AN EFFECTIVE ANALYTIC FORMULA FOR THE NUMBER OF
DISTINCT IRREDUCIBLE FACTORS OF A POLYNOMIAL

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ABSTRACT. We obtain an effective analytic formula, with explicit constants,
for the number of distinct irreducible factors of a polynomial \( f \in \mathbb{Z}[x] \). We use
an explicit version of Mertens’ theorem for number fields to estimate a related
sum over rational primes. For a given \( f \in \mathbb{Z}[x] \), our result yields a finite list of
primes that certifies the number of distinct irreducible factors of \( f \).

1. Introduction

In this note we establish an effective analytic formula (Theorem 2) for the number
of distinct irreducible factors of a polynomial \( f \in \mathbb{Z}[x] \). Our error bounds are
unconditional and explicit in terms of their dependence upon \( f \). To this end,
we first introduce Theorem 1, which is of independent interest since it relates a
Mertens-type sum over number fields to a weighted sum over rational primes in an
explicit manner that does not involve the residue of a Dedekind zeta function.

1.1. A Mertens-type sum. Let \( K \) be a number field of degree \( d \) with ring of
integers of \( \mathcal{O}_K \). Let \( N(p) \) denote the norm of a prime ideal \( p \subset \mathcal{O}_K \)
and \( p \) a rational prime. The primitive element theorem says that \( K = \mathbb{Q}(\alpha) \)
in which \( g \in \mathbb{Z}[x] \) is irreducible with root \( \alpha \) and leading coefficient \( c \). The
degree of \( g \) is \( d \) [17, p. 47] and the discriminant \( D_g \) of \( g \) is nonzero (Lemma 4).

Theorem 1. Let \( K = \mathbb{Q}(\alpha) \), in which \( g \in \mathbb{Z}[x] \) is irreducible with root \( \alpha \), leading
coefficient \( c \), and degree \( d \). Define \( D_g = |c|^{(d-1)(d-2)}|D_g| \). For \( x > \max\{2, \sqrt{D_g}\} \),

\[
\sum_{p \leq x} \frac{\omega_g(p)}{p} = \sum_{N(p) \leq x} \frac{1}{N(p)} + A_g, \quad M_K(x) \leq M_Q(|c|) + M_Q(\sqrt{D_g}) + 0.64,
\]

in which \( \omega_g(p) \) is the number of solutions to \( g(x) \equiv 0 \pmod{p} \),

\[
|A_g| \leq d \left( M_Q(|c|) + M_Q(\sqrt{D_g}) + 0.64 \right), \quad \text{and } \quad M_Q(x) = \sum_{p \leq x} \frac{1}{p}. \tag{1}
\]

Rosser–Schoenfeld bounded \( M_Q(x) \) explicitly [14, (3.20)]; see [12] below. It is
known that \( M_E(x) = \log \log x + O(1) \), where the \( O(1) \) term depends upon the
residue of the corresponding Dedekind zeta function at \( s = 1 \) [7, 13]. Theorem 1

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avoids this inconvenience and reduces the computation of a Mertens-type sum over rational primes, with an explicit error bound.

1.2. **An effective analytic formula.** Recall that $f \in \mathbb{Z}[x]$ is **primitive** if the greatest common divisor of its coefficients is 1. A nonconstant polynomial in $\mathbb{Z}[x]$ is **irreducible** in $\mathbb{Z}[x]$ if and only if it is primitive and irreducible in $\mathbb{Q}[x]$. Gauss’ primitivity lemma ensures that the product of primitive polynomials is primitive, so we may assume that each irreducible factor of a given $f \in \mathbb{Z}[x]$ is primitive.

Suppose that $f = f_1 f_2 \cdots f_k \in \mathbb{Z}[x]$, in which $f_1, f_2, \ldots, f_k \in \mathbb{Z}[x]$ are irreducible; they are uniquely determined up to ordering. Since the Euclidean algorithm reveals any common factors of $g, g' \in \mathbb{Z}[x]$, we may assume that the $f_i$ are distinct without loss of generality. Equivalently, the discriminant $D_f$ of $f$ is nonzero. Under these circumstances, we provide an analytic formula for $k$. Our result is unconditional and explicit. The error term depends only upon the degree and discriminant of $f$.

**Theorem 2.** Suppose that $f = f_1 f_2 \cdots f_k \in \mathbb{Z}[x]$ is a product of distinct, irreducible nonconstant polynomials $f_1, f_2, \ldots, f_k \in \mathbb{Z}[x]$. Let $f$ have degree $d \geq 1$ and leading coefficient $c$. Write $D_f = |c|^{(d-1)(d-2)}|D_f|$, in which $D_f$ is the discriminant of $f$. For $x \geq \max\{2, |D_f|, \sqrt{|D_f|}\}$,

$$\left|\frac{1}{\log \log x} \sum_{p \leq x} \frac{\omega_f(p)}{p} - k\right| \leq \frac{dM_Q(|D_f|) + A + B(x) + C}{\log \log x},$$

in which

$$A \leq d(M_Q(|c|) + M_Q(\sqrt{|D_f|}) + 0.64),$$

$$B(x) \leq \frac{2}{\log x} \left( \frac{\Lambda \sqrt{|D_f|}}{0.36232} \left( 0.55d^2 + 44.86d \right) + 2d \right), \quad \text{and}$$

$$C \leq d \left( \gamma + 1.02d - 0.02 + \frac{d-1}{2} \log D_f \right).$$

Here $\gamma = 0.57721 \ldots$ is the Euler–Mascheroni constant, $M_Q$ is given by (1), and

$$\Lambda = e^{28.2d + 5(d + 1)^{\frac{d+4}{2}}} |D_f| (\log |D_f|)^d.$$

Theorem 2 produces a finite list of primes that certifies $f \in \mathbb{Z}[x]$ has exactly $k$ distinct irreducible factors: take $x \geq \max\{2, |D_f|, \sqrt{|D_f|}\}$ such that the right-hand side of (2) is less than 0.5. Although this is not yet practical, Theorem 2 is a valuable proof of concept, and several avenues for improvement are discussed in Section 5. Table 1 exhibits the behavior of the main term in Theorem 2.

The proof of Theorem 2 relies upon Theorem 1 and its proof requires recent work on explicit Mertens’ theorems for number fields and residue bounds for Dedekind zeta functions [7]. We must consider the fields $\mathbb{K}_i$ generated by the roots of $f_i$ and quantify the resulting error in terms of the degree, discriminant and leading coefficient of $f$. Moreover, each step must be uniform and explicit.

