Reality Conditions for Spin Foams

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Abstract

An idea of reality conditions in the context of spin foams (Barrett-Crane models) is developed. The square of areas are the most elementary observables in the case of spin foams. This observation implies that simplest reality conditions in the context of the Barrett-Crane models is that the all possible scalar products of the bivectors associated to the triangles of a four simplex be real. The continuum generalization of this is the area metric reality constraint: the area metric is real iff a non-degenerate metric is real or imaginary. Classical real general relativity (all signatures) can be extracted from complex general relativity by imposing the area metric reality constraint. The Plebanski theory can be modified by adding a Lagrange multiplier to impose the area metric reality condition to derive classical real general relativity. I discuss the $SO(4, C)$ BF model and $SO(4, C)$ Barrett-Crane model. It appears that the spin foam models in 4D for all the signatures are the projections of the $SO(4, C)$ spin foam model using the reality constraints on the bivectors.

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1 Introduction

The Ashtekar self-dual formalism [21] in its most intuitive form relates to complex general relativity. A set of conditions referred to as reality conditions [21] is to be imposed to go to real general relativity. The reality conditions not only impose the reality of the physics, but also the Lorentzian signature [21]. The goal of this article is to introduce analogous idea of reality conditions for the covariant version of quantum gravity - the Barrett-Crane model [11].

The Barrett-Crane model [11] is defined by a set of constraints defined on the bivectors associated to the triangles of a four simplex. The constraints need to be realized at the quantum level to define a quantum state of a tetrahedron or to assign an amplitude to a four simplex. Let us make the bivectors to be complex which corresponds to $SO(4, C)$ general relativity. Then we need a set of conditions to extract Barrett-Crane models for real general relativity. In the Barrett-Crane models the elementary physical observables are the square of area eigenvalues associated to the triangles. So the simplest reality conditions in Barrett-Crane models is to be defined in terms of the squares of the areas. This is equivalent to expecting the inner product of a bivector with itself to be real. For real general relativity we also need the square of areas corresponding to the sum of the bivectors of the triangles to be real. As will be explained in this article this is equivalent to expecting the inner product of the bivectors corresponding to any pair of triangles to be real.
The continuum generalization of the squares of the areas of triangles of simplicial manifold is the Area metric. It can be shown that the reality of an area metric is equivalent to the reality of a geometry. An area metric can be defined as an inner product of a bivector two-form field with itself. Because of this, the reality of an area metric can be imposed using a Lagrange multiplier in the Plebanski formulation of SO(4, C) general relativity. The Barrett-Crane model corresponds to the discrete analog of the Plebanski formalism. Analogously, it can be shown that the reality of the bivector inner products discussed before is the discrete equivalent of the area metric reality constraint.

An attempt by me to rigorously develop and unify the various models for the Lorentzian general relativity was made in Ref. [18]. The attempt was made to derive the two models by directly solving the Barrett-Crane constraints. The Barrett-Crane cross-simplicity constraint operator was explicitly written using the Gelfand-Naimarck representation theory of SL(2, C) [19]. But after numerous attempts I could not obtain any solution for the constraint. But the efforts in this research lead to the development of the reality conditions for the spin foam models. Then as will be discussed in this article the Barrett-Crane models for real general relativity theories for all signatures appears to be related to that of the SO(4, C) general relativity through the quantum version of the discretized area metric reality condition. In this way we have a unified understanding of the Barrett-Crane models for the four dimensional real general relativity theories for all signatures (non-degenerate) and the SO(4, C) general relativity.

The layout of this article is as follows:

- Section two: I review the Plebanski formulation [2] of SO(4, C) general relativity starting from vectorial actions.
- Section three: I discuss the area metric reality constraint. After solving the Plebanski (simplicity) constraints [2], I show that, the area metric reality constraint requires the space-time metric to be real or imaginary for the non-denegerate case. I modify the vectorial Plebanski actions by adding a Lagrange multiplier to impose the reality constraint.
- Section four: I discuss the discretization of the area metric reality constraint on the simplicial manifolds in the context of the Barrett-Crane theory [11].
- Section five: I discuss the spin foam model for the SO(4, C) BF theory.
- Section six: I discuss the SO(4, C) Barrett-Crane model.
- Section seven: Using the bivector scalar product reality constraint the Barrett-Crane models for the real general relativity for all signatures and SO(4, C) general relativity are discussed in a unified manner.

Footnote:

1 The Barrett-Crane model based on the propagators on the null-cone [14] is an exception to this. This needs to be carefully investigated.
Appendices: I discuss the necessary representation theories. I discuss the area metric reality constraint for arbitrary metrics. I also discuss the field theory over group formalism for the $SO(4, C)$ general relativity.

2 $SO(4, C)$ General Relativity

Plebanski’s work [2] on complex general relativity presents a way of recasting general relativity in terms of bivector 2-form fields instead of tetrad fields [4] or space-time metrics. It helped to reformulate general relativity as a topological field theory called the BF theory with a constraint (for example Reisenberger [20]). Originally Plebanski’s work was formulated using spinors instead of vectors. The vector version of the work can be used to formulate spin foam models of general relativity [20], [8]. Understanding the physics behind this theory simplifies with the use of spinors. Here I would like to review the Plebanski theory for a $SO(4, C)$ general relativity on a four dimensional real manifold starting from vectorial actions.

Let me define some notations to be used in this article. I would like to use the letters $i, j, k, l, m, n$ as $SO(4, C)$ vector indices, the letters $a, b, c, d, e, f, g, h$ as space-time coordinate indices, the letters $A, B, C, D, E, F$ as spinorial indices to do spinorial expansion on the coordinate indices.

In the cases of Riemannian and $SO(4, C)$ general relativity the Lie algebra elements are the same as the bivectors. On arbitrary bivectors $a^{ij}$ and $b^{ij}$, I define

\[ a \land b = \frac{1}{2} \epsilon_{ijkl} a^{ij} b^{kl} \quad \text{and} \]
\[ a \cdot b = \frac{1}{2} \eta_{ij} \eta_{kl} a^{ij} b^{kl}. \]

Consider a four dimensional manifold $M$. Let $A$ be a $SO(4, C)$ connection 1-form and $B^{ij}$ a complex bivector valued 2-form on $M$. I would like to restrict myself to non-degenerate general relativity in this and the next section by assuming $b = \frac{1}{4!} \epsilon^{abcd} B_{ab} \land B_{cd} \neq 0$. Let $F$ be the curvature 2-form of the connection $A$. I define real and complex continuum $SO(4, C)$ BF theory actions as follows,

\[ S_{cBF}(A, B_{ij}) = \int_M \epsilon^{abcd} B_{ab} \land F_{cd} \quad \text{and} \]

\[ S_{rBF}(A, B_{ij}, \bar{A}, \bar{B}_{ij}) = \Re \int_M \epsilon^{abcd} B_{ab} \land F_{cd}. \]

The $S_{cBF}$ is considered as a holomorphic functional of it’s variables. In $S_{rBF}$ the variables $A, B_{ij}$ and their complex conjugates are considered as independent variables. The wedge is defined in the Lie algebra coordinates. The field

\[ a \land b = \frac{1}{2} \epsilon_{ijkl} a^{ij} b^{kl} \quad \text{and} \]
\[ a \cdot b = \frac{1}{2} \eta_{ij} \eta_{kl} a^{ij} b^{kl}. \]
equations corresponding to the extrema of these actions are

\[ D_{[a} B_{bc]} = 0 \quad \text{and} \quad F_{cd} = 0. \]

BF theories are topological field theories. It is easy to show that the local variations of solutions of the field equations are gauged out under the symmetries of the actions \(^1\).

The Plebanski actions for \(SO(4, C)\) general relativity is got by adding a constraint term to the BF actions. First let me define a complex action \(^2\),

\[ S_{cGR}(A, B_{ij}, \phi) = \int_M \left[ \varepsilon^{abcd} B_{ab} \wedge F_{cd} + \frac{1}{2} b \phi^{abcd} B_{ab} \wedge B_{cd} \right] d^4x, \quad (3) \]

and a real action

\[ S_{rGR}(A, B_{ij}, \phi, \bar{A}, \bar{B}_{ij}, \bar{\phi}) = \text{Re} S_C(A, B_{ij}, \phi). \quad (4) \]

The complex action is a holomorphic functional of it’s variables. Here \(\phi\) is a complex tensor with the symmetries of the Riemann curvature tensor such that \(\phi^{abcd} \epsilon_{abcd} = 0\). The \(b\) is inserted to ensure the invariance of the actions under coordinate change.

The field equations corresponding to the extrema of the actions \(S_C\) and \(S\) are

\[ D_{[a} B^{ij}_{bc]} = 0, \quad (5a) \]

\[ \frac{1}{2} \varepsilon^{abcd} F_{ij}^{cd} = b \phi^{abcd} B_{ij}^{cd} \quad \text{and}, \quad (5b) \]

\[ B_{ab} \wedge B_{cd} - b \epsilon_{abcd} = 0, \quad (5c) \]

where \(D\) is the covariant derivative defined by the connection \(A\). The field equations for both the actions are the same.

Let me first discuss the content of equation (5c) called the simplicity constraint. The \(B_{ab}\) can be expressed in spinorial form as

\[ B_{ab}^{ij} = B_{AB}^{ij} \epsilon_{\hat{A} \hat{B}} + B_{\hat{A} \hat{B}}^{ij} \epsilon_{AB}, \]

where the spinor \(B_{AB}\) and \(B_{\hat{A}\hat{B}}\) are considered as independent variables. The tensor

\[ P_{abcd} = B_{ab} \wedge B_{cd} - b \epsilon_{abcd} \]

has the symmetries of the Riemann curvature tensor and it’s pseudoscalar component is zero. In appendix A the general ideas related to the spinorial decomposition of a tensor with the symmetries of the Riemann Curvature tensor have been summarized. The spinorial decomposition of \(P_{abcd}\) is given by

\[ P_{abcd} = B_{(AB} \wedge B_{CD)} \epsilon_{\hat{A}\hat{B}} \epsilon_{\hat{C}\hat{D}} + B_{(\hat{A}\hat{B})} \wedge B_{(\hat{C}\hat{D})} \epsilon_{AB} \epsilon_{CD} + \frac{\hat{b} \delta_{[a} \delta_{b]} d}{6} \epsilon_{CD} \epsilon_{\hat{A}\hat{B}} + B_{AB} \wedge B_{\hat{A}\hat{B}} \epsilon_{\hat{A}\hat{B}} \epsilon_{CD} + \epsilon_{AB} \epsilon_{\hat{A}\hat{B}} \epsilon_{\hat{C}\hat{D}}, \]
where $\hat{b} = B_{AB} \wedge B^{AB} + B_{\dot{A}\dot{B}} \wedge B^{\dot{A}\dot{B}}$. Therefore the spinorial equivalents of the equations are

$$
\begin{align*}
B_{(1 \wedge B_{CD})} &= 0, \\
B_{(\dot{A}\dot{B} \wedge B_{\dot{C}\dot{D}})} &= 0, \\
B_{AB} \wedge B^{AB} + B_{\dot{A}\dot{B}} \wedge B^{\dot{A}\dot{B}} &= 0 \quad \text{and} \\
B_{AB} \wedge B_{\dot{A}\dot{B}} &= 0.
\end{align*}
$$

These equations have been analyzed by Plebanski [2]. The only difference between my work (also Reisenberger [20]) and Plebanski’s work is that I have spinorially decomposed on the coordinate indices of $B$ instead of the vector indices. But this does not prevent me from adapting Plebanski’s analysis of these equations as the algebra is the same. From Plebanski’s work, we can conclude that the above equations imply $B^{ij}_{ab} = \theta^{i}_{a} \theta^{j}_{b}$ where $\theta^{i}_{a}$ are a complex tetrad.

