HOMOLOGICALLY MAXIMIZING GEODESICS IN
CONFORMALLY FLAT TORI

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ABSTRACT. We study homologically maximizing timelike geodesics in conformally flat tori. A causal geodesic $\gamma$ in such a torus is said to be homologically maximizing if one (hence every) lift of $\gamma$ to the universal cover is arclength maximizing. First we prove a compactness result for homologically maximizing timelike geodesics. This yields the Lipschitz continuity of the time separation of the universal cover on strict sub-cones of the cone of future pointing vectors. Then we introduce the stable time separation $l$. As an application we prove relations between the concavity properties of $l$ and the qualitative behavior of homologically maximizing geodesics.

1. INTRODUCTION

Here, we present a version of Mather theory for maximizing geodesics on conformally flat Lorentzian tori. More general Lorentzian manifolds will be treated in [1]. The source for the techniques we employ are [2] and [3].

Consider a real vector space $V$ of dimension $m < \infty$ and $\langle ., . \rangle_1$ a nondegenerate symmetric bilinear form on $V$ with signature $(-,+,\ldots,+)$. Set $\| . \|_1 := \sqrt{\langle ., . \rangle_1}$. Further let $\Gamma \subseteq V$ be a co-compact lattice and $f : V \to (0,\infty)$ a smooth $\Gamma$-invariant function. The Lorentzian metric $\overline{g} := f^2 \langle ., . \rangle_1$ then descends to a Lorentzian metric on the torus $V/\Gamma$. Denote the induced Lorentzian metric by $g$. Choose a time-orientation of $(V,\langle ., . \rangle_1)$. This time-orientation induces a time-orientation on $(V/\Gamma, g)$ as well. Note that $(V/\Gamma, g)$ is vicious ([4] p. 137) and the universal cover $(V,\overline{g})$ is globally hyperbolic ([4] p. 65). According to [5] proposition 2.1, $(V/\Gamma, g)$ is geodesically complete in all three causal senses. Fix a norm $\| . \|$ on $V$ and denote the dual norm by $\| . \|^*$. We define $B_r(x) := \{ y \in V | \| y - x \| < r \}$. Note that $\| . \|$ induces a metric on $V/\Gamma$. For a subset $A \subseteq V$ we write $\text{dist}(x, A)$ to denote the distance of the point $x \in V$ to $A$ relative to $\| . \|$. Further denote by $\Xi$ the positive oriented causal vectors of $(V,\langle ., . \rangle_1)$, i.e. the vectors $v \in V \setminus \{ 0 \}$ with $\langle v, v \rangle_1 \leq 0$ and positive time-oriented. For $\varepsilon > 0$ set $\Xi_\varepsilon := \{ v \in \Xi | \text{dist}(v, \partial \Xi) \geq \varepsilon \| v \| \}$.

Let $I$ be any (bounded or unbounded) interval in the reals. A causal geodesics $\gamma : I \to V/\Gamma$ of $(V/\Gamma, g)$ is said to be homologically maximizing if one (hence every) lift $\overline{\gamma} : I \to V$ is arclength maximizing in $(V,\overline{g})$ (for simplicity we will only consider future pointing curves) in the following sense: For every compact subinterval $[a, b] \subseteq I$ the curve $\gamma|_{[a,b]}$ is arclength maximizing among all causal curves connecting $\gamma(a)$ to $\gamma(b)$. In section [2] we will prove a compactness result for homologically maximizing timelike geodesics. Using this compactness result we will then deduce the Lipschitz continuity of the time separation of $(V,\overline{g})$ on...
{(x, y) ∈ V × V | y − x ∈ T } for every ε > 0. Here we use the term time separation as a synonym for the Lorentzian distance function (\cite{4} definition 4.1).

In section \[3\] we show the existence of the stable version of the time separation \(d(V, \mathcal{T})\), i.e., \(l(v) = \lim_{n \to \infty} \frac{d(x, x + nv)}{n}\) exists for all \(x \in V\) and \(v \in \mathcal{T}\) and is independent of \(x\). We will call \(l\) the stable time separation of \((V/\Gamma, g)\). Furthermore for any \(\varepsilon > 0\) there exists a constant \(K(\varepsilon) < \infty\) such that \(|d(x, x + v) − l(v)| ≤ K(\varepsilon)|\) for all \(x \in V\) and \(v \in \mathcal{T}\). The stable time separation constitutes the Lorentzian version of the stable norm on \(H_1(M, \mathbb{R})\) of a compact Riemannian manifold \((M, g_R)\).

The strategy of deduction we follow is taken from \cite{2}. Even in the Riemannian case, the mentioned estimate on \(|d(x, x + v) − l(v)|\) is not obvious.

In section \[4\] we relate concavity properties of \(l\) to the existence and the asymptotic properties of homologically maximizing geodesics. More precisely, we show that for any homologically maximizing geodesic \(\gamma: \mathbb{R} \to V/\Gamma\) there exists a support function \(\alpha\) of \(\Gamma\) such that all accumulation points of sequences of rotation vectors of subarcs of \(\gamma\) lie in the intersection \(\alpha^{-1}(1) \cap \Gamma^{-1}(1)\). Conversely, for any support function \(\alpha\) of \(\Gamma\) we can find a homologically maximizing timelike geodesic such that the limits of rotation vectors of \(\gamma\) lie in \(\alpha^{-1}(1) \cap \Gamma^{-1}(1)\).

A causal curve \(\gamma: [s, t] \to V/\Gamma\) is said to be \((G, \varepsilon)\)-timelike if there exist \(a, b \in I\) such that \(\gamma(b) − \gamma(a) \in \mathcal{T}\) and \(\|\gamma(b) − \gamma(a)\| \geq G\).

(ii) Let \(F < \infty\). A causal curve \(\gamma: I \to V/\Gamma\) is said to be \(F\)-almost maximal if

\[L^0(\gamma|_{[s, t]}) \geq d(\gamma(t), \gamma(s)) − F\]

for one (hence every) lift \(\gamma\) to \(V\) and all \([s, t] \subseteq I\).

**Proposition 2.2.** For every \(\varepsilon > 0\) and \(F < \infty\) there exist constants \(\delta > 0\) and \(0 < K < \infty\) such that for all \(G < \infty\), all \(F\)-almost maximal \((G, \varepsilon)\)-timelike curves \(\gamma: I \to V/\Gamma\) and all \(s < t \in I\) with \(\|\gamma(t) − \gamma(s)\| \geq K\) we have

\[\gamma(t) − \gamma(s) \in \mathcal{T}\delta.\]

Before giving the proof of proposition \[2.2\] we review some applications.

Choose an orthonormal basis \(\{e_1, \ldots, e_m\}\) of \((V, \langle , \rangle_1)\). Note that the translations \(x \mapsto x + v\) are conformal diffeomorphisms of \((V, \mathcal{T})\) for all \(v \in V\). Then the \(\mathcal{T}\)-orthogonal frame field \(x \mapsto (\langle e_1, \ldots, e_m\rangle)\) on \(V\) descends to a \(g\)-orthogonal frame field on \(V/\Gamma\). In this way it makes sense to speak of a tangent vector \(w \in T(V/\Gamma)\) as belonging to \(\mathcal{T}\) or \(\mathcal{T}_\varepsilon\) for \(\varepsilon > 0\).

**Theorem 2.3.** For every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that for all future pointing homologically maximizing geodesics \(\gamma: I \to V/\Gamma\) with \(\gamma(t_0) \in \mathcal{T}_\varepsilon\) for some \(t_0 \in I\),
we have
\[ \dot{\gamma}(t) \in \mathcal{I}_\delta \]
for all \( t \in I \).

Theorem 2.3 is a direct consequence of proposition 2.2 together with the continuity of the geodesic flow and the invariance of the set of lightlike vectors under the geodesic flow. This has the following immediate consequence.

**Corollary 2.4.** Let \( \varepsilon > 0 \) and \( G < \infty \). Then any limit curve of a sequence of homologically maximizing \((G, \varepsilon)\)-timelike curves in \((V/\Gamma, g)\) is timelike.

Note that limit curves are understood in the sense of the limit curve lemma (14.2). The corollary resembles the generalized timelike co-ray condition in \([7]\). It requires that any co-ray to a given timelike ray is again timelike. Following \([7]\) it was proved in \([6]\) that the generalized timelike co-ray condition implies the Lipschitz continuity of the Busemann function associated to a given timelike ray.