**Structure.** This paper is structured as follows. The preliminaries are covered in Section 2. Section 3 contains the proof of Theorem 1. The proof of Theorem 2 is the focus of Section 4. Finally, we consider future avenues of research in Section 5.
Table 1. $F(x) = \frac{1}{\log x} \sum_{p \leq x} \frac{\omega(p)}{p}$, the main term in Theorem 2.

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2. Preliminaries

In this section, we review results needed for the proof of Theorem 2. Subsection 2.1 concerns several important notions from elimination theory, while Subsection 2.2 presents explicit unconditional bounds on Dedekind zeta residues. We explore a Mertens-type sums for algebraic number fields in Subsection 2.3. Finally, Subsection 2.4 contains a few remarks about a function related to the prime zeta function.

2.1. Resultants and discriminants. Although the material in this subsection is classical, it is surprisingly difficult to find it all stated in one convenient reference. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$ be complex polynomials of positive degrees $n$ and $m$, respectively. Their **resultant** $R(f(x), g(x); x)$ is the determinant of the $(m+n) \times (m+n)$ **Sylvester matrix**

$$
\begin{bmatrix}
  a_n & 0 & \cdots & 0 & b_m & 0 & \cdots & 0 \\
  a_{n-1} & a_n & \cdots & 0 & b_{m-1} & b_m & \cdots & 0 \\
  a_{n-2} & a_{n-1} & \cdots & 0 & b_{m-2} & b_{m-1} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_0 & a_1 & \cdots & b_0 & b_1 & \cdots & \vdots & \vdots \\
  0 & a_0 & \cdots & 0 & b_0 & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & a_1 & \vdots & \cdots & \vdots & b_1 \\
  0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0
\end{bmatrix}
$$

(3)

We often suppress the variable $x$ and write $R(f, g)$ instead of $R(f(x), g(x); x)$. Since $[5]$ has $m$ columns with the coefficients of $f$ and $n$ columns with the coefficients of $g$, it follows that $R(\alpha f, \beta g) = \alpha^m \beta^n R(f, g)$ for $\alpha, \beta \in \mathbb{C}$. The definition ensures that $R(f, g) \in \mathbb{Z}$ whenever $f, g \in \mathbb{Z}[x]$.

We now need an important fact about polynomials. Suppose $f \in \mathbb{Z}[x]$ and $f = AB$, in which $A, B \in \mathbb{Q}[x]$. Gauss’ lemma ensures that $f = ab$, where $a, b \in \mathbb{Z}[x]$ are of the form $a = \mu A$ and $b = \mu^{-1} B$ for some $\mu \in \mathbb{Q} \setminus \mathbb{Z}$ [5, Prop. 5, p. 303].

**Lemma 3.** Let $f, g \in \mathbb{Z}[x]$ have positive degree. Then $R(f, g) = 0$ if and only if $f$ and $g$ have a common divisor in $\mathbb{Z}[x]$ of positive degree.

**Proof.** Let $P_i$ denote the $\mathbb{C}$-vector space of polynomials of degree at most $i - 1$. Then the matrix in (3) represents the linear map $(u, v) \mapsto uf + vg$ from $P_m \times P_n$ to $P_{m+n}$ with respect to the corresponding monomial bases (listed in order of
descending degree). Thus, $R(f, g) = 0$ if and only if $f, g$ have a common divisor in $\mathbb{C}[x]$ of positive degree, that is, if and only if $f, g$ share a common root in $\alpha \in \mathbb{C}$. Since $\alpha$ is algebraic and $f, g \in \mathbb{Z}[x]$, the preceding is equivalent to asserting that $f = m_\alpha F$ and $g = m_\alpha G$, in which $m_\alpha \in \mathbb{Q}[x]$ denotes the minimal polynomial of $\alpha$ over $\mathbb{Q}$ and $F, G \in \mathbb{Q}[x]$. The remarks from the preceding paragraph imply that $R(f, g) = 0$ if and only if $f, g$ have a common divisor in $\mathbb{Z}[x]$ of positive degree. \[\square\]

For $f, g \in \mathbb{C}[x]$ with leading coefficients $a_n$ and $b_m$, respectively,

$$R(f, g) = a_n^m b_m^n \prod_{1 \leq i, j \leq m} (\lambda_i - \mu_j),$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_m$ are the roots of $f$ and $g$ in $\mathbb{C}$, counted by multiplicity \[\text{[5] Ex. 31c, p. 621}]. For $c \in \mathbb{C}$ and $h \in \mathbb{C}[x]$, \[\text{[4]}\] implies

$$R(fg, h) = R(f, h)R(g, h) \quad \text{and} \quad R(f(cx), g(cx)) = c^{mn}R(f, g).$$ (5)

The discriminant

$$D(f; x) = \frac{(-1)^{n(n-1)/2}}{a_n} R(f(x), f'(x); x)$$

of $f \in \mathbb{C}[x]$ with degree $n$ and leading coefficient $a_n$ is a homogeneous polynomial of degree $2n - 2$ in the coefficients of $f$. The choice of sign ensures that $D(f; x) \geq 0$ if the roots of $f$ are real. We often write $D_f$ or $D_{f(x)}$ instead of $D(f; x)$.

If $f(\alpha) = 0$ and $D_\alpha \neq 0$, then $D_f \neq 0$. If $f(\alpha) = 0$, then $D_f = 0$, and $f(\alpha) = 0$. Lemma \[\text{[3]}\] implies that $f(\alpha) = 0$ if and only if $f$ and $f'$ share a common divisor in $\mathbb{Z}[x]$ of positive degree; that is, if and only if $f$ has a repeated root in $\mathbb{C}$. This yields the following.

**Lemma 4.** If $f \in \mathbb{Z}[x]$ is irreducible and has positive degree, then $D_f \neq 0$.

The definition and \[\text{[4]}\] yield

$$D_f = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2,$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of $f$, repeated according to multiplicity. For $\alpha \in \mathbb{C}$, \[\text{[5]}\] implies

$$D_{f(\alpha x)} = \alpha^n D_f(x) \quad \text{and} \quad D_{gh} = D_g D_h R_{g,h}^2.$$ (6)

If $f, g \in \mathbb{Z}[x]$ and $g|f$, then $D_f|D_f$. If, in addition, $D_f \neq 0$, then $|D_g| \leq |D_f|$.