Equations (6a) are not modified by changing the signs of $B_{AB}$ or/and $B_{\dot{A}\dot{B}}$. These are equivalent to replacing $B_{ab}$ by $-B_{ab}$ or $\pm \frac{1}{2} \epsilon_{abcd}B_{cd}$ which produce three more solution of the equations [5], [20].

The four solutions and their physical nature were discussed in the context of Riemannian general relativity by Reisenberger [20]. It can be shown that equation (5) is equivalent to the zero torsion condition [3]. Then $A$ must be the complex Levi-Civita connection of the complex metric $g_{ab} = \theta_{a} \cdot \theta_{b}$ on $M$. Because of this the curvature tensor $F^{cd}_{ab} = F^{ij}_{ab} \theta^{i}_{a} \theta^{j}_{b}$ satisfies the Bianchi identities. This makes $F$ to be the $SO(4, \mathbb{C})$ Riemann Curvature tensor. Using the metric $g_{ab}$ and it’s inverse $g^{ab}$ we can lower and raise coordinate indices.

Let me assume I have solved the simplicity constraint, and $dB = 0$. Substitute in the action $S$ the solutions $B^{ij}_{ab} = \pm \frac{1}{2} \epsilon^{i}_{abcd}B_{cd}$ and $A$ the Levi-Civita connection for a complex metric $g_{ab} = \theta_{a} \cdot \theta_{b}$. This results in a reduced action which is a function of the metric only,

$$
S(g_{ab}) = \mp \int d^{4}x bF,
$$

where $F$ is the scalar curvature $F^{ab}_{cd}$, and $b^{2} = \det(g_{ab})$. This is simply the Einstein-Hilbert action for $SO(4, \mathbb{C})$ general relativity.

The solutions $\pm \frac{1}{2} \epsilon^{i}_{abcd}B_{cd}$ do not correspond to general relativity [5], [20]. If $B^{ij}_{ab} = \pm \frac{1}{2} \epsilon^{i}_{abcd}B_{cd}$, we obtain a new reduced action,

$$
S(\theta) = \mp \text{Re} \int d^{4}x \epsilon^{abcd}F_{abcd},
$$

which is zero because of the Bianchi identity $\epsilon^{abcd}F_{abcd} = 0$. So there is no other field equation other than the Bianchi identities.

\footnote{For a proof please see footnote-7 in Ref. [20].}
3 Plebanski theory with Reality Constraint

3.1 Reality Constraint for $b \neq 0$

Let the bivector 2-form field $B_{ij}^{ab} = \pm \theta_i^a \theta_j^b$ and the space-time metric $g_{ab} = \delta_{ij} \theta_a^i \theta_i^b$. Then, the area metric \[ A_{abcd} = B_{ab} \cdot B_{cd} \] is defined by

\[ A_{abcd} = \frac{1}{2} \eta_{ik} \eta_{jl} B_{ij}^{ab} B_{kl}^{cd} \]

Consider an infinitesimal triangle with two sides as real coordinate vectors $X^a$ and $Y^b$. Its area $A$ can be calculated in terms of the coordinate bivector $Q_{ab} = \frac{1}{2} [X^a, Y^b]$ as follows

\[ A^2 = A_{abcd} Q^{ab} Q^{cd}. \]

In general $A_{abcd}$ defines a metric on coordinate bivector fields: \[ < \alpha, \beta > = A_{abcd} \alpha^{ab} \beta^{cd} \]

where $\alpha^{ab}$ and $\beta^{cd}$ are arbitrary bivector fields.

Consider a bivector 2-form field $B_{ij}^{ab} = \pm \theta_i^a \theta_j^b$ on the real manifold $M$ defined in the last section. Let $\theta_a^i$ be non-degenerate complex tetrads. Let $g_{ab} = g_{ab}^R + ig_{ab}^I$, where $g_{ab}^R$ and $g_{ab}^I$ are the real and the imaginary parts of $g_{ab} = \theta_a^i \theta_i^b$.

**Theorem 1** The area metric being real

\[ \text{Im}(A_{abcd}) = 0, \]

is the necessary and the sufficient condition for the non-degenerate metric to be real or imaginary.

**Proof.** Equation \[ \text{Im}(A_{abcd}) = 0, \]

is equivalent to the following:

\[ g_{ac}^R g_{db}^I = g_{ad}^R g_{cb}^I. \]

From equation \[ g_{ac}^R g_{db}^I = g_{ad}^R g_{cb}^I, \]

the necessary part of our theorem is trivially satisfied. Let $g$, $g^R$ and $g^I$ be the determinants of $g_{ab}$, $g_{ab}^R$ and $g_{ab}^I$ respectively. The consequence of equation \[ g_{ac}^R g_{db}^I = g_{ad}^R g_{cb}^I, \]

is that $g = g^R + g^I$. Since $g \neq 0$, one of $g^R$ and $g^I$ is non-zero.

Let me assume $g^R \neq 0$ and $g_{ac}^R$ is the inverse of $g_{ab}^R$. Let me multiply both the sides of equation \[ g_{ac}^R g_{db}^I = g_{ad}^R g_{cb}^I, \]

by $g_{ac}^R$ and sum on the repeated indices. We get $4g_{db}^I = g_{db}^I$, which implies $g_{db}^I = 0$. Similarly we can show that $g^I \neq 0$ implies $g_{db}^R = 0$. So we have shown that the metric is either real or imaginary iff the area metric is real.

Since an imaginary metric essentially defines a real geometry, we have shown that the area metric being real is the necessary and the sufficient condition for real geometry (non-degenerate) on the real manifold $M$. In one of the appendix I discuss this for any dimensions and rank of the space-time metric.
To understand the nature of the four volume after imposing the area metric reality constraint, consider the determinant of both the sides of the equation
\[ g_{ab} = \theta_a \cdot \theta_b, \]
\[ g = b^2, \]
where \( b = \frac{1}{4!} \epsilon^{abcd} B_{ab} \wedge B_{cd} \neq 0. \) From this equation we can deduce that \( b \) is not sensitive to the fact that the metric is real or imaginary. But \( b \) is imaginary if the metric is Lorentzian (signature \( + + + - \) or \( - - - + \)) and it is real if the metric is Riemannian or Kleinian (\( + + + +, - - - -, - - ++ \)).

The signature of the metric is directly related to the signature of the area metric \( A_{abcd} = g_{a[c}g_{d]b}. \) It can be easily shown that for Riemannian, Kleinian and Lorentzian geometries the signatures type of \( A_{abcd} \) are \( (6,0), (4,2) \) and \( (3,3) \) respectively.

Consider the Levi-Civita connection
\[ \Gamma^a_{bc} = \frac{1}{2} g^{ad} [\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}] \]
defined in terms of the metric. From the expression for the connection we can clearly see that it is real even if the metric is imaginary. Similarly the Riemann curvature tensor
\[ F^a_{bcd} = \partial_c \Gamma^a_{bd} + \Gamma^e_{[c} \Gamma^a_{de]} \]
is real since it is a function of \( \Gamma^a_{bc} \) only. But \( F^a_{bc} = g^{de} F^a_{bce} \) and the scalar curvature are real or imaginary depending on the metric.

In background independent quantum general relativity models, areas are fundamental physical quantities. In fact the area metric contains the full information about the metric up to a sign \(^4\). If \( B_{ab}^R \) and \( B_{ab}^L \) (vectorial indices suppressed) are the self-dual and the anti-self dual parts of an arbitrary \( B_{ab}^{ij} \), one can calculate the left and right area metrics as
\[ A_{abcd}^L = B_{ab}^L \cdot B_{cd}^L - \frac{1}{4!} \epsilon_{efgh} B_{ef}^L \cdot B_{gh}^L \epsilon_{abcd} \]
and
\[ A_{abcd}^R = B_{ab}^R \cdot B_{cd}^R + \frac{1}{4!} \epsilon_{efgh} B_{ef}^R \cdot B_{gh}^R \epsilon_{abcd} \]
respectively \(^\underline{20}. \) These metrics are pseudo-scalar component free. Reisenberger has derived Riemannian general relativity by imposing the constraint that the left and right area metrics be equal to each other \(^\underline{20}. \) This constraint is equivalent to the Plebanski constraint \( B_{ab} \wedge B_{cd} - \epsilon_{abcd} = 0. \) I would like to take this one step further by utilizing the area metric to impose reality constraints on \( SO(4,C) \) general relativity.

\(^4\)For example, please see the proof of theorem 1 of Ref:\(^\underline{3}. \)
3.2 Plebanski Action with the reality constraint.

Next, I would like to proceed to modify $SO(4, C)$ general relativity actions defined before to incorporate the area metric reality constraint. The new actions are defined as follows:

$$S_c(A, B, \bar{B}, \phi, q) = \int_M \varepsilon^{abcd} B_{ab} \wedge F_{cd} d^4x + C_S + C_R,$$

and

$$S_r(A, B, \bar{A}, \bar{B}, \phi, \bar{\phi}, q) = \text{Re} S(A, B, \bar{B}, \phi, q),$$

where

$$C_S = \int_M \frac{b}{2} g^{abcd} B_{ab} \wedge B_{cd} d^4x,$$

and

$$C_R = \int_M \frac{|b|}{2} g^{abcd} \text{Im} (B_{ab} \cdot B_{cd}) d^4x.$$  

The field $g^{abcd}$ is the same as in the last section. The field $q^{abcd}$ is real with the symmetries of the Riemann curvature tensor. The $C_R$ is the Lagrange multiplier term introduced to impose the area metric reality constraint.

The field equations corresponding to the extrema of the actions under the $A$ and $\phi$ variations are the same as given in section two. They impose the condition $B_{ij}^{ab} = \pm \theta_a^i \theta_b^j$ or $\pm * \theta_a^i \theta_b^j$ and $A$ be the Levi-Civita connection for the complex metric. The field equations corresponding to the extrema of the actions under the $q^{abcd}$ variations are $\text{Im}(B_{ab} \cdot B_{cd}) = 0$. This, as we discussed before, imposes the condition that the metric $g_{ab} = \theta_a \cdot \theta_b$ be real or imaginary.

Let me assume I have solved the simplicity constraint, the reality constraint and $dB = 0$. Substitute the solutions $B_{ij}^{ab} = \pm \theta_a^i \theta_b^j$ and $A$ the Levi-Civita connection for a real or imaginary metric $g_{ab} = \theta_a \cdot \theta_b$ in the action $S$. This results in a reduced action which is a function of the tetrad $\theta_a^i$ only,

$$S(\theta) = \mp \text{Re} \int d^4x \varepsilon^{abcd} F_{abcd}$$

where $F$ is the scalar curvature $F_{ab}$. Recall that $F$ is real or imaginary depending on the metric. This action reduces to Einstein-Hilbert action if both the metric and space-time density are simultaneously real or imaginary. If not, it is zero and there is no field equation involving the curvature $F_{cd}$ tensor other than the Bianchi identities.

If $B_{ij}^{ab} = \pm * \theta_a^i \theta_b^j$, we get a new reduced action,

$$S(\theta) = \mp \text{Re} \int d^4x \varepsilon^{abcd} F_{abcd},$$

which is zero because of the Bianchi identity $\varepsilon^{abcd} F_{abcd} = 0$. So there is no other field equation other than the Bianchi identities.

