Umbral Calculus and Frobenius-Euler Polynomials

by
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Abstract
In this paper, we study some properties of umbral calculus related to Appell sequence. From those properties, we derive new and interesting identities of Frobenius-Euler polynomials.

1 Introduction
Let $\mathbb{C}$ be the complex number field. For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials are defined by the generating function to be

$$
\frac{1 - \lambda}{e^t - \lambda} e^{xt} = e^{H(x|\lambda)t} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{t^n}{n!}, \quad \text{(see [7–11])},
$$

with the usual convention about replacing $H_n(x|\lambda)$ by $H_n(\lambda)$.

In the special case, $x = 0$, $H_n(0|\lambda) = H_n(\lambda)$ are called the $n$-th Frobenius-Euler numbers. By (1), we get

$$
H_n(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(\lambda) x^l = (H(\lambda) + x)^n, \quad \text{(see [1, 2, 3, 4])},
$$

with the usual convention about replacing $H^n(\lambda)$ by $H_n(\lambda)$.

Thus, from (1) and (2), we note that

$$(H(\lambda) + 1)^n - \lambda H_n(\lambda) = (1 - \lambda)\delta_{0,n},$$

where $\delta_{n,k}$ is the kronecker symbol (see [6,7]).

For $r \in \mathbb{Z}_+$, the Frobenius-Euler polynomials of order $r$ are defined by the
generating function to be
\[
\left( \frac{1 - \lambda}{e^t - \lambda} \right) e^{xt} = \left( \frac{1 - \lambda}{e^t - \lambda} \right) \times \cdots \times \left( \frac{1 - \lambda}{e^t - \lambda} \right) e^{xt}
\]
\(r\)-times
\(= \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}.
\]

In the special case, \(x = 0\), \(H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)\) are called the \(n\)-th Frobenius-Euler numbers of order \(r\) (see [6,7]).

From (3), we can derive the following equation:
\[
H_n^{(r)}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(\lambda) x^l,
\]
(4)

and
\[
H_n^{(r)}(\lambda) = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \ldots, l_r} H_{l_1}(\lambda) \cdots H_{l_r}(\lambda).
\]
(5)

By (4) and (5), we see that \(H_n^{(r)}(x|\lambda)\) is a monic polynomial of degree \(n\) with coefficients in \(\mathbb{Q}(\lambda)\).

Let \(\mathbb{P}\) be the algebra of polynomials in the single variable \(x\) over \(\mathbb{C}\) and let \(\mathbb{P}^*\) be the vector space of all linear functionals on \(\mathbb{P}\). As is known, \(\langle L|p(x) \rangle\) denotes the action of the linear functional \(L\) on a polynomial \(p(x)\) and we remind that the addition and scalar multiplication on \(\mathbb{P}^*\) are respectively defined by
\[
\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle, \langle cL|p(x) \rangle = c\langle L|p(x) \rangle,
\]
where \(c\) is a complex constant (see [5, 8]).

Let \(\mathbf{F}\) denote the algebra of formal power series:
\[
\mathbf{F} = \{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k | a_k \in \mathbb{C} \}, \quad (\text{see } [5, 8]).
\]
(6)

The formal power series define a linear functional on \(\mathbb{P}\) by setting
\[
\langle f(t)|x^n \rangle = a_n, \text{ for all } n \geq 0.
\]
(7)

Indeed, by (6) and (7), we get
\[
\langle t^k|x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0), \quad (\text{see } [5, 8]).
\]
(8)
This kind of algebra is called an umbral algebra. The order $O(f(t))$ of a nonzero power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. A series $f(t)$ for which $O(f(t)) = 1$ is said to be an invertible series (see [5, 8]). For $f(t), g(t) \in \mathbf{F}$ and $p(x) \in \mathbb{P}$, we have

$$
\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle = \langle g(t)|f(t)p(x) \rangle, \quad (\text{see} \ [5]).
$$

(9)

One should keep in mind that each $f(t) \in \mathbf{F}$ plays three roles in the umbral calculus: a formal power series, a linear functional and a linear operator. To illustrate this, let $p(x) \in \mathbb{P}$ and $f(t) = e^{yt} \in \mathbf{F}$. As a linear functional, $e^{yt}$ satisfies $\langle e^{yt}|p(x) \rangle = p(y)$. As a linear operator, $e^{yt}$ satisfies $e^{yt}p(x) = p(x + y)$ (see [5]). Let $s_n(x)$ denote a polynomial in $x$ with degree $n$. Let us assume that $f(t)$ is a delta series and $g(t)$ is an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k|s_n(x) \rangle = n!\delta_{n,k}$ for all $n, k \geq 0$ (see [3, 8]). This sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$. If $s_n(x) \sim (1, f(t))$, then $s_n(x)$ is called the associated sequence for $f(t)$. If $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called the Appell sequence.