**Lemma 5.** If $f = f_1 f_2 \cdots f_k \in \mathbb{Z}[x]$ is a product of distinct irreducible polynomials $f_1, f_2, \ldots, f_k \in \mathbb{Z}[x]$ of positive degree, then $R(f_i, f_j)|D_f$ for $i \neq j$.

**Proof.** Since $f_1, f_2, \ldots, f_k \in \mathbb{Z}[x]$ are distinct, irreducible, and have positive degree, Lemma \[\text{[3]}\] ensures that $R(f_i, f_j) \neq 0$ for $i \neq j$. Moreover, $D_f \neq 0$ since $f$ has no repeated factors. Induction and \[\text{[4]}\] imply $R(f_i, f_j)|D_f$ for $i \neq j$. \[\square\]

The final lemma of this subsection concerns the number of solutions $\omega_f(p)$ to $f(x) \equiv 0 \pmod{p}$ for a rational prime $p$. Note that if $f \in \mathbb{Z}[x]$ is reducible, then $\omega_f(p) > \deg f$ is possible. For example, consider $3x - 6$ modulo 3.

**Lemma 6.** If $f \in \mathbb{Z}[x]$ is irreducible and $\deg f \geq 1$, then $\omega_f(p) \leq \deg f$ for all $p$. 

Proof. If deg $f \geq p$, then $\omega_f(p) \leq p \leq \deg f$. Now suppose that $\deg f < p$. Since $f$ is irreducible, at least one coefficient of $f$ is nonzero modulo $p$, and hence $f$ has positive degree modulo $p$. Lagrange’s theorem ensures that $\omega_f(p) \leq \deg f$. \qed

2.2. Dedekind zeta residues. Let $K$ be a number field of degree $n_K$ with ring of algebraic integers $\mathcal{O}_K$, and let $\Delta_K$ denote the discriminant of $K$. In what follows, $p \subset \mathcal{O}_K$ denotes a prime ideal, $\mathfrak{a} \subset \mathcal{O}_K$ an ideal (not necessarily prime), and $N(\mathfrak{a})$ the norm of $\mathfrak{a}$. The Dedekind zeta function $\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} N(\mathfrak{a})^{-s}$ of $K$ is analytic on $\text{Re} s > 1$ and extends meromorphically to $\mathbb{C}$, except for a simple pole at $s = 1$. The analytic class number formula asserts that the residue of $\zeta_K(s)$ at $s = 1$ is

$$\kappa_K = \frac{2 \pi^r (2\pi)^{r_2} h_K R_K}{w_K \sqrt{|\Delta_K|}},$$

in which $r_1$ is the number of real places of $K$, $r_2$ is the number of complex places of $K$, $w_K$ is the number of roots of unity in $K$, $h_K$ is the class number of $K$, and $R_K$ is the regulator of $K$ \[9\]. The nontrivial zeros of $\zeta_K(s)$ lie in the critical strip $0 < \text{Re} s < 1$, in which there might also exist an exceptional, real zero $0 < \beta < 1$. The Generalized Riemann Hypothesis (GRH), which remains unproven, asserts that the nontrivial zeros of $\zeta_K(s)$ satisfy $\text{Re} s = \frac{1}{2}$ and that no exceptional zero exists.

We require unconditional bounds for the residue $\kappa_K$ of $\zeta_K(s)$ at $s = 1$. If $d = 1$, then $K = \mathbb{Q}$ and hence $\zeta_\mathbb{Q}(s)$ is the Riemann zeta function, for which $\kappa_\mathbb{Q} = 1$. Consequently, we restrict our attention to the case $d \geq 2$.

Lemma 7. Let $g \in \mathbb{Z}[x]$ be irreducible with degree $d \geq 2$ and leading coefficient $c$. If $g(\alpha) = 0$ and $K = \mathbb{Q}(\alpha)$, then

$$0.36232 \sqrt{D_g} \leq \kappa_K \leq \left( \frac{e \log D_g}{2(d-1)} \right)^{d-1},$$

(7)

in which $D_g = |c|^{(d-1)(d-2)}|D_g|$.

Proof. Let $g \in \mathbb{Z}[x]$ be irreducible with degree $d \geq 2$ and leading coefficient $c$. Then $c^{d-1} g(x) = h(cx)$, in which $h \in \mathbb{Z}[x]$ is monic, irreducible, and has degree $d$. Let $\alpha \in \mathbb{C}$ be such that $g(\alpha) = 0$. Then $h(c\alpha) = 0$ and $K = \mathbb{Q}(\alpha) = \mathbb{Q}(c\alpha)$: the degree $n_K$ of the number field $K$ is $d$. Since the discriminant of $c^{d-1} g(x)$ is $(c^{d-1})^{2d-2} D_g$ and the discriminant of $h(cx)$ is $c^{d(d-1)} D_h$, it follows that

$$D_h = c^{(d-1)(d-2)} D_g.$$  

(8)

From \[12\] Prop. I.2.12] observe that

$$1 \leq \left[ \mathcal{O}_K : \mathbb{Z}[c\alpha] \right] = \left( \frac{\det \mathbb{Z}[c\alpha]}{\det \mathcal{O}_K} \right)^2 = \frac{|D_h|}{|\Delta_K|},$$

and hence

$$|\Delta_K| \leq |D_h| = D_g.$$ \quad \text{(9)}

Since $n_K = d \geq 2$, Louboutin \[10\, Thm. 1\] provides the first inequality in

$$\kappa_K \leq \left( \frac{e \log |\Delta_K|}{2(n_K-1)} \right)^{n_K-1} \leq \left( \frac{e \log D_g}{2(d-1)} \right)^{d-1}.$$ \quad \text{(10)}
The lower bound follows from [7]; the argument is short and reproduced here. Since \( n_K = r_1 + 2r_2 \geq 2 \), we have \( 2^{r_1} (2\pi)^{r_2} \geq 2^2 (2\pi)^0 = 4 \). Friedman [6, Thm. B] proved that \( R_{K/w_K} \geq 0.09058 \), improving on Zimmert [20] (see [9, Thm. 7, p. 273]). Thus,  

\[
\kappa_K \geq \frac{2^{r_1} (2\pi)^{r_2} R_{K/w_K}}{\sqrt{|\Delta_K|}} > \frac{4 \cdot 0.09058}{\sqrt{|\Delta_K|}} = \frac{0.36232}{\sqrt{|\Delta_K|}} \geq \frac{0.36232}{\sqrt{D_g}}.
\]

This concludes the proof. □

Remark 8. With respect to \( D_g \), there are asymptotically superior lower bounds available (with explicit constants). For example, one can obtain  

\[
\kappa_K > \frac{0.015744605}{d!D_g^{1/d}}
\]

from Stark [16] by tracking the constants involved [7]. However, the dependence upon \( d = n_K \) is poor, so we use the degree-independent lower bound in [7] instead.

