STRONG SOLUTION TO THE MULTIDIMENSIONAL STOCHASTIC BURGERS EQUATION

EVELINA SHAMAROVA AND ALBERTO OHASHI

Abstract. We prove the existence and uniqueness of a global strong adapted solution to the multidimensional stochastic Burgers equation in the space $C([0,T] \times \mathbb{R}^n)$ without gradient-type assumptions on the force or the initial condition. The solution is $C^2$ in $x \in \mathbb{R}^n$ and $\alpha$-Hölder continuous in $t \in [0,T]$ for some $\alpha < \frac{1}{2}$. Our approach is based on an interplay between forward-backward SDEs and PDEs. Moreover, we show that as the viscosity goes to zero, the solution of the viscous stochastic Burgers equation converges uniformly to the local strong adapted solution of the inviscid stochastic Burgers equation.

1. Introduction

In this article, we study the multidimensional stochastic Burgers equation

$$y(t,x) = h(x) + \int_0^t \left[ \nu \Delta y(s,x) - (y, \nabla)y(s,x) + f(s,x,y) \right] ds + \int_0^t g(s,x) dB_s$$

(1)

on $[0,T] \times \mathbb{R}^n$, where $h$ is the random initial condition, $\nu > 0$ is the viscosity, and $B_t$ is a $d$-dimensional Brownian motion. We obtain the existence and uniqueness of a global strong adapted solution to (1) in the space $C([0,T],\mathbb{R}^n)$ without any gradient-type assumptions on $f$, $g$, and $h$. Our results also hold for the case when $\mathbb{R}^n$ is replaced with the $n$-dimensional torus.

In the last two decades much activities have been focused on the problem of Burgers turbulence [3, 5, 6, 7, 16, 19, 20], that is, the study of solutions to a Burgers equation with a random initial condition or force. The interest in Burgers turbulence is motivated by its emerging applications in cosmology, fluid dynamics, superconductors, etc. Zel’dovich [28] proposed to use the multidimensional Burgers equation to study the formation of large scale structures in the Universe. Kardar, Parisi, and Zhang [23] showed that the Burgers equation with the random force can be used to study the dynamics of interfaces. Blatter et al. [8] used the Burgers equation to model vortices in high-temperature superconductors. An informative survey on Burgers turbulence is contained in [5].

In the deterministic case, the global existence and uniqueness of a classical solution to the multidimensional Burgers equation is known due to the results of Ladyzhenskaya et al. [25], and follows as a particular case of a more general theory for systems of quasilinear parabolic PDEs. In [25], the initial-boundary value problem on a bounded domain is studied in Chapter VII, while the existence and

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uniqueness for a Cauchy problem follows from the diagonalization argument described in Chapter V (see also Theorem 1 below).

As for the stochastic case, due to the motivation by Burgers turbulence and the asymptotics for many nonlinear dissipative systems, the one-dimensional stochastic Burgers equation has been intensely investigated over the last two decades in a variety of contexts and based on different techniques. The literature is vast, so we refer the reader to the series of works [9, 11, 12, 15, 17], and references therein. The stochastic multidimensional case, when the force and the initial data are of the gradient form, has been investigated by some authors. Iturriaga and Khanin [21] studied stationary solutions for a multidimensional spatially periodic inviscid Burgers equation with a random force of the gradient form. Boritchev [3] considered a multidimensional generalized spatially periodic stochastic Burgers equation under assumption that the stochastic force and the solution itself are of the gradient form. Other attempts have been made in the gradient case by interpreting solutions as stochastic distributions in the sense of white-noise analysis (see, e.g., Assing [2] and references therein). Frequently, the analysis in the multidimensional gradient case can be simplified due to the Cole-Hopf transformation [26]. We remark that unlike the aforementioned articles as well as most of the literature on the stochastic Burgers equation, in the present article, we consider the non-gradient case for both, the stochastic force and the initial condition.

To our knowledge, the only work dealing with the non-gradient force and the initial condition is the article by Brzezniak et al. [10], where the authors prove the existence and uniqueness of a global strong solution to the multidimensional stochastic Burgers equation in the $L_p$ space with the number $p$ bigger than the space dimension. We emphasize that in our work, the unique global strong solution to the multidimensional stochastic Burgers equation is proved to exist in $C([0,T] \times \mathbb{R}^n)$. Moreover, the solution is twice continuously differentiable in $x$, $\alpha$-Hölder continuous in $t$ with $\alpha < \frac{1}{2}$ (both in the classical sense), and adapted.

Our strategy is based on the use of forward-backward SDEs (FBSDEs), associated to a random PDE equivalent to (1), and on some results on quasilinear parabolic PDEs and a priori estimates from the monograph of Ladyzhenskaya et al. [25]. The stochastic and deterministic techniques complement each other allowing to deal with solutions non-differentiable in time, arising from the FBSDEs, while PDEs provide a priori estimates needed for the global existence. Additionally, FBSDEs are used for various limiting procedures and for obtaining the adaptedness of the solution. We remark that our approach essentially differs from the approach of [10].

In this work, we also study the local inviscid limit of (1). The vanishing viscosity in hydrodynamics problems, even on a short time interval, has always been of interest. As such, Ebin and Marsden [14] proved the convergence of local Sobolev-space-valued solutions of the Navier-Stokes equation to local solutions of the Euler equation. Golovkin [18] and Ladyzhenskaya [24] obtained the aforementioned convergence uniformly in space and time. It is known that, even if the initial data and the force are smooth, the inviscid Burgers equation develops discontinuities (shocks) at a finite time, and, therefore, fails to have a global classical solution. Thus, one cannot expect a global uniform approximation of inviscid solutions by
viscous. We remark that our result on the local inviscid limit confirms the numerical evidence reported in the physics literature on Burgers turbulence (see, e.g., Bec and Khanin [5], and references therein).

The inviscid limit of (1) is, again, studied by means of the associated random forward.backward system. Namely, for the inviscid stochastic Burgers equation, we obtain the existence and uniqueness of a local strong adapted solution differentiable in $x$ and $\alpha$-Hölder continuous in $t$ with $\alpha < \frac{1}{2}$, and, most importantly, the uniform convergence (in $(t, x)$, as $\nu \to 0$) of the solutions of (1) to the local inviscid solution.

The organization of our paper is as follows. Section 2 is dedicated to the multi-dimensional stochastic Burgers equation. First, we reduce SPDE (1) to a random PDE. Then, in Subsection 2.1, we state the existence and uniqueness theorem for the multidimensional stochastic Burgers equation. First, we reduce SPDE (1) to a random PDE. Then, in Subsection 2.1, we state the existence and uniqueness theorem for the random PDE when the random parameter $\omega$ is frozen. Further, in Subsection 2.2, we prove the existence of a local adapted solution to the random PDE by means of the associated FBSDEs. Finally, in Subsections 2.3 and 2.4, we prove the global existence and uniqueness. Section 3 is dedicated to the study of the local inviscid limit of viscous stochastic Burgers equation (1).

2. Global existence and uniqueness for the stochastic Burgers equation

In this section, we show that under assumptions (A1)-(A3) below, SPDE (1) possesses a unique global strong adapted solution $y(t, x)$ which is $C^2$ in $x$ and $\frac{\beta}{2}$-Hölder continuous in $t$ with $\beta \in (0, 1)$. Since the spatially periodic case will follow as a consequence of uniqueness, we will consider (1) only on $[0, T] \times \mathbb{R}^n$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{F}_0$ be a $\sigma$-algebra independent of the natural filtration of $B_t$ such that $h(x), x \in \mathbb{R}^n$, is $\mathcal{F}_0$-measurable. Further let

$$\mathcal{F}_t^B = \sigma\{B_s, 0 \leq s \leq t\} \lor \mathcal{F}_0 \lor \mathcal{N}$$

be the filtration generated by $B_t$, $\mathcal{F}_0$, and augmented with $\mathbb{P}$-null sets $\mathcal{N}$.

Assume the following:

(A1) $f(t, x, y)$ is an $\mathbb{R}^n$-valued deterministic function of class $C^{\frac{\beta}{2}+2}_0([0, T] \times \mathbb{R}^{2n})$ (see Remark 2), i.e., it is $C^2$ in $(x, y)$ and $\frac{\beta}{2}$-Hölder continuous in $t$, $\beta \in (0, 1)$. Moreover, there exists a constant $L > 0$ such that $|f(t, x, y)| \leq L(1 + |y|)$, and $|f(t, x, y) - f(t, x, y')| \leq L|y - y'|$ for all $t \in [0, T]$, $x, y, y' \in \mathbb{R}^n$.

Further, the first and second order derivatives of $f$ in $x$ and $y$ are bounded on $[0, T] \times \mathbb{R}^n \times [-D, D]^n$ for any constant $D > 0$. Furthermore, $f(0, x, 0)$ is compactly supported.

(A2) $g(t, \cdot) = (g_1(t, \cdot), \ldots, g_d(t, \cdot)), t \in [0, T]$, is an $\mathcal{F}_t^B$-adapted stochastic process with values in the Sobolev space $H^k(\mathbb{R}^{2n})$, $k = \left[\frac{n}{2}\right] + 5$, such that $E \int_0^T \|g(s, \cdot)\|_{H^k(\mathbb{R}^{2n})}^p ds < \infty$ for some $p > \frac{4}{1-\beta}$. For each $x \in \mathbb{R}^n$, the value $g_i(t, x), i = 1, 2, \ldots, d$, is defined due to the Sobolev embedding $H^k(\mathbb{R}^n) \hookrightarrow C^{2+\beta}_0(\mathbb{R}^n)$.

(A3) For each $x \in \mathbb{R}^n$, $h(x)$ is an $\mathcal{F}_0$-measurable random variable such that the function $h : \mathbb{R}^n \to \mathbb{R}^n$ belongs to the Hölder space $C^{2+\beta}_0(\mathbb{R}^n)$ for all $\omega \in \Omega$.

Remark 1. By the Sobolev embedding theorem [1] (Theorem 5.4, p. 97), $H^k(\mathbb{R}^n) \hookrightarrow C^{2+\beta}_0(\mathbb{R}^n)$. Therefore, in Assumption (A2), $g_i(t, x)$ is viewed as an element of $C^{2+\beta}_0(\mathbb{R}^n)$ due to this embedding.
Remark 2. The space $C^d_b([0,T] \times \mathbb{R}^m)$, where $l > 0$ is an integer, is defined as the (Banach) space of functions $\varphi(t, u)$ possessing the finite norm

$$
\|\varphi\|_{C^d_b([0,T] \times \mathbb{R}^m)} = \|\varphi\|_{C^0_b([0,T] \times \mathbb{R}^m)} + \sup_{u \in \mathbb{R}^m} |\varphi|^2_f,
$$

where the Hölder constant $[\varphi]^2_f$ is defined as

$$
[\varphi]^2_f = \sup_{t,t' \in [0,T], t \neq t'} \frac{|\varphi(t, u) - \varphi(t', u)|}{|t - t'|^\frac{2}{f}},
$$

and $C^0_b([0,T] \times \mathbb{R}^m)$ is the space of functions $\varphi(t, u)$, which are bounded and continuous in $(t, u) \in [0,T] \times \mathbb{R}^m$ together with their derivatives with respect to $u$ up to the $l$th order.

Remark 3. The requirements that $f(0, x, 0)$ is compactly supported is needed to satisfy a compatibility condition (on the boundary) in initial-boundary value problem (6) below.

Remark 4. The Hölder space $C^{2+\beta}(\mathbb{R}^n)$, $\beta \in (0, 1)$, is understood as the (Banach) space with the norm

$$
\|h\|_{C^{2+\beta}(\mathbb{R}^n)} = \|h\|_{C^2(\mathbb{R}^n)} + [\nabla^2 h]_\beta, \text{ where } [\psi]_\beta = \sup_{x,y \in \mathbb{R}^n, 0 < |x-y|<1} \frac{|\psi(x) - \psi(y)|}{|x-y|^{\beta}}.
$$

Lemma 1. Under Assumption (A2), there is a version $\eta(t, x)$ of the stochastic integral $\int_0^t g(s, x)dB_s$ which belongs to the space $C^{2,4}_b([0,T] \times \mathbb{R}^n)$.

Proof. Let $\tilde{\eta}(t, \cdot)$ be a $\frac{2}{d}$-Hölder continuous version of the stochastic integral $\int_0^t g(s, \cdot)dB_s$, considered in the Hilbert space $H^k(\mathbb{R}^n)$. Namely, there exists a random variable $K(\omega) > 0$ such that

$$
\|\tilde{\eta}(t, \cdot) - \tilde{\eta}(t', \cdot)\|_{H^k(\mathbb{R}^n)} \leq K(\omega)|t - t'|^{\frac{2}{d}} \text{ a.s.}
$$

The process $\eta(t, \cdot)$ exists by Kolmogorov’s continuity theorem in Hilbert spaces ([22], p. 31). Indeed, it suffices to note that by the Burkholder-Gundy-Davis inequality,

$$
\mathbb{E}\left|\int_0^t g(s, \cdot)dB_s - \int_0^{t'} g(s, \cdot)dB_s\right|^p_{H^k(\mathbb{R}^n)} \leq c_p |t - t'|^{\frac{2}{d} - 1 - \frac{1}{2p}} \mathbb{E}\left|\int_0^T g(s, \cdot)\right|^p_{H^k(\mathbb{R}^n \times \mathbb{R}^n)} ds,
$$

where $p > \frac{4}{1-d}$ is the number from Assumption (A2).

Let $\pi_x$ be the evaluation map $\mathbb{C}(\mathbb{R}^n) \to \mathbb{R}^n$, $\pi_x \varphi = \varphi(x)$, and let $j$ denote the Sobolev embedding $H^k(\mathbb{R}^n) \to C^0_b(\mathbb{R}^n)$. Define $\eta(t, x) = j \tilde{\eta}(t, \cdot)|_x$. Since the operator $\pi_x \circ j : H^k(\mathbb{R}^n) \to \mathbb{R}^n$ is continuous (and its norm is bounded by the embedding constant), we obtain that for each $x$, $\eta(t, x) = \int_0^t g(s, x)dB_s$. Moreover, for each $t \in [0, T]$, $\eta(t, \cdot) \in C^0_b(\mathbb{R}^n)$ a.s., and as a continuous function $[0, T] \to C^0_b(\mathbb{R}^n)$, $\eta \in C^0_b([0, T], \mathbb{R}^n)$ a.s. Further, by (2), $\sup_{x \in \mathbb{R}^n} |\eta|^4_f$ is bounded. This implies that $\eta \in C^{2,4}_b([0,T] \times \mathbb{R}^n)$ a.s. $\square$

Remark 5. According to Lemma 1, there exists a set of full $\mathbb{P}$-measure where $\eta$ belongs to class $C^{2,4}_b([0, T] \times \mathbb{R}^n)$. In what follows, this set will be denoted by $\Omega_0$. 
Lemma 2. The substitution

$$\hat{y}(t, x) = y(t, x) - \eta(t, x)$$

transforms (1) to the following Burgers-type equation with random $F_t^B$-adapted coefficients:

$$\begin{align*}
\partial_t \hat{y}(t, x) &= \nu \Delta \hat{y}(t, x) - (\eta(t, x) + \hat{y}, \nabla)\hat{y}(t, x) + F(t, x, \hat{y}), \\
\hat{y}(0, x) &= h(x),
\end{align*}$$

where

$$F(t, x, \hat{y}) = f(t, x, \hat{y} + \eta(t, x)) + \nu \Delta \eta(t, x) - (\hat{y} + \eta, \nabla)\eta(t, x).$$

Proof. The proof is straightforward. \qed

In what follows, we will be concerned with both Problems (1) and (4) on $\Omega_0$. First, we consider the case when $\eta(t, x)$ and $h(x)$ are bounded uniformly in $\omega \in \Omega_0$ in the norms of $C^{2,4}_b$ and $C^2_b$, respectively. It will be referred to below as the bounded case. The case when we are in Assumptions (A1)–(A3) will follow by a limiting procedure and will be referred to below as the general case.

The results of Subsections 2.2 and 2.3 hold only in the bounded case. Our main result is Theorem 4 on the global existence and uniqueness of the strong solution to (1), proved in Subsection 2.4 under Assumptions (A1)–(A3), will follow from the results of Subsections 2.1, 2.2, and 2.3.

2.1 Global existence and uniqueness for a frozen $\omega$

The results of this subsection are due to Ladyzhenskaya et al. [25].