\footnote{Also for $B_{ij}^{ab} = \pm * \theta_a^i \theta_b^j$, it can be verified that the reality constraint implies that the metric $g_{ab} = \theta_a \cdot \theta_b$ be real or imaginary.}
4 Discretization

4.1 BF theory

Consider that a continuum manifold is triangulated with four simplices. The discrete equivalent of a bivector two-form field is the assignment of a bivector $B^i_j$ to each triangle $b$ of the triangulation. Also the equivalent of a connection one-form is the assignment of a parallel propagator $g_{ij}$ to each tetrahedron $e$. Using the bivectors and parallel propagators assigned to the simplices, the actions for general relativity and BF theory can be rewritten in a discrete form [7]. The real $SO(4, C)$ BF action can be discretized as follows [6]:

$$S(B_b, g_e) = \text{Re} \sum_b B^i_j b \ln H_{bij}. \quad (14)$$

The $H_b$ is the holonomy associated to the triangle $b$. It will be quantized to get an spin foam model later as done by Ooguri in section five.

4.2 Barrett–Crane Constraints

The bivectors $B_i$ associated with the ten triangles of a four simplex in a flat Riemannian space satisfy the following properties called the Barrett-Crane constraints [11]:

1. The bivector changes sign if the orientation of the triangle is changed.

2. Each bivector is simple.

3. If two triangles share a common edge, then the sum of the bivectors is also simple.

4. The sum of the bivectors corresponding to the edges of any tetrahedron is zero. This sum is calculated taking into account the orientations of the bivectors with respect to the tetrahedron.

5. The six bivectors of a four simplex sharing the same vertex are linearly independent.

6. The volume of a tetrahedron calculated from the bivectors is real and non-zero.

The items two and three can be summarized as follows:

$$B_i \wedge B_j = 0 \forall i, j,$$

where $A \wedge B = \varepsilon_{IJKL} A^{IJ} B^{KL}$ and the $i, j$ represents the triangles of a tetrahedron. If $i = j$, it is referred to as the simplicity constraint. If $i \neq j$ it is referred as the cross-simplicity constraints.

Barrett and Crane have shown that these constraints are sufficient to restrict a general set of ten bivectors $E_b$ so that they correspond to the triangles of a
geometric four simplex up to translations and rotations in a four dimensional flat Riemannian space [11].

The Barrett-Crane constraints theory can be easily extended to the $SO(4, C)$ general relativity. In this case the bivectors are complex and so the volume calculated for the sixth constraint is complex. So we need to relax the condition of the reality of the volume.

We would like to combine the area metric reality constraint with the Barrett-Crane Constraints. For this we must find the discrete equivalent of the area metric reality condition. For this let me next discuss the area metric reality condition in the context of three simplices and four simplices. I would like to show that the discretized area metric reality constraint combined with the Barrett-Constraint constraint requires the complex bivectors associated to a three or four simplex to describe real flat geometries.

### 4.2.1 Tetrahedron

Consider a tetrahedron $t$. Let the numbers 0 to 3 denote the vertices of the tetrahedron. Let me choose the 0 as the origin of the tetrahedron. Let $B_{ij}$ be the complex bivector associated with the triangle $0ij$ where $i$ and $j$ denote one of the vertices other than the origin and $i < j$. Let $B_0$ be the complex bivector associated with the triangle 123. Then similar to Riemannian general relativity [11], the Barrett-Crane constraints$^6$ for $SO(4, C)$ general relativity imply that

\begin{align}
B_{ij} &= a_i \wedge a_j, \\
B_0 &= -B_{12} - B_{23} - B_{34},
\end{align}

where $a_i$, $i = 1$ to 3 are linearly independent complex four vectors associated to the links $0i$ of the three simplex. Let me choose the vectors $a_i$, $i = 1$ to 3 to be the complex vector basis inside the tetrahedron. Then the complex 3D metric inside the tetrahedron is

$$g_{ij} = a_i \cdot a_j,$$

where the dot is the scalar product on the vectors. This describes a flat complex three dimensional geometry inside the tetrahedron. The area metric is given by

$$A_{ijkl} = g_{ik}g_{lj}.$$

The coordinates of the vectors $a_i$ are simply

$$a_1 = (1, 0, 0),$$
$$a_2 = (0, 1, 0),$$
$$a_3 = (0, 0, 1).$$

Because of this all of the six possible scalar products made out of the bivectors $B_{ij}$ are simply the elements of the area metric. From the discussion of the last

$^6$We do not require to use the fifth Barrett-Crane constraint since we are only considering one tetrahedron of a four simplex.
section the reality of the area metric simply requires that the metric $g_{ij}$ be real or imaginary. Since $B_0$ is also defined by equation (15b), its inner product with itself and other bivectors are real. Thus in the context of a three simplex, the discrete equivalent of the area metric reality constraint is that the all possible scalar products of bivectors associated with the triangles of a three simplex be real.

4.2.2 Four Simplex

In the case of a four simplex $s$ there are six bivectors $B_{ij}$. There are four $B_0$ type bivectors. Let $B_i$ denote the bivector associated to the triangle made by connecting the vertices other than the origin and vertex $i$. The Barrett-Crane constraints imply equation (15a) with $i, j = 1$ to 4. There is one equation for each $B_i$ similar to equation (15b). Now the metric $g_{ij} = a_i \cdot a_j$ describes a complex four dimensional flat geometry inside the four simplex $s$. Now assuming we are dealing with non-degenerate geometry, the reality of the geometry requires the reality of the area metric. Similar to the three dimensional case, the components of the area metric are all of the possible scalar products made out of the bivectors $B_{ij}$. The scalar products of the bivectors $B_i$ among themselves or with $B_{ij}$’s are simple real linear combinations of the scalar products made from $B_{ij}$’s. So one can propose that the discrete equivalent of the area metric reality constraint is simply the condition that the scalar product of these bivectors be real. Let me refer to the later condition as the bivector scalar product reality constraint.

\textbf{Theorem 2} The necessary and sufficient conditions for a four simplex with real non-degenerate flat geometry are 1) The $SO(4, C)$ Barrett-Crane constraints\footnote{The $SO(4, C)$ Barrett-Crane constraints differ from the real Barrett-Crane constraints by the following: \begin{enumerate} \item The bivectors are complex, and \item The condition for the reality of the volume of tetrahedron is not required. \end{enumerate}} and 2) The reality of all possible bivector scalar products.

\textbf{Proof.} The necessary condition can be shown to be true by straight forward generalization of the arguments given by Barrett and Crane\cite{Barrett2001} and application of the discussions in the last paragraph. The sufficiency of the conditions follow from the discussion in the last paragraph.

5 Spin foam of the $SO(4, C)$ BF model

Consider a four dimensional submanifold $M$. Let $A$ be a $SO(4, C)$ connection 1-form and $B^i$ a complex bivector valued 2-form on $M$. Let $F$ be the curvature 2-form of the connection $A$. Then the real continuum BF theory action defined...
in section two is,
\[ S_{BF}(A, B_{ij}, \bar{A}, \bar{B}_{ij}) = \text{Re} \int_M B \wedge F, \tag{17} \]
where \( A, B_{ij} \) and their complex conjugates are considered as independent free variables. This classical theory is a topological field theory. This property also holds on spin foam quantization as will be discussed below.

The Spin foam model for the \( SO(4, C) \) BF theory action can be derived from the discretized BF action by using the path integral quantization as illustrated in Ref.\[6\] for compact groups. Let \( \Delta \) be a simplicial manifold obtained by a triangulation of \( M \). Let \( g_e \in SO(4, C) \) be the parallel propagators associated with the edges (three-simplices) representing the discretized connection. Let \( H_b = \prod_{e \supset b} g_e \) be the holonomies around the bones (two-simplices) in the four dimensional matrix representation of \( SO(4, C) \) representing the curvature. Let \( B_b \) be the \( 4 \times 4 \) antisymmetric complex matrices corresponding to the dual Lie algebra of \( SO(4, C) \) corresponding to the discrete analog of the \( B \) field. Then the discrete BF action is
\[ S_d = \text{Re} \sum_{b \in M} \text{tr}(B_b \ln H_b), \]
which is considered as a function of the \( B_b \)'s and \( g_e \)'s. Here \( B_b \) the discrete analog of the \( B \) field are \( 4 \times 4 \) antisymmetric complex matrices corresponding to dual Lie algebra of \( SO(4, C) \). The \( \ln \) maps from the group space to the Lie algebra space. The trace is taken over the Lie algebra indices. Then the quantum partition function can be calculated using the path integral formulation as,
\[ Z_{BF}(\Delta) = \int \prod_b dB_b d\bar{B}_b \exp(iS_d) \prod_e dg_e = \int \prod_b \delta(H_b) \prod_e dg_e, \tag{18} \]
where \( dg_e \) is the invariant measure on the group \( SO(4, C) \). The invariant measure can be defined as the product of the bi-invariant measures on the left and the right \( SL(2, C) \) matrix components. Please see appendix A and B for more details. Similar to the integral measure on the \( B \)'s an explicit expression for the \( dg_e \) involves product of conjugate measures of complex coordinates.

Now consider the identity
\[ \delta(g) = \frac{1}{64\pi^8} \int d\omega \text{tr}(T_\omega(g))d\omega, \tag{19} \]
where the \( T_\omega(g) \) is a unitary representation of \( SO(4, C) \), where \( \omega = (\chi_L, \chi_R) \) such that \( n_L + n_R \) is even, \( d\omega = |\chi_L \chi_R|^2 \). The details of the representation theory is discussed in appendix B. The integration with respect to \( d\omega \) in the above equation is interpreted as the summation over the discrete \( n \)'s and the integration over the continuous \( \rho \)'s.
By substituting the harmonic expansion for \( \delta(g) \) into the equation (18) we can derive the spin foam partition of the \( SO(4, C) \) BF theory as explained in Ref:[1] or Ref:[6]. The partition function is defined using the \( SO(4, C) \) intertwiners and the \( \{15\omega\} \) symbols.

The relevant intertwiner for the BF spin foam is

\[
i_e = \gamma_{1} \gamma_{2} \gamma_{3} \gamma_{4} \omega_{5} \omega_{15} \omega_{1}
\]

The nodes where the three links meet are the Clebsch-Gordan coefficients of \( SO(4, C) \). The Clebsch-Gordan coefficients of \( SO(4, C) \) are just the product of the Clebsch-Gordan coefficients of the left and the right handed \( SL(2, C) \) components. The Clebsch-Gordan coefficients of \( SL(2, C) \) are discussed in the references [19] and [31].

The quantum amplitude associated with each simplex \( s \) is given below and can be referred to as the \( \{15\omega\} \) symbol,

\[
\{15\omega\} = \omega_{12} \omega_{13} \omega_{14} \omega_{15} \omega_{24} \omega_{25} \omega_{34} \omega_{35} \omega_{45} \omega_{1} \omega_{15} \omega_{12} \omega_{13} \omega_{14} \omega_{15} \omega_{24} \omega_{25} \omega_{34} \omega_{35} \omega_{45}
\]

The final partition function is

\[
Z_{BF}(\Delta) = \int_{\{\omega_b, \omega_e\}} \prod_b d\omega_b 64\pi^8 \sum_s Z_{BF}(s) \prod_b d\omega_b \prod_e d\omega_e,
\]

where the \( Z_{BF}(s) = \{15\omega\} \) is the amplitude for a four-simplex \( s \). The \( d\omega_b = |\chi_L \chi_R|^2 \) term is the quantum amplitude associated with the bone \( b \). Here \( \omega_e \) is the internal representation used to define the intertwiners. Usually \( \omega_e \) is replaced by \( i_e \) to indicate the intertwiner. The set \( \{\omega_b, \omega_e\} \) of all \( \omega_b \)'s and \( \omega_e \)'s is usually called a coloring of the bones and the edges. This partition function may not be finite in general.