2.3. The function \( M_K(x) \). For a number field \( K \), define the Mertens-type sum  

\[
M_K(x) = \sum_{N(p) \leq x} \frac{1}{N(p)},
\]

where the sum runs over the prime ideals \( p \) of \( O_K \). If \( K = \mathbb{Q} \), then the sum reduces to the classical Mertens sum  

\[
M_\mathbb{Q}(x) = \sum_{p \leq x} \frac{1}{p} \leq \log \log x + C_\mathbb{Q} + \frac{1}{(\log x)^2},
\]

in which \( C_\mathbb{Q} = 0.26149\ldots \) is the Meissel–Mertens constant and (12) holds unconditionally for \( x > 1 \) [14 (3.20)]. For \( K \neq \mathbb{Q} \), the first two authors [7] used an ideal-counting estimate of Sunley [18, 19] to prove unconditionally for \( x \geq 2 \) that  

\[
M_K(x) = \log \log x + C_K + B_K(x),
\]

in which  

\[
\gamma + \log \kappa_K - d_K \leq C_K \leq \gamma + \log \kappa_K
\]

and  

\[
|B_K(x)| \leq \frac{2 \Upsilon_K}{\log x},
\]

where  

\[
\Upsilon_K = \left( \frac{(n_K + 1)^2}{2 \kappa_K (n_K - 1)} \Lambda_K + 1 \right) + \frac{0.55 \Lambda_K n_K (n_K + 1)}{\kappa_K} + n_K + 40.31 \frac{\Lambda_K n_K}{\kappa_K}
\]

and  

\[
\Lambda_K = e^{28.2n_K + 5(n_K + 1)} \frac{5(n_K + 1)}{2} |\Delta_K|^{-\frac{1}{n_K + 1}} (\log |\Delta_K|)^{n_K}.
\]

The Dedekind zeta residue \( \kappa_K \) in [10] can be explicitly bounded by Lemma [7].
2.4. The function $P(x)$. For $x \geq 1$, define

$$P(x) = \sum_{2 \leq k \leq x} \frac{P(k)}{k},$$

in which $P(s) = \sum_p p^{-s}$ is the prime zeta function. Then $P(1) = 0$ and $P(n)$ tends monotonically to 0.31571845... as $n \to \infty$; see Table 2. That $\lim_{x \to \infty} P(x)$ exists follows from an old result of Euler (1781) [15, (2.12), p. 133]:

$$P(x) < \sum_{k \geq 2} \frac{1}{k} \sum_p \frac{1}{p^k} < \sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=2}^{\infty} \frac{1}{n^k} = \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k} = 1 - \gamma.$$  

A convenient upper bound is $P(x) \leq 0.32$ for all $x \geq 2$.

3. Proof of Theorem $\star$

The proof of Theorem $\star$ requires the next lemma, which handles the monic case.

**Lemma 9.** Let $h(x) \in \mathbb{Z}[x]$ be monic and irreducible with degree $d \geq 1$. Let $\alpha \in \mathbb{C}$ be such that $h(\alpha) = 0$ and let $K = \mathbb{Q}(\alpha)$. For $x > \sqrt{|D_h|}$,

$$\left| M_{\mathbb{Z}}(x) - \sum_{p \leq x} \frac{\omega_h(p)}{p} \right| < (M_{\mathbb{Z}}(\sqrt{|D_h|}) + 0.64)d.$$

**Proof.** Suppose $h(x) \in \mathbb{Z}[x]$ is monic and irreducible with degree $d \geq 1$. Let $\alpha \in \mathbb{C}$ be such that $h(\alpha) = 0$ (that is, $h$ is the minimal polynomial of $\alpha$ over $\mathbb{Q}$) and consider the number field $K = \mathbb{Q}(\alpha)$, which has degree $d$ [17, p. 47]. Let $O_K$ denote the ring of algebraic integers in $K$; by construction it contains $\alpha$. For each rational prime $p$, the ideal $pO_K$ factors uniquely (up to reordering the factors) as

$$pO_K = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r},$$

in which $p_1, p_2, \ldots, p_r \subset O_K$ are distinct prime ideals [17, Thm. 5.6]. Then $N(p_i) = |O_K/\mathbb{Z}| = p^{f_i}$, in which $f_i$ is the inertia degree of $p$ at $p_i$, and

$$e_1f_1 + e_2f_2 + \cdots + e_rf_r = d;$$

see [12, Prop. 8.2, p. 46]. Since each prime ideal $p \subset O_K$ can occur in the factorization for only one rational prime $p$ and, moreover, $p \leq N(p) \leq p^d$ [17, Thm. 5.14c], the number of prime ideals of norm $p^k$ is at most $d/k$.

The Dedekind factorization criterion relates the factorization of $pO_K$ to the factorization of $h$ over $\mathbb{F}_p$ [3, Prop. 25, p. 27]. If $p \nmid [O_K : \mathbb{Z}[\alpha]]$, then

$$h(x) = h_1(x)^{e_1}h_2(x)^{e_2} \cdots h_r(x)^{e_r}$$

| $x$ | $P(x)$ | $x$ | $P(x)$ |
|-----|--------|-----|--------|
| 1   | 0      | 6   | 0.3136222260 |
| 2   | 0.2261237100 | 7   | 0.3148056307 |
| 3   | 0.2843779231 | 8   | 0.3153133064 |
| 4   | 0.3036262081 | 9   | 0.3155360250 |
| 5   | 0.3107772116 | 10  | 0.3156353854 |

Table 2. Approximate values of $P(x)$.  

in \(\mathbb{F}_p[x]\), where \(h_1, h_2, \ldots, h_r \in \mathbb{F}_p[x]\) are distinct and irreducible with \(\deg h_i = f_i\). If \(a \in \mathbb{F}_p\) and \(h(a) = 0\), then there is a unique index \(i\) such that \(h_i(x) = x - a\). Hence \(f_i = \deg h_i = 1\) and \(N(p_i) = p\). Conversely, \(N(p_i) = p\) implies \(\deg h_i = f_i = 1\) and hence \(h_i(x) = x - a\) for some unique \(a \in \mathbb{F}_p\). We conclude that \(\omega_h(p)\) equals the number of prime ideals of norm \(p\) in the prime ideal factorization of \(p\mathcal{O}_K\).