Consider problem (4) with the function $F$ not necessary given by (5) and satisfying some regularity assumptions. Define the Hölder constant

$$[F]^{\beta}_y = \sup_{x, x' \in \mathbb{R}^n, 0 < |x - x'| < 1} \frac{|F(t, x, y) - F(t, x', y)|}{|x - x'|^{\beta}}.$$ 

$[F]^{\beta}_y$ is defined in similarly to $[F]^{\beta}_x$. We have the following result.

Theorem 1. Let the random functions $F$, $\eta$, and $h$ satisfy assumptions (i)–(iii) on $\Omega_0$.

(i) $h : \mathbb{R}^n \to \mathbb{R}^n$ belongs to the Hölder space $C^{2+\beta}_b(\mathbb{R}^n)$.

(ii) The partial derivatives $\partial_t F$, $\partial_x F$, $\partial_y F$, $\partial_t \eta$, $\partial_x \eta$ are continuous.

(iii) The following estimates are satisfied on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$: $|F(t, x, y)| < C(\omega)(1 + |y|)$, $|\eta(t, x)| + |\eta(x, y)| + |\eta(x)| < C_1(\omega)$, and $[F]^2 + |F|^2 + [F]^{\beta} < C_2(\omega)(1 + |y|)$, where $C(\omega)$, $C_1(\omega)$, $C_2(\omega)$ are positive random variables that are finite on $\Omega_0$, and $\beta \in (0, 1)$.

Then, for each $\omega \in \Omega_0$, there exists a unique $C^{1,2}_b$-solution $y(t, x)$ to problem (4). Moreover, the global bound for $y(t, x)$ depends only on $C(\omega)$, $T$, and on the bound for $|h(x)|$; the global bound for $\partial_x y(t, x)$ depends on $\nu$, $C(\omega)$, $T$, and on the bounds for $|h(x)|$ and $|\nabla h(x)|$. 

Proof. Consider (4) for a fixed $\omega \in \Omega_0$ (frozen random parameter). Furthermore, consider the following initial-boundary value problem

\begin{equation}
\begin{aligned}
\partial_t y(t, x) &= \nu \Delta y(t, x) - (y + \eta(t, x), \nabla) y(t, x) + F(t, x, y), \\
y(0, x) &= h(x) \zeta(x), \quad y(t, x) \bigg|_{\partial \Omega} = 0,
\end{aligned}
\end{equation}

where $B_r$ is an open ball of radius $r > 1$, $\partial B_r$ is its boundary, and $\zeta(x)$ is a smooth function such that $\zeta(x) = 1$ if $x \in B_{r-1}$, $\zeta(x) = 0$ if $x \notin B_r$, $\zeta(x)$ decays from 1 to 0 along the radius on $B_r \setminus B_{r-1}$ in a way that $\nabla^l \zeta$, $l = 1, 2, 3$, does not depend on $r$. Here $B_{r-1}$ is an open ball of radius $r - 1$. The class of initial-boundary value problems for systems of quasilinear parabolic PDEs, which includes problem (6), was considered by Ladyzhenskaya et al. in [25] (Theorem 7.1, p. 596). Namely, Theorem 7.1 of [25] implies that problem (6) possesses a unique solution on $[0, T] \times B_r$ which belongs to the parabolic Hölder space $C^{1+\frac{\beta}{2}, 2+\beta}$. We remind that if $Q \subset \mathbb{R}^n$ is a bounded domain, then the norm in $C^{1+\frac{\beta}{2}, 2+\beta}(Q)$ is defined as follows:

$$
\|y\|_{C^{1+\frac{\beta}{2}, 2+\beta}(Q)} = \|y\|_{C^{1, 2}(Q)} + \sup_{x \in Q} |\partial_t y| + \sup_{x \in Q} |\partial_x y| + \sup_{x \in Q} |\partial_{xx} y| + \sup_{x \in Q} |\partial_{xxx} y| + \sup_{x \in Q} |\partial_{xxxx} y|.
$$

Furthermore, by Theorem 5.1 (p. 586, estimates of Hölder norms) and Theorem 6.1 (p. 592, gradient estimate), the $C^{1+\frac{\beta}{2}, 2+\beta}(Q)$-norm of the solution $y(t, x)$ possesses a bound that depends only on $C(\omega)$, $C_1(\omega)$, $C_2(\omega)$, and on the Hölder norm $\|h\|_{C^{1,2}(\mathbb{R}^n)}$. It is important to emphasize that the bound for this norm does not depend on the radius $r$ of the ball.

Moreover, by Theorem 6.1 from [25], the bound for the gradient $\partial_x y(t, x)$ depends on $\nu$, $C(\omega)$, $T$, and the bounds for $|h(x)|$ and $|\nabla h(x)|$. Also, according to the results of § 7 from [25] (p. 596), the bound for $y(t, x)$ depends only on $C(\omega)$, $T$, and the bound for $|h(x)|$.

To prove the existence of a solution to (4), we employ the diagonalization argument similar to the one presented in [25] (p. 493) for the case of one equation. Take a closed ball $\bar{B}_R$ of radius $R$. Let $y_r(t, x)$ be the solution to problem (6) in the ball $B_{r+1}$. We assume that $y_r(t, x)$ is extended by 0 outside of $B_{r+1}$. Since the Hölder norms $\|y_r\|_{C^{1+\frac{\beta}{2}, 2+\beta}(\mathbb{R}^n)}$ possess a bound not depending on $r$, then by the Arzelà-Ascoli theorem, the family of functions $y_r(t, x)$, parametrized by $r$, is relatively compact in $C^{1, 2}(Q)$. Therefore, we can find a sequence $\{y_{r_n}\}$ which converges in $C^{1, 2}(Q)$. Then, we can find a further subsequence $\{y_{r_n}^{(1)}\}$ that converges in $C^{1, 2}(Q)$. Proceeding this way, we find a subsequence $\{y_{r_n}^{(k)}\}$ that converges in $C^{2, 1}(Q)$. It remains to note that the diagonal subsequence $\{y_{r_n}^{(n)}\}$ converges at each point of $Q$ to a function $y(t, x)$, while its derivatives $\partial_t y_{r_n}^{(n)}$, $\partial_x y_{r_n}^{(n)}$, and $\partial_{xx} y_{r_n}^{(n)}$ converge to the corresponding derivatives of $y(t, x)$. Clearly, $y(t, x)$ is a solution to (4). It is bounded in the $C^{1, 2}$-norm for each fixed $\omega \in \Omega_0$, i.e., it belongs to $C^{1, 2}(Q)$. Indeed, the $C^{1, 2}$-norm of each function $y_{r_n}^{(n)}(t, x)$ has the same bound. Moreover, the bound for $y(t, x)$ depends only on $C(\omega)$, $T$, and the bound for $|h(x)|$; the bound for $\partial_x y(t, x)$ depends on $\nu$, $C(\omega)$, $T$, and the bounds for $|h(x)|$ and $|\nabla h(x)|$. 


For the proof of uniqueness, we assume there are two solutions to problem (4), \(y_1, y_2 \in C^{1,2}_b([0, T] \times \mathbb{R}^n)\), and let \(y = y_1 - y_2\). Then, \(y(t, x)\) solves the problem

\[
\begin{aligned}
\partial_t y(t, x) &= \nu \Delta y(t, x) - (\eta(t, x) + y_1, \nabla) y(t, x) + (\nabla_2 F(t, x) + \partial_x y_2) y(t, x) = 0, \\
y(0, x) &= 0,
\end{aligned}
\]

where \(\nabla_2 F(t, x) = \int_0^1 \nabla_2 F(t, x, \lambda y_1 + (1 - \lambda) y_2) d\lambda\) with \(\nabla_2 F(t, x, y) = \partial_y F(t, x, y)\). From here, by Theorem 2.6 from [25] (p. 19), \(F\) and uniqueness of a local 1 Corollary such that \(F\) respect to filtrations other than \(F\) below to different problems of type (4), where the coefficients are adapted with \(F\) subsection will be proved with respect to a filtration \(F\). Moreover, the force term \(F\) for \(y(t, x)\) which only depends on \(\tilde{\eta}, T, M\) bounds in the spaces \(C\) does not depend on \(\omega\).

Remark 6. The proof of uniqueness holds without assuming the differentiability in \(t\) of the functions \(F\) and \(\eta\).

**Corollary 1.** Let Assumptions of Theorem 1 hold, and let the bound \(\tilde{C} = C(\omega)\) does not depend on \(\omega\). Furthermore, we let \(|h(x)|\) and \(|\nabla h(x)|\) possess deterministic bounds \(M_1\) and \(M_2\), respectively. Then, there exists a global deterministic bound for \(y(t, x)\) which only depends on \(\tilde{C}, T, M_1, M_2\), as well as a global deterministic bound for \(\partial_x y(t, x)\) which depends on \(\nu, \tilde{C}, T, M_1, M_2\).

### 2.2 Local existence and uniqueness in the bounded case

Throughout Subsection 2.2 we assume that \(\eta, F, \) and \(h\) possess deterministic bounds in the spaces \(C^{1,2}_b([0, T] \times \mathbb{R}^n), C^{1,2}_b([0, T] \times \mathbb{R}^{2n}),\) and \(C^2(\mathbb{R}^n)\), respectively. Moreover, the force term \(F\) is not assumed to have form (5). Also, the results in this subsection will be proved with respect to a filtration \(F_t^B\) which can be, in particular, \(\mathcal{F}_t^B\), or a different filtration. This will be helpful in applications of Theorem 2 below to different problems of type (4), where the coefficients are adapted with respect to filtrations other than \(\mathcal{F}_t^B\). In Theorem 2 below, we prove the existence and uniqueness of a local \(\mathcal{F}_t^B\)-adapted \(C^{1,2}_b\)-solution to (4). First, by doing the time change \(\tilde{y}(t, x) = y(T - t, x)\), we transform (4) to the backward equation

\[
\tilde{y}(t, x) = h(x) + \int_t^T [\nu \Delta \tilde{y}(s, x) - (\tilde{\eta}(t, x) + \tilde{y}, \nabla) \tilde{y}(s, x) + \tilde{F}(s, x, y)] ds
\]

with \(\tilde{F}(t, x, y) = F(T - t, x, y)\) and \(\tilde{\eta}(t, x) = \eta(T - t, x)\).

The following lemma will be useful.

**Lemma 3.** Let \(W_t\) be a one-dimensional Brownian motion and let \(\mathcal{B}\) be a \(\sigma\)-algebra independent of the (augmented) natural filtration \(\mathcal{F}_t^W\) of \(W_t\). Assume that \(\Phi_t\) is \(\mathcal{F}_t^W \vee \mathcal{B}\)-adapted and such that \(\mathbb{E} \int_0^t |\Phi_s|^2 ds < \infty, t > 0\). Then, \(\mathbb{E} \left[ \int_0^t \Phi_s dW_s | \mathcal{B} \right] = 0\) a.s.

**Proof.** Let \(0 = s_1 < \ldots < s_n = t\) be a partition. Note that for a simple \(\mathcal{F}_t^W \vee \mathcal{B}\)-adapted integrand \(\Phi = \sum_i \Phi_i 1_{[s_{i-1}, s_i]}\), it holds that

\[
\mathbb{E} \left[ \int_0^t \Phi_s dW_s | \mathcal{B} \right] = \mathbb{E} \left[ \sum_i \Phi_i (W_{s_{i+1}} - W_{s_i}) | \mathcal{B} \right] = \sum_i \mathbb{E} \left[ \Phi_i (W_{s_{i+1}} - W_{s_i}) | \mathcal{F}_{s_i}^W \vee \mathcal{B} \right] | \mathcal{B} = 0.
\]

Further, we note that if a sequence \(\{\Phi_i^{(n)}\}\) of simple \(\mathcal{F}_t^W \vee \mathcal{B}\)-adapted integrands is such that \(\mathbb{E} \int_0^t (\Phi_s^{(n)} - \Phi_s)^2 ds \to 0\), then by the conditional Jensen’s inequality and Itô’s isometry, \(\mathbb{E} \left[ \int_0^t (\Phi_s^{(n)} - \Phi_s) dW_s | \mathcal{B} \right]^2 \to 0\). \(\square\)
Define the constants $K$, $C$, and $C_1$ below

\begin{equation}
K = \sup_{\Omega_0 \times \mathbb{R}^n} |\nabla h| + \sup_{\Omega_0 \times [0,T] \times \mathbb{R}^n} |\partial_x \eta| + \sup_{\Omega_0 \times [0,T] \times \mathbb{R}^n} \left( |\partial_x F| + |\partial_y F| \right),
\end{equation}

\begin{equation}
C = \sup_{\Omega_0 \times \mathbb{R}^n} |\nabla^2 h| + \sup_{\Omega_0 \times [0,T] \times \mathbb{R}^n} |\partial_{xx} \eta| + \sup_{\Omega_0 \times [0,T] \times \mathbb{R}^n} \left( |\partial_{xx}^2 F| + |\partial_{xy}^2 F| \right),
\end{equation}

\begin{equation}
C_1 = \sup_{\Omega_0 \times \mathbb{R}^n} |h| + \sup_{\Omega_0 \times [0,T] \times \mathbb{R}^n} |\eta| + \sup_{\Omega_0 \times [0,T] \times \mathbb{R}^n} |F|.
\end{equation}

**Theorem 2.** Let, for each fixed $(x, y) \in \mathbb{R}^{2n}$, $\eta(t, x)$ and $F(t, x, y)$ be $\mathcal{F}_t^\tau$-adapted. Further let for almost each $\omega \in \Omega$, $\eta(t, x)$, $F(t, x, y)$, and $h(x)$ belong to classes $C^{1,2}_b([0, T] \times \mathbb{R}^n)$, $C^{0,2}_b([0, T] \times \mathbb{R}^{2n})$, and $C^2_b(\mathbb{R}^n)$, respectively, and possess deterministic bounds with respect to the norms of these spaces. Then, there exists a constant $\gamma_{K, C}$, depending only on $K$ and $C$, such that on $[T - \gamma_{K, C}, T]$ there exists an $\mathcal{F}_{T - t}^\tau$-adapted $C^{1,2}_b$-solution $\tilde{y}(t, x)$ to equation (7).

**Proof.** Everywhere below, $\gamma_i, \mu_i, i = 1, 2, 3, \ldots$, are positive deterministic constants that may depend only on $p$, $K$, $C$, $C_1$ but do not depend on $\nu$. Furthermore, the constants $\gamma_K, \gamma_K, \gamma_K, \gamma_K, \gamma_K, \gamma_{K, C}$ are positive and deterministic, that depend either on $K$, or on $K$ and $C$, and determine the length of the interval. Without loss of generality, these $\gamma_{K}$- or $\gamma_{K, C}$-type constants are assumed to be smaller than 1.

We prove the existence of $\mathcal{F}_{T - t}^\tau$-adapted $C^{1,2}_b$-solution to (7) by means of the associated FBSDEs (see [13], [27]):

\begin{equation}
\begin{aligned}
X_t^{\tau, x} &= x - \int_\tau^t \left( \tilde{\eta}(s, X_s^{\tau, x}) + Y_s^{\tau, x} \right) ds + \sqrt{2\nu}(W_t - W_\tau), \\
Y_t^{\tau, x} &= h(X_t^{\tau, x}) + \int_\tau^t \tilde{F}(s, X_s^{\tau, x}, Y_s^{\tau, x}) ds - \int_\tau^t Z_s^{\tau, x} dW_s,
\end{aligned}
\end{equation}

where $W_t$ is an $n$-dimensional Brownian motion independent of the filtration $\mathcal{F}_{T - t}^\tau$, and the upper index $\tau, x$ means that the process $X_t^{\tau, x}$ starts at $x$ at time $\tau > 0$. Note that $F(t, x, y)$ and $\tilde{\eta}(t, x)$ are adapted with respect to the backward filtration $\mathcal{F}_{T - t}^\tau$. For each $\tau \in (0, T)$, we define the filtrations

\[(\mathcal{F}_{\tau, t}^W)_{\tau \leq t \leq T} = \sigma\{W_s - W_\tau, s \in [\tau, t]\} \cup \mathcal{N} \quad \text{and} \quad (\mathcal{G}_t^\tau)_{\tau \leq t \leq T} = \mathcal{F}_{\tau, t}^W \vee \mathcal{F}_{T - \tau}^\tau,
\]

where $\mathcal{N}$ denote the set of $\mathbb{P}$-null sets.

In what follows, when it does not lead to misunderstanding, we will often skip the upper index $\tau, x$ in $(X_t^{\tau, x}, Y_t^{\tau, x}, Z_t^{\tau, x})$ and similar processes to simplify notation.

Symbol $\mathbb{E}_{\tau}$ will denote the conditional expectation with respect to $\mathcal{F}_{T - \tau}^\tau$.