It is well known that the BF theories are topological field theories. A priori one cannot expect this to be true for the case of the BF spin foam models because of the discretization of the BF action. For the spin foam models of the BF theories for compact groups, it has been shown that the partition functions are triangulation independent up to a factor [13]. This analysis is purely based on spin foam diagrammatics and is independent of the group used as long the
BF spin foam is defined formally by equation (18) and the harmonic expansion in equation (19) is formally valid. So one can apply the spin foam diagrammatics analysis directly to the $SO(4, C)$ BF spin foam and write down the triangulation independent partition function as

$$Z'_{BF}(\Delta) = \tau^{n_4-n_3} Z_{BF}(\Delta)$$

using the result from [13]. In the above equation $n_4, n_3$ is number of four bubbles and three bubbles in the triangulation $\Delta$ and

$$\tau = \delta_{SO(4,C)}(I) = \frac{1}{64\pi^8} \int d^2\omega.$$ 

The above integral is divergent and so the partition functions need not be finite. The normalized partition function is to be considered as the proper partition function because the BF theory is supposed to be topological and so triangulation independent.

6 The $SO(4, C)$ Barrett-Crane Model

6.1 Introduction

My goal here is to systematically construct the Barrett-Crane model of the $SO(4, C)$ general relativity. In the previous section I discussed the $SO(4, C)$ BF spin foam model. The basic elements of the BF spin foams are spin networks built on graphs dual to the triangulations of the four simplices with arbitrary intertwiners and the principal unitary representations of $SO(4, C)$ discussed in appendix B. These closed spin networks can be considered as quantum states of four simplices in the BF theory and the essence of these spin networks is mainly gauge invariance. To construct a spin foam model of general relativity these spin networks need to be modified to include the Plebanski Constraints in the discrete form.

A quantization of a four-simplex for the Riemannian general relativity was proposed by Barrett and Crane [11]. The bivectors $B_i$ associated with the ten triangles of a four-simplex in a flat Riemannian space satisfy the properties called the Barrett-Crane constraints$^8$. They have been listed in section 4.2 which are repeated below for convenience:

1. The bivector changes sign if the orientation of the triangle is changed.

2. Each bivector is simple.

3. If two triangles share a common edge, then the sum of the bivectors is also simple.

---

$^8$ I would like to refer the readers to the original paper [11] for more details.
4. The sum of the bivectors corresponding to the edges of any tetrahedron is zero. This sum is calculated taking into account the orientations of the bivectors with respect to the tetrahedron.

5. The six bivectors of a four-simplex sharing the same vertex are linearly independent.

6. The volume of a tetrahedron calculated from the bivectors is real and non-zero.

The items two and three can be summarized as follows:

\[ B_i \wedge B_j = 0 \quad \forall i, j, \]

where \( A \wedge B = \varepsilon_{IJKL}A^{IJ}B^{KL} \) and the \( i, j \) represents the triangles of a tetrahedron. If \( i = j \), it is referred to as the simplicity constraint. If \( i \neq j \) it is referred as the cross-simplicity constraints.

Barrett and Crane have shown that these constraints are sufficient to restrict a general set of ten bivectors \( E_b \) so that they correspond to the triangles of a geometric four-simplex up to translations and rotations in a four dimensional flat Riemannian space.

The Barrett-Crane constraints theory can be trivially extended to the \( SO(4, C) \) general relativity. In this case the bivectors are complex and so the volume calculated for the sixth constraint is complex. So we need to relax the condition of the reality of the volume.

A quantum four-simplex for Riemannian general relativity is defined by quantizing the Barrett-Crane constraints \(^{[11]}\). The bivectors \( B_i \) are promoted to the Lie operators \( \hat{B}_i \) on the representation space of the relevant group and the Barrett-Crane constraints are imposed at the quantum level. A four-simplex has been quantized and studied in the case of the Riemannian general relativity before \(^{[11]}\). All the first four constraints have been rigorously implemented in this case. The last two constraints are inequalities and they are difficult to impose. This could be related to the fact that the Riemannian Barrett-Crane model reveal the presence of degenerate sectors \(^{[25], [26]}\) in the asymptotic limit \(^{[25]}\) of the model. For these reasons here after I would like to refer to a spin foam model that satisfies only the first four constraints as an essential Barrett-Crane model, While a spin foam model that satisfies all the six constraints as a rigorous Barrett-Crane model.

Here I would like to derive the essential \( SO(4, C) \) Barrett-Crane model. For this one must deal with complex bivectors instead of real bivectors. The procedure that I would like to use to solve the constraints can be carried over directly to the Riemannian Barrett-Crane model. This derivation essentially makes the derivation of the Barrett-Crane intertwiners for the real and the complex Riemannian general relativity more rigorous.
6.2 The SO(4,C) intertwiner

The group $SO(4,C)$ is locally isomorphic to $\frac{SL(2,C) \times SL(2,C)}{Z_2}$. An element $B$ of the Lie algebra space of $SO(4,C)$ can be split into the left and the right handed $SL(2,C)$ components,

$$ B = B_L + B_R. \quad (21) $$

There are two Casimir operators for $SO(4,C)$ which are $\varepsilon_{IJKL} B^{IJ} B^{KL}$ and $\eta_{IK} \eta_{JL} B^{IJ} B^{KL}$, where $\eta_{IK}$ is the flat Euclidean metric. In terms of the left and right handed split I can expand the Casimir operators as

$$ \varepsilon_{IJKL} B^{IJ} B^{KL} = B_L \cdot B_L - B_R \cdot B_R \quad \text{and} $$

$$ \eta_{IK} \eta_{JL} B^{IJ} B^{KL} = B_L \cdot B_L + B_R \cdot B_R, $$

where the dot products are the trace in the $SL(2,C)$ Lie algebra coordinates.

The bivectors are to be quantized by promoting the Lie algebra vectors to Lie operators on the unitary representation space of $SO(4,C) \approx SL(2,C) \times SL(2,C)/Z_2$. The relevant unitary representations of $SO(4,C)$ are labeled by a pair $(\chi_L, \chi_R)$ such that $n_L + n_R$ is even (appendix B). The elements of the representation space $D_{\chi_L} \otimes D_{\chi_R}$ are the eigen states of the Casimirs and on them the operators reduce to the following:

$$ \varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 - \chi_R^2}{2} \hat{I} \quad \text{and} \quad (22) $$

$$ \eta_{IK} \eta_{JL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 + \chi_R^2 - 2}{2} \hat{I}. \quad (23) $$

The equation (22) implies that on $D_{\chi_L} \otimes D_{\chi_R}$ the simplicity constraint $B \wedge B = 0$ is equivalent to the condition $\chi_L = \pm \chi_R$. I would like to find a representation space on which the representations of $SO(4,C)$ are restricted precisely by $\chi_L = \pm \chi_R$. Since a $\chi$ representation is equivalent to $-\chi$ representations [19], $\chi_L = +\chi_R$ case is equivalent to $\chi_L = -\chi_R$ [19].

The Barrett-Crane intertwiner for Riemannian general relativity has been systematically quantized in Ref.[18]. Since the representation theory for $SO(4,C)$ is similar to that of $SO(4,R)$, the systematic derivation can be generalized to $SO(4,C)$ general relativity\(^9\).

The components of the Barrett-Crane intertwiner $|\Psi\rangle \in \bigotimes_i D_{\chi_i} \otimes D^*_{\chi_i}$ can

\(^9\)Readers can refer to the preprint Ref.[22] for details of derivations of $SO(4,C)$ Barrett-Crane intertwiner.
be written using the Gelfand-Naimarck representation theory \[19\] as:

$$|\Psi\rangle = \int_{CS^3} \chi_1, \chi_2, \chi_3, \chi_4 \, dn.$$

Since $SL(2, C) \approx CS^3$, using the following graphical identity:

$$\int_{SL(2, C)} dg = \int \chi_1, \chi_2, \chi_3, \chi_4 \, d\chi,$$

the Barrett-Crane solution can be rewritten as

$$|\Psi\rangle = \int \chi_1, \chi_2, \chi_3, \chi_4 \, d\chi,$$

which emerges as an intertwiner in the familiar form in which Barrett and Crane proposed it for the Riemannian general relativity. It can be clearly seen that the simple representations for $SO(4, R)$ ($J_L = J_R$) has been replaced by the simple representation of $SO(4, C)$ ($\chi_L = \pm \chi_R$).

All the analysis done until for the $SO(4, C)$ Barrett-Crane theory can be directly applied to the Riemannian Barrett-Crane theory. The correspondences between the two models are listed in the following table:\[10\]:

| Property          | $SO(4, R)$ BC model | $SO(4, C)$ BC model |
|-------------------|--------------------|---------------------|
| Gauge group       | $SO(4, R) \approx \frac{SL(2,C) \otimes SL(2,C)}{Z_2}$ | $SO(4, C) \approx \frac{SU(2) \otimes SU(2)}{Z_2}$ |
| Representations   | $J_L, J_R$         | $\chi_L, \chi_R$   |
| Simple representations | $J_L = J_R$                  | $\chi_L = \pm \chi_R$   |
| Homogenous space  | $S^3 \approx SU(2)$ | $CS^3 \approx SL(2, C)$ |

\[10\] BC stands for Barrett-Crane. For $\chi_L$ and $\chi_R$ we have $n_L + n_R = \text{even.}$
6.3 The Spin Foam Model for the $SO(4, C)$ General Relativity.

The $SO(4, C)$ Barrett-Crane intertwiner derived in the previous section can be used to define a $SO(4, C)$ Barrett-Crane spin foam model. The amplitude $Z_{BC}(s)$ of a four-simplex $s$ is given by the $\{10\chi\}_{SO(4, C)}$ symbol given below:

$$\{10\chi\}_{SO(4, C)} = \chi_{12} \chi_{25} \chi_{34} \chi_{35} \chi_{45} \chi_{14}, \quad (24)$$

where the circles are the Barrett-Crane intertwiners. The integers represent the tetrahedra and the pairs of integers represent triangles. The intertwiners use the four $\chi$’s associated with the links that emerge from it for its definition in equation (24). In the next subsection, the propagators of this theory are defined and the $\{10\chi\}$ symbol is expressed in terms of the propagators in the subsubsection that follows it.

The $SO(4, C)$ Barrett-Crane partition function of the spin foam associated with the four dimensional simplicial manifold with a triangulation $\Delta$ is

$$Z(\Delta) = \sum_{\{\chi_b\}} \left( \prod_b \frac{d_{\chi_b}^2}{64\pi^8} \right) \prod_s Z(s), \quad (25)$$

where $Z(s)$ is the quantum amplitude associated with the 4-simplex $s$ and the $d_{\chi_b}$ adopted from the spin foam model of the $BF$ theory can be interpreted as the quantum amplitude associated with the bone $b$.