The same proof (with some obvious modifications) works in the present situation as well.

**Theorem 2.5.** For all \( \varepsilon > 0 \) there exists an \( L = L(\varepsilon) < \infty \) such that the time separation \( d \) of \((V, \mathcal{I})\) is \( L \)-Lipschitz on \( \{(x, y) \in V \times V | y - x \in \mathcal{I}_\varepsilon\} \).

Now we proceed to the proof of proposition 2.2.

First note the following fact. There exist constants \( c > 0 \) and \( C < \infty \) such that
\[ c\|w\| \text{dist}(v, \partial \mathcal{I}) \leq |\langle v, w \rangle_1| \quad \text{and} \quad |v|^2 \leq C\|v\| \text{dist}(v, \partial \mathcal{I}) \]
for all \( v, w \in \mathcal{I} \).

Since \( \langle , \rangle_1 \) is non-degenerate there exist constants \( c' > 0 \) and \( C' < \infty \) with
\[ c'\|v\| \leq \|\langle , \rangle_1\|' \leq C'\|v\| \]
for all \( v \in V \). For the first inequality in (2.1) note that for every \( w \in \mathcal{I} \) the orthogonal complement of \( w \) relative to \( \langle , \rangle_1 \), denoted by \( w^\perp \), is a spacelike hyperplane.

Since \( \langle , \rangle_1 \) is non-degenerate, we have \( \text{dist}(v, \partial \mathcal{I}) \leq \text{dist}(v, w^\perp) \) for every \( v \in \mathcal{I} \).

Consider for \( v \in V \) a \( v_0 \in w^\perp \) with \( \|v - v_0\| = \text{dist}(v, w^\perp) \). Note that we have
\[ |\langle v - v_0, w \rangle_1| = \|v - v_0\| \|\langle , \rangle_1\|' \].

Consequently we get
\[ \|\langle v, v \rangle_1\| = |\langle v_0 - v, w \rangle_1| \geq c'\|w\| \|v - v_0\| = c'\|w\| \text{dist}(v, w^\perp). \]

Next consider a \( v_1 \in \partial \mathcal{I} \) with \( \|v - v_1\| = \text{dist}(v, \partial \mathcal{I}) \). Note that \( \|v_1\| \leq 2\|v\| \). Then we have
\[ -(v, v)_1 = -(v, v)_1 + (v_1, v_1)_1 = -2 \int_0^1 (1 - t)v_1 + tv, v - v_1 dt \leq 2C' \sup_{t \in [0,1]} \|v - v_1\| \leq 4C'\|v\| \|v - v_1\| \quad \text{for } t \in [0,1] \].

Since \( \mathcal{I} \) contains no linear subspaces we can choose \( \eta > 0 \) such that
\[ \sum_{i \leq N} \|v_i\| \geq \eta \sum_{i \leq N} \|v_i\| \]
for any finite set \( \{v_i\}_{1 \leq i \leq N} \subset \mathcal{I} \). Note that this implies that we have
\[ L^\|\|'(\gamma) \geq \eta \|\gamma(b) - \gamma(a)\| \]
for all \( \gamma \in \mathcal{I} \).
Proof. For Corollary 2.8. we have
\[ \text{dist}(v, w, \partial \mathcal{T}) \geq \text{dist}(v, \partial \mathcal{T}) + \text{dist}(w, \partial \mathcal{T}) \geq \varepsilon(\|v\| + \|w\|) \]
\[ \geq \varepsilon\|v + w\|. \]

\[ \square \]
Lemma 2.9. Let $\lambda_1 < \infty$ and set $\mu := \frac{c \eta}{4\lambda_1^2}$. Then we have

$$|v + w|_1 \geq \lambda_1(|v|_1 + |w|_1)$$

for all $v, w \in \Sigma$ with $||v|| \leq \mu ||w||$ and $\text{dist}(w, \partial \Sigma) \leq \mu \text{dist}(v, \partial \Sigma)$.

Proof. Using (2.1), (2.5) and (2.2) we get

$$|v + w|_1 \geq \sqrt{c \text{dist}(v + w, \partial \Sigma)}||v + w|| \geq \sqrt{c \eta \text{dist}(v, \partial \Sigma)}||w||$$

$$= \frac{\sqrt{c}}{2} \left( \sqrt{\text{dist}(v, \partial \Sigma)||w||} + \sqrt{\text{dist}(v, \partial \Sigma)||w||} \right)$$

$$\geq \lambda_1 \sqrt{c \text{dist}(v, \partial \Sigma)}||w|| + \sqrt{c \text{dist}(w, \partial \Sigma)||w||} \geq \lambda_1(|v|_1 + |w|_1).$$

Note that the time separation $d$ of $(V, \mathcal{F})$ satisfies

$$\inf f |v|_1 \leq d(x, x + v) \leq \sup f |v|_1$$

for all $v \in \Sigma$ and $x \in V$.

Lemma 2.10. Let $\kappa', F' < \infty$ and $\varepsilon' > 0$ be given. Then there exists $K' := K'(\kappa', F', \varepsilon') < \infty$ such that for all $G' < \infty$ and all future pointing $(G', \varepsilon')$-timelike $F'$-almost maximal curves $\gamma: [s, t] \rightarrow V/\Gamma$ with $\|\gamma(t) - \gamma(s)\| \geq K'$, we have

$$\text{dist}(\gamma(t), \gamma(s)) \geq \kappa'.$$

Proof. Assume that the claim is false. Then there exist $\kappa', F' < \infty$, $\varepsilon' > 0$, a sequence $G'_n < \infty$ and a sequence of $(G'_n, \varepsilon')$-timelike and $F'$-almost maximal curves $\gamma_n: [s_n, t_n] \rightarrow V/\Gamma$ with

$$\|\gamma_n(t_n) - \gamma_n(s_n)\| \geq n \text{ and } \text{dist}(\gamma_n(t_n) - \gamma_n(s_n), \partial \Sigma) \leq \kappa'.$$

Choose $[a_n, b_n] \subseteq [s_n, t_n]$ with $\|\gamma(b_n) - \gamma(a_n)\| \geq G'_n$ and $\gamma(b_n) - \gamma(a_n) \in \Sigma_{\varepsilon'}$. If $\varepsilon'\|\gamma(b_n) - \gamma(a_n)\| > \kappa'$ the contradiction is obvious. We have

$$\text{dist}(\gamma(b_n) - \gamma(a_n), \partial \Sigma) \geq \varepsilon'\|\gamma(b_n) - \gamma(a_n)\| > \kappa'.$$

Since $\text{dist}(\gamma(t_n) - \gamma(s_n), \partial \Sigma) \geq \text{dist}(\gamma(b_n) - \gamma(a_n), \partial \Sigma)$, we get $\text{dist}(\gamma(t_n) - \gamma(s_n), \partial \Sigma) > \kappa'$.

Therefore we can assume that $\varepsilon'G'_n \leq \varepsilon'\|\gamma(b_n) - \gamma(a_n)\| \leq \kappa'$. Note that we can further assume that $G'_n = \varepsilon'$, since we have $\|\gamma(b_n) - \gamma(a_n)\| \geq \text{dist}(\gamma(b_n) - \gamma(a_n), \partial \Sigma)$.

First we consider the case that $\|\gamma_n(t_n) - \gamma_n(a_n)\|$ is unbounded. We can pass to a subsequence and assume that $\|\gamma_n(t_n) - \gamma_n(a_n)\| \rightarrow \infty$. We can assume that $\gamma_n$ is parameterized by $\|\|\text{-arclength}$. With (2.3) we know that $L\|\|\|\gamma_n[n_t, t_n]\|\| = t_n - a_n \rightarrow \infty$ for $n \rightarrow \infty$. By shifting the parameter we can assume that $a_n \equiv 0$.