**Step 1. Existence and uniqueness of the local solution to (9).** Boundedness of $\mathbb{E}_{\tau} |Y_t^{\tau, x}|^p$. To obtain the existence and uniqueness, we use the fixed-point argument for FBSDEs (9). The argument is similar to the one developed in [13].

Introduce the space $S([\tau, T], \mathbb{R}^n)$ of $\mathcal{G}_t^\tau$-adapted stochastic processes $\xi_t$ such that $\mathbb{E} \sup_{t \in [\tau, T]} |\xi_t|^2 < \infty$. Define the map $S([\tau, T], \mathbb{R}^n) \rightarrow S([\tau, T], \mathbb{R}^n)$, $Y_t = \Gamma(Y_t)$, where the map $\Gamma$ is given by the equations

\begin{equation}
\begin{aligned}
X_t &= x - \int_\tau^t \left( \tilde{\eta}(s, X_s) + \tilde{Y}_s \right) ds + \sqrt{2\nu}(W_t - W_\tau), \\
Y_t &= h(X_T) + \int_\tau^t \tilde{F}(s, X_s, Y_s) ds - \int_\tau^t Z_s dW_s.
\end{aligned}
\end{equation}

Namely, given $\tilde{Y}_t$, the process $X_t$ is the unique $\mathcal{G}_t^\tau$-adapted solution to the forward SDE in (10). Further, $(Y_t, Z_t)$ is the unique $\mathcal{G}_t^\tau$-adapted solution to the backward SDE in (10). Note that from the assumptions of the theorem, it follows that the
solution $Y_t$ to the BSDE in (10) always belongs to $\mathcal{S}([\tau, T], \mathbb{R}^n)$. Indeed, by Itô's formula,
\[|Y_t|^2 + \int_t^T |Z_s|^2 ds = |h(X_T)|^2 + 2 \int_t^T (\bar{F}(s, X_s, Y_s), Y_s) ds - \int_t^T (Y_s, Z_s dW_s).\]
Furthermore, due to the Burkholder-Gundy-Davis inequality, we have the following estimate for the last term
\[
(11) \quad \mathbb{E} \sup_{t \in [\tau, T]} \left| \int_t^T (Y_s, Z_s dW_s) \right| \leq \varepsilon \mathbb{E} \sup_{t \in [\tau, T]} |Y_t|^2 + \gamma_1 \mathbb{E} \int_\tau^T |Z_s|^2 ds,
\]
where $\varepsilon > 0$ is a small number. The existence of the fixed point for the map $\Gamma$ follows from Itô's formula and from an estimate of type (11). Namely, take $\hat{Y}_t^1, \hat{Y}_t^2 \in \mathcal{S}([\tau, T], \mathbb{R}^n)$, and define $Y_t^1 = \Gamma(\hat{Y}_t^1)$, $Y_t^2 = \Gamma(\hat{Y}_t^2)$. Applying Itô's formula to $|Y_t^2 - Y_t^1|^2$, and then taking the $\mathbb{E}\sup_{t \in [\tau, T]}$ operation of the both parts, we obtain
\[
\mathbb{E} \sup_{t \in [\tau, T]} |Y_t^2 - Y_t^1|^2 \leq K^2 \mathbb{E} |X_T^2 - X_T^1|^2 + 3K \int_\tau^T \mathbb{E} (|X_s^2 - X_s^1|^2 + |Y_s^2 - Y_s^1|^2) ds
+ \mathbb{E} \sup_{t \in [\tau, T]} \left| \int_t^T (Y_s^2 - Y_s^1, (Z_s^2 - Z_s^1) dW_s) \right|;
\]
\[
\int_\tau^T \mathbb{E}|Z_s^2 - Z_s^1|^2 ds \leq K^2 \mathbb{E} |X_T^2 - X_T^1|^2 + 3K \int_\tau^T \mathbb{E} (|X_s^2 - X_s^1|^2 + |Y_s^2 - Y_s^1|^2) ds,
\]
where $(X_t^1, Z_t^1)$ and $(X_t^2, Z_t^2)$ are obtained from (10) via $\hat{Y}_t^1$ and $\hat{Y}_t^2$, respectively. Following the same strategy as in the proof of Theorem 1.1 in [13], from the above inequalities, we obtain that there exists a constant $\tilde{\gamma}_K > 0$, depending only on $K$, such that whenever $T - \tau \leq \tilde{\gamma}_K$, the map $\Gamma$, defined by (10), becomes a contraction map. In particular, the constant $\tilde{\gamma}_K$ is the same for all intervals $[\tau, T]$ whose length does not exceed $\tilde{\gamma}_K$. This proves that FBSDEs (9) possess the unique $\mathcal{G}_t$-adapted solution $(X_t^{\tau,x}, Y_t^{\tau,x}, Z_t^{\tau,x})$ on $[\tau, T]$ whenever $T - \tau \leq \tilde{\gamma}_K$.

Let us prove the boundedness of $\mathbb{E}_\tau |Y_t^{\tau,x}|^p$. Note that, for $p \geq 2$,
\[
(|g|)^p h = p|g|^{p-2}(g, h); \quad (|g|^p)''h_1h_2 = p(p-2)|g|^{p-4}(g, h_1)(g, h_2) + p|g|^{p-2}(h_1, h_2).
\]
The BSDE in (9) and Itô’s formula imply
\[
(12) \quad \mathbb{E}_\tau |Y_t|^p + p(p - 2) \int_t^T \mathbb{E}_\tau [|Y_s|^{p-4} \sum_{i=1}^n |(Z_s^i, Y_s)|^2] ds
+ p \int_t^T \mathbb{E}_\tau [|Y_s|^{p-2}|Z_s|^2] ds = \mathbb{E}_\tau |h(X_T)|^p + 2p \int_t^T \mathbb{E}_\tau [|Y_s|^{p-2}(\bar{F}(s, X_s, Y_s), Y_s)] ds.
\]
By assumptions of the theorem, Young's inequality, and Gronwall's lemma,
\[
(13) \quad \mathbb{E}_\tau |Y_t|^p \leq \gamma_2 \quad \text{and} \quad |Y_t^{\tau,x}|^p \leq (\gamma_2)^\frac{2}{p}.
\]
Note that $\gamma_2$ only depends on $C_1$ and $p$, and does not depend on $\nu$. Moreover, $\gamma_2$ is the same for all $(\tau, x) \in [T - \tilde{\gamma}_K, T] \times \mathbb{R}^n$.

**Step 2. Differentiability of the FBSDEs solution in $x$.** Boundedness of $\mathbb{E}_\tau |\partial X_t^{\tau,x}|^p$ and $\mathbb{E}_\tau |Y_t^{\tau,x}|^p$. First, we prove that the map $[T - \tilde{\gamma}_K, T] \times \mathbb{R}^n \to C([T - \tilde{\gamma}_K, T] \times \mathbb{R}^n)$, $(\tau, x) \to (X_t^{\tau,x}, Y_t^{\tau,x})$ is continuous a.s. This continuity will be required, in particular, for the proof of the differentiability of $(X_t^{\tau,x}, Y_t^{\tau,x})$ with respect to $x$. Extend $X_s^{\tau,x}$ to $[T - \tilde{\gamma}_K, T]$ by $x$, and $Y_s^{\tau,x}$ by $\mathbb{E}[Y_t^{\tau,x} | \mathcal{G}_s^x]$. By Corollary A.6
from [13] (p. 266), there exists a constant \( \gamma_K < \gamma_K \) such that for any \( x, x' \in \mathbb{R}^n \), 
\( \tau, \tau' \in [T - \gamma_K, T] \),

\[
\mathbb{E} \sup_{t \in [T - \gamma_K, T]} |X_t^{\tau,x} - X_t^{\tau',x'}|^2 + \mathbb{E} \sup_{t \in [T - \gamma_K, T]} |Y_t^{\tau,x} - Y_t^{\tau',x'}|^2 \leq \gamma_3 (|x - x'|^2 + (1 + |x|^2)|\tau - \tau'|^p),
\]

where \( p \geq 1 \). In (14), we pick \( p \) such that \( \frac{p}{2} < 1 \). Then by Kolmogorov’s continuity criterion in Banach spaces (see [22], p. 31), there exists a continuous modification of the map \([T - \gamma_K, T] \times \mathbb{R}^n \rightarrow C([T - \gamma_K, T] \times \mathbb{R}^2n), (\tau, x) \mapsto (X^{\tau,x}, Y^{\tau,x})\). In what follows, we let \((X^{\tau,x}, Y^{\tau,x})\) to be this continuous modification. In other words, there exists a set \( \Omega_1, \mathbb{P}(\Omega_1) = 1 \), such that the map \([T - \gamma_K, T] \times \mathbb{R}^n \rightarrow C([T - \gamma_K, T] \times \mathbb{R}^2n), (\tau, x) \mapsto (X^{\tau,x}, Y^{\tau,x})\) is continuous for all \( \omega \in \Omega_1 \). In particular, the map \( (\tau, x) \mapsto Y^{\tau,x} \) is continuous for all \( \omega \in \Omega_1 \).

Now we proceed with the proof of differentiability. For any function \( \alpha(x) \), we define \( \Delta^k_\delta \alpha(x) = \delta^{-1} (\alpha(x + \delta e_k) - \alpha(x)) \), \( k = 1, \ldots, n \), where \( \{e_k\}_{k=1}^n \) denotes the orthonormal basis in \( \mathbb{R}^n \). In particular, \( \Delta^k_\delta X_t = \delta^{-1} (X_t^{\tau,x+\delta e_k} - X_t^{\tau,x}), k = 1, \ldots, n \), and \( \Delta^k_\delta Y_t, \Delta^k_\delta Z_t \) are defined similarly. For a function \( \Phi \) (which can be any of the functions \( F, h, \eta \), or their gradients with respect to the spatial variables), we define 
\[
\nabla_1 \Phi(t, u, v) = \partial_u \Phi(t, u, v), \quad \nabla_2 \Phi(t, u, v) = \partial_v \Phi(t, u, v).
\]

Furthermore, we define

\[
\nabla_1^k \Phi_t = \int_0^1 \nabla_1 \Phi(t, X_t + \lambda \Delta^k_\delta X_t, Y_t) d\lambda; \quad \nabla_2^k \Phi_t = \int_0^1 \nabla_2 \Phi(t, X_t, Y_t + \lambda \Delta^k_\delta Y_t) d\lambda
\]

and note that \( \nabla_1^k \Phi_t = \int_0^1 \nabla_1 \Phi(t, (1 - \lambda)X_t^{\tau,x} + \lambda X_t^{\tau,x+\delta e_k}, Y_t) d\lambda \), and similar for \( \nabla_2^k \Phi_t \). In case of just one spatial variable (like in \( h \) or \( \eta \)) we write \( \nabla \) instead of \( \nabla_1 \) and \( \nabla^k \) instead of \( \nabla_1^k \). Note that

\[
\Delta^k_\delta \Phi_t = \nabla_1^k \Phi_t \Delta^k_\delta X_t + \nabla_2^k \Phi_t \Delta^k_\delta Y_t.
\]

It is immediate to verify that the triple \((\Delta^k_\delta X_t, \Delta^k_\delta Y_t, \Delta^k_\delta Z_t)\) satisfies the FBSDEs

\[
\begin{aligned}
\Delta^k_\delta X_t &= e_k - \int_t^T (\Delta^k_\delta Y_s + \nabla^k \eta \Delta^k_\delta X_s) ds, \\
\Delta^k_\delta Y_t &= \nabla^k h_T \Delta^k_\delta X_T + \int_t^T (\nabla^k \bar{F}_s \Delta^k_\delta X_s + \nabla^k \bar{F}_s \Delta^k_\delta Y_s) ds - \int_t^T \Delta^k_\delta Z_s dW_s,
\end{aligned}
\]

on the same time interval \([\tau, T]\), where we proved the existence and uniqueness of a solution to (9). Now consider the FBSDEs with respect to \((\partial_k X_t, \partial_k Y_t, \partial_k Z_t)\)

\[
\begin{aligned}
\partial_k X_t &= e_k + \int_t^T (\partial_k Y_s + \nabla \eta \partial_k X_s) ds \\
\partial_k Y_t &= \nabla h(X_T) \partial_k X_T + \int_t^T (\nabla \bar{F}_s \partial_k X_s + \nabla \bar{F}_s \partial_k Y_s) ds - \int_t^T \partial_k Z_s dW_s,
\end{aligned}
\]

where 
\( \nabla \eta = \nabla \eta(s, X_s), \nabla h = \nabla h(X_T), \nabla \bar{F} = \nabla \bar{F}(s, X_s, Y_s), i = 1, 2 \), and 
\( \partial_k \) denotes the partial derivative with respect to \( x_k \). The existence and uniqueness of a \( G^T \)-adapted solution to (18) on \([\tau, T]\) can be proved similarly to (9). By now, we only know that \((\partial_k X_t, \partial_k Y_t, \partial_k Z_t)\) is the solution to (18), and we would like to prove that this triple is the partial derivative of \((X_t, Y_t, Z_t)\) with respect to \( x_k \).
First we prove that \( E_\tau |\partial X_t|^p \) and \( E_\tau |\partial_k Y_t|^p, \ p \geq 2 \), are bounded. Itô’s formula together with the BSDE in (18) imply

\[
E_\tau |\partial_k Y_t|^p + p(p-2) \int_t^T E_\tau [ |\partial_k Y_s|^{p-4} \sum_{j=1}^n |(\partial_k Z^{j}_{s}, \partial_k Y_s)|^2 ] \, ds \\
+ p \int_t^T E_\tau [ |\partial_k Y_s|^{p-2} |\partial_k Z_s|^2 ] \, ds = E_\tau [ \nabla h(X_T) \partial_k X_T |^p ] \\
+ 2p \int_t^T E_\tau [ |\partial_k Y_s|^2 (\nabla_1 \hat{F}(s,X_s, Y_s) \partial_k X_s + \nabla_2 \hat{F}(s,X_s, Y_s) \partial_k Y_s, \partial_k Y_s)] \, ds.
\]

From here, by the forward SDE in (18) and Young’s inequality, it follows that for all \( t \in [\tau,T] \), \( E_\tau |\partial_k Y_t|^p \leq \gamma_4 (1 + \int_\tau^T E_\tau |\partial_k Y_s|^p \, ds) \), which together with the forward SDE in (18), implies that there exists a constant \( \hat{\gamma}_K < \hat{\gamma}_K \) such that for \( \tau \in [T-\hat{\gamma}_K, T] \) and \( t \in [\tau,T] \),

\[
\max \{ E_\tau |\partial_k X_t|^p; E_\tau |\partial_k Y_t|^p \} \leq \gamma_5.
\]

By the same argument, FBSDEs (17) imply

\[
\max \{ E_\tau |\Delta^k_X X_t|^p; E_\tau |\Delta^k_Y Y_t|^p \} \leq \gamma_5.
\]

Note that the constants \( \gamma_4 \) and \( \gamma_5 \) depend only on \( K \). Let us prove that as \( \delta \to 0 \),

\[
\sup_{t \in [\tau,T]} \{ |\Delta^k_X X_t - \partial_k X_t|^2 + |\Delta^k_Y Y_t - \partial_k Y_t|^2 \} \right. \\
+ \left. \int_\tau^T E_\tau |\Delta^k_X Z_t - \partial_k Z_t|^2 \, dt \to 0
\]
as.s. Let \( \zeta_X(\delta,t) = \Delta^k_X X_t - \partial_k X_t \). Similarly, we define \( \zeta_Y(\delta,t) \) and \( \zeta_Z(\delta,t) \). FBSDEs for the triple \( (\zeta_X(\delta,t), \zeta_Y(\delta,t), \zeta_Z(\delta,t)) \) take the form

\[
\begin{cases}
\zeta_X(\delta,t) = \int_t^T (\zeta_Y(\delta,s) + \nabla^{\delta,k} \eta_s \zeta_X(\delta,s) + \xi^X_s) \, ds, \\
\zeta_Y(\delta,t) = \nabla^{\delta,k} h_T \zeta_X(\delta,T) - \int_t^T (\nabla_1^{\delta,k} \hat{F}_s \zeta_X(\delta,s) + \nabla_2^{\delta,k} \hat{F}_s \zeta_Y(\delta,s) + \xi^Y_s) \, ds \\
\zeta_Z(\delta,t) = \int_t^T \zeta_Z(\delta,s) \, ds + \xi^Z_t,
\end{cases}
\]

where \( \xi^X_s = (\nabla^{\delta,k} \eta_s - \nabla \eta_s) \partial_k X_s, \xi^Y_s = (\nabla^{\delta,k} \hat{F}_s - \nabla \hat{F}_s) \partial_k X_s + (\nabla^{\delta,k} \hat{F}_s - \nabla \hat{F}_s) \partial_k Y_s, \)

and \( \xi^Z_t = (\nabla^{\delta,k} h_T - \nabla h_T) \partial_k X_T \). Note that \( \nabla^{\delta,k} \eta_s, \nabla^{\delta,k} h_T, \) and \( \nabla^{\delta,k} \hat{F}_s, \ i = 1, 2, \) are bounded by \( K \), which follows from (15). Then, by standard arguments, on the interval \( [\tau,T] \) whose length is smaller than \( \hat{\gamma}_K \), one has the estimate

\[
E_{\tau} |\zeta_Y(\delta,t)|^2 + \int_t^T |\zeta_X(\delta,s)|^2 \, ds \leq \gamma_6 (E_\tau |\xi^X_T|^2 + \int_t^T (|\xi^X_s|^2 + |\xi^Y_s|^2) \, ds).
\]

By (19) and the conditional dominated convergence theorem, the right-hand side of the above inequality converges to zero a.s. This means that \( (X^{\tau,x}_t, Y^{\tau,x}_t, Z^{\tau,x}_t) \) is differentiable with respect to \( x \) in the sense of (21). Since \( Y^{\tau,x}_t \) is \( F_{\tau} \)-measurable, then for each fixed \( \tau \) and \( x \), it is differentiable in \( x \) a.s. Furthermore, estimate (19) implies that, a.s.,

\[
|\partial_x Y^{\tau,x}_t| < \sqrt{\gamma_5}.
\]

This holds for all \( (\tau,x) \in [T-\hat{\gamma}_K, T] \times \mathbb{R}^n \). Moreover, \( \gamma_5 \) does not depend on \( \nu \).