6.4 The Features of the $SO(4, C)$ Spin Foam

- Areas: The squares of the areas of the triangles (bones) of the triangulation are given by $\eta_{IK} \eta_{JL} B_{IJ} B_{KL}$. The eigen values of the squares of the areas in the $SO(4, C)$ Barrett-Crane model from equation (25) are given by

$$\eta_{IK} \eta_{JL} B_{bIJ} B_{bKL} = (\chi^2 - 1) \hat{I}$$

$$= \left( \frac{n^2}{2} - \rho^2 - 1 + i \rho m \right) \hat{I}.$$ 

One can clearly see that the area eigen values are complex. The $SO(4, C)$ Barrett-Crane model relates to the $SO(4, C)$ general relativity. Since in the $SO(4, C)$ general relativity the bivectors associated with any two dimensional flat object are complex, it is natural to expect that the areas
defined in such a theory are complex too. This is a generalization of the concept of the space-like and the time-like areas for the real general relativity models: Area is imaginary if it is time-like and real if it is space-like.

- Propagators: Laurent and Freidel have investigated the idea of expressing simple spin networks as Feynman diagrams [32]. Here we will apply this idea to the $SO(4, C)$ simple spin networks. Let $\Sigma$ be a triangulated three surface. Let $n_i \in CS^3$ be a vector associated with the $i^{th}$ tetrahedron of the $\Sigma$. The propagator of the $SO(4, C)$ Barrett-Crane model associated with the triangle $ij$ is given by

$$G_{\chi_{ij}}(n_i, n_j) = Tr(T_{\chi_{ij}}(g(n_i))T_{\chi_{ij}}^\dagger(g(n_j)))$$

$$= Tr(T_{\chi_{ij}}(g(n_i)g^{-1}(n_j))),$$

where $\chi_{ij}$ is a representation associated with the triangle common to the $i^{th}$ and the $j^{th}$ tetrahedron of $\Sigma$. If $X$ and $Y$ belong to $CS^3$ then

$$tr\left(g(X)g(Y)^{-1}\right) = 2X.Y,$$

where $X.Y$ is the Euclidean dot product and $tr$ is the matrix trace. If $\lambda = e^t$ and $\frac{1}{\lambda}$ are the eigen values of $g(X)g(Y)^{-1}$ then,

$$\lambda + \frac{1}{\lambda} = 2X.Y$$

$$X.Y = \cosh(t).$$

From the expression for the trace of the $SL(2, C)$ unitary representations, (appendix A, [19]) I have the propagator for the $SO(4, C)$ Barrett-Crane model calculated as

$$G_{\chi_{ij}}(n_i, n_j) = \frac{\cos(p_{ij}\eta_{ij} + n_{ij}\theta_{ij})}{|\sinh(\eta_{ij} + i\theta_{ij})|^2},$$

where $\eta_{ij} + i\theta_{ij}$ is defined by $n_i.n_j = \cosh(\eta_{ij} + i\theta_{ij})$. Two important properties of the propagators are listed below.

1. Using the expansion for the delta on $SL(2, C)$ I have

$$\delta_{CS^3}(X,Y) = \delta_{SL(2, C)}(g(X)g^{-1}(Y))$$

$$= \frac{1}{8\pi^4} \int \bar{\chi}\chi Tr(T_{\chi}(g(X)g^{-1}(Y)))d\chi,$$

where the suffix on the deltas indicate the space in which it is defined. Therefore

$$\int \bar{\chi}\chi G_{\chi}(X,Y)) = 8\pi^4\delta_{CS^3}(X,Y).$$
2. Consider the orthonormality property of the principal unitary representations of $SL(2, \mathbb{C})$ given by
\[
\int_{CS^3} T_{z_1}^{\chi_1}(g(X)) T_{z_2}^{\chi_2}(g(X)) dX = \frac{8\pi^4}{\chi_1 \chi_2} \delta(\chi_1 - \chi_2) \delta(z_1 - \hat{z}_1) \delta(z_2 - \hat{z}_2),
\]
where the delta on the $\chi$'s is defined up to a sign of them. From this I have
\[
\int_{CS^3} G_{\chi_1}(X,Y) G_{\chi_2}(Y,Z) dY = \frac{8\pi^4}{\chi_1 \chi_2} \delta(\chi_1 - \chi_2) G_{\chi_1}(X,Z).
\]

- The $\{10\chi\}$ symbol can be defined using the propagators on the complex three sphere as follows:
\[
Z(s) = \int_{x_k \in CS^3} \prod_{i<j} T_{\chi_{ij}}(g(x_i)g(x_j)) \prod_k dx_k,
\]
\[
eq \int_{x_k \in CS^3} \prod_{i<j} G_{\chi_{ij}}(x_i,x_j) \prod_k dx_k,
\]
where $i$ denotes a tetrahedron of the four-simplex. For each tetrahedron $k$, a free variable $x_k \in CS^3$ is associated. For each triangle $ij$ which is the intersection of the $i$'th and the $j$'th tetrahedron, a representation of $SL(2, \mathbb{C})$ denoted by $\chi_{ij}$ is associated.

- Discretization Dependence and Local Excitations: It is well known that the BF theory is discretization independent and is topological. The spin foam for the $SO(4, \mathbb{C})$ general relativity is got by imposing the Barrett-Crane constraints on the BF Spin foam. After the imposition of the Barrett-Crane constraints the theory loses the discretization independence and the topological nature. This can be seen in many ways.

  - The simplest reason is that the $SO(4, \mathbb{C})$ Barrett-Crane model corresponds to the quantization of the discrete $SO(4, \mathbb{C})$ general relativity which has local degrees of freedom.

  - After the restriction of the representations involved in BF spin foams to the simple representations and the intertwiners to the Barrett-Crane intertwiners, various important identities used in the spin foam diagrammatics and proof of the discretization independence of the BF theory spin foams in Ref\[13\] are no longer available.

  - The BF partition function is simply gauge invariant measure of the volume of space of flat connections. Consider the following harmonic expansion of the delta function which was used in the derivation of the $SO(4, \mathbb{C})$ BF theory:
\[
\delta(g) = \frac{1}{8\pi^4} \int d\omega \text{tr}(T_\omega(g)) d\omega.
\]
Imposition of the Barrett-Crane constraints on the BF theory spin
foam, suppresses the terms corresponding to the non-simple repre-
sentations. If only the simple representations are allowed in the right
hand side, it is no longer peaked at the identity. This means that
the partition function for the \( SO(4, C) \) Barrett-Crane model involves
contributions only from the non-flat connections which has local in-
formation.

- In the asymptotic limit study of the \( SO(4, C) \) spin foams in section
four of Ref:42 the discrete version of the \( SO(4, C) \) general relativity
(Regge calculus) is obtained. The Regge calculus action is clearly
discretization dependent and non-topological.

- The real Barrett-Crane models that are discussed in the next section are
the restricted form of the \( SO(4, C) \) Barrett-Crane model. The above rea-
oning can be applied to argue that they are also discretization dependent.

7 Spin Foams for 4D Real General Relativity
and reality constraints

7.1 The Formal Structure of Barrett-Crane Intertwiners

Let me briefly discuss the formal structure of the Barrett-Crane intertwiner
of the \( SO(4, C) \) general relativity for the purpose of the developing spin foam
models for real general relativity theories. It has the following elements:

- A gauge group \( G \),
- A homogenous space \( X \) of \( G \),
- A \( G \) invariant measure on \( X \) and,
- A complete orthonormal set of functions which call as \( T \)–functions which
are maps from \( X \) to the Hilbert spaces of a subset of unitary representa-
tions of \( G \):
  \[ T_{\rho} : X \rightarrow D_{\rho}, \]
where \( \rho \) is a representation of \( G \). The \( T \)–functions correspond to the
various unitary representations under the transformation of \( X \) under \( G \).
The \( T \)–functions are complete in the sense that on the \( L^2 \) functions on \( X \)
they define invertible Fourier transforms. The \( T \)– functions are written
using its components in a linear vector basis of representation \( D_{\rho} \).

Formally Barrett-Crane intertwiners are quantum states \( \Psi \) associated to
closed simplicial two surfaces defined as an integral of a outer product of \( T \)–functions
on the space \( X \):
\[
\Psi = \int_X \prod_{\rho} T_{\rho}(x) d_x x \in \prod_{\rho} D_{\rho}.
\]
It can be seen that $\Psi$ is gauge invariant under $G$ because of the invariance of the measure $dXx$.

### 7.2 The Real Barrett-Crane Models

Consider a four-simplex with complex bivectors $B_i$, $i = 1$ to $10$ associated with its triangles. The discrete equivalent of the area metric reality constraint is the bivector scalar product reality constraint. Then the bivector scalar product reality constraint requires

$$\text{Im}(B_i \wedge B_j) = 0 \quad \forall i, j.$$ 

I would like to formally reduce the Barrett-Crane models for real general relativity from that of the $SO(4, C)$ Barrett-Crane model by using the bivector scalar products reality constraint. Precisely I plan to use the following three ideas to reduce the Barrett-Crane models:

1. The formal structure of the reduced intertwiners should be the same as that of the $SO(4, C)$ Barrett-Crane model,

2. The eigen value of the Casimir corresponding to the square of the area of any triangle must be real. I would like to refer to this as the self-reality constraint$^{11}$,

3. The eigen values of the square of area Casimir corresponding to the representations associated with the internal links of the intertwiner must be real. I would like to refer to this as the cross-reality constraint.

The first idea sets a formal ansatz for the reduction process. The simple and symmetric nature of the $SO(4, C)$ (or $SO(4, R)$) Barrett-Crane intertwiner and, the work done in Ref:[32], Ref:[17] and Ref:[33] can be considered as evidences for the formal structure to be the general form of structure of intertwiners for all signatures. Here we assume this idea as a hypothesis.

The square of the area of a triangle is simply the scalar product of the bivector of a triangle with itself. Second condition is the quantum equivalent of the reality of the scalar product of a bivector associated with a triangle with itself. Once the second condition is imposed the third condition is the quantum equivalent of the reality of the scalar product of the two bivectors of any two triangle of a tetrahedron$^{12}$.

---

$^{11}$I would like to mention that the areas being real necessarily does not mean that the bivectors must also be real.

$^{12}$We have ignored to impose reality of the scalar products of the bivectors associated to any two triangles of the same four simplex which intersect at only at one vertex. This is because these constraints appears not to be needed for a formal extraction of the Barrett-Crane models of real general relativity from that of $SO(4, C)$ general relativity described in this section. Imposing these constraints may not be required because of the enormous redundancy in the bivector scalar product reality constraints. This issue need to be carefully investigated.
My goal is to use the above principles to derive reduced Barrett-Crane models and later one can convince oneself by identifying and verifying that the Barrett-Crane constraints are satisfied for a subgroup of \( SO(4, C) \) for each of the reduced model.

In general by reducing a certain Hilbert space associated with the representations of a group \( G \) by some constraints, the resultant Hilbert space need not contain the states gauge invariant under \( G \). In that case one can look for gauge invariance states under subgroups of \( G \). In our case we will find that the suitable quantum states extracted by adhering to the above principles are gauge symmetry reduced versions of \( SO(4, C) \) Barrett-Crane states. They are gauge invariant only under the real subgroups of \( SO(4, C) \).

