i||)
\]

\[
x_i = \cot\left(\frac{\theta_i}{2}\right) \epsilon_i
\]

The following mapping is obtained \([24]\)

\[
\mathcal{R}_{\epsilon(i)} = I_3 + \sin(\theta_i) \begin{bmatrix} \epsilon_i \end{bmatrix}_\times + (1 - \cos(\theta_i)) \begin{bmatrix} \epsilon_i \end{bmatrix}_\times^2 \quad \forall i = 1, 2, \ldots, n
\]

(12)

**Remark 1.** Recall the definition of the normalized Euclidean distance in (4). From (12) and Definition 1, one finds \(-1 \leq \text{Tr}\{\mathcal{R}_{\epsilon(i)}\} \leq 3\) such that \(\text{Tr}\{\mathcal{R}_{\epsilon(i)}\} \rightarrow -1 \) as \( \epsilon_i \rightarrow \infty \) and \(\text{Tr}\{\mathcal{R}_{\epsilon(i)}\} \rightarrow 3 \) as \( \epsilon_i \rightarrow 0 \).

**Definition 2.** (Fast adaptation) Based on Definition 1 and Remark 1, define the following positive function \( \psi : \mathbb{SO}(3) \rightarrow \mathbb{R}_+ \)

\[
\psi(\epsilon_i) = \frac{k_p}{1 + \text{Tr}\{\mathcal{R}_{\epsilon(i)}\}}
\]

(13)

The value of the function \( \psi(\epsilon_i) \) in (13) becomes increasingly aggressive with \( \psi(\epsilon_i) \rightarrow +\infty \) as \( \epsilon_i \rightarrow \pm \infty \) and \( \psi(\epsilon_i) \rightarrow k_p/4 \) as \( \epsilon_i \rightarrow 0 \), visit \([24]\). The behavior of the proposed function in (13) is illustrated in Fig. 2.

**IV. NONLINEAR OBSERVER DESIGN**

Consider the following nonlinear observer design:
where $k_w \in \mathbb{R}$, $k_p \in \mathbb{R}$, $\Gamma \in \mathbb{R}^{3 \times 3}$, and $\alpha_i \in \mathbb{R}$ are positive constants, $W_\Omega$ and $W_V$ are correction factors, and $\hat{b}_\Omega$ and $\hat{b}_V$ are the estimates of $b_\Omega$ and $b_V$, respectively.

**Theorem 1.** Consider the true motion kinematics in (6), landmark measurements $(y_i = R^T (P_i - P))$ for all $i = 1, 2, \ldots, n$, and angular velocity measurement $\Omega_m = \Omega + \hat{b}_\Omega$, and translational velocity measurement $V_m = V + \hat{b}_V$ as in (7). Assume that Assumption 1 is met. Consider the observer design to be as in (14). Define the set

$$S = \{ (e_1, e_2, \ldots, e_n) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 | e_i = 0_i, \forall i = 1, 2, \ldots n \}$$

Then

1) $e_i$ in (10) exponentially approaches $S$, and
2) the error in attitude and position $\hat{R} \to R_e$ and $\hat{P} \to P_e$ as $t \to \infty$ where $R_e \in SO(3)$ refers to a constant matrix and $P_e \in \mathbb{R}^3$ refers to a constant vector.

**Proof.** From (8), one has

$$\dot{T} = \dot{\hat{T}}^{-1} + \hat{T}\hat{T}^{-1}$$

where $\hat{T}^{-1} = -T^{-1}\dot{T}^{-1}$. Thereby, the error dynamics of (10) are equivalent to

$$\begin{bmatrix} \dot{e}_i \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{p}_i \\ 0 \end{bmatrix} - \hat{T} \begin{bmatrix} \dot{P}_i \\ 1 \end{bmatrix} - \hat{T} \hat{\theta}_i$$

From (16), one finds

$$\dot{T} \begin{bmatrix} \hat{b}_U \\ \hat{b}_V \end{bmatrix} \hat{T}^{-1} = \left[ \hat{R} \hat{\theta}_\Omega \right] \quad \text{in } \text{st} (3)$$

where for $x \in \mathbb{R}^3$ and $R \in SO(3)$, $[Rx]_x = R [x]_x R^T$. Accordingly, the result in (18) is equivalent to

$$\dot{T} \begin{bmatrix} \hat{b}_U \\ \hat{b}_V \end{bmatrix} \hat{T}^{-1} = \left[ \begin{bmatrix} \hat{R} & 0_{3 \times 3} \\ \hat{P} & \hat{R} \end{bmatrix} \begin{bmatrix} \hat{b}_\Omega \\ \hat{b}_V \end{bmatrix} \right]$$

which shows that

$$\dot{T} \begin{bmatrix} \hat{b}_U \\ \hat{b}_V \end{bmatrix} \hat{T}^{-1} T \hat{p}_i = \begin{bmatrix} -\hat{R} [y_i]_{x} & \hat{R} \end{bmatrix} \begin{bmatrix} \hat{b}_\Omega \\ \hat{b}_V \end{bmatrix}$$

Therefore, it can be concluded that the error dynamics are

$$\dot{e}_i = \hat{p}_i - [-\hat{R} [y_i]_{x} \hat{R}] \begin{bmatrix} \hat{b}_\Omega \\ \hat{b}_V \end{bmatrix}$$

Consider the candidate Lyapunov function $V = V(e_1, \ldots, e_n, \hat{b}_\Omega, \hat{b}_V)$ defined as follows:

$$V = \sum_{i=1}^{n} \frac{1}{2 \alpha_i} e_i^T e_i + \frac{1}{2} \hat{b}_\Omega^T \hat{b}_\Omega - \frac{1}{2} \hat{b}_V^T \hat{b}_V$$