According to the limit-curve lemma there exists a subsequence of $\{\gamma_n[0, t_n]\}_{n \in \mathbb{N}}$ converging uniformly on compact sets to a future pointing curve $\gamma_\infty: [0, \infty) \rightarrow V/\Gamma$. It is classical that $\gamma_\infty$ is $F'$-almost maximal as well (3) proposition 14.3). The fact that there exists a $T_0 \geq 0$ with

$$\|\gamma_\infty(T_0) - \gamma_\infty(0)\| \in \left[ \varepsilon', \kappa' \right] \text{ and } \gamma_\infty(T_0) - \gamma_\infty(0) \in \Sigma_{\varepsilon'}$$

is ensured by the uniform convergence and the continuity of the functions $v \mapsto ||v||$ and $v \mapsto \text{dist}(v, \partial \Sigma)$. With the same argument we get that

$$\text{dist}(\gamma_\infty(T) - \gamma_\infty(0), \partial \Sigma) \leq \kappa'.$$
for all $T \geq 0$.

Set $\lambda_1 := \frac{2 \sup f}{\inf f}$ and $\mu := \frac{c \eta}{4T^2}$. Choose $T_1 > T_0$ such that we have

$$\dist(\gamma_\infty(T) - \gamma_\infty(T_1), \partial \Sigma) \leq \mu \varepsilon'^2 \leq \mu \dist(\gamma_\infty(T_1) - \gamma_\infty(0), \partial \Sigma)$$

for all $T > T_1$. Further choose $T_2 > T_1$ with

$$\|\gamma_\infty(T_2) - \gamma_\infty(T_1)\| \geq \mu \|\gamma_\infty(T_1) - \gamma_\infty(0)\|$$

and

$$\|\gamma_\infty(T_2) - \gamma_\infty(0)\| \geq \frac{4F'^2}{c \varepsilon'^2 \inf f} + 1.$$ 

On the one hand we get

$$\inf \frac{f}{2} |\gamma_\infty(T_2) - \gamma_\infty(0)|_1 \geq \sup f(|\gamma_\infty(T_2) - \gamma_\infty(T_1)|_1 + |\gamma_\infty(T_1) - \gamma_\infty(0)|_1)$$

with lemma 2.9. On the other hand, using (2.6), we have

$$\inf f |\gamma_\infty(T_2) - \gamma_\infty(0)|_1 \geq \inf f \sqrt{c \dist(\gamma_\infty(T_2) - \gamma_\infty(0), \partial \Sigma) \|\gamma_\infty(T_2) - \gamma_\infty(0)\|}$$

$$\geq \inf f \sqrt{c \dist(\gamma_\infty(T_1) - \gamma_\infty(0), \partial \Sigma) \|\gamma_\infty(T_2) - \gamma_\infty(0)\|}$$

$$\geq \inf f \varepsilon' \sqrt{c \|\gamma_\infty(T_2) - \gamma_\infty(0)\|} \geq 2F'.$$

Therefore we get

$$L^\infty(\|0, T_2\|) \geq \inf f |\gamma_\infty(T_2) - \gamma_\infty(0)|_1 - F'$$

$$> \frac{\inf f}{2} |\gamma_\infty(T_2) - \gamma_\infty(0)|_1$$

$$\geq \sup f(|\gamma_\infty(T_2) - \gamma_\infty(T_1)|_1 + |\gamma_\infty(T_1) - \gamma_\infty(0)|_1)$$

$$\geq L^\infty(\|0, T_1\|) + L^\infty(\|T_1, T_2\|).$$

Consequently the sequence $\|\gamma_n(t_n) - \gamma_n(a_n)\|$ has to be bounded.

In the other case $\|\gamma_n(b_n) - \gamma_n(s_n)\| \to \infty$, we obtain an analogous contradiction. Since

$$\|\gamma_n(t_n) - \gamma_n(s_n)\| \leq \|\gamma_n(b_n) - \gamma_n(s_n)\| + \|\gamma_n(t_n) - \gamma_n(a_n)\|,$$

we have a contradiction to the assumption that $\|\gamma_n(t_n) - \gamma_n(s_n)\| \to \infty$ for $n \to \infty$. This finishes the proof. \qed

Denote by

$$\text{diam}(\Gamma, \|\cdot\|) := \frac{1}{2} \inf \left\{ \max_{1 \leq i \leq m} \|k_i\| \; \langle k_1, \ldots, k_m \rangle \subseteq \Gamma \right\},$$

where $\langle k_1, \ldots, k_m \rangle \subseteq \Gamma$. Since $\Gamma$ is a co-compact lattice there exists for all $x, y \in V$ a $l_{x, y} \in \Gamma$ with

$$\|x - (y + l_{x, y})\| \leq \text{diam}(\Gamma, \|\cdot\|).$$

Lemma 2.11. There exists $D < \infty$ such that for all $x, y \in V$ there exists $k_{x, y} \in \Gamma$ with $\|x - (y + k_{x, y})\| \leq D$ and $y + k_{x, y} \in x + \Sigma$.

Proof. Choose $v \in \Sigma \setminus \partial \Sigma$ and $\varepsilon > 0$ such that we have $B_\varepsilon(v) \subseteq \Sigma$. Since $\Sigma$ is a cone we have $B_{\lambda \varepsilon}(\lambda v) \subseteq \Sigma$ for all $\lambda \geq 0$. Choose $\text{diam}(\Gamma, \|\cdot\|) < \Lambda < \infty$. Then we have $B_{\text{diam}(\Gamma, \|\cdot\|)}(\Lambda v) \subseteq \Sigma$. Set $k_{x, y} := l_{x + \Lambda v, y}$ and $D := \text{diam}(\Gamma, \|\cdot\|) + \Lambda \|v\|$. \qed
Proof of proposition 2.3. Let $F, G < \infty, \varepsilon > 0$ and $\gamma : I \to V/\Gamma$ be a $F$-almost maximal $(G, \varepsilon)$-timelike curve.

(i) Choose $[a', b'] \subset I$ with $\|\gamma(b') - \gamma(a')\| \geq G$ and $\gamma(b') - \gamma(a') \in \mathfrak{T}_\varepsilon$. Set $G_0 := \max \left\{ \frac{2F}{\sqrt{\varepsilon \inf f}} : \varepsilon \right\}$. If $G \geq G_0$ we get

$$
\inf f\|\gamma(b') - \gamma(a')\| \geq \inf f \sqrt{c \cdot \text{dist}(\gamma(b'), \gamma(a'), \partial\mathfrak{T})} \|\gamma(b') - \gamma(a')\| \\
\geq \inf f \sqrt{c} \varepsilon \|\gamma(b') - \gamma(a')\| \geq \inf f \sqrt{c} \varepsilon G \geq 2F.
$$

For any partition $\{\gamma_{[a,b_i]}\}_{1 \leq i \leq N}$ of $\gamma_{[a',b']}$ we have

$$
\sup f \sum |\gamma(b_i) - \gamma(a_i)| \geq \sum L^\varepsilon(\gamma_{[a,b_i]}) = L^\varepsilon(\gamma_{[a',b']}) \\
\geq \inf f \|\gamma(b') - \gamma(a')\| - F \\
\geq \inf f \frac{1}{2} |\gamma(b') - \gamma(a')|.
$$

Apply lemma 2.3 to $v_i := \gamma(b) - \gamma(a)$ with $\lambda_0 := \frac{\inf f}{2 \sup f}$ and $\varepsilon > 0$ as above. Consequently there exist $\varepsilon_0 > 0$ and $[a, b] \subset [a', b']$ with $\|\gamma(b) - \gamma(a)\| \in [\varepsilon, 2G_0]$ (note that $\text{dist}(v, \partial\mathfrak{T}) \leq \|v\|$ for all $v \in \mathfrak{T}$) and $\gamma(b) - \gamma(a) \in \mathfrak{T}_{\varepsilon_0}$.