Let \( P \) be a formal projector which reduces the Hilbert space \( D_{\chi_L} \otimes D_{\chi_R} \) to a reduced Hilbert space such that the reality constraints are satisfied. Let me assume as an ansatz that now the complex three sphere is replaced by its subspace \( X \) due to projection. Now I expect, the projected \( SO(4, C) \) Barrett-Crane intertwiner is spanned by the following states for all \( \chi \) satisfying the reality constraints:

\[
\Psi_X = \int_{x \in X} \prod_i PT_{\chi_i}(g(x)) \tilde{d}g(x),
\]

where \( \tilde{d}g(n) \) is the reduced measure of \( d\!\!\!g(n) \) on \( X \). The imposition of the self-reality constraints expressed at the quantum level sets \( \rho_i \) or \( n_i \) to be zero on each vertex of the \( SO(4, C) \) Barrett-Crane intertwiner. Let me rewrite the projected intertwiner as follows.

\[
\Psi_X = \int_{x, y \in X} \prod_{1, 2} PT_{\chi_1}(g(x)) \delta_X(x, y) \prod_{3, 4} PT_{\chi_1}(g(y)) dX g(x) dX g(y),
\]

where \( \delta_X(x, y) \) is the delta function on \( X \). Since \( X \) is a subspace of \( SL(2, C) \) a harmonic expansion can be derived for \( \delta(x, y) \) using the unitary representations of \( SL(2, C) \). Since the intertwiner must obey the cross reality constraint the harmonic expansion must only contain simple representations of \( SL(2, C) \) (\( \rho \) or \( n \) is zero).

For the Fourier transform defined by \( PT_{\chi}(g(x)) \) to be complete and orthonormal I must have

\[
\int_{\chi \in Q} \bar{\chi} tr(PT_{\chi}(g(x))PT_{\chi}(g(y))) d\chi = \delta_X(x, y),
\]

where \( Q \) is the set of all simple representations\(^{13} \) of \( SL(2, C) \) required for the expansion. Only the simple representations of \( SL(2, C) \) must be used to satisfy the cross-reality constraints. Thus, the number of reduced intertwiners derivable is directly related to the possible solutions for this equation (subjected to Barrett-Crane constraints).

\(^{13}\)One could also call the simple representations of \( SL(2, C) \) as the real representations since it corresponds to the real areas and the real homogenous spaces. But I will avoid this to avoid any possible confusion.
The equation of a complex three sphere is
\[ x^2 + y^2 + z^2 + t^2 = 1. \]

There are four different topologically different maximally connected real subspaces of \( CS^3 \) such that the harmonic (Fourier) expansions on these spaces use the simple representations of \( SL(2, C) \) only. They are namely, the three sphere \( S^3 \), the real hyperboloid \( H^+ \), the imaginary hyperboloid \( H^- \) and the Kleinien hyperboloid\(^4\) \( K^3 \). Each of these subspace \( X \) are maximal real subspaces of \( CS^3 \). They are all homogenous under the action of a maximal real subgroup\(^5\) \( G_X \) of \( SO(4, C) \). There exists a \( G_X \) invariant measure \( d^X(x) \). The reduced bivectors acting on the functions on \( X \) effectively take values in the Lie algebra of \( G_X \). Since the measure \( d^X(n) \) is invariant, the reduced intertwiner is gauge invariant. So the intertwiner \( \Psi_X \) must correspond to the quantum general relativity for the group \( G_X \).

Let the coordinates of \( n = (x, y, z, t) \) be restricted to real values here after in this section. Let me discuss the various reduced intertwiners:

1. \( \rho = 0 \) case: This uses only the \( \chi = (0, n) \) representations only. This corresponds to \( X = S^3 \), satisfying
\[ x^2 + y^2 + z^2 + t^2 = 1, \]
which is invariant under \( SO(4, R) \). So this case corresponds to the Riemannian general relativity. The appropriate projected \( T \)–functions are the representation matrices of \( SU(2) \approx S^3 \) and the reduced measure is the Haar measure of \( SU(2) \). The intertwiner I get is the Barrett-Crane intertwiner for the Riemannian general relativity. Here the \( \chi' \)'s has been replaced by the \( J' \)'s and the complex three sphere by the real three sphere. The case of going from the \( SO(4, C) \) Barrett-Crane model to the Riemannian Barrett-Crane model is intuitive. It is a simple process of going from complex three sphere to its subspace the real three sphere.

2. \( n = 0 \) case: This uses \( \chi = (\rho, 0) \) representations only: This corresponds to \( X \) as a space-like hyperboloid (only one sheet) with \( G_X = SO(3, 1, R) \):
\[ x^2 + y^2 + z^2 - t^2 = 1. \]

The intertwiner now corresponds to the Lorentzian general relativity. This intertwiner was introduced in \( \text{[14]} \). The unitary representations of the Lorentz group on the real hyperboloid have been studied by Gelfand and Naimark \( \text{[19]} \), from which the \( T \)–functions are
\[ T_\rho(x)[\xi] = [\xi, x]^{\frac{1}{2}} \rho^{-1}, \]
\(^4\)By Kleinien hyperboloid I refer to the space described by \( x^2 + y^2 - z^2 - t^2 = 1 \) for real \( x, y, z \) and \( t \).
\(^5\)The real group is maximal in the sense that there is no other real topologically connected subgroup of \( SO(4, C) \) that is bigger.
where $\xi \in$ null cone intersecting $t = 1$ plane in the Minkowski space. Here $\xi$ replaces $(z_1, z_2)$ in the $T$–function $T_\chi(g(x))(z_1, z_2)$ of the $SO(4, C)$ Barrett–Crane Model. An element $g \in SO(3, 1)$ acts as a shift operator as follows:

$$gT_\rho(x)[\xi] = T_\rho(gx)[\xi] = T_\rho(x)[g^{-1}\xi].$$

This intertwiner was first introduced in [14].

3. Combination of $(0, n)$ and $(\rho, 0)$ representations: There are two possible models corresponding to this case. One of them has $X$ as the Kleinian hyperboloid defined by

$$x^2 + y^2 - z^2 - t^2 = 1,$$

with $G_X = SO(2, 2, R)$. Here the $X$ is isomorphic to $SU(1, 1) \approx SL(2, R)$. The intertwiner now corresponds to Kleinian general relativity $(+−−−$ signature). The $T$–functions are of the form $T_\chi(k(n))(z_1, z_2)$ where $z_1$ and $z_2$ takes real values only (please refer to appendix $C$), $\chi \neq 0$ and $k$ is an isomorphism from the Kleinian hyperboloid to $SU(1, 1)$ defined by

$$k(n) = \begin{bmatrix} x - iy & z - it \\ z + it & x + iy \end{bmatrix}.$$ 

The representations corresponding to the $n = 0$ and $\rho = 0$ cases are qualitatively different. The representations corresponding to $\rho \neq 0$ are called the continuous representations and those to $n \neq 0$ are called the discrete representations. The action of $g \in SO(2, 2, R)$ on the $T$–functions is

$$gT_\chi(k(x)) = T_\chi(k(g(x))),$$

where $g(x)$ is the result of action of $g$ on $x \in X$.

4. The second model using both $(0, n)$ and $(\rho, 0)$ representations: This corresponds to the time-like hyperboloid with $G_X = SO(3, 1)$,

$$x^2 - y^2 - z^2 - t^2 = 1,$$

where two vectors that differ just by a sign are identified as a single point of the space $X$. The corresponding spin foam model has been introduced by Barrett and Crane [11]. It has been derived using a field theory over group formalism by Rovelli and Perez [16]. Similar to the previous case, I have both continuous and discrete representations, with the $T$–functions given by

$$T_\rho(x)[\xi] = [\xi, x]^{*\rho-1},$$

$$T_n(x)[l(a, \xi)] = \exp(-2i\theta)\delta(a, \xi),$$

26
where the \(l(a, \xi)\) is an isotropic line\(^{16}\) on the imaginary hyperboloid along direction \(\xi\) going through a point \(a\) on the hyperboloid and the \(\theta\) is the distance between \(l(a, \xi)\) and \(l(x, \xi)\) given by \(\cos \theta = a \cdot x\), where the dot is the Lorentzian scalar product. I have for \(g \in SO(3, 1, R)\),

\[
gT_n(x)[l(a, \xi)] = T_n(x)[l(a, g\xi)] \\
= T_n(g^{-1}x)[l(a, \xi)],
\]

and the action of \(g\) on continuous representations are defined similar to equation \(^{20}\). The corresponding spin foam model has been introduced and investigated before by Rovelli and Perez \(^{16}\).

In the table below the representations and the homogenous spaces associated with various Barrett-Crane intertwiners (models) in four dimensions have been summarized.

| Model          | Representations | Homogenous Space          |
|---------------|----------------|---------------------------|
| \(SO(4, C)\) model | \(\chi_L = \pm \chi_R = \frac{n}{2} + i\rho\) | Complex three-sphere       |
| \(SO(4, R)\) model | \(\rho = 0\) Discrete irreps | Real three-sphere          |
| \(SO(3, 1)\) model | \(n = 0\) Continuous irreps | Space-Like Hyperboloid     |
| \(SO(3, 1)\) model | \(n = 0 \oplus \rho = 0\) | Time-Like Hyperboloid      |
| \(SO(2, 2)\) model | \(n = 0 \oplus \rho = 0\) | Kleinian Hyperboloid       |

From the table, it can be clearly seen that the representations used for the intertwiners for real general relativity are various possible combinations of representations of \(SL(2, C)\) simply restricted by the reality condition \(n\rho = 0\). Also all the homogenous spaces of the intertwiners of the real general relativity theories are simply the all possible real cross-sections (maximal) of the complex three sphere. From this point of view the intertwiners for the real general relativity theories listed in the table makes a complete set.

### 7.3 The Area Eigenvalues

Using the \(T\)–functions described above, the intertwiners for real general relativity can be constructed. Using these intertwiners, spin foam models (Barrett-Crane) for the real general relativity theories of the various different signatures can be constructed. The square of the area of a triangle of a four-simplex for all signatures associated with a representation \(\chi\) is described by the same formula\(^{17}\),

\[
\eta_{IK}\eta_{JL}\hat{B}^{IJ}\hat{B}^{KL} = (\chi^2 - 1) \hat{I} \\
= \left(\frac{n^2}{2} - \rho^2 - 1\right) \hat{I},
\]

\(^{16}\)A line on an imaginary hyperboloid \(^{19}\) is the intersection of a 2-plane of the Minkowski space with it. The line is called isotropic if the Lorentzian distance between any two points on it is zero. An isotropic line \(l\) is described by the equation \(x = s\xi + x_0\), \(x\) is the variable point on \(l\), \(x_0\) is any fixed point on \(l\), and \(\xi\) is a null-vector. For more information please refer to \(^{19}\).

\(^{17}\)Please refer to the end of appendix C regarding the differences between the Casimers of \(SL(2, C)\) and \(SU(1, 1)\).
where only of \(n\) and \(\rho\) is non-zero. The square of the area is negative or positive depending on whether \(\rho\) or \(n\) is non-zero. The negative (positive) sign corresponds to a time-like (space-like) area.