Now let us show that the map \( [T-\hat{\gamma}_K, T] \times \mathbb{R}^n \to C([T-\hat{\gamma}_K, T],\mathbb{R}^{2n}), \)

\( (\tau,x) \mapsto (\partial_x X^{\tau,x}_t, \partial_x Y^{\tau,x}_t) \) is continuous in \( (\tau,x) \) a.s. for some constant \( \hat{\gamma}_K < \hat{\gamma}_K \).
By Corollary A.6 from [13] (p. 266), there exists a constant $\tilde{\gamma}_K < \gamma_K$ such that for any $\tau, \tau' \in [T - \tilde{\gamma}_K, T]$, $p \geq 1$,
\[
\mathbb{E} \sup_{t \in [T - \tilde{\gamma}_K, T]} |\partial_h X_t^{\tau,x} - \partial_h X_t^{\tau',x}|^{2p} + \mathbb{E} \sup_{t \in [T - \tilde{\gamma}_K, T]} |\partial_y Y_t^{\tau,x} - \partial_y Y_t^{\tau',x}|^{2p} \leq \gamma_8 |\tau - \tau'|^p,
\]
where $\partial_h X_t^{\tau,x}$ is extended to $[T - \tilde{\gamma}_K, \tau]$ by $e_k$, and $\partial_y Y_t^{\tau,x}$ by $\mathbb{E}[\partial_h Y_t^{\tau,x} | \mathcal{G}_T^\tau]$. Further, by Theorem A.2 from [13] (p. 258), there exists a constant $\gamma_9 < \tilde{\gamma}_K$ such that
\[
\mathbb{E} \sup_{t \in [T - \tilde{\gamma}_K, T]} |\partial_h X_t^{\tau,x} - \partial_h X_t^{\tau',x}|^{2p} + \mathbb{E} \sup_{t \in [T - \tilde{\gamma}_K, T]} |\partial_y Y_t^{\tau,x} - \partial_y Y_t^{\tau',x}|^{2p} \leq \gamma_9 |\tau - \tau'|^p.
\]

Therefore, by Kolmogorov’s continuity criterion in Banach spaces [22], there exists a set $\Omega_2$, $\mathbb{P}(\Omega_2) = 1$, and a version of the map $[T - \tilde{\gamma}_K, T] \times \mathbb{R}^n \to C([T - \tilde{\gamma}_K, T] \times \mathbb{R}^{2n})$, $(\tau, x) \mapsto (\partial_h X_{\tau,x}, \partial_y Y_{\tau,x})$ which is continuous for all $\omega \in \Omega_2$, in particular, there exists a continuous version of the map $(\tau, x) \mapsto \partial_h Y_{\tau,x}$. This, in turn, implies that there exists a set $\Omega_3$, $\mathbb{P}(\Omega_3) = 1$, that does not depend on $\tau$ and $x$, such that $Y_{\tau,x}$ possesses a continuous derivative $\partial_h Y_{\tau,x}$ for all $\omega \in \Omega_3$.

**Step 3. Second order differentiability of the FBSDEs solution in $x$. Boundedness of $\mathbb{E}_\tau |\partial_{ik} Y_{\tau,x}|^2$.** Consider FBSDEs for the triple $(\partial_{ik}^2 X_t, \partial_{ik}^2 Y_t, \partial_{ik}^2 Z_t)$

\[
\begin{align*}
(\partial_{ik}^2 X_t &= - \int_t^\tau (\partial_{ik}^2 X_s + \nabla \eta_2 \partial_{ik}^2 X_s + \chi^X_s) ds, \\
(\partial_{ik}^2 Y_t &= \mathbb{E}_\tau |\partial_{ik}^2 X_t| + \int_t^\tau (\nabla_1 F_t \partial_{ik}^2 X_s + \nabla_2 F_t \partial_{ik}^2 Y_s + \chi^Y_Y) ds - \int_t^\tau \partial_{ik}^2 Z_s dW_s + \bar{\chi}_{ik}^Y,
\end{align*}
\]
where

\[
\begin{align*}
\chi^X_s &= \nabla_2 \eta_t \partial_h X_s \partial_h X_s; & \bar{\chi}^Y_{ik} &= \mathbb{E}_\tau |\partial_{ik}^2 X_t| \partial_h X_t; \\
\chi^Y_s &= \nabla_1 F_t \partial_{ik}^2 X_s + \nabla_2 F_t \partial_{ik}^2 Y_s + \nabla_1 \partial_{ik}^2 X_s \partial_h X_s \partial_h Y_s + \nabla_2 \partial_{ik}^2 Y_s (\partial_h X_s \partial_h Y_s + \partial_h X_s \partial_h Z_s).
\end{align*}
\]

As in the previous step, so far we do not know if $(\partial_{ik}^2 X_t, \partial_{ik}^2 Y_t, \partial_{ik}^2 Z_t)$ is the second order partial derivative $\partial_{ik}^2$ of $(X_t, Y_t, Z_t)$, and treat it just as a solution of (24). Remark that in (24), the triples $(X_t, Y_t, Z_t)$ and $(\partial_h X_t, \partial_h Y_t, \partial_h Z_t)$ are assumed to be known from the previous steps, and, moreover, (19) holds true. Let us first prove that if $(\partial_{ik}^2 X_t, \partial_{ik}^2 Y_t, \partial_{ik}^2 Z_t)$ is a solution to (24), then $\mathbb{E}_\tau |\partial_{ik}^2 Y_t|^2$ is bounded.
Indeed, Itô’s formula implies

\[
|\partial_{\delta k}^2 Y_t|^2 + \int_t^T |\partial_{\delta k}^2 Z_s|^2 ds = |\nabla h(X_T)\partial_{\delta k}^2 X_T + \check{\chi}_T|^2 \\
+ 2 \int_t^T (\nabla_1 \check{F}_s \partial_{\delta k}^2 X_s + \nabla_2 \check{F}_s \partial_{\delta k}^2 Y_s + \chi_s \partial_{\delta k}^2 Y_s) ds + \int_t^T (\partial_{\delta k}^2 Y_s, \partial_{\delta k}^2 Z_s dW_s).
\]

From here, by using the forward SDE in (24), a.s.,

\[
\mathbb{E}_\tau |\partial_{\delta k}^2 Y_t|^2 \leq \mu_1 (1 + \int_t^T \mathbb{E}_\tau |\partial_{\delta k}^2 Y_s|^2 ds + \int_t^T \mathbb{E}_\tau (|\chi_s|^2 + |\chi_s'|^2) ds + \mathbb{E}_\tau |\check{\chi}_T|^2).
\]

Note that by (19) and the assumptions of the theorem, \(\mathbb{E}_\tau |\check{\chi}_T|^2\) and \(\int_t^T \mathbb{E}_\tau (|\chi_s|^2 + |\chi_s'|^2) ds\) are bounded and the bound does not depend on \(\nu\). This implies that there exists a constant \(\bar{\gamma}_{K,C} < \bar{\gamma}_K\), depending only on \(K\) and \(C\), such that, a.s.,

\[
\max\{\mathbb{E}_\tau |\partial_{\delta k}^2 X_t|^2, \mathbb{E}_\tau |\partial_{\delta k}^2 Y_t|^2\} \leq \mu_2
\]

Consequently,

\[
|\partial_{\delta k}^2 Y_t|^2 \leq \sqrt{\mu_2} \quad \text{a.s.}
\]

We remark that \(\mu_2\) depends only on \(K\) and \(C\), and does not depend on \(\nu\). Moreover, (28) holds uniformly in \((\tau, x) \in [T - \bar{\gamma}_{K,C}, T]\).

Now let us prove the existence of the second derivative. Note that FBSDEs (24) have a similar structure with FBSDEs (18). The difference is only in the presence of the terms \(\chi_t^X, \chi_t^Y,\) and \(\check{\chi}_T^Y\) which do not depend on the solution \((\partial_{\delta k}^2 X_t, \partial_{\delta k}^2 Y_t, \partial_{\delta k}^2 Z_t)\).

Thus, similar to Step 2, there exists a unique solution to (24) on the same short time interval \([\tau, T]\) as in the previous step, i.e., whose length is smaller than \(\bar{\gamma}_K\). Further, applying the operation \(\Delta^1_k\) to FBSDEs (18), using formula (16), and noticing that for any functions \(\alpha_1(x)\) and \(\alpha_2(x)\), \(\Delta^1_k [\alpha_1(x) \alpha_2(x)] = \Delta^1_k \alpha_1(x) \alpha_2(x) + \alpha_1(x + \delta e_i) \Delta^1_k \alpha_2(x),\) we obtain the FBSDEs for the triple \((\Delta^1_k \partial_{\delta k} X_t, \Delta^1_k \partial_{\delta k} Y_t, \Delta^1_k \partial_{\delta k} Z_t)\):

\[
\begin{align*}
\Delta^1_k \partial_{\delta k} X_t &= - f^1_T (\Delta^1_k \partial_{\delta k} X_s + \nabla \check{h}_{s,\delta} \Delta^1_k \partial_{\delta k} X_s + \zeta_{s}^X) ds, \\
\Delta^1_k \partial_{\delta k} Y_t &= \nabla h_{T,\delta} \Delta^1_k \partial_{\delta k} X_T + \int_t^T \left( \nabla_1 \check{F}_s \Delta^1_k \partial_{\delta k} X_s + \nabla_2 \check{F}_s \Delta^1_k \partial_{\delta k} Y_s + \zeta_{s}^Y \right) ds \\
&\quad - \int_t^T \Delta^1_k \partial_{\delta k} Z_s dW_s + \zeta_{s}^Y,
\end{align*}
\]

where \(\nabla \check{h}_{s,\delta} = \nabla \check{h}(s, X_s^{\tau,x+\delta e_i}), \nabla \check{F}_s = \nabla \check{F}(s, X_s^{\tau,x+\delta e_i}, Y_s^{\tau,x+\delta e_i}), i = 1, 2,\)

\(\nabla h_{T,\delta} = \nabla h(X_T^{\tau,x+\delta e_i}),\)

\(\zeta_{s}^X = \nabla^{i,1} \nabla \check{h}_{s,\delta} \Delta^1_k X_s \partial_{\delta k} X_s, \zeta_{s}^Y = \nabla^{i,1} \nabla h_{T,\delta} \Delta^1_k X_T \partial_{\delta k} X_T,\)

and \(\zeta_{s}^Y = \nabla^{i,1} \nabla_1 \check{F}_s \Delta^1_k X_s \partial_{\delta k} X_s + \nabla^{i,1} \nabla_2 \check{F}_s \Delta^1_k Y_s \partial_{\delta k} Y_s + \nabla^{i,1} \nabla_1 \check{F}_s \partial_{\delta k} X_s \partial_{\delta k} X_s + \nabla^{i,1} \nabla_2 \check{F}_s \partial_{\delta k} Y_s \partial_{\delta k} Y_s + \nabla^{i,1} \nabla_1 \check{F}_s \partial_{\delta k} X_s \partial_{\delta k} Z_s + \nabla^{i,1} \nabla_2 \check{F}_s \partial_{\delta k} Y_s \partial_{\delta k} Z_s + \nabla^{i,1} \nabla_1 \check{F}_s \partial_{\delta k} X_s \partial_{\delta k} Z_s + \nabla^{i,1} \nabla_2 \check{F}_s \partial_{\delta k} Y_s \partial_{\delta k} Z_s + \nabla^{i,1} \nabla_1 \check{F}_s \partial_{\delta k} X_s \partial_{\delta k} Z_s \)

Let \(\zeta_X(\delta, t) = \Delta^1_k \partial_{\delta k} X_t - \partial_{\delta k}^2 X_t, \zeta_Y(\delta, t) = \Delta^1_k \partial_{\delta k} Y_t - \partial_{\delta k}^2 Y_t, \zeta_Z(\delta, t) = \Delta^1_k \partial_{\delta k} Z_t - \partial_{\delta k}^2 Z_t.\) We show that as \(\delta \to 0,\)

\[
\sup_{t \in [\tau, T]} \mathbb{E}_\tau \{ |\zeta_X(\delta, t)|^2 + |\zeta_Y(\delta, t)|^2 \} + \mathbb{E}_\tau \int_{\tau}^{T} |\zeta_Z(\delta, s)|^2 ds \to 0, \quad \text{a.s.}
\]
for $\tau \in [T - \tilde{\gamma}_{K,C}, T]$. As in the previous step, we will do it by means of the FBSDEs for the triple $(\zeta_X(\delta, t), \zeta_Y(\delta, t), \zeta_Z(\delta, t))$ which take the form

$$
\begin{cases}
\zeta_X(\delta, t) = - \int_t^T \left( \zeta_Y(\delta, x) + \nabla \eta_{s,\delta} \zeta_X(\delta, s) + \theta_s^X \right) ds \\
\zeta_Y(\delta, t) = \nabla h_{T,\delta} \zeta_X(\delta, T) + \int_t^T \left( \nabla_1 \tilde{F}_{s,\delta} \zeta_X(\delta, s) + \nabla_2 \tilde{F}_{s,\delta} \zeta_Y(\delta, s) + \theta_s^Y \right) ds
\end{cases}
$$

where $\theta_s^Y = (\nabla Y_s^T X_T \partial_0 X_T - \nabla^2 h_{T,\delta} \partial_0 X_T \partial_0 X_T) + (\nabla h_{T,\delta} - \nabla h_T) \partial_{ik}^2 X_T$;

$\theta_s^X = (\nabla^2 h_{T,\delta} \partial_0 X_T \partial_0 X_T - \nabla^2 \eta_{s,\delta} \partial_0 X_T \partial_0 X_T) + (\nabla \eta_{s,\delta} - \nabla \eta_{t,\delta}) \partial_{ik}^2 X_s$;

$\tilde{Y}_t = \nabla_1 \tilde{F}_s \partial_0 X_s \partial_0 X_s - \nabla_2 \tilde{F}_s \partial_0 X_s \partial_0 X_s + \nabla_3 \tilde{F}_s \partial_0 X_s \partial_0 X_s - \nabla_4 \tilde{F}_s \partial_0 X_s \partial_0 X_s$.

Since the first and second order derivatives of the functions $h$, $\tilde{F}$, and $\tilde{\eta}$ are a.s. bounded and continuous (by the assumptions of the theorem), and the map $x \mapsto (X_t^{T,x}, Y_t^{T,x})$ is a.s. continuous, then by (19), (20), (27), and the conditional dominated convergence theorem, as $\delta \to 0$, $E_x[|\tilde{Y}_T|^2 + \int_0^T (|\tilde{X}_s^T|^2 + |\tilde{Y}_s^T|^2) ds] \to 0$, a.s. This holds for $\tau \in [T - \tilde{\gamma}_{K,C}, T]$. Therefore, by standard arguments, including an application of Itô's formula to $|\zeta_Y(\delta, t)|^2$ and a subsequent taking the conditional expectations $E_x$, we obtain (30). This proves the existence of the second derivatives $\partial_{ik}^2 X_t^{T,x}$ and $\partial_{ik}^2 Y_t^{T,x}$ with respect to $x$ in the sense of (30) on the interval $[\tau, T]$, whose length is smaller than $\tilde{\gamma}_{K,C}$. This also proves that for every $\tau \in [T - \tilde{\gamma}_{K,C}, T]$ and $x \in \mathbb{R}^n$, the second derivative $\partial_{ik}^2 Y_t^{T,x}$ exists a.s.