Let $s_n(x) \sim (g(t), f(t))$. Then we see that

$$
h(t) = \sum_{k=0}^{\infty} \frac{< h(t)|s_k(x) >}{k!}g(t)^k, \quad h(t) \in \mathbf{F},
$$

(10)

$$
p(x) = \sum_{k=0}^{\infty} \frac{< g(t)f(t)^k|p(x) >}{k!}s_k(x), \quad p(x) \in \mathbb{P},
$$

(11)

$$
f(t)s_n(x) = ns_{n-1}(x), \quad < f(t)|p(\alpha x) >= < f(\alpha t)|p(x)>,
$$

(12)

and

$$
\frac{1}{g(f(t))}e^{yt} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!}t^k, \quad \text{for all } y \in \mathbf{C},
$$

(13)

where $\tilde{f}(t)$ is the compositional inverse of $f(t)$ (see [8]). In this paper, we study some properties of umbral calculus related to Appell sequence. For those properties, we derive new and interesting of Frobenius-Euler polynomials.
2 Frobenius-Euler polynomials and Umbral Calculus.

By (3) and (13), we see that
\[
H_n^{(r)}(x|\lambda) \sim \left( \frac{e^t - \lambda}{1 - \lambda} \right)^r. \tag{14}
\]
Thus, by (14), we get
\[
\left< \left( \frac{e^t - \lambda}{1 - \lambda} \right)^r t^k \Big| H_n^{(r)}(x|\lambda) \right> = n! \delta_{n,k}. \tag{15}
\]
Let
\[
\mathbb{P}_n(\lambda) = \{ p(x) \in \mathbb{Q}(\lambda)[x] \mid \deg p(x) \leq n \}. 
\]
Then it is an \((n + 1)\)-dimensional vector space over \(\mathbb{Q}(\lambda)\).

So we see that \(\{ H_0^{(r)}(x|\lambda), H_1^{(r)}(x|\lambda), \ldots, H_n^{(r)}(x|\lambda) \} \) is a basis for \(\mathbb{P}_n(\lambda)\). For \(p(x) \in \mathbb{P}_n(\lambda)\), let
\[
p(x) = \sum_{k=0}^{n} C_k H_k^{(r)}(x|\lambda), \quad (n \geq 0). \tag{16}
\]
Then, by (14), (15) and (16), we get
\[
\left< \left( \frac{e^t - \lambda}{1 - \lambda} \right)^r t^k \Big| p(x) \right> = \sum_{l=0}^{n} C_l \left< \left( \frac{e^t - \lambda}{1 - \lambda} \right)^r t^k \Big| H_l^{(r)}(x|\lambda) \right> \tag{17} = \sum_{l=0}^{n} C_l l! \delta_{l,k} = k! C_k.
\]
From (17), we have
\[
C_k = \frac{1}{k!} \left< \left( \frac{e^t - \lambda}{1 - \lambda} \right)^r t^k \Big| p(x) \right> = \frac{1}{k!} \left< \left( \frac{e^t - \lambda}{1 - \lambda} \right)^r D^k p(x) \right> \tag{18} = \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} < e^{jt} | D^j p(x) > = \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} < e^{jt} D^j p(x) > \]
\[
= \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} < t^j | D^j p(x + j) >.
\]
Therefore, by (16) and (18), we obtain the following theorem.
Theorem 1. For $p(x) \in \mathbb{P}_n(\lambda)$, let

$$p(x) = \sum_{k=0}^{n} C_k H^{(r)}_k(x).$$

Then we have

$$C_k = \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j),$$

where $Dp(x) = \frac{dp(x)}{dx}$.