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A Unitary Representations of \(SL(2,\mathbb{C})\)
The Representation theory of \(SL(2,\mathbb{C})\) was developed by Gelfand and Naimarck [12]. Representation theory of \(SL(2,\mathbb{C})\) can be developed using functions on \(\mathbb{C}^2\) which are homogenous in their arguments\(^{18}\). The space of functions \(D_\chi\) is defined as functions \(f(z_1, z_2)\) on \(\mathbb{C}^2\) whose homogeneity is described by

\[
f(a z_1, a z_2) = a^{\chi_1 - 1} a^{\chi_2 - 1} f(z_1, z_2),
\]

for all \(a \neq 0\), where \(\chi\) is a pair \((\chi_1, \chi_2)\). The linear action of \(SL(2,\mathbb{C})\) on \(\mathbb{C}^2\) defines a representation of \(SL(2,\mathbb{C})\) denoted by \(T_\chi\). Because of the homogeneity of functions of \(D_\chi\), the representations \(T_\chi\) can be defined by its action on the functions \(\phi(z)\) of one complex variable related to \(f(z_1, z_2) \in D_\chi\) by

\[
\phi(z) = f(z, 1).
\]

There are two qualitatively different unitary representations of \(SL(2,\mathbb{C})\): the principal series and the supplementary series, of which only the first one is relevant to quantum general relativity. The principal unitary irreducible representations of \(SL(2,\mathbb{C})\) are the infinite dimensional. For these \(\chi_1 = -\bar{\chi}_2 = \frac{n+i\rho}{2}\), where \(n\) is an integer and \(\rho\) is a real number. In this article I would like to label the representations by a single complex number \(\chi = \frac{n}{2} + i\frac{\rho}{2}\), wherever necessary. The \(T_\chi\) representations are equivalent to \(T_{-\chi}\) representations [19].

Let \(g\) be an element of \(SL(2,\mathbb{C})\) given by

\[
g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},
\]

where \(\alpha, \beta, \gamma\) and \(\delta\) are complex numbers such that \(\alpha \delta - \beta \gamma = 1\). Then the \(D_\chi\) representations are described by the action of a unitary operator \(T_\chi(g)\) on the square integrable functions \(\phi(z)\) of a complex variable \(z\) as given below:

\[
T_\chi(g)\phi(z) = (\beta z_1 + \delta)^{\chi_1 - 1} (\bar{\beta} \bar{z}_1 + \bar{\delta})^{-\bar{\chi}_1 - 1} \phi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right),
\]

(27)

This action on \(\phi(z)\) is unitary under the inner product defined by

\[
(\phi(z), \eta(z)) = \int \phi(z)\eta(z)d^2z,
\]

\(^{18}\)These functions need not be holomorphic but infinitely differentiable may be except at the origin \((0, 0)\).
where $d^2z = \frac{i}{2} dz \wedge d\bar{z}$ and I would like to adopt this convention everywhere. Completing $D_\chi$ with the norm defined by the inner product makes it into a Hilbert space $H_\chi$.

Equation (27) can also be written in kernel form [15],

$$T_\chi(g)\phi(z_1) = \int T_\chi(g)(z_1, z_2)\phi(z_2) d^2 z_2,$$

Here $T_\chi(g)(z_1, z_2)$ is defined as

$$T_\chi(g)(z_1, z_2) = (\beta z_1 + \delta)\chi^{-1}(\bar{\beta} \bar{z}_1 + \bar{\delta})^{-1}\delta(z_2 - g(z_1)),$$  \hspace{1cm} (28)

where $g(z_1) = \frac{az_1 + \gamma}{\beta z_1 + \delta}$. The Kernel $T_\chi(g)(z_1, z_2)$ is the analog of the matrix representation of the finite dimensional unitary representations of compact groups. An infinitesimal group element, $a$, of $SL(2, \mathbb{C})$ can be parameterized by six real numbers $\varepsilon_k$ and $\eta_k$ as follows [40]:

$$a \approx I + \frac{i}{2} \sum_{k=1}^{3} (\varepsilon_k \sigma_k + \eta_k i \sigma_k),$$

where the $\sigma_k$ are the Pauli matrices. The corresponding six generators of the $\chi$ representations are the $H_k$ and the $F_k$. The $H_k$ correspond to rotations and the $F_k$ correspond to boosts. The bi-invariant measure on $SL(2, \mathbb{C})$ is given by

$$dg = \left(\frac{i}{2}\right)^3 \frac{d^2 \beta d^2 \gamma d^2 \delta}{|\delta|^2} = \left(\frac{i}{2}\right)^3 \frac{d^2 \alpha d^2 \beta d^2 \gamma}{|\alpha|^2}.$$ 

This measure is also invariant under inversion in $SL(2, \mathbb{C})$. The Casimir operators for $SL(2, \mathbb{C})$ are given by

$$\hat{\mathcal{C}} = \det \begin{bmatrix} \hat{X}_3 & \hat{X}_1 - i \hat{X}_2 \\ \hat{X}_1 + i \hat{X}_2 & -\hat{X}_3 \end{bmatrix}$$

and its complex conjugate $\bar{\mathcal{C}}$ where $X_i = F_i + iH_i$. The action of $C$ ($\hat{C}$) on the elements of $D_\chi$ reduces to multiplication by $\chi^2 - 1$ ($\chi_2^2 - 1$). The real and imaginary parts of $C$ are another way of writing the Casimirs. On $D_\chi$ they reduce to the following

$$\text{Re}(\hat{C}) = \left(-\rho^2 + \frac{n^2}{4} - 1\right) \hat{I},$$

$$\text{Im}(\hat{C}) = \rho n \hat{I}.$$ 

The Fourier transform theory on $SL(2, \mathbb{C})$ was developed in Ref. [19]. If $f(g)$ is a square integrable function on the group, it has a group Fourier transform defined by

$$F(\chi) = \int f(g) T_\chi(g) dg,$$  \hspace{1cm} (29)
where is $F(\chi)$ is linear operator defined by the kernel $K_\chi(z_1, z_2)$ as follows:

$$F(\chi)\phi(z) = \int K_\chi(z, \bar{z})\phi(\bar{z})d^2\bar{z}.$$ 

The associated inverse Fourier transform is

$$f(g) = \frac{1}{8\pi^4} \int \text{Tr}(F(\chi)T_\chi(g^{-1}))\chi\bar{\chi}d\chi,$$  

(30)

where the $\int d\chi$ indicates the integration over $\rho$ and the summation over $n$. From the expressions for the Fourier transforms, I can derive the orthonormality property of the $T_\chi$ representations,

$$\int_{SL(2, \mathbb{C})} T^*_{\chi_1}(g)T_{\chi_2}(g)dg = \frac{8\pi^4}{\chi_1\chi_1}\delta(\chi_1 - \chi_2)\delta(z_1 - \bar{z}_1)\delta(z_2 - \bar{z}_2),$$

where $T^*_\chi$ is the Hermitian conjugate of $T_\chi$.

The Fourier analysis on $SL(2, \mathbb{C})$ can be used to study the Fourier analysis on the complex three sphere $CS^3$. If $x = (a, b, c, d) \in CS^3$ then the isomorphism $g : CS^3 \rightarrow SL(2, \mathbb{C})$ can be defined by the following:

$$g(x) = \begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}.$$

Then, the Fourier expansion of $f(x) \in L^2(CS^3)$ is given by

$$f(x) = \frac{1}{8\pi^4} \int \text{Tr}(F(\chi)T_\chi(g(x)^{-1}))\chi\bar{\chi}dx$$

and its inverse is

$$F(\chi) = \int f(g)T_\chi(g(x))dx,$$

where the $dx$ is the measure on $CS^3$. The measure $dx$ is equal to the bi-invariant measure on $SL(2, \mathbb{C})$ under the isomorphism $g$.

The expansion of the delta function on $SL(2, \mathbb{C})$ from equation (30) is

$$\delta(g) = \frac{1}{8\pi^4} \int \text{tr} [T_\chi(g)] \chi\bar{\chi}d\chi.$$  

(31)

Let me calculate the trace $\text{tr} [T_\chi(g)]$. If $\lambda = e^{i\rho+i\theta}$ and $\frac{1}{\lambda}$ are the eigen values of $g$ then

$$\text{tr} [T_\chi(g)] = \frac{\lambda^{\chi_1}\bar{\lambda}^{\chi_2} + \lambda^{-\chi_1}\bar{\lambda}^{-\chi_2}}{|\lambda - \lambda^{-1}|^2},$$

which is to be understood in the sense of distributions [19]. The trace can be explicitly calculated as

$$\text{tr} [T_\chi(g)] = \frac{\cos(\eta \rho + n\theta)}{2|\sinh(\eta + i\theta)|^2}.$$  

(32)
Therefore, the expression for the delta on $SL(2, C)$ explicitly is

$$
\delta(g) = \frac{1}{8\pi^2} \sum_n \int d\rho (n^2 + \rho^2) \frac{\cos(\rho \eta + n\theta)}{|\sinh(\eta + i\theta)|^2}. \quad (33)
$$

Let us consider the integrand in equation (30). Using equation (29) in it we have

$$
Tr(F(\chi)T_\chi(g^{-1}))\bar{\chi}\bar{\chi} = \chi\bar{\chi} \int f(\hat{g})Tr(T_\chi(\hat{g})T_{\chi}(g^{-1}))d\hat{g}
\quad = \chi\bar{\chi} \int f(\hat{g})Tr(T_{\chi}(\hat{gg}^{-1}))d\hat{g}. \quad (34)
$$

But, since the trace is insensitive to an overall sign of $\chi$, so are the terms of the Fourier expansion of the $L^2$ functions on $SL(2, C)$ and $CS^3$.

**B Unitary Representations of $SO(4, C)$**

The group $SO(4, C)$ is related to its universal covering group $SL(2, C) \times SL(2, C)$ by the relationship $SO(4, C) \approx \frac{SL(2, C) \times SL(2, C)}{Z_2}$. The map from $SO(4, C)$ to $SL(2, C) \times SL(2, C)$ is given by the isomorphism between complex four vectors and $GL(2, C)$ matrices. If $X = (a, b, c, d)$ then $G : C^4 \rightarrow GL(2, C)$ can be defined by the following:

$$
G(X) = \left[ \begin{array}{cc} a + ib & c + id \\ -c + id & a - ib \end{array} \right].
$$

It can be easily inferred that $\det G(X) = a^2 + b^2 + c^2 + d^2$ is the Euclidean norm of the vector $X$. Then, in general a $SO(4, C)$ rotation of a vector $X$ to another vector $Y$ is given in terms of two arbitrary $SL(2, C)$ matrices $g_L \ A \ B$, $g_R \ A' \ B' \in SL(2, C)$ by

$$
G(Y)^{AA'} = g_L \ A \ B \ g_R \ A' \ B' \ G^{AB}(X),
$$

where $G^{AB}(X)$ is the matrix elements of $G(X)$. The above transformation does not differentiate between $(L_A^R, R_A^B)$ and $(-L_A^R, -R_A^B)$ which is responsible for the factor $Z_2$ in $SO(4, C) \approx \frac{SL(2, C) \times SL(2, C)}{Z_2}$.

The unitary representation theory of the group $SL(2, C) \times SL(2, C)$ is easily obtained by taking the tensor products of two Gelfand-Naimarck representations of $SL(2, C)$. The Fourier expansion for any function $f(g_L, g_R)$ of the universal cover is given by

$$
f(g_L, g_R) = \frac{1}{64\pi^8} \int \chi_L \bar{\chi}_L \chi_R \bar{\chi}_R F(\chi_L, \chi_R) T_\chi(g_L^{-1})T_{\chi}(g_R^{-1})d\chi_L d\chi_R,
$$

where $\chi_L = \frac{\eta_L + i\rho_L}{2}$ and $\chi_R = \frac{\eta_R + i\rho_R}{2}$. The Fourier expansion on $SO(4, C)$ is given by reducing the above expansion such that $f(g_L, g_R) = f(-g_L, -g_R)$. From equation (32) I have

$$
tr[T_\chi(-g)] = (-1)^n tr[T_\chi(-g)],
$$

31
where \( \chi = \frac{n+i\rho}{2} \). Therefore

\[
f(-g_L,-g_R) = \frac{1}{8\pi^4} \int \chi_L \bar{\chi}_L \chi_R \bar{\chi}_R F(\chi_L,\chi_R)(-1)^{n_L+n_R} T_x(g_L^{-1}) T_x(g_R^{-1}) d\chi_L d\chi_R.
\]

This implies that for \( f(g_L,g_R) = f(-g_L,-g_R) \), I must have \((-1)^{n_L+n_R} = 1\). From this, I can infer that the representation theory of \( SO(4,C) \) is deduced from the representation theory of \( SL(2,C) \times SL(2,C) \) by restricting \( n_L+n_R \) to be even integers. This means that \( n_L \) and \( n_R \) should be either both odd numbers or even numbers. I would like to denote the pair \((\chi_L,\chi_R) (n_L+n_R \text{ even})\) by \( \omega \).