If \(p > \sqrt{|D_h|}\), then \(p \mid [\mathcal{O}_K : \mathbb{Z}[\alpha]]\) since

\[
[\mathcal{O}_K : \mathbb{Z}[\alpha]] = \frac{\det \mathbb{Z}[\alpha]}{\det \mathcal{O}_K} = \sqrt{\frac{|D_h|}{|\Delta_K|}} \leq \sqrt{|D_h|}
\]

by [12, Sec. 1.2]. Consequently, \(x \geq \sqrt{|D_h|}\) implies

\[
\sum_{p \leq x} \frac{\omega_h(p)}{p} = \sum_{p \leq \sqrt{|D_h|}} \frac{\omega_h(p)}{p} + \sum_{\sqrt{|D_h|} < p \leq x} \frac{\omega_h(p)}{p}
\]

\[
= \sum_{p \leq \sqrt{|D_h|}} \frac{\omega_h(p)}{p} + \sum_{\sqrt{|D_h|} < p \leq x} \frac{1}{N(p)}
\]

\[
= M_K(x) + \sum_{p \leq \sqrt{|D_h|}} \frac{\omega_h(p)}{p} - \sum_{N(p) \leq \sqrt{|D_h|}} \frac{1}{N(p)} - \sum_{\sqrt{|D_h|} < p' \leq x} \frac{1}{N(p')}.
\]

First, note that

\[
|\star| \leq \max \left\{ \frac{1}{N(p)} \sum_{p \leq \sqrt{|D_h|}} \frac{\omega_h(p)}{p}, \sum_{N(p) \leq \sqrt{|D_h|}} \frac{\omega_h(p)}{p} \right\}
\]

because the two sums in [18] are both nonnegative. Since \(P(y) \leq 0.32\) for \(y \geq 2\),

\[
\sum_{N(p) \leq \sqrt{|D_h|}} \frac{1}{N(p)} \leq \sum_{p' \leq \sqrt{|D_h|}} \frac{d/p'}{p'^2} = d \sum_{p \leq \sqrt{|D_h|}} \frac{1}{p} + d \sum_{2 \leq \ell \leq d} \frac{1}{\ell p'^\ell} \\
\leq d M_K(\sqrt{|D_h|}) + d P(\sqrt{|D_h|}) \\
\leq d (M_K(\sqrt{|D_h|}) + 0.32).
\]

Lemma [14] ensures that \(\omega_h(p) \leq \deg f_i\) for all \(p\). Thus,

\[
\sum_{p \leq \sqrt{|D_h|}} \frac{\omega_h(p)}{p} \leq d \sum_{p \leq \sqrt{|D_h|}} \frac{1}{p} = d M_K(\sqrt{|D_h|}).
\]

To bound [18], we take the greater of the two estimates above.

Since the number of prime ideals of norm \(p'^\ell\) is at most \(d/\ell\),

\[
|\star| = \sum_{\sqrt{|D_h|} < p' \leq x} \frac{1}{N(p')} \leq d \sum_{\ell=2}^d \frac{1}{\ell} \sum_{p \leq x} \frac{1}{p'^\ell} = d P(x) \leq 0.32 d.
\]
Combining these observations and using the triangle inequality yields the result. □

We are now ready for the proof of Theorem 1.

Proof of Theorem 1. Let \( g \in \mathbb{Z}[x] \) be irreducible with degree \( d \geq 1 \) and leading coefficient \( c \). Then

\[
c^{d-1}g(x) = h(cx),
\]

in which \( h \in \mathbb{Z}[x] \) is monic, irreducible, and has degree \( d \). Let \( \alpha \in \mathbb{C} \) be such that \( g(\alpha) = 0 \). Then \( h(c\alpha) = 0 \) and \( \mathbb{K} = \mathbb{Q}(\alpha) = \mathbb{Q}(c\alpha) \).

For \( p \nmid c \), (19) gives \( \omega_g(p) = \omega_h(p) \). For \( p \mid c \), Lemma 6 yields \( \omega_g(p), \omega_h(p) \leq d \). Thus,

\[
\left| \sum_{p \leq x} \frac{\omega_g(p)}{p} - \sum_{p \leq x} \frac{\omega_h(p)}{p} \right| \leq \sum_{p \mid c} \frac{\max\{\omega_g(p), \omega_h(p)\}}{p} \leq d \sum_{p \mid c} \frac{1}{p} \leq d \mathcal{M}_\mathbb{Q}(|c|).
\]

Since the discriminant of \( c^{d-1}g(x) \) is \( (c(d-1))^{2d-2}D_g \) and the discriminant of \( h(cx) \) is \( c^{d(d-1)}D_h \), it follows from (19) that

\[
|D_h| = |c|^{(d-1)(d-2)}|D_g| = D_g.
\]

For \( x \geq \sqrt{|D_h|} \), Lemma 9 provides

\[
\left| \mathcal{M}_\mathbb{K}(x) - \sum_{p \leq x} \frac{\omega_g(p)}{p} \right| = \left| \mathcal{M}_\mathbb{K}(x) - \sum_{p \leq x} \frac{\omega_h(p)}{p} \right| + \left| \sum_{p \leq x} \frac{\omega_h(p)}{p} - \sum_{p \leq x} \frac{\omega_g(p)}{p} \right|
\leq d\left( \mathcal{M}_\mathbb{Q}(|c|) + \mathcal{M}_\mathbb{Q}(\sqrt{|D_h|}) + 0.64 \right).
\]

To complete the proof, use (21) and rewrite the preceding in terms of \( D_g \). □

Remark 10. The term \( \mathcal{M}_\mathbb{Q}(|c|) \) in (1) can be improved. Return to (20) and observe that the maximal order of \( \sum_{p \mid c} 1/p \log \log |c| \) since if \( c = \prod_{p \leq y} p \), then \( c = e^{(1+o(1))y} \) and hence \( y = (1+\alpha(1)) \log c \). Consequently, \( \sum_{p \mid c} 1/p = \sum_{p \leq y} 1/p \sim \log \log y \sim \log \log \log c \). This can even be made explicit with [2] Cor. 2.1:

\[
\log c = \sum_{p \leq y} \log p = \theta(y) \leq (1+1.93378 \times 10^{-8})y.
\]

On the other hand, \( \mathcal{M}_\mathbb{Q}(|c|) \sim \log \log |c| \) so the dependence upon \( c \) is weak already and the improvement is not worth pursuing.