The time derivative of (22) becomes

$$\dot{V} = \sum_{i=1}^{n} \frac{1}{2 \alpha_i} e_i^T \dot{e}_i - \frac{1}{2} \hat{b}_\Omega^T \hat{b}_\Omega - \frac{1}{2} \hat{b}_V^T \hat{b}_V$$

Replacing $W_\Omega$, $W_V$, $\hat{b}_\Omega$, $\hat{b}_V$, and $\hat{p}_i$ with their definitions in (14) leads to

$$\dot{V} \leq -\sum_{i=1}^{n} \frac{\psi(e_i)}{\alpha_i} ||e_i||^2 - k_w \sum_{i=1}^{n} e_i^2$$

From (24), $V$ is negative for all $e_i \neq 0$ and $V$ is equal to zero at $e_i = 0_{3 \times 1}$. Thus, the inequality in (24) shows that $e_i$ is regulated exponentially to the set $S$ in (15). In view of Barbatai Lemma, $V$ is negative, continuous, and converges to zero indicating that $\hat{b}_\Omega$ and $\hat{b}_V$ are bounded. As such, $\hat{R} \to R_e$ and $\hat{P} \to P_e$ as $t \to \infty$ completing the proof.

The discrete implementation of the observer in (14) is given by

$$\begin{align*}
T[k+1] &= T[k] \exp \left( \begin{bmatrix} \Omega_m[k] & -\hat{b}_\Omega[k] & -W_\Omega[k] \\ V_m[k] & -\hat{b}_V[k] & -W_V[k] \end{bmatrix} \right) \\
\theta_i &= 2 \tan^{-1}(||[e_i]||), \quad x_i = \cot(\theta_i) e_i[k] \\
\hat{R}_e(i) &= I_3 + \sin(\theta_i) [x_i]_x + (1 - \cos(\theta_i)) [x_i]_x^2 \\
\psi(e_i) &= \frac{\psi(e_i)}{\alpha_i} \\
\hat{p}_i[k+1] &= \hat{p}_i[k] - \Delta t \psi(e_i) e_i[k], \quad \forall i = 1, 2, \ldots, n \\
\hat{b}_\Omega[k+1] &= \hat{b}_\Omega[k] - \Delta t \sum_{i=1}^{n} \frac{1}{\alpha_i} [y_i]_{x_i} \hat{R}_e[i] e_i[k] \\
\hat{b}_V[k+1] &= \hat{b}_V[k] - \Delta t \sum_{i=1}^{n} \frac{1}{\alpha_i} \hat{R}_e[i] e_i[k] \\
W_\Omega &= -\sum_{i=1}^{n} \hat{b}_\Omega[k] e_i[k] \\
W_V &= -\sum_{i=1}^{n} \hat{b}_V[k] e_i[k]
\end{align*}$$

(25)
V. Simulation Results

This section reveals the robustness of the proposed nonlinear observer with fast adaptation for SLAM on the Lie group of \( SLAM_n(3) \). Consider the following set of data, initialization parameters, and measurement bias:

\[
\begin{align*}
\Omega &= [0, 0, 0.3]^{\top} \text{(rad/sec)} \\
V &= [2.5, 0, 0]^{\top} \text{(m/sec)} \\
R(0) &= I_3 \\
P(0) &= [0, 0, 6]^{\top} \\
p_1 &= [7, 7, 0]^{\top} \\
p_2 &= [-7, 7, 0]^{\top} \\
p_3 &= [7, -7, 0]^{\top} \\
p_4 &= [-7, -7, 0]^{\top} \\
b_{\Omega} &= [0.09, -0.15, -0.1]^{\top} \text{(rad/sec)} \\
b_V &= [0.09, 0.06, -0.07]^{\top} \text{(m/sec)}
\end{align*}
\]

Consider the initial estimates of attitude, position, and landmark locations to be

\[
\begin{align*}
\hat{R}(0) &= I_3, \quad \hat{P}(0) = 0_{3 \times 1} \\
\hat{p}_1 (0) &= \hat{p}_2 (0) = \hat{p}_3 (0) = \hat{p}_4 (0) = 0_{3 \times 1}
\end{align*}
\]

Consider selecting the design parameters as follows: \( \alpha_i = 0.1, \Gamma = 30I_3, k_p = 1, \) and \( k_w = 2, \) while the initial estimates of the biases are \( \hat{b}_{\Omega} (0) = \hat{b}_V (0) = 0_{3 \times 1} \) for all \( i = 1, 2, 3, 4. \)

Fig. 3 illustrates the output performance of the proposed observer against the true trajectory. The true performance is depicted in black center-line with final destinations depicted as black circles. The estimated performance is shown in red dash-line and blue center-line, while the estimated positions of the final destinations are depicted as red and blue stars \( \star. \)

Fig. 3 reveals strong estimation capabilities of the proposed observer in localizing the unknown pose of the vehicle as well as mapping the unknown environment.

Fig. 4 reveals asymptotic and fast convergence of \( e_i \) to the origin from large error in initialization. Likewise, Fig. 5 demonstrates fast convergence of \( ||p_i - \hat{p}_i|| \) from large error in initialization to the close neighborhood of the origin.

VI. Conclusion

A nonlinear observer for Simultaneous Localization and Mapping (SLAM) modeled on the Lie group of \( SLAM_n(3) \) is proposed. The observer follows the structure of the true SLAM problem. The proposed observer compensates for the unknown bias attached to angular and translational velocities. The proposed observer can be easily implemented on a vehicle given the availability of velocity and landmark measurements. Numerical results revealed the observer’s ability to concurrently map the unknown environment and obtain the vehicle’s pose.

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Fig. 5. Evolution of error trajectories of $e_i$. 

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