There are two Casimir operators available for \( SO(4,C) \), namely \( \varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} \) and \( \eta_{IK} \eta_{JL} \hat{B}^{IJ} \hat{B}^{KL} \). The elements of the representation space \( D_{\chi_L} \otimes D_{\chi_R} \) are the eigen states of the Casimirs. On them, the operators reduce to the following:

\[
\varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 - \chi_R^2}{2} \quad \text{and} \quad \eta_{IK} \eta_{JL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 + \chi_R^2 - 2}{2}.
\]

(C) **Unitary Representations of \( SU(1,1) \)**

The unitary representations of \( SU(1,1) \approx SL(2,R) \), given in Ref:[41], is defined similar to that of \( SL(2,C) \). The main difference is that the \( D_\chi \) are now functions \( \phi(z) \) on \( C^1 \). The representations are indicated by a pair \( \chi = (\tau, \varepsilon) \), \( \varepsilon \) is the parity of the functions (\( \varepsilon \) is 0 for even functions and \( \frac{1}{2} \) for odd functions) and \( \tau \) is a complex number defining the homogeneity:

\[
\phi(az) = |a|^{2\tau} \text{sgn}(a)^{2\varepsilon} \phi(z),
\]

where \( a \) is a real number. Because of homogeneity the \( D_\chi \) functions can be related to the infinitely differentiable functions \( \phi(e^{i\theta}) \) on \( S^1 \) where \( \theta \) is the coordinate on \( S^1 \). The representations are defined by

\[
T_\chi(g) \phi(e^{i\theta}) = (\beta e^{i\theta} + \bar{\alpha})^{\tau+\varepsilon} (\bar{\beta} e^{-i\theta} + \alpha)^{\tau-\varepsilon} \phi(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}).
\]

There are two types of the unitary representations that are relevant for quantum general relativity: the continuous series and the discrete series. For the continuous series \( \chi = (i\rho - \frac{1}{2}, \varepsilon) \), where \( \rho \) is a non-zero real number. Let me denote the continuous series representations with suffix or prefix \( c \), for example \( T^c_\chi \).

There are two types of discrete series representations which are indicated by signs \( \pm \). They have their respective homogeneity as \( \chi_{\pm} = (l, \varepsilon_l^\pm) \) where \( \varepsilon_l^\pm = \pm 1 \) is defined by the condition \( l \pm \varepsilon_l^\pm \) is an integer. Let me denote
the representations as $T^+_l$ and $T^-_l$. The $T^+_l$ ($T^-_l$) representations can be re-expressed as linear operators on the functions $\phi_+(z)(\phi_-(z))$ on $C^1$ that are analytical inside (outside) the unit circle. The $T^+_l(g)$ are defined as

$$T^+_l(g)\phi_\pm(z) = |\beta z + \bar{\alpha}|^{2l} \phi_\pm\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right).$$

The inner products are defined by

$$(f_1, f_2)_c = \frac{1}{2\pi} \int_0^{2\pi} f_1(e^{i\theta})\overline{f_2(e^{i\theta})} d\theta,$$

$$(f_1, f_2)^+_l = \frac{1}{\Gamma(-2l - 1)} \int_{|z|<1} (1 - |z|)^{-2l - 2} f_1(z) f_2(z) \frac{dz d\bar{z}}{2\pi i},$$

$$(f_1, f_2)^-_l = \frac{1}{\Gamma(-2l - 1)} \int_{|z|>1} (1 - |z|)^{-2l - 2} f_1(z) f_2(z) \frac{dz d\bar{z}}{2\pi i}.$$

The Fourier transforms are defined for the unitary representations by

$$F_c(\chi) = \int f(g) T^c_\chi(g) dg,$$

$$F^+_l(l) = \int f(g) T^+_l(g) dg, \quad \text{and}$$

$$F^-_l(l) = \int f(g) T^-_l(g) dg,$$

where $dg$ is the bi-invariant measure on the group.

The inverse Fourier transform is defined by

$$f(g) = \frac{1}{4\pi^2} \left\{ \sum_{l \in \mathbb{Z}} \sum_{\varepsilon} \frac{1}{\rho Tr[F_\rho(\chi)T^c_\rho]} \left[ Tr[F(l)(T^+_l(g)) + F_-^l(l)(T^-_l(g))]\right] \right\}.$$

The $T_{(\tau, \varepsilon)}$ is equivalent to $T_{(-\tau-1, \varepsilon)}$. The Casimir operator for the $T_\chi$ representations (all) can be defined similar to $SU(2)$ and its eigen values are

$$C = \tau(\tau + 1),$$

where the $\tau$ comes from $\chi = (\tau, \varepsilon)$. The $\tau$ in this section is related to the $\chi$ in the representations of $SL(2, C)$ by $\chi = \tau + \frac{1}{2}$. The expressions for the Casimirs of the two groups differ by a factor of 4.

### D Reality Constraint for Arbitrary Metrics

Here we analyze the area metric reality constraint for a metric $g_{ac}$ of arbitrary rank in arbitrary dimensions, with the area metric defined as $A_{abcd} = g_{a[c}g_{d]}b$. Let the rank of $g_{ac}$ be $r$. 

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If the rank \( r = 1 \) then \( g_{ab} \) is of form \( \lambda_a \lambda_b \) for some complex non zero co-vector \( \lambda_a \). This implies that the area metric is zero and therefore not an interesting case.

Let me prove the following theorem.

**Theorem 3** If the rank \( r \) of \( g_{ac} \) is \( \geq 2 \), then the area metric reality constraint implies the metric is real or imaginary. If the rank \( r \) of \( g_{ac} \) is equal to 1, then the area metric reality constraint implies \( g_{ac} = \eta \alpha_a \alpha_b \) for some complex \( \eta \neq 0 \) and real non-zero co-vector \( \alpha_a \).

The area metric reality constraint implies
\[
g^{R}_{ac}g^{I}_{db} = g^{R}_{ad}g^{I}_{cb}. \tag{38}
\]
Let \( g_{AC} \) be a \( r \) by \( r \) submatrix of \( g_{ac} \) with a non zero determinant, where the capitalised indices are restricted to vary over the elements of \( g_{AC} \) only. Now we have
\[
g^{R}_{AC}g^{I}_{DB} = g^{R}_{AD}g^{I}_{CB}. \tag{39}
\]
From the definition of the determinant and the above equation we have
\[
\det(g_{AC}) = \det(g^{R}_{AC}) + \det(g^{I}_{AC}).
\]
Since \( \det(g_{AC}) \neq 0 \) we have either \( \det(g^{R}_{AC}) \) or \( \det(g^{I}_{AC}) \) not equal to zero. Let me assume \( g^{R}_{AC} \neq 0 \). Then contracting both the sides of equation (39) with the inverse of \( g^{R}_{AC} \) we find \( g^{I}_{DB} \) is zero. Now from equation (38) we have
\[
g^{R}_{AC}g^{I}_{db} = g^{R}_{ad}g^{I}_{CB} = 0. \tag{40}
\]
Since the Rank of \( g^{R}_{AC} \geq 2 \) we can always find a \( g^{R}_{AC} \neq 0 \) for some fixed \( A \) and \( C \). Using this in equation (40) we find \( g^{I}_{db} \) is zero. Now consider the following:
\[
g^{R}_{AC}g^{I}_{db} = g^{R}_{ad}g^{I}_{CB}. \tag{41}
\]
we can always find a \( g^{R}_{AC} \neq 0 \) for some fixed \( A \) and \( C \). Using this in equation (41) we find \( g^{I}_{db} = 0 \). So we have shown that if \( g^{R}_{ac} \neq 0 \) then \( g^{I}_{db} = 0 \). Similarly if we can show that if \( g^{I}_{ac} \neq 0 \) then \( g^{R}_{db} = 0 \).

**E Field Theory over Group and Homogenous Spaces.**

One of the problems with the Barrett-Crane model for general relativity is its dependence on the discretization of the manifold. A discretization independent model can be defined by summing over all possible discretizations. With a proper choice of amplitudes for the lower dimensional simplices the BF spin foams can be reformulated as a field theory over a group (GFT). Similarly, the Barrett-Crane models can be reformulated as a field theory over the homogenous space of the group. Consider a tetrahedron. Let a group element
Let a real field \( \phi(g_1, g_2, g_3, g_4) \) invariant under the exchange of its arguments be associated with the tetrahedron. Let the field be invariant under the simultaneous (left or right) action of a group element \( g \) on its variables. Then the kinetic term is defined as

\[
K.E = \int \prod_{i=1}^{4} dg_i \phi_i^2.
\]

To define the potential term, consider a four-simplex. Let \( g_i \), where \( i = 1 \) to 10 be the group elements associated with its ten triangles. With each tetrahedron \( e \) of the four-simplex, associate a \( \phi \) field which is a function of the group elements associated with its triangles. Denote it as \( \phi_e \). Then the potential term is defined as

\[
P.E = \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \prod_{e=1}^{5} \phi_e,
\]

where \( \lambda \) is an arbitrary constant.

Now the action for a GFT can be defined as

\[
S(\phi) = K.E + P.E = \int \prod_{i=1}^{4} dg_i \phi_i^2 + \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \prod_{e=1}^{5} \phi_e.
\]

The action has two terms, namely the kinetic term and the potential terms. The Partition function of the GFT is

\[
Z = \int D\phi e^{-S(\phi)}.
\]

Now, an analysis of this partition function yields the sum over spin foam partitions of the four dimensional BF theory for group \( G \) for all possible triangulations. From the analysis of the GFT we can easily show that this result is valid for \( G = SO(4, C) \) with the unitary representations defined in the appendix B.

Let us assume \( \phi \) is invariant only under the simultaneous action of an element of a subgroup \( H \) of \( G \). Then, if \( G = SO(4, R) \) and \( H = SU(2) \) we get GFTs for the Barrett-Crane model\(^{19} \) [24]. Similarly, if \( G = SL(2, C) \) and \( H = SU(2) \) or \( SU(1, 1) \), we can define GFT for the Lorentzian general relativity [10], [15]. The representation theories of \( SO(4, C) \) and \( SL(2, C) \) has similar structure to those of \( SO(4, R) \) and \( SU(2) \) respectively. So the GFT with \( G = SO(4, C) \) and \( H = SL(2, C) \) should yield the sum over triangulation formulation of the \( SO(4, C) \) Barrett-Crane model. The details of this analysis and its variations will be presented elsewhere.

\(^{19}\)Depending on whether we are using the left or right action of \( G \) on \( \phi \), we get two different models that differ by amplitudes for the lower dimensional simplices [24].
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