(ii) Let $s < a < b < t \in I$. If $\|\gamma(b) - \gamma(s)\| \geq \nu := \frac{4(f_0 + 2inf_f^2)\varepsilon_0 \varepsilon}{4 \cdot 2\mu G_0}$ we get

$$
\inf f\|\gamma(b) - \gamma(s)\| \geq \inf f \sqrt{c \cdot \text{dist}(\gamma(b) - \gamma(s), \partial\mathfrak{T})} \|\gamma(b) - \gamma(s)\| \\
\geq \inf f \sqrt{c} \varepsilon_0 \varepsilon \|\gamma(b) - \gamma(s)\| \geq 2(F + 1).
$$

Consequently we have

$$
\sup f\|\gamma(a) - \gamma(s)\| + \|\gamma(b) - \gamma(a)\| > \frac{\inf f}{2} |\gamma(b) - \gamma(s)|.
$$

Set $\lambda_1 := \frac{2 \sup f}{\inf f}$. Recall the definition of $\mu := \frac{\varepsilon_0 \varepsilon}{4 \sqrt{c} \varepsilon_0 \varepsilon}$. If $\|\gamma(a) - \gamma(s)\| \geq \mu (2G_0)$ we get

$$
\text{dist}(\gamma(a) - \gamma(s), \partial\mathfrak{T}) \geq \mu \text{dist}(\gamma(b) - \gamma(a), \partial\mathfrak{T}) \geq \mu \varepsilon_0 \varepsilon
$$

with lemma 2.3. Note that we have

$$
\|\gamma(a) - \gamma(s)\| \geq \|\gamma(b) - \gamma(s)\| - \|\gamma(b) - \gamma(a)\| \geq \nu - 2G_0.
$$

Consequently we get that if

$$
\sup_{s' \in I, s' < a} \|\gamma(b) - \gamma(s')\| \geq \max \{\nu - 2G_0, 2\mu G_0\} := H_0
$$

there exists $s \in I, s < a$ with

$$
|\gamma(a) - \gamma(s)| = H_0 \text{ and } \gamma(a) - \gamma(s) \in \mathfrak{T}_{\delta_0}
$$

for $\delta_0 := \frac{\mu \varepsilon_0 \varepsilon}{H_0 \sqrt{c}}$. In the same way we obtain the existence of a parameter $t \in I, t > b$ with $\|\gamma(t) - \gamma(b)\| = H_0$ and $\gamma(t) - \gamma(b) \in \mathfrak{T}_{\delta_0}$, if we have that $\sup_{t' \in I, t' > b} \|\gamma(t') - \gamma(b)\| \geq H_0$.

(iii) Define $\kappa' := 3D, G' := 2\mu G_0, F' := F$ and $\varepsilon' := \delta_0$ and $K_0 := K'(3D, 2\mu G_0, F, \delta_0)$, according to lemma 2.10. Then there exists $\tau \in I, b < \tau$ with

$$
\|\gamma(\tau) - \gamma(b)\| = K_0 \text{ and dist}(\gamma(\tau) - \gamma(b), \partial\mathfrak{T}) \geq 3D
$$

if $\sup_{t' \in I, t' > b} \|\gamma(t') - \gamma(b)\| \geq K_0$. Analogously, if we have $\sup_{s' \in I, s' < a} \|\gamma(a) - \gamma(s')\| \geq K_0$, there exists $\sigma \in I, \sigma < a$ with

$$
\|\gamma(a) - \gamma(\sigma)\| = K_0 \text{ and dist}(\gamma(a) - \gamma(\sigma), \partial\mathfrak{T}) \geq 3D.
$$
We saw in step (ii) that for intervals \([b, t] \subseteq I\) with \(\| \gamma(t) - \gamma(b) \| = H_0\) we have
\[
\gamma(t) - \gamma(b) \in \Xi_{b, t}.
\]
We will carry this over to all intervals \([s, t] \subset I\) with \(\| \gamma(t) - \gamma(s) \| \) sufficiently large via the following cut-and-paste argument.

Note first that it suffices to consider the case \(\sup_{t' \in I, t' > b} \| \gamma(t') - \gamma(b) \| \geq K_0\).

The case \(\sup_{s' \in I, s' < a} \| \gamma(a) - \gamma(s') \| \geq K_0\) can be reduced to the former by considering \(\gamma_{\text{inv}}(t) := \gamma(t')\) and the opposite time-orientation on \((V / \Gamma, g)\).

Therefore we can assume that \(\sup_{t' \in I, t' > b} \| \gamma(t') - \gamma(b) \| \geq K_0\). Choose \(\tau \in I\), \(b < \tau\) with \(\| \gamma(\tau) - \gamma(b) \| = K_0\). We have \(\text{dist}(\gamma(\tau) - \gamma(b), \partial \Sigma) \geq 3 D\). Let \([s, t] \subset I\) be an interval mutually disjoint to \([a, \tau]\). We can choose future pointing curves \(\zeta: [a, b] \to V / \Gamma (i = 1, \ldots, 6)\) with \(L^{i, \|z\|}(\zeta_{1, 2, 4, 5}) \leq D\) such that
\[
\gamma' := [\gamma|_{[a, b]} * \zeta_1 * \gamma|_{[s, t]} * \zeta_2 * \gamma|_{[t, s]} * \zeta_3
\]
is future pointing and homotopic with fixed endpoints to \(\gamma|_{[a, t]}\) if \(s \geq \tau\) and such that
\[
\gamma'' := [\gamma|_{[s, t]} * \zeta_4 * \gamma|_{[a, b]} * \zeta_5 * \gamma|_{[t, a]} * \zeta_6
\]
is future pointing and homotopic with fixed endpoints to \(\gamma|_{[s, \tau]}\) if \(t \leq a\).

This can be seen as follows: Assume first that \(s \geq \tau\). Choose future pointing curves \(\zeta_1, \zeta_2\) with \(L^{i, \|z\|}(\zeta_i) \leq D\) connecting \(\gamma(b)\) with \(\gamma(s)\) and \(\gamma(t)\) with \(\gamma(\tau)\). Now consider a lift \(\tilde{\gamma}\) of \(\gamma\) to \(V\) and a lift \(\tilde{\zeta}_1\) \([a_i, b_i] \to V / \Gamma (i = 1, \ldots, 6)\) with \(\tilde{\zeta}_1\). Let \(q\) be the terminal point of \(\tilde{\zeta}_1\). Then we have
\[
\tilde{\gamma}(t) - q = [\gamma(\tau) - \gamma(t)] + \sum_{i=1}^{2} [\zeta_i(b_i) - \zeta_i(a_i)]
\]
By construction we have \(\| \sum_{i=1}^{2} [\zeta_i(b_i) - \zeta_i(a_i)] \| \leq 2 D\). Since dist \((\gamma(\tau) - \gamma(t), \partial \Sigma) \geq 3 D\) we get \(\tilde{\gamma}(t) - q \in \Xi\). Choose a future pointing curve \(\zeta_3\) : \([a_3, b_3] \to V / \Gamma\) with \(\zeta_3(a_3) = \gamma(s)\) and \(\zeta_3(b_3) - \zeta_3(a_3) = \tilde{\gamma}(t) - q\). This completes the construction of \(\gamma'\).

If \(t \leq a\) choose future pointing curves \(\zeta_4, \zeta_5\) with \(L^{i, \|z\|}(\zeta_{4, 5}) \leq D\) connecting \(\gamma(t)\) with \(\gamma(a)\) and \(\gamma(b)\) with \(\gamma(t)\). Consider a lift \(\tilde{\gamma}\) of \(\gamma\) to \(V\) and a lift \(\tilde{\zeta}_2\) of \(\zeta_4 * \gamma|_{[a, b]} * \zeta_5 * \gamma|_{[t, a]}\) starting at \(\tilde{\gamma}(b)\). Let \(q\) be the terminal point of \(\tilde{\zeta}_2\). Then we have
\[
\tilde{\gamma}(\tau) - q = [\gamma(\tau) - \gamma(t)] + \sum_{i=1}^{5} [\zeta_i(b_i) - \zeta_i(a_i)]
\]
By construction we have \(\| \sum_{i=1}^{5} [\zeta_i(b_i) - \zeta_i(a_i)] \| \leq 2 D\). Since dist \((\gamma(\tau) - \gamma(b), \partial \Sigma) \geq 3 D\) we get \(\tilde{\gamma}(\tau) - q \in \Xi\). Choose a future pointing curve \(\zeta_6\) : \([a_6, b_6] \to V / \Gamma\) with \(\zeta_6(a_6) = \gamma(a)\) and \(\zeta_6(b_6) - \zeta_6(a_6) = \tilde{\gamma}(\tau) - q\).