From Theorem 1, we note that

$$p(x) = \frac{1}{1-\lambda} \sum_{k=0}^{n} \binom{r}{j} (-\lambda)^{r-j} D^k p(j) H^{(r)}(x|\lambda).$$

Let us consider the operator $\tilde{\triangle}_\lambda$ with $\tilde{\triangle}_\lambda f(x) = f(x+1) - \lambda f(x)$ and let $J_\lambda = \frac{1}{1-\lambda} \tilde{\triangle}_\lambda$. Then we have

$$J_\lambda(f)(x) = \frac{1}{1-\lambda} \{ f(x+1) - \lambda f(x) \}. \quad (19)$$

Thus, by (19), we get

$$J_\lambda(H^{(r)}_n(x|\lambda)) = \frac{1}{1-\lambda} \{ H^{(r)}_n(x+1|\lambda) - \lambda H^{(r)}_n(x|\lambda) \}. \quad (20)$$

From (20), we can derive

$$\sum_{n=0}^{\infty} \{ H^{(r)}_n(x+1|\lambda) - \lambda H^{(r)}_n(x|\lambda) \} \frac{t^n}{n!} = \left( \frac{1-\lambda}{e^t-\lambda} \right)^{r-1} e^{(x+1)t} - \lambda \left( \frac{1-\lambda}{e^t-\lambda} \right)^{r-1} e^{xt} \quad (21)$$

By (20) and (21), we get

$$J_\lambda(H^{(r)}_n(x|\lambda)) = H^{(r-1)}_n(x|\lambda). \quad (22)$$

From (22), we have

$$J_\lambda^{(r)}(H^{(r)}_n(x|\lambda)) = J_\lambda^{(r-1)}(H^{(r-1)}_n(x|\lambda)) = \cdots = H^{(0)}_n(x|\lambda) = x^n,$$
and
\[ J_{x}(x^n) = J_{x}^{r} H_n^{(0)}(x|\lambda) = H_n^{(-r)}(x|\lambda) = J_{x}^{2r} H_n^{(r)}(x|\lambda). \tag{23} \]

For \( s \in \mathbb{Z}_+ \), from (22), we have
\[ J_{s}(H_n^{(r)}(x|\lambda)) = H_n^{(-r-s)}(x|\lambda). \tag{24} \]

On the other hand, by (13), (14) and (22),
\[ J_{s}(H_n^{(r)}(x|\lambda)) = \left( e^{\lambda} - \frac{1}{1 - \lambda} \right)^{s} (H_n^{(r)}(x|\lambda)) \tag{25} \]
\[ = \frac{1}{(1 - \lambda)^s} ((1 - \lambda) + \sum_{k=1}^{\infty} \frac{t^k}{k!})^s (H_n^{(r)}(x|\lambda)). \]

Thus, by (25), we get
\[ J_{s}(H_n^{(r)}(x|\lambda)) = \sum_{m=0}^{s} \binom{s}{m} \frac{1}{(1 - \lambda)^m} \sum_{l=m}^{\infty} \sum_{k_1 + \cdots + k_m = l}^{\infty} \frac{1}{k_1! \cdots k_m!} D^l(H_n^{(r)}(x|\lambda)) \tag{26} \]
\[ = \sum_{m=0}^{\min\{s, n\}} \frac{s}{m} \sum_{l=m}^{\infty} \frac{1}{l!} \sum_{k_1 + \cdots + k_m = l}^{\infty} \binom{l}{k_1, \ldots, k_m} H_{n-1}^{(r)}(x|\lambda) \]
\[ = \sum_{l=0}^{\min\{s, n\}} \binom{n}{l} \sum_{m=0}^{\min\{s, n\}} \frac{s}{m} \sum_{k_1 + \cdots + k_m = l}^{\min\{s, n\}} \binom{l}{k_1, \ldots, k_m} H_{n-1}^{(r)}(x|\lambda) \]
\[ + \sum_{l=\min\{s, n\}+1}^{n} \binom{n}{l} \sum_{m=0}^{\min\{s, n\}} \frac{s}{m} \sum_{k_1 + \cdots + k_m = l}^{\min\{s, n\}} \binom{l}{k_1, \ldots, k_m} H_{n-1}^{(r)}(x|\lambda) \]