Let us prove the continuity of the map $[T - \tilde{\gamma}_{K,C}, T] \times \mathbb{R}^n \to C([T - \tilde{\gamma}_{K,C}, T] \times \mathbb{R}^n)$, $(\tau, x) \mapsto \partial_{ik}^2 Y_t^{T,x}$ for some constant $\gamma_{K,C} < \tilde{\gamma}_{K,C}$. We do it similarly to the proof of continuity of the map $(\tau, x) \mapsto \partial_0 Y_t^{T,x}$ in Step 2. Namely, by Corollary A.6 from [13] (p. 266), there exists a constant $\gamma_{K,C} < \tilde{\gamma}_{K,C}$ such that for any $\tau, \tau' \in [T - \tilde{\gamma}_{K,C}, T]$, $p \geq 1$,

$$
E \sup_{t \in [T - \tilde{\gamma}_{K,C}, T]} |\partial_{ik}^2 X_t^{T,x} - \partial_{ik}^2 X_t'^{T,x}|^2 + E \sup_{t \in [T - \tilde{\gamma}_{K,C}, T]} |\partial_{ik}^2 Y_t^{T,x} - \partial_{ik}^2 Y_t'^{T,x}|^2 \leq \mu_3 |\tau - \tau'|^p,
$$

where $\partial_{ik}^2 X_t^{T,x}$ is extended to $[T - \tilde{\gamma}_{K,C}, T]$ by 0, and $\partial_{ik}^2 Y_t^{T,x}$ by $E[\partial_{ik}^2 Y_t^{T,x} | G_t]$. Further, by Theorem A.7 from [13] (p. 267), there exists a constant $\gamma_{K,C} < \tilde{\gamma}_{K,C}$, such that

$$
E \sup_{t \in [T - \tilde{\gamma}_{K,C}, T]} |\partial_{ik}^2 Y_t^{T,x} - \partial_{ik}^2 Y_t'^{T,x}|^2 \leq \mu_4 \left( E \left[ \left( (\nabla h(X_t^{T,x}) - \nabla h(X_t'^{T,x})) \partial_{ik}^2 X_t^{T,x} \right)^2 \right] + \left( (\tilde{\zeta}^{X,x}_T - \tilde{\zeta}^{X,x}_T') \right)^2 + E \left[ \int_0^T \left( (\nabla_1 \tilde{F}(s, X_s^{T,x}, Y_s^{T,x}) - \nabla_1 \tilde{F}(s, X_s'^{T,x}, Y_s'^{T,x})) \partial_{ik}^2 X_s^{T,x} \right)^2 ds \right] + \int_0^T \left( (\nabla_2 \tilde{F}(s, X_s^{T,x}, Y_s^{T,x}) - \nabla_2 \tilde{F}(s, X_s'^{T,x}, Y_s'^{T,x})) \partial_{ik}^2 Y_s^{T,x} \right)^2 ds + \left( \tilde{\zeta}_T^{X,x} - \tilde{\zeta}_T^{X,x}' \right)^2 \right)^{1/2},
$$

where $\tilde{\zeta}_T^{X,x}$, $\tilde{\zeta}_T^{Y,x}$ and $\tilde{\zeta}_T^{X,x}'$ are defined by (25) via $(X_t^{T,x}, Y_t^{T,x})$ and their derivatives in $x$, while $\zeta_{s}^{X,x}$, $\zeta_{s}^{Y,x}$ and $\tilde{\zeta}_T^{X,x}$ are defined in the similar manner via $(X_t^{T,x}, Y_t^{T,x})$. 
From here, by the boundedness of the first and second order derivatives of the functions $h$, $\bar{\eta}$, and $\bar{\tilde{F}}$, as well as by (14), (19), (23), and (27), it follows that

$$
E \sup_{t \in [T - \bar{\gamma}_{K,C}, T]} |\partial_{ik}^2 Y_t^{r,x} - \partial_{ik}^2 \bar{Y}_t^{r,x}|^2 \leq \mu_5 |x - x'|^{2p}.
$$

Therefore, by Kolmogorov’s continuity theorem, there exists a set $\Omega_4$ of full $\mathbb{F}$-measure and a version of the map $[T - \bar{\gamma}_{K,C}, T] \times \mathbb{R}^n \to C([T - \bar{\gamma}_{K,C}, T] \times \mathbb{R}^n)$, $(\tau, x) \mapsto \partial_{ik}^2 Y_t^{r,x}$ which is continuous for all $\omega \in \Omega_4$. In particular, there exists a continuous version the map $(\tau, x) \mapsto \partial_{ik}^2 Y_t^{r,x}$. This implies that there exists a set $\Omega_5$ of full $\mathbb{F}$-measure, that does not depend on $\tau$ and $x$, such that for all $\omega \in \Omega_5$, $Y_t^{r,x}$ is twice continuously differentiable in $x$, and, moreover, the derivatives of $Y_t^{r,x}$ up to the second order are bounded.

Step 4. Solution to random PDE (7). Define $y(\tau, x, \omega) = Y_t^{r,x}(\omega)$ for each $\omega \in \Omega_5$. Since $y(\tau, x) \in G^r_t$-measurable, it is $F_{t, \gamma} - \mathbb{F}$-measurable. Let us prove that there exists a set $\Omega_6$ of full $\mathbb{F}$-measure and a constant $\gamma_{K,C} < \bar{\gamma}_{K,C}$ such that

$$
Y_t^{r,x} = \tilde{y}(t, X_t^{r,x})
$$

for all $\omega \in \Omega_6$, $\tau \in [T - \bar{\gamma}_{K,C}, T]$, $t \in [\tau, T]$, and $x \in \mathbb{R}^n$. Note that $Y_t^{r,x} = Y_t^{t, X_t^{r,x}}$, where $(X_t^{t, \xi}, Y_t^{t, \xi}, Z_t^{t, \xi})$ denotes the solution to (9) such that $X_t^{t, \xi} = \xi$ (remark, that the contraction argument developed in Step 1 holds if we replace $x$ by a $G^r_t$-measurable random variable $\xi$ such that $X_t^{t, \xi} = \xi$). It suffices to show that for any $G^r_t$-measurable random variable $\xi$, it holds that $Y_t^{t, \xi} = \tilde{y}(t, \xi)$, a.s. Let $\xi = \sum_{i=1}^{\infty} x_i 1_{A_i}$, $x_i \in \mathbb{R}^n$, be a simple $G^r_t$-measurable random variable, where $A_i \in G^r_t$ are disjoint sets covering $\Omega$. By Theorem A.2 from [13], on $A_i \cap \Omega_1$, $Y_t^{r,x} = Y_t^{r,\xi}$ for each $i$. Therefore, for a simple random variable $\xi$ it hold that, $Y_t^{r,\xi} = \tilde{y}(t, \bar{\xi})$ a.s. For an arbitrary $G^r_t$-measurable random variable $\xi$, take a simple random variable $\xi$ such that $|\xi - \bar{\xi}| < \varepsilon$ uniformly in $\omega$, where $\varepsilon > 0$ is small arbitrary fixed number. Next, we note that (20) implies $|\tilde{y}(\tau, x) - \tilde{y}(\tau, x')| = |Y_{r,x}^{r,x} - Y_{r,x}^{r,x'}| < \gamma_5 |x - x'|$ on a set of full $\mathbb{F}$-measure that can be chosen independent of $x$ by continuity of the map $x \mapsto Y_{r,x}^{r,x}$ for $\omega \in \Omega_1$. Therefore, a.s., $|\tilde{y}(\tau, \xi) - \tilde{y}(\tau, \bar{\xi})| < \gamma_5 \varepsilon$. Together with Theorem A.2 from [13] this implies that $E|Y_{r,x}^{r,\xi} - \tilde{y}(\tau, \xi)|^2 < \mu_6 \varepsilon^2$. Therefore, $Y_t^{r,\xi} = \tilde{y}(t, \xi)$ a.s., and, consequently, (32) is fulfilled on a set $\Omega_{t, \gamma, x} \subset \Omega$ of full $\mathbb{F}$-measure that may depend on $t$, $\tau$, and $x$.

By continuity of the maps $(\tau, x, t) \mapsto (X_{t, \xi}^{r,x}, Y_{t, \xi}^{r,x})$ and $(x, t) \mapsto \tilde{y}(t, x)$, there exists a set $\Omega_6$ of full $\mathbb{F}$-measure such that identity (32) holds for all $\omega \in \Omega_6$ and all $(\tau, x, t) \in [T - \bar{\gamma}_{K,C}, T] \times \mathbb{R}^n \times [T - \bar{\gamma}_{K,C}, T]$.

Let us prove that $\tilde{y}(t, x)$ is a solution to (7). The proof is similar to the proof of Theorem 3.2. in [27] (p. 213). However, we give the proof here since we deal with the random coefficient case. Define $L \mu = \nu \Delta u + (u + \bar{\eta}, \nabla)u$. We have

$$
\tilde{y}(t + h, x) - \tilde{y}(t, x) = [\tilde{y}(t + h, x) - \tilde{y}(t + h, X_t^{t+h})] + [\tilde{y}(t + h, X_t^{t+h}) - \tilde{y}(t, x)].
$$

Since $\tilde{y}$ is of class $C_b^{0,2}$, we can apply Itô’s formula to the first term. Further, by (9) and (32), we substitute the second term with $- \int_t^{t+h} F(s, X_s^{t,x}, \tilde{y}(s, X_s^{t,x})ds + ...$
Corresponding solutions cannot be applied here since \( F^\text{M} \) to obtain, by virtue of Corollary 1, a common deterministic bound.

Proof. The idea of the proof is to use the mollifications (in \( t \)) \( \eta_m \) and \( F_m \) for \( \eta \) and \( F \) to obtain, by virtue of Corollary 1, a common deterministic bound \( M \) for the corresponding solutions \( \bar{y}_m \), as well as a deterministic bound \( M_1 \) for their gradients \( \partial_y \bar{y}_m \). Further, we prove that \( M \) is the bound for the solution \( \bar{y} \) to equation (7), while \( M_1 \) is the bound for its gradient \( \partial_y \bar{y} \), which allows to obtain the global existence.

Step 1. Existence of a local \( \mathcal{F}^B_{T-t} \)-adapted solution to (7). Remark that Theorem 2 cannot be applied here since \( F(t,x,y) \), as well as its first and second order derivatives in \( x \), have linear growth in \( y \), and, therefore, \( F \) does not belong to class \( C^{1,2}_b([0,T] \times \mathbb{R}^{2n}) \).

\[
\int_t^{t+\eta} Z_{s,x}^t dW_s. \text{ Thus, we obtain that, a.s.,}
\]

\[
\bar{y}(t+h,x) - \bar{y}(t,x) = - \int_t^{t+\eta} \mathcal{L}\bar{y}(t+h,X_s^t,x)ds - \sqrt{2\nu} \int_t^{t+\eta} \nabla \bar{y}(t+h,X_s^t,x)dW_s
\]

\[
- \int_t^{t+\eta} F(s,X_s^t,x,\bar{y}(s,X_s^t))ds + \int_t^{t+\eta} Z_{s,x}^t dW_s.
\]

Fix a partition \( \mathcal{P} = \{ \tau = t_0 < t_1 < \cdots < t_n = T \} \). Taking the conditional expectation \( E_\tau \) of the both parts and summing up, we obtain

\[
(33) \quad \bar{y}(\tau,x) - h(x) = E_\tau \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\mathcal{L}\bar{y}(t_{i+1},X_{s,x}^{t_i}) + F(s,X_s^t,x,\bar{y}(s,X_s^t))ds,
\]

a.s. Indeed, the conditional expectation of the stochastic integrals is zero by Lemma 3. Note that the expression under the integral sign is bounded, a.s., since \( \mathcal{L}\bar{y}(t,x) \) is bounded by what was proved in the previous steps, and \( F(s,x,y) \) is bounded by the assumption. Moreover, \( \mathcal{L}\bar{y}(t,x) \) and \( F(s,X_s^t,x,\bar{y}(s,X_s^t)) \) are a.s. continuous in \((t,x)\). Letting the mesh of \( \mathcal{P} \) in (33) go to zero, by the conditional bounded convergence theorem, we obtain that there exists a set \( \Omega_x \subset \Omega \) (that may depend on \( x \)) of full \( \mathcal{P} \)-measure such that \( \bar{y}(t,x) \) solves (7) on \( t \in [T - \gamma_{K,C}, T] \) for each \( \omega \in \Omega_x \). Since \( \bar{y}(t,x) \), \( \mathcal{L}\bar{y}(t,x) \) and \( F(t,x,\bar{y}(t,x)) \) are continuous in \((t,x)\) for every \( \omega \in \Omega_5 \), we conclude that the set \( \Omega_x \), where (7) holds, can be chosen independent of \( x \). Therefore, \( \bar{y}(t,x) \) solves (7) on \( [T - \gamma_{K,C}, T] \times \mathbb{R}^n \) a.s. Finally, by (13), (22), (28), and equation (7), we conclude that, a.s., \( \bar{y} \in C^{1,2}_b \). The theorem is proved. □

Corollary 2. Under assumptions of Theorem 2, there exists a constant \( \gamma_{K,C} \) such that for each \( \tau \in [T - \gamma_{K,C}, T] \) there exists a unique \( \mathcal{G}^{-}_t \)-adapted solution \((X^\tau_t,x, Y^\tau_t,x, Z^\tau_t,x)\) to FBSDEs (9) as well as its first and second derivative processes \((\partial_y X^\tau_t,x, \partial_y Y^\tau_t,x, \partial_y Z^\tau_t,x)\) and \((\partial_y^2 X^\tau_t,x, \partial_y^2 Y^\tau_t,x, \partial_y^2 Z^\tau_t,x)\), respectively. Moreover, the solution \( \bar{y}(\tau,x) \) to (7) is a \( C^{1,2}_b \)-version of \( Y^\tau_t,x \).

2.3 Global existence in the bounded case

In this subsection we will prove the global existence of a \( C^{1,2}_b \)-solution to (7) under (A1)–(A3) and the assumption that \( \eta \) and \( h \) are bounded uniformly in \( \omega \in \Omega_0 \) in the norms of \( C^{0,4}_b \) and \( C^2_b \), respectively. The force term \( F \) is assumed to be given by (5). The set of full \( \mathcal{P} \)-measure \( \Omega_0 \) is as defined in Remark 5.

Theorem 3. Let (A1)–(A3) hold. Further let \( \eta(t,x) \) and \( h(x) \) be bounded uniformly in \( \omega \in \Omega_0 \) with respect to the norms of \( C^{0,4}_b \) and \( C^2_b \), respectively. Assume that \( F(t,x,y) \) is given by (5). Then, there exists a unique \( \mathcal{F}^B_{T-t} \)-adapted solution to backward equation (7) which belongs to class \( C^{1,2}_b \) for all \( \omega \in \Omega_0 \).

Proof. The idea of the proof is to use the mollifications (in \( t \)) \( \eta_m \) and \( F_m \) for \( \eta \) and \( F \) to obtain, by virtue of Corollary 1, a common deterministic bound \( M \) for the corresponding solutions \( \bar{y}_m \), as well as a deterministic bound \( M_1 \) for their gradients \( \partial_y \bar{y}_m \). Further, we prove that \( M \) is the bound for the solution \( \bar{y} \) to equation (7), while \( M_1 \) is the bound for its gradient \( \partial_y \bar{y} \), which allows to obtain the global existence.