Set \(F_0 := F + \sup f \sqrt{C} K_0\). We claim that \(\gamma'\) and \(\gamma''\) are \(F_0\)-almost maximal. Indeed we have
\[
L^g(\gamma') \geq L^g(\gamma|_{[a, t]}) - L^g(\gamma|_{[b, \tau]}) \geq d(\tilde{\gamma}(a), \tilde{\gamma}(t)) - L^g(\gamma|_{[b, \tau]}) - F
\geq d(\tilde{\gamma}(a), \tilde{\gamma}(t)) - (F + \sup f \sqrt{C} K_0).
\]
Analogously we get \(L^g(\gamma') \geq d(\tilde{\gamma}(s), \tilde{\gamma}(\tau)) - (F + \sup f \sqrt{C} K_0)\).
Set $v := [\gamma(b) - \gamma(a)] + [\zeta_1(b_1) - \zeta_1(a_1)]$ and $w := \gamma(t) - \gamma(s)$. Note that we have $\|v\| \leq 2G_0 + D$ and $\text{dist}(v, \partial \Sigma) \geq \varepsilon_0 \varepsilon$. If $\|v + w\| \geq \frac{4(F_0 + 1)^2}{\inf f^2 c \varepsilon_0 \varepsilon}$ we get
\[
\inf f|v + w|_1 \geq \inf f \sqrt{c \text{dist}(v + w, \partial \Sigma)}\|v + w\|
\geq \inf f \sqrt{c \varepsilon_0 \varepsilon \|v + w\|} \geq 2(F_0 + 1).
\]
Consequently we have
\[
\sup f(|v|_1 + |w|_1) > \inf f \frac{2}{\|v + w\|}.
\]
Set $\lambda_1 := \frac{2\sup f}{\inf f}$. Recall the definition of $\mu := \frac{\varepsilon_0 \varepsilon}{4 \lambda_1 \varepsilon}$. If $\|w\| \geq \mu (2G_0 + D)$ we get
\[
\text{dist}(w, \partial \Sigma) > \mu \text{dist}(v, \partial \Sigma) \geq \mu \varepsilon_0 \varepsilon
\]
with lemma 2.9.

Set
\[
K := \frac{4}{\eta} \max \left\{ \mu(2G_0 + D), \frac{4(F_0 + 1)^2}{\inf f^2 c \varepsilon_0 \varepsilon}, G_0 + K_0 \right\}.
\]

Let $[s, t] \subset I$ with $\|\gamma(t) - \gamma(s)\| \geq K$. We want to show that there exists $\delta > 0$ such that $\gamma(t) - \gamma(s) \in \Sigma_\delta$. If $\|\gamma(t) - \gamma(s)\| \geq 2K$ we can partition $[s, t]$ into mutually disjoint subintervals $[s_i, t_i]$ with $\|\gamma(t_i) - \gamma(s_i)\| \in [K, 2K]$. If we have $\gamma(t_i) - \gamma(s_i) \in \Sigma_\delta$ for all $i$ and some $\delta > 0$, we get $\gamma(t) - \gamma(s) \in \Sigma_\delta$, using corollary 2.8. Therefore we can assume that $\|\gamma(t) - \gamma(s)\| \in [K, 2K]$. If we have $[s, t] \setminus (a, \tau_0) = \emptyset$, we are done since we can then apply the above cut-and-paste argument and obtain $\text{dist}(\gamma(t) - \gamma(s), \partial \Sigma) \geq \mu \varepsilon_0 \varepsilon$. Then we have
\[
\gamma(t) - \gamma(s) \in \Sigma_\delta,
\]
for $\delta' := \frac{\mu \varepsilon_0 \varepsilon}{\inf f^2 c \varepsilon_0 \varepsilon}$.

If we have $[s, t] \setminus (a, \tau_0) \neq \emptyset$, then $[s, a]$ or $[\tau, t] \neq \emptyset$. By the choice of $K$, we have
\[
\max\{\|\gamma(a) - \gamma(s)\|, \|\gamma(t) - \gamma(\tau)\|\} \geq \frac{1}{\eta} \max \left\{ \mu(2G_0 + D), \frac{4(F_0 + 1)^2}{\inf f^2 c \varepsilon_0 \varepsilon} \right\}.
\]

Recall that we have $\gamma(b) - \gamma(a) \in \Sigma_{\varepsilon_0}$ and $\gamma(\tau_0) - \gamma(b) \in \Sigma_{\varepsilon'}$ for $\varepsilon' := \frac{\mu \varepsilon_0}{K_0}$. Again with the above cut-and-paste argument we obtain
\[
(2.7) \quad \gamma(t) - \gamma(\tau) \text{ or } \gamma(a) - \gamma(s) \in \Sigma_{\delta'}
\]
for $\delta'' := \min\{\delta', \varepsilon_0, \varepsilon'\}$. Note the following fact. Let $v, w \in \Sigma$ with $\|w\| \leq \|v\|$ and $v \in \Sigma_{\varepsilon/2}$. Then we have $v + w \in \Sigma_{\varepsilon/2}$. Combining this with (2.7) we get
\[
\gamma(t) - \gamma(s) \in \Sigma_\delta,
\]
for $\delta := \frac{1}{2}\delta''$. This finishes the proof.

3. The Stable Time Separation

**Proposition 3.1.** There exists a positively homogenous concave function $t: \Sigma \to [0, \infty)$ such that:

1. For every $\varepsilon > 0$ there exists a $K(\varepsilon) > 0$ such that
   \[
   |t(v) - d(x, x + v)| \leq K(\varepsilon)
   \]
   for all $v \in \Sigma_\varepsilon$ and all $x \in V$.
2. $\inf f|v|_1 \leq |t(v)| \leq \sup f|v|_1$ for all $v \in \Sigma$.
3. $|t(v + w)| \geq |t(v)| + |t(w)|$ for all $v, w \in \Sigma$. 


The following lemma is an adapted version of lemma 1 in [2].

**Lemma 3.2.** Let \( K < \infty \) and \( f : [0, \infty) \to [0, \infty) \) be a \( L \)-Lipschitz continuous function such that

1. \( f(2t) - 2f(t) \leq K, \)
2. \( zf(t) - f(zt) \leq K \) for \( z = 2, 3 \)

and all \( t \geq 0 \). Then there exists an \( a \in \mathbb{R} \), such that

\[
|f(t) - at| \leq 2K,
\]

for all \( t \geq 0 \).

**Proof.** We have

\[
z^n f(t) \leq f(z^n t) + K \sum_{k=1}^{n} z^k
\]

for all \( t \geq 0 \), \( z = 2, 3 \), and integers \( n \in \mathbb{Z}_{\geq 0} \). This implies

\[
\frac{f(t)}{t} < \frac{f(z^n t)}{z^n t} + \frac{2K}{t}.
\]

From \( f(t) \leq Lt + f(0) \) we get the existence of \( \limsup_{n \to \infty} \frac{f(z^n t)}{z^n t} =: a_z(t) \). Choose for \( \varepsilon > 0 \) an integer \( r \in \mathbb{Z}_{\geq 0} \) such that \( \frac{f(z^n t)}{z^n t} \geq a_z(t) - \varepsilon \) and \( \frac{2K}{z^n t} \leq \varepsilon \). Then we have

\[
\frac{f(z^n t + t)}{z^n t} \geq \frac{f(z^n t)}{z^n t} - \frac{2K}{z^n t} \geq a_z(t) - 2\varepsilon
\]

for all \( n \geq 0 \). Therefore the sequence \( \left\{ \frac{f(z^n t)}{z^n t} \right\}_n \) converges to \( a_z(t) \).

We claim that \( a_z(t) \) is independent of \( t \geq 0 \) and \( z = 2, 3 \). We have

\[
\left| \frac{f(2^n t)}{2^n t} - \frac{f(3^n s)}{3^n s} \right| \leq \left| \frac{f(2^n t)}{2^n t} - \frac{f(3^n s)}{2^n t} \right| + \left| \frac{f(3^n s)}{2^n t} - \frac{f(3^n s)}{3^n s} \right|
\]

\[
\leq L \frac{|2^n t - 3^n s|}{2^n t} + \frac{|2^n t - 3^n s|}{2^n t} \frac{|f(3^n s)|}{3^n s}
\]

for all \( s, t \geq 0 \). Since \( \liminf_{m,n \to \infty} \frac{|2^n t - 3^n s|}{2^n t} = 0 \) we get \( a_2(t) = a_3(s) =: a \).