4. Proof of Theorem 2

The proof of Theorem 2 requires two additional lemmas (Subsection 4.5). After that, we estimate three quantities which arise: \( A \) in Section 4.2, \( B(x) \) in Section 4.3, and \( C \) in Section 4.4. Finally, we wrap things up in Section 4.5.
4.1. Preliminary lemmas. The next two lemmas set up the final estimates needed for the proof of Theorem 3. Recall that \( \omega_f(p) \) denotes the number of solutions to \( f(x) \equiv 0 \pmod{p} \). The first lemma concerns the additive structure of \( \omega_f(p) \).

**Lemma 11.** Suppose that \( f = f_1 f_2 \cdots f_k \in \mathbb{Z}[x] \) is a product of distinct, irreducible nonconstant polynomials \( f_1, f_2, \ldots, f_k \in \mathbb{Z}[x] \). If \( p > |D_f| \), then

\[
\omega_f(p) = \omega_{f_1}(p) + \cdots + \omega_{f_k}(p).
\]

**Proof.** Each zero of \( f \) in \( \mathbb{Z}/p\mathbb{Z} \) is a zero of some \( f_i \). Thus,

\[
\omega_f(p) \leq \omega_{f_1}(p) + \cdots + \omega_{f_k}(p)
\]
holds for any \( p \). Since every zero of \( f_i \) in \( \mathbb{Z}/p\mathbb{Z} \) is a zero of \( f \), it suffices to show that \( f_i \) and \( f_j \) have no common zeros in \( \mathbb{Z}/p\mathbb{Z} \) if \( i \neq j \) and \( p > |D_f| \). If \( i \neq j \), then there are \( u_{ij}, v_{ij} \in \mathbb{Z}[x] \) such that

\[
|u_{ij} f_i + v_{ij} f_j| = R(f_i, f_j),
\]
in which \( R(f_i, f_j) \in \mathbb{Z} \) is the resultant of \( f_i \) and \( f_j \). Since \( f_i, f_j \in \mathbb{Z}[x] \) are distinct and irreducible, they share no common factors over \( \mathbb{C} \), so \( R(f_i, f_j) \neq 0 \) [3 Prop. 5, Sec. 3.6]. If \( f_i, f_j \) have a common zero in \( \mathbb{Z}/p\mathbb{Z} \), then \( |u_{ij} f_i + v_{ij} f_j| = 0 \) (Lemma 4). Thus, \( p > |D_f| \) (Lemma 5), and hence \( p \leq |D_f| \) since \( D_f \neq 0 \) (Lemma 3). Thus, \( p > |D_f| \) implies \( \omega_f(p) \) is an equality. \( \square \)

The second lemma is an asymptotic estimate for a special sum over \( \omega_f(p) \).

**Lemma 12.** Suppose that \( f = f_1 f_2 \cdots f_k \in \mathbb{Z}[x] \) is a product of distinct, irreducible, nonconstant polynomials \( f_1, f_2, \ldots, f_k \in \mathbb{Z}[x] \) of degrees \( d_1, d_2, \ldots, d_k \geq 1 \), respectively. For \( x \geq \sqrt{|D_f|} \),

\[
\left| \sum_{p \leq x} \frac{\omega_f(p)}{p} - k \log \log x \right| \leq d \mathcal{M}_Q(|D_f|) + A + B(x) + C,
\]

in which

\[
d = d_1 + d_2 + \cdots + d_k,
\]

and

\[
A = \sum_{i=1}^{k} |A_{f_i}|, \quad B(x) = \sum_{i=1}^{k} |B_{K_i}(x)|, \quad \text{and} \quad C = \sum_{i=1}^{k} |C_{K_i}|.
\]

Here \( A_{f_i}, B_{K_i}(x), \) and \( C_{K_i} \) are given by \( \{1\}, \{15\}, \) and \( \{13\} \), respectively.

**Proof.** For each \( i = 1, 2, \ldots, k \), let \( \alpha_i \) be a root of \( f_i \) and define \( K_i = \mathbb{Q}(\alpha_i) \), which has degree \( d_i \) since each \( f_i \) is irreducible. Observe that \( D_f \neq 0 \) (since \( f \) has no repeated roots in \( \mathbb{C} \))

\[
k \leq d \quad \text{and} \quad 1 \leq |D_{f_1}|, |D_{f_2}|, \ldots, |D_{f_k}| \leq |D_f|.
\]

Without loss of generality, let \( c \geq 1 \) denote the leading coefficient of \( f \) and let \( c_1, c_2, \ldots, c_k \geq 1 \) denote the leading coefficients of \( f_1, f_2, \ldots, f_k \), respectively. Then \( c = c_1 c_2 \cdots c_k \) and \( c_1, c_2, \ldots, c_k \leq c \).

If \( x > |D_f| \), then Lemma \( \{14\} \) implies

\[
\sum_{p \leq x} \frac{\omega_f(p)}{p} = \sum_{p \leq |D_f|} \frac{\omega_f(p)}{p} + \sum_{|D_f| < p \leq x} \frac{\omega_f(p)}{p}.
\]
\[ \sum_{p \leq |D_f|} \frac{\omega_f(p)}{p} + \sum_{|D_f| < p \leq x} \left( \sum_{i=1}^{k} \frac{\omega_f_i(p)}{p} - \sum_{p \leq |D_f|} \frac{\omega_f(p)}{p} \right) \]

## Term I

Observe that

\[ |I| = \left| \sum_{p \leq |D_f|} \frac{\omega_f(p) - (\omega_f_1(p) + \omega_f_2(p) + \cdots + \omega_f_k(p))}{p} \right| \]

\[ \leq \sum_{p \leq |D_f|} \frac{\omega_f_1(p) + \omega_f_2(p) + \cdots + \omega_f_k(p)}{p} \quad \text{by (22)} \]

\[ \leq \sum_{p \leq |D_f|} \frac{d_1 + d_2 + \cdots + d_k}{p} \quad \text{by Lemma [0]} \]

\[ \leq \sum_{p \leq |D_f|} \frac{d}{p} \quad \text{by (25)} \]

\[ \leq d M_Q(|D_f|). \]