Step 1. Existence of a local \( \mathcal{F}^B_{T-t} \)-adapted solution to (7). Remark that Theorem 2 cannot be applied here since \( F(t,x,y) \), as well as its first and second order derivatives in \( x \), have linear growth in \( y \), and, therefore, \( F \) does not belong to class \( C^{0,2}_b([0,T] \times \mathbb{R}^{2n}) \).
Let \( \eta_m(t, x) \) and \( f_m(t, x, y) \) be the mollifications of \( \eta(t, x) \) and \( f(t, x, y) \) in \( t \) given by \( \eta_m(t, x) = \int_{-\infty}^{\infty} \eta(t - s, x) \varphi_{\frac{1}{m}}(s)ds \) and \( f_m(t, x, y) = \int_{-\infty}^{\infty} f(t - s, x, y) \varphi_{\frac{1}{m}}(s)ds \), where \( \varphi_{\frac{1}{m}}(t) \) is the standard mollifier supported on \([-\varepsilon, \varepsilon]\). To have the functions \( \eta_m \) and \( f_m \) well-defined, we extend \( \eta(t, x) \) to \((T, \infty)\) by \( \eta(T, x) \), and to \((-\infty, 0)\) by \( \eta(0, x) = 0 \). The function \( f \) is assumed to be extended to \((-\infty, 0) \cup (T, \infty)\) in the similar manner. Define

\[
F_m(t, x, y) = f_m(t, x, y + \eta_m(t, x)) + \nu \Delta \eta_m(t, x) - (y + \eta_m, \nabla)\eta_m(t, x).
\]

Consider the equations

\begin{align*}
\bar{y}(t, x) &= h(x) + \int_{t}^{T} \left[ \nu \Delta \bar{y}(s, x) - \bar{\eta}_m(t, x) + \bar{y}, \nabla \bar{y}(s, x) + \bar{F}_m(s, x, \bar{y}) \right] ds,
\end{align*}

where \( \bar{\eta}_m(t, x) = \eta_m(T - t, x) \) and \( \bar{F}_m(t, x, y) = \bar{F}_m(T - t, x, y) \). Note that \( F(t, x, y) \) and \( \eta(t, x) \), satisfy Assumption (iii) of Theorem 1, where the constant \( C(\omega) \) is deterministic (we denote it by \( \bar{C} \)). Observe that the mollifications \( F_m(t, x, y) \) and \( \eta_m(t, x) \) also satisfy Assumption (iii) with the same constant \( \bar{C} \). Hence, by Theorem 1, for every fixed \( \omega \in \Omega_0 \) (frozen parameter \( \omega \)), there exists a unique \( C_1^b \)-solution \( \bar{y}_m(t, x) \) to (34) on \([0, T]\). Moreover, by Corollary 1, the global bound \( M \) for \( \bar{y}_m(t, x) \) on \( \Omega_0 \) is deterministic and depends only on \( \bar{C} \), \( T \), and on the deterministic bound for \( [h(x)] \). Now let \( \xi_M(y) = \xi_M(y) \), where \( \xi_M(y) \) is a smooth cutting function for the ball \( B_M \) of radius \( M \) centered at the origin, defined as follows: \( \xi_M(y) = 1 \) if \( y \in B_M \), \( \xi_M(y) = 0 \) if \( y \notin B_M \), and \( \xi_M(y) \) decays from one to zero along the radius of the ball such that \( \nabla \xi_M \) and the \( \nabla^2 \xi_M \) are bounded and continuous. We modify \( F \) and \( F_m \) by introducing \( \xi_M(y) \) instead of \( y \):

\begin{align*}
F^M_m(t, x, y) &= f_m(t, x, y + \eta_m(t, x)) + \nu \Delta \eta_m(t, x) - (\xi_M(y) + \eta_m, \nabla)\eta_m(t, x),
\end{align*}

\begin{align*}
F^M(t, x, y) &= f(t, x, y + \eta(t, x)) + \nu \Delta \eta(t, x) - (\xi_M(y) + \eta, \nabla)\eta(t, x).
\end{align*}

Further, we note that the solution \( \bar{y}_m(t, x) \) to (34) is also the solution to

\begin{align*}
\bar{y}(t, x) &= h(x) + \int_{t}^{T} \left[ \nu \Delta \bar{y}(s, x) - \bar{\eta}_m(t, x) + \bar{y}, \nabla \bar{y}(s, x) + \bar{F}_m(s, x, \bar{y}) \right] ds,
\end{align*}

where \( \bar{F}_m(t, x, y) = F^M_m(T - t, x, y) \). Moreover, by Theorem 1, the \( C_1^b \)-solution to (37) is unique. Therefore, for all \( \omega \in \Omega_0 \), the unique \( C_1^b \)-solution to (34) and (37) is the same. Further, we note that since \( \eta_m(t, x) = \int_{t}^{T} \xi_M(y) + \eta_m, \nabla)\eta_m(t, x), \) then \( \eta_m(t, x) \) and \( F_m(t, x, y) \) are adapted with respect to the filtration \( \mathcal{F}^{B_m}_{t+\frac{1}{m}} \) \( t \geq 0 \) for each fixed \( x \) and \( y \). Introduce constants \( K \) and \( C \) by formulas (8) with respect to \( F^M \) and \( \eta \), and the constants \( K_m \) and \( C_m \) by the same formulas with respect to \( F^M_m \) and \( \eta_m \), and note that \( K_m \leq K \) and \( C_m \leq C \) for all \( m \). Theorem 2, applied to (37) with respect to the filtration \( \mathcal{F}^{B}_{T-1} = \mathcal{F}^{B}_{T-1+\frac{1}{m}} \) implies that for each \( m \), (37) possesses an \( \mathcal{F}^{B}_{T-1+\frac{1}{m}} \)-adapted \( C_1^b \)-solution on the interval \([T - \gamma_{C,K}, T]\). By uniqueness (Theorem 1), the aforementioned solution coincides with \( \bar{y}_m(t, x) \). We remark that the short time interval \([T - \gamma_{C,K}, T]\) is the same for all \( m \).
Consider the equation
\begin{equation}
\bar{y}(t, x) = h(x) + \int_t^T [\nu \Delta \bar{y}(s, x) - (\bar{y}(t, x) + \bar{y})], \nabla \bar{y}(s, x) + \bar{F}^M(s, x, \bar{y})] ds,
\end{equation}
where \( \bar{F}^M(t, x, y) = F^M(t - x, y) \). By Theorem 2, there exists an \( F^B_{\tau,x} \)-adapted \( C^1_x \)-solution \( \bar{y}(t, x) \) to (38) on \( [T - \gamma_{C,K}, T] \). We use the associated FBSDEs to show that there exists a constant \( \gamma'_{K,C} < \gamma_{K,C} \) such that as \( m \to \infty \),
\begin{equation}
\mathbb{E}[|\bar{y}_m(\tau, x) - \bar{y}(\tau, x)|^2 + |\partial_x \bar{y}_m(\tau, x) - \partial_x \bar{y}(\tau, x)|^2] \to 0.
\end{equation}
for each \( (\tau, x) \in [T - \gamma'_{K,C}, T] \times \mathbb{R}^n \).

Let \( (X^1_{T,x,m}, Y^1_{T,x,m}, Z^1_{T,x,m}) \) and \( (X^2_{T,x}, Y^2_{T,x}, Z^2_{T,x}) \) be \((\tau, x, t)\)-continuous versions of the solutions to the FBSDEs associated to (37) and (38) on \([T - \gamma_{K,C}, T]\) whose existence was stated in Corollary 2. We consider FBSDEs (40) and (41) with respect to the same filtration \( \mathcal{F}^{\tau, m_0} = \mathcal{F}^W \vee \mathcal{F}^B_{T - t + \frac{1}{m_0}} \) and for \( m > m_0 \). Moreover, by Corollary 2, \( \bar{y}_m(\tau, x) \) and \( \bar{y}(\tau, x) \) are \( C^0_{\tau,x} \)-versions of \( Y_{T,x}^1 \) and, respectively, \( Y_{T,x}^2 \). By Theorem A.2 from [13] (p. 258), there exists a constant \( \gamma'_{K,C} < \gamma_{K,C} \) such that for \( \tau \in [T - \gamma'_{K,C}, T] \) we have the estimates
\begin{align}
\mathbb{E} \sup_{t \in [\tau, T]} |X^m_t - X_t|^2 + \mathbb{E} \sup_{t \in [\tau, T]} |Y^m_t - Y_t|^2 \\
\leq \mu_1 \mathbb{E} \int_{\tau}^{T} (|\bar{F}^M - \bar{F}^M| + |\bar{y}_m - \bar{y}|)(t, X^m_t, Y^m_t) dt,
\end{align}
\begin{align}
\mathbb{E} \sup_{t \in [\tau, T]} |\partial_k Y^m_t - \partial_k Y_t|^2 \\
\leq \mu_2 \left( \mathbb{E} \int_{\tau}^{T} \left| (\nabla_1 \bar{F}^m(s, X^m_s, Y^m_s) - \nabla_1 \bar{F}^M(s, X_s, Y_s)) \partial_k X_s \right| ds \\
+ \mathbb{E} \int_{\tau}^{T} \left| (\nabla_2 \bar{F}^m(s, X^m_s, Y^m_s) - \nabla_2 \bar{F}^M(s, X_s, Y_s)) \partial_k Y_s \right| ds \\
+ \mathbb{E} \int_{\tau}^{T} \left| (\nabla \bar{y}_m(s, X^m_s) - \nabla \bar{y}(s, X_s)) \partial_k X_s \right| ds + \left| (\nabla h(X^m_s) - \nabla h(X_T)) \partial_k X_T \right| \right).
\end{align}
As before, we skipped the upper index \( \tau, x \) for simplicity of notation. Estimates (42) and (43) show that convergence (39) holds for all \( (\tau, x) \in [T - \gamma'_{K,C}, T] \times \mathbb{R}^n \). Therefore, by (39) and the continuity of \( \bar{y}(\tau, x) \) and \( \partial_x \bar{y}(\tau, x) \) in \( \tau \) and \( x \),
\begin{equation}
|\bar{y}(\tau, x)| < M \quad \text{and} \quad |\partial_x \bar{y}(\tau, x)| < M_1 \quad \text{a.s.}
\end{equation}
Moreover, the set of full \( \mathbb{P} \)-measure, where (44) holds, can be chosen independent of \( \tau \) and \( x \). This implies that \( \xi(T) = \bar{y} \) and, therefore, \( \bar{y}(t, x) \) is a solution to (7).

Step 2. Global existence and uniqueness for equation (7). Without loss of generality, \( \gamma_{K,C} = \gamma'_{K,C} \), i.e., bounds (44) hold, a.s., for all \( (\tau, x) \in [T - \gamma_{K,C}, T] \times \mathbb{R}^n \). Let \( K_1 = K + M_1 \), where \( K \) is defined by (8). Consider equation (7) on the interval \([T - \gamma_{K,C} - \gamma_{K_1,C}, T - \gamma_{K,C}] \) with the final condition \( \bar{y}(T - \gamma_{K,C}, x) \).
Corollary 3. Let (A1)–(A3) hold, and let \( \nu \in (0, \nu_0) \). Then, the constants \( M \) and \( M_1 \), defined in the proof of Theorem 3, are global bounds for \( \bar{y}(t,x) \) and \( \partial_x \bar{y}(t,x) \), respectively. Moreover, \( M \) only depends on \( T, L, \nu_0, \|h(x)\| \), and the constant \( \bar{C} \) with the property \( |F(t,x,y)| \leq \bar{C}(1 + |y|) \). Since \( \bar{E} \) is given by (5), the constant \( \bar{C} \) depends on \( L, \nu_0 \), and \( \|\eta(t,x)\|_{C_b^{0,2}} \). □

Remark 7. Note that \( \bar{C} \) is an increasing function of \( \|\eta(t,x)\|_{C_b^{0,2}} \). Therefore, in Corollary 3, any constant bigger than \( \|\eta(t,x)\|_{C_b^{0,2}} \) will also determine a bound for \( \bar{y} \).

Corollary 4. Consider problem (7) with deterministic functions \( \eta \in C_b^{2,4}([0,T] \times \mathbb{R}^n) \), \( h \in C_b^{2+\beta} (\mathbb{R}^n) \), and \( F(t,x,y) \) given by (5). Then, there exists a unique deterministic \( C_b^{1,2} \)-solution to (7).

Proof. First, consider (7) on the short-time interval \([T - \gamma_{K,C}, T]\), where \( \gamma_{K,C} \) is defined by (8). By Corollary 2, the solution \( \bar{y}(t,x) \) to (7) on \([T - \gamma_{K,C}, T]\) is a \( C_b^{0,2} \)-version of \( Y_{\gamma_{K,C}} \). Therefore, \( \bar{y}(t,x) \) is deterministic with probability 1. Namely, for each \((t,x) \in [T - \gamma_{K,C}, T] \times \mathbb{R}^n\), there exists a set \( \Omega_{t,x} \), \( P(\Omega_{t,x}) = 1 \), such that \( y(t,x,\omega_1) = \bar{y}(t,x,\omega_2) \) for all \( \omega_1, \omega_2 \in \Omega_{t,x} \). By the \((t,x)\)-continuity of \( \bar{y}(t,x) \), we can find a set \( \Omega_{t} \), \( P(\Omega_{t}) = 1 \), which does not depend on \( t \) and \( x \), such that \( \bar{y}(t,x) \) is deterministic on \( \Omega_{t} \). By modifying \( y(t,x) \) on \( \Omega_{t} \), we obtain a deterministic solution \( \bar{y}(t,x) \) on \([T - \gamma_{K,C}, T] \times \mathbb{R}^n\). Further, the proof of Theorem 3 implies that we can construct a unique deterministic solution \( \bar{y}(t,x) \) on \([0,T] \times \mathbb{R}^n\). □

2.4 Global existence in the general case

Now we prove our main result which is the global existence and uniqueness of solution to stochastic Burgers equation (1).

Theorem 4. Let (A1)–(A3) hold. Then, there exists a unique \( F_t^B \)-adapted \( C_b^{0,2} \)-solution to SPDE (1).

Proof. Fix \( \omega_0 \in \Omega_0 \). Everywhere below, \( k_i(\omega_0) \), \( i = 1, 2, \ldots \), are positive deterministic constants that may depend on \( \omega_0 \).

Equation (7), evaluated at \( \omega_0 \), can be regarded as a deterministic equation. By Corollary 4, there exists a unique deterministic solution \( \bar{y}(t,x,\omega_0) \) to (7). Moreover,
\( \bar{y}(t, x, \omega_0) \) and \( \partial_x \bar{y}(t, x, \omega_0) \) are bounded, and since \( \omega_0 \) is fixed, the corresponding bounds \( M(\omega_0) \) and \( M_1(\omega_0) \) are regarded as deterministic.

Define the sequence of stopping times

\[
T_N = T \land \inf \{ t \in (0, T) : \| \eta(t, \cdot) \|_{C^3_b(\mathbb{R}^n)} > N \},
\]

where \( N > 0 \) is an integer. Note that since \( \eta \in C^{0,4}_b([0, T] \times \mathbb{R}^n) \) on \( \Omega_0 \), then the stopping time \( T_N \) is non-zero on \( \Omega_0 \). Further, we define

\[
\eta_N(t, x) = \eta(t \land T_N, x) \quad \text{and} \quad h_N(x) = h(x) 1_{\{ \| h \|_{C^3_b(\mathbb{R}^n)} \leq N \}}.
\]

Note that for each \( \omega \in \Omega_0 \), \( \| \eta_N \|_{C^{0,4}_b([0, T] \times \mathbb{R}^n)} \leq N \) and \( \| h_N \|_{C^3_b(\mathbb{R}^n)} \leq N \).

Let \( F_N(t, x, y) \) be defined by (5) via \( \eta_N \). Note that as \( N \to \infty \), \( \eta_N(t, x) \to \eta(t, x) \), \( h_N(x) \to h(x) \), and \( F_N(t, x, y) \to F(t, x, y) \) on \( \Omega_0 \times [0, T] \times \mathbb{R}^n \). Consider the backward equation

\[
\bar{y}(t, x) = h_N(x) + \int_t^T \left[ \mu \Delta \bar{y}(s, x) - (\bar{\eta}_N(t, x) + \bar{y}, \nabla) \bar{y}(s, x) + \bar{F}_N(s, x, \bar{y}) \right] ds,
\]

where \( \bar{F}_N(t, x, y) = F_N(T - t, x, y) \) and \( \bar{\eta}_N(t, x) = \eta_N(T - t, x) \). By Theorem 3, there exists a unique global \( \mathcal{F}_{T-\tau}^B \)-adapted solution \( \bar{y}_N(t, x) \) which belongs to class \( C^{1,2}_b \) for all \( \omega \in \Omega_0 \). Let us prove that

\[
\lim_{N \to \infty} \bar{y}_N(t, x, \omega_0) = \bar{y}(t, x, \omega_0).
\]

If (48) is proved, then \( \bar{y}(t, x) \) will be \( \mathcal{F}_{T-\tau}^B \)-adapted. Indeed, \( \omega_0 \in \Omega_0 \) was fixed arbitrarily and \( \bar{y}_N(t, x) \) is \( \mathcal{F}_{T-\tau}^B \)-adapted.