Define \( f(t) - at =: \delta(t) \). Then we get

\[
|f(2^n t) - 2^n(at + \delta(t))| \leq \sum_{k=1}^{n} 2^k K \leq 2^{n+1} K
\]

and

\[
\left| \frac{f(2^n t)}{2^n t} - \left( a + \frac{\delta(t)}{t} \right) \right| < \frac{2K}{t}.
\]

Passing to the limit \( n \to \infty \), we obtain the lemma. \( \square \)

**Lemma 3.3.** For all \( \varepsilon > 0 \) there exists a \( K' = K'(\varepsilon) < \infty \) such that

\[
d(x, x + zv) \geq zd(x, x + v) - K'
\]

for \( z = 2, 3 \) and all \( x, v \in V \) with \( v \in \mathbb{S}_\varepsilon \).
Proof. Recall the definition of \( \text{diam}(\Gamma, ||\cdot||) \). Choose \( l \in \Gamma \) with \( x + l \in B_{\text{diam}(\Gamma, ||\cdot||)}(x + \nu) \). Using theorem 2.3 we have
\[
d(x, x + 2\nu) \geq d(x, x + \nu) + d(x + \nu, x + 2\nu) \\
\geq d(x, x + \nu) + d(x + l, x + l + \nu) - 2L(\varepsilon)\|v - l\| \\
= 2d(x, x + \nu) - 2L(\varepsilon)\text{diam}(\Gamma, ||\cdot||).
\]
The other case follows analogously. \( \square \)

Lemma 3.4 ([2, lemma 2]). Let \( \tau: [a, b] \to V \) be a continuous curve. Then there exist \( k \leq [m/2] \) and \( k \)-many mutually disjoint closed subintervals \( [s_i, t_i] \) with \( i \leq k \subseteq [a, b] \) with
\[
\sum_{i=1}^{k} [\tau(t_i) - \tau(s_i)] = \frac{1}{2} [\tau(b) - \tau(a)].
\]

Lemma 3.5. For all \( \varepsilon > 0 \) there exists a \( K'' = K''(\varepsilon) \). Choose \( v \) with \( \|v\| \geq (2m + 1)D \). Let \( \tau: [a, b] \to V \) be a maximal curve from \( x \) to \( x + 2\nu \). Choose intervals \( [s_i, t_i] \subseteq [a, b] \) as in lemma 3.4. Next choose points \( \tau(s_i)^* \) and \( \tau(t_i)^* \) in the \( \Gamma \)-orbit of \( x \) such that \( \tau(s_i)^* \in (\tau(s_i) + \Sigma) \cap B_D(\tau(s_i)) \) and \( \tau(t_i)^* \in (\tau(t_i) + \Sigma) \cap B_D(\tau(t_i)) \). Applying the transformations in \( \Gamma \) that map \( \tau(s_i)^* \) to \( \tau(t_{i-1})^* \), resp. \( \tau(s_1)^* \) to \( x \), yields:
\[
d(x, x + v) \geq \sum_{i=1}^{k} [\tau(s_i)^* - \tau(s_i)] + [\tau(t_i)^* - \tau(t_i)] \\
\leq d(x, x + v) + \frac{L(\varepsilon)}{2} \sum_{i=1}^{k} \text{dist}(\tau(s_i), \tau(s_i)^*) + \text{dist}(\tau(t_i), \tau(t_i)^*) \\
\leq d(x, x + v) + mL(\varepsilon/2)D.
\]

Since we can repeat the argument with \( \tau|_{[s_i, t_i]} \) replaced by \( \tau|_{[t_{i-1}, s_i]} \) we get after summing of the results:
\[
d(x, x + 2\nu) = \sum_{i=1}^{k} d(\tau(s_i), \tau(t_i)) + d(\tau(t_{i-1}), \tau(s_i)) \\
\leq 2d(x, x + v) + 2mL(\varepsilon/2)D
\]
\( \square \)

Proof of proposition 3.7. Lemma 3.4 and 3.5 ensure that lemma 3.2 can be applied to \( f_\varepsilon(t) := d(x, x + tv) \). Then there exists an \( a_\varepsilon(v) \) with \( |d(x, x + tv) - a_\varepsilon(v)t| \leq 2\max\{K', K''\} =: K \). In fact \( a_\varepsilon(v) \) does not depend on \( x \) since for \( x' \in V \) we have
\[
\left| \frac{d(x', x' + nv)}{n} - a_\varepsilon(v) \right| \leq \frac{K + 2L(\varepsilon)D}{n}.
\]
This shows the independence of \( a_x(v) \) of \( x \) as well as the uniform convergence on compact subsets of \( \Sigma_x \) of \( \frac{1}{q}d(x, x + nv) \) to \( a_x(v) \). Set \( l(v) := a_x(v) \). The estimate \( (1) \) then follows from the definition. Property \((2)\) follows directly from the estimate:

\[
\inf f|v|_1 \leq d(x, x + v) \leq \sup f|v|_1
\]

The inverse triangle inequality follows readily from the inverse triangle inequality for the time separation. For the positive homogeneity note that by the uniform convergence on compact subsets of \( \Sigma_x \) it suffices to consider rational factors \( \eta = \frac{p}{q} \).

Then by considering subsequences we get

\[
l(\eta v) = \lim \frac{1}{n}d(x, x + \frac{p}{q}nv) = \lim \frac{1}{qn}d(x, x + pnv)
\]

\[
= \lim \frac{p}{qn}d(x, x + nv) = \frac{p}{q} \lim \frac{1}{n}d(x, x + nv) = \eta l(v).
\]

\[\square\]

4. The Rotation Vector

Define the rotation vector of a future pointing curve \( \gamma: [a, b] \rightarrow V/\Gamma \):

\[
\rho(\gamma) := \frac{1}{l(\gamma(b) - \gamma(a))} [\gamma(b) - \gamma(a)]
\]

**Theorem 4.1.** Let \( \varepsilon > 0 \) and \( \gamma: \mathbb{R} \rightarrow V/\Gamma \) be a homologically maximizing geodesic with \( \dot{\gamma}(t_0) \in \Sigma_x \) for some \( t_0 \in \mathbb{R} \). Then there exists a support function \( \alpha \) of \( \Gamma \) such that for all neighborhoods \( U \) of \( \alpha^{-1}(1) \cap \Gamma^{-1}(1) \) there exists a \( K = K(\varepsilon, U) > 0 \) such that for all \( s < t \in \mathbb{R} \) with \( \|\gamma(t) - \gamma(s)\| \geq K \), we have

\[
\rho(\gamma|_{[s,t]}) \in U.
\]

**Lemma 4.2.** Let \( \varepsilon, \delta > 0 \), \( F, G < \infty \) and \( n \in \mathbb{N} \) be given. Then there exists a \( K = K(\varepsilon, \delta, F, G, n) < \infty \) such that for all \( k \leq n \), all \( T \geq K \) and all \( F \)-almost maximal \((G, \varepsilon)\)-timelike curves \( \gamma: \mathbb{R} \rightarrow V \), the following holds.

Given \( k \) many intervals \([t_i, t_i + \sigma_i]\) with disjoint interiors and \( L^g(\gamma|_{[t_i, t_i + \sigma_i]}) = T \) we have

\[
l(\sum \rho(\gamma|_{[t_i, t_i + \sigma_i]})) \leq k + 1 + \delta.
\]

**Proof.** Assume that the arcs \( \gamma|_{[t_i, t_i + \sigma_i]} \) are indexed in increasing order. W.l.o.g. we can suppose that \( \|\gamma(t_i + \sigma_i) - \gamma(t_{i+1})\| \leq D \). If this is not the case we can repeat the cut-and-paste operation from the proof of proposition 2.2. Choose \( \delta > 0 \) according to theorem 2.2. Notice that in this case for any interval \([s_\infty, t_\infty] \) with \( \|\gamma(t_\infty) - \gamma(s_\infty)\| = (2n - 1)D \), we have \( \text{dist}(\gamma(t_\infty) - \gamma(s_\infty), \partial \Sigma) \geq (2n - 1)D \). Choose one such interval disjoint from \([t_i, t_i + \sigma_i]\). The new curve, resulting from the cut-and-paste operation, will be \( L^g(\gamma|_{[s_\infty, t_\infty]}) - F \)-almost maximal. Define \( A := \max L^g(\gamma|_{[t_i + \sigma_i, t_{i+1}]})) \).