## Term II

Since \(|c_i| \leq |c|\) and \(d_i \leq d\), it follows that

\[ D_{f_i} \leq D_f \quad \text{(26)} \]

for \(i = 1, 2, \ldots, k\). For \(x \geq \sqrt{D_f} \geq \max_{1 \leq i \leq k} \sqrt{D_{f_i}}\), Theorem [1] ensures that

\[ II = \sum_{i=1}^{k} \sum_{p \leq x} \frac{\omega_f(p)}{p} \]

\[ = \sum_{i=1}^{k} (M_{Q_i}(x) + A_{f_i}) \quad \text{by Theorem [1]} \]

\[ = k \log \log x + \sum_{i=1}^{k} (A_{f_i} + B_{K_i}(x) + C_{K_i}) \quad \text{by (13)} \]

\[ = k \log \log x + A + B(x) + C. \]

Putting this all together, we obtain \(\square\).

To complete the proof of Theorem [2] we must estimate \(A\), \(B(x)\), and \(C\).

### 4.2. Estimating \(A\)

Since \(M_Q\) and \(P\) are increasing, Theorem [1] and (26) provide

\[ A = \sum_{i=1}^{k} |A_{f_i}| \leq \sum_{i=1}^{k} d_i \left( M_Q(|c_i|) + M_Q(\sqrt{D_{f_i}}) + 0.64 \right) \]


\[
\leq \left( \sum_{i=1}^{k} d_i \right) \left( M_Q(|c|) + M_Q(\sqrt{D_f}) + 0.64 \right) \\
\leq d \left( M_Q(|c|) + M_Q(\sqrt{D_f}) + 0.64 \right).
\]

4.3. Estimating \( B(x) \). The degree \( d_i \) of each irreducible factor \( f_i \) of \( f \) equals the degree \( n_{K_i} \) of the number field \( K_i \) generated by a root of \( f_i \); that is, \( n_{K_i} = d_i \) for \( i = 1, 2, \ldots, k \). Suppose for now that each \( d_i \geq 2 \). Then (6) and (17) imply

\[
\Lambda_{K_i} = e^{28.2n_{K_i}+5} (n_{K_i} + 1) \frac{5n_{K_i}+5}{2} |\Delta_{K_i}|^{\frac{1}{n_{K_i}}} (\log |\Delta_{K_i}|)^{n_{K_i}} \leq e^{28.2d_i+5} (d_i + 1) \frac{5d_i+5}{2} |D_{f_i}|^{\frac{1}{d_i}} (\log |D_{f_i}|)^{d_i} \leq e^{28.2d_i+5} (d_i + 1) \frac{5d_i+5}{2} |D_f| (\log |D_f|)^d
\]

since \( D_{f_i} \mid D_f \) and \( D_f \neq 0 \) (see Lemma 3 and the discussion below (6)). Define

\[
\Lambda = e^{28.2d+5} (d + 1) \frac{5d+5}{2} |D_f| (\log |D_f|)^d,
\]

which satisfies \( 0 \leq \Lambda_{K_i} \leq \Lambda \) for \( i = 1, 2, \ldots, k \) and vanishes if \( D_f = 1 \).

From (14), (15), and (16),

\[
|B_{K_i}| \leq \frac{2}{\log x} \left[ \left( \frac{(n_{K_i} + 1)^2}{2n_{K_i}(n_{K_i} - 1)} \Lambda_{K_i} + 1 \right) + 0.55 \Lambda_{K_i} n_{K_i} + 40.31 \Lambda_{K_i} \right]
\leq \frac{2}{\log x} \left[ \frac{\Lambda}{K_{K_i}} \left( \frac{(n_{K_i} + 1)^2}{2(n_{K_i} - 1)} + 0.55 n_{K_i} + 40.31 n_{K_i} \right) + 1 + n_{K_i} \right]
\leq \frac{2}{\log x} \left[ \frac{\Lambda \sqrt{D_f}}{0.36232} \left( \frac{(d_i + 1)^2}{2(d_i - 1)} + 0.55 d_i(d_i + 1) + 40.31 d_i \right) + 1 + d_i \right]
\leq \frac{2}{\log x} \left[ \frac{\Lambda \sqrt{D_f}}{0.36232} \left( 0.5(d_i + 7) + 0.55 d_i(d_i + 1) + 40.31 d_i \right) + 1 + d_i \right]
\]

for \( d_i \geq 2 \) and \( x \geq 2 \). In the final inequality, we used the fact that

\[
\frac{(x + 1)^2}{2(x - 1)} \leq \frac{x + 7}{2} \quad \text{for} \quad x \geq 2.
\]

This inequality is equivalent to \( 2 - 2(x - 1)^{-1} \geq 0 \), which holds for \( x \geq 2 \). We use this argument to ensure that the choice \( d_i = 1 \) is permissible below.

If \( d_i = 1 \), then \( K_i = \mathbb{Q} \) and \( D_{f_i} = 1 \) (the discriminant of a nonconstant linear polynomial is 1). Recall from (12) that \( M_Q(x) = \log \log x + C_Q + B_Q(x) \), in which \( C_Q = 0.2614972 \ldots \) is the Meissel–Mertens constant and \( |B_Q(x)| \leq 2(\log x)^{-2} \) for \( x > 1 \) (13), (3.17), (3.20)). Since

\[
\frac{2}{(\log x)^2} \leq \frac{4}{\log x}, \quad \text{for} \quad x \geq 2,
\]

it follows that (24) holds if \( d_i = 1 \) (even if \( |D_f| = 1 \), so \( \Lambda = 0 \)). Thus, (25) yields

\[
B(x) = \sum_{i=1}^{k} |B_{K_i}(x)| \\
\leq \sum_{i=1}^{k} \frac{2}{\log x} \left[ \frac{\Lambda \sqrt{D_f}}{0.36232} \left( 0.5(d_i + 7) + 0.55 d_i(d_i + 1) + 40.31 d_i \right) + 1 + d_i \right]
\]
\[
\leq \frac{2}{\log x} \left( \sum_{i=1}^{k} \left[ \frac{\Lambda \sqrt{D_f}}{0.36232} (0.5d_i + 3.5 + 0.55 d_i(d + 1) + 40.31d_i) \right] + k + d \right)
\]

\[
\leq \frac{2}{\log x} \left( \left[ \frac{\Lambda \sqrt{D_f}}{0.36232} (0.5d + 3.5k + 0.55 d(d + 1) + 40.31d) \right] + 2d \right)
\]

\[
= \frac{2}{\log x} \left( \left[ \frac{\Lambda \sqrt{D_f}}{0.36232} (0.55d^2 + 44.86d) \right] + 2d \right).
\]