Thus, we aim to show (48). Let \( (X_t, Y_t, Z_t) \) be the unique \( \mathcal{F}_t^W \)-adapted solution to

\[
\begin{aligned}
X_t & = x - \int_t^T \left( \bar{\eta}(s, X_s, \omega_0) + Y_s \right) ds + \sqrt{2\nu}(W_t - W_s), \\
Y_t & = h(X_T, \omega_0) + \int_t^T \bar{F}_N(s, X_s, Y_s, \omega_0) ds - \int_t^T Z_s dW_s,
\end{aligned}
\]

and \( \tau \in [T - \gamma_{K_{\omega_0}}, T] \), where the constant \( \gamma_{K_{\omega_0}} \) is defined similar to the constant \( \gamma_K \) whose existence was established in Step 1 of the proof of Theorem 2. Since we deal with deterministic FBSDEs, the constant \( K_{\omega_0} \) is defined as follows:

\[
K_{\omega_0} = \sup_{\mathbb{R}^n} | \nabla h(x, \omega_0) | + \sup_{[0, T] \times \mathbb{R}^n} | \partial_x \eta(t, x, \omega_0) | + \sup_{[0, T] \times \mathbb{R}^n} \left( | \partial_x F^M(\omega_0)(t, x, y, \omega_0) | + | \partial_y F^M(\omega_0)(t, x, y, \omega_0) | \right).
\]

Further, let \( (X_t^N, Y_t^N, Z_t^N) \) be the \( \mathcal{F}_t^W \)-adapted solutions to the FBSDEs

\[
\begin{aligned}
X_t^N & = x - \int_t^T \left( \bar{\eta}_N(s, X_s^N, \omega_0) + Y_s^N \right) ds + \sqrt{2\nu}(W_t - W_s), \\
Y_t^N & = h_N(X_T^N, \omega_0) + \int_t^T \bar{F}_N(s, X_s^N, Y_s^N, \omega_0) ds - \int_t^T Z_s^N dW_s,
\end{aligned}
\]

where \( \tau \in [T - \gamma_{K_{\omega_0}}, T] \), and \( \bar{F}_N(\omega_0) \) is defined by (36) via \( \eta_N \) instead of \( \eta \). Remark that the constant \( \gamma_{K_{\omega_0}} \) can be chosen the same for FBSDEs (49) and (51). Indeed, the length \( \gamma_{K_{\omega_0}} \) is determined only by the constant \( K_{\omega_0} \). We can take \( K_{\omega_0} \) bigger than the right-hand side of (50) by estimating the derivatives \( \partial_x F^M(\omega_0) \) and \( \partial_y F^M(\omega_0) \) via \( L, M(\omega_0) \), and \( | h \|_{C^{0,3}_b} \) (see the definition of \( F \) by (36)). Since for all \( N \), \( | \eta_N \|_{C^{0,3}_b} \leq \| \eta \|_{C^{0,3}_b}, | h_N(x) \| \leq | h(x) | \), and \( | \partial_x \eta_N(t, x) | \leq | \partial_x \eta(t, x) | \), then the constant \( \gamma_{K_{\omega_0}} \) for FBSDEs (49) and (51) can be chosen the same.
Note that by Theorem 1, $M(\omega_0)$ is also the bound for $\bar{y}_N(t,x,\omega)$ for all $N$. Further, the processes $Y^\tau_{t,x}$ and $Y^{N\tau}_{t,x}$, obtained from the solutions to (49) and (51), are deterministic, a.s., and by Corollaries 2 and 4, $\bar{y}(t,x,\omega)$ is a deterministic $C^{0,2}_b$-version of $Y^\tau_{t,x}$, while $\bar{y}_N(t,x,\omega_0)$ is a deterministic $C^{0,2}_b$-version of $Y^{N\tau}_{t,x}$. Now we again apply Theorem 1.3 from [13] (p. 218), to conclude that there exists a constant $\gamma_{K_\omega} < \gamma_{K_\omega}$ such that for $\tau \in [T - \gamma_{K_\omega}, T]$ we have the estimate

$$\|\bar{y}_N(t,x,\omega_0) - \bar{y}(t,x,\omega_0)\|^2 \leq k_3(\omega_0)\left(\mathbb{E}\|h - h_N\|^2(X^\tau_{t,x},\omega_0)\right)$$

$$+ \mathbb{E} \int_{\tau}^{T} \left(\|\tilde{F}^M(\omega_0) - \tilde{F}^M_N(\omega_0)\|^2 + |\bar{y} - \bar{y}_N|^2(t, X^\tau_{t,x}, Y^{\tau}_{t,x}, \omega_0) dt\right).$$

By (13), $\mathbb{E}|Y^\tau_{t,x}|^p \leq k_2(\omega_0)$. Therefore, the integrand of the second term on the right-hand side of (52) is bounded by an integrable function, and we can pass to the limit under the integral sign. This implies that convergence (48) holds on $[T - \gamma_{K_\omega}, T]$.

Now let $T_1 = T_1(\omega_0) = T - \gamma_{K_\omega}$. Consider the equations

$$\bar{y}(t) = \bar{y}_N(T_1, t, x, \omega_0) + \int_{t}^{T_1} \left[\mu \Delta \bar{y}(s, x) - (\bar{y}_N(s, x, t, \omega_0) + \bar{y}, \nabla)\bar{y}(s, x) + \tilde{F}^M_N(\omega_0)\right] ds,$$

where $\tilde{F}^M_N(\omega_0)$ and $F^M(\omega_0)$ are evaluated at $(s, x, \bar{y}(s, x), \omega_0)$. Now since at $\omega_0$, $\|\bar{y}_N\|_{C^{0,2}_b} \leq \|\bar{y}\|_{C^{0,2}_b}$ and $\|h_N\|_{C^1} \leq \|h\|_{C^1}$, then, by Theorem 1, the global bound $\partial_x \tilde{y}(t, x, \omega_0)$ can be chosen the same as for $\partial_x \tilde{y}(t, x, \omega_0)$, i.e. $M_1(\omega_0)$. Define $K^{(1)}_{\omega_0} = K_{\omega_0} + M_1(\omega_0)$ (the choice of $K_{\omega_0}$ was discussed above and involves the estimate of the right-hand side of (50)). Similar to (52), there exists a constant $\gamma_{K^{(1)}_{\omega_0}}$, depending only on $K^{(1)}_{\omega_0}$, such that on $[T_1 - \gamma_{K^{(1)}_{\omega_0}}, T_1]$, it holds that

$$\|\bar{y}_N(t,x,\omega_0) - \bar{y}(t,x,\omega_0)\|^2 \leq k_3(\omega_0)\left(\mathbb{E}\|\bar{y}_N(T_1, X_T, \omega_0) - \bar{y}(T_1, X_T, \omega_0)\|^2\right)$$

$$+ \mathbb{E} \int_{T_1}^{T} \left(\|\tilde{F}^M(\omega_0) - \tilde{F}^M_N(\omega_0)\|^2 + |\bar{y} - \bar{y}_N|^2(t, X_T, Y_T, \omega_0) dt\right).$$

This proves convergence (48) on $[T - \gamma_{K^{(1)}_{\omega_0}} - \gamma_{K_\omega}, T]$. Note that $\gamma_{K^{(1)}_{\omega_0}}$ can be chosen as the length of each successive short-time interval starting from the second since $M_1(\omega_0)$ is the global bound for $\partial_x \tilde{y}(t, x, \omega_0)$. Therefore, convergence (48) holds on $[0, T] \times \mathbb{R}^n$. Since $\omega_0 \in \Omega_0$ was fixed arbitrary, then $\bar{y}_N(t,x,\omega_0) \rightarrow \bar{y}(t,x,\omega)$ on $[0, T] \times \mathbb{R}^n \times \Omega_0$. This implies that for each $x \in \mathbb{R}^n$, $\bar{y}(t,x)$ is $F^B_{T-t}$-adapted. By Lemma 2, $y(t,x) = \bar{y}(T-t,x) + \eta(t,x)$ is the solution to (1). This solution is defined everywhere on $\Omega_0$ and it is $F^B_{T-t}$-adapted. Since $\bar{y}(t,x)$ is of class $C^{3,4}_b$ and $\eta(t,x)$ is of class $C^{2,4}_b$, then $y(t,x)$ is of class $C^{2,4}_b$. The theorem is proved.

**Corollary 5.** Let (A1)–(A3) hold. Further let $h(x)$, $f(t,x,y)$, and $g(t,x)$, $t \in [0, T]$, are spatially periodic in $x$. Then, the solution $y(t,x)$ to (1) on $[0, T]$ is spatially periodic.

**Proof.** The periodicity of $y(t,x)$ is a consequence of uniqueness. □
3. The inviscid limit

Here we investigate the local behavior of the solution to (1) when the viscosity \( \nu \) goes to zero. Throughout this section, the \( C_b^{0,2} \)-norm of the function \( h(x) \) is assumed bounded in \( \omega \). At first, the term \( \eta(t,x) \) will be also assumed bounded in \( \omega \) in the \( C_b^{0,4} \)-norm. This will allow to prove that the local inviscid limit for equation (4) exists on \([0, \beta_{K,C}]\), where \( \beta_{K,C} \) is a small constant that depends only on the constants \( K \) and \( C \), defined by (8). Further, an application of the above result to \( \eta = \eta_N \), where \( \eta_N \) is given by (46), will imply that the inviscid limit to SPDE (1) exists on \([0, T_N \wedge \beta_{K,C}]\), where the stopping time \( T_N \) is defined by (45).

Everywhere below \( \beta_i, i = 1, 2, \ldots \), denote positive constants, and \( E^B_\gamma \) denote the conditional expectation with respect to \( F^B_\gamma \). The set of full \( \mathbb{P} \)-measure \( \Omega_0 \) is defined, as before, in Remark 5.

**Lemma 4.** Assume (A1) – (A3). Further assume that \( \|h\|_{C^2_b} \) are \( \|\eta\|_{C_b^{0,4}} \) are bounded in \( \omega \in \Omega_0 \). Then, for all \( \omega \in \Omega_0 \), the system of forward-backward random equations

\[
\begin{align*}
X^\tau,x,0_t &= x - \int_t^\tau (\bar{\eta}(s, X^\tau,x,0_s) + Y^\tau,x,0_s) \, ds \\
Y^\tau,x,0_t &= h(X^\tau,x,0_t) + \int_t^\tau M(s, X^\tau,x,0_s, Y^\tau,x,0_s) \, ds
\end{align*}
\]

possesses a unique \((\tau,x,t)\)-continuous solution \((X^\tau,x,0_t, Y^\tau,x,0_t)\) on \([T - \gamma_{K,C}, T]\), where the constant \( \gamma_{K,C} \) is defined by Theorem 2. In (53), \( \bar{\eta}(t,x) = \eta(T-t,x), \) \( F(M(t,x,y) = F^{M}(T-t,x,y) \) with \( F^{M}(t,x,y) \) given by (36), and the constant \( M \) is the deterministic bound (independent of \( \nu \in (0,\nu_0] \)) for the solution \( \bar{y}(t,x) \) to equation (7) whose existence was established by Corollary 3.

**Proof.** For each fixed \( \omega \in \Omega_0 \), we consider the map \( C([\tau,T], \mathbb{R}^n) \to C([\tau,T], \mathbb{R}^n) \), \( Y^\tau,x = \Gamma(Y^\tau,x) \),

\[
\begin{align*}
X^\tau,x_t &= x - \int_t^\tau (\bar{\eta}(s, X^\tau,x_s) + \bar{Y}^\tau,x_s) \, ds \\
Y^\tau,x_t &= h(X^\tau,x_t) + \int_t^\tau M(s, X^\tau,x_s, Y^\tau,x_s) \, ds
\end{align*}
\]

Given \( \bar{Y}^\tau,x \), the process \( X^\tau,x_t \) is obtained as the unique solution of the forward equation in (54), and then, \( Y^\tau,x_t \) is obtained as the unique solution of the backward equation. Forward-backward system (54) is a particular case of FBSDEs (10).

Moreover, the fixed point argument simplifies in comparison to (10) due to the absence of stochastic integrals. The unique fixed point of \( \Gamma \) exists at least on the same time interval as for the stochastic case considered in Step 1 of the proof of Theorem 2, and, therefore, it exists for all \( \tau \in \gamma_{K,C}, T \]. This fixed point is the unique solution \((X^\tau,x,0_t, Y^\tau,x,0_t)\) to (53). Further, we note that (14) can be proved for (53) without involving expectations, i.e., for each \( \omega \in \Omega_0 \). This implies the \((\tau,x,t)\)-continuity of the solution \((X^\tau,x,0_t, Y^\tau,x,0_t)\).

For each \((t,x,\omega) \in [T - \gamma_{K,C}, T] \times \mathbb{R}^n \times \Omega_0 \), we define

\[
\bar{y}_0(t,x) = Y_t^{\tau,x,0}.
\]

Let for any viscosity \( \nu \in (0,\nu_0] \), \( \bar{y}_0(t,x) \) denote the unique \( F^B_{t-\tau} \)-adapted \( C_b^{1,2} \) solution to (7). In the lemma below, we would like to treat \( \nu \) as a “time” parameter and \( \bar{y}_0 : [0,\nu_0] \times \Omega \to C_b([T - \gamma_{K,C}, T] \times \mathbb{R}^n) \), \( (\nu,\omega) \mapsto \bar{y}_0(\cdot,\cdot) \), as a stochastic process with values in \( C_b([T - \gamma_{K,C}, T] \times \mathbb{R}^n) \).
LEMMA 5. Assume (A1)–(A3). Further assume that \( ||h||_{C^2} \) are \( ||\eta||_{C^0,1} \) are bounded in \( \omega \in \Omega_0 \). Then, there exists a constant \( \gamma^{(1)}_{K,C} < \gamma_{K,C} \) such that there is a continuous version of

\[
\bar{y} : [0,\nu_0] \times \Omega \to C_b([T - \gamma^{(1)}_{K,C}, T] \times \mathbb{R}^n), \quad (\nu, \omega) \mapsto \bar{y}_\nu(\cdot, \cdot).
\]

Proof. Let \( (X_t^{\tau,x,\nu}, Y_t^{\tau,x,\nu}, Z_t^{\tau,x,\nu}) \) be the solution to (9) associated to \( \nu \in (0,\nu_0) \).

We would like to prove that as \( \nu \to \bar{\nu} \), \( (X_t^{\tau,x,\nu}, Y_t^{\tau,x,\nu}, Z_t^{\tau,x,\nu}) \) converges to \( (X_t^{\tau,x,\bar{\nu}}, Y_t^{\tau,x,\bar{\nu}}, Z_t^{\tau,x,\bar{\nu}}) \), if \( \bar{\nu} > 0 \), or to \( (X_t^{\tau,x,0}, Y_t^{\tau,x,0}, 0) \), if \( \bar{\nu} = 0 \), in the space \( S([\tau, T], \mathbb{R}^n) \times S([\tau, T], \mathbb{R}^n) \times S([\tau, T], \mathbb{R}^{n \times n}) \), where \( S([\tau, T], \mathbb{R}^{n \times n}) \) is the space of all progressively measurable \( \mathbb{R}^{n \times n} \)-valued processes \( \xi_s \) such that \( \mathbb{E} \int_T^T |\xi_s|^2 ds < \infty \).