Now let \( \Sigma' \) be a proper sub-cone of \( \Sigma \), \( v_i \in \Sigma' \ (0 \leq i \leq k) \) with \( l(v_i) = 1 \) and \( l(\sum v_i) = k + 1 + r \). Further let \( \lambda_i > 0 \) and \( \lambda \leq \min \lambda_i \). Then

\[
l(\sum \lambda_i v_i) \geq \lambda l(\sum v_i) \geq \lambda(\sum \rho(\gamma|_{[t_i, t_i + \sigma_i]})) \geq \lambda(k + 1 + r)
\]

and for \( \lambda_i = l(\gamma(t_i + \sigma_i) - \gamma(t_i)) \)

\[
l(\sum \gamma(t_i + \sigma_i) - \gamma(t_i)) \geq \lambda l(\sum \rho(\gamma|_{[t_i, t_i + \sigma_i]})).
\]
By proposition 3.1 there exists a $K < \infty$, depending only on $F, G$ and $\varepsilon$, such that $|l(\gamma(t) - \gamma(s)) - L^\beta(\gamma)| \leq K$ for all $s,t \in \mathbb{R}$. For $T > K$ set $\lambda := T - K$. Then we conclude

$$(k + 1 + r)(T - K) \leq \sum L^\beta(\gamma_{|t_i, t_{i+1} + \sigma_i})$$

$$\leq L^\beta(\gamma_{|t_i, t_{k+1}}) = (k + 1)T + k(A + K).$$

Solving for $r$ shows

$$r \leq \frac{1}{T - K}((2k + 1)K + kA).$$

Increasing $T$ sufficiently we conclude the assertion. \hfill \Box

**Lemma 4.3.** Let $\{W_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of $\Gamma^{-1}(1)$ such that $W_{n+1} \subset cone(W_n)$. Assume further that there exists a sequence $\delta_n \downarrow 0$ such that for any $k$-tuple of pairwise different $v_i \in W_n$, $l(\sum v_i) \leq k + 1 + \delta_n$ holds. Then there exists a supporting hyperplane $E$ of $\Gamma^{-1}(1)$ such that for any neighborhood $U$ of $E \cap \Gamma^{-1}(1)$ the intersection cone$(W_n) \cap \Gamma^{-1}(1)$ is eventually contained in $U$.

**Proof.** Let $\{v_0, \ldots, v_k\} \subset W_n$ and $t_0, \ldots, t_k \geq 0$ with $\sum t_i = 1$. Since $l(v_i) = 1$ we conclude

$$k + 1 + \delta_n \geq l(\sum v_i) \geq l(\sum t_i v_i) + \sum l((1 - t_i)v_i)$$

$$= l(\sum t_i v_i) + \sum (1 - t_i) = l(\sum t_i v_i) + k.$$

Therefore $l(\sum t_i v_i) \leq 1 + \delta_n$. By Carathéodory’s theorem the convex hull of $W_n$ is the union of all simplices with vertices in $W_n$. Consequently the closure of the convex hull of $W_n$ is disjoint from $\Gamma^{-1}(1 + 2\delta_n, \infty))$. By the inverse triangle inequality, proposition 3.1 (ii) and the assumption $W_{n+1} \subset cone(W_n)$, the sets $W_n$ are uniformly bounded. Then there exists an affine hyperplane $E_n$ separating the convex hull of $W_n$ from $\Gamma^{-1}(1 + 2\delta_n, \infty))$. Now consider a limit hyperplane $E$ of $\{E_n\}_{n \in \mathbb{N}}$. Since $\lim \delta_n = 0$, $E$ is a supporting hyperplane of $\Gamma^{-1}(1)$. The assertion now follows easily. \hfill \Box

**Proof of theorem 4.1** It suffices to prove the assertion for parameter intervals of the form $[i2^n, (i + 1)2^n]$ for $i \in \mathbb{Z}$ and $n \in \mathbb{N}$. Set $W_n := \{\rho(\gamma_{|[i2^n, (i + 1)2^n]}): i \in \mathbb{Z}\}$. By lemma 4.2 there exists a sequence $\delta_n \downarrow 0$ such that for all $\{v_0, \ldots, v_k\} \subset W_n$ ($k \leq m$) holds $l(v_0 + \ldots + v_k) \leq k + 1 + \delta_n$. Since any vector $\gamma((i + 1)2^n) - \gamma((i + 2)^n)$ is the sum of two vectors of the form $\gamma((j + 1)2^n) - \gamma(j2^n)$, the convex cone over $W_n$ contains $W_{n+1}$. This establishes the assumptions of lemma 4.3 and therefore the assertion. \hfill \Box

Recall the definition of the dual cone $\mathfrak{S}^\ast := \{\alpha \in V^* | \alpha|_{\mathfrak{S}} \geq 0\}$ of $\mathfrak{S}$. We call a function $h: V \to \mathbb{R}$ $\alpha$-equivariant, if

$$h(x + k) = h(x) + \alpha(k)$$

for all $x \in V$ and $k \in \Gamma$.

**Remark 4.4.** For $\alpha \in V^*$ there exists an $\alpha$-equivariant time function if and only if $\alpha \in (\mathfrak{S}^\ast)^2$. Furthermore, the existence of an $\alpha$-equivariant time function is equivalent to the existence of a smooth $\alpha$-equivariant temporal function (a $C^1$-function is a temporal function if the Lorentzian gradient is always timelike and past pointing).
Let $\alpha \in V^*$ and $h: V \to \mathbb{R}$ be an $\alpha$-equivariant $C^1$-function. Then the 1-form $dh$ descends to a 1-form $\omega_h$ on $V/\Gamma$.

**Definition 4.5.** Let $\alpha \in (T^*)^0$ and $\tau: V \to \mathbb{R}$ be an $\alpha$-equivariant temporal function.

1. Define for $\sigma \in \mathbb{R}$:
   
   $$h_\tau(\sigma) := \sup \{ L^g(\gamma) | \gamma \text{ future pointing}, \int_\gamma \omega_\tau = \sigma \}$$

2. A homologically maximizing curve $\gamma: I \to V/\Gamma$ is said to be $\alpha$-almost maximal if there exists a constant $F < \infty$ such that
   
   $$L^g(\gamma|_{[s,t]}) \geq h_\tau(\int_{\gamma|_{[s,t]}} \omega_\tau) - F$$

for all $s < t \in I$.

**Remark 4.6.**

(i) Note that the definition of $\alpha$-almost maximality does not depend on the choice of $\alpha$-equivariant temporal function. A different choice of $\alpha$-equivariant time function only yields a different constant $F'$.

(ii) In Riemannian geometry the notion of $\alpha$-almost minimality (analogue of $\alpha$-almost maximality) is now replaced by the notions of calibrations and calibrated curves (compare [8]). The Lorentzian versions of calibrations and calibrated curves will be introduced in [1].

Define the dual stable time separation $\Gamma^*: T^* \to \mathbb{R}$, $\alpha \mapsto \min \{ \alpha(v) | l^*(v) = 1 \}$.

**Theorem 4.7.**

1. For every $\alpha \in (T^*)^0$ there exists an $\alpha$-almost maximal timelike geodesic $\gamma: \mathbb{R} \to V/\Gamma$.
2. Let $\alpha \in T^*$ with $\Gamma^*(\alpha) = 1$. Then for every neighborhood $U$ of $\alpha^{-1}(1) \cap l^{-1}(1)$ there exists a $K = K(\alpha, U) < \infty$ such that
   
   $$\rho(\gamma|_{[s,t]}) \in U$$

for all $\alpha$-almost maximal future pointing curves $\gamma: \mathbb{R} \to V/\Gamma$ and every $s < t \in \mathbb{R}$ with $\|\gamma(t) - \gamma(t)\| \geq K$.

**Corollary 4.8.** $(V/\Gamma, g)$ contains infinitely many geometrically distinct homologically maximizing timelike geodesics $\gamma: \mathbb{R} \to V/\Gamma$ with the additional property that the limit

$$\lim_{t \to \infty} \rho(\gamma|_{[s,s+t]}) =: v$$

exists uniformly in $s \in \mathbb{R}$. The $v$ are exposed points of $\Gamma^{-1}(1)$.