4.4. Estimating $C$. Recall from (14) that

\[
\gamma - d_i + \log \kappa_{K_i} \leq C_{K_i} \leq \gamma + \log \kappa_{K_i},
\]

for $i = 1, 2, \ldots, k$. Therefore,

\[
|C_{K_i}| \leq \gamma + d_i + |\log \kappa_{K_i}|.
\]

For $d_i \geq 2$, the residue bounds (10), (11), and the inequality (26), provide

\[
\frac{0.36232}{\sqrt{D_f}} \leq \frac{0.36232}{\sqrt{D_{f_i}}} \leq \kappa_{K_i} \leq \left( \frac{e \log D_{f_i}}{2(d_i - 1)} \right)^{d_i-1} \leq \left( \frac{e \log D_f}{2} \right)^{d-1}.
\]

If $d_i = 1$, then $K_i = Q$, $\kappa_{K_i} = 1$, $D_{f_i} = 1$, and $D_f \geq D_{f_i} \geq 1$. Thus,

\[
\frac{0.36232}{\sqrt{D_f}} \leq \kappa_{K_i} \leq \max \left\{ \left( \frac{e \log D_f}{2} \right)^{d-1} \right\}
\]

where the maximum in (28) is 1 for $D_f = 1, 2$ only. Using (28), for $i = 1, 2, \ldots, k$ and $d_i \geq 1$, we have

\[
|\log \kappa_{K_i}| \leq \max \left\{ \log \left( \frac{\sqrt{D_f}}{0.36232} \right), (d-1) \log \max \left\{ \left( \frac{e \log D_f}{2} \right)^{d-1} \right\} \right\}
\]

\[
\leq \left\{ \begin{array}{ll}
\max \left\{ 1.01523 + \frac{1}{2} \log D_f, 0 \right\} & \text{if } D_f = 1, 2, \\
\max \left\{ 1.01523 + \frac{1}{2} \log D_f, (d-1) \left( \log \frac{36232}{D_f} + \log \log D_f \right) \right\} & \text{if } D_f \geq 3,
\end{array} \right.
\]

\[
\leq (d-1) \left( 1.01523 + \frac{1}{2} \log D_f \right) + \left\{ \begin{array}{ll}
\max \left\{ 1.01523 + \frac{1}{2} \log D_f, 0.30686 + \frac{1}{2} \log D_f \right\} & \text{if } D_f = 1, 2, \\
\max \left\{ 1.01523 + \frac{1}{2} \log D_f, (d-1) \left( \log \frac{36232}{D_f} + \log \log D_f \right) \right\} & \text{if } D_f \geq 3,
\end{array} \right.
\]

\[
\leq (d-1) \left( 1.02 + \frac{1}{2} \log D_f \right),
\]

since $\log D_f < \sqrt{D_f}$ because $D_f \geq 1$.

Since $k \leq d$, and because of (28), we conclude

\[
C = \sum_{i=1}^{k} |C_{K_i}| \leq \sum_{i=1}^{k} (\gamma + d_i + |\log \kappa_{K_i}|) \leq d \gamma + d + (d-1) \sum_{i=1}^{k} \left( 1.02 + \frac{1}{2} \log D_f \right)
\]

\[
\leq d \left( \gamma + 1 + (d-1)1.02 + \frac{d-1}{2} \log D_f \right)
\]

\[
= d \left( \gamma + 1.02d - 0.02 + \frac{d-1}{2} \log D_f \right).
\]
4.5. Wrapping things up. Now return to (24) and use the bounds on $A$, $B(x)$, and $C$ obtained in Subsections 4.2, 4.3, and 4.4, respectively. This yields the desired bound (2) of Theorem 2 and completes the proof.

□

5. Conclusion

A natural number can be proven composite without factoring it. In a similar manner, Theorem 2 permits us to certify the number of distinct irreducible factors of $f \in \mathbb{Z}[x]$; compute how large $x$ must be so that the error in (2) is less than $\frac{1}{2}$.

We have not made a concerted effort to obtain the best possible error bounds. Our main point is that recent advances on explicit Mertens’ theorems for number fields [7] finally permit completely explicit results such as our Theorems 1 and 2. Some improvements are required to make this approach computationally feasible.

5.1. The Generalized Riemann Hypothesis. The Generalized Riemann Hypothesis will lead to much stronger error terms in the number-field analogues of Mertens’ theorems. In particular, Theorem 2 might be brought into the realm of computational feasibility, especially if done in conjunction with the following idea.

5.2. Alternate for Theorem 2 One might use a more rapidly divergent series in place of $\sum_{N(p) \leq x} N(p)^{-1} \sim \log \log x$. For $x \geq 2$, the first two authors [7] proved

$$\left| \sum_{N(p) \leq x} \frac{\log N(p)}{N(p)} - \log x \right| \leq \Upsilon_K,$$

with $\Upsilon_K$ as in (16). One hopes for an effective version of Theorem 2 of the form

$$k = \frac{1}{\log x} \sum_{p \leq x} \frac{\omega_f(p) \log p}{p} + O\left( \frac{1}{\log x} \right).$$

This possibility is suggested by an inexplicit result of Nagell [11], whose error term was improved by Bantle [1]:

$$\sum_{p \leq x} \frac{\omega_f(p) \log p}{p} = \log x + O(1), \quad (29)$$

Diamond–Halberstam provide an alternate proof [4, Prop. 10.1], and they use this result in their sieve methods. Moreover, Halberstam–Richert use a generalized version (see equation $(\Omega(\kappa, L))$ on [8, p. 142]) of (29) throughout their treatise on sieve methods [8]. Therefore, results like Theorem 1 are of independent interest, for example, in the development of explicit sieve methods.

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