As before, sometimes we skip the upper index \((\tau, x)\) (but keep \( \nu \)). We have

\[
\begin{cases}
X_t^\nu - X_t^\bar{\nu} = \int_0^t [\hat{\eta}(s, X_s^\nu) - \hat{\eta}(s, X_s^\bar{\nu}) + Y_s^\nu - Y_s^\bar{\nu}] ds + (\sqrt{2\nu} - \sqrt{2\bar{\nu}})(W_t - W_{\tau}), \\
Y_t^\nu - Y_t^\bar{\nu} = h(X_t^\nu) - h(X_t^\bar{\nu}) + \int_t^T (\hat{F}^M(s, X_s^\nu, Y_s^\nu) - \hat{F}^M(s, X_s^\bar{\nu}, Y_s^\bar{\nu})) ds \\
- \int_t^T (Z_s^\nu - Z_s^\bar{\nu}) dW_s.
\end{cases}
\]

Note that \( Z_t^{\tau,x,0} = 0 \). Pick an integer \( p > 1 \). By Gronwall’s inequality, the forward SDE implies

\[
\sup_{t \in [\tau, T]} \mathbb{E}^B [|X_t^\nu - X_t^\bar{\nu}|^{2p}] \leq \beta_1 [(T - \tau)^{2p} \sup_{t \in [\tau, T]} \mathbb{E}^B |Y_t^\nu - Y_t^\bar{\nu}|^{2p}] ds + (T - \tau)^p |\nu - \bar{\nu}|^p.
\]

Itô's formula applied to the BSDE in (57) gives

\[
\begin{align*}
\mathbb{E}^B |Y_t^\nu - Y_t^\bar{\nu}|^{2p} &\leq \mathbb{E}^B [h(X_t^\nu) - h(X_t^\bar{\nu})]^{2p} \\
&+ 4p \int_t^T \mathbb{E}^B [Y_s^\nu - Y_s^\bar{\nu}]^{2p-2} (\hat{F}^M(s, X_s^\nu, Y_s^\nu) - \hat{F}^M(s, X_s^\bar{\nu}, Y_s^\bar{\nu})) ds.
\end{align*}
\]

From (58) and (59) it follows that there exists a positive constant \( \gamma^{(1)}_{K,C} < \gamma_{K,C} \) such that for all \( t \in [T - \gamma^{(1)}_{K,C}, T] \),

\[
\sup_{t \in [\tau, T]} \mathbb{E}^B |Y_t^{\tau,x,\nu} - Y_t^{\tau,x,\bar{\nu}}|^{2p} \leq \beta_2 |\nu - \bar{\nu}|^p.
\]

Note that the constant on the right-hand side of (60) does not depend on \( \tau \) and \( x \). Therefore,

\[
\mathbb{E} \sup_{t \in \mathbb{R}^n, \tau \in [T - \gamma^{(1)}_{K,C}, T]} |y_\nu(\tau, x) - y_\bar{\nu}(\tau, x)|^{2p} \leq \beta_2 |\nu - \bar{\nu}|^p.
\]

By Kolmogorov’s continuity theorem ([22], p. 31), there is an a.s. \( \nu \)-continuous version of the \( C_b([T - \gamma^{(1)}_{K,C}, T] \times \mathbb{R}^n) \)-valued stochastic process \( \bar{y} : [0,\nu_0] \times \Omega \to C_b([T - \gamma^{(1)}_{K,C}, T] \times \mathbb{R}^n), (\nu, \omega) \mapsto \bar{y}_\nu(\cdot, \cdot) \). \hfill \Box

Theorem 5 below states the existence of the local inviscid limit of equation (7) in the case when \( ||\eta||_{C^0,1} \) is bounded in \( \omega \in \Omega_0 \).

THEOREM 5. Assume (A1)–(A3). Further assume that \( ||h||_{C^2} \) are \( ||\eta||_{C^0,1} \) are bounded in \( \omega \in \Omega_0 \). Then, there exists a positive constant \( \beta_{K,C} < \gamma^{(1)}_{K,C} \), that depends only on \( K \) and \( C \), such that \( \bar{y}_0(t, x) \), defined by (55), is the unique \( \mathcal{F}^B_{t-} \)-adapted
C^{1,1}_b$-solution to equation (7) with $\nu = 0$ on $[T - \beta_{K,C}, T]$. Moreover, as $\nu \to 0$, a.s., $\bar{y}_\nu(t, x) \to \bar{y}_0(t, x)$ uniformly in $(x, t) \in \mathbb{R}^n \times [T - \beta_{K,C}, T]$, where $\bar{y}_\nu$ is the $\nu$-continuous process defined by (56).

Proof. Let us prove that for each fixed $x \in \mathbb{R}^n$ and $\tau \in [T - \beta_{K,C}, T]$, we can take a limit in (7) as $\nu \to 0$ in the space $L_2(\Omega)$, where $\beta_{K,C}$ is an appropriate small constant. Note that the proof of the differentiability of the FBSDE solution (Step 2 of the proof of Theorem 2) holds for the case $\nu = 0$ (with $Z_t^{x,0} = 0$). Therefore, $(X_t^{x,0}, Y_t^{x,0})$ is differentiable in $x$, and $(\partial_{\bar{y}_\nu} X_t^{x,0}, \partial_{\bar{y}_\nu} Y_t^{x,0})$ satisfies (18) with $\nu = 0$. FBSDEs for the triple $(\partial_{\bar{y}_\nu} X_t^\nu - \partial_{\bar{y}_0} X_t^\nu, \partial_{\bar{y}_\nu} Y_t^\nu - \partial_{\bar{y}_0} Y_t^\nu, \partial_{\bar{y}_\nu} Z_t^\nu)$ take the form

\begin{align}
\begin{cases}
\partial_{\bar{y}_\nu} X_t^\nu - \partial_{\bar{y}_0} X_t^\nu = -J_t \left( \nabla \bar{y}(s, X_s^\nu) (\partial_{\bar{y}_\nu} X_s^\nu - \partial_{\bar{y}_0} X_s^\nu) + \partial_{\bar{y}_\nu} Y_s^\nu - \partial_{\bar{y}_0} Y_s^0 + \xi_s^X(s) \right) ds \\
\partial_{\bar{y}_\nu} Y_t^\nu - \partial_{\bar{y}_0} Y_0^\nu = \nabla h(X_0^\nu, X_T^\nu)(\partial_{\bar{y}_\nu} X_T^\nu - \partial_{\bar{y}_0} X_T^\nu) + J_T \left( \nabla F M(s, X_s^\nu, Y_s^\nu) (\partial_{\bar{y}_\nu} X_s^\nu - \partial_{\bar{y}_0} X_s^\nu) \\
+ \nabla^2 F M(s, X_s^\nu, Y_s^\nu) (\partial_{\bar{y}_\nu} Y_s^\nu - \partial_{\bar{y}_0} Y_s^0) + \xi_\nu(s) \right) ds + \int_T^T \partial_{\bar{y}_\nu} Z_s^\nu dW_s + \xi_\nu^Y(s)
\end{cases}
\end{align}

where $\xi_s^X(s) = -\nabla \bar{y}(s, X_s^\nu) - \nabla \bar{y}(s, X_s^0)) \partial_{\bar{y}_\nu} X_s^\nu, \xi_\nu^Y(s) = \nabla h(X_T^\nu) - \nabla h(X_T^0)) \partial_{\bar{y}_\nu} X_T^\nu$.

From (62), by standard arguments, we obtain that there exists a constant $\beta_{K,C} < \gamma_{K,C}^{(1)}$ such that for all $\tau \in [T - \beta_{K,C}, T],$

\begin{align}
\sup_{\tau \in [\tau, T]} \mathbb{E}^\beta \left[ \left| \partial_{\bar{y}_\nu} Y_t^{x,\nu} - \partial_{\bar{y}_0} Y_t^{x,0} \right|^2 \right] \leq \beta_3 \mathbb{E}^\beta \left\{ \int_T^{T - \beta_{K,C}} (|\xi_s^X(s)|^2 + (|\xi_\nu^Y(s)|^2) ds + |\xi_\nu^Y|^2 ) \right\}.
\end{align}

By (19), (58), (60), the right-hand side of (63) tends to zero as $\nu \to 0$. Therefore,

\begin{align}
\mathbb{E} \sup_{x \in \mathbb{R}^n, \tau \in [T - \beta_{K,C}, T]} |\partial_{x} \bar{y}_\nu(\tau, x) - \partial_{x} \bar{y}_0(\tau, x)|^2 \to 0.
\end{align}

By Corollary 3, the bounds for $\bar{y}_\nu(t, x)$ and $\partial_{x} \bar{y}(t, x)$ do not depend on $\nu \in (0, \nu_0]$. Therefore, as $\nu \to 0,$

\begin{align}
\mathbb{E} \sup_{x \in \mathbb{R}^n, \tau \in [T - \beta_{K,C}, T]} |(\bar{y}_\nu, \nabla) \bar{y}_\nu(t, x) - (\bar{y}_0, \nabla) \bar{y}_0(t, x)|^2 \\
\leq \mathbb{E} \sup_{x \in \mathbb{R}^n, \tau \in [T - \beta_{K,C}, T]} ((\bar{y}_\nu - \bar{y}_0, \nabla) \bar{y}_\nu(t, x))^2 + ((\bar{y}_0, \nabla) (\bar{y}_\nu - \bar{y}_0)(t, x))^2 \to 0
\end{align}

Finally, by (22), $\Delta \bar{y}_\nu(t, x)$ is bounded uniformly in $\nu \in (0, \nu_0]$ and $(t, x) \in [T - \beta_{K,C}, T] \times \mathbb{R}^n$. This implies that as $\nu \to 0,$

\begin{align}
\nu \mathbb{E} \sup_{x \in \mathbb{R}^n, \tau \in [T - \beta_{K,C}, T]} |\Delta \bar{y}_\nu(t, x)|^2 \to 0.
\end{align}

Now equation (7), together with Lemma 5, (64), and (65) imply that there exists a set $\Omega$ of full $\mathbb{P}$-measure such that for all $(t, x, \omega) \in [T - \beta_{K,C}, T] \times \mathbb{R}^n \times \Omega,$

\begin{align}
\bar{y}_0(t, x) = h(x) + \int_t^T \left[ (\bar{y}_\nu, \nabla) \bar{y}_\nu(s, x) + \bar{F}(s, x, \bar{y}_\nu(t, x)) \right] ds.
\end{align}
Note that each solution $\bar{y}_\nu(t,x)$ is bounded by the constant $M$ that does not depend on $\nu \in (0,\nu_0)$. Therefore, $(\bar{y}_0(t,x))$ is bounded by the same constant by Lemma 5. This implies that $\bar{y}_0(t,x)$ is also a solution to

$$
(67) \quad \bar{y}_0(t,x) = h(x) + \int_0^T \left[ (\bar{y}_0, \nabla)\bar{y}_0(s,x) + \bar{F}(s,x,\bar{y}_0(t,x)) \right] ds,
$$

where $\bar{F}(s,x,y) = F(T-s,x,y)$. Furthermore, by Lemma 5, for the $\nu$-continuous version of the process $\bar{y} : [0,\nu_0] \times \Omega \to C([T - \beta_{K,C},T] \times \mathbb{R}^n)$, $(\nu, \omega) \mapsto \bar{y}_\nu$, it holds that on $\Omega$,

$$
\sup_{x \in \mathbb{R}^n, \tau \in [T - \beta_{K,C},T]} |\bar{y}_\nu(\tau, x) - \bar{y}_0(\tau, x)| \to 0, \quad \text{as } \nu \to 0.
$$

This, in particular, implies that $\bar{y}_0(t,x)$ is $\mathcal{F}_{T-\tau}^B$-adapted.

Assume, there is another local solution $\tilde{y}_0(t,x)$ that satisfies (7) with $\nu = 0$ for $(t,x) \in [T - \beta_{K,C},T] \times \mathbb{R}^n$ on some set $\tilde{\Omega}$ of full $\mathbb{P}$-measure. Let for each $\omega \in \tilde{\Omega}$, $X_{t,x}^{\nu}$ be the solution to

$$
(68) \quad \tilde{X}_{t,x}^{\nu} = x - \int_0^t \left( \bar{\eta}(s, \tilde{X}_{s,x}^{\nu}) + \tilde{y}_0(s, \tilde{X}_{s,x}^{\nu}) \right) ds.
$$

Then, it is straightforward to verify that $(X_{t,x}^{\nu}, \tilde{y}_0(t,x), \tilde{X}_{t,x}^{\nu})$ is a solution to (53). Indeed, it suffices to note that $\partial_t \tilde{y}_0(t,x) = (\partial_t \tilde{X}_{t,x}^{\nu}, \nabla)\tilde{y}_0(t,x)$ and compute $\partial_t \tilde{X}_{t,x}^{\nu}$ via (68). By the uniqueness of solution to (53) on $[T - \beta_{K,C} \wedge \beta_{K,C},T]$, we conclude that $\tilde{y}_0(t,x) = \bar{y}_0(t,x)$ on $\tilde{\Omega} \cap \Omega \cup [T - \beta_{K,C} \wedge \beta_{K,C},T] \times \mathbb{R}^n$. The theorem is proved.

The following theorem is our main result about the local inviscid limit of stochastic Burgers equation (1).

**Theorem 6.** Assume (A1)–(A3). Further, we assume that $\|h\|_{C^2}$ is bounded in $\omega \in \Omega_0$. Then, there exists a stopping time $S$, positive a.s., such that on $[0,S]$ there exists an $\mathcal{F}_t^B$-adapted $C^{1,1}_b$-solution $y_0(t,x)$ to the inviscid stochastic Burgers equation

$$
(69) \quad y(t,x) = h(x) + \int_0^t \left[ f(s,x,y) - (y, \nabla)g(s,x) \right] ds + \int_0^t g(s,x) dB_s.
$$

Moreover, if $\tilde{y}_0(t,x)$ is another solution to (69) on $[0,\tilde{S}]$, where $\tilde{S} > 0$ is a stopping time, then, a.s., $\tilde{y}_0(t,x) = y_0(t,x)$ on $[0, S \wedge \tilde{S}]$. Furthermore, if $y_0(t,x)$ is a $C^{1,2}_b$-solution to (1) (whose existence is established by Theorem 4), then there exists a $\nu$-continuous version of the $C_b([0, S] \times \mathbb{R}^n)$-valued stochastic process $y : [0,\nu_0] \times \Omega \to C_b([0, S] \times \mathbb{R}^n)$, $(\nu, \omega) \mapsto y_\nu$. For this version, in particular, it holds that $\lim_{\nu \to 0} y_\nu(t,x) = y_0(t,x)$ a.s., where the limit is uniform in $(x, t) \in \mathbb{R}^n \times [0,S]$.

**Proof.** Let for a fixed $N$, $T_N$ be the stopping time given by (45). Further let $\eta_N$ be defined by (46) and $F_N$ be defined by (5) via $\eta_N$. Define the constants $K_N$ and $C_N$ by (8) via $\eta_N$, $F_N$, and $h$.

Let $\tilde{y}_\nu(t,x), \nu \in (0,\nu_0]$, be the unique $\mathcal{F}_{T-t}^B$-adapted $C^{1,2}_b$-solution to (7) with $\eta = \eta_N$ and $F = F_N$ Then, for each $\nu \in (0,\nu_0]$, $y_\nu(t,x) = \tilde{y}_\nu(T-t,x) + \eta_N(t,x)$ is
the unique $\mathcal{F}_t^B$-adapted $C_b^{\frac{3}{2}}$-solution to
\begin{equation}
y(t, x) = h(x) + \int_0^t \left[ f(s, x, y) - (y, \nabla)y(s, x) + \nu \Delta y(s, x) \right] ds + \eta_N(t, x).
\end{equation}

By Lemma 5, we can choose $\bar{y}_\nu$ to be the $\nu$-continuous version of the process
$\tilde{y}. : [0, \nu_0] \times \Omega \to C_0([T - \beta_{K_\nu,C_N}, T] \times \mathbb{R}^n), (\nu, \omega) \mapsto \tilde{y}_\nu$, where $\tilde{y}_0(t, x)$ is the unique $\mathcal{F}_t^B$-adapted $C_b^{1,1}$-solution to (7) with $\nu = 0$, $\eta = \eta_N$, and $F = F_N$ on $[T - \beta_{K_\nu,C_N}, T]$. Furthermore, it is straightforward to verify that $y_0(t, x) = \bar{y}_0(t - T, x) + \eta_N(t, x)$ is a unique $\mathcal{F}_t^B$-adapted $C_b^{\frac{3}{2}}$-solution to
\begin{equation}
y(t, x) = h(x) + \int_0^t \left[ f(s, x, y) - (y, \nabla)y(s, x) \right] ds + \eta_N(t, x)
\end{equation}
on $[0, \beta_{K_\nu,C_N}]$. Consequently, $y_\nu$, $\nu \in [0, \nu_0]$, is a $\nu$-continuous version of the process $y. : [0, \nu_0] \times \Omega \to C_0([0, \beta_{K_\nu,C_N}] \times \mathbb{R}^n), (\nu, \omega) \mapsto \tilde{y}_\nu$ on some set $\Omega$ of full $\mathbb{P}$-measure. Define the stopping time $S = \beta_{K_\nu,C_N} \wedge T_N$. Then, on $\Omega$, $\lim_{\nu \to 0} y_\nu(t, x) = y_0(t, x)$, and the limit is uniform in $(t, x) \in [0, S] \times \mathbb{R}^n$. It remains to note that on $[0, S] \times \mathbb{R}^n$, $\eta_N(t, x) = \eta(t, x)$, and, therefore, $y_\nu(t, x)$ is a solution to (1) and $y_0(t, x)$ is a solution to (69). To show the local uniqueness, we note that by the previous arguments, the solution to (70) with $\eta_N(t, x) = \eta(t \wedge S \wedge T_N, x)$ on $[0, \beta_{K_\nu,C_N}]$ is unique. The theorem is proved.

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