**Lemma 4.9.** Let $\alpha \in (T^*)^0$ and $\tau: V \to \mathbb{R}$ an $\alpha$-equivariant time function.

(i) Then there exists an $\varepsilon = \varepsilon(\alpha) > 0$, such that

$$y - x \in T_{\varepsilon}$$

for all $x, y \in V$ with $y - x \in T_{\varepsilon}$, $\tau(y) - \tau(x) \geq 2$ and $2d(x, y) \geq h_\tau(\tau(y) - \tau(x))$.

(ii) There exists a constant $I = I(\alpha) < \infty$ such that

$$h_\tau(\sigma_1) + h_\tau(\sigma_2) - I \leq h_\tau(\sigma_1 + \sigma_2) \leq h_\tau(\sigma_1) + h_\tau(\sigma_2)$$

for all $\sigma_1, \sigma_2 \geq 0$. 
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(iii) We have
\[
\lim_{\sigma \to \infty} \frac{\sigma}{h_\tau(\sigma)} = \Gamma^*(\alpha).
\]

Proof. (i) Clear from the fact that \(\tau\) is an \(\alpha\)-equivariant time function and theorem 2.3

(ii) The right side of the inequality is clear from the observation that any curve \(\gamma: [a, b] \to V/\Gamma\) with \(\int_\gamma \omega_\tau = \sigma_1 + \sigma_2\) is the concatenation of two curves \(\gamma_i: [a_i, b_i] \to V/\Gamma\) \((i = 1, 2)\) with \(\int_{\gamma_i} \omega_\tau = \sigma_i\).

For the other inequality we reverse this procedure. Let \(\gamma_i: [a_i, b_i] \to V/\Gamma\) be future pointing curves with \(\int_{\gamma_i} \omega_\tau = \sigma_i\) \((i = 1, 2)\). Choose a future pointing curve \(\gamma\) from \(\gamma_1(b_1)\) to \(\gamma_2(a_2)\) of \(\|\cdot\|\)-arclength bounded by \(D\). Then for sufficiently large \(\sigma_2\) (the other cases can be absorbed into the constant \(I\)) there exists a \(b'_2 \in [a_2, b_2]\) with \(\int_{\gamma_1 \cdot \gamma_2(b_2) \cdot [a_2, b'_2]} \omega_\tau = \sigma_1 + \sigma_2\). Using (i), the distance \(\|\gamma_2(b_2) - \gamma_2(b'_2)\|\) is uniformly bounded by some \(I'(\alpha) < \infty\). The claim now follows by noting that \(d(x, x + v) \leq \sup f|v|_1\).

(iii) Choose \(v_\alpha \in \Gamma^{-1}(1)\) with \(\alpha(v_\alpha) = 1\). Then for all \(x \in V\)
\[
\Gamma^*(\alpha) = \lim_{n \to \infty} \frac{n}{d(x, x + n v_\alpha)} \Gamma^*(\alpha) \geq \limsup_{n \to \infty} \frac{n \Gamma^*(\alpha)}{h_\tau(\tau(x + n v_\alpha) - \tau(x))}.
\]
The difference \(\tau(x + n v_\alpha) - \tau(x) - n \alpha(v_\alpha)\) is uniformly bounded. The continuity of \(h_\tau\) and part (ii) then imply
\[
\limsup_{\sigma \to \infty} \frac{\sigma}{h_\tau(\sigma)} \leq \Gamma^*(\alpha).
\]

It remains to prove the opposite inequality
\[
\liminf_{\sigma \to \infty} \frac{\sigma}{h_\tau(\sigma)} \geq \Gamma^*(\alpha).
\]

By proposition 4.1 there exists a \(K = K(\alpha) < \infty\) such that \(l(\sigma v') \geq d(x, x + \sigma v') - K\) for all \(\sigma \geq 0\) and all \(v' \in \Gamma^{-1}(1) \cap \alpha^{-1}(1)\). But then we have
\[
\sigma \geq \Gamma^*(\alpha) l(\sigma v') \geq \Gamma^*(\alpha) h_\tau(\tau(x + \sigma v') - \tau(x)).
\]
The claim now follows by noting again that the difference \(\tau(x + n v') - \tau(x) - n \alpha(v')\) is uniformly bounded. □

Proof of theorem 4.7. (i) Let \(\tau: V \to \mathbb{R}\) be an \(\alpha\)-equivariant temporal function. Consider a sequence of homologically maximizing timelike pregeodesics \(\gamma_\alpha: [-T_n, T_n] \to V/\Gamma\) parameterized by \(\|\cdot\|\)-arclength such that \(\int_{\gamma_\alpha} \omega_\tau = 2n\) and \(L^\theta(\gamma_\alpha) = h_\tau(2n)\). The sequence \(T_n\) is obviously unbounded. The limit curve lemma implies that \(\{\gamma_\alpha\}\) contains a converging subsequence. Denote the limit curve by \(\gamma: \mathbb{R} \to V/\Gamma\). The \(\alpha\)-maximality of the \(\gamma_\alpha\) imply that the \(\gamma_\alpha\) are uniformly timelike. \(\gamma\) is therefore timelike as well. The homological maximality of \(\gamma\) follows from the homological maximality of the \(\gamma_\alpha\). \(\gamma\) is an \(\alpha\)-almost maximal curve. Note that by lemma 4.3

(ii) for any interval \([a, b] \subset [-T_n, T_n]\) we have
\[
L^\theta(\gamma_\alpha) = L^\theta(\gamma_\alpha|[-T_n, a]) + L^\theta(\gamma_\alpha|[a, b]) + L^\theta(\gamma_\alpha|[b, T_n]) = h_\tau(2n)
\geq h_\tau \left( \int_{\gamma_\alpha|[-T_n, a]} \omega_\tau \right) + h_\tau \left( \int_{\gamma_\alpha|[a, b]} \omega_\tau \right) + h_\tau \left( \int_{\gamma_\alpha|[b, T_n]} \omega_\tau \right) - 2F.
\]
This implies already \(L^\theta(\gamma|[a, b]) \geq h_\tau \left( \int_{\gamma|[a, b]} \omega_\tau \right) - 2F\).
The first step is to note that for all \( \alpha \)-almost maximal curves \( \gamma \) and all \( \alpha \)-equivariant temporal functions \( \tau \), the ratio

\[
\int_{\gamma|[s,t]} \omega_{\tau} \frac{d}{\ell^0(\gamma|[s,t])} \rightarrow 1
\]

for \( \|\gamma(t) - \gamma(s)\| \rightarrow \infty \). Lemma 4.9 (iii) implies \( \lim_{\sigma \rightarrow \infty} \sigma = 1 \). Since \( \gamma \) is \( \alpha \)-almost maximal, we conclude

\[
\lim_{\|\gamma(t) - \gamma(s)\| \rightarrow \infty} \frac{h_{\tau}(\int_{\gamma|[s,t]} \omega_{\tau})}{\ell^0(\gamma|[s,t])} = 1.
\]

This implies (4.1).

By lemma 4.9 (i), there exists an \( \varepsilon(\alpha) > 0 \) such that for \( \|\gamma(t) - \gamma(s)\| \) sufficiently large, \( \gamma(t) - \gamma(s) \in \mathfrak{T}_\varepsilon(\alpha) \). But then

\[
L^0(\gamma|[s,t]) - K(\varepsilon) \leq \ell(\gamma(t) - \gamma(s)) \leq \alpha(\gamma(t) - \gamma(s)).
\]

Now let \( U \) be a neighborhood of \( \alpha^{-1}(1) \cap \Gamma^{-1}(1) \) and \( \delta = \delta(U) > 0 \) such that for all \( h \in \Gamma^{-1}(1) \setminus U \), \( \alpha(h) \geq 1 + \delta \). If \( \rho(\gamma(t) - \gamma(s)) \notin U \), we get

\[
\alpha(\gamma(t) - \gamma(s)) \geq (1 + \delta)\ell(\gamma(t) - \gamma(s)) \geq (1 + \delta)(L^0(\gamma|[s,t]) - K(\varepsilon)).
\]

If \( \gamma(t) - \gamma(s) \) is not bounded from above, this contradicts the above conclusion:

\[
\frac{\int_{\gamma|[s,t]} \omega_{\tau}}{L^0(\gamma|[s,t])} \rightarrow 1.
\]

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