Therefore, by (24) and (26), we obtain the following theorem.
Theorem 2. For any \( r, s \geq 0 \), we have

\[
H_n^{(r-s)}(x|\lambda) = \sum_{l=0}^{\min\{s,n\}} \binom{n}{l} \sum_{m=0}^{l} \binom{s}{m} \sum_{k_1+\cdots+k_m=l} \binom{l}{k_1, \ldots, k_m} H_{n-l}^{(r)}(x|\lambda)
\]

\[
+ \sum_{l=\min\{s,n\}+1}^{n} \binom{n}{l} \min\{s,n\} \binom{s}{m} \sum_{k_1+\cdots+k_m=l} \binom{l}{k_1, \ldots, k_m} H_{n-l}^{(r)}(x|\lambda).
\]

Let us take \( s = r - 1 (r \geq 1) \) in Theorem 2. Then we obtain the following corollary.

Corollary 3. For \( n \geq 0, r \geq 1 \), we have

\[
x^n = \sum_{l=0}^{\min\{r-1,n\}} \binom{n}{l} \sum_{m=0}^{l} \binom{r-1}{m} \sum_{k_1+\cdots+k_m=l} \binom{l}{k_1, \ldots, k_m} H_{n-l}^{(r)}(x|\lambda)
\]

\[
+ \sum_{l=\min\{r-1,n\}+1}^{n} \binom{n}{l} \min\{r-1,n\} \binom{r-1}{m} \sum_{k_1+\cdots+k_m=l} \binom{l}{k_1, \ldots, k_m} H_{n-l}^{(r)}(x|\lambda).
\]

Let us take \( s = r (r \geq 1) \) in Theorem 2. Then we obtain the following corollary.

Corollary 4. For \( n \geq 0, r \geq 1 \), we have

\[
x^n = \sum_{l=0}^{\min\{r,n\}} \binom{n}{l} \sum_{m=0}^{l} \binom{r}{m} \sum_{k_1+\cdots+k_m=l} \binom{l}{k_1, \ldots, k_m} H_{n-l}^{(r)}(x|\lambda)
\]

\[
+ \sum_{l=\min\{r,n\}+1}^{n} \binom{n}{l} \min\{r,n\} \binom{r}{m} \sum_{k_1+\cdots+k_m=l} \binom{l}{k_1, \ldots, k_m} H_{n-l}^{(r)}(x|\lambda).
\]
Now, we define the analogue of Stirling numbers of the second kind as follows:

\[
S_\lambda(n, k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^{k-j} n^j, \quad (n, k \geq 0).
\]  

(27)

Note that \( S_1(n, k) = S(n, k) \) is the stirling number of the second kind.

From the definition of \( \tilde{\triangle}_\lambda \), we have

\[
\tilde{\triangle}_\lambda^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} f(k) \]  

(28)

By (27) and (28), we get

\[
S_\lambda(n, k) = \frac{1}{k!} \tilde{\triangle}_\lambda^k 0^n, \quad (n, k \geq 0). \]  

(29)

Let us take \( s = 2r \). Then we have

\[
J_x^n = H_n^{(-r)}(x|\lambda)
\]

\[
= \sum_{l=0}^{\min\{2r,n\}} \binom{n}{l} \sum_{m=0}^{l} \frac{2r}{(1-\lambda)^m} \sum_{k_1 \geq 1} \ldots \sum_{k_m \geq 1} \binom{l}{k_1, \ldots, k_m} H_{n-1}^{(r)}(x|\lambda),
\]

and

\[
J_x^n = \left( \frac{1}{1-\lambda} \right)^r \left( \frac{\tilde{\triangle}_\lambda}{1-\lambda} \right)^r (x^n) = \frac{1}{(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} (x+j)^n.
\]  

(31)
By (30) and (31), we get

\[
\frac{1}{(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} (x+j)^n = \frac{1}{(1-\lambda)^r} \hat{\Delta}^r x^n
\]  

(32)

\[
= \sum_{l=0}^{\min\{2r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \binom{2r}{m} \sum_{k_1+\ldots+k_m=l \atop k_j \geq 1} \binom{l}{k_1,\ldots,k_m} \right\} H_{n-l}^{(r)}(x|\lambda)
\]

\[
+ \sum_{l=\min\{2r,n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{l} \binom{2r}{m} \sum_{k_1+\ldots+k_m=l \atop k_j \geq 1} \binom{l}{k_1,\ldots,k_m} \right\} H_{n-l}^{(r)}(x|\lambda).
\]

Let us take \(x = 0\) in (32). Then we obtain the following theorem.

**Theorem 5.**

\[
\frac{r!}{(1-\lambda)^r} S_\lambda(n,r) = \frac{r!}{(1-\lambda)^r} \frac{\hat{\Delta}^0 0^n}{r!}
\]

\[
= \sum_{l=0}^{\min\{2r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \binom{2r}{m} \sum_{k_1+\ldots+k_m=l \atop k_j \geq 1} \binom{l}{k_1,\ldots,k_m} \right\} H_{n-l}^{(r)}(\lambda)
\]

\[
+ \sum_{l=\min\{2r,n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{l} \binom{2r}{m} \sum_{k_1+\ldots+k_m=l \atop k_j \geq 1} \binom{l}{k_1,\ldots,k_m} \right\} H_{n-l}^{(r)}(\lambda)
\]

\[
= \sum_{m=0}^{\min\{r,n\}} \binom{r}{m} \sum_{k_1+\ldots+k_m=n \atop k_j \geq 1} \binom{n}{k_1,\ldots,k_m}.
\]
Let us consider $s = 2r - 1$ in the identity of Theorem 2. Then we have

$$J_{\lambda}^{r-1} x^n = H_n^{(r-1)}(x|\lambda)$$

(33)

$$= \sum_{l=0}^{\min\{2r-1, n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{2r-1}{(1-\lambda)^m} \sum_{k_1+\cdots+k_m=t} \left( k_1, \cdots, k_m \right) \right\} H_{n-l}^{(r)}(x|\lambda)$$

$$+ \sum_{l=\min\{2r-1, n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{\min\{2r-1, n\}} \frac{2r-1}{(1-\lambda)^m} \sum_{k_1+\cdots+k_m=t} \left( k_1, \cdots, k_m \right) \right\} H_{n-l}^{(r)}(x|\lambda)$$

$$= \frac{1}{(1-\lambda)^{r-1-j}} \sum_{j=0}^{r-1} \binom{r-1}{j} (-\lambda)^{r-1-j} (x+j)^n = \frac{1}{(1-\lambda)^{r-1-j}} \Delta_{\lambda}^{r-1} x^n.$$

Let us take $x = 0$ in (33). Then we obtain the following theorem.

**Theorem 6.** For $n \geq 0$ and $r \geq 1$, we have

$$\frac{(r-1)!}{(1-\lambda)^{r-1}} S_\lambda(n, r-1) = \frac{(r-1)!}{(1-\lambda)^{r-1}} \Delta_{\lambda}^{r-1} 0^n$$

$$= \sum_{l=0}^{\min\{2r-1, n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{2r-1}{(1-\lambda)^m} \sum_{k_1+\cdots+k_m=t} \left( k_1, \cdots, k_m \right) \right\} H_{n-l}^{(r)}(\lambda)$$

$$+ \sum_{l=\min\{2r-1, n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{\min\{2r-1, n\}} \frac{2r-1}{(1-\lambda)^m} \sum_{k_1+\cdots+k_m=t} \left( k_1, \cdots, k_m \right) \right\} H_{n-l}^{(r)}(\lambda)$$
Remark. Note that

\[
\frac{(r-1)!}{(1-\lambda)^{r-1}} S_\lambda(n, r-1)
\]

\[
= \sum_{l=0}^{\min\{r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{r^m}{(1-\lambda)^m} \sum_{k_1+\cdots+k_m=l \atop k_j \geq 1} \binom{l}{k_1, \ldots, k_m} \right\} H_{n-l}(\lambda)
\]

\[
+ \sum_{l=\min\{r,n\}+1}^{n} \left( \binom{n}{l} \min\{r,n\} \frac{r^m}{(1-\lambda)^m} \sum_{k_1+\cdots+k_m=l \atop k_j \geq 1} \binom{l}{k_1, \ldots, k_m} \right) H_{n-l}(\lambda)